Continuous ensembles and the $\chi$-capacity of infinite-dimensional channels

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1 Introduction

This paper is devoted to systematic study of the classical capacity (more precisely, a closely related quantity – the $\chi$-capacity) of infinite dimensional quantum channels, following [7], [9], [17]. While major attention in quantum information theory up to now was paid to finite dimensional systems, there is an important and interesting class of Gaussian channels, see e.g. [8], [5], [16] which act in infinite dimensional Hilbert space. Although many questions for Gaussian Bosonic systems with finite number of modes can be solved with finite dimensional matrix techniques, a general underlying Hilbert space operator analysis is indispensable.

Moreover, it was observed recently [17] that Shor’s proof of global equivalence of different forms of the famous additivity conjecture is related to weird discontinuity of the $\chi$-capacity in the infinite dimensional case. All this calls for a mathematically rigorous treatment involving specific results from the operator theory in a Hilbert space and measure theory.

There are two important features essential for channels in infinite dimensions. One is the necessity of the input constraints (such as mean energy constraint for Gaussian channels) to prevent from infinite capacities (although considering input constraints was recently shown quite useful also in the study of the additivity conjecture for channels in finite dimensions [9]). Another is the natural appearance of infinite, and, in general, “continuous” state ensembles understood as probability measures on the set of all quantum states. By using compactness criteria from probability theory and operator theory we can show that the set of all generalized ensembles with the average in a compact set of states is itself a compact subset of the set of all probability
measures. With this in hand we give a sufficient condition for existence of an optimal generalized ensemble for a constrained quantum channel. This condition can be verified in particular in the case of Bosonic Gaussian channels with constrained mean energy. In the case of convex constraints we give a characterization of the optimal generalized ensemble extending the “maximal distance property” [13], [9].

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators in $\mathcal{H}$, $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators with the trace norm $\| \cdot \|_1$ and $\mathfrak{S}(\mathcal{H})$ the closed convex subset of $\mathfrak{T}(\mathcal{H})$ consisting of all density operators (states) in $\mathcal{H}$, which is complete separable metric space with the metric defined by the norm. We shall use the fact that convergence of a sequence of states to a state in the weak operator topology is equivalent to convergence of this sequence to this state in the trace norm [1]. A closed subset $\mathcal{K}$ of states is compact if and only if for any $\varepsilon > 0$ there is a finite dimensional projector $P$ such that $\text{Tr} \rho P \geq 1 - \varepsilon$ for all $\rho \in \mathcal{K}$ [14].

A finite collection $\{\pi_i, \rho_i\}$ of states $\rho_i$ with the corresponding probabilities $\pi_i$ is conventionally called ensemble. The state $\bar{\rho} = \sum \pi_i \rho_i$ is called the average of the ensemble.

We refer to [2], [13] for definitions and facts concerning probability measures on separable metric spaces. In particular we denote $\text{supp}(\pi)$ support of measure $\pi$ as defined in [13].

Definition. We call generalized ensemble an arbitrary Borel probability measure $\pi$ on $\mathfrak{S}(\mathcal{H})$. The average\footnote{Also called barycenter of the measure $\pi$.} of the generalized ensemble $\pi$ is defined by the Pettis integral

$$\bar{\rho}(\pi) = \int_{\mathfrak{S}(\mathcal{H})} \rho \pi(\rho) \, d\rho.$$ 

Using the result of [1] it is possible to show that the above integral exists also in Bochner sense [6].

The conventional ensembles correspond to measures with finite support.

Denote by $\mathcal{P}$ the convex set of all probability measures on $\mathfrak{S}(\mathcal{H})$ equipped with the topology of weak convergence [2]. It is easy to see (due to the result of [1]) that the mapping $\pi \mapsto \bar{\rho}(\pi)$ is continuous in this topology.
Lemma 1. The subset of measures with finite support is dense in the set of all measures with given average \( \bar{\rho} \).

A proof of this statement is given in the appendix A.

In what follows \( \log \) denotes the function on \([0, +\infty)\), which coincides with the usual logarithm on \((0, +\infty)\) and vanishes at zero. If \( A \) is a positive finite rank operator in \( \mathcal{H} \), then the entropy is defined as

\[
H(A) = \text{Tr} A (I \log \text{Tr} A - \log A),
\]

where \( I \) is the unit operator in \( \mathcal{H} \). If \( A, B \) two such operators then the relative entropy is defined as

\[
H(A \| B) = \text{Tr} (A \log A - A \log B + B - A)
\]

provided \( \text{ran} A \subseteq \text{ran} B \), and \( H(A \| B) = +\infty \) otherwise (throughout this paper ran denotes the closure of the range of an operator in \( \mathcal{H} \)).

These definitions can be extended to arbitrary positive \( A, B \in \mathcal{S}(\mathcal{H}) \) with the help of the following lemma [11]:

Lemma 2. Let \( \{P_n\} \) be an arbitrary sequence of finite dimensional projectors monotonously increasing to the unit operator \( I \). The sequences \( \{H(P_n A P_n)\} \), \( \{H(P_n A P_n \| P_n B P_n)\} \) are monotonously increasing and have the limits in the range \([0, +\infty]\) independent of the choice of the sequence \( \{P_n\} \).

