A BINARY ADDITIVE EQUATION INVOLVING FRACTIONAL POWERS

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1. Introduction

It is well-known that the number of integers \( n \leq x \) that can be expressed as sums of two squares is \( O(x(\log x)^{-1/2}) \). On the other hand, Deshouillers \[2\] showed that when \( 1 < c < \frac{4}{3} \), every sufficiently large integer \( n \) can be represented in the form

\[
[m_1^c] + [m_2^c] = n, \tag{1}
\]

with integers \( m_1, m_2 \); henceforth, \([\theta]\) denotes the integral part of \( \theta \). Subsequently, the range for \( c \) in this result was extended by Gritsenko \[3\] and Konyagin \[5\]. In particular, the latter author showed that (1) has solutions in integers \( m_1, m_2 \) for \( 1 < c < \frac{3}{2} \) and \( n \) sufficiently large.

The analogous problem with prime variables is considerably more difficult, possibly at least as difficult as the binary Goldbach problem. The only progress in that direction is a result of Laporta \[6\], which states that if \( 1 < c < \frac{17}{11} \), then almost all \( n \) (in the sense usually used in analytic number theory) can be represented in the form (1) with primes \( m_1, m_2 \). Recently, Balanzario, Garaev and Zuazua \[1\] considered the equation

\[
[m_c^c] + [p_c^c] = n, \tag{2}
\]

where \( p \) is a prime number and \( m \) is an integer. They showed that when \( 1 < c < \frac{17}{11} \), this hybrid problem can be solved for almost all \( n \). It should be noted that in regard to the range of \( c \), this result goes even beyond Konyagin’s. On the other hand, when \( c \) is close to 1, one may hope to solve (2) for all sufficiently large \( n \), since the problem is trivial when \( c = 1 \). The main purpose of the present note is to address this issue. We establish the following theorem.

**Theorem 1.** Suppose that \( 1 < c < \frac{16}{15} \). Then every sufficiently large integer \( n \) can be represented in the form (2).

The main new idea in the proof of this theorem is to translate the additive equation (2) into a problem about Diophantine approximation. The same idea enables us to give also a simple proof of a slightly weaker version of the result of Balanzario, Garaev and Zuazua. For \( x \geq 2 \), let \( E_c(x) \) denote the number of integers \( n \leq x \) that cannot be represented in the form (2). We prove the following theorem.

**Theorem 2.** Suppose that \( 1 < c < \frac{3}{2} \) and \( \varepsilon > 0 \). Then

\[
E_c(x) \ll x^{3(1-1/c)+\varepsilon}.
\]

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We remark that Theorem 1 is hardly best possible. It is likely that more sophisticated exponential sum estimates and/or sieve techniques would have allowed us to extend the range of $c$. The resulting improvement, however, would have been minuscule; thus, we decided not to pursue such ideas.

**Notation.** Most of our notation is standard. We use Landau’s $O$-notation, Vinogradov’s $\ll$-symbol, and occasionally, we write $A \asymp B$ instead of $A \ll B \ll A$. We also write $\{\theta\}$ for the fractional part of $\theta$ and $||\theta||$ for the distance from $\theta$ to the nearest integer. Finally, we define $e(\theta) = \exp(2\pi i \theta)$.

### 2. Proof of Theorem 1: initial stage

In this section, we only assume that $1 < c < 2$. We write $\gamma = 1/c$ and set

$$X = \left(\frac{1}{7}n\right)\gamma, \quad X_1 = \frac{5}{4}X, \quad \delta = \gamma X^{1-c}. \tag{3}$$

If $n$ is sufficiently large, it has at most one representation of the form (2) with $X < p \leq X_1$. Furthermore, such a representation exists if and only if there is an integer $m$ satisfying the inequality

$$(n - \lfloor p^c \rfloor)^\gamma \leq m < (n + \lfloor p^c \rfloor)^\gamma. \tag{4}$$

We now proceed to show that such an integer exists, if $p$ satisfies the conditions

$$X < p \leq X_1, \quad \{p^c\} < \frac{1}{2}, \quad 1 - \frac{5}{6}\delta < \left\{ (n - p^c)^\gamma \right\} < 1 - \frac{2}{3}\delta. \tag{5}$$

