Landscape analysis of an improved power method for tensor decomposition

Joe Kileel
UT Austin

Timo Klock
Simula Research Laboratory

João M. Pereira
UT Austin

Abstract

In this work, we consider the optimization formulation for symmetric tensor decomposition recently introduced in the Subspace Power Method (SPM) of Kileel and Pereira. Unlike popular alternative functionals for tensor decomposition, the SPM objective function has the desirable properties that its maximal value is known in advance, and its global optima are exactly the rank-1 components of the tensor when the input is sufficiently low-rank. We analyze the non-convex optimization landscape associated with the SPM objective. Our analysis accounts for working with noisy tensors. We derive quantitative bounds such that any second-order critical point with SPM objective value exceeding the bound must equal a tensor component in the noiseless case, and must approximate a tensor component in the noisy case. For decomposing tensors of size $D \times m$, we obtain a near-global guarantee up to rank $\tilde{O}(D \lfloor m/2 \rfloor)$ under a random tensor model, and a global guarantee up to rank $O(D)$ assuming deterministic frame conditions. This implies that SPM with suitable initialization is a provable, efficient, robust algorithm for low-rank symmetric tensor decomposition. We conclude with numerics that show a practical preferability for using the SPM functional over a more established counterpart.

1 Introduction

From applied and computational mathematics [29] to machine learning [39] and multivariate statistics [32] to many-body quantum systems [47], high-dimensional data sets arise that are naturally organized into higher-order arrays. Frequently these arrays, known as hypermatrices or tensors, are decomposed into low-rank representations. In particular, the real symmetric CANDECOMP/PARAFAC (CP) decomposition [15] is often appropriate:

$$T = \sum_{i=1}^{K} \lambda_i a_i^{\otimes m}. \quad (1)$$

Here, we are given $T$, a real symmetric tensor of size $D \times m$. The goal is to expand $T$ as a sum of $K$ rank-1 terms, coming from scalar/vector pairs $(\lambda_i, a_i) \in \mathbb{R} \times \mathbb{R}^D$. Importantly, the number of terms $K$ must be minimal possible for the given tensor, in which case $K$ is called the rank of the input $T$.

When $m > 2$ and $K = O(D^m)$, CP decompositions are generically unique (up to permutation and scaling) by fundamental results in algebraic geometry [12]. An actionable interpretation [4] is that CP decomposition infers well-defined latent variables $\{(\lambda_i, a_i) : i \in [K]\}$ encoded by $T$. Indeed in learning applications, where symmetric tensors are formed from statistical moments (higher-order covariances) or multivariate derivatives (higher-order Hessians), CP decomposition has enabled parameter estimation for mixtures of Gaussians [22, 38], generalized linear models [37], shallow neural networks [21, 26, 48], deeper networks [19, 20, 33], hidden Markov models [6], among others.

Unfortunately, CP decomposition is NP-hard in the worst case [25]. In fact, it is believed to possess a computational-to-statistical gap [8], with efficient algorithms expected to exist for random tensors.
We label the problem PM-P. These properties were shown for SPM-P. There are at least two motivating reasons to study the optimization landscape of SPM-P. In [27], it is proven that projected gradient ascent applied to SPM-P is a particular polynomial optimization problem of degree $O(D^{m/2})$. For this reason, and due to its non-convexity and high-dimensionality, CP decomposition has served theoretically as a key testing ground for better understanding mysteries of non-convex optimization landscapes. To date, this focus has been on the non-convex program

$$\max_{x \in \mathbb{R}^D : \|x\|_2 = 1} \langle T, x^{\otimes m} \rangle.$$  

We label the problem PM-P standing for Power Method Program, because projected gradient ascent applied to PM-P corresponds to the Shifted Symmetric Higher-Order Power Method of Kolda and Mayo [31]. Important analyses of PM-P include Ge and Ma’s [23] and the earlier [5] on overcomplete tensors, as well as [36] for low-rank tensors, and [34] which studied PM-P assuming the tensor components form a unit-norm tight frame with low incoherence.

In this paper, we perform an analysis of the non-convex optimization landscape associated with the Subspace Power Method (SPM) for computing symmetric tensor decompositions. The first and third authors introduced SPM in [27]. This method is based on the following non-convex program:

$$\max_{x \in \mathbb{R}^D : \|x\|_2 = 1} F_A(x),$$  

where $F_A(x) := \|P_A(x^{\otimes n})\|_2$, $n := \lceil m/2 \rceil$, $A := \text{Span}\{a_1^{\otimes n}, \ldots, a_K^{\otimes n}\}$, and $P_A : (\mathbb{R}^D)^{\otimes n} \rightarrow A$ is orthogonal projection with respect to Frobenius inner product.

Note that SPM-P is a particular polynomial optimization problem of degree $2n$ on the unit sphere. There are at least two motivating reasons to study the optimization landscape of SPM-P. Firstly, it was observed in numerical experiments in [27] that the SPM algorithm is competitive within its applicable rank range of $K = O(D^{m/2})$. It gave a roughly one-order of magnitude speed-up over the decomposition methods in [28] as implemented in Tensorlab [45], while matching the numerical stability of FOOBi [16]. Thus SPM is a practical algorithm. Secondly, from a theory standpoint, the program SPM-P has certain desirable properties which PM-P lacks. Specifically for an input tensor $T = \sum_{i=1}^K \lambda_i a_i^{\otimes m}$ with rank $K \leq D^{[m/2]}$ and Zariski-generic $1$ Zariski-generic means that the failure set can be described by the vanishing non-zero polynomials [24], so in particular, has Lebesgue measure $0$. These properties were shown for SPM-P in [27], but both fail for PM-P [see Figure 1]. Thus, SPM-P is more relevant theoretically than PM-P as a test problem for non-convex CP decomposition.

Prior theory. In [27], it is proven that projected gradient ascent applied to SPM-P initialized at almost all starting points with a constant explicitly-bounded step size, must converge to a second-order critical point of SPM-P at a power rate or faster. However this left open the possible existence

\footnote{Zariski-generic means that the failure set can be described by the vanishing non-zero polynomials [24], so in particular, has Lebesgue measure $0$.}
of spurious second-order critical points, i.e., second-order points with reasonably high objective value that are not global maxima (unequal to, and possibly distant from, each CP component \( \pm a_i \)). Such critical points could pose trouble for the successful optimization of SPM-P. Furthermore all theory for SPM-P in [21] was restricted to the clean case: that is, when the input tensor \( \hat{T} \) is exactly of a sufficiently low rank \( K \). The analysis for PM-P in [23, 34, 36] also assume noiseless inputs. However, tensors arising in practice are invariably noisy due, e.g., to sampling or measurement errors.

Main contributions. We perform a landscape analysis of SPM-P by characterizing all second-order critical points, using suitable assumptions on \( a_1, \ldots, a_K \). Under deterministic frame conditions on \( a_1, \ldots, a_K \), which are satisfied by mutually incoherent vectors, near-orthonormal systems, and random ensembles of size \( K = O(D) \), Theorem 7 shows that all second-order critical points of SPM-P coincide exactly with \( \pm a_i \). Theorem 16 shows the same result for overcomplete random ensembles of size \( K = o(D^{m/2}) \), however requiring an additional superlevel set condition to exclude maximizers with vanishing objective values. Both results extend to noisy tensor decomposition, where SPM is applied to a perturbation \( \hat{T} \approx T \). In this setting, second-order critical points with objective values exceeding \( O(\|T - T\|_F) \) are \( O(\|T - T\|_F) \)-near to one of the components \( \pm a_i \). We also show in Lemma 9 that spurious local maximizers (with lower objective values) do exist in the noisy case. The results imply a clear separation of the functional landscape between near-global maximizers, with objective values close to 1, and spurious local maximizers with small objective value. Hence, the SPM objective can be used to validate a final iterate of projected gradient ascent in the noisy case. In Theorem 18 we combine our landscape analysis with bounds on error propagation incurred during SPM’s deflation steps. This gives guarantees for end-to-end tensor decomposition using SPM.

Lastly, we expose the relation between PM-P and SPM-P. Specifically, SPM-P can be expressed as PM-P with the appropriate insertion of the inverse of the Grammian \( (G_n)_{ij} := \langle a_i, a_j \rangle^m \). The resulting de-biasing effect on the local maximizers with respect to the components \( \pm a_i \) (cf. Figure 1) is responsible for many advantages of SPM-P. Along the way, we state a conjecture about the minimal eigenvalue of \( G_n \) when \( a_i \) are i.i.d. uniform on the sphere, which may be of independent interest.

2 Notation

Vectors and matrices. When \( x \) is a vector, \( \|x\|_p \) is the \( \ell_p \)-norm (\( p \in \mathbb{R}_{\geq 1} \cup \{\infty\} \)). For \( x, y \in \mathbb{R}^D \), the entrywise (or Hadamard) product is \( x \odot y \in \mathbb{R}^D \), and the entrywise power is \( x^{\odot s} := x \odot \ldots \odot x \in \mathbb{R}^D \) \((s \in \mathbb{N})\). When \( M \) is a matrix, \( \|M\|_2 \) is the spectral norm. If \( M \) is real symmetric, \( \mu_j(M) \) is the eigenvalue of \( M \) that is the \( j \)-th largest in absolute value. We denote the identity by \( \mathrm{id}_D \in \mathbb{R}^{D \times D} \).

Tensors. A real tensor of length \( D \) and order \( m \) is an array of size \( D \times D \times \ldots \times D \) \((m \text{ times})\) of real numbers. Write \( T^m_D := (\mathbb{R}^D)^{\odot m} \cong \mathbb{R}^{D^m} \) for the space of tensors of size \( D \times D \times \ldots \times D \). Meanwhile, \( \text{Sym} (T^m_D) \subseteq T^m_D \) is the subspace of symmetric tensors (i.e., tensors unchanged by any permutation of indices). The Frobenius inner product and norm are denoted by \( \langle \cdot, \cdot \rangle \) and \( \|\cdot\|_F \), respectively. Given any linear subspace of tensors \( A \subseteq T^m_D \), let \( P_A : T^m_D \to A \) denote the orthogonal projector onto \( A \) with respect to \( \langle \cdot, \cdot \rangle \). In the case \( A = \text{Sym}(T^m_D) \), the projector \( P_{\text{sym}(T^m_D)} \) is the symmetrization operator. \( \text{Sym} : T^m_D \to \text{Sym}(T^m_D) \). Given \( T \in T^m_D \) and \( S \in T^{m^2}_D \), the tensor (or outer) product is \( T \otimes S \in T^{m_1+m^2} \), defined by \((T \otimes S)_{i_1,\ldots,i_{m_1+1},j_1,\ldots,j_{m_2}} := T_{i_1,\ldots,i_{m_1+1}} S_{j_1,\ldots,j_{m_2}} \). For \( T \in T^m_D \) and \( S \in T^m_D \), the tensor power is \( T^{\odot s} := T \odot \ldots \odot T \in T^{ms}_D \). For \( T \in T^{m_1}_D \) and \( S \in T^{m_2}_D \), the contraction \( T \cdot S \) is defined by \((T \cdot S)_{i_1,\ldots,i_{m_1+1},j_1,\ldots,j_{m_2}} := \sum_{j_1,\ldots,j_{m_2}} T_{i_1,\ldots,i_{m_1+1}} S_{j_1,\ldots,j_{m_2}} \). Let \( \text{Reshape} (T, [d_1, \ldots, d_i]) \) be the function that reshapes the tensor \( T \) to have dimensions \( d_1, \ldots, d_i \), as in corresponding Matlab/NumPy commands.

Other. The unit sphere in \( \mathbb{R}^D \) is \( S^{D-1} \), and \( \text{Unif}(S^{D-1}) \) is the associated uniform probability distribution. Given a function \( f : \mathbb{R}^D \to \mathbb{R} \), the Euclidean gradient and Hessian matrix at \( x \in \mathbb{R}^D \) are \( \nabla f(x) \in \mathbb{R}^D \) and \( \nabla^2 f(x) \in \text{Sym}(T^2_x) \). The Riemannian gradient and Hessian with respect to \( S^{D-1} \) at \( x \in S^{D-1} \) are \( \nabla_{g_{S^{D-1}}} f(x) \) and \( \nabla^2_{g_{S^{D-1}}} f(x) \) (see [1]). Write \( \text{Span} \) for linear span, \( |K| := \{1, \ldots, K\} \), and \(|A| \) for the cardinality of a finite set \( A \). Lastly, we use asymptotic notation freely.
3 Symmetric tensor decomposition via Subspace Power Method

In this section, we outline the tensor decomposition method SPM of [27], and provide basic insights on the program SPM-P. Throughout we assume that \( m \geq 3 \) is an integer and define \( n := \lfloor m/2 \rfloor \).

**SPM algorithm.** The input is a tensor \( \hat{T} \in \text{Sym}(T_D^m) \), with the promise that \( \hat{T} \approx T = \sum_{i=1}^{K} \lambda_i a_i \otimes a_i \) Zariski-generic and \( K \leq (D^n + 1) - D \) if \( m \) is even and \( K \leq D^n \) if \( m \) is odd.

As a first step, SPM obtains the orthogonal projector \( P_A : T_D^m \to \hat{A} \) that projects onto the column span of \( \text{Reshape}(\hat{T}, [D^n, D^{m-n}]) \), by using matrix singular value decomposition. Provided that \( \hat{T} \approx T \), the associated subspace approximation error \( \hat{A} \approx A = \text{span}\{a_i \otimes a_i : i \in [K]\} \) defined by

\[
\Delta_A := \|P_A - \hat{A}\|_F = \sup_{T \in T_D^m, \|T\|_F = 1} \|P_A(T) - \hat{A}(T)\|_F,
\]

can be bounded as follows. (Note that by (11) Lem. 2.3, we know \( \Delta_A \leq 1 \) a priori.)

**Lemma 1** (Error in subspace). Let \( m \geq 3 \), \( n = \lfloor m/2 \rfloor \), \( T \in \text{Sym}(T_D^n) \) and assume that \( M := \text{Reshape}(T, [D^n, D^{m-n}]) \) has exactly \( K \) nonzero singular values \( \sigma_1(M) \geq \ldots \geq \sigma_K(M) > 0 \). Let \( \hat{T} \in \text{Sym}(T_D^n) \), \( \hat{M} := \text{Reshape}(T, [D^n, D^{m-n}]) \). Assume \( \Delta_M := \|M - \hat{M}\|_2 < \sigma_K(M) \). Then

\[
\left\| \text{Im}(M) - \text{Im}(\hat{M}) \right\|_2 \leq \frac{\Delta_M}{\sigma_K(M) - \Delta_M},
\]

where \( \text{Im}(M) \subseteq \mathbb{R}^{D^n} \) denotes the image of \( M \) and \( \text{Im}(\hat{M}) \subseteq \mathbb{R}^{D^n} \) denotes the subspace spanned by the \( K \) leading left singular vectors of \( M \). In particular, if \( T = \sum_{i=1}^{K} \lambda_i a_i \otimes a_i \), \( \hat{A} = \text{span}\{a_i \otimes a_i : i \in [K]\} \), \( \text{Im}(A) = K \), and \( \hat{A} \) is the subspace spanned by \( K \) leading tensorized left singular vectors of \( \hat{M} \), the right-hand side of (3) upper-bounds \( \Delta_A \).

**Remark 2.** If \( T = \sum_{i=1}^{K} \lambda_i a_i \otimes a_i \), one coefficient \( \lambda_i \) is small and the vectors \( \{a_i : i \in [K]\} \) are not too correlated, then the flattened tensor \( \text{Reshape}(T, [D^n, D^{m-n}]) \) has a small eigenvalue. This makes estimating the corresponding eigenvector sensitive to noise. See Remark S.33 in the appendix.

Given \( \hat{A} \), SPM seeks one tensor component \( a_i \) by solving the noisy variant of (nSPM-P) defined by

\[
\max_{x \in \mathbb{S}^{D-1}} F_{\hat{A}}(x), \quad \text{where} \quad F_{\hat{A}}(x) := \|P_{\hat{A}}(x \otimes n)\|_F^2.
\]

Starting from a random initial point \( x_0 \sim \text{Unif}(\mathbb{S}^{D-1}) \), the projected gradient ascent iteration

\[
x \leftarrow x + \gamma P_{\hat{A}}(x \otimes n) - (n-1)P_{\hat{A}}(x \otimes n), \quad \|x + \gamma P_{\hat{A}}(x \otimes n) - (n-1)P_{\hat{A}}(x \otimes n)\|_2^2,
\]

with a constant step-size \( \gamma \), is guaranteed to converge to a second-order critical point of (nSPM-P) almost surely by [27]. Here we require that the step-size \( \gamma \) is less than an explicit upper bound given in [27]. Denoting by \( \hat{a}_i \) the final iterate obtained by SPM, we accept the candidate approximate tensor component \( \hat{a}_i \) if \( F_{\hat{A}}(\hat{a}_i) \) is large enough; otherwise we draw a new starting point \( x_0 \) and re-run (4).

Next given \( \hat{a}_i \), SPM evaluates a deflation formula based on Wedderburn rank reduction [14] from matrix algebra to compute the corresponding weight \( \lambda_i \). Then, we update the tensor \( \hat{T} \leftarrow \hat{T} - \lambda_i a_i \otimes a_i \).

To finish the tensor decomposition, SPM performs the projected gradient ascent and deflation steps \( K \) times to compute all of the tensor components and weights \( \{(\lambda_i, \hat{a}_i) : i = 1\} \).

**Preparatory material about nSPM-P.** The goal of this paper is to show that second-order critical points of nSPM-P with reasonable function value must be near the global maximizers \( \pm a_1, \ldots, \pm a_K \) of SPM-P under suitable incoherence assumptions on the rank-one components \( a_1, \ldots, a_K \). Naturally, the optimality conditions for nSPM-P play an important part in this analysis.

**Proposition 3** (Optimality conditions). Let \( x \in \mathbb{S}^{D-1} \) be first and second-order critical for nSPM-P. Then for each \( z \in \mathbb{S}^{D-1} \) with \( z \perp x \), we have

\[
P_{\hat{A}}(x) - P_{\hat{A}}(x \otimes n) = F_{\hat{A}}(x), \quad F_{\hat{A}}(x) \geq n\|P_{\hat{A}}(x \otimes n)\|_F^2 + (n-1)\langle P_{\hat{A}}(x \otimes n), x \otimes n^{-2} 2 \rangle.
\]

Furthermore, for any \( y \in \mathbb{S}^{D-1} \) we have

\[
F_{\hat{A}}(x) \geq n\|P_{\hat{A}}(x \otimes n^2)\|_F^2 + (n-1)\langle P_{\hat{A}}(x \otimes n), x \otimes n^{-2} y \rangle - 2(n-1)F_{\hat{A}}(x)\langle x, y \rangle.
\]
In the analysis later, we make frequent use of expressing the objective $F_A(x)$ using the Gram matrix

$$G_n \in \text{Sym}(T_2^n) \quad \text{defined by} \quad (G_n)_{ij} := \langle a_i^\otimes n, a_j^\otimes n \rangle = \langle a_i, a_j \rangle^n. \quad (8)$$

Under linear independence of the tensors $a_1^\otimes n, \ldots, a_K^\otimes n$, which is implied by our assumptions made later, the inverse $G_n^{-1}$ exists and the noiseless program $\text{nSPM-P}$ can be expressed as follows.

**Lemma 4.** Let $A := \{a_1, \ldots, a_K\} \in \mathbb{R}^{D \times K}$ and $\{a_i^\otimes n : i \in [K]\}$ be linearly independent. We have

$$F_A(x) = \|P_A(x^\otimes n)\|_F^2 = \left((A^T x)^\otimes n\right)^\top G_n^{-1} \left((A^T x)^\otimes n\right). \quad (9)$$

**Lemma 4** exposes the relation between $\text{nSPM-P}$ and $\text{PM-P}$. While $\text{PM-P}$ can be rewritten $(T, x^\otimes m) = \left((A^T x)^\otimes n, (A^T x)^\otimes n\right)$ if $m$ is even, $\text{nSPM-P}$ takes into account correlations among the tensors $a_1^\otimes n, \ldots, a_K^\otimes n$ and inserts the Grammian $G_n^{-1}$ into (9). Consequently, correlations among the tensor components are considered in $\text{nSPM-P}$ without any a priori knowledge of the tensors $a_1^\otimes n, \ldots, a_K^\otimes n$.

In the special case of orthonormal systems, or more generally systems that resemble equiangular tight frames [13, 40], the $\text{nSPM-P}$ and $\text{PM-P}$ objectives coincide up to shift and scaling.

**Lemma 5.** Assume there exist $\rho \in (-1, 1) \setminus \{\frac{1}{K-1}\}$ and $M \in \mathbb{R}$ such that $\langle a_i, a_j \rangle^n = \rho$ for all $i \neq j$ and $\sum_{i \in [K]} \langle x, a_i \rangle^n = M$ for all $x \in \mathbb{S}^{D-1}$. Denote $A := \{a_1, \ldots, a_K\} \in \mathbb{R}^{D \times K}$. Then

$$F_A(x) = (1 - \rho)^{-1} \|A^\top x\|_{2^n}^2 - (1 - \rho)^2 + K \rho (1 - \rho)^{-1} \rho M^2. \quad (10)$$

## 4 Main results

In this section, we present the main results about local maximizers of the $\text{nSPM-P}$ program. Section 4.1 is tailored to low-rank tensor models with $K = \mathcal{O}(D)$ components that satisfy certain deterministic frame conditions. Section 4.2 then considers the overcomplete case $K = \tilde{\mathcal{O}}(D^{1/m^2})$ in an average case scenario, where $a_1, \ldots, a_K$ are modeled as independent copies of an isotropic random vector.

### 4.1 Low-rank tensors under deterministic frame conditions

Motivated by frame constants in frame theory [13], we measure the incoherence of the ensemble $a_1, \ldots, a_K$ by scalars $\rho_s \in \mathbb{R}_{\geq 0}$, which are defined via

$$\rho_s := \sup_{x \in \mathbb{S}^{D-1}} \sum_{i=1}^K |\langle x, a_i \rangle|^s - 1. \quad (11)$$

They satisfy the order relation $\rho_s \leq \rho_{s'}$ for $s' \leq s$, due to $\|a_i\|_2 = 1$, and can be related to extremal eigenvalues of Grammians $G_s$ and $G_{\lfloor s/2 \rfloor}$ as shown in the following result.

**Lemma 6.** Let $\{a_i : i \in [K]\} \subseteq \mathbb{S}^{D-1}$ and $(G_{s})_{ij} := \langle a_i, a_j \rangle^s$ for $s \in \mathbb{N}$. Then

$$1 - \rho_s \leq \mu_K(G_s) \leq \mu_1(G_s) \leq 1 + \rho_s \leq \mu_1(G_{\lfloor s/2 \rfloor}). \quad (12)$$

The characterization in Lemma 6 allows to compute bounds for $\rho_s$ for low-rank tensors with mutually incoherent components or rank-$\mathcal{O}(D)$ tensor with random components. We provide details on this in Remark 8 below, but first state the main guarantee about local maximizers using $\rho_2$ and $\rho_n$.

**Theorem 7** (Main deterministic result). Let $\{a_i : i \in [K]\} \subseteq \mathbb{S}^{D-1}$ and $A = \text{Span}\{a_i^\otimes n : i \in [K]\}$. Let $\hat{A} \subseteq \text{Sym}(T_2^n)$ be a perturbation of $A$ with $\Delta_A = \|P_A - P_{\hat{A}}\|_{F \rightarrow F}$. Let

$$\tau := \frac{1}{6} - n^2 \rho_2 \rho_n \quad \text{and} \quad \Delta_0 := \frac{2\tau}{2 + 4\tau + 3n^2}. \quad (13)$$

Then, if $\Delta_A < \Delta_0$, the program $\text{nSPM-P}$ has exactly $2K$ second-order critical points in the superlevel set where

$$F_{\hat{A}}(x) \geq \frac{2 + 2\tau + 3n^2}{2\tau} \Delta_A. \quad (14)$$
Each of these critical points is a strict local maximizer for $F_{\hat{A}}$. Further for each such point $x^*$, there exists unique $i \in [K]$ and $s \in \{-1, 1\}$ such that

$$
\|x^* - sa_i\|_2^2 \leq \frac{2\Delta A}{n}.
$$

In the noiseless case ($\Delta A = 0$), if $\tau \geq 0$ then there are precisely $2K$ second-order critical points of SPM-P with positive functional value, and they are the global maximizers $sa_i$ ($i \in [K], s \in \{-1, 1\}$).

