Vertex operators and character varieties

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Abstract

We prove some combinatorial conjectures extending those proposed in [13, 14]. The proof uses a vertex operator due to Nekrasov, Okounkov, and the first author [4] to obtain a "gluing formula" for the relevant generating series, essentially reducing the computation to the case of \( CP^1 \) with three punctures.

1 Introduction

Let \( g \geq 0 \) be a non-negative integer. For a partition \( \lambda \in \mathcal{P} \) let

\[
\mathcal{H}_\lambda(u; z, w) := \prod_{s \in \lambda} \frac{\prod_{i=1}^g (z^{2a(s)+1} - u_i w^{2l(s)+1})(z^{2a(s)+1} - u_i^{-1} w^{2l(s)+1})}{(z^{2a(s)+2} - w^{2l(s)})(z^{2a(s)} - w^{2l(s)+2})},
\]

where the product is over all cells \( s \) of \( \lambda \) with \( a(s) \) and \( l(s) \) its arm and leg length, respectively and \( z, w, u = (u_1, \ldots, u_g) \) are independent variables. This is a generalization of the hook function of [13] which we recover by setting \( u_i = 1 \) for \( i = 1, \ldots, g \). When \( g = 1 \) there is only one variable \( u_i \), which we will just denote by \( u \).

Given nonnegative integers \( g, k \), let us define

\[
\Omega(u; z, w) := \sum_{\lambda} \mathcal{H}_\lambda(u; z, w) \tilde{H}_\lambda[X_1] \cdots \tilde{H}_\lambda[X_k]
\]

(1.0.1)

where \( |\lambda| \) is the size of the partition \( \lambda \), and \( \tilde{H}_\lambda[X_i] \) is the modified Macdonald polynomial defined below, in an infinite set of variables \( X_i = \{x_{i1}, x_{i2}, \ldots\} \), with the usual parameters specialized to \( (q, t) = (z^2, w^2) \). If \( k = 0 \) we will include in the definition of \( \Omega \) a factor of \( T^{|\lambda|} \), where \( T \) is a formal parameter, to keep track of the degree. Consider the power series

\[
\mathbb{H}(u; z, w) := (z^2 - 1)(1 - w^2) \log \Omega(u; z, w).
\]
where \( \text{Log} \) is the *plethystic logarithm*, defined below. If necessary for clarification we will add the subscript \( g \) or \( g, k \) in this order.

Let us define coefficients

\[
\mathbb{H}(u; z, w) = \sum_{\lambda} \mathbb{H}_\lambda(u; z, w) m_\lambda[X_1] \cdots m_\lambda[X_k]
\]  

(1.0.2)

where the sum is over all \( \lambda = (\lambda^1, \ldots, \lambda^k) \in \mathcal{P}^k \). Note that \( \mathbb{H}_\lambda(u; z, w) = 0 \) unless \( |\lambda^i| = n \) for every \( i \).

For \( k = 0 \) there are no symmetric functions and the polynomials are simply indexed by their degree \( n \); we will denote them by \( \mathbb{H}_{(n)}(u; z, w) \). Note that this notation is consistent, in the sense that the degree \( n \) term for \( k = 0 \) is indeed the coefficient of \( m_{(n)}(X) \) for \( k = 1 \). To see this it is enough to note that if we set \( X = (T, 0, \cdots) \) then \( \tilde{H}_\lambda(X) = T^{[\lambda]} \) for all \( \lambda \) and \( m_{(n)}(X) \) vanishes unless \( \lambda = (n) \) for some \( n \) when it equals \( T^n \).

The specializations \( \mathbb{H}_\lambda(z, w) := \mathbb{H}_\lambda(1, \ldots, 1; z, w) \) are precisely the coefficients of \([14]\) where the following conjecture was put forward.

**Conjecture 1.0.1.** (\([14]\) [Conj. 1.2.1])

1. The rational function \( \mathbb{H}_\lambda(z, w) \) is in fact a polynomial, and \( \mathbb{H}_\lambda(-z, w) \) has nonnegative integer coefficients.

2. Moreover, the mixed Hodge polynomial of the character variety \( \mathcal{M}_\lambda \) of a Riemann surface of genus \( g \) with \( k \) punctures and generic semi-simple conjugacy classes of type \( \lambda \) is given by

\[
H_v(\mathcal{M}_\lambda; q, t) = (t \sqrt{q})^{|\lambda|} \mathbb{H}_\lambda \left( -\frac{1}{\sqrt{q}}, \frac{t}{\sqrt{q}} \right).
\]

In certain special cases in which the mixed Hodge polynomials are known, Conjecture \([1.0.1]\) reduces to a purely combinatorial identity. For instance, in \([13]\) [Conj. 1.1.2], the authors conjectured an explicit formula in the special case of genus one, and no punctures:

\[
\mathbb{H}_1(z, w) = (z - w)^2 \frac{T}{1 - T}.
\]  

(1.0.3)

Notice the positivity after substituting \( z = -z \). Similarly, combining \([1.0.3]\) with Conjecture 4.2 of \([15]\), which relies on known facts on the geometry of the Hilbert scheme of points on \( \mathbb{C}^* \times \mathbb{C}^* \), we expect the following more involved identity to hold:

\[
\mathbb{H}_{1,\psi}(z, w) = \mathbb{H}_{1,\psi}(1; z, w) = (z - w)^2 (z^2 + w^2 - z^2 w^2) \frac{T}{1 - T},
\]  

(1.0.4)

where

\[
\mathbb{H}_{1,\psi}(u; z, w) := (z^2 - 1)(1 - w^2) \log \sum_{\lambda} \mathbb{H}_\lambda(u; z, w) \psi_\lambda(1, w^2) T^{[\lambda]}
\]

and

\[
\psi_\lambda(q, t) := 1 + (1 - q)(1 - t) B_\lambda(q, t), \quad B_\lambda(q, t) := \sum_{(i,j) \in \lambda} q^{j-1} t^{j-1}.
\]
We note that the \( \psi_\lambda(q, t) \)'s are well known to be the eigenvalues of the modified Macdonald operator \( D_0 \) of \([2]\) described below.

In this paper we prove the following more general versions of the identities \((1.0.3)\) and \((1.0.4)\) (see §5 for the proof).

**Theorem 1.0.2.** The following identities for \( g = 1 \) hold

\[
H_1(u; z, w) = (z - u^{-1}w)(z - uw)\frac{T}{1 - T},
\]

\[
H_{1,\lambda}(u; z, w) = (z - uw)(z - u^{-1}w)(z^2 + w^2 - z^2 w^2)\frac{T}{1 - T}.
\]

In particular, the special cases \((1.0.3)\) and \((1.0.4)\), obtained by setting \( u = 1 \), also hold.

**Remark 1.0.3.** The authors have recently learned of an independent proof equation \((1.0.5)\) due to Rains and Warnaar. Also, these type of identities are related to work of Nekrasov and Okounkov \([22]\) and are discussed in the physics literature; see for example \([16]\) and \([1]\)(1.4).

