Semisimple Modules Relative to A Semiradical Property

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Abstract
In this paper, we introduce the concept of s.p-semisimple module. Let S be a semiradical property, we say that a module M is s.p-semisimple if for every submodule N of M, there exists a direct summand K of M such that K ≤ N and N / K has S. we prove that a module M is s.p-semisimple module if and only if for every submodule A of M, there exists a direct summand B of M such that A = B + C and C has S. Also, we prove that for a module M is s.p-semisimple if and only if for every submodule A of M, there exists an idempotent e ∈ End(M) such that e(M) ≤ A and (1 - e)(A) has S.

Keywords: Semiradical (radical) property, Semisimple modules, t-semisimple modules.

1. Introduction
Throughout this paper, all rings are associative with identity and all modules are unitary left R-modules. Let A be a submodule of a module M. A is called an essential submodule of M (denoted by A ≤ e M) if A ∩ B ≠ 0, ∀ 0 ≠ B ≤ M. A submodule B of M is called a closed submodule of M if B has no proper essential extension. A module M is called an extending module if every submodule of M is essential in a direct summand. Equivalently, every closed submodule of M is a direct summand, see [1], [2], [3].

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Let $M$ be a module. Recall that the socle of $M$ (denoted by $\text{Soc}(M)$) is the sum of all simple submodules of $M$, a module $M$ is called a semisimple if $\text{Soc}(M) = M$. Equivalently a module $M$ is semisimple if and only if every submodule is a direct summand of $M$, see [1], [4]. Recall that the Jacobson radical of $M$ (denoted by $J(M)$) is the intersection of all maximal submodules of $M$. If $M$ has no maximal submodule, we write $J(M) = M$, see [5].

Let $x \in M$. Recall that $\text{ann} \,(x) = \{ r \in R: rx = 0 \}$. For a module $M$, the singular submodule is defined as follows $Z(M) = \{ x \in M \mid \text{ann} \, x \leq e \, R \}$ or equivalently, $Ix = 0$ for some essential left ideal $I$ of $R$. If $Z(M) = M$, then $M$ is called a singular module. If $Z(M) = 0$, then $M$ is called a nonsingular module. The second singular (or Goldie torsion) submodule of a module $M$ (denoted by $Z_2(M)$) is defined by $Z(M / Z(M)) = Z_2(M) / Z(M)$, see [1], [6].

A submodule $A$ of a module $M$ is called $t$-essential submodule (denoted by $A \leq \text{tes} \, M$) if for any submodule $B$ of $M$, $A \cap B \leq Z_2(M)$ implies $B \leq Z_2(M)$. A module $M$ is called $t$-semisimple if for every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $K \leq \text{tes} \, N$, see [5], [7].

A property $S$ is called a radical property if:
1- for every module $M$, there exists a submodule (denoted by $S(M)$) such that $a$- $S(M)$ has $S$.
$b$- $A \leq S(M)$, for every submodule $A$ of $M$ such that $A$ has $S$.
2- If $f: M \rightarrow N$ is an epimorphism and $M$ has $S$, then $N$ has $S$.
3- $S(M / S(M)) = 0$ for every $R$-module $M$, see [8].

A property $S$ is called a semiradical property if it satisfies conditions 1 and 2, see [8].

It's known that each of the following two properties is a radical property, see [8].

1- $S = Z_2$. For a module $M$, $S(M) = Z_2(M)$, the second singular of $M$.
2- $S = \text{Snr}$. For a module $M$, $\text{Snr}(M)$ is a submodule of $M$ such that $a_1$- $J(\text{Snr}(M)) = \text{Snr}(M)$ (i.e. $\text{Snr}(M)$ has no maximal submodule).
$b_2$- $A \leq \text{Snr} \,(M)$, for every submodule $A$ of $M$ such that $J(A) = A$, see [8].

While each of the following two properties is a semiradical property (but it is not radical property), see [8].

1- $S = Z$. For a module $M$, $S(M) = Z \,(M)$, the singular submodule of $M$.
2- $S = \text{Soc}$. For a module $M$, $S(M) = \text{Soc}(M) = \sum_{A \text{ is simple}} A$.

Let $S$ be a semiradical property. It is known that
1- $M$ has $S$ if and only if $S(M) = M$.
2- $S(S(M)) = S(M)$.
3- If $M = \bigoplus_{i \in I} M_i$, then $S(M) = \bigoplus_{i \in I} S(M_i)$, where $I$ is any index set.
4- if $S(M) = 0$, then $S(A) = 0, \forall \, A \leq M$.
5- For any short exact sequence $0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$, if $S(M) = 0$ and $S(K) = 0$, then $S(N) = 0$, see [8].

