GAMES OF FIXED RANK:
A HIERARCHY OF BIMATRIX GAMES

RAVI KANNAN AND THORSTEN THEOBALD

Abstract. We propose a new hierarchical approach to understand the complexity of the open problem of computing a Nash equilibrium in a bimatrix game. Specifically, we investigate a hierarchy of bimatrix games \((A, B)\) which results from restricting the rank of the matrix \(A + B\) to be of fixed rank at most \(k\). For every fixed \(k\), this class strictly generalizes the class of zero-sum games, but is a very special case of general bimatrix games. We show that even for \(k = 1\) the set of Nash equilibria of these games can consist of an arbitrarily large number of connected components. While the question of exact polynomial time algorithms to find a Nash equilibrium remains open for games of fixed rank, we can provide polynomial time algorithms for finding an \(\varepsilon\)-approximation.

1. Introduction

Models of non-cooperative game theory serve to analyze situations of strategic interactions. Driven by current developments in auction theory as well as in equilibria models for the internet, the basic model of a Nash equilibrium has recently attracted much attention (see for example the survey by Papadimitriou [14] or the recent papers [1, 4, 15, 17]).

In [13], von Neumann and Morgenstern introduced the model of zero-sum games, which are described by a single \(m \times n\)-matrix \(A\). These games always possess an equilibrium, and the set of all equilibria (which is a polyhedral set and thus in particular connected) can be computed efficiently using linear programming (see, e.g., [3]).

Nash investigated the model of bimatrix games \((A, B)\) (and more generally \(N\)-player games) [11, 12], in which the gain of one player does not necessarily agree with the loss of the other player, thus adding much expressive power to the model of zero-sum games. By Nash’s results any bimatrix game has at least one equilibrium. However, it is still not known whether an equilibrium can be computed in polynomial time, and that question has been named by Papadimitriou to be the most concrete open question on the boundary of \(P\) [14]. Even approximability in polynomial time is not known; for quasi-polynomial time approximation algorithms see Lipton, Markakis, and Mehtat [8].

Thus, it will be of interest to impose restrictions on bimatrix games which while preserving expressive power of the games may admit simple polynomial time algorithms. Recently, Lipton et al. [8] investigated games where both payoff matrices \(A, B\) are of fixed rank \(k\). They showed that in this restricted model a Nash equilibrium can be found in polynomial time. However, for a fixed rank \(k\), the expressive power of that model is limited; in particular, most zero-sum games do not belong to that class.

Part of this work was done while the second author was a Feodor Lynen fellow of the Alexander von Humboldt Foundation at Yale University.
In this paper, we propose and investigate a related model based on low-rank restrictions, but which is a strict superset of the model of zero-sum games. The viewpoint we start with is that in a zero-sum game, the sum of the payoff matrices \( C := A + B \in \mathbb{R}^{m \times n} \) is the zero matrix, which for our purposes we consider as a matrix of rank 0. In a general bimatrix game the rank of \( C \) can take any value up to \( \min\{m, n\} \). Here, we consider the hierarchy given by the class of games in which we restrict \( C \) to be of rank at most \( k \) for some given \( k \). We call these games \emph{rank} \( k \)-\emph{games}.

**Our contributions.** We show that the expressive power of fixed rank-games is significantly larger than that of zero-sum games. In order to provide this separation, we exhibit a sequence of \( d \times d \)-games of rank 1 whose number of connected components of equilibria exceeds any given constant. Our lower bound for the maximal number of Nash equilibria of a \( d \times d \)-game is linear in \( d \). This bound is not tight.

Although the problem of finding a Nash equilibrium in a game of fixed rank is a very special case of the problem of finding a Nash equilibrium in an arbitrary bimatrix game, we do not know if there exists an exact polynomial time algorithm for this problem. Note that the problem strictly generalizes linear programming (see, e.g., [3, Ch. 13.2] for the equivalence of linear programming and zero-sum games).

However, we provide approximation results for two approximation models. Firstly, we propose a model of \( \varepsilon \)-approximation for rank \( k \)-games. Using existing results from quadratic optimization, we show that we can approximate Nash equilibria of constant rank-games in polynomial time, with an error relative to a natural upper bound on the “maximum loss” of the game (as defined in Section 4.1).

Secondly, we present a polynomial time approximation algorithm for \emph{relative} approximation (with respect to the payoffs in a Nash equilibrium) provided that the matrix \( C \) has a nonnegative decomposition.

2. **Preliminaries**

We consider an \( m \times n \)-bimatrix game with payoff matrices \( A, B \in \mathbb{Z}^{m \times n} \). Let

\[
S_1 = \{ x \in \mathbb{R}^m : \sum_{i=1}^m x_i = 1, \ x \geq 0 \} \quad \text{and} \quad S_2 = \{ y \in \mathbb{R}^n : \sum_{j=1}^n y_j = 1, \ y \geq 0 \}
\]

be the sets of mixed strategies of the two players, and let \( \overline{S}_1 = \{ x \in \mathbb{R}^m : \sum_{i=1}^m x_i = 1 \} \) and \( \overline{S}_2 = \{ y \in \mathbb{R}^n : \sum_{j=1}^n y_j = 1 \} \) denote the underlying linear subspaces. The first player (the row player) plays \( x \in S_1 \) and the second player (the column player) plays \( y \in S_2 \). The payoffs for player 1 and player 2 are \( x^T A y \) and \( x^T B y \), respectively.

