Quasi-inertial ellipsoidal flows in relativistic hydrodynamics

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We search for non-trivial relativistic solutions of the hydrodynamic equations with quasi-inertial flows such as in the Bjorken-like models. The problem is analyzed in general and the known results are reproduced by a method proposed. A new class of 3D anisotropic analytic solutions with quasi-inertial property is found. An ellipsoidal generalization of the spherically symmetric Hubble flow with constant pressure is proposed as a particular case. The relativistic expansion of finite systems into vacuum is also described within this class. A region of applicability and possible utilization of the new solutions for processes of A+A collisions is discussed.

\textbf{I. INTRODUCTION}

The equations of the relativistic hydrodynamics have highly nonlinear nature and, therefore, only a few analytical solutions are known until now. For the first time one-dimensional, or (1+1), analytical solution for Landau initial conditions - hot pion gas in Lorentz contracted thin disk \cite{1}, has been developed by Khalatnikov \cite{2}. The equation of state (EoS) was chosen as ultrarelativistic one: \( p = c_s^2 \varepsilon, c_s^2 = 1/3 \). It is noteworthy that according to that solution the longitudinal flows developed to the end of hydrodynamic expansion, at freeze-out, are quasi-inertial: \( v \approx x_L/t \). Much later, in the papers \cite{3} for the same EoS have been found the infinite (1+1) boost-invariant solution, for finite systems the similar approach was developed in Ref.\cite{4}. The property of quasi-inertia preserves in these solutions during the \textit{whole} stage of the evolution. Bjorken \cite{5} utilized these solution as the basis of the hydrodynamic model of ultra-relativistic A+A collisions.

The spherically symmetric variant of a such kind of flows with the Hubble velocity distribution, \( v = r/t \), has been considered in Ref. \cite{6}. Some generalization of these results was proposed in a case of the Hubble flow for EoS of massive gas with conserved particle number in Ref. \cite{7} and for the cylindrically symmetric boost-invariant expansion with a constant pressure in \cite{8}.

All these solutions were used for an analysis of ultra-relativistic heavy ion collisions. Since the longitudinal boost-invariance in a fairly wide rapidity region is not observed even at RHIC, as it was expected, the Hubble-like models are also used now for a description of the experimental data \cite{9}. It is naturally, however, that, unlike to the Hubble type flows, the velocity gradients should be different in different directions since there is an initial asymmetry between longitudinal and transverse directions in central A+A collisions and, in addition, between in-plane and off-plane transverse ones in non-central collisions. In this letter we make a general analysis of the hydrodynamic equations for the quasi-inertial flows aiming to find a new class of analytical solutions with 3D asymmetric relativistic flows.

\textbf{II. GENERAL ANALYSIS}

Let us start from the equations of relativistic hydrodynamics:

\[
\partial_{\nu} T^{\mu\nu} = 0, \tag{1}
\]

where the energy-momentum tensor corresponds to a perfect fluid:

\[
T^{\mu\nu} = (\varepsilon + p) u^{\mu} u^{\nu} - p \cdot g^{\mu\nu} \tag{2}
\]

We can attempt to find a particular class of solutions and therefore have to make some simplifications of \eqref{1}. We do not fix EoS at this stage.

\begin{itemize}
  \item Let us put the condition of quasi-inertiality
    \[
    u^{\nu} \partial_{\nu} u^{\mu} = 0 \tag{3}
    \]

    which means that flow is accelerationless in the rest systems of each fluid element; this property holds for the known Bjorken (boost-invariant) and Hubble flows.

    Then, we find that \( u^{\mu}[(\varepsilon + p)\partial_{\nu} u^{\nu} + u^{\nu} \partial_{\nu} \varepsilon] + [u^{\mu} u^{\nu} \partial_{\nu} p - \partial^{\mu} p] = 0 \). Contracting this equation with \( u_{\mu} \) we get

    \[
    (\varepsilon + p)\partial_{\nu} u^{\nu} + u^{\nu} \partial_{\nu} \varepsilon = 0. \tag{4}
    \]

    Obviously, the remaining equation to satisfy is:

    \[
    u^{\mu} u^{\nu} \partial_{\nu} p - \partial^{\mu} p = 0 \tag{5}
    \]

    The task is to find solution of the system \eqref{4} and \eqref{5}. As one can see, the number of equations exceeds the number of independent variables. So, the equations must be self-consistent in order to have nontrivial solutions.

