Renormalized HVBK dynamics for Superfluid Helium Turbulence

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Abstract

We review the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) equations for superfluid Helium turbulence and discuss their implications for recent measurements of superfluid turbulence decay.

A new Hamiltonian formulation of these equations renormalizes the vortex line velocity to incorporate finite temperature effects. These effects also renormalize the coupling constant in the mutual friction force between the superfluid and normal fluid components by a factor of $\rho_s/\rho$ (the superfluid mass fraction) but they leave the vortex line tension unaffected. Thus, the original HVBK form is recovered at zero temperature and its mutual friction coefficients are renormalized at nonzero temperature. The HVBK equations keep their form and no new parameters are added. However, a temperature dependent trade-off does arise between the mutual friction coupling and the vortex line tension.

The renormalized HVBK equations obtained via this new Hamiltonian approach imply a dynamical equation for the space-integrated vortex tangle length, which is the quantity measured by second sound attenuation experiments in superfluid turbulence. A Taylor-Proudman theorem also emerges for the superfluid vortices that shows the steady vortex line velocity becomes columnar under rapid rotation.

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1 HVBK equations

Recent experiments establish the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) equations as a leading model for describing superfluid Helium turbulence. See Nemirovskii and Fiszdon [1995] and Donnelly [1999] for authoritative reviews. See Henderson and Barenghi [2000] for a recent fluid mechanics study of steady cylindrical Couette flow using computer simulations of the incompressible HVBK equations.

In the Galilean frame of the normal fluid with velocity $v_n$, the HVBK equations may be expressed as follows, upon ignoring thermal diffusivity and viscosity,

$$\begin{align*}
\frac{\partial \rho}{\partial t} &= -\text{div}J, \\
\frac{\partial J}{\partial t} &= -\partial_k T^k, \\
\frac{\partial S}{\partial t} &= -\text{div}(Sv_n) + R/T, \\
\rho_s \frac{\partial v_s}{\partial t} + \rho_s (v_s \cdot \nabla)v_s &= -\rho_s \nabla(\mu - \frac{1}{2}|v_s - v_n|^2) + \rho_s f,
\end{align*}$$

and summing over pair of upper and lower repeated indices. One may consult, e.g., Bekarevich and Khalatnikov (BK) [1961] and Donnelly [1999] to compare these equations with the form they take in the reference frame of the superfluid.

**Notation.** Here $\rho$ and $\rho_s$ denote the total and superfluid mass densities, respectively. The entropy density of the normal fluid is $S$, its temperature is denoted $T$, and $\rho_n = \rho - \rho_s$ denotes its mass density. The superfluid velocity is denoted $v_s$ and $J = \rho_s v_s + \rho_n v_n$ is the total momentum density. In the entropy equation $R$ is the rate that heat is produced by the phenomenological friction and reactive forces in $f$, which must be specified to close the theory. We also denote

- **Stress tensor:** $T^k_i = \rho_s v_{si} v_{ki}^s + \rho_n v_{ni} v_{ki}^n + (P + \lambda \omega) \delta^k_i - \lambda_i \omega^k$,
- **Euler’s pressure law:** $P = -\varepsilon_0 + TS + \rho \mu$,
- **Superfluid First Law:** $d\varepsilon_0 = \mu d\rho + TdS + (J - \rho v_n) \cdot d(v_s - v_n) + \lambda \cdot d\omega$.

In the superfluid First Law, $\omega = \text{curl} v_s$ is the superfluid vorticity with magnitude $|\omega| = \hat{\omega} \cdot \omega$, where $\hat{\omega} = \omega/|\omega|$ is its unit vector. BK [1961] takes the energy density $\varepsilon_0$ to depend on the magnitude of the superfluid vorticity, $|\omega|$, as

$$\varepsilon_0 = \frac{\rho_s \kappa |\omega|}{4\pi} \ln \frac{R}{a}. \quad (3)$$

This is the energy per unit length of a superfluid vortex line, $\rho_s (\kappa^2/4\pi) \ln (R/a)$, with quantum of circulation $\kappa = \hbar/m \simeq 10^{-3} (\text{cm}^2/\text{sec})$ and ratio $R/a$ of mean distance between vortices $R$ to effective vortex radius $a$, times the vortex length per unit volume, $|\omega|/\kappa$. Hence, we find

$$\lambda = \frac{\partial \varepsilon_0}{\partial |\omega|} \frac{\partial |\omega|}{\partial \omega} = |\lambda| \hat{\omega} \quad \text{with} \quad |\lambda|/\rho_s \approx \frac{\kappa}{4\pi} \ln \frac{R}{a} = \lambda_0, \quad (4)$$

1In making this comparison it is useful to recall the Galilean transformation of the chemical potential, $\mu' = \mu - \frac{1}{2}|v_s - v_n|^2$, where $\mu'$ is evaluated in the superfluid frame and $\mu$ in the normal-fluid frame. See Putterman [1974] for a clear discussion of the role of Galilean transformations in superfluid hydrodynamics.
where one ignores the derivative of \( R \approx \sqrt{\kappa/|\omega|} \) inside the logarithm. The appearance of \( \lambda \) in the stress tensor \( T^k_i \) shifts the pressure \( P \), and \( \text{div} T \) introduces an additional force \( -\omega \cdot \nabla \lambda \equiv -\rho_s T \) into the motion equation. The quantity \( T \) is called the “vortex line tension.”

