DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF
SO(p + 1, p)

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Abstract. For p odd, the Lie group \(G^\circ = \text{SO}_0(p + 1, p + 1)\) has a family of unitary degenerate principal series representations realized on the space of real \((p + 1) \times (p + 1)\) skew symmetric matrices, similar to the Stein’s complementary series for \(\text{SL}(2n, \mathbb{C})\) or Speh’s representation for \(\text{SL}(2n, \mathbb{R})\). We consider their restriction on the subgroup \(G_0 = \text{SO}_0(p + 1, p)\) and prove that they are still irreducible and is equivalent to (a unitarization of) the principal series representation of \(G\), and also irreducible under a maximal parabolic subgroup of \(G\).

1. Introduction

In the present paper we shall study the unitarity of degenerate principal series representations of the group \(G = \text{SO}(p + 1, p)\) induced from certain maximal parabolic subgroup for odd \(p = 2q - 1\).

In the case of the group \(\text{SO}_0(n, n)\), or more generally \(\text{SU}(n, n; F)\), for \(F = \mathbb{R}, \mathbb{C}, \mathbb{H}\), Johnson [15] has determined the range of unitarity of the representations; for \(\text{SU}(n, n, F)\), \(n\) even, he found certain complementary series. Some generalizations of these results were obtained in [20, 19, 28, 12] for a larger class of groups by using computations in the compact picture. The analysis of these representations in the non-compact picture has been done in [2].

We shall prove that the restriction of the complementary series of \(G^\circ = \text{SO}_0(p + 1, p + 1)\) to the opposite maximal parabolic subgroup of \(G\) is irreducible, and in particular the restriction to the identity component \(G_0\) of \(G\) is irreducible. We shall use mostly the non-compact realization of the principal series. The proof relies on both Euclidean and nilpotent Fourier transform.

The restriction of the degenerate principal series representations of \(\text{SO}(n, n)\) to the subgroups \(\text{SO}(n, m) \times \text{SO}(n - m)\) for \(m < n\) has been studied earlier by Lee-Loke [17] in the compact picture, the representations of \(\text{SO}(n, m) \times \text{SO}(n - m)\) appearing are of the form \(\tau \times \tau'\), and the representations \(\tau\) are degenerate principal series. It might be true that the representations \(\tau\) of \(\text{SO}(n, m)\) are also irreducible under corresponding the maximal parabolic subgroup, as we show here for \(m = n - 1\). We mention also that there has been quite some study of complementary series representations for semisimple Lie groups. In [1] a large class of complementary series representations is constructed with parabolic subgroups being cuspidal and maximal; our case of \(\text{SO}(p + 1, p)\) here is however not cuspidal. For the groups

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SO\((p, q)\) some complementary series similar to ours can be constructed by using the
branching of holomorphic representations of SU\((p, q)\) to SO\((p, q)\). However those
constructions works only for \(q - p > 2\) (see [13] Theorem 3.1, [24] Theorem 5.2),
and thus they do not cover our present case of SO\((p + 1, p)\). Our result about the
irreducibility of the restriction to SO\((p+1, p)\) of representations of SO\(0(p + 1, p + 1)\) is
in a way similar to the Kirillov conjecture, now a theorem [3, 21] for GL\(_{p+1}(\mathbb{R})\) and
GL\(_{p}(\mathbb{R}) \times \mathbb{R}^p\). See also [22] on the study of the restriction of complementary series
of SO\((n, 1)\) to SO\((n - 1, 1)\), [13] on branching of highest weight representations, and
[14] the classification of finitely decomposable representations of G\(^{\sharp}\) under G. We
note also that the irreducibility result under the the parabolic group P can possibly
be also proved abstractly by using the Mackey theory on induced representations.
However we present a relatively elementary and direct proof, in particular it also
yields a decomposition of the representation under the subgroup NSp\((q - 1, \mathbb{R})\).

2. Preliminaries

2.1. The group G\(^{\sharp}\) = SO\(_0(p + 1, p + 1)\) and G = SO\((p + 1, p)\).

Let \(M_{p,q}\) be the space of real \(p \times q\)-matrices, and denote \(M_{p} = M_{p,p}\). Denote
\(X = X_{p} = \{X \in M_{p}, X = -X^t\}\) the subspace of skew-symmetric real matrices. We
will also use the short-hand notation \(X^{-t} = (X^t)^{-1}\), for an invertible \(X\). Denote
\(I_{n}\) the identity matrix in \(M_{n}\) and \(I_{p,q} = \text{diag}(-I_{p}, I_{q})\).

Let \(p > 1\) and G\(^{\sharp}\) = SO\(_0(p + 1, p + 1)\) be the identity component of SO\((p+1, p + 1)\) = \(\{g \in M_{2p+2}; \text{det} g = 1, \ g^t I_{p+1,p+1} g = I_{p+1,p+1}\}\), and G = SO\((p + 1, p)\) realized
as the subgroup of G\(^{\sharp}\) via:

\[
G = \{\text{diag}(g, 1) \in G^{\sharp}\} \subset G^{\sharp}.
\]

The group G has two connected components and we denote its identity component
by \(G_0\).

Elements \(g\) of G\(^{\sharp}\) will be written as \(2 \times 2\) block matrices

\[
g = \begin{pmatrix}
g_{1,1} & g_{1,2} \\
g_{2,1} & g_{2,2}
\end{pmatrix},
\]

with each entry being in \(M_{p+1}\). The Lie algebra \(g^{\sharp}\) of G\(^{\sharp}\) has the decomposition
\(g^{\sharp} = \mathfrak{t}^{\sharp} \oplus \mathfrak{p}^{\sharp}\) with \(\mathfrak{t}^{\sharp} = \mathfrak{so}(p + 1) \oplus \mathfrak{so}(p + 1)\) with respect to the Cartan involution
\(g \rightarrow g^{-t}\). The group

\[
K^{\sharp} = \{\text{diag}(k_1, k_2); \ k_1, k_2 \in \text{SO}(p + 1)\} = \text{SO}(p + 1) \times \text{SO}(p + 1),
\]

is a maximal compact subgroup of G\(^{\sharp}\) with Lie algebra \(\mathfrak{k}^{\sharp}\). Correspondingly \(g = \mathfrak{t} \oplus \mathfrak{p}, \ \mathfrak{t} = \mathfrak{so}(p + 1) \oplus \mathfrak{so}(p)\) and

\[
K_0 = \{\text{diag}(k_1, k_2, 1); \ k_1 \in \text{SO}(p + 1), k_2 \in \text{SO}(p)\} \sim \text{SO}(p + 1) \times \text{SO}(p)
\]

is a maximal compact subgroup of \(G_0\), while

\[
K = \{\text{diag}(k_1, k_2, \text{det} k_2); \ k_1 \in \text{SO}(p + 1), k_2 \in \text{O}(p)\} \sim \text{SO}(p + 1) \times \text{O}(p)
\]

is a maximal compact subgroup of G. Note that \(K = \{I_{p+1,p+1}, I_{2p+2}\} \times K_0 \sim \mathbb{Z}_2 \times P\).