**Corollary 1.0.4.** Conjecture 4.2 of \([15]\) is true.

**Proof.** This is just a reformulation of \((1.0.4)\) in view of \((1.0.3)\). □

The conjecture Conjecture 1.0.1 is purely topological in the sense that the underlying Riemann surface of genus \( g \) enters only via its fundamental group. When incorporating the structure of this surface as a smooth projective algebraic curve \( X \) we need the data coming from the action of Fobenius on its \( l \)-adic cohomology. The variables \( u_i \) above correspond to the eigenvalues of the Frobenius automorphism acting on the first cohomology group of \( X \). More precisely, let \( X \) be a smooth projective curve over \( \mathbb{F}_q \) and let \( \alpha_1, \ldots, \alpha_g, q/\alpha_1, \ldots, q/\alpha_g \) be the eigenvalues of Frobenius on \( H^1_{\text{et}}(X, \mathbb{Q}_l) \). The following extension of Conjecture 1.0.1 was essentially formulated by Mozgovoy \([20]\)(Conj. 3.2) for \( k = 0 \).

**Conjecture 1.0.5.** With the above notation we have

1. The coefficients \( H_\lambda(u; -z, w) \) are polynomials in \( u, z, w \) with non-negative integer coefficients.

2. The number of stable Higgs bundles on \( X \) of rank \( n \) and degree coprime to \( n \) is given by \( q^{2(1+q^{-1})n^2} \prod_{i=1}^g (\alpha_i/\sqrt{q}; 1, \sqrt{q}) \), where \( \alpha := (\alpha_1, \ldots, \alpha_g) \).

We have tested part 1 of this conjecture with the help of MAPLE for the the ranges of values shown in Table 1. The \( g = 0 \) case of course does not involve the new parameters \( u_i \), and so the first entries of this table are simply confirming the original conjectures of \([14]\) for these values. The columns which include all values of \( n \) follow from the Cauchy product formula for modified Macdonald polynomials, and Theorem 1.0.2.

Consider part 1 of Conjecture 1.0.5 for \( k = 0 \) and \( n \leq 2 \). Let \( P \) be the polynomial

\[
P(z, w) := \prod_{i=1}^g (z - u_i w)(z - u_i^{-1} w).
\]
Table 1: Tested ranges for Conjecture 1.0.5

| $g$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
|-----|---|---|---|---|---|---|---|---|---|---|
| $k$ | ≤2 | 3 | ≤7 | 0 | 1 | 2 | 3 | 0 | 1 | 2 |
| $n$ | <∞ | ≤7 | ≤3 | <∞ | ≤6 | ≤5 | ≤4 | ≤6 | ≤5 | ≤4 |

Note that $H_{1}(1) = P$ satisfies the claim.

It is straightforward to verify that

$$
\frac{H_{2}(u; z, w)}{P(z, w)} = \frac{P(z, w^4)}{(z^2 - w^2)(1 - w^4)} - \frac{P(z^2, w)}{(z^2 - w^2)(z^4 - 1)} - \frac{1}{2} \frac{P(z, w)}{(z^2 - 1)(1 - w^2)} - \frac{1}{2} \frac{P(z, -w)}{(z^2 + 1)(1 + w^2)}.
$$

**Proposition 1.0.6.** The rational function $H_{2}(u; z, w)/P(z, w)$ is a polynomial in $z, w$.

**Proof.** Note that $P$ is a generic polynomial of $z, w$ with the property that it is fixed by $(z, w) \mapsto (w, z)$ and $(z, w) \mapsto (-z, -w)$. So to prove that $H_{2}(2)$ is a polynomial it is enough to check that the expression, say $C_{m,n}(z, w)$, on the right hand side of (1.0.8) for $P(z, w) := z^m w^n + z^n w^m$, where $m, n$ run over all pairs of non-negative integers with $m + n$ even, is a polynomial.

To verify this we can form the generating series, say for the case of $m, n$ both even,

$$
C(z, w; x, y) := \sum_{r,s \geq 0} C_{2r,2s}(z, w) x^r y^s.
$$

The result is the rational function

$$
C(z, w; x, y) := \frac{1}{(z^2 - w^2)(1 - w^4)(1 - z^2 y)(1 - w^8)} + \frac{1}{(z^2 - w^2)(1 - w^4)(1 - z^2 y)(1 - w^8 x)} + \frac{1}{(z^2 - w^2)(z^4 - 1)(1 - z^2 x)(1 - w^4 y)} + \frac{1}{(z^2 - w^2)(z^4 - 1)(1 - z^2 x)(1 - w^4 x)} + \frac{1}{2} \frac{1}{(z^2 - 1)(1 - w^2)(1 - z^2 y)(1 - w^2 x)} + \frac{1}{2} \frac{1}{(z^2 - 1)(1 - w^2)(1 - z^2 y)(1 - w^2 x)} + \frac{1}{2} \frac{1}{(z^2 + 1)(1 + w^2)(1 - z^2 x)(1 - w^4)} + \frac{1}{2} \frac{1}{(z^2 + 1)(1 + w^2)(1 - z^2 x)(1 - w^4)}.
$$

Now we can put this expression in a computer and verify that the denominator of $C$ is

$$(1 - z^2 x)(1 - z^2 y)(1 - w^2 x)(1 - w^2 y)(1 - z^6 x)(1 - z^6 y)(1 - w^6 x)(1 - w^6 y)$$

and this implies that the power series coefficients $C_{2r,2s}(z, w)$ are polynomial. A similar argument proves the analogous statement for the case where $m, n$ are both odd.

**Remark 1.0.7.** To verify that the coefficients of $C_{m,n}(z, w)$ are non-negative appears to be quite tricky.
We can now check part 2 of Conjecture 1.0.5 by specializing $H_2(u; z, w)$ at $z = 1$ and comparing the result to the calculation of the number of points over $\mathbb{F}_q$ of the Higgs moduli space by Schiffmann [23]. After some tedious manipulations we find that $H_2(u; 1, w) = Q(w)A(w)$ where $Q(w) := P(1, w)$ and

$$A(w) := \frac{Q(w^3)}{(1 - w^2)(1 - w^4)} + \frac{1}{4} \frac{(w^2 - 3)Q(w)}{(1 - w^2)^2} + \frac{1}{2} \frac{2gQ(w) - w\partial Q/\partial w}{1 - w^2} - \frac{1}{4} \frac{Q(-w)}{1 + w^2}. \quad (1.0.8)$$

**Proposition 1.0.8.** Part 2 of Conjecture 1.0.5 holds for $n \leq 2$.

**Proof.** As mentioned, we compare the specialization $H_2(\alpha/\sqrt{q}; 1, \sqrt{q})$ to the formula obtained by Schiffmann in [23] (specifically the examples given right after Theorem 1.6). This is immediate for $n = 1$ and routine for $n = 2$ using (1.0.8). \[\square\]

Our first result, Theorem 5.0.4, is an inductive formula for the power series $\Omega(u; z, w)$ associated to a Riemann surface obtained by gluing along the punctures of two other surfaces. This reduces the computation of $\Omega(u; z, w)$ to the case of genus zero with three punctures, as often happens in TQFT-type formulas. It would be very desirable to establish the gluing relations at $u = 1$ on the character variety side. As an application we obtain a proof of Theorem (1.0.2).

