ON THE POSSIBLE TEMPERATURES FOR FLOWS ON A UHF ALGEBRA

KLAUS THOMSEN

1. Introduction

A flow on a $C^*$-algebra $A$ is a continuous one-parameter group $\alpha = (\alpha_t)_{t \in \mathbb{R}}$ of automorphisms of $A$. In quantum statistical mechanics such a flow is sometimes interpreted as the time evolution of the observables of a physical system and the equilibrium states of the system are then given by states $\phi$ of $A$ that satisfy the trace-like condition

$$\phi(ab) = \phi(b\alpha_\beta(a))$$

for all $a, b \in A$ with $a$ analytic for $\alpha$, $\beta$. The number $\beta$ for which this holds is interpreted as the inverse temperature of the system in the state given by $\phi$ and $\phi$ is said to be a $\beta$-KMS state for $\alpha$. The $C^*$-algebra $A$ of observables in such a model is often a UHF algebra and it is therefore of interest to determine the KMS states and the possible inverse temperatures which can occur for flows on a UHF algebra. For such flows it follows from work by Powers and Sakai in the 70’s, $\text{PS}$, that when the flow is approximately inner the set of possible inverse temperatures is the whole real line $\mathbb{R}$, and Powers and Sakai conjectured in $\text{PS}$ that all flows on a UHF algebra are approximately inner. That this is not the case was proved by Matui and Sato in $\text{MS2}$, following work by Kishimoto on the AF algebra case, $\text{Ki2}$. Presently it is therefore an open question which sets of real numbers can occur as the set of inverse temperatures for a flow on a UHF algebra, and it is the purpose with this paper to contribute to the possible solution of this problem. Although our main aim is to handle UHF algebras, the methods we develop work well for the much larger class of simple unital AF algebras and we proceed therefore in that generality.

In the examples of Kishimoto, and Matui and Sato, the set of possible inverse temperatures is as small as it can be, consisting only of 0, cf. Remark 3.2 in $\text{Th3}$. Here we will show that a variant of the method developed in $\text{Th3}$ can be used to construct flows on any infinite dimensional simple unital AF algebra such that the set of inverse temperatures is any given compact set of real numbers containing zero. The method is a natural extension of ideas and methods from work by Bratteli, Elliott, Herman in $\text{BEH}$, Kishimoto in $\text{Ki2}$ and Matui and Sato in $\text{MS2}$. As in all these papers we depend on results from
the classification of simple $C^*$-algebras. Besides the classification of AF algebras, which was fundamental already in [BEH], we apply the recent work by Castillejos, Evington, Tikuisis, White and Winter in [CETWW], where the classification results we use in turn are based on work by Elliott, Gong, Lin and Niu, [GLN], [EGLN], among others. However, when the algebra in question is UHF there is an alternative route to the desired result which we describe Remark 3.12.

Our results answer Question 2.6 in [Th3] positively, but Question 6.2 in [Th2] only partially. What remains is to pass from compact to closed subsets of the real line. It may be that the methods we use can be developed to achieve this, but some fundamental changes are necessary because when the set of inverse temperatures contains arbitrarily large numbers the flow must have ground states, and this is not possible for periodic flows whose fixed point algebra is simple, cf. Remark 3.5 in [BEH].

Acknowledgement I am grateful to Y. Sato for comments on the first version of this paper which helped me navigate in the literature on the classification of simple $C^*$-algebras. The work was supported by the DFF-Research Project 2 'Automorphisms and Invariants of Operator Algebras', no. 7014-00145B.

2. Statement of results

**Theorem 2.1.** Let $A$ be a unital infinite dimensional simple AF $C^*$-algebra. For each non-empty closed face $F$ in the tracial state space $T(A)$ of $A$ and for each compact set $K \subseteq \mathbb{R}$ of real numbers containing 0 there is a $2\pi$-periodic flow $\alpha$ on $A$ such that

- the fixed point algebra $A^\alpha$ of $\alpha$ is a simple AF algebra,
- there is a $\beta$-KMS state for $\alpha$ if and only if $\beta \in K$,
- for $\beta \in K \setminus \{0\}$ the simplex of $\beta$-KMS states for $\alpha$ is affinely homeomorphic to $F$, and
- all traces of $A$ are $\alpha$-invariant and the simplex of 0-KMS states for $\alpha$ is an affinely homeomorphic to $T(A)$.

By taking $F$ to consist of a single extremal point in $T(A)$ we get the following

**Corollary 2.2.** Let $A$ be a unital infinite dimensional simple AF $C^*$-algebra. For each compact set $K \subseteq \mathbb{R}$ of real numbers containing 0 there is a $2\pi$-periodic flow $\alpha$ on $A$ such that

- the fixed point algebra $A^\alpha$ of $\alpha$ is a simple AF algebra,
- there is a $\beta$-KMS state for $\alpha$ if and only if $\beta \in K$,
- for $\beta \in K \setminus \{0\}$ the $\beta$-KMS state is unique, and
- all traces of $A$ are $\alpha$-invariant and the simplex of 0-KMS states for $\alpha$ is an affinely homeomorphic to $T(A)$.
ON THE POSSIBLE TEMPERATURES

We say that a UHF algebra $U$ is of infinite type when there is a natural number $m \geq 2$ such that $M_m(\mathbb{C})$ embeds unitally into $U$ for all $k \in \mathbb{N}$. When this holds there is a UHF algebra $U_1$ such that $U \simeq U \otimes U_1$ and we can therefore proceed exactly as in Section 5 of [11,3] to obtain the following additional corollaries. In the formulation we denote by $S_\beta^\alpha$ the simplex of $\beta$-KMS states for a flow $\alpha$.

