Minimal generating sets for matrix monoids

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Abstract

In this paper, we determine minimal generating sets for several well-known monoids of matrices over certain semirings. In particular, we find minimal generating sets for the monoids consisting of: all \( n \times n \) boolean matrices when \( n \leq 8 \); the \( n \times n \) boolean matrices containing the identity matrix (the reflexive boolean matrices) when \( n \leq 7 \); the \( n \times n \) boolean matrices containing a permutation (the Hall matrices) when \( n \leq 8 \); the upper, and lower, triangular boolean matrices of every dimension; the \( 2 \times 2 \) matrices over the semiring \( \mathbb{N} \cup \{-\infty\} \) with addition \( \oplus \) defined by \( x \oplus y = \max(x, y) \) and multiplication \( \otimes \) given by \( x \otimes y = x + y \) (the max-plus semiring); the \( 2 \times 2 \) matrices over any quotient of the max-plus semiring by the congruence generated by \( t = t + 1 \) where \( t \in \mathbb{N} \); the \( 2 \times 2 \) matrices over the min-plus semiring and its finite quotients by the congruences generated by \( t = t + 1 \) for all \( t \in \mathbb{N} \); and the \( n \times n \) matrices over \( \mathbb{Z}/n\mathbb{Z} \) relative to their group of units.

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1 Introduction

In this paper we find minimum cardinality generating sets for several well-known finite monoids of matrices over semirings. The topic of determining such minimum cardinality generating sets for algebraic objects is classical, and has been studied extensively in the literature; see, for example, [4, 14, 15, 18, 24, 28, 48, 56]. In this paper we are principally concerned with monoids of matrices over the boolean semiring \( \mathbb{B} \); we also present some results
about monoids of min-plus and max-plus matrices of dimension 2, and matrices of arbitrary dimension over the rings \( \mathbb{Z}/n\mathbb{Z}, n \in \mathbb{N}^+ \).

If \( S \) is a semigroup, then the least cardinality of a generating set for \( S \) is often called the \textbf{rank} of \( S \) and is denoted by \( \text{d}(S) \). A \textbf{semiring} is a set \( S \) with two operations, \( \oplus \) and \( \otimes \), such that \((S, \oplus)\) forms a commutative monoid with identity \( e \), \((S, \otimes)\) forms a monoid, \( e \otimes x = x = x \otimes e = e \) for all \( x \in S \), and multiplication distributes over addition. We refer to \( e \) and \( f \) as the \textbf{zero} and \textbf{one} of the semiring respectively. One natural example of a semiring is the natural numbers \( \mathbb{N} \); note that in this paper \( 0 \in \mathbb{N} \) and we write \( \mathbb{N}^+ \) for the set of positive natural numbers. Another well-known example is the \textbf{boolean semiring} \( \mathbb{B} \). This is the set \( \{0, 1\} \) with addition defined by

\[
0 \oplus 1 = 1 \oplus 0 = 1 \oplus 1 = 1 \quad \text{and} \quad 0 \oplus 0 = 0
\]

and multiplication defined as usual for the real numbers 0 and 1. If \( n \in \mathbb{N}^+ \), then we denote by \( M_n(\mathbb{B}) \) the monoid consisting of all \( n \times n \) matrices with entries in \( \mathbb{B} \). The semiring \( \mathbb{B} \) is one of the simplest examples of a semiring, and the matrix monoids \( M_n(\mathbb{B}) \) for \( n \in \mathbb{N} \) have been widely studied in the literature since the 1960s to the present day; see, for example, \([5, 6, 8, 9, 11, 12, 21, 37, 40, 42, 54, 55, 57, 58, 59, 63, 67]\). If \( \alpha \) is a binary relation on the set \( \{1, \ldots, n\} \), then we can define an \( n \times n \) matrix \( A_\alpha \) with entries in the boolean semiring \( \mathbb{B} \) such that the entry in row \( i \), column \( j \) of \( A_\alpha \) is 1 if and only if \( (i, j) \in \alpha \). The function that maps every binary relation \( \alpha \) to the corresponding \( A_\alpha \) is an isomorphism between the monoid of binary relations on \( \{1, \ldots, n\} \) and \( M_n(\mathbb{B}) \). Functions are a special type of binary relations, and composition of functions coincides with composition of binary relations when applied to functions. As such the monoid \( M_n(\mathbb{B}) \) can be thought of as a natural generalisation of the full transformation monoid consisting of all functions from \( \{1, \ldots, n\} \) to itself, under composition of functions. In comparison to the full transformation monoid and its peers, such as the symmetric inverse monoid or the so-called diagram monoids, whose structures are straightforward to describe, \( M_n(\mathbb{B}) \) has a rich and complex structure. For example, every finite group appears as a maximal subgroup of \( M_n(\mathbb{B}) \) for some \( n \in \mathbb{N} \) (see \([13, 47, 55]\)), and the Green’s structure of \( M_n(\mathbb{B}) \) is highly complicated; neither the number of \( J \)-classes of \( M_n(\mathbb{B}) \) nor the largest length of a chain of \( J \)-classes is known for \( n \geq 9 \). In this landscape it is perhaps not surprising that the minimum sizes \( \text{d}(M_n(\mathbb{B})) \) of generating sets for the monoids \( M_n(\mathbb{B}) \) were previously unknown for \( n \geq 6 \). On the other hand, a description of a minimal generating set for \( M_n(\mathbb{B}) \) was given by Devadze in 1968 \([17]\) (see Theorem 3.1.1). There is no proof in \([17]\) that the given generating sets are minimal, a gap that was filled by Konieczny in 2011 \([39]\). The minimal generating sets given by Devadze and Konieczny are specified in terms of representatives of certain \( J \)-classes of \( M_n(\mathbb{B}) \), the exact number of which is difficult to compute, and grows very quickly with \( n \); see Corollary 3.1.8.

The monoid \( M_n(\mathbb{B}) \) has several natural submonoids, such as the monoids: \( M_n^{\text{id}}(\mathbb{B}) \) consisting of the \textbf{reflexive boolean matrices} (that is matrices containing the identity matrix); \( M_n^\mathbb{S}(\mathbb{B}) \) consisting of all \textbf{Hall matrices} (matrices containing a permutation); and \( UT_n(\mathbb{B}) \) of \textbf{upper triangular boolean matrices}. Each of these submonoids has been extensively studied in their own right; see, for example, \([\text{Gaysin2021a}, 8, 12, 38, 41, 53, 58, 62, 63, 66]\). Many other submonoids of \( M_n(\mathbb{B}) \) have also been investigated, although we do not consider these submonoids in this article; for a recent example, see \([10]\). Unlike \( M_n(\mathbb{B}) \), \( M_n^{\text{id}}(\mathbb{B}) \) is \( J \)-trivial, and so has precisely \( |M_n^{\text{id}}(\mathbb{B})| = 2^{n^2-n} \) \( J \)-classes. Similarly, \( A, B \in M_n^\mathbb{S}(\mathbb{B}) \) are \( J \)-related if and only if one can be obtained from the other by permuting the rows and columns. However, the size of \( M_n^\mathbb{S}(\mathbb{B}) \) is not known for \( n \geq 8 \). This question was raised by Kim Ki Hang \([37, \text{Problem 13}]\) in 1982 and recently raised again in \([\text{Gaysin2020a}]\); see \([49]\) for the known values. In contrast, the upper-triangular boolean matrix monoid \( UT_n(\mathbb{B}) \) is easily seen to have size \( 2^{\frac{n(n+1)}{2}} \). The monoid \( UT_n(\mathbb{B}) \) appears to have been primarily studied in the context of varieties; see for instance \([11, 66]\). It is somewhat surprising that there appears to be no description of the unique minimal generating set of \( UT_n(\mathbb{B}) \) in the literature, in particular because this minimal generating set is more straightforward to determine than that of the other submonoids of \( M_n(\mathbb{B}) \) we consider (see Section 3.5).

We also consider monoids of matrices over certain \textbf{tropical} semirings. The \textbf{min-plus semiring} \( K^\infty \) is the set \( \mathbb{N} \cup \{\infty\} \), with \( \oplus = \min \) and \( \otimes \) extending the usual addition on \( \mathbb{N} \) so that \( x \otimes \infty = \infty \otimes x = \infty \) for all \( x \in K^\infty \). The multiplicative identity of \( K^\infty \) is 0, and the additive identity is \( \infty \).
The **max-plus semiring** $\mathbb{K}^{-\infty}$ is the set $\mathbb{N} \cup \{-\infty\}$ with $\oplus = \max$ and $\otimes$ extending the usual addition on $\mathbb{N}$ so that $x \otimes -\infty = -\infty \otimes x = -\infty$ for all $x \in \mathbb{K}^{-\infty}$. The one of $\mathbb{K}^{-\infty}$ is 0 and the zero is $-\infty$.

These semirings give rise to two infinite families of finite quotients. The **min-plus semiring with threshold** $t$, denoted $\mathbb{K}^{\infty}_t$, is the set $\{0, 1, \ldots, t, \infty\}$ with operations $\oplus = \min$ and $\otimes$ defined by

$$a \otimes b = \begin{cases} \min(t, a + b) & a \neq \infty \text{ and } b \neq \infty \\ \infty & a = \infty \text{ or } b = \infty. \end{cases}$$

The **max-plus semiring with threshold** $t$, denoted $\mathbb{K}^{-\infty}_t$, is constructed analogously; its elements are $\{-\infty, 0, 1, \ldots, t\}$, addition is max, and multiplication is defined by $a \otimes b = \min(t, a + b)$ for all $a, b \in \mathbb{K}^{-\infty}_t$. For arbitrary $t \in \mathbb{N}$, the semirings $\mathbb{K}^{\infty}_t$ and $\mathbb{K}^{-\infty}_t$ with threshold $t$ can also be defined as the quotient of the corresponding infinite semirings $\mathbb{K}^{\infty}$ and $\mathbb{K}^{-\infty}$ by the congruence generated by $(t, t + 1)$. The max-plus and min-plus tropical semirings are often defined in the literature to be $\mathbb{R} \cup \{-\infty\}$ or $\mathbb{R} \cup \{\infty\}$, respectively. The monoids $M$ of matrices over these semirings are uncountably infinite, and so every generating set for such a monoid has cardinality $|M|$. This is one rationale for considering the semirings $\mathbb{K}^{-\infty}$ and $\mathbb{K}^{\infty}$ in this paper. Another reason we consider this unorthodox definition is that the results in this paper concerning monoids of matrices over $\mathbb{K}^{-\infty}$ and $\mathbb{K}^{\infty}$ arose initially from computational experiments in a reimplementation of the Froidure-Pin Algorithm [23, 52] in [45], and the original implementation by Jean-Eric Pin in [52] included support for the monoids we consider here. Note that under the standard definition, the min-plus and max-plus semirings are isomorphic under the map $x \mapsto -x$; this is not the case under our definition. There is a significant amount of literature on matrices over the tropical semirings; see for example [16, 27, 30, 32, 33, 34, 60, 61]. However, there is relatively little literature on the classical semigroup-theoretic properties of the tropical matrix monoids like Green’s relations or generating sets; indeed descriptions of the Green’s relations were only published relatively recently (see [27, 32]). This may simply be because the monoids are rather complex and difficult to work with.

The final semiring that we are concerned with is $\mathbb{K}^{-\infty} = \mathbb{R} \cup \{-\infty\}$, which is isomorphic under the map $x \mapsto -x$; this is not the case under our definition. There is a significant amount of literature on matrices over the tropical semirings; see for example [16, 27, 30, 32, 33, 34, 60, 61]. However, there is relatively little literature on the classical semigroup-theoretic properties of the tropical matrix monoids like Green’s relations or generating sets; indeed descriptions of the Green’s relations were only published relatively recently (see [27, 32]). This may simply be because the monoids are rather complex and difficult to work with.

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monoid obtained by adjoining an identity to $S$. Then
\[
\begin{align*}
sLt \text{ if and only if } S^1s &= S^1t \\
sRt \text{ if and only if } ss^1 &= ts^1 \\
sJt \text{ if and only if } S^1sS^1 &= S^1ts^1.
\end{align*}
\]

Finally, we define Green’s $H$-relation as the intersection of $L$ and $R$. We write $X_s$ for the Green’s $X$-class of $s$, and $S/X$ for the set of Green’s $X$-classes of $S$, where $X \in \{L, R, J, H\}$. There is a natural partial order on certain Green’s classes given by
\[
\begin{align*}
R_s \leq R_t & \text{ if and only if } ss^1 \subseteq ts^1 \\
L_s \leq L_t & \text{ if and only if } S^1s \subseteq S^1t \\
J_s \leq J_t & \text{ if and only if } S^1sS^1 \subseteq S^1ts^1.
\end{align*}
\]

Further background on Green’s relations can be found in [29].

2.2 Matrix semigroups

Given a semiring $S$ and $m, n \in \mathbb{N}^+$, we may study the set of $m \times n$ matrices over $S$; we will denote this by $M_{m,n}(S)$. In particular, we are interested in the multiplicative monoid of square matrices of dimension $n \in \mathbb{N}^+$ over $S$, denoted by $M_n(S)$. There are a number of common features of such matrix monoids. As usual, we let $0 \in S$ denote the additive identity of $S$ and $1$ denote the multiplicative identity. The following definitions are entirely analogous to those from the standard linear algebra of vector spaces over fields.

Let $A \in M_n(S)$. We write $A_{is}$ to denote the $i$th row of $A$, $A_{si}$ to denote the $i$th column, and $A_{ij}$ to denote the $j$th entry of the $i$th row. A **linear combination** of rows is a sum of scalar multiples of the rows, where both operations are defined componentwise in the usual way. The **row space** $\Lambda(A)$ of $A$ is the set of all linear combinations of the rows of $A$. An **isomorphism** of row spaces is a linear bijection; that is, a bijection $T : \Lambda(A) \rightarrow \Lambda(B)$ such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in \Lambda(A)$ and for all $\alpha, \beta \in S$. A set of rows is **linearly independent** if no row can be written as a linear combination of other rows; this coincides with the usual definition when $S$ is a field. A **spanning set** for a row space is a set of rows which every element of the row space may be written as a linear combination of. A **row basis** of $A$ is then a linearly independent spanning set for the row space of $A$. **Column spaces** $P(A)$ and **column bases** $\rho(A)$ are defined dually.

The importance of row and column bases arises from the following well-known results:

**Proposition 2.2.1.** Let $A, B \in M_n(S)$. Then the following hold:

(i) $\Lambda(AB) \subseteq \Lambda(B)$ and $P(AB) \subseteq P(A)$.

(ii) $L_A \leq L_B$ if and only if $\Lambda(A) \subseteq \Lambda(B)$ and $R_A \leq R_B$ if and only if $P(A) \subseteq P(B)$.

**Proof.** We will only prove the statements related to row spaces and $L$-classes; the other statements may be proved in a dual fashion. (i). Suppose that $A = [\alpha_{ij}]$ and $B = [\beta_{ij}]$. If $AB = [\gamma_{ij}]$, then
\[
\gamma_{ij} = \alpha_{i1}\beta_{1j} + \alpha_{i2}\beta_{2j} + \cdots + \alpha_{in}\beta_{nj} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}.
\]

It follows that the $i$th row of $AB$ is
\[
(\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}) = \left(\sum_{k=1}^{n} \alpha_{ik}\beta_{k1}, \sum_{k=1}^{n} \alpha_{ik}\beta_{k2}, \ldots, \sum_{k=1}^{n} \alpha_{ik}\beta_{kn}\right) = \sum_{k=1}^{n} \alpha_{ik}B_{ks}.
\]

This notation for row and column spaces and bases arises from their application in acting semigroup algorithms; see [20] for some details although not in the particular context of matrix semigroups.
Thus the $i$th row of $AB$ is a sum of rows of $B$, and so $\Lambda(AB) \subseteq \Lambda(B)$.

(ii). If $L_A \leq L_B$, then there exists $X \in M_n(\mathbb{S})$ such that $A = XB$; hence $\Lambda(A) = \Lambda(XB) \subseteq \Lambda(B)$ by (i). Conversely, if $\Lambda(A) \subseteq \Lambda(B)$, then every row of $A$ can be expressed as a linear combination of rows of $B$. For $1 \leq i \leq n$, let $A_{is} = \sum_{j=1}^{n} x_{ij}B_{js}$, and define $X = [x_{ij}]$. Then $A = XB$, and hence $L_A \leq L_B$. $\square$

The following corollary now follows immediately from Proposition 2.2.1.

Corollary 2.2.2. Let $A,B \in M_n(\mathbb{S})$. Then $A\mathcal{J}B$ if and only if $\Lambda(A) = \Lambda(B)$, and $A\mathcal{P}B$ if and only if $P(A) = P(B)$.

The following lemma provides a useful connection between the $\mathcal{J}$-relation and row spaces; see Theorem 3.1.3 for a strengthening in the case of $M_n(\mathbb{B})$. The proof of this lemma is essentially the same as for boolean matrices, as found in [37, Theorem 1.3.3, forward direction]. Note that this theorem and proof is in terms of Green’s $\mathcal{D}$-relation (see [29]), but for the finite monoids $M_n(\mathbb{S})$, $\mathcal{D} = \mathcal{J}$.

Proposition 2.2.3. Let $S$ be a finite semiring. If two matrices in $M_n(\mathbb{S})$ are $\mathcal{J}$-related, then they have isomorphic row spaces.

The group of units of $M_n(\mathbb{S})$ is the group of invertible matrices (units) in $M_n(\mathbb{S})$; the identity matrix is always a unit. The group of units is always the maximal class in the $\mathcal{J}$-order of $M_n(\mathbb{S})$.

For any semiring $S$, the symmetric group embeds into the group of units of $M_n(\mathbb{S})$ by the map

$$
\phi : \alpha \mapsto [a_{ij}],
$$

$$
a_{ij} = \begin{cases}
1 & i\alpha = j \\
0 & \text{otherwise}
\end{cases}
$$

The image of this embedding will often simply be referred to as $S_n$ or the symmetric group when context prevents ambiguity. Elements of this embedding $S_n$ are called permutation matrices; multiplying by a permutation matrix on the left permutes rows of a matrix, and multiplying on the right permutes columns. Similarly, we may define a transformation matrix to be one which contains a single 1 in every row; these are the images of the obvious extension of $\phi$ to the full transformation monoid $T_n$.

Two matrices $A, B \in M_n(\mathbb{S})$ are similar if each can be obtained from the other by permuting the rows and/or columns. Note that similar matrices are $\mathcal{J}$-related in $M_n(\mathbb{S})$, by multiplying on the left and/or right by permutation matrices.

A non-unit matrix $A \in M_n(\mathbb{S})$ is prime if $A = BC$ implies $B$ or $C$ is a unit. The prime matrices of $M_n(\mathbb{S})$ are immediately below the group of units in the $\mathcal{J}$-order.

