Unitary Quantum Field Theory on the Noncommutative Minkowski space

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Abstract: This is the written version of a talk I gave at the 35th Symposium Ahrensboop in Berlin, Germany, August 2002. It is an exposition of joint work with S. Doplicher, K. Fredenhagen, and Gh. Piacitelli. The violation of unitarity found in quantum field theory on noncommutative spacetimes in the context of the so-called modified Feynman rules is linked to the notion of time ordering implcitely used in the assumption that perturbation theory may be done in terms of Feynman propagators. Two alternative approaches which do not entail a violation of unitarity are sketched. An outlook upon our more recent work is given.

1 Introduction

Noncommutative spacetimes are studied for various reasons, one of them being the Gedankenexperiment that Heisenberg’s uncertainty relation along with the laws of classical gravity leads to a restriction as to the best possible localization of an event in spacetime. The idea is that the simultaneous measurement of two or more spacetime directions with an arbitrarily high precision would require an arbitrarily high energy which in the end would result in forming a horizon, cf. for instance [2]. Another motivation is based on string theory [3].

The mathematical model on which our analysis is founded was defined in [2], where continuous spacetime is replaced by a noncommutative *-algebra \( \mathcal{E} \) generated by Hermitean noncommutative coordinate-operators \( q_0, \ldots, q_3 \) with \( [q^\mu, q^\nu] = iQ^{\mu\nu} \) subject to “quantum conditions”,

\[
Q_{\mu\nu}Q^{\mu\nu} = 0, \quad \left( \frac{1}{4} Q_{\mu\nu} Q_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \right)^2 = \lambda_P^8 I, \quad [q_\rho, Q_{\mu\nu}] = 0
\]

where \( \lambda_P \) is the Planck length. The quantum conditions are Poincaré invariant, and entail that for any state \( \omega \) in the domain of the \( [q_\mu, q_\nu] \), the uncertainties \( \Delta_\omega q_\mu = \)}
\[ \sqrt{\omega(q^2) - \omega(q_n)^2} \] fulfill the following space-time uncertainty relations \[2\],

\[ \Delta q_0 \cdot (\Delta q_1 + \Delta q_2 + \Delta q_3) \geq \lambda_P, \quad \Delta q_1 \cdot \Delta q_2 + \Delta q_1 \cdot \Delta q_3 + \Delta q_2 \cdot \Delta q_3 \geq \lambda_P^2 \] (2)

Note that while it is obvious that for spacelike noncommutativity the first of the above uncertainty relations is trivial, it is not clear whether for lightlike noncommutativity \[3\], where \( Q_{\mu\nu} Q^{\mu\nu} = Q_{\mu\nu} Q_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = 0 \), such uncertainty relations hold at all, since in the above framework both right hand sides would be zero, cf. \[3\] p.199.

The regular realizations of the quantum conditions lead to the Weyl-Wigner calculus of ordinary quantum mechanics. In particular, the product in \( \mathcal{E} \) is given by the twisted convolution,

\[ f(q) g(q) = \int d^4k \, d^4l \, \tilde{f}(k) \tilde{g}(l) \, e^{-\frac{i}{2}k_0l_0} \, e^{i(k_\mu + l_\mu)q^\mu} = \int d^4k \, e^{ik_\mu q^\mu} \, (f \ast g) \hat{}(k) \] (3)

with \( f \in \mathcal{F} L^1(\mathbb{R}^4) \), \( \tilde{f} = \mathcal{F}^{-1} f \), where \( \mathcal{F} \) is the ordinary Fourier transform, and

\[ f \ast g(x) = \int dx_1 dx_2 \, e^{2i(x - x_1)Q^{-1}(x_2 - x)} \, f(x_1) \, g(x_2) \] (4)

In the literature \[3\] is often thought of as being defined by the inverse Fourier transform of the Moyal star product instead of the product \( \hat{} \).

It should also be stressed that in the approach followed here, \( Q \) is not a fixed matrix, but that by the quantum conditions, the joint spectrum \( \Sigma \) of the operators \( Q_{\mu\nu} \) is homeomorphic to the non-compact manifold \( T S^2 \times \{1, -1\} \) and that \( \mathcal{E} \) is a trivial bundle over \( \Sigma \). The full Poincaré group acts as automorphisms on \( \mathcal{E} \), and derivatives may be defined as the infinitesimal generators of translations. Note also that the evaluation in a point \( f(q) \rightarrow f(a) \) is not a positive functional on \( \mathcal{E} \). For all details see \[2\].

