INFINITENESS OF ZERO MODES FOR THE PAULI OPERATOR
WITH SINGULAR MAGNETIC FIELD

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Abstract. We establish that the Pauli operator describing a spin-1/2 two-
dimensional quantum system with a singular magnetic field has, under certain
conditions, an infinite-dimensional space of zero modes, possibly, both spin-up
and spin-down, moreover there is a spectral gap separating the zero eigenvalue
from the rest of the spectrum. In particular, infiniteness takes place if the
field has infinite flux, which settles this previously unknown case of Aharonov-
Casher theorem.

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1. Introduction

The presence of zero modes, eigenfunctions with zero eigenvalues, is a typical fea-
ture for two-dimensional spin 1/2 quantum systems involving magnetic fields. Such
eigenvalues were first found by Landau (see [15]) for the Pauli operator with con-
stant magnetic field, and the multiplicity turned out to be infinite. Later Aharonov
and Casher [3] calculated the number of zero modes for a bounded compactly sup-
ported magnetic field, and this number turned out to be finite and determined by
the total flux of the field. The conditions on the magnetic field were gradually re-
laxed, see [16, 5], until in [8] the case of measure-valued magnetic fields was settled
and an Aharonov-Casher type formula was established for a magnetic field being
a regular measure with finite total variation, thus producing a finite number of
zero modes. On the other hand, a weak perturbation of the constant magnetic
field leaves the space of zero modes infinite-dimensional ([14]). In the paper by
Shigekawa [19] it was established that if the field is sufficiently locally regular and
separated from zero at infinity (or tends at infinity to zero sufficiently slowly) then,
again, the space of zero modes is infinite-dimensional. On the other hand, an ex-
ample in [8] had shown that if the total variation of the field is not finite, there
may be no zero modes at all, even if the total flux of the field, defined as a condi-
tionally convergent integral, is nonzero. Under some rather restrictive conditions,
infiniteness of zero modes was established for a periodic magnetic field, see [6], [7].

A new type of magnetic fields was recently considered in relation to the study
of zero modes by Geyler and Grishanov in [9]. They have studied a system of
equal Aharonov-Bohm magnetic solenoids placed at the points of an infinite double-
periodical lattice in the plane. Neither of the previous results apply for this case, the
field being very singular and the total flux being infinite. Nevertheless, the authors of [9] proved that such field produces an infinite-dimensional zero energy subspace. Moreover, both spin-up and spin-down null subspaces are infinite-dimensional. This property is proved to be stable when one adds a constant (positive) magnetic field, of arbitrary size for the spin-down component, and not too large for the spin-up component. Further on, in [10] this result was extended to certain perturbations of this periodic structure.

Not so much is known about the rest of the spectrum of the Pauli operator. For the Landau operator, with constant magnetic field, the spectrum consists of Landau levels, eigenvalues with infinite multiplicity placed at the points of an arithmetical progression. Under a weak perturbation of the field, these eigenvalues, except the lowest one, may split, producing a cluster of the discrete spectrum around the Landau levels (see [19]). Thus zero remain to be an isolated point of the spectrum. On the other hand, a weak magnetic field without a background constant field leaves the whole positive semi-axis belonging to the spectrum, so no spectral gap arises. If the magnetic field grows unboundedly at infinity, the whole spectrum, except zero, is discrete (see, again [19]). Under rather restricting conditions the presence of the spectral gap was established in [6], [7] for a periodic field. However in a more or less general case this question is still open.

In the present paper we study the zero modes of the Pauli operator with a non-regular magnetic field with an infinite total flux. The typical example of the fields in question is a, probably infinite, discrete configuration of AB solenoids on the background of a more regular magnetic field. The Pauli operator for a strongly singular field is not essentially self-adjoint, there is an ongoing discussion on which self-adjoint extension of the Pauli operator in the presence of AB solenoids more adequately describes the real physical situation – see [1], [4], [22] and references therein. It turns out that depending on which approximation to AB field by more regular fields is chosen, with simultaneous adjustment of some other physical parameters, different self-adjoint extensions can arise. Our main analysis deals with the so called maximal extension. Its advantage is its invariance with respect to singular gauge transformations reducing the AB fluxes. We handle also another extension considered in the paper [8], also gauge invariant, but with different spectral properties. We discuss the relations of these two extensions in Sect. 2, as well as describe the connections of the study of zero modes with problems in the theory of analytical functions.

In Sect. 3 we find rather general conditions for the infiniteness of zero modes and for zero being an isolated point in the spectrum, with a possibility to estimate the size of the spectral gap. We start by settling the long-standing hypothesis (see the discussion in [8]) that a field of constant direction with infinite total flux produces infinitely many zero modes. Further on, we show that this infiniteness is preserved under addition of a field with different direction, having a finite flux. This establishes Aharonov-Casher theorem for the case of an infinite total flux of the field. We pass then to the case when this `wrong' component may have an infinite flux. Here, we suppose that the flux of the field through any disk of a fixed size is non-negative, at least far enough from the origin, moreover the flux of the averaged field is infinite. This requirement, together with some additional local conditions, grants that the spin-down zero subspace is infinite-dimensional. If, additionally, the above local fluxes are separated from zero, then zero is an isolated point of the spectrum of the Pauli operator. For regular fields, this condition prevents the infiniteness of the spin-up zero modes, since spin inversion corresponds to changing the sign of the field. However, if the discrete component of the field is large enough, in other words, if sufficiently many Aharonov-Bohm solenoids are present, then it
turns out that the main condition can be satisfied for the spin-up component of the 'maximal' operator as well, so there are infinitely many spin-up zero modes too. We also explain how the results change when we pass to the self-adjoint Pauli operator considered in [8]. Here the situation with both spin-up and spin-down zero modes does not appear. We conclude Sect.3, by some examples, in particular the case of a periodic and quasi-periodic magnetic field fits into the general approach, and the results of [6] and [7] are substantially extended.

Further on, we pass to the situation when the general results are not sufficient, since the main condition of positivity of local flux may be violated. Supposing that the magnetic field in question is a perturbation of some initial field where a quadratic lower estimate for the potential is known, we establish such quadratic estimate for the potential of the perturbed field, thus ensuring the infiniteness of zero modes, but, probably, without the spectral gap. Among others, the constant one, AB-lattice, probably, on the background of a constant field, a periodic or quasi-periodic field with some mild local regularity may serve as the unperturbed field. Admissible perturbations are rather general, in particular they allow existence of arbitrary large regions on the plane with field having 'wrong' direction. In the end we discuss some examples where the perturbation theorems can be applied. The estimates for the potential obtained on this way of reasoning, may be useful in the further study of the perturbation of the Pauli operator by an electric field.

The starting point of our study was an attempt to understand the possibility of perturbing the results of [9], by means of changing the intensities and positions of the AB solenoids. We thank V. Geyler who attracted our attention to this kind of problems. Further on, when it turned out that much more general situations can be taken care of, the proof of the crucial theorem 3.2 appeared in the process of discussions with F. Nazarov. We highly appreciate also the discussions with B. Berndtsson on the spectral gaps and with L. Erdős about the definition of the Pauli operator. The second author (N.Sh.) was supported by the stipend from the Swedish Royal Academy of Sciences. Both authors thank the Mittag-Leffler Institute for hospitality when the work on the paper was in its most active phase.

2. Definition of the operator

We identify the real two-dimensional space \( \mathbb{R}^2 \) with co-ordinates \( x = (x_1, x_2) \) with the complex plane \( \mathbb{C} \), setting \( z = x_1 + ix_2 \); as usual, \( \bar{\partial} = \partial_{\bar{z}} = (\partial_1 - i\partial_2)/2 \) and the Lebesgue measure will be denoted by \( dx \).

Formally, the Pauli operator in \( L^2(\mathbb{R}^2) \), with gyro-magnetic ratio \( g = 2 \), is defined as the square of the Dirac operator

\[
D = \sigma \cdot (-i\nabla + A) = (\sigma_1(-i\partial_1 + A_1) + \sigma_2(-i\partial_2 + A_2)).
\]

Here \( \sigma_1, \sigma_2 \) are the Pauli matrices, \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), and \( A_j \) are real functions, components of the magnetic potential \( A = (A_1, A_2) \). So, \( \mathcal{P} = D^2 = -(\sigma \cdot (\nabla + iA))^2 \).

Introducing the notations \( \Pi_j = -i\partial_j + A_j \), \( Q_\pm = \Pi_1 \pm i\Pi_2 \), we can represent the Pauli operator \( \mathcal{P} \) as

\[
\begin{pmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{pmatrix} = \begin{pmatrix} Q_-Q_+ & 0 \\ 0 & Q_+Q_- \end{pmatrix}.
\]

Formally, the operators \( Q_\pm \) are adjoint to each other.

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\(^1\)Different sign conventions are used in the literature. We follow the sign choice made in [8], which is the opposite to the one made in, say, [5].
The magnetic field is defined as $\mathbf{B} = \text{curl} \mathbf{A} = \partial_1 A_2 - \partial_2 A_1$, and it is considered in the classical physics as the only actual physical reality, the potential being merely a mathematical fiction. This is the fact also in the quantum physics, provided the magnetic field (and therefore the potential) are not too singular. The latter statement means that if for two magnetic potentials $\mathbf{A}_1, \mathbf{A}_2$ the magnetic field (and therefore the potential) are not too singular, the latter is a mathematical fiction. This is the fact also in the quantum physics, provided the classical physics as the only actual physical reality, the potential being merely a mathematical fiction. This is the fact also in the quantum physics, provided the classical physics as the only actual physical reality, the potential being merely a mathematical fiction.

For magnetic fields possessing local singularities, the one which we are going to study further on, this description is not satisfactory, as it was explained in [8], and another approach, based upon the scalar potential, was proposed. We will use the potential $\Psi = \Psi_{\text{cont}} + \Psi_{\text{disc}}$, a solution of the equation $\Delta \Psi = \mu$ in the sense of distributions. The corresponding vector potential is defined as $\mathbf{A} = (A_1, A_2) = \text{sggrad} \Psi = (\partial_2 \Psi, -\partial_1 \Psi)$, again in the sense of distributions.

The quadratic form (2.2), under certain regularity conditions, can be transformed to

$$p[\Psi] = p_+ [\psi_+] + p_- [\psi_-] = 4 \int |\partial (e^{-\Psi} \psi_+)|^2 e^{2\Psi} dx + 4 \int |\partial (e\Psi \psi_-)|^2 e^{-2\Psi} dx. \quad (2.3)$$

For a field $\mu$ with singularities, is is the form (2.3) that is used for defining the Pauli operator. The decomposition of the measure $\mu = \mu_{\text{cont}} + \mu_{\text{disc}}$ leads to a similar decomposition of the potential, $\Psi = \Psi_{\text{disc}} + \Psi_{\text{cont}}$. We will use the potential $\Psi_{\text{cont}}$ constructed in [8]. This is a function satisfying the equation $\Delta \Psi_{\text{cont}} = \mu_{\text{cont}}$ in the sense of distributions. It is established in [8] that such a function exists and possesses certain regularity properties, in particular, $\exp(\pm 2\Psi_{\text{cont}}) \in L^p_{1,\text{loc}}(\mathbb{R}^2)$, $\nabla \Psi_{\text{cont}} \in L^p_{p,\text{loc}}, p < 2$. 

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If only the continuous part of the measure is present, the natural domain for the form (2.3) consists of all functions $\psi_\pm$ for which (2.3) is finite. Although this definition is rather implicit, this domain possesses an easily describable core: the space of functions $\psi_\pm$ for which $e^{-\Psi}\psi_+$ and $e^{\Psi}\psi_-$ are smooth functions with compact support (see [8]).

We assume next that only the discrete part of the measure $\mu = \mu_{\text{disc}}$ is present,

$$\mu_{\text{disc}} = 2\pi \sum_{\lambda \in \Lambda} \alpha_\lambda \delta(x - \lambda), \quad x \in \mathbb{R}^2. \quad (2.4)$$

The support $\Lambda$ of $\mu_{\text{disc}}$ will be supposed to be a discrete set, without finite accumulation points, moreover, uniformly discrete:

$$\text{dist}(\lambda, \Lambda \setminus \lambda) \geq r_0, \quad r_0 > 0, \quad \text{for any } \lambda \in \Lambda. \quad (2.5)$$

Each component of the discrete measure is an Aharonov-Bohm (AB) solenoid (see [2]) with flux $2\pi \alpha_\lambda$ and intensity $\alpha_\lambda$. We consider the case of one solenoid first. The AB magnetic potential corresponding to one term in (2.4), $B = 2\pi \alpha \delta(x - x^0)$; $x^0 = (x^0_1, x^0_2)$, with intensity $\alpha$ can be chosen as $A(x) = \left(-\alpha \frac{x_2 - x^0_2}{r}, \alpha \frac{x_1 - x^0_1}{r}\right), \quad r = |x - x^0|$. The corresponding scalar potential is $\Psi(x) = \Psi(z) = \alpha \ln|z - z^0|$ (here $z^0 = x^0_1 + x^0_2$; from now on, the complex picture is more convenient.) Thus the expressions $\exp(\pm\Psi)$ have singularities of the form $|z - z^0|^{\pm \alpha}$ at $z^0$.

Formally, the Pauli operator with AB field admits gauge transformations. For an integer $m$, we set $\phi(z) = \exp(-im \arg(z - z^0))$. Then the multiplication by $\phi$ transforms the AB Pauli operator with intensity $\alpha$ to the one with intensity $\alpha + m$. The potential correspondingly transforms as $\Psi \mapsto \Psi|z - z^0|m$. Whether this transformation is a unitary equivalence of operators depends on how the self-adjoint operator corresponding to the form (2.3) is defined.

To make our description of such self-adjoint operators more precise, we introduce the following notations. In what follows, the notations $\partial, \partial$ have the meaning of derivatives in the sense of the space of distributions $\mathcal{D}'(\mathbb{R}^2)$. For a closed set $E$, we denote by $\partial_E, \partial_E$ the derivatives in the sense of $\mathcal{D}'(\mathbb{R}^2 \setminus E)$.