BK [1961] assigned the following form to the phenomenological coupling force \( f \) appearing in the superfluid velocity equation in (1),

\[
f = (v_L - v_s) \times \omega , \quad v_L = v_n - \rho_s (\alpha s^0 + \beta \hat{\omega} \times s^0) ,
\]

where \( s^0 = v_0^\ell - v_n \), \( v_0^\ell = v_s + \rho - \frac{1}{\rho_s} \text{curl} \lambda \),

with “vortex velocity” \( v_L \) and “slip velocity” \( s^0 \) introduced as auxiliary quantities. The HVBK equations written as (1) in the normal-fluid frame conserve the energy,

\[
E = \int \left[ \frac{1}{2} \rho |v_n|^2 + (J - \rho v_n) \cdot v_n + \varepsilon_0 \right] d^3x .
\]

The BK [1961] form of the phenomenological force \( f \) also implies the dissipative heating rate,

\[
R = (J - \rho v_n + \text{curl} \lambda) \cdot \omega \times (v_L - v_n) ,
\]

which is Galilean invariant and positive. Substituting this form of \( f \) into the superfluid motion equation in (1) and taking its curl provides the following equation for the superfluid vorticity, \( \omega = \text{curl} v_s \),

\[
\partial_t \omega = \text{curl} (v_L \times \omega) .
\]

This vorticity equation implies the HVBK superfluid Kelvin circulation theorem,

\[
\frac{d}{dt} \int_S \omega \cdot dS = \frac{d}{dt} \oint_{\partial S(v_L)} v_s \cdot dx = 0 .
\]

The Kelvin formula (10) expresses conservation of the flux of superfluid vorticity through any surface \( S \) whose boundary \( \partial S \) moves with the velocity \( v_L \), so \( v_L \) may be regarded as the local velocity of a vortex line. Equivalently, the Kelvin formula expresses conservation of superfluid velocity circulation around any loop that moves with the vortex line velocity \( v_L \).

In a key phenomenological step that closed the theory, BK [1961] assigned the undetermined functions \( \alpha \) and \( \beta \) in the force \( f \) and its auxiliary vortex line velocity \( v_L \) as

\[
1 + \alpha \rho_s = \frac{B' \rho_n}{2\rho} , \quad \beta \rho_s = \frac{B \rho_n}{2\rho} .
\]

The dimensionless coefficients \( B \) and \( B' \) were introduced earlier in Hall and Vinen (HV) [1956] to parameterize the Gorter-Mellink [1949] “mutual friction” force \( B \) and its reactive component \( (B') \). Hence the name, HVBK equations for this closure.

The assignments in BK [1961] of the undetermined functions \( \alpha, \beta \) in (11), as well as the vortex slip velocity \( s^0 \) in (1) are designed to reproduce the phenomena observed in HV.
and yet still conserve mass, momentum and energy. This phenomenological approach used in BK [1961] is indeterminate, however, in the sense that some freedom still remains in making these assignments. The HVBK equations (1) that result from this approach do possess the desired conservation laws for mass, momentum and energy. And they also possess a Kelvin theorem for the circulation of superfluid velocity. However, because of the indeterminacy inherent in the phenomenological approach, the HVBK equations (1) are not unique in possessing these properties. An alternative assignment of the vortex slip velocity is

\[ s = v_\ell - v_n \text{ with } v_\ell = \bar{v} + \rho^{-1} \text{curl } \lambda, \text{ with } \bar{v} = \rho^{-1}(\rho_s v_s + \rho_n v_n). \]  

(12)

The mean velocity \( \bar{v} \) also figures prominently in Hills and Roberts [1977] discussion of the HVBK equations. As we shall see, the alternative expression (12) for the auxiliary vortex slip velocity in terms of the mean velocity \( \bar{v} \) arises naturally in the Hamiltonian derivation of a slightly modified set of HVBK equations. These equations possess the same formal conservation and circulation properties as HVBK, modulo redefining the vortex slip velocity as \( s \) rather than \( s^0 \). The vortex slip velocity \( s \) in (12) is defined relative to the Galilean frame of the normal fluid, which is present only at finite temperature. The HVBK \( s^0 \) in (6) is the limit of the vortex slip velocity \( s \) for zero temperature, at which no normal fluid remains. The vortex slip velocity \( s \) in (12) is a slight modification of \( s^0 \) in (6) necessary to incorporate finite temperature effects, without changing the form of the HVBK theory, obtained from a Hamiltonian derivation of these equations in the normal fluid frame. In the Hamiltonian framework, the energy-momentum conservation laws and Kelvin circulation theorem are all natural consequences. Moreover, the velocities \( v_\ell \) and \( v_n \) are identified as being dual to the momenta given by \( \rho v_s \) and \( \rho_n(v_n - v_s) \), respectively.

**Outline.** We shall use a Hamiltonian approach with Lie-Poisson brackets to derive the expression (12) for the vortex line velocity \( v_\ell \) at finite temperature from first principles by using the energy \( E \) in (7) as the Hamiltonian. The momenta conjugate to the velocities \( v_\ell \) and \( v_n \) shall be our basic dynamical variables. The finite temperature vortex line velocity \( v_\ell \) and slip velocity \( s \) determined this way turn out to be

\[ v_\ell = \rho^{-1}(J + \text{curl } \lambda), \text{ and } s = v_\ell - v_n \simeq \left(\frac{\rho_s}{\rho}\right)(v_\ell^0 - v_n). \]  

(13)

At zero temperature, \( \rho \to \rho_s \) and these reduce to the BK [1961] phenomenological expressions with \( v_\ell^0 \) given by (9). Thus, the finite temperature corrections found by using the Hamiltonian approach renormalize the HVBK slip velocity in the mutual friction force \( f \) by the factor \( \rho_s/\rho \) (the superfluid mass fraction). Aside from this renormalization, the vortex line tension is left unaffected by this renormalization, the superfluid vortex equation keeps its form and no new parameters are added.

Technical details of deriving this renormalized theory from its Hamiltonian and Lie-Poisson brackets are given in the Appendix.
Main results. We shall use the superfluid vortex dynamics for the renormalized HVBK equations obtained via this Hamiltonian approach to write a dynamical equation for the space-integrated total vortex tangle length, which is the quantity measured in the Oregon experiments on superfluid turbulence reported in Skrbek, Niemela and Donnelly [1999].

