A study on $r$-regular and $l$-regular near-rings

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Abstract
In this paper, by studying $r$-regular near-rings and $m$-regular near-rings, we proved some characterizations of $m$-regular near-rings, $r$-regular near-rings with IFP. We introduced the term $l$-regular near-ring and proved some results.

Keywords
$m$-regular near-ring, $r$-regular near-ring, $l$-regular near-ring, IFP.

AMS Subject Classification
16Y30.

1. Introduction
Development of the concept of near-rings is highly shaped by the inventive research on Ring theory. In ring theory, Roos $^{[14]}$ defined the concept of regularity and this notion was enforced and developed to Near-rings and several mathematicians gave a various characterization of near-rings such as Bell $^{[2]}$, Steve Ligh $^{[7]}$, YV Reddy and CVLN Murthy $^{[13]}$, Ramakotaiah $^{[10, 11]}$, Dheena $^{[5]}$, S Suryanarayanan and N Ganesan $^{[18]}$, Atagün, Akin and Kamacı, Hüseyin and Taştekin, Ismail and SEZGİN, Ashlan $^{[1]}$, Yong UK Cho $^{[3]}$ and Christian Lompjerzy Matczuk $^{[8]}$ developed the concept of semicentral idempotents for near-rings and rings. Especially, in ideal theory, Pairote Yiarayong $^{[20]}$ developed a strong relationship on various kinds of prime ideals in near-rings. Wendt Gerhard $^{[19]}$ investigated minimal ideals and primitivity in Right near-rings. Recently, S Ramkumar and T Manikantan $^{[12]}$ established the notion of the extension of a fuzzy soft set over a near-ring.

2. Preliminaries
For necessary definitions and basic results, the author follows $^{[9]}$. In this Preliminaries section, We recall the required definitions and results as follows.

Definition 2.1. A triplet $(\mathcal{R}, +, \cdot)$ is referred to as The, Right near-ring where

1. $\mathcal{R}$ holds the properties of a "Group" under addition.
2. $\mathcal{R}$ holds the properties of a "Semi-group" under multiplication.
3. $(t^1 + q^1).s^1 = t^1.s^1 + q^1.s^1, \forall t^1, q^1, s^1 \in \mathcal{R}$ (right distributive law).

Moreover in this paper, we consider Right near-ring$(\mathcal{R}, +, .)$ and we designate a right near-ring as $\mathcal{R}$ unless and otherwise mentioned. We write $t^1.s^1$ to denote $t^1.s^1$ for any two elements $t^1$ and $s^1$ in a near-ring $\mathcal{R}$.

Example 2.2. Let $(\mathcal{R}, +)$ where $\mathcal{R} = \{i^1, p^1, q^1, r^1\}$ be a Klein’s four group with addition and product tables mentioned below is an example for a near-ring. [see Pilz, p408 (12)(0,7,13,9)]
Definition 2.3. Let $\mathcal{R}$ is referred to as "Zero-symmetric near-ring (ZSN)" if $k0 = 0$ $\forall k \in \mathcal{R}$ i.e. $\mathcal{R} = \mathcal{R}_0$.

In the above example 2.2, $(\mathcal{R}, +, \cdot)$ is a ZSN and we denote it as $\mathcal{R} \in \eta_0$.

Definition 2.4. Let $\mathcal{D}$ be a subgroup of $\mathcal{R}$ is said to be $\mathcal{R}$-subgroup $(\mathcal{R}$-SG) if $\mathcal{R} \subseteq \mathcal{D}$.

If $S, T \subseteq \mathcal{R}$ then we define $ST = \{st/s \in S, t \in T\}$.

We, now designate a normal subgroup as NSG.

Definition 2.5. Let $\mathcal{I}$ be a NSG of $(\mathcal{R}, +)$ is referred to as the left ideal of $\mathcal{R}$ if $\forall t, p \in \mathcal{R}, \forall s \in \mathcal{I}, t(p+s) - tp \in \mathcal{I}$.

Definition 2.6. Let $\mathcal{I}$ be a NSG of $(\mathcal{R}, +)$ is referred to as the right ideal of $\mathcal{R}$ if $\exists \mathcal{R} \supseteq \mathcal{I}$.

Definition 2.7. Let $\mathcal{I}$ be a NSG of $(\mathcal{R}, +)$ is referred to as ideal(two-sided ideal) if it satisfies both the definitions of left ideal and a right ideal of $\mathcal{R}$.

Proposition 2.8. [9, proposition 1.34(c)]

For a $\mathcal{R} \in \eta_0$, every ideal is a $\mathcal{R}$-SG of $\mathcal{R}$.

Definition 2.9. Assume that $\mathcal{F}$ is a non-void subset in $\mathcal{R}$. Then $\{l/s \in I\}$ be the family of all left ideals which contain $\mathcal{F}$. $L = \bigcap_{x \in L} L_x$ is the smallest one among all left ideal containing $\mathcal{F}$ can be referred as "left ideal generated by $\mathcal{F}$".