2 Perturbation Theory

In \[2\] the free field on \( \mathcal{E} \) was formally defined as

\[ \phi(q) := \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2k_0} \, (a(k) \otimes e^{-ikq} + a^*(k) \otimes e^{ikq}) \bigg|_{k_0 = \sqrt{k^2 + m^2}} = \int d^4k \, \hat{\phi}(k) \, e^{ikq} \] (5)

where \( \phi(q + x) := U(x) \phi(q) U(x)^{-1} \) is to be understood as an operator valued distribution, \( f \mapsto \int dx \, f(x) \phi(q + x) \). From \[3\] we deduce that products of fields are nonlocal, rendering the perturbative definition of an interacting quantum field theory difficult, since it is for instance far from obvious how the time ordering or time-zero fields should be defined. Moreover, no equivalent of Osterwalder-Schrader-positivity has yet been proved, and it is unclear how a Euclidean version of the theory could be related to a quantum field theory in the Minkowski regime. Let us now consider the so-called modified Feynman rules \[3\] as one possible approach to treating interactions on noncommutative spacetimes.

2.1 Modified Feynman rules

Starting point of this approach is the action functional

\[ S[\phi] = \int d^4q \left( \mathcal{L}_0(q) + \lambda \phi(q)^n \right) = \int d^4x \left( \mathcal{L}_0(x) + \lambda \phi \ast \ldots \ast \phi(x) \right) \] (6)

\[ \text{In fact, since I gave this talk, we have found that the Euclidean approach and the one on Minkowski space cannot be easily related as certain tadpoles which are finite in the Euclidean regime cease to be so on Minkowski space.} \]
where $\int d^4q$ is the trace on $\mathcal{E}$. In the literature, the twisted convolution (or the Moyal star product) is often taken at a particular point $\sigma \in \Sigma$, and thereby Lorentz invariance is broken explicitly.

Since $\int d^4q f(q)g(q) = \int d^4x f(x)g(x)$, the free action is the same as in ordinary quantum field theory, whereas the interaction is nonlocal, and given by

$$\lambda \int d^4k_1 \ldots d^4k_n \hat{\phi}(k_1) \ldots \hat{\phi}(k_n) e^{-\frac{i}{2} \sum_{i<j} k_i \sigma k_j} \delta^{(4)}(\sum k_i)$$

It is therefore tempting to assume that the ordinary perturbative setup can be used, where Feynman propagators serve as internal lines. The nonlocality of the interaction is then taken into account by simply adding adequate twisting factors $\exp \left(-\frac{i}{2} \sum_{i<j} k_i \sigma k_j\right)$ at the vertices.

Unfortunately, as was shown in [4], this setup leads to a violation of unitarity, since in $\phi^3$-selfinteracting theory at second order perturbation theory the optical theorem does not hold, i.e. $2 \text{Im} \quad \neq | \quad |^2$, unless spacelike or lightlike noncommutativity is assumed.

In [1] we have linked this phenomenon to the definition of time-ordering by comparing the modified Feynman rules with two alternative approaches which independently of the chosen quantum conditions do not entail a violation of unitarity.

### 2.2 Hamiltonian approach

The first of these approaches was already given in [2] and is based on the introduction of a Hamiltonian,

$$H(t) = \int_{q_0=t} d^3\vec{q} \mathcal{H}(q) = \int_{x_0=t} d^3\vec{x} \left( \mathcal{H}_0(x) + \mathcal{H}_1(x) \right)$$

where $\int_{q_0=t} d^3\vec{q}$ is defined as a positive weight on $\mathcal{E}$, $\mathcal{H}_0(x)$ is the ordinary free Hamiltonian and the interaction Hamiltonian is given in terms of the twisted convolution, $\mathcal{H}_I(x) = \lambda : \phi \ast \cdots \ast \phi(x) :$. As in ordinary field theory, the corresponding $S$-Matrix is then defined as a formal power series in the coupling constant,

$$S = I + \sum_{r=1}^{\infty} \frac{(-i)^r}{r!} \int dt_1 \ldots dt_r \mathbf{T} H(t_1) \ldots H(t_r)$$

where the time ordering $\mathbf{T} H(t_1) \ldots H(t_r)$ is defined with respect to the parameter times $t_1, \ldots, t_r$, and therefore separated from the nonlocal products. Since the Hamiltonian is symmetric, i.e. $H(t)^* = H(t)$, the $S$-matrix is obviously unitary (up to possible violations arising in the renormalization procedure).

