EQUIVARIANT GROMOV-WITTEN INVARIANTS

Alexander B. GIVENTAL *

Department of Mathematics
University of California
Berkeley, California 94720-3840 USA

The objective of this paper is to describe some construction and applications of the equivariant counterpart to the Gromov-Witten (GW) theory, i.e. intersection theory on spaces of (pseudo-) holomorphic curves in (almost-) Kahler manifolds.

Given a Killing action of a compact Lie group $G$ on a compact Kahler manifold $X$, the equivariant GW-theory provides, as we will show in Section 3, the equivariant cohomology space $H^*_G(X)$ with a Frobenius structure (see [2]). We discuss applications of the equivariant theory to the computation ([7],[11]) of quantum cohomology algebras of flag manifolds (Section 5), to the simultaneous diagonalization of the quantum cup-product operators (Sections 7,8), and to the $S^1$-equivariant Floer homology theory on the loop space $LX$ (see Section 6 and [10],[10]).

In Sections 9 — 11 we combine the general theory developed in Sections 1 — 6 with the fixed point localization technique [3] in order to prove the mirror conjecture (in the form suggested in [10]) for projective complete intersections. We manage to interpret in rigorous terms of stable maps [3] all the results from [9,10] which were motivated by geometry of loop spaces of projective complete intersections. In particular, we prove in Section 11 the following

**Theorem.** Consider the Picard-Fuchs differential equation

$$\left(\frac{d}{dt}\right)^4 I = 5e^t(5\frac{d}{dt} + 1)(5\frac{d}{dt} + 2)(5\frac{d}{dt} + 3)(5\frac{d}{dt} + 4) I$$

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satisfied by the periods
\[ I(t) = \int_{t_0^3} \frac{du_0 \wedge ... \wedge du_4}{d(u_0 + ... + u_4) \wedge d(u_0...u_4)} \]
of the non-vanishing holomorphic 3-forms on the Calabi-Yau 3-folds \( Y_t \) with Hodge numbers \( h^{2,1} = 1, \ h^{1,1} = 101 \) given by the affine equations \( Y_t : u_0 + ... + u_4 = 1, u_0...u_4 = e^t. \)

Pick the basis \( I_0, ..., I_3 \) of solutions to this differential equation determined by

\[ I_0(t) + I_1(t)P + I_2(t)P^2 + I_3(t)P^3 = \sum_{d=0}^{\infty} e^{(P+d)t} \frac{\prod_{m=1}^{5d} (5P + m)}{\prod_{m=1}^{d} (P + m)^5} \quad (\text{mod } P^4). \]

Introduce the new variable \( T(t) = I_1(t)/I_0(t) \).

Then

\[ \frac{I_0}{I_0} + \frac{I_1}{I_0}(t(T))P + \frac{I_2}{I_0}(t(T))P^2 + \frac{I_3}{I_0}(t(T))P^3 = e^{PT} + \frac{P^2}{5} \sum_{d=1}^{\infty} n_d d^3 \sum_{k=1}^{\infty} e^{(P+kd)t} \frac{1}{(P + kd)^2} \quad (\text{mod } P^4), \]

where the components of the RHS form the basis of solutions to the differential equation

\[ \left( \frac{d}{dT} \right)^2 \frac{1}{K(e^T)} \left( \frac{d}{dT} \right)^2 J = 0 \quad \text{with} \quad K(q) = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1 - q^d}, \]

and \( n_d \) is the virtual number of degree \( d \) rational curves in \( \mathbb{C}P^4 \) situated on a generic degree 5 hypersurface \( X \), a Calabi-Yau 3-fold with Hodge numbers \( h^{2,1} = 101, \ h^{1,1} = 1 \).

Analogous results hold for any non-singular Calabi-Yau 3-dimensional projective complete intersection \( X \).

The virtual numbers of rational curves on a Calabi-Yau 3-fold \( X \) are defined in several equivalent ways in the quantum cohomology theory (see [14, 13]) and are equal to the algebraic number of such curves on \( X \) provided with a generic almost Kahler structure. It is known that for generic quintic hypersurfaces \( X \subset \mathbb{C}P^4 \) the virtual number \( n_d \) coincide with the number of the degree \( d \) rational curves in \( \mathbb{C}P^4 \) situated in \( X \) at least for \( d \leq 24 \). The number \( n_1 = 2875 \) of straight lines on a generic quintic 3-fold has been known since the last century, \( n_2 = 609250 \) and \( n_3 = 317206375 \) were found (see [13]) several years ago, while
$n_4 = 242467530000$ was predicted in [15] and confirmed in [3] (as an illustration of a method that allows in principle to find each $n_d$). The simultaneous description of all the numbers $n_d$ given in the theorem was conjectured in [15] on the basis of physical ideas of mirror symmetry between the Calabi-Yau manifolds $X$ and $Y$ whose Hodge diamonds happened to be mirror-symmetric to one another.

As far as we know, our Theorem and its generalization to Calabi-Yau projective complete intersections given in Section 11 provide the first examples of Calabi-Yau manifolds for which predictions of the mirror symmetry are verified for rational curves of all degrees.

The results of Sections 9 – 11 can be immediately carried over to complete intersections in products of projective spaces. The method can be also applied to complete intersections in general toric varieties where however some generalization of our algebraic formalism and some refinement in foundations of the equivariant Gromov – Witten theory would be necessary.

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1 Moduli spaces of stable maps

It was M. Gromov [8] who first suggested to construct (and constructed some) topological invariants of a symplectic manifold $X$ as bordism classes of spaces of pseudo-holomorphic curves in $X$. Recently M. Kontsevich [3] suggested the concept of stable maps which gives rise to an adequate compactification of these spaces. We recall here some basic facts from [3] about these compactifications.

Let $(C, p)$ be a compact connected complex curve with only double singular points and with $n$ ordered non-singular marked points $(p_1, ..., p_n)$. Two holomorphic maps $(C, p) \to X$, $(C', p') \to X$ to an almost-Kahler manifold $X$ are called equivalent if they can be identified by a holomorphic isomorphism $(C, p) \to (C', p')$. A holomorphic map $(C, p) \to X$ is called stable if it does not have infinitesimal automorphisms. In other words, a map is unstable if either it is constant on a genus 0 irreducible component of $C$ with $< 3$ special (= marked
or singular) points or if $C$ is a torus, carries no marked points and the map is constant. A stable map may have a non-trivial finite automorphism group.

According to Gromov’s compactness theorem [8], any sequence of holomorphic maps $C \to X$ of a nonsingular compact curve $C$ has a subsequence Hausdorff-convergent to a holomorphic map $\hat{C} \to X$ of (may be reducible) curve $\hat{C}$ of the same genus $g$ and representing the same total homology class $d \in H_2(X, \mathbb{Z})$. A refinement of this theorem from [3] says that equivalence classes of stable maps $C \to X$ with given $g, n, d$ form a single compact Hausdorff space — the moduli space of stable maps — which we denote $X_{g,n,d}$. Here $g = \dim H^1(C, \mathcal{O}) = 1 - \chi(C \setminus C_{\text{sing}})$.

In the case $X = \text{pt}$ the moduli spaces coincide with Deligne-Mumford compactifications $\mathcal{M}_{g,n}$ of moduli spaces of genus $g$ Riemannian surfaces with $n$ marked points. They are compact nonsingular orbifolds (i.e. local quotients of nonsingular manifolds by finite groups) and thus bear the rational fundamental cycle which allows one to build up intersection theory. In general, the moduli spaces $X_{g,n,d}$ are singular and may have “wrong” dimension, and the idea of the program started in [3, 4] is to provide $X_{g,n,d}$ with virtual fundamental cycles insensitive to perturbations of the almost-Kahler structure on $X$. In some nice cases however the spaces $X_{g,n,d}$ are already nonsingular orbifolds of the “right” dimension.

A compact complex manifold is called ample if it is a homogeneous space of its Lie algebra of holomorphic vector fields.

**Theorem 1.1 ([3, 6])** If $X$ is ample then all non-empty moduli spaces $X_{n,d}$ of genus 0 stable maps are compact nonsingular complex orbifolds of “right” dimension $\langle c_1(T_x), d \rangle + \dim \mathbb{C} X + n - 3$.

Additionally, there are canonical morphisms $X_{n,d} \to X_{n-1,d}$, $X_{n,d} \to \mathcal{M}_{0,n}$, $X_{n,d} \to X^n$ between the moduli spaces $X_{n,d}$ called forgetful, contraction and evaluation (and defined by forgetting one of the marked points, forgetting the map and evaluating the map at marked points respectively). We refer to [3, 4] for details of their construction.

In the rest of this paper we will stick to ample manifolds; we comment however on which results are expected to hold in greater generality. A number of recent preprints by B. Behrend – B. Fantechi, J.Li – G. Tian, T. Fukaya – K. Ono shows that Kontsevich’s “virtual fundamental cycle” program is being realized successfully and leaves no doubts that these
generalizations are correct. Still some verifications are necessary in order make them precise theorems.

## 2 Equivariant correlators

The Gromov-Witten theory borrows from quantum field theory the name \((quantum)\) correlators for numerical topological characteristics of the moduli spaces \(X_{n,d}\) (characteristic numbers) and borrows from bordism theory the construction of such correlators as integrals of suitable wedge-products of various universal cohomology classes (characteristic classes of the GW theory) over the fundamental cycle.

We list here some such characteristic classes.

1. Pull-backs of cohomology classes from \(X^n\) by the evaluation maps \(e_1 \times \cdots \times e_n : X_{g,n,d} \to X^n\) at the marked points.

2. Any polynomial of the first Chern classes \(c^{(1)}, \ldots, c^{(n)}\) of the line bundles over \(X_{g,n,d}\) consisting of tangent lines to the mapped curves at the marked points. One defines these line bundles (by identifying the Cartesian product of the forgetful and evaluation maps \(X_{n+1,d} \to X_{n,d} \times X\) with the \emph{universal stable map} over \(X_{n,d}\)) as normal line bundles to the \(n\) embeddings \(X_{n,d} \to X_{n+1,d}\) defined by the \(n\) marked points of the universal stable map. We will call these line bundles the \emph{universal tangent lines} at the marked points.

3. Pull-backs of cohomology classes of the Deligne-Mumford spaces by contraction maps \(\pi : X_{n,d} \to M_{0,n}\). We will make use of the classes \(A_I := A_{i_1, \ldots, i_k}\) Poincare-dual to fundamental cycles of fibers of forgetful maps \(M_{0,n} \to M_{0,k}\).

We define the \emph{GW-invariant}

\[
A_I(\phi_1, \ldots, \phi_n)_{n,d} := \int_{X_{n,d}} \pi^* A_I \wedge e^*_1 \phi_1 \wedge \cdots \wedge e^*_n \phi_n.
\]

It has the following meaning in enumerative geometry: it counts the number of pairs “a degree-\(d\) holomorphic map \(\mathbb{C}P^1 \to X\) with given \(k\) points mapped to given \(k\) cycles, a configuration of \(n-k\) marked points mapped to the \(n-k\) given cycles”.
Suppose now that the ample manifold $X$ is provided with a Killing action of a compact Lie group $G$. Then $G$ act also on the moduli spaces of stable maps. The evaluation, forgetful and contraction maps are $G$-equivariant, and one can define correlators $A_I\langle \phi_1, \ldots, \phi_n \rangle_{n,d}$ of equivariant cohomology classes of $X$.

The equivariant cohomology $H^*_G(M)$ of a $G$-space $M$ is defined as the ordinary cohomology $H^*(M_G)$ of the homotopic quotient $M_G = EG \times_G M$ — the total space of the $M$-bundle $p : M_G \to BG$ associated with the universal principal $G$-bundle $EG \to BG$. The characteristic class algebra $H^*(BG) = H^*_G(pt)$ plays the role of the coefficient ring of the equivariant theory (so that $H^*_G(M)$ is a $H^*_G(pt)$-module). If $M$ is a compact manifold with smooth $G$-action, the push-forward $p_* : H^*_G(M) \to H^*_G(pt)$ ("fiberwise integration") provides the equivariant cohomology of $M$ with intersection theory with values in $H^*_G(pt)$.

We introduce the equivariant GW-invariants, $A_I(\langle \phi_1, \ldots, \phi_n \rangle_{n,d})$, with values in $H^*(BG)$, where $\phi_1, \ldots, \phi_n \in H^*_G(X)$. Values of such invariants on fundamental cycles of maps $B \to BG$ are accountable for enumeration of rational holomorphic curves in families of complex manifolds with the fiber $X$ associated with the principal $G$-bundles over a finite-dimensional manifold $B$.

### 3 The WDVV equation

One of the main structural results about Gromov-Witten invariants — the composition rule [19],[18] — expresses all genus-0 correlators via the 3-fixed point ones, (we denote them $\langle \phi_1, \ldots, \phi_n \rangle_{n,d}$ since the corresponding $A_I = 1$) satisfying additionally the so-called Witten-Dijkgraaf-Verlinde-Verlinde equation. We will see here that the same result holds true for equivariant Gromov-Witten invariants (at least in the ample case).

Following [20], introduce the potential

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_d q^d \langle t, \ldots, t \rangle_{n,d}.$$  

It is a formal function of $t \in H^*_G(X)$ with values in the coefficient ring $\Lambda = H^*_G(pt, \mathbb{C}[[q]])$.

Here $\mathbb{C}[[q]]$ stands for some completion of the group algebra $\mathbb{C}[H_2(X, \mathbb{Z})]$ so that the symbol $q^d = q_1^{d_1} \ldots q_k^{d_k}$ represents the class $(d_1, \ldots, d_k)$ in the lattice $\mathbb{Z}^k = H_2(X, \mathbb{Z})$ of $2$-cycles. Fundamental classes of holomorphic curves in $X$ have non-negative coordinates with respect
to a basis of Kahler forms so that the formal power series algebra $\mathbb{C}[[q]]$ can be taken on the role of the completion. Strictly speaking, the formula $\mathbb{I}$ defines $F$ up to a quadratic polynomial of $t$ since the spaces $X_{n,0}$ are defined only for $n \geq 3$.

Denote $\nabla$ the gradient operator with respect to the equivariant intersection pairing $\langle \ , \ \rangle$ on $H^*_G(X)$. It is defined over the field of fractions of $H^*_G(pt)$. The WDVV equation is an identity between third directional derivatives of $F$. It says that

$$\langle \nabla F_{\alpha, \beta}, \nabla F_{\gamma, \delta} \rangle$$

is totally symmetric with respect to permutations of the four directions $\alpha, \beta, \gamma, \delta \in H^*_G(X)$.

**Theorem 3.1** The WDVV equation holds for ample $X$.

Notice that

$$\langle \nabla \int_X a \wedge t, \nabla \int_X b \wedge t \rangle = \langle a, b \rangle$$

has geometrical meaning of integration $\int_{\Delta \subset X \times X} a \otimes b$ over the diagonal in $X \times X$.

