A Quadratic Optimization Framework for Credit Portfolio

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Abstract. A novel quadratic optimization framework for credit portfolio is introduced when the portfolio risk is measured by Conditional Value-at-Risk (CVaR). This method is formulated in terms of the Lagrange multiplier method subjected under an artificial quadratic error term, which is comparable to the amount or cost of total portfolio adjustment, as the necessary constraint. The route toward the optimal portfolio state can be searched from the initial portfolio state via a continuation process through the maximally three-parameter space described by the total portfolio budget, the increment of the total return or the tolerance of the additional risk, and the total portfolio adjustment cost.

Keywords: credit portfolio, conditional value-at-risk, quadratic optimization, Lagrange multiplier method, dimensional reduction, derivative-at-risk, return-to-risk index.

1. Introduction

The loss distribution of a credit portfolio is usually far from the standard Gaussian. Rather, it is highly non-symmetric and fat-tailed with large skewness and kurtosis. This implies that the usual mean–variance analysis is not suitable for the credit portfolio optimization.

Such an optimization can be formulated as a linear programming (LP) problem \([1, 2, 3, 4]\) when the risk measure is Conditional Value-at-Risk (CVaR) as a “natural coherent alternative to VaR” (see \([5, 6, 7, 8]\)). However, the portfolio optimization is essentially a nonlinear problem if its risk is measured by any type of Value-at-Risk (VaR), so it is just natural to employ a nonlinear method to handle such a problem.

The standard implementation of a nonlinear portfolio optimization problem, as it appears, may be cumbersome when there are too many assets or asset groups in a portfolio because the associated Jacobian matrix is usually dense for any portfolio of a fully nontrivial asset correlation. Moreover, we may also have to deal with non-smooth or discontinuous loss distributions, in which cases discontinuous partial derivatives in the Jacobian matrix should be properly defined.

In this paper, we newly introduce an alternative optimization framework for the credit portfolio optimization by maximizing the return-to-risk index when CVaR is used as the risk measure. The formulation introduced here is expressed in terms of the Lagrange multiplier method subjected under an artificially introduced small enough error term, comparable to the total infinitesimal change of asset or asset-group allocation in a portfolio, as the necessary constraint, so this optimization framework will be referred to as a quadratic optimization method.

The essential idea is to define a map between the weight distribution space, which may be high-dimensional for any portfolio optimization, to a three-dimensional parameter space spanned by the total portfolio budget, the total portfolio return or the total portfolio risk, and the total reallocation cost: this mapping process is a sort of dimensional reduction. Throughout the overall optimization procedure, this new method makes the use of the Derivative-at-Risk, denoted by DaR, of which key properties are discussed in \([9]\).
2. Preliminaries

2.1. Time series of a portfolio

Suppose that a credit portfolio consists of $N$ different assets or asset groups for $N \geq 2$. Asset groups are characterized by a collection of assets which are completely or closely related (i.e. the correlation between assets in an asset group is considered to be close to the unity) within themselves. Let us say that a random variable $X$ represents the value of the total portfolio. The value of each asset or asset group is denoted by random variables $X^{(j)}_k$ for $j = 1, 2, \cdots, N$, where $k$ denotes a parameter representing the time step.

A time series for a virtual scenario of portfolio, starting from the initial portfolio allocation, can be simply expressed in the following matrix form, which has $(N+1)$ columns if the last column is the value of the total portfolio:

\[
\begin{bmatrix}
X^{(1)}_0 & X^{(2)}_0 & \cdots & X^{(N)}_0 & X_0 \\
X^{(1)}_1 & X^{(2)}_1 & \cdots & X^{(N)}_1 & X_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
X^{(1)}_k & X^{(2)}_k & \cdots & X^{(N)}_k & X_k \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}
\]

(2.1)

where $k = 0$ means the initial or present status and $X_k = \sum_{j=1}^{N} X^{(j)}_k$. A finite or possibly the infinite number of rows correspond to time steps within a time horizon, for instance, 6 months, 1 year, and 2 years, etc., when the values of each assets or asset groups are estimated. Without the loss of generality, we may assume that the first $N$ columns are linearly independent, namely, there is no asset or asset group of our interest that is a composite of the other assets or asset groups.

2.2. Return

In real practices, the return is closely related with the interest rate. Let us say that the return of each asset or asset group is denoted by $r^{(j)}$ for $j = 1, 2, \cdots, N$. Then, the relation between individual returns and the total return is simply given by

\[
r = \sum_{j=1}^{N} r^{(j)} w^{(j)},
\]

(2.2)

where $w^{(j)}_0 = \frac{X^{(j)}_0}{X_0}$ is the initial weight of the $j$-th asset or asset group for $j = 1, 2, \cdots, N$ with respect to the total portfolio. For the convenience of sake, the return of each asset or asset group is assumed to be unchanged during the optimization process, but it may be actually adjusted in the process of portfolio reallocation, in general.

2.3. Probability distribution of loss

The loss distribution of a portfolio is obtained by counting the frequencies of the loss in the time series, where the loss of the $j$-th individual asset or asset group is defined by $L^{(j)}_k = X^{(j)}_0 - X^{(j)}_k$ for $j = 1, 2, \cdots, N$ and $k = 1, 2, \cdots$ (accordingly, $L_k = X_0 - X_k$ for the whole portfolio). The correlations between different assets or asset groups are naturally deduced from the set of times series. The key
assumption is that there exist the convergent loss distribution function and the associated correlation coefficients, which are fixed, regardless of the proportion of each asset or asset group in a portfolio.

