Donkin-Koppinen filtration for general linear supergroup

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Abstract

In this article we consider a generalization of Donkon-Koppinen filtrations for coordinate superalgebras of general linear supergroups. More precisely, if $G = GL(m|n)$ is a general linear supergroup of (super)degree $(m|n)$, then its coordinate superalgebra $K[G]$ is a natural $G \times G$-supermodule. For any finitely generated ideal $\Gamma \subseteq \Lambda \times \Lambda$ the largest supersubmodule $O_{\Gamma}(K[G])$, whose all composition factors are $L(\lambda) \otimes L(\mu)$ with $(\lambda, \mu) \in \Gamma$, has a decreasing filtration $O_{\Gamma}(K[G]) = V_0 \supseteq V_1 \supseteq \ldots$, such that $\bigcap_{i \geq 0} V_i = 0$ and $V_i/V_{i+1} \simeq V_- (\lambda_i)^* \otimes H^0_-(\lambda_i)$. Here $H^0_-(\lambda)$ and $V_- (\lambda)$ are couniversal and universal $G$-supermodules of highest weight $\lambda \in \Lambda$ respectively (see [5]). We apply this result to describe adjoint action invariants of $G$.

Introduction

Let $G$ be a reductive algebraic group defined over an algebraically closed field $K$. The group $G \times G$ acts on $G$ by $(g, (g_1, g_2)) \mapsto g_1^{-1}gg_2, g, g_1, g_2 \in G$. This induces a structure of a rational $G$-bimodule on $K[G]$. Donkin (and previously Donkin) proved that $K[G]$ has an increasing $G$-bimodule filtration $0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$, such that $\bigcup_{i \geq 1} V_i = K[G]$ and $V_i/V_{i-1} \simeq V(\lambda_i)^* \otimes H^0(\lambda_i)$, $\lambda_i \in X(T)^+$ [10]. Besides, if $k > l$, then either $\lambda_k \leq \lambda_l$ or $\lambda_k$ and $\lambda_l$ are not comparable.

In the present article we generalize this result for the general linear supergroup $G = GL(m|n)$. In contrast to the classical case the $G$-supersubmodule $K[G]$ has not increasing filtrations as above. In fact, the set of highest weights $\Lambda$ of simple $G$-supermodules has not minimal elements. Nevertheless, one can prove that for any finitely generated ideal $\Gamma \subseteq \Lambda \times \Lambda$ the largest supersubmodule $O_{\Gamma}(K[G])$, whose all composition factors are $L(\lambda) \otimes L(\mu)$ with $(\lambda, \mu) \in \Gamma$, has a decreasing filtration $O_{\Gamma}(K[G]) = V_0 \supseteq V_1 \supseteq \ldots$, such that $\bigcap_{i \geq 0} V_i = 0$ and $V_i/V_{i+1} \simeq V_-(\lambda_i)^* \otimes H^0_-(\lambda_i)$. Here $H^0_-(\lambda) = ind^G_B K_\lambda$ is an induced supermodule of highest weight $\lambda$, $V_-(\lambda)$ is a Weyl supermodule of highest weight $\lambda$, and $B^-$ is a Borel supersubgroup of $G$ consisting of all lower triangular matrices and $K_\lambda$ is a natural (even) one dimensional $B^-$-supermodule of weight $\lambda$ (see [5] for more definitions). In the last section we use the above filtrations to describe adjoint action invariants of $G$.

1 Preliminary definitions and notations

Let $G$ be an affine supergroup. In other words, $G$ is a (representable) functor from the category of commutative superalgebras $SAlg_K$ to the category of groups $Gr$, such that $G(A) = \text{Hom}_{SAlg_K}(K[G], A), A \in SAlg_K$. The Hopf superalgebra $K[G]$ is called coordinate superalgebra of $G$. The category of (left) $G$-supermodules with even morphisms
is denoted by $G - \text{mod}$. The category $G - \text{mod}$ is equivalent to the category of (right) $K[G]$-supermodules with even morphisms \([1][5]\). If $V \in G - \text{mod}$, then its coaction map is denoted by $\rho_V$. From now on we use Sweedler’s notations : $\rho_V(v) = \sum v_1 \otimes f_2, v, v_1 \in V, f_2 \in K[G]$. In what follows $\text{Hom}_G(V, W)$ is a (superspace) of all (not necessary even) morphisms between $G$-supermodules $V$ and $W$. In particular, $\text{Hom}_{G-\text{mod}}(V, W) = \text{Hom}_G(V, W)_0$. In the same way, $\text{Ext}^i_G(V, ?) = R^i\text{Hom}_G(V, ?), i \geq 0$. Any $\text{Ext}^i_G(V, W)$ has a superspace structure with $\langle \text{Ext}^i_G(V, W) \rangle = R^i\text{Hom}_G(V, W)\epsilon, \epsilon = 0, 1$. Moreover, $\text{Ext}^i_G(V^c, W) \simeq \text{Ext}^i_G(V, W^c) = \text{Ext}^i_G(V, W)^c$, where $V^c$ is conjugated to $V$ \([5]\).

Let $K[c_{ij}|1 \leq i, j \leq m + n]$ be a commutative superalgebra freely generated by the elements $c_{ij}$, where $|c_{ij}| = 0$ iff $1 \leq i, j \leq m$ or $m + 1 \leq i, j \leq m + n$, otherwise $|c_{ij}| = 1$. Denote the generic matrix $(c_{ij})_{1 \leq i, j \leq m + n}$ by $C$. Set

$$C_{00} = (c_{ij})_{1 \leq i, j \leq m}, C_{01} = (c_{ij})_{1 \leq i \leq m, m + 1 \leq j \leq m + n},$$

$$C_{10} = (c_{ij})_{m + 1 \leq i \leq m + n, 1 \leq j \leq n}, C_{11} = (c_{ij})_{m + 1 \leq i \leq m + n, m + 1 \leq j \leq m + n}$$

and $d_1 = \det(C_{00}), d_2 = \det(C_{11})$. The general linear supergroup $GL(m|n)$ is an algebraic supergroup, whose coordinate (Hopf) superalgebra $K[GL(m|n)]$ is isomorphic to $K[c_{ij}|1 \leq i, j \leq m + n|d_1, d_2]$. The comultiplication and the counit of $K[GL(m|n)]$ are given by $\delta_{GL(m|n)}(c_{ij}) = \sum_{1 \leq t \leq m + n} c_{it} \otimes c_{tj}$ and by $\epsilon_{GL(m|n)}(c_{ij}) = \delta_{ij}$ respectively. For the definition of the antipode see \([5][12]\).

Let $\mathcal{C}$ be a $K$-abelian and locally artinian Grothendieck category (see \([3][4]\) for definitions). Assume that all simple objects in $\mathcal{C}$ are indexed by the elements of a partially ordered set $(\Lambda, \leq)$. Let $L(\Lambda)$ be a simple object and $I(\lambda)$ be its injective envelope, $\lambda \in \Lambda$. The costandard object $\nabla(\lambda)$ is defined as largest subobject of $I(\lambda)$ whose composition quotients are $L(\mu), \mu \leq \lambda$. For example, if $\mathcal{C}$ is a highest weight category in the sense of \([3]\), then the costandard objects coincide with the first members of good filtrations of injective envelopes. Denote by $C_f$ the full subcategory of $\mathcal{C}$ consisting of all finite objects. If all $\nabla(\lambda)$ belong to $C_f$ and $C_f$ has a duality $\tau$, preserving simple objects, then one can define standard objects in $\mathcal{C}$ by $\Delta(\lambda) = \tau(\nabla(\lambda)), \lambda \in \Lambda$. Any costandard object $\nabla(\lambda)$ is uniquely defined by the following universal property. If $W \in \mathcal{C}$ such that $\text{soc}(W) = L(\lambda)$ and all other composition quotients of $W$ are $L(\mu), \mu < \lambda$, then $W$ is isomorphic to a subobject of $\nabla(\lambda)$. Symmetrically, if $W/\text{rad}W \simeq L(\lambda)$ and all other composition quotients of $\text{rad}W$ are $L(\mu), \mu < \lambda$, then $W$ is isomorphic to a factor of $\Delta(\lambda)$.

2 Hochschild-Serre spectral sequences

Let $G$ be an affine supergroup and $N$ is a normal supersubgroup of $G$. The dur $K$-sheaf $\tilde{G}/N$ is an affine supergroup (see Theorem 6.2, \([6]\)). Moreover, $K[\tilde{G}/N] \simeq K[G]^N$. To simplify our notations we denote $G/N$ just by $G/N$. If $V \in G - \text{mod}$, then $V^N$ has a natural structure of a $G/N$-supermodule. More precisely, by Proposition 3.1 from \([5]\), $V$ is embedded (as a supercomodule) into a direct sum of several copies of $K[G]$ and $K[G]^c$. It implies that $V^N$ is the largest supersubmodule of $G$-supermodule $V$ whose coefficient space $cf(V)$ lies in $K[G]^N$. The canonical epimorphism $G \rightarrow G/N$ is denoted by $\pi$. Following notations of \([5]\) we have the restriction functor $\pi_0 : G/N - \text{mod} \rightarrow G - \text{mod}$. The proof of the following lemma can be copied from Lemma 6.4, II, \([7]\).
Lemma 2.1 The functor $V \rightarrow V^N$ is left exact and right adjoint to $\pi_0$.

