Velocity excitations and impulse responses of strings — Aspects of continuous and discrete models

Georg Essl*

Media Lab Europe
Sugar House Lane
Dublin 8, Ireland

(Received September 16, 2018)

Abstract

This paper discusses aspects of the second order hyperbolic partial differential equation associated with the ideal lossless string under tension and it's relationship to two discrete models. These models are finite differencing in the time domain and digital waveguide models. It is known from the theory of partial differential operators that in general one has to expect the string to accumulate displacement as response to impulsive excitations. Discrete models should be expected to display comparable behavior. As a result it is shown that impulsive propagations can be interpreted as the difference of step functions and hence how the impulsive response can be seen as one case of the general integrating behavior of the string. Impulsive propagations come about in situations of time-symmetry whereas step-function occur as a result of time-asymmetry. The difference between the physical stability of the wave equation, which allows for unbounded growth in displacement, and computational stability, that requires bounded growth, is derived.

PACS numbers: 43.40.Cw, 43.58.Ta, 43.20.Bi, 43.75.-z, 43.60.-c, 43.60.Ac

* Electronic mail: georg@mle.media.mit.edu
I. INTRODUCTION

Our purpose here is to discuss aspects of the relationship of the solution of the one-dimensional second order wave equation to two discrete models thereof. The first discrete model is the digital waveguide model in one spatial dimension. The second discrete model is a finite difference model in the time domain. In particular we will also discuss how this relationship explains different behavior between the discrete models. This relationship has drawn much attention recently [1, 2, 3, 4, 5, 6, 7, 8, 9].

It is shown that the finite difference model can account for solutions of the wave equation and that these solutions are physically meaningful.

This allows for a direct interpretation of recent results by Karjalainen and Erkut [7] from the fundamental solution of the wave equation. Karjalainen and Erkut gave the restricting conditions necessary to make finite difference models in the time domain and digital waveguide models connect.

Regarding the stability behavior of the discrete models, the continuous stability is discussed and it is shown that physically stable responses of the string may appear unstable in a discrete model or signal-processing sense.

The paper is structured as follows. First known derivations of the solution of the wave equation is given, both classically and via the theory of fundamental solutions of its partial differential operators. A discrete comparison of the finite difference [10] and the digital waveguide model [29] follows. Smith’s text [9] provides the authoritative summary in of digital waveguides with respect to the continuous derivations given earlier. We conclude with implications of these observations.

II. SOLUTION OF THE WAVE EQUATION IN ONE DIMENSIONS

The results in this section are well-known. They are repeated here to facilitate arguments in the following sections. Of concern is the general digital simulation of a string under force. The free ideal string is well-described by the the $1 + 1$ dimensional d’Alembertian operator on a scalar field [30]:

\[
\Box y(x, t) \overset{\text{def}}{=} \left( \frac{\partial^2}{\partial x^2} - c^2 \frac{\partial^2}{\partial t^2} \right)y(x, t)
\]  

(1)
An external force leads to the inhomogeneous case of the same equation:

$$\Box y(x,t) = f(x,t) \quad (2)$$

Without loss of generality we assume that $c = 1$ for our discussion and hence one gets the factored form of the d’Alembertian:

$$\Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) \quad (3)$$

The factored form suggests the substitution $\xi = x - t$ and $\eta = x + t$ we obtain the canonical form of the wave equation:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{4} f \left( \frac{\xi + \eta}{2}, \frac{\eta - \xi}{2} \right) = \phi(\xi, \eta) \quad (4)$$

This can be directly integrated, yielding

$$u = \int_0^\eta \int_{\xi_0}^\xi \phi(u,w) \, du \, dw + h_1(\xi) + h_2(\eta) \quad (5)$$

where $h_1$, and $h_2$ are “constants of integration”. However, this notation hides that either one of these constants has been integrated over with regards to its parameter. Also these are not uniquely defined functions from a yet undefined functional space but rather any function from suitable family of functions. The derivation procedure suggests that $h_1(\cdot)$ and $h_2(\cdot)$ are one and twice differentiable everywhere in the solution space, but which one is twice differentiable depends in what order the solution has been integrated over.

The solution of the homogeneous case with initial conditions $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$ is well known to correspond to d’Alembert’s solution:

$$y(x,t) = \frac{1}{2} \left( f(x + ct) + f(x - ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \quad (6)$$

Hence initial displacements travel left and right. Initial velocity, however smears over a widening domain of influence.
Alternatively this results can be derived using the theory of partial differential operators. Writing $L = \Box$ and $X = (x, t)$ one arrives at the generic partial differential operator equation:

$$Lu(X) = k(X) \quad (7)$$

Here it is important to note that it is no longer required that $u(\cdot)$ is in the class of twice differentiable functions $C^2$, but rather that $u(\cdot)$ and $k(\cdot)$ are distributions or generalized functions. What this means in detail we will have to defer to expositions available elsewhere. For our purpose it is interest to note that in the theory of distributions jumps and discontinuity are gracefully and meaningfully included in the formalism including the definition derivatives of entities like the Dirac-delta $\delta(\cdot)$ over the space of suitable functions testing for this property. In addition it makes continuous convolutions a central operation to the calculation of continuous solution, which make it’s treatment very similar to the study of discrete models which is regularly used for digital waveguide models.

