Global existence and exponential stability of coupled Lamé system with distributed delay and source term without memory term

Salah Boulaaras¹,²* and Nadjat Doudi³

Abstract
In this paper, we prove the global existence and exponential energy decay results of a coupled Lamé system with distributed time delay, nonlinear source term, and without memory term by using the Faedo–Galerkin method. In addition, an appropriate Lyapunov functional, more general relaxation functions, and some properties of convex functions are considered.

MSC: 35L90; 74G25; 35B40; 26A51

Keywords: Viscoelastic term; Delay function; Lyapunov functional; Distributed delay; Exponential decay

1 Introduction
In this work, we study the following Lamé system in $\Omega \times \mathbb{R}^+$:

\[
\begin{align*}
&u_{tt} - \Delta_e u + k_1 u_t + \int_{\tau_1}^{\tau_2} \mu_1(\varrho) u_t(x, t - \varrho) \, d\varrho = f_1(u, v), \\
v_{tt} - \Delta_e v + k_2 v_t + \int_{\tau_1}^{\tau_2} \mu_2(\varrho) v_t(x, t - \varrho) \, d\varrho = f_2(u, v), \\
&u(x, t) = v(x, t) = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
u(x, 0) = v(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad u_t(x, 0) = u_1(x), \\
v_t(x, 0) = v_1(x), \quad x \in \Omega, \\
(u_t(x, -t), v_t(x, -t)) = (f_0(x, t), g_0(x, t)) \quad \text{in } \Omega \times (0, r_2),
\end{align*}
\]

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n \ (n = 1, 2, 3)$ with smooth boundary $\partial \Omega$. The elasticity differential operator $\Delta_e$ is given by

\[\Delta_e u = \mu \Delta u + (\mu + \lambda) \nabla (\text{div } u),\]

and the Lamé constants $\mu$ and $\lambda$ satisfy the following conditions:

$\mu > 0, \quad \mu + \lambda > 0.$

*Correspondence: S.Boulaaras@qu.edu.sa

¹Department of Mathematics, College of Sciences and Arts, ArRas, Qassim University, Buraidah, Kingdom of Saudi Arabia
²Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Oran, Algeria

Full list of author information is available at the end of the article.
The parameters $k_1, k_2, \tau_1,$ and $\tau_2$ are positive constants, with $\tau_1 < \tau_2$. The functions $\mu_1, \mu_2 : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ are bounded. The functions $f_1(u, v)$ and $f_2(u, v)$, which represent the source terms, will be specified later.

After several authors have studied the problems of coupled systems and hyperbolic systems, their stability is associated with velocities and is proven under some given conditions (see, for example, [1–11]). In recent years, several authors have been interested in studying the existence and stability for Lamé systems, we refer to [12–14]. The Lamé system with localized nonlinear damping and a general decay result of energy have been considered by some recent works (see, for example, [12, 14], and [15]). Bchatnia et al. in [16] investigated Lamé systems with past history. Then, Taouaf et al. in [17] established the well-posedness and asymptotic stability for the Lamé system with internal distributed delay.

Beniani et al. [13] proved the well-posedness and exponential stability of the following coupled Lamé system:

$$
\begin{align*}
\alpha u_t + \Delta u + \int_0^t g_1(t-s) \Delta u(s) \, ds - \mu_1 \Delta u_t = 0, \\
\beta v_t + \Delta v + \int_0^t g_2(t-s) \Delta v(s) \, ds - \mu_2 \Delta v_t = 0.
\end{align*}
$$

(1.2)

After that, Baoweie et al. in [18] considered the same problem with more general assumption on the relaxation functions. They established an explicit and general decay result, which are optimal, to the system.

Boulaaras et al. in [14] considered the previous problem with a source term, where under some suitable conditions on the initial data and the relaxation functions, they proved an asymptotic stability result of global solution taking into account that the kernel is not necessarily decreasing.

In the present work, we prove the existence and general decay results of problem (1.1) with respect to the presence of distributed term delay in order to ensure fast stability under some given conditions. We establish the exponential energy decay results to the system by using an appropriate Lyapunov functional.

This paper is organized as follows. In the second section, we give some preliminaries related to problem (1.1). In Sect. 3, we prove the global existence by using Faedo–Galerkin method. In the fourth section, we prove our main result of exponential energy decay.

2 Preliminaries

In this section, we provide some materials and necessary assumptions which we need in the proof of our results. We use the standard Lebesgue and Sobolev spaces with their scaler products and norms. For simplicity, we would write $\| \cdot \|$ instead of $\| \cdot \|_2$.

(A1) For the source terms $f_1$ and $f_2$, we take

$$
f_1(u, v) = \alpha (u + v)^p + \beta |u|^{\frac{p-1}{2}} |u|^{|p+1|}, \quad \forall (u, v) \in \mathbb{R}^n,
$$

and

$$
f_2(u, v) = \alpha (u + v)^p + \beta |v|^{\frac{p-1}{2}} |v|^{|p+1|}, \quad \forall (u, v) \in \mathbb{R}^n,
$$

with $\alpha, \beta > 0$. Clearly,

$$
u f_1(u, v) + v f_2(u, v) = (p + 1)F(u, v), \quad \forall (u, v) \in \mathbb{R}^n,
$$

(2.1)
where

\[ F(u, v) = \frac{1}{(p + 1)} \left[ \alpha |u + v|^{p+1} + 2\beta |uv|^{\frac{p+1}{2}} \right], \quad \forall (u, v) \in \mathbb{R}^n, \]  

(2.2)

and

\[ f_1(u, v) = \frac{\partial F}{\partial u}, \quad f_2(u, v) = \frac{\partial F}{\partial v}. \]  

(2.3)

If \( n = 1, 2; \ p \geq 3, \) if \( n = 3; \ p = 3. \) \hspace{1cm} (2.4)

So, we have the embedding

\[ H_0^1(\Omega) \hookrightarrow L^q(\Omega) \hspace{1cm} \text{for} \ 2 \leq q \leq \frac{2n}{n-2} \hspace{0.5cm} \text{if} \ n \geq 3 \hspace{0.5cm} \text{or} \ q \geq 2 \hspace{0.5cm} \text{if} \ n = 1, 2 \]

and

\[ L^r \hookrightarrow L^q \hspace{1cm} \text{for} \ q < r. \]

Let \( c_i \) be the same embedding constant, so we have

\[ \|v\|_q \leq c_i \|
abla v\|_2, \quad \|v\|_q \leq c_i \|v\|_r \hspace{1cm} \text{for} \ v \in H_0^1(\Omega). \]  

(2.5)

As in many papers, we introduce the following new variables:

\[
\begin{align*}
&z(x, \rho, \varrho, t) = u_t(x, t - \varrho \rho), \\
y(x, \rho, \varrho, t) = v_t(x, t - \varrho \rho),
\end{align*}
\]

then we obtain

\[
\begin{align*}
&\varrho z_t(x, \rho, \varrho, t) + z_t(x, \rho, \varrho, t) = 0, \\
z(x, 0, \varrho, t) = u_t(x, t),
\end{align*}
\]

(2.6)

and

\[
\begin{align*}
&\varrho y_t(x, \rho, \varrho, t) + y_t(x, \rho, \varrho, t) = 0, \\
y(x, 0, \varrho, t) = v_t(x, t).
\end{align*}
\]

(2.7)

Consequently, problem (1.1) is equivalent to

\[
\begin{align*}
u_{tt} - \Delta_x u + k_1 u_t + \int_{\Omega} \mu_1(\varrho) z(x, 1, \varrho, t) \, d\varrho &= f_1(u, v), \\
v_{tt} - \Delta_x v + k_2 v_t + \int_{\Omega} \mu_2(\varrho) y(x, 1, \varrho, t) \, d\varrho &= f_2(u, v), \\
\varrho z_t(x, \rho, \varrho, t) + z_t(x, \rho, \varrho, t) &= 0, \\
\varrho y_t(x, \rho, \varrho, t) + y_t(x, \rho, \varrho, t) &= 0,
\end{align*}
\]

(2.8)
with the initial data and boundary conditions

\[
\begin{aligned}
(u(x,0), v(x,0)) & = (u_0(x), v_0(x)) \quad \text{in } \Omega, \\
(u_t(x,0), v_t(x,0)) & = (u_1(x), v_1(x)) \quad \text{in } \Omega, \\
(u_t(x,-t), v_t(x,-t)) & = (f_0(x,t), g_0(x,t)) \quad \text{in } \Omega \times (0, \tau_2), \\
\frac{\partial u}{\partial n} & = 0 \quad \text{in } \partial \Omega \times (0, \infty), \\
(u(x,t), v(x,t)) & = (f_0(x,\rho,\varrho), g_0(x,\rho,\varrho)) \quad \text{in } \Omega \times (0,1) \times (0, \tau_2),
\end{aligned}
\]  

(2.9)

where

\[(x, \rho, \varrho, t) \in \Omega \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty) .\]

The energy associated with problem (2.8) is defined by

\[
E(t) = \frac{1}{2} \left[ \|u_t\|^2 + \|v_t\|^2 + \mu \left( \|\nabla u\|^2 + \|\nabla v\|^2 \right) + (\lambda + \mu) \left( \|\text{div} u\|^2 + \|\text{div} v\|^2 \right) \right] \\
+ \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \left[ |\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 + |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2 \right] d\varrho \, d\rho \, dx \\
- \int_\Omega F(u, v) \, dx. 
\]  

(2.10)

First, we prove in the following theorem, the result of energy identity.