Definition 2.10. Assume that an ideal $\mathcal{A}$ of $\mathcal{R}$ is termed to "principal ideal" if $\mathcal{A}$ is generated by one component.

If an ideal $\mathcal{A}$ which is generated by an element ‘$a$’, then $\mathcal{A}$ is symbolized by $(a)$.

If a left ideal $\mathcal{A}$ is generated by a single component ‘$a$’, then $\mathcal{A}$ is symbolized by $(a)$.

If the right ideal $\mathcal{A}$ is generated by a single component ‘$a$’, then $\mathcal{A}$ is symbolized by $(a)$.

Definition 2.11. The center of a near-ring $\mathcal{R}$ is defined as $\mathcal{C} = \{x \in \mathcal{R} : nx = xn, \forall n \in \mathcal{R}\}$.

Elements in $\mathcal{C}$ are said to be central.

Definition 2.12. A component ‘$p$’ is termed as an idempotent element of $\mathcal{R}$ if $p^2 = p$, for $p \in \mathcal{R}$.

Table 1. Addition table

|   | $i^1$ | $i^1$ | $p^1$ | $q^1$ | $r^1$ |
|---|-------|-------|-------|-------|-------|
| $i^1$ | $i^1$ | $p^1$ | $q^1$ | $r^1$ |
| $p^1$ | $p^1$ | $i^1$ | $q^1$ | $r^1$ |
| $q^1$ | $q^1$ | $r^1$ | $i^1$ | $p^1$ |
| $r^1$ | $r^1$ | $q^1$ | $p^1$ | $i^1$ |

Table 2. Product table

|   | $i^1$ | $i^1$ | $p^1$ | $p^1$ | $q^1$ | $q^1$ | $r^1$ | $r^1$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|
| $i^1$ | $i^1$ | $i^1$ | $i^1$ | $i^1$ | $i^1$ | $i^1$ | $i^1$ | $i^1$ |
| $p^1$ | $p^1$ | $p^1$ | $p^1$ | $p^1$ | $p^1$ | $p^1$ | $p^1$ | $p^1$ |
| $q^1$ | $q^1$ | $q^1$ | $q^1$ | $q^1$ | $q^1$ | $q^1$ | $q^1$ | $q^1$ |
| $r^1$ | $r^1$ | $r^1$ | $r^1$ | $r^1$ | $r^1$ | $r^1$ | $r^1$ | $r^1$ |

Definition 2.13. A non-zero element ‘$t$’ in $\mathcal{R}$ is termed as nilpotent, if $\exists k \in \mathcal{R}$ which is greater than or equal to 2 such that $t^k = 0$.

Definition 2.14. A subset $\mathcal{G}$ of $\mathcal{R}$ is referred to as "nil" if for all $t \in \mathcal{G}$ are nilpotent.

Definition 2.15. The set $0 : \Delta = \{t \in \mathcal{R} : tx = 0, \forall x \in \Delta\}$, where $\Delta$ be a subset of $\mathcal{R}$, is known as the annihilator of $\Delta$.

If $\Delta = \{\delta\}$, then $(0 : \Delta)$ is denoted by $(0 : \delta)$.

Corollary 2.16. [9, corollary 1.43 (a)] For any $\delta \in \mathcal{R}$, $(0 : \delta)$ is a "left ideal" of $\mathcal{R}$.

Corollary 2.17. [9, corollary 1.43 (b)] If $\Delta$ is a $\mathcal{R}$-SG of $\Gamma$, then the annihilator $(0 : \Delta)$ is an ideal in $\mathcal{R}$.

According to [2, 5, 9], let $\mathcal{R}$ is identified as Insertion of Factors Property (IFP), supposing that $ts = 0 \implies tps = 0$, $\forall t, s, p \in \mathcal{R}$. The above-mentioned near-ring Example 2.2 is an example for IFP near-ring.

Proposition 2.18. [9, proposition 9.3] The following affirmations are equivalent:

- $\mathcal{R}$ has the insertion of factors property (IFP).
- $(0 : s)$ is an ideal of $\mathcal{R}$, $\forall s \in \mathcal{R}$.
- Let $\mathcal{I} = (0 : \mathcal{G})$, for all subsets $\mathcal{G}$ of $\mathcal{R}$, $\mathcal{I}$ is an ideal.

Definition 2.19. For each component $k \in \mathcal{R}$, if $k^2 = 0 \implies k = 0$, then $\mathcal{R}$ is known as reduced near-ring.

Lemma 2.20. [5, lemma 2.8] For each $d, l \in \mathcal{R}$, which is a reduced near-ring then $dlt = dtl$ where $t^2 = t, t \in \mathcal{R}$.

Proposition 2.21. [9, proposition 9.37] If $\mathcal{R} \in \eta_0$ is having no non-zero nilpotent components, then $\mathcal{R}$ satisfies the IFP.

Definition 2.22. For each component $c \in \mathcal{R}$, if $\mathcal{R}c = \mathcal{R}c^2$ then $\mathcal{R}$ is known as "left bi potent".