Proposition 2.1 (Proposition 6.6, II, [5]) For $M \in G/N - \text{mod}, V, U \in G - \text{mod}, \dim U < \infty$, we have the following spectral sequences:

1) $E_2^{n,m} = Ext^n_{G/N}(M, Ext^m(U, V)) \Rightarrow Ext^{n+m}_{G}(M \otimes U, V)$.

2) $E_2^{n,m} = Ext^n_{G}(M, H^m(N, V)) \Rightarrow Ext^{n+m}_{G}(M, V)$.

3) $E_2^{n,m} = H^n(G/N, H^m(N, V)) \Rightarrow H^{n+m}(G, V)$.

Proof. The statements 2) and 3) are deduced from 1). To check it one has to set $U = K$ for the statement 2) and then $M = K$ for the statement 3). By Proposition 2.1, [5], $\text{Hom}_N(U, V) \cong (U^* \otimes V)^N$ and by Lemma 1.1

$$\text{Hom}_{G/N}(M, (U^* \otimes V)^N) \cong \text{Hom}_{G}(M, U^* \otimes V) \cong \text{Hom}_{G}(M \otimes U, V).$$

In particular, the functor $V \rightarrow \text{Hom}_N(U, V)$ is left exact and takes injective $G$-supermodules to injective $G/N$-supermodules. Theorem 7, III, [5] completes the proof.

3 Adjoint and coadjoint actions

Let $G$ be an affine supergroup. We have a right action of $G \times G$ on $G$ by $(g, (g_1, g_2)) \mapsto g_1^{-1}gg_2, g, g_1, g_2 \in G(A), A \in SAlg_{gK}$. Its dual morphism $\rho : K[G] \rightarrow K[G] \otimes K[G] \otimes 2$ is defined as

$$\rho : f \mapsto \sum (-1)^{|f_1||f_2|} f_2 \otimes s_G(f_1) \otimes f_3, f \in K[G].$$

Compose with the diagonal inclusion $G \rightarrow G \times G$ we obtain a right adjoint action of $G$ on itself. Its dual morphism coincides with

$$\nu_1 : f \mapsto \sum (-1)^{|f_1||f_2|} f_2 \otimes s_G(f_1)f_3,$$

(see [5]).

Let $V$ be a $L$-supermodule and $W$ be a $H$-supermodule, where $L$ and $H$ are affine supergroups. The superspace $V \otimes W$ has natural structure of a $L \times H$-supermodule by

$$\rho_{V \otimes W}(v \otimes w) = \sum (-1)^{|w_1||f_2|} v_1 \otimes w_1 \otimes f_2 \otimes h_2,$$

where $\rho_V(v) = \sum v_1 \otimes f_2, \rho_W(w) = \sum w_1 \otimes h_2$. Let $\sigma$ be a Hopf superalgebra anti-isomorphism of $K[L]$ and $\tau$ be a Hopf superalgebra anti-isomorphism of $K[H]$. It is clear that $\sigma \otimes \tau$ is an anti-isomorphism of Hopf superalgebra $K[L \times H]$. We have dualities $V \rightarrow V^{<\sigma>}, W \rightarrow W^{<\tau>}$ and $U \rightarrow U^{<\sigma \otimes \tau>}$ of the categories $L - \text{smod}, H - \text{smod}$ and $L \times H$-smod correspondingly (see [5] for more definitions).

Lemma 3.1 In the above notations, if $V$ and $W$ are finite dimensional, then $V^{<\sigma> \otimes W^{<\tau>}}$ and $(V \otimes W)^{<\sigma \otimes \tau>}$ are canonically isomorphic as $L \times H$-supermodules. In addition, if $V'$ is a $L$-supermodule and $W'$ is a $H$-supermodule, then $(V \otimes W) \otimes (V' \otimes W')$ and $(V \otimes V') \otimes (W \otimes W')$ are canonically isomorphic as $L \times H$-supermodules, where $(V \otimes W)$ and $(V' \otimes W')$ are considered as $L \times H$-supermodules, but $(V \otimes V')$ and $(W \otimes W')$ are considered as supermodules with respect to the diagonal action of $L$ and $H$ respectively.
Proposition 3.1
where dimensional superspace of parity $\epsilon$

\[ G = GL(m|n). \]
In what follows the anti-isomorphism of $K[G]$, defined by $c_{ij} \mapsto (-1)^{i[j][j]} c_{ji}$, $1 \leq i, j \leq m+n$, is denoted by $\tau$. The Borel supersubgroups of $G$ consisting of lower (respectively, upper) triangular matrices is denoted by $B^{-}$ (respectively, by $B^{+}$). The supergroup $T = B^{-} \cap B^{+}$ is a maximal torus in $G$, called standard. The category $T$-mod is semisimple. Any simple $T$-supermodule $V$ is one dimensional and uniquely defined by its character $\lambda \in X(T) = Z^{m+n}$, and by its parity $\epsilon = 0, 1$. We denote it by $K^{\lambda}$. The induced supermodules $H^{0}(G/B^{-}, K^{\lambda})$ and $H^{0}(G/B^{+}, K^{\lambda})$ are denoted by $H^{0}(\lambda)$ and $H_{0}^{+}(\lambda)$ correspondingly. The Weyl supermodules $H^{0}(\lambda)^{<\tau>}$ and $H_{0}^{+}(\lambda)^{<\tau>}$ are denoted by $V_{-}(\lambda)$ and $V_{+}(\lambda)$ (all details can be found in [5]). All arguments of [5] can be symmetrically repeated for $B^{+}$. In particular, the category $G - mod$ is a highest weight category with respect to the inverse dominant order on $X(T)$ (see Remark 5.3, [5]). Precisely, the corresponding costandard and standard objects are $H^{0}_{-}(\lambda)$ and $V_{+}(\lambda)$ respectively, where $\epsilon = 0, 1$, and $\lambda \in X(T)^{-}$. Remind that $H^{+}(\lambda) = H^{0}_{+}(\lambda) = H^{0}_{-}(\lambda)^{1+} \subset V_{+}(\lambda) = V_{+}(\lambda)^{c}$ and

\[ X(T)^{-} = \{ \lambda = (\lambda_{1}, \ldots, \lambda_{m+n}) \in Z^{m+n} | \lambda_{1} \leq \cdots \leq \lambda_{m}, \lambda_{m+1} = \cdots = \lambda_{m+n} \}. \]

The parity $\epsilon$ coincides with the parity of the one dimensional supersubspace $H^{0}_{-}(\lambda)^{c}$ (respectively, with the parity of the one dimensional supersubspace $V_{+}(\lambda^{c})^{\lambda}$). The following lemma is now obvious.

**Lemma 3.2** For any $\lambda \in X(T)^{+}$ and $\epsilon = 0, 1$, we have natural isomorphisms $H^{0}_{-}(\lambda^{c})^{*} \simeq V_{+}(-\lambda^{c})$ and $V_{-}(\lambda^{c})^{*} \simeq H^{0}_{+}(-\lambda^{c})$.

It is clear that

\[ V_{+}(\lambda^{c}) \otimes V_{-}(\mu^{c}) \simeq V_{+}(\lambda^{c}) \otimes V_{+}(\mu) \simeq V_{+}(\lambda) \otimes V_{-}(\mu^{c}) \]

and

\[ H^{0}_{+}(\lambda^{c}) \otimes H^{0}_{-}(\mu^{c}) \simeq H^{0}_{+}(\lambda^{c}) \otimes H^{0}_{-}(\mu) \simeq H^{0}_{+}(\lambda) \otimes H^{0}_{-}(\mu^{c}), \]

where $\pi = \epsilon + \epsilon'$ (mod 2).

**Proposition 3.1** For any $\lambda_{1}, \mu_{1} \in X(T)^{-}, \lambda_{2}, \mu_{2} \in X(T)^{+}, \epsilon, \epsilon' = 0, 1$, a superspace $Ext_{G \times G}^{i}(V_{+}(\lambda_{1}^{c}) \otimes V_{-}(\lambda_{2}), H_{0}(\mu_{1}^{c}) \otimes H_{0}(\mu_{2}))$ is not equal to zero iff $i = 0$ and $\lambda_{1} = \mu_{1}, \lambda_{2} = \mu_{2}$. In the last case $Hom_{G \times G}(V_{+}(\lambda_{1}^{c}) \otimes V_{-}(\lambda_{2}), H_{0}(\mu_{1}^{c}) \otimes H_{0}(\mu_{2}))$ is an one dimensional superspace of parity $\epsilon + \epsilon'$ (mod 2).