Under these assumptions, one has

$$X^{1-c} = (n - X^c)^{\gamma - 1} < (n - p^c)^{\gamma - 1} \leq (n - X_1^c)^{\gamma - 1} < 1.1X^{1-c}.$$

Hence,

$$(n - \lfloor p^c \rfloor)^\gamma = (n - p^c)^\gamma \left(1 + \gamma\{p^c\}(n - p^c)^{-1} + O(n^{-2})\right)$$

$$< (n - p^c)^\gamma + \frac{1}{2}\gamma(n - p^c)^{\gamma - 1} + O(n^{\gamma - 2})$$

$$< (n - p^c)^\gamma + 0.55\delta + O(\delta n^{-1})$$

$$< \left[ (n - p^c)^\gamma \right] + 1 - 0.1\delta,$$

and

$$(n + 1 - \lfloor p^c \rfloor)^\gamma = (n - p^c)^\gamma \left(1 + \gamma(1 + \{p^c\})(n - p^c)^{-1} + O(n^{-2})\right)$$

$$\geq (n - p^c)^\gamma + \gamma(n - p^c)^{\gamma - 1} + O(n^{\gamma - 2})$$

$$> (n - p^c)^\gamma + \delta + O(\delta n^{-1})$$

$$> \left[ (n - p^c)^\gamma \right] + 1 + 0.1\delta.$$

Consequently, conditions (5) are indeed sufficient for the existence of an integer $m$ satisfying (4). It remains to show that there exist primes satisfying the inequalities in (5). To this end, it suffices to show that

$$\sum_{X < p \leq X_1} \Phi(p^c)\Psi((n - p^c)^\gamma) > 0 \tag{6}$$

for some smooth, non-negative, $1$-periodic functions $\Phi$ and $\Psi$ such that $\Phi$ is supported in $(0, 1/2)$ and $\Psi$ is supported in $\left(1 - \frac{5}{6}\delta, 1 - \frac{2}{3}\delta\right)$. 

Let \( \psi_0 \) be a non-negative \( C^\infty \)-function that is supported in \([0, 1]\) and is normalized in \( L^1 \): \( \| \psi_0 \|_1 = 1 \). We choose \( \Phi \) and \( \Psi \) to be the 1-periodic extensions of the functions
\[
\Phi_0(t) = \psi_0(2t) \quad \text{and} \quad \Psi_0(t) = \psi_0(6\delta^{-1}(t - 1) + 5),
\]
respectively. Writing \( \hat{\Phi}(m) \) and \( \hat{\Psi}(m) \) for the \( m \)th Fourier coefficients of \( \Phi \) and \( \Psi \), we can report that
\[
\hat{\Phi}(0) = \frac{1}{2}, \quad |\hat{\Phi}(m)| \ll_r (1 + |m|)^{-r} \quad \text{for all} \ r \in \mathbb{Z},
\]
\[
\hat{\Psi}(0) = \frac{1}{6}\delta, \quad |\hat{\Psi}(m)| \ll_r (1 + \delta|m|)^{-r} \quad \text{for all} \ r \in \mathbb{Z}.
\]
Replacing \( \Phi(p^c) \) and \( \Psi((n - p^c)\gamma) \) on the left side of (8) by their Fourier expansions, we obtain
\[
\sum_{X < p \leq X_1} \Phi(p^c)\Psi((n - p^c)\gamma) = \sum_{h \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \sum_{X < p \leq X_1} \hat{\Phi}(h)\hat{\Psi}(j) e(hp^c + j(n - p^c)\gamma). \tag{8}
\]
Set \( H = X^\varepsilon \) and \( J = X^{c-1+\varepsilon} \), where \( \varepsilon > 0 \) is fixed. By (7) with \( r = [\varepsilon^{-1}] + 2 \), the contribution to the the right side of (8) from the terms with \( |h| > H \) or \( |j| > J \) is bounded above by a constant depending on \( \varepsilon \). Thus,
\[
\sum_{X < p \leq X_1} \Phi(p^c)\Psi((n - p^c)\gamma) = \frac{1}{12}\delta (\pi(X_1) - \pi(X)) + O(\delta R + 1),
\]
where \( \pi(X) \) is the number of primes \( \leq X \) and
\[
R = \sum_{|h| \leq H} \sum_{|j| \leq J} \left| \sum_{X < p \leq X_1} e(hp^c + j(n - p^c)\gamma) \right|.
\]
Thus, it suffices to show that
\[
\sum_{X < p \leq X_1} e(hp^c + j(n - p^c)\gamma) \ll X^{2 - c - 3\varepsilon} \tag{9}
\]
for all pairs of integers \((h, j)\) such that \( |h| \leq H, |j| \leq J, \) and \((h, j) \neq (0, 0)\).

