N = 2 Conformal Supergravity from Twistor-String Theory

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Abstract

A chiral superfield strength in N = 2 conformal supergravity at linearized level is obtained by acting two superspace derivatives on N = 4 chiral superfield strength which can be described in terms of N = 4 twistor superfields. By decomposing SU(4)_R representation of N = 4 twistor superfields into the SU(2)_R representation with an invariant U(1)_R charge, the surviving N = 2 twistor superfields contain the physical states of N = 2 conformal supergravity. These N = 2 twistor superfields are functions of homogeneous coordinates of weighted complex projective space WCP^{3|4} where the two weighted fermionic coordinates have weight −1 and 3.
1 Introduction and Summary

There exist two descriptions for twistor-string vertex operators, topological B-model of \( \text{CP}^{3|4} \) \cite{1} and open string version of twistor-string theory \cite{2}. A twistor space function of definite homogeneity describes a massless particle state in Minkowski spacetime of definite helicity \cite{3, 4}. The twistor space fields describing \( \mathcal{N} = 4 \) conformal supergravity are homogeneous function of bosonic and fermionic variables of \( \text{CP}^{3|4} \) \cite{5}. The vertex operators describe supermultiplets whose bottom component is independent of fermionic variables \( \psi^A \) where \( A = 1, 2, 3, 4 \) and top component is quartic in \( \psi^A \) of \( \text{CP}^{3|4} \). Then one can determine \( SU(4)_R \) representation for each component field in \( \psi^A \)-expansion by tensor product since \( \psi^A \) transforms \( 4 \) under that representation.

By looking at the \( SU(4)_R \) representations for component fields, each supermultiplet possesses \( 8 \) bosonic and \( -8 \) fermionic degrees of freedom. As a result, four Lorentz scalar twistor functions have the maximal helicity \( 2, 2, 0, 0 \) and the minimal helicity \( 0, 0, -2, -2 \). For the remaining eight fermionic twistor supermultiplets, the maximal helicity is given by \( 3/2 \) and \( 1/2 \) and the minimal helicity is \( -1/2 \) and \( -3/2 \). Since there are \( 12 \) supermultiplets, \( 96 \) bosonic and \( -96 \) fermionic degrees of freedom are present. This indicates that the on-shell degrees of freedom in \( \mathcal{N} = 4 \) conformal supergravity should reflect this observation.

The basic variable of \( \mathcal{N} = 4 \) linearized conformal supergravity \cite{6, 7, 8} is characterized by a chiral superfield strength which is a Lorentz scalar. One of the lessons in \cite{5} is that they interpreted this quantity as a coupling of the abelian gauge field, relevant to one of the physical vertex operators, to the boundary of an open string world sheet \cite{2} in \( \mathcal{N} = 4 \) superspace. The fermionic coordinates \( \theta^A_a \) where \( a \) is a spinor index in this chiral Minkowski superspace are connected by above fermionic coordinates \( \psi^A \) of \( \text{CP}^{3|4} \) via twistor equations. By plugging the solutions for the equations of motion on various conformal supergravity fields into a chiral superfield strength which has all the component fields in \( \theta^A_a \)-expansion, it was possible to write the \( 12 \) twistor supermultiplets in terms of physical states of \( \mathcal{N} = 4 \) conformal supergravity.

What happens for \( \mathcal{N} = 1 \) conformal supergravity \cite{9}? In this case, since a chiral superfield strength has three spinor indices, a coupling of the abelian gauge field to the boundary of an open string world sheet should be modified by multiplying a triple product of \( \mathcal{N} = 4 \) super derivatives, contracted with \( SU(4)_R \) indices, which has also three spinor indices. The twistor space fields describing \( \mathcal{N} = 1 \) conformal supergravity are homogeneous function of bosonic and fermionic variables of \( \text{WCP}^{3|2}(1, 1, 1, 1\vert 1, 3) \) \cite{10}. See also \cite{11}. Four Lorentz scalar functions have the maximal helicity \( 2, 2, 0, 0 \) and the minimal helicity \( 3/2, 3/2, -1/2, -1/2 \).
For the two fermionic supermultiplets, the maximal helicity is given by $3/2$ and $1/2$ and the minimal helicity is given by $-1/2$ and $-3/2$.

The presence of a \textit{weighted} fermionic coordinate of weight 3 of $\text{WCP}^{3|2}$ in these fermionic supermultiplets which have an unweighted fermionic index was crucial to match with the full structure of the physical states for $\mathcal{N} = 1$ conformal supergravity. As a result, each bosonic supermultiplet possesses 1 bosonic and $-1$ fermionic degrees of freedom. Each fermionic supermultiplet possesses 2 bosonic and $-2$ fermionic degrees of freedom. Therefore, there exist 8 bosonic and $-8$ fermionic degrees of freedom by summing up each bosonic and fermionic degree of freedom. This is consistent with the number of on-shell degrees of freedom in $\mathcal{N} = 1$ conformal supergravity.

The immediate question is to ask how the physical states of $\mathcal{N} = 2$ conformal supergravity \cite{7, 8} occur in twistor-string theory. In the $\mathcal{N} = 2$ conformal supergravity side, the number of spinor index of chiral superfield strength is equal to 2 with two $SU(2)$ internal indices that are antisymmetric. Then a coupling of the abelian gauge field to the boundary in the twistor-string theory should be changed by multiplying \textit{two} $\mathcal{N} = 4$ super derivatives, contracted with $SU(4)_R$ indices. Four Lorentz scalar twistor functions will have the maximal helicity $2, 2, -1, -1$ and the minimal helicity $1, 1, -2, -2$. For the three fermionic twistor supermultiplets, the maximal helicity will be $3/2$ and 0 and the minimal helicity will be 0 and $-3/2$. Each bosonic and fermionic supermultiplet has 4 bosonic and $-4$ fermionic degrees of freedom. There exist 20 bosonic and $-20$ fermionic degrees of freedom which will be consistent with the number of on-shell degrees of freedom in $\mathcal{N} = 2$ conformal supergravity.

Then what is corresponding twistor-string theory which will contain the physical states of $\mathcal{N} = 2$ conformal supergravity? As suggested in \cite{10}, the Calabi-Yau supermanifold should contain the bosonic submanifold $\text{CP}^3$ of $\text{CP}^{3|4}$ and the fermionic submanifold should have two fermionic coordinates of weight 1 which will participate in $\mathcal{N} = 2$ chiral Minkowski superspace. The superconformal algebra $SU(2, 2|2)$ will act on these two fermionic coordinates as well as four bosonic ones. In order to require the Calabi-Yau supermanifold condition, the equality of the sum of bosonic weights and the sum of fermionic weights, the sum of the remaining fermionic weights should be equal to 2. When we consider \textit{two odd} fermionic weights, the simplest one is given by 3 and $-1$.

Therefore, one can add homogeneous bosonic and fermionic coordinates of $\text{WCP}^{3|2}$ we have considered for $\mathcal{N} = 1$ conformal supergravity to two fermionic coordinates of weights 1, $-1$ and make a weighted complex projective space $\text{WCP}^{3|4}(1, 1, 1, 1|1, 1, -1, 3)$ where the two fermionic coordinates are \textit{weighted} and their values are $-1$ and 3 respectively. Then we
will see that twistor space fields corresponding to the physical states of $\mathcal{N} = 2$ conformal supergravity are functions of homogeneous coordinates of this Calabi-Yau supermanifold \(^1\).

In section 2, by an appropriate decomposition of $SU(4)_R$ representation of $\mathcal{N} = 4$ twistor fields into $SU(2)_R$ representation of $\mathcal{N} = 2$ twistor fields, the complete structure of helicity states with $SU(2)_R$ representation is given. In section 3, by studying the $\mathcal{N} = 2$ linearized conformal supergravity, the spectrum of massless helicity states is identified with the twistor-string theory description given in section 2. In section 4, we will make some comments on the case of $\mathcal{N} = 3$ conformal supergravity from twistor-string theory and a mirror symmetry of the above Calabi-Yau supermanifold.

\section{Vertex operators and spectrum of massless fields in Minkowski spacetime}

In the open twistorial string theory \([2]\), the worldsheet action depends on the homogeneous coordinates

$$Z^I = (\lambda^a, \mu^a, \psi^A)$$

of WCP\(^{3/4}\) and conjugate super twistor variables $Y_I$. The physical states are given by conformal dimension 1 vertex operator. For example, the conformal supergravity multiplet can be described by this dimension 1 vertex operator: $Y_I f^I(Z)$ and $g_I(Z) \partial Z^I$. For the multiplets $f^I$ of weight 1, they transform as $f^I \rightarrow t f^I$ under the transformation $Z^I \rightarrow t Z^I$ where $t$ is real. In other words, these $f^I$ carry $GL(1)$ charge 1. For the fermionic multiplets of weight $-1$, they scale as $f^{A=3} \rightarrow t^{-1} f^{A=3}$ and $f^{A=4} \rightarrow t^{3} f^{A=4}$ under the above transformation respectively. They have $GL(1)$ charge $-1$ and 3. Similarly, the multiplets $g_I$ of weight $-1$, they transform as $g_I \rightarrow t^{-1} g_I$ and other fermionic multiplets scale as $g_{A=3} \rightarrow t g_{A=3}$ and $g_{A=4} \rightarrow t^{-3} g_{A=4}$ respectively. The $GL(1)$ charge of $g_I$ is opposite to those of $f^I$.

Let us first consider the twistor field $f^I(Z)$ which is a function of four bosonic and four fermionic variables $\lambda^a, \mu^a$ and $\psi^A$. Let us denote half of fermionic variables by $\psi$ and $\chi$ which

\(^1\)Naively, one can think of WCP\(^{3/3}\)(1,1,1,1,1,1,2) space by adding one even fermionic weight 2 as well as two odd fermionic weights of 1. According to \([12]\), this space can be obtained from both CP\(^{3/3}\) space and WCP\(^{3/4}\) space by the fermionic dimensional reduction. Although WCP\(^{3/4}\) space cannot be obtained from CP\(^{3/4}\) through the fermionic dimensional reduction directly, since WCP\(^{3/4}\) space is related to WCP\(^{3/3}\) space, eventually WCP\(^{3/4}\) is associated with CP\(^{3/4}\) space through WCP\(^{3/3}\) space. It might be interesting to see how WCP\(^{3/3}\) Calabi-Yau supermanifold can provide $\mathcal{N} = 2$ conformal supergravity spectrum.

\(^2\)In this section, the twistor fields $f^I(Z)$ and $g_I(Z)$ are $\mathcal{N} = 2$ contents. Unfortunately, later, we also denote them by $\mathcal{N} = 4$ twistor fields. Their relations will be clear when we discuss about the identifications with twistor fields in section 3.
will participate in the $\mathcal{N} = 2$ chiral Minkowski superspace, through the twistor equation, and the remaining variables by $\alpha$ and $\beta$ which are weighted differently:

$$\psi^{A=1} = \psi, \quad \psi^{A=2} = \chi; \quad \psi^{A=3} = \alpha, \quad \psi^{A=4} = \beta.$$ 

Since the $f^I(Z)$, for each $I = a, \dot{a}, A = 1, A = 2$, is homogeneous in $Z^I$ of degree 1, there exist four bosonic and two fermionic helicity states, each of helicity $3/2$. Here a massless state in Minkowski spacetime has a helicity $1 + \frac{\text{deg.}}{2} [3, 4]$ when we put $\psi^A = 0$ and do not take the spinor index. For $A = 3$, the $f^{A=3}(Z)$ is homogeneous in $Z^I$ of degree $-1$, there is one fermionic helicity state $1/2$, by above formula. For $A = 4$, the $f^{A=4}(Z)$ is homogeneous in $Z^I$ of degree 3, there is one fermionic helicity state $5/2$ which is greater than 2. Both spinor index $a$ and $\dot{a}$ provides a helicity $1/2$ and $-1/2$. Then this intermediate consideration leads to two bosonic states of helicity 2 and two of helicity 1.

For the fermions, the index $A = 1, 2$ transforms as 2 of the $SU(2)_R$ group of R-symmetries and there exist two states of helicity $3/2$, one state of helicity $1/2$ and one state of helicity $5/2$ respectively. According to the arguments of [5], after taking account of the gauge invariance and the constraint, there exist two bosonic states of helicity 2 and two fermionic states of helicity $3/2$, one fermionic state of helicity $1/2$ and one fermionic state of helicity $5/2$.

The action of $SU(2,2|2)$ on the coordinates $(\lambda^a, \mu^{\dot{a}}, \psi, \chi)$ of $\text{WCP}^{3|4}$ is generated by $6 \times 6$ supertraceless matrices where the trace of $4 \times 4$ upper-left part is equal to the trace of $2 \times 2$ lower-right part. The bosonic conformal algebra $SU(2,2)$ is represented by the 15 matrices which lie in the $4 \times 4$ upper-left part. A bosonic generator which takes the form $\psi^A \frac{\partial}{\partial \psi^A}$ where $A = 1, 2$ for the chiral $U(1)_R$ transformation (which does not exist for $\mathcal{N} = 4$ conformal supergravity) is represented by a diagonal supertraceless matrix. The 8 spinorial and 8 special conformal symmetry generators which have the following form $\lambda^a \frac{\partial}{\partial \psi^A}$, $\psi^A \frac{\partial}{\partial \xi^A}$, $\psi^A \frac{\partial}{\partial \mu^{\dot{a}}}$, and $\mu^{\dot{a}} \frac{\partial}{\partial \psi^A}$ where $A = 1, 2$ are represented by $2 \times 4$ lower-left part and $4 \times 2$ upper-right part. Moreover three $SU(2)$ generators which are necessary to close the anticommutators between the spinorial generators and special conformal generators are represented by $2 \times 2$ lower-right part.

