Near Horizon Supergravity Superspace†

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Abstract
We present a construction of the superspace of maximally supersymmetric $adS_{p+2} \times S^{d-p-2}$ near-horizon geometry based entirely on the supergravity constraints of which the bosonic space is a solution. Besides the geometric superfields, i.e. the vielbeine and the spinconnection, we also derive the isometries of the superspace together with the compensating tangent space transformations to all orders in $\theta$.

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1 Introduction and motivation

Brane solutions to supergravity theories have been considered \([1]\) as interpolating solutions between two maximally supersymmetric vacua, i.e. flat space related to the brane at infinity and \(adS_{p+2} \times S^{d-p-2}\), together with non-trivial forms, which is the near-horizon geometry of the brane solution. Both solutions can be understood as superspaces. The algebraic fact that the (super)isometries of the \(adS_{p+2} \times S^{d-p-2}\) (super)space form an extended (super)conformal group in \(p+1\) dimensions is the basic argument in favor of the \(adS/(S)\text{CFT}\) conjecture of Maldacena \([2]\), which was originally proposed in connection to branes. The conjecture relates superstring theory/supergravity on \(adS_{p+2} \times S^{d-p-2}\) to a superconformal field theory which lives on the \(p+1\) dimensional boundary of the \(adS\) space. The most elaborated example is certainly Type IIB string theory on \(adS_5 \times S^5\) related to 4-dimensional \(N=4\) super Yang-Mills theory.

One way to realise the superconformal field theory in \(p+1\) dimensions is to consider a super \(p\)-brane probe in the near-horizon \(adS_{p+2} \times S^{d-p-2}\) background. The actions of such probes consist of a (modified) ‘Dirac’-type term (for \(D\)-\(p\)-branes this is the Born-Infeld action) and a Wess-Zumino term, both defined in terms of the pull-backs to the world volume of the geometric superfields of the background, i.e. the supervielbeine and super \(p+2\)-form. By construction these actions are invariant under the rigid superisometries of the background and under local symmetries, i.e. diffeomorphisms of the world volume and \(\kappa\)-symmetry. By fixing the local symmetries of super \(p\)-brane probe actions in their \(\kappa\) near-horizon backgrounds, the complete set of isometries of the background are realized non-linearly on the world volume of the brane. The bosonic part, which leads to a conformal field theory on the brane was carried out in \([3]\).

To construct the superconformal theory on the brane, however, one needs to know the complete supergeometry of the background. Therefore, we have to construct the geometric superfields to \(all\; orders\) in anticommuting superspace coordinates \(\theta\), because these higher order \(\theta\) terms will constitute the interactions on the world volume, i.e. in the superconformal field theory. The general form of the super \(p\)-brane actions in terms of the geometric superfields have been constructed but the proof of invariances of the action only made use of the constraints that these superfields satisfy. Although the complete geometric superfields were known for flat superspace they have only recently been constructed to all orders in \(\theta\) for the near-horizon superspace \(adS_{p+2} \times S^{d-p-2}\) \([4, 5, 6]\). These constructions were based on a supercoset \(G/H\) construction where \(G\) is the relevant extended superconformal group and the stability group \(H\) is the \(\text{product}\) group \(SO(p+1,1) \times SO(d-p-2)\). Also the superisometries have been obtained in a closed form to all orders in \(\theta\) in \([7]\) and therefore in principle we can obtain the full superconformal symmetry on the brane by gauge fixing the local diffeomorphisms and \(\kappa\)-symmetry.

The complete geometric data of the background can also be constructed entirely in supergravity superspace, for a certain class of backgrounds. This class is characterised by the fact that all covariant geometric superfields become covariantly constant. Essentially this restricts us to the above mentioned vacua, which have been shown to be exact in the framework of supergravity \([8]\). As expected the supergravity superspace construction, i.e. the solution of the supertorsion and curvature constraints to all orders in \(\theta\) yields a completely equivalent description of the supergeometry as the coset construction, which has been shown in \([9]\) for the 11-dimensional vacua. The translation between the two approaches by interpreting the supergravity constraints as Maurer-Cartan equations for the supergroup \(G\) in \([10]\).

