THE EQUIVALENCE OF OPTIMAL PERSPECTIVE FORMULATION AND SHOR’S SDP FOR QUADRATIC PROGRAMS WITH INDICATOR VARIABLES

SHAONING HAN, ANDRÉS GÓMEZ AND ALPER ATAMTÜRK

Abstract. In this paper, we compare the strength of the optimal perspective reformulation and Shor’s SDP relaxation. We prove that these two formulations are equivalent for quadratic optimization problems with indicator variables.

Keywords. Mixed-integer quadratic optimization, semidefinite programming, perspective formulation, indicator variables, convexification.

December 2021

1. INTRODUCTION

We consider the optimization problem with a convex quadratic objective with indicators:

\[
\begin{align*}
\text{(Q)} & \quad \min_{x, y} a'x + b'y + y'Qy \\
\text{s.t.} & \quad y_i(1 - x_i) = 0 \quad \forall i \in [n] \\
& \quad x \in \{0, 1\}^n, \ y \in \mathbb{R}_+^n \\
& \quad (x, y) \in \mathcal{X} \subseteq \mathbb{R}^n \times \mathbb{R}^n,
\end{align*}
\]

where \(a\) and \(b\) are \(n\)-dimensional vectors, \(Q \in \mathbb{R}^{n \times n}\) is a symmetric positive semidefinite (PSD) matrix, \([n] := \{1, 2, \ldots, n\}\), and \(\mathcal{X}\) is a closed convex set that incorporates other unspecified constraints arising in applications. For each \(i \in [n]\), the complementarity constraint \(y_i(1 - x_i) = 0\), along with the indicator variable \(x_i \in \{0, 1\}\), is used to state that \(y_i = 0\) whenever \(x_i = 0\). Numerous applications, including portfolio optimization \([8]\), optimal control \([18]\), image segmentation \([24]\), signal denoising \([6]\) are either formulated as \((\text{Q})\) or can be relaxed to \((\text{Q})\). While special cases of \((\text{Q})\) with a diagonal
Q or M-matrix $Q$ can be solved in polynomial time [3], in general, $(QI)$ is \textit{NP}-hard [21].

Building strong convex relaxations of $(QI)$ is instrumental in solving it effectively. There is a substantial body of research on the perspective formulation of convex univariate functions with indicators [1, 11, 12, 14, 20, 22, 31]. When $Q$ is diagonal, $y' Q y$ is separable and the perspective formulation provides the convex hull of the epigraph of $y' Q y$ with indicator variables by strengthening each term $Q_{ii} y_i^2$ with its perspective counterpart $Q_{ii} y_i^2 / x_i$, individually.

Another powerful approach for nonconvex quadratic optimization problems is semidefinite programming (SDP) reformulation, first proposed by Shor [28]. Specifically, a convex relaxation is constructed by introducing a rank-one matrix $Z$ representing $zz'$, where $z$ is the decision vector, and then forming the semidefinite relaxation $Z \succeq zz'$. Such SDP relaxations have been widely utilized in numerous applications, including max-cut problems [19], hidden partition problems of finding clusters in large network datasets [25], matrix completion problems [2, 10], power systems [13], robust optimization problems [1], Sufficient conditions for exactness of SDP relaxations [e.g. 9, 23, 26, 30, 29] and stronger rank-one conic formulations [4, 5] are also given in the literature.

These two approaches have been studied extensively in literature. While it is known that Shor’s SDP formulation is at least as strong as the optimal perspective formulation [11], the other direction has not been explored. We show in this note that these two formulations are, in fact, equivalent. The equivalence makes the perspective formulation the favorable choice as it is much smaller than Shor’s SDP and easier to solve.

\textbf{Notation.} Throughout, we adopt the following convention for division by 0: given $a \geq 0$, $a^2 / 0 = \infty$ if $a \neq 0$ and $a^2 / 0 = 0$ if $a = 0$. For a vector $v$, diag$(v)$ denotes the diagonal matrix $V$ with $V_{ii} = v_i$ for each $i$.

