Viability of a viscous $\Lambda$WDM model: Near equilibrium condition, mathematical stability and entropy production

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Abstract:

In this paper we study the viability of an exact solution recently found in [N. Cruz, E. González, and J. Jovel Phys. Rev. D 105, 024047 (2022)], which describe the evolution of a viscous warm ADM model at late times, where the DM component obey a polytropic EoS and experiment a dissipation with a bulk viscosity proportional to its the energy density, leading a behavior very similar to the $\Lambda$CDM model for a small dissipation, evolving also to a de Sitter type expansion at the very far future. In the present work, the viability of this solution lies in the fulfillment of the three following conditions: the near equilibrium condition, that it is assumed in the Eckart’ theory of non-perfect fluids; the mathematical stability of the solution under small perturbations, and the positiveness of the entropy production. We explore the conditions on the range of parameters of the model that allow to fulfill the three conditions at the same time, founding that a warm DM component, compatible with previous results derived from the cosmological data, and a very small viscosity, is a viable model from the thermodynamic point of view, which also behaves very close to the standard model, with the same asymptotic de Sitter expansion.

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I. INTRODUCTION

In the current cosmology the observational evidence suggests that almost the total energy density of the Universe is compound by the dark sector, roughly classified into 30% of dark matter (DM) and 70% of dark energy (DE) [1–4]. The DM is responsible of the structure formation in the Universe while the DE is responsible of the recent accelerating expansion of the Universe [5]. The most simple model that includes these components and fits very well the cosmological data is the $\Lambda$CDM model [1, 2], where the DE is modeled by a positive cosmological constant (CC) $\Lambda$ and the DM is described as a pressureless fluid known as cold DM (CDM). However, this model is not absent of problems from the theoretical and observational point of view. For example, the CC problem, where the value of the CC differs from theoretical field estimations in 60-120 order of magnitude than the observed value [6–8]; also, measurements of the Hubble parameter at the current time, $H_0$, presents a discrepancy of 4.4$\sigma$ between the measurements obtained from Planck CMB and the locally measurements obtained by A. G. Riess et al. [9].

In order to try to overcome some of these problems, dissipative effects can be considered as a more realistic way of treating cosmic fluids. In a homogeneous and isotropic universe, the dissipative process is usually characterized by a bulk viscosity. In this sense, some authors have considered the non-inclusion of the CC in order to alleviate the CC problem, explaining the late time acceleration behavior of the Universe through a dissipative viscous fluid [10–21], as a natural choice since the effect of the bulk viscosity is to produce a negative pressure that leads to an acceleration in the universe expansion [22]. Also in [23, 24] the authors discuss the $H_0$ tension problem in the context of a dissipative fluids as a good chance to construct new cosmological models with non-ideal fluids.

On the other hand, bulk viscosity seems to be significant in the cosmic evolution. For example, many observational properties of disk galaxies can be represented by a dissipative DM component [25, 26]. For neutralino CDM, the bulk viscous pressure is present in the CDM fluid through the energy transferred from the CDM fluid to the radiation fluid [27]. Some authors propose that bulk viscosity can produce different cooling rates of the components of the cosmic fluid [28, 30], or may be the result of non-conserving particle interactions [31]. Even more, from Landau and Lifshitz [32], the bulk viscosity can be interpreted from the macroscopic point of view, as the existence of slow processes to restore the equilibrium state. At perturbative level, viscous fluid dynamics provides also a simple and accurate framework with the purpose of extend the description into the nonlinear

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regime \cite{33}. Therefore, since the nature of the DM it is unknown up to date and dissipative effect can not be discarded, it is of physical interest to explore how a viscous DM behaves in the ΛCDM model.

In the study of dissipative fluids it is necessary to develop a relativistic thermodynamic theory out of equilibrium, being Eckart the first to develop it \cite{34}, with a similar model proposed by Landau and Lifshitz \cite{32}. However, it was later shown that the Eckart’s theory was a non-causal theory \cite{35, 36}. A causal theory was proposed by Israel and Stewart (IS) \cite{37, 38}, which is reduced to the Eckart’s theory when the relaxation time for the bulk viscous effects are negligible \cite{39}. Therefore, considering that the IS theory presents a much greater mathematical difficulty than the Eckart’s theory, this last one is considered as a first approximation in order to study viscous cosmology \cite{40, 45}. It is important to mention that, in these theories the bulk viscous pressure II has to be lower than the equilibrium pressure \( p \) of the dissipative fluid, i.e.,

\[
 l = \frac{\Pi}{|p|} \ll 1, \tag{1}
\]

known as the near equilibrium condition.

According to Maartens, in the context of dissipate inflation \cite{39}, the condition to have an accelerated expansion due only to the negativeness of the viscous pressure \( II \) in the Eckart and IS theories enters into direct contradiction with the near equilibrium condition given by Eq. (1). In this sense, as has been proposed in \cite{40, 47} that if a positive CC is consider in Eckart or IS theory the near equilibrium condition could be preserved in some regime. In addition, it was showed by J. Hua and H. Hu \cite{43} that a dissipative DM in the Eckart’s theory with CC has a significantly better fitting with the cosmological data than the ΛCDM model, which indicates that this model is competitive in order to fit the combined SNe Ia + CMB + BAO + H(z) data. Nevertheless, the inclusion of the CC implies to abandon the idea of unified DM models with dissipation, whose advantage is to avoid the CC problem.