**Corollary 2.3.** Let $U$ be a UHF algebra of infinite type and let $K$ be a compact set of real numbers containing 0. Let $\mathbb{I}$ be a finite or countably infinite collection of intervals in $\mathbb{R}$ such that $I = \mathbb{R}$ for at least one $I \in \mathbb{I}$. For each $I \in \mathbb{I}$ choose a compact metrizable Choquet simplex $S_I$ and for $\beta \in K$ set $\mathbb{I}_\beta = \{ I \in \mathbb{I} : \beta \in I \}$. There is a $2\pi$-periodic flow $\alpha$ on $U$ such that

- there is $\beta$-KMS state for $\alpha$ if and only if $\beta \in K$,
- for each $I \in \mathbb{I}$ and $\beta \in (I \cap K) \setminus \{ 0 \}$ there is a closed face $F_I$ in $S_\beta^\alpha$ strongly affinely isomorphic to $S_I$, and
- for each $\beta \in K \setminus \{ 0 \}$ and each $\omega \in S_\beta^\alpha$ there is a unique norm-convergent decomposition

$$\omega = \sum_{I \in \mathbb{I}_\beta} \omega_I,$$

where $\omega_I \in \{ t\mu : t \in [0,1], \mu \in F_I \}$.

**Corollary 2.4.** Let $U$ be a UHF algebra of infinite type and let $K$ be a compact set of real numbers containing 0. There is a $2\pi$-periodic flow $\alpha$ on $U$ such that $S_\beta^\alpha = \emptyset$ if and only if $\beta \notin K$, and for $\beta, \beta' \in K$ the simplexes $S_\beta^\alpha$ and $S_{\beta'}^\alpha$ are not strongly affinely isomorphic unless $\beta = \beta'$.

3. PROOFS

3.1. Setting the stage. Let $A$ be an infinite dimensional simple unital AF algebra. The tracial state space $T(A)$ of $A$ is a compact metrizable Choquet simplex which we denote by $\Delta$. We shall handle $\Delta$ as an abstract simplex and when an element $x \in \Delta$ is considered as a trace on $A$ we denote it by $\text{tr}_x$. The $K_0$-group $K_0(A)$ is a simple non-cyclic dimension group and we denote by $K_0(A)^+$ the semi-group of positive elements of $K_0(A)$. The unit $1$ in $A$ represents an element $[1] \in K_0(A)$ which is an order unit in $K_0(A)$. To simplify notation we set

$$(K_0(A), K_0(A)^+, [1]) = (H, H^+, u).$$

Each element $x \in \Delta$ defines in a canonical way a homomorphism $\text{tr}_x : H \to \mathbb{R}$, and since $A$ is AF the state space

$$\{ \phi \in \text{Hom}(H, \mathbb{R}) : \phi(H^+) \subseteq [0, \infty), \phi(u) = 1 \}$$

of $(H, H^+, u)$ is affinely homeomorphic to $\Delta$ via the map $\Delta \ni x \mapsto \text{tr}_x$. Let $\text{Aff}(\Delta)$ denote the space of real-valued continuous affine functions.
on \( \Delta \). There is then a homomorphism
\[
\theta : H \to \text{Aff}(\Delta)
\]
defined such that \( \theta(h)(x) = \text{tr}_{x^*}(h) \). By Theorem 4.11 in [GH]

- \( \theta(H) \) is norm-dense in \( \text{Aff}(\Delta) \), and
- \( H^+ = \{ h \in H : \theta(h)(x) > 0 \ \forall x \in \Delta \} \cup \{ 0 \} \).

In several instances it is convenient to consider weights instead of states and for weights we adopt the same conventions as in [Th3]. In particular traces and KMS weights are always non-zero, densely defined and lower semi-continuous.

3.2. The construction. Let \( K \) be the compact set of real numbers from Theorem [22] and set \( L = \{ e^{\beta} : \beta \in K \} \). Let \( A(\Delta \times L) \) denote the real vector space of functions \( f \in C_R(\Delta \times L) \) for which the map \( \Delta \ni x \mapsto f(x, t) \) is affine for all \( t \in L \); clearly a norm-closed subspace of \( C_R(\Delta \times L) \). Set
\[
G = \oplus_z H,
\]
and define \( \Sigma : G \to A(\Delta \times L) \) such that
\[
\Sigma(g)(x, t) = \sum_{m \in \mathbb{Z}} \theta(g_m)(x)t^{-m}.
\]
Let \( F \) be a non-empty closed face in \( \Delta \) and set
\[
G^+ = \{ g \in G : \Sigma(g)(x, t) > 0 \ \forall (x, t) \in (F \times L) \cup (\Delta \times \{ 1 \}) \} \cup \{ 0 \};
\]
a semi-group in \( G \) which turns \( G \) into a partially ordered abelian group. We aim to show that \( G \) is a simple dimension group.

For \( (k, l) \in \mathbb{Z}^2 \) let \( g_{k,l} \in C_R(L) \) be the function
\[
g_{k,l}(t) = t^k - t^l.
\]

Lemma 3.1. \( \text{Span}\{g_{k,l} : (k, l) \in \mathbb{Z}^2\} \) is dense in \( \{ h \in C_R(L) : h(1) = 0 \} \).