In our treatment here we will closely follow Hulshof’s and Joshi and Wassermann’s lecture notes and also Edwards’ text which are more accessible than, for example the technical survey by Egorov, Komech and Shubin. The interested reader is pointed to the latter for statements of necessary theorems as well as proofs or detailed references to original proofs.

The *fundamental solution* of the equation the effect of the operator $L$ on the distribution $u$ when it sees as input a Dirac-delta (as noted earlier this is not a function, but a distribution). In the digital signal processing literature $u$ is called the impulse response of $L$. In a dynamical sense it is the response to the inhomogeneous equation where the external force distribution is a Dirac-delta $\delta$. Of immediate interest are solution forward in time the discussion is restricted to fundamental solutions in the positive half-plane with respect to time. This will be indicated by the superscript $+$ to the symbol $E$ for the fundamental solution:

$$L^+E (X) = \delta(X) \quad (8)$$

It can be proved that the solution of both the homogeneous equation with initial value
data and the inhomogeneous equation with some external force distribution can be recovered from the fundamental solution \[16\]. This is done, in analogy to the impulse response convolution in digital signal processing \[17\] by convolution of the fundamental solution with the force distribution and the initial value data \(u(x, 0) = f(\cdot)\) and \(u_t(x, 0) = g(\cdot)\). In the literature these are call Cauchy data.

The fundamental solution \[14, 15\] of the one-dimensional wave equation can be derived to be:

\[
E^+(x, t) = \frac{1}{2} H(t) [H(x + t) - H(x - t)]
\]  

Here \(H(\cdot)\) is the Heaviside distribution, which is the distributional integral of the Dirac-delta distribution \(\delta(\cdot)\). In conventional functional form the Heaviside step-“function” can be written as \[19\]:

\[
H(x) = \begin{cases} 
0 & x < 0, \\
\frac{1}{2} & x = 0, \\
1 & x > 0.
\end{cases}
\]  

The interpretation of this equation is indeed important because it indicates, that the “system response” of a wave operator to an input impulse are not isolated traveling wave pulses but traveling step-distributions.

It may be convenient to think of the fundamental solution as the “distributional continuous impulse response”. This makes sense because the solution of equation can be recovered by convolution of the fundamental solution with the Cauchy data. The continuous convolution has the familiar form \[14, 16\]:

\[
u(x) = (E^+ * f)(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} E^+(x - s)f(s) \, ds
\]  

where \(f(x)\) is Cauchy data in one variable.

If both data and the fundamental solution are in two dimensions (as is possibly the case for external force distributions), one need to convolve over both variables.
Specifically it can be shown (see [15, 16]) that for the set of Cauchy data \( u(x, 0) = f(x) \), \( u_t(x, 0) = g(x) \) and \( Lu(x, t) = k(x, t) \) one gets the complete solution:

\[
u(x, t) = u_f(x, t) + u_g(x, t) + u_k(x, t) \tag{12}\]

with the convolutions:

\[
u_f(x, t) = \mathcal{E}_t^f(\cdot, t) * f(x) \tag{13}\]
\[
u_g(x, t) = \mathcal{E}_t^g(\cdot, t) * g(x) \tag{14}\]
\[
u_k(x, t) = \mathcal{E}_t^k * k(x, t) \tag{15}\]

Performing the convolutions yields the solution which is equivalent to the conventional inhomogeneous initial value solution of the wave equation [15]:

\[
y(x, t) = \frac{1}{2} (f(x + t, 0) + f(x - t, 0)) + \frac{1}{2} \int_{x-t}^{x+t} g(s, 0) ds \tag{16} + \frac{1}{2} \int_{t}^{t} \int_{x-(t-\tau)}^{x+(t-\tau)} k(s, \tau) ds d\tau
\]

For proofs of uniqueness see Egorov, Komech and Shubin [16]. In the absence of external forces this reduces to the initial value solution [6].

The theory of generalized functions for partial differential operators explains why equations [3], derived for the forced case, and [6], derived for the homogeneous initial value case have similar structure. The inhomogeneous case can in a generalized sense be made to include the homogeneous initial value problem (also called Cauchy problem). For example, if we symbolically write \( \tilde{k}(x, t) = \delta(t)k(x, t) \) one sees that the external force response matches the response to the initial velocity.

\[
u_g(x, t) = \frac{1}{2} \int_{x-t}^{x+t} g(s, 0) ds \tag{17}\]
\[
u_k(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \tilde{k}(s, 0) ds \tag{18}\]
Hence an external force distribution which is impulsive in time $\tilde{k}(x, \cdot)$ is indistinguishable from the equivalent initial velocity distribution $g(x)$. Conversely it is noteworthy that initial values do not prescribe a state of the string alone, but also prescribe a sudden onset of such state. Hence initial values are not necessarily a free-field solution of the wave equation, that is the state of a string in the absence of force. Rather impulsive onset states act like external forces.