In order to prove the non-equivariant version of the WDVV equation one interprets the 4-point correlators $A_{1234}(\alpha, \beta, \gamma, \delta)_{4,d}$ which are totally symmetric in $\alpha, \beta, \gamma, \delta$ as integrals over the fibers $\Gamma_\lambda$ of the contraction map $\pi : X_{4,d} \to M_4 = \mathbb{C}P^1$ and specializes the cross-ratio $\lambda$ to 0, 1 or $\infty$. Stable maps corresponding to generic points of, say, $\Gamma_0$ are glued from a pair of maps $f_1 : (\mathbb{C}P^1, p_1, p_2, a_1) \to X$, $f_2 : (\mathbb{C}P^1, p_3, p_4, a_2) \to X$ of degrees $d_1 + d_2 = d$ with three marked points each, satisfying the diagonal condition $f_1(a_1) = f_2(a_2)$. One can treat such a pair as a point in $X_{3,d_1} \times X_{d_2,3}$ situated on the inverse image $\Gamma_{d_1,d_2}$ of the diagonal $\Delta \subset X \times X$ under the evaluation map $e_3 \times e_3$. The glueing map $\sqcup_{d_1+d_2=d} \Gamma_{d_1,d_2} \to \Gamma$ is an isomorphism at generic points and therefore it identifies the analytic fundamental cycles. This means that

$$A_{1234}(\alpha, \beta, \gamma, \delta)_{4,d} = \sum_{d_1+d_2=d} \langle \nabla \langle \alpha, \beta, t \rangle_{3,d_1}, \nabla \langle \gamma, \delta, t \rangle_{3,d_2} \rangle .$$

The above argument applies to the correlators $A_{1234}(\alpha, \beta, \gamma, d, t, \ldots, t)_{n+4,d}$ with additional marked points and therefore it identifies the analytic fundamental cycles. This means that

$$A_{1234}(\alpha, \beta, \gamma, d)_{4,d} = \sum_{d_1+d_2=d} \langle \nabla \langle \alpha, \beta, t \rangle_{3,d_1}, \nabla \langle \gamma, \delta, t \rangle_{3,d_2} \rangle .$$

The above argument applies to the correlators $A_{1234}(\alpha, \beta, \gamma, d, t, \ldots, t)_{n+4,d}$ with additional marked points and gives rise to

$$\langle \nabla F_{\alpha, \beta}, \nabla F_{\gamma, \delta} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d} q^{d} A_{1234}(\alpha, \beta, \gamma, \delta, t \ldots t)_{4+k,d}$$

(4)
which is totally symmetric in \( \alpha, \beta, \gamma, \delta \).

Ampleness of \( X \) is used here only in order to make sure that the moduli spaces have fundamental cycles and that the diagonal in \( X \times X \) consists of regular values of the evaluation map \( e_3 \times e_3 \).

In order to justify the above argument in the equivariant situation, it is convenient to reduce the problem to the case of tori actions (using maximal torus of \( G \)) and use the De Rham version of equivariant cohomology theory.

For a torus \( G = (S^1)^r \) acting on a manifold \( M \) the equivariant De Rham complex \( [1] \) consists of \( G \)-invariant differential forms on \( M \) with coefficients in \( \mathbb{C}[u_1, \ldots, u_r] = H^*_G(pt) \), provided with the coboundary operator \( d_G = d + \sum_{s=1}^r u_s i_s \) where \( i_s \) are the operators of contraction by the vector fields generating the action. Applying the ordinary Stokes formula to \( G \)-invariant forms and \( G \)-invariant chains we obtain well-defined functionals \( H^*_G(M) \to \mathbb{C}[u] \) of integration over invariant cycles. The identity \( [1] \) follows now from the obvious \( G \)-invariance of the analytic varieties \( \Gamma_\lambda, \Gamma, \Gamma_{d_1,d_2} \).

A similar argument proves a composition rule that reduces computation of all equivariant correlators \( A_f(\ldots) \) to that of \( \langle \ldots \rangle \).

4 Ample vector bundles

The following construction was designed by M. Kontsevich in order to extend the domain of applications of WDVV theory to complete intersections in ample Kahler manifolds.

Let \( E \to X \) be an ample bundle, that is, a holomorphic vector bundle spanned by its holomorphic sections. For stable \( f : (C, p) \to X \) (of degree \( d \), with \( n \) marked points), the spaces \( H^0(C, f^*E) \) form a holomorphic vector bundle \( E_{n,d} \) over the moduli space \( X_{n,d} \). If \( f \) is glued from \( f_1 \) and \( f_2 \) as in the proof of \( [1] \), then \( H^0(C, f^*E) = \ker(H^0(C_1, f_1^*E) \oplus H^0(C_2, f_2^*E) e_{12}^1 e_2^*E = e_1^*E) \) where \( e_i : H^0(C_i, f_i^*E) \to e_i^*E \) is defined by evaluation of sections at the marked point \( a_i \).

This allows one to construct a solution \( F \) to the WDVV equation starting with an ample \( G \)-equivariant bundle \( E \) and any invertible \( G \)-equivariant multiplicative characteristic class \( c \) (the total Chern class would be a good example).

Redefine

\[
\langle a, b \rangle := \int_X a \wedge b \wedge c(E)
\]
\[ \langle t, \ldots, t \rangle_{n,d} := \int_{X_{n,d}} e_1^* t \wedge \cdots \wedge e_n^* t \wedge c(E_{n,d}), \]

\[ F(t) = \sum_{n=0}^{\infty} \frac{1}{k!} \sum_d q^d \langle t, \ldots, t \rangle_{n,d}. \]

Then \( \langle \nabla F_{\alpha,\beta}, \nabla F_{\gamma,\delta} \rangle \) is totally symmetric in \( \alpha, \beta, \gamma, \delta \).

This construction bears a limit procedure from the total Chern class to the (equivariant) Euler class, and the limit of \( F \) corresponds to the GW-theory on the submanifold \( X' \subset X \) defined by an (equivariant) holomorphic section \( s \) of the bundle \( E \). Namely, the section \( s \) induces a holomorphic section \( s_{n,d} \) of \( E_{n,d} \), and the (equivariant) Euler class \( \text{Euler} E_{n,d} \) becomes represented by some cycle \( [X'_{n,d}] \) situated in the zero locus \( X'_{n,d} := s_{n,d}^{-1}(0) \) of the induced section. The variety \( X'_{n,d} \) consists of stable maps to \( X' \), the Euler cycle \( [X'_{n,d}] \) plays the role of the virtual fundamental cycle in \( X'_{n,d} \), and the correlators

\[ \langle t, \ldots, t \rangle_{n,d} := \int_{X_{n,d}} e_1^* t \wedge \cdots \wedge e_n^* t \wedge \text{Euler} E_{n,d} = \int_{[X'_{n,d}]} e_1^* t \wedge \cdots \wedge e_n^* t \]

are correlators of GW-theory on \( X' \) between the classes \( t \) which come from the ambient space \( X \).

Another solution of the WDVV-equation can be obtained from the bundles \( E'_{d,k} := H^1(C, f^* E^*) \): one should put \( \langle a, b \rangle := \int_X a \wedge b \wedge c^{-1}(E^*), \langle t, \ldots, t \rangle_{n,d} = \int_{X_{n,d}} e_1^* t \wedge \cdots \wedge e_n^* t \wedge c(E_{n,d}) \) for \( d \neq 0 \) and \( \langle t, \ldots, t \rangle_{n,0} = \int_{X_{n,0}} e_1^* t \wedge \cdots \wedge e_n^* t \wedge c(E^*) \).

## 5 Quantum cohomology

One interprets the WDVV equation as the associativity identity for the \textit{quantum cup-product} on \( H_G^*(X) \) defined by

\[ \langle \alpha \ast \beta, \gamma \rangle = F_{\alpha,\beta,\gamma}. \]

It is a deformation of the ordinary cup-product (with \( t \) and \( q \) in the role of parameters) in the category of (skew)-commutative algebras \textit{with unity}:

\[ \langle \alpha \ast 1, \gamma \rangle = \langle \alpha, \gamma \rangle. \quad (5) \]

Indeed, the push-forward by the forgetful map \( \pi : X_{n,d} \to X_{n-1,d} \) (with \( n \geq 3 \)) sends \( 1 \in H_G^*(X_{n,d}) \) to 0 unless \( d = 0 \) and \( k = 3 \) in which case \( X_{n,d} = X \) and \( X_{n-1,d} \) is not defined.
The structure usually referred in the literature as the quantum cohomology algebra corresponds to the restriction of the deformation $\ast_{t,q}$ to $t = 0$. As it is shown in [4], in many cases the function $F$ can be recovered on the basis of WDVV-equation from the structural constants $F_{\alpha,\beta,\gamma}|_{t=0}(q)$ of the quantum cohomology algebra due to the following symmetry of the potential $F$. Let $u \in H^2_G(X)$ and $(u_1, ..., u_k)$ be its coordinates with respect to the basis of the lattice $(\mathbb{Z}^k)^* = H^2(X) = H^2_G(X)/H^2_G(pt)$ (so that $u_i \in H^*_G(pt)$). Then

$$ (F_{\alpha,\beta,\gamma})_u = \sum_{i=1}^k u_i q_i \partial F_{\alpha,\beta,\gamma}/\partial q_i \quad \forall \alpha, \beta, \gamma \in H^*_G(X). \quad (6) $$

The identity (6) follows from the obvious push-forward formula $\pi_* u = d_i u_i$.

The symmetry (6) can be interpreted in the way that the quantum deformation of the cup-product restricted to $t = 0$ is equivalent to the deformation with $q = 1$ and $t$ restricted to the 2-nd cohomology of $X$ (in the equivariant setting it is better however to keep both parameters in place — see Sections 7, 8).

In this paper, we will use the term quantum cup-product for the entire $(q,t)$-deformation and reserve the name quantum cohomology algebra for the restriction of the quantum cup-product to $t = 0$.

I have heard some complaints about such terminology because it allows many authors to compute quantum cohomology algebras without even mentioning the deformation in $t$-directions. There are some indications however that (despite the equivalence (6)) the $q$-deformation has a somewhat different nature than the $t$-deformation. The loop space approach [9] and our computations in Sections 9 – 11 seem to emphasize this distinction.

Quantum cohomology algebras of the classical flag manifolds have been computed in [7], [11] on the basis of several conjectures about properties of $U_n$-equivariant quantum cohomology (see also [4] where a slightly different formalism was applied). The answer (in terms of generators and relations) for complete flag manifolds $U_n/T^n$ is strikingly related to conservation laws of Toda lattices. The conjectures named in [4] the product, induction and restriction properties and describing behavior of equivariant quantum cohomology under some natural constructions, were motivated by interpretation of the quantum cohomology in terms of Floer theory on the loop space $LX$. Although a construction of the equivariant counterpart of the Floer - Morse theory on $LX$ remains an open problem, the three
conjectured axioms can be justified within the Gromov-Witten theory. This was done by B.Kim [12]. The induction and restriction properties follow directly from definitions given in this paper and hold for the entire quantum deformation (not only at $t = 0$), while the “product” axiom that the $G_1 \times G_2$-equivariant quantum cohomology algebra of $X_1 \times X_2$ is the tensor product of the $G_i$-equivariant quantum cohomology algebras of the factors $X_i$ has been verified in [12] for ample manifolds.

Behavior of the quantum cup-product at $t \neq 0$ under the Cartesian product operation on the target manifolds is much more complicated than the operation of the tensor product.

6 Floer theory and $D$-modules

Structural constants $\langle \alpha \ast \beta, \gamma \rangle$ of the quantum cup-product are derivatives $\partial_3 F_{\alpha,\gamma}$ of the same function. This allows to interpret the WDVV-equation as integrability condition of some connections $\nabla_h$ on the tangent bundle $T_H$ of the space $H = H^\ast(X,\mathbb{C})$. Namely, put $t = \sum t_\alpha p_\alpha$ where $p_1 = 1, p_2, \ldots, p_N$ is a basis in $H$ and define

$$\nabla_h = \hbar d - \sum (p_\alpha \ast) dt_\alpha \wedge : \Omega^0(T_H) \to \Omega^1(T_H)$$

where $p_\alpha \ast$ are operators of quantum multiplication by $p_\alpha$. Then $\nabla_h \circ \nabla_h = 0$ for each value of the parameter $\hbar$. Notice that the integrability condition that reads “the system of differential equations $\hbar \partial_\alpha I = p_\alpha \ast I$ has solutions $I \in \Omega^0(TH)$” is actually obtained as a somewhat combinatorial statement (the WDVV-equation) about coefficients of the series $F$.

In [9], [10] we attempted to improve this unsatisfactory explanation of the integrability property by describing a direct geometrical meaning of the solutions $I$ in terms of $S^1$-equivariant Floer theory on the loop space $LX$. Briefly, the universal covering $\widetilde{LX}$ carries the action of the covering transformation lattice $\pi_2(X)$ with generators $q_1, \ldots, q_k$ and the $S^1$-action by rotation of loops which preserves natural symplectic forms $\omega_1, \ldots, \omega_k$ on $LX$ and thus defines corresponding Hamiltonians $H_1, \ldots, H_k$ on $\widetilde{LX}$ (the action functionals). The Duistermaat–Heckman forms $w_i + \hbar H_i$ (here $\hbar$ is the generator of $H^\ast_{S^1}(pt)$) are equivariantly closed, and operators $p_i$ of exterior multiplication by these forms have the following Heisenberg commutation relations with the covering transformations:

$$p_i p_j - p_j p_i = \hbar q_j \delta_{ij}.$$
Conjecturally, this provides $S^1$-equivariant Floer cohomology of $\tilde{LX}$ with a $D$-module structure which is equivalent to the above system of differential equations (restricted to $t = 0, q \neq 0$) and reduces to the quantum cohomology algebra in the quasi-classical limit $\hbar = 0$ (see [4, 3]).

In this section we describe solutions to $\nabla_\hbar I = 0$ by imitating the $S^1$-equivariant Floer theory (which is still to be constructed) within the framework of Gromov–Witten theory. This construction turns out to be crucial in our proof in Section 11 of the mirror conjecture for Calabi-Yau projective complete intersections.

One may think of the graph of an algebraic loop $\mathbb{C}P^1 \setminus \{0, \infty\} \to X$ of degree $d$ as of a stable map $\mathbb{C}P^1 \to X \times \mathbb{C}P^1$ of bidegree $(d, 1)$. Our starting point consists in interpretation of the moduli space $L_d(X)$ of such stable maps as a degree-$d$ approximation to $\tilde{LX}$ and application of equivariant Gromov–Witten theory to the action of $S^1$ on the second factor $\mathbb{C}P^1$ with the fixed points $\{0, \infty\}$.

In the theorem below we assume $X$ to be ample. It is natural to expect however that the theorem holds true whenever the non-equivariant Gromov–Witten theory works for $X$ since the $S^1$-action is non-trivial only on the factor $\mathbb{C}P^1$ which is ample on its own.

Let $\langle , \rangle$ be the Poincare pairing on $H = H^*(X, \mathbb{C})$. The equivariant cohomology algebra $H^*_S(X \times \mathbb{C}P^1)$ is isomorphic to $H \otimes_{\mathbb{C}} \mathbb{C}[p, \hbar]/(p(p - \hbar))$ with the $S^1$-equivariant pairing

$$\langle \varphi, \psi \rangle = \frac{1}{2\pi i} \oint \frac{\langle \varphi, \psi \rangle dp}{p(p - \hbar)}.$$  

Localization in $\hbar$ allows to introduce coordinates $\varphi = tp/\hbar + \tau(h - p)/\hbar$, $\tau, t \in H$, diagonalizing the equivariant pairing:

$$((\tau, t), (\tau', t')) = \frac{\langle t, t' \rangle - \langle \tau, \tau' \rangle}{\hbar}.$$  

The potential $F(t, \tau, h, q, q_0)$ satisfying the equivariant WDVV-equation for $X \times \mathbb{C}P^1$ expands as

$$F = F^{(0)} + q_0 F^{(1)} + q_0^2 F^{(2)} \ldots$$  

according to contributions of stable maps of degree $0, 1, 2, \ldots$ with respect to the second factor. Denote $F = F(t, q)$ the potential ([4]) of the GW-theory for $X$.

**Theorem 6.1** (a) $F^{(0)} = (F(t, q) - F(\tau, q))/\hbar.$  

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(b) The matrix \((\Phi_{\alpha\beta}) := (\partial^2 F^{(1)}/\partial \tau_\alpha \partial t_\beta)\) is a fundamental solution of \(\nabla_{\pm \hbar} I = 0\):

\[-\hbar \frac{\partial}{\partial \tau_\gamma} \Phi = p_\gamma(t) \Phi,
\]

\[\hbar \frac{\partial}{\partial t_\gamma} \Phi^* = p_\gamma(\tau) \Phi^* ,\]

where \(p_\gamma = (p_\alpha^\beta)_\gamma, \gamma = 1, \ldots, N,\) are matrices of quantum multiplication by \(p_1 = 1, \ldots, p_N,\) and \(\Phi^*\) is transposed to \(\Phi\).

