Continuous-Variable Quantum Games ★

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Abstract

We investigate the quantization of games in which the players can access to a continuous set of classical strategies, making use of continuous-variable quantum systems. For the particular case of the Cournot’s Duopoly, we find that, even though the two players both act as “selfishly” in the quantum game as they do in the classical game, they are found to virtually cooperate due to the quantum entanglement between them. We also find that the original Einstein-Podolsky-Rosen state contributes to the best profits that the two firms could ever attain. Moreover, we propose a practical experimental setup for the implementation of such quantum games.

Key words: quantum games, entanglement, continuous-variable quantum system
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Introduction

Quantum game theory was initiated by a paper of D. A. Meyer[1], discovering that a player can always beats his classical opponent by adopting quantum strategies. Consequently the quantization of the famous Prisoners’ Dilemma was presented by J. Eisert et al.[2]. They showed that this game ceases to pose a dilemma if quantum strategies are allowed for. By far, investigations on

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multiplayer quantum games[3,4], as well as many interesting aspects on quantum games[5,6,7,8,9,10,11], were also presented. Recently, we investigated the role of entanglement in quantum games and found that the properties of the quantum Prisoners’ Dilemma change discontinuously when its entanglement varies[12,13]. Besides the theoretical investigations, the first experimental realization of quantum games was also successfully accomplished on our NMR quantum computer[13].

However most of the current investigations on quantum games focus on games in which the players have finite number of strategies. In real-life society and economy, many cases should be represented by games in which the players can access to a continuous set of strategies[14]. Thus games with a continuous set of strategies are quite familiar and are very important. Considering the intimate connection between the theory of games and the theory of quantum communication[2], quantization of games with continuum strategic space therefore deserves thorough investigations, for both theoretical and practical reasons.

In this paper we investigate the quantization of games with continuous strategic space. A classic instance of such games is the Cournot’s Duopoly[15], which is also a cornerstone of modern economics. The classical game exhibits a dilemma-like situation, i.e. the unique Nash equilibrium is inferior to the Pareto Optimal, just like in the Prisoners’ Dilemma[2]. We give a quantum structure of Cournot’s Duopoly, and show that — in the context of this structure — the players can escape the frustrating dilemma-like situation if the structure involves a maximally entangled state. We also observe the transition of the game from purely classical to fully quantum, as the game’s entanglement increases from zero to maximum. The profits at quantum Nash equilibrium increases monotonously as the game’s entanglement increases. In the maximally entangled game, the profits at the unique Nash equilibrium is exactly the Pareto Optimal (the best result they can ever attain while maintaining the symmetry of the game), and the dilemma-like situation in the classical Cournot’s Duopoly is completely resolved. Moreover we propose a practical experimental setup for the quantum Cournot’s Duopoly, within the capability of nowadays optical technology.

1 Classical Cournot’s Duopoly

A duopoly is a case in which two firm monopolize the market of a certain commodity without a third competitor. In a simple scenario of Cournot’s model for this duopoly, firm 1 and firm 2 simultaneously decide the quantities $q_1$ and $q_2$, respectively, of a homogeneous product they want to put on the market. Let $Q$ be the total quantity, i.e. $Q = q_1 + q_2$, and $P(Q)$ the price.
A formula of \( P(Q) \) is \( P(Q) = a - Q \) for \( Q \leq a \) while \( P(Q) = 0 \) for \( Q > a \). This is a simple but reasonable reflection of the fact that the more product is put on the market, the less the price will be, and when the total quantity is extremely large, the product will be nearly worthless. Assume that the unit cost of producing the product is a constant \( c \) with \( c < a \). Then the profits can be written as

\[
\begin{align*}
 u_j(q_1, q_2) &= q_j [P(Q) - c] \\
 &= q_j [a - c - (q_1 + q_2)] \\
 &= q_j [k - (q_1 + q_2)],
\end{align*}
\]

with \( k = a - c \) being a constant and \( j = 1, 2 \). It is easy to find that the unique Nash equilibrium of the game is

\[
q^*_1 = q^*_2 = \frac{k}{3}.
\]

At this equilibrium the profits are

\[
\begin{align*}
 u_1(q^*_1, q^*_2) &= u_2(q^*_1, q^*_2) = \frac{k^2}{9},
\end{align*}
\]

However this equilibrium fails to be the optimal solution. It is easy to check that if the two firms can cooperate and restrict their quantities to

\[
q'_1 = q'_2 = \frac{k}{4},
\]

they can both acquire higher profits

\[
\begin{align*}
 u_1\left(\frac{k}{4}, \frac{k}{4}\right) &= u_2\left(\frac{k}{4}, \frac{k}{4}\right) = \frac{k^2}{8}.
\end{align*}
\]

In fact, this is the highest profits they can ever attained while remaining the symmetry of the game. But they can never escape the Nash equilibrium since unilateral deviation will decrease the individual profit. This dilemma-like situation again exhibits the conflict between individual rationality and collective rationality, just like in the classical Prisoners’ Dilemma[2].
Fig. 1. The quantum structure of the Cournot’s Duopoly.

