Taming the ultra-violet divergences of Quantum Field Theories (QFTs) by defining them on a lattice has been an invaluable tool in the study of both high-energy and condensed matter physics. However, in chiral QFTs, such as the Standard Model, the gauge fields couple to left- and right-handed modes differently. Defining lattice chiral QFTs has presented an ongoing challenge. Nielsen and Ninomiya [1] first pointed out that, for naïve free band theories, the periodic nature of band structure implies that any gauge field must couple to right- and left-handed modes in the same way. Since then, numerous approaches have tried to sidestep this no-go result. In this paper, we present a numerical demonstration of a novel but simple lattice regularization method that uses interactions with a disordered Higgs field to avoid the no-go theorem while still remaining amenable to simulation.

This work follows many other approaches to the lattice chiral QFT problem. One of the most successful approaches to chiral Lattice gauge theory is the class of Ginsparg-Wilson theories [2], which often involve non-on-site actions of the gauge symmetry. Another remarkable class of theories are the Overlap-Fermion approaches [3–6], which compute correlation functions as the overlaps of successive ground states. However, it is unclear if these solutions possess finite dimensional Hilbert spaces for finite space volumes. The domain wall [7, 8] technique is the predecessor to what we employ, though in that approach the gauge fields propagate in one higher dimension than the fermions.

The approach used in this paper, proposed previously in Refs. [9–12], follows the ‘mirror fermion’ approach [13–16], in which one first creates a lattice regularization of both the anomaly-free chiral theory and its mirror conjugate, and then introduces interactions to gap out only the mirror theory without breaking the gauge symmetry if the chiral theory is free of all quantum anomalies. Instead of numerically intractable fermion-fermion interactions, we couple the mirror theory to a Higgs field driven into a symmetry-preserving, disordered, gapped phase.

We present a numerical treatment of a novel non-perturbative lattice regularization of a 1 + 1d SU(2) Chiral Gauge Theory. Our approach follows recent proposals that exploit the newly discovered connection between anomalies and topological (or entangled) states to show how to create a lattice regularization of any anomaly-free chiral gauge theory. In comparison to other methods, our regularization enjoys on-site fermions and gauge action, as well as a physically transparent fermion Hilbert space. We follow the ‘mirror fermion’ approach, in which we first create a lattice regularization of both the chiral theory and its mirror conjugate and then introduce interactions that gap out only the mirror theory. The connection between topological states and anomalies shows that such interactions exist if the chiral theory is free of all quantum anomalies. Instead of numerically intractable fermion-fermion interactions, we couple the mirror theory to a Higgs field to avoid the no-go theorem while still remaining amenable to simulation.

Our approach begins by creating a lattice model with two edges (Figure 1). The system is gapped in the bulk with the only low-energy excitations localized at edges; one edge will be described by the chiral QFT and the other will be described by its mirror conjugate QFT. To do so, we construct a space-time lattice Integer Quantum Hall (IQH) state with filling fraction ν = 1 extending in the t, x, and w directions, with open boundary conditions in the w direction and periodic boundary conditions otherwise (see appendix A for an explicit construction). We
We can stack IQH states to form a system of $n$ right-moving modes at $/D_w$ and a left moving mode at $/D_l$, respectively. We can repeat this construction with charge $-1$ and a left-moving mode at $/D_l$.

We couple the fermions on the right edge to a Higgs field that gaps out the right edge, leaving only the mirror edge. We Couple the fermions on the right edge to a Higgs field that gaps out the right edge, leaving only the mirror edge. Entangled states generically possess gapless edge modes [31], and so the system must be in an unentangled (product) state near the now-gapped mirror edge. As there are no further gapless excitations in the bulk, we conclude that the entire bulk must be in a trivial, gapped phase. Following the preceding discussion, this implies that the chiral theory, which contains all remaining gapless modes, must be anomaly-free.

Confirming that a theory is free of all anomalies is not generally an easy task. The cancellation of all Adler-Bell-Jackiw (ABJ) [32,33] type anomalies can be ensured using the usual anomaly cancellation conditions [10,34], which examine the Lie Algebra of the gauge group $G$ to provide powerful constraints. However, anomalies beyond those detectable from the Lie Algebra can still occur (e.g. [35]) which result from the non-trivial homotopic structure of the gauge group. For our $1 + 1d$ system defined on the edge of a $2 + 1d$ bulk, we ought to naively require that $\pi_n(g) = 0$, $n \leq 3$.

