Abstract

The notions of formal contexts and concept lattices, although introduced by Wille only ten years ago [Wille], already have proven to be of great utility in various applications such as data analysis and knowledge representation. In this paper we give arguments that Wille’s original notion of formal context, although quite appealing in its simplicity, now should be replaced by a more semantic notion. This new notion of formal context entails a modified approach to concept construction. We base our arguments for these new versions of formal context and concept construction upon Wille’s philosophical attitude with reference to the intensional aspect of concepts. We give a brief development of the relational theory of formal contexts and concept construction, demonstrating the equivalence of concept-lattice construction [Wille] with the well-known completion by cuts [MacNeille]. Generalization and abstraction of these formal contexts offers a powerful approach to knowledge representation.

Contents

1 Conceptual Collectives 5
2 Concept Construction 7
3 Contextual Fibration 12
4 Contextual Summation 16
5 Future Work 19
A The Central Adjointness of Logic 22
   A.1 Upper/Lower Operators ................................................................. 23
   A.2 Direct/Inverse Derivation .............................................................. 24
B Properties of Derivation 25
C Wille’s Concept Lattice Construction 28
Introduction

Classification of knowledge into ordered systems has played an important role in the history of science from the very beginning. More recently, classification has become quite important in computer science in general (knowledge representation and databases) and programming languages in particular. The fundamental task in conceptual classification is the search for conceptual structures (classes) naturally inherent in the problem domain and the construction of class hierarchies. These class hierarchies tend to be of two kinds: generalization/specialization inheritance hierarchies and whole/part containment hierarchies. Both linear and network representations of knowledge have been used for classification. Such knowledge representation systems can have both a structural, taxonomic aspect and an assertional, logical aspect. Although this paper emphasizes the structural aspect, the assertional aspect also can be represented. Taxonomies of structured conceptual descriptions originated in the KL-ONE knowledge representation system of Woods and others [Woods]. As Woods has pointed out, however, the semantics of the conceptual structures in KL-ONE and related systems is not totally clear. Although many researchers have identified concepts with the notion of predicate in first-order logic, Woods argues for the need to represent intensional concepts. According to Woods, such intensions cannot be represented in first-order logic, and cannot be thought of as the classes of traditional knowledge representation systems. Although Woods uses the formal notion of abstract conceptual description as a means to logically represent both the intensional and the extensional aspects of concepts, the simpler notion of a formal context as championed by Rudolf Wille [Wille] is a more elegant alternative. In this paper we give arguments that Wille’s original notion of formal context, although quite appealing in its simplicity and elegance, now should be replaced by a more semantic notion.

The basic constituents in conceptual classification and knowledge representation are entities or objects corresponding to real-world objects, and ways of describing these in terms of attributes or properties. In programming languages attributes represent the data and operations of a data type. The relationship between entities and attributes is a has relationship called a context. Formal concept analysis, a new approach to classification and knowledge representation initiated by Wille, starts with the primitive notion of a formal context. A formal context is a triple \( \langle X_0, X_1, \mu \rangle \) consisting of two sets \( X_0 \) and \( X_1 \) and a binary relation \( \mu \subseteq X_0 \times X_1 \) between \( X_0 \) and \( X_1 \). Intuitively, the elements of \( X_0 \) are thought of as entities or objects, the elements of \( X_1 \) are thought of as properties, characteristics or attributes that the entities might have, and \( x_0 \mu x_1 \) asserts that “the entity \( x_0 \) has the attribute \( x_1 \).” One should take note of the strict segregation between entities on the one hand and attributes on the other. From an extensional point-of-view a formal context \( \langle X_0, X_1, \mu \rangle \) is a base set of entities \( X_0 \) with an indexed collection of subsets \( \{ \mu x_1 \mid x_1 \in X_1 \} \). Then a second formal context \( \langle X_1, X_2, \nu \rangle \) would be extensionally interpreted as an indexed collection of indexed collections of subsets of \( X_0 \): \( \{ \{ \mu x_1 \mid x_1 \in \nu x_2 \} \mid x_2 \in X_2 \} \). Obviously, higher-order types are implicitly represented here. Simple relational composition of formal contexts \( \langle X_0, X_2, \mu \circ \nu \rangle \) corresponds extensionally to indexed unions: \( \{ (\mu \circ \nu) x_2 \mid x_2 \in X_2 \} = \bigcup_{x_1 \in \nu x_2} \mu x_1 \). Relational implication \( \langle X_1, X_2, \mu \setminus \nu \rangle \), of two formal contexts \( \langle X_0, X_1, \mu \rangle \) and \( \langle X_0, X_2, \nu \rangle \) over the same base set of entities \( X_0 \), extensionally indexes extensional containment: \( (\mu \setminus \nu) x_2 = \{ x_1 \mid \mu x_1 \subseteq \nu x_2 \} \).

As Wille has explained, formal concept analysis is based upon the understanding that a concept is a unit of thought consisting of two parts: its extension and its intension. Within a certain restricted context or scope (a type-like notion is implicit here), the extent of a concept is a subset \( \phi \in 2^{X_0} \) consisting of all entities or objects belonging to the concept — you as an individual person belong to the concept ‘living person’, whereas the intent of a concept is a subset \( \psi \in 2^{X_1} \) which includes all attributes or properties shared by the entities — all ‘living persons’ share the attribute ‘can breathe’. A concept of a given context will
consist of an extent/intent pair \((\phi, \psi)\). These conceptual structures of Wille, called formal concepts, start to address the problems mentioned by Woods. The notion of a formal concept is a valuable first step toward a mathematical representation for the class concept from certain other classification domains — object-oriented programming and knowledge representation systems. Class hierarchies are formally represented by concept lattices. The foundation for formal concept analysis has relied in the past upon a set-theoretic model of conceptual structures. This model has been applied to both data analysis and knowledge representation. We argue here that an enriched order-theoretic model for conceptual structures provides for an improved foundation for formal concept analysis and knowledge representation.

Of central importance in concept construction are two derivation operators which define the notion of "sharing" or "commonality". For any subsets \(\phi \in 2^{X_0}\) and \(\psi \in 2^{X_1}\), we define
\[
\phi \overset{\mu}{\leftarrow} \{x_1 \in X_1 \mid x_0 \mu x_1 \text{ for all } x_0 \in \phi\}
\]
\[
\psi \overset{\mu}{\rightarrow} \{x_0 \in X_0 \mid x_0 \mu x_1 \text{ for all } x_1 \in \psi\}
\]
To demand that a concept \((\phi, \psi)\) be determined by its extent and its intent means that the extent should contain precisely those attributes shared by all entities in the extent \(\phi \overset{\mu}{\rightarrow} = \psi\), and vice-versa, that the extent should contain precisely those entities sharing all attributes in the intent \(\phi = \psi \overset{\mu}{\rightarrow}\). Concepts are ordered by generalization/specialization: one concept is more specialized (and less general) than another \((\phi, \psi) \leq_{\text{CL}} (\phi', \psi')\) when its intent contains the other’s intent \(\psi \supseteq \psi'\), or equivalently, when the opposite ordering on extents occurs \(\phi \subseteq \phi'\). Concepts with this generalization/specialization ordering form a concept hierarchy for the context. The concept hierarchy is a complete lattice \(\text{CL}(X_0, X_1, \mu)\) called the concept lattice of \((X_0, X_1, \mu)\). The meets and joins in \(\text{CL}(X_0, X_1, \mu)\) can be described as follows:
\[
\bigwedge_{k \in K}(\phi_k, \psi_k) = \left(\bigcap_{k \in K} \phi_k, \left(\bigcup_{k \in K} \psi_k\right) \overset{\mu}{\rightarrow}\right)
\]
\[
\bigvee_{k \in K}(\phi_k, \psi_k) = \left(\bigcup_{k \in K} \phi_k \overset{\mu}{\rightarrow}, \bigcap_{k \in K} \psi_k\right)
\]
The join of a collection of concepts represents what the concepts have in ‘common’ or ‘share’, and the top of the concept hierarchy represents all entities (the universal concept). The information presented in Table 1 and originally described in [Wille] gives a limited context for the planets of our solar system. The entities \(X_0 = \{\text{Me, V, E, Ma, J, S, U, N, P}\}\) are the planets and the attributes \(X_1 = \{\text{ss, sm, sl, dn, df, my, mn}\}\) are the seven scaled properties relating to size, distance from the sun, and existence of moons, with abbreviations

| entities       | attributes |
|----------------|------------|
| Me — Mercury   | ss — size |
| V — Venus      | sm — size |
| E — Earth      | sl — size |
| Ma — Mars      | dn — dis |
| J — Jupiter    | df — dis |
| S — Saturn     | my — moo |
| U — Uranus     | mn — moo |
| N — Neptune    |           |
| P — Pluto      |           |

The table itself represents the has relationship \(\mu \subset X_0 \times X_1\). The fact \(x_0 \mu x_1\) that the \(x_0\)th object has the \(x_1\)th attribute is indicated by a ‘\(\times\)’ in the \(x_0\)th \(x_1\)th entry in Table 1. The concepts for this planetary context are listed in Table 2.

Entities generate concepts. There is a function \(X_0 \overset{i_0}{\rightarrow} \text{CL}(X_0, X_1, \mu)\) called the generator function which maps each entity \(x_0 \in X_0\) to its associated concept \(i_0(x_0) \overset{\mu}{=} (x_0 \overset{\mu}{\rightarrow}, x_0 \overset{\mu}{\rightarrow})\). Similarly, attributes generate concepts by means of a generator function \(X_1 \overset{i_1}{\rightarrow} \text{CL}(X_0, X_1, \mu)\) which maps attributes \(x_1 \in X_1\) to their
Table 1: A contextual relationship for planets

| concept description          | extent $\phi$ | intent $\psi$ |
|-----------------------------|--------------|---------------|
| “everything”                | $X_0$        | $\emptyset$   |
| “with moon”                 | $\{E, Ma, J, S, U, N, P\}$ | $\{my\}$ |
| “small size”                | $\{Me, V, E, Ma, P\}$ | $\{ss\}$ |
| “small with moon”           | $\{E, Ma, P\}$ | $\{ss, my\}$ |
| “far”                       | $\{J, S, U, N, P\}$ | $\{df, my\}$ |
| “near”                      | $\{Me, V, E, Ma\}$ | $\{ss, dn\}$ |
| “Plutoness”                 | $\{P\}$     | $\{ss, df, my\}$ |
| “medium size”               | $\{U, N\}$  | $\{sm, df, my\}$ |
| “large size”                | $\{J, S\}$  | $\{sl, df, my\}$ |
| “near with moon”            | $\{E, Ma\}$ | $\{ss, dn, my\}$ |
| “moonless”                  | $\{Me, V\}$ | $\{ss, dn, mn\}$ |
| “nothing”                   | $\emptyset$ | $X_1$         |

Table 2: Concept lattice $\text{CL}(X_0, X_1, \mu)$ for the planetary relationship
The first section explains with examples the meaning of collective entities and collective attributes. In the second section we discover the hidden relationship opposite to the original has relationship. Section three defines our new version of formal context, introduces contextual closure, and describes formal concepts from this new standpoint. In section four we explain a new notion of order-theoretic sum which centralizes our distributed version of formal contexts, and we state and prove the Equivalence Theorem which relates the concept lattice construction with Dedekind-MacNeille completion. Finally, in section five we indicate some areas of new research.

1 Conceptual Collectives

The first step that we take in the analysis of concept construction is the observation, already made by Wille, that a formal context, although defined a priori in terms of sets and relations, has order relationships on entities and attributes induced by the corresponding generator map into the concept lattice.
This same approach can be used to define order relations on both source and target sets [Wille]:

1. $\mathcal{X}^\mu_0 \equiv (X_0, \leq^\mu_0)$, where the order relation $\leq^\mu_0$ is defined by $x'_0 \leq^\mu_0 x_0$ iff $i_0(x'_0) \leq_{\text{CL}} i_0(x_0)$ iff $x'_0 \mu \supseteq x_0 \mu$, so that $\leq^\mu_0 = \mu \leftarrow \mu$; $\mathcal{X}^\mu_0$ is the largest source order for which $\mu$ is closed on the left.

2. $\mathcal{X}^\mu_1 \equiv (X_1, \leq^\mu_1)$, where the order relation $\leq^\mu_1$ is defined by $x'_1 \leq^\mu_1 x_1$ iff $i_1(x'_1) \leq_{\text{CL}} i_1(x_1)$ iff $\mu x_1 \subseteq \mu x'_1$, so that $\leq^\mu_1 = \mu \rightarrow \mu$; $\mathcal{X}^\mu_1$ is the largest target order for which $\mu$ is closed on the right.