We shall also study the restriction of the renormalized HVBK equations for the incompressible case, in which \( \rho_n \) and \( \rho_s \) are constants and one takes \( \nabla \cdot \mathbf{v}_n = 0 \) and \( \nabla \cdot \mathbf{v}_s = 0 \). Finally, we shall demonstrate the invariance of the forms of these equations upon transforming into a rotating frame. The Coriolis force in such a rotating frame couples to the vortex line velocity \( \mathbf{v}_L \), which of course differs from both the superfluid velocity and the normal velocity. We shall derive a Taylor-Proudman theorem for steady superfluid vortices under rapid rotation. According to this superfluid Taylor-Proudman theorem, the vortex line velocity becomes columnar under sufficiently rapid rotation. That is, the lateral vortex line velocity is nondivergent and independent of the axial coordinate, and the axial velocity decouples from the lateral motion. Therefore, under sufficiently rapid rotation, the superfluid vortex filaments will straighten and become parallel to the axis of rotation as they approach a steady state.

Numerical implications. This renormalization of the vortex line element slip velocity in the HVBK equations from \( s^0 \rightarrow s \simeq s^0 \rho_s/\rho \) is sensitive to temperature, but it does not affect the vortex line tension. Therefore, a temperature sensitive trade-off arises between mutual friction and vortex line tension that may be worth testing in numerical simulations such as those reported in Henderson and Barenghi [2000]. The HVBK equations are thought to break down in the presence of strong counterflow. However, as general conservation laws there is no mechanism in the equations that would signal this breakdown. A rotating Rayleigh-Besnard experiment might be useful in testing the range of validity of the HVBK equations (Barenghi, private communication). Such an experiment might also indicate how these equations should be modified in the presence of strong counterflow.

Experimental implications. Temperature sensitivity of the coupling between the superfluid vortices and the normal fluid component is an area of intense current investigation in superfluid Helium turbulence, see Donnelly [1999]. One would like to know whether the \( \rho_s/\rho \) renormalization of the mutual friction forces relative to the vortex line tension would matter significantly in comparisons of the predictions of the HVBK equations with modern experiments in Helium turbulence at low, but finite temperatures.

Superfluid vortex dynamics. To begin addressing this issue, we may use the superfluid vorticity equation for the renormalized HVBK equations obtained in the Appendix via the Hamiltonian approach to write an explicit equation for the dynamics of Vinen’s vortex length density \( L = |\omega|/\kappa \). In the superfluid turbulence decay experiments reported by Skrbek, Niemela and Donnelly [1999] the spatial integral of this quantity is measured as a function of time to decrease over six decades as \( t^{-3/2} \). The integrated vortex length measured
in these experiments is predicted by the renormalized HVBK equations to be governed by the superfluid vorticity dynamics alone.

Upon including mutual friction, the superfluid vortex dynamics for the renormalized HVBK equations is expressed as, cf. equation (9),

$$\partial_t \omega = \text{curl} (v_L \times \omega), \quad (14)$$

in which the renormalized total vortex line velocity given by, cf. equation (5),

$$v_L = v_\ell - \frac{B' \rho_n}{2\rho} s - \frac{B \rho_n}{2\rho} \hat{\omega} \times s, \quad \text{where} \quad s = v_\ell - v_n, \quad (15)$$

and its Hamiltonian limit is found to be

$$v_\ell = \tilde{v} + \rho^{-1} \text{curl} \lambda, \quad \text{with} \quad \tilde{v} = \rho^{-1} J \quad \text{and} \quad \lambda = \lambda \hat{\omega}. \quad (16)$$

Thus, relative to the Hamiltonian approach, the terms in $B$ and $B'$ are additional velocities introduced by phenomenology, while $v_\ell$ is the vortex line velocity in the absence of mutual friction.

The HVBK superfluid vorticity equation implies the following dynamics for the integrated vortex length measured in the turbulence decay experiments,

$$\frac{d}{dt} \int L d^3 x = \int \hat{\omega} \cdot \partial_t \omega / \kappa \, d^3 x$$

\[= \int L \hat{v} \cdot (\hat{\omega} \times \text{curl} \hat{\omega}) \, d^3 x - \int L \frac{B \rho_n}{2\rho^2} (\hat{\omega} \times \text{curl} \lambda \hat{\omega}) \cdot (\hat{\omega} \times \text{curl} \hat{\omega}) \, d^3 x
\]

\[+ \int L (\hat{n} \times \hat{\omega}) \cdot ((\hat{\omega} \times v_\ell) + \frac{B \rho_n}{2\rho} (v_\ell - v_n)) \, dS. \quad (17)\]

Here $\hat{\omega}$ is the unit vector tangent to a superfluid vortex filament, so $\kappa = (\hat{\omega} \times \text{curl} \hat{\omega})$ is its local curvature. The transport and damping of the vortex tangle length is proportional to this local curvature. The effective velocity $\tilde{v}$ in the transport term is given by

$$\tilde{v} = \nabla - \frac{B' \rho_n}{2\rho} (\nabla - v_n) - \frac{B \rho_n}{2\rho} \hat{\omega} \times (\nabla - v_n). \quad (18)$$

According to the last term in (17), vortex length is created or destroyed at the boundary, unless the vortex filaments approach it in the normal direction, so that $\hat{n} \times \hat{\omega} = 0$.

Formula (17) for the evolution of the total superfluid vortex length presents a trade-off between the mass-weighted velocity $\nabla$ and the local induction velocity (or filament curvature) $\hat{\omega} \times \text{curl} \hat{\omega}$, in the interior of the domain. This trade-off in the interior competes with the
process of creation and destruction at the boundary. For example, in counterflow turbulence, the superfluid moves toward the heater at the boundary, so the term in $v$ would tend to be nonzero. In contrast, for grid turbulence, $v$ is small, so this term would tend to contribute less. This formula governs the dynamics of the experimentally measured quantity $\int L \, d^3x$.