We explain now the way of defining the Pauli operator, proposed in [8]. For $\alpha \in [-1/2, 1/2)$, one accepts as a domain of the form (2.3) the space of such functions $\psi \in L_2(\mathbb{R}^2)$ for which the derivatives $\partial(e^{-\Psi}\psi_+)$ and $\partial(e^{\Psi}\psi_-)$ (thus taken in the sense of distributions in $\mathcal{D}'(\mathbb{R}^2)$) are functions, and the form is finite:

$$p[\psi] < \infty. \quad (2.6)$$

With such domain, which we denote here by $\mathcal{D}_{\text{EV}}(\alpha)$, the form (2.3) is closed and defines the self-adjoint operator which we denote by $\mathcal{P}_{\text{EV}} = \mathcal{P}_{\text{EV}}(\alpha)$. For $\alpha$ outside the above interval, the operator is defined by means of the gauge transformation. For a given $\alpha$, let $\alpha^*$ be the unique number in the interval $[-1/2, 1/2)$ such that $\alpha^* - \alpha = m$ is an integer. Then the Pauli operator $\mathcal{P}_{\text{EV}}(\alpha)$ is defined as $\mathcal{P}_{\text{EV}}(\alpha) = \exp(im \arg(z - z^0))\mathcal{P}_{\text{EV}}(\alpha^*) \exp(-im \arg(z - z^0))$. With this definition, the operator is automatically gauge invariant. However, for $\alpha \notin [-1/2, 1/2)$, the description of the domain does not agree with (2.6). In fact, if the distributional $\partial(e^{-\Psi}\psi_+)$ is a function, for $\Psi = |z - z^0|^{\alpha^*}$, the gauge transformation leads to the expression $\partial((z - z^0)m e^{-\Psi}\psi_+)$, which is not necessary a function, it may contain the $\delta$-distributions and its derivatives. This might be considered as a minor inconvenience, however it leads to the unnatural absence of invariance of the number of zero modes under the change of sign of the magnetic field, as can be seen from the version of the Aharonov-Casher theorem in [8] (or, more easily, from the non-symmetry of the main interval $[-1/2, 1/2)$, chosen arbitrarily - see [18] for more details).
We consider, along with the above operator, an alternative one. For a given \( \alpha \) we define \( P_{\text{max}} = P_{\text{max}}(\alpha) \) as the operator corresponding to the quadratic form

\[
P_{\text{max}}[\psi] = P_{\text{max}}[\psi^+] + P_{\text{max}}[\psi^-]
\]

defined on such functions \( \psi \) that the derivatives in (2.7) (thus understood in the sense of \( \mathcal{D}'(\mathbb{R}^2 \setminus \{z^0\}) \)) are functions and \( p_{\text{max}}[\psi] \) is finite. The operator \( P_{\text{max}} \) is again gauge invariant (see, again, [18] for corresponding calculations).

Both constructions can be carried over to the case of a finite or infinite system of AB solenoids placed at the points of a discrete set \( \Lambda \) of the plane, as in (2.4). We say that the vector-function \( A(x) = (A_1(x), A_2(x)) \) is a vector potential for the magnetic field (2.4) if \( \mu_{\text{disc}} = \text{curl} A \) in the sense of \( \mathcal{D}'(\mathbb{R}^2) \). The function \( \Psi_{\text{disc}} \) satisfying the Poisson equation \( \Delta \Psi_{\text{disc}} = \mu_{\text{disc}} \) is the scalar potential. As above, we define the quadratic form \( p \) by the expression in (2.3). The gauge transformations enable changing all intensities \( \alpha_\lambda \) by arbitrary integers. If \( m_\lambda, \lambda \in \Lambda \), is a collection of integers then the gauge transformation changing the intensity at the point \( \lambda \) by \( 2\pi m_\lambda \) consists in the multiplication by a function \( W(z) \). The function \( W(z) \), as proposed in [8], equals \( L(z)/|L(z)| \), \( L(z) = F(z)G(z) \) where \( F(z) \) is an analytical function having zeros of order \( m_\lambda \) at the points \( \lambda \) where \( m_\lambda > 0 \), and \( G(z) \) has zeros of order \( -m_\lambda \) at the points \( \lambda \) where \( m_\lambda < 0 \). For any collection of \( \alpha_\lambda \), by adding proper \( m_\lambda \), one obtains the reduced intensities \( \alpha_\lambda' \in [-1/2, 1/2] \), used for defining the operator. As in the case of a single AB solenoid, the operator \( P_{\text{EV}}(\{\alpha_\lambda, \lambda \in \Lambda\}) \) is defined as the one gauge equivalent to \( P_{\text{EV}}(\{\alpha_\lambda', \lambda \in \Lambda\}) \), the latter determined by the quadratic form (2.3) on all functions \( \psi \) for which this form is finite. Alternatively, the maximal operator \( P_{\text{max}}(\{\alpha_\lambda, \lambda \in \Lambda\}) \) is defined for any set of (non-integer) \( \alpha_\lambda \) by the quadratic form \( p_\Lambda \) in (2.7) on the functions for which this latter form is finite.

Finally, when both discrete and continuous components of the measure \( \mu \) are present, the discrete one located at the points \( \lambda \) of a discrete set \( \Lambda \), the Pauli operators \( P_{\text{EV}} \) and \( P_{\text{max}} \) are defined in a similar way, with only difference that the scalar potential \( \Psi \) is now the sum of the potentials \( \Psi_{\text{disc}} \) and \( \Psi_{\text{cont}} \) corresponding to the discrete and continuous parts of the measure \( \mu \). We do not touch upon the question on which of these operators (if any) describes the actual physical picture. We keep however in mind that for a continuous measure as a field these operators coincide.

In the general situation it is hard to describe the domain of these two operators explicitly. However, more can be said about the null subspace of these operators, in other words, about the zero modes. Note, first of all, that the quadratic form of \( p_{\text{max}} \) is an extension of the form \( p_{\text{EV}} \). Since both forms are non-negative, this implies that \( \text{Ker}(P_{\text{EV}}) \subseteq \text{Ker}(P_{\text{max}}) \). This can also be seen from the direct description of the zero modes. If a function \( \psi = (\psi^+, \psi^-) \) lies in the null subspace of the operator \( P_{\text{EV}} \) or \( P_{\text{max}} \) then \( \alpha \) must annule the corresponding quadratic forms \( p_{\text{EV}}, p_{\text{max}} \), which means

\[
\partial(e^{-\Psi_+}) = 0, \partial(e^{\Psi_-}) = 0
\]

for \( P_{\text{EV}} \) and

\[
\tilde{\partial}_\Lambda(e^{-\Psi_+}) = 0, \tilde{\partial}_\Lambda(e^{\Psi_-}) = 0
\]

for \( P_{\text{max}} \). Both (2.8) and (2.9) mean that the function \( f_+ = e^{-\Psi_+} \) must be analytical, \( f_- = e^{\Psi_-} \) must be anti-analytical, but on different sets. For the operator \( P_{\text{EV}} \), by (2.8), these functions must be entire functions of variables \( z, \bar{z} \) respectively. On the other hand, for the operator \( P_{\text{max}} \), by (2.8) these functions
may have poles at the points \( \lambda \in \Lambda \), but not too strong ones, so that still after the multiplication by \( \exp(\pm \Psi) \), they get into \( L_2 \).

To make things more concrete we suppose that from the very beginning the gauge transformation is made, so that all intensities \( \alpha_\lambda \) are in the interval \((0, 1)\) for \( \mathcal{P}_{\max} \). In this case, the function \( e^\Psi \) behaves as \(|z - \lambda|^{\alpha_\lambda}\) near the point \( \lambda \in \Lambda \). Therefore the condition \( \partial_z f_- = 0 \), together with \( f_- \exp(-\Psi) \in L_2 \) leads to anti-analyticity of \( f_- \) at the points of \( \Lambda \) as well: near a point \( \lambda \in \Lambda \) the function \( f_-^2 \) must be summable with weight having a singularity of the form \(|z - \lambda|^{-2\alpha_\lambda}\), therefore the possible singularity of \( f_- \) is removable. On the other hand, for the spin-up component \( \psi_+ \), the function \( f_+ \) has to be holomorphic outside \( \Lambda \) but near the points of \( \Lambda \) it must belong to \( L_2 \) with weight \(|z - \lambda|^{2\alpha_\lambda}\), which, due to \( \alpha_\lambda \in (0, 1) \), allows \( f_+ \) to have a simple pole at \( \lambda \). This asymmetry can be reversed by changing the normalization of the discrete part of the measure: by mean of a gauge transformation we can decrease all intensities by 1 thus arriving at the measure \( \mu' \) with negative discrete part having the intensity \( \alpha'_\lambda = (\alpha_\lambda - 1) \in (-1, 0) \) at the point \( \lambda \in \Lambda \). Then it is for the spin-up component that the null subspace is generated by entire functions, and for the spin-down one by meromorphic functions with simple poles. We can, moreover, take the first normalization when studying the spin-down component and the second one for the spin-up component, thus only entire functions will be involved. With this last agreement accepted, the derivatives involved in the forms do not depend in the space of distributions where they are considered, so we can painlessly omit the corresponding subscripts in our notations.

To compare the null subspaces of two operators under consideration, we suppose first that all intensities \( \alpha_\lambda \) lie in \((0, \frac{1}{2})\). In this case for the spin-down component \( \mathcal{P}_- \) null subspaces coincide, being in both cases generated by entire anti-analytical functions. The null subspace of \( \Psi_{+\max} \) may be larger than the null subspace of \( \Psi_{+\EV} \), since the latter subspace may involve meromorphic functions \( f_+ \) in addition to entire functions for \( \mathcal{P}_{+\EV} \). If some of the intensities \( \alpha_\lambda \) lie in \([ -\frac{1}{2}, 0)\), both \( \mathcal{P}_{+\max} \) and \( \mathcal{P}_{-\max} \) may have zero modes generated by meromorphic functions, so both null subspaces may turn out to be larger than the ones for \( \mathcal{P}_{+\EV} \). See, again, [18] for more detailed comparison of these two self-adjoint extensions.

Now we make some more observations about the part \( \Psi_{\text{disc}} \) of the potential, the one responsible for the discrete part of the measure. It follows from the uniform discreteness condition (2.5) that the discrete set \( \Lambda \) has a density not higher than that of a regular lattice, more exactly,

\[
N(R) \equiv \sharp\{ \lambda \in \Lambda, \lambda < R \} = O(R^2), r \to \infty. \tag{2.10}
\]

Consider the sum

\[
\Psi_{\text{disc}} (z) = \alpha_0 \log |z| + \sum_{\lambda \in \Lambda, \lambda \neq 0} \alpha_\lambda \left( \log \left| 1 - \frac{z}{\lambda} \right| + \Re \left( \frac{z}{\lambda} + \frac{1}{2} \left( \frac{z}{\lambda} \right)^2 \right) \right); \tag{2.11}
\]

if \( \lambda = 0 \) does not belong to the set \( \Lambda \), the first term in (2.11) is omitted. The series converges uniformly on any compact set in \( \mathbb{C} \) not containing the points in \( \Lambda \), moreover the Laplace operator can be applied term-wise, so (2.11) produces the required potential.

The particular case of a special interest is the one of a purely discrete measure, a regular lattice, with all intensities equal,

\[
\Lambda = \{ \lambda_{m_1m_2} \} = \omega_1 m_1 + \omega_2 m_2, \quad \alpha_{\lambda_{m_1m_2}} = \alpha \in (0, 1), \tag{2.12}
\]
where \(\omega_1, \omega_2\) are complex numbers with non-real \(\omega_1/\omega_2\). In for this configuration, as it was noticed in [9] the potential \(\Psi\) is closely related to the Weierstrass \(\sigma\)-function

\[
\sigma(z) = z \prod_{\lambda \neq 0, \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{1}{2} \left(\frac{z}{\lambda}\right)^2\right),
\]

so that \(\tilde{\Psi}_0(z) = \alpha \log |\sigma(z)|\) can serve as a potential for the magnetic field (2.4). It was established in [17] that, the potential \(\tilde{\Psi}_0\) possesses a very special property:

\[
\tilde{\Psi}_0(z) = \alpha (\Re(\nu z^2) + m|z|^2 + \rho(z)),
\]

where \(\nu\) is a certain coefficient determined by the lattice, \(m = \frac{1}{4\pi}, S\) being the area of the elementary cell, and \(\rho\) a \(\Lambda\)-periodic function, with proper logarithmic singularities at the points of the lattice. We do not care about the value of \(\nu\), explicitly given in [17, 9]. Anyway, the first summand in (2.14) is a harmonic function; we subtract it and for a regular lattice we will consider

\[
\Psi_0(z) = \alpha (m|z|^2 + \rho(z)),
\]

with \(\rho\) determined by (2.14). With such potential \(\Psi_0\), for any entire function \(f(z)\) subject to \(|f(z)| \leq C \exp(\gamma |z|^2)\) with \(\gamma < \alpha m\), the function \(\psi_-(z) = \exp(-\alpha (m|z|^2 + \rho(z)))f(z)\) belongs to \(L_2\). This observation made in [9] proves infiniteness of zero modes for \(\mathcal{P}_{-\max}\). Passing to the operator \(\mathcal{P}_+\), we make the gauge transformation reducing all fluxes to \(2\pi (\alpha - 1)\). For the reduced operator, the function \(\Psi_{0+}(z) = (\alpha - 1)(m|z|^2 + \rho(z))\) serves as a potential, and thus any entire function \(f_+\) of the variable \(z\) produces the \(L_2\) zero mode \(\psi_+ = \exp((\alpha - 1)(m|z|^2 + \rho(z)))f_+\) of the operator \(\mathcal{P}_{+\max}\). As it is explained above, if \(\alpha\) lies in the interval \((0, \frac{1}{2})\), infiniteness of zero modes holds also for \(\mathcal{P}_{-\EV}\), and the operator \(\mathcal{P}_{+\EV}\), as follows easily from the properties of \(\Psi_0\), has no zero modes. In the case \(\alpha \in [-\frac{1}{2}, 0)\) the spin-up and spin-down components change their roles. In Sect.4, we will show that such estimates for the potential are preserved under certain types kinds of the perturbations of the regular AB lattice, thus, in particular, providing us with examples of magnetic fields possessing arbitrarily large regions with 'wrong' direction of the field, but, nevertheless, with infinitely many zero modes.

So, under our normalization conditions, the study of zero modes is reduced to the study of existence of entire functions which, after being multiplied by a certain weight get into \(L_2\). This study can be done by means of explicit estimates for the potential \(\Psi\), like in [3, 5, 16, 8, 9] or by indirect methods, cf. [19]. We are going to combine both approaches.