    We see that Eq.\eqref{4} can be rewritten in the form:

    \[
    u^{\mu} \partial_{\mu} \varepsilon = -F(\varepsilon)(\partial_{\nu} u^{\nu}) \tag{6}
    \]

    where \( F(\varepsilon) = \varepsilon + p \), and Eq.\eqref{5} as the following:

    \[
    p'(\varepsilon)(u^{\mu} u^{\nu} \partial_{\nu} \varepsilon - \partial^{\mu} \varepsilon) = 0, \tag{7}
    \]

    supposing EoS in the form \( p = p(\varepsilon) \). If \( p'(\varepsilon) \neq 0 \)

    \[
    u^{\mu} F(\varepsilon)(\partial_{\nu} u^{\nu}) + \partial^{\mu} \varepsilon = 0. \tag{8}
    \]
Normally $F(\varepsilon) \neq 0$, and we can divide the last equation by $F(\varepsilon)$ and introduce the function $\Phi(\varepsilon)$ by the definition 
\[
\frac{1}{\varepsilon + m(\varepsilon)} = \Phi'(\varepsilon),
\]
so that
\[
\partial^\mu \Phi(\varepsilon) = -u^\mu (\partial_\nu u^\nu) \quad (9)
\]

Then, the conditions of consistency of equations (11) and (12) can be written as:
\[
\partial^\lambda (u^\mu \partial_\nu u^\nu) = \partial^\mu (u^\lambda \partial_\nu u^\nu) \quad (10)
\]

In general case, there are 6 independent equations.

Finally, the relativistic hydrodynamics of quasi-inertial flows is described by the equations (4) and (10) for the hydrodynamic velocities $u^\mu$, and the equations (9) for the energy density: one should use derivative 
\[
\frac{1}{\varepsilon + m(\varepsilon)} = \Phi'(\varepsilon)
\]
at any EoS $p = p(\varepsilon)$ to find function $\varepsilon(x)$. A serious problem is, however, to find non-trivial solutions for the field $u^\mu(x)$ of hydrodynamic 4-velocities.

III. GRADIENT-LIKE VELOCITY ANSATZ

One can try to satisfy to Eqs. (3), (10) for velocity profile by a use of gradient-like representation for it, namely,
\[
\begin{align*}
    u^\mu &= \partial^\mu \phi, \quad (11) \\
    \partial_\mu \phi \partial^\mu \phi &= 1 \quad (12)
\end{align*}
\]

Then one can check that (13) is satisfied automatically, and (10) leads to:
\[
\partial^\lambda (\partial^\mu \phi \cdot \square \phi) = \partial^\mu (\partial^\lambda \phi \cdot \square \phi) \quad (13)
\]

Thus, gradient-like velocity ansatz (11) reduce the problem to equations (12), (13).

One can see that the above equation can be, in particular, reduced to:
\[
\square \phi = F(\phi) \quad (14)
\]

with any real function $F$ that have to be solved together with (12). Note that if $F(\phi) = a + b \phi$ then (13) is the linear inhomogeneous partial differential equations (PDE) and its any solution is a partial solution $\phi_{ih}$ of inhomogeneous PDE, plus general solution $\phi_h$ of correspondent homogenous PDE (if $b \neq 0$):
\[
\begin{align*}
    \phi_{ih} &\sim (t^2 - x^2), \quad b = 0 \\
    \phi_{ih} &\sim \frac{x^2}{t}, \quad b \neq 0
\end{align*}
\]

and
\[
\phi_h = \int d^4 p \delta(p^2 - b) f(p) e^{ipx}
\]

where $f(p)$ is arbitrary function with properties $f^*(k) = f(-k)$. Then the problem is reduced to a solution of the

nonlinear integral equation (12) for $f(p)$. If $a = b = 0$, the only potential $\phi = c + b_0 t + \sum c_i x_i (i = 1, 2, 3)$ with the constrain on the constants $c_i$ of $\sum c_i^2 = 1$ is satisfied to these equations. It describes a relativistic motion of a medium as the whole. It is an open problem whether there are analytical solutions at $a \neq 0$ and/or $b \neq 0$.