**Proof.** Moduli spaces of bidegree-\((d,0)\) stable maps to \(X \times \mathbb{C}P^1\) coincide with \(X_{n,d} \times \mathbb{C}P^1\). This implies (a) and shows that the WDVV-equation for \(F\) modulo \(q_0\) follows from the WDVV-equation for \(F\). Part (b) follows now directly from the WDVV-equation for \(F\) modulo \(q_0^2\) and from

\[\hbar \frac{\partial}{\partial t_1} \Phi_{\alpha\beta} = \Phi_{\alpha\beta} = -\hbar \frac{\partial}{\partial \tau_1} \Phi_{\alpha\beta}\]

due to (5) and (6). Here \(\partial/\partial t_1, \partial/\partial \tau_1\) are derivatives in the direction \(1 \in H^*(X)\) of the identity components of \(t\) and \(\tau\) respectively.

The following corollary is obtained by expressing equivariant correlators \(\Phi_{\alpha\beta}\) via localization of equivariant cohomology classes of moduli spaces \(L_d(X)\) to fixed points of the \(S^1\) action.

Define

\[\psi_{\alpha\beta} = \sum_{n=0}^{\infty} \frac{\hbar}{n!} \sum_d q^d \langle p_\beta^{\hbar+c} : t, \ldots, t, p_\alpha \rangle_{n+2,d}\]  \hspace{1cm} (7)

where \(c\) is the first Chern class of the line bundle over \(X_{k,d}\) introduced in Section 1 as “the universal tangent line at the first marked point”, and \(\langle \frac{p_\beta^{\hbar+c}}{\hbar+c} : p_\alpha \rangle_{2,0} := \langle p_\alpha, p_\beta \rangle\).

**Corollary 6.2** \(\hbar \partial \psi/\partial t_\gamma = p_\gamma(t) \psi,\) i.e., the matrix \(\psi\) is (another) fundamental solution of \(\nabla_{\hbar} I = 0\).

**Proof.** A fixed point in \(L_d(X)\) is represented by a stable map \(C_0 \cup \mathbb{C}P^1 \cup C_\infty \rightarrow X \times \mathbb{C}P^1\) where \(\varphi_i : C_i \rightarrow X \times \{i\}\) are stable maps of degrees \(d_1 + d_2 = d\) connected by a “constant loop” \(\mathbb{C}P^1 \xrightarrow{\sim} \{x\} \times \mathbb{C}P^1\). Thus components of \(L_d(X)^{S^1}\) can be identified with submanifolds in
\(X_{d_1,k_1+1} \times X_{d_2,k_2+1}\) defined by the diagonal constraint \(e_1(\varphi_0) = e_1(\varphi_\infty)\), with \(h^2(h + c(0))(h - c(\infty))\) to be the equivariant Euler class of the normal bundle. This gives rise to

\[
h^2 \Phi_{\alpha\beta} = \sum_{d} e^{d \tau} \tilde{\psi}_{\alpha\epsilon}(\tau, -h) \eta^{\epsilon\epsilon'} \psi_{\epsilon'\beta}(t, h)
\]

(8)

where \(\sum \eta^{\epsilon\epsilon'} p_\epsilon \otimes p_{\epsilon'}\) is the coordinate expression of the diagonal cohomology class of \(X \subset X \times X\).

We give here several reformulations which will be convenient for computation of quantum cohomology algebras in Sections 9 – 10.

Consider the specialization of the connection \(\nabla_h\) to the parameter subspace corresponding to the deformation of the quantum cup-product along the 2-nd cohomology (this is accomplished by putting first \(t = 0\) and then replacing \(q^d\) by \(\exp(d, t)\) where \(t = (t_1, ..., t_k)\) represents coordinates on \(H^2(X)\) with respect to the basis \(p^{(1)}, ..., p^{(k)} \in H^2(X)\). In this new setting put

\[s_{\alpha,\beta} := \sum_{d} e^{d t} \langle p_{\beta} \frac{e^{d t}}{h + c}, p_\alpha \rangle\]

where \(pt := \sum p^{(i)} t_i\).

**Corollary 6.3.** The matrix \((s_{\alpha,\beta}(t))\) is a fundamental solution to

\[\nabla_h s = 0 : \quad h \frac{\partial}{\partial t_i} s = p^{(i)} * s.\]

**Proof.** One should combine Corollary 6.2 with iterative applications of the following generalized symmetries (5), (6):

\[
\langle f(c), ..., 1 \rangle_{n+1,d} = \langle \frac{f(0) - f(c)}{c}, ..., \rangle_{n,d},
\]

\[
\langle f(c), ..., p^{(i)} \rangle_{n+1,d} = d_i \langle f(c), ..., \rangle_{n,d} + \langle p^{(i)} \frac{f(0) - f(c)}{c}, ..., \rangle_{n,d}.
\]

Here \(f(c)\) is a function of \(c\) with values in \(H^*(X)\).

The symmetries are easily verified on the basis of the following geometrical properties of universal tangent lines:

(i) Consider the push-forward along the map \(\pi : X_{n+1,d} \to X_{n,d}\) (forgetting the last marked point). It is easy to see that the difference \(\pi^*(c) - c\) between the Chern class of the
universal tangent line at the 1-st marked point and the pull-back of its counterpart from $X_{n,d}$ is represented by the fundamental cycle of the section $i : X_{n,d} \to X_{n+1,d}$ defined by the first marked point.

(ii) $i^*(c) = c$.

In particular $\pi_* \left( \frac{1}{\hbar + c} \right) = 1/[\hbar(\hbar + c)]$.

**Corollary 6.4.** Consider the functions

$$s_\beta := \sum_d e^{\hbar t} \langle p_\beta e^{\beta t/\hbar}, 1 \rangle_{2,d}.$$  

Let $P(h\partial/\partial t, \exp t, \hbar)$ be a differential operator annihilating simultaneously all the functions $s_\alpha$. Then the relation $P(p^{(1)}, \ldots, p^{(k)}, q_1, \ldots, q_k, 0) = 0$ holds in the quantum cohomology algebra of $X$ (we assume here that $P$ depends only on non-negative powers of $\hbar$).

The functions $s_\beta$ — “the first components of the vector-solutions $s_{\alpha,\beta}$” — generate a left $\mathcal{D}$-module with the solution locally constant sheaf described by the flat connection $\nabla_\hbar$ and with the characteristic variety isomorphic to the spectrum of the quantum cohomology algebra.

All results of this section extend literally to the equivariant setting and/or to the generalization to ample vector bundles described in Section 4. We will apply them in this extended form in Sections 9–11.

**Remarks.** 1) The universal formula (7) for solutions of $\nabla_\hbar I = 0$ was perhaps discovered independently by several authors. I first learned this formula from R. Dijkgraaf. It can also be found in [2] in the axiomatic context of conformal topological field theory. One can prove it directly from a recursion relation (in the spirit of WDVV-equation) for so-called gravitational descendents — correlators involving the first Chern classes of the universal tangent lines (or, in a slightly different manner, by describing explicitly the divisor in $X_{n,d}$ representing $c$). Our approach provides an interpretation of (7) in terms of fixed point localization in equivariant cohomology.

2) One can generalize our theorem to bundles over $\mathbb{C}P^1$ with the fiber $X$. This seems to indicate that a straightforward “open-string” approach to $S^1$-equivariant Floer theory on $\widetilde{LX}$ would be more powerful and flexible than the approximation by Gromov–Witten theory on $X \times \mathbb{C}P^1$ described above.
3) Although the theorem provides a geometrical interpretation of solutions to $\nabla_h I = 0$, it does not eliminate the combinatorial nature of the integrability condition. Indeed, the theorem is deduced from an equivariant WDVV-equation which in its turn can be interpreted as an integrability condition. Of course one can explain it using the $S^1 \times S^1$-equivariant WDVV-equation on $(X \times \mathbb{C}P^1) \times \mathbb{C}P^1$, etc. It would be interesting to find out whether this process converges.

7 Frobenius structures

In [2], B. Dubrovin studied geometrical structures defined by solutions of WDVV-equations on the parameter space and reduced classification of generic solutions to the classification of trajectories of some Euler-like non-autonomous Hamiltonian systems on $\mathfrak{so}_N^*$. We show here how this approach to equivariant Gromov–Witten theory yields analogous Hamiltonian systems on the affine Lie coalgebras $\tilde{\mathfrak{so}}_N$.

The quantum cup-product on $H = H_G^*(X)$ considered as an $N$-dimensional vector space over the field of fractions $K$ of the algebra $H_G^*(pt)$ defines a formal Frobenius structure on $H$. The structure consists of the following ingredients.

1. A symmetric $K$-bilinear inner product $\langle \ , \ \rangle$,

2. a (formal) function $F : H \to K$ whose third directional derivatives $\langle a * b, c \rangle := F_{a,b,c}$ provide tangent spaces $T_t H$ with the Frobenius algebra structure (i.e. associative commutative multiplication $*$ satisfying $\langle a * b, c \rangle = \langle a, b * c \rangle$).

3. The constant vector field $\mathbb{1}$ of unities of the algebras $(T_t H, *)$ whose flow preserves the multiplication $*$ (i.e. $L_\mathbb{1} (\ast) = 0$).

4. Grading: In the non-equivariant case axiomatically studied by B. Dubrovin it can be described by the Euler vector field $E$, such that the tensor fields $\mathbb{1}$, $*$ and $\langle \ , \ \rangle$ are homogeneous (i.e. are eigen-vectors of the Lie derivative $L_E$) of degrees $-1, 1$ and $D$ respectively (where $D = \dim_{\mathbb{C}} X$ in the models arising from the GW-theory). In the equivariant GW-theory this grading axiom should be slightly modified since the grading of the structural ring $H_G^*(pt)$ is non-trivial and thus the natural Euler operator $L_E$ is $\mathbb{C}$-linear but not $K$-linear.
The fact that the multiplication $\ast$ is defined on tangent vectors to $H$ means that the algebra $(\Omega^0(T_H), \ast)$ can be naturally considered as the algebra $K[L]$ of regular functions on some subvariety $L \subset T^*H$ in the cotangent bundle. A point $t \in H$ is called semi-simple if the algebra $(T_tH, \ast)$ is semi-simple, that is if $L \cap T^*_tH$ consists of $N$ linearly independent points.

Flatness of the connection (defined on $T_H$)

$$\nabla_h = h\partial - \sum p_\alpha \ast dt_\alpha$$

implies $[$4$]$ that $L$ is a Lagrangian submanifold in $T^*H$ near a semisimple $t$. Following $[$2$]$, introduce local canonical coordinates $(u_1, \ldots, u_N)$ such that the sections $(du_1, \ldots, du_N)$ of $T^*H$ are the $N$ branches of $L$ near $t$, and transform the connections $\nabla_h$ to these local coordinates and to a (suitably normalized) basis $f_1, \ldots, f_N$ of vector fields on $H$ diagonalizing the $\ast$-product.

The result of this transformation can be described as follows.

(a) The basis $\{f_i\}$ can be normalized in a way that in the transformed form

$$\nabla_h = h\partial - hA^1 \land -D^1 \land$$

of the connection $\nabla_h$ with $D^1 = \text{diag}(du_1, \ldots, du_N)$, and $A_{ij} = V_{ij}(u)d(u_i - u_j)/(u_i - u_j)$ for all $i \neq j$, we will have additionally $A_{ii} = 0 \forall i$.

(b) The vector field $\mathbb{I}$ in the canonical coordinates assumes the form $\sum k \partial_k$ where $\partial_k := \partial/\partial u_k$ are the canonical idempotents of the $\ast$-product:

$$\partial_i \ast \partial_j = \delta_{ij}\partial_j.$$  

(c) The (remaining part of the) integrability condition $\nabla_h^2 = 0$ reads $d(A^1) = A^1 \land A^1$ or

$$\partial_i \phi_{\alpha}^j = \phi_{\alpha}^i V_{ij}/(u_i - u_j), \ i \neq j,$$  

where $(\phi_{\alpha}^i)$ is the transition matrix, $\partial/\partial t_\alpha = \sum_i \phi_{\alpha}^i f_i$; it can be reformulated as compatibility of the PDE system $[$12$]$ for $(\phi_{\alpha}^i)$ completed by

$$\sum_k \partial_k \phi_{\alpha}^j = 0.$$  

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(d) The Frobenius property \( \langle a \ast b, c \rangle = \langle a, b \ast c \rangle \) of the \( \ast \)-product shows that the diagonalizing basis \( \{ f_i \} \) is orthogonal, that its normalization by \( \langle f_i, f_j \rangle = \delta_{ij} \) obeys \( A_{ii} = 0 \) and, additionally, implies anti-symmetry \( A_{ij} = -A_{ji} \), or

\[
V_{ij} = -V_{ji}.
\]

(14)

The presence of the grading axiom (4) of Frobenius structures over \( K = \mathbb{C} \) allows B.Dubrovin to describe anti-symmetric matrices \( V = (V_{ij}) \in so_N^* \) satisfying the integrability conditions \((12)\) and \((13)\) in quasi-homogeneous canonical coordinates (i.e. \( L_E u_i = u_i \) so that \( E = \sum u_k \partial_k \)) as trajectories of \( N \) commuting non-autonomous Hamiltonian systems (see [2]):

\[
\partial_i V = \{ H_i, V \}
\]

where the Poisson-commuting non-autonomous quadratic Hamiltonians \( H_i \) on \( so_N^* \) are given by

\[
H_i = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij} V_{ji}}{u_i - u_j}.
\]

Consider now the following model modification of the grading axiom: \( K = \mathbb{C}[[\lambda^{\pm 1}]] \), \( \deg \lambda = 1 \). In quasi-homogeneous canonical coordinates \( (u_1, \ldots, u_N, \lambda) \) the Euler vector field takes then on

\[
L_E = \sum_k u_k \partial_k + \lambda \partial_\lambda.
\]

(15)

Introduce the connection operator

\[
\nabla = \lambda \partial_\lambda - V \in \hat{so}_N^*
\]

and the quadratic Hamiltonians on the Poisson manifold \( \hat{so}_N^* \)

\[
\mathcal{H}_i(\nabla) = \oint H_i(V) \frac{d\lambda}{\lambda}.
\]

(16)

**Proposition 7.1** The Hamiltonians \( \mathcal{H}_1, \ldots, \mathcal{H}_N \) are in involution. The operator \( \nabla \) of a Frobenius manifold over \( K \) satisfies the non-autonomous system of Hamiltonian equations

\[
\partial_i \nabla = \{ \mathcal{H}_i, \nabla \}, \ i = 1, \ldots, N.
\]

(17)
The columns $\phi_i = (\phi^i_\alpha)$ of the transition matrix are eigen-functions of the connection operator $\nabla$:

$$\nabla \phi_\alpha := (\lambda \partial_\lambda - V) \phi_\alpha = \left( \frac{n}{2} - \deg t_\alpha + 1 \right) \phi_\alpha.$$  \hfill (18)

**Proof.** It can be obtained by a straightforward calculation quite analogous to that in [2].

In our real life the model equations (15–18) describe the structure of Frobenius manifolds over each semi-simple orbit of the grading Euler field in the ground parameter space. This parameter space is the spectrum of the coefficient algebra $H^*_\text{pt}(\mathbb{C}) \otimes \mathbb{C}[q_1^{\pm 1}, \ldots, q_k^{\pm 1}]$ (its field of fractions can be taken on the role of the ground field $K$). An orbit of the Euler vector field in this parameter space is semi-simple if the corresponding $\mathbb{C}$-Frobenius algebras are semi-simple.

The equations (15–18) over semi-simple Euler orbits should be complemented by the additional symmetries (6).

In the next section we will show how the canonical coordinates of the axiomatic theory of Frobenius structures emerge from localization formulas in equivariant Gromov–Witten theory.