In particular one can write the solution of the wave equation for the inhomogeneous initial value problem in the following simple form [18]:

$$u(x, t) = E^+ * g(x) + E^+_t * f(x) + \int_0^t E^+_{t-s} * k(x, s) \, ds$$  \hspace{1cm} (19)

Next we note the differentiation of distributions on step-functions (see also the Appendix A):

$$\langle H', \phi \rangle = \langle \delta, \phi \rangle$$  \hspace{1cm} (20)

observe, using this relationship, that the derivative of the fundamental solution (9) are two propagating Dirac-deltas, here again in symbolic notation:

$$E^+_t = \frac{1}{2} H(t) \left( \delta(x + t) + \delta(x - t) \right)$$  \hspace{1cm} (21)

For the current discussion it remains only to point out that the first contribution of (19) look like propagating impulses under differentiation.

From (19) we see that the solution of the wave equation will only stay on the characteristic lines $\xi$ and $\eta$ for a restricted class solutions of the wave equation. Only for very exceptional cases of initial velocities $g$ and external forces $k$ will the solution not integrate into the domain. Rather generically one ought to expect them to integrate into the inside of the forward characteristic cone $x \pm t \geq 0$ as depicted in Figure 1.

The condition in which integration into the interior of the characteristic cone does not occur will be discussed in the discrete case and in this context has been discovered by Karjalainen and Erkut [7]. For this to be appropriate, care needs to be taken when assuming the impulse response of the system to have a particular form, otherwise one ought to expect
both contributions at the same time. Additionally, in a physically realistic situation, when it cannot be guaranteed that an excitation is of purely displacement-type, one ought to expect that the dynamics of the system to integrate and persist over the whole inside of the characteristic cone. It also is worthwhile to point out that this is the mathematically consistent solution of the wave equation \[16\]. Hence, assuming that the wave equation is a reasonable model for a given physical situation, one ought to expect such a behavior to exist and be observable.

As a note, we state, that it is in fact well known that the Huygens’ Principle, the isolated propagation of wave fronts, only holds for d’Alembertians of odd spatial dimensions greater or equal to 3 \[11, 16\] meaning that only in this case does the fundamental solution concentrate on the characteristic cone \(t^2 = |x|^2\). Other cases, including one spatial dimension \[33\], have a wave influence inside the characteristic cone. The idea that the solution would concentrate on the characteristic cone for all dimensions was held for a long time by mathematicians working on the wave-equation until Hadamard opened the development and the situation is since well understood \[20\].

Regarding Huygens’ principle, a short popular exposition can be found in \[21\] whereas a comprehensive technical exposition can be found in \[22\].

III. COMPARISON OF LEAPFROG AND WAVEGUIDE SOLVERS

Next the leapfrog finite difference scheme \[10\] will be compared with the digital waveguide method \[9\]. The full treatment of derivations will not be repeated here and the reader is referred to these sources for details.

Now let us discuss the properties of the so-called leap-frog finite difference molecule for the wave equation. The explicit time-stepping equation reads \[9, 10\]:

\[
y(n+1, m) = y(n, m-1) + y(n, m+1) - y(n-1, m) \tag{22}
\]

where the relationship between the discrete time step \(T\) and the spatial discretization \(X\) is chosen to satisfy \(c = X/T\). In this case the leapfrog molecule can be shown to be consistent at sampling points with the wave equation \[10\]. It can also be shown that waveguide solutions are solutions of the leapfrog \[9\]. As Karjalainen observes, the converse does not hold \[6\].
For future discussion we will use the following symbols:

\[ y_+ = y(n + 1, m) \]  \hspace{2cm} (23)
\[ y_> = y(n, m + 1) \]  \hspace{2cm} (24)
\[ y_< = y(n, m - 1) \]  \hspace{2cm} (25)
\[ y_- = y(n - 1, m) \]  \hspace{2cm} (26)

The condition that an impulse at the root of the molecule will only create responses along the characteristics of the wave can be expressed by the condition \( y_+ = 0 \), i.e. there is no data within the characteristic domain of the molecule.

Hence the non-integrating molecule condition reads:

\[ y_< + y_> - y_- = 0 \]  \hspace{2cm} (27)

From this we get the relationship of waves on the characteristics to their sum:

\[ y_< + y_> = y_- \]  \hspace{2cm} (28)

The updating rules for waveguides\(^{[34]}\) are:

\[ y_l(n, m) = y_l(n - 1, m + 1) \]  \hspace{2cm} (29)
\[ y_r(n, m) = y_r(n - 1, m - 1) \]  \hspace{2cm} (30)

with the external force rule:

\[ y_l(n, m) = \frac{1}{2} f(n, m) \]  \hspace{2cm} (31)
\[ y_r(n, m) = \frac{1}{2} f(n, m) \]  \hspace{2cm} (32)

The wave reconstruction rule is:

\[ y(n, m) = y_l(n, m) + y_r(n, m) \]  \hspace{2cm} (33)
in response to an external force function $f(n, m)$. If we take an impulse of height $y_-$ at time $n - 1$ we get:

$$y_l(n, m - 1) = y_l(n - 1, m) = f(n - 1, m) = \frac{1}{2}y_-$$

(34)

$$y_r(n, m + 1) = y_r(n - 1, m) = f(n - 1, m) = \frac{1}{2}y_-$$

(35)

The reconstructed wave using (33) is zero everywhere except at:

$$y(n, m - 1) = y_l(n, m - 1) = \frac{1}{2}y_-$$

(36)

$$y(n, m + 1) = y_r(n, m + 1) = \frac{1}{2}y_-$$

(37)

$$y(n, m) = y_l(n - 1, m) + y_r(n - 1, m) = y_-$$

(38)

and we see that the non-integrating case of the leapfrog (28) is satisfied with $y_0 = 1/2y_-$ and $y_0 = 1/2y_-$. 

