PLURIPOTENTIAL MONGE-AMPÈRE FLOWS
IN BIG COHOMOLOGY CLASSES

QUANG-TUAN DANG

Abstract. We study pluripotential complex Monge-Ampère flows in big cohomology classes on compact Kähler manifolds. We use the Perron method, considering pluripotential subsolutions to the Cauchy problem. We prove that, under natural assumptions on the data, the upper envelope of all subsolutions is continuous in space and semi-concave in time, and provides a unique pluripotential solution with such regularity. We apply this theory to study pluripotential Kähler-Ricci flows on compact Kähler manifolds of general type as well as on stable varieties.

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Introduction

The primary goal of this paper is to study pluripotential complex Monge-Ampère flows motivated by the Minimal Model Program (MMP for brevity) in algebraic geometry, whose aim is the (birational) classification of projective manifolds. In a recent celebrated work, Birkar-Cascini-Hacon-Mckernan [BCHM10] showed the existence of minimal models for a large class of varieties which are called varieties of general type. J. Song and G. Tian [ST12, ST17] have recently proposed an analytic analogue making use of (twisted) Kähler-Ricci flows on compact Kähler manifolds.

Let \(X\) be a compact Kähler manifold of dimension \(n\) equipped with a Kähler form \(\hat{\omega}\). The (normalized) Kähler-Ricci flow on \(X\) starting at \(\hat{\omega}\) is the solution to the following evolution equation

\[
\frac{\partial \theta_t}{\partial t} = -\text{Ric}(\theta_t) - \lambda \theta_t, \quad \theta|_{t=0} = \hat{\omega},
\]

(0.1)

where the sign of \(\lambda \in \mathbb{R}\) depends on that of the first Chern class \(c_1(K_X)\). Solving the normalized Kähler-Ricci flow (0.1) turns out to be equivalent to solving the

Date: January 4, 2022.
2020 Mathematics Subject Classification. 53C44, 32W20, 58J35.
Key words and phrases. Parabolic Monge-Ampère equation, big cohomology class, Kähler-Ricci flow.

This work is partially supported by the ANR projects GRACK and PARAPLUI.
scalar complex Monge-Ampère flow
\[
\begin{cases}
(\omega_t + dd^c \varphi_t)^n = e^{\theta_t + \lambda \varphi_t + h(t,x)} dV
\\
\omega_t + dd^c \varphi_t > 0,
\end{cases}
\]
where \( h \) is a smooth density, and \( \omega_t \in \{\theta_t\} \in H^{1,1}(X,\mathbb{R}) \) is fixed.

Since the MMP requires one to work on singular varieties, it is necessary to develop a fine theory dealing with weak solutions. One has indeed to deal with similar complex Monge-Ampère flows with various degeneracies: the reference forms \( \omega_t \) are no longer Kähler and the densities \( h \) is no longer smooth, with integrability properties that depend on the type of singularities. A parabolic viscosity approach has been developed recently in [EGZ16], which requires the densities to be continuous and has a limited scope of applications. The first elements of a parabolic pluripotential theory has been laid down in [GLZ21a, GLZ20] which are the parabolic analogues of the pioneering work of Bedford and Taylor in the local case [BT76, BT82]. We extend here this theory so as to be able to deal with big cohomology classes.

Assumptions and Notations. Before going further and stating the main results of the paper, let us fix some notations. Let \( X \) be a compact Kähler manifold of dimension \( n \). We let \( X_T := (0,T) \times X \) denote the real \((2n+1)\)-dimensional manifold with \( T \in (0, +\infty] \). We focus mostly on finite time intervals i.e. \( T < +\infty \). The parabolic boundary of \( X_T \) is denoted by
\[
\partial X_T := \{0\} \times X.
\]
We fix \( \theta \) a smooth closed \((1,1)\)-form representing a big cohomology class. We let \( \Omega \) denote the ample locus of \( \theta \),
\[
\Omega := \text{Amp}(\theta)
\]
which is a non empty Zariski open subset of \( X \). We also set \( \Omega_T := (0,T) \times \Omega \).

We assume that \( (\omega_t)_{t \in [0,T]} \) is a smooth family of closed \((1,1)\)-forms on \( X \) such that
\[
g(t)\theta \leq \omega_t, \quad \forall t \in [0,T),
\]
where \( g(t) \) is an increasing smooth positive function on \([0,T]\).

Throughout the article we assume that there exists a Kähler form \( \Theta \) such that
\[
(0.2)
-\Theta \leq \omega_t, \dot{\omega}_t, \ddot{\omega}_t \leq \Theta.
\]
We let \( dV \) denote a smooth volume form on \( X \). We shall always assume that
\begin{itemize}
  \item \( 0 \leq f \in L^p(X,dV) \) for some \( p > 1 \), and \( f \) is strictly positive almost everywhere;
  \item \( F : [0,T] \times X \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous on \([0,T] \times X \times \mathbb{R} \);
  \item the function \( r \mapsto F(\cdot,r) \) is increasing in \( r \);
  \item the function \( F \) is uniformly Lipschitz in \((t,x) \in [0,T] \times \mathbb{R} \), i.e. there exists a constant \( \kappa_F > 0 \) such that for all \( t,t' \in [0,T], x \in X, r,r' \in \mathbb{R}, \)
\[
|F(t,x,r) - F(t',x,r')| \leq \kappa_F(|t-t'| + |r-r'|);
\]
  \item the function \( (t,r) \mapsto F(t,\cdot,r) \) is convex.
\end{itemize}
With the assumptions above, we consider the complex Monge-Ampère flow:
\[
\text{(CMAF)} \quad dt \wedge (\omega_t + dd^c \varphi_t)^n = e^{\varphi_t + F(t,\cdot,\varphi_t)} f dV \wedge dt
\]
on \( X_T \). Note that the equation (CMAF) should be understood in the weak sense of measures in \((0,T) \times \Omega \) (see Section 1.3).

The existence of the weak Kähler-Ricci flow is often proved by using approximation arguments and a priori estimates (cf. [ST17, GLZ20]). Big cohomology classes
can not be approximated by Kähler ones so this approach breaks down in our case. We shall instead use the Perron method, inspired by [GLZ21a], considering the upper envelope $U$ of all pluripotential subsolutions to the Cauchy problem. We prove that this upper envelope is locally uniformly semi-concave in time:

**Theorem A.** Let $\varphi_0$ be a $\omega_0$-psh function with minimal singularities. Then the upper envelope $U$ of all subsolutions to (CMAF) with initial data $\varphi_0$ is a pluripotential solution to (CMAF) which is locally uniformly Lipschitz and locally uniformly semi-concave in $t \in (0, T)$.

We prove Theorem A by following the arguments of [GLZ21a] in the local context:
- we first show that the upper envelope of all subsolutions is locally uniformly Lipschitz in $t$ (Theorem 2.7) and that it is itself a pluripotential subsolution;
- we then show that the envelope is locally uniformly semi-concave (Theorem 2.13);
- we finally apply a balayage process and use the analogue result in the local context [GLZ21a] to conclude the proof.

We prove in Theorem 3.6 that the envelope $U$ in Theorem A has minimal singularities and is continuous in $(0, T) \times \Omega$ under an extra assumption:

$$\dot{\omega}_t \leq A\omega_t, \ t \in [0, T),$$

for some positive constant $A$. We also show that $U$ is the unique pluripotential solution with such regularity by establishing the following comparison principle:

**Theorem B.** Let $\varphi$ (resp. $\psi$) be a pluripotential subsolution (resp. supersolution) to (CMAF) with initial data $\varphi_0$ (resp. $\psi_0$). We assume that $\psi$ is locally uniformly semi-concave in $t \in (0, T)$ and $\psi$ is continuous in $(0, T) \times \Omega$. We assume moreover that for each $t$, $\psi_t$ has minimal singularities. Then $\varphi \leq \psi$ on $[0, T) \times X$ if $\varphi_0 \leq \psi_0$.

The assumption that $\psi_t$ has minimal singularities means that for each $t \in (0, T)$, there exists a constant $C_t$ such that $|\psi_t - V_{\omega_t}|$ is bounded by $C_t$, where $V_{\omega_t}$ is the largest negative $\omega_t$-psh function. The proof of Theorem B is provided in Section 3.2, generalizing some ideas from [GLZ20].

Starting from a Kähler form $\omega_0$, it follows from [Cao85, Tsu88, TZ06] that the (smooth) normalized Kähler-Ricci flow exists in $(0, T)$ where

$$T := \sup\{t > 0 : e^{-t}\{\omega_0\} + (1 - e^{-t})c_1(K_X) \text{ is Kähler}\}.$$  

The maximal existence time $T$ is finite unless $K_X$ is nef (numerically effective).

It is an interesting question to know how to define the flow for $t > T$. This was formulated in [FIK03, Section 10, Question 8] and a precise conjecture was made in [BT12]. Note that, if $X$ is of general type, i.e. $K_X$ is big, then for any $t > T$ the cohomology class $e^{-t}\{\omega_0\} + (1 - e^{-t})c_1(K_X)$ remains big but are no longer nef, thus one can not hope to make sense of the flow in the classical one. It was proved in [Tô21] that the flow can be continued through $T$ in the viscosity sense and it eventually converges to the unique singular Kähler-Einstein metric on $\text{Amp}(K_X)$. Using the tools developed above, we establish the pluripotential analogue of the main result of [Tô21]:

**Theorem C.** Let $X$ be a compact $n$-dimensional Kähler manifold of general type. Then the normalized pluripotential Kähler-Ricci flow emanating from a Kähler metric $\omega_0$,

$$\frac{\partial \theta_t}{\partial t} = -\text{Ric}(\theta_t) - \theta_t,$$

exists for all time. It coincides with the smooth flow on $(0, T)$ and deforms $\omega_0$ towards the unique singular Kähler-Einstein metric $\omega_{KE}$ on $\text{Amp}(K_X)$, as $t \to +\infty$. 
We actually establish a more general result allowing to run the flow from an arbitrary closed positive current with bounded potential; (see Theorem 4.1). We can also continue the pluripotential Kähler-Ricci flow for all time when $K_X$ is pseudoeffective (see Section 4.2).

In the last part of the paper we study pluripotential Kähler-Ricci flows on Kähler varieties $X$ with semi-log canonical singularities (the most general class of singularities appearing in the log MMP) and ample canonical line bundle.

It has been shown by R. Berman and H. Guenancia [BG14] that $X$ admits a unique Kähler-Einstein current $\omega_{KE}$ in the class $c_1(K_X)$ which is smooth in the regular locus $X_{\text{reg}}$. We apply our theory to run the pluripotential (normalized) Kähler-Ricci flow on $X$ and recover the canonical metric $\omega_{KE}$ as the long time limit of the flow. More precisely, we have the following:

**Theorem D.** Let $X$ be a projective complex algebraic variety with semi-log canonical singularities such that $K_X$ is ample. Then the normalized pluripotential Kähler-Ricci flow emanating from a Kähler metric $\omega_0$,\[ \frac{\partial \theta_t}{\partial t} = -\text{Ric}(\theta_t) - \theta_t, \]
exists for all time. It deforms $\omega_0$ towards the unique singular Kähler-Einstein metric $\omega_{KE}$ on $X_{\text{reg}}$, as $t \to +\infty$.

Again we actually show that the flow can be run from an arbitrary positive closed current with bounded potentials (see Theorem 4.10).

For varieties of general type with log terminal singularities the pluripotential Kähler-Ricci flow (with non continuous data) was constructed in [GLZ20, Section 5.1]. A similar result has been obtained in the recent work [CGLS19], where the authors have extended the approach of Song-Tian [ST17] to the case of $\mathbb{Q}$-factorial projective varieties with log canonical singularities: establishing higher order a priori estimates, they obtain a good notion of weak Kähler-Ricci flow which is smooth in the regular locus of variety.

**Organization of the paper.** In Section 1 we provide some backgrounds on pluripotential theory in big cohomology classes. In Section 2 we study the regularity properties of the envelope of pluripotential subsolutions. In Section 3 we shall prove Theorem A and Theorem B. We study in Section 4 the pluripotential normalized Kähler-Ricci flow on compact Kähler manifolds of general type (resp. stable varieties) and prove Theorem C (resp. Theorem D).

**Acknowledgements.** The author would like to thank his advisors Vincent Guedj and Hoang-Chinh Lu for constant help and encouragement. We are truly grateful to Henri Guenancia and Ahmed Zeriahi for several interesting discussions. We thank Tät-Dat Tô for useful conversations on his results in [Tô21]. We also thank the referee for giving numerous valuable comments which really improved the presentation of the paper.

1. **Preliminaries**

In this section we recall necessary definitions and backgrounds. Let $X$ be a compact Kähler manifold of complex dimension $n$, and $\Theta$ be a Kähler metric on $X$. We let $H^{1,1}(X, \mathbb{R})$ denote the Bott-Chern cohomology of $d$-closed real $(1,1)$-forms (or currents) modulo $\partial \bar{\partial}$-exact ones.

1.1. **Monge-Ampère operators in big cohomology classes.**
1.1.1. **Big cohomology classes.** Let $\theta$ be a smooth real closed $(1,1)$-form on $X$. An upper semi-continuous function $\varphi : X \to [-\infty, +\infty)$ is called $\theta$-plurisubharmonic ($\theta$-psh for short) if in any local holomorphic coordinates $\varphi$ can be written as the sum of a psh and a smooth function, and

$$\theta +dd^c\varphi \geq 0,$$

in the weak sense of currents, where $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$.

We let $\text{PSH}(X, \theta)$ denote the set of all $\theta$-psh functions on $X$ which are not identically $-\infty$. This set is endowed with the weak topology which coincides with the $L^1$-topology. By Hartogs’ lemma $\varphi \mapsto \sup_X \varphi$ is continuous in the $L^1$-topology.

By the $dd^c$-lemma any closed positive $(1,1)$-current $T$ cohomologous to $\theta$ can be written as $T = \theta + dd^c \varphi$ for some $\theta$-psh function $\varphi$ which is moreover unique up to an additive constant.

If $T$ and $T'$ are two closed positive $(1,1)$-currents on $X$ which are cohomologous, then $T$ is said to be less singular than $T'$ if their global potentials satisfy $\varphi' \leq \varphi + O(1)$ (then we also say that $\varphi$ is less singular than $\varphi'$). A positive current $T$ is now said to have minimal singularities if it is less singular than any other positive current in its cohomology class.

**Definition 1.1.** A $\theta$-psh function $\varphi$ is said to have minimal singularities if it is less singular than any other $\theta$-psh function on $X$.

Such $\theta$-psh functions always exists, one can consider, following Demailly, the upper envelope

$$V_\theta := \sup\{\varphi : \varphi \in \text{PSH}(X, \theta), \text{and } \varphi \leq 0\}.$$  

Observe that $V_\theta^*$ is a $\theta$-psh function satisfying $V_\theta^* \leq V_\theta$, hence $V_\theta = V_\theta^*$ is a $\theta$-psh function with minimal singularities.

The cohomology class $\alpha = \{\theta\} \in H^{1,1}(X, \mathbb{R})$ is said to be big if there exists a closed $(1,1)$-current

$$T_+ = \theta + dd^c \varphi_+,$$

cohomologous to $\theta$ such that $T_+$ is strictly positive i.e $T_+ \geq \varepsilon_0 \Theta$ for some constant $\varepsilon_0 > 0$.

A function $u$ has analytic singularities if it can locally be written as

$$u = \frac{c}{2} \log \sum_{j=1}^N |f_j|^2 + h,$$

where the functions $f_j$ are holomorphic, $h$ is smooth and $c$ is a positive constant.

In the sequel we always assume that the class $\alpha = \{\theta\}$ is big. By Demailly’s regularization theorem [Dem92], any $\theta$-psh function $u$ can be approximated from above by a sequence of $(\theta + \varepsilon_j \omega)\text{-psh}$ functions $(u_j)$ with analytic singularities. Applying this to the potential $\varphi_+$ of a Kähler current $T_+ = \theta + dd^c \varphi_+$, one can moreover assume that the function $\varphi_+$ has analytic singularities. Such a current $T_+$ is then smooth on a Zariski open subset, this motivates the following:

**Definition 1.2.** The ample locus $\text{Amp}(\alpha)$ of $\alpha$ is the set of $x \in X$ such that there exists a Kähler current with analytic singularities which is smooth around $x$.

It follows from the Noetherian property of closed analytic subsets that $\text{Amp}(\alpha)$ is a Zariski open set. Note that any $\theta$-psh function $\varphi$ with minimal singularities is locally bounded on the ample locus $\text{Amp}(\alpha)$ since it has to satisfy $\varphi_+ \leq \varphi + O(1)$. Moreover, $\varphi_+$ does not have minimal singularities unless $\alpha$ is a Kähler class (cf. [Bou04, Proposition 2.5]).
By the above analysis, there exists a \( \theta \)-psh function \( \chi \) on \( X \) with analytic singularities such that, for some \( \delta_0 > 0 \),
\[
\theta + dd^c \chi \geq 2\delta_0 \Theta.
\]
Subtracting a large constant, we can always assume that \( \chi \leq 0 \), thus \( \chi \leq V_\Theta \).
Moreover, \( \chi \) is smooth in the ample locus \( \text{Amp}(\alpha) \), and \( \chi(x) \to -\infty \) as \( x \to \partial \Omega \) (cf. [Bou04, Theorem 3.17]).

1.1.2. Full Monge–Ampère mass. In [BEGZ10], the authors defined the non-pluripolar product \( T \mapsto \langle T^n \rangle \) of any closed positive \((1,1)\)-current \( T \in \alpha \), which is shown to be well-defined as a positive measure on \( X \) putting no mass on pluripolar sets. In particular given a \( \theta \)-psh function \( \varphi \), one can define its non-pluripolar Monge–Ampère product by
\[
\text{MA}_{\theta}(\varphi) := \langle (\theta + dd^c \varphi)^n \rangle.
\]
From now we denote the non-pluripolar Monge–Ampère product \( (\theta + dd^c \varphi)^n \) instead of \( \langle (\theta + dd^c \varphi)^n \rangle \). By definition the total mass of \( \text{MA}(\varphi) \) is less than or equal to the volume \( \text{Vol}(\alpha) \) of the class \( \alpha \):
\[
\int_X \text{MA}(\varphi) \leq \text{Vol}(\alpha) := \int_X \text{MA}(V_\Theta).
\]
A particular class of \( \theta \)-psh functions that appears naturally is the one for which the last inequality is an equality. We will say that such functions (or the associated currents) have full Monge–Ampère mass. For example, \( \theta \)-psh functions with minimal singularities have full Monge–Ampère mass (cf. [BEGZ10, Theorem 1.16]), but the converse is not true.

We let \( c_1 \) be the normalizing constant such that \( 2^n e^{c_1} f dV \) has total mass equal to \( \text{Vol}(\alpha) \). We have the following:

**Theorem 1.3 ([BEGZ10, Theorem 4.1]).** There exists a unique \( \theta \)-psh function \( \rho \) with full Monge–Ampère mass such that
\[
(\theta + dd^c \rho)^n = 2^n e^{c_1} f dV,
\]
and normalized by \( \sup_X \rho = 0 \). Moreover, there exists a constant \( M > 0 \) only depending on \( \theta, dV, \) and \( p > 1 \) such that
\[
\rho \geq V_\Theta - M||f||_p^{1/n}.
\]

1.2. Parabolic potentials. In this section we define the parabolic pluripotential objects in big cohomology classes necessary for our study. These are mainly taken from [GLZ21a, GLZ20] but we need to be more precise when dealing with unbounded functions. Let \( \omega = (\omega_t)_{t \in [0,T]} \) be a smooth family of closed real \((1,1)\)-forms satisfying the assumptions in Introduction.

**Definition 1.4.** We let \( \mathcal{P}(X_T,\omega) \) denote the set of functions \( \varphi : X_T \to [-\infty, +\infty) \) such that
\begin{itemize}
  \item \( \varphi \) is upper semi-continuous on \( X_T \) and \( \varphi \in L^1_{\text{loc}}(X_T) \);
  \item for each \( t \in (0, T) \) fixed, the slice \( \varphi_t : x \mapsto \varphi(t, x) \) is \( \omega_t \)-psh on \( X \);
  \item for any compact subinterval \( J \subset (0, T) \), there exists a positive constant \( \kappa = \kappa_J(\varphi) \) such that
\end{itemize}
\[
\partial_t \varphi \leq \kappa - \kappa(\rho + \chi),
\]
in the sense of distributions on \( J \times \Omega \), where \( \rho, \chi \) are defined in (1.2), (1.1).

We would like to have an interpretation of the last condition. For any compact subset \( K \subset \Omega \), there exists a constant \( C = C(K) > 0 \) such that
\[
\partial_t (t, x) \leq C, \forall (t, x) \in J \times K.
\]
Hence for every $x \in K$, the function $t \mapsto \varphi(t, x) - Ct$ is decreasing in $J$, so the partial derivative $\partial_2 \varphi$ exists for almost everywhere $t \in J$ (see e.g. [KK96, Theorem 2.1.8]).

**Lemma 1.5.** Let $\varphi_0$ be an $\omega_0$-psh function and $\varphi \in \mathcal{P}(X_T, \omega)$. If $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0$, then the extension $\varphi : [0, T) \times X \to [-\infty, +\infty)$ is upper semi-continuous in $[0, T) \times X$.

**Proof.** It suffices to prove that the extension $\varphi$ is upper semi-continuous at $(0, x_0)$ for any $x_0 \in X$. Let $(t_j, x_j) \in X_T$ be a sequence which converges to $(0, x_0)$. We will show that

$$\limsup_{j \to +\infty} \varphi(t_j, x_j) \leq \varphi_0(x_0).$$

Since $\varphi$ is bounded from above we can assume the functions $\varphi_t$ are negative. Let $h_t$ be a smooth local potential for $\omega_t$ in an open neighborhood $B$ of $x_0$ i.e. $dd^c h_t = \omega_t$. Up to replacing $\varphi_t$ by $\varphi_t + h_t$, we may assume that the functions $\varphi_t$ are psh and negative on $B$. Fix $r$ so small that $B(x, 2r) \subset B$. For any $\delta \in [0, r)$, there exists $j_0$ such that $x_j \in B(x_0, \delta)$ for all $j \geq j_0$, hence $B(x_0, r) \subset B(x_j, r + \delta)$. We have

$$\varphi(t_j, x_j) \leq \frac{1}{\text{Vol}(B(x_j, r + \delta))} \int_{B(x_j, r + \delta)} \varphi(t_j, x) dV \leq \frac{1}{\text{Vol}(B(x_j, r + \delta))} \int_{B(x_0, r)} \varphi(t_j, x) dV.$$

Since $\limsup_j \varphi_t(x) \leq \varphi_0(x)$ for all $x \in X$, Fatou’s lemma implies that

$$\lim_{j \to +\infty} \varphi(t_j, x_j) \leq \frac{\text{Vol}(B(x_0, r))}{\text{Vol}(B(x_0, r + \delta))} \frac{1}{\text{Vol}(B(x_0, r))} \int_{B(x_0, r)} \varphi_0(x) dV(x).$$

Now we first let $\delta \to 0$ and then $r \to 0$ to conclude the proof. \hfill \Box

**Definition 1.6.** We say that $\varphi \in \mathcal{P}(X_T, \omega)$ has minimal singularities if $\varphi_t - V_{\omega_t}$ is bounded for each $t \in (0, T]$ fixed.