In this paper, $S$ is a semiradical property, unless otherwise stated.
2- s.p - semisimple modules

In this section, we introduce the concept of s.p-semisimple modules and give the basic properties of this module. Also, we illustrate it with some examples.

**Definition 2.1.** Let S be a semiradical property. We say that a module M is s.p - semisimple module if for each submodule N of M, there exists a direct summand K of M such that K ≤ N and N / K has S.

**Remarks and Examples 2.2.**
1. Every semisimple module is s.p - semisimple. The converse is not true in general.
   **Proof.** Let N be a submodule of a semisimple module M, then N is a direct summand of M, by [4]. Let K = N, hence S(N / K) = S(N / N) = S(0) = 0 ≅ N / K. Thus M is s.p - semisimple. For example $\mathbb{Z}_6$ as $\mathbb{Z}_6$-module is s.p - semisimple module.

   For the converse, Let S = Second singularity. Consider module $\mathbb{Z}_4$ as $\mathbb{Z}$-module. Since $\mathbb{Z}_4$ is singular, then every submodules of $\mathbb{Z}_4$ is singular, by [1]. Therefore, $\mathbb{Z}_2(N) = \mathbb{Z}(N) = N$, ∀ N ≤ $\mathbb{Z}_4$. Let K = 0, hence $\mathbb{Z}_2(N / 0) \cong \mathbb{Z}_2(N) = \mathbb{Z}(N) = N \cong N / 0$. So N / 0 has S, ∀ N ≤ $\mathbb{Z}_4$. Thus $\mathbb{Z}_4$ is s.p - semisimple. Clearly that $\mathbb{Z}_4$ is not semisimple.

2. Let S be a hereditary property and M be a module. If M has S, then M is s.p - semisimple.
   **Proof.** Let N be a submodule of M and A ≤ N. Since M is s.p - semisimple, then there exists a direct summand K of M such that K ≤ A and A / K has S. By modular law, K is a direct summand of N. Thus N is s.p - semisimple.

3. Let S = singularity. Consider module Q as Z-module. Clearly, that Q is nonsingular. Hence, $\mathbb{Z}(Q) = 0$. Let N = 3Z. Since Q is indecomposable, then 0 is the only direct summand contained in 3Z. So S(3Z / 0 ) ≅ S(3Z) = $\mathbb{Z}(3Z) = 0$. Thus Q is not s.p - semisimple module.

**Proposition 2.3.** Every submodule of s.p - semisimple module M is s.p – semisimple, For every property S.
   **Proof.** Let N be a submodule of M and A ≤ N. Since M is s.p - semisimple, then there exists a direct summand K of M such that K ≤ A and A / K has S. By modular law, K is a direct summand of N. Thus N is s.p - semisimple.

**Proposition 2.4.** Let M be an indecomposable module and S be an assumed. Then M is s.p - semisimple if and only if every proper submodule of M has S.
   **Proof.** ⇒) Let N be a proper submodule of M. Since M is s.p - semisimple, then there exists a direct summand K of M such that K ≤ N and N / K has S. But M is an indecomposable. Therefore, K = 0. Hence S(N) ≅ S(N / 0) = S(N / K) = N / K = N / 0 ≅ N. Thus N has S.
   ⇐) Clear.

   Let S be a semiradical property. Recall that S is called a cohereditary property, if S(M) = 0 is closed under homomorphic images of M for every module M, see [8].

**Proposition 2.5.** Let S be a cohereditary property and let M be a module. If S(M) = 0. Then M is semisimple if and only if M is s.p - semisimple.
Proof. ⇒) Clear.

⇐) Let \( N \) be a submodule of \( M \). Since \( M \) is s.p-semisimple, then there exists a direct summand \( K \) of \( M \) such that \( K \leq N \) and \( N / K \) has \( S \). But \( S(M) = 0 \), therefore \( S(N) = 0 \), by [8]. Since \( S \) is cohereditary property, then \( S(N / K) = 0 \). Hence \( N = K \) is a direct summand of \( M \). Thus \( M \) is semisimple.

Remark 2.6. Let \( S \) be a hereditary property and \( M \) be a module. If \( S(M) = M \), then \( M / N \) is s.p-semisimple module, for each submodule \( N \) of \( M \).

Proof. Let \( N \) be a submodule of \( M \) and \( S(M) = M \), then \( M / N \) has \( S \), by [8]. Thus by 2.2-2, \( M / N \) is s.p-semisimple module.

Proposition 2.7. Let \( M \) be s.p-semisimple module. Then every submodule \( N \) of \( M \) such that \( S(N) = 0 \) is a direct summand of \( M \). The converse is true if \( S(M) = 0 \).