Our main tool turns out to be a vertex operator discovered by Nekrasov, Okounkov, and the first author in [4], described in Theorem 3.0.1 below. The result states

$$\Gamma(u)\tilde{H}_\lambda = \sum_\mu N_{\lambda,\mu}(u; q, t)\tilde{H}_\mu, \quad (1.0.9)$$

where $\Gamma(u)$ is a composition of a homomorphism with a multiplication map with an explicit description in the power sum basis, and $N_{\lambda,\mu}(u)$ is an explicit product whose diagonal entries satisfy

$$\frac{N_{\lambda,\mu}(uz^{-1}w^{-1}; z^2, w^2)}{N_{\lambda,\mu}(z^{-1}w^{-1}; z^2, w^2)} = \mathcal{H}_{1,\lambda}(u; z, w).$$

This operator has a natural geometric interpretation in terms of the torus-equivariant $K$-theory of $\text{Hilb}_n \mathbb{C}^2$, the Hilbert scheme of points in the complex plane. The connection with modified Macdonald polynomials is due to Haiman’s study of the isomorphism of Bridgeland King and Reid in this case, viewing the Hilbert scheme as a resolution of singularities [3, 9, 10]. This operator extends to $K$-theory some previous work due to Okounkov and the first author, who introduced a family of geometrically defined operators on the cohomology

$$\bigoplus_{n \geq 0} H^*(\text{Hilb}_n S, \mathbb{Q}),$$

for a general smooth quasiprojective surface $S$, and proved an explicit formula in terms of Nakajima’s Heisenberg operators [5, 21]. The original operator of [5] is recovered in the case when the surface $S = \mathbb{C}^2$ by essentially taking the Jack polynomial limit of the Macdonald polynomials. See [4] for details.
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2 Preliminaries on partitions and symmetric functions

First, we recall some notations on partitions and symmetric functions, which can be found in Macdonald’s book [19]. For more on the plethystic operations, we also recommend [2, 11].

2.1 Partitions

A partition \( \mu \) is a nonstrictly decreasing sequence of nonnegative integers eventually stabilizing to zero. Its length \( l(\mu) \) is the number of nonzero terms, and the norm \( |\mu| \) is the sum of the entries. For each \( i \geq 1 \), we let \( m_i(\mu) \) denote the number of times \( i \) appears in \( \mu \). For any partition, we have the corresponding Young diagram consisting of its boxes, namely the ordered pairs \( s = (i, j) \) such that \( 1 \leq j \leq \mu_i \). For any ordered pair \( (i, j) \), we define the arm, leg, and hook length in \( \mu \) by the formula

\[
a_{\mu}(s) = \mu_i - j, \quad l_{\mu}(s) = \mu'_j - i, \quad h_{\mu}(s) = a(s) + l(s) + 1
\]

where \( \mu' \) is the conjugate partition to \( \mu \), i.e. the partition whose columns are the rows of \( \mu \) in the Young diagram. Notice that the arm and leg length make sense even if the ordered pair \( s \) is not a box of \( \mu \), in which case they may take negative values. Another important statistic we will need is

\[
n(\mu) = \sum_i (i-1)\mu_i.
\]

For instance, if \( \mu = [5, 3, 3] \), then the Young diagram would be

```
  +---+---+---+
  |   |   |   |
  +---+---+---+
  |   |   |   |
  +---+---+---+
  |   |   |   |
```

The conjugate partition would be \( \mu' = [3, 3, 3, 1, 1] \), and we would have

\[
l(\mu) = 3, \quad |\mu| = |\mu'| = 9, \quad n(\mu) = 9.
\]

The arm, leg, and hook length of the box \( s = (1, 2) \) are given by

\[
a_{\mu}(s) = 3, \quad l_{\mu}(s) = 2, \quad h_{\mu}(s) = 6.
\]

An \( r \)-core is a partition such that none of its hook numbers are divisible by \( r \). For instance, the partition \( \lambda = (5, 3, 1, 1) \) is a 3-core, because its hook numbers are

\[1, 1, 1, 2, 2, 2, 4, 5, 5, 8].\]
none of which are divisible by 3. We now recall the bijection between $r$-cores and lattice vectors of \([8]\). Given an $r$-core $\lambda = (\lambda_1, \lambda_2, \ldots)$ let $n = (n_0, \ldots, n_{r-1}) \in \mathbb{Z}^r$ be the vector with coordinates

$$ n_k := \left\lfloor \frac{\lambda_k - i}{r} \right\rfloor + 1, \quad i = \min\{\nu | \lambda_\nu - \nu \equiv k \mod r\}. $$

Set $v := rn + \rho$, where $\rho := (-k, \ldots, k) \in \mathbb{Z}^r$ and $r = 2k + 1$. This yields a one-to-one correspondence between $r$-cores and integer vectors $v = (v_{-k}, \ldots, v_k)$, such that

$$ v_i \equiv i \mod r, \quad -k \leq i \leq k, \quad v_{-k} + \cdots + v_k = 0. $$

Under this correspondence

$$ |\lambda| = \frac{1}{2r}(v_{-k}^2 + \cdots + v_k^2) - \frac{r^2 - 1}{24} $$

and if $v'$ is the vector corresponding to the dual partition $\lambda'$ of $\lambda$ we have

$$ v'_\alpha = -v_{-\alpha}, $$

where the indices are read modulo $r$.

For example, if $\lambda = (5, 3, 1, 1)$ is viewed as a 3-core, then $n = (0, 2, -2)$ and hence $v = (-1, 6, -5)$. We check that indeed $10 = |\lambda| = ((-1)^2 + 6^2 + (-5)^2)/6 - (3^2 - 1)/24$.

Remark 2.1.1. The vector $v$ associated to an $r$-core coincides with the $V$-coding of \([12]\) up to re-ordering (his indices are in the range $0 \leq i \leq r - 1$ whereas we are using $-k \leq i \leq k$). The vector $n$ is also the weight vector in \([17]\) \S 4] associated to $\lambda$.

