Homotopic Curve Shortening and the Affine Curve-Shortening Flow

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Abstract
We define and study a discrete process that generalizes the convex-layer decomposition of a planar point set. Our process, which we call homotopic curve shortening (HCS), starts with a closed curve (which might self-intersect) in the presence of a set $P \subset \mathbb{R}^2$ of point obstacles, and evolves in discrete steps, where each step consists of (1) taking shortcuts around the obstacles, and (2) reducing the curve to its shortest homotopic equivalent.

We find experimentally that, if the initial curve is held fixed and $P$ is chosen to be either a very fine regular grid or a uniformly random point set, then HCS behaves at the limit like the affine curve-shortening flow (ACSF). This connection between HCS and ACSF generalizes the link between “grid peeling” and the ACSF observed by Eppstein et al. (2017), which applied only to convex curves, and which was studied only for regular grids.

We prove that HCS satisfies some properties analogous to those of ACSF: HCS is invariant under affine transformations, preserves convexity, and does not increase the total absolute curvature. Furthermore, the number of self-intersections of a curve, or intersections between two curves (appropriately defined), does not increase. Finally, if the initial curve is simple, then the number of inflection points (appropriately defined) does not increase.

1 Introduction

Let $S^1$ be the unit circle. In this paper we call a piecewise-smooth function $\gamma : [0, 1] \to \mathbb{R}^2$ a path, and a piecewise-smooth function $\gamma : S^1 \to \mathbb{R}^2$ a closed curve, or simply a curve. If $\gamma$ is injective then the curve or path is said to be simple. We say that two paths or curves $\gamma, \delta$ are $\varepsilon$-close to each other if their Fréchet distance is at most $\varepsilon$, i.e. if they can be re-parametrized such that for every $t$, the Euclidean distance between the points $\gamma(t), \delta(t)$ is at most $\varepsilon$.

1.1 Shortest Homotopic Curves

Let $P$ be a finite set of points in the plane, which we regard as obstacles. Two curves $\gamma, \delta$ that avoid $P$ are said to be homotopic if there exists a way to continuously transform $\gamma$ into $\delta$ while avoiding $P$ at all times. And two paths $\gamma, \delta$ that avoid $P$ (except possibly at the endpoints) and satisfy $\gamma(0) = \delta(0), \gamma(1) = \delta(1)$ are said to be homotopic if there exists a way to continuously transform $\gamma$ into $\delta$, without moving their endpoints, while avoiding $P$ at all times (except possibly at the endpoints). We extend these definitions to the case where $\gamma$ avoids obstacles but $\delta$ does not, by requiring the continuous transformation of $\gamma$ into $\delta$ to avoid obstacles at all times except possibly at the last moment.
Then, for every curve (resp. path) $\gamma$ in the presence of obstacles there exists a unique shortest curve (resp. path) $\delta$ that is homotopic to $\gamma$. The problem of computing the shortest path or curve homotopic to a given piecewise-linear path or curve, under the presence of polygonal or point obstacles, has been studied extensively [6, 8, 11, 17, 25, 26].

### 1.2 The Affine Curve-Shortening Flow

In the affine curve-shortening flow, a smooth curve $\gamma \subset \mathbb{R}^2$ varies with time in the following way. At each moment in time, each point of $\gamma$ moves perpendicularly to the curve, towards its local center of curvature, with instantaneous velocity $r^{-1/3}$, where $r$ is that point’s radius of curvature at that time. See Figure 1.

The ACSF was first studied by Alvarez et al. [2] and Sapiro and Tannenbaum [27]. It differs from the more usual curve-shortening flow (CSF) [9, 13], in which each point is given instantaneous velocity $r^{-1}$. Unlike the CSF, the ACSF is invariant under affine transformations. It has applications in computer vision [9].

