THE SIDON CONSTANT FOR HOMOGENEOUS POLYNOMIALS

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ABSTRACT. The Sidon constant for the index set of nonzero $m$-homogeneous polynomials $P$ in $n$ complex variables is the supremum of the ratio between the $\ell^1$ norm of the coefficients of $P$ and the $H^{\infty}(\mathbb{D}^n)$ norm of $P$. We present an estimate which gives the right order of magnitude for this constant, modulo a factor depending exponentially on $m$. We use this result to show that the Bohr radius for the polydisc $\mathbb{D}^n$ is bounded from below by a constant times $\sqrt{\log n}/n$.

1. INTRODUCTION

This note presents an estimate on the Sidon constant $S(m, n)$ for the index set of homogeneous polynomials of degree $m$ in $n$ complex variables. The result is optimal in the sense that the exact value of $S(m, n)$ is determined up to a factor depending exponentially on $m$. We will use this estimate to find the precise asymptotic behavior of the $n$-dimensional Bohr radius, which was introduced and studied by H. Boas and D. Khavinson [2].

The Sidon constant $S(m, n)$ for the index set $\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : |\alpha| = m\}$ is defined in the following way. Let

$$P(z) = \sum_{|\alpha|=m} c_{\alpha} z^\alpha$$

be a homogeneous polynomial of degree $m$ in $n$ complex variables. We let $\mathbb{D}^n$ denote the unit polydisc in $\mathbb{C}^n$ and set

$$\|P\|_{\infty} = \sup_{z \in \mathbb{D}^n} |P(z)| \quad \text{and} \quad \|P\|_1 = \sum_{|\alpha|=m} |c_{\alpha}|;$$

then $S(m, n)$ is the smallest constant $C$ such that the inequality $\|P\|_1 \leq C\|P\|_{\infty}$ holds for every $P$. It is plain that $S(1, n) = 1$ for all $n$, and this case is therefore excluded from our discussion. Our main result is the following estimate.

Theorem 1. There exists an absolute constant $C$ such that the Sidon constant $S(m, n)$ satisfies

$$S(m, n) \leq C^m \sqrt{\frac{n^{m-1}}{(m-1)!}}$$

when $n > m^2 > 1$.

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The Sidon constant $S(m, n)$ is effectively the same as the unconditional basis constant for the monomials of degree $m$ in $H^\infty(D^n)$; the latter is larger than $S(m, n)$ by a factor not exceeding 2. This and similar unconditional basis constants were studied in [6], where it was established that $S(m, n)$ is bounded from above and below by $n^{(m-1)/2}$ times constants depending only on $m$. The more precise estimate

$$S(m, n) \leq C^m n^{m-1},$$

with $C$ an absolute constant, can be extracted from [7]. By Hölder’s inequality, (2) also follows from an interesting inequality of H. Queffélec [11], which says that the $\ell^{2m/(m+1)}$ norm of the coefficients of a homogeneous polynomial $P$ of degree $m$ is bounded by $\|P\|_\infty$ times a certain precise constant depending only on $m$. A more direct deduction of (2) is implicit in the work of F. Bohnenblust and E. Hille [3]; this approach has inspired our proof of (1).

Note that we also have the following trivial estimate:

$$S(m, n) \leq \sqrt{n + \frac{m-1}{m}},$$

which is a consequence of the Cauchy–Schwarz inequality along with the fact that the number of different monomials of degree $m$ in $n$ variables is $\binom{n+m-1}{m}$. Comparing (1) and (3), we see that our estimate gives a nontrivial result only in the range $\log n > m$. Using the Salem–Zygmund inequality for random trigonometric polynomials (see [10, p. 68]), one may check that the estimates (3) and (1) together give the right value for $S(m, n)$, up to a factor less than $c^m$ with $c < 1$ an absolute constant.