Definition 2.23. For each component $k \in \mathcal{R}$, there is a component $l$ in $\mathcal{R}$ such that $k = kl$ then $\mathcal{R}$ is known as "regular near-ring (RN)"

Definition 2.24. For each component $p \in \mathcal{R}$, there is a component $l$ in $\mathcal{R}$ such that $p = lp^2$, then $\mathcal{R}$ is known as "left strongly regular near-ring (left SRN)"

According to [15], for each component $q \in \mathcal{R}$, there is a component $l$ which is an idempotent in $\mathcal{R}$ such that $q = ql, l \in \langle q \rangle$, then $\mathcal{R}$ is known as "$r$-regular near-ring (r-RN)"

Theorem 2.25. [15, Theorem 2.8] If $\mathcal{R}$ is r-RN with 1 and has IFP then $a = al$ implies $a = la$, where $l$ is an idempotent in $\mathcal{R}, l \in \langle a \rangle$.

Theorem 2.26. [15, Theorem 2.9] Let $\mathcal{R}$ be a r-RN which satisfies IFP with 1 then $\mathcal{R}$ is reduced.
Lemma 2.27. [9] [17] Let \( \mathcal{R} \in \eta_0 \) have IPF if and only if \( \mathcal{R} \) is an ideal where \( \mathcal{R} = (0 : \mathcal{S}) \), for all subsets \( \mathcal{S} \) of \( \mathcal{R} \).

Lemma 2.28. [5, lemma 1]
If a near-ring \( \mathcal{R} \in \eta_0 \) is reduced then for any \( 0 \neq a \in \mathcal{R} \)
1. \( \mathcal{R} \setminus A(a) \) is reduced and the residue class \( \mathcal{R} \) of a mod\( A(a) \) is a nonzero divisor where \( A(a) = \{ x \in \mathcal{R} : xa = 0 \} \).
2. \( k_1k_2...k_n = 0 \) implies \( (k_1)(k_2)...(k_n) = 0 \) for any \( k_1, k_2, ..., k_n \).

Theorem 2.29. [5, Theorem 1]
Let a near-ring \( \mathcal{R} \) be reduced. If \( \mathcal{S} \) is a nonvoid multiplicative subsemigroup of \( \mathcal{R} \) such that \( 0 \notin \mathcal{S} \), then a completely prime ideal \( \mathcal{D} \) exists in \( \mathcal{R} \) such that \( \mathcal{D} \cap \mathcal{S} = \emptyset \).

3. Characterization of ”r-regular near-rings”.

The principal object ”m-regular near-ring” was cited by G.Gopala Krishna Moorthy, R. Veega, and S. Geetha [6] and proved some results. In this section, we have a new characterization.

According to [6] for each component \( k \in \mathcal{R} \), there is a component \( l \in \mathcal{R} \) such that \( k = kl^m \) where \( m \geq 1 \) is a fixed integer, then \( \mathcal{R} \) is known as ”m-regular near-ring(m-RN)”.

Lemma 3.1. [6, lemma 3.10] Let \( \mathcal{R} \) be a m-RN, \( a \in \mathcal{R} \) and \( a = ab^m \). Then
- The idempotents are \( ab^m \) and \( b^m a \).
- \( ab^m \mathcal{R} = a \mathcal{R} \) & \( \mathcal{R}b^m a = \mathcal{R}a \).

Let \( \mathcal{D} \) subset of \( \mathcal{R} \) then \( \sqrt{\mathcal{D}} = \{ x \in \mathcal{R} : x^k \in \mathcal{D} \) for some \( k \geq 1 \} \)

Definition 3.2. Let \( \mathcal{D} \) be an ideal of \( \mathcal{R} \) is known as Semi-Prime Ideal(S-PI) supposing that for all ideals \( \mathcal{J} \) of \( \mathcal{R} \), \( \mathcal{J}^2 \subseteq \mathcal{D} \) implies \( \mathcal{J} \subseteq \mathcal{D} \).

Theorem 3.3. Let \( \mathcal{R} \in \eta_0 \) be a m-RN, r-RN with unity, and has IPF. Then \( \mathcal{C} = \sqrt{\mathcal{C}} \) where \( \mathcal{C} \) is \( \mathcal{R} \)-SG of \( \mathcal{R} \).

Proof. Assume that \( \mathcal{C} \) is a \( \mathcal{R} \)-SG of \( \mathcal{R} \). Let \( p \in \mathcal{C} \) implies \( p \mathcal{C} \in \mathcal{C} \) which implies \( p \in \sqrt{\mathcal{C}} \) hence, we get \( \mathcal{C} \subseteq \sqrt{\mathcal{C}} \).

Now let \( p \in \sqrt{\mathcal{C}} \Rightarrow p^k \in \mathcal{C} \).

By using the definition of m-RN, lemma 3.1 and theorem 2.25, we have \( p = pl^mp = (l^mp) = (l^m)p = l^mp^2 \).