**Lemma 2.1** Assume that

\[
\int_{\tau_1}^{\tau_2} |\mu_i(\varrho)| \, d\varrho < k_i, \quad i = 1, 2.
\]  

(2.11)

Then the energy modified defined by (2.10) satisfies, along the solution \((u, v, z, y)\) of (2.8), the estimate

\[
\frac{d}{dt} E(t) \leq \left[ k_1 - \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \, d\varrho \right] \|u_t\|^2 - \left[ k_2 - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, d\varrho \right] \|v_t\|^2 \\
\leq 0. 
\]  

(2.12)

Proof First multiplying equation (2.8) by \(u_t\) and integrating by parts over \(\Omega\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \|u_t\|^2 + \mu \|\nabla u\|^2 + (\lambda + \mu) \|\text{div} u\|^2 \right] + k_1 \|u_t\|^2 \\
+ \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z(x, 1, \varrho, t) \, d\varrho \, dx \\
= \int_\Omega u_t f_1(u, v) \, dx. 
\]  

(2.13)
Similarly, multiplying equation (2.8) by \( v_t \) and integrating over \( \Omega \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| v_t \|^2 + \mu \| \nabla v \|^2 + (\lambda + \mu) \| \text{div} \ v \|^2 \right) + k_1 \| v_t \|^2
\]

\[
+ \int_{\Omega} v_t \int_{\Omega_1} |\mu_2(\rho)| \gamma(x, 1, \rho, t) \, d\Omega \, dx
\]

\[
= \int_{\Omega} v_t f_2(u, v) \, dx. \tag{2.14}
\]

Multiplying equation (2.8) by \( |\mu_1(\rho)| z(x, \rho, \rho, t) \) and integrating by parts over \( \Omega \times (0, 1) \times (t_1, t_2) \), we obtain

\[
\int_{\Omega} \int_{0}^{1} \int_{t_1}^{t_2} \phi |\mu_1(\rho)| z(x, \rho, \rho, t) z_t(x, \rho, \rho, t) \, d\rho \, d\rho \, dx
\]

\[
= - \int_{\Omega} \int_{0}^{1} \int_{t_1}^{t_2} |\mu_1(\rho)| z(x, 1, \rho, t) z_t(x, 1, \rho, t) \, d\rho \, d\rho \, dx,
\]

therefore

\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \int_{0}^{1} \int_{t_1}^{t_2} \phi |\mu_1(\rho)| |z(x, \rho, \rho, t)|^2 \, d\rho \, d\rho \, dx \right)
\]

\[
= - \frac{1}{2} \int_{\Omega} \int_{0}^{1} \int_{t_1}^{t_2} |\mu_1(\rho)| |z(x, \rho, \rho, t)|^2 \, d\rho \, d\rho \, dx \tag{2.15}
\]

Similarly, multiplying the fourth equation of (2.8) by \( |\mu_2(\rho)| \gamma(x, \rho, \rho, t) \) and integrating over \( \Omega \times (0, 1) \times (t_1, t_2) \), we obtain

\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \int_{0}^{1} \int_{t_1}^{t_2} \phi |\mu_2(\rho)| |\gamma(x, \rho, \rho, t)|^2 \, d\rho \, d\rho \, dx \right)
\]

\[
= - \frac{1}{2} \int_{\Omega} \int_{0}^{1} \int_{t_1}^{t_2} |\mu_2(\rho)| |\gamma(x, \rho, \rho, t)|^2 \, d\rho \, d\rho \, dx \tag{2.16}
\]

For the source term, we have

\[
\int_{\Omega} u f_1(u, v) \, dx + \int_{\Omega} v f_2(u, v) \, dx
\]

\[
= \int_{\Omega} u \left( \alpha |u + v|^{p-1}(u + v) + \beta |u|^{p-1} u \right) \frac{\partial}{\partial \tau} u^{\frac{p-1}{2}}
\]

\[
+ \int_{\Omega} v \left( \alpha |u + v|^{p-1}(u + v) + \beta |v|^{p-1} v \right) \frac{\partial}{\partial \tau} v^{\frac{p-1}{2}}
\]

\[
= \int_{\Omega} \left( \alpha |u + v|^{p-1}(u + v)(u_t + v_t) + \beta (|u|^{p-3} u u_t) |v|^{p-1} \right) \frac{\partial}{\partial \tau} u^{\frac{p-1}{2}}
\]

\[
+ \beta (|v|^{p-3} v v_t) |u|^{p-1} \right) \frac{\partial}{\partial \tau} v^{\frac{p-1}{2}} \right) \frac{\partial}{\partial \tau} dx. \tag{2.17}
\]
\[
\frac{d}{dt} \int_\Omega \left( \frac{\alpha}{p+1} |u + v|^{p+1} + \frac{2\beta}{p+1} |uv|^{p+1} \right) \, dx = \frac{d}{dt} \int_\Omega F(u, v) \, dx.
\]

By collecting the previous equations (2.13)–(2.17), we get
\[
\begin{align*}
\frac{d}{dt} E(t) &= -k_1 \|u_t\|^2 - \int_\Omega u_t \int_{t_1}^{t_2} \mu_1(\epsilon) z(x, 1, \epsilon, t) \, d\epsilon \, dx \\
&\quad - k_2 \|v_t\|^2 - \int_\Omega v_t \int_{t_1}^{t_2} \mu_2(\epsilon) y(x, 1, \epsilon, t) \, d\epsilon \, dx \\
&= \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_1(\epsilon)| |z(x, 1, \epsilon, t)|^2 \, d\epsilon \, dx \right) + \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_2(\epsilon)| |y(x, 1, \epsilon, t)|^2 \, d\epsilon \, dx \right).
\end{align*}
\]

Using Young’s inequality, we obtain
\[
\int_\Omega u_t \int_{t_1}^{t_2} \mu_1(\epsilon) z(x, 1, \epsilon, t) \, d\epsilon \, dx \leq \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_1(\epsilon)| \, d\epsilon \right) \|u_t\|^2 + \frac{1}{2} \int_\Omega \int_{t_1}^{t_2} |\mu_1(\epsilon)| |z(x, 1, \epsilon, t)|^2 \, d\epsilon \, dx,
\]

similarly
\[
\int_\Omega v_t \int_{t_1}^{t_2} \mu_2(\epsilon) y(x, 1, \epsilon, t) \, d\epsilon \, dx \leq \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_2(\epsilon)| \, d\epsilon \right) \|v_t\|^2 + \frac{1}{2} \int_\Omega \int_{t_1}^{t_2} |\mu_2(\epsilon)| |y(x, 1, \epsilon, t)|^2 \, d\epsilon \, dx.
\]

This completes the proof. \(\Box\)

## 3 Global existence

**Theorem 3.1** (Global existence) Let \((u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2\), \((u_1, v_1) \in (H_0^1(\Omega))^2\) and \((f_0, g_0) \in (H^1(\Omega \times (0,1) \times (t_1, t_2)))^2\) satisfying the compatibility condition
\[
(f_0(\cdot, 0), g_0(\cdot, 0)) = (u_1, v_1).
\]
Assume that (A1)–(A2) hold. Then, problem (2.8)–(2.9) admits a weak solution such that \(u, v \in L^\infty(0, \infty; H^2(\Omega) \cap H_0^1(\Omega)), u_t, v_t \in L^\infty(0, \infty; H_0^1(\Omega)),\) and \(u_{tt}, v_{tt} \in L^2(0, \infty; H_0^1(\Omega)).\)

Throughout this section we assume \((u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2\), \((u_1, v_1) \in (H_0^1(\Omega))^2\) and \((f_0, g_0) \in H^2(\Omega; H^1(0,1)) \cap H_0^1(\Omega; H^1(0,1)).\) We employ the Galerkin method to construct a global solution. Let \(T > 0\) be fixed and denote by \(V_k\) the space generated by \([w_1, w_2, \ldots, w_k]\), where the set \([w_k, k \in \mathbb{N}]\) is a basis of \(H^2(\Omega) \cap H_0^1(\Omega).\) Now, we define for \(1 \leq j \leq k\) the sequence \(\phi_j(x, \rho)\) as follows:
\[
\phi_j(x, 0) = w_j.
\]