If $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\text{loc}}(\Omega_T)$ the product

$$(\omega_t + dd^c \varphi_t)^n$$

is well defined as a positive measure in $\Omega$ as follows from the works of Bedford-Taylor [BT76, BT82]. This Monge-Ampère measure extends trivially over $X$ since $\Omega$ is a Zariski open subset in $X$. Since $\omega_t \leq \Theta$ for $0 \leq t \leq T$, the positive Borel measures $(\omega_t + dd^c \varphi_t)^n$ have uniformly bounded masses on $X$:

$$(1.4) \quad \int_X (\omega_t + dd^c \varphi_t)^n \leq \int_X (\Theta + dd^c \varphi_t)^n \leq \int_X \Theta^n.$$

These can be considered as a family of currents of degree $2n$ in the real $(2n + 1)$-dimensional manifold $X_T = (0, T) \times X$. We have the following:

**Lemma 1.7.** Let $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\text{loc}}(\Omega_T)$ and $\gamma$ be a continuous test function in $\Omega_T$. Then $t \mapsto \int_{\Omega} \gamma(t, \cdot)(\omega_t + dd^c \varphi_t)^n$ is a Borel bounded measurable function in $(0, T)$, and

$$\sup_{0 < t < T} \left| \int_{\Omega} \gamma(t, \cdot)(\omega_t + dd^c \varphi_t)^n \right| \leq (\max_{\partial \Omega} \gamma) \int_X \Theta^n.$$

**Proof.** For the first statement, the proof is identical to the corresponding one in the local context; see [GLZ21b, Lemma 2.2]. The second one follows from the inequality (1.4) above. \hfill \Box
This shows that $dt \wedge (\omega_t + dd^c \varphi_t)^n$ is well-defined as a positive Borel measure in $X_T$.

**Definition 1.8.** Fix $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\text{loc}}(\Omega_T)$. The map

$$\gamma \mapsto \int_{X_T} \gamma dt \wedge (\omega_t + dd^c \varphi_t)^n := \int_0^T dt \left( \int_\Omega \gamma(t, \cdot) (\omega_t + dd^c \varphi_t)^n \right)$$

defines a positive $(2n+1)$-current on $\Omega_T$, hence on $X_T$, denoted by $dt \wedge (\omega_t + dd^c \varphi_t)^n$, which can be identified with a positive Radon measure on $X_T$.

The following is a parabolic analogue of the convergence result of Bedford-Taylor [BT76, BT82].

**Lemma 1.9.** Assume that $(\varphi^j)$ is a monotone sequence of functions in $\mathcal{P}(X_T, \omega)$ which converges almost everywhere to a function $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\text{loc}}(\Omega_T)$ on $X_T$. Then

$$dt \wedge (\omega_t + dd^c \varphi^j_t)^n \to dt \wedge (\omega_t + dd^c \varphi_t)^n$$
in the sense of measures on $\Omega_T$.

**Proof.** The proof is similar to that of [GLZ20, Proposition 1.12] but, because of its crucial role in the sequel, we give the details here.

Let $\gamma(t, x)$ be a continuous test function in $\Omega_T$. By definition we have, for any $j$,

$$\int_{\Omega_T} \gamma(t, \cdot) dt \wedge (\omega_t + dd^c \varphi^j_t)^n = \int_0^T dt \left( \int_\Omega \gamma(t, \cdot) (\omega_t + dd^c \varphi^j_t)^n \right).$$

We now apply Bedford-Taylor’s convergence theorem (see e.g. [GZ17, Theorem 3.23]) to infer that, for any $t \in (0, T)$,

$$\int_\Omega \gamma(t, \cdot) (\omega_t + dd^c \varphi^j_t)^n \to \int_\Omega \gamma(t, \cdot) (\omega_t + dd^c \varphi_t)^n.$$

On the other hand, Lemma 1.7 yields, for all $t \in (0, T)$,

$$\left| \int_\Omega \gamma(t, \cdot) (\omega_t + dd^c \varphi^j_t)^n \right| \leq \max_\Omega |\gamma(t, \cdot)| \text{Vol}(\omega_t) \leq C(\gamma) \int_X \Theta^n.$$

The result follows from Lebesgue dominated convergence theorem. \hfill \Box

We say that $\varphi : X_T \to \mathbb{R}$ is locally uniformly semi-concave (resp. semi-convex) in $\Omega_T = (0, T) \times \Omega$ if for any compact subset $J \times K \subset (0, T) \times \Omega$ there exists $\kappa = \kappa(\varphi, J, K) > 0$ (resp. $\kappa < 0$) such that for all $x \in K$, the function $t \to \varphi(t, x) - \kappa t^2$ is concave (resp. convex) in $t \in J$. For any $x \in \Omega$ fixed, the left and right derivatives,

$$\partial_t^+ \varphi(t, x) = \lim_{s \to 0^+} \frac{\varphi(t+s, x) - \varphi(t, x)}{s},$$

and

$$\partial_t^- \varphi(t, x) = \lim_{s \to 0^-} \frac{\varphi(t+s, x) - \varphi(t, x)}{s},$$

exist for all $t \in (0, T)$, and they coincide when $\partial_t \varphi(t, x)$ exists.

Let $\ell$ denote the Lebesgue measure on $\mathbb{R}$ and $\mu$ denote a positive Borel measure on $X$. We have the following result whose proof is identical to that of [GLZ21a, Lemma 1.12].

**Proposition 1.10.** Let $\varphi : \Omega_T \to \mathbb{R}$ be a continuous function which is locally uniformly semi-concave in $(0, T)$. Then $(t, x) \to \partial_t^- \varphi(t, x)$ is upper semi-continuous while $(t, x) \to \partial_t^+ \varphi(t, x)$ is lower semi-continuous in $\Omega_T$. In particular, $\partial_t^+ \varphi$ and $\partial_t^- \varphi$ coincide and are continuous in $\Omega_T \setminus E$, where $E$ is a Borel set with $\ell \otimes \mu$ measure zero.
The following convergence results play a key role in the sequel. We omit their proofs and refer the reader to [GLZ21a, Section 2].

**Proposition 1.11.** Let $D$ be a bounded open subset in $\mathbb{R}^n$, $J \subset \mathbb{R}$ be a bounded open interval, and $0 < f \in L^p(D)$ with $p > 1$. Let $(v_j)$ be a sequence of Borel functions in $J \times D$ such that $(e^{v_j})$ is uniformly bounded in $L^1(J \times D, dt \wedge dV)$. Assume that for any $x \in D$, $v_j(., x)$ converges to a bounded Borel function $v(., x)$ in the sense of distributions on $J$ and for all $\eta \in C_0^\infty(J \times D)$

\[
\sup_{j \in \mathbb{N}, x \in D} \left| \int_J \eta(t, x)v_j(t, x)dt \right| < +\infty.
\]

Then for any positive smooth test function $\eta \in C_0^\infty(J \times D)$,

\[
\lim_{j \to +\infty} \int_{J \times D} \eta(t, x)e^{v_j(t,x)}f(x)dt \wedge dV \geq \int_{J \times D} \eta(t, x)e^{v(t,x)}f(x)dt \wedge dV.
\]

**Proof.** See [GLZ21a, Proposition 2.6].

**Proposition 1.12.** Let $(f_j)$ be a sequence of positive functions converging to $f$ in $L^1(X_T, \ell \otimes \mu)$. Let $(\varphi_j)$ be a sequence of functions in $\mathcal{P}(X_T, \omega) \cap L^\infty(\Omega_T)$ which

- converges $\ell \otimes \mu$-almost everywhere in $X_T$ to a function $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty(\Omega_T)$;
- is locally uniformly semi-concave in $(0, T) \times \Omega$.

Then the limit $\lim_{j \to +\infty} \varphi_j(t, x)$ exists and is equal to $\varphi(t, x)$ for $\ell \otimes \mu$-almost every $(t, x) \in \Omega_T$, and

\[
h(\varphi_j) f_j \ell \otimes \mu \to h(\varphi) f \ell \otimes \mu,
\]

in the weak sense of measures on $\Omega_T$, for all $h \in C^0(\mathbb{R}, \mathbb{R})$.

**Proof.** We refer the reader to [GLZ21a, Proposition 2.9] (see also [GLZ20, Theorem 1.14]).

### 1.3. Pluripotential subsolutions/supersolutions.

We assume that $T < +\infty$. As explained in the introduction, we assume here that $g(t)\theta \leq \omega \leq \Theta$, where $\theta$ is a big $(1, 1)$-form, $g$ is a smooth increasing positive function in $t \in [0, T]$, and $\Theta$ is a Kähler form.

Let us emphasize here that by comparison with [GLZ21a, GLZ20], for an element $\varphi \in \mathcal{P}(X_T, \omega)$ the weak derivative $\partial_t \varphi(t, \cdot)$ is merely locally bounded from above in $\Omega$ but $\varphi$ is not locally uniformly Lipschitz in $(0, T)$. This is natural as we are dealing with quasi-psh functions which are bounded from above but not from below.

Before defining pluripotential subsolutions (supersolutions), we need to make sense of the quantity $\dot{\varphi}_t = \partial_t \varphi(t, \cdot)$, in order to define the right-hand side of (CMAF). By the definition of $\mathcal{P}(X_T, \omega)$, for any compact subset $K \subset \Omega$, $J \Subset (0, T)$, there exists a constant $C = C_{K, J} > 0$ such that $Ct - \varphi(t, x)$ is increasing in $t \in J$ for every $x \in K$. Thus, for every $x \in K$, $\partial_t \varphi_t(x)$ is well defined for almost every $t \in J$ (see e.g. [KK96, Lemma 1.2.8]). This implies that the right-hand side of (CMAF) is well-defined almost everywhere in $\Omega_T$ (using Fubini’s theorem). This analysis motivates the following:

**Definition 1.13.** We say that a parabolic potential $\varphi \in \mathcal{P}(X_T, \omega)$ is a pluripotential subsolution to (CMAF) on $X_T$ if

- for each $t \in (0, T)$ fixed, the $\omega_t$-psh function $\varphi(t, \cdot)$ is locally bounded in $\Omega$
- the inequality

\[
(\omega_{1+t} + dd^c\varphi_t)^n \wedge dt \geq e^{\dot{\varphi}_t + F(t, \cdot, \varphi_t)} f dV \wedge dt
\]

holds in the sense of measures in $(0, T) \times \Omega$. 

Definition 1.14. We say that a parabolic potential \( \varphi \in \mathcal{P}(X_T, \omega) \) is a pluripotential supersolution to \( \text{(CMAF)} \) on \( X_T \) if
\[
\begin{itemize}
\item for each \( t \in (0, T) \) fixed, the \( \omega_t \)-psh function \( \varphi(t, \cdot) \) is locally bounded in \( \Omega \),
\item the inequality
\[
(\omega_t + dd^c \varphi_t)^n \wedge dt \leq e^{\varphi_t + F(t, \cdot ; \varphi_t)} d\nu_t \wedge dt
\]
holds in the sense of measures in \( (0, T) \times \Omega \).
\end{itemize}
\]

Remark 1.15. In these definitions the left-hand side is well-defined by using Bedford-Taylor’s theory (see Definition 1.8).

Lemma 1.16. Let \( \varphi \in \mathcal{P}(X_T, \omega) \) be a parabolic potential such that the restriction of \( \varphi \) to \( \{t\} \times \Omega \) is an \( \omega_t \)-psh function which is locally bounded on \( \Omega \). Then
\[
1) \varphi \text{ is a pluripotential subsolution to \( \text{(CMAF)} \) if and only if for a.e. } t \in (0, T),
\]
\[
(1.6) \quad (\omega_t + dd^c \varphi_t)^n \geq e^{\partial_t \varphi_t + F(t, \cdot ; \varphi_t)} d\nu_t,
\]
in the sense of measures in \( \Omega \).
\[
2) \varphi \text{ is a pluripotential supersolution to \( \text{(CMAF)} \) if and only if for a.e. } t \in (0, T),
\]
\[
(1.7) \quad (\omega_t + dd^c \varphi_t)^n \leq e^{\partial_t \varphi_t + F(t, \cdot ; \varphi_t)} d\nu_t,
\]
in the sense of measures in \( \Omega \).

Proof. We shall prove the result for subsolutions. The proof for supersolutions is similar.

We first assume that \( (1.6) \) holds for almost every \( t \). Let \( \eta \) be a positive continuous test function in \( (0, T) \times \Omega \). We thus obtain
\[
\int_\Omega \eta(t, \cdot)(\omega_t + dd^c \varphi_t)^n \geq \int_\Omega \eta(t, \cdot)e^{\partial_t \varphi_t + F(t, \cdot ; \varphi_t)} d\nu_t.
\]

Integrating with respect to \( t \), we get
\[
\int_0^T \int_\Omega \eta(\omega_t + dd^c \varphi_t)^n \wedge dt \geq \int_0^T \int_\Omega \eta e^{\partial_t \varphi_t + F(t, \cdot ; \varphi_t)} d\nu dt,
\]
hence \( \varphi \) is a pluripotential subsolution to \( \text{(CMAF)} \).

Conversely, assume that \( \varphi \) is a pluripotential subsolution to \( \text{(CMAF)} \). We consider positive test functions that can be decomposed as
\[
\eta(t, x) = \lambda(t) \xi_j(x),
\]
where \( (\xi_j) \) is a sequence of positive test functions on \( \Omega \) which generates a dense subspace of the space \( C^0_c(\Omega) \) in \( C^0 \)-topology. It follows from Fubini’s theorem that
\[
\int_0^T \left( \int_\Omega \xi_j(x)(\omega_t + dd^c \varphi_t)^n \right) dt \geq \int_0^T \left( \int_\Omega \xi_j(x)e^{\partial_t \varphi_t + F(t, \cdot ; \varphi_t)} d\nu(x) \right) dt.
\]

Hence for any \( j \) there exists a subset \( E_j \) which has full measure in \( (0, T) \) so that for all \( t \in (0, T) \)
\[
(1.8) \quad \int_\Omega \xi_j(x)(\omega_t + dd^c \varphi_t)^n \geq \int_\Omega \xi_j(x)e^{\partial_t \varphi_t + F(t, \cdot ; \varphi_t)} d\nu(x).
\]
If we set \( E := \cap_j E_j \), then \( E \) has full measure in \( (0, T) \). Moreover, the inequality \( (1.8) \) holds for all \( t \in E \) and for all \( j \). Let \( \xi \) be an arbitrary positive continuous function in \( \Omega \). We can approximate this function by convex combinations of the \( \xi_j \), we infer that for all \( t \in E \),
\[
\int_\Omega \xi(x)(\omega_t + dd^c \varphi_t)^n \geq \int_\Omega \xi(x)e^{\partial_t \varphi_t + F(t, \cdot ; \varphi_t)} d\nu(x),
\]
from which \( (1.6) \) follows. \( \square \)
Lemma 1.17. Let $\varphi, \psi \in \mathcal{P}(X_T, \omega)$ be two pluripotential subsolutions to (CMAF). Then
\[
1_{\{\varphi \geq \psi\}} \partial_t \max(\varphi, \psi) = 1_{\{\varphi \geq \psi\}} \partial_t \varphi,
\]
almost everywhere in $\Omega_T$, and
\[
(\omega + dd^c \max(\varphi, \psi))^\wedge dt \geq 1_{\{\varphi \geq \psi\}}(\omega + dd^c \varphi)^\wedge dt
\]
in the sense of measures in $\Omega_T$.

Proof. Fix $K \subset \Omega$ and $J \subset (0, T)$. Then there exists a constant $C = C_{K, J} > 0$ such that, for any $x \in K$ fixed, $Ct - \varphi(t, x)$ and $Ct - \psi(t, x)$ are increasing in $t \in J$. These functions thus have derivatives in $t$ almost everywhere on $J$ (see e.g. [KK96, Theorem 1.2.8]). Hence the first equality follows from [GT83, Lemma 7.6].

The second inequality is a simple consequence of the elliptic maximum principle in the local context (see e.g. [GZ17, Theorem 3.27]). $\square$

Lemma 1.18. For every $\lambda \in [(1 + \delta_0 g(0))^{-1}, 1]$, the function $\lambda g(0)(\rho + \chi)/2$ is $\omega_0$-psh. In particular, there exists a uniform constant $C_0 > 0$ such that
\[
\lambda g(0)\frac{\rho + \chi}{2} - C_0 \leq \varphi_0.
\]

Recall here that $\chi$ is a fixed $\theta$-psh function with analytic singularities such that $\theta + dd^c \chi \geq 2\delta_0 \Theta$, for some $\delta_0 > 0$.

Proof. By hypothesis (0.2), we first observe that
\[
\omega_0 + dd^c \lambda g(0)\frac{\rho + \chi}{2} = \frac{\lambda}{2} (\omega_0 + g(0)dd^c \rho) + \frac{\lambda}{2} (\omega_0 + g(0)dd^c \chi) + (1 - \lambda)\omega_0
\geq \frac{\lambda}{2} g(0)(\theta + dd^c \rho) + \frac{\lambda}{2} g(0)(\theta + dd^c \chi) + (1 - \lambda)\omega_0
\geq \lambda g(0)\delta_0 \Theta - (1 - \lambda)\Theta \geq 0
\]
where the last inequality follows from the choice of $\lambda$. Thus the function $\lambda g(0)(\rho + \chi)/2$ is $\omega_0$-psh. Since $\varphi_0$ is $\omega_0$-psh with minimal singularities, there exists a constant $C_0 > 0$ such that $\lambda g(0)(\rho(x) + \chi(x))/2 - C_0 \leq \varphi_0(x)$ for all $x \in X$. $\square$

2. The envelope of subsolutions

2.1. Definition.

Definition 2.1. A Cauchy datum for (CMAF) is a $\omega_0$-psh function $\varphi_0 : X \to \mathbb{R}$ with minimal singularities. We say $\varphi \in \mathcal{P}(X_T, \omega)$ is a subsolution to the Cauchy problem:
\[
(\omega_t + dd^c u_t)^\wedge = e^{\partial_t u_t + F(t, u_t)} f dV, \quad u_t \big|_{\{0\} \times X} = \varphi_0
\]
if $\varphi$ is a pluripotential subsolution to (CMAF) such that $\limsup_{t \to 0} \varphi(t, x) \leq \varphi_0(x)$ for all $x \in X$.

We let $\mathcal{S}_{\varphi_0, F}(X_T)$ denote the set of pluripotential subsolutions to the Cauchy problem above.

Lemma 2.2. The set $\mathcal{S}_{\varphi_0, F}(X_T)$ is non-empty, uniformly bounded from above on $X_T$, and stable under finite maxima.

Proof. Fix $\lambda \in [(1 + \delta_0 g(0))^{-1}, 1]$. Consider, for any $(t, x) \in X_T$,
\[
(2.1) \quad m(t, x) := \lambda g(t)\frac{\rho(x) + \chi(x)}{2} - C_1(t + 1),
\]
where \( \rho \) and \( \chi \) are defined in (1.2) and (1.1), the uniform constant \( C_1 > 0 \) will be chosen later. By hypothesis (0.2) on \( \omega_t \) we have
\[
\frac{\lambda}{2} (\omega_t + g(t) dd^c \chi) + (1 - \lambda) \omega_t \geq \frac{\lambda}{2} g(t)(\theta + dd^c \chi) + (1 - \lambda) \omega_t \\
\geq \lambda (t) \delta_0 \Theta - (1 - \lambda) \Theta \\
= |\lambda(1 + \delta_0 g(0)) - 1| \Theta \geq 0.
\]
since \( g(t) \) is increasing. Therefore, we obtain
\[
(\omega_t + dd^c \mu)^n = \left( \frac{\lambda}{2} (\omega_t + g(t) dd^c \rho) + \frac{\lambda}{2} (\omega_t + g(t) dd^c \chi) + (1 - \lambda) \omega_t \right)^n \\
\geq \left( \frac{\lambda}{2} g(t)(\theta + dd^c \rho) \right)^n = (\lambda g(t))^n e^{c_1} f dV.
\]
We set
\[
C_1 = C_0 + M_F + |n \log(g(T))| + |c_1|,
\]
it thus follows from (2.2) that
\[
\exp(\partial_t \mu + F(t, \cdot, \mu(\cdot))) f dV = \exp(\lambda g(t)(\rho + \chi)/2 - C_1 + F(t, \cdot, \mu(\cdot))) f dV \\
\leq \exp(n \log(\lambda g(t)) + c_1) f dV \\
\leq (\omega_t + dd^c \mu)^n
\]
using in the first inequality that \( g \) is increasing in \( t \in [0, T] \), \( \sup X \rho = \sup X \chi = 0 \). It follows moreover from the choice of \( C_1 \) and Lemma 1.18 that \( \mu(0, \cdot) \leq \varphi_0 \) on \( X \), hence \( \mu \in \mathcal{S}_{\varphi_0,f,F}(X_T) \).

Let now \( \varphi \in \mathcal{S}_{\varphi_0,f,F}(X_T) \) such that \( \varphi \geq \omega \). Set with \( \mu := f dV \). Consider the set
\[
G := \{ x \in X : \omega(t, x) > -M \},
\]
for \( M > 0 \) so large that \( \mu(G) > \frac{\mu(X)}{2} \). We observe that for every \( t \in (0, T) \),
\[
\varphi_t(x) \geq \omega(t, x) \geq \omega(t, x) > -M, \forall x \in G.
\]
Set \( -m_F = \inf_{[0,T] \times X} F(t, x, -M) > -\infty \). Since \( F(\cdot, \cdot, r) \) is non-decreasing in \( r \) we obtain
\[
\int_G e^{\varphi_t - m_F} d\mu \leq \int_G e^{\varphi_t + F(t, \cdot, \varphi_t)} d\mu \leq \int_G (\omega_t + dd^c \varphi_t)^n \leq \int_G (\Theta + dd^c \varphi_t)^n \leq \int_X \Theta^n.
\]
On the other hand, it follows from Jensen’s inequality that
\[
\exp \left( \int_G \varphi_t d\mu / \mu(G) \right) \leq \int_G e^{\varphi_t d\mu / \mu(G)} \mu(G).
\]
Combining these two estimates we get
\[
\int_G \varphi_t d\mu \leq \mu(G) \log \left( \frac{e^{m_F} \int_X \Theta^n}{\mu(G)} \right) \leq \mu(X) \log \left( \frac{2 e^{m_F} \int_X \Theta^n}{\mu(X)} \right) =: C.
\]
We then infer that the function \( t \mapsto \int_G \varphi_t d\mu - Ct \) is non-increasing in \((0, T)\), hence
\[
\int_G \varphi_t d\mu \leq \int_G \varphi_0 d\mu + Ct \leq \int_G \varphi_0 d\mu + CT.
\]
On the other hand, it follows from [GZ17, Proposition 8.5] that there exists a uniform constant \( C' \) (only depending on \( \mu \)) such that
\[
\int_X (\psi - \sup \psi) d\mu \geq -C', \text{ for all } \psi \in PSH(X, \Theta).
\]
Thus for each \( t \in (0, T) \),
\[
-C' \leq \int_X (\varphi_t - \sup \varphi_t) \, d\mu \leq \int_G (\varphi_t - \sup \varphi_t) \, d\mu \leq C'' - \mu(G) \sup \varphi_t.
\]
We deduce that \( \sup_X \varphi_t \) is uniformly bounded from above.

The stability under finite maxima follows immediately from Lemma 1.17. \( \square \)

From now on, we let \( M_0 > 0 \) denote a uniform upper bound of all pluripotential subsolutions \( \varphi \) to (CMAF) such that \( \varphi \geq \underline{u} \) on \( X_T \), and set
\[
MF := \sup_{X_T} F(\cdot, \cdot, M_0).
\]

Lemma 2.2 allows us to define the upper envelope of subsolutions:

**Definition 2.3.** We let \( U = U_{\varphi_0, f, F, X_T} := \sup \{ \varphi \in \mathcal{S}_{\varphi_0, f, F}(X_T) : \underline{u} \leq \varphi \leq M_0 \} \) denote the upper envelope of all subsolutions.