3. **Bounds on exponential sums**

In this section, we establish estimates for bilinear exponential sums, which we shall need in the proof of (8). Our first lemma is a variant of van der Corput’s third-derivative estimate (see [4] Corollary 8.19).

**Lemma 3.** Suppose that \( 2 \leq F \leq N^{3/2}, N < N_1 \leq 2N, \) and \( 0 < \delta < 1 \). Let \( f \in C^3[N, N_1] \) and suppose that we can partition \([N, N_1]\) into \( O(1) \) subintervals so that on each subinterval one of the following sets of conditions holds:

i) \( \delta FN^{-2} \ll |f''(t)| \ll FN^{-2} \);

ii) \( \delta FN^{-3} \ll |f'''(t)| \ll FN^{-3}, |f''(t)| \ll \delta FN^{-2} \).

Then
\[
\sum_{N < n \leq N_1} e(f(n)) \ll \delta^{-1/2} (F^{1/6}N^{1/2} + F^{-1/3}N).
\]
Proof. Let \( \eta \) be a parameter to be chosen later so that \( 0 < \eta \leq \delta \) and let \( I \) be one of the subintervals of \([N,N_1]\) mentioned in the hypotheses. If i) holds in \( I \), then by \[4, Corollary 8.13\],

\[
\sum_{n \in I} e(f(n)) \ll \delta^{-1/2} (F^{1/2} + NF^{-1/2}).
\]  

(10)

Now suppose that ii) holds in \( I \). We subdivide \( I \) into two subsets:

\[
I_1 = \{ t \in I : \eta FN^{-2} \leq |f''(t)| \ll \delta FN^{-2}\}, \quad I_2 = I \setminus I_1.
\]

Since \( f'' \) is monotone on \( I \), the set \( I_1 \) consists of at most two intervals and \( I_2 \) is (possibly empty) subinterval of \( I \). If \( I_2 = [a, b] \), then there is a \( \xi \in (a, b) \) such that

\[ f''(b) - f''(a) = (b - a)f''(\xi) \implies b - a \ll \eta \delta^{-1} N. \]

Thus, by \[4, Corollary 8.13\] and \[4, Corollary 8.19\],

\[
\sum_{n \in I_1} e(f(n)) \ll \eta^{-1/2} (F^{1/2} + NF^{-1/2}),
\]  

(11)

\[
\sum_{n \in I_2} e(f(n)) \ll \eta \delta^{-4/3} F^{1/6} N^{1/2} + \eta^{1/2} \delta^{-2/3} F^{-1/6} N.
\]  

(12)

Combining (10) - (12), we get

\[
\sum_{N < n \leq N_1} e(f(n)) \ll \eta^{-1/2} (F^{1/2} + NF^{-1/2}) + \eta \delta^{-4/3} N^{1/2} F^{1/6} + \eta^{1/2} \delta^{-2/3} NF^{-1/6}.
\]  

(13)

We now choose

\[
\eta = \delta \max (F^{-1/3}, F^{2/3} N^{-1}).
\]

With this choice, (13) yields

\[
\sum_{N < n \leq N_1} e(f(n)) \ll \delta^{-1/2} (F^{1/6} N^{1/2} + F^{-1/3} N) + \delta^{-1/3} (F^{5/6} N^{-1/2} + F^{-1/6} N^{1/2}),
\]

and the lemma follows on noting that, when \( F \ll N^{3/2}, \)

\[
F^{-1/6} N^{1/2} \ll F^{-1/3} N, \quad F^{5/6} N^{-1/2} \ll F^{1/6} N^{1/2}.
\]

\( \square \)

Next, we turn to the bilinear sums needed in the proof of \[39\]. From now on, \( X, X_1, N, H, J \) have the same meaning as in \[39\] and \( \varepsilon \) is subject to \( 0 < \varepsilon < \frac{1}{5} \left( \frac{16}{15} - c \right) \).

