An Averaging Consensus Algorithm and Its Stopping Rule over Noisy Undirected Networks of MIMO Linear Symmetric Agents

Kenta Hanada¹, Takayuki Wada³, Izumi Masubuchi², Toru Asai³, and Yasumasa Fujisaki²

¹Graduate School of Information Science and Technology, Osaka University
1-1 Yamadaoka, Suita, Osaka 565-0871, Japan
E-mail: {k-hanada, t-wada, fujisaki}@ist.osaka-u.ac.jp
²Graduate School of System Informatics, Kobe University
1-1 Rokkodai, Nada, Kobe, Hyogo 657-8501, Japan
E-mail: msb@harbor.kobe-u.ac.jp
³Graduate School of Engineering, Nagoya University
Furo-cho, Chikusa, Nagoya, Aichi 464-8603, Japan
E-mail: asai@nuem.nagoya-u.ac.jp

Abstract

A stochastic averaging consensus algorithm is considered for an agent system over a noisy undirected network with multi-input/multi-output (MIMO) linear symmetric agents. The convergence of the algorithm is investigated, which gives an explicit relation between the number of iterations and the closeness of the agreement, i.e., a stopping rule. The result is illustrated through a numerical example.

1 Introduction

An averaging consensus is a basic distributed algorithm which works over a multi-agent networked systems and gives the average value of initial states of the agents. In this algorithm, each agent communicates only with its neighbors and exchanges information iteratively. Since communication noise is unavoidable and usually interferes with the information exchange, several stochastic consensus algorithms have been proposed under noisy environment [1, 2]. Stopping rules, which provide a relation between the explicit number of iterations and the quality of the consensus, have also been established for stochastic noisy environment, where the agents are assumed to be first order systems [3] and higher order systems [4, 5]. The latter employs a linear symmetric system [6].

Most multi-agent algorithms are considered, where each agent is a single-input/single-output (SISO) sub-system. In order to represent more complex model, multi-input/multi-output (MIMO) agents should be considered. In fact, the consensusability of MIMO multi-agent systems is studied for noise free case [7].

In this paper, we tackle a multi-agent system over noisy undirected networks, where each agent is a MIMO linear symmetric system. We give a communication gain to achieve the averaging consensus and the stopping rule under an appropriate assumption for the symmetric system. These results include the existing ones [3] and [4] as special cases if the agents are first order systems or SISO symmetric systems.

2 Multi-Agent Systems

Let us consider $N$ linear symmetric agents having the same dynamics

$$
\begin{align*}
x_i(k+1) &= Ax_i(k) + B u_i(k), \\
y_i(k) &= C x_i(k), \\
A &= A^T, \\
B &= C^T, \\
i &\in \mathcal{V},
\end{align*}
$$

where $x_i(k) \in \mathbb{R}^n$ is the state, $u_i(k) \in \mathbb{R}^m$ is the input, $y_i(k) \in \mathbb{R}^m$ is the output, $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of agents, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $N \in \mathbb{N}$, and $k \in \mathbb{N}$. We assume that there exists a positive definite matrix $R = R^T \in \mathbb{R}^{m \times m}$ such that $0 < R \leq I_m$ and $0 \leq A - \kappa BRC < I_n$ for all $\kappa \in (0, 1]$, where $I_m \in \mathbb{R}^{m \times m}$ is the identity matrix. That is, each agent can be stabilized by an arbitrary low positive-definite gain output feedback. This assumption is equivalent to three inequalities

$$
0 < R \leq I_m, \\
0 \leq A \leq I_n, \\
0 \leq A - BRC \leq (1 - \eta) I_n
$$

for some $\eta \in (0, 1]$. This $\eta$ will be used later. We remark that the MIMO linear symmetric agent includes SISO one investigated in [4] when $B$ and $C$ are vectors and $R = 1$. It can be also the standard first order agent when $A = B = C = R = 1$ with $\eta = 1$.

