On Picard Type Theorems and Entire Solutions of Differential Equations

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Abstract. We give a connection between the Picard type theorem of Polya-Saxer-Milliox and characterization of entire solutions of a differential equation and then their higher dimensional extensions, which leads further results on both (ordinary and partial) differential equations and Picard type theorems.

Key words. Entire function, Picard’s Theorem, Picard type theorem, ordinary differential equation, partial differential equation.

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In the recent paper [8], a connection/equivalence was established between Picard’s theorem in complex analysis and characterization of entire solutions of an ordinary differential equation, which was further extended to partial differential equations (see [8] for the details). Recall that Picard’s theorem asserts that an entire function, i.e., a complex-valued function differentiable in the complex plane \( \mathbb{C} \), omitting two complex numbers must be constant. It also implies, by a simple transform, the meromorphic version of the theorem that a meromorphic function in \( \mathbb{C} \) omitting three distinct values must be constant. Picard’s theorem is among the most striking results in complex analysis and plays a decisive role in the development of the theory of entire and meromorphic functions and other applications. We refer to [12] for an exposition (the history, methods, and references) of the theorem and [1], [2], [4], [5], [7], [14], etc. for various proofs.

We call a theorem a Picard type theorem if it asserts that an entire function under certain conditions must reduce to constant. The observation in [8] suggests a heuristic: A Picard type theorem is apt to imply a characterization of entire solutions of a differential equation, and vice versa. (This may also be extended to the meromorphic case.)

Inspired by this heuristic, we present, in this article, a connection between the well-known Picard type theorem of Polya-Saxer-Milliox (see [11] and [10]; see also [6]) and a characterization of entire solutions of an ordinary differential equation, which provides a different approach to the existing results and leads
further results on both (ordinary and partial) differential equations and Picard type theorems (cf. below).

**Theorem A** (Polya-Saxer-Millioux) *An entire function $f$ must be constant if $f$ omits 0 and $f'$ omits 1 in $\mathbb{C}$.*

We note that Theorem A has undergone various extensions over the years; especially it also holds for meromorphic functions (see [6]). In this paper we however only consider entire functions, which serves our main purpose for the above mentioned connections, although Theorem 1 below holds also for meromorphic functions, which is clear from the proof given below.

We also note, by a simple linear transform, that the values 0 and 1 in Theorem A can be replaced by any two complex numbers $a$ and $b \neq 0$, respectively; but, the assumption $b \neq 0$ cannot be dropped, as seen from the nonconstant entire function $f = e^z$, for which both $f$ and $f'$ omit 0.

We first give the characterization of entire solutions of the differential equation in the following

**Theorem 1.** Let $a(z)$ be an entire function in $\mathbb{C}$ and let $P(z_1, z_2)$ be an entire function in $\mathbb{C}^2$ divisible by $(z_1 - c)(z_2 - d)$ in the ring of entire functions, where $c, d$ with $d \neq 0$ are complex numbers. Then an entire solution $f$ in $\mathbb{C}$ of the differential equation

$$f' + a(z)P(f, f') = 0$$

(1)

must be constant.

We will see that Theorem 1 implies Theorem A immediately; as a matter of fact, Theorem 1 and Theorem A are equivalent in the sense that one implies the other.

**Theorem 1 $\implies$ Theorem A.** Since $f$ satisfies that $f \neq 0$ and $f' \neq 1$, the function $a(z) := \frac{f'}{f''(f - 1)}$ is entire. Clearly, $f' - a(z)f(f' - 1) = 0$, which is an equation of the form (1) with $P(z_1, z_2) = z_1(z_2 - 1)$. Thus, $f$ must be constant by Theorem 1.

**Theorem A $\implies$ Theorem 1.** Since $P$ is divisible by $(z_1 - c)(z_2 - d)$ in the ring of entire functions, we can write $P(z_1, z_2) = (z_1 - c)(z_2 - d)g(z_1, z_2)$, where $g$ is an entire function in $\mathbb{C}^2$. It follows from (1) that

$$f'(z) = -a(z)(f(z) - c)(f'(z) - d)g(f(z), f'(z)).$$

(2)

It is clear that $f'$ cannot assume $d$, since otherwise, the right hand side of (2) is 0 while the left hand side is nonzero, a contradiction. It is also easy to see that $f$ cannot assume $c$, since otherwise the right hand side of (2) would have a zero (coming from a zero of $f - c$) with multiplicity strictly greater than that of the same zero of the left hand side (due to the derivative, which decreases the multiplicity), which is absurd. Thus, $f$ omits $c$ and $f'$ omits $d$. By Theorem A, $F = \frac{f - c}{d}$ and thus $f$ must be constant.

□
Study of solutions of differential equations in complex variables has a long history. The connection between Picard type theorems and characterizations of entire solutions of differential equations may help discover further results in both directions, as seen below.

