HIGHER COHOMOLOGIES OF COMMUTATIVE MONOIDS

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Abstract. Extending Eilenberg-Mac Lane’s methods, higher level cohomologies for commutative monoids are introduced and studied. Relationships with pre-existing theories (Leech, Grillet, etc.) are stated. The paper includes a cohomological classification for symmetric monoidal groupoids and explicit computations for cyclic monoids.

1. Introduction and summary

In [19, Chapter X, §12], Mac Lane explains how to define, for each integer \( r \geq 0 \), the \( r \)th level cohomology groups of a (skew) commutative DGA-algebra (differential graded augmented algebra) over a commutative ring \( K \), say \( D \). Take the commutative DGA-algebra \( B^r(D) \), obtained by iterating \( r \) times the reduced bar construction on \( D \), and then, for any \( K \)-module \( A \), define

\[
H^n(D, r; A) = H^n(\text{Hom}_K(B^r(D), A)), \quad n = 0, 1, \ldots,
\]

where \( \text{Hom}_K(B^r(D), A) \) is the cochain complex obtained by applying the functor \( \text{Hom}_K(\cdot, A) \) to the underlying chain complex of \( K \)-modules \( B^r(D) \).

This process may be applied, for example, when \( D = \mathbb{Z}G \) is the group ring of an abelian group \( G \), regarded as a trivially graded DGA-ring, augmented by \( \alpha : \mathbb{Z}G \to \mathbb{Z} \) with \( \alpha(x) = 1 \) for all \( x \in G \). Thus, the Eilenberg-Mac Lane \( r \)th level cohomology groups of the abelian group \( G \) with coefficients in an abelian group \( A \) are defined by

\[
H^n(G, r; A) = H^n(\mathbb{Z}G, r; A).
\]

In particular, the first level cohomology groups \( H^n(G, 1; A) = H^n(G, A) \) are the ordinary cohomology groups of \( G \) with coefficients in the trivial \( G \)-module \( A \) [19, Chapter IV, Corollary 5.2]. These \( r \)th level cohomology groups of abelian groups were studied primarily with interest in Algebraic Topology. For instance, they have a topological interpretation in terms of the Eilenberg-Mac Lane spaces \( K(G, r) \), owing to the isomorphisms \( H^n(G, r; A) \cong H^n(K(G, r), A) \) [7, Theorem 20.3]. However, they early found application in solving purely algebraic problems. For example, we could recall that central group extensions of \( G \) by \( A \) are classified by cohomology classes in \( H^2(G, 1; A) \), while abelian group extensions of \( G \) by \( A \) are classified by cohomology classes in \( H^3(G, 2; A) \) [8, §26, (26.2), (26.3)]; or that second level cohomology classes in \( H^2(G, 2; A) \) classify braided monoidal categorical groups [14, Theorem 3.3], while third level cohomology classes in \( H^3(G, 3; A) \) classify Picard categories [21, II, Proposition 5].

Here, we introduce a generalization of Eilenberg-Mac Lane’s theory for abelian groups to commutative monoids. The obtained \( r \)th level cohomology groups of a commutative monoid \( M \), denoted by

\[
H^n(M, r; A),
\]

enjoy many desirable properties, whose study this work and its companion paper [5] are mainly dedicated to. In our development, the role of coefficients is now played by abelian group objects \( A \) in the comma category of commutative monoids over \( M \). We call them \( \mathbb{H}M - \text{modules} \) since,
as a result by Grillet [12, Chapter XII, §2], they are the same as abelian group valued functors
\( \mathcal{A} : \mathbb{H}M \to \mathbf{Ab} \) on the small category \( \mathbb{H}M \), whose set of objects is \( M \) and set of arrows \( M \times M \), with \( (x, y) : x \to xy \).

For any given commutative monoid \( M \), the category of chain complexes of \( \mathbb{H}M \)-modules is
an abelian category. In Section 2, we show that it is also a symmetric monoidal category, with a
distributive tensor product \( \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B} \), and whose unit object is \( Z \), the concentrated in degree zero
complex defined by the constant \( \mathbb{H}M \)-module given by the abelian group \( Z \) of integers. Hence,
commutative \textit{DGA-algebras over} \( \mathbb{H}M \) arise as internal commutative monoids \( \mathcal{A} \) in the symmetric
monoidal category of complexes of \( \mathbb{H}M \)-modules, endowed with a morphism of internal monoids
\( \mathcal{A} \to Z \).

Quite similarly as for ordinary commutative DGA-algebras over a commutative ring, a reduced bar construction \( \mathcal{A} \to B(\mathcal{A}) \) works on these DGA-algebras over \( \mathbb{H}M \). Thus, \( B(\mathcal{A}) \) is
obtained from \( \mathcal{A} \) by first totalizing the double complex of \( \mathbb{H}M \)-modules
\[
\bigoplus_{p \geq 0} \mathcal{A}/Z \otimes_{\mathbb{H}M} (p \text{ factors}) \cdots \otimes_{\mathbb{H}M} \mathcal{A}/Z ,
\]
and then enriching the (suitably graded) totalized complex of \( \mathbb{H}M \)-modules with a multiplicative
structure by a shuffle product. We do this in Section 3 where we also define, for any \( \mathbb{H}M \)-module
\( \mathcal{B} \), the \( r \)th level cohomology groups of \( \mathcal{A} \) with coefficients in \( \mathcal{B} \) by
\[
H^n(\mathcal{A}, r; \mathcal{B}) = H^n(\text{Hom}_{\mathbb{H}M}(B'(\mathcal{A}), \mathcal{B})), \quad n = 0, 1, \ldots .
\]

Next, in Section 4 we briefly study free \( \mathbb{H}M \)-modules. These arise as a left adjoint construction
to a forgetful functor from the category of \( \mathbb{H}M \)-modules to the comma category of sets over the
underlying set of \( M \). In particular, in Section 5 we introduce the free \( \mathbb{H}M \)-module on the
identity map \( id_M : M \to M \), denoted by \( \mathcal{Z} M \). This becomes a (trivially graded) commutative
DGA-algebra over \( \mathbb{H}M \) and then, for each integer positive \( r \), we define the \( r \)th level cohomology
groups of a commutative monoid \( \mathcal{M} \) with coefficients in a \( \mathbb{H}M \)-module \( \mathcal{A} \) by
\[
H^n(\mathcal{M}, r; \mathcal{A}) = H^n(\mathcal{Z} M, r; \mathcal{A}).
\]

When \( M = G \) is an abelian group, for any integer \( r \geq 0 \), \( B'(\mathbb{Z} G) \) is isomorphic to the constant
DGA-algebra over \( \mathbb{H}G \) defined by the Eilenberg-Mac Lane DGA-ring \( B'(\mathbb{Z} G) = A_N(\mathbb{Z}, G, r) \) in [7, §14]). Hence, for any abelian group \( A \), viewed as a constant \( \mathbb{H}G \)-module, the cohomology groups
\( H^n(G, r; A) \) defined as in (2) are naturally isomorphic to those by Eilenberg and Mac Lane in [14],
which, recall, compute the cohomology groups of the spaces \( K(G, r) \) as \( H^n(G, r; A) \equiv H^n(K(G, r), A) \). In the companion paper [5] we show that, for any commutative monoid \( M \), there are isomorphisms
\[
H^n(M, r; A) \cong H^n(\mathcal{Z} M, r; A),
\]
where \( H^n(\mathcal{Z} M, A), n \geq 0, \) are Gabriel-Zisman cohomology groups [10, Appendix II] of the
underlying simplicial set of the simplicial monoid \( \mathcal{Z} M \), obtained by iterating the \( \mathcal{W} \) construction
on the constant simplicial monoid defined by \( M \).

An analysis of the complex \( B(\mathcal{Z} M) \), for \( M \) any commutative monoid, leads us in Proposition 6 to identify the cohomology groups \( H^n(M, 1; A) \) with the standard cohomology groups
\( H^n(M, A) \) by Leech [16]. Recall that Leech cohomology groups of a (not necessarily commutative)
monoid \( M \) take coefficients in abelian group valued functors on the category \( \mathbb{D}M \), whose
objects are the elements of \( M \) and arrows triples \( (x, y, z) : y \to xyz \). When the monoid \( M \)
is commutative, there is a natural functor \( \mathbb{D}M \to \mathbb{H}M \) which is the identity on objects and
carries a morphism \( (x, y, z) \) of \( \mathbb{D}M \) to the morphism \( (y, xz) \) of \( \mathbb{H}M \). Via this functor, every
\( \mathbb{H}M \)-module \( \mathcal{A} \) is regarded as a \( \mathbb{D}M \)-module and we prove that, for any commutative monoid
\( M \) and \( \mathbb{H}M \)-module \( \mathcal{A} \), there are natural isomorphisms
\[
H^n(M, 1; A) \cong H^n(M, A), \quad n = 0, 1, \ldots .
\]
For any \( r \geq 2 \), we show explicit descriptions of the complexes \( \mathcal{B}'(ZM) \) truncated at dimensions \( \leq r + 3 \), which are useful both for theoretical and computational interests concerning the cohomology groups \( H^n(M, r; A) \) for \( n \leq r + 2 \). Some conclusions here summarize as follows:

- \( H^0(M, r; A) \cong H^0(M, 1; A) \cong H^0(M, A) \cong A(e) \),
- \( H^n(M, r; A) = 0 \), for \( 0 < n < r \),
- \( H^1(M, r; A) \cong H^1(M, 1; A) \cong H^1(M, A) \),
- \( H^{r+1}(M, r; A) \cong H^{r+1}(M, 2; A) \cong H^2(M, A) \).

where \( H^n(M, A) \) denotes the \( n \)-th cohomology group by Grillet \cite{grillet1951,grillet1952}.

- \( H^4(M, 2; A) \cong H^2(M, A) \),
- \( H^{r+2}(M, r; A) \cong H^5(M, 3; A) \), for \( r \geq 3 \).

There are natural inclusions \( H^3(M, A) \subseteq H^5(M, 3; A) \subseteq H^3(M, A) \).

Most of these cohomology groups above have known algebraic interpretations. For example, elements of \( H^1(M, 1; A) = H^1(M, A) \) are derivations \cite{grillet1951} Chapter II, 2.7. Cohomology classes in \( H^2(M, 1; A) = H_2^1(M, A) \) are isomorphism classes of group coextensions \cite{grillet1951} Chapter V, §2 (or \cite{.AddRange2} Theorem 2), while elements of \( H^3(M, 2; A) = H_2^3(M, A) \) classify abelian group coextensions \cite{grillet1951} Chapter V, §4. Cohomology classes in \( H^3(M, 1; A) = H_3^1(M, A) \) are equivalence classes of monoidal abelian groupoids \cite{grillet1951} Theorem 4.31, elements of \( H^4(M, 2; A) = H_3^2(M, A) \) are equivalence classes of braided monoidal abelian groupoids \cite{grillet1951} Theorem 4.51, and elements of \( H^3(M, A) \) are equivalence classes of strictly commutative monoidal abelian groupoids \cite{grillet1951} Theorem 3.11. Thus, among them, only the cohomology groups \( H^3(M, 3; A) \) are pending of interpretation, and we solve this in Section \ref{section6}. Here we give a natural interpretation of the cohomology classes in \( H^5(M, 3; A) \) in terms of equivalence classes of symmetric monoidal abelian groupoids, that is, groupoids \( \mathcal{M} \), whose isotropy groups \( \text{Aut}_\mathcal{M}(x) \) are all abelian, endowed with a monoidal structure by a tensor functor \( \otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \), a unit object \( I \), and coherent associativity, unit and commutativity constraints \( a : (x \otimes y) \otimes z \cong x \otimes (y \otimes z) \), \( l : I \otimes x \cong x \), and \( c : x \otimes y \cong y \otimes x \) which satisfy the symmetry condition \( c^2 = id \). The classification of symmetric monoidal abelian groupoids we give extends that, above refereed, by Sinh in \cite{sinh1994} II, Proposition 5 for Picard categories.

In last Section \ref{section7} we compute the cohomology groups \( H^n(M, r; A) \), for \( n \leq r + 2 \), when \( M \) is any cyclic monoid.

2. Commutative differential graded algebras over \( \mathbb{H}M \)

Throughout this paper \( M \) denotes a commutative multiplicative monoid, whose unit is \( e \).

As noted in the introduction, in \cite{grillet1951} Chapter XII, §2 Grillet observes that the category of abelian group objects in the slice category of commutative monoids over \( M \), \( \text{CMon}_M \downarrow M \), is equivalent to the category of abelian group valued functors on the small category \( \mathbb{H}M \), whose object set is \( M \) and arrow set \( M \times M \), where \( (x, y) : x \to xy \). Composition is given by \( (xy, z)(x, y) = (x, yz) \), and the identity of an object \( x \) is \( (x, e) \). The category of functors from \( \mathbb{H}M \) into the category of abelian groups will be denoted by \( \mathbb{H}M\text{-Mod} \)

and called the category of \( \mathbb{H}M \)-modules. An \( \mathbb{H}M \)-module, say \( \mathcal{A} : \mathbb{H}M \to \text{Ab} \), then consists of abelian groups \( \mathcal{A}(x) \), one for each \( x \in M \), and homomorphisms \( y_* : \mathcal{A}(x) \to \mathcal{A}(xy) \), one for each \( x, y \in M \), such that, for any \( x, y, z \in M, y_*(yz) = (yz)_* : \mathcal{A}(x) \to \mathcal{A}(xyz) \) and, for any \( x \in M, e_* = id_{\mathcal{A}(x)} : \mathcal{A}(x) \to \mathcal{A}(xe) = \mathcal{A}(x) \).
For instance, let

\[(3) \quad Z : \mathbb{H}M \to \mathbb{Ab}, \quad x \mapsto Z(x) = \mathbb{Z}\{x\},\]

be the \(\mathbb{H}M\)-module which associates to each element \(x \in M\) the free abelian group on the generator \(x\), and to each pair \((x, y)\) the isomorphism of abelian groups \(y_* : Z(x) \to Z(xy)\) given on the generator by \(y_*x = xy\). This is isomorphic to the \(\mathbb{H}M\)-module defined by the constant functor on \(\mathbb{H}M\) which associates the abelian group of integers \(\mathbb{Z}\) to any \(x \in M\).

For two \(\mathbb{H}M\)-modules \(A\) and \(B\), a morphism between them (i.e., a natural transformation) \(f : A \to B\) consists of homomorphisms \(f_x : A(x) \to B(x)\), such that, for any \(x, y \in M\), the square below commutes.

\[
\begin{array}{ccc}
A(x) & \xrightarrow{f_x} & B(x) \\
\downarrow{y_*} & & \downarrow{y_*} \\
A(xy) & \xrightarrow{f_{xy}} & B(xy)
\end{array}
\]

The category of \(\mathbb{H}M\)-modules is abelian and we refer to [19, Chapter IX, §3] for details. Recall that the set of morphisms between two \(\mathbb{H}M\)-modules \(A\) and \(B\), denoted by \(\text{Hom}_{\mathbb{H}M}(A, B)\), is an abelian group by objectwise addition, that is, if \(f, g : A \to B\) are morphisms, then \(f + g : A \to B\) is defined by setting \((f + g)_x = f_x + g_x\), for each \(x \in M\). The zero \(\mathbb{H}M\)-module is the constant functor \(0 : \mathbb{H}M \to \mathbb{Ab}\) defined by the trivial abelian group 0, and the direct sum of two \(\mathbb{H}M\)-modules \(A\) and \(B\) is given by taking direct sum at each object, that is, \((A \oplus B)(x) = A(x) \oplus B(x)\).

Similarly, all limits and colimits (in particular, kernels, images, cokernels, etc.) in the category \(\mathbb{H}M\)-Mod are pointwise constructed.

**Remark 2.1.** Every abelian group \(A\) defines a constant \(\mathbb{H}M\)-module, equally denoted by \(A\), such that \(A(x) = A\) and \(y_* = \text{id}_A : A(x) \to A(xy)\), for any \(x, y \in M\). In this way, the category of abelian groups becomes a full subcategory of the category of \(\mathbb{H}M\)-modules.

When \(M = G\) is an abelian group, then this inclusion \(\mathbb{Ab} \hookrightarrow \mathbb{H}G\text{-Mod}\) is actually an equivalence of categories. In the other direction, we have the functor associating to each \(\mathbb{H}G\)-module \(A\) the abelian group \(A(e)\), and there is natural isomorphism of \(\mathbb{H}G\)-modules \(A \cong A(e)\) whose component at each \(x \in G\) is the isomorphism of abelian groups \(x_*^{-1} : A(x) \to A(e)\).

### 2.1. Tensor product of \(\mathbb{H}M\)-modules

For any two \(\mathbb{H}M\)-modules \(A\), \(B\), their **tensor product**, denoted by \(A \otimes_{\mathbb{H}M} B\), is the \(\mathbb{H}M\)-module defined as follows: It attaches to any \(x \in M\) the abelian group defined by the coequalizer sequence of homomorphisms

\[
\bigoplus_{uvw = x} \mathbb{Z}(u) \otimes A(v) \otimes B(w) \xrightarrow{\phi} \bigoplus_{zt = x} A(z) \otimes B(t) \xrightarrow{\psi} (A \otimes_{\mathbb{H}M} B)(x),
\]

where, for any two abelian groups \(A\) and \(B\), \(A \otimes B\) denotes their tensor product as \(\mathbb{Z}\)-modules, the direct sum on the left is taken over all triples \((u, v, w) \in M^3\) such that \(uvw = x\), the direct sum on the middle is over all pairs \((z, t) \in M^2\) with \(zt = x\), and the homomorphisms \(\phi\) and \(\psi\) are defined by

\[
\phi(u \otimes a_v \otimes b_w) = u_* a_v \otimes b_w \in A(uv) \otimes B(w),
\]

\[
\psi(u \otimes a_v \otimes b_w) = a_v \otimes u_* b_w \in A(v) \otimes B(uw),
\]

for all \(u, v, w \in M\) with \(uvw = x\), \(a_v \in A(v)\), and \(b_w \in B(w)\). For any pair \((x, y) \in M^2\), the homomorphism

\[
y_* : (A \otimes_{\mathbb{H}M} B)(x) \to (A \otimes_{\mathbb{H}M} B)(xy)
\]

is given on generators by

\[
y_*(a_z \otimes b_t) = y_* a_z \otimes b_t = a_z \otimes y_* b_t, \quad (a_z \in A(z), b_t \in B(t), zt = x).
\]
If \( f : A \to A' \) and \( g : B \to B' \) are morphisms of \( \mathbb{H}M \)-modules, then there is an induced one \( f \otimes g : A \otimes_{\mathbb{H}M} B \to A' \otimes_{\mathbb{H}M} B' \) such that, for each \( x \in M \), the homomorphism
\[
(f \otimes g)_x : (A \otimes_{\mathbb{H}M} B)(x) \to (A' \otimes_{\mathbb{H}M} B')(x)
\]
is given on generators by
\[
(f \otimes g)_x(a_z \otimes b_t) = f_z a_z \otimes g_t b_t, \quad (a_z \in A(z), b_t \in B(t), zt = x).
\]
Thus, we have a distributive tensor functor
\[
- \otimes_{\mathbb{H}M} - : \mathbb{H}M\text{-Mod} \times \mathbb{H}M\text{-Mod} \to \mathbb{H}M\text{-Mod}.
\]
Further, there are canonical isomorphisms of \( \mathbb{H}M \)-modules
\[
I_A : \mathbb{Z} \otimes_{\mathbb{H}M} A \cong A, \quad c_{A,B} : A \otimes_{\mathbb{H}M} B \cong B \otimes_{\mathbb{H}M} A,
\]
respectively defined by the formulas
\[
I_{zt}(z \otimes a_t) = z a_t, \quad c_{zt}(a_z \otimes b_t) = b_t \otimes a_z, \quad a_{pzt}(a_y \otimes (b_z \otimes c_t)) = (a_y \otimes b_z) \otimes c_t,
\]
which are easily proven to be natural and coherent in the sense of [17, Theorem 5.1]. Therefore, \( \mathbb{H}M\text{-Mod} \) is a symmetric monoidal category. We will usually treat the constraints above as identities, so we think of \( \mathbb{H}M\text{-Mod} \) as a symmetric strict monoidal category.