**Remark 8.** Using Lemma 6 we identify two situations where $\rho_2$ and $\rho_n$ can be bounded from above.

1. **Mutually incoherent ensembles.** Let $a_1, \ldots, a_K$ have mutual incoherence $\rho := \max_{i \neq j} |\langle a_i, a_j \rangle|$. Using Gershgorin’s circle theorem, we obtain

$$
\rho_s \leq \mu_1(G_{\lfloor s/2 \rfloor}) - 1 \leq \max_{i \neq j} \sum_{s \neq i} |\langle a_i, a_j \rangle|^{|s/2|} \leq (K - 1)\rho^{s/2}
$$

for any $s \in \mathbb{N}_{\geq 2}$, which implies that the conditions of Theorem 7 are satisfied if $K\rho$ is sufficiently small. This setting is comparable to the analysis for PM-P in [34]. Moreover, if $K = D$ and $a_1, \ldots, a_K$ are mutually orthogonal, then $\rho_s = 0$ for each $s \geq 2$. Therefore Theorem 7 holds with $\tau = 1/6$ for orthogonally decomposable tensors [7].

2. **Low-rank random ensembles.** Let $a_1, \ldots, a_K$ be independent copies of an isotropic unit-norm random vector $a$ with sub-Gaussian norm $O(1/\sqrt{D})$ (e.g., $a_i \sim \text{Unif}(\mathbb{S}^{D-1})$). With high probability, the ensemble can achieve arbitrarily small $\rho_2$, provided that $K \leq CD$ for a sufficiently small constant $C > 0$. The proof of this fact relies on $A = [a_1 \ldots a_K]$ satisfying, with high probability, the so-called $(K, \delta)$ restricted isometry property (Thm. 5.65) as defined in Definition 10 in the next section. We note that despite requiring conditions milder than those in the unit-norm tight frame analysis for PM-P in [34], we achieve a comparable scaling of $K = O(D)$.

In the noiseless case where $\Delta A = 0$, Theorem 7 shows that all local maximizers coincide with global maximizers $\pm a_1, \ldots, \pm a_K$, provided the tensor components are sufficiently incoherent to ensure $\tau \geq 0$. In the noisy case, all local maximizers with objective values $F_{\hat{A}}(x) \geq C(n, \tau)\Delta A$ are $O(\Delta A)$-close to the global optimizers of the noiseless objective $F_{\hat{A}}$, where the constant $C(n, \tau)$ increases as the incoherence of the vectors $a_1, \ldots, a_K$ shrinks. Unfortunately, the presence of spurious local maximizers of $F_{\hat{A}}$ with small objective values cannot be avoided under a deterministic noise model, as the next result shows.

**Lemma 9.** Let $\delta \in (0, 1)$, $a \in \mathbb{S}^{D-1}$ and $A = \text{span}\{a_n\}$. Then there exists a subspace $\hat{A} \subseteq \text{Sym}(T_B^n)$ with $\dim(\hat{A}) = 1$ and $\|P_A - P_{\hat{A}}\|_{F \to F} = \delta$ such that nSPM-P possesses a strict local maximizer of objective value exactly $\delta^2$.

### 4.2 Average case analysis of overcomplete tensors

The overcomplete case with $K = \tilde{O}(D^{m/2})$ falls outside the range of Theorem 7 because $\tau$ in (13) becomes negative when $K \gg D$. Instead, our analysis for the overcomplete case relies on $A = \{a_1, \ldots, a_K\} \in \mathbb{R}^{D \times K}$ obeying the $(p, \delta)$-restricted isometry property (RIP) for $p = O(D/\log(K))$.

**Definition 10.** Let $A \in \mathbb{R}^{D \times K}, 1 \leq p \leq K$ be an integer, and $\delta \in (0, 1)$. We say that $A$ is $(p, \delta)$-RIP if every $D \times p$ submatrix $A_p$ of $A$ satisfies $\|A_p - \text{Id}_p\|_2 \leq \delta$.

A consequence of the RIP, which is particularly useful in the analysis of overcomplete tensor models, is that the correlation coefficients $\{\langle a_i, x \rangle : i \in [K]\}$ can naturally be split into two groups.

**Lemma 11** (RIP-induce partitioning of correlation coefficients). Suppose that $A = \{a_1 \ldots a_K\} \in \mathbb{R}^{D \times K}$ satisfies the $(p, \delta)$-RIP for $p = \delta \cdot D/\log(K)$]. Let $\tilde{c}_\delta := (1 + \delta)/\delta$. Then for all $x \in \mathbb{S}^{D-1}$ there is a subset of indices $\mathcal{I}(x) \subseteq [K]$ with cardinality $p$ such that

$$
1 - \delta \leq \sum_{i \in \mathcal{I}(x)} \langle a_i, x \rangle^2 \leq 1 + \delta \quad \text{and} \quad \langle a_i, x \rangle^2 \leq \tilde{c}_\delta \frac{\log(K)}{D} \text{ for } i \notin \mathcal{I}(x),
$$

Let us now collect all assumptions needed to analyze the overcomplete case.
We now present our main theorem about local maximizers of well-posed and the smallest eigenvalue of $\alpha$.

In particular, if $\gamma$ satisfies some constant $\kappa$, and suppose that $\lim_{\gamma \to \infty} \epsilon K = 0$. Assume $a_1, \ldots, a_K \in \mathbb{R}^D$ satisfy $A_1$ and $A_2$ for some $\epsilon, c_1, c_2 > 0$. Then there exist $\delta_0$, depending on $n$, such that if $\delta < \delta_0$, $D > D_0$, and $\Delta_A \leq \Delta_0$, the program SNPM-P has exactly $2K$ second-order critical points in the superlevel set

$$\{ x \in S^{D-1} : F_A(x) \geq C\epsilon K + 5\Delta_A \}.$$
Each of these critical points is a strict local maximizer for $F_A$. Further for each such point $x^*$, there exists unique $i \in [K]$ and $s \in \{-1, 1\}$ such that
\[
\|x^* - sa_i\|_2^2 \leq \frac{2\Delta_A}{n}. \tag{19}
\]
In particular, in the noiseless case ($\Delta_A = 0$), the second-order critical points in the superlevel set (18) are exactly the global maximizers $sa_i$ ($i \in [K], s \in \{-1, 1\}$).

By Theorem 16 all non-degenerate local maximizers in the superlevel set (18) are close to global maximizers of the noiseless objective $F_A$. In the noiseless case $\Delta_A = 0$, they coincide with global maximizers and the right hand side in (18) tends to 0 as $D \to \infty$. Hence, we have a clear separation between the global maximizers $\pm a_1, \ldots, \pm a_K$ and degenerate local maximizers with vanishing objective value, so that the objective value acts as a certificate for the validity of an identified maximizer.

Theorem 16 is not a global guarantee because a random starting point $x_0 \sim \text{Unif}(S^{D-1})$, used for starting the projected gradient ascent iteration (2), is, with high probability, not contained in (18).

Remark 17 (Objective value at random initialization). For a random sample $x_0 \sim \text{Unif}(S^{D-1})$ in the clean case $\Delta_A = 0$, we empirically observe the objective value $F_A(x_0) \approx CK/D^n$ as illustrated in Figure 3. As such, we conjecture
\[
\mathbb{E}_{x_0 \sim \text{Unif}(S^{D-1})}[F_A(x_0)] = O\left(\frac{K}{D^n}\right).
\]
Comparing the objective to the level set condition in Theorem 16,
\[
\left\{ x \in S^{D-1} : F_A(x) \geq C \frac{K \log^3(K)}{D^n} + 5\Delta_A \right\}, \tag{20}
\]
a random starting point $x_0$ therefore falls short of satisfying (20) by a $\log^3(K)$-factor only. In this sense, Theorem 16 furnishes a “near-global” guarantee for inSPM-P in the random overcomplete case.

4.3 End-to-end tensor decomposition

By combining our landscape analyses with bounds for error propagation during deflation, we obtain a theorem about end-to-end tensor decomposition using the SPM algorithm. That is, under the conditions of Theorem 7 or 16 a tweaking of SPM (Algorithm S.1 in the appendix) recovers the entire CP decomposition exactly in the noiseless case ($\hat{T} = T$). In the noisy regime, it obtains an approximate CP decomposition, and we bound the error in terms of $\Delta_A$. Due to space constraints, we leave precise descriptions of Algorithm S.1 and our deflation bounds to the supplementary material.

Theorem 18 (Main result on end-to-end tensor decomposition). Let $T = \sum_{i=1}^K \lambda_i a_i \otimes m \in \text{Sym}(T_D^m)$ and $M := \text{Reshape}(T, [D^n, D^{m-n}])$. Let $\sigma_1(M) \geq \ldots \geq \sigma_K(M)$ be the singular values of $M$, and assume $\sigma_K(M) > 0$. For other tensor $T \in \text{Sym}(T_D^m)$, let $\hat{M} = \text{Reshape}(T, [D^n, D^{m-n}])$, assume $\Delta_M := \|M - \hat{M}\|_2 \leq \frac{1}{2} \sigma_K(M)$ and let $\hat{\Delta}_A = \frac{\Delta_M}{\sigma_K(M) - \Delta_M}$. Suppose that $T$ satisfies the assumptions of either Theorem 7 or Theorem 16 define $\Delta_0$, as in the corresponding theorem statement and let $\ell(\Delta_A)$ be the corresponding level set threshold. Then there exist constants $C_1, C_2$, not depending on $\hat{T}$ or $\Delta_M$, such that if we define $\Delta_A := C_1 \Delta_A + C_2 \sqrt{\Delta_A}$ the following holds. Assume

In both theorem statements, the level set threshold depends on $\Delta_A$. 

Figure 3: The empirical average of $F_A(x)$ over 10000 trials of a random ensemble $a_1, \ldots, a_K, x$ consisting of $K + 1$ independent copies of the random vector $a \sim \text{Unif}(S^{D-1})$ for different scalings of $K$ and $D$. The shaded areas indicate plus/minus one empirical standard deviation.
Recovering individual tensor components $a_i$ using PM-P and nSPM-P in the noiseless case;

PM-P and nSPM-P applied to noisy tensors;

Error of individual recovery with nSPM-P after $k$ deflations.

Figure 4: Numerical results for symmetric tensor decomposition using PM-P versus nSPM-P. The shaded areas indicate plus/minus one standard deviation. The experiments are described in Section 5.

In particular, in the noiseless case ($\Delta_0 = 0$), Algorithm 1 returns the exact CP decomposition of $\hat{T}$.

We conclude that the Subspace Power Method, with an initialization scheme for nSPM-P that gives $x \in \mathbb{R}^{D^2}$ where $F_\lambda(x) > \ell(\Delta_A)$, is a guaranteed algorithm for low-rank tensor decomposition.

5 Numerical experiments

Here we present numerical experiments that corroborate the theoretical findings of Section 4. We illustrate that SPM identifies exact tensor components in the noiseless case (in contrast to PM), and that SPM behaves robustly in noisy tensor decomposition. We use the implementation of SPM of the first and third authors, available at https://github.com/joaompereira/SPM, which is licensed under the MIT license. All of our experiments presented below may be conducted on a standard laptop computer within a few hours. For further numerical experiments, we refer the reader to [27], where SPM was tested in a variety of other scenarios, justifying it as a possible replacement for state-of-the-art symmetric tensor decomposition methods such as FOobi [16] or Tensorlab [45].

Global optimizers and noise robustness. In the first set of experiments, we are interested in the recovery of individual tensor components $a_i$ for different $D, K$ and $m$ from noisy approximately low rank tensors. With $m = 2n$ as the tensor order, we create noiseless tensors with $a_i \sim \text{Unif}(S^{D-1})$ as the tensor components and $\lambda_i = \sqrt{D^m/K} \hat{\lambda}_i$ as the tensor weights, where $\hat{\lambda}_i \sim \text{Unif}(1/2, 2)$. This way, the variance of each entry of the tensor is about 1. We construct noisy tensors $\hat{T} \approx T$ by adding independent copies of $\epsilon \sim \mathcal{N}(0, m!\sigma^2)$ to each entry of the tensor and then project onto
We presented a quantitative picture for the optimization landscape of a recent formulation of the Subspace Power Method [27] is guaranteed to converge to second-order critical points, by combining with analysis of deflation, it follows that SPM (with sufficient initialization) is provable robust results for only by analyzing the program with the superlevel condition in Theorem 16. This would give a fully global guarantee. Compared to the usual power method functional, the novelty of the SPM functional is the de-biasing role played by the inverse of a Grammian matrix recording correlations between rank-1 tensors (recall Eq. (8)). This Grammian matrix is responsible for many of the SPM functional’s desirable properties, particularly when the inverse is utilized in the deflation process. The figure illustrates that projected gradient ascent applied to nSPM-P always converges to the global maxima in the noiseless case (up to numerical error), provided the scaling is $K = O(D^{m/2}) = O(D^2)$ and the constant adheres to the rank constraints pointed out in [27, Prop. 4.2]. This is in agreement with Theorem 16 which shows local maxima with sufficiently large objective have to coincide with global maximizers. In the noisy case, Figure 4b illustrates the robustness of nSPM-P giving an error of $O(\sigma)$. In contrast to the PM-P objective suffers from a large bias and does not recover exact tensor components. The effect of the bias even dominates errors induced by moderately-sized entrywise noise.

Deflation with SPM. In the second experiment, we recover complete tensors by using the deflation procedure described in [27]. We are mostly interested in the noisy case, since deflation with exact $a_i$’s, as identified by nSPM-P in the noiseless case, does not induce any additional error.

We test $n$th order tensors ($n = 2$) with $D = 40$, $K = 200$ and $6$th order tensors ($n = 3$) with $D = 15$, $K = 200$, vary the noise level $\sigma$, and construct random tensors as in the previous experiment. Figure 4c plots the average error over 100 repetitions of successfully recovered tensor components, where the $x$-axis ranges from the first recovered component at 1 to the last component at index 300. The figure illustrates that nSPM-P combined with modified deflation allows for recovering all tensor components up to an error of $O(\sigma)$. Surprisingly, we do not observe error propagation, despite the fact that noisy recovered tensor components are being used within each deflation step.

6 Conclusion

We presented a quantitative picture for the optimization landscape of a recent formulation of the non-convex, high-dimensional problem of symmetric tensor decomposition. We identified different assumptions on the tensor components $a_1, \ldots, a_K$ and bounds on the rank $K$ so that all second-order critical points of the optimization problem with sufficiently high functional value must equal one of the input tensor’s CP components. In Theorem 7 the assumptions were deterministic frame and low-rank conditions, while in Theorem 16 the hypotheses were random components and an overcomplete rank. Our proofs accommodated noise in the input tensor’s entries, and we obtained robust results for only by analyzing the program nSPM-P. Our analysis has algorithmic implications. As the Subspace Power Method of [27] is guaranteed to converge to second-order critical points, by combining with analysis of deflation, it follows that SPM (with sufficient initialization) is provable under our assumptions. In Theorem 18 we gave guarantees for end-to-end decomposition using SPM.

Compared to the usual power method functional, the novelty of the SPM functional is the de-biasing role played by the inverse of a Grammian matrix recording correlations between rank-1 tensors (recall Eq. (8)). This Grammian matrix is responsible for many of the SPM functional’s desirable properties, but it also complicated our analysis. We showed that there are theoretical and numerical advantages in using the SPM functional for tensor decomposition over its usual power method counterpart.

This paper suggests several directions for future research:

- Why do we see no error propagation when using deflation and sequential solves of nSPM-P for CP decomposition? Can errors accumulate if we choose the noise deterministically?

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Supplementary material

In these appendices, we supply proofs for the statements made in the main body of the paper. Results and equations solely within this supplementary material are labeled with a leading S.

Organization.

Section A Overview of main proofs
Section B Setup and notation
Section C Proofs of lemmas in Sections 3 and 4
Section D Derivation of Riemannian derivatives and optimality conditions for nSPM-P
Section E Preparatory results on random ensembles over the sphere and Conjecture 15
Section F Technical tools for the proofs of Theorems 7 and 16
Section G Proof of Theorem 7
Section H Proof of Theorem 16
Section I Deflation bounds and proof of Theorem 18

A Overview of main proofs

Before commencing with the exact details, let us first give a high-level overview of the main steps in the proofs of Theorems 7 and 16. We recall the definition \( A := \{a_1, \ldots, a_K\} \in \mathbb{R}^{D \times K} \). The proofs consist of three steps. We begin by plugging in equations solely within this supplementary material are labeled with a leading S.

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**B Setup and notation**

**Additional notation.** Besides the notation in Section 2, we will use the following in the rest of the supplementary materials.

- For tensors $S, T$, we abbreviate their tensor product via concatenation, $ST := S \otimes T$, and the tensor power by $S^n := S^\otimes n$. (It will be clear from context when concatenation denotes matrix multiplication instead.)
- We use $\text{Vec}(T)$ to denote the vectorization of a tensor $T \in T_D$, i.e., $\text{Vec}(T) := \text{Reshape}(T, D^n)$. Also, $\text{Tensor}(-) := \text{Reshape}(-, [D, \ldots, D])$ indicates the inverse mapping, which tensorizes a vector.
- We denote the (columnwise) Khatri-Rao power by a bullet: if $a \in \mathbb{R}^D$ then $a^{\bullet n} := \text{Vec}(a^{\otimes n}) \in \mathbb{R}^{Dn}$. If $A = [a_1 | \ldots | a_K] \in \mathbb{R}^{D \times K}$ then $A^{\bullet n} := [a_1^{\bullet n} | \ldots | a_K^{\bullet n}] \in \mathbb{R}^{Dn \times K}$.

- We write the Tucker tensor product as follows: for $T \in T_D$ and $M^{(1)}, \ldots, M^{(n)} \in \mathbb{R}^{E \times D}$, the Tucker product is denoted $T \times_1 M^{(1)} \times_2 \ldots \times_n M^{(n)} \in T_E^n$ and defined by

\[
(T \times_1 M^{(1)} \times_2 \ldots \times_n M^{(n)})_{i_1, \ldots, i_n} := \sum_{j_1=1}^D \cdots \sum_{j_n=1}^D T_{j_1, \ldots, j_n} M^{(1)}_{i_1, j_1} \cdots M^{(n)}_{i_n, j_n}.
\]

The next definition serves to collect together the key quantities appearing later in our proofs.

**Definition S.1.** Let $\{a_i : i \in [K]\} \subseteq \mathbb{S}^{D-1}$ be a given set of unit-norm vectors. Write $A = [a_1 | \ldots | a_K] \in \mathbb{R}^{D \times K}$. Consider the corresponding set of tensors $\{a^n_i : i \in [K]\}$, and the subspace $\mathcal{A} = \text{Span}(\{a^n_i : i \in [K]\}) \subseteq \text{Sym}(T_D)$. Let $P_A : T_D \rightarrow \mathcal{A}$ denote orthogonal projection onto $\mathcal{A}$. Write $A^{\bullet n} := [\text{Vec}(a^n_1) | \ldots | \text{Vec}(a^n_K)] \in \mathbb{R}^{Dn \times K}$ for the $n$-th columnwise Khatri-Rao power of $A$.

1. Define the $n$-th Grammian matrix to be $G_n = (A^{\bullet n})^T A^{\bullet n} \in \text{Sym}(T_D^2)$. Thus

\[
(G_n)_{ij} := \langle a^n_i, a^n_j \rangle = \langle a_i, a_j \rangle^n \quad \forall \ i, j \in [K].
\]  

(S.3)

2. Let $x \in \mathbb{S}^{D-1}$. Define the **correlation coefficients** of $x$ with respect to $\{a_i : i \in [K]\}$ to be $\zeta = \zeta(x) := A^\top x \in \mathbb{R}^K$. Thus

\[
\zeta_i := \langle x, a_i \rangle \quad \forall \ i \in [K].
\]  

(S.4)

3. Let $x, y \in \mathbb{S}^{D-1}$ and $s \in \{0, 1, \ldots, n\}$. Define the **correlation coefficients** of $x^s y^{n-s}$ with respect to $\{a^n_i : i \in [K]\}$ to be $\eta = \eta(x^s y^{n-s}) = (A^{\bullet n})^\top \text{Vec}(x^s y^{n-s}) \in \mathbb{R}^K$. Thus

\[
\eta_i := \langle x^s y^{n-s}, a^n_i \rangle = \langle x, a_i \rangle^s \langle y, a_i \rangle^{n-s} \quad \forall \ i \in [K].
\]  

(S.5)

4. Let $x \in \mathbb{S}^{D-1}$. Define the expansion coefficients of $P_A(x^n)$ with respect to $\{a^n_i : i \in [K]\}$ to be $\sigma = \sigma(x) \in \mathbb{R}^K$ so that the following expansion holds

\[
P_A(x^n) = \sum_{i=1}^K \sigma_i a^n_i.
\]  

(S.6)

(If $a^n_1, \ldots, a^n_K$ are linearly independent, then Eq. (S.6) uniquely defines $\sigma$; otherwise, we fix once and for all a choice of $\sigma$ so that (S.6) holds.)

5. Let $x \in \mathbb{S}^{D-1}$. For $i \in [K]$ and $s \in \{0, \ldots, n\}$, define the **remainder term** for $P_A(x^n)$ with respect to $x^{n-s} a^n_i$ to be $R_{i,s} = R_{i,s}(x) \in \mathbb{R}$ so that the following equation holds

\[
\langle P_A(x^n), x^{n-s} a^n_i \rangle =: \sigma_i e^{n-s} + R_{i,s}.
\]  

(S.7)

Last, define the maximal $s$-th **remainder size to be** $R_s = R_s(x) := \max_{i \in [K]} |R_{i,s}| \in \mathbb{R}$.

**Remark S.2.** Arguably all the quantities in Definition S.1 are natural to consider, with the possible exception of $R_{i,s}$ and $R_s$. In fact these terms play a key role in our proofs; see inequality (S.39) in Proposition S.11. We call $R_{i,s}$ a "remainder" because, under the assumptions of Theorems 7 and 16, as we will see it holds that $\langle P_A(x^n), x^{n-s} a^n_i \rangle \approx \sigma_i e^{n-s}$, whence $R_{i,s}$ and $R_s$ are small.
C  Proofs of lemmas in Sections 3 and 4

In this section, we prove the various isolated results stated in Sections 3 and 4 of the main body.