Let \( C^{(i)} \) denote the \( i \)-th row of a matrix \( C \) (as a row vector), and let \( C_{(j)} \) denote the \( j \)-th column of \( C \) (as a column vector). A pair of mixed strategies \( (\overline{x}, \overline{y}) \) is a Nash equilibrium if

\[
(2.1) \quad \overline{x}^T A \overline{y} \geq x^T A y \quad \text{and} \quad \overline{x}^T B \overline{y} \geq x^T B y
\]

for all mixed strategies \( x, y \). Equivalently, \( (\overline{x}, \overline{y}) \) is a Nash equilibrium if and only if

\[
(2.2) \quad \overline{x}^T A \overline{y} = \max_{1 \leq i \leq m} A^{(i)} \overline{y} \quad \text{and} \quad \overline{x}^T B \overline{y} = \max_{1 \leq j \leq n} x^T B_{(j)}.
\]
2.1. Economic interpretation of low-rank games. If $A + B = 0$ then the game is called a zero-sum game. The economic interpretation of a zero-sum game is “What is good for player 1 is bad for player 2”. In order to describe game-theoretic situations which are close to that behavior, we consider a model where $a_{ij} + b_{ij}$ is a function which depends only on $i$ and $j$

$$a_{ij} + b_{ij} = f(i, j)$$

and where $f$ is a simple function. If $f : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \mathbb{Z}$ is an additive function, $f(i, j) = u_i + v_j$ with constants $u_1, \ldots, u_m, v_1, \ldots, v_n$, then there is an equivalent zero-sum game, i.e., a game having the same set of Nash equilibria. Namely, define the payoff matrices $A'$ and $B'$ by

$$a'_{ij} = a_{ij} - v_j, \quad b'_{ij} = b_{ij} - u_i.$$ 

That is, $A'$ results from $A$ by adding the column vector $(v_1, \ldots, v_n)^T$ to the $j$-th column $(1 \leq j \leq n)$ and $B'$ results from $B$ by adding the row vector $(u_1, \ldots, u_m)$ to the $i$-th row $(1 \leq i \leq m)$. Now

$$x^T A' y - x^T A y = x^T A y - \sum_{j=1}^{n} v_j y_j - x^T A y + \sum_{j=1}^{n} v_j y_j = x^T A y - x^T A y$$

and a similar relation w.r.t. $B$ holds. So the zero-sum game $(A', B')$ has the same Nash equilibria as $(A, B)$. We remark that the case $v_j = 0$ yields the row-constant games introduced in [5].

If $f$ is a multiplication function, $f(i, j) = u_i v_j$ with constants $u_1, \ldots, u_m, v_1, \ldots, v_n$, this is a rank 1-game. If $f$ is a sum of $k$ multiplication functions, this is a game of rank at most $k$.

Rank-1 games also occur under the term “multiplication games” in the paper [2] by Bulow and Levin.

2.2. Approximate Nash equilibria. We also consider approximate equilibria. To define them, suppose $x$ is not necessarily an optimal strategy for player 1 given that player 2 has played $y$. Then the “loss” for player 1 (from optimum) is $\max_i A^{(i)} y - x^T A y$. Similarly, if $y$ is not optimal for player 2 given that the first player has played $x$, the loss for player 2 would be $\max_j x^T B^{(j)} - x^T B y$. We will use the total of these two losses – i.e.,

$$\ell(x, y) = \max_i A^{(i)} y + \max_j x^T B^{(j)} - x^T (A + B) y$$

as a measure of how much $(x, y)$ is off from equilibrium. For a matrix $X \in \mathbb{R}^{m \times n}$ let $|X| = \max_{1 \leq i \leq m, 1 \leq j \leq n} |x_{ij}|$.

**Definition 2.1.** For $\varepsilon \geq 0$, a pair $(x, y)$ of mixed strategies is an $\varepsilon$-approximate equilibrium if

$$\ell(x, y) \leq \varepsilon |A + B|.$$ 

Note that the term $|A + B|$ on the right hand side provides a stronger approximation model compared to the term $|A| + |B|$. Also observe that $|A + B|$ is an upper bound for
the term $x^T(A + B)y$. For a game with $A - B \neq 0$, a pair of strategies is an exact equilibrium if and only if it is a 0-approximate equilibrium. Besides the notion of “absolute” approximation in Definition 2.1 in Section 4.2 we will also consider a notion of “relative” approximation.

**Lemma 2.2.** Suppose $(\bar{x}, \bar{y})$ is an $\varepsilon$-approximate equilibrium. Then

$$x^T A \bar{y} + \bar{x}^T B y - \bar{x}^T (A + B) \bar{y} \leq \varepsilon |A + B|$$

for any other mixed strategies $x, y$.

Also, conversely, if a pair of mixed strategies $(\bar{x}, \bar{y})$ satisfies (2.4) then it is an $\varepsilon$-approximate equilibrium.

**Proof.** The proof follows from the equivalence of the statements (2.1) and (2.2). □

2.3. **Approximation of games by low rank games.** If the matrix $C = A + B$ of a bimatrix game is “close” to a game with rank $k$, then the game can be approximated by a rank $k$-game $(A', B')$ in such a way that the Nash equilibria of the original game $(A, B)$ remain approximate Nash equilibria in the game $(A', B')$.