Theorem 1, equivalent to the Picard type theorem of Polya-Saxer-Milloux but in the form on solutions of the differential equation, leads naturally consideration of the related higher order differential equations

\[ f^{(n)}(z) + a(z)P(f, f') = 0 \]  \hspace{1cm} (3)

for \( n \geq 2 \), in view of the known fact that Theorem A still holds with \( f^{(n)} \) replacing \( f' \) (see e.g. [6]). It is tempting to expect a result similar to Theorem 1 would hold for the equation (3). However, entire solutions of (3) are not necessarily constant any more. In fact, when \( n \geq 2 \) the equation \( f^{(n)}(z) + a(z)f'(z) - 1 = 0 \) has a nonconstant entire solutions \( f(z) = z \). We will see that there is a generality behind this and a complete characterization of entire solutions can be given, which allows extensions to more general partial differential equations and can yield further results on Picard type theorems. We consider

\[ \sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f_{z_j}(z)}{\partial z_1 \cdots \partial z_n} + a(z)P(f(z), f_{z_j}(z)) = 0 \]  \hspace{1cm} (4)

where \( z = (z_1, z_2, \cdots, z_n) \) in \( \mathbb{C}^n \), \( \alpha = (\alpha_1, \cdots, \alpha_n) \) is a multi-index with \( \alpha_j \geq 0 \) and \(|\alpha| = \alpha_1 + \cdots + \alpha_n \), \( a_\alpha \) are polynomials in \( \mathbb{C}^n \) and \( f_{z_j} = \frac{\partial f}{\partial z_j} \) (1 \( \leq j \leq n \)).

**Theorem 2.** Let \( a(z) \) be a nonzero entire function in \( \mathbb{C}^n \) and let \( P(x, y) \) be a nonzero entire function in \( \mathbb{C}^2 \) divisible by \((x - c)(y - d)\) in the ring of entire functions in \( \mathbb{C}^2 \) for two complex numbers \( c, d \) with \( d \neq 0 \). Then an entire solution \( f \neq c \) in \( \mathbb{C}^n \) of the partial differential equation (4) is given by \( f(z) = dz + \phi \), where \( \phi \) is an entire function in \( z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n \). In particular, \( f(z) = dz + A \) if \( n = 1 \), where \( A \) is a constant.

In proving Theorem 2, we will utilize the following known properties of

\[ m(r, f) := \int_{S_n(r)} \log^+ |f| \eta_n \]

for a nonconstant entire function \( f \) in \( \mathbb{C}^n \), where \( \log^+ x = \max\{0, \log x\} \) and \( \eta_n \) is the usual positive volume form on the sphere \( S_n(r) := \{z \in \mathbb{C}^n : |z| = r\} \) normalized so that the total volume of the sphere is 1 (see e.g. [13]; cf. also [3]):

(I) \( m(r, 1/f) \leq m(r, f) + O(1) \), which follows from Jensen’s formula or the Nevanlinna first fundamental theorem in several complex variables;

(II) \( m(r, \frac{\partial^{n+1} f(z)}{\partial z_1 \cdots \partial z_n}) = S(r, f) \), where \( S(r, f) \) denotes a quantity satisfying that \( S(r, f) = O\{\log(rm(r, f))\} \) as \( r \to \infty \), which follows from the logarithmic derivative lemma in several complex variables, where the symbol \( \hat{=} \) means that the equality holds outside a set of \( r \) of finite Lebesgue measure.
Proof of Theorem 2. If \( f_{z_j} \equiv d \), the theorem already holds. We assume in the following that \( f_{z_j} \not\equiv d \) and will derive a contradiction.

By the assumption on \( P \), we can write \( P(x, y) = (x - c)(y - d)g(z) \), where \( g \) is a nonzero entire function in \( \mathbb{C}^2 \). Thus, the given equation can be written as

\[
\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f_{z_j}(z)}{\partial^{|\alpha|} z_1 \cdots \partial^{|\alpha|} z_n} = b(z)(f(z) - c)(f_{z_j}(z) - d),
\]

(5)

where \( b(z) = -a(z)g(f(z), f_{z_j}(z)) \). We now write (5) as

\[
\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f_{z_j}(z)}{f_{z_j}(z) - d} = b(z)(f(z) - c).
\]

and

\[
\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f_{z_j}(z)}{f(z) - c} = b(z)(f_{z_j}(z) - d)
\]

in view of the fact that \( f \not\equiv c \). We then obtain by Property (II) that

\[
m(r, b(f - c)) = S(r, f_{z_j}) = S(r, f)
\]

and

\[
m(r, b(f_{z_j} - d) = S(r, f),
\]

which implies, by Property (I), that

\[
m(r, \frac{f_{z_j} - d}{f - c}) = m(r, \frac{b(f_{z_j} - d)}{b(f - c)})
\]

\[
\leq m(r, \frac{b(f_{z_j} - d)}{b(f - c)}) + m(r, \frac{1}{b(f - c)})
\]

\[
\leq m(r, b(f_{z_j} - d)) + m(r, b(f - c)) + O(1) = S(r, f).
\]