The binary relation from the original context is a closed relation w.r.t. these induced orders

$$\mathcal{X}^\mu_0 \subseteq \mathcal{X}^\mu_1.$$  

The source and target orders induced by the concept lattice of the planetary context are displayed in Table 3.

| Source order | Target order |
|--------------|--------------|
| Me V E Ma J S U N P | ss sm sl dn df my mn |
| x x x x x x x | x x x x x |

Table 3: Induced orders for the planetary context

Intuitively, for the source set the order $\leq^\mu_0$ specifies implicational information between entities: an order relationship $x'_0 \leq^\mu_0 x_0$ exists between two entities $x'_0$ and $x_0$ when $x'_0$ has all the attributes of $x_0$ (and possibly some others). In a sense, $x_0$ represents its collection of individual attributes $x_0 \mu \subseteq x'_0 \mu \subseteq X_1$ — in fact, we can regard the entity $x_0$ as being a kind of collective attribute of $x'_0$. In particular, any entity is a collective attribute of itself. By the same token, for the target set the order $\leq^\mu_1$ specifies implicational information between attributes: an order relationship $x_1 \leq^\mu_1 x'_1$ exists between two attributes $x_1$ and $x'_1$ when entities which have attribute $x_1$ also have attribute $x'_1$; $x_1$ represents its collection of individual entities $\mu x_1 \subseteq \mu x'_1 \subseteq X_0$, we regard the attribute $x_1$ as being a kind of collective entity of $x'_1$, and any attribute is a collective entity of itself. These dual relationships are picture in Figure 1. Here we see more evidence of the interchangeability of entities and attributes, arguing for a certain kind of inherent blending or integration of the two notions. Ultimately we will argue for the total blending or integration of entities and attributes, but in a very structured fashion which allows for a locally relative distinction.

Since the order $\mathcal{X}^\mu_0$ is a legitimate relationship between entities and (collective) attributes, we can define direct and inverse derivation along the identity relationship $\mathcal{X}^\mu_0$. These are identical with the upper and lower operators on $\mathcal{X}^\mu_0$. Suppose $\phi \subseteq X_0$ is a subset of entities. What is the meaning of the upper operator applied to $\phi$? The upper operator applied to $\phi$ returns the closed-above subset

$$\phi^\mu_{\mathcal{X}^\mu_0} = \phi^\supseteq_{\mathcal{X}^\mu_0}$$

$$= \{ x'_0 \in X_0 \mid x_0 \leq^\mu_0 x'_0 \text{ for all } x_0 \in \phi \}$$

$$= \{ x'_0 \in X_0 \mid i_0(x_0) \leq_{\text{CL}} i_0(x'_0) \text{ for all } x_0 \in \phi \}$$

$$= \{ x'_0 \in X_0 \mid x_0 \mu \supseteq x'_0 \mu \text{ for all } x_0 \in \phi \}.$$
which consists of all collective attributes of all entities in $\phi$. An entity in $\phi$ represents a certain ‘type of commonality’ for all the entities in $\phi$. We wish to emphasize the obvious analogy with the definition of the direct derivation $\phi$ along $\mu$, the main and only distinction being that direct derivation $\phi$ returns all individual attributes of all entities in $\phi$, whereas the upper operator $\phi$ returns all collective attributes of all entities in $\phi$. A dual discussion can be given for inverse derivation where existence of collective entities in the target set is observed. Formal contexts and their various constituents are illustrated in Figure 2.

To summarize the discussion above, given any relation $X_0 \mu X_1$ used to represent concepts in knowledge representation, intuitively each element $x_0 \in X_0$ is both an individual entity following the standard interpretation and a collective attribute representing the collection $x_0 \in 2^{X_1}$ of individual attributes, and dually, each element $x_1 \in X_1$ is both an individual attribute following the standard interpretation and a collective entity representing the collection $\mu x_1 \in 2^{X_0}$ of individual entities. We will use this point of view in order to understand the appropriate course of action for concept construction when order information is specified a priori for both source (entities) and target (attribute) sets.

2 Concept Construction

The second step is crucial! We use Wille’s philosophical position regarding the extension/intension duality of concepts, and argue that the entity and attribute orders are themselves local relationships which should be used simultaneously with the original relationship of the context. The argument here centers around the viewpoint that entities can be seen as collective attributes and dually that attributes can be seen as collective entities — whence the title of the paper.

Formal contexts have an order-theoretic nature, in the sense that at least an implicit order exists on both
source set (entities) and target set (attributes). We can respect this observation by defining a formal context a priori in terms of orders and order-closed relations, effectively changing from the set-theoretic to the order-theoretic realm. This order-theoretic realm replaces sets with orders and replaces ordinary relations with closed relations (other enriched realms will be considered in a subsequent paper, where a more formal and abstract analysis is given). Let us use the point of view espoused in the first subsection above in order to understand the appropriate course of action for concept construction when order information is specified a priori for both source (entities) and target (attribute) sets. Let the closed relation \( X_0 \supseteq X_1 \) represent a formal context in the enriched order-theoretic realm. The order information \( x_0 \preceq X_0 \) is interpreted as “the entity \( x_0 \) is a collective attribute of \( x' \)”. By closure of the relation \( \mu \) at the source order \( X_0 \), any \( \mu \)-attribute of \( x_0 \) is an \( \mu \)-attribute of \( x' \). Now given a subset \( \phi \subseteq X_0 \), when computing the common shared attributes of elements of \( \phi \) during concept construction, it seems appropriate to consider not only application of the direct derivation operator
\[
2^{X_0} \xrightarrow{\phi_{\mu}^\Leftarrow} (2^{X_1})^{\text{op}}
\]
getting the order filter \( \phi_{\mu}^\Leftarrow \in 2^{X_1} \) of all shared individual attributes, but also application of the upper operator
\[
2^{X_0} \xrightarrow{\phi_{\mu}^\Rightarrow} (2^{X_0})^{\text{op}}
\]
getting the order filter \( \phi_{\mu}^\Rightarrow \in 2^{X_0} \) of all shared collective attributes. This pair of order filters satisfies the filter assertion
\[
\phi_{\mu}^\Leftarrow \circ \mu \leq \phi_{\mu}^\Rightarrow
\]
since \( \phi \circ \phi_{\mu}^\Leftarrow \circ \mu = \phi \circ (\phi \setminus X_0) \circ \mu \leq X_0 \circ \mu = \mu \).

To recapitulate, if we start with a single order ideal in \( 2^{X_0} \), the direct phase of concept construction returns two assertationally constrained order filters, one in \( 2^{X_0} \) and one in \( 2^{X_1} \). Such constrained pairs of

| knowledge representation | individual entity | order | collective attribute | relation | individual attribute |
|--------------------------|-------------------|-------|----------------------|----------|----------------------|
| linguistics              | ? is-a animal     | ?     | ?                    | ?        | ?                    |
| planets                  | Earth equiv-to    | Mars  | has size:small       |          |                     |
| database                 | Smith is-a        | engineer | works-for | Aerospace, Inc. |             |
| generic database         | employee          | works-for | company | located-in | city               |

Table 4: Examples of collective attributes

| Roget’s thesaurus | individual entity | relation | collective entity | order | individual attribute |
|-------------------|-------------------|----------|-------------------|-------|----------------------|
| word-string       | occurs-in         | RIT-category sub-type | RIT-class |
| “toast”           |                   | 324: Cooking         |         |
| planets           | Pluto has         | distance:far         | implies | moon:yes             |
|                   | Felix member      | cat sub-type         | mammal  |                     |

Table 5: Examples of collective entities
order filters provide a necessary structural constraint on the intensional aspect of concepts: the intent of a concept is a pair \((\psi_0, \psi_1)\), an order filter of collective attributes \(\psi_0 \in 2^{X_0}\) and an order filter of individual attributes \(\psi_1 \in 2^{X_1}\) subject to the filter assertion

\[\psi_0 \circ \mu \preceq \psi_1.\]

The need of this assertional constraint for “type summability” is discussed below in the order-theoretic realm. It places a restriction upon filter pairs, allowing only certain admissible pairs, and is described by the slogan

The (image of the) collective component is contained in the individual component.

Continuing the argument above, in the inverse phase of concept construction, we start from the intent — the common attributes of a concept. For this inverse phase, since there are (at least) three relationships \(\mu, \preceq X_0\) and \(\preceq X_1\), there are (at least) three relevant operators:

1. the inverse derivation operator

\[
(2^{X_1})^{op} \xrightarrow{(\psi_1)^{op}_\mu} 2^{X_0^{op}}
\]

which when applied to the order filter \(\psi_1 \in 2^{X_1}\) of individual attributes returns the order ideal \((\psi_1)^{\preceq}_\mu \in 2^{X_0^{op}}\) of all individual entities which share all of the individual attributes in \(\psi_1\),

2. the lower operator

\[
(2^{X_1})^{op} \xrightarrow{(\psi_1)^{l}_{X_1}} 2^{X_0^{op}}
\]

which when applied to the order filter \(\psi_1 \in 2^{X_1}\) of individual attributes returns the order ideal \((\psi_1)^{1}_{X_1} \in 2^{X_0^{op}}\) of all collective entities which share all of the individual attributes in \(\psi_1\), and

3. the lower operator

\[
(2^{X_0})^{op} \xrightarrow{(\psi_1)^{l}_{X_0}} 2^{X_0^{op}}
\]

which when applied to the order filter \(\psi_0 \in 2^{X_0}\) of collective attributes returns the order ideal \((\psi_0)^{1}_{X_0} \in 2^{X_0^{op}}\) of all individual entities which share all of the collective attributes in \(\psi_0\).

Since we are again constructing commonality, it is appropriate to take the meet \((\psi_0)^{1}_{X_0} \wedge (\psi_1)^{\preceq}_\mu\) of the two order ideals in \(2^{X_0^{op}}\); this consists of all entities in common with both collective attributes in \(\psi_0\) and individual attributes in \(\psi_1\). We end up with the pair of order ideals \((\psi_0)^{1}_{X_0} \wedge (\psi_1)^{\preceq}_\mu, (\psi_1)^{1}_{X_1}\).

SOMETHING IS WRONG! — this pair does NOT necessarily satisfy the ideal assertion

\[\mu \circ (\psi_1)^{1}_{X_1} \preceq (\psi_0)^{1}_{X_0} \wedge (\psi_1)^{\preceq}_\mu\]

which is the codification of the admissibility slogan for the extensional aspect of concepts, that “the (image of the) collective component is contained in the individual component”. We note that “half” of this constraint does hold:

\[\mu \circ (\psi_1)^{1}_{X_1} \preceq (\psi_1)^{\preceq}_\mu\]

since \(\mu \circ (\psi_1)^{1}_{X_1} \circ \psi_1 = \mu \circ (X_1/\psi_1) \circ \psi_1 \leq \mu \circ X_1 = \mu\). In order to satisfy the full assertional constraint, we need an appropriate factor (order ideal) \(\alpha\) which will restrict the collective component of the extent in
Negations of relations are described in Table 6. When source and target orders are the induced correspondence with the restriction of the individual component of the extent by \((\psi_0)^1_{\mathcal{X}_0}\). Then the full assertional constraint would be
\[
\mu \odot (\alpha \land (\psi_1)^1_{\mathcal{X}_1}) \preceq (\psi_0)^1_{\mathcal{X}_0} \land (\psi_1)^\leq_{\mu}
\]
For this to hold, a sufficient condition on the factor \(\alpha\) is the partial (half) assertional constraint
\[
\mu \odot \alpha \preceq (\psi_0)^1_{\mathcal{X}_0}
\]
The maximal order ideal satisfying this constraint is \(\alpha \overset{df}{=} \mu \land (\psi_0)^1_{\mathcal{X}_0} = \mu \land (\mathcal{X}_0/\psi_0) = (\mathcal{X}_0/\mu) \land \psi_0 = (\psi_0)^{\leq}_{\mu \land \mathcal{X}_0}\). We interpret this as the inverse derivation of the order filter of collective attributes \(\psi_0\) along the source negation \(\mathcal{X}_1 \overset{\mu}{\to} \mathcal{X}_0\). The source negation is the largest relation \(\mathcal{X}_1 \overset{\nu}{\to} \mathcal{X}_0\) that is opposite to \(\mu\) and satisfies the partial asymmetric orthogonal constraint \(\mu \odot \nu \preceq \mathcal{X}_0\).