2.3 Minimal generating sets

Given a semigroup $S$, we may ask what the minimum cardinality is of a generating set for $S$. This is known as the rank of $S$ and is denoted by $d(S)$. The same notion exists for monoid generating sets; the rank of a monoid as a monoid is either one less than the rank as a semigroup (if the group of units is trivial), or the same otherwise. We therefore consider only the rank of monoids as semigroups in this paper. A generating set $X$ for $S$ is irredundant if no subset of $X$ generates $S$, and we say that any irredundant generating set of cardinality $d(S)$ is a minimal generating set. If $d(S) \in \mathbb{N}$, then every generating set of cardinality $d(S)$ is irredundant and hence minimal; this does not hold when $d(S)$ is infinite. We will also say that a generator $x \in X$ is irredundant if $X \setminus \{x\}$ does not generate $S$.

An element $x$ of a monoid $M$ with identity $e$ is decomposable if $x$ may be written as a product of elements in $M \setminus \{e, x\}$, and indecomposable otherwise. Note that a different definition of indecomposable elements exists in the literature. Under that definition, an element $x$ of a semigroup $S$ is decomposable if $x \in S^2 = \{st : s, t \in S\}$,
and indecomposable otherwise. These definitions are closely related but not equivalent, and our definition will be more useful for the purposes of this paper.

Note that every indecomposable element of a monoid \( M \) must be contained in every generating set for \( M \); hence the minimal generating set of a monoid is a superset of the indecomposable elements.

We will use the following straightforward lemma repeatedly to show that certain sets generate submonoids of \( M_n(\mathbb{B}) \).

**Lemma 2.3.1.** Let \( S \) be a finite semigroup and let \( X \) be a subset of \( S \), such that for every element \( x \in X \) with \( \mathcal{J} \)-class \( J_x \), we have \( J_x \subseteq \langle X \rangle \). If every element \( s \in S \) that is not \( \mathcal{J} \)-related to an element of \( X \) can be written as a product of elements of \( S \), none of which are \( \mathcal{J} \)-related to \( s \) in \( S \), then \( X \) generates \( S \).

**Proof.** Let \( s \in S \). Then either \( s \) is in a \( \mathcal{J} \)-class of an element belonging to \( X \), in which case we are done, or \( s \) may be written as a product of elements which are not \( \mathcal{J} \)-related to \( s \), and hence lie strictly above \( s \) in the \( \mathcal{J} \)-order. This same argument applies to each of those elements. Since \( S \) has finitely many \( \mathcal{J} \)-classes, this process must eventually terminate. However, the process of decomposing elements may only terminate if each element to be decomposed lies in the \( \mathcal{J} \)-class of an element of \( X \); hence \( s \in \langle X \rangle \).

We also have the following proposition, used to prove that certain generating sets are minimal.

**Proposition 2.3.2.** Let \( S \) be a finite semigroup. Suppose that \( X \) is an irredundant generating subset of \( S \) that contains at most one element from each \( \mathcal{J} \)-class of \( S \). Then \( X \) has minimum cardinality, i.e. \( d(S) = |X| \).

**Proof.** Let \( x \in X \). It suffices to show that any generating set for \( S \) contains an element of \( J_x \).

Let \( U \) be the union of the \( \mathcal{J} \)-classes of \( S \) that lie strictly above \( J_x \) in the \( \mathcal{J} \)-order. If \( x \) is written as a product of elements of \( S \), then it is clear that those elements are contained in \( J_x \cup U \). Furthermore, since \( X \) generates \( S \), which contains \( U \), and since no element of \( U \) can be written as a product involving \( x \), it follows that \( U \subseteq \langle X \setminus \{x\} \rangle \). Therefore \( \langle U \rangle \leq \langle X \setminus \{x\} \rangle \). The assumption that \( X \) is irredundant implies that \( x \not\in \langle X \setminus \{x\} \rangle \), and so \( x \not\in \langle U \rangle \). Therefore, if \( x \) is written as a product of elements of \( S \), then at least one of those elements is contained in \( J_x \). In other words, any generating set for \( S \) contains an element of \( J_x \).

## 3 Boolean matrix monoids

In this section, we compute minimal generating sets for the full boolean matrix monoid \( M_n(\mathbb{B}) \) and some of its natural submonoids. In order to define these submonoids, we must introduce the concept of matrix containment: given two matrices \( A, B \in M_n(\mathbb{B}) \), we say that \( A \) is contained in \( B \) if for all \( 1 \leq i, j \leq n \), \( A_{ij} = 1 \) implies that \( B_{ij} = 1 \); that is, \( B \) contains a \( 1 \) in every position in which \( A \) does.

We consider the submonoids:

(i) \( M_n^{id}(\mathbb{B}) \), of matrices containing the identity (reflexive matrices),

(ii) \( M_n^{S}(\mathbb{B}) \), of matrices containing a permutation matrix (Hall matrices),

(iii) \( UT_n(\mathbb{B}) \) and \( LT_n(\mathbb{B}) \), of upper- and lower-triangular boolean matrices.

As mentioned above, there is a natural isomorphism between binary relations on \( \{1, \ldots, n\} \) and \( M_n(\mathbb{B}) \). The monoid \( M_n^{id}(\mathbb{B}) \) arises as the image of the reflexive binary relations under this isomorphism, and this is the primary reason for studying \( M_n^{id}(\mathbb{B}) \). However, it may also be viewed as the monoid of matrices containing a certain pattern of \( 1 \)'s - namely the identity matrix. Similarly, \( M_n^{S}(\mathbb{B}) \) arises as those matrices describing a family of subsets which satisfy Hall’s marriage condition, but also as those matrices containing a permutation matrix.

We compute the largest known ranks of these monoids, which are presented in Table 1 along with whether that rank was previously known or is practically computable by brute force. The rank of \( UT_n(\mathbb{B}) \) is given by the triangular numbers plus 1; this may have already been known but we were unable to find a reference in the
Table 1: The ranks of certain matrix monoids over \( \mathbb{B} \); a * denotes that the rank was already known or is computable by brute force, and a ? indicates that the value is unknown. For \( M^T_n(\mathbb{B}) \), see the proposition below and the discussion preceding it. Note that the ranks given are to generate the monoids as semigroups. The values for \( d(M_n(\mathbb{B})) \) and \( d(M^d_id_n(\mathbb{B})) \) can be found in the OEIS: [50, 51].

There are several obvious relationships between columns in Table 1; we prove that these relationships hold for all \( n \in \mathbb{N}^+ \).

Another submonoid of \( M_n(\mathbb{B}) \), which has attracted recent attention, is the Gossip monoid \( G_n \); see [7, 21] for some recent results and [2, 25] for the problem which inspired \( G_n \). This is typically defined as the monoid generated by the so-called \textit{phone-call} matrices \( C[i,j] \), where

\[
C[i,j]_{kl} = \begin{cases} 
1 & \text{if } k = l \text{ or } \{i, j\} = \{k, l\} \\
0 & \text{otherwise}
\end{cases}
\]

and where \( 1 \leq i, j \leq n \). It is straightforward to show that the phone-call matrices form a minimal generating set for \( G_n \): since every element of \( G_n \) contains the identity matrix and at least two more 1s, the phone-call matrices are indecomposable, and hence contained in any generating set.

There are a number of other submonoids of \( M_n(\mathbb{B}) \) that arise as generalisations of the Hall monoid. If we define the containment closure \( \bar{S} \) of a subsemigroup \( S \leq M_n(\mathbb{B}) \) to be the set of matrices \( A \in M_n(\mathbb{B}) \) containing some element of \( S \), then for any choice of subsemigroup \( S \), \( S \leq \bar{S} \leq M_n(\mathbb{B}) \). In particular, the submonoids \( M^d_id_n(\mathbb{B}) \) and \( M^S_n(\mathbb{B}) \) are of this type. It remains an open problem to determine minimal generating sets for arbitrary containment closures. This problem seems difficult; a more tractable problem may be to restrict \( S \) to subgroups of \( S_n \) or submonoids of transformation matrices. In particular, if we choose \( S \) to be the set of transformation matrices, we obtain the monoid \( M^T_n(\mathbb{B}) \) of matrices containing a transformation matrix. The following is an unpublished result of Marcel Jackson.

**Proposition ([31]).** For \( n \geq 2 \), every minimal generating set for \( M^T_n(\mathbb{B}) \) consists of a set of representatives of the \( J \)-classes of \( M_n(\mathbb{B}) \) which contain a prime matrix, together with a minimal generating set for \( S_n \), and two matrices similar to

\[
\begin{pmatrix} 
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 
\end{pmatrix}
\text{ and }
\begin{pmatrix} 
1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 
\end{pmatrix}.
\]
For $n > 2$, a minimal generating set for $S_n$ has size 2 (for example, elements corresponding to a transposition and an $n$-cycle); for $n \leq 2$ one element is sufficient to generate $S_n$.

### 3.1 Preliminaries

It is known that $d(M_n(\mathbb{B}))$ grows super-exponentially with $n$; see Corollary 3.1.8. In contrast, the subsemigroup of $M_n(\mathbb{B})$ generated by all the regular elements can be generated by the four matrices

$$
T = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix},
U = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix},
E = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix},
F = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

(3.1)

see [57] for further details. We will continue to use these names for these matrices throughout Section 3. Note that together $T$ and $U$ minimally generate $S_n$ for $n \geq 2$; they correspond to the transposition $(1\ 2)$ and the $n$-cycle $(1\ 2\ \ldots\ n)$ respectively. All results which involve $T$ and $U$ in this paper continue to hold if $T$ and $U$ are replaced with any two elements which together generate $S_n$.

We call any matrix similar to $E$ an elementary matrix. In particular, the elementary matrix which consists of the identity matrix with an additional 1 in position $j$ of the $i$th row will be denoted by $E^{i,j}$. If $J_E$ is the $\mathcal{J}$-class of $E$ in $M_n(\mathbb{B})$, then it is easy to show that $J_E = \{E^{i,j} : 1 \leq i, j \leq n\}$ and we call $J_E$ the elementary $\mathcal{J}$-class.

**Theorem 3.1.1 (Devadze’s Theorem [39]).** For $n > 2$, the set $\{T, U, E, F\} \cup P$ is a generating set for $M_n(\mathbb{B})$ of minimum cardinality, where $P$ is any set containing one matrix from every $\mathcal{J}$-class of $M_n(\mathbb{B})$ which contains a prime matrix.

By Devadze’s Theorem, in order to determine the rank of $M_n(\mathbb{B})$ it is sufficient to compute a set of representatives of the $\mathcal{J}$-classes of $M_n(\mathbb{B})$ containing a prime matrix; see Section 3.2 for details of this computation.

As well as the description of a minimal generating set for $M_n(\mathbb{B})$, there are certain additional concepts attached to $M_n(\mathbb{B})$ which do not apply to matrices over arbitrary semirings. Many of these arise from the observation that we may view a vector $v \in \mathbb{B}^n$ as the characteristic function of a subset $s(v)$ of $\{1, \ldots, n\}$, where $i \in s(v)$ if and only if $v_i = 1$. Then we define the union of two vectors $v, w$ to be $s^{-1}(s(v) \cup s(w))$. Note that the union and sum of two vectors coincide in $\mathbb{B}^n$. In fact, since there is a single non-zero element of $\mathbb{B}$, unions, sums, and linear combinations all coincide. Given $v, w \in \mathbb{B}^n$, we also say that $v$ is contained in $w$, and write that $v \leq w$, if $s(v) \subseteq s(w)$.

It is straightforward to verify that if $A \in M_n(\mathbb{B})$, then there is a unique row basis $\lambda(A)$, which is the unique minimal generating set for $A(A)$ under union, consisting of the non-zero $\leq$-minimal rows. The dual statement for column spaces also holds. This yields the following more practical counterpart to Corollary 2.2.2 as an immediate consequence of that corollary.

**Proposition 3.1.2.** Let $A, B \in M_n(\mathbb{B})$. Then $A \mathcal{J} B$ in $M_n(\mathbb{B})$ if and only if $\lambda(A) = \lambda(B)$, and $A \mathcal{R} B$ in $M_n(\mathbb{B})$ if and only if $\rho(A) = \rho(B)$. 


This is key to our strategy for computing a minimal generating set for \( j_1 \) in position two matrices.

Note that transposition gives an anti-automorphism from \( M_n \) of the \( L \)- and \( R \)-structure of \( M_n \) grows extremely rapidly with \( n \), as shown in Table 2 so that it quickly becomes infeasible to compute the \( L \)- and \( R \)-structure of \( M_n \). Note that transposition gives an anti-automorphism from \( M_n \) to itself which exchanges \( L \)- and \( R \)-classes, and hence determining the \( L \)- and \( R \)-structure only requires computation of one of the relations.

Even determining the possible cardinalities of row spaces of matrices in \( M_n \) is a hard problem: see [5, 40, 42, 59, 67]. This is in stark contrast to the row-spaces of matrices over fields, which have dimension a power of the cardinality of the field. For example, the matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix} \in M_n(\mathbb{B})
\]

has row space cardinality 3.

It is significantly more difficult to determine the \( J \)-relation in \( M_n(\mathbb{B}) \) than the \( L \)- or \( R \)-relation. Indeed, the problem of determining whether two matrices are \( J \)-related in \( M_n(\mathbb{B}) \) is NP-hard; see [22, Theorem 2.7] and [43]. The \( J \)-relation on \( M_n(\mathbb{B}) \) is characterised by row space embeddings in the following theorem; a function \( f : \Lambda(A) \rightarrow \Lambda(B) \) is a row space embedding if it respects containment and non-containment, i.e. \( f(v) \leq f(w) \) if and only if \( v \leq w \) for all \( v, w \in \Lambda(A) \).

**Theorem 3.1.3 (Zaretskii’s Theorem [65])**. Let \( A, B \in M_n(\mathbb{B}) \). Then \( J_A \leq J_B \) if and only if there exists a row space embedding \( f : \Lambda(A) \rightarrow \Lambda(B) \).

Zaretskii’s Theorem reduces the problem of determining the \( J \)-order of \( M_n(\mathbb{B}) \) to the problem of digraph embedding, but it should be noted that the size of \( \Lambda(A) \) is bounded above by \( 2^n \), and equality is possible. Computing the \( J \)-structure of \( M_n(\mathbb{B}) \) is challenging; see [6]. The largest \( n \) for which the number of \( J \)-classes of \( M_n(\mathbb{B}) \) is known is 8; see [5].

Considering rows as subsets of \( \{1, \ldots, n\} \), it is natural to consider when rows of a matrix \( A \in M_n(\mathbb{B}) \) are contained in other rows of \( A \). We will call \( A \) row-trim if no non-zero row of \( A \) is contained in another row. Column-trim is defined dually. We say that \( A \) is trim if it is both row-trim and column-trim.

Our interest in trim matrices is due to the following result.

**Lemma 3.1.4 ([39, Lemma 3.1])**. Every prime matrix in \( M_n(\mathbb{B}) \) is trim.

**Proof**. Let \( A \in M_n(\mathbb{B}) \) be prime. Suppose some non-zero row indexed \( k \) of \( A \) is contained in a row indexed \( l \) of \( A \). Define \( X \in M_n(\mathbb{B}) \) to be the matrix such that \( X_{ij} = 1 \) precisely if \( A_{j*} \leq A_{i*} \). Then \(XA = A \), and since \( |X_{is}| \geq 2 \), \( X \notin S_n \), a contradiction. A dual argument shows that no non-zero column of \( A \) is contained in another column.

It is difficult to enumerate prime matrices directly, but comparatively simple to enumerate trim matrices. This is key to our strategy for computing a minimal generating set for \( M_n(\mathbb{B}) \).

The technique used to define the matrix \( X \) in the proof of Lemma 3.1.4 will be useful throughout. Given two matrices \( A, B \in M_n(\mathbb{B}) \), we say that the greedy left multiplier of \( (A, B) \) is the matrix \( C \) containing a 1 in position \( j \) of row \( i \) if and only if \( A_{j*} \leq B_{i*} \). The greedy right multiplier of \( (A, B) \) is the matrix \( D \)

| \( n \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) |
| --- | --- | --- | --- | --- | --- | --- | --- |
| \( |M_n(\mathbb{B})|/\mathcal{L}| \) | 2 | 7 | 55 | 1324 | 120633 | 42299663 | ? |

Table 2: The number of \( \mathcal{L} \)- or \( \mathcal{R} \)-classes in \( M_n(\mathbb{B}) \)
containing a 1 in position $j$ of row $i$ if and only if $A_{si} \leq B_{sj}$. Observe that if $\lambda(A)$ is contained in $\Lambda(B)$, then every row $v$ of $A$ may be written as the linear combination of those rows of $B$ which are contained in $v$; hence $A = CB$ where $C$ is the greedy left multiplier of $(A, B)$. Combining this observation with Corollary 2.2.2 yields the following lemma:

**Lemma 3.1.5.** For any $A, B \in M_n(\mathbb{B})$ and $C$ the greedy left multiplier of $(A, B)$, the following are equivalent:

(i) $A = CB$
(ii) $L_A \subseteq L_B$
(iii) $\Lambda(A) \subseteq \Lambda(B)$.

The dual statement holds for greedy right multipliers, Greens $\mathcal{R}$-order, and column spaces.

A similar property to being trim is being **reduced**. A matrix $A \in M_{m,n}(\mathbb{B})$ is **row-reduced** if no row of $A$ can be written as a union of other rows of $A$. **Column-reduced** is defined dually. We say that $A$ is **reduced** if it is both row-reduced and column-reduced. Since no row of a trim matrix is contained in another row, it follows that no row can be expressed as a union of other rows. A dual argument applies to columns, and hence we have the following lemma.

**Lemma 3.1.6.** Every trim matrix is reduced.

The following lemma describes the $\mathcal{J}$-relation on reduced matrices in $M_n(\mathbb{B})$.

**Lemma 3.1.7** ([54, Theorem 1.8]). Let $A, B \in M_n(\mathbb{B})$ be reduced. Then $A \mathcal{J} B$ if and only if $A$ and $B$ are similar.

Due to the previous lemma, reduced matrices are particularly convenient to compute with. A linear reduction to graph isomorphism, described in Section 3.2, shows that the problem of determining whether two reduced matrices are $\mathcal{J}$-related has at most the same complexity as graph isomorphism. A recent paper of Babai claims that this complexity is at most quasi-polynomial; see [1]. The previous lemma also implies that the $\mathcal{J}$-class of any prime matrix $P$ consists of prime matrices similar to $P$, and we refer to such a $\mathcal{J}$-class as a **prime $\mathcal{J}$-class**. Note that a prime $\mathcal{J}$-class therefore contains at most $(n!)^2$ elements. This observation, combined with Devadze’s Theorem, has the following corollary, which was previously known but does not appear to have been published.