### 8 Fixed point localization

We consider here the case of a circle $T^1$ acting by Killing transformations on a compact Kahler manifold $X$ with isolated fixed points only. The case of tori actions with isolated fixed points requires only slight modification of notations which we leave to the reader. Our results are rigorous for ample $X$ (which includes homogeneous Kahler spaces of compact Lie group and their maximal tori) while applications to general toric manifolds (which are typically not ample) yet to be justified.

It is the Borel localization theorem that reduces computations in torus-equivariant cohomology to computations near fixed points. Let $\{p_\alpha\}, \alpha = 1, \ldots, N$, be the fixed points of the action. We will denote with the same symbols $p_\alpha$ the equivariant cohomology class of $X$ which restricts to $1 \in H^*_\text{pt}(p_\alpha)$ at $p_\alpha$ and to $0$ at all the other fixed points. These classes are well-defined over the field of fractions $\mathbb{C}(\lambda)$ of the coefficient ring $H^*_\text{pt}(\mathbb{C}) = \mathbb{C}[\lambda]$ and form the basis of canonical idempotents in the semi-simple algebra $H^*_\text{pt}(X, \mathbb{C}(\lambda))$. The equivariant
Poincare pairing reduces to \( \langle p_\alpha, p_\beta \rangle = \delta_{\alpha,\beta}/e_\beta \) where \( e_\alpha \in \mathbb{C}[\lambda] \) is the equivariant Euler class of the normal “bundle” \( T_{p_\alpha}X \to p_\alpha \) to the fixed point.

The results described below apply to the setting of Section 4 of a manifold \( X \) provided with an ample vector bundle in which case \( e_\alpha \)'s should be modified accordingly.

The same localization theorem reduces computation of GW-invariants to that near the fixed point set (orbifold) in the moduli spaces \( X_{n,d} \). A fixed point in the moduli space is represented by a stable map to \( X \) of a (typically reducible) curve \( C \) such that each component of \( C \) is mapped to (the closure of) an orbit of the complexified action \( T_C : X \). Any such an orbit is either one of the fixed points \( p_\alpha \) or isomorphic to \((\mathbb{C} - 0)\) connecting two distinct fixed points \( p_\alpha \) and \( p_\beta \) corresponding to 0 and \( \infty \). Respectively, there are two types of components of \( C \):

(i) Each component of \( C \) which carries 3 or more special points must be mapped to one of the fixed points \( p_\alpha \).

(ii) All other components are multiple covers \( z \mapsto z^d \) of the non-constant orbits, and their special points may correspond only to \( z = 0 \) or \( \infty \).

The combinatorial structure of such a stable map can be described by a tree whose edges correspond to chains of components of type (ii) and should be labeled by the total degree of this chain as a curve in \( X \), and vertices correspond to the ends of the chains. The ends may carry 0 or 1 marked point, or correspond to a (tree of) type-(i) components with 1 or more marked points and should be labeled by the indices of these marked points and by the target point \( p_\alpha \).

The fixed stable maps with different combinatorial structure belong to different connected components of the fixed point orbifold in \( X_{n,d} \).

The results below are based on the observation that a stable map with the first \( k \geq 3 \) marked points in a given generic configuration (i.e. with the given generic value of the contraction map \( X_{n,d} \to \mathcal{M}_{0,k} \)) must have in an irreducible component \( C_0 \) in the underlying curve \( C \) which contains this given configuration of \( k \) special points, (so that the corresponding first \( k \) marked points are located on the branches outgoing these special points of \( C_0 \)). The cause is hidden in the definition of the contraction map (see \([3, 4]\)).

We will call the component \( C_0 \) special.

The observation applied to a fixed stable map of the circle action allows to subdivide all fixed point components in \( X_{3+n,d} \) into \( N \) types \( p_i \) according to the fixed points \( p_i \) where the
special component is mapped to. We introduce the superscript notation \((...)^i\) for the contribution (via Borel’s localization formulas) of type-\(p_i\) components into various equivariant correlators. For example,

\[
F^{i}_{\alpha\beta\gamma} = \sum_{n} \frac{1}{n!} \sum_{d} q^d(p_{\alpha}, p_{\beta}, p_{\gamma}, t, \ldots, t)^i_{3+n,d}
\]

where \(t = \sum_{\alpha=1}^{N} t_{\alpha} p_{\alpha}\) is the general class in \(H_T^*(X, \mathbb{C}(\lambda))\), so that \(F_{\alpha\beta\gamma} = \sum_i F^{i}_{\alpha\beta\gamma}\).

We introduce also the notations

- \(\Psi_{i\alpha\beta}\) — for contributions to \(e_i F^{i}_{\alpha\beta}\) of those fixed points which have the third marked point situated directly on the special component \(C_0\) (it is convenient here to introduce the normalizing factor \(e_i \in H_T^*(pt)\), the Euler class of the normal “bundle” to the fixed point \(p_i\) in \(X\));
- \(\Psi_{i\alpha} := \Psi_{i\alpha 1} = \sum_{\beta} \Psi_{i\alpha\beta}\);
- \(D_{i\alpha}\) — for contributions to \(e_i F^{i}_{\alpha ii}\) of those fixed points which have the second and third marked points situated directly on the special component;
- \(\Delta^i\) — for contributions to \(e_i F^{i}_{iii}\) of those fixed points which have the first three marked points situated directly on the special component;
- \(u_i = t_{i+}\) contributions to \(e_i F^{i}_{ii}\) of all those fixed point components in \(X_{2+n,d}\) for which the first two marked points belong to the same vertex of the tree describing the combinatorial structure.

The correlators \(u_i\) can be also interpreted as contributions to the genus-1 equivariant correlators

\[
\sum_{n} \frac{1}{n!} \sum_{d} q^d(t, \ldots, t)_{n,d}
\]

with \textit{given} complex structure of the elliptic curve of those \(T\)-invariant classes which map the (only) genus-1 component of the curve \(C\) to the fixed point \(p_i\).

\textbf{Theorem 8.1.} (a) The functions \(u_1(t), \ldots, u_N(t)\) are the canonical coordinates of the Frobenius structure on \(H_G^*(X, \mathbb{C}(\lambda))\).
(b) The functions $D^i_\alpha(t)$ are eigen-values of the quantum multiplication by $p_\alpha$: $du_i = \sum_\alpha D^i_\alpha dt_\alpha$.

(c) The transition matrix $(\Psi^i_\alpha)$ provides simultaneous diagonalization of the quantum cup product: $F^i_{\alpha\beta\gamma} = \Psi^i_\alpha D^i_\beta \Psi^i_\gamma$ and obeys the following orthogonality relations:

$$\sum_i \Psi^i_\alpha \Psi^i_\beta = \delta_{\alpha\beta}/e_\beta, \sum_\alpha \Psi^i_\alpha \Psi^j_\alpha = \delta_{ij}.$$  

Proof. We first apply the localization formula

$$A_{1234}(\ldots)_{n,d} = \sum_i A_{1234}(\ldots)_{n,d}^i$$

to the 4-point equivariant correlators with the fixed cross-ratio $z$ of the 4 marked points and only after this specialize the cross-ratio to 0,1 or $\infty$. This gives rise to the local WDVV-identities

$$\Psi^i_\alpha \Psi^i_\beta \Psi^i_\gamma \Psi^i_\delta \text{ is totally symmetric in } \alpha, \beta, \gamma, \delta$$

which is independent of the global WDVV-equation. When combined with the global identities

$$A_{1234}(\mathbb{1}, p_\alpha, p_\beta, p_\gamma)_{n,d} = \langle p_\alpha, p_\beta, p_\gamma \rangle_{n,d}$$

they yield the orthogonality relation $\sum_i \Psi^i_\alpha \Psi^j_\beta = \delta_{\alpha\beta}/e_\beta$ and localization formulas

$$F_{\alpha\beta\gamma} = \sum_i \Psi^i_\alpha \Psi^i_\beta \Psi^i_\gamma$$

for the structural constants of the quantum cup-product.

A similar argument with > 4-point correlators $A_{12345\ldots}(\ldots)^i$ proves the diagonalization

$$\langle p_\alpha * p_\beta, p_\gamma \rangle = \sum_i \Psi^i_\alpha D^i_\beta \Psi^i_\gamma / e_i ,$$
\[ \langle p_\alpha \ast p_\beta \ast p_\gamma, p_\delta \rangle = \sum_i \Psi^i_\alpha D^i_\beta D^i_\gamma \Psi^i_\delta / e^i \]

and the identities
\[ \Psi^i_\alpha \beta = \Psi^i_\alpha D^i_\beta, \quad (\Delta^i)^{-1} = \sum_\alpha \Psi^i_\alpha. \]

Finally, the identity \( du_i = \sum_\alpha D^i_\alpha dt_\alpha \) follows directly from the definition of \( u_i \) and implies that \( u_1, \ldots, u_N \) are the canonical coordinates of the Frobenius structure.

### 9 Projective complete intersections

We are going to describe explicitly solutions of the differential equations arising from quantum cohomology of projective complete intersections. Let \( X \) be such a non-singular complete intersection in \( Y := \mathbb{C}P^n \) given by \( r \) equations of the degrees \( (l_1, \ldots, l_r) \). If \( l_1 + \ldots + l_r = n + 1 \) then \( X \) is a Calabi-Yau manifold and its quantum cohomology is described by the mirror conjecture. In this and the next sections we study respectively the cases \( l_1 + \ldots + l_r < n \) and \( l_1 + \ldots + l_r = n \) when the 1-st Chern class of \( X \) is still positive. In the case \( l_1 + \ldots + l_r > n + 1 \) (which from the point of view of enumerative geometry can be considered as “less interesting” for rational curves generically occur only in finitely many degrees) the “mirror symmetry” problem of hypergeometric interpretation of quantum cohomology differential equations remains open.

Let \( E_d \) be the Euler class of the vector bundle over the moduli space \( Y_{2,d} \) of genus 0 degree \( d \) stable maps \( \phi : (C, x_0, x_1) \to \mathbb{C}P^n \) with two marked points, with the fiber \( H^0(C, \phi^* H^{l_1} \oplus \ldots \oplus \phi^* H^{l_r}) \) where \( H^1 \) is the \( l \)-th tensor power of the hyperplane line bundle over \( \mathbb{C}P^n \).

Consider the class
\[ S_d(h) := \frac{1}{h + c_1^{(0)}} \quad E_d \in H^*(Y_{2,d}) \]
where \( c_1^{(0)} \) is the 1-st Chern class of the “universal tangent line at the marked point \( x_0 \)”, and \( e_0, e_1 \) are the evaluation maps. Due to the factor \( E_d \) this class represents the push forward along \( X_{2,d} \to Y_{2,d} \) of the class \( 1/(h + c_1^{(0)}) \) \( \in H^*(X_{2,d}) \) (by the very construction of \( X_{2,d} \) in Section 4).

In the cohomology algebra \( \mathbb{C}[P]/(P^{n+1}) \) of \( \mathbb{C}P^n \), consider the class
\[
S(t, \hbar) := e^{Pt/\hbar} \sum_{d=0}^{\infty} e^{dt}(e_0)_*(S_d(\hbar))
\]

where \((e_0)_*\) represents the push-forward along the evaluation map (and for \(d = 0\), when \(Y_{2,d}\) is not defined, we take \(Euler (\oplus_j H^{\otimes j})\) on the role of \((e_0)_*S_0\)).

Considered as a function of \(t\), \(S\) is a curve in \(H^*(\C P^n)\) whose components are solutions of the differential equation we are concerned about. Indeed, according to Section 6 a similar sum represents the solutions of the quantum cohomology differential equation for \(X\), and \(S\) is just the push-forward of that sum from \(H^*(X)\) to \(H^*(Y)\). (Strictly speaking \(S\) carries information only about correlators between those classes which come from the ambient projective space; also if \(X\) is a surface \(\text{rk} \, H_2(X)\) can be greater than 1 and \(S\) mixes information about the curves of different degrees in \(X\) when they have the same degree in \(Y\).)

**Theorem 9.1.** Suppose that \(l_1 + \ldots + l_r < n\). Then

\[
S = e^{Pt/\hbar} \sum_{d=0}^{\infty} e^{dt} \prod_{m=0}^{d_l}(l_1P + m\hbar) \ldots \prod_{m=0}^{d_l}(l_rP + m\hbar) \prod_{m=1}^{d}(P + m\hbar)^{n+1}.
\]

The formula coincides with those in [9, 10] (found by analysis of toric compactifications of spaces of maps \(\C P^1 \to \C P^n\)) for solutions of differential equations in \(S^1\)-equivariant cohomology of the loop space.

**Corollary 9.2.** (see [9, 10]) The components \(s := \langle P^i, S\rangle\), \(i = 0, \ldots, n - r\), of \(S\) form a basis of solutions to the linear differential equation

\[
(h \frac{d}{dt})^{n+1-r} s = e^t \prod_{j=1}^{r} l_j \prod_{m=1}^{l_j-1} \hbar (l_j \frac{d}{dt} + 1) \ s .
\]

This implies (combine [10] with [7]) that the solutions have an integral representation of the form

\[
\int_{\gamma^{n-r} \subset X'} e^{(u_0 + \ldots + u_n)/\hbar} \frac{du_0 \wedge \ldots \wedge du_n}{dF_0 \wedge dF_1 \wedge \ldots \wedge dF_r}
\]

where

\[
F_0 = u_0 \ldots u_n, \quad F_1 = u_1 + \ldots + u_1, \quad F_2 = u_{1+1} + \ldots + u_{1+l_2}, \ldots, \quad F_r = u_{l_1+1+ \ldots + l_r-1+1} + \ldots + u_{l_1+ \ldots + u_{l_r}}
\]

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and the “mirror manifolds” $X'_t$ are described by the equations

$$X'_t = \{(u_0, ..., u_n) \mid F_0(u) = e^t, \ F_1(u) = 1, ..., \ F_r(u) = 1\}.$$  

This proves for $X$ the mirror conjecture in the form suggested in [10].

**Corollary 9.3.** If $\dim \mathbb{C} X \neq 2$ the cohomology class $p$ of hyperplane section satisfies in the quantum cohomology of $X$ the relation

$$p^{n+1-r} = l_1^{l_1} ... l_r^{l_r} q^{l_1 + ... + l_r - r}.$$  

When $X$ is a surface the same relation holds true in the quotient of the quantum cohomology algebra which takes in account only degrees of curves in the ambient $\mathbb{CP}^n$ (we leave to figure out a precise description of this quotient to the reader; quadrics $\mathbb{CP}^1 \times \mathbb{CP}^1$ in $\mathbb{CP}^3$ provide a good example: $(p_1 + p_2)^3 = 4q(p_1 + p_2) \mod p_1^2 = q = p_2^2$.)

This corollary is consistent with the result of A. Beauville [21] describing quantum cohomology of complete intersections with $\sum l_j \leq n + 1 - \sum (l_j - 1)$ and with results of M. Jinzenji [22] on quantum cohomology of projective hypersurfaces ($r = 1$) with $l_1 < n$.

**Corollary 9.4.** The number of degree $d$ holomorphic maps $\mathbb{CP}^1 \to X^{n-r} \subset \mathbb{CP}^n$, which send $0$ and $\infty$ to two given cycles and send $n + 1 - r$ given points in $\mathbb{CP}^1$ to $n + 1 - r$ given generic hyperplane sections, is equal to $l_1^{l_1} ... l_r^{l_r}$ times the number of degree $d-1$ maps which send $0$ and $\infty$ to the same cycles and $l_1 + ... + l_r - r$ given points — to $l_1 + ... + l_r - r$ given hyperplane sections.

This is the enumerative meaning of Corollary 9.3; of course in this formulation numerous general position reservations are assumed.

**Control examples.** 1. $l_1 = ... = l_r = 1$: The above formulas for quantum cohomology and for solutions of the differential equations in the case of a hyperplane section give rise to the same formulas with $n := n - 1$.