C.1 Proof of Lemma 1

Proof. We note that (3) is valid for any matrices $W, \tilde{W} \in \mathbb{R}^{p \times q}$, such that the rank of $W$ is $r$. That is, defining $\Delta_W := \| W - \tilde{W} \|_2 < \sigma_r(W)$ and $\text{Im}(W), \text{Im}(\tilde{W})$ as in the statement, we have

$$\left\| P_{\text{Im}(W)} - P_{\text{Im}(\tilde{W})} \right\|_2 \leq \frac{\Delta_W}{\sigma_r(W) - \Delta_W}. \quad (S.8)$$

We first show the result if $W, \tilde{W}$ are symmetric matrices. In that case, we denote the eigenvalue decomposition of $W, \tilde{W}$ as

$$W = (V_1 \ V_2) \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad \text{and} \quad \tilde{W} = (\tilde{V}_1 \ \tilde{V}_2) \begin{pmatrix} \hat{\Lambda}_1 & 0 \\ 0 & \hat{\Lambda}_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix},$$

where $\Lambda_i$ and $\hat{\Lambda}_i$ are diagonal matrices with the largest $K$ eigenvalues, in magnitude, of $W$ and $\tilde{W}$ on the diagonal, respectively. Letting $\Sigma_1$ be a diagonal matrix with the singular values of $W$, we have $|\Lambda_1| = |\Sigma_1|$, and the same holds for $\tilde{W}$. From the eigenvector decomposition, we can write $P_{\text{Im}(W)} = V_1 V_1^T$ and $P_{\text{Im}(\tilde{W})} = \tilde{V}_1 \tilde{V}_1^T$. Lemma 2.3 in [11] implies that

$$\left\| P_{\text{Im}(W)} - P_{\text{Im}(\tilde{W})} \right\|_2 = \| V_1 \tilde{V}_2 \|_2.$$ 

Then (S.8) holding for symmetric matrices follows from

$$\Delta_W = \| W - \tilde{W} \|_2 \geq \| V_1^T (W - \tilde{W}) \tilde{V}_2 \|_2 = \| \Lambda_1 V_1^T \tilde{V}_2 - V_1^T \tilde{V}_2 \hat{\Lambda}_2 \|_2 \geq \| \Lambda_1 V_1^T \tilde{V}_2 \|_2 - \| V_1^T \tilde{V}_2 \hat{\Lambda}_2 \|_2 \geq \sigma_r(W) \| V_1^T \tilde{V}_2 \|_2 - \| \hat{\Lambda}_2 \|_2 \| V_1^T \tilde{V}_2 \|_2 \geq (\sigma_r(W) - \Delta_W) \left\| P_{\text{Im}(W)} - P_{\text{Im}(\tilde{W})} \right\|_2.$$ 

Here we used the following facts:

- $\| W - \tilde{W} \|_2 \geq \| V_1^T (W - \tilde{W}) \tilde{V}_2 \|_2$ since $V_1$ and $V_2$ have orthonormal columns.
- Since $\tilde{V}_2^T V_1 (\Lambda_1^2 - \sigma_r(W)^2 \text{Id}_r) V_1^T \tilde{V}_2$ is positive semi-definite, we have

$$\| \Lambda_1 V_1^T \tilde{V}_2 \|_2 = \sqrt{\| \Lambda_1 V_1^T \tilde{V}_2 \|^2} \geq \sigma_r(W) \sqrt{\| \tilde{V}_2^T V_1 \|_2} \| V_1^T \tilde{V}_2 \|_2 = \sigma_r(W) \| V_1^T \tilde{V}_2 \|_2.$$

- By Weyl’s inequality [46], it holds $\| \hat{\Lambda}_2 \| \leq \Delta_W$.

Now, if $W, \tilde{W}$ are not symmetric, we apply the result for the symmetric matrices

$$H = \begin{pmatrix} 0 & W^T \\ W^T & 0 \end{pmatrix} \quad \text{and} \quad \hat{H} = \begin{pmatrix} 0 & \tilde{W}^T \\ \tilde{W}^T & 0 \end{pmatrix}.$$

Denote the singular vector decomposition of $W$ and $\tilde{W}$ by $W = U \Sigma V^T$ and $\tilde{W} = \hat{U} \hat{\Sigma} \hat{V}^T$. We can further split $U = (U_1, U_2), \Sigma = \text{Blockdiag}(\Sigma_1, \Sigma_2)$ and $V = (U_1, U_2)$, where $U_1, \Sigma_1, V_1$ correspond to the largest $K$ singular values of $W$, and split $\hat{U}, \hat{\Sigma}, \hat{V}$ analogously. The eigendecomposition of $H$ is related to the singular vector decomposition of $W$ as follows

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} U & \frac{1}{\sqrt{2}} U \\ \frac{1}{\sqrt{2}} V & -\frac{1}{\sqrt{2}} V \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} U^T \\ \frac{1}{\sqrt{2}} V^T \end{pmatrix}, \quad (S.9)$$

and an analogous relation holds between $\hat{H}$ and $\tilde{W}$. It can be checked that this is an eigendecomposition by using that $W = U \Sigma V^T$ to show that the RHS in (S.9) is equal to $H$, and use that $U$ and $V$ have orthonormal columns to show

$$\begin{pmatrix} \frac{1}{\sqrt{2}} U & \frac{1}{\sqrt{2}} U \\ \frac{1}{\sqrt{2}} V & -\frac{1}{\sqrt{2}} V \end{pmatrix}^T \begin{pmatrix} \frac{1}{\sqrt{2}} U \\ \frac{1}{\sqrt{2}} V \end{pmatrix} = \text{Id}_{2r}.$$
Thus, if $W$ has rank $r$, $H$ has rank $2r$, and since $H$ is symmetric,

$$\|P_{\text{Im}(H)} - P_{\text{Im}_{2r}(H)}\|_2 \leq \frac{\Delta_H}{\sigma_{2r}(H) - \Delta_H} \quad (S.10)$$

Regarding the right-hand side, denote $\delta W = \hat{W} - W$. Equation (S.9) implies $\sigma_{2r}(H) = |\lambda_{2r}(H)| = \sigma_r(W)$, while

$$\Delta_H = \|H - \hat{H}\|_2 = \left\| \begin{array}{cc} 0 & \delta W^T \\ \delta W & 0 \end{array} \right\|_2 = \left\| \begin{array}{cc} 0 & \delta W \delta W^T \\ \delta W^T & 0 \end{array} \right\|_2^{\frac{1}{2}} = \max\{\|\delta W\delta W^T\|_2, \|\delta W^T\delta W\|_2\}^{\frac{1}{2}} = \|\delta W\|_2 = \Delta_W$$

On the left-hand side, we have

$$P_{\text{Im}(H)} = \left( \frac{1}{\sqrt{2}} \begin{array}{c} U_1 \\ -\frac{1}{\sqrt{2}} V_1 \end{array} \right) \left( \frac{1}{\sqrt{2}} U_1^T \\ -\frac{1}{\sqrt{2}} V_1^T \right) = \left( \begin{array}{cc} U_1^T & 0 \\ 0 & V_1^T \end{array} \right),$$

thus

$$\|P_{\text{Im}(H)} - P_{\text{Im}_{2r}(H)}\|_2 = \left\| \begin{array}{cc} U_1^T - \hat{U}_1^T \\ \hat{V}_1^T - V_1^T \end{array} \right\|_2 = \max\left\{ \left\|P_{\text{Im}(W)} - P_{\text{Im}_{r}(\hat{W})}\right\|_2, \left\|P_{\text{Im}_{r}(\hat{W}^T)} - P_{\text{Im}_{r}(W^T)}\right\|_2 \right\}$$

Replacing these in (S.10), we show (S.8) when $W, \hat{W}$ are not symmetric.

$$\|P_{\text{Im}(W)} - P_{\text{Im}_{r}(\hat{W})}\|_2 \leq \|P_{\text{Im}(H)} - P_{\text{Im}_{2r}(\hat{H})}\|_2 \leq \frac{\Delta_H}{\sigma_{2r}(H) - \Delta_H} = \frac{\Delta_W}{\sigma_r(W) - \Delta_W}.$$ 

The bound for $\Delta_A$ follows from

$$\Delta_A = \sup_{T \in \mathcal{T}_D^n, \|T\|_F = 1} \left\|P_A(T) - P_A(T)\right\|_F = \sup_{u \in \mathbb{R}^{D^n}, \|u\|_2 = 1} \left\|P_{\text{Im}(\mathcal{M})}(u) - P_{\text{Im}_{K}(\mathcal{M})}(u)\right\|_2 = \left\|P_{\text{Im}(\mathcal{M})} - P_{\text{Im}_{K}(\mathcal{M})}\right\|_2.$$ 

### C.2 Proof of Lemma 4

**Proof.** This is the case $n = s$ in (S.11) in Lemma S.3 below, as $\eta(x^n) = \zeta(x)^{\otimes n} = (A^T x)^{\otimes n}$. 

**Lemma S.3.** Assume $\{a_i: i \in [K]\} \subseteq \text{Sym}(\mathcal{T}_D^n)$ is linearly independent. Then in the setup of Definition 5.1 we have the following formulas in terms of the inverse Gramian:

$$\|P_A(x^n y^{n-s})\|_F^2 = \eta(x^n y^{n-s})^T G_n^{-1} \eta(x^n y^{n-s}) \quad \text{for all } x, y \in S^{D-1}, \quad (S.11)$$

and

$$\sigma(x) = G_n^{-1} \eta(x^n) = G_n^{-1} (\zeta(x)^{\otimes n}) \quad \text{for all } x \in S^{D-1}. \quad (S.12)$$

**Proof.** Let $A^{**} := \{\text{Vec}(a_i^{\otimes n}) | i \in [K]\} \subseteq \mathbb{R}^{D^n \times K}$. Up to reshapings, $P_A$ is projection onto the column space of $A^{**}$. Since $A^{**}$ has full column rank by assumption, projection onto the column space of $A^{**}$ is represented by the matrix

$$A^{**} ((A^{**})^T A^{**})^{-1} (A^{**})^T = A^{**} G_n^{-1} (A^{**})^T.$$ 

It follows that for $T \in \mathcal{T}_D^n$, we have

$$\|P_A(T)\|_F^2 = \langle P_A(T), T \rangle = \text{Vec}(T)^T A^{**} G_n^{-1} (A^{**})^T \text{Vec}(T). \quad (S.13)$$

Substituting $T = x^n y^{n-s}$ in (S.13) and using $(A^{**})^T \text{Vec}(T) = \eta(x^n y^{n-s})$ yields (S.11), while substituting $T = x^n$ into (S.14) gives (S.12). This completes the calculation. 

\[\square\]
C.3 Proof of Lemma 5

Proof. Write \( \zeta := \zeta(x) = A^\top x. \) From \( \langle a_i^n, a_j^n \rangle = \langle a_i, a_j \rangle^n = \rho \) whenever \( i \neq j \) and \( \langle a_i^n, a_i^n \rangle = \langle a_i, a_i \rangle^n = 1 \) for each \( i, \) we can write the Grammian of \( a_1^n, \ldots, a_K^n \) as

\[
G_n = (1 - \rho)I_K + \rho I_K \otimes I_K, \tag{S.15}
\]

where \( I_K \in \mathbb{R}^D \) denotes the all-ones vector. By the Sherman-Morrison inversion formula,

\[
G_n^{-1} = (1 - \rho)I_K + \rho I_K \otimes I_K)^{-1} = \frac{1}{1 - \rho} \left( I_K + \frac{\rho}{(1 - \rho)^2} I_K \otimes I_K \right)^{-1} = \frac{1}{1 - \rho} I_K - \frac{\rho}{(1 - \rho)^2 + K\rho(1 - \rho)} I_K \otimes I_K.
\]

Using that \( \sum_{i \in [K]} \langle x, a_i \rangle^n = M \) uniformly on \( \mathbb{S}^{D-1}, \) we have \( \langle I_K, \zeta(x) \rangle \leq M. \) By writing \( F_A \) as in Lemma 4, we can complete the proof:

\[
F_A(x) = \zeta \otimes_n G_n^{-1} \zeta = \frac{1}{1 - \rho} \| \zeta \|^2_{2n} - \rho \left( \frac{1}{2} K \sum_{i=1}^K (\langle a_i, x \rangle^n)^2 \right) = \frac{1}{1 - \rho} \| \zeta \|^2_{2n} - \frac{\rho M^2}{(1 - \rho)^2 + K\rho(1 - \rho)}. \tag*{\square}
\]

C.4 Proof of Lemma 6

Proof. By Gershgorin’s circle theorem and the fact \( \| a_i \|_2 = 1, \) the eigenvalue \( \mu_\ell(G_{s/2}) \) for each \( \ell = 1, \ldots, K \) adheres to

\[
|\mu_\ell(G_{s/2}) - 1| \leq \sum_{i \neq \ell} \| a_i \|_2^2 \leq \rho_s, \quad \text{hence} \quad \mu_\ell(G_{s/2}) \leq 1 + \rho_s \quad \text{and} \quad \mu_K(G_{s/2}) \geq 1 - \rho_s.
\]

First, we suppose that \( s \) is even. By a classic result in frame theory [13 Prop. 3.6.7], the extremal eigenvalues of the Grammian \( G_{s/2} \) satisfy

\[
\mu_K(G_{s/2}) \leq \sum_{i=1}^K \langle T, a_i^{s/2} \rangle^2 \leq \mu_1(G_{s/2}), \quad \text{for all unit-norm} \ T \in \text{span}\{a_i^{s/2} : i \in [K]\}. \tag{S.16}
\]

The right-most inequality of (S.16) implies

\[
\sum_{i=1}^K \langle T, a_i^{s/2} \rangle^2 = \sum_{i=1}^K \left( P_{\text{span}\{a_i^{s/2} : i \in [K]\}}(T), a_i^{s/2} \right)^2 \leq \mu_1(G_{s/2}) \quad \text{for all unit-norm} \ T \in \mathbb{T}_{s/2}^D. \tag{S.17}
\]

Substituting \( T = x^{s/2} \) into (S.17) and using that \( s \) is even, we obtain

\[
\sum_{i=1}^K |\langle x, a_i \rangle|^2 = \sum_{i=1}^K (\langle x^{s/2}, a_i^{s/2} \rangle^2) \leq \mu_1(G_{s/2}) \quad \text{for all} \ x \in \mathbb{S}^{D-1},
\]

whence \( \rho_s \leq \mu_1(G_{s/2}) - 1. \) If instead \( s \) is odd, we may similarly derive the upper bound \( \rho_s \leq \mu_1(G_{s/2}), \) so that we have the following relation in general

\[
1 - \rho_s \leq \mu_K(G_s) \leq \mu_1(G_{s/2}) \leq 1 + \rho_s \leq \mu_1(G_{s/2}). \tag*{\square}
\]

C.5 Proof of Lemma 9

Proof. This argument relies on Lemma 4, which is proven independently in the next section. Choose \( b \in \mathbb{S}^{D-1} \) with \( b \perp a. \) Define \( S := \sqrt{1 - \delta^2} a^n + b^n \in \text{Sym}(T_B^n), \) and consider the corresponding subspace \( \tilde{A} := \text{span}\{S\}. \)
For each $T \in \mathcal{T}_B^D$ with $\|T\|_F = 1$, using $\|a^n\|_F = \|b^n\|_F = 1$, $\langle a^n, b^n \rangle = 0$ and $\|S\|_F = 1$,
\[
\left\| P_A(T) - P_A(T) \right\|^2_F = \left\| (a^n, T)a^n - \sqrt{1 - \delta^2}a^n - \delta b^n, T \right\|_F^2 = \left\| \delta^2 a^n + \delta \sqrt{1 - \delta^2}b^n, T \right\|_F^2 + \left\| \delta \sqrt{1 - \delta^2}a^n - \delta b^n, T \right\|_F^2 = \delta^2 (a^n, T)^2 + \delta^2 (b^n, T)^2.
\]
This quadratic is maximized over the unit sphere in $\mathcal{T}_B^D$ at any unit-norm $T \in \text{span}\{a^n, b^n\}$, where its value is $\delta^2$. It follows $\|P_A - P_A\|_{F \to F} = \delta$.

Also, we have
\[
F_A(b) = \langle S, b^n \rangle S - \langle S, b^n \rangle^2 = \delta^2.
\]
We now verify that $b$ is a local maximizer of $F_A$ by checking the optimality conditions. By (S.18),
\[
\nabla_{S^{D-1}} F_A(b) = 2nP_A(b^n) \cdot b^{n-1} - 2nF_A(b)b
= 2n\langle S, b^n \rangle S \cdot b^{n-1} - 2n\delta^2 n
= 2n(-\delta)(-\delta b) - 2n\delta^2 b
= 0,
\]
which shows that $b$ is a stationary point of $F_A$. Now taking any unit norm $z \perp b$ and using (S.19),
\[
z^T \nabla_{S^{D-1}}^2 F_A(b)z = 2n^2 \left\| P_A(b^{n-1}z) \right\|^2_F + 2n(n-1)\langle P_A(b^n), b^{n-2}z^2 \rangle - 2nF_A(b)
= 2n^2 \left\| (S, b^{n-1}z) S \right\|_F^2 + 2n(n-1)\langle (S, b^n) S, b^{n-2}z^2 \rangle - 2n\delta^2
= -2n(n-1)\delta \sqrt{1 - \delta^2} \langle a, b \rangle^{n-2} \langle a, z \rangle^2 - 2n\delta^2
\leq -2n\delta^2
< 0,
\]
where we used the fact $n \geq 2$ in third equality. Thus, the Riemannian Hessian of $F_A$ is strictly negative-definite at $b$. Therefore, $b$ is a strict local maximizer of $F_A$ as we wanted.  

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### D Derivation of Riemannian derivatives and optimality conditions for nSPM-P

In this section, we derive the Riemannian derivatives and optimality conditions for nSPM-P.

#### D.1 Riemannian derivatives of $F_A$ on the sphere

**Lemma S.4.** Let $x, z \in S^{D-1}$ with $z \perp x$. The Riemannian gradient and Hessian of $F_A$ satisfy
\[
\nabla_{S^{D-1}} F_A(x) = 2nP_A(x^n) \cdot x^{n-1} - 2nF_A(x)x, \tag{S.18}
\]
\[
z^T \nabla_{S^{D-1}}^2 F_A(x)z = 2n^2 \| P_A(x^{n-1}z) \|^2_F + 2n(n-1)\langle P_A(x^n), x^{n-2}z^2 \rangle - 2nF_A(x). \tag{S.19}
\]

**Remark S.5.** In this lemma $z$ is a tangent vector to the unit sphere at $x$, since $z \perp x$. For convenience we work with normalized tangent vectors, which is why there is the further assumption that $\|z\|_2 = 1$. This is without loss of generality: second-order criticality for nSPM-P is the condition that the quadratic form on the tangent space given by $\nabla_{S^{D-1}}^2 F_A(x)$ is negative semi-definite. By homogeneity of the quadratic form, this is verified by just checking normalized tangent vectors.

**Proof.** We first calculate the Euclidean gradient and Hessian of $F_A$. Let $U_1, \ldots, U_K \in \text{Sym}(\mathcal{T}_B^D)$ form an orthonormal basis of $\mathcal{A}$. Then for all $S, T \in \mathcal{T}_B^D$,
\[
P_A(T) = \sum_{i=1}^K \langle U_i, T \rangle U_i, \quad \langle P_A(T), S \rangle = \sum_{i=1}^K \langle U_i, T \rangle \langle U_i, S \rangle, \quad \text{and} \quad \|P_A(T)\|^2_F = \sum_{i=1}^K \langle U_i, T \rangle^2.
\]
Also, direct calculations verify that if $T \in \text{Sym}(T_{\mathcal{P}}^D)$, then $\nabla \langle T, x^n \rangle = nT \cdot x^{n-1}$ and $\nabla^2 \langle T, x^n \rangle = n(n-1)T \cdot x^{n-2}$ (see [20] Lem. 3.1, Lem. 3.3)). Therefore,

\[\nabla F_\mathcal{A}(x) = \nabla \|P_\mathcal{A}(x^n)^2\| = \sum_{i=1}^{K} \nabla \langle U_i, x^n \rangle^2\]

\[= 2 \sum_{i=1}^{K} (U_i, x^n) \nabla \langle U_i, x^n \rangle = 2n \sum_{i=1}^{K} (U_i, x^n) U_i \cdot x^{n-1} = 2nP_\mathcal{A}(x^n) \cdot x^{n-1},\]

and

\[\nabla^2 F_\mathcal{A}(x) = \sum_{i=1}^{K} \nabla^2 \langle U_i, x^n \rangle^2 = 2 \sum_{i=1}^{K} (\nabla \langle U_i, x^n \rangle) (\nabla \langle U_i, x^n \rangle)^\top + 2 \sum_{i=1}^{K} (U_i, x^n) \nabla^2 \langle U_i, x^n \rangle\]

\[= 2n^2 \sum_{i=1}^{K} (U_i \cdot x^{n-1})(U_i \cdot x^{n-1})^\top + 2(n-1) \sum_{i=1}^{K} (U_i, x^n) U_i \cdot x^{n-2}.\]

Thus,

\[z^\top \nabla^2 F_\mathcal{A}(x) z = 2n^2 \sum_{i=1}^{K} (U_i, x^{n-1} z)^2 + 2(n-1) \sum_{i=1}^{K} (U_i, x^n) (U_i, x^{n-2} z^2)\]

\[= 2n^2 \|P_\mathcal{A}(x^{n-1} z)\|_F^2 + 2(n-1) \|P_\mathcal{A}(x^n), x^{n-2} z^2\|.\]

We now calculate the Riemannian gradient and Hessian of $F_\mathcal{A}$. In general, for a twice-differentiable function $g: \mathbb{R}^n \rightarrow \mathbb{R}$, the Riemannian gradient and Hessian of the restriction of $g$ to the unit sphere $S^{D-1}$ are related to the Euclidean counterparts of $g$ as follows (see [1] Ex. 3.6.1 and [2] Eq. (10), Sec. 4.2):\[
\nabla_{S^{D-1}} g(x) = (I - xx^\top) \nabla g(x),
\]

and\[
\nabla_{S^{D-1}}^2 g(x) = (I - xx^\top) (\nabla^2 g(x) - (x^\top \nabla g(x)) \Delta D) (I - xx^\top).
\]

Applying this to $F_\mathcal{A}$, and using $x^\top \nabla F_\mathcal{A}(x) = 2nP_\mathcal{A}(x^n) \cdot x^n = 2nF_\mathcal{A}(x)$, $(I - xx^\top)$ $z = z$ (since $z \perp x$) and $z^\top z = 1$ (since $z \in S^{D-1}$), yields the result. \[\square\]

### D.2 Proof of Proposition 4

**Proof.** The first-order (stationary point) condition is given by setting the Riemannian gradient (S.18) to zero. The second-order condition is the requirement that the Riemannian Hessian be negative semi-definite, i.e., that (S.19) is non-positive for all unit norm $z \perp x$. Dividing by $2n$ and rearranging yields (S.19) and (S.20) respectively.

It remains to show (S.21). Assume $x \in S^{D-1}$ is first and second-order critical for $n\text{SPM-Pr}$ for each $y \in S^{D-1}$. For each $x \in S^{D-1}$, let $\alpha, \beta \in [0, 1]$, $z \in S^{D-1}$ be such that $\beta = \sqrt{1 - \alpha^2}$, $z \perp x$ and $y = \alpha x + \beta z$. Then,

\[\|P_\mathcal{A}(x^{n-1} y)\|_F^2 = \|\alpha P_\mathcal{A}(x^n) + \beta P_\mathcal{A}(x^{n-1} z)\|_F^2\]

\[= \alpha^2 \|P_\mathcal{A}(x^n)\|_F^2 + \beta^2 \|P_\mathcal{A}(x^{n-1} z)\|_F^2 + 2\alpha\beta \langle P_\mathcal{A}(x^n), P_\mathcal{A}(x^{n-1} z) \rangle\]

\[= \alpha^2 F_\mathcal{A}(x) + \beta^2 \|P_\mathcal{A}(x^{n-1} z)\|_F^2 + 2\alpha\beta \langle P_\mathcal{A}(x^n), x^{n-1} z \rangle\]

\[= \alpha^2 F_\mathcal{A}(x) + \beta^2 \|P_\mathcal{A}(x^{n-1} z)\|_F^2 + 2\alpha\beta \langle F_\mathcal{A}(x), x z \rangle\]

\[= \alpha^2 F_\mathcal{A}(x) + \beta^2 \|P_\mathcal{A}(x^{n-1} z)\|_F^2 + 2\alpha\beta \langle P_\mathcal{A}(x^n), x^{n-1} z \rangle + \beta^2 (P_\mathcal{A}(x^n), x^{n-2} z^2)\]

\[= \alpha^2 F_\mathcal{A}(x) + \beta^2 (P_\mathcal{A}(x^n), x^{n-2} z^2),\]

where we used the stationary point condition (S.19) in (S.20) and $x \perp z$ in (S.21). By similar logic,

\[\langle P_\mathcal{A}(x^n), x^{n-2} y^2 \rangle = \langle P_\mathcal{A}(x^n), x^{n-2} (\alpha x + \beta z)^2 \rangle\]

\[= \alpha^2 \|P_\mathcal{A}(x^n)\|_F^2 + 2\alpha\beta \langle P_\mathcal{A}(x^n), x^{n-1} z \rangle + \beta^2 (P_\mathcal{A}(x^n), x^{n-2} z^2)\]

\[= \alpha^2 F_\mathcal{A}(x) + \beta^2 (P_\mathcal{A}(x^n), x^{n-2} z^2),\]
Finally, we substitute (S.21) and (S.22) into (7), and use the second-order condition (6):

\[ n \|P_A(x^{n-1}y)\|_F^2 + (n-1)\langle P_A(x^n), x^{n-2}y^2 \rangle \]

\[ = (2n-1)\alpha^2 F_A(x) + \beta^2 \left( n \|P_A(x^{n-1}z)\|_F^2 + (n-1)\langle P_A(x^n), x^{n-2}z^2 \rangle \right) \]

\[ \leq (2n-1)\alpha^2 F_A(x) + \beta^2 F_A(x) \]

\[ = (2n-1)\alpha^2 F_A(x) + (1-\alpha^2)F_A(x) \]

\[ = (1+(2n-2)\alpha^2)F_A(x). \quad \text{(S.23)} \]

Substituting \( \alpha = \langle x, y \rangle \) into (S.23) gives (7) as desired. \( \square \)

E Preparatory results on random ensembles over the sphere and Conjecture 15

In this section, we prove the results on random ensembles over the sphere that are used in the analysis of the overcomplete random tensor model. Specifically, we prove Lemma 11 and Proposition 13.