3. Zero modes and the spectral gap. Methods of the theory of subharmonic functions

In this section we establish the infiniteness of zero modes under rather general conditions. We start by proving this for the magnetic field with constant sign and infinite flux, and then relax the positivity restriction in different ways. Further on we find conditions for the existence of the spectral gap.

We suppose that the general conditions on the measure as formulated in the previous section are fulfilled.

**Theorem 3.1.** Let \(\mu\) be non-negative locally finite Borel measure on \(\mathbb{C}\), with the support of the discrete part not having finite accumulation points, and the normalization agreements of the previous Section be fulfilled. Then the operators \(\mathcal{P}_{-\EV}\), \(\mathcal{P}_{-\max}\) have infinitely many zero modes.

As it is explained in the previous section, it is sufficient to establish the following fact about (anti-)analytical functions, which is valid for any non-negative measure. Theorem 3.1 follows from it, with an obvious replacement of \(\mu\) by \(2\mu\).
Theorem 3.2. Let $\Psi$ be a subharmonic function such that for the measure $\mu = \Delta \Psi$,
\[
\mu(\mathbb{C}) = \int_{\mathbb{C}} d\mu(z) = \infty. \tag{3.1}
\]
Then the spaces of entire analytical and anti-analytical functions $f$ such that
\[
\int_{\mathbb{C}} |f(z)|^2 e^{-\Psi} dx < \infty \tag{3.2}
\]
are infinite-dimensional.

Proof. Of course, it suffices to establish just one part, say, about analytical functions, which we are going to do, for convenience of references. The strategy of proving the theorem is the following. We chose a sequence of points $z_n$, $n = 1, \ldots$, in a special way. For any given $n$ a collection of functions $f_k$, $k = 1, \ldots, n$ satisfying (3.2) will be constructed in such way that $f_k(z_l) = 0$, $k < l$, $f_k(z_k) \neq 0$. Such system of functions is, obviously, linearly independent.

We denote by $D(z, R)$ the open disc centered at $z$ with radius $R$. All the measures in the proof are supposed to be non-negative.

We take $z_1 = 0$. Then we chose $R_1$ so that $\mu(D(0, R_1)) \geq 46$. Then for each $k > 1$ we find $R_k$ so that $\mu(\Omega_k) \geq 20$, $\Omega_k = D(0, R_k) \setminus D(0, R_{k-1})$, $\Omega_1 = D(0, R_1)$. This can be done due to infiniteness of (3.1). Then we take any $z_k$, strictly inside $\Omega_k$, not in the support of $\mu_{\text{disc}}$, $k \geq 2$.

Now we fix $n$ and find a sufficiently small positive $\delta$ such that the disks $D(z_k, \delta)$, $k = 1, \ldots, n$, lie strictly inside respective $\Omega_k$. Then we fix measure $\mu_0 \leq \mu$ supported in $\Omega_1$ such that $\mu_0(\Omega_1) = 26$. Denote by $\Psi_0$ the logarithmic potential of measure $\mu_0$, $\Psi_0(z) = (2\pi)^{-1} \int \ln|z-w| d\mu_0(w)$. The function $\Psi_0$ behaves as $\Psi_0(z) \sim \frac{\pi}{|z|}$ as $|z| \to \infty$.

We set further $\Psi_1 = \Psi - \Psi_0$, $\mu_1 = \mu - \mu_0 \geq 0$, so $\Delta \Psi_1 = \mu_1$ and, by our construction,
\[
\mu_1(\Omega_k) \geq 20, \, k = 1, \ldots, n. \tag{3.3}
\]

Further on we chose measures $\nu_k \leq \mu_1$, $k = 1, \ldots, n$, supported in respective $\Omega_k$, and such that $\nu_k(\Omega_k) = 20$, and denote by $U_k$ the logarithmic potential of the measure $\nu_k$, with the same asymptotic behavior $U_k \sim \frac{10}{|z|} \ln|z|$ for large $|z|$. For a positive $h < \delta/2$, we denote by $\nu'_k$, $k = 1, \ldots, n$, the measures supported in the respective disks $D(z_k, h)$ and coinciding there with $\frac{20}{\pi h^2}$ times the Lebesgue measures, so that the logarithmic potentials $U'_k$ of these measures have the same asymptotic behavior for large $|z|$ as $U_k$, $U'_k(z) \sim \frac{10}{|z|} \ln|z|$. We denote by $U^h(z)$ the subharmonic function
\[
U^h(z) = \Psi_1(z) + \sum_{k=1}^n (U'_k(z) - U_k(z)). \tag{3.4}
\]

By our choice of measures, $U_k$ and $U'_k$ differ controllably for large $z$. In fact,
\[
U'_k(z) - U_k(z) = \frac{1}{2\pi} \int_{\Omega_k} \ln \left| 1 - \frac{w}{z} \right| (d\nu'_k - d\nu_k), \tag{3.5}
\]
and since for $|z| \geq 2(R_n + \delta)$, $|w| < R_n$, we have $\ln |1 - w/z| < \ln 2$, so $|U'_k(z) - U_k(z)| < 20 \ln 2(2\pi)^{-1} < 15$. Adding up such estimates for all $k$, we obtain
\[
|U^h(z) - \Psi_1(z)| \leq 15n \tag{3.6}
\]
for large $|z|$, $|z| \geq 2(R_n + \delta)$. Therefore, for any non-negative function $v$, and any $R \geq 2(R_n + \delta)$,
\[
\int_{R \leq |z| \leq 2R} v \exp(-\Psi_1) dx \leq e^{15n} \int_{R \leq |z| \leq 2R} v \exp(-U^h) dx, \tag{3.7}
\]
For $z$ small, $|z| \leq 2(R_n + \delta)$ but lying outside the disks $D(z_k, \delta)$, we note that each function $U'_k(z)$, being the logarithmic potential of a measure supported in the disk $D(z_k, h)$, $h \leq \delta/2$, is bounded by some constant depending on $\delta$ and $R_n$, $|U'_k(z)| \leq C(d, R_n)$. The potential $U_k$, being the logarithmic potential of a measure supported in the disk $D(0, R_n)$, is not necessarily bounded from below but it is bounded from above, again, by some constant depending on $\delta$ and $R_n$, $U_k(z) \leq C(\delta, R_n)$ for $|z| \leq 2(R_n + \delta)$. This gives us

$$U^h(z) - \Psi_1(z) = \sum(U'_k(z) - U_k(z)) \geq -2nC(\delta, R_n), |z| \leq 2(R_n + \delta), z \notin \bigcup_{k < n} D(z_k, \delta).$$  

(3.8)

Next we fix a function $\varphi \in C_k^\infty(\mathbb{C})$ such that $\varphi$ vanishes in the disks $D(z_k, \delta)$, $k < n$, but $\varphi(z) = 1$ in $D(z_n, \delta)$. To the function $\partial\varphi$ we apply the theorem by Hörmander, see [13], Theorem 4.4.2, on the solutions of $\partial$-equation in weighed spaces: we find a function $g = g_h$ solving the equation $\partial g_h = \partial\varphi$ such that

$$\int_{\mathbb{C}} |g_h|^2 e^{-U_h} \frac{d^2}{(1 + |z|^2)^2} dx \leq \int_{\mathbb{C}} |\partial\varphi|^2 e^{-U_h} dx.$$  

(3.9)

We recall now that $\partial\varphi(z) = 0$ in the disks $D(z_k, \delta)$. Therefore the estimate (3.9) gives

$$\int_{\mathbb{C}} |g_h|^2 e^{-U_h} \frac{d^2}{(1 + |z|^2)^2} dx \leq \int_{\mathbb{C} \cup D(z_k, \delta)} |\partial\varphi|^2 e^{-U_h} dx.$$  

(3.10)

Now we use the estimates (3.6) and (3.8) which enable us to replace in the right-hand side of (3.10) the weight $\exp(-U_h)$ by $\exp(-\Psi_1)$. We obtain therefore the inequality

$$\int_{\mathbb{C}} |g_h|^2 e^{-\Psi_1} dx \leq C \int_{\mathbb{C} \cup D(z_k, \delta)} |\partial\varphi|^2 e^{-\Psi_1} dx = K$$  

(3.11)

with a constant $C$ depending on $n, \delta, R_n$ but not depending on $h$. The left-hand side in (3.11) can be estimated from below for large $R$, using (3.7), which gives

$$\int_{R \leq |z| \leq 2R} |g_h|^2 e^{-\Psi_1} dx \leq (1 + R^2)^2 K,$$  

(3.12)

as well as

$$\int_{|z| \geq 2(R_n + \delta)} |g_h|^2 e^{-\Psi_1} \frac{d^2}{(1 + |z|^2)^2} dx \leq 2K,$$  

(3.13)

so that the weighted norms of $g_h$ over any annulus $R < |z| < 2R$ are bounded uniformly in $h$ (of course, the bound may depend on $R$). Recalling now that $\varphi$ is a smooth function with compact support, we deduce from (3.12) that the weighted $L_2$ norms of $g_h$ over the annuli are bounded uniformly in $h$ as well.

We set $f_h = g_h - \varphi$, $\partial f_h = \partial g_h - \partial\varphi = 0$, thus $f_h$ is an entire function, moreover,

$$\int_{R \leq |z| \leq 2R} |f_h|^2 dx \leq \left( \int_{R \leq |z| \leq 2R} |f_h|^2 e^{-\Psi_1} dx \right)^{1/2} \left( \int_{R \leq |z| \leq 2R} e^{\Psi_1} dx \right)^{1/2} \leq C(R)K$$  

(3.14)

It follows from (3.14) that in any annulus $R < |z| < 2R$ the family of entire functions $\{f_h\}, 0 < h < \delta$, has bounded $L_1$-norms, therefore it has bounded $C^N$-norms of any order $N$ in a smaller annulus (see Theorem 1.2.4 in [13]) and thus, by maximum principle, bounded $C^N$-norms in any disk $|z| < R$. Therefore, by Montel’s theorem (see, e.g., [12], Theorem 15.2.5) this family is compact with respect to uniform convergence on compacts: there exists a sequence $h_i \to 0$ and an entire function
f such that $f_h$ converges to $f$ uniformly on any compact. This implies that the sequence of functions $g_h$ converges to $g = f - \varphi$ uniformly on any compact. But now note that for a fixed $k$, the potential $U'_k(z)$ equals

$$U'_k(z) = \frac{10}{\pi} \ln h + 5(1 - h^2)\chi^2$$

inside the disk $D(z_k, h)$, while all the other terms in $U^h$, see (3.4), do not depend on $h$ or are uniformly bounded in $h$. Therefore $\exp(-U^h(z))$ has the order $h^{-10/\pi}$ in $D(z_k, h)$, and since the constant in (3.11) is independent of $h$, the sequence $g_h(z_k)$ may only have 0 as its limit value. So, $g(z_k) = 0$ and $f(z_k) = g(z_k) + \varphi(z_k)$ equals 0 for $k < n$ and 1 for $k = n$. This function $f$ is the one we are looking for, because

$$\int_C |f|^2 e^{-\Psi} \, dx = \int_C |f|^2 e^{-\Psi - \Psi_0} \, dx,$$

and the finiteness of the latter integral follows from the estimate

$$e^{-\Psi_0(z)} \leq C \exp\left( -\frac{1}{2\pi} \int \Delta \Psi_0 \ln |z| \right) \leq C \exp(-13/\pi \ln |z|) < C|z|^{-4}$$

for large $|z|$, so that $\int_C |f|^2 e^{-\Psi} \, dx \leq C \int |f|^2 e^{-\Psi_1} (1 + |z|^2)^{-2} \, dx$, which is finite due to (3.13). So we have found the function $f_n$. The functions $f_k$, $k < n$ are constructed in the same way, just the function $\varphi$ has to be chosen to be equal 1 in the disk $D(z_k, \delta)$ and vanishing in $D(z_{k'}, \delta), k' \neq k$. \hfill \Box

Having established Theorems 3.2 and 3.1 for a non-negative measure, we have as our next goal extending the results to measures having a negative part. The general requirement here is that the negative part $\mu_-$ is in a certain, each time concretely defined, sense weaker than the positive part $\mu_+$. For the rest of the section we suppose that the measure $\mu_+$ has finite flux, $\mu_+(\mathbb{R}^2) = \mu_+(C) = \infty$.

Let us first discuss what may be the obstacle for an entire function $f$ with finite $\int |f|^2 e^{-2\Psi} \, dx$ to be quadratic summable with the weight $e^{-2\Psi_+ + 2\Psi_-}$, where $\Psi_-$ is a potential for $\mu_-$. It may turn out that $\Psi_-$ grows at infinity, so a certain extra decay of $f$ is required. On the other hand, the local singularities of the potential $\Psi_-$ can only be negative, and they would not cause any trouble since the introduction of the weight $e^{2\Psi_-}$ can only improve the convergence of the integral of $|f|^2$.

So, the easiest result in this direction concerns the case when we can explicitly estimate the growth of $\Psi_+$ and then take care of the corresponding term in the weight.

**Corollary 3.3.** Suppose that $\mu_-$ has compact support. Then the statements of Theorems 3.2 and 3.1 hold for $\mu = \mu_+ - \mu_-$.

**Proof.** Let $\mu_-(C) = 2\pi \Phi, \Phi > 0$. Then the logarithmic potential $\Psi_-(z)$ of $\mu_-$ grows at infinity as $\Phi \ln |z|$. Let $N$ be some integer larger than $\Phi$. Take $N$ points $z_1, z_2, \ldots, z_N$ such that each $z_k$ is not a common zero for the space $\mathcal{L}$ of entire functions $f$ with finite $\int |f|^2 e^{-2\Psi^+} \, dx$. The latter can, surely, be achieved, and this, in particular, means that $e^{-2\Psi^+}$ belongs to $L_1$ near $z_k$. The conditions $f(z_k) = 0$, $k = 1, \ldots, N$ define a subspace $\mathcal{L}_N$ of co-dimension $N$ in $\mathcal{L}$, so $\mathcal{L}_N$ is infinite-dimensional. Now fix a polynomial $p(z)$ having simple zeros at the points $z_k$. The polynomial grows as $|z|^N$ at infinity, therefore all functions of the form $g = p(z)^{-1} f$, $f \in \mathcal{L}_N$ are entire and have finite integral $\int |g|^2 e^{-2\Psi^++2\Psi^-} \, dx$. \hfill \Box

Relaxing the condition of the compactness of $\text{supp} \mu_-$, we suppose only that $\mu_-(C) = 2\pi \Phi$ is finite.