2. $n = 5, r = 1, l = 2$: $X$ is the Plucker embedding of the grassmanian $Gr_{4,2}$. Its quantum cohomology algebra is described by the relations $c_1^3 = 2c_1c_2, \ c_2^2 - c_2c_1^2 + q = 0$ between the Chern classes of the tautological plane bundle. For the 1-st Chern class $p = -c_1$ of the determinant line bundle we deduce the relation $p^5 = 4pq$ prescribed by Corollary 9.3.
We will deduce Theorem 9.1 from its equivariant generalization. Consider the space $\mathbb{C}^{n+1}$ provided with the standard action of the $(n+1)$-dimensional torus $T$. The equivariant cohomology algebra of $\mathbb{C}^{n+1}$ coincides with the algebra of characteristic classes $H^*(BT^{n+1}) = \mathbb{C}[\lambda_0, \ldots, \lambda_n]$. The equivariant cohomology algebra of the projective space $(\mathbb{C}^{n+1} - 0)/\mathbb{C}^\times$ in these notations is identified with $\mathbb{C}[p, \lambda]/((p - \lambda_0)\ldots(p - \lambda_n))$ and the push-forward $H^*_T(\mathbb{C}P^n) \to H^*_T(pt)$ is given by the residue formula

$$f(p, \lambda) \mapsto \frac{1}{2\pi i} \oint \frac{f(p, \lambda)dp}{(p - \lambda_0)\ldots(p - \lambda_n)}.$$  

Here $-p$ can be considered as the equivariant 1-st Chern class of the Hopf line bundle provided with the natural lifting of the torus action. We will use $\phi_i := \prod_{j \neq i} (p - \lambda_j)$, $i = 0, \ldots, n$, as a basis in $H^*_T(\mathbb{C}P^n)$.

Consider the $T$-equivariant vector bundle $\bigoplus_{j=1}^r H^{d_j}$ and provide it with the fiberwise action of the additional $r$-dimensional torus $T'$. The equivariant Euler class of this bundle is equal to $(l_1p - \lambda'_1)\ldots(l_rp - \lambda'_r)$ where $\mathbb{C}[\lambda] = H^*(BT')$.

Introduce the equivariant counterpart $S'$ of the class $S$ in the $T \times T'$-equivariant cohomology of $\mathbb{C}P^n$. This means that we use the equivariant class $p$ instead of $P$ and replace the Euler classes $E_d$ and $c_1(0)$ by their equivariant partners.

**Theorem 9.5.** Let $l_1 + \ldots + l_r < n$. Then

$$S' = e^{pt/\hbar} \sum_{d=0}^\infty e^{dt} \frac{\prod_{i=0}^r (l_ip - \lambda'_i + \hbar m)\ldots\prod_{i=0}^r (l_rp - \lambda'_r + \hbar m)}{\prod_{i=0}^r (p - \lambda_0 + \hbar m)\ldots\prod_{i=0}^r (p - \lambda_n + \hbar m)}.$$  

Theorem 9.1 follows from Theorem 9.5 by putting $\lambda = 0, \lambda' = 0$ which corresponds to passing from equivariant to non-equivariant cohomology.

The vector-function $S'$ satisfies the differential equation

$$\Pi_{i=0}^r (\hbar \frac{d}{dt} - \lambda_i) S' = e^{\Pi_{i=0}^r (l_i \hbar \frac{d}{dt} - \lambda'_i + \hbar m)\ldots\Pi_{i=0}^r (l_r \hbar \frac{d}{dt} - \lambda'_r + \hbar m)} S'.$$

We intend to prove Theorem 9.5 by means of localization of $S'$ to the fixed point set of the torus $T$ action on the moduli spaces $Y_{2,d}$. As it is shown in [3], all correlators of the equivariant theory on $\mathbb{C}P^n$ are computable at least in principle, and in practice the computation reduces to a recursive procedure which can be understood as a summation over
trees and can be also formulated as a non-linear fixed point (or critical point) problem. We will see below that in the case of correlators $\langle \phi_i, S' \rangle$ certain reasons of a somewhat geometrical character cause numerous cancellations between trees so that the recursive procedure reduces to a “summation over chains” and respectively to a linear recurrence equation. The formula of Theorem 9.5 is simply the solution to this equation.

In the proof of Theorem 9.5 below we write down all formulas for for $r = 1$ (it serves the case when $X$ is a hypersurface in $\mathbb{C}P^n$ of degree $l < n$). The proof for $r > 1$ differs only by longer product formulas.

Let us abbreviate $c_1^{(0)}$ as $c$, denote $E_d'$ the equivariant Euler class of the vector bundle over $Y_{2,d}$ whose fiber over the point $\psi: (C, x_0, x_1) \rightarrow Y = \mathbb{C}P^n$ consists of holomorphic sections of the bundle $\psi^*(H^l)$ vanishing at $x_0$, and introduce the following equivariant correlator:

$$Z_i := \sum_{d=0}^{\infty} q^d \int_{Y_{2,d}} e_0^*(\phi_i) \frac{1}{h + c} E_d'.$$

We have

$$\langle \phi_i, S' \rangle = e^{\lambda_i t/h} (l\lambda_i - \lambda')(Z_i|_{q=e^{t}})$$

**Proposition 9.6.**

$$Z_i = 1 + \sum_{d>0} \left( \frac{q}{h^{n+1-l}} \right)^d \int_{Y_{2,d}} \frac{(-c)^{(n+1-l)d-1}}{1 + c/h} E_d' \ e_0^*(\phi_i) .$$

*Proof.* We have just dropped first several terms in the geometrical series $1/(h + c)$ since their degree added with the degrees of other factors in the integral over $Y_{2,d}$ is still less than the dimension of $Y_{2,d}$. It is important here that all the equivariant classes involved including $\phi_i$ are defined in the equivariant cohomology over $\mathbb{C}[\lambda, \lambda']$ without any localization.

It is a half of the geometrical argument mentioned above. The other half comes from the description of the fixed point set in $Y_{2,d}$ given in [11].

Consider a fixed point of the torus $T$ action on $Y_{2,d}$. It is represented by a holomorphic map of a possibly reducible curve with complicated combinatorial structure and with two marked points on some components. Each component carrying 3 or more special points is
mapped to one of the $n + 1$ fixed points of $T$ on $\mathbb{C}P^n$, and the other components are mapped (with some multiplicity) onto the lines joining the fixed points and connect the point-mapped components in a tree-like manner.

In the Borel localization formula for $\int e^*_0(\phi_i)\ldots$ the fixed point will have zero contribution unless the marked point $x_0$ is mapped to the $i$-th fixed point in $\mathbb{C}P^n$ (since $\phi_i$ has zero localizations at all other fixed points.

Consider a fixed point curve $C$ whose marked point $x_0$ is indeed mapped to the $i$-th fixed point in $\mathbb{C}P^n$. There are two options

(i) the marked point $x_0$ is situated on an irreducible component of $C$ mapped with some degree $d'$ onto the line joining the $i$-th fixed point with the $j$-th fixed point in $\mathbb{C}P^n$ with $i \neq j$;

(ii) the marked point $x_0$ is situated on a component of $C$ mapped to the $i$-th fixed point and carrying two or more other special points.

Consider first the option (ii) and the contribution of such a connected component of the fixed point set in $Y_{2,d}$ to the Borel localization formula for $\int c^{(n+1-l)d-1} \ldots$. The connected component itself is the (product of the) Deligne-Mumford configuration space of, say, $s + 1$ special points: the marked point $x_0$, may be the marked point $x_1$, and respectively $s - 1$ or $s$ endpoints of other components of $C$ mapped onto the lines outgoing the $i$-th fixed point in $\mathbb{C}P^n$.

**Lemma 9.7.** The type (ii) fixed point component in $Y_{2,d}$ has zero contribution to the Borel localization formula for $\int_{Y_{2,d}} c^{(n+1-l)d-1} \ldots$

**Proof.** Restriction of the class $c$ from $Y_{2,d}$ to the type (ii) fixed point component coincides with the 1-st Chern class of the line bundle on the Deligne-Mumford factor $\mathcal{M}_{0,s+1}$ of the component defined as “the universal tangent line as the marked point $x_0$” and is thus nilpotent in the cohomology of the component. Since the number of straight lines in a curve of degree $d$ does not exceed $d$ we find that the dimension $s - 2$ of the factor $\mathcal{M}_{0,s+1}$ is less than $d$ which in its turn does not exceed $(n + 1 - l)d - 1$ for $d > 0$ (because we assumed that $n + 1 - l \geq 2$).

Consider now the option (i). The irreducible component $C'$ of the curve $C = C' \cup C''$ carrying the marked point $x_0$ is mapped with the multiplicity $d' \leq d$ onto the line joining $i$-th fixed point in $\mathbb{C}P^n$ with the $j$-th one while the remaining part $C'' \to \mathbb{C}P^n$ of the map represents a fixed point in $Y_{2,d-d'}$. Moreover, the normal space to the fixed point component at
the type (i) point (the equivariant Euler class of the normal bundle occurs in the denominator of the Borel localization formula) is the sum of (a) such a space $N'_{i}$ for $C'' \rightarrow \mathbb{C}P^{n}$, (b) the space $N'$ of holomorphic vector fields along the map $C' \rightarrow \mathbb{C}P^{n}$ vanishing at the fixed point $j$ factorized by infinitesimal reparametrizations of $C'$, (c) the tensor product $L$ of the tangent lines to $C'$ and $C''$ at their intersection point. Since the space $V$ of holomorphic sections of $H^{1}$ restricted to $C$ (and vanishing at $x_{0}$) admits a similar decomposition $V' \oplus V''$, we arrive to the following linear recursion relation for $Z_{i}$.

**Proposition 9.8.** Put $z_{i}(Q, \hbar) := Z_{i}(\hbar^{(n+1-l)}Q, \hbar)$. Then

$$z_{i}(Q, \hbar) = 1 + \sum_{j \neq i} \sum_{d' > 0} Q^{d'} \text{Coeff}_{i}^{j}(d') z_{j}(Q, (\lambda_{j} - \lambda_{i})/d')$$

where

$$\text{Coeff}_{i}^{j}(d') = \frac{[l(\lambda_{j} - \lambda_{i})/d]^{(n+1-l)d-1} \text{Euler}(N')}{1 + (\lambda_{i} - \lambda_{j})/d\hbar} \text{Euler}(V') \phi_{i}|_{p=\lambda_{i}} \cdot$$

**Proof.** Here $(\lambda_{i} - \lambda_{j})/d'$ is the localization of $c$, and the key point is that the equivariant Chern class of the line bundle $L$ over $Y_{2,d-d'}$ is what we would denote $\hbar + c$ for the moduli space $Y_{2,d-d'}$ but with $\hbar = (\lambda_{j} - \lambda_{i})/d'$. This is how the recursion for the correlators $z_{i}$ becomes possible. The rest is straightforward.

**Remark.** Our reduction to the linear recursion relation can be interpreted in the following more geometrical way: contributions of all non-isolated fixed points cancel out with some explicit part of the contribution from isolated fixed points; the latter are represented by chains of multiple covers of straight lines connecting the two marked points.

Let us write down explicitly the factor $\text{Coeff}_{i}^{j}(d')$ from Proposition 9.8 (compare with [3]).

$$\text{Coeff}_{i}^{j}(d') = \frac{\prod_{m=1}^{d} ((\lambda_{i} - \lambda_{j})/d')[(\lambda_{j} - \lambda_{i})/d]^{(n+1-l)d-1}}{d(1 + (\lambda_{i} - \lambda_{j})/d\hbar)\prod_{m=0}^{d} \prod_{m=1}^{d} (\alpha, m) \neq (j, d)(\lambda_{i} - \lambda_{\alpha} + m(\lambda_{j} - \lambda_{i})/d)}$$

(here the product in the numerator is $\text{Euler}(V')$, the denominator — it is essentially $\text{Euler}(N')$ where however the cancellation with $\phi_{i}|_{p=\lambda_{i}}$ is taken care of — has been computed using the exact sequence $0 \rightarrow C \rightarrow \mathbb{C}^{n+1} \otimes H \rightarrow T_{Y} \rightarrow 0$ of vector bundles over $Y = \mathbb{C}P^{n}$, and the “hard-to-explain” extra-factor $d$ is due to the orbifold structure of the moduli spaces (the $d$-multiple map of $C'$ onto the $(ij)$-line in $\mathbb{C}P^{n}$ has a discrete symmetry of order $d$).
\[
= \frac{1}{[(\lambda_i - \lambda_j)/h + d]} \frac{\prod_{m=1}^{l}(\frac{(\lambda_i - \lambda_j)d + m}{\lambda_i - \lambda_j})}{\prod_{n=0}^{d} m_{=1} (\alpha, m) \neq (j, d) (\frac{(\lambda_i - \lambda_j)d + m}{\lambda_i - \lambda_j})}.
\]

Now it is easy to check

**Proposition 9.9.** The correlators \( z_i(Q, 1/\omega) \) are power series \( \sum C_i(d)Q^d \) in \( Q \) with coefficients \( C_i(d) \) which are reduced rational functions of \( \omega \) with poles of the order \( \leq 1 \) at \( \omega = d'/(\lambda_j - \lambda_i) \) with \( d' = 1, \ldots, d \). The correlators \( z_i \) are uniquely determined by these properties, the recursion relations of Proposition 9.8 and the initial condition \( C_i(0) = 1 \).

The proof of Theorem 9.5 is completed by the following

**Proposition 9.10.** The series

\[
z_i = \sum_{d=0}^{\infty} Q^d \frac{\prod_{m=1}^{l}(\sum_{\alpha=1}^{\lambda_i} \omega + m)}{d! \prod_{\alpha \neq 1}^{\lambda_i} \prod_{m=1}^{d} (\sum_{\alpha=1}^{\lambda_i} \omega + m)}
\]

satisfy all the conditions of Proposition 9.10.

Proof. The recursion relation is deduced by the decomposition of the rational functions of \( \omega \) into the sum of simple fractions (or, equivalently, from the Lagrange interpolation formula for each numerator through its values at the roots of the corresponding denominator).

## 10 Complete intersections with \( l_1 + \ldots + l_r = n \)

Let \( X \subset Y = \mathbb{CP}^n \) be a non-singular complete intersection given by equations of degrees \( (l_1, \ldots, l_r) \) with \( l_1 + \ldots + l_r = n \). There are only two points where our proof of Theorem 9.1 would fail for such \( X \). One of them is the Lagrange interpolation formula in the proof of Proposition 9.10. Namely, the rational functions of \( \omega \) there are not reduced — the degree \( dl \) of the numerator is equal to the degree \( dn \) of the corresponding denominator. The other one is Lemma 9.7. Namely, we have the following lemma instead.

**Lemma 10.1.** The type (ii) fixed point component in \( Y_{2,d} \) makes zero contribution via Borel localization formulas to \( \int_{Y_{2,d}} c^{d-1} \ldots \) unless it consists of maps \( (C' \cup C'', x_0, x_1) \to Y \) where \( C' \) is mapped to a fixed point in \( \mathbb{CP}^n \) and carries both marked points, and \( C'' \) is a disjoint union of \( d \) irreducible components (intersecting \( C' \) at \( d \) special points) mapped (each
with multiplicity 1) onto straight lines outgoing the fixed point. All type (ii) components make zero contribution to \( \int_{Y_{2,d}} c^d \).

Let us modify the results of Section 9 accordingly. As we will see, the LHS in Theorem 9.5 is now only proportional to the RHS, and we will compute the proportionality coefficient (a series in \( q \)) directly.

