Inertial and fluctuational effects on the motion of a Bose superfluid vortex

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We study the motion of a vortex under the influence of a harmonic force in an approximately two dimensional trapped Bose-condensed gas. The Hall-Vinen-Iordanskii equations, modified to include a fluctuational force and an inertial mass term, are solved for the vortex motion. The mass of the vortex has a strong influence on the time it takes the vortex to escape the trap. Since the vortex mass also depends on the trap size we have an additional dependence on the trap size in the escape time which we compare to the massless case.

PACS numbers:

INTRODUCTION

Quantum vortices appear in a large range of physical systems ranging from laboratory condensed matter systems like superconductors,¹ and neutral superfluids (³He superfluid,² superfluid ⁴He, and cold atomic gases³), to exotic excitations playing a role in the early universe.⁴ Understanding the motion of these vortices is key to understanding the properties of such systems.

The motion of superfluid vortices has been studied since the 1950’s.⁵ Their dynamics has often been described by some form of Hall-Vinen-Iordanskii (HVI) equation⁶ ⁷, which essentially describes the different forces acting on a vortex. However until recently the form and even existence of both the forces⁷ ⁸ and the vortex mass⁸ have been controversial. Recently Thompson and Stamp⁹ derived an equation of motion for a vortex in a boson superfluid, starting from a low energy effective field theory for the superfluid. In the limit of low frequency motion, which turns out to be a semiclassical limit, this equation of motion reduced to a modified HVI equation, in which both an inertial term and a Markovian noise term are added to the original HVI equation. However it was also found⁹ that at higher frequencies, strong departures occurred from the HVI equation, and the vortex coordinate then obeys a different equation which shows strong retardation effects, with highly non-Markovian correlations.

In this paper we show how the HVI equations, modified to include the inertial and noise terms, can be used to discuss the dynamics of a single vortex in a cold BEC gas. At the temperatures and frequencies so far employed in experiments on such gases, it turns out we are in the well within the semiclassical regime. Vortices in trapped Bose-Einstein condensates were first created in the lab by Matthews et al.¹¹, and since then single vortices and lattices of vortices in BECs have been studied in various contexts¹² ¹⁵. The motion of a single vortex in a trapped BEC is the subject of this paper. As predicted by Rokhsar¹⁶ and observed later observed experimentally¹¹ ¹² single vortices created in an optical trap spiral out from the centre of the trap until they decay at the edge of the condensate.

There have been fairly extensive theoretical studies of vortex dynamics in BECs; however these have not included an inertial term. Jackson et al.¹⁷ and Svidzinsky and Fetter¹⁸ used the Gross-Pitaevskii equation to calculate the precession velocity of the vortex. Fedichev and Shlyaponikov¹⁹ used a version of the HVI equations with a term describing dissipation, derived by considering scattering of the single particle excitations from a stationary vortex core and a background superflow due to the interaction of the vortex with the boundary of the condensate. More recently Jackson et al.²⁰ performed numerical simulations, and interpreted the results in terms of an HVI equation with no inertial mass term or backflow. Duine et al.²¹ used a stochastic form of the Gross-Pitaevskii equation at finite temperature to derive a purely dissipative equation for the motion of the vortex, including a thermal noise term.

In what follows we study the motion of a vortex in a trap using the form of the two-dimensional HVI equation derived by Thompson and Stamp¹⁰ in the semiclassical limit. We assume the vortex is in a harmonic potential caused by the trap. We solve these equations of motion for the motion of the vortex and calculate the time it takes the vortex to escape the trap. We discuss how the escape time depends on the radius of the trap and on the temperature, and we discuss the role that the vortex inertial mass plays in these results. We also compare the results to previous work.

EQUATION OF MOTION

A quantum vortex can be described either by looking at the N-particle wave-function of the system, or by defining a vortex reduced density matrix \( \rho(r, r'; t) = \langle r | \hat{\rho}(t) | r' \rangle \), in which the other degrees of freedom of the superfluid have been averaged over. If we wish to find an equation of motion for the vortex itself, we are then obliged to derive this from the dynamics of \( \rho(r, r'; t) \). This density matrix is conveniently rewritten in terms of a 'centre of mass' variable \( R = (r + r')/2 \) and a quan-
tum fluctuation variable $\xi = (r - r')/2$. In the 'classical limit' where the characteristic frequency $\Omega$ of the vortex dynamics is low (such that $\hbar \Omega \ll kT$, where $T$ is the temperature), the quantum fluctuations $\xi(t)$ become negligible, and we then expect a set of modified HVI equations to be valid for now is a semiclassical vortex coordinate $\mathbf{R}(t)$. In a 2-dimensional Bose superfluid we have the form