The supergravity superspace description of the background could be required in some cases. E.g. the supercoset \(G/H\) construction of the \(adS_{p+2} \times S^{d-p-2}\) superspace yields the metric as a product metric. Sometimes it could be nice to have the metric in Cartesian coordinates, in which the metric is manifestly invariant with respect to the directions along the brane and to those transverse to the brane. Especially the \(R\)-symmetry of the extended superconformal group is manifest in these coordinates \([9]\). Interpreting the supergravity
constraints as Maurer Cartan equations in these coordinates yield a soft supergroup \( G \) with structure functions, which are covariantly constant. Fortunately the construction in \([7]\) does not require the structure functions to be constant but only covariantly constant.

Here we will extend the development in \([7]\) and construct the geometric superfields and the isometries only relying on supergravity constraints and not referring to any coset construction.

## 2 Supergravity in Superspace

We will derive the superspace with coordinates

\[
Z^\Lambda = \{ x^\mu, \theta^\dot{\alpha} \}.
\]  

Supergravity in superspace is described in terms of geometric superfields, i.e. the supervielbeine and the spinconnection

\[
E^\Lambda = dZ^\Lambda E^{\Lambda}(Z), \quad \Omega^\Sigma = dZ^\Lambda \Omega^{\Lambda;\Sigma}. \tag{2}
\]

We consider Lorentz superspaces where the spinconnection is determined entirely in terms of \( \Omega^{ab} \) \([10]\).

From these geometric superfields we can derive the torsion and the curvature

\[
T^{\Lambda} = dE^{\Lambda} - E^{\Sigma} \Omega_{\Sigma}^{\Lambda}, \quad R^{\Sigma}_{\Lambda} = d\Omega^{\Sigma}_{\Lambda} - \Omega^{\Sigma}_{\Lambda;\Pi} \Omega^{\Pi}_{\Pi}. \tag{3}
\]

These covariant superfields satisfy the Bianchi-identities

\[
D^\Lambda = -E^{\Sigma} R^{\Sigma}_{\Lambda}, \quad D R^{\Sigma}_{\Lambda} = 0. \tag{4}
\]

The local symmetries of the supergravity theory in superspace are given by

- **Superdiffeomorphisms** (with parameters \( \Xi^\Lambda \))
  \[
  \delta Z^\Lambda = -\Xi^\Lambda(Z), \quad \delta \Phi_n = L_{\Xi} \Phi_n, \tag{5}
  \]
  where \( L_{\Xi} \) is the super Lie-derivative along the super vectorfield \( \Xi \equiv \Xi^\Lambda \frac{\partial}{\partial x^\Lambda} \) and \( \Phi_n \) a generic \( n \)-form.

- **Tangent-space rotations** (with parameter \( L^{ab} \))
  \[
  \delta E^a = E^b L_b^a(Z), \quad \delta E^\alpha = -\left( \frac{1}{4} L^{ab}(Z) \Gamma_{ab} E^\alpha \right), \\
  \delta \Omega^{ab} = dL^{ab} - L^c_{\ a} \Omega^b_{\ c} + \Omega^a_{\ c} L^{cb}. \tag{6}
  \]

The geometric superfields and parameters are polynomials in \( \theta \) with \( x \)-dependent coefficients. To obtain the fieldcontent and symmetries of the supergravity theory, it is clear that not all these fields can be independent. Therefore one imposes covariant constraints on the supertorsion and curvature to reduce the fieldcontent, consistent with the Bianchi-identities \([7]\). For all relevant theories the constraints are known. The 11-dimensional constraints e.g. have been derived in \([10]\) and the 10-dimensional ones can be found in \([11]\). They yield on-shell supergravity in both cases. Besides the vielbeine and spinconnection there are also a number of forms present which we ignore for the moment.