2 Quantum Cournot’s Duopoly

We now investigate the quantization of the classical Cournot’s Duopoly. The extension of a classical game into the quantum domain is usually done by setting up a Hilbert space, assigning the possible outcomes of each classical strategy to certain quantum states. The quantum strategies are therefore operations on the quantum states, and the payoffs are read out from the final measurement. Since different classical strategies are completely distinguishable, it is then natural to require that the Hilbert space used to have at least the same number of distinguishable states as that of the different classical strategies. Therefore in the cases that the strategic space is a continuum, one needs a Hilbert space with a continuous set of orthogonal bases, i.e. the Hilbert space of a continuous-variable quantum system.

In this paper, we would like to make use of two single-mode electromagnetic fields, of which the quadrature amplitudes have a continuous set of eigenstates. Fig. 1 shows the quantum structure of the game. The game starts from state $|\psi_i\rangle = \hat{J}(\gamma) |\text{vac}\rangle_1 |\text{vac}\rangle_2$, (6) which is the tensor product of two single-mode vacuum states of two electromagnetic fields. This state consequently undergoes a unitary operation $\hat{J}(\gamma)$, which is known to both firms (the meaning of $\gamma$ will be presented later). In order to maintain the symmetry of this game, $\hat{J}(\gamma)$ should be symmetric with respect to the interchange of the two electromagnetic fields. At this stage, the state of the game is

$$|\psi_f\rangle = \hat{J}(\gamma) (\hat{D}_1 \otimes \hat{D}_2) \hat{J}(\gamma) |\text{vac}\rangle_1 |\text{vac}\rangle_2.$$ (7)
It is straightforward to set the final measurement be corresponding to the observables \( \hat{X}_j = (\hat{a}_j^\dagger + \hat{a}_j) / \sqrt{2} \) (the “position” operators) of firm \( j \), where \( \hat{a}_j^\dagger \) (\( \hat{a}_j \)) is the creation (annihilation) operator of firm \( j \)’s electromagnetic field. This measurement is usually done by the homodyne measurement. In order to reduce the noise of the result (since a coherent state is not an eigenstate of the “position” operator), we shall squeeze the reference light beam in the homodyne measurement. In this paper we focus on the idea case that the reference light beam is infinitely squeezed, then the noise is reduced to zero. Let \( \tilde{x}_j \) be the measurement result, then the individual quantity is determined by \( q_j = \tilde{x}_j \), and hence the profit by

\[
u^Q_j(\hat{D}_1, \hat{D}_2) = u_j(\tilde{x}_1, \tilde{x}_2),
\]

where superscript “\( Q \)” denotes “quantum”.

The classical Cournot’s Duopoly should be included as a subset of the quantum structure, so that the quantum game is comparable with the original classical one[2]. Indeed the classical game is faithfully represented as long as \( \hat{J}(\gamma) = \hat{J}(\gamma)^\dagger = I \) (the identity operators). Let \( \hat{P}_j = i(\hat{a}_j^\dagger - \hat{a}_j) / \sqrt{2} \) be “momentum” operator of firm \( j \)’s electromagnetic field. We can see that if the players are restricted to choose their strategies from the sets

\[
S_j = \{ \hat{D}_j(x_j) = \exp(-ix_j\hat{P}_j) \mid x_j \in [0, \infty) \}, \quad j = 1, 2
\]

then the final state is

\[
|\psi_f\rangle = (\exp(-ix_1\hat{P}_1) |\text{vac}_1\rangle) \otimes (\exp(-ix_2\hat{P}_2) |\text{vac}_2\rangle).
\]

And the consequent final measurement gives the results \( q_1 = x_1 \) and \( q_2 = x_2 \), thus the original classical game is recovered. We hence obtain that the set \( S_j \) is exactly the quantum analog of classical strategic space.

