The spin Hall effect of radiofrequency waves in magnetic-fusion devices

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In inhomogeneous media, electromagnetic-wave rays deviate from the trajectories predicted by the leading-order geometrical optics. This effect, called the spin Hall effect of light, is typically neglected in ray-tracing codes used for modeling waves in plasmas. Here, we demonstrate that the spin Hall effect can be significant for radiofrequency waves in toroidal fusion plasmas. For example, an electron-cyclotron wave beam can deviate by as large as ten wavelengths (∼0.1 m) relative to the lowest-order ray trajectory in the poloidal direction. We calculate this displacement using gauge-invariant ray equations of extended geometrical optics, and we also compare our theoretical predictions with full-wave simulations.

Introduction.— Precise modeling of radiofrequency (RF) waves in magnetic-fusion experiments is essential for many applications, including cyclotron heating, current drive [1–3], and suppression of tearing modes [4, 5]. Waves with short wavelengths, particularly those in the electron-cyclotron and lower-hybrid frequency range, are commonly modeled with ray-tracing codes [6–11], which are based on geometrical optics (GO) [12, 13]. In the traditional, lowest-order GO, the evolution of the ray coordinate x and momentum k (wavevector) are governed by the Hamiltonian that equals the local dispersion function and ignores the local gradients of the medium parameters [14]. However, the evolution of the wave polarization in an inhomogeneous medium produces corrections that make rays deviate from the trajectories predicted by this lowest-order Hamiltonian [15, 16]. This effect, which is similar to the spin–orbital interaction in quantum mechanics, is known as the spin Hall effect (SHE) of light. It has been studied in various contexts [17–26], but only general theoretical investigation has been carried out in the plasma-physics context so far [27–29]. The importance of the SHE for practical plasma simulations has not been explored in detail, and the common ray-tracing codes used in fusion research ignore the SHE entirely [30].

Here, we show that the SHE can be significant for RF waves in fusion plasmas. We calculate this effect using equations of “extended GO” (XGO) as formulated in Refs. 27 and 29. By comparing XGO predictions with two-dimensional (2-D) full-wave (FW) simulations, we show that the ray equations that account for the SHE describe waves in fusion plasmas more accurately than the traditional ray equations. For three-dimensional (3-D) toroidal plasmas, we show that electromagnetic waves in the electron-cyclotron frequency range can deviate by as large as ten wavelengths in the poloidal plane relative to the GO predictions. Displacements of this size can be important in practice, for example, because processes like collisionless absorption and mode conversion can be very sensitive to the spatial distribution of the wave energy.

Basic equations.— Let us start by briefly restating the derivation of the XGO ray equations [28, 29]. Consider a multicomponent wave field Ψ on a given space (or spacetime) xμ and suppose that this wave is governed by an equation of the form DΨ = 0, where D is some linear dispersion operator generally of integro-differential form. Assume that the wave has an eikonal form Ψ(xμ) = e^iθ(xμ)ψ(xμ). Here, θ is a fast real phase and kμ = ∂/∂xμ is the associated wavevector, which is generally a field on xμ. (The symbol ≈ denotes definitions, kμ = ∂/∂xμ, and ∂μ = ∂/∂(kμ).) The multicomponent function ψ represents a slow complex envelope governed by Êψ = 0, where Ê = e^−iθDψ. We assume that the least scale L of the envelope dynamics is much smaller than the local wavelength λ ∼ 2π/|k|, yielding a small parameter ε = λ/L ≪ 1 (the “GO parameter”). Then, assuming Euclidean or pseudo-Euclidean (e.g., Minkowski) coordinates for simplicity, one obtains [29]

\[ Ê ≈ D(xμ, kμ(xμ)) - iVμ∂μ - i(∂μVμ)/2. \]  

(1)

Here, the “local dispersion tensor” D, which is a matrix function on the ray phase space (xμ, kμ), is the Weyl symbol of Ê, and Vμ = (∂μD(xμ, kμ))kμ=¯kμ(xμ). We assume that the Hermitian part of the dispersion tensor, DH ≡ (D + D†)/2, is O(1), while its anti-Hermitian part, DA ≡ (D − D†)/2i, is O(ε), so one can use Vμ(xμ) ≈ (∂μDH(xμ, kμ(xμ)))kμ=¯kμ(xμ). Then, DA determines local damping, while the wave propagation is determined entirely by DH, namely, as follows.