For \(j = 1, \ldots, p + 1\), let

\[
H_j = \begin{pmatrix}
0 & X \\
X^t & 0
\end{pmatrix} \in \mathfrak{p}^{\sharp}\ 	ext{ where } \ X = \text{diag}(0, \cdots, 0, 1, 0, \cdots, 0),
\]
with 1 on the $j$th position. Then $\mathfrak{t}^\sharp := \mathbb{R} H_1 + \cdots + \mathbb{R} H_{p+1}$ and $\mathfrak{t} := \mathbb{R} H_1 + \cdots + \mathbb{R} H_p$ are maximal abelian subspaces of $\mathfrak{p}^\sharp$ and $\mathfrak{p}$. Let $\{ \xi_j \}$ be the dual basis of $\{ H_j \}$. The positive root systems of $(\mathfrak{g}^\sharp, \mathfrak{t}^\sharp)$ and $(\mathfrak{g}, \mathfrak{t})$ are \( \{ \xi_j \pm \xi_k, 1 \leq j < k \leq p+1 \} \) and $\{ \xi_j \pm \xi_k, 1 \leq j < k \leq p \}$.

2.2. The maximal parabolic subgroups $P^\sharp$ and $P$.

We fix the elements:

$$\xi^\sharp = H_1 + \cdots + H_{p+1} \in \mathfrak{p}^\sharp \quad \text{and} \quad \xi = H_1 + \cdots + H_p \in \mathfrak{p}.$$ 

Let $\mathfrak{a}^\sharp = \mathbb{R} \xi^\sharp$, $\mathfrak{a} = \mathbb{R} \xi$. The root space decomposition of $\mathfrak{g}^\sharp$ under $\xi^\sharp$ and $\mathfrak{g}$ under $\xi$ is then

$$\mathfrak{g}^\sharp = \mathfrak{n}^\sharp_2 + \mathfrak{m}^\sharp + \mathfrak{a}^\sharp + \mathfrak{n}^\sharp_2,$$

and for the negative root spaces:

$$\mathfrak{n}^\sharp = \mathfrak{n}^\sharp_2 \quad \text{and} \quad \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{n}^\sharp.$$

The Lie algebra $\mathfrak{n}^\sharp$ is abelian and $\mathfrak{n}$ is a 2-step nilpotent Lie sub-algebra of $\mathfrak{g}$. Elements $n_{(z,v)}$ will simply be written as $(z,v)$. The Lie bracket in $\mathfrak{n}$ is given, via the above identification $\mathfrak{n} = X_p \oplus \mathbb{R}^p$, is

$$[z_1 + v_1, z_2 + v_2] = v_1 v_2^\prime - v_2 v_1^\prime.$$

Thus $\mathfrak{n}$ is the free nilpotent Lie algebra with $p$ generators (over $\mathbb{R}$).

The centralizer of $\mathfrak{a}^\sharp$ in $\mathfrak{g}^\sharp$ is

$$\mathfrak{m}^\sharp \oplus \mathfrak{a}^\sharp = \{ l^\sharp_{(X,Y)}, X = X^\sharp, Y^\sharp = -Y \in M_{p+1} \}, \quad l^\sharp_{(X,Y)} := \begin{pmatrix} Y & X \\ X & -Y \end{pmatrix},$$

identified with $\mathfrak{gl}(p+1)$ via $l^\sharp_{(X,Y)} \mapsto X + Y \in \mathfrak{gl}(p+1)$, whereas the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is

$$\mathfrak{m} \oplus \mathfrak{a} = \{ \text{diag}(l_{(x,y)},0); \ x = x^\sharp, y^\sharp = -y \in M_p \}, \quad l_{(x,y)} := \begin{pmatrix} y & 0 & x \\ 0 & 0 & 0 \\ x & 0 & y \end{pmatrix},$$

identified with $\mathfrak{gl}(p)$. Note that

$$\mathfrak{gl}(p) \sim \mathfrak{m} \oplus \mathfrak{a} \subset \mathfrak{m}^\sharp \oplus \mathfrak{a}^\sharp \sim \mathfrak{gl}(p+1).$$
Let $M^t$, $A^t$, $N^t$ and $\tilde{N}^t$ be the simply connected subgroup of $G^t$ with Lie algebras $m^t$, $a^t$, $n^t$ and $\tilde{n}^t$ respectively. Let $P^t = M^t A^t N^t$ and $\tilde{P}^t = M^t A^t \tilde{N}^t$ be the corresponding maximal parabolic subgroups of $G^t$. Similarly we define the connected subgroups $P_0 = M_0 A N$ and $\tilde{P}_0 = M_0 A \tilde{N}$ of $G_0$ with Lie algebra $m + a + n$ and $m + a + \tilde{n}$. Note that the centralizer of $a^t$ in $K^t$ is
\[ Z_{K^t}(a^t) = K^t \cap M^t = \{ \text{diag}(k_1, k_1); \ k_1 \in SO(p+1) \} \sim SO(p+1) , \]
and the centralizer of $a$ in $K$ is
\[ Z_K(a) = K \cap M = \{ \text{diag}(k_2, \det k_2, \det k_2); \ k_2 \in O(p) \} \sim O(p) , \]
and the centralizer of $a$ in $K_0$ is the connected component of the identity of $Z_K(a)$. We set:
\[ M = Z_K(a)M_0 \quad \text{and} \quad P = Z_K(a)P_0 = MAN \quad \text{and} \quad \tilde{P} = Z_K(a)\tilde{P}_0 = MA\tilde{N} . \]
The group $M^t A^t$ is isomorphic to the matrix group $GL_{p+1}^+ = \{ h \in M_{p+1}, \det h > 0 \}$ via:
\begin{equation}
(2.2) \quad h \in GL_{p+1}^+ \mapsto k_0 \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} k_0 \in M^t A^t \quad \text{and} \quad k_0 = 2^{-\frac{t}{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .
\end{equation}
Restricting this isomorphism to $\{ \text{diag}(h, \det h), \ h \in GL_p \}$ and to $\{ \text{diag}(h, \det h), \ h \in GL_{p+1}^+ \}$, we obtain an isomorphism between $MA$ and $GL_p$ and between $M_0 A$ and $GL_p^+$. Thus, using the isomorphisms just above as identifications, $P^t$, $P$ and $P_0$ can be described as the semi-direct product:
\[ P^t = GL_{p+1}^+ N^t \quad \text{and} \quad P = GL_p N \quad \text{and} \quad P_0 = GL_p^+ N . \]