Proof. Assume that \( N \) is a submodule of \( M \) such that \( S(N) = 0 \). Then there exists a direct summand \( K \) of \( M \) such that \( K \leq N \) and \( N / K \) has \( S \). Let \( M = K \oplus K_1 \), for some submodule \( K_1 \) of \( M \). By modular law, \( N = K \oplus (N \cap K_1) \). Since \( N \cap K_1 \leq N \) and \( S(N) = 0 \), then \( S(N \cap K_1) = 0 \), by [8]. Since \( N / K = (K \oplus (N \cap K_1)) / K \cong (N \cap K_1) / 0 \cong N \cap K_1 \), by the second isomorphism theorem, then \( S(N / K) = 0 \). But \( S(N / K) = N / K \), therefore \( N / K = 0 \). Thus \( N = K \) is a direct summand of \( M \).

Conversely, let \( S(M) = 0 \) and \( N \) be a submodule of \( M \). Then \( S(N) = 0 \), by [8]. By our assumption \( N \), is a direct summand of \( M \). Therefore \( M \) is semisimple. Thus by 2.2-1, \( M \) is s.p-semisimple module.

Proposition 2.8. Let \( M = A + S(M) \) be s.p-semisimple module. Then there exists a direct summand \( B \) of \( M \) such that \( B \leq A \), \( M = B + S(M) \) and \( A / B \) has \( S \).

Proof. Assume that \( M \) is s.p-semisimple module. Then there exists a direct summand \( B \) of \( M \) such that \( B \leq A \) and \( A / B \) has \( S \). Let \( M = B \oplus C \), for some submodule \( C \) of \( M \). Then \( A = B \oplus (C \cap A) \), by modular law. But \( A / B \cong (C \cap A) \), by the second isomorphism theorem, therefore \( (C \cap A) \) has \( S \). Since \( (C \cap A) \) has \( S \), then \( (C \cap A) \leq S(M) \). Thus \( M = A + S(M) = B + (C \cap A) + S(M) \) and hence \( M = B + S(M) \).

Proposition 2.9. Let \( S \) be a hereditary property and \( M = M_1 \oplus M_2 \) be a module such that \( M_1 \) has \( S \) and \( M_2 \) is semisimple. Then \( M \) is s.p-semisimple module.

Proof. Let \( N \) be a submodule of \( M \). Since \( M_2 \) is semisimple, then \( N \cap M_2 \) is a direct summand of \( M_2 \). But, \( M_2 \) is a direct summand of \( M \), therefore \( N \cap M_2 \) is a direct summand of \( M \). By the second isomorphism theorem, \( M / M_2 = (M_1 \oplus M_2) / M_2 \cong M_1 \). Since \( M_1 \) has \( S \), then \( M / M_2 \) has \( S \). But \( N / (N \cap M_2) \cong (N + M_2) / M_2 \leq M / M_2 \) and \( S \) hereditary property. So \( N / (N \cap M_2) \) has \( S \). Thus \( M \) is s.p-semisimple module.

Corollary 2.10. Let \( S \) be a hereditary property and \( M \) be a module. If \( M = S(M) \oplus M_1 \), where \( M_1 \) is a semisimple module, then \( M \) is s.p-semisimple module.
Proof. Clear.

**Proposition 2.11.** Let \( M = M_1 \oplus M_2 \) be a module such that \( R = \text{Ann}(M_1) + \text{Ann}(M_2) \). If \( M_1 \) and \( M_2 \) are s.p - semisimple modules, then \( M \) is s.p - semisimple module.

**Proof.** Let \( N \) be a submodule of \( M = M_1 \oplus M_2 \). Since \( R = \text{Ann}(M_1) + \text{Ann}(M_2) \), then by the same argument of the proof [9, prop.4.2, CH.1], \( N = N_1 \oplus N_2 \), where \( N_1 \leq M_1 \) and \( N_2 \leq M_2 \). Since \( M_i \) is s.p - semisimple for \( i = 1, 2 \), then there exist direct summands \( K_i \) of \( M_i \) such that \( K_i \) is a submodule of \( N_i \) and \( N_i / K_i \) has S (\( i = 1, 2 \)). Let \( M_i = K_i \oplus L_i \), for some submodule \( L_i \) of \( M_i \). Therefore \( M = M_1 \oplus M_2 = (K_1 \oplus L_1) \oplus (K_2 \oplus L_2) = (K_1 \oplus K_2) \oplus (L_1 \oplus L_2) \). Hence \((K_1 \oplus K_2) \oplus (L_1 \oplus L_2) \) is a direct summand of \( M \) and \( (K_1 \oplus K_2) \leq N_1 \oplus N_2 = N \). Now since \( N_i / K_i \) has S \( (i = 1, 2) \), then by [8], \((N_1 / K_1) \oplus (N_2 / K_2) \) has S. But \((N_1 / K_1) \oplus (N_2 / K_2) \) is \((N_1 \oplus N_2) / (K_1 \oplus K_2) \), by [10, p. 33], hence \((N_1 \oplus N_2) / (K_1 \oplus K_2) \) has S. Thus \( M \) is s.p - semisimple module.