**Corollary 3.4.** Suppose that $\mu_-(C) < \infty$. Then the statements of of Theorems 3.2 and 3.1 hold for $\mu = \mu_+ - \mu_-$. 
Proof. We choose the potential $\Psi_-$ for the measure $\mu_-$ in the form:

$$
\Psi_-(z) = \frac{1}{2\pi} \int_{D(z, 5)} \ln |z - w| d\mu_-(w) + \frac{1}{2\pi} \int_{C \setminus D(z, 5)} \ln \frac{2|z - w|}{|w|} d\mu_-(w) = U_0(z) + U_1(z),
$$

(3.15)

For $|z| > 5$ we split $U_1$ into two terms, $U_1 = U' + U''$, where the first term corresponds to integration over the disk $|w - z| < |z|/4$ and the second one to the integration over the rest of the plane. In the disk, the expression $\ln \frac{2|z - w|}{|w|}$ is negative, and $U' < 0$. To estimate the second term we note that $\frac{2|w| - |z|}{|w|} \in [2/3, 5]$ for $|w| \geq 2|z|$, and $\frac{2|w| - |z|}{|w|} \in [\frac{1}{3}, \frac{1}{4}]$ for $5 \leq |w| \leq 2|z|$. Thus the integrand in $U''$ majorizes by $C_1 + C_2 \ln |z|$ and therefore $|U''(z)| \leq (C_1 + C_2 \ln |z|)\mu(\{|z| \geq 5\})$. A similar logarithmic estimate, with coefficient $\mu(D(0, 5))$, holds for $U_0$ for large $|z|$. Thus, as a whole, we have $\Psi_-(z) \leq C \log |z|$, and the proof is concluded exactly as the one for Corollary 3.3.

If the negative part of the measure $\mu$ is infinite, infiniteness of zero modes can still be established supposing that $\mu$ becomes non-negative after an averaging, however we need some additional local regularity conditions.

Further on consider the following conditions for the signed measure $\mu = \mu_+ - \mu_-$, $\mu_\pm \geq 0$.

Condition 3.5. There exist constants $r_0 > 0$ and $\theta_0 \in (0, 1)$ such that $\mu_+(D(z, r_0)) \leq 2\pi \theta_0$ for any disk in $\mathbb{R}^2 = \mathbb{C}$ with radius $r_0$.

Note that Condition 3.5 implies that if AB solenoids are present, their intensities lie in the interval $(0, \theta_0)$ for $P_{\text{max}}$.

In particular cases we also suppose that $\mu_+$ and/or $\mu_-$ satisfy

Condition 3.6. There is a constant $A_1$ and a radius $R_1$ such that for any disk $D(z, R_1)$

$$
\int_{D(z, R_1)} |\ln |z - w|| d\mu_\pm(w) \leq A_1.
$$

(3.16)

In particular, Condition 3.6 is satisfied if the measure $\mu_\pm$ is absolutely continuous with respect to the Lebesgue measure $dx$ and the corresponding densities belong uniformly to $L^p_{\text{loc}}$ for some $p > 1$. Note also that if this condition is fulfilled for some $R_1$, it holds for any other $R_1$, with a different constant $A_1$.

For a measure $\mu$ satisfying Condition 3.5, we consider a potential $\Psi(z)$, a solution of the equation $\Delta \Psi = \mu$, as well as the potentials of the measures $\mu_\pm$, $\Delta \Psi_\pm = \mu_\pm$, $\Psi = \Psi_+ - \Psi_-$. The potential $\Psi$ (as well as $\Psi_\pm$) is determined not uniquely but up to an arbitrary harmonic function.

The first elementary fact we establish concerns measures satisfying Condition 3.6. Let $\chi$ be a smooth non-negative function $\chi \in C^\infty_0(D(0, R))$, $\chi = \chi(|z|)$ for some $R$, such that $\int_{D(0, R)} \chi = 1$, and we set $\Psi_R = \Psi * \chi$, so that $\Delta \Psi_R = \mu_R = \mu * \chi$. We set also $\Psi_\pm, R = \Psi_\pm * \chi$, $\Delta \Psi_\pm, R = \mu_\pm, R = \mu_\pm * \chi$.

Lemma 3.7. Suppose that the measure $\mu_-$ or $\mu_+$ satisfies Condition 3.6. Then there is a constant $C = C(R, R_1, A_1)$ such that, with the corresponding sign $\pm$, $$|\Psi_\pm(z) - \Psi^*_\pm(z)| \leq C.$$  

(3.17)

Proof. Denote by $D$ the disk $D(z, R)$ and by $D'$ the concentric disk with twice as large radius. Split the measure $\mu_\pm$ into the sum of the measure $\mu'_\pm$ supported in $D'$ and $\mu_\pm$ supported outside this disk. Correspondingly, the potential $\Psi_\pm$ splits into the sum of $\Psi'_\pm = \mu'_\pm * G_0$ and $\Psi_\pm = \Psi_\pm - \Psi'_\pm$, $G_0(z) = (2\pi)^{-1} \ln |z|$. The function
$\tilde{\Psi}_+$ is harmonic in $D'$, therefore $\tilde{\Psi}_+ \ast \chi = \tilde{\Psi}_+$ in the disk $D$, in particular, at the point $z$. The potentials $\Psi_+$ and $\Psi_+ \ast \chi$ are bounded, by (3.16). This proves the Lemma.

Next we will study the potentials of measures for which $\mu_+$ satisfies Condition 3.5, and another component, $\mu_-$ satisfies Condition 3.6. Fix some $R$. For a fixed point $z_0 \in \mathbb{C}$ denote by $D_D, D_D', D_2$ the disks with center at $z_0$ and radii, respectively, $r_0/2, r_0, r_0 + R, r_0 + 2R$. We fix a non-negative mollifier $\chi \in C^\infty_0(D(0, R))$ as above and set $\Psi_R(z) = \chi \ast \Psi, \Psi_{\pm, R} = \chi \ast \Psi_{\pm}$. We prove now our main local estimate.

**Proposition 3.8.** For a fixed $z$, suppose that, Condition 3.5 is satisfied for $\mu_+$, and Condition 3.6 is satisfied for $\mu_-$ in $4R$-neighborhood of $z$. Then there exist constants $C_0 = C_0(r_0, R, \theta_0, A_1), C_1 = C_1(r_0, R, \theta_0, A_1)$, and $C_2 = C_2(r_0, R, \theta_0, A_1)$ such that

$$\int_{D_D} e^{-\Psi(z)} |f(z)|^2 dx \leq C_0 \int_{D_D'} e^{-\Psi_R(z)} |f(z)|^2 dx + C_1 \int_{D_D} e^{-\Psi_R(z)} |\partial f(z)|^2 dx$$

(3.18) and

$$\int_{D_D} e^{\Psi_R(z)} |f(z)|^2 dx \leq C_2 \int_{D_D'} e^{\Psi_R(z)} |f(z)|^2 dx$$

(3.19)

for any function $f \in L^2(D_D')$, as soon as the inequalities make sense. The derivative $\partial$ in (3.18) can be replaced by $\overline{\partial}$.

**Proof.** For brevity, we prove the inequalities for the disks centered at the origin, noticing that the constants in all estimates below depend only on $r_0, R, \theta_0, A_1$.

First, due to Lemma 3.7, we can restrict ourselves to a non-negative measure $\mu$ since the negative part of $\mu$ contributes to the estimates only with a constant factor when passing from $e^{-\Psi}$ to $e^{-\Psi_R}$.

We split the measure $\mu$ into the sum $\mu = \mu' + \nu$, so that $\mu'$ is supported in the disk $D_2$ and $\nu$ is supported outside this disk. The potential $\Psi$ splits into two parts, $\Psi = \Psi' + H$, where $\Psi' = G * \mu'$ is the Newton potential of the measure $\mu'$, $G(z) = 1/(2\pi) \ln |z/R|$, and $H(z) = \Psi(z) - \Psi'(z)$ is a harmonic function inside the disk $D_2$.

Correspondingly, the smoothed potential $\Psi_R$ splits into two terms, $\Psi_R = \Psi_R' + H_R$, where $\Psi_R' = \Psi' \ast \chi, H_R = H \ast \chi$. Note that since $H$ is harmonic inside $D_2$, the functions $H$ and $H_R$ coincide inside $D_1$. The function $\Psi_R = \mu' * G * \chi$ is bounded in $D_1, |\Psi_R'| \leq c_1 = c_1(r_0, R, A_1, \theta_0)$.

Let $g(z)$ be a function, anti-analytical in $D_1$, such that $H(z) = -\ln(|g(z)|)$.

Then we have for any $f$

$$\int_{D_D} e^{-\Psi(z)} |f(z)|^2 dx = \int_{D_D} e^{-\Psi'(z)} |f(z)|^2 dx,$$

$$\int_{D_D} e^{-\Psi_R(z)} |f(z)|^2 dx = \int_{D_D} e^{-\Psi_R'(z)} |f(z)|^2 dx,$$

$$\int_{D_D} e^{-\Psi_R(z)} |\partial f(z)|^2 dx = \int_{D_D} e^{-\Psi_R'(z)} |\partial f(z)|^2 dx.$$

So, denoting $u = fg$ and taking into account the boundedness of $\Psi_R(z)$, we see that it is sufficient to establish the estimate

$$\int_{D_D} e^{-\Psi(z)} |u|^2 dx \leq C \int_{D_1} |u|^2 + |\partial u|^2 dx$$

(3.20)

To prove (3.20), we split $\mu'$ into further two parts, $\mu' = \mu_0 + \mu_1$ where $\mu_0$ is supported in $D$ and $\mu_1$ in $D_1 \setminus D$. The function $\Psi_1 = \mu_1 * G$ is bounded in $D_0$. In
fact, the distance between points in $D_0$ and in the support of $\mu_2$ lies between $r_0/2$ and $R + 2r_0$, therefore

$$|\Psi_1(z)| \leq \max\{|\ln(r_0/(2R)|, |\ln(R + 2r_0)/R)|\mu(D_1)| \leq C.$$ 

If $\mu_0 = 0$, the required inequality is now obvious. Otherwise, in order to estimate the contribution of $\Psi_0 = \mu_0 * G$, we apply Jensen’s inequality:

$$e^{-2\mu(D)} \int_{D} G(z-w) \frac{d\mu(w)}{\mu(D)} \leq \int_{D} e^{-2\mu(D)} G(z-w) \frac{d\mu(w)}{\mu(D)} = R^{2\mu(D)} \int_{D} |z-w|^{-\rho(D)/2} \frac{d\mu(w)}{\mu(D)}.$$ 

So, for the left-hand side in (3.20) we have

$$\int_{D_0} e^{-2\Psi_0(z)} |u(z)|^2 dx \leq C \int_{D} \left[ \int_{D_0} |u(z)|^2 |z-w|^{-\rho(D)/2} dx \right] \frac{d\mu(w)}{\mu(D)}.$$ 

In the inner integral we apply the H"older inequality, taking into account that $\mu(D)/\pi < 2\theta_0 < 2$:

$$\int_{D_0} |u(z)|^2 |z-w|^{-\rho(D)/2} dx \leq ||u||^2_{L_{2\rho}(D_0)} ||z-w|^{-\rho(D)/2}| ||L_{\rho}(D_0) \leq C ||u||^2_{L_{2\rho}(D_0)},$$

provided $q \in (1, \infty)$ is chosen so that $\theta_0 q' < 1$, therefore the norm of $|z-w|^{-\rho(D)/2}$ in (3.22) is finite. The second integration in (3.21) gives then

$$\int_{D_0} e^{-2\Psi_0(z)} |u(z)|^2 dx \leq C ||u||^2_{L_{2\rho}(D_0)}.$$ 

Finally we apply the Sobolev type embedding theorem in the disk $D_0$:

$$||u||^2_{L_{2\rho}(D_0)} \leq C ||u||^2_{L_{2\rho}(D_0)} + ||\partial u||^2_{L_{2\rho}(D_0)}$$

and recall (3.23) and Lemma 3.7. This proves (3.20), and therefore (3.18). The inequality (3.19) follows immediately from Lemma 3.7. Obvious changes establish Lemma for $D$.

De-localizing (3.18), (3.19) leads to the following fundamental fact.

**Proposition 3.9.** Suppose that for all $z$ the measure $\mu_+$ satisfies Condition 3.5, and $\mu_-$ satisfies Condition 3.6, $\Psi(z)$ is a potential for the measure $\mu$ and $\Psi_R$ is the smoothened potential $\Psi_R = \Psi * \chi$. Then, with some constants $C_0, C_1, C_2$ depending only on $r_0, R_1, \theta_0, A_1$.

$$\int_{C} e^{-2\Psi(z)} |f(z)|^2 dx \leq C_0 \int_{C} e^{-2\Psi_R(z)} |f(z)|^2 dx + C_1 \int_{C} e^{-2\Psi_R(z)} |\partial f(z)|^2 dx,$$ 

and

$$\int_{C} e^{-2\Psi_R(z)} |\partial f(z)|^2 dx \leq C_2 \int_{C} e^{-2\Psi(z)} |\partial f(z)|^2 dx,$$

as soon as the inequalities (3.24), resp., (3.25), make sense. Again, $\partial$ can be replaced by $\partial$.

**Proof.** We take a covering of $C$ by the disks $D$ with radius $r_0/2$ such that the concentric disks with radius $R + r_0/2$ form a covering with finite multiplicity $x$. Then we write the estimate (3.18) for each disk $D$, and sum these inequalities. This leads us to (3.24). To prove (3.25), consider the usual splitting $\Psi = \Psi_+ - \Psi_-$. The function $\Psi_+$ is subharmonic, therefore $\exp(-2\Psi_+, R) \leq \exp(-2\Psi_+)$, so it is sufficient to establish

$$\int_{C} e^{2\Psi_-(z)} |h(z)|^2 dx \leq C_2 \int_{C} e^{2\Psi(z)} |h(z)|^2 dx,$$
where \( h(z) = \exp(-\Psi_+/\overline{\partial} f(z)) \). The latter inequality follows immediately from its localized version (3.19).

**Proposition 3.10.** Suppose that the measure \( \mu_- \) satisfies Condition 3.6 Then

\[
\int_C e^{-2\Psi_R(z)}|h(z)|^2\,dx \leq C_2 \int_C e^{-2\Psi(z)}|h(z)|^2\,dx,
\]

(3.26)

as soon as (3.26), makes sense.