**Definition 2.3.** Let $(A, B)$ be an $(m \times n)$-game and $C = A + B$. If a matrix $C' \in \mathbb{R}^{m \times n}$ satisfies $|C - C'| < \varepsilon (|A + B|)$ then the game $(A', B')$ with $A' = A + \frac{1}{2}(C' - C)$, $B' = B + \frac{1}{2}(C' - C)$ $\varepsilon$-approximates $(A, B)$.

Note that $A' + B' = C'$.

Under the perturbation of the game, Nash equilibria of the original game are approximate equilibria of the perturbed game:

**Theorem 2.4.** Let $(A', B')$ be an $\varepsilon$-approximation of the game $(A, B)$ and $\varepsilon < 1$. If $(\bar{x}, \bar{y})$ is a Nash equilibrium of the game $(A, B)$, then $(\bar{x}, \bar{y})$ is a $3\varepsilon$-approximate Nash equilibrium for the game $(A', B')$.

**Proof.** The loss $\ell'(\bar{x}, \bar{y})$ for $(\bar{x}, \bar{y})$ with respect to the perturbed game $(A', B')$ satisfies

$$\ell'(\bar{x}, \bar{y}) \leq \varepsilon + \max_i (A_i - A)_{(i)} \bar{y} + \max_j (B_j - B)_{(j)} \bar{x} - \bar{x}^T (C' - C) \bar{y}$$

$$\leq \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \varepsilon = 3\varepsilon$$

□

We can apply the Singular Value Decomposition (SVD) to approximate the matrix $C$ by a matrix of some given rank $k$. The approximation factor in Theorem 2.4 is then a function of the singular values of $C$.

3. **The expressive power of low rank games**

3.1. **The geometry of Nash equilibria.** One measure for the expressive power of a game-theoretic model is the number of Nash equilibria it can have (depending on the number of strategies $m, n$). For simplicity, we will concentrate on the case $d := m = n$. If the Nash equilibria are not isolated, then we might count the number of connected components, but we will mainly concentrate on non-degenerate games in which there exist only a finite number of Nash equilibria.
GAMES OF FIXED RANK

Note that the usual definition of a non-degenerate game is slightly stronger than just requiring isolated Nash equilibria (see the discussion in [19]).

**Definition 3.1.** A bimatrix game is called non-degenerate if the number of the pure best responses of player 1 to a mixed strategy \( y \) of player 2 never exceeds the cardinality of the support \( \text{supp} y := \{ j : y_j \neq 0 \} \) and if the same holds true for the best pure responses of player 2.

If \( d \leq 4 \), then a non-degenerate \( d \times d \)-game can have at most \( 2^d - 1 \) Nash equilibria, and this bound is tight (see [6, 14]). For \( d \geq 5 \), determining the maximal number of a non-degenerate \( d \times d \)-game is an open problem (see [18]). Based on McMullen’s Upper Bound Theorem for polytopes, Keiding [6] gave an upper bound of \( \Phi_{d, 2d} - 1 \), where

\[
\Phi_{d,k} := \begin{cases} 
\frac{k}{k-\frac{d}{2}} \left(\frac{k-d}{k-\frac{d}{2}}\right) & \text{if } d \text{ even}, \\
2 \left(\frac{k-d+1}{k-\frac{d}{2}}\right) & \text{if } d \text{ odd}.
\end{cases}
\]

A simple class of configurations which yield an exponential lower bound of \( 2^d - 1 \) is the game where the payoff matrices of both players are the identity matrix \( I_d \) (see [16]).

The best known lower bound was given by von Stengel [18], who showed that for even \( d \) there exists a non-degenerate \( d \times d \)-game having

\[
\tau(d) := f(d/2) + f(d/2 - 1) - 1
\]

Nash equilibria, where \( f(n) := \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k} \). Asymptotically, \( \tau \) grows as \( \tau(d) \sim 0.949(1 + \sqrt{2})^d \).

If the ranks of \( A \) and \( B \) are bounded by a fixed constant, then the number of Nash equilibria is bounded polynomially in \( d \):

**Theorem 3.2.** For any \( d \times d \)-bimatrix game \((A, B)\) in which the ranks of both \( A \) and \( B \) are bounded by a fixed constant \( k \), the number of connected components of the Nash equilibria is bounded by \( \left(\frac{d}{k+1}\right)^2 \).

In particular, for a non-degenerate game the number of Nash equilibria is at most \( \left(\frac{d}{k+1}\right)^2 \), i.e., that number is bounded polynomially in \( d \).

**Proof.** Let \( A \) and \( B \) be of rank at most \( k \). The column space of \( Ay \) has dimension at most \( k \). By applying Caratheodory’s Theorem on the columns of \( Ay \), it was shown in [8, Theorem 4] that for every Nash equilibrium \((x, y)\) there exists a Nash equilibrium \((\mathbf{r}, \mathbf{y}')\) in which the second player plays at most \( k + 1 \) pure strategies with positive probability. The same argument can be used to bound the number of pure strategies which are used by player 1. It follows from that argument that there exists a continuous path from the original Nash equilibrium to the Nash equilibrium with small support.