We can thus define the entropy and the relative entropy as

\[
H(A) = \lim_{n \to +\infty} H(P_n A P_n); \quad H(A \| B) = \lim_{n \to +\infty} H(P_n A P_n \| P_n B P_n).
\]

As it is well known, the properties of the entropy for infinite and finite dimensional Hilbert spaces differ quite substantially: in the latter case the entropy is bounded continuous function on \( \mathcal{S}(\mathcal{H}) \), while in the former it is discontinuous (lower semicontinuous) at every point, and infinite “most everywhere” in the sense that the set of states with finite entropy is a first category subset of \( \mathcal{S}(\mathcal{H}) \) [18].
3 The \( \chi \)-capacity of constrained channels

Lemma 2 implies, in particular, that the nonnegative function \( \rho \mapsto H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi))) \) is measurable on \( \mathfrak{S}(\mathcal{H}) \). Hence the functional

\[
\chi_{\Phi}(\pi) = \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi)))\pi(d\rho)
\]

is well defined on the set \( \mathcal{P} \) (with the range \([0; +\infty]\)).

**Proposition 1.** The functional \( \chi_{\Phi}(\pi) \) is lower semicontinuous on \( \mathcal{P} \). If \( H(\Phi(\bar{\rho}(\pi))) < \infty \), then

\[
\chi_{\Phi}(\pi) = H(\Phi(\bar{\rho}(\pi))) - \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho))\pi(d\rho).
\]

**Proof.** Let \( \{P_n\} \) be an arbitrary sequence of finite dimensional projectors monotonously increasing to the unit operator \( I \). We show first that the functionals

\[
\chi_{\Phi}^n(\pi) = \int_{\mathfrak{S}(\mathcal{H})} H(P_n\Phi(\rho)P_n\|P_n\Phi(\bar{\rho}(\pi))P_n)\pi(d\rho)
\]

are continuous.

We have

\[
\text{ran}(P_n\Phi(\rho)P_n) \subseteq \text{ran}(P_n\Phi(\bar{\rho}(\pi))P_n)
\]

for \( \pi \)-almost all \( \rho \). Indeed, closure of the range is orthogonal complement to the null subspace of a Hermitian operator, and for null subspaces the opposite inclusion holds obviously. It follows that

\[
H(P_n\Phi(\rho)P_n\|P_n\Phi(\bar{\rho}(\pi))P_n) = \text{Tr}((P_n\Phi(\rho)P_n)\log(P_n\Phi(\rho)P_n)
\]

\[- (P_n\Phi(\rho)P_n)\log(P_n\Phi(\bar{\rho}(\pi))P_n) + P_n\Phi(\bar{\rho}(\pi))P_n - P_n\Phi(\rho)P_n)
\]

for \( \pi \)-almost all \( \rho \). By using (11) we have

\[
\chi_{\Phi}^n(\pi) = - \int_{\mathfrak{S}(\mathcal{H})} H(P_n\Phi(\rho)P_n)\pi(d\rho) + \int_{\mathfrak{S}(\mathcal{H})} \text{Tr}(P_n\Phi(\rho))\log\text{Tr}(P_n\Phi(\rho))\pi(d\rho)
\]

\[- \int_{\mathfrak{S}(\mathcal{H})} \text{Tr}(P_n\Phi(\rho)P_n)\log(P_n\Phi(\bar{\rho}(\pi))P_n)\pi(d\rho)
\]

\[+ \int_{\mathfrak{S}(\mathcal{H})} \text{Tr}(P_n\Phi(\bar{\rho}(\pi)))\pi(d\rho) - \int_{\mathfrak{S}(\mathcal{H})} \text{Tr}(P_n\Phi(\rho))\pi(d\rho).
\]
It is easy to see that the two last terms cancel while the central term can be transformed in the following way

\[- \int \frac{\text{Tr}(P_n \Phi(\rho) P_n) \log(P_n \Phi(\bar{\rho}(\pi)))}{\mathcal{E}(\mathcal{H})} \pi(d\rho) \]

\[= - \text{Tr} \int \frac{(P_n \Phi(\rho) P_n) \log(P_n \Phi(\bar{\rho}(\pi)))}{\mathcal{E}(\mathcal{H})} \pi(d\rho) \]

\[= H(P_n \Phi(\bar{\rho}(\pi))) - \text{Tr}(P_n \Phi(\bar{\rho}(\pi))) \log \text{Tr}(P_n \Phi(\bar{\rho}(\pi))). \]

Hence

\[\chi^n_\Phi(\pi) = H(P_n \Phi(\bar{\rho}(\pi))) - \text{Tr}(P_n \Phi(\bar{\rho}(\pi))) \log \text{Tr}(P_n \Phi(\bar{\rho}(\pi))) \]

\[- \int \frac{H(P_n \Phi(\rho) P_n) \pi(d\rho)}{\mathcal{E}(\mathcal{H})} + \int \frac{\text{Tr}(P_n \Phi(\rho)) \pi(d\rho)}{\mathcal{E}(\mathcal{H})}. \quad (4)\]

Continuity and boundedness of the quantum entropy in the finite dimensional case and similar properties of the function \( \rho \mapsto \text{Tr}(P_n \Phi(\rho) \log \text{Tr}(P_n \Phi(\rho))) \) imply continuity of the functionals \( \chi^n_\Phi(\pi) \).

By the monotonous convergence theorem (in what follows, m.c.-theorem) \([10],[6]\) the sequence of functionals \( \chi^n_\Phi(\pi) \) is monotonously increasing and pointwise converges to \( \chi_\Phi(\pi) \). Hence the functional \( \chi_\Phi(\pi) \) is lower semicontinuous.