2.4. Risk measures

2.4.1. Value-at-risk (VaR)

VaR of the total portfolio for its loss $Z = X_0 - X$ is defined in terms of its loss distribution function $P(\cdot)$ by

$$ \text{VaR}_\beta^{(\text{loss})}(X) \equiv \text{VaR}_\beta(Z) = \inf_{Y \in \mathcal{A}} \{ P(Z \mid Z \leq Y) \geq \beta \} \quad (2.3) $$

for a given confidence level $0 \leq \beta \leq 1$, where $\mathcal{A}$ is the admissible set of the portfolio loss, typically the set of real numbers. Typical values for the confidence level $\beta$ used in practice are $0.95$, $0.99$, $0.995$, $0.999$, etc. within a given time horizon. There are also many other risk measures associated with VaR.

2.4.2. Conditional value-at-risk (CVaR) for loss

CVaR of the total portfolio is defined in terms of the underlying VaR by

$$ \text{CVaR}_\beta^{(\text{loss})}(X) = \frac{\text{CVaR}_\beta^{(\text{loss})+}(X) + (\beta^* - \beta)\text{VaR}_\beta^{(\text{loss})}(X)}{1 - \beta}, \quad (2.4) $$

where

$$ \beta^* = P \left( Z \mid Z < \text{VaR}_\beta^{(\text{loss})}(X) \right), \quad (2.5a) $$

$$ \text{CVaR}_\beta^{(\text{loss})+}(X) = E \left[ Z \mid Z \geq \text{VaR}_\beta^{(\text{loss})}(X) \right] = \int_{\text{VaR}_\beta^{(\text{loss})}}^{+\infty} ZP(Z)dZ. \quad (2.5b) $$

Note that $\text{CVaR}_\beta^{(\text{loss})+}(\cdot)$ is called the tail conditional expectation. Either when the loss distribution is continuous or when $\beta$ does not split any of atoms in the loss distribution, in particular, we have $\beta^* = \beta$, so that $\text{CVaR}_\beta^{(\text{loss})}(\cdot) = \text{CVaR}_\beta^{(\text{loss})+}(\cdot)$. Obviously, $\text{CVaR}_\beta^{(\text{loss})}(\cdot)$ is strictly greater than or equal to $\text{VaR}_\beta^{(\text{loss})}(\cdot)$.

2.4.3. Risk contribution and Derivative-at-Risk (DaR)

For the purpose of portfolio optimization, it is useful to consider the risk contributions of individual assets or asset groups with respect to the underlying risk measure of the total portfolio. This is feasible if the total risk is measured from a time series of assets or asset groups in a portfolio.

When the total risk is measured by CVaR, the risk contribution of the $j$-th asset or asset group is given by

$$ \text{CVaR}_{\beta}^{(\text{loss})+(j)}(X) = E \left[ Z^{(j)} \mid Z = \text{CVaR}_{\beta}^{(\text{loss})}(X) \right], \quad (2.6) $$

where $Z^{(j)} = X^{(j)}_0 - X^{(j)}$ is the loss of the $j$-th asset or asset group for $j = 1, 2, \cdots, N$. Therefore, we should have

$$ \sum_{j=1}^{N} \text{CVaR}_{\beta}^{(\text{loss})+(j)}(X) = E \left[ Z \mid Z = \text{CVaR}_{\beta}^{(\text{loss})}(X) \right] = \text{CVaR}_{\beta}^{(\text{loss})}(X). \quad (2.7) $$
The underlying VaR-type risk measure is simple scale-invariant, namely, \( \text{CVaR}_\beta((1 + \delta a)X) = (1 + \delta a)\text{CVaR}_\beta(X) \), where \(|\delta a| \ll 1\), so it follows that

\[
\text{CVaR}^{(\text{loss})}_{\beta}(X) = X^{(j)} \frac{\partial \text{CVaR}_\beta(X)}{\partial X^{(j)}} = w^{(j)} \frac{\partial \text{CVaR}_\beta(X)}{\partial w^{(j)}},
\]

where the partial derivatives, named as Derivative-at-Risk (DaR), exist. In such a case, \( \frac{\partial \text{CVaR}_\beta(X)}{\partial w^{(j)}} \) exists in the distribution sense, in other words, in the sense of weak derivative.

Because CVaR is piecewise linear with respect to the proportion of individual assets or asset groups in a portfolio, its associated partial derivatives are all piecewise constant functions. Accordingly, the relation between CVaR and DaR is derived as follows:

\[
\text{CVaR}^{(\text{loss})}_{\beta}(X) = \sum_{j=1}^{N} w^{(j)} \text{DaR}^{(j)}_{\beta}(X),
\]

(2.9)

where \( \text{DaR}^{(j)}_{\beta}(X) = \frac{\partial \text{CVaR}^{(\text{loss})}_{\beta}(X)}{\partial w^{(j)}} \).