2.2. Tensor product of complexes of \( \mathbb{H}M \)-modules. The (positive) complexes of \( \mathbb{H}M \)-modules
\[
A = \cdots \to A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0
\]
and the morphisms between them also form an abelian symmetric monoidal category, where the tensor product \( A \otimes_{\mathbb{H}M} B \) of two complexes of \( \mathbb{H}M \)-modules \( A \) and \( B \) is the graded \( \mathbb{H}M \)-module whose component of degree \( n \) is
\[
(A \otimes_{\mathbb{H}M} B)_n = \bigoplus_{p+q=n} A_p \otimes_{\mathbb{H}M} B_q,
\]
and whose differential \( \partial^\otimes \), at any \( x \in M \),
\[
\partial^\otimes_x : (A \otimes_{\mathbb{H}M} B)_n(x) \to (A \otimes_{\mathbb{H}M} B)_{n-1}(x),
\]
is defined on generators by the Leibniz formula
\[
\partial^\otimes_x (a_z \otimes b_t) = \partial_x a_z \otimes b_t + (-1)^p a_z \otimes \partial_x b_t.
\]
for all \( z, t \in M \) such that \( zt = x \), \( a_z \in A_p(z) \), \( b_t \in B_q(t) \), and \( p, q \geq 0 \) such that \( p + q = n \).

In this monoidal category, the unit object is \( \mathbb{Z} \), defined in [38], regarded as a complex concentrated in degree zero. The structure constraints
\[
I_A : \mathbb{Z} \otimes_{\mathbb{H}M} A \cong A, \quad c_{A,B} : A \otimes_{\mathbb{H}M} B \cong B \otimes_{\mathbb{H}M} A,
\]
are respectively defined by the formulas
\[
I_{xy}(x \otimes a_y) = x a_y, \quad c_{xy}(a_x \otimes b_y) = (-1)^q b_y \otimes a_x, \quad a_{xyz}(a_x \otimes (b_y \otimes c_z)) = (a_x \otimes b_y) \otimes c_z,
\]
for any \( x, y, z \in M \), \( a_x \in A_p(x) \), \( b_y \in B_q(y) \), and \( c_z \in C_r(z) \). As for \( \mathbb{H}M \)-modules, we will treat these constraints as identities.
2.3. Commutative differential graded algebras over $\mathbb{H}M$. A commutative differential graded algebra (DG-algebra) $A$ over $\mathbb{H}M$ is defined to be a commutative monoid in the symmetric monoidal category of complexes of $\mathbb{H}M$-modules, see [13] Chapter VII, §3. Hence, $A$ is a complex of $\mathbb{H}M$-modules equipped with a multiplication morphism of complexes $\circ : A \otimes_{\mathbb{H}M} A \to A$ satisfying the associativity $\circ(\circ \otimes id) = (id \otimes \circ)$ and the commutativity $\circ c = 0$, and a unit morphism of complexes $\iota : \mathbb{Z} \to A$ satisfying $\circ(\iota \otimes id_A) = I_A$. We write

$$1 = \iota_x(e) \in A_0(e)$$

and, for any $x, y \in M$, $a_x \in A_p(x)$, and $a_y \in A_q(y)$,

$$a_x \circ a_y = \circ_{xy}(a_x \otimes a_y) \in A_{p+q}(xy),$$

so that the algebra structure on the complex $A$ gives us multiplication homomorphisms of abelian groups

$$A_p(x) \otimes A_q(y) \to A_{p+q}(xy), \quad a_x \otimes a_y \mapsto a_x \circ a_y,$$

and a unit $1 \in A_0(e)$, satisfying

\begin{align}
(5) \quad x_x a_y \circ a_z &= x_x (a_y \circ a_z) = a_y \circ x_x a_z, \\
(6) \quad a_x \circ a_y &= (-1)^{pq} a_y \circ a_x, \\
(7) \quad 1 \circ a_x &= a_x = a_x \circ 1, \\
(8) \quad a_x \circ (a_y \circ a_z) &= (a_x \circ a_y) \circ a_z, \\
(9) \quad \partial_{xy}(a_x \circ a_y) &= \partial_x a_x \circ a_y + (-1)^p a_x \circ \partial_y a_y,
\end{align}

for all $x, y, z \in M$, $a_x \in A_p(x)$, $a_y \in A_q(y)$, and $a_z \in A_r(z)$.

In these terms, a morphism $f : A \to B$ of commutative DG-algebras over $\mathbb{H}M$ is a morphism of complexes of $\mathbb{H}M$-modules such that $f_{xy}(a_x \circ a_y) = f_x a_x \circ f_y a_y$, and $f_e(1) = 1$.

The category of commutative DG-algebras over $\mathbb{H}M$ is symmetric monoidal. The tensor product of two of them $A \otimes_{\mathbb{H}M} B$ is given by their tensor product as complexes of $\mathbb{H}M$-modules endowed with multiplication such that, for $u, v, x, y \in M$, $a_u \in A_p(u)$, etc.,

$$(a_u \otimes b_x) \circ (a_y \otimes b_z) = (a_u \circ a_y) \otimes (b_x \circ b_z)$$

and with unit $1 \otimes 1 \in (A \otimes_{\mathbb{H}M} B)_0(e)$. Observe that the canonical isomorphisms in (4) are actually of DG-algebras whenever the data $A$, $B$ and $C$ therein are DG-algebras over $\mathbb{H}M$.

Commutative DG-algebras over $\mathbb{H}M$ which are concentrated in degree zero are the same as commutative monoids in the symmetric monoidal category of $\mathbb{H}M$-modules, and they are simply called algebras over $\mathbb{H}M$ or $\mathbb{H}M$-algebras. For example, $\mathbb{Z}$ is an $\mathbb{H}M$-algebra with multiplication the unit constraint $I : \mathbb{Z} \otimes_{\mathbb{H}M} \mathbb{Z} \cong \mathbb{Z}$ and unit the identity $id : \mathbb{Z} \to \mathbb{Z}$. In other words, $\mathbb{Z}$ is an $\mathbb{H}M$-algebra whose unit is $e \in \mathbb{Z}e$ and whose multiplication homomorphisms $\mathbb{Z}(x) \otimes \mathbb{Z}(y) \to \mathbb{Z}(xy)$ are given by $mx \circ ny = (mn)xy$, where $mn$ is multiplication of $m$ and $n$ in the ring $\mathbb{Z}$.

The augmented case is relevant. A commutative differential graded augmented algebra (DGA-algebra) $A$ over $\mathbb{H}M$ is a commutative DG-algebra over $\mathbb{H}M$ as above equipped with a homomorphism of commutative DG-algebras (the augmentation) $\epsilon : A \to \mathbb{Z}$. Such an augmentation is entirely determined by its component of degree 0, which is a morphism of $\mathbb{H}M$-algebras $\epsilon : A_0 \to \mathbb{Z}$ such that $\epsilon \partial = 0$. Morphisms of commutative DGA-algebras over $\mathbb{H}M$ are those of commutative DG-algebras which are compatible with the augmentations (i.e., $\epsilon f = \epsilon$).

Remark 2.2. When $M = G$ is a group, the equivalence between the category of abelian groups and the category of $\mathbb{H}G$-modules, described in Remark 2.1, is symmetric monoidal and, therefore, produces an equivalence between the category of commutative DGA-rings and the category of commutative DGA-algebras over $\mathbb{H}G$. Thus every commutative DGA-ring $A$ defines a constant commutative DGA-algebra over $\mathbb{H}G$, equally denoted by $A$, and each commutative DGA-algebra
over $\mathbb{H}G$, $\mathcal{A}$, gives rise to the DGA-ring $\mathcal{A}(e)$, which comes with a natural isomorphism of DGA-algebras $\mathcal{A} \cong \mathcal{A}(e)$ whose component at each $x \in G$ is the isomorphism of augmented chain complexes $x^{-1}_*: \mathcal{A}(x) \to \mathcal{A}(e)$.

3. The Bar construction on commutative DGA-algebras over $\mathbb{H}M$

Let $\mathcal{A}$ be any given commutative DGA-algebra over $\mathbb{H}M$. As we explain below, $\mathcal{A}$ determines a new commutative DGA-algebra over $\mathbb{H}M$, denoted by $\mathcal{B}(\mathcal{A})$ and called the bar construction on $\mathcal{A}$.

Previously to describe $\mathcal{B}(\mathcal{A})$, let us introduce complexes of $\mathbb{H}M$-modules $\bar{\mathcal{A}}$, $\mathcal{S}\bar{\mathcal{A}}$, and $T^p\mathcal{S}\bar{\mathcal{A}}$ for each integer $p \geq 0$, and a double complex of $\mathbb{H}M$-modules $T^\bullet \mathcal{S}\bar{\mathcal{A}}$, as follows:

The reduced complex $\bar{\mathcal{A}} = \cdots \to \bar{\mathcal{A}}_2 \xrightarrow{\partial} \bar{\mathcal{A}}_1 \xrightarrow{\partial} \bar{\mathcal{A}}_0$ is defined to be the cokernel of the unit morphism $\iota: \mathbb{Z} \to \mathcal{A}$. That is, $\bar{\mathcal{A}} = \cdots \to \mathcal{A}_2 \xrightarrow{\partial} \mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0/\mathbb{Z}$. Note that $\iota$ embeds $\mathbb{Z}$ as a direct summand of the underlying complex $\mathcal{A}$, since, being $\iota: \mathcal{A} \to \mathcal{S}$ the augmentation, $\epsilon = id_{\mathcal{Z}}$. We will use below the following notation: For any $x \in \mathcal{A}$, let $\bar{\epsilon}(x)$ be any given commutative DGA-algebra over $\mathbb{H}A$. Previously to describe $\mathcal{B}(\mathcal{A})$, let us introduce complexes of $\mathbb{H}M$-modules $\bar{\mathcal{A}}$, $\mathcal{S}\bar{\mathcal{A}}$, and $T^p\mathcal{S}\bar{\mathcal{A}}$ for each integer $p \geq 0$, and a double complex of $\mathbb{H}M$-modules $T^\bullet \mathcal{S}\bar{\mathcal{A}}$, as follows:

The reduced complex $\bar{\mathcal{A}} = \cdots \to \bar{\mathcal{A}}_2 \xrightarrow{\partial} \bar{\mathcal{A}}_1 \xrightarrow{\partial} \bar{\mathcal{A}}_0$ is defined to be the cokernel of the unit morphism $\iota: \mathbb{Z} \to \mathcal{A}$. That is, $\bar{\mathcal{A}} = \cdots \to \mathcal{A}_2 \xrightarrow{\partial} \mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0/\mathbb{Z}$. Note that $\iota$ embeds $\mathbb{Z}$ as a direct summand of the underlying complex $\mathcal{A}$, since, being $\iota: \mathcal{A} \to \mathcal{S}$ the augmentation, $\epsilon = id_{\mathcal{Z}}$. We will use below the following notation: For any $x \in \mathcal{A}$, let $\bar{\epsilon}(x)$ be any given commutative DGA-algebra over $\mathbb{H}A$. Previously to describe $\mathcal{B}(\mathcal{A})$, let us introduce complexes of $\mathbb{H}M$-modules $\bar{\mathcal{A}}$, $\mathcal{S}\bar{\mathcal{A}}$, and $T^p\mathcal{S}\bar{\mathcal{A}}$ for each integer $p \geq 0$, and a double complex of $\mathbb{H}M$-modules $T^\bullet \mathcal{S}\bar{\mathcal{A}}$, as follows:

The reduced complex $\bar{\mathcal{A}} = \cdots \to \bar{\mathcal{A}}_2 \xrightarrow{\partial} \bar{\mathcal{A}}_1 \xrightarrow{\partial} \bar{\mathcal{A}}_0$ is defined to be the cokernel of the unit morphism $\iota: \mathbb{Z} \to \mathcal{A}$. That is, $\bar{\mathcal{A}} = \cdots \to \mathcal{A}_2 \xrightarrow{\partial} \mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0/\mathbb{Z}$. Note that $\iota$ embeds $\mathbb{Z}$ as a direct summand of the underlying complex $\mathcal{A}$, since, being $\iota: \mathcal{A} \to \mathcal{S}$ the augmentation, $\epsilon = id_{\mathcal{Z}}$. We will use below the following notation: For any $x \in \mathcal{A}$, let $\bar{\epsilon}(x)$ be any given commutative DGA-algebra over $\mathbb{H}A$.
The double complex of $\mathbb{H}M$-modules

$$T^*SA = \cdots \rightarrow T^2SA \xrightarrow{\partial} T^1SA \xrightarrow{\partial} T^0SA$$

is then constructed, thanks to the multiplication $\circ$ in $A$, by the morphisms of complexes of $\mathbb{H}M$-modules $\partial^0 : T^pSA \rightarrow T^{p-1}SA$, which are of degree $-1$ (so that $\partial^0 \partial^0 = -\partial^0 \partial^0$) and defined, at any $x \in M$, by the homomorphisms

$$\partial^0 : (T^pSA)_n(x) \rightarrow (T^{p-1}SA)_{n-1}(x)$$

given on generators as in (11) by

$$\partial^0[a_{x_1}] \cdots [a_{x_p}] = \varepsilon_{x_1}(a_{x_1}) x_{1*}[a_{x_2}] \cdots [a_{x_p}]$$

$$+ \sum_{i=1}^{p-1} (-1)^{e_i} [a_{x_1}] \cdots [a_{x_{i-1}}] [a_{x_i} \circ a_{x_{i+1}}] [a_{x_{i+1}}] \cdots [a_{x_p}]$$

$$+ (-1)^{e_p} \varepsilon_{x_p}(a_{x_p}) x_{p*}[a_{x_1}] \cdots [a_{x_{p-1}}]$$

(recall the notation $\varepsilon$ from (10), and note that the first (resp. last) summand in the above formula is zero whenever the degree $r_1$ of $a_{x_1}$ in the chain complex $A(x_1)$ (resp. $(r_p$ of $a_{x_p}$)) is higher than zero).

All in all, we are now ready to present the bar construction $B(A)$. As a graded $\mathbb{H}M$-module

$$B(A) = \cdots \rightarrow B(A)_2 \xrightarrow{\partial} B(A)_1 \xrightarrow{\partial} B(A)_0$$

is defined by the $\mathbb{H}M$-modules

$$B(A)_n = \bigoplus_{p \geq 0} (T^pSA)_n.$$ 

Notice that $\partial^0 B(A)_n \subseteq B(A)_{n-1}, \partial^0 B(A)_n \subseteq B(A)_n - 1$, and that $(\partial^0 + \partial^0)^2 = 0$. Thus, $B(A)$ becomes a complex of $\mathbb{H}M$-modules with differential

$$\partial = \partial^0 + \partial^0 : B(A)_n \rightarrow B(A)_{n-1}.$$ 

**Proposition 3.1.** $B(A)$ is a commutative DGA-algebra over $\mathbb{H}M$, with multiplication

$$\circ : B(A) \otimes_{\mathbb{H}M} B(A) \rightarrow B(A)$$

defined, for integers $m, n \geq 0$ and $x, y \in M$, by the homomorphisms of abelian groups

$$\circ : B(A)_m(x) \otimes B(A)_n(y) \rightarrow B(A)_{m+n}(xy)$$

given by the shuffle products

$$[a_{x_1}] \cdots [a_{x_p}] \circ [a_{x_{p+1}}] \cdots [a_{x_{p+q}}] = \sum_{\sigma} (-1)^{e(\sigma)} [a_{x_{\sigma(1)}}] \cdots [a_{x_{\sigma(p+q)}}]$$

for any $x_i \in M$ and $a_{x_i} \in A_{r_i}(x_i), i = 1, \ldots, p+q$, such that $x_1 \cdots x_p = x, x_{p+1} \cdots x_{p+q} = y, p + \sum_{i=1}^{p} r_i = m, \text{ and } q + \sum_{j=1}^{q} r_{p+j} = n$, where the sum is taken over all $(p,q)$-shuffles $\sigma$ and, for each $\sigma$, the exponent of the sign is $e(\sigma) = \sum_{i=1}^{p} (1 + r_i) (1 + r_{p+j})$ summed over all pairs $(i, p+j)$ such that $\sigma(i) > \sigma(p+j)$.

The unit is $[1] \in B(A)_0(e)$, that is, the unit morphism $1 : \mathbb{Z} \rightarrow B(A)$ is the isomorphism of $\mathbb{H}M$-modules $1 : \mathbb{Z} \cong B(A)_0$ given by $1_x(x) = x_*$, for any $x \in M$, and the augmentation $\epsilon : B(A) \rightarrow \mathbb{Z}$ is defined by the isomorphism of $\mathbb{H}M$-modules $\epsilon = \epsilon^{-1} : B(A)_0 \cong \mathbb{Z}$ such that $\epsilon_x(x_*) = x$, for any $x \in M$.

**Proof.** We give an indirect proof, by using that the category of $\mathbb{H}M$-modules is closely related to the category $ZM$-Mod, of ordinary modules over the monoid ring $\mathbb{Z}M$.

There is an exact faithful functor $\Gamma : \mathbb{H}M$-Mod $\rightarrow ZM$-Mod, which carries any $\mathbb{H}M$-module $A$ to the $ZM$-module defined by the abelian group $\Gamma A = \bigoplus_{x \in M} A(x)$, with $M$-action of an
element \( y \in M \) on an element \( a_x \in \mathcal{A}(x) \) given by \( y a_x = y_s a_x \in \mathcal{A}(xy) \). This functor \( \Gamma \) is left adjoint to the functor which associates to any \( ZM \)-module \( A \) the constant on objects \( \mathbb{H}M \)-module defined by the underlying abelian group \( A \), with \( y_s : A \to A \), for any \( y \in M \), the homomorphism of multiplication by \( y \) \([15]\).

It is plain to see that \( \Gamma \) is a symmetric strict monoidal functor, that is, \( \Gamma Z = ZM \), for any \( \mathbb{H}M \)-modules \( A \) and \( B \), \( \Gamma (A \otimes_{\mathbb{H}M} B) = \Gamma A \otimes_{ZM} \Gamma B \), and it carries the associativity, unit, and commutativity constraints of the monoidal category of \( \mathbb{H}M \)-modules to the corresponding ones of the category of \( ZM \)-modules. Then, the same properties hold for the induced functor \( \Gamma \) from the symmetric monoidal category of complexes of \( \mathbb{H}M \)-modules to the the symmetric monoidal category of complexes of \( ZM \)-modules. It follows that \( \Gamma \) transform commutative monoids in the category of complexes of \( \mathbb{H}M \)-modules (i.e. commutative DG-algebras over \( \mathbb{H}M \)) to commutative monoids in the category of \( ZM \)-modules (i.e., commutative DG-algebras over \( ZM \)), and therefore \( \Gamma \) also transform commutative DGA-algebras over \( \mathbb{H}M \) to commutative DGA-algebras over the monoid ring \( ZM \).

Now, given \( A \), a commutative DGA-algebra over \( \mathbb{H}M \), let \( B(\Gamma A) \) be the commutative DGA-algebra over \( ZM \) obtained by applying the ordinary Eilenberg-Mac Lane bar construction on \( \Gamma A \) \([19\text{ Chapter X, Theorem 12.1}]\). A direct comparison shows that \( B(\Gamma A) = \Gamma B(A) \) as complexes of \( ZM \)-modules, and also that its multiplication, unit, and augmentation are, respectively, just the morphisms

\[
\begin{align*}
B(\Gamma A) \otimes_{ZM} B(\Gamma A) & \xrightarrow{\Gamma \otimes \Gamma} \Gamma B(A) \otimes_{ZM} B(\Gamma A) = B(\Gamma A) , \\
ZM = \Gamma Z & \xrightarrow{\Gamma \circ} \Gamma B(A) = B(\Gamma A), \quad B(\Gamma A) = \Gamma B(A) \xrightarrow{\Gamma} \Gamma Z = ZM.
\end{align*}
\]

Then, as \( B(\Gamma A) \) is actually a commutative DGA-algebra over \( ZM \), it follows that the equalities

\[
\begin{align*}
\Gamma(\circ (\circ \otimes id_{B(A)})) &= \Gamma(\circ (id_{B(A)} \otimes \circ)), \quad \Gamma(\circ c_{B(A), B(A)}) = \Gamma\circ, \\
\Gamma(\circ (\circ \otimes id_{B(A)})) &= \Gamma l_{B(A)}, \quad \Gamma(\circ (\circ \otimes \epsilon)) = \Gamma(\epsilon), \quad \Gamma(\epsilon) = \Gamma id_{\mathbb{Z}}.
\end{align*}
\]

hold. Therefore, the result, that is, that \( B(\Lambda A) \) is a commutative DGA-algebra over \( \mathbb{H}M \), follows since the functor \( \Gamma \) is faithful. \( \square \)

**Remark 3.2.** Observe, as in \([2\text{ §7}]\), that the shuffle product \( \circ \) on \( B(A) \) can also be expressed by the recursive formula below, where \( \alpha = [a_{x_1}] \cdots [a_{x_p}] \in B(A)_r(x), \beta = [b_{y_1}] \cdots [b_{y_q}] \in B(A)_s(y), a_x \in \mathcal{A}_n(z) \) and \( b_t \in \mathcal{A}_n(t) \).

\[
[\alpha \mid a_z] \circ [\beta \mid b_t] = [[\alpha \mid a_z] \circ \beta \mid b_t] + (-1)^{r(s+1)}[\alpha \circ [\beta \mid b_t] \mid a_z]
\]

Let us stress the *suspension* morphism of complexes of \( \mathbb{H}M \)-modules, of degree 1 (hence satisfying \( \partial S = -S \partial \)),

\[
S : \mathcal{A} \to B(A) ,
\]

which is defined, at any \( x \in M \), by \( S_x a_x = [a_x] \in B(A)(x) \), for any chain \( a_x \) of \( \mathcal{A}(x) \).