Let \( M \) be an R- module. Recall that \( M \) is called a duo-module if every submodule of \( M \) is fully invariant, see [11].

**Proposition 2.12.** Let \( M = \bigoplus_{i \in I} M_i \) be a duo module. Then \( M \) is s.p - semisimple modules if and only if for \( M_i \) is s.p - semisimple module \( \forall \ i \in I \).

**Proof.** Since \( M \) is s.p - semisimple, then by prop.2.3, \( M_i \) is s.p - semisimple, \( \forall \ i \in I \). Conversely, let \( M = \bigoplus_{i \in I} M_i \) be a module such that \( M_i \) is s.p - semisimple, \( \forall \ i \in I \). Let \( N \leq M \), then \( N = N \cap M = N \cap (\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} (N \cap M_i) \), by [12,lem.2.1]. Let \( N_i = N \cap M_i \), \( \forall \ i \in I \), then \( N_i \leq M_i \), \( \forall \ i \in I \). Since \( M_i \) is s.p - semisimple, then there exists \( K_i \) a direct summand of \( M_i \) such that \( K_i \) is a submodule of \( N_i \) and \( N_i / K_i \) has S \( \forall \ i \in I \). Hence \((\bigoplus_{i \in I} N_i) / (\bigoplus_{i \in I} K_i) \) is \( \bigoplus_{i \in I} (N_i / K_i) \) has S, by [10]. Thus \( M = \bigoplus_{i \in I} M_i \) is s.p - semisimple.

Let \( M_1 \) and \( M_2 \) be R- modules. \( M_1 \) is called \( M_2 \)-projective if for every submodule \( N \) of \( M_2 \) and any homomorphism \( f : M_1 \to M_2 / N \), there is a homomorphism \( g : M_1 \to M_2 \) such that \( \pi \circ g = f \). where \( \pi : M_2 \to M_2 / N \) is the natural epimorphism, see [13].

\[
\begin{array}{c}
M_1 \\
\downarrow g \\
M_2 \\
\downarrow \pi \\
M_2 / N \\
\end{array}
\]

\[
\begin{array}{c}
\Rightarrow f \\
\Rightarrow 0 \\
\end{array}
\]

\( M_1 \) and \( M_2 \) are called relatively projective if \( M_1 \) is \( M_2 \)-projective and \( M_2 \) is \( M_1 \)-projective.

We know that for a module \( M = A \oplus B \). \( A \) is B-projective if and only if for every submodule \( C \) of \( M \) such that \( M = B + C \), there exists a submodule \( D \) of \( C \) such that \( M = B \oplus D \), see [14].

**Proposition 2.13.** Let \( S \) be a hereditary property. Let \( M_1 \) and \( M_2 \) be s.p - semisimple modules such that \( M_1 \) and \( M_2 \) are relative projective. Then \( M = M_1 \oplus M_2 \) is s.p - semisimple.
Let \( N \) be a submodule of \( M \). Since \((N + M_2) \cap M_1 \leq M_1\) and \( M_1 \) is s.p - semisimple, then there exists a direct summand \( A_1 \) of \( M \) such that \( A_1 \leq (N + M_2) \cap M_1 \) and \(((N + M_2) \cap M_1) / A_1\) has \( S \). Let \( M_1 = A_1 \oplus B_1 \), for some submodule \( B_1 \) of \( M_1 \). Hence \((N + M_2) \cap M_1 = A_1 \oplus (N + M_2) \cap M_1 \cap B_1\), by modular law. Since by the second isomorphism theorem, \((N + M_2) \cap M_1) / A_1 \cong (N + M_2) \cap M_1 / B_1\), then \((N + M_2) \cap B_1\) has \( S \), by [8]. Therefore \( M = M_1 \oplus M_2 = A_1 \oplus B_1 \oplus M_2 = (N + M_2) \cap M_1 \cap B_1 + M_2 = N + M_2 + B_1 + M_2 = N + (M_2 \oplus B_1)\). Since \((N + B_1) \cap M_2 \leq M_2\) and \( M_2\) is s.p - semisimple, then there exists a direct summand \( A_2 \) of \( M_2 \) such that \( A_2 \leq (N + B_1) \cap M_2\) and \(((N + B_1) \cap M_2) / A_2\) has \( S \). Let \( M_2 = A_2 \oplus B_2\), for some submodule \( B_2 \) of \( M_2\) then \((N + B_1) \cap M_2 = A_2 \oplus ((N + B_1) \cap M_2) \cap B_2\), by modular law. By the second isomorphism theorem, \(((N + B_1) \cap M_2) / A_2 \cong ((N + B_1) \cap M_2) \cap B_2\), then \((N + B_1) \cap M_2 \cap B_2 = (N + B_1) \cap B_2\) has \( S \), by [8]. Thus \( M = N + (M_2 \oplus B_1) = N + A_2 + B_2 + B_1 = N + (B_1 \oplus B_2)\). Since \( M = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)\) and \( M_1\) and \( M_2\) are relative projective, then \( A_1\) is \( B_j\) - projective and \( A_2\) is \( B_j\) - projective for \( j = 1, 2\), by [9, prop. 2.1.6]. So by [15, prop.2.1.7], \( A_1\) is \( B_1 \oplus B_2\)-projective and \( A_2\) is \( B_1 \oplus B_2\) - projective. Hence \( A_1 \oplus A_2\) is \( B_1 \oplus B_2\)-projective, by [15,prop.2.1.6]. Hence, there exists \( X \leq N\) such that \( M = X \oplus B_1 \oplus B_2\), by [14, lem. 5].