### 2.2 Symmetric functions

Let $\Lambda_R = \Lambda_R[\mathcal{X}]$ denote the ring of symmetric functions in an infinite set of variables $\mathcal{X}$ over a ring $R$, and let $p_\mu, m_\mu, h_\mu, e_\mu, s_\mu \in \Lambda = \Lambda_Q$ denote the power sum, monomial, complete, elementary, and Schur symmetric polynomial bases indexed by $\mu$ respectively. We have the standard inner product on $\Lambda$ described in a few different ways by

$$ (p_\mu, p_\nu) = \delta_{\mu, \nu} z_\mu, \quad (h_\mu, m_\nu) = \delta_{\mu, \nu}, \quad (s_\mu, s_\nu) = \delta_{\mu, \nu}, $$

where

$$ z_\mu = \prod_{i \geq 1} \hat{i}^{m_i} m_i!, \quad m_i := m_i(\mu). $$

A homomorphism of $\Lambda_R$ into another $R$-algebra may be defined by specifying its value on the algebra generators $p_i$ for $i$ a nonnegative integer. A particularly useful example of such a homomorphism is that of plethystic substitution. Let $(R, \lambda)$ be a $\lambda$-ring, and let $A \mapsto A^{(k)}$ denote the corresponding Adams homomorphisms. For instance, if $A$ is an element of a ring of rational functions, polynomials, or Laurent polynomials in some sets of variables, we will let $A^{(k)}$ denote the element obtained from $A$ by substituting $x = x^k$ for each variable $x$. Another example is the ring $\Lambda_R$. In this case, we extend the operations on $R$ by setting
$p_i^{(k)} = p_{ik}$ for the power sum generators, which is compatible with specialization to finitely many variables. For instance, we would have

$$A = \frac{1 - q}{1 - t} \implies A^{(k)} = \frac{1 - q^k}{1 - t^k}.$$  

We use the usual language that

$$\lambda(TA) = \sum_{n \geq 0} \lambda^n(A)(-T)^n \in R[[T]]$$

for some formal variable $T$. Now consider the plethystic exponential

$$\exp(TA) := \lambda(-TA) = \exp \left( \sum_{k \geq 1} \frac{A^{(k)}T^k}{k} \right). \quad (2.2.2)$$

For instance, if $R = \Lambda[X]$, we would have that

$$\exp(TX) = \exp(Tp_1) = \sum_{n \geq 0} T^n h_n \in \Lambda[X][[T]].$$

It has an inverse called the plethystic logarithm defined as follows: if

$$F = 1 + \sum_{n \geq 1} A_n T^n,$$

then we let

$$\log(F) = \sum_{n \geq 1} V_n T^n,$$

where we define coefficients $U_n, V_n$ by

$$V_n = \frac{1}{n} \sum_{d|n} \mu(d) U^{(d)}_{n/d}, \quad \log(F) =: \sum_{n \geq 1} U_n \frac{T^n}{n}.$$  

See [13] for details.

In the case of polynomials or Laurent polynomials, we will let $\lambda(A)$ denote the evaluation at $T = 1$ of the rational function with power series

$$\lambda(TA) = \sum_{i \geq 0} (-T)^i \lambda^i(A) \in R[[T]],$$

provided that it exists. A simple example is

$$\lambda(q - 2qt^{-2}) = \frac{1 - q}{(1 - qt^{-2})^2}.$$  

In the case of symmetric functions, if $f \in \Lambda$ has no constant term, we may also refer to its exponential $\exp(f)$, leaving off the formal variable $T$. In this case, we have not defined an honest element of $\Lambda$, but it does make sense to consider a pairing such as $(\exp(f), g)$. 

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because only finitely many terms in the sum (2.2.2) contribute. For that reason, it only really
defines an element of the dual space.

For any element \( A \in R \), we have a homomorphism
\[
\Lambda \to R, \quad p_k \mapsto A^{(k)}.
\]
We denote the image of \( f \in \Lambda \) by \( f[A] \). The reason for this notation is that
\[
f[x_1 + \cdots + x_n] = f(x_1, \ldots, x_n),
\]
the evaluation of \( f \) on some finite set of variables \( x_i \). Under this map, we have \( e_i[A] = \lambda_i(A) \). It is called a plethystic homomorphism because if and \( f, g \in \Lambda \) are the characters of representations of the general linear group corresponding to some polynomial functors \( F, G \) of vector spaces, then \( f[g] \) is the character of the composition \( F \circ G \).

A second homomorphism is the following operator, which will play a central role in this
paper: if \( A \in R \), then
\[
\Gamma_m^+(A) : \Lambda_R \to \Lambda_R, \quad f \mapsto f[X + mA],
\]
where \( X \) in place of \( p_1 = p_1[X] \). The variable \( m \) may either be taken as a number, or may be viewed as a symbol that is not affected by the plethystic operation, i.e. \( m^{(k)} = m \).

It is not hard to verify that
\[
\Gamma_m^+(A) = \exp \left( m \sum_{i=1} A^{(i)} \partial_i \right),
\]
where \( \partial_k \) is the operator differentiation by \( p_k \) on \( \Lambda_R = R[p_1, p_2, \ldots] \). Combining equations (2.2.4) and (2.2.2), we can see that its dual operator under the standard inner product is given by a multiplication operator:
\[
(\Gamma_m^+(A)f, g) = (f, \Gamma_m^+(A)g), \quad \Gamma_m^-(A)f = \text{Exp}(mA) f.
\]
We will also write
\[
\Gamma_{\pm}(A) = \Gamma_{\pm}^1(A), \quad \Gamma_{\pm} = \Gamma_{\pm}(1)
\]
so that \( \Gamma_m^+(A) = \Gamma_{\pm}(A)^m \) for integers \( m \). For this paper, an expression of the form \( \Gamma_{\pm}(A)\Gamma_{\pm}(B) \)
will be called a vertex operator. It follows easily from the definitions that
\[
\Gamma_{\pm}(A)\Gamma_{\pm}(B) = \text{Exp}(AB)\Gamma_{\pm}(B)\Gamma_{\pm}(A)
\]
provided both sides are convergent as power series. A useful example is
\[
\Gamma_{\pm}(x^{-1})\Gamma_{\pm}(y) = (1 - yx^{-1})^{-1}\Gamma_{\pm}(y)\Gamma_{\pm}(x^{-1}) \in \text{End}(\Lambda[[x^{\pm1}]][[y]]).
\]

A third plethystic homomorphism is
\[
\Upsilon_A f = f[AX].
\]

For instance, we have
\[
x^d = \Upsilon_{x^d} : p_k \mapsto x^d p_k,
\]
where \( d \) is the operator that multiplies a homogeneous polynomial \( f \) of degree \( k \) by \( k \). A second common example is the usual automorphism

\[
\omega = (-1)^d \Gamma_{-1} : p_k \mapsto (-1)^{k-1} p_k
\]

It follows from definitions that we have the commutation relations

\[
\Gamma_+(B) \Upsilon_A = \Upsilon_A \Gamma_+(AB), \quad \Upsilon_A \Gamma_-(B) = \Gamma_-(AB) \Upsilon_A, \quad \Upsilon_A \Upsilon_B = \Upsilon_{AB}. \tag{2.2.9}
\]

Let \( \Lambda_{q,t} = \Lambda_{Q(q,t)} \), with the Macdonald inner product

\[
(f, g)_{q,t} = (f, \Upsilon_A g), \quad A = \frac{1 - q}{1 - t} \tag{2.2.10}
\]

or in the traditional notation,

\[
(p_{\mu}, p_{\nu})_{q,t} = \delta_{\mu,\nu} z_{\mu} \prod_i \frac{1 - q^i}{1 - t^i}. \tag{2.2.11}
\]

Let \( P_{\mu} = P_{\mu}[X; q, t] \in \Lambda_{q,t} \) denote the usual Macdonald polynomials, and let \( J_{\mu} \) be the integral form defined by

\[
J_{\mu}[X; q, t] = \prod_{s \in \mu} (1 - q^{\mu(s)})(1 - t^{\mu(s)}) P_{\mu}[X; q, t], \tag{2.2.12}
\]

whose expansion in the monomial basis has coefficients which are integer valued polynomials in \( q, t \).