However, it does not yet show how to obtain the $t^{-3/2}$ decrease seen in this quantity by Skrbek, Niemela and Donnelly [1999] in their experiments on decay of turbulence.

Suppose the main source of decay were the term labeled “damping by curvature” in formula (17) and the flow were isothermal and incompressible. This would imply

$$\frac{1}{\langle L \rangle} \frac{d}{dt} \langle L \rangle = - c_0(T) \lambda_0 \frac{\langle L/R^2 \rangle}{\langle L \rangle} = - \frac{3}{2 (t + t_0)} ,$$

(19)

where $\lambda_0 = \lambda/\rho_s = (\kappa/4\pi) \ln(b/a)$ is the quantum vortex constant, $t_0$ is a time shift in the experimental analysis, $c_0(T) \equiv B \rho_0 \rho_s/(2 \rho^2)$ and angle brackets $\langle \cdot \rangle$ denote spatial integral over the measurement domain. In particular,

$$\langle L \rangle \equiv \int L \, d^3x , \quad \langle L/R^2 \rangle \equiv \int L |\hat{\omega} \times \text{curl} \hat{\omega}|^2 \, d^3x .$$

(20)

The measured $t^{-3/2}$ decrease in $\langle L \rangle$ implies via formula (19) that the length-weighted mean curvature of the vortex tangle $\langle L/R^2 \rangle / \langle L \rangle$ decays due to mutual friction as $t^{-1}$. Thus, on the average as the vortex tangle decays, the vortices tend to straighten, under the effects of mutual friction damping.

**Preservation of helicity versus preservation of vortex length.** The helicity, or linkage number for the superfluid vorticity is defined as

$$\Lambda = \int (v_s \cdot \omega) \, d^3x .$$

(21)

The helicity satisfies an evolution equation obtained from the superfluid vortex dynamics,

$$\frac{d\Lambda}{dt} = - \int (\hat{n} \cdot \omega) \left( \mu - \frac{1}{2} v_n^2 - v_s \cdot (v_L - v_n) \right) \, dS - \int (\hat{n} \cdot v_L) (v_s \cdot \omega) \, dS .$$

(22)

Therefore, even with mutual friction, helicity is created and destroyed only on the boundary. Moreover, helicity will be preserved, provided both $\omega$ and $v_L$ are tangential at the boundary. The former condition, however, is the opposite of that required for the creation and destruction of vortex length at the boundary to cease. Therefore, no equilibrium should be expected that preserves both the helicity and the vortex length in a superfluid.

**Superfluid vortex equilibria are not ABC flows.** The steady equilibrium solutions of the superfluid vorticity dynamics satisfy

$$\text{curl} \, (v_L \times \omega) = 0 .$$

(23)

For example, a steady equilibrium exists when $\omega$ and $v_L$ are parallel. Note that these “super-Beltrami flows” are not eigenfunctions of the curl. Therefore, they are not Arnold-Beltrami-Childress (ABC) flows, as occur for the Euler equations.
2 Incompressible renormalized HVBK flows

To express the renormalized HVBK equations in the incompressible limit, we begin by re-collecting the compressible equations and abbreviating $|v_s - v_n|^2 = v_{sn}^2$,

\[
\begin{align*}
\partial_t S &= - \text{div}(Sv_n) + R/T, \\
\partial_t \rho &= - \text{div}(\rho_s v_s + \rho_n v_n), \\
\rho_s (\partial_t v_s + (v_s \cdot \nabla) v_s) &= - \rho_s \nabla (\mu - \frac{1}{2}v_{sn}^2) + \rho_s (v_L - v_s) \times \omega, \\
\partial_t (\rho_s v_{si} + \rho_n v_{ni}) &= - \partial_k (\rho_n v_{ni} v_i^k + \rho_s v_{si} v_s^k) - \partial_i P - \partial_k \tau_i^k, \\
\tau_i^k &= \epsilon_{klm} \partial_l (v_{si} \lambda_m) - \lambda_i \omega_k + \delta_i^k \lambda \cdot \omega.
\end{align*}
\]

As we have seen, finite temperature effects renormalize the total vortex line velocity as $v_L = v_\ell - \rho_n \rho (B_2 \hat{\omega} \times s + B'_2 s)$, where $s = v_\ell - v_n$.

\begin{equation}
(24)
\end{equation}

and the Hamiltonian part of the line velocity (with corrections for finite temperature) is defined as

\begin{equation}
(25)
\end{equation}

To the extent that $\rho$, $\rho_s$, $\rho_n$ and $S$ all may be taken as constants for a given temperature and the heating rate $R$ is negligible, then the velocities $v_n$ and $v_s$ are incompressible, i.e.,

\begin{equation}
(26)
\end{equation}

In this situation, the pressure $P$ may be obtained from the Poisson equation,

\begin{equation}
- \nabla^2 (P + \lambda \cdot \omega) = \text{div}(\rho_s (v_s \cdot \nabla) v_s + \rho_n (v_n \cdot \nabla) v_n - \omega \cdot \nabla \lambda),
\end{equation}

\begin{equation}
(27)
\end{equation}

found from the divergence of the total momentum equation. Combining the superfluid motion equation with total momentum conservation results in an equation for the normal fluid velocity in the incompressible case

\begin{equation}
(28)
\end{equation}

where $v_L$ is given in equation (24). We set $P' \equiv P + \lambda \cdot \omega$ and take it as the total pressure. (One also could have absorbed $\lambda \cdot \omega$ into $P$ earlier, by including it in Euler's pressure law.) Since $\lambda = |\lambda| \hat{\omega}$ and $\hat{\omega}$ is a unit vector, we find for constant $\rho_s$ the standard relation for the vortex line tension denoted as $T$. Namely,