Under either the CSF or the ACSF, a simple curve remains simple, and its length decreases strictly with time ([13], [27], resp.). Furthermore, a pair of disjoint curves, run simultaneously, remain disjoint at all times ([28], [4], resp.). More generally, the number of intersections between two curves never increases ([3], [4], resp.). The total absolute curvature of a curve decreases strictly with time and tends to $2\pi$ ([20, 21], [4], resp.). The number of inflection points of a simple curve does not increase with time ([3], [4], resp.). Under the CSF, a simple curve eventually becomes convex and then converges to a circle as it collapses to a point [20, 21]. Correspondingly, under the ACSF, a simple curve becomes convex and then converges to an ellipse as it collapses to a point [4].

When the initial curve is not simple, a self-intersection might collapse and form a singularity that lasts for an instant. Unfortunately, unlike for the case of the CSF, for the ACSF no rigorous results have been obtained for self-intersecting curves [4]. Still, ACSF computer simulations can be run on curves that have self-intersections or singularities with little difficulty.

### 1.3 Relation to Grid Peeling

Let $P$ be a finite set of points in the plane. The convex-layer decomposition (also called the onion decomposition) of $P$ is the partition of $P$ into sets $P_1, P_2, P_3, \ldots$ obtained as follows: Let $Q_0 = P$. Then, for each $i \geq 1$ for which $Q_{i-1} \neq \emptyset$, let $P_i$ be the set of vertices of the
convex hull of \( Q_{i-1} \), and let \( Q_i = Q_{i-1} \setminus P_i \). In other words, we repeatedly remove from \( P \) the set of vertices of its convex hull. See [5, 12, 15, 16].

Eppstein et al. [18], following Har-Peled and Lidický [23], studied grid peeling, which is the convex-layer decomposition of subsets of the integer grid \( \mathbb{Z}^2 \). Eppstein et al. found an experimental connection between ACSF for convex curves and grid peeling. Specifically, let \( \gamma \) be a fixed convex curve. Let \( n \) be large, let \((\mathbb{Z}/n)^2\) be the uniform grid with spacing \( 1/n \), and let \( P_n(\gamma) \) be the set of points of \((\mathbb{Z}/n)^2\) that are contained in the region bounded by \( \gamma \). Then, as \( n \to \infty \), the convex-layer decomposition of \( P_n(\gamma) \) seems experimentally to converge to the ACSF evolution of \( \gamma \), after the time scale is adjusted appropriately. They raised the question whether there is a way to generalize the grid peeling process so as to approximate ACSF for non-convex curves as well.

1.4 Our Contribution

In this paper we describe a generalization of the convex-layer decomposition to non-convex, and even non-simple, curves. We call our process homotopic curve shortening, or HCS. Under HCS, an initial curve evolves in discrete steps in the presence of point obstacles. We find that, if the obstacles form a uniform grid, then HCS shares the same experimental connection to ACSF that grid peeling does. Hence, HCS is the desired generalization sought by Eppstein et al. [18]. We also find that the same experimental connection between ACSF and HCS (and in particular, between ACSF and the convex-layer decomposition) holds when the obstacles are distributed uniformly at random, with the sole difference being in the constant of proportionality.

Although the experimental connection between HCS and ACSF seems hard to prove, we do prove that HCS satisfies some simple properties analogous to those of ACSF: HCS is invariant under affine transformations, preserves convexity, and does not increase the total absolute curvature. Furthermore, the number of self-intersections of a curve, or intersections between two curves (appropriately defined), does not increase. Finally, if the initial curve is simple, then the number of inflection points (appropriately defined) does not increase.

2 Homotopic Curve Shortening

Let \( P \) be a finite set of obstacle points. A \( P \)-curve (resp. \( P \)-path) is a curve (resp. path) that is composed of straight-line segments, where each segment starts and ends at obstacle points.

Homotopic curve shortening \( (\text{HCS}) \) is a discrete process that starts with an initial \( P \)-curve \( \gamma_0 \) (which might self-intersect), and at each step, the current \( P \)-curve \( \gamma_n \) is turned into a new \( P \)-curve \( \gamma_{n+1} = \text{HCS}_P(\gamma_n) \).