To prove \( (3) \) note that lemma 2 implies

\[\lim_{n \to +\infty} H(P_n \Phi(\bar{\rho}(\pi))) P_n = H(\Phi(\bar{\rho}(\pi)))\]

and

\[\lim_{n \to +\infty} \int \frac{H(P_n \Phi(\rho) P_n) \pi(d\rho)}{\mathcal{E}(\mathcal{H})} = \int \frac{H(\Phi(\rho)) \pi(d\rho)}{\mathcal{E}(\mathcal{H})}\]

due to m.c.-theorem. For every \( \rho \) the sequence \( \{\text{Tr}(P_n \Phi(\rho))\} \) is in \([0,1]\) and converges to 1, therefore \( \lim_{n \to +\infty} \text{Tr}(P_n(\rho)) \log \text{Tr}(P_n(\rho)) = 0 \), in particular the second term in \( (4) \) tends to 0. Since \( |x \log x| < 1 \) for all \( x \in (0,1] \), the last term also tends to 0 by dominated convergence theorem, so passing to the limit \( n \to \infty \) in \( (4) \) gives \( (3) \). \( \square \)

Let \( \mathcal{H}, \mathcal{H}' \) be a pair of separable Hilbert spaces which we shall call correspondingly input and output space. A channel \( \Phi \) is a linear positive trace preserving map from \( \mathcal{F}(\mathcal{H}) \) to \( \mathcal{F}(\mathcal{H}') \) such that the dual map \( \Phi^* : \mathcal{B}(\mathcal{H}') \to \mathcal{B}(\mathcal{H}) \)
Let $\mathcal{A}$ be an arbitrary subset of $\mathcal{G}(\mathcal{H})$. We consider constraint on input ensemble $\{\pi_i, \rho_i\}$, defined by the requirement $\tilde{\rho} \in \mathcal{A}$. The channel $\Phi$ with this constraint is called the $\mathcal{A}$-constrained channel. We define the $\chi$-capacity of the $\mathcal{A}$-constrained channel $\Phi$ as

$$\bar{C}(\Phi; \mathcal{A}) = \sup_{\tilde{\rho} \in \mathcal{A}} \chi_\Phi(\{\pi_i, \rho_i\}),$$

where

$$\chi_\Phi(\{\pi_i, \rho_i\}) = \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\tilde{\rho})).$$

Throughout this paper we shall consider the constraint sets $\mathcal{A}$ such that

$$\bar{C}(\Phi; \mathcal{A}) < +\infty.$$  

The subset of $\mathcal{P}$, consisting of all measures $\pi$ with the average state $\tilde{\rho}(\pi)$ in a subset $\mathcal{A} \subseteq \mathcal{G}(\mathcal{H})$, will be denoted $\mathcal{P}_\mathcal{A}$. Lemma 1 and proposition 1 imply

**Corollary 1.** The $\chi$-capacity of $\mathcal{A}$-constrained channel $\Phi$ can be defined by

$$\bar{C}(\Phi; \mathcal{A}) = \sup_{\pi \in \mathcal{P}_\mathcal{A}} \chi_\Phi(\pi).$$

**Proof.** The definition (5) is a similar expression in which the supremum is over all measures in $\mathcal{P}_\mathcal{A}$ with finite support. By lemma 1 we can approximate arbitrary measure $\pi$ in $\mathcal{P}_\mathcal{A}$ by a sequence $\{\pi_n\}$ of measures in $\mathcal{P}_\mathcal{A}$ with finite support. By proposition 1, $\liminf_{n \to +\infty} \chi_\Phi(\pi_n) \geq \chi_\Phi(\pi)$. It follows that the supremum over all measures in $\mathcal{P}_\mathcal{A}$ coincides with the supremum over all measures in $\mathcal{P}_\mathcal{A}$ with finite support. □

### 4 Compact constraints

It is convenient to introduce the following notion. An unbounded positive operator $H$ in $\mathcal{H}$ with discrete spectrum of finite multiplicity will be called a $\mathcal{H}$-operator. Let $Q_n$ be the spectral projector of $H$ corresponding to the lowest $n$ eigenvalues. Following [7] we shall denote

$$\text{Tr}_\rho H = \lim_{n \to \infty} \text{Tr}_\rho Q_n H,$$
where the sequence on the right side is monotonously nondecreasing. It was shown in [7] that

\[ \mathcal{K} = \{ \rho : \text{Tr}\rho H \leq h \}, \]  

where \( H \) is an \( \mathcal{H}_0 \)-operator, is a compact subset of \( \mathcal{S}(\mathcal{H}) \).

**Lemma 3.** Let \( \mathcal{A} \) be a compact subset of \( \mathcal{S}(\mathcal{H}) \). Then there exist an \( \mathcal{H}_0 \)-operator \( H \) and a positive number \( h \) such that \( \text{Tr}\rho H \leq h \) for all \( \rho \in \mathcal{A} \).

**Proof.** By the compactness criterion from [14] for any natural \( n \) there exists a finite rank projector \( P_n \) such that \( \text{Tr}\rho P_n \geq 1 - n^{-3} \) for all \( \rho \) in \( \mathcal{A} \). Without loss of generality we may assume that \( \bigvee_{k=1}^{+\infty} P_k(\mathcal{H}) = \mathcal{H} \), where \( \bigvee \) denotes closed linear span of the subspaces. Let \( \hat{P}_n \) be the projector on the finite dimensional subspace \( \bigvee_{k=1}^{n} P_k(\mathcal{H}) \). Thus

\[ H = \sum_{n=1}^{+\infty} n(\hat{P}_{n+1} - \hat{P}_n) \]

is an \( \mathcal{H}_0 \)-operator satisfying

\[ \text{Tr}\rho H = \sum_{n=1}^{+\infty} n\text{Tr}\rho(\hat{P}_{n+1} - \hat{P}_n) \leq \sum_{n=1}^{+\infty} n\text{Tr}\rho(I_{\mathcal{H}} - \hat{P}_n) \leq \sum_{n=1}^{+\infty} n^{-2} = h \]

for arbitrary state \( \rho \) in the set \( \mathcal{A} \).  \( \square \)

**Proposition 2.** The set \( \mathcal{P}_A \) is a compact subset of \( \mathcal{P} \) if and only if the set \( \mathcal{A} \) is a compact subset of \( \mathcal{S}(\mathcal{H}) \).