So the error above was an ERROR OF OMISSION — there is a hidden relationship in the opposite direction to \(\mu\) that also must be considered. This hidden relationship provides for a fourth operator active in the inverse phase of concept construction above. Actually, in order for derivation to work correctly in both the direct and inverse phases of concept construction, we must use a relation no larger than the negation of \(\mu\) [Kent], the relation \(\mathcal{X}_1 \overset{\mu}{\to} \mathcal{X}_0\) defined by
\[
\neg \mu \overset{df}{=} (\mu \land \mathcal{X}_0) \land (\mathcal{X}_1/\mu)
\]
This is the largest relation \(\mathcal{X}_1 \overset{\nu}{\to} \mathcal{X}_0\) which is opposite to \(\mu\) and satisfies the full symmetric orthogonal constraints
\[
\mu \odot \nu \preceq \mathcal{X}_0 \quad \text{and} \quad \nu \odot \mu \preceq \mathcal{X}_1
\]
Note that \(x_0 \mu x_1\) implies \(\downarrow x_0 \subseteq \mu x_1\) and \(\uparrow x_1 \subseteq x_0 \mu\). Since \(\mu \land \mathcal{X}_0 = \{(x_1, x_0) \mid x_1 \in \mathcal{X}_1, x_0 \in \mathcal{X}_0, \mu x_1 \subseteq \downarrow x_0\}\) and \(\mathcal{X}_1/\mu = \{(x_1, x_0) \mid x_1 \in \mathcal{X}_1, x_0 \in \mathcal{X}_0, x_0 \mu \subseteq \uparrow x_1\}\), negation is a kind of contrapositive of \(\mu\) defined pointwise by
\[
\neg \mu \overset{df}{=} \{(x_1, x_0) \mid x_1 \in \mathcal{X}_1, x_0 \in \mathcal{X}_0, \mu x_1 \subseteq \downarrow x_0 \text{ and } x_0 \mu \subseteq \uparrow x_1\}.
\]

Negations of some special relations are described in Table 6. When source and target orders are the induced

| relation                | negation                |
|------------------------|-------------------------|
| \(\mathcal{X}_0 \overset{\mu}{\to} \mathcal{X}_1\) | \(\neg (\mu \lor \nu) = \neg \mu \land \neg \nu\) |
| bottom                 | \(\neg \bot = \top\)    |
| identity               | \(\mathcal{X} \overset{\alpha}{\to} \mathcal{X}\) |
| complement             | \(\neg \mathcal{Z} = \{(x, x') \mid (\forall y \in \mathcal{X}) x \leq_{\mathcal{X}} y \text{ or } y \leq_{\mathcal{X}} x'\} = \{(x, x') \mid \mathcal{X} = \uparrow x \cup \downarrow x'\}\) |
| ideal                  | \(\neg \phi = \phi \setminus \mathcal{X} = \phi^1_{\mathcal{X}}\) |
| filter                 | \(1 \overset{\psi}{\to} \mathcal{X}\) |

\(\neg \psi = \mathcal{X} \setminus \psi = \psi^1_{\mathcal{X}}\)
orders $X_0^\mu$ and $X_0^\mu$, negation is

$$\neg \mu = \mu \setminus X_0^\mu = X_0^\mu \cap \mu = \mu \setminus \mu \setminus \mu.$$  \hfill (1)

Let us recall the initial discussion in Section 1 about the order-theoretic constructions induced by Wille’s concept lattice construction. In addition to the original relation (source-target), and the two induced orders (source-source and target-target), the only other definable relation with this data is a relation opposite to $\mu$ (target-source), a relation closed w.r.t. the induced orders

$$X_1^\mu \supseteq X_0^\mu,$$  \hfill (2)

which is defined either by

$$x_1 \supseteq x_0 \iff i_1(x_1) \leq x_0 \iff \mu x_1 \subseteq x_0$$

or by

$$x_1 \supseteq x_0 \iff i_1(x_1) \leq x_0 \iff x_0 \setminus x_1$$

Proposition 1 Negation $X_1^\mu \supseteq X_0^\mu$ is the opposite relation induced by the concept lattice

$$\neg \mu = \overline{\mu}.$$  

The negation of the original planetary relationship, which is the opposite relation induced by the concept lattice of the planetary context, is displayed in Table 7. Intuitively, the negation relation $\neg \mu$ specifies implicative information. From one viewpoint, a negation relationship $x_1 \neg \mu x_0$ iff $x_1 (\mu \setminus X_0^\mu) x_0$ exists between individual attribute $x_1$ and individual entity $x_0$ when any individual entities having individual attribute $x_1$ also have collective attribute $x_0$. Since $x_1$ represents its collection of individual entities as a collective entity, we can say that $x_1 \neg \mu x_0$ when collective entity $x_1$ has collective attribute $x_0$. Arguing from the dual position, a negation relationship $x_1 \neg \mu x_0$ iff $x_1 (X_0^\mu \setminus \mu)$ $x_0$ exists between individual attribute $x_1$ and individual entity $x_0$ when any individual attributes of individual entity $x_0$ are also attributes of collective entity $x_1$. Since $x_0$ represents its collection of individual attributes as a collective attribute, we can say that $x_1 \neg \mu x_0$ when $x_0$ is a collective attribute of collective entity $x_1$ (the same meaning as before).

| Me | V | E | Ma | J | S | U | N | P |
|----|---|---|----|---|---|---|---|---|
| ss | × | × |     |    |    |    |    |    |
| sm | × | × |     |    |    |    |    |    |
| sl |   |   |     |    |    |    |    |    |
| dn |   |   |     |    |    |    |    |    |
| df |   |   |     |    |    |    |    |    |
| my |   |   |     |    |    |    |    |    |
| mn | × | × |     |    |    |    |    |    |
3 Contextual Fibration

Now we are at a crucial point in our analysis of concept construction. The resolution of the above problem requires a change of viewpoint — a shift in our conceptual framework. In order to make visible and explicit the hidden relationship of a context, we must define contexts as follows. A formal context $\mathcal{X} = \langle \mathcal{X}_0, \mu_00, \mu_{10}, \mathcal{X}_1 \rangle$ is a pair of orders $\mathcal{X}_0 = \langle X_0, \leq_0 \rangle$ and $\mathcal{X}_1 = \langle X_1, \leq_1 \rangle$, and a pair of oppositely directed closed relations $\mathcal{X}_0 \overset{\mu_00}{\rightarrow} \mathcal{X}_1$ and $\mathcal{X}_1 \overset{\mu_{10}}{\rightarrow} \mathcal{X}_0$ between them, which satisfy the orthogonal constraints $\mu_00 \circ \mu_{10} \preceq \mathcal{X}_0$ and $\mu_{10} \circ \mu_00 \preceq \mathcal{X}_1$. The four components of a formal context can be arranged in a matrix

$$
\mathcal{X} = \begin{pmatrix}
\mathcal{X}_0 \\
\mu_00 \\
\mu_{10} \\
\mathcal{X}_1
\end{pmatrix}.
$$

Formal contexts in this sense are quite general. Let $\langle \mathcal{X}, t \rangle$ be any pair consisting of an order $\mathcal{X} = \langle X, \leq_X \rangle$ and a monotonic function $\mathcal{X} \overset{t}{\rightarrow} \mathcal{T}$ from $\mathcal{X}$ to the binary order $\mathcal{T}$ defined by $\mathcal{T} = \langle \{0, 1\}, \sqsubseteq \rangle$, where $0 \sqsubseteq 1$ and $1 \sqsubseteq 0$. This pair specifies a formal context

$$
\mathcal{X}^t = \begin{pmatrix}
\mathcal{X}_0 \\
\mu_00 \\
\mu_{10} \\
\mathcal{X}_1
\end{pmatrix}
$$

called the $t$-partition of $\mathcal{X}$, whose components are defined by

$$
\begin{align*}
\mathcal{X}_0 & \overset{\mu_00}{\rightarrow} \mathcal{X}_0 & t^{-1}(0 \sqsubseteq 0) &= \{(x_0', x_0) \mid x_0', x_0 \in X_0, x_0' \leq_X x_0\} \\
\mathcal{X}_1 & \overset{\mu_{10}}{\rightarrow} \mathcal{X}_1 & t^{-1}(1 \sqsubseteq 1) &= \{(x_1', x_1) \mid x_1', x_1 \in X_1, x_1 \leq_X x_1'\} \\
\mathcal{X}_0 & \overset{\mu_00}{\rightarrow} \mathcal{X}_1 & t^{-1}(0 \sqsubseteq 1) &= \{(x_0, x_1) \mid x_0 \in X_0, x_1 \in X_1, x_0 \leq_X x_1\} \\
\mathcal{X}_0 & \overset{\mu_{10}}{\rightarrow} \mathcal{X}_1 & t^{-1}(1 \sqsubseteq 0) &= \{(x_1, x_0) \mid x_1 \in X_1, x_0 \in X_0, x_1 \leq_X x_0\}.
\end{align*}
$$

The monotonic function $t$, called a tag or index function, indicates from which component order an element in the sum originates $t(x_0) = 0$ and $t(x_1) = 1$, and functions as a partition or fibration $\{X_0, X_1\}$ of the underlying set $X = X_0 + X_1$.

Sum orders define contextual fibrations: given any two orders $\mathcal{X}_0$ and $\mathcal{X}_1$, the disjoint union order $\mathcal{X}_0 + \mathcal{X}_1$ and the linear sum order $\mathcal{X}_0 \oplus \mathcal{X}_1$ define the contexts

$$
\begin{pmatrix}
\mathcal{X}_0 \\
\downarrow \\
\mathcal{X}_1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\mathcal{X}_0 \\
\top \\
\downarrow \\
\mathcal{X}_1
\end{pmatrix},
$$

respectively. Product orders define contextual fibrations: given any order $\mathcal{X}$, the binary product projection $\mathcal{X} \times \mathcal{X} \overset{\pi_0}{\rightarrow} \mathcal{X}$ and the Boolean product projection $\mathcal{X} \times \mathcal{X} \overset{\pi_1}{\rightarrow} \mathcal{X}$ define the contexts

$$
\begin{pmatrix}
\mathcal{X}_0 \\
\mathcal{X}_1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\mathcal{X}_0 \\
\mathcal{X}_1
\end{pmatrix},
$$

respectively, with off-diagonal entries being the identity relation.

Of special significance in the analysis of concept construction is the contextual closure of a binary relation $\mathcal{X}_0 \overset{\mu}{\rightarrow} \mathcal{X}_1$. This is the formal context

$$
\mathcal{X}^\mu = \begin{pmatrix}
\mathcal{X}_0^\mu \\
\downarrow \\
\mathcal{X}_1^\mu
\end{pmatrix} = \begin{pmatrix}
\mathcal{X}_0^\mu \\
\downarrow \\
\mathcal{X}_1^\mu
\end{pmatrix} = \begin{pmatrix}
\mu_00 \cap \mu_00 \\
\downarrow \\
\mu_{10} \cap \mu_{10}
\end{pmatrix}.
$$

$^1$The complexity of this definition, when compared with Wille’s original set-theoretic definition, is quite noticeable — but the main argument of this paper is that this complexity is unavoidable, if we are committed to the philosophical principles that (1) intensions should be represented explicitly, and (2) conceptual structures are coherent units of thought consisting of an extensional aspect and an intensional aspect which determine each other by means of the notion of commonality or sharing.
consisting of the induced orders $X^0_\mu \overset{\text{df}}{=} \mu/\mu$ and $X^1_\mu \overset{\text{df}}{=} \mu/\mu$, the given relation $\mu$ which is closed w.r.t. these orders, and the negation relation $\neg\mu$. The contextual closure is the largest formal context containing $\mu$. Contextual closure transforms the combinatorial object $(X_0, X_1, \mu)$, the original notion of formal context of Wille, into the algebraic and potentially abstract object $X^\mu$ — thereby tremendously increasing the power, flexibility and expressibility of the basic mathematical object under study. The central result of this paper called the Equivalence Theorem, demonstrates the equivalence of concept lattice construction and Dedekind-MacNeille completion by using contextual closure.