$$M_\nu \frac{d^2 \mathbf{R}}{dt^2} - \rho \kappa \mathbf{v}_n \times \frac{\mathbf{d} \mathbf{R}}{dt} + D_0(T) \frac{\mathbf{d} \mathbf{R}}{dt} = F_{\text{huc}}(t) + F_{\text{ap}}(t) \quad (1)$$

in a frame of reference where $\mathbf{v}_n = \mathbf{v}_s = 0$ (ie., both he normal current and supercurrents are zero; for the more general case where they are arbitrary, see ref. [10]). Here $F_{\text{ap}}$ is the applied force on the vortex, $\rho$ is the fluid density, $\kappa = \frac{\hbar}{m}$ is the quantum of circulation, $D_0(T)$ is the temperature-dependent longitudinal damping coefficient, and $F_{\text{huc}}(t)$ is a fluctuating force with the high-$T$ Markovian correlator

$$\langle F_{\text{huc}}(t) F_{\text{huc}}(s) \rangle = \chi(T) \delta(t-s) \quad (2)$$

in which the angled brackets denote an average over an ensemble of identically prepared systems. Finally, $M_\nu$ is the geometry-dependent vortex mass. For a circular trap geometry with a radius $R_0 \gg a_0$, the vortex core radius, the vortex mass is $\frac{\hbar}{m}$. We assume that the vortex is in a harmonic well, so that

$$F_{\text{ap}}(t) = M_\nu \omega_o^2 \mathbf{R}(t) \equiv \mathbf{k}_o \cdot \mathbf{R}(t) \quad (4)$$

in which $\omega_o$ is the trap frequency, and $\mathbf{k}_o$ the ‘spring constant’. In this case the equation of motion can be simplified by defining the complex position variable $\mathbf{R} \equiv R^x + i R^y$, the complex ‘dissipation’ $\Gamma \equiv \frac{D_0}{M_\nu} - i \frac{\sigma \kappa}{M_\nu}$, a normalized noise constant $\sigma = \frac{\chi}{M_\nu}$, and a normalized complex fluctuation variable $\xi(t)$, where $\sigma \xi(t) = [F_{\text{huc}}(t) + i F_{\text{huc}}^{\dagger}(t)]$. The equation of motion then becomes

$$\frac{d^2 \mathbf{R}}{dt^2} + \Gamma \frac{d \mathbf{R}}{dt} - M_\nu \omega_o^2 \mathbf{R} = \sigma \xi(t). \quad (5)$$

This equation is most easily solved by considering the 4-dimensional phase space position vector $Q$

$$Q(t) = \left[ \begin{array}{c} \dot{\mathbf{R}}(t) \\ \mathbf{R}(t) \end{array} \right]. \quad (6)$$

In terms of $Q$ the solution to equation (5) can be written,

$$Q(t) = e^{-\mathcal{M}t} Q(0) + \sigma e^{-\mathcal{M}t} \int_0^t ds \ e^{\mathcal{M}s \left( \langle \xi(s) \rangle_0 \right)} \quad (7)$$

where $\mathcal{M}$ is the matrix

$$\mathcal{M} = \left( \begin{array}{cc} \Gamma & -\omega_o^2 \\ -1 & 0 \end{array} \right). \quad (8)$$

From the solution (7) we may immediately calculate the correlator,

$$\langle Q(t') Q^\dagger(t) \rangle = e^{-\mathcal{M}t} \left[ \langle Q(0) Q^\dagger(0) \rangle + 2 \sigma^2 \int_0^t ds e^{\mathcal{M}s} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{\mathcal{M}s} \right] e^{-\mathcal{M}t} \quad (9)$$

Here a superscript dagger denotes the matrix hermitian conjugate and it is assumed that $t' \geq t$.

**ESCAPE FROM TRAP**

To get an estimate of the time it takes the vortex to escape from the trap we look at $\langle |R(t)|^2 \rangle$ as a function of time. Since the eigenvalues of $\mathcal{M}$ are $\frac{1}{2}(\Gamma \pm \Delta)$ where $\Delta = \sqrt{\Gamma^2 + 4 \omega_o^4}$ is the discriminant of the characteristic polynomial, we see that the leading order term is for large $t$ is $\sim \exp(2Re\Delta t)$ where $Re\Delta$ denotes the real part of $\Delta$. The full leading order term for $\langle |R(t)|^2 \rangle$ in the long time limit is

$$\langle |R(t)|^2 \rangle = \frac{2\sigma^2 |\Delta + \Gamma|^2}{|\Delta|^4 \left( \frac{2D_0}{M_\nu} + Re\Delta \right)} e^{2Re\Delta t}. \quad (11)$$

Note that any contributions from the initial velocity or position of the vortex are subdominant at large times $t \gg |\Delta|$.