2. \textbf{Optimal perspective formulation vs. Shor’s SDP}

In this section we analyze two well-known convex formulations: the optimal perspective formulation and Shor’s SDP. We first introduce the two formulations and then show that they are equivalent for $(QI)$.

Splitting $Q$ into some diagonal PSD matrix $D = \text{diag}(d)$ and a PSD residual, i.e., $Q - D \succeq 0$, one can apply the perspective reformulation to each diagonal term, by replacing $D_{ii} y_i^2$ with $D_{ii} y_i^2 / x_i$, to get a valid convex relaxation of $(QI)$—after relaxing integrality constraints in $x$ and dropping
the complementarity constraints $y_i(1 - x_i) = 0$:

$$\begin{align*}
\min_{(x,y) \in \mathbb{R}^{2n}} & \quad a'x + b'y + y'(Q - \text{diag}(d))y + \sum_{i \in [n]} d_i \frac{y_i^2}{x_i} \\
\text{s.t.} & \quad 0 \leq x \leq 1, \ y \geq 0 \\
& \quad (x, y) \in \mathcal{X} \subseteq \mathbb{R}^n \times \mathbb{R}^n.
\end{align*}$$

(2)

In certain applications, such as the sensor placement [16], the single-period unit commitment [17] and the $\ell_2$-penalized least square regression [27], vector $d$ is immediate from the context. Thus, in such cases, (2) directly delivers a strong relaxation of (1). For cases where a decomposition of $Q$ is not immediate, several approaches are proposed in literature [e.g. 15, 32] to obtain a desirable $d$. Because different decompositions usually yield different relaxations, the relaxation quality of (2) relies on the choice of vector $d$. Introducing a symmetric matrix variable $Y$, Dong et al. [11] describe an optimal perspective relaxation for (QI):

$$\begin{align*}
\min_{(x,y) \in \mathbb{R}^{2n}} & \quad a'x + b'y + \langle Q, Y \rangle \\
\text{s.t.} & \quad Y - yy' \succeq 0 \\
& \quad y_i^2 \leq Y_{ii}x_i \quad \forall i \in [n] \\
& \quad 0 \leq x \leq 1, \ y \geq 0 \\
& \quad (x, y) \in \mathcal{X} \subseteq \mathbb{R}^n \times \mathbb{R}^n.
\end{align*}$$

(OptPersp)

(3)

Note that by adding integrality constraints on $x$ in (OptPersp), one obtains a mixed-integer SDP problem, which can be solved by a branch-and-bound algorithm. What is more, the resulting mixed-integer program is equivalent to the original model (QI). Indeed, if $x$ is integral, then (3c) either reduces to $y_i = 0$ or is implied (3b). In any case, $Y$ and $y$ are linked only through (3b), which must hold as an equality at the optimal solution because $Q \succeq 0$.

The authors [11] show that (OptPersp) is optimal in the sense that every perspective relaxation of the form (2) is dominated by (OptPersp), and moreover, there indeed exists a decomposition of $Q$ such that the resulting perspective formulation (2) is equivalent to (OptPersp).
Proposition 1 (Theorem 3 in [11]). $(\text{OptPersp})$ is equivalent to the following max-min optimization problem:

$$
\max_{d \in \mathbb{R}^n} \min_{(x,y) \in \mathbb{R}^{2n}} a'x + b'y + y'(Q - \text{diag}(d))y + \sum_{i \in [n]} d_i \frac{y_i^2}{x_i}
$$

\[\text{s.t. } d \geq 0, \ Q - \text{diag}(d) \succeq 0\]
\[0 \leq x \leq 1, \ y \geq 0\]
\[(x, y) \in X \subseteq \mathbb{R}^n \times \mathbb{R}^n.\]