For this set of the symmetric agents, we introduce interactions

$$
\begin{align*}
u_i(k) &= \tau(k) R \sum_{j \in \mathcal{N}_i} (z_{ij}(k) - y_i(k)), \\
z_{ij}(k) &= y_j(k) + w_{ij}(k), \\
i &\in \mathcal{V}, \\
j &\in \mathcal{N}_i,
\end{align*}
$$

where $\tau(k)$ is a time-varying function, and $w_{ij}(k)$ is the communication noise.
where \( r(k) \in \mathbb{R} \) is the communication gain to be determined later, \( z_{ij}(k) \in \mathbb{R}^m \) is the information which agent \( i \) receives from the agent \( j \) (\( i \neq j \)), and \( w_{ij}(k) \in \mathbb{R}^m \) is the communication noise which satisfies \( E[w_{ij}(k)] = 0, \text{ Cov}[w_{ij}(k)] \leq V \), where \( E[a] \) and \( \text{Cov}[a] \) denote the expectation and the covariance of a random variable \( a \) and \( V = V^\top > 0 \). We assume that the random variables are i.i.d. with respect to \( i, j \), and \( k \). Furthermore, \( \mathcal{N}_i \subseteq \mathcal{V} \setminus \{i\} \) is the set of agents that can communicate with the agent \( i \), which introduces a graph. We assume that the graph is undirected and connected.

The averaging consensus problem of the multi-agent system (1) with the noisy interactions (3) is to achieve

\[
\lim_{k \to \infty} \mathbb{P} \left( \exists i \in \mathcal{V}, \text{ s.t. } \left\| x_i(k) - \frac{1}{N} \sum_{i=1}^{N} x_i(k) \right\| \geq \epsilon \right) = 0
\]

for all \( \epsilon > 0 \), where \( \mathbb{P} \) is the probability measure on the noise sequence. The goal of this paper is to give a consensus algorithm and its stopping rule for this noisy network of the MIMO linear symmetric agents.

### 3 Convergence Analysis

Let us define the graph Laplacian \( L \) whose \( \{i, j\} \)-th element \( l_{ij} \) is set as \(-1 \) if \( j \in \mathcal{N}_i \), \( |\mathcal{N}_i| \) if \( j = i \), or \( 0 \) otherwise. Since the given graph is undirected and connected, \( L = L^\top \), \( L 1_N = 0 \), and rank \( L = N - 1 \), where \( 1_N \) is the \( N \)-dimensional vector whose elements are all \( 1 \). In fact, \( L \) has only real eigenvalues which are defined as \( 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N \). The multi-agent system is then represented as

\[
x(k+1) = ((I_N \otimes A) - r(k) (L \otimes BRC)) x(k) + r(k) (I_N \otimes BR) \tilde{w}(k) 1_N(k)
\]

where \( x(k) = [ x_1^\top(k) \ x_2^\top(k) \ \cdots \ x_N^\top(k) ]^\top \in \mathbb{R}^{Nn} \), \( W(k) \in \mathbb{R}^{Nn \times N} \) is the matrix whose \( \{i, j\} \)-th block element is \( w_{ij}(k) \) if \( j \in \mathcal{N}_i \) or \( 0 \) otherwise, and \( \otimes \) denotes the Kronecker product. Employing a state coordinate transformation

\[
\begin{bmatrix}
\xi_1(k) \\
\xi_2(k)
\end{bmatrix} =
\begin{bmatrix}
S^\top \otimes I_n \\
\frac{1}{\sqrt{N}} \otimes I_n
\end{bmatrix} x(k),
\]

\[
x(k) =
\begin{bmatrix}
S \otimes I_n \\
\frac{1}{\sqrt{N}} \otimes I_n
\end{bmatrix}
\begin{bmatrix}
\xi_1(k) \\
\xi_2(k)
\end{bmatrix}
\]

with an orthonormal complement \( S \in \mathbb{R}^{N \times (N-1)} \) of \( \frac{1}{N} 1_N^\top / \sqrt{N} \), we obtain

\[
\begin{align*}
\xi_1(k+1) &= ((I_N \otimes A) - r(k) (S^\top L S \otimes BRC)) \xi_1(k) + r(k) (I_{N-1} \otimes BR) \tilde{w}_1(k), \\
\xi_2(k+1) &= \Lambda \xi_2(k) + r(k) B R \tilde{w}_2(k),
\end{align*}
\]

where \( \tilde{w}_1(k) \) and \( \tilde{w}_2(k) \) satisfy \( E[\tilde{w}_1(k)] = 0, E[\tilde{w}_2(k)] = 0, \text{ Cov}[\tilde{w}_1(k)] \leq N(I_{N-1} \otimes V), \text{ and Cov}[\tilde{w}_2(k)] \leq NV \).