For this paper, we take the Lie Group $G = SU(2)$. The simplest $SU(2)$ representation that satisfies the anomaly cancellation conditions is $1_R \oplus (0_R) \oplus (1/2L)^4$, where subscripts indicate a collection of left- or right-movers. Topologically, $SU(2) \simeq S^3$, and so while $\pi_1(SU(2)) = \pi_2(SU(2)) = 0$, $\pi_3(SU(2)) = \mathbb{Z}$. Fortunately, this simply reflects the possibility of a Wess-Zumino-Witten (WZW) [36] term, and corresponds to the perturbative ABJ anomalies which are absent in our model by design.

Having built a lattice theory that gives rise to a chiral theory and its mirror conjugate, we now must choose the $SU(2)$-symmetric interactions which will gap out the mirror theory. Fermion-fermion interactions would render the problem numerically intractable. Instead, we use an $SU(2)$ Higgs field. We consider the field to be condensed $|\phi(x)| = 1$, leaving a non-linear $\sigma$-model. In contrast to the usual, symmetry-breaking $\phi = \text{const.}$ Higgs configurations, we drive the Higgs field into a disordered, symmetry-preserving phase with zero spatial average $L^{-2} \int d^2x \phi(x) = 0$. We demand that $\phi$ fluctuates smoothly, with a correlation length $\xi \gg 1$ that remains finite when we scale the system size. Capturing the fluctuations of this dynamical Higgs field is precisely why
our calculation must be done on a spacetime lattice.

The choice of dynamical Higgs field is of central importance. If at any point $\phi$ fluctuates too rapidly, a low-energy fermion mode may be localized there. An ideal disordered configuration would have $|\nabla \phi(x)| = \text{const.} > 0$. In the lattice model, we first choose a random $\phi(x)$ and then smooth it, taking care to apply the most smoothing in regions of largest $|\nabla \phi|$. This nonlinear smoothing process leads to a $\phi$ with nearly constant but nonzero $|\nabla \phi|$. We note that our approach resembles previous notions of condensing fluctuations [32], as we condense $\phi$ to set $|\phi| = \text{const.}$ then condense $\nabla \phi$ to set $|\nabla \phi| = \text{const.}$

We can now assemble the full lattice model. For our chosen SU(2) representation, we need 8 left-moving and 8 right-moving modes, which we collect into $\Psi_L, \Psi_R$, respectively. For a given $\phi \in SU(2)$, denote by $\Theta_{L,R}[\phi]$ the block diagonal representation of $\phi$ acting on the left- and right-moving modes, respectively. Then the total Lagrangian is

$$
\Psi^I \partial^I \Psi = \Psi^I \left( \begin{array}{cc} \mathcal{D}_R & g \Theta_R^\dagger \Theta_L \\ g \Theta_L^\dagger \Theta_R & \mathcal{D}_L \end{array} \right) \Psi
$$

where $\Psi^I = (\Psi^I_R, \Psi^I_L)$, $\mathcal{D}_{L,R}$ are the spacetime Lagrangian matrices for the individual IQH states described above, and we have introduced a coupling $g$. Here, $\Theta_{L,R}[\phi(x, t)]$ encode the $1_R \oplus (0_R)^5 \oplus (1/2_L)^4$ action of the Higgs field, though we have suppressed the Higgs dependence above. Note that $\mathcal{D}$ is itself a Lagrangian, not a Lagrangian density, so the fermion action is just $S[\Psi^I, \Psi] = \Psi^I \mathcal{D} \Psi$.

The full partition function of our system is now:

$$
Z = \int D\phi e^{-S_H[\phi]} \int D\Psi^I D\Psi \exp[-\Psi^I \mathcal{D} \Psi]
$$

where $S_H[\phi] = -U[\phi]$ is the action for the Higgs and we have neglected a proportionality constant. Performing the full integral is intractable. Instead we adopt a semiclassical approach and perform the calculation for a few $\phi$ configurations produced via the nonlinear smoothing process. These configurations provide a representative sample. Moreover, because we have chosen $S_H[\phi]$ to give rise to a disordered phase, any calculation is spatially self-averaging. To perform the calculation, we simply create a Higgs configuration and then assemble and calculate the small eigenvalues of the matrix $\mathcal{D}$.