Proof. We have the exact sequence of algebraic supergroups

\[ 1 \rightarrow G \rightarrow G \times G \rightarrow G \rightarrow 1, \]

where the epimorphism $G \times G \rightarrow G$ is dual to the monomorphism $K[G] \rightarrow K[G] \otimes K[G]$ defined as $f \mapsto 1 \otimes f$. The kernel of this epimorphism coincides with $G \times 1 \simeq G$. Combining Proposition 3.2, [5], with Lemma 3.1 we have

\[ Ext_{G \times G}^{i}(V_{+}(\lambda_{1}^{c}) \otimes V_{-}(\lambda_{2}), H_{0}(\mu_{1}^{c}) \otimes H_{0}(\mu_{2})) \simeq \]
\[ H^i(G \times G, V_+(\lambda_1^\vee) \otimes V_-(\lambda_2)^* \otimes H_+^0(\mu_1^\vee) \otimes H_+^0(\mu_2)) \simeq \]
\[ H^i(G \times G, V_+(\lambda_1^\vee) \otimes H_+^0(\mu_1^\vee) \otimes V_-(\lambda_2)^* \otimes H_-^0(\mu_2)). \]

On the other hand,
\[ H^i(G \times 1, V_+(\lambda_1^\vee) \otimes H_+^0(\mu_1^\vee) \otimes V_-(\lambda_2)^* \otimes H_+^0(\mu_2)) \simeq \]
\[ Ext^i_G(V_+(\lambda_1^\vee), H_+^0(\mu_1^\vee)) \otimes (V_-(\lambda_2)^* \otimes H_+^0(\mu_2)) = 0, \]
provided \( i > 0 \) (see [5], Theorem 5.5). Proposition 2.1 and the standard spectral sequence arguments infer that
\[ Ext^i_G(V_+(\lambda_1^\vee), H_+^0(\mu_1^\vee)) \otimes Ext^j_G(V_-(\lambda_2), H_+^0(\mu_2)) \]
where in the second tensor multiplier \( G \) is identified with \( 1 \times G \). It remains to refer to the universal properties of standard/costandard objects.

Let \( G_1, G_2 \) be affine supergroups and let \( H_1 \leq G_1, H_2 \leq G_2 \) be their (closed) super-subgroups.

**Lemma 3.3** (see Lemma 3.8, [7], part I) If \( V_i \) is a \( H_i \)-supermodule, \( i = 1, 2 \), then there is a canonical isomorphism \( \text{ind}_{H_1 \times H_2}^{G_1 \times G_2} V_1 \otimes V_2 \simeq \text{ind}_{H_1}^{G_1} V_1 \otimes \text{ind}_{H_2}^{G_2} V_2 \) of \( G_1 \times G_2 \)-supermodules.

**Proof.** The isomorphism is induced by the map \( V_1 \otimes V_2 \otimes K[G_1] \otimes K[G_2] \to V_1 \otimes K[G_1] \otimes V_2 \otimes K[G_2] \) defined as \( v_1 \otimes v_2 \otimes f_1 \otimes f_2 \mapsto (-1)^{|v_2||f_1|} v_1 \otimes f_1 \otimes v_2 \otimes f_2 \). In fact, by Proposition 3.3, [5], and Lemma 3.1
\[ \text{ind}_{H_1 \times H_2}^{G_1 \times G_2} V_1 \otimes V_2 \simeq (V_1 \otimes V_2 \otimes (K[G_1] \otimes K[G_2]))_{H_1 \times H_2} \simeq \]
\[ ((V_1 \otimes K[G_1])_{H_1} \otimes (V_2 \otimes K[G_2])_{H_2} \simeq \text{ind}_{H_1}^{G_1} V_1 \otimes \text{ind}_{H_2}^{G_2} V_2. \]

Here, for any affine supergroup \( L, K[L] = K[L]_l \) is a \( L \)-supermodule via
\[ f \mapsto \sum (-1)^{|f_1||f_2|} f_1 \otimes s_L(f_1), \delta_L(f) = \sum f_1 \otimes f_2. \]

The character group \( X(T \times T) \) is identified with \( X(T) \times X(T) \). It is partially ordered by \( (\lambda, \lambda') \leq (\mu, \mu') \) iff \( \lambda \geq \lambda', \mu \leq \mu' \). This ordering corresponds to the Borel supersubgroup \( B^+ \times B^- \) of \( G \times G \). The epimorphism \( K[B^+ \times B^-] \to K[T \times T] \) is split and \( K[T \times T] \) can be canonically identified with a Hopf supersubalgebra of \( K[B^+ \times B^-] \) generated by the group-like elements \( c_i^{\pm 1} \otimes c_j^{\pm 1}, 1 \leq i, j \leq m + n \). Denote the kernel of this epimorphism by \( J \). Let \( V \) be a \( B^+ \times B^- \)-supermodule. Take a weight \( (\lambda, \lambda') \in X(T \times T) \) and a \( \mathbb{Z}_2 \)-homogeneous vector \( v \in V_{(\lambda, \lambda')} \). As in Proposition 5.3, [5], we have
\[ \rho_V(v) = v \otimes c^{(\lambda, \lambda')} + y, y \in \bigoplus_{(\mu, \mu') < (\lambda, \lambda')} V_{(\mu, \mu')} \otimes J, \]
where \( c^{(\lambda, \lambda')} = \prod_{1 \leq i \leq m+n} c_i^{\lambda_i} \otimes \prod_{1 \leq i \leq m+n} c_i^{\lambda'_i} \). In particular, any simple \( B^+ \times B^- \)-supermodule is one dimensional and isomorphic to \( K^{(\lambda, \lambda')} \simeq K^{\lambda} \otimes K^{\lambda'} \). Besides, if a \( B^+ \times B^- \)-supermodule is generated by a \( (\mathbb{Z}_2 \text{-homogeneous}) \) vector \( v \) of weight \( (\lambda, \lambda') \),
then $V_{(\mu, \mu')} \neq 0$ implies $(\mu, \mu') \leq (\lambda, \lambda')$ and $\dim V_{(\lambda, \lambda')} = 1$. Using Proposition 5.4, \cite{5}, one can easily check that the morphism of superalgebras

$$K[G \times G] \xrightarrow{(\pi_+ \otimes \pi_-) \delta_{G \times G}} K[B^+ \times B^-] \otimes K[B^- \times B^+],$$

is an inclusion, where $\pi_+ : K[G] \to K[B^+ \times B^-]$ and $\pi_- : K[G] \to K[B^- \times B^+]$ are canonical epimorphisms. Now, everything is prepared to prove the following lemmas.

**Lemma 3.4** The supermodule $H^0_+ (\lambda') \otimes H^0_- (\lambda')$ runs over all costandard objects in the category of $G \times G$-supermodules, whenever $(\lambda, \lambda')$ runs over $X(T)^- \times X(T)^+$ and $\epsilon = 0, 1$. Besides, the simple socle of $H^0_+ (\lambda') \otimes H^0_- (\lambda')$ is isomorphic to $L(-\lambda') \otimes L(\lambda')$.

**Proof.** Word-by-word repetition of the proofs of Propositions 5.5 and 5.6, \cite{5}.

**Lemma 3.5** The supermodule $V_+(\lambda') \otimes V_-(\lambda')$ runs over all standard objects in the category of $G \times G$-supermodules, whenever $(\lambda, \lambda')$ runs over $X(T)^- \times X(T)^+$ and $\epsilon = 0, 1$.

**Proof.** It obviously follows by Lemma 3.1.

As in \cite{5} a $G \times G$-supermodule $V$ is called restricted iff for any $(\lambda, \lambda') \in X(T)^- \times X(T)^+$, we have $\dim \text{Hom}_{G \times G}(V_+(\lambda) \otimes V_-(\lambda'), V) < \infty$ and the set

$$\hat{V} = \{ (\lambda, \lambda') \in X(T)^- \times X(T)^+ | \text{Hom}_{G \times G}(V_+(\lambda) \otimes V_-(\lambda'), V) \neq 0 \}$$

does not contain infinite decreasing chains.

**Lemma 3.6** A restricted $G \times G$-supermodule $V$ has a costandard (or good) filtration with quotients $H^0_+ (\lambda') \otimes H^0_- (\lambda')$ iff for all $(\lambda, \lambda') \in X(T)^- \times X(T)^+$ one of the following equivalent conditions hold:

1. $\text{Ext}^i_{G \times G}(V_+(\lambda) \otimes V_-(\lambda'), V) = 0$;
2. $\text{Ext}^i_{G \times G}(V_+(\lambda) \otimes V_-(\lambda'), V) = 0$ for any $i \geq 1$.