(3.1)
Then, we may extend $\phi(x,0)$ by $\phi(x,\rho)$ over $L^2(\Omega \times [0,1])$ and denote by $Z_k$ the space generated by $\{\phi_1,\phi_2,\ldots,\phi_k\}$. We construct approximate solutions $(u^k, v^k, z^k, y^k)$ $k = 1, 2, \ldots$ in the form

$$
\begin{align*}
    u^k(t) &= \sum_{j=1}^{k} g_{jk}(t)w_j(x), \\
    z^k(t) &= \sum_{j=1}^{k} c_{jk}(t)\phi_j(x,\rho,\varrho), \\
    v^k(t) &= \sum_{j=1}^{k} h_{jk}(t)w_j(x), \\
    y^k(t) &= \sum_{j=1}^{k} d_{jk}(t)\phi_j(x,\rho,\varrho),
\end{align*}
$$

where $g_{jk}, h_{jk}, c_{jk}$, and $d_{jk}$, $j = 1, 2, \ldots$, are determined by the following ordinary differential equations:

$$
\begin{align*}
    \langle u^k_j, w_j \rangle + \mu \langle \nabla u^k, \nabla w_j \rangle + (\lambda + \mu) \langle \text{div} u^k, \text{div} w_j \rangle + k_1 \langle u^k, w_j \rangle \\
    + \int_{t_0}^{t} \mu_1(\varrho) \langle z^k(x,1,\varrho,t) \partial \varrho w_j, w_j \rangle = \langle f_1(u^k), w_j \rangle, \quad 1 \leq j \leq k, \\
    \langle v^k_j, w_j \rangle + \mu \langle \nabla v^k, \nabla w_j \rangle + (\lambda + \mu) \langle \text{div} v^k, \text{div} w_j \rangle + k_2 \langle v^k, w_j \rangle \\
    + \int_{t_0}^{t} \mu_2(\varrho) \langle y^k(x,1,\varrho,t) \partial \varrho w_j, w_j \rangle = \langle f_2(u^k), w_j \rangle, \quad 1 \leq j \leq k, \\
    \langle \phi^k_j \rangle + \|z^k\|_{L^2} \langle \phi_j \rangle = 0, \quad 1 \leq j \leq k, \\
    \langle \phi^k_j \rangle + \|y^k\|_{L^2} \langle \phi_j \rangle = 0, \quad 1 \leq j \leq k,
\end{align*}
$$

and

$$
\begin{align*}
    u^k(0), u_1(k) &= (u_{0k}, u_{1k}), \\
    v^k(0), v_1(k) &= (u_{0k}, u_{1k}), \\
    z^k(x,0,\varrho,t) &= u^k_1(x,t), \\
    y^k(x,0,\varrho,t) &= v^k_1(x,t).
\end{align*}
$$

Suppose that

$$
w_j \in H^2(\Omega).
$$

We choose $u^k_0, v^k_0, u^k_1$ and $v^k_1 \in [w_1, \ldots, w_k]$, where

$$
\begin{align*}
    u^k(0) &= u^k_0 = \sum_{j=1}^{k} (u_0,w_j) w_j \rightarrow u_0 \quad \text{in } H^2(\Omega) \cap H^1_0(\Omega) \text{ as } k \rightarrow +\infty, \\
    u^k_1(0) &= u^k_1 = \sum_{j=1}^{k} (u_1,w_j) w_j \rightarrow u_1 \quad \text{in } H^1_0(\Omega) \text{ as } k \rightarrow +\infty, \\
    v^k(0) &= v^k_0 = \sum_{j=1}^{k} (v_0,w_j) w_j \rightarrow v_0 \quad \text{in } H^2(\Omega) \cap H^1_0(\Omega) \text{ as } k \rightarrow +\infty, \\
    v^k_1(0) &= v^k_1 = \sum_{j=1}^{k} (v_1,w_j) w_j \rightarrow v_1 \quad \text{in } H^1_0(\Omega) \text{ as } k \rightarrow +\infty, \\
    z^k(\rho,0) &= z^k_0 = \sum_{j=1}^{k} (f_0,\phi_j) \phi_j \rightarrow f_0 \quad \text{in } L^2(\Omega \times (0,1) \times (\tau_1,\tau_2)) \text{ as } k \rightarrow +\infty,
\end{align*}
$$
\[ y^k(\rho, 0) = y_0^k = \sum_{j=1}^{k} (g_{ij}) \phi_j \rightarrow g_0 \quad \text{in} \ L^2 \left( \Omega \times (0, 1) \times (\tau_1, \tau_2) \right) \quad \text{as} \ k \rightarrow +\infty. \quad (3.12) \]

By virtue of the theory of ordinary differential equations, system (3.4)–(3.12) has a unique local solution which is extended to a maximal interval \([0, T_k]\) (with \(0 < T_k \leq +\infty\)). We can utilize a standard compactness argument for the limiting procedure.

The first estimate.

**Lemma 3.2** There exists a constant \(T > 0\) such that the approximate solution satisfies, for all \(k \geq 1\):

\[
\begin{align*}
\text{\(u^k, v^k\) & bounded in} \ L^\infty(0, T; H^1_0(\Omega)), \quad (3.13) \\
\text{\(u^k_t, v^k_t\) & bounded in} \ L^\infty(0, T; L^2(\Omega)), \quad (3.14) \\
\text{\(z^k(x, \rho, \varrho, t), y^k(x, \rho, \varrho, t)\) & bounded in} \ L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))). \quad (3.15)
\end{align*}
\]

**Proof** Multiplying the first and second equations of (3.4) by \((g_{jk}^\rho)\) and \((h_{jk}^\rho)\) respectively and summing with respect to \(j\), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \|u^k_t\|^2 + \mu \|\nabla u^k\|^2 + (\lambda + \mu) \|\text{div} u^k\|^2 \right] &+ \mu \|\nabla v^k_t\|^2 + (\lambda + \mu) \|\text{div} v^k\|^2 \\
+ \int_\Omega u^k_t \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z^k(x, 1, \varrho, t) \frac{d\varrho}{dx} + \int_\Omega v^k_t \int_{\tau_1}^{\tau_2} \mu_2(\varrho) y^k(x, 1, \varrho, t) \frac{d\varrho}{dx} \\
+ k_1 \|u^k_t\|^2 + k_2 \|v^k_t\|^2 &= \{f_1(u^k, v^k), u^k_t\} + \{f_2(u^k, v^k), v^k_t\}, \quad 1 \leq j \leq k.
\end{align*}
\]

Multiplying (3.4) by \(|\mu_1(\varrho)\)(\(c_{jk}\)) and \(|\mu_2(\varrho)\)(\(d_{jk}\)) respectively, iterating over \(\Omega \times (0, 1) \times (\tau_1, \tau_2)\), and summing with respect to \(j\), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|z^k\|^2 d\varrho d\rho dx \\
+ \frac{1}{2} \frac{d}{dt} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|y^k\|^2 d\varrho d\rho dx \\
= -\frac{1}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|z^k(x, 1, \varrho, t)\|^2 d\rho dx + \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u^k_t\|^2 \\
- \frac{1}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|y^k(x, 1, \varrho, t)\|^2 d\rho dx + \frac{1}{2} \left( \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v^k_t\|^2.
\end{align*}
\]

By summing (3.16)–(3.17), we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \|u^k_t\|^2 + \mu \|\nabla u^k\|^2 + (\lambda + \mu) \|\text{div} u^k\|^2 \right] &+ \mu \|\nabla v^k_t\|^2 + (\lambda + \mu) \|\text{div} v^k\|^2 \\
+ \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|z^k(x, \rho, \varrho, t)\|^2 d\varrho d\rho dx \\
+ \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|y^k(x, \rho, \varrho, t)\|^2 d\varrho d\rho dx.
\end{align*}
\]
By using Holder and Young’s inequalities, we have

\[
\frac{1}{2} \frac{d}{dt} \left[ \| u^k(t) \|^2 + \mu \| \nabla u^k(t) \|^2 + (\lambda + \mu) \| \text{div} u^k(t) \|^2 \right]
+ \| v^k(t) \|^2 + \mu \| \nabla v^k(t) \|^2 + (\lambda + \mu) \| \text{div} v^k(t) \|^2
+ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \int_{\Omega} \| |u_1(\varphi)| \| z^k(x, \rho, t) \|^2 \, d\vartheta \, d\rho
+ \| |u_2(\varphi)| \| y^k(x, \rho, t) \|^2 \, d\vartheta \, d\rho
+ \left[ k_1 - \left( \int_{\Omega} \int_{\Omega} |u_1(\varphi)| \, d\vartheta \right) \right] \| u^k \|^2
+ \left[ k_2 - \left( \int_{\Omega} \int_{\Omega} |u_2(\varphi)| \, d\vartheta \right) \right] \| v^k \|^2
\leq \int_{\Omega} \left( u^k \cdot f_1(u^k, v^k) + v^k \cdot f_2(u^k, v^k) \right) \, dx.
\]