Proof. Note that \( t g_{k,l}(t) = g_{k+1, l+1}(t) \). It follows that \( \text{Span}\{g_{k,l} : (k, l) \in \mathbb{Z}^2\} \) contains all functions of the form \( P(t)(t - t^{-1}) \) where \( P \) is a polynomial. It follows therefore from Weierstrass’ theorem that the closure of \( \text{Span}\{g_{k,l} : (k, l) \in \mathbb{Z}^2\} \) contains all functions in \( C_R(L)(t - t^{-1}) \). Let \( h \in C_R(L) \) satisfy that \( h(1) = 0 \), and let \( \epsilon > 0 \). There is a function \( h_1 \in C_R(L) \) such that \( h_1(t) = 0 \) for all \( t \) in a neighborhood of 1 and \( \sup_{t \in L} |h(t) - h_1(t)| \leq \epsilon \). Since
\[
h_1(t) = \frac{h_1(t)}{t - t^{-1}}(t - t^{-1})
\]
it follows first that \( h_1 \) is in the closure of \( \text{Span}\{g_{k,l} : (k, l) \in \mathbb{Z}^2\} \), and then that so is \( h \).
\[\square\]
When $h \in H$ and $k, l \in \mathbb{Z}$, $k \neq l$, we denote in the following by $[[h]]_{k,l}$ the element of $G$ such that

$$
([[h]]_{k,l})_i = \begin{cases} 
    h, & i = -k \\
    -h, & i = -l \\
    0, & i \notin \{-k, -l\}
\end{cases},
$$

and by $[[h]] \in G$ the element with $[[h]]_0 = h$ and $[[h]]_i = 0$ when $i \neq 0$.

**Lemma 3.2.** $\Sigma(G)$ is dense in $A(\Delta \times L)$.

**Proof.** Note that $\Sigma([[h]]_{k,l})(x, t) = \theta(h)(x)g_{k,l}(t)$ and $\Sigma([[h]])(x, t) = \theta(h)(x)$. Hence $\Sigma(G)$ contains all functions of the form

$$
\theta(h)(x)(t^k - t^l),
$$

where $k, l \in \mathbb{Z}$, as well as all functions of the form $\theta(h)(x)$ when $h \in H$. Since $\theta(H)$ is dense in $\text{Aff}(\Delta)$ it follows from Lemma 3.1 that the closure $\overline{\Sigma(G)}$ of $\Sigma(G)$ contains all functions of the form $(x, t) \mapsto a(x)f(t)$ where $a \in \text{Aff}(\Delta)$ and $f \in C_\mathcal{R}(L)$. A wellknown partition of unity argument shows then that $A(\Delta \times L) = \overline{\Sigma(G)}$. \hfill \Box

**Lemma 3.3.** Let $a \in \text{Aff}(F)$. For each $x_0 \in \Delta \setminus F$ there is a function $b \in \text{Aff}(\Delta)$ such that $b|_F = a$ and $b(x_0) > a(y)$ for all $y \in F$.

**Proof.** Choose $u, v, w \in \mathbb{R}$ such that $u \leq a(y) \leq v$ for all $y \in F$. By the Hahn-Banach separation theorem there is a $c \in \text{Aff}(\Delta)$ such that $c(y) \leq 0$ for all $y \in F$ and $c(x_0) > 0$. Multiplying by a positive number we can arrange that $c(x_0) > v - u$. Set $C = \sup_{x \in \Delta} c(x)$. Then $c(y) \leq a(y) - u \leq v - u + C$ for all $y \in F$ and $c(x) \leq v - u + C$ for all $x \in \Delta$. It follows from Edwards separation theorem, Theorem 3 in [AE] or Corollary 7.7 in [AE], that there is a $b' \in \text{Aff}(\Delta)$ such that $b'|_F = a - u$ and $c \leq b' \leq v - u + C$ on $\Delta$. Set $b = b' + u$. Then $b|_F = a$ and $b(x_0) \geq c(x_0) + u > v$. \hfill \Box

**Lemma 3.4.** $G$ is a simple dimension group.

**Proof.** It is easy to see that $G$ is unperforated and that $G^+ \cap (-G^+) = \{0\}$. Since $(F \times L) \cup (\Delta \times \{1\})$ is compact it is also clear that every non-zero element of $G^+$ is an order unit for $G$, so what remains is to show that $G$ has the Riesz interpolation property. Let $c^i, d^i, i \in \{1, 2\}$, be elements of $G$ such that $c^i \leq d^i$ for all $i, j \in \{1, 2\}$. If $c^i = d^i$ for some $i', j'$, set $z = c^{i'}$. Then $c^i \leq z \leq d^j$ for all $i, j$. Assume instead that $c^i < d^j$ for all $i, j$. Then $\Sigma(c^i)(x, t) < \Sigma(d^j)(x, t)$ for all $i, j$ and all $(x, t) \in (F \times L) \cup (\Delta \times \{1\})$. Define $c, d \in C_\mathcal{R}(\Delta \times L)$ such that

$$
c(x, t) = \max\{\Sigma(c^1)(x, t), \Sigma(c^2)(x, t)\}
$$

and

$$
d(x, t) = \min\{\Sigma(d^1)(x, t), \Sigma(d^2)(x, t)\}.
$$

Let $\delta > 0$. If $\delta$ is small enough

$$
c(x, t) + \delta < d(x, t) - \delta
$$
for all \((x,t) \in (F \times L) \cup (\Delta \times \{1\})\). Fix \(t \in L \setminus \{1\}\). As a closed face in the simplex \(\Delta\) the set \(F\) is itself a Choquet simplex and hence the space \(\text{Aff}(F)\) has the Riesz interpolation property for the usual order, as well as for the strict order, cf. Lemma 3.1 in [EHS]. It follow therefore that there is a function \(a_t \in \text{Aff}(F)\) such that

\[
c(y,t) + \delta < a_t(y) < d(y,t) - \delta
\]

for all \(y \in F\). When \(t = 1\) it follows in the same way that there is a function \(a_1 \in \text{Aff}(\Delta)\) such that

\[
c(x,1) + \delta < a_1(x) < d(x,1) - \delta
\]

for all \(x \in \Delta\). We can then construct a finite cover \(V_i, i = 1,2,\ldots,N\), of \(L\) by open sets and elements \(t_i \in V_i\) with \(t_1 = 1\) such that

\[
|c(x,t) - c(x,t_i)| \leq \frac{\delta}{2}
\]

and

\[
|d(x,t) - d(x,t_i)| \leq \frac{\delta}{2}
\]

for all \(t \in V_i, \) all \(x \in \Delta\) and all \(i\). We arrange, as we can, that \(1 \notin V_j, j \geq 2\). For \(i \in \{2,3,\ldots,N\}\) we use Lemma 3.3 to get \(b_i \in \text{Aff}(\Delta)\) such that \(b_i|_F = a_{t_i}\), and we set \(b_1 = a_1\). Let \(\{\varphi_i\}_{i=1}^N\) be a partition of unity on \(L\) which is subordinate to \(\{V_i\}_{i=1}^N\). Define \(a \in A(\Delta \times L)\) such that

\[
a(x,t) = \sum_{i=1}^N b_i(x) \varphi_i(t).
\]