Such as Mac Lane did in \([19\text{ Chapter X, §12}]\) for ordinary commutative DGA-algebras over a commutative ring, the cohomology of a commutative DGA-algebra over \( \mathbb{H}M \) can be defined in “stages” or “levels”. If \( A \) is any commutative DGA-algebra over \( \mathbb{H}M \), then \( B(A) \) is again a commutative DGA-algebra over \( \mathbb{H}M \), so an iteration is possible to form \( B^r(A) \) for each integer \( r \geq 1 \). Hence, we define the *\( r \)th level cohomology groups of \( A \) with coefficients in a \( \mathbb{H}M \)-module \( B \), denoted by \( H^r(A, r; B) \), as

\[
H^r(A, r; B) = \text{Hom}_{\mathbb{H}M}(B^r(A), B), \quad n = 0, 1, \ldots ,
\]

where \( \text{Hom}_{\mathbb{H}M}(B^r(A), B) \) is the cochain complex obtained by applying the functor \( \text{Hom}_{\mathbb{H}M}(-, B) \) to the underlying chain complex of \( \mathbb{H}M \)-modules \( B(A) \).
Remark 3.3. When the bar construction above is applied on the constant DGA-algebra over \(\mathbb{H}M\) defined by a commutative DGA-ring \(A\), the result is just the constant DGA-algebra over \(\mathbb{H}M\) defined by the commutative DGA-ring obtained by applying on \(A\) the Eilenberg-Mac Lane reduced bar construction. Hence, the notation \(B(A)\) is not confusing.

If \(\mathcal{A}\) and \(\mathcal{B}\) are commutative DGA-algebras over \(\mathbb{H}M\), then we say that two morphisms of DGA-algebras \(f: \mathcal{A} \to \mathcal{B}\) and \(g: \mathcal{B} \to \mathcal{A}\) form a contraction whenever \(fg = id_{\mathcal{B}}\), and there exists a homotopy of morphisms of complexes \(\Phi: gf \Rightarrow id_{\mathcal{B}}\) satisfying the conditions

\[
\Phi g = 0, \ f\Phi = 0, \ \Phi \Phi = 0. 
\]

Paralleling the proof by Eilenberg and MacLane of [7] Theorem 12.1, one proves the following:

Lemma 3.4. If \(f: \mathcal{A} \to \mathcal{B}\) and \(g: \mathcal{B} \to \mathcal{A}\) form a contraction of commutative DGA-algebras over \(\mathbb{H}M\), then the induced \(B(f): B(\mathcal{A}) \to B(\mathcal{B})\) and \(B(g): B(\mathcal{B}) \to B(\mathcal{A})\) also form a contraction.

4. Free \(\mathbb{H}M\)-modules

Let \(\text{Set}_{\downarrow M}\) be the comma category of sets over the underlying set of \(M\); that is, the category whose objects \(S = (S, \pi)\) are sets \(S\) endowed with a map \(\pi: S \to M\), and whose morphisms are maps \(\varphi: S \to T\) such that \(\pi \varphi = \pi\). There is a forgetful functor

\[
\mathcal{U}: \mathbb{H}M\text{-Mod} \to \text{Set}_{\downarrow M},
\]

which carries any \(\mathbb{H}M\)-module \(\mathcal{A}\) to the disjoint union set

\[
\mathcal{U}\mathcal{A} = \bigcup_{x \in M} \mathcal{A}(x) = \{(x, a_x) \mid x \in M, \ a_x \in \mathcal{A}(x)\},
\]

endowed with the projection map \(\pi: \mathcal{U}\mathcal{A} \to M\), \(\pi(x, a_x) = x\). A morphism \(f: \mathcal{A} \to \mathcal{B}\) is sent to the map \(\mathcal{U}f: \mathcal{U}\mathcal{A} \to \mathcal{U}\mathcal{B}\) given by \(\mathcal{U}f(x, a_x) = (x, f_x a_x)\). There is also a free \(\mathbb{H}M\)-module functor

\[
\mathcal{Z}: \text{Set}_{\downarrow M} \to \mathbb{H}M\text{-Mod},
\]

which is defined as follows: If \(S\) is any set over \(M\), then \(\mathcal{Z}S\) is the \(\mathbb{H}M\)-module such that, for each \(x \in M\),

\[
\mathcal{Z}S(x) = \mathbb{Z}\{(u, s) \in M \times S \mid u \pi(s) = x\}
\]

is the free abelian group with generators all pairs \((u, s)\), where \(u \in M\) and \(s \in S\), such that \(u \pi(s) = x\). We usually write \((e, s)\) simply by \(s\); so that each element of \(s \in S\) is regarded as an element \(s \in \mathcal{Z}S(\pi s)\). For any \(x, y \in M\), the homomorphism

\[
y_s: \mathcal{Z}S(x) \to \mathcal{Z}S(xy)
\]

is defined on generators by \(y_s(u, s) = (uy, s)\). If \(\varphi: S \to T\) is any map of sets over \(M\), the induced morphism \(\mathcal{Z}\varphi: \mathcal{Z}S \to \mathcal{Z}T\) is given, at each \(x \in M\), by the homomorphism \((\mathcal{Z}\varphi)_x: \mathcal{Z}S(x) \to \mathcal{Z}T(x)\) defined on generators by \((\mathcal{Z}\varphi)_x(u, s) = (u, \varphi s)\).

Proposition 4.1. The functor \(\mathcal{Z}\) is left adjoint to the functor \(\mathcal{U}\). Thus, for \(S\) any set over \(M\), to each \(\mathbb{H}M\)-module \(\mathcal{A}\) and each list of elements \(a_s \in \mathcal{A}(\pi s)\), one for each \(s \in S\), there is a unique morphism of \(\mathbb{H}M\)-modules \(f: \mathcal{Z}S \to \mathcal{A}\) with \(f_\pi s(a_s) = a_s\) for every \(s \in S\).

Proof. At any set \(S\) over \(M\), the unit of the adjunction is the map

\[
\nu: S \to \mathcal{U}\mathcal{Z}S = \{(x, a_x) \mid x \in M, \ a_x \in \mathcal{Z}S(x)\}, \ s \mapsto (\pi s, s).
\]

If \(\mathcal{A}\) is a \(\mathbb{H}M\)-module and \(\varphi: S \to \mathcal{U}\mathcal{A}\) is any map over \(M\), then, the unique morphism of \(\mathbb{H}M\)-modules \(f: \mathcal{Z}S \to \mathcal{A}\) such that \((\mathcal{U}f)\nu = \varphi\) is determined by the equations \(f_x(u, s) = u \varphi(s)\), for any \(x \in M\) and \((u, s) \in M \times S\) with \(u \pi(s) = x\). \(\square\)
The category $\mathbf{Set} \downarrow M$ has a symmetric monoidal structure, where the tensor product of two sets over $M$, say $S$ and $T$, is the cartesian product set of $S \times T$ with $\pi(s, t) = \pi(s) \pi(t)$. The unit object is provided by the unitary set $\{e\}$ with $\pi(e) = e \in M$, and the associativity, unit, and commutativity constraints are the obvious ones. Hereafter, the category $\mathbf{Set} \downarrow M$ will be considered with this monoidal structure\[1\].

**Proposition 4.2.** The free $\mathbb{H}M$-module functor $\mathbb{H} \rightarrow \mathbf{Set}$ is symmetric monoidal, that is, there are natural and coherent isomorphisms of $\mathbb{H}M$-modules

$$\mathcal{Z}(S \times T) \cong \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T, \quad \mathcal{Z}\{e\} \cong \mathbb{Z},$$

for $S$ and $T$ any sets over $M$.

**Proof.** For $S$, $T$ any given sets over $M$, the isomorphism $f : \mathcal{Z}(S \times T) \cong \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T$ is the morphism of $\mathbb{H}M$-modules such that, for any $(s, t) \in S \times T$, $f_{\pi(s,t)}(s, t) = s \otimes t$. Observe that, for any $x \in M$, the abelian group $\mathcal{Z}(S \times T)(x)$ is free with generators the elements $(u, s, t) = u_*(s, t)$, with $u \in M$, $s \in S$, and $t \in T$, such that $u \pi(s) \pi(t) = x$, while $(\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T)(x)$ is the abelian group generated by the elements $(u, s) \otimes (v, t) = u_*s \otimes v_*t$, with $u, v \in M$, $s \in S$, and $t \in T$, such that $u \pi(s) v \pi(t) = x$, with the relations $u_*s \otimes v_*t = (uv)_*(s \otimes t)$. Then, the homomorphism $f_x : \mathcal{Z}(S \times T)(x) \rightarrow (\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T)(x)$, which acts on elements of the basis by $f_x(u_*(s, t)) = u_*(s \otimes t)$, is clearly an isomorphism of abelian groups.

The isomorphism $f : \mathcal{Z}\{e\} \cong \mathbb{Z}$ is the morphism of $\mathbb{H}M$-modules such that $f_e(e) = e$. Observe that, for any $x \in M$, the isomorphism $f_x$ is the composite

$$\mathcal{Z}\{e\}(x) = \mathcal{Z}\{(u, e) \mid ue = x\} = \mathcal{Z}\{(x, e)\} \cong \mathcal{Z}\{x\} \cong \mathbb{Z}(x).$$

It is straightforward to see that the isomorphisms $f$ above are natural and coherent, so that $\mathcal{Z}$ is actually a symmetric monoidal functor. \[\Box\]

**Corollary 4.3.** For $S$ and $T$ any two sets over $M$, the tensor product $\mathbb{H}M$-module $\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T$ is free on the set of elements $s \otimes t$, $s \in S$, $t \in T$, with $\pi(s \otimes t) = \pi(s) \pi(t)$.

Since the functor $\mathcal{Z}$ is symmetric monoidal, it transports commutative monoids in $\mathbf{Set} \downarrow M$ to commutative monoids in $\mathbb{H}M$-Mod, that is, to algebras over $\mathbb{H}M$. As a commutative monoid in the symmetric monoidal category $\mathbf{Set} \downarrow M$ is merely a commutative monoid over $M$, that is, a commutative monoid $S$ endowed with a homomorphism $\pi : S \rightarrow M$, the corollary below follows.

**Corollary 4.4.** If $S$ is a commutative monoid over $M$, then the free $\mathbb{H}M$-module $\mathcal{Z}S$ is an algebra over $\mathbb{H}M$. The multiplication morphism $\circ : \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}S \rightarrow \mathcal{Z}S$ is the composite

$$\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}S \cong \mathcal{Z}(S \times S) \xrightarrow{Zm} \mathcal{Z}S,$$

where $m : S \times S \rightarrow S$ is the homomorphism of multiplication in $S$, $m(s, s') = ss'$, and the unit morphism $\iota : \mathcal{Z} \rightarrow \mathcal{Z}S$ is the composite $\mathcal{Z} \cong \mathcal{Z}\{e\} \xrightarrow{Z1} \mathcal{Z}S$, where $\iota : \{e\} \rightarrow S$ is the trivial homomorphism mapping the unit of $M$ to the unit of $S$.

5. The cohomology groups $H^n(M, r; A)$

Let us consider the commutative monoid $M$ over itself with $\pi = id_M : M \rightarrow M$. Then, by Corollary 4.3, the free $\mathbb{H}M$-module $\mathcal{Z}M$ is an algebra over $\mathbb{H}M$. Explicitly, this is described as follows: For each $x \in M$,

$$\mathcal{Z}M(x) = \mathcal{Z}\{(u, v) \mid uv = x\}$$

\[1\]The category $\mathbf{Set} \downarrow M$ has a different monoidal structure where the tensor product is given by the fibre-product $S \times_M T$ with $\pi(s, t) = \pi(s) = \pi(t)$. 

is the free abelian group with generators all pairs \((u, v) \in M \times M\) such that \(uv = x\). For any \(x, y \in M\), the homomorphism \(y_* : \mathcal{Z}(x) \to \mathcal{Z}(xy)\) is given on generators by \(y_*(u, v) = (yu, v)\), and the homomorphism of multiplication

\[
\circ : \mathcal{Z}(x) \otimes \mathcal{Z}(y) \to \mathcal{Z}(xy)
\]

is defined on generators by \((u, v) \otimes (w, t) \mapsto (u, v) \circ (w, t) = (uw, vt)\), for any \(u, v, w, t \in M\) such that \(uv = x\) and \(wt = y\). The unit is \((e, e) \in \mathcal{Z}(e)\). We see each element \(x \in M\) as an element of \(\mathcal{Z}(x)\) by means of the identification \(x = (e, x)\), so that any generator \((u, v)\) of \(\mathcal{Z}(x)\) can be write as \(u, v\).

By Proposition 14 if \(A\) is any \(\mathbb{H}M\)-module, for any list of elements \(a_x \in A(x)\), one for each \(x \in M\), there is an unique morphism of \(\mathbb{H}M\)-modules \(f : \mathcal{Z}(M) \to A\) such that each homomorphism \(f_x : \mathcal{Z}(x) \to A(x)\) verifies that \(f_x(x) = a_x\) (explicitly, \(f_x\) acts on generators by \(f_x(u, v) = u, a_x\)). Furthermore, it is plain to see that, if \(A\) is an algebra over \(\mathbb{H}M\), then \(f\) is a morphism of algebras if and only if \(a_x = 1\) and \(a_x \circ a_y = a_{xy}\) for all \(x, y \in M\).

Hereafter, we regard \(\mathcal{Z}(M)\) as a commutative DGA-algebra over \(\mathbb{H}M\) with the trivial grading, that is, with \((\mathcal{Z}(M))_n = 0\) for \(n > 0\) and \((\mathcal{Z}(M))_0 = \mathcal{Z}(M)\), and with augmentation the morphism of \(\mathbb{H}M\)-algebras

\[
\epsilon : \mathcal{Z}(M) \to \mathbb{Z},
\]

such that, for any \(x \in M\), \(\epsilon(x) = x \in \mathbb{Z}(x)\). Then, we define, for each integer \(r \geq 1\), the \(r\)th level cohomology groups of the commutative monoid \(M\) with coefficients in a \(\mathbb{H}M\)-module \(A\) by

\[
H^n(M, r; A) = H^n(\mathcal{Z}(M), r; A), \quad n = 0, 1, \ldots,
\]

or, in other words,

\[
H^n(M, r; A) = H^n(\text{Hom}_{\mathbb{H}M}(B'((\mathcal{Z}(M)), A)),
\]

where \(\text{Hom}_{\mathbb{H}M}(B'((\mathcal{Z}(M)), A)\) is the cochain complex obtained by applying the abelian group valued functor \(\text{Hom}_{\mathbb{H}M}(-, A)\) to the neglected chain complex of \(\mathbb{H}M\)-modules \(B'((\mathcal{Z}(M))\).

Remark 5.1. When \(M = G\) is an abelian group, \(\mathcal{Z}(G)\) is isomorphic to the constant DGA-algebra over \(\mathbb{H}G\) defined by the commutative DGA-ring \(\mathbb{Z}(G)\) (see Remark 29), which is itself isomorphic to the trivially graded DGA-ring defined by the group ring \(\mathbb{Z}[G]\) with augmentation the ring homomorphism \(\alpha : \mathbb{Z}(G) \to \mathbb{Z}\) such that \(\alpha(x) = 1\) for any \(x \in G\). To see this, observe that \(\mathbb{Z}(G)\) is the commutative ring whose underlying abelian group is freely generated by the elements of the form \((x^{-1}, x), \quad x \in G\), with multiplication such that \((x^{-1}, x) \circ (y^{-1}, y) = ((xy)^{-1}, xy)\), and unit \((e, e) = e\). The map \((x^{-1}, x) \mapsto x\) clearly determines a ring isomorphism between \(\mathbb{Z}(G)\) and the group ring \(\mathbb{Z}[G]\), which is compatible with the corresponding augmentations.

Hence, for any integer \(r \geq 1\), \(B'((\mathbb{Z}(G)) \cong B'(\mathbb{Z}(G))\) (see Remark 33) and therefore for any abelian group \(A\), regarded as a constant \(\mathbb{H}G\)-module, there are natural isomorphisms

\[
\text{Hom}_{\mathbb{H}G}(B'((\mathbb{Z}(G)), A) \cong \text{Hom}_{\mathbb{H}M}(B'((\mathcal{Z}(M)), A), A)
\]

showing that the \(r\)th level cohomology groups \(H^n(G, r; A)\) in [10] agree with those by Eilenberg and Mac Lane in [7], which compute the cohomology of the spaces \(K(G, r)\) by means of natural isomorphisms \(H^n(K(G, r), A) \cong H^n(G, r; A)\).

From here on, this section is dedicated to show explicit cochain descriptions for some of these cohomology groups, starting with those of first level

\[
H^n(M, 1; A) = H^n(\text{Hom}_{\mathbb{H}M}(\mathcal{Z}(M), A)).
\]

Let us analyze the underlying complex \(\mathcal{B}(\mathcal{Z}(M))\). For any integer \(n \geq 1\),

\[
\mathcal{B}(\mathcal{Z}(M))_n = \mathcal{Z}(M) \otimes_{\mathbb{H}M} (n \text{ factors}) \otimes_{\mathbb{H}M} \mathcal{Z}(M),
\]

\[2\]The commutative DGA-rings \(B'((\mathbb{Z}(G))\) are denoted by \(A_{\mathbb{Z}}(G, r)\) in [7].
where \( \overline{ZM} = ZM/\langle e \rangle = ZM/\mathbb{Z}\{e\} \cong ZM^* \) is a free \( \mathbb{H}M \)-module on \( M^* = M \setminus \{e\} \) with \( \pi : M^* \to M \) the inclusion map. Then, by construction and Proposition 4.2, we have that

- The \( \mathbb{H}M \)-module \( B(\mathbb{Z}M)_0 \) is free on the unitary set \( \{[1]\} \) with \( \pi[1] = e \) and, for any \( n \geq 1 \), \( B(\mathbb{Z}M)_n \) is a free \( \mathbb{H}M \)-module generated by the set over \( M \) consisting of \( n \)-tuples of elements of \( M \)
  \[ \alpha_n = [x_1|\cdots|x_n], \quad \text{with } \pi\alpha_n = x_1\cdots x_n, \]
  which we call generic \( n \)-cells of \( B(\mathbb{Z}M) \), with the relations \( \alpha_n = 0 \) whenever some \( x_i = e \).

- The differential \( \partial : B(\mathbb{Z}M)_n \to B(\mathbb{Z}M)_{n-1} \) is the morphism of \( \mathbb{H}M \)-modules such that, for each \( x \in M \) and any generic \( n \)-cell \( [x_1|\cdots|x_n] \) with \( x_1\cdots x_n = x \),
  \[
  \partial x [x_1|\cdots|x_n] = x_1 [x_2|\cdots|x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1|\cdots|x_ix_{i+1}|\cdots|x_n] \\
  + (-1)^n x_n [x_1|\cdots|x_{n-1}].
  \]

Hence, Proposition 4.1 gives the following.