**Proof.** Let the set \( \mathcal{P}_A \) be compact. The set \( \mathcal{A} \) is the image of the set \( \mathcal{P}_A \) under the continuous mapping \( \pi \mapsto \bar{\rho}(\pi) \), hence it is compact.

Let the set \( \mathcal{A} \) be compact. By lemma 3 there exists an \( \mathcal{H}_0 \)-operator \( H \) such that \( \text{Tr}\rho H \leq h \) for all \( \rho \) in \( \mathcal{A} \). For arbitrary \( \pi \in \mathcal{P}_A \) we have

\[ \int_{\mathcal{S}(\mathcal{H})} (\text{Tr}\rho H)\pi(d\rho) = \text{Tr} \left( \int_{\mathcal{S}(\mathcal{H})} \rho\pi(d\rho) \right) H = \text{Tr}\bar{\rho}(\pi)H \leq h \]  

(10)

The existence of the integral on the left side and the first equality follows from m.c.-theorem, since by (8) the function \( \text{Tr}\rho H \) is the limit of nondecreasing sequence of continuous bounded functions \( \text{Tr}\rho Q_n H \).

Let \( \mathcal{K}_\varepsilon = \{ \rho : \text{Tr}\rho H \leq h\varepsilon^{-1} \} \). The set \( \mathcal{K}_\varepsilon \) is a compact subset of \( \mathcal{S}(\mathcal{H}) \) for any \( \varepsilon \). By (10) for any measure \( \pi \) in \( \mathcal{P}_A \) we have

\[ \pi(\mathcal{S}(\mathcal{H}) \setminus \mathcal{K}_\varepsilon) = \int_{\mathcal{S}(\mathcal{H}) \setminus \mathcal{K}_\varepsilon} \pi(d\rho) \leq \varepsilon h^{-1} \int_{\mathcal{S}(\mathcal{H}) \setminus \mathcal{K}_\varepsilon} (\text{Tr}\rho H)\pi(d\rho) \leq \varepsilon \]  

(11)

By Prokhorov’s theorem [12] the (obviously closed) set \( \mathcal{P}_A \) is compact.  \( \square \)
We will use the following notions, introduced in [17]. The sequence of ensembles \( \{ \pi^k_i, \rho^k_i \} \) with the averages \( \bar{\rho}^k \in \mathcal{A} \) is called an approximating sequence if
\[
\lim_{k \to +\infty} \chi_{\Phi}(\{ \pi^k_i, \rho^k_i \}) = \bar{C}(\Phi; \mathcal{A}).
\]
The state \( \bar{\rho} \in \mathcal{A} \) is called an optimal average state if it is a partial limit of a sequence of average states for some approximating sequence of ensembles. Compactness of the set \( \mathcal{A} \) implies that the set of optimal average states is not empty.

**Theorem.** If the restriction of the output entropy \( H(\Phi(\rho)) \) to the set \( \mathcal{A} \) is continuous at least at one optimal average state \( \bar{\rho}_0 \in \mathcal{A} \) then there exist an optimal generalized ensemble \( \pi^* \) in \( \mathcal{P}_\mathcal{A} \) such that \( \text{supp} \pi^* \subseteq \text{Extr} \mathcal{G}(\mathcal{H}) \) and
\[
\bar{C}(\Phi; \mathcal{A}) = \chi_{\Phi}(\pi^*) = \int_{\mathcal{G}(\mathcal{H})} H(\Phi(\rho)) \| \Phi(\bar{\rho}(\pi^*))) \pi^*(d\rho).
\]

**Proof.** We will show first that the function
\[
\pi \mapsto \int_{\mathcal{G}(\mathcal{H})} H(\Phi(\rho)) \pi(d\rho)
\]
is well defined and lower semicontinuous on the set \( \mathcal{P}_\mathcal{A} \).

By lemma 2 the function \( H(\Phi(\rho)) \) is a pointwise limit of the monotonously increasing sequences of functions
\[
f_n(\rho) = \text{Tr} \left( (P_n \Phi(\rho) P_n) \left( I \log \text{Tr}(P_n \Phi(\rho) P_n) - \log(P_n \Phi(\rho) P_n) \right) \right),
\]
which are continuous and bounded on \( \mathcal{G}(\mathcal{H}) \). Hence the function \( H(\Phi(\rho)) \) is measurable and the m.c.-theorem implies
\[
\int_{\mathcal{G}(\mathcal{H})} H(\Phi(\rho)) \pi(d\rho) = \lim_{n \to +\infty} \int_{\mathcal{G}(\mathcal{H})} f_n(\rho) \pi(d\rho).
\]
The sequence of continuous functionals
\[
\pi \mapsto \int_{\mathcal{G}(\mathcal{H})} f_n(\rho) \pi(d\rho)
\]
is nondecreasing. Hence its pointwise limit is lower semicontinuous.