**Lemma 2.4.** There exists \( \varphi \in \mathcal{S}_{\varphi_0, f, F}(X_T) \) such that for any \( x \in X \),
\[
\lim_{t \to 0} \varphi(t, x) = \varphi_0(x).
\]

**Proof.** We set \( \delta = \delta_0 g(0). \) For any \( (t, x) \in [0, \delta] \times X \), consider
\[
\varphi(t, x) = (1 - \alpha_t) \varphi_0(x) + \alpha_t g(0) \frac{\rho(x) + \chi(x)}{2} + nt(\log(\delta_0^{-1} t) - 1) - Ct,
\]
where \( \alpha_t = \delta^{-1} t \), the functions \( \rho, \chi \) are defined in (1.2), (1.1), and
\[
C := MF + \delta^{-1} \sup_X \left( g(0) \frac{\rho + \chi}{2} - \varphi_0 \right) - \min(c_1, 0).
\]

Lemma 1.18 with \( \lambda = 1 \) ensures that \( C < +\infty. \) We compute
\[
\omega_t + dd^c \varphi = (1 - \alpha_t) (\omega_0 + dd^c \varphi_0) + \frac{\alpha_t}{2} (\omega_0 + g(0) dd^c \rho)
\]
\[
+ \frac{\alpha_t}{2} (\omega_0 + g(0) dd^c \chi) + \omega_t - \omega_0.
\]

Since \( \omega_t + t\Theta \) is increasing, we have \( \omega_t - \omega_0 \geq -t\Theta \), hence
\[
\frac{\alpha_t}{2} (\omega_0 + g(0) dd^c \chi) + \omega_t - \omega_0 \geq \frac{\alpha_t g(0)}{2} (\theta + dd^c \chi) + \omega_t - \omega_0
\]
\[
\geq \alpha_t g(0) \delta_0 \Theta - t\Theta = 0.
\]

Computing the time derivative we obtain
\[
\partial_t \varphi(t, \cdot) = (\delta_0 g(0))^{-1} \left( g(0) \frac{\rho + \chi}{2} - \varphi_0 \right) + n \log(\delta_0^{-1} t) - C.
\]

Hence, for all \( t \in (0, \delta) \) we have, by the choice of the constant \( C \),
\[
(\omega_t + dd^c \varphi)^n \geq \left( \frac{\alpha_t g(0)}{2} (\theta + dd^c \rho) \right)^n = (\delta_0^{-1} t)^n e^{c_1 f dV}
\]
\[
\geq e^{\partial_t \varphi + F(t, \cdot, \varphi)) f dV}.
\]

We divide \([0, T]\) into \( N \) small intervals of the same length \([T_k, T_{k+1}], k = 0, ..., N - 1\) so that \([T_{k+1} - T_k] \leq \delta := \delta_0 g(0), T_0 = 0 \) and \( T_N = T \). For \( t \in [T_k, T_{k+1}] \) we define
\[
\varphi^{(k)}(t, \cdot) := (1 - \alpha_t^{(k)}) \varphi_{T_k} + \alpha_t^{(k)} g(T_k) \frac{\rho + \chi}{2} - C^{(k)}(t - T_k)
\]
\[
+ n(t - T_k)(\log(\delta_0^{-1} (t - T_k)) - 1),
\]
where \( \alpha_t^{(k)} = \delta^{-1}(t - T_k) \), and
\[
C^{(k)} = M_F + \delta^{-1} \sup_{\lambda} \left( g(T_k) \frac{\rho + \chi}{2} - \varphi_{T_k} \right) - \min(c_1, 0).
\]
The subsolution constructed in the proof of Lemma 2.2 (see (2.1)) ensures that \( C^{(k)} < +\infty \) is a uniform positive constant. The same arguments as above ensure that \( \psi^{(k)} \) is a pluripotential subsolution to (CMAF) in \([T_k, T_{k+1}] \times X\). Gluing these functions, we get our desired pluripotential subsolution defined on \([0, T) \times X\). It is also clear from the definition that \( \varphi(t, \cdot) \) converges to \( \varphi_0 \) in \( L^1(X, dV) \) as \( t \to 0^+ \).

\[ \square \]

2.2. Lipschitz regularity in time. In this section, we study the regularity in time \( t \) of the Perron upper envelope by adapting some arguments in [GLZ21a, Section 4].

**Proposition 2.5.** For all \( 0 < S < T \) we have \( U_{\varphi_0, f, F, X_S} = U_{\varphi_0, f, F, X_T} \) in \( X_S \).

**Proof.** Set \( U_T := U_{\varphi_0, f, F, X_T} \) and \( U_S := U_{\varphi_0, f, F, X_S} \). We can assume that \( |T - S| \leq \frac{\delta_0 g(0)}{2} \), since if we can show that \( U_T = U_S \) for such \( S \) we can restart the process to prove that \( U_S = U_{S'} \) for \( S - \frac{\delta_0 g(0)}{2} < S' < S \).

It suffices to show that \( U_T = U_{U_T} \) because the reverse inequality is clear. Fix \( \varphi \in \mathcal{S}_{\varphi_0, f, F}(X_S) \). Fix \( 0 < t_0 < S \) such that \( T - t_0 < \delta_0 g(0) \). Set, for \( (t, x) \in (t_0, T) \times X \),
\[
\psi(t, x) = (1 - \alpha_t) \varphi(t_0, x) + \alpha_t g(t_0) \frac{\rho(x) + \chi(x)}{2} - C(t - t_0) + n(t - t_0)(\log[\delta_0^{-1}(t - t_0)] - 1),
\]
where \( \alpha_t = (\delta_0 g(t_0))^{-1} (t - t_0) < 1 \), the functions \( \rho, \chi \) are defined in (1.2), (1.1), and
\[
C := M_F + (\delta_0 g(t_0))^{-1} \sup_{\lambda} \left( g(t_0) \frac{\rho + \chi}{2} - \varphi_{t_0} \right) + |c_1|.
\]
From (2.1), with \( \lambda = 1 \) and \( t = t_0 \), we see that \( C < +\infty \) is a uniform constant. From (2.1) again we see that \( \partial_t \psi(t, x) \) satisfies (1.5). We compute
\[
\omega_t + dd^c \psi_t = (1 - \alpha_t) (\omega_{t_0} + dd^c \varphi_{t_0}) + \frac{\alpha_t}{2} (\omega_{t_0} + g(t_0) dd^c \rho) + \frac{\alpha_t}{2} (\omega_{t_0} + g(t_0) dd^c \chi) + \omega_t - \omega_{t_0},
\]
Since \( \omega_t + t \Theta \) is increasing, we have \( \omega_t - \omega_{t_0} \geq -(t - t_0) \Theta \) for all \( t \geq t_0 \). It thus follows that
\[
\frac{\alpha_t}{2} (\omega_{t_0} + g(t_0) dd^c \chi) + \omega_t - \omega_{t_0} \geq \frac{\alpha_t g(t_0)}{2} (\theta + dd^c \rho) + \omega_t - \omega_{t_0} \geq \alpha_t g(t_0) \delta_0 \Theta - (t - t_0) \Theta = 0.
\]
Hence, for all \( t \in [t_0, T) \),
\[
(\omega_t + dd^c \psi_t)^n \geq \left( \frac{\alpha_t g(t_0)}{2} (\theta + dd^c \rho) \right)^n = (\delta_0^{-1}(t - t_0))^n e^c dV
\]
by the choice of the constant \( C \). Therefore the function
\[
(t, x) \mapsto u(t, x) := \begin{cases} \varphi(t, x), & \text{if } t \in [0, t_0) \\ \psi(t, x), & \text{if } t \in [t_0, T) \end{cases}
\]
is a pluripotential subsolution to (CMAF) in \([0, T) \times X\) by using Lemma 1.16. We thus have \( u \in \mathcal{S}_{\varphi_0, f, F}(X_T) \) since \( u(0, \cdot) = \varphi_0 \). This yields \( u \leq U_T \) in \([0, T) \times X\). In particular \( \varphi \leq U_T \in [0, t_0) \times X \), and it follows that \( U_S \leq U_T \) on \([0, t_0) \times X \) by
taking supremum over all subsolutions. We now let $t_0 \to S$ to obtain $U_s \leq U_T$ in $X_S$.

Next we introduce the mixed type inequality:

**Lemma 2.6.** Let $\theta_1, \theta_2$ be two closed smooth $(1, 1)$-forms on $X$ with big cohomology classes. Let $\varphi_1$ ($\varphi_2$ resp.) be a bounded $\theta_1$-psh ($\theta_2$-psh resp.) function such that

\[
(\theta_1 + dd^c \varphi_1)^n \geq e^{f_1} \mu \quad \text{and} \quad (\theta_2 + dd^c \varphi_2)^n \geq e^{f_2} \mu
\]

where $f_1, f_2$ are bounded measurable functions and $\mu$ is a positive Radon measure with $L^1$ density with respect to Lebesgue measure. Then, for any $\lambda \in (0, 1)$,

\[
(\lambda(\theta_1 + dd^c \varphi_1) + (1 - \lambda)(\theta_2 + dd^c \varphi_2))^n \geq e^{\lambda f_1 + (1 - \lambda) f_2} \mu.
\]

**Proof.** The proof is the same as that of [GLZ21a, Lemma 2.10] using the convexity of the exponential together with the mixed Monge-Ampère inequalities; see e.g. [Din09].

**Theorem 2.7.** There exists a uniform constant $L_U > 0$ such that for all $(t, x) \in X_T$,

\[
(2.6) \quad |t| \partial U(t, x)| \leq L_U - L_U(\rho(x) + \chi(x)).
\]

**Proof.** Let $\varphi \in \mathcal{S}_{\rho_0, f, F}(X_T)$ such that $\rho \geq \mu$ on $X_T$, where $\mu$ is defined in (2.1). Fix $0 < T' < T$ and $\varepsilon_0 > 0$ so small that $(1 + \varepsilon_0)T' < T$. Set, for all $(t, x) \in X_{T'}$, $s \in (1 - \varepsilon_0, 1 + \varepsilon_0),$

\[
u^*(t, x) := \alpha_s \varphi(st, x) + (1 - \alpha_s)g(t)\rho(x) + \chi(x) - \frac{C}{2} |s - 1|(t + 1),
\]

where $\rho, \chi$ are defined in (1.2), (1.1), $\alpha_s = 1 - A|s - 1|$, and $C = C_0(A + 2) + \kappa_F T + AM_p + (A + 2)C_1(T + 1) + (A + 2)M_0$. The constant $C_1$ is defined in (2.3), and the constant $A$ will be chosen later that depends only on $T$. We will show that $u^* \in \mathcal{S}_{\rho_0, f, F}(X_T)$. We compute

\[
\omega_t + dd^c u^*(t, \cdot) = \frac{\alpha_s}{s} \omega_{st} + dd^c \varphi_{st}
\]

\[
+ \alpha_s \omega_t - \frac{\alpha_s}{s} \omega_{st} + (1 - \alpha_s) \left( \omega_t + g(t)dd^c \frac{\rho + \chi}{2} \right).
\]

Since $\nu$ is a subsolution to (CMAF), we have for almost every $t \in (0, T'),$

\[
(s^{-1}(\omega_{st} + dd^c \varphi_{st}))^n \geq e^{-n \log s + t \varphi(st, \cdot) + \log f(t, \cdot, \varphi(st, \cdot))} f dV.
\]

Recalling the definition of $\rho$ and $\omega_t \geq \theta$, we also have

\[
(2.7) \quad \left( \frac{1}{2}(\omega_t + g(t)dd^c \rho) \right)^n \geq \left( \frac{g(t)}{2}(\theta + dd^c \rho) \right)^n = e^{n \log g(t) + c_1} f dV.
\]

On the other hand, since $\omega_t \geq -\Theta$, we have

\[
\alpha_s \omega_t - \frac{\alpha_s}{s} \omega_{st} = \frac{\alpha_s}{s} (\omega_t - \omega_{st}) + \alpha_s (1 - s^{-1}) \omega_t
\]

\[
\geq -\alpha_s |s^{-1} - 1| \Theta - \alpha_s |s^{-1} - 1| \Theta
\]

\[
\geq -(t + 1)s^{-1} |s - 1| \Theta
\]

\[
\geq -(2T + 2)|s - 1| \Theta,
\]

where the last line follows from $s \geq 1/2$. Recall that $\theta + dd^c \chi \geq 2\delta_0 \Theta$ for some $\delta_0 > 0$. If we choose $A \geq 2(T + 1)(\delta_0 g(0))^{-1}$, then

\[
\alpha_s \omega_t - \frac{\alpha_s}{s} \omega_{st} + (1 - \alpha_s) \left( \omega_t + g(t)dd^c \chi \right) \geq -(2T + 2)|s - 1| \Theta + A |s - 1| g(t) \delta_0 \Theta \geq 0,
\]
since $g$ is increasing in $t$. Combining these estimates with the mixed Monge-Ampère inequality (Lemma 2.6) we obtain

\[
(\omega_t + dd^c u^*(t, \cdot))^n \geq (\alpha_s(s^{-1} \omega_{st} + dd^c \phi_{st}) + (1 - \alpha_s)(2^{-1}(\omega_t + dd^c \rho)))^n
\geq \exp(\alpha_s \partial_t \phi(st, \cdot) + \alpha_s F(st, \cdot, \varphi(st, \cdot))
- \alpha_s n \log s + (1 - \alpha_s)(n \log g(t) + c_1) f dV
\geq \exp(\partial_t u^*(t, \cdot) + F(t, \cdot, u^*(t, \cdot)) f dV.
\]

(2.8)

where the last line follows from the choice of $C$ as we now explain. Indeed, observe that

\[
\alpha_s \partial_t \phi(st, \cdot) = \partial_t u^*(t, \cdot) + C|s - 1| - (1 - \alpha_s)g(t)^{\rho + \chi \over 2}
\geq \partial_t u^*(t, \cdot) + C|s - 1|.
\]

(2.9)

Since $g$ is non-decreasing, we also have

\[
g(ts) - g(t) \leq \kappa_g|s - 1| \leq \kappa_g T \omega_0 g(0)^{-1} g(t).
\]

It thus follows that $g(ts) \leq \gamma(t)$ where $\gamma = 1 + \varepsilon_0 \kappa_g T g(0)^{-1}$. Choosing $\varepsilon_0$ small enough at the beginning we can ensure that $\gamma(1 + \delta_0 g(0))^{-1} < 1$. Up to increasing $A$ so large that $\frac{A}{A + 2} \geq \gamma(1 + \delta_0 g(0))^{-1}$, hence

\[
\left(1 - \frac{\alpha_s}{s}\right) \varphi_{st} = \left(1 - \frac{\alpha_s}{s}\right) (\varphi_{st} - M_0) + \left(1 - \frac{\alpha_s}{s}\right) M_0
\geq (A + 2)|s - 1|(\varphi_{ts} - M_0)
\geq (A + 2)|s - 1| \left(\frac{A}{(A + 2)\gamma} g(ts)^{\rho + \chi \over 2} - C(t + 1) - M_0\right)
\geq A|s - 1|g(t)^{\rho + \chi \over 2} - ((A + 2)C(t + 1) + (A + 2)M_0)|s - 1|,
\]

where the first inequality follows from the elementary one $1 - \frac{\alpha_s}{s} \leq (A + 2)|s - 1|$, while the second inequality follows from $\varphi \geq \varphi_{st}$ We infer

\[
\varphi_{st} \geq \frac{\alpha_s}{s} \varphi_{st} + (1 - \alpha_s)g(t)^{\rho + \chi \over 2} - C|s - 1| \geq u^*(t, \cdot).
\]

Using this and the assumption that $F$ is non-decreasing in $r$ and uniformly Lipschitz in $t$, we get

\[
\alpha_s F(st, \cdot, \varphi(st, \cdot)) = F(st, \cdot, \varphi(st, \cdot)) - (1 - \alpha_s) F(st, \cdot, \varphi(st, \cdot))
\geq F(t, \cdot, \varphi_{ts}(\cdot)) - \kappa_F t|s - 1| - |s - 1| AM_F
\geq F(t, \cdot, u^*(t, \cdot)) - \kappa_F T + AM_F|s - 1|.
\]

(2.10)

Combining (2.9), (2.10), and the definition of $C$, we obtain the last inequality in (2.8). Hence $u^*$ is a pluripotential subsolution to (CMAF) by Lemma 1.16. We now take care of the initial values. For any $x \in X$ we have

\[
u^*(0, x) = \varphi_0(x) - C|s - 1| + (1 - \alpha_s)g(0)^{\rho(x) + \chi(x) \over 2} - \left(1 - \frac{\alpha_s}{s}\right) \varphi_0(x)
\leq \varphi_0(x) - C|s - 1| + A|s - 1|g(0)^{\rho(x) + \chi(x) \over 2} - (A + 2)|s - 1|\varphi_0(x)
\leq \varphi_0(x) - C|s - 1| + (A + 2)|s - 1| \left(\frac{A}{A + 2} g(0)^{\rho(x) + \chi(x) \over 2} - \varphi_0(x)\right)
\leq \varphi_0(x)
\]

where the last line follows again from the choice of $C$. Thus, for any $x \in X$ we also get $\limsup_{t \to 0} u^*(t, x) \leq \varphi_0(x)$. Therefore $u^* \in S_{\varphi_0, f, F}(X_T)$, so $u^* \leq U$ in $X_T$. 
We thus obtain
\[ \frac{\alpha_s}{s} c(st, x) + (1 - \alpha_s)g(t) \frac{\rho(x) + \chi(x)}{2} - C|s - 1|(t + 1) \leq U(t, x). \]
We now take the supremum over all subsolutions \( \varphi \in \mathcal{S}_{\varphi_0, f, F}(X_T) \) to get
\[ \frac{\alpha_s}{s} U(st, x) + A|s - 1|g(t)(\rho(x) + \chi(x)) - C|s - 1|(t + 1) \leq U(t, x), \quad \forall (t, x) \in X_T. \]
Letting \( s \to 1 \), we infer, for all \( (t, x) \in X_T \) that
\[ t[\partial_t U(t, x)] \leq C(T + 1) + AM_0 - Ag(T)(\rho(x) + \chi(x)). \]
We can now define \( L_U := Ag(T) + C(T + 1) + AM_0 \). Finally, letting \( T' \to T \) and applying Proposition 2.5 we finish the proof. \( \square \)

2.3. Convergence at initial time. We define the upper semi-continuous (u.s.c) regularization \( U^* \) of \( U \) by the formula
\[ U^*(t, x) = \limsup_{\forall \Omega \ni (t, y) \to (0, x)} U(s, y), \quad (t, x) \in X_T. \]

We then prove that the upper envelope has the right initial values:

**Theorem 2.8.** The upper semi-continuous regularisation of the upper envelope \( U := U_{\varphi_0, f, X_T} \) satisfies, for all \( x \in \Omega \),
\[ \lim_{\Omega \ni (t, y) \to (0, x)} U^*(t, y) = \varphi_0(x). \]

**Proof.** Thanks to Lemma 2.4, it suffices to show that for all \( x \in \Omega \),
\[ \limsup_{\Omega \ni (t, y) \to (0, x)} U^*(t, y) \leq \varphi_0(x). \]

Theorem 2.7 ensures that for \( y \in \Omega \) fixed, the upper envelope \( U(\cdot, y) \) is locally Lipschitz in \((0, T)\). Arguing exactly as in the proof of [GLZ1, Lemma 1.7] we can show that \( U^*(t, \cdot) = (U_t)^* \) for all \( t \in (0, T) \), where \((U_t)^*\) denotes the u.s.c regularization of \( u_t \) (t fixed) in the \( x \)-variable only. It thus remains to prove that, for all \( x \in \Omega \),
\[ \limsup_{t \to 0} U^*_t(x) \leq \varphi_0(x). \]

Fixing \( M > 0 \), we set \( G := \{ u_T > -M \} \), where \( u \) is defined as in (2.1). We claim that there exists a constant \( C > 0 \) (also depending on \( M \)) such that, for all \( t \in (0, T) \),
\[ (2.11) \quad \int_G U^*_t f dV \leq \int_G \varphi_0 f dV + Ct. \]

Fix \( t_0 \in (0, T) \). By Choquet’s lemma, there exists a sequence \( \{ \varphi^j \} \) in \( \mathcal{S}_{\varphi_0, f, F}(X_T) \) such that
\[ \lim_{j \to +\infty} \varphi^j_{t_0} = U^*_{t_0} \quad \text{in} \quad X. \]

Since the set \( \mathcal{S}_{\varphi_0, f, F}(X_T) \) is stable under finite maximum, we can moreover assume that the sequence \( \{ \varphi^j \} \) is increasing with \( \varphi^j \leq M_0 \) on \( X \). It follows from (2.4) that
\[ \int_G \varphi^j f dV \leq \int_G \varphi_0 f dV + Ct, \quad \forall t \in (0, T), \]
for a constant \( C = C(M) > 0 \) independent of the sequence \( \{ \varphi^j \} \). For \( t = t_0 \), letting \( j \to +\infty \), we obtain
\[ \int_G U^*_{t_0} f dV \leq \int_G \varphi_0 f dV + Ct_0, \]
thanks to a classical theorem of Lelong (see e.g. [GZ17, Proposition 1.40]). Note that the sequence \( \{\varphi'_t\} \) depends on \( t_0 \), but the constant \( C \) does not. Therefore the claim (2.11) follows.

Let now \( u_0 \in \text{PSH}(X, \omega_0) \) be any cluster point of \( U^*_t \) as \( t \to 0 \). We can assume that \( U^*_t \) converges to \( u_0 \) in \( L^q(X, dV) \) for any \( q > 1 \). Then \( U^*_tf \) converges to \( u_0f \) in \( L^1(X) \). Thus, the claim above ensures that

\[
\int_G u_0fdV \leq \int_G \varphi_0fdV.
\]

We infer that \( u_0 \leq \varphi_0 \) almost everywhere on \( G \) with respect to \( fdV \), hence everywhere on \( G \) by the assumption on \( f \). Letting \( M \to +\infty \), we thus conclude that \( \limsup_{t\to0}U^*_t = \varphi_0 \) on \( \Omega \).

\[\square\]

2.4. The envelope is a subsolution. We now consider the set of subsolutions which are locally uniformly Lipschitz.

**Definition 2.9.** Let \( \kappa \) be a fixed positive constant. We let \( S^\kappa_{\varphi_0,f,F}(X_T) \) denote the set of all functions \( \varphi \in S_{\varphi_0,f,F}(X_T) \) such that, for all \( t \in (0, T), x \in \Omega \),

\[
t_0\partial_t\varphi(t, x) \leq \kappa - \kappa(\rho(x) + \chi(x)).
\]

Set

\[
U^\kappa := U^\kappa_{\varphi_0,f,F,X_T} := \sup\\{\varphi : \varphi \in S^\kappa_{\varphi_0,f,F}(X_T)\}.
\]

**Proposition 2.10.** For all \( 0 < S < T \) we have \( U^\kappa_{\varphi_0,f,F,X_S} = U^\kappa_{\varphi_0,f,F,X_T} \) in \( X_S \).

**Proof.** The proof is the same as that of Proposition 2.5.

**Theorem 2.11.** We have, for all \( \kappa > 0 \) and \( (t, x) \in X_T \),

\[
t_0\partial_tU^\kappa(t, x) \leq L_U - L_U(\rho(x) + \chi(x)),
\]

where \( L_U \) is the constant defined in Theorem 2.7.

**Proof.** The proof is the same as that of Theorem 2.7. In fact, if \( \varphi \in S^\kappa_{\varphi_0,f,F}(X_T) \) then the function \( u^\kappa \) in the proof of Theorem 2.7 satisfies

\[
t_0\partial_tu^\kappa \leq \alpha_\kappa(\kappa - \kappa(\rho + \chi)) \leq \kappa - \kappa(\rho + \chi)
\]

because \( 0 < \alpha_\kappa \leq 1 \). It follows that \( u^\kappa \in S^\kappa_{\varphi_0,f,F}(X_T) \) and we argue as in the proof of Theorem 2.7 to conclude.