Now, we want to show that \( N \cap (B_1 \oplus B_2)\) has \( S\). Since \((N+M_2) \cap B_1 = ((N + (A_2 \oplus B_2)) \cap B_1\) has \( S\) and \((N + B_2) \cap B_1 \leq ((N + (A_2 \oplus B_2)) \cap B_1\), then \((N + B_2) \cap B_1\) has \( S\). Since \((N + B_1) \cap B_2\) has \( S\), then \((N \oplus B_2) \cap B_1 \oplus (N \oplus B_1) \cap B_2\) has \( S\), by [8]. But by [15,lem.3.2], \(N \cap (B_1 \oplus B_2) \leq (N \oplus B_2) \cap B_1 \oplus (N \oplus B_1) \cap B_2\). Therefore, \(N \cap (B_1 \oplus B_2)\) has \( S\). Thus \( M\) is s.p - semisimple module.

Let \( M\) be an \( R\)-module. \( M\) is said to have the summand intersection property (briefly SIP) if the intersection of any two direct summands of \( M\) is a direct summand of \( M\), see [16].

**Proposition 2.14.** Let \( M\) be s.p - semisimple module. If for any two direct summand \( A\) and \( B\) of \( M\), \( S(A \cap B) = 0\), then \( M\) has SIP.

**Proof.** Let \( A\) and \( B\) be direct summands of \( M\). Since \( M\) is s.p - semisimple, then there exists a direct summand \( N\) of \( M\) such that \( N \leq A \cap B\) and \((A \cap B) / N\) has \( S\). Let \( M = N \oplus N_1\), for some submodule \( N_1\) of \( M\), then \( A \cap B = N \oplus (N_1 \cap (A \cap B))\). Hence by the second isomorphism theorem, \((A \cap B) / N = [N \oplus (N_1 \cap (A \cap B))] / N \cong N_1 \cap (A \cap B) / A \cap B\). Since \( S(A \cap B) = 0\), then \( S(N_1 \cap (A \cap B)) = 0\), by [8]. So \( S(A \cap B) / N = 0\). But \((A \cap B) / N\) has \( S\), therefore \( A \cap B = N\). Hence \( A \cap B\) is a direct summand of \( M\). Thus \( M\) has SIP.

Let \( R\) be an integral domain. Recall that an \( R\)- module \( M\) is called a torsion free module if \( \text{ann}(x) = 0\), for all \( x \neq 0 \in M\), see [1].

**Theorem 2.15.** Let \( R\) be an integral domain and \( M\) be a torsion free module and s.p - semisimple module. Then for every \( m \in M\), either \( Rm\) is a direct summand of \( M\) or \( Rm\) has \( S\).

**Proof.** Let \( 0 \neq m \in M\). Then there exists a direct summand \( K\) of \( M\) such that \( K \leq Rm\) and \( Rm / K\) has \( S\). Let \( M = K \oplus H\), for some submodule \( H\) of \( M\). Then \( Rm = K \oplus (Rm \cap H)\), by modular law. But \( Rm / K \cong Rm \cap H\), by the second isomorphism theorem. Therefore \( Rm \cap H\) has \( S\).