Now for nonzero $\psi^A$, one can expand $f^I(Z)$ in powers of $\psi^A$: $f^I(\lambda, \mu, \psi) = f^I_0(\lambda, \mu) + f^I_{1A}(\lambda, \mu)\psi^A + f^I_{2AB}(\lambda, \mu)\psi^A\psi^B + \cdots$ where $f^I_k$ is homogeneous in $\lambda, \mu$ with degree $(1 - k)$ and provides a massless state of helicity $(3/2 - k/2)$ when we ignore the angular momentum carried by the index $I$. One can read off each helicity state and $SU(2)_R$ representation as one considers for nonzero $\psi^A$. As we have observed, there exist one fermionic state of helicity $1/2$ characterized by the twistor field $f^{A=3}(Z)$. In order to make comparison with $\mathcal{N} = 4$ twistor field contents, it is better to add a helicity $1/2$ by differentiating $f^{A=3}(Z)$ with respect to
the third fermionic coordinate $\psi^{A=3} = \alpha$ and making a twistor field which is homogeneous of degree 0 resulting in a massless state of helicity 1.

We list the full structure of helicity states described by the $\mathcal{N} = 2$ field $f^I(Z)$ by taking account of angular momentum, the gauge invariance and the constraint

$$\begin{align*}
\lambda^a f_a & : (2, 1), \left(\frac{3}{2}, 2\right), (1, 1), \\
\mu^\dot{a} f_{\dot{a}} & : (2, 1), \left(\frac{3}{2}, 2\right), (1, 1), \\
f^{A=1,2} & : \left(\frac{3}{2}, 2\right), (1, 3 \oplus 1), \left(\frac{1}{2}, 2\right), \\
\partial_{\alpha} f^{A=3} & : (1, 1), \left(\frac{1}{2}, 2\right), (0, 1),
\end{align*}$$

where the first element is the helicity and the second element is the $SU(2)_R$-representation of $R$-symmetries and $\partial_{\alpha} = \frac{\partial}{\partial a_{\alpha}} = \frac{\partial}{\partial \dot{a}_{\alpha}}$. Let us explain what we obtained in detail. The field contents in each $f^I(Z)$ are exactly the truncation of $\mathcal{N} = 4$ twistor fields satisfying $SU(2)_R$ symmetry with same $U(1)_R$ charge for $\mathcal{N} = 4$ twistor fields under the $SU(4)_R \to SU(2) \times SU(2)_R \times U(1)_R$. For example, the helicity state $(\frac{3}{2}, 2)_{-1}$ with $U(1)_R$ charge $-1$ of $\mathcal{N} = 4$ twistor field $\lambda^a f_a(Z)$ breaks into $(\frac{3}{2}, (1, 2)_{-1}) + (\frac{3}{2}, (2, 1)_{1})$ where the second element has the following representation $(SU(2), SU(2)_R)_{U(1)_R}$. Then the only state $(\frac{3}{2}, (1, 2)_{-1})$ preserving $U(1)_R$ charge survives. This is denoted by $(\frac{3}{2}, 2)$ above in a simplified notation. One can analyze other helicity states from $\mathcal{N} = 4$ description and how to break into $\mathcal{N} = 2$ field contents with correct quantum numbers.

In particular, the $\mathcal{N} = 4$ fields $f^A(Z)$ contain two different $SU(4)_R$ representations for each helicity: $(1, 15 \oplus 1), (\frac{1}{2}, 20 \oplus 4), (0, 10 \oplus 6)$. In other words, they are divided into two parts. Higher dimensional representations 15 and 20 which have nonnegative Weyl weights enter into $f^{A=1,2}(Z)$ of $\mathcal{N} = 2$ twistor fields through the breaking $SU(4)_R \to SU(2) \times SU(2)_R \times U(1)_R$ while lower dimensional representations 1, $\bar{4}$ and 6 which have negative Weyl weights enter into $\partial_{\alpha} f^{A=3}(Z)$ which is homogeneous in $Z^I$ of degree 0 resulting in a massless state of helicity 1.

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3For a generic field $\Phi_{\mu_1 \mu_2 \cdots \mu_n}$, the flat space action contains $\int d^4x \Phi_{\mu_1 \cdots \mu_n} \partial_{\lambda_1} \cdots \partial_{\lambda_n} \Phi_{\nu_1 \cdots \nu_n} + \cdots$. By assumption [8], the canonical mass dimension of $\Phi$ is equal to $2 - p$. If $\Phi$ does not carry world indices ($n = 0$), then its Weyl weight $w$ is equal to its canonical weight. If $n \neq 0$, a new field $\Phi_{\mu_1 \cdots \mu_n} e_{\alpha_1}^{\mu_1} \cdots e_{\alpha_n}^{\mu_n}$ is defined with $\bar{n} = 0$ and $\bar{w} = 2 - p$. In this case $w = 2 - p - n$ where $w(e^A_0) = 1$. For example, the $SU(4)_R$ representation 15, $V^B_{\mu A}$, has a canonical dimension 1 because it has second order derivatives and its Weyl weight is equal to $1 - n = 1 - 1 = 0$. For the representation 20, $\xi^{1C}_{[AB]}$, its Weyl weight is equal to its canonical weight 3/2(it has first order derivative) because it does not have world indices.

4For the gravitino $e^a_{\mu}$, the canonical weight is equal to 0 because it has fourth order derivatives while its Weyl weight is equal to $0 - n = 0 - 1 = -1$. For the gravitino $\pi^{0}_{\mu A}$, the Weyl weight is equal to $-1/2$ since
A zero total number of on-shell (dynamical) degree of freedom can be checked by counting each bosonic and fermionic degree of freedom in the spectrum. Each row in (2.1) has bosonic degree of freedom 2 and fermionic degree of freedom −2. There exist five columns. Therefore, there are 10 bosonic and −10 bosonic degrees of freedom. Their sum is equal to zero. Also note that the fermionic state of helicity 5/2 characterized by $f^{A=4}(Z)$ implies higher spin which is greater than 2. This cannot be obtained from $\mathcal{N} = 4$ conformal supergravity, by same reasons [10].

Let us consider the twistor field $g_I(Z)$ and the Lorentz scalars $(\lambda^a g_a, \mu^a g_a, \partial_\alpha g^a, \partial_\beta g^\dot{a})$ are homogeneous of degree $(0, 0, -2, -2)$ respectively. By counting of the gauge invariance and the constraint, we are left with two twistor fields of degree −2 leading to two massless states of helicity 0. The field $g_{A=1,2}$ is homogeneous of weight −1 describing massless states of helicity 1/2 while the field $g_{A=3}$ is homogeneous of weight 1 describing massless states of helicity 3/2 and the field $g_{A=4}$ is homogeneous of weight −3 describing massless states of helicity −1/2. As for the case of $f^{A=4}(Z)$, this $g_{A=4}(Z)$ is not allowed for the spectrum. As we have observed in $f^I(Z)$, we need to act some fermionic differentiation on $g_I(Z)$ in order to see the truncation of $\mathcal{N} = 4$ twistor fields manifestly. Then the complete structure of helicity states by the field $g_I(Z)$ are summarized by

\[
\begin{align*}
\partial_\alpha \partial_\beta \partial_\alpha g^a & : (-1, 1), \left( -\frac{3}{2}, 2 \right), \left( -2, 1 \right), \\
\partial_\alpha \partial_\beta \partial_\dot{a} g^\dot{a} & : (-1, 1), \left( -\frac{3}{2}, 2 \right), \left( -2, 1 \right), \\
\partial_\alpha \partial_\beta g_{A=1,2} & : \left( -\frac{1}{2}, 2 \right), \left( -1, 3 \oplus 1 \right), \left( -\frac{3}{2}, 2 \right), \\
\partial_\beta g_{A=3} & : (0, 1), \left( -\frac{1}{2}, 2 \right), \left( -1, 1 \right),
\end{align*}
\]

where $\partial_\beta = \frac{\partial}{\partial \theta^\beta} = \frac{\partial}{\partial \bar{\theta}^\beta}$. The role of $\partial_\alpha$ and $\partial_\beta$ is to add the degree 1, −3 to any twistor fields respectively. The field contents in each $g_I(Z)$ are exactly the truncation of $\mathcal{N} = 4$ twistor fields satisfying $SU(2)_R$ symmetry for the invariant $U(1)_R$ charge. In this case, also the $\mathcal{N} = 4$ fields $g_A(Z)$ contain two different representations for each helicity: $(0, \mathbf{10} \oplus \mathbf{6}), (\mathbf{20} \oplus \mathbf{4}), (\mathbf{15} \oplus \mathbf{1})$.

Higher dimensional representations $\mathbf{20}$ and $\mathbf{15}$ of $SU(4)_R$ enter into $\partial_\alpha \partial_\beta g_{A=1,2}(Z)$ of the canonical weight is 1/2 because it has third order derivatives and $n = 1$. For $f^{AB}_{\mu \nu}$, the Weyl weight is equal to −1 because its canonical weight is 1 because it has second order derivatives and $n = 2$.

Of course, the twistor-string theory, in general, contains more information than conformal supergravity theory has. We would like to see how the $\mathcal{N} = 2$ conformal supergravity spectrum appears in the context of twistor-string theory, as a first step. It is an open problem to study the full twistor-string theory without ignoring the state of helicity 5/2 and it might lead to an interaction of conformal supergravity theory with other matter multiplets for this extra state.
\( \mathcal{N} = 2 \) twistor fields while lower dimensional representations 6, 4, and 1 of \( SU(4)_R \) enter into \( \partial_3 g_{A=3}(Z) \). We will see the detailed analysis in next section. The massless fields described by (2.2) have the opposite helicities and \( U(1)_R \) charges from those described by (2.1). In this case, there are also 10 bosonic and \(-10\) fermionic degrees of freedom. Therefore, a zero total number of on-shell (dynamical) degree of freedom is given by 20 bosonic and \(-20\) fermionic degrees of freedom (their sum is equal to zero), by counting each bosonic and fermionic degree of freedom in the spectrum of \( f^I(Z) \) and \( g_I(Z) \).

We will prove that the spacetime fields described by the twistor fields \( f^I(Z) \) and \( g_I(Z) \) we have constructed contain the physical states of \( \mathcal{N} = 2 \) conformal supergravity in next section.

### 3 Linearized spectrum of \( \mathcal{N} = 2 \) conformal supergravity and twistor description

We will describe how \( \mathcal{N} = 2 \) chiral superfield strength arises from corresponding \( \mathcal{N} = 4 \) chiral superfield strength, determine the spectrum of massless helicity states, and compare those with the results obtained from twistor-string description.

#### 3.1 A chiral superfield strength in \( \mathcal{N} = 2 \) conformal supergravity

The linearized \( \mathcal{N} = 4 \) conformal supergravity can be described by a chiral scalar superfield \( \mathcal{W}^{\mathcal{N}=4}(x, \theta) \) whose Weyl weight 0 and whose component \( \theta \)-expansion has the following explicit form \([5, 7]\)

\[
\mathcal{W}^{\mathcal{N}=4}(x, \theta) = \cdots + (\theta^2)^{(AB)} E_{(AB)} + (\theta^2)^{(ab)} T^{[AB]}_{(ab)} + \ldots
\]

which satisfies the constraint

\[
\epsilon^{ABCD} D^a_{(AB} D_{(a} D_{b} D_{c)} D^b_{C)} \mathcal{W}^{\mathcal{N}=4} = \epsilon_{EFGH} D^a_{(AB} D^b_{C} D^c_{[E} D^d_{F]} \mathcal{W}^{\mathcal{N}=4} \tag{3.1}
\]

and the chirality of \( \mathcal{W}^{\mathcal{N}=4} \) is imposed: it does not contain the \( \overline{\theta}^a_A \)-dependence. Due to this constraint equation, some of higher \( \theta \)-components of \( \mathcal{W}^{\mathcal{N}=4} \) are expressed in terms of derivatives of fields whose complex conjugates occur in the lower \( \theta \)-components. The \( SU(4)_R \) irreducible
representations \((\theta^2)^{(AB)}, (\theta^2)^{(ab)}, (\theta^3)^{(abc)}, \ldots, (\theta^6)^{[AB](ab)}\) transform as 10, 6, 4, 20, 15, 1, 20', 20, 4, \(\overline{10}, 6\) of SU(4)_R respectively. In particular, \(s_{[AB]}^{(4)}\) and \(d_{[AB]}^{(C)}\) are traceless.

Let us construct \(\mathcal{N} = 2\) chiral field strength superfield \(W_{ab}^{AB}(x, \theta)\) where \(A\) and \(B\) are SU(2) indices and antisymmetric in terms of above \(W^{N=4}\), by acting two super derivatives on \(W^{N=4}\) contracted with SU(4)-invariant epsilon tensor and putting the other fermionic coordinates to zero,

\[
W_{ab}^{\mathcal{N}}(x, \theta) = \epsilon^{12AB}D_{aA}D_{bB}W^{N=4}(x, \theta)\bigg|_{\theta^3 = \theta^4 = 0} \tag{3.3}
\]

where we fix SU(2) indices 1 and 2.