Next, we consider Shor’s SDP relaxation for problem $(\text{QI})$:

$$
\min a'x + b'y + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}Z_{ij}
$$

\[\text{s.t. } y_i - Z_{i,i+n} = 0 \quad \forall i \in [n] \quad (5b)\]
\[(\text{Shor})\]
\[x_i - Z_{i+n,i+n} = 0 \quad \forall i \in [n] \quad (5c)\]
\[Z - \begin{pmatrix} y' & x' \end{pmatrix} \begin{pmatrix} y & x \end{pmatrix} \succeq 0 \quad (5d)\]
\[0 \leq x \leq 1, \ y \geq 0\]
\[(x, y) \in X' \subseteq \mathbb{R}^n \times \mathbb{R}^n, \quad (5f)\]

where $Z \in \mathbb{R}^{2n \times 2n}$ such that $Z_{ij}$ is a proxy for $y_i^2$, $Z_{i+n,i+n}$ is a proxy for $x_i^2$, $Z_{i,i+n}$ is a proxy for $x_i y_i$, $i \in [n]$, and $Z_{ij}$ is a proxy for $y_i y_j$ for $1 \leq i, j \leq n$.

It is known that $(\text{Shor})$ is at least as strong as $(\text{OptPersp})$ [12], as constraints $(5c)$ are implied by the positive definiteness of some $2 \times 2$ principal minors of $(5d)$. We show below that the two formulations are, in fact, equivalent. As $(\text{OptPersp})$ is a much smaller formulation than $(\text{Shor})$, the equivalence makes it the favorable choice.

Theorem 1. $(\text{OptPersp})$ is equivalent to $(\text{Shor})$.

Proof. First we verify $(\text{Shor})$ is at least as strong as $(\text{OptPersp})$ by checking that constraints $(3c)$–$(3d)$ are implied by $(5d)$. Let $Y_{ij} = Z_{ij}$ for any $i, j \in [n]$. By Schur Complement Lemma,

$$Z - \begin{pmatrix} y' & x' \end{pmatrix} \begin{pmatrix} y & x \end{pmatrix} \succeq 0 \iff \begin{pmatrix} 1 & y' & x' \\ y & Z \end{pmatrix} \succeq 0.
$$

Since $Y$ is a principal submatrix of $Z$, we have

$$\begin{pmatrix} 1 & y' \\ y & Y \end{pmatrix} \succeq 0 \iff Y - yy' \succeq 0.$$
Moreover, constraint (5d) also implies that for any $i \in [n]$,
\[
\begin{pmatrix}
Z_{ii} & Z_{i,i+n} \\
Z_{i,i+n} & Z_{i+n,i+n+n}
\end{pmatrix} \succeq 0.
\] (6)

After substituting $Y_{ii} = Z_{ii}$, $x_i = Z_{i+n,i+n+n}$, and $y_i = Z_{i,i+n}$, we find that (6) implies $Y_{ii}x_i \geq y_i^2$ in (OptPersp), concluding the argument.

We next prove that (OptPersp) is at least as strong as (Shor), by showing that for any given feasible solution $(x, y, Y)$ of (OptPersp), it is possible to construct a feasible solution of (Shor) with same objective value. First, we rewrite $Z$ in the form $Z = YU + V$. For a fixed feasible solution $(x, y, Y)$ of (OptPersp) consider the optimization problem
\[
\lambda^* := \min_{\lambda, U, V} \lambda
\] (7a)
\[
\text{s.t. } \begin{pmatrix}
1 & y' & x' \\
y & Y & U \\
x & U' & V
\end{pmatrix} + \lambda I \succeq 0
\] (7b)
\[
U_{ii} = y_i, \quad \forall i \in [n] \quad (7c)
\]
\[
V_{ii} = x_i, \quad \forall i \in [n], \quad (7d)
\]
where $I$ is the identity matrix. Observe that if $\lambda^* \leq 0$, then an optimal solution of (7) satisfies (5d) and thus induces a feasible solution of (Shor).

