Non-axisymmetric relativistic Bondi-Hoyle accretion onto a Schwarzschild black hole

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Abstract

We present the results of an exhaustive numerical study of fully relativistic non-axisymmetric Bondi-Hoyle accretion onto a moving Schwarzschild black hole. We have solved the equations of general relativistic hydrodynamics with a high-resolution shock-capturing numerical scheme based on a linearized Riemann solver. The numerical code was previously used to study axisymmetric flow configurations past a Schwarzschild hole. We have analyzed and discussed the flow morphology for a sample of asymptotically high Mach number models. The results of this work reveal that initially asymptotic uniform flows always accrete onto the hole in a stationary way which closely resembles the previous axisymmetric patterns. This is in contrast with some Newtonian numerical studies where violent flip-flop instabilities were found. As discussed in the text, the reason can be found in the initial conditions used in the relativistic regime, as they can not exactly duplicate the previous Newtonian setups where the instability appeared. The dependence of the final solution with the inner boundary condition as well as with the grid resolution has also been studied. Finally, we have computed the accretion rates of mass and linear and angular momentum.

Key words Accretion — Hydrodynamics — Methods: numerical — Relativity — Shock waves
1 Introduction

In a previous paper (Font and Ibáñez 1997, hereafter Paper I) we studied the morphology and dynamics of relativistic Bondi-Hoyle accretion in axisymmetric flows past a Schwarzschild black hole. The main conclusion of that work was to extend the validity of the Bondi-Hoyle accretion picture (Hoyle and Lyttleton, 1939; Bondi and Hoyle, 1944) to the relativistic regime, finding that the matter is always accreted onto the hole in a stationary way. Furthermore, if the flow was, initially, supersonic, the main feature of the accretion pattern was the presence of a shock cone in the solution. At the same time, we checked the validity of our numerical code revisiting an existing previous calculation (Petrich et al., 1989) using more accurate numerical techniques specifically designed to capture discontinuities. In the present investigation we have extended those studies to account for non-axisymmetric configurations. We have only considered uniform flows at infinity focussing on studying whether or not the flow pattern reaches, ultimately, a stationary state. The main purpose of this work is, then, to extend previous non-axisymmetric Newtonian computations to the realm of general relativity. As far as we know, this has not been studied so far. In particular, we plan to address if non-steady patterns, so prominent in some of the 2D non-axisymmetric simulations in the classical regime, also arise here.

During the last years, a large number of wind accretion simulations past a gravitating body have been performed in Newtonian hydrodynamics (see, e.g., the list of references in Paper I). One of the most interesting features, only revealed by numerical simulations, was the appearance of unstable accretion patterns, contrary to the theoretical (and simplified) Bondi-Hoyle (1944) accretion picture. This highly non-steady behaviour on the wake of the accretor was only found in 2D non-axisymmetric simulations, especially when the resolution employed was fine enough. These unstable patterns are characterized by the shock wave moving from side to side (the so-called flip-flop instability) and also by the appearance of transient phases of disc formation where the angular momentum accreted by the central object significantly increases. This kind of behaviour has been found not only assuming local density and velocity gradients (Fryxell and Taam, 1988; Taam and Fryxell, 1989) but also when considering accretion of uniform flows at infinity (Matsuda et al., 1991). However, in detailed 3D computations performed more recently (Ruffert and Arnett, 1994; Ruffert, 1994a,b, 1995, 1996) the accretion cone remains quite stable and no sign of flip-flop instability appears unless the flow is assumed to have explicit density gradients at infinity (Ruffert, 1997).

In all previous studies it was shown that the key parameter which controls the appearance of the instability was the size of the accretor. In particular, this size should be a very small fraction of the accretion radius in order to find any evidence of flip-flop. For instance, in Sawada et al. (1989) and Matsuda et al. (1991) computations, the flip-flop instability appeared, in 2D, for a central size object of $\approx 0.0625r_a$, where $r_a$ is the accretion radius, the natural length scale of this problem, defined as

$$r_a = \frac{2GM}{v_\infty^2}.$$  \hspace{1cm} (1)