The modified Macdonald inner product \([2]\), is a variation of (2.2.11) which is symmetric in \( q, t \) defined by

\[
(f, g)_* = (f, \Upsilon_{-M} g) = (\Upsilon_{1-t} f, \Upsilon_{1-t} g)_{q,t}, \quad M = (1 - q)(1 - t), \tag{2.2.13}
\]

or, in the power sum basis by

\[
(p_{\mu}, p_{\nu})_* = \delta_{\mu,\nu} (-1)^{\mu(*)} z_{\mu} \prod_i (1 - q^{\mu_i})(1 - t^{\mu_i}). \tag{2.2.14}
\]

This differs from the usual notation \([2]\) by a sign of \((-1)^d\), but is more natural for the purposes of our paper. The modified Macdonald polynomials are defined by

\[
\tilde{H}_{\mu} = \Gamma^{-n(\mu)} \Gamma_{(1-t)^{-1}} J_{\mu}[X; q, t^{-1}] \tag{2.2.15}
\]

and it follows easily that they are orthogonal with respect to the inner product (2.2.14). Their norms are given by

\[
(\tilde{H}_{\mu}, \tilde{H}_{\nu})_* = \delta_{\mu,\nu} (-1)^{\mu(*)} \prod_{s \in \mu} (q^{\mu(s)} - t^{\mu(s)+1})(t^{\mu(s)} - q^{\mu(s)+1}). \tag{2.2.16}
\]

Unlike the usual Macdonald polynomials, these polynomials have the following symmetry property in \( q, t \):

\[
\tilde{H}_{\mu} = \tilde{H}_{\mu^*}\bigg|_{q=t, t=q}
\]
As explained in [2], they are eigenfunctions of the modified Macdonald operator, i.e.

\[ D_0 \tilde{H}_\mu = \psi_{\mu}(q, t) \tilde{H}_\mu, \quad (2.2.17) \]

where

\[ \psi_{\mu}(q, t) = 1 + MB_{\mu}(q, t), \quad B_{\mu}(q, t) = \sum_{(i, j) \in \mu} q^{-1} t^{-1}, \]

and

\[ \sum_{x \in \mathbb{Z}} x^k D_k := \Gamma(-x) \Gamma(M x^{-1}) \in \text{End}(\Lambda_{q, t})[[x^\pm 1]]. \]

The modified Macdonald polynomials arose in Garsia, Haiman, and Procesi’s work on the \( n! \) conjecture [7, 9], in which they arise as the Frobenius character of the Garsia-Haiman module. In the course of the proof, they were given the following geometric interpretation:

let \( \text{Hilb}_n \mathbb{C}^2 \) denote the Hilbert scheme of points on \( \mathbb{C}^2 \), which is a smooth complex algebraic variety parametrizing ideals \( I \in \mathbb{C}[x, y] \) such that \( \dim_\mathbb{C}(\mathbb{C}[x, y]/I) = n \). There is an action of a two-dimensional torus \( T \) on this space, by pulling back ideals from the action \( T \rightleftharpoons \mathbb{C}^2, \quad (q, t) \cdot (x, y) = (q^{-1} x, t^{-1} y) \).

The fixed points correspond to Young diagrams of norm \( n \), parametrizing ideals generated by monomials:

\[ \left( \text{Hilb}_n \mathbb{C}^2 \right)^T = \left\{ I_\mu : |\mu| = n \right\}, \quad I_\mu = (x^{\mu_1}, x^{\mu_2-1} y, \ldots, y^{\mu_l}) \subset S = \mathbb{C}[x, y]. \]

There is a bundle \( \mathcal{P} \) on \( \text{Hilb}_n \mathbb{C}^2 \) of rank \( n! \) called the Procesi bundle, that is equivariant under the action of \( T \), equipped with a fiberwise action of the symmetric group \( S_n \) in such a way that every fiber is isomorphic to the regular representation. The fiber \( \mathcal{P}_\mu \) over any fixed point \( I_\mu \in \text{Hilb}_n \mathbb{C}^2 \) is therefore a representation of \( S_n \times T \), that turns out to be isomorphic to the Garsia-Haiman module \( R_\mu \), and the \( n! \) conjecture amounts to the statement that \( \dim_\mathbb{C} R_\mu = n! \).

The modified Macdonald polynomial \( \tilde{H}_\mu \) arises as the Frobenius characteristic of this module, meaning that if

\[ \text{ch} R_\mu = \sum_A a_{\lambda, \mu}(q, t) \chi_{\lambda} \]

describes the decomposition into irreducibles over \( S_n \), then

\[ \tilde{H}_\mu = \sum_A a_{\lambda, \mu}(q, t) s_{\lambda} \]

is the description of \( \tilde{H}_\mu \) in the Schur basis. For instance, for \( n = 2 \) we would have

\[ \tilde{H}_{[2]} = s_{[2]} + q s_{[1,1]}, \quad \tilde{H}_{[1,1]} = s_{[2]} + t s_{[1,1]}, \]

showing that both modules have one copy of the trivial representation and one copy of the alternating representation, but with different torus actions on each component.
3 The Vertex operator

Several useful quantities have translates on either side. First, consider the equivariant Euler characteristic of a pair of monomial ideals

\[ \chi_{\mu,\nu} = \sum_{i \geq 0} (-1)^i \text{ch} \text{Ext}^i_S(I_{\mu}, I_{\nu}) \in \mathbb{Z}(q, t), \]

where \( \text{ch} \) refers to the torus character. The Euler characteristic is a well-behaved quantity that may be computed in any resolution of either ideal, which leads to the following nice expression:

\[ \chi_{\mu,\nu} = M I_{\nu} I_{\mu}, f(q, t) = f(q^{-1}, t^{-1}), \quad (3.0.18) \]

where we are using \( I_\mu \) to denote both the ideal and its torus character \( I_\mu = M^{-1} - B_\mu \), noticing that \( M^{-1} \) is the character of \( S \).