**Theorem 5.2.** For any \( \mathbb{H}M \)-module \( \mathcal{A} \), the cohomology groups \( H^n(M,1;\mathcal{A}) \) can be computed as the cohomology groups of the cochain complex of normalized 1st level cochains of \( M \) with values in \( \mathcal{A} \),

\[
(17) \quad C(M,1;\mathcal{A}) : 0 \to C^0(M,1;\mathcal{A}) \xrightarrow{\partial^0} C^1(M,1;\mathcal{A}) \xrightarrow{\partial^1} C^2(M,1;\mathcal{A}) \xrightarrow{\partial^2} \cdots,
\]

where

- \( C^0(M,1;\mathcal{A}) = \mathcal{A}(e) \), and for \( n \geq 1 \), \( C^n(M,1;\mathcal{A}) \) is the abelian group, under pointwise addition, of functions
  \[
  f : M^n \to \bigcup_{x \in M} \mathcal{A}(x)
  \]
  such that \( f(x_1,\ldots,x_n) \in \mathcal{A}(x_1\cdots x_n) \) and \( f(x_1,\ldots,x_n) = 0 \) whenever some \( x_i = e \),

- \( \partial^0 = 0 \), and for \( n \geq 1 \), the coboundary \( \partial^n : C^n(M,1;\mathcal{A}) \to C^{n+1}(M,1;\mathcal{A}) \) is given by
  \[
  (\partial^n f)(x_1,\ldots,x_{n+1}) = x_1 f(x_2,\ldots,x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1,\ldots,x_i x_{i+1},\ldots,x_{n+1}) \\
  + (-1)^{n+1} x_{n+1} f(x_1,\ldots,x_n).
  \]

Let us now recall that the so-called Leech cohomology groups [16] of a (not necessarily commutative) monoid \( M \), which we denote here by \( H^n(M,A) \), take coefficients in \( \mathbb{D}M \)-modules, that is, in abelian group valued functors on the category \( \mathbb{D}M \), whose set of objects is \( M \) and set of arrows \( M \times M \times M \), where \( (x,y,z) : y \to xyz \). Composition is given by \( (u,x,y,v)(x,y,z) = (ux,y,zv) \), and the identity morphism of any object \( x \) is \( (e,e,e) : x \to x \). For any \( \mathbb{D}M \)-module \( \mathcal{A} : \mathbb{D}M \to \mathbb{Ab} \), if we write \( \mathcal{A}(x,y,z) = x^* z^* : \mathcal{A}(y) \to \mathcal{A}(xyz) \), then we see that \( \mathcal{A} \) consists of abelian groups \( \mathcal{A}(x) \), one for each \( x \in M \), and homomorphisms

\[
{x_* : \mathcal{A}(y) \to \mathcal{A}(xy), \quad x^* : \mathcal{A}(y) \to \mathcal{A}(yx),}
\]

for each \( x,y \in M \), such that the equations below hold.

\[
x_* y_* = (xy)_* : \mathcal{A}(z) \to \mathcal{A}(xyz), \quad y^* x^* = (xy)^* : \mathcal{A}(z) \to \mathcal{A}(zxy),
\]
\[
e_* = e^* = id_{\mathcal{A}(x)} : \mathcal{A}(x) \to \mathcal{A}(x), \quad x y^* = y^* x_* : \mathcal{A}(z) \to \mathcal{A}(xyz).
\]

When the monoid \( M \) is commutative, as it is in our case, there is a full functor \( \mathbb{D}M \to \mathbb{H}M \), which is the identity on objects and carries a morphism \( (x,y,z) : y \to xyz \) of \( \mathbb{D}M \) to the morphism \( (y,xz) : y \to xyz \) of \( \mathbb{H}M \). Composition with this functor induces a full embedding of \( \mathbb{H}M \)-Mod into \( \mathbb{D}M \)-mod, whose image consists of the symmetric \( \mathbb{D}M \)-modules, that is, those
satisfying that \( x_a = x^*: A(y) \to A(xy) \), for all \( x, y \in M \) \cite{16} Chapter II, 7.15. Thus, \( H_\mathbb{M} \)-modules and symmetric \( \mathbb{D}M \)-modules are the same thing.

As a direct inspection shows that, for any \( H_\mathbb{M} \)-module \( A \), the cochain complex \( C(M, 1; A) \) in \cite{17} coincides with the standard normalized cochain complex of \( M \) with coefficients in \( A \) by Leech \cite{16} Chapter II, 2.12], next theorem follows.

**Proposition 5.3.** For any \( H_\mathbb{M} \)-module \( A \), there are natural isomorphisms

\[
H^n(M, 1; A) \cong H^n_\mathbb{M}(M, A), \quad n = 0, 1, \ldots
\]

We now analyze the complex of \( H_\mathbb{M} \)-modules \( B'(ZM) \) for \( r \geq 2 \) any integer. By construction,

- \( B'(ZM)_0 \) is the free \( H_\mathbb{M} \)-module on the unitary set consisting of the 0-tuple

\[
[ \ ] \quad \text{with } [ ] = e,
\]

which we call the generic 0-cell of \( B'(ZM) \),

and, for \( n \geq 1, \)

\[
B'(ZM)_n = \bigoplus_{p+\sum n_i = n} B^{r-l}(ZM)_{n_1} \otimes_{H_\mathbb{M}} \cdots \otimes_{H_\mathbb{M}} B^{r-l}(ZM)_{n_p}.
\]

Since \( B^{r-l}(ZM)_0 = 0 \) while, for \( n_i \geq 1, B^{r-l}(ZM)_{n_i} = B^{r-l}(ZM)_{n_i} \), it follows by induction on \( r \) that

- \( B'(ZM)_n = 0 \) for \( 0 < n < r \),

and that, for any \( r \leq n, \)

\[
B'(ZM)_n = \bigoplus_{n_1, \ldots, n_p \geq r - 1, \sum n_i = n} B^{r-l}(ZM)_{n_1} \otimes_{H_\mathbb{M}} \cdots \otimes_{H_\mathbb{M}} B^{r-l}(ZM)_{n_p}.
\]

Then, if we denote by \(| \cdot | \) the symbol \( | \cdot | \) used for the tensor product in the construction of \( B'(ZM) \) from \( B^{r-l}(ZM) \), by Proposition \cite{12} and induction, we see that

- \( B'(ZM)_n, \) for \( r \leq n, \) is a free \( H_\mathbb{M} \)-module generated by the set over \( M \) consisting of all \( p \)-tuples, which we call generic \( n \)-cells of \( B'(ZM) \),

\[
\alpha_n = [\alpha_{n_1}|, \alpha_{n_2}|, \ldots|, \alpha_{n_p}], \quad \text{with } \pi\alpha_n = \pi\alpha_{n_1} \cdots \pi\alpha_{n_p},
\]

of generic \( n \)-cells of \( B^{r-l}(ZM) \), such that \( n_i \geq r - 1 \) and \( p + \sum n_i = n \), with the relations \( \alpha_n = 0 \) whenever some \( \alpha_{n_i} = 0 \).

Let us stress that a generic \( n \)-cell \( \alpha_n \) of any \( B'(ZM) \) is actually a generator of the abelian group \( B'(ZM)_n(\pi\alpha_n) \). Indeed, for each \( x \in M, B'(ZM)_n(x) \) is the free abelian group generated by the elements \( u_\alpha\alpha_n \) with \( u \) an element of \( M \) and the \( \alpha_n \) any non-zero generic \( n \)-cell of \( B'(ZM) \) such that \( u\pi\alpha_n = x \). Arbitrary elements of the groups \( B'(ZM)_n(x) \), are referred as \( n \)-chains of \( B'(ZM) \).

For any \( r \geq 1, \) the multiplication \( \circ \) of \( B'(ZM) \) is given by the morphism of \( H_\mathbb{M} \)-modules

\[
\circ : B'(ZM)_n \otimes_{H_\mathbb{M}} B'(ZM)_m \to B'(ZM)_{n+m}
\]

which, according to Proposition \cite{4.1} are determined on generic cells by the shuffle product

\[
[\alpha_{n_1}|, \ldots, \alpha_{n_p}] \circ [\alpha_{n_{p+1}}|, \ldots, \alpha_{n_{p+q}}] = \sum_{\sigma} (-1)^{e(\sigma)} [\alpha_{n_{\sigma^{-1}(1)}}, \ldots, \alpha_{n_{\sigma^{-1}(p+q)}}],
\]

where the sum is taken over all \( (p, q) \)-shuffles \( \sigma \) and \( e(\sigma) = \sum (1 + n_i)(1 + n_{p+j}) \) summed over all pairs \((i, p + j)\) such that \( \sigma(i) > \sigma(p + j) \). In particular, for \( r = 1, \)

\[
[\alpha_1| \ldots | \alpha_n] \circ [\alpha_{n+1}| \ldots | \alpha_{n+m}] = \sum_{\sigma} (-1)^{e(\sigma)} [\alpha_{\sigma^{-1}(1)}| \ldots | \alpha_{\sigma^{-1}(n+m)}],
\]
where the sum is taken over all \((n, m)\)-shuffles \(\sigma\) and \(e(\sigma)\) is the sign of the shuffle.

Then, for \(r \geq 2\),

- the boundary \(\partial : B'(\mathbb{Z}M)_n \to B'(\mathbb{Z}M)_{n-1}\) is the morphism of \(\mathbb{H}M\)-modules recursively defined, on any generic \(n\)-cell \(\alpha_n = [\alpha_{n_1}, \cdots, \alpha_{n_p}]\) of \(B'(\mathbb{Z}M)\) with \(\pi\alpha_n = x\) and \(\pi\alpha_n = x_i\), by

\[
\partial x \alpha_n = -\sum_{i=1}^p (-1)^{e_i-1}[\alpha_{n_1}, \cdots, \alpha_{n_{i-1}}, \partial x_i \alpha_n, \alpha_{n_{i+1}}, \cdots, \alpha_{n_p}]
\]

\[
+ \sum_{i=1}^{p-1} (-1)^{e_i}[\alpha_{n_1}, \cdots, \alpha_{n_{i-1}}, \alpha_n, \alpha_{n_{i+1}}, \cdots, \alpha_{n_p}],
\]

where the exponents \(e_i\) of the signs are \(e_i = i + \sum n_i\).

In the above formula, the term \(\partial x \alpha_n\), which refers to the differential of \(\alpha_n\) in \(B'^{-1}(\mathbb{Z}M)\), or \(\alpha_n, \alpha_{n+1}\), is not in general a generic cell of \(B'^{-1}(\mathbb{Z}M)\) but a chain; the term is to be expanded by linearity.

Recall now that we have the embedding suspensions \(\mathbb{Z} : B'^{-1}(\mathbb{Z}M) \hookrightarrow B'(\mathbb{Z}M)\), through which we identify any generic \((n-1)\)-cell \(\alpha_{n-1}\) of \(B'^{-1}(\mathbb{Z}M)\) with the generic \(n\)-cell \(S\alpha_{n-1} = [\alpha_{n-1}]\) of \(B'(\mathbb{Z}M)\). Hence, by induction, one proves that any generic \(n\)-cell of any \(B'(\mathbb{Z}M)\) can be uniquely written in the form

\[
\alpha_n = [x_1, x_2, \cdots, x_{m-1}, x_m]
\]

with \(x_i \in M, 1 \leq m, 1 \leq k_i \leq r\), and \(r + \sum_{i=1}^{m-1} k_i = n\). So written, we have \(\pi\alpha_n = x_1 \cdots x_m\), and \(\alpha_n = 0\) if \(x_i = e\) for some \(i\). Observe that if some \(k_i = r\), then \(n \geq 2r\). Indeed, the generic \(n\)-cells of lowest \(n\) appearing in \(B'(\mathbb{Z}M)\) but not in \(B'^{-1}(\mathbb{Z}M)\) are those generic \(2r\)-cells of the form \([x_1, x_2]\). Thus, via the suspension morphism, \(B'^{-1}(\mathbb{Z}M)_{n-1}\) is identified with \(B'(\mathbb{Z}M)_n\) for \(r \leq n < 2r\), while \(B'^{-1}(\mathbb{Z}M)_{n-1} \subsetneq B'(\mathbb{Z}M)_n\) for \(n \geq 2r\). In particular, we have the commutative diagram of suspensions

\[
\begin{array}{ccccccc}
B(\mathbb{Z}M)_4 & \longrightarrow & B(\mathbb{Z}M)_3 & \longrightarrow & B(\mathbb{Z}M)_2 & \longrightarrow & B(\mathbb{Z}M)_1 & \longrightarrow & B(\mathbb{Z}M)_0 \\
\downarrow s & & \downarrow s & & \downarrow s & & \downarrow s & & \downarrow 0 \\
B^2(\mathbb{Z}M)_5 & \longrightarrow & B^2(\mathbb{Z}M)_4 & \longrightarrow & B^2(\mathbb{Z}M)_3 & \longrightarrow & B^2(\mathbb{Z}M)_2 & \longrightarrow & 0 \\
\downarrow s & & \downarrow s & & \downarrow s & & \downarrow s & & \downarrow 0 \\
B^3(\mathbb{Z}M)_6 & \longrightarrow & B^3(\mathbb{Z}M)_5 & \longrightarrow & B^3(\mathbb{Z}M)_4 & \longrightarrow & B^3(\mathbb{Z}M)_3 & \longrightarrow & 0 \\
\downarrow s^{r-3} & & \downarrow s^{r-3} & & \downarrow s^{r-3} & & \downarrow s^{r-3} & & \downarrow 0 \\
B^r(\mathbb{Z}M)_{r+3} & \longrightarrow & B^r(\mathbb{Z}M)_{r+2} & \longrightarrow & B^r(\mathbb{Z}M)_{r+1} & \longrightarrow & B^r(\mathbb{Z}M)_r & \longrightarrow & 0 \\
\end{array}
\]

where in the bottom row is \(r \geq 3\), and

- \(B^2(\mathbb{Z}M)_4\) is the free \(\mathbb{H}M\)-module on the set of suspensions of the non-zero generic \(3\)-cells \([x_1, x_2, x_3]\) of \(B(\mathbb{Z}M)\) together the non-zero generic \(4\)-cells

\[
[x_1, x_2],
\]

with \(\pi [x_1, x_2] = x_1 x_2\), and whose differential is \((x = x_1 x_2)\)

\[
\partial x [x_1, x_2] = [x_1 x_2] - [x_2 x_1].
\]

- \(B^3(\mathbb{Z}M)_5\) is the free \(\mathbb{H}M\)-module on the set of suspensions of the non-zero generic \(4\)-cells \([x_1, x_2, x_3, x_4]\) of \(B(\mathbb{Z}M)\) together the non-zero generic \(5\)-cells

\[
[x_1, x_2, x_3], [x_1, x_2, x_3].
\]
with \( \pi[x_1|x_2|x_3] = x_1 x_2 x_3 = \pi[x_1|x_2|x_3] \), and whose differential is \( (x = x_1 x_2 x_3) \)
\[
\partial_2[x_1|x_2|x_3] = -x_2[x_1|x_3] + [x_1|x_2|x_3] - x_3[x_1|x_2] + [x_1|x_2|x_3]
\]
\[
+ [x_1|x_2|x_3] - [x_2|x_1|x_3] + [x_2|x_3|x_1],
\]
\[
\partial_2[x_1|x_2|x_3] = -x_3[x_2|x_3] + [x_1 x_2 x_3] - x_2[x_1|x_3]
\]
\[- x_1[x_2|x_3] + [x_1|x_3|x_2] - [x_3|x_1|x_2].
\]

\( B^2(ZM)_6 \) is the free \( \mathbb{H}M \)-module on the set of double suspensions of the non-zero generic 4-cells \([x_1|x_2|x_3|x_4] \) of \( B(ZM) \), together with the suspensions of the non-zero generic 5-cells \([x_1|x_2|x_3] \) and \([x_1|x_2|x_3|x_4] \) of \( B^2(ZM) \), and the non-zero generic 6-cells
\[
[x_1|x_2|x_3|x_4],
\]
with \( \pi[x_1|x_2] = x_1 x_2 \), whose differential is \( (x = x_2 x_3) \)
\[
\partial_2[x_1|x_2] = -[x_1|x_2] - [x_2|x_1].
\]

Therefore, from Proposition 4.1, we get the following.

**Theorem 5.4.** For any \( \mathbb{H}M \)-module \( A \), the cohomology groups \( H^n(M; r; A) \), for \( n \leq r + 2 \), are isomorphic to the cohomology groups of the truncated cochain complexes of normalized \( r \)th level cochains of \( M \) with values in \( A \), \( C(M; r; A) \),
\[
\begin{array}{cccccccc}
C(M, r; A) : & 0 & \longrightarrow & C^0(M, r; A) & \longrightarrow 0 & \cdots & 0 & \longrightarrow C^r(M, r; A) \\
& & & \nearrow & & \searrow & & \\
& & & C^{r+1}(M, r; A) & \longrightarrow C^{r+2}(M, r; A) & \longrightarrow C^{r+3}(M, r; A) \\
\end{array}
\]

where \( C^0(M, r; A) = A(e) \), and the remaining non-trivial parts occur in the commutative diagram
\[
\begin{array}{cccccccc}
0 & \longrightarrow & C^1(M, 1; A) & \longrightarrow & C^2(M, 1; A) & \longrightarrow & C^3(M, 1; A) & \longrightarrow & C^4(M, 1; A) \\
0 & \longrightarrow & C^2(M, 2; A) & \longrightarrow & C^3(M, 2; A) & \longrightarrow & C^4(M, 2; A) & \longrightarrow & C^5(M, 2; A) \\
0 & \longrightarrow & C^3(M, 3; A) & \longrightarrow & C^4(M, 3; A) & \longrightarrow & C^5(M, 3; A) & \longrightarrow & C^6(M, 3; A) \\
0 & \longrightarrow & C^r(M, r; A) & \longrightarrow & C^{r+1}(M, r; A) & \longrightarrow & C^{r+2}(M, r; A) & \longrightarrow & C^{r+3}(M, r; A) \\
\end{array}
\]

where in the bottom row is \( r \geq 3 \), and

- \( C^4(M, 2; A) \) is the abelian group, under pointwise addition, of pairs of functions \( (g, \mu) \), where
\[
g : M^3 \rightarrow \bigcup_{x \in M} A(x) \quad \mu : M^2 \rightarrow \bigcup_{x \in M} A(x),
\]
with \( g(x, y, z) \in A(xyz) \) and \( \mu(x, y) \in A(xy) \), which are normalized in the sense that they take the value 0 whenever some of their arguments are equal to the unit \( e \) of the monoid.

- The coboundary \( \partial : C^3(M, 2; A) = C^2(M, 1; A) \rightarrow C^4(M, 2; A) \) acts on a normalized 2-cochain \( f \) of \( M \) in \( A \) by \( \partial f = (g, \mu) \), where
\[
g(x, y, z) = -x_* f(y, z) + f(xy, z) - f(x, yz) + z_* f(xy),
\]
\[
\mu(x, y) = f(x, y) - f(y, x).
\]
Corollary 5.6. For any $n \geq 1$, $H^0(M, 2; A) \cong A(e)$.

Corollary 5.7. For any $r \geq 2$, $H^r(M, r; A) \cong H^r(M, 2; A)$.

Corollary 5.8. For any $r \geq 2$, $H^{r+1}(M, r; A) \cong H^3(M, 2; A)$, and there is a natural monomorphism $H^r(M, 2; A) \hookrightarrow H^4(M, 1; A)$.

Corollary 5.9. For any $r \geq 3$, $H^{r+2}(M, r; A) \cong H^5(M, 3; A)$, and there is a natural monomorphism $H^r(M, 3; A) \hookrightarrow H^4(M, 2; A)$.