The definition of \( \gamma' = \text{HCS}_P(\gamma) \) for a given \( P \)-curve \( \gamma \) is as follows. Let \((p_0, \ldots, p_{m-1})\) be the circular list of obstacle points visited by \( \gamma \). Call \( p_i \) nailed if \( \gamma \) goes straight through \( p_i \), i.e. if \( \angle p_{i-1}p_ip_{i+1} = \pi \).\(^1\) Let \((q_0, \ldots, q_{k-1})\) be the circular list of nailed vertices of \( \gamma \). Suppose first that \( k \geq 1 \). Then \( \gamma' \) is obtained through the following three substeps:

1. **Splitting.** We split \( \gamma \) into \( k \) \( P \)-paths \( \delta_0, \ldots, \delta_{k-1} \) at the nailed vertices, where each \( \delta_i \) goes from \( q_i \) to \( q_{i+1} \).

2. **Shortcutting.** For each non-endpoint vertex \( p_i \) of each \( \delta_i \), we make the curve avoid \( p_i \) by taking a small shortcut. Specifically, let \( \varepsilon > 0 \) be sufficiently small, and let \( C_{p_i} \) be a

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\(^1\) All indices in circular sequences are modulo the length of the sequence.
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Figure 2  Computation of a single step of HCS: Given a $P$-curve $\gamma$ (blue), we first identify its nailed vertices (purple). In this case, the two nailed vertices split $\gamma$ into two paths $\delta_0, \delta_1$. In each $\delta_i$ we take a small shortcut around each intermediate vertex (red). Then we replace each $\delta_i$ by the shortest path homotopic to it, obtaining the new $P$-curve $\gamma' = \text{HCS}_P(\gamma)$ (green).

circle of radius $\varepsilon$ centered at $p_i$. Let $e_i$ be the segment $p_{i-1} p_i$ of $\delta_i$. Let $x_i = e_i \cap C_{p_i}$ and $y_i = e_{i+1} \cap C_{p_i}$. Then we make the path go straight from $x_i$ to $y_i$ instead of through $p_i$. Call the resulting path $\rho_i$, and let $\rho$ be the curve obtained by concatenating all the paths $\rho_i$.

3. Shortening. Each $\rho_i$ in $\rho$ is replaced by the shortest $P$-path homotopic to it. The resulting curve is $\gamma'$.

If $\gamma$ has no nailed vertices ($k = 0$) then $\gamma'$ is obtained by performing the shortcutting and shortening steps on the single closed curve $\gamma$. Figure 2 illustrates one HCS step on a sample curve.

The process terminates when the curve collapses to a point. If the initial curve $\gamma_0$ is the boundary of the convex hull of $P$, then the HCS evolution of $\gamma_0$ is equivalent to the convex-layer decomposition of $P$.

3  Experimental Connection Between ACSF and HCS

Our experiments on a variety of curves show that HCS, using $P = (\mathbb{Z}/n)^2$ as the obstacle set, approximates ACSF at the limit as $n \to \infty$, just as grid peeling approximates ACSF for convex curves. Furthermore, we find that the connection between ACSF and HCS also holds if the uniform grid $(\mathbb{Z}/n)^2$ is replaced by a random point set, though with a different constant of time proportionality. See Figure 3 for an example.

4  Properties of Homotopic Curve Shortening

We prove that HCS satisfies some properties analogous to those of ACSF.

- Theorem 4.1. HCS is invariant under affine transformations.

- Theorem 4.2. Under HCS, once a curve becomes the boundary of a convex polygon, it stays that way.

The total absolute curvature of a piecewise-linear curve $\gamma$ with vertices $(p_0, \ldots, p_{m-1})$ is the sum of the exterior angles $\sum_{i=0}^{m-1} (\pi - |\angle p_{i-1} p_i p_{i+1}|)$. 

Figure 3 Left: Initial curve (blue) and simulated ACSF result after the curve’s length reduced to 70% of its original length (red). Right: Comparison between ACSF approximation (red), HCS with $n = 10^7$ uniform-grid obstacles (green), and HCS with $n = 10^7$ random obstacles (yellow) on a small portion of the curve.