Here, $M$ is the mass of the accreting object, $G$ is the gravitational constant and $v_\infty$ is the asymptotic velocity of the fluid. They also found that the flow was stable if the size was larger than $\approx 0.125r_a$. More recently, Benensohn, Lamb and Taam (1997) have performed detailed 2D non-axisymmetric computations with a smaller accretor of size $0.0375r_a$ finding unstable behaviour as well.
In addition, there are other parameters that, combined with the size of the accretor, can play a role in the development of the instability. In particular, one of these is the Mach number of the flow. The larger the Mach number at infinity is, the more turbulent the wake becomes. Finally, another extremely important consideration is the position of the shock, namely if it is attached (tail shock) or detached (bow shock) with respect to the central object. This is basically controlled by the values of the asymptotic Mach number and the adiabatic index of the gas. For low Mach number and high adiabatic index flows (say 5/3), the shock is mainly detached at the front while the opposite trend produces a tail shock. All previous studies where the instability appeared needed the formation of a detached shock in front of the accretor. As the evolution proceeded, this detached shock transformed into a dome-shaped shock and, eventually, this was followed by a violent flip-flop instability (Sawada et al., 1989; Matsuda et al., 1991; Livio et al., 1991; see also Ruffert, 1997). In particular, Livio et al. (1991) demonstrated, using a simplified analytic treatment, the existence of a wake instability to small deflections when the shock opening angle is large. This can happen even for an initially homogeneous flow if the accretion process has low efficiency. However, the flow can be stabilized if the accreting object is larger than some critical dimension (Livio et al., 1991).

According to these previous classical results, there are several factors that can lead to the non-appearance of this instability in the relativistic regime when the accretor is a black hole. First, and most important, in this regime there is a physical minimum value for the size of the accreting object, given by the gravitational (Schwarzschild) radius. This, of course, places some constraints in the possibility of extending previous Newtonian studies to very small values of the accretor, assuming that we also need an asymptotically supersonic flow (in order to have any shock wave) and avoiding, at the same time, flow velocities larger than the speed of light. The second factor has to do with the position of the shock. As we will show below (see also Paper I), all models that we have evolved are characterized by the presence of a tail shock in the rear part of the hole. Hence, according to the Newtonian studies, this would not help the development of the flip-flop instability.

The paper is organized as follows: in next Section (§2) we briefly present the system of equations of general relativistic hydrodynamics written as a hyperbolic system of conservation laws. The numerical code and the initial setup are also described here. The results of the simulations are presented and analyzed in Section §3. Finally, Section §4 summarizes the main conclusions of this work.

2 Equations, initial setup and numerical issues

2.1 Equations

In order to study non-axisymmetric patterns in two dimensions we have restricted ourselves to an infinitesimally thin disk in the equatorial plane of the black hole. Therefore, we are using \((r, \phi)\) coordinates, instead of \((r, \theta)\) used previously in Paper I. Although this configuration is somehow artificial, it suffices to try to understand the stability of the flow. This kind of setup has also been used in Newtonian simulations, assuming flow past an infinite cylinder (Fryxell and Taam, 1988; Taam and Fryxell, 1989; Benensohn et al., 1997).

With the same definitions introduced in Paper I and using geometrized units \((G = c = 1\) with \(c\) the speed of light), the equations of (adiabatic) general relativistic hydrodynamics can be written, in the equatorial plane \((\theta = \pi/2)\) of the Schwarzschild metric, as
\[ \frac{\partial \mathbf{U}(\mathbf{w})}{\partial t} + \frac{\partial \mathbf{F}^r(\mathbf{w})}{\partial r} + \frac{\partial \mathbf{F}^\phi(\mathbf{w})}{\partial \phi} = \mathbf{S}(\mathbf{w}). \] (2)

In this equation, the vector of primitive variables is defined as
\[ \mathbf{w} = (\rho, v_r, v_\phi, \varepsilon) \] (3)

where \( \rho \) and \( \varepsilon \) are, respectively, the rest-mass density and the specific internal energy, related to the pressure \( p \) via an equation of state which we chose that of an ideal gas law
\[ p = (\gamma - 1)\rho\varepsilon. \] (4)

with \( \gamma \) being the constant adiabatic index. In addition, \( v_r \) and \( v_\phi \) denote the radial and azimuthal covariant components of the velocity. On the other hand, the vector of unknowns (evolved quantities) in equation (2) is
\[ \mathbf{U}(\mathbf{w}) = (D, S_r, S_\phi, \tau). \] (5)

The explicit relations between both sets of variables are
\[ D = \rho W \] (6)
\[ S_j = \rho h W^2 v_j \ (j = r, \phi) \] (7)
\[ \tau = \rho h W^2 - p - D. \] (8)

The quantity \( W \) stands for the Lorentz factor, which satisfies \( W = (1 - v^2)^{-1/2} \) with \( v^2 = \gamma_{ij} v^i v^j \), where \( v^i \) is the 3-velocity of the fluid, defined, for the case of a zero shift vector, according to \( v^i = \frac{u^i}{W} \) and \( \gamma_{ij} \) are the spatial components of the spacetime metric where the fluid evolves. The quantity \( h \) appearing in equations (7) and (8) is the specific enthalpy, defined as \( h = 1 + \varepsilon + p/\rho \).

