The $\kappa$-Fréchet–Urysohn property for locally convex spaces

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Abstract

A topological space $X$ is $\kappa$-Fréchet–Urysohn if for every open subset $U$ of $X$ and every $x \in U$ there exists a sequence in $U$ converging to $x$. We prove that every $\kappa$-Fréchet–Urysohn Tychonoff space $X$ is Ascoli. We apply this statement and some of known results to characterize the $\kappa$-Fréchet–Urysohn property in various important classes of locally convex spaces. In particular, answering a question posed in [7] we obtain that $C_p(X)$ is Ascoli iff $X$ has the property ($\kappa$).

Keywords: $\kappa$-Fréchet–Urysohn, Ascoli space, $C_p(X)$, $C_k(X)$, Banach space, weak topology

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1. Introduction

Following Arhangel’kii, a topological space $X$ is said to be $\kappa$-Fréchet–Urysohn if for every open subset $U$ of $X$ and every $x \in U$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq U$ converging to $x$. Clearly, every Fréchet–Urysohn space is $\kappa$-Fréchet–Urysohn. In [10, Theorem 3.3] Liu and Ludwig showed that a topological space $X$ is $\kappa$-Fréchet–Urysohn if and only if $X$ is a $\kappa$-pseudo open image of a metric space. Below we give another characterization of $\kappa$-Fréchet–Urysohn spaces, see 2.1. It is known that there are $\kappa$-Fréchet–Urysohn spaces which are not $k$-spaces, and there are sequential spaces which are not $\kappa$-Fréchet–Urysohn, see [10] or Proposition 2.6 below.

Let $X$ be a Tychonoff (=completely regular and Hausdorff) space. Denote by $C_k(X)$ and $C_p(X)$ the space $C(X)$ of all real-valued continuous functions on $X$ endowed with the compact-open topology and the pointwise topology, respectively. Following [2], $X$ is called an Ascoli space if every compact subset $K$ of $C_k(X)$ is evenly continuous (i.e., if the map $(f, x) \mapsto f(x)$ is continuous as a map from $K \times X$ to $\mathbb{R}$). In [4] we noticed that $X$ is Ascoli if and only if every compact subset of $C_k(X)$ is equicontinuous. The classical Ascoli theorem [3, Theorem 3.4.20] states that every $k$-space is Ascoli.

In [14, Theorem 2.1], Sakai characterized those spaces $C_p(X)$ which are $\kappa$-Fréchet–Urysohn. Recall that a family $\{A_i\}_{i \in I}$ of subsets of a set $X$ is said to be point-finite if the set $\{i \in I : x \in A_i\}$ is finite for every $x \in X$. A family $\{A_i\}_{i \in I}$ of subsets of a topological space $X$ is called strongly point-finite if for every $i \in I$, there exists an open set $U_i$ of $X$ such that $A_i \subseteq U_i$ and $\{U_i\}_{i \in I}$ is point-finite. Following Sakai [14], a topological space $X$ is said to have the property $(\kappa)$ if every sequence of pairwise disjoint finite subsets of $X$ has a strongly point-finite subsequence.

Theorem 1.1 ([14]). The space $C_p(X)$ is $\kappa$-Fréchet–Urysohn if and only if $X$ has the property $(\kappa)$.
A characterization of the spaces $C_k(X)$ which are $\kappa$-Fréchet–Urysohn is given in \[15\].

In \[7\] we proved the following theorem.

**Theorem 1.2 (\[7\]).** If $C_p(X)$ is Ascoli, then it is $\kappa$-Fréchet–Urysohn.

However, the question (see \[7, Question 2.4\]) of whether every $\kappa$-Fréchet–Urysohn space $C_p(X)$ is Ascoli remained open. In this short note we answer this question in the affirmative using the following somewhat unexpected result.

**Theorem 1.3.** Each $\kappa$-Fréchet–Urysohn space $X$ is Ascoli.

Now Theorems 1.1-1.3 immediately imply the following characterization of spaces $C_p(X)$ which are Ascoli.

**Corollary 1.4.** Let $X$ be a Tychonoff space. Then $C_p(X)$ is Ascoli if and only if $X$ has the property $(\kappa)$.

Denote by $D(\Omega)$ the space of test functions over an open subset $\Omega$ of $\mathbb{R}^n$. In \[5\] we proved that $D(\Omega)$ and the strong dual $D'(\Omega)$ of $D(\Omega)$, the space of distributions, are not Ascoli. Therefore, by Theorem 1.3 $D(\Omega)$ and $D'(\Omega)$ are not $\kappa$-Fréchet–Urysohn spaces. Below we apply Theorem 1.3 and some of the main results from \[1, 4, 5, 6, 8\] to characterize the $\kappa$-Fréchet–Urysohness in various important classes of locally convex spaces.