Another important point of the near equilibrium condition given by the Eq. (1) is the fact that we need to have a non-zero equilibrium pressure for the dissipative fluid, discarding the possibility of a CDM, which leads to consider a dissipative warm DM (WDM) \cite{48, 53} in order to satisfy the near equilibrium condition. The possibility of a WDM has been contrasted with observations \cite{48, 50, 54, 55} as a candidate for DM (for reviews of DM candidates see e.g. \cite{56, 60}) in response to the fact that its nature remains unclear. Some other candidates could be active neutrinos, axions or supersymmetric (SUSY) particles (like gravitinos, neutralinos or axinos) \cite{48}. In the case of active neutrinos, they are too light to be consistent with some observations \cite{61, 63}. Therefore, an interesting WDM particle hypothesis implies an extension of the standard model of particle physics by three sterile (right-handed, gauge singlet) neutrinos \cite{48, 65, 66}, produced via mixing with active neutrinos in the early Universe \cite{48, 67, 71}.

It is important to mention that, for cosmologies with perfect fluids there are not entropy production because these fluids are in equilibrium and their thermodynamics are reversible. But, for cosmologies with non-perfect fluid, where irreversible process exists, we need to imposed a positive entropy production during the cosmic evolution \cite{19, 72, 73}. On the other hand, for analytical solutions, it is of importance from the physical point of view to study the stability of the solutions under small perturbations. This criteria has been considered previously, for example by Maartens \cite{39}, for inflationary solutions, obtaining a constrain upon the equation of state (EoS) of the dissipative fluid. The purpose is then that the constrains obtained from the mathematical stability criteria can be compared with the constrains obtained from the fulfillment of the near equilibrium condition and the positiveness of the entropy production, giving some important clues about the physical behavior of the analytical solutions.

The near equilibrium condition, mathematical stability and entropy production has been previously discussed in the literature. The near equilibrium condition was studied, for example, in \cite{73} for the IS theory with gravitational constant \( G \) and \( A \) that vary over time; while in \cite{70} it was studied in the Eckart and IS theories for the case of a dissipative Boltzmann gas and without the inclusion of a CC. The mathematical stability was studied, in particular, in \cite{77} in the IS theory for a universe filled with one viscous fluid, whose bulk viscosity obeys a power law in the energy density and without the inclusion of a CC; while in \cite{78} it was studied in the de Sitter phase of cosmic expansion when the source of the gravitational field is a viscous fluid. The entropy production was studied in \cite{72} in the Eckart and IS theories for a dissipative DE; while in \cite{19} the authors study the entropy production in the full IS theory with a matter content represented by a one dissipative fluid component, and the kinematics and thermodynamics properties of the solutions are discussed (the entropy production in cosmological viscous fluids has been more widely studied, and more references can be found in \cite{79, 84}).

The aim of this paper is to study an analytical solution obtained in \cite{55}, for a flat Friedman-Lemaître-Robertson-Walker (FLRW) universe, dominated by a dissipative DM modeled by the barotropic EoS \( p = (\gamma - 1)\rho \), where \( \rho \) is the energy density of the dissipative DM and \( \gamma \) is known as barotropic index, and a DE modeled by the CC, in the framework of the Eckart’s theory. This solution was obtained using the expression \( \xi = \xi_0/\rho \) for the bulk viscosity, where \( \xi_0 > 0 \) is a bulk viscous constant, and was studied in the context of the late and early-times singularities. This solution has the particularity that, for a positive CC, behave very similarly to the ΛCDM model for all the cosmic time when \( \xi_0 \to 0 \), without singularity towards the past in a asymptotic behavior known as “soft-Big Bang” \cite{86, 87}, and with an asymptotic de Sit-
ter expansion towards the future. This last behaviour is of interest because the solution tends to the de Sitter expansion regardless of the value of $\xi_0$ and $\gamma$, as long as $\xi_0 < \gamma/3H_0$. Therefore, we focus our study to the late-time behaviour of this solution assuming that $\gamma \neq 1$ but close to 1, which represents a dissipative $\Lambda$WDM model with the same asymptotic late-time behaviour that the $\Lambda$CDM model. In particular, we study the near equilibrium condition, the mathematical stability and the entropy production of this solution in order to find the constrains that these criteria impose on the model’s free parameters, focusing in the possibility to have a range of them satisfying all of these conditions. We discuss how the presence of a CC and WDM favour the fulfillment of the above criteria. It is important to mention that the bulk viscosity coefficient $\xi$ depends, particularly, on the temperature and pressure of the dissipative fluid [88].

Therefore, a natural election for the bulk viscosity of the temperature and pressure of the dissipative fluid [88].

In this section we briefly resume a de Sitter-like solution in Eckart's theory with CC.

Obtained with the border condition $H(t=0) = H_0$, and $a(t=0) = 1$, which leads to

$$H(t) = \frac{H_0}{\sqrt{\Omega_{\Lambda_0}}} \left(\left(\sqrt{\Omega_{\Lambda_0}} + 1\right) e^{3\gamma H_0 t \sqrt{\Omega_{\Lambda_0}}} - \sqrt{\Omega_{\Lambda_0}} + 1\right),$$

with $E = H(t)/H_0$. It is important to note that Eq. (1) is the usual de Sitter solution, written in its dimensionless form, and naturally appears in this dissipative scenario. On the other hand, the exact analytical solution ($\dot{H} = 0$) found takes the following expression in terms of the dimensionless parameters

$$\tau = \frac{\Omega_{\xi_0} \sqrt{\Omega_{\Lambda_0}} \log\left(\frac{(1-\Omega_{\Lambda_0})\gamma - \Omega_{\xi_0}}{\Omega_{\xi_0}}\right)}{3\sqrt{\Omega_{\Lambda_0}} \left(\gamma^2 - \Omega_{\xi_0}^{-1}\right)} + \frac{\gamma \log\left(\frac{\Omega_{\xi_0}^{-1}}{\Omega_{\Lambda_0}}\right)}{3\sqrt{\Omega_{\Lambda_0}} \left(\gamma^2 - \Omega_{\xi_0}^{-1}\right)},$$

where $\Omega_{\Lambda_0} = \Lambda/(3H_0^2)$. Note that Eq. (2) tends asymptotically at very late times ($t \to \infty$) to the de Sitter solution $H_{ds} = H_0/\sqrt{\Omega_{\Lambda_0}}$.

