THE CORRELATION CONSTANT OF A FIELD

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Abstract. We study the correlation of edges, vectors or elements to be in a randomly chosen spanning tree or a basis, respectively. Here we follow the guideline of Huh and Wang and introduce as a measure an invariant that is called the correlation constant of a graph, vector configuration, matroid or field. It follows from one of their results that these correlation constants are numbers between 0 and 2. Here, we show that the correlation constant of every field is at least $\frac{8}{7}$. In our proof we explicitly construct vector configurations and matroids with positively correlated elements.

1. Introduction

This article deals with a basic question which appears in both the theory of graphs and finite geometries and ask how strongly independence of edges in a graph or vectors in a configuration is correlated.

First we consider graphs and their edges. Let $G$ be a finite connected graph, $i$, $j$ two edges of $G$ and consider a uniform distribution on the spanning trees of $G$. We denote by $\Pr(i \in T)$ the probability that the edge $i$ is in a randomly chosen tree $T$.

**Question 1.** What is the correlation between the probabilities $\Pr(i \in T)$ and $\Pr(j \in T)$ for distinguishable edges $i$ and $j$?

From Kirchhoff’s law in an electrical network Brooks, Smith, Stone and Tutte derive an equation [BSST40, Equation (2.34)] which implies that the above events are negatively correlated, i.e., the covariance $\Pr(i, j \in T) - \Pr(i \in T) \cdot \Pr(j \in T)$ is negative. This plays a central role in Tutte’s characterization of graphs with a constant number of spanning trees through any two edges; see [Tut74].

Now we take a look at vectors. Given a field $\mathbb{K}$, let $\mathcal{P}$ be a vector configuration in a $\mathbb{K}$-vector space with a uniform distribution on the basis formed by vectors of $\mathcal{P}$. The central number of this article is the correlation constant $\beta(\mathcal{P}) = \max_{v, w \in \mathcal{P}} \frac{\Pr(v, w \in B)}{\Pr(v \in B) \cdot \Pr(w \in B)}$ of the configuration $\mathcal{P}$, where $B$ is a randomly chosen basis. Huh and Wang [HW17] asked for the following.

**Question 2.** How large can the correlation constant (for a given field) be?

The correlation constant $\beta_{\mathbb{K}}$ of a field is the supremum of all correlation constants taken over all vector configurations. The aim of this article is to give an explicit lower bound on this correlation constant.

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The common language of graphs and finite vector configurations is matroid theory. The monographs of Oxley \[Oxl11\] and White \[Whi86\] serve as the foundation for this article. A matroid $M$ is a non-empty collection $\mathcal{B}$ of subsets of a finite set $E$ with the property that for every pair $B, B' \in \mathcal{B}$ and any element $e \in B \setminus B'$ an element $e' \in B' \setminus B$ exists such that $B \setminus \{e\} \cup \{e'\} \in \mathcal{B}$. The set $E$ is called the ground set, the sets in $\mathcal{B}$ are the bases of the matroid $M$ and a loop is an element that does not occur in any basis. In the following we will assume that the ground set $E$ is $[n] = \{1, 2, \ldots, n\}$. The questions above lead to the following definition.

**Definition 1.** Let both $i$ and $j$ be elements of $M$ that are not loops. Then we define

$$
\beta(M; i, j) := \frac{b \cdot b_{ij}}{b_i \cdot b_j}.
$$

Where $b$, $b_i$, $b_j$ and $b_{ij}$ are the numbers of bases of $M$, the number of bases containing $i$, $j$ or $i, j$, respectively. Assume that $M$ has at least two non-loops, then the correlation constant of $M$ is the number $\beta(M) := \max_{i, j} \beta(M; i, j)$, where the maximum ranges over all non-loops $i, j$ of $M$.

Question 1 asks about the correlation constant of graphical matroids, while Question 2 is about the correlation constant of $K$-representable matroids.