Recalling that the Lagrangian density gives rise to low-energy modes of the form $\psi^I(\partial^I \pm i \partial_\perp) \psi$, we see that $\mathcal{D}$ is not Hermitian. The eigenvalues of $\mathcal{D}$, denoted $\lambda_n$, will generically be complex. However, we can quickly see how to interpret the eigenvalues. In momentum space for small $\omega$ and $k$, the low energy theories are of the form $\psi^I(i\omega - H(k)) \psi$, with $H(k) = \pm k$. From this we can quickly read off the meaning of the eigenvalues at low frequency $\omega \ll 1$: the real part of the eigenvalue corresponds to the energy, while the imaginary part is the variation in time. Hence any gapless excitations can be identified by their $\omega \rightarrow 0$ limits. In turn, a gap is just a region around zero in the complex plane devoid of
While we have seen that there is a spectral gap \((\det(\mathcal{D}) \neq 0)\) in the fermionic sector of the mirror theory, we must now discuss why the partition function is not wiped out by the integral over Higgs configurations. Since \(\mathcal{D}\) is not Hermitian, this determinant may be complex and its phase can fluctuate. Fortunately, anomalous behavior is prevented by the anomaly-free conditions discussed earlier: first we note that \(\pi_i(SU(2)) = 0\), \(i = 0, 1, 2\), so there are no topologically distinct sectors of Higgs field configurations or topological defects; secondly, the presence of a perturbative WZW term is prevented by the anomaly cancellation condition.

There is a symmetry that controls the phase of the partition function. Consider first the time reflection symmetry that sends \(\phi(x, t) \rightarrow \phi(x, -t)\) or, equivalently, reverses the lattice in the time direction and leaves \(\phi(x, t)\) unchanged. Because the imaginary parts of eigenvalues come only from the temporal hopping, this has the effect of complex conjugation \(\lambda_n \rightarrow \lambda_n^*\). This implies that so long as the action for \(\phi\) is time-reflection symmetric, we may replace the determinant by its real part in the partition function:

\[
Z = \int D\phi e^{-S[\phi]}\Gamma[\det(\mathcal{D})]
\]

Hence the partition function is real, despite involving a manifestly non-Hermitian Lagrangian.

Finally, we wish to highlight the difficulty of writing a field-theoretic description for the mirror edge in this model. The naïve approach of employing a theory of Weyl fermions and a Higgs field is spoiled by the Higgs potential: a Higgs configuration with \(|\nabla \phi| = \text{const.} > 0\) can be realized as minimizing the potential \(U[\phi] = \int dx dt [-a|\nabla \phi|^2 + b|\nabla \phi|^4 + ...\) with \(a, b > 0\). However, we do not know at present how to evaluate such a field integral. Any more elegant attempt is likely to be complicated, as there is no mass term capable of giving a mass to the fermion fields without breaking symmetry. Whether a low-energy description can be found in terms of fields radically different than the naïve fermion fields—or in terms of a CFT without a Lagrangian description—remains to be seen.

In summary, the method that we have demonstrated in this paper leads to a particularly simple lattice regularization for chiral QFTs. Both fermions and the gauge symmetry action are entirely onsite, and the Hilbert space associated to any one site is well-defined and physically clear. We consider this model’s inability to regularize most anomalous theories a great feature; this method brings the physical nature of quantum anomalies nearer to heuristic focus. Crucially, the use of a Higgs field, as opposed to fermion-fermion interactions, renders this method very promising for further numerical studies.

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Appendix A: Lattice Model

Here we give the explicit construction of the spacetime Lagrangian. We first build a spatial, lattice model of a single IQH state before extending the model to give a spacetime structure. Then, we stack copies of these spacetime IQH models to give our full 16-flavor model.

The spatial component of our lattice is provided by the lattice integer Quantum Hall (IQH) states shown in figure 4 (a) extending in the $w-x$ plane. Note that hopping around any plaquette gives a phase of $\pi$, hence we have exactly one flux quantum per site. This model is tuned so that the chemical potential is exactly between the Landau levels, giving rise to a $\nu = 1$ state. We give the $x$ direction periodic boundary conditions but leave open boundaries in the $w$ direction. The open boundary conditions give rise to two low-energy modes: a right-moving excitation on the right hand edge and a left-moving excitation on the left hand edge. Denoting this matrix as $H_R$, the $q = -1$ state is just obtained by $H_L = H_R^*$. We can then add a time hopping to our spatial system to create a spacetime lattice as shown in figure 4; the twin linear modes are precisely the edge modes.

