A Comment on the Sums $\sum_{n \in \mathbb{Z}} \frac{(-1)^{nk}}{(an+1)^k}$

Vivek Kaushik

Abstract

We recall a proof of Euler's identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ involving the evaluation of a double integral. We extend the method to find Hurwitz Zeta series of the form $S(k, a) = \sum_{n \in \mathbb{Z}} \frac{(-1)^{nk}}{(an+1)^k}$, where $a \in \mathbb{N} \setminus \{1\}$ and $k \in \mathbb{N}$. In particular, we consider a general $k$-dimensional integral over $(0, 1)^k$ that equals the series representation $S(k, a)$. Then we use an algebraic change of variables that diffeomorphically maps $(0, 1)^k$ to a $k$-dimensional hyperbolic polytope. We interpret the integral as a sum of two probabilities, and find explicit representations of such probabilities with combinatorial techniques.

1 Introduction

In this article, we evaluate Hurwitz Zeta Series of the form

$$S(k, a) = \sum_{n \in \mathbb{Z}} \frac{(-1)^{nk}}{(an+1)^k}, \quad a \in \mathbb{N} \setminus \{1\}, \quad k \in \mathbb{N}. \quad (1.1)$$

The values of such series can be obtained through standard techniques from Fourier Analysis and complex variables. Some specific examples of $S(k, a)$ are found in [1, 2, 11], with the case $a = 4$ being the focal point of [3, 6–8].

In particular, we provide an alternative method using multiple integration. We generalize the double integral proof of

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (1.2)$$

by Zagier and Kontsevich [9, p. 8 - 9], which is as follows. Consider the integral

$$I_2 = \int_{(0,1)^2} \frac{1}{\sqrt{x_1 x_2(1 - x_1 x_2)}} \, dx_2 \, dx_1. \quad (1.3)$$

We convert the integrand into a geometric series,

$$\frac{1}{\sqrt{x_1 x_2(1 - x_1 x_2)}} = \sum_{n \geq 0} (x_1 x_2)^{n-1/2}. \quad (1.4)$$

Replacing the integrand with the geometric series representation, we find

$$I_2 = \int_{(0,1)^2} \sum_{n \geq 0} (x_1 x_2)^{n-1/2} \, dx_1 \, dx_2 \quad (1.5)$$

$$= \sum_{n \geq 0} \int_{(0,1)^2} (x_1 x_2)^{n-1/2} \, dx_1 \, dx_2$$

$$= \sum_{n \geq 0} \frac{1}{(n + 1/2)^2}$$

$$= \sum_{n \geq 0} \frac{4}{(2n + 1)^2}$$

$$= 4S(2, 2),$$

1
where the interchanging of sum and integral in (1.5) follows from the Monotone Convergence Theorem. On the other hand, the change of variables

$$x_1 = \frac{\xi_1^2 (1 + \xi_2^2)}{1 + \xi_1^2}, \quad x_2 = \frac{\xi_2^2 (1 + \xi_1^2)}{1 + \xi_2^2},$$

(1.6)

has Jacobian Determinant

$$\det \frac{\partial (x_1, x_2)}{\partial (\xi_1, \xi_2)} = \frac{4\sqrt{x_1 x_2} (1 - x_1 x_2)}{(1 + \xi_1^2)(1 + \xi_2^2)}$$

(1.7)

and diffeomorphically maps \((0, 1)^2\) to the hyperbolic triangle

$$\mathbb{H}^2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \xi_2 < 1, \xi_1, \xi_2 > 0\}.$$  

(1.8)

Hence,

$$I_2 = \int_{\mathbb{H}^2} \frac{4}{(1 + \xi_1^2)(1 + \xi_2^2)} \, d\xi_2 \, d\xi_1 = \int_0^\infty \frac{4 \cot^{-1}(\xi_1)}{1 + \xi_1^2} \, d\xi_1 = \int_0^\infty \frac{2\pi - 4 \tan^{-1}(\xi_1)}{1 + \xi_1^2} \, d\xi_1 = \frac{\pi^2}{2}.$$

Thus,

$$S(2, 2) = \frac{\pi^2}{8}.$$

Finally, we can write

$$\sum_{n \geq 1} \frac{1}{n^2} = \sum_{n \geq 1} \frac{1}{(2n)^2} + \sum_{n \geq 0} \frac{1}{(2n + 1)^2} = \sum_{n \geq 1} \frac{1}{(2n)^2} + S(2, 2) = \frac{1}{4} \zeta(2) + \frac{\pi^2}{8},$$

from which we may deduce

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

using simple algebra.