If $V$ has a good filtration, then the multiplicity of a factor $H^0_+ (\lambda') \otimes H^0_- (\lambda')$ is equal to $\dim \text{Hom}_{G \times G}(V_+(\lambda) \otimes V_-(\lambda'), V)_0$ (see Remark 5.5 in \cite{5}).

**Corollary 3.1** Any injective $G \times G$-supermodule has a good filtration with quotients $H^0_+ (\lambda') \otimes H^0_- (\lambda')$. In particular, $G \times G$-mod is a highest weight category with poset $(X(T)^- \times \{0, 1\}) \times X(T)^+$ ordered as above.

**Remark 3.1** Let $\delta = (\delta_1, \ldots, \delta_\ell)$ be an element of the set $\{+, -\}^\ell$. Set $G^\ell = \underbrace{G \times \cdots \times G}_{\ell}$.

Mimic the above arguments one can prove that $G^\ell$-mod is a highest weight category with poset $(X(T)^{\delta_1} \times \{0, 1\}) \times \cdots \times X(T)^{\delta_\ell}$, whose costandard and standard objects are $H^0_{\delta_1} (\lambda_1') \otimes \cdots \otimes H^0_{\delta_\ell} (\lambda_\ell')$ and $V_{\delta_1} (\lambda_1') \otimes \cdots \otimes V_{\delta_\ell} (\lambda_\ell')$ respectively.

**Proposition 3.2** (see \cite{7}, Part II, Proposition 4.20) The superalgebra $K[G]$, considered as a $G \times G$-supermodule via $\rho$, satisfies the following conditions:

1. $\text{Ext}^i_{G \times G}(V_+(\lambda) \otimes V_-(\lambda'), K[G]) = 0$ for all $i \geq 1, (\lambda, \lambda') \in X(T)^- \times X(T)^+$.
2. \( \text{Hom}_{G \times G}(V_+(\lambda^e) \otimes V_-(\lambda'), K[G]) \neq 0 \) iff \( \epsilon = 0 \) and \( \lambda = -\lambda' \).

Besides, \( \dim \text{Hom}_{G \times G}(V_+(\lambda) \otimes V_-(\lambda'), K[G]) = 1 \).

Proof. By Proposition 2.1 we have a spectral sequence
\[
H^n(1 \times G, H^m(G \times 1, H^0_-(\lambda') \otimes H^0_+(\lambda') \otimes K[G])) \Rightarrow H^{m+n}(G \times G, H^0_+(\lambda') \otimes H^0_-(\lambda') \otimes K[G]).
\]
On the left side \( K[G] \) is isomorphic to \( K[G]_{|1} \) as a \( G = G \times 1 \)-supermodule. As \( K[G]_{|1} \simeq K[G] \) is injective (see [3]), this sequence degenerates and yields isomorphism
\[
H^n(1 \times G, (H^0_-(\lambda') \otimes K[G])^G \otimes H^0_+(\lambda')) \simeq \text{Ext}^n_{G \times G}(V_+(\lambda^e) \otimes V_-(\lambda'), K[G]).
\]
Finally, \( (H^0_-(\lambda') \otimes K[G])^G \simeq \text{Ind}_G^H H^0_-(\lambda') = H^0_-(\lambda') \) and the space on left is isomorphic to
\[
H^n(G, H^0_-(\lambda') \otimes H^0_+(\lambda')) \simeq \text{Ext}^n_{G}(V_+(\lambda^e), H^0_+(\lambda')).
\]
Thus \( \text{Ext}^n_{G \times G}(V_+(\lambda^e) \otimes V_-(\lambda'), K[G]) \neq 0 \) iff \( n = 0, \lambda = -\lambda' \). Besides, in the last case \( \text{Hom}_{G \times G}(V_+(\lambda^e) \otimes V_-(\lambda'), K[G]) \) is one dimensional and even iff \( \epsilon = 0 \).

4 Few results about filtrations

Let \( \mathcal{C} \) be a highest weight category from Section 1, with a duality \( \tau \). We call \( \tau \) a Chevalley duality. In what follows we assume that all costandard objects are finite and Schurian, that is \( \text{End}_\mathcal{C}(\nabla(\lambda)) = K \) for any \( \lambda \in \Lambda \). For a weight \( \lambda \in \Lambda \) we denote by \( (\lambda) \) the (possible infinite) interval \( \{ \mu \in \Lambda| \mu \leq \lambda \} \). The open interval \( \{ \mu | \mu < \lambda \} \) is denoted by \( (\lambda) \). Let \( \Gamma \subseteq \Lambda \) and \( M \in \mathcal{C} \). We say that \( M \) belongs to \( \Gamma \) iff all composition factors of \( M \) are \( L(\lambda) \) with \( \lambda \in \Gamma \). Any \( N \in \mathcal{C} \) contains the largest subobject that belongs to \( \Gamma \). We denote it by \( O_\Gamma(N) \). Symmetrically, \( N \) contains a unique minimal subobject \( O^\Gamma(N) \) such that \( N/O^\Gamma(N) \) belongs to \( \Gamma \). For example, \( \nabla(\lambda) = O_(\lambda)(I(\lambda)) \), \( \lambda \in \Lambda \). Finally, denote by \( [M : L(\lambda)] \) the supremum of multiplicities of a simple object \( L(\lambda) \) in composition series of all finite subobjects of \( M \).

The full subcategory consisting of all objects \( M \) with \( O_\Gamma(M) = M \) is denoted by \( \mathcal{C}[\Gamma] \). It is obvious that \( O_\Gamma \) is a left exact functor from \( \mathcal{C} \) to \( \mathcal{C}[\Gamma] \) and it commutes with direct sums. The functor \( O^\Gamma : \mathcal{C} \to \mathcal{C} \) also commutes with direct sums, but it is not right exact in general. In fact, for any exact sequence
\[
0 \to X \to Y \to Z \to 0
\]
we see that \( O^\Gamma(Y) \to O^\Gamma(Z) \) is an epimorphism and \( O^\Gamma(X) \subseteq X \cap O^\Gamma(Y) \). Nevertheless, it is possible that \( O^\Gamma(X) \) is a proper subobject of \( X \cap O^\Gamma(Y) \).

A subset \( \Gamma \subseteq \Lambda \) is called ideal, if \( \mu \leq \lambda \) implies \( \mu \in \Gamma \), provided \( \lambda \in \Gamma \). If \( \Gamma = \bigcup_{1 \leq k \leq s}(\lambda_k) \), we say that \( \Gamma \) is finitely generated (by the elements \( \lambda_1, \ldots, \lambda_s \)). We suppose that any finitely generated ideal \( \Gamma \subseteq \Lambda \) has a decreasing chain of finitely generated subideals \( \Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \ldots \), such that \( \Gamma \setminus \Gamma_k \) is a finite set for all \( k \geq 0 \) and \( \bigcap_{k \geq 0} \Gamma_k = \emptyset \). In particular, if \( \Gamma \subseteq \Lambda \) is a finitely generated ideal, then \( \Gamma \) is at most countable. From now on \( \Gamma \) is a finitely generated ideal, unless otherwise stated.
Example 4.1 The category $G - \text{mod}$ satisfies all the above conditions. In fact, any $\lambda \in X(T)^{+}$ has finitely many predecessors $\mu < \lambda$ such that there is not $\pi \in X(T)^{+}$ between $\lambda$ and $\mu$. More precisely, if $\mu$ is a such predecessor and $\sum_{1 \leq i \leq m} \mu_{i} < \sum_{1 \leq i \leq m} \lambda_{i}$, then $\mu < \lambda'$, where $\lambda' = (\lambda_{1}, \ldots, \lambda_{m-1}, \lambda_{m} - 1, \lambda_{m+1} + 1, \lambda_{m+2}, \ldots, \lambda_{m+n})$. It implies that either $\mu = \lambda'$ or $\mu_{+} \leq \lambda_{+}, \mu_{-} \leq \lambda_{-}$. It remains to notice that for any (ordered) partition $\pi$ there are only finitely many partitions of the same length, less or equal $\pi$. Repeating these arguments as many times as we need, one can prove that $G^t - \text{mod}$ satisfies all the above conditions for any root data $(X(T)^{+_{t}} \times \{0, 1\}) \times \ldots \times X(T)^{+_{t}}$.

The subcategory $C[\Gamma]$ is a highest weight category with costandard objects $\nabla(\lambda)$ and finite injective envelopes $I_{\Gamma}(\lambda) = O_{\Gamma}(I(\lambda)), \lambda \in \Gamma$. By Theorem 3.9 from [3] we have $Ext^{i}_{C}(M, N) = Ext^{i}_{C[\Gamma]}(M, N)$, for any $M, N \in C[\Gamma]$.