Integrating over \((0, t), 0 < t < T_k\), we obtain

\[
\| u^k(t) \|^2 + \mu \| \nabla u^k(t) \|^2 + (\lambda + \mu) \| \text{div} u^k(t) \|^2 + \| v^k(t) \|^2
+ \mu \| \nabla v^k(t) \|^2 + (\lambda + \mu) \| \text{div} v^k(t) \|^2
+ \int_{0}^{t} \int_{\Omega} \| |u_1(\varphi)| \| z^k(x, \rho, t) \|^2 + \| |u_2(\varphi)| \| y^k(x, \rho, t) \|^2 \, d\vartheta \, d\rho
+ \left[ k_1 - \left( \int_{\Omega} \int_{\Omega} |u_1(\varphi)| \, d\vartheta \right) \right] \int_{0}^{t} \| u^k(s) \|^2 \, ds
+ \left[ k_2 - \left( \int_{\Omega} \int_{\Omega} |u_2(\varphi)| \, d\vartheta \right) \right] \int_{0}^{t} \| v^k(s) \|^2 \, ds
\leq C_0 + c \int_{0}^{t} \int_{\Omega} \left[ u^k \cdot f_1(u^k, v^k) + v^k \cdot f_2(u^k, v^k) \right] \, dx \, ds,
\]

where

\[
C_0 = C \left( \| u^k \|_{H^1(\Omega)} \| v^k \|_{H^1(\Omega)} \| \frac{d}{dt} u^k \|_{L^2(\Omega)} \| \frac{d}{dt} v^k \|_{L^2(\Omega)}, \right.
\]
is a positive constant. We just need to estimate the right-hand terms of (3.20). Applying Hölder’s inequality, Sobolev’s embedding theorem, and Young’s inequality, we obtain

\[
\left| \int_0^t \int_{\Omega} u^k(s) f_1(u^k(s), v^k(s)) \, ds \right| \\
\leq C \int_0^t \int_{\Omega} \left( |u^k(s)|^p + |v^k(s)|^p + |\nabla u^k(s)|^p + |\nabla v^k(s)|^p \right) |u^k_t(s)| \, ds \\
\leq C \int_0^t \int_{\Omega} \left( \|u^k(s)\|_{2^p}^p + \|v^k(s)\|_{2^p}^p \right. \\
\quad + \left. \|\nabla u^k(s)\|_{p}^p \|\nabla v^k(s)\|_{p}^p \right) |u^k_t(s)| \, ds \\
\leq C \int_0^t \int_{\Omega} \left( \|u^k_t(s)\|_2^2 + \|\nabla u^k(s)\|_2^2 \|\nabla v^k(s)\|_2^2 \right) \, ds \\
\quad + \|\nabla u^k(s)\|_{p-1}^{p-1} \|\nabla v^k(s)\|_{p-1}^{p-1} \, ds,
\]

(3.21)

when we have used in (3.21) the Sobolev imbedding in (2.5) and the fact when \( n = 3 \) then \( 2p = 3(p - 1) = \frac{3(p+1)}{2} = 6 \).

Likewise, we obtain

\[
\left| \int_0^t \int_{\Omega} v^k(s) f_2(u^k(s), v^k(s)) \, ds \right| \\
\leq C \int_0^t \int_{\Omega} \left( \|v^k_t(s)\|_2^2 + \|\nabla u^k(s)\|_2^2 \|\nabla v^k(s)\|_2^2 \right) \, ds \\
\quad + \|\nabla u^k(s)\|_{p+1}^{p+1} \|\nabla v^k(s)\|_{p+1}^{p+1} \, ds.
\]

(3.22)

Let

\[
X_k(t) = \|u^k_t(t)\|_2^2 + \|v^k_t(t)\|_2^2 + \|\nabla u^k(t)\|_2^2 + \|\nabla v^k(t)\|_2^2.
\]

(3.23)

From assumptions of Lemma 2.1, we can find positive constants such that

\[
X_k(t) + c_1 \|\text{div} u^k(t)\|_2^2 + c_1 \|\text{div} v^k(t)\|_2^2 \\
\quad + c_2 \int_0^t \int_{\Omega} \rho \left[ ||\mu_1(\varphi)|| \|\nabla^2 (x, \rho, \varphi, t)\|_2^2 + c_2 |\mu_2(\varphi)|| \|\nabla^2 (x, \rho, \varphi, t)\|_2^2 \right] \, d\varphi \, d\rho \\
\quad + c_3 \int_0^t \|u^k_t(s)\|_2^2 \, ds + c_3 \int_0^t \|v^k_t(s)\|_2^2 \, ds \\
\leq C_0 + c \int_0^t \int_{\Omega} (X_k(t))^p \, dx \, ds
\]

(3.24)

Particulary, we have

\[
X_k(t) \leq C_0 + c \int_0^t \int_{\Omega} (X_k(t))^p \, dx \, ds.
\]

(3.25)
Using Gronwall-type inequality, we can get
\[ X_k(t) \leq \left[ C_0 - (p - 1)Ct \right]^{1/(p-1)}. \] (3.26)

Thus, we deduce from (3.26) that there exists a time \( T > 0 \) such that
\[ X_k(t) \leq C_1, \quad \forall t \in [0, T], \] (3.27)
where \( C_1 \) is a positive constant independent of \( k \). Then inequality (3.27) established the first two parts of lemma. The last part of lemma immediately follows from (3.24). \( \square \)

**The second estimate:** First, we are going to estimate \( u_{tt}^k(0) \) and \( v_{tt}^k(0) \). Testing the first and second equations in (3.4) by \( g_j^\prime(t) \) and \( h_j''(t) \) respectively and taking \( t = 0 \), we obtain
\[
\left\| u_{tt}^k(0) \right\|_2^2 + \left\| v_{tt}^k(0) \right\|_2^2 
\leq \left\| \Delta u_0^k \right\|_2^2 + \left\| \Delta v_0^k \right\|_2^2 + c \left\| \text{div } u_0^k \right\|_2^2 + c \left\| \text{div } v_0^k \right\|_2^2 + \left\| \Delta u_0^k(0) \right\|_2^2 
+ \left\| \Delta v_0^k(0) \right\|_2^2 + c \int_{\Omega} \int_{\tau_1}^{\tau_2} \left| \mu_1(\varrho) \right| \left| \Delta x_j^k(x, 1, \varrho, 0) \right|^2 \, d\varrho \, dx 
+ c \int_{\Omega} \int_{\tau_1}^{\tau_2} \left| \mu_2(\varrho) \right| \left| \Delta y_j^k(x, 1, \varrho, 0) \right|^2 \, d\varrho \, dx. \] (3.28)

From (3.7)–(3.12), we have
\[
\left\| u_{tt}^k(0) \right\|_2^2 + \left\| v_{tt}^k(0) \right\|_2^2 \leq C. \] (3.29)

In order to calculate the second estimate, we take the derivatives of the first and second equations of system (3.4) with respect to \( t \), we get
\[
\begin{align*}
\left[u_{tt}^k, w_j\right] + \mu \left(\nabla u_j^k, \nabla w_j\right) + (\lambda + \mu) \left(\text{div } u_j^k, \text{div } w_j\right) \\
+ k_1 [u_{tt}^k, w_j] + \left\{ \int_{\tau_1}^{\tau_2} \left| \mu_1(\varrho) \right| x_j^k(x, 1, \varrho, t) \, d\varrho, w_j \right\} = \left\{ Df_1(u_k, v_k), w_j \right\}, \quad 1 \leq j \leq k
\end{align*}
\]
and
\[
\begin{align*}
\left[v_{tt}^k, w_j\right] + \mu \left(\nabla v_j^k, \nabla w_j\right) + (\lambda + \mu) \left(\text{div } v_j^k, \text{div } w_j\right) \\
+ k_2 [v_{tt}^k, w_j] + \left\{ \int_{\tau_1}^{\tau_2} \left| \mu_2(\varrho) \right| y_j^k(x, 1, \varrho, t) \, d\varrho, w_j \right\} = \left\{ Df_2(u_k, v_k), w_j \right\}, \quad 1 \leq j \leq k.
\end{align*}
\]
Multiplying by \( (g_j^\prime(t)) \) and \( (h_j''(t)) \) respectively and summing with respect to \( j \) from 1 to \( k \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ \left\| u_{tt}^k(t) \right\|^2 + \left\| \nabla u_j^k(t) \right\|^2 + (\lambda + \mu) \left\| \text{div } u_j^k(t) \right\|^2 \right] + k_1 \left\| u_{tt}^k(t) \right\|^2 
+ \int_{\Omega} u_{tt}^k(t) \int_{\tau_1}^{\tau_2} \left| \mu_1(\varrho) \right| x_j^k(x, 1, \varrho, t) \, d\varrho \, dx 
\]
\[
= \int_{\Omega} Df_1(u_k, v_k) u_{tt}^k(t) \, dx, \quad 1 \leq j \leq k, \tag{3.30}
\]
\[
\frac{1}{2} \frac{d}{dt} \left[ \left\| v_{tt}^k(t) \right\|^2 + \left\| \nabla v_j^k(t) \right\|^2 + (\lambda + \mu) \left\| \text{div } v_j^k(t) \right\|^2 \right] + k_2 \left\| v_{tt}^k(t) \right\|^2 
+ \int_{\Omega} v_{tt}^k(t) \int_{\tau_1}^{\tau_2} \left| \mu_2(\varrho) \right| y_j^k(x, 1, \varrho, t) \, d\varrho \, dx
\]
\[
= \int_{\Omega} Df_2(u_k, v_k) v_{tt}^k(t) \, dx, \quad 1 \leq j \leq k.
\]
and