**Proof.** To prove (3.26), note that the function \( \Psi_+ \) is subharmonic, therefore \( \exp(-2\Psi_+/R) \leq \exp(-2\Psi_+) \), so it is sufficient to establish

\[
\int_C e^{2\Psi-R(z)}|f(z)|^2\,dx \leq C_2 \int_C e^{2\Psi(z)}|f(z)|^2\,dx,
\]

where \( f(z) = \exp(-\Psi_+)|h(z)| \). The latter inequality follows immediately from Lemma 3.7.

Now we can establish our next theorem on zero modes. We recall that for the operator \( \mathcal{P}_{\text{max}} \) we accept normalization of the discrete part of the measure \( \mu \) so that all intensities of AB-solenoids lie in \((0,1)\), while for the operator \( \mathcal{P}_{\text{EV}} \) these intensities lie in \([-\frac{1}{2}, \frac{1}{2}]\), with 0 excluded.

**Theorem 3.11.** Suppose that the measure \( \mu_- \) satisfies Condition 3.6, \( \mu \) satisfies Condition 3.5, moreover, for a certain \( R_0 > 0 \), \( A(z) = \mu(D(z), R_0) \) \( \geq 0 \) for \( |z| \) large enough and \( \int_C A(z)\,dx = \infty \). Then the spin-down components of the Pauli operators \( \mathcal{P}_{\text{max}}, \mathcal{P}_{\text{EV}} \) have an infinite-dimensional null subspace. If, additionally a stronger condition

\[
A(z) \geq A_0 > 0
\]

(3.27)

is satisfied for all \( |z| \) large enough then the point zero is an isolated point in the spectrum of \( \mathcal{P}_{\text{EV}}, \mathcal{P}_{\text{max}} \) and the spectral gap above zero is estimated from below by some constant depending on \( r_0, R_0, A_0, A_1 \).

**Proof.** Note first that due to our normalization agreement, see Sect.2, the null subspace in both \( \text{EV} \) and \( \text{max} \) cases is generated by entire functions. Therefore we suppress the corresponding subscript. Let \( \Psi \) be a potential for \( \mu \). The null subspace of the operator \( \mathcal{P}_- \) consists of the functions of the form \( u \in L_2(\mathbb{C}) \) such that \( u = \exp(-\Psi)f \), with an entire (anti-analytical) function \( f(z) \). Take some \( R > R_0 \) and a mollifier \( \chi_0 \) supported in the disk \( D(0, R - R_0) \). Set \( \chi = \chi_0 * \chi_{R_0} \), where \( \chi_{R_0} \) is the characteristic function of \( D(0, R_0) \). Under the conditions of the first part of the theorem, the potential \( \Psi_R(z) = \Psi * \chi \) is subharmonic outside a compact set. If \( f \) is an entire function, the second term on the right-hand side in (3.24) vanishes, so we obtain the estimate

\[
\int_C e^{-2\Psi(z)}|f(z)|^2\,dx \leq C_0 \int_C e^{-2\Psi_R(z)}|f(z)|^2\,dx.
\]

(3.28)

This inequality implies that if for some entire function \( f \) the function \( \exp(-\Psi_R)f \) belongs to \( L_2 \) then \( \exp(-\Psi)f \) also belongs to \( L_2 \). However the space of the functions satisfying the former condition is infinite-dimensional by Corollary 3.3. Therefore the space of entire functions \( f \) with \( \exp(-\Psi)f \in L_2 \) is also infinite-dimensional.

To prove the existence of the spectral gap, we note first of all that under the conditions of the second part of the theorem we can, by increasing \( R_0 \), have \( A(z) > A_0/2 \) for all \( z \), so, that \( \Psi_R \) is strictly subharmonic. From Hörmander’s theorem (Lemma 4.4.1 in [13]) on solutions of the \( \overline{\partial} \)-equation in weighted spaces (we apply
this Lemma with \( \partial \) replacing \( \overline{\partial} \), it follows that for any function \( g \in L_2 \), there exists a solution \( f \) of the equation \( \partial f = g \) such that
\[
\int_{\mathbb{C}} e^{-2\Psi_R(z)} |f(z)|^2 dx \leq c \int_{\mathbb{C}} e^{-2\Psi_R(z)} |g(z)|^2 dx.
\]
(3.29)
Substituting this inequality into (3.24), and using (3.26), we obtain
\[
\int_{\mathbb{C}} e^{-2\Psi_R(z)} |f(z)|^2 dx \leq c' \int_{\mathbb{C}} e^{-2\Psi_R(z)} |g(z)|^2 dx.
\]
(3.30)
We set \( e^{-\Psi(z)} f(z) = u(z) \), \( e^{-\Psi(z)} g(z) = v(z) \), and recalling that \( g = \partial f \) and \( Q_- = e^{-\Psi} \partial e^{\Psi} \) we get the inequality
\[
\|u\|_{L^2}^2 \leq C\|v\|_{L^2}^2, \quad v = Q_- u,
\]
(3.31)
where \( v \) is an arbitrary function in \( L_2 \) and \( u \) is a certain function in \( L_2 \) satisfying \( Q_- u = v \). Let \( h(z) \) be the projection of \( u \) onto the subspace \( \mathcal{N} \) in \( L_2 \) orthogonal to all solutions \( w \) of the equation \( Q_- w = 0 \); we still have \( v = Q_- h \). The left-hand side of (3.31) can only decrease if we replace there \( u \) by \( h \). So, on the subspace \( \mathcal{N} \) which is the spectral subspace of the Pauli operator \( \mathcal{P}_- = Q_- Q_- \), corresponding to the nonzero spectrum, the norm of the function \( h \) is majorated by \( p_- [h] \), the value of the quadratic form of the operator \( \mathcal{P}_- \) on the function \( h \). This exactly means that the nonzero spectrum of \( \mathcal{P}_- \) is separated from zero, and the width of the spectral gap is controlled by the constant \( C \) in (3.31).

The restrictions imposed in Theorem 3.11 on the negative part of \( \mu \) can be relaxed.

**Corollary 3.12.** Suppose that the measure \( \mu \) satisfies conditions of the first part of Theorem 3.11 and \( v \) is a finite non-negative measure. Then for the measure \( \mu - \nu \) the zero energy subspace is infinite-dimensional.

The Corollary follows immediately from the main theorem and Corollary 3.4.

If there are arbitrarily large regions where the field vanishes, we cannot use the second part of Theorem 3.11 to establish the presence of the spectral gap, even for a non-negative measure since such regions violate the condition (3.27). The following statement shows that if there are such regions then, actually, zero cannot be an isolated point in the spectrum. We suspect that the result is known to specialists, but we could not find a reference, so we present a proof, for the sake of completeness.

**Proposition 3.13.** Suppose that for any \( R > 0 \) there exists a disk \( D \) with radius \( R \) where the measure \( \mu \) is zero. Then zero is not an isolated point in the spectrum of the Pauli operators \( \mathcal{P}_- \) and \( \mathcal{P}_+ \).

Note that we do not suppose anything about the nature of zero as a point in the spectrum.

**Proof.** To justify the statement, we find, for any \( \epsilon > 0 \), a function \( \phi \) orthogonal to the null subspace of the operator \( \mathcal{P}_- \) (or \( \mathcal{P}_+ \)) such that \( p_- [\phi] \), resp., \( p_+ [\phi] \) is smaller than \( \epsilon \|\phi\|^2 \). To do this (for \( \mathcal{P}_- \), for example), we fix a non-trivial smooth function \( u_0 \geq 0 \), with compact support in the unit disk \( D_1 \). For some constant \( C \), the estimate \( \|u_0\|_{L^2} \leq C \|\nabla u_0\|_{L^2} \) holds. Now, let \( D \) be a disk with radius \( R > \epsilon^{-1} \) and center at \( z_0 \), such that the restriction of \( \mu \) to \( D \) is zero, and \( u \) be the function in \( C^0_\infty \) obtained from \( u_0 \) by the dilation and shift, \( u(z) = u_0(R^{-1}(z - z_0)) \). The function \( u \) satisfies
\[
\|u\|_{L^2} \leq C R^{-2} \|\nabla u\|_{L^2}
\]
(3.32)
Since the magnetic field is regular (in fact, it zero) on the support of \( u \), the function \( u \) belongs to the domain of the operator \( \mathcal{P}_- \).
Denote by $v$ the projection of $u$ to the null space of the operator $\mathcal{P}_-$. The function $v$ also belongs to the domain of the operator $\mathcal{P}_-$ (the function $v$ is zero if, in particular, the null space is trivial.) So, $\phi = u - v$ also belongs to the domain of $\mathcal{P}_-$ and is orthogonal to $\text{Ker}(\mathcal{P}_-)$; $\phi$ is nontrivial since $v$, being a zero mode, cannot have compact support (unless $v = 0$). The norm of the function $\phi$ in $L_2(\mathbb{R}^2)$ is not greater than the norm of $u$ in $L_2(\mathbb{R}^2)$, or, what is the same, than $||u||_{L_2(\mathcal{D})}$. At the same time, for the quadratic form of the operator $\mathcal{P}_-$, we have

$$p_\phi = p_- u = \int_\mathcal{D} \exp(-2\Psi) \partial_z (\exp(\Psi) u)^2 dx.$$ 

However the magnetic field $\mu = \Delta \Psi$ vanishes in $\mathcal{D}$, so $\Psi$ is a harmonic function in $\mathcal{D}$. Taking into account that $u$ has its support in $\mathcal{D}$, by means of the usual partial integration we obtain that $p[u] = \int_\mathcal{D} |\nabla u|^2 dx$. Now, the inequality (3.32) gives us

$$||\phi||_{L_2(\mathbb{R}^2)}^2 \leq cR^{-2} p_- [\phi], \quad (3.33)$$

for the function $\phi$ orthogonal to the null subspace of $\mathcal{P}$. Supposing that there are disks of arbitrary size $R$, not intersecting the support of $\mu$ we obtain that in the neighborhood of the point zero there are infinitely many points of the spectrum of $\mathcal{P}_-$. □

The reasoning in the proof of Theorem 3.11 does not apply directly to the operator $\mathcal{P}_{\text{max}}$, if, for example, the measure $\mu$ is continuous. In fact, when we pass to $\mathcal{P}_+$, we have to replace $\mu$ by $-\mu$ and $\partial$ by $\bar{\partial}$. The latter change is not that essential. However the measure $-\mu$ does not satisfy the condition $-\mu(\mathcal{D}(z, R_0)) \geq 0$, this quantity is negative and the line of reasoning breaks down in several places.

The case when the game can be saved for the operator $\mathcal{P}_{\text{max}}$ is the one with a measure $\mu$ consisting of the system of Aharonov-Bohm solenoids placed at the points of a discrete set $\Lambda$, with intensities $\alpha_\lambda$, $\alpha_\lambda \in (0, 1)$, on the background of a continuous field $\nu$ such that both $\nu_+$ and $\nu_-$ satisfy Condition 3.6:

$$\mu = \mu_{AB} + \nu, \mu_{AB} = 2\pi \sum_{\lambda \in \Lambda} \alpha_\lambda \delta(z - \lambda). \quad (3.34)$$

In this case, as explained in Sect. 2, the measure $-\mu$ can be reduced by a gauge transform to the one containing AB-fluxes with intensities $(1 - \alpha_\lambda)$, i.e., to the measure

$$\tilde{\mu} = \tilde{\mu}_{AB} - \nu, \tilde{\mu}_{AB} = 2\pi \sum_{\lambda \in \Lambda} (1 - \alpha_\lambda) \delta(z - \lambda). \quad (3.35)$$

If the set $\Lambda$ is infinite, does not have large gaps, and the numbers $\alpha_\lambda$ are separated both from 0 and 1, it is possible that both measures $\mu, \tilde{\mu}$ satisfy conditions of Theorem 3.11. This leads to infiniteness of zero modes and the spectral gap for both $\mathcal{P}_{\text{max}}$ and $\mathcal{P}_{\text{max}}$.

We formulate a special case, where Conditions on the measures are expressed in more geometrical terms:

**Condition 3.14.** There exist positive numbers $s^0, R_0$ such that any disc $\mathcal{D}(z, R_0)$ contains at least one point in $\Lambda$, any disk $\mathcal{D}(\lambda, r^0)$, $\lambda \in \Lambda$, contains no points in $\Lambda$, other than $\lambda$, and all intensities $\alpha_\lambda$, $\lambda \in \Lambda$, satisfy $\alpha_\lambda \in (\theta_0, 1 - \theta_0)$ for a certain $\theta_0 > 0$.

**Corollary 3.15.** Suppose that the measure $\mu$ has the form (3.34) its discrete part satisfies Condition 3.14. Suppose also that both positive and negative parts of the continuous part $\nu$ in (3.34) satisfy Condition 3.6, moreover, for $|z|$ large enough, $\theta_0 \geq \nu_\pm(\mathcal{D}(z, R_0))$. Then both operators $\mathcal{P}_{\text{max}}$ have an infinite-dimensional null subspace. If, moreover, $\theta_0 \geq \nu_\pm(\mathcal{D}(z, R_0)) \geq A_0 > 0$, both operators possess a
spectral gap. The size of the gap is determined by the numbers $r^0, R_0, \theta_0, A_1,$ and $a_0.$

The Corollary above covers, among other cases, a purely discrete measure, i.e., an infinite configuration of AB solenoids satisfying Condition 3.14, as well as such a configuration on the background of a constant or 'almost constant' magnetic field. For a regular lattice $\Lambda$ and equal intensities and a constant background field, the infiniteness of zero modes was established in [9], [10].

On the other hand, if the measure $\mu$ is continuous (thus $P_{\text{max}} = P_{EV}$), so that there are no AB solenoids, we can establish the spectral gap also for $P_+.$

**Corollary 3.16.** Let the measure $\mu$ satisfy Condition 3.5, $\mu(D(z, R_0)) \geq A_0 > 0$ for $|z|$ large enough, and both $\mu_{\pm}$ satisfy Condition 3.6. Then the operator $P_+$ has no zero modes and possesses a spectral gap.