The leapfrog will “integrate” whenever condition (28) is not satisfied. To study the behavior within the characteristic domain it is first assumed that the elements on the characteristic of the molecule $y_<$ and $y_>$ are unaltered. That is, the same waves as before travel outward in the molecule. This leaves us to study an altered relationship between $y_+$ and $y_-$. 

Let $y_-$ be the difference of $y_0$, the molecule value for the non-integrating case (28), and $\tilde{y}_-$, an assumed contribution to the interior of the characteristic domain. Then we get:

$$y_+ = y_< + y_> - (y_0^0 + \tilde{y}_-)$$

(39)

$$0 = y_< + y_> - y_-$$

(40)

Subtracting (40) from (39) we get:

$$y_+ = \tilde{y}_-$$

(41)

Hence the response at at the interior point of the characteristic domain is constant with regards to the contribution of the incoming wave that violates the non-integration condition (28).
To study how this behavior, one can illustrate the response of the leapfrog to an initial

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & \\
1 & 1 & 1 & \\
1 & 1 & \\
1 & \\
\end{array}
\] (42)

and compare it to an excitation which observes (28). \( y_\leq = y_\geq = 1 \) and \( y_- = 2 \):

\[
\begin{array}{cccccc}
1 & \\
1 & 1 & \\
1 & 1 & \\
1 & 1 & \\
1 & 2 & \\
\end{array}
\] (43)

Observe that (43) appears to be the sum of traveling histories and they are time-
symmetric around the intersection point. A time-symmetric solution is an equal contribution
to the solution traveling forward and backward in time and their sum yielding the complete
solution.

D’Alembert’s solution (6) can be used to investigate this observation when writing it in
the form following Alpert, Greengard and Hagstrom[23):

\[
y(x, t) + y(x, -t) = \frac{1}{2} \left( f(x + ct) + f(x - ct) \right) \\
+ \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \\
+ \frac{1}{2} \left( f(x - ct) + f(x + ct) \right) \\
+ \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds
\] (44)

Taking the time-symmetric sum we get:

\[
y(x, t) + y(x, -t) = f(x + ct) + f(x - ct)
\] (45)
Similarly, by taking the difference, one finds the time-asymmetric case:

\[ y(x, t) - y(x, -t) = \frac{1}{c} \int_{x-ct}^{x+ct} g(s) \, ds \]  

(46)

In the discrete case it is easy to see this property preserved in the leapfrog case:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & \\
1 & & \\
0 & 0 & \\
-1 & & \\
-1 & -1 & \\
-1 & -1 & -1 \\
1 & 1 & \\
1 & 1 & \\
2 & & \\
1 & 1 & \\
1 & 1 & \\
\end{array}
\]  

(47)

While (47) nicely illustrates the time-symmetry and the “interference” of waves at the interaction point, (48) is insightful as it clearly shows the properties of a velocity excitation. The displacement vanishes at the interaction moment, while the temporal slope is maximal. It should be made clear, that vanishing of data at one time-step in the leap-frog simulation does not imply velocity solutions. This can be seen if a positive and a negative impulsive wave cross, creating a time-symmetric situation that will not integrate inside the domain:

\[
\begin{array}{ccc}
-1 & 1 & \\
-1 & 1 & \\
0 & & \\
1 & -1 & \\
1 & -1 & \\
\end{array}
\]  

(49)

We observe that this situation does satisfy the time-symmetric equation (45).
Alternatively similar results can be derived by discretizing the initial velocity $g(x)$ directly using a matching center difference scheme \[12\]:

$$y^+ - y^- = g$$ \hspace{1cm} (50)

and including an arbitrary background field one gets:

$$y^+ = \frac{1}{2}(f^- + f^+) + g$$ \hspace{1cm} (51)

where we use the notation $f^+$ and $f^-$ to denote initial displacement wave contributions aligned with the left-right branch of the leapfrog-molecule. Observe that (51) does satisfy the same integration (41) and non-integration (28) conditions. Hence a velocity contribution $g$ can be interpreted as any violation of the rule of the sum of incoming traveling waves.

This behavior has been observed earlier. Karjalainen observed that an asymmetric pair of impulses need to be fed into a leapfrog motivated junction formulations to avoid integration behavior \[24\]. The subsequent physical interpretation is derived in \[7\] from a center-difference time-discrete velocity excitation. An interpretation of this result follows next.