**Lemma 2.1.** The following inclusions of Lie algebras hold:
\[ m \oplus a \subset m^t \oplus a^t \quad \text{and} \quad m \oplus a \oplus n \subset m^t \oplus a^t \oplus n^t . \]
We have:
\[ M_0 A \subset MA \subset M^t A^t \quad \text{and} \quad P_0 \subset P \subset P^t . \] Moreover we have a factorization of $\exp(n_{(z,v)}) \in N$ in $GL_{p+1}^+ = M^t A^t$,
\begin{equation}
(2.3) \quad \exp n_{(z,v)} = m \exp n_{M(z,v)}^t ,
\end{equation}
where $m \in M^t$ corresponds to $\begin{pmatrix} I_p & v \\ 0 & 1 \end{pmatrix} \in SL_{p+1}$ and
\begin{equation}
(2.4) \quad M(z,v) := \begin{pmatrix} z & \frac{1}{2}v \\ -\frac{1}{2}v^t & 0 \end{pmatrix} \in \mathfrak{X}_{2d} ,
\end{equation}
is viewed as an element of $n^t$.

**Proof.** The first relation is in (2.1). We can write $n_{(z,v)} = t_{(X,Y)}^t + n_{W}^t$ with:
\[ X = \begin{pmatrix} 0 & \frac{1}{2}v \\ \frac{1}{2}v^t & 0 \end{pmatrix} , \quad Y = \begin{pmatrix} 0 & \frac{1}{2}v \\ -\frac{1}{2}v^t & 0 \end{pmatrix} , \quad W = \begin{pmatrix} z & \frac{1}{2}v \\ -\frac{1}{2}v^t & 0 \end{pmatrix} . \]
This shows $n \subset m^t \oplus n^t$ and implies $m \oplus a \oplus n \subset m^t \oplus a^t \oplus n^t$. The group inclusions follow immediately. Easy matrix computations give the equality (2.3). \qed

Note that the adjoint action of $MA = GL_p^+$ and $GL_p$ on $N$ is
\begin{equation}
(2.5) \quad g \cdot (z,v) = (gzg^t, gv) .
\end{equation}
3. Degenerate principal series representations

Throughout the paper we assume $p$ is odd and we write $2q = p + 1$.

3.1. Principal series of $G^\sharp = \text{SO}_0(p + 1, p + 1)$.

Let $\mu \in \mathbb{C}$ and consider the induced representation $I^\sharp(\mu)$ of $G^\sharp$ from the following character on $P^\sharp$
\[
\chi^\sharp_\mu : me^{t \xi^\sharp} n \mapsto e^{(\mu + \rho^\sharp)t},
\]
where $\rho^\sharp = q(2q - 1)$ is the half sum of the positive root of $\text{ad}\xi^\sharp$. In terms of $P^\sharp = \text{GL}_2^+ \mathbb{R}^N$ this is
\[
(3.1) \quad \chi^\sharp_\mu : l n \mapsto \det(l)^{\frac{\mu + \rho^\sharp}{2q}}, \quad l \in \text{GL}_2^+.
\]
This representation can be realized on the space of Haar measurable functions $f(g)$ on $G^\sharp$ such that
\[
(3.2) \quad f(gln) = \det(l)^{\frac{\mu + \rho^\sharp}{2q}} f(g), \quad l n \in P^\sharp,
\]
and
\[
(3.3) \quad f|_{K^\sharp} \in L^2(K^\sharp).
\]
See [16]. The group $G^\sharp$ acts on $I^\sharp(\mu)$ by the left regular action and we denote the representation by $(I^\sharp(\mu), \pi^\sharp_\mu)$. The condition (3.2) implies that $f \in I^\sharp(\mu)$ is invariant under the right action of $K^\sharp \cap M^\sharp$ and can therefore be identified as functions on $K^\sharp/K^\sharp \cap M^\sharp$. However $K^\sharp/K^\sharp \cap M^\sharp$ can be realized as $\text{SO}(2q)$ since the group $K^\sharp$ acts on $\text{SO}(2q)$ by
\[
K^\sharp \ni \text{diag}(k_1, k_2) : \left\{ \begin{array}{c}
\text{SO}(2q) \\
\text{SO}(2q)
\end{array} \rightarrow \text{diag}(k_1ak_2^{-1}) \right.,
\]
and the isotropy group of the identity matrix $I_{2q} \in \text{SO}(2q)$ is $K^\sharp \cap M^\sharp$. Thus condition (3.3) can be equivalently replaced by
\[
(3.4) \quad f|_{K^\sharp/K^\sharp \cap M^\sharp} \in L^2(K^\sharp/K^\sharp \cap M^\sharp) = L^2(\text{SO}(2q)).
\]

We denote by $(I^\sharp_{K^\sharp}(\mu), \pi^\sharp_{K^\sharp})$ the space of $K^\sharp$-finite elements. We will need its decomposition under $K^\sharp$. Recall, see e.g. [15], that each irreducible representation of $\text{SO}(2q)$ is determined by a $q$-tuple of integers:
\[
\mathbf{m} = (m_1, \cdots, m_q), \quad m_1 \geq \cdots \geq m_{q-1} \geq |m_q|.
\]

We write $\mathcal{V}_m$ for the representation space of $\mathbf{m}$. Thus $I^\sharp_{K^\sharp}(\mu)$ is the same as the space $L^2(\text{SO}(2q))_{K^\sharp}$ of $K^\sharp$-finite elements in $L^2(\text{SO}(2q))$ and we have [15]:
\[
(3.4) \quad I^\sharp_{K^\sharp}(\mu) \sim L^2(\text{SO}(2q))_{K^\sharp} = \sum_{\mathbf{m}} \mathcal{V}_m \otimes \mathcal{V}^*_m,
\]
where $\mathcal{V}^*_m$ stands for the dual representation of $\mathcal{V}_m$.

Each function in the space $I^\sharp(\mu)$ is also uniquely determined by its restriction to $\bar{N}^\sharp$. The $G^\sharp$-action in this realization is referred as $\bar{N}^\sharp$-realization. It is [10] p. 169 the space $L^2(\bar{N}^\sharp, e^{2\pi(\mu + \rho^\sharp)H^\sharp})$ where the function $H^\sharp$ is defined by $t = H^\sharp(\tilde{n})$ using the Iwasawa decomposition of $\tilde{n} = kme^{t\xi^\sharp}n_+ \in K^\sharp P^\sharp$. Note however that the $L^2$-norm in $L^2(K)$ or $L^2(\bar{N}^\sharp, e^{2\pi(\mu + \rho^\sharp)H^\sharp})$ is $G^\sharp$-invariant only for purely imaginary $\mu$, $\mu \in i\mathbb{R}$.
3.2. Zeta distribution and complementary series $C^\sharp(\nu)$ of $G^\sharp$.