We extend this proof to find arbitrary \(S(k, a)\). In particular, we evaluate the integral

$$I_{k,a} = \frac{1}{a^k} \int_{(0,1)^k} \frac{(x_1 \ldots x_k)^{-1+1/a} + (x_1 \ldots x_k)^{-1/a}}{1 - (-1)^k x_1 \ldots x_k} \, dx_1 \ldots dx_k,$$

the generalization of (1.3) in two ways. The first way will be to convert the integrand into a geometric series and show that it is equal to \(S(k, a)\). On the other hand, we will use a change of variables generalizing (1.6).

### 2 Evaluation of \(I_{k,a}\)

#### 2.1 From Integral to Series

We will evaluate

$$I_{k,a} = \frac{1}{a^k} \int_{(0,1)^k} \frac{(x_1 \ldots x_k)^{-1+1/a} + (x_1 \ldots x_k)^{-1/a}}{1 - (-1)^k x_1 \ldots x_k} \, dx_1 \ldots dx_k,$$

(2.1)

the generalization of (1.3) in two ways.
Theorem 2.1.1. We have

\[ I_{k,a} = S(k,a). \]

Proof. First, we convert the integrand into a geometric series as such

\[ \frac{1}{a^k} \frac{(x_1 \ldots x_k)^{-1+1/a} + (x_1 \ldots x_k)^{-1/a}}{1 - (-1)^k x_1 \ldots x_k} = \sum_{n \geq 0} \frac{(-1)^n}{a^k} (x_1 \ldots x_k)^{n-1+1/a} + (x_1 \ldots x_k)^{n-1/a}. \]  

(2.2)

Replacing this geometric series representation with the integrand in \( I_{k,a} \) we obtain

\[ I_{k,a} = \int_{(0,1)^k} \sum_{n \geq 0} \frac{(-1)^n}{a^k} (x_1 \ldots x_k)^{n-1+1/a} + (x_1 \ldots x_k)^{n-1/a} \, dx_1 \ldots dx_k \]

\[ = \sum_{n \geq 0} \int_{(0,1)^k} \frac{(-1)^n}{a^k} (x_1 \ldots x_k)^{n-1+1/a} + (x_1 \ldots x_k)^{n-1/a} \, dx_1 \ldots dx_k \]

\[ = \sum_{n \geq 0} \frac{(-1)^n}{a^k(n+1/a)^k} + \sum_{n \geq 0} \frac{(-1)^n}{a^k(n-1/a+1)^k} \]

\[ = \sum_{n \geq 0} \frac{(-1)^n}{(an+1)^k} + \sum_{n \geq 0} \frac{(-1)^n}{(an+a-1)^k} \]

\[ = \sum_{n \geq 0} \frac{(-1)^n}{(an+1)^k} + \sum_{n \leq -1} \frac{(-1)^n}{(an+1)^k} \]

\[ = \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{(an+1)^k} = S(k,a), \]

where the interchanging of sum and integral in (2.3) follows from the Monotone Convergence Theorem.

\[ \square \]

2.2 From Integral to Hyperbolic Polytope

Now, we evaluate \( I_{k,a} \) directly. We use the change of variables

\[ x_i = \frac{\xi_i^a(1+\xi_{i+1}^a)}{1+\xi_i^a}, \quad i \in \{1, \ldots, k\}. \]

(2.4)

where we cyclically index mod \( k \), that is, we have \( \xi_{k+1} := \xi_1 \).

Theorem 2.2.1. The change of variables from (2.4) has Jacobian Determinant

\[ \det \frac{\partial(x_1, \ldots, x_k)}{\partial(\xi_1, \ldots, \xi_k)} = \begin{cases} a(\xi_1)^{a-1} & k = 1 \\ a^k(\xi_1 \cdots \xi_k)^{a-1} (1 - (-1)^k(\xi_1 \cdots \xi_k)^a) & \text{else.} \end{cases} \]

and diffeomorphically maps \((0,1)^k\) to the hyperbolic polytope

\[ \mathbb{H}^k = \{ (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k : \xi_i \xi_{i+1} < 1, \xi_i > 0, \, i \in \{1, \ldots, k\} \}. \]

Proof. The case \( k = 1 \) is trivial. The case \( k = 2 \) recovers the change of variables in (1.6) from the introduction; one may see that the stated results in (1.7) and (1.8) corroborate the theorem.