**Theorem 2.12.** The upper envelope \( U \) is a pluripotential subsolution to (CMAF) in \( X_T \).

**Proof.** We will first show that \( U^\kappa = (U^\kappa)^* \) is a subsolution to (CMAF). Indeed, Choquet’s lemma implies that there exists a sequence \( \{\varphi^j\} \) in \( S^\kappa_{\varphi_0,f,F}(X_T) \) such that

\[
(U^\kappa)^* = \left( \sup_{j \in \mathbb{N}} \varphi^j \right)^* \text{ in } X_T.
\]

Since \( S^\kappa \) is stable under finite maximum, we can assume that the sequence \( \{\varphi^j\} \) is non-decreasing. We now claim that

\[
dt \wedge (\omega_t + dd^c\varphi^j)^n \to dt \wedge (\omega_t + dd^c(U^\kappa)^*)^n
\]

in the sense of measures in \( (0, T) \times \Omega \).

Let \( K \) be a relatively compact open subset of \( \Omega \) and \( J \) be a compact interval of \( (0,T) \). Then there exists a constant \( C = C(J,K) > 0 \) such that for all \( j \in \mathbb{N}, \varphi^j(t,x) - C t \) is decreasing in \( t \in J \), for any \( x \in K \). Moreover, the sequence of functions \( \varphi^j \) increases towards \( u \), so for any \( x \in K \), \( u(t,x) - Ct \) is decreasing in \( t \).
Thus for each \( x \in K \), there exists a countable subset \( E_x \subset J \) such that \( u(\cdot, x) \) is continuous on \( J \setminus E_x \). Now set
\[
E := \{ (t, x) \in J \times K : t \in E_x \}.
\]
Note that \( E \) has zero \((2n+1)\)-dimensional Lebesgue measure by using Fubini’s theorem. Let \( N \) be the set of \( t \in J \) such that \( E_t = \{ x \in K : (t, x) \in E \} \) has positive Lebesgue measure. We must have that \( N \) has zero Lebesgue measure. Thus for any \( t \in J' := J \setminus N \), the set \( E_t \) has zero Lebesgue measure, and \( \lim_{s \to t} u(s, x) = u(t, x) \) for all \( x \in K \setminus E_t \). Fixing \( (t, x) \in J' \times K \), we want to show that
\[
\limsup_{(s, y) \to (t, x)} u(s, y) \leq (U^*_t)^*(x),
\]
where the upper semicontinuous regularization on the RHS is in the \( x \)-variable only. Since the problem is local we may assume that the functions \( u \) where the upper semicontinuous regularization on the RHS is in the \( x \)-variable and negative in a neighborhood \( B(x, 2r) \subset K \). Fix \( \delta \in (0, r) \). For \( y \) so close to \( x \) that \( B(x, r) \subset B(y, r + \delta) \) we have
\[
\varphi_j(s, y) \leq \frac{1}{\Vol(B(y, r + \delta))} \int_{B(y, r + \delta)} \varphi_j(s, z) dV(z)
\]
\[
\leq \frac{1}{\Vol(B(y, r + \delta))} \int_{B(y, r + \delta)} u(s, z) dV(z).
\]
Letting \( j \to +\infty \) we get
\[
u(s, y) \leq \frac{1}{\Vol(B(y, r + \delta))} \int_{B(y, r + \delta)} u(s, z) dV(z)
\]
\[
\leq \frac{\Vol(B(x, r))}{\Vol(B(y, r + \delta)) \Vol(B(x, r))} \int_{B(x, r)} u(s, z) dV(z).
\]
Since \( \lim_{s \to t} u(s, z) = u(t, z) \) for almost every \( z \in B(x, r) \subset K \), Fatou’s lemma yields
\[
\limsup_{(s, y) \to (t, x)} u(s, y) \leq \frac{\Vol(B(x, r))}{\Vol(B(x, r + \delta)) \Vol(B(x, r))} \int_{B(x, r)} u(t, z) dV(z).
\]
Now, we first let \( \delta \to 0 \) and then \( r \to 0 \) to obtain the desired inequality (2.12) by definition of \( U^n \). The reverse inequality is clear, hence we get the equality. Therefore, for each \( \kappa \in \mathbb{R} \) we have that \( \varphi_j \) increase almost everywhere towards \( (U^*_t)^* = (U^*_t)_{\kappa}^* \) on \( K \), so Bedford-Taylor’s convergence theorem yields
\[
(\omega_t + dd^c \varphi_j^n) \to (\omega_t + dd^c (U^*_t)^n)
\]
in the weak sense of measures in \( K \). Thus the claim follows directly from Fubini’s theorem.

On the other hand, for each \( x \in K \) fixed, the sequence \( \{ \partial_t \varphi(t, x) + F(t, x, \varphi(t, x)) \} \) converges to \( \partial_t (U^*_t)^*(t, x) + F(t, x, (U^*_t)^*(t, x)) \) in the sense of distributions in \( J \), with the later being bounded in \( J \times K \). Applying Proposition 1.11 we obtain
\[
\lim_{j \to +\infty} e^{\partial_t \varphi_j + F(t, \varphi_j)} f dt \wedge dV \geq e^{\partial_t (U^*_t)^* + F(t, (U^*_t)^*)} f dt \wedge dV
\]
in the weak sense of measures in \( J \times K \). It thus follows that \( (U^*_t)^* \) is a pluripotential substitution to \( \text{(CMAF)} \) in \( \Omega_T \), and hence \( (U^*_t)^* \in S^\omega_{\varphi_0, f, F}(X_T) \). We thus deduce that \( U^\omega = (U^*_t)^* \).

We have shown that, for some \( \kappa > 0 \), \( U^\omega = U^\kappa \) for all \( \kappa > \kappa_0 \) (by Theorem 2.11). It thus remains to prove that \( U = U^\kappa \) in \( X_T \). We first assume that
\[
\varphi_0 = P_{\omega_0} h := \sup \{ \psi \in \text{PSH}(X, \omega_0) : \psi \leq h \}
\]
for some continuous function \( h \). Fix \( 0 < S < T \), \( s > 0 \) sufficiently small, and \( \varphi \in S_{\varphi_0, f, F}(X_T) \). For \( (t, x) \in [0, S] \times X \), we define

\[
u^s(t, x) := \alpha_s \varphi(t + s, x) + (1 - \alpha_s)g(t + s) \frac{\rho(x) + \chi(x)}{2} - Cs(t + 1) - \eta(s),
\]

where \( \alpha_s = 1 - (\delta_0g(0))^{-1}s, \eta(s) := \sup_X (\alpha_s\varphi - h) \) and

\[
C = (\delta_0g(0))^{-1}C_1(T + 1) + 2\delta_0^{-1}M_F + n|\log(g(T))| + |c_1|,
\]

with \( C_1 > 0 \) defined in (2.3). We compute

\[
\omega_t + dd^c u^s(t, \cdot) = \alpha_s(\omega_{t+s} + dd^c \varphi_{t+s}) + \frac{1 - \alpha_s}{2}(\omega_{t+s} + g(t + s)dd^c \rho) + \frac{1 - \alpha_s}{2}(\omega_{t+s} + g(t + s)dd^c \chi) + \omega_t - \omega_{t+s},
\]

It follows from the assumption (0.2) that

\[
\omega_t - \omega_{t+s} \geq -s\Theta.
\]

Since \( \theta + dd^c \chi \geq 2\delta_0\Theta \) we thus obtain

\[
\frac{1 - \alpha_s}{2}(\omega_{t+s} + dd^c \chi) + \omega_t - \omega_{t+s} \geq \frac{(\delta_0g(0))^{-1}}{2}s g(t + s)(\theta + dd^c \rho) - s\Theta \geq 0.
\]

We thus get

\[
(\omega_t + dd^c u^s(t, \cdot))^n \geq (\alpha_s(\omega_{t+s} + dd^c \varphi_{t+s}) + (1 - \alpha_s)g(t + s)(\theta + dd^c \rho)/2)^n \geq e^{\alpha_s(\partial_t \varphi_{t+s} + F(t + s, \cdot, \varphi_{t+s})) + (1 - \alpha_s)\log(g(t + s) + c_1)} fdV
\]

where we apply Lemma 2.6 in the last line. Since the function \( (t, r) \mapsto F(t, \cdot, r) \) is uniformly Lipschitz, it follows that

\[
\alpha_s F(t + s, \cdot, \varphi_{t+s}) = F(t + s, \cdot, \varphi_{t+s}) - (1 - \alpha_s)F(t + s, \cdot, \varphi_{t+s}) \geq F(t, \cdot, \varphi_{t+s}) - \kappa F s - (\delta_0g(0))^{-1}s M_F.
\]

Since \( \varphi \geq \underline{u} \) on \( X_T \) we have

\[
(1 - \alpha_s)\varphi_{t+s} \geq (1 - \alpha_s)g(t + s) \frac{\rho + \chi}{2} - (\delta_0g(0))^{-1}s C_1(T + 1).
\]

Consequently, it follows from the choice of \( C \) that \( \varphi_{t+s} \geq u^s(t, \cdot) \) for all \( t \in [0, S] \). Therefore,

\[
\alpha_s F(t + s, \cdot, \varphi_{t+s}) \geq F(t, \cdot, u^s(t, \cdot)) - s(\kappa F + (\delta_0g(0))^{-1}M_F),
\]

since the function \( r \mapsto F(\cdot, \cdot, r) \) is increasing. Observe now that

\[
\alpha_s \partial_t \varphi_{t+s} = \alpha_s \partial_t \nu^s + Cs - \alpha_s g(t + s) \frac{\rho + \chi}{2}.
\]

It thus follows from the choice of \( C \) and the estimates above that

\[
(\omega_t + dd^c u^s(t)^n) \geq e^{\alpha_s \partial_t \nu^s + F(t, \cdot, u^s(t))} fdV,
\]

which means that \( u^s \) is a pluripotential subsolution to (CMAF). By definition of \( u^s \) we have \( u^s(0, \cdot) \leq h \) on \( X \) since \( \sup_X \rho = \sup_X \chi = 0 \). Since \( u^s(0, \cdot) \) is \( \omega_0 \)-psh, we infer \( u^s(0, \cdot) \leq \varphi_0 = P_{\omega_0} h \) on \( X \). It follows that \( u^s \in S_{\varphi_0, f, F}(X_S) \), and hence \( u^s \in S_{\varphi_0, f, F}(X_S) \) for some \( \kappa > 0 \) large enough. Therefore, \( u^s \leq U^\kappa = U^{\omega_0} \) in \( X_S \) by Proposition 2.10. On the other hand it follows from Hartogs’ Lemma that \( \lim_{s \to 0} \eta(s) \leq 0 \). Letting \( s \to 0 \) we get \( \varphi \leq U^{\omega_0} \) in \( X_S \). Finally, letting \( S \to T \) to obtain \( \varphi \leq U^{\omega_0} \), so \( U \leq U^{\omega_0} \) on \( X_T \) (see Proposition 2.5). Therefore \( U = U^{\omega_0} \) is the maximal subsolution to (CMAF) with initial data \( \varphi_0 \).

We now remove the extra assumption on \( \varphi_0 \). Let \( \{h_j\} \) be a sequence of continuous functions decreasing to \( \varphi_0 \). Then \( \varphi_0^j = P_{\omega_0}(h_j) \) is a decreasing sequence of \( \omega_0 \)-psh functions converging to \( \varphi_0 \). We thus obtain that the upper envelope
We have $U^j := U_{\varphi_{0}^j,F,X_T}$ is also a subsolution to (CMAF) by the previous arguments. We also provide a uniform Lipschitz constant for $U^j$. Since $\varphi_{0}^j$ decreases to $\varphi_{0}$, we have that $U^j$ decreases to some $V \in \mathcal{P}(X_T)$ which is a subsolution to (CMAF) and $U \leq V$. On the other hand we see that $V \equiv (0) \leq \varphi_{0}$. Hence $V = U$. □

2.5. The envelope is locally uniformly semi-concave in time.

**Theorem 2.13.** There exists a uniform constant $C_U > 0$ such that

$$t^2 \partial_t^2 U(t, x) \leq C_U - C_U (\rho(x) + \chi(x)), $$

in the sense of distributions in $X_T$.

**Proof.** Fix $0 < T' < T$ and $\varepsilon_0 > 0$ small enough such that $(1 + \varepsilon_0)T' < T$, $s \in [1 - \varepsilon_0, 1 + \varepsilon_0]$. Set, for any $(t, x) \in X_{T'}$,

$$u^s(t, x) := \frac{s^{-1}U(st, x) + sU(s^{-1}t, x)}{2} + (1 - \alpha_s)\frac{\rho(x) + \chi(x)}{2} - C|s - 1|^2(t + 1),$$

where $\alpha_s = 1 - A(s - 1)^2$ for $A > 0$ a uniform constant to be chosen later, and

$$C := (A + 1)C_0 + \kappa_F T + A n \log(g(T)) + c_1.$$

We are going to prove that $u^s$ is a subsolution to (CMAF). We compute

$$\omega_t + dd^c u^s(t, \cdot) = \frac{1}{2} \left( \omega_{st} + dd^c U_{st} \right) + s(\omega_{s-1} + dd^c U_{s-1}) \right) + (1 - \alpha_s) \left( \frac{\omega_t + g(t) dd^c \rho}{2} \right).$$

Consider, for $s \in [1 - \varepsilon_0, 1 + \varepsilon_0], \quad h(s) := \frac{1}{2}(\omega_{st} + \omega_{s-1}).$

We have $h(1) = h'(1) = 0$, and $|h''(s)| \leq (T + 2)^2 \Theta$ on $[1 - \varepsilon_0, 1 + \varepsilon_0]$. Hence

$$\alpha_s \omega_t - \frac{\alpha_s}{2} \left( \frac{1}{2} (\omega_{st} + s \omega_{s-1}) + (1 - \alpha_s) \frac{\omega_t + g(t) dd^c \rho}{2} \right).$$

Recall that $\chi$ is a $\theta$-psh function on $X$ such that $\theta + dd^c \chi \geq 2\delta_0 \Theta$. If we take $A > 0$ such that $A \geq (T + 2)^2 (\delta_0 g(0))^{-1}$ then

$$\alpha_s \omega_t - \frac{\alpha_s}{2} \left( \frac{1}{2} (\omega_{st} + s \omega_{s-1}) + (1 - \alpha_s) \frac{\omega_t + g(t) dd^c \rho}{2} \right) \geq 0,$$

hence,

$$\omega_t + dd^c u^s(t, \cdot) \geq \frac{1}{2} \left( \omega_{st} + dd^c U_{st} \right) + s(\omega_{s-1} + dd^c U_{s-1}) \right) + (1 - \alpha_s) \frac{\omega_t + g(t) dd^c \rho}{2}.$$

It follows from Theorem 2.12 that $U$ is a pluripotential subsolution to (CMAF). We then have for almost every $t \in (0, T')$

$$(s^{-1} (\omega_{st} + dd^c U_{st})^n) \geq e^{\theta U_{st} + F(st, U_{st}) - n \log s} f dV,$$

and

$$(s(\omega_{s-1} + dd^c U_{s-1})^n) \geq e^{\theta U_{s-1} + F(s^{-1} t, U_{s-1}) + n \log s} f dV.$$

Combining these together with Lemma 2.6, we obtain

$$(\omega_t + dd^c u^s(t, \cdot))^n \geq e^{a(s) + a(s^{-1})(1 - \alpha_s) (n \log g(t) + c_1)} f dV,$$

where

$$a(s) = \frac{\alpha_s}{2} (\partial_t U_{st} + F(st, U_{st})).$$
Since $F$ is a convex function in $r$ we get
\begin{equation}
\frac{1}{2} F(st, \cdot, U_{st}) + \frac{1}{2} F(s^{-1}t, \cdot, U_{s^{-1}t}) \geq F\left( t, \frac{U(st, \cdot) + U(s^{-1}t, \cdot)}{2} \right).
\end{equation}

Now we use the same arguments as in Theorem 2.7 to show that for each $t \in [0, T)$,
\begin{equation}
U(st, \cdot) + U(s^{-1}t, \cdot) \geq u^s(t, \cdot).
\end{equation}

It is equivalent to show that
\begin{equation}
\frac{1}{2} \left[ (1 - s^{-1} \alpha_s) U_{st} + (1 - s \alpha_s) U_{s^{-1}t} \right] \geq (1 - \alpha_s) g(t) \frac{\rho + \chi}{2} - C(t + 1)(s - 1)^2.
\end{equation}

The left-hand side can be rewritten as
\[
\frac{1}{2} \left[ (1 - s^{-1}(1 - A(s - 1)^2)) U_{st} + (1 - s(1 - A(s - 1)^2)) U_{s^{-1}t} \right] = \frac{1}{2} \left[ (s - 1)(U_{st} - U_{s^{-1}t}) + (A - 1)s^{-1}(s - 1)^2 U_{st} + As(s - 1)^2 U_{s^{-1}t} \right].
\]

By Theorem 2.7, we have
\[
(s - 1)(U_{st} - U_{s^{-1}t}) \geq 2(s - 1)^2 (L_U(\rho + \chi) - L_U).
\]

The same arguments as in the proof of Theorem 2.7 give
\[
(A - 1)s^{-1} U_{st} \geq A U_{st} \geq (A - L_U/g(0)) g(t) \frac{\rho + \chi}{2} - C_1(T + 1),
\]
\[
As U_{s^{-1}t} \geq (A + 1) U_{s^{-1}t} \geq (A - L_U/g(0)) g(t) \frac{\rho + \chi}{2} - C_1(T + 1).
\]

where $A$ is large enough so that $(A - L_U/g(0))/(A + 1) \geq (1 + \delta_0 g(0))^{-1} (\gamma = 1/c_0 \kappa g(T g(0)^{-1}))$. Combining these estimates, it follows from the choice of $C$ that (2.16) holds. Since $F$ is non-decreasing in $r$ and uniformly Lipschitz in $t$, it follows from (2.14) and (2.15) that
\[
\frac{1}{2} F(st, \cdot, U_{st}) + \frac{1}{2} F(s^{-1}t, \cdot, U_{s^{-1}t}) \geq F\left( t, \frac{U(st, \cdot) + U(s^{-1}t, \cdot)}{2} \right) - \kappa_F t \left( s + s^{-1} - 1 \right)
\]
\[
\geq F(t, \cdot, u^s(t, \cdot)) - \kappa_F T(s - 1)^2,
\]

hence
\[
\frac{\alpha_s}{2} \left( F(st, \cdot, U_{st}) + F(s^{-1}t, \cdot, U_{s^{-1}t}) \right) \geq F(t, \cdot, u^s(t, \cdot)) - \kappa_F T(s - 1)^2 - A M F(s - 1)^2.
\]

Therefore, we obtain
\[
a(s) + a(s^{-1}) (n \log g(t) + c_1) \geq \partial_x u^s(t, \cdot) + F(t, \cdot, u^s(t, \cdot)).
\]

On the other hand, the choice of $C$ ensures, for any $(t, x) \in X_{T'}$, that
\[
u^s(0, x) \leq \varphi_0(x) - C(s - 1)^2 + (1 - \alpha_s) g(0) \frac{\rho(x) + \chi(x)}{2} - \left( 1 - \frac{s + s^{-1}}{2} \alpha_s \right) \varphi_0(x)
\]
\[
\leq \varphi_0(x) - C(s - 1)^2 + A(s - 1)^2 g(0) \frac{\rho(x) + \chi(x)}{2} - (A + 1)(s - 1)^2 \varphi_0(x)
\]
\[
\leq \varphi_0(x) - C(s - 1)^2 + (A + 1)(s - 1)^2 A g(0) \frac{\rho(x) + \chi(x)}{2} - \varphi_0(x)
\]
\[
\leq \varphi_0(x) - C(s - 1)^2 + (A + 1) C_0(s - 1)^2 \leq \varphi_0(x).
\]
Therefore, we conclude that \( u^* \in S_{\phi_n,f,F}(X_T) \), so we obtain for any \((t,x) \in X_T\), that
\[
\alpha_s \frac{s^{-1}U(st,x) + sU(s^{-1}t,x)}{2} - U(t,x) + A/2(s-1)^2(\rho(x) + \chi(x)) \\
\leq C(T+1)(s-1)^2,
\]
hence
\[
\frac{s^{-1}U(st,x) + sU(s^{-1}t,x)}{2} - U(t,x) + A(s-1)^2(\rho(x) + \chi(x)) \\
\leq (C(T+1) + 2AM_0)(s-1)^2.
\]
From this, we obtain for all \((t,x) \in X_T\),
\[
\frac{U(st,x) + U(s^{-1}t,x)}{2} - U(t,x) + A(s-1)^2(\rho(x) + \chi(x)) \\
\leq (C(T+1) + (2A+1)M_0 + 2L_U - L_U(\rho(x) + \chi(x)))(s-1)^2.
\]
Letting \( s \to 1 \) yields for all \((t,x) \in X_T\),
\[
i^2\partial_t^2 U(t,x) \leq (C(T+1) + (2A+1)M_0 + 5L_U) - (A + L_U)(\rho(x) + \chi(x)).
\]
We finally let \( T' \to T \) and apply Proposition 2.5 to complete the proof. \( \square \)

3. Existence and Uniqueness

3.1. Existence of solutions. We shall prove in this section that \( U_{\phi_n,f,F,X_T} \) is the unique pluripotential solution to the Cauchy problem (see Definition 2.1).

**Theorem 3.1.** The upper envelope \( U := U_{\phi_n,f,F,X_T} \) is a pluripotential solution to the Cauchy problem for the parabolic complex Monge-Ampère equation (CMAF) in \( X_T \). Moreover, \( U \) is locally uniformly semi-concave in \((0,T) \times \Omega \).

**Proof.** We have shown in Theorem 2.13 that \( U \) is locally uniformly semi-concave in \( t \in (0,T) \), and \( U \in S_{\phi_n,f,F}(X_T) \) and it satisfies the initial condition. It remains to show that \( U \) solves the parabolic equation (CMAF). We apply a local balayage process to modify the function \( U \) on a given "small ball" \( B \subset \Omega \) by constructing a new \( \omega_t \)-psh function \( U_B \) so that it satisfies the local Monge-Ampère flow \( (\omega_t + dd^c U_B)^n = e^\omega U_B(t,z) + F(t,z,U_B(t,z)) f dV \) on \( B_T = (0,T) \times B \), \( U_B \geq U \) on \( B_T \) and \( U_B = U \) on \( X_T \setminus B_T \).