By the assumption the restriction of the function \(H(\Phi(\rho))\) to the set \(\mathcal{A}\) is continuous at some optimal average state \(\bar{\rho}_0\). The continuity of the mapping \(\pi \mapsto \bar{\rho}(\pi)\) implies that the restriction of the functional \(\pi \mapsto H(\Phi(\bar{\rho}(\pi)))\) to the set \(\mathcal{P}_A\) is continuous at any point \(\pi_0\) such that \(\bar{\rho}(\pi_0) = \bar{\rho}_0\). Hence \(H(\Phi(\bar{\rho}(\pi))) < +\infty\) for any point \(\pi\) in the intersection of \(\mathcal{P}_A\) with some neighbourhood of \(\pi_0\). For every such point \(\pi\) the relation \(\text{(3)}\) holds. Therefore the restriction of the functional \(\chi\Phi(\pi)\) to the set \(\mathcal{P}_A\) is upper semicontinuous, and by proposition 1 it is continuous at any point \(\pi_0\) in \(\mathcal{P}_A\) such that \(\bar{\rho}(\pi_0) = \bar{\rho}_0\).

Let \(\{\pi_i^n, \rho_i^n\}\) be an approximating sequence of ensembles with the corresponding sequence of average states \(\bar{\rho}^n\) converging to the state \(\bar{\rho}_0\). Decomposing each state of the ensemble \(\{\pi_i^n, \rho_i^n\}\) into a countable convex combination of pure states we obtain the sequence \(\{\hat{\pi}_j^n, \hat{\rho}_j^n\}\) of generalized ensembles consisting of countable number of pure states with the same sequence of the average states \(\bar{\rho}^n\). Let \(\hat{\pi}^n\) be the sequence of measures ascribing value \(\hat{\pi}_j^n\) to the set \(\{\hat{\rho}_j^n\}\) for each \(j\). It follows that

\[
\chi\Phi(\hat{\pi}_n) = \sum_j \hat{\pi}_j^n H(\Phi(\hat{\rho}_j^n)\|\Phi(\bar{\rho}^n)) \geq \sum_i \pi_i^n H(\Phi(\rho_i^n)\|\Phi(\bar{\rho}^n)) = \chi\Phi(\{\pi_i^n, \rho_i^n\}),
\]

where the inequality follows from convexity of the relative entropy. By construction \(\text{supp}\hat{\pi}^n \subseteq \text{Extr}\mathcal{G}(\mathcal{H})\) for each \(n\). By proposition 2 there exists a subsequence \(\hat{\pi}_{nk}\), converging to some measure \(\pi^*\) in \(\mathcal{P}_A\). Since the set \(\text{Extr}\mathcal{G}(\mathcal{H})\) of all pure states is closed subset of \(\mathcal{G}(\mathcal{H})^2\) we have \(\text{supp}\pi^* \subseteq \text{Extr}\mathcal{G}(\mathcal{H})\) due to theorem 6.1 in [13]. It is clear that \(\bar{\rho}(\pi^*) = \bar{\rho}_0\) and, hence, as shown above, the restriction of the functional \(\chi\Phi(\pi)\) on the set \(\mathcal{P}_A\) is continuous at the point \(\pi^*\). This, the approximating property of the sequence \(\{\pi_i^n, \rho_i^n\}\) and \(\text{(12)}\) implies

\[
\bar{C}(\Phi; \mathcal{A}) = \lim_{k \to \infty} \chi\Phi(\{\pi_i^{nk}, \rho_i^{nk}\}) \leq \lim_{k \to \infty} \chi\Phi(\hat{\pi}_{nk}) = \chi\Phi(\pi^*).
\]

Since the converse inequality follows from corollary 1, we obtain \(\bar{C}(\Phi; \mathcal{A}) = \chi\Phi(\pi^*),\) which means that the measure \(\pi^*\) is an optimal generalized ensemble for the \(\mathcal{A}\)-constrained channel \(\Phi\). \(\square\)

\[\text{The set } \text{Extr}\mathcal{G}(\mathcal{H}) \text{ is described by the inequality } H(\rho) \leq 0, \text{ and due to lower semicontinuity of the quantum entropy it is closed.}\]
Corollary 2. For arbitrary state \( \rho_0 \) with \( H(\Phi(\rho_0)) < +\infty \) there exists a generalized ensemble\(^3\) \( \pi_0 \) such that \( \bar{\rho}(\pi_0) = \rho_0 \) and

\[
\chi_\Phi(\rho_0) \equiv \sup_{\sum_i \pi_i \rho_i = \rho_0} \chi_\Phi(\{\pi_i, \rho_i\}) = \int \mathcal{H} H(\Phi(\rho)\|\Phi(\rho_0)) \pi_0(d\rho).
\]

**Proof.** It is sufficient to note that the condition of the theorem holds trivially for \( \mathcal{A} = \{\rho_0\} \). □

In the finite dimensional case we obviously have

\[
\bar{C}(\Phi; \mathcal{A}) = \chi_\Phi(\bar{\rho}),
\]

where \( \bar{\rho} \) is the average state of any optimal ensemble. The generalization of this relation to the infinite dimensional case is closely connected with the question of existence of the optimal generalized ensemble.

Corollary 3. If an optimal generalized ensemble for the \( \mathcal{A} \)-constrained channel \( \Phi \) exists, then the equality (13) holds for some optimal average state \( \bar{\rho} \) for the \( \mathcal{A} \)-constrained channel \( \Phi \).

If the equality (13) holds for some optimal average state \( \bar{\rho} \) for the \( \mathcal{A} \)-constrained channel \( \Phi \) with \( H(\Phi(\bar{\rho})) < +\infty \) then there exists an optimal generalized ensemble for the \( \mathcal{A} \)-constrained channel \( \Phi \).