Lemma 4. Suppose that \( 1 < c < \frac{6}{5} - 6\varepsilon, M < M_1 \leq 2M, 2 \leq K < K_1 \leq 2K, \) and

\[
M \ll X^{1-2c/3-\varepsilon}.
\]  

(14)

Further, suppose that \( h, j \) are integers with \( |h| \leq H, |j| \leq J, (h, j) \neq (0, 0), \) and that the coefficients \( a_m \) satisfy \( |a_m| \leq 1. \) Then

\[
\sum_{M < m \leq M_1} \sum_{K < k \leq K_1 \atop X < mk \leq X_1} a_m e(hm^e k^c + j(n - m^e k^c)') \ll X^{2-c-4\varepsilon}.
\]
Proof. We shall focus on the case \( j \neq 0 \), the case \( j = 0 \) being similar and easier. We set
\[
y = jn^\gamma, \quad x = y^{-1}hn, \quad T = T_m = n^\gamma m^{-1} \ll K.
\]
With this notation, we have
\[
f(k) = f_m(k) = hm\epsilon k^c + j(n - m\epsilon k^c)^\gamma = y\alpha(kT_m^{-1}),
\]
where
\[
\alpha(t) = \alpha(t; x) = xt^c + (1 - t^c)^\gamma.
\]
We have
\[
f''(k) = yT^{-2}\alpha''(kT^{-1}), \quad f'''(k) = yT^{-3}\alpha'''(kT^{-1}),
\]
and
\[
\alpha''(t) = (c - 1)t^{c-2}(cx - (1 - t^c)^{\gamma-2}), \quad \alpha'''(t) = -(c - 1)(2c - 1)t^{2c-3}(1 - t^c)^{\gamma-3} + (c - 2)t^{-1}\alpha''(t).
\]
Moreover, by virtue of (3),
\[
\frac{1}{2} < (kT^{-1})^{c} \leq \frac{1}{2}(1.25)^c < \frac{4}{9}
\]
whenever \( X < mk \leq X_1 \).
Let \( \delta_0 = X^{-\varepsilon/10} \). If \( |x| \geq \delta_0^{-1} \), then by (16), (17), and (19),
\[
|f''(k)| \asymp |xy|K^{-2} \asymp |h|nK^{-2} \quad \implies \quad JX^{1-\varepsilon}K^{-2} \ll |f''(k)| \ll JXK^{-2}.
\]
Thus, by Lemma 3 with \( \delta = X^{-\varepsilon} \), \( F = JX \) and \( N = K \),
\[
\sum_{M < m \leq M_1} \sum_{K < k \leq K_1} \sum_{X < mk \leq X_1} a_m e(f_m(k)) \ll MX^{\varepsilon/2}(X^{(c+\varepsilon)/6}K^{1/2} + KX^{-c/3}). \quad (20)
\]
Note that we need also to verify that \( JX \leq K^{3/2} \). This is a consequence of (14).
Suppose now that \( |x| \leq \delta_0^{-1} \). The set where \( |\alpha''(kT^{-1})| \geq \delta_0 \) consists of at most two intervals. Consequently, we can partition \([K, K_1] \) into at most three subintervals such that on each of them we have one of the following sets of conditions:
\[
i) \quad \delta_0 |y|K^{-2} \ll |f''(k)| \ll \delta_0^{-1}|y|K^{-2};
\]
\[
ii) \quad |y|K^{-3} \ll |f'''(k)| \ll |y|K^{-3}, \quad |f''(k)| \ll \delta_0 |y|K^{-2}.
\]
Thus, by Lemma 3 with \( \delta = \delta_0^2 \), \( F = \delta_0^{-1}|y| \asymp \delta_0^{-1}|y|X \), and \( N = K \),
\[
\sum_{M < m \leq M_1} \sum_{K < k \leq K_1} \sum_{X < mk \leq X_1} a_m e(f_m(k)) \ll MX^{\varepsilon/10}(X^{(c+2\varepsilon)/6}K^{1/2} + KX^{-1/3}). \quad (21)
\]
Again, we have \( \delta_0^{-1}|y|X \leq JX^{1+\varepsilon/10} \ll K^{3/2} \), by virtue of (14).
Combining (20) and (21), we obtain the conclusion of the lemma, provided that \( c < \frac{4}{3} - 5\varepsilon \) and
\[
M \ll X^{3-7c/3-10\varepsilon}.
\]
Once again, the latter inequality is a consequence of (14).
\[\Box\]
Lemma 5. Suppose that $1 < c < \frac{16}{15} - 2\varepsilon$, $M < M_1 \leq 2M$, $K < K_1 \leq 2K$, and
\[ X^{2c-2+9\varepsilon} \ll M \ll X^{3-2c-9\varepsilon}. \] (22)
Further, suppose that $h, j$ are integers with $|h| \leq H$, $|j| \leq J$, $(h, j) \neq (0, 0)$, and that the coefficients $a_m, b_k$ satisfy $|a_m| \leq 1$, $|b_k| \leq 1$. Then
\[ \sum_{M < m \leq M_1} \sum_{K < k \leq K_1} \sum_{X < m k \leq X_1} a_m b_k e(hm^c k^c + j(n - m^c k^c)\gamma) \ll X^{-c-4\varepsilon}. \]