**Proposition 10.2.** Put \( z_i(Q, h) := Z_i(hQ, h) \). Then

\[
z_i(Q, h) = 1 + \sum_{d>0} Q^d \text{Coeff } i(d) + \sum_{j \neq i} \sum_{d' > 0} Q^d \text{Coeff } j(d') z_j(Q, (\lambda_j - \lambda_i)/d')
\]

where \( \text{Coeff } i(d) \) is equal to the contribution of type (ii) fixed point components to \( \int_{Y_{2,d}} (c - 1) E'_d e_0^*(\phi_i) \), and

\[
\text{Coeff } j(d) = 1 \left( \frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i} + m \right).
\]

**Corollary 10.3.** The correlators \( z_i(Q, 1/\omega) \) are power series \( \sum_d C_i(d)Q^d \) with coefficients

\[
C_i(d) = P_d(\omega, \lambda, \lambda')/\Pi_{\alpha \neq i}(\lambda_i - \lambda_\alpha)^d
\]

where \( P_d = P_d^0 \omega^{nd} + ... \) is a polynomial in \( \omega \) of degree \( nd \). The correlators \( z_i \) are uniquely determined by these properties, the recursion relations of Proposition 10.2 and the initial conditions

\[
\sum_d \text{Coeff } i(d)Q^d = \sum_d Q^d P_d^0 \frac{\Pi_{\alpha \neq i}(\lambda_i - \lambda_\alpha)^d}{d!}.
\]

**Proposition 10.4.** The series

\[
z'_i = \sum_{d=0}^{\infty} Q^d \frac{\Pi_{\alpha=1} \Pi_{d=1}^a ((l_a \lambda_i - \lambda'_a)^d \omega + m)}{d! \Pi_{\alpha \neq i} \Pi_{m=1}^d ((\lambda_i - \lambda_\alpha)^d \omega + m)}
\]

satisfy the requirements of Corollary 10.3 with the initial condition

\[
\sum_d Q^d \frac{\Pi_{\alpha=1} \Pi_{d=1}^a ((l_a \lambda_i - \lambda'_a)^d \omega + m)}{d! \Pi_{\alpha \neq i} \Pi_{m=1}^d ((\lambda_i - \lambda_\alpha)^d \omega + m)} = \exp \{ Q \frac{\Pi_{\alpha} (l_a \lambda_i - \lambda'_a)^d}{\Pi_{\alpha \neq i} (\lambda_i - \lambda_\alpha)^d} \}.
\]
Now let us compute $\operatorname{Coeff}_{i}(d)$ using the description of type (ii) fixed point components given in Lemma 10.1.

**Proposition 10.5.** Contribution of the type (ii) fixed point components to
$\sum_{d} Q^{d} \int_{Y_{2,d}} (-c)^{d-1} E'_{d} \delta_{i}$ is
\[
\exp \left\{ \frac{\prod_{a} (l_{a} \lambda_{i} - \lambda'_{a})^{l_{a}}}{\prod_{\alpha \neq i} (\lambda_{i} - \lambda_{\alpha})} \right\} \exp \left\{ -Q \prod_{1} l_{1}! \ldots l_{r}! \right\}.
\]

**Proof.** Each fixed point component described in Lemma 10.1 is isomorphic to the Deligne-Mumford configuration space $M_{0,d+2}$. Our computation is based on the following known formula (see for instance [3]) for correlators between Chern classes of universal tangent lines at the marked points:
\[
\int_{M_{0,k}} 1 \prod_{a} \frac{w_{1}^{c_{1}(1)} \ldots w_{k}^{c_{1}(k)}}{w_{1} \ldots w_{k}} = \frac{(1/w_{1} + \ldots 1/w_{k})^{k-3}}{w_{1} \ldots w_{k}}.
\]

Consider the type (ii) fixed point component specified by the following combinatorial structure of stable maps: $d$ degree 1 irreducible components join the $i$-th fixed point with the fixed points with indices $j_{1}, \ldots, j_{d}$. Using the above formula and describing explicitly the normal bundle to this component in $Y_{2,d}$ and localization of the Euler class $E'_{d}$ we arrive to the following expression for the contribution of this component to $\int_{Y_{2,d}} (-c)^{d-1} \delta_{i} E'_{d}$:
\[
\prod_{s=1}^{d} \frac{\prod_{a=1}^{s} l_{a} \lambda_{i} - \lambda'_{a} + m(\lambda_{j_{s}} - \lambda_{i})}{(\lambda_{i} - \lambda_{j_{s}}) \prod_{\alpha \neq j_{s}, i} (\lambda_{j_{s}} - \lambda_{\alpha})}.
\]

Summation over all type (ii) components in all $Y_{2,d}$ with weights $Q^{d}$ gives rise to
\[
\exp \left\{ -Q \sum_{j \neq i} \frac{\prod_{a} l_{a} \lambda_{i} - \lambda'_{a} + m(p - \lambda_{i})}{\prod_{\alpha \neq j} (p - \lambda_{\alpha})} \right\} |_{p = \lambda_{j}}.
\]

The exponent can be understood as a sum of residues at $p \neq \lambda_{i}, \infty$ and is thus opposite to the sum
\[
\prod_{a} (l_{a} \lambda_{i} - \lambda'_{a})^{l_{a}} / \prod_{\alpha \neq i} (\lambda_{i} - \lambda_{\alpha})
\]
of residues at $\infty$ and $\lambda_{i}$.

**Corollary 10.6.** $z_{i}(Q, 1/\omega) = z'_{i}(Q, \omega) \exp(-l_{1}! \ldots l_{r}! Q).$

**Proof.** Multiplication by a function of $Q$ does not destroy the recursion relation of Proposition 10.2 but changes the initial condition.
We have proved the following

**Theorem 10.7.** Suppose $l_1 + \ldots + l_r = n$. Then

$$S' = e^{(p-l_1!\ldots l_r!e^t)/\hbar} \sum_{d=0}^{\infty} e^{dt} \frac{\prod_{i=1}^{d_1} (l_1p - \lambda_i' + mh) \ldots \prod_{j=1}^{d_r} (l_rp - \lambda_r' + mh)}{\prod_{i=1}^{d} (p - \lambda_i + mh)}.$$ 

$$S = S'|_{\lambda_i = 0, \lambda_r' = 0} = e^{(p-l_1!\ldots l_r!e^t)/\hbar} \frac{\prod_{j=1}^{d_1} (l_j P + mh)}{\prod_{m=1}^{d} (P + mh)^{n+1}} \mod P^{n+1}.$$ 

**Corollary 10.8.** Let $D = \hbar d/dt + l_1!\ldots l_r!e^t$. Then

$$D^{n+1-r} S = l_1\ldots l_r e^t \prod_{j=1}^{d_1} (l_j D + \hbar) \ldots (l_j D + (l_j - 1)\hbar) S.$$ 

**Corollary 10.9.** In the quantum cohomology algebra of $X$ the class $p$ of hyperplane sections satisfies the following relation (with the same reservation in the case $\dim X \leq 2$ as in Corollary 9.3):

$$(p + l_1!\ldots l_r!q)^{n+1-r} = l_1^{l_1'} \ldots l_r^{l_r'} q(p + l_1!\ldots l_r!q)^{n-r}.$$ 

**Control examples.** 1. $X = pt$ in $\mathbb{C}P^1$ $(n = 1, r = 1, l = 1)$. The above relation takes on $p + q = q$, or $p = 0$. Since $P^2 = 0$, we also find from Theorem 10.7 that $S = P \exp(-e^t) \sum_d e^{dt}/d! = P$, or $\langle 1, S \rangle = 1$ as it should be for the solution of the differential equation $\hbar d/dx s = 0$ that arises from quantum cohomology of the point.

2. $X = \mathbb{C}P^1$ embedded as a quadric into $\mathbb{C}P^2$ $(n = 2, r = 1, l = 2)$. We get $(p + 2q)^2 = 4q(p + 2q)$, or $p^2 = 4q^2$. Taking into account that $p$ is twice the generator in $H^2(\mathbb{C}P^1)$ and the line in $\mathbb{C}P^1$ has the degree 2 in $\mathbb{C}P^2$ we conclude that this is the correct relation in the quantum cohomology of $\mathbb{C}P^1$. This example was the most confusing for the author: predictions of the loop space analysis \cite{9} appeared totally unreliable because they gave a wrong answer for the quadric in $\mathbb{C}P^2$. As we see now, the loop space approach gives correct results if $l_1 + \ldots + l_r < n$ and require “minor” modification (by the factor $\exp(-l_1!\ldots l_r!q/\hbar)$) in the boundary cases $l_1 + \ldots l_r = n$; the quadric on the plane happens to be one of such cases.

3. $n = 3, r = 1, l = 3$. We have $(p + 6q)^3 = 27q(p + 6q)^2$, or $p^3 = 9qp^2 + 6q^2p + 27 \cdot 28q^3$. In particular, $\langle p * p, p \rangle = 9q \langle p, p \rangle + 6q^2 \langle p, 1 \rangle + 27 \cdot 28q^3 \langle 1, 1 \rangle = 27q + 0 + 0$ which indicates that there should exist 27 discrete lines on a generic cubical surface in $\mathbb{C}P^3$. 

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11 Calabi-Yau projective complete intersections

Let $L_d(Y)$ denote, as in Section 6, the moduli space of stable maps $\psi: \mathbb{C}P^1 \to \mathbb{C}P^n \times \mathbb{C}P^1$ of bidegree $(d, 1)$ with 2 marked points mapped to $\mathbb{C}P^n \times \{0\}$ and $\mathbb{C}P^n \times \{\infty\}$ respectively. Let $E_d$ denote the equivariant Euler class of the vector bundle over $L_d(Y)$ with the fiber $H^0(\mathbb{C}P^1, \psi^*(E))$ where $E$ is the bundle on $\mathbb{C}P^n \times \mathbb{C}P^1$ induced from our ample bundle $\oplus a H_a$ by the projection to the first factor.

Consider the equivariant correlator

$$\Phi = \int_{Y} \text{Euler}^{-1}(E) S'(t, \hbar) S'({\tau}, -\hbar) = \sum_{d, d'} e^{dt} e^{d'} \sum_{i} \Pi_{a} \left( l_{a} \lambda_{i} - \lambda'_{a} \right) \int_{Y_{d}} E'_{d} \frac{e^{pt/\hbar} e^0_{a}(\phi_{i})}{\hbar + c} \int_{Y_{d'}} E'_{d'} \frac{e^{-pt/\hbar} e^0_{a}(\phi_{i})}{-\hbar + c}.$$ 

In the case $l_1 + ... + l_r < n$ it is easy to check using the explicit formula for $S'$ from Theorem 9.5 that

$$\Phi = \frac{1}{2\pi i} \int e^{p(t-\tau)/\hbar} \left[ \sum_{d} e^{dt} \frac{\Pi_{a} \Pi_{m=0}^{l_{a}}(l_{a} p - \lambda'_{a} - m \hbar)}{\Pi_{j=0}^{n} \Pi_{m=0}^{d}(p - \lambda_{j} - m \hbar)} \right] dp.$$ 

This is an equivariant version of a formula found in [9] in the context of loop spaces and toric compactifications of spaces of rational maps. Namely, consider the projective space $L'_d$ of $(n + 1)$-tuples of polynomials in one variable of degree $\leq d$ each, up to a scalar factor (notice that $L'_d$ has the same dimension $d(n + 1) + n$ as $L_d$). It inherits the component-wise action of the torus $T^{n+1}$ and the action of $S^1$ by the rotation of the variable (“rotation of loops”). Integration over the equivariant fundamental cycle in $L'_d$ is given by the residue formula

$$f(p, \lambda, \hbar) \mapsto \frac{1}{2\pi i} \int \frac{f dp}{\Pi_{j=0}^{n} \Pi_{m=0}^{d}(p - \lambda_{j} - m \hbar)}.$$ 

Consider the equivariant vector bundle over $L'_d$ such that substitution of the $(n + 1)$ polynomials into $r$ (invariant) homogeneous equations in $\mathbb{C}P^n$ of degrees $l_1, ..., l_r$ produces a section of this bundle. The equivariant Euler class of the bundle is

$$\mathcal{E}'_d = \Pi_{a=1}^{r} \Pi_{m=0}^{l_{a}}(l_{a} p - \lambda'_{a} - m \hbar).$$
The formula for $\Phi$ indicates that there should exist a close relation between the spaces $L_d$ and $L'_d$. This relation is described in the following lemma whose proof will be given in the end of this Section.

**The Main Lemma.** There exists a natural $S^1 \times T^{n+1}$-equivariant map $\mu : L_d \to L'_d$. Denote $-p$ the equivariant 1-st Chern class of the Hopf bundle over $L'_d$ induced by $\mu$ to $L_d$. Then

$$\Phi(u, \tau) = \sum_{d} e^{dT} \int_{L_d} e^{p(t-\tau)/\hbar} \mathcal{E}_d.$$

Define $\Phi'(q, z, \hbar) := \Phi|_{t = \tau + z\hbar}, q = e^{\tau}$. (By the way the limit of the series $\Phi'$ at $\hbar = 0$ has the topological meaning of what is called in [7] the *generating volume function*, and the meaning of this limit procedure in terms of differential equations satisfied by $\Phi$ is the *adiabatic approximation*.)

**Corollary 11.1.** $\Phi'(q, z) := \sum_{d} q^{d} \int_{L_d} e^{pz} \mathcal{E}_d =$

$$= \frac{1}{2\pi i} \oint e^{pz} \sum_{d} \prod_{j=0}^{n} \prod_{m=0}^{d}(p - \lambda_j - mh) dp$$

where $E_d = \mu_*(\mathcal{E}_d)$ is a polynomial (of degree $< (n+1)d$) of all its variables.

**Proof.** The integrals $E^{(k)} = \int_{L_d} p^{k}\mathcal{E}_d$, $k = 0, \ldots, \dim L'_d$, which determine the push-forward $\mu_*(\mathcal{E})$ are polynomials in $(\lambda, \lambda', \hbar)$. The matrix $\int_{L'_d} p^{d+\dim L'_d-j}$ is triangular with all eigenvalues equal to 1. This means that there exists a unique polynomial in $p$ with coefficients polynomial in $(\lambda, \lambda', \hbar)$ which represents the push-forward with any given polynomials $E^{(k)}(\lambda, \lambda', \hbar)$.

The last argument also proves

**Proposition 11.2.** Suppose that a series

$$s = \sum_{d} q^{d} \frac{P_d(p, \lambda, \lambda', \hbar)}{\prod_{j=0}^{n} \prod_{m=0}^{d}(p - \lambda_j - mh)}$$

with coefficients $P_d$ which are polynomials of $p$ of degree $\leq \dim L_d$ has the property that for every $k = 0, 1, 2, \ldots$ the q-series $\oint sp^k dp$ has polynomial coefficients in $(\lambda, \lambda', \hbar)$. Then the coefficients of all $P_d$ are polynomials of $(\lambda, \lambda', \hbar)$, and vice versa.

The coefficient $E_d(p, \lambda, \lambda', \hbar)$ in the series $\Phi'$ has the total degree $(l_1 + \ldots + l_r)d + r$ according to the dimension of the vector bundle whose Euler class it represents. Consider the following operations with the series $\Phi$: 35
(i) multiplication by a series of \( e^t \) and/or \( e^\tau \);
(ii) simultaneous change of variables \( t \mapsto t + f(e^t), \tau \mapsto \tau + f(e^\tau) \).
(iii) multiplication by \( \exp[C(f(e^t) - f(e^\tau))/\hbar] \) (here the factor \( C \) should be a linear function of \((\lambda, \lambda')\) in order to obey homogeneity).

**Proposition 11.3.** The property of the series \( \Psi \) to generate polynomial coefficients \( E_d(p, \lambda, \lambda'h) \) is invariant with respect to the operations (i), (ii), (iii).

**Proof.** The polynomiality property of coefficients in \( \Phi' \) is equivalent, due to Proposition 11.2, to the fact that for all \( k \) the \( q \)-series \( (\partial/\partial z)^k |_{z=0} \Phi' \) has polynomial coefficients.

Multiplication by a series of \( q \) does not change this property, which proves the invariance with respect to multiplication by functions of \( e^\tau \).