Formal contexts can be compared. For any two contexts

$$X = \left( \begin{array}{ccc} X_0 & \mu_{01} & X_0 \\ \mu_{10} & X_1 \end{array} \right) \quad \text{and} \quad \mathcal{Y} = \left( \begin{array}{ccc} \mathcal{Y}_0 & \nu_{01} & \mathcal{Y}_0 \\ \nu_{10} & \mathcal{Y}_1 \end{array} \right),$$

a map of formal contexts $\mathcal{Y} \xrightarrow{f} X$ from context $\mathcal{Y}$ to context $X$ is a pair $f \overset{\text{df}}{=} (f_0, f_1)$ of monotonic functions between component orders $\mathcal{Y}_0 \xrightarrow{f_0} X_0$ and $\mathcal{Y}_1 \xrightarrow{f_1} X_1$, which satisfy both of the symbolic conditions

$$\mathcal{Y}_0 \xrightarrow{\nu_{01}} \mathcal{Y}_1 \quad \mathcal{Y}_0 \xrightarrow{\nu_{10}} \mathcal{Y}_1 \quad f_0 \downarrow \preceq \downarrow f_1 \quad f_0 \downarrow \preceq \downarrow f_1 \quad X_0 \xrightarrow{\mu_{01}} X_1 \quad X_0 \xrightarrow{\mu_{10}} X_1$$

expressed formally as two sets of equivalent conditions

$$f_0^\circ \circ \nu_{01} \circ f_0^\circ \leq \mu_{01} \quad \nu_{01} \circ f_0^\circ \leq f_0^\circ \circ \mu_{01} \quad \nu_{01} \leq f_0^\circ \circ \mu_{01} \circ f_1^\circ \quad f_1^\circ \circ \nu_{10} \circ f_1^\circ \leq \mu_{10} \quad \nu_{10} \circ f_1^\circ \leq f_1^\circ \circ \mu_{10} \quad \nu_{10} \leq f_1^\circ \circ \mu_{10} \circ f_0^\circ$$

Maps of formal contexts preserve the has relationships: if $y_0$ has attribute $y_1$ w.r.t. the relationship $\mathcal{Y}_0 \xrightarrow{\nu_{01}} \mathcal{Y}_1$, symbolically $y_0 \nu_{01} y_1$, then $f_0(x_0)$ has attribute $f_1(x_1)$ w.r.t. the relationship $X_0 \xrightarrow{\mu_{01}} X_1$, symbolically $f_0(y_0) \mu_{01} f_1(y_1)$. Similarly for the opposite direction.

Formal contexts and their maps form the category $\text{Cxt}$. There is a fully-faithful embedding functor called $\text{inclusion-of-identity-contexts}$

$$\text{Ord} \xrightarrow{\text{Inc}} \text{Cxt}$$

which maps orders $\mathcal{X}$ to identity contexts $\text{Inc}(\mathcal{X}) \overset{\text{df}}{=} \left( \begin{array}{cc} \mathcal{X} & \mathcal{X} \\ \mathcal{X} & \mathcal{X} \end{array} \right)$

and maps monotonic functions $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ to their doubles $\text{Inc}(\mathcal{Y}) \xrightarrow{\text{Inc}(f)} \text{Inc}(\mathcal{X})$, defined as the pair $\text{Inc}(f) \overset{\text{df}}{=} (f, f)$.

1. **Opposite:** The opposite involution, taking contexts

$$\mathcal{X} = \left( \begin{array}{cc} \mathcal{X}_0 & \mu_{01} \\ \mu_{10} & \mathcal{X}_1 \end{array} \right) \quad \text{to their opposites} \quad \mathcal{X}^\text{op} = \left( \begin{array}{cc} \mathcal{X}_0^\text{op} & \mu_{01}^\text{op} \\ \mu_{10}^\text{op} & \mathcal{X}_1^\text{op} \end{array} \right),$$

interchanges individual attributes with individual entities and interchanges collective attributes with collective entities.
2. **Terminal:** There is a very simple formal context

\[ 1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]

having exactly one entity and one related attribute. It consists of two copies of the unit order \( X_0 = 1 = X_1 \), plus two copies of the identity relation \( \mu_{01} = 1 = \mu_{10} \). This context is terminal, since from any formal context \( X \) there is a unique map of contexts \( X \rightarrow 1 \) to the terminal context 1, consisting of the pair of unique monotonic functions to the unit order \( \top \).

3. **Inverse image:** For any contexts

\[ X = \begin{pmatrix} X_0 & \mu_{01} \\ \mu_{10} & X_1 \end{pmatrix} \]

and any pair \( f = \langle f_0, f_1 \rangle \) of monotonic functions \( Y_0 \xrightarrow{f_0} X_0 \) and \( Y_1 \xrightarrow{f_1} X_1 \), there is an inverse image context

\[ f^{-1}(X) = \begin{pmatrix} Y_0 & f_0 \circ \mu_{01} \circ f_1 \\ f_1 \circ \mu_{10} \circ f_0 & Y_1 \end{pmatrix}, \]

with canonical map of formal contexts \( f^{-1}(X) \xrightarrow{\ell} X \).

4. **Product:** Given any pair of formal contexts

\[ Y = \begin{pmatrix} Y_0 & \nu_{01} \\ \nu_{10} & Y_1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_0 & \mu_{01} \\ \mu_{10} & X_1 \end{pmatrix} \]

the context

\[ Y \times X = \begin{pmatrix} Y_0 \times X_0 & \nu_{01} \times \mu_{01} \\ \nu_{10} \times \mu_{10} & Y_1 \times X_1 \end{pmatrix} \]

is the product context.

5. **Meet:** Given any pair of formal contexts

\[ X' = \begin{pmatrix} X'_0 & \mu'_{01} \\ \mu'_{10} & X'_1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_0 & \mu_{01} \\ \mu_{10} & X_1 \end{pmatrix} \]

over the same pair of underlying sets \( X_0 \) and \( X_1 \), the context

\[ X' \wedge X = \begin{pmatrix} X'_0 \wedge X_0 & \mu'_{01} \wedge \mu_{01} \\ \mu'_{10} \wedge \mu_{10} & X'_1 \wedge X_1 \end{pmatrix} \]

is the meet context over the same pair.

**Proposition 2** The category of formal contexts \( \text{Cxt} \) is complete; limits exist for all diagrams. It is also involutorial and fibered.

Let us take stock of our current situation. In order to define derivation in a coherent fashion in concept construction, closely following the philosophy that a concept consists of an extent and an intent that determine each other, and respecting any and all relationships that are actually present, we have been forced to change our starting framework — our notion of a formal context. It seems appropriate that we should start
we should have started with a pair \((φ, ψ)\) described by an order ideal of individual entities \(φ \in 2^{X_0^{op}}\) and an order ideal of collective entities \(ψ \in 2^{X_1}\), subject to the ideal assertions \(μ_{01} ▷ φ_1 ≤ φ_0\) and \(μ_{10} ▷ ψ_0 ≤ ψ_1\). A formal concept will consist of a quadruple \([φ_0, φ_1, ψ_0, ψ_1]\), a pair of extent-intent pairs with extent and intent described by

\[
\begin{align*}
\text{extent} & \quad \{ \text{individual} \quad φ_0 \in 2^{X_0^{op}} \quad \text{collective} \quad φ_1 \in 2^{X_1^{op}} \} \quad \text{satisfying ideal assertions} \quad \{ μ_{01} ▷ φ_1 ≤ φ_0 \quad μ_{10} ▷ ψ_0 ≤ ψ_1 \}, \\
\text{intent} & \quad \{ \text{collective} \quad ψ_0 \in 2^{X_0} \quad \text{individual} \quad ψ_1 \in 2^{X_1} \} \quad \text{satisfying filter assertions} \quad \{ ψ_0 ▷ μ_{01} ≤ ψ_1 \quad ψ_1 ▷ μ_{10} ≤ ψ_0 \}.
\end{align*}
\]

The direct and inverse phases of derivation each consist of four operations. The component operators for derivation data flow are described as follows.

- **direct derivation**
  \[
  \begin{align*}
  \left( \begin{array}{c}
  X_0 \xrightarrow[u]{u} X_1 \\
  \mu_{10} \xrightarrow[u]{u} X_1
  \end{array} \right) \quad \left( \begin{array}{c}
  \left( \begin{array}{c}
  X_0 \xrightarrow[u]{u} X_1 \\
  \mu_{10} \xrightarrow[u]{u} X_1
  \end{array} \right)\, u
  \end{array} \right)
  \end{align*}
  \]
  
  \[
  \begin{align*}
  \text{collective intent of individual extent} & \quad \text{collective intent of collective extent} \\
  \text{individual intent of individual extent} & \quad \text{individual intent of collective extent}
  \end{align*}
  \]

- **inverse derivation**
  \[
  \begin{align*}
  \left( \begin{array}{c}
  X_0 \xleftarrow[u]{u} X_1 \\
  \mu_{10} \xleftarrow[u]{u} X_1
  \end{array} \right) \quad \left( \begin{array}{c}
  \left( \begin{array}{c}
  X_0 \xleftarrow[u]{u} X_1 \\
  \mu_{10} \xleftarrow[u]{u} X_1
  \end{array} \right)\, u
  \end{array} \right)
  \end{align*}
  \]
  \[
  \begin{align*}
  \text{individual extent of collective intent} & \quad \text{individual extent of individual intent} \\
  \text{collective extent of individual intent} & \quad \text{collective extent of individual intent}
  \end{align*}
  \]

The data flow in the two phases of concept construction is illustrated in Figure 3. The requirement that conceptual extent and conceptual intent determine each other is expressed by the constraining definitions

\[
\begin{align*}
\text{extent} & \quad \{ φ_0 \overset{df}{=} (ψ_0)_{X_0} \land (ψ_1)_{μ_{01}} \land κ_{X_0} \land κ_{μ_{01}} \land κ_{X_1}, \quad \text{intent} & \quad \{ ψ_0 \overset{df}{=} (φ_0)_{X_0} \land (φ_1)_{μ_{10}} \land κ_{X_0} \land κ_{μ_{10}} \land κ_{X_1}, \quad
\end{align*}
\]

![Figure 3: Data flow in derivation](image-url)
4 Contextual Summation

A coherent approach to concept construction is the definition of an appropriate notion of a sum of a formal context

\[ \mathcal{X} = \left( \mathcal{X}_0 \rightarrow \mu_{01} \mathcal{X}_1 \right). \]

The formal context \( \mathcal{X} \) (pair of orthogonal relations \( \mathcal{X}_0 \rightarrow \mu_{01} \mathcal{X}_1 \) and \( \mathcal{X}_1 \rightarrow \mu_{10} \mathcal{X}_0 \)) specifies external constraints between the component orders \( \mathcal{X}_0 \) and \( \mathcal{X}_1 \) in two senses: either a source constraint or a target constraint.

Two relations \( \mathcal{X}_0 \xrightarrow{\rho_0} \mathcal{Y} \) and \( \mathcal{X}_1 \xrightarrow{\rho_1} \mathcal{Y} \) from the component source orders \( \mathcal{X}_0 \) and \( \mathcal{X}_1 \) to a common target order \( \mathcal{Y} \) satisfy the external source constraints specified by the formal context when

\[ \mu_{01} \circ \rho_1 \leq \rho_0 \quad \text{and} \quad \mu_{10} \circ \rho_0 \leq \rho_1. \]  

(3)

When \( \mathcal{Y} = 1 \) the two relations are order ideals \( \rho_0 \in 2^{\mathcal{X}_0 \op} \) and \( \rho_1 \in 2^{\mathcal{X}_1 \op} \), which satisfy the ideal assertions 3.

Two relations \( \mathcal{W} \xrightarrow{\sigma_0} \mathcal{X}_0 \) and \( \mathcal{W} \xrightarrow{\sigma_1} \mathcal{X}_1 \) from a common source order \( \mathcal{W} \) to the component target orders \( \mathcal{X}_0 \) and \( \mathcal{X}_1 \) satisfy the external target constraint specified by the formal context when

\[ \sigma_0 \circ \mu_{01} \leq \sigma_1 \quad \text{and} \quad \sigma_1 \circ \mu_{10} \leq \sigma_0. \]  

(4)

When \( \mathcal{W} = 1 \) the two relations are order filters \( \sigma_0 \in 2^{\mathcal{X}_0} \) and \( \sigma_1 \in 2^{\mathcal{X}_1} \), which satisfy the filter assertions 4.