Let $R$ be a class of objects from $C$. An increasing filtration

$0 = M_{0} \subseteq M_{1} \subseteq M_{2} \subseteq \ldots$

of an object $M$ such that $M_{i}/M_{i-1} \in R, i \geq 1$, and $\bigcup_{i \geq 1} M_{i} = M$, is called increasing $R$-filtration. If $M$ has decreasing filtration

$M = M_{0} \supseteq M_{1} \supseteq M_{2} \supseteq \ldots$

such that $M_{i}/M_{i+1} \in R, i \geq 0$, and $\cap_{i \geq 0} M_{i} = 0$, then we call it decreasing $R$-filtration. For example, any injective envelope $I(\lambda)$ has an increasing $\nabla$-filtration, where $\nabla = \{\nabla(\lambda) | \lambda \in \Lambda\}$. To complete our notations, let us denote the class $\{\Delta(\lambda) | \lambda \in \Lambda\}$ by $\Delta$. Remind that $L(\lambda) = \Delta(\lambda)/\text{rad}(\lambda)$ and all other composition factors $L(\mu)$ of $\Delta(\lambda)$ satisfy $\mu < \lambda$.

Lemma 4.1 If $M$ belongs to $\Gamma$, then $Ext^{i}_{C}(M, \nabla(\lambda)) \neq 0$, for $i > 0$, infers that there is a composition factor $L(\mu)$ of $M$ such that $\mu > \lambda$.

Proof. Without loss of generality one can suppose that $\lambda \in \Gamma$. Consider the short exact sequence

$0 \rightarrow \nabla(\lambda) \rightarrow I_{\Gamma}(\lambda) \rightarrow Q \rightarrow 0,$

where $Q$ has a $\nabla$-filtration with quotients $\nabla(\mu), \mu > \lambda$. The fragment of long exact sequence

$\ldots \rightarrow Ext^{i-1}_{C}(M, Q) \rightarrow Ext^{i}_{C}(M, \nabla(\lambda)) \rightarrow 0$

shows that $Ext^{i-1}_{C}(M, Q) \neq 0$. The induction on $i$ implies that $Hom_{C}(M, \nabla(\mu)) \neq 0$ for some $\mu > \lambda$.

Lemma 4.2 The category $C[\Gamma]$ has enough projectives.

Proof. Denote $\tau(I_{\Gamma}(\lambda))$ by $P_{\Gamma}(\lambda), \lambda \in \Gamma$. It is an easy exercise to prove that $P_{\Gamma}(\lambda)$ is a projective cover of $L(\lambda)$. We leave it for the reader.

It is obvious that any $P_{\Gamma}(\lambda)$ has the $\Delta$-filtration which is Chevalley dual to the corresponding $\nabla$-filtration of $I_{\Gamma}(\lambda), \lambda \in \Gamma$. The following lemma is the symmetric variant of Lemma 4.1.

Lemma 4.3 Let $M \in C[\Gamma]$. If $Ext^{i}_{C}(\Delta(\lambda), M) \neq 0$, for $i > 0$, then there is composition factor $L(\mu)$ of $M$ such that $\mu > \lambda$. 

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Corollary 4.1 (compare with Theorem 3.11 from [3]) For any $\lambda, \mu \in \Lambda$ and $i > 0$, we have $\Ext^i_C(\Delta(\lambda), \nabla(\mu)) = 0$.

Corollary 4.2 Let $M \in \mathcal{C}$. If $\Ext^1_C(\Delta(\lambda), M) = 0$ (respectively, $\Ext^1_C(M, \nabla(\lambda)) = 0$) for all $\lambda \in \Lambda$, then the same still holds for $O_{\Gamma}(M)$ (respectively, for $M/\Gamma(M)$).

Proof. If $\Ext^1_C(\Delta(\lambda), O_{\Gamma}(M)) \neq 0$, then $\lambda \in \Gamma$. It remains to consider the following fragment of long exact sequence

$$\Hom_C(\Delta(\lambda), M/O_{\Gamma}(M)) \to \Ext^1_C(\Delta(\lambda), O_{\Gamma}(M)) \to 0$$

and notice that $\Hom_C(\Delta(\lambda), M/O_{\Gamma}(M)) = 0$.

An object $M \in \mathcal{C}[\Gamma]$ is called $\Gamma$-restricted iff $[M : L(\lambda)]$ is finite for any $\lambda \in \Gamma$. If $\Gamma = \bigcup_{1 \leq j \leq k}(\pi_j)$ and $\lambda \in \Gamma$, we denote by $(\lambda, \Gamma)$ (respectively, by $[\lambda, \Gamma)$) the finite set $\bigcup_{1 \leq j \leq k}[\lambda, \pi_j]$ (respectively, the finite set $\bigcup_{1 \leq j \leq k}[\lambda, \pi_j]$). The following theorem generalizes Corollary 4.1.

Theorem 4.1 Assume that a $\Gamma$-restricted object $M$ has an increasing (decreasing) $\Delta$-filtration. Then $\Ext^i_C(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$ and $i \geq 1$. Moreover, if we assume that $M$ has an increasing (decreasing) $\nabla$-filtration, then $\Ext^i_C(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$ and $i \geq 1$.

Proof. We consider the case of decreasing $\Delta$-filtration only, all other cases are similar. Without loss of generality, one can assume that $\lambda \in \Gamma$. As the set $A = (\lambda, \Gamma)$ is finite, there is a finite subobject $N \subseteq M$ such that $[N : L(\nu)] = [M : L(\nu)]$ for any $\nu \in A$. For sufficiently large $k$ it holds that $N \cap M_k = 0$, where $M_k$ is the $k$-th member of the corresponding decreasing $\Delta$-filtration of $M$. In particular, $[M_k : L(\nu)] = 0$ for $\nu \in A$. We have the fragment of long exact sequence

$$\ldots \to \Ext^i_C(M/M_k, \nabla(\lambda)) \to \Ext^i_C(M, \nabla(\lambda)) \to \Ext^i_C(M_k, \nabla(\lambda)) \to \ldots$$

It remains to notice that $\Ext^i_C(M/M_k, \nabla(\lambda)) = 0$ by Corollary 4.1 and $\Ext^i_C(M_k, \nabla(\lambda)) = 0$ by Lemma 4.1.

Lemma 4.4 Let $M$ be a $\Gamma$-restricted object such that $\Ext^1_C(\Delta(\lambda), M) = 0$ (respectively, $\Ext^1_C(M, \nabla(\lambda)) = 0$) for all $\lambda \in \Lambda$. Then for all $\lambda \in \Lambda$ and $i > 1$ we have $\Ext^i_C(\Delta(\lambda), M) = 0$ (respectively, $\Ext^i_C(M, \nabla(\lambda)) = 0$).

Proof. As in Theorem 4.1 any increasing chain $\pi < \pi_1 < \pi_2 < \ldots$ in $\Gamma$ has cardinality at most $[\pi, \Gamma]$. Again, one can assume that $\lambda \in \Gamma$. We work in the category $\mathcal{C}[\Gamma]$. The short exact sequence

$$0 \to Q \to P_{\Gamma}(\lambda) \to \Delta(\lambda) \to 0$$

induces

$$\ldots \to \Ext^{i-1}_C(Q, M) \to \Ext^i_C(\Delta(\lambda), M) \to 0.$$ 

Since $Q$ has a $\Delta$-filtration with factors $\Delta(\mu)$, where $\mu > \lambda$, one can argue by induction on $i$ and the partial order. The proof of the second statement is similar.

Theorem 4.2 Let $M$ be a $\Gamma$-restricted object and $\Ext^1_C(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$. Then $M$ has a decreasing $\nabla$-filtration. The symmetrical statement is also true, that is if $\Ext^1_C(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$, then $M$ has an increasing $\Delta$-filtration.
Proof. Suppose that $\text{Ext}^1_C(\Delta(\lambda), M) = 0$ for all $\lambda \in \Lambda$. By our assumption there is a decreasing chain of finitely generated ideals

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots$$

such that $\Gamma \setminus \Gamma_k$ is finite for any $k \geq 0$ and $\bigcap_{k \geq 0} \Gamma_k = \emptyset$. For the sake of shortness we denote $O_{\Gamma_k}(M)$ just by $M_k$. We have decreasing chain of subobjects

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots,$$

where any quotient $M/M_k$ is finite and $\bigcap_{1 \leq k} M_k = 0$. In fact, the socle of any $M/M_k$ belongs to $\Gamma \setminus \Gamma_k$ and therefore, it is finite. Thus $M/M_k$ can be embedded into a finite sum of finite indecomposable injectives from $C[\Gamma]$. Consider the following fragment of long exact sequence

$$\text{Hom}_C(\Delta(\mu), M/M_k) \rightarrow \text{Ext}^1_C(\Delta(\mu), M_k) \rightarrow 0 \rightarrow \text{Ext}^1_C(\Delta(\mu), M/M_k) \rightarrow \text{Ext}^2_C(\Delta(\mu), M_k).$$

If $\text{Ext}^1_C(\Delta(\mu), M_k) \neq 0$, then $\mu \in \Gamma_k$. On the other hand, the socle of $M/M_k$ does not belong to $\Gamma_k$, that is $\text{Hom}_C(\Delta(\mu), M/M_k) = 0$. In particular, $\text{Ext}^1_C(\Delta(\mu), M_k) = 0$ for any $\mu$. By Lemma 4.4 $\text{Ext}^2_C(\Delta(\mu), M_k) = 0$ for any $\mu$ also and therefore, $\text{Ext}^2_C(\Delta(\mu), M/M_k) = 0$. Since $M_{k-1}/M_k = O_{\Gamma_{k-1}}(M/M_k)$, one can repeat the above arguments to obtain that $\text{Ext}^1_C(\Delta(\mu), M_{k-1}/M_k) = 0$ for all $\mu$. Finally, any object $M_{k-1}/M_k$ is finite and we conclude the proof by the standard arguments from [7, 9].