For \(y \in F\),

\[
a(y,t) = \sum_{i=1}^N a_{t_i}(y) \varphi_i(t) \leq \sum_{i=1}^N (d(y,t_i) - \delta) \varphi_i(t)
\]

\[
\leq \sum_{i=1}^N (d(y,t) - \frac{\delta}{2}) \varphi_i(t) = d(y,t) - \frac{\delta}{2},
\]

and similarly \(c(y,t) + \frac{\delta}{2} \leq a(y,t)\). For \(x \in \Delta\) we find that

\[
c(x,1) + \delta < a_1(x) = a(x,1) < d(x,1) - \delta.
\]

It follows that

\[
\Sigma(c^i)(x,t) < a(x,t) < \Sigma(d^i)(x,t)
\]

for all \(i,j\) and all \((x,t) \in (F \times L) \cup (\Delta \times \{1\})\). Let \(\epsilon > 0\). By Lemma 3.4 there is a \(g \in G\) such that

\[
\sup_{(x,t) \in \Delta \times L} |\Sigma(g)(x,t) - a(x,t)| \leq \epsilon.
\]

If \(\epsilon\) is small enough it follows that \(c^i \leq g \leq d^i\) in \(G\) for all \(i,j \in \{1,2\}\). \(\square\)
Define an automorphism $\rho$ of $(G, G^+)$ such that
\[ \rho((g_n)_{n \in \mathbb{Z}}) = (g_{n+1})_{n \in \mathbb{Z}}. \]

It follows from Lemma 3.4 and [EHS] that there is a simple AF algebra $B$ whose $K_0$-group and dimension range is isomorphic to $(G, G^+)$. Furthermore, it follows from [E] that $B$ is stable and that there is an automorphism $\gamma$ of $B$ such that $\gamma_* = \rho$ under the identification $K_0(B) = G$. Using Lemma 3.4 in [Th3] we choose $\gamma$ such that it has the following additional properties:

**Additional properties 3.5.**

(1) The restriction map $\mu \mapsto \mu|_B$ is a bijection from the traces $\mu$ on $B \rtimes_\gamma \mathbb{Z}$ onto the $\gamma$-invariant traces on $B$.

(2) $B \rtimes_\gamma \mathbb{Z}$ is stable.

(3) $B \rtimes_\gamma \mathbb{Z}$ is $\mathbb{Z}$-stable.

Set $C = B \rtimes_\gamma \mathbb{Z}$. No power of $\gamma$ is inner since $\gamma^k \neq \text{id}_G$ for $k \neq 0$ and it follows therefore from [K1] that $C$ is simple. The Pimsner-Voiculescu exact sequence shows that $K_1(C) = 0$ since $\text{id}_G - \gamma_*$ is injective and that
\[ K_0(C) = \text{coker}(\text{id}_G - \gamma_*) = G/((\text{id}_G - \gamma_*) (G)). \]

Under this identification the map $K_0(B) \rightarrow K_0(C)$ induced by the inclusion $B \subseteq C$ is the quotient map $q : G \rightarrow G/((\text{id}_G - \gamma_*) (G))$. Hence $G^*/((\text{id}_G - \gamma_*) (G)) \subseteq K_0(C)^*$.

**Lemma 3.6.** Let $d \in G$. Then $\sum_{n \in \mathbb{Z}} d_n = 0$ if and only if $d = (\text{id}_G - \gamma_*) (g)$ for some $g \in G$.

**Proof.** The proof of Lemma 4.4 in [Th3] works ad verbatim. \hfill \Box

It follows from Lemma 3.6 that we can define an injective homomorphism $S : G/((\text{id}_G - \gamma_*) (G)) \rightarrow H$ such that
\[ S(q(g)) = \sum_{n \in \mathbb{Z}} g_n \quad \forall g \in G. \]

$S$ is surjective since $S(q([[h]])) = h$ when $h \in H$, and hence an isomorphism with inverse $S^{-1}$ given by $S^{-1}(h) = q([[h]])$. Let $p \in B$ be a projection such that $[p] = [[u]]$ in $G$.

### 3.3. The Elliott invariants of $pCp$ and $A$ are isomorphic.

For $(y, t) \in F \times L$ and $x \in \Delta$ the maps $G \ni g \mapsto \Sigma(g)(y, t)$ and $G \ni g \mapsto \Sigma(g)(x, 1)$ are positive homomorphisms that take $[[u]]$ to 1. This implies the following, cf. Lemma 3.5 in [Th3].

**Lemma 3.7.** For $(y, t) \in F \times L$ and $x \in \Delta$ there are traces $\tau_{y, t}$ and $\tau_x$ on $B$ such that $\tau_{y, t}(p) = \tau_x(p) = 1$, and
\[ \tau_{y, t}(g) = \sum_{n \in \mathbb{Z}} \theta(g_n)(y)t^{-m} \]
and

$$\tau_x(g) = \sum_{m \in \mathbb{Z}} \theta(g_m)(x)$$

for all $g \in G$.

Lemma 3.8. Set

$$G^{++} = \{0\} \cup \left\{g \in G : \sum_{m \in \mathbb{Z}} \theta(g_m)(x) > 0 \ \forall x \in \Delta\right\}.$$ 

Then $K_0(C)^+ = G^{++}/(1d_G - \gamma_+)(G)$ and $S$ takes $K_0(C)^+$ onto $H^+$.