Any formal context \( \mathcal{X} \), which specifies such collections of external constraints, can be internalized or centralized as a sum order \( \oplus \mathcal{X} = \langle X, \leq_X \rangle \) consisting of the disjoint union of elements \( X = X_0 + X_1 \) with order relation \( \leq_X \) defined by

\[
x'_0 \leq_X x_0 \iff x'_0 \leq x_0 \\
x_0 \leq_X x_1 \iff x_0 \mu_{01} x_1 \\
x_1 \leq_X x_0 \iff x_1 \mu_{10} x_0 \\
x_1 \leq_X x'_1 \iff x_1 \leq x'_1
\]

for all elements \( x_0 \in X_0 \) and \( x_1 \in X_1 \). The coproduct injections for the underlying disjoint union are monotonic functions \( \mathcal{X}_0 \xrightarrow{i_0} \oplus \mathcal{X} \xleftarrow{i_1} \mathcal{X}_1 \), which satisfy the defining conditions

\[
i_0 \circ i_0 = \mathcal{X}_0 \quad i_0 \circ i_1 = \mu_{01} \\
i_0 \circ i_0 = \mu_{10} \quad i_1 \circ i_1 = \mathcal{X}_1 \\
(i_0 \circ i_0) \vee (i_1 \circ i_1) = \oplus \mathcal{X}
\]

“suborder disjointness equations”

“covering equation”

The sum of the terminal context

\[
1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

is the binary order \( \mathcal{2} = \oplus 1 \).

Given any pair of relations \( \mathcal{X}_0 \xrightarrow{\rho_0} \mathcal{Y} \xleftarrow{\rho_1} \mathcal{X}_1 \) which satisfy the external source constraints \( \mu_{01} \circ \rho_1 \leq \rho_0 \) and \( \mu_{10} \circ \rho_0 \leq \rho_1 \) specified by the formal context, there is a unique “mediating” relation \( \oplus \mathcal{X} \xrightarrow{\rho} \mathcal{Y} \), symbolized by \( \rho = [\rho_0, \rho_1] \) and called the relative copairing of \( \rho_0 \) and \( \rho_1 \), which satisfies the rules

\[
i_0 \circ [\rho_0, \rho_1] = \rho_0 \\
i_1 \circ [\rho_0, \rho_1] = \rho_1
\]
Just define $[\rho_0, \rho_1] = (i_0^\rho \circ \rho_0) \lor (i_1^\rho \circ \rho_1)$. These properties say that the sum order $\oplus_X$ is a coproduct relative to the external constraints specified by the formal context. Clearly, the copairing operator $[,]$ is an order-isomorphism: $\rho_0 \preceq \rho'_0$ and $\rho_1 \preceq \rho'_1$ implies $[\rho_0, \rho_1] \preceq [\rho'_0, \rho'_1]$. Then $[\rho_0, \rho_1] \land [\rho'_0, \rho'_1] = [\rho_0 \land \rho'_0, \rho_1 \land \rho'_1]$. For any relation $\mathcal{Y} \overset{\sigma}{\rightarrow} \mathcal{Z}$ it is immediate that $[\rho_0, \rho_1] \circ \sigma = [\rho_0 \circ \sigma, \rho_1 \circ \sigma]$. An alternate definition of copairing, in terms of implications instead of product, is given by $[\rho_0, \rho_1] = (i_0^\rho \setminus \rho_0) \land (i_1^\rho \setminus \rho_1)$. The “overlap” of the $\rho_0$-part and the $\rho_1$-part of the source pairing is the relation $i_0^\rho \circ \rho_0 = (i_0^\rho \circ \rho_0) \land (i_1^\rho \circ \rho_1)$. When $\mathcal{Y} = \mathcal{1}$ the copairing operator defines an isomorphism

$$2^{X_{\text{op}}} \times_\mu 2^{Y_{\text{op}}} \cong 2^{\oplus_X^{\text{op}}}.$$  

Dually, given any pair of relations $X_0 \overset{\sigma}{\rightarrow} W \overset{\sigma'}{\rightarrow} X_1$, which satisfy the external target constraints $\sigma_0 \circ \mu_{01} \preceq \sigma_1$ and $\sigma_1 \circ \mu_{10} \preceq \sigma_0$ specified by the formal context, there is a unique “mediating” relation $W \overset{\sigma'}{\rightarrow} \oplus_X$, symbolized by $\sigma = (\sigma_0, \sigma_1)$ and called the relative pairing of $\sigma_0$ and $\sigma_1$, which satisfies the rules

$$(\sigma_0, \sigma_1) \circ i_0^\sigma = \sigma_0$$

$$(\sigma_0, \sigma_1) \circ i_1^\sigma = \sigma_1$$

Just define $(\sigma_0, \sigma_1) = (\sigma_0 \circ i_0^\sigma) \lor (\sigma_1 \circ i_1^\sigma)$. These properties say that the sum $\oplus_X$ is a product relative to the external constraints specified by the formal context. Clearly, the pairing operator $(,)$ is an order-embedding and preserves meets. For any relation $\mathcal{V} \overset{\tau}{\rightarrow} \mathcal{W}$ it is immediate that $\rho \circ (\sigma_0, \sigma_1) = (\rho \circ \sigma_0, \rho \circ \sigma_1)$. An alternate definition of pairing, in terms of implications instead of product, is given by $(\sigma_0, \sigma_1) = (\sigma_0 - i_0^\sigma) \land (\sigma_1 - i_1^\sigma)$. When $\mathcal{W} = \mathcal{1}$ the pairing operator defines an isomorphism

$$2^{X_0} \times_\mu 2^{Y_1} \overset{(,)\text{-iso}}{\cong} 2^{\oplus_X}.$$  

There is a canonical monotonic function

$$\oplus_X \overset{T}{\rightarrow} \oplus 1$$

from the sum order $\oplus_X$ to the binary order. This canonical function, called the tag or index function, indicates from which component order an element in the sum originates: $\tau(x_0) = 0$ and $\tau(x_1) = 1$, and functions as a partition (fibration) of the underlying set $X$. The components of the distributed context, which are used in the summation (centralization) process, are recoverable by the definitions

$$X_0 \overset{\text{def}}{=} \tau^{-1}(0) = \{x \mid x \in X, \tau(x) = 0\}$$

$$X_1 \overset{\text{def}}{=} \tau^{-1}(1) = \{x \mid x \in X, \tau(x) = 1\}$$

$$X_0 \overset{\leq_0}{\rightarrow} X_1 \overset{\text{def}}{=} \tau^{-1}(0 \leftrightarrow 0) = \{(x'_0, x_0) \mid x'_0, x_0 \in X_0, x'_0 \leq_X x_0\}$$

$$X_1 \overset{\preceq_1}{\rightarrow} X_0 \overset{\text{def}}{=} \tau^{-1}(1 \leftrightarrow 1) = \{(x_1, x'_1) \mid x_1, x'_1 \in X_1, x_1 \leq_X x'_1\}$$

$$X_0 \overset{\mu_{00}}{\rightarrow} X_1 \overset{\text{def}}{=} \tau^{-1}(0 \leftrightarrow 1) = \{(x_0, x_1) \mid x_0 \in X_0, x_1 \in X_1, x_0 \leq_X x_1\}$$

$$X_0 \overset{\mu_{10}}{\rightarrow} X_1 \overset{\text{def}}{=} \tau^{-1}(1 \leftrightarrow 0) = \{(x_1, x_0) \mid x_1 \in X_1, x_0 \in X_0, x_1 \leq_X x_0\}$$

The point of view that we foster in this paper is that summation and fibration are inverse transformations between specification of formal contexts as matrices of relations

$$X = \begin{pmatrix} X_0 & \mu_{01} \\ \mu_{10} & X_1 \end{pmatrix}$$

and specification of formal contexts as indexed orders $\oplus_X \overset{T}{\rightarrow} \oplus 1$.  

17
Consider any map of formal contexts $\mathcal{Y} \xrightarrow{f} \mathcal{X}$. By using the monotonicity of $f_0$ and $f_1$, the defining conditions for the map $f = \langle f_0, f_1 \rangle$ and the orthogonality conditions satisfied by the target context, it is straightforward to show that the sum function $Y_0 \cup Y_1 \xrightarrow{\oplus f} X_0 \cup X_1$ defined by $\oplus f \overset{\text{df}}{=} f_0 + f_1$ is a monotonic function between the sum orders $\oplus \mathcal{Y} \xrightarrow{\oplus \mathcal{f}} \oplus \mathcal{X}$. Since the monotonic index function $\oplus \mathcal{f} \xrightarrow{\circ \mathcal{f}} \oplus \mathcal{X}$ of the terminal map $\tau = \oplus \mathcal{f} \cdot \oplus \mathcal{T}$ (similarly for $\mathcal{Y}$), the sum function $\oplus f$ preserves partitions, in the sense that $\oplus \mathcal{T} \overset{\mathcal{f}}{=} \oplus f \cdot \oplus \mathcal{T}$. In the reverse direction, any monotonic function $\oplus \mathcal{f} \xrightarrow{\circ \mathcal{f}} \oplus \mathcal{X}$ resolves into two functions $Y_0 \xrightarrow{f_0} X_0$ and $Y_1 \xrightarrow{f_1} X_1$ defined by pullback (restriction to inverse image) of the component orders $X_0$ and $X_1$. The monotonicity of $f$ implies that $f_0$ and $f_1$ are monotonic functions $\mathcal{Y}_0 \xrightarrow{f_0} \mathcal{X}_0$ and $\mathcal{Y}_1 \xrightarrow{f_1} \mathcal{X}_1$, and implies the defining conditions $\nu_0 \leq f_0 \circ \mu_0 \circ f_0^o$ and $\nu_1 \leq f_1 \circ \mu_1 \circ f_1^o$. In summary, the category of formal contexts is isomorphic to the category of orders over the binary order

\[
\text{Cxt} \cong \text{Ord}/\mathcal{Z}.
\]

Define a summation functor $\text{Cxt} \xrightarrow{\oplus} \text{Ord}$ to be the composite $\oplus \overset{\text{df}}{=} \cdot \circ \partial_0$, where $\text{Ord}/\mathcal{Z} \xrightarrow{\partial_0} \text{Ord}$ is the projection functor mapping monotonic functions $\mathcal{X} \xrightarrow{i} \mathcal{Z}$ to their source order $\mathcal{X}$. Summation is left adjoint $\oplus \dashv \text{Inc}$ to the inclusion functor $\text{Ord} \xrightarrow{\text{Inc}} \text{Cxt}$ with adjunction unit context map $\mathcal{X} \xrightarrow{i} \text{Inc}(\oplus \mathcal{X})$ consisting of the pair $i = (i_0, i_1)$ of sum injection functions.

For any formal context

\[
\mathcal{X} = \left( \begin{array}{cc} \mathcal{X}_0 & \mu_01 \\ \mu_10 & \mathcal{X}_1 \end{array} \right)
\]

we can construct the Dedekind-MacNeille completion of the sum $\text{DM}(\oplus \mathcal{X})$ whose elements, called concepts, are formalized as extent-intent pairs $\langle \phi_0, \phi_1, (\psi_0, \psi_1) \rangle$: the extent pair of order ideals satisfies the ideal assertions $\mu_01 \circ \phi_1 \leq \phi_0$ and $\mu_10 \circ \phi_0 \leq \phi_1$ and the extent constraints $\phi_0 = (\psi_0)_{\mathcal{X}_0} \wedge (\psi_1)_{\mathcal{X}_1}$; and the intent pair of order filters satisfies the filter assertions $\phi_0 \circ \mu_01 \leq \psi_1$ and $\psi_1 \circ \mu_10 \leq \psi_0$ and the intent constraints $\psi_0 = (\phi_0)_{\mathcal{X}_0} \wedge (\phi_1)_{\mathcal{X}_1} \circ \mu_01$ and $\psi_1 = (\phi_0)_{\mathcal{X}_0} \wedge (\phi_1)_{\mathcal{X}_1} \circ \mu_10$. The embedding generator functions $\mathcal{X}_0 \xrightarrow{i_0} \text{DM}(\oplus \mathcal{X}) \xrightarrow{i_1} \mathcal{X}_1$, which are the compositions of coproduct injection followed by Dedekind-MacNeille completion, are described in detail as the following concepts (extent-intent pairs):

| generating | extent           | intent            |
|------------|------------------|-------------------|
| element    | $\phi_0$         | $\phi_1$          |
| $x_0 \in \mathcal{X}_0$ | $\downarrow x_0 \wedge (x_0)_{\mu}^{\ominus}$, $(x_0)_{\mu}^{\ominus}$ | $\uparrow x_0$, $x_0$ |
| $x_1 \in \mathcal{X}_1$ | $\mu x_1$       | $\downarrow x_1$, $\mu x_1$, $\uparrow x_1 \wedge (x_1)_{\mu}^{\ominus}$ |

The generated concept $i_0(x_0) = \langle \downarrow x_0 \wedge (x_0)_{\mu}^{\ominus}, (x_0)_{\mu}^{\ominus}, \uparrow x_0, x_0 \mu \rangle$ can start from the pair $\langle \downarrow x_0, 0 \rangle \in 2^{x_0 \wedge x_0} \times 2^{x_0 \mu}$, and the generated concept $i_1(x_1) = \langle \mu x_1, \downarrow x_1 \rangle$, $\langle \downarrow x_1, \mu x_1, \uparrow x_1 \wedge (x_1)_{\mu}^{\ominus} \rangle$ can start from the pair $\langle 0, \uparrow x_1 \rangle \in 2^{x_0 \wedge x_1}$.