Figure 4 HCS might cause disjoint curves to intersect, or a simple curve to self-intersect.

Theorem 4.3. Under HCS, the total absolute curvature of a curve never increases.

If $\gamma, \delta$ are disjoint $P$-curves, then $HCS_P(\gamma), HCS_P(\delta)$ are not necessarily disjoint. Similarly, if $\gamma$ is a simple $P$-curve, then $HCS_P(\gamma)$ is not necessarily simple. See Figure 4.

Curves $\gamma, \delta$ are called disjoinable if they can be made into disjoint curves by performing on them an arbitrarily small perturbation. Similarly, a curve $\gamma$ is called self-disjoinable if it can be turned into a simple curve by an arbitrarily small perturbation. (Akitaya et al. [1] recently found an $O(n \log n)$-time algorithm that can decide, in particular, whether a given curve is self-disjoinable.)

An intersection between two curves, or between two portions of one curve, is called transversal, if at the point of intersection both curves are differentiable and their normal vectors are not parallel at that point. If all intersections between curves $\gamma_1$ and $\gamma_2$ are transversal, then we say that $\gamma_1, \gamma_2$ are themselves transversal. Similarly, if all self-intersections of $\gamma$ are transversal, then we say that $\gamma$ is self-transversal. (Transversal and self-transversal curves are sometimes called generic, see e.g. [10].)
If $\gamma$ is self-transversal, we denote by $\chi(\gamma)$ the number of self-intersections of $\gamma$. If $\gamma$ is not self-transversal, then we define $\chi(\gamma)$ as the minimum of $\chi(\hat{\gamma})$ among all self-transversal curves $\hat{\gamma}$ that are $\varepsilon$-close to $\gamma$, for all small enough $\varepsilon > 0$. Hence, $\chi(\gamma) = 0$ if and only if $\gamma$ is self-disjoinable. We define similarly the number of intersections $\chi(\gamma_1, \gamma_2)$ between two curves. Then, $\gamma_1$ and $\gamma_2$ are disjoinable if and only if $\chi(\gamma_1, \gamma_2) = 0$. (Fulek and Tóth recently proved that the problem of computing $\chi(\gamma)$ is NP-hard [19].)

▶ Theorem 4.4. Under HCS, the intersection and self-intersection numbers never increase.

We say that an obstacle set $P$ is in general position if no three points of $P$ lie on a line. Note that if $P$ is in general position then there are no nailed vertices in HCS.

▶ Theorem 4.5. Under HCS with obstacles in general position, a simple curve stays simple, and a pair of disjoint curves stay disjoint.

Let $\gamma$ be a simple piecewise-linear curve with vertices $(v_0, \ldots, v_{n-1})$. An inflection edge of $\gamma$ is an edge $v_i v_{i+1}$ such that the previous and next vertices $v_{i-1}, v_{i+2}$ lie on opposite sides of the line through $v_i, v_{i+1}$. Let $\varphi(\gamma)$ be the number of inflection edges of $\gamma$. Note that $\varphi(\gamma)$ is always even, since every inflection edge lies either after a sequence of clockwise vertices and before a sequence of counterclockwise vertices, or vice versa.

If $\gamma$ is not simple but self-disjoinable, then we define $\varphi(\gamma)$ as the minimum of $\varphi(\gamma')$ over all simple piecewise-linear curves $\gamma'$ that are $\varepsilon$-close to $\gamma$, for all sufficiently small $\varepsilon > 0$.

▶ Theorem 4.6. Under HCS on a self-disjoinable curve, the curve’s number of inflection edges never increases.

5 Discussion

One of the reasons continuous curve-shortening flows were introduced and studied was to overcome the shortcomings of the Birkhoff curve-shortening process ([7], see also e.g. [14]), specifically the fact that it might cause the number of curve intersections to increase [22, 24]. As we have shown, HCS is a discrete process that overcomes this flaw without introducing analytical difficulties, at least in the plane. It would be interesting to check whether HCS can be applied on more general surfaces.

For more details see our full paper at [arXiv:1909.00263].

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