\begin{equation}
\omega \cdot \nabla \lambda = - \lambda_0 \rho_s \omega \times \text{curl} \hat{\omega} \equiv \rho_s T,
\end{equation}

\begin{equation}
(29)
\end{equation}

where $\lambda_0 = \lambda/\rho_s = (\kappa/4\pi) \ln(b/a)$ is a constant.
<p><strong>Remark.</strong> We note that the quantity $T$ known as the vortex line tension first appears in the normal fluid equation, as a reaction to the presence of the superfluid. The standard convention for introducing the mutual friction force has the effect of shifting $T$ into the superfluid equation. By action and reaction, though, $T$ could appear in either equation. These equations of motion must be completed by providing an equation of state relation for the quantity $\mu - \frac{1}{2} v_s^2$. BK [1961] assumes a law of partial pressures,

$$P_n = \rho_n \rho' = P' - P_s \quad \text{and} \quad P_s = \rho_s \rho' = \rho_s \mu - \frac{1}{2} \rho_s v_s^2.$$  \hspace{1em} (30)

In this case, the renormalized HVBK motion equations for incompressible flow reduce to

$$\partial_t v_s + (v_s \cdot \nabla) v_s = - \frac{1}{\rho} \nabla P' - \rho_n \rho F_{n,s} + T,$$  \hspace{1em} (31)

$$\partial_t v_n + (v_n \cdot \nabla) v_n = - \frac{1}{\rho} \nabla P' + \rho_s \rho F_{n,s}.$$  \hspace{1em} (32)

In these superfluid motion equations, the renormalized mutual friction force $F_{n,s}$ is defined as the sum (with $\omega = \text{curl } v_s$)

$$F_{n,s} = (s^0 \times \omega) + \left( \frac{\rho_s}{\rho} \right) F_{n,s}^0,$$  \hspace{1em} (33)

where $F_{n,s}^0 = \left( \frac{B}{2} \omega \times s^0 + \frac{B'}{2} s^0 \right) \times \omega$.

Here $F_{n,s}^0$ is the HVBK mutual friction force without any finite temperature corrections. To acquire these formulas, we used the relations for the incompressible case,

$$s = v_\ell - v_n = \frac{\rho_s}{\rho} (v_s + \lambda_0 \text{curl } \omega - v_n) = \frac{\rho_s}{\rho} (v_\ell - v_n) = \frac{\rho_s}{\rho} s^0,$$  \hspace{1em} (34)

with $s^0 = v_s + \lambda_0 \text{curl } \omega - v_n$, and we eliminated $v_L$ by using the relation

$$- \rho_s (v_L - v_s - \lambda_0 \text{curl } \omega) = \rho_n s + \frac{\rho_s \rho_n}{\rho} \left( \frac{B}{2} \omega \times s + \frac{B'}{2} s \right).$$  \hspace{1em} (35)

In equation (33) for $F_{n,s}$, the quantity $(s^0 \times \omega)$ is the Hamiltonian reactive force (which could be naturally absorbed into Vinen’s $B'$ parameter) and $F_{n,s}^0$ is the phenomenological mutual friction force defined according to the standard convention as in BK [1961] and Donnelly [1999]. The finite-temperature corrections contribute an overall factor of $\rho_s/\rho$ to the standard zero-temperature expression $F_{n,s}^0$ for the phenomenological mutual friction force. No new parameters are added, but a temperature dependent trade-off is identified between the renormalized mutual friction coupling and the vortex line tension, since the vortex line tension remains unaffected by the finite-temperature corrections.

In the isothermal case, the motion equations are closed by the Poisson equation for $P$, since the other coefficients ($B$, $B'$, $\rho_n/\rho$, etc.) are specified functions of temperature and they may be taken as constants, for an isothermal incompressible superfluid flow.
Note that equations (31-32) may be rewritten with $\omega_n = \text{curl} \mathbf{v}_n$ as

$$\partial_t \mathbf{v}_s + \nabla \mu_s = \mathbf{v}_n \times \omega + \frac{\rho_s}{\rho} \mathbf{s}^0 \times \omega - \frac{\rho_n \rho_s}{\rho^2} \mathbf{F}_{ns}^0 \quad \text{with} \quad \mu_s = P'/\rho + \frac{1}{2}v_s^2, \quad (36)$$

$$\partial_t \mathbf{v}_n + \nabla \mu_n = \mathbf{v}_n \times \omega_n + \frac{\rho_s}{\rho} \mathbf{s}^0 \times \omega + \frac{\rho^2}{\rho^2} \mathbf{F}_{ns}^0 \quad \text{with} \quad \mu_n = P'/\rho + \frac{1}{2}v_n^2. \quad (37)$$

These equations imply an equation for the velocity difference,

$$\partial_t (\mathbf{v}_s - \mathbf{v}_n) + \frac{1}{2} \nabla (v_s^2 - v_n^2) = \mathbf{v}_n \times (\omega - \omega_n) - \frac{\rho_s}{\rho} \mathbf{F}_{ns}^0 \quad \text{with} \quad \omega_n = \text{curl} \mathbf{v}_n. \quad (38)$$

and there is no tendency for mutual friction to cause any alignment in the vorticities of the superfluid and its normal component. Instead, the curl $\hat{\omega}$ part of $\mathbf{F}_{ns}^0 \neq 0$ would break any such alignment, if it were to form spontaneously. Indeed, alignments sufficient for steady solutions are

$$\mathbf{s}^0 \times (\nabla \mu_s \times \nabla \mu_n) = 0, \quad \mathbf{s}^0 \times \omega = 0 \quad \text{and} \quad \mathbf{s}^0 \times \mathbf{v}_n = 0, \quad \text{with} \quad \mathbf{s}^0 \equiv \mathbf{v}_s + \lambda_0 \text{curl} \hat{\omega} - \mathbf{v}_n, \quad (39)$$

provided $\mu_s$ and $\mu_n$ are functionally unrelated. Thus, the steady equilibrium alignments imposed by mutual friction involve $\mathbf{v}_n$, $\mathbf{v}_s$ and curl $\hat{\omega}$, as well as the independent gradients of $\mu_s$ and $\mu_n$. For example, one class of equilibria has $\mathbf{s}^0$, $\mathbf{v}_n$, $\omega$ all aligned tangent to intersections of level surfaces of $\mu_n$ and $\mu_s$.