Indeed, we choose complex coordinates \( z = (z_1, \ldots, z_n) \) identifying \( B \) with the complex unit ball \( \mathbb{B} \subset \mathbb{C}^n \). We can write \( \omega_t = dd^c g_t \) in a local holomorphic coordinate chart \( B \subset X \), for some smooth local potential \( g_t \). Set \( f = f_0 z^{-1} \in L^p(\mathbb{B}) \) and \( d\tilde{V} \) is the restriction of the volume \( dV \) to \( B \). We consider the following complex Monge-Ampère flow
\[
(3.1) \quad dt \wedge (dd^c u_t)^n = e^\partial_t u(t,z) + \tilde{F}(t,z,u(t,z)) \tilde{f} d\tilde{V} \wedge dt
\]
in \( \mathbb{B}_T := (0,T) \times \mathbb{B} \) with the Cauchy-Dirichlet boundary data \( h \) being the restriction of \( U \) defined on the parabolic boundary of \( \mathbb{B}_T \) denoted by \( \partial \mathbb{B}_T := \{0,T\} \times \partial \mathbb{B}\} \cup \{(0) \times \mathbb{B}\} \). Here \( \tilde{F}(t,x,r) = F(t,x,r-g_t(x)) - \partial_r g_t \) satisfies the same assumptions as \( F \). We have shown that \( h \) is locally uniformly Lipschitz (see Theorem 2.7) and locally uniformly semi-concave (see Theorem 2.13) i.e. for all \( 0 < T' < T \), and for all \( (t,z) \in (0,T') \times \partial \mathbb{B} \), there exist constants \( L \) and \( C = C(T') \) such that
\[
(3.2) \quad t|\partial_t h(t,z)| \leq L, \quad t^2 |\partial^2 h(t,z)| \leq C.
\]
Using mollifiers we can find a sequence \( h^j \) of continuous functions on \( [0,T) \times \partial \mathbb{B} \) such that \( h^j \) decreases pointwise to \( h \). The function \( h^j \) is thus the Cauchy-Dirichlet boundary data satisfying the same assumption (3.2) as \( h \).
Then it follows from [GLZ21a, Theorem 6.4] that there exists a sequence of functions $u^j$ solving (3.1) with the boundary data $h^j$. Moreover, $u^j$ is locally uniformly semi-concave in $t \in [0,T)$. Since $h^j$ decreases to $U \circ z^{-1}$ on $\partial_p B_T$, so $U \circ z^{-1} \leq u^j$ and the sequence $u^j$ decreases to some function $v$. The function $v$ solves (3.1) by using Proposition 1.12, and $\limsup_{t\to0} \psi(t,z) \leq U_0 \circ z^{-1}$ in $B$. But the comparison principle (see [GLZ21a, Theorem 6.5]) ensures that $U \circ z^{-1} \leq v$ in $B_T$. Hence $\lim_{t\to0} \psi(t,z) = U_0 \circ z^{-1}$. We then define

$$U_B(t,x) = \begin{cases} v(t,z(x)) & \text{in } B_T \\ U(t,x) & \text{in } X_T \setminus B_T. \end{cases}$$

as required. We infer that $U_B$ belongs to the set $S_{\phi_0,f,F}(X_T)$, the maximal property ensures that $U_B \leq U$, hence equality. Since $B$ is an arbitrary ball in $\Omega$, this shows that $U$ solves (CMAF) on $\Omega_T$, hence on $X_T$.

Moreover, by Theorem 2.7 and Theorem 2.13, $U$ is locally uniformly uniformly Lipschitz and semi-concave in $t$.

3.2. The comparison principle. We first establish a version of the comparison principle which requires relatively strong regularity assumptions:

**Proposition 3.2.** Let $\phi$ (resp. $\psi$) be a subsolution (resp. supersolution) to (CMAF) with initial value $\phi_0$ (resp. $\psi_0$). We assume that

1. $\phi$ is $C^1$ in $t$ and continuous on $(0,T) \times \Omega$,
2. $\psi$ is locally uniformly semi-concave in $t$,
3. $\phi \rightarrow \phi_0$ and $\psi_t \rightarrow \psi_0$ in $L^1(X)$, as $t \to 0$,
4. for any $t \in [0,T)$, $\psi_t$ has minimal singularities,
5. the function $(t,x) \mapsto \psi(t,x)$ is continuous on $[0,T) \times \Omega$.

Then

$$\phi_0 \leq \psi_0 \Rightarrow \phi \leq \psi \quad \text{in } X_T.$$

**Proof.** Fix $0 < T' < T$, in particular $T' < +\infty$. We shall prove that $\phi \leq \psi$ on $[0,T'] \times X$. The result thus follows by letting $T' \to T$. We fix $\lambda, \varepsilon > 0$ sufficiently small. Set for $(t,x) \in [0,T'] \times X$,

$$\phi_{\lambda}(t,x) := (1 - \lambda)\phi(t,x) + \lambda g(t) \rho(x) + \chi(x),$$

where $\rho, \chi$ are $\theta$-psh functions defined in (1.2), (1.1). One can moreover impose $\chi < 0$ to be smooth in the ample locus $\Omega = \text{Amp}\{\theta\}$, with analytic singularities, and such that $\chi(x) \to -\infty$ as $x \to \partial \Omega$. We will show that $\phi_{\lambda} \leq \psi$ and we then let $\lambda \to 0$ to conclude the proof. Set

$$w(t,x) := \phi_{\lambda}(t,x) + \lambda g(0)\delta_0 \chi(x) - \psi(t,x) - 3\varepsilon t.$$ 

Observe that by Lemma 1.5, this function is upper semi-continuous on $[0,T'] \times \Omega$. By the assumption d) we have $\phi_{\lambda}(t,\cdot) \leq \psi(t,\cdot) + O(1)$ for each $t$. Since $\chi$ is continuous in $\Omega$ and tends to $-\infty$ on $\partial \Omega$, we have that $w$ tends to $-\infty$ on $\partial \Omega$. Hence $w$ attains its maximum at some point $(t_0, x_0) \in [0,T'] \times \Omega$.

We want to show that $w(t_0, x_0) \leq 0$. Assume by contradiction that it is not the case i.e $w(t_0, x_0) > 0$, with $t_0 > 0$. The set

$$K := \{x \in \Omega : w(t_0, x) = w(t_0, x_0)\}$$

is a compact subset of $\Omega$ since $w(t_0, x)$ tends to $-\infty$ as $x \to \partial \Omega$. The classical maximum principle ensures for all $x \in K$ that

$$(1 - \lambda)\partial_{\psi} \phi(t_0, x) \geq \partial_{\psi}^r \psi(t_0, x) + 3\varepsilon,$$

since $g'(t) \geq 0$ for all $t$. The partial derivative $\partial_{\psi} \phi(t, x)$ is continuous in $\Omega$ by assumption. Since the function $t \mapsto \psi(t,x)$ is locally uniformly semi-concave, for
any $t \in (0, T)$, the left derivative $\partial_t^- \psi(t, \cdot)$ is upper semi-continuous in $\Omega$ (see Proposition 1.10). We can thus find $\eta > 0$ small enough that, by introducing the open set containing $K$,
$$D := \{ x \in \Omega : w(t_0, x) > w(t_0, x_0) - \eta \} \Subset \Omega.$$We have for all $x \in D$,
$$(3.3) \quad (1 - \lambda)\partial_t \psi(t_0, x) > \partial_t^- \psi(t_0, x) + 2\varepsilon.$$Set $u := \varphi(x(t_0, \cdot)) + \lambda g(0)\delta_{x_0}^{\varphi}$ and $v = \psi(x(t_0, \cdot))$. We observe that
$$\frac{\lambda \omega_x + dd^c g(t_x)}{2} + \lambda g(0) dd^c \chi \geq \lambda g(0) \delta_{x_0} + \lambda g(0) dd^c \chi \geq 0.$$Since $\varphi$ is a pluripotential subsolution to (CMAF), we infer by using Lemma 2.6 that
$$(\omega_{t_0} + dd^c u)^a \geq (1 - \lambda)(\omega_{t_0} + dd^c \varphi_{t_0}) + \lambda(g(t_0)\theta + g(t_0) dd^c \rho)/2)^a \geq e^{(1-\lambda)(\partial_t \varphi(t_0, \cdot) + F(t_0, \varphi(t_0, \cdot)))} + \lambda(n \log g(t) + c_1) dV$$in the weak sense of measures in $D$. Choosing $\lambda$ so small that
$$\lambda < \min_{[0, T]} \{ (M_F + |n \log g(t) + c_1|)^{-1} \}.$$It thus follows from (3.3) and the increasing property of $F$ that
$$(\omega_{t_0} + dd^c u)^a \geq e^{0^+ \psi(t_0, \cdot) + F(t_0, \psi(t_0, \cdot))} dV$$in the weak sense of measures in $D$. On the other hand, $\psi$ is a pluripotential supersolution to (CMAF), thus
$$(\omega_{t_0} + dd^c v)^a \leq e^{0^- \psi(t_0, \cdot) + F(t_0, \psi(t_0, \cdot))} dV$$in the weak sense of measures in $D$. The last two inequalities yield
$$(\omega_{t_0} + dd^c u)^a \geq e^{F(t_0, \psi(t_0, \cdot))} (\omega_{t_0} + dd^c v)^a.$$Shrinking $D$ if necessary, we can assume that $u(x) > v(x)$ for any $x \in D$. We thus get
$$(\omega_{t_0} + dd^c u)^a \geq e^c (\omega_{t_0} + dd^c v)^a$$in the sense of measures in $D$. Consider now $\tilde{u} := u + \min_{\partial D}(v - u)$. We observe that $v \geq \tilde{u}$ on $\partial D$, hence the elliptic comparison principle (see Proposition 3.3) yields
$$\int_{\{v < \tilde{u}\} \cap D} e^c (\omega_{t_0} + dd^c v)^a \leq \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + dd^c u)^a \leq \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + dd^c v)^a.$$Thus $\tilde{u} \leq v$ almost everywhere in $D$ with respect to the measure $(\omega_{t_0} + dd^c v)^a$. It thus follows from the domination principle (Proposition 3.3) that $\tilde{u} \leq v$ everywhere in $D$. In particular,
$$(4.4) \quad u(x) - v(x) + \min_{\partial D}(v - u) \leq \tilde{u}(x) - v(x) \leq 0.$$On the other hand, since $K \cap \partial D = \emptyset$, we get $w(t_0, x) \leq w(t_0, x_0)$, for all $x \in \partial D$, hence
$$u(x) - v(x) < u(x_0) - v(x_0),$$contradicting (4.4). Therefore, we must have $t_0 = 0$, hence
$$(1 - \lambda)\varphi + \lambda g(t)\frac{\varphi + \chi}{2} + \lambda g(0)\chi - \psi - 3\varepsilon t \leq \lambda \sup_{X} \left( g(0)\frac{\varphi + \chi}{2} - \varphi_0 \right),$$
in \([0, T] \times \Omega\). The right-hand side is finite thanks to Lemma 1.18. Letting \(\lambda \to 0\) we obtain \(\varphi \leq \psi + 3\varepsilon T\) in \([0, T'] \times \Omega\), hence in \([0, T'] \times X\). We thus conclude the proof by letting \(\varepsilon \to 0\) and \(T' \to T\).

\[\Box\]

**Proposition 3.3.** Fix a nonempty open subset of \(D \subseteq \Omega\). Let \(u, v\) be \(\theta\)-psh functions, which are bounded in a neighborhood of \(D\), such that

\[
\limsup_{D \ni x \to \partial D} (u - v)(x) \geq 0.
\]

Assume that \(v\) has minimal singularities. Then

\[
\int_{\{u < v\} \cap D} MA_{\theta}(v) \leq \int_{\{u < v\} \cap D} MA_{\theta}(u).
\]

Moreover, if \(MA_{\theta}(u)\{u < v\} \cap D\) = 0 then \(u \geq v\) in \(D\).

**Proof.** Fix \(\varepsilon > 0\). For each \(C > 0\) we set \(u^C := \max(u, V_{\theta} - C)\). Then the function \(\max(u^C, v - \varepsilon)\) is \(\theta\)-psh with minimal singularities and coincides with \(u^C\) in a neighborhood of \(\partial D\). The boundary condition means that for any \(\varepsilon > 0\) the subset \(\{u^C < v - \varepsilon\}\) is compact in \(D\). Let \(\{u^C < v - \varepsilon\} \subseteq D' \subseteq D\). We claim that

\[
\int_{D'} (\theta + dd^c u^C)^n = \int_{D'} (\theta + dd^c \max(u^C, v - \varepsilon))^n.
\]

Indeed, set \(w := \max(u^C, v - \varepsilon)\), using local regularization of plurisubharmonic functions, we observe that \((\theta + dd^c w)^n - (\theta + dd^c u^C)^n = dd^c S\) in the sense of currents on \(D\), where \(S := (w - u^C)((\theta + dd^c w)^{n-1} + \cdots + (\theta + dd^c u^C)^{n-1})\) is a well-defined current with compact support in \(D\). Pick any test function \(\gamma\) which is identically 1 in a neighborhood of the support of \(S\). Then

\[
\int_{D'} dd^c S = \int_{D'} \gamma dd^c S = \int_{D'} S \wedge dd^c \gamma = 0,
\]

where we have known that \(dd^c \gamma = 0\) on the support of \(S\). This implies (3.5).

On the other hand, we apply [GZ17, Theorem 3.27] to get

\[
\int_{\{u^C < v - \varepsilon\}} (\theta + dd^c \max(u^C, v - \varepsilon))^n = \int_{\{u^C < v - \varepsilon\}} (\theta + dd^c v)^n.
\]

Combining this together with the equality (3.5), we obtain

\[
\int_{\{u^C < v - \varepsilon\} \cap D'} MA_{\theta}(v) = \int_{D'} MA_{\theta}(\max(u^C, v - \varepsilon)) - \int_{\{u^C \geq v - \varepsilon\}} MA_{\theta}(\max(u^C, v - \varepsilon))
\]

\[
\leq \int_{D'} MA_{\theta}(u^C) - \int_{\{u^C > v - \varepsilon\}} MA_{\theta}(\max(u^C, v - \varepsilon))
\]

\[
\leq \int_{D'} MA_{\theta}(u^C) - \int_{\{u^C > v - \varepsilon\}} MA_{\theta}(u^C),
\]

which implies

\[
\int_{\{u^C < v - \varepsilon\} \cap D'} MA_{\theta}(v) \leq \int_{\{u^C \leq v - \varepsilon\} \cap D} MA_{\theta}(u^C)
\]

since \(D'\) was taken arbitrarily. Letting \(\varepsilon \to 0\) and then \(C \to +\infty\) we obtain the required inequality.

Arguing as in [BEGZ10, Corollary 2.5], we complete the last statement. Let \(\chi\) be a \(\theta\)-psh function defined in (1.1). Since \(v\) has minimal singularities, we may also assume that \(\chi \leq v\). For any \(\varepsilon > 0\) small enough, consider \(v_\varepsilon := (1 - \varepsilon) v + \varepsilon \chi\), hence

\[
\limsup_{D \ni x \to \partial D} (u(x) - v_\varepsilon(x)) \geq 0.
\]

Then

\[
\varepsilon^n \int_{\{u < v_\varepsilon\}} MA_{\theta}(\chi) \leq \int_{\{u < v_\varepsilon\}} MA_{\theta}(v_\varepsilon) \leq \int_{\{u < v_\varepsilon\}} MA_{\theta}(u) = 0,
\]

since \(\chi \leq v\).
since $\chi \leq v$ implies that $\{u < v_\varepsilon\} \subset \{u < v\}$. On the other hand, $\text{MA}_\theta(\chi)$ dominates Lebesgue measure. We deduce that $u \geq v_\varepsilon$ almost everywhere with respect to Lebesgue measure in the open set $D$, hence everywhere in $D$. The result follows by letting $\varepsilon \to 0$. 

We now slightly relax the hypothesis e) in Proposition 3.2.

**Proposition 3.4.** Let $\varphi$ (resp. $\psi$) be a pluripotential subsolution (resp. supersolution) to (CMAF) with initial value $\varphi_0$ (resp. $\psi_0$). We assume that

a) $\varphi$ is $C^1$ in $t$ and continuous on $(0, T) \times \Omega,$

b) $\psi$ is locally uniformly semi-concave in $t,$

c) $\varphi_t \to \varphi_0$ and $\psi_t \to \psi_0,$ as $t \to 0,$

d) for any $t \in (0, T),$ $\varphi_t$ has minimal singularities,

e') the function $(t, x) \mapsto \psi(t, x)$ is continuous on $(0, T) \times \Omega.$

Then

$$\varphi_0 \leq \psi_0 \Rightarrow \varphi \leq \psi \quad \text{in} \, X_T.$$ 

**Proof.** We proceed as in the proof of Theorem 2.7. We fix $s > 0$ sufficiently small and consider

$$v^s(t, x) = \psi(t + s, x) + Cs(t + 1) - Cs \log \delta_0^{-1}s,$$

and

$$u^s(t, x) := \alpha_s \varphi(t, x) + (1 - \alpha_s)g(t)\rho(x) + \chi(x) - Cs(t + 1).$$

Here $\alpha_s = 1 - Ap \in (0, 1),$ $A > 0$ is determined in Lemma 3.5, the functions $\rho, \chi$ are defined in (1.2), (1.1) and $C$ is a positive constant which will be chosen later. We want to show that for $C > 0$ large enough, $u^s$ is a subsolution while $v^s$ is a supersolution to (CMAF) and $u^s(0, \cdot) \leq v^s(0, \cdot).$ We can then apply Proposition 3.2 and let $s \to 0$ to complete the proof. We first observe that

$$\omega_{t+s} + dd^cu^s = \alpha_s(\omega_t + dd^c\varphi_t) + \frac{1 - \alpha_s}{2}(\omega_t + g(t)dd^c\rho)$$

$$+ \frac{1 - \alpha_s}{2}(\omega_t + \rho(t)dd^c\chi) + \omega_{t+s} - \omega_t.$$ 

By assumptions, we see that for $s > 0 \omega_{t+s} - \omega_t \geq -s\theta.$ Since $\theta + dd^c\chi \geq 2\delta_0\theta$ we thus obtain

$$\frac{1 - \alpha_s}{2} + (\omega_t + g(t)dd^c\chi) + \omega_{t+s} - \omega_t \geq Asg(t)\delta_0\theta - s\theta \geq 0.$$ 

Hence,

$$(\omega_{t+s} + dd^c\psi_t)^n \geq (\alpha_s(\omega_t + dd^c\varphi_t) + (1 - \alpha_s)g(t)(\theta + dd^c\rho)/2)^n$$

$$\geq e^{\alpha_s(\partial_t\varphi_t + F(\psi(t, \cdot))) + (1 - \alpha_s)(n \log g(t) + c_1)} f dV$$

applying Lemma 2.6 in the last line. Since $\alpha_s = 1 + O(s)$ and $F$ is bounded from above and locally uniformly Lipschitz, by choosing $C > 0$ large enough, depending on $\delta_0, \kappa_F, M_F, c_1$ (see Theorem 2.7), we have

$$(\omega_{t+s} + dd^c\psi_t)^n \geq e^{\theta s + F(\psi(t, \cdot))} f dV.$$ 

On the other hand, since $\psi$ is a supersolution, we have

$$(\omega_{t+s} + dd^c\psi_t)^n \leq e^{\theta s - C_3 + F(t, \cdot, \psi(t, \cdot))} f dV$$

$$\leq e^{\theta s - C_3 + F(t, \cdot, \psi(t, \cdot)) + \kappa_F s} f dV$$

$$\leq e^{\theta s + F(t, \cdot, \psi(t, \cdot))} f dV.$$
where the second line follows from the Lipschitz condition and the increasing monotonicity of $F$, the last line follows from the choice of $C$. Up to increasing $C > 0$ it follows from Lemma 3.5 that for any $x \in X$,
\[
 u^s(0, x) \leq (1 - As)\psi(0, x) + Ag(0)s(\rho + \chi)/2 \\
\leq \psi_s(x) + Cs - Cs \log Ag(0)s = v^s(0, x).
\]

It then follows from Proposition 3.2 that $u^s(t, x) \leq v^s(t, x)$ for all $(t, x) \in X_r$. Letting $s \to 0$, we finish the proof. \hfill \Box

**Lemma 3.5.** With the same assumptions of $\psi$ as in Proposition 3.4. Then there exist uniform constants $A > 0$, $C > 0$, and $t_0 > 0$ small enough such that for all $(t, x) \in (0, t_0) \times X$,

\[
\psi(t, x) \geq (1 - A)\psi_0(x) + C(t \log(Ag(0)t) - t) + Ag(0)t(\rho(x) + \chi(x))/2.
\]

**Proof.** The proof is similar to that of [GLZ20, Lemma 3.14]. We recall that the function $F$ satisfies the Lipschitz condition i.e., there exists a constant $\kappa_F > 0$ such that, for all $t, t' \in [0, T/2]$, $x \in X$, $r \in \mathbb{R}$,

\[
|F(t, x, r) - F(t', x, r)| \leq \kappa_F|t - t'|.
\]

Set $t_0 = \min(1, A, T/4)$ with $A > 0$ under control. Fix $s > 0$ sufficiently small and consider for $(t, x) \in (0, t_0) \times X$,

\[
u^s(t, x) := (1 - At)\psi_s(x) + Atg(s)(\rho(x) + \chi(x))/2 + n(t \log Ag(0)t - t) - Ct,
\]

\[
v^s(t, x) := \psi(t + s, x) + 2\kappa Fs,
\]

where $\rho, \chi$ are $\theta$-psh functions on $X$ defined in (1.2),(1.1), and $C$ is a positive constant to be chosen later. We see that $u^s$ is of class $C^1$ in $t$, and for any $t \in (0, t_0)$ fixed, $u^s(t, \cdot)$ is continuous in $\Omega$ since $\rho$ is continuous in $\Omega$ (see e.g. [GZ17, Theorem 12.23]). Now let $A \geq (\delta g(0))^{-1}$. One has, for any $t \in (0, t_0)$,

\[
\omega_{t+s} + dd^c u^s(t, \cdot) = (1 - At)\omega_s + dd^c \psi_s + At(\omega_s + g(s)dd^c \rho)/2 + 2At(\omega_s + g(s)dd^c \chi)/2 + \omega_{t+s} - \omega_s.
\]

By hypothesis (0.2), we have $\omega_{t+s} - \omega_s \geq -t\Theta$. Since $\theta + dd^c \chi \geq 2\delta_\theta \Theta$, we thus obtain

\[
At(\omega_s + g(s)dd^c \chi)/2 + \omega_{t+s} - \omega_s \geq Atg(s)\delta_\theta \Theta - t\Theta \geq 0,
\]

hence

\[
(\omega_{t+s} + dd^c u^s)^n \geq (At(\omega_s + g(s)dd^c \rho)/2)^n \geq (Ag(0)t)^n \epsilon^s f dV,
\]

since $g(s) \geq 0$. We now choose $C = As\sup_X(\rho + \chi - 2\varphi_0)/2 + M_F - \min(c, 1)$, hence

\[
(\omega_{t+s} + dd^c u^s)^n \geq e^{\partial_t u^s(t, \cdot) + F(t, s, v^s(t, \cdot))} f dV.
\]

It follows from the definition that $u^s(t, \cdot)$ converges in $L^1(X, dv)$ to $u^s(0, \cdot) = \psi_s$. On the other hand, since $\psi$ is a supersolution to (CMAF), we have

\[
(\omega_{t+s} + dd^c v^s)^n \leq e^{\partial_t v^s(t, \cdot) + F(t, s, \psi(t+s, \cdot))} f dV = e^{\partial_t v^s(t, \cdot) + F(t, s, \psi(t+s, \cdot))} f dV.
\]

Since the function $F$ is Lipschitz in $t$ and is increasing in $r$, for all $t, s \in (0, t_0), x \in X$,

\[
F(t + s, x, \psi(t + s, x)) \leq F(t, x, \psi(t + s, x)) + \kappa Fs
\]

\[
\leq F(t, x, v^s(t, x)) + \kappa Fs,
\]

this yields

\[
(\omega_{t+s} + dd^c v^s)^n \leq e^{\partial_t v^s(t, \cdot) + F(t, s, v^s(t, \cdot))} f dV.
\]
Since for each \( s \) that \( v_s \) is continuous on \( \Omega \) hence \( v^s \) is continuous on \([0,t_0) \times \Omega\) and it is clear that \( v^s(t,\cdot) \) converges to \( v^s(0,\cdot) = \psi_s \) in \( L^1(X,dV) \) as \( t \to 0 \). We moreover see that for each \( t \), \( v^s(t,\cdot) \) has minimal singularities because \( \psi_s \) has. We can now apply Proposition 3.2 and get \( u^s \leq v^s \) on \((0,t_0) \times X\). Letting \( s \to 0 \) we have for all \((t,x) \in (0,t_0) \times X\),
\[
(1 - At)\psi_0(x) + Ag(0)t(\rho(x) + \chi(x))/2 + C(t \log(Ag(0)t) - t) \leq \psi(t,x),
\]
as desired. \( \square \)

### 3.3. Space regularity

In this section, we shall use the extra assumption (3.6)
\[
\varpi_t \leq A\varpi_t, \quad \forall t \in [0,T),
\]
for some constant \( A > 1 \).