Proof. Let $g = (g_m)_{m \in \mathbb{Z}} \in G^{++}\{0\}$ and set $D = \{n \in \mathbb{Z} : g_n \neq 0\}$. For each $n \in D$ we use Lemma 3.1 to find a finite set $I_n \subset \{(k,l) \in \mathbb{Z}^2 : k \neq l\}$ and real number $a^n_{k,l} \in \mathbb{R}$ such that

$$\sum_{n \in D} \theta(g_n)(x) [t^n - 1] + \sum_{n \in D} \sum_{(k,l) \in I_n} a^n_{k,l} \theta(g_n)(x)(t^k - t^l) \geq - \frac{1}{3} \sum_{m \in \mathbb{Z}} \theta(g_m)(x)$$

for all $(x,t) \in \Delta \times L$. Since $\theta(H)$ is dense in $\text{Aff}(\Delta)$ we can subsequently find element $w^n_{k,l} \in H$ such that

$$\sum_{n \in D} \sum_{(k,l) \in I_n} \theta(w^n_{k,l})(x)(t^k - t^l) \geq \sum_{n \in D} \sum_{(k,l) \in I_n} a^n_{k,l} \theta(g_n)(x)(t^k - t^l) - \frac{1}{3} \sum_{m \in \mathbb{Z}} \theta(g_m)(x)$$

for all $(x,t) \in \Delta \times L$. Set

$$g^+ = g + \sum_{n \in D} \sum_{(k,l) \in I_n} [[w^n_{k,l}]]^{k,l}.$$
Then $\sum_{m \in \mathbb{Z}} g_m^+ = \sum_{m \in \mathbb{Z}} g_m$ because $\sum_{m \in \mathbb{Z}} \left( \frac{[u^m]}{[k,l]} \right)_m = 0$ for all $(k,l)$ and $n$. It follows that $g^+ - g \in (\text{id}_G - \gamma_*) (G)$ by Lemma 3.6. Furthermore,

$$\sum_{m \in \mathbb{Z}} \theta(g_m)(x)t^{-m} = \sum_{m \in \mathbb{Z}} \theta(g_m)(x)t^{-m} + \sum_{n \in \mathcal{D}} \sum_{(k,l) \in I_n} \theta(w^n_{k,l})(x)(t^k - t^l)$$

$$\geq \sum_{m \in \mathbb{Z}} \theta(g_m)(x)t^{-m} + \sum_{n \in \mathcal{D}} \sum_{(k,l) \in I_n} a^n_{k,l} \theta(g_m)(x)(t^k - t^l) - \frac{1}{3} \sum_{m \in \mathbb{Z}} \theta(g_m)(x)$$

$$= \sum_{m \in \mathbb{Z}} \theta(g_m)(x) + 1 - 1$$

$$= \sum_{m \in \mathbb{Z}} \theta(g_m)(x) > 0$$

for all $(x,t) \in \Delta \times L$, which implies that $g^+ \in G^+$. This shows that $G^+ / (\text{id}_G - \gamma_*) (G) \subseteq G^+ / (\text{id}_G - \gamma_*) (G) \subseteq K_0(C)^+$. Let $z \in K_0(C)^+ \setminus \{0\} \subseteq G^+ / (\text{id}_G - \gamma_*) (G)$ and choose $g \in G$ such that $q(g) = z$. Let $x \in \Delta$. The trace $\tau_x$ from Lemma 3.7 is $\gamma$-invariant and there is therefore a trace $\tau'_x$ on $C$ such that $\tau'_x | B = \tau_x$. Then $\tau'_x(z) > 0$ since $z > 0$ in $K_0(C)$ and $C$ is simple. Hence

$$0 < \tau'_x(z) = \tau_x(g) = \sum_{m \in \mathbb{Z}} \theta(g_m)(x).$$

It follows that $g \in G^{++}$ and $z \in G^+ / (\text{id}_G - \gamma_*) (G)$. We conclude that $K_0(C)^+ = G^+ / (\text{id}_G - \gamma_*) (G)$ and $S(K_0(C)^+) \subseteq H^+$. When $h \in H^+ \setminus \{0\}$, set $z = \frac{1}{[h]}$. Then $z \in G^{++}$, $q(z) \in K_0(C)^+$ and $S(q(z)) = h$. \hfill \square

**Lemma 3.9.** Let $\phi : G \to \mathbb{R}$ be a positive homomorphism such that $\phi([\{u\}]) = 1$. Assume that $\phi \circ \gamma_* = s \phi$ for some $s > 0$. Then $s \in L$ and when $s \neq 1$ it follows that $\phi = \tau_{y,s}$ for some $y \in F$, and when $s = 1$ it follows that $\phi = \tau_{x,s}$ for some $x \in \Delta$.

**Proof.** We claim that there is continuous linear map $\phi' : A(\Delta \times L) \to \mathbb{R}$ such that $\phi = \phi' \circ \Sigma$. Since $\theta(H)$ is dense in $\text{Aff}(\Delta)$ there is a $c \in H$ such that $0 < \theta(c)(x) < N^{-1}$ for all $x \in \Delta$. Then $0 \leq N[[c]] \leq [[u]]$ in $G$ and hence $0 \leq \phi([[c]]) \leq \frac{1}{N} \phi([[u]]) = \frac{1}{N}$. Assume $g \in G$ and that $\Sigma(g) = 0$. Then $\pm g + [[c]] \in G^+$ and hence $-\frac{1}{N} \leq \phi(g) \leq \frac{1}{N}$. Letting $N \to \infty$ we conclude that $\phi(g) = 0$, and it follows that there is a homomorphism $\phi' : \Sigma(G) \to \mathbb{R}$ such that $\phi' \circ \Sigma = \phi$. Let $h \in \Sigma(G)$; say $h = \Sigma(g)$, and let $k, l \in \mathbb{N}$ be natural numbers such that $|h(x,t)| < \frac{1}{N}$ for all $(x,t) \in \Delta \times L$. \hfill \square
Then $k[[u]] \pm lg \in G^+$ and hence