## 3 Rotating frame renormalized HVBK equations

We transform to a rotating frame with relative velocities denoted with an asterisk as $\mathbf{v}_s^* = \mathbf{v}_s - \mathbf{R}(x)$, etc., and curl $\mathbf{R} = 2\Omega$. After a calculation involving Legendre transformations, we obtain the Hamiltonian for the relative motion, cf. the Hamiltonian in (51) of the Appendix,

$$h(M^*, \rho, S, \mathbf{u}, \mathbf{A}^*, n) = \int \left\{ \frac{1}{2} \rho |\mathbf{v}_s^* + \mathbf{R}(x)|^2 + (\mathbf{M}^* - \rho \mathbf{A}^* - \rho \mathbf{v}_n^*) \cdot \mathbf{v}_n^* \right. \right.$$

$$\left. + \epsilon_0 (\rho, S, \mathbf{v}_s^* - \mathbf{v}_n^*, \omega^* + 2\Omega) - \mathbf{R} \cdot \left[ \rho (\mathbf{v}_s^* + \mathbf{R}) + (\rho - n)(\mathbf{A}^* - \mathbf{R}) \right] \right\} d^3x. \quad (40)$$

Here $\mathbf{M}^* - \rho \mathbf{A}^* = \mathbf{J}^* = \mathbf{J} - \rho \mathbf{R}$ and $\mathbf{v}_s = \mathbf{u} - (\mathbf{A}^* - \mathbf{R})$. The equations resulting from the Lie-Poisson bracket (55) of the Appendix in these relative variables keep their forms and the condition $n = \rho$ is still preserved. We conclude with the following three remarks.

**Superfluid Coriolis force couples to the vortex line velocity.** The Hamiltonian evolution equation for the superfluid velocity in the rotating frame is expressed as

$$\partial_t \mathbf{v}_s^* + (\mathbf{v}_s^* \cdot \nabla) \mathbf{v}_s^* = -\nabla (\mu - \frac{1}{2} |\mathbf{v}_s^* - \mathbf{v}_n^*|^2 - \frac{1}{2} |\mathbf{R}|^2) + (\mathbf{v}_s^* - \mathbf{v}_n^*) \times \omega^* + \mathbf{v}_s^* \times 2\Omega. \quad (41)$$
The last term is the Coriolis force and it involves the relative vortex line velocity. The curl of this equation yields
\[
\partial_t (\omega^* + 2\Omega) = \text{curl} \left( \mathbf{v}_\ell^* \times (\omega^* + 2\Omega) \right). \tag{42}
\]
The form of the vortex dynamics equation is invariant under passing to a steadily rotating frame, and the superfluid Coriolis force contains the renormalized vortex line velocity, rather than the superfluid velocity. Therefore, this is not merely a kinematic force. The vortex line velocity appearing in the superfluid Coriolis force includes the interaction between the vortex lines and the superfluid component. It also includes the interaction with the normal component, since \( \mathbf{v}_\ell \) depends on the relative momentum density and contains finite temperature effects. The superfluid Coriolis force is essential in the spin up problem in He-II, see, e.g., Reissenegger [1993].

**Superfluid Taylor-Proudman theorem.** For steady, or slow motions and rapid rotation we have
\[
0 = \text{curl} \left( \mathbf{v}_\ell^* \times 2\Omega \right). \tag{43}
\]
If the rotation is uniform (\( \nabla \Omega = 0 \)) and oriented vertically (\( \Omega = |\Omega|\hat{z} \)) this becomes
\[
0 = 2|\Omega| \left( \partial_z \mathbf{v}_\ell^* - \hat{z} \text{div} \mathbf{v}_\ell^* \right) = 2|\Omega| \left( \partial_z v_{\ell x}^* , \partial_z v_{\ell y}^* , -\partial_x v_{\ell x}^* - \partial_y v_{\ell y}^* \right)^T , \tag{44}
\]
where ( \( \cdot \)^T denotes transpose of a row vector into a column vector. Thus, for steady, or slow motions and rapid uniform rotation, we find that vortex line motion is columnar. That is, the lateral vortex line velocity is nondivergent and independent of the axial coordinate, and the axial velocity decouples from the lateral motion. Therefore, under sufficiently rapid rotation, the superfluid vortex filaments will straighten and become parallel to the axis of rotation as they approach a steady state. However, they may still undergo nondivergent motion in the lateral plane. This superfluid Taylor-Proudman theorem explains why steady superfluid vortices tend to be aligned with the rotation axis under rapid uniform rotation. The same conclusion applies, if the velocity \( \mathbf{v}_\ell^* \) in the Hamiltonian formulation is replaced by the phenomenological relative velocity \( \mathbf{v}_L^* = \mathbf{v}_L - \mathbf{R} \). Similar considerations are discussed in Sonin [1987] from a more microscopic viewpoint.