In [5], the authors introduced a family of classes in the (equivariant) K-theory of \( \text{Hilb}_m S \times \text{Hilb}_n S \) for any smooth quasiprojective surface \( S \) together with a line bundle on it, and proved that its Euler characteristic defines a vertex operator in Nakajima’s Heisenberg operators. In the case when \( S = \mathbb{P}^2 \) with the above torus action and trivial bundle, it is the class of an honest an equivariant bundle \( E \) of rank \( m + n \) on \( \text{Hilb}_m \mathbb{P}^2 \times \text{Hilb}_n \mathbb{P}^2 \), whose fibers map be described in terms of Ext-groups after extending ideals to sheaves on \( \mathbb{P}^2 \). The torus characters of the fibers of this bundle over a fixed point \((I_\mu, I_\nu) \in \text{Hilb}_m \mathbb{P}^2 \times \text{Hilb}_n \mathbb{P}^2 \) are given by

\[ E_{\mu,\nu} = E_{\mu,\nu}(q, t) = \sum_{s \in \mu} q^{-a_+(s)} \mu_+(s)^{+1} + \sum_{s \in \nu} q^{a_-(s)} \nu_-(s)^{+1} t^{l_+(s)}. \quad (3.0.19) \]

Now for any pair of partitions, let us define

\[ N_{\lambda,\mu} = N_{\lambda,\mu}(u; q, t) = (-u)^{\nu_0} q^{\rho_0} \rho(\mu) \lambda(u E_{\lambda,\mu}). \]

If only one partition is specified, we will assume that \( \nu = \mu \). If no variable is specified, we will assume that \( u = 1 \). The modified Macdonald inner product can now be written as

\[ (\tilde{H}_\lambda, \tilde{H}_\mu) = \delta_{\lambda,\mu} N_\lambda. \]

On the other hand, we also have that

\[ \mathcal{G}_{1,1}(u; z, w) = \frac{N_{1,1}(u^{-1} w^{-1}; z^2, w^2)}{N_{1,1}(z^{-1} w^{-1}; z^2, w^2)} \]

Now define the following vertex operator introduced in [4]:

\[ \Gamma(u) := \Gamma_+ \left( \frac{u^{-1} - 1}{(1 - q)(1 - t)} \right) \Gamma_+ (1 - uqt) \quad (3.0.20) \]

In our notation, their main result states that
Theorem 3.0.1. (4) We have that

\[
\left( \Gamma(u)\tilde{H}_\lambda, \tilde{H}_\mu \right)_s = \left( \Gamma_+ (1 - uqt) \tilde{H}_\lambda, \Gamma_+ (1 - u^{-1}) \tilde{H}_\mu \right)_s = N_{\lambda,\mu}(u).
\]

The first equality follows from the properties in (2.2.9) and (2.2.5), as well as the definition of the inner product (2.2.13). The proof given uses tools such as the Fourier transform in symmetric functions as well as some formulas due to Cherednik, and will not be reproduced here. In fact, this operator is most naturally defined as acting on the equivariant \( K \)-theory of \( \text{Hilb}^n \mathbb{C}^2 \). The point is that this operator extends the cohomological vertex operator of [5] in the case where the smooth surface is \( \mathbb{C}^2 \), in sense that we can recover it by setting \( q = e^{\epsilon_1}, t^{\epsilon_2}, u^a \) and taking the limit

\[
\lim_{a \to 0} \Upsilon_{1,q} \Gamma(u)\Upsilon_{1,q}^{-1}.
\]

Under this limit, the modified Macdonald symmetric functions tend to the Jack symmetric functions with parameter \(-\epsilon_1/\epsilon_2\).

For instance, let us calculate the case \( \lambda, \mu = [2,1], [1,1] \). We have

\[
\tilde{H}_{[2]} = s_{[2]} + qs_{[1,1]} = \frac{1 + q}{2} p_1^2 + \frac{1 - q}{2} p_2,
\]

\[
\tilde{H}_{[1,1]} = s_{[2]} + ts_{[1,1]} = \frac{1 + t}{2} p_1^2 + \frac{1 - t}{2} p_2.
\]

We can calculate

\[
\Gamma_+ (1 - uqt) \tilde{H}_{[2]} = \frac{1 + q}{2} p_1^2 + \frac{1 - q}{2} p_2 + (1 - uqt)(1 + q)p_1 + (1 - uqt)(1 - uq^2t),
\]

\[
\Gamma_+ (1 - u^{-1}) \tilde{H}_{[1,1]} = \frac{1 + t}{2} p_1^2 + \frac{1 - t}{2} p_2 - (1 - u^{-1})(1 + t)p_1 + (1 - u^{-1})(1 - u^{-1}t).
\]

Taking the inner product gives

\[
u^{-2}t(1 - u)(1 - ut)(1 - uq^2t^{-1})(1 - uqt) = N_{[2],[1,1]}(u).
\]

4 Macdonald identities

Before proving the main theorems, it is useful to consider the identities in some special cases. An interesting specialization of (1.0.5) is the Euler specialization: \( z = \sqrt{q}, w = 1/\sqrt{q} \). We get

\[
\log \left( \sum_{\lambda} \prod_{\sigma \in \Pi} \frac{(1 - u \omega^\lambda(\sigma))(1 - u^{-1} \omega^\lambda(\sigma))}{(1 - \omega^\lambda(\sigma))^2} T^{\sigma|\lambda} \right) = \frac{(q - u)(q - u^{-1})}{(q - 1)^2} \frac{T}{1 - T}.
\]
Setting \( u = q^r \) for \( r \) a positive integer this becomes

\[
\sum_{\lambda} \prod_{s \in \lambda} \frac{(1 - q^{h(s)+r})(1 - q^{h(s)-r})}{(1 - q^{h(s)})^2} \tau^{[\lambda]} = \prod_{j=1}^{r-1} \prod_{n \geq 1} \left( (1 - q^{r-j}T^n)(1 - q^{r+j}T^n)(1 - T^n) \right),
\]

(4.0.22)
as one easily checks that

\[
\frac{(q - q^r)(q - q^{-r})}{(q - 1)^2} = - \sum_{j=1}^{r-1} \left( jq^{r-j} + jq^{-r+j} + 1 \right).
\]

The identities (4.0.22) are a \( q \)-analogue of certain identities from [22]. We give an independent proof of these identities below.

Note that the series in (4.0.22) visibly gets restricted to partitions that are \( r \)-cores; i.e., partitions with no hook of length \( r \). We can give a direct proof of (4.0.22) for \( r = 2 \). Indeed, the 2-cores are known to be the partitions of the form \( \lambda = (m, m-1, \ldots, 2, 1) \) for some positive integer \( m \). A calculation shows that in this case

\[
\prod_{s \in \lambda} \frac{(1 - q^{h(s)+2})(1 - q^{h(s)-2})}{(1 - q^{h(s)})^2} = (-1)^m \frac{q^{m+1} - q^{-m}}{q - 1},
\]

(4.0.23)
and hence (4.0.22) becomes Jacobi’s triple product identity in the form

\[
\sum_{m \geq 0} (-1)^m \sum_{k=-m}^m q^k T^{m(m+1)/2} = \prod_{n \geq 1} \left( (1 - qT^n)(1 - q^{-1}T^n)(1 - T^n) \right).
\]

For general \( r \) (4.0.22) follows from the Macdonald identity for the affine Lie algebra \( A_{r-1}^{(1)} \). Indeed, if we specialize the right hand side of [13] (0.4) so that \( e^\alpha = q \) for all simple roots \( \alpha \) of \( A_{r-1} \) (hence in general \( e^\alpha \) is mapped to \( q^{ht(\alpha)} \), where \( ht(\alpha) \) is the height of \( \alpha \)) we obtain the right hand side of (4.0.22) as there are \( j \) roots of height \( r - j \) for \( j = 1, 2, \ldots, r-1 \). To complete the proof of (4.0.22) we would need an analogue of (4.0.23). This is given in Proposition 4.0.2, which is a consequence of the main result of [6].