The corresponding fluxes in the radial and azimuthal directions are, respectively
\[ \mathbf{F}^r(\mathbf{w}) = (Dv^r, S_r v^r + p, S_\phi v^r, (\tau + p)v^r) \] (9)
\[ \mathbf{F}^\phi(\mathbf{w}) = (Dv^\phi, S_r v^\phi, S_\phi v^\phi + p, (\tau + p)v^\phi) \] (10)

and the corresponding vector of sources \( \mathbf{S}(\mathbf{w}) \) is
\[ \mathbf{S}(\mathbf{w}) = \left( Dv^r \left[ \frac{M}{\alpha r^2} - \frac{2\alpha}{r} \right], -\frac{M}{r^2} (\tau + D) + \alpha \left[ \frac{1}{r} S_\phi v^\phi - \frac{2}{r^2} S_r v^r \right], S_\phi v^r \left[ \frac{M}{\alpha r^2} - \frac{2\alpha}{r} \right], -\frac{2}{r} S_r - Dv^r \left[ \frac{M}{\alpha r^2} - \frac{2\alpha}{r} \right] \right) \] (11)

where \( \alpha = \sqrt{1 - \frac{2M}{r}} \).

### 2.2 Initial setup

The models that we have evolved numerically are summarized in Table 1. Some of them were already evolved in axisymmetry in Paper I. Now, we have added some higher Mach number models and we have not considered subsonic and low Mach number flows. According to Newtonian studies, this is the appropriate initial condition to look for wake instabilities. We must also mention that, contrary to the axisymmetric results, it was now extremely difficult to evolve low Mach number models without corrupting the asymptotic initial values in the upstream region of the flow. In this case, we have chosen a different initial condition.
part of the domain, all fluid variables should increase monotonically. In practice, we found that
the local velocity and Mach number in the upstream direction fell below the initial values. This
was still the case even when we placed the outer boundary further out (although the evolution
could be accurately followed for much longer times). This sort of behaviour has not appeared
in the high Mach number models, as we will show in detail below. This inherent difficulty in
approximate the asymptotic conditions to a finite distance has also been pointed out in previous
Newtonian simulations (Ruffert and Arnett, 1994; Benensohn et al., 1997).

The flow pattern is completely characterized by some initial conditions at infinity upstream the
hole, namely the velocity \(v_\infty\), sound speed \(c_{s\infty}\) and adiabatic index, \(\gamma\). The first two parameters
fix the asymptotic Mach number, \(M_\infty\). The covariant components of the initial velocity are given
in terms of its asymptotic value:

\[
v_r = \frac{1}{\sqrt{\gamma_{rr}}} v_\infty \cos\phi
\]

\[
v_\phi = -\frac{1}{\sqrt{\gamma_{\phi\phi}}} v_\infty \sin\phi
\]

where \(\gamma_{rr} = 1 - \frac{2M}{r}\) and \(\gamma_{\phi\phi} = \frac{1}{r^2}\).

As mentioned in the introduction, it is convenient to stress the differences that our initial setup
has with respect to those of classical simulations where the tangential wake instability arose. For
this purpose, we plot in Fig. 1 the ratio between the size of the accretor, \(2M\), the Schwarzschild
radius, and the accretion radius, as a function of the asymptotic flow velocity. For the accretion
radius we are using the same definition of Paper I

\[
r_a = \frac{M}{(v_\infty^2 + c_{s\infty}^2)}.
\]

The curved lines represent that quotient for different values of the sound speed at infinity. The
solid line corresponds to zero sound speed and if we move upward on the plot, we go to increasingly
larger values. The last three curves correspond to the thermodynamically maximum permitted
values, \(\sqrt{\gamma - 1}\), for \(\gamma = 1.1, 4/3\) and \(5/3\), respectively. On the other hand, the straight lines
represent the fluid Mach number (divided by 5) as a function of the velocity, parametrized for the
same set of sound speeds. Therefore, the lines are drawn accordingly with the same line style,
with the exception of the solid straight line which corresponds to \(c_{s\infty} = 0.05\) (instead of zero
sound speed). From that figure, one can see that, in order to have tiny accretors, one has to
move to the lower left corner. However, in that region, the asymptotic initial velocity has to be
quite small (Newtonian flow). Furthermore, in order to have an initial supersonic, or even better,
hypersonic flow, the initial sound speed has to be considerably small. This kind of initial data is
quite difficult to evolve for any numerical code, as the internal energy density is very close to zero.
On the other hand, if we move to larger values of the velocity, for the sample of sound speeds
considered here, the size of the accretor has to be huge. In the limiting case of maximum sound
speed and maximum velocity, the minimum size of the accretor is larger than \(2r_a\) for \(\gamma = 1.1\) and
\(4/3\) and even larger than \(3r_a\) for \(\gamma = 5/3\). According to this, and having in mind the classical
results, it would not be too surprising that the relativistic patterns would appear much more
stable.