**Relative total momentum is not conserved for rotating compressible flows.** Since the Hamiltonian depends explicitly on spatial position, instead of conserving relative total momentum, we have the balance
\[
\partial_t J_i^* + \partial_j T_i^{*j} = - \left. \frac{\partial h}{\partial \mathbf{v}_i} \right|_{\text{explicit}} = \frac{\rho}{2} \partial_i |\mathbf{R}|^2 , \tag{45}
\]
where \( h \) is the Hamiltonian density in equation (40). This relative momentum balance is the effect of centrifugal force. Here we have dropped terms proportional to \( \rho - n \), since \( \rho = n \) is still preserved in a rotating frame. Consequently, the stress tensor in the relative momentum equation also keeps its form in passing to a rotating frame, although the total relative momentum is no longer conserved if the flow is compressible.
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Appendix: Lie-Poisson Hamiltonian formulation

Conservation of the number of quantum vortices moving through superfluid $^4$He (and across the streamlines of the normal fluid component) is expressed by

$$\frac{d}{dt} \int_S \omega \cdot \hat{n} dS = 0,$$

(46)

where the superfluid vorticity $\omega$ is the areal density of vortices and $\hat{n}$ is the unit vector normal to the surface $S$ whose boundary $\partial S$ moves with the vortex line velocity $v_\ell$. When $\omega = \text{curl} \, v_s$ this is equivalent to a vortex Kelvin theorem

$$\frac{d}{dt} \oint_{\partial S(v_\ell)} v_s \cdot d\mathbf{x} = 0,$$

(47)

which in turn implies the fundamental relation

$$\partial_t v_s - v_\ell \times \omega = \nabla \mu.$$

(48)

The superfluid velocity naturally splits into $v_s = u - A$, where $u = \nabla \phi$ and (minus) the curl of $A$ yields the superfluid vorticity $\omega$. The phase $\phi$ is then a regular function without singularities. This splitting will reveal that the Hamiltonian dynamics of superfluid $^4$He with vortices may be expressed as an invariant subsystem of a larger Hamiltonian system in which $u$ and $A$ have independent evolution equations.

We begin by defining a phase frequency in the normal velocity frame as

$$\partial_t \phi + v_n \cdot \nabla \phi = \nu.$$

(49)

The mass density $\rho$ and the phase $\phi$ are canonically conjugate in the Hamiltonian formulation. Therefore, one may set $\nu = -\delta h/\delta \rho$ for a given Hamiltonian $h$ and the phase gradient $u = \nabla \phi$ satisfies

$$\partial_t u + v_n \cdot \nabla u + (\nabla v_n)^T \cdot u = -\nabla \frac{\delta h}{\delta \rho},$$

(50)
where \((\mathbf{v}_n)^T\) denotes transpose, so that \((\nabla \mathbf{v}_n)^T \cdot \mathbf{u} = u_j \nabla v_n^j\). The mass density \(\rho\) satisfies the dual equation

\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}_n) = -\nabla \cdot \left( \frac{\delta h}{\delta \mathbf{u}} \right). \tag{51}
\]

Perhaps not surprisingly, the rotational and potential components of the superfluid velocity \(\mathbf{v}_s = \mathbf{u} - \mathbf{A}\) satisfy similar equations, but the rotational component is advected by the vortex line velocity \(\mathbf{v}_\ell\), instead of the normal velocity \(\mathbf{v}_n\). Absorbing all gradients into \(\mathbf{u}\) yields

\[
\partial_t \mathbf{A} + \mathbf{v}_\ell \times \omega = 0. \tag{52}
\]

Taking the difference of the equations for \(\mathbf{u}\) and \(\mathbf{A}\) then recovers equation (48) as

\[
\partial_t \mathbf{v}_s - \mathbf{v}_\ell \times \omega = -\nabla \left( \mathbf{v}_n \cdot \mathbf{u} + \frac{\delta h}{\delta \rho} \right) \quad \text{with} \quad \mathbf{v}_s = \mathbf{u} - \mathbf{A}, \tag{53}
\]

in which one uses regularity of the phase \(\phi\) to set \(\text{curl} \mathbf{u} = 0\). It remains to determine \(\mathbf{v}_\ell\) from the Hamiltonian formulation. Including the additional degree of freedom \(\mathbf{A}\) associated with the vortex lines allows them to move relative to both the normal and super components of the fluid, and thereby introduces an additional reactive force without introducing any additional inertia. This Hamiltonian approach thus yields renormalized HVBK equations.

**Proposition:** Upon splitting the superfluid velocity into \(\mathbf{v}_s = \mathbf{u} - \mathbf{A}\) (with \(\mathbf{u} = \nabla \phi\) so that \(\omega = -\text{curl} \mathbf{A}\)) the (renormalized) HVBK equations in the Galilean frame of the normal fluid form an invariant subsystem of a Lie-Poisson Hamiltonian system,

\[
\frac{\partial f}{\partial t} = \{f, h\} \quad \text{with} \quad f, h \in (M, \rho, S, \mathbf{u}, \mathbf{A}, n), \tag{54}
\]

and Lie-Poisson bracket given by

\[
\{f, h\} = -\int \left\{ \frac{\delta f}{\delta M_j} \left[ (M_k \partial_j + \partial_k M_j) \frac{\delta h}{\delta M_k} + \rho \partial_j \frac{\delta h}{\delta \rho} + S \partial_j \frac{\delta h}{\delta S} + (\partial_k u_j - u_k, j) \frac{\delta h}{\delta u_k} \right] \\
+ \left[ \frac{\delta f}{\delta \rho} \partial_k \rho + \frac{\delta f}{\delta S} \partial_k S + \frac{\delta f}{\delta u_j} (u_k \partial_j + u_j, k) \right] \frac{\delta h}{\delta M_k} + \left[ \frac{\delta f}{\delta \rho} \partial_k \frac{\delta h}{\delta u_k} + \frac{\delta f}{\delta u_j} \frac{\delta h}{\delta \rho} \right] \\
- \frac{\delta f}{\delta A_j} \left[ \partial_j \frac{\delta h}{\delta n} + \frac{A_{j, k} - A_{k, j}}{n} \frac{\delta h}{\delta A_k} \right] - \frac{\delta f}{\delta n} \partial_k \frac{\delta h}{\delta A_k} \right\} d^3x. \tag{55}
\]