**Proof.** Let $\Psi$ be the potential of the measure $\mu$, $\Psi_R$ its averaging, so that $\Delta \Psi_R = \mu * \chi$, is a measure absolutely continuous with respect to Lebesgue measure with positive density separated from zero. As it follows from Proposition 3.10 and the conditions imposed on $\mu$, for any function $f$,

$$\int \exp(2\Psi)|f|^2 dx \geq C \int \exp(2\Psi_R)|f|^2 dx.$$  

(3.36)

Now for the smooth function $\Psi_R$ we use the commutational relation which gives

$$\int_c |(e^{\Psi_R(z)} u) \partial(e^{-\Psi_R(z)} u)|^2 dx = \int_c e^{-2\Psi_R(z)} |\partial_z(e^{\Psi_R(z)} u)|^2 dx + \frac{4}{c} \int |u|^2 d\mu_R.$$  

(3.37)

And this establishes our statement. $\square$

An important special case of the above considerations concerns periodic magnetic fields. Such configurations attracted interest in early 80-s. In the papers [6] and [7] for the case of a rational flux of the field over an elementary cell of the lattice the infiniteness of zero modes was proved as well as the existence of the spectral gap, and nothing has been done since. We show below that the above restriction is irrelevant.

**Corollary 3.17.** Let the measure $\mu$ be periodic with respect to some lattice in the plane. Suppose that Condition 3.6 is satisfied for both positive and negative parts of the measure $\mu$ and, moreover, the measure of one cell $f$ of the lattice is positive. Then there are infinitely many zero modes for the Pauli operator $P$ and a spectral gap.

In fact, for a periodic measure, the Conditions 3.5 and positivity of $\mu(D(z, R_0))$ for large $|z|$ obviously follow from the positivity of the measure of the cell, and this, by Theorem 3.11, establishes both infiniteness of zero modes and spectral gap for $P_-$. The presence of a spectral gap for $P_+$ follows from Corollary 3.16.

For a periodic magnetic field the infiniteness of zero modes may be proved also in the following way. Let $\Lambda$ be the lattice of periods of the measure $\mu$, with the elementary cell $f$. Let $\sigma(z)$ be the Weierstrass function of the lattice $\Lambda$ defined in (2.13). As it is explained in Sect.2, with proper $\nu$, the function $\Psi_0 = \ln |\sigma(z)| - R(\nu z^{-2})$ equals $m|z|^2 + \rho(z)$, $m = \frac{1}{2\pi}$, with a periodic function $\rho(z)$ having singularities of the form $(2\pi)^{-2} \ln |z - \lambda|$ near each point $\lambda$ of the lattice $\Lambda$, $|f|$ being the area of the elementary cell. Consider the potential

$$\Psi_\mu(z) = \int_f \exp(\Psi_0(z-w)) d\mu(w) = \int_f |z-w|^2 d\mu(w) + \int_f \rho(z-w) d\mu(w),$$  

(3.38)

with integration over the elementary cell $f$ of $\Lambda$. Since $\Delta \Psi_0 = \sum \delta(z-\lambda)$, the function $\Psi$ is a potential for the measure $\mu$. From the Condition 3.6 for $\mu_{\pm}$, it
follows that the second term in (3.38) is a periodic bounded function and positivity of the flux $\Phi$ over the elementary cell produces the growth of the first term in (3.38) as $\Phi m|z|^2$ Therefore any function having the form $u = \exp(-\Psi_{\mu}(z)) f$ with an entire function $f(z)$ growing not faster that $\exp(\gamma |z|^2)$, $c < \Phi m$, is a zero mode for the periodic magnetic field $\nu$.

More generally, we can consider a quasi-periodic magnetic field. Let $\mu = \sum \mu_k$, $k = 1, \ldots, N$, so that each measure $\mu_k$ is periodic with respect to its own lattice $\Lambda_k$. Suppose also that Condition 3.6 is satisfied for each measure $\mu_k$, and the sum $\Phi = \sum_k \frac{\Phi_k}{|F_k|}$, $|F_k|$ being the area of the cell of the lattice of periods for $\mu_k$, is positive. Then the reasoning used for a single periodic field goes through. Again, the smoothened field $\mu * \chi$ is a measure with a bounded positive density, which guarantees, as in Corollary 3.17, infiniteness of zero modes and the presence of a spectral gap. Moreover, the explicitly constructed potential, being the sum of potentials $\Psi_{\mu_k}$, grows as $c|z|^2$.

Of course if the sum is negative, then the potential for the measure $\mu = \sum \mu_k$ is majorated by $-c|z|^2$, and this leads to infinitely many zero modes for the Pauli operator $P_+$. We note here that the fields above can be, by Corollary 3.12, perturbed by any finite measure thus preserving the infiniteness of zero modes.

In the following sections we consider more strong perturbations of the field configurations described above. These perturbation preserve the quadratic lower estimate for the potential, thus guaranteeing infiniteness of zero modes, however we will see that they may destroy the spectral gap.

4. Perturbations of the Field and Zero Modes

From now on we consider a more special situation than in the previous section. Suppose that for a certain magnetic field $\mu_0$ we know that there exists a potential $\Psi_0$ satisfying the growth condition $\Psi_0(z) \geq \gamma |z|^2$ for $|z|$ large enough, with some $\gamma > 0$. Then any entire analytical function $f$ growing at infinity slower than $\exp(\gamma - \epsilon)|z|^2$ satisfies $f \exp(-\Psi) \in L_2$, and thus the null subspace of the operator $P_-$ is infinite-dimensional. Of course, this does not guarantee the presence of the spectral gap, and, moreover, the spectral gap may fail to exist, as we show below. Now, let $\mu_1$ be another magnetic field with a potential $\Psi_1(z)$ satisfying the estimate $\Psi_0(z) - \Psi_1(z) = o(|z|^2)$ or even $|\Psi_0(z) - \Psi_1(z)| \leq C \gamma |z|^2$ with $\gamma < \gamma_0$, for large $|z|$. Then, of course, the potential $\Psi_1$ grows at infinity sufficiently fast so that the infiniteness of the number of zero modes for the Pauli operator with field $\mu_1$ is granted. More generally, the estimate for $\Psi_0$ and the inequality for the difference $\Psi_0(z) - \Psi_1(z)$ may contain some singular terms, as it happens in the presence of A-B solenoids.

There are several types of the magnetic fields that we can take as the unperturbed field $\mu_0$. One obvious example is the constant magnetic field described by the measure $d\mu_0 = Bdx$, proportional to the Lebesgue measure – here the potential has the form $\Psi_0 = \frac{B}{2}|z|^2$. As shown in the end of the previous section, a more general field, a periodic field with mild local regularity properties (Condition 3.6) with nonzero flux through the cell, or the even more general one, a quasi-periodic field with nonzero $\sum_k \frac{\Phi_k}{|F_k|}$, also possess potentials subject to required growth conditions.

Another kind of the starting point of our study can the observation made in [9]. Let $\Lambda = \Lambda_0$ be a regular, periodic lattice with periods $\omega_1, \omega_2$ and all intensities are equal, $\alpha_\lambda = \alpha \in (0, 1)$, $\lambda \in \Lambda$. The scalar potential $\Psi_{0+}(z)$ is defined in (2.15). Since $\alpha$ lies between 0 and 1, the function $u(z) = f(z) \exp(-\Psi_{0+}(z)) = f(z) \exp(-\alpha |z|^2) \exp(-\alpha \rho(z))$ belongs to $L_2$ as soon as the entire function $u(z)$ grows at infinity not faster than $\exp \gamma |z|^2$, $\gamma < \alpha m$. Of course, there are a lot
of such entire functions $v(z)$, in particular, all polynomials fit. This proves the infiniteness of zero modes for $P_{\text{max}}$ or, under the condition $\alpha \in (0, \frac{1}{2})$, for $P_{\text{EV}}$.

Taking into account the possibility of applying the gauge transformations discussed in the end of Sect. 2, the same reasoning applies to the operator $P_{\text{max}}$. Again we can perturb the (discrete now) measure $\mu_0$ by a measure having a potential with a slower growth than $\Psi_0$ and with controlled logarithmic singularities.

So, in order to determine which perturbations of the initial field $\mu_0$ preserve infiniteness of zero modes, one has to know which measures possess potentials with prescribed control over the behavior at infinity and, if needed, at singular points. A number of such results exist in the literature, see, e.g., [11], however they are not sufficient in our situation since they require an extra decay of the measure at infinity, the condition we aim to avoid. Moreover, they do not usually take into account the possible cancellation of the contribution of the positive and negative parts of the perturbing measure. In this section we present some results on the estimates for potentials for certain classes of measures.

The situation we consider first is the one when the whole (or a part of) measure $\mu_0$ is re-arranged, more exactly, this measure is replaced by its image under some mapping of the plane. Under such re-arrangement, large regions with field having 'wrong direction' may arise, so that the positivity conditions of the general theorems are substantially broken. Nevertheless, the infiniteness of zero modes is preserved. We consider the case of $\mu$ being a continuous measure with $\mu_{\pm}$ satisfying Condition 3.6. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a Borel measurable mapping and $\mu^*$ be the measure induced by this mapping: for a Borel set $E$, $\mu^*(E)$ equals $\mu(\Phi^{-1}(E))$.

**Proposition 4.1.** Suppose that the mapping $\Phi$ satisfies the condition
\[
|\Phi(w) - \lambda| \leq a|w|^\tau, \quad \tau < 1
\] (4.1)
for some $a \geq 0$, for $|w|$ large enough. Let $\mu_{\pm}^*$ also satisfy Condition 3.6. Fix some $R > 0$, such that (4.1) is satisfied for $|w| > R$ and define the function
\[
\Psi^*(z) = \int_{|w| < R} \Re \left[ \ln(1 - z/\Phi(w)) - \ln(1 - z/w) \right] d\mu(w) + \int_{|w| \geq R} \Re \left[ \ln(1 - z/\Phi(w)) - \ln(1 - z/w) + z(\Phi(w)^{-1} - w^{-1}) \right] d\mu(w)
\] (4.2)

Then the function $\Psi^*$ satisfies the equation $\Delta \Psi^* = 2\pi(\mu^* - \mu)$ and, moreover
\[
|\Psi^*(z)| \leq C|z|^{1+\tau}.
\] (4.3)

**Proof.** The fact that the integral converges is obvious. To check that it satisfies the Poisson equation, it is sufficient to notice that for a bounded domain $\Omega$ containing the point $z$,
\[
\int_{\Omega} \Re \ln(1 - z/\Phi(w)) d\mu(w) = \int_{\Phi(\Omega)} \Re \ln(1 - z/w') d\mu^*(w').
\] (4.4)

To prove the crucial inequality (4.3) we have to estimate only the second integral in (4.2) since the first one may grow at most logarithmically at infinity.

For a fixed $z$, we split the plane into three regions
\[
\Omega_0 = \{ w \in \mathbb{C}, \ R < |w| \leq |z|/2 \}, \ \Omega_1 = \{ w \in \mathbb{C}, \ |z|/2 < |w| \leq 2|z| \}, \ \Omega_2 = \{ w \in \mathbb{C}, \ 2|z| < |w| \},
\]
and estimate the corresponding integrals $I_0, I_1, I_2$ separately.
For the integrand in $I_2$, we have

$$\ln \left(1 - \frac{z}{\Phi(w)} \right) = \ln (1 - \frac{z}{w}) + z \left(\frac{1}{\Phi(w)} - \frac{1}{w} \right) = \left(\ln \left(1 - \frac{z}{\Phi(w)} \right) + \frac{z^2}{2\Phi(w)^2} \right) - \left(\ln (1 - \frac{z}{w}) + \frac{z^2}{2w^2} \right) \geq \frac{1}{2} \left[ \frac{z^2}{\Phi(w)^2} - \frac{z^2}{w^2} \right],$$

and correspondingly, $I_2$ splits into $I_{21}$ and $I_{22}$. For $|w| > 2|z|$, supposing $|z|$ is large enough, we have $|z/\Phi(w)| < 3/4$.

Consider the function $h(\xi) = \ln(1 - \xi) + \xi + \xi^2/2$, analytical in the open unit disk. For $|\xi_1|, |\xi_2| \leq \frac{2}{3}$, we have

$$|h(\xi_1) - h(\xi_2)| \leq |\xi_1 - \xi_2| \max_{t \in [\xi_1, \xi_2]} |h'(t)| \leq 4|\xi_1 - \xi_2| \max(|\xi_1|^2, |\xi_2|^2).$$

We set here $\xi_1 = \frac{z}{\Phi(w)}$, $\xi_2 = \frac{w}{w}$, obtaining from (4.6)

$$|h \left(\frac{z}{\Phi(w)} \right) - h \left(\frac{w}{w} \right)| \leq 4 \left| \frac{z}{\Phi(w)} - \frac{w}{w} \right| \max \left( \left| \frac{z}{\Phi(w)} \right|^2, \left| \frac{w}{w} \right|^2 \right) \leq C|z|^3 \left| \frac{1}{\Phi(w)} - \frac{1}{w} \right| |w|^{-2} \leq C|z|^3 |w|^{-4+\tau}.