Seymour and Welsh \[SW75, Conjecture 4\] conjectured that the correlation constant of any matroid is bounded by one, i.e., the elements of a matroid are negatively correlated. About fifty years ago Rota conjectured that the coefficients of the characteristic polynomial of a matroid are log-concave, the coefficients $a_k$ satisfy $a_k^2 \geq a_{k1} \cdot a_{k+1}$. Seymour and Welsh \[SW75\] claimed that their conjecture implies log-concavity of the coefficients. However, they found a rank four binary matroid on eight elements with correlation constant $\frac{48 \cdot 12}{20 \cdot 28} = \frac{36}{35} \approx 1.02$; cf. Example 9 below. This counterexample has been originally published as a note added in proof \[SW75\].

Recently, Adiprasito, Huh and Katz \[AHK15\] proved Rota’s conjecture as a key they introduced a version of the Hodge-Riemann bilinear relations for the Chow ring of a matroid. Surveys to their remarkable techniques are \[AHK17\] and \[Bak18\]. Huh and Wang \[HW17\] developed a variant of their combinatorial approach using a Hodge-Riemann form over a Möbius algebra which is generated by the variables associated with the elements in the ground set of a matroid. A bilinear pairing between two of these variables is given by the numbers $b_{ij}$ from above if $i \neq j$ and 0 otherwise. They claim in \[HW17\] that the Hodge-Riemann form of a simple matroid has exactly one positive eigenvalue and that a proof will be part of \[HW\]. Cauchy’s interlacing for symmetric matrices shows that this property is preserved when restricting to a well chosen 3-dimensional subspace. The obtained form has a positive determinant, which is $2(r - 1)^2 \cdot b_i b_j b_{ij} - r(r - 1) \cdot b_{ij}^2$. Hence the correlation constant of every matroid is bounded by two.

As mentioned before, the aim of this article is the following new lower bound on the correlation constant $\beta_K$ of a field, i.e., the suprema of correlation constants for $\mathbb{K}$-representable matroids.

**Theorem 2.** The correlation constant $\beta_K$ of the field $\mathbb{K}$ satisfies $\frac{8}{7} \leq \beta_K \leq 2$. 
This statement is a central part of Theorem 11. As a tool we introduce another invariant $\alpha(M)$ of a matroid $M$; cf. Definition 3. In Theorem 6 we show that this $\alpha$-ratio is an upper bound for the correlation constant of a matroid with positive correlation.

We apply Theorem 6 to the large class of sparse paving matroids whose $\alpha$-ratio is bounded by one. Hence we conclude that elements in these matroids are negatively correlated. It is conjectured by Mayhew, Newman, Welsh and Whittle in [MNWW11, Conjecture 1.6] that almost all matroids are sparse paving. This conjecture has its origin in a question of Welsh stated in [Wel71].

We continue by constructing a sequence of matroids $M_k$ that we derive from a matroid $M$, such that the sequence of correlation constants $\beta(M_k)$ converges monotonically to $\alpha(M)$; see Lemma 8. This completes our comparison of the $\alpha$-ratio and the correlation constant of a matroid.

In the last section, we present new examples of matroids with positive correlation and apply Theorem 6 and Lemma 8 from the previous section. In particular, we deduce Theorem 11 and that the correlation constant of any field is at least $\frac{8}{7} \approx 1.14$.

### 2. The $\alpha$-Ratio and the Correlation Constant of a Matroid

For our next steps we introduce further notation. We denote by $b_i^j = b_i - b_{ij}$ the number of bases of $M$ containing $i$ and not $j$. Similar we define $b_j^i = b_j - b_{ij}$ and $b_j^i = b - b_i - b_j^i$. Now we are able to give a definition of a second ratio.

**Definition 3.** Let $i, j$ neither be loops, coloops nor parallel elements in $M$, then we define

$$\alpha(M; i, j) := \frac{b_j^i \cdot b_{ij}}{b_i^j \cdot b_j^i}.$$ 

Let $M$ be a matroid that has at least a valid pair of elements, then the $\alpha$-ratio $\alpha(M)$ of $M$ is the maximum $\max_{i,j} \alpha(M; i, j)$ over all valid pairs of elements $i, j$.