The known quasi-inertial solutions correspond to $F(\phi) = n/\phi$ in Eq. (13). The value $n = 1$ generates gradient ansatz $\phi = \sqrt{t^2 - x^2 - \varepsilon^2}$ that gives the (1+1) boost-invariant Bjorken expansion along axis $z$, $v = z/t$; $n = 2$ leads to $\phi = \sqrt{t^2 - x^2 - y^2}$, and, correspondingly, to the two-dimensional (1+2) Hubble-like flow with cylindrical symmetry; at $n = 3$ one can get solution of (13) for $\phi$ in the form $\phi = \tau \pm \sqrt{t^2 - x^2 - y^2 - z^2}$ describing spherically symmetric Hubble flow $u^\mu = x^\mu/\tau$. The equation (14) has the form $\partial^\mu \Phi(\varepsilon) = n u^\mu / \tau^2$ where number of space coordinates is equal to $n$. Then the energy density is described by the following expression
\[
\frac{\varepsilon}{\varepsilon + p(\varepsilon)} = \ln(\frac{T_0}{\tau})^n. \quad (17)
\]

IV. RELATIVISTIC ELLIPSOIDAL SOLUTIONS

One more possibility to satisfy to Eqs. (5) or (17) besides of the gradient-like flows is to suppose a constant pressure in the EoS: $p(\varepsilon) = const$. This possibility was first used in (8) as physically corresponding to a thermodynamic state of the system in the softest point with the velocity of sound $c_s^2 = 0$. Such a state could be associated with the first order phase transition. In A+A collisions it corresponds, probably, to transition between hadron and quark-gluon matter at SPS energies. The solution proposed in (8) has the cylindrical symmetry in the transverse plane and the longitudinal boost invariance:
\[
\begin{align*}
    u_\mu &= \gamma(\frac{t}{r}, \frac{x}{r}, \frac{y}{r}, \frac{z}{\tau}), \\
    \gamma &= (1 - v^2)^{-1/2}
\end{align*}
\]

with transverse radius, $r = \sqrt{x^2 + y^2}$,
\[
v = \frac{\alpha}{1 + \alpha \tau}
\]
describes axially symmetric transverse flow.

The above solution has, however, a limited region of applicability since the boost invariance is not expected at SPS energies and can be used only in a small mid-rapidity interval (11), it is not reached even at RHIC energies (11). Most important, however, is that in non-central collisions there is no axial symmetry and, therefore, one needs in transversely asymmetric solutions to describe the elliptic flows in these collisions, e.g., $v_2$ coefficients. Now we propose a new class of analytic solutions of the relativistic hydrodynamics for 3D asymmetric flows.
First we construct the ansatz for normalized 4-velocity:

\[ u^\mu = \left( \frac{t}{\sqrt{t^2 - \sum a_i^2(t)x_i^2}}, \frac{a_k(t)x_k}{\sqrt{t^2 - \sum a_i^2(t)x_i^2}} \right) \]

where the Latin indexes denote spatial coordinates and \( a_i \) are functions of time only. In this case a set of nonequal \( a_i \) induces 3D elliptic flow with velocities \( u_i = a_i(t)x_i/t \): at any time \( t \) the absolute value of velocity is constant, \( v^2 = \text{const} \), at an elliptic surface \( \sum a_i^2 x_i^2 = \text{const} \). Note that this solution is not gradient-like, so we follow in the specific way of a further analysis starting from \( \text{Eq. 3} \).