2. Proof of Theorem 1.3

We start from the following characterization of $\kappa$-Fréchet–Urysohn spaces. The closure of a subset $A$ of a topological space $X$ is denoted by $\overline{A}$ or $\text{cl}_X(A)$.

**Theorem 2.1.** A topological space $X$ is $\kappa$-Fréchet–Urysohn if and only if each point $x \in X$ is contained in a dense $\kappa$-Fréchet–Urysohn subspace of $X$.

**Proof.** The necessity is clear. To prove sufficiency, fix an open subset $U$ of $X$ and a point $x \in \overline{U}$. Let $Y$ be a dense $\kappa$-Fréchet–Urysohn subspace of $X$ containing $x$. Then $V := U \cap Y$ is an open subset of $Y$. We claim that $x \in \text{cl}_Y(V)$. Indeed, if $W \subseteq Y$ is an open neighborhood of $x$ in $Y$, take an open $W' \subseteq X$ such that $W = W' \cap Y$. Then the set $W' \cap U$ is open in $X$. Since $Y$ is dense in $X$ the set $(W' \cap U) \cap Y = (W' \cap Y) \cap (U \cap Y) = W \cap V$ is not empty. Thus $x \in \text{cl}_Y(V)$ and the claim is proved. Finally, since $Y$ is $\kappa$-Fréchet–Urysohn there is a sequence $\{y_n\}_{n \in \mathbb{N}} \subseteq V \subseteq U$ converging to $x$. \hfill $\Box$

**Corollary 2.2.** Let $Y$ be a dense subset of a homogeneous space (in particular, a topological group) $X$. If $Y$ is $\kappa$-Fréchet–Urysohn, then $X$ is also a $\kappa$-Fréchet–Urysohn.

**Proof.** Fix arbitrarily $y_0 \in Y$. Let $x \in X$. Take a homeomorphism $h$ of $X$ such that $h(y_0) = x$. Then $x \in h(Y)$ and $h(Y)$ is a $\kappa$-Fréchet–Urysohn space. Therefore, each element of $X$ is contained in a dense $\kappa$-Fréchet–Urysohn subspace of $X$ and Theorem 2.1 applies. \hfill $\Box$

In \[10, Theorem 4.1\] Liu and Ludwig proved that the product of a family of bi-sequential spaces is $\kappa$-Fréchet–Urysohn. Note that any countable product of bi-sequential spaces is bi-sequential, see \[12, Proposition 3.D.3\]. On the other hand, countable products of $W$-spaces are $W$-spaces (\[9, Theorem 4.1\]) and there are $W$-spaces which are not bi-sequential (\[8, Example 5.1\]). Taking into account that bi-sequential spaces and $W$-spaces are Fréchet–Urysohn spaces, the next corollary essentially generalizes Theorem 4.1 of \[10\].
Corollary 2.3. Let \( \{X_i : i \in I\} \) be a family of topological spaces such that \( \prod_{i \in I'} X_i \) is Fréchet–Urysohn for any countable subset \( I' \) of \( I \). Then the space \( X = \prod_{i \in I} X_i \) is \( \kappa \)-Fréchet–Urysohn.

**Proof.** For every \( z = (z_i) \in X \), set
\[
\sigma(z) := \{ x = (x_i) \in X : \{ i : x_i \neq z_i \} \text{ is finite} \}.
\]

Clearly, \( \sigma(z) \) is a dense subspace of \( X \). Proposition 2.6 of \([7]\) states that \( \sigma(z) \) is Fréchet–Urysohn. By Theorem 2.1, \( X \) is \( \kappa \)-Fréchet–Urysohn. \( \square \)

Below we prove Theorem 1.3.

**Proof of Theorem 1.3.** Suppose for a contradiction that \( X \) is not an Ascoli space. Then there exists a compact set \( K \) in \( C_k(X) \) which is not equicontinuous at some point \( z \in X \). Therefore there is \( \varepsilon_0 > 0 \) such that for every open neighborhood \( U \) of \( z \) there exists a function \( f_U \in K \) for which the open set \( W_{f_U} := \{ x \in U : |f_U(x) - f_U(z)| > \varepsilon_0 \} \) is not empty. Set
\[
W := \bigcup \{ W_{f_U} : U \text{ is an open neighborhood of } z \}.
\]

Then \( W \) is an open subset of \( X \) such that \( z \in \overline{W} \setminus W \). As \( X \) is \( \kappa \)-Fréchet–Urysohn, there is a sequence \( \{x_n : n \in \mathbb{N}\} \subseteq W \) converging to \( z \). For every \( n \in \mathbb{N} \), choose an open neighborhood \( U_n \) of \( z \) such that \( x_n \in W_{f_{U_n}} (\subseteq U_n) \) and, therefore,
\[
|f_{U_n}(x_n) - f_{U_n}(z)| > \varepsilon_0 \quad (\text{for all } n \in \mathbb{N}). \tag{2.1}
\]