The roles of \( t \) and \( \tau \) can be interchanged by the substitutions \( p \mapsto p + \hbar d, \hbar \mapsto -\hbar \) in each summand of \( \Phi \). This proves the invariance with respect to multiplication by functions of \( e^t \).

The operation (ii) transforms \( \Phi' \) to

\[
\frac{1}{2\pi i} \sum_d q^d e^{\varphi(q)} \oint \exp \left\{ \frac{z\hbar + f(qe^{z\hbar}) - f(q)}{\hbar} \right\} \frac{E_d(p)}{\prod_j \prod_m (p - \lambda_j - mh)} dp.
\]

Since the exponent is in fact divisible by \( \hbar \), the derivatives in \( z \) at \( z = 0 \) still have polynomial coefficients. This proves the invariance with respect to (ii). The case of the operation (iii) is analogous.

We are going to use the above polynomiality and invariance properties of the correlator \( \Phi \) in order to describe quantum cohomology of Calabi-Yau complete intersections in \( \mathbb{C}P^n \) (in which case \( l_1 + \ldots + l_r = n + 1 \)). We will use this polynomiality in conjunction with recursion relations based on the fixed point analysis of Sections 9, 10. The result can be roughly formulated in the following way: the hypergeometric functions of Theorem 9.5 in the case \( l_1 + \ldots + l_r = n + 1 \) can be transformed to the correlators \( S' \) by the operations (i), (ii), (iii). Notice that in the Calabi–Yau case all our formulas are homogeneous with the grading \( \deg q = 0, \deg p = \deg \hbar = \deg \lambda = \deg \lambda' = 1, \deg z = -1 \). In particular the transformations (i)–(iii) also preserve the degree \( \dim L_d \) of the coefficients \( E_d \) in \( \Phi' \). In the “positive” case \( l_1 + \ldots + l_r \leq n \) where \( \deg q = n + 1 - \sum l_a > 0 \) the transformations (i)–(iii) in fact increase degrees of the coefficients \( E_d \) and are “not allowed”. The only exception is
the operation (iii) with \( f(q) = \text{const} \) \( q \) in the case \( l_1 + \ldots + l_r = n \) when \( \deg q = 1 \). In Section 10 we found the right constant to be \(-l_1!\ldots l_r!\).

Consider now the correlator \( \Phi \),

\[
\Phi = \sum_{i} \Pi_a(l_a \lambda_i - \lambda'_a) e^{\lambda_i (t-\tau)/\hbar} Z_i(e^t, \hbar) Z_i(e^{\tau}, -\hbar),
\]
in the Calabi-Yau case \( l_1 + \ldots + l_r = n + 1 \) (see Section 9 for a definition of \( Z_i \)).

**Proposition 11.4.** (1) The coefficients of the power series \( Z_i(q, \hbar) = \sum_d q^d C_i(d) \) are rational functions

\[
C_i(d) = \frac{P^{(i)}_d}{d! \hbar^d \Pi_{j \neq i} \Pi_{m=1}^d (\lambda_i - \lambda_j + m \hbar)}
\]

where \( P^{(i)}_d \) is a polynomial in \((\hbar, \lambda, \lambda')\) of degree \((n + 1)d\).

(2) The polynomial coefficients \( E_D(p) \) in \( \Phi' \) are determined by their values

\[
E_D(\lambda_i + d \hbar) = \Pi_a(l_a \lambda_i - \lambda'_a) P^{(i)}_d(h) P^{(i)}_{D-d}(\hbar)
\]
at \( p = \lambda_i + d \hbar, \ i = 0, \ldots, n, \ d = 0, \ldots, D \).

(3) The correlators \( Z_i(q, \hbar) \) satisfy the recursion relation

\[
Z_i(q, \hbar) = 1 + \sum_d \frac{q^d \Pi_{\alpha \neq i} \Pi_{m=1}^d (l_a \lambda_i - \lambda'_a + m(\lambda_j - \lambda_i)) / d}{d! \hbar^d \Pi_{j \neq i} \Pi_{m=1}^d (\lambda_i - \lambda_j + m \hbar)} Z_j(q(\lambda_j - \lambda_i) / d, (\lambda_j - \lambda_i) / d)
\]

where \( R_{i,d} = R^{(0)}_{i,d} h^d + R^{(1)}_{i,d} h^{d-1} + \ldots \) is a polynomial of \((\hbar, \lambda, \lambda')\) of degree \( d \), and

\[
\text{Coeff}_{f_i^j(d)} = \frac{\Pi_a l_a \Pi_{m=1}^d (l_a \lambda_i - \lambda'_a + m(\lambda_j - \lambda_i) / d)}{d! \Pi_{\alpha \neq i} \Pi_{m=1}^d (\lambda_i - \lambda_\alpha + m(\lambda_j - \lambda_i) / d)}
\].

For any given \( \{R_{i,d}\} \) these recursion relations have a unique solution \( \{Z_i\} \).

(4) Consider the class \( \mathcal{P} \) of solutions \( \{Z_i\} \) to these recursion relations which give rise to polynomial coefficients \( E_d \) in the corresponding \( \Phi \). A solution from \( \mathcal{P} \) is uniquely determined by the first two coefficients \( R^{(0)}_{i,d}, R^{(1)}_{i,d}, i = 0, \ldots, n, \ d = 1, \ldots, \infty \), of its initial condition (that is by the first two terms in the expansion of \( Z_i = Z^{(0)}_i + Z^{(1)}_i / \hbar + \ldots \) as power series in \( 1/\hbar \)).

(5) The class \( \mathcal{P} \) is invariant with respect to the following operations:

(a) simultaneous multiplication \( Z_i \mapsto f(q)Z_i \) by a power series of \( q \) with \( f(0) = 1 \);
(b) changes $Z_i(q, \hbar) \mapsto e^{\lambda_f(q)/\hbar}Z_i(qe^{f(q)}, \hbar)$ with $f(0) = 0$;
(c) multiplication $Z_i \mapsto \exp(Cf(q)/\hbar)Z_i$ where $C$ is a linear function of $(\lambda, \lambda')$ and $f(0) = 0$.

Proof. (3) We have

$$Z_i = 1 + \sum_{d>0} q^d \left[ \sum_{k=0}^{d-1} h^{-k-1} \int_{Y_{2,d}} E_d' e_0^d(\phi_i)(-c)^k \right] + \sum_{d>0} q^d h^{-d} \int_{Y_{2,d}} E_d' e_0^d(\phi_i) \frac{(-c)^d}{h+c}$$

where the integrals of the last sum have zero contributions from the type (ii) fixed point components (Lemmas 9.7, 10.1). Thus these integrals have a recursive expression identical to those of Sections 9 and 10. The terms of the double sum constitute the initial condition $\{R_{i,d}\}$. The recursion relations have the form of the decomposition of rational functions of $\hbar$ (coefficients at powers of $Q = q/\hbar$) into the sum of simple fractions in the case when degrees of numerators exceed degrees of denominators. This proves existence and uniqueness of solutions to the recursion relations.

(1) follows directly from the form and topological meaning of the recursion relations.

(2) follows from the definition of $\Phi$ in terms of $Z_i$.

(4) Perturbation theory: Suppose that two solutions from the class $P$ have the same initial condition up to the order $(d-1)$ inclusively. Then (2) shows that corresponding $E_k$ for these solutions coincide for $k < d$ and the variation $\delta E_d(p)$ vanishes at $p = \lambda_i + k \hbar$ for $0 < k < d$. This means that the polynomial $\delta E_d$ is divisible by $\Pi_j \Pi_{m=1}^{d-1} (p - \lambda_j - m \hbar)$. On the other hand (1) and (2) imply that the variation $\delta R_{i,d}$ of the initial condition satisfies

$$\delta R_{i,d}(\hbar) \Pi_a (l_a \lambda_i - \lambda'_a) \Pi_{j \neq i} \Pi_{m=1}^{d-1} (\lambda_i - \lambda_j + m \hbar) = \delta E_d|_{p=\lambda_i+h\hbar}$$

(since $R_{i,0} = 1$) and thus $\delta R_{i,d}$ is divisible by $\hbar^{d-1}$. Since $\delta R_{i,d}$ is a degree $d$ polynomial, it leaves only the possibility

$$\delta R_{i,d} = \delta R_{i,d}^{(0)} \hbar^d + \delta R_{i,d}^{(1)} \hbar^{d-1}.$$ 

Thus if two class $P$ solutions coincide in orders $\hbar^0, \hbar^{-1}$ then $\delta R_{i,d} = 0$, and thus the very solutions coincide.

(5) The operations (a), (b), (c) give rise to the operations of type (i)-(iii) for corresponding polynomials $E_d$. Thus it suffices to show that the operations (a), (b), (c) transform a solution $\{Z_i\}$ of the recursion relations to another solution.
Consider in our recursion relation the coefficient $\hbar^{-d}q^d \text{Coeff}_{i}(d)$ responsible for the simple fraction with the denominator $(\lambda_i - \lambda_j + d\hbar)$. The operations (a), (b), (c) cause respectively the following modifications in this coefficient:

\[
q^d \mapsto f(q)q^d/f(Q),
\]

\[
q^d \mapsto q^d \exp\left\{\frac{\lambda_i f(q)}{\hbar} + df(q) - \frac{d\lambda_j f(Q)}{(\lambda_j - \lambda_i)}\right\},
\]

\[
q^d \mapsto q^d \exp\left\{C\frac{f(q)}{\hbar} + C\frac{df(Q)}{\lambda_i - \lambda_j}\right\},
\]

where $Q = (\lambda_j - \lambda_i)q/d\hbar = q - (\lambda_i - \lambda_j + d\hbar)q/d\hbar$. In the case of the change (b), additionally, the argument $q$ in $Z_j$ on the RHS of the recursion relation gets an extra-factor $\exp[f(q) - f(Q)]$.

The difference $Q - q$ and the exponents vanish at $\hbar = (\lambda_j - \lambda_i)/d$. This means that

\[
\frac{q^d}{l_i - \lambda_j + d\hbar} \mapsto \frac{q^d}{\lambda_i - \lambda_j + d\hbar} + \text{terms without the pole}.
\]

The latter terms give contributions to a new initial condition, while the coefficient $\hbar^{-d}q^d \text{Coeff}_{i}(d)$ does not change. It is easy to see that the required properties of the initial condition (that the degree of $R_{i,d}(\hbar)$ does not exceed $d$ and $R_{i,0} = 1$) are also satisfied under our assumptions about $f$ (for those contributions involve $\hbar$ only in the combination $q/\hbar$).

Let us consider now the hypergeometric series

\[
Z^*_i = \sum_{d=0}^{\infty} q^d \frac{\prod_{a=1}^{r} \prod_{m=1}^{l_a} (l_a \lambda_i - \lambda'_a - m\hbar)}{\prod_{a=0}^{r} \prod_{m=1}^{l_a} (\lambda_i - \lambda_a + m\hbar)}
\]

where $l_1 + \ldots + l_r = n + 1$.

It is straightforward to see that $\{Z^*_i\}$ satisfy the recursion relations of Proposition 11.4(3) (see the proof of Proposition 9.10) and that the formulas of Proposition 11.4(2) generate corresponding

\[
\Phi^* = \frac{1}{2\pi i} \int e^{p(t - \tau)/\hbar} \sum_{d=0}^{\infty} e^{d\tau} \frac{\prod_{a=1}^{r} \prod_{m=0}^{l_a} (l_a p - \lambda'_a - m\hbar)}{\prod_{a=0}^{r} \prod_{m=0}^{l_a} (p - \lambda_i - m\hbar)} dp
\]

with polynomial numerators. Thus $\{Z_i\}$ is a solution from the class $\mathcal{P}$. 39
Computation of the first two terms in the initial condition gives

\[ Z_i^*(0) = f(q) = \sum_{d=0}^{\infty} \frac{(l_1d)!... (l_rd)!}{(d!)^{n+1}} q^d, \]

\[ Z_i^*(1) = \lambda_i \sum_a l_a[g_a(q) - g_1(q)] + \left( \sum_\alpha \lambda_\alpha g_1(q) - \sum_a \lambda'_a g_a(q) \right) \]

where

\[ g_l = \sum_{d=1}^{\infty} q^d \prod_{a} (l_a d)! \frac{1}{(d!)^{n+1}} \left( \sum_{m=1}^{ld} \frac{1}{m} \right). \]

Let us compare these initial conditions with those for \( \{Z_i\} \).

**Proposition 11.5.** \( Z_i^{(0)} = 1, Z_i^{(1)} = 0. \)

**Proof.** The first statement follows from the definition of \( Z_i \) while the second means that

\[ \int_{Y_{1,d}} E'_{d} e_0^*(\phi_i) = 0 \] for all \( d > 0 \). It is due to the fact that the class \( E'_{d} e_0^*(\phi_i) \) is a pull-back from \( Y_{1,d} \). (In fact we have just repeated an argument proving (5) from Section 5 and thus the proposition can be deduced from general properties of quantum cohomology.)

Combining the last two propositions we arrive to the following

**Theorem 11.6.** The hypergeometric solution \( \{Z_i^*(q,h)\} \) coincides with the solution \( \{Z_i(Q,h)\} \) up to transformations (a),(b),(c). More precisely, perform the following operations with \( \{Z_i\} \)

1) put

\[ Q = q \exp\left\{ \sum_a l_a[g_a(q) - g_1(q)]/f(q) \right\}, \]

2) multiply \( Z_i(Q(q),h) \) by

\[ \exp\left\{ \frac{1}{f(q)h} [\sum_a (l_a \lambda_i - \lambda'_a) g_a(q) - (\sum_\alpha (\lambda_i - \lambda_\alpha)) g_1(q)] \right\}, \]

3) multiply all \( Z_i \) simultaneously by \( f(q) \).

Then the resulting functions coincide with hypergeometric series \( Z_i^*(q,h) \).

**Proof.** The three steps correspond to consecutive applications of operations of type (b),(c) and (a) to \( \{Z_i\} \) and transform the initial condition of Proposition 11.5 to that for \( \{Z_i^*\} \). According to Proposition 11.4 this transforms the whole solution \( \{Z_i\} \) to \( \{Z_i^*\} \).
Consider the solutions 
\[ s_i = e^{\lambda_i T / \hbar} Z_i(e^T, \hbar) \]
to the equivariant quantum cohomology differential equations.

**Corollary 11.7.** The operations
1) change 
\[ T = t + \sum \lambda \left[ g_a(e^t) - g_1(e^t) \right] / f(e^t), \]
2) multiplication by 
\[ f(e^t) \exp \left\{ \left[ g_1(e^t) \left( \sum \lambda \right) - \sum \lambda a g_a(e^t) \right] / (hf(e^t)) \right\} \]
transform \( \{ s_i \} \) to the hypergeometric solutions 
\[ s_i^* = e^{pt / \hbar} \sum_d e^{dt} \Pi_a \Pi_m (l_a p - \lambda_a m h) \Pi_a \Pi_m (p - \lambda_a m h) \bigg|_{p = \lambda_i} \]
of the differential equation
\[ \Pi_a (h \frac{d}{dt} - \lambda_a) s^* = e^t \Pi_a \Pi_m (h l_a \frac{d}{dt} - \lambda_a + m h) s^* . \]

For \( \lambda' = 0, \lambda_0 + ... + \lambda_n = 0 \) the solutions \( s_i^* \) have the following integral representation:
\[ \int_{\Gamma \subset \{ F_0(u) = e^t \}} \frac{u_0^{\lambda_0} ... u_n^{\lambda_n} du_0 \wedge ... \wedge du_n}{F_1(u) ... F_r(u) dF_0} \]

where 
\[ F_1 = (1-u_1 - ... - u_1), \quad F_2 = (1-u_{1+1} - ... - u_{1+2}), \quad ..., \quad F_r = (1-u_{1+...+r-1+1} - ... - u_{1+...+r}) \]
and \( F_0 = u_0 ... u_n \).