E.1 Proof of Lemma 11

Proof. Fix \( x \in S^{D-1} \), and assume without loss of generality that the indices are ordered according to \( \langle a_1, x \rangle^2 \geq \ldots \geq \langle a_K, x \rangle^2 \). We claim that the index set \( \mathcal{I}(x) = \{p\} \) satisfies the specified conditions. Denote by \( A_p \in \mathbb{R}^{D \times p} \) the submatrix of \( A \) consisting of the first \( p \) columns. The \( (p, \delta) \)-RIP implies

\[ 1 + \delta = (1+\delta)\|x\|_2^2 \geq x^\top A_p A_p^\top x = \sum_{i=1}^p \langle a_i, x \rangle^2 \geq (1-\delta)\|x\|_2^2 = 1 - \delta. \quad \text{(S.24)} \]

Hence, \( \mathcal{I}(x) \) satisfies the first condition. As for the second condition, by the ordering assumption,

\[ \sum_{i=1}^p \langle a_i, x \rangle^2 \geq p(a_p, x)^2. \quad \text{(S.25)} \]

Inequalities (S.24) and (S.25) imply \( \langle a_p, x \rangle^2 \leq (1+\delta)p^{-1} \). Then again by the ordering relation,

\[ \max_{i \in \mathcal{I}(x)} \langle a_i, x \rangle^2 \leq \langle a_p, x \rangle^2 \leq \frac{1+\delta}{p} \leq \frac{1+\delta \log(K)}{c_\delta K D} = \frac{c_\delta \log(K)}{D}. \]

E.2 Proof of Proposition 13

Now consider a random ensemble of independent copies \( a_1, \ldots, a_K \) of the random vector \( a \sim \text{Unif}(S^{D-1}) \). Before showing Proposition 13 which states that for this ensemble the assumptions hold with high probability for all \( n \), and \( A \) holds with high probability if \( n = 2 \), we will recall these assumptions. Write \( A := [a_1 \ldots [a_K] \in \mathbb{R}^{D \times K} \).

A1 There exists \( c_\delta > 0 \), depending only on \( \delta \), such that \( A \) is \( \lfloor c_\delta D/\log(K) \rfloor, \delta \)-RIP.

A2 There exists \( c_1 > 0 \), independent of \( K, D \), such that \( \max_{i \neq j} \langle a_i, a_j \rangle^2 \leq c_1 \log(K)/D \).

A3 There exists \( c_2 > 0 \), independent of \( K, D \), such that \( \|G_n^{-1}\|_2 \leq c_2 \).

The proof for A3 is lengthy and more technically involved (even though we only prove it for the case of \( n = 2 \)). So, we first give the arguments for A1 A2.

Lemma S.6 A1 A2 in Proposition 13. Let \( a_1, \ldots, a_K \) be independent copies of the random vector \( a \sim \text{Unif}(S^D) \), and define \( A := [a_1 \ldots [a_K] \in \mathbb{R}^{D \times K} \). Assume \( \log(K) = o(D) \).

1. Fix \( \delta \in (0,1) \). There exists a universal constant \( C > 0 \) and constants \( D_0 \in \mathbb{N} \), \( c_\delta > 0 \) depending only on \( \delta \) such that, if \( D \geq D_0 \) and we put \( p := \lfloor c_\delta D/\log(K) \rfloor \), then \( A \) is \( (p, \delta) \)-RIP with probability at least \( 1 - 2\exp(-C\delta^2 D) \).

2. There exists a universal constant \( c_1 > 0 \) such that

\[ \mathbb{P} \left( \max_{i,j \neq j} \langle a_i, a_j \rangle^2 \leq c_1 \frac{\log(K)}{D} \right) \geq 1 - \frac{1}{K}. \]
Proof. 1. By [42] Thm. 5.65, A satisfies the \((p, \delta)\)-RIP with probability at least \(1 - 2 \exp(-C\delta^2 D)\), provided that \(D \geq C'\delta^{-2} p \log(eK/p)\), where \(C', C'' > 0\) are universal constants. Thus it suffices to check the latter inequality holds if \(c_4, D_0\) are appropriately chosen. Because \(\log K = o(D)\), for each fixed \(c_3 > 0\), there exists \(D_0\) such that \(p > e\) whenever \(D \geq D_0\). In this case, we have

\[
C'\delta^{-2} p \log(eK/p) < C'\delta^{-2} p \log(K) = C'\delta^{-2} \frac{[c_3 D/\log(K)] \log(K)}{c_3 D},
\]

where the last inequality in (S.26) is because \(p > e\) implies \(\log(K) < \frac{1}{2}c_3 D\). We note that the right-most quantity in (S.26) is bounded above by \(D\), as desired, if we choose \(c_3 \leq 2\delta^2/(3C').\)

2. Let \(d_{ij} := |\langle a_i, a_j \rangle|\). By rotational symmetry, \(d_{ij} \overset{d}{=} |a_{11}|\) for \(i \neq j\) in distribution, where \(a_{11} := \langle a_1, e_1 \rangle\) denotes the first coordinate of \(a_1\). By [43] Thm. 3.4.6, \(a_{11}\) is a sub-Gaussian random variable with sub-Gaussian norm \(\|a_{11}\|_{\psi_2} \leq c_1' D^{-2} \frac{\delta}{c_1'}\) for a universal constant \(c_1' > 0\). Using this, a union bound and a tail bound for sub-Gaussian random variables [43] Eq. 2.14,

\[
\mathbb{P} \left( \max_{i \neq j} d_{ij} \leq t \right) = 1 - \mathbb{P} \left( \text{there exists } i \neq j \text{ s.t. } d_{ij} > t \right) \geq 1 - \sum_{i=1}^{K} \sum_{j=i+1}^{K} \mathbb{P} \left( d_{ij} > t \right)
\]

\[
= 1 - \frac{K(K-1)}{2} \mathbb{P} \left( |a_{11}| > t \right) > 1 - K^2 \exp \left( -\frac{t^2 D}{c_1'^2} \right)
\]

for a universal constant \(c_1'' > 0\). We finish by setting \(t = \sqrt{3c_1'' \log(K)/D}\) and \(c_1 = 3c_1''\).

Next we come to A3 for a random ensemble on the sphere when \(n = 2\).

**Proposition S.7** (A3 in Proposition 13). Let \(K, D \in \mathbb{N}\) satisfy \(1 \leq K \leq D^2\). Let \(a_1, \ldots, a_K\) be independent copies of the random vector \(a \sim \text{Unif}(S^{D-1})\). Define the Grammian \(G_2 \in \text{Sym}(T_K^2)\) by \((G_2)_{ij} := \langle a_i, a_j^* \rangle = \langle a_i, a_j \rangle^2\). Then there exists a universal constant \(C > 0\) such that the minimal eigenvalue of \(G_2\) satisfies

\[
\mu_K(G_2) \geq 1 - (C+1) \frac{\sqrt{K}}{D} \log \left( \frac{eD}{\sqrt{K}} \right), \quad \text{with probability at least } 1 - C \left( \frac{eD}{\sqrt{K}} \right)^{C \sqrt{K}} \quad (S.27)
\]

In particular, if \(K = o(D^2)\), then for each constant \(c_2 > 1\), we have \(\|G_2^{-1}\|_2 \leq c_2\) with probability tending to 1 as \(D \rightarrow \infty\).

The statement is implied by \((p, \delta)\)-RIP with \(p = K\) for matrices like \([a_1^2 \ldots a_K^2]\) (columns are Khatri-Rao squares of vectors). Such RIP statements have been analyzed in [17]. The main technical ingredients are [3] Thm. 3.3, which proves RIP for matrices containing columnwise sub-exponential random vectors, and the fact that certain vectorized Khatri-Rao products of sub-Gaussian random vectors are sub-exponential with favorable sub-exponential norm. For completeness, we include a self-contained proof of Proposition S.7. First, here is a version of [3] Thm. 3.3 tailored to our needs.

**Proposition S.8** (Tailored version of [3] Thm. 3.3). Let \(K, D \in \mathbb{N}\) satisfy \(1 \leq K \leq D^2\). Let \(X_1, \ldots, X_K \in \mathbb{R}^{D^2}\) be independent copies of a sub-exponential random vector \(X \in \mathbb{R}^{D^2}\) with \(O(1)\) sub-exponential norm (i.e., upper bounded independently of \(D\) and \(K\)). Furthermore, assume that \(\|X\|_2 = D\) almost surely. Define \(M := \frac{1}{D} [X_1 \ldots X_K] \in \mathbb{R}^{D^2 \times K}\). Then there exists a universal constant \(C > 0\) such that

\[
\|M^T M - \mathbb{I}_K\|_2 \leq C \frac{\sqrt{K}}{D} \log \left( \frac{eD}{\sqrt{K}} \right), \quad \text{with probability at least } 1 - C \left( \frac{eD}{\sqrt{K}} \right)^{C \sqrt{K}} \quad (S.28)
\]

In particular, in this event the minimal eigenvalue of the associated Grammian satisfies

\[
\mu_K(M^T M) \geq 1 - C \frac{\sqrt{K}}{D} \log \left( \frac{eD}{\sqrt{K}} \right). \quad (S.29)
\]

**Proof.** The first part of the statement is [3] Thm. 3.3, in the special case (relating their notation to our notation) \(n = D^2, m = K, N = K, r = 1, K = 1, K' = 1 + \epsilon\) for any \(\epsilon \in (0, 1)\) and \(\theta' \rightarrow 0\), where we note that \(M\) satisfies \((K, \delta)\)-RIP if and only if \(\|M^T M - \mathbb{I}_K\|_2 \leq \delta\), since \(M\) has \(K\) columns. The second part follows immediately from the definition of the spectral norm. \(\square\)
We briefly indicate how this is done. We first note that all random vectors thus, we focus on the Grammian associated with the shifted and centered random variables this implies to deduce Proposition S.7, we apply Proposition S.8 to a centered and scaled version of the Khatri-Rao squares $a_i^{\bullet 2} = \text{Vec}(a_i^T)$, following [17].

**Proof of Proposition S.7** Consider the Khatri-Rao square $A^{\bullet 2} := [a_1^{\bullet 2} \ldots a_K^{\bullet 2}] \in \mathbb{R}^{D^2 \times K}$ and note that $G_2 = (A^{\bullet 2})^T A^{\bullet 2}$. Instead of working with $a_1^{\bullet 2}, \ldots, a_K^{\bullet 2}$, we introduce auxiliary variables

$$X_i := \sqrt{\frac{D}{D-1}} \text{Vec}(D a_i^T - \text{Id}_D) \in \mathbb{R}^{D^2},$$

in order to be able to apply Proposition S.8. Then $X_1, \ldots, X_K$ are independent copies of the random vector $X \in \mathbb{R}^{D^2}$ given by

$$X := \sqrt{\frac{D}{D-1}} \text{Vec}(D a - \text{Id}_D), \quad \text{where } a \sim \text{Unif}(S^{D-1}). \quad \text{(S.30)}$$

Note that $\mathbb{E}[X_i] = \sqrt{\frac{D}{D-1}} (\text{Vec}(D \mathbb{E}[a_i^2] - \text{Id}_D)) = 0$. Using $a_i \sim S^{D-1}$, we also have

$$\|X_i\|_2^2 = \frac{D}{D-1} \|D a_i a_i^T - \text{Id}_D\|_2^2 = \frac{D}{D-1} \left( \|D a_i a_i^T\|_2^2 - 2 \langle D a_i a_i^T, \text{Id}_D \rangle + \|\text{Id}_D\|_2^2 \right)$$

$$= \frac{D}{D-1} \left( D^2 \|a_i\|_2^4 - 2D \|a_i\|_2^2 + D \right) = \frac{D}{D-1} \left( D^2 - 2D + D \right) = D^2,$$

so that $\|X_i\|_2^2 = D$ as required in Proposition S.8.

Define $M := \frac{1}{D}[X_1 \ldots X_K] \in \mathbb{R}^{D^2 \times K}$. For each $i, j \in [K]$, again using $a_i, a_j \sim S^{D-1}$, we have

$$\langle (M^T M)_{i,j} \rangle = \frac{1}{D(D-1)} \langle \langle D a_i a_i^T - \text{Id}_D, D a_j a_j^T - \text{Id}_D \rangle \rangle_F$$

$$= \frac{1}{D(D-1)} \left( D^2 \langle a_i a_i^T, a_j a_j^T \rangle_F - D a_i a_i^T \langle a_j a_j^T, \text{Id}_D \rangle_F + \langle \text{Id}_D, \text{Id}_D \rangle_F \right)$$

$$= \frac{1}{D(D-1)} \left( D^2 \langle a_i, a_j \rangle^2 - D \|a_i\|_2^2 D \|a_j\|_2^2 + D \right)$$

$$= \frac{D}{D-1} \langle a_i, a_j \rangle^2 - \frac{1}{D-1}.$$

Hence, $M^T = \frac{D}{D-1} G_2 - \frac{1}{D-1} I_K I_K^T$, which may be rewritten $G_2 = \frac{D-1}{D} M^T M + \frac{1}{D} I_K I_K^T$. This implies

$$\mu_K(G_2) \geq \frac{D-1}{D} \mu_K(M^T M). \quad \text{(S.31)}$$

Thus, we focus on the Grammian associated with the shifted and centered random variables $X_1, \ldots, X_K$. To apply Proposition S.8 to $M^T M$, it remains to show that the sub-exponential norm of $X$ in (S.30) is bounded by some universal constant, independent of $K, D$.

We briefly indicate how this is done. We first note that all random vectors $\sqrt{D} a_1, \ldots, \sqrt{D} a_K$ have the so-called convex concentration property for some universal constant $C' > 0$, see [17] Thm. 6, Def. 7, Thm. 8]. Following [17] and the references therein, this can be used to prove the Hanson-Wright type inequality

$$\mathbb{P} \left( \|D(a_i^T Y a_i - \mathbb{E}[a_i^T Y a_i])\| > t \right) \leq 2 \exp \left( -C' \min \left( \frac{t^2}{\|Y\|_F^2}, \frac{t}{\|Y\|_2^2} \right) \right), \quad \text{(S.32)}$$

for all (deterministic) $Y \in \mathbb{R}^{D \times D}$ and $t > 0$. Taking now an arbitrary unit-norm vector $y \in \mathbb{R}^{D^2}$, denoting $Y \in \mathbb{R}^{D \times D}$ as the matrix satisfying $y = \text{vec}(Y)$, and using $\mathbb{E}[aa^T] = \frac{1}{D} \text{Id}_D$, we have

$$\langle X, y \rangle = \sqrt{\frac{D}{D-1}} \langle D a a^T - \text{Id}_D, Y \rangle$$

$$= \sqrt{\frac{D}{D-1}} \langle D a a^T - D \mathbb{E}[a a^T], Y \rangle$$

$$= \sqrt{\frac{D}{D-1}} \langle D(a^T Y a - \mathbb{E}[a^T Y a]) \rangle. \quad \text{(S.33)}$$
Combining (S.33), (S.32) and \( \|Y\|_2 \leq \|Y\|_F = 1 \), we obtain

\[
\mathbb{P}\left( (X, y) > \sqrt{\frac{D}{D-1}} t \right) \leq 2 \exp(-C' \min(t^2, t)).
\]

This implies \((X, y)\) is sub-exponential with sub-exponential norm \( C' \sqrt{D/(D-1)} = O(1) \). Taking a supremum over all unit-norm vectors \( y \in \mathbb{R}^D \) does not change this bound, since it is independent of \( y \), and so the random vector \( X \) has sub-exponential norm \( O(1) \). Thus, Proposition S.8 applies and implies there exists a universal constant \( C > 0 \) such that (S.29) holds with the probability in (S.28).

We now notice that (S.27) follows by substituting (S.29) into (S.31) and using that which gives (S.35). Equation (S.36) then follows by setting \( \frac{1}{D} \leq \frac{\sqrt{K}}{D} \log(eD/\sqrt{K}) \), because then

\[
\mu_K(G_2) \geq (1 - \frac{1}{D}) \left( 1 - C' \sqrt{\frac{K}{D}} \log\left( \frac{eD}{\sqrt{K}} \right) \right) \geq 1 - (C + 1) \sqrt{\frac{K}{D}} \log\left( \frac{eD}{\sqrt{K}} \right). \tag{S.34}
\]

To conclude, we justify the last sentence in Proposition S.7 where \( K = o(D^2) \). It only remains to note that, in this case, the right-most quantity in (S.34) tends to 1 as \( D \to \infty \). However, this holds because \( \frac{\sqrt{K}}{D} \to 0^+ \) by \( K = o(D^2) \), and \( \lim_{x \to 0^+} x \log(\frac{1}{x}) = 0 \) by L'Hôpital's rule.

As mentioned in the main body of the paper, this proof technique cannot be immediately applied to the case \( n > 2 \) because some key technical results are missing. We are not aware of extensions of Proposition S.8 to random variables with tails heavier than subexponential random variables. Further we do not know how to show that an auxiliary variable \( \hat{z} \) formed from higher-order vectorized tensors \( a_i^n \) satisfies a suitable tail bound, which would be needed in an extended Proposition S.8.

\section*{F Technical tools for proving Theorems 7 and 16}

In this section we prove the key technical tools to be used in the proofs of our landscape theorems.

\textbf{Lemma S.9} (Subspace perturbation effect). Let \( A, \hat{A} \) be any two subspaces of \( \text{Sym}(T^D_B) \), and define

\[
\Delta_A := \|P_A - P_{\hat{A}}\|_{F \to F} = \sup_{T \in \text{Sym}(T^D_B), \|T\|_F = 1} \|P_A(T) - P_{\hat{A}}(T)\|_F.
\]

Then for all \( S, T \) in \( T^D_B \), we have

\[
|\langle P_A(T) - P_{\hat{A}}(T), S \rangle| \leq \Delta_A \|\text{Sym}(T)\|_F \|\text{Sym}(S)\|_F \leq \Delta_A \|T\|_F \|S\|_F, \tag{S.35}
\]

and

\[
\|P_A(T)\|_F^2 - \|P_{\hat{A}}(T)\|_F^2 \leq \Delta_A \|\text{Sym}(T)\|_F^2 \leq \Delta_A \|T\|_F^2. \tag{S.36}
\]

\textbf{Proof.} Using the fact \( A, \hat{A} \subseteq \text{Sym}(T^D_B) \), the Cauchy-Schwarz inequality, and the definition of \( \Delta_A \),

\[
|\langle P_A(T) - P_{\hat{A}}(T), S \rangle| = \|P_A(\text{Sym}(T)) - P_{\hat{A}}(\text{Sym}(T)) S\|_F \|S\|_F \leq \Delta_A \|\text{Sym}(T)\|_F \|S\|_F,
\]

which gives (S.35). Equation (S.36) then follows by setting \( S = T \) in (S.35), and using \( \langle P_A(T) - P_{\hat{A}}(T), T \rangle = \|P_A(T)\|_F^2 - \|P_{\hat{A}}(T)\|_F^2 \) which holds since \( P_A \) and \( P_{\hat{A}} \) are orthogonal projectors.

Next we recall the remainder terms for \( P_A(x^n) \) with respect to \( x^{n-s} a_i^s \) (S.7). These are defined by

\[
\langle P_A(x^n), x^{n-s} a_i^s \rangle = : \sigma_i \xi_i^{n-s} + R_{i,s}. \tag{S.37}
\]

\textbf{Lemma S.10.} The following alternative expressions for the remainder terms hold:

1. For any \( x \in \mathbb{S}^{D-1}, i \in [K] \) and \( 0 \leq s \leq n \), we have

\[
R_{i,s} = \langle P_A(x^n), x^{n-s} a_i^s \rangle - \sigma_i \xi_i^{n-s}. \tag{S.37}
\]
2. For any $x \in S^{D-1}$ and $i \in [K]$, we have
\[ R_{i,n} = \zeta_i^n - \sigma_i. \] 

(S.38)

In the first step of the proofs of our two main results, we derive lower bounds for $\| A^T x \|_{\infty}$, where $x$ is a second-order critical point. For both theorems, this step is based on the following inequality.

Proposition S.11 (Tool #1). For any second-order critical point $x$ of $\nSPM-P$ and $i \in [K]$, it holds
\[ (1 + 2(n-1)\zeta_i^2)F(x) \geq (2n-1)\sigma_i\zeta_i^n - n\zeta_i^{n-2}R_{i,n} + (n-1)R_{i,2} - (4n-2)\Delta_A. \] 

(S.39)

Proof. We substitute $y = a_i$ into the inequality (7), which is a consequence of $x$ being first and second-order critical for $\nSPM-P$
\[ (1 + 2(n-1)\zeta_i^2)F(x) \geq n \| P_A(x^{n-1} a_i) \|_F^2 + (n-1)\| P_A(x^n), x^{n-2}a_i^2 \|. \] 

(S.40)

Then by the subspace perturbation results (S.35) and (S.36), we have
\[ F_A(x) = \| P_A(x^n) \|_F^2 \leq \| P_A(x^{n-1} a_i) \|_F^2 + \Delta_A = F_A(x) + \Delta_A, \] 

(S.41)
\[ \| P_A(x^{n-1} a_i) \|_F^2 \geq \| P_A(x^{n-1} a_i) \|_F^2 - \Delta_A \geq \| P_{\text{Span}(a_i^\infty)}(x^{n-1} a_i) \|_F^2 - \Delta_A \]
\[ = \zeta_i^{2n-2} - \Delta_A = \zeta_i^{2n-2} - (R_{i,n} + \sigma_i) - \Delta_A, \] 

(S.42)
\[ \langle P_A(x^n), x^{n-2}a_i^2 \rangle \geq \langle P_A(x^n), x^{n-2}a_i^2 \rangle - \Delta_A = R_{i,2} + \sigma_i\zeta_i^{n-2} - \Delta_A. \] 

(S.43)

Here we used $\text{Span}(a_i^\infty) \subseteq A$ in the second inequality of (S.42), the identity (S.38) in the second equality of (S.42), and the identity (S.37) in the equality of (S.43). Substituting (S.41), (S.42) and (S.43) into (S.40) completes the proof.\[ \square \]

In the second part of both proofs, we show concavity holds in a spherical cap near each $s a_i$, $i \in [K], s \in \{ -1, 1 \}$. As a starting point in both arguments, we use the next statement to upper bound the eigenvalues of the Riemannian Hessian.