We will compute the right hand side of (3.3) together with (3.1) explicitly. Let us first consider the quadratic term in \(\theta\) of \(W^{N=4}\), \((\theta^2)^{(CD)}\), transforming 10 under the SU(4)_R when we act two super derivatives on it, together with \(E^{(CD)}\). There are two contributions but they are cancelled each other. That is, one obtains

\[
\epsilon^{12AB}D_{aA}D_{bB} (\theta^2)^{(CD)} E^{(CD)}\bigg|_{\theta^3 = \theta^4 = 0} = 0. \tag{3.4}
\]

Next let us compute the contributions from the quadratic term in \(\theta\) of \(W^{N=4}\), \((\theta^2)^{(cd)}\), transforming 6 under the SU(4)_R. The result can be written as \(\epsilon^{12EF}\epsilon_{CDEF}\delta^c_d\) by realizing that \((\theta^2)^{(cd)} = \epsilon_{CDEF}\theta^E(\theta^4)F\). By combining this with the component field \(T^{(cd)}\), one gets

\[
\epsilon^{12AB}D_{aA}D_{bB} (\theta^2)^{(cd)} T^{(CD)}\bigg|_{\theta^3 = \theta^4 = 0} = \delta^1_{(c}\delta^2_{D]} T^{(cd)}_{(ab)}. \tag{3.5}
\]

Here the role of two super derivatives acting on the fermionic coordinates and putting \(\theta^3 = \theta^4 = 0\) projects out the representation 6, \(T^{(cd)}\), of SU(4)_R and leaves invariant 1, \(T^{(12)}_{(ab)}\), of SU(2)_R.

Let us compute the contributions from cubic terms in \(\theta\) of \(W^{N=4}\), \((\theta^3)^{(cde)}\), which has the following expression \(\epsilon^{CDEF}\theta^D(\theta^E\theta^d\theta^c)F\), transforming 4 under the SU(4)_R. It is easy to check \(\epsilon^{12AB}D_{aA}D_{bB}(\theta^3)^{(cde)} = \delta^a_c\delta^b_d\theta^c(\theta^4)\) by computing the action of super derivatives 6. Then by combining this with the component field \((\partial\eta)^{(cde)}\), one obtains

\[
\epsilon^{12AB}D_{aA}D_{bB} (\theta^3)^{(cde)} (\partial\eta)^{(D)}_{(cde)}\bigg|_{\theta^3 = \theta^4 = 0} = \theta^c(\partial\eta)^{(D)}_{(ab)c} \delta^1_{(c}\delta^2_{D]}. \tag{3.6}
\]

In this case, the only state represented by 2, \((\partial\eta)^{(D)}_{(ab)c}\) where \(D = 1\) or 2, of SU(2)_R is survived by acting some projection on 4 of SU(4)_R. The next term \((\theta^4)^{(EF)} = \epsilon_{CDEF}(\theta^2)^{EF}\theta^Gc\) we consider transforms as \(20\) under the SU(4)_R and the contributions can be written as

\[\text{(Let us emphasize that after differentiating the } \theta \text{'s under the super derivative } D_{aA} \text{ and putting the condition } \theta^3 = \theta^4 = 0, \text{ the remnants of } \theta^{A=1,2} \text{ are the fermionic coordinates of } N = 2 \text{ superspace.}}\]
\[ \epsilon^{12FG} \epsilon_{CD} \delta^{C}_{a} \theta^{E}_{b} \] and \[ \epsilon^{12EF} \epsilon_{CD} \delta^{C}_{a} \theta^{G}_{b} \]. The first term combined with the component field \( \xi^{[CD]}_{CE} \) becomes \( \delta^{[1]}_{C} \theta^{A}_{E} \xi^{[CD]}_{AB} \). Note that when we restrict the condition of \( \theta^{3} = \theta^{4} = 0 \), the SU(4) index A becomes the SU(2) index. That is, the value of index A is 1 or 2.

What happens for the second term? The whole expression will be \( \epsilon^{12EF} \epsilon_{CD} \theta^{G}_{E} \xi^{[CD]}_{b} \). When the index G is equal to 1 or 2, the index E can be C or D in the product of two epsilon tensors. However, this contribution vanishes because there exists a relation \( \xi^{[CD]}_{bC} = 0 \).

Eventually, we end up with

\[
\epsilon^{12AB} D_{aA} D_{bB} (\theta^{3})^{C}_{E} \xi^{[CD]}_{CE} \bigg|_{\theta^{3} = \theta^{4} = 0} = \theta^{A}_{E} \xi^{[CD]}_{b} \delta^{1}_{C} \delta^{2}_{D}.
\]

After projecting out, the representation of \( 2, \xi^{[12]}_{bA} \) where \( A = 1 \) or 2, of SU(2) \( R \) remains. Originally it was a state represented by \( 20 \) of SU(4) \( R \).

We move on the quartic term in \( \theta \) of \( W^{N = 4} \). First let us consider the term \( (\theta^{4})^{C}_{D} \) transforming \( 15 \) of SU(4) \( R \). By factorizing this into \( (\theta^{2})^{C}_{E} \) and \( (\theta^{2})^{CD}_{E} \), as the two super derivatives act on only the last factor, the result can be written as \( (\theta^{2})^{C}_{E} (\partial V)^{E}_{C} (\delta D)_{[ab]} \delta^{1}_{C} \delta^{2}_{D} \) where \( C = 1, 2 \) by using the property of (3.5). On the other hand, when the two super derivatives act on the first and second factors separately, half of the terms contain \( \delta^{2}_{D} \) and this will lead to the contribution \( (\partial V)^{A}_{A} \). This is zero from the irreducibility of SU(4) tensor. The rest of the terms consists of \( (\theta^{2})^{C}_{ab} (\partial V)^{D}_{bc} \delta^{1}_{D} \delta^{2}_{E} \). Then one obtains

\[
\epsilon^{12AB} D_{aA} D_{bB} (\theta^{4})^{C}_{D} (\partial V)^{D}_{E} \bigg|_{\theta^{3} = \theta^{4} = 0} = (\theta^{2})^{A}_{ab} \xi^{[CD]}_{E} \delta^{1}_{C} \delta^{2}_{D}.
\]

Therefore \( (\partial V)^{D}_{ab} \) where the indices A, D are 1 or 2 transforms as \( 3 + 1 \) under the SU(2) \( R \).

For the singlet \( (\theta^{4})^{[cd]ef} = \epsilon_{CD} \theta^{E} \theta^{F} \) of SU(4) \( R \), the contribution can be written as

\[
\epsilon^{12AB} D_{aA} D_{bB} (\theta^{4})^{[cd]ef} W_{def} \bigg|_{\theta^{3} = \theta^{4} = 0} = (\theta^{2})^{[CD]}_{cd} W_{ab} \delta^{1}_{[C} \delta^{2}_{D]}.
\]

By rewriting \( (\theta^{4})^{[CD]}_{E} \) as a factorized form \( \epsilon_{EF} \epsilon_{GH} (\theta^{2})^{CG} (\theta^{2})^{DH} \), the only nonzero contribution occurs when each super derivative \( D \) acts on each \( (\theta^{2}) \) factor because we have seen that as the two super derivatives act on only one factor, there is no contribution from (3.4). Among four terms, the only nonzero contribution arises from the expression \( \delta^{1}_{[E} \delta^{2}_{F]} (\theta^{2})^{CD}_{ab} \). The other three terms vanish when we restrict to the case of \( \theta^{3} = \theta^{4} = 0 \) since the SU(4) index C or D should be equal to \( E \) or \( F \) and the irreducibility condition implies \( \delta^{1}_{[C} \delta^{2}_{E]} = 0 \). Therefore, we are led to

\[
\epsilon^{12AB} D_{aA} D_{bB} (\theta^{4})^{[CD]}_{EF} d^{[CD]}_{EF} \bigg|_{\theta^{3} = \theta^{4} = 0} = (\theta^{2})^{CD}_{ab} d^{EF}_{CD} \delta^{1}_{[C} \delta^{2}_{D]}.
\]
The original representation $20'$ of $SU(4)_R$ is reduced to a state by $1$, $d_{[CD]}^{[12]}$ where the indices $C$ and $D$ are 1 or 2, of $SU(2)_R$.

Now let us consider the next term $(\theta^5)^{[CD]}_E$ of $20$ under the $SU(4)_R$. Let us factorize this into $(\theta^2)^{[CD]}_E$ and $(\theta^3)^{cde}$. There are three possibilities for nontrivial contributions. When the two super derivatives act on the first $(\theta^2)$ factor, the result turns out to be $\epsilon^{12CD}(\theta^3)^{c}_{E(ab)}$, according to the computations (3.5). However, after restricting to the condition of $\theta^3 = \theta^4 = 0$, this will be the product of three $\theta$'s contracted with the three indices of epsilon tensor. Then the contribution becomes zero. The next nontrivial contribution is the case where all the super derivatives are acting on the last factor $(\theta^3)$. Based on the previous computation (3.6), the result can be written as $(\theta^2)^{[CD]}_E [\theta^1(\delta^2_E\partial_\alpha)\xi_{[CD]}^{\alpha}]$. In order to simplify this, let us recall that the quadratic expression of $\theta$ has the following terms: $\epsilon^{CD}(\theta^2)_E$ and $\epsilon_{bc}(\theta^2)^{CD}$. Then by multiplying an extra $\theta$, one obtains $\epsilon^{CD}(\theta^3)^{E}_{[(a}\partial_{b)]\xi_{[CD]}^{\alpha}}\xi_{[CD]}^{\alpha}\delta^1_{E F}$. Finally, the last contribution arises when the each super derivative acts on each factor separately. Two of six terms contain $\epsilon_{ab}$ and so this becomes zero. Two of them can be written as $\delta^C_E$ and this can be combined with the component field $\xi_{[CD]}^{\alpha}$. However, since $\xi_{[CD]}^{\alpha} = 0$ from the irreducibility of $SU(4)_R$, there is no contribution. Two of them can be written as $\epsilon^{12CF}\epsilon^{EFGH}(\theta^2)^{DH}\theta^G\delta^C_E$. One can write this as two $(\theta^3)^{H}_{a}\epsilon^{DG}$ and $(\theta^3)^{D}_{a}\epsilon^{HG}$-dependent terms. In both cases, they contain $\epsilon^C_E$ which will reduce to zero contribution by same reasoning. Therefore, we are led to

$$\epsilon^{12AB} D_{aA}D_{bB} (\theta^5)^{[CD]}_E [\theta^1(\delta^2_E\partial_\alpha)\xi_{[CD]}^{\alpha}]|_{\theta^3 = \theta^4 = 0} = (\theta^3)^{E}_{[(a}\partial_{b)]\xi_{[CD]}^{\alpha}}\xi_{[CD]}^{\alpha}\epsilon^{CD}\delta^1_{E F}.$$

One can factorize the next term $(\theta^5)^{C(\text{def})}_E$ of 4 into two factors: $(\theta^2)^{CD}(\theta^3)_D^{cde}$. When the two super derivatives act on the first factor, the contribution will be zero by (3.4). Let us consider when the two super derivatives act on the second factor. According to the computations done before (3.6), one gets $(\theta^2)^{CD}\theta^E\delta^1_{DF}\delta^2_{E F} (\partial \rho)_{Cabc}$. In order to simplify this, let us recall that the cubic expression of $\theta$ has the following terms and it can be decomposed into: $\epsilon^{CE}(\theta^3)^{cD}$ and $\epsilon^{DE}(\theta^3)^{cC}$. As a result, one obtains $\epsilon^{12}(\theta^3)^C (\partial \rho)_{(ab)cC}$. Of course, there are other expressions which can be easily reduced to this. Moreover, there is an expression when each super derivative acts on each factor. The only nontrivial part can be obtained from the case where the $SU(4)$ index $A$ is equal to $C$ while the index $B$ is equal to $E$, $F$ or $G$. In this case, also the final result can be written as the one we wrote above. Therefore, we arrive at the final contribution as follows:

$$\epsilon^{12AB} D_{aA}D_{bB} (\theta^5)^{C(\text{def})}_E (\partial \rho)_{C(\text{def})}|_{\theta^3 = \theta^4 = 0} = \epsilon^{12}(\theta^3)^C (\partial \rho)_{(ab)cC}.$$

By writing down the explicit dependence of $\theta$'s where $(\theta^6)_E$ transforms as $10$ under the $SU(4)_R$, there exist 30 terms. Six of them have a structure of $\epsilon_{ab}$ and this will vanish by
symmetrizing the indices \( a \) and \( b \). Twelve of them contain three product of \( \theta \)'s contracted with \( SU(4) \)-invariant epsilon tensor and this also vanishes because at the final step we put \( \theta^3 = \theta^4 = 0 \). The remaining terms take the form of \( \epsilon^{12IF} \epsilon_{EFG} \epsilon_{HIJ} \theta_a^I \theta^J_b \theta^G_c \). From this, the last three \( \theta \)'s can be written in terms of both \( (\theta^3)_b^i \) and \( (\theta^3)_b^{ij} \). By multiplying the first factor \( \theta^F \), all these expressions have \( \epsilon_{ab} \) and there are no contributions at all. Therefore, we conclude that 7

\[
\epsilon^{12AB} D_{aA} D_{bB} (\theta^6)_{(CD)} \frac{\partial}{\partial \theta^a} \frac{\partial}{\partial \theta^b} \overline{E}^{(CD)} |_{\theta^3 = \theta^4 = 0} = 0.
\]

Next terms can be manipulated similarly. When we express \( (\theta^6)_{(CD)(cd)} \) explicitly, it can be written as \( (\theta^2)^{EG}(\theta^2)^{FH}(\theta^2)^{GJ}_{GH} \epsilon_{CDEFG} \). As we did before, let us act the super derivatives on this. According to previous computation (3.4), when the two derivatives act on the first two \( (\theta^2) \) factors, the result is zero. The first nontrivial contribution can be obtained from the case where each super derivative acts on the second and third \( (\theta^2) \) factor respectively. Among four terms, two terms vanish because the remaining \( \theta \)'s contain \( SU(4) \) index 3 or 4. The remaining terms can be written as \( \delta^{1G}_{[G} \delta_{J]} \epsilon_{CDEFG}(\theta^2)^{EG}(\theta^2)^{FJ}_{cd} \epsilon_{cb} \). By manipulating the \( \theta \)'s, the quartic term in \( \theta \) can be written as \( (\theta^4)(\epsilon^{EF} \epsilon^{GJ} + \epsilon^{EJ} \epsilon^{GF}) \epsilon_{cd} \). Therefore, one arrives at \( (\theta^4) \partial_{a(a} \partial_{b)b} \overline{T}^{(ab)(CD)} \epsilon_{CDEFG} \). The second nontrivial contribution arises from the case where the two super derivatives act on only the last \( (\theta^2) \) factor. The result turned out to be \( \delta^{1G}_{[G} \delta_{H]2} \epsilon_{CDEFG}(\theta^2)^{EG}(\theta^2)^{FH} \epsilon_{da} \epsilon_{cb} \) which is equal to the previous one. Then, one gets the final expression

\[
\epsilon^{12AB} D_{aA} D_{bB} (\theta^6)_{[CD](cd)} \frac{\partial}{\partial \theta^a} \frac{\partial}{\partial \theta^b} \overline{T}^{[CD]} |_{\theta^3 = \theta^4 = 0} = (\theta^4) \partial_{a(a} \partial_{b)b} \overline{T}^{(ab)[CD]} \epsilon_{CDEFG} \quad \text{12}.
\]