**Corollary 11.8.** The hypergeometric class \( S^*(t, \hbar) \in H^*(\mathbb{C}P^n) = \mathbb{C}[P]/(P^{n+1}), \)
\[ S^* = e^{pt / \hbar} \sum_d e^{dt} \Pi_a \Pi_m (l_a P + m h) \Pi_m (P + m h)^{n+1} \]
whose \( n + 1 - r \) non-zero components are solutions to the Picard-Fuchs equation
\[ \left( \frac{d}{dt} \right)^{n+1-r} s^* = l_1 ... l_r e^{pt} \Pi_m (l_a \frac{d}{dt} + m) s^* \]
for the integrals
\[ \int_{\gamma^{n-r} \subset X_t} \frac{du_0 \wedge ... \wedge du_n}{dF_0 \wedge dF_1 \wedge ... \wedge dF_r}, \]
(here \(X'_t = \{(u_0, ..., u_n) | F_0(u) = e', F_1(u) = 0, ..., F_r(u) = 0\}) are obtained from the class \(S\) (describing the quantum cohomology \(\mathcal{D}\)-module for the Calabi-Yau complete intersection \(X^{n-r} \subset \mathbb{C}P^n\)),
\[ S = e^{PT/h} \sum_d e^{dT}(e_0)_* \left( \frac{E_d}{h + c^{(0)}_1} \right), \]
by the change
\[ T = t + \sum_a l_a [g_a(e') - g_1(e')]/f(e') \]
followed by the multiplication by \(f(e')\).

Proof. Corollary 11.7 shows that for \(\lambda' = 0\), \(\sum \lambda_a = 0\) these change and multiplication transform the corresponding equivariant classes \(S'\) and \(S'^*\) to one another. The class \(-\varphi\) in the formula for \(s'_r\) in Corollary 11.7 is the equivariant Chern class of the Hopf line bundle over \(\mathbb{C}P^n\). In the limit \(\lambda = 0\) it becomes \(-\varphi\) while \(S'\) and \(S'^*\) transform to their non-equivariant counterparts \(S\) and \(S^*\).

Remarks. 1) Notice that the components \(S^*_0\) and \(S^*_1\) in
\[ S^* = l_1...l_r [P^r S^*_0(t) + P^{r+1} S^*_1(t) + ... + P^n S^*_n(t)] \]
are exactly \(f(e')\) and \(tf(e') + \sum_a l_a [g_a(e') - g_1(e')]\) respectively. Thus the inverse transformation from \(S^*\) to \(S\) consists in division by \(S^*_0\) followed by the change \(T = S^*_1(t)/S^*_0(t)\) in complete accordance with the recipe \([14, 17, 9]\) based on the mirror conjecture.

2) According to \([16]\) the \((n - r)\)-dimensional manifolds \(X'_t\) admit a Calabi-Yau compactification to the family \(\bar{X}'_t\) of mirror manifolds of the Calabi-Yau complete intersection \(X^{n-r} \subset \mathbb{C}P^n\). The Picard-Fuchs differential equation from Corollary 11.8 describes variations of complex structures for \(\bar{X}'_t\). This proves the mirror conjecture (described in detail in \([17]\)) for projective Calabi-Yau complete intersections and confirms the enumerative predictions about rational curves and quantum cohomology algebras made there (and in some other papers) on the basis of the mirror conjecture.

3) The description \([14]\) of the quantum cohomology algebra of a Calabi-Yau 3-fold in terms of the numbers \(n_d\) of rational curves of all degrees \(d\) (see for instance \([9]\) for the description of the corresponding class \(S\) in these terms) has been rigorously justified in \([13]\).
Combining these results with Corollary 11.8 we arrive to the theorem formulated in the introduction.

Proof of The Main Lemma.

In our construction of the map \( \mu : L_d \to L'_d \) we will denote \( L_d \) the moduli space of stable maps \( C \to \mathbb{C}P^n \times \mathbb{C}P^1 \) of bidegree \((d,1)\) with no marked points (it also has dimension \( d(n+1) + n \)). The construction works for any given number of marked points but produces a map which is the composition of \( \mu \) with the forgetful map. In this form it applies to the submanifold of stable maps with two marked points confined over 0 and \( \infty \) in \( \mathbb{C}P^1 \) (this submanifold is what we denoted \( L_d \) in the formulation of The Main Lemma).

Let \( \psi : C \to \mathbb{C}P^n \times \mathbb{C}P^1 \) be a stable genus 0 map of bidegree \((d,1)\). Then \( C = \overset{r}{\underset{i=1}{\cup}}C_i \) where \( C_0 \) is isomorphic to \( \mathbb{C}P^1 \) and \( \psi|_{C_0} \) maps \( C_0 \) onto the graph of a degree \( d' \leq d \) map \( \mathbb{C}P^1 \to \mathbb{C}P^n \), and for \( i = 1,\ldots,r \) the bidegree \((d_i,0)\) map \( \psi|_{C_i} \) sends \( C_i \) into the slice \( \mathbb{C}P^n \times \{p_i\} \) where \( p_i \neq p_j \) and \( d_1 + \ldots + d_r = d - d' \).

The map \( \mu : L_d \to L'_d \) assigns to \([\psi]\) the \((n+1)\)-tuples \((f_0g : f_1g : \ldots : f_ng)\) of polynomials (= binary forms) on \( \mathbb{C}P^1 \) where \( g \) is the polynomial of degree \( d - d' \) with roots \((p_1,...,p_r)\) of multiplicities \((d_1,...,d_r)\) and the tuples \((f_0 : \ldots : f_n)\) of degree \( d' \) polynomials (with no common roots, including \( \infty \)) is the one that describes the map \( \psi|_{C_0} \).

In order to prove that the map \( \mu \) is regular let us give it another, more invariant description.

Denote \( \hat{L}_d \) the moduli space of bidegree \((d,1)\) stable maps with an extra-marked point and pull back to \( \hat{L}_d \) the line bundle

\[
H := \text{Hom}(\pi_1^* \mathcal{O}_{\mathbb{C}P^n}(1), \pi_2^* \mathcal{O}_{\mathbb{C}P^1}(d))
\]

by the evaluation map \( e : \hat{L}_d \to \mathbb{C}P^n \times \mathbb{C}P^1 \) (where \( \pi_i \) are projections to the factors). Consider the push-forward sheaf \( H^0 := R^0 \pi_* e^*(H) \) of the locally free sheaf \( e^*H \) along the forgetful map \( \pi : \hat{L}_d \to L_d \). To a small neighborhood \( U \subset L_d \), it assigns the \( \mathcal{O}_U \)-module \( H^0(\pi^{-1}(U), e^*H) \) of sections of \( e^*H \).

Claim. 1) \( H^0 \) is a rank 1 locally free sheaf on \( L_d \).

2) The fiber at \([\psi]\) of the corresponding line bundle can be identified with

\[
H^0(C_0, (\psi|_{C_0})^*(H) \otimes \mathcal{O}(-[p_1])^{\otimes d_1} \otimes \ldots \otimes \mathcal{O}(-[p_r])^{\otimes d_r}).
\]
3) The kernel of the natural map

\[ h : H^0(C, \psi^*\pi^* \mathcal{O}_{\mathbb{C}P^n}(1)) \rightarrow H^0(C, \psi^*\pi^* \mathcal{O}_{\mathbb{C}P^1}(d)) = H^0(\mathbb{C}P^1, \mathcal{O}(d)) \]

defined by a nonzero vector in this fiber consists of the sections vanishing identically on \( C \).

Using this, we pick \( n + 1 \) independent sections of \( \mathcal{O}_{\mathbb{C}P^n}(1) \) (that is homogeneous coordinates on \( \mathbb{C}P^n \)), define corresponding sections of \( e^*\pi^* \mathcal{O}_{\mathbb{C}P^n}(1) \) and apply the map \( h \). By this we obtain a degree 1 map from the total space of the line bundle \( H^0 \) to the linear space \( \mathbb{C}^{n+1} \otimes H^0(\mathbb{C}P^1, \mathcal{O}(d)) \). Since the homogeneous coordinates on \( \mathbb{C}P^n \) nowhere vanish simultaneously, we obtain a natural map \( L_d \rightarrow L'_d = \text{Proj}(\mathbb{C}^{n+1} \otimes H^0(\mathbb{C}P^1, \mathcal{O}(d))) \) which sends \([\psi]\) to \((f_0g : ... : f_ng)\) and conclude that \( \mu \) is regular.

The remaining statements of The Main Lemma are proved by looking at localizations of the equivariant class \( p \) at the \( S^1 \times T^{n+1} \)-fixed points in \( L'_d \) and \( L_d \) (in this paragraph we use the notation \( L_d \) for the same space as in the formulation of The Main Lemma). The fixed points in \( L'_d \) are represented by the vector-monomials \((0 : ... : 0 : x^{d'} : 0 : ... : 0)\) where \( p \) localizes to \( \lambda_i + d'h \). A fixed point in \( L_d \) is represented by \( \psi \) with \( \psi(C_0) = (0 : ... : 0 : 1 : 0 : ... : 0) \), \( r = 2 \), \( p_0 = 0 \), \( p_1 = \infty \) and the maps \( \psi|C_k : C_k \rightarrow \mathbb{C}P^n \), \( k = 1, 2 \) representing \( T^{n+1} \)-fixed points respectively in \( Y_{2,d'} \) and \( Y_{2,d-d'} \) such that their (say) second marked points are mapped to the point \( \psi(C_0) \). This implies that the class \( \mu^*(p) \) localizes to \( \lambda_i + d'h \) at such a fixed point and thus the pull back of \( p \) to the fixed point set

\[ \{[\psi] \in Y_{2,d'} \times Y_{2,d-d'}|(e_2 \times e_2)([\psi]) \in \Delta \subset Y \times Y \]

of the \( S^1 \)-action on \( L_d \) coincides with the pull back through the common marked point of the \( T^{n+1} \)-equivariant class \( p + d'h \) on the diagonal \( \Delta = \mathbb{C}P^n \). Now localizations of \( \int_{L_d} e^{\mu(t-\tau)} \mathcal{E}_d \) to the fixed points of \( S^1 \)-action identify the form of the correlator \( \Phi \) given in The Main Lemma with the definition of \( \Phi \) as the convolution of \( S'(t, \hbar) \) and \( S'(\tau, -\hbar) \).

In order to justify the claim we need to compute the space of global sections of the sheaf \( e^*(H) \) over the formal neighborhood of the fiber \( \pi^{-1}([\psi]) \) of the forgetful map \( \pi : \hat{L}_d \rightarrow L_d \). The fiber itself is isomorphic to the tree-like genus 0 curve \( C \). Let \((x_j, y_j), j = 1, ..., N \geq r \) be some local parameters on irreducible components of \( C \) near the singular points such
that $\varepsilon_j = x_j y_j$ are local coordinates on the orbifold $L_d$ near $[\psi]$ (one should add some local coordinates $\varepsilon'$ on the stratum $\varepsilon_1 = \ldots \varepsilon_N = 0$ of stable maps $C \to \mathbb{C}P^n$ in order to construct a complete local coordinate system on $L_d$). Such a description of local coordinates on $L_d$ follows from the very construction of the moduli spaces of stable maps to ample manifolds; we refer the reader to [3, 6] for details.

A line bundle over the neighborhood of $C \subset \hat{L}_d$ can be specified by the set

$$u_j(x_j^{\pm 1}, \varepsilon), v_j(y_j^{\pm 1}, \varepsilon), \ j = 1, \ldots, N,$$

of non-vanishing functions describing transition maps between trivializations of the bundle inside and outside the neighborhoods (with local coordinates $(x_j, y_j, \varepsilon_1, \ldots, \hat{\varepsilon}_j, \ldots, \varepsilon_N, \varepsilon')$ of the double points.

Let us consider first the following model case. Suppose that $C$ consists of $r + 1$ irreducible components ($C_0, C_1, \ldots, C_r$) such that each $C_j$ with $j > 0$ intersects $C_0$ at some point $p_j$. Let $x_j$ be the local parameter on $C_0$ near $p_j$, and the line bundle (of the degree $-d_j \leq 0$ on $C_j$) be specified by $v_j = y_j^{-d_j}$.

In the neighborhood of $p_j$ a section of such a bundle is given by a function $s(x_j, y_j, \hat{\varepsilon}_j)$ satisfying

$$s = y_j^{-d_j} s_j(y_j^{-1}, \varepsilon)$$

where the function $s_j$ represents the section in the trivialization over the neighborhood of $C_j - p_j$. Here $\hat{\varepsilon}_j$ means that $\varepsilon_j$ is excluded from the set of coordinates $\varepsilon$ (remember that $\varepsilon_j = x_j y_j$). This implies that $s_j = \varepsilon_j^{d_j} f_j(y_j^{-1} \varepsilon_j, \varepsilon)$ where $f_j$ is some regular function. Thus this section in the neighborhood of $p \in C_0$ is given by a function $s(x_j, \varepsilon) = x_j^{d_j} f_j(x_j, \varepsilon)$ with zero of order $d_j$ at $x_j = 0$, and the restriction of this section to the neighborhood of $C_j$ is determined by $s$.

In other words, the $\mathbb{C}[[\varepsilon]]$-module of global sections in the formal neighborhood of $C$ identifies with the module of global sections on $C_0$ for the line bundle given by the loops $x_j^{-d_j} u_j$ instead of $u_j$ (this corresponds to the subtraction of the divisor $\sum d_j[p_j]$).

The more general situation where $v_j$ is the product of $y_j^{-d_j}$ with an invertible function $w_j(y_j, x_j, \hat{\varepsilon}_j)$ preserves the above conclusion with $w^{-1} s = x_j^{d_j} f_j(x_j, \varepsilon)$ instead of $s$.

Obviously, the above computation bears dependence on additional parameters.

Now we apply our model computation to the neighborhood of a general tree-like curve $C$ inductively by decomposing the tree into simpler ones starting from the root component.
$C_0$. We conclude that the $\mathbb{C}[[\varepsilon]]$-module of sections of the bundle $e^*(H)$ is identified with the module of sections of some line bundle over the product of $C_0$ with the polydisk with coordinates $(\varepsilon_1, ..., \varepsilon_r, ..., \varepsilon_N, \varepsilon')$, and that this line bundle is $e^*(H)$ for $C_0$ (given by the loops $u_j$ in our current notations) twisted by the loops $x_j^{-d_j}$ in the punctured neighborhoods of the points $(p_1, ..., p_r)$, where $(d_1, ..., d_r)$ are the degrees of the maps $\psi|C_j : C_j \rightarrow \mathbb{C}P^n$ (in the notations of the claim so that $d_1 + ... + d_r = d - d'$).

This implies that the $\mathbb{C}[[\varepsilon]]$-module $H^0$ of global sections can be identified with the module of those global sections of the degree $d - d'$ locally free sheaf $(\psi|C_0)^* (H) \otimes \mathbb{C}[[\varepsilon]]$ which have zeroes of order $d_j$ at $p_j$ for $j = 1, ..., r$. In particular

1) $H^0$ is a free $\mathbb{C}[[\varepsilon]]$-module of rank 1,

2) $H^0 \otimes_{\mathbb{C}[[\varepsilon]]} (\mathbb{C}[[\varepsilon]]/\langle \varepsilon \rangle)$ is the 1-dimensional space $H^0|\psi$ described in the claim, and

3) non-zero vectors in $H^0|\psi$ represent sections of $\psi^*(H)$ over $C$ non-zero on $C_0$ (and thus their product with a non-zero on $C_0$ section of $\psi^* \pi_1^*(\mathcal{O}_{\mathbb{C}P^n}(1))$ can not vanish identically on $C_0$.)

Factorization by the discrete group $\text{Aut}(\psi)$ preserves (1−3) with $\mathbb{C}[[\varepsilon]]$ replaced by $\mathbb{C}[[\varepsilon]]^\text{Aut}(\psi)$. 

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