**Corollary 3.1.8.** The size $d(M_n(\mathbb{B}))$ of a minimal generating set for $M_n(\mathbb{B})$ grows super-exponentially with $n$.

**Proof.** This follows from the fact that there at least $2^{\frac{n^2}{4} - O(n)}$ prime boolean matrices in $M_n(\mathbb{B})$ (see [37, Theorem 2.4.1]) and each prime $\mathcal{J}$-class contains at most $(n!)^2$ elements; hence there are super-exponentially many prime $\mathcal{J}$-classes.

The prime matrices of $M_n(\mathbb{S})$ sit directly below the group of units in the $\mathcal{J}$-order on $M_n(\mathbb{S})$ for any semiring $\mathbb{S}$. In the case of $M_n(\mathbb{B})$, there is an additional $\mathcal{J}$-class immediately below $S_n$: the $\mathcal{J}$-class $J_E$ of the elementary matrix $E$ from (3.1). The next result shows that in fact these are all of the $\mathcal{J}$-classes immediately below $S_n$.

Let $\beta_n$ denote the set $\{\lambda(A) : A \in M_n(\mathbb{B}) \setminus S_n\}$ of all possible proper row subspaces of elements of $M_n(\mathbb{B})$, and let $\mathbb{B}^n$ denote the space of all boolean vectors of length $n$. Note that the matrices in $M_n(\mathbb{B})$ with row space equal to $\mathbb{B}^n$ are the permutation matrices.

**Theorem 3.1.9** (cf. Theorem 5.1 in [9]). Let $A \in M_n(\mathbb{B}) \setminus S_n$. Then $\lambda(A)$ is maximal with respect to containment in $\beta_n$ if and only if $A$ is prime or elementary.

We may now prove a slightly stronger form of Devadze’s Theorem.
Theorem 3.1.10 (Devadze’s Theorem [39]). For \( n > 2 \), any minimal generating set for \( M_n(\mathbb{B}) \) is given by \( \{ T', U', E', F' \} \cup P \), where \( T' \) and \( U' \) generate \( S_n \), \( E' \) is elementary, \( F' \) is a matrix similar to \( F \), and \( P \) is a set of representatives of the prime \( \mathcal{J} \)-classes of \( M_n(\mathbb{B}) \). Conversely, any such set generates \( M_n(\mathbb{B}) \) minimally.

Proof. The converse follows immediately from Theorem 3.1.1 noting that \( A \in \langle A', T', U' \rangle \) for any similar matrices \( A, A' \in M_n(\mathbb{B}) \).

Let \( X \) be a minimal generating set for \( M_n(\mathbb{B}) \). Since \( \det(S_n) = 2 \), and \( X \) must contain generators of the group of units \( S_n \), it follows that \( X \) contains two elements which together generate \( S_n \), which we denote by \( T' \) and \( U' \). By [39, Lemma 4.2], \( X \) also contains a set \( P \) of representatives of the prime \( \mathcal{J} \)-classes of \( M_n(\mathbb{B}) \). By [39, Lemma 4.5], and the fact that the elementary \( \mathcal{J} \)-class lies immediately below the group of units, \( X \) must also contain an elementary matrix, say \( E' \). It only remains to show that \( X \) must contain a matrix similar to \( F \). Since none of the elements of \( P \), nor \( E' \), contain a zero row, \( P \cup \{ E' \} \) does not generate \( F \), and \( X \) must contain a matrix with at least one zero row. Since matrices similar to \( F \) have the maximal row spaces amongst matrices containing zero rows, \( X \) must contain a matrix similar to \( F \). \( \square \)

3.2 The full boolean matrix monoid

As mentioned above, in order to compute minimal generating sets for \( M_n(\mathbb{B}) \) it is sufficient to compute sets of representatives of the prime \( \mathcal{J} \)-classes of \( M_n(\mathbb{B}) \). In this section we describe how such a computation may be performed.

The kernel of a function \( f : X \rightarrow Y \) is the equivalence relation containing a pair \( (a, b) \) if and only if \( f(a) = f(b) \). We call a function \( \phi : M_n(\mathbb{B}) \rightarrow M_n(\mathbb{B}) \) a canonical form if \( \ker \phi = \mathcal{J} \). Given a canonical form \( \phi \), the image \( \text{im} \phi \) is a set of representatives of the \( \mathcal{J} \)-classes of \( M_n(\mathbb{B}) \), and the image \( P_\phi = \phi(\{ A \in M_n(\mathbb{B}) : A \text{ prime} \}) \) is a set of representatives of the prime \( \mathcal{J} \)-classes of \( M_n(\mathbb{B}) \).

We wish to enumerate \( P_\phi \) for some canonical form \( \phi \). Roughly speaking, our strategy is to enumerate efficiently as small a superset \( Q_\phi \subseteq \text{im} \phi \) of \( P_\phi \) as is practical, and then to filter \( Q_\phi \) to remove the non-prime matrices. The full image \( \text{im} \phi \) is a set of representatives for the \( \mathcal{J} \)-classes of \( M_n(\mathbb{B}) \); Table 3 demonstrates the growth of \( \text{im} \phi \) for \( n = 1, \ldots, 8 \). The number of trim matrices in Breen form in \( M_n(\mathbb{B}) \) is comparable to the number of \( \mathcal{J} \)-classes for \( n \leq 8 \). Since it is infeasible to compute the image of each of the \( 2^n \) elements of \( M_n(\mathbb{B}) \) for \( n \geq 7 \), we instead compute the images of a smaller set of matrices of a particular form, which contains at least one matrix of every \( \mathcal{J} \)-class of \( M_n(\mathbb{B}) \).

Definition 3.2.1 ([6, Proposition 3.6]). We say that a matrix \( A \in M_{m,n}(\mathbb{B}) \) is in Breen form if it has all of the following properties:

(i) \( A \) is reduced,

(ii) all non-zero rows of \( A \) are at the bottom,

(iii) all non-zero columns of \( A \) are at the right,

(iv) the non-zero rows of \( A \) as binary numbers are a strictly increasing sequence, as are the columns

(v) all ones in the first non-zero row of \( A \) are on the right,

(vi) all ones in the first non-zero column of \( A \) are at the bottom,

(vii) every non-zero row has at least as many ones as the first non-zero row.

This definition appears as a proposition in [6]; Breen defines a matrix in \( M_n(\mathbb{B}) \) to be in standard form if the matrix has minimal value as a binary number in its \( \mathcal{J} \)-class, and proves that such a matrix has the properties of Definition 3.2.1 [6, Proposition 3.6]. This leads directly to the following proposition:

Proposition 3.2.2. In every \( \mathcal{J} \)-class of \( M_n(\mathbb{B}) \), there exists a matrix in Breen form.
Table 3: The sizes of: the monoid $M_n(\mathbb{B})$, the set $B_n$ of matrices in $M_n(\mathbb{B})$ in Breen form; the set $TB_n$ of trim matrices in Breen form in $M_n(\mathbb{B})$; the image $\phi(TB_n)$ of the trim matrices in Breen form under any canonical form $\phi$; and the set $M_n(\mathbb{B})/\mathcal{J}$ of $\mathcal{J}$-classes of $M_n(\mathbb{B})$ (see [5, 6] for details of the last).

| $n$ | $|M_n(\mathbb{B})|$ | $|B_n|$ | $|TB_n|$ | $|\phi(TB_n)|$ | $|M_n(\mathbb{B})/\mathcal{J}|$ |
|-----|-------------------|--------|----------|----------------|----------------|
| 1   | 2                 | 2      | 2        | 2              | 2              |
| 2   | 16                | 4      | 3        | 3              | 3              |
| 3   | 512               | 13     | 5        | 5              | 11             |
| 4   | 65 536            | 146    | 12       | 10             | 60             |
| 5   | 33 554 432        | 7 549  | 141      | 32             | 877            |
| 6   | 68 719 476 736    | 1 660 301 | 34 014  | 394            | 42 944         |
| 7   | $5.6 \times 10^{14}$ | 1 396 234 450 | 7 876 125 | 17 120 845  | 4 256 203 214 |
| 8   | $1.8 \times 10^{19}$ | ? | 18 409 121 852 | 17 120 845  | 4 256 203 214 |

In contrast, being in Breen form is not enough to guarantee that a matrix is minimal. Consequently there is not necessarily a unique matrix in Breen form in any given $\mathcal{J}$-class of $M_n(\mathbb{B})$, as the following example demonstrates.

Example 3.2.3. Let $A, B \in M_n(\mathbb{B})$ be the matrices defined by

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$ 

Then $A$ and $B$ are in the Breen form of Definition 3.2.1. Swapping rows 1 and 2 and columns 3 and 4 shows that $A$ and $B$ are similar, and hence are $\mathcal{J}$-related in $M_n(\mathbb{B})$.

There are significantly fewer than $2^{n^2}$ trim matrices in Breen form in $M_n(\mathbb{B})$, as shown in Table 3, and so it is feasible to enumerate the matrices in Breen form for several values of $n$ for which it is not feasible to enumerate the matrices in $M_n(\mathbb{B})$. Given the set $B_n$ of matrices in Breen form in $M_n(\mathbb{B})$, and a canonical form $\phi$, the image $Q_\phi = \phi(B_n)$ contains $P_\phi$. Later in this section, we will discuss several methods of filtering $Q_\phi$ to obtain the prime matrices.

While theoretically any canonical form $\phi$ is sufficient, the complexity of computing $\phi$ is important. We now describe how to obtain canonical forms $\Phi_n$ that are practical to compute with, using a reduction to bipartite graphs.

Given a matrix $A \in M_n(\mathbb{B})$, we may form the vertex-coloured bipartite graph $\Gamma(A)$ with vertices $\{1, \ldots, 2n\}$, colours

$$\text{col}(v) = \begin{cases} 0 & \text{if } 1 \leq v \leq n, \\ 1 & \text{if } n < v \leq 2n, \end{cases}$$

and an edge from $i$ to $j + n$ if and only if $A_{ij} = 1$. The numbers $\{1, \ldots, n\}$ represent indices of rows and the numbers $\{n + 1, \ldots, 2n\}$ represents indices of columns in the matrix $A$.

It is easy to see that $\Gamma$ is a bijection from $M_n(\mathbb{B})$ to the set $B_{n,n}$ of bipartite graphs with two parts of size $n$, where one part is coloured 0 and the other coloured 1. We call a function $\psi_n : B_{n,n} \rightarrow B_{n,n}$ a canonical form for $B_{n,n}$ if the equivalence classes of ker $\psi$ are the graph-theoretical colour-preserving isomorphism classes of $B_{n,n}$. Computing canonical forms for graphs is a well-studied problem; for a recent article see [44] and the
Proof. We must show that \( \ker \Phi_n = \Gamma^{-1} \psi_n \Gamma \) are canonical forms when restricted to reduced matrices.

The functions \( \Phi_n = \Gamma^{-1} \psi_n \Gamma \) are canonical forms when restricted to reduced matrices.

Given \( v \in \mathbb{B}^n \), it will convenient to denote the number represented by \( v \) in binary as \( \text{num}(v) \). We will also write \( \text{vec} \) for \( \text{num}^{-1} \).

We now describe how to backtrack through matrices in Breen form to find a superset of prime matrices \( Q_{\Phi_n} \). This may be seen as a depth-first traversal of a (non-rooted) tree, with nodes \( m \times n \) matrices \( (m \leq n) \) and leaves \( n \times n \) matrices.

Algorithm 3.2.5. Backtracking for canonical forms.

Input: A natural number \( n \).

Output: A set \( Q_{\Phi_n} \), with \( P_{\Phi_n} \subseteq Q_{\Phi_n} \subseteq \text{im} \Phi_n \).

1. Assume that we are at a node \( A \) of dimension \( m \times n \), and the index of the first non-zero row of \( A \) is \( f \leq m \).

2. If \( m = n \), the non-zero columns of \( A \) form a strictly increasing sequence under \( \text{num} \), and \( A \) is column-reduced, then add \( \Phi_n(A) \) to \( X \), the set of matrices to return.

3. If \( m < n \), then for each \( x \in \{ \text{num}(A_{ms}) + 1, \ldots, 2^n - 1 \} \), if:

   (i) \( \text{vec}(x) \) does not contain \( A_{ls} \) for any \( 1 \leq l \leq m \), and
   (ii) \( \text{vec}(x) \) has at least as many ones as \( A_{fs} \), and
   (iii) for all column indices \( 1 \leq i < j \leq m \) such that \( A_{si} = A_{sj} \), if \( \text{vec}(x)_i = 1 \) then \( \text{vec}(x)_j = 1 \),

   then set the current node to be the matrix obtained from \( A \) by adjoining \( \text{vec}(x) \) as the last row, and return to step 1.

4. after every \( x \) has been processed, return to the previous node (if any) and carry out step 3 for the next \( x \) at that node (if any).

\footnote{In fact, a slightly-modified version of bliss which avoids repeated memory allocation was used; this is available at \url{https://github.com/digraphs/bliss}}
Having initialised step 1 with each $m \times n$ matrix ($m \geq 1$) consisting of $m - 1$ zero rows followed by a row containing some non-zero number of 1s on the right, return $X$.

**Lemma 3.2.6.** The output of Algorithm 3.2.5 with input $n$, is a subset of $\text{im} \Phi_n$ containing $P_{\Phi_n}$. Moreover, Algorithm 3.2.5 does not compute $\Phi_n(A)$ of any matrix $A$ not in Breen form.

**Proof.** We will prove that the $A$s of step 4 for which $\Phi_n(A)$ are calculated are precisely the set of trim matrices in Breen form; since every prime matrix is trim (Lemma 3.1.4) and every $J$-class contains a matrix in Breen form, this implies that the output contains $P_{\Phi_n}$. We will first prove that each such $A$ is in Breen form.

Define a matrix to be in *quasi-Breen form* if it satisfies each property of Definition 3.2.1 other than being column reduced and the non-zero columns forming a strictly-increasing sequence. We will prove that each node $A$ visited in the algorithm is trim and in quasi-Breen form. Note that the matrices that the algorithm is initialised with are all trim and in quasi-Breen form, so we must simply prove that passing from a node $A$ which is trim and in quasi-Breen form to a node $A'$ by adding a row $\text{vec}(x)$ in step 3 preserves these properties.

Trimness is preserved due to condition (i) of step 3. Since all trim matrices are reduced, $A'$ is also reduced. The conditions on non-zero rows being at the bottom and forming a strictly increasing sequence of binary numbers are satisfied by choosing $x$ from the range $\{\text{num}(A_m^*) + 1, \ldots, 2^n - 1\}$. The conditions on non-zero columns being on the right follows from requirement (iii) of step 3; note that this requirement also forces the columns to appear in (not-necessarily-strictly) increasing order. The first non-zero column contains the most significant digit of the rows as binary numbers, and since the rows of $A'$ are increasing the set of rows with that digit equal to 1 must be contiguous and at the end of $A'$. Hence all of the ones in the first non-zero column of $A'$ are at the bottom.

Since each leaf $A$ visited is trim and in quasi-Breen form, the two conditions of step 2, that the non-zero columns form a strictly increasing sequence and that $A$ is column-reduced, guarantee that $A$ is in standard form. Hence, we only calculate the canonical form $\Phi_n(A)$ if $A$ is trim and in Breen form.

It remains to prove that every trim matrix $A \in M_n(\mathbb{B})$ in Breen form is visited as a node in the enumeration. First, note that the first $m$ rows of any trim, standard-form matrix $A \in M_n(\mathbb{B})$ form an $m \times n$ trim matrix in quasi-Breen form. Also, the zero-rows of $A$ together with the first non-zero row form one of the matrices with which step 1 is initialised. It simply remains to show that from each $m \times n$ node $X$ consisting of the first $m$ rows of $A$, we visit the $(m + 1) \times n$ node $X'$ consisting of the first $m + 1$ rows of $A$. It is easy to verify that each of the three conditions in step 3 is satisfied by row $m + 1$ of $A$, and hence $X'$ is visited.

Given $\Phi_n$ and $Q_{\Phi_n}$, the final step is to detect when an element of $Q_{\Phi_n}$ is prime. We present two algorithms for doing so, in Algorithm 3.2.7 and Algorithm 3.2.11.

**Algorithm 3.2.7.** Filtering canonical forms by row spaces.

**Input:** A set $Q_{\Phi_n}$, containing the images $P_{\Phi_n}$ of the prime matrices of $M_n(\mathbb{B})$ under $\Phi_n$, and not containing any permutation matrices.

**Output:** The set $P_{\Phi_n}$.

1. Compute $X = \{\Lambda(A\alpha) : A \in Q_{\Phi_n} \cup \{E\}, \alpha \in S_n\}$
2. For every $A \in Q_{\Phi_n}$, and for every $\Lambda(B\beta) \in X$, if $A \neq B$ and $\Lambda(A) \not< \Lambda(B\beta)$ then discard $\Lambda(A\alpha)$ from $X$ for all $\alpha \in S_n$.
3. Output the set of non-elementary elements $A$ such that $\Lambda(A)$ remains in $X$ after the previous step.

**Lemma 3.2.8.** The output of Algorithm 3.2.7 is a set of representatives of prime $J$-classes.

**Proof.** Since the $J$-class of a prime matrix consists only of similar matrices, the set $\{\Lambda(A\alpha) : A \in P_{\Phi_n}, \alpha \in S_n\} \subset X$ contains all row spaces of prime matrices in $M_n(\mathbb{B})$, and hence so does $X$. Similarly, $X$ contains the row space of every elementary matrix. Hence, the elements that are maximal in $X$ are precisely the maximal elements of $\beta_n = \{\Lambda(A) : A \in M_n(\mathbb{B}) \setminus S_n\}$ and thus by Theorem 3.1.9 correspond to primes and elementary matrices. Since $\Lambda(A)$ remains in $X$ after step 2 precisely when $\Lambda(A)$ (and $\Lambda(A\alpha)$, for all $\alpha \in S_n$) is maximal in $X$, the output is $P_{\Phi_n}$. \qed
Table 4: The number of row spaces generated during Algorithm 3.2.7 when given input $Q_{\Phi_n}$, the output of Algorithm 3.2.5.

| $n$ | $|Q_{\Phi_n}|$ | $|X|$ |
|-----|---------------|------|
| 3   | 6             | 91   |
| 4   | 11            | 588  |
| 5   | 33            | 8194 |
| 6   | 395           | 570 636 |
| 7   | 34 015        | 342 915 296 |
| 8   | 17 120 845   | ?    |

Table 5: Runtimes of the algorithms described in this section, for $3 \leq n \leq 8$. Runtimes for $n = 1, 2$ were excluded as they are essentially immediate after accounting for system factors. Matrices were prefiltered by generators of $M_{n-1}(B)$ extended as in Lemma 3.2.9, in particular a good balance was found to be taking those extended matrices with the 13 largest row spaces amongst the extended row spaces. Each algorithm was executed on a cluster of 60 2.3GHz AMD Opteron 6376 cores with 512GB of ram. Algorithm 3.2.11 involves launching GAP multiple times, which took approximately 6s in each run. For $n = 8$, Algorithm 3.2.7 with prefiltering was run on a machine with 32 AMD Opteron 6276 cores and 192GB of RAM.

| $n$ | Algorithm 3.2.5 | Algorithm 3.2.7 | Algorithm 3.2.11 | prefiltering | Algorithm 3.2.7 with prefiltering | Algorithm 3.2.11 with prefiltering |
|-----|-----------------|-----------------|------------------|--------------|---------------------------------|----------------------------------|
| 3   | 5.4ms           | 12ms            | 6.5s             | 23ms         | 11ms                            | 6.5s                             |
| 4   | 5.6ms           | 13ms            | 7.0s             | 25ms         | 11ms                            | 6.7s                             |
| 5   | 9.2ms           | 16ms            | 7.2s             | 33ms         | 16ms                            | 7.3s                             |
| 6   | 150ms           | 216ms           | 8.6s             | 80ms         | 171ms                           | 8.7s                             |
| 7   | 55s             | 1h 20m          | 1m 11s           | 28s          | 1h 9m                           | 1m 8s                            |
| 8   | 30h 52m         | -               | -                | 5h 30m       | approx. 30 days                 | 3.9 days                         |

Note that step 2 of Algorithm 3.2.7 requires $O(|Q_{\Phi_n}| |X|)$ comparisons. The size of $X$ grows extremely rapidly with $n$ as shown in Table 4, and so this algorithm is only suitable for small $n$.