The unitarity of $(I^\sharp(\mu), \pi_\mu)$ for $\mu$ outside the standard unitary range $\mu \in \mathbb{R}$ has been completely determined in [14] in the $K$-finite realization; see also [20, 28]. Let $\mu \in (0, q)$. There exists a $g^\sharp$-invariant (i.e. $g^\sharp$ acts as skew Hermitian operators) positive definite inner product $(\cdot, \cdot)_\mu$ on the space $I^\sharp_{K^\flat}(\mu)$ of $K^\flat$-finite elements of the principal series of $I^\sharp(\mu)$ (see [15, Theorem 7.5] and [2, section 7]). We will denote the unitary representation of $G^\sharp$ on the completion of $I^\sharp_{K^\flat}(\mu)$ as $(C^\sharp(\mu), \pi^\sharp_\mu)$.

The non-compact picture has been further studied in [2] and we recall it briefly here. We identify $N^\sharp$ with $X_{2q}$ via $Z \mapsto \exp \tilde{n}_Z^\sharp$ and we consider the following (formally defined) linear form on the Schwarz space $S(X_{2q})$

$$Z_s(h) = \gamma_{s, 2q} \int_{X_{2q}} h(x) |\text{Pf}(x)|^s dx, \quad \gamma_{s, 2q} = \frac{\pi^{\frac{2}{2} + 2q - 1}}{\Pi_{j=0}^{q-1} \Gamma(\frac{2q - 1 - j}{2})},$$

for $s \in \mathbb{C}$ with sufficiently large $\Re s$ where $\text{Pf}$ denotes the Pfaffian polynomial. This defines $\{Z_s\}$ a family of tempered distributions which admits a holomorphic continuation to the whole complex plane and whose Fourier transform satisfies the following functional equality:

$$Z_{2q, s - (2q - 1)}(\mathcal{F} h) = Z_{2q, -s}(h), \quad s \in \mathbb{C};$$

here the Fourier transform on $X_{2q}$ is given by:

$$\mathcal{F} h(\zeta) = \int_{M_{2q}^s} h(x) e^{2\pi i(x, \zeta)} dx, \quad h \in S(X_{2q}).$$

The inner product in the $N^\sharp$-realization is given by:

$$(f, g)_\mu := (f, Z_s g)_{L^2(X)} = Z_s(f \ast \tilde{g}) \quad \text{with} \quad s = \frac{\mu}{q} - (2q - 1),$$

where $\tilde{g} : x \mapsto \tilde{f}(-x)$, initially defined on the Schwarz space $S(X_{2q})$. From (3.5), we see that $(f, f)_\mu = Z_{-\frac{\mu}{q}}(|\mathcal{F} f|^2)$ for any $f \in S(X_{2q})$; this makes sense because for $\mu \in (0, q)$, $Z_{-\frac{\mu}{q}}$ is a locally integrable function. The completion of $S(X_{2q})$ is

$$C^\sharp(\mu) = \{ f \in S'(X_{2q}); \ |\text{Pf}|^{-\frac{\mu}{q}} \mathcal{F} f \in L^2(X_{2q}) \},$$

(3.6)

(Note that the condition $f \in S'(X_{2q})$ can be deduced also from $|\text{Pf}|^{-\frac{\mu}{q}} \mathcal{F} f \in L^2(X_{2q})$).

We summarize the results in the following

**Proposition 3.1.** Suppose $\mu \in (0, q)$. There exists a $(g^\sharp, \pi^\sharp)$-invariant positive definite inner product $(\cdot, \cdot)_\mu$ on $I^\sharp_{K^\flat}(\mu)$. Its completion defines a unitary representation of $G^\sharp$. In the non-compact realization the Hilbert space is described by (3.6).

The operator

$$(\mathcal{F}_\mu : f \in S(X_{2q}) \mapsto \phi = \mathcal{F}^{-1}(|\text{Pf}|^{-\frac{\mu}{q}} \mathcal{F} f) \in L^2(X_{2q}),$$

then extends to a unitary operator from $C^\sharp(\mu)$ onto $L^2(X_{2q})$. We denote the corresponding representation on $L^2(X_{2q})$ by

$$\tilde{\pi}^\sharp_\mu = \mathcal{F}_\mu \pi^\sharp_\mu \mathcal{F}^{-1};$$

(3.8)

We obtain a simple description of the action of $(\tilde{P}^\sharp, \tilde{\pi}^\sharp_\mu)$ (see [2 sections 7 and 8]):

$$\tilde{\pi}^\sharp_\mu(\exp \tilde{n}_Z^\sharp)\phi(W) = \phi(W - Z)$$

(3.9)
and for \( g = me^{i\xi} \in M^2 A^2 \) identified with an element of \( GL_{p+1}^+ \):
\[
(3.10) \quad \tilde{\pi}_\mu^* \phi(W) = e^{-\rho \xi} \phi(g^t W g).
\]

### 3.3. Principal series \( I(\nu) \) of \( G \)

Let \( I(\nu) \) be the principal series induced from the following character of \( P = GL_p N \):
\[
\chi_\nu : l_n \mapsto |\det(l)|^{\frac{\nu + \rho}{\rho}} ,
\]
where \( \rho = \frac{p^2}{2} \) is the half sum of the positive roots of \( \text{ad} \xi \), with the similar condition as in (3.2) and (3.3). That is, the representation space is realized as Haar measurable functions \( f(g) \) on \( G \) satisfying
\[
(3.11) \quad f(g ln) = |\det(l)|^{-\frac{\nu + \rho}{\rho}} f(g),
\]
and
\[
(3.12) \quad f|_K \in L^2(K) \quad \text{or equivalently} \quad f|_K \in L^2(K/K \cap M);
\]
the group \( G \) acts on \( I(\nu) \) by the left regular action and we denote this action by \((I(\nu), \pi_\nu)\). We denote by \((I_K(\nu), \pi_\nu)\) the space of \( K \)-finite elements.