Since for a given support of the Nash equilibria, the set of Nash equilibria with that support is a polyhedral set, the number of connected components of the Nash equilibria of game \((A, B)\) is at most \( \left(\frac{d}{k+1}\right)^2 \). \( \square \)

Now we show that the expressive power of fixed rank-games is significantly higher than the expressive power of zero-sum games. In order to show this, we prove that the number
of Nash equilibria of a rank 1-game can exceed any given constant and give a linear lower bound.

**Theorem 3.3.** For any \( d \in \mathbb{N} \) there exists a non-degenerate \( d \times d \)-game of rank 1 with at least \( 2d - 1 \) many Nash equilibria.

The following questions remain unsolved.

**Open problem 3.4.** Is the maximal number of Nash equilibria for non-degenerate \( d \times d \)-games of rank \( k \) smaller than the maximal number of Nash equilibria of non-degenerate \( d \times d \)-games of arbitrary rank? Is the maximal number of Nash equilibria for non-degenerate \( d \times d \)-games of rank \( k \) polynomially bounded in \( d \)?

In order to prove Theorem 3.3, we use the following representation of Nash equilibria introduced by Mangasarian [9].

**Definition 3.5.** For an \( m \times n \)-bimatrix game \((A, B)\), the polyhedra \( \overline{P} \) and \( \overline{Q} \) are defined by

\[
\overline{P} = \{ (\overline{x}, v) \in \mathbb{R}^m \times \mathbb{R} : \overline{x} \geq 0, \overline{x}^T B \leq 1^T v, 1^T \overline{x} = 1 \}, \tag{3.2}
\]

\[
\overline{Q} = \{ (\overline{y}, u) \in \mathbb{R}^n \times \mathbb{R} : A \overline{y} \leq 1^T u, \overline{y} \geq 0, 1^T \overline{y} = 1 \}. \tag{3.3}
\]

A pair of mixed strategies \((\overline{x}, \overline{y})\) \(\in S_1 \times S_2\) is a Nash equilibrium if and only if there exist \(u, v \in \mathbb{R}\) such that \((\overline{x}, v) \in \overline{P}\), \((\overline{y}, u) \in \overline{Q}\) and for all \(i \in \{1, \ldots, m+n\}\), the \(i\)-th inequality of \(\overline{P}\) or \(\overline{Q}\) is binding. Here, \(u\) and \(v\) represent the payoffs of player 1 and player 2, respectively. For \(i \in \{1, \ldots, m\}\) we call the inequality \(x_i \geq 0\) the \(i\)-th nonnegativity inequality of \(P\), and for \(j \in \{1, \ldots, n\}\) we call the inequality \(\overline{x}^T B_{(j)} \leq u\) the \(j\)-th best response inequality of \(P\). And analogously for \(Q\).

### 3.2. A class of low rank games with arbitrarily many Nash equilibria

We construct a sequence \((A_d, B_d)\) of \(d \times d\)-games of rank 1 in which all pairs \((i, i)\) of pure strategies \((1 \leq i \leq d)\) are Nash equilibria. For convenience of notation, we will omit the index \(d\) in the notation of the game. In order to achieve the desired properties, we enforce that for every \(i \in \{1, \ldots, d\}\) the element \(a_{ii}\) is the maximal element in the \(i\)-th column of \(A\) and the element \(b_{ii}\) is the maximal element in the \(i\)-th row of \(B\).

Let us begin with an auxiliary sequence of games \((\overline{A}, \overline{B})\). Let \(\overline{A}, \overline{B} \in \mathbb{R}^{d \times d}\) be defined by

\[
\overline{a}_{ij} = \overline{b}_{ij} = -(i - j)^2. \tag{3.4}
\]

Then for every \(i \in \{1, \ldots, d\}\) the element \(\overline{a}_{ii}\) is the largest element in the \(i\)-th column of \(\overline{A}\), and the element \(\overline{b}_{ii}\) is the largest element in the \(i\)-th row of \(B\). Expanding (3.4) shows that both \(\overline{A}\) and \(\overline{B}\) can be written as the sum of three rank 1-matrices; since \(\overline{A} = \overline{B}\), it follows that the game \((\overline{A}, \overline{B})\) is a rank 3-game.

In order to transform \((\overline{A}, \overline{B})\) into a rank 1-game, we observe that adding a constant column vector to a column of \(A\) or adding a constant row vector to a row of \(B\) does
not change the set of Nash equilibria. For \( j \in \{1, \ldots, d\} \), we add the constant vector 
\((2j^2, \ldots, 2j^2)^T\) to the \( j \)-column of \( \mathbf{A} \), and for \( i \in \{1, \ldots, d\} \) we add the constant vector 
\((2i^2, \ldots, 2i^2)\) to the \( i \)-th row of \( \mathbf{B} \). Let \( A, B \in \mathbb{R}^{n \times d} \) be the resulting matrices, i.e.,
\[
(3.5) \quad a_{ij} = 2ij - i^2 + j^2, \quad b_{ij} = 2ij + i^2 - j^2.
\]
Since \( A + B = (4ij)_{i,j} \), the matrix \( A + B \) is of rank 1. Note that the game \((A, B)\) is
symmetric, i.e., \( A = B^T \).

**Lemma 3.6.** For any mixed strategy \( x \in S_1 \) there are at most two pure best responses for
player 2. And for any mixed strategy \( y \in S_2 \) there are at most two pure best responses for
player 1. 