After combining all the nonzero contributions we obtained so far, the component field \( \theta \)-expansion of \( \mathcal{N} = 2 \) chiral superfield strength \( \mathcal{W}^{AB}_{ab}(x, \theta) \) where two \( SU(2)_R \) indices \( A \) and \( B \) are antisymmetric is given by

\[
\mathcal{W}^{AB}_{ab}(x, \theta) = T^{[AB]}_{(ab)} + \theta^c [A (\partial \eta^B)_{(ab)c} + \theta^c [AB]_{(ab)c} + \theta^c [AB]_{(ab)B_c} + (\theta^2)^{C[A} (\partial V^B)_{(ab)c} + (\theta^2)^{[AB]} F^{(CD)}_{(ab)} + (\theta^2)^{[AB]} (\partial A)_{(ab)} + (\theta^2)^{[AB]} d^{[AB]}_{(CD)}]

+ (\theta^2)^{[A} (\partial_{b)} \overline{\xi}^{B]}_{(CD)} \epsilon^{CD} + (\theta^2)^{[A} (\partial_{b)} \overline{\xi}^{B]}_{(CD)} \epsilon^{CD} + (\theta^2)^{C}[A (\partial \rho)_{(ab)c} \epsilon^{AB}

+ (\theta^4) \epsilon^{AB} \epsilon_{CDEFG} \overline{T}^{(ab)(CD)} \quad \text{(3.7)}
\]

7One can prove this also by writing \( (\theta^6)_{(CD)} \) as \( \epsilon^{EFG} \epsilon^{HIJ} (\theta^2)^{EH} (\theta^2)^{FJ} (\theta^2)^{GJ} \). Then the nontrivial contribution can be obtained by acting two super derivatives on each \( (\theta^2) \) factor. In other words, we have \( \epsilon^{12AB} \epsilon^{EF} \epsilon^{HIJ} (\theta^2)^{EH} (\theta^2)^{FJ} \{ D_a A (\theta^2)^{FI} \}[D_b B (\theta^2)^{GJ}] \). After simplifying this, one gets an expression of \( \epsilon^{EFG} \epsilon^{HIJ} (\theta^2)^{EH} (\theta^2)^{J}_{(ab)} \). The last factor \( (\theta^2)^{J}_{(ab)} \) contains \( \epsilon_{ab} \) and \( (\theta^2)_{ab} \). Then the \( \epsilon_{ab} \) does not contribute when we make an antisymmetrization and there exists a relation \( (\theta^2)^{EH}(\theta^2)_{ab} = 0 \). As a result, the whole contribution is zero.
which was written in [7] using four spinor notation and has Weyl weight \( w = 1 \). The \( SU(2)_R \) irreducible representations \( \theta^a, \theta^a_A, \ldots, (\theta^a) \) transform as \( 2, 2, 3, 3, 3, 2, 2, 1 \), of \( SU(2)_R \) respectively. This is not an unrestricted chiral superfield because some of higher \( \theta \)-components are expressed in terms of derivatives of fields whose complex conjugates occur in the lower \( \theta \) sector. One can see this constrained condition from \( N = 4 \) description. From the constraint equation (3.2) in \( N = 4 \) superspace, one sees immediately that \( D^a D^b \mathcal{W}'_{ab}^{12} = \mathcal{D}_a \mathcal{D}_b \mathcal{W}'_{ab}^{12} \).

Then the linearized \( N = 2 \) conformal supergravity can be described off-shell by a chiral field strength superfield \( \mathcal{W}_{ab}^{AB} \) (3.7) that satisfies the constraint [7, 8]

\[
D^a D^b \mathcal{W}_{ab}^{AB} = \mathcal{D}_a \mathcal{D}_b \mathcal{W}_{ab}^{AB},
\]

where the \( N = 2 \) superspace derivatives are given by \( D^a_A = \frac{\partial}{\partial \theta^a_A} + \overline{\theta}^a_{\dot{a}} \partial_{\dot{a}a}, \mathcal{D}_a = \frac{\partial}{\partial \theta^a_A} \) where \( a, \dot{a}, A = 1, 2 \). One can find the prepotential \( V \) for \( \mathcal{W}_{ab}^{AB} \) by solving the constraints (3.8) directly or adding the constraints (3.8) in the action together with Lagrange multiplier and eliminating \( \mathcal{W}_{ab}^{AB} \). Then one obtains \( \mathcal{W}_{ab}^{AB} = \epsilon^{AB} \mathcal{D}^4 D^2 \partial \mathcal{V} \) [13].

The constraint equation (3.8) ensures that the reduced superfield (3.7) contains 24 + 24 independent bosonic and fermionic components, as does the Weyl multiplet. Then the condition for \( \mathcal{W}_{ab}^{AB} \) to be chiral, \( \mathcal{T}^C \mathcal{W}_{ab}^{AB} (x^{\dot{a}A}, \theta^a_A, \overline{\theta}^a_{\dot{a}}) = 0 \) implies that \( \mathcal{W}_{ab}^{AB} \) does not depend on \( \overline{\theta}^a_{\dot{a}} \). The field \( \mathcal{W}_{ab}^{AB} \) has an analog in \( \mathcal{N} = 1 \) super Yang-Mills theory [14, 15] where, at the linearized level, the theory is described by a chiral superfield \( \mathcal{W}_a (\mathcal{D}_a \mathcal{W}_b = 0) \) satisfying the additional constraint equation \( D^a \mathcal{W}_a = \mathcal{D}_a \mathcal{W}_a \). Some of components in the \( \theta \)-expansion of \( \mathcal{W}_a \) are real. For example, this constraint implies that the auxiliary field \( D^a \mathcal{W}_a \) at \( \theta = 0 \) is real. Other components obey Bianchi identities. Similarly, one can think of the auxiliary fields \( F_{ab}^{\dot{a}A} = \epsilon^{AB} D_A D_B \mathcal{W}_{ab}^{\dot{a}B} \) at \( \theta = 0 \) and this is real according to (3.8) and the Weyl multiplet \( F_{abcd} = \epsilon^{AB} D_A D_B \mathcal{W}_{cd}^{ab} \) at \( \theta = 0 \) satisfies the Bianchi identities.

Since the square of the Weyl multiplet, contracted over all indices, is a chiral scalar multiplet with Weyl weight \( w = 2 \). The highest component, a Lorentz scalar, has Weyl weight \( w = 4 \). Note that the Weyl weights for \( \theta^a_A \) and \( D^a_A \) are \( -1/2 \) and \( 1/2 \) respectively. Then, the action for \( \mathcal{N} = 2 \) conformal supergravity at the linearized level [6, 7, 8] is given by

\[
S = \int d^4x \int d^3\theta \, \mathcal{W}^2, \quad \mathcal{W}^2 = \epsilon_{AB} \epsilon^{CD} \mathcal{W}_{ab}^{AB} \mathcal{W}_{cd}^{CD}.
\]

By computing the \( \theta \)-integrals with correct normalization, this action leads to the following component action [7]

\[
S = \int d^4x \left[ 4 \eta^{[AB]} (\partial a) \partial b \mathcal{T}_{(ab)[AB]} - \frac{1}{2} \epsilon^{[AB]} \partial a \mathcal{C}_{a[AB]} + \frac{3}{2} d^2 + \frac{1}{2} (\partial V)^A_{(ab)B} (\partial V)^{(ab)B}_A + \frac{1}{4} C_{abcd} C^{abcd} - \frac{1}{8} (\partial A)_{(ab)} (\partial A)^{(ab)}_A \right].
\]

(3.10)
Here $d$ is an auxiliary nonpropagating pseudo-scalar field. In Table 1, we present some properties of these field contents. The complete generalized component action [7] for $N = 2$ conformal supergravity was found by constructing the non-linear $N = 2$ Weyl multiplet and applying the $N = 2$ density formula.

### 3.2 Physical spectrum of $N = 2$ conformal supergravity

To find out the spectrum of the theory, it is necessary to study the solutions of equations of motion. For a single antisymmetric tensor field $T_{\mu\nu}^{AB} = T_{[\mu\nu]}^{[AB]} = (\sigma_{\mu\nu})^{(ab)} T_{(ab)}^{[AB]}$, the linearized equation of motion from above action (3.10) is given by $\partial^a \partial^b T_{(ab)}^{[AB]} = 0$ and the solution becomes, as given in [5],

$$
T_{(ab)}^{[AB]} = \int d^4 k \delta(k^2) e^{ik \cdot x} \left[ \pi_a \pi_b \left( T_{-1}^{[AB]}(k) + i \frac{X_0}{k_0} T_{-1}^{d[AB]}(k) \right) + \pi_a \tilde{\tau}_b \eta_0^{[AB]}(k) \right].
$$

(3.11)

In this case, the number of on-shell degrees of freedom is 3 described by three different helicities $-1, -1', 0$. For two spinors $\xi_{\mu C}^{[AB]}$, the equation of motion is given by $\partial^a \xi_{\mu C}^{[AB]} = 0$ and the solution describes a massless state of helicity $-1/2$.

For two gravitinos $\eta^{Aa}_\mu$, the linearized equation of motion is $\partial^b \partial^a (\epsilon \partial^\mu b) \eta^{Aa}_\mu = 0$ and the solution is given by [5]

$$
\eta^{Aa}_\mu = \sigma^b \int d^4 k \delta(k^2) e^{ik \cdot x} \left[ \pi_a \pi_b \tau_b \left( \eta^{-\frac{1}{2}} A(k) + \frac{i}{k_0} \eta^{-\frac{1}{2}} \frac{X_0}{k_0} \right) + \pi_a \tilde{\tau}_b \eta^{Aa} \eta^{Aa} \left( k \right) + \tilde{\tau}_a \tilde{\eta} e^{Aa} \right],
$$

where the number of on-shell degrees of freedom of conformal gravitinos is $-4$ since there exist four independent helicity states. For three $SU(2)$ gauge fields $V_{\mu A}^B$, the equation of motion is obtained and the number of on-shell degrees of freedom of the gauge vector is equal to 2 by the helicity states $-1$ and 1.

Finally, for graviton $\epsilon_{\mu a \bar{a}}$, the equation of motion is $\partial^b \partial^a (\epsilon \partial^\mu b) \sigma^{\mu b}_{bb} e_{\mu a \bar{a}} = 0$ and the solution [5] can be written as

$$
e_{\mu a \bar{a}} = \sigma^b \int d^4 k \delta(k^2) e^{ik \cdot x} \left[ \pi_a \pi_b \tau_b \left( e_{-2} \left( k \right) + \frac{i}{k_0} e_{-2} \left( k \right) \right) + \pi_a \tilde{\tau}_b \eta^{Aa} e_{-1} \left( k \right) + \tilde{\tau}_a \tilde{\tau}_b \tilde{\pi}_a \tilde{\pi}_b \left( e_{2} \left( k \right) + \frac{i}{k_0} e_{2} \left( k \right) \right) \right]
$$

where the gauge transformation is used. The number of on-shell degrees of freedom of the Weyl graviton is 6. For theories involving higher derivatives, the counting of states is nontrivial. The number of on-shell degree of freedom $\nu$ [8] from path-integral methods which are different from a canonical approach [9] can be found as follows:

$$
\nu(T_{[AB]}^{ab}) = \nu(T_{[AB]}^{\bar{a}b}) = 3, \ \nu(\xi_{[AB]}^{aC}) = \nu(\xi_{[AB]}^{\bar{a}C}) = -1, \ \nu(\eta_{[AB]}^{aA}) = \nu(\eta_{[AB]}^{\bar{a}A}) = -4,
$$

$$
\nu(V_{\mu A}^B) = 2, \ \nu(A_\mu) = 2, \ \nu(d) = 0, \ \nu(e_{\mu}^{a\bar{a}}) = 6.
$$
Then a zero total number of on-shell degree of freedom \(^8\) can be checked by multiplying the above each degree of freedom by the number of each type of the field in the spectrum

\[ \nu_{\text{total}} = 6 \times 1 - 2 \times 2 - 8 \times 2 + 2 \times 3 + 2 \times 1 + 0 \times 1 + 6 \times 1 = +20 - 20 = 0. \]

We summarize the various aspects of \( \mathcal{N} = 2 \) conformal supergravity in Table 1.