The homogeneous space \( K/K \cap M \) can be realized as the Stiefel manifold of rank \( p \) isometries:
\[
K/K \cap M = V_{p+1,p} = \{ x \in M_{p+1,p}; x^t x = I_p \},
\]
where the group \( K \) acts transitively via:
\[
K \ni \text{diag}(k_1, k_2) : \begin{cases}
V_{p+1,p} &\mapsto V_{p+1,p} \\
x &\mapsto k_1 x k_2^{-1}
\end{cases},
\]
and \( K \cap M \) is the isotropy group of \( \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \in V_{p+1,p} \). The elements \( f \) in \( I(\nu) \) then satisfy
\[
f|_{K/K \cap M} \in L^2(K/K \cap M) = L^2(V_{p+1,p}).
\]

We will need the multiplicity free decomposition of \( L^2(V_{p+1,p}) \) under \( K_0 = \text{SO}(p+1) \times \text{SO}(p) \). Let us recall that each irreducible representation of \( \text{SO}(p) \) is determined by a \((q-1)\)-tuples of integers
\[
\underline{n} = (n_1, \cdots, n_{q-1}), \quad n_1 \geq \cdots \geq n_{q-1} \geq 0,
\]
and we write \( W_{\underline{n}} \) for the representation space. Given a representation \( \underline{m} \) of \( \text{SO}(p+1) \) we write \( \underline{m} \succeq \underline{n} \) if \( \underline{n} \) appears in the irreducible decomposition of \( \underline{m} \) under \( \text{SO}(p) \). It is a classical result, see e.g. \[10, 26, 13\] that \( \underline{n} \) appears in \( \underline{m} \) multiplicity free. This implies that the space of \( K_0 \)-finite elements of \( L^2(V_{p+1,p}) \) is decomposed under \( K_0 \) as follows:
\[
(3.13) \quad I_K(\nu) \sim L^2(V_{p+1,p})_{K_0} = \sum_{\underline{m} \succeq \underline{n}} V_{\underline{m}} \otimes W_{\underline{n}},
\]
and this decomposition is multiplicity free.
3.4. The restriction map $R$.

We shall consider simply the restriction of functions in $I^t(\mu)$ to $G \subset G^t$. To clarify its definition we note first that the space $I^t_{K^t}(\mu)$ of $K^t$-finite functions are smooth functions on $G^t$. Thus the restriction map

$$R : I^t_{K^t}(\mu) \rightarrow C^\infty(G), \quad Rf(g) = f(g), \ g \in G$$

makes sense. In the $K^t$-realization of $I^t(\mu)$, we have $Rf \in L^2(K)$ for any $K^t$-finite elements $f \in L^2(SO(2q))$.

Our main observation is the following:

**Proposition 3.2.** Let $\nu, \mu \in \mathbb{C}$ such that $\nu = \frac{p}{p+1}\mu$. The restriction map $R$ is a $G$-equivariant isomorphism from $I^t_{K^t}(\mu)$ onto $I_K(\nu)$ in the sense that

$$R\pi^\sharp_\mu(g)f = \pi_\nu(g)Rf, \ f \in I^t(\mu), \ g \in G,$$

and it is unitary as a map from $I^t_{K^t}(\mu) \sim L^2(SO(2q))_{K^t}$ onto $I_K(\nu) \sim L^2(V_{p+1,q})_K$.

**Proof.** Let $f \in I^t_{K^t}(\mu)$. By Lemma 2.1 one check easily for $ln \in P \subset P^t \chi^t_\mu(ln) = \chi_\nu(ln)$. Together with (3.12), it implies that $Rf$ satisfies (3.11). Moreover (3.3) implies (3.12). So $Rf \in I(\nu)$ and $R\pi^\sharp_\mu(g)f = \pi_\nu(g)Rf$ for any $g \in G$. As $f$ is $K^t$-finite, $Rf$ is also $K$-finite.

The decompositions (3.13) and (3.4) show the rest of the claim. \hfill $\square$

Using Proposition 3.2 we get that restriction to $G$ of the complementary series $C^t(\mu)$ defines a unitarizable representation of $G$, which we write as $C(\nu)$, whose $K$-finite elements are the same as $I_K(\nu)$, by Proposition 3.2, i.e,

$$C(\nu) = R\mathcal{C}^t(\mu), \ C_K(\nu) = I_K(\mu), \ \nu = \frac{p}{p+1}\mu, \ \mu \in (0, q).$$

The main result of this paper is the following theorem which states that restriction $C(\nu)$ to the maximal parabolic subgroup $\tilde{P}$ of $G$ is irreducible.

**Theorem 3.3.** Let $\nu = \frac{p}{p+1}\mu$ with $\mu \in (0, q)$. Then restriction to $G$ of the complementary series $(C^t(\mu), G^t)$ defines a unitarizable irreducible representation $C(\nu)$. It is the unitarization of the principal series representation $(I_K(\nu), G)$ realized in the non-compact picture. Moreover, it remains irreducible when restricted to the maximal parabolic subgroup $GL_p \tilde{N} = \tilde{P}$.

The irreducibility under $G_0$ in the above statement is essentially proved in [17]. Indeed let $\tilde{K} = \text{Spin}(p+1) \times \text{Spin}(p)$. The representation $C(\nu)$ is treated as representation of $\text{Spin}(p+1, p)$ and it is proved [17] 12.2.1 that the $(\mathfrak{g}, \tilde{K})$-module of $C(\nu)$ is irreducible. However the representations of $\tilde{K}$ in $C(\nu)$ descend to the representation of $K_0$ and thus the $(\mathfrak{g}, K)$-module is the same as $(\mathfrak{g}, K)$-module $C_K(\nu)$, and the latter is then irreducible.

To prove the rest of Theorem 3.3 we will use the non-compact picture. As $\tilde{P} \subset P^t$ we can find the action of $P$ on $(\bar{\pi}^\sharp_\mu, L^2(\mathcal{X}_{2q}))$.

**Lemma 3.4.** The representation of $\tilde{P}$ on $(\bar{\pi}^\sharp_\mu, L^2(\mathcal{X}_{2q}))$ is unitarily equivalent to the representation $(\pi, L^2(N_p))$ given by:

$$\pi(\bar{n}_0) \cdot \phi(\bar{n}) = \phi(\bar{n}_0^{-1}\bar{n})$$
for an element $n_0 \in \bar{N}$ and for an element $g \in GL_p$:

$$\pi(g)\phi(n) = |\det g|^{-\frac{p}{2}} \phi(g^{-1} \cdot n),$$

(3.15) where the action of $GL_p$ on $\bar{N}$ is given by (2.4).

**Proof.** Let us consider the unitary isomorphism:

$$L^2(N_p) \rightarrow L^2(\mathcal{X}_2q) \quad \text{given by} \quad \psi(z,v) = \phi(M(z,v)).$$

It is easy to check (3.15). Now for $(z_o, v_o) \in N_p$, using (2.3), we compute:

$$\left(\exp \bar{n}^\sharp_M(z_o, v_o) \bar{m}^\sharp_h\right) \cdot \phi(M(z, v)) = \phi(-M(z_o, v_o) + h^1M(z, v)h)$$

and direct computations show

$$-M(z_o, v_o) + h^1M(z, v)h = M((z_o, v_o)^{-1}(z, v))$$

so the action of $\bar{N}$ is given by (3.14).

So by Schur’s Lemma, Theorem 3.3 is proved once we have shown the following proposition:

**Proposition 3.5.** Let $T$ be a bounded operator on $L^2(\bar{N})$ commuting with the action $\pi$ of $P$ defined in Lemma 3.4. Then $T$ is the scalar multiple of the identity.