2.5. Return-to-risk index: return on risk-adjusted capital (RORAC)

It is customary that we may expect more return from more risk. In order to measure the performance level of assets, asset groups, and the whole portfolio, it is desirable to adopt a performance measure that is defined by the ratio between the return and the risk as follows:

\[
\mathcal{I} \equiv \frac{rX_0}{\text{CVaR}^{(\text{loss})}_{\beta}(X)},
\]

(2.10a)

\[
\mathcal{I}_j \equiv \frac{r^{(j)}X^{(j)}_0}{\text{CVaR}^{(\text{loss})}_{\beta}(X^{(j)})} \quad \text{for} \quad j = 1, 2, \ldots, N,
\]

(2.10b)

for the total portfolio and each individual asset or asset group, respectively. This index is typically a nonlinear function of the weight of individual asset or asset group for a general risk measure, but it becomes a rational function made of a ratio between two piecewise linear functions with respect to the weight functions for CVaR.

2.5.1. Relation between DaR and the individual returns at the optimal return-to-risk index

Let us now examine a property that relates DaR and the returns at the optimal return-to-risk index when CVaR is used for the risk measure. Assuming that the return-to-risk index is a differentiable function of the weights even in the distribution sense, we should have

\[
\frac{\partial \mathcal{I}}{\partial w^{(j)}} = 0 \iff \frac{r^{(j)}}{r} = \frac{\text{DaR}^{(j)}_{\beta}(X)}{\text{CVaR}^{(\text{loss})}_{\beta}(X)},
\]

(2.11)

for \( j = 1, 2, \ldots, N \) via the weighted summation at the optimal state.
2.6. Diversification index

The ratio of the total portfolio risk with respect to the sum of the risk of assets or asset groups is defined as the diversification index $D_\beta(X)$:

$$D_\beta(X) = \frac{\text{CVaR}_\beta^{(\text{loss})}(X)}{\sum_{j=1}^{N} \text{CVaR}_\beta^{(\text{loss})}(X^{(j)})} = \frac{\sum_{j=1}^{N} w^{(j)} \text{DaR}_{\beta}^{(j)}(X)}{\sum_{j=1}^{N} \text{CVaR}_\beta^{(\text{loss})}(X^{(j)})}. \quad (2.12)$$

This quantity measures the degree of correlation between the loss distributions of individual assets or asset groups. As a special case, if the assets or asset groups are completely correlated, then they all behave like a single asset such that $\text{CVaR}_{\beta}^{(\text{loss})}(X^{(j)}) = \text{CVaR}_{\beta}^{(\text{loss})}(X)$ for $j = 1, 2, \cdots, N$, so that $D_\beta(X)$ should be 1, but this critical case is excluded in our discussion. In general, $D_\beta(X)$ is a positive number, nontrivially farther below from 1.

3. Quadratic optimization framework

The fundamental question is how to adjust the initial asset allocation in order to maximize or to minimize a given objective function. More specifically, what should be the most fair way to enhance the performance of a portfolio from the current asset allocation for a given amount of the total portfolio adjustment?

Let us begin our main discussion by denoting $w^{(j)}_+$ as the adjusted weight of the $j$-th asset of asset group in a portfolio after an infinitesimal amount of the asset or asset group adjustment process, $w^{(j)}_-$ as the original weight of the $j$-th asset of asset group in a portfolio, and $\delta w^{(j)}_+ = w^{(j)}_+ - w^{(j)}_-$ for $j = 1, 2, \cdots, N$. Also, we define that $w_+ = \left( w^{(1)}_+, w^{(2)}_+, \cdots, w^{(N)}_+ \right)^T$ and $\delta w_+ = \left( \delta w^{(1)}_+, \delta w^{(2)}_+, \cdots, \delta w^{(N)}_+ \right)^T$ for the whole portfolio: $X^{(j)}_+, X_+, r_+, \delta r_+, I^{(j)}_+, \delta I_+$ are defined all in the same fashion. It is additionally assumed that each weight component is nonzero, namely $w^{(j)}_+ \neq 0$ for $j = 1, 2, \cdots, N$, because there is no need to consider reallocating the assets or asset groups of no contribution for any practical purpose.

3.1. Objective functions

Assuming that the amount of weight adjustment is small enough, we introduce the following objective functions for the single-step optimization procedure in the forward sense:

- Risk minimization

$$\min_{\delta w} \frac{\delta \text{CVaR}_{\beta}^{(\text{loss})}(X_+)}{X_0}, \quad (3.1)$$

where

$$\delta \text{CVaR}_{\beta}^{(\text{loss})}(X_+) = \sum_{j=1}^{N} \text{DaR}_{\beta}^{(j)}(X_-) \delta w^{(j)}_+, \quad (3.2)$$
assuming that $\delta w^{(j)}$ is small enough so that the partial derivative $\frac{\partial \text{CVaR}^{(\text{loss})}(X)}{\partial w^{(j)}}$ stays unchanged before and after the adjustment of an infinitesimal asset allocation. Note that dividing by $X_0$ is just for the purpose of nondimensionalization.