We show next that this is, indeed, the case.

Let $\tilde{Y} = \begin{pmatrix} 1 & y' \\ y & Y \end{pmatrix}$ and consider the SDP dual of (7):
\[
\lambda^* = \max_{R, s, t, z} -\langle \tilde{Y}, R \rangle - \sum_i (2x_i z_i + t_i x_i + 2s_i y_i) \] (8a)
\[
\text{s.t. } \begin{pmatrix}
R & z' \\
z & \text{diag}(s)
\end{pmatrix} \succeq 0
\] (8b)
\[
\text{Tr}(R) + \sum_i t_i = 1,
\] (8c)
where $R, z, \text{diag}(s)$, $\text{diag}(t)$ are the dual variable associated with $\tilde{Y} + \lambda I$, $x, U$, and $V + \lambda I$, respectively. Note that we abuse the symbol $I$ to represent the identity matrices of different dimensions. One can verify that the strong duality holds for (7) since $\lambda$ can be an arbitrary positive number to ensure that the matrix inequality holds strictly. Because the off-diagonal elements of $U$ and $V$ do not appear in the primal objective function and constraints other than (7d), the corresponding dual variables are zero.
Note that to show $\lambda^* \leq 0$, it is sufficient to consider a relaxation of (8a). Therefore, dropping (8c), it is sufficient to show that
\[
\langle \tilde{Y}, R \rangle + \sum_i (2x_i z_i + t_i x_i + 2s_i y_i) \geq 0
\] (9)
for all $t \geq 0, s, z, R$ satisfying (8b).

Observe that if $t_i = 0$, then $s_i = z_i = 0$ in any solution satisfying (8b). In this case, all such terms indexed by $i$ vanish in (9). Therefore, it suffices to prove (9) holds for all $t > 0$.

For $t > 0$, by Schur Complement Lemma, (8b) is equivalent to
\[
R \succeq \left( \frac{z'}{\text{diag}(s)} \right) \text{diag}^{-1}(t) \left( \frac{z}{\text{diag}(s)} \right)
\] (10)
Moreover, since $\tilde{Y} \succeq 0$, we find that
\[
\langle \tilde{Y}, R \rangle \geq \langle \tilde{Y}, \left( \frac{z'}{\text{diag}(s)} \right) \text{diag}^{-1}(t) \left( \frac{z}{\text{diag}(s)} \right) \rangle + \sum_i (2x_i z_i + t_i x_i + 2s_i y_i) \geq 0
\] (11)
holds for all $t > 0, s, z$ and $R$ satisfying (10). A direct computation shows that
\[
\left( \frac{z'}{\text{diag}(s)} \right) \text{diag}^{-1}(t) \left( \frac{z}{\text{diag}(s)} \right) = \begin{pmatrix}
\frac{\sum_i z_i^2}{t_i} & \frac{z_1 s_1}{t_1} & \cdots & \frac{z_n s_n}{t_n} \\
\frac{z_1 s_1}{t_1} & \frac{s_1^2}{t_1} & & \\
\vdots & & \ddots & \\
\frac{z_n s_n}{t_n} & & & \frac{s_n^2}{t_n}
\end{pmatrix}
\]
with all off-diagonal elements equal to 0, except for the first row/column. Thus, (11) reduces to the separable expression
\[
\sum_i \left( \frac{z_i^2}{t_i} + \frac{2z_i s_i y_i}{t_i} + \frac{s_i^2}{t_i} Y_{ii} + 2x_i z_i + x_i t_i + 2y_i s_i \right) \geq 0.
\]
For each term, we have
\[
\begin{align*}
\frac{z_i^2}{t_i} + \frac{2z_i s_i y_i}{t_i} + \frac{s_i^2}{t_i} Y_{ii} + 2x_i z_i + x_i t_i + 2y_i s_i & = (z_i^2 + 2z_i s_i y_i + s_i^2 Y_{ii})/t_i + x_i t_i + 2x_i z_i + 2y_i s_i \\
& \geq 2 \sqrt{x_i (z_i^2 + 2z_i s_i y_i + s_i^2 Y_{ii})} + 2x_i z_i + 2y_i s_i \\
& \geq 0,
\end{align*}
\]
where the first inequality follows from the inequality between the arithmetic and geometric mean $a + b \geq 2\sqrt{ab}$ for $a, b \geq 0$. The last inequality holds trivially if $2x_iz_i + 2y_is_i \geq 0$; otherwise, we have