Suppose \( k > 2 \). Note that

\[ \frac{\partial x_i}{\partial \xi_j} = \begin{cases} \frac{a\xi_i^{a-1}(1+\xi_{i+1}^a)}{(1+\xi_i^a)^2} & j = i \\ \frac{a\xi_i^{a-1}(-1)^j \xi_i^{a-1}}{(1+\xi_i^a)^2} & j \equiv i + 1 \mod k \cdot \\ 0 & \text{else} \end{cases} \]
These are the entries of the Jacobian matrix \( \frac{\partial (x_1, \ldots, x_k)}{\partial (\xi_1, \ldots, \xi_k)} \) corresponding to (2.4). Using cofactor expansion along the first row, we find that
\[
\det \frac{\partial (x_1, \ldots, x_k)}{\partial (\xi_1, \ldots, \xi_k)} = \frac{\partial x_1}{\partial \xi_1} \det(A) - \frac{\partial x_2}{\partial \xi_2} \det(B),
\]
where \( A \) and \( B \) are \((k-1) \times (k-1)\) matrices with entries
\[
A_{ij} = \left[ \frac{\partial x_{i+1}}{\partial \xi_j} \right]_{i,j \neq 1}, \quad B_{ij} = \left[ \frac{\partial x_{i+1}}{\partial \xi_j} \right]_{i \neq 1, j \neq 2}.
\]
It can be seen that \( A \) is lower triangular, and \( B \) is upper triangular. Hence their determinants are easy to calculate using cofactor expansions on the top row of \( A \) and the bottom row of \( B \), respectively. The result will simplify down to the claimed Jacobian determinant.

For the second statement, it can be shown that (2.4) is a \( C^1 \) bijective map from \((0, 1)^k\) to \( \mathbb{H}^k \) with \( \det \frac{\partial (x_1, \ldots, x_k)}{\partial (\xi_1, \ldots, \xi_k)} \neq 0 \) in \( \mathbb{H}^k \). The Inverse Function Theorem guarantees on any local neighborhood in \( \mathbb{H}^k \), we will have
\[
\frac{\partial (x_1, \ldots, x_k)^{-1}}{\partial (\xi_1, \ldots, \xi_k)} = \frac{\partial (\xi_1, \ldots, \xi_k)}{\partial (x_1, \ldots, x_k)}.
\]

**Remark.** When \( a = 2 \), if we instead were to make the substitution \( x_i = \sqrt{\frac{\xi_i^2 (1 + \xi_{i+1}^2)}{1 + \xi_i^2}} \) and then \( \xi_i = \tan(u_i) \) will result in us obtaining
\[
x_i = \frac{\sin(u_i)}{\cos(u_{i+1})},
\]
which is Calabi’s trigonometric change of variables, considered in all of [3][8][11]. Hence, we may view (2.4) as an algebraic generalization of Calabi’s change of variables.

Hence, our two theorems and the change of variables formula imply
\[
S(k, a) = \int_{\mathbb{R}^k} \frac{1}{(1 + \xi_1^2) \cdots (1 + \xi_k^2)} \ d\xi_1 \cdots d\xi_k
= \int_{\mathbb{R}^k} \frac{1}{1 + \xi_1^2} \ d\xi_1 \cdots d\xi_k + \int_{\mathbb{R}^k} \frac{\xi_1 \cdots \xi_k}{(1 + \xi_1^2) \cdots (1 + \xi_k^2)} \ d\xi_1 \cdots d\xi_k \quad (2.5)
\]
We wish to evaluate (2.5) by mimicking the combinatorial analysis used in [11] p. 592 - 599]