**Proof.** Let \( y \) be a mixed strategy of player 2 with support \( J := \{j_1, \ldots, j_k\} \). We assume
that there exists a 3-element subset \( I = \{i_1, i_2, i_3\} \subset \{1, \ldots, d\} \) such that
\[
(3.6) \quad (Ay)_{i_1} = (Ay)_{i_2} = (Ay)_{i_3} \geq (Ay)_i \quad \text{for all } i \notin I.
\]
The equations in (3.6) imply that for all distinct \( i, i' \in I \) we have
\[
\sum_{j \in J} (2ij - i^2 + j^2) y_j = \sum_{j \in J} (2i'j - i'^2 + j^2) y_j,
\]
which, using \( \sum_{j \in J} y_j = 1 \), is equivalent to \( 2(i - i') \sum_{j \in J} j y_j = (i^2 - i'^2) \). Hence,
\( 2 \sum_{j \in J} j y_j = (i + i') \). The left hand side of this equation is independent of \( i \). Therefore
there cannot be more than two indices in \( I \) such that this equation is satisfied for all pairs
of these indices. The proof of the other statement is symmetric. \( \square \)

**Lemma 3.7.** Each of the two polyhedra \( \mathcal{P} \) and \( \mathcal{Q} \) has \( \frac{d}{2}(d^2 + 5) \) vertices, which come in
two classes:

1. There exists a \( j \in \{1, \ldots, d\} \) such that the best response inequality of \( \mathcal{Q} \) with index
   \( j \) is binding and all nonnegativity inequalities of \( \mathcal{Q} \) but the one with index \( j \) are
   binding (\( d \) vertices).
2. There exist \( j_1, j_2 \in \{1, \ldots, d\} \), \( j_1 < j_2 \) and \( i \in \{j_1, \ldots, j_2 - 1\} \) such that the best
   response inequalities with indices \( i \) and \( i + 1 \) are binding and all nonnegativity
   inequalities except those with indices \( j_1, j_2 \) are binding (altogether \( \sum_{k=1}^{d-1} k(d-k) \)
   vertices).

And similarly for \( \mathcal{P} \).

**Proof.** We consider the polyhedron \( \mathcal{Q} \). By Lemma 3.6, at most two best response inequalities
can be binding at a vertex of \( \mathcal{Q} \).

If there is a single binding best response inequality, say, with index \( i \), then, at a vertex \( v \),
at least \( d - 1 \) of the nonnegativity inequalities must be binding, and therefore there exists
a single index \( j \) such that \( y_j \) is nonzero; hence \( y_i = 1 \). Now the condition \( v \in \mathcal{Q} \) implies
\( a_{ij} \geq a_{ij'} \) for all \( j' \in \{1, \ldots, d\} \), and it suffices to observe that for a fixed \( j \) the value \( a_{ij} \)
is maximized for \( i = j \), and this defines indeed a vertex.
Now assume that there are two binding best response inequalities \( i_1 \) and \( i_2 \) with \( i_1 < i_2 \). Then there are at most two nonzero components of \( y \), say \( y_{j_1} \) and \( y_{j_2} \). We can assume that \( j_1 \neq j_2 \) since otherwise we are in the situation discussed before.

We claim that \( i_1 \) and \( i_2 \) are neighboring indices. Otherwise there would exist an \( i' \) with \( i_1 < i' < i_2 \). Now, similar to the calculations in the proof of Lemma 3.6, the property \( i' + i_2 > i_1 + i_2 \) implies that \( 2(j_1y_{j_1} + j_2y_{j_2}) = (i_1 + i_2) < (i' + i_2) \) and therefore
\[
(Ay)_{i'} > (Ay)_{i_1} = (Ay)_{i_2}.
\]

This contradicts \( v \in \overline{Q} \).

Now let \( i_2 = i_1 + 1 \). Computing the solutions for \( y_{j_1} \) and \( y_{j_2} \) of the equations
\[
2j_1y_{j_1} + 2j_2y_{j_2} = i_1 + i_2,
\]
\[
y_{j_1} + y_{j_2} = 1
\]
yields
\[
y_{j_1} = \frac{2j_2 - (i_1 + i_2)}{2(j_2 - j_1)}, \quad y_{j_2} = \frac{(i_1 + i_2) - 2j_1}{2(j_2 - j_1)},
\]
which in connection with \( y \geq 0 \) shows \( j_1 \leq i_1 \) and \( j_2 > i_1 \).

It remains to show that the stated pairs indeed define vertices. In order to prove this, we have to show that for \( i' < i_1 \) or \( i'' > i_2 \) we obtain \( (Ay)_{i'} < (Ay)_{i_1} \), which follows in the same way as in the case \( i_1 < i' < i_2 \) that was discussed before.

Now summing up over all the possibilities proves the stated number. \( \Box \)

Corollary 3.8. A pair of mixed strategies \((x, y)\) is a Nash equilibrium of the game \((A, B)\) if and only if \( x = y = e_i \) for some unit vector \( e_i \), \( 1 \leq i \leq d \), or \( x = y = \frac{1}{2}(e_i + e_{i+1}) \) for some \( i \in \{1, \ldots, d-1\} \).