For the second statement it is enough to prove that all subobjects $O^{\Gamma_k}(M)$ are finite. In fact, $O^{\Gamma_k}(M)$ contains a finite subobject $N$ such that $[N : L(\mu)] = [O^{\Gamma_k}(M) : L(\mu)]$ for all $\mu \in \Gamma \setminus \Gamma_k$. In particular, $O^{\Gamma_k}(M)/N$ belongs to $\Gamma_k$, that is $N = O^{\Gamma_k}(M)$. The final arguing is the same as above.

Corollary 4.3 Assume that $M$ is $\Gamma$-restricted and we have an exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow S \rightarrow 0.$$

Theorems 4.1 and 4.2 imply

1. If both $M$ and $N$ have decreasing $\nabla$-filtrations, then $S$ has a decreasing $\nabla$-filtration.
2. If both $M$ and $S$ have increasing $\Delta$-filtrations, then $N$ has an increasing $\Delta$-filtration.
3. If $M$ has a decreasing (respectively, increasing) $\nabla$ (respectively, $\Delta$)-filtration then any its direct summand has a decreasing (respectively, increasing) $\nabla$ (respectively, $\Delta$)-filtration.
4. If $M$ has a decreasing $\nabla$-filtration, then an object $\nabla(\lambda)$ appears exactly $(M : \nabla(\lambda)) = \dim \text{Hom}_C(\Delta(\lambda), M)$ times as a factor of it. Besides, $(M : \nabla(\lambda)) = (N : \nabla(\lambda)) + (S : \nabla(\lambda))$, provided $N$ has a decreasing $\nabla$-filtration.
5. If $M$ has an increasing $\Delta$-filtration, then an object $\Delta(\lambda)$ appears exactly $(M : \Delta(\lambda)) = \dim \text{Hom}_C(M, \nabla(\lambda))$ times as a factor of it. Moreover, $(M : \Delta(\lambda)) = (N : \Delta(\lambda)) + (S : \Delta(\lambda))$, provided $S$ has an increasing $\Delta$-filtration.
5 Donkin-Koppinen filtration and coadjoint action invariants

Lemma 5.1 The superalgebra $K[G]$ is a $\Lambda$-restricted $G \times G$-supermodule, where $\Lambda = (X(T)^- \times \{0,1\}) \times X(T)^+$. Proof. It is enough to prove that $O_{\Gamma}(K[G])$ is $\Gamma$-restricted for any (finitely) generated ideal $\Gamma$. By Corollary 4.2, Theorem 4.2, Corollary 4.3 and Proposition 3.2 we have

$$[O_{\Gamma}(K[G]) : L(-\lambda') \otimes L(\lambda')] = \dim \text{Hom}_{G \times G}(P_{\Gamma}(-\lambda', \lambda'), O_{\Gamma}(K[G])) \leq \sum_{(\mu', \mu') \in [(\lambda', \lambda'), \Gamma]} \dim \text{Hom}_{G \times G}(V_+(\mu') \otimes V_-(\mu'), O_{\Gamma}(K[G])) \leq \frac{[((\lambda', \lambda'), \Gamma)]}{2}.$$

The following theorem is now obvious.

Theorem 5.1 For any $\Gamma$ the $G \times G$-supersubmodule $O_{\Gamma}(K[G])$ has an decreasing (good) filtration $O_{\Gamma}(K[G]) = V_0 \supseteq V_1 \supseteq V_2 \supseteq \ldots$, such that $V_k/V_{k+1} \simeq V_-(\lambda_k)^* \otimes H^0_\mu(\lambda_k), k \geq 0$. Moreover, for any pair of indexes $k < l$ we have either $\lambda_k > \lambda_l$ or these weights are not comparable each to other.

Remark 5.1 By Proposition 3.2 a supermodule $V_-(\lambda)^* \otimes H^0_\mu(\lambda)$ appears as a factor of some good filtration of $O_{\Gamma}(K[G])$ iff $(-\lambda, \lambda) \in \Gamma$ and $(O_{\Gamma}(K[G]) : V_-(\lambda)^* \otimes H^0_\mu(\lambda)) = 1$. For the obvious reason the above filtration can be called Donkin-Koppinen filtration (see [10] and also notice in the end of this article).

Consider the coadjoint action $\nu_G$ of $G$ on $K[G]$. Define supersubalgebra $R = K[G]^G$ of coadjoint (rational) invariants. By definition, $R = \{ f \in K[G] | \nu_G(f) = f \otimes 1 \}$. As $\nu_G$ is dual to the adjoint action of $G$ on itself, these invariants can be also called adjoint. Moreover, an rational function $f \in K[G]$ belongs to $R$ iff for any $A \in SAlg_k$, $g_1, g_2 \in G(A)$ we have $f(g_1^{-1}g_2g_1) = f(g_1)$. In other words, invariants from $R$ are absolute ones.

The subalgebra of $R$, generated by polynomial invariants, is denoted by $R_{pol}$.

Let $H$ be an algebraic supergroup and $V \in H - mod, \dim V < \infty$. Fix a $Z_2$-homogeneous basis of $V$, say $v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}$, where $|v_i| = 0$ iff $1 \leq i \leq p$, otherwise $|v_i| = 1$. Set $\rho_V(v_i) = \sum_{1 \leq j \leq p+q} v_j \otimes f_{ji}, 1 \leq i \leq p + q$. Denote by $Tr(\rho)$ (or by $Tr(V)$, see [11]) the supertrace $\sum_{1 \leq i \leq p} f_{ii} - \sum_{p+1 \leq i \leq p+q} f_{ii}$. It is well known that $Tr(\rho)$ does not depend on the choice of $Z_2$-homogeneous basis of $V$ [11, 12, 13].

Lemma 5.2 The supertrace $Tr(\rho)$ belongs to the superalgebra of (co)adjoint invariants $K[H]^H$.

Proof. Denote by $S$ the Hopf supersubalgebra of $K[H]$, generated by the elements $f_{ij}$. We have the natural Hopf superalgebra epimorphism $K[GL(p|q)] \to S$, induced by the map $c_{ij} \mapsto f_{ij}$. It remains to refer to [12].

Consider $K[G]$ as a $T$-supermodule. Notice that this action is a restriction of the coaction $\rho$ on the (super)subgroup $1 \times T \subseteq T \times T$. Remind that a $G$-supermodule $V$ is semisimple as a $T$-supermodule and $V = \bigoplus_{\lambda \in X(T)} V_\lambda$, where $V_\lambda$ is a direct sum of one dimensional $T$-supermodules of weight $\lambda$. In particular, any element $f \in K[G]$ can be
represented as a sum $\sum_{\lambda \in X(T)} f_{\lambda}$, where $f_{\lambda} \in K[G]_{\lambda}$. A non-zero summand $f_{\lambda}$ of $f$ is called leading if $\lambda$ is maximal among all $\mu$ with $f_{\mu} \neq 0$. The corresponding weight $\lambda$ is also called leading.

**Lemma 5.3** Let $V$ be a $G$-supermodule. Choose a $\mathbb{Z}_2$-homogeneous basis $v_1, \ldots, v_t$ of $V$ such that $v_i \in V_{\lambda(i)}$ (it is possible that $\lambda(i) = \lambda(j)$ for $i \neq j$). Set $\rho_V(v_i) = \sum_{1 \leq j \leq t} v_j \otimes c_{ji}, 1 \leq i \leq t$. Then $c_{ji} \in K[G]_{\lambda(i)}, 1 \leq i, j \leq t$.

Proof. Straightforward calculations.