\(^1\)The indices are split into curved ones \( \Lambda = \{ \mu, \dot{\alpha} \} \), with \( \mu \) bosonic and \( \dot{\alpha} \) fermionic. The flat indices are denoted as \( \Lambda = \{ a, \alpha \} \).
3 The supergeometry and isometries near the horizon

In this section we will solve the supergravity constraints and the super Killing equations to all orders in \( \theta \). We will restrict to superspaces with vanishing gravitino. Then the constraints read in general

\[
T^a = \frac{1}{2} E^\beta E^\alpha T_{a\beta}^\alpha, \\
T^\alpha = E^\beta E^\alpha T_{a\beta}^\alpha, \\
R_{ab} = \frac{1}{2} E^\beta E^\gamma R_{cd} \dot{a}^\beta \dot{b} + \frac{1}{2} E^\beta E^\alpha R_{a\alpha b}^\beta,
\]

where \( R_{ab}^{cd} \) is the curvature of the bosonic space.

To obtain a complete superspace description we have to solve these equations. In general this can be done order by order in \( \theta \), which seems very tedious. However we can apply the following trick. Consider the transformation

\[
Z^A = \{x^\mu, \theta^a\} \to Z_t^A = \{x^\mu, t \theta^a\}.
\]

We denote superfields as functions of rescaled \( \theta^a \)'s with a subscript \( t \). Taking the derivative with respect to \( t \) of \( E_t \) and \( \Omega_t \) leads to the coupled first-order equations in \( t \),

\[
\partial_t E^\alpha_t = (d\theta^a E_{t;\alpha \hat{\lambda}} - \dot{\theta}^a E_{t;\alpha \hat{\lambda}} + \dot{\theta}^a E_{t;\alpha \hat{\lambda}}^\Sigma T_{t;\Sigma \hat{\lambda}} + E_{t;\alpha \hat{\lambda}}^\Sigma T_{t;\Sigma \hat{\lambda}}), \\
\partial_t \Omega_t^{ab} = (d\theta^a \Omega_{t;\alpha \hat{b}} - \dot{\theta}^a \Omega_{t;\alpha \hat{b}} + \dot{\theta}^a \Omega_{t;\alpha \hat{b}}^\Sigma R_{t;\Sigma \hat{b}} + \Omega_{t;\alpha \hat{b}}^\Sigma R_{t;\Sigma \hat{b}} + \Omega_{t;\alpha \hat{b}} + \delta \Omega_{t;\alpha \hat{b}}\Omega_{t;\alpha \hat{b}}).
\]

To solve these equations we make the assumptions (gauge choices)

\[
\dot{\theta}^a E_{t;\alpha \hat{\lambda}} = \dot{\theta}^a \Omega_{t;\alpha \hat{\lambda}} = 0, \quad \dot{\theta}^a E_{t;\alpha \hat{b}} = \dot{\theta}^a e_{\alpha \hat{b}}(x) \equiv \Theta^\alpha.
\]

We get using (7),

\[
\partial_t E^\alpha_t = (d\Theta + \frac{1}{4} \Omega_t \cdot \Gamma \Theta)^\alpha - \Theta^\beta E^\alpha T_{a\beta}^\alpha, \\
\partial_t \Omega_t^{ab} = \Theta^\beta E^\alpha R_{a\beta}^{\alpha }.
\]

These equations can be solved order by order in \( t \), by taking multiple derivatives w.r.t. \( t \) and considering the initial conditions

\[
E_{t=0}^a = 0, \quad E_{t=0} = e^a(x), \quad \Omega_{t=0}^{ab} = \omega^{ab}(x),
\]

where \( e^a \) and \( \omega^{ab} \) are the vierbein and spinconnection of the bosonic background. The explicit solution is written in closed form as

\[
E^\alpha_t = \left( \frac{\sinh tM}{M^2} \right)^\alpha \Theta^\beta E^\alpha T_{a\beta}^\alpha, \\
\Omega_t^{ab} = \omega^{ab} - \Theta^\alpha R_{a\beta}^{\alpha ab} \left( \frac{\sinh tM/2}{M^2} \right)^\beta.
\]

where

\[
D\Theta^\alpha = (d\Theta + \frac{1}{4} \omega \cdot \Theta)^\alpha - \Theta^\beta E^\alpha T_{a\beta}^\alpha, \quad (M^2)^\alpha_\beta = -\frac{1}{4}(\Gamma_{ab})^\alpha_\beta \Theta^\gamma \Theta R_{\delta \beta}^{ab} - T_{a\gamma}^\alpha \Theta^\delta R_{\delta \beta}^{ab}.
\]