Therefore we get

$$|I_{21}| \leq C|z|^3 \int_{|w|>2|z|} |w|^{-4+\tau} d\mu(w) \leq C|z|^{1+\tau}. \quad (4.7)$$

Further on, in order to estimate $I_{22}$, we make the transformation

$$\frac{1}{2} \left( \frac{z}{w} \right)^2 - \frac{1}{2} \left( \frac{z}{\Phi(w)} \right)^2 = \frac{1}{2} z^2 \left( w - \Phi(w) \right) \frac{w + \Phi(w)}{w^2 \Phi(w)^2},$$

which gives

$$\left| \frac{1}{2} \left( \frac{z}{w} \right)^2 - \frac{1}{2} \left( \frac{z}{\Phi(w)} \right)^2 \right| \leq C|z|^2 |w|^{-3+\tau},$$

and therefore

$$|I_{22}(z)| \leq C|z|^2 \int_{\Omega_2} |w|^{-3+\tau} d\mu(w) \leq C a |z|^{1+\tau}. \quad (4.8)$$

Taken together, (4.7) and (4.8) give

$$|I_2(z)| \leq C|z|^{1+\tau}. \quad (4.9)$$

We pass to estimating $I_1$. We write the integrand now as

$$\Re [\ln(1 - z/\Phi(w)) - \ln(1 - z/w) + z(\Phi(w)^{-1} - w^{-1})] = \ln |\Phi(w)/w| + \ln \left| \frac{\Phi(w) - z}{w - z} \right| + \Re (z(F(w)^{-1} - w^{-1})). \quad (4.10)$$

In the first term on the right in (4.10), we have $|1 - \Phi(w)/w| \leq 1/2$, therefore $|\ln |\Phi(w)/w|| = |\ln \left|(1 - (\Phi(w) - w)/w)\right|| \leq C|w|^{-1+\tau}$, and we get the estimate by $|z|^{1+\tau}$ for the integral over $\Omega_1$. The integral over $\Omega_1$ of the second term in (4.10) we split into the sum of $I_{11}$ and $I_{12}$, the former being the integral over the domain $\Omega_{11} : |w - z| \geq 4a|z|^\tau$ and the latter over $\Omega_{12} = \Omega_1 \setminus \Omega_{11}$. For $I_{11}$, we have $\frac{\Phi(w) - w}{|w - z|} \leq \frac{1}{2}$, so

$$I_{11} \leq C \int_{\Omega_{11}} \left| \frac{\Phi(w) - w}{w - z} \right| d\mu(w) \leq C|z|^\tau \int_{\Omega_{11}} |w - z|^{-1} d\mu(w) \leq C|z|^{1+\tau}.$$
For the integral over $\Omega_{12}$ we apply the rough estimate $|\ln \left| \frac{\Phi(w) - z}{w - z} \right| | \leq |\ln \phi(w) - z| + |\ln |w - z||$ and integrate each term separately. Since the integration here is performed over the disk with radius $4a|z|^\gamma$, the conditions on the measures $\mu$ and $\mu^*$ imply that $I_{12}$ can be estimated by $|z|^{2\gamma} |\ln |z|| = O(|z|^{1+\gamma}).$

Finally, for $I_0$ note that for $w \in \Omega_0$ we have $\left| \frac{\Phi(w) - w}{w - z} \right| \leq \frac{2a|w|^\gamma}{|z|} \leq 1/2$ for $|z|$ sufficiently large, therefore

$$|I_0(z)| \leq \int_{\Omega_0} |\ln \left| \Phi(w)w^{-1}||d\mu(w) + \int_{\Omega_0} |\ln \left| \frac{\Phi(w) - z}{w - z} ||d\mu(w)$$

$$+ |z| \int_{\Omega_0} |w^{-1} - \Phi(w)^{-1}|d\mu(w) \leq C \int_{\Omega_0} |w|^{-1+\gamma} |d\mu(w)$$

$$+ C \int_{\Omega_0} |w|^{1-\gamma} |d\mu(w) + C|z| \int_{\Omega_0} |w|^{-1+\gamma} |d\mu(w). \quad (4.11)$$

All integrals in this expression are estimated by $CA|z|^{1+\gamma}$.

Following the general pattern described in the beginning of the Section, we arrive at the following case where the infiniteness of zero modes is granted. Note that the somewhat complicated conditions imposed on the unperturbed measure are aimed to cover the interesting cases described in the beginning of the Section.

**Theorem 4.2.** Let $\mu_0$ be a measure, with discrete part $\mu_{0, \text{disc}} = \sum_{\lambda \in \Lambda_0} \alpha_\lambda \delta(z - \lambda)$ supported on the set $\Lambda_0$ satisfying (2.5), with $0 < \alpha_\lambda \leq \theta_0 < 1$ for the operator $\mathcal{P}_{\max}$ and $\alpha_\lambda \in [-1/2, 1/2]$ for $\mathcal{P}_{EV}$. Let $\lambda_0(z)$ be the point in $\Lambda_0$ closest to $z$ (or any of such points) and $\alpha_0(z)$ be $\alpha_{\lambda_0(z)}$. Suppose that the measure $\mu_0$ admits a potential $\Psi_0$ subject to the estimate

$$\Psi_0(z) - \alpha(z) \ln |z - \lambda_0(z)| \geq \gamma |z|^2, \gamma > 0 \quad (4.12)$$

for sufficiently large $|z|$ (if $\Lambda$ is empty, $\ln |z - \lambda_0(z)|$ in (4.12) is replaced by zero). Let $\mu$ be another measure and $\Phi$ be a Borel measurable mapping so that the conditions of Proposition 4.1 are satisfied. Then for the measure $\mu_0 + \mu - \mu^*$ there exists a potential $\Psi^*$ also satisfying (4.12) with some $\gamma', 0 < \gamma' < \gamma$, and thus there are infinitely many zero modes for the Pauli operator $\mathcal{P}_{\max}$, resp., $\mathcal{P}_{EV}$ with the field $\mu_0 + \mu - \mu^*$.

The version of Proposition 4.1 for the case of a discrete perturbing measure $\mu$ is also valid. We only give the formulation here, the proof being practically the same as above.

**Proposition 4.3.** Let $\mu$ be a discrete measure $\mu = 2\pi \sum_{\lambda \in \Lambda} \alpha_\lambda \delta(z - \lambda)$ such that $|\alpha_\lambda| \leq \theta_0 < 1$ and the discrete set $\Lambda$, the support of $\mu$, satisfies the uniform discreteness condition (2.5). Let each point $\lambda \in \Lambda$ move to a new position $\lambda'$ so that the mapping $\Phi : \lambda \mapsto \lambda'$ transforms $\Lambda$ to another discrete set $\Lambda'$, also satisfying (2.5), moreover, for any $\lambda' \in \Lambda'$,

$$|\alpha'(\lambda')| = \left| \sum_{\lambda \in \Phi^{-1}(\lambda')} \alpha_\lambda \right| \leq \theta_0.$$

Suppose finally that $\Phi$ satisfies (4.1). We define the measure $\mu' = 2\pi \sum_{\lambda' \in \Lambda'} \alpha'_\lambda \delta(z - \lambda')$ and the function

$$\Psi(z) = \frac{1}{2\pi} \sum_{\lambda' \in \Lambda'} \sum_{\lambda \in \Phi^{-1}(\lambda')} \alpha_\lambda \left[ \ln |\lambda' - z| - \ln |\lambda - z| + \Re(z(\lambda'^{-1} - \lambda^{-1})) \right]. \quad (4.13)$$
Define also $\alpha(z)$ as $\alpha_{\lambda(z)}$ where $\lambda(z)$ is the point in $\Lambda$, closest to $z$, and, similarly $\alpha'(z) = \alpha'(\lambda'(z))$ where $\lambda'(z)$ is the point in $\Lambda'$, closest to $z$. Then the function $\Psi$ satisfies the Poisson equation $\Delta \Psi = \mu' - \mu$ and

$$|\Psi(z) - \alpha'(z)| \ln|z - \lambda'(z)| + \alpha(z) \ln|z - \lambda(z)|| \leq C|z|^{1+\gamma}. \quad (4.14)$$

The discrete version of Theorem 4.2 is now formulated as following.

**Theorem 4.4.** Suppose that $\mu_0$ is a measure satisfying conditions of Theorem 4.2. Let $\mu$ be a discrete measure and $\Phi$ a mapping such that the conditions of Proposition 4.3 are satisfied. Suppose finally that the union of the sets $\Lambda_0, \Lambda, \Lambda'$ also satisfies (2.5).

Then the measure $\mu_0 + \mu' - \mu$ possesses the potential $U(z)$ subject to

$$U(z) - \alpha_0(z) \ln|z - \lambda_0(z)| + \alpha'(z) \ln|z - \lambda'(z)| - \alpha(z) \ln|z - \lambda(z)| \geq \gamma'|z|^2, \quad 0 < \gamma' < \gamma$$

for $|z|$ large enough, and thus the Pauli operators $P_{\text{max}}$, resp., $P_{\text{max}}$ have infinitely many zero modes.

We will give later some examples how Theorems 4.2 and 4.4 can be applied in interesting concrete situations.

Further on we show that the quadratic growth of the potential and the property of the null subspace to be infinite-dimensional are stable also under certain additive perturbations of the magnetic field. The perturbation of the field is a signed measure $\mu$ such that its discrete part $\mu_{\text{disc}}$ with weights $\beta_\lambda$ satisfies conditions of Proposition 4.3 and the continuous part $\mu_{\text{cont}}$ satisfies

$$\int_{|w - z| < r_0} |\ln|w - z||d\mu_{\text{c}}(w) \leq B \quad (4.16)$$

for some $B$, for all $z \in \mathbb{C}$ ($r_0$ is the constant in (2.5)). To evaluate the size of the measure $\mu$, the following characteristics will be used:

$$\omega(r) = \omega(r, \mu) = |\mu|(D(0, r)),$$

where, recall, $D(0, r)$ is the disk with center at the origin and radius $r$, and

$$M(r) = \int_{r_1 \leq |w| \leq r} w^{-2}d\mu(w)$$

for some fixed $r_1$. To such measure $\mu$ we associate the potential $\Psi_\mu(z)$ defined as

$$\frac{1}{2\pi} \left[ \int_{|w| \leq R} \ln|1 - z/w|d\nu(w) + \int_{|w| > R} \left( \ln\left|1 - \frac{z}{w}\right| + \Re\left(\frac{z}{w} + \frac{1}{2} \left( \frac{z}{w} \right)^2\right) \right) d\nu(w) \right]. \quad (4.17)$$

The value of $R$ will be chosen later. Note, moreover, that changing $R$, we add a harmonic function, the real part of a second degree polynomial to $\Psi_\mu$. For a given point $z \in \mathbb{C}$, there may be only one point $\lambda \in \Lambda$ in the $r^0/2$-neighborhood of $z$. If such point exists we denote it by $\lambda(z)$ and set $\beta(z) = \beta_{\lambda(z)}$ and $L(z) = \beta(z) \ln|z - \lambda(z)|$, otherwise $L(z)$ is set to be zero.

Now we formulate our main estimate.

**Proposition 4.5.** Under the above conditions the following estimates hold.

1. Suppose that for $r$ large enough

$$\omega(r) \leq C(r^{2-\tau}), \tau > 0. \quad (4.18)$$

Then

$$|\Psi_\mu(z) - L(z)| \leq C'(|z| + 1)^{2-\tau} \ln|z|.$$
(2) Suppose that for $r$ large enough,
\[ \omega(r) \leq cr^2 \]  
(4.19)
and
\[ |M(r)| \leq \epsilon, r > R_0. \]  
(4.20)
Then
\[ |\Psi_\mu(z) - L(z)| \leq C\epsilon \ln(1/\epsilon)(|z| + 1)^2. \]  
(4.21)

Note that in both cases the estimate can be written as
\[ |\Psi_\mu(z) - L(z)| \leq C\tilde{\omega}(|z|) \ln(\sqrt{\tilde{\omega}(|z|)}|z|^{-1}), \]
where $\tilde{\omega}(r)$ is the majorant for $\omega(r)$ in the proposition.

Before starting to prove the Proposition, we explain, that, in the first case, the measure $\mu$ is supposed to be rather sparse, so that the $|\mu|$–measure of the disk $D(0, r)$ grows slower than the area of the disk. In the second case, the measure of the disk may grow as fast as its area, with a small factor, however an additional condition (4.20) is imposed. This latter condition is satisfied if the measure $\mu$ is more or less uniformly distributed in all directions or at least, has zero second circular harmonic. On the other hand, if $\mu$ is the Lebesgue measure restricted to some angle in $\mathbb{C}$ not coinciding with the whole plane or a half-plane, then the condition (4.20) is violated.

Proof. We prove the parts (1) and (2) of the proposition simultaneously. Similar to the proof of Proposition 4.3, we divide the plane with the disk $|w| < R$ removed into the same three parts, $\Omega_0 = \{R < |w| < \frac{1}{2}|z|\}$, $\Omega_1 = \{\frac{1}{2}|z| \leq |w| < 2|z|\}$, and $\Omega_2 = \{|w| \geq \frac{1}{2}|z|\}$. Correspondingly, the integral in (4.17) splits into the sum of three integral which we denote by $I_0, I_1, I_2$.

For $I_0$ we have
\[
|I_0| = \left| \int_{\Omega_0} \ln |1 - z/w| d\mu(w) + \int_{\Omega_0} \Re(z/w) d\mu(w) + \frac{1}{2} \int_{\Omega_0} \Re \left( (z/w)^2 \right) d\mu(w) \right|
\leq \int_{\Omega_0} \ln |1 - z/w| |d\mu|(w) + \left( \int_{\Omega_0} |z/w| |d\mu|(w) + \frac{1}{2} \right) \left| \Re \left( z^2 \int_{\Omega_0} w^{-2} d\mu(w) \right) \right|. \tag{4.22}
\]

For $w \in \Omega_0$, we have $|z - w| \geq |z|/2$ and $|1 - z/w| \leq 2|z/w|$, and therefore
\[
\int_{\Omega_0} \ln |1 - \frac{z}{w}| |d\mu|(w) \leq \int_{\Omega_0} \ln \frac{2z}{w} |d\mu|(w) = \int_{R} \ln \frac{2z}{w} |d\omega(r)|
= 2\pi \ln 4 \omega(z) - \ln \left( \frac{|z|}{R} \omega(r) \right) + 2\pi \int_{R} \frac{r}{2|z|} \omega(r) dr. \tag{4.23}
\]

The conditions (4.18), resp., (4.19), imply then that the expression in (4.23) is majorated by $C\tilde{\omega}(|z|)$.

The second term on the right-hand side in (4.22) is estimated simply by
\[
\int_{|w| < \frac{1}{2}|z|} \Re \left( \frac{z}{w} \right) |d\mu|(w) \leq C|z| \int_{R} r^{-1} d\omega(r) \leq C\tilde{\omega}(|z|). \]
It is in the last term that the treatment of two cases is different. In the case (1),
\[
\left| \Re \left( z^2 \int_{\Omega_0} w^{-2} d\mu(w) \right) \right| \leq C|z|^2 \int_{R} |r|^{-2} d\omega(r) \leq C\tau^{2-\tau}.
\]
In the case (2) the estimate by absolute value is not sufficient, so we act differently:
\[
\left| \Re \left( z^2 \int_{\Omega_0} w^{-2} d\mu(w) \right) \right| \leq |z|^2 \int_{\Omega_0} w^{-2} d\mu(w) \leq |z|^2 |M(|z|/2)| \leq \epsilon|z|^2.
\]
Taken together, the last three inequalities give \(|I_0| \leq C\omega(|z|)\).
To estimate \(I_1\), we split the integration set \(\Omega_1\) into the disk \(D = \{|w-z| < r^0/4\}\) and \(\Omega_1' = \Omega_1 \setminus D\), correspondingly, \(I_1 = I_{10} + I_{11}\). If there are no points of \(\Lambda\) in the \(r^0/2\)-neighborhood of \(z\), so that the measure \(\mu\) is continuous in \(D\), then \(I_{10}\) is estimated by a constant by (4.16). If there is a (unique) point \(\lambda(z) \in \Lambda\) in the above disk, then this contribution equals \(\beta(z) \ln |z-\lambda(z)|\) plus some bounded term coming from the continuous part of the measure. In the notation of the Proposition, in any case, \(|I_{10} - L(|z|)| \leq C\).