Our application of Theorem 1 to the asymptotic behavior of the Bohr radius for the polydisc will further illuminate the significance of (1). Following [2], we now let $K_n$ be the $n$-dimensional Bohr radius, i.e., the largest positive number $r$ such that all polynomials $\sum \alpha \mathbf{c}_\alpha \mathbf{z}^\alpha$ satisfy

$$\sup_{z \in D^n} \left| \sum \mathbf{c}_\alpha \mathbf{z}^\alpha \right| \leq \sup_{z \in D^n} \left| \sum \mathbf{c}_\alpha \mathbf{z}^\alpha \right|.$$

The classical Bohr radius $K_1$ was studied and estimated by H. Bohr [4] himself, and it was shown independently by M. Riesz, I. Schur, and F. Wiener that $K_1 = 1/3$. In [2], the two inequalities

$$\frac{1}{3} \sqrt{\frac{1}{n}} \leq K_n \leq 2 \sqrt{\frac{\log n}{n}}$$

were established for $n > 1$. The paper of Boas and Khavinson aroused new interest in the Bohr radius and has been a source of inspiration for many subsequent papers. For some time (see for instance [1]) it was thought that the left-hand side of (4) could not be improved. However, using (2), A. Defant and L. Frerick [7] showed that

$$K_n \geq c \sqrt{\frac{\log n}{n \log \log n}},$$

holds for some constant $c > 0$.

Using Theorem 1 we will prove the following estimate.
Theorem 2. The $n$-dimensional Bohr radius $K_n$ satisfies

$$K_n \geq \gamma \sqrt{\frac{\log n}{n}}$$

for an absolute constant $\gamma > 0$.

Combining this result with the right inequality in (4), we conclude that

$$K_n = b(n) \sqrt{\frac{\log n}{n}}$$

with $0 < \gamma \leq b(n) \leq 2$. It is possible to extract from our methods a numerical value for $\gamma$ larger than 0.2, cf. the concluding remark of Section 5.

2. Preliminaries on multilinear forms

The transformation of a homogeneous polynomial to a corresponding multilinear form will play a crucial role in the proof of Theorem 1. We denote by $B$ an $m$-multilinear form in $\mathbb{C}^n$, i.e., given $m$ points $z^{(1)}, \ldots, z^{(m)}$ in $\mathbb{C}^n$, we set

$$B(z^{(1)}, \ldots, z^{(m)}) = \sum_{\beta} b_\beta z^{(1)}_{\beta_1} \cdots z^{(m)}_{\beta_m}.$$ 

We may express the coefficients as $b_\beta = B(e^{\beta_1}, \ldots, e^{\beta_m})$, where $\{e^i\}_{i=1,\ldots,n}$ is the canonical base of $\mathbb{C}^n$. The form $B$ is symmetric if for every permutation $\sigma$ of the set \{1, 2, \ldots, $m$\},

$B(z^{(1)}, \ldots, z^{(m)}) = B(z^{(\sigma(1))}, \ldots, z^{(\sigma(m))}).$ 

If we restrict a symmetric multilinear form to the diagonal $P(z) = B(z, \ldots, z)$, then we obtain a homogeneous polynomial. The converse is also true: Given a homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree $m$, by polarization, we may define the symmetric $m$-multilinear form $B : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ by setting

$$(5) \quad B(z^{(1)}, \ldots, z^{(m)}) = \frac{1}{2^m m!} \sum_{\epsilon_1, \epsilon_2, \ldots, \epsilon_m} \epsilon_1 \epsilon_2 \cdots \epsilon_m P\left( \sum_{i=1}^m \epsilon_i z^{(i)} \right)$$ 

so that $B(z, \ldots, z) = P(z)$. In what follows, $B$ will denote the symmetric $m$-multilinear form obtained in this way from $P$.

We will consider the analogous norms for symmetric multilinear forms as those introduced above. This means that we set

$$\|B\|_\infty = \sup_{D^n \times \cdots \times D^n} |B(z^{(1)}, \ldots, z^{(m)})| \quad \text{and} \quad \|B\|_1 = \sum_{|\beta| = m} |b_\beta|.$$ 