Now, \( p = l^mpp = l^m(l^mp^2) = l^m(l^{m+1}p^2) = \cdots = l^{(k-1)m}p^k \subseteq \mathcal{R} \).

Hence, we get \( \sqrt{\mathcal{C}} \subseteq \mathcal{C} \).

Thus, \( \mathcal{C} = \sqrt{\mathcal{C}} \) where \( \mathcal{C} \) is \( \mathcal{R} \)-SG of \( \mathcal{R} \).

Definition 3.4. For each component \( p, t \) in a m-RN \( \mathcal{R} \) is referred to have IPF if \( pt = 0 \) then \( pl^mt = 0 \), for some \( l \in \mathcal{R} \) and \( m \geq 1 \) is a fixed integer.

Theorem 3.5. If \( \mathcal{R} \in \eta_0 \) be a m-RN, r-RN in which all the idempotents are central then \( \mathcal{R} \) is reduced.

Proof. Suppose \( p \in \mathcal{R} \) such that \( p^2 = 0 \).

By using the definition of m-RN, and lemma 3.1, \( p = pl^m p = l^mp^2 = l^mp = 0 \).

Therefore, \( \mathcal{R} \) is reduced.

Theorem 3.6. If \( \mathcal{R} \in \eta_0 \) be a m-RN, r-RN in which all the idempotents are central then \( \mathcal{R} \) satisfies IPF.

Proof. Let \( t, p \in \mathcal{R} \) such that \( tp = 0 \).

Now, \((pt)^2 = (pt)(pt) = p(tp)t = p0 = 0 \).

By the theorem 3.5, \( pt = 0 \).

For \( m \geq 1 \), a fixed integer, consider \( (l^mp)^2 = (l^m)(l^mp) = t(l^m)p = t^mp = t^m0 = 0 \).

By the theorem 3.5, \( t^mp = 0 \).

Hence \( \mathcal{R} \) has IPF.

Theorem 3.7. If \( \mathcal{R} \in \eta_0 \) be a m-RN, r-RN in which all the idempotents are central then \( \mathcal{R} \)-SG is an ideal.

Proof. Let \( \mathcal{R} \) be r-RN in which all idempotents are central.

By the definition of r-RN and By the theorem 2.25, we have \( a = ea, e^2 = e, e \in (a) \).

Let \( a \in \mathcal{R} \). Since, by the definition of m-RN, we have \( a = ab^m a \) where \( m \geq 1 \), a fixed integer and By the lemma 3.1, \( b^m a \) is idempotent.

Let \( b^m = e \) then by using the lemma 3.1, \( \mathcal{R} e = \mathcal{R} b^m a = \mathcal{R} a \).

Claim: \( 0 : \mathcal{F} = \{ y \in \mathcal{R} : xy = 0 \forall s \in \mathcal{G} \} \).

Now, \( (c - ce) e = ce - ce^2 = ce - ce = 0 \forall c \in \mathcal{R} \).

By the theorem 3.6, \( \mathcal{R} \) has IFP, \( (c - ce) e = 0 \forall c \Rightarrow \mathcal{R} e \in (0 : \mathcal{F}) \).

Let \( y \in (0 : \mathcal{F}) \Rightarrow sy = 0, \forall s \in \mathcal{G} \).

\( \Rightarrow syx^m y = 0 \).

Now, \( yx^m = (yx^m) e \in \mathcal{G} \Rightarrow yx^m (yx^m) e = 0 \).

\( \Rightarrow yx^m y = 0, \forall s \in \mathcal{G} \).

Therefore, \( 0 : \mathcal{F} = \mathcal{R} e = \mathcal{R} b^m a = \mathcal{R} a \).

By the lemma 2.27, \( 0 : \mathcal{F} \) become an ideal, for any subset of \( \mathcal{R} \) of \( \mathcal{R} \).

\( \mathcal{R} \) become an ideal.

Thus, every \( \mathcal{R} \)-SG is an ideal of \( \mathcal{R} \).

Theorem 3.8. If \( \mathcal{R} \in \eta_0 \) be a m-RN, r-RN in which all the idempotents are central then \( \mathcal{R} \) is semi-prime near-ring.

Proof. Let us define an ideal \( \mathcal{D} \) in \( \mathcal{R} \) such that \( pt \in \mathcal{D} \) for \( p, t \in \mathcal{R} \).

Let \( \mathcal{G} \) be \( \mathcal{R} \)-SG of \( \mathcal{R} \).

Then by the theorem 3.7, \( \mathcal{G} \) is an ideal of \( \mathcal{R} \) and suppose that \( \mathcal{D}^2 \subseteq \mathcal{G} \).

Since \( \mathcal{R} \) is zero-symmetric, \( \mathcal{R} \mathcal{D} \subseteq \mathcal{D} \).

If \( p \in \mathcal{D} \), then \( p = pt^m p \in \mathcal{D} \mathcal{D} \subseteq \mathcal{D} \subseteq \mathcal{D}^2 \subseteq \mathcal{G} \).

\( \mathcal{D} \subseteq \mathcal{G} \).