Using Algorithm 3.2.5 and Algorithm 3.2.7, minimal generating sets for $n \leq 7$ may be obtained.

For $n = 8$ Algorithm 3.2.7 is no longer sufficient, and it is necessary to use heuristics to reduce the size of the input set $Q_{\Phi_n}$. The simplest way to do this is to select a small subset $Y \subset Q_{\Phi_n}$ of matrices with large row spaces, generate the row spaces $Z = \{\Lambda(A\alpha) : A \in Y, \alpha \in S_n\} \subset X$, and check whether $\Lambda(B)$ is properly contained in any element of $Z$ for each element $B \in Q_{\Phi_n}$. It is also worthwhile to filter $Q_{\Phi_n}$ by checking containment in the row spaces of all the column permutations of a known set of prime matrices, such as those described in the following lemma.

**Lemma 3.2.9.** Let $A$ be a prime matrix in $M_{n-1}(B)$. Extend $A$ to a matrix $B \in M_n(B)$ by adding a row of zeros at the bottom and a column of zeros on the right, then setting $B_{n,n} = 1$. Then $B$ is prime in $M_n(B)$.

**Proof.** Let $C$ be a matrix with row space maximal in $\beta_n$, such that $\Lambda(B) \subseteq \Lambda(C)$. The last row of $B$ must also be in $C$ since it is a minimal in $B^n$. Now $\Lambda(A)$ has a unique basis of $n - 1$ rows which must be contained in both $B$ and $C$, so we have $\lambda(B) = \lambda(C)$; hence $\Lambda(B)$ is maximal in $\beta_n$. Since $B$ is not elementary, it is prime.

For $n = 8$ prefiltering by some of the large row spaces and extended prime matrices is enough to obtain a minimal generating set, although the computation is extremely lengthy; see Table 5 for some details. A significant improvement can be obtained by using Zaretskii’s Theorem.
Let \( A \in M_n(\mathbb{B}) \). The **graph** of \( \Lambda(A) \) is the directed graph with vertices \( \Lambda(A) \) and an edge from \( v \) to \( w \) if and only if \( v \leq w \).

It is an immediate corollary of Zaretskii’s Theorem that \( J_A \leq J_B \) if and only if there exists a homomorphic embedding of the graph of \( \Lambda(A) \) into the graph of \( \Lambda(B) \) which respects non-edges. An efficient and optimised search for such embeddings is implemented in \( [3] \).

For practical computational purposes, it is useful to add extra structure to the row space graphs to guide searches for embeddings. The **augmented graph** of \( \Lambda(A) \) is the disjoint union of the graph of \( \Lambda(A) \) with the empty graph on the vertices \( C = \{ c_i : 1 \leq i \leq n \} \), with an edge from \( v \in \Lambda(A) \) to \( c_i \) if and only if \( v_i = 1 \).

**Lemma 3.2.10.** Let \( Q \) be a superset of a canonical set of prime matrices \( P \) which does not contain a permutation matrix, and let \( A \in Q \). Then \( A \) is not prime or elementary if and only if for some \( B \in (Q \cup \{ E \}) \setminus \{ A \} \) there exists a digraph embedding \( \phi \) from the augmented graph of \( \Lambda(A) \) into the augmented graph of \( \Lambda(B) \) which permutes \( \{ c_i : 1 \leq i \leq n \} \) and respects non-adjacency.

**Proof.** Let \( A \) not be prime or elementary; then \( \Lambda(A) \) is contained in the row space of some column permutation \( \alpha \) of some \( B \in Q \setminus \{ A \} \). Then the embedding that extends the map \( c_i \rightarrow c_{\alpha^{-1} i} \) has the properties required. Conversely, if such a map \( \phi \) exists, the permutation \( \alpha \) induced by the restriction to \( C \) has the property that \( \Lambda(B\alpha^{-1}) \) contains \( \Lambda(A) \), and hence by Theorem 3.1.9 \( A \) is not prime or elementary.

Note that such an embedding \( \phi \) must also map a vector containing \( i \) ones to another containing \( i \) ones in order to preserve adjacency and non-adjacency with the set \( C \).

We can therefore use the following improved algorithm to filter canonical supersets of prime matrices.

**Algorithm 3.2.11.** Filtering canonical forms by digraph embeddings.

**Input:** A set \( Q_{\Phi_n} \), containing the images \( P_{\Phi_n} \) of the prime matrices of \( M_n(\mathbb{B}) \) under \( \Phi_n \), and not containing any permutation matrices.

**Output:** The set \( P_{\Phi_n} \).

1. Generate the set \( G \) of augmented graphs of row spaces of matrices in \( Q \).

2. For every \( K, L \in G \), if there exists an embedding of \( K \) into \( L \) as in Lemma 3.2.10 then discard \( K \) from \( G \).

3. Output \( X \subset Q \), the set of non-elementary elements \( A \) with corresponding graphs remaining in \( G \) after the previous step.

Note that this is in effect the same computation as in Algorithm 3.2.7; it replaces a brute-force search through all permutations of columns with a guided search for an appropriate permutation. It is also superior in that no more data has to be computed, unlike Algorithm 3.2.7 where new row spaces must be produced for each column permutation. Additionally, information about the graphs can be reused (in particular their automorphism group). However, this is not as useful as it might seem, since the automorphism group of prime row spaces appears to almost always be trivial.

Using this method of filtration, a minimal generating sets for \( M_n(\mathbb{B}) \), \( 6 \leq n \leq 8 \) have been computed; the size of such generating sets is contained in Table 4. It seems unlikely that these methods can produce minimal generating sets for \( n > 8 \). The generating sets obtained from this algorithm are obtainable at \( [46] \), along with code to produce them.

### 3.3 Reflexive boolean matrices

An interesting submonoid of \( M_n(\mathbb{B}) \) is the monoid \( M_n^{id}(\mathbb{B}) \) of reflexive boolean matrices, that is, boolean matrices with an all-1 main diagonal. Minimal generating sets for \( M_n^{id}(\mathbb{B}) \) are significantly larger than those of \( M_n(\mathbb{B}) \).

**Theorem 3.3.1.** The unique minimal monoid generating set for \( M_n^{id}(\mathbb{B}) \) consists of the set of elementary matrices in \( M_n^{id}(\mathbb{B}) \) together with the set of indecomposable trim matrices in \( M_n^{id}(\mathbb{B}) \).
In order to prove this theorem, we must understand the $\mathcal{J}$-relation on $M_n^{id}(\mathbb{B})$. Let $A,B$ be two matrices belonging to $M_n^{id}(\mathbb{B})$. Observe that since $A,B$ both contain the identity matrix, the product $AB$ must contain both every row of $A$ and every column of $B$; hence $A \leq AB$ and $B \leq AB$. This leads to the following well-known lemma:

**Lemma 3.3.2.** $M_n^{id}(\mathbb{B})$ is $\mathcal{J}$-trivial.

**Proof.** If $A \not\mathcal{J} B$, then $A \leq B$ and $B \leq A$ with respect to containment. Hence $A = B$. \qed

Note that it also follows that $AB$ has at least as many 1s as the maximum number of 1s in $A$ or $B$. It also follows from the lemma that any decomposable element $A$ is decomposable into a product of elements not $\mathcal{J}$-related to $A$.

We now prove several results relating to decomposability of elements in $M_n^{id}(\mathbb{B})$.

**Lemma 3.3.3.** Every matrix in $M_n^{id}(\mathbb{B})$ that is neither trim nor elementary is decomposable in $M_n^{id}(\mathbb{B})$.

**Proof.** Let $A \in M_n^{id}(\mathbb{B})$ be neither trim nor elementary. Since $A$ is not trim, it is either not row-trim or not column-trim (or both). Suppose that $A$ is not row-trim; then there exist some distinct $1 \leq i, j \leq n$ such that the $i$th row $A_{i*}$ is contained in the $j$th row $A_{j*}$. Let $B$ be the matrix obtained by setting entry $i$ of row $j$ of $A$ to be equal to 0, that is

$$B_{kl} = \begin{cases} 0 & k = j \text{ and } l = i \\ A_{kl} & \text{otherwise}. \end{cases}$$

Since $i \neq j$ and $A$ is reflexive, so too is $B$. Now $B_{j*} \cup A_{i*} = A_{j*}$, and so $A = E_{j,i}B$. By Lemma 3.3.2 neither $E_{j,i}$ nor $B$ is $\mathcal{J}$-related to $A$.

If, instead, $A$ is column trim, then the same argument applied to the transpose $A^T$ demonstrates that $A^T$ may decomposed in such a way, and thus $A$ may also. \qed

**Corollary 3.3.4.** An indecomposable element of $M_n^{id}(\mathbb{B})$ is either trim or elementary.

**Lemma 3.3.5.** Elementary matrices are indecomposable in $M_n^{id}(\mathbb{B})$.

**Proof.** Elementary matrices are precisely those matrices in $M_n^{id}(\mathbb{B})$ containing $n+1$ ones. As noted above, the product of two matrices in $M_n^{id}(\mathbb{B})$ contains at least as many 1s as the maximum number of 1s in a factor. It follows that if an elementary matrix is decomposable it is decomposable into other elementary matrices. However, it is routine to verify that the number of 1s in a product of any two distinct elementary matrices is at least $n+2$, and that each reflexive elementary matrix is idempotent. Hence elementary matrices are indecomposable. \qed

**Proof of Theorem 3.3.1.** Let $T$ denote the set of indecomposable trim matrices, and $\mathcal{E}$ denote the set of reflexive elementary matrices. By Lemma 3.3.2 and Corollary 3.3.4 it follows from Lemma 2.3.1 that $(T \cup \mathcal{E}) = M_n^{id}(\mathbb{B})$. Since each element of $T \cup \mathcal{E}$ is indecomposable by definition or by Lemma 3.3.5, $T \cup \mathcal{E}$ must be contained in any generating set for $M_n^{id}(\mathbb{B})$, and hence is minimal. \qed

We now discuss how we may compute minimal generating sets for $M_n^{id}(\mathbb{B})$. Since it is easy to enumerate the reflexive elementary matrices, the problem is determining the set of indecomposable trim matrices in $M_n^{id}(\mathbb{B})$. The following lemma gives a method for testing indecomposability.

**Lemma 3.3.6.** A trim matrix $A \in M_n^{id}(\mathbb{B})$ is decomposable if and only if it may be written as a product $A = BC$ of matrices $B,C \in M_n^{id}(\mathbb{B}) \setminus \{I,A\}$ where the rows of $C$ are intersections of rows of $A$, and $B$ is the greedy left multiplier of $(C,A)$. 

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Proof. The reverse direction is immediate. For the forward direction, suppose that \( A \) is decomposable; that is, there exist matrices \( Y, Z \in M_n(\mathbb{B}) \setminus \{ I, A \} \) such that \( A = YZ \). For \( 1 \leq i \leq n \), define \( K_i \) to be the set of positions of 1s in the \( i \)th column of \( Y \), and define \( C \) to be the matrix with rows \( C_{i*} = \cap_{j \in K_i} A_{j*} \). Now since \( A = YZ \), for all \( j \in K_i \) we have \( Z_{i*} \leq A_{j*} \). It follows that \( Z_{i*} \leq C_{i*} \leq A_{j*} \) for all \( j \in K_i \). The row \((YC)_{i*}\) consists of the union of those rows \( C_{j*} \) such that \( Y_{ij} = 1 \). The condition that \( Y_{ij} = 1 \) is equivalent to \( i \in K_j \); note the change in indices from previous uses of \( K \). For each such \( j \), we have \( C_{j*} \leq A_{j*} \) by the inequality above. It follows that \( YC \leq A \). But from the same inequality above, we then have \( A = YZ \leq YC \leq A \); hence \( A = YC \). By Lemma 3.1.5, \( A = BC \) where \( B \) is the greedy left multiplier of \((C,A)\). It remains to show that \( B,C \in M_n(\mathbb{B}) \setminus \{ I, A \} \). Since the greedy left multiplier \( B \) is the maximal (by containment) matrix whose product with \( C \) is \( A \), we have \( Y \leq B \) and hence \( B \) is not the identity matrix. Similarly, \( Z \leq C \) and so \( C \) is not the identity matrix. Since \( A \) is trim, if either \( B \) or \( C \) were equal to \( A \) then the product \( BC \) would have more 1s than \( A \) does, but \( A = BC \). \( \square \)

In order to determine whether a trim matrix \( A \) is decomposable it therefore suffices to:

1. generate all matrices \( C \) whose rows are intersections of rows of \( A \), and for each \( C \)
2. test whether the product \( BC \) of \( C \) with the greedy left multiplier \( B \) of \((A,C)\) is equal to \( A \), and \( B,C \not\in \{I,A\} \).

If no such matrices \( B,C \) are found, then \( A \) is indecomposable. This method avoids computing unnecessary products, but still quickly becomes infeasible to use for all trim matrices in \( M_n(\mathbb{B}) \).

In order to reduce the time spent checking matrices using Lemma 3.3.6, we would like to only check the smallest necessary set of representatives of some equivalence relation. The most obvious choice is to take canonical representatives in \( M_n(\mathbb{B}) \). However, the following example illustrates that two reduced reflexive matrices can belong to the same \( J \)-class of \( M_n(\mathbb{B}) \) while only one of them is decomposable in \( M_n(\mathbb{B}) \); hence we can not use the same canonical forms \( \Phi \) as were used in the algorithms in Section 3.2.

Example 3.3.7. Let

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
\end{pmatrix}.
\]

Then \( A \) and \( B \) belong to \( M_5(\mathbb{B}) \), and \( B \) may be obtained by exchanging columns 3 and 5 of \( A \). However, it can be shown that \( A \) is indecomposable whilst

\[
B = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
\end{pmatrix}.
\]

While the canonical forms \( \Phi \) from Section 3.2 cannot be used, the following lemma shows that it is still possible to reduce the space of matrices that must be checked.

Lemma 3.3.8. Let \( A,B \in M_5(\mathbb{B}) \) be such that \( A = P^{-1}BP \) for some permutation matrix \( P \in S_n \) (i.e. \( A \) is obtained by permuting the rows and columns of \( B \) by the same permutation). Then \( A \) is decomposable in \( M_5(\mathbb{B}) \) if and only if \( B \) is decomposable in \( M_5(\mathbb{B}) \).
Proof. Suppose that $B = XY$ for $X,Y \in M_n^\text{id}(\B) \setminus \{I,B\}$. Then $P^{-1}XP, P^{-1}YP \in M_n^\text{id}(\B) \setminus \{I,A\}$ and $P^{-1}XPP^{-1}YP = A$. The proof of the other direction is dual, since $PAP^{-1} = B$. \hfill \square

Hence it is sufficient to consider representatives of the equivalence which relates any two matrices which are similar under the same row and column permutation. In order to compute these representatives, we modify the construction of the bipartite graphs $\Gamma$ of Section 3.2.

Given a matrix $A \in M_n^\text{id}(\B)$, we form the vertex-coloured tripartite graph $\Gamma_\text{id}(A)$ with vertices $\{1, \ldots, 3n\}$, colours

$$\text{col}(v) = \begin{cases} 
0 & \text{if } 1 \leq v \leq n, \\
1 & \text{if } n < v \leq 2n, \\
2 & \text{if } 2n < v \leq 3n,
\end{cases}$$

an edge from $i$ to $j + n$ if and only if $A_{ij} = 1$ for $1 \leq i \leq n$, and an edge from $i + 2n$ to $i$ and $i + n$ for $1 \leq i \leq n$. The numbers $\{1, \ldots, n\}$ represent indices of rows, and the numbers $\{n+1, \ldots, 2n\}$ represents indices of columns in the matrix $A$. The additional vertices $\{2n + 1, \ldots, 3n\}$, adjacent to both the corresponding row and column nodes, force an isomorphism of $\Gamma_\text{id}(A)$ to induce the same permutation on rows and columns of $A$ in the same way that a permutation was induced in Lemma 3.2.4. As before, we may obtain canonical forms $\Psi_n$ for the graphs $\Gamma_\text{id}(A)$ via bliss. Since $\Gamma_\text{id}$ is clearly injective, we may compute the functions $\Xi_n = \Gamma_\text{id}\Psi_n\Gamma_\text{id}^{-1}$. It is easy to show that the equivalence classes $\ker\Xi$ are precisely the classes of matrices which are similar under permuting rows and columns by the same permutation.

As in Section 3.2, we wish only to enumerate matrices of a certain form. A similar argument to [6, Proposition 3.6] shows that by permuting the rows and columns of matrices by the same permutation, matrices in $M_n^\text{id}(\B)$ can be put in the following reflexive Breen form:

(i) all 1s in the first row of $A$ are on the left,

(ii) no row has fewer ones than the first row,

(iii) for each row $A_{is}$, if $A_{ij} = 1$ then for each $l \in \{1, \ldots, j\}$ there exists $k \in \{1, \ldots, i\}$ such that $A_{kl} = 1$.

As in Example 3.2.3 it is possible for two distinct matrices in this form to have the same value under $\Xi_n$. Now, given $\Xi_n$, a similar backtrack search to Algorithm 3.2.5 allows appropriate representatives of matrices to be enumerated.