Set \( S := \{x_n : n \in \mathbb{N}\} \cup \{z\} \). Then \( S \) is a compact subset of \( X \). Denote by \( p \) the restriction map \( p : C_k(X) \to C_k(S), p(f) = f|_S \). Then \( p(K) \) is a compact subset of the Banach space \( C_k(S) \). Applying the Ascoli theorem to the compact space \( S \) we obtain that the sequence \( \{p(f_{U_n})\}_{n \in \mathbb{N}} \subseteq p(K) \) is equicontinuous at \( z \in S \) and, therefore, there is an \( N \in \mathbb{N} \) such that
\[
|f_{U_n}(x_i) - f_{U_n}(z)| < \frac{\varepsilon_0}{2} \quad \text{for all } i \geq N \text{ and } n \in \mathbb{N}.
\]

In particular, for \( i = n = N \) we obtain \( |f_{U_n}(x_N) - f_{U_n}(z)| < \frac{\varepsilon_0}{2} \). But this contradicts (2.1). Thus \( X \) is an Ascoli space. \( \square \)

The next corollary strengthens Theorem 1.3 of \([7]\).

**Corollary 2.4.** Let \( X \) be a Čech-complete space. Then \( C_p(X) \) is Ascoli if and only if \( X \) is scattered.

**Proof.** If \( C_p(X) \) is Ascoli, then \( X \) is scattered by Theorem 1.3 of \([7]\). Conversely, if \( X \) is scattered, then, by Corollary 3.8 of \([14]\), \( X \) has the property \((\kappa)\). Thus, by Corollary 1.4, \( C_p(X) \) is Ascoli. \( \square \)

Let \( E \) be a locally convex space over a field \( F \), where \( F = \mathbb{R} \) or \( \mathbb{C} \), and let \( E' \) the dual space of \( E \). If \( E \) is a Banach space, denote by \( B \) the closed unit ball of \( E \) and set \( B_w := (B, \sigma(E, E')|_B) \), where \( \sigma(E, E') \) is the weak topology on \( E \).

**Corollary 2.5.** (i) If \( E \) is a Banach space, then \( B_w \) is \( \kappa \)-Fréchet–Urysohn if and only if \( E \) does not contain an isomorphic copy of \( \ell_1 \).

(ii) A Fréchet space \( E \) over \( F \) is \( \kappa \)-Fréchet–Urysohn in the weak topology if and only if \( E = F^N \) for some \( N \leq \omega \).

(iii) If \( X \) is a \( \mu \)-space and a \( k_{\mathbb{R}} \)-space, then \( C_k(X) \) is \( \kappa \)-Fréchet–Urysohn in the weak topology if and only if \( X \) is discrete.

(iv) The weak* dual space of a metrizable barrelled space \( E \) is \( \kappa \)-Fréchet–Urysohn if and only if \( E \) is finite-dimensional.
PROOF. (i) Theorem 1.9 of [5] or Theorem 6.1.1 and Corollary 1.7 of [6] state that $B_w$ is Ascoli if and only if $B_w$ is Fréchet–Urysohn if and only if $E$ does not contain an isomorphic copy of $\ell_1$. Now Theorem 1.3 applies.

(ii) Corollary 1.7 of [6] states that $E$ is Ascoli in the weak topology if and only if $E = F^N$ for some $N \leq \omega$. This result and Theorem 1.3 imply the desired.

(iii) Corollary 1.9 of [6] states that $\mathcal{C}(X)$ is Ascoli in the weak topology if and only if $X$ is discrete. Now the assertion follows from Theorem 1.3 and the fact that any product of metrizable spaces is $\kappa$-Fréchet–Urysohn (see Fact 1.2 of [7]).

(iv) Corollary 1.14 of [6] states that the weak* dual space of $E$ is Ascoli if and only if $E$ is finite-dimensional, and Theorem 1.3 applies.

Now we consider direct locally convex sums of locally convex spaces. The simplest infinite direct sum of lcs is the space $\varphi$, the direct locally convex sum $\bigoplus_{n \in \mathbb{N}} E_n$ with $E_n = F$ for all $n \in \mathbb{N}$. It is well known that $\varphi$ is a sequential non-Fréchet–Urysohn space, see Example 1 of [8].

**Proposition 2.6.** An infinite direct sum of (non-trivial) locally convex spaces is not $\kappa$-Fréchet–Urysohn. In particular, $\varphi$ is not a $\kappa$-Fréchet–Urysohn space.