**Remark:** Using the non-compact pictures, one can show the unitarity of $C(\nu)$.

Indeed using Lemma 2.1 it is not difficult to show that the Knapp-Stein intertwiner $A^\sharp_I(\nu)$ and $I^\sharp_I(\nu)$ for the series $I^\sharp_I(\mu)$ and $I(\nu)$ satisfies:

$$A^\sharp_I(\nu)(\exp \bar{n}^\sharp_M(z_o, v_o)) = A^\sharp_I(\nu)(\exp \bar{n}^\sharp_M(z_o, v_o)),$$

(3.16) and the properties of $A^\sharp_I(\nu)$ described in [2] imply that $I^\sharp_I(\nu)$ is unitarizable.

Knapp-Stein intertwiners are (nilpotent) convolution operator with very singular kernels. By [2], $A^\sharp_I$ is an abelian convolution with a power of the Pfaffian. It can be computed using geometric means that the kernel of $A^\sharp_I$ is of the form $Q(z, v)^s = \det(z + \frac{1}{4}vv^*)^s$ for $(z, v) \in N$ and it is only through some elementary but tricky matrix computations that it can be linked with the abelian convolution with some power of the Pfaffian as in (3.10).

4. **Proof of Proposition 3.5**

We will use some well-known results for the Plancherel formula and von Neumann algebras of left regular representations of $\bar{N}$ on $L^2(\bar{N})$; see [27] Chapt.14 and [7] for general locally compact groups [4] for the case of nilpotent groups. We describe first the support of the Plancherel measure described in [4] [8] [24].

4.1. **The support of the Plancherel measure.** To ease notation we write elements in $\bar{n}$ or $\bar{N}$ as $n$ instead of $\bar{n}$. We may identify them with elements of $\mathcal{X}_2q$ by (2.3); on $\mathcal{X}_2q$ we consider the standard inner product $(Z, W) = \frac{1}{2} \text{Tr}ZW^*$. Hence we have equipped $\bar{n}$ with an inner product and we can now identify the dual $n^*$ with $\bar{n}$. The dual action of $g \in GL_p$ on $n^* = \bar{n}$ will be written as $g \ast n$.

We fix a generic point $o_n^* = (z_{o_n^*}, v_{o_n^*})$, the element of $n^* \sim \bar{n}$ defined using (2.3) by:

$$M(o_n^*) = J_q \quad \text{where} \quad J_q = \text{diag}(J, \ldots, J) \in M_{2q} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
It is easy to see that a representative of a coadjoint orbit can be chosen of the form \((z, v) \in n_v^*\) with \(zv = 0\), that is, the vector \(v\) being in the kernel of the matrix \(z\). Let \(O\) be the collection of those representatives \((z, v)\) with \(M(z, v)\) non-singular:

\[
O := \{ (z, v) \in n_v^* \; , \; zv = 0 \; \text{ and } \; \det M(z, v) \neq 0 \} .
\]

It is easy to see that \(O\) is the following union of \(SO(p)\)-orbits of certain ”diagonal representatives”:

\[
O := \text{SO}(p)\Delta \ast o_{n^*} \quad \text{where} \quad \Delta = \{ (\text{diag}(d_1I_2, \ldots, d_{q-1}I_2, d_q) \; , \; d_j \in \mathbb{R}^* ) \} \subset \text{GL}_p .
\]

Any \(w \in O \subset n^*\) then induces an irreducible unitary representation \(\lambda_w\) of \(\bar{n}\) and \(N\) on \(L^2(\mathbb{R}^{q-1})_w \cong L^2(\mathbb{R}^{q-1})\). We describe the representation of \(n\) very briefly for the element \(o_{n^*} = (z_{o_{n^*}}, v_{o_{n^*}} )\). The construction for general \(w \in O\) can be done similarly by using the equivariant action of \(\text{GL}_p\) on \(n^*\). As the writing of \(w \in O\) as \(\text{GL}_p \cdot o_{n^*}\) is not unique, there is a certain ambiguity here but it is harmless for our proof.

The element \(o_{n^*}\) defines a splitting (or a complex structure) of \(\mathbb{R}^{2(q-1)} = \mathbb{R}^{q-1} + \mathbb{R}^{q-1}\). The space \(n\) is decomposed as

\[
(4.1) \quad n = X_p \oplus \mathbb{R}^p = n_0 \oplus h
\]

where \(n_0 := (z_{o_{n^*}} \cap X_p ) \oplus \mathbb{R} v_{o_{n^*}}\) while \(h := \mathbb{R} z_{o_{n^*}} \oplus \mathbb{R}^{q-1} + \mathbb{R}^{q-1}\) is the Heisenberg algebra. There exists a unique representation \((\lambda_{o_{n^*}}, L^2(\mathbb{R}^{q-1}))\) of \(N\) whose restriction to \(\exp \mathbb{R} o_{n^*}\) is given by the character \(\exp i2\pi o_{n^*}\).

The Plancherel formula for \(L^2(N)\) is given by

\[
(4.2) \quad \| f \|^2_{L^2(N)} = \int_{O} \| \hat{f}(w) \|^2_{2} dw(w), \quad f(0) = \int_{O} \text{Tr}(\hat{f}(w)) dw(w) ,
\]

where we have denoted the group Fourier transform of a function \(f\) by

\[
\hat{f}(w) = \int_{N} f(g) \lambda_w(g) dg ,
\]

and its Hilbert-Schmidt norm by \(\| \hat{f}(w) \|_2\) i.e. in \(HS_w := L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})^* \). The Plancherel measure \(d\lambda\) can be explicitly computed but we will not need it here. In otherwords, the regular action of \(N \times N\) on \(L^2(N)\) is decomposed as

\[
(4.3) \quad L^2(N) = \int_{O} \lambda_{o_{n^*}} \oplus \lambda_{o_{n^*}}^{*} dw(w) ,
\]

where \(\lambda_{o_{n^*}}^*\) is the contragradient of \(\lambda_{o_{n}}\).

4.2. **Proof of Proposition 3.5.** Let \(\Omega := \{ (z, v)n ; \; \det(M(z, v)) \neq 0 \}\). Clearly \(\Omega\) is open and dense in \(n\), and which strictly contained in \(\Omega \subset O\). By elementary matrix computations, it can be also described as the orbit of \(o_{n^*}\):

**Lemma 4.1.** \(\text{GL}_p\) acts transitively on \(\Omega\) and we have \(\Omega = \text{GL}_p \cdot o_{n^*} = \text{GL}_p / \text{Sp}(q-1, \mathbb{R})\).