$$0 \leq \phi(k[[u]] \pm lg) = k \pm l\phi(h).$$

It follows that $|\phi'(h)| \leq \frac{\phi}{h}$, proving that $\phi'$ is Lipshitz continuous. Since $\Sigma(G)$ is dense in $A(\Delta \times L)$ by Lemma 3.2, it follows that $\phi'$ extends by continuity to a continuous linear map $\phi' : A(\Delta \times L) \to \mathbb{R}$, proving the claim. Let $T : A(\Delta \times L) \to A(\Delta \times L)$ denote the operator

$$T(\psi)(x, t) = t\psi(x, t).$$

Since $\Sigma \circ \gamma_+(g)(x, t) = t\Sigma(g)(x, t)$ for all $g \in G$ and all $(x, t) \in \Delta \times L$, we find that

$$s\phi'(\Sigma(g)) = s\phi(g) = \phi(\gamma_+(g)) = \phi' \circ \Sigma \circ \gamma_+(g) = \phi'(T(\Sigma(g))).$$

It follows therefore from Lemma 3.2 that

$$s\phi' = \phi' \circ T.$$  \tag{3.1}

When $a \in \text{Aff}(\Delta)$ and $f \in C_\mathbb{R}(L)$ we denote by $a \otimes f$ the function $\Delta \times L \ni (x, t) \mapsto a(x)f(t)$. Assume $a(x) \geq 0$ for all $x \in \Delta$ and that $f(t) \geq 0$ for all $t \in L$. It follows then from Lemma 3.2 and the definition of $G^+$ that $a \otimes f$ can be approximated by elements from $\Sigma(G^+)$, implying that $\phi'(a \otimes f) \geq 0$. There is therefore a bounded Borel measure $\mu_a$ on $L$ such that

$$\phi'(a \otimes f) = \int_L f \, d\mu_a \quad \forall f \in C_\mathbb{R}(L).$$

It follows from (3.1) that

$$\int_L sf(t) \, d\mu_a(t) = \int_L tf(t) \, d\mu_a$$

for all $f \in C_\mathbb{R}(L)$, and hence that $\mu_a(L \setminus \{s\}) = 0$. Since $\phi \neq 0$, not all $\mu_a$ can be zero and we conclude therefore that $s \in L$ and that

$$\phi'(a \otimes f) = \lambda(a)f(s)$$  \tag{3.2}

for some $\lambda(a) \geq 0$ and all $f \in C_\mathbb{R}(L)$. Every element of $\text{Aff}(\Delta)$ is the difference between two positive elements and it follows therefore that for all $a \in \text{Aff}(\Delta)$ there is a real number $\lambda(a)$ such that (3.2) holds for all $f \in C_\mathbb{R}(L)$. The resulting map $a \mapsto \lambda(a)$ is clearly linear and positive, and $\lambda(1) = 1$ since $1 = \phi([[u]]) = \phi'(1)$. This implies that there is an $x_0 \in \Delta$ such that $\lambda(a) = a(x_0)$ for all $a \in \text{Aff}(\Delta)$. The elements of $\{a \otimes f : a \in \text{Aff}(\Delta), f \in C_\mathbb{R}(L)\}$ span a dense set in $A(\Delta \times L)$ and we find therefore that

$$\phi'(h) = h(x_0, s) \quad \forall h \in A(\Delta \times L).$$ \tag{3.3}

To see that $x_0 \in F$ when $s \neq 1$, assume for a contradiction that $x_0 \notin F$. By Lemma 3.3 we can find $b \in \text{Aff}(\Delta)$ such that $b|_F = 0$ and $b(x_0) > 0$, and since $s \neq 1$ we can find $f \in C_\mathbb{R}(L)$ such that $f(1) = 0$ and $f(s) = 1$. The function $b \otimes f$ vanishes on $(F \times L) \cup (\Delta \times \{1\})$. By Lemma 3.2 and
the definition of \( G^* \) this implies that \( \phi'(b \otimes f) = 0 \), which contradicts (3.3) since \( b(x_0) f(s) > 0 \). Hence \( x_0 \in F \) when \( s \neq 1 \). It follows that

\[
\phi(g) = \phi'(\Sigma(g)) = \sum_{m \in \mathbb{Z}} \theta(g_m)(x_0)s^{-m},
\]

and that \( x_0 \in F \) when \( s \neq 1 \).

**Lemma 3.10.** Let \( p \in B \) be a projection representing \([u]\) \in G = \( K_0(B) \). The Elliott invariants of \( A \) and \( p(B \rtimes_{\gamma} \mathbb{Z})p \) are isomorphic.

**Proof.** Since \( K_1(A) = K_1(pCp) = 0 \) it suffices to supplement the isomorphism \( S : (K_0(C), [[u]]) \to (H, u) \) of partially ordered groups with order unit with an affine homeomorphism \( T : T(pCp) \to T(A) \) such that

\[
T(\tau)_* (S(z)) = \tau_*(z) \quad \forall (\tau, z) \in (T(pCp)) \times K_0(C).
\]