### 2.3 Hyperbolic Polytope and Combinatorics

We write \([m] := \{1, \ldots, m\}\) for \( m \in \mathbb{N} \). Let \( \Xi_i \) for \( i \in [k] \) be independent and identically distributed with density function
\[
f_{\Xi_i}(\xi_i) = \frac{\frac{a}{2} \sin \left( \frac{\pi}{2} \xi_i \right)}{1 + \xi_i^a}, \quad \xi_i \geq 0.
\]
Similarly, let \( \Theta_i \) for \( i \in [k] \) be independent and identically distributed with density function
\[
f_{\Theta_i}(\theta_i) = \frac{\frac{a}{2} \sin \left( \frac{\pi}{2} \theta_i \right) \theta_i^{a-2}}{1 + \theta_i^a}, \quad \theta_i \geq 0.
\]

**Theorem 2.3.1.** For each \( i \in [k] \), both \( f_{\Xi_i}(\xi_i) \) and \( f_{\Theta_i}(\theta_i) \) are valid density functions.

**Proof.** We recall the cumulative distribution function for \( \Xi_i \) and \( \Theta_i \) are
\[
F_{\Xi_i}(t) = \int_0^t f_{\Xi_i}(\xi_i) \ d\xi_i \quad (2.6)
\]
\[
F_{\Theta_i}(t) = \int_0^t f_{\Theta_i}(\theta_i) \ d\theta_i. \quad (2.7)
\]
The claim is equivalent to showing \( \lim_{t \to \infty} F_{X_i}(t) = 1 \) and \( \lim_{t \to \infty} F_{\Theta_i}(t) = 1 \).

According to Gradshteyn and Ryzhik [10 Section 2.142], we see
\[
\int \frac{1}{1 + x^a} \, dx = \left\{ \begin{array}{ll}
\frac{a}{a} \sum_{j=0}^{a/2-1} P_j(x) \cos \left( \frac{2j+1}{a} \pi \right) + Q_j(x) \sin \left( \frac{2j+1}{a} \pi \right) & \text{a even} \\
\frac{1}{a} \log(1 + x) - \frac{a}{2} \sum_{j=0}^{a/2-1} P_j(x) \cos \left( \frac{2j+1}{a} \pi \right) + Q_j(x) \sin \left( \frac{2j+1}{a} \pi \right) & \text{a odd}
\end{array} \right.,
\]
where
\[
P_j(x) = \frac{1}{2} \log \left( x^2 - 2x \cos \left( \frac{2j+1}{a} \pi \right) + 1 \right)
\]
\[
Q_j(x) = \arctan \left( \frac{x - \cos \left( \frac{2j+1}{a} \pi \right)}{\sin \left( \frac{2j+1}{a} \pi \right)} \right).
\]
It can be shown upon plugging in \( x = 0 \), and converting the cosines and sines into complex exponentials, that the right hand side of (2.8) is \( -\frac{a}{a} \csc(\pi/a) \). It also can be shown that as \( x \to \infty \), the right hand side of (2.8) approaches 0. Observing these facts will allow us to deduce \( \lim_{t \to \infty} F_{X_i}(t) = 1 \).

The second result follows from making the substitution \( \theta_i \mapsto 1/\theta_i \) in the defining integral representation presented in (2.7) and deducing \( F_{\Theta_i}(t) = 1 - F_{\Xi_i}(1/t) \).

Our main goal is to evaluate
\[
Pr(\Xi_{i+1} < 1, i \in [k]) + Pr(\Theta_i \Theta_{i+1} < 1, i \in [k]),
\]
where \( \Xi_{k+1} := \Xi_1, \Theta_{k+1} := \Theta_1 \). In words, (2.9) is the sum of the probability that all \( \Xi_i \) have cyclically consecutive products less than 1 and the probability that all \( \Theta_i \) have cyclically consecutive products less than 1. It is easy to see through (2.9) that \( S(k, a) \) is precisely the product of \((\frac{\sin(\frac{\pi}{a})}{a})^k \) and (2.9).

We begin with the easy case in calculating (2.9).