The key identity linking the main series (4.0.21) and that of the Macdonald identities is the following.

**Proposition 4.0.2.** For an \( r \)-core \( \lambda \) and associated vector \( v \).

\[
\prod_{s \in \lambda} \frac{(1 - q^{h(s)+r})(1 - q^{h(s)-r})}{(1 - q^{h(s)})^2} = \frac{V(q^v, \ldots, q^n)}{V(q^2v, \ldots, q^n)},
\]

(4.0.24)
where \( V \) is the Vandermonde determinant.

**Proof.** It follows from [6][Theorem 1] by taking \( \tau(n) := 1 - q^n \). \( \square \)

Once we have (4.0.24) then (4.0.22) becomes the Macdonald identity [13] (0.4) for the affine Lie algebra \( A_{r-1}^{(1)} \) specialized to \( e^\alpha = q \) for all simple roots. To see this note that the
lattice $M$ of \[18\] (0.4) consists of the vectors $\mu = rn$ with $n = (n_0, \ldots, n_{r-1}) \in \mathbb{Z}^r$ satisfying $n_0 + n_1 + \cdots + n_{r-1} = 0$. The half-sum of the positive roots $\rho$ is $(-k, \ldots, k)$ and

$$
\frac{1}{2r} \left( ||rn + \rho||^2 - ||\rho||^2 \right) = \frac{1}{2r} (v_k^2 + \cdots + v_k^2) - \frac{r^2 - 1}{24},
$$

where $(v_k, \ldots, v_k) := rn + \rho$. We claim that the quantity $\chi(\mu)$ of Macdonald’s matches the right hand side of (4.0.24). The specialization we are considering sends $e^\nu$ to $q^{(\nu, \nu)}$ and hence sends the numerator of $\chi(\mu)$ to

$$
\sum_{\sigma \in S_n} \text{sgn}(\sigma) q^{(\sigma \nu, \rho)} = q^{-r(r-1)/2} V(q^{\nu_k}, \ldots, q^{\nu_k}).
$$

Similarly, the denominator of $\chi(\mu)$ equals $q^{-r(r-1)/2} V(q^{-k}, \ldots, q^{-k})$ proving our claim.

It is interesting to see what happens to the vertex operator in this special case, i.e. when $t = q^{-1}, u = q^{-r}$. In this case we get

$$
\Gamma'(q) := \left. \Gamma_{q^{-1} \Gamma_{r^{-1}}} \right|_{t=q^{-1}} \Gamma_{q^{-1} \Gamma_{r^{-1}}} = \Gamma_{-1} \left( 1 + q + \cdots + q^{-r} \right) \Gamma_{+} \left( -1 - q^{-1} - \cdots - q^{1-r} \right).
$$

By Theorem 3.0.1, we must have

$$
\left( \Gamma'(q) s_{\lambda}, s_{\mu} \right) = q^{-r |\lambda|} \prod_{\alpha \in \Lambda} \frac{(1 - q^{r^{\alpha}(s) + l(s) + 1}) \prod_{\alpha \in \Lambda} (1 - q^{-r^{\alpha}(s) - l(s) - 1})}{a_{\alpha}(q^{-1}) a_{\alpha}(q)},
$$

which can be checked to agree with the left hand side of (4.0.24) at $\lambda = \mu$. It should therefore be possible to show that the left hand side agrees with the right hand side of (4.0.24) in this case.

We prove something more general:

**Proposition 4.0.3.** If $\lambda$ is an $r$-core with associated vector $v$, then

$$
\left( \Gamma_{-} (x_1 + \cdots + x_r) \Gamma_{+} (-x_1^{-1} - \cdots - x_r^{-1}) s_{\lambda}, s_{\mu} \right) = \frac{\det(A^\nu)}{\det(A^\rho)}
$$

where $A_{i,j}^\nu = x_{j}^{v_i}$.  

**Proof.** Using (2.2.7), we may write

$$
\Gamma_{-} (x_1 + \cdots + x_r) \Gamma_{+} (-x_1^{-1} - \cdots - x_r^{-1}) = \left( \prod_{i \neq j} (1 - x_i x_j^{-1})^{-1} \right) \psi(x_1) \cdots \psi(x_r) \quad (4.0.25)
$$

where

$$
\psi(x) = \sum_{i \in \mathbb{Z}} x_i \psi_i = \Gamma_{-}(x) \Gamma_{+}^{-1}(x^{-1})
$$
is the Bernstein vertex operator. Notice that

\[
\prod_{i<j}(1 - x_j x_i)^{-1} = x_1^{r-1} x_2^{r-2} \cdots x_{r-1} \det(A^r)^{-1}
\]

The components \( \psi_i \) can be described in the Schur basis as follows: for any partition \( \mu \), consider its Maya diagram

\[
m(\mu) = \{ \mu_1, \mu_2 - 1, \mu_3 - 2, \ldots \} \subset \mathbb{Z}.
\]

If \( \mu \) is an \( r \)-core, then its Maya diagram has a particularly simple shape: it satisfies

\[
j \in m(\mu), \quad i < j, \quad i = j \pmod{r} \Rightarrow i \in m(\mu).
\]

If \( i \in \mathbb{Z} - m(\mu) \), let \( \tilde{\mu} \) denote the unique partition with

\[
m(\tilde{\mu}) + 1 = \{ i \} \cup m(\mu)
\]

We have that

\[
\psi_i s_\mu = \begin{cases} \pm s_\tilde{\mu} & i \notin m(\tilde{\mu}) \\ 0 & \text{otherwise.} \end{cases}
\]

The sign is determined by the position of the insertion of \( i \).

Now expand

\[
x_1^{r-1} x_2^{r-2} \cdots x_{r-1} (\psi(x_1) \cdots \psi(x_r) s_{\lambda}, s_{\lambda}) = \sum_{I = (i_1, \ldots, i_r) \in \mathbb{Z}^r} a_I x_1^{i_1} \cdots x_r^{i_r}
\]

as a power series in \( x_i \). Since \( \lambda \) is an \( r \)-core, we can see that \( I \) consists of \( r \) distinct entries with distinct remainders modulo \( r \), and that \( a_I = 0 \) for all but one possible underlying set of \( I \). Every possible reordering of this set contributes, leaving us with precisely \( \det(A^r) \). \( \Box \)

Now setting \( x_i = q_i^{-1} \), we obtain an independent proof of (4.0.24).

5 Proof of the main identities Theorem 1.0.2

Proof. We start by proving the following recursive formula:

Theorem 5.0.4. We have that

\[
\Omega_{g+1,k}(u; q, t) = \varphi_{u_{g+1}}(\Omega_{k+2}(u; q, t))
\]

where

\[
\varphi_u : \Lambda[X_{k+1}] \otimes \Lambda[X_{k+2}] \to \mathbb{Q}(q, t)
\]

is the linear map satisfying

\[
\varphi_u(f[X_{k+1}] \otimes g[X_{k+2}]) = (\Gamma(u)f, g)_{\lambda}.
\]
Proof. We have
\[
\varphi_{u^1+1}(\Omega_{k+2}(u; q, t)) = \sum \frac{N_\lambda(u_1) \cdots N_\lambda(u_g)}{N_\lambda} \hat{H}_\lambda[X_1] \cdots \hat{H}_\lambda[X_k] \left( \Gamma(u_{g+1}) \hat{H}_\lambda \right) N_\lambda =
\]
\[
\sum \frac{N_\lambda(u_1) \cdots N_\lambda(u_{g+1})}{N_\lambda} \hat{H}_\lambda[X_1] \cdots \hat{H}_\lambda[X_k] = \Omega_{g+1,k}(u; q, t)
\]
using Theorem [3.0.1].