**Proof.** The first assertion is obvious while the second one follows from corollary 2. □

**Remark.** The continuity condition in the theorem is essential, as it is shown in Appendix B. It is possible to show that this condition holds automatically if the set \( \mathcal{A} \) is convex with a finite number of extreme points with finite output entropy. We conjecture that this condition holds for arbitrary convex compact set \( \mathcal{A} \) due to the special properties of optimal average states in this case, considered in [17].

**Proposition 3.** Let \( H' \) be a \( \mathcal{H} \)-operator on the space \( \mathcal{H}' \) such that

\[
\text{Tr} \exp(-\beta H') < +\infty \quad \text{for all} \quad \beta > 0
\]

and \( \text{Tr} \Phi(\rho)H' \leq h' \) for all \( \rho \in \mathcal{A} \). Then there exists an optimal generalized ensemble for the \( \mathcal{A} \)-constrained channel \( \Phi \).

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\(^3\)In what follows we can consider the generalized ensembles as measures supported by the set of pure states

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Proof. We will show that under the condition of the lemma the restriction of the output entropy $H(\Phi(\rho))$ on the set $A$ is continuous, which implies validity of the condition of the theorem.

Let $\rho'_\beta = (\text{Tr} \exp(-\beta H'))^{-1} \exp(-\beta H')$ be a state in $\mathcal{G}(\mathcal{H}')$. For arbitrary $\rho$ in $A$ we have

$$H(\Phi(\rho)\|\rho'_\beta) = -H(\Phi(\rho)) + \beta \text{Tr}\Phi(\rho)H' + \log \text{Tr} \exp(-\beta H') \quad (15)$$

Let $\rho_n$ be an arbitrary sequence of states in $A$ converging to the state $\rho$. By using (15) and lower semicontinuity of the relative entropy we obtain

$$\limsup_{n \to \infty} H(\Phi(\rho_n)) = H(\Phi(\rho)) + \liminf_{n \to \infty} H(\Phi(\rho_n))\|\rho'_\beta) + \limsup_{n \to \infty} \beta \text{Tr}\Phi(\rho_n)H' - \beta \text{Tr}\Phi(\rho)H' \leq H(\Phi(\rho)) + \beta h'.$$

By tending $\beta$ in the above inequality to zero we can establish the upper semicontinuity of the restriction of the function $H(\Phi(\rho))$ to the set $A$. The lower semicontinuity of this function follows from the lower semicontinuity of the entropy [18]. Hence the restriction of the function $H(\Phi(\rho))$ on the set $A$ is continuous. □

The condition of proposition 3 is fulfilled for Gaussian channels with the power constraint of the form (9) where $H = R^T \epsilon R$ is the many-mode oscillator Hamiltonian with nondegenerate energy matrix $\epsilon$, and $R$ are the canonical variables of the system. We give a brief sketch of the argument which can be made rigorous by taking care of unboundedness of the canonical variables. Indeed, let

$$R' = KR + K_E R_E$$

be the equation of the channel in the Heisenberg picture, where $R_E$ are the canonical variables of the environment which is in the Gaussian state with zero mean and the correlation matrix $\alpha_E$ [8]. Taking $H' = c[R^T R - IS\alpha_E K_E^T K_E]$, we have $\Phi'(H') = cR^T K^T K R$, and we can always choose a positive $c$ such that $\Phi'(H') \leq H$. Moreover, $H'$ satisfies the condition (14). Thus the conditions of proposition 3 can be fulfilled in this case.

Conjecture. For arbitrary Gaussian channel with the power constraint an optimal generalized ensemble is given by a Gaussian measure supported by the set of pure Gaussian states with arbitrary mean and a fixed correlation matrix.

This conjecture was stated in [8] for attenuation/amplification channel with classical noise. For the case of pure attenuation channel characterized
by the property of zero minimal output entropy the validity of this conjecture
was established in [5].

5 Convex constraints

In the case of convex constraint set there are further special properties, such
as uniqueness of the output of optimal average state, see [17]. The following
lemma is a generalization of Donald’s identity [4].

Lemma 4. For arbitrary measure π in \( \mathcal{P} \) and arbitrary state \( \sigma \) in \( \mathcal{S}(\mathcal{H}) \) the following identity holds

\[
\int_{\mathcal{S}(\mathcal{H})} H(\rho\|\sigma) \pi(d\rho) = \int_{\mathcal{S}(\mathcal{H})} H(\rho\|\bar{\rho}(\pi)) \pi(d\rho) + H(\bar{\rho}(\pi)\|\sigma). \tag{16}
\]

Proof. We first notice that in the finite dimensional case Donald’s iden-
tity

\[
\sum_i \pi_i H(\rho_i\|\sigma) = \sum_i \pi_i H(\rho_i\|\bar{\rho}(\pi)) + H(\bar{\rho}(\pi)\|\sigma)
\]

holds for not necessarily normalized positive operators with the generalized
definition of the relative entropy [2]. This can be obviously extended to
generalized ensembles in finite-dimensional Hilbert space, giving (16) for this
case. Thus this relation holds for the operators \( P_n\rho P_n, P_n\sigma P_n \), where \( P_n \) is
an arbitrary sequence of finite projectors increasing to \( I_{\mathcal{H}} \). Passing to the
limit \( n \to \infty \) and referring to the m.c.-theorem, we obtain (16) in infinite-
dimensional case. □

The following proposition is a generalization of the “maximal distance
property”, cf. proposition 1 in [9].