We are particularly interested in the case of $m = 1$ of Eq. (2) with a positive CC, where the following de Sitter-like solution ($\dot{H} = 0$) are found

$$E_{ds} = \sqrt{\Omega_{\Lambda_0}},$$

obtained with the border condition $H(t=0) = H_0$, and where $\Omega_{\xi_0} = 3\xi_0 H_0$ and $\gamma = tH_0$ are the dimensionless bulk viscous constant and cosmic time, respectively. The above solution is an implicit relation of $E(\tau)$. According to [88], Eq. (6) presents a future singularity in a finite time known as Big-Rip [145, 149, 152] when $\Omega_{\xi_0} > \gamma$. In this singularity we have an infinite $a$, $\dot{a}$ and $a$, and, therefore, the Ricci scalar diverges. Also, is discussed that one interesting behavior of this solution can be obtained if we considered the opposite condition, i.e.,

$$\Omega_{\xi_0} < \gamma,$$

which leads to a universe with a behavior very similar to the $\Lambda$CDM model, which coincide as $\Omega_{\xi_0} \to 0$, as can be seen in Fig. 4 where we have numerically found the behavior of $E$ as a function of $\tau$ from Eq. (6), taking into account the condition (7) with $\gamma = 1.002$, $\Omega_{\xi_0} = 0.001$. Thus, we focus our study to the late-times, where in Sec. IV we study the viability of the solution at late-times, where in Sec. IV, we present some conclusions and final discussions.
and $\Omega_{\Lambda_0} = 0.69$. For a comparison, we also plotted the $\Lambda$CDM model.

Note that the solution [6] tends asymptotically for $\tau \to \infty$ to the usual de Sitter solution [5], which can be seen in the Fig. 1. Therefore, for the condition given by Eq. (7) and for $\gamma \neq 1$ but close to 1, solution [6] represent a viscous $\Lambda$WDM model with a late-time behaviour very similar to the $\Lambda$CDM model and with the same asymptotic de Sitter expansion.

![Fig. 1](image)

**FIG. 1.** Numerical behavior of $E(\tau)$ obtained from Eq. (6) at late times, for $\Omega_{\Lambda_0} = 0.69$, $\Omega_{\xi_0} = 0.001$ and $\gamma = 1.002$. For a comparison, we also plotted the $\Lambda$CDM model obtained from Eq. (5).

### III. NEAR EQUILIBRIUM CONDITION, MATHEMATICAL STABILITY AND ENTROPY PRODUCTION

In what follows we found the main expressions in terms of $\gamma$, and the dimensionless quantities $E$, $\Omega_{\xi_0}$ and $\Omega_{\Lambda_0}$, that arises from the near equilibrium condition, the mathematical stability and the entropy production.

#### A. Near equilibrium condition

As it was previously discussed, in the Eckart’s theory it is necessary to fulfill the near equilibrium condition [1]. Following Maartens [39] and according to the expression

$$\frac{\ddot{a}}{a} = -\frac{1}{6} \left[ \rho + 3 (p + \Pi) \right] + \frac{\Lambda}{3},$$

the condition to have an accelerated expansion driven only by the negativeisness of the bulk viscous pressure $\Pi$, imposing $\ddot{a} > 0$ and taking $\Lambda = 0$, is

$$-\Pi > p + \frac{\rho}{3}.$$  (9)

This last result implies that the viscous stress is greater than the equilibrium pressure $p$ of the fluid, i.e., the near equilibrium condition is not fulfilled because in order to obtain an accelerated expansion the fluid has to be far from equilibrium. This situation could be change if a positive CC is included [46, 47]. In this case the condition $\ddot{a} > 0$ on Eq. (8) leads to

$$-\Pi > -\frac{2\Lambda}{3} + p + \frac{\rho}{3},$$

i.e., the near equilibrium condition could be fulfilled in some regime, because from Eq. (10) the viscous stress not necessarily is greater than the equilibrium pressure $p$. The near equilibrium condition given by Eq. (11) can be rewritten in terms of the dimensionless parameters, using the expression $\Pi = -3H\xi$, which is the viscous pressure in the Eckart’s theory [34], and the EoS of the DM component, obtaining

$$l = \left| \frac{E(\tau)\Omega_{\xi_0}}{\gamma - 1} \right| \ll 1.$$  (11)

From the above equation is clear to see that for a CDM component with $\gamma = 1$ it is not possible to satisfy the near equilibrium condition, and only for some kind of WDM with $\gamma > 1$ it is possible to find some constraints for the parameters of the models that can fulfill this condition. On the other hand, note that in the above expression the solution given by $E(\tau)$ drives the behavior of $l$ as a function of the cosmic time $\tau$. In this sense, and since $E(\tau)$ is a decreasing function of time, the constraints on $\Omega_{\xi_0}$ are more restrictive as we look forward.