### 3.3 Identification with \( \mathcal{N} = 2 \) twistor fields from \( \mathcal{N} = 4 \) twistor fields

We will prove that the physical states of \( \mathcal{N} = 2 \) conformal supergravity in four dimensions are equal to the spacetime fields described by the twistor fields \( f^I \) and \( g_I \). Let us start with the identification of the chiral superfield \( \mathcal{W}^{AB}_{ab} \) in the linearized conformal supergravity with the twistor fields

\[
\mathcal{W}^{12}_{ab}(x, \theta) = \left( \epsilon^{12AB} D_{aA} D_{bB} \int_{\mathbf{D}_{x,\theta}} g_I dZ^I \right) \bigg|_{\theta^3 = \theta^4 = 0}
\]

where the \( \mathcal{N} = 4 \) twistor fields \( g_I(Z) \) are given in [5] explicitly and the differentials \( dZ^I \) are \( dZ^I = (d\lambda^a, d\mu^\dot{a}, d\psi^A) = (d\lambda^a, d\lambda_b x^\dot{b}, d\lambda_c \theta^A) \). Here we used the twistor equations by introducing a bosonic spinor \( \lambda_a \)

\[
(\mu^\dot{a}, \psi^A) = (x^{a\dot{a}} \lambda_a, \theta^A \lambda_a).
\]

(3.12)

Through these, the fermionic coordinates \( \psi^A \) of CP\(^{3|4} \) are related to the fermionic coordinates \( \theta^A \) of chiral Minkowski superspace. By using the expression of \( dZ^I \), one can write down

\[
\mathcal{W}^{12}_{ab}(x, \theta) = \epsilon^{12AB} D_{A(a} D_{bB)} \int d\lambda^c \left[ g_c(Z) + x_{c\dot{c}} g^\dot{c}(Z) + \theta^A g_A(Z) \right] \bigg|_{\theta^3 = \theta^4 = 0}
\]

(3.13)

where it is understood that the homogeneous coordinates \( Z^I \) are viewed as functions of \( \lambda^a \) for fixed \( x \) and \( \theta \) through the twistor equations (3.12). What does this equation (3.13) mean? The left hand side of (3.13) provides the solutions for the conformal supergravity fields in momentum space using the result of previous subsection while the right hand side of (3.13) describes the \( \mathcal{N} = 2 \) twistor fields. Our aim is to demonstrate the precise relations between the physical states of \( \mathcal{N} = 2 \) conformal supergravity and the vertex operators of twistor-string theory using (3.13).

---

\(^8\)The number of off-shell degree of freedom \( n \) [8] can be found as follows: \( n(T^{[AB]}_{ab}) = 3 \), \( n(\epsilon^{[AB]}_{\mu\dot{a}C}) = -2 \), \( n(\eta^{\dot{a}A}_I) = -4 \), \( n(V^B_{\mu A}) = 3 \), \( n(A_\mu) = 3 \), \( n(d) = 1 \), \( n(\epsilon^{\dot{a}A}_\mu) = 5 \). Then a zero total number of off-shell degree of freedom can be checked by multiplying the above each degree of freedom by the number of each type of the field in the spectrum \( n_{\text{total}} = 6 \times 1 - 4 \times 2 - 8 \times 2 + 3 \times 3 + 3 \times 1 + 1 \times 1 = +24 - 24 = 0. \)
| States in $\mathcal{N} = 2$ CSG | $U(1)_R$ charge | Helicity | $SU(2)_R$ rep. | $SU(4)_R$ rep. |
|---------------------------------|-----------------|----------|----------------|----------------|
| $T_{(ab)}^{[AB]} = e^{AB}T_{(ab)}$ | 2               | $-1, -1', 0$ | 1              | 6              |
| $\xi_{aC}^{[AB]} = e^{AB}\xi_{aC}$ | 1               | $-\frac{1}{2}$ | 2              | 20             |
| $\eta_{\mu}^{Aa}$ | 1              | $-\frac{3}{2}, -\frac{3}{2}', -\frac{1}{2}, \frac{3}{2}$ | 2 | 4 |
| $V_{\mu A}^B$ | 0               | 1, -1      | 3              | 15             |
| $A_\mu$ | 0               | 1, -1      | 1              | -              |
| $d_{[AB]}^{[CD]} = e_{AB}e^{CD}d$ | 0               | none       | 1              | 20'            |
| $e_{\mu}^{\dot{a}}$ | 0               | 2, 2', 1, $-1, -2, -2'$ | 1 | 1 |
| $T_{\mu A}^{\dot{a}}$ | $-1$            | $\frac{3}{2}, \frac{3}{2}', \frac{1}{2}, -\frac{3}{2}$ | 2 | 4 |
| $\xi_{[AB]}^{[CD]} = e_{AB}\xi_{[CD]}^{[AB]}$ | $-1$            | $\frac{1}{2}$ | 2 | 20 |
| $T_{(ab)}^{[AB]} = e^{AB}T_{(ab)}$ | $-2$           | $1, 1', 0$ | 1              | 6              |

Table 1: The $U(1)_R$ charges, helicities and $SU(2)_R$ representations of physical states in $\mathcal{N} = 2$ conformal supergravity (CSG) in four dimensions. In the first column, we present the various field contents of $\mathcal{N} = 2$ CSG with $SU(2)_R$-invariant epsilon tensor. The $SU(4)_R$ $R$-symmetry of $\mathcal{N} = 4$ conformal supergravity is broken to $SU(2) \times SU(2)_R \times U(1)_R$. Each $SU(2)_R$ representation of $\mathcal{N} = 2$ conformal supergravity corresponds to the representation $SU(2) \times SU(2)_R \times U(1)_R$ with invariant $U(1)_R$ charge. For example, in the branching rule of $6 \to (1, 1)_2 + (1, 1)_{-2} + (2, 2)_0$, the $U(1)_R$ invariant representation is $(1, 1)_2$ which is denoted by $U(1)_R$ charge 2 and $SU(2)_R$ representation 1 separately in the first row. All other $SU(2)_R$ representations, 20, 4, 15, 20', 1, 4, 20 can be broken to the appropriate $SU(2)_R$ representation with invariant $U(1)_R$ charge as above. This will be discussed in detail later.
The equation of motion for $W_{ab}^{AB}$ from the superspace action is given by

$$D^a_A D^b_E W_{ab}^{AB} = 0.$$ (3.14)

Let us check that the above superspace function $W_{ab}(x, \theta)$ (3.13) which can be written as twistor fields $g_I$ satisfies (3.14). Suppose that there is a function $\phi(Z)$ that is any function of $Z$. Then one can transform it to a function $\phi'(\lambda, x, \theta)$ by writing $\mu$ and $\psi$ in terms of $\lambda$ through the relation (3.12). By the chain rule, one gets $D^a_A \phi' = \lambda^a \left( \frac{\partial}{\partial \mu^A} + \theta^b \frac{\partial}{\partial \psi^b} \right) \phi'$ where $\phi' = \phi(\lambda^c, x^\epsilon c, \theta^4 d).$ Since $\lambda^a \lambda_a = 0$, it is easy to see that $D^a_A D_B a \phi' = 0$. As observed in [5], inside of an integral (3.13), there are the factors of $x$ and $\theta$ which are linear in $x$ and $\theta$, the commutators between $D^2_A D_B a$ and $x$ or $\theta$ vanish and we are left with $D^2_A D_B a D^b_C D^b_d \int g_I dZ^I = 0$ implying that (3.13) satisfies (3.14).

Let us determine the $\mathcal{N} = 2$ twistor fields $g_I(Z)$ in terms of $\mathcal{N} = 2$ chiral superfield $W_{ab}^{AB}$ which has a simple relation (3.3) with $\mathcal{N} = 4$ chiral superfield providing corresponding twistor fields. Let us start with $\mathcal{N} = 4$ twistor field [5]

$$g_a(Z) = \cdots - \left( \frac{i \lambda^a \sigma^b \theta^b_{\theta \partial} \theta^4 \theta^4}{\kappa^2} \right) \left[ (\psi^2)_{AB} T^{[AB]}_{-1} + (\psi^3)_{A} \lambda^4 a + (\psi^4) \lambda^4 e \right].$$ (3.15)

From this, one can take the product of super derivatives $D_c^\beta D_d^\gamma$ acting on $g_a(Z)$ and the nontrivial terms arise from the $\theta^3 \theta^4$-dependent terms and the result is given by 9

$$\epsilon^{12CD} D_c^\beta D_d^\gamma \left. g_a(Z) \right|_{\theta^3 = \theta^4 = 0} = \left( - \frac{i \lambda^a \sigma^b \theta^b_{\theta \partial} \theta^4 \theta^4}{\kappa^2} \right) \lambda_c \lambda_d \left[ T^{[AB=12]}_{-1} + \psi \lambda^4 a + \lambda \lambda^4 e \right].$$ (3.16)

What does this result mean? We have determined the second term on the right hand side of (3.13) in terms of $\mathcal{N} = 2$ twistor fields. Then how one can relate this to $\mathcal{N} = 2$ chiral superfield $W_{ab}^{AB}$? After we are plugging the result (3.16) into the right hand side of (3.13), then the left hand side of (3.13) will tell us about the component fields for $\mathcal{N} = 2$ chiral superfield $W_{ab}^{AB}$. In this way, one can determine all the identifications between the spacetime fields described by twistor fields and the physical states of conformal supergravity.

In section 2, the helicity and $SU(2)_R$ representation for $\partial_\alpha \partial_\beta \theta^a g_a$ was given in (2.2). One can identify the right hand side of (3.16) with the twistor description in section 2, term by term. For example, the massless state characterized by $(0, 1)$ before taking $\partial_\alpha \partial_\beta$ on $\theta^a g_a$ corresponds to $\lambda_c \lambda_d T^{[AB=12]}_{-1}$ which denotes the twistor field for the spacetime field.

---

9Since we keep the notation for $\mathcal{N} = 4$ quantity by $f^I(Z)$ and $g_I(Z)$, the corresponding $\mathcal{N} = 2$ quantity can be obtained by acting super derivatives on $\mathcal{N} = 4$ twistor fields and putting the other fermionic coordinates to zero. Therefore, the results depend on only $\theta^3$ and $\theta^4$. 

16
\( \lambda_c \lambda_d T_{-1}^{[AB=12]} \) of helicity zero \(^{10}\) and is a singlet \( \mathbf{1} \) under \( SU(2)_R \)-representation \(^{11}\). The fact that the above spacetime field has a helicity 0 is obvious because \( T_{-1}^{[AB=12]} \) itself represents a helicity \(-1\) and the role of \( \lambda_c \lambda_d \) inside the integral of (3.13) increases the helicity +1. Then how one can understand this \( SU(2)_R \) representation? It is known that the \( SU(4)_R \)-representation \( \mathbf{6} \) of \( \mathcal{N} = 4 \) conformal supergravity is decomposed into \([16]\)

\[
\mathbf{6} \rightarrow (\mathbf{1}, \mathbf{1})_2 + (\mathbf{1}, \mathbf{1})_{-2} + (\mathbf{2}, \mathbf{2})_0
\]  

(3.17)

under the \( SU(4)_R \rightarrow SU(2) \times SU(2)_R \times U(1)_R \). The only invariant quantity through the above procedure (truncation of one can understand this \( SU(2)_R \)-representation is \( (\mathbf{1}, \mathbf{1})_2 \) which is represented by \( \hat{T}_{-1}^{[AB=12]} \) or \( \hat{T}_{-1}^{[AB=12]} \). Note that when we take \( \partial^a \) in both sides of (3.16), the prefactor \(-i \lambda^a g_a^2 \) in the right hand side is gone \([5]\). If we are using \( \partial_a \partial_b \) on \( g_a(Z) \) rather than the super derivatives, then the factor \( \lambda_c \lambda_d \) in the right hand side of (3.16) will disappear.

What about other terms in the right hand side of (3.16)? The next massless state \((-\frac{1}{2}, \mathbf{2}) \) for \( \partial_a g^a \) given in section 2 should correspond to both \( \lambda_c \lambda_d \hat{\eta}_{-\frac{1}{2}}^{A=2} \) and \( \lambda_c \lambda_d \hat{\eta}_{-\frac{1}{2}}^{A=1} \) in (3.16). One can combine and write them as \( \hat{\psi}_A \hat{\eta}_{-\frac{1}{2}}^A \) where index \( A \) is a representation \( \mathbf{2} \) under the \( SU(2)_R \)-representation since the presence of \( \lambda_c \lambda_d \) in the spinor integral of (3.16) makes an increase of helicity +1. The representation \( \mathbf{2} \) of \( SU(2)_R \) can be understood as follows. More precisely, the \( SU(4)_R \)-representation \( \mathbf{4} \) is decomposed into

\[
\mathbf{4} \rightarrow (\mathbf{1}, \mathbf{2})_1 + (\mathbf{2}, \mathbf{1})_{-1}
\]  

(3.18)

under the \( SU(4)_R \rightarrow SU(2) \times SU(2)_R \times U(1)_R \). Then the state \( \hat{\eta}_{-\frac{1}{2}}^A \) above corresponds to \( (\mathbf{1}, \mathbf{2})_1 \).

Finally, the massless state \((-1, \mathbf{1}) \) corresponds to the last term in (3.16). In this case, one can express it as \( \hat{\psi}_A \hat{\psi}_B \dot{\hat{e}}_{-1} \epsilon^{AB} \) which has a helicity \(-1\) and is a singlet \( \mathbf{1} \) where the indices \( A \) and \( B \) are \( SU(2)_R \) indices.

So far, we have considered the contributions from \( g^a(Z) \) in (3.16). Now we move on those from \( g_a(Z) \). Let us consider \( \mathcal{N} = 4 \) twistor field \([5]\)

\[
g_a(Z) = \cdots + \lambda_a \left[ (\psi^2)_{[AB]} \hat{T}_{-1}^{[AB]} + (\psi^3)_{A} \hat{\eta}_{-\frac{1}{2}}^{A} + (\psi^4) \dot{\hat{e}}_{-2} \right].
\]  

(3.19)

\(^{10}\)Recall that the hatted fields stand for twistor fields and unhatted fields stand for supergravity fields.

