ON BLOWUP OF SECANT VARIETIES OF CURVES

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Abstract. In this paper, we show that for a nonsingular projective curve and a positive integer $k$, the $k$-th secant bundle is the blowup of the $k$-th secant variety along the $(k-1)$-th secant variety. This answers a question raised in the recent paper of the authors on secant varieties of curves.

1. Introduction. Throughout the paper, we work over an algebraically closed field $k$ of characteristic zero. Let $C$ be a nonsingular projective curve of genus $g \geq 0$, and $L$ be a very ample line bundle on $C$. The complete linear system $|L| := \mathbb{P}(H^0(C, L))$ embeds $C$ into a projective space $\mathbb{P}^r$. For an integer $k \geq 0$, the $k$-th secant variety $\Sigma_k = \Sigma_k(C, L) \subseteq \mathbb{P}^r$ is the Zariski closure of the union of $(k+1)$-secant $k$-planes to $C$.

Assume that $\deg L \geq 2g + 2k + 1$. Then the $k$-th secant variety $\Sigma_k$ can be defined by using the secant sheaf $E_{k+1,L}$ and the secant bundle $B^k(L)$ as follows. Denote by $C_m$ the $m$-th symmetric product of $C$. Let $\sigma_{k+1}: C_k \times C \to C_{k+1}$ be the morphism sending $(\xi, x)$ to $\xi + x$, and $p: C_k \times C \to C$ the projection to $C$. The secant sheaf $E_{k+1,L}$ on $C_{k+1}$ associated to $L$ is defined by

$$E_{k+1,L} := \sigma_{k+1,*}p^* L,$$

which is a locally free sheaf of rank $k + 1$. Notice that the fiber of $E_{k+1,L}$ over $\xi \in C_{k+1}$ can be identified with $H^0(\xi, L|\xi)$. The secant bundle of $k$-planes over $C_{k+1}$ is

$$B^k(L) := \mathbb{P}(E_{k+1,L}),$$

equipped with the natural projection $\pi_k: B^k(L) \to C_{k+1}$. We say that a line bundle $\mathcal{L}$ on a variety $X$ separates $m+1$ points if the natural restriction map $H^0(X, \mathcal{L}) \to H^0(\xi, \mathcal{L}|\xi)$ is surjective for any effective zero-cycle $\xi \subseteq X$ with $\text{length}(\xi) = m + 1$.

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Notice that a line bundle $L$ is globally generated if and only if $L$ separates 1 point, and $L$ is very ample if and only if $L$ separates 2 points. Since $\deg L \geq 2g + k$, it follows from Riemann–Roch that $L$ separates $k + 1$ points. Then the tautological bundle $\mathcal{O}_{B^k(L)}(1)$ is globally generated. We have natural identifications

$$H^0(B^k(L), \mathcal{O}_{B^k(L)}(1)) = H^0(C_{k+1}, E_{k+1}) = H^0(C, L),$$

and therefore, the complete linear system $|\mathcal{O}_{B^k(L)}(1)|$ induces a morphism

$$\beta_k : B^k(L) \rightarrow \mathbb{P}^r = \mathbb{P}(H^0(C, L)).$$

The $k$-th secant variety $\Sigma_k = \Sigma_k(C, L)$ of $C$ in $\mathbb{P}^r$ can be defined to be the image $\beta_k(B^k(L))$. Bertram proved that $\beta_k : B^k(L) \rightarrow \Sigma_k$ is a resolution of singularities (see [1, Section 1]).

It is clear that there are natural inclusions

$$C = \Sigma_0 \subseteq \Sigma_1 \subseteq \cdots \subseteq \Sigma_{k-1} \subseteq \Sigma_k \subseteq \mathbb{P}^r.$$

The preimage of $\Sigma_{k-1}$ under the morphism $\beta_k$ is actually a divisor on $B^k(L)$. Thus there exists a natural morphism from $B^k(L)$ to the blowup of $\Sigma_k$ along $\Sigma_{k-1}$. Vermeire proved that $B^1(L)$ is indeed the blowup of $\Sigma_1$ along $\Sigma_0 = C$ ([3, Theorem 3.9]). In the recent work [2], we showed that $B^k(L)$ is the normalization of the blowup of $\Sigma_k$ along $\Sigma_{k-1}$ ([2, Proposition 5.13]), and raised the problem asking whether $B^k(L)$ is indeed the blowup itself ([2, Problem 6.1]). The purpose of this paper is to give an affirmative answer to this problem by proving the following:

**Theorem 1.1.** Let $C$ be a nonsingular projective curve of genus $g$, and $L$ be a line bundle on $C$. If $\deg L \geq 2g + 2k + 1$ for an integer $k \geq 1$, then the morphism $\beta_k : B^k(L) \rightarrow \Sigma_k(C, L)$ is the blowup of $\Sigma_k(C, L)$ along $\Sigma_{k-1}(C, L)$.

To prove the theorem, we utilize several line bundles defined on symmetric products of the curve. Let us recall the definitions here and refer the reader to [2] for further details. Let

$$C^{k+1} = C \times \cdots \times C \quad \text{($k+1$ times)}$$

be the $(k + 1)$-fold ordinary product of the curve $C$, and $p_i : C^{k+1} \rightarrow C$ be the projection to the $i$-th component. The symmetric group $\mathfrak{S}_{k+1}$ acts on $p_1^*L \otimes \cdots \otimes p_{k+1}^*L$ in a natural way: a permutation $\mu \in \mathfrak{S}_k$ sends a local section $s_1 \otimes \cdots \otimes s_{k+1}$ to $s_{\mu(1)} \otimes \cdots \otimes s_{\mu(k+1)}$. Then $p_1^*L \otimes \cdots \otimes p_{k+1}^*L$ is invariant under the action of $\mathfrak{S}_{k+1}$, so it descends to a line bundle $T_{k+1}(L)$ on the symmetric product $C_{k+1}$ via the quotient map $q : C^{k+1} \rightarrow C_{k+1}$. We have $q^*T_{k+1}(L) = p_1^*L \otimes \cdots \otimes p_{k+1}^*L$. Define a divisor $\delta_{k+1}$ on $C_{k+1}$ such that the associated line bundle $\mathcal{O}_{C_{k+1}}(\delta_{k+1}) = \det (\sigma_{k+1,*}(\mathcal{O}_{C_{k+1} \times C}))^\ast$. Let

$$A_{k+1,L} := T_{k+1}(L)(-2\delta_{k+1})$$

be a line bundle on $C_{k+1}$. When $k = 0$, we use the convention that $T_1(L) = E_{1,L} = L$ and $\delta_1 = 0$.

The main ingredient in the proof of Theorem 1.1 is to study the positivity of the line bundle $A_{k+1,L}$. Some partial results and their geometric consequences have been discussed in [2, Lemma 5.12 and Proposition 5.13]. Along this direction, we establish the following proposition to give a full picture in a general result describing the positivity of the line bundle $A_{k+1,L}$. This may be of independent interest.
Proposition 1.2. Let $C$ be a nonsingular projective curve of genus $g$, and $L$ be a line bundle on $C$. If $\deg L \geq 2g + 2k + \ell$ for integers $k, \ell \geq 0$, then the line bundle $A_{k+1,L}$ on $C_{k+1}$ separates $\ell+1$ points.

In particular, if $\deg L \geq 2g + 2k$, then $A_{k+1,L}$ is globally generated, and if $\deg L \geq 2g + 2k + 1$, then $A_{k+1,L}$ is very ample.

2. Proof of the main theorem. In this section, we prove Theorem 1.1. We begin with showing Proposition 1.2.

Proof of Proposition 1.2. We proceed by induction on $k$ and $\ell$. If $k = 0$, then $A_{1,L} = L$ and $\deg L \geq 2g + \ell$. It immediately follows from Riemann–Roch that $L$ separates $\ell+1$ points. If $\ell = 0$, then $\deg L \geq 2g + 2k$. By [2, Lemma 5.12], $A_{k+1,L}$ separates 1 point.