#### B. Mathematical stability

In this section we explore the mathematical stability of the solution in order to find possible new constraint upon the main free parameters of the model. To do so, we investigate the behavior of a perturbed solution of the form

$$H_{\delta}(t) = H(t) + h(t), \quad |h(t)| \ll 1,$$  (12)

where $H(t)$ correspond to the unperturbed solution and $h(t)$ is a small perturbation function. Introducing Eq. (12) in Eq. (2), we obtain the following differential equation, in our dimensionless notation, for $h(t)$:

$$\dot{h} - \frac{9\Omega_{\xi_0}}{2} \left( E^2 - \frac{2\gamma}{3\Omega_{\xi_0}} \right) h = 0.$$  (13)

The above expression is a differential equation for the perturbation function $h(t)$ and must satisfy, in order to the solution $H(t)$ be mathematically stable, that $h \to 0$ when $t \to \infty$.

#### C. Entropy production

The First law of thermodynamics is given by

$$TdS = dU + pdV,$$  (14)
where $T$, $S$, $U$, $V$ are the temperature, entropy, internal energy, and the three dimensional volume of the cosmic fluid. The internal energy of the fluid and the physical three dimensional volume of the Universe are given respectively by $U = \rho V$ and $V = V_0 a^3$ (where $V_0$ is the volume at the present time). With these, we get from Eq. (14) the Gibbs equation [72]

\[ dS = - \left( \frac{\rho + p}{T n^2} \right) dn + \frac{dp}{T n}, \quad (15) \]

where $n = N/V$ is the number of particle density. The following integrability condition must hold on the thermodynamical variables $\rho$ and $n$

\[ \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial S}{\partial n} \right) \right]_n = \left[ \frac{\partial}{\partial n} \left( \frac{\partial S}{\partial \rho} \right) \right]_\rho, \quad (16) \]

then, we considered the thermodynamic assumption in which the temperature is a function of the number of particles density and the energy density, i.e., $T(n, \rho)$. With this, the above integrability condition become in [72, 75]

\[ n \frac{\partial T}{\partial n} + (\rho + p) \frac{\partial T}{\partial \rho} = T \frac{\partial \rho}{\partial \rho}. \quad (17) \]

We study the case of a perfect fluid and a viscous fluid separately in order to compare our result with the model without viscosity.

For a perfect fluid the particle 4-current is taken to be $n^\alpha_\rho = 0$, where “;” accounts for the covariant derivative, which together with the conservation equation, we have the following expressions for the particle density and the energy density, respectively

\[ \dot{n} + 3Hn = \frac{\dot{N}}{N} = 0, \quad (18) \]

\[ \dot{\rho} + 3H(\rho + p) = 0. \quad (19) \]

Assuming that the the energy density depends on the temperature and the volume, i.e., $\rho(T, V)$ [72], we have the following relation:

\[ \frac{d\rho}{da} = \frac{\partial \rho}{\partial T} \frac{dT}{da} + 3n \frac{\partial \rho}{\partial n}. \quad (20) \]

Using Eqs. [19], [20], and the EoS, it can be shown (as in [72]) that the temperature from [17] is directly proportional to the internal energy, and is given by

\[ \frac{T}{T_0} = \frac{\rho}{\rho_0} \frac{a^3}{U} = \frac{U}{U_0}. \quad (21) \]

Additional to this, from Eq. [15], together with Eqs. [18], [19], and the EoS, we have $dS = 0$ or, consequently, $dS/dt = 0$, which imply that there is no entropy production in the cosmic expansion, i.e. the fluid is adiabatic.

For a viscous fluid an average 4-velocity is chosen in which there is no particle flux [88]; so, in this frame, the particle 4-current is taking again as $n^\alpha_\rho = 0$ and the equation [18] is still valid. On the other hand, from the Eckart’s theory, we have the following conservation equation [34]:

\[ \dot{\rho} + 3H(\rho + p + \Pi) = 0, \quad (22) \]

which together with the Eq. (18) and the EoS, Eq. (15) give us the follow expression for the entropy production [72, 74], written in a dimensionless form,

\[ \frac{dS}{d\tau} = \frac{3E^2 \Omega_{\xi_0} \rho}{nT}. \quad (23) \]

Therefore, the entropy production in the viscous expanding universe is, in principle, always positive and we recovered the behavior of a perfect fluid when $\Omega_{\xi_0} = 0$. As we will see later, this positiveness require some constrains under the free parameters of the solution, specifically, in the expression for the temperature of the dissipative fluid.

\section*{IV. VIABILITY OF THE EXACT SOLUTION}

In this section we study the viability of the exact solution [6] under the condition [7] in terms of the fulfillment at the same time of the near equilibrium condition, the mathematical stability, and the positiveness of the entropy production. For that end, we focus our analysis in two defined late-time epochs of validity for the solution: (i) the actual time $\tau = 0$ for which $E = 1$, and (ii) the very late-times $\tau \to \infty$ for which $E = \sqrt{\Omega_{\Lambda_0}}$. We will extend the analysis for $0 < \tau < \infty$ and for $\tau < 0$.

It is important to mention that the asymptotic behaviour of the solution [6], given by the de Sitter solution [9] when the condition [7] holds, leads to a universe dominated only by the cosmological constant in which the dissipative WDM is diluted due to the universe expansion, as can be seen by evaluating the Eq. (5) in the Friedmann equation

\[ H^2 = \frac{\rho}{3} + \frac{\Lambda}{3}, \quad (24) \]

which leads to $\rho = 0$. Therefore, in this asymptotic behaviour we do not have a dissipative fluid to study the near equilibrium condition and the entropy production. Nevertheless, we can study these two conditions as an asymptotic regime of the solution while $\rho \to 0$.