Proposition S.12 (Tool #2). For any $x, z \in S^{D-1}$ with $z \perp x$, the Riemannian Hessian of $F_A$ satisfies, for any $i \in [K]$, 
\[ \frac{1}{2n} z^T \nabla^2_{S^{D-1}} F_A(x) z \leq n \| P_A(x^{n-1} z) \|_F^2 + (n-1)\| P_A(x^n), x^{n-2}z^2 \| - \zeta_i^{2n} + 4\Delta_A. \] 

(S.44)

Proof. We first show that for all integers $0 \leq s \leq n$
\[ \| \text{Sym}(x^{n-s} f^s) \|_F = \left( \frac{n}{s} \right)^{-1/2}. \] 

(S.45)

Let $R \in SO(D)$ (special orthogonal group). Then we have (denoting Tucker product on the RHS)
\[ \| \text{Sym}(x^{n-s} f^s) \|_F = \| \text{Sym}(x^{n-s} f^s) \times_1 R \times_2 \ldots \times_n R \|_F \] 

(S.46)
by rotational invariance of Frobenius norm. Furthermore,
\[ \text{Sym}(x^{n-s} f^s) \times_1 R \times_2 \ldots \times_n R = \text{Sym}(x^{n-s} f^s) \times_1 R \times_2 \ldots \times_n R = \text{Sym}(R x)^{n-s}(R z)^s \] 

(S.47)
by a direct calculation. Inserting (S.47) into (S.46), and choosing $R$ appropriately,
\[ \| \text{Sym}(x^{n-s} f^s) \|_F = \| \text{Sym}(e_1^{n-s} e_2^s) \|_F \]
where $e_1, e_2$ are the first two standard basis vectors of $\mathbb{R}^D$. However, $\text{Sym}(e_1^{n-s} e_2^s)$ is the tensor whose $(i_1, \ldots, i_n)$-entry equals $\binom{n}{s}^{-1}$ if $(i_1, \ldots, i_n)$ consists of $n-s$ ones and $s$ twos (in some order), and equals 0 otherwise. Hence,
\[ \| \text{Sym}(e_1^{n-s} e_2^s) \|_F = \sqrt{\binom{n}{s} \cdot \binom{n}{s}^{-2}} = \binom{n}{s}^{-1/2}. \]
We now recall some basics about geodesic convexity on the sphere. The standard notion of concavity for twice-differentiable functions on the sphere is as follows.

Let \( y \in \mathbb{S}^{D-1} \) and \( r \in (0, 1) \), define the spherical cap with center \( y \) and height \( r \) by\n\[
B_r(y) := \{ x \in \mathbb{S}^{D-1} : \langle x, y \rangle \geq 1 - r \} \subseteq \mathbb{S}^{D-1}.
\]

Given distinct points \( x_1, x_2 \in B_r(y) \), the geodesic segment connecting \( x_1 \) to \( x_2 \) is the curve \( c : [0, 1] \rightarrow B_r(y) \) defined by\n\[
c(t) := \cos(t\theta)x_1 + \sin(t\theta)x^\perp_2 \quad \text{for} \quad 0 \leq t \leq 1. \tag{S.49}\]

Here \( \theta := \cos^{-1}(\langle x_1, x_2 \rangle) \in (0, \pi) \) is the angle between \( x_1 \) and \( x_2 \), and \( x^\perp_2 \) is the component of \( x_2 \) that is orthogonal to \( x_1 \) (normalized to lie on the unit sphere) given by\n\[
x^\perp_2 := \frac{x_2 - \langle x_1, x_2 \rangle x_1}{\|x_2 - \langle x_1, x_2 \rangle x_1\|_2} = \frac{-\langle x_1, x_2 \rangle}{\sqrt{1 - \langle x_1, x_2 \rangle^2}} x_1 + \frac{1}{\sqrt{1 - \langle x_1, x_2 \rangle^2}} x_2 \in \mathbb{S}^{D-1}.\]

(If \( x_1 = x_2 \), the geodesic segment connecting \( x_1 \) to \( x_2 \) is the constant curve at \( x_1 \).)

The standard notion of concavity for twice-differentiable functions on the sphere is as follows.

**Definition S.13.** Spherical caps and geodesics are defined on the unit sphere as follows.

1. For \( y \in \mathbb{S}^{D-1} \) and \( r \in (0, 1) \), define the spherical cap with center \( y \) and height \( r \) by
   \[
   B_r(y) := \{ x \in \mathbb{S}^{D-1} : \langle x, y \rangle \geq 1 - r \} \subseteq \mathbb{S}^{D-1}.
   \]

2. Given distinct points \( x_1, x_2 \in B_r(y) \), the geodesic segment connecting \( x_1 \) to \( x_2 \) is the curve \( c : [0, 1] \rightarrow B_r(y) \) defined by
   \[
   c(t) := \cos(t\theta)x_1 + \sin(t\theta)x^\perp_2 \quad \text{for} \quad 0 \leq t \leq 1. \tag{S.49}\]

Here \( \theta := \cos^{-1}(\langle x_1, x_2 \rangle) \in (0, \pi) \) is the angle between \( x_1 \) and \( x_2 \), and \( x^\perp_2 \) is the component of \( x_2 \) that is orthogonal to \( x_1 \) (normalized to lie on the unit sphere) given by
\[
x^\perp_2 := \frac{x_2 - \langle x_1, x_2 \rangle x_1}{\|x_2 - \langle x_1, x_2 \rangle x_1\|_2} = \frac{-\langle x_1, x_2 \rangle}{\sqrt{1 - \langle x_1, x_2 \rangle^2}} x_1 + \frac{1}{\sqrt{1 - \langle x_1, x_2 \rangle^2}} x_2 \in \mathbb{S}^{D-1}.\]

(If \( x_1 = x_2 \), the geodesic segment connecting \( x_1 \) to \( x_2 \) is the constant curve at \( x_1 \).)

The standard notion of concavity for twice-differentiable functions on the sphere is as follows.

**Definition S.14.** Let \( F \) be a real-valued function defined on an open subset of \( \mathbb{S}^{D-1} \) containing the spherical cap \( B_r(y) \). Assume that \( F \) is twice-differentiable.

1. We say \( F \) is geodesically strictly concave on \( B_r(y) \) if the Riemannian Hessian of \( F \) is negative definite throughout \( B_r(y) \),
   \[
   z^T \nabla^2_{\mathbb{S}^{D-1}} F(x) z < 0, \quad \forall x \in B_r(y) \forall z \in \mathbb{S}^{D-1} \text{ with } x \perp z. \tag{S.50}\]

2. Let \( \mu > 0 \). We say \( F \) is geodesically \( \mu \)-strongly concave on \( B_r(y) \) if the Riemannian Hessian of \( F \) satisfies the following eigenvalue bound throughout \( B_r(y) \),
   \[
   z^T \nabla^2_{\mathbb{S}^{D-1}} F(x) z \leq -\mu, \quad \forall x \in B_r(y) \forall z \in \mathbb{S}^{D-1} \text{ with } x \perp z. \tag{S.51}\]

The terminology in Definition S.14 is justified by the following standard lemma.

**Lemma S.15 (Restricting to geodesics).** Let \( F \) be a real-valued function defined on an open subset of \( \mathbb{S}^{D-1} \) containing the spherical cap \( B_r(y) \). Assume that \( F \) is twice-differentiable.

1. If \( F \) is geodesically strictly concave on \( B_r(y) \), then for all geodesic segments \( c : [0, 1] \rightarrow B_r(y) \) with image contained in \( B_r(y) \) and \( c(0) \neq c(1) \), the pulled-back function \( F \circ c : [0, 1] \rightarrow \mathbb{R} \) is strictly concave on \( [0, 1] \).
2. Let \( \mu > 0 \). If \( F \) is geodesically \( \mu \)-strongly concave on \( B_r(y) \), then for all geodesic segments \( c : [0, 1] \to B_r(y) \) with image contained in \( B_r(y) \) and \( c(0) \neq c(1) \), the pulled-back function \( F \circ c : [0, 1] \to \mathbb{R} \) is \( \mu L(c)^2 \)-strongly concave on \([0, 1]\). Here \( L(c) := \int_0^1 \|c'(t)\| dt = \cos^{-1}(\langle c(0), c(1) \rangle) \) denotes the length of \( c \).

**Proof.** See [9, Thm. 11.19(3), Def. 11.3] and [9, Thm. 11.19(2), Def. 11.5] respectively.

The usual implications of geodesic concavity are as follows.

**Lemma S.16.** Assume the setup of Lemma S.15

1. If \( F \) is geodesically strictly concave on \( B_r(y) \), there exists at most one point \( x \in B_r(y) \) where \( \nabla_{S^{D-1}} F(x) = 0 \). Such a point \( x \) is automatically a strict local maximizer of \( F \).

2. Let \( \mu > 0 \). If \( F \) is geodesically \( \mu \)-strongly concave on \( B_r(y) \), then for all geodesics segments \( c \) as in the lemma above, \( F \circ c \) is upper-bounded by a concave quadratic function via:

\[
(F \circ c)(t) \leq (F \circ c)(0) + (F \circ c)'(0) t - \frac{\mu L(c)^2}{2} t^2, \quad \forall t \in [0, 1].
\]

**Proof.** 1. We first show that for all geodesic segments \( c : [0, 1] \to B_r(y) \) with \( c(0) \neq c(1) \), the derivative \( (F \circ c)' \) is strictly decreasing on \([0, 1]\). To see this, note that since \( F \) is twice-differentiable, \( F \circ c \) is twice-differentiable; and since \( c \) is a geodesic it holds

\[
(F \circ c)''(t) = \dot{c}(t)^T \nabla_{S^{D-1}} F(c(t)) \dot{c}(t),
\]

by [9, Eq. (5.30), page 106]. The right-hand side of \((S.52)\) is strictly negative by \((S.50)\) and the fact \( \dot{c}(t) \neq 0 \) (note \( \|\dot{c}(t)\|_2 = \cos^{-1}(\langle c(0), c(1) \rangle) \) and we are assuming \( c(0) \neq c(1) \)). Thus \((F \circ c)''\) is strictly negative on \([0, 1]\). Hence by the mean value theorem, \((F \circ c)'\) is strictly decreasing on \([0, 1]\).

Now, to deduce item 1 in the lemma, assume for a contradiction that there exist two distinct points \( x_1, x_2 \in B_r(y) \) where the Riemannian gradient vanishes. Let \( c \) be the geodesic segment connecting \( x_1 \) to \( x_2 \). Again by [9, Eq. (5.30), page 106],

\[
(F \circ c)'(t) = \dot{c}(t)^T \nabla_{S^{D-1}} F(c(t)).
\]

By the vanishing Riemannian gradient assumption, this implies \((F \circ c)'(0) = 0\) and \((F \circ c)'(1) = 0\). But that contradicts the fact that \((F \circ c)'\) is strictly decreasing. So item 1 follows.

2. By Taylor’s theorem applied to \( F \circ c \), for each \( t \in [0, 1] \) there exists \( \xi \in [0, t] \) such that

\[
(F \circ c)(t) = (F \circ c)(0) + (F \circ c)'(0) t + \frac{(F \circ c)''(\xi)}{2} t^2. \tag{S.53}
\]

From \((S.52)\), \((S.51)\) and \( \|\dot{c}(\xi)\|_2 = L(c) \), we get \((F \circ c)''(\xi) \leq -\mu L(c)^2\). Insert this into \((S.53)\).

The next tool is how we complete the proofs of our main results, by showing a second-order critical point with sufficiently large functional value must land in a spherical cap around one of \( sa_i \).

**Proposition S.17** (Tool #3). Let \( \{a_i : i \in [K]\} \subseteq S^{D-1} \) be any system of vectors, let \( \mathcal{A} = \text{Span}\{a_i^n : i \in [K]\} \subseteq \text{Sym}(T^n_{S^n}) \) be the corresponding subspace of tensors and let \( \hat{\mathcal{A}} \subseteq \text{Sym}(T^n_{S^n}) \) be another subspace of tensors which is a perturbation of \( \mathcal{A} \) with approximation error \( \Delta_A \). Let \( 0 < r \leq R < 1 \), so there is an inclusion of spherical caps \( B_r(sa_i) \subseteq B_R(sa_i) \). Assume \( F_{\hat{\mathcal{A}}} \) is geodesically \( n \)-strongly concave on the inner cap \( B_r(sa_i) \), the height of the inner cap satisfies \( r > \Delta_A/n \), and \( F_{\hat{\mathcal{A}}} \) is geodesically strictly concave on the outer cap \( B_R(sa_i) \). Then, in \( B_R(sa_i) \) there exists a unique first-order critical point \( x^* \) of \text{nSPM-P}. Further, this point is a strict local maximizer for \text{nSPM-P} on \( S^{D-1} \). Finally, the distance between \( x^* \) and \( sa_i \) is upper-bounded independently of \( R \) and \( r \) via:

\[
\|x^* - sa_i\|_2^2 \leq \frac{2\Delta_A}{n}. \tag{S.54}
\]
We start by relating the auxiliary scalars \( R \) and \( s \). Thus (S.54) follows from (S.57) as desired. We can quickly obtain the rest of the statement, and see that

\[
\begin{align*}
\text{Proof.} & \quad \text{For each } R, \\
& \quad \text{We prove our deterministic theorem. Here we define } \zeta, \\
& \quad \text{In this section we prove our deterministic theorem. Here we define } \zeta, \\
& \quad \text{Theorem 7} \\
& \quad \text{Proof of Theorem 7} \\
& \quad \text{Proof.} \\
& \quad \text{For then from the assumption that } r > \Delta_A/\kappa, \text{ we get } 1 - \langle x^\star, sa_i \rangle < r. \text{ Hence } x^\star \text{ lies strictly in the interior of the inner cap } B_r(sa_i). \text{ Therefore } x^\star \text{ is a local maximizer of } F \text{ on } S^{D-1}. \text{ By Lemma S.16 a) and strict concavity, } x^\star \text{ is in fact a strict local maximizer. This ends the proof.} \quad \square
\end{align*}
\]

G Proof of Theorem 7

In this section we prove our deterministic theorem. Here we define \( \zeta, \sigma \) and \( R_{i,s} \) as in Definition S.1.

G.1 Bounds on \( R_{i,s} \) and \( \| G_n^{-1} \|_2 \)

We start by relating the auxiliary scalars \( R_{i,s} \) to the frame constants \( \rho_s \).

Lemma S.18. For each \( i \in [K] \text{ and } s \in [n], \) we have

\[
\begin{align*}
|R_{i,s}| & \leq \max_{t \neq i} |\sigma_t \zeta_t^{n-s}| \rho_s \leq \| \sigma \circ \zeta^{n-s} \|_\infty \rho_s. \\
\text{If } s \text{ is even, then we have a slight refinement:} \\
\min_{\ell \neq i} \sigma_\ell \zeta_\ell^{n-s} \rho_s & \leq R_{i,s} \leq \max_{\ell \neq i} \sigma_\ell \zeta_\ell^{n-s} \rho_s. \\
\end{align*}
\]

Proof. For each \( i \) and \( s \), use the definition of \( R_{i,s} \), the triangle inequality and the definition of \( \rho_s \):

\[
| R_{i,s} | = | \sum_{\ell \neq i} \sigma_\ell \zeta_\ell^{n-s} (a_i, a_\ell)^s | \leq \sum_{\ell \neq i} |\sigma_\ell \zeta_\ell^{n-s}| |(a_i, a_\ell)^s| \leq \max_{\ell \neq i} |\sigma_\ell \zeta_\ell^{n-s}| \sum_{\ell \neq i} |(a_i, a_\ell)^s| \\
\leq \max_{\ell \neq i} |\sigma_\ell \zeta_\ell^{n-s}| \sum_{\ell = 1}^{K} |(a_i, a_\ell)^s| - 1 \leq \max_{\ell \neq i} |\sigma_\ell \zeta_\ell^{n-s}| \rho_s \leq \| \sigma \circ \zeta^{n-s} \|_\infty \rho_s.
\]

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We begin the proof of Theorem 7 by showing that any constrained second-order critical point
without loss of generality that
Our starting point is Eq. (S.39) in Proposition S.11, which says that for each \( R \)
Corollary S.20. \( \sigma \)
Firstly using (S.63), the fact \( R \)
\( \sigma \)
\( \| \)
Fix a second-order critical point \( x \)
Proposition S.21.
This follows immediately from (S.61), and the bound 
\[
\| G_n^{-1} \|_2 \leq \frac{1}{1 - \rho_n} \leq \frac{36}{35},
\]
Proof. This follows immediately from \( (S.61) \), and the bound \( 1 - \rho_n \leq \mu_n(G_n) \) in Lemma 6 \( \Box \)
G.2 Lower bound on maximum correlation coefficient
We begin the proof of Theorem 7 by showing that any constrained second-order critical point \( x \) has at least one large correlation coefficient.

Proposition S.21. Consider a system \( \{ a_i : i \in [K] \} \) that satisfies the assumptions of Theorem 7. For a second-order critical point \( x \) of \( nSPM-P \) then either \( F_A(x) = 0 \) or we have
\[
\| \zeta \|_\infty \geq 1 - \left( 1 + \frac{n}{2(n-1)} \right) \frac{\rho_2}{1 + \rho_2} - \frac{n}{2(n-1)} \frac{\rho_n}{(1 - 2\rho_n)(1 + \rho_2)} - \frac{4n - 2}{2(n-1)} \frac{\Delta_A}{F_A(x)}.
\]
Proof. Fix a second-order critical point \( x \in S^{D-1} \) of \( nSPM-P \) and let \( \zeta = \zeta(x) = A^T x \in \mathbb{R}^K \).
Since we can change \( a_i \) to \(-a_i\) without altering the function \( F_A \) or the constants \( \rho_i \), we may assume without loss of generality that \( \zeta \geq 0 \) for each \( \ell \) \[ \in [K] \). We may also assume that \( F_A(x) \neq 0 \).
Our starting point is Eq. (S.39) in Proposition S.11 which says that for each \( \ell \) \[ \in [K] \) we have
\[
(1 + 2(n-1)\zeta_i^2) F_A(x) \geq (2n - 1)\sigma_i \zeta_i^{n-2} + n\zeta_i^{n-2}R_{i,n} + (n-1)R_{i,2} - (4n - 2)\Delta_A.
\]
(S.62)
Since we want to deduce that \( \| \zeta \|_\infty \) is large, we fix \( \ell \) \[ \in [K] \) to be an index such that the first (main) term on the right-hand side of (S.62) is maximal; that is, let \( \ell \) \[ \in \text{argmax}_{\ell \in [K]} \sigma_i \zeta_i^{n-2} \).
Now notice
\[
0 < F_A(x) = (P_A(x^n), x^n) = \left( \sum_{\ell=1}^K \sigma_i a_i^n, x^n \right) = \sum_{\ell=1}^K \sigma_i \zeta_i^n \leq \sum_{\ell=1}^K \sigma_i \zeta_i^{n-2} \sum_{\ell=1}^K \zeta_i^2
\]
(S.63)
from which it follows that \( \sigma_i \zeta_i^{n-2} > 0 \). In (S.62), our goal is to bound \( F_A(x) \) on the left-hand side from above, and \( R_{i,n}, R_{i,2} \) on the right-hand side from below, all by quantities involving \( \zeta_i \).
Firstly using (S.63), the fact \( \sigma_i \zeta_i^{n-2} > 0 \) and the definition of \( \rho_2 \), we have
\[
F_A(x) \leq \sum_{\ell=1}^K \sigma_i \zeta_i^n \leq \sigma_i \zeta_i^{n-2} \sum_{\ell=1}^K \zeta_i^2 = \sigma_i \zeta_i^{n-2}(1 + \rho_2).
\]
(S.64)
Then by Lemma 3.18, Eq. (S.59) with \( s = n \) we have
\[
R_{i,2} \geq \min_{\ell \neq i} \sigma \zeta^n - \rho_2, \quad (S.65)
\]
while by (S.58) with \( s = n \) we have
\[
\zeta^n - R_{i,n} \geq -\|\sigma\|_\infty \rho_n. \quad (S.66)
\]
To further bound the right-hand sides in (S.65) and (S.66) in terms of \( \zeta_i \), we now show that \( \sigma_i \approx \|\sigma\|_\infty \) and that \( \min_{\ell \neq i} \sigma \zeta^n - \rho_2 \leq -\sigma_i \zeta^n - \rho_2 \).

**Showing \( \sigma_i \approx \|\sigma\|_\infty \):** Let \( j \in [K] \) be an index such that \( |\sigma_j| = \|\sigma\|_\infty \). We note that \( \sigma_j \geq 0 \), because if \( \sigma_j < 0 \), then using \( \zeta_j \geq 0 \) and (S.58) we would have
\[
\|\sigma\|_\infty = |\sigma_j| \leq |\sigma_j - \zeta_j^n| = |R_{j,n}| \sum_{\ell \neq j} \rho \leq |R_{j,n}| + |R_{i,n}| \leq 2\sigma_j \rho_n.
\]
This contradicts the assumption \( \rho_n < 1 \), so indeed \( \sigma_j \geq 0 \). Now consider the three facts:

1. \( \sigma_i \zeta^n - \sigma_j \zeta^n \) by definition of \( i \);
2. \( \sigma_j \geq \sigma_i \) by \( \sigma_j \geq 0 \) and definition of \( j \);
3. \( \zeta_i \geq 0 \) and \( \zeta_j \geq 0 \).

It follows that \( \zeta_i \geq \zeta_j \). Using this and the bound for the auxiliary constants (S.58), we can estimate:
\[
\sigma_j - \sigma_i \leq \sigma_j - \zeta_j^n + \zeta_i^n - \sigma_i = -R_{j,n} + R_{i,n} \leq |R_{j,n}| + |R_{i,n}| \leq 2\sigma_j \rho_n.
\]
Rearranging gives
\[
\sigma_j \geq \sigma_i \geq (1 - 2\rho_n)\sigma_j = (1 - 2\rho_n) \|\sigma\|_\infty. \quad (S.67)
\]
By the assumptions in Theorem 7 and (S.61), in particular \( \rho_n / (1 - 2\rho_n) > 0 \). Hence multiplying throughout by \( \rho_n / (1 - 2\rho_n) \) in (S.67) yields
\[
\rho_n \frac{\sigma_i}{1 - 2\rho_n} \leq \rho_n \|\sigma\|_\infty,
\]
and then substituting into (S.66) we obtain
\[
\zeta_i^n - R_{i,n} \geq -\zeta_i^n - \rho_n \geq -\zeta_i^n - \frac{\rho_n}{1 - 2\rho_n} \sigma_i. \quad (S.68)
\]

**Showing \( \min_{\ell \neq i} \sigma \zeta^n - \rho_2 \leq -\sigma_i \zeta^n - \rho_2 \):** Assume to the contrary that \( \min_{\ell \neq i} \sigma \zeta^n - \rho_2 < -\sigma_i \zeta^n - \rho_2 \) and let \( k \) be an index attaining the minimum on the left-hand side. Since \( \sigma \zeta^n > 0 \) and \( \zeta^n \geq 0 \), it must be that \( \sigma_k < 0 \). Using \( \sigma_k < 0 \) and \( \zeta_k \geq 0 \), we estimate
\[
|\sigma_k| \leq |\sigma_k - \zeta_k^n| = |R_{k,n}| \leq \|\sigma\|_\infty \rho_n.
\]
This implies
\[
\|\sigma \zeta^n - \rho_2 \|_\infty = \max \{\sigma_i \zeta^n, -\sigma_k \zeta^n\} = -\sigma_k \zeta^n \leq \|\sigma\|_\infty \zeta^n - \rho_n.
\]
Similarly to the previous part, using \( -\sigma_k \zeta^n > \sigma_k \zeta^n \) we can see \( \zeta_k \geq \zeta_j \), where \( j \) is again the index such that \( \sigma_j = \|\sigma\|_\infty \). Hence, as before, we have
\[
\sigma_j - \sigma_k = \sigma_j - \zeta_j^n + \zeta_k^n - \sigma_k = -R_{j,n} + R_{k,n} \leq 2\sigma_j \rho_n,
\]
which rearranges to \( \sigma_k \geq (1 - 2\rho_n)\sigma_j \). Since we have previously shown \( \sigma_j \geq 0 \) and we know \( \rho_n < 1/2 \) by (S.61), this is in contradiction with \( \sigma_k < 0 \). Thus it follows that \( \min_{\ell \neq i} \sigma \zeta^n - \rho_2 \geq -\sigma_i \zeta^n - \rho_2 \) as desired. Substituting into (S.65) yields
\[
R_{i,2} \geq \min_{\ell \neq i} \sigma \zeta^n - \rho_2 \geq -\sigma_i \zeta^n - \rho_2. \quad (S.69)
\]
We can now complete the proof of the proposition. Substituting the bounds (S.64), (S.68), and (S.69) into the second-order criticality inequality (S.62), we get

\[(1 + 2(n - 1)\zeta_i^2)(1 + \rho_2)\sigma_i\zeta_i^{n-2} \geq (2n - 1)\sigma_i\zeta_i^{n-2} - n\frac{\rho_n}{1 - 2\rho_n}\sigma_i\zeta_i^{n-2} - (n - 1)\rho_2\sigma_i\zeta_i^{n-2} - (4n - 2)\Delta_A.\]

Dividing by \((1 + \rho_2)\sigma_i\zeta_i^{n-2} > 0\), subtracting 1 and then dividing by \(2(n - 1)\), this simplifies to

\[\zeta_i^2 \geq \frac{1}{1 + \rho_2} - \frac{n}{2(n - 1)} \left(\frac{\rho_n}{(1 - 2\rho_n)(1 + \rho_2)} + \frac{\rho_2}{1 + \rho_2}\right) - \frac{4n - 2}{2(n - 1)} \frac{\Delta_A}{(1 + \rho_2)\sigma_i\zeta_i^{n-2}}.\]

We replace the denominator in the last term by \(F_A(x)\) by re-using the bound \(F_A(x) \leq \sigma_i\zeta_i^{n-2}(1 + \rho_2)\) (S.64). Then we rearrange the terms and substitute \(||\zeta||_\infty^2 \geq \zeta_i^2\). This gives the first inequality in the proposition. The second stated inequality is an immediate consequence of the first, where we simplify each term up to a constant factor using \(\frac{n}{2(n - 1)} \leq 1, 1 + \rho_2 \geq 1, 1 - 2\rho_n > \frac{1}{2}\) and \(\frac{4n - 2}{2(n - 1)} \leq 3\). \(\square\)

G.3 Concavity

In the next proof step we analyze the Riemannian Hessian and show that it is strictly negative definite.