**Lemma 5.4** Let $V$ be a $G$-supermodule. Choose a $\mathbb{Z}_2$-homogeneous basis $e_1, \ldots, e_{m+n}$, where $|e_i| = 0$ iff $1 \leq i \leq m$, otherwise $|e_i| = 1$, and such that $\rho_E(e_i) = \sum_{1 \leq j \leq m+n} e_j \otimes c_{ji}, 1 \leq i \leq m + n$. Denote by $Ber(E)$ the one dimensional $G$-supermodule corresponding to the group-like element (berezinian) $Ber((c_{ij})) = det(C_{00} - C_{01}C_{11}^{-1})$. It is clear that $Ber(E) \in K[G]_\emptyset$, where $\emptyset = (1^m | -1^n) = (1, \ldots, 1 | -1, \ldots, -1)$.

Let $I(\tau)$ be a set of all maps $I : \tau \to m + n$. One can consider any $I \in I(\tau)$ as a multi-index $(i_1, \ldots, i_r)$, where $i_k = I(k), 1 \leq k \leq r$. For $I, J \in I(\tau)$ we set $x_{IJ} = (-1)^{s(I,J)} c_{IJ}$, where $c_{IJ} = \prod_{1 \leq k \leq \tau} c_{i_kj_k}, s(I,J) = \sum |i_k| (\sum_{s \leq l} |i_s| + |j_s|)$ and $|i| = |e_i|$. The superspace $E^{\otimes r}$ has a basis $e_I = e_{i_1} \otimes \ldots \otimes e_{i_r}, I \in I(\tau)$. It is known that $E^{\otimes r}$ has the natural structure of $G$-supermodule, as well as exterior and symmetric powers $\Lambda^r(E)$ and $S^r(E)$ respectively [15].

**Lemma 5.5** The structure of $G$-supermodule on $E^{\otimes r}$ is given by

$$\rho_{E^{\otimes r}}(e_I) = \sum_{J \in I(\tau)} e_J \otimes x_{IJ}.$$ 

Proof. Straightforward calculations.

Denote by $LI(\tau)$ the subsets of $I(\tau)$, consisting of all multi-indexes $I$ such that for some $k \leq r$ we have $i_1 < \ldots < i_k \leq m < j_{k+1} \leq \ldots \leq j_r$. Analogously, denote by $ST(\tau)$ the subsets of $I(\tau)$, consisting of all multi-indexes $I$ such that for some $k \leq r$ we have $i_1 \leq \ldots \leq i_k \leq m < j_{k+1} < \ldots < j_r$. Let $\pi_1 : E^{\otimes r} \to \Lambda^r(E)$ and $\pi_2 : E^{\otimes r} \to S^r(E)$ are canonical epimorphisms. Then $\pi_1(e_I)$ ($\pi_2(e_I)$) form a basis of $\Lambda^r(E)$ (respectively, basis of $S^r(E)$), when $I$ runs over $LI(\tau)$ (respectively, over $ST(\tau)$). Remind that the symmetric group $S_r$ acts on $E^{\otimes r}$ as $e_{I\sigma} = (-1)^{s(I,\sigma)} e_{I\sigma}$, where

$$s(I, \sigma) = |\{(k, l) | 1 \leq k < l \leq r, \sigma(k) > \sigma(l), |i_{\sigma(k)}| = |i_{\sigma(l)}| = 1\}|,$$

see [16]. If we replace $E$ by $E^c$ but preserve our notations, then this action turns into $e_{I\sigma} = (-1)^{s'(I,\sigma)} e_{I\sigma}$, where

$$s'(I, \sigma) = |\{(k, l) | 1 \leq k < l \leq r, \sigma(k) > \sigma(l), |i_{\sigma(k)}| = |i_{\sigma(l)}| = 0\}|.$$

As in [16] one can define two actions of $S_r$ on the set of monomials $x_{IJ}$. Precisely, $x_{I\sigma} = (-1)^{s(I,\sigma)} x_{I\sigma}$ and $x_{I\sigma} = (-1)^{s'(I,\sigma)} x_{I\sigma}$. It can be easily checked that $x_{I\sigma} x_{J\rho} = x_{IJ}$ and $x_{I\sigma} x_{J\rho} = x_{IJ}$. 

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Lemma 5.5 For any $r \geq 0$ we have

$$Tr(\Lambda^r(E)) = \sum_{I \in \mathcal{L}(r)} (-1)^{|I|} \sum_{\sigma \in \text{Stab}(I) \setminus S_r} x_{I \sigma, I},$$

and

$$Tr(S^r(E)) = \sum_{I \in \mathcal{S}(r)} (-1)^{|I|} \sum_{\sigma \in \text{Stab}(I) \setminus S_r} x_{I \sigma, I},$$

where $|I| = \sum_{1 \leq k \leq m+n} |i_k| = |e_I|$. Proof. We consider only the first equality, the second one is symmetrical. It is clear that for any $I \in \mathcal{L}(r)$ the vector $\pi_1(e_I)$ appears in $\rho(\pi_1(e_I))$ as follows:

$$\sum_{\sigma \in \text{Stab}(I) \setminus S_r} \pi_1(e_{I\sigma}) \otimes x_{I \sigma, I}.$$ 

It remains to notice that $\pi_1(e_{I\sigma}) = (-1)^{s(I, \sigma)} \pi_1(e_I)$.

Corollary 5.2 An invariant $C_r = Tr(\Lambda^r(E))$ has the unique leading summand of weight $(1^r, 0^{m-r})$ iff $r \leq m$, otherwise this summand has weight $(1^m | r - m, 0^{n-1})$.

Consider the dual $G$-supermodule $E^*$. By definition, $\rho(c^*_{ij}) = \sum_{1 \leq j \leq m+n} e^*_j \otimes c^*_ji$, where $e^*_j \delta_{ik} = c^*_ji. e^*_j = (-1)^{j[(j+i)l]} sG(c_{ij}), 1 \leq i, k \leq m + n [5]$. Denote $Tr(S^r(E^*))$ by $D_r$.

Corollary 5.3 An invariant $D_r$ has the unique leading summand of weight $(0^m | 0^{n-r}, (-1)^r)$ iff $r \leq n$, otherwise this summand has weight $(0^m | n-r, (-1)^n)$.

Proof. Any $e^*_j$ has weight $(0, \ldots, 0, -1, 0, \ldots, 0)$. Since the lowest weight, appearing in $S^r(E)$, is $(0^m | 0^{n-r}, 1^r)$ in the case $r \leq n$, otherwise it is $(0^m | r - n, 1^n)$, we are done by Corollary 5.1.

Let $f$ be an invariant from $R$. Denote by $M$ the $G \times G$-supersubmodule generated by $f$. Since $M$ is finite dimensional, the ideal

$$\Gamma = \bigcup_{(\mu^a, \mu^b), [M : L(-\mu^a) \otimes L(\mu^b)] \neq 0} ((\mu^a, \mu^b))$$

is finitely generated. Fix a Donkin-Koppinen filtration of $O_G(K[G])$, say $O_G(K[G]) = V_0 \supseteq V_1 \supseteq V_2 \supseteq \ldots$, as in Theorem 5.1.

Lemma 5.6 For any $t \geq 0$ the superspace $(V_t/V_{t+1})^G$ is even and one dimensional. Moreover, a non-zero leading vector from $(V_t/V_{t+1})^G$ has the unique leading summand of weight $\lambda_t$.

Proof. A non-zero basic vector $g$ of $(V_t/V_{t+1})^G \simeq Hom_G(V_-(\lambda_t), H^0_-(\lambda_t))$ can be represented as a sum $\sum_{\mu \leq \lambda_t} \sum_{1 \leq i \leq k_\mu} \phi_{i, \mu} \otimes v_{i, \mu}$, where $\phi_{i, \mu} \in V_-(\lambda_t)^*$ and $v_{i, \mu}$ runs over a basis of $H^0_-(\lambda_t)$. Besides, $g_\mu = \sum_{1 \leq i \leq k_\mu} \phi_{i, \mu} \otimes v_{i, \mu}$ with respect to the right $T$-action, $k_{\lambda_t} = 1$ and $\phi_{1, \lambda_t}(V_-(\lambda_t)) \neq 0$.
Corollary 5.4. The superalgebra $R$ is pure even.

Let $\lambda = (\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n}) \in X(T)^+$. As in [5] we denote $(\lambda_1, \ldots, \lambda_m)$ and $(\lambda_{m+1}, \ldots, \lambda_{m+n})$ by $\lambda_+$ and $\lambda_-$ correspondingly. The invariant

$$f_\lambda = \text{Ber}(E)^{\lambda_{m+n}} \prod_{1 \leq s \leq n-1} (\text{Ber}(E)D_s)^{\lambda_{m+n-s} - \lambda_{m+n-1}} \prod_{1 \leq t \leq m-1} C_t^{\lambda_t - \lambda_{t+1}},$$

has the unique leading summand of weight $\lambda$.