Let \( : R \to Rm\) be a map defined by \( f(r) = rm\), for each \( r \in R\). It is easy to see that \( f\) is an epimorphism and \( \text{Ker}(f) = \text{ann}(m)\). By the first isomorphism theorem, \( R / \text{ann}(m) \cong Rm\). Since \( M\) is torsion free module, then \( \text{ann}(m) = 0\). Thus \( R \cong Rm\). But \( R\) is indecomposable.
Therefore, $Rm$ is indecomposable. Implies that either $Rm = K$ or $Rm = Rm \cap H$. Thus either $Rm$ is a direct summand of $M$ or $Rm$ has $S$.

**Proposition 2.16.** Let $R$ be an indecomposable ring and $M$ be a projective module. If $M$ is s.p-semisimple module, then for every $m \in M$, either $Rm$ is a direct summand of $M$ or $Rm$ has $S$.

**Proof.** Assume that $M$ is a projective and s.p-semisimple module and let $m \in M$. Then there exists a direct summand $K$ of $M$ such that $K \leq Rm$ and $Rm / K$ has $S$. Let $M = K \oplus H$ for some submodule $H$ of $M$, then $Rm = K \oplus (H \cap Rm)$, by modular law. But $Rm / K \cong H \cap Rm$, by the second isomorphism theorem. Therefore, $H \cap Rm$ has $S$.

Now, let $f: R \to Rm$ be a map defined by $f(r) = rm$, for all $r \in R$. It is clear that $f$ is an epimorphism map. Let $P: Rm \to K$ be the projection map. Clearly, $Pof: R \to K$ is an epimorphism. Since $M$ is projective, then $K$ is projective by [4]. Therefore, $\text{Ker}(Pof) = 0$ or $\text{Ker}(Pof) = R$. $\text{Ker}(Pof) = f^{-1}(Rm \cap H) = f^{-1}(Rm \cap H)$. So either $Rm \cap H = 0$ or $Rm \cap H = R$. Thus $Rm = K$ or $Rm \cap H = Rm$ has $S$.

3- **Characterization of s.p-semisimple Modules**

In this section, we give various characterizations of s.p-semisimple modules.

We start with the following theorem.

**Theorem 3.1.** Let $M$ be a module. Then the following statements are equivalent

1- $M$ is s.p-semisimple module.

2- For every submodule $A$ of $M$, there exists a decomposition $M = B \oplus C$ such that $B \leq A$ and $A \cap C$ has $S$.

3- For every submodule $A$ of $M$, $A = A_1 \oplus A_2$, where $A_1$ is a direct summand of $M$ and $A_2$ has $S$.

**Proof.**

1$\Rightarrow$2) Let $A$ be a submodule of $M$. Since $M$ is s.p-semisimple, then there exists a direct summand $B$ of $M$ such that $B \leq A$ and $A / B$ has $S$. Let $M = B \oplus C$, where $C$ is a submodule of $M$. Then $A = B \oplus (C \cap A)$, by modular law. By the second isomorphism theorem, $A / B \cong (C \cap A)$. Thus $A / B \cong C \cap A$.

2$\Rightarrow$3) Let $A$ be a submodule of $M$. By (2), there exists a decomposition $M = B \oplus C$ such that $B \leq A$ and $A \cap C$ has $S$. By modular law, $A = B \oplus (C \cap A)$. Let $A_2 = A \cap C$ has $S$.

3$\Rightarrow$1) Let $A$ be a submodule of $M$. By (3), $A = A_1 \oplus A_2$, where $A_1$ is direct summand of $M$ and $A_2$ has $S$. By the second isomorphism theorem, $A / A_1 \cong A_2$. So $A / A_1$ has $S$. Thus $M$ is s.p-semisimple.

**Proposition 3.2.** A module $M$ is s.p-semisimple if and only if for every submodule $A$ of $M$ there exists a direct summand $B$ of $M$ such that $A = B + C$, where $C$ is a submodule of $M$ has $S$.

**Proof.**

$\Rightarrow$ It is clear by Theorem 3.1.

$\Leftarrow$ Let $A$ be a submodule of $M$. By our assumption, there exists a direct summand $B$ of $M$ such that $A = B + C$ and $C$ has $S$. Let $M = B \oplus D$, for some submodule $D$ of $M$, then $A = B \oplus (A \cap D)$, by modular law. Hence, $(A / B) = (B + C) / B \cong C / (B \cap C)$, by the second isomorphism theorem. But $C$ has $S$, then $C / (B \cap C)$ has $S$. This implies that $A / B$ has $S$. Thus $M$ is s.p-semisimple.
Proposition 3.3. A module $M$ is s.p - semisimple if and only if for each submodule $A$ of $M$, there exists an idempotent $e \in \text{End}(M)$ such that $e(M) \leq A$ and $(1-e)(A)$ has $S$.