So, any \( \mathcal{R} \)-SG of \( \mathcal{R} \) is a S-PI.
Specifically, \( \{0\} \) is a S-PI and hence \( R \) is a semi-prime near-ring.

**Example 3.9.** Let us define \( R \) on \( Z_6 = \{0, 1, 2, 3, 4, 5\} \) with addition and product tables.[see Pilz, p409 (24)(3, 5, 3, 1, 1)]

Addition is modulo 6.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

Then \( (R, +, .) \) is a r-RN and also m-RN.

### 4. Characterization of ”l-regular near-rings”.

On studying the concepts of r-regular near-ring in [15, 16], the term l-regular near-ring was introduced. Yong Uk Cho [4] introduced semicentral idempotents and developed some results in the concept of reducibility in near-ring and we extended this notion of semicentral idempotent to the generalized regular near-rings namely r-regular near-ring(r-RN) and l-regular near-rings(l-RN).

We introduce the term ”l-regular near-ring(l-RN)” as follows:

**Definition 4.1.** For each element \( q \in \mathbb{R} \), there is a component \( l \) which is an idempotent in \( R \) such that \( q = lq, l \in |q| \), then \( R \) is known as ”l-regular near-ring(l-RN)”.

**Definition 4.2.** For each element \( p^2 = p \in R \) is referred to be left semicentral idempotent(left-SCI) if \( R \circ p = p \circ R \).

**Definition 4.3.** For each element \( q^2 = q \in \mathbb{R} \) is referred to be right semicentral idempotent(right-SCI) if \( q \circ R = q \circ q \).

**Definition 4.4.** For each element \( e^2 = e \in \mathbb{R} \) is referred to be central idempotent( CI) if \( ek = ke \) for all \( k \in \mathbb{R} \).

**Theorem 4.5.** Let \( R \in \eta_2 \), r-RN with 1 and has IFP. Then every left-SCI is right-SCI.

**Proof.** Since by the theorem 2.25, \( q = qe \) implies \( q = eq \) for all \( q \in R \).

Let \( R \in \eta_2 \), r-RN with 1 and has IFP.

Now for each \( q \in \mathbb{R} \) \( \exists e^2 = e \in \mathbb{R} \) such that \( q = qe, e \in |q| \subseteq |q| \).

Since \( (1-e)e = 0 \iff (1-e)qe = 0 \forall q \in \mathbb{R} \).

\( qe - eqe = 0 \iff qe = eqe \iff e \) is left-SCI.

By the theorem 2.25, \( qe = eqe = eq \iff eq = eq \iff e \) is right-SCI.

Thus, every left-SCI is right-SCI.

**Corollary 4.6.** Let \( R \in \eta_0 \), r-RN with 1 and has IFP. Then \( R \) is central.

**Theorem 4.7.** Let \( R \in \eta_0 \) be l-RN with 1 and has IFP. Then for any idempotent is left-SCI.

**Proof.** Let \( R \in \eta_0 \), l-RN with 1 and has IFP.

Now for each \( q \in \mathbb{R} \) \( \exists e^2 = e \in \mathbb{R} \) such that \( q = qe, e \in |q| \subseteq |q| \).

Since \( (1-e)e = 0 \implies (1-e)qe = 0 \forall q \in \mathbb{R} \).

\( qe - eqe = 0 \implies q = eqe \implies e \) is left-SCI.

Thus, for any idempotent is left semicentral idempotent(left-SCI).

In the above theorems 4.5, 4.7 and corollary 4.6, the concepts of unity and reducibility is essential.

**Example 4.8.** Consider a near-ring on the group \( Z_6 = \{0, 1, 2, 3, 4, 5\} \) with addition and product table given below.[see Pilz, p410 (53)(0, 1, 4, 3, 4, 1)]

Addition is modulo 5.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

This near-ring is r-RN and also l-RN. This near-ring is ZSN, reduced without unity.

It is clear that the idempotent elements 2 and 5 are not central. This near-ring \( R \) is right-SCI but not left-SCI. (for an element \( 1 \in R \) such that \( 2.1 \neq 1.2, 1 \).

**Example 4.9.** Any regular near-ring(RN) is r-RN and l-RN. Let us consider \( R \) on the group \( Z_5 = \{0, 1, 2, 3, 4\} \) with addition and product tables. [see Pilz, p408, (7)(0, 1, 4, 3, 4)]

Addition is modulo 5.

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 1 | 2 | 3 | 4 | 0 |
| 2 | 3 | 4 | 0 | 1 |
| 3 | 4 | 0 | 1 | 2 |
| 4 | 0 | 1 | 2 | 3 |

Then \( (R, +, .) \) is a RN.

**Remark 4.10.** In the above mentioned example 4.9, the near-ring \( R \) is left-SCI but not right-SCI(for an element \( 1 \in R \) such that \( 2.1 \neq 2.1.2 \).
Theorem 4.11. For a near-ring \( \mathcal{R} \) is l-RN then \( \mathcal{R} = \mathcal{R}l \).