- **Return maximization**

$$\max_{\delta w} \sum_{j=1}^{N} r^{(j)} \delta w^{(j)}.$$  \hspace{1cm} (3.3)

- **Alternative return-to-risk index maximization**

$$\max_{\delta w} \frac{\delta r_\pm}{\delta \text{CVaR}_{\beta}^{(\text{loss})}(X_\pm)} = \max_{\delta w} \frac{X_0 \sum_{j=1}^{N} r^{(j)} \delta w^{(j)}_\pm}{\sum_{j=1}^{N} \text{DaR}^{(j)}_{\beta}(X_-) \delta w^{(j)}_\pm}. \hspace{1cm} (3.4)$$

The following constraints may be specified in the optimization procedure:

- **Budget constraint:**

$$\sum_{j=1}^{N} \delta w^{(j)} = \delta \alpha$$ \hspace{1cm} (3.5)

for a constant $\delta \alpha$, where

$$\sum_{j=1}^{N} w^{(j)}_- = 1, \quad \sum_{j=1}^{N} w^{(j)}_+ = 1 + \delta \alpha.$$ \hspace{1cm} (3.6)

- **Constraint on the total return (except for return maximization) in the forward sense:**

$$\sum_{j=1}^{N} r^{(j)} \delta w^{(j)}_\pm = \delta \gamma,$$ \hspace{1cm} (3.7)

where

$$\sum_{j=1}^{N} r^{(j)} w^{(j)}_\pm = r_\pm.$$ \hspace{1cm} (3.8)

- **Constraint on the total risk (except for risk minimization):**

$$\frac{1}{X_0} \sum_{j=1}^{N} \text{DaR}^{(j)}_{\beta}(X_-) \delta w^{(j)}_\pm = \delta \gamma.$$ \hspace{1cm} (3.9)
• Constraint on (weighted) adjustment cost:

\[
\sum_{j=1}^{N} c^{(j)} \delta w_{\pm}^{(j)} = \delta c^2, \quad (3.10)
\]

where \( c^{(j)} \) is the coefficient of adjustment cost of each asset or asset group for \( j = 1, 2, \cdots, N \).

We may define the cost constraint in the \( l_p \) sense for any \( p > 0 \), but \( p = 2 \) is chosen, according to which the optimization method is referred to be quadratic.

In this method, the adjustment cost constraint must be taken for any types of optimization whereas the other types of constraints may be excluded. For the optimization of the return-to-risk indices, either constraint on the total risk and/or on the total return can be specified. The coefficients of adjustment cost may be subject to further modeling.

3.2. Return maximization or risk minimization

3.2.1. Linearized optimization under the constraints of the infinitesimal amount of asset adjustments

A linearized optimization framework is now introduced, which becomes a basic building block to perform the quadratic optimization for multiple time intervals. The whole optimization procedure is completed by a continual sequence of linearized optimizations.

Each step of linearized optimizations is formulated in terms of the Lagrange multiplier method. When the cost adjustment constraint is defined in the sense of \( l_2 \) norm, the Lagrange multiplier formulation is written by

\[
L = \sum_{j=1}^{N} f^{(j)} \delta w_{\pm}^{(j)} - \left( \sum_{j=1}^{N} \delta w_{\pm}^{(j)} - \delta \alpha \right) s - \left( \sum_{j=1}^{N} h^{(j)} \delta w_{\pm}^{(j)} - \delta \gamma \right) t - \left( \sum_{j=1}^{N} c^{(j)} \delta w_{\pm}^{(j)}^2 - \delta c^2 \right) p, \quad (3.11a)
\]

where

\[
f^{(j)} = \begin{cases} r^{(j)} & \text{for return maximization,} \\ \frac{\text{DaR}_\beta^{(j)}(X_-)}{X_0} & \text{for risk minimization,} \end{cases} \quad (3.11b)
\]

\[
h^{(j)} = \begin{cases} \frac{\text{DaR}_\beta^{(j)}(X_-)}{X_0} & \text{for return maximization,} \\ \frac{r^{(j)}}{X_0} & \text{for risk minimization,} \end{cases} \quad (3.11c)
\]

\[
\delta \gamma = \begin{cases} \frac{\text{CVaR}_\beta^{(i)}(X_-)}{X_0} & \text{for return maximization,} \\ \frac{\delta r_{\pm}}{X_0} & \text{for risk minimization,} \end{cases} \quad (3.11d)
\]

\( \delta \alpha \) is the budget adjustment ratio and \( c^{(j)} \) is a measure for the total cost of weight adjustments in the \( l_2 \) sense for \( j = 1, 2, \cdots, N \). We assume that the initial weight of each asset or asset group is nonzero, namely, \( w_{0\pm}^{(j)} \neq 0 \) for \( j = 1, 2, \cdots, N \).

Then, the single-step optimization is solved as follows:

\[
\frac{\partial L}{\partial \delta w_{\pm}^{(j)}} = f^{(j)} - s - h^{(j)} t - 2c^{(j)} \delta w_{\pm}^{(j)} p = 0 \quad \text{for} \quad j = 1, 2, \cdots, N, \quad (3.12a)
\]
\[
\frac{\partial L}{\partial s} = \sum_{j=1}^{N} \delta w_{\pm}^{(j)} - \delta \alpha = 0, \tag{3.12b}
\]
\[
\frac{\partial L}{\partial t} = \sum_{j=1}^{N} h^{(j)} \delta w_{\pm}^{(j)} - \delta \gamma = 0, \tag{3.12c}
\]
\[
\frac{\partial L}{\partial p} = \sum_{j=1}^{N} c^{(j)}^2 \delta w_{\pm}^{(j)}^2 - \delta c^2 = 0, \tag{3.12d}
\]