**Theorem 3.6.** Under the extra assumption (3.6), the envelope \( U \) has minimal singularities and \( U_t \) is moreover continuous in \( \Omega \), for each \( t \in (0,T) \).

**Proof.** We first show that for each \( t, U_t \) has minimal singularities. Observe that
\[
\eta_t := e^{-At}\varpi_t
\]
is decreasing in \( t \). By [BEGZ10, Theorem 6.1], for each \( t \in [0,T) \) there exists a unique \( \eta_t \)-psh function \( \phi_t \) with full Monge-Ampère mass such that
\[
(\eta_t + dd^c\phi_t)^n = e^{\varpi_t + c_1}fDV,
\]
for \( c_1 > 0 \) so that \( \sup_{X} \varphi_0 = 0 \). For \( 0 < s \leq t \) we have
\[
(\eta_s + dd^c\phi_s)^n \geq (\eta_t + dd^c\phi_t) = e^{\varpi_t + c_1}fDV.
\]
It follows that \( \phi_t \) is a subsolution to \( (\eta_s + dd^c\phi_s)^n = e^{\varpi_s + c_1}fDV, \) so a classical comparison principle (see e.g. [BEGZ10, Proposition 6.3]) ensures that \( \phi_t \leq \phi_s \) for \( s \leq t \). Therefore the function \( t \mapsto \phi_t(x) \) is decreasing for all \( x \in X \). Since \( t \mapsto \eta_t \) is decreasing in \( t \), we may assume that \( \eta_t \geq \theta' \) for some big \((1,1)\)-form \( \theta' \). As explained in Section 1.3, \( \phi_t = \partial_t\phi_t \) is well-defined almost everywhere and negative. Set for any \((t,x) \in X_T\),
\[
u(t,x) := e^{At}\phi_t(x) - C_2(t+1)
\]
where \( C_2 > 0 \) is a constant to be chosen later. Since \( \sup_{X} \varphi_0 = 0 \) and \( \phi_t \) is decreasing in \( t \), we have \( \phi_t \leq 0 \) for all \( t \in [0,T) \). We infer for almost every \( t \in [0,T) \),
\[
u(t,\cdot) = e^{At}\phi_t + Ae^{At}\phi_t - C_2 \leq \phi_t - C_2,
\]
hence
\[
(\omega_t + dd^c\nu_t)^n = e^{nAt + c_1 + \psi}fDV \geq e^{\psi(t) + F(t)}fDV,
\]
where for the last inequality we have chosen \( C_2 > 0 \) so big that \( C_2 > nAT + c_1 + M_F \). Hence \( u_t \) is a subsolution to (CMAF). Moreover, we can choose \( C_2 > 0 \) so big that \( u(0,\cdot) = \phi_0 - C_2 \leq \varphi_0 \) since \( \varphi_0 \) is a \( \eta_0 \)-psh function with minimal singularities. Therefore, \( u \) is a subsolution to the Cauchy problem, i.e. \( u \in \mathcal{S}_{\varphi_0,f,F}(X_T) \). By [BEGZ10, Theorem 6.1] we have \( \phi_t \geq V_{\eta_t} - C(t) \) for some time-dependent constant \( C(t) \). Since \( V_{\varpi_t} = e^{At}V_{\eta_t} \), we thus infer \( u_t \geq V_{\varpi_t} - C(t) \), hence \( U_t \geq V_{\varpi_t} - C(t) \) for all \( t \in [0,T) \).

It remains to show that for each \( t \in (0,T) \), \( U_t \) is continuous in \( \Omega \). We have that for each \( t > 0 \), \( \partial_t U + F(t,\cdot,U_t) \leq k - k(\rho + \chi) + M_F \) on \( X \), for some constant \( k > 0 \). The continuity of \( U_t \) in \( \Omega \) thus follows from [Dan21, Theorem 3.2]. \( \square \)

We now show that the solution constructed in Theorem 3.1 is unique:

**Theorem 3.7.** Let \( \Phi \) be a pluripotential solution to the Cauchy problem for (CMAF) with initial data \( \varphi_0 \). Assume that (3.6) holds and
• \( \Phi \) is locally uniformly semi-concave in \((0, T)\);
• for each \( t \), \( \Phi_t \) has minimal singularities;
• \( \Phi_t \) is continuous in \( \Omega \).

Then \( \Phi = U \).

**Proof.** Since \( \Phi \) is locally uniformly Lipschitz in \( t \) we infer that \( \Phi \) is continuous on \((0, T) \times \Omega \). We would like to apply Proposition 3.4 but \( U \) is not \( C^1 \) in \( t \). We are going to regularize it by taking convolution in \( t \) as in [GLZ20, Proposition 3.16]. Fix \( 0 < T' < T \), \( s > 0 \) near 1. Set, for any \((t, x) \in X_{T'}\),

\[
V^s(t, x) := \frac{\alpha_s}{s} U(st, x) + (1 - \alpha_s) \eta(t) \frac{\rho(x) + \chi(x)}{2} - C|s - 1|(t + 1),
\]

where \( \alpha_s, C \) are defined as in the proof of Theorem 2.7 so that \( V^s \in \mathcal{S}_{F^\eta,F^P}(X_{T'}) \).

Let \( \eta \) be a smooth function with compact support in \([-1, 1]\) such that \( \int \eta(t) dt = 1 \). Set, for \( \varepsilon > 0 \) small, \( \eta_\varepsilon(t) = \varepsilon^{-1} \eta(t/\varepsilon) \), and we define for any \((t, x) \in X_{T'}\),

\[
u^\varepsilon(t, x) := \int \Phi^s(t, x) \eta_\varepsilon(s - 1) ds - B\varepsilon(t + 1).
\]

Using the arguments as in [GLZ20, Proposition 3.16], we will show that \( \nu^\varepsilon \) is a pluripotential subsolution to \((\text{CMAF})\) which is \( C^1 \) in \( t \). Indeed, since \( V^s \) is a pluripotential subsolution to \((\text{CMAF})\), Lemma 1.16 yields, for any \( t \in (0, T) \),

\[
(\omega_t + dd^c \nu^\varepsilon(t, \cdot))^{1/n} \geq \exp(\partial_t V^s + F(t, \cdot, V^s(t, \cdot))) f^1/n.
\]

We know that the function \( A \mapsto (\det A)^{1/n} \) is concave on the convex cone of non negative hermitian matrices. It follows from Jensen’s inequality that, for any \( t \in (0, T) \),

\[
(\det(\omega_t + dd^c \nu^\varepsilon))^{1/n} = \left( \det \left( \int (\omega_t + dd^c V^s) \eta_\varepsilon(s - 1) ds \right) \right)^{1/n} \\
\geq \left( \int \left( \frac{1}{n} (\partial_t V^s(t, \cdot) + F(t, \cdot, V^s(t, \cdot))) \right) \eta_\varepsilon(s - 1) ds \right)^{1/n} \\
\geq \exp \left( \int \frac{1}{n} (\partial_t V^s + F(t, \cdot, V^s(t, \cdot))) \eta_\varepsilon(s - 1) ds \right)^{1/n} \\
\geq \exp \left( \frac{1}{n} \left( \partial_t \nu^\varepsilon(t, \cdot) + B \varepsilon + F(t, \cdot, \int \nu^\varepsilon(t, \cdot) \eta_\varepsilon(s - 1) ds) \right) \right)^{1/n} \\
\geq \exp \left( \frac{1}{n} \left( \partial_t \nu^\varepsilon(t, \cdot) + F(t, \cdot, \nu^\varepsilon) \right) \right)^{1/n}.
\]

The second line follows from the Main Theorem in [GLZ19], the third and fourth ones follow from the convexity of the exponential and \( F \), and the last one follows from the monotonicity of \( F \). Using Lemma 1.16 again, we infer that \( \nu^\varepsilon \) is a subsolution to \((\text{CMAF})\).

On the other hand, it follows from the proof of Theorem 2.7 that \( V^s(0, x) \leq \varphi_0 \) on \( X \). Hence \( \nu^\varepsilon(0, \cdot) \leq \varphi_0 \) on \( X \) by taking \( B > 0 \) large enough. We can thus apply Proposition 3.4 to obtain \( \nu^\varepsilon \leq \Phi \) on \([0, T'] \times \Omega \), hence on \([0, T'] \times X \). Letting \( \varepsilon \to 0 \) and \( T' \to T \) we get \( U \leq \Phi \) on \([0, T) \times X \). Hence the equality holds. \( \square \)

4. Applications

We apply the tools we have developed in related geometrical settings. We first define and study the longtime behavior of the normalized Kähler-Ricci flow on manifolds of general type. We next prove the existence of a longtime solution of the on manifolds with nonnegative Kodaira dimension. We then analyze the
4.1. **Manifolds of general type.** We study in this section the (normalized) Kähler-Ricci flow on a manifold of general type. We try to run such flow in a weak sense beyond the maximal existence time by using the results obtained in the previous section. We then study the long-time behavior of this flow.

Let \((X, \omega_0)\) be a compact Kähler manifold of general type i.e. the canonical divisor \(K_X\) is big, and \(\omega_0\) is a Kähler form. The normalized Kähler-Ricci flow is the evolution equation:

\[
\frac{\partial \vartheta_t}{\partial t} = -\text{Ric}(\vartheta_t) - \vartheta_t, \quad \vartheta|_{t=0} = \omega_0.
\]

Let \(T\) be the maximal existence time of smooth flows which is defined by

\[
T := \sup \{ t > 0 : e^{-t}\{\omega_0\} + (1 - e^{-t})c_1(K_X) \text{ is Kähler} \}.
\]

Note that \(T = +\infty\) if and only if the canonical divisor \(K_X\) is nef. In this case, the normalized Kähler-Ricci flow exists in the classical (smooth) sense and converges to a singular Kähler-Einstein metric (cf. [Tsu88, TZ06]).

When \(K_X\) is not nef, the flow has a finite time singularity at \(T < +\infty\). The limiting class at \(T\) of the flow \(\alpha_T := \lim_{t \to T} \{\vartheta_t\} = e^{-T}\{\omega_0\} + (1 - e^{-T})c_1(K_X)\) is big and nef. In [CT15, Theorem 1.5], Collins and Tosatti showed that the flow \(\vartheta_t\) exists on the maximal time interval \([0, T)\) and develops singularities precisely on the Zariski closed set \(X \setminus \text{Amp}(\alpha_T)\) as \(t \to T^-\). For \(t > T\), the cohomology class \(\{\vartheta_t\}\) is still big, but no longer nef, we can not continue the flow in the classical sense.

In [FIK03, Section 10], Feldman, Ilmanen and Knopf have asked the question: can one define and construct weak solutions of the Kähler-Ricci flow beyond the singular time? In [BT12, Theorem 4], Boucksom and Tsuji have constructed the normalized Kähler-Ricci flow on smooth projective varieties with pseudoeffective canonical class for all times. They used the discretization of the Kähler-Ricci flow and some algebro-geometric tools. In the end, they have conjectured the same result for the case of general Kähler manifolds (see [BT12, Conjecture 1]). Tô [Tô21] used the viscosity theory to show that the weak Kähler-Ricci flow exists for all time in the viscosity sense and converges to the unique singular Kähler-Einstein metric in the class \(c_1(K_X)\) constructed in [EGZ09, BEGZ10].

In this section we show that the normalized Kähler-Ricci flow can be extended through a finite time singularity and understood in the weak sense of parabolic pluripotential theory developed in Section 3. A compact Kähler manifold of general type turns out to be projective by a classical result of Moishezon. Our result thus gives an alternative approach to the existence of weak Kähler-Ricci flows previously obtained by Boucksom-Tsuji and Tô. The main point in our proof is that the flow survives through a finite time singularity provided that the limiting class is big. The argument is thus working also on a general Kähler manifold with pseudoeffective canonical class as will be shown later in Section 4.2.

Now let \(\theta\) be a smooth closed \((1, 1)\)-form representing \(c_1(K_X)\). Set \(\omega_t := e^{-t}\omega_0 + (1 - e^{-t})\theta\). Since \(dV_X\) is a smooth volume form on \(X\), \(-\text{Ric}(dV) \in -c_1(K_X)\), and so there is a smooth function \(f\) such that \(\theta = -\text{Ric}(dV_X) + dd^c f\). We then define

\[
\mu = e^f dV
\]
which is a smooth positive volume with \( \theta = \text{Ric}(\mu) \). Thus the normalized Kähler-Ricci flow (4.1) can be written as the complex Monge-Ampère flow

\[
\left( \omega_t + dd^c \varphi_t \right)^n = e^{\varphi_t + \varphi_t} d\mu.
\]

It has been shown in [BEGZ10, Theorem 6.1] that there exists a unique \( \theta \)-psh function \( \varphi_{KE} \) with minimal singularities such that

\[
(\theta + dd^c \varphi_{KE})^n = e^{\varphi_{KE}} d\mu.
\]

The current \( \omega_{KE} := \theta + dd^c \varphi_{KE} \) is called the singular Kähler-Einstein metric. It has bounded potentials and is smooth in \( \Omega := \text{Amp}(K_X) \), where it satisfies

\[
\text{Ric}(\omega_{KE}) = -\omega_{KE}.
\]

We are going to show that the (normalized) pluripotential Kähler-Ricci flow (4.2) exists for all time and continuously deforms any initial Kähler form \( \omega_0 \) towards \( \omega_{KE} \) on \( \text{Amp}(K_X) \), as \( t \to +\infty \). The result even holds for an initial datum \( S_0 \) which is a positive current with bounded potentials:

**Theorem 4.1.** Let \( \varphi_0 \) be a bounded \( \omega_0 \)-psh function. Then there exists a unique pluripotential solution \( \varphi \) to (4.2) with initial data \( \varphi_0 \) for \( t > 0 \). Furthermore, \( \varphi_t \) converges exponentially fast towards \( \varphi_{KE} \) on \( \text{Amp}(K_X) \), as \( t \to +\infty \).

**Proof.** Recall that \( \theta \) is a smooth representative of \( c_1(K_X) \). Since \( \omega_0 \) is a Kähler form, there exists a small constant \( c > 0 \) such that \( \omega_0 \geq c \theta \). Hence \( \omega_t = e^{-t} \omega_0 + (1 - e^{-t}) \theta \geq g(t) \theta \), where \( g(t) = e^{-t} + 1 - e^{-t} \) is a smooth (strictly) positive function with \( g'(t) = e^{-t}(1 - c) > 0 \) for \( c > 0 \) small enough. The existence of the unique pluripotential solution follows from the results in Section 3.

It thus remains to study its long-term behavior. We first establish a lower bound for the solution \( \varphi \) by constructing a subsolution to the Cauchy Problem. Set, for any \( (t, x) \in (0, T) \times X \),

\[
u(t, x) := e^{-t} \varphi_0(x) + (1 - e^{-t}) \varphi_{KE}(x) + h(t),
\]

for a \( \mathcal{C}^1 \) function \( h \) to be chosen later so that \( \nu \) is a subsolution to the Cauchy Problem. We observe that for all \( t > 0 \),

\[
\omega_t + dd^c \nu_t = e^{-t}(\omega_0 + dd^c \varphi_0) + (1 - e^{-t})(\theta + dd^c \varphi_{KE}) \geq 0
\]

in the weak sense of currents, so \( \nu_t \) is \( \omega_0 \)-psh and

\[
(\omega_t + dd^c \nu_t)^n \geq (1 - e^{-t})^n (\theta + dd^c \varphi_{KE})^n = e^{n \log(1 - e^{-t})} e^{\varphi_{KE}} dV.
\]

On the other hand \( \partial_t \nu_t + \nu_t = \varphi_{KE} + h'(t) + h(t) \) hence \( \nu \) is a subsolution if

\[
n \log(1 - e^{-t}) \geq h(t) + h'(t).
\]

We thus choose \( h \) to be the unique solution of the ODE:

\[
h(t) + h'(t) = n \log(1 - e^{-t}), h(0) = 0.
\]

We compute \( (e^t h(t))' = e^t (h(t) + h'(t)) = ne^t \log(1 - e^{-t}) \), hence

\[
h(t) = ne^{-t} \left[ \int e^t \log(1 - e^{-t}) dt \right] = ne^{-t} \left[ (e^t - 1) \log(e^t - 1) - te^t + C \right],
\]

for some constant \( C \). Since \( h(0) = 0 \) the constant \( C \) must be zero, hence

\[
h(t) = ne^{-t} \left[ (e^t - 1) \log(e^t - 1) - te^t \right] = O(te^{-t}) \quad \text{as} \quad t \to \infty.
\]

It follows from the comparison principle that \( \nu \leq \varphi \) hence

\[
\varphi_{KE}(x) + e^{-t}(\varphi_0(x) - \varphi_{KE}(x)) + h(t) \leq \varphi(t, x).
\]
For the upper bound, we argue as in the proof of [T621, Theorem 4.4]. Since the cohomology class of $\theta$ is big, we can find a $\theta$-psh function $\chi_0$ with analytic singularities such that 
\[ \theta + dd^c \chi_0 \geq \varepsilon \omega_0 \]
for some small constant $\varepsilon > 0$. We can assume that $\varepsilon \leq 1$. Replacing $\chi_0$ by $\chi_0 - \sup_X \chi_0$, we can always assume that $\chi_0 \leq 0$ hence $\chi_0 \leq \omega_0$. We then have
\[
\omega_t + dd^c \varphi_t = e^{-t}(\omega_0 - \varepsilon^{-1} dd^c \chi_0) + (1 - e^{-t})\theta + dd^c(\varphi_t + \varepsilon^{-1} e^{-t} \chi_0)
\]
\begin{equation}
(4.5)
\end{equation}
Set $g(t) = 1 + (\varepsilon^{-1} - 1)e^{-t}$ and $u(t, x) = \varphi_t(x) + e^{-t}(\varepsilon^{-1} \chi_0(x) - C)$ for a constant $C > 0$ to be chosen later. It follows from (4.5) that 
\[ (g(t)\vartheta + dd^c u_t)^n \geq (\omega_t + dd^c \varphi_t)^n \geq e^{2\varepsilon t + \varphi_t + \varepsilon^{-1}\mu} = e^{u_t + u_0} \mu, \]
Let $\phi_0$ be a $\varepsilon$-psh function with minimal singularities. We can find a constant $C > 0$ such that $\phi_0 - \varepsilon^{-1} \chi_0 \geq \varphi_0 - C$. Therefore $u$ is a subsolution of the following Cauchy problem
\begin{equation}
(4.6)
\end{equation}
where $\phi$ denotes the pluripotential solution to (4.6). Thus the comparison principle (Proposition 3.4) yields $u \leq \phi$ on $[0, \infty) \times \Amp(K_X)$, i.e.
\[ \varphi(t, x) + e^{-t}(\varepsilon^{-1} \chi_0(x) - C) \leq \phi(t, x). \]
On the other hand, the function $\phi_t$ converges to $\varphi_{KE}$ on $\Amp(K_X)$ as $t \to +\infty$ by Lemma 4.2 below. Combining this with (4.4), we infer that $\varphi_t$ converges to $\varphi_{KE}$ on $\Amp(K_X)$ as $t \to +\infty$.

\begin{lemma}
The solution $\phi_t$ of (4.6) converges locally exponentially fast towards $\varphi_{KE}$ on $\Amp(K_X)$ as $t \to +\infty$.
\end{lemma}

\begin{proof}
Set 
\[ \tilde{\phi}_t := g(t)^{-1}(\phi_t - a(t)), \]
where $g(t) = 1 + (\varepsilon^{-1} - 1)e^{-t}$, and $a$ is the unique solution of the ODE: $a(t) + a'(t) = n \log g(t)$, with $a(0) = 0$. An easy computation shows that $a(t) = O(t e^{-t})$. Now the flow (4.6) becomes
\begin{equation}
(4.7)
\end{equation}
We now normalize in time $\psi(t, \cdot) = \tilde{\phi}(s(t), \cdot)$, where $s(t)$ is the unique solution of the ODE $s'(t) = g(s(t))$ with $s(0) = 0$. Then the flow (4.6) can be written as
\begin{equation}
(4.8)
\end{equation}
We set for any $(t, x) \in (0, +\infty) \times X$, $v(t, x) := e^{-t} \psi_0(x) + (1 - e^{-t}) \varphi_{KE}(x) + h(t)$, where $h$ is the unique solution to the ODE $h'(t) + h(t) = n \log(1 - e^{-t})$, with $h(0) = 0$. As in the proof of Theorem 4.1 we can check that $u$ is a subsolution to the Cauchy problem for the flow (4.8).

On the other hand, since $\varphi_{KE}$ is a $\theta$-psh function with minimal singularities we can choose a constant $C > 0$ such that $\varphi_{KE} + C \geq \psi_0$. Set for $(t, x) \in (0, +\infty) \times X$,
\[ v(t, x) := \varphi_{KE}(x) + Ce^{-t}. \]
One can check that $v$ is supersolution to (4.8). Therefore, the comparison principle yields
\[ e^{-t}v_0(x) + (1 - e^{-t})v_{KE}(x) + O(te^{-t}) \leq v(t, x) \leq v_{KE}(x) + C e^{-t}, \]
which implies $\psi_t \to v_{KE}$ on $\text{Amp}(K_X)$ as $t \to +\infty$. So does the flow $\phi_t$ since $s(t) \to +\infty$, $g(t) \to 1$ and $a(t) \to 0$ as $t \to +\infty$. $\square$

**Remark 4.3.** The uniqueness of the flow (4.2) follows directly from Theorem 3.7.

### 4.2. Extending the Kähler-Ricci flow through finite time singularities.

In this subsection, we apply our results to prove the existence of the (pluripotential) Kähler-Ricci flow on manifolds through a finite time singularities. In particular, the answer of Feldman-Ilmanen-Knopf’s question is affirmative also in this case.

Let $(X, \omega_0)$ be a compact Kähler manifold of dimension $n$ and consider the Kähler-Ricci flow with initial data $\omega_0$.

\begin{equation}
\frac{\partial \theta}{\partial t} = -\text{Ric}(\theta), \quad \theta|_{t=0} = \omega_0.
\end{equation}

The maximal existence time $T$ of the flow is defined by

\[ T := \sup\{t > 0 : \{\omega_0\} + tc_1(K_X) \text{ is Kähler}\}. \]

Suppose that $T < \infty$ ($K_X$ is not nef). Then the limiting class $\{\theta_T\} := \{\omega_0\} + tc_1(K_X)$ is nef, but not Kähler. If we assume moreover that $\int_X \theta_T^2 > 0$, then the class $\{\theta_T\}$ is big by a fundamental theorem of Demailly and Paun [DP04, Theorem 2.12]. Since the set of big cohomology classes is open, there is a constant $c > 0$ so small that the class $\{\theta_T\}$ is big for $t \in [0, T + c]$. We can prove the existence of a pluripotential solution of the flow on $[0, T + c]$.

**Theorem 4.4.** Let $(X, \omega_0)$ be a compact Kähler manifold. Assume that the solution $\theta(t)$ of the Kähler-Ricci flow (4.9) starting at $\omega_0$ exists on the maximal time interval $[0, T)$ with $T < \infty$, and that the limiting class $\{\omega_0\} + tc_1(K_X)$ is big. Then the pluripotential Kähler-Ricci flow starting with $\omega_0$ exists for $t \in [0, T + c)$ for some small $c > 0$.