We now define the average and the deviation of the states of all agents at \( k \) as

\[
\bar{x}(k) = \left( \frac{1}{N} \otimes I_n \right) x(k),
\]

\[
\tilde{x}(k) = x(k) - (I_N \otimes I_n) \bar{x}(k) = \left( I_N - \frac{1}{N} 1_N^\top \otimes I_n \right) x(k).
\]

Then we see that

\[
\bar{x}(k) = \frac{1}{\sqrt{N}} \xi_2(k), \quad \tilde{x}(k) = (S \otimes I_n) \xi_1(k),
\]

which implies that \( \| \tilde{x}(k) \| = \| \xi_1(k) \| \). That is, we can consider the convergence of \( \tilde{x}(k) \) as that of \( \xi_1(k) \).

The following lemma is the key tool which will be used in the proof of the main result.

**Lemma 1.** For the undirected connected graph, its graph Laplacian \( L \) satisfies

\[
\left\| (I_{N-1} \otimes A) - \frac{1}{\lambda_N} (S^\top L S \otimes BRC) \right\| \leq 1 - \frac{\lambda_2}{\lambda_N},
\]

where \( \| \cdot \| \) denotes the spectral norm.

**Proof.** From (2) and \( \lambda_2 I_{N-1} \leq S^\top L S \leq \lambda_N I_{N-1} \),

\[
0 \leq I_{N-1} \otimes (A - BRC)
\]

\[
\leq (I_{N-1} \otimes A) - \frac{1}{\lambda_N} (S^\top L S \otimes BRC)
\]

\[
\leq I_{N-1} \otimes \left( A - \frac{\lambda_2}{\lambda_N} BRC \right)
\]

\[
\leq I_{N-1} \otimes \left( 1 - \frac{\lambda_2}{\lambda_N} \right) I_n
\]

holds, which concludes Lemma 1.

Now, let us select the communication gain as

\[
r(k) = \frac{1}{\eta \lambda_2} (k_0 + k), \quad k_0 \geq \frac{\lambda_N}{\eta \lambda_2} - 1,
\]

where \( k_0 \in \mathbb{N} \). Then we have the main result.

**Theorem 1.** For given constants \( \alpha \in (0, \infty), \beta \in (0, \infty), \) and \( \gamma \in (0, 1) \), select \( k_f \in \mathbb{N} \) which satisfies

\[
k_f \geq \max(\tau_1, \tau_2),
\]

\[
\tau_1 = \left( \frac{1}{\alpha} - 1 \right) k_0 + 1, \quad \tau_2 = \frac{N(N-1)}{\beta \gamma^2 \eta^2 \lambda_2^2} \text{ Tr}(V) - k_0 + 1,
\]

where \( \text{Tr} \) is the trace of the matrix. Then the deviation \( \tilde{x}(k_f) \) satisfies

\[
P ( \| \tilde{x}(k) \| \leq \alpha \| \tilde{x}(1) \| + \beta ) \geq 1 - \gamma
\]

for any initial state \( x(1) \). Furthermore, for each \( k \in \mathbb{N} \), the average \( \bar{x}(k) \) satisfies

\[
E[\bar{x}(k)] = A^{k-1} \bar{x}(1),
\]

\[
E[\| \bar{x}(k) - E[\bar{x}(k)] \|^2] = \text{ Tr}(\text{Cov}[\bar{x}(k)]) \leq \frac{\pi^2}{6 \eta \gamma^2 \lambda_2^2} \text{ Tr}(V)
\]

for any initial state \( x(1) \).
Proof. Let us introduce $\Gamma(k)$ and $\Phi(k, \ell)$ as

$$
\Gamma(k) = (I_{N-1} \otimes A) - r(k) \left( S^T LS \otimes BRC \right),
$$

$$
\Phi(k, \ell) = \left\{ \begin{array}{ll}
\Gamma(k - 1) \cdots \Gamma(2) \cdots \Gamma(\ell) & \text{if } k > \ell,

I_{(N-1)n} & \text{otherwise},
\end{array} \right.
$$

where Lemma 1 says that $\|\Gamma(k)\| \leq (k_0 + k - 1)/(k_0 + k)$ and thus $\|\Phi(k, \ell + 1)\| \leq (k_0 + \ell)/(k_0 + k - 1)$. Then $\xi_1(k)$ can be expressed as

$$
\xi_1(k) = \Phi(k, 1) \xi_1(1) + \sum_{\ell=1}^{k-1} r(\ell) \Phi(k, \ell + 1) (I_{N-1} \otimes B) \bar{w}_1(\ell).
$$