\[
\frac{1}{2} \frac{d}{dt} \left[ \| v^h_n \|^2 + \mu \| \nabla v^h_n \|^2 + (\lambda + \mu) \| \text{div } v^h_n \|^2 \right] + k_j \| v^h_n \|^2
\]

\[+ \int_\Omega v^h_n(t) \int_{t_1}^{t_2} |\mu_2(\varrho)| y^j_t(x, 1, \varrho, t) \, d\varrho \, dx\]

\[= \int_\Omega Df_2(u_k, v_k) v^h_n(t) \, dx, \quad 1 \leq j \leq k.\]  

(3.31)

Differentiating the third and fourth equations in (3.4) with respect to \( t \), we get

\[
\begin{align*}
\langle \varrho z^j_{\rho \alpha} + \frac{\partial}{\partial \rho} z^j_{\rho \alpha}, \phi_j \rangle &= 0, \quad 1 \leq j \leq k, \\
\langle \varrho y^j_{\rho \alpha} + \frac{\partial}{\partial \rho} y^j_{\rho \alpha}, \phi_j \rangle &= 0, \quad 1 \leq j \leq k.
\end{align*}
\]

Multiplying by \(|\mu_1(\varrho)| \varrho^j_{\mu \rho}\) and \(|\mu_2(\varrho)| \varrho^j_{\mu \rho}\) respectively, integrating over \((0, 1) \times (t_1, t_2)\), and summing over \( j \) from 1 to \( k \), it follows that

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \int_{t_1}^{t_2} \int_\Omega e|\mu_1(\varrho)| |z^j_t(x, \rho, \varrho, t)|^2 \, d\varrho \, d\rho \, dx
\]

\[+ \frac{1}{2} \int_0^1 \int_{t_1}^{t_2} \int_\Omega e|\mu_1(\varrho)| \frac{d}{d\varrho} |z^j_t(x, \rho, \varrho, t)|^2 \, d\varrho \, d\rho \, dx = 0,
\]

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \int_{t_1}^{t_2} \int_\Omega e|\mu_2(\varrho)| |y^j_t(x, \rho, \varrho, t)|^2 \, d\varrho \, d\rho \, dx
\]

\[+ \frac{1}{2} \int_0^1 \int_{t_1}^{t_2} \int_\Omega e|\mu_2(\varrho)| \frac{d}{d\varrho} |y^j_t(x, \rho, \varrho, t)|^2 \, d\varrho \, d\rho \, dx = 0,
\]

then we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \int_{t_1}^{t_2} \int_\Omega e|\mu_1(\varrho)| |z^j_t(x, \rho, \varrho, t)|^2 \, d\varrho \, d\rho \, dx
\]

\[+ \frac{1}{2} \int_{t_1}^{t_2} |\mu_1(\varrho)| |z^j_t(x, 1, \varrho, t)|^2 \, d\varrho \, dx
\]

\[- \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_1(\varrho)| \, d\varrho \right) \| v^h_n(t) \|^2 \, dx = 0.\]  

(3.32)

and

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \int_{t_1}^{t_2} \int_\Omega e|\mu_2(\varrho)| |y^j_t(x, \rho, \varrho, t)|^2 \, d\varrho \, d\rho \, dx
\]

\[+ \frac{1}{2} \int_{t_1}^{t_2} |\mu_2(\varrho)| |y^j_t(x, 1, \varrho, t)|^2 \, d\varrho \, dx
\]

\[- \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_2(\varrho)| \, d\varrho \right) \| v^h_n(t) \|^2 \, dx = 0.\]  

(3.33)
Taking the sum of (3.30), (3.31),(3.32), and (3.33), we get

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \nu^k_t \|^2 + \mu \| \nabla u^k_t \|^2 + (\lambda + \mu) \| \text{div} u^k_t \|^2 + \| v^k_t \|^2 + \mu \| \nabla v^k_t \|^2 + (\lambda + \mu) \| \text{div} v^k_t \|^2 \right] \\
+ \frac{1}{2} \frac{d}{dt} \int_0^1 \int_{\Gamma_1} \int_{\Omega_1} e |\mu_1(\varphi)| |z^k_t(x, \rho, \varphi, t)|^2 \, d\varphi \, d\rho \, dx \\
+ \int_0^1 \int_{\Gamma_1} \int_{\Omega_1} e |\mu_2(\varphi)| |y^k_t(x, \rho, \varphi, t)|^2 \, d\varphi \, d\rho \, dx \\
+ k_1 \| u^k_t \|^2 + k_2 \| v^k_t \|^2 + \frac{1}{2} \int_0^1 \int_{\Gamma_1} |\mu_1(\varphi)| |z^k_t(x, 1, \varphi, t)|^2 \, d\varphi \, dx \\
+ \frac{1}{2} \int_{\Omega} \int_{\tau_1} |\mu_2(\varphi)| |y^k_t(x, 1, \varphi, t)|^2 \, d\varphi \, dx \\
= \frac{1}{2} \left( \int_{\tau_1} |\mu_1(\varphi)| \, d\varphi \right) \| u^k_t(t) \|^2 + \frac{1}{2} \left( \int_{\tau_1} |\mu_2(\varphi)| \, d\varphi \right) \| v^k_t(t) \|^2 \\
- \int_{\Omega} u^k_t(t) \int_{\tau_1} |\mu_1(\varphi)| z^k_t(x, 1, \varphi, t) \, d\varphi \, dx \\
- \int_{\Omega} v^k_t(t) \int_{\tau_1} |\mu_2(\varphi)| y^k_t(x, 1, \varphi, t) \, d\varphi \, dx \\
+ \int_{\Omega} Df_1(u_k, v_k) u^k_t(t) \, dx + \int_{\Omega} Df_2(u_k, v_k) v^k_t(t) \, dx.
\]

Using Cauchy–Schwarz and Young’s inequalities, we conclude

\[
\left| \int_{\Omega} u^k_t(t) \int_{\tau_1} |\mu_1(\varphi)| z^k_t(x, 1, \varphi, t) \, d\varphi \, dx \right| \\
\leq \frac{1}{2} \left( \int_{\tau_1} |\mu_1(\varphi)| \, d\varphi \right) \| u^k_t(t) \|^2 \\
+ \frac{1}{2} \int_{\tau_1} |\mu_1(\varphi)| \| z^k_t(x, 1, \varphi, t) \|^2 \, d\varphi. \tag{3.35}
\]

Similarly,

\[
\left| \int_{\Omega} v^k_t(t) \int_{\tau_1} |\mu_2(\varphi)| y^k_t(x, 1, \varphi, t) \, d\varphi \, dx \right| \\
\leq \frac{1}{2} \left( \int_{\tau_1} |\mu_2(\varphi)| \, d\varphi \right) \| v^k_t(t) \|^2 \\
+ \frac{1}{2} \int_{\tau_1} |\mu_2(\varphi)| \| y^k_t(x, 1, \varphi, t) \|^2 \, d\varphi. \tag{3.36}
\]

For the source term

\[
\int_{\Omega} Df_1(u_k, v_k) u^k_t(t) \, dx \\
\leq C \left[ \left( \| u^k \|^{p-1} + \| v^k \|^{p-1} \right) \| u^k_t \| + \left( \| u^k \|^{p-1} + \| v^k \|^{p-1} \right) \| v^k_t \| \right] \| u^k_t \| \\
\leq C \left[ \| u^k \|^{2(p-1)} + \| v^k \|^{2(p-1)} + \| u^k_t \|^2 + \| v^k_t \|^2 \right] \| u^k_t \|_2. \tag{3.37}
\]
After simplification, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \|u_n^k(t)\|^2 + \|\nabla u_n^k\|^2 + (\lambda + \mu) \|\nabla v_n^k\|^2 + \|v_n^k\|^2 + \mu \|\nabla v_n^k\|^2 + (\lambda + \mu) \|\text{div} v_n^k\|^2 \right]
\]

and

\[
\int_\Omega D_f^2(u_n, v_n) v_n^k(t) \, dx \\
\leq C \left[ (\|u_n^k\|^2 + \|\nabla u_n^k\|^2 + (\lambda + \mu) \|\nabla v_n^k\|^2 + \|v_n^k\|^2 + \mu \|\nabla v_n^k\|^2 + (\lambda + \mu) \|\text{div} v_n^k\|^2) \right] \|v_n^k\|_2^2
\]