It will be important for us to be able to relate the norms of $P$ and $B$. It is plain that $\|P\|_\infty \leq \|B\|_\infty$. On the other hand, it was proved by L. Harris [9] that we have, for non-negative integers $m_1, \cdots, m_k$ with $m_1 + \cdots + m_k = m$,

$$(6) \quad |B(z^{(1)}, \ldots, z^{(1)}, \ldots, z^{(k)}, \ldots, z^{(k)})| \leq \frac{m_1! \cdots m_k!}{m_1^{m_1} \cdots m_k^{m_k}} \frac{m^m}{m!} \|P\|_\infty,$$ 

this result can be obtained from the polarization formula (5).
To compare the $\| \cdot \|_1$ norms, observe that the coefficients $b_\beta$ of $B$ can be computed from the corresponding coefficient of $P$: $b_\beta = c_\alpha / h(\beta)$, where $h(\beta)$ is the number of different words that can be formed with the letters in $\beta$. The corresponding $\alpha_j$ is the number of times any of the indices $\beta_i$ equals $j$. It is therefore clear that

$$\sum_\alpha |c_\alpha| = \sum_\beta |b_\beta|,$$

or, in other words, $\|P\|_1 = \|B\|_1$.

3. The tetrahedral part of a homogeneous polynomial

A polynomial $Q(z) = \sum_\alpha c_\alpha z^\alpha$ is said to be tetrahedral if $c_\alpha$ is nonzero only if $\max_j \alpha_j \leq 1$; thus no term in $Q$ contains a factor of degree 2 or higher in any of the variables $z_1, \ldots, z_n$. Now set $E = \{(\alpha_1, \ldots, \alpha_n) : |\alpha| = m, \alpha_i \leq 1, \forall i = 1, \ldots, n\}$. Then $T(P) = \sum_{\alpha \in E} c_\alpha z^\alpha$ is the tetrahedral part of $P$ and $R(P) = P - T(P)$ is the remainder corresponding to monomials containing a higher order power in at least one of the variables $z_1, \ldots, z_n$.

In the next lemma, $p_1, p_2, \ldots$ are the prime numbers, listed by increasing order, and $\text{sinc } x = (\sin x) / x$.

**Lemma 1.** We have $\|T(P)\|_\infty \leq \kappa^m \|P\|_\infty$ for every homogeneous polynomial $P$ of degree $m$, where the constant $\kappa$ can be taken as

$$\kappa = \left( \prod_{k=1}^\infty \text{sinc } \frac{\pi}{p_k} \right)^{-1} = 2.209 \ldots .$$

**Proof.** We will need the counting function for the prime numbers, which will be denoted by $\omega(x)$, in order not to confuse it with the number $\pi$. We begin by constructing some auxiliary functions. Set $Q = [0, 1]^{\omega(m)}$, let $t = (t_1, \ldots, t_{\omega(m)})$ denote a point in $Q$, and let $d\mu$ be Lebesgue measure on $Q$. Define

$$r_m(t) = c_m \exp \left( 2\pi i \left( \frac{t_1}{2} + \frac{t_2}{3} + \cdots + \frac{t_{\omega(m)}}{p_{\omega(m)}} \right) \right),$$

where

$$c_m = \prod_{k=1}^{\omega(m)} \left( \frac{p_k}{2\pi k} \left( e^{\frac{2\pi i}{p_k}} - 1 \right) \right)^{-1}.$$

The functions $r_m : Q \to \mathbb{C}$ have the following properties:

(i) $\int_Q r_m(t) \, d\mu(t) = 1$,

(ii) $\int_Q r_m^k(t) \, d\mu(t) = 0$ for all $k = 2, \ldots, m$,

(iii) $|r_m(t)| \leq \kappa$ for all $t$ in $Q$ and all $m > 1$.

It is immediate that (i) and (ii) are satisfied. We note that (iii) also holds, because $|r_m(t)| \equiv |c_m|$ and

$$|c_m|^{-2} = \prod_{k=1}^{\omega(m)} \left( \frac{p_k^2}{(2\pi k)^2} \left| e^{\frac{2\pi i}{p_k}} - 1 \right|^2 \right) = \prod_{k=1}^{\omega(m)} \text{sinc}^2 \frac{\pi}{p_k}.$$
By properties (i) and (ii),

\[ T(P)(z) = \int_{Q^n} P(z_1 r(t^1), \ldots, z_n r(t^n)) \, d\mu(t^1) \cdots d\mu(t^n), \]

and so, by property (iii), \( |P(z_1 r(t^1), \ldots, z_n r(t^n))| \leq \kappa^n \|P\|_\infty \) for every \( z \) in \( \mathbb{D}^n \).