To the zeroth order in ε, Eq. (1) gives

\[ DH(xμ, kμ(xμ))ψ(xμ) = 0, \]  

(2)

so ψ is the eigenvector of DH corresponding to an eigenvalue Λ that is zero on the solution at ε → 0. To eliminate mode conversion, which makes the analysis more complicated [29], let us assume that DH(xμ, kμ) has only one eigenvalue that is zero at kμ = kμ(xμ); we call it an active mode. Then, Eq. (2) can be restated as follows. Consider the corresponding unit eigenvector η,

\[ DHη = Λη, \quad η†η = 1, \quad Λ = η†DHη, \]  

(3)

where all quantities are considered as functions of (xμ, kμ). Then, Λ(xμ, kμ(xμ)) = 0 serves as
an approximate dispersion relation, and \( \psi(x^\mu) = \eta(x^\mu, k_\mu(x^\mu))a(x^\mu) \), where \( a(x^\mu) \) is a scalar amplitude. To separate the phase dynamics from the amplitude dynamics completely, we attribute the whole phase that the envelope may have to \( \theta \) and \( \eta \), so the function \( a \) is real by definition. In particular, note that \( \eta \) is defined only up to \( e^{i\alpha} \), where \( \alpha \) is real and slow but otherwise arbitrarily. This constitutes a gauge symmetry of XGO, with \( \varphi \) being the gauge potential (see below).

To the first order in \( \epsilon \), Eq. (1) gives an amplitude equation

\[
(\Lambda - U)a - i[V^\mu \partial_\mu + (\partial_\mu V^\mu)/2 - \Gamma]a = 0. 
\]

(4)

Here, \( \Lambda \) is as in Eq. (3), \( \Gamma = \eta^i D_{\Lambda} \eta, U = \text{Im}[(\partial_\mu \eta^i)V^\mu \eta], \) and \( \eta = \eta(x^\mu, k_\mu(x^\mu)) \), so \( \partial_\mu \) applies to both its arguments. Since \( a \) is real by definition, the imaginary part of Eq. (1) gives a dispersion relation, and \( \eta \) is provided by a generic eigensolver, having its phase arbitrary makes computing an approximate dispersion relation, and \( \psi(x^\mu) = \eta(x^\mu, k_\mu(x^\mu))a(x^\mu) \), where \( \psi_\perp = \mathcal{O}(\epsilon) \) is perpendicular to \( \eta \). Then, projecting Eq. (1) on \( \eta \) gives [29]

\[
\sum_m \eta^\dagger_m (\partial_\mu D_H \eta)_m \eta^\dagger_m (\partial_\mu D_H \eta) \frac{\Lambda_m}{\Lambda_m}, 
\]

(11a)

and \( F^{\mu\nu} = 2 \text{Im} \sum_m \eta^\dagger_m (\partial_\mu D_H \eta)_m \eta^\dagger_m (\partial_\nu D_H \eta) \frac{\Lambda_m}{\Lambda_m} \). (11b)

Lowering the indices \( \mu \) and \( \nu \) on the right-hand side of Eq. (11b) yields the corresponding components of \( \Lambda \) with lower and mixed indices. Here, \( \eta_m \) and \( \Lambda_m \) are the unit eigenvectors and the corresponding eigenvalues of \( D_H \) that correspond to modes with \( \Lambda_m \neq \Lambda \); we call them passive modes. These equations have the benefit of “numerical gauge invariance” in that changing \( \eta \rightarrow e^{i\alpha} \eta \) leaves both \( U_0 \) nor \( F \) intact. It is also seen from Eqs. (11) that the SHE is amplified when one or more of the passive modes is in resonance with the active mode, namely, \( \Lambda_m \approx 0 \). In this regime, one can also expect mode conversion, i.e., tunneling of the wave action between separate dispersion surfaces. However, this tunneling scales with \( \Lambda_m \) exponentially [13, 33], while the SHE scales with \( \Lambda_m \) algebraically, so it can be amplified at small \( \Lambda_m \), while the mode conversion remains negligible.

**Waves in cold magnetized plasma.**—Let us apply the above results to a cold-plasma model, which is commonly used for ray tracing in fusion devices [2]. For simplicity, let us neglect the ion response, plasma flows, and dissipation, and let us also assume that the waves have a fixed frequency \( \omega \), as usual. Then, the linear-wave equations can be written as \( \hat{D} \Psi = 0 \), where \( D = D_H = H(x, k) - \omega \),

\[
\hat{H}(x, -i\partial_k) = \begin{pmatrix} -i\Omega(x) & \omega p(x) & 0 \\ -i\omega p(x) & 0 & i\epsilon \partial_k x \\ 0 & -i\epsilon \partial_k x & 0 \end{pmatrix}, 
\]