Proposition 4. Let \( \mathcal{A} \) be convex subset of \( \mathcal{S}(\mathcal{H}) \). A measure \( \pi \in \mathcal{P}_\mathcal{A} \) is
optimal generalized ensemble for the \( \mathcal{A} \)-constrained channel \( \Phi \) if and only if

\[
\int_{\mathcal{S}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi))) \mu(d\rho) \leq \int_{\mathcal{S}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi))) \pi(d\rho) = \chi_\Phi(\pi) \tag{17}
\]

for arbitrary measure \( \mu \in \mathcal{P}_\mathcal{A} \).
Proof. Let inequality (17) holds for arbitrary measure \( \mu \in \mathcal{P}_A \). By lemma 4 we have

\[
\chi_\Phi(\mu) \leq \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\mu)))\mu(d\rho) + H(\Phi(\bar{\rho}(\mu))\|\Phi(\bar{\rho}(\pi)))
\]

\[
= \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi)))\mu(d\rho) \leq \chi_\Phi(\pi),
\]

which implies optimality of the measure \( \pi \).

Conversely, let \( \pi \) be an optimal generalized ensemble for the \( A \)-constrained channel \( \Phi \) and \( \mu \) be an arbitrary measure in \( \mathcal{P}_A \). By convexity of the set \( \mathcal{A} \) the measure \( \pi_\eta = \eta \mu + (1 - \eta)\pi \) is also in \( \mathcal{P}_A \) for arbitrary \( \eta \in (0, 1) \). Using lemma 4 we have

\[
\chi_\Phi(\pi_\eta) = \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi_\eta)))\pi_\eta(d\rho)
\]

\[
= \eta \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi_\eta)))\mu(d\rho) + (1 - \eta)\chi_\Phi(\pi) + (1 - \eta)H(\Phi(\bar{\rho}(\pi))\|\bar{\rho}(\pi_\eta)).
\]

The optimality of \( \pi \) and nonnegativity of the relative entropy imply

\[
\int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi_\eta)))\mu(d\rho) - \chi_\Phi(\pi) \leq \eta^{-1}(\chi_\Phi(\pi_\eta) - \chi_\Phi(\pi)) \leq 0. \quad (18)
\]

By lemma 4 and lower semicontinuity of the relative entropy

\[
\liminf_{\eta \to 0} \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi_\eta)))\mu(d\rho)
\]

\[
= \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\mu)))\mu(d\rho) + \liminf_{\eta \to 0} H(\Phi(\bar{\rho}(\mu))\|\bar{\rho}(\pi_\eta))
\]

\[
\geq \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\mu)))\mu(d\rho) + H(\Phi(\bar{\rho}(\mu))\|\bar{\rho}(\pi))
\]

\[
= \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi)))\mu(d\rho).
\]

Then (18) implies

\[
\int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi)))\mu(d\rho) - \chi_\Phi(\pi)
\]

\[
\leq \liminf_{\eta \to 0} \int_{\mathcal{E}(\mathcal{H})} H(\Phi(\rho)\|\Phi(\bar{\rho}(\pi_\eta)))\mu(d\rho) - \chi_\Phi(\pi)
\]

\[
\leq \liminf_{\eta \to 0} \eta^{-1}(\chi_\Phi(\pi_\eta) - \chi_\Phi(\pi)) \leq 0. \quad \square
\]
6 Appendices

A. Proof of lemma 1. We first notice that $\text{supp}(\pi) \subseteq U$, where $U$ a closed convex subset of $\mathcal{G}(\mathcal{H})$ implies

$$\bar{\rho}(\pi) \in U. \quad (19)$$

This is obvious for arbitrary measure $\pi$ with finite support. By theorem 6.3 in [13] the set of such measures is dense in $\mathcal{P}$. The continuity of the mapping $\pi \mapsto \bar{\rho}(\pi)$ completes the proof of (19).

Let now $\pi$ be an arbitrary measure in $\mathcal{P}$. Since $\mathcal{G}(\mathcal{H})$ is separable we can, for each $n \in \mathbb{N}$, find a sequence $\{A^n_i\}$ of Borel sets of diameters less than $1/n$ such that $\mathcal{G}(\mathcal{H}) = \bigcup_{i=1}^{+\infty} A^n_i$, $A^n_i \cap A^n_j = \emptyset$ for $j \neq i$. Find a number $m = m(n)$ such that $\sum_{i=m+1}^{+\infty} \pi(A^n_i) < 1/n$. Consider the finite collection of Borel set $\{\hat{A}_i^n\}_{i=1}^{m+1}$, where $\hat{A}_i^n = A^n_i$ for all $i = 1, m$ and $\hat{A}^n_{m+1} = \bigcup_{i=m+1}^{+\infty} A^n_i$. We have

$$\bar{\rho}(\pi) = \sum_{i=1}^{m+1} \int_{\hat{A}_i^n} \rho \pi(d\rho) = \sum_{i=1}^{m+1} \pi^n_i \rho^n_i, \quad (20)$$

where $\pi^n_i = \text{Tr} \int_{\hat{A}_i^n} \rho \pi(d\rho) = \pi(\hat{A}_i^n)$ and $\rho^n_i = (\pi(\hat{A}_i^n))^{-1} \int_{\hat{A}_i^n} \rho \pi(d\rho)$ (without loss of generality we assume $\pi^n_i > 0$). Let $\pi^n$ be the probability measure on $\mathcal{G}(\mathcal{H})$, ascribing the value $\pi^n_i$ to the point $\rho^n_i$. Equality (20) implies $\bar{\rho}(\pi^n) = \bar{\rho}(\pi)$. Since $\pi^n$ has finite support for each $n$, to prove the assertion of the lemma it is sufficient to show that $\pi^n$ tends to $\pi$ in the weak topology as $n$ tends to $+\infty$. By theorem 6.1 in [13] to establish the above convergence it is sufficient to show that