Proof. As in the proof of Lemma 4, we shall focus on the case $j \neq 0$. By symmetry, we may assume that $M \geq X^{1/2}$. We set
\[ y = jn^\gamma, \quad x = y^{-1}hn, \quad T = n^\gamma. \]

With this notation, we have
\[ f(k, m) = hm^c k^c + j(n - m^c k^c)\gamma = y\alpha(mkT^{-1}), \]
where $\alpha(t)$ is the function defined in \[15\].

By Cauchy’s inequality and \[4\] Lemma 8.17,
\[
\left| \sum_{M < m \leq M_1} \sum_{K < k \leq K_1} a_m b_k e(f(k, m)) \right|^2 \ll \frac{X}{Q} \sum_{|q| \leq Q} \sum_{K < k \leq 2K} \left| \sum_{m \in I(k, q)} e(g(m; k, q)) \right| \ll \frac{X^2}{Q} + \frac{X}{Q} \sum_{0 < |q| \leq Q} \sum_{K < k \leq 2K} \left| \sum_{m \in I(k, q)} e(g(m; k, q)) \right|, \tag{23}
\]
where $g(m; k, q) = f(k + q, m) - f(k, m)$, $Q = J^2 X^{6\varepsilon}$, and $I(k, q)$ is a subinterval of $[M, M_1]$ such that
\[ X < mk, m(k + q) \leq X_1 \]
for all $m \in I(k, q)$. We remark that the right inequality in \[22\] ensures that $Q \ll KX^{-\varepsilon}$. When $q \neq 0$, we write
\[ g(m; k, q) = y T^{-1} \int_{mk}^{mk+q} \alpha'(t T^{-1}) \, dt = qy \int_0^1 \beta(m(k + \theta q)T^{-1}) \frac{d\theta}{k + \theta q}, \]
where $\beta(t) = t\alpha'(t)$. Introducing the notation
\[ z_\theta = z_\theta(k, q) = y(q + \theta q)^{-1}, \quad U_\theta = U_\theta(k, q) = T(k + \theta q)^{-1} \ll M, \]
we find that
\[ g''(m) = \int_0^1 z_\theta U_\theta^{-2} \beta''(m U_\theta^{-1}) \, d\theta, \quad g''(m) = \int_0^1 z_\theta U_\theta^{-3} \beta''(m U_\theta^{-1}) \, d\theta, \]
and
\[ \beta''(t) = (c - 1)t^{c-2}(c^2 x + (1 - t)\gamma^{-3}(c + (c - 1)t)) \tag{24} \]
\[ \beta''(t) = (c - 1)(2c - 1)t^{2c-3}(1 - t)^{-\gamma^{-4}}((c - 1)t^{c - c} + (c - 2)t^{-1} \beta''(t)) \tag{25} \]
Let $\delta_0 = X^{-\varepsilon/10}$. If $|x| \geq \delta_0^{-1}$, then by \[24\] and a variant of \[19\],
\[ |g''(m)| \asymp |qxy|(XM)^{-1} \quad \implies \quad |q|JX^{-\varepsilon} M^{-1} \ll |g''(m)| \ll |q|JM^{-1}. \]