If $k$ is sufficiently large, then $M \cap V_{k+1}^+ = 0$. In particular, all weights of $f$ are among the weights of $\bigoplus_{0 \leq t \leq k} V_t/V_{t+1}$. Assume that $f \in V_t \setminus V_{t+1}, t \leq k$. By Lemma 5.5 $f$ has a leading summand of weight $\lambda_t$. By Corollary 4.2 and Remark 5.1 the invariant $f^* = f_{\lambda_t}$ also belongs to $V_t \setminus V_{t+1}$. It infers that the leading summand of $f_{\lambda_t}$ coincides with the above leading summand of $f$ up to a non-zero scalar. We call $f^* = f_{\lambda_t}$ a companion of $f$.

Denote by $A$ the free Laurent polynomial algebra $K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$. We have epimorphism of (super)algebras $\phi : K[G] \to A$, defined by $c_{ij} \mapsto \delta_{ij}x_i$ if $i \leq m$, otherwise $c_{ij} \mapsto \delta_{ij}y_i$. The algebra $A$ can be obviously identified with $K[T]$, that is $A$ has an obvious $T$-supermodule structure. Moreover, for any $\lambda \in X(T)$ the epimorphism $\phi$ takes $K[G]_\lambda$ to $A_\lambda$.

Theorem 5.2. (Chevalley’s restriction theorem) The restriction of $\phi$ on $R$ is a monomorphism.

Proof. Let $f \in R \setminus 0$. It is sufficient to prove that the image of a leading term of $f$ is not zero. In particular, if the image of the leading summand of its companion is not zero, then we are done. Since $A$ is an integral domain, it remains to notice that the images of the leading summands of $C_1, \ldots, C_m, D_1, \ldots, D_{n-1}$ and $\text{Ber}(E)$ are non-zero and use the above product representation of $f^*$.

Denote the images of $C_r$ and $D_r$ in $A$ by $c_r$ and $d_r$, respectively. It is clear that

$$c_r(x_1, \ldots, x_m, y_1, \ldots, y_n) = c_r(x|y) = \sum_{0 \leq i \leq \min\{r, m\}} (-1)^r - i \sigma_i(x)p_{r-i}(y),$$

where $\sigma_i(x)$ and $p_j(y)$ are elementary and complete symmetric functions correspondingly.

The subalgebra of $A$ consisting of polynomials $f(x|y) = f(x_1, \ldots, x_m, y_1, \ldots, y_n)$ symmetric in the $x$ and $y$ separately and such that, $\frac{\partial}{\partial t}(f|_{x_t = y_t}) = 0$, is denoted by $A_s$. If $\text{char} K = 0$, then they have already been considered in [13] and in [17] (as pseudosymmetric and supersymmetric polynomials respectively). Moreover, it was proved that the subalgebra $A_s$ is generated by the elements $c_r$. Since $\phi(R_{pol}) \subseteq A_s$ (see [13]) it implies that $\phi(R_{pol}) = A_s$ and $R_{pol}$ is generated by the elements $C_r$. If $\text{char} K = p > 0$ it is still open question: what are the generators of $R_{pol}$?

Let $\mathcal{V} = \{V\}$ be a collection of polynomial $G$-supermodules. It is called good iff for any $\lambda \in X(T)^+$ there is $V \in \mathcal{V}$ such that $\lambda$ is the highest weight of $V$. For example, the collection of all simple polynomial $G$-supermodules is good. As in [5] denote the largest polynomial supersubmodule of $H_0^0(\lambda)$ by $\nabla(\lambda)$. Remind that $\nabla(\lambda) \neq 0$ iff $L(\lambda)$ is polynomial if $\lambda \in X(T)^{++}$. The subset $X(T)^{++} \subseteq X(T)^+$ is completely described in [2]. The collection $\{\nabla(\lambda)\}_{\lambda \in X(T)^{++}}$ is also good.
Lemma 5.7 Denote by $X(T)^{++}_{\geq 0}$ the set \{\(\lambda \in X(T)^{++}| \lambda_m, \lambda_{m+n} \geq 0\)\} (it does not imply \(\lambda \in X(T)^{++}\)). Then for any \(\mu \in X(T)^{++}_{\geq 0}\) the set \(\langle \mu | \cap X(T)^{++}_{\geq 0}\) is finite.

Proof. Use the arguments from Example 4.1 and induction on \(|\mu_+|\).

Theorem 5.3 The algebra \(R_{\text{pol}}\) is generated (as a vector space) by \(\text{Tr}(V)\), where \(V\) runs over a good collection of \(G\)-supermodules.

Proof. Consider \(f \in R_{\text{pol}}\). As above, \(M\) is a \(G \times G\)-supersubmodule generated by \(f\) and \(M \subseteq \text{O}_T(K[G])\) for a finitely generated ideal \(\Gamma\). Besides, \(M \subseteq V_i, t \leq k\) and \(M \cap V_{k+1} = 0\) for a Donkin-Koppinen filtration \(\{V_i\}_{i \geq 0}\) of \(\text{O}_T(K[G])\). The ideal generated by \((\text{finite} \text{ly} \text{ many})\) \(\mu\) with \(f\mu \neq 0\) denote by \(\Gamma'\). It is obvious that \(\Gamma'\) is also generated by leading weights of \(f\). Since \(M|_{1 \times G}\) contains a polynomial \(G\)-supersubmodule (generated by \(f\)), in the quotient \(V_i/V_{i+1} \cong V_{-}(\lambda_i)^* \otimes H^0(\lambda_i)\) the right hand side factor \(H^0(\lambda_i)\) has non-zero polynomial part. In other words, \(\lambda_i \in X(T)^{++}\) and there is \(V \in \mathcal{V}\) whose highest weight is \(\lambda_i\). Again, \(\text{Tr}(V) \in V_i \setminus V_{i+1}\) and for a non-zero scalar \(a\) all weights of polynomial invariant \(f - a\text{Tr}(V)\) belong to \((\Gamma' \cap X(T)^{++}_{\geq 0}) \setminus \{\lambda_i\}\). Lemma 5.7 concludes the proof.

Let \(V\) be a \(G\)-supermodule with a basis as in Lemma 5.3. The (Laurent) polynomial \(\chi(V) = \sum_{1 \leq i \leq t}(-1)^{r_i} \phi(c_{ii})\) is called formal supercharacter of \(V\). It is clear that \(\chi(V) = \sum_{\lambda \in X(T)}(\dim(V_\lambda) - \dim(V_\lambda)_1)x^\lambda y^{\lambda^\vee}\), where \(x^\lambda = \prod_{1 \leq i \leq m}x_i^{\lambda_i}\) and \(y^{\lambda^\vee} = \prod_{m+1 \leq i \leq m+n}y_i^{\lambda_i} \cdot \prod_{i \leq j \leq m+n}y_i^{\lambda_j}\).

Example 5.1 (see [14]) Set \(m = n = 1\). A simple polynomial \(GL(1|1)\)-supermodule is at most two dimensional. More precisely, set \(X(T)^{++} = \bigcup_{r \geq 0} X(T)^{++}_r\), where \(X(T)^{++}_r = \{\lambda = (\lambda_1|\lambda_2) \in X(T)^{++}| \lambda \in \Gamma\}\). If \(p | r\), then \(X(T)^{++}_r = \{(i, r - i)| 0 \leq i \leq r\}\) and \(L(i) = L((i)|r - i)\) is (even) one dimensional. In particular, \(\text{Tr}(L(i)) = x_1^iy_1^{r-i}\) obviously belongs to \(A_2\). Otherwise, \(X(T)^{++}_r = \{(i, r - i)| 1 \leq i \leq r\}\) and \(L(i) = L((i)|r - i)\) is a two dimensional supermodule. By definition, its highest vector is even and the second basic vector is odd of weight \((i - 1)|r - i + 1\). Thus \(\text{Tr}(L(i)) = x_1^iy_1^{r-i} - x_1^{r-i}y_1^r \in A_2\).

A homogeneous polynomial \(f(x|y) = \sum_{\lambda \in X(T)} a_\lambda x^\lambda y^{\lambda^\vee}\) is said to be \(p\)-balanced iff for any \(\lambda\) with \(a_\lambda \neq 0\) and any \(1 \leq i \leq m < j \leq m+n\) it satisfies \(p(\lambda_i + \lambda_j)\).

Hypothesis 5.1 The algebra \(\phi(R_{\text{pol}})\) coincides with \(A_2\). Moreover, it is generated by all \(c_r\) and by all \(p\)-balanced polynomials symmetric in \(x\) and \(y\) separately.

Problem 5.1 What are the preimages of the \(p\)-balanced polynomials from \(A_2\)?

Acknowledgements

This work was supported by RFFI 07-01-00392 and by INDAM.

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