\[
\sqrt{x_i(z_i^2 + 2z_is_iy_i + s_i^2Y_{ii})} \geq -(x_i + y_is_i)
\]
\[
\iff x_i(z_i^2 + 2z_is_iy_i + s_i^2Y_{ii}) \geq (x_i + y_is_i)^2
\]
\[
\iff x_i z_i^2(1 - x_i) + s_i^2(x_iY_{ii} - y_i^2) \geq 0. \quad (\text{as } 0 \leq x_i \leq 1 \text{ and } x_iY_{ii} \geq y_i^2)
\]

In conclusion, $\lambda^* \leq 0$ and this completes the proof.

3. Conclusion

In this short note, we prove that Shor’s SDP and optimal perspective reformulation have the equal strength for convex quadratic optimization problems with indicator variables. The smaller size of the optimal perspective makes it advantageous to solve as a relaxation compared to Shor’s SDP. For solving (QI) with integral constraints, it may be preferable to use conic quadratic formulation (2) with an optimal $d^*$, which can be retrieved as optimal dual variables of (3) with respect to constraints (3c).

Acknowledgments

Andrés Gómez is supported, in part, by grants 1930582 and 1818700 from the National Science Foundation. Alper Atamtürk is supported, in part, by NSF AI Institute for Advances in Optimization Award 2112533, NSF grant 1807260, and DOD ONR grant 12951270.

References

[1] Aktürk, M. S., Atamtürk, A., and Gürel, S. (2009). A strong conic quadratic reformulation for machine-job assignment with controllable processing times. Operations Research Letters, 37(3):187–191.
[2] Alfakih, A. Y., Khandani, A., and Wolkowicz, H. (1999). Solving euclidean distance matrix completion problems via semidefinite programming. Computational Optimization and Applications, 12(1-3):13–30.
[3] Atamtürk, A. and Gómez, A. (2018). Strong formulations for quadratic optimization with M-matrices and indicator variables. Mathematical Programming, 170(1):141–176.
[4] Atamtürk, A. and Gómez, A. (2019). Rank-one convexification for sparse regression. arXiv preprint arXiv:1901.10334.
[5] Atamtürk, A. and Gómez, A. (2020). Supermodularity and valid inequalities for quadratic optimization with indicators. arXiv preprint arXiv:2012.14633.
[6] Bach, F. (2019). Submodular functions: from discrete to continuous domains. *Mathematical Programming*, 175(1-2):419–459.

[7] Ben-Tal, A., El Ghaoui, L., and Nemirovski, A. (2009). *Robust Optimization*, volume 28. Princeton University Press.

[8] Bienstock, D. (1996). Computational study of a family of mixed-integer quadratic programming problems. *Mathematical Programming*, 74(2):121–140.

[9] Burer, S. and Ye, Y. (2019). Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs. *Mathematical Programming*, pages 1–17.

[10] Candes, E. J. and Plan, Y. (2010). Matrix completion with noise. *Proceedings of the IEEE*, 98(6):925–936.

[11] Dong, H., Chen, K., and Linderoth, J. (2015). Regularization vs. relaxation: A conic optimization perspective of statistical variable selection. *arXiv preprint arXiv:1510.06083*.