We can similarly define a decomposition of symmetric \( m \)-multilinear forms. Let \( F \) be the set of multiindices \( F = \{ (\beta_1, \ldots, \beta_m) : 1 \leq \beta_i \leq n \text{ and all indices } \beta_k \text{ are pairwise disjoint} \} \). Then we may decompose \( B = T(B) + R(B) \), where

\[ T(B)(z^{(1)}, \ldots, z^{(m)}) = \sum_{\beta \in F} b_\beta z^{(1)}_{\beta_1} \cdots z^{(m)}_{\beta_m}, \]

Clearly, if \( P \) is a homogeneous polynomial and \( B \) its corresponding symmetric multilinear form, then \( T(P) \) has \( T(B) \) as the corresponding multilinear form.

4. PROOF OF THEOREM 1

Since

\[ \|P\|_1 = \|R(P)\|_1 + \|T(P)\|_1, \]

it will suffice to obtain appropriate estimates for each of the norms \( \|R(P)\|_1 \) and \( \|T(P)\|_1 \). The two lemmas below together give the required bound for \( \|P\|_1 \).

We begin by estimating \( \|R(P)\|_1 \) in the range \( n > m^2 \).

**Lemma 2.** For a homogeneous polynomial \( P \) of degree \( m \) and \( n > m^2 > 1 \), we have

\[ \|R(P)\|_1 \leq \sqrt{2me \frac{n^{m-1}}{(m-1)!}} \|P\|_\infty. \tag{7} \]

**Proof.** We begin by observing that the number of monomials \( z^\alpha \) in \( n \) variables of degree \( m \) with \( \max_j \alpha_j > 1 \) is \( \binom{n+m+1}{m} - \binom{n}{m} \). Thus the Cauchy–Schwarz inequality gives

\[ \|R(P)\|_1 \leq \sqrt{\binom{n+m+1}{m} - \binom{n}{m}} \|P\|_\infty. \]

The result follows from this because

\[ \binom{n+m+1}{m} - \binom{n}{m} \leq \frac{n^m}{m!} \left[ (1 + \frac{m}{n})^m - (1 - \frac{m}{n})^m \right] \]

\[ \leq \frac{n^m}{m!} \left[ e^{m^2/n} - e^{-m^2/n} \right] \leq 2em \frac{n^{m-1}}{(m-1)!}. \]

We turn next to the most challenging case, which is that of the tetrahedral part \( T(P) \). Now the Cauchy–Schwarz inequality does not work because there are too many coefficients. We will transfer the problem to \( T(B) \) and use instead a special form of the multilinear Khinchine inequality, which can be traced back to [5]. The precise formulation of the result to be used is
in [8, Theorem 3.2.2]. In the theorem below, \( \{ \epsilon_i \}_{i=1}^{\infty} \) denotes a Rademacher sequence of random variables, i.e., the \( \epsilon_i \) are i.i.d and \( \mathbb{P}(\epsilon_i = 1) = \mathbb{P}(\epsilon_i = -1) = 1/2 \).

**Theorem 3 (Hypercontractivity).** Let \( X \) be a homogeneous chaos of order \( m \):

\[
X = \sum_{1 \leq i_1 < \cdots < i_m \leq n} x_{i_1, \ldots, i_m} \epsilon_{i_1} \cdots \epsilon_{i_m},
\]

with \( x_{i_1, \ldots, i_m} \in \mathbb{C} \). Then

\[
\left( \mathbb{E}(|X|^2) \right)^{1/2} \leq e^{m} \mathbb{E}(|X|).
\]

With this theorem we can prove the following.

**Lemma 3.** For every homogeneous polynomial \( P \) of degree \( m \) with \( m < n \), we have

\[
\|T(B)\|_1 \leq (e\kappa)^m \left( \frac{n - 1}{m - 1} \right)^{1/2} \|P\|_\infty,
\]

where \( \kappa \) is the constant from Lemma [7].