\subsection*{A. Near equilibrium condition of the exact solution}

Note that the near equilibrium condition [11], considering that $1 < \gamma \leq 2$, can be rewritten as

\[ E(\tau) \ll \frac{\gamma - 1}{\Omega_{\xi_0}} = \frac{\gamma}{\Omega_{\xi_0}} - \frac{1}{\Omega_{\xi_0}}, \quad (25) \]

expression that tells us that as a long as $E = H/H_0$ be much smaller than $(\gamma - 1)/\Omega_{\xi_0}$, the solution must be
near the thermodynamic equilibrium. This open the possibility that the solution is able to fulfill this condition considering that \( E(t) \), given by Eq. (9), decrease asymptotically to the de Sitter solution [3] as \( \tau \to \infty \) under the condition [7], as we can be seen in Fig. 1. Furthermore, at the actual time \( \tau = 0 \), the condition (25) leads to

\[
\Omega_{\xi_0} \ll \gamma - 1,
\]

and for the very late times \( \tau \to \infty \) leads to

\[
\sqrt{\Omega_{\xi_0}} \ll \frac{\gamma - 1}{\Omega_{\xi_0}} = \frac{\gamma}{\Omega_{\xi_0}} - \frac{1}{\Omega_{\xi_0}}.
\]

The fulfillment of the condition (26) implies the fulfillment of the condition (27), because \( 0 < \Omega_{\xi_0} \leq 1 \) and from the condition (26) we get \( 1 \ll (\gamma - 1)/\Omega_{\xi_0} \). Also, note that the fulfillment of the condition (26) implies the fulfillment of the condition (7). Hence, the fulfillment of the condition (26) implies the fulfillment of the near equilibrium condition from \( 0 \leq \tau < \infty \) and the condition [7] for which the solutions (6) behave as the de Sitter solution at very late times. It is important to note that the fulfillment of the near equilibrium condition depends only on the values of \( \gamma \) and \( \Omega_{\xi_0} \), and not in the values of \( \Omega_{\Lambda_0} \), with the characteristic that for a value of \( \gamma \) more closer to 1 (CDM) a smaller value of \( \Omega_{\xi_0} \) must be considered. Even more, for \( 1 < \gamma \leq 2 \) we can see that \( \Omega_{\xi_0} < 1 \). On the other hand the condition (26) is independent of the behaviour of the solution because the election \( E(\tau = 0) = 1 \) is arbitrary, but, this not implies that the condition (27) be always fulfilled because this condition depends on the behaviour of the solution. In this sense note that if we do not satisfy the condition (7) in Eq. (11), then \( E \) diverges and the solution will be far from near equilibrium in a finite time in the Big-Rip scenario.

For \( \tau < 0 \) it is still possible to fulfill the near equilibrium condition (25), as we mentioned above, while \( E \) be much smaller than \( (\gamma - 1)/\Omega_{\xi_0} \). In Fig. 2 we depict the near equilibrium condition (11) as a function of \( \Omega_{\xi_0} \) and \( E \) for the fixed values of \( \Omega_{\Lambda_0} = 0.69 \) and \( \gamma = 1.002 \). The red zone represent the values for which we are far from the near equilibrium and the green line represent the near equilibrium condition when \( \Omega_{\xi_0} = 0.001 \) and the red zone represent the values for which we are far from the near equilibrium condition (\( l > 1 \)).

**B. Mathematical stability of the exact solution**

To analyze the mathematical stability we use the Eqs. (2) and (13), changing the integration variable from \( t \) to \( E \), obtaining in our dimensional notation the expression

\[
\frac{dh}{dE} = \frac{\Omega_{\Lambda_0} \Omega_{\xi_0} + 2\gamma E - 3E^2 \Omega_{\xi_0}}{(E^2 - \Omega_{\Lambda_0})(\gamma - E \Omega_{\xi_0})} h,
\]

whose integration leads to

\[
h(E) = C (\gamma - E \Omega_{\xi_0}) \left( E^2 - \Omega_{\Lambda_0} \right),
\]

being \( C \) and integration constant. From this equation we can see that the perturbation function is zero when \( E = \pm \sqrt{\Omega_{\Lambda_0}} \) and when \( E = \gamma/\Omega_{\xi_0} \). Furthermore, using the second derivative criteria we can see that

\[
E_{\min} = \frac{\sqrt{\gamma^2 + 3\Omega_{\xi_0}^2 \Omega_{\Lambda_0}}}{3 \Omega_{\xi_0}},
\]

and

\[
E_{\max} = \frac{\gamma + \sqrt{\gamma^2 + 3\Omega_{\xi_0}^2 \Omega_{\Lambda_0}}}{3 \Omega_{\xi_0}},
\]

are the points where the function (29) have a relative minimum and maximum, respectively. Note that the argument in the square root of Eqs. (30) and (31) as \( \rho^{3/2} \) and the equilibrium pressure behaves as \( \rho \) and, therefore, considering that in this case \( \rho \) grows to the past, then the dissipative pressure grows more quickly than the equilibrium pressure and \( \xi_0 \) acts as a modulator of this growth.
are always positive. Therefore, the perturbation function \( h \) is a decreasing function when \(-\infty < E < E_{\text{min}}\) and \( E_{\text{max}} < E < \infty \), and an increasing function when \( E_{\text{min}} < E < E_{\text{max}}\).