Let \( \tau \in T(pCp) \). By Proposition 4.7 in \([CP]\) there is a unique trace \( \tau' \) on \( C \) such that \( \tau'|_{pCp} = \tau \). Then \( \tau'|_B \) is a \( \gamma \)-invariant trace on \( B \) such that \( \tau'(p) = 1 \). It follows from Lemma 3.9 that there is an \( x \in \Delta \) such that \( \tau'|_{B_x} = \tau_x \). Since \( x \) is uniquely determined by \( \tau \) we can define \( T : T(pCp) \to T(A) \) such that \( T(\tau) = tr_x \). Let \( e \in K_0(A) \), \( 0 \leq e \leq [1] \). Then \( 0 \leq [[e]] \leq [[u]] \) in \( G \) and hence \([e] \) is represented by a projection \( e' \) from \( pBp \in pCp \). We find that

\[
tr_{x*}(e) = \theta(e)(x) = tr_{x*}([[e]]) = \tau'|_{B_x}([[e]]) = \tau(e');
\]

an equality which shows that \( T \) is affine and continuous. To see that \( T \) is surjective let \( x \in \Delta \). The trace \( \tau_x \) from Lemma 3.7 is \( \gamma \)-invariant and defines therefore a trace \( \tau' \) on \( C \) such that \( \tau'|_B = \tau_x \). Then \( \tau'(p) = \tau_x([[u]]) = 1 \) and hence \( \tau'|_{pCp} \in T(pCp) \) and \( T(\tau'|_{pCp}) = tr_x \). To see that \( T \) is also injective consider to traces \( \tau_i \), \( i = 1, 2 \), in \( T(pCp) \) and assume that \( T(\tau_1) = T(\tau_2) \). Then \( \tau_1|_{B_x} = \tau_2|_{B_x} \) on \( G \) and hence \( \tau_1|_B = \tau_2|_B \) by Lemma 3.5 in \([Th3]\). It follows from the first of the additional properties (3.3) that \( \tau_1 = \tau_2 \) and hence also that \( \tau_1 = \tau_1|_{pCp} = \tau_2|_{pCp} = \tau_2 \). We conclude that \( T \) is an affine homeomorphism. Consider \( \tau \in T(pCp) \) and \( e \in K_0(A) \), \( 0 \leq e \leq 1 \). Then \( T(\tau) = tr_x \) for some \( x \in \Delta \) and, as observed above, \([e] \) in \( G \) is represented by a projection \( e' \) in \( pBp \) such that \( tr_{x*} = \tau(e') \). Hence

\[
T(\tau)_* (S(S^{-1}(e))) = tr_{x*} (S(q([[e]]))) = tr_{x*}(e) = \tau(e')
= \tau_* (q([[e]])) = \tau_* (S^{-1}(e)).
\]

Since the elements of \( \{S^{-1}(e) : e \in K_0(A) \} \) generate \( K_0(pCp) \) as a group, we conclude that (3.4) holds. \( \square \)

3.4. **Conclusion.**

**Lemma 3.11.** \( A \) is *-isomorphic to \( p(B \rtimes_{\gamma} \mathbb{Z})p \).
Proof. By Lemma 3.10 it remains only to show that $A$ and $pCp$ belong to a class of simple $C^*$-algebras for which the Elliott invariant is complete. That this is the case follows from [CETWW] and [TW] as in the proof of Theorem 4.12 in [Th3]. □

Remark 3.12. Assume that $A$ is UHF. Then the algebras $pCp$ and $A$ occurring in the last proof only have one trace state and there is an alternative way to deduce the isomorphism $A \cong pCp$. The path through the literature takes more space to explain, but presents presumably a shorter argument: By Corollary 6.2 in [MS2] it suffices to show that both algebras are unital, separable, simple, infinite dimensional, nuclear, quasi-diagonal, satisfy the UCT and have strict comparison. Many of these properties are well-known for both algebras. What remains is to explain why they are quasi-diagonal and have strict comparison. Since $pCp$ and $A$ are both exact, simple and unital it follows from Corollary 4.6 in [Ro] that $\mathcal{Z}$-stability implies strict comparison and hence it suffices to argue that both algebras are quasi-diagonal and $\mathcal{Z}$-absorbing. Concerning quasi-diagonality it is well-known that AF algebras have this property and that it is inherited by subalgebras, so a straightforward application of [Br] shows that both algebras are quasi-diagonal.

Corollary 3.13. $B \rtimes_\gamma \mathbb{Z}$ is $*$-isomorphic to $A \otimes \mathbb{K}$.

Proof. Thanks to Lemma 3.10 this follows now from the second of the additional properties 3.5 and Brown’s theorem, [B]. □

The dual action on $B \rtimes_\gamma \mathbb{Z}$ is defined by restriction a $2\pi$-periodic flow on $p(B \rtimes_\gamma \mathbb{Z})p$ which we denote by $\hat{\gamma}$.

Lemma 3.14. For $\beta \in \mathbb{R}$ there is a $\beta$-KMS state for $\hat{\gamma}$ on $p(B \rtimes_\gamma \mathbb{Z})p$ if and only if $\beta \in K$. For $\beta \in K \setminus \{0\}$ the simplex of $\beta$-KMS states for $\hat{\gamma}$ is affinely homeomorphic to $F$ and for $\beta = 0$ it consists of all trace states on $p(B \rtimes_\gamma \mathbb{Z})p$ and hence is affinely homeomorphic to $\Delta$.