\[6\]
Thus, by Lemma 3 with $\delta = X^{-\varepsilon}$, $F = |q|JM$ and $N = M$,
\[
\sum_{m \in I(k,q)} e(g(m; k, q)) \ll (|q|J)^{1/6} M^{2/3} X^{\varepsilon/2}. \tag{26}
\]
Note that we need also to verify that $F \leq M^{3/2}$, which holds if
\[
M \gg X^{6(c-1)+12\varepsilon}. \tag{27}
\]
Suppose now that $|x| \leq \delta_0^{-1}$. We then deduce from (24) and (25) that
\[
|\beta''(mU_\theta^{-1})| \ll \delta_0^{-1}, \quad |\beta'''(mU_\theta^{-1})| \ll \delta_0^{-1},
\]
whence
\[
|\beta''(mU_\theta^{-1})| = |\beta''(mU_\theta^{-1})| + O(|q|K^{-1}\delta_0^{-1}) = |\beta''(mU_\theta^{-1})| + O(\delta_0^3).
\]
We now note that the subset of $[M, M_1]$ where $|\beta''(mU_\theta^{-1})| \geq \delta_0$ consists of at most two intervals. Consequently, we can partition $[M, M_1]$ into at most three subintervals such that on each of them we have one of the following sets of conditions:
\begin{enumerate}
  \item $\delta_0 |qy|(XM)^{-1} \ll |g''(m)| \ll \delta_0^{-1} |qy|(XM)^{-1};$
  \item $|qy|X^{-1}M^{-2} \ll |g'''(m)| \ll |qy|X^{-1}M^{-2}, \quad |g''(m)| \ll \delta_0 |qy|(XM)^{-1}.$
\end{enumerate}
Thus, Lemma 3 with $\delta = \delta_0^2$, $F = \delta_0^{-1}|qy|M$, and $N = M$ yields (26), provided that (27) holds.

Combining (24) and (28), we get
\[
\left| \sum_{M < m \leq M_1} \sum_{K < k \leq K_1} \sum_{X < mk \leq X_1} a_m b_k e(f(k, m)) \right|^2 \ll X^2 Q^{-1} + X^{2+\varepsilon/2} (QJ)^{1/6} M^{-1/3}. \tag{28}
\]
In view of our choice of $Q$, the conclusion of the lemma follows from (28), provided that
\[
M \gg X^{7.5(c-1)+10\varepsilon}.
\]
Both (27) and the last inequality follow from the assumption that $M \geq X^{1/2}$ and the hypothesis $c < \frac{16}{15} - 2\varepsilon$.

We close this section with a lemma that will be needed in the proof of Theorem 2.

**Lemma 6.** Suppose that $1 < c < 2$, $2 \leq X < X_1 \leq 2X$, and $0 < \delta < \frac{1}{4}$. Let $S_\delta$ denote the number of integers $n$ such that $X < n \leq X_1$ and $\|n^c\| < \delta$. Then
\[
S_\delta \ll \delta (X_1 - X) + \delta^{-1/2} X^{c/2}.
\]

**Proof.** Let $\Phi$ be the 1-periodic extension of a smooth function that majorizes the characteristic function of the interval $[-\delta, \delta]$ and is majorized by the characteristic function of $[-2\delta, 2\delta]$. Then
\[
S_\delta \leq \sum_{X < n \leq X_1} \Phi(n^c) = \sum_{X < n \leq X_1} \tilde{\Phi}(0) + \sum_{h \neq 0} \tilde{\Phi}(h) \sum_{X < n \leq X_1} e(hn^c). \tag{29}
\]
If $h \neq 0$, [4] Corollary 8.13] yields
\[
\sum_{X < n \leq X_1} e(hn^c) \ll |h|^{1/2} X^{c/2},
\]
whence
\[
\sum_{h \neq 0} \hat{\Phi}(h) \sum_{X < n \leq X_1} e(hn^c) \ll X^{c/2} \sum_{h \neq 0} |\hat{\Phi}(h)||h|^{1/2} \\
\ll X^{c/2} \sum_{h \neq 0} \frac{\delta|h|^{1/2}}{(1 + \delta|h|)^2} \ll \delta^{-1/2}X^{c/2}.
\] (30)