Proof. By the definition of l-RN, then \( l = el \), since \( e^2 = e, e \in |l| \).
\[ l \in \mathcal{R}l \implies l \in \mathcal{R}l \]
Therefore \( \mathcal{R} = \mathcal{R}l \).

Theorem 4.12. For a near-ring \( \mathcal{R} \) is l-RN then \( (0 : u) = (0 : \mathcal{R}u) = (0 : \mathcal{R}) \), \( \forall u \in \mathcal{R} \).

Proof. Since \( \mathcal{R} \) is l-RN, \( u \in \mathcal{R}u \).
Let \( x \in (0 : \mathcal{R}u) \).
Now \( xu = 0 \implies xu = 0 \implies (0 : \mathcal{R}u) = (0 : u) \).
Let \( x \in (0 : u) \) then \( xu = 0 \implies xu = 0 \implies x \in (0 : \mathcal{R}u) \implies (0 : u) \subseteq (0 : \mathcal{R}u) \).
Therefore \( (0 : u) = (0 : \mathcal{R}) \).
By the theorem 4.11, \( (0 : u) = (0 : \mathcal{R}u) = (0 : \mathcal{R}) \).

Theorem 4.13. Let a near-ring \( \mathcal{R} \) be l-RN. Then every principal ideal is generated by an idempotent.

Proof. Let \( c \in \mathcal{R} \). Consider a principal ideal generated by \( c \).
If \( \mathcal{R} \) is l-RN, \( \mathcal{R} = uc, u^2 = u, u \in (c) \subseteq (c) \implies (c) \subseteq (u) \).
Therefore \( (c) = (u) \).

Example 4.14. Let us consider \( \mathcal{R} \) on \( \mathbb{Z}_5 = \{0, 1, 2, 3, 4, 5\} \) with addition and product table given below.[see Pilm. p409 (24)(3, 5, 3, 1, 1)]
Addition is modulo 6.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

The only ideals of \( \mathcal{R} \) are \( \{0\} \), \( \{0, 2, 4\} \) and \( \{0, 1, 2, 3, 4, 5\} \).
This near-ring \( (\mathcal{R}, +, \cdot) \) is both r-RN and l-RN.

Theorem 4.15. Let a near-ring \( \mathcal{R} \) be l-RN. Then \( \mathcal{R} \) has no nonzero nil ideals.

Proof. Suppose \( A \) be a nonzero nil ideal in \( \mathcal{R} \).
Let \( 0 \neq a \in A \) and \( a = ea, e \in |a| \), \( e^2 = e \).
By the theorem 4.13, \( e \in (a) \subseteq A \).
\[ \implies \ \\
\] is nilpotent, which is a conflict to ‘\( e \)' is idempotent.
Thus, \( \mathcal{R} \) has no nonzero nil ideals.

Theorem 4.16. For a near-ring \( \mathcal{R} \in \mathcal{R} \) is l-RN and every \( \mathcal{R} \)-subgroup is an ideal of \( \mathcal{R} \) then \( \mathcal{R} \) is left SRN.

Proof. Suppose that \( \mathcal{R} \) is l-RN and every \( \mathcal{R} \)-subgroup is an ideal of \( \mathcal{R} \).
By proposition 2.8, \( a = ea, e^2 = e, e \in |a| \subseteq (a) = \mathcal{R}a \).
\[ \implies e = na, \text{for some } n \in \mathcal{R} \]
Therefore \( a = ea = nna = na^2 \) for some \( n \in \mathcal{R} \).
Hence \( \mathcal{R} \) is left SRN.

Theorem 4.17. For a near-ring \( \mathcal{R} \in \eta_0 \) is l-RN with 1 then \( \mathcal{R} \) is reduced.

Proof. Let \( t \in \mathcal{R} \) and \( t^2 = 0 \implies t \in (0 : t) \implies \langle t \rangle \subseteq (0 : t) \).
Suppose \( \mathcal{R} \) is l-RN, then \( t = et, e^2 = e, e \in |t| \subseteq (t) \subseteq (0 : t) \implies et = 0 \).
Therefore \( t = 0 \).
Hence \( \mathcal{R} \) is reduced.

Theorem 4.18. For a near-ring \( \mathcal{R} \in \eta_0 \) is l-RN with 1 and has IFP then \( d = ed \) implies \( d = de \) where ‘\( e \)' is an idempotent.

Proof. Suppose \( \mathcal{R} \) is l-RN with 1 and has IFP.
Now \( d \in \mathcal{R} \exists e^2 = e \in \mathcal{R} \implies d = ed, e \in |d| \subseteq |d| \).
Since \( (1 - e) e = 0 \implies (1 - e) de = 0 \forall d \in \mathcal{R} \implies de - ede = 0 \implies de = ed = ed = d \) [by the lemma 2.20].
Therefore \( d = ed \) implies \( d = de \).

Definition 4.19. Let \( \mathcal{R} \) is referred to as weakly regular near-ring(WRN) if \( A^2 = A \) for every ideal \( A \) of \( \mathcal{R} \).