(3.38)

Taking into account (3.35)–(3.38) into (3.34), we get

\[
\frac{1}{2} \frac{d}{dt} \left[ \|u_n^k\|^2 + \|\nabla u_n^k\|^2 + (\lambda + \mu) \|\nabla v_n^k\|^2 + \|v_n^k\|^2 + \mu \|\nabla v_n^k\|^2 + (\lambda + \mu) \|\text{div} v_n^k\|^2 \right]
\]

\[
+ \frac{1}{2} \frac{d}{dt} \int_0^1 \int_\Omega \int_{t_1}^{t_2} \varphi^2 |\mu_1(\varphi)| |z_n^k(x, \varphi, \rho, t)|^2 \, d\varphi \, d\rho \, dx
\]

\[
+ \frac{1}{2} \int_0^1 \int_{t_1}^{t_2} \int_\Omega \varphi |\mu_1(\varphi)| |y_n^k(x, \varphi, \rho, t)|^2 \, d\varphi \, d\rho \, dx
\]

\[
+ k_1 \|u_n^k\|^2 + k_2 \|v_n^k\|^2 + \frac{1}{2} \int_{t_1}^{t_2} \|\mu_2(\varphi)\| |y_n^k(x, 1, \varphi, t)|^2 \, d\varphi
\]

\[
\leq \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_1(\varphi)| \, d\varphi \right) \left( \int_{t_1}^{t_2} |u_n^k(\varphi)| \, d\varphi \right)^2 + \frac{1}{2} \left( \int_{t_1}^{t_2} |\mu_2(\varphi)| \, d\varphi \right) \left( \int_{t_1}^{t_2} |v_n^k(\varphi)| \, d\varphi \right)^2
\]

\[
+ \frac{1}{2} \int_{t_1}^{t_2} |\mu_1(\varphi)| \, d\varphi \right) \left( \int_{t_1}^{t_2} |u_n^k(\varphi)| \, d\varphi \right)^2 + \frac{1}{2} \int_{t_1}^{t_2} |\mu_2(\varphi)| \, d\varphi \right) \left( \int_{t_1}^{t_2} |v_n^k(\varphi)| \, d\varphi \right)^2
\]

\[
+ \|u_n^k(t)\|^2 + \|v_n^k(t)\|^2 + c.
\]
Integrating (3.39) over \((0, t)\), we get

\[
\left[ \| u^k(t) \|^2 + \mu \| \nabla u^k(t) \|^2 + (\lambda + \mu) \| \text{div} u^k(t) \|^2 + \| v^k(t) \|^2 \right] \\
\times \left[ \int_0^1 \int_\Omega \int_{\mathcal{F}} \mu_1(\varphi) \| z^k(\varphi) \|^2 \, d\varphi \, d\rho \, dx \right] \\
+ \frac{1}{2} \int_0^1 \int_\Omega \int_{\mathcal{F}} \mu_1(\varphi) \| y^k(\varphi) \|^2 \, d\varphi \, d\rho \, dx \\
+ \left( k_1 - \left( \int_{\mathcal{F}} \mu_1(\varphi) \right) \right) \int_0^t \| u^k_1(s) \|^2 \, ds \\
+ \left( k_2 - \left( \int_{\mathcal{F}} \mu_2(\varphi) \right) \right) \int_0^t \| v^k_1(s) \|^2 \, ds \\
\leq \left[ \| u^k_1(0) \|^2 + \mu \| \nabla u^k_1(0) \|^2 + (\lambda + \mu) \| \text{div} u^k_1(0) \|^2 + \| v^k_1(0) \|^2 \right] \\
\times \left[ \int_0^1 \int_\Omega \int_{\mathcal{F}} \mu_1(\varphi) \| z^k(\varphi) \|^2 \, d\varphi \, d\rho \, dx \right] \\
+ \frac{1}{2} \int_0^1 \int_\Omega \int_{\mathcal{F}} \mu_1(\varphi) \| y^k(\varphi) \|^2 \, d\varphi \, d\rho \, dx \\
+ \int_0^t \| u^k_1(s) \|^2 \, ds + \int_0^t \| v^k_1(s) \|^2 \, ds + c.
\]

Taking

\[
y^k(t) = \| u^k(t) \|^2 + \| v^k(t) \|^2
\]

and by using Gronwall’s inequality, we conclude that

\[
u^k_1 \text{ and } v^k_1 \text{ are bounded in } L^\infty(0, T; H^1(\Omega)), \tag{3.40}
\]

\[
u^k_1 \text{ and } v^k_1 \text{ are bounded in } L^\infty(0, T; L^2(\Omega)), \tag{3.41}
\]

\[
\begin{align*}
z^k_1 \text{ and } y^k_1 & \text{ are bounded in } L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))) \\
& \tag{3.42}
\end{align*}
\]

Applying Dunford–Pettis’ theorem, we deduce from (3.13), (3.14), (3.15), (3.40), (3.41), and (3.42), replacing the sequence \( u_k \) with a subsequence if necessary, that

\[
u^k \rightarrow \nu, \nu^k \rightarrow \nu \text{ weak-star in } L^\infty(0, T; H^1(\Omega)), \tag{3.43}
\]

\[
z^k \rightarrow z, y^k \rightarrow y \text{ weak-star in } L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))), \tag{3.44}
\]

\[
u^k_1 \rightarrow \nu_1, \nu^k_1 \rightarrow \nu_1 \text{ weak-star in } L^\infty(0, T; H^1(\Omega)), \tag{3.45}
\]

\[
u^k_1 \rightarrow \nu_1, \nu^k_1 \rightarrow \nu_1 \text{ weak-star in } L^\infty(0, T; L^2(\Omega)), \tag{3.46}
\]

\[
z^k_1 \rightarrow z_1, y^k_1 \rightarrow y_1 \text{ weak-star in } L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))). \tag{3.47}
\]
Corollary 3.3  The sequences of approximate solutions \( \{ u_k, v_k \} \) satisfy, as \( k \to \infty \),

\[
\begin{align*}
&f_1(u_k, v_k) \to f_1(u, v) \text{ strongly in } L^\infty(0, T; L^2(\Omega)), \\
&f_2(u_k, v_k) \to f_2(u, v) \text{ strongly in } L^\infty(0, T; L^2(\Omega)).
\end{align*}
\]  

(3.48)

Proof  The proof is similar to that of [11]. 

We can complete the proof of theorem as in [2].

4 General decay

In this section we prove that the solution of problem (2.8)–(2.9) decays generally to a trivial solution using the energy method and a suitable Lyapunov functional.

In the following, we present our main stability result.

Theorem 4.1  The solution of (2.8) satisfies, for two positive constants \( \alpha, \beta \), the estimate

\[
E(t) \leq \beta e^{-\alpha t}, \quad \forall t \geq 0.
\]  

(4.1)

To prove the desired result, we create a Lyapunov functional equivalent to \( E \). For this, we define some functions that allow us to construct this Lyapunov function.