Assume that $k \geq 1$ and $\ell \geq 1$. Let $z$ be a length $\ell+1$ zero-dimensional subscheme of $C_{k+1}$. We aim to show that the natural restriction map

$$ r_{z,k+1,L} : H^0(C_{k+1}, A_{k+1,L}) \rightarrow H^0(z, A_{k+1,L}|_z) $$

is surjective. We can choose a point $p \in C$ such that $X_p$ contains a point in the support of $z$, where $X_p$ is the divisor on $C_{k+1}$ defined by the image of the morphism $C_k \rightarrow C_{k+1}$ sending $\xi$ to $\xi + p$. Let $y := z \cap X_p$ be the scheme-theoretic intersection, and $\mathcal{I}_x := (\mathcal{I}_z : \mathcal{I}_y)$, which defines a subscheme $x$ of $z$ in $C_{k+1}$, where $\mathcal{I}_z$ and $\mathcal{I}_y$ are ideal sheaves of $z$ and $X_p$ in $C_{k+1}$, respectively. We have the following commutative diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
\mathcal{I}_x(-X_p) \\
\downarrow \\
\mathcal{I}_z \\
\downarrow \\
\mathcal{O}_{C_{k+1}}(-X_p) \\
\downarrow \\
\mathcal{O}_x(-X_p) \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
\mathcal{I}_y \\
\mathcal{O}_{C_{k+1}} \\
\mathcal{O}_x \\
\mathcal{O}_y \\
0
\end{array}
$$

where all rows and columns are short exact sequences. By tensoring with $A_{k+1,L}$ and taking the global sections of last two rows, we obtain the commutative diagram with exact sequences

$$
\begin{array}{c}
0 \\
\downarrow \quad r_{x,k+1,L}(-p) \\
0 \\
\downarrow \quad r_{z,k+1,L} \\
0 \\
\downarrow \quad r_{y,k,L}(-2p)
\end{array}
\quad
\begin{array}{c}
H^0(A_{k+1,L}(-X_p)) \\
\downarrow \quad r_{x,k+1,L}(-p) \\
H^0(A_{k+1,L}) \\
\downarrow \quad r_{z,k+1,L} \\
H^0(A_{k+1,L}|_{X_p}) \\
\downarrow \quad r_{y,k,L}(-2p) \\
0
\end{array}
\quad
\begin{array}{c}
H^0(A_{k+1,L}(-X_p)|_z) \\
\downarrow \quad r_{x,k+1,L}(-p) \\
H^0(A_{k+1,L}|_z) \\
\downarrow \quad r_{z,k+1,L} \\
H^0(A_{k+1,L}|_y) \\
\downarrow \quad r_{y,k,L}(-2p) \\
0
\end{array}
$$

in which we use the fact that $H^1(A_{k+1,L}(-X_p)) = 0$ (see the proof of [2, Lemma 5.12]). Note that $A_{k+1,L}(-X_p) = A_{k+1,L}(-p)$ and $A_{k+1,L}|_{X_p} \cong A_{k,L}(-2p)$, where we identify $X_p = C_k$.

Since $\text{length}(y) \leq \text{length}(z) = \ell + 1$ and $\deg L(-2p) \geq 2g + 2(k-1) + \ell$, the induction hypothesis on $k$ implies that $r_{y,k,L}(-2p)$ is surjective. On the other hand,
if \( x = \emptyset \), which means that \( z \) is a subscheme of \( X_p \), then trivially \( r_{x,k+1,L(-p)} \) is surjective. Otherwise, suppose that \( x \neq \emptyset \). By the choice of \( X_p \), we know that \( y \) is not empty, and therefore, we have \( \text{length}(x) \leq \text{length}(z) - 1 = \ell \). Now, \( \text{deg } L(-p) \geq 2g + 2k + (\ell - 1) \), so the induction hypothesis on \( \ell \) implies that \( L(-p) \) separates \( \ell \) points. In particular, \( r_{x,k+1,L(-p)} \) is surjective. Hence \( r_{x,k+1,L} \) is surjective as desired.

\[ \square \]

**Lemma 2.1.** Let \( \varphi : X \to Y \) be a finite surjective morphism between two varieties. If \( \varphi^{-1}(q) \) is scheme theoretically a reduced point for each closed point \( q \in Y \), then \( \varphi \) is an isomorphism.