Note that the ideal pair $\langle \uparrow x_0, x_0 \mu \rangle$ satisfies the ideal assertion $\uparrow x_0 \circ \mu \leq x_0 \mu$, and the filter pair $\langle \mu x_1, \downarrow x_1 \rangle$ satisfies the filter assertion $\mu \circ (\downarrow x_1) \leq \mu x_1$.

**Theorem 2 [Equivalence]** The concept lattice of a relation is (isomorphic to) the Dedekind-MacNeille completion (Figure 4) of the sum of the contextual closure of the relation:

\[
\text{CL}(X_0, X_1, \mu) \cong \text{DM}(\oplus \mathcal{X}^\mu).
\]

18
**Proof.** Assume that \( \langle \phi, \psi \rangle \) is a concept w.r.t. \( \text{CL}(X_0, X_1, \mu) \), hence satisfying the closure identities \( \phi = \psi^\omega \) and \( \psi = \phi^\omega \). We will show that \( \langle [\phi, \mu \wedge \dot{\phi}], (\psi/\mu, \psi) \rangle \) is a concept w.r.t. \( \text{DM}(\oplus X^\mu) \). The isomorphism between concepts will then be

\[
\langle \phi, \psi \rangle \longleftrightarrow \langle [\phi, \mu \wedge \dot{\phi}], (\psi/\mu, \psi) \rangle
\]

We need to verify the closure identities

\[
\mu \wedge \dot{\phi} = (\psi/\mu)^{\wedge X_0^\mu} \wedge \psi^\mu = \psi^\mu
\]

\[
\mu \wedge \dot{\phi} = (\psi/\mu)^{\wedge X_1^\mu} \wedge \psi^1 = \psi^1
\]

\[
\mu \wedge \dot{\phi} = (\psi/\mu)^{\wedge X_0^\mu} \wedge (\mu \wedge \dot{\phi})^{\wedge \mu} = \phi^\mu_{X_0^\mu}
\]

\[
\psi/\mu = \phi^\mu_{X_1^\mu}
\]

(see Appendix B). These hold, since \( \mu \wedge \dot{\phi} = \mu \wedge \dot{\psi}_{\mu} = \mu \wedge (\mu/\psi) = (\mu \wedge \mu)/\psi = X^\mu_1/\psi = \psi^1 \) and \( \psi/\mu = \phi^\mu_{X_1^\mu} = (\phi \wedge \mu)/\mu = \phi \wedge (\mu/\mu) = \phi \wedge X^\mu_0 = \phi^\mu_{X_0^\mu} \). We also need to verify the assertions

\[
\mu \circ (\mu \wedge \dot{\phi}) \leq \phi
\]

\[
\mu \circ (\mu \wedge \dot{\phi}) \leq \phi
\]

These hold by modus ponens, and orthogonality constraints \( \mu \circ \neg \leq \phi \) and \( \psi \circ \neg \leq (\psi/\mu) \).

The sum order of the contextual closure of the original planetary relationship is displayed in Table 8. This table is a matrix sum of the block in Table 1 representing the original relationship, the two blocks in Table 3 representing the induced orders, and the block in Table 7 representing the negation of the original planetary relationship. The concepts in the Dedekind-MacNeille completion of the planetary context (the sum of the contextual closure of the planetary relationship) are listed in Table 9. A comparison of these concepts with the concepts in the concept lattice of the planetary relationship as listed in Table 2 will confirm their isomorphism.

## 5 Future Work

In a follow-up paper we generalize formal contexts to a distributed version more suitable for knowledge representation called formal situations. There we give an equivalent categorical rendition known as distributed orders. In this follow-up paper we incorporate Woods’s notions of abstract conceptual descriptions, subsumption, extended quantifiers, etc. At the same time we rationalize the assertional/terminological distinction — A-boxes versus T-boxes.

In a second paper we abstract formal situations and their concept construction from the special order-theoretic realm to the general realm of semiadditive Heyting categories. In this abstraction, formal situations and distributed orders become distributed monads, and derivation operators become a kind of logical
Table 8: Sum $\oplus X^\mu$ of the contextual closure of the original planetary relationship

| concept description | extent | intent |
|---------------------|--------|--------|
|                      | $\phi_0$ | $\phi_1$ | $\psi_0$ | $\psi_1$ |
| "everything"        | $X_0$   | $X_1$   | $0$      | $0$      |
| "with moon"         | $\{E, Ma, J, S, U, N, P\}$ | $\{sm, sl, df, my\}$ | $0$ | $\{my\}$ |
| "small size"        | $\{Me, V, E, Ma, P\}$ | $\{ss, dn, mn\}$ | $0$ | $\{ss\}$ |
| "small with moon"   | $\{E, Ma, P\}$     | $\emptyset$  | $0$      | $\{ss, my\}$ |
| "far"               | $\{J, S, U, N, P\}$ | $\{sm, sl, df\}$ | $0$ | $\{df, my\}$ |
| "near"              | $\{Me, V, E, Ma\}$ | $\{dn, mn\}$ | $0$ | $\{ss, dn\}$ |
| "Plutoness"         | $\{P\}$             | $\emptyset$  | $\{P\}$ | $\{ss, df, my\}$ |
| "medium size"       | $\{U, N\}$          | $\{sm\}$    | $\{U, N\}$ | $\{sm, df, my\}$ |
| "large size"        | $\{J, S\}$          | $\{sl\}$    | $\{J, S\}$ | $\{sl, df, my\}$ |
| "near with moon"    | $\{E, Ma\}$         | $\emptyset$  | $\{E, Ma\}$ | $\{ss, dn, my\}$ |
| "moonless"          | $\{Me, V\}$         | $\{mn\}$    | $\{Me, V\}$ | $\{ss, dn, mn\}$ |
| "nothing"           | $\emptyset$          | $\emptyset$  | $X_0$    | $X_1$    |

Table 9: Dedekind-MacNeille completion DM($\oplus X^\mu$) for the sum
negation. We resolve distributed monads into the two constructions of distribution (matrices) and monad-enrichment, and identify the direct/inverse derivation operators of concept construction and the upper/lower operators of Dedekind-MacNeille completion with the two dual implication operators from relational logic [Lawvere, Kent]. Since the structural aspect of the mathematics here is very close to a Grothendieck topos, the topos nature of formal concept analysis needs to be investigated.

Initial applications have been carried out in terms of C++ software which implements the semantic version of formal context defined in this paper, the modified approach to concept lattice construction, and query processing against a lattice, in a windows environment on a personal computer or a work station. In a companion paper we have abstracted the related but distinct approach of Pawlak to classification and predicate approximation using rough sets [Pawlak]. The mathematics shows an intimate connection between the two approaches. Further work needs to be done on extending the new approach for concept construction to conceptual scaling, the situation more common in both object-orientation, database and knowledge representation, where multi-valued attributes exist.
A closed relation $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y}$ between two orders is a binary relation $\alpha \subseteq X \times Y$ between the underlying sets which is closed on the left w.r.t. $\mathcal{X}$ in the sense that $x' \leq_X x$ and $x \alpha y$ implies $x' \alpha y$ for all $x', x \in X$ and $y \in Y$, and closed on the right w.r.t. $\mathcal{Y}$ in the dual sense that $x \alpha y$ and $y \leq_Y y'$ implies $x \alpha y'$ for all $x \in X$ and $y, y' \in Y$. Clearly, an alternate description is that a closed relation $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y}$ is a monotonic function to the special Boolean order $\mathcal{Y}^{op} \xleftarrow{\beta} 2$. Closed relations between $\mathcal{X}$ and $\mathcal{Y}$ are ordered by subset inclusion (homset order) in the product powerset: two closed relations $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y}$ are ordered $\alpha \leq \beta$ when $\alpha \subseteq \beta$ as subsets $\alpha, \beta \in \mathcal{P}(X \times Y)$. Any closed relation $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y}$ has an opposite closed relation $\mathcal{Y}^{op} \xrightarrow{\alpha^{op}} \mathcal{X}^{op}$ defined by $y \alpha^{op} x$ iff $x \alpha y$. Two closed relations $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y}$ and $\mathcal{Y} \xrightarrow{\beta} Z$ with matching target and source, respectively, are said to be composable. The composition of two such closed relations is defined by $\alpha \circ \beta \overset{df}{=} \{(x, z) \mid (\exists y \in Y) x \alpha y \text{ and } y \beta z\}$ — ordinary relational composition. For every preorder $\mathcal{X} = (X, \leq_X)$ the order relation $\leq_X \subseteq X \times X$ is a closed relation $\mathcal{X} \xrightarrow{\leq_X} \mathcal{X}$, defined by $x \alpha x'$ when $x \leq_X x'$, which is the identity relation at $\mathcal{X}$ w.r.t. relational composition: $\mathcal{X} \circ r = r$ for any relation $\mathcal{X} \xrightarrow{r} \mathcal{Y}$. Composition preserves homset order. The opposite operator is an involution.

For any monotonic function $\mathcal{X} \xrightarrow{h} \mathcal{Y}$ there are two associated closed relations: its direct graph $\mathcal{X} \xrightarrow{h^p} \mathcal{Y}$ defined by $h^p \overset{df}{=} \{(x, y) \mid h(x) \leq_Y y\}$, and its inverse graph $\mathcal{Y} \xrightarrow{h^q} \mathcal{X}$ defined by $h^q \overset{df}{=} \{(y, x) \mid y \leq_X h(x)\}$. These relations form an adjoint pair $h^p \dashv h^q$. The graph of a composite map $\mathcal{X} \xrightarrow{f \circ g} \mathcal{Z}$ is the composition of the component graphs, in both the direct sense $(f \cdot g)^p = f^p \circ g^p$ and the inverse sense $(f \cdot g)^q = g^q \circ f^q$. The graph of the identity function $\mathcal{X} \xrightarrow{\text{Id}_X} \mathcal{X}$ is the identity relation, in both the direct sense $\text{Id}_X^p = \mathcal{X}$ and the inverse sense $\text{Id}_X^q = \mathcal{X}$.

The central adjointness for both classical and intuitionistic logic is the adjunction between conjunction and implication:

\[
\begin{align*}
\text{conjunction} & \quad \text{implication} \\
(\ ) \wedge p & \quad (\ ) \vdash p
\end{align*}
\]

as verified by the adjointness equivalence

\[
q \wedge p \vdash r \quad \text{iff} \quad q \vdash r \iff p
\]

for any three propositional symbols $p, q, r \in 2 = \{0, 1\}$. One reason why this adjointness is so central is that it has several powerful analogs in other contexts. We are especially interested in the relational analog, as illustrated in Table 10, and the central adjointness in the relational context. To extend this analogy to implication, we need a notion of relational implication. Unlike the case of propositional logic where the combining form of conjunction is symmetric, in relational logic the combining form of composition is asymmetric. This implies existence of two (related, but nonequivalent) relational implications.

| Traditional Logic | Relational Logic |
|-------------------|------------------|
| propositions $p$  | relations $\mathcal{X} \xrightarrow{\alpha} \mathcal{Y}$ |
| conjunction $p \land q$ | composition $\alpha \circ \beta$ |
| entailment $p \vdash q$ | order $\alpha \leq \beta$ |
| implication $p \Rightarrow q$ | implication $\beta \setminus \alpha$ |

Table 10: Analogies between traditional and relational logic
\[
\alpha/\gamma \overset{\text{df}}{=} \{ (x, z) \mid (\forall y \in Y) \ z \gamma y \text{ implies } x \phi y \} \\
\beta \setminus \alpha \overset{\text{df}}{=} \{ (z, y) \mid (\forall x \in X) \ x \beta z \text{ implies } x \phi y \}
\]
for any three closed relations \( X \xrightarrow{\beta} Z \), \( Z \xrightarrow{\gamma} Y \) and \( X \xrightarrow{\alpha} Y \). These implications are closed relations \( X \xrightarrow{\alpha/\gamma} Z \) and \( Z \xrightarrow{\beta \setminus \alpha} Y \). The relationship between the two forms of implication is expressible in terms of the opposite involution
\[
\alpha/\gamma = (\gamma \circ \alpha \circ \gamma)^{\text{op}} \quad \text{and} \quad \beta \setminus \alpha = (\alpha \circ \beta \circ \alpha)^{\text{op}}.
\]
The central adjointness for relational logic is the adjunction between relational composition and relational implication:

\[
\begin{array}{ccc}
\text{relational} & & \text{relational} \\
\text{composition} & \Rightarrow & \text{implication} \\
\beta \circ ( ) & \Rightarrow & \beta \setminus ( ) \\
( ) \circ \gamma & \Rightarrow & ( )/\gamma
\end{array}
\]
as verified by the adjointness equivalences
\[
\gamma \leq \beta \setminus \alpha \quad \text{iff} \quad \beta \circ \gamma \leq \alpha \quad \text{iff} \quad \beta \leq \alpha/\gamma.
\]