**Proposition S.22.** Consider a system \(\{a_i : i \in [K]\}\) that satisfies the assumptions in Theorem 7. For any \(x, z \in S^{D-1}\) with \(z \perp x\), the Riemannian Hessian of \(F_A\) satisfies

\[z^T \nabla^2_{S^{D-1}} F_A(x)z \leq -2n + 2n^2 \frac{2 - \rho_n}{1 - \rho_n} (1 - ||\zeta||_\infty^2) + 2n(3n - 2) \frac{\rho_n}{1 - \rho_n} + 8n\Delta_A. \quad (S.70)\]

**Proof.** Let \(i \in [K]\) be such that \(||\zeta|| = ||\zeta||_\infty\) where \(\zeta = \zeta(x) = A^T x\). Our starting point is (S.44) in Proposition S.12 which reads

\[\frac{1}{2n} z^T \nabla^2_{S^{D-1}} F_A(x)z \leq n \left\| P_A(x^{n-1}z) \right\|_F^2 + (n - 1) \left( P_A(x^n), x^{n-2}z^2 \right) - \zeta_i^2 \geq \zeta_i^2 + 4\Delta_A. \quad (S.71)\]

Expanding \(P_A(x^n) = \sum_{\ell=1}^K \sigma_i a_i a_i^T\) as in Definition S.1 we can upper-bound the second term on the right-hand side of (S.71) as follows by applying Holder’s inequality and the definition of \(\rho_n\):

\[\left( P_A(x^n), x^{n-2}z^2 \right) = \sum_{\ell=1}^K \sigma_i \zeta_i^{n-2}(a_\ell, z)^2 = \sum_{\ell=1}^K \zeta_i^{n-2}(a_\ell, z)^2 + \sum_{\ell=1}^K (\sigma_i - \zeta_i^{n-2})(a_\ell, z)^2 \]

\[\leq \sum_{\ell=1}^K \zeta_i^{n-2}(a_\ell, z)^2 + \left\| \sigma - \zeta_i^{n-2} \right\|_\infty \sum_{\ell=1}^K \zeta_i^{n-2}(a_\ell, z)^2 \]

\[\leq \sum_{\ell=1}^K \zeta_i^{n-2}(a_\ell, z)^2 + \left\| \sigma - \zeta_i^{n-2} \right\|_\infty \left( \sum_{\ell=1}^K \zeta_i^{n-2}(a_\ell, z)^2 \right)^2 \]

\[\leq \sum_{\ell=1}^K \zeta_i^{n-2}(a_\ell, z)^2 + \left\| \sigma - \zeta_i^{n-2} \right\|_\infty (1 + \rho_n). \quad (S.72)\]

Further we can quickly bound the second term in the right-hand side of (S.72). Firstly by (S.38) and (S.58), we have \(||\sigma - \zeta_i^{n-2}||_\infty \leq 1 + ||\sigma - \zeta_i^{n-2}||_\infty \leq 1 + ||\sigma||_\infty \rho_n \Rightarrow ||\sigma||_\infty \leq (1 - \rho_n)^{-1}||\sigma||_\infty\). Then we can bound \(||\sigma||_\infty\) via

\[||\sigma||_\infty = ||\sigma - \zeta_i^{n-2} + \zeta_i^{n-2}||_\infty \leq 1 + ||\sigma - \zeta_i^{n-2}||_\infty \leq 1 + ||\sigma||_\infty \rho_n \Rightarrow ||\sigma||_\infty \leq (1 - \rho_n)^{-1}.\]

Putting the last two sentences together gives us

\[||\sigma - \zeta_i^{n-2}||_\infty (1 + \rho_n) \leq ||\sigma||_\infty \rho_n (1 + \rho_n) \leq \frac{1 + \rho_n}{1 - \rho_n} \rho_n. \quad (S.73)\]

As for the first term on the right-hand side of (S.72), we first split the sum:

\[\sum_{\ell=1}^K \zeta_i^{n-2}(a_\ell, z)^2 = \zeta_i^{n-2}(a_i, z)^2 + \sum_{\ell \neq i} \zeta_i^{n-2}(a_\ell, z)^2. \quad (S.74)\]
We use Bessel’s inequality and the assumptions \( x, z \in \mathbb{S}^{D-1} \) and \( x \perp z \) to get
\[
\zeta_i^{2n-2} \langle a_i, z \rangle^2 = \zeta_i^{2n-2}(\langle a_i, z \rangle^2 + \langle a_i, x \rangle^2 - \zeta_i^2) \leq \zeta_i^{2n-2}(\|a_i\|^2 - \zeta_i^2) \leq \zeta_i^{2n-2}(1 - \zeta_i^2).
\]
(S.75)

Then we estimate
\[
\sum_{\ell \neq i} \zeta_i^{2n-2} \langle a_\ell, z \rangle^2 \leq \sum_{\ell \neq i} \zeta_i^{2n-2} = K \zeta_i^{2n-2} - \zeta_i^2 \leq 1 + \rho_{2n-2} - \zeta_i^{2n-2} \leq 1 + \rho_n - \zeta_i^{2n-2},
\]
where the last inequality used \( 2n - 2 \geq n \). Finally we substitute (S.75) and (S.76) into (S.74) to get
\[
\sum_{\ell = 1}^K \zeta_i^{2n-2} \langle a_\ell, z \rangle^2 \leq 1 - \zeta_i^{2n} + \rho_n.
\]
(S.77)

Combining (S.72), (S.77), (S.73) we arrive at:
\[
\langle P_A(x^n), x^{n-2} z \rangle^2 \leq 1 - \zeta_i^{2n} + \rho_n + \frac{1 + \rho_n}{1 - \rho_n} \rho_n.
\]
(S.78)

At this point, we turn to the first term on the right-hand side of (S.71). Recalling Lemma S.3 Definition S.1 and Corollary S.20 we may write
\[
\|P_A(x^{n-1} z)\|_F^2 = \eta(x^{n-1} z)^\top G_n^{-1} \eta(x^{n-1} z),
\]
(S.79)

where \( \eta(x^{n-1} z)_\ell := \langle x^{n-1} z, q_n^\ell \rangle = \zeta_i^{n-1} \langle a_\ell, z \rangle \). Clearly (S.79) is less than or equal to
\[
\|G_n^{-1}\|_2 \|\eta(x^{n-1} z)\|_2^2 = \|G_n^{-1}\|_2 \sum_{\ell = 1}^K \zeta_i^{2n-2} \langle a_\ell, z \rangle^2.
\]

Re-using the just-proven (S.77) as well as Corollary S.20 it follows that
\[
\|P_A(x^{n-1} z)\|_F^2 \leq \|G_n^{-1}\|_2 \sum_{\ell = 1}^K \zeta_i^{2n-2} \langle a_\ell, z \rangle^2 \leq \frac{1 - \zeta_i^{2n} + \rho_n}{1 - \rho_n}.
\]
(S.80)

To complete the proof, we plug (S.80) and (S.78) into (S.71) and rearrange terms to get (S.70). □

As a corollary of S.22, we can now prove the existence of exactly one local maximizer within spherical caps of the global maximizers \( \pm a_i' \)'s of the clean objective \( F_A \).

**Corollary S.23.** Consider a system \( \{a_i : i \in [K]\} \) that satisfies the assumptions in Theorem 7. Define the height \( r_+ \in (0, 1) \) by
\[
1 - r_+ := \left( 1 - \frac{1 - \rho_n}{n - 2 - \rho_n} + \frac{3n - 2}{n} \frac{\rho_n}{2 - \rho_n} + \frac{4(1 - \rho_n) \Delta A}{n(2 - \rho_n)} \right)^{\frac{1}{3n}}.
\]

Then the 2K spherical caps \( B_{r_+}(a_1), B_{r_+}(-a_1), B_{r_+}(a_2), \ldots, B_{r_+}(-a_K) \) are disjoint. Further for each \( s \in \{-1, +1\} \) and \( i \in [K] \), there exists exactly one first-order critical point \( x_s \) of \( F_A \) in \( B_{r_+}(sa_i) \). Moreover \( x_s \) is a strict local maximum of \( F_A \) (hence second-order critical), and
\[
\|x_s - sa_i\|_2^2 \leq \frac{2 \Delta A}{n}.
\]

**Proof.** First, we check that \( r_+ \in (0, 1) \). Indeed \( r_+ > 0 \) is clear from (S.61), and \( r_+ < 1 \) is equivalent to \( (3n - 1) \rho_n + 4(1 - \rho_n) \Delta A < 1 \) which holds by the assumptions in Theorem 7. Also
\[
1 - r_+ \leq (1 - \frac{1}{3n} + 2 \rho_n + \frac{2}{n} \Delta A)^{\frac{1}{3n}} \text{ as stated, by the bound on } \rho_n \text{ (S.61).}
\]
Next, we check that the $2K$ spherical caps are disjoint. For a contradiction, suppose there exist distinct $(s, i), (s', i') \in \{-1, 1\} \times [K]$ and $x \in S^{D-1}$ such that $x \in B_{r_+}(sa_i) \cap B_{r_+}(sa_{i'})$. Thus $(x, sa_i) \geq 1-r_+$ and $(x, sa_{i'}) \geq 1-r_+$. By the triangle inequality with respect to geodesic distances on the sphere, we know $\arccos (sa_i, sa_{i'}) \leq \arccos (sa_i, x) + \arccos (x, sa_{i'}) \leq 2 \arccos (1-r_+)$, hence by the double angle formula for cosine $(sa_i, sa_{i'}) \geq 2(1-r_+)^2 - 1$. But note that this gives a lower bound on $\rho_2$: letting $y \in S^{D-1}$ be the midpoint of the arc joining $sa_i$ and $sa_{i'}$, we have

$$\rho_2 \geq \langle y, sa_i \rangle^2 + \langle y, sa_{i'} \rangle^2 - 1 = \langle sa_i, sa_{i'} \rangle^2 - 1 = \langle sa_i, sa_{i'} \rangle \geq 2(1-r_+)^2 - 1,$$

where the second equality is again by the double angle formula. Therefore

$$\rho_2 \geq 2(1-r_+)^2 - 1 \geq 2 \left(1 - \frac{1}{2n}\right)^2 - 1 > 2 \left(1 - \frac{1}{2n}\right) - 1 = 1 - \frac{1}{n},$$

which contradicts the implied upper bound on $\rho_2$ \cite{S60}. So the $2K$ spherical caps are disjoint.

For the remaining statements we apply Proposition \cite{S17}. Thus it suffices to check that $F_{\hat{A}}$ is strictly $n$-concave on $B(sa_i, 1-\frac{\Delta_A}{n})$ and strictly concave in the interior of $B_{r_+}(sa_i)$. To verify these, we will use Proposition \cite{S22} to upper-bound the eigenvalues of the Riemannian Hessian of $F_{\hat{A}}$.

So for the strict $n$-concavity condition, we want to show:

$$-2n + 2n^2 \frac{2 - \rho_n}{1 - \rho_n} (1 - (1 - \frac{\Delta_A}{n})^{2n}) + 2n(3n - 2) \frac{\rho_n}{1 - \rho_n} + 8n\Delta_A < -n. \quad (S.81)$$

Since $(1-x)^{2n} \geq 1 - 2nx$ for $x \in \mathbb{R}$, we have

$$\left(1 - \frac{\Delta_A}{n}\right)^{2n} \geq 1 - 2\Delta_A,$$

whence \eqref{S1} is implied by

$$-2n + 4n^2 \frac{2 - \rho_n}{1 - \rho_n} \Delta_A + 2n(3n - 2) \frac{\rho_n}{1 - \rho_n} + 8n\Delta_A < -n. \quad (S.82)$$

Rearranging terms, Eq. \eqref{S2} is equivalent to

$$(4n(2 - \rho_n) + 8(1 - \rho_n)) \Delta_A + (6n - 3)\rho_n < 1. \quad (S.83)$$

However indeed, \eqref{S3} holds as a consequence of the assumptions in Theorem \cite{S7}.

As for the concavity condition on $B_{r_+}(sa_i)$, we want to show:

$$-2n + 2n^2 \frac{2 - \rho_n}{1 - \rho_n} (1 - (1 - r_+)^{2n}) + 2n(3n - 2) \frac{\rho_n}{1 - \rho_n} + 8n\Delta_A \leq 0. \quad (S.84)$$

However upon substituting in the definition of $r_+$, we see that the left-hand side of \eqref{S4} is precisely 0, i.e., $r_+$ was chosen to be maximal so that \eqref{S4} holds. This completes the proof of the corollary. \hfill $\square$

### G.4 Completing the proof of Theorem \cite{S7}

Finally, we can conclude the proof of Theorem \cite{S7} by combining the previous three steps.

**Proof of Theorem \cite{S7}** Let $\ell = \frac{2n^2 + 2}{2\pi} \Delta_A + \Delta_A$. It suffices to verify the following two claims:

1. Each second-order critical point $x^*$ of $\text{nsPM-P}$ satisfying $F_{\hat{A}}(x^*) > \ell$ must lie in one of the spherical caps $B_{r_+}(a_1), B_{r_+}(-a_1), \ldots, B_{r_+}(-a_K)$ from Corollary \cite{S23}.
2. The level $\ell$ satisfies $\ell < 1 - \Delta_A$.

It is easy to see that these statements imply Theorem \cite{S7}. By the first claim and Corollary \cite{S23} there are at most $2K$ second-order critical points $x^*$ with $F_{\hat{A}}(x^*) > \ell$. They are all strict local maximizers of $F_{\hat{A}}$. For each such point, there exists unique $i \in [K], s \in \{-1, +1\}$ with $\|x^* - sa_i\|^2 \leq \frac{2\Delta_A}{n}$. Meanwhile by the second claim, for each $s \in \{-1, +1\}, i \in [K]$ the maximizer $x^*$ of $F_{\hat{A}}$ in
\(B_{r_s}(sa_i)\) from Corollary \[S.23\] satisfies \(F_A(x) \geq F_A(sa_i) \geq F_A(sa_i) - \Delta_A = 1 - \Delta_A > \ell\). Thus there are at least \(2K\) second-order critical points in the superlevel set, and the theorem follows.

Now we verify the first claim. Letting \(\zeta = A^\top x^*\), we want \(\|\zeta\|_\infty \geq 1 - r_+\). Indeed using the bound

\[
\frac{\Delta_A}{F_A(x^*)} \leq \frac{\Delta_A}{F_A(x^*) - \Delta_A} < \frac{\Delta_A}{\ell - \Delta_A} = \frac{2\tau}{2 + 3n^2} = \frac{2\tau}{3n^2} \frac{3n^2}{2 + 3n^2} \leq \frac{2\tau}{3n^2} \frac{3n^2}{2 + 3n^2 + 4\tau} = \frac{2\tau}{3n^2} \left(1 - \frac{2}{2 + 3n^2 + 4\tau}\right) = \frac{2}{3n^2} (\tau - \Delta_0)
\]

where we used \(\frac{a + c}{b + c} \geq \frac{a}{b}\) if \(b \geq a, 0 < c\), Proposition \[S.21\] gives

\[
\|\zeta\|_\infty ^2 \geq 1 - 2\rho_2 - 2\rho_n - 3 \frac{\Delta_A}{F_A(x)} \geq 1 - 2\rho_2 - 2\rho_n - \left(\frac{2}{n^2}\right)^2 (\tau - \Delta_A)
\]

\[
= 1 - 2\rho_2 - 2\rho_n - \left(\frac{1}{6} - \Delta_A - n^2\rho_2 - (n^2 + n)\rho_n\right)
\]

\[
= 1 - \frac{\rho_n}{\rho_n + \rho_n + \frac{2}{n^2}\Delta_A}
\]

\[
\geq \left(1 - \frac{1}{3n} + 2\rho_n + \frac{2}{n}\Delta_A\right)^{\frac{1}{2}}
\]

\[
\geq (1 - r_+)^2.
\]

Note that the penultimate line used \((1 - x)^{\frac{1}{2}} \leq 1 - \frac{x}{n}\) for \(x \leq 1\), and the last line was a part of Corollary \[S.23\].

Next we check the second claim. Substituting the definition of \(\ell\), we require

\[
\frac{3n^2 + 2}{2\tau} \Delta_A + \Delta_A < 1 - \Delta_A \iff \Delta_A < \frac{1}{\frac{3n^2 + 2}{2\tau} + 2} = \Delta_0.
\]

But this is guaranteed by the assumption \(\Delta_A < \Delta_0\). The proof of Theorem \[7\] is finished. \(\square\)

\section{Proof of Theorem \[16\]}

In this section we prove our random overcomplete theorem. Here we define \(\zeta, \sigma\) and \(R_{i,s}\) as in Definition \[1.1\]. We recall the shorthand \(\varepsilon_K = K \log^n(K)/D^n\). Furthermore, as required by the statement of Theorem \[16\], we assume that conditions hold \[A1, A3\] hold.

\subsection{Bounds on \(R_{i,s}\)}

A central part of the proof is to bound the remainder \(|R_{i,s}|\). We start with a general result that bounds inner products with the basis coefficients \(\sigma\). Then we deduce a first bound on \(R_{i,s}\).

\textbf{Lemma \[S.24\].} Assume that conditions \[A2\] and \[A3\] hold. For any vector \(\xi \in \mathbb{R}^K\), we have

\[
|\sigma^\top \xi| \leq \sqrt{c_2 F_A(x)} \|\xi\|_2.
\]

\[\text{S.85}\]

In particular, letting \(R_s = \max_i |R_{i,s}|\) as in Definition \[1.1\] we have

\[
R_s \leq \sqrt{c_2 \varepsilon_K^n F_A(x)} \varepsilon_K^{\frac{n}{n-s}} \|\xi\|_2^{n-s}.
\]

\[\text{S.86}\]
Further, there exists a constant $C$ where the last equation follows from the definition of $G_n^{-1}$. Then as a corollary, we obtain a bound on the remainders $R_i,s$ depending only on $n$.

The restricted isometry property plays an important part in our analysis of the overcomplete regime.

In the next statement, we present the inequalities that we will use which rely on the RIP assumption.

For $(S.86)$, we note that $R_i,s = \sigma^T \xi$, where $\xi \in \mathbb{R}^K$ is given by $\xi_j = 1(j \neq i) \zeta_j^{n-s} \langle i, a_j \rangle$ (each $j \in [K]$). Substituting this into $(S.85)$ gives $(S.86)$, because

$$\|\xi\|_2 = \sqrt{\sum_{j : j \neq i} c_j^{2n-2s} \langle i, a_j \rangle^{2s}}$$

$$\leq \sqrt{\left( \sum_{j : j \neq i} c_j^{2n} \right)^{2n-2s} \left( \sum_{j : j \neq i} \langle i, a_j \rangle^{2n} \right)^{n-2s}}$$

$$\leq \|\zeta\|_2^{n-s} \left( K \left( \frac{c_1 \log(K)}{D} \right)^n \right)^{\frac{2}{n}}$$

$$= c_2^{2} \varepsilon^{-\frac{2}{n}} \|\zeta\|_2^{n-s},$$

where the last equation follows from the definition of $\varepsilon_K$. This completes the proof of the lemma. □

The restricted isometry property plays an important part in our analysis of the overcomplete regime. In the next statement, we present the inequalities that we will use which rely on the RIP assumption. Then as a corollary, we obtain a bound on the remainders $R_i,s$ that will be more useful than $(S.86)$.

**Lemma S.25 (Consequences of RIP).** Assume that conditions $A1, A3$ hold. Let $x \in S^{D-1}$ and let $I = I(x) \subseteq [K]$ satisfy the conclusions of Lemma [7]. Then it holds

$$| F_A(x) - \sum_{i \in I} \sigma_i \zeta_i^n | \leq c_2 \varepsilon K \| F_A(x) \|_K$$.

Further, there exists a constant $C_1$, depending only on $n$, $c_1$, $c_2$ and $c_3$, and another constant $C_2$, depending only on $n$ and $c_3$, such that we have

$$\|\zeta\|_2^n \leq F_A(x) + C_1 \varepsilon K F_A(x) + C_2 \varepsilon K$$.

**Proof.** To show $(S.87)$, start with the identity $F_A(x) = \sum_{i=1}^{K} \sigma_i \zeta_i^n$, then apply the inner product bound $(S.85)$ with $\zeta \in \mathbb{R}^K$ given by $\zeta_j = 1(j \notin I) \zeta_j^n$, and then use Lemma [7] based on RIP:

$$| F_A(x) - \sum_{i \in I} \sigma_i \zeta_i^n | = \sum_{i \notin I} \sigma_i \zeta_i^n$$

$$\leq \sqrt{c_2 F_A(x) \left( \sum_{i \notin I} \zeta_i^{2n} \right)}$$

$$\leq \sqrt{c_2 F_A(x) \left( K \left( \frac{c_1 \log(K)}{D} \right)^n \right)}$$

$$= \sqrt{c_2 \varepsilon K F_A(x) \| F_A(x) \|_K}$$. 

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Next to show (S.88), split \( \| \zeta \|_{2n}^2 \) into a sum over \( I \) and \( [K] \setminus I \):

\[
\| \zeta \|_{2n}^2 = \sum_{i \in I} \zeta_i^{2n} + \sum_{i \notin I} \zeta_i^{2n}.
\] (S.89)

As before, the sum over \( [K] \setminus I \) is upper-bounded using the second conclusion in Lemma [11]

\[
\sum_{i \notin I} \zeta_i^{2n} \leq \tilde{c}_3 \varepsilon_K.
\] (S.90)

As for the sum over \( I \) in (S.89), we can relate this to \( F_A(x) \) by using

\[
\sum_{i \in I} \zeta_i^{2n} = \sum_{i \in I} \sigma_i \zeta_i^{n} + \sum_{i \in I} R_{i,n} \zeta_i^{n}.
\] (S.91)

Indeed we have already proven that the first sum on the right-hand side of (S.91) is approximately \( F_A(x) \); more precisely (S.87) holds. We can see that the other sum is small by using our initial bound on \( R_s \) (S.86) together with the first conclusion of Lemma [11]:

\[
\left| \sum_{i \in I} R_{i,n} \zeta_i^n \right| \leq R_n \sum_{i \in I} |\zeta_i|^n \leq R_n \sum_{i \in I} \zeta_i^2 \leq R_n (1 + \delta)
\]

\[
\leq (1 + \delta) \sqrt{c_2 \varepsilon_K^3 F_A(x) \varepsilon_K} \leq 2 \sqrt{c_2 \varepsilon_K^3 F_A(x) \varepsilon_K}.
\]

Putting the last three sentences together via the triangle inequality gives

\[
\left| \sum_{i \in I} \zeta_i^{2n} - F_A(x) \right| \leq \sqrt{c_2 \varepsilon_K^3 F_A(x) \varepsilon_K} + 2 \sqrt{c_2 \varepsilon_K^3 F_A(x) \varepsilon_K}.
\]

Combining (S.89), (S.90), (S.91), we conclude that we may set

\[
C_1 = \tilde{c}_3 \quad \text{and} \quad C_2 = \sqrt{c_2 \varepsilon_K^3} + 2 \sqrt{c_2 \varepsilon_K^3},
\]

and (S.88) follows as desired. \( \square \)

**Corollary S.26.** Assume that conditions [A1, A3] hold. Let \( x \in S^{D-1} \) be arbitrary and \( s \in \{0, \ldots, n\} \). Then there exists a constant \( \tilde{C}_s \), depending only on \( s, n, c_1, c_2 \) and \( \tilde{c}_3 \), such that

\[
R_s \leq \tilde{C}_s \varepsilon_K^{\frac{1}{2n}} \max\{\varepsilon_K, F_A(x)\}^{1 - \frac{s}{2n}}.
\] (S.92)

**Proof.** Plugging (S.88) into (S.86) yields

\[
R_s \leq \sqrt{c_2 \varepsilon_K^3 F_A(x) \varepsilon_K} \left( F_A(x) + C_1 \varepsilon_K F_A(x) + C_2 \varepsilon_K \right)^{\frac{1}{2n}} \leq \tilde{C}_s \varepsilon_K^{\frac{1}{2n}} \max\{\varepsilon_K, F_A(x)\}^{1 - \frac{s}{2n}},
\]

where \( \tilde{C}_s = \sqrt{c_2 \varepsilon_K^3 (1 + C_1 + C_2)^{\frac{1}{2n}}} \). \( \square \)

### H.2 Lower bound on maximum correlation coefficient

We now provide a lower bound on \( \| \zeta \|_\infty \) depending on \( \varepsilon_K F_A(x) \).