**Proof.** Let $\eta$ be a smooth representative of the class $\{\theta_{T+c}\}$, and set

\[ \chi = \frac{1}{T+c}(\eta - \omega_0) \in c_1(K_X); \]

\[ \omega_t = \omega_0 + t\chi = \frac{1}{T+c}((T + c - t)\omega_0 + t\eta) \in \{\omega_0\} + tc_1(K_X). \]

Fix a volume form $dV$ on $X$ with $dd^c \log V = \chi$. Then the Kähler-Ricci flow can be written as the complex Monge-Ampère flow

\[ (\omega_t + dd^c \varphi_t)^n = e^{\varphi_t}dV, \quad \varphi(0) = 0. \]

Since $\omega_0$ is a Kähler form, there exists a small constant $c \in (0, 1)$ such that $\omega_0 \geq c\eta$.

Hence $\omega_t \geq g(t)\eta$ for $t \in [0, T']$, where $g(t) = (T+c)^{-1}(c(T+c)+t(1-c))$ is a positive increasing function. Theorem 3.1 can be applied (with $F(t, x, r) \equiv 0$, $f = 1$) and guarantees the existence of a pluripotential solution to the Monge-Ampère flow on $X_{T+c}$. $\square$

Using the same argument as above the pluripotential Kähler-Ricci flow can be continued as long as the class $\{\omega_0\} + tc_1(K_X)$ is big. If $X$ has nonnegative Kodaira dimension, then $c_1(K_X)$ is pseudoeffective, and hence the class $\{\omega_0\} + tc_1(K_X)$ is big for any $t > 0$. In particular, the flow is volume non-collapsing at a finite time singularity, as emphasized by Collins-Tosatti (see [CT15, Proposition 4.2]). We thus obtain a longtime pluripotential solution:
Theorem 4.5. Let \((X, \omega_0)\) be a compact Kähler manifold with nonnegative Kodaira dimension. Then the pluripotential Kähler-Ricci flow starting with \(\omega_0\) exists for \(t \in [0, \infty)\).

Proof. Fix \(T < \infty\). Let \(\eta\) be a smooth representative of the class \(\{\theta_T\}\), and set

\[
\chi = \frac{1}{T}(\eta - \omega_0) \in \mathcal{C}_1(K_X);
\]

\[\omega_t = \omega_0 + t\chi = \frac{1}{T}((T-t)\omega_0 + t\eta) \in \{\omega_0\} + t\mathcal{C}_1(K_X).\]

Fix a volume form \(dV\) on \(X\) with \(dd^c \log V = \chi\). Then the Kähler-Ricci flow can be written as the complex Monge-Ampère flow

\[
(\omega_t + dd^c \phi_t)^n = e^{\phi_t} dV, \quad \phi(0) = 0.
\]

Since \(\omega_0\) is a Kähler form, there exists a small constant \(c \in (0, 1)\) such that \(\omega_0 \geq c\eta\). Hence \(\omega_t \geq g(t)\eta\) for \(t \in [0, T]\), where \(g(t) = T^{-1}(cT + t(1 - c))\) is a positive increasing function. Again, by Theorem 3.1 there exists a pluripotential solution \(U = U_{\varphi_0,f,F,X_T}\) with \(\varphi_0 = 0, f = 1, F = 0\).

We next claim that \(U_t\) has minimal singularities for each \(t \in (0, T)\). The proof of the claim is very similar to that of Theorem 3.6, but for completeness we provide the details below. We first observe that \(\tilde{\omega}_t \leq \omega_t\) for all \(t > 0\), yielding that

\[\eta_t := t^{-1}\omega_t\]

is decreasing in \(t\). By [BEGZ10, Theorem 6.1], for each \(t \in (0, T)\) there exists a unique \(\eta_t\)-psh function \(\phi_t\) with full Monge-Ampère mass such that

\[(\eta_t + dd^c \phi_t)^n = e^{\phi_t} dV.\]

For \(0 < s \leq t\) we have

\[(\eta_t + dd^c \phi_t)^n \geq (\eta_s + dd^c \phi_s)^n = e^{\phi_s} dV.\]

It follows that \(\phi_t\) is a subsolution to \((\eta_t + dd^c \phi_t)^n = e^{\phi_t} f dV\), so the comparison principle (see e.g. [BEGZ10, Proposition 6.3]) ensures that \(\phi_s \leq \phi_t\) for \(s \leq t\). Therefore the function \(t \mapsto \phi_t(x)\) is decreasing for all \(x \in X\). As explained in Section 1.3, \(\phi_t = \partial_t \phi_0\) is well-defined almost everywhere on \(X_T\). Set \(u(t,x) := t\phi_t(x) + n(t \log t - t)\). We infer, for almost every \((t, x) \in (0, T) \times X\),

\[u(t,x) = t\phi_t(x) + \phi_t(x) + n(t \log t - t),\]

hence

\[(\omega_t + dd^c u_t)^n = e^{n \log t + \phi_t} dV \geq e^{\phi_t} dV.\]

Moreover, since \(u(0, \cdot) = 0 = \varphi_0\), we have that \(u\) is a subsolution to the Cauchy problem, i.e. \(u \in \mathcal{S}_{\varphi_0,f,F}(X_T)\) with \(\varphi_0 = 0, F \equiv 0\). By [BEGZ10, Theorem 6.1] we have \(\phi_t \geq V_{\eta_t} - C(t)\) for some time-dependent constant \(C(t)\). Since \(V_{\eta_t} = tV_{\eta_0}\), we thus infer \(u_t \geq V_{\eta_0} - C'(t)\), hence \(U_t \geq V_{\eta_0} - C'(t)\) for all \(t \in (0, T)\). This completes the proof of the claim.

If \(T' > T\), then by the above arguments there exist a pluripotential solution \(U' = U_{\varphi_0,f,F,X_{T'}}\) of the flow. Both \(U\) and \(U'\) satisfy the assumptions in Theorem 3.7, hence \(U = U'\) on \(X_{T'}\). We can thus glue all these solutions to get a longtime solution of the flow, finishing the proof.

\[\square\]

4.3. Stable varieties.
Log canonical pairs. A pair \((X, D)\) is by definition a complex normal compact projective variety carrying a Weil \(\mathbb{Q}\)-Cartier \(D\) (not necessarily effective). We will say that the pair \((X, D)\) is a log canonical (lc) pair if \(K_X + D\) is \(\mathbb{Q}\)-Cartier, and if for some (or equivalently any) log resolution \(\pi : X' \to X\), we have

\[
K_{X'} = \pi^*(K_X + D) + \sum a_iE_i
\]

where \(E_i\) are either exceptional divisors or components of the strict transform of \(D\), and the coefficients \(a_i\) satisfy the inequality \(a_i \geq -1\).

When \(D \equiv 0\), we say that \(X\) has log canonical singularities.

Semi-log canonical singularities. We give here a short overview of the notion of semi-log canonical singularities and stable varieties. We refer to the survey [Kov13, §5, 6] and the references therein for more details.

In the sequel, \(X\) will be a reduced and equidimensional scheme of finite type over \(\mathbb{C}\) unless stated otherwise, and we set \(n := \dim_{\mathbb{C}} X\). In order to study the normalized Kähler-Ricci flow, one needs a canonical sheaf (or a canonical divisor). Let us stress that the dualizing sheaf, even if it exists, is not necessarily a line bundle. A scheme (variety) \(X\) is Cohen-Macaulay if for every \(x \in X\) the depth of \(\mathcal{O}_{X,x}\), denoted by \(\text{depth}(\mathcal{O}_{X,x})\), is equal to its Krull dimension. If \(X\) is Cohen-Macaulay, then \(X\) admits a dualizing sheaf \(\omega_X\).

We say that \(X\) is Gorenstein if \(X\) is Cohen-Macaulay (\(X\) admits a dualizing sheaf \(\omega_X\)) and \(\omega_X\) is a line bundle. A scheme (variety) \(X\) is called \(G_1\) if it is Gorenstein in codimension 1, which means that there exists an open subset \(U \subset X\) such that \(\text{codim}_X (X \setminus U) \geq 2\) and \(U\) is Gorenstein.

We say that \(X\) satisfies the \(S_2\) condition of Serre if for all \(x \in X\), we have \(\text{depth}(\mathcal{O}_{X,x}) \geq \min\{\dim \mathcal{O}_{X,x}, 2\}\). This condition is equivalent to saying that for each closed subset \(i : Z \hookrightarrow X\) of codimension at least two, the natural map \(\mathcal{O}_X \to i_* \mathcal{O}_X\vert_Z\) is an isomorphism.

We now want to have an interpretation of \(\omega_X\) in terms of Weil divisor. If \(X\) satisfies the conditions \(G_1\) and \(S_2\), and \(U\) is a Gorenstein open subset whose complement has codimension at least 2, we may define the “canonical+dualizing” sheaf \(\omega_U\) as the determinant of the cotangent bundle, i.e., the sheaf of top differential forms, \(\omega_U = \det \Omega_U\). One can then define the canonical sheaf \(\omega_X\) by \(\omega_X = j_* \omega_U\) where \(j : U \hookrightarrow X\) is the open embedding.

As \(U\) is non-singular, \(\omega_U\) is a line bundle, hence corresponds to a Cartier divisor. Let \(K_U := \sum a_iK_i\) be a Weil divisor associated to this Cartier divisor such that for all \(i\), \(K_i\) does not contain any component of \(X_{\text{sing}}\) of codimension 1. Let \(K_i\) denote the closure of \(K_i\) and

\[
K_X := \sum a_iK_i.
\]

Since \(\text{codim}_X(U) \geq 2\), this is the unique Weil divisor for which \(K_X \vert_U = K_U\). We see that the divisorial sheaf

\[
\mathcal{O}_X(K_X) := \{ f \in K(X) : K_X + \text{div}(f) \geq 0 \}
\]

is reflexive, and coincides with \(\omega_U = \omega_X \vert_U\), hence the \(S_2\) condition implies that

\[
\omega_X \simeq \mathcal{O}_X(K_X).
\]

Remark 4.6. The condition \(G_1\) guarantees the existence of the canonical sheaf \(\omega_X\), and the condition \(S_2\) ensures its uniqueness. When \(X\) is projective, we know that it admits a dualizing sheaf, as it is reflexive, it coincides with \(\omega_X\) by the \(S_2\) condition.
We let $\omega_X^{[m]}$ denote the $m$-th reflexive power of the canonical sheaf $\omega_X$ (defined by $\omega_X^{[m]} := (\omega_X^{\otimes m})^{**}$). The same arguments above yield $\omega_X^{[m]} \simeq O_X(mK_X)$. Thus the Weil divisor $K_X$ is $\mathbb{Q}$-Cartier if and only if $\omega_X^{[m]}$ is a line bundle for some $m > 0$. From now on we work with the canonical divisor $K_X$ instead of its associated canonical sheaf $\omega_X$.

We say that a closed point $x \in X$ is double crossing if it is locally analytically isomorphic to the singularity

$$\{0 \in (z_0z_1 = 0) \subset \mathbb{C}^{n+1}\}.$$  

A scheme $X$ is called demi-normal if it satisfies the $S_2$ condition and has only double crossing singularities in codimension 1. We now give the definition of semi-log canonical models:

**Definition 4.7.** We say that $X$ has semi-log canonical (slc) singularities if $K_X$ is $\mathbb{Q}$-Cartier and there exist two Zariski open sets $U, V$ such that

- $X = U \cup V$,
- $U$ is a normal variety with log canonical singularities,
- $V$ has only double crossing points.

We mention that semi-log canonical models may not be normal varieties. Let $\mu : X^n \to X$ be a normalization of $X$. We emphasize again that $X$ is not irreducible in general, so its normalization is defined to be the disjoint union of the normalization of its irreducible components. The conductor ideal sheaf

$$\mathcal{I}_C := \text{Ann}_{\mathcal{O}_X}(\mu_* \mathcal{O}_{X^n}/\mathcal{O}_X)$$

is defined to be the largest ideal sheaf on $X$ that is also an ideal sheaf on $X^n$. If we consider the affine case where $A^n$ is the integral closure of some integral ring $A$, then one can see that the annihilator $\text{Ann}_A(A^n/A) := \{a \in A : aA^n \subset A\}$ is the largest ideal in $A$ that is also an ideal in $B$.

For the case of schemes (varieties), we let $\mathcal{I}_{C_{X^n}}$ denote the corresponding conductor ideal sheaf on $X^n$, and we define the conductor subscheme as $C_X := \text{Spec}_X(\mathcal{O}_X/\mathcal{I}_C)$ on $X$ and $C_{X^n} := \text{Spec}_{X^n}(\mathcal{O}_{X^n}/\mathcal{I}_{C_{X^n}})$ on $X^n$. If $X$ is seminormal (i.e. every finite morphism $X' \to X$, with $X'$ is reduced, that is a bijection on points is an isomorphism) and $S_2$, then one can show that these subschemes have pure codimension 1 hence they define Weil divisors which are moreover reduced (cf. [KSS10, 4.5]).

If $X$ is semi-normal and $K_X$ is $\mathbb{Q}$-Cartier, then we have the following relation

$$\mu^*K_X = K_{X^n} + C_{X^n}.$$  

Under the previous seminormality and $S_2$ assumptions, the $G_1$ condition is equivalent to the demi-normality. In other words, we may alternatively define slc models as follows:

**Definition/Proposition 4.8.** A scheme $X$ has semi-log canonical singularities if and only if

- $X$ is $G_1$ and $S_2$,
- $K_X$ is $\mathbb{Q}$-Cartier (of index $m$),
- The pair $(X^n, C_{X^n})$ is log-canonical.

Note that there are many schemes satisfying the $S_2$ condition and the seminormality but not demi-normality. For instance, a reduced scheme consisting of the three axes in $\mathbb{A}^3$ does not have double crossings in codimension 1, but is both $S_2$ and seminormal.

We can finally give the definition of stable variety:
Definition 4.9. A projective variety $X$ is called stable if

- $X$ has semi-log canonical singularities,
- $K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

From Definition 4.7, we can see that $X$ is a stable variety if $K_X$ is ample and $X = U \cup V$, where $U, V$ are Zariski open sets, $U$ is a normal variety with log canonical singularities, and $V$ has only double crossing singularities.

4.4. Convergence of NKRF on stable varieties. Let $X$ be a complex projective variety with semi-log canonical singularities such that $K_X$ is ample (stable variety). We now consider the normalized Kähler-Ricci flow starting at any Kähler metric $\omega_0$ on $X$, this is the evolution following equation:

$$
\frac{\partial \theta_t}{\partial t} = -\text{Ric}(\theta_t) - \theta_t, \quad \theta|_{t=0} = \omega_0.
$$

(4.11)

After passing to a suitable resolution of singularities, we may as well assume that $X$ is smooth if we study the setting of log pairs $(X, D)$, where $D = \sum_{i=1}^N a_i D_i$ is the $\mathbb{Q}$-divisor on $X$ with simple normal crossing (snc), where the role of the canonical line bundle is played by the log canonical line bundle $K_X + D$ (which occurs as the pull-back to the resolution of the original canonical line bundle). In this setting the original variety has semi-log canonical singularities precisely when the log pair $(X, D)$ is log canonical (lc) in the usual sense of Minimal Model Program (MMP), i.e. the coefficients of $D$ are at most equal to one (but negative coefficients allowed). Let us mention that even if the original canonical line bundle is ample, the corresponding log canonical line bundle is merely semi-ample and big on the resolution, since it is trivial along the exceptional divisors of the corresponding resolution. The initial data $\omega_0$ may now assume to be a smooth semi-positive $(1,1)$ form with big cohomology class.

Let $X$ be a compact Kähler manifold and $(X, D)$ be a log canonical pair such that $K_X + D$ is semi-ample and big (i.e., $(K_X + D)^n > 0$). We fix $\theta$ a smooth representative of the class $c_1(K_X + D)$. It has been shown in [BG14, Theorem C] that there exists a unique closed positive current $\omega_{KE} = \theta + dd^c \psi_{KE}$ in $c_1(K_X + D)$ which is smooth on a Zariski open set $U$ of $X$ and satisfies

$$
\text{Ric}(\omega_{KE}) = -\omega_{KE} + [D]
$$

in the sense of currents on $X$. The current $\omega_{KE}$ is called the singular Kähler-Einstein metric.

Our aim is to prove the existence and the convergence of the pluripotential solutions of the normalized Kähler-Ricci flow like the previous one, this is the content of the following theorem:

Theorem 4.10. Let $S_0$ be a positive closed current with bounded potentials. Then the normalized Kähler-Ricci flow starting with $S_0$ admits a unique pluripotential solution defined on $[0, +\infty) \times X$. Furthermore, the pluripotential normalized Kähler-Ricci flow converges towards $\omega_{KE}$ on $\text{Amp}(K_X + D)$, as $t \to +\infty$.

Observe that Theorem 4.10 implies Theorem D in the introduction: indeed, if $Y$ is a projective variety with semi-log canonical singularities such that $K_Y$ is ample (stable variety) and $\pi : (X, D) \to Y$ is a log resolution of the normalization (endowed with its conductor), then the exceptional locus of $\pi$ is contained in the complement of the ample locus of $K_X + D$.

Remark 4.11. A similar result has been obtained in [CGLS19, Theorem 1.3] with a very different approach. These authors generalize the a priori estimates of Song-Tian [ST17] to the case of $\mathbb{Q}$-factorial projective varieties with log canonical singularities. They also show that if $X$ is stable, then the normalized Kähler-Ricci
flow (4.11) has a unique maximal weak solution on \([0, +\infty)\) which is smooth in \((0, +\infty) \times X_{\text{reg}}\) and converges to the singular Kähler-Einstein metric \(\omega_{KE}\) both in the sense of currents and in the \(C^\infty(X_{\text{reg}})\)-topology as \(t\) tends to infinity.

Our approach allows one to treat more general equations, avoiding any projectivity assumption on the variety nor any integrality on the initial cohomology class, and applies to big classes for which no smooth deformation is available.

**Proof of Theorem 4.10.** By definition, \(D = \sum_{i=1}^N a_i D_i\) is a simple normal crossings \(\mathbb{R}\)-divisor with \(a_i \in (-\infty, 1]\) and defining section \(s_i\). The normalized Kähler-Ricci flow (4.11) can be written as the following complex Monge-Ampère flow

\[
(\omega_t + dd^c \varphi_t)^n = e^{\varphi_t + \varphi_0} d\mu_t,
\]

where \(\omega_t := e^{-t} \omega_0 + (1 - e^{-t}) \theta\), and \(d\mu\) is a measure on \(X\) which is of the form

\[
d\mu = \frac{dV_X}{\prod_{i=1}^N |s_i|^{2a_i}} = fdV_X
\]

where \(s_i\) are non-zero sections of \(\mathcal{O}_X(D_i)\), \(\cdot, | \cdot |\) are smooth hermitian metrics on \(\mathcal{O}_X(D_i)\), and \(dV_X\) is a smooth volume form on \(X\). We let \(D_\text{K} := \cup_{a_i=1} D_k\) denote the "non-klt" locus.

**Step 1: constructing a subsolution.** We let \(\Omega\) denote the ample locus of the class \(\{\theta\}\). Since the latter is big, there exists a \(\theta\)-psh function \(\chi_0\) such that

\[
\tilde{\theta} := \theta + dd^c \chi_0 \geq \omega_{\text{K X}} \text{ on } \Omega \text{ for some } \delta \geq 0 \text{ and } \chi_0 \to -\infty \text{ near } \partial \Omega.
\]

Up to multiplying by a positive constant we can assume that \(|s_i|^2 \leq 1/e\) so that \(-\log(|s_i|^2) \geq 1\) out of \(D_i\). Note also that \(dd^c(-\log(|s_i|^2))\) extends as a smooth real \((1,1)\)-form on \(X\) whose cohomology class is \(2\pi c_1(D)\). We compute

\[
-dd^c(\log(\lambda - \log(|s_i|^2))) = -\frac{dd^c(\lambda - \log(|s_i|^2))}{\lambda - \log(|s_i|^2)} + \frac{ds_i \wedge d^c s_i}{|s_i|^2(\lambda - \log(|s_i|^2))^2}.
\]

The second term is a semipositive \((1,1)\)-form. Since \(-\log(|s_i|^2)\) goes to \(\infty\) near \(D_j\), we infer that \(\tilde{\theta} - dd^c(\log(\lambda - \log |s_i|^2))\) is positive on \(\Omega \setminus D_j\) when \(\lambda\) is big enough. Replacing \(\tilde{\theta}\) by \(\frac{1}{\delta}\tilde{\theta}\) and increasing \(\lambda\) if necessary one has \(\frac{1}{\delta}\tilde{\theta} - dd^c(\log(\lambda - \log |s_i|^2)) > 0\), hence \(\sum \frac{1}{\delta}\tilde{\theta} - dd^c(\log(\lambda - |s_i|^2))\) defines a Kähler form on \(\Omega \setminus D\). Then for suitable positive constants \(\lambda, A\) the function

\[
v := -2 \sum_{i=1}^N \log(\lambda - \log |s_i|^2) + \chi_0 - A
\]

is a subsolution of the complex Monge-Ampère equation:

\[
(\theta + dd^c \psi_{KE})^n = e^{\psi_{KE}} dV_X \prod_{i=1}^n |s_i|^{2a_i}.
\]

By the arguments above, we get the lower bound (see also [BG14, 5.5.2]):

\[
\psi_{KE} \geq \chi_0 - \sum_{a_i=1} \log(\lambda - \log |s_k|^2) - A
\]

for some uniform constant \(A > 0\). Here the hermitian metrics \(|\cdot|\) are chosen conveniently.

Since \(\tilde{\theta}\) is semi-positive we see that for all \(t\), \(\omega_t \geq c\tilde{\theta}\) for some \(c > 0\) small enough. For simplicity, we may assume that \(c = 1\). We can check that

\[
u(t, x) := \psi_{KE}(x) - C_0 e^{-t}
\]

is a subsolution to (4.12), for a constant \(C_0 > 0\) so large that \(u(0, \cdot) \leq \varphi_0\).
Step 2: the approximating flows. We now establish the existence of the flow \((4.12)\) by an approximation argument using ideas from [DL15, Theorem 4.5]. Fix \(T < +\infty\). The difficulty is that the density \(f = \Pi_1|s_i|^{-2\alpha_i}\) is not in \(L^p\), \(p > 1\), (not even in \(L^1\)) since some of the coefficients \(a_j\) might be equal to 1. For each \(j \in \mathbb{N}\), Theorems A and B provide a unique \(\varphi_{t,j} \in P(X_T, \omega)\) such that

\[
(\omega_t + dd^c \varphi_{t,j})^n = e^{\varphi_{t,j} + \varphi_t} \min(f, j) dV_X, \quad \varphi_{0,j} = \varphi_0.
\]

Since \(\omega_t\) is the pull-back of a smooth family of Kähler forms, we have

\[-A\omega_t \leq \omega_t \leq A\omega_t,

for a uniform constant \(A > 0\). We can proceed as in the proof of Theorem 2.7 and Theorem 2.13 to establish the following uniform bounds: for each \(T \in (0, +\infty)\) and any compact \(K \subset \Omega \setminus Dc_\epsilon\), there is a constant \(C(T, K)\) such that

\[
t|\partial_t \varphi_{t,j}(x)| \leq C(T, K), \quad t^2|\partial_t^2 \varphi_{t,j}(x)| \leq C(T, K), \quad \forall (t, x) \in (0, T) \times K.
\]

Indeed, on \((0, T)\) the forms \(\omega_t\) satisfy \(\omega_t \geq g(t)\theta\), where \(g(t) = c_0 > 0\) is a constant. The function \(F\) in our case is defined by \(r \mapsto F(t, x, r) \equiv r\) which satisfies the assumptions in the introduction. More precisely, we have the following:

**Proposition 4.12.** Let \(J = [a, b]\) be a compact interval of \((0, T)\). There exist uniform constants \(C_0, C_1, C_2 > 0\) such that for all \(j \in \mathbb{N}\), \(t \in J\),

1. \(C_0 \geq \varphi_{t,j}(x) \geq \psi_{KE}(x) - C_0 e^{-t}\),
2. \(|\partial_t \varphi_{t,j}| \leq C_1 + \sum_{a_k=1}^{\alpha_k=1} \log(-\log |s_k|^2) - \chi_0\),
3. \(\partial_t^2 \varphi_{t,j} \leq C_2 + \sum_{a_k=1}^{\alpha_k=1} \log(-\log |s_k|^2) - \chi_0\).