**Proof.** Put

\[
F = \{ (i_1, \ldots, i_m), 1 \leq i_1, \ldots, i_m \leq n \text{ pairwise distinct} \},
\]

\[
F_{i_1} = \{ (i_2, \ldots, i_m) : (i_1, \ldots, i_m) \in F \},
\]

\[
\tilde{F}_{i_1} = \{ (i_2, \ldots, i_m) \in F_{i_1}, i_2 < \cdots < i_m \}.
\]

We may write

\[
\|T(B)\|_1 = \sum_{(i_1, \ldots, i_m) \in F} |B(e^{i_1}, \ldots, e^{i_m})|
\]

\[
= \sum_{i_1=1}^{n} \sum_{(i_2, \ldots, i_m) \in F_{i_1}} |B(e^{i_1}, e^{i_2}, \ldots, e^{i_m})|
\]

\[
= \sum_{i_1=1}^{n} \sum_{(i_2, \ldots, i_m) \in \tilde{F}_{i_1}} (m - 1)! |B(e^{i_1}, e^{i_2}, \ldots, e^{i_m})|
\]

\[
\leq \left( \frac{n - 1}{m - 1} \right)^{1/2} \sum_{i_1=1}^{n} \left( \sum_{(i_2, \ldots, i_m) \in \tilde{F}_{i_1}} (m - 1)! |B(e^{i_1}, e^{i_2}, \ldots, e^{i_m})| \right)^{1/2}.
\]

By Theorem [3],

\[
\left( \sum_{(i_2, \ldots, i_m) \in \tilde{F}_{i_1}} (m - 1)! |B(e^{i_1}, e^{i_2}, \ldots, e^{i_m})| \right)^{1/2} \leq e^{m-1} \mathbb{E} \left( \left| \sum_{(i_2, \ldots, i_m) \in \tilde{F}_{i_1}} (m - 1)! B(e^{i_1}, e^{i_2}, \ldots, e^{i_m}) \epsilon_{i_2} \cdots \epsilon_{i_m} \right| \right).
\]
Summing over $i_1$, we get

\begin{equation}
\|T(B)\|_1 \leq e^{m-1}\left(\frac{n-1}{m-1}\right)^{1/2}\sup_{z \in D^n} \sum_{i_1=1}^{n} \sum_{(i_2, \ldots, i_m) \in F_{i_1}} B(e^{i_1}, e^{i_2}, \ldots, e^{i_m}) z_{i_2} \cdots z_{i_m}.
\end{equation}

We introduce the notation

$$\lambda_{i_1}(z) = \frac{\left| \sum_{(i_2, \ldots, i_m) \in F_{i_1}} B(e^{i_1}, e^{i_2}, \ldots, e^{i_m}) z_{i_2} \cdots z_{i_m} \right|}{\sum_{(i_2, \ldots, i_m) \in F_{i_1}} B(e^{i_1}, e^{i_2}, \ldots, e^{i_m}) z_{i_2} \cdots z_{i_m}}$$

and obtain from (8)

\begin{align*}
\|T(B)\|_1 & \leq e^{m-1}\left(\frac{n-1}{m-1}\right)^{1/2}\sup_{z \in D^n} \sum_{(i_2, \ldots, i_m) \in F} B(e^{i_1}, e^{i_2}, \ldots, e^{i_m}) \lambda_{i_1}(z) z_{i_2} \cdots z_{i_m} \\
& \leq e^{m-1}\left(\frac{n-1}{m-1}\right)^{1/2}\sup_{(z^{(1)}, z^{(2)}) \in D^n \times D^n} |T(B)(z^{(1)}, z^{(2)}, \ldots, z^{(2)})| \\
& \leq e^{m}\left(\frac{n-1}{m-1}\right)^{1/2}\|T(P)\|_\infty \\
& \leq e^{m}\kappa^m\left(\frac{n-1}{m-1}\right)^{1/2}\|P\|_\infty,
\end{align*}

where in the last two steps we used Harris’s inequality (6) and Lemma 1.

5. PROOF OF THEOREM 2

For the proof of Theorem 2, we need the following lemma of F. Wiener (see [2]).

**Lemma 4.** Let $P$ be a polynomial in $n$ variables and $P = \sum_{m \geq 0} P_m$ its expansion in homogeneous polynomials. If $\|P\|_\infty \leq 1$, then $\|P_m\|_\infty \leq 1 - |P_0|^2$ for every $m > 0$.