Considering that the solution \( [6] \) is a decreasing function with the time, which tends to \( \gamma/\Omega_{\xi_0} \) when \( \tau \to -\infty \) and to \( \sqrt{\Omega_{\Lambda_0}} \) when \( \tau \to \infty \), this last one as a function as we fulfill the condition \([7]\), we can conclude that for \( \Omega_{\xi_0} \neq 0 \), value for which the point \( E_{\text{min}} \) given by the Eq. \([30]\) is negative, the perturbative function is bounded above by \( h(E_{\text{max}}) \) and is zero when \( \tau \to \pm \infty \), i.e., the solution \([6]\) is mathematically stable. On the other hand, when \( \Omega_{\xi_0} \to 0 \), the Eq. \([29]\) becomes

\[
h(E) \to C \gamma \left( E^2 - \Omega_{\lambda_0} \right),
\]

function that has only a relative minimum for \( E = \sqrt{\Omega_{\lambda_0}} \). Therefore, in this case the perturbation function \( h \) is stable only for \( E \to \sqrt{\Omega_{\lambda_0}} \), i.e., for \( \tau \to \infty \) (very late-times) and unstable for \( E \to \infty \). This is an expected behavior because the solution \([6]\) in this case tends to a solution with a singularity towards the past very similar to the \( \Lambda \)CDM model, which coincides when \( \gamma = 1 \).

![FIG. 3. Behavior of \( h \), given by Eq. \( [29] \), for \( 0 \leq \Omega_{\xi_0} \leq 0.1 \) and \( \sqrt{\Omega_{\lambda_0}} \leq E \leq 10 \). We also consider the fixed values of \( \Omega_{\lambda_0} = 0.69 \) and \( \gamma = 1.002 \).](image)

The behaviour in which the exact solution \([6]\) becomes mathematically unstable when \( \Omega_{\xi_0} \to 0 \) seems to be not compatible with the near equilibrium condition \([25]\). In particular, for \( E = 1 \) \( (\tau = 0) \) from equation \([29]\) we can see that \( h(E) \gg 1 \) for a certain values of \( C \) and \( \Omega_{\lambda_0} \), because from the condition \([26]\) \( 1 \gg \gamma - \Omega_{\xi_0} \). Despite this fact we can always ensure the mathematically stability and the fulfillment of the near equilibrium condition for this solution at late times because the fulfillment of the condition \([26]\) implies the fulfillment of the condition \([7]\) and, therefore, the solution \([6]\) tends to the de Sitter solution \([5]\) for which the perturbation function \([29]\) tends to zero. Even more, as a long as \( \Omega_{\xi_0} \neq 0 \), then the perturbation function remains bounded above and the solution is mathematically stable. It is important to note that if we do not fulfill the condition \([7]\), then we have a Big Rip singularity in a future finite time which is mathematically unstable.

In Fig. 3 we present the behavior of perturbation function \( h \), given by Eq. \([29]\), as a function of \( E \) and \( \Omega_{\xi_0} \), for the particular values of \( \gamma = 1.002 \) and \( \Omega_{\lambda_0} = 0.69 \). We use as an initial condition the arbitrary value of \( h(E = 1) = 1 \times 10^{-5} \), for which \( C = 1 \times 10^{-5}/((\gamma - \Omega_{\xi_0})(1 - \Omega_{\lambda_0})) \).

### C. Entropy production of the exact solution

In order to obtain the entropy production of the dissipative fluid from the Eq. \([23]\), we need to find their temperature from the Eq. \([17]\). Rewriting the conservation Eq. \([22]\) in the form

\[
\frac{d \rho}{da} = -\frac{3 \rho}{a} (\gamma - 3 H \xi_0),
\]

we can rewrite Eq. \([20]\) as

\[
\rho (\gamma - 3 H \xi_0) = \frac{a}{3} \frac{\partial \rho}{\partial a} - a \frac{\partial \rho}{\partial T}.
\]

Then, from Eq. \([17]\), remembering that in the Eckart’s theory \( p \to p + \Pi \), we have

\[
\frac{\partial T}{\partial a} + \rho (\gamma - 3 H \xi_0) \frac{\partial T}{\partial \rho} = T \left[ (\gamma - 1) - 3 H \xi_0 - 3 \xi_0 \rho \frac{\partial H}{\partial \rho} \right],
\]

expression with together Eq. \([34]\) leads to

\[
\frac{dT}{T} = -3 \frac{da}{a} \left[ (\gamma - 3 H \xi_0) - 1 - 3 \xi_0 \rho \frac{\partial H}{\partial \rho} \right].
\]

Thus, using Eqs. \([24]\) and \([33]\) we get, from Eq. \([36]\), the following expression:

\[
\frac{dT}{T} = \frac{d \rho}{\rho} \left[ 1 - \frac{2 \sqrt{3 (\rho + \Lambda + \xi_0) \gamma - \sqrt{3 (\rho + \Lambda) \xi_0}}}{2 \sqrt{3 (\rho + \Lambda) \xi_0}} \right],
\]

which has the following solution in our dimensionless notation:

\[
\ln \left( \frac{T}{T_0} \right) = \ln \left( \frac{\rho}{\rho_0} \right)
+ 2 \Omega_{\xi_0} \Omega_{\lambda_0} \left[ \frac{1}{2} \frac{\sqrt{\Omega_{\xi_0}}}{\Omega_{\lambda_0}} \arctanh \left( \frac{\gamma - \Omega_{\xi_0}}{\Omega_{\lambda_0}} \right) - \frac{\gamma - \Omega_{\xi_0}}{\Omega_{\lambda_0}} \right]
- \gamma \ln \left( \frac{\rho}{\rho_0} \right) + [\gamma (2 + \gamma) - \Omega_{\lambda_0} \Omega_{\xi_0}] \ln \left( \frac{\gamma - \Omega_{\xi_0}}{\Omega_{\lambda_0} \xi_0} \right).
\]