Proof. \(\Rightarrow\) Let $A$ be a submodule of $M$. Since $M$ is s.p - semisimple, then there exists a decomposition $M = B \oplus C$ such that $B \leq A$ and $A \cap C$ has $S$, by th.3.1, 1-2. Let $e : M \rightarrow B$ be the projection map. Clearly that $e^2 = e$ and $C = (1 - e)(M)$. Claim that $(1-e)(A) = (1-e)M \cap A$. To show that, let $m \in (1-e)(A)$, then there is a $a \in A$ such that $m = (1 - e)(a) = a - e(a)$. Therefore $m \in A$ and hence $m \in (1-e)(M) \cap A$. Thus $(1-e)(A) \leq (1-e)(M) \cap A$. Now, let $n \in (1-e)(M) \cap A$, then $n = e(M) \cap A$ and $n \in A$. Hence, there is $k \in M$ such that $n = (1 - e)(k) = k - e(k)$. So $n = e(k) = k \in A$. then $n \in (1-e)(A)$. Thus $A \cap C = A \cap (1-e)(M) = (1-e)(A)$. Thus $(1-e)A$ has $S$.

\(\Leftarrow\) Let $A$ be a submodule of $M$ and $e \in \text{End}(M)$ be an idempotent such that $e(M) \leq A$ and $(1-e)A$ has $S$. Claim that $M = e(M) \oplus (1-e)(M)$. To show that, let $x \in M$, then $x = x + e(x) - e(x) = e(x) + x - e(x) = e(x) + (1-e)(x)$. Thus $M = e(M) + (1-e)(M)$.

Now, let $y \in e(M) \cap (1-e)(M)$, then $y = e(m_1)$ and $y = (1-e)(m_2)$, for some $m_1, m_2 \in M$. So $y = e(m) = e(m_1) = e(1-e)(m_2) = e(m_2) - e(m_2) = 0$, then $y = e(m_1) = 0$. Thus $M = e(M) \oplus (1-e)(M)$. Let $B = e(M) \leq A$ and $C = (1-e)(M)$. Therefore $M = B \oplus C$ and $A \cap C = A \cap (1-e)M = (1-e)A$ has $S$. Thus $M$ is s.p - semisimple, by Theorem 3.1.

Let $M$ be a module and $N$ be a submodule of $M$. Recall that a submodule $K$ of $M$ is called an $S$-generalized supplement of $N$ in $M$, if $M = N + K$ and $N \cap K \leq S(K)$, see [17].

Let $M$ be a module. Recall that $M$ is called an $S$-generalized supplemented module (or briefly $S$-GS module), if every submodule of $M$ has $S$-generalized supplement in $M$, where $S$ is semiradical property on modules, see [17].

Proposition 3.4. Every s.p - semisimple module $M$ is $S$-GS supplemented module.

Proof. Let $M$ is s.p - semisimple module and $N$ be a submodule of $M$, then there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N / K$ has $S$. Hence, $M = K \oplus K_1$, for some submodule $K_1$ of $M$. But $K \leq N$, therefore $M = N + K_1$. So by modular law, $N = K \oplus (N \cap K_1)$, then by the second isomorphism theorem, $N / K \cong N / K_1$ has $S$. Thus $N \cap K_1 \leq S(K)$ by [8].

Proposition 3.5. Let $M$ be s.p-semisimple module. If $M = N + K$, where $N$ is a direct summand of $M$, then $N$ contains an $S$-generalized supplement submodule of $K$ in $M$.

Proof. Since $M$ is an s.p - semisimple, then by Theorem 3.1.1-3, $N \cap K = A \oplus B$, where $A$ is a direct summand of $M$ and $B$ has $S$. Let $M = A \oplus C$, for some submodule $C$ of $M$. Hence, $N = A \oplus (N \cap C)$, by modular law. Let $A_1 = N \cap C$, then $M = N + K = (A + A_1) + K$. But $A \leq K$. Therefore, $M = K + A_1$. Now we want to show $K \cap A_1 \leq S(A_1)$. Since $N \cap K = (A \oplus A_1) \cap K = A \oplus (K \cap A_1)$, by modular law. Let $A_1 = N \oplus A_1 \rightarrow A_1$ be the projection map. So we have $K \cap A_1 = (A \oplus (K \cap A_1) = (N \cap K) = (A \oplus B) = (B)$. But $B$ has $S$. Therefore, $K \cap A_1$ has $S$, by [8]. Hence, $K \cap A_1 \leq S(A_1)$. Thus $A_1$ is an $S$-generalized supplement submodule of $K$ in $M$ and $A_1$ is contained in $N$.