**Proof.** Note that \( \varphi \) is proper, injective, and unramified. Then it is indeed a classical result that \( \varphi \) is an isomorphism. Here we give a short proof for reader’s convenience. The problem is local. We may assume that \( X = \text{Spec } B \) and \( Y = \text{Spec } A \) for some rings \( A, B \). We may regard \( A \) as a subring of \( B \). For any \( g \in Y \), let \( p := \varphi^{-1}(q) \in X \). It is enough to show that the localizations \( A' := A_{m_p} \) and \( B' := B_{m_p} \) are isomorphic. Let \( m'_p, m_p \) be the maximal ideals of the local rings \( A', B' \), respectively. The assumption says that \( m_p A' = m'_p. \) We have

\[
B'/A' \otimes_{A'} A'/m'_q = B'/\left( (m'_pB' + A') = B'/(m'_pA') = 0. \right.
\]

By Nakayama lemma, we obtain \( B'/A' = 0 \).

We keep using the notations used in the introduction. Recall that \( C \) is a non-singular projective curve of genus \( g \geq 0 \), and \( L \) is a very ample line bundle on \( C \). Consider \( \xi \in C_k \) and \( x \in C \), and let \( \xi := \xi_k + x \in C_{k+1} \). The divisor \( \xi \) spans a \( k \)-secant \((k - 1)\)-plane \( \mathbb{P}(H^0(\xi, L|_{\xi})) \) to \( C \) in \( \mathbb{P}(H^0(C, L)) \), and it is naturally embedded in the \((k + 1)\)-secant \( k \)-plane \( \mathbb{P}(H^0(\xi, L|_{\xi})) \) spanned by \( \xi \). This observation naturally induces a morphism

\[
\alpha_{k,1} : B^{k-1}(L) \times C \to B^k(L).
\]

To see it in details, we refer to [1, p.432, line -5]. We define the relative secant variety \( Z = Z_{k-1} \) of \((k - 1)\)-planes in \( B^k(L) \) to be the image of the morphism \( \alpha_{k,1} \). The relative secant variety \( Z \) is a divisor in the secant bundle \( B^k(L) \), and it is the preimage of \((k - 1)\)-th secant variety \( \Sigma_{k-1} \) under the morphism \( \beta_k \). It plays the role of transferring the codimension two situation \((\Sigma_k, \Sigma_{k-1})\) into the codimension one situation \((B^k(L), Z)\). We collect several properties of \( Z \).

**Proposition 2.2** ([2, Proposition 3.15, Theorem 5.2, and Proposition 5.13]). Recall the situation described in the diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & B^k(L) \\
\downarrow \pi_k & & \downarrow \beta_k \\
C_{k+1} & \subseteq & \mathbb{P}^r = \mathbb{P}(H^0(C, L))
\end{array}
\]

Let \( H \) be the pull back of a hyperplane divisor of \( \mathbb{P}^r \) by \( \beta_k \), and let \( I_{\Sigma_{k-1} | \Sigma_k} \) be the ideal sheaf on \( \Sigma_k \) defining the subvariety \( \Sigma_{k-1} \). Then one has

\[
1. \quad \mathcal{O}_{B^k(L)}((k + 1)H - Z) = \pi_k^*A^1_{k+1,L}.
\]

\[
2. \quad R^i\beta_k_*\mathcal{O}_{B^k(L)}(-Z) = \begin{cases} 
I_{\Sigma_{k-1} | \Sigma_k} & \text{if } i = 0 \\
0 & \text{if } i > 0.
\end{cases}
\]

\[
3. \quad I_{\Sigma_{k-1} | \Sigma_k} \cdot \mathcal{O}_{B^k(L)} = \mathcal{O}_{B^k(L)}(-Z).
\]
As a direct consequence of the above proposition, we have an identification
\[ H^0(C_{k+1}, A_{k+1,L}) = H^0(\Sigma_k, I_{\Sigma_{k-1}}(k+1)). \]