Let us now recall that the so-called Grillet cohomology groups $H^n(M, A)$, for $1 \leq n \leq 3$, can be computed as the cohomology groups of the truncated cochain complex $C^n_0(M, A)$, called the complex of (normalized on $e \in M$) symmetric cochains on $M$ with values in $A$ [12] Chapters XII, XIII, XIV, where

- $C^0_0(M, A)$ consists of normalized functions $f : M \to \bigcup_{x \in M} A(x)$, with $f(x) \in A(x)$.
- $C^1_0(M, A)$ consists of normalized functions $f : M^2 \to \bigcup_{x \in M} A(x)$, with $f(x, y) \in A(xy)$, such that $f(x, y) = f(y, x)$.
- $C^2_0(M, A)$ consists of normalized functions $f : M^3 \to \bigcup_{x \in M} A(x)$ with $f(x, y, z) \in A(xyz)$, such that
  \[ f(x, y, z) + f(z, y, x) = 0, \quad f(x, y, z) + f(y, z, x) + f(z, x, y) = 0. \]
\begin{itemize}
\item $C^4_c(M, \mathcal{A})$ consists of normalized functions $f : M^4 \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$ with $f(x, y, z, t) \in \mathcal{A}(xyzt)$, such that
\begin{align*}
f(x, y, y, x) = 0, \quad f(t, z, y, x) + f(x, y, z, t) = 0, \\
f(x, y, z, t) - f(y, z, t, x) + f(z, t, x, y) - f(t, x, y, z) = 0, \\
f(x, y, z, t) - f(y, y, z, t) + f(y, z, x, t) - f(y, z, t, x) = 0.
\end{align*}
\end{itemize}

- the coboundary homomorphisms are defined by
\begin{align*}
\delta^1 f(x, y) &= -x_* f(y) + f(xy) - y_* f(x), \\
\delta^2 f(x, y, z) &= -x_* f(y, z) + f(xy, z) - f(x, yz) + z_* f(x, y), \\
\delta^3 f(x, y, z, t) &= -x_* f(y, z, t) + f(xy, z, t) - f(x, yz, t) + f(x, yzt) - t_* f(x, y).
\end{align*}

There is natural injective cochain map
\begin{align}
0 &\xrightarrow{i_1 = \text{id}} C^1_c(M, \mathcal{A}) \xrightarrow{\delta^1} C^2_c(M, \mathcal{A}) \\
&\xrightarrow{i_2} C^2_c(M, A) \xrightarrow{\delta^2} C^3_c(M, A) \xrightarrow{i_3} C^3_c(M, A) \xrightarrow{\delta^3} C^4_c(M, A) \\
&\xrightarrow{i_4} C^4_c(M, A) \xrightarrow{\delta^4} C^5_c(M, A) \xrightarrow{i_5} C^5_c(M, A) \xrightarrow{\delta^5} C^6_c(M, A),
\end{align}
which is the identity map, $i_1(f) = f$, on symmetric 1-cochains, the map $i_2(f) = -f$ on symmetric 2-cochains, and on symmetric 3- and 4-cochains is defined by the simple formulas $i_3(f) = (f, 0)$ and $i_4(f) = (-f, 0, 0, 0)$, respectively. The only non-trivial verification here concerns the equality $\delta^3 i_3 = i_2 \delta^2$, that is, $\delta^3(f, 0) = (-\delta^3 f, 0, 0, 0)$, for any $f \in C^3_c(M, A)$, but it easily follows from Lemma 5.10 below.

**Lemma 5.10.** Let $f : M^4 \rightarrow \bigcup_{x \in M} \mathcal{A}(x)$ be a function with $f(x, y, z) \in \mathcal{A}(xyz)$. Then $f$ satisfies the symmetry conditions
\begin{align}
(22) & \quad f(x, y, z) + f(z, y, x) = 0, \quad f(x, y, z) + f(y, z, x) + f(z, x, y) = 0, \\
(23) & \quad f(x, y, z) - f(y, x, z) + f(y, z, x) = 0 \quad \text{if and only if it satisfies either (23) or (24) below.} \\
(24) & \quad f(x, y, z) - f(x, z, y) + f(z, x, y) = 0
\end{align}

**Proof.** The implications $(22) \Rightarrow (23)$ and $(22) \Rightarrow (24)$ are easily seen. To see that $(23) \Rightarrow (22)$, observe that, making the permutation $(x, y, z) \rightarrow (z, y, x)$, equation (23) is written as $f(y, x, z) = 0$. If we carry this to (22), we obtain
\begin{align*}
f(x, y, z) - f(y, x, z) + f(z, y, x) + f(y, x, z) &= f(x, y, z) + f(z, y, x) = 0,
\end{align*}
that is, the first condition in (22) holds. But then, we get also the second one simply by replacing the term $f(y, x, z)$ with $-f(z, x, y)$ in (23). The proof that $(24) \Rightarrow (22)$ is parallel. $\square$

**Proposition 5.11.** For any $\mathbb{H}$-module $\mathcal{A}$, the injective cochain map (21) induces natural isomorphisms
\begin{align*}
H^1_c(M, \mathcal{A}) &\cong H^1(M, 1; \mathcal{A}), \\
H^2_c(M, \mathcal{A}) &\cong H^3(M, 2; \mathcal{A}),
\end{align*}
and a natural monomorphism
\begin{align*}
H^3_c(M, \mathcal{A}) &\hookrightarrow H^5(M, 3; \mathcal{A}).
\end{align*}

**Proof.** From diagram (21), it follows directly that $\ker \delta^1 = \ker \delta^3$ and $i_2 \text{Im} \delta^1 = \text{Im} \delta^3$. Further, $i_2 \ker \delta^2 = \ker \delta^4$, since the condition $\delta^4 f = 0$ on a cochain $f \in C^4_c(M, 3; \mathcal{A}) = C^2_c(M, 1; \mathcal{A})$ implies the symmetry condition $f(x, y, z) = f(y, x, z)$, then,
\begin{align*}
H^1_c(M, \mathcal{A}) &\cong \ker \delta^1 = \ker \delta^3 \cong H^3(M, 3; \mathcal{A}) \cong H^1(M, 1; \mathcal{A}),
\end{align*}
and
\[ H_3^2(M, A) = \ker \delta^2 / \operatorname{Im} \delta^1 \cong \frac{i_2 \ker \delta^2}{\operatorname{Im} \delta^1} = \frac{\ker \partial^4}{\ker \partial^3} \cong H^4(M, 3; \mathcal{A}) \cong H^3(M, 2; \mathcal{A}). \]

To prove that the induced homomorphism \( H_3^2(M, A) \to H^5(M, 3; \mathcal{A}) \) is injective, suppose \( f \in C_3^2(M, A) \) is a symmetric 3-cochain such that \( i_3 f = \partial^4 g \) for some \( g \in C^4(M, 3; \mathcal{A}) = C^2(M, 1; \mathcal{A}) \). This means that the equalities
\[
f(x, y, z) = x \cdot g(y, z) - g(x, yz) + g(x, yz) - z \cdot g(x, y), \quad 0 = g(x, y) - g(y, x),
\]
hold. Then, \( g \in C_3^2(M, A) \) is a symmetric 2-cochain, and \( f = -\delta^2 g \) is actually a symmetric 2-coboundary. It follows that the injective map \( i_3 : \ker \delta^3 \hookrightarrow \ker \partial^3 \) induces a injective map in cohomology \( H^3C_3(M, A) \to H^5C(M, 3; A) \), as required. \( \square \)

To complete the list of relationships between the cohomology groups \( H^n(M, r; A) \) with those already known in the literature, let us note that a direct comparison of the cochain complex \textsuperscript{(19)} with the cochain complex in \textsuperscript{(2)} (6), which computes the lower commutative cohomology groups \( H^n(M, A) \), gives the following.

**Proposition 5.12.** For any \( \mathbb{M} \)-module \( A \), there are natural isomorphisms
\[
H^1(M, 1; A) \cong H_3^1(M, A), \quad H^3(M, 2; A) \cong H_3^2(M, A), \quad H^4(M, 2; A) \cong H_3^3(M, A).
\]

### 6. Cohomology classification of symmetric monoidal abelian groupoids

This section is dedicated to showing a precise classification for symmetric monoidal abelian groupoids, by means of the 3rd level cohomology groups of commutative monoids \( H^5(M, 3; A) \).

Symmetric monoidal categories have been studied extensively in the literature and we refer to Mac Lane \textsuperscript{[17]} and Saavedra \textsuperscript{[20]} for the background. Recall that a groupoid is a small category all whose morphisms are invertible. A groupoid \( \mathcal{M} \) is said to be abelian if its isotropy (or vertex) groups \( \operatorname{Aut}_\mathcal{M}(x), x \in \operatorname{Ob} \mathcal{M} \), are all abelian. We will use additive notation for abelian groupoids. Thus, the identity morphism of an object \( x \) of an abelian groupoid \( \mathcal{M} \) will be denoted by \( 0_x \); if \( a : x \to y, b : y \to z \) are morphisms, their composite is written by \( b + a : x \to z \), while the inverse of \( a \) is \(-a : y \to x\).

A symmetric monoidal abelian groupoid
\[
\mathcal{M} = (\mathcal{M}, \otimes, I, a, l, r, c)
\]
consists of an abelian groupoid \( \mathcal{M} \), a functor \( \otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) (the tensor product), an object \( I \) (the unit object), and natural isomorphisms \( a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z) \), \( l_x : I \otimes x \to x \), \( r_x : x \otimes I \to x \) (called the associativity and unit constraints, respectively) and \( c_{x,y} : x \otimes y \to y \otimes x \) (the symmetry), such that the four coherence conditions below hold.

\[
\begin{align*}
(25) \quad & a_{x,y,z,t} + a_{x \otimes y, z,t} = (0_x \otimes a_{y,z,t}) + a_{x,y \otimes z,t} + (a_{x,y,z} \otimes 0_t), \\
(26) \quad & (0_x \otimes l_y) + a_{x,l,y} = r_x \otimes 0_y, \\
(27) \quad & (0_y \otimes c_{x,z}) + a_{y,x,z} + (c_{x,y} \otimes 0_z) = a_{y,z,x} + c_{x,y,z} + a_{x,y,z}, \\
(28) \quad & c_{y,x} + c_{x,y} = 0_{x \otimes y}.
\end{align*}
\]

For further use, we recall that, in any symmetric monoidal abelian groupoid \( \mathcal{M} \), the equalities below hold (see \textsuperscript{(14)} Propositions 1.1 and 2.1]).

\[
\begin{align*}
(29) \quad & l_x \otimes y + a_{1,x,y} = l_x \otimes 0_y, \quad 0_x \otimes r_y + a_{x,y,1} = r_x \otimes y, \\
(30) \quad & l_x + c_{x,1} = r_x, \quad r_x + c_{l,x} = l_x.
\end{align*}
\]
Example 6.1 (2-dimensional crossed products). Every 3rd level 5-cocycle \((g, \mu) \in Z^5(M, 3; \mathcal{A})\), gives rise to a symmetric monoidal abelian groupoid

\[
\mathcal{A} \rtimes_{g, \mu} M = (\mathcal{A} \rtimes_{g, \mu} M, \otimes, I, a, l, r, c),
\]

that should be thought of as a 2-dimensional crossed product of \(M\) by \(\mathcal{A}\), and it is built as follows: Its underlying groupoid has as set of objects the set \(M\); if \(x \neq y\) are different elements of the monoid \(M\), then there are no morphisms in \(\mathcal{A} \rtimes_{g, \mu} M\) between them, whereas its isotropy group at any \(x \in M\) is \(\mathcal{A}(x)\).

The tensor product \(\otimes : (\mathcal{A} \rtimes_{g, \mu} M) \times (\mathcal{A} \rtimes_{g, \mu} M) \to \mathcal{A} \rtimes_{g, \mu} M\) is given on objects by multiplication in \(M\), so \(x \otimes y = xy\), and on morphisms by the group homomorphisms

\[
\otimes : \mathcal{A}(x) \times \mathcal{A}(y) \to \mathcal{A}(xy), \quad a_x \otimes a_y = y_+a_x + y_-a_x.
\]

The unit object is \(I = e\), the unit of the monoid \(M\), and the structure constraints are

\[
a_{x,y,z} = g(x, y, z) : (xy)z \to x(yz),
\]

\[
c_{x,y} = \mu(x, y) : xy \to yx,
\]

\[
l_z = 0_z : e_x = x \to x
\]

\[
r_z = 0_x : x_e = x \to x,
\]

which are easily seen to be natural since \(\mathcal{A}\) is an abelian group valued functor. The coherence conditions \((25), (27), \text{ and } (28)\) follow from the 5-cocycle condition \(\partial^5(h, \mu) = (0, 0, 0, 0, 0)\), while the coherence condition \((26)\) holds due to the normalization conditions \(h(x, e, y) = 0\).

If \(\mathcal{M}, \mathcal{M}'\) are symmetric monoidal abelian groupoids, then a symmetric monoidal functor \(F = (F, \varphi, \varphi_0) : \mathcal{M} \to \mathcal{M}'\) consists of a functor between the underlying groupoids \(F : \mathcal{M} \to \mathcal{M}'\), natural isomorphisms \(\varphi_{x,y} : Fx \otimes Fy \to F(x \otimes y)\), and an isomorphism \(\varphi_0 : I \to FI\), such that the following coherence conditions hold:

\[
\varphi_{x,y,z} + (0_{Fx} \otimes \varphi_{y,z}) + a_{Fx,Fy,Fz}^F = F(a_{x,y,z}) + \varphi_{x,y,z} + (\varphi_{x,y} \otimes 0_{Fz}),
\]

\[
F(l_z) + \varphi_{1,x} + (\varphi_{0 \otimes Fx}) = l_{Fx}^F, \quad F(r_z) + \varphi_{x,1} + (0_{Fx} \otimes \varphi_0) = r_{Fx}^F,
\]

\[
\varphi_{y,x} + c_{Fx,Fy}^F = F(c_{x,y}) + \varphi_{x,y}.
\]

Suppose \(F' : \mathcal{M} \to \mathcal{M}'\) is another symmetric monoidal functor. Then, a symmetric monoidal isomorphism \(\theta : F \Rightarrow F'\) is a natural isomorphism between the underlying functors, \(\theta : F \Rightarrow F'\), such that the following coherence conditions hold:

\[
\varphi_{x,y,z}' + (\theta_x \otimes \theta_y') + \varphi_{x,y} = \theta_{x,y} + \varphi_{x,y}, \quad \theta_1 + \varphi_0 = \varphi_0'.
\]

With compositions defined in a natural way, symmetric monoidal abelian groupoids, symmetric monoidal functors, and symmetric monoidal isomorphisms form a 2-category \([10, \text{Chap. V, §1}]\). A symmetric monoidal functor \(F : \mathcal{M} \to \mathcal{M}'\) is called a symmetric monoidal equivalence if it is an equivalence in this 2-category.

Our goal is to show a classification for symmetric monoidal abelian groupoids, where two symmetric monoidal abelian groupoids connected by a symmetric monoidal equivalence are considered the same, as stated in the theorem below. Recall that any homomorphism of monoids \(i : M \to M'\) induces a functor \(\mathbb{H}M \to \mathbb{H}M'\) in a obvious way, and then, by composition with it, a functor \(i^* : \mathbb{H}M\text{-Mod} \to \mathbb{H}M'\text{-Mod}.

Theorem 6.2 (Classification of Symmetric Monoidal Abelian Groupoids). (i) For any symmetric monoidal abelian groupoid \(\mathcal{M}\), there exist a commutative monoid \(M\), a \(\mathbb{H}M\text{-module} \mathcal{A}\), a 3rd level 5-cocycle \((g, \mu) \in Z^5(M, 3; \mathcal{A})\), and a symmetric monoidal equivalence

\[
\mathcal{A} \rtimes_{g, \mu} M \simeq \mathcal{M}.
\]
(ii) For any two 3rd level 5-cocycles \((g, \mu) \in Z^5(M, 3; A)\) and \((g', \mu') \in Z^5(M', 3; A')\), there is a symmetric monoidal equivalence

\[ A \times_{g, \mu} M \simeq A' \times_{g', \mu'} M' \]

if and only if there exist an isomorphism of monoids \(i : M \cong M'\) and a natural isomorphism \(\psi : A \cong i^*A'\), such that the equality of cohomology classes below holds.

\[ [g, \mu] = \psi^{-1}i^*[g', \mu'] \in H^5(M, 3; A) \]

Proof. (i) Let \(M = (\mathcal{M}, \otimes, I, \alpha, \beta, \rho, \mu, \nu)\) be any given symmetric monoidal abelian groupoid.

By the coherence theorem \[17\], there is no loss of generality in assuming that \(M\) is itself strictly unitary, that is, where both unit constraints \(I\) and \(\rho\) are identities. Then, we observe that \(\mathcal{M}\) is symmetric monoidal equivalent to another one that is totally disconnected, that is, where there is no morphism between different objects. Indeed, by the generalized Brandt’s theorem \[13, \text{Chapter 6, Theorem 2}\], there is a totally disconnected groupoid, say \(M'\), with an equivalence of groupoids \(M \to M'\). Hence, by Saavedra \[20, \text{I, 4.4}\], we can transport the symmetric monoidal structure along this equivalence so that \(M'\) becomes a strictly unitary symmetric monoidal abelian groupoid and the equivalence a symmetric monoidal one.

Hence, we assume that \(\mathcal{M}\) is totally disconnected and strictly unitary. Then, a triplet \((M, \mathcal{A}, (g, \mu))\), such that \(\mathcal{A} \times_{g, \mu} M = \mathcal{M}\) as symmetric monoidal abelian groupoids, can be defined as follows:

- The monoid \(M\). Let \(M = \text{Ob}\mathcal{M}\) be the set of objects of \(\mathcal{M}\). The tensor functor \(\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}\) determines a multiplication on \(M\), simply by \(xy = x \otimes y\), for any \(x, y \in M\). Since \(\mathcal{M}\) is strictly unitary, this multiplication on \(M\) is unitary with \(e = I\), the unit object of \(\mathcal{M}\). Moreover, it is associative and commutative since \(\mathcal{M}\) being totally disconnected implies that \((xy)z = x(yz)\) and \(xy = yx\). Thus, \(M\) becomes a commutative monoid.

- The \(\mathbb{H}M\)-module \(\mathcal{A}\). For each \(x \in M = \text{Ob}\mathcal{M}\), let \(\mathcal{A}(x) = \text{Aut}_{\mathcal{M}}(x)\) be the vertex group of the underlying abelian groupoid at \(x\). Since the diagrams below commute, due to the naturality of the structure constraints and the symmetry,

\[
\begin{array}{ccc}
(xy)z & \xrightarrow{\alpha_{x,y,z}} & x(yz) \\
(a_x \otimes a_y) \otimes a_z & \xrightarrow{\alpha_x \otimes (a_y \otimes a_z)} & a_x \otimes (a_y \otimes a_z) \\
(xy)z & \xrightarrow{\alpha_{x,y,z}} & x(yz) \\
(a_x \otimes a_y) \otimes a_z & \xrightarrow{a_x \otimes (a_y \otimes a_z)} & a_x \otimes (a_y \otimes a_z)
\end{array}
\]

it follows that the equations below hold.

\[ (34) \quad (a_x \otimes a_y) \otimes a_z = a_x \otimes (a_y \otimes a_z), \quad a_x \otimes a_y = a_y \otimes a_x, \quad 0_x \otimes a_x = a_x, \]

Then, if we write \(y_x : \mathcal{A}(x) \to \mathcal{A}(xy)\) for the homomorphism such that \(y_xa_x = 0_y \otimes a_x = a_x \otimes 0_y\), the equalities

\[
(yz)_x(a_x) = 0_y \otimes a_x = (0_y \otimes 0_z) \otimes a_x \quad (33) \quad 0_y \otimes (0_z \otimes a_x) = y_x(z_a a_x), \\

\]

\[
e_x a_x = 0_e \otimes a_x = a_x \quad (34)
\]

implies that the assignments \(x \to \mathcal{A}(x), (x, y) \to y_x : \mathcal{A}(x) \to \mathcal{A}(xy)\), define an \(\mathbb{H}M\)-module. Observe that this \(\mathbb{H}M\)-module \(\mathcal{A}\) determines indeed the tensor product \(\otimes\) of \(\mathcal{M}\), since

\[
a_x \otimes a_y = (a_x + 0_x) \otimes (0_y + a_y) = (a_x \otimes 0_y) + (0_x \otimes a_y) = y_xa_x + xsa_y.
\]

- The 3rd level 5-cocycle \((g, \mu) \in Z^5(M, 3; A)\). The associativity constraint and the symmetry of \(\mathcal{M}\) can be written in the form \(a_{x,y,z} = g(x, y, z)\) and \(c_{x,y} = \mu(x, y)\), for some given lists \((g(x, y, z) \in \mathcal{A}(xyz))_{x, y, z \in M}\) and \((\mu(x, y) \in \mathcal{A}(xy))_{x, y \in M}\). Since \(\mathcal{M}\) is strictly unitary, equations
in (26) and (29) give the normalization conditions \( g(x, e, y) = 0 = g(e, x, y) = g(x, y, e) \) for \( g \), while equations in (30) imply the normalization conditions \( \mu(x, e) = 0 = \mu(e, x) \) for \( \mu \). Thus, \( (g, \mu) \in C^3(M, 3; A) \) is a 3rd level 5-cochain. By the coherence conditions (24), (27), and (28) we have that

\[
\begin{aligned}
g(x, y, z) + g(xy, z, y) &= x_s g(y, z, y) + g(x, yz, y) + t_s g(x, y, z) \\
y_s \mu(x, z) + g(y, x, z) + z_s \mu(x, y) &= g(y, z, x) + \mu(x, yz) + g(x, y, z), \\
\mu(x, y) + \mu(y, x) &= 0,
\end{aligned}
\]

and combining the last two equations we also have

\[-y_s \mu(x, z) + g(y, x, z) - z_s \mu(y, x) = g(y, z, x) - \mu(yz, x) + g(x, y, z).\]

Hence, we obtain the required cocycle condition \( \partial^3(g, \mu) = (0, 0, 0) \). Since a direct comparison shows that \( M = A \ltimes_{g, \mu} M \) as symmetric monoidal abelian groupoids, the proof of this part is complete.