Let us define the ‘escape time’ $\tau_e$ by setting $\langle |R(\tau_e)|^2 \rangle = R_0^2$, i.e., the time it takes for the vortex to move to the edge of the circular container; we then find

$$\tau_e = A \ln \left( \frac{R_0^2}{C} \right) = \frac{a_0}{4} \sqrt{\frac{\pi \rho_s}{K}} \left[ \ln \left( \frac{R_0}{a_0} \right) \right]^2 \ln \left( \frac{\sqrt{K^3 \pi \rho_s a_0 R_0^2}}{\chi} \right) \quad (12)$$

where we have defined

$$A = \frac{1}{Re\Delta} \quad (13)$$

$$C = \frac{2\sigma^2 |\Delta + \Gamma|^2}{|\Delta|^4 \left( \frac{2D_0}{M_\nu} + Re\Delta \right)}. \quad (14)$$

The approximations we have made only give the leading order dependence of $\tau_e(R_0)$ on $R_0$, for large $R_0$. Three remarks are in order:

(i) In two dimensions $\chi \sim T^3$ and $D_0 \sim T^4$, so at low temperature we have

$$\tau_e = A_o \ln \left( \frac{R_o}{BT^5} \right) \quad (15)$$
where $B$ is a constant independent of $R_o$ and $A_o = A(T = 0)$ which is proportional to $\sqrt{\ln(R_o)}$ for large $R_o$.

(ii) We can also drop the inertial term completely (i.e., let $M_v \to 0$); this is equivalent to solving the HVI equations (1) with an added noise term, but no inertial term. The exact solution for the displacement $\mathcal{R}(t)$ with the initial conditions $\mathcal{R}(0) = 0$ is then

$$\langle |\mathcal{R}(t)|^2 \rangle \xrightarrow{M_v \to 0} \frac{\chi}{k_o D_o} \left[ \exp \left( \frac{2k_o D_o t}{D_o^2 + \rho^2 \kappa^2} \right) - 1 \right]$$

which leads to the escape time

$$\tau^0_e = \left( \frac{D_o^2 + \rho^2 \kappa^2}{2k_o D_o} \right) \ln \left( 1 + \frac{k_o D_0 R_o^2}{\chi} \right) \sim T^{-4} \ln \left( \frac{R_o^2}{CT} \right)$$

where $C$ is a constant. Thus the presence of an inertial term in the equation of motion leads to a completely different temperature dependence in the escape time.

(iii) We can compare these results to the work of Duine et al. [21], who have $\tau_e \sim T^{-1} \ln \left( \frac{R_o}{\tau_{\text{min}}} \right)$, where $G$ is independent of both $T$ and $R_o$, and to that of Fedichev and Shlyapnikov [19], who find $\tau_e \sim T^{-1} \ln \left( \frac{R_o}{\tau_{\text{min}}} \right)$, where $R_{\text{min}}$ the initial radial position of the vortex. We see that our result for $\tau_e$ has a different dependence on both the trap radius $R_o$ and the temperature $T$. The difference in the dependence on $R_o$ arises because the inertial mass of the vortex depends on the trap radius. The difference in the temperature dependence here, compared to that of Fedichev and Shlyapnikov, comes (a) because they have no inertial mass term (b) because our stochastic force (which is absent from their equation) depends strongly on temperature, and (c) because Fedichev and Shlyapnikov have a different temperature dependence in $D_0(T)$ from that used here. While Duine et al. [21] do have a stochastic force term in their equation of motion, they have no inertial mass term.

Finally, we note that the most obvious way in which calculations of this kind can be tested is in experiments on ‘pancake’ quasi-2d Bose condensates, of the kind investigated in MIT and Paris [14-15]. In this context, we emphasize that there are several limitations to our calculation. First, we have ignored the effect of the boundary of the condensate. Fetter and Svidzinsky [3] introduced a background superflow to the equation of motion to cancel the normal component of the superfluid at the boundary of the condensate, and this will presumably affect our result. Second, the Thompson-Stamp derivation [10] assumed that local deviations from the mean superfluid density were small. This assumption does not always hold for Bose-Einstein condensates, and it will be interesting to see how the results may be modified for BECs in the extreme compressible limit. Finally, we note that at sufficiently low temperatures, we expect serious departures from the modified HVI equations, coming from the non-local terms found by Thompson and Stamp - experimental probes of this regime will be of great interest, particularly in view of the long-standing controversy over the correct equations of motion for a quantum vortex [2–4].

**CONCLUSIONS**

Using the modified form for the HVI equations that was found by Thompson and Stamp in the low-frequency semiclassical limit [10], we have studied the motion of a superfluid vortex in a harmonic trap. We calculated the life-time of the vortex in a trap for this case, and find that the temperature dependence of the trap escape time is different than that found in previous calculations [3, 21], even when the inertia of the vortex is neglected. The vortex inertia also has a significant influence on this escape time. Experimental tests of results like this will be possible on trapped cold BECs, and will allow significant tests of the theory of vortex dynamics (which have been very difficult until now).

This work was supported by NSERC, by CIFAR, and by PITP.

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