### A.1 Upper/Lower Operators

A closed-below subset (order ideal) of an order \( X = (X, \leq) \) is a subset \( \phi \subseteq X \) satisfying the condition: \( x' \leq_X x \) and \( x \in \phi \) implies \( x' \in \phi \) for all \( x, x' \in X \). Clearly, an order ideal can be expressed alternately as a closed relation \( X \xrightarrow{\phi} 1 \) or as a monotonic function \( X^{\text{op}} \xrightarrow{\phi} 2 \). So the set of order ideals of \( X \) is the special exponential order \( 2^{X^{\text{op}}} \). Dually, a closed-above subset (order filter) of \( X \) is a subset \( B \subseteq X \) satisfying the condition: \( x \in B \) and \( x \leq_X x' \) implies \( x' \in B \) for all \( x, x' \in X \). An order filter is a closed relation \( 1 \xrightarrow{\phi} X \) or a monotonic function \( X^{\text{op}} \xleftarrow{\phi} 2 \). So the set of order filters of \( X \) is the special exponential order \( 2^X \). If \( X \xrightarrow{\phi} 1 \) is an order ideal and \( 1 \xleftarrow{\psi} X \) is an order filter, then the relational composition \((X^{\phi \circ \psi} X) = \phi \times \psi \) is the Cartesian product of \( \phi \) and \( \psi \) as an endorelation. The relational interior of \( \phi \circ \psi \) is the intersection of \( A \) and \( \psi \) as a relational comonoid.

Given any order \( X \), the upper operator \( \left( \right)^u_X \) maps any subset \( \phi \subseteq X \) to its subset of upper bounds
\[
\phi^u_X \overset{\text{df}}{=} \{ x_2 \in X \mid x_1 \leq x_2 \text{ for all } x_1 \in \phi \} \\
= \{ x_2 \in X \mid (\forall x_1 \in X) \ x_1 \in \phi \text{ implies } x_1 \leq x_2 \} \\
= \{ x_2 \in X \mid \phi \subseteq \downarrow x_2 \} \\
= \bigcup_{x_1 \in \phi} \uparrow x_1,
\]
which is an order filter. The upper operator is invariant w.r.t. closure below, \( (\downarrow \phi)^u_X = \phi^u_X \) for any subset \( \phi \subseteq X \), and hence we can restrict application of the operator to just order ideals \( \phi \in 2^{X^{\text{op}}} \), so that \( 2^{X^{\text{op}}} (\downarrow \phi)^u_X = (2^X)^{\text{op}} \). It is important to observe the fact that the upper operator is a special case of relational implication
\[
\phi^u_X = \phi \setminus X.
\]
The varying quantity \( \phi \) is in the contravariant position w.r.t. implication. Dually, the lower operator \( \left( \right)^l_X \) maps any subset \( \psi \subseteq X \) to its subset of lower bounds
\[
\psi^l_X \overset{\text{df}}{=} \{ x_1 \in X \mid x_1 \leq x_2 \text{ for all } x_2 \in \psi \} \\
= \{ x_1 \in X \mid (\forall x_2 \in X) \ x_2 \in \psi \text{ implies } x_1 \leq x_2 \} \\
= \{ x_1 \in X \mid \psi \subseteq \downarrow x_2 \} \\
= \bigcap_{x_2 \in \psi} \uparrow x_2,
\]
an order ideal, is invariant w.r.t. closure above, \((\uparrow \psi)^1_X = \psi^1_X\) for any subset \(\psi \subseteq X\), and hence when restricted to just order filters \(\psi 2^X\) has type \((2^X)_{\text{op}} \frac{\Rightarrow}{\Rightarrow} 2^{X_{\text{op}}}\). The lower operator is also a special case of relational implication

\[\psi^1_X = X \setminus \psi.\]

The upper and lower operators form an adjoint pair of monotonic functions

\[ (\_)^u_X \dashv (\_)^l_X.\]

These operators can be expressed in terms of elements as follows:

\[\phi^1_X = \bigcap \{\uparrow x \mid x \in \phi\} = \bigcap \{x^u_X \mid x \in \phi\}\]
\[\psi^1_X = \bigcap \{\downarrow y \mid y \in \psi\} = \bigcap \{y^l_X \mid y \in \psi\}\]

### A.2 Direct/Inverse Derivation

Let \(X_0 \overset{\mu}{\rightarrow} X_1\) by any closed relation. The direct derivation (or intent) operator \((\_)^\Rightarrow_\mu\) maps any subset \(\phi \subseteq X_0\) to the subset

\[\phi^\Rightarrow_\mu \overset{\text{df}}{=} \{x_1 \in X_1 \mid x_0 \mu x_1 \text{ for all } x_0 \in \phi\}\]
\[= \{x_1 \in X_1 \mid (\forall x_0 \in X_0) \ x_0 \in \phi \implies x_0 \mu x_1\},\]

an order filter of \(X_1\). The direct derivation operator is invariant w.r.t. closure below, \((\downarrow \phi)^\Rightarrow_\mu = \phi^\Rightarrow_\mu\) for any subset \(\phi \subseteq X_0\), and hence we can restrict application of the operator to just order ideals \(\phi \in 2^{X_{\text{op}}}_0\), so that \(2^{X_{\text{op}}}_0 \frac{\Rightarrow}{\Rightarrow} (2^X_1)_{\text{op}}\). The intuitive interpretation in terms of formal concept analysis is that \(\phi^\Rightarrow_\mu\) is the collection of all individual attributes that the entities in \(\phi\) share, or have in common. It is important to observe the fact that, just as for the upper operator in the more homogeneous case of a single poset, the direct derivation operator is also a special case of relational implication

\[\phi^\Rightarrow_\mu = \phi \setminus \mu.\]

Dually, the inverse derivation (or extent) operator \((\_)^\Leftarrow_\mu\) maps any subset \(\psi \subseteq X_1\) to the subset

\[\psi^\Leftarrow_\mu \overset{\text{df}}{=} \{x_0 \in X_0 \mid x_0 \mu x_1 \text{ for all } x_1 \in \psi\}\]
\[= \{x_0 \in X_0 \mid (\forall x_1 \in X_1) \ x_1 \in \psi \implies x_0 \mu x_1\},\]

an order ideal of \(X_0\), is invariant w.r.t. closure above, \((\uparrow \psi)^\Leftarrow_\mu = \psi^\Leftarrow_\mu\) for any subset \(\psi \subseteq X_1\), and hence when restricted to just order filters \(\psi \in 2^X_1\) has type \((2^X_1)_{\text{op}} \frac{\Rightarrow}{\Rightarrow} 2^{X_{\text{op}}}_0\). The intuitive interpretation in terms of formal concept analysis is that \(\psi^\Leftarrow_\mu\) is the collection of all individual entities that share the attributes in \(\psi\). The inverse derivation operator is also a special case of relational implication

\[\psi^\Leftarrow_\mu = \mu \setminus \psi.\]

The direct and the inverse derivation operator form an adjoint pair of monotonic functions

\[ (\_)^\Rightarrow_\mu \dashv (\_)^\Leftarrow_\mu.\]
B Properties of Derivation

The following facts concerning sum orders are useful in determining derivation along pairings and copairings.

1. For any relations \( W \xrightarrow{x} X_0, W \xrightarrow{y} X_1 \) and \( W \xrightarrow{z} Z \)
   \[ z \setminus (x, y) = (z \setminus x, z \setminus y) \]

2. For any relations \( Z \xrightarrow{z} X_0, Z \xrightarrow{y} X_1, W \xrightarrow{x'} X_0 \) and \( W \xrightarrow{y'} X_1 \)
   \[ (x', y') \setminus (x, y) = (x' \setminus x) \land (y' \setminus y) \]

3. For any relations \( X_0 \xrightarrow{x} Z, X_1 \xrightarrow{y} Z, X_0 \xrightarrow{x'} W \) and \( X_1 \xrightarrow{y'} W \)
   \[ [x, y] \setminus [x', y'] = (x \setminus x') \land (y \setminus y') \]

4. For any relations \( X_0 \xrightarrow{x} Z, X_1 \xrightarrow{y} Z \) and \( W \xrightarrow{z} Z \)
   \[ [x, y] \setminus z = [x \setminus z, y \setminus z] \]

The following facts concerning relational implication are useful in determining derivation along inverse image relations.

1. For any monotonic function \( Y \xrightarrow{f} X \) and any relations \( Y \sigma \xrightarrow{W} X_0, Y \rho \xrightarrow{Z} X_1 \)
   \[ \sigma \setminus (f \circ \rho) = (f \circ \sigma) \setminus \rho \]

2. For any monotonic function \( Y \xrightarrow{f} X \) and any relations \( Z \sigma \xrightarrow{W} X_0, Z \rho \xrightarrow{X} X_1 \)
   \[ \sigma \setminus (\rho \circ f^\circ) = (\sigma \setminus \rho) \circ f^\circ \]

Basic properties of derivation are classified according to (1) type: order/continuity versus structure, and (2) varying quantity: ideal/filter versus relation.

1. Order/Continuity:

   (a) Ideal/Filter:

   **Order** Derivation, either direct or inverse, is contravariant in the subset argument
   \[ \phi_1 \preceq_2 \phi_2 \] implies \[ (\phi_1)_\mu \preceq_2 (\phi_2)_\mu \]
   \[ \psi_1 \preceq_2 \psi_2 \] implies \[ (\psi_1)_\mu \preceq_2 (\psi_2)_\mu \]

   **Continuity** Derivation is continuous in the subset argument
   \[ (\bigvee_{i \in I} \phi_i)_\mu = \bigwedge_{i \in I} (\phi_i)_\mu \]
   \[ (\bigwedge_{j \in J} \psi_j)_\mu = \bigvee_{j \in J} (\psi_j)_\mu \]
   In particular, when the index set is empty \( I = \emptyset \) derivation is
   \[ (\bot_{\preceq_2})_\mu = X_1 = \top_{\preceq_2} \]
   \[ (\bot_{\preceq_2})_\mu = \emptyset_\mu = X_0 = \top_{\preceq_2} \]
   and when the index set is two \( I = 2 \) derivation is
   \[ (\phi_1 \lor_2 \phi_2)_\mu = \phi_1 \lor_2 \phi_2 \]
   \[ (\psi_1 \lor_2 \psi_2)_\mu = \psi_1 \lor_2 \psi_2 \]

   (b) Relation:

25
Order\, Derivation, either direct or inverse, is covariant in the relation argument: for any two parallel contexts $X_0 \xrightarrow{\mu} X_1$ which are order as $\mu \leq \mu'$, derivation is ordered by
\[
\phi_{\mu}^\rightarrow \preceq_{2x_i} \phi_{\mu'}^\rightarrow \quad \psi_{\mu}^< \preceq_{2x_0^{\text{op}}} \psi_{\mu'}^<.
\]

Continuity\, Derivation is continuous in the relational argument: for any collection of parallel contexts $\{X_0 \xrightarrow{\mu_i} X_1 \mid i \in I\}$ derivation along the meet is expressed as
\[
\phi_{\wedge \mu_i}^\rightarrow = \bigwedge_{2x_i} (\phi_{\mu_i}^\rightarrow \mid i \in I), \quad \psi_{\mu_i}^< = \bigwedge_{2x_0^{\text{op}}} (\psi_{\mu_i}^< \mid i \in I).
\]
In particular, when the index set is empty $I = \emptyset$, and the meet is the top relation $X_0 \xrightarrow{T} X_1$ defined by $T = X_0 \times X_1$, derivation is
\[
\phi_T^\rightarrow = X_1 = \bigwedge_{2x_i} \phi_{\mu_i}^\rightarrow, \quad \psi_T^< = X_0 = \bigwedge_{2x_0^{\text{op}}} \psi_{\mu_i}^<
\]
and when the index set is two $I = 2$, and the meet is the relation $X_0 \xrightarrow{\mu \wedge \mu'} X_1$, derivation is
\[
\phi_{\mu \wedge \mu'}^\rightarrow = \phi_{\mu}^\wedge \bigwedge_{2x_i} \phi_{\mu'}^\rightarrow, \quad \psi_{\mu \wedge \mu'}^< = \psi_{\mu}^\wedge \bigwedge_{2x_0^{\text{op}}} \psi_{\mu'}^<.
\]

2. Structure\, Some general constructions on closed relations are listed below, along with the expression of their derivation operators in terms of component derivation and basic relational operators.