hence,
\[
\delta w_{\pm}^{(j)} = \frac{f^{(j)} - \frac{h^{(j)}}{2pc^{(j)}^2}}{2pc^{(j)}^2} \quad \text{for} \quad j = 1, 2, \ldots, N, \tag{3.13a}
\]
\[
\begin{pmatrix}
U \\
V \\
W
\end{pmatrix}
\begin{pmatrix}
s \\
t
\end{pmatrix}
= \begin{pmatrix}
G - 2p\delta \alpha \\
H - 2p\delta \gamma
\end{pmatrix}, \tag{3.13b}
\]
\[
s = \frac{(G - 2p\delta \alpha)W - (H - 2p\delta \gamma)V}{UW - V^2} = \frac{2(\delta \gamma V - \delta \alpha W)p + GW - HV}{UW - V^2}, \tag{3.13c}
\]
\[
t = \frac{(H - 2p\delta \gamma)U - (G - 2p\delta \alpha)V}{UW - V^2} = \frac{2(\delta \alpha V - \delta \gamma U)p + HU - GV}{UW - V^2}, \tag{3.13d}
\]

where
\[
U = \sum_{j=1}^{N} \frac{1}{c^{(j)}^2}, \tag{3.14a}
\]
\[
V = \sum_{j=1}^{N} \frac{h^{(j)}}{c^{(j)}^2}, \tag{3.14b}
\]
\[
W = \sum_{j=1}^{N} \frac{h^{(j)}^2}{c^{(j)}^2}, \tag{3.14c}
\]
\[
G = \sum_{j=1}^{N} \frac{f^{(j)}}{c^{(j)}^2}, \tag{3.14d}
\]
\[
H = \sum_{j=1}^{N} \frac{f^{(j)}h^{(j)}}{c^{(j)}^2}. \tag{3.14e}
\]

Therefore, we have
\[
\delta w_{\pm}^{(j)} = \begin{cases} 
\frac{f^{(j)} - \frac{2(\delta \gamma V - \delta \alpha W)p + GW - HV}{UW - V^2}}{2pc^{(j)}^2} & \text{if } w_{\pm}^{(j)} \neq 0, \\
0 & \text{if } w_{\pm}^{(j)} = 0,
\end{cases} \tag{3.15a}
\]
\[
Q = \sum_{j=1}^{N} f^{(j)} \delta w_{\pm}^{(j)} = \frac{F - \frac{H^2U + G^2W - 2GHV}{UW - V^2}}{2p} + \frac{(GW - HV)\delta \alpha + (HU - GV)\delta \gamma}{UW - V^2}, \tag{3.15b}
\]
where

\[
F = \sum_{j=1}^{N} \frac{f(j)^2}{c(j)^2}.
\]  

(3.16)

Substituting (3.15a) to (3.12d), the condition for \( p \) in terms of a quadratic equation is obtained by

\[
4a_2p^2 + a_0 = 0,
\]  

(3.17)

where

\[
a_2 = \frac{U\delta\gamma^2 + W\delta\alpha^2 - 2V\delta\alpha\delta\gamma}{UW - V^2} - \delta c^2,
\]  

(3.18a)

\[
a_0 = F - \frac{G^2W + H^2U - 2GHV}{UW - V^2},
\]  

(3.18b)

upon beautiful simplification. It turns out that the linear term in the quadratic equation for \( p \) vanishes out. Also, Note that

\[
UW \geq V^2,
\]  

(3.19a)

\[
UF \geq G^2,
\]  

(3.19b)

\[
WF \geq H^2,
\]  

(3.19c)

hence,

\[
U\delta\gamma^2 + W\delta\alpha^2 - 2V\delta\alpha\delta\gamma \geq 0,
\]  

(3.20a)

\[
G^2W + H^2U - 2GHV \geq 0.
\]  

(3.20b)

\( a_0 \) is always nonnegative because it is calculated by squaring all the terms that do not involve \( p \) for the expression of \( \delta w_\pm(j) \) in (3.15a).

It is required that

\[
\delta c^2 > \frac{U\delta\gamma^2 + W\delta\alpha^2 - 2V\delta\alpha\delta\gamma}{UW - V^2}
\]  

(3.21)

in order to make \( p \) real-valued so that any possible degenerate cases, where there are infinitely many solutions, are excluded. \( \delta c \) should be always nonzero to proceed the optimization process when the portfolio is in any sub-optimal states even though the other two path parameters \( \delta \alpha \) and \( \delta \gamma \) are allowed to be zero.