We then have the following algorithm for finding a minimal generating set for $M_n^\text{id}(\B)$:

Algorithm 3.3.9. Computing the minimal generating set for $M_n^\text{id}(\B)$

**Input:** A natural number $n$.

**Output:** A minimal generating set for $M_n^\text{id}(\B)$.

1. Enumerate the trim reflexive boolean matrices in reflexive Breen form using the analogue of Algorithm 3.2.5, storing canonical representatives under a row and column permutation in a set $S$.

2. Filter out the decomposable matrices in $S$ using Lemma 3.3.6, leaving a set $T$ of trim matrices that are not decomposable.

3. Return $T$ together with the reflexive elementary matrices.

Using this algorithm, we can calculate the sizes of minimal generating sets up to $n = 6$; these are contained in Table 1.

There are a small number of matrices in $M_7^\text{id}(\B)$ for which the approach based on Lemma 3.3.6 is too inefficient; we chose to use a different method to test 12 matrices in total. Observe that a matrix $A$ is decomposable into a product of generators $X_1X_2\ldots X_k$ from some minimal generating set if and only if $\alpha A\alpha^{-1} = \alpha(X_1X_2\ldots X_{k-1})\alpha^{-1}\alpha X_k\alpha^{-1}$ for all permutations $\alpha$. Since the set $S$ of Algorithm 3.3.9 contains
\(\alpha X_k \alpha^{-1}\) for some \(\alpha \in S_n\), we can detect if \(A\) is decomposable by testing whether \(\alpha A \alpha^{-1} = CB\) for any \(\alpha \in S_n\), \(B \in S\) and with \(C\) the greedy left multiplier of \((\alpha A \alpha^{-1}, B)\). A brute force approach based on this observation is sufficient to filter the 12 difficult matrices for \(n = 7\). Although this suggests that the brute-force method is superior to that of Lemma 3.3.6, this is not in practice the case; for most matrices the approach based on Lemma 3.3.6 is far more efficient. The 12 matrices that are particularly non-susceptible to this method for \(n = 7\) have many more combinations of intersections of rows than the other matrices (on average, roughly 10 000 times more).

### 3.4 Hall matrices

The Hall monoid is the submonoid of \(M_n(B)\) consisting of matrices which contain a permutation matrix. These matrices correspond to instances of the Hall marriage problem that have a solution, and are thus referred to as Hall matrices; see [58, 8, 63, 12] for further reading. We will denote the Hall monoid by \(M^S_n(B)\). For convenience, we shall often simply say that a Hall matrix contains a permutation when it contains the corresponding permutation matrix.

The main result of this section is the following theorem. Unlike the monoid of reflexive boolean matrices \(M_{\text{id}}^n(B)\), the Hall monoid \(M^S_n(B)\) has minimal generating sets that are strongly related to the minimal generating sets for \(M_n(B)\).

**Theorem 3.4.1.** Every minimal generating set for \(M^S_n(B)\) is obtained by removing a matrix similar to \(F\) from a minimal generating set for \(M_n(B)\). That is, every minimal generating set for \(M^S_n(B)\) consists of a set of representatives \(P\) of the prime \(J\)-classes of \(M_n(B)\) together with a minimal set of generators for the group of units and an elementary matrix.

In order to prove this theorem, we will need the following classical result, restated in our context.

**Theorem 3.4.2** (Hall’s Marriage Theorem [26, Theorem 1]). Let \(A \in M_n(B)\). Then \(A\) is a Hall matrix if and only if every union of \(k\) rows contains at least \(k\) ones, for \(1 \leq k \leq n\).

We shall say that a subset \(X\) of the rows of a matrix satisfies the Hall condition if the union of the rows in \(X\) contains at least \(|X|\) ones, and that a matrix satisfies the Hall condition if every subset of the rows satisfies the Hall condition.

Similarly to the case for reflexive matrices, the fact that Hall matrices contain permutations gives useful information on products of matrices. Given two matrices \(A, B \in M^S_n(B)\), both \(A\) and \(B\) contain a permutation matrix; hence \(AB\) contains a row-permuted copy of \(B\) and a column-permuted copy of \(A\). It follows that \(AB\) contains at least as many 1s as the maximum number of 1s in \(A\) or \(B\).

The \(J\)-relation for \(M^S_n(B)\) is easily described by the following lemma.

**Lemma 3.4.3** ([8, Theorem 2]). Two matrices \(A, B \in M^S_n(B)\) are \(J\)-related in \(M^S_n(B)\) if and only if they are similar.

**Proof.** The reverse direction is clear. For the forward direction, suppose that \(A = SBT\) and \(B = UAV\) for matrices \(S, T, U, V \in M_n^S(B)\). Then \(BT\) contains a column-permuted copy of \(B\), and hence \(A\) contains a row- and column-permuted copy of \(B\). Since, similarly, \(B\) contains a row- and column-permuted copy of \(A\), it follows that \(A\) is a row- and column-permutation of \(B\).

We will prove Theorem 3.4.1 through a series of lemmas. The first thing to prove is that every element specified actually belongs to \(M^S_n(B)\); this is clear for elements of the group of units and elementary matrices but less clear for prime matrices.

**Lemma 3.4.4.** Every prime matrix is Hall.
Proof. Let \( A \in M_n(\mathbb{B}) \) be prime. By Theorem 3.4.2 it suffices to show that there is no subset of rows of \( A \) of size \( k \) such that the union of these rows contains fewer than \( k \) ones. This is equivalent to not containing any \( k \times (n-k+1) \) submatrix containing only 0. As a consequence of the discussion after [9, Definition 2.4], the set of rows of \( A \) containing more than a single 1 do not contain any \( k \times (n-k) \) submatrix containing only 0; hence \( A \) does not contain a \( k \times (n-k+1) \) submatrix, and \( A \in M_n^S(\mathbb{B}) \).

In order to apply Lemma 2.3.1 we must be able to decompose certain elements of \( M_n^S(\mathbb{B}) \) into products of elements that lie above the given element in the \( \mathcal{J} \)-order. The following two lemmas are the first steps in finding such a decomposition.

**Lemma 3.4.5.** Let \( A \) be a non-trim matrix belonging to \( M_n^S(\mathbb{B}) \). Then there exist \( B, C \in M_n^S(\mathbb{B}) \) such that \( A = CB \), and neither \( B \) nor \( C \) is \( \mathcal{J} \)-related to \( A \).

**Proof.** We may assume that \( A \) contains the identity permutation, since \( A \) is decomposable into such matrices \( B \) and \( C \) if and only if every matrix similar to \( A \) is decomposable into matrices similar to \( B \) and \( C \). The lemma now follows using the same proof as in Lemma 3.3.3.

It will be convenient in the following proofs to define \( e_i \) to be the boolean vector of length \( n \) with a single 1 in position \( i \), for \( 1 \leq i \leq n \).

**Lemma 3.4.6.** Let \( A \) be a trim matrix belonging to \( M_n^S(\mathbb{B}) \) which is not prime, elementary, or a permutation matrix. Then there exist \( B \in M_n^S(\mathbb{B}) \) and \( C \in M_n(\mathbb{B}) \) such that \( A = CB \), neither \( B \) nor \( C \) is \( \mathcal{J} \)-related to \( A \) in \( M_n^S(\mathbb{B}) \), and \( C \) contains a 1 in every column.

**Proof.** As in Lemma 3.4.5, we may assume that \( A \) contains the identity permutation. Since \( A \) is not prime, elementary, or a permutation matrix, it has non-maximal row space in \( \beta_n = \{ \Lambda(A) : A \in M_n(\mathbb{B}) \setminus S_n \} \). Let \( B \) be a maximal non-permutation matrix in \( M_n(\mathbb{B}) \) whose row space contains the row space of \( A \).

Letting \( C \in M_n(\mathbb{B}) \) be the greedy left multiplier of \( (A, B) \), we have \( A = CB \) by Lemma 3.1.5. Suppose that \( C \) is \( \mathcal{J} \)-related to \( A \); then by Lemma 3.4.3 \( C \) is similar to \( A \). Then since \( A \) is trim, so too is \( C \). Since \( B \) was chosen to be a non-permutation matrix, the product \( CB \) contains more 1s than \( C \). But \( A = CB \), and \( C \) is similar to \( A \), a contradiction. Hence \( C \) is not \( \mathcal{J} \)-related to \( A \). Since \( \Lambda(A) \subseteq \Lambda(B) \), \( B \) is not \( \mathcal{J} \)-related to \( A \) in \( M_n^S(\mathbb{B}) \) and thus is not \( \mathcal{J} \)-related to \( A \) in \( M_n^S(\mathbb{B}) \).

If \( C \) contains a 1 in every column, we are done. However, there is no reason that this must be the case.

Suppose that the columns of \( C \) indexed by \( X = \{ c_1, c_2, \ldots, c_k \} \) do not contain a 1, and that \( B \) contains the permutation \( \alpha \). For \( 1 \leq i \leq k \), define \( B'_{c_i} = e_{\alpha(c_i)} \). Define the remaining rows of \( B' \) to be the corresponding rows of \( B \). Then \( B' \) contains \( \alpha \). Let \( C' \) be the greedy left multiplier of \( (A, B') \); then for all \( 1 \leq i \leq k \), \( C'_{\alpha(c_i)c_i} = 1 \) since \( A_{\alpha(c_i)c_i} = 1 \). Note that since the columns indexed by \( X \) of \( C \) did not contain a 1, \( \Lambda(A) \) is in fact a subset of the subspace of \( \Lambda(B) \) generated by the rows indexed by \( \{ 1, \ldots, n \} \setminus X \); hence \( \Lambda(A) \subseteq \Lambda(B) \) and thus by Lemma 3.1.5, \( A = C'B' \). Hence we have found \( B', C' \in M_n(\mathbb{B}) \) such that \( A = C'B', B' \in M_n^S(\mathbb{B}) \), and every column of \( C' \) contains a 1. We must finally prove that \( B' \) and \( C' \) are not \( \mathcal{J} \)-related to \( A \) in \( M_n^S(\mathbb{B}) \).

Suppose that \( A \not\mathcal{J} B' \); then \( B' \) is similar to \( A \), and hence \( B' \) is trim. Since \( C \) contains a 1 in every row, and \( C' \) contains all 1s of \( C \) together with at least one more, \( C' \) is not the identity matrix. As above, it follows that \( C'B' \) contains more 1s than \( B' \) does; this is a contradiction. Hence \( A \) is not \( \mathcal{J} \)-related to \( B' \). Now suppose that \( A \not\mathcal{J} C' \). Then \( C' \) is similar to \( A \), and thus trim. In order to apply the previous argument to \( C' \), we must prove that \( B' \) is not a permutation matrix. Since \( B \) was not a permutation matrix, and \( B' \) was obtained by replacing some rows of \( B \) by rows \( e_{\alpha(c_i)} \) containing a single 1, the only way that \( B' \) may be a permutation matrix is if the non-replaced rows of \( B \) - indexed by \( \{ 1, 2, \ldots, n \} \setminus X \) - all contained a single 1. But since \( C \) contains 1s only in those columns, the product \( A = CB \) could then only contain 1s in at most \( n - |X| \) columns, and thus would not be a Hall matrix. Hence, \( B' \) is not a permutation matrix and a similar argument to that which showed that \( A \) is not \( \mathcal{J} \)-related to \( B' \) applies to \( A \) and \( C' \), to show that \( A \) is not \( \mathcal{J} \)-related to \( C' \) in \( M_n^S(\mathbb{B}) \).
While Lemma 3.4.6 does not provide the decomposition we require in order to use Lemma 2.3.1, it provides a decomposition into two matrices that are close to having the correct properties. This motivates the following definition. Given a row-trim matrix $A \in M_n(\mathbb{B})$, define the core $A^\circ$ of $A$ to be the submatrix of $A$ consisting of those rows containing at least two 1s. We say that a subset of rows of $A$ violating the Hall condition is a maximal violator of $A$ if it has largest cardinality amongst such subsets, and that a matrix is $k$-deficient if $k$ is the cardinality of a maximal violator of $A^\circ$. The following lemma shows that we are justified in considering maximal violators of $A^\circ$ rather than $A$.

**Lemma 3.4.7.** If $A$ is a row-trim matrix belonging to $M_n(\mathbb{B})$, then $A$ has a subset of rows violating the Hall condition if and only if $A^\circ$ has a subset of rows violating the Hall condition.

**Proof.** Suppose $A$ has rows $X = \{r_1, r_2, \ldots, r_k\}$ containing a single 1. Since $A$ is row-trim, no other row may contain a 1 in the columns in which the rows in $X$ have a 1; hence we may permute the rows and columns of $A$ so that the rows containing a single 1 form a identity matrix $I_k$ as the $k \times k$ bottom-right block. Now $A^\circ$ forms the first $n - k$ rows, and it is clear that the union of a maximal violator of $A^\circ$ with the last $k$ rows forms a maximal violator of $A$, and that restricting a maximal violator of $A$ to the first $n - k$ rows induces a maximal violator of $A^\circ$.

We will demonstrate that we can iteratively reduce the $k$-deficiency of the matrix $C$ in the statement of Lemma 3.4.6 until it is 0-deficient and therefore belongs to $M_n^S(\mathbb{B})$.

**Lemma 3.4.8.** Suppose that $A$ is a trim matrix belonging to $M_n^S(\mathbb{B})$ containing the identity permutation, and that $A = CB$ where $B$ belongs to $M_n^S(\mathbb{B})$, $C \in M_n(\mathbb{B})$ is a $k$-deficient matrix containing a 1 in every column, and neither $B$ nor $C$ are $\mathcal{F}$-related to $A$ in $M_n^S(\mathbb{B})$. Then there exist $T \in M_n^S(\mathbb{B})$ and $S \in M_n(\mathbb{B})$ such that $A = ST$, $S$ contains a 1 in every column, neither $S$ nor $T$ are $\mathcal{F}$-related to $A$ in $M_n^S(\mathbb{B})$, and $S$ is at most $(k - 1)$-deficient.

**Proof.** Since $C$ is $k$-deficient, there is some maximal violator of $C^\circ$, indexed in $C$ by $W = \{w_1, w_2, \ldots, w_k\} \subset \{1, \ldots, n\}$. Since $C$ contains a 1 in every column, $k < n$. We will show how we can construct new matrices from $C$ and $B$ so that the rows indexed by $W$ satisfy the Hall condition. It will be useful in this proof to denote the complement of a set $Z \subseteq \{1, 2, \ldots, n\}$ in $\{1, 2, \ldots, n\}$ by $Z^C$. For the sake of brevity, we will also not distinguish between indices of rows and the rows themselves when the distinction is not important; the same is true of columns. Thus we may talk about the rows $W$ rather than the rows indexed by $W$.

Let $X = \{x_1, \ldots, x_l\} \subset \{1, \ldots, n\}$ denote the indices of those columns of $C$ that do not contain any 1s in the rows $W$. By multiplying $C$ by an appropriate permutation matrix on the right, and $B$ by the inverse of this permutation matrix on the left, we may assume that $X^C \subset W$, and that $X \cap W = \{x_1, \ldots, x_{m-1}\}$, where $m = k+1-l + 1$. That is, the 1s that occur in rows $W$ occur in a subset of the columns $W$, and the complement of this subset in $W$ is labelled by $\{x_1, \ldots, x_{m-1}\}$. Consider the remaining $n - k$ column indices $\{x_m, \ldots, x_k\}$.

Let $D$ be the $(n-k) \times (n-k)$ submatrix of $C$ consisting of rows $W^C$ of $C$ and columns $\{x_m, \ldots, x_k\}$. Suppose that $D$ contains maximal violator $V$. The rows in $V$ cannot all be rows of $C$ which contained a single 1: since $A$ is trim, $C$ is row trim and hence each row of $C$ containing a single 1 contains a distinct 1. It follows that $V$ contains at least one row of $C^\circ$. But then the union of $W$ with the rows of $C^\circ$ corresponding to rows of $V$ is a maximal violator of $C^\circ$ with larger cardinality than $W$, a contradiction. Hence $D$ satisfies the Hall condition, and therefore $D$ contains a permutation matrix. This defines a bijection $f : W^C \to \{x_m, \ldots, x_k\}$ such that for all $i \in W^C$, $f(i)$ is the column containing a 1 in row $i$ of the permutation matrix. For all $i \in W^C$, we define row $i$ of $S$ to be $e_{f(i)}$, and row $f(i)$ of $T$ to be $A_{i^a}$; then $(ST)_{i^a} = A_{i^a}$. Observe that since $C_{i,f(i)} = 1$, row $B_{f(i)}$ is contained in row $A_{i^a}$ and hence also contained in row $T_{i^a}$.

For all $j \in X^C$, we define $T_{j^a} = B_{j^a}$; then for all $i \in W$, $(CT)_{i^a} = A_{i^a}$. It remains to define rows $W$ of $S$ and rows $\{x_1, \ldots, x_{m-1}\}$ of $T$. Since $B$ is a Hall matrix, it contains a permutation matrix $U$ with corresponding permutation $u$. The rows $X^C \cup \{x_m, \ldots, x_k\}$ of $T$ contain the corresponding rows of $U$. For all $j \in \{x_1, \ldots, x_k\}$, consider the cycle in the permutation $u$ starting at $j$. If $u(j) \in W$, then define $T_{j^a} = e_{u(j)}$. Otherwise, $u(j) \not\in W$, and we may construct the tuple $(u^1(j), u^2(j), \ldots, u^q(j), iu^{q+1}(j))$ where $u^i(j) \not\in W^C$ for
1 \leq i \leq q$, and $u^{q+1}(j) \in W$. Then we define $T_{js} = e_{u^{q+1}(j)}$. By the assumptions on $X$, for $1 \leq i \leq q$ we have $u^i(j) \in \{x_m, \ldots, x_1\}$, and by the definitions of rows $\{x_m, \ldots, x_1\}$ of $T$, we have that $T_{u^i(j), u^i(q)} = 1$. For each such tuple, we modify $U$ by defining $U_{js} = e_{u^{q+1}(j)}$, and $U_{u^i(q), s} = e_{u^{q+1}(j)}$ for $1 \leq i \leq q$. Note that this modification is well-defined, and in particular does not depend on the order in which the modification is carried out, since $u$ is a permutation and thus decomposes into disjoint cycles. After this modification, $T$ is fully defined and contains the permutation matrix $U$.

For each $j \in \{x_1, \ldots, x_{m-1}\}$, we have $T_{js} = e_w$ for some $w \in W$, and hence there is a 1 in position $(w, j)$ of the greedy left multiplier of $(A, T)$. Define the rows $W$ of $S$ to be the corresponding rows of the greedy left multiplier of $(A, T)$. Then the union $y$ of the rows $W$ of $S$ contains ones in columns $\{x_1, \ldots, x_{m-1}\}$. Note that since the rows of $T$ indexed by $X^C$, the set of columns that had a 1 in some row of $W$, are equal to the corresponding rows of $B$, we have $S_{ws} \geq C_{ws}$ for all $w \in W$. Hence $y$ contains $(n-l)+m=k$ ones, and so the rows $W$ satisfy the Hall condition. Since every other row of $S$ contains a single distinct 1, it follows that $S$ is at most $(k-1)$-deficient.