(ii) Suppose there exist an isomorphism of monoids \( i : M \cong M' \) and a natural isomorphism \( \psi : A \cong i^* A' \) such that \( \psi_*[g, \mu] = i^*[g', \mu'] \in H^5(M, 3; i^* A') \). This implies that there is a 3rd level 4-cochain \( f \in C^4(M, 3; i^* A') = C^3(M, 1; i^* A') \) such that

\[
\begin{aligned}
\psi_{xy} g(x, y, z) &= g(ix, iy, iz) + (ix)_s f(y, z) - f(xy, z) + f(x, yz) - (iz)_s f(x, y), \\
\psi_{xy} \mu(x, y) &= \mu(iy, x) - f(x, y) + f(y, x).
\end{aligned}
\]

Then, a symmetric monoidal isomorphism

\[
F(f) = (F, \varphi, \varphi_0) : A \ltimes_{g, \mu} M \rightarrow A' \ltimes_{g', \mu'} M'.
\]

can be defined as follows: The underlying functor acts by \( F(a_x : x \rightarrow x) = (\psi_x a_x : ix \rightarrow ix) \). The constraints of \( F \) are given by \( \varphi_{x,y} = f(x, y) : (ix)(iy) \rightarrow i(xy) \), and \( \varphi_0 = 0 : e \rightarrow ie = e \). So defined, it is easy to see that \( F \) is an isomorphism between the underlying groupoids. The naturality of the isomorphisms \( \varphi_{x,y} \), that is,

\[
\psi_{xy}(x_a a_y + y_a a_y) + \varphi_{x,y} = \varphi_{x,y} + (ix)_s \psi_y a_y + (iy)_s \psi_x a_x
\]

for \( a_x \in A(x), a_y \in A(y) \), holds owing to the commutativity of \( A'(i(xy)) \) and the naturality of \( \psi : A \cong i^* A' \), which says that

\[
\psi_{xy}(x_a a_y) = (ix)_s \psi_y a_y.
\]

The coherence conditions (31) and (32) are obtained as a consequence of equations (35) and (36), respectively, whereas the conditions in (32) trivially follow from the normalization conditions \( f(x, e) = 0_{ix} = f(e, x) \).

Conversely, suppose we have \( F = (F, \varphi, \varphi_0) : A \ltimes_{g, \mu} M \rightarrow A' \ltimes_{g', \mu'} M' \) a symmetric monoidal equivalence. By [1] Lemma 18, there is no loss of generality in assuming that \( F \) is strictly unitary in the sense that \( \varphi_0 = 0 : e \rightarrow e = Fe \). As the underlying functor establishes an equivalence between the underlying groupoids, and these are totally disconnected, it is an isomorphism.

We write \( i : M \cong M' \) for the bijection established by \( F \) between the object sets; that is, such that \( ix = Fx, x \in M \). Then, \( i \) is actually an isomorphism of monoids, since the existence of the structure isomorphisms \( \varphi_{x,y} : (ix)(iy) \rightarrow i(xy) \) implies \( (ix)(iy) = i(xy) \).

Let us write now \( \psi_x : A(x) \cong A'(ix) \) for the isomorphism such that \( F \alpha_x = \psi_x \alpha_x \), for each automorphism \( \alpha_x \in A(x) \), and \( x \in M \). The naturality of the automorphisms \( \varphi_{x,y} \) tell us that the equalities (37) hold. In particular, when \( \alpha_x = 0_x \), we obtain the equation (35) and so \( \psi : A \cong i^* A' \) is indeed a natural isomorphism.

Finally, if we write\( f(x, y) = \varphi_{x,y} \), for each \( x, y \in M \), we have a 3rd level 4-cochain \( f(F) = (f(x, y) \in A'(i(xy)))_{x,y \in M} \), since the equations \( f(x, e) = 0_{ix} = f(e, x) \) hold due to (32). Equations (35) and (36) follow from to the coherence equations (31) and (32). This means
that \( \psi_*(g, \mu) = \iota^*(g', \mu') + \partial^4 f \) and, therefore, we have that \( \psi_* [g, \mu] = \iota^* [g', \mu'] \in H^5 (M, 3; i^* \mathcal{A}') \), whence \( [g, \mu] = \psi_*^{-1} \iota^* [g', \mu'] \in H^5 (M, 3; \mathcal{A}) \).

### 7. Cohomology of cyclic monoids

In this section we compute the cohomology groups \( H^n (C, r; \mathcal{A}) \), for \( n \leq r + 2 \), when \( C \) is any cyclic monoid. The method we employ follows similar lines to the one used by Eilenberg and Mac Lane in [8] §14 and §15, for computing higher level cohomology of cyclic groups, though the generalization to monoids is highly nontrivial.

#### 7.1. Cohomology of finite cyclic monoids.

The structure of finite cyclic monoids was first stated by Frobenius [9]. Briefly, let us recall that if \( \equiv \) is any not equality congruence on the additive monoid \( \mathbb{N} = \{0, 1, \ldots\} \) of natural numbers, then the least \( m \geq 0 \) such that \( m \equiv x \) for some \( x \neq m \) is called the index of the congruence, and the least \( q \geq 1 \) such that \( m \equiv m + q \) is called its period. Hence,

\[
x \equiv y \quad \text{if and only if either} \quad x = y < m, \quad \text{or} \quad x, y \geq m \quad \text{and} \quad x \equiv y \quad \text{mod} \quad q.
\]

The quotient \( \mathbb{N} / \equiv \) is called the cyclic monoid of index \( m \) and period \( q \), and denoted here \( C_{m, q} \). As \( \mathbb{N} \) is a free monoid on the generator \( 1 \), every finite cyclic monoid is isomorphic to a proper quotient of \( \mathbb{N} \) and, therefore, to a monoid \( C_{m, q} \) for some \( m \) and \( q \).

From now on, \( C = C_{m, q} \) denotes the finite cyclic monoid of index \( m \) and period \( q \). We assume that \( m + q \geq 2 \), so that \( C \) is not the zero monoid.

Since every element of \( C \) can be written uniquely in the form \( [x] \) with \( 0 \leq x < m + q \), this monoid can be described as the set

\[
C = \{0, 1, \ldots, m, m + 1, \ldots, m + q - 1\},
\]

with addition

\[
x \oplus y = \varphi (x + y),
\]

where \( \varphi : \mathbb{N} \to C \) is the projection map given by

\[
\varphi (x) = \begin{cases} 
  x & \text{if} \quad x < m + q \\
  x - kq & \text{if} \quad m + kq \leq x < m + (k + 1)q.
\end{cases}
\]

To any pair \( x, y \in C \), we can associate the useful integer

\[
s (x, y) = \frac{ (x + y) - (x \oplus y) }{ q },
\]

which satisfies \( s (x, y) \geq 1 \) if \( x + y \geq m + q \), whereas \( s (x, y) = 0 \) if \( x + y < m + q \). It follows directly from the associativity in \( C \) that the cocycle property below holds.

\[
(39) \quad s (y, z) + s (x, y \oplus z) = s (x \oplus y, z) + s (x, y).
\]

Next, we construct a specific commutative DGA-algebra over \( \mathbb{H} C \), denoted by

\[
\mathcal{R} = \mathcal{R} (C),
\]

which is homologically equivalent to \( \mathcal{B} (\mathbb{Z} C) \) but algebraically simpler and more lucid. For each integer \( k = 0, 1, \ldots \), let us choose unitary sets over \( C \), \( \{ v_k \} \) and \( \{ w_k \} \), with

\[
(40) \quad \pi v_k = \varphi (km), \quad \pi w_k = \varphi (km + 1),
\]

and define

\[
\begin{align*}
\mathcal{R}_{2k} &= \text{the free} \ \mathbb{H} C\text{-module on} \ \{ v_k \}, \\
\mathcal{R}_{2k+1} &= \text{the free} \ \mathbb{H} C\text{-module on} \ \{ w_k \}.
\end{align*}
\]
The augmentation \( \alpha : \mathcal{R}_0 \rightarrow \mathbb{Z} \), the differential \( \partial : \mathcal{R}_n \rightarrow \mathcal{R}_{n-1} \), and the multiplication \( \circ : \mathcal{R} \otimes_{\mathcal{C}} \mathcal{R} \rightarrow \mathcal{R} \) are determined by the equations

\[
\alpha v_0 = 1, \quad \partial v_{k+1} = (m+q)(m+q-1)_* w_k - m(m-1)_* w_k, \quad \partial w_k = 0,
\]

(42)

\[
v_k \circ v_l = \binom{k+l}{k} v_{k+l}, \quad w_k \circ w_l = 0, \quad v_k \circ w_l = \binom{k+l}{k} w_{k+l} = w_l \circ v_k,
\]

and the unit is \( v_0 \).

**Proposition 7.1.** \( \mathcal{R} \), defined as above, is a commutative DGA-algebra over \( \mathbb{C} \).

**Proof.** By Proposition 4.1, the mapping \( v_{k+1} \mapsto \partial v_{k+1} \), determines a morphism of \( \mathbb{C} \)-modules \( \mathcal{R}_{2k+2} \rightarrow \mathcal{R}_{2k+1} \) since

\[
(m+q-1)_* \pi w_k \equiv (m+q-1)_* \varphi(km+1) = \varphi(m+q+km) = \varphi(m+km) = \pi v_{k+1},
\]

(43)

\[
(m-1)_* \pi w_k \equiv (m-1)_* \varphi(km+1) = \varphi(m+km) = \pi v_{k+1},
\]

and therefore \( \partial v_{k+1} \in \mathcal{R}_{2k+1}(\pi v_{k+1}) \). Similarly, by Proposition 4.1, we see that the formulas in Proposition 7.2 determine a multiplication morphism of \( \mathbb{C} \)-modules since \( \varphi(km) \oplus \varphi(lm) = \varphi((k+l)m) \) and \( \varphi(km) \oplus \varphi(lm+1) = \varphi((k+l)m+1) \). Associativity condition \( \mathbb{C} \) follows from the equality on combinatorial numbers

\[
\binom{k+l+t}{k} + \binom{l+t}{l} = \frac{(k+l+t)!}{k!l!t!} = \binom{k+l+t}{k} + \binom{k+l}{l},
\]

while condition \( \mathbb{C} \) holds thanks to the equality

\[
\binom{k+l-1}{k-1} + \binom{k+l-1}{k} = \binom{k+l}{k},
\]

and the remaining conditions in \( \mathbb{C} \) are quite obviously verified. \( \square \)

In next proposition we shall define a morphism \( f : \mathcal{B}(\mathcal{C}) \rightarrow \mathcal{R} \). Previously, observe that the graded \( \mathbb{C} \)-module \( \{ \mathcal{R}_n \} \) admits another structure of commutative graded algebra over \( \mathbb{C} \) (although it does not respect the differential structure), whose multiplication is determined by the simpler formulas

\[
v_k \bullet v_l = v_{k+l}, \quad w_k \bullet w_l = 0, \quad v_k \bullet w_l = w_{k+l} = w_l \bullet v_k.
\]

**Proposition 7.2.** A morphism \( f : \mathcal{B}(\mathcal{C}) \rightarrow \mathcal{R} \), of DGA-algebras over \( \mathbb{C} \), may be defined by the recursive formulas

\[
\begin{align*}
f[ | ] & = v_0, \\
f[x] & = x((x-1)_* w_0), \\
f[x \mid y] & = \begin{cases} 
0 & \text{if } x + y < m + q, \\
((x \oplus y)_* m)_* \left( \sum_{i=0}^{s(x,y)-1} (iq)_* v_1 \right) & \text{if } x + y \geq m + q,
\end{cases}
\end{align*}
\]

(44)

\[
f[x \mid y \mid \sigma] = f[x \mid y] \bullet f[\sigma],
\]

where \( \sigma = [z] \cdots \) is any cell of dimension 1 or greater.

**Proof.** This is divided into four parts. Note first that, from the inequalities

\[
m + s(x,y)q \leq (x \oplus y) + s(x,y)q = x + y < 2m + 2q - 1,
\]

it follows that \( s(x,y)q < m + 2q - 1 \). Therefore, for any \( 0 \leq i < s(x,y) \), we have \( iq = \varphi(iq) \in \mathcal{C} \) and the formula above for \( f[x \mid y] \) is well defined.

**Part 1.** We prove in this step that the assignment in (44) extends to a morphism of complexes of \( \mathbb{C} \)-modules. This follows from Proposition 4.1 since one verifies recursively that

\[
f[x_1 \mid \cdots \mid x_n] \in \mathcal{R}_n(x_1 \oplus \cdots \oplus x_n)
\]
as follows: The case when \( n = 0 \) is obvious. When \( n = 1 \), it holds since \( w_0 \in \mathcal{R}_1 \) and \((x_1 - 1) \oplus \pi w_0 = (x_1 - 1) \oplus 1 = x_1\), and for \( n = 2 \) since \( v_1 \in \mathcal{R}_2 \) and
\[
((x_1 \oplus x_2) \cdot m) \oplus \pi v_1 = ((x_1 \oplus x_2) \cdot m) \oplus m = x_1 \oplus x_2.
\]

Then, for \( n \geq 3 \), induction gives
\[
f[x_1 \cdots x_n] = f[x_1|x_2] \cdot f[x_3|\cdots|x_n] \in \mathcal{R}_2(x_1 \oplus x_2) \cdot \mathcal{R}_{n-2}(x_3 \oplus \cdots \oplus x_n) \subseteq \mathcal{R}_n(x_1 \oplus \cdots \oplus x_n).
\]

**Part 2.** We prove now that \( \partial f = f \partial \).

For a 1-cell \([x]\) of \( \mathcal{B}(\mathcal{C}) \), we have \( \partial f[x] = x((x - 1) \cdot \partial w_0) \not\equiv 0 = f \partial [x] \).

For a 2-cell \([x | y]\), we have
\[
f \partial [x | y] = x_\ast f[y] - f[x \oplus y] + y_\ast f[x].
\]

To compare with \( \partial f[x | y] \), we shall distinguish three cases:

- **Case** \( x + y < m + q \). In this case \( \partial f[x | y] = 0 \), and also
\[
f \partial [x | y] = y((x + y - 1) \cdot w_0) - (x + y)((x + y - 1) \cdot w_0) + x((x + y - 1) \cdot w_0) = 0.
\]

- **Case** \( x + y \geq m + q \) and \( x \oplus y = m \). Here, \((x - 1) \oplus y = m + q - 1 = x \oplus (y - 1) \). Then,
\[
\partial f[x | y] = \sum_{i=0}^{s(x,y)-1} (m + q)((iq \oplus (m + q - 1)) \cdot w_0) - m((iq \oplus (m - 1)) \cdot w_0)
\]
\[
= (m + q)((m + q - 1) \cdot w_0) - m((m - 1) \cdot w_0)
\]
\[
+ \sum_{i=1}^{s(x,y)-1} (m + q)((m + q - 1) \cdot w_0) - m((m - 1) \cdot w_0)
\]
\[
= (m + q)((m + q - 1) \cdot w_0) - m((m - 1) \cdot w_0) + (s(x,y) - 1)q((m + q - 1) \cdot w_0)
\]
\[
= (m + q)((m + q - 1) \cdot w_0) - m((m - 1) \cdot w_0)
\]
\[
= (x + y)((m + q - 1) \cdot w_0) - m((m - 1) \cdot w_0) = f \partial [x | y].
\]

- **Case** \( x + y \geq m + q \) and \( x \oplus y > m \). In this case, \((x - 1) \oplus y = (x \oplus y) - 1 = x \oplus (y - 1)\), whence
\[
\partial f[x | y] = \sum_{i=0}^{s(x,y)-1} ((x \oplus y) \cdot m) \oplus iq_\ast \partial v_1 - \sum_{i=0}^{s(x,y)-1} ((x \oplus y) \cdot m) \oplus ((iq \oplus (m + q - 1)) \cdot w_0)
\]
\[
- \sum_{i=0}^{s(x,y)-1} m((x \oplus y) \cdot m) \oplus (iq \oplus (m - 1)) \cdot w_0) = \sum_{i=0}^{s(x,y)-1} (m + q)((x \oplus y) - 1) \cdot w_0)
\]
\[
= (y - (x \oplus y) + x)((x \oplus y) - 1) \cdot w_0) = f \partial [x | y] .
\]

For a 3-cell \([x | y | z]\), we have to prove that \( f \partial [x | y | z] = 0 \), or, equivalently, that
\[
(45) \quad x_\ast f[y | z] + f[x | y \oplus z] = z_\ast f[x | y] + f[x \oplus y | z].
\]

Since \( x \oplus (y \oplus z) = x \oplus y \oplus z + s(x,y \oplus z)q \), it follows that
\[
x \oplus ((y \oplus z) \cdot m) = ((x \oplus y \oplus z) \cdot m) \oplus q(s(x,y \oplus z)q),
\]
whenever \( y \oplus z \geq m \). Then, we can write

\[
x_* f[y \mid z] = \begin{cases} 
0, & \text{if } s(y, z) = 0, \\
(x \oplus ((y \oplus z) - m)_*)^{s(y, z) - 1} \left( \sum_{i=0}^{s(y, z) - 1} \varphi(q)_i v_1 \right), & \text{if } s(y, z) \geq 1,
\end{cases}
\]

\[
x_* f[y \mid z] = \begin{cases} 
0, & \text{if } s(y, z) = 0, \\
((x \oplus y \oplus z) - m)_*^{s(y, z) - 1} \left( \sum_{i=0}^{s(y, z) - 1} \varphi(s(x, y \oplus z)q + iq)_i v_1 \right), & \text{if } s(y, z) \geq 1.
\end{cases}
\]

As

\[
f[x \mid y \oplus z] = \begin{cases} 
0, & \text{if } s(x, y \oplus z) = 0, \\
((x \oplus y \oplus z) - m)_*^{s(x, y \oplus z) - 1} \left( \sum_{i=0}^{s(x, y \oplus z) - 1} \varphi(q)_i v_1 \right), & \text{if } s(x, y \oplus z) \geq 1,
\end{cases}
\]

one concludes the formula

\[
x_* f[y \mid z] + f[x \mid y \oplus z] = \begin{cases} 
0, & \text{if } s(y, z) = 0 = s(x, y \oplus z), \\
((x \oplus y \oplus z) - m)_*^{s(x, y \oplus z) - 1} \left( \sum_{i=0}^{s(x, y \oplus z) - 1} \varphi(q)_i v_1 \right), & \text{otherwise},
\end{cases}
\]

Similarly, one sees that

\[
z_* f[x \mid y] + f[x \oplus y \mid z] = \begin{cases} 
0, & \text{if } s(x, y) = 0 = s(x \oplus y, z), \\
((x \oplus y \oplus z) - m)_*^{s(x, y \oplus z) - 1} \left( \sum_{i=0}^{s(x, y \oplus z) - 1} \varphi(q)_i v_1 \right), & \text{otherwise},
\end{cases}
\]

and the equality in (45) follows by comparison using (39).

Finally, for a cell \( [x \mid y \mid z \mid t] = [x \mid y \mid z \mid \tau] \) of dimension higher than 3 we use the formulas

\[
(46) \quad \partial(\sigma \mid x \mid b) = [\partial(\sigma \mid x) \mid b] + [a \mid \partial(x \mid b)]
\]

which holds for any even chain \( a \) and any other chain \( b \) of \( B(\mathbb{Z}C) \), and

\[
(47) \quad \partial(c \bullet d) = c \bullet \partial d,
\]

which holds for any chains \( c, d \in \mathcal{R} \). Thus, as we know that \( f \partial[x \mid y \mid z] = 0 \), induction gives

\[
\partial(f[x \mid y \mid z \mid \tau]) = \partial(f[x \mid y \mid z]) = f[x \mid y \mid \partial(z \mid \tau)]
\]

Part 3. Here we show that \( f \) preserves products. It is enough to prove that \( f(\sigma \circ \tau) = f(\sigma) \circ f(\tau) \) for cells \( \sigma = [x_1 \mid \cdots \mid x_n] \) and \( \tau = [y_1 \mid \cdots \mid y_{n'}] \) of \( B(\mathbb{Z}C) \).