Since $E[\xi_1(k)] = \Phi(k, 1) \xi_1(1)$, we have

$$
\|E[\xi_1(k)]\| \leq \frac{k_0}{k_0 + k - 1} \|\xi_1(1)\| \leq \alpha \|\xi_1(1)\|
$$

if $k \geq \tau_1$. We also have

$$
\text{Tr} \left( \text{Cov}[\xi_1(k)] \right)
= \text{Tr} \left( E \left[ \sum_{\ell=1}^{k-1} r^2(\ell) \Phi(k, \ell + 1) (I_{N-1} \otimes B) \bar{w}_1(\ell) \right. \right.
+ \left. \bar{w}_1(\ell)^\top (I_{N-1} \otimes B)^\top \Phi(k, \ell + 1)^\top \right]
\right)
\leq N (N - 1) \text{Tr}(V) \sum_{\ell=1}^{k-1} r^2(\ell) \|\Phi(k, \ell + 1)\|^2 \|B\|^2
\leq \frac{N(N-1)}{\eta^2 \lambda_2} \frac{k - 1}{k_0 + k - 1} \text{Tr}(V)
\leq \frac{N(N-1)}{\eta^2 \lambda_2} \frac{1}{k_0 + k - 1} \text{Tr}(V) \leq \beta^2 \gamma
$$

if $k \geq \tau_2$. Since Markov’s inequality says that

$$
P \left( \|\xi_1(k)\| \leq \|E[\xi_1(k)]\| + \frac{\text{Tr} \left( \text{Cov}[\xi_1(k)] \right)}{\gamma} \right) \geq 1 - \gamma
$$

holds, we see that the first part of Theorem 1 has been proved. Furthermore, for each $k \in \mathbb{N}$, we have

$$
\bar{x}(k) = \frac{1}{\sqrt{N}} \left\{ A^{k-1} \xi_2(1) + \sum_{\ell=1}^{k-1} r(\ell) A^{k-1-\ell} B R \bar{w}_2(\ell) \right\}.
$$

Since $E[\bar{w}_2(k)] = 0$, we obtain

$$
E[\bar{x}(k)] = \frac{1}{\sqrt{N}} A^{k-1} \xi_2(1) = A^{k-1} \bar{x}(1).
$$

We also obtain

$$
\text{Tr} \left( \text{Cov}[\bar{x}(k)] \right)
= \frac{1}{N} \text{Tr} \left( E \left[ \sum_{\ell=1}^{k-1} r^2(\ell) A^{k-1-\ell} B R \bar{w}_2(\ell) \right.ight.
\bar{w}_2(\ell)^\top (BR)^\top \left( A^{k-1-\ell} \right)^\top \right]
\right)
\leq \text{Tr}(V) \sum_{\ell=1}^{k-1} r^2(\ell) \|A^{k-1-\ell}\|^{2} \|BR\|^2
\leq \text{Tr}(V) \sum_{\ell=1}^{k-1} \frac{1}{\eta^2 \lambda_2^2} \frac{1}{(k_0 + \ell)^2} \leq \frac{\pi^2}{6 \eta^2 \lambda_2^2} \text{Tr}(V),
$$

which concludes the second part of Theorem 1.

Note that this result includes the SISO case [4]. In fact, since the noise $w_i(k)$ is a scalar in the SISO case, $V$ becomes a scalar $v^2$. Thus the theorem derived in [4] can basically be obtained from Theorem 1. Furthermore, when each agent is the standard first order subsystem, we see that $\text{Tr}(\text{Cov}[\bar{x}(k)]) = \text{Var}[\bar{x}(k)]$. Thus the theorem derived in [3] can also be derived from the second part of Theorem 1.