Note that the numbers occurring in Definition 1 and Definition 3 are the numbers of bases in deletions and contractions of the two elements $i$ and $j$.

As a first example for these definitions let us take a look at uniform matroids. A matroid whose collection of bases is formed by all $r$-sets of $[n]$ is called the uniform matroid $U_{r,n}$. Clearly the number of its bases is given by the binomial coefficient $\binom{n}{r}$. The class of uniform matroids is minor closed, i.e., deletions and contractions are again uniform matroids.

**Example 4.** Let $1 < r < n$. We get for every pair $i, j$ in the uniform matroid $U_{r,n}$

$$\alpha(U_{r,n}, i, j) = \frac{(r - 1) \cdot (n - r - 1)}{r \cdot (n - r)} \quad \text{and} \quad \beta(U_{r,n}, i, j) = \frac{n \cdot (r - 1)}{r \cdot (n - 1)}$$

and hence we have $0 < \alpha(U_{r,n}) < \beta(U_{r,n}) < 1$.

We do not define an $\alpha$-ratio for the uniform matroids $U_{0,n}$, $U_{1,n}$ and $U_{n,n}$. In general, each pair of elements in a matroid is either parallel, contains a loop or a coloop if and only if the matroid is a direct sum of the form $U_{1,n_1} \oplus U_{0,n_2} \oplus U_{n_3,n_3}$. 
We start now with analyzing the relation between the $\alpha$-ratio and the correlation constant of a matroid.

**Proposition 5.** Let $M = M_1 \oplus M_2$ be a disconnected matroid and $i, j, \ell$ are neither loops, coloops nor pairwise parallel. If $i$ and $j$ are in the same connected component $M_1$, then we have

$$\alpha(M; i, j) = \alpha(M_1; i, j) \quad \text{and} \quad \beta(M; i, j) = \beta(M_1; i, j).$$

Further, if $i$ and $\ell$ are disconnected in $M$ then we have the independency

$$\alpha(M; i, \ell) = \beta(M; i, \ell) = 1.$$

**Proof.** The number $b(M)$ of bases of a disconnected matroid $M = M_1 \oplus M_2$ decomposes into the product $b(M_1) \cdot b(M_2)$.

For elements $i, j$ in $M_1$ and $\ell$ $M_2$ we obtain the factorizations

$$b_i(M) = b_i(M_1) \cdot b(M_2), \quad b_{ij}(M) = b_{i,j}(M_1) \cdot b(M_2), \quad b^1_i(M) = b^1_i(M_1) \cdot b(M_2),$$

$$b^0_i(M) = b^0_i(M_1) \cdot b^0(M_2), \quad b^1_i(M) = b^1_i(M_1) \cdot b(M_2), \quad b^0_i(M) = b^0_i(M_1) \cdot b^0(M_2).$$

Substitution into the definitions provides the desired equations. \qed

Now let us proceed and include connected matroids in our considerations.

**Theorem 6.** Let $M$ be a matroid, $i$ and $j$ be neither loops nor parallel. Then one of the following four conditions holds

$$0 < \alpha(M; i, j) < \beta(M; i, j) < 1 \quad \text{or} \quad 1 = \alpha(M; i, j) = \beta(M; i, j) \quad \text{or} \quad 0 = \alpha(M; i, j) = \beta(M; i, j).$$

In particular, for a matroid with positive correlated elements the $\alpha$-ratio is an upper bound for its correlation constant.