**Proof.** Let $L = \bigoplus_{i \in I} E_i$ be the direct locally convex sum of an infinite family $\{E_i\}_{i \in I}$ of locally convex spaces. It is well known that every $E_i$ can be represented as a direct sum $F \oplus E'_i$. Therefore $L$ contains $\varphi$ as a direct summand. Since the projection of $L$ onto $\varphi$ is open and the $\kappa$-Fréchet–Urysohn property is preserved under open maps (see Proposition 3.3 of [7]), it is sufficient to show that $\varphi$ is not a $\kappa$-Fréchet–Urysohn space.

We consider elements of $\varphi$ as functions from $\mathbb{N}$ to $F$ with finite support. Recall that the sets of the form

$$\{f \in \varphi : |f(n)| < \varepsilon_n \text{ for every } n \in \mathbb{N}\},$$

(2.2)

where $\varepsilon_n > 0$ for all $n \in \mathbb{N}$, form a basis at 0 of $\varphi$ (see for example [13, Example 1]). For every $n, k \in \mathbb{N}$, set

$$U_{n,k} := \left\{ f \in \varphi : |f(1)| > \frac{1}{2n} \text{ and } |f(n)| > \frac{1}{2k} \right\},$$

and set $U := \bigcup_{n,k \in \mathbb{N}} U_{n,k}$. It is easy to see that all the sets $U_{n,k}$ are open in $\varphi$ and $0 \notin U_{n,k}$. Hence $U$ is an open subset of $\varphi$ such that $0 \notin U$. To show that $\varphi$ is not $\kappa$-Fréchet–Urysohn, it suffices to prove that (A) $0 \in U$, and (B) there is no a sequence in $U$ converging to 0.

(A) Let $W$ be a basic neighborhood of zero in $\varphi$ of the form (2.2). Choose an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon_1$, and take $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon_n$. It is clear that $U_{n,k} \cap W$ is not empty. Thus $0 \in U$.

(B) Suppose for a contradiction that there is a sequence $S = \{f_j\}_{j \in \mathbb{N}}$ in $U$ converging to 0. For every $j \in \mathbb{N}$, take $n_j, k_j \in \mathbb{N}$ such that $f_j \in U_{n_j,k_j}$. Since $f_j \to 0$, the definition of $U_{n,k}$ implies that $\frac{1}{2n_j} < |f_j(1)| \to 0$, and hence $n_j \to \infty$. Without loss of generality we can assume that $1 \leq n_1 < n_2 < \cdots$. For every $n \in \mathbb{N}$, define $\varepsilon_n = \frac{1}{4k_j}$ if $n = n_j$ for some $j \in \mathbb{N}$, and $\varepsilon_n = 1$ otherwise. Set

$$V := \{ f \in \varphi : |f(n)| < \varepsilon_n \text{ for every } n \in \mathbb{N}\}.$$

Then, $V$ is a neighborhood of 0, and the construction of $U_{n,k}$ implies that $V \cap U_{n_j,k_j} = \emptyset$ for every $j \in \mathbb{N}$. Thus $S \cap V = \emptyset$ and hence $f_j \not\to 0$, a contradiction.

Recall that a strict $(LF)$-space $E$ is the direct limit $E = \text{s-ind}_{\mathbb{N}} E_n$ of an increasing sequence

$$E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots$$
of Fréchet (= locally convex complete metric linear) spaces in the category of locally convex spaces and continuous linear maps. The space $D(\Omega)$ of test functions is one of the most famous and important examples of strict $(LF)$-spaces.

**Corollary 2.7.** A strict $(LF)$-space $E$ is $\kappa$-Fréchet–Urysohn if and only if $E$ is a Fréchet space.

**Proof.** Theorem 1.2 of [5] states that $E$ is an Ascoli space if and only if $E$ is a Fréchet space or $E = \varphi$. Now the assertion follows from Theorem 1.3 and Proposition 2.6. □

One of the most important classes of locally convex spaces is the class of free locally convex spaces. Following [11], the free locally convex space $L(X)$ on a Tychonoff space $X$ is a pair consisting of a locally convex space $L(X)$ and a continuous map $i : X \to L(X)$ such that every continuous map $f$ from $X$ to a locally convex space $E$ gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is essentially unique.

**Corollary 2.8.** Let $X$ be a Tychonoff space. Then $L(X)$ is a $\kappa$-Fréchet–Urysohn space if and only if $X$ is finite.

**Proof.** It is well known that $L(D)$ over a countably infinite discrete space $D$ is topologically isomorphic to $\varphi$. By Theorem 1.2 of [6], $L(X)$ is an Ascoli space if and only if $X$ is a countable discrete space. This fact, Theorem 1.3 and Proposition 2.6 immediately imply the assertion. □

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