Lemma 4.2  Let \((u, v, z, y)\) be a solution of problem (2.8). Then the functional

\[
\varphi(t) = \int_\Omega u(t)u_t(t) \, dx + \int_\Omega v(t)v_t(t) \, dx
\]  

(4.2)

satisfies the estimate

\[
\varphi'(t) \leq \left( 1 + \frac{Ck_1^2}{\mu} \right) \| u_t(t) \|^2 + \left( 1 + \frac{Ck_2}{\mu} \right) \| v_t(t) \|^2 - \frac{\mu}{2} \| \nabla u(t) \|^2 - \frac{\mu}{2} \| \nabla v(t) \|^2 \\
- (\lambda + \mu) \| \text{div} \, u(t) \|^2 - (\lambda + \mu) \| \text{div} \, v(t) \|^2 \\
+ \frac{Ck_1}{\mu} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x, 1, \varrho, t)|^2 \, d\varrho \, dx \\
+ \frac{Ck_2}{\mu} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\nabla y(x, 1, \varrho, t)|^2 \, d\varrho \, dx \\
+ (p + 1) \int_\Omega F(u(t), v(t)) \, dx.
\]  

(4.3)

Proof  Taking the derivative of (4.2), we obtain

\[
\varphi'(t) = \int_\Omega |u_t(t)|^2 \, dx + \int_\Omega u(t)u_{tt}(t) \, dx + \int_\Omega |v_t(t)|^2 \, dx + \int_\Omega v(t)v_{tt}(t) \, dx.
\]
From problem (2.8) and using integration by parts, we get

\[ \varphi'(t) = \| u_t(t) \|^2 + \| v_t(t) \|^2 \]
\[ + \int_\Omega u(t) \left( \Delta_x u - k_1 u_t - \int_{\tau_1}^{\tau_2} \mu_1(\varphi) z(x, 1, \varphi, t) d\varphi + f_1(u, v) \right) dx \]
\[ + \int_\Omega v(t) \left( \Delta_x v - k_2 \Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\varphi) y(x, 1, \varphi, t) d\varphi + f_2(u, v) \right) dx \]
\[ = \| u_t(t) \|^2 + \| v_t(t) \|^2 - k_1 \int_\Omega u u_t dx - k_2 \int_\Omega v v_t dx \quad (4.4) \]

By using Hölder’s and Young’s inequalities, we have

\[ \int_\Omega u(t) \int_{\tau_1}^{\tau_2} \mu_1(\varphi) z(x, 1, \varphi, t) d\varphi dx \]
\[ \leq \frac{\varepsilon}{2} \| u(t) \|^2 + \frac{1}{2\varepsilon} \int_\Omega \left( \int_{\tau_1}^{\tau_2} \left| \mu_1(\varphi) \right| z(x, 1, \varphi, t) d\varphi \right)^2 dx \]
\[ \leq \frac{C\varepsilon}{2} \| \nabla u(t) \|^2 + \frac{1}{2\varepsilon} \int_\Omega \left( \int_{\tau_1}^{\tau_2} \left| \mu_1(\varphi) \right| d\varphi \right) \int_\Omega \int_{\tau_1}^{\tau_2} \left| \mu_1(\varphi) \right| \left| \nabla z(x, 1, \varphi, t) \right|^2 d\varphi \]
\[ \leq \frac{C\varepsilon}{2} \| \nabla u(t) \|^2 + \frac{k_1}{2\varepsilon} \int_\Omega \int_{\tau_1}^{\tau_2} \left| \mu_1(\varphi) \right| \left| z(x, 1, \varphi, t) \right|^2 d\varphi. \quad (4.5) \]

Similarly,

\[ \int_\Omega v(t) \int_{\tau_1}^{\tau_2} \mu_1(\varphi) y(x, 1, \varphi, t) d\varphi dx \]
\[ \leq \frac{C\varepsilon}{2} \| \nabla v(t) \|^2 + \frac{k_2}{2\varepsilon} \int_\Omega \int_{\tau_1}^{\tau_2} \left| \mu_1(\varphi) \right| \left| y(x, 1, \varphi, t) \right|^2 d\varphi. \quad (4.6) \]

Young’s and Poincaré’s inequalities give

\[ k_1 \int_\Omega u(t) u_t(t) dx \leq \frac{C\varepsilon}{2} \| \nabla u(t) \|^2 + \frac{k_1^2}{2\varepsilon} \| u_t(t) \|^2 \quad (4.7) \]

and

\[ k_2 \int_\Omega v(t) \nabla v_t(t) dx \leq \frac{C\varepsilon}{2} \| \nabla v(t) \|^2 + \frac{k_2^2}{2\varepsilon} \| v_t(t) \|^2. \quad (4.8) \]

For the source term, we have

\[ \int_\Omega u(t) f_1(u, v) dx + \int_\Omega v(t) f_2(u, v) dx = (p + 1) \int_\Omega F(u, v) dx. \quad (4.9) \]

Combining equations (4.4)–(4.9) and taking \( \varepsilon = \frac{\mu}{\pi^2} \), thus, our proof is completed. \( \square \)
Lemma 4.3 Let \((u, v, z, y)\) be a solution of problem (2.8). Then the functional

\[
I(t) = \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} \left[ |\mu_{1}(\varphi)| \left| z(x, \rho, \varphi, t) \right|^{2} + |\mu_{2}(\varphi)| \left| y(x, \rho, \varphi, t) \right|^{2} \right] dx \, d\varphi \, d\rho \tag{4.10}
\]
satisfies the estimate

\[
I'(t) \leq -e^{-\omega \tau_{3}} \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} \left[ |\mu_{1}(\varphi)| \left| z(x, 1, \varphi, t) \right|^{2} \right] dx \, d\varphi \, d\rho
- e^{-\omega \tau_{3}} \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} |\mu_{2}(\varphi)| \left| y(x, 1, \varphi, t) \right|^{2} dx \, d\varphi \, d\rho
+ k_{1} \left\| u_{1}(t) \right\|^{2} + k_{2} \left\| v_{1}(t) \right\|^{2} - I(t). \tag{4.11}
\]

Proof Differentiating (4.10) with respect to \(t\), we get

\[
\frac{d}{dt} I(t) = 2 \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} |\mu_{1}(\varphi)| z(x, \rho, \varphi, t) z_{t}(x, \rho, \varphi, t) dx \, d\varphi \, d\rho
+ 2 \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} |\mu_{2}(\varphi)| y(x, \rho, \varphi, t) y_{t}(x, \rho, \varphi, t) dx \, d\varphi \, d\rho.
\]

By using (2.6)–(2.7), we have

\[
\frac{d}{dt} I(t) = -\int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} \left[ |\mu_{1}(\varphi)| \frac{\partial}{\partial \rho} |z(x, \rho, \varphi, t)|^{2} \right] dx \, d\varphi \, d\rho
+ |\mu_{2}(\varphi)| \frac{\partial}{\partial \rho} \left( e^{-\omega \varphi} |z(x, \rho, \varphi, t)|^{2} \right) dx \, d\varphi \, d\rho
- \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} \left[ |\mu_{1}(\varphi)| \left| z(x, 1, \varphi, t) \right|^{2} + |\mu_{2}(\varphi)| \left| y(x, 1, \varphi, t) \right|^{2} \right] dx \, d\varphi \, d\rho.
\]

Thus,

\[
\frac{d}{dt} I(t) = -\int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} |\mu_{1}(\varphi)| \left| z(x, 1, \varphi, t) \right|^{2} dx \, d\varphi + \left( \int_{\tau_{1}}^{\tau_{2}} |\mu_{1}(\varphi)| d\varphi \right) \left\| u_{1}(x, \varphi, t) \right\|^{2}
- \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} |\mu_{2}(\varphi)| \left| y(x, 1, \varphi, t) \right|^{2} dx \, d\varphi + \left( \int_{\tau_{1}}^{\tau_{2}} |\mu_{2}(\varphi)| d\varphi \right) \left\| v_{1}(x, \varphi, t) \right\|^{2}
- \int_{\Omega} \int_{0}^{1} \int_{\tau_{1}}^{\tau_{2}} e^{-\omega \varphi} \left[ |\mu_{1}(\varphi)| \left| z(x, \rho, \varphi, t) \right|^{2} + |\mu_{2}(\varphi)| \left| y(x, \rho, \varphi, t) \right|^{2} \right] dx \, d\varphi \, d\rho.
\]

Since \(e^{-\omega \varphi}\) is a decreasing function over \((\tau_{1}, \tau_{2})\), the desired estimate (4.11) follows immediately from (2.11).

Proof of Theorem 4.1 Now, we define the following functional:

\[
\mathcal{F}(t) = NE(t) + \varphi(t) + N_{2} I(t), \tag{4.12}
\]
where \( N \) and \( N_2 \) are positive constants. It is easy to prove \( F(t) \) and \( E(t) \) are equivalent, namely there exist two positive constants \( \kappa_1 \) and \( \kappa_2 \) such that

\[
\kappa_1 E(t) \leq F(t) \leq \kappa_2 E(t). \tag{4.13}
\]

From Lemmas 4.2 and 4.3, we have, for any \( t \geq 0 \),

\[
F'(t) \leq -N\left[ k_1 - \left( \int_{\Omega_1} |\mu_1(\varrho)| d\varrho \right) \right] \| u_t \|^2 - N\left[ k_2 - \left( \int_{\Omega_1} |\mu_2(\varrho)| d\varrho \right) \right] \| v_t \|^2 \\
+ \left( 1 + \frac{Ck_1}{\mu} \right) \| u_t(t) \|^2 + \left( 1 + \frac{Ck_2}{\mu} \right) \| v_t(t) \|^2 - \frac{\mu}{2} \| \nabla u(t) \|^2 - \frac{\mu}{2} \| \nabla v(t) \|^2 \\
- (\lambda + \mu) \| \text{div} \ u(t) \|^2 - (\lambda + \mu) \| \text{div} \ v(t) \|^2 \\
+ \frac{Ck_1}{\mu} \int_{\Omega} \int_{\tau_1} |\mu_1(\varrho)| \| \nabla z(x, 1, \varrho, t) \|^2 d\varrho \ dx \\
+ \frac{Ck_2}{\mu} \int_{\Omega} \int_{\tau_1} |\mu_2(\varrho)| \| \nabla y(x, 1, \varrho, t) \|^2 d\varrho \ dx \\
+ (p + 1) \int_{\Omega} F(u(t), v(t)) \ dx \tag{4.14}
\]

\[
- N_2 e^{-\tau_2} \int_{\Omega} \int_{\tau_1} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 \ dx \ d\varrho \\
- N_2 e^{-\tau_2} \int_{\Omega} \int_{\tau_1} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 \ dx \ d\varrho \\
- N_2 \int_{\Omega} \int_{0}^{1} \int_{\tau_1} \varrho e^{-\varrho \rho} \left[ |\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 \\
+ |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2 \right] \ dx \ d\varrho \ d\rho \\
+ N_2 k_1 \| u_t(t) \|^2 + N_2 k_2 \| v_t(t) \|^2.
\]