Once we have two real solutions for \( p \), we choose only one of them so that it satisfies the condition of the objective function. In particular, when \( \delta \alpha = 0 \), this quadratic optimization problem is understood as finding the maximum and the minimum, which are unique respectively on the crosscut between an \((N - 1)\)-dimensional hyperplane and the surface of an \(N\)-dimensional ellipsoid centered at the origin. Accordingly, one of the increment of the objective function, denoted by \( \sum_{j=1}^{N} f(j) \delta w_\pm(j) \), should be positive whereas the other should be negative. (Indeed, both increments of the objective function may have the same sign if \( \delta \alpha \) is nonzero.) In any cases, we take the larger increment of the objective function to find the desired local optimal state for the return maximization and the smaller one for the risk minimization. Note that the range of \( \delta w_\pm(j) \) for \( j = 1, 2, \cdots, N \) stays positive real-valued.
3.2.2. Extension to numerical continuation
The quadratic optimization is conducted by a sequence of aforementioned linearized optimizations for infinitesimal adjustment costs. For given constraints, all parameters are divided into multiple subdivisions that make a discretized approximation associated with a pre-assigned numerical continuation path, expressed by the following increasing sequences:

\[ \{(\alpha_l, \gamma_l, c_l) \mid 0 = \alpha_0 < \alpha_l < \alpha_M = \alpha, 0 = \gamma_0 < \gamma_l < \gamma_M = \gamma, 0 = c_0 < c_l < c_M \}_{l=0}^{M}, \]

where \((\delta\alpha_l, \delta\gamma_l, \delta c_l) = (\alpha_l - \alpha_{l-1}, \gamma_l - \gamma_{l-1}, c_l - c_{l-1})\) for \(l = 1, 2, \ldots, M\) should be small enough for each single-step linearized optimization, so that it can satisfy the increment constraint (3.21).

At the end of each single-step optimization between subintervals of the continuation path, the total risk of the whole portfolio, as well as the risk contribution of each asset or asset group, should be updated by re-sorting the time series of the adjusted total portfolio. When the iterative procedures go through all subdivisions in the cost parameters, a numerical continuation for the quadratic optimization is completed.

4. Optimal paths
For optimal paths, the weight adjustment parameter \(c\) is the only independent variable, and the other two parameters, \(\alpha\) and \(\gamma\) should be subject to \(c\).

4.1. Parameter curve maximizing the return or minimizing the risk
4.1.1. Non-fixed total budget
In order to determine the parameter curve maximizing the return or minimizing the risk, we calculate the sensitivity of the objective function \(Q\) with respect to three path parameters \((\delta\alpha, \delta\gamma, \delta c)\) as follows:

\[
\frac{\partial Q}{\partial \delta\alpha} = \pm \frac{W \delta\alpha - V \delta\gamma}{UW - V^2} A \sqrt{\frac{UW - V^2}{UW - V^2}} + \frac{GW - HV}{UW - V^2},
\]

(4.1a)

\[
\frac{\partial Q}{\partial \delta\gamma} = \pm \frac{W \delta\alpha - V \delta\gamma}{UW - V^2} A \sqrt{\frac{UW - V^2}{UW - V^2}} + \frac{GV - HU}{UW - V^2},
\]

(4.1b)

\[
\frac{\partial Q}{\partial \delta c} = \pm A \delta c \sqrt{\frac{UW - V^2}{UW - V^2}},
\]

(4.1c)

where

\[ A = \sqrt{a_0}. \]

Hence, the direction of the steepest ascent path is determined by

\[(\delta\alpha, \delta\gamma, \delta c) \parallel \left( \frac{\partial Q}{\partial \delta\alpha}, \frac{\partial Q}{\partial \delta\gamma}, \frac{\partial Q}{\partial \delta c} \right), \]

(4.3)

equivalently,

\[
(UW - V^2)\kappa_1 = V \kappa_2 - W \kappa_1 \pm \frac{HV - GW}{A} \sqrt{1 - \frac{U \kappa_1^2 + W \kappa_1^2 - 2V \kappa_1 \kappa_2}{UW - V^2}},
\]

(4.4a)

\[
(UW - V^2)\kappa_2 = V \kappa_1 - U \kappa_2 \pm \frac{HU - GV}{A} \sqrt{1 - \frac{U \kappa_1^2 + W \kappa_1^2 - 2V \kappa_1 \kappa_2}{UW - V^2}},
\]

(4.4b)
where
\[ \kappa_1 = \frac{\delta \alpha}{\delta c}, \quad \kappa_2 = \frac{\delta \gamma}{\delta c}. \] (4.5)

Therefore, we should have
\[ \kappa \equiv \frac{\kappa_2}{\kappa_1} = \frac{\delta \gamma}{\delta \alpha} = \frac{(HV - GW)V + (UW - V^2 + W)(HU - GV)}{(HU - GV)V + (UW - V^2 + U)(HV - GW)}, \] (4.6a)

\[ \kappa_1 = \pm \frac{1}{\sqrt{\frac{(UW - V^2 + W - 2V\kappa)^2}{(HV - GW)^2}a_0 + \frac{U\kappa^2 + W - 2V\kappa}{UW - V^2}}} \] (4.6b)

We take the sign in the way that the objective function should be the most maximized or minimized.

4.1.2. Fixed total budget: \( \delta \alpha \equiv 0 \)

Setting \( \kappa_1 \equiv 0 \) from (4.4b), we have
\[ (UW - V^2)\kappa_2 = -U\kappa_2 + \frac{HU - GV}{A} \sqrt{1 - \frac{U\kappa_2^2}{UW - V^2}}, \] (4.7)

hence,
\[ \kappa_2 = \pm \frac{1}{\sqrt{\frac{(UW - V^2 + U)^2}{(HU - GV)^2}a_0 + \frac{U}{UW - V^2}}} \] (4.8)

Likewise, the criterion to choose the plus or minus sign is the same as before.