As in [8] page 99, a term \( T = \pm [t_1 \mid \cdots \mid t_{n+n'}] \) in the shuffle product (18) of \( \sigma \) and \( \tau \) is called mixed whenever there exists an index \( i \) such that \( t_{2i-1} \) is an \( x \) of \( \sigma \) and \( t_{2i} \) an \( y \) of \( \tau \), or vice versa. Choose the first index \( i \) for each mixed \( T \), and let \( T' \) be the term obtained from \( T \) by interchanging \( t_{2i-1} \) with \( t_{2i} \). Since \( f[x, y] \) is symmetric,

\[
f(T) = f[t_1 \mid t_2] \cdots \cdot f[t_{2i-1} \mid t_{2i}] \cdots f[t_{2i-1-n} \mid t_{2i}] = f[T']
\]

Since \( T \) and \( T' \) have opposite signs, the results cancel and \( f(\sigma \circ \tau) = \sum f(T) \), with summation taken only over the unmixed terms, and where the sign of each term due the shuffle is always plus. If \( n = 2r + 1 \) and \( n' = 2r' + 1 \) are both odd, there are no unmixed terms, so \( f(\sigma \circ \tau) = 0 \) in agreement with the fact that \( f(\sigma) \circ f(\tau) = 0 \) (since \( w_k \circ w_l = 0 \)). If \( n = 2r \) and \( n' = 2r' \)
$2r'$ are both even, the unmixed terms $T$ are obtained by taking all shuffles of the $r$ pairs $(x_1, x_2), \ldots, (x_{2r-1}, x_{2r})$ through the pairs $(y_1, y_2), \ldots, (y_{2r'-1}, y_{2r'})$. For any such shuffle

$$f(T) = f[x_1 \mid x_2] \cdot \cdots \cdot f[x_{2r-1} \mid x_{2r}] \cdot f[y_1 \mid y_2] \cdot \cdots \cdot f[y_{2r'-1} \mid y_{2r'}] = f(\sigma) \cdot f(\tau)$$

and the number of such shuffles is $\binom{r + r'}{r}$, hence

$$f(\sigma \circ \tau) = \binom{r + r'}{r} f(\sigma) \cdot f(\tau) = f(\sigma) \circ f(\tau),$$

as desired. For $n = 2r$ and $n' = 2r' + 1$, the unmixed terms $T$ are as above but with the last argument $y_{2r'+1}$ always at the end. Hence, for each of them

$$f(T) = f[x_1 \mid x_2] \cdot \cdots \cdot f[x_{2r-1} \mid x_{2r}] \cdot f[y_1 \mid y_2] \cdot \cdots \cdot f[y_{2r'-1} \mid y_{2r'}] \cdot f[y_{2r'+1}] = f(\sigma) \cdot f(\tau),$$

and therefore $f(\sigma \circ \tau) = \binom{r + r'}{r} f(\sigma) \cdot f(\tau) = f(\sigma) \circ f(\tau)$. The remaining case $n = 2r + 1$ and $n' = 2r'$ is treated similarly.

**Proposition 7.3.** A morphism $g : \mathcal{R} \to B(\mathcal{C})$, of DGA-algebras over $\mathbb{K}C$, may be defined by the recursive formulas

$$
\begin{aligned}
gv_0 &= [\,], \\
gw_k &= [gwk \mid 1], \\
gwk+1 &= \sum_{t< m+q} (m+q-t-1)_s [gw_k \mid t] - \sum_{s<m} (m-s-1)_s [gw_k \mid s].
\end{aligned}
$$

**Proof.** Part 1. We show here that the assignment in (48) extends to a morphism of complexes of $\mathbb{K}C$-modules. By Proposition 4.1, we have to verify that $gw_k \in B(\mathcal{C})_{2k}(\varphi(km))$ and $gw_k \in B(\mathcal{C})_{2k+1}(\varphi(km + 1))$. Clearly $gv_0 = \lbrack \, \rbrack \in B(\mathcal{C})_0(0)$. Assume that $gv_k \in B(\mathcal{C})_{2k}(\varphi(km))$. Then, we have

$$gw_k = [gw_k \mid 1] \in B(\mathcal{C})_{2k+1}(\varphi(km) \oplus 1) = B(\mathcal{C})_{2k+1}(\varphi(km + 1)),$$

as required. Moreover, for any $t < m + q$ and $s < m$,

$$(m+q-t-1)_s [gw_k \mid t], (m-s-1)_s [gw_k \mid s] \in B(\mathcal{C})_{2k+2}(\varphi((k+1)m)),$$

since

$$(m+q-t-1) \oplus \varphi(km + 1) \oplus t = \varphi((k+1)m) = (m-s-1) \oplus \varphi((k+1)m) \oplus s.$$

Whence $gw_{k+1} \in B(\mathcal{C})_{2k+2}(\varphi((k+1)m))$.

**Part 2.** Here we shall prove, as an auxiliary result, that

$$gw_k \circ [\,] = gw_k, \quad gw_k \circ [1] = 0,$$

where $\circ = \circ_1$ is the shuffle product \[13\] of $B(\mathcal{C})$. Clearly $gv_0 \circ [\,] = [\,] \circ [\,] = [\,] = [gv_0 \mid 1] = gw_0$. Assuming the result for $gv_k$, we have

$$gw_k \circ [\,] = gw_k \circ [\,] \circ [1] = gw_k \circ ([1 \mid 1] - [1 \mid 1]) = 0,$$

from where, in addition, it follows that, for any $t \in C$,

$$[gw_k \mid t] \circ [1] = [gw_k \mid t \mid 1] - [gw_k \circ [1] \mid t] = [gw_k \mid t \mid 1],$$

whence

$$gv_{k+1} \circ [1] = \sum_{t<m+q} (m+q-t-1)_s [gw_k \mid t \mid 1] - \sum_{s<m} (m-s-1)_s [gw_k \mid s \mid 1]$$

$$= [gw_{k+1} \mid 1] = gw_{k+1}.$$

**Part 3.** We now prove recursively that $\partial g = g\partial$. 
For argument $w_0$ is immediate: $\partial gw_0 = \partial[1] = 0$. For argument $v_{k+1}$, first observe that $\partial gw_k = 0$ gives, for ant $t \in C$,

\[
\partial[gw_k \mid t] = \partial[gv_k \mid 1 \mid t] + [gv_k \mid \partial[1 \mid t]] = [\partial gw_k \mid t] + [gv_k \mid \partial[1 \mid t]] = [gv_k \mid \partial[1 \mid t]] = 1_* [gv_k \mid t] - [gv_k \mid 1 \oplus t] + t_* [gv_k \mid 1] = 1_* [gv_k \mid t] - [gv_k \mid 1 \oplus t] + t_* gw_k.
\]

Then,

\[
\partial gw_{k+1} = \sum_{t < m+q} (m+q-t-1)_* \partial gw_k \mid t] - \sum_{t < m} (m-t-1)_* \partial gw_k \mid t] = \sum_{t < m+q-1} (m+q-t)_* [gv_k \mid t] - (m+q-t-1)_* [gv_k \mid 1+t] + (m+q-1)_* gw_k + 1_* [gv_k \mid m+q-1] - [gv_k \mid m] + (m+q-1)_* gw_k - \sum_{t < m} (m-t)_* [gv_k \mid t] - (m-t-1)_* [gv_k \mid 1+t] + (m-1)_* gw_k + (m+q) ((m+q-1)_* gw_k) - m ((m-1)_* gw_k) = gw_k \odot gw_{l+1}.
\]

And for argument $w_{k+1}$,

\[
\partial gw_{k+1} = \frac{\partial gw_{k+1} \odot [1] = (m+q) ((m+q-1)_* gw_k) - m ((m-1)_* gw_k)) \odot [1] = 0.}

\textbf{Part 4.} Here we show that $g$ preserves products by proving that $g(a \odot b) = ga \odot gb$ for $a, b \in \{v_k, w_l\}$. For the case when $a = w_k$ and $b = w_l$, we have

\[
gw_k \odot gw_l = gv_k \odot [1] \odot gw_l = 0 = gw_k \odot gw_l.
\]

To prove the remaining cases, first observe that if $gw_k \odot gw_l = g(v_k \odot v_l)$ for some $k$ and $l$, then

\[
gw_k \odot gw_l = gw_k \odot [1] \odot gw_l = gw_k \odot gw_l \odot [1] = g(v_k \odot v_l) \odot [1] = \left( \begin{array}{c} k+l \end{array} \right) gw_{k+l} \odot [1] = \left( \begin{array}{c} k+l \end{array} \right) gw_k \odot gw_l = g(w_k \odot v_l).
\]

Next, we show that $gw_k \odot gw_l = g(v_k \odot v_l)$ by induction. The case when $k = 0$ or $l = 0$ is immediate, since $gw_0 = [1]$. Now, using that, for any $t, s \in C$,

\[
[gw_k \mid t] \odot [gw_l \mid s] = [gw_k \mid t] \odot gw_l \mid s] + [gw_k \odot gw_l \mid s, t],
\]
we have

\[
g_{v_{k+1}} \circ g_{v_{l+1}} = \sum_{s<m+q} (m+q-s-1)_s \left[ \sum_{t<m+q} (m+q-t-1)_t (gw_k \circ gw_l) \bigg| s \right] \\
- \sum_{s<m+q} (m+q-s-1)_s \left[ \sum_{t<m} (m-t-1)_t (gw_k \circ gw_l) \bigg| s \right] \\
+ \sum_{t<m+q} (m+q-t-1)_t \left[ gw_k \circ \sum_{s<m+q} (m+q-s-1)_s (gw_l) \bigg| t \right] \\
- \sum_{t<m} (m+q-t-1)_t \left[ gw_k \circ \sum_{s<m} (m+q-s-1)_s (gw_l) \bigg| s \right] \\
+ \sum_{s<m} (m+q-t-1)_t \left[ gw_k \circ \sum_{s<m+q} (m+q-s-1)_s (gw_l) \bigg| s \right]
\]

and then, by induction,

\[
g_{v_{k+1}} \circ g_{v_{l+1}} = \\
= \sum_{s<m+q} (m+q-s-1)_s (gw_{k+1} \circ gw_l) \bigg| s + \sum_{t<m+q} (m+q-t-1)_t (gw_k \circ gw_{l+1}) \bigg| t \\
- \sum_{s<m} (m+q-s-1)_s (gw_{k+1} \circ gw_l) \bigg| s - \sum_{t<m} (m+q-t-1)_t (gw_k \circ gw_{l+1}) \bigg| t \\
= \binom{k+l+1}{k+1} \left( \sum_{s<m+q} (m+q-s-1)_s (gw_{k+l+1} \bigg| s) - \sum_{s<m} (m+q-s-1)_s (gw_{k+l+1} \bigg| s) \right) \\
+ \binom{k+l+1}{k} \left( \sum_{t<m+q} (m+q-t-1)_t (gw_{k+l+1} \bigg| t) - \sum_{t<m} (m+q-t-1)_t (gw_{k+l+1} \bigg| t) \right) \\
= \binom{k+l+1}{k+1} g_{v_{k+l+2}} + \binom{k+l+1}{k} g_{v_{k+l+2}} = \binom{k+l+2}{k+1} g_{v_{k+l+2}} \\
= g(v_{k+1} \circ v_{l+1}).
\]

Now, we are ready to establish the following key result.

**Theorem 7.4.** The morphisms \( f : B(ZC) \rightarrow R \) and \( g : R \rightarrow B(ZC) \), as defined above, form a contraction.
Proof. Part 1. We start by showing that the composite $fg$ is the identity. Clearly $fgv_0 = f[ ] = v_0$. Then, induction gives

$$fgv_k = f(gv_k \circ [1]) = fgv_k \circ f[1] = v_k \circ v_0 = w_k,$$

$$fgv_{k+1} = \sum_{t<m+q} (m+q-t-1)_s f[gv_k[1 | t]] - \sum_{s<m} (m-s-1)_s f[gv_k|s]$$

$$= \sum_{t<m+q} (m+q-t-1)_s (f[gv_k] \bullet f[1 | t]) - \sum_{s<m} (m-s-1)_s (f[gv_k] \bullet f[1 | s]),$$

$$= v_k \bullet f[1 | m+q-1] = v_k \bullet v_1 = v_{k+1}.$$ 

Part 2. Here, we describe the composite $gf$. Clearly $gf[ ] = [ ]$ and $gf[ ] = x((x-1)_s [1]).$ For those 2-cells $[x | y]$ such that $x+y < m+q$ we have $gf[ ] = 0$, and, as we prove below, the effect of $gf$ on the 2-cells $[x | y]$ with $x+y \geq m+q$ is described by the formula

$$gf[ ] = \sum_{t=x+y-m-q}^{m+q-1} (x+y-t-1)_s [1 | t] + \sum_{i=1}^{r-1} (m+r-t-1)_s [1 | t]$$

$$- \sum_{t=0}^{m-1} (m+r-t-1)_s [1 | t] + \sum_{t=(i-1)q+r}^{s(x,y)-1} \sum_{t=iq+r}^{r-1} (m+iq+r-t-1)_s [1 | t]$$

$$+ \sum_{i=1}^{s(x,y)-1} \sum_{t=m}^{m+q-1} (m+iq+r-t-1)_s [1 | t],$$

where we write $x+y = m+s(x,y)q+r$ with $0 \leq r < q$ (so that $x \oplus y = m+r$). Concerning the two last terms, note that $(s(x,y)-1)q+r < m+q$ whenever $s(x,y) \geq 2$, since $m+s(x,y)q+r = x+y < 2m+2q$.

In effect, by definition of $f$ and $g$, we have

$$gf[ ] = \sum_{i=0}^{s(x,y)-1} \left( \sum_{t=0}^{m+q-1} \varphi(m+(i+1)q+r-t-1)_s [1 | t] - \sum_{t=0}^{m-1} \varphi(m+iq+r-t-1)_s [1 | t] \right).$$

Then, since for any $i \geq 1$ and $t < r$ is $\varphi(m+(i+1)q+r-t-1) = \varphi(m+iq+r-t-1)$, we see that

$$gf[ ] = \sum_{t=r}^{m+q-1} (m+q+r-t-1)_s [1 | t] + \sum_{t=0}^{r-1} (m+r-t-1)_s [1 | t] - \sum_{t=0}^{m-1} (m+r-t-1)_s [1 | t]$$

$$+ \sum_{i=1}^{s(x,y)-1} \left( \sum_{t=r}^{m+q-1} \varphi(m+(i+1)q+r-t-1)_s [1 | t] - \sum_{t=r}^{m-1} \varphi(m+iq+r-t-1)_s [1 | t] \right)$$

$$= \sum_{t=0}^{r-1} \sum_{t=r}^{m+q-1} \varphi(m+(i+1)q+r-t-1)_s [1 | t] - \sum_{t=0}^{m-1} \sum_{t=r}^{m-1} \varphi(m+iq+r-t-1)_s [1 | t],$$
from where \( (50) \) follows thanks to the equalities
\[
\sum_{t=r}^{m+q-1} \varphi(m + (i+1)q + r - t - 1)_* [1 \mid t] = \sum_{l=0}^{i-1} \sum_{t=lp+r}^{i-1 (l+1)q+r-1} (m + (l+1)q + r - t - 1)_* [1 \mid t] + \sum_{t=iq+r}^{m+q-1} (m + (i+1)q + r - t - 1)_* [1 \mid t],
\]
\[
\sum_{t=r}^{m-1} \varphi(m + iq + r - t - 1)_* [1 \mid t] = \sum_{l=1}^{i-1} \sum_{t=(l-1)q+r}^{i-1 lq+r-1} (m + lq + r - t - 1)_* [1 \mid t] + \sum_{t=(i-1)q+r}^{m-1} (m + iq + r - t - 1)_* [1 \mid t].
\]

Finally, to complete the description of the composite \( gf \), for generic cells \([x \mid y \mid \sigma]\) of dimensions greater than 2 we have the formula
\[
gf[x \mid y \mid \sigma] = [gf[x, y] \mid gf[\sigma]].
\]
In effect, as \( gf[x \mid y \mid \sigma] = g(f[x, y] \circ f[\sigma]) \), by linearity, it suffices to observe that, for any \( k \geq 1 \),
\[
g(v_1 \circ w_k) = [gv_1 \mid gw_k], \quad g(v_1 \circ v_k) = [gv_1 \mid gv_k],
\]
or, equivalently, that \( gw_{k+1} = [gv_1 \mid gw_k] \) and \( gv_{k+1} = [gv_1 \mid gv_k] \). But these last equations are immediate for \( k = 1 \), and for higher \( k \) by a straightforward induction.

**Part 3.** We establish here a homotopy \( \Phi \) from \( gf \) to the identity, which is determined by the recursive formulas
\[
\Phi[\ ] = 0,
\]
\[
\Phi[x \mid y \mid \sigma] = \Phi[x \mid y \mid \sigma] + [gf[x \mid y] \mid \Phi[\sigma]].
\]
Since, for any \( t < x \) in \( C \), \((x - t - 1) \oplus 1 \oplus t = x \), we see that \( \pi \Phi[x] = x \) and then, by recursion, that \( \pi \Phi[x \mid y \mid \sigma] = x \oplus y \oplus \pi[\sigma] \). Hence, by Proposition 4.1, the formulas above determine an endomorphism of the complex of \( H\)-modules \( B(\mathcal{C}) \), which is of differential degree +1.

Next, we prove that \( \Phi : gf \Rightarrow id \) is actually a homotopy:

For a 1-cell \([x] \) is \( \Phi \partial [x] = 0 \), and
\[
\partial \Phi[x] = \sum_{t \leq x} (x - t)_*[t] - (x - t - 1)_*[1 + t] + (x - 1)_*[1] = -[x] + x((x - 1)_*[1]) = -[x] + gf[x],
\]
as required.

For a 2-cell \([x \mid y] \) we have
\[
(\partial \Phi + \Phi \partial)[x \mid y] = \sum_{t \leq x} (x - t - 1)_*[1 \mid t] + (x - t - 1)_*[1 \mid t] - \sum_{t \leq y} ((x + y) - t - 1)_*[1 \mid t] + \sum_{t \leq y} ((x - t - 1) \oplus y)_*[1 \mid t]
\]
\[
= \sum_{t \leq x} (x - t)_*[t] - (x - t - 1)_*[1 + t] + \sum_{t \leq x} (x - t - 1)_*[1 \mid t] + \sum_{t \leq y} ((x + y) - t - 1)_*[1 \mid t] + \sum_{t \leq y} ((x - t - 1) \oplus y)_*[1 \mid t]
\]
\[
= -[x, y] + \sum_{t < x} (x - t - 1)_*[1 \mid t] + \sum_{t < y} (x \oplus (y - t - 1))_*[1 \mid t] - \sum_{t < x \oplus y} ((x + y) - t - 1)_*[1 \mid t].
\]
If \( s(x, y) = 0 \) then, for any \( t < x \), \( t \oplus y = t + y \) and \( x \oplus (y-t-1) = x + y - t - 1 = (x \oplus y) - t - 1 \). Therefore

\[
\sum_{t < x}(x - t - 1)_* [1 \mid t \oplus y] + \sum_{t \leq y} (x + y - t - 1)_* [1 \mid t] - \sum_{t < x+y} (x + y - t - 1)_* [1 \mid t] = 0,
\]
and, since \( gf[x \mid y] = 0 \), it follows that \( (\partial \Phi + \Phi \partial)[x \mid y] = -[x \mid y] + gf[x \mid y] \), as required.