Proof. By the characterization of the vertices in Lemma 3.7, the Nash equilibria come in two classes. If for some \( i \in \{1, \ldots, d\} \) both players play the \( i \)-th pure strategy, then this gives a Nash equilibrium. Moreover, for every \( i \in \{1, \ldots, d-1\} \), if both players only use the \( i \)-th and the \((i + 1)\)-th pure strategy, there exists a Nash equilibrium. It is easy to check that in this situation, both players play both of their pure strategies with probability \( \frac{1}{2} \). \( \Box \)

Combining Theorem 3.8 for rank 1-games with von Stengel’s result, we obtain the following lower bound for rank \( k \)-games.

Corollary 3.9. For odd \( d \geq 3 \) and \( k \leq d \), there exists a \( d \times d \)-game of rank \( k \) with at least \( \tau(k-1) \cdot (2(d-k) + 1) \) Nash equilibria, where \( \tau \) is defined as in (3.1). For fixed \( k \), this sequence converges to \( \infty \) as \( d \) tends to \( \infty \).

Proof. We construct a \( d \times d \)-game \((A, B)\) of rank \( k \) with
\[
A = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}
\]
where \( A', B' \in \mathbb{R}^{k-1} \times \mathbb{R}^{k-1} \) define a \((k-1) \times (k-1)\)-game with \( \tau(k-1) \) Nash equilibria, which exists by von Stengel’s construction. Moreover, let \( A'', B'' \in \mathbb{R}^{d-k+1} \times \mathbb{R}^{d-k+1} \) define a \((d-k+1) \times (d-k+1)\)-game of rank 1 with \( 2(d-k+1) - 1 \) Nash equilibria based
on the construction in Theorem \[ \text{5.3} \]. Then the game \((A, B)\) is of rank \(k\) and has at least \(\tau(k - 1) \cdot (2(d - k) + 1)\) Nash equilibria. \qed

Remark 3.10. Generalizing the construction in (3.4), for a mapping \(g : \{1, \ldots, d\} \to \mathbb{R}\) and a polynomial \(p = \sum_{i=0}^{n} a_i x^i\) of degree \(n\), the matrix \(C \in \mathbb{R}^{d \times d}\) defined by
\[ c_{ij} = p(g(i) - g(j)) \]
has rank at most \(\frac{1}{2}(n + 1)(n + 2)\). This follows immediately from applying the Binomial Theorem on \(p(g(i) - g(j))\),
\[ p(g(i) - g(j)) = \sum_{k=0}^{n} a_k \sum_{l=0}^{k} \binom{k}{l} g(i)^l(-g(j))^{k-l}, \]
and observing that the rank of \(C\) is bounded by the number of terms in this expansion.

4. Approximation Algorithms

4.1. \(\varepsilon\)-approximating Nash equilibria of low rank games. For general bimatrix games, no polynomial time algorithm for \(\varepsilon\)-approximating a Nash equilibrium is known. In a related model to ours, \[ \text{5} \] has provided the first subexponential algorithm for finding an approximate equilibrium.

Here, we show the following result for our restricted class of bimatrix games.

Theorem 4.1. Let \(k\) be a fixed constant and \(\varepsilon > 0\). If \(A + B\) is of rank \(k\) then an \(\varepsilon\)-approximate Nash equilibrium can be found in time \(\text{poly}(L, 1/\varepsilon)\), where \(L\) is the bit length of the input.

Set
\[ Q = \begin{pmatrix} 0 & \frac{1}{2}(A + B) \\ \frac{1}{2}(A^T + B^T) & 0 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} x \\ y \end{pmatrix} \]
so that the quadratic form \(x^T (A + B)y\) can be written as \(\frac{1}{2} z^T Q z\) with a symmetric matrix \(Q\). We assume that \(A + B\) has rank \(k\) for a fixed constant \(k\); thus \(Q\) has rank \(2k\). Since the trace of the matrix \(Q\) is zero, this matrix is either the zero matrix or an indefinite matrix. Hence, in the case \(Q \neq 0\) the quadratic form defined by \(Q\) is indefinite.

We use the following straightforward formulation of a Nash equilibrium as a solution of a system of linear and quadratic inequalities.

Lemma 4.2. A pair of mixed strategies \(z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}_1 \times \mathcal{S}_2\) is a Nash equilibrium if and only if there exists an \(s \in \mathbb{R}\) such that
\[ z^T Q z \geq s \]
\[ s \geq (A^{(i)} | B_{(j)}^T) z \quad \text{for all} \quad i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}. \]
Since \(z^TQz \leq s\) in any feasible solution of this optimization problem, we have \(z^TQz = s\) for any feasible solution. Hence, the Nash equilibria are exactly the optimal solutions of the quadratic optimization problem

\[
\text{(QP :)} \quad \min s - z^TQz
\]

\[
(4.1) \quad s \geq \left(\begin{array}{c|c}
A(i) & B_T(j) \\
\end{array}\right)z \quad \text{for all } i \in \{1, \ldots, m\}, \ j \in \{1, \ldots, n\}, \ z \in S_1 \times S_2.
\]

Vavasis has shown the following polynomial approximation result for quadratic optimization problems with compact polyhedral feasible set \([20, 21]\).