One can easily checks that (10) is satisfied.

Having derived the geometric superfields \( E^A_t \) and \( \Omega_{t}^{ab} \), we are also interested in the transformations which leave these fields invariant. A specific subset of the symmetries (7)–(10) with rigid parameters will leave the above superfields invariant, this subset are the Killing supervector \( \Xi^A \) and the compensating tangent-space rotation \( L^{ab} \). We first derive the higher
order $\theta$ terms. The super Killing equations are given by (vanishing transformations of the solution)

$$0 = \mathcal{L}_{\Xi} E^\Lambda + E^{\Sigma} L_{\Xi}^\Lambda = D\tilde{\Xi}^\Lambda - \tilde{\Xi}^\Lambda E^{\Sigma} T_{\Sigma\Delta} \tilde{\Lambda} + E^{\Sigma} \tilde{L}_{\Sigma}^\Lambda, \quad (15)$$

$$0 = \mathcal{L}_{\Xi} \Omega^{ab} + dL^{ab} - L^a \Omega^{cb} + \Omega^a_{\cdot c} L^{cb} = D\tilde{\Xi}^\Lambda - \tilde{\Xi}^\Lambda E^{\Lambda} R_{\Delta \Sigma}^{ab}, \quad (16)$$

where

$$\tilde{L}_{\Sigma}^\Lambda = \Xi^\Lambda \Omega_{\Delta \Sigma}^\Lambda + L_{\Sigma}^\Lambda, \quad \tilde{\Xi}^\Lambda = \Xi^\Lambda E_{\Lambda}^\Lambda,$$

$$D\tilde{\Xi}^\Lambda = d\tilde{\Xi}^\Lambda - \Xi^\Lambda \Omega_{\Sigma}^\Lambda, \quad D\tilde{L}^{ab} = d\tilde{L}^{ab} - \tilde{L}^a \Omega^{cb} + \Omega^{a}_{\cdot c} \tilde{L}^{cb}. \quad (17)$$

It turns out that we can solve the equations for $\tilde{\Xi}^\Lambda$ and $\tilde{L}^{ab}$ to all orders in $\theta$ and therefore we derive the Killing supervector $\Xi^\Lambda$ and compensating transformation $L^{ab}$. Now we again rescale $\theta$’s in the parameters and derive them with respect to $t$. Using the constraints (17), the Killing equations (16) and also (10) we obtain again a coupled set of first order equations in $t$,

$$\partial_t \tilde{\Xi}^a_t = -\Theta^b \tilde{\xi}^a T_{\alpha \beta}^b + \frac{1}{4}(\tilde{L} \cdot \Gamma \Theta)^a, \quad \partial_t \tilde{L}^{ab}_t = -\Theta^b \tilde{\xi}^a R_{\alpha \beta}^{ab} \quad (18)$$

This set of equations can be solved in terms of the initial conditions

$$\tilde{\Xi}^{a}_{(t=0)} = \xi^a = \xi^\mu e^a_\mu, \quad \tilde{\Xi}^{a}_{(t=0)} = \tilde{\epsilon}^a = \epsilon^a \mathcal{K}(x)_\alpha^\alpha, \quad \tilde{L}^{ab}_{(t=0)} = \epsilon^{ab} + \xi^\mu \omega^a_{\mu b}, \quad (19)$$