We are now ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let
\[ b: \bar{\Sigma}_k := \text{Bl}_{\Sigma_{k-1}} \Sigma_k \longrightarrow \Sigma_k \]
be the blowup of \( \Sigma_k \) along \( \Sigma_{k-1} \) with exceptional divisor \( E \). As \( I_{\Sigma_{k-1}}|_{\Sigma_k} \cdot \mathcal{O}_{B^k(L)} = \mathcal{O}_{B^k(L)}(-Z) \) (see Proposition 2.2), there exists a morphism \( \alpha \) from \( B^k(L) \) to the blowup \( \bar{\Sigma}_k \) fitting into the following commutative diagram
\[ B^k(L) \xrightarrow{\alpha} \bar{\Sigma}_k \]
\[ \downarrow \beta_k \]
\[ \Sigma_k. \]

We shall show that \( \alpha \) is an isomorphism.

Write \( V := H^0(\Sigma_k, I_{\Sigma_{k-1}}(k+1)) \). As proved in [2, Theorem 5.2], \( I_{\Sigma_{k-1}}|_{\Sigma_k} \) is globally generated by \( V \). This particularly implies that on the blowup \( \bar{\Sigma}_k \) one has a surjective morphism \( V \otimes \mathcal{O}_{\bar{\Sigma}_k} \to b^* \mathcal{O}_{\Sigma_k}(k+1)(-E) \), which induces a morphism \( \gamma: \bar{\Sigma}_k \longrightarrow \mathbb{P}(V) \).

On the other hand, one has an identification \( V = H^0(C_{k+1}, A_{k+1,L}) \) by Proposition 2.2. Recall from Proposition 1.2 that \( A_{k+1,L} \) is very ample. So the complete linear system \( |V| = |A_{k+1,L}| \) on \( C_{k+1} \) induces an embedding
\[ \psi: C_{k+1} \longrightarrow \mathbb{P}(V). \]
Also note that \( \alpha^*(b^* \mathcal{O}_{\Sigma_k}(k+1)(-E)) = \beta_k^* \mathcal{O}_{\Sigma_k}(k+1)(-Z) = \pi_k^* A_{k+1,L} \) by Proposition 2.2. Hence we obtain the following commutative diagram
\[ B^k(L) \xrightarrow{\alpha} \bar{\Sigma}_k \]
\[ \downarrow \pi_k \]
\[ C_{k+1} \xrightarrow{\psi} \mathbb{P}(V). \]

Take an arbitrary closed point \( x \in \bar{\Sigma}_k \), and consider its image \( x' := b(x) \) on \( \Sigma_k \). There is a nonnegative integer \( m \leq k \) such that \( x' \in \Sigma_m \setminus \Sigma_{m-1} \subseteq \Sigma_k \). In addition, the point \( x' \) uniquely determines a degree \( m+1 \) divisor \( \xi_{m+1,x'} \) on \( C \) in such a way that \( \xi_{m+1,x'} = \Lambda \cap C \), where \( \Lambda \) is a unique \((m+1)\)-secant \( m \)-plane to \( C \) with \( x' \in \Lambda \) (see [2, Definition 3.12]). By [2, Proposition 3.13], \( \beta_k^{-1}(x') \cong C_{k-m} \) and \( \pi_k(\beta_k^{-1}(x')) = \xi_{m+1,x'} + C_{k-m} \subseteq C_{k+1} \). Consider also \( x'' := \gamma(x) \) which lies in the image of \( \psi \). As \( \psi \) is an embedding, we may think \( x'' \) as a point of \( C_{k+1} \). Now, through forming fiber products, we see scheme-theoretically
\[ \alpha^{-1}(x) \subseteq \pi_k^{-1}(x'') \cap \beta_k^{-1}(x'). \]
However, the restriction of the morphism \( \pi_k \) on \( \beta_k^{-1}(x') \) gives an embedding of \( C_{k-m} \) into \( C_{k+1} \). This suggests that \( \pi_k^{-1}(x'') \cap \beta_k^{-1}(x') \) is indeed a single reduced point, and so is \( \alpha^{-1}(x) \). Finally by Lemma 2.1, \( \alpha \) is an isomorphism as desired. \( \square \)
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