**Proof.** The correlation constant of $M$ in terms of $b^i, b^0_i, b^0_j$ and $b_{ij}$ is

$$\beta(M; i, j) = \frac{(b^i + b^0_i + b^0_j + b_{ij}) \cdot b_{ij}}{(b^i + b_{ij}) \cdot (b^0_i + b_{ij})} = \frac{b^i \cdot b_{ij} + (b^0_i + b^0_j + b_{ij}) \cdot b_{ij}}{b^i \cdot b^0_i + (b^0_i + b^0_j + b_{ij}) \cdot b_{ij}}.$$

Clearly $b_{ij} = 0$ implies the equality $\alpha(M; i, j) = \beta(M; i, j) = 0$. The expression $\frac{(b^i + b^0_i + b_{ij}) \cdot b_{ij}}{b^i \cdot b^0_i}$ is positive whenever $b_{ij}$ does not vanish and the above Equation (1) is equivalent to

$$\frac{(b^i + b^0_i + b_{ij}) \cdot b_{ij}}{b^i \cdot b^0_i} \cdot (1 - \beta(M; i, j)) = \beta(M; i, j) - \alpha(M; i, j).$$

This confirms that exactly one of the four claimed cases applies. \qed
We will now take a look at a large class of matroids. A matroid $S$ is called sparse paving if and only if every $r$-subset of $[n]$ is either a basis or a circuit of $S$. It is conjectured that almost all matroids are sparse paving; see [MNWW11, Conjecture 1.6],[Oxl11, Conjecture 15.5.10].

**Proposition 7.** The $\alpha$-ratio of a sparse paving matroid $S$ satisfies $\alpha(S) \leq \beta(S) \leq 1$.

**Proof.** Let $S$ be a sparse paving $r$-matroid on $n$ elements. The deletion of an element and contraction of another leads to a sparse paving matroid of rank $r - 1$ on $n - 2$ elements. Such a matroid has at least $\frac{n-r-1}{r-1} \binom{n-2}{r-2}$ bases if $2r \leq n$ and at least $\frac{n-1}{r-1} \binom{n-2}{r-2}$ bases otherwise. This follows from [MNRRInVF12, Theorem 4.8] and the fact that the class of sparse paving matroids is dually closed. These numbers give a lower bound on $b^i_j$ and $b^j_i$. Clearly, $b^{ij}$ is the number of bases of a $r$-matroid and $b_{ij}$ of a $(r-2)$-matroid on $n-2$ elements, hence these numbers are bounded from above by the corresponding binomial coefficients. Applying these bounds we get the following estimations.

\[
\alpha(S) \leq \left( \frac{r - 1}{n - r - 1} \right)^2 \frac{(n-2)}{r-2} \leq \frac{(r - 1) \cdot (n-r)}{r \cdot (n-r-1)} \leq 1 \text{ if } 2r \leq n \text{ and }
\]
\[
\alpha(S) \leq \left( \frac{n - r - 1}{r - 1} \right)^2 \frac{(n-2)}{r-2} \leq \frac{r \cdot (n-r-1)}{(r-1) \cdot (n-r)} < 1 \text{ if } 2r > n.
\]

From $\alpha(S) \leq 1$ and Theorem 6 the claim follows. $\Box$

Now we want to construct a sequence of matroids $M_k$ with the property that $\beta(M_k)$ converges to $\alpha(M)$. Let $i, j$ be a valid pair of elements of the $r$-matroid $M$ on $n$ elements and $M_k$ the matroid that is obtained from $M$ by adding $k-1$ parallel copies of each element other than $i$ and $j$. The $r$-matroid $M_k$ consists of $k \cdot (n-2) + 2$ elements, and the sequence of those matroids fulfills the desired property.