\(^{11}\)Let us describe this more explicitly and how it works. From the relation (3.13), one sees that \( \square W_{cd} = \int d\lambda^a \lambda_c \epsilon^{12CD} D_{cD} D_{dD} \partial_a g^a \). Now the lowest term of the left hand side together with (3.7) can be written as \( \int d^4 k \delta(k^2) e^{ikx} \pi_c \pi_d T_{-1}^{[AB=12]}(k) \) by substituting the solution of equation of motion (3.11) into (3.7). On the other hand, the lowest term of right hand side of above relation can be written as, from the result (3.16), \( \int d\lambda^a \lambda_c \lambda_d \hat{T}_{-1}^{[AB=12]} \) which is equal to \( \lambda_c \lambda_d T_{-1}^{[AB=12]}(x) \) up to a numerical factor. This is a precise correspondence between the twistor field \( T_{-1}^{[AB=12]} \) and spacetime field \( T_{-1}^{[AB=12]} \).
By acting two super derivatives on this, there exists the following relation which looks similar to (3.16) but the other component fields of ‘dipole’ fields occur

\[ \epsilon^{12CD} D_{eC} D_{dD} g_a(Z) \big|_{\theta^4=0} = \lambda_a (\lambda_c \lambda_d) \left[ \bar{T}_{-\frac{1}{2}}^{[AB=12]} + \psi \eta_{-\frac{1}{2}}^{A=2} + \chi \hat{\theta}_{-\frac{1}{2}}^{A=1} + \psi \hat{\epsilon}_{-2} \right]. \]  

(3.20)

The factor \( \lambda_a \) in (3.20) can be removed when we consider \( \partial^a g_a(Z) \) instead of \( g_a(Z) \). Now it is easy to see the exact identification in section 2 by following the analysis in previous paragraph. The massless states \((0, 1), (-\frac{1}{2}, 2), (-1, 1)\) given in section 2 correspond to the component twistor fields together with \( \lambda, \lambda_d \) in (3.20).

Now we are left with the contributions from \( \theta^A \hat{g}_A(Z) \) in (3.16). One has to be careful about these contributions because the super derivative can act on the fermionic coordinate \( \theta^A \) also. In this computation, let us start with other type of \( \mathcal{N} = 4 \) twistor field \([5]\)

\[ g_A(Z) = \cdots + \psi^B \hat{T}_{0[AB]} + \psi^B \hat{E}_{0[AB]} + (\psi^2)_{[AB]} \bar{T}_{-\frac{1}{2}}^{B=2} + (\psi^2)_{[BC]} \bar{T}_{-\frac{3}{2}A}^{[BC]} + (\psi^3)_{A} \hat{e}_{-1} + (\psi^3)_{B} \hat{V}_{-1}^{B} + (\psi^4) \hat{\eta}_{-\frac{3}{2}A}. \]  

(3.21)

One takes two super derivatives and has to pick up the \( \theta^A \theta^4 \)-dependent terms in above. Then it turns out

\[ \epsilon^{12CD} D_{eC} D_{dD} g_{A=1}(Z) \big|_{\theta^4=0} = (\lambda_c \lambda_d) \left[ \left( \bar{T}_{-\frac{1}{2}}^{B=2} + \bar{T}_{-\frac{3}{2}A}^{[BC=12]} \right) + \psi \hat{V}_{-1}^{B=1} \right] + \psi \hat{e}_{-1} + \psi \hat{\eta}_{-\frac{3}{2}A}. \]  

(3.22)

One has to understand this \( \mathcal{N} = 2 \) twistor field in the context of (3.13) and there exists a factor \( \theta^A_{e=1} \) as well as (3.22) in the integrand. Once again, after we are plugging the result (3.22) into the right hand side of (3.13), then the left hand side of (3.13) will tell us about the particular component fields for \( \mathcal{N} = 2 \) chiral superfield \( \mathcal{W}_{ab}^{[AB]} \). Let us describe each term in detail. The \( SU(4)_R \)-representation \( 20 \) represented by \( \xi_{[aC]}^{[AB=12]} \) can be decomposed into

\[ 20 \rightarrow (1,2)_1 + (3,2)_1 + (2,1)_3 + (1,2)_{-3} + (2,1)_{-1} + (2,3)_{-1}. \]  

(3.23)

Then \((1, 2)_1\) which has \( U(1)_R \) charge 1 describes a twistor field \( \xi_{-\frac{3}{2}A=1}^{[BC=12]} \). In other words, the index \( A \) plays the role of \( 2 \) of \( SU(2)_R \) representation. The other element of \( 2 \) will be seen

\[ \frac{12}{12} \text{How one realizes the correspondence between twistor field and conformal supergravity field? There is a relation given by (3.13). By substituting the solution of } \xi_{[aC]}^{[AB]} \text{ in terms of momentum space into the left hand side of (3.13), then there is an expression of } \theta^D_c \int d^4k \eta \lambda_a \lambda_b \xi_{-\frac{3}{2}A=12}. \text{ On the other hand, the right hand side of (3.13) has a term } \int d^4k \theta^D_c \lambda_a \lambda_b \xi_{-\frac{3}{2}A=12}. \text{ Now one differentiates both expressions with respect to } \theta^D_c. \text{ Then one obtains the following relation: } \int d^4k \eta \lambda_a \lambda_b \xi_{-\frac{3}{2}A=12}^{[AB=12]} = \int d^4k \lambda_a \lambda_b \xi_{-\frac{3}{2}A=12}^{[AB=12]}. \text{ In this way, one gets a precise relation between twistor fields and spacetime fields.} \]
from the description of \( g_{A=2}(Z) \) below. Since \( \eta_{-\frac{1}{2}} \) has a representation \((1,2)_1\) according to (3.18), one can construct the sum of these two twistor fields as \( \mathcal{N} = 2 \) twistor field, \( \xi_{-\frac{1}{2},A=1} \) because they have common helicity and \( SU(2)_R \) representation.

Under the \( SU(4)_R \rightarrow SU(2) \times SU(2)_R \times U(1)_R \), the \( SU(4)_R \)-representation 15 characterized by \( V^B_{\mu A} \) is decomposed into

\[
15 \rightarrow (1,1)_0 + (1,3)_0 + (3,1)_0 + (2,2)_2 + (2,2)_{-2}.
\]  

(3.24)

Therefore, the representation \((1,1)_0 + (1,3)_0\) preserving \( U(1)_R \) charge corresponds to \( \hat{V}^B_{-1,A} \).

We can also combine two terms in \( \chi \)-dependent terms and express as \( \hat{V}^B_{-1,A=1} \). Also from the branching rule of \( \mathbf{4} \) of \( SU(4)_R \), the representation \((1,2)_{-1}\) gives a description for \( \hat{\eta}^{BC=12}_{-\frac{1}{2},A=1} \). For the description of helicity on these fields, it is easy to check.

Let us consider when the index \( A \) in \( g_{A}(Z) \) is equal to 2. Similarly, one can do for \( g_{A=2}(Z) \):

\[
\epsilon^{12CD} D_{cD} D_{dD} g_{A=2}(Z) \bigg|_{\theta^3=\theta^4=0} = (\lambda_c \lambda_d) \left[ \left( \eta^B=1_{\frac{1}{2}} + \xi^{[BC=12]}_{-\frac{1}{2},A=2} \right) + \psi \left( \hat{\eta}^{B=2}_{-1,A=2} + \hat{\eta}^{B=2}_{-1,A=2} \right) \right].
\]  

(3.25)

The first two terms can be combined \( \mathcal{N} = 2 \) twistor field \( \xi_{-\frac{1}{2},A=2} \) because they share common helicity and \( SU(2)_R \) representation and moreover the two terms in \( \psi \)-terms can be written as \( \hat{V}^{B=2}_{-1,A=2} \). It is more clear if we combine the two results (3.22) and (3.25) in a single covariant way. It becomes \( \xi_{-\frac{1}{2},A} + \psi^B \hat{V}^{B}_{-1,A} + \psi^B \psi^C \hat{\eta}^{BC}_{-\frac{1}{2},A} \epsilon_{BC} \) which corresponds to the massless states \((-\frac{1}{2},2), (-1,3 \oplus 1), (-\frac{3}{2},2)\) for \( \partial_\alpha \partial_\beta g_{A=1,2} \) given in section 2. Note that if we use \( \partial_\alpha \partial_\beta \) on \( g_{A=1,2}(Z) \) rather than the super derivatives, then the factor \( \lambda_c \lambda_d \) in the right hand side of (3.22) or (3.25) disappears.

Now let us consider when the index \( A = 3 \) for \( g_{A}(Z) \). In (3.13), one inserts the quadratic spinors \( \lambda_a \lambda_b \) in front of \( \mathcal{N} = 1 \) twistor field \( g_{A=3}(Z) \) in the integrand which is homogeneous of degree \(-1\) and divide into those factors. Then the quantity \( \lambda_a \lambda_b g_{A=3}(Z) \) is homogeneous of degree 1 which is the same degree of \( \mathcal{N} = 2 \) twistor field we introduced in section 2 and the quantity \( \frac{d \chi^A g_{A=3}}{\lambda_a \lambda_b} \) is homogeneous of degree \(-1\). The integral appearing in (3.13) makes sense because the whole expression is homogeneous of degree zero. Finally, one can construct the following \( \mathcal{N} = 2 \) object, by acting one super derivative \( D_{d,B=3} \) on the factor \( \theta^{A=3}_c \) and the other super derivative \( D_{d,D=4} \) on the \( g_{A=3}(Z) \),

\[
\epsilon^{123D} \lambda_a \lambda_b D_{dD} g_{A=3}(Z) \bigg|_{\theta^3=\theta^4=0} = (\lambda_a \lambda_b \lambda_d) \left[ \left( \hat{T}_{0[AB=34]} + \hat{E}_{0(AB=34)} \right) + \psi \left( \hat{\eta}^{B=2}_{1,A=3} + \xi^{[BC=23]}_{-\frac{1}{2},A=3} \right) \right] + \chi \left( \hat{\eta}^{B=2}_{1,A=3} + \xi^{[BC=23]}_{-\frac{1}{2},A=3} \right) + \psi \chi \left( \hat{\eta}^{B=3}_{-1,A=3} + \hat{V}^{B=3}_{-1,A=3} \right).
\]  

(3.26)
When we act two super derivatives on \( g_{A=3}(Z) \) completely, due to the factor \( \theta_A^{A=3} \) in front of it, the contribution will be zero after we put \( \theta^3 = \theta^4 = 0 \). If one wants to compare this twistor field with the spacetime field, one should insert the above result into the spinor integral given in (3.13). According to the branching rule of \( \mathbf{10} \) of \( SU(4)_R \) into

\[
\mathbf{10} \rightarrow (2,2)_0 + (3,1)_2 + (1,3)_{-2}
\]

under the \( SU(2) \times SU(2)_R \times U(1)_R \), the twistor field \( \hat{E}_{0(AB=34)} \) describes one of the element for the representation \( (3,1)_2 \) preserving \( U(1)_R \) charge. From the breaking pattern (3.17), \( \hat{T}_{0[AB=34]} \) which can be written as \( \epsilon^{12CD} \hat{T}_{0[CD]} = \hat{T}_{0[12]} \) corresponds to the representation \( (1,1)_2 \).

So we combine these two terms as \( \hat{T}_{0[AB=34]} \) because they share same helicity and \( SU(2)_R \) representation. Next term can be understood from (3.18) and the twistor field \( \hat{s}^{B=2}_{-3,4} \) describes one of the element for the representation \( (1,2)_1 \). Similarly, the twistor field \( \hat{s}^{[BC=23]}_{-4,3} \) represents one of the element for \( (1,2)_1 \) from the branching rule (3.23). One can write \( \hat{T}_{0[AB=34]} + \psi^A\hat{\eta}^{1/2,4} + \psi^B\psi^C\hat{e}^{1/2,4} \epsilon_{BC} \) in covariant way to which the massless states \( (0,1), (-1/2,2), (-1,1) \) given in section 2 correspond.

In addition to a chiral superfield \( W^{AB}_{ab} \), there is also an anti chiral superfield \( \overline{W}^{ab}_{AB} \) which depends on both \( \hat{x}^{\hat{a}} = x^{\hat{a}} + \theta^{\hat{a}}\hat{\eta}_{-\hat{A}} \) and \( \hat{\eta}^\lambda = \hat{\eta}_{\lambda} \). Then the relation between spacetime fields and twistor fields can be written as

\[
\overline{W}^{ab}_{12}(\hat{x}, \theta) = \epsilon_{12AB}\overline{D}^{A}(\hat{D}^{B}) \int d\hat{\theta} \left[ \overline{g}_c(Z) + \hat{x}_{\hat{a}}\hat{g}^{\hat{c}}(Z) + \hat{\eta}_{\hat{c}}\hat{g}^{\hat{C}}(Z) \right] \bigg|_{\gamma_3 = \gamma_4 = 0}
\]

where the complex conjugate of \( Z^I, \overline{Z}_I \), is given by \( \overline{Z}_I = (\overline{x}_{a}, \overline{\theta}_a = \hat{x}^{a\hat{a}}\overline{x}^\hat{a}, \overline{\psi}_A = \overline{\theta}_{\hat{A}}\overline{\lambda}_a) \) using corresponding twistor equations. The dual fields \( \hat{g}^{I}(Z) \) appearing in the integrand of right hand side have an explicit form, through Fourier-like transform, in terms of \( f^I(Z) \) [5]. Fourier transform maps the \( f^I(Z) \) field to a dual field \( \hat{g}^{I}(Z) \). Inside of the integrand of this defining equation, there is a factor \( e^{ZK}\overline{Z}_K \). When we differentiate \( \hat{g}^{I}(Z) \) with respect to \( \overline{Z}_I \), the factor \( Z^I \) appears in the integrand, due to this exponential factor. Therefore, the action of two superspace derivatives \( \overline{D}^{A} \)'s on \( \hat{g}^{I}(Z) \) leads to the two product of \( Z^I \)'s in the integrand, in general. Since this integral contains an integral over the fermionic coordinates, the nonzero contribution will take place only if \( f^I(Z) \) does not have these fermionic coordinates(in the present case, \( \theta^3 \) or \( \theta^4 \)) in \( \theta^{\hat{A}} \)-expansion. In other words, we simply put the condition of \( \theta^3 = \theta^4 = 0 \) on the \( f^I(Z) \)'s in order to get \( \mathcal{N} = 2 \) twistor fields from \( \mathcal{N} = 4 \) field contents.