**Proposition 4.3.** Let \(\min \{\frac{1}{2}x^TQx + q^Tx : Ax \leq b\}\) be a quadratic optimization problem with compact support set \(\{x \in \mathbb{R}^n : Ax \leq b\}\), and let the rank \(k\) of \(Q\) be a fixed constant. If \(x^*\) and \(x^\#\) denote points minimizing and maximizing the objective function \(f(x) := \frac{1}{2}x^TQx + q^Tx\) in the feasible region, respectively, then one can find in time \(\text{poly}(L, 1/\varepsilon)\) a point \(x^{\diamond}\) satisfying

\[
f(x^{\diamond}) - f(x^*) \leq \varepsilon(f(x^\#) - f(x^*)) ,
\]

where \(L\) is the bit length of the quadratic problem. Such a point \(x^{\diamond}\) is called an \(\varepsilon\)-approximation of the quadratic problem.

**Proof of Theorem 4.1.** The feasible region of the quadratic program \((4.1)\) is unbounded. Since the value of \(z^TQz\) is at most \(|A + B|\) for any feasible solution \(z\) and since the objective value for a Nash equilibrium is 0, we can add the constraint \(s \leq |A + B|\) to \((4.1)\), which makes the feasible region compact. Denote the resulting quadratic optimization problem by \(\text{QP}'\) and recall that the approximation ratio of the quadratic program depends on the maximum objective value in the feasible region.

By Proposition 4.3, we can compute in polynomial time an \(\varepsilon\)-approximation \((z^{\diamond}, s^{\diamond})\) with \(z^{\diamond} = (x^{\diamond}, y^{\diamond})\) of \(\text{QP}'\). Since the optimal value of \(\text{QP}'\) is 0, we have

\[
s^{\diamond} - (z^{\diamond})^TQz^{\diamond} = f(z^{\diamond}, s^{\diamond}) \leq \varepsilon f(z^\#, s^\#) \leq \varepsilon|A + B| .
\]

Hence, \((x^{\diamond}, y^{\diamond})\) is an \(\varepsilon\)-approximate Nash equilibrium of the game \((A, B)\). \(\square\)

**Remark 4.4.** The proof in [20] computes an \(LDL^T\) factorization of the matrix \(Q\) defining the quadratic form and then constructs a sufficiently fine grid in the fixed-dimensional space. Since the quadratic form \(x^TQy\) is bilinear, we can also directly apply an \(LDU^T\) factorization on the matrix of the bilinear form.

### 4.2. Relative approximation in case of a nonnegative decomposition.

The right hand side in Definition 2.1 of an approximate Nash equilibrium depends only on \(\varepsilon\) and on \(|A + B|\). Since different Nash equilibria in the same game can differ strongly in their payoffs, we introduce a notion of *relative approximation* with respect to a Nash payoff which takes into account these differences.

Consider the quadratic problem \((4.1)\). In a Nash equilibrium \((x, y) \in S_1 \times S_2\) there exists an \(s \in \mathbb{R}\) such that \((x, y, s)\) is a feasible solution to \((4.1)\); in this situation \(s\) coincides with
the sum of the payoffs of the two players. In the relative approximation, we aim at finding pairs of strategies \((x, y)\) for which there exists an \(s \in \mathbb{R}\) such that \((x, y, s)\) is feasible and

\[
s - x^T(A + B)y \leq \rho s.
\]

Using our notion of loss, by observing \(s = \max_i A^{(i)}x + \max_j x^TB^{(j)}\) for an optimally chosen \(s\), this means

\[
\ell(x, y) \leq \rho(\max_i A^{(i)}x + \max_j x^TB^{(j)}).
\]

We provide an efficient approximation algorithm for the case that \(C = A + B\) has a known decomposition of the form

\[
C = \sum_{i=1}^{k} u^{(i)}(v^{(i)})^T
\]

with non-negative vectors \(u^{(i)}\) and \(v^{(i)}\).

**Theorem 4.5.** If \(C\) has a known decomposition of the form (4.2) then for any given \(\varepsilon > 0\) a relatively approximate Nash equilibrium with approximation ratio \(1 - \frac{1}{(1 + \varepsilon)^2}\) can be computed in time \(\text{poly}(\mathcal{L}, 1/\log(1 + \varepsilon))\), where \(\mathcal{L}\) is the bit length of the input.

Let \(z_i = x^Tu^{(i)}\), \(w_i = (v^{(i)})^Ty\). We put a grid on each of the \(z_i\) and on each of the \(w_i\) in a geometric progression: denoting by \((z_i)_{\text{min}} = \min_{x \in S_1} x^Tu^{(i)}\) and \((z_i)_{\text{max}} = \max_{x \in S_1} x^Tu^{(i)}\) the minimum and the maximum possible value for \(z_i\), we partition the interval \([z_i)_{\text{min}}, (z_i)_{\text{max}}]\) into the intervals \([(z_i)_{\text{min}}, (1 + \varepsilon)(z_i)_{\text{min}}]\), \([(1 + \varepsilon)(z_i)_{\text{min}}, (1 + \varepsilon)^2(z_i)_{\text{min}}]\), and so on. And analogously for the \(w_i\).

For every cell we construct a linear program which “approximates” the quadratic program (4.1). Let the intervals of a grid cell be \([\alpha_i, (1 + \varepsilon)\alpha_i]\) and \([\beta_i, (1 + \varepsilon)\beta_i]\), i.e.,

\[
\alpha_i \leq z_i \leq (1 + \varepsilon)\alpha_i,
\]

\[
\beta_i \leq w_i \leq (1 + \varepsilon)\beta_i.
\]

Then for any pair of strategies \((x, y)\) \(\in S_1 \times S_2\) falling into that cell, the quadratic form \(x^TCy\) satisfies

\[
\sum_{i=1}^{k} \alpha_i\beta_i \leq x^TCy \leq (1 + \varepsilon)^2\sum_{i=1}^{k} \alpha_i\beta_i,
\]

where the left inequality uses that all the values in the decomposition are nonnegative.