Definition 4.20. Let an ideal \( \mathcal{D} \) of \( \mathcal{R} \) is referred to as "Completely Prime Ideal(CPI) if \( kl \in \mathcal{D} \) implies \( k \in \mathcal{D} \) or \( l \in \mathcal{D} \).

Definition 4.21. Let an ideal \( \mathcal{D} \) of \( \mathcal{R} \) is referred to as "3-Prime Ideal(3-PI) if \( kn^l \in \mathcal{D} \) implies \( k \in \mathcal{D} \) or \( l \in \mathcal{D} \) for every \( n^l \in \mathcal{R} \).

Theorem 4.22. Let a near-ring \( \mathcal{R} \) be l-RN. Then \( \mathcal{R} \) is WRN.

Proof. Let \( \mathcal{D} \) be an ideal of \( \mathcal{R} \) and \( a \in \mathcal{D} \).
\[ a = ea, e^2 = e, e \in |a| \subseteq \mathcal{D} \subseteq \mathcal{D}, \mathcal{D} = \mathcal{D}^2. \]
But \( \mathcal{D}^2 \subseteq \mathcal{D} \), therefore \( \mathcal{D} = \mathcal{D}^2. \)
Thus, \( \mathcal{R} \) is WRN.

Theorem 4.23. Let a near-ring \( \mathcal{R} \) be l-RN. Then \( \mathcal{R} \) has no nonzero nilpotent ideal.

Proof. Suppose \( J \) be a nonzero nilpotent ideal in \( \mathcal{R} \).
Then \( J^k = (0) \) for some \( k \) which is greater than or equal to 2.
By the theorem 4.22, every ideal in \( \mathcal{R} \) is idempotent i.e., \( J = J^2. \)
\[ J^k = J^{k-2} J = J^{k-4} J^2 J \]
Continuing in this way we get \( J = (0) \).
It is a contradiction.
Thus \( \mathcal{R} \) has no nonzero nilpotent ideal.

Theorem 4.24. Let a near-ring \( \mathcal{R} \) be l-RN with left unity then every CPI is a maximal
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Proof. Let $\mathcal{C}$ be a CPI of $\mathfrak{R}$
Suppose $\mathcal{C} \subseteq \mathfrak{M} \subseteq \mathfrak{R}$ then $\exists a \in \mathfrak{M} \setminus \mathcal{C}$
Now $a = ea, e^2 = e, e \in |a| \subseteq |\mathcal{C}| \subseteq |\mathfrak{M}| \Rightarrow e \in \mathfrak{M}$
$(1 - e)a = 0 \in \mathcal{C} \Rightarrow 1 - e \in \mathcal{C} \subseteq \mathfrak{M} \Rightarrow 1 - e \in \mathfrak{M}$.
Let $c \in \mathfrak{R}$ then $c = (1 - e + e) = (1 - e)e + ec \in \mathfrak{M}$.
Therefore $\mathfrak{R} = \mathfrak{M}$.

Hence $\mathcal{C}$ is a maximal ideal.

Theorem 4.25. Let $\mathfrak{R}$ be a l-RN with left unity and has IFP then every 3-PI is maximal.

Proof. Let $\mathcal{C}$ be a 3-PI of $\mathfrak{R}$.
Assume $\mathcal{C} \subseteq \mathfrak{M} \subseteq \mathfrak{R}$.
Let $c \in \mathfrak{M} \setminus \mathcal{C}$.
Now $c = ec, e^2 = e, e \in |c| \subseteq |\mathcal{C}|$.
$(1 - e)c = 0$.
Since $\mathfrak{R}$ has IFP, $(1 - e)nc = 0 \forall n \in \mathfrak{R}$.
$(1 - e)c = 0 \subseteq \mathcal{C} \Rightarrow 1 - e \in \mathcal{C} \subseteq \mathfrak{M} \Rightarrow 1 - e \in \mathfrak{M}$.
For any $x$ in $\mathfrak{R}$, $x = ex + (1 - e)x \in \mathfrak{M}$.
Therefore $\mathfrak{R} = \mathfrak{M}$.

Thus $\mathcal{C}$ is maximal ideal.

Theorem 4.26. If a near-ring $\mathfrak{R}$ is l-RN then every ideal $I$ of $\mathfrak{R}$ is l-RN.

Proof. Suppose $\mathfrak{R}$ is l-RN, then $a = ea, e^2 = e, e \in |a|$.
Assume that $I$ is an ideal $\mathfrak{R}$.
Let $a \in I$ then $a = ea, e \in |a| \subseteq I$.
Therefore $I$ is l-RN.

Theorem 4.27. For a near-ring $\mathfrak{R} \in \eta_0$ with identity.

1. $\mathfrak{R}$ is l-RN and has IFP.

2. $\mathfrak{R}$ is reduced and every CPI is maximal.

Proof. (1) $\Rightarrow$ (2)
Suppose $\mathfrak{R}$ is l-RN.
By theorem 4.17, $\mathfrak{R}$ is reduced and by theorem 4.24, it is proved.