The condition \( \text{Eq. 3} \) of accelerationlessness in this case is reduced to the ordinary differential equation (ODE) for the functions \( a_i(t) \):

\[ \frac{da_i}{dt} = \frac{a_i - a_i^2}{t}, \]

the general solution of which is:

\[ a_i(t) = \frac{t}{t + T_i}, \]

where \( T_i \) is some set of 3 parameters (integration constants) having the dimension of time. The different values \( T_1, T_2 \) and \( T_3 \) results in anisotropic 3D expansion with the elliptic flows.

The equation \( \text{Eq. 1} \) is satisfied since we assume the constant pressure profile: \( p = p_0 = \text{const} \). The next step is to find solution of Eq. \( \text{1} \) for energy density \( \varepsilon \). Taking into account that \( \partial_\mu u^\mu = \sum a_i/\tilde{\tau} \), where

\[ \tilde{\tau} = \sqrt{t^2 - \sum a_i^2 x_i^2}, \]

one can get

\[ (\varepsilon + p_0) \sum a_i(t) + t \partial_t \varepsilon + \sum a_i(t) x_i \partial_i \varepsilon = 0. \]

General solution of the equation is

\[ \varepsilon + p_0 = \frac{F_\varepsilon(\frac{x_1}{t + T_1}, \frac{x_2}{t + T_2}, \frac{x_3}{t + T_3})}{(t + T_1)(t + T_2)(t + T_3)} \]

where \( F_\varepsilon \) is an arbitrary function of its variables. If one fixes the parameters \( T_i \) that define the velocity profile, then the function \( F_\varepsilon \) is completely determined by the initial conditions for the enthalpy profile, say, at the initial time \( t = 0: \varepsilon(t = 0, x) + p_0 = F_\varepsilon(\frac{x_1}{T_1}, \frac{x_2}{T_2}, \frac{x_3}{T_3})/T_1T_2T_3 \).

If some value, associated with a quantum number or with particle number in a case of chemically frozen evolution are conserved \( \text{10} \) then one should add the corresponding equation to the basic ones. Such an equation has the standard form \( \text{12} \):

\[ n \partial_\mu u^\mu + u^\mu \partial_\mu n = 0 \]

where \( n \) is associated with density of the correspondent conserved value, e.g., with the baryon or particle densities. A general structure of this equation is similar to what Eq. \( \text{21} \) has and, therefore, the solution looks like as \( \text{25} \):

\[ n = \frac{F_n(\frac{x_1}{t + T_1}, \frac{x_2}{t + T_2}, \frac{x_3}{t + T_3})}{(t + T_1)(t + T_2)(t + T_3)} \]

where the function \( F_n \) is an arbitrary function of its arguments and can be fixed by the initial conditions for (particle) density \( n: n(t = 0, x)T_iT_jT_3 = F_n(\frac{x_1}{T_1}, \frac{x_2}{T_2}, \frac{x_3}{T_3}) \).

To establish a behavior of other thermodynamic values we use link between different thermodynamic potentials \( \varepsilon = T_s - \mu + \mu n \) and utilize the thermodynamic equations based on the free energy density \( f(n, T) = \varepsilon - TS - \mu n - \mu p \). Since the volume is fixed (it is unit) the free energy depends on \( T \) and \( n \) only, \( df = -sdT + \mu dn \), and the chemical potential \( \mu = f_{,n}/T = \text{const} \) and the entropy density \( s = -f_{,T|n=\text{const}} \).

In a case of chemically equilibrated expansion of the ultrarelativistic gas when the particle number is uncertain and is defined by the conditions and parameters of the thermodynamic equilibrium, e.g., by the temperature \( T \), the chemical potential \( \mu \equiv 0 \) (we suppose here that there is no other conserved values associated with charges, or the corresponding chemical potentials are zero close to zero). Then \( f = -p_0 = \text{const}, df = -sdT = 0 = 0 \) that means the temperature \( T = \text{const} \) for such a system and the entropy \( s = (\varepsilon(t, x) + p_0)/T \) where \( \varepsilon(t, x) \) is defined by \( \text{25} \).