Finally, concerning the numerical grid, the spatial numerical domain, \((r, \phi)\), has been covered
by a canonical grid of \(200 \times 80\) numerical zones. The radial and angular discretizations lie,
respectively, in the interval \(r_{min} \leq r \leq r_{max}\) and \(0 \leq \phi \leq 2\pi\), where \(r_{min}\) and \(r_{max}\) depend on the
particular model. The specific values are given in Table 1. Note that, with the coordinates we
are using, \( r_{\text{min}} \) will always be slightly larger than \( 2M \) in order to avoid divergences at the horizon in some hydrodynamical quantities such as, e.g., the Lorentz factor. For the angular direction we have used an equally spaced grid while in the radial direction we have employed the Schwarzschild tortoise coordinate defined by \( r_* = r + 2M \ln(r/2M - 1) \).

### 2.3 Numerical issues

We have solved the general relativistic hydrodynamic equations with the same numerical code described in Paper I. We briefly summarize its main features here. The code belongs to the so called TVD (Total Variation Diminishing) schemes (Harten, 1983) and makes use of a linearized Riemann solver (Roe, 1981) as an accurate procedure to capture discontinuities. It also uses a monotonic upstream reconstruction procedure (van Leer, 1979) to extrapolate the variables at the cell interfaces to solve for the Riemann problem and a third order Runge-Kutta scheme, which preserves the TVD property, to update the solution in time. Further details of the code can be found in Paper I and Font et al. (1994).

The boundary conditions employed in the simulations are, basically, the same as we used in Paper I. For the innermost radial boundary we have considered now two different sets of conditions, namely outflow, where all variables are linearly extrapolated to the boundary zones, and absorbing conditions, where the velocities are reset to zero and the density and internal energy are fixed to very small values. In practice, we have not found any difference between using one condition or the other. Both types of conditions give the maximum accretion rates of mass and momentum (Benensohn et al., 1997). For the outermost radial boundary we have, again, imposed different conditions depending on the upstream or downstream position of the flow with respect to the hole. In the upstream part, the initial asymptotic values for all variables have been used at every time step. For the downstream part, we have performed a linear extrapolation. This boundary condition turns out to be the most critical one as, if the Mach number of the flow is not high enough, unphysical information can be reflected back to the computational domain. This could be the reason why we did not succeed in evolving low Mach number models. In future work we plan to use characteristic conditions in this boundary to see if they work better. Finally, we do not need to impose boundary conditions on the angular direction as no boundaries exist.

We have tried different angular resolutions to analyze the convergence of the numerical scheme. We chose model MB2 of Table 1 for this purpose, as it showed signs of unstable patterns in the wake. This resolution study was motivated by the recent work by Benensohn et al. (1997) where the unstable patterns only appeared if the number of angular zones to describe the entire circle was fine enough (roughly of the order of 200 zones). We performed 4 runs with 40, 80, 160 and 320 angular zones. In all cases the initial conditions were identical and the flow was completely uniform and totally symmetric with respect to the \( \phi = 0 \) line. With 40 zones the flow started stable and symmetric but after some time, due to purely numerical reasons, mainly accumulated roundoff error, an unstable pattern developed which even broke the wake. This was also visible with 80 angular zones as we will show below. However, the numerical instability appeared later and was much less violent (meaning that the solution was converging). With 160 and 320 zones we have not found any difference (even looking at the specific numbers in the data files) between the two regions above and below \( \phi = 0 \). For practical purposes we have always used a grid of 80 angular zones as most of the models showed good late time behaviour with this resolution.
3 Results

3.1 Flow morphology

The evolution of the models listed in Table 1 is discussed here. In Figs. 2 and 3 we show the final state of the first six models. We plot in Fig. 2 isocontours of the logarithmic of the density normalized to the asymptotic value while Fig. 3 shows isocontours of the total velocity of the flow. We note in both figures that all models present two clearly defined different regions upstream and downstream the accreting hole. In the upstream part of the flow, the isocontours are almost perfectly rounded, indicating slight deviation from spherical accretion. This is clearly demonstrated in Fig. 4 where we plot a radial cut along the line passing through the points \( \phi = 0 \) and \( \pi \) for several quantities and only for model MB3 (the rest of models show a completely similar behaviour). We can not directly compare the curves for the density and total velocity with the analytic values expected for spherical accretion (see, e.g., Hawley, Smarr and Wilson 1984):

\[
\rho(r) = \frac{k}{r^2 \sqrt{2M/r}} \tag{15}
\]

\[
v(r) = \sqrt{\frac{2M}{r}} \tag{16}
\]

where \( k \) is a constant. The reason is that we are assuming a fixed value for the velocity at the edge of the grid. For zero velocity at infinity this value is the free-fall velocity at the given radius (assuming that this radius is smaller than the transonic radius). However, the velocity pattern is clearly spherical as, just by simply adding a radial term to the exact velocity, of, e.g., the form \((r - r_{\text{min}})^{1/3}/a\), where \( a \) is a constant, it is possibly to get the numerical spherical solution. It is worth to mention, however, that the maximum values for the velocity at the innermost point for all models completely agree with the expected free-fall values as can be read off Table 2. From Fig. 4 one can also notice that, at the rear part of the hole, the density has always its maximum values, noticeably larger than the values at the front part. This means that the majority of the material is accreted onto the hole precisely at its rear part. The velocity plot reaches a zero value at the stagnation point behind the hole. All material inside the radius delimited by this point is ultimately captured by the hole.