In the set \(\Omega_1'\), outside the disk \(D\), we again estimate the absolute value of the integral by
\[
|I_{11}'| \leq \int_{\Omega_1'} |\ln|1 - z/w||d\mu(w)| + \int_{\Omega_1'} |z/w|d\mu(w) + \frac{1}{2} \int_{\Omega_1'} z^2 w^{-2} d\mu(w). \tag{4.24}
\]
The second and the third terms in (4.24) are estimated by \(|z|^{2-\tau}\) in the same way as the similar terms in \(I_0\).

More trouble we have with the first term in (4.24), and the estimates we get for this term are the worst ones. We have \(|\ln|1 - z/w||| \leq C + |\ln|z/w||\) and the first term here contributes with \(O(\omega(|z|))\) to \(I_{11}'\). We also have
\[
\int_{z \in \Omega_1', |z-w| \geq |z|/2} \left| \ln \frac{z-w}{z} \right| \leq C\omega(|z|).
\]
So it remains to evaluate
\[
I_{11} = \int_{r^0/4 < |z-w| \leq |z|/2} |\ln|z-w||/z|d\mu(w).
\]
We split the region \(r^0/4 < |z-w| \leq |z|/2\) into annuli \(U_k = \{kr^0/4 < |z-w| \leq (k+1)r^0/4\}\), \(k=1,2,\ldots\) (the last one may be somewhat larger). Denote by \(S_k\) the \(|\mu|\)-measure of the annulus \(U_k\). We have, of course,
\[
\sum S_k \leq |\mu|(\Omega_1') \leq \omega(2|z|).
\]
At the same time, since each annulus \(U_k\) can be covered by no more than \(8k\pi\) disks with radius \(r^0/4\), the condition (4.16) implies that \(S_k \leq \gamma k\), \(\gamma = 8\pi B_0\). Now, we majorize the integral \(I_{11}\) by the sum of integrals over annuli \(U_k\) and estimate it from above in the terms of \(S_k\):
\[
I_{11} \leq \sum \ln \left( \frac{4|z|}{kr_0} \right) S_k. \tag{4.25}
\]
The sequence \(\ln \left( \frac{4|z|}{kr_0} \right)\) decreases in \(k\). Therefore, if we replace \(S_1\) by its largest possible value \(\gamma\) with simultaneous decreasing of the rest of \(S_k\) to some new non-negative values, keeping the sum the same, the sum in (4.25) can only increase. We perform the same operation with \(S_2\), making it equal \(2\gamma\), then with \(S_3\) and so on, until all 'small' \(S_k\), \(k < N\) have their maximal possible values, \(S_k = k\gamma\), the next
one, $S_N$ is smaller than $N^\gamma$, and the rest are zeros. Since the sum of $S_k$ is not greater than $\omega(2|z|)$, we have $\gamma N (N + 1)/2 \leq \omega(2|z|)$, so $N \leq (2\omega(2|z|)/\gamma)^{1/2}$. In this way we have reduced the task of estimating $I_{11}$ to evaluating the sum

$$\sum_{k=1}^{(2\omega(2|z|)/\gamma)^{1/2}} k \ln \left( \frac{|z|}{kr_0^3} \right).$$

This sum can be estimated by the integral

$$\gamma \int_{r_0/4}^{(2\omega(2|z|)/\gamma)^{1/2}} t \ln \left( \frac{|z|}{tr_0^3} \right) dt.$$

After the substitution $t = s|z|/r_0$, the integral transforms to

$$C \gamma |z|^2 \int_0^{(2\omega(2|z|)/\gamma)^{1/2}/|z|} s \ln(1/s) ds,$$

which is estimated directly since $\int_0^s s \ln(1/s) ds = O(\epsilon^2 \ln(1/\epsilon))$, $\epsilon \to 0$. Thus we obtain

$$I_{11} \leq C \omega(|z|) \ln \left( \frac{\sqrt{\omega(|z|)}{|z|}}{|z|} \right).$$

Collecting this inequality with previously found estimates for other contributions to $I_1$, we obtain

$$|I_1(z) - L(z)| \leq C \omega(|z|) \ln \left( \frac{\sqrt{\omega(|z|)}{|z|}}{|z|} \right).$$

The term $I_2$ is the easiest one. Since $|z/w| < 1/2$ in $\Omega_2$, we have

$$|\ln(1 - z/w) + \Re(z/w + (z/w)^2/2)| \leq \frac{2}{3} |z/w|^3.$$

Therefore

$$|I_2| \leq \frac{2}{3} |z|^3 \int_{\Omega_2} |w|^{-3} d|\mu|(w) \leq C |z|^3 \int_{2|z|}^{\infty} r^{-4} \omega(r) dr,$$

which gives the required estimate in both cases.

The Proposition we have proved leads to the following Theorem.

**Theorem 4.6.** Let $\mu_0$ be a measure subject to conditions of Theorem 4.2. Suppose that the perturbation $\mu$ satisfies conditions of Proposition 4.5. We suppose also that the number $\epsilon$ in the case (2) of the Theorem is small enough, so that $C_\nu \epsilon \ln(1/\epsilon) < \gamma$, where $C_\nu$ is the constant in (4.21). Suppose finally that the union of the discrete sets $\Lambda_0$ and $\Lambda$ satisfy the condition (2.5), probably, with different $r^0$. Then the measure $\mu^* = \mu_0 + \mu$ possesses a potential $\Psi^*$ satisfying

$$\Psi^*(z) - \alpha^*(z) \ln(d^*(z)) \geq \gamma^* |z|^2, \quad \gamma^* > 0 \tag{4.26}$$

where $d^*(z)$ is the distance from $z$ to the nearest point of the support of the discrete part of measure $\mu^*$ and $\alpha^*(z)$ is the measure $\mu$ of this point (or any of such points, if there are several of them). In this situation the Pauli operators $P_{\max}$ resp. $P_{-EV}$ have infinitely many zero modes.

Although Theorem 3.11 provides us with very general conditions for infiniteness of zero modes, it applies only in such cases when there are no arbitrarily large regions in the plane where the field is negative - this would destroy the subharmonicity of the averaged potential. The perturbation results enable us to establish the infiniteness of zero modes in certain situations when such regions are present: they arise as a result of perturbations allowed by the theorems in this section.

The starting point has to be a magnetic field where an exponential estimate of the form (4.2) is already known. After this, we can apply allowed types of perturbation again and again, as long as at each step the perturbation satisfies the
general conditions of this Section, in other words, is weak enough, in the proper sense.

The examples below do not exhaust all kinds of perturbations given by our theorems but rather illustrate their possibilities.

Example 4.7. Let $\mu_0$ be a measure for which the conditions of Theorem 4.6 are satisfied. Let $\Omega$ be a set in $\mathbb{C}$ such that the $|\mu_0| (\Omega \cap D(0, R)) = O(R^{2-\tau})$ for some $\tau > 0$ and $\mu_0$ be the measure $\mu_0$ restricted to $\Omega$. Then for the magnetic field $\mu_0 - B\mu_1$, for any positive $S$ the operator Pauli has infinitely many zero modes. In fact, Theorem 4.6, part (1) applies here. If $B > 1$, the measure $\mu_0 - B\mu_1$ has in $\Omega$ the sign opposite to the one of $\mu_0$. The domain $\Omega$ can be rather large. In particular, for the examples of $\mu_0$ discussed above, the set $\Omega$ may be the domain $\{|x_2| \leq C(1 + |x_1|)^\tau\}$, with some $\tau < 1$, thus admitting rather large negative-field regions. Another case is $\Omega$ being the union of disks $\Omega_k$ of radii $R_k$ tending to infinity as $k \to \infty$, such that the combined area of the disks $\Omega_k$ fitting into the disk $D(0, R)$ is majorated by $R^{2-\tau}$.

If the field to be perturbed is more regular, even stronger perturbations are allowed.

Example 4.8. Let $\mu_0$ be a regular lattice of Aharonov-Bohm potentials, possibly on the background of the constant magnetic field, or a periodic (quasi-periodic) measure satisfying Condition 3.6 with nonzero $\sum_j \tau_j$, so that the potential with quadratic growth exists. Let $\Omega = \Omega_- \cup \Omega_+$ be the double angle in the plane,

$$\Omega = \{\arg z \in (\theta_1, \theta_2)\} \cup \{\arg z \in (\theta_1 + \pi, \theta_2 + \pi)\},$$

and $\mu_{\Omega_j}$ be the restriction of $\mu_0$ to the angles $\Omega_j$. Then, in the notations of Example 4.7, Theorem 4.6, part (2), establishes the required estimate for the potential of the measure $\mu - B\mu_{\Omega_+} + B\mu_{\Omega_-}$ and therefore the infiniteness of zero modes for the Pauli operator, as soon as the size of the angle, $\theta_2 - \theta_1$, is small enough. This smallness guarantees the fulfillment of the condition (4.19), while the symmetry of the domain $\Omega$ leads to fulfillment of (4.20). Instead of symmetry, we can require that the domain $\Omega$ consists of four angles and is invariant with respect to rotation by $\pi/2$. Then, if the size of the angles is small enough, the perturbation $-B\mu_{\Omega}$ again satisfies conditions of the perturbation theorems. Of course, one can change the field not necessarily in two (four) angles but in any domain symmetric with respect to the rotation by $\pi$, as long as the area of the portion of the domain in the disk $D(0, R)$ grows not faster than $cR^2$, with $c$ small enough.

The most efficient pattern for applying the re-arrangement Theorems 4.2 and 4.4 is the following. Suppose that we have a decomposition of the plane $\mathbb{C}$ into the union of disjoint sets $\Omega_j$ so that the diameter of $\Omega_j$ is not greater than $C|z_j|^\tau$, $\tau < 1$, where $z_j$ is some point in $\Omega_j$. Let $\mu$ a perturbing measure satisfying conditions of Theorems 4.2 or 4.4, such that $\mu(\Omega_j) = 0$. Then we can define the mapping $\Phi$ as $\Phi(\Omega_j) = \{z_j\}$, so that the whole set $\Omega_j$ is mapped into one point. This mapping induces the zero measure $\mu^*$. The perturbation theorems immediately produce the estimates for the potential of the measure $\mu$ together with infiniteness of zero modes. This reasoning illustrates that positive and negative parts of the perturbation can cancel each other even if they lie rather far apart.

The sets $\Omega_j$ can be chosen rather arbitrarily. The following construction may be useful.

Lemma 4.9. Let $\tau \in (0, 1)$. Then there exists a family of squares $Q_j$, with centers at $z_j$ and sides $d_j$ covering the plane and having no common interior points, such that

$$c|z_j|^\tau \leq d_j \leq C|z_j|^\tau,$$

(4.27)
with some constants $c, C$, for all squares, except the one containing the origin.

**Proof.** We start with the unit square $Q_0$ centered at the origin. Surround $Q_0$ by eight equal squares which will be called the first layer, so the first nine squares form a square with side 3, and condition (4.27) is satisfied, with, say, constants $c = 1/4, C = 4$. Further, inductively, having already a square in the plane filled with squares $Q_j$, so that the inequalities (4.27), we surround this square by squares of the same size as in the last layer, if for the new layer (4.27) still holds. In the opposite case, if there are $N$ squares in the last layer, the new layer will be composed by the squares with the side twice as large as in the last layer (if $N$ is even) or with the side $2N/(N + 1)$ times larger (if $N$ is odd). One can easily check that this construction preserves the inequality (4.27), and repeating it we get the required covering.

To show how the above construction works, we consider the case of a continuous measure.

**Example 4.10.** Let $\mu$ be a continuous measure satisfying the conditions of Theorem 4.2 with the following property: for any square $Q$ with center at $z$, as in Lemma 4.9, $\mu(Q) - B|Q| \leq C|z|^{2\tau - \epsilon}$, for some $\epsilon > 0$ with a positive constant $B$, $|Q|$ denoting the area of the square. We set $\mu_0$ being $B$ times the Lebesgue measure; it possesses the required quadratically growing potential and will serve as the unperturbed field. The difference, $\mu - \mu_0$, will be represented as a sum of two terms, $\mu_1 + \mu_2$. We set $\mu_1 = \mu - \mu(Q_j)/|Q_j|\,dx$ on the square $Q_j$, and $\mu_2 = (\mu(Q_j)/|Q_j| - B)\,dx$. Then the perturbation $\mu_1$ satisfies conditions of Theorem 4.2, with $\Phi$ mapping the whole square $Q_j$ to the single point $z_j \in Q_j$, and $\mu_2$ satisfies conditions of the Theorem 4.6.

Note that in the situation of the example, the regions where the field points in the ‘wrong direction’, can be very large. In fact, one can choose $\mu$ so that for each square $Q_j$ in our covering, $\mu > 0$ in the narrow strip near the boundary, with width $a|z_j|^{\tau}$, with an arbitrarily small fixed $a$, and $\mu$ is negative in the main part of the square.

A similar property holds for discrete measures

**Example 4.11.** Let $\Lambda$ be a regular lattice and $\alpha_\lambda, \lambda \in \Lambda$ is the collection of intensities, $\alpha_\lambda \in [-1/2, 1/2)$. We suppose that for any square $Q$ with center at $z$ (or only for squares $Q_j$ constructed in Lemma 4.9),

$$\sum_{\lambda \in \Lambda \cap Q} \alpha_\lambda - B|Q| \leq C|z|^{2\tau - \epsilon}, B > 0$$

for some $\epsilon > 0$. Then, similarly to the previous example, the system of $AB$ solenoids with intensities $\alpha_\lambda$ placed at the points of $\Lambda$ can be obtained by a re-arrangement perturbation and an additive perturbation of the regular $AB$ lattice with equal intensities. This proves infiniteness of zero modes for the operator $P_{-EV}$ for such configuration of the field. Moreover, we can even suppose that initially $\Lambda$ is not a regular lattice but just a discrete set, in proper sense, almost uniformly distributed in the plane. This situation is taken care of by additional perturbation consisting in moving the points of $\Lambda$ to the points of a regular lattice and then making one more additive perturbation to dispose of the points which cannot be moved. We do not go into details here.

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