Local Quantum Observables in the Anti-deSitter - Conformal QFT Correspondence

Karl-Henning Rehren
Institut für Theoretische Physik, Universität Göttingen, 37073 Göttingen, Germany
(March 15, 2000)

Quantum field theory on $d+1$-dimensional anti-deSitter space-time admits a re-interpretation as a quantum field theory with conformal symmetry on $d$-dimensional Minkowski space-time. This conjecture originally emerged from string theory considerations. Here, it is proven in a general framework by an explicit identification between the local observables of the two corresponding theories.

PACS numbers: 11.10.Cd, 11.25.Hf, 04.62.+v

I. THE ADS-CFT CORRESPONDENCE

Quantum field theory on anti-deSitter space-time (AdS) has received an important impact from string theory. Evidence was found [1] to the effect that certain theories on $d+1$-dimensional AdS equivalently describe conformally invariant quantum field theories (CFT) on $d$-dimensional Minkowski space-time. In particular, the higher-dimensional AdS theory can be recovered from the lower-dimensional CFT. So the correspondence is “holomorphic” in the sense of [5] where the influence of a black hole horizon on quantum fields in the ambient bulk space was discussed.

This correspondence has attracted much attention, as it suggests a wealth of implications for quantum gravity and for gauge theories in physical (four-dimensional) space-time. For a comprehensive review, see [2].

A large portion of the work on the AdS-CFT correspondence crucially involves “stringy” pictures (branes, duality, M-theory) when comparing contributions to the relevant path integrals and/or correlation functions. The correspondence is, however, claimed to be a model-independent feature of quantum field theory [3].

So the question arises as to whether it can be understood in more basic terms not relying on string theory. In fact, one main ingredient, the coincidence of the space-time symmetry groups, is even a purely classical one, long known to physicists: both the isometry group of AdS in $d+1$ dimensions and the conformal group of Minkowski space-time in $d$ dimensions are SO($2,d$).

It is the aim of this letter to show that it is indeed possible to understand (and to prove) the AdS-CFT correspondence in a general quantum field theoretical set-up. We shall first give a brief introduction to this set-up in Sect. II. It is certainly the most general one to incorporate the two fundamental principles of relativistic Covariance and Causality in quantum theory.

II. LOCAL OBSERVABLES

The prime objects of consideration in a quantum theory are the quantum observables, represented as self-adjoint operators on a Hilbert space whose elements are the vector states in which the system can be prepared. The real expectation values of the observables in various states (e.g., the vacuum state) predict the statistical outcome of any measurement.

In a relativistic quantum theory, in contrast to quantum mechanics, observables have the property of localization, compatible with Locality and Covariance. Locality is a consequence of Einstein Causality and means that observables which are localized at spacelike distance commute with each other. Covariance requires that the space-time symmetry group acts (by unitary operators $U(g)$) on the localization of observables according to its geometric significance, thereby preserving any algebraic relations among them. In the AdS-CFT case at hand this group is the AdS resp. conformal group SO($2,d$).

In conventional quantum field theory, the above features are usually encoded by quantum fields: objects $\phi(x)$ localized at the points $x$ in space-time, commuting at spacelike distance and transforming under some relativistically covariant transformation law. Due to the singular nature of quantum fields, these are not Hilbert space operators, but become operators after smearing with a test function. The best localization of observables is therefore an open region in space-time which contains the support of a test function.

The choice of quantum fields used for the description of a relativistic quantum system is to a large extent a matter of convenience. It has been recognized long ago [4] that different quantum fields may well describe the same quantum system. Prime examples are the equivalence of the Sine-Gordon and Thirring model [5], the re-interpretation of Chern-Simons theories in terms of models with Yukawa interaction [6], and the duality of certain supersymmetric Yang-Mills theories [7].

One is thus led to the conclusion [8] that what determines the physical interpretation of a quantum theory are not the individual quantum fields but the algebras of Hilbert space operators which are generated by localized field operators. Theories with possibly different equations of motion may well be equivalent if they only generate the same system of local algebras. The existence of generating fields is not even required if the local algebras can be specified by any other consistent prescription.

To conclude, the knowledge of localization is sufficient
III. ADS-CFT RESUMED

We adopt the set-up, sketched above, of algebras of localized observables, based on fundamental principles generally accepted. It applies to any physically reasonable relativistic quantum field theory, including certain string theories \[13\]. We shall show that it is the most natural set-up to establish the AdS-CFT correspondence. For it is this structure which is preserved by the correspondence. In contrast, the description in terms of specific fields and Lagrangeans will in general not be preserved.

As we are aiming at a general and intrinsic description of the AdS resp. CFT theories, we shall deal here with their local algebras and assume that they comply with the requirements of Covariance and Locality, as well as the additional but obvious property of Isotony: an observable localized in some region \(O\) is localized in any larger region also.