Using similar arguments to the analysis given in \( g_I(Z) \), one obtains the \( \mathcal{N} = 2 \) twistor field \( f^I(Z) \). From \( \mathcal{N} = 4 \) twistor fields [5], one writes

\[
\lambda^a f_a(Z) \bigg|_{\theta^3 = \theta^4 = 0} = \hat{c}_2 + \psi \hat{\eta}_{\hat{A}}^{2,4} + \hat{\lambda} \hat{\eta}_{\hat{A}}^{2,4} + \psi \hat{\chi} \hat{T}_{1}^{[AB=34]}.
\]

(3.27)
How one does understand this result? Let us concentrate on the first term of (3.27). One can see this from Fourier-like transform. The nonzero contribution of fermionic integral will arise from the exponential factor because the first term of (3.27) does not depend on the fermionic coordinate $\psi^A$. Then the dependence on $\bar{\psi}_A$ of dual field $\bar{g}^a(Z)$ has a term like $(\bar{\psi}^i)$ because the integrals over $\psi^A$ in Fourier-like transform pick up $(\psi^i)$ term. When we go back to the expression of $g_a(Z)$ (3.15) and make a complex conjugation of it, then the Lorentz scalar $\partial_a \bar{g}^a(Z)$ will have a term like $(\bar{\psi}^i) \hat{\epsilon}_2$. Therefore, one leads to the first term of (3.27). Other terms can be described similarly. Once we identify the relation between $f^I(Z)$ and $\hat{g}^I(Z)$ explicitly, then the spacetime interpretation of $f^I(Z)$ (i.e., a precise relation between hatted twistor fields and unhatted spacetime fields) can be read off from those for $\hat{g}^I(Z)$ which is related to $g_I(Z)$.

As we have seen in section 2, the massless state given by $(2, 1)$ describes $\hat{e}'_2$ which is a singlet of $SU(2)_R$ and the states $(\frac{3}{2}, 2)$ corresponds to both $\hat{\psi}_1$, $A = 1$ and $\hat{\psi}_1$, $A = 2$. The branching rule of $\mathcal{N} = 4$ result [5], one requires that other two fermionic coordinates $\theta^3$ and $\theta^4$ vanish. Then one gets

$$
\mu^A f_a(Z) \big|_{\theta^3=\theta^4=0} = \hat{e}_2 + \psi \hat{\gamma}^4, A = 1 + \chi \hat{\gamma}^4, A = 2 + \psi \chi \hat{\gamma}^4 [AB=34].
$$

This looks like (3.27) exactly except that the independent fields for given helicity are replaced by other elements of ‘dipole’ fields. Let us consider the first term of (3.28). The nonzero contribution of Fourier-like transform will arise from the exponential factor because the first term of (3.28) does not depend on the fermionic coordinate $\psi^A$. Then the dependence on $\bar{\psi}_A$ of dual field $\bar{g}^a(Z)$ has a term like $(\bar{\psi}^i)$. From the expression of $g_a(Z)$ (3.19), the Lorentz scalar $\partial_a \bar{g}^a(Z)$ will have a term like $(\bar{\psi}^i) \hat{\epsilon}_2$. Therefore, one leads to the first term of (3.28). The exact identification in section 2 by following the analysis in previous paragraph can be done. That is, the massless states $(2, 1), (\frac{3}{2}, 2), (1, 1)$ given in section 2 correspond to the component twistor fields in (3.28).

For the vertex operator with fermionic index, one can compute the following quantity from
\[ N = 4 \text{ result } [5]. \] That is,

\[
\left. f^{A=1}(Z) \right|_{\theta^3 = \theta^4 = 0} = \hat{\eta}_{\frac{1}{2}}^{A=1} + \psi(\hat{e}_1 + \hat{V}_{1,A=1} B = 1) + \psi(\hat{V}_{1,A=1} B = 2) + \psi(\hat{e}_1 + \hat{V}_{1,A=1} B = 2) + \psi(\hat{V}_{1,A=1} B = 1) + \psi(\hat{V}_{1,A=1} B = 1). (3.29)\]

One can also understand each term above, through Fourier-like transform. By looking at the \((\psi^4)\) term of \(g_{A=1}(Z)\) (3.21) and taking a complex conjugation with opposite helicity, one arrives at the first term of (3.29). Here a twistor field \(\hat{\xi}_{\frac{3}{2}}^{A=1} [BC=12]\) arises from the representation \((\mathbf{1}, \mathbf{2})_{-1}\), according to the branching rule of \(\mathbf{20}\) of \(SU(4)_R\). See for example, (3.23). Note that the twistor field \(\hat{\eta}_{\frac{1}{2}, B=2}^{A=1}\) has same representation \((\mathbf{1}, \mathbf{2})_{-1}\) from the branching rule of \(\mathbf{4}\) of \(SU(4)_R\). The analysis for \(15\) of \(SU(4)_R\) representation (3.24) to the twistor field \(\hat{V}_{1,A=1} B = 1\) and \(\hat{V}_{1,A=1} B = 2\) holds here. The two terms in \(\psi\) and the two terms in \(\psi\chi\) in (3.29) can be written as a single term \(\hat{V}_{1,A=1} B = 1\) and \(\hat{\xi}_{\frac{3}{2}}^{A=1} [BC=12]\) respectively because they have same helicity and \(SU(2)_R\) representation.

Let us compute also other component from \(N = 4\) result

\[
\left. f^{A=2}(Z) \right|_{\theta^3 = \theta^4 = 0} = \hat{\eta}_{\frac{1}{2}}^{A=2} + \psi(\hat{V}_{1,A=2} B = 1) + \chi(\hat{V}_{1,A=2} B = 2) + \psi(\hat{V}_{1,A=2} B = 1) + \psi(\hat{V}_{1,A=2} B = 2) + \chi(\hat{V}_{1,A=2} B = 1). (3.30)\]

One can write the two results (3.29) and (3.30) in \(SU(2)_R\) covariant way in order to compare with the description given in section 2. That is, the \(N = 2\) twistor field can be summarized by

\[
f^{A}(Z) = \hat{\eta}_{\frac{1}{2}}^{A} + \psi^{B} \hat{V}_{1,A} + \psi^{C} \hat{\xi}_{\frac{3}{2}}^{A} \epsilon_{BC}. \]

Each massless state coincides with \((\mathbf{3}, \mathbf{2}), (\mathbf{1}, \mathbf{3} \oplus \mathbf{1}), (\mathbf{1}, \mathbf{2})\) respectively given in section 2. The helicities are encoded in the subscript of twistor fields and \(SU(2)_R\) representation can be read off easily.

Let us move on the final case. One can multiply \(\lambda_a \lambda_b\) in the integrand of Fourier-like transform and divide \(\lambda_a \lambda_b\). The expression \(\frac{f^{A=3}(Z)}{\lambda_a \lambda_b}\) is homogeneous of degree \(-1\) which is the same degree for \(N = 2\) twistor field. Then one can compute the following \(N = 2\) object which has a super derivative \(D_{a,A=3}\), compared with a twistor field \(g_{A=3}(Z)\) on which a super derivative \(D_{a,A=4}\) acts,

\[
\epsilon^{12D4} \frac{1}{\lambda_a \lambda_b} D_{aD} f^{A=3}(Z) \left|_{\theta^3 = \theta^4 = 0} \right. = \frac{1}{\lambda_b} \left[ (\hat{e}_1 + \hat{V}_{1,A=3} B = 3) + \psi(\hat{\eta}_{\frac{1}{2}, B=1}^{A=3} + \hat{\xi}_{\frac{3}{2}}^{A=3} [BC=13] + \hat{V}_{1,A=3} B = 2) + \psi(\hat{\eta}_{\frac{1}{2}, B=2}^{A=3} + \hat{\xi}_{\frac{3}{2}}^{A=3} [BC=23] + \hat{V}_{1,A=3} B = 1) \right].
\]

Note that when we take the differentiation on \(\hat{g}^{I=3}(Z)\) with respect to \(\bar{\theta}_4\) or \(\bar{\psi}_{A=4}\), then a factor \(\theta^4\) appears in the integrand of defining equation, due to the exponential factor. Then the fermionic integration over \(\hat{\psi}^A\) acts on \(\theta^4 e^{zK \zeta_K} f^{I=3}(Z)\) in the integrand. Therefore, the nonzero contribution of this integral occurs when the fermionic integration over \(\hat{\psi}^{A=3}\) (which
is equivalent to the differentiation $D_{a,D=3}$ acts on the function $f^{A=3}(Z)$, as above. Moreover, the nonzero contribution in the integrand has a term $\psi\chi\bar{\psi}\chi$. By computing the integration over $\psi^A$, the resulting expression has a coefficient of $\bar{\psi}\chi$. Therefore, the fermionic independent term of above equation $\hat{e}_1 + \hat{V}_{1,A=3}$ can be obtained from the $\psi^A$-expansion of $g_{A=3}(Z)$ (3.21) with opposite helicity: the term which has a factor $\psi\chi\beta$.

The twistor field $\hat{E}_{\theta(AB=34)}$ describes the representation $(3,\mathbf{T})_{-2}$ in the breaking pattern of 10 of $SU(4)_R$ which can be read off from (3.26). By redefining the two terms in $\psi$ which have common $SU(2)_R$ represenation and $U(1)_R$ charge due to the analysis of (3.29) and the two terms in $\chi$ which have the same quantum number also, $(\mathbf{T},2)_{-1}$, one can write this as follows: $\hat{\eta}_{\bullet,B=1}$ and $\hat{\eta}_{\bullet,B=2}$. Similar redefinition for the last term can be applied also by similar description given in $g_{A=3}(Z)$ above. Then, we obtain the simplified form $\hat{e}_1 + \psi^A\hat{\eta}_{\bullet,A} + \psi_A\psi_B\hat{T}_{0}^{AB}$ which is coincident with the massless states $(1,1), (\frac{1}{2},2), (0,1)$ given in section 2.

4 Concluding remarks

In this paper, a chiral superfield strength in $\mathcal{N} = 2$ conformal supergravity at linearized level was obtained by acting two superspace derivatives on $\mathcal{N} = 4$ chiral superfield strength. By decomposing $SU(4)_R$ representation of $\mathcal{N} = 4$ twistor superfields into the $SU(2)_R$ representation of $\mathcal{N} = 2$ twistor superfields with an invariant $U(1)_R$ charge, the physical states of $\mathcal{N} = 2$ conformal supergravity can be read off from the surviving $\mathcal{N} = 2$ twistor superfields. These $\mathcal{N} = 2$ twistor superfields are functions of homogeneous coordinates of $\text{WCP}^{3|4}$.

We will describe two relevant subjects, the construction of $\mathcal{N} = 3$ conformal supergravity in twistor-string theory and a mirror symmetry of Calabi-Yau super manifold $\text{WCP}^{3|4}$, in the remaining section.

Let us first make some comments on $\mathcal{N} = 3$ conformal supergravity. One can apply our analysis given in previous sections to $\mathcal{N} = 3$ conformal supergravity. The $\mathcal{N} = 3$ chiral superfield strength $[6,7,8]$ can be constructed from $\mathcal{N} = 4$ one similarly and it is given by

$$\mathcal{W}^{ABC}_{a}(x,\theta) = \epsilon^{ABC}\mathcal{W}_{a}(x,\theta) = \epsilon^{ABCD}D_{aD}\mathcal{W}^{N=4}(x,\theta)|_{\theta^4=0}$$

explicitly. Then we expect to have the following $\theta$-expansion: $\mathcal{W}^{123}_{a}(x,\theta) = \Lambda_{4a} + \cdots + (\theta^6)^{\rho\sigma\tau}\partial^{\rho}\partial^{\sigma}\partial^{\tau}\Lambda^{4a}$. The $\mathcal{N} = 3$ Weyl multiplet is a Lorentz spinor with $\Lambda_{4a}$ as its lowest $\theta$ component $\Lambda_{4a}$. This can be done by computing the above relation, as we have done for $\mathcal{N} = 2$ conformal supergravity. The field contents have extra spinor field $\Lambda_{\lambda A}$ and complex
scalar $E_{(AB)}$ as well as $N = 2$ field contents with different multiplicities we have discussed so far. The constraint equation for $N = 3$ conformal supergravity can be read off from the $N = 4$ relation (3.2). That is,

$$
\epsilon^{ABC} D_{Da} D_E^b D_A^a \mathcal{W}_b = \epsilon^{abcdef} \partial_b^c \partial_a^e \mathcal{W}^f.
$$

Since the square of the Weyl multiplet, contracted over all indices, is a chiral scalar multiplet with Weyl weight $w = 1$. The highest component, a Lorentz scalar has Weyl weight $w = 4$. The action at the linearized level is given by

$$
S = \int d^4x \int d^8 \theta \mathcal{W}^2, \quad \mathcal{W}^2 = \epsilon^{ABC} \epsilon^{DEF} \mathcal{W}_{\mu ABC} \mathcal{W}_{\mu DEF}.
$$

By computing the $\theta$-integrals, the bosonic action is obtained. The extra piece consists of $\Lambda_a \partial^a \partial a \bar{\Lambda}_a$ and $F a \partial^a \bar{F}^A$. Then the physical spectrum of conformal supergravity for these extra piece can be obtained by solving the equations of motion. As in $N = 4$ conformal supergravity, a spinor field $\Lambda_a$ has a helicity $-1/2, -1/2, 1/2$ while complex scalar field $E_A$ has zero helicity.