The explicit difference between this approach and the modified Feynman rules was sketched in [4]. For instance, at second order in $\phi^3$-interaction, the Hamiltonian approach would yield the following contribution to the $S$-matrix,

$$\int dt_1 dt_2 \theta(t_1 - t_2) \int d^3\vec{x} \int d^3\vec{y} : \phi(x)^* : \phi(y)^* :$$

$$+ \int dt_1 dt_2 \theta(t_2 - t_1) \int d^3\vec{y} \int d^3\vec{x} : \phi(y)^* : \phi(x)^* :$$

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(10)
(11)
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with the Heaviside function \( \theta \). In order to calculate expectation values of the above explicitly, we apply the Wick theorem and pick up all possible contractions, using ordinary formulas such as

\[
\langle \Omega | \hat{\phi}(k_1) \hat{\phi}(k_2) | \Omega \rangle = (2\pi)^{-4} \Delta_+(k_2) \delta(k_1+k_2).
\]

One immediately finds that some contractions will involve only pointwise products, while some will involve twisted convolutions. The former are referred to as planar contributions while the latter are called nonplanar.

The important observation then is that contrary to the assumption made in the context of the modified Feynman rules we do not have Feynman propagators in terms which involve twistings. For instance, we find the following nonplanar contribution to the fish graph:

\[
\theta \cdot \Delta_+ \ast \Delta_+ + (1 - \theta) \cdot \Delta_- \ast \Delta_- \quad \text{where the symbol } \ast \text{ is used instead of } \ast \text{ to indicate that the twisted convolution is to be taken with respect to } 2Q. \quad \text{While the corresponding planar graph } \theta \cdot \Delta_+^2 + (1 - \theta) \cdot \Delta_-^2 \quad \text{indeed yields the square of the Feynman propagator, } \Delta_F^2, \text{ this is not the case for the nonplanar contribution,}
\]

\[
\theta \cdot \Delta_+ \ast \Delta_+ + (1 - \theta) \cdot \Delta_- \ast \Delta_- \neq \Delta_F \ast \Delta_F \quad \text{(12)}
\]

unless the Heaviside function \( \theta \) may pass the twisted convolution, in the sense that

\[
\theta \cdot \Delta_+ \ast \Delta_+ = \theta \Delta_+ \ast \theta \Delta_+.
\]

This is not generally true, but may be done only in the case of spacelike or lightlike noncommutativity. In these cases, the assumption that Feynman propagators serve as internal lines is compatible with the requirement that the theory be unitary \([1]\). For additional explicit calculations in this framework see \([6]\).

### 2.3 Yang Feldman equation

A covariant approach to perturbation theory on the noncommutative Minkowski space was given in \([1]\). It is based on the field equation and results in a direct perturbative definition of the interacting field \([7]\). As early as 1952 this approach was already used in the analysis of nonlocal field theories \([8]\). The idea is to solve the field equation

\[
(\Box + m^2)\phi(q) = -\lambda\phi(q)^n - 1 \quad \text{(13)}
\]

perturbatively by

\[
\phi(q) = \sum_{k=0}^{\infty} \lambda^k \phi_k(q) \quad \text{(14)}
\]

Identifying \( \phi_0(q) \) with the incoming field \( \phi_{in}(q) \), we have at \( k \)-th order

\[
\phi_k(q) = \int dy \Delta_{ret}(y) \sum_{k_1+\cdots+k_{n-1}=k-1} \phi_{k_1}(q-y) \cdots \phi_{k_{n-1}}(q-y) \quad \text{(15)}
\]

with ordinary retarded propagators \( \Delta_{ret} \). As in the Hamiltonian approach, the time ordering is thus separated from the nonlocal products. The graph theory of the above construction is given by rooted trees with \( n - 1 \) branches at each vertex and with retarded propagators connecting different vertices. Unitarity in this context means that the interacting field must be Hermitean, which it obviously is if we assume that the incoming field is Hermitean.

As an explicit example let us again consider \( \phi^3 \)-theory, where we have only 2 branches at each vertex, such that at first order, the interacting field is