In the remaining part of the paper, we would like to present the “minimal” extension of the classical Cournot’s Duopoly into the quantum domain: we maintain the strategic space unexpanded (\( S_j \) for firm \( j \)) and only extend the initial state \( |\psi_i\rangle \) to be superposition or entangled state (\( \hat{J}(\gamma) \neq I \) and \( \hat{J}(\gamma)^\dagger \neq I \)). The classical game is a subset of this “minimal” extension in the sense that the quantum game turns back to the original classical form when the initial state is not entangled. In this extension, if the game of non-zero entanglement exhibits any features not seen in the classical game, we can be sure that these features are completely attributed to the quantum entanglement. However it does not rule out the possibility of getting a quantum version of Cournot’s Duopoly by extending both the states and the strategic space.
The entangling operator $\hat{J}(\gamma)$ is given by

$$\hat{J}(\gamma) = \exp\{-\gamma(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)\} = \exp\{i\gamma(\hat{X}_1 \hat{P}_2 + \hat{X}_2 \hat{P}_1)\}, \quad (11)$$

Hence the initial state is

$$|\psi_i\rangle = \exp\{-\gamma(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)\} |\text{vac}\rangle_1 |\text{vac}\rangle_2, \quad (12)$$

which is exactly the two-mode squeezed vacuum state used to teleport continuous quantum variables[16,17,18] and to demonstrate the violation of Bell’s inequalities for continuous variable systems[19]. $\gamma \geq 0$ is known as the squeezing parameter and can be reasonably regarded as a measure of entanglement. Note that in the infinite squeezing limit $\gamma \to \infty$, the initial state approximates the EPR state, i.e. $\lim_{\gamma \to \infty} |\psi_i\rangle = \int |x, -x\rangle dx = |\text{EPR}\rangle[16,17,18]$.

Detailed calculation gives

$$\hat{J}(\gamma)^\dagger \hat{D}_1(x_1) \hat{J}(\gamma) = \exp\{-ix_1(\hat{P}_1 \cosh \gamma + \hat{P}_2 \sinh \gamma)\},$$

$$\hat{J}(\gamma)^\dagger \hat{D}_2(x_2) \hat{J}(\gamma) = \exp\{-ix_2(\hat{P}_2 \cosh \gamma + \hat{P}_1 \sinh \gamma)\}.$$  

Therefore

$$\hat{J}(\gamma)^\dagger [\hat{D}_1(x_1) \otimes \hat{D}_2(x_2)] \hat{J}(\gamma)$$

$$= \exp\{-i(x_1 \cosh \gamma + x_2 \sinh \gamma)\hat{P}_1\} \exp\{-i(x_2 \cosh \gamma + x_1 \sinh \gamma)\hat{P}_2\}.$$ 

From this we obtain the final state

$$|\psi_f\rangle = \exp\{-i(x_1 \cosh \gamma + x_2 \sinh \gamma)\hat{P}_1\} |\text{vac}\rangle_1$$

$$\otimes \exp\{-i(x_2 \cosh \gamma + x_1 \sinh \gamma)\hat{P}_2\} |\text{vac}\rangle_2. \quad (13)$$

The final measurement gives the respective quantities of the two firms

$$q_1 = x_1 \cosh \gamma + x_2 \sinh \gamma,$$

$$q_2 = x_2 \cosh \gamma + x_1 \sinh \gamma.$$ 

For convenience, we directly denote the strategy in the quantum game by $x_1$ and $x_2$ when the strategies are $\hat{D}_1(x_1)$ and $\hat{D}_2(x_2)$, respectively. Referring equation (1) the quantum profits for them are
Fig. 2. The profits at quantum Nash equilibrium as a function of the squeezing parameter $\gamma$, which can be reasonably regarded as a measure of entanglement.

\[
u^Q_1(x_1, x_2) = u_1(q_1, q_2) = (x_1 \cosh \gamma + x_2 \sinh \gamma) \times [k - e^\gamma (x_1 + x_2)],
\]

\[
u^Q_2(x_1, x_2) = u_2(q_1, q_2) = (x_2 \cosh \gamma + x_1 \sinh \gamma) \times [k - e^\gamma (x_1 + x_2)].
\] (14)

Solving the Nash equilibrium gives the unique one

\[x_1^* = x_2^* = \frac{k \cosh \gamma}{1 + 2e^{2\gamma}}.
\] (15)

And the profits at this equilibrium are

\[
u^Q_1(x_1^*, x_2^*) = \nu^Q_2(x_1^*, x_2^*) = \frac{k^2 e^\gamma \cosh \gamma}{(3 \cosh \gamma + \sinh \gamma)^2}.
\] (16)

It would be interesting to review how the profits at Nash equilibrium vary with respect to the measure of entanglement $\gamma$, as depicted in Fig. 2. It shows that the more entangled the game’s state is, the higher profits the two firms can obtain. When the game is not entangled, i.e. $\gamma = 0$, the quantum game goes back to the original classical form. However in maximally entangled game, i.e. in the infinite squeezing limit $\gamma \to \infty$, the initial state $|\psi_i\rangle \to |\text{EPR}\rangle$ and $\nu^Q_1(x_1^*, x_2^*) = \nu^Q_2(x_1^*, x_2^*) \to k^2/8$, which is the best situation the two firms can ever achieve. This novel feature indicates that the original Einstein-Podolsky-Rosen state enables the two firms to best cooperate and therefore to be best rewarded, and the dilemma-like situation in the classical game is completely removed.