(a) Ideal/Filter:

Generators\, Derivation, either direct or inverse, can be generated (constructed) from elements using intersection
\[
\phi_{\mu}^\rightarrow = \bigwedge_{2x_i} \{x_0 \mu \mid x_0 \in \phi\} = \bigwedge_{2x_i} \{x_0 \mu_{i}^\rightarrow \mid x_0 \in \phi\} = \bigwedge_{2x_0^{\text{op}}} \{\mu x_1 \mid x_1 \in \psi\} = \bigwedge_{2x_0^{\text{op}}} \{(x_1)_{\mu_{i}^<} \mid x_1 \in \psi\}.
\]

(b) Relation:

Identity\, Derivation across the identity relation $X \xrightarrow{\text{id}} X$ reduces to the upper/lower operators
\[
\phi_X^\rightarrow = \phi_X^\leftarrow, \quad \psi_X^< = \psi_X^1,
\]
showing Dedekind-MacNeille completion to be a special case of concept-lattice construction.

Map\, Derivation along the direct graph $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ of a monotonic function $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ factors as direct/inverse ideal image and upper/lower operator

direct \begin{align*}
\psi_{\mathcal{X}}^f &= \psi \land f^\circ = ((f^\circ \circ \psi) \land \mathcal{X}) = (f^\circ \circ \psi)^\mathcal{X} \\
\end{align*}

inverse \begin{align*}
\phi_{\mathcal{X}}^f &= f^\circ \land \phi = f^\circ \circ (\mathcal{X} \land \phi) = f^\circ \circ \phi^\mathcal{X}
\end{align*}

Derivation along the inverse graph $\mathcal{X} \xrightarrow{f^\circ} \mathcal{Y}$ of a monotonic function $\mathcal{Y} \xrightarrow{f^\circ} \mathcal{X}$ factors as direct/inverse filter image and upper/lower operator

direct \begin{align*}
\phi_{\mathcal{Y}}^{f^\circ} &= \phi \land f^\circ = (\phi \land \mathcal{X}) \circ f^\circ = \phi^\mathcal{Y} \circ f^\circ \\
\end{align*}

inverse \begin{align*}
\psi_{\mathcal{Y}}^{f^\circ} &= f^\circ \land \psi = \mathcal{X} \land (\psi \circ f^\circ) = (\psi \circ f^\circ)^\mathcal{X}
\end{align*}

Inverse Image\, Given any pair of monotonic functions $\mathcal{Y}_0 \xrightarrow{f_0} \mathcal{X}_0$ and $\mathcal{Y}_1 \xrightarrow{f_1} \mathcal{X}_1$ derivation along the inverse image $\mathcal{Y}_0 \xrightarrow{f_0^{\text{op}}} \mathcal{X}_1$ of a relation $X_0 \xrightarrow{\mu} X_1$ factors as
The following special derivation inequalities are used in the proof of the Equivalence theorem. For any relation $\mathcal{X}_0 \xrightarrow{\mu,\nu} \mathcal{X}_0$ factors as existential/universal filter quantification and upper/lower operator

**Negation** Derivation along the source negation of a relation $\mathcal{X}_1 \xleftarrow{\mu,\nu} \mathcal{X}_0$ factors as existential/universal ideal quantification and upper/lower operator

**Pairing** Derivation along a pairing of two relations $\mathcal{X}_0 \xrightarrow{\mu,\nu} \mathcal{Y} + \mathcal{Z}$ is the pairing of filters and the meet of ideals

**CoPairing** Derivation along a copairing of two relations $\mathcal{Y} + \mathcal{Z} \xrightarrow{\mu,\nu} \mathcal{X}_1$ is the meet of filters and the pairing of ideals

The following special derivation inequalities are used in the proof of the Equivalence theorem. For any closed relation $\mathcal{X}_0^\mu \xrightarrow{\mu} \mathcal{X}_0^\mu$ between induced orders,
| relation      | direct derivation operator | inverse derivation operator |
|--------------|----------------------------|-----------------------------|
| equality     | \(\phi^= = \phi^n\)       | \(\psi^= = \psi^n\)        |
| identity     | \(\phi^i = \{x_1 \in X \mid x_1 = x_1 \text{ for all } x_0 \in \phi\}\) | \(\psi^i = \{x_0 \in X \mid x_0 = x_1 \text{ for all } x_1 \in \psi\}\) |
| bottom       | \(\phi^\downarrow = \{x_1 \in X_1 \mid (\forall x_0 \in X_0) x_0 \notin \phi\}\) | \(\psi^\downarrow = \{x_0 \in X_0 \mid (\forall x_1 \in X_1) x_1 \notin \psi\}\) |
| complement   | \(\phi^\uparrow = X \setminus \phi\) | \(\psi^\uparrow = X \setminus \psi\) |

Table 11: Special relations

**direct**
- “the direct derivation is within
  - the upper bounds of the universal ideal quantification
    \(\phi^\mu = \phi \wedge \mu \leq (\mu \wedge \phi) \wedge (\mu \wedge \mu) = (\mu \wedge \phi) \wedge \mu\)
  - hence \(\phi^\mu = \phi^\mu \wedge (\mu \wedge \phi) \wedge \mu\)

**inverse**
- “the inverse derivation is within
  - the lower bounds of the universal filter quantification
    \(\psi^\mu = \mu \setminus \psi \leq (\mu \setminus \mu) \setminus (\psi \setminus \mu) = (\psi \setminus \mu) \setminus \mu\)
  - hence \(\psi^\mu = (\psi \setminus \mu) \setminus \mu\)

**upper**
- “the upper bounds are within
  - the direct derivation along negation of the universal ideal quantification
    \(\phi^{\mu^\downarrow} = \phi \wedge \mu^{\downarrow} \leq (\mu \wedge \phi) \wedge (\mu \wedge \mu^{\downarrow}) = (\mu \wedge \phi) \wedge \mu^{\downarrow}\)
  - hence \(\phi^{\mu^\downarrow} = \phi^{\mu^\downarrow} \wedge (\mu \wedge \phi) \wedge \mu^{\downarrow}\)

**lower**
- “the lower bounds are within
  - the inverse derivation along negation of the universal filter quantification
    \(\psi^{\mu^\uparrow} = (\mu \setminus \mu) \setminus \psi \leq \mu \setminus (\psi \setminus \mu) = (\psi \setminus \mu) \setminus \mu\)
    \(\psi^{\mu^\uparrow} = (\psi \setminus \mu) \setminus \mu\)

C Wille’s Concept Lattice Construction

**Proposition 3** Let \(X_0\) and \(X_1\) be any pair of orders. The concept lattice for the top relation \(X_0 \rightarrow X_1\) is the unit:

\[
\text{CL}(X_0, X_1, \top) = \{(X_0, X_1)\} = 1
\] (5)
Proposition 4 Let $Y_0 \xrightarrow{f_0} X_0$ and $Y_1 \xrightarrow{f_1} X_1$ be any pair of surjective monotonic functions and $X_0 \overset{\mu}{\rightarrow} X_1$ be any closed relation. The concept lattice for the inverse image relation $Y_0 \xrightarrow{f_0 \circ \mu \circ f_1^D} Y_1$ consists of inverse ideal image of extents:

$$\text{CL}_{\text{ext}}(Y_0, Y_1, f_0 \circ \mu \circ f_1^D) = \{ f_0 \circ \phi_0 \mid \phi_0 \in \text{CL}_{\text{ext}}(X_0, X_1, \mu) \}$$  \hfill (6)

Corollary 1 Let $X_0 \overset{\mu}{\rightarrow} X_1$ and $Y_0 \overset{\nu}{\rightarrow} Y_1$ be any closed relations. The concept lattice for the product relation $X_0 \times Y_0 \overset{\mu \times \nu}{\rightarrow} X_1 \times Y_1$ consists of products of extents:

$$\text{CL}_{\text{ext}}(X_0 \times Y_0, X_1 \times Y_1, \mu \times \nu) = \{ \phi_0 \times \phi_1 \mid \phi_0 \in \text{CL}_{\text{ext}}(X_0, X_1, \mu), \psi_0 \in \text{CL}_{\text{ext}}(Y_0, Y_1, \nu) \}$$  \hfill (7)

Thus, the lattice of a product is the product of the lattices

$$\text{CL}_{\text{ext}}(X_0 \times Y_0, X_1 \times Y_1, \mu \times \nu) \cong \text{CL}_{\text{ext}}(X_0, X_1, \mu) \times \text{CL}_{\text{ext}}(Y_0, Y_1, \nu)$$  \hfill (7)

Proposition 5 Let $X \overset{\mu}{\rightarrow} Y$ and $X \overset{\nu}{\rightarrow} Z$ be any closed relations, and let $X^{(\mu, \nu)} \rightarrow Y + Z$ be their pairing (called apposition in [Wille]). The concept lattice for the pairing consists of meets of extents:

$$\text{CL}_{\text{ext}}(X, Y + Z, (\mu, \nu)) = \{ \alpha \land \beta \mid \alpha \in \text{CL}_{\text{ext}}(X, Y, \mu), \beta \in \text{CL}_{\text{ext}}(X, Z, \nu) \}$$  \hfill (8)

Proof. By the above facts, a concept of the pairing can be described as a triple $\langle \phi, (\psi, \zeta) \rangle$ where $\phi \in 2^{\text{X} \rightarrow \text{Y}}$, $\psi \in 2^{\text{Y} \rightarrow \text{Z}}$ and $\zeta \in 2^{\text{Z} \rightarrow \text{Z}}$ are subsets satisfying the conditions $\phi^\rightarrow_\mu = \psi$, $\phi^\rightarrow_\nu = \zeta$ and $\phi = \psi^\rightarrow_\mu \land \zeta^\rightarrow_\nu$. By defining $\alpha = \psi^\rightarrow_\mu$ and $\beta = \zeta^\rightarrow_\nu$, we see that any pairing concept is of the form given by description 8, since $\langle \alpha, \phi \rangle \in \text{CL}(X, Y, \mu)$ and $\langle \beta, \zeta \rangle \in \text{CL}(X, Z, \nu)$. On the other hand, assume that $\langle \alpha, \phi \rangle \in \text{CL}(X, Y, \mu)$ and $\langle \beta, \zeta \rangle \in \text{CL}(X, Z, \nu)$ are arbitrary concepts in the component lattices. Since $(\alpha \land \beta)^\rightarrow_\mu \leq \alpha^\rightarrow_\mu = \alpha$ and $(\alpha \land \beta)^\rightarrow_\nu \leq \beta^\rightarrow_\nu = \beta$, we have $\alpha \land \beta = (\alpha \land \beta)^\rightarrow_\mu \land (\alpha \land \beta)^\rightarrow_\nu$. So the triple $\langle \phi, (\psi, \zeta) \rangle$ where $\phi \overset{\text{df}}{=} \alpha \land \beta$, $\psi \overset{\text{df}}{=} \phi^\rightarrow_\mu$ and $\zeta \overset{\text{df}}{=} \phi^\rightarrow_\nu$ are subsets satisfying the pairing conditions. \qed

29
References

[CarWal] A. Carboni and R.F.C. Walters, Cartesian bicategories I. Journal of Pure and Applied Algebra 49 (1987) 11–32.

[Kent] R.E. Kent, Dialectical logic: the process calculus. To appear in Studia Scientiarum Mathematicarum Hungarica (1992).

[Lawvere] F.W. Lawvere, Metric spaces, generalized logic, and closed categories. Seminario Mathematico E. Fisico. Rendiconti. Milan. 43 (1973) 135–166.

[MacNeille] H.M. MacNeille, Partially ordered sets. Transactions of the American Mathematical Society 42 (1937) 90–96.

[Pawlak] Z. Pawlak, Rough Sets. International Journal of Information and Computer Science 11 (1982), 341–356.

[Wille] R. Wille, Restructuring lattice theory: an approach based on hierarchies on concepts. In Ordered Sets ed. I. Rival Reidel, Dordrecht-Boston (1982) 445–470.

[Woods] W.A. Woods, Understanding subsumption and taxonomy: a framework for progress, in Principles of Semantic Networks ed. J. Sowa (1991) 45–94.