Using the more obvious rule that
\[
\Omega_{g_1+g_2}([u_1, \ldots, u_{g_1+g_2}]; q, t)[X_1, \ldots, X_{k_1-1}, Y_1, \ldots, Y_{k_2-1}] =
\]
\[
\varphi \left( \Omega_{g,k_1}([u_1, \ldots, u_{g_1}]; q, t) \otimes \Omega_{g_2,k_2}([u_{g_1+1}, \ldots, u_{g_1+g_2}]; q, t) \right)
\]
(5.0.26)
where
\[
\varphi (f[X_k]; g[X_{k_i}]) = (f, g)_* \]
we have expressed \( \Omega_k \) in terms of the power series \( \Omega_{0,3} \), i.e. the power series associated to a pair of pants in TQFT language.

By contracting from genus 0 with 2 punctures to genus 1 with no punctures, we have
\[
\Omega_{1,0} = \text{Tr} T^d \Gamma(u).
\]
Let
\[
F(x) = \text{Tr} T^d \Gamma_- \left( \frac{u^{-1} - 1}{(1 - q)(1 - t)} \right) T x \Gamma_+ (1 - uqt)
\]
so that \( F(1) = \Omega_{1,0} \). Then we have
\[
F(x) = \text{Tr} \Gamma_- \left( \frac{u^{-1} - 1}{(1 - q)(1 - t)} T x \right) T^d \Gamma_+ (1 - uqt) =
\]
\[
\text{Tr} T^d \Gamma_+ (1 - uqt) \Gamma_- \left( \frac{u^{-1} - 1}{(1 - q)(1 - t)} T x \right) =
\]
\[
\text{Exp} \left( \frac{(u^{-1} - 1)(1 - uqt)}{(1 - q)(1 - t)} T x \right) F(T x),
\]
and
\[
F(0) = \text{Tr} T^d \Gamma_+ (1 - uqt) = \text{Exp} \left( \frac{T}{1 - T} \right)
\]
Solving this recurrence, we find that
\[
F(x) = \text{Exp} \left( \frac{(u^{-1} - 1)(1 - uqt)}{(1 - q)(1 - t)} x \frac{T}{1 - T} + \frac{T}{1 - T} \right).
\]
Putting \( x = 1 \) and recalling the definition of \( \mathbb{H}(u; q, t) \) we obtain
\[
\mathbb{H}_1(u; q, t) = \left( -(u^{-1} - 1)(1 - uqt) - (1 - q)(1 - t) \right) \frac{T}{1 - T} =
\]

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\[-u^{-1}(1 - uq)(1 - ut)\frac{T}{1 - T}.\]

This proves (1.0.5).

Next, we prove the more difficult second formula (1.0.6). Let us define

\[
\Omega'_k(u; q, t) = \sum_{\lambda} T^{[\lambda]} \tilde{H}_1[\lambda] \cdots \tilde{H}_k[\lambda] \frac{N_{\lambda}}{N_{A}} N_{\lambda} \psi_\mu(q, t),
\]

as well as the corresponding terms \(H'_k(u; q, t), \tilde{H}'_{n, \lambda}(u; z, w)\) in the obvious way.

First, we will need the following lemma.

**Lemma 5.0.5.** We have

\[
[x^0] \text{Exp} \left( x(uqt - 1) \frac{1}{1 - T} + x^{-1}(u^{-1} - 1) \frac{T}{1 - T} \right) = \text{Exp} \left( (1 - u^{-1})(1 - uqt) \frac{T}{1 - T} \right)
\]

(5.0.27)

as power series in \(\mathbb{C}(q, t)((u))[[T]]\).

**Proof.** Let

\[
A_N = \text{Exp} \left( x(uqt - 1) \frac{1 - T^N}{1 - T} + x^{-1}(u^{-1} - 1) \frac{T - T^N}{1 - T} \right) \in \mathbb{C}(q, t, u, x, T),
\]

and let \(\mathcal{T}A_N\) denote the corresponding element of \(\mathbb{C}(q, t)((u))[[T]]\). Then we have that

\[
[x^0] \mathcal{T}A_N = \sum_{p \in \{0, 1/u, T/u, T^2/u, \ldots\}} \mathcal{T} \text{Res}_{x=p} x^{-1} A_N.
\]

We can check that for every \(j\) we have

\[
\lim_{n \to \infty} \text{ldeg}_{x}[T^j] \text{Res}_{x=T^n/u} x^{-1} A_N = 0
\]

by a bound that is independent of \(N\), where \(\text{ldeg}_{x}\) is the degree of the leading term in \(u\). We can also check that

\[
\text{Res}_{x=0} x^{-1} A_N = u^{-N},
\]

which tends to zero as \(N\) becomes large. From this we find that the left hand side of (5.0.27) is given by

\[
\lim_{N \to \infty} [x^0] \mathcal{T}A_N = \sum_{n \geq 0} \lim_{N \to \infty} \mathcal{T} \text{Res}_{x=T^n/u} x^{-1} A_N = (1 - u) \text{Exp} \left( (qt - u + 1 - u^{-1}) \frac{T}{1 - T} \right) \sum_{n \geq 0} u^n \frac{(qtT; T)_n}{(T; T)_n}.
\]

The lemma now follows by using the \(q\)-binomial theorem

\[
\sum_n u^n \frac{(a; T)_n}{(T; T)_n} = \frac{(au; T)_\infty}{(u; T)_\infty}.
\]

\(\square\)
Recall that $\psi_\mu(q,t)$ is the eigenvalue of the Macdonald operator from equation (2.2.17). Then using an argument similar to the proof of the first identity we have

$$
\Omega'_{1,0}(u; q, t) = \text{Tr} T^d \Gamma(u) D_0 = \\
[x^0] \text{Tr} T^d \Gamma \left( \frac{u^{-1} - 1}{M} \right) \Gamma_+ (1 - uqt) \Gamma_- (-x) \Gamma_+ \left( Mx^{-1} \right) = \\
[x^0] \text{Exp} \left( \frac{(u^{-1} - 1)(1 - uqt)}{M} \frac{T}{1 - T} \right) \text{Exp} \left( -M \frac{T}{1 - T} \right) \times \\
\text{Exp} \left( x(uqt - 1) \frac{1}{1 - T} + x^{-1}(u^{-1} - 1) \frac{T}{1 - T} \right) \text{Exp} \left( \frac{T}{1 - T} \right).
$$

The result follows after applying lemma 5.0.5 to the part that depends on $x$. □

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