On the other hand, downstream the accretor, all models show the presence of a well-defined tail shock that, starting very close to the axis, propagates all the way up to the outer boundary of the domain. This shock wave starts at the axis as a consequence of the gravitational bending that the matter feels, which originates the collision and piling-up of the upper and lower parts of the flow. Then, after developing the Bondi-Hoyle accretion column, the shock starts to continuously open to wider angles (due to the large pressure gradient between the inner and outer parts of the wake) until it finally accommodates itself in a fixed position. The particular models that we have considered are always characterized by a tail-shock. This shock never propagates to the upstream part of the flow. As we can see in both figures the wake appears to be completely symmetric with respect to the \( \phi = 0 \) line for all models. As mentioned earlier, only slight asymmetries develop for \( \gamma = 4/3 \) models (MB2 and MB3). This is purely a numerical instability. Results with larger resolutions (160 or 320 angular zones), not shown here, reveal a fully converged and symmetric pattern. However, with inadequate resolutions, this numerical instability can trigger asymmetric patterns which can, eventually, broke the wake. These asymmetries are more clearly noticeable in the velocity contours of Fig. 3.
The final model in Table 1, model UA2, is plotted in Figs. 5 and 6. Fig. 5 shows isocontours of density and velocity while Fig. 6 shows radial cuts along the line connecting $\phi = 0$ and $\phi = \pi$. This model differs from the rest in having a larger value for the asymptotic sound speed. This value is very close to its maximum given by $\sqrt{\gamma - 1}$, $\gamma$ being the adiabatic index of the fluid. Hence, this is an ultrarelativistic model, both from the thermodynamic and kinematic points of view (note that the asymptotic initial velocity is also very close to one). The isocontour plots reveal a completely similar flow pattern to those of Figs. 2 and 3. Note that the steady-state solution is achieved here at much earlier times. Clearly there is a correlation between flow velocity and stability: the larger the speed, the more stable the pattern becomes (and the larger the size of the accretor is). Let us also mention that model UA2 is a large accretor, with $r_{\text{min}} = 2.12 r_a$, which can help in obtaining extremely stable patterns. The stability of the large speed flows has also been confirmed with accretion rate plots as we will show below.

In Table 2 we present the main quantitative results concerning the evolution of these models. Comparing the morphology of the flow with the axisymmetric configurations of Paper I we notice that the shock opening angles are now smaller and, somehow, closer to the expected analytic values at large distances from the object

$$\theta_a = \sin^{-1} \frac{1}{\mathcal{M}_\infty^R}. \tag{17}$$

We can also compute this analytic estimate using the relativistic definition for the Mach number (Königl, 1980)

$$\mathcal{M}_\infty^R = \frac{W}{W_s} \mathcal{M}_\infty \tag{18}$$

where $W$ is the Lorentz factor of the flow and $W_s$ is the Lorentz-like factor for the sound speed, i.e., $W_s = (1 - c_s^2)^{-1/2}$, with the sound speed defined as $c_s^2 = \gamma p/\rho h$. With this definition, the expected values for the mildly relativistic models (MA2-MC3) are slightly modified: from 11.5 to 10 for models labeled ‘2’ and from 5.7 to 4.9 for models labeled ‘3’. The corresponding value for UA2, which has a larger asymptotic initial velocity, is now much different changing from 19.5 to 7.4. In either case, the agreement with the exact solution is not too good. There should also be mentioned that all the computed angles are quite similar despite the different Mach numbers. The dependence of the shock opening angle with the asymptotic Mach number is plotted in Fig. 7.

The position of the stagnation point presents some similitudes and some differences with respect to the axisymmetric models. The first thing we notice is that the position has very little dependence on the incident Mach number, as can be seen comparing models labeled ‘2’ with models labeled ‘3’. On the other hand there is a clear dependence with the adiabatic index of the flow. The position decreases as $\gamma$ increases. This behaviour was also found in the axisymmetric models. Comparing with those models of Paper I we find now considerable smaller values. All models that can be directly compared (MA2, MB2 and MC2) are now roughly a factor 2 smaller. This is however not the case for model UA2.