(2) $\Rightarrow$ (1)
Suppose $\mathfrak{R} \in \eta_0$ is reduced and every CPI is maximal.
Since $\mathfrak{R} \in \eta_0$ is reduced, $ab = 0 \Rightarrow ba = 0$.
Consider $nba = n(ba) = n0 = 0 \Rightarrow (nb)a = 0 \Rightarrow anb = 0 \forall n \in \mathfrak{R}$.
Therefore $\mathfrak{R}$ has IFP.
Let $0 \neq a \in \mathfrak{R}$, by the lemma 2.28, $\mathfrak{K} = \mathfrak{R} \setminus A(a)$ is reduced and $\bar{a}$ is not a zero divisor.
Also, every CPI of $\mathfrak{K}$ is a maximal ideal in $\mathfrak{K}$.
Let $Q$ be the multiplicative subsemigroup generated by an element $\bar{a} - \bar{1} \bar{a}$ where $\bar{1} \in |a|$.
If not, by the theorem 2.29, there exists a CPI $\bar{P}$ with $\bar{P} \cap Q = \emptyset$.
Suppose $|a| \subseteq \bar{P}$ then $\bar{a} \in \bar{P}$.
$\Rightarrow \bar{a} - \bar{1} \bar{a} \in \bar{P}$.
$\Rightarrow \bar{a} - \bar{1} \bar{a} \in \bar{P} \cap Q$, it is a contradiction to the fact that $\bar{P} \cap Q = \emptyset$.
Suppose $|a| \nsubseteq \bar{P}$ and $\bar{P}$ is maximal, we have $\mathfrak{K} = \bar{P} + |a|$. $\bar{P} = \bar{a} + \bar{t}$ where $\bar{a} \in \bar{P}, \bar{t} \in |a|$.
$\Rightarrow \bar{a} - \bar{t} \bar{a} = \bar{a} \bar{a} \in \bar{P}$.
$\Rightarrow \bar{a} - \bar{t} \bar{a} \in \bar{P} \cap Q$, it is a contradiction to the fact, $\bar{P} \cap Q = \emptyset$.
Thus $0 \in Q$.
Now $0 = (\bar{a} - \bar{t} \bar{a})(\bar{a} - \bar{t} \bar{a}) \cdots (\bar{a} - \bar{t} \bar{a}) = (\bar{1} - \bar{t}) \bar{a}, \bar{a} \in |a|$.
Since $\bar{a}$ is not zero divisor, $(\bar{1} - \bar{t}) = 0 \Rightarrow \bar{1} = \bar{t}, t \in |a|$.
Hence $(1 - t)a \in A(a) \Rightarrow (1 - t)a = 0, t \in |a|, t^2 = t \Rightarrow a = ta, t^2 = t, t \in |a|$.
Therefore $\mathfrak{R}$ is l-RN.

Definition 4.28. Let a near-ring $\mathfrak{R}$ is referred to as “Left Quasi Duo(LQD)” if every maximal left ideal of $\mathfrak{R}$ is two-sided ideal.

Theorem 4.29. For a near-ring $\mathfrak{R} \in \eta_0$ is the LQD with left unity $l$, $\mathfrak{R}$ is l-RN then $\mathfrak{R} = \{q\} + (0 : q)$.

Proof. Since $\mathfrak{R}$ is l-RN, then $q = tq, t^2 = t, t \in |q| \subseteq |q|$.
$\Rightarrow q \subseteq (q)q.
Then (q)q \subseteq (q)q.
And we have $|q\rangle q \subseteq (q)q$.
Therefore $\bar{q} = \langle q \rangle q$.
Suppose that $\mathfrak{R} \neq \langle q \rangle + (0 : q)$.
Then there exists a maximal left ideal $\mathfrak{C}$ such that $\langle q \rangle + \langle 0 : q \rangle \subseteq \mathfrak{C}$.
Since $\mathfrak{R}$ is LQD, $\mathfrak{C}$ is a two-sided ideal.
Since $q \in \mathfrak{C}, (q)q \subseteq \mathfrak{C} \subseteq \bar{q} = \langle q \rangle q$.
Therefore $\mathfrak{C}q = \langle q \rangle q$.
Therefore $\mathfrak{C}q = \langle q \rangle q \subseteq \mathfrak{C}$.
Therefore $s \in \{q\}$ such that $q = sq, s \in \{q\}$.
$\Rightarrow (1 - s)q = 0 \Rightarrow 1 - s \in \{0 : q\}$.
Therefore $1 = s + (1 - s) \in \{q\} + \{0 : q\} \subseteq \mathfrak{C}$.
It is a contradiction.
Therefore $\mathfrak{R} = \langle q \rangle + (0 : q)$.

5. Conclusion

In mathematics, several researchers are working on algebra. Recently as an application of near-rings, mathematicians used planar near-rings, near-rings of polynomials, and other near-rings to expand designs and codes. In this publication, we made an effort to develop the concept of regular near-rings and generalized regular near-rings.

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