Thus

\[
F'(t) \leq -\left( N\sigma_1 - \left( 1 + \frac{Ck_1^2}{\mu} \right) - N_2 k_1 \right) \| u_t \|^2 - \left( N\sigma_2 - \left( 1 + \frac{Ck_2^2}{\mu} \right) - N_2 k_2 \right) \| v_t \|^2 \\
- \frac{\mu}{2} \| \nabla u(t) \|^2 - \frac{\mu}{2} \| \nabla v(t) \|^2 - (\lambda + \mu) \| \text{div} \ u(t) \|^2 - (\lambda + \mu) \| \text{div} \ v(t) \|^2 \\
- \left( N_2 e^{-\tau_2} - \frac{Ck_1}{\mu} \right) \int_{\Omega} \int_{\tau_1} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 \ dx \ d\varrho \\
- \left( N_2 e^{-\tau_2} - \frac{Ck_2}{\mu} \right) \int_{\Omega} \int_{\tau_1} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 \ dx \ d\varrho \tag{4.15}
\]

\[
+ (p + 1) \int_{\Omega} F(u(t), v(t)) \ dx \\
- N_2 \int_{\Omega} \int_{0}^{1} \int_{\tau_1} \varrho e^{-\varrho \rho} \left[ |\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 \\
+ |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2 \right] \ dx \ d\varrho \ d\rho.
\]
where

$$\sigma_1 = \left[ k_1 - \left( \int_{t_1}^{t_2} |\mu_1(\varphi)| \, d\varphi \right) \right], \quad \sigma_2 = \left[ k_2 - \left( \int_{t_1}^{t_2} |\mu_2(\varphi)| \, d\varphi \right) \right].$$

First, we take $N_2$ large such that

$$N_2 e^{\tau_2} - \frac{C k_1}{\mu} > 0 \quad \text{and} \quad N_2 e^{\tau_2} - \frac{C k_2}{\mu} > 0.$$

We choose $N > 0$ large enough so that

$$N \sigma_1 - \left( 1 + \frac{C k^2_1}{\mu} \right) - N_2 k_1 > 0$$

and

$$N \sigma_2 - \left( 1 + \frac{C k^2_2}{\mu} \right) - N_2 k_2 > 0$$

and $F \sim E$. Thus we arrive at

$$F'(t) \leq -\alpha F(t),$$

which yields, by integration,

$$F(t) \leq F(0) e^{-\alpha t}, \quad \forall t \geq 0.$$

The use of $F \sim E$ gives

$$E(t) \leq \beta e^{-\alpha t}, \quad \forall t \geq 0.$$

Acknowledgements
The first author would like to thank all the professors of the mathematics department at the University of Annaba in Algeria, especially his Professors/Scientists Pr. Mohamed Haiour, Pr. Ahmed-Salah Chibi, and Pr. Azzedine Benchettah for the important content of masters and PhD courses in pure and applied mathematics which he received during his studies. Moreover, he thanks them for the additional help they provided to him during office hours in their office about the few concepts/difficulties he had encountered, and he appreciates their talent and dedication for their postgraduate students currently and previously. In addition, the idea and research project in this paper was presented by the first author (Pr. Salah Boulaaras) and was carried out by both authors. The authors are grateful to the anonymous referees for the careful reading and their important observations/suggestions for the sake of improving this paper.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this manuscript. The authors declare that they have no competing interests.

Consent for publication
Not applicable.
Authors' contributions
The authors contributed equally in this article. They have all read and approved the final manuscript.

Author details
1 Department of Mathematics, College of Sciences and Arts, ArRas, Qassim University, Buraidah, Kingdom of Saudi Arabia
2 Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, Ahmed Benbella, Oran, Algeria. 3 Laboratory of Operator Theory and PDEs: Foundations and Applications, Department of Mathematics, Faculty of Exact Sciences, University of El Oued, Box. 789, El Oued, 39000, Algeria.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 October 2020 Accepted: 20 November 2020 Published online: 30 November 2020

References
1. Mezouar, N., Boulaaras, S.: Global existence and exponential decay of solutions for generalized coupled non-degenerate Kirchhoff system with a time varying delay term. Bound. Value Probl. 2020, 90 (2020). https://doi.org/10.1186/s13661-020-01390-9
2. Boulaaras, S., Guefaifia, R., Mezouar, N., Alghamdi, A.M.: Global existence and decay for a system of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms. J. Funct. Spaces 2020, Article ID 5085101 (2020). https://doi.org/10.1155/2020/5085101
3. Ouchenane, D., Boulaaras, S., Alharbi, A., Cherrif, B.: Blow up of coupled nonlinear Klein–Gordon system with distributed delay, strong damping, and source terms. J. Funct. Spaces 2020, Article ID 5297063 (2020). https://doi.org/10.1155/2020/5297063
4. Doudi, N., Boulaaras, S.: Global existence combined with general decay of solutions for coupled Kirchhoff system with a distributed delay term. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 114, 204 (2020). https://doi.org/10.1007/s13398-020-00938-9
5. Gala, S., Ragusa, M.A.: Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices. Appl. Anal. 95(6), 1271–1279 (2016)
6. Gala, S., Liu, Q., Ragusa, M.A.: A new regularity criterion for the nematic liquid crystal flows. Appl. Anal. 91(9), 1741–1747 (2012)
7. Guliyev, V.S., Guliyev, R.V., Omarova, M.N., Ragusa, M.A.: Schrödinger type operators on local generalized Morrey spaces related to certain nonnegative potentials. Discrete Contin. Dyn. Syst., Ser. B 25(2), 671–690 (2020)
8. Lui, G.: Well-posedness and exponential decay of solutions for a transmission problem with distributed delay. Electron. J. Differ. Equ. 2017(174), 1 (2017)
9. Wu, S-T: On decay and blow up of solutions for a system of nonlinear wave equations. J. Math. Anal. Appl. 394, 360–377 (2012)
10. Mezouar, N., Boulaaras, S.: Global existence and decay of solutions of a singular nonlocal viscoelastic system with damping terms. Topol. Methods Nonlinear Anal. 56(1), 283–312 (2020). https://doi.org/10.12775/TMNA.2020.014
11. Xiaosen, H., Mingxin, W.: Global existence and blow-up of solutions for a system of nonlinear viscoelastic wave equations with damping and source. Nonlinear Anal. 71, 5427–5450 (2009)
12. Bchatnia, A., Daoulatli, M.: Behavior of the energy for Lamé systems in bounded domains with nonlinear damping and external force. Electron. J. Differ. Equ. 2013(1), 1 (2013)
13. Beniani, A., Taouaf, N., Benaissa, A.: Well-posedness and exponential stability for coupled Lamé system with viscoelastic term and strong damping. Comput. Math. Appl. 75(12), 4397–4404 (2018)
14. Boulaaras, S., Ouchenane, D.: General decay for a coupled Lamé system of nonlinear viscoelastic equations. Math. Methods Appl. Sci. 43(4), 1717–1735 (2020)
15. Boulaaras, S.: A well-posedness and exponential decay of solutions for a coupled Lamé system with viscoelastic term and logarithmic source terms. Appl. Anal. (2019). https://doi.org/10.1080/00036811.2019.1648793
16. Bchatnia, A., Guesmia, A.: Well-posedness and asymptotic stability for the Lamé system with infinite memories in a bounded domain. Math. Control Relat. Fields 4(4), 451–463 (2014)
17. Taouaf, N., Amroun, N., Benaissa, A., Beniani, A.: Well-posedness and asymptotic stability for the Lamé system with internal distributed delay. Math. Morav. 22(1), 31–41 (2018)
18. Feng, B., Hajjej, Z., Balegh, M.: Existence and general decay rate estimates of a coupled Lamé system only with viscoelastic dampings. Math. Methods Appl. Sci. 1(18) (2020, in press). https://doi.org/10.1002/mma.6586