As noted above, $(ST)_{is} = A_{is}$ for all $i \in W^C$. For $i \in W$, we have that $S_{is} \geq C_{is}$; since $(CT)_{is} = A_{is}$ we must have $(ST)_{is} \geq A_{is}$. By the definition of the greedy left multiplier, $(ST)_{is} \leq A_{is}$ for all $i \in W$. Hence $A = ST$.

We wish to now show that $S$ contains a 1 in every column, and that $A$ is not $\mathcal{J}$-related to $S$ or to $T$. That $S$ contains a 1 in every column follows from the union $y$ of the rows $W$ containing 1s in each of the columns $W$, and the other rows being defined via the bijection $f$. That $T$ is not $\mathcal{J}$-related to $A$ follows from the same argument as above; since $A$ is trim so would be $B$, but $S$ contains at least one row with more than one 1, a contradiction. In order to prove that $S$ is not $\mathcal{J}$-related to $A$, we must as above prove that $T$ is not a permutation matrix. Similarly to the argument above, we note that if $T$ is a permutation matrix then each row of $T$ belonging to $X^C$ would contain a single one, but these are equal to the corresponding rows of $B$. Hence the union of rows $W$ of $CB$ would contain precisely $n-l < k$ ones, and thus $A$ would not be Hall. It follows that $T$ is not a permutation matrix, and the argument proceeds as above.

Corollary 3.4.9. Let $A$ be a matrix belonging to $M^S_n(\mathbb{B})$ which is not prime, elementary, or a permutation matrix. Then $A$ can be decomposed as a product $A = CB$ of matrices $B, C \in M^S_n(\mathbb{B})$, such that neither $B$ nor $C$ is $\mathcal{J}$-related to $A$ in $M^S_n(\mathbb{B})$.

Proof. We may assume that $A$ contains the identity permutation, since the lemma holds for $A$ if and only if it holds for all matrices similar to $A$. If $A$ is not trim then this follows directly from Lemma 3.4.5. Otherwise, $A$ is trim and by Lemma 3.4.6 there exist $B \in M^S_n(\mathbb{B})$ and $C \in M_n(\mathbb{B})$ such that $A = CB$, neither $B$ nor $C$ is $\mathcal{J}$-related to $A$ in $M^S_n(\mathbb{B})$, and $C$ contains a 1 in every column. If $C$ also belongs to $M^S_n(\mathbb{B})$, then we are done; otherwise $C$ is $k$-deficient for some $1 \leq k \leq n$. By iteratively replacing $B$ and $C$ with the matrices constructed in Lemma 3.4.8, we must eventually construct $B$ and $C$ with the properties above and where $C$ is $0$-deficient, and thus Hall by Lemma 3.4.7.

We may now prove Theorem 3.4.1.

Proof of Theorem 3.4.1. Let $G \subseteq M^S_n(\mathbb{B})$ consist of a set of representatives $P$ of the prime $\mathcal{J}$-classes of $M_n(\mathbb{B})$ together with two matrices that generate $S_n$ and an elementary matrix. By Lemma 3.4.3 for any element $x \in G$ with $\mathcal{J}$-class $J_x$ in $M^S_n(\mathbb{B})$, we have $J_x \subseteq \langle G \rangle$. It follows from Corollary 3.4.9 and Lemma 2.3.1 that $G$ generates $M^S_n(\mathbb{B})$. Since any generating set for $M_n(\mathbb{B})$ requires a matrix from every prime $\mathcal{J}$-class, two generators for $S_n$, and an elementary matrix, and each of these lie in $M^S_n(\mathbb{B})$, we must require these matrices in any generating set for $M^S_n(\mathbb{B})$; hence $G$ is minimal. The same argument shows that every minimal generating set is obtained in this way.

3.5 Triangular boolean matrices

We now turn our attention to the monoids of upper and lower triangular boolean matrices, denoted by $UT_n(\mathbb{B})$ and $LT_n(\mathbb{B})$ respectively. These matrices have a particularly simple minimal generating set:
Lemma 3.5.1. The unique minimal generating set $G_U$ for $UT_n(\mathbb{B})$ consists of those elementary matrices and matrices similar to $F$ which are upper triangular, together with the identity matrix. The dual statement holds for $LT_n(\mathbb{B})$.

Proof. We will prove the lemma for $UT_n(\mathbb{B})$; the result for $LT_n(\mathbb{B})$ follows via the anti-isomorphism given by transposition. We will first construct any matrix $A \in UT_n(\mathbb{B})$ as a product of matrices in $G_U$. We iteratively define a product $A(i), 0 \leq i \leq n$ where $A(0)$ is the identity and $n$ is the number of $1$s in $A$. For the $i$th $1$ contained in $A$ (ordered by row then column), in row number $x_i$ and column number $y_i$, we obtain $A(i)$ from $A(i-1)$ by left-multiplying by $E^{x_i\cdot y_i}$. Let the zero rows of $A$ have indices $\{z_1, \ldots, z_k\}$. Note that the matrices in $UT_n(\mathbb{B})$ similar to $F$ are precisely those matrices obtained by deleting a single 1 from an identity matrix. We define $A(n+j)$ to be the matrix obtained by left multiplying $A(n+j-1)$ by the element of $X$ with a zero row in position $z_j$. Then $A(n+k) = A$.

It is routine to show that for each matrix $A \in G_U$, if $A$ is written as a product $A = BC$ in $UT_n(\mathbb{B})$ then one of $B$ or $C$ must be equal to $A$; it follows that any generating set for $UT_n(\mathbb{B})$ must contain all matrices in $G_U$. Hence $G_U$ is the unique minimal generating set for $UT_n(\mathbb{B})$.

Corollary 3.5.2. For $n \geq 2$, the ranks of $UT_n(\mathbb{B})$ and $LT_n(\mathbb{B})$ are $T_n + 1$, where $T_n$ is the $n$th triangular number $T_n = n(n+1)/2$.

Proof. Elementary matrices in $UT_n(\mathbb{B})$ are precisely those in the form of an identity matrix together with an additional 1 in some position above the main diagonal; hence there are $T_{n-1}$ elementary matrices in $UT_n(\mathbb{B})$. There are $n$ matrices similar to $F$ in $UT_n(\mathbb{B})$, and hence $T_{n-1} + n = T_n$ non-identity elements of the minimal generating set.

4 Tropical matrices

Recall that $\mathbb{K}_\infty$ denotes the min-plus semiring where addition is min and multiplication is $+$, and the min-plus semiring with threshold $\mathbb{K}_\infty^t$ is the quotient of $\mathbb{K}_\infty$ by the least congruence containing $(t, t+1)$. Similarly, $\mathbb{K}_\infty^\infty$ denotes the max-plus semiring and $\mathbb{K}_\infty^\infty$ is the quotient of $\mathbb{K}_\infty^\infty$ by the least congruence containing $(t, t+1)$. Generating sets for $M_2(\mathbb{K}_\infty^\infty)$, $M_2(\mathbb{K}_\infty^t)$, $M_2(\mathbb{K}_\infty^\infty)$ and $M_2(\mathbb{K}_\infty^\infty)$ were found in [19]; see also Theorem 4.1.1 and Corollary 4.1.2. In this section, we show that the generating sets from [19] are, in fact, minimal. Minimal generating sets are not known for such matrices of arbitrary dimension, and results in this direction seem unlikely. For instance, if $t = 0$, then $M_n(\mathbb{K}_\infty^\infty)$ is isomorphic to the full boolean matrix monoid $M_n(\mathbb{B})$ via the semiring isomorphism $\phi : \mathbb{K}_\infty^\infty \to \mathbb{B}$ which maps $\phi(\infty) = 0$ and $\phi(0) = 1$. Finding minimal generating sets for $M_n(\mathbb{B})$ is the subject of Section 3.2 and it seems reasonable to say that this is somewhat difficult. In particular, any general theorem on minimal generating sets for $M_n(\mathbb{K}_\infty^\infty)$ must include Devadze’s Theorem (Theorem 3.1.10) as a special case, and so it seems likely that any such theorem would not provide explicit generators but rather would require the computation of those $\mathcal{J}$-classes immediately below the group of units as in Section 3.2.

4.1 Min-plus matrices

Theorem 4.1.1 ([19] Theorem 1.1]). The monoid $M_2(\mathbb{K}_\infty^\infty)$ of $2 \times 2$ min-plus matrices is generated by the matrices:

$$A(i) = \begin{pmatrix} i & 0 \\ 0 & \infty \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \infty \\ \infty & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} \infty & \infty \\ \infty & 0 \end{pmatrix}$$

where $i \in \mathbb{N} \cup \{\infty\}$.

Corollary 4.1.2 ([19] Corollary 1.2]). Let $t \in \mathbb{N}^+$ be arbitrary. Then the finite monoid $M_2(\mathbb{K}_\infty^t)$ of $2 \times 2$ min-plus matrices is generated by the $t+4$ matrices:

$$A(i) = \begin{pmatrix} i & 0 \\ 0 & \infty \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \infty \\ \infty & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} \infty & \infty \\ \infty & 0 \end{pmatrix}$$
where \( i \in K^\infty_f \).

**Theorem 4.1.3.** The generating sets of Theorem 4.1.1 and Corollary 4.1.2 are irredundant and minimal.

We will prove in the following sequence of lemmas that the given generating sets are irredundant. In the infinite case, the cardinality of the given generating set then guarantees that it is minimal. In the finite case, it suffices, by Proposition 2.3.2, to show that the generating set contains at most one element from each \( J \)-class of the monoid.

**Lemma 4.1.4.** Let \( S \in \{K^\infty\} \cup \{K^\infty_t : t \in N^+\} \). Then any generating set for \( M_2(S) \) contains a matrix similar to \( B \) and a matrix similar to \( A(i) \), for each \( i \in S \).

**Proof.** It is easy to show that if a matrix \( X \in M_n(S) \) satisfies the property that when written as a product \( X = YZ \) for \( Y, Z \in M_n(S) \), one of \( Y \) or \( Z \) must be similar to \( X \), then the same property holds for all matrices similar to \( X \). It follows that we may prove the lemma by establishing this property for \( B \) and \( A(i) \), since any product of matrices equal to one of \( A(i) \) or \( B \) must contain a matrix similar to that generator.

Fix \( i \in N \cup \infty \), and suppose that \( A(i) \) is written as the product of two matrices in \( M_2(R) \). That is, suppose that

\[
\begin{pmatrix}
i & 0 \\
0 & \infty
\end{pmatrix} = \begin{pmatrix}a & b \\
c & d\end{pmatrix} \begin{pmatrix}u & v \\
w & x\end{pmatrix}
\]

where \( a, b, c, d, u, v, w, x \in R \). It follows by the definitions of matrix multiplication, \( \oplus \), and \( \otimes \) that

1. \((a + u = i \) and \( b + w \geq i) \) or \((b + w = i \) and \( a + u \geq i) \),
2. \(a = v = 0 \) or \( b = x = 0 \),
3. \(c = u = 0 \) or \( d = w = 0 \), and
4. \((c = \infty \) or \( v = \infty) \) and \((d = \infty \) or \( x = \infty) \).

Suppose that (2) is satisfied by \( a = v = 0 \). Then \( c = \infty \) by (4). It follows that \( d = w = 0 \) by (3), and so \( x = \infty \) by (4). Thus by (1), either \( u = i \) and \( b \geq i \), or \( b = i \) and \( u \geq i \), that is,

\[
\begin{pmatrix}u & v \\
w & x\end{pmatrix} = \begin{pmatrix}i & 0 \\
0 & \infty\end{pmatrix} \text{ or } \begin{pmatrix}a & b \\
c & d\end{pmatrix} = \begin{pmatrix}0 & i \\
\infty & 0\end{pmatrix}.
\]

Instead, if we suppose that (2) is satisfied by \( b = x = 0 \), then either

\[
\begin{pmatrix}u & v \\
w & x\end{pmatrix} = \begin{pmatrix}0 & \infty \\
i & 0\end{pmatrix} \text{ or } \begin{pmatrix}a & b \\
c & d\end{pmatrix} = \begin{pmatrix}i & 0 \\
0 & \infty\end{pmatrix}.
\]

In particular, if \( A(i) \) is written as the product of two matrices in \( M_2(S) \), then one of those matrices is similar to \( A(i) \).

A similar argument based on writing \( B \) as the product of two elements completes the proof. \( \square \)

Since every generating set for \( M_2(K^\infty) \) contains a matrix similar to \( A(i) \) for every \( i, M_2(K^\infty) \) is not finitely generated.

**Corollary 4.1.5.** The generating sets given in Theorem 4.1.1 and Corollary 4.1.2 are irredundant.

**Proof.** By Lemma 4.1.4, any generating set for \( M_2(S) \), where \( S \in \{K^\infty\} \cup \{K^\infty_t : t \in N^+\} \), contains a matrix similar to \( A(i) \) for each \( i \in S \). Since the only such element in the given generating set is \( A(i) \) itself, it follows that the generators \( A(i) \) are irredundant. The same argument applies to \( B \).

It is straightforward to show that if a matrix with \( \infty \) occurring twice in one row is expressed as a product of two matrices, then one of those matrices has a row where \( \infty \) occurs twice. Therefore, in order to generate such matrices, a generator with a row containing two occurrences of \( \infty \) is required. In the generating sets of Theorem 4.1.1 and Corollary 4.1.2 \( C \) is the only such generator, so it is irredundant. \( \square \)
Since $M_2(\mathbb{K}^\infty)$ is not finitely generated, it follows from the corollary that the generating set for $M_2(\mathbb{K}^\infty)$ given in Theorem 4.1.1 is minimal. The proof of Theorem 4.1.3 is therefore complete in the case of $\mathbb{K}^\infty$.

**Lemma 4.1.6.** If $t \in \mathbb{N}^+$ is arbitrary, then the generating set for $M_2(\mathbb{K}^\infty_t)$ given in Corollary 4.1.2 consists of $\mathcal{J}$-non-equivalent elements.

**Proof.** By Proposition 2.2.3, it is sufficient to demonstrate that the row spaces of the generators are non-isomorphic. In order to do so, we first observe several facts about spanning sets of these row spaces.

It is routine to verify that the rows $(\infty 0), (1 \infty), (0 \infty)$ are indecomposable, in the sense that every linear combination of rows equal to one of these rows must contain a non-zero scalar multiple of that row. It follows that $(\infty 0)$ is contained in any spanning set for $\Lambda(C)$ and so $\{(\infty 0)\}$ is the unique minimal spanning set for $\Lambda(C)$. Similarly, $(\infty 0)$ and $(1 \infty)$ are contained in any spanning set for $\Lambda(B)$, making $\{(\infty 0), (1 \infty)\}$ the unique minimal spanning set for $\Lambda(B)$.

To determine that $\Lambda(A(i))$ has a unique minimal spanning set, we note that if $(x 0) \in \Lambda(A(i))$, then we can write

$$(x 0) = a(i 0) \oplus b(0 \infty)$$

for some $a, b \in \mathbb{K}^\infty_t$. From the second column, we have $a = 0$; hence $x \leq i$. If $(i 0)$ is written as a sum of rows in $\Lambda(A(i))$,

$$(i 0) = (c, d) \oplus (e, f),$$

then one of $d$ or $f$ is 0; without loss of generality we can assume $d = 0$. By the previous observation, we have $c \leq i$, and by assumption $\min(c, e) = i$; hence $c = i$. It follows that $(i 0)$ is contained in any spanning set for $\Lambda(A(i))$; hence the rows of $A$ form the unique minimal spanning set of $\Lambda(A(i))$. Since $\Lambda(C)$ is spanned by the single row $(\infty 0)$ while $\Lambda(A(i))$ and $\Lambda(B)$ cannot be spanned by a single element, $\Lambda(C)$ is not isomorphic to $\Lambda(A(i))$ or $\Lambda(B)$. Any isomorphism from $\Lambda(A(i))$ to $\Lambda(B)$ must map the rows of $A(i)$ to the rows of $B$ as these form unique minimal spanning sets, but since the sets

$$\{a(i 0) : a \in \mathbb{K}^\infty_t\}, \quad \{a(0 \infty) : a \in \mathbb{K}^\infty_t\}$$

both contain $t$ unique rows, while the sets

$$\{a(1 \infty) : a \in \mathbb{K}^\infty_t\}, \quad \{a(\infty 0) : a \in \mathbb{K}^\infty_t\}$$

contain $t - 1$ and $t$ rows respectively, no such isomorphism exists.

It remains to show that $\Lambda(A(i))$ and $\Lambda(A(j))$ are non-isomorphic for $i \neq j$. Since

$$(a 0) = 0(i 0) + a(0 \infty)$$

for $a \in \mathbb{K}^\infty_t$ such that $a \leq i$, together with the observation above we have $(a 0) \in \Lambda(A(i))$ if and only if $a \leq i$. It follows that there are precisely $i + 1$ rows $y \in \Lambda(A(i))$ such that there exists $a \in \mathbb{K}^\infty_t$ with $ay = (i 0)$. Similarly there is precisely one row $z \in \Lambda(A(i))$ such that there exists $b \in \mathbb{K}^\infty_t$ with $bz = (0 \infty)$. Since any isomorphism from $\Lambda(A(i))$ to $\Lambda(A(j))$ must map the rows of $A(i)$ to the rows of $A(j)$, but there are $j + 1$ such rows for one of the rows of $A(j)$, there is no such isomorphism.

This completes the proof of Corollary 4.1.2.

For any $t \in \mathbb{N}^+$, there is a quotient map $\phi_t : \mathbb{K}^\infty \longrightarrow \mathbb{K}^\infty_t$ under which $\phi_t(x) = \min(x, t)$ for all $x \in \mathbb{K}^\infty \setminus \{\infty\}$ and $\phi_t(\infty) = \infty$; this is a semiring homomorphism. It is routine to verify that the map $\psi_t : M_n(\mathbb{K}^\infty) \longrightarrow M_n(\mathbb{K}^\infty_t)$ defined by entrywise application of $\phi_t$ is a semigroup homomorphism. Let $A, B$ be two elements in the generating set of Theorem 4.1.1. Consider a finite quotient $\mathbb{K}^\infty_t$ of $\mathbb{K}^\infty$ to which all entries of both $A$ and $B$ belong. The previous lemma shows that $A$ is not $\mathcal{J}$-equivalent to $B$ in $M_n(\mathbb{K}^\infty)$. But $\psi_t(A) = A$ and $\psi_t(B) = B$ and so $\psi_t(A)$ is not $\mathcal{J}$-equivalent to $\psi_t(B)$ in $M_n(\mathbb{K}^\infty_t)$. It follows that $A$ and $B$ are not $\mathcal{J}$-equivalent in $M_n(\mathbb{K}^\infty)$. This proves the following corollary.