$$\lim_{n \to +\infty} \int_{\mathcal{G}(\mathcal{H})} f(\rho) \pi^n(d\rho) = \int_{\mathcal{G}(\mathcal{H})} f(\rho) \pi(d\rho)$$

for arbitrary bounded uniformly continuous function $f(\rho)$ on $\mathcal{G}(\mathcal{H})$. Let $M_f = \sup_{\rho \in \mathcal{G}(\mathcal{H})} |f(\rho)|$. For arbitrary $\varepsilon > 0$ let $n_\varepsilon$ be such that $\varepsilon n_\varepsilon > 2M_f$ and

$$\sup_{\rho \in U(n_\varepsilon)} f(\rho) - \inf_{\rho \in U(n_\varepsilon)} f(\rho) < \varepsilon$$

for arbitrary closed ball $U(n_\varepsilon)$ of diameter $1/n_\varepsilon$. Let $n \geq n_\varepsilon$. By construction the set $\hat{A}_i^n$ is contained in some ball $U_i(n)$ for each $i = 1, m$. By (19) the
state $\rho^n_i$ lies in the same ball $U_i(n)$. Hence we have

$$\left| \int_{\mathcal{H}} f(\rho)\pi^n(d\rho) - \int_{\mathcal{H}} f(\rho)\pi(d\rho) \right|$$

$$\leq \sum_{i=1}^{m+1} \int_{\hat{A}^n_i} |f(\rho) - f(\rho_i)|\pi(d\rho)$$

$$\leq \varepsilon \sum_{i=1}^{m} \pi(\hat{A}^n_i) + 2M_f\pi(\hat{A}^n_{m+1}) < 2\varepsilon.$$ 

for all $n \geq n_\varepsilon$. □

**B. Example of a channel without optimal generalized ensembles.**

Consider Abelian von Neumann algebra $l_\infty$ and its predual $l_1$. Let $\Phi$ be the noiseless channel on $l_1$. Consider the sequence of states

$$\rho_n = \{1 - q_n, \frac{q_n}{n}, \frac{q_n}{n}, ..., \frac{q_n}{n}, 0, 0, ...\},$$

where $q_n$ is a sequence of numbers in $[0, 1]$, which will be defined below. Note that in this case $\chi_\Phi(\rho_n) = H(\rho_n) = h_2(q_n) + q_n \log n$, where $h_2(x) = -x \log x - (1 - x) \log(1 - x)$. We will show later that there exists the sequence $q_n$ such that $\lim_{n \to +\infty} q_n = 0$ while the corresponding sequence $\chi_\Phi(\rho_n) = H(\rho_n)$ monotonously increases to 1. Let $q_n$ be such a sequence and $\mathcal{A}$ be the closure of the sequence $\rho_n$, which obviously consists of states $\rho_n$ and pure state $\rho_* = \lim_{n \to +\infty} \rho_n = \{1, 0, 0, ...\}$. By definition and the above monotonicity $\bar{C}(\Phi; \mathcal{A}) = \lim_{n \to +\infty} \chi_\Phi(\rho_n) = 1$ while $\rho_*$ is the only optimal average state for the $\mathcal{A}$-constrained channel $\Phi$ and $\chi_\Phi(\rho_*) = H(\rho_*) = 0$. So we have $\bar{C}(\Phi; \mathcal{A}) > \chi_\Phi(\rho_*)$ and corollary 3 implies that there is no optimal ensemble for the $\mathcal{A}$-constrained channel $\Phi$.

Let us construct the sequence $q_n$ with the above properties. Consider the strongly increasing function $f(x) = x(1 - \ln x)$ on $[0, 1]$. It is easy to see that $f'(x) = -\ln x$ and $f([0, 1]) = [0, 1]$. Let $f^{-1}$ be the converse function and $g(x) = xf^{-1}(\ln 2/x)$ for all $x \geq 1$. Note that the function $g(x)$ is implicitly defined by the equation

$$g(1 - \ln(g(x))) = \ln 2. \quad (21)$$

Using this it is easy to see that the function $g(x)$ satisfies the following differential equation

$$\ln(g(x))g' = g/x. \quad (22)$$

Since $g(x)/x = f^{-1}(\ln 2/x)$ we have $g(x)/x \in [0, 1]$. This with (21) and (22) implies $g(x) \in [0, 1]$, $\lim_{x \to +\infty} g(x) = 0$ and $g'(x) < 0$ correspondingly.
Consider the function \( H(x) = h_2(g(x)) + g(x) \log x \). By (21) and (22) with the above observations we have
\[
\lim_{x \to +\infty} H(x) = (\ln 2)^{-1} \lim_{x \to +\infty} g(x) \ln x = 1
\]
and
\[
H'(x) = (\ln 2)^{-1} \left( g'(x) \ln(1 - g(x)) - g'(x) \ln g(x) + g'(x) \ln x + g(x)/x \right)
\]
\[
= (\ln 2)^{-1} g'(x) \log(1 - g(x)) > 0, \quad \forall x > 1.
\]
It follows that \( H(x) \) is an increasing function on \([1, +\infty)\), tending to its upper bound 1 at infinity. Setting \( q_n = g(n) \) we obtain the sequence with the desired properties.

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