We then deduce that

\[
m(r, \frac{d}{f - c}) \leq m(r, \frac{d}{f - c}) + O(1)
\]

\[
\leq m(r, \frac{d - f_{z_j}}{f - c}) + m(r, \frac{f_{z_j}}{f - c}) + O(1) = S(r, f).
\]

Changing the equation (5) to

\[
\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f_{z_j}(z)}{f_{z_j}(z) - d} \frac{1}{f - c} = b(z)
\]
we obtain that \( m(r, b) = S(r, f) \). We then change the equation (5) to

\[
\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} f_{z_j}(z)}{f_{z_j}(z) - d} \frac{1}{b} = f - c,
\]

from which and Properties (I) and (II) again we deduce that

\[
m(r, f) = m(r, f - c) + O(1)
\]

\[
\leq m(r, \frac{\partial^{|\alpha|} f_{z_j}(z)}{f_{z_j}(z) - d}) + m(r, \frac{1}{b})
\]

\[
= S(r, f) = O\{\log(rm(r, f))\}
\]

and then that \( m(r, f) = O(\log r) \), which implies that \( f \) is a polynomial. Then, the left hand side of (6), as a linear combination of polynomials, must be a polynomial, which implies that the entire function \( b(z) \), as a quotient of two polynomials, must be a polynomial, too. But, it is evident that the degree of the left hand side of (6) (due to higher order derivatives) is strictly lower than that of the left hand side, which is impossible. This completes the proof. □

The above proof of Theorem 2 can be pushed over to even more general situation where \( f_{z_j} \) is replaced by a partial derivative \( \frac{\partial^{|I|} f(z)}{\partial^{i_1} z_1 \cdots \partial^{i_n} z_n} \) of any order or a sum of the form

\[
Df = \sum_{|I|=1}^{m} a_I \frac{\partial^{|I|} f(z)}{\partial^{i_1} z_1 \cdots \partial^{i_n} z_n}
\]

(6)

with \( I = (i_1, \cdots, i_n) \) and \( a_I \)'s being polynomials in \( C^n \) (or any expression formed by \( f \) and finitely many partial derivatives of \( f \) of any orders so that the proof can go through). We include the following theorem for partial differential equations

\[
\sum_{|\alpha|=1}^{m} a_\alpha \frac{\partial^{|\alpha|} Df(z)}{\partial^{a_1} z_1 \cdots \partial^{a_n} z_n} + a(z)P(f(z), Df(z)) = 0.
\]

(7)

Theorem 3. Let \( a(z) \) be a nonzero entire function in \( C^n \) and let \( P(x, y) \) be a nonzero entire function in \( C^2 \) divisible by \( (x - c)(y - d) \) in the ring of entire functions in \( C^2 \) for two complex numbers \( c, d \) with \( d \neq 0 \). Then an entire function \( f \neq c \) in \( C^n \) is a solution of the partial differential equation (7) if and only if \( f \) is a solution of \( Df(z) = d \).

The sufficiency of Theorem 3 is clear in view of (5) with \( f_{z_j} \) replaced by \( Df \). The proof of the necessity of Theorem 3 is identical to that of Theorem 2 by replacing \( f_{z_j} \) there with \( Df \). We thus omit the details.

Theorem 3 can yield some old and new Picard type theorems. In particular, Theorem 3 generalizes the result of Polya-Saxer-Milliox in the case \( n = 1 \) and thus provides a different approach to the Picard type theorem.
Corollary 4. Suppose that \( f \) is an entire function in \( \mathbb{C}^n \). If \( f \) omits 0 and \( Df \) omits 1, then \( Df \) is identically zero.

Examples. (i) Take \( n = 1 \) and \( Df = f^{(m)} \) in Corollary 4, then the conclusion \( Df \equiv 0 \) implies, by integration, that \( f \) is a polynomial and thus \( f \) must be constant since \( f \) omits 0. Thus, Corollary 4 gives the Picard type theorem of Polya-Saxer-Milliox that an entire function \( f \) must be constant if \( f \) omits 0 and \( f^{(m)} \) omits 1.

(ii) Take \( n = 1 \) and \( Df = f^{(m)} - f^{(m-1)} \), where \( m \) is any positive integer. Then under the condition of Corollary 4, we must have \( Df = 0 \), i.e., \( f^{(m+1)} - f^{(m)} = 0 \). Solving this linear differential equation directly, we obtain that \( f = c_0 + c_1 z + \cdots + c_{m-1} z^{m-1} + c_m z^m \), where \( c_j \)'s are constants. But, \( f \) does not assume 0; thus we must have that \( f = c_0 \), a constant.