where $\xi^\mu$, $\epsilon^a$ and $\epsilon^{ab}$ are the bosonic killing vector and spinor and the compensating tangent-space rotation. They have been reviewed in various coordinates in [7]. The spinor $\epsilon^a$ is a constant spinor related to the Killing spinor by the $x$-dependent matrix $\mathcal{K}$. The complete solution to the super Killing equations can then be written again in closed form as

$$\tilde{\Xi}^a_t = (\cosh t \mathcal{M} \tilde{\epsilon})^a + \left(\frac{\sinh \mathcal{M} \mathcal{M} \tilde{\epsilon}}{\mathcal{M} \mathcal{M}^2} \mathcal{B} \Theta\right)^a, \quad (20)$$

This completes the derivation of the “covariant” Killing superfields to all orders in $\theta$, which are obtained at $t = 1$. In a special gauge, the Killing spinor gauge [4], one can easily ‘invert’ the covariant Killing superfields to the Killing supervector and compensating tangent space parameter acting on the coordinates and supervielbeine. The Killing spinor gauge consists of choosing the $e(x)_\alpha^a$ in (10) to be $\mathcal{K}(x)_\alpha^a$, defined in (19) by the Killing spinor of the bosonic geometry, hence its name. In this gauge we have the following simplification

$$E^a_{\alpha} = e^a_\mu(x), \quad E^a_{\alpha} = 0, \quad \Omega^{ab}_{\mu} = \omega^{a b}_{\mu}(x), \quad (21)$$

i.e. the bosonic background is not affected by higher order $\theta$ corrections. We derive the following explicit expressions $\Xi^\Lambda$ and $\Xi^\mu$, which give the variations of the superspace coordinates

$$-\delta \theta^a = \Xi^a = [\mathcal{K}^{-1} \mathcal{M} \coth \mathcal{M} \epsilon] \tilde{\epsilon}^a + (\mathcal{K}^{-1} \mathcal{B} \Theta)^a, \quad (22)$$

$$-\delta x^a = -\Xi^a = \epsilon^a - \Theta^{\alpha \beta} T_{\alpha \beta}^a \left[\frac{\tanh \mathcal{M}/2}{\mathcal{M}}\right] \epsilon e^a_\mu \quad (22)$$

This completes the derivation of the covariant Killing superfields to all orders in $\theta$, which are obtained at $t = 1$. In a special gauge, the Killing spinor gauge [4], one can easily ‘invert’ the covariant Killing superfields to the Killing supervector and compensating tangent space parameter acting on the coordinates and supervielbeine. The Killing spinor gauge consists of choosing the $e(x)_\alpha^a$ in (10) to be $\mathcal{K}(x)_\alpha^a$, defined in (19) by the Killing spinor of the bosonic geometry, hence its name. In this gauge we have the following simplification

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$$-\delta x^a = -\Xi^a = \epsilon^a - \Theta^{\alpha \beta} T_{\alpha \beta}^a \left[\frac{\tanh \mathcal{M}/2}{\mathcal{M}}\right] \epsilon e^a_\mu \quad (22)$$
and also the compensating tangent space rotation can also be derived. To have a complete description of the near-horizon superspace, we also have to define the non-trivial superforms of the supergravity theory. The superforms $F$ are given in supergravity see e.g. [10, 11] for the 11-dimensional and 10-dimensional supergravity. They can be written in covariant form i.e.

$$F = E^{A_n} \ldots E^{A_1} F_{A_1 \ldots A_n}. \quad (23)$$

For the near-horizon solutions these covariant components $F_{A_1 \ldots A_n}$ are covariantly constant and do not get higher order $\theta$ corrections [8]. Therefore since we know the bosonic geometry and forms and we have derived $E^A$ to all orders in $\theta$ we know the complete superforms. In principle these forms have to be guessed in the supercoset approach, but they are then uniquely defined by the fact that they have to satisfy the appropriate Bianchi identity. This concludes the derivation of the complete near-horizon superspace, forms and the superisometries in supergravity. Although the supercoset methods used in [4]–[7] give a very nice description, one can also use supergravity superspace only.

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