IV. SINGULAR PROPAGATION FROM INTEGRATION

The non-integrating condition can be algorithmically enforced by a method discovered by Karjalainen and Erkut \[6, 7\]. Hence we will call this the Karjalainen-Erkut condition. The rule is to present the excitation through a feed-forward filter \[6\]:

$$H(z) = 1 - z^{-2}$$ \hspace{1cm} (52)

which can be derived from physical conditions by using a center difference velocity term \[7\].

We observe that the Karjalainen-Erkut condition \(52\) creates two impulses from one and those impulses are center symmetric and sign-inverted. If we calculate those two impulse responses separately and then create the sum, we see that the impulsive propagating solution
comes about as the *difference of two Heaviside distributions*. The pulses are represented by their sign only as the amplitudes are assumed to be matched:

\[
\begin{array}{cccccc}
+ & + & + & + & + \\
+ & + & + & + & + \\
+ & + & + & + & + \\
+ & + & + & + & + \\
+ & + & + & + & + \\
\end{array}
\]

(53)

\[
\begin{array}{cccccc}
0 & - & - & - & 0 \\
0 & - & - & - & 0 \\
0 & - & - & - & 0 \\
0 & - & - & - & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

(54)

\[
\begin{array}{cccccc}
+ & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & + \\
\end{array}
\]

(55)

Hence we see that in an impulse-response interpretation of the leap-frog, a Heaviside integration over the characteristic cone of influence is sensible and the Karjalainen-Erkut condition ensures that each Heaviside integration is matched with a delayed sign-inverted response that cancels all interior integration of the first impulse to leave unaltered the traveling impulse solution.

V. **EFFECTS OF THE BOUNDARY**

Next the effect of imposing spatial boundary conditions is studied. For this it is assumed that the solution of the wave equation is only meaningful and defined for a compact domain
The length of the domain is denoted by $L = |\Omega|$. For the boundary of the domain we write $\partial \Omega$ and the interior of the domain is defined by the quotient $\Omega \setminus \partial \Omega$. On each distinct point of the boundary $\partial \Omega$ we impose one boundary condition. Fixed ends $u(\partial \Omega) = 0$ we call Dirichlet boundary conditions, whereas open ends $u_t(\partial \Omega) = 0$ we call Neumann boundary conditions. Note that a circular domain $u(|\Omega|) = u(0)$ is a periodic unbounded domain.

The behavior at the boundary can be conveniently studied by extension of the domain. If the boundary is of Dirichlet type, the value of $u$ needs to vanish at the boundary and hence the extension needs to be odd. In the case of Neumann conditions $u$ needs to be even. As the resulting infinite extension is periodic in $2L$ this extension can be interpreted as a periodic unbounded domain of this length \[13\]. We will denote the original domain by subscript $0$ and extended domains by indices $n \in \mathbb{Z} \setminus 0$. The periodicity implies $\Omega_m + 2 = \Omega_m$ for all $m \in \mathbb{Z}$.

The following discussion will be restricted to the behavior in response to the velocity term $g$ in \[6\]. Observe that with periodicity we can write the integral as the sum of contributions of the periodic domains:

$$
\int_{\xi=x-t}^{\eta=x+t} g(s) \, ds = \sum_m \int_{\Omega_m \geq \xi, \eta} g(s) \, ds
$$

(56)

Hence we integrate over all contributions above the characteristic lines from an excitation point of the periodic domain.

With Dirichlet conditions one gets the odd extension \[25\]:

$$
g(x) = -g(2m|\Omega| - x)
$$

(57)

and for Neumann conditions we get the even extension:

$$
g(x) = g(2m|\Omega| - x)
$$

(58)

with $x \in \Omega_0$ and $m \in \mathbb{Z} \setminus 0$.

Integrating up to the point where the characteristic lines have reached $2|\Omega|$ one gets for Dirichlet boundary conditions:
\[ u(\Omega) = \int_{\Omega_0} g(s) \, ds - \int_{\Omega_{\pm 1}} g(-s) \, ds = 0 \] (59)

Hence integral contributions cancel every \(2|\Omega|\) and the maximum amplitude is bounded by the integral of \(g(\cdot)\) over the original domain \(\Omega_0\).

The same procedure for Neumann boundary conditions leads to:

\[ u(\Omega) = \int_{\Omega_0} g(s) \, ds + \int_{\Omega_{\pm 1}} g(-s) \, ds \]
\[ = 2 \int_{\Omega_0} g(s) \, ds \] (60)

Figure 3 shows the behavior for a string tied at the ends (Dirichlet conditions) after an initial impulsive distribution. It reveals many properties of the effect of the boundary on the integration of velocities under Dirichlet boundary conditions. It shows the odd-periodic extension of the domain \(\Omega_0\) to \(\Omega_n\), \(n \in \mathbb{Z}\), it also shows the cancellation and constructive interference effect of overlapping integration regions. It also shows the \(2|\Omega|\) cancellation of waves. Erkut and Karjalainen\[26\] (compare their Figure 7) reported numerical simulations using the leapfrog molecule with comparable results, which hence matches the situation of the continuous model.

**A. Linear Growth of Displacement**

Observe that the Neumann condition leads to a linear increase in the displacement as a response to velocity or force data being present.

The difference between the Dirichlet condition and the Neumann condition can be interpreted as the difference between an alternating sum and an accumulative sum.