[12] Dong, H. and Linderoth, J. (2013). On valid inequalities for quadratic programming with continuous variables and binary indicators. In Goemans, M. and Correa, J., editors, *Proceedings of IPCO 2013*, page 169–180, Berlin. Springer.

[13] Fattahi, S., Ashraphijuo, M., Lavaei, J., and Atamtürk, A. (2017). Conic relaxations of the unit commitment problem. *Energy*, 134:1079–1095.

[14] Frangioni, A. and Gentile, C. (2006). Perspective cuts for a class of convex 0–1 mixed integer programs. *Mathematical Programming*, 106(2):225–236.

[15] Frangioni, A. and Gentile, C. (2007). SDP diagonalizations and perspective cuts for a class of nonseparable miqp. *Operations Research Letters*, 35:181–185.

[16] Frangioni, A., Gentile, C., Grande, E., and Pacifieri, A. (2011). Projected perspective reformulations with applications in design problems. *Operations research*, 59(5):1225–1232.

[17] Galiana, F. D., Motto, A. L., and Bouffard, F. (2003). Reconciling social welfare, agent profits, and consumer payments in electricity pools. *IEEE Transactions on Power Systems*, 18(2):452–459.

[18] Gao, J. and Li, D. (2011). Cardinality constrained linear-quadratic optimal control. *IEEE Transactions on Automatic Control*, 56(8):1936–1941.

[19] Goemans, M. X. and Williamson, D. P. (1995). Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the ACM (JACM)*, 42(6):1115–1145.

[20] Günlük, O. and Linderoth, J. (2010). Perspective reformulations of mixed integer nonlinear programs with indicator variables. *Mathematical
[21] Han, S., Gómez, A., and Atamtürk, A. (2020). 2x2-convexifications for convex quadratic optimization with indicator variables. arXiv preprint arXiv:2004.07448.

[22] Hijazi, H., Bonami, P., Cornuéjols, G., and Ouorou, A. (2012). Mixed-integer nonlinear programs featuring “on/off” constraints. Computational Optimization and Applications, 52:537–558.

[23] Ho-Nguyen, N. and Kilınç-Karzan, F. (2017). A second-order cone based approach for solving the trust-region subproblem and its variants. SIAM Journal on Optimization, 27(3):1485–1512.

[24] Hochbaum, D. S. (2001). An efficient algorithm for image segmentation, Markov random fields and related problems. Journal of the ACM, 48:686–701.

[25] Javanmard, A., Montanari, A., and Ricci-Tersenghi, F. (2016). Phase transitions in semidefinite relaxations. Proceedings of the National Academy of Sciences, 113(16):E2218–E2223.

[26] Jeyakumar, V. and Li, G. (2014). Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. Mathematical Programming, 147(1-2):171–206.

[27] Pilanci, P., Wainwright, M. J., and El Ghaoui, L. (2015). Sparse learning via boolean relaxations. Mathematical Programming, 151:63–87.

[28] Shor, N. Z. (1987). Quadratic optimization problems. Soviet Journal of Computer and Systems Sciences, 25:1–11.

[29] Wang, A. L. and Kilınç-Karzan, F. (2020). The generalized trust region subproblem: solution complexity and convex hull results. Mathematical Programming, pages 1–42.

[30] Wang, A. L. and Kilınç-Karzan, F. (2021). On the tightness of SDP relaxations of QCQPs. Mathematical Programming, pages 1–41.

[31] Wu, B., Sun, X., Li, D., and Zheng, X. (2017). Quadratic convex reformulations for semicontinuous quadratic programming. SIAM Journal on Optimization, 27:1531–1553.

[32] Zheng, X., Sun, X., and Li, D. (2014). Improving the performance of miqp solvers for quadratic programs with cardinality and minimum threshold constraints: A semidefinite program approach. INFORMS Journal on Computing, 26(4):690–703.