If chemically frozen evolution takes place, the chemical potential associated with conserved particle number is not zero and describes the deviation from chemical equilibrium in relativistic systems. The solution of differential equation \( n f_{,n|T=\text{const}} = -p_0 = f(n, T) = nc(T) - p_0 \), where \( c(T) \) is some function of the temperature. Then it follows directly from the thermodynamic identities that

\[ \varepsilon(t, x) + p_0 = n(t, x)(c(T) - Tc'(T)) \]

Since the structures of general solutions for \( n \) and \( \varepsilon \) are found as \( \text{20} \) and \( \text{21} \), the temperature profile has the form

\[ T(t, x) = F_T(\frac{x_1}{t + T_1}, \frac{x_2}{t + T_2}, \frac{x_3}{t + T_3}) \]

where \( F_T \) is some function of its arguments that is defined by the initial conditions for \( \varepsilon \) and \( n \) and also by EoS \( \varepsilon = \varepsilon(n, T) \). The latter can be fixed by a choice of the function \( c(T) \) in Eq. \( \text{28} \). If the initial enthalpy density profile is proportional to the particle density profile, \( F_n(\frac{x_1}{T_1}, \frac{x_2}{T_2}, \frac{x_3}{T_3}) \sim F_\varepsilon(\frac{x_1}{T_1}, \frac{x_2}{T_2}, \frac{x_3}{T_3}) \), then \( T = \text{const} \) (and so \( \mu = c(T) = \text{const} \)) during the system’s evolution for any function \( c(T) \) except the linear one: \( c(T) = a - bT \) when \( T \) is not defined by the equation \( \text{28} \). In the last case \( n = (\varepsilon + p_0)/\mu, s = bn/a \). In another particular case which corresponds to EoS \( \varepsilon + p_0 = anT \) with \( c(T) = -aT \ln(bT) \) one can get:

\[ T(t, x) = (\varepsilon + p_0)/(an), s(t, x) = an(\ln(bT) + 1) \]
V. GENERALIZATION OF THE HUBBLE-LIKE FLOWS

Let us describe some important particular solutions of the equations for relativistic ellipsoidal flows. If one defines the initial conditions on the hypersurface of constant time, say \( t = 0 \), then \( t \) is a natural parameter of the evolution. Such a representation of the solutions similar to the Bjorken and Hubble ones with velocity field \( v_i = a_i x_i / t \) has property of an infinite velocity increase at \( x \to \infty \). A real fluid, therefore, can occupy only the space-time region where \( |v| < 1 \), or \( \bar{v}^2 > 0 \). To guarantee the energy-momentum conservation of the system during the evolution, all thermodynamic densities have to be zero at the boundary of the physical region, otherwise one should consider the boundary as the massive shell \( \Sigma \). Hence in the standard hydrodynamic approach the enthalpy and particle density must be zero at the surface defined by \( |v(t, x)| = 1 \) at any time \( t \). One of a simple form of such a solution (for the case of particle number conservation) can be obtained from (25), (27) by choosing \( F_{n, n} \sim \exp(-b_n^2 \bar{v}_n^2) \):

\[
\varepsilon(t) + p_0 = \frac{C_z}{\prod_i (t + T_i)} \exp(-b_z^2 \bar{v}_z^2),
\]

\[
n = \frac{C_n}{\prod_i (t + T_i)} \exp(-b_n^2 \bar{v}_n^2),
\]

where \( \bar{v} \) is defined by (28), and the constants \( C_z, C_n, b_z, b_n \) are determined by the initial conditions as described in the previous section. As one can see, the enthalpy density tends to zero when \( |x| \) becomes fairly large approaching the boundary surface defined by \( |v(t, x)| = 1 \), in the other words, when \( \bar{v} \to 0 \). Thus the physically inconsistent situation when massive fluid elements move with the velocity of light at the surface \( \bar{v} = 0 \) is avoided. Of course, in such a solution one has to put a constant pressure to be zero, \( p_0 = 0 \).