**Proposition S.27.** Suppose that \( x \) is a second-order critical point of \( nSPM-P \) and \( F_A(x) \geq \varepsilon_K \). Then there exists a constant \( \tilde{C}_2 \) depending only on \( n, c_1, c_2, \tilde{c}_3, \delta \), and another constant \( \tilde{C}_n \), depending only on \( n, c_1, c_2, \tilde{c}_3 \), such that we have

\[
\| \zeta \|_\infty^2 \geq 1 - \frac{2n - 1}{2n - 2} \frac{\delta}{1 + \delta} - \tilde{C}_2 \left( \frac{\varepsilon_K}{F_A(x)} \right)^{\frac{1}{2}} - \tilde{C}_n \left( \frac{\varepsilon_K}{F_A(x)} \right)^{\frac{1}{2}} - 3 \frac{\Delta_A}{F_A(x)}.
\] (S.93)

\[
\geq 1 - \frac{2n - 1}{2n - 2} \frac{\delta}{1 + \delta} - (\tilde{C}_2 + \tilde{C}_n) \left( \frac{\varepsilon_K}{F_A(x)} \right)^{\frac{1}{2}} - 3 \frac{\Delta_A}{F_A(x)}.
\]
Remark S.28. Note that while the statement is for a constrained second-order critical point of $F_A$, (S.93) contains $F_A(x)$. This is intentional: later we use this result together with a bound on $F_A(x)$.

Proof. Our starting point is Eq. (S.39) in Proposition S.11 which says that for each $i \in [K]$ we have

$$(1 + 2(n - 1)\zeta_i^2) F_A(x) \geq (2n - 1)\sigma_i\zeta_i^{n-2} + n\zeta_i^{n-2}R_i,n + (n - 1)R_{i,2} - (4n - 2)\Delta_A.$$ 

Multiplying both sides by $\zeta_i^2$ and using that $1 + 2(n - 1)\|\zeta_i\|_\infty^2 \geq 1 + 2(n - 1)\zeta_i^2$, we have

$$\zeta_i^2 (1 + 2(n - 1)\|\zeta_i\|_\infty^2) F_A(x) \geq (2n - 1)\sigma_i\zeta_i^{n} + \zeta_i^2 (n\zeta_i^{n-2}R_i,n + (n - 1)R_{i,2} - (4n - 2)\Delta_A).$$

(S.94)

Now let $I = I(x) \subseteq [K]$ satisfy the conclusions of Lemma S.11 and sum the inequalities (S.94) as $i$ ranges over $I$. Denoting $\|\zeta_i\|_{2,I} = \sum_{i \in I} \zeta_i^2$, this gives

$$\|\zeta\|_{2,I}^2 \geq (2n - 1) \sum_{i \in I} \sigma_i \zeta_i^n + \sum_{i \in I} \zeta_i^2 (n\zeta_i^{n-2}R_i,n + (n - 1)R_{i,2} - (4n - 2)\Delta_A).$$

(S.95)

We lower-bound each of the sums on the right-hand side in turn. Firstly by (S.87),

$$(2n - 1) \sum_{i \in I} \sigma_i \zeta_i^n \geq (2n - 1) \left( F_A(x) - \sqrt{c_2^N F_A(x)} \epsilon_K \right),$$

(S.96)

Secondly by combining the triangle inequality, the trivial bound $|\zeta_i^{n-2}| \leq 1$, Corollary S.26 with $s = 2$, and the assumption $F_A(x) \geq \epsilon_K$, we have

$$\sum_{i \in I} \zeta_i^2 (n\zeta_i^{n-2}R_i,n + (n - 1)R_{i,2} - (4n - 2)\Delta_A) \geq -\|\zeta\|_{2,I}^2 (nR_n + (n - 1)R_2 + (4n - 2)\Delta_A) \geq -\|\zeta\|_{2,I}^2 \left( n\tilde{C}_n \sqrt{\epsilon_K F_A(x)} + (n - 1)\tilde{C}_2 \epsilon_K F_A(x)^{\frac{4}{5}} + (4n - 2)\Delta_A \right).$$

(S.97)

Inserting (S.96) and (S.97) into (S.95) gives

$$\|\zeta\|_{2,I}^2 \geq (2n - 1) \left( F_A(x) - \sqrt{c_2^N F_A(x)} \epsilon_K \right) - \|\zeta\|_{2,I}^2 \left( n\tilde{C}_n \sqrt{\epsilon_K F_A(x)} + (n - 1)\tilde{C}_2 \epsilon_K F_A(x)^{\frac{4}{5}} + (4n - 2)\Delta_A \right).$$

Isolating $\|\zeta\|_{2,I}$, and using $1 - \delta \leq \|\zeta\|_{2,I} \leq 1 + \delta$ and $\frac{4n - 2}{2n - 2} \leq 3$, yields (S.93) as desired where

$$\tilde{C}_2 = \frac{1}{1 - \delta} \frac{\sqrt{c_2^N}}{2n - 2} + \frac{n}{2n - 2} \tilde{C}_n \quad \text{and} \quad \tilde{C}_n = \frac{1}{2} \tilde{C}_2.$$

This completes the proof of the proposition. \(\square\)

H.3 Concavity

We now show that $F_A$ is strictly geodesically concave in a spherical cap around each $\pm a_i$.

Proposition S.29. There exist constants $C$, $\Delta_0$ and $D_0$, where $C$ and $\Delta_0$ depend only on $n$, $c_2$, and $D_0$ depends additionally on $c_1$, $\epsilon_3$, such that $C < 1$ and if $\Delta_A < \Delta_0$, and $D \geq D_0$, then for all $i \in [K]$ and $s \in \{-1, 1\}$, it holds

1. $F_A$ is strictly concave on the spherical cap $B_{r_+} (sa_i)$, where

$$r_+ := 1 - \left( 1 - C (0.99 - 4\Delta_A)^2 \right)^{\frac{1}{n}}.$$

2. There exists exactly one local maximum in each spherical cap $B_{r_+} (sa_i)$, and denoting it by $x_*$, we have

$$\|x_* - sa_i\|_2^2 \leq \frac{2\Delta_A}{n}.$$  

(S.98)
Proof. We first note that Lemma S.25 implies that, for all \( y \in \mathbb{S}^{D-1} \),
\[
\sum_{i=1}^{K} \langle y, a_i \rangle^{2n} \leq \|\zeta(y)\|_{2n}^2 \leq F_A(y) + C \sqrt{\varepsilon_K} \max\{\varepsilon_K, F_A(y)\} \leq 1 + O(\sqrt{\varepsilon_K}),
\]
thus since \( \lim_{D \to \infty} \varepsilon_K = 0 \), for all \( \mu \in (0, 1) \) there exist \( D_0 \) such that if \( D \geq D_0 \),
\[
\sum_{i=1}^{K} \langle y, a_i \rangle^{2n} \leq 1 + \mu, \quad \text{uniformly for all } y \in \mathbb{S}^{D-1}. \quad (S.99)
\]

Let \( i \in [K] \), \( s \in \{-1, 1\} \) and \( y \perp x \) arbitrary and recall (S.44),
\[
\frac{1}{2n} y^T \nabla_{\mathbb{S}^{D-1}}^2 F_A(x)y \leq n \|P_A(x^{n-1}y)\|^2 + (n-1)\langle P_A(x^n), x^{n-2}y^2 \rangle - \zeta^{2n} + 4\Delta_A. \quad (S.100)
\]

The second term can be bounded using Lemma S.9
\[
\left| \langle P_A(x^n), x^{n-2}y^2 \rangle \right| \leq \left| \langle x^n, x^{n-2}y^2 \rangle \right| + \left| \langle P_A(x^n) - x^n, x^{n-2}y^2 \rangle \right|
\leq 0 + \|P_A(x^n)\|_F \|\text{Sym}(x^{n-2}y^2)\|
\leq \frac{\sqrt{2}}{n(n-1)} \sqrt{1 - \|P_A(x^n)\|^2_F} \leq \frac{2}{n-1} \sqrt{1 - \langle x^n, y \rangle^{2n}}.
\]

The first term in (S.100) is slightly more involved. We first use Lemma S.3 and A3 to write
\[
\|P_A(x^{n-1}y)\|^2_F = \beta(x^{n-1}y)^T G_n^{-1} \beta(x^{n-1}y) \leq c_2 \|\beta(x^{n-1}y)\|^2_F.
\]

Then, by leveraging the fact that \( y \perp x \), we further get
\[
\|\beta(x^{n-1}, y)\|_2^2 = \sum_{j=1}^{K} (a_j, x)^{2n-2} \langle a_j, y \rangle^2 \leq (a_i, x)^{2n-2} \langle a_i, y \rangle^2 + \sum_{j \neq i}^{K} (a_j, x)^{2n-2} \langle a_j, y \rangle^2
\leq (a_i, x)^{2n-2} (\langle a_i \rangle^2 - (a_i, x)^2) + \sum_{j \neq i}^{K} (a_j, x)^{2n-2} \langle a_j, y \rangle^2
= (a_i, x)^{2n-2} - (a_i, x)^{2n} + \sum_{j \neq i}^{K} (a_j, x)^{2n-2} \langle a_j, y \rangle^2,
\]
where \( (a_i, x)^2 + (a_i, x)^2 \leq \|a_i\|^2 \) is a consequence of Bessel’s inequality. On one hand, since the function \( f(x) = (1 - x)^{\frac{n}{n-1}} - (1 - x) \) is concave in \([0, 1]\), and \( f'(0) = \frac{1}{n} \), we have \( f(x) \leq \frac{x}{n} \), which implies \( (a_i, x)^{2n-2}(1 - \langle a_i, x \rangle^2) \leq \frac{4(\langle a_i, a \rangle)^{2n}}{n} \). On the other hand, using Hölder’s inequality with \( p = n \) and \( q = \frac{n}{n-1} \) satisfying \( 1/p + (n-1)/n = 1 \), and Equation (S.99), we get
\[
\sum_{j \neq i}^{K} (a_j, x)^{2n-2} \langle a_j, y \rangle^2 \leq \left( \sum_{j \neq i}^{K} (a_j, x)^{2n} \right)^{\frac{n-1}{n}} \left( \sum_{j \neq i}^{K} (a_j, y)^{2n} \right)^{\frac{1}{n}}
\leq (1 + \mu)^{\frac{n}{n}} (1 + \mu - (a_i, x)^{2n})^{\frac{n-1}{n}}
\leq (1 + \mu) \left( 1 + \mu - (a_i, x)^{2n} \right)^{\frac{n-1}{n}}.
\]

Combining the estimates and writing \( \eta := 1 - \langle a_i, x \rangle^{2n} \) for short, we obtain
\[
\frac{1}{2n} y^T \nabla_{\mathbb{S}^{D-1}}^2 F_A(x)y \leq \|G_n^{-1}\|_2 \left( \eta + (n + \mu) \langle \eta + \mu \rangle^{\frac{n-1}{n}} \right) + (n-1)\sqrt{\eta} - 1 + \eta + 4\Delta_A
\leq \|G_n^{-1}\|_2 \left( \eta + (n + \mu) \langle \eta + \mu \rangle^{\frac{n-1}{n}} \right) + (n-1)\sqrt{\eta} - 1 + \eta + 4\Delta_A
\leq \|G_n^{-1}\|_2 \left( 1 + n + \mu \right) \sqrt{\eta} + (n + \mu) \mu^{\frac{n-1}{n}} + n\sqrt{\eta} + 4\Delta_A - 1
\leq (n + (n + \mu + 1)c_2) \sqrt{\eta} + (n + \mu)c_2 \mu^{\frac{n-1}{n}} + 4\Delta_A - 1.
\]
Now choosing \( \mu \) (and consequently \( D_0 \)) such that \((n + \mu)c_2\mu^{\frac{n-1}{n}} \leq 0.01\), noticing that \( \mu c_2 \leq (n + \mu)c_2\mu^{\frac{n-1}{n}} \), and setting \( C = \left(\frac{1}{n+0.01(n+1)c_2}\right)^2 \), we get
\[
\frac{1}{2n} y^T \nabla^2_{S^D} F_A(x) y \leq \sqrt{\frac{\mu}{n} + 4\Delta_A} - 0.99, \tag{S.101}
\]
and thus, by rearranging terms, we get that \( F_A \) is strictly geodesically concave in \( B_{r_+}, sa_i \).

We now define
\[
\tilde{r}_+ := 1 - \left(1 - C (0.49 - 4\Delta_A)^2\right)^\frac{1}{2n},
\]
and notice that for all \( x \in B_{r_+}(sa_i) \) and \( y \in S^{D-1} \), \((S.101)\) implies that \( y^T \nabla^2_{S^D} F_A(x) y \leq -n \). We also have
\[
1 - 2^{n} \sqrt{1 - C (0.49 - 4\Delta_A)^2} > 1 - \left(1 - \frac{C}{2n} (0.49 - 4\Delta_A)^2\right) = 0.49^2 \frac{C}{2n} \left(1 - \frac{4}{0.49} \Delta_A\right)^2 \geq 0.49^2 \frac{C}{2n} \left(1 - \frac{8}{0.49} \Delta_A\right) > \frac{\Delta_A}{n},
\]
where the last equality follows if we choose \( \Delta_0 = 0.49^2 \frac{C}{2n} / \left(\frac{1}{n} + 3.92 \frac{C}{2n}\right) \), since \( \Delta_A < \Delta_0 \).

Therefore, applying Proposition \(S.17\) we obtain that it exists exactly one local maximum \( x_+ \in B_{r_+}(sa_i) \), and \( x_+ \) satisfies \(S.98\).

Finally, since \( F_A \) is strictly geodesically concave on \( B_{r_+}(sa_i) \), Proposition \(S.17\) implies that there exists at most one local maximum in \( B_{r_+}(sa_i) \), thus since \( B_{r_+}(sa_i) \subseteq B_{r_+}(sa_i) \), \( x_+ \) is also the only local maximum of \( B_{r_+}(sa_i) \).

**H.4 Completing the proof**

We now have all the pieces to conclude the argument for Theorem \(16\).

**Proof of Theorem \(16\)** Suppose that \( x \) is a local maximum and \( F_A(x) \geq C\varepsilon_K + 5\Delta_A \), where \( C \) will be defined below. Our proof strategy is to show that this implies that \( x \) is in one of the \( 2K \) spherical caps defined in Proposition \(S.29\). The rest of the proof then follows from Proposition \(S.29\) since each spherical cap contains exactly one local maximum.

Denote \( \tilde{C} = \tilde{C}_2 + \tilde{C}_n \) from the statement of Proposition \(S.27\) and define \( C \) so that \( C \geq 1 \). Therefore \( F_A(x) \geq \varepsilon_K \) and Proposition \(S.27\) implies
\[
\|\xi\|^2 \geq 1 - 2n - \frac{\varepsilon_K}{F_A(x)} - \tilde{C} \left(\frac{\varepsilon_K}{F_A(x)}\right)^\frac{1}{n} - 3 \frac{\Delta_A}{F_A(x)}.
\]
By the AM-GM inequality, we have
\[
\tilde{C} \left(\frac{\varepsilon_K}{F_A(x)}\right)^\frac{1}{n} \leq \frac{1}{n} \left(4^{-1} \tilde{C} n \frac{\varepsilon_K}{F_A(x)} + \frac{1}{4} (n - 1)\right),
\]
thus
\[
\tilde{C} \left(\frac{\varepsilon_K}{F_A(x)}\right)^\frac{1}{n} + 3 \frac{\Delta_A}{F_A(x)} \leq \frac{1}{F_A(x)} \left(4^{-1} \tilde{C} n \varepsilon_K + 3\Delta_A\right) + \frac{1}{4} - \frac{1}{4n}.
\]
We now define \( C := \frac{1}{4} \tilde{C} n \), thus having
\[
F_A(x) \geq F_A(x) - \Delta_A \geq C\varepsilon_K + 4\Delta_A \geq \frac{4}{3} \left(\frac{4^{-1} \tilde{C} n \varepsilon_K + 3\Delta_A}{n}\right),
\]
and
\[
\tilde{C} \left(\frac{\varepsilon_K}{F_A(x)}\right)^\frac{1}{n} + 3 \frac{\Delta_A}{F_A(x)} \leq \frac{3}{4} + \frac{1}{4} - \frac{1}{4n} \leq 1 - \frac{1}{4n}.
\]
If we now choose $\delta_0$ such that $\frac{2n-1}{2n-2} < \frac{1}{8n}$, we have that if $\delta < \delta_0$, then

$$\|\zeta\|_\infty^2 \geq 1 - \frac{2n - 1}{2n - 2} \frac{\delta}{1 + \delta} - \left(1 - \frac{1}{4n}\right) \geq \frac{1}{8n}. \quad (S.102)$$

Note that we always have $F_A(x) \geq \|\zeta\|_\infty^2$. To show this, define $i$ such that $\|\zeta\|_\infty = |\zeta_i|$. Then $F_A(x) = \|P_A(x^n)\|^2 \geq \|P^\eta_{\alpha_i}(x^n)\|^2 = \zeta_i^{2n} = \|\zeta\|_\infty^{2n}$. Therefore, $(S.102)$ implies that $F_A(x) \geq (\frac{1}{8n})^n$, and since $\lim_{D \to \infty} \varepsilon_K = 0$, for any $\nu \in (0, 1)$, there exists $D_0$ such that if $D \geq D_0$, then

$$\tilde{C} \left(\frac{\varepsilon_K}{F_A(x)}\right)^{\frac{1}{4}} \leq 8n\tilde{C} \varepsilon_k^{\frac{1}{4}} \leq \frac{1}{2} \nu.$$ 

Furthermore, choosing $\delta_0$ such that $\frac{2n-1}{2n-2} < \frac{1}{8n}$, we obtain, by applying Proposition S.27 again, that for all $\nu \in (0, 1)$ there exists $D_0$ and $\delta_0$ such that, if $D \leq D_0$ and $\delta \leq \delta_0$,

$$\|\zeta\|_\infty^2 \geq 1 - \nu - 3 \frac{\Delta_A}{F_A(x)}. \quad (S.103)$$

Simply to improve the implicit constant, we can choose $\nu = 0.01$, and notice that $F_A(x) \geq 4\Delta_A$, to get $\|\zeta\|_\infty^2 \geq 0.96/4$, and consequently, $F_A(x) \geq (0.96/4)^n$. Plugging this in $(S.103)$, we obtain

$$\|\zeta\|_\infty^2 \geq 1 - \nu - 3(4/0.96)^n \Delta_A.$$ 

We now further choose $\Delta_0$ and a smaller $\nu$ such that, if $\Delta_A < \Delta_0$, then

$$\sqrt{1 - C (0.99 - 4\Delta_A)^2} \leq 1 - \frac{C}{n} (0.99 - 4\Delta_A)^2 \leq 1 - \frac{C}{n} (0.98 - 7.92 \Delta_A) \leq 1 - \nu - 3(4/0.96)^n \Delta_A \leq \|\zeta\|_\infty^2. \quad (S.104)$$

This then implies that $x$ must lie in one of the caps defined in Proposition S.29, and the rest of the result follows from Proposition S.29. For $(S.104)$ to hold, it suffices to choose $\nu = \min(0.49 \frac{C}{n}, 0.01)$ and $\Delta_0 = 0.49 \frac{C}{n} / (3(4/0.96)^n + 7.92 \frac{C}{n})$. 

\section{Deflation bounds and proof of Theorem 18}

Suppose we want to decompose the tensor $\hat{T}$, which is an estimate of the low rank tensor $T = \sum_{i=1}^N \lambda_i a_i^{\otimes m}$. We will consider the following slight modification of the SPM algorithm [27].
We analyze Algorithm S.1 assuming the conditions of Theorem 7 or 16. The conclusion will be

Algorithm S.1 SPM with modified deflation step

Input: \( \hat{T} \in \mathcal{T}_D^n \)
Hyper-parameters: \( \alpha, \tau \in \mathbb{R}^+ \)
Returns: \( (\hat{a}_k, \hat{\lambda}_k)_{k \in [K]} \)

\[ M \leftarrow \text{Reshape}(T, [D^n, D^{n-n}]) \]

Set \( K \) as the number of singular values of \( M \) exceeding \( \alpha \).

Set \( \hat{M}_K \) as the rank-\( K \) truncation of the SVD of \( M \).

Let \( \hat{A} = \text{colspan}(\hat{U}) \) and \( \hat{A}_1 = \hat{A} \)

for \( k = 1, \ldots, K \) do

Obtain \( \hat{a}_k \) by applying POWER METHOD [27] on functional \( F_{\hat{A}_k} \) until convergence.

Repeat last step with new initializations until \( F_{\hat{A}_k}(\hat{a}_k) \geq \tau \).

if \( k = 1 \) then \( \hat{a}_1 \leftarrow \hat{a}_1 \)

else Obtain \( \hat{a}_k \) by applying POWER METHOD on \( F_{\hat{A}} \) with \( \hat{a}_k \) as starting point until convergence.

\[
\hat{\lambda}_k \leftarrow \frac{1}{\text{vec}(\hat{a}_k^\otimes n)^T (\hat{M}_K^T)^\dagger \text{vec}(\hat{a}_k^\otimes m-n)}
\] (S.105)

if \( k < K \) then

\[
\hat{A}_{k+1} \leftarrow \hat{A}_k \cap ((\hat{M}_K^T)^\dagger \text{vec}(\hat{a}_k^\otimes m-n))^\bot
\]

We analyze Algorithm S.1 assuming the conditions of Theorem 7 or 16. The conclusion will be Theorem [18]. We note the sign and permutation ambiguity there are inherent to the CP decomposition.

A result on the stability of matrix pseudoinversion is needed; we include a proof for completeness.