Since \(\hat{\Phi}(0) \leq 4\delta\), the lemma follows from (29) and (30). \(\square\)

4. Proof of Theorem 1: conclusion

Suppose that \(1 < c < \frac{16}{15}\) and \(0 < \varepsilon < \frac{1}{2}(\frac{16}{15} - c)\). To prove (9), we recall Vaughan’s identity in the form of [4, Proposition 13.4]. We can use it to express the sum in (9) as a linear combination of \(O(\log^2 X)\) sums of the form
\[
\sum_{M < m \leq M_1} \sum_{K < k \leq K_1} a_m b_k e\left(hm^ck^c + j(n - m^ck^c)\gamma\right),
\]
where either
i) \(|a_m| \ll m^{\varepsilon/2}, b_k = 1, \text{ and } M \ll X^{2/3}\); or
ii) \(|a_m| \ll m^{\varepsilon/2}, |b_k| \ll k^{\varepsilon/2}, \text{ and } X^{1/3} \ll M \ll X^{2/3}\).

A sum subject to conditions ii) is \(\ll X^{2-c-3.5\varepsilon}\) by Lemma 5. A sum subject to conditions i) can be bounded using Lemma 4 if (14) holds and using Lemma 5 if (14) fails. In either case, the resulting bound is \(\ll X^{2-c-3.5\varepsilon}\). Therefore, each of the \(O(\log^2 X)\) terms in the decomposition of (9) is \(\ll X^{2-c-3.5\varepsilon}\). This establishes (9) and completes the proof of the theorem.

5. Proof of Theorem 2

We can cover the interval \((x^{1/2}, x]\) by \(O((\log x)^3)\) subintervals of the form \((N, N_1]\), with \(N_1 = N(1 + (\log N)^{-2})\). Thus, it suffices to show that
\[
Z_c(N) \ll N^{3-3/c+5\varepsilon/6},
\] (31)
where \(Z_c(N)\) is the number of integers \(n\) in the range
\[
N < n \leq N(1 + (\log N)^{-2})
\]
that cannot be represented in the form (2).

As in the proof of Theorem 1 we derive solutions of (2) from solutions of (4). We set \(\gamma = 1/c, \eta = (\log N)^{-2},\) and write
\[
N_1 = (1 + \eta)N, \quad X = \left(\frac{1}{2}N\right)^\gamma, \quad X_1 = (1 + \eta)X, \quad \delta = \gamma X^{1-c}.
\]
Suppose that \(N < n \leq N_1\) and \(X < p \leq X_1\). Then
\[
(1 - \eta)\delta < \gamma (n - p^c)^{\gamma-1} < (1 + 2\eta)\delta.
\]
Assuming that \(p\) satisfies the inequalities
\[
4\eta < \{p^c\} < 1 - 4\eta, \quad 1 - \delta - \eta\delta < \{(n - p^c)^\gamma\} < 1 - \delta + \eta\delta,
\] (32)
we deduce that
\[
(n - [p^e])^\gamma < (n - p^e)^\gamma + (1 - 4\eta)(1 + 2\eta)\delta + O(\delta n^{-1})
\]
\[
< [(n - p^e)^\gamma] + 1 - \eta \delta,
\]
\[
(n + 1 - [p^e])^\gamma > (n - p^e)^\gamma + (1 + 4\eta)(1 - \eta)\delta + O(\delta n^{-1})
\]
\[
> [(n - p^e)^\gamma] + 1 + \eta \delta.
\]

In particular, a prime \( p \), \( X < p \leq X_1 \), that satisfies \( (32) \) yields a solution \( m \) of \( (1) \) and a representation of \( n \) in the form \( (2) \).

Let \( \Phi \) be the 1-periodic extension of a smooth function \( \Phi_0 \) that majorizes the characteristic function of \([6\eta, 1 - 6\eta]\) and is majorized by the characteristic function of \([4\eta, 1 - 4\eta]\). Further, let \( \Psi \) be the 1-periodic extension of
\[
\Psi_0(t) = \psi_0((2\eta \delta)^{-1}(t - 1 + \delta) + \frac{1}{2}),
\]
where \( \psi_0 \) is the function appearing in the proof of Theorem \( (11) \). Then \( \Psi_0 \) is supported inside \([1 - \delta - \eta \delta, 1 - \delta + \eta \delta]\) and the Fourier coefficients of \( \Psi \) satisfy
\[
\hat{\Psi}(0) = 2\eta \delta, \quad |\hat{\Psi}(h)| \ll \eta \delta(1 + \eta \delta|h|)^{-r} \quad \text{for all } r \in \mathbb{Z}.
\]