The alert reader may argue that the quantum structure proposed here can be simulated purely classically. For instance, firm 1 and firm 2 may each communicate their choice $(x_1, x_2)$ to the judge using ordinary telephone lines. The judge then computes the payoffs according to equation (14). However this
Fig. 3. Practical setup based on optical experiments for the quantum structure of Cournot’s Duopoly (shown in Fig. 1). \( \hat{D}_j \) is the strategic move adopted by firm \( j \), and \( M_j \) denotes the measurement of firm \( j \’s. \nabla

implementation loses comparability with the original game, because once the judge has to change the payoff functions according to \( \gamma \), at the same time he changes the game itself, since in the original game the payoffs are always given by the same procedure. Yet the quantum structure maintains the comparability since the payoff are given by the same procedure independent of \( \gamma \): measuring the quadrature amplitudes, taking the results as quantities, and calculating the profits according to the classical payoff functions (in equation (1)). Further, any classical “simulation” maintaining the comparability should have to introduce direct interaction between strategies of the players (firms), because the final quantity of a single firm should contain the strategy of its opponent (see equation (14)). This direct interaction, however, is prohibited since the game is a static one. While in the quantum structure the entanglement provides an “invisible channel” for the two firms to affect each other, rather than to directly interact.

3 Experimental Setup

So far all the investigations are in theory, we now want to construct a practical setup of the quantum Cournot’s Duopoly, based on feasible optical experiments. The setup is shown in Fig. 3. BS\(_1\), BS\(_2\) and BS\(_3\) are beam splitters. The operator of a beam splitter is \[ \hat{B}(\theta, \phi) = \exp\{\frac{\theta}{2}(\hat{a}_1^\dagger \hat{a}_2 e^{i\phi} - \hat{a}_1 \hat{a}_2^\dagger e^{-i\phi})\}, \] (17)

with the amplitude reflection and transmission coefficients \( t = \cos \theta/2 \) and \( r = \sin \theta/2 \), and \( \phi \) being the phase difference between the reflected and transmitted fields. Here we set

\[
\begin{align*}
\hat{B}(\pi/2, 3\pi/2) &= BS_1 = BS_2 = \hat{B}(\pi/2, 3\pi/2), \\
\hat{B}(\pi/2, \pi/2) &= BS_3 = BS_3^\dagger.
\end{align*}
\]

\( \hat{S}(\gamma) = \exp\{-i\gamma(\hat{a}^\dagger + \hat{a})^2/2\} \) is squeezing operator, which can be implemented by parametric down-conversion inside a nonlinear crystal[22]. Detailed calcu-
lation yields that

\[
BS_1 e^{-i\frac{\gamma}{2} (\hat{a}_1^2 + \hat{a}_2^2)} e^{-i\frac{\gamma}{2} (\hat{a}_2^2 + \hat{a}_1^2)} |\text{vac}\rangle |\text{vac}\rangle = e^{-\gamma (\hat{a}_1^\dagger \hat{a}_1^\dagger - \hat{a}_2 \hat{a}_2)} |\text{vac}\rangle |\text{vac}\rangle,
\]

\[
BS_3 e^{-i\frac{\gamma}{2} (\hat{a}_1^2 + \hat{a}_2^2)} e^{-i\frac{\gamma}{2} (\hat{a}_2^2 + \hat{a}_1^2)} BS_2 = e^{\gamma (\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_2 \hat{a}_1)}.
\]

Therefore the operations of \( \hat{J}(\gamma) \) and \( \hat{J}(\gamma)^\dagger \) are constructed. \( \hat{D}_1(x_1) \) and \( \hat{D}_2(x_2) \) can be realized by simple phase-space displacements on each single electromagnetic field. The setup in Fig. 3 hence faithfully represents the quantum structure in Fig. 1.

4 Conclusion

We investigate the quantization of games with continuum strategic space, making use of continuous-variable quantum systems. For the particular case of Cournot’s Duopoly, we construct a quantum structure of it, and proposed the extension from classical to quantum, which we refer to be “minimal”. The classical game is shown to be a subset of the quantum game, and therefore they can be compared in an unbiased manner. We observed novel features in the quantum Cournot’s Duopoly, which are completely due to quantum entanglement. In the quantum game two firms virtually cooperate even though they still behave selfishly. This virtual cooperation, as well as the profits, increases as the entanglement increases. For the maximal entanglement limit, the initial state is precisely the original Einstein-Podolksy-Rosen state, and enables the two firms to cooperate best and to be best rewarded. Further we proposed a feasible experimental scheme to demonstrate the quantum Cournot’s Duopoly, showing the game's procedure of preparing initial states, executing strategic moves and reading out payoffs (profits). This scheme is within the capability of current optical technology, and could be implemented indeed.

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