Lemma S.30. Let \( W, \hat{W} \in \mathbb{R}^{p \times q} \) and suppose that \( W \) has exactly \( r \) nonzero singular values \( \sigma_1(W) \geq \ldots \geq \sigma_r(W) > 0 \) and \( \Delta_W := \| W - \hat{W} \|_2 < \sigma_r(W) \). Define \( \hat{W}_r \) as the SVD of \( \hat{W} \) truncated at the \( r \)-th singular value. Then

\[
\| W^\dagger - \hat{W}_r^\dagger \|_2 \leq \frac{\Delta_W}{\sigma_r(W)} - \Delta_W \left( \frac{2}{\sigma_r(W)} + \frac{1}{\sigma_r(W) - \Delta_W} \right).
\]

Proof. We denote the singular value decomposition of matrices \( W, \hat{W} \in \mathbb{R}^{p \times q} \) as

\[
W = (U_1 \quad U_2) \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right) \left( \begin{array}{c} V_1^T \\ V_2^T \end{array} \right) \quad \text{and} \quad \hat{W} = (\hat{U}_1 \quad \hat{U}_2) \left( \begin{array}{cc} \hat{\Sigma}_1 & 0 \\ 0 & \hat{\Sigma}_2 \end{array} \right) \left( \begin{array}{c} \hat{V}_1^T \\ \hat{V}_2^T \end{array} \right),
\]

where \( \Sigma_1 \) and \( \hat{\Sigma}_1 \) are diagonal matrices with the largest \( r \) singular values of \( W \) and \( \hat{W} \) on the diagonal, respectively. We have \( \| W^\dagger - \hat{W}_r^\dagger \|_2 = \left\| U_1 \Sigma_1^{-1} V_1^T - \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T \right\|_2 \). Then

\[
U_1 \Sigma_1^{-1} V_1^T - \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T = (U_1 U_1^T - \hat{U}_1 \hat{U}_1^T) U_1 \Sigma_1^{-1} V_1^T + \hat{U}_1 \hat{U}_1^T (U_1 \Sigma_1^{-1} V_1^T - \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T)
\]

\[
= (U_1 U_1^T - \hat{U}_1 \hat{U}_1^T) U_1 \Sigma_1^{-1} V_1^T + \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T (\hat{W}^T U_1 \Sigma_1^{-1} V_1^T - \hat{V}_1 \hat{V}_1^T),
\]

where we used \( U_1 \hat{U}_1^T \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T = \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T = \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T \hat{V}_1^T \) and

\[
\hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T W^T = \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^T (\hat{V}_1 \hat{U}_1^T + \hat{V}_2 \hat{U}_2^T) = \hat{U}_1 \hat{U}_1^T.
\]

Furthermore,

\[
\hat{W}^T U_1 \Sigma_1^{-1} V_1^T - \hat{V}_1 \hat{V}_1^T = \hat{W}^T U_1 \Sigma_1^{-1} V_1^T - V_1 V_1^T + (V_1 V_1^T - \hat{V}_1 \hat{V}_1^T)
\]

\[
= (\hat{W}^T - V_1 U_1^T) U_1 \Sigma_1^{-1} V_1^T + (V_1 V_1^T - \hat{V}_1 \hat{V}_1^T)
\]

\[
= (\hat{W}^T - W^T) U_1 \Sigma_1^{-1} V_1^T + (V_1 V_1^T - \hat{V}_1 \hat{V}_1^T).
\]
We then have
\[
\|W^t - \hat{W}_i\|_2 = \left\| U_1 \Sigma_1^{-1} V_1^t - \hat{U}_i \hat{\Sigma}_1^{-1} \hat{V}_1^t \right\|_2 \\
= \left\| (U_1 U_1^t - \hat{U}_1 \hat{U}_1^t) U_1 \Sigma_1^{-1} V_1^t + \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^t (W^t - W^t) U_1 \Sigma_1^{-1} V_1^t + \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^t (V_1 V_1^t - \hat{V}_1 \hat{V}_1^t) \right\|_2 \\
\leq \left\| U_1 U_1^t - \hat{U}_1 \hat{U}_1^t \right\|_2 \left\| U_1 \Sigma_1^{-1} V_1^t \right\|_2 + \left\| \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^t \right\|_2 \left\| W^t - W^t \right\|_2 \left\| U_1 \Sigma_1^{-1} V_1^t \right\|_2 + \left\| \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^t \right\|_2 \left\| V_1 V_1^t - \hat{V}_1 \hat{V}_1^t \right\|_2 \\
\leq \frac{\Delta_W}{\sigma_r(W) - \Delta_W} \left( \frac{2}{\sigma_r(W)} + \frac{1}{\sigma_r(W) - \Delta_W} \right). 
\]
The last line uses Wedin’s bound and \( \left\| \hat{U}_1 \hat{\Sigma}_1^{-1} \hat{V}_1^t \right\|_2 = \frac{1}{\sigma_r(W)} \leq \frac{1}{\sigma_r(W) - \Delta_W} \), which comes from Weyl’s inequality.

Next we control the propagation of error on the weights and intermediate subspaces in Algorithm S.1

**Proposition S.31.** Define \( T, \hat{T} \) as in Algorithm S.1 with \( M := \text{Reshape}(T, [D^n, D^{m-n}]) \) and its singular values \( \sigma_1(M) \geq \ldots \geq \sigma_K(M) > 0 \). Let \( \hat{M} := \text{Reshape}(\hat{T}, [D^n, D^{m-n}]) \), and suppose that \( \Delta_M := \|\hat{M} - M\|_2 < \sigma_K(M) \). Then the error for \( \hat{\lambda}_k \) obtained by Algorithm S.1 is bounded as follows:

\[
\left| \frac{1}{\lambda_k} - \frac{1}{\hat{\lambda}_k} \right| \leq \frac{1}{\sigma_K(M)} \sqrt{2m} \|a_k - \hat{a}_k\| + \frac{\Delta_M}{\sigma_K(M) - \Delta_M} \left( \frac{2}{\sigma_K(M)} + \frac{1}{\sigma_K(M) - \Delta_M} \right). \tag{S.106}
\]

Additionally, let \( \mathcal{A}_k := \text{span}\{a_i^{\otimes n}, i \in \{k, \ldots, K\}\} \), \( A_{[k-1]} \) the submatrix of \( A \) with columns in \([k-1]\), \( G_{d,[k-1]} = (A_{[k-1]}^T)^T (A_{[k-1]}^T) = (A_{[k-1]}^T A_{[k-1]})^{\otimes k} \) and \( \varphi_{d,[k-1]} = \sqrt{\|G_{d,[k-1]}\|_2} \). Then the error of the deflated subspace is bounded by

\[
\left\| P_{\mathcal{A}_k} - P_{\hat{\mathcal{A}}_k} \right\|_2 \leq \frac{\Delta_M}{\sigma_K(M) - \Delta_M} \left( 1 + \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2} \|G_{m-n}\|_2 \left( \frac{2}{\sigma_K(M)} + \frac{1}{\sigma_K(M) - \Delta_M} \right) + \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2} \frac{1}{\sigma_K(M) - \Delta_M} \sqrt{(m-n) \sum_{k=1}^{K-1} \|a_k - \hat{a}_k\|^2}. \right. \tag{S.107}
\]

**Proof.** We start by showing that (S.105) holds in the clean case, that is,

\[
\lambda_k = \frac{1}{\text{vec}(a_k^{\otimes n})^T (M^T)^T \text{vec}(a_k^{\otimes m-n})^T}.
\]

We have

\[
M = \sum_{i=1}^K \lambda_i \text{vec}(a_i^{\otimes n}) \text{vec}(a_i^{\otimes m-n})^T = A^{\otimes n} A(A^{\otimes m-n})^T,
\]

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_K) \). Therefore, \( (M^T)^T = ((A^{\otimes n})^T)^T \Lambda^{-1} (A^{\otimes m-n})^T \) and

\[
(A^{\otimes n})^T (M^T)^T A^{\otimes m-n} = \Lambda^{-1}. \tag{S.108}
\]

Denoting by \( e_k \) the \( k \)-th canonical basis vector, we have \( \text{vec}(a_k^{\otimes n}) = (A^{\otimes n}) e_k \), thus

\[
\text{vec}(a_k^{\otimes n})^T (M^T)^T \text{vec}(a_k^{\otimes m-n}) = e_k^T ((A^{\otimes n})^T (M^T)^T A^{\otimes m-n}) e_k = e_k^T \Lambda^{-1} e_k = \frac{1}{\lambda_k}.
\]
We first show that if

Then

that, for all $\hat{\lambda}_i \in \sigma(A_{k:i} \otimes m-n)$.

To show this, consider the following equation:

$$
\|A_{k:i} \otimes m-n \otimes n - \hat{A}_i \otimes m-n \otimes n \|_F + \|A_{k:i} \otimes m-n - \hat{A}_i \otimes m-n \|_F \leq \sqrt{(m-n)^2 + \sqrt{n}} \|a_i - \hat{a}_i\| \leq \sqrt{2m\|a_i - \hat{a}_i\|},
$$

and the proof of (S.105) is complete. Regarding (S.107), we have that

$$
\hat{A}_k = \hat{A} \cap \bigcap_{i=1}^{k-1} ((\hat{M}_K^t)^t \text{vec}(\hat{a}_i^{\otimes m-n}))^\perp = \hat{A} \cap \text{colspan}((\hat{M}_K^t)^t \hat{A}_{[k-1]}^{*,m-n})^\perp
$$

and

$$
\hat{A} = \text{colspan}(\hat{M}_K) = \text{colspan}((\hat{M}_K^t)^t) \supseteq \text{colspan}((\hat{M}_K^t)^t \hat{A}_{[k-1]}^{*,m-n}).
$$

We first show that if $\hat{T} = T$, $\hat{A}_k = A_k$, that is,

$$
\mathcal{A} \cap \text{colspan}((M^t)^t \hat{A}_{[k-1]}^{*,m-n})^\perp = \text{span}\{a_i^{\otimes n}, i \in \{k, \ldots, K\}\}.
$$

It is enough to show that $\dim(\hat{A}_k) = K - k + 1 = \dim(A_k)$ and that $\mathcal{A}_k \subset \hat{A}_k$. We have that $\dim(\mathcal{A}) = K$, $\dim(\text{colspan}((\hat{M}_K^t)^t \hat{A}_{[k-1]}^{*,m-n})) = k - 1$, thus (S.110) implies that $\dim(\hat{A}_k) = K - k + 1$. To show that $\mathcal{A}_k \subset \mathcal{A} \cap \text{colspan}((M^t)^t \hat{A}_{[k-1]}^{*,m-n})^\perp$, note that,

$$
\mathcal{A}_k = \text{span}\{a_i^{\otimes n}, i \in \{k, \ldots, K\}\} = \text{colspan}(A_{[k:K]}^{*,m-n}) \subset \text{colspan}(A^{*,m-n}) = \mathcal{A},
$$

where $\{k : K\} = \{k, \ldots, K\}$. On other hand, $\text{colspan}(A_{[k:K]}^{*,m-n}) \subset ((M^t)^t \hat{A}_{[k-1]}^{*,m-n})^\perp$ is equivalent to, for all $i \in \{k : K\}$, $j \in \{k - 1\}$, $\text{vec}(a_i^{\otimes n})^t (M^t)^t \text{vec}(a_j^{\otimes m-n}) = 0$. In fact, (S.108) implies that, $\text{vec}(a_i^{\otimes n})^t (M^t)^t \text{vec}(a_j^{\otimes m-n}) = (\Lambda^{-1})_{ij} = 0$ for all $i \in \{k : K\}$, $j \in \{k - 1\}$, as we wanted to show.

Let $U = (U_1, U_2), \hat{U} = (\hat{U}_1, \hat{U}_2)$ be orthogonal matrices such that the columns of $U_1$ and $\hat{U}_1$ are orthonormal basis for $\mathcal{A}$ and $\hat{A}$, respectively. Moreover, since $\text{colspan}((\hat{M}_K^t)^t \hat{A}_{[k-1]}^{*,m-n}) \subset \hat{A}$, there exist orthogonal matrices $O = (O_1, O_2)$ and $\hat{O} = (\hat{O}_1, \hat{O}_2)$ such that $\text{colspan}(U_1 O_1) = \text{colspan}((\hat{M}_K^t)^t \hat{A}_{[k-1]}^{*,m-n})$ and $\text{colspan}(\hat{U}_1 \hat{O}_1) = \text{colspan}((\hat{M}_K^t)^t \hat{A}_{[k-1]}^{*,m-n})$, respectively. We then have $\mathcal{A}_k = \text{colspan}(U_1 O_2) \hat{A}_k = \text{colspan}(\hat{U}_1 \hat{O}_2)$, thus:

$$
\|P_{\mathcal{A}_k} - P_{\hat{A}_k}\|_2 = \|U_1 O_2 O_2^T U_1^\top - \hat{U}_1 \hat{O}_2 \hat{O}_2^T \hat{U}_1^\top\|_2
\leq \|U_1 U_1^\top - \hat{U}_1 \hat{U}_1^\top\|_2 + \|U_1 O_1 O_1^T U_1^\top - \hat{U}_1 \hat{O}_1 \hat{O}_1^T \hat{U}_1^\top\|_2
\leq \|U_1 U_1^\top - \hat{U}_1 \hat{U}_1^\top\|_2 + \|U_1 O_1 O_1^T U_1^\top - \hat{U}_1 \hat{O}_1 \hat{O}_1^T \hat{U}_1^\top\|_2.
$$

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The first term is bounded by Lemma 1 while the second term is bounded using [11] Theorem 2.5:
\[
\left\| U_1 O_1 \tilde{O}_1 U_1^T - U_1 \tilde{O}_1 \tilde{O}_1^T \tilde{U}_1^T \right\|_2 \leq \left\| (M^\top)^{\dagger} A_{m-n}^* - (\tilde{M}_{K}^\top)^{\dagger} \hat{A}_{m-n}^* \right\|_2
\leq \left\| (M^\top)^{\dagger} A_{m-n}^* - (\tilde{M}_{K}^\top)^{\dagger} \hat{A}_{m-n}^* \right\|_2.
\]

Let \( \eta_1 \geq \cdots \geq \eta_K \) be the singular values of \((M^\top)^{\dagger} A_{m-n}^*\), and \( \nu_1 \geq \cdots \geq \nu_{k-1} \) the singular values of \((M^\top)^{\dagger} \hat{A}_{m-n}^*\). We have \(\| (M^\top)^{\dagger} A_{m-n}^* \|_2 = \frac{1}{\eta_K} \) and \(\| (M^\top)^{\dagger} \hat{A}_{m-n}^* \|_2 = \frac{1}{\nu_{k-1}}\). Moreover, by [11] Theorem 1, we have \(\nu_{k-1} \geq \eta_K\), thus
\[
\left\| (M^\top)^{\dagger} A_{m-n}^* \right\|_2 \leq \left\| (M^\top)^{\dagger} \hat{A}_{m-n}^* \right\|_2.
\]
Furthermore \(\textbf{[S.108]}\) implies that \((M^\top)^{\dagger} A_{m-n}^* = \Lambda(A^*)^T\), therefore
\[
\left\| (M^\top)^{\dagger} A_{m-n}^* \right\|_2 \leq \left\| \Lambda(A^*)^T \right\|_2 \leq \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2}.
\]

On the other hand, we have
\[
\left\| (M^\top)^{\dagger} A_{m-n}^* - (\tilde{M}_{K}^\top)^{\dagger} \hat{A}_{m-n}^* \right\|_2 \leq \left\| (M^\top)^{\dagger} - (\tilde{M}_{K}^\top)^{\dagger} \right\|_2 A_{m-n}^* \right\|_2 + \left\| (\tilde{M}_{K}^\top)^{\dagger} \hat{A}_{m-n}^* \right\|_2 A_{m-n}^* \right\|_2
\]
and \(\left\| (M^\top)^{\dagger} \right\|_2 = \frac{1}{\sigma_K(M_{K})} \leq \frac{1}{\sigma_K(M_{K}) - \Delta_M}\). Finally,
\[
\left\| A_{m-n}^* - \hat{A}_{m-n}^* \right\|_2 \leq \left\| A_{m-n}^* - \hat{A}_{m-n}^* \right\|_2 = \sqrt{\sum_{i=1}^{k-1} \left\| a_i \otimes m-n - \hat{a}_i \otimes m-n \right\|_F^2}
\leq \sqrt{(m-n) \sum_{i=1}^{k-1} \| a_i - \hat{a}_i \|_2^2},
\]
where we used \(\textbf{[S.109]}\) in the last line.

We can now prove our guarantee for end-to-end symmetric tensor decomposition using SPM.

**Proof of Theorem 18** First, to show that Algorithm \textbf{[S.1]} picks the correct tensor rank \(K\), we use the assumptions of Theorem \textbf{[18]} and Weyl’s inequality to obtain

\[
\sigma_K(\hat{M}) \geq \sigma_K(M) - \Delta_M > \alpha > \Delta_M \geq \sigma_{K+1}(\hat{M}).
\]

Then, we show the bound on \(\| \hat{a}_k - a_k \| \) using induction on \(k\). When \(k = 1\), we have \(\hat{A}_1 = \hat{A}\). Then, Lemma \textbf{[1]} implies that \(\| P_\hat{A} - P_A \| \leq \Delta_A\). Since \(\hat{a}_1\) was returned by the Power Method, it is a second order critical point of \(F_{\hat{A}}\), and Algorithm \textbf{[S.1]} asserts that \(F_{\hat{A}}(\hat{a}_1) > \tau \geq \ell(\hat{\Delta}_A) \geq \ell(\Delta_A)\). Therefore \(\hat{a}_1\) is a second order critical point in the level set stated in Theorem \textbf{[7]} \(\textbf{[16]}\) which implies that there exists \(\pi(1) := i \in [K]\) and \(s_1 \in \{-1, 1\}\) such that \(\| \hat{a}_1 - s_1 a_{\pi(1)} \| \leq \sqrt{\frac{2\Delta_A}{n}}\). If \(k > 1\), let \(\pi([k-1]) = \{ \pi(i), i \in [k-1] \}\) and \(A_k = \text{span}\{ a_i^{\otimes n}, i \in [K] \setminus \pi([k-1]) \}\). Since the subspace
We first show that for all matrices \( \lambda \) where we set

\[
\text{deflation step in Algorithm S.1} \text{ does not depend on the sign of } \hat{a}_i, \ i \in [k-1], \text{ we can flip the sign in the bound provided in Proposition S.31.}
\]

\[
\left\| P_{A_k} - P_{\hat{A}_k} \right\|_2 \leq \frac{\Delta_M}{\sigma_K(M) - \Delta_M} \left( 1 + \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2 \|G_{m-n}\|_2} \left( \frac{2}{\sigma_K(M)} + \frac{1}{\sigma_K(M) - \Delta_M} \right) \right)
\]

\[
+ \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2} \frac{1}{\sigma_K(M) - \Delta_M} \left( (m-n) \sum_{i=1}^{k-1} \|a_{\pi(i)} - s_i \hat{a}_i\|_2 \right)
\]

\[
\leq \hat{\Delta}_A \left( 1 + \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2 \|G_{m-n}\|_2} \frac{4}{\sigma_K(M)} \right)
\]

\[
+ \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2} \frac{2}{\sigma_K(M)} \sqrt{\frac{2(m-n)(k-1)}{n}} \hat{\Delta}_A
\]

\[
\leq \hat{\Delta}_A \left( 1 + \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2 \|G_{m-n}\|_2} \frac{4}{\sigma_K(M)} \right)
\]

\[
+ \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2} \frac{2}{\sigma_K(M)} \sqrt{\frac{2(m-n)K}{n}} \hat{\Delta}_A
\]

\[
= \hat{\Delta}_A,
\]

where we set

\[
C_1 = 1 + \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2 \|G_{m-n}\|_2} \frac{4}{\sigma_K(M)} \quad \text{(S.111)}
\]

and

\[
C_2 = \max_{i \in [K]} |\lambda_i| \sqrt{\|G_n\|_2} \frac{2}{\sigma_K(M)} \sqrt{\frac{2(m-n)K}{n}} \quad \text{(S.112)}
\]

Then, since \( F_{\hat{A}_k}(\hat{a}_k) > \tau \geq \ell(\hat{\Delta}_A) \), and \( \hat{a}_k \) is a second order critical point of \( \hat{A}_k \), Theorem 7 / 16 implies that there exists \( \pi(k) \in [K] \setminus \pi([k-1]) \) and \( s_k \in \{-1, 1\} \), such that \( \|a_k - s_k a_{\pi(k)}\| \leq \sqrt{\frac{2\hat{\Delta}_A}{n}} \). In particular, it is shown in the proof of both theorems that \( \hat{a}_k \) lies in a spherical cap centered around \( s_k a_{\pi(k)} \) where \( F_{\hat{A}_k} \) is concave. However, in both theorem statements, the radius of the spherical cap where concavity holds is a decreasing function of \( \hat{\Delta}_A \), therefore \( \hat{a}_k \) is also in a spherical cap centered around \( s_k a_{\pi(k)} \) where \( F_{\hat{A}_k} \) is concave. Applying the POWER METHOD with the functional \( F_{\hat{A}_k} \) using \( \hat{a}_k \) as a starting point will then converge for the local maxima in this concave region, and the bound for \( \|a_k - s_k a_{\pi(k)}\| \) follows.

Finally, the error bound for \( \hat{\Delta}_k \) follows from the bound on \( \|\hat{a}_k - s_k a_{\pi(k)}\| \) and Proposition S.31.

We conclude with a bound on the error in terms of the scalar coefficients \( \lambda_i \), which follows from a bound on \( \sigma_K(M) \).

**Lemma S.32.** With \( \sigma_K(M) \) defined and under the same conditions of Theorem 18, it holds

\[
\sigma_K(M) \geq \frac{\min_i |\lambda_i|}{\sqrt{\|G_n^{-1}\|_2 \|G_{m-n}^{-1}\|_2}}. \quad \text{(S.113)}
\]

**Proof.** Let \( \mu_j(A) \) denote the \( j \)-th algorithm (in descending order) of \( A \), that is \( \mu_1(A) \geq \mu_2(A) \geq \cdots \).

We first show that for all matrices \( A_{n \times n} \) and \( B_{n \times n} \) such that \( m \geq n \) and \( B \) is symmetric and positive definite, we have

\[
\mu_n(A^T BA) \geq \mu_n(B)\mu_n(A^T A)
\]

In fact, we have \( A^T BA = \mu_n(B)A^T A + A^T (B - \mu_n(B)I)A \), therefore by Weyl’s inequality [46]:

\[
\mu_n(A^T BA) \geq \mu_n(B)\mu_n(A^T A) + \mu_n(A^T (B - \mu_n(B)I)A) \geq \mu_n(B)\mu_n(A^T A).
\]
Applying this twice, we have
\[ \sigma_K(M) = \sigma_K(A^\bullet n \Lambda (A^\bullet m-n)^\top) = \sqrt{\mu_K(A^\bullet n \Lambda (A^\bullet m-n)^\top) A^\bullet m-n \Lambda (A^\bullet n)^\top}, \]
\[ \geq \sqrt{\mu_K(G_{m-n}) \mu_K(A^\bullet n A^\bullet m-n^2)^\top}, \]
\[ \geq \min_i |\lambda_i| \sqrt{\mu_K(G_{m-n}) \mu_K(A^\bullet n (A^\bullet n)^\top)} = \min_i |\lambda_i| \sqrt{\sigma_K(G_{n}) \sigma_K(G_{m-n})}. \]

Now the lemma follows from \( \sigma_K(G_{n}) = 1/\|G_{n}^{-1}\|_2. \)

**Remark S.33 (Small \( \lambda_i \)).** Lemma S.32 indicates that the smaller \( \min_i |\lambda_i| \) is, the larger is the error of the decomposition obtained by SPM. For instance, the lemma can be used to obtain the bound
\[ \hat{\Delta}_A \leq \frac{\Delta_M \sqrt{\|G_{n}^{-1}\|_2 \|G_{m-n}^{-1}\|_2}}{\min_i |\lambda_i| - \Delta_M \sqrt{\|G_{n}^{-1}\|_2 \|G_{m-n}^{-1}\|_2}}, \]
which suggests a smaller \( \min_i |\lambda_i| \) increases the subspace error \( \Delta_A \). This is also evident in the error arising from deflation. Substituting Lemma S.32 into the definition of \( C_1 \) and \( C_2 \) in Theorem 18,
\[ C_1 \leq 1 + 4 \max_i |\lambda_i| \sqrt{\kappa(G_{n}) \kappa(G_{m-n})}, \]
and
\[ C_2 \leq 2 \max_i |\lambda_i| \sqrt{\kappa(G_{n}) \|G_{m-n}^{-1}\|_2} \sqrt{2(m-n)K \frac{n}{m}}. \]

Here we denoted the condition number of a matrix \( A \) by \( \kappa(A) := \|A\|_2 \|A^{-1}\|_2 \).

Finally, the issue also shows up in the bound of the tensor coefficient error of Theorem 18
\[ \left| \frac{s_k^m}{\lambda_{\pi(k)}} - \frac{1}{\lambda_k} \right| \leq \sqrt{\|G_{n}^{-1}\|_2 \|G_{m-n}^{-1}\|_2 \min_i |\lambda_i|} \left( 2 \sqrt{m/n \hat{\Delta}_A} + 4 \hat{\Delta}_A \right). \]