For the grid cell, we consider the linear program

\[
\begin{align*}
\min s - \sum_{i=1}^{k} \alpha_i\beta_i \\
\alpha_i &\leq x^Tu^{(i)} \leq (1 + \varepsilon)\alpha_i, \\
\beta_i &\leq (v^{(i)})^Ty \leq (1 + \varepsilon)\beta_i, \\
s &\geq \left(A^{(i)} | B^{(j)}_{ij}\right)z \quad \text{for all } i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}, \\
(x, y) &\in S_1 \times S_2, \ s \in \mathbb{R}.
\end{align*}
\]
In at least one of the cells there exists a Nash equilibrium. The linear program corresponding to that cell yields a solution with

$$(4.4) \quad \sum_{i=1}^{k} \alpha_i \beta_i \leq s \leq (1 + \varepsilon)^2 \left( \sum_{i=1}^{k} \alpha_i \beta_i \right).$$

Hence, by the left inequality in (4.3) and the right inequality in (4.4) we have

$$x^T Cy \geq \sum_{i=1}^{k} \alpha_i \beta_i \geq \frac{s}{(1 + \varepsilon)^2}.$$

We conclude

$$s - x^T Cy \leq s \left( 1 - \frac{1}{(1 + \varepsilon)^2} \right),$$

which shows Theorem 4.5.

References

[1] I. Barany, S. Vempala, and A. Vetta. Nash equilibria in random games. In Proc. 46th IEEE Foundations of Computer Science (Pittsburgh, PA), 2005.
[2] J. Bulow and J. Levin. Matching and price competition. Preprint, 2003.
[3] G.B. Dantzig. Linear Programming and Extensions. Princeton Univ. Press, Princeton, NJ, 1963.
[4] C. Daskalakis, P.W. Goldberg, and C.H. Papadimitriou. The complexity of computing a Nash equilibrium. Electronic Colloquium on Computational Complexity. Report TR05–115, 2005.
[5] K. Isaacson and C.B. Millham. On a class of Nash-solvable bimatrix games and some related Nash subsets. Naval. Res. Logist. Quarterly 23:311–319, 1980.
[6] H. Keiding. On the maximal number of Nash equilibria in an $n \times n$ bimatrix game. Games Econom. Behavior 21:148–160, 1997.
[7] C.E. Lemke and J.T.Howson. Equilibrium points of bimatrix games. J. Soc. Indust. Appl. Math. 12:413–423, 1964.
[8] R.J. Lipton, E. Markakis, and A. Mehta. Playing large games using simple strategies. In Proc. ACM Conf. on Electronic Commerce (San Diego, CA), 36-41, 2003.
[9] O.L. Mangasarian. Equilibrium points of bimatrix games. J. Soc. Industr. Appl. Math. 12:778–780, 1964.
[10] A. McLennan and I.-U. Park. Generic $4 \times 4$ two person games have at most 15 Nash equilibria. Games Econom. Behavior 26:111-130, 1997.
[11] J. Nash. Equilibrium points in $n$-person games. Proc. Amer. Math. Soc. 36:48-49, 1950.
[12] J. Nash. Non-cooperative games. Annals of Mathematics 54:286–295, 1951.
[13] J. von Neumann and O. Morgenstern. Theory of Games and Economic Behavior. Princeton University Press, Princeton, NJ, 1944.
[14] C.H. Papadimitriou. Algorithms, games and the Internet. In Proc. 33rd ACM Symp. Theory of Computing, Chersonissos, Kreta, 749–753, 2001.
[15] C.H. Papadimitriou and T. Roughgarden. Computing equilibria in multi-player games In Symp. on Discrete Algorithms (Vancouver, BC), 82–91, 2005.
[16] T. Quint and M. Shubik. A bound on the number of Nash equilibria in a coordination game. Economic Letters 77:323–327, 2002.
[17] R. Savani and B. von Stengel. Exponentially many steps for finding a Nash equilibrium in a bimatrix game. In Proc. 45th IEEE Foundations of Computer Science (Rome), 258–257, 2004.
[18] B. von Stengel. New maximal numbers of equilibria in bimatrix games. Discrete Comput. Geom. 21:557–568, 1999.
[19] B. von Stengel. Computing equilibria for two-person games. In R.J. Aumann, S. Hart (Hrsg.), *Handbook of Game Theory*, North-Holland, Amsterdam, 2002.

[20] S. Vavasis. Approximation algorithms for indefinite quadratic programming. Technical Report 91-1228, Dept. of Computer Science, Cornell University (Ithaca, NY), 1991.

[21] S. Vavasis. Approximation algorithms for indefinite quadratic programming. *Math. Program.* 57:279–311, 1992.

R. Kannan: Dept. of Computer Science, Yale University, P.O. Box 208285, New Haven, CT 06520–8285, USA
E-mail address: kannan@cs.yale.edu

T. Theobald: Institut für Mathematik, MA 6–2, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany
E-mail address: theobald@math.tu-berlin.de