4.1.3. Fixed total return or risk: \( \delta \gamma \equiv 0 \)

Setting \( \kappa_2 \equiv 0 \) from (4.4a), we have
\[ (UW - V^2)\kappa_1 = -W\kappa_1 + \frac{HV - GW}{A} \sqrt{1 - \frac{W\kappa_1^2}{UW - V^2}}, \] (4.9)

hence,
\[ \kappa_1 = \pm \frac{1}{\sqrt{\frac{(UW - V^2 + W)^2}{(HV - GW)^2}a_0 + \frac{W}{UW - V^2}}} \] (4.10)

Again, the sign choice criterion is identical.

4.2. Parameter curve maximizing the return-to-risk index

The parameter curve maximizing the return-to-risk index may not be the same as the one with the return maximization or with the risk minimization because the return does not have to be maximized or the risk does not have to be minimized in order to maximize the return-to-risk index along a
parameter path. The parameter path aligned with the steepest ascent direction of the return-to-risk index (3.4) is determined by

$$
\delta w_{\pm}^{(j)} \parallel r^{(j)} \text{CVaR}_\beta(X_-) - r_- \text{DaR}_\beta^{(j)}(X_-)
$$

(4.11)

for \( j = 1, 2, \ldots, N \), which imposes an additional constraint that \( \sum_{j=1}^N w^{(j)}_j \) is constant, whether the objective of optimization is the return maximization or the risk minimization. Therefore, we have

$$
\delta w_{\pm}^{(j)} = \frac{\text{CVaR}_\beta(X_-) r^{(j)} - r_- \text{DaR}_\beta^{(j)}(X_-)}{\text{CVaR}_\beta(X_-) \sum_{j=1}^N r^{(j)} - r_- \sum_{j=1}^N \text{DaR}_\beta^{(j)}(X_-)} \delta \alpha
$$

(4.12)

for \( j = 1, 2, \ldots, N \), where

$$
\delta \gamma = \left\{ \begin{array}{ll}
\sum_{j=1}^N \left\{ \text{CVaR}_\beta(X_-) r^{(j)} - r_- \text{DaR}_\beta^{(j)}(X_-) \right\} \text{DaR}_\beta^{(j)}(X_-) \delta \alpha & \text{under the total risk constraint,} \\
\sum_{j=1}^N \left\{ \text{CVaR}_\beta(X_-) r^{(j)} - r_- \text{DaR}_\beta^{(j)}(X_-) \right\} r^{(j)} X_0 & \\
\sum_{j=1}^N \left\{ \text{CVaR}_\beta(X_-) r^{(j)} - r_- \text{DaR}_\beta^{(j)}(X_-) \right\} \delta \alpha & \text{under the total return constraint,} \\
\sum_{j=1}^N \left\{ \text{CVaR}_\beta(X_-) r^{(j)} - r_- \text{DaR}_\beta^{(j)}(X_-) \right\}^2 c^{(j)} & \\
\sum_{j=1}^N \left\{ \text{CVaR}_\beta(X_-) r^{(j)} - r_- \text{DaR}_\beta^{(j)}(X_-) \right\}^2 c^{(j)} & \end{array} \right. 
$$

(4.13a)

Two different signs correspond to maximizing or minimizing the increment of the objective functions, respectively, because the Lagrange multiplier method itself does not distinguish those two. We need to take only one of the signs depending on the purpose of optimization.

### 5. Numerical results from actual credit portfolio data

We apply this quasilinear optimization method for a real corporate credit portfolio data obtained from Hana Bank. The bank’s credit portfolio risk data are produced by CreditMetrics. The loss distribution for the credit portfolio is provided during one year time horizon.

Figure 1 and Figure 2 show the return-to-risk index maximization result, with CVaR as the risk measure, following the optimal path under the condition of non-fixed total budget in Section 4.2. The total loss distribution goes far shifted to the left as the optimization proceeds. For the purpose of the return-to-risk maximization, the total budget seems to keep decreasing. Although the total return attains the peak value when the total weight adjustment is about 0.025 and then decreases, the total risk clearly is reduced (See Figure 7). The return-to-risk index curve in Figure 7a is supposedly the efficient frontier, which is the ultimate upper bound of all possible such curves starting from the same initial portfolio state, within the range of numerical errors.

In Figure 3 and Figure 4, the CVaR minimization result is provided following the parameter path under the condition of the fixed total budget in Section 4.1.2. Once the total weight adjustment grows to be about 0.016, the portfolio reaches a steady state for both of the total return and the total risk, so those curves sharply flatten (See Figure 7). In Figure 5 and Figure 6, the CVaR minimization result is presented assuming that the total return is constant by following the path as in Section 4.1.3. Similar to the case of CVaR minimization under the fixed total budget condition, the return-to-risk index smoothly flattens when the value of total weight adjustment reaches about 0.023. The convergence
results shown for the risk minimization are specific to a certain initial portfolio state and may not be generally expected.