In the Dirichlet case the sign of the area integrated over alternates with periodicity \(|\Omega|\) and hence any finite bounded signal \(g(\cdot)\) will produce an alternating sum which is bounded similarly but infinitely periodic.

In the Neumann the signs match and hence the area integrated over increases with every iteration over the domain by the integral over the finite bounded signal \(g(\cdot)\). Once the support of \(g(\cdot)\) has been exhausted, this obviously corresponds to a linear increase.
This is however, not an unphysical situation. This corresponds to constant kinetic energy being present in the string and hence implies that the energy is bound. This can easily be understood as linear increasing displacement implies constant velocity, which in turn implies constant energy. Hence energy is conserved\cite{13}. It can be interpreted as a string moving at constant velocity, which is meaningful as the Neumann conditions imply that the string is not tied down at the boundaries. Hence linear buildup in a displacement-like wave variable is energy-conserving\cite{35}.

Numerically this is still an undesirable situation because even in the absence of numerical imprecision, the dynamic range of numbers are bounded and hence an infinite increase cannot be represented.

The case of Neumann boundary conditions is interesting because it highlights the difference between notions of stability as customary in discrete signal literature\cite{17} and stability in physical situations. The Neumann displacement response to any bounded input will be unbounded and hence is evidently not bounded-input bounded-output (BIBO) stable, see Oppenheim and Schafer\cite{17}, p. 20. We suggest that this BIBO-unstable but energy-conserving system be called \textit{physically stable}. The BIBO-instability is a discrete computational problem and not one of the physical situation\cite{36}.

VI. IMPLICATIONS

This paper discussed the linear lossless wave equation and its relationship to discrete models, namely a finite difference scheme called leapfrog, and the digital waveguide method. It is shown that the waveguide model corresponds to the finite difference scheme in the absence of integration. In the continuous case, integration can be expected to occur when initial velocities or external forces are present. In this light the observed integrating behavior of finite difference discretization in the time domain using the leapfrog molecule displays results consistent with the continuous model. Here we assume that the wave equation is at least in principle physically meaningful for the modeled situation. If this is the case one should expect consistent behavior of the related discrete models.

In relation to this argument, a use of waveguide discretization that does not include contributions inside the characteristic cone, does not include the integrating behavior of the model equation. In general both integrating and non-integrating responses are to be
expected and hence should be present unless they can be explicitly excluded for physical reasons.

This also implies that the impulse response in just one variable in general will not carry the full dynamics of the string. Hence any assumption of the non-integrating impulse response in one variable in the construction of physical models might contain deviations for it only covers a reduced set of the solution space.

Acknowledgments

Much thanks to Sile O’Modhrain for her support and input. The author also has much gratitude to send to Matti Karjalainen and Cumhur Erkut for stimulating discussions relating to this topic. I am also grateful for their sending of reprints and the graceful sharing of novel unpublished manuscripts. This work was made possible by the kindness of employment of Media Lab Europe and access to its academic resources.

APPENDIX A: PROPERTIES OF DISTRIBUTIONS

Let \( f \) be a distribution on a real open interval \( \Omega \) and let \( \phi \) be in the the set of test functions \( \mathcal{D}(\Omega) \) then one has \([14]\):

\[
\langle \phi, f' \rangle = \int_\Omega f\phi' \, d\mu = -\langle \phi', f \rangle \quad (A1)
\]

and for arbitrary derivatives:

\[
\langle \phi, \partial^p f \rangle = (-1)^{|p|} \langle \partial^p \phi, f \rangle \quad (A2)
\]

the Dirac delta \( \delta \) has the property:

\[
\langle \phi, \delta \rangle = \phi(0) \quad (A3)
\]

hence returns the value of \( \phi \) at 0. By the differentiation rule the higher order derivatives of the Dirac delta returns the higher order derivatives at 0 with alternating sign:
\[ \langle \phi, \partial^p \delta \rangle = -(1)^{|p|} \partial^p \phi(0) \]  

(A4)

Let \( H \) be the Heaviside distribution. It is defined as 15:

\[ \langle \phi, H \rangle = \int_{-\infty}^{\infty} H(s) \phi(s) \, ds = \int_{0}^{\infty} \phi(s) \, ds \]  

(A5)

It hence permits the positive part of \( \phi \) over the domain. The derivative of the Heaviside distribution \( H \) yields (using (A1) and (A4)) the Dirac-delta:

\[ \langle \phi, H' \rangle = -\langle \phi', H \rangle = \phi(0) = \langle \phi, \delta \rangle \]  

(A6)

APPENDIX B: THE WAVE EQUATION AND FIRST ORDER SYSTEMS

In order to derive the relationship between the wave equation to first order systems, we discuss two forms of such systems, namely, two transport equations in one variable and two transport equations in a mixed pair of variables.