As it follows from an analysis of a behavior of the thermodynamic values in the previous section, the temperature is constant if \( b_z = b_n \), otherwise one can choose the temperature approaching zero at the system’s boundary, e.g., for EoS which is linear in temperature, the latter has the form

\[
T = \text{const} \quad b_z = b_n
\]

\[
T \sim e^{-(b_z^2 - b_n^2) \bar{v}_z^2} \to 0, \quad |v(x)| \to 1 \quad b_z > b_n
\]

according to (29).

Note that in the region of non-relativistic velocities, \( v^2 = \sum a_i^2 x_i^2 \ll 1 \) the space distributions of the thermodynamical quantities (31), (32) has the Gaussian profile:

\[
\varepsilon + p_0 \approx \frac{C_z}{\prod_i (t + T_i)} e^{-b_z^2 \sum a_i^2 \bar{v}_z^2},
\]

\[
n \approx \frac{C_n}{\prod_i (t + T_i)} e^{-b_n^2 \sum a_i^2 \bar{v}_n^2}.
\]

The forms of solutions (33) are similar to what was found in Ref. [31] as the elliptic solutions of the non-relativistic hydrodynamics equations. In this sense the solution proposed could be considered as the generalization (at vanishing pressure) of the corresponding non-relativistic solutions allowing one to describe relativistic expansion of the finite system into vacuum.

One can note that the case of equal flow parameters \( T_i = 0 \) and \( b_z = b_n = 0 \) induces formally Hubble-like velocity profile with the behavior of the density and enthalpy similar to (17) at \( n=3, p = p_0 \), and with the substitution \( \tau \to t \).

The direct physical generalization of the Hubble solution for asymmetric case should be associated with the hypersurfaces of the pseudo-proper time \( \bar{\tau} \) rather than with time \( t \), that eliminates the problem of infinite velocities: \( v^2 = \sum a_i^2 x_i^2 \ll 1 \) at any hypersurface \( \bar{\tau}^2 = \text{const} > 0 \). It can be reached if one chooses the function \( F_z \) and \( F_n \) in (25), (27) in the form

\[
F = \left( \frac{t}{\bar{\tau}} \right)^3.
\]

Then the generalized Hubble solution is

\[
u^\mu = \left\{ \frac{t}{\bar{\tau}}, \frac{a_1 x_1}{\bar{\tau}}, \frac{a_2 x_2}{\bar{\tau}}, \frac{a_3 x_3}{\bar{\tau}} \right\},
\]

\[
\varepsilon + p_0 = \frac{C_z}{\prod_i (t + T_i)} \left( \frac{a_1 a_2 a_3}{\bar{\tau}^3} \right),
\]

\[
n = \frac{C_n}{\prod_i (t + T_i)} \left( \frac{a_1 a_2 a_3}{\bar{\tau}^3} \right),
\]

where \( \bar{\tau} = \sqrt{t^2 - \sum a_i^2 x_i^2} \), \( a_i \equiv a_i(t) = t / (t + T_i) \) and constants are: \( C_z = T_1 T_2 T_3 (\varepsilon(0, 0, 0) + p_0), C_n = T_1 T_2 T_3 n(0, 0, 0) \). Again, the proportionality between \( \varepsilon \) and \( n \) results in the temperature to be a constant during the evolution. If all parameters \( T_i \) are equal to each other, then \( a_i \) are also equal and solution (34) just corresponds to spherically symmetric Hubble flow (at constant pressure) and \( \bar{\tau} \) is the proper time of fluid element, \( \bar{\tau} = \tau \). Note that comparing to the standard representation of the Hubble solution the origin of a time scale is shifted, \( t \to t + T_i \), and therefore the singularity at \( t = 0 \) is absent. Thus, if this solution is applied to a description of heavy ion collision, \( T_i \) should be interpreted as the initial proper time of thermalization and hydrodynamic expansion to which the origin of a time scale is shifted, typically \( \tau_0 = T_i \sim 1 \text{fm}/c \). As to a general case of an asymmetric expansion, the minimal parameter \( T_i \) can be considered as the initial time \( t \) (at \( x = 0 \)) of the beginning of the hydrodynamic evolution. In analogy with the Hubble flow the initial conditions in asymmetric case (35) can be ascribed to the hypersurface \( \sigma : \bar{\tau} = \text{const} \) so that \( t_\sigma(x = 0) = 0 \). Note, that such a hypersurface at \( |x| \to \infty \) tends to the hyperbolical hypersurface \( \tau = \text{const} \) since \( t_\sigma(x) \to \infty \) in this limit and so all \( a_i \to 1 \).