### 3.2 Accretion rates

We have computed the accretion rates of mass, radial momentum (drag rate) and angular momentum. The results are summarized in Table 2 and in Figs. 8-13. These quantities are excellent indicators to see if the solution ever approaches to a steady-state. In particular, monitoring the angular momentum rate during the whole simulation is interesting to see if any transient accretion disc forms.
Following the same procedure of Paper I we now compute the mass accretion rate in the equatorial plane according to

\[ \dot{m} = -\int Dv^r r^2 d\phi \]  (19)

To compute the radial and angular momentum rates we have followed Petrich et al. (1989) defining the variation in time of the \( i \) momentum as

\[ \dot{p}^i = -\int_{\partial V} T^{ij} \sqrt{-g} d\Sigma_j + \int_V (\text{source})^i \sqrt{-g} dV \]  (20)

where the flux integral is evaluated along the surface \( \partial V \) of a given volume \( V \) and the integral including the source terms is a full volume integral. In practice, if we have the outer boundary sufficiently far out, one can neglect the second integral as the source terms go to zero in flat space. However, as this is not the case in our finite numerical domain, we have always added the contribution of the sources in our results. In the above expression, \( T^{ij} \) are the spatial components of the stress-energy tensor and \( g \) is the determinant of the four metric.

After some algebra, the final expressions we use are

\[ \dot{p}^r = -r_c^2 \int_0^{2\pi} [S^\phi v^\phi + p(1 - \frac{2M}{r_c})] d\phi + \int_{r_{min}}^{r_c} \int_0^{2\pi} \left[ -\frac{M}{\alpha} (\tau + D) + \alpha r (S^\phi v^\phi - 2S_r v^r) \right] dr d\phi \]  (21)

for the radial momentum rate, and

\[ \dot{p}^\phi = -r_c^2 \int_0^{2\pi} S^\phi v^\phi d\phi + \int_{r_{min}}^{r_c} \int_0^{2\pi} S^\phi v^\phi (\frac{M}{\alpha} - 2\alpha) dr d\phi \]  (22)

for the angular momentum rate. In these expressions, \( r_c \) is the critical radius at which we have computed the rates. In practice, the mass accretion rate has been obtained at the accretion radius for all models except for model UA2 where it was computed at 5\( M \) (the accretion radius is inside the horizon in this case). The momentum rates have been calculated at three different parts of the grid just to check the influence of the source terms integral. These three radii are \( r_{min} \), \( r_a \) and \( r_{max} \). We only plot the results corresponding to the accretion radius. All mass accretion rates have been normalized to the canonical value proposed by Petrich et al. (1989). We can, thus, compare our results for the mass accretion rates with those previously found in axisymmetric computations. In addition, the radial momentum rates for models MA2-MC3 have been scaled to one and the values corresponding to model UA2 are displayed with respect to the same scale factor.

The evolution in time of the different accretion rates appears in Figs. 8-13. Fig. 8 shows the mass accretion rates for models MA2 to MC3 and Fig. 9 shows the same quantity for model UA2. This mass rate has been scaled by the same canonical value used in Paper I. The scaled radial momentum rates for models MA2 to MC3 are plotted in Fig. 10 while Fig. 11 shows this quantity for model UA2. Finally, the angular momentum accretion rates for the different models appear in Figs. 12 and 13.

From these figures we can again conclude that all models approach to a remarkably well defined final steady-state. All the conclusions derived in the axisymmetric simulations are still valid here. We find again that, given a fixed value of \( \mathcal{M}_\infty \), the mass accretion rate increases as \( \gamma \) goes from 1.1 to 5/3. The dependence with \( \mathcal{M}_\infty \) for a fixed value of \( \gamma \) is, however, not too clear, and the
numerical values show only small discrepancies. This is particularly true for models MB2 and MB3 which give the same final value. As in the axisymmetric simulations of Paper I, the model with an almost maximum sound speed, model UA2, displays much larger values. Comparing the values of Table 2 for the mass accretion rates with the axisymmetric results, we found that, for models MA2, MB2 and MC2 now we get roughly 10 times larger values. For model UA2 the difference is even bigger, a factor 16.

The radial momentum accretion rate plots clearly show the same behaviour than the mass accretion plots. After some initial time the solution relaxes to a constant value. This happens after $t \approx 150M$ for all models with low sound speed (MA2-MC3) and much earlier, $t \approx 35M$, for model UA2. As mentioned previously, we have computed the radial momentum rate at three different locations. Although we do not plot it here, we have verified that, as expected, for larger values of the radial coordinate, the integral containing the source terms in equation (18) becomes less relevant and the solution converges to a constant value, both in radius and time.