(iii) While \( Df \) is identically zero in the conclusion of Corollary 4, the function \( f \) itself is not necessarily constant. Take \( f = e^{z_1 - z_2} \), a nonconstant entire function in \( \mathbb{C}^2 \). But, \( f \) omits 0 and \( Df := f_{z_1} + f_{z_2} \equiv 0 \) omits 1, satisfying the conditions of Corollary 4. The entire functions \( f \) are however characterized by the conclusion of Corollary 4, i.e., \( Df = f_{z_1} + f_{z_2} = 0 \). In fact, solving this partial differential equation directly by using its characteristic equations, one obtains that \( f = g(z_1 - z_2) \), where \( g \) is an entire function in \( \mathbb{C} \). But \( f \) does not assume 0; thus \( g = e^h \) for an entire function \( h \). Hence \( f = e^{h(z_1 - z_2)} \).

Proof of Corollary 4. For any fixed \( 1 \leq j \leq n \), the function \( a(z) := \frac{\partial Df}{\partial z_j} \) is entire. We have that
\[
\frac{\partial Df}{\partial z_j} = af(Df - 1).
\]
If \( a \neq 0 \), then by the necessary condition of Theorem 3, we have that \( Df = 1 \), a contradiction to the assumption that \( Df \) omits 1. Thus, \( a \equiv 0 \), i.e., \( \frac{\partial Df}{\partial z_j} = 0 \).

We have this equality for all \( 1 \leq j \leq n \), which implies that \( Df = C \) is constant. Since \( f \) omits 0 we have that \( f = e^g \) for an entire function \( g \) in \( \mathbb{C}^n \). By the definition of \( Df \), it is clear that the equality \( Df = C \) becomes \( e^g \partial g = C \), where \( Q \) is a polynomial in some partial derivatives of \( g \) with polynomial coefficients. We claim that \( Q \) must be identically zero. If not, \( g \) then cannot be constant by the definition of \( Q \) and we can write the above equality to \( e^g = \frac{Q}{\partial g} \), which implies, by applying Properties I and II, that \( m(r, e^g) = m(r, \frac{Q}{\partial g}) = O(m(r, g)) + O(\log r) \) as \( r \to \infty \). But, whenever \( g \) is not constant it always holds that \( m(r, g) = o(T(r, e^g)) \) by the several complex variable version of the Clunie’s lemma (see [3], p.88). We then have that \( m(r, e^g) = o(m(r, e^g)) + O(\log r) \). This is impossible unless \( g \) is constant, a contradiction. This shows the claim, i.e., \( Q = 0 \) and thus \( C = 0 \). Therefore, \( Df \) is identically zero. \( \square \)

References

[1] L. V. Ahlfors, Conformal Invariants: topics in geometrical function theory, McGraw-Hill, New York, 1973.
[2] E. Borel, Sur les zéros des functions entières, *Acta Math.* 20 (1897) 357-396.

[3] D. C. Chang, B. Q. Li and C. C. Yang, On composition of meromorphic functions in several complex variables, *Forum Mathematicum* 7(1995), 77-94.

[4] B. Davis, Picard’s theorem and Brownian motion, *Trans. Amer. Math. Soc.* 23 (1975) 353-362.

[5] W. H. J. Fuchs, *Topics in the Theory of Functions of One Complex Variable*, D. Van Nostrand, Princeton, NJ, 1967.

[6] W. K. Hayman, Picard Values of Meromorphic Functions and their Derivatives, *Annals of Mathematics*, 70(1959), 9-42.

[7] J.L. Lewis, Picard’s theorem and Richman’s theorem by way of Harnack’s inequality, *Proc. Amer. Math. Soc.* 122(1994), 199-206.

[8] B.Q. Li, On Picard’s theorem, *J. Math. Anal. Appl.* 460(2018), 561-564.

[9] B.Q. Li, A logarithmic derivative lemma in several complex variables and its applications, *Trans. Amer. Math. Soc.* 363 (2011), 6257-6267.

[10] H. Milloux, Extension d’un théorème de M.R. Nevanlinna et applications, *Act. Scient. et Ind.* no. 888, 1940.

[11] W. Saxer, Sur les valeurs exceptionelles des dérivées successives des fonctions méromorphes, *C.R. Acad. Sci. Paris* 182(1926), 831-833.

[12] S.L. Segal, *Nine Introductions in Complex Analysis*, North Holland, 1981.

[13] A. Vitter, The lemma of the logarithmic derivative in several complex variables, *Duke Math. J.* 44(1977), 89-104.

[14] G.Y. Zhang, Curves, domains and Picard’s theorem, *Bull. London Math. Soc.* 34(2002), 205-211.

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