Proposition 3.6. Let $S$ be a hereditary property and $M$ be a module. Then the following statements are equivalent

1- $M$ is s.p - semisimple module.
2- Every submodule $N$ of $M$ has $S$-generalized supplement $K$ in $M$ such that $N \cap K$ is a direct summand of $N$. 
Proof. 1⇒2) Let N be a submodule of M. Then by the same argument of proof of Proposition 3.4. N has an S-generalized supplement.

2⇒1) Let N be a submodule of M. Then by our assumption N has an S-generalized supplement K in M such that N ∩ K is a direct summand of N. Hence M = N + K and N ∩ K ≤ S(K). Let N = (N ∩ K) ⊕ L, for some submodule L of N. Then M = (N ∩ K) + L + K = L + K. But, L ∩ K = N ∩ K ∩ L = 0. Therefore, M = L ⊕ K. By the second isomorphism theorem, N / L ≅ N ∩ K. Since N ∩ K ≤ S(K) and S is hereditary property, then N ∩ K has S by [8] and hence N / L has S. Thus M is s.p-semisimple.

Proposition 3.7. Let M be a module. If M is S-GS supplemented module, then M / S(M) is a semisimple module.

Proof. Let N / S(M) be a submodule of M / S(M). Since M is S-GS supplemented, then there exists a submodule K of M such that M = N + K and N ∩ K ≤ S(K). Then M / S(M) = (N + K) / S(M) = N / S(M) + (K + S(M)) / S(M). Since (N / S(M)) ∩ ((K + S(M)) / S(M)) = [(N ∩ K) + S(M)] / S(M), by modular law and N ∩ K ≤ S(K) ≤ S(M), by [17]. Then (N ∩ K) + S(M) = S(M). Therefore M / S(M) = (N / S(M)) ⊕ ((K + S(M) / S(M)). Thus M / S(M) is semisimple.

Corollary 3.8. Let M be a module. If M is S-GS supplemented module, then M / S(M) is s.p-semisimple module.

Proof. It is clear by Proposition. 3.7 and 2.2-1.

Proposition 3.9. Let M be s.p-semisimple module. Then every submodule N of M has an S-generalized supplement which is a direct summand of M.

Proof. Let N be a submodule of M, then there exists a decomposition M = A ⊕ B such that A ≤ N and N ∩ B has S, by Theorem 3.1, 1-2. Clearly M = N + B and N ∩ B ≤ S(B). Thus B is an S-generalized supplement of N which is a direct summand of M.

Let M be an R- module. Recall that M is called π-projective (or co-continuous) if for every two submodules U, V of M with U + V = M there exists f ∈ End(M) with Im (f) ≤ U and Im (1− f) ≤ V, see [18].

Proposition 3.10. Let S be a hereditary property and a module M be a π-projective module. Then M is s.p - semisimple if and only if M is S-GS module.

Proof. ⇒) It is clear by Proposition 3.4.

⇐) Let N be a submodule of M. Since M is S-GS module, then there exists a submodule K of M such that M = N + K and N ∩ K ≤ S(K). Since M is π- projective, then there exists an idempotent e ∈ End (M) such that Im (e) ≤ N and Im (1− e) ≤ K. But by the same proof of Proposition 3.3 we have N(1- e) = N ∩ (1− e)M ≤ N ∩ K ≤ S(K) and S is hereditary property, therefore N(1- e) has S. Thus by Proposition 3.3 M is s.p - semisimple.

Conclusion

In this work, the concept of s.p-semisimple module is introduced and studied. We also conclude the following:
1. Every semisimple module is s.p – semisimple. However, the converse is not true. Let S = Second singularity. Consider module Z₄ as Z-module. Since Z₄ is singular, then every submodules of Z₄ is singular, by [1]. Therefore, \( Z_2(N) = Z(N) = N \), \( \forall N \leq Z_4 \), let \( K = 0 \), hence \( Z(Z(N \mod 0)) = Z_2(N) = Z(N) = N \mod 0 \). So \( N \mod 0 \) has S, \( \forall N \leq Z_4 \). Thus Z₄ is s.p - semisimple. Clearly, that Z₄ is not semisimple.

2. Let \( M = \bigoplus_{i \in I} M_i \) be a duo module. Then M is s.p - semisimple modules if and only if \( M_i \) is s.p - semisimple module \( \forall i \in I \).

3. Let S be a hereditary property. If \( M_1 \) and \( M_2 \) are s.p - semisimple modules such that \( M_1 \) and \( M_2 \) are relative projective. Then \( M = M_1 \bigoplus M_2 \) is s.p - semisimple.

4. Every s.p - semisimple module M is S-GS supplemented module.

5. Let S be a hereditary property and a module M be a π-projective module. Then M is s.p - semisimple if and only if M is S-GS module.

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