**Proof.** Let \((z, v) \in O\). The diagonalization of \(z\) provides a \(g \in SO(p)\) such that \(g \cdot (z, v) = (w, u)\) with \(u = (u_1, \ldots, u_p)\) and

\[
w = \text{diag} \left( \begin{array}{cccc} 0 & w_1 & \cdots & w_{q-1} \\ -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -w_{q-1} & 0 & \cdots & 0 \end{array} \right) .
\]

We compute on the one hand \(\det M(w, u) = (w_1 \cdots w_{q-1} u_p)^2\) and on the other hand \(\det = \det M(g(z, v)) = \det g \det M(z, v) = \det M(z, v) \neq 0\). Namely \(u_p, w_1, \ldots, w_n \neq 0\)
0. We solve now the equation \( g(w, u) = o_{n^*} \) with \( g \in GL_p \) of the form \( g = \begin{pmatrix} A & 0 \\ B & c \end{pmatrix} \), viz,

\[
AZ_1 A^t = J_0 , \quad AZ_1 B + AY c = 0 , \quad cu_p = 1 ,
\]

which is easy to see to have a solution, e.g. by taking \( c = v_p^{-1} \), \( B = -v_p^{-1} Z_1^{-1} Y \) and

\[
A = \text{diag} (\text{sgn} w_1|w_1|^2, |w_1|^2, \ldots, \text{sgn} w_{q-1}|z_{q-1}|^2, |z_1|^2) .
\]

The isotropic subgroup in \( GL_p \) of \( o_{n^*} \) consists of \( g = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL_p \) with \( B = 0, C = 0, D = 1 \) and \( A \) such that \( AJ_{q-1} A^t = J_{q-1} \). Thus it is a realization of the symplectic group \( Sp(q - 1, \mathbb{R}) \) and we have \( \Omega = GL_p/Sp(q - 1, \mathbb{R}) . \)

There exists [9] a representation \((\tau, L^2(\mathbb{R}^{q-1}), Mp(q-1, \mathbb{R}))\) of the double cover \( Mp(q-1, \mathbb{R}) \) of \( Sp(q - 1, \mathbb{R}) \) such that

\[
(4.4) \quad \tau(\tilde{g}) \lambda_{o_{n^*}}(n) \tau(\tilde{g})^* = \lambda_{o_{n^*}}(g \cdot n) , \quad \tilde{g} \in Mp(q-1, \mathbb{R}) ,
\]

with \( \tilde{g} \in Mp(q - 1, \mathbb{R}) \mapsto g \in Sp(q - 1, \mathbb{R}) \) the double covering. Furthermore the representation \((\tau, Mp(q-1, \mathbb{R}), L^2(\mathbb{R}^{q-1}))\) is a sum of two irreducible inequivalent representations [9] Theorem 4.56],

\[
(4.5) \quad L^2(\mathbb{R}^{q-1}) = L^2_0(\mathbb{R}^{q-1}) \oplus L^2_1(\mathbb{R}^{q-1})
\]
of even an odd functions.

We can now prove Proposition 3.5.

Proof of Proposition 3.5. Let \( T \) be a bounded operator on \( L^2(N) \) commuting with the action \( \pi \) of \( \bar{P} \) defined in Lemma 3.4.

As \( T \) commutes with the left translation, by the Plancherel Theorem [7], there exists a measurable field \( \{ \hat{T}(w), w \in \mathcal{O} \} \) of bounded operators on \( L^2(\mathbb{R}^p_{\mathbb{Z}}) \) such that

\[
(4.6) \quad \hat{T}\hat{f}(w) = \hat{f}(w)\hat{T}(w) , \quad f \in \mathcal{S}(N) ;
\]

this measurable field of operator is unique (up to a \( \nu \)-negligible set).

Let \( w = g \ast o_{n^*} \) with \( g \in SO(p)\Delta \). By the orbit method, the representations \( \lambda_w \) and \( n \mapsto \lambda_{o_{n^*}}(gn) \) are unitarily equivalent: there exists a unitary operator \( A_w \) such that \( A_w \lambda_w(n) = \lambda_{o_{n^*}}(gn)A_w \) and, for \( f \in \mathcal{S}(N) \), we compute with the change of variable \( n_1 = gn_1 \):

\[
\hat{f}(w) = A_w^{-1} \int_N f(n) A_w^{-1} \lambda_{o_{n^*}} (gn) A_w dn = A_w^{-1} \int_N (\pi(g)) f(n_1) \lambda_{o_{n^*}} (n_1) dn_1 A_w
\]

\[
(4.7) \quad = A_w^{-1} \pi(g) \hat{f}(o_{n^*}) A_w .
\]

Now as \( T \) commutes with \( \pi(g) \), we obtain easily:

\[
\hat{T}\hat{f}(w) = A_w^{-1} (T(\pi(g)f))^\ast (o_{n^*}) A_w .
\]

Using \(4.6\) and the uniqueness of \( \{ \hat{T}(w), w \in \mathcal{O} \} \), we obtain for \( \nu \)-almost all \( w \in \mathcal{O} \),

\[
(4.8) \quad \hat{T}(w) = A_w^{-1} \hat{T}(o_{n^*}) A_w .
\]

We may assume that \( \hat{T}(o_{n^*}) \) exists and that relation \(4.5\) holds for all \( w \in \mathcal{O} \).
In the same way, we consider \( \hat{g} \in \text{Mp}(q-1, \mathbb{R}) \) and the equivalence relation \((4.4)\). Proceeding just as above, we obtain:

\[(4.9) \quad \hat{T}(o_{n'}) = \tau(\hat{g})\hat{T}(o_{n'})\tau(\hat{g})^{-1}.\]

It follows then from the irreducible decomposition \((4.3)\) that \( \hat{T}(o_{n'}) \) is constant on each space, namely

\[
\hat{T}(o_{n'}) = c_0I + c_1U
\]

where \( U \) is the reflection,

\[Uh(x) = h(-x), \quad h \in L^2(\mathbb{R}^{q-1}), \quad x \in \mathbb{R}^{q-1}.\]

By the Plancherel Theorem \([7]\), there exists a bounded operator \( T_1 : L^2(N) \to L^2(N) \) which commutes with the left translations and satisfies

\[(4.10) \quad \hat{T}_1f(w) = \hat{f}(w)A_w^{-1}UA_w.\]

for all \( w \in \mathcal{O} \) and \( f \in \mathcal{S}(N) \). Because of \((1.8)\) and \((1.9)\), we have \( T = c_0I + c_1T_1 \).

So the proof of Proposition \(5.5\) will be over once we have shown that \( c_1 = 0 \) and for this it suffices to show that \( T_1 \) does not commute with the action of \((\pi, \text{GL}_p)\).

As \( T_1 \) is bounded on \( L^2(N) \) and commutes with left translation, it is a convolution operator with a tempered kernel: there exists \( \kappa \in \mathcal{S}'(N) \) such that \( T_1f = f * \kappa \) for any \( f \in \mathcal{S}(N) \). We claim that \( \kappa \) is not invariant under \( \text{GL}_p \) and this shows that \( T_1 \) does not commute with the action of \((\pi, \text{GL}_p)\).