The numerical procedure is first-order accurate with respect to $\delta c$. Nevertheless, it can be exact as long as all discontinuities of DaR are precisely identified. The actual total cumulative numerical error bound for this numerical scheme is estimated by

$$
\epsilon_{\text{max}} = O \left( \sum_{l=1}^{M} \sum_{j=1}^{N} \left| \left( \text{DaR}_{\beta}^{(j)}(X_l) - \text{DaR}_{\beta}^{(j)}(X_{l-1}) \right) \left( w^{(j)}_l - w^{(j)}_{l-1} \right) \right| \right).
$$

(5.1)

Table I. Total cumulative numerical errors.

| Objective Function                               | Total Cumulative Numerical Error |
|--------------------------------------------------|----------------------------------|
| Return-to-risk maximization                      | 0.001365                         |
| Risk minimization (fixed total budget)           | 0.0009469                        |
| Risk minimization (fixed total return)           | 0.0009559                        |

6. Discussion and concluding remarks

Some criticism may arise at the preliminary stage that we do not deal with individual distribution functions for the purpose of portfolio optimization: the initial portfolio data at hand is just a realized or projected instance of all possible plausible scenario for the behavior of a portfolio. However, our approach here is acceptable, as it is, in the sense that the main objective is to find the best possible way to adjust the amount of each asset or asset group in a portfolio only on the basis of the realized history or a virtual scenario of the portfolio. The essentially same idea has been earlier used in [1, 2, 3, 4] as well, so we do not intend that our discussion goes further to get an ultimate answer for the portfolio optimization starting from the pre-assigned distribution for the value of the whole portfolio along with a well-defined correlation between its component assets or asset groups. Even for the latter (ultimate) case, the portfolio distribution and its correlation should be modeled in most occasions, according to an empirical scenario for portfolio performance, otherwise pre-mentioned. Therefore, the way of our confined discussion is meaningful.

A small neighborhood around a certain portfolio state needed to proceed an optimization procedure can be any type of small ball measured by $l_p$ for $p > 0$ centered at the state in the weight distribution space. When $p = 1$, the Lagrange multiplier method is nothing but the usual LP formulation. When $p = 2$, which is the case in this paper, the required analytical procedure precisely ends up with explicit a quadratic closed form, which greatly enhances the computational cost. The size of ellipsoid should be taken small enough, in principle, to search the right optimal path even if an occasional jumping over happens between piecewise linear hyperplane segments of different normal vectors. However, taking too much small ellipsoid may cause the computational time to be excessive if every such jumping over is tried to be avoided.

The path independence of the optimal solution in the parameter space is not necessarily guaranteed, in general, because the optimal solution path may vary depending on how to choose the parameter path. This is more or less natural when the number of assets or asset groups in a portfolio is large enough greater than three because this optimization procedure is conducted through a dimensional
Figure 1. The change of the loss distribution along the return-to-risk index maximization procedure when the total weight adjustment is 0, 0.05, and 0.1.
Figure 2. The change of the weight distribution of the credit portfolio before and after the return-to-risk index maximization when the total weight adjustment is 0.1.

reduction, mapping from the weight distribution space of assets or asset groups in a portfolio, which is spanned by \( (w^{(j)})_{j=1}^N \), to the three-dimensional parameter space of the total weight adjustment, the amount of the total return or risk change, and the total reallocation cost, which is spanned by \((\alpha, \gamma, c)\).

The perfect smoothness of the optimal solution path is not necessarily guaranteed, either, when the probability distribution function is made of a discrete set of raw time series. This affects the non-smoothness of the optimal state functions, such as the return-to-risk ratio, the total portfolio risk, the total portfolio return, and the diversification index, etc., with respect to the amount of change in the total portfolio adjustment cost.

The reallocation costs for individual assets or asset groups should be understood all comparable each other in the sense that

\[
0 \ll \frac{c^{(j)}}{c^{(i)}}
\]

for all pairs of \((i, j)\) where \(i, j = 1, 2, \ldots, N\). The reallocation cost may be affected from various factors, such as the liquidity of each asset or asset group, reallocation or transaction fee, and so on.
Figure 3. The change of the loss distribution along the risk minimization procedure when the total weight adjustment is 0, 0.05, and 0.1 under the constraint of the fixed total budget.

In case that the reallocation cost is a nontrivial function of the weight, we may need to justify that the reallocation cost is slowly varied with respect to the amount of weight readjustment.

The optimization framework discussed in this paper is valid for general other types of statistical information, not only for the time series of which probability distribution is skewed or fat-tailed. This method can be a useful replacement for various portfolio optimization problems, which do not belong to the proper realm of the mean–variance analysis.

The return maximization may not always give the optimal results in the desired directions of risk states. Depending on the risk structure of portfolio, maximizing the return may sometimes lead to unfavorable excessive increase of the total portfolio risk, and vice versa.

The cost minimization as an objective in the optimization procedure and the dependence of optimal states on the confidence level are definitely worth further investigation. There are much room for in-depth studies, as well, when stochastic returns and non-static risk distributions are considered.
Acknowledgements

This paper was initiated from a credit portfolio optimization project conducted at the Hana Bank, Korea from April to July in 2008. The author wishes to thank Prof. Frank C. Park, Roger H. Kim, Junsuk Her, and Prof. Stanislav Uryasev for their kind collaboration and discussion with the author.

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Figure 6. The change of the weight distribution of the credit portfolio before and after the risk minimization under the constraint of the fixed total return when the total weight adjustment is 0.1.
Figure 7. Comparison between return-to-risk index maximization (−), risk minimization under the fixed total budget (−−), and risk minimization under the fixed total return (−−).