Let us denote the bijective correspondence between local algebras \(A(W)\) of observables localized in each open \(d\)-dimensional space-time region \(W\) if

\[ A(O) = \text{span}\{\phi: \phi \text{ is an observable localized in } O\} \]

This assignment has to comply with Covariance and Locality.

Thus, to prove the AdS-CFT correspondence, we have to establish a prescription specifying the algebras \(B(W)\) of local AdS observables for suitable AdS regions \(W\) if the algebras \(A(K)\) of local CFT observables for suitable Minkowski regions \(K\) are given, and vice versa. This prescription must pass on Locality and Covariance from the given theory to the new theory in correspondence.

As the discussion of quantum fields in the presence of a gravitational horizon \[14\] underlying the holographic picture suggests, the set of all operators representing observables should be the same for both theories, and act on the same Hilbert space. Moreover, the conformal space-time should play the role of a horizon in AdS space-time.

Indeed, the \(d + 1\)-dimensional AdS space-time given as

\[ \text{AdS}_{1,d} = \{\xi \in \mathbb{R}^{d+2}: \xi_0^2 - \xi_1^2 - \ldots - \xi_d^2 + \xi_{d+1}^2 = R^2\} \]

has a “boundary” at spacelike infinity, and the induced (properly rescaled) metric of the boundary is that of \(d\)-dimensional conformal Minkowski space-time. The action of the AdS group on \(\text{AdS}_{1,d}\) preserves the boundary, and acts on it like the conformal group in \(d\)-dimensional Minkowski space-time.

The law of causal propagation between the bulk of AdS and its boundary suggest how to find the prescription to identify localized observables between the two theories \[14\]. Namely, let \(K\) be a causally complete open and convex region in Minkowski space-time, – a convenient choice is a double-cone, i.e., the intersections of a future-directed and a past-directed light-cone. It uniquely determines a wedge-shaped region \(W\) of AdS which consists of all points at which one can receive signals from some point of \(K\), and from which one can send signals to some other point of \(K\) (the “causal completion” of \(K\) in AdS). Conversely, the boundary region \(K\) is recovered from this AdS region \(W\) by taking its intersection with the boundary of AdS.

We omit proofs of the geometric facts mentioned here and in the sequel; the reader may find details in \[14\]. It is largely sufficient to visualize \(\text{AdS}_{1,d}\) in suitable coordinates as a full cylinder \(\mathbb{R} \times B^d\) whose axis \(\mathbb{R}\) represents time (possibly periodic, see below), and whose boundary \(\mathbb{R} \times S^{d-1}\) represents spacelike infinity. \((B^d\) is a ball, and \(S^{d-1}\) is a sphere.) Double-cones \(K\) are inscribed into the boundary, and the wedges \(W\) look like actual wedges “chopped into the cylinder” (cf. Fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{adscft.png}
\caption{Wedge regions in AdS and corresponding double-cones in the boundary (in Penrose coordinates)}
\end{figure}

Let us denote the bijective correspondence between AdS wedges and boundary (Minkowski) double-cones by \(K = \iota(W) \leftrightarrow W = \iota^{-1}(K)\). Then the specification

\[ A(K) := B(W) \quad \text{if} \quad K = \iota(W), \]

determines a system of local algebras \(A(K)\) of observables on Minkowski space-time, given the system of local algebras \(B(W)\) on AdS.

This identification preserves Covariance. For if a wedge \(W\) is transformed under the AdS group, then its intersection \(K\) with the boundary undergoes a conformal transformation, and vice versa. More concretely, if \(g\) stands for an element of the AdS group, and \(\hat{g}\) for its induced
conformal transformation of the boundary, then
\[ \iota(gW) = gK \quad \text{if} \quad K = \iota(W), \]

implying the correct transformation of the observables:
\[ U(g)A(K)U(g)^{-1} = U(g)B(W)U(g)^{-1} = B(gW) = A(\iota(gW)) = A(\iota(gK)). \]

In particular, the conformal symmetry is implemented by the same unitary Hilbert space representation \( U \) of \( \text{SO}(2, d) \) as the AdS symmetry.

The identification also preserves Isotony: One has
\[ \iota(W_1) \subset \iota(W_2) \quad \text{if} \quad W_1 \subset W_2 \]

for obvious reasons. Since the given algebras \( B(W) \) satisfy Isotony, this implies
\[ A(K_1) \subset A(K_2) \quad \text{if} \quad K_1 \subset K_2. \]

Finally, also Locality is preserved: Namely, \( \iota \) maps pairs of causally complementary AdS regions onto pairs of causally complementary boundary regions (the causal complement \( X' \) of a region \( X \) consists of all points which are spacelike separated from any point in \( X \)):
\[ \iota(W') = K' \quad \text{if} \quad \iota(W) = K. \]

If now \( K = \iota(W) \) and \( K \) are spacelike separated, then \( K' \) is a subset of \( K' = \iota(W') \). Observables localized in \( K \) and \( K' \) are thus identified with operators in \( B(W) \) and \( B(W') \), and hence commute as required by Locality.