In our calculation, we assumed that the two edges were effectively decoupled. Naively, this would require $L_w \gg 1$, but for eigenvalue computations $L_w = 2$ is in fact sufficient. With $L_w = 2$, momentum spectrum of the chiral theory is slightly affected by the Higgs fluctuations on the mirror edge. This effect may be reduced by increasing $L_w$, at significant computational cost. The eigenvalues of the $L_w = 2$ system are shown in figure 4; the twin linear modes are precisely the edge modes.

We can then add a time hopping to our spatial system to create a spacetime lattice as shown in figure 4. In momentum space this hopping leads to a term $T = \ell(e^{i\omega} - 1)$. We set the strength of the time hopping term $t$ so that

$$\text{Physics Letters B 105, 219 (1981)}$$
$$\text{Phys. Rev. D 25, 2649 (1982)}$$
$$\text{Physics Letters B 302, 62 (1993)}$$
$$\text{Physics Letters B 393, 360 (1997)}$$
$$\text{Nucl. Phys. Proc. Suppl. 29BC}$$
$$\text{Phys. Rev. B 91}$$
$$\text{Phys. Rev. B 177, 2426 (1969)}$$
$$\text{Nucl. Phys. B 324 (1989).}$$
$$\text{Erratum: Phys. Rev. Lett. B331, no.3-4, 449 (1994)], arXiv:hep-lat/9403014 [hep-lat].}$$
$$\text{arXiv:1412.4784 [cond-mat.str-el].}$$

1. H. Nielsen and M. Ninomiya, Physics Letters B 105, 219 (1981)
2. P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25, 2649 (1982)
3. R. Narayanan and H. Neuberger, Physics Letters B 302, 62 (1993)
4. R. Narayanan and H. Neuberger, Physics Letters B 393, 360 (1997)
5. M. Luscher, Nucl. Phys. B549, 295 (1999) arXiv:hep-lat/9811032 [hep-lat]
6. M. Luscher, Theory and experiment heading for new physics. Proceedings, 38th course of the International School of subnuclear physics, Erice, Italy, August 27-September 5, 2000. Subnucl. Ser. 38, 41 (2002)
7. B. Kaplan, Physics Letters B 288, 342 (1992)
8. Y. Shamir, Nuclear Physics B 406, 90 (1993)
9. X.-G. Wen, Chin. Phys. Lett. 30, 111101 (2013) arXiv:1305.1045 [hep-lat]
10. J. Wang and X.-G. Wen, (2013), arXiv:1307.7480 [hep-lat]
11. Y.-Z. You and C. Xu, Phys. Rev. B 91, 125147 (2015) arXiv:1412.4784
12. D. M. Grabowska and D. B. Kaplan, Phys. Rev. Lett. 116, 211602 (2016)
13. E. Eichten and J. Preskill, Nuclear Physics B 268, 179 (1986)
14. I. Montvay, Nucl. Phys. Proc. Suppl. 29BC, 159 (1992) arXiv:hep-lat/9205023 [hep-lat]
15. J. Giedt and E. Poppitz, JHEP 10, 076 (2007) arXiv:hep-lat/0701004 [hep-lat]
16. D. M. Grabowska and D. B. Kaplan, Phys. Rev. D 94, 114504 (2016)
17. M. F. L. Golterman, D. N. Petcher, and E. Rivas, Nucl. Phys. B395, 596 (1993) arXiv:hep-lat/9206010 [hep-lat]
18. L. Lin, Phys. Lett. B324, 418 (1994) [Erratum: Phys. Lett.B331,no.3-4,449(1994)], arXiv:hep-lat/9403014 [hep-lat]
19. C. Chen, J. Giedt, and E. Poppitz, JHEP 04, 131 (2013) arXiv:1211.6947 [hep-lat]
20. T. Banks and A. Dabholkar, Phys. Rev. D46, 4016 (1992) arXiv:hep-lat/9204017 [hep-lat]
21. X.-G. Wen, Phys. Rev. D88, 045013 (2013) arXiv:1303.1803 [hep-th]
22. J. Wang, L. H. Santos, and X.-G. Wen, Phys. Rev. B91, 195134 (2015) arXiv:1403.5256 [cond-mat.str-el]
23. Y. BenTov, JHEP 07, 034 (2015) arXiv:1412.0154 [cond-mat.str-el]
24. V. Ayyar and S. Chandrasekharan, Phys. Rev. D 93, 081701 (2016)
25. V. Ayyar and S. Chandrasekharan, Journal of High Energy Physics 2016, 58 (2016)
26. Y. BenTov and A. Zee, Phys. Rev. D93, 065036 (2016) arXiv:1505.04312 [hep-th]
27. L. Kong and X.-G. Wen, (2014), arXiv:1405.5858 [cond-mat.str-el]
28. X. Chen, Z. C. Gu, and X. G. Wen, Phys. Rev. B82, 155138 (2010) arXiv:1001.3833 [cond-mat.str-el]
29. X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science 338, 1604 (2012)
30. X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B84, 235141 (2011) arXiv:1106.4752 [cond-mat.str-el]
31. There are plenty of exceptions to this, particularly when the mirror edge may be given a gapped topological order (e.g. 33, 39). Though such exceptions may lead to unique special cases, the generic, naive approach we present requires a trivially entangled bulk.
32. J. S. Bell and R. Jackiw, Il Nuovo Cimento A (1965-1970) 60, 47 (1969)
33. S. L. Adler, Phys. Rev. 177, 2426 (1969)
34. M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory: 1995 ed. (Westview, Boulder, CO, 1995).
35. E. Witten, Physics Letters B 117, 324 (1982)
36. J. Wess and B. Zimino, Physics Letters B 37, 95 (1971)
37. Y.-Z. You and C. Xu, Phys. Rev. B91, 125147 (2015) arXiv:1412.4784 [cond-mat.str-el]
38. X. Chen, L. Fidkowski, and A. Vishwanath, Physical Review X 3, 041016 (2013) arXiv:1305.5855 [cond-mat.str-el]
39. E. Witten, Nuclear Physics B 223, 422 (1983)
FIG. 4. (color online) (a): Spatial IQH state hopping model. Fermion sites are shown as spheres, with hopping terms as links. A yellow link indicates a hopping of +1, a red link a hopping of \(-1\), and a green link a hopping of \(+i\) in the direction of the arrow and \(-i\) in the opposite direction. Hopping around any plaquette generates a phase of \(\pi\), hence with 1 fermion per site this is a \(\nu = 1\) IQH state. (b): Spacetime lattice with \(L_w = 2\). Each spatial component is just a slice of a IQH state shown in (a), while the purple links represent a Hopping of \(ti\) in only the direction of the double arrows. In addition, each site is given the onsite chemical potential \(-t\psi_i^\dagger \psi_i\), where we later set \(t = 3\). The Hopping matrix corresponding to this model is precisely our spacetime Lagrangian. (c) Dispersion relation for the spatial lattice with \(L_w = 2\). (d) Complex Eigenvalues of the spacetime hopping model.