To show our claim, we first compute \( T_1f(0) \) for \( f \in \mathcal{S}(\tilde{N}) \). For this we will use the Plancherel formula (see \((4.2)\)):

\[(4.11) \quad T_1f(0) = \int_{\mathcal{O}} \text{Tr}(\hat{T}_1f(w)) \, \text{d}w.\]

Now by \((4.7)\) and \((4.10)\), for \( w = g * o_{n'} \), we have:

\[(4.12) \quad \text{Tr}(\hat{T}_1f(w)) = \text{Tr}(\hat{f}(w)A_w^{-1}UA_w) = \text{Tr}(A_w\hat{f}(w)A_w^{-1}U) = \text{Tr}(\pi(g)f(o_{n'})U).\]

So we just want to compute the trace of \( \hat{f}(o_{n'})U \) on the Hilbert space \( L^2(\mathbb{R}^{q-1}) \).

This can be derived from the standard formulas \((23\) Chapt. XII, \S6, \([23\) Chapt. II, \S2-\S3\)) for the Weyl transform.

Indeed considering the decomposition \((4.1)\), we write the elements of \( \mathfrak{n} \) as \( h + h^\perp \) where \( h \in \mathfrak{h} \) and \( h^\perp \in \mathfrak{n}_0 \). Integrating \( f \) over \( \mathfrak{n}_0 \), we obtain the function \( F \) with \( \mathfrak{h} \):

\[F(h) = \int_{\mathfrak{n}_0} f(h + h^\perp)dh^\perp.\]

We now identify \( \mathfrak{h} \) with the Heisenberg group and we write the elements of \( \mathfrak{h} \) as \( h = (x, y, t) \in \mathbb{R}^{q-1} \times \mathbb{R}^{q-1} \times \mathbb{R} \). It is clear that \( \hat{f}(o_{n'}) \) coincides with the Schrödinger representation of the Heisenberg group at \( F \). From \((23\) Chapt. XII, \S6.3\), this shows that \( \hat{f}(o_{n'}) \) is the integral operator on \( L^2(\mathbb{R}^{q-1}) \) with kernel

\[K_h(x, y) := c \int_{\mathbb{R}^{q-1} \times \mathbb{R}} e^{2\pi i \frac{1}{2}w \cdot (y + x) + t} F(u, y - x, t) \, du \, dt, \quad x, y \in \mathbb{R}^{q-1},\]

where \( c = c_q \) is a known constant (our \( t \) here corresponds to \( \frac{1}{2} t \) in \((23\) Chapt. XII, \S6.3, (91)))). So the kernel of the operator \( \hat{f}(o_{n'})U \) is \( K_h(-x, y) \) and we can now
compute the trace of the operator:
\[
\text{Tr} \left( f(o_{n^*})U \right) = \int_{\mathbb{R}^{q-1}} K_b(-x,x)dx = e \int_{\mathbb{R}^{q-1}} \int_{\mathbb{R}} e^{2\pi i t} F(u,2x,t)dudt dx .
\]
Thus we have obtained:
\[
\text{Tr} \left( \hat{f}(o_{n^*})U \right) = C(Ff)(o_{n^*}) ,
\]
for some non-zero known constant $C$ where we have denoted by $F$ the Euclidean Fourier transform on $n$, that is,
\[
Ff(\zeta,\nu) = \int_{n} f(z,v)e^{2\pi i (\zeta z + \nu v)}dvdz ,
\]
where we have used the canonical Euclidean scalar products on $\mathcal{X}_{q-1}$ and $\mathbb{R}^{q-1}$.

Now by (4.12), this shows that for any $w \in \mathcal{O}$, we have:
\[
\text{Tr} \left( \hat{f}(w)U \right) = C(Ff)(w) .
\]
By (4.11), we obtain:
\[
\int f(n)\kappa(n^{-1})dn = T_1 f(0) = C \int_{\mathcal{O}} F(w)du(w) .
\]
Hence the support of $\mathcal{F}\kappa(-1)$ is included in $\overline{\mathcal{O}}$. But $\overline{\mathcal{O}}$ is invariant under $n \mapsto n^{-1}$ but not invariant under $GL_p$ since $\mathcal{O}$ is strictly included in $\Omega = GL_p \cdot o_{n^*}$ (see Lemma 3.11). So $\kappa$ is not $GL_p$-invariant. This shows that $T_1$ does not commute with $(\pi, GL_p)$ and concludes the proof of Proposition 3.5. \hfill \Box

4.3. Decomposition of $L^2(N)$ under $\tilde{N} \times \text{Sp}(q-1,\mathbb{R})$. We note that the above proof also yields a decomposition of $L^2(N)$ under the action of $\tilde{N} \times \text{Sp}(q-1,\mathbb{R})$. Consider first the reference point $w = o_{n^*}$ and the space $L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})$ with $N$ acting on the left factor by $\lambda_\omega$. Note that the metaplectic representation $\tau$ on $L^2(\mathbb{R}^{q-1})$, by its definition, defines a unitary representation of $\tau \otimes \tau^*$ of $\text{Sp}(q-1,\mathbb{R})$ on $L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})$, viewed as Hilbert-Schmidt operators, by
\[
(\tau \otimes \tau^*)(g)T = \tau(g)T \tau^*(g) .
\]
The group $\tilde{N} \times \text{Sp}(q-1,\mathbb{R})$ acts on $L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1})$ and we denote the corresponding representation, by $\lambda_\omega \times \tau_\omega$.

Using the decomposition of $L^2(\mathbb{R}^{q-1})$ into even and odd functions $L^2(\mathbb{R}^{q-1})_i$, $i = 0,1$, we have:
\[
L^2(\mathbb{R}^{q-1}) \otimes L^2(\mathbb{R}^{q-1}) = L^2(\mathbb{R}^{q-1})_0 \otimes L^2(\mathbb{R}^{q-1})_0 \oplus L^2(\mathbb{R}^{q-1})_0 \otimes L^2(\mathbb{R}^{q-1})_1 ,
\]
and we obtain the $\tilde{N} \times \text{Sp}(q-1,\mathbb{R})$-irreducible decomposition:
\[
\lambda_\omega \times \tau_\omega = (\lambda_\omega \times \tau_\omega)_0 + (\lambda_\omega \times \tau_\omega)_1 .
\]
Clearly this construction can be done for any $\omega$. We have then

**Corollary 4.2.** The space $L^2(N)$ is decomposed under $\tilde{N} \times \text{Sp}(q-1,\mathbb{R})$ as
\[
L^2(N) = \int_{\mathcal{O}} (\lambda_\omega \times \tau_\omega)_0 + (\lambda_\omega \times \tau_\omega)_1 dw(w) .
\]
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