On the other hand, integrating Eq. \([33]\) with the help of Eq. \([24]\), we obtain in our dimensionless notation the
expression

\[ \ln a^3 = \frac{2\Omega_{\xi_0} \sqrt{1 - \frac{\rho}{\rho_0}} \left[ \text{arctanh} \left( \frac{\sqrt{\Omega_{\xi_0}}}{E} \right) - \text{arctanh} \left( \sqrt{\Omega_{\xi_0}} \right) \right]}{\gamma^2 - \Omega_{\Lambda_0} \Omega_{\xi_0}^2} \]

\[ + \frac{\gamma - \Omega_{\Lambda_0} \Omega_{\xi_0}^2}{\gamma^2 - \Omega_{\Lambda_0} \Omega_{\xi_0}^2} \left( \gamma - \Omega_{\xi_0} \right) \ln \left( \frac{\rho}{\rho_0} \right) + 2 \gamma \ln \left( \frac{\gamma - E \Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}} \right), \]

from which we can express the Eq. (38) in terms of the scale factor as follows:

\[ \ln \left( \frac{T}{T_0} \right) = \ln \left( \frac{\rho}{\rho_0} \right) + \ln \left( \frac{\gamma - E \Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}} \right) + \ln a^3. \]

Hence, the temperature of the dissipative fluid as a function of the scale factor is given by

\[ T = T_0 \left( \frac{\rho}{\rho_0} \right)^{\frac{E - E \Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}}} a^3, \]

which reduced to the expression for the temperature of a perfect fluid [21] when \( \Omega_{\xi_0} = 0 \).

Note that it is possible to obtain an expression for the temperature of the dissipative fluid that does not depend on \( \rho \), by combining the Eqs. (39) and (41), which leads to

\[ T = T_0 \left( \frac{E + \sqrt{\Omega_{\Lambda_0}}}{E - \sqrt{\Omega_{\Lambda_0}}} \right)^{\frac{\gamma - \Omega_{\xi_0}}{\Omega_{\xi_0}}} \left( \frac{E - \sqrt{\Omega_{\Lambda_0}}}{1 + \sqrt{\Omega_{\Lambda_0}}} \right)^{\frac{\Omega_{\xi_0}}{\Omega_{\Lambda_0}}} \left( \frac{\Omega_{\xi_0}}{\sqrt{\Omega_{\Lambda_0}}} \right)^{\frac{\gamma}{1 + \sqrt{\Omega_{\Lambda_0}}}} \]

and from which we can see that the temperature is always positive in two cases: (i) when \( \gamma - E \Omega_{\xi_0} > 0 \) and \( \gamma - \Omega_{\xi_0} > 0 \) or (ii) when \( \gamma - E \Omega_{\xi_0} < 0 \) and \( \gamma - \Omega_{\xi_0} < 0 \). Therefore, if we fulfill the near equilibrium condition [23], then we obtain a positive expression for the temperature, since the condition (i), and from the same condition, we also fulfill the condition (ii) from which the solution [6] tends asymptotically to the future at the usual de Sitter solution. Note that the condition (ii) implies that the fluid is far from the near equilibrium and the solution [6] has a Big-Rip singularity. On the other hand, considering that the solution [6] is a decreasing function with the time when the condition [7] holds, then the cubic term in the Eq. (42) is also a decreasing function; thus, considering that \( E(\tau \to \infty) \to \sqrt{\Omega_{\Lambda_0}} \), a decreasing temperature with the time requires that

\[ \frac{a^3 (\gamma - \Omega_{\xi_0}^2 \Omega_{\Lambda_0})}{\gamma} \left( E - \sqrt{\Omega_{\Lambda_0}} \right)^{\frac{\Omega_{\xi_0}}{\gamma}} \to 0, \]

which is only possible, considering that \( a(\tau \to \infty) \to \infty \), when the exponent of the power law for the scale factor is negative, i.e., if

\[ \Omega_{\Lambda_0} < \frac{\gamma (\gamma - 1)}{\Omega_{\xi_0}}, \]

which is always true because \( 0 < \Omega_{\Lambda_0} \leq 1 \) and the fulfillment of the condition (7) implies that \( 1 < \gamma/\Omega_{\xi_0} \), as well as the fulfillment of the condition (26) implies that \( 1 \ll (\gamma - 1)/\Omega_{\xi_0} \). It is important to note that if \( \gamma = 1 \), then the exponent of the power law for the scale factor is always positive and the temperature is an increasing function with the time.

In the Fig. 4 we present the numerical behaviour of the temperature \( T \) of the dissipative fluid, given by the Eq. (41), as a function of the scale factor \( a \). We rewrite \( E \) as a function of the energy density \( \rho \) from Eq. (24) and we use the expression for \( \rho \) given by the Eq. (39). For the free parameters we use the values of \( T_0 = 1, \Omega_{\xi_0} = 0.001, \Omega_{\Lambda_0} = 0.69, \) and \( \gamma = 1.002 \). We also present the behaviour of the temperature when \( \gamma = 1 \). It is important to mention that the difference between the initial value of the temperature for WDM case and his final value is 0.0102895 for a scale factor that is 3.6 times more bigger than the actual size of the universe, this is the result to be close to the near equilibrium condition, that makes that temperature decrease very slowly to zero.

FIG. 4. Numerical behavior of \( T \), given by Eq. (41), for \( 0.5 \leq a \leq 3.5 \). We also consider the fixed values of \( T_0 = 1, \Omega_{\xi_0} = 0.001, \) and \( \Omega_{\Lambda_0} = 0.69 \).

Now, with the temperature of the dissipative fluid given by Eq. (41), we can calculate the entropy from Eq. (23), which in our dimensionless notation, and using also that from Eq. (18) \( n = n_0 a^{-\gamma} \), takes the following form:

\[ \frac{dS}{d\tau} = \frac{3E^2 \Omega_{\xi_0} \rho}{nT} = \frac{3E^2 \rho_0 \Omega_{\xi_0} (\gamma - \Omega_{\xi_0})}{n_0 T_0 (\gamma - E \Omega_{\xi_0})}. \]
then we have a null entropy production. On the other hand, as well as the temperature of the dissipative fluid, the entropy production goes to infinity when we do not satisfy the condition (7) in a finite time in the Big-Rip singularity.