The boost-invariant (1+1) solutions are also contained in general ellipsoidal solutions (25), (27) for quasi-inertial flow. To get it one has to choose functions \( F_z \) and \( F_n \).
in the form with another power: 3 → 1; it leads to the same form of solution as with replacement \( \tau^3 \rightarrow \tau^1 \). The next step is to suppress the transverse flow by setting \( T_1 \rightarrow \infty, T_2 \rightarrow \infty \) (as usual, \( x_1 \) and \( x_2 \) denotes coordinates in the transverse plane and \( x_3 \) is the longitudinal axis), while the parameter \( T_3 \) is finite. This limit approach gives us \( a_1 = a_2 = 0 \) and \( \tau \rightarrow \tau = \sqrt{(t + T_3)^2 - x_3^2} \) and results directly in the Bjorken solution at \( T_3 = 0 \). Since \( T_t \) is a shift of a time scale to the beginning of hydrodynamic expansion, it is naturally to consider \( T_3 \neq 0 \) as this was discussed above for the Hubble-like solution. This value transforms as \( T_3 \rightarrow T_3' = T_3 / \gamma \) at Lorentz boosts along axis \( x_3 \).

Note, that if one does not change the power 3 → 1 in (36), the particle density (and enthalpy) behavior will differ from the boost-invariant one as the following:

\[
n \sim \tau^{-1} \rightarrow n \sim \tau^{-1} \left( 1 - \frac{x_3^2}{(t + T_3)^2} \right)^{-1} \quad (37)
\]

It is also the solution of (1+1) relativistic hydrodynamics at \( p = const \) but it obviously violates the boost-invariance: the particle and energy densities are not constants at any hypersurface and their analytic forms are changed in new coordinates after Lorentz boosts.

It is worthily to emphasize that the physical solutions with non-zero constant pressure have a limited region of applicability in time-like direction: if one wants to continue the solutions to asymptotically large times, then \( (\epsilon + p_0)_{t \rightarrow \infty} \approx \frac{c^2}{\gamma} \rightarrow 0 \), and this results in non-physical asymptotical behavior \( \epsilon \rightarrow -p_0 \), unless we set \( p_0 = 0 \). Therefore, it is naturally to utilize such kind of solutions in a region of the first order phase transition, characterized by the constant temperature and soft EoS, \( c^2_s = \partial p / \partial \epsilon \approx 0 \), or at the final stage of the evolution that always corresponds to the quasi-inertial flows.

VI. CONCLUSIONS

A general analysis of quasi-inertial flows in the relativistic hydrodynamics is done. The known analytical solutions, like the Hubble and Bjorken ones, are reproduced from approach developed. A new class of analytic solutions for 3D relativistic expansion with anisotropic flows is found. The ellipsoidal generalization of the spherically symmetric Hubble flow is considered within this class. These solutions can also describe the relativistic expansion of the finite systems into vacuum. They can be utilized for a description of the matter evolution in central and non-central ultra-relativistic heavy ion collisions, especially during deconfinement phase transition and the final stage of evolution of hadron systems. Also, the solutions can serve as a test for numerical codes describing 3D asymmetric flows in the relativistic hydrodynamics.

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