**Theorem 2.3.2.** Suppose \( \Xi_i, \Theta_i < 1 \) for each \( i \in [k] \). Then (2.9) is equal to
\[
\left( \int_0^1 \frac{a}{1 + \xi^a} \, d\xi \right)^k + \left( \int_0^1 \frac{\pi}{\theta^{a-2}} \, d\theta \right)^k
\]

**Proof.** Clearly for any \( i \in [k] \), we have \( \Xi_i \Xi_{i+1}, \Theta_i \Theta_{i+1} < 1 \). Hence (2.9) is equal to
\[
\int_{(0,1)^k} f_{\Xi_i}(\xi_1) \cdots f_{\Xi_i}(\xi_k) \, d\xi_1 \cdots d\xi_k + \int_{(0,1)^k} f_{\Theta_i}(\theta_1) \cdots f_{\Theta_i}(\theta_k) \, d\theta_1 \cdots d\theta_k,
\]
from which the result immediately follows. \( \square \)

The nontrivial case is when there exists \( i \in [k] \) such that \( \Xi_i, \Theta_i \geq 1 \). We wish to set up an explicit integral representation of (2.9) in this case.

**Theorem 2.3.3.** Suppose \( \Xi_i, \ldots, \Xi_k, \Theta_1, \ldots, \Theta_k \) satisfy the conditions as described by their respective probability terms in (2.9). Suppose further \( r_1, \ldots, r_n \in [k] \) are distinct with \( 1 \leq r_{n} \leq \cdots \leq \Xi_{r_1} \) and \( 1 \leq \Theta_{r_n} \leq \ldots \leq \Theta_{r_1} \). Then for distinct \( i, j \in [n] \), we have \( r_i \) and \( r_j \) are pairwise cyclically nonconsecutive, that is, \( |r_i - r_j| \notin \{1, k - 1\} \). In addition, \( n \leq [k/2] \).

**Proof.** The proofs of these statements are identical to those in [11 Theorem 3.2, 3.3]. \( \square \)

We now use a mechanism to set up the integral corresponding to (2.9) if \( r_1, \ldots, r_n \in [k] \) satisfy the first statement of the previous theorem. For each \( j \in [n] \), define \( \alpha_j \) to be the number of \( \Xi_z \) (or \( \Theta_z \) from \( \{\Xi_{r_1}, \ldots, \Xi_{r_n}\} \) (or \( \{\Theta_{r_1}, \ldots, \Theta_{r_n}\} \)) with the property that \( \text{sup}(\Xi_z) = 1/\Xi_{r_z} \) (or \( \text{sup}(\Theta_z) = 1/\Theta_{r_z} \)). In words, \( \alpha_j \) counts the number of bounds of the form \( 0 < \Xi_{z} < 1/\Xi_{r_z} \) (or \( 0 < \Theta_{z} < 1/\Theta_{r_z} \)) that will appear when we set up the integral for the first probability term (or second probability term) in (2.9).

**Theorem 2.3.4.** We have
\[
\alpha_j = 2 - \delta(k, 2) - \sum_{m=0}^{j-1} \delta(|r_m - r_j|, 2) + \delta(|r_m - r_j|, k - 2),
\]
where \( \delta(a, b) = 1 \) if \( a = b \) and 0 else.

5
Proof. The proof is identical to that of [11] Theorem 3.5. □

Now we are ready to set up an integral representation for \((2.9)\) if \(r_1, \ldots, r_k \in [k]\) satisfy the first statement in Theorem 2.3.3.

**Theorem 2.3.5.** If \(r_1, \ldots, r_k \in [k]\) satisfy the first statement in Theorem 2.3.3, we have \((2.9)\) is equal to the sum of the integrals

\[
J_{r_1, \ldots, r_n} = (\psi(1))^{k-n - \sum_{j=1}^n \alpha_j} \int_{\xi_{r_1}} \int_{\theta_{r_1}} \frac{\left(\frac{\sin \left(\frac{\xi}{\alpha_j}\right)}{\xi}\right)^n \left(\frac{\sin \left(\frac{\theta}{\alpha_j}\right)}{\theta}\right)^{\alpha_j} \cdots \left(\frac{\sin \left(\frac{\xi}{\alpha_j}\right)}{\xi}\right)^{\alpha_n}}{(1 + \xi_{r_1}) \cdots (1 + \xi_{r_1})} \ d\xi_{r_1} \cdots d\theta_{r_1}
\]

\[
K_{r_1, \ldots, r_n} = (\phi(1))^{k-n - \sum_{j=1}^n \alpha_j} \int_{\theta_{r_1}} \int_{\theta_{r_1}} \frac{\left(\frac{\sin \left(\frac{\theta}{\alpha_j}\right)}{\theta}\right)^{\alpha_j} \cdots \left(\frac{\sin \left(\frac{\theta}{\alpha_j}\right)}{\theta}\right)^{\alpha_n}}{(1 + \theta_{r_1}) \cdots (1 + \theta_{r_1})} \ d\theta_{r_1} \cdots d\theta_{r_1},
\]

where \(\psi(t)\) and \(\phi(t)\) are the cumulative distribution functions defined in \((2.9)\), \(2.7\), respectively.