**Corollary 4.1.7.** The generating set given in Theorem 4.1.1 consists of $\mathcal{J}$-non-equivalent elements.
4.2 Max-plus matrices

The semirings $\mathbb{K}^\infty$ and $\mathbb{K}^{-\infty}$ are in certain ways similar to each other; it is only in the interaction of their respective additions and multiplication operations that they display significantly different behaviour. The result of this is that the process of proving that certain generating sets for $M_2(\mathbb{K}^{-\infty})$ and $M_2(\mathbb{K}_t^{-\infty})$ are minimal is essentially the same as in the case of $M_2(\mathbb{K}^\infty)$ and $M_2(\mathbb{K}_t^\infty)$; it only differs in some details.

**Theorem 4.2.1** ([19]). The monoid $M_2(\mathbb{K}^{-\infty})$ of $2 \times 2$ max-plus matrices is generated by the matrices

$$X(i) = \begin{pmatrix} i & 0 \\ 0 & -\infty \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -\infty & -\infty \\ -\infty & 0 \end{pmatrix}, \quad W(j, k) = \begin{pmatrix} 0 & j \\ k & 0 \end{pmatrix},$$

where $i \in \mathbb{K}^{-\infty}$ and $j, k \in \mathbb{N}$ such that $1 \leq j \leq k$.

**Corollary 4.2.2** ([19]). Let $t \in \mathbb{N}^+$ be arbitrary. Then the finite monoid $M_2(\mathbb{K}_t^{-\infty})$ of $2 \times 2$ max-plus matrices is generated by the $(t^2 + 3t + 8)/2$ matrices

$$X(i) = \begin{pmatrix} i & 0 \\ 0 & -\infty \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -\infty & -\infty \\ -\infty & 0 \end{pmatrix}, \quad W(j, k) = \begin{pmatrix} 0 & j \\ k & 0 \end{pmatrix},$$

where $i \in \mathbb{K}_t^{-\infty}$ and $j, k \in \{1, \ldots, t\}$ with $j \leq k$.

**Theorem 4.2.3.** The generating sets of Theorem 4.2.1 and Corollary 4.2.2 are irredundant and minimal.

The proof of this theorem follows the same pattern as the proof of Theorem 4.1.3.

**Lemma 4.2.4.** Let $\mathbb{S} \subseteq \{\mathbb{K}^{-\infty} \cup \{\mathbb{K}_t^{-\infty} : t \in \mathbb{N}^+\}$. Then any generating set for $M_2(\mathbb{S})$ contains matrices similar to each matrix other than $Z$ in the appropriate generating set from Theorem 4.2.1 or Corollary 4.2.2.

**Proof.** As in the proof of Lemma 4.1.4 it suffices to show that if any of the generators described is expressed as a product of two matrices, then one of those factors must be similar to the generator. We will only prove this for $W(j, k)$ as an illustrative example; the other proofs are similar.

Writing

$$W(j, k) = \begin{pmatrix} 0 & j \\ k & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix},$$

for some $a, b, c, d, u, v, w, x \in \mathbb{S}$, we obtain the conditions

1. $(a = u = 0$ and $b + w \leq 0)$ or $(b = w = 0$ and $a + u \leq 0)$,
2. $(a + v = j$ and $b + x \leq j)$ or $(b + x = j$ and $a + v \leq j)$,
3. $(c + u = k$ and $d + w \leq k)$ or $(d + w = k$ and $c + u \leq k)$,
4. $(c = v = 0$ and $d + x \leq 0)$ or $(d = x = 0$ and $c + v \leq 0)$.

First assume that in (1), we have $a = u = 0$. If we also assume that in (4) we have $c = v = 0$, then from (2) we have $b + x = j$ and hence both $b$ and $x$ are not equal to $-\infty$. It follows from (1) that $b = w = 0$ or $w = -\infty$. In the latter case, (3) cannot be satisfied. In the former, it follows from (3) that $d = k$. However, then (4) cannot be satisfied as $d + x > 0$ unless $x = -\infty$, which we noted above was not the case. Hence in (4), we must have $d = x = 0$. We now consider three cases:

1. If $w > 0$, then by (1) we have $b = -\infty$. Hence, by (2), $v = j$ since $a = 0$. It follows from (4) that $c = -\infty$ and thus by (3) $u = k$, and the second factor is similar to $W(j, k)$.
2. If $w = -\infty$, then from (3) we have $c = k$, since $u = 0$. It follows that $v = -\infty$ (by (4)), and thus that $b = j$ (by (2)). The first factor is then similar to $W(j, k)$.
3. If \( w = 0 \), then by (1) we have \( b \leq 0 \). It follows from (2) that \( v = j \), and hence from (4) that \( c = -\infty \).

However, it is then impossible to satisfy (3), and so this case cannot arise.

In each case that may arise, one factor is similar to \( W(i, j) \). It now remains to consider the case of \( b = w = 0 \) in (1); the proof of this case is dual to argument above.

Since every generating set for \( M_2(\mathbb{K}^{-\infty}) \) contains a matrix similar to \( X(i) \) for every \( i \), \( M_2(\mathbb{K}^{-\infty}) \) is not finitely generated.

Note that the previous lemma fails for \( Z \), since

\[
Z = \begin{pmatrix} -\infty & -\infty \\ -\infty & 0 \end{pmatrix} = \begin{pmatrix} -\infty & -\infty \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\infty & 0 \\ -\infty & 0 \end{pmatrix}.
\]

In order to show that \( Z \) is an irredundant generator we use the following lemma, the proof of which is elementary.

**Lemma 4.2.5.** Let \( S \in \{\mathbb{K}^{-\infty} \} \cup \{\mathbb{K}_t^{-\infty} : t \in \mathbb{N}^+\} \). If a matrix \( A \in M_2(S) \) which has a row in which both entries are \(-\infty\) is written as a product \( A = BC \) of two matrices \( B, C \in M_2(R) \), then one of \( B \) or \( C \) has a row in which both two entries are \(-\infty\).

**Corollary 4.2.6.** The generating sets given in Theorem 4.2.1 and Corollary 4.2.2 are irredundant.

**Proof.** This follows from Lemma 4.2.4 and Lemma 4.2.5 in the same way as in Corollary 4.1.5.

As in the previous section, it follows from the corollary that the generating set for \( M_2(\mathbb{K}^{-\infty}) \) given in Theorem 4.2.1 is minimal.

In order to demonstrate that the generating sets of Theorem 4.2.1 consist of \( \mathcal{F} \)-non-equivalent elements, we establish the following result on row bases of max-plus matrices.

**Lemma 4.2.7.** For any \( S \in \{\mathbb{K}^{-\infty} \} \cup \{\mathbb{K}_t^{-\infty} : t \in \mathbb{N}^+\} \) and matrix \( A \in M_n(S) \), \( A \) has a unique row-basis consisting of the set of those rows \( A_{is} \) not expressible as a linear combination of rows distinct from \( A_{is} \).

**Proof.** We first note that \( \oplus \) and \( \otimes \) by an element in \( R \setminus \{\{-\infty\}\} \) preserves or increases the order of any element in \( S \). It follows that if \( aA_{is} \) is a term in some linear combination equal to a row \( x \), for some \( a \in S \setminus \{-\infty\} \), then \( x_j \geq A_{ij} \) for \( 1 \leq j \leq n \). Let \( W \) be those rows of \( A \) distinct from \( A_{is} \), and let \( \langle W \rangle \) denote the row space spanned by these rows. Suppose that \( A_{is} \notin \langle W \rangle \). Then for any row \( x \in \Lambda(A) \setminus \langle W \rangle \), \( x \) can be expressed as a linear combination with a non-trivial term involving \( A_{is} \); hence \( A_{ij} \leq x_j \) for \( 1 \leq j \leq n \). It follows that \( A_{is} \) cannot be written as a linear combination involving \( x \) in a non-trivial way unless \( x = A_{is} \). Thus \( A_{is} \notin \left\langle \Lambda(A) \setminus \{A_{is}\} \right\rangle \). It follows that any row \( A_{is} \) which cannot be written as a linear combination of distinct other rows of \( A \) must be included in any spanning set for \( \Lambda(A) \). We must also show that such rows do span \( \Lambda(A) \).

If \( X \) is any set of rows of \( A \) and \( A_{xs} \in X \) is such that \( A_{xs} \notin \langle X \setminus \{A_{xs}\} \rangle \), then as usual \( \langle X \rangle = \langle X \setminus \{A_{xs}\} \rangle \). We must establish that if we also have \( A_{ys} \in X \setminus \{A_{xs}\} \) such that \( A_{ys} \notin \langle X \setminus \{A_{ys}\} \rangle \), then \( A_{ys} \notin \langle X \setminus \{A_{xs}, A_{ys}\} \rangle \); the result then follows by repeatedly removing such rows from \( A \).

Suppose that some term \( aA_{ys} \) appears in a linear combination equal to \( A_{xs} \), where \( a \in S \setminus \{-\infty\} \). This implies that \( A_{ys} \leq A_{xs} \) in every component, and for some component the inequality is strict since \( A_{ys} \neq A_{xs} \). It follows that \( A_{xs} \) cannot be involved in a non-trivial way in any linear combination equal to \( A_{ys} \), and hence \( A_{ys} \notin \langle X \setminus \{A_{xs}, A_{ys}\} \rangle \). Suppose instead that no such term \( aA_{ys} \) exists; that is, \( A_{xs} \notin \langle X \setminus \{A_{xs}, A_{ys}\} \rangle \). Then the same linear combination of elements of \( X \setminus \{A_{ys}\} \) that is equal to \( A_{ys} \), possibly with \( A_{xs} \) replaced by the linear combination of elements in \( X \setminus \{A_{xs}, A_{ys}\} \) that is equal to \( A_{xs} \), witnesses that \( A_{ys} \notin \langle X \setminus \{A_{xs}, A_{ys}\} \rangle \).

**Lemma 4.2.8.** The generating sets given in Corollary 4.2.2 consist of \( \mathcal{F} \)-non-equivalent elements.

**Proof.** Fix \( t \in \mathbb{N}^+ \). By Proposition 2.2.3 it suffices to show that the generating sets consist of elements with non-isomorphic row spaces. By Lemma 4.2.7 each row space has a unique basis; it follows that any isomorphism between these row spaces must map one unique basis to another. We calculate the number of distinct scalar multiples of each basis element, which is an isomorphism invariant:
• \( \Lambda(Y) \) has unique basis \{ (1 \infty), (-\infty 0) \}, with \( t+1 \) and \( t+2 \) distinct scalar multiples (including themselves) respectively;

• \( \Lambda(Z) \) has unique basis \{ (-\infty 0) \}, with \( t+2 \) distinct scalar multiples;

• \( \Lambda(X(i)) \), for any \( i \in \mathbb{K}^{t-\infty} \), has unique basis \{ (i 0), (0 -\infty) \}, with \( t+2 \) distinct scalar multiples each;

• \( \Lambda(W(j,k)) \), for \( 1 \leq j \leq k \leq t \), has unique basis \{ (0 j), (k 0) \}, with \( t+2 \) distinct scalar multiples each.

It follows that the only possible isomorphisms between row spaces of distinct generators are between \( \Lambda(W(j,k)) \), \( \Lambda(W(p,q)) \), \( \Lambda(X(i)) \), and \( \Lambda(X(l)) \) for some \( (p,q) \neq (j,k) \) and \( l \neq i \). We note that in \( \Lambda(X(i)) \) the sum of the two basis rows is equal to one of the basis rows, while the same is not true in \( \Lambda(W(j,k)) \); hence there can be no isomorphism between row spaces of these types of generators. Furthermore, \( \Lambda(X(i)) \) is non-isomorphic to \( \Lambda(X(j)) \) for \( i \neq j \) since \( i \) is the largest value such that for some choice of \( g_1 \) and \( g_2 \) being distinct basis rows, \( ig_1 \oplus g_2 = g_2 \), and this is an isomorphism invariant.

We show that \( \Lambda(W(j,k)) \) is not isomorphic to \( \Lambda(W(p,q)) \) in a similar way. Let \( x_1, x_2 \) be distinct basis rows of \( W(j,k) \). Then there exist least elements \( a, b \in \{ 1, \ldots, t \} \) such that \( ax_1 \oplus x_2 = ax_1 + bx_2 \). By examination of the basis given above, the lesser of \( a \) and \( b \) (if there is one) is equal to \( j \) and the greater is equal to \( k \) (or all of the values are equal). Since \( \{a,b\} \) is an isomorphism invariant, there is no isomorphism between \( \Lambda(W(j,k)) \) and \( \Lambda(W(p,q)) \) for \( (j,k) \neq (p,q) \).

By Proposition \ref{prop:isomorphism_invariant} this completes the proof of Corollary \ref{cor:non_isomorphic}. As in the previous section, a corollary of the previous lemma is that the generating set given in Theorem \ref{thm:generating_set} also consists of \( J \)-non-equivalent elements.

### 5 Matrices over \( \mathbb{Z}_n \)

Perhaps surprisingly, there appears to be no known general characterisation of a minimal generating set of the multiplicative group \( U_n \) of integers coprime to \( n \), modulo \( n \). Any description of a minimal generating set for \( M_k(\mathbb{Z}_n) \) for general \( k \) would provide such a minimal generating set for \( U_n \), by taking \( k = 1 \) and restricting the generators to those of the group of units. This leads us to consider the so-called relative rank of \( M_k(\mathbb{Z}_n) \) with respect to the group of units \( GL_k(\mathbb{Z}_n) \). For a semigroup \( S \) and subset \( A \subseteq S \), the relative rank of \( S \) with respect to \( A \), denoted \( d(S : A) \), is defined to be the minimum cardinality of any subset \( B \) such that \( A \cup B \) generates \( S \). In this section, we will determine \( d(M_k(\mathbb{Z}_n) : GL_k(\mathbb{Z}_n)) \) by proving the following analogues to Devadze’s Theorem (see Theorem \ref{thm:devadze}).

**Theorem 5.1.1.** Let \( k, n \in \mathbb{N}^+ \). The monoid \( M_k(\mathbb{Z}_n) \) is generated by any set consisting of \( GL_k(\mathbb{Z}_n) \) and one element from each of the \( J \)-classes immediately below \( GL_k(\mathbb{Z}_n) \) in the \( J \)-order of \( M_k(\mathbb{Z}_n) \). For example, \( M_k(\mathbb{Z}_n) \) is generated by the set

\[
GL_k(\mathbb{Z}_n) \cup \begin{cases}
(p \mod n & 0 & \cdots & 0) \\
0 & 1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{cases} : p \text{ is a prime divisor of } n \text{ in } \mathbb{N}
\]

For each prime divisor \( p \) of \( n \), in \( \mathbb{N} \), we write

\[
X_p = \begin{pmatrix}
p \mod n & 0 & \cdots & 0 \\
0 & 1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]
and denote by $X_n$ the set $\{X_p : p$ is a prime divisor of $n\}$.

Note that the entry $p \mod n$ is, of course, simply $p$ unless $p = n$. Since the generating sets defined in Theorem 5.1.1 contain precisely one element from each $\mathcal{J}$-class immediately below the group of units, those elements are irredundant. By Proposition 2.3.2 this proves the following theorem:

**Theorem 5.1.2.** Let $k, n \in \mathbb{N}^+$, let $G$ be a generating set of minimum cardinality for $GL_k(\mathbb{Z}_n)$, and let $P$ be a set of representatives of the $\mathcal{J}$-classes immediately below $GL_k(\mathbb{Z}_n)$ in the $\mathcal{J}$-order of $M_k(\mathbb{Z}_n)$. Then $G \cup P$ is a minimal generating set for $M_k(\mathbb{Z}_n)$. In particular,

$$d(M_k(\mathbb{Z}_n) : GL_k(\mathbb{Z}_n)) = |\{p : p$ is a prime divisor of $n\}|.$$ 

We note that Theorem 5.1.1 and Theorem 5.1.2 hold trivially for $n = 1$ as 1 has no prime divisors and for any $k \in \mathbb{N}^+$, $|M_k(\mathbb{Z}_1)| = 1$. The further results and proofs in this section may also be stated in such a way as to hold for $n = 1$, but this is not necessary and results in slightly less tidy statements.

In order to prove Theorem 5.1.1 we will show that each $\mathcal{J}$-class of $M_k(\mathbb{Z}_n)$ contains a matrix in a certain standard form; we will then show how to generate these standard forms and hence generate the full matrix monoid.

In order to demonstrate that these standard forms exist in $M_k(\mathbb{Z}_n)$, we first prove the following elementary but useful lemma, without any claim of originality.

**Lemma 5.1.3.** Let $a \in \mathbb{Z}_n \setminus U_n$, with $\gcd(a, n) = d > 1$. Then there exists $b \in U_n$ such that $ab = d \mod n$.

**Proof.** By Bezout’s identity, there exist some $x, y \in \mathbb{Z}$ such that $ax + ny = d$; equivalently $\frac{a}{d}x + \frac{n}{d}y = 1$. It follows that $\frac{n}{d}$ and $x$ are coprime. For every $k \in \mathbb{Z}$, we also have $\frac{a}{d}(x + k\frac{n}{d}) + \frac{n}{d}(y - k\frac{n}{d}) = 1$. It is therefore sufficient to find a value of $x + k\frac{n}{d}$ which is coprime to $n$. Since $x$ and $\frac{n}{d}$ are coprime, by setting $k$ to be the product of those prime factors of $n$ that do not appear in $x$ or in $\frac{n}{d}$ we have that each prime factor of $n$ appears in precisely one of the terms $x$ and $k\frac{n}{d}$; hence no prime factor of $n$ divides the sum $x + k\frac{n}{d}$. \qed

**Lemma 5.1.4.** For $n > 1$, every matrix $Y \in M_k(\mathbb{Z}_n)$ can be written as a product $AXB$, for two matrices $A, B \in GL_k(\mathbb{Z}_n)$ and a diagonal matrix $X \in J_Y$ of the form

$$\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & d_1 & \cdots \\
0 & \cdots & 0 & d_m
\end{pmatrix},$$

for some $d_1, \ldots, d_m \in \mathbb{Z}_n \setminus \{0\}$, where $d_1, \ldots, d_m | n$ in $\mathbb{N}$, and $d_1 \geq \cdots \geq d_m$.