(12)

and \( \Psi(t, x) = (v, E, B)^T \) is a 9-dimensional column vector (the symbol \( T \) denotes transposition) that includes the properly normalized velocity \( v \), the wave electric field \( E \), and the wave magnetic field \( B \) [34]. Also, \( \times \) denotes vector product, as usual; \( \omega_p = |4\pi n^2/m|^{1/2} \) is the plasma frequency; \( q \), \( m \), and \( n(x) \) are the electron charge, mass, and background density, correspondingly; \( \Omega(x) = q B(x)/(m c) \) is the gyrofrequency; \( B(x) \) is the background magnetic field; and \( \epsilon \) is the speed of
light. In fusion applications of the electron waves governed by Eq. (12), one typically has $\omega \sim \omega_p \sim \Omega$, so we attribute this range, loosely, as the electron-cyclotron range. The eigenvectors of $H$ are the same as those of $H$, so $H \eta = (\Lambda m + \omega) \eta$.

**SHE in a plasma slab.**—First, let us consider waves propagating perpendicular to the magnetic field in a plasma slab, with coordinates denoted $x \equiv (x, y, z)$. There are three modes in this case: an O mode with $\omega > \omega_p$ and two X modes, with $\omega < \omega_{uh}$ and $\omega > \omega_{uh}$, respectively, where $\omega_{uh}$ is the upper-hybrid frequency, $\omega_{uh} = (\Omega^2 + \omega_p^2)^{1/2}$. We assume a homogeneous magnetic field along the z axis, with magnitude $|B| = 0.5 \, T$, so $\Omega \approx 8.8 \times 10^{18} \, s^{-1}$. We also adopt $\omega_p(x) = \omega_{p,0}(1 + x/L_0)$, with $L_0 = 1 \, m$ and $n(x = 0) = 10^{19} \, m^{-3}$, so $\omega_{p,0} \approx 1.8 \times 10^{18} \, s^{-1}$. The centers of the wave beams simulated pass through $x_0 = (0, 0, 0)$, where the wavevector is $k_0 = (-200, 0, 0) \, m^{-1}$, so $\omega \sim 0.03$. The ray-tracing simulations are performed, separately for O waves and X waves, using Eqs. (9) and (11). Deviations of the ray trajectory from the x axis in this geometry, if any, are entirely due to the SHE.

We also compare our XGO ray tracing with FW simulations, which we perform using the finite-difference time-domain method described in Ref. [35]. For simplicity, we assume that the system is uniform along the z axis ($\partial_z = 0$), so 2-D modeling is enough. For the FW simulations, the initial field is taken to be

$$\Psi = \eta e^{i k_0 \cdot x} \exp \left[-(x - x_0)^2/(2\sigma^2)\right] + O(\epsilon),$$

(13)

with $\sigma = 3.5 \, cm$. The (nonnegligible) term $O(\epsilon)$ is specified in Supplemental Material. For comparison with the ray-tracing simulations, “the” wave coordinate $x$ is defined as that of the maximum of the beam action density $I \equiv |\Psi|^2$, and the same applies to “the” wavevector $k$. The spatial grid size is $\Delta x = \Delta y = 8.0 \times 10^{-5} \, m \ll \lambda$. The temporal grid size is $\Delta t = 7.8 \times 10^{-6} \, ns$. Then, the phase space velocity of the wave is about $0.1(\Delta x/\Delta t) < \Delta x/\Delta t$, which ensures numerical stability [36].

As an example, we present typical results of numerical simulations for an X wave with $\omega < \omega_{\text{uh}}$ in Fig. 1. The wave packet moves roughly along the $-x$ axis with increasing wavenumber $k_x$. The trajectories from XGO and GO are very close to each other yet distinguishable at high resolution. As seen in the zoomed-in plot [Fig. 1(c)], the separation $\zeta$ between them is about 3 mm, and FW simulations are in better agreement with XGO than they are with GO. As could be expected from Eqs. (9), this separation constitutes about 1% of the ray path $\ell$, so $\zeta/\ell \sim \epsilon$. For all three modes, the comparison between XGO and FW simulations is presented in Fig. 2. In GO, all three rays would travel along $y = 0$, so the horizontal displacement of the X-wave rays is entirely due to the SHE. The O wave does not exhibit the SHE because its polarization vector $\eta$ remains parallel to the z-axis and has a fixed phase, so $\partial_y \eta \equiv 0$ and $\text{Im}(\partial^y \eta^0) \partial^y \eta \equiv 0$, so $U_0 \equiv 0$ by Eq. (6) and $F \equiv 0$ by Eqs. (10). Notably, this difference between the SHE for X and O waves is consistent with the fact that these modes have different Chern numbers $C$, which are interals of the Berry curvatures and represent waves’ topological invariants [37–39]. The O wave has $C = 0$, the X wave with $\omega < \omega_{\text{uh}}$ has $C = 2$, and the X wave with $\omega > \omega_{\text{uh}}$ has $C = -1$.