Proof. We already know the integral bounds for \(\xi_{r_1}, \ldots, \xi_{r_n}\). We already know there are \(\sum_{j=1}^n \alpha_j\) bounds of the form \(0 < \Xi < 1/\Xi_{r_j}\). This means there are \(k - n - \sum_{j=1}^n \alpha_j\) bounds of the form \(0 < U_t < 1\).

Explicitly, the first probability term in \((2.9)\) is

\[
\int_{\xi_{r_1}}^{1/\xi_{r_1}} \int_{\theta_{r_1}}^{1/\theta_{r_1}} \cdots \int_{\xi_{r_1}}^{1/\xi_{r_1}} \frac{\xi_{r_1}}{\xi_{r_1}} \cdots \frac{\xi_{r_1}}{\theta_{r_1}} \ d\xi_{r_1} \cdots d\theta_{r_1}
\]

where \(dV\) is the product of the differentials \(d\xi_{r_1}, \ldots, d\theta_{r_k}\) in the appropriate order as dictated by the integral bounds. It follows that \((2.10)\) is equal to \(J_{r_1, \ldots, r_n}\) upon evaluating the innermost integrals.

A similar argument can be used to show that the second probability in \((2.9)\) is equal to \(K_{r_1, \ldots, r_n}\). □

Our theorems and \((2.5)\) give the following result

\[
\sum_{n \geq 0} \frac{(-1)^n}{(an + 1)^k} = \left(\int_0^1 \frac{\sin \left(\frac{\theta}{a}\right)}{1 + \xi} d\xi\right)^k + \left(\int_0^1 \frac{\sin \left(\frac{\theta}{a}\right)}{1 + \theta} d\theta\right)^k + \sum_{n=1}^{\lceil k/2 \rceil} \sum_{\{r_1, ..., r_n\} \in [k]^n \setminus \{r_i \in \{0, 1, k-1\}, i \neq j \in [n]\}} J_{r_1, ..., r_n} + K_{r_1, ..., r_n},
\]

where \(J_{r_1, ..., r_n}\) and \(K_{r_1, ..., r_n}\) are defined as in the previous theorem.

**References**

1. Paul Bourgade, Takahiko Fujita, and Marc Yor, *Euclidean formula* for \((2n)\) and products of Cauchy variables, Electronic Communications in Probability 12 (2007), 73–80, DOI 10.1214/ECP.v12-1244.

2. Junesang Choi, *Evaluation of Certain Alternating Series*, Honam Mathematical J. 36 (2014), 263–273.

3. Frits Beukers, Eugenio Calabi, and Johan AC Kolk, *Sums of generalized harmonic series and volumes*, Nieuw Archief voor Wiskunde 11 (1993), 561-573.

4. FMS Lima, *New definite integrals and a two-term dilogarithm identity*, Indagationes Mathematicae 23 (2012), no. 1, 1-9.

5. Joseph D'Avanzo and Nikolai Krylov, *ζ(n) via hyperbolic functions*, Involves, a Journal of Mathematics 3 (2010), no. 3, 289–296.

6. Noam David Elkies, *On the Sums \(\sum_{n=1}^{\infty} (4k+1)^{-n}\)*, The American Mathematical Monthly 110 (2003), no. 7, 561-573.

7. Zurab Silagadze, *Sums of generalized harmonic series for kids from five to fifteen*, arXiv preprint arXiv:1003.3602 (2010).

8. ———, *Comment on the sums \(S(n)\)* = \(\sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n}\), Georgian Math. J. 19 (2012), no. 3, 587–595.

9. Don Zagier and Maxim Kontsevich, *Periods*, Springer, 2001.

10. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh (Daniel and Moll Zwillinger Victor H., ed.), Academic Press, 2007.

11. Vivek Kaushik and Daniele Ritelli, *Evaluation of harmonic sums with integrals*, Quart. Appl. Math. 76 (2018), no. 3, 577–600, DOI 10.1090/qam/1499.