4 Numerical Example

Let us consider a multi-agent system with $N = 4$. We set $m = 2$, $n = 5$, and

$$
A = \begin{bmatrix}
0.667 & 0.071 & -0.132 & -0.007 & 0.068 \\
0.071 & 0.667 & 0.101 & -0.108 & 0.077 \\
-0.132 & 0.101 & 0.667 & 0.034 & -0.169 \\
-0.007 & -0.108 & 0.034 & 0.667 & -0.188 \\
0.068 & 0.077 & -0.169 & -0.188 & 0.667
\end{bmatrix},
$$

$$
B = C^\top = \begin{bmatrix}
0.400 & 0.200 \\
0.040 & 0.050 \\
0.030 & 0.030 \\
0.020 & 0.010 \\
0.010 & 0.080
\end{bmatrix},
R = \begin{bmatrix}
0.550 & 0.100 \\
0.100 & 0.800
\end{bmatrix},
$$

where the assumption (2) holds with $0 < \eta \leq 0.164$. The initial state was chosen as

$$
x_1(1) = \begin{bmatrix}
1 & 2 & 3 & 4 & 5
\end{bmatrix}^\top,
$$

$$
x_2(1) = \begin{bmatrix}
6 & 7 & 8 & 9 & 10
\end{bmatrix}^\top,
$$

$$
x_3(1) = \begin{bmatrix}
11 & 12 & 13 & 14 & 15
\end{bmatrix}^\top,
$$

$$
x_4(1) = \begin{bmatrix}
16 & 17 & 18 & 19 & 20
\end{bmatrix}^\top.
$$

We see the graph Laplacian as

$$
L = \begin{bmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix},
$$

which is a ring topology, where $\lambda_N = 4.0$ and $\lambda_2 = 2.0$. The upper bound $V$ of $\text{Cov}[w_{ij}(k)]$ was set as

$$
V = \begin{bmatrix}
1.7 & 1.0 & 0.3 & 0.2 \\
1.0 & 1.5 & 0.4 & 1.2 \\
0.3 & 0.4 & 1.0 & 1.4 \\
0.2 & 1.2 & 1.4 & 3.0
\end{bmatrix},
$$

for all $i$, $j$, and $k$. The other parameters were chosen as $\eta = 0.1638$, $\alpha = 0.1$, $\beta = 0.5$ and $\gamma = 0.2$. 

- 48 -
The number of iterations $k$ for (4) is 16,094. Since $\|\tilde{x}(1)\|$ is 25,000, the theorem says $P(\|\tilde{x}(16,094)\| \leq 3.000) \geq 0.8$.

We executed 1,000 times with different noise sequences. Figs. 1, 2, and 3 show that the behavior of the state $x(k)$, average $\bar{x}(k)$, and the deviation $\tilde{x}(k)$ is consistent with Theorem 1.

According to Fig. 2, each element of the average moves around each element of the above vector when the number of iterations $k$ is sufficiently large. These results are consistent with Theorem 1.

## 5 Concluding Remarks

We have proposed the averaging consensus algorithm and stopping rule for the multi-agent system over noisy undirected networks of MIMO multi-agent systems. The theorem establishes the relation between the closeness of the agreement and the number of iterations explicitly with a probabilistic guarantee, which gives a stopping rule for the averaging consensus algorithm.

### Acknowledgment

This research was supported by JST CREST Grant Number JPMJCR15K2, Japan.

### References

[1] S. Kar and J. M. F. Moura, "Distributed Consensus Algorithms in Sensor Networks With Imperfect Communication: Link Failures and Channel Noise", IEEE Transactions on Signal Processing, Vol. 57, No. 1, pp. 355–369, 2009.

[2] T. Li and J. F. Zhang, "Consensus Conditions of Multi-Agent Systems With Time-Varying Topologies and Stochastic Communication Noises," in IEEE Transactions on Automatic Control, vol. 55, no. 9, pp. 2043–2057, Sept. 2010.

[3] R. Morita, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki, "Multi-agent consensus with noisy communication: Stopping rules based on network graphs", IEEE Transactions on Control of Network Systems, vol. 19, No. 4, pp. 358–365, 2016.

[4] K. Hanada, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki, "Convergence analysis of stochastic consensus over noisy networks of linear symmetric agents", The 48th International Symposium on Stochastic Systems Theory and Its Applications, 2016.

[5] K. Hanada, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki, "Stochastic Consensus over Time-Varying Networks of Linear Symmetric Agents", Transactions of the Institute of Systems, Control and Information Engineers, vol. 31, No. 1, pp. 28–35, 2018.

[6] J. C. Willems, "Realization of systems with internal passivity and symmetry constraints", Journal of the Franklin Institute, vol. 301, No. 6, pp. 605 – 621, 1976.

[7] G. Gu, F. Liu, and X. Chen, "Consensus control and feedback graph co-design for MIMO discrete-time multi-agent systems", Journal of Control and Decision, Vol. 1, No. 1, pp. 18–33, 2014.