**Proof.** We first prove (1). For the lower bound, we can check that the function \(u\) in (4.15) is also a subsolution to (4.16). For the upper bound, we pick \(C > 0\) so big that \(\omega_t^0 \leq e^C f dV\) for all \(t \in [0, T]\). The domination principle (see e.g. [BEGZ10, Corollary 2.5]) yields \(\varphi_{t,j} \leq C\) holds everywhere for all \(t, j\).

We next prove (2). Fix \(\varepsilon_0 > 0\) such that \((1 + \varepsilon_0)b < T\). For all \(t \in J\) and \(s \in (1 - \varepsilon_0, 1 + \varepsilon_0)\) there exists a constant \(A_1 > 0\) such that

\[
\omega_t \geq (1 - A_1|s - 1|)\omega_{t,k}.
\]

For \(s\) small enough we set

\[
\lambda_s := \frac{1 - s}{s}, \quad \alpha_s := s(1 - \lambda_s)(1 - A_1|s - 1|) \in (0, 1),
\]

hence \(\gamma_s := \lambda_s / (1 - \alpha_s) \geq \varepsilon_1 > 0\). Shrinking \(\varepsilon_1\) we may assume that \(\gamma_s \omega_t \geq \varepsilon_1 \theta\).

Let \(v_1\) be a solution to the following equation

\[
(\varepsilon_1 \theta + dd^c v_1)^n = e^{v_1} f dV_X.
\]

The same argument in the Step 1 yields

\[
v_1 \geq \varepsilon_1 \chi_0 - \varepsilon_1 \sum_{a_k=1}^{\alpha_k=1} \log(-\log |s_k|^2) - A,
\]

for some uniform constant \(A > 0\). For any \((t, x) \in J \times X\) we set

\[
u^*(t, x) := \frac{\alpha_s}{s} \varphi_{t,j}(ts, x) + (1 - \alpha_s)v_1(x) - C|s - 1|e^{-t},
\]
for $C > 0$ to be chosen later. We have

\[
(\omega_t + dd^c u^*(t, \cdot))^n = \left[ (1 - \lambda_s)\omega_t + \frac{\alpha_s}{s} dd^c \varphi_{ts} + (1 - \lambda_s)\omega_t + (1 - \alpha_s) dd^c v_1 \right]^n \\
\geq \left[ \alpha_s (\varphi_{ts} + dd^c \varphi_{ts,j}) + (1 - \alpha_s)(\gamma_s \omega_t + dd^c v_1) \right]^n \\
\geq e^{\alpha_s \left( \partial_t \varphi_j (ts_j) + \varphi_j (ts_j) \right) + (1 - \alpha_s) v_1} \min(f, j) dV \\
= e^{\partial_t u^*(t, \cdot) + u^*(t, \cdot)} \min(f, j) dV
\]

where we use (4.17) in the second line and Lemma 2.6 in the third one. Therefore $u^*$ is a subsolution to (4.16). Since $\varphi_0$ is bounded we can choose $C > 0$ so large that $u^*(0, \cdot) \leq \varphi_0$ on $X$. Hence the comparison principle (Proposition 3.4) ensures that for any $j$, $u^* \leq \varphi_j$ in $J \times X$, i.e.,

\[
\frac{\alpha_s}{s} \varphi_j (ts, x) + (1 - \alpha_s) v_1 - C|s - 1| e^{-t} \leq \varphi_j (t, x), \quad \forall (t, x) \in J \times X.
\]

Letting $s \to 1$ we infer for all $(t, x) \in J \times X$,

\[
|\partial_t \varphi_j (t, x)| \leq C_1 - C_1 v_1(x),
\]

for a uniform constant $C_1 > 0$.

To prove (3) we argue as above. Set for any $(t, x) \in J \times X$,

\[
v^*(t, x) := \alpha_s \frac{s^{-1} \varphi_j (ts, x) + s \varphi_j (ts^{-1}, x)}{2} + (1 - \alpha_s) v_1 (x) - C|s - 1| e^{-t},
\]

for a constant $C > 0$ so large that $v^*(0, \cdot) \leq \varphi_0$. We can check as above that $v^*$ is a subsolution to (4.16). By the same arguments we can obtain the estimate (3). \hfill $\Box$

We now finish the proof of Step 2. For $t \in (0, T)$ fixed, $\varphi_{t,j}$ is decreasing as $j \to \infty$ by the comparison principle (Proposition 3.4). It follows from Proposition 4.12 that

\[
\varphi_{t,j}(x) \geq \psi_{KE}(x) - C_0, \quad \forall (t, x) \in [0, T) \times X,
\]

for a large constant $C_0 > 0$. It has been shown in [BEGZ10, Theorem 4.2] that $\psi_{KE} \in \mathcal{E}(X, \omega_t)$ for each $t$ since $0 \leq \omega_t \leq \omega_t$, hence $\varphi_{t,j} \in \mathcal{E}(X, \omega_t)$. We want to prove that $\lim_j \varphi_{t,j} = \bar{\varphi}_j$ is a solution to the flow (4.12).

Fix a compact sub-interval $J \subseteq (0, T)$, a compact subset $K \subseteq (\Omega \setminus D_\rho)$. Proposition 4.12 implies that there exists a constant $C = C_j > 0$ such that the function $t \to \varphi_j (t, x) - Ct^2$ is concave in $J$, for all $x \in K$. Moreover, the function $x \to \varphi_j (t, x)$ is $\omega_t$-plsh and uniformly bounded on $K$ for all $j$. We obtain the same properties for the limiting function $\varphi_j (t, x)$ by letting $j \to +\infty$. It follows from Proposition 1.10 that $\bar{\varphi}_j$, $\bar{\varphi}$ are well-defined and $\lim_j \varphi_j (t, \cdot) = \bar{\varphi}(t, \cdot)$. Consider

\[
G := \{ x \in X : f(x) > M \} \cup \{ x \in X : -v_1(x) > M \},
\]

where $v_1$ is a solution to (4.19). Since $-v_1$ is locally bounded outside a divisor, we can choose $M > 0$ so large that $G$ has small Monge-Ampère capacity $\text{Cap}_\theta(G) < \varepsilon$ for some Kähler form $\Theta$ and for any $\varepsilon > 0$. Hence for all $t \in J$, $j \in \mathbb{N}$, we have that $\varphi_{t,j}$ is uniformly bounded from above on $X \setminus G$. Therefore Lebesgue dominated convergence theorem ensures that

\[
\lim_{j \to +\infty} \int_J \int_{X \setminus G} e^{\bar{\varphi}_{t,j} + \varphi_{t,j}} \min(f, j) dV dt = \int_J \int_{X \setminus G} e^{\bar{\varphi} + \psi_{KE}} f dV dt.
\]

Using the notations from [DL15, Section 2], it follows from [DL15, Theorem 2.9] that, for all $t \in (0, T)$, $j \in \mathbb{N}$,

\[
\int_G (\omega_t + dd^c \varphi_{t,j})^n \leq \text{Cap}_{\psi_{KE} - C_0}(G) \leq h(\varepsilon),
\]
for some continuous function $h : [0, +\infty) \to [0, +\infty)$ with $h(0) = 0$. Hence

$$
\int_J \int_X e^{\hat{\varphi} + \phi_t} f dV dt \geq \int_J \int_{X \setminus G} e^{\hat{\varphi} + \phi_t} f dV dt = \lim_{j \to +\infty} \int_J \int_{X \setminus G} (\omega_t + dd^c \varphi_{t,j})^n
$$

$$
= \lim_{j \to +\infty} \int_0^T \int_X (\omega_t + dd^c \varphi_{t,j})^n - \lim_{j \to +\infty} \int_J \int_{X \setminus G} (\omega_t + dd^c \varphi_{t,j})^n
$$

$$
\geq \int_J \int_X \omega^n_t - Th(\varepsilon).
$$

Letting $\varepsilon \to 0$ we obtain $\int_J \int_X e^{\hat{\varphi} + \phi_t} f dV dt \geq \int_X \omega^n_t$.

On the other hand, since $(\omega_t + dd^c \varphi_{t,j})^n$ converges to $(\omega_t + dd^c \varphi_t)^n$, Fatou’s lemma yields

$$dt \wedge (\omega_t + dd^c \varphi_t)^n \geq e^{\hat{\varphi} + \phi_t} f dV dt
$$

in the sense of measures in $(0, T) \times X$, whence equality. This implies that $\varphi$ is a solution to (4.12).

**Proposition 4.13.** For each $t$, the solution $\varphi_t$ of (4.12) is continuous on $\Omega \setminus D_t$.

**Proof.** It follows from Proposition 4.12 that

$$e^{\hat{\varphi} + \phi_t} f \leq \exp \left( C + \sum_{s_k=1} \log(-\log|s_k|^2) - \chi_0 - \sum_i \log(|s_i|^2) \right).$$

The proof thus follows from [Dan21, Theorem 3.2].

**Step 3: convergence at time zero.** Using similar arguments as in the proof of Theorem 2.8, we are going to check that the solution $\varphi_t$ of the equation (4.12) converges pointwise towards $\varphi_0$ as $t \to 0^+$.

Arguing as at the beginning of the proof of Theorem 4.1, we can check that

$$\varphi(t, x) \geq u(t, x) := e^{-t} \varphi_0(x) + (1 - e^{-t})\psi_{KE}(x) + h(t), \quad \forall (t, x) \in (0 + \infty) \times X,$$

where $\psi_{KE}$ is the solution of (4.14) and $h(t) = n e^{-t} [(e^t - 1) \log(e^t - 1) - te^t]$. It thus remains to show that for all $x \in X$, $\lim_{t \to 0} \varphi_t(x) \leq \varphi_0(x)$.

Fix $T < +\infty$ and consider

$$G := \{x \in X : u(T, x) > -M\},$$

where $M > 0$ is a constant such that $\mu(G) > \mu(X)/2$ (recall that $\psi_{KE}$ is smooth outside a divisor). Observe that $\varphi(t, x) \geq u(t, x) \geq u(T, x) > -M$ for all $x \in G$, $t \in (0, T)$. Following the proof of Theorem 2.8, we obtain as in (2.11) that

$$\int_G \varphi_t d\mu \leq \int_G \varphi_0 d\mu + Ct,$$

for a constant $C > 0$ depending on $G$.

Let now $u_0 \in PSH(X, \omega_0)$ be any cluster point of $\varphi_t$ as $t \to 0$. We can assume that $\varphi_t$ converges to $u_0$ in $L^q(X, dV)$ for any $q > 1$. On the other hand, $d\mu = \prod |s_i|^{-2a_i} dV_X$ has density $f = \prod |s_i|^{-2a_i} \in L^p_{loc}(X \setminus D)$ for any $p > 1$. Hence, $\varphi_t f$ converges to $u_0 f$ in $L^1(K)$ for any compact subset $K$ of $X \setminus D$. Thus, the claim above ensures that

$$\int_G u_0 f dV \leq \int_G \varphi_0 f dV.$$

We infer that $u_0 \leq \varphi_0$ almost everywhere on $G$ with respect to $dV$, hence everywhere on $G$. Letting $M \to +\infty$, we conclude that $\limsup_{t \to 0} \varphi_t = \varphi_0$ on $X \setminus \mathcal{D}$, hence on the whole $X$. 

Step 4: uniqueness of the flow. By the previous steps, we have shown that there exists a solution $\varphi$ to (4.12) with initial data $\varphi_0$. This function satisfies the following properties:

- $\varphi$ is locally uniformly semi-concave in $t$,
- $(t,x) \rightarrow \varphi(t,x)$ is continuous on $(0, +\infty) \times U$, where $U := \Omega \setminus D_{(c)}$,
- $\varphi_t \rightarrow \varphi_0$ pointwise as $t \rightarrow 0^+$.

We are going to show that such a solution is unique. Let $\Phi$ be a solution to (4.12) with the same properties as above. We shall prove that $\varphi \leq \Phi$ on $[0, +\infty) \times X$, whence equality. The proof follows step by step from the uniqueness result obtained in Section 3.2.

Step 4.1. Assume moreover that:

1. $\varphi$ is $C^1$ in $t$,
2. $\Phi$ is continuous on $[0, +\infty) \times U$.

Since $\theta$ is semi-positive we fix $c > 0$ such that $\omega_t \geq c \theta$ for all $t$. For simplicity we again assume that $c = 1$. Let $\chi$ be a $\theta/2$-psh function with analytic singularities such that $\chi$ is smooth in $U$, $\chi = -\infty$ on $\partial U$, and $\sup_X \chi = 0$. We will use this function in order to apply the classical maximum principle in $U$. The standard strategy is to replace $\varphi$ by $(1 - \lambda)\varphi + \lambda \chi$. Nevertheless, the time derivative $\varphi_t$ may blow up as $t \rightarrow 0$ so we need another auxiliary function. Let $\rho \in \text{PSH}(X, \theta/2)$ be the unique solution to

$$
(\theta/2 + dd^c \rho)^n = e^\rho d\mu,
$$

normalized by $\sup_X \rho = 0$, where $d\mu = \prod_i |f_i|^{-2\omega_t} dV$. It follows from [Dan21, Corollary 3.5] that $\rho$ is continuous in $U$.

Fix $0 < T < +\infty$. For $\varepsilon, \lambda > 0$ small enough we set

$$
w(t, x) := (1 - \lambda)\varphi(t, x) + \lambda(\rho(x) + \chi(x)) - \Phi(t, x) - 3\varepsilon t, \quad \forall (t, x) \in (0, T) \times X.
$$

By Lemma 1.5, this function is upper semi-continuous on $[0, T] \times U$. Since $\rho + \chi$ is a $\theta$-psh function which is continuous in $U$ and tends to $-\infty$ on $\partial U$, the function $w$ attains its maximum at some point $(t_0, x_0) \in [0, T] \times U$.

We want to show that $w(t_0, x_0) \leq 0$. Assume by contradiction that it is not the case i.e., $w(t_0, x_0) > 0$ with $t_0 > 0$. The set

$$
K := \{x \in U : w(t_0, x) = w(t_0, x_0)\}
$$

is a compact subset of $U$ since $w(t_0, x)$ tends to $-\infty$ as $x \rightarrow \partial U$. The classical maximum principle ensures that for all $x \in K$,

$$
(1 - \lambda)\partial_t \varphi(t_0, x) \geq \partial_t^{-} \Phi(t_0, x) + 3\varepsilon.
$$

The partial derivative $\partial_t \varphi(t, x)$ is continuous on $U$ by assumption. Since the function $t \rightarrow \Phi(t, x)$ is locally uniformly semi-concave, for any $t \in (0, T)$, the left derivative $\partial_t^{-} \Phi(t, \cdot)$ is upper semi-continuous in $\Omega$ (see Proposition 1.10). We can thus find $\eta > 0$ small enough that, by introducing the open set containing $K$,

$$
D := \{x \in U : w(t_0, x) > w(t_0, x_0) - \eta\} \subseteq U.
$$

We have for all $x \in D$

$$
(1 - \lambda)\partial_t \varphi(t_0, x) > \partial_t^{-} \Phi(t_0, x) + 2\varepsilon.
$$
Set \( u := (1 - \lambda)c(t_0, \cdot) + \lambda(\rho + \chi) \) and \( v = \Phi(t_0, \cdot) \). Since \( \varphi \) is a pluripotential solution to (4.12), using Lemma 2.6 we infer that
\[
(\omega_{t_0} + d\lambda u)^n \geq [(1 - \lambda)(\omega_{t_0} + d\lambda \varphi(t_0)) + \lambda(\theta + d\lambda \rho)]^n
\geq e^{(1-\lambda)(\theta \varphi(t_0) + \varphi(t_0)) + \lambda \rho} d\mu
\geq e^{\delta \varphi(t_0) + \delta \varphi(t_0)} d\mu.
\]

On the other hand, \( \Phi \) is a pluripotential solution to (4.12), hence
\[
(\omega_{t_0} + d\lambda v)^n \leq e^{\delta \varphi(t_0) + \delta \varphi(t_0)} d\mu
\]
in the weak sense of measures in \( D \). The last two inequalities yield
\[
(\omega_{t_0} + d\lambda u)^n \geq e^{\varphi + 3\tau} (\omega_{t_0} + d\lambda v)^n.
\]
We then repeat the arguments as in the proof of Proposition 3.2 to obtain a contradiction. Therefore, we must have \( t_0 = 0 \), hence
\[
(1 - \lambda) \varphi + \lambda(\rho + \chi) - \Phi - 3\varepsilon \leq \lambda \sup_x ((\rho + \chi) - \varphi_0),
\]
in \([0, T] \times U\). Letting \( \lambda \to 0 \) we obtain \( \varphi \leq \Phi + 3\varepsilon \) in \([0, T] \times U\), hence in \([0, T] \times X\). We thus finish the proof by letting \( \varepsilon \to 0 \) and \( T \to +\infty\).

**Step 4.2.** We next remove the continuity assumption on \( \Phi \) in Step 4.1.

Fixing \( s > 0 \) small enough, we set
\[
u^s(t, x) := e^{-s} \varphi_t(x) + (1 - e^{-s}) \psi_{KE}(x) + h(s)
\]
where \( h \) is defined as in the proof of Theorem 4.1. We observe that
\[
\omega_{t+s} = e^{-s} \omega_t + (1 - e^{-s}) \theta, \quad \forall \ t \in [0, +\infty),
\]

hence
\[
(\omega_{t+s} + d\lambda u^s)^n = e^{-s}(\omega_t + d\lambda \varphi_t) + (1 - e^{-s})(\theta + d\lambda \psi_{KE})]^n
\geq e^{\theta - s}(\omega_t + d\lambda \varphi_t) + (1 - e^{-s})(\theta + d\lambda \psi_{KE}) d\mu
\]

where the last inequality follows from Lemma 2.6. Since \( h(s) \leq 0 \) for \( s > 0 \) we have
\[
(\omega_{t+s} + d\lambda u^s)^n \geq e^{\theta - s} d\mu.
\]

On the other hand,
\[
(\omega_{t+s} + d\lambda v^s)^n = e^{\theta - s} d\mu,
\]
where \( v^s(t, x) := \Phi(t + s, x) \) for \((t, x) \in (0, +\infty) \times X\). By Lemma 4.14 we have \( u^s(0, x) \leq v^s(0, x) \) for all \( x \in X \). Since \( v^s \) is continuous on \([0, +\infty) \times U\), it follows from Step 4.1 that
\[
u^s(t, x) \leq v^s(t, x), \quad (t, x) \in [0, +\infty) \times X.
\]

Letting \( s \to 0 \) we thus obtain \( \varphi \leq \Phi \) on \([0, +\infty) \times X\).

**Lemma 4.14.** For all \((t, x) \in (0, T) \times X\),
\[
\Phi_t(x) \geq e^{-\tau} \varphi_0(x) + (1 - e^{-t}) \psi_{KE}(x) + h(t),
\]

where \( h \) is the unique solution to the ODE: \( h'(t) + h(t) = \log(1 - e^{-t}), \ h(0) = 0 \).

**Proof.** Fix \( \varepsilon > 0 \), and consider
\[
u^s(t, x) = e^{-\tau} \Phi_t + (1 - e^{-t}) \psi_{KE} + h(t).
\]

A direct computation shows that
\[
(\omega_{t+s} + d\lambda u^s)^n = \left( e^{-\tau}(\omega_t + d\lambda \Phi_t) + (1 - e^{-t})(\theta + d\lambda \psi_{KE}) \right)^n
\geq e^{\log(1 - e^{-\tau}) + \psi_{KE}} d\mu.
\]
where we have used $\omega_\cdot + dd^c\Phi_\cdot \geq 0$. Since $h'(t) + h(t) = n \log(1 - e^{-t})$ we have

$$(\omega_{t+} + dd^c w_\cdot)^n \geq e^{\partial_i w_\cdot + w_\cdot} \mu.$$ 

It is also clear from the definition that $w^\varepsilon(t, \cdot)$ converges in $L^1(X)$ to $w^\varepsilon(0, \cdot) = \Phi_\cdot$ as $t \to 0^+$. On the other hand, $w^\varepsilon$ is $C^1$ in $t$ and $\Phi_{t+} \cdot$ is continuous on $[0, +\infty) \times U$.

We can thus apply Step 4.1 to obtain $w^\varepsilon(t, x) \leq \Phi(t + \varepsilon, x)$. The proof follows by letting $\varepsilon \to 0$. □

**Step 4.3.** We are now ready to treat the general case by removing the extra assumption on $\varphi$.

For $s > 0$ near 1 we set, for any $(t, x) \in (0, T) \times X$

$$V^s(t, x) := \frac{\alpha_s}{s} \varphi(ts, x) + (1 - \alpha_s)v_1(x) - C|s - 1|e^{-t},$$

where $\alpha_s$ is defined as in (4.18), and $v_1$ is a solution to (4.19). For $C > 0$ large enough, the proof of Proposition 4.12 ensures that $V^s$ is a subsolution to (4.12) that satisfies $V^s(0, \cdot) \leq \varphi_0$ on $X$. Let $\{\eta_s\}_{s > 0}$ be a family of smoothing kernels in $\mathbb{R}$ approximating the Dirac mass $\delta_0$. For $s > 0$ small enough we define

$$\varphi^s(t, x) := \int_\mathbb{R} V^s(t, x) \eta_s(s - 1) ds$$

We proceed as in the proof of Theorem 3.7 to show that $\varphi^s - O(\varepsilon)$ is again a subsolution and apply the previous step to conclude.

**Step 5: the long-term behavior of the flow.** It remains to establish the convergence at $t = +\infty$. We have seen that

$$u(t, x) := e^{-t} \varphi_0 + (1 - e^{-t})\psi_{KE}(x) + h(t)$$

is a subsolution to (4.12). The comparison principle (see Step 4) yields for any $t > 0, x \in X$,

$$\psi_{KE}(x) - C(t + 1)e^{-t} \leq u(t, x) \leq \varphi(t, x)$$

for some uniform constant $C > 0$.

For the upper bound, since $\bar{\theta} = \theta + dd^c \chi_0$ is a Kähler current we can fix a constant $A > 0$ such that $\omega_t \leq (1 + A)\bar{\theta}$ on $\Omega$, thus $\omega_t \leq (1 + Ae^{-t})\bar{\theta}$ for all $t$. Set

$$v(t, x) := (1 + Ae^{-t})\psi_{KE}(x) + Be^{-t}$$

where $B$ is chosen so that $v_0 \geq \varphi_0$. Thus the function $v$ is a supersolution to the Cauchy problem for the parabolic equation

$$((1 + Ae^{-t})\bar{\theta} + dd^c w_t)^n \leq e^{\tilde{\psi}_t + \psi_t + nAe^{-t}}$$

with initial data $\varphi_0$, while $w(t, x) := \varphi(t, x) - Ae^{-t}$ is a subsolution to this equation since

$$((1 + Ae^{-t})\bar{\theta} + dd^c w_t)^n \geq (\omega_t + dd^c \varphi_t)^n = e^{\tilde{\psi}_t + \psi_t} f dV = e^{\tilde{\psi}_t + \psi_t} f dV.$$ 

The comparison principle thus yields

$$\varphi(t, x) \leq (1 + Ae^{-t})\psi_{KE}(x) + C'e^{-t},$$

as desired. □
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