Since \( \iota \) is a bijection, the prescription can be reversed, specifying the system of algebras \( B(W) \) by the given local algebras \( A(K) \):
\[ B(W) := A(K) \quad \text{if} \quad W = \iota^{-1}(K). \]

By the same arguments as before, Covariance, Isotony and Locality hold for \( B(W) \) if they hold for \( A(K) \).

We emphasize that we have reduced the problem of establishing the AdS-CFT correspondence to a completely geometric one. It is not necessary to proceed from a specific quantum field theory in order to understand why it admits a holographic re-interpretation.

We now turn to discuss some physical implications of the correspondence thus established.

### A. Change of the physical interpretation

Although the pair of corresponding theories shares the same set of local observables as operators on the same Hilbert space, they have different physical interpretations. This possibility is familiar from quantum mechanics where the state space is always a separable Hilbert space and the set of all observables are the functions of position and momentum.

The physical interpretation arises from the assignment of observables to localization regions, and the consequent correlations in the expectation values of observables in various geometric arrangements. A re-assignment completely changes the interpretation. For instance, the description of a scattering experiment would require the determination of correlations between observables at asymptotically large distances. As notions like “spacelike infinity” are not preserved by the bijection \( \iota \), the computation of asymptotic correlations yields entirely different results in corresponding theories.

Furthermore, the one-parameter subgroups of \( \text{SO}(2, d) \) describing time translations in the AdS and conformal interpretations do not coincide. Therefore, also notions like dynamics, energy and entropy change their meaning under the AdS-CFT correspondence.

### B. Pointlike AdS and extended CFT observables

Not even the concept of a point is preserved by the correspondence (which should not be a surprise since corresponding theories live in space-times of different dimension). For instance, arbitrarily small double-cones in the boundary correspond to wedges close to infinity in AdS, which always have infinite volume. That an observable can be written as a field \( \phi(x) \) smeared with a test function, or as some function of field operators, may be true in AdS, but not in the corresponding CFT, or vice versa. Thus, a description in terms of fields may fail in one of the two theories. This is an instance where the advantage of thinking in terms of extended observables and the description of their time evolution by an automorphism group in contrast to fields and equations of motion, is clearly exhibited.

We want to demonstrate that the identification of localized observables implies that AdS observables localized in finite AdS regions (in particular proper AdS fields) correspond to genuinely extended CFT observables. In the argument we assume the dimension of AdS to be \( d + 1 > 1 + 1 \) (the case \( d = 1 \) is very special and has been discussed elsewhere).

Let \( X \) be a bounded region in AdS. Pick some wedge \( W \) which contains \( X \) and consider the family of wedges \( W_i \) contained in \( W \) which are spacelike to \( X \). One finds that the corresponding boundary double-cones \( K_i = \iota(W_i) \) are contained in \( K = \iota(W) \) and cover its \( t = 0 \) surface.

Let \( B(X) \) denote the algebra of AdS observables localized in \( X \). It belongs to \( B(W) \) and commutes with all observables in \( B(W_i) \), hence as a CFT observable it belongs to \( A(K) \) and commutes with all observables in \( A(K_i) \). It commutes in particular with all boundary fields smeared over a neighborhood of a Cauchy surface of \( K \).

Now, if the algebra \( A(K) \) were generated by the family of subalgebras \( A(K_i) \), we would find that \( B(X) \) belongs to \( A(K) \) and commutes with every operator in \( A(K) \), and
therefore is a commutative algebra. Its elements can be classical observables only. This conclusion applies, e.g., if the CFT is completely described by fields with a causal dynamical law, since the fields along the Cauchy surface generate all observables localized in $K$.

Reversing the argument, we conclude that the quantum observables in $B(X)$ (e.g., AdS field operators smeared within $X$) correspond to CFT observables in $A(K)$ which are not generated by the family of subalgebras $A(K)$ covering the $t = 0$ surface of the double-cone $K$. They are thus genuinely extended CFT observables. In particular, the CFT cannot be completely described by its fields with a dynamical law.

The extended observables of the CFT, whose presence is implied by the above argument, might be Wilson loop operators in nonabelian gauge theories. While observable fields fail to generate all quantum observables of the CFT, gauge invariant nonlocal “functions” of gauge fields could account for the rest.