Hence,
\[
\sum_{X < p \leq X_1} \Phi(p^e) \Psi((n - p^e)^\gamma) = \sum_{h \in \mathbb{Z}} \sum_{X < p \leq X_1} \Phi(p^e) \hat{\Psi}(h) e(h(n - p^e)^\gamma)
\]
\[
= \hat{\Psi}(0) \sum_{X < p \leq X_1} \Phi(p^e) + \mathcal{R}(n)
\]
\[
= 2\eta \delta((\pi(X_1) - \pi(X) + O(S)) + \mathcal{R}(n)).
\]

Here,
\[
\mathcal{R}(n) = \sum_{h \neq 0} \hat{\Psi}(h) \sum_{X < p \leq X_1} \Phi(p^e) e(h(n - p^e)^\gamma)
\]
and \( \mathcal{S} \) is the number of integers \( m \) such that \( X < m \leq X_1 \) and \( \|m^e\| < 6\eta \). By Lemma \( (6) \)
\[
\mathcal{S} \ll \eta(X_1 - X) + \eta^{-1/2}X^{e/2} \ll \eta^2 X.
\]

Combining \( (34), (35) \) and the Prime Number Theorem, we find that
\[
\sum_{X < p \leq X_1} \Phi(p^e) \Psi((n - p^e)^\gamma) \gg X^{2-\varepsilon}(\log X)^{-5}
\]
for any \( n, N < n \leq N_1 \), for which we have
\[
\mathcal{R}(n) \ll X^{2-e-\varepsilon/12}.
\]

Since the sum on the right side of \( (36) \) is supported on the primes \( p \) satisfying \( (32), (31) \) will follow if we show that \( (37) \) holds for all but \( O\left(X^{3-3\gamma+5e/6}\right) \) integers \( n \in (N, N_1] \).

Set \( H = X^{e-1+e/6} \). By \( (33) \) with \( r = 2 + [2e^{-1}] \), the contribution to \( \mathcal{R}(n) \) from terms with \( |h| > H \) is bounded. Consequently,
\[
Z_{e}(N) \ll X^{-2+e/6} \sum_{N < n \leq N_1} \mathcal{R}_1(n)^2,
\]

\[\text{9}\]
where
\[ R_1(n) = \sum_{0 < |h| \leq H} \left| \sum_{X < p \leq X_1} \Phi(p^c)e(h(n - p^c)\gamma) \right|. \]

Appealing to Cauchy’s inequality and the Weyl–van der Corput lemma [4, Lemma 8.17], we obtain
\[
Z_c(N) \ll X^{c-3+\varepsilon/3} \sum_{0 < |h| \leq H} \sum_{N < n \leq N_1} \left| \sum_{X < p \leq X_1} \Phi(p^c)e(h(n - p^c)\gamma) \right|^2 \\
\ll X^{c-2+\varepsilon/3} Q^{-1} \sum_{0 < |h| \leq H} \sum_{|q| \leq Q} \sum_{X < p \leq X_1} \left| \sum_{N < n \leq N_1} e(f(n)) \right|,
\]

where \( Q \leq \eta X \) is a parameter at our disposal and
\[ f(n) = qh\left((n - p^c)\gamma - (n - (p + q)^c)\gamma\right). \]

We choose \( Q = \eta X^{1-\varepsilon/6} \). Then
\[ |qh|N^{-1} \ll |f'(n)| \ll |qh|N^{-1} \ll \eta < \frac{1}{2}, \]
so [4, Corollary 8.11] and the trivial bound yield
\[ \sum_{N < n \leq N_1} e(f(n)) \ll N(1 + |qh|)^{-1}. \]

We conclude that
\[ Z_c(N) \ll NX^{c-2+2\varepsilon/3} \sum_{0 < |h| \leq H} \sum_{|q| \leq Q} (1 + |qh|)^{-1} \ll NX^{2c-3+5\varepsilon/6}. \]

This establishes (31) and completes the proof of the theorem.

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