**Lemma 8.** The sequence $\beta(M_k; i, j)$ converges monotonically to $\alpha(M; i, j)$.

**Proof.** The numbers of bases of the deletions and contractions of the matroid $M_k$ satisfy the following equations due to the fact that we have $k$ choices to form a basis for every element in $M$ that is neither $i$ nor $j$.

\[
b_{ij}(M_k) = k^r \cdot b_{ij}(M), \quad b^i_j(M_k) = k^{r-1} \cdot b^i_j(M)
\]
\[
b^j_i(M_k) = k^{r-1} \cdot b^j_i(M), \quad b^{ij}(M_k) = k^r \cdot b^{ij}(M).
\]

Hence $\alpha(M; i, j) = \alpha(M_k; i, j)$ and Equation (1) turns into

\[
\beta(M_k; i, j) = \frac{k^2 \cdot (b^{ij} \cdot b_{ij}) + (k \cdot b^i_j + k \cdot b^j_i + b_{ij}) \cdot b_{ij}}{k^2 \cdot (b^i_j \cdot b^j_i) + (k \cdot b^i_j + k \cdot b^j_i + b_{ij}) \cdot b_{ij}}
\]

which converges clearly to $\alpha(M; i, j)$. The monotonicity can be read off from the numerator of the derivative with respect to $k_i$, which is $(b^{ij} \cdot b_{ij} - b^i_j \cdot b^j_i) \cdot (k^2 \cdot b^i_j + k^2 \cdot b^j_i + 2k \cdot b_{ij}) \cdot b_{ij}$. $\Box$
3. Examples of matroids with positive correlations

Our aim in this section is to construct examples of (representable) matroids with a positive correlation.

Let $p$ be a prime number, $\mathbb{F}_p$ the prime field of characteristic $p$ and $r \geq 2$ an integer. Consider the following vector configuration in $\mathbb{F}_p^r$ given by the $2 + p \cdot (r - 1)$ vectors:

$$e_1, \ v = \sum_{\ell=2}^{r} e_\ell \quad \text{and} \quad v_{k,\ell} = k \cdot e_1 + e_\ell \ \text{for} \ 1 < \ell \leq r \ \text{and} \ 0 \leq k < p.$$  

Let $M_{r,p}$ denote the corresponding realizable $r$-matroid, with two special elements. The element $i$ that corresponds to the vector $e_1$, and the element $j$ that corresponds to $v$.

To the best of the author’s knowledge the following example is the only published example of a matroid with positive correlated elements.

Example 9. The matroid $M_{4,2}$ is the example given by Seymour and Welsh. Its correlation constant is $\beta(M_{4,2}) = \frac{36}{35}$.

We now determine the numbers of bases $b^i_j$, $b_{ij}$, $b^{ij}$ and $b^r_j$ of all combinations of deletions and contractions of the two elements $i$ and $j$ in the matroid $M_{r,p}$.

The projection to the last $r - 1$ coordinates corresponds to the contraction of $i$. The obtained vector configuration consists of the all ones vector which is the projection of $v$, and $p$ copies of each of the $r - 1$ standard vectors. Deleting the vector $v$ leads to $p$ choices of each standard vector. Hence, $b^r_j = p^{r-1}$.

If we contract $j$, then in each basis exactly one of the $r - 1$ standard vectors is not appearing and therefore $b_{ij} = (r - 1) \cdot p^{r-2}$.

Note that for every index $\ell$ the three vectors $v_{k_1,\ell}$, $v_{k_2,\ell}$, $v_{k_3,\ell}$ are dependent. Hence, an index maximally appears twice in a basis. A consequence is that each basis that does not contain $e_1$ and $v$ consists of exactly one pair of vectors $v_{k_1,\ell}$, $v_{k_2,\ell}$ for an index $2 \leq \ell \leq r$ and values $0 \leq k_1, k_2 < p$. There are $r - 1$ possibilities for the index $\ell$ and $\frac{p(p-1)}{2}$ choices for $k_1 \neq k_2$. The vector $e_1$ lies in the span of $v_{k_1,\ell}$ and $v_{k_2,\ell}$ and with the arguments from before we get that we have $p^{r-2}$ choices for the other elements to form a basis. We conclude that $b^{ij} = (r - 1) \cdot \frac{(p-1)}{2} \cdot p^{r-1}$.