**Proof of Theorem 2.** We assume that $\sup_{z \in D^n} |\sum c_\alpha z^\alpha| \leq 1$. Observe that for all $z$ in $rD^n$,

$$\sum |c_\alpha z^\alpha| \leq |c_0| + \sum_{m>1} r^m \sum_{|\alpha|=m} |c_\alpha|.$$

When $m > \log n$, we use (3) and Lemma 4 and obtain the estimate

\begin{equation}
\sum_{m>\log n} r^m \sum_{|\alpha|=m} |c_\alpha| \leq \sum_{m>\log n} r^m \sqrt{\binom{n+m-1}{m}} (1 - |c_0|^2),
\end{equation}

whence

\begin{equation}
\sum_{m>\log n} r^m \sum_{|\alpha|=m} |c_\alpha| \leq \sum_{m>\log n} r^m (2e)^m \max(1, n/m)^{m/2} (1 - |c_0|^2).
\end{equation}
If we take into account the estimate
\[
\frac{(\log n)^m}{n} \leq m!
\]
(obtained by a calculus argument), then Theorem 1 and Lemma 4 give
\[
\sum_{m<\log n} r^m \sum_{|\alpha|=m} |c_\alpha| \leq \sum_{m<\log n} r^m \sqrt{m} \frac{C^m}{n \log n} \frac{m/2}{(1-|c_0|^2)}.
\]
(11)

If we now choose \( r \leq \varepsilon \sqrt{\frac{\log n}{n}} \) with \( \varepsilon \) small enough and combine (10) and (11), we obtain
\[
\sum |c_\alpha z^\alpha| \leq |c_0| + (1-|c_0|^2)/2 \leq 1
\]
whenever \( |c_0| \leq 1 \). Thus the theorem is proved with \( \gamma = \varepsilon \).

A closer examination of this proof shows that a better choice would be to use Theorem 1 only when \( m < (2 + 2 \log \kappa)^{-1} \log n \). By this approach and taking into account the estimates from Lemmas 2 and 3, we get
\[
b(n) \geq \frac{1}{\sqrt{2e(1+\log \kappa)}} + o(1)
\]
when \( n \to \infty \). By also doing a meticulous analysis of (9) for “small” \( n \) and keeping in mind that \( S(1,n) = 1 \), one may arrive at a numerical value for \( \gamma \) which is larger than 0.2.

\textbf{REFERENCES}

[1] H. P. Boas, \textit{Majorant series}, J. Korean Math. Soc. 37 (2000), 321–337.
[2] H. P. Boas and D. Khavinson, \textit{Bohr's power series theorem in several variables}, Proc. Amer. Math. Soc. 125 (1997), 2975–2979.
[3] H. F. Bohnenblust and E. Hille, \textit{On the absolute convergence of Dirichlet series}, Ann. of Math. (2) 32 (1931), 600–622.
[4] H. Bohr, \textit{A theorem concerning power series}, Proc. London Math. Soc. 13 (1914), 1–5.
[5] A. Bonami, \textit{Étude des coefficients de Fourier des fonctions de \( L^p(G) \)}, Ann. Inst. Fourier (Grenoble) 20 (1970), 335–402.
[6] A. Defant, J. C. Díaz, D. García, and M. Maestre, \textit{Unconditional basis and Gordon–Lewis constants for spaces of polynomials}, J. Funct. Anal. 181 (2001), 119–145.
[7] A. Defant and L. Frerick, \textit{A logarithmic lower bound for multi-dimensional Bohr radii}, Israel J. Math. 152 (2006), 17–28.
[8] V. H. de la Peña and E. Giné, \textit{Decoupling}, Probability and Its Applications (New York), Springer-Verlag, New York, 1999.
[9] L. A. Harris, \textit{Bounds on the derivatives of holomorphic functions of vectors}, Analyse fonctionnelle et applications (Comptes Rendus Colloq. Analyse, Inst. Mat., Univ. Federal Rio de Janeiro, Rio de Janeiro, 1972), Hermann, Paris, 1975, pp. 145–163. Actualités Aci. Indust., No. 1367.
[10] J.-P. Kahane, \textit{Some Random Series of Functions}, Second edition, Cambridge Studies in Advanced Mathematics, 5. Cambridge University Press, Cambridge, 1985.
[11] H. Queffélec, \textit{Harald Bohr's vision of Dirichlet series; old and new results}, J. Anal. 3 (1995), 43–60.
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