The complex eigenvalue spectrum of \(\mathcal{V}_R\) is shown in Figure 4d. For any fixed \(\omega\), the corresponding eigenvalues form a line parallel to the real axis. The broader, circular complex structure is then produced by the \(e^{i\omega} - 1\) terms.

For our full model, we need 8 copies each of \(\mathcal{V}_R\) and \(\mathcal{V}_L\), together with the interaction discussed in the main body of the paper. The total form is then:

\[
\Psi^\dagger \mathcal{V} \Psi = \Psi^\dagger \left( \mathbb{1}_8 \otimes \mathcal{V}_R + g \Theta_L^\dagger \Theta_L \right) \Psi
\]

where \(\Theta_L,R[\phi(x)]\) are the block-diagonal matrices corresponding to the \(SU(2)\) representation \(1_R \oplus (0_R)^5 \oplus (1/2_L)^4\).

Appendix B: Larger Lattice Calculation

The main obstacle to increasing system size is the large number of chiral-edge gapless modes that must be found near \(\omega = \pi\), the real part of the eigenvalue is more negative than the bandgap of the system (note that since \(\mathcal{L} = p\psi - \mathcal{H}\), a negative real part is a positive energy). For our purposes, \(t = 3\) suffices. The one-flavor spacetime Lagrangian is then just \(\mathcal{D}_{L,R} = T \oplus (-H_{L,R})\), where \(\oplus\) is the usual Kronecker sum.

Fig. 5 shows these results and demonstrates that with \(\xi \approx 8\), \(\Delta \approx .35\), and we are again well in the thermodynamic regime for \(L \gtrsim 40\).