The last remaining case deals with the deletion of $i$ and contraction of $j$. A set of vectors $\{v_{k,\ell} | 2 \leq \ell \leq r\}$ forms a basis with $v$ in the original vector configuration if and only if the sum $\sum_{\ell=2}^{r} k_\ell$ does not vanish. Clearly this sum depends on the characteristic. In characteristic $p$ there are $p^{r-2} \cdot (p-1)$ of these bases. There are further bases that contain a pair of vectors $v_{k_1,\ell}$, $v_{k_2,\ell}$ for an index $\ell$ and omit one of the other $r - 2$ indices. These are $(r - 1)(r - 2) \cdot \frac{p(p-1)}{2} \cdot p^{r-3}$ additional bases. In total this leads to $b^r_j = \frac{p^{r-2}(p-1)^2}{2} \cdot (2 + (r - 1)(r - 2))$. We summarize our results.

Lemma 10. Let $r \geq 4$ and $p$ be a prime number. The matroid $M_{r,p}$ has a positively correlated pair of elements. Its $\alpha$-ratio is $\alpha(M_{r,p}) = \frac{(r-1)^2}{2 + (r-1)(r-2)}$. This ratio is maximal $\frac{8}{7}$ for $r = 5$. 
With arguments as above, the configuration in (2) embedded in the rational vector space \( \mathbb{Q}^r \) yields

\[
b_{ij} = (r - 1)p^{r-2}, \quad b^{ij} = (r - 1) \cdot \frac{p^{r-1}}{2}, \quad b^i_j = p^{r-1} - 1 + (r - 1)(r - 2) \frac{-1}{2} p^{r-2},
\]

as the sum \( \sum_{t=2}^r k_t = 0 \) if and only if all the non negative summands vanish. Hence the \( \alpha \)-ratio of this matroid converges to \( \frac{(r-1)^2}{2+(r-1)(r-2)} \) as \( p \to \infty \).

Theorem 11. The following inequalities hold for any class \( C \) of \( r \)-matroids that is closed under parallel extensions and contains a matroid with a positive correlation.

\[
1 < \sup_{M \in C} \alpha(M) = \sup_{M \in C} \beta(M) \leq 2 \cdot \frac{r - 1}{r}.
\]

In particular, the correlation constant \( \beta_K \) of the field \( K \) satisfies \( \frac{8}{7} \leq \beta_K \leq 2 \).

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References

[AHK15] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. Preprint arXiv:1511.02888, 2015.

[AHK17] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory of matroids. Notices Amer. Math. Soc., 64(1):26–30, 2017.

[Bak18] Matthew Baker. Hodge theory in combinatorics. Bull. Amer. Math. Soc. (N.S.), 55(1):57–80, 2018.

[BSST40] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte. The dissection of rectangles into squares. Duke Math. J., 7:312–340, 1940.

[HW] June Huh and Botong Wang. Mason’s conjecture and the hodge–riemann relations for matroids. In preparation.

[HW17] June Huh and Botong Wang. Enumeration of points, lines, planes, etc. Acta Math., 218(2):297–317, 2017.

[MNRrInVF12] Criel Merino, Steven D. Noble, Marcelino Ramí rez Ibáñez, and Rafael Villarroel-Flores. On the structure of the \( h \)-vector of a paving matroid. European J. Combin., 33(8):1787–1799, 2012.

[MNWW11] Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle. On the asymptotic proportion of connected matroids. European J. Combin., 32(6):882–890, 2011.

[Oxl11] James Oxley. Matroid theory, volume 21 of Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, second edition, 2011.

[SW75] P. D. Seymour and D. J. A. Welsh. Combinatorial applications of an inequality from statistical mechanics. Math. Proc. Cambridge Philos. Soc., 77:485–495, 1975.

[Tut74] W. T. Tutte. A problem on spanning trees. Quart. J. Math. Oxford Ser. (2), 25:253–255, 1974.

[Wel71] D. J. A. Welsh. Combinatorial problems in matroid theory. In Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), pages 291–306. Academic Press, London, 1971.
Neil White, editor. *Theory of matroids*, volume 26 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1986.

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