**SHE in a toroidal plasma.**—The SHE, which accumulates over time, can result in more significant $\zeta/\ell$ when the group velocity is small. To illustrate this, let us consider the propagation of electromagnetic waves in a toroidal plasma. We assume

$$\omega_p(x) = \omega_{p,0} \left[\exp\left(-\frac{(R - R_0)^2}{2\sigma_R^2} - \frac{z^2}{2\sigma_z^2}\right) + d\right],$$

(14)

FIG. 1. Simulations of an X wave with $\omega < \omega_{\text{uh}}$ in the $(x, y)$ plane. Shown are snapshots at $t = 0$ and $t = 4 \, ns$. (a) Re $B_z$, (b) $I$, (c) a zoom-in on the green rectangle from (b). The ray trajectories from the GO (blue curve), XGO (red curve), and FW (green discs) simulations.

where $R \equiv (x^2 + y^2)^{1/2}$, $\sigma_R^2 = 0.1 \, m^2$, $\sigma_z^2 = 1 \, m^2$, $d = 0.01$, and the density maximum is located at $(R_0, z_0) = (1, 0) \, m$, where $\omega_p = \omega_{p,0} = 1.8 \times 10^{11} \, s^{-1}$, same as before. We also assume that the magnetic field is toroidal, specifically, aligned with the unit-vector field $(-y/R, x/R, 0)$.
and \( \Omega(x) = \Omega_0 R_0 / R(x) \), with \( \Omega_0 = 8.8 \times 10^{10} \text{ s}^{-1} \), also same as before.

Consider a wave starting at \( x_0 = (0, 1.45, 0) \text{ m} \) with initial wavenumber \( k_0 = (-330, 150, -250) \text{ m}^{-1} \) and \( \omega \approx 3.1 \times 10^{10} \text{ s}^{-1} \), which is the lowest-frequency mode in the system. The numerical results are presented in Fig. 3 and show that the separation between the XGO and GO trajectories is as large as \( \zeta \sim 0.1 \text{ m} \), which is about ten wavelengths. Such a large deviation can significantly affect resonant absorption of radiofrequency waves in fusion applications. Therefore, typical ray-tracing codes that are based on GO instead of XGO are at risk of missing important physics, even if \textit{usually} the SHE is less pronounced than in our example.

\textbf{Conclusions.}— Here, we present the first systematic study of the SHE for magnetized plasmas relevant for fusion applications. We start with the XGO formulation of the SHE and derive a gauge-invariant form of the ray equations that describe SHE for general waves [Eq. (9)]. We also express the right-hand side in a form that is better suited for simulations due to its “numerical gauge invariance” [Eqs. (11)]. Then, we perform ray-tracing simulations based on these equations for electromagnetic waves in a cold magnetized collisionless electron plasma slab and compare them with 2-D FW simulations. We show that the FW simulations are in better agreement with XGO, which retains the SHE, than they are with GO simulations, which ignores the SHE. Finally, we present an example of a large SHE in toroidal plasma, where a wave beam deviates from the GO trajectory by a distance of roughly ten wavelengths. Such a large deviation can significantly affect resonant absorption of radiofrequency waves in fusion applications. Therefore, typical ray-tracing codes that are based on GO instead of XGO are at risk of missing important physics, even if \textit{usually} the SHE is less pronounced than in our example.

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FIG. 3. The propagation of an RF wave in the electron-cyclotron range in (a) the \((x, y)\) plane and (b) the \((R, z)\) plane. The gray scale indicates the local density or \(\omega_p\) (a) in the plane \(z = 0\) and (b) in any toroidal cross section. The stars mark the initial position of the rays. The GO and XGO rays are shown as blue and red curves, respectively.

FIG. 4. (a) The local GO parameter \(\epsilon = \lambda / L\) along the GO and XGO trajectories. The local length scale is calculated as \(L(x) \equiv \min(|\omega_p / \nabla \omega_p|, |\Omega / \nabla \Omega|)\). (b) The frequency \(\omega_m\) of the passive mode closest to the active mode evaluated on the XGO trajectory \((x(t), k(t))\), in units \(\omega_p\), as a function of the ray path. Since \(\omega_m / \omega - 1 \sim 1\), mode conversion is not to be expected.
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Quasioptical schemes [29, 40] do not infer the envelope evolution from the rays but rather evolve the envelope independently. As a result, the envelope is allowed to drift away from the reference ray and SHE is automatically accounted for as long as this effect is not too large.

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