It is also interesting to look at the mass and radial momentum accretion rates plots to compare the stability of the different models. First thing to notice is that both types of figures display the same behaviour. Focusing on models MA2-MC3, we can see that those with $\gamma = 1.1$ are the most smooth and stable ones. As $\gamma$ increases to $4/3$ there are some small and irregular oscillations. Finally, for $\gamma = 5/3$, the oscillations around the central equilibrium value are clearly of constant period and their amplitude is fairly small. We have also found that models MB2 and MB3 with higher resolution (160 angular zones) show the same periodic feature than models MC2 and MC3. On the other hand, our ultrarelativistic model, UA2, shows, apart from an oscillatory phase around $110M$, a very smooth constant pattern. This is due to the large velocities and large size of this accretor, and is also in agreement with the rest of $\gamma = 1.1$ models (MA2 and MA3).

Finally, we have computed the angular momentum accretion rates. By looking at Figs. 12 and 13 one can clearly notice that the flow never succeeds in getting high angular momentum values, except for some large amplitude oscillations at late times for model MB3 and at an intermediate time for model UA2. This means that no transient accretion discs actually form, which would have not been the case if any flip-flop instability would have developed. All models show, basically, the same behaviour: an initial constant, almost zero value, followed by a period of oscillation around that central value. In some cases, as model MC3, this oscillatory phase shows signs of growing linearly, although at the time the simulation was stopped, the amplitude of the oscillations was still small. In particular, models MA2, MA3, MC2 and MC3 display very small maximum amplitudes. Models with $\gamma = 4/3$, MB2 and MB3, show, however, larger oscillations and, in particular, model MB3 shows a clear deviation from zero at very late times. As mentioned previously, the reason for this is again purely numeric and directly related to the unresolved wake patterns obtained with our canonical grid size. The corresponding plot for model UA2 appears in Fig. 13. Although at some point of the evolution it gets some large oscillations, it eventually settles down to small values.

4 Conclusions

We have performed a detailed numerical study of relativistic Bondi-Hoyle accretion onto a moving Schwarzschild black hole. We have extended our previous work to include non-axisymmetric configurations. This has been motivated by the possibility of finding unstable accretion patterns in the flow. We have only considered flows which are initially uniform. We have found that, contrary to Newtonian simulations, the non-axisymmetric solution is always stable to tangential
oscillations. However, this statement should be relaxed, as the corresponding initial setup in our relativistic wind simulations can not reproduce, for physical reasons, the one used in classical simulations. In particular, we have considered values of the accretion radius which, when used in the classical regime, would have not given rise to any instability as well. This may indicate that, in the relativistic regime, a uniform flow accreting onto a black hole could always remain symmetric and stationary. It might be worthwhile to go to much larger values of the asymptotic Mach number to see if any unstable pattern actually develops. These are, however, quite challenging simulations that would push the code to its limits, as the asymptotic sound speed should be considerably small.

We have checked the stability and stationarity of our relativistic accretion patterns by analyzing the broad morphology of the flow with isocontour plots as well as by integrating, over all time of the simulation, global quantities as the accretion rates of mass and momentum. In all cases considered, we have found that the mass and radial momentum accretion rates approach asymptotically to very accurate constant values in time. In addition, the angular momentum accretion rate always remains in very low values, indicating the total absence of rotating flows around the hole which could have triggered any tangential instability.

It maybe worthwhile to mention that, though we have restricted in the present work to homogeneous flows, the code is capable of evolving more complicated non-axisymmetric configurations where the flow has some amount of angular momentum at infinity. We are currently working in a parametric study of this kind of flows (Pons et al., 1998). In particular, we plan to gradually increase the angular momentum of the incident flow and study how it is transfered onto the hole via alternate spiral shocks. At the same time, we also plan to add rotation to the hole – a feature that the actual code used in this work already incorporates – extending thus the relativistic Bondi-Hoyle accretion to moving Kerr black holes. Ultimately, we plan to perform these simulations in three spatial dimensions.