**Proof.** It is straightforward to show that there are units in $GL_k(\mathbb{Z}_n)$ which, via left multiplication, apply the elementary row operations of adding one row to another, subtracting one row from another, or exchanging rows in any matrix $Y \in M_k(\mathbb{Z}_n)$. This is familiar from linear algebra; the only “missing” standard row operation is scaling by an arbitrary scalar, which is not always represented by an element of $GL_k(\mathbb{Z}_n)$. However, scaling some row by an element of the group $U_n$ does still correspond to left multiplication by an element of $GL_k(\mathbb{Z}_n)$. The dual statements also hold for column operations. Fix $Y \in M_K(\mathbb{Z}_n)$. It is sufficient to show that elementary row and column operations can “diagonalise” $Y$ to obtain $X$; this implies that $Y = AXB$ for $A, B \in GL_k(\mathbb{Z}_n)$, and, since multiplication by units preserves $\mathcal{J}$-classes, that $X \in J_Y$. The diagonalisation procedure applied is essentially that of Gaussian elimination, but it is not altogether obvious that such a process applies in this case. This is what we now demonstrate.

Suppose that the entry $Y_{ij}$ is non-zero. Define $g$ to be the greatest common divisor of the entries of $Y_{is}$. By repeatedly subtracting one column from another to obtain a new matrix $Y'$, we may apply the Euclidean
algorithm so that some entry $Y''_{ik}$ in the $i$th row is equal to $g$. By subtracting column $k$ from column $j$ an appropriate number of times, we may assume that $Y''_{ij} = g$. Similarly, if $h$ is defined as the greatest common divisor of $g$ and the entries of $Y_{sj}$, by repeatedly subtracting appropriate rows we may produce a matrix $Y'''$ with $Y'''_{ij} = h$. Since $h$ divides every entry in row $i$ and column $j$ of $Y'''$, we may then use row and column subtractions to clear each non-zero entry in row $i$ and column $j$ of $Y'''$ other than $Y'''_{ij}$. Repeating this process produces a matrix $V$ with at most one non-zero entry in each row and column. For each row of $V$ with a non-zero entry $x$ coprime to $n$, we may scale that row by the inverse of $x$ in $U_n$ by left multiplication by some unit; this produces a matrix $W$ with each entry either 1 or non-coprime with $n$. By Lemma 5.1.3, if row $i$ of $W$ contains the non-zero entry $a$ where $gcd(a,n) = d > 1$, there exists some $b \in U_n$ with $ab = d \mod n$. We can therefore left-multiply by some unit to scale row $i$ so that the non-zero entry in row $i$ of the product is $d$. By repeating this for each such row, and then permuting rows and columns, the matrix $X$ with the required diagonal form is obtained.

We say that matrices in the form of Lemma 5.1.4 are in “standard diagonal form”. The row space $Λ(X)$ of such a standard diagonal form matrix can easily be seen to be isomorphic (as an additive group) to the direct product $G(X) = \mathbb{Z}_{n/d_1} \times \cdots \times \mathbb{Z}_{n/d_m}$. Note that for any prime divisor $p$ of $n$, $X_p$ is in the form of Lemma 5.1.4. Furthermore, by considering the prime power decomposition of $G(X_p)$, it is straightforward to determine that there are no other matrices $X$ of the same form with $G(X) \cong G(X_p)$. Hence there are no other matrices $X$ of the same form with $Λ(X) \cong Λ(X_p)$. By Proposition 2.2.3, this observation yields the following lemma.

**Lemma 5.1.5.** Let $p$ be a prime divisor of $n$. Then $X_p$ is the unique standard diagonal form matrix in $J_{X_p}$.

**Corollary 5.1.6.** Let $p_1$ and $p_2$ be distinct prime divisors of $n$. Then $X_{p_1}$ and $X_{p_2}$ are not $\mathcal{J}$-related.

In the following two lemmas, we show that the example set given in Theorem 5.1.1 indeed generates $M_k(\mathbb{Z}_n)$.

**Lemma 5.1.7.** Let $p$ be a prime divisor of $n$. Then $J_{X_p} \subseteq \langle GL_k(\mathbb{Z}_n) \cup \{Y\} \rangle$ for all $Y \in J_{X_p}$.

**Proof.** Let $Y \in J_{X_p}$. By Lemmas 5.1.4 and 5.1.5, $J_{X_p} = \{AX_pB : A, B \in GL_k(\mathbb{Z}_n)\}$; in particular, $Y = EX_pF$ for some $E, F \in GL_k(\mathbb{Z}_n)$. Hence

$$J_{X_p} = \{A(E^{-1}YF^{-1})B : A, B \in GL_k(\mathbb{Z}_n)\} \subseteq \langle GL_k(\mathbb{Z}_n) \cup \{Y\} \rangle.$$  

**Lemma 5.1.8.** Let $\mathcal{Y} \subseteq M_k(\mathbb{Z}_n)$ be a set of representatives of the $\mathcal{J}$-classes $\{J_{X_p} : p$ is a prime divisor of $n\}$. Then $M_k(\mathbb{Z}_n) = \langle GL_k(\mathbb{Z}_n) \cup \mathcal{Y} \rangle$.

**Proof.** By Lemma 5.1.7, $\mathcal{X}_n \subseteq \langle GL_k(\mathbb{Z}_n) \cup \mathcal{Y} \rangle$. Since the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & \\ & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

is a product of matrices in $\mathcal{X}_n$, and row- and column-permutations are represented by left and right multiplication by units, $\langle GL_k(\mathbb{Z}_n) \cup \mathcal{Y} \rangle$ contains every diagonal matrix that differs from the identity matrix by having a 0 or a prime divisor of $n$ in exactly one place on the main diagonal. It is straightforward to see that any matrix in standard diagonal form may be expressed as a product of such matrices. The result now follows by Lemma 5.1.4.

In order to complete the proof of Theorem 5.1.1 it remains to show that the $\mathcal{J}$-classes $J_{X_p}$ lie immediately below the group of units for all $X_p \in \mathcal{X}_n$. Since any generating set for $M_k(\mathbb{Z}_n)$ must contain a representative of each $\mathcal{J}$-class immediately below the group of units, the set $\mathcal{X}_n$ must contain a representative of each $\mathcal{J}$-class immediately below the group of units. It only remains to prove that all elements of $\mathcal{X}_n$ lie in such a maximal $\mathcal{J}$-class.
Lemma 5.1.9. The $\mathcal{J}$-classes $J_{X_p}$ are precisely the $\mathcal{J}$-classes immediately below the group of units in the $\mathcal{J}$-order on $M_k(\mathbb{Z}_n)$.

Proof. Observe that as an additive group, $\Lambda(X_p)$ is a maximal subgroup of $\Lambda(I)$, where $I$ is the identity matrix. Let $J$ be a $\mathcal{J}$-class with $J_{X_p} \leq J \leq GL_k(\mathbb{Z}_n)$, and suppose that $J$ lies immediately below $GL_k(\mathbb{Z}_n)$. We wish to show that $J = J_{X_p}$. By Lemma 5.1.8, $J \in \{J_{X_q} : X_q \in X_n\}$ as otherwise $J$ would not be generated by $X_n \cup GL_k(\mathbb{Z}_n)$. Let $q$ be the prime divisor of $n$ such that $J = J_{X_q}$. It follows that $X_p = AX_qB$ for some matrices $A, B \in M_k(\mathbb{Z}_n)$. The row space $\Lambda(AX_qB) = \Lambda(X_p)$ is an additive subgroup of $\Lambda(X_qB)$; since $\Lambda(X_p)$ is a maximal subgroup, and $|\Lambda(X_qB)| < |\Lambda(I)|$, it follows that $\Lambda(X_p) = \Lambda(X_qB)$. Right multiplication by $B$ is therefore a surjective homomorphism from $\Lambda(X_q)$ to $\Lambda(X_p)$, but by Lagrange’s Theorem no such homomorphism exists for $p \neq q$; hence $J = J_p$. □

References

[1] L. Babai. “Graph Isomorphism in Quasipolynomial Time”. 2016. arXiv:1512.03547 URL: http://arxiv.org/abs/1512.03547.

[2] B. Baker and R. Shostak. “Gossips and telephones”. In: Discrete Math. 2.3 (1972), pp. 191–193.

[3] J. D. Beule et al. Digraphs - GAP package, Version 1.4.1 May 2021. DOI: https://doi.org/10.5281/zenodo.596465 URL: https://digraphs.github.io/Digraphs.

[4] M. J. J. Branco, G. M. S. Gomes, and P. V. Silva. “On the semigroup rank of a group”. In: Semigroup Forum 99.3 (2019), pp. 568–578.

[5] M. Breen and D. Hume. “On the Cardinalities of Row Spaces of Boolean Matrices”. In: Semigroup Forum 62 (2001), pp. 331–332.

[6] M. Breen and D. Hume. “Properties of a standard form for a Boolean matrix”. In: Lin. Alg. Appl. 254.1-3 (1997). Proceeding of the Fifth Conference of the International Linear Algebra Society, pp. 49–65.

[7] A. E. Brouwer, J. Draisma, and B. J. Frenk. “Lossy gossip and composition of metrics”. In: Discrete Comput. Geom. 53.4 (2015), pp. 890–913.

[8] K. K. H. Butler. “The semigroup of Hall relations”. In: Semigroup Forum 9.3 (1974), pp. 253–260.

[9] D. D. Caen and D. A. Gregory. “Primes in the semigroup of boolean matrices”. In: Lin. Alg. Appl. 37 (1981), pp. 119–134.

[10] C. Carvalho and B. Martin. “The lattice and semigroup structure of multipermutations”. 2021. arXiv:2103.06012 [math.RA].

[11] H. H. Cho. “Prime Boolean matrices and factorizations”. In: Lin. Alg. Appl. 190 (1993), pp. 87–98.

[12] H. H. Cho. “Regular matrices in the semigroup of Hall matrices”. In: Linear Algebra Appl. 191 (1993), pp. 151–163.

[13] A. H. Clifford. “A proof of the Montague-Plemmons-Schein theorem on maximal subgroups of the semigroup of binary relations”. In: Semigroup Forum 1 (1970), pp. 272–275.

[14] L. Colarte and R. M. Miró-Roig. “Minimal set of binomial generators for certain Veronese 3-fold projections”. In: J. Pure Appl. Algebra 224.2 (2020), pp. 768–788.

[15] F. Dalla Volta and A. Lucchini. “Generation of almost simple groups”. In: J. Algebra 178.1 (1995), pp. 194–223.

[16] L. Daviaud, M. Johnson, and M. Kambites. “Identities in upper triangular tropical matrix semigroups and the bicyclic monoid”. In: J. Algebra 501 (2018), pp. 503–525.

[17] H. M. Devadze. “Generating sets of the semigroup of all binary relations in a finite set, (Russian)”. In: Doklady Akad. Nauk BSSR 12 (1968), pp. 765–768.
[18] I. Dimitrova et al. “Ranks of monoids of endomorphisms of a finite undirected path”. In: Bull. Malays. Math. Sci. Soc. 43.2 (2020), pp. 1623–1645.
[19] J. East, J. Jonušas, and J. D. Mitchell. “Generating the monoid of $2 \times 2$ matrices over max-plus and min-plus semirings”. 2020. arXiv: 2009.10372.
[20] J. East et al. “Computing finite semigroups”. In: Journal of Symbolic Computation 92 (2019), pp. 110–155.
[21] P. Fenner, M. Johnson, and M. Kambites. “NP-completeness in the gossip monoid”. In: Internat. J. Algebra Comput. 28.4 (2018), pp. 653–672.
[22] P. J. Fenner. “Some algorithmic problems in monoids of Boolean matrices”. PhD thesis. Manchester, UK: The University of Manchester, 2018.
[23] V. Froidure and J.-É. Pin. “Algorithms for computing finite semigroups”. In: Foundations of computational mathematics (Rio de Janeiro, 1997). Springer, Berlin, 1997, pp. 112–126.
[24] T. Gelander and R. Slutsky. “On the minimal size of a generating set of lattices in Lie groups”. In: J. Lie Theory 30.1 (2020), pp. 33–40.
[25] A. Hajnal, E. C. Milner, and E. Szemerédi. “A Cure for the Telephone Disease”. In: Canadian Math. Bull. 15.3 (1972), pp. 447–450.
[26] P. Hall. “On Representatives of Subsets”. In: Journal of the London Mathematical Society s1-10.1 (1935), pp. 26–30.
[27] C. Hollings and M. Kambites. “Tropical matrix duality and Green’s $\mathcal{D}$ relation”. In: J. Lond. Math. Soc. (2) 86.2 (2012), pp. 520–538.
[28] D. F. Holt and C. M. Roney-Dougal. “Minimal and random generation of permutation and matrix groups”. In: J. Algebra 387 (2013), pp. 195–214.
[29] J. M. Howie. Fundamentals of semigroup theory. Vol. 12. London Mathematical Society Monographs. New Series. Oxford Science Publications. New York: The Clarendon Press Oxford University Press, 1995. ISBN: 0-19-851194-9.
[30] Z. Izhakian and S. W. Margolis. “Semigroup identities in the monoid of two-by-two tropical matrices”. In: Semigroup Forum 80.2 (2010), pp. 191–218.
[31] M. Jackson. Personal communication. July 2021.
[32] M. Johnson and M. Kambites. “Green’s $\mathcal{J}$-order and the rank of tropical matrices”. In: Journal of Pure and Applied Algebra 217.2 (2013), pp. 280–292.
[33] M. Johnson and M. Kambites. “Idempotent tropical matrices and finite metric spaces”. In: Adv. Geom. 14.2 (2014), pp. 253–276.
[34] M. Johnson and N. M. Tran. “Geometry and algorithms for upper triangular tropical matrix identities”. In: J. Algebra 530 (2019), pp. 470–507.
[35] T. Junttila and P. Kaski. bliss. Version 0.73. Sept. 2015. URL: http://www.tcs.hut.fi/Software/bliss/.
[36] T. Junttila and P. Kaski. “Engineering an efficient canonical labeling tool for large and sparse graphs”. In: Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments and the Fourth Workshop on Analytic Algorithms and Combinatorics. Ed. by D. Applegate et al. SIAM, 2007, pp. 135–149.
[37] K. H. Kim. Boolean matrix theory and applications. Vol. 70. Monographs and textbooks in pure and applied mathematics. New York: Marcel Dekker, 1982.
[38] K. H. Kim and F. W. Roush. “Linear representations of semigroups of Boolean matrices”. In: Proceedings of the American Mathematical Society 63.2 (1977), pp. 203–207.
[39] J. Konieczny. “A proof of Devadze’s theorem on generators of the semigroup of Boolean matrices”. In: Semigroup Forum 83.2 (2011), pp. 281–288.
J. Konieczny. “On cardinalities of row spaces of Boolean matrices”. In: *Semigroup Forum* 44 (1992), pp. 393–402.

J. R. Li and Y. F. Luo. “On the finite basis problem for the monoids of triangular Boolean matrices”. In: *Algebra Universalis* 65.4 (2011), pp. 353–362.

W. Li and M.-C. Zhang. “On Konieczny’s conjecture of Boolean matrices”. In: *Semigroup Forum* 50.1 (1995), pp. 37–58.

G. Markowsky. “Ordering $\mathcal{D}$-classes and computing Schein rank is hard”. In: *Semigroup Forum* 44.1 (1992), pp. 373–375.

B. D. McKay and A. Piperno. “Practical graph isomorphism, II”. In: *Journal of Symbolic Computation* 60 (2014), pp. 94–112.

J. D. Mitchell et al. *libsemigroups*. Version 2.0.0. May 2021. DOI: https://doi.org/10.5281/zenodo.4813083 URL: https://github.com/libsemigroups/libsemigroups/

J. D. Mitchell and F. Smith. *Code and data files*. 2020. URL: https://github.com/BoolMat/BoolMat

J. S. Montague and R. J. Plemmons. “Maximal subgroups of the semigroup of relations”. In: *Journal of Algebra* 13.4 (1969), pp. 575–587.

J. Moor. “Generating sets for $F_2$ and its automorphism group”. In: *J. Algebra* 159.2 (1993), pp. 488–499.

OEIS Foundation. *The On-Line Encyclopedia of Integer Sequences*. 2020. URL: https://oeis.org/A227414

OEIS Foundation. *The On-Line Encyclopedia of Integer Sequences*. 2021. URL: https://oeis.org/A346686

OEIS Foundation. *The On-Line Encyclopedia of Integer Sequences*. 2021. URL: https://oeis.org/A346687

J.-É. Pin. *Semigroupe*. Version 2.01. Apr. 2009. URL: https://www.irif.fr/~jep/Logiciels/Semigroupe2.0/semigroupe2.html

J.-É. Pin and H. Straubing. “Monoids of upper triangular matrices”. In: *Colloq. Math. Soc. János Bolyai* 39 (1985), pp. 259–272.

R. J. Plemmons and M. T. West. “On The Semigroup Of Binary Relations”. In: *Pacific Journal of Mathematics* 35.3 (1970).

R. J. Plemmons and B. M. Schein. “Groups of binary relations”. In: *Semigroup Forum* 1 (1970), pp. 267–271.

G. Rosenberger. “Minimal generating systems of a subgroup of $SL(2, \mathbb{C})$”. In: *Proc. Edinburgh Math. Soc.* (2) 31.2 (1988), pp. 261–265.

F. W. Roush and K. H. Kim. “On generating regular elements in the semigroup of binary relations”. In: *Semigroup Form* 14 (1977), pp. 29–32.

Š. Schwarz. “The semigroup of fully indecomposable relations and Hall relations”. In: *Czechoslovak Math. J.* 23.1 (1973), pp. 151–163.

H. Shaofang. “Gaps in the Cardinalities of Row Spaces of Boolean Matrices.” In: *Semigroup Forum* 56 (1998), pp. 158–186.

Y. Shitov. “The complexity of tropical matrix factorization”. In: *Adv. Math.* 254 (2014), pp. 138–156.

I. Simon. “On semigroups of matrices over the tropical semiring”. In: *RAIRO Inform. Théor. Appl.* 28.3-4 (1994), pp. 277–294.

H. Straubing. “On finite $\mathcal{J}$-trivial monoids”. In: *Semigroup Forum* 19.2 (1980), pp. 107–110.
[63] Y. Tan. “The semigroup of Hall matrices over distributive lattices”. In: *Semigroup Forum* 61.2 (2000), pp. 303–314.

[64] D. Wilding, M. Johnson, and M. Kambites. “Exact rings and semirings”. In: *J. Algebra* 388 (2013), pp. 324–337.

[65] K. A. Zaretskii. “The Semigroups of Binary Relations”. In: *Mat. Sb. (N.S.)*, 61.103 (1963), pp. 291–305.

[66] W. T. Zhang, Y. F. Luo, and N. Wang. “Finite basis problem for involution monoids of unitriangular boolean matrices”. In: *Algebra Universalis* 81.1 (2020), Paper No. 7, 23.

[67] M. Zivkovic. “Row Space Cardinalities”. In: *Semigroup Forum* 73.3 (2006), pp. 404–426.