A generic version of a system of inhomogeneous first order hyperbolic equations reads:

\[
\begin{align*}
    a \frac{\partial y}{\partial x} + b \frac{\partial y}{\partial t} &= h_1(x) + h_2(t) \quad \text{(B1)} \\
    c \frac{\partial y}{\partial x} + d \frac{\partial y}{\partial t} &= h_3(x) + h_4(t) \quad \text{(B2)}
\end{align*}
\]

For simplicity assume that the force terms are separated in the independent dimensions. Then a second order version is usually derived taking the derivative of one equation with respect to \( t \) and the other one with respect to \( x \). The cross-term \( y_{xt} \) can be eliminated and one gets two equations:

\[
\begin{align*}
    \frac{b}{d} \frac{\partial^2 y}{\partial t^2} - \frac{c}{a} \frac{\partial^2 y}{\partial x^2} &= \frac{1}{a} \frac{\partial}{\partial t} h_2(t) - \frac{1}{a} \frac{\partial}{\partial x} h_3(x) \quad \text{(B3)} \\
    -\frac{a}{c} \frac{\partial^2 y}{\partial t^2} + \frac{d}{b} \frac{\partial^2 y}{\partial x^2} &= \frac{1}{b} \frac{\partial}{\partial x} h_1(x) - \frac{1}{b} \frac{\partial}{\partial t} h_4(t) \quad \text{(B4)}
\end{align*}
\]
The key observation is that one second order equation (B3) or (B4) is not strictly equal to the system of first order equations (B1) and (B2). It is only equal up to two functions (whichever got eliminated, \( h_1, h_4 \) or \( h_2, h_3 \)). They are equivalent up to two "constants of integration".

For systems of first order linear equations in two independent variables a related proof holds. An intuitive interpretation is that in fact for first order equations of the type:

\[
\begin{align*}
    u_x + w_t &= g_1(t) \\
    w_x + u_t &= g_2(x)
\end{align*}
\] (B5) (B6)

one sees that the reduction to second order equations in \( u \) by differentiating (B5) with respect to \( x \) and (B6) with respect to \( t \) one gets:

\[
\begin{align*}
    u_{xx} - u_{tt} &= 0 \\
    -w_{xx} + w_{tt} &= \dot{g}_1(t) - g_2'(x)
\end{align*}
\] (B7) (B8)

Note that differentiation eliminated \( g_1 \) and \( g_2 \) in one case and hence the homogeneous wave equation is again indistinguishable for both the homogeneous and a class of inhomogeneous systems of first order equations and in this sense they are equivalent only up to two functions.

[1] J. Bensa, S. Bilbao, R. Kronland-Martinet, and J. O. Smith, “The simulation of piano string vibration: From physical models to finite difference schemes and digital waveguides,” J. Acoust. Soc. Am. 114(2), 1095–1107 (2003).

[2] S. D. Bilbao, “Wave and Scattering Methods for the Numerical Integration of Partial Differential Equations,” Ph.D. thesis, Stanford University, 2001.

[3] S. Bilbao, ”Spectral Analysis of Finite Difference Meshes” retrieved online on January 2, 2004 at http://ccrma-www.stanford.edu/~jos/vonn/vonn.pdf (unpublished).

[4] S. Bilbao and J. O. Smith, “Finite Difference Schemes and Digital Waveguide Networks for the Wave Equation: Stability, Passivity, and Numerical Dispersion,” IEEE T. Speech Audi. P. 11(3), 255–265 (2003).
[5] C. Erkut and M. Karjalainen, “Finite Difference Method vs. Digital Waveguide Method in String Instrument Modeling and Synthesis,” in Proceedings of the International Symposium on Musical Acoustics (ISMA 2002) (National Autonomous University of Mexico, Mexico City, Mexico, 2002), pp. 9–13.

[6] M. Karjalainen, “Mixed Physical Modeling: DWG + FDTD + WDF,” in Proceedings of the 2003 IEEE Workshop on Applications of Signal Processing to Audio and Acoustics (IEEE, New Paltz, New York, 2003), pp. 225–228.

[7] M. Karjalainen and C. Erkut, "Digital Waveguides vs. Finite Difference Structures: Equivalence and Mixed Modeling,” manuscript, accepted for publication in EURASIP J. Appl. Sig. P. (unpublished).

[8] A. Krishnaswamy and J. O. Smith, “Methods for Simulating String Collisions with Rigid Spatial Obstacles,” in Proceedings of the 2003 IEEE Workshop on Applications of Signal Processing to Audio and Acoustics (IEEE, New Paltz, New York, 2003), pp. 233–236.

[9] J. O. Smith, "Digital Waveguide Modeling of Musical Instruments,” Draft of online manuscript, available at http://ccrma-www.stanford.edu/~jos/waveguide/ (unpublished).

[10] W. F. Ames, Numerical Methods for Partial Differential Equations, 3 ed. (Academic Press, San Diego, 1992).

[11] R. Courant and D. Hilbert, Methoden der Mathematischen Physik I, II, 3rd german ed. (Springer, Berlin, Germany, 1968).

[12] E. Kreyszig, Advanced Engineering Mathematics, 8th ed. (John Wiley & Sons, New York, 1999).

[13] E. M. Stein and R. Shakarchi, Princeton Lectures in Analysis I: Fourier Analysis (Princeton University Press, Princeton, New Jersey, 2003).