On the other hand, CFT fields correspond to AdS observables attached to infinity, which might just be suitably renormalized limits of AdS fields. More enthralling is the possibility to identify some AdS degrees of freedom, which collectively restore the crossing symmetry of conformal operator product expansions obtained by an AdS prescription, as strings – thus making contact with the original conjectures.

C. Global structure of space-time

The bijection $\iota$ between double-cones and wedges (Sect. III) pertains to proper conformal Minkowski space-time and projective AdS space-time which is the AdS hyperboloid with antipodal points $\xi$ and $-\xi$ identified: $\text{PAdS}_{1,d} = \text{AdS}_{1,d}/(\xi \sim -\xi)$. One may still formulate the AdS theory on $\text{AdS}_{1,d}$, but then one finds $B(W) = B(-W)$; antipodal wedges have the same observables.

A CFT on proper conformal Minkowski space-time cannot describe any interaction since its observables commute also at timelike separation, hence any causal influence is bound to lightlike geodesic propagation. This implies that observables in the corresponding theory on projective AdS commute unless their localizations are connected by a causal geodesic (see also [8] for an independent argument to the same effect). But if causal influence only propagates along geodesics, then no process like the decay of a particle (with non-geodesic trajectory due to recoil) is possible. Hence, the AdS theory will also not describe a system with interaction.

Theories of physical interest thus rather “live” on covering spaces of projective AdS and of conformal Minkowski space-time, respectively, where the closed timelike curves of these manifolds are unwound. The bijection $\iota$ generalizes to corresponding covering spaces, and especially the universal coverings of both spaces (disregarding the case $d = 1$ which is again peculiar).

This allows for possible anomalous dimensions of conformal fields and nontrivial timelike commutation relations, and evades the above conclusion of geodesic propagation and absence of interaction on AdS.

IV. CONCLUSION

We have obtained the proper identification of local quantum observables which underlies the “holographic” correspondence between quantum field theory on $d + 1$-dimensional anti-deSitter space-time and $d$-dimensional conformal quantum field theory. It simply reflects the geometric law of causal propagation between AdS space-time and its boundary. But it suffices to define one theory in terms of the other, and entails a complete reinterpretation of the physical content.

We conclude from this result, among other things, that AdS fields correspond to genuinely extended CFT observables. These can be a hint at conformal gauge theories.

Strings play no particular role in the present explanation of the AdS-CFT correspondence. But it is conceivable that they re-appear as “collective” AdS variables required by crossing symmetry of the corresponding CFT.

\[\begin{align*}
1. & \quad J. \text{Maldacena, Adv. Theor. Math. Phys. 2} (1998) 231. \\
2. & \quad E. \text{Witten, Adv. Theor. Math. Phys. 2} (1998) 253. \\
3. & \quad S. S. \text{Gubser, I.R. \text{Klebanov}, A.M. Polyakov, Phys. Lett. B428} (1998) 105. \\
4. & \quad G. \text{‘t Hooft, gr-qc/9310026.} \\
5. & \quad \text{A. Salam Festschrift, World Scientific, 1993, p. 284.} \\
6. & \quad H.-J. \text{Borchers, Nuovo Cimento 15} (1960) 784. \\
7. & \quad S. \text{Coleman, Phys. Rev. D11} (1975) 2088. \\
8. & \quad M. \text{L"{u}scher, Nucl. Phys. B326} (1989) 557. \\
9. & \quad C. \text{Montonen, D. \text{Olive, Phys. Lett. B72} (1977) 117.} \\
10. & \quad \text{N. Seiberg, E. \text{Witten, Nucl. Phys. B426} (1994) 19.} \\
11. & \quad \text{R. Haag, D. \text{Kastler, J. Math. Phys. 5} (1964) 848.} \\
12. & \quad \text{D. Buchholz, R. \text{Haag, hep-th/9910243, to appear in J. Math. Phys.}} \\
13. & \quad \text{J. \text{Dimock: J. Math. Phys. 41} (2000) 40.} \\
14. & \quad \text{K.-H. \text{Rehren, hep-th/9905179, to appear in Ann. H. Poinc.}} \\
15. & \quad \text{K.-H. \text{Rehren, hep-th/9910074, to appear in: “Quantum Theory and Symmetries”, Goslar 1999 proceedings.}} \\
16. & \quad \text{M. \text{Bertola, J. Bros, U. Moschella, R. \text{Schaefler, hep-th/9908140.}}} \\
17. & \quad \text{L. \text{Hofmann, A.C. Petkou, W. \text{R"{u}hl, hep-th/0002154.}}} \\
18. & \quad \text{D. \text{Buchholz, M. \text{Florig and S.J. \text{Summers, Class. Qu. Grav. 17} (2000) L31.}}} 
\end{align*}\]