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References

[1] Banyuls, F., Font, J.A., Ibáñez, J.M., Martí, J.M., and Miralles, J.A., 1997, ApJ, 476, 221
[2] Benensohn, J.S., Lamb, D.Q., and Taam, R.E., 1997, ApJ, 478, 723
[3] Bondi, H., and Hoyle, F. 1944, MNRAS, 104, 273
[4] Font, J.A., Ibáñez, J.M., Marquina, A., and Martí, J.M. 1994, A&A, 282, 304
[5] Font, J.A., and Ibáñez, J.M., 1997, ApJ (in press) (Paper I in the text)
[6] Fryxell, B.A., and Taam, R.E. 1988, ApJ, 335, 862
[7] Harten, A. 1983, J. Comput. Phys., 49, 357
[8] Hawley, J.F., Smarr, L.L., and Wilson, J.R. 1984, ApJ, 277, 296
[9] Hoyle, F., and Lyttleton, R.A., 1939, Proc. Cambridge Phil. Soc., 35, 405
[10] Königl, A. 1980, Phys. Fluids 23, 1083
[11] Livio, M., Soker, N., Matsuda, T., and Anzer, U. 1991, MNRAS, 253, 633
[12] Matsuda, T., Inoue, M., and Sawada, K. 1987, MNRAS, 226, 785
[13] Matsuda, T., Ishii, T., Sekino, N., Sawada, K., Shima, E., Livio, M., and Anzer, U. 1992, MNRAS, 255, 183
[14] Matsuda, T., Sekino, N., Sawada, K., Shima, E., Livio, M., Anzer, U., and Börner, G. 1991, A&A, 248, 301
[15] Petrich, L.I., Shapiro, S.L., Stark, R.F., and Teukolsky, S.A. 1989, ApJ, 336, 313 (PSST in the text)
[16] Pons, J.A., Font, J.A., Ibáñez, J.M., Martí, J.M. and Miralles, J.A., 1998 in preparation.
[17] Roe, P.L., 1981, J. Comput. Phys., 43, 357
[18] Ruffert, M., and Arnett, D. 1994, ApJ, 427, 351
[19] Ruffert, M. 1994a, ApJ, 427, 342
[20] Ruffert, M. 1994b, A&ASS, 106, 505
[21] Ruffert, M. 1995, A&ASS, 113, 133
[22] Ruffert, M. 1996, A&A, 311, 817
[23] Ruffert, M. 1997, A&A, 317, 793
[24] Sawada K., Matsuda, T., Anzer, U., Börner, G., and Livio, M. 1989, A&A, 231, 263
[25] Taam, R.E., and Fryxell, B.A. 1989, ApJ, 339, 297
[26] Taam, R.E., Fu, A., and Fryxell, B.A. 1991, ApJ, 371, 696
[27] van Leer, B. 1979, J. Comput. Phys., 32, 101
\[ M \text{radial values of the computational domain} \quad t = \frac{\dot{m}}{\dot{m}_\text{can}} \]

\[ \dot{m}/\dot{m}_\text{can} \quad \hat{p}^r \quad \hat{p}^\phi \]

The asterisk in models MB2 and MB3 indicates that the results have been obtained with 160 angular zones. \( \theta_a \) is the analytical value for the shock opening angle at large distances according to \( \sin^{-1} \frac{1}{M_\infty} \)

\( \rho = \pi, \rho^d_{\max} \) is the corresponding maximum in the density downstream \( \phi = 0, v^u_{\max} \) is the maximum upwind velocity, \( v^d_{\max} \) is the maximum downwind velocity, \( M^u_{\max} \) is the maximum value of the upwind Mach number and \( M^d_{\max} \) is its corresponding maximum value in the downwind direction. In addition \( \dot{m}/\dot{m}_\text{can} \) is the normalized mass accretion rate (to the canonical value used in Paper I) and \( \hat{p}^r \) and \( \hat{p}^\phi \) are the scaled rates of radial and angular momentum, respectively. For the latter quantity we only indicate the maximum values.
Figure 1: Minimum size of the accretor in units of the accretion radius as a function of the asymptotic flow velocity. The curved lines are parametrized for different sound speed values, specifically, 0 (solid), 0.1 (dotted), 0.31 (dashed), 0.57 (long dashed) and 0.81 (dot-dashed). In addition, the straight lines represent the asymptotic Mach number (over 5) for the same set of sound speeds (except the solid line which corresponds to $c_{s\infty} = 0.05$).
Figure 2: Isocontours of the logarithm of the scaled density at the final time of the evolution ($t = 750M$) for models MA2 to MC3. The dotted line indicates the position of the accretion radius. All models show a well-defined tail shock and a spherical upstream accretion pattern.
Figure 3: Isocontours of the total velocity at the final time of the evolution \((t = 750M)\) for models MA2 to MC3.
Figure 4: Radial plot of density and velocity for Model MB3. The left panels show the upstream part of the flow while the right panels show the downstream part. The upstream solution clearly indicates spherical flow plus a constant velocity added.

Figure 5: Isocontours of the logarithm of the scaled density (left) and total velocity (right) at the final time of the evolution ($t = 200M$) for model UA2. The accretion radius, dotted line, is inside the horizon.
Figure 6: Radial plot of density and velocity for Model UA2. The left panels show the upstream part of the flow while the right panels show the downstream part.

Figure 7: Shock opening angle versus asymptotic Mach number. The solid line represents the analytic values at large distance from the hole. The symbols indicate the numerical estimations for the different models.
Figure 8: Normalized mass accretion rate evolution for models MA2-MC3.

Figure 9: Normalized mass accretion rate evolution for model UA2.
Figure 10: Normalized radial momentum accretion rate evolution for models MA2-MC3.

Figure 11: Normalized radial momentum accretion rate evolution for model UA2.
Figure 12: Angular momentum accretion rate evolution for models MA2-MC3.

Figure 13: Angular momentum accretion rate evolution for model UA2.