\[
\phi_1(q) = \int dy \Delta_{ret}(y) \phi_0(q-y) \phi_0(q-y) \quad \text{(16)}
\]
and at second order,
\[
\phi_2(q) = \int dy \, \Delta_{\text{ret}}(y) \left( \phi_0(q-y)\phi_1(q-y) + \phi_1(q-y)\phi_0(q-y) \right) \tag{17}
\]
\[
= \int dy \, \Delta_{\text{ret}}(y) \int dz \, \Delta_{\text{ret}}(z) \left( \phi_0(q-y) \phi_0(q-y-z) \phi_0(q-y+z) \right) \tag{18}
\]
\[
+ \phi_0(q-y-z) \phi_0(q-y-z) \phi_0(q-y) \right) \tag{19}
\]
\[
= \int dy \, \Delta_{\text{ret}}(y) \int dz \, \Delta_{\text{ret}}(z) \left( \phi_0(q-y) \phi_0(q-y-z) \phi_0(q-y+z) \right) \tag{21}
\]
\[
= \int dy \, \Delta_{\text{ret}}(y) \int dz \, \Delta_{\text{ret}}(z) \left( \phi_0(q-y) \phi_0(q-y-z) \phi_0(q-y+z) \right) \tag{22}
\]
\[
= \int dy \, \Delta_{\text{ret}}(y) \int dz \, \Delta_{\text{ret}}(z) \left( \phi_0(q-y) \phi_0(q-y-z) \phi_0(q-y+z) \right) \tag{23}
\]
Again, loop graphs appear when products of fields are Wick-ordered. On the ordinary Minkowski spacetime, these graphs are known as Dyson’s double graphs, since they involve both retarded propagators \(\Delta_{\text{ret}}\) as well as propagators \(\Delta^{(1)} := \Delta_+ + \Delta_-\). On a noncommutative spacetime they cease to be equivalent to Feynman graphs. As an example consider again the fish graph which may in terms of the twisted convolution be written as
\[
\int d^4k e^{ikq} \int d^4xe^{-ikx} \cdot \left( \int dy \, \Delta_{\text{ret}}(y) \int dz \, \left( \Delta_+ \cdot \Delta_{\text{ret}}(z) + \Delta_{\text{ret}} \cdot \Delta_+(z) \right) \phi_0(x-y-z) \right) \tag{21}
\]
\[
+ \int dy \, \Delta_{\text{ret}}(y) \int dz \, \left( \Delta_- \ast \Delta_{\text{ret}}(z) + \Delta_{\text{ret}} \ast \Delta_+(z) \right) \phi_0(x-y-z) \right) \tag{22}
\]
\[
= \int dy \, \Delta_{\text{ret}}(y) \int dz \, \left( \Delta_+ \ast \Delta_{\text{ret}}(z) + \Delta_{\text{ret}} \ast \Delta_+(z) \right) \phi_0(x-y-z) \right) \tag{23}
\]
Thus we have again found a planar and a nonplanar contribution. And while in the planar contribution the time ordering from the retarded propagator may be absorbed into a Feynman propagator,
\[
\Delta_{\text{ret}}(\Delta_+ + \Delta_-) = -i \Delta_F^2 + i \Delta_-^2 \tag{24}
\]
this is not the case for the nonplanar contribution, where an additional product of retarded and advanced propagators appears,
\[
\Delta_{\text{ret}} \ast \Delta_+ + \Delta_- \ast \Delta_{\text{ret}} = -i \Delta_F \ast \Delta_F + i \Delta_- \ast \Delta_- + i \Delta_{\text{ret}} \ast \Delta_{\text{av}} \tag{25}
\]
Precisely this term, which is not present in the context of the modified Feynman rules, is needed to render the theory unitary [1]. It is absent if spacelike or lightlike noncommutativity is assumed, since then \(\theta \Delta \ast (1-\theta)\Delta = \theta(1-\theta) \cdot \Delta \ast \Delta = 0\). We may thus conclude again that only in these special cases, unitarity is compatible with the assumption that the time ordering may be absorbed in Feynman propagators alone, while the Yang-Feldman approach always renders a unitary theory.

### 3 Outlook

In the above discussion I have not stated what the correct definition of the Wick product of fields should be. But since some tadpoles remain finite on noncommutative spacetimes [3], we should expect them to be different from those appearing in the ordinary case.
As an example let us consider a 3-fold product of fields. In ordinary quantum field theory, all tadpoles would be infinite, and the Wick product defined as

\[
\phi(x_1)\phi(x_2)\phi(x_3) := \phi(x_1)\phi(x_2)\phi(x_3) - \Delta_+(x_1-x_2)\phi(x_3)
\]

(26)

\[
-\Delta_+(x_1-x_3)\phi(x_2) - \Delta_+(x_2-x_3)\phi(x_1)
\]

(27)

would be well-defined at coinciding points. However, on a noncommutative spacetime, where we are interested in products such as

\[
\phi(q+x_1)\phi(q+x_2)\phi(q+x_3)
\]

(28)

the subtraction corresponding to the first term in (27) yields the following nonplanar expression,

\[
\int dk \Delta_+(x_1-x_3-Qk) \tilde{\phi}(k)e^{ik(q+x_2)}
\]

(29)

which remains well-defined at coinciding points \(x_i = x\) and therefore does not need to be subtracted. Moreover, it is nonlocal in the sense that it cannot be written as a product of a distribution and a field, and therefore, it should not be subtracted. The precise definition of Wick products where only infinite, and (in the sense suggested above) local terms are subtracted, as well as the proof of the adequate Wick theorem are part of our current research. We hope to be able to shortly produce our results on these questions as well as comment on the surprising consequences for the so-called ultraviolet/infrared mixing problem [9].

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