Using $SU(3)_R$-invariant epsilon tensor $\epsilon^{ABC}$, one can write $N = 3$ conformal supergravity field contents [8] from the field contents of $N = 4$ conformal supergravity as follows:

$$
\epsilon^{ABCD} \Lambda^D = \epsilon^{ABC} \Lambda, \quad \epsilon^{ABCD} E_{DE} = \epsilon^{ABC} E_E, \quad \epsilon^{ABCD} T_{(ab)A}, \quad \epsilon^{ABD} \xi_{aCD},
\eta^a_{\mu}, \quad V^B_{\mu A}, \quad A_\mu, \quad \epsilon^{ABC} \epsilon^{DEF} a_C, \quad \epsilon^a_{\mu}, \quad \bar{\eta}^b_{\mu A}, \quad \epsilon_{ABD} \xi^{CDE},
\epsilon_{ABC} \bar{T}^C_{(ab)}, \quad \epsilon_{ABCD} \bar{E}^{DE} = \epsilon_{ABC} \bar{E}^D, \quad \epsilon_{ABCD} \bar{\Lambda}^D = \epsilon_{ABC} \bar{\Lambda}
$$

where the $N = 4$ fields $\Lambda^{ABC} \equiv \epsilon^{ABCD} \Lambda^D$ and $E^{ABC} \equiv \epsilon^{ABCD} E_{DE}$ are written as in dual form which will make the truncation of $N = 4$ conformal supergravity more clear. Under the breakings of $SU(4)_R$ representations of $N = 4$ theory into $SU(3)_R \times U(1)_R$,

$$
\mathbf{4} \to \mathbf{3}_{-1} + \mathbf{1}_3, \quad \mathbf{10} \to \mathbf{1}_6 + \mathbf{3}_2 + \mathbf{6}_{-2}, \quad \mathbf{6} \to \mathbf{3}_{-2} + \mathbf{3}_2
$$

the above $N = 3$ conformal supergravity $\Lambda, E, T_{(ab)A}$ describe $1_3, \mathbf{3}_2, \mathbf{3}_2$ by realizing the correct $U(1)_R$ charge respectively. A zero total number of on-shell degree of freedom can

13Similarly, for the conjugated representations $\bar{\Lambda}, \bar{E}^D, \bar{T}^C_{(ab)}$, one can identify $1_{-3}, \mathbf{3}_{-2}, \mathbf{3}_{-2}$ with them respectively. For the breaking patterns $20 \to \mathbf{3}_1 + \mathbf{3}_2 + \mathbf{6}_{-2} + \mathbf{8}_0$, $4 \to \mathbf{3}_1 + \mathbf{1}_3$, $15 \to \mathbf{1}_6 + \mathbf{3}_4 + \mathbf{3}_{-4} + \mathbf{8}_0$, the above conformal supergravity fields $\xi_{aCD}, \eta^a_{\mu A}, V^B_{\mu A}, A_\mu$ correspond to $\mathbf{3}_1 + \mathbf{6}_1, \mathbf{3}_1, \mathbf{8}_0, \mathbf{1}_0$ respectively. For $\bar{T}^C_{(ab)}$, one identifies them with $\mathbf{3}_{-1}$ and $\mathbf{3}_{-1} + \mathbf{6}_{-1}$ respectively. Finally, under the breaking pattern $20' \to \mathbf{6}_{-4} + \mathbf{8}_0$, the field $d_F^E$ describes a representation $\mathbf{8}_0$. 

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be checked \(^{14}\) by multiplying each degree of freedom by the number of each type of the field in the spectrum \(\nu_{\text{total}} = +48 - 48 = 0\).

The structure of the off-shell multiplets of \(\mathcal{N} = 3\) conformal supergravity is in correspondence with the structure of the on-shell massive supermultiplet of extended supersymmetry \([7]\). This can be done by establishing a correspondence between the states of each particular spin. The massive supermultiplets are described by an antisymmetric tensor representation of \(Sp(6)\) and by decomposing these representations into the \(SU(3)_R\) subgroup, the number of states for given spins are determined. For example, the number of spin 1 states can be described by \(15 \rightarrow 1 + 8 + 3 + \overline{3}\) under \(Sp(6) \rightarrow SU(3)_R\). This is in agreement with the spectrum of conformal supergravity: 1 \(U(1)\) and 8 \(SU(3)_R\) vectors and 3 antiself-dual and 3 self-dual antisymmetric tensors.

For \(\mathcal{N} = 3\) conformal supergravity, at least there should be present three fermionic coordinates of weight 1 which will play a role for three fermionic coordinates in \(\mathcal{N} = 3\) superspace. Then the only possible Calabi-Yau supermanifold for four dimensional fermionic submanifold is nothing but \(\mathbb{CP}^{3|4}\). The action of \(\mathcal{N} = 3\) superconformal algebra \(SU(2,2|3)\) on the homogeneous coordinates \((\lambda^a, \mu^a, \psi, \chi, \alpha)\) of \(\mathbb{CP}^{3|4}\) can be generated by \(7 \times 7\) supertraceless matrices where the trace of \(4 \times 4\) upper-left corner is equal to the trace of \(3 \times 3\) lower-right corner. By checking the breaking patterns \(SU(4)_R \rightarrow SU(3)_R \times U(1)_R\), and reading off the correct \(U(1)_R\) charge, one gets the full spectrum of massless fields. Similar but different symmetry breaking pattern \(SU(4)_R \rightarrow SU(3) \times U(1)_R\) preserving only \(\mathcal{N} = 1\) supersymmetry appears in the context of marginal deformations of \(\mathcal{N} = 4\) super Yang-Mills theory from open/closed twistor-string theory \([17]\).

The helicity states by the \(\mathcal{N} = 3\) twistor fields \(f^I(Z)\) can be summarized as follows:

\[
\begin{align*}
\lambda^a f_a & : (2, 1), \left(\frac{3}{2}, \overline{3}\right), (1, 3), \left(\frac{1}{2}, 1\right), \\
\mu^\dot{a} f_\dot{a} & : (2, 1), \left(\frac{3}{2}, \overline{3}\right), (1, 3), \left(\frac{1}{2}, 1\right), \\
f^{A=1,2,3} & : \left(\frac{3}{2}, 3\right), (1, 8 \oplus 1), \left(\frac{1}{2}, \overline{3} + 6\right), (0, 3), \\
\partial_\beta f^{A=4} & : (1, 1), \left(\frac{1}{2}, \overline{3}\right), (0, 3), \left(-\frac{1}{2}, 1\right).
\end{align*}
\]

In the last column elements which do not exist in \(\mathcal{N} = 2\) conformal supergravity we have discussed so far, one can see the presence of twistor spinor fields \(\Lambda^{A=4}_I, \overline{\Lambda}^{A=4}_I\), three complex

\(^{14}\)For the extra fields, it is known that the number of on-shell degrees of freedom is given by \(\nu(E_E) = \nu(E_D) = 1\) and \(\nu(\Lambda) = \nu(\overline{\Lambda}) = -3\) \([8]\) which can be seen also from the helicity counting above.

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scalars \( \mathcal{E}_0^{(A,B=4)} \), and other kind of spinor field \( \tilde{\Lambda}^{A=4} \) for the appropriate helicity and \( SU(3)_R \) representation. Each row has bosonic degree of freedom 4 and fermionic degree of freedom \(-4\). There exists six columns. Therefore, there are 24 bosonic and \(-24\) bosonic degrees of freedom. It is rather straightforward to determine twistor fields from \( \mathcal{N} = 4 \) ones explicitly 15.

Similarly, the helicity states by the \( \mathcal{N} = 3 \) twistor fields \( g_A(Z) \) can be summarized by

\[
\begin{align*}
\partial_\beta \partial_ag^a & : (-\frac{1}{2},1), \ (-1,\bar{3}), \ (-\frac{3}{2},3), \ (-2,1), \\
\partial_\beta \partial_ag^a & : (-\frac{1}{2},1), \ (-1,\bar{3}), \ (-\frac{3}{2},3), \ (-2,1), \\
\partial_3 g_{A=1,2,3} & : (0,\bar{3}), \ (-\frac{1}{2},3+\bar{3}), \ (-1,8 \oplus 1), \ (-\frac{3}{2},\bar{3}), \\
g_{A=4} & : (\frac{1}{2},1), \ (0,\bar{3}), \ (-\frac{1}{2},3), \ (-1,1).
\end{align*}
\]

It is evident from the corresponding (3.13) with a single \( D_{4a} \) that by chain rule one can calculate the objects acting on \( g_b(Z), g_b(Z), \, \text{and} \, \theta^4_A g_A(Z) \). When we compute the last one, there is no contribution from \( \theta^4_A D_{4a} g_{A=4}(Z) \) because at the final expression we put \( \theta^4 = 0 \). This is the reason why there is a super derivative acting on \( g_{A=1,2,3}(Z) \) except \( g_{A=4}(Z) \) above helicity states. In this case, the twistor spinor fields \( \hat{\Lambda}^{A=4}_{-\frac{1}{2},A=4}, \hat{\Lambda}'^{A=4}_{-\frac{1}{2},A=4} \), three complex scalars \( \hat{E}^{(0,A,B=4)} \), and other kind of spinor field \( \hat{\Lambda}^{A=4}_{\frac{1}{2},A=4} \) for the appropriate helicity and \( SU(3)_R \) representation appear in the first columns.

There are also 24 bosonic and \(-24\) fermionic degrees of freedom. A zero total number of on-shell(dynamical) degree of freedom is given by 48 bosonic and \(-48\) fermionic degrees of freedom, by counting each bosonic and fermionic degree of freedom in the full spectrum.

We will consider the second subject. The weighted projective superspace \( \text{WCP}^{3,4} \) has an extension of a linear sigma model description in terms of four bosonic homogeneous coordinates of weight 1 and four fermionic homogeneous coordinates of weights 1, 1, \(-1\), and 3, respectively. The geometry of the linear sigma model with a given complexified Kahler class

\footnote{For example, let us compute \( \lambda^a f_a(Z) \big|_{\theta^4 = 0} \). By putting the condition of \( \theta^4 = 0 \) into the \( \mathcal{N} = 4 \) expression \( \lambda^a f_a(Z) \), one gets \( \hat{\epsilon}_2 + \psi^A \tilde{\eta}^A_{+,-} + \psi^A \psi^B \epsilon_{ABC} \tilde{T}_{1}^{(C,A)} + \psi^A \psi^B \psi^C \epsilon_{ABC} \tilde{T}_{1}^{(D=4)} \) where \( \psi^A \) transforms as \( 3 \) under the \( SU(3)_R \). That is, \( \psi^1 = \psi^3 \), \( \psi^2 = \chi \), \( \psi^3 = \alpha \). Let us concentrate on the first term. One can see this from Fourier-like transform. The nonzero contribution of fermionic integral will arise from the exponential factor. Then the dependence on \( \tilde{\psi}^A \) of dual field \( \tilde{g}^a(Z) \) has a term like \( (\psi^A) \tilde{\epsilon}_2 \) because the integrals over \( \psi^A \) in Fourier-like transform pick up \( (\psi^A) \) term. What about the top component? That comes from the component field with a term like \( (\tilde{\psi}^A) \) in dual field \( \tilde{g}^a(Z) \) because the fermionic integral gives a nonzero result for a quartic term in \( \psi^A \). Three of them arises from \( f_a(Z) \) above and one of them from the exponential factor.}
The parameter can be analyzed by solving the D-term constraint. The super Landau Ginzburg B-model mirror of this Calabi-Yau supermanifold is given by the path integral for the holomorphic sector

$$\int \prod_{I=1}^{4} dY_{I} \prod_{J=1}^{4} dX_{J} d\eta_{J} d\chi_{J} \delta \left( \sum_{I=1}^{4} Y_{I} - X_{1} - X_{2} + X_{3} - 3X_{4} - t \right) \times \exp \left[ \sum_{I=1}^{4} e^{-Y_{I}} + \sum_{J=1}^{4} e^{-X_{J}} \left( 1 + \eta_{J} \chi_{J} \right) \right].$$

The super Landau Ginzburg model has 7 bosonic (1 degree of freedom is removed by delta function constraint) and 8 fermionic degrees of freedom. The mirror manifold has the same super dimension \(7 - 8 = -1\) as the original one \(3 - 4 = -1\) by mirror symmetry.

With the same spirit of [18, 19], it would be interesting to take A-model on \(\text{WCP}^{3|4}\) which is mapped to B-model on the same space by S-duality and study the B-model mirror of the topological A-model on \(\text{WCP}^{3|4}\). It would be also interesting to see where they lead to some hypersurface equation satisfied by mirror Calabi-Yau supermanifold and its mirror geometry.

Recently, the topological B-model on fattened complex manifolds [12], which are an extensions of ordinary manifolds with additional dimensions by even nilpotent coordinates, has been studied. It would be interesting to see this kind of exotic supermanifolds corresponding to \(\text{WCP}^{3|4}\). A map between a Kahler spacetime foam in conformal supergravity and twistor space carrying D1 brane charge was found [20]. It is also interesting to see how this mapping arises in our construction. As an open subset of the weighted projective space, the so-called enhanced super twistor space [21] has been found in the construction of extended self-dual super Yang-Mills hierarchies. It is also interesting to see how they appear in \(\mathcal{N} \leq 4\) conformal supergravity sector.

Acknowledgments

I would like to thank N. Berkovits, A.D. Popov, P. Svrcek and E. Witten for discussions. This research was supported by a grant in aid from the Monell Foundation through Institute for Advanced Study, by SBS Foundation, and by Korea Research Foundation Grant (KRF-2002-015-CS0006).
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