[14] R. E. Edwards, Functional Analysis: Theory and Applications (Dover, Mineola, New York, 1995).
[15] J. Hulshof, "Linear Partial Differential Equations," retrieved online on December 19, 2003 at http://www.cs.vu.nl/~jhulshof/NOTES/pdv.ps. See also http://www.cs.vu.nl/~jhulshof/NOTES/waveheat.ps (unpublished).

[16] Y. V. Egorov, A. I. Komech, and M. A. Shubin, Elements of the Modern Theory of Partial Differential Equations (Springer, Berlin, 1999).

[17] A. V. Oppenheim and R. W. Schafer, Discrete-Time Signal Processing (Prentice Hall, Englewood Cliffs, New Jersey, 1989).

[18] M. S. Joshi and A. J. Wassermann, "Lecture Notes for IIB Partial Differential Equations" retrieved online on November 15, 2003 at http://www.damtp.cam.ac.uk/user/dmas2/public_ps/pdewj.ps (unpublished).

[19] Handbook of Mathematical Functions, 10th printing ed., edited by M. Abramowitz and I. A. Stegun. (US National Bureau of Standards, Washington, D.C., 1972).

[20] L. Gårding, “Hyperbolic Equations in the Twentieth Century,” Séminaires et Congrès 3, 37–68 (1998).

[21] A. P. Veselov, retrieved on November 15, 2003 at http://www.lboro.ac.uk/departments/ma/preprints/papers02/02-49.pdf (unpublished).

[22] J. P. Zubelli, Topics on wave propagation and Huygens’ principle (Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janero, 1997), pp. x+83.

[23] B. Alpert, L. Greengard, and T. Hagstrom, “An Integral Evolution Formula for the Wave Equation,” J. Comp. Phys. 162, 536–543 (2000).

[24] M. Karjalainen, “Time-Domain Physical Modeling Real-Time Synthesis Using Mixed Modeling Paradigms,” in Proceedings of the Stockholm Music Acoustics Conference (Royal Institute of Technology, Stockholm, Sweden, 2003), pp. 393–396.

[25] K. F. Graff, Wave Motion in Elastic Solids (Dover, New York, 1991).

[26] C. Erkut and M. Karjalainen, “Virtual Strings Based on a 1-D FDTD Waveguide Model,”
in Proceedings of the Audio Engineering Society 22nd International Conference (Audio Engineering Society, Espoo, Finland, 2002), pp. 317–323.

[27] P. R. Cook, Real Sound Synthesis for Interactive Applications (A K Peters, Ltd., Natick, Massachusetts, 2002).

[28] L. Schwartz, A Mathematician Grappling with His Century (Birkhäuser, Basel, 2001).

[29] For a friendly introductory exposition see Cook[27].

[30] We limit our discussion to this particular form. A note regarding the relationship of this equation to pairs of first order equations can be found in Appendix 11.

[31] It is noteworthy to mention that the nomenclature for “d’Alembert’s solution” is not universally agreed upon. Some sources use it to describe the general form of the integration of the homogeneous equation in the absence of initial value or force (also called Cauchy) data (like [12]) whereas in other sources, for example [16], it refers to the solution including Cauchy (initial, boundary, external force) data. We will use it to denote the solution of the initial value problem.

[32] Laurent Schwartz, the originator of the theory of distributions recently wrote an autobiography [28] that includes a readable account of the historical development of his discovery. We quote two statements from his book that are of interest relating to this article. He writes, when describing the prehistory of his discovery [28, p. 212]: “One of the most important precursors of distributions was the electrical engineer Heaviside.” In a section headed Vibrating strings, harmonic functions he writes [28, p. 218]: “We had learned that the general solution of this equation is of the form $u(t, x) = f(x + vt) + g(x - vt)$, where $f$ and $g$ are arbitrary functions of one variable. Naturally, this presupposes that $f$ and $g$ are $C^2$, so as to be able to differentiate them. What should one think of a function $u$ which would be analogous except that $f$ and $g$ would be merely $C^1$ (continuous), or not even continuous? Is it a wave or not? I was obsessed by this question for some time, then I stopped thinking about it and relegated the question to a corner of my mind for future reflection.”

[33] The case of two spatial dimensions is relevant, but is not in the main thrust of this paper’s discussion. An illustration of the one and two-dimensional wake of the wave-equation can be found in Graff [25], p. 220, Fig. 4.4.
[34] Strictly speaking digital waveguide synthesis can be formulated in various ways. We will not use any arguments from transmission-line theory here. For treatment of those aspects we refer to [9].

[35] Evidently this argument is valid for any amplitude, also small amplitude oscillations, for which the wave-equation is valid, as constant displacement does not alter curvature.

[36] Note that the velocity response is in fact BIBO-stable and hence treatment of the problem in a velocity variable will not suffer this problem.
\[ \xi = x - t \quad \eta = x + t \]

FIG. 1: The characteristic cone of the one-dimensional wave equation.

FIG. 2: Leapfrog computational molecule for the one-dimensional wave equation.

FIG. 3: Sum of velocity domains. \( \Omega_0 \) is the original string domain and \( \Omega_n \) with \( n \in \mathbb{Z} \setminus 0 \) are domains created by continuation of the domain obeying the boundary condition \( u(\partial \Omega) = 0 \).