In Fig. 5, we show the numerical behavior of the entropy production, given by Eq. (45), as a function of the time $\tau$ for $\Omega_{\xi_0} = 0.001$, $\Omega_{\Lambda_0} = 0.69$, and $\gamma = 1.002$. We also show the behavior of the entropy production when $\Omega_{\xi_0} = 2.4$. For the last one the near equilibrium condition is not satisfied, since this value enter in to contradiction with Eq. (7), and the entropy production diverge in a finite time given by $\text{SS}5$

$$
\tau_s = \frac{2\Omega_{\xi_0} \log \left( \frac{1 - \sqrt{\Omega_{\Lambda_0}}}{1 + \sqrt{\Omega_{\Lambda_0}}} \Omega_{\xi_0} \Omega_{\Lambda_0} (1 - \Omega_{\Lambda_0}) \right)}{3 \left( \gamma^2 - \Omega_{\xi_0} \Omega_{\Lambda_0} \right)},
$$

which according to our parameters, this is equal to $\tau_s = 0.346571$, which is roughly equivalent to 4.98626 Gyr from the present time.

V. CONCLUSIONS AND FINAL DISCUSSIONS

We have discussed throughout this work the near equilibrium condition, the mathematical stability and entropy production of a cosmological model filled with a dissipative WDM, where the bulk viscosity is proportional to the energy density, and a positive CC, which is described by an exact solution previously found in $\text{SS}5$. Assuming the condition given by Eq. (7), this solution behaves very similar to the standard model for small values of $\Omega_{\xi_0}$, as we can see in Fig. 1 avoiding the appearance of a future singularity in a finite time (Big-Rip).

We have shown that the presence of the CC together with a small viscosity from the expression (11), and considering the condition given by the Eq. (7), leads to a near equilibrium regime for the WDM component. Besides, the stability in the Hubble parameter is also fulfilled when Eq. (26) is taken account.

The WDM component has a temperature which decrease very slowly with the cosmic expansion as a result of being close to the near equilibrium condition, contrary to the non physically behaviour found for the dust case ($\gamma = 1$), where the temperature increase with the scale factor. Besides, we have shown that the second law of thermodynamics is fulfilled as long we satisfied the conditions (7) and (25) (near equilibrium condition). On the contrary, the entropy production would diverges in a finite time if a Big-Rip singularity occurs.

In order to fulfilled the three criteria discussed in this work we need to have then $\Omega_{\xi_0} < 1$ and a WDM component with a EoS satisfying the constraint given in $\text{SS}5$.

It is important to mention that in our WDM model we need to have $\Omega_{\xi_0} \ll \gamma - 1$. For small values of $\Omega_{\xi_0}$ (in particularly $\Omega_{\xi_0} = 0.001$), the model enter into agreement with some previous results found, for example in $\text{SS}4$, where cosmological bounds on the EoS for the DM were found, and the inclusion of the CC is considered. The bounds for a constant EoS are $-1.50 \times 10^{-6} < \omega < 1.13 \times 10^{-6}$ ($\omega = \gamma - 1$) if there is no entropy production and $-8.78 \times 10^{-3} < \omega < 1.86 \times 10^{-3}$ if the adiabatic speed of sound vanishes, both at 3$\sigma$ of confidence level. Another example can be found in $\text{SS}4$ where, using WMAP+BAO+HO observations, the EoS at the present time is given by $\omega = 0.0005$. Also the EoS for sterile neutrinos (a dark matter candidate), discussed in $\text{SS}3$, that arising from thermal distributions and the non-thermal distribution, has a barotropic index bounded by $0 < \omega < 0.35$.

It is important to highlight that the asymptotic behavior of the exact solution at the infinite future, given by (1), corresponds to the the usual de Sitter solution, which indicates that this solution describe a dissipative WDM that could reproduce the same asymptotic behavior of the standard model. As long as the exact solution tends to this value, the near equilibrium condition Eq. (1) can be satisfied. Of course, the de Sitter solution has a constant temperature, according to Eq. (17) and, since in this case we have a null density and null pressure of the fluid, the entropy production is zero, according to Eq. (26).

We have shown in this work that, previously to any constraining from the cosmological data, the study of thermodynamics consistences required by the Eckart approach such as the near equilibrium condition and the entropy production, as much as the mathematical stability, leads to important constraints on the cosmological parameter, such as the given one by Eq. (26), which implies the necessity of WDM, in agreement with some previous results found in $\text{SS}3$-$\text{SS}4$. On the other hand, the constrain (7) tell us that Big-Rip singularities are avoiding at late times, if the near equilibrium condition is preserved. Even though, the exact solution explored behaves very similar to the standard model, and open
the possibility of more realistic fluid description of the DM containing dissipation processes, with the Eckart’s framework giving us physically important clues about the EoS of this component ($\gamma$) and the size of dissipation involved ($\Omega_{\xi_0}$).

Finally, the main contribution of the present study is to show that the inclusion of a CC allows to fulfill the near equilibrium condition of a dissipative WDM component, which is not possible if the accelerated expansion is due only to the negativeness of the bulk pressure, renouncing to a unified DM model but gaining a solution which behaves very close to the standard model, with the same asymptotic behavior. Besides, the fulfillment of the near equilibrium condition implies a positive entropy production. Therefore, our exact solution describe a physically viable model for a dissipative WDM component.

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