Proof. Let $P : B \rtimes_\gamma \mathbb{Z} \to B$ be the canonical conditional expectation. Set $t = e^{-\beta}$ and let $y \in F$. If $\beta \in K$ it follows from Lemma 3.5 in [Th3] that $\tau_{y,t}$ is a trace on $B$ such that $\tau_{y,t} \circ \gamma = e^{-\beta} \tau_{y,t}$. Then $\tau_{y,t} \circ P$ is a $\beta$-KMS state for $\hat{\gamma}$ by Lemma 3.1 in [Th3]. It follows that there is a $\beta$-KMS state for $\hat{\gamma}$ for all $\beta \in K$. Conversely, if there is a $\beta$-KMS state for $\hat{\gamma}$ it follow from Theorem 2.4 in [Th1] that there is also a $\beta$-KMS weight for the dual action on $B \rtimes_\gamma \mathbb{Z}$ and then from Lemma 3.1 in [Th3] that there is a trace $\tau$ on $B$ such that $\tau \circ \gamma = e^{-\beta} \tau$ and $\tau(p) = 1$. Then $\tau_* : G \to \mathbb{R}$ is a positive homomorphism such that $\tau_* \circ \gamma_* = e^{-\beta} \tau_*$ and $\tau_*([u]) = 1$. 
It follows from Lemma 3.9 that $\beta \in K$. When $\beta \in K\{0\}$ and $y \in F$ it follows from Lemma 3.1 in [Th3] that we can define a $\beta$-KMS state on $p(B \times \gamma, Z)p$ by $\tau_{y,e^{-\beta}} \circ P_{p(B, \gamma)}$. The resulting map, from $F$ to the simplex of $\beta$-KMS weights for $\gamma$ on $p(B \times \gamma, Z)p$, is clearly continuous, affine and injective. To see that it is surjective, let $\omega$ be a $\beta$-KMS state for $\gamma$. By Theorem 2.4 in [Th1] and Lemma 3.1 in [Th3] there is trace $\tau$ on $B$ such that $\tau \circ \gamma = e^{-\beta} \tau$, $\tau(p) = 1$ and $\omega = \tau \circ P_{p(B, \gamma)}$. It follows from Lemma 3.9 that $\tau_\ast = \tau_{y,e^{-\beta}}$ on $G$ and then from Lemma 3.5 in [Th3] that $\tau = \tau_{y,e^{-\beta}}$ for some $y \in F$, and we conclude that $\omega$ is the image of $y \in F$ under the map we consider. This shows that the map is an affine homeomorphism. The same argument shows that the simplex of 0-KMS states for $\gamma$ is affinely homeomorphic to $\Delta$, and finally it follows from the first of the additional properties 3.5 combined with either Proposition 4.7 in [CP] or Theorem 2.4 in [Th1], that all trace states of $p(B \times \gamma, Z)p$ are $\gamma$-invariant.

Theorem 2.1 follows immediately from Lemma 3.14 and Lemma 3.11.

References

[AE] L. Asimow and A.J. Ellis, Convexity theory and its applications in functional analysis, Academic Press, 1980.

[BEH] O. Bratteli, G. Elliott and R.H. Herman, On the possible temperatures of a dynamical system, Comm. Math. Phys. 74 (1980), 281-295.

[BR] O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics I + II, Texts and Monographs in Physics, Springer Verlag, New York, Heidelberg, Berlin, 1979 and 1981.

[B] L. Brown, Stable isomorphism of hereditary subalgebras of $C^*$-algebras, Pacific J. Math. 71 (1977), 335-348.

[Br] N. Brown, AF embeddability of crossed products of AF algebras by the integers, J. Func. Anal. 60 (1998), 150-175.

[CETWW] J. Castillejos, S. Evington, A. Tikuisis, S. White and W. Winter, Nuclear dimension of simple $C^*$-algebras, arXiv:1901.05853v3, Invent. Math., to appear.

[CP] J. Cuntz and G. K. Pedersen, Equivalence and traces on $C^*$-algebras, J. Func. Analysis 33 (1979), 135-164.

[Ed] D.A. Edwards, Minimum-stable wedges of semi-continuous functions, Math. Scand. 19 (1966), 15-26.

[EHS] E. Effros, D. Handelman and C. L. Shen, Dimension groups and their affine representations, Amer. J. Math. 102 (1980), 385-407.

[E] G. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, J. Algebra 38 (1976), 29-44.

[EGLN] G. A. Elliott, G. Gong, H. Lin and Z. Niu, On the classification of simple amenable $C^*$-algebras with finite decomposition rank, II, preprint, arXiv:1507.08437.

[GLN] G. Gong, H. Lin, and Z. Niu, Classification of finite simple amenable $Z$-stable $C^*$-algebras, arXiv:1501.00135v6.

[GH] K.R. Goodearl and D.E. Handelman, Metric Completions of Partially Ordered Abelian Groups, Indiana Univ. Math. J. 29 (1980), 861-895.

[Ki1] A. Kishimoto, Outer Automorphisms and Reduced Crossed Products of Simple $C^*$-Algebras, Comm. Math. Phys. 81 (1981), 429-435.
14 KLAUS THOMSEN

[Ki2] A. Kishimoto, Non-commutative shifts and crossed products, J. Func. Analysis 200 (2003), 281-300.

[MS1] H. Matui and Y. Sato, Strict comparison and $\mathbb{Z}$-absorption for nuclear $C^*$-algebras, Acta Math. 209 (2012), 179-196.

[MS2] H. Matui and Y. Sato, Decomposition rank of UHF-absorbing $C^*$-algebras, Duke Math. J. 163 (2014), 2687-2708.

[PS] R.T. Powers and S. Sakai, Existence of ground states and KMS states for approximately inner dynamics, Comm. Math. Phys. 39 (1975), 273-288.

[Ro] M. Rørdam, The stable rank and the real rank of $\mathbb{Z}$-absorbing $C^*$-algebras, Inter. J. Math. 15 (2004), 1065-1084.

[Th1] K. Thomsen, KMS weights on graph $C^*$-algebras, Adv. Math. 309 (2017) 334–391.

[Th2] K. Thomsen, Phase transition in the CAR algebra, Adv. Math. 372 (2020); arXiv:1612.04716v5.

[Th3] K. Thomsen, On the possible temperatures for flows on an AF-algebra, arXiv:2011.06377v3.

[TW] A. Toms and W. Winter, Strongly self-absorbing $C^*$-algebras, Trans. Amer. Math. Soc. 358 (2007), 3999-4029.

Email address: matkt@math.au.dk

DEPARTMENT OF MATHEMATICS, AARHUS UNIVERSITY, NY MUNKEGADE, 8000 AARHUS C, DENMARK