If \( s(x, y) > 0 \), the composite \( gf[x \mid y] \) has been computed in (50) and, writing as there \( x + y = m + s(x, y)q + r \) with \( 0 \leq r < q \), we have

\[
\sum_{t < x} (x - t - 1)_*[1 \mid t \oplus y] = \sum_{i=1}^{s(x, y)-1} \sum_{t=m}^{m+q-1} (m + iq + r - t - 1)_*[1 \mid t]
+ \sum_{t=y}^{x+y-m-q} (x + y - t - 1)_*[1 \mid t] + \sum_{t=m}^{m+r-1} (m + r - t - 1)_*[1 \mid t].
\]

Similarly, we have

\[
\sum_{t < y} (x \oplus (y-t-1))_*[1 \mid t] = \sum_{i=1}^{s(x, y)-1} \sum_{t=(i-1)q+r}^{iq+r-1} (m + iq + r - t - 1)_*[1 \mid t]
+ \sum_{t=x+y-m-q}^{y-1} (x + y - t - 1)_*[1 \mid t] + \sum_{t=0}^{r-1} (m + r - t - 1)_*[1 \mid t],
\]
and

\[
\sum_{t < x \oplus y} ((x \oplus y) - t - 1)_*[1 \mid t] = \sum_{t=0}^{m+r-1} (m + r - t - 1)_*[1 \mid t].
\]

Hence, a direct comparison with (50) gives that \( (\partial \Phi + \Phi \partial)[x \mid y] = -[x \mid y] + gf[x \mid y] \), as required.

Finally, we prove that \( (\partial \Phi + \Phi \partial)(\tau) = -\tau + gf(\tau) \) if \( \tau \) is a cell of dimension 3 or greater. To do so, previously observe that, for any generic cell \( \gamma \) of \( B(ZC) \), we have

\[
(53) \quad \partial[gf[x \mid y] \mid \Phi(\gamma)] = [gf[x \mid y] \mid \partial \Phi(\gamma)].
\]

To prove it, by linearity, it suffices to check that \( \partial[gv_1 \mid 1 \mid \beta] = [gv_1 \mid \partial[1 \mid \beta]] \), for any generic cell \( \beta \):

\[
\partial[gv_1 \mid 1 \mid \beta] = \partial[gv_1 \mid 1 \mid \beta] + [gv_1 \mid \partial[1 \mid \beta]]
= [\partial gv_1 \mid \beta] + [gv_1 \mid \partial[1 \mid \beta]] = [\partial gv_1 \mid \beta] + [gv_1 \mid \partial[1 \mid \beta]] = [gv_1 \mid \partial[1 \mid \beta]].
\]

Now, according to the definition in (52), on chains \( c \) of \( B(ZC) \) of dimensions 2 or greater, we can write \( \Phi(c) = \Phi_1(c) + \Phi_2(c) \), where \( \Phi_1 \) and \( \Phi_2 \) are the morphisms of \( HC \)-modules given
on generic cells by $\Phi_1[x | y | \sigma] = [\Phi[x | y | \sigma]$ and $\Phi_2[x | y | \sigma] = [gf[x | y | \Phi(\sigma)]$. Then, for the generic cell $\tau = [x | y | z | \rho]$, as
\[ \partial\tau = [\partial[x | y | z | \rho] - [x | \partial[y | z | \rho]] = [\partial[x | y | z | \rho] + [x | y | \partial[z | \rho]], \]
we have
\[ \Phi\partial(\tau) = \Phi_1[\partial[x | y | z | \rho] - \Phi_2[\partial[x | y | z | \rho]] + \Phi_2[\partial[x | y | z | \rho] + \Phi_2[\partial[x | y | \partial[z | \rho]] \]
\[ = [\Phi\partial[x | y | z | \rho] - [\Phi[x | y | z | \rho]] + [gf[x | y | \Phi[z | \rho]] + [gf[x | y | \Phi\partial[z | \rho]], \]

since $f\partial[x | y | z] = 0$ by (14). Furthermore, by using (10) and (53), we have
\[ \partial\Phi(\tau) = [\partial\Phi[x | y | z | \rho] + [gf[x | y | \Phi[z | \rho]], \]

whence, by the already proven above and induction on the dimension of $\rho$, we get
\[ (\partial\Phi + \Phi\partial)(\tau) = [\partial\Phi[x | y | z | \rho] + [gf[x | y | \Phi[z | \rho]] + [gf[x | y | \Phi\partial[z | \rho]] + [gf[x | y | \Phi\partial[z | \rho]], \]

as required.

This completes the proof of Theorem 7.4 since the conditions in (14) are easily verified.  

If $A$ is any $\mathcal{H}C$-module, by Proposition 5.3 the first level cohomology groups $H^n(C, 1; A)$ are precisely Leeoh cohomology groups $H^n_\mathcal{H}(C, A)$. Hence, by Theorem 7.4 these can be computed as $H^n_\mathcal{H}(C, A) = H^n\mathcal{H}(C, A)$. Since, by Proposition 4.4 there are natural isomorphisms
\[ \mathcal{H}(C, A) \cong A(\varphi(km)), \quad \mathcal{H}(C, A) \cong A(\varphi(km + 1)), \]
we obtain the following already known result (see [4, Theorem 5.1]) for a general result computing Leeoh cohomology groups for finite cyclic monoids.

**Proposition 7.5** (4, Corollary 5.6). Let $C = C_{m, q}$ be the cyclic monoid of index $m$ and period $q$. Then, for any $\mathcal{H}C$-module $A$ and any integer $k \geq 0$, there is a natural exact sequence of abelian groups
\[ 0 \rightarrow H^{k+1}_L(C, A) \rightarrow \mathcal{H}(\varphi(km + 1)) \rightarrow \mathcal{H}(\varphi(km + m)) \rightarrow H^{k+2}_L(C, A) \rightarrow 0, \]
where $\partial$ is given by $\partial(a) = (m + q)(m + q - 1)a - m(m - 1,a)$. 

Thus, for instance, if $A$ is any abelian group, regarded as a constant $\mathcal{H}C$-module, then the homomorphism $\partial : A \rightarrow A$ is multiplication by $q$, that is, $\partial(a) = qa$. Therefore, for all $k \geq 0,
\[ H^{k+1}_L(C, A) \cong \text{Ker}(q : A \rightarrow A), \]
\[ H^{k+2}_L(C, A) \cong \text{Coker}(q : A \rightarrow A). \]

We consider now the $r$th level cohomology groups of $C = C_{m, q}$ with $r \geq 2$. By Theorem 1.4 and an iterated use of Lemma 3.4 we conclude that the complexes of $\mathcal{H}C$-modules $B^r(\mathcal{H}C)$ and $B^{r-1}(\mathcal{R})$ are homotopy equivalent. Therefore, for any $\mathcal{H}C$-module $A$, there are natural isomorphisms
\[ H^n(\mathcal{C}, r, A) \cong H^n(\mathcal{H}(\mathcal{B}^{r-1}(\mathcal{R}), A)). \]
An analysis of the complexes $B^{n-1}(R)$ tell us that $B^{n-1}(R)_n = 0$ for $0 < n < r$, and that we have the diagram of suspensions

$$
\begin{array}{ccccccc}
R_4 & \rightarrow & R_3 & \rightarrow & R_2 & \rightarrow & R_1 & \rightarrow & 0 \\
s & \downarrow & s & \downarrow & s & \downarrow & s & \downarrow & s \\
B(R)_5 & \rightarrow & B(R)_4 & \rightarrow & B(R)_3 & \rightarrow & B(R)_2 & \rightarrow & 0 \\
&s & \downarrow & s & \downarrow & s & \downarrow & s & \downarrow \\
B^2(R)_6 & \rightarrow & B^2(R)_5 & \rightarrow & B^2(R)_4 & \rightarrow & B^2(R)_3 & \rightarrow & 0 \\
\end{array}
$$

where

- $B(R)_4$ is the free $\mathbb{H}C$-module on the binary set consisting of the suspension of the 3-cell $w_1$ of $R$ and the 4-cell $[w_0 | w_0]$ with $\pi[w_0 | w_0] = \varphi(2)$, whose differential is $\partial([w_0 | w_0]) = w_0 \circ w_0 = 0$,

- $B(R)_5$ is the free $\mathbb{H}C$-module on the set consisting of the suspension of the 4-cell $v_2$ of $R$ together the 5-cell $[w_0 | v_1]$, $[v_1 | w_0]$ with $\pi[w_0 | v_1] = m \oplus 1 = \pi[v_1 | w_0]$, and whose differential is

  $$\partial[w_0 | v_1] = w_1 - (m + q)((m + q - 1)_* [w_0 | w_0]) + m((m - 1)_* [w_0 | w_0]),$$

  $$\partial[v_1 | w_0] = -w_1 - (m + q)((m + q - 1)_* [w_0 | w_0]) + m((m - 1)_* [w_0 | w_0]).$$

- $B^2(R)_6$ is the free $\mathbb{H}C$-module on the set consisting of the double suspension of the 4-cell $v_2$ of $R$, the suspension of the 5-cells $[w_0 | v_1]$ and the $[v_1 | w_0]$ of $B(R)_5$, and the 6-cell $[w_0 \parallel w_0]$ with $\pi[w_0 \parallel w_0] = \varphi(2)$, whose differential is $\partial[w_0 | w_0] = 0$.

Then, by Proposition 4.1, there are natural isomorphisms

$$\Hom_{\mathbb{H}C}(B(R)_2, A) \cong A(1), \quad \Hom_{\mathbb{H}C}(B(R)_4, A) \cong A(m \oplus 1) \times A(\varphi(2)),$$
$$\Hom_{\mathbb{H}C}(B(R)_3, A) \cong A(m), \quad \Hom_{\mathbb{H}C}(B(R)_5, A) \cong A(\varphi(2m)) \times A(m \oplus 1) \times A(m \oplus 1),$$
$$\Hom_{\mathbb{H}C}(B^2(R)_6, A) \cong A(\varphi(2m)) \times A(m \oplus 1) \times A(m \oplus 1) \times A(\varphi(2)).$$

In these terms the truncated complex $\Hom_{\mathbb{H}C}(B(R), A)$ is written as

$$0 \rightarrow A(1) \xrightarrow{\partial^1} A(m) \xrightarrow{\partial^2} A(m \oplus 1) \times A(\varphi(2)) \xrightarrow{\partial^3} A(\varphi(2m)) \times A(m \oplus 1) \times A(m \oplus 1),$$

where the coboundaries are given by

$$\partial^1(a) = -(m + q)((m + q - 1)_* a) + m((m - 1)_* a),$$

$\partial^2 = 0$ is the morphism zero, and

$$\partial^3(a, b) = \left( -(m + q)((m + q - 1)_* a) + m((m - 1)_* a), a - (m + q)((m + q - 1)_* b) + m((m - 1)_* b), -a - (m + q)((m + q - 1)_* b) + m((m - 1)_* b) \right).$$
while the truncated complex $\text{Hom}_{\mathbb{H}C}(B^2(\mathcal{R}), \mathcal{A})$ is written as

\begin{equation}
(55) \quad 0 \to A(1) \xrightarrow{\partial_1} A(m) \xrightarrow{\partial_2} A(m+1) \times A(\varphi(2)) \xrightarrow{\partial_3} A(\varphi(2m)) \times A(m+1) \times A(\varphi(2)),
\end{equation}

where $\partial_1$ and $\partial_2$ are the same as above whereas $\partial_3$ acts by

$$
\partial_3(a, b) = \left( (m+q)((m+q-1)a) - m((m-1)a), \\
-a + (m+q)((m+q-1)b) - m((m-1)b), \\
a + (m+q)((m+q-1)b) - m((m-1)b), \quad 0 \right).
$$

Then, as an immediate consequence of (54) and (55), we have

**Theorem 7.6.** Let $C = C_{m,q}$ be the cyclic monoid of index $m$ and period $q$. Then, for any $\mathbb{H}C$-module $A$, there is a natural exact sequence of abelian groups

$$
0 \to H^2(C; 2; A) \to A(1) \xrightarrow{\partial} A(m) \to H^3(C; 2; A) \to 0
$$

where $\partial(a) = (m+q)((m+q-1)a) - m((m-1)a)$, and natural isomorphisms

$$
H^4(C, 2; A) \cong H^3(C, 3; A) \cong \left\{ b \in A(\varphi(2)) \mid (m+q)^2\varphi(2m+q-2), b = m^2\varphi(2m-2), b, \right. \\
\left. 2(m+q)(m+q-1), b = 2m(m-1), b, \right\}.
$$

Note that in the case when the cyclic monoid is of index $m = 1$, the above description of $H^4(C; 2; A)$ adopts the simpler form

$$
H^4(C, 2; A) \cong \left\{ b \in A(\varphi(2)) \mid (q+1)^2q, b = b, \right. \\
\left. 2(q+1)q, b = 2b, \right\},
$$

while when $m \geq 2$,

$$
H^4(C, 2; A) \cong \left\{ b \in A(\varphi(2)) \mid (2mq + q^2)\varphi(2m-2), b = b, \right. \\
\left. 2(m+q)(m+q-1), b = 2m(m-1), b, \right\}.
$$

**Corollary 7.7.** For any finite cyclic monoid $C$, any integer $r \geq 1$, and any $\mathbb{H}C$-module $A$, there are natural isomorphisms

$$
H^{r+1}(C, r; A) \cong H^2(C, A) \cong H^2(C, A).
$$

**Proof.** A direct comparison of the exact sequence in Theorem 7.6 with the sequence in Proposition 5.11 for the case when $k = 0$, gives $H^3(C; 2; A) \cong H^2(C; A)$. Then, the result follows since $H^3(C; 2; A) \cong H^2(C; A)$ by Proposition 5.11 and $H^{r+1}(C, r; A) \cong H^3(C; 2; A)$ by Corollary 5.8. $\square$

**Corollary 7.8.** For any finite cyclic monoid $C$, any integer $r \geq 2$, and any $\mathbb{H}C$-module $A$, there are natural isomorphisms

$$
H^{r+2}(C, r; A) \cong H^2(C, A).
$$

**Proof.** By Corollary 5.9 $H^{r+2}(C, r; A) \cong H^2(C, 3; A)$, for any $r \geq 3$. Since, by Theorem 7.6 $H^3(C, 3; A) \cong H^4(C, 2; A)$, the result follows by Proposition 5.12. $\square$

For instance, if $A$ is any abelian group viewed as a constant $\mathbb{H}C$-module, then $H^4(C, 2; A)$ is isomorphic to the subgroup of $A$ consisting of those elements $b$ such that

$$
\begin{array}{c}
(m+q)^2b = m^2b, \\
2qb = 0,
\end{array} \quad \iff \quad \begin{array}{c}
2mq + q^2b = 0, \\
2qb = 0,
\end{array} \quad \iff \quad \begin{array}{c}
q^2b = 0, \\
2qb = 0,
\end{array} \quad \iff \quad \begin{array}{c}
(2q, q^2) b = 0,
\end{array}
$$

where $(2q, q^2) = \gcd(2, q)$ is the greatest common divisor of 2 and $q$. This leads to the following isomorphism, which is analogous to the proven by Eilenberg- Mac Lane for the third abelian cohomology group of the cyclic group $C_q$ with coefficients in $A$ [8, §21].
Corollary 7.9. For any finite cyclic monoid $C$, any integer $r \geq 2$, and any abelian group $A$, there is a natural isomorphism
\[ H^{r+2}(C; r; A) \cong \text{Hom}_{\text{Ab}}(\mathbb{Z}/(2q^2)\mathbb{Z}, A). \]

7.2. Cohomology of the infinite cyclic monoid.

In this subsection we focus on the additive monoid of natural numbers $\mathbb{N}$. As before, we start by introducing a commutative DGA-algebra over $\mathbb{H}C_\infty$, simpler than $B(\mathbb{Z}C_\infty)$.

For each integer $k = 0, 1, \ldots$, let us choose unitary sets over $\mathbb{N}$.

Then, as consequence of Theorem 7.10, we recover the computation by Leech of the $\mathbb{H}$-algebra morphisms $f : B(\mathbb{Z}C_\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to B(\mathbb{Z}C_\infty)$, determined by the formulas
\[
\begin{align*}
\{ f[1] \} &= v_0, \\
\{ f[x] \} &= x(x-1)_s w_0.
\end{align*}
\]

which form a contraction.

Proof. It is plain to see that above assignments determine well defined morphisms of DGA-algebras over $\mathbb{H}C_\infty$. To prove that they form a contraction, we limit ourselves to describe the homotopy $\Phi : g f \Rightarrow id$, by the formula below, because the details are parallel and much more simpler than those in the proof of Theorem 7.10.

It is obvious that $R_0 \cong \mathbb{Z}$, and the multiplication on $R$ is by determined by the rules $v_0 \circ v_0 = v_0$, $v_0 \circ w_0 = w_0$ and $w_0 \circ w_0 = 0$.

Theorem 7.10. There are DGA-algebra morphisms $f : B(\mathbb{Z}C_\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to B(\mathbb{Z}C_\infty)$, determined by the formulas
\[
\begin{align*}
\{ f[1] \} &= v_0, \\
\{ f[x] \} &= x(x-1)_s w_0.
\end{align*}
\]

which form a contraction.

Proof. It is plain to see that above assignments determine well defined morphisms of DGA-algebras over $\mathbb{H}C_\infty$. To prove that they form a contraction, we limit ourselves to describe the homotopy $\Phi : g f \Rightarrow id$, by the formula below, because the details are parallel and much more simpler than those in the proof of Theorem 7.10.

By Proposition 7.8, there are isomorphisms $H^n(C_{\infty}; 1; A) \cong H^n_{\mathbb{L}}(C_{\infty}; A)$, for any $\mathbb{H}C_{\infty}$-module $A$. Then, as consequence of Theorem 7.10 we recover the computation by Leech of the cohomology groups of the monoid $C_{\infty}$ [16, Theorem 6.8].

Proposition 7.11. For any $\mathbb{H}C_{\infty}$-module $A$, there are natural isomorphisms
\[ H^n_{\mathbb{L}}(C_{\infty}, A) \cong A(0), \quad H^n_{\mathbb{L}}(C_{\infty}, A) \cong A(1), \]
and for every $n \geq 2$, $H^n_{\mathbb{L}}(C_{\infty}, A) = 0$.

We now pay attention to the second level cohomology groups of $C_{\infty}$. By Theorem 7.10 and Lemma 7.4, $H^n(C_{\infty}, 2; A) \cong H^n(\text{Hom}_{\mathbb{H}C_{\infty}}(B(\mathbb{R}), A))$. An analysis of $B(\mathbb{R})$ tells us that
\[
\begin{align*}
B(\mathbb{R})_{2k} &= \text{the free } \mathbb{H}C_{\infty}-\text{module on } \{v_k\}, \\
B(\mathbb{R})_{2k+1} &= 0,
\end{align*}
\]

where, recall, $\pi v_k = k$; the augmentation is the canonical isomorphism $B(\mathbb{R})_0 \cong \mathbb{Z}$ and the product is given by
\[ v_k \circ v_l = \binom{k+l}{k} v_{k+l}. \]

Hence,
Proposition 7.12. For any \( \mathbb{H}C_\infty \)-module \( A \), and any integer \( k \geq 0 \),
\[
H^{2k}(C_\infty, 2; A) \cong A(k), \quad H^{2k+1}(C_\infty, 2; A) = 0.
\]

From Corollary 5.8 it follows that

Corollary 7.13. For any \( \mathbb{H}C_\infty \)-module \( A \), and any integer \( r \geq 2 \),
\[
H^{r+1}(C_\infty, r; A) = 0.
\]

We finish by specifying the 3rd level 5-cohomology group of \( C_\infty \).

Proposition 7.14. For any \( \mathbb{H}C_\infty \)-module \( A \), and any integer \( r \geq 3 \), there is a natural isomorphism
\[
H^{r+2}(C_\infty, r; A) \cong \{ a \in A(2) \mid 2a = 0 \}.
\]

Proof. By Corollary 5.9, \( H^{r+2}(C_\infty, r; A) \cong H^5(C_\infty, 3; A) \). An analysis of \( B^2(\mathcal{R}) \) tell us that \( B^2(\mathcal{R})_1 = B(\mathcal{R})_3 = 0 \), \( B^2(\mathcal{R})_5 = B(\mathcal{R})_4 \) is the free \( \mathbb{H}C_\infty \)-module on \( \{v_1\} \), where \( \pi v_2 = 2 \), \( B^2(\mathcal{R})_6 \) is the free \( \mathbb{H}C_\infty \)-module on \( \{(v_1 \parallel v_1)\} \), with \( \pi [v_1 \parallel v_1] = 2 \), and the differential is \( \partial [v_1 \parallel v_1] = -2v_2 \).

Whence, for any \( \mathbb{H}C_\infty \)-module \( A \), \( H^5(C_\infty, 3; A) \cong \{ a \in A(2) \mid 2a = 0 \} \).

\[\square\]

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