**Remarks:** Here \(M\) is the total momentum density, the total mass density is \(\rho\) and the entropy density is \(S\). We shall interpret the density \(n\) later, after we develop the Hamiltonian equations of motion. It shall emerge that \(n = \rho\) is an invariant condition and, hence,

\[
M - n \mathbf{A} = J = \mathbf{p} + \rho \mathbf{v}_n, \tag{56}
\]

for \(n = \rho\), where \(\mathbf{p} = J - \rho \mathbf{v}_n = \rho_s (\mathbf{v}_s - \mathbf{v}_n)\) is the relative momentum density of the superfluid in the Galilean frame of the normal fluid. The momentum density associated with the vortex fluid will be \(\mathbf{N} = -n \mathbf{A}\). The Hamiltonian will be the energy \(E\) in (7).
The Lie-Poisson bracket in the Proposition appeared first in Holm and Kupershmidt [1987] in a study of various approximate equations for the dynamics of multicomponent superfluids with charged condensates. The mathematical structure of this Lie-Poisson bracket and its association with the dual of a certain Lie algebra is discussed in Holm and Kupershmidt [1987]. Our re-interpretation of this Poisson bracket introduced and studied earlier shall now yield an extension of the HVBK equations that enables the vortex line velocity $v_\ell$ and hence the vortex reactive force and mutual friction force to be expressed at finite temperature. Identifying this Poisson bracket as being dual to a Lie algebra establishes that it satisfies the Jacobi identity, $\varepsilon^{ijk}\{f_i, \{f_j, f_k\}\} = 0$. The term in the Poisson bracket responsible for the reactive vortex force will turn out to be $\{A_i, A_j\} \neq 0$. The Poisson bracket $\{v_{si}, v_{sj}\}$ would vanish (as does $\{u_i, u_j\} = 0$) and thus the reactive vortex force would be absent, in any Hamiltonian formulation for which $A$ and $n$ were not independent degrees of freedom from $M, \rho, S$. Volovik and Dotsenko [1979, 1980] obtain a different result and provide no Lie-algebraic justification.

A Lagrangian formulation of these equations is also available. However, it involves an equation for $\partial_l/\partial \nu$ about which nothing is known physically.

**Corollary #1:** The Lie-Poisson bracket is equivalent to the following separate Hamiltonian matrix forms for the dynamical equations

\[
\frac{\partial}{\partial t} \begin{bmatrix} M_i \\ S \\ \rho \\ u_i \end{bmatrix} = - \begin{bmatrix} M_j \partial_i + \partial_j M_i & S \partial_i & \rho \partial_i & \partial_j u_i - u_{j,i} \\ \partial_j S & 0 & 0 & 0 \\ \partial_j \rho & 0 & 0 & \partial_j \\ u_j \partial_i + u_{i,j} & 0 & \partial_i & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta M_j \\ \delta h/\delta S \\ \delta h/\delta \rho \\ \delta h/\delta u_j \end{bmatrix},
\]

and, upon defining $N = -nA$,

\[
\frac{\partial}{\partial t} \begin{bmatrix} N_i \\ n \end{bmatrix} = - \begin{bmatrix} N_j \partial_i + \partial_j N_i & n \partial_i \\ \partial_j n & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta N_j \\ \delta h/\delta n \end{bmatrix}.
\]

These are individually expressed as

\[
\begin{align*}
\partial_t S &= \{S, h\} = - \text{div}(S \delta h/\delta M), \\
\partial_t n &= \{n, h\} = - \text{div}(n \delta h/\delta N), \\
\partial_t \rho &= \{\rho, h\} = - \text{div}(\rho \delta h/\delta M + \delta h/\delta u), \\
\partial_t u &= \{u, h\} = - \nabla(\delta h/\delta \rho + (\delta h/\delta M) \cdot u) + (\delta h/\delta M) \times \text{curl} u, \\
\partial_t (N/n) &= \{(N/n), h\} = - \nabla(\delta h/\delta n + (\delta h/\delta N) \cdot (N/n)) + (\delta h/\delta N) \times \text{curl} (N/n), \\
\partial_t (M_j + N_j) &= \{M_j + N_j, h\} = - \partial_k T_j^k.
\end{align*}
\]

**Corollary #2:** Consider a translation invariant Hamiltonian density with dependence

\[
h(M, \rho, S, n, v_s, \omega, A),
\]
where \( \mathbf{v}_s = \mathbf{u} - \mathbf{A} \), \( \mathbf{A} = -\mathbf{N}/n \) and \( \omega = \text{curl} \mathbf{v}_s \). The stress tensor \( T^k_j \) is expressed in terms of this Hamiltonian as

\[
T^k_j = M_j \frac{\partial h}{\partial M_k} + v_{s} \left( \frac{\partial h}{\partial v_{sk}} + \left( \text{curl} \frac{\partial h}{\partial \omega} \right)_{k} \right) - v_{s,l,j} \epsilon_{mlk} \frac{\partial h}{\partial \omega_{m}} + \delta_{j}^{k} P - A_{j} \frac{\partial h}{\partial A_{k}} \bigg|_{\mathbf{v}_s} .
\]

where

\[
P = M_{i} \frac{\partial h}{\partial M_{i}} + \rho \frac{\partial h}{\partial \rho} + S \frac{\partial h}{\partial S} + n \frac{\partial h}{\partial n} - h ,
\]

as in the Euler relation for pressure.

Remark. One notes many parallels and correspondences among these equations. Note especially the expected similarities in the equations for \( \mathbf{u} \) and \( \mathbf{N}/n \). Recall that \( \mathbf{A} = -\mathbf{N}/n \), so that the superfluid velocity is given by \( \mathbf{v}_s = \mathbf{u} - \mathbf{A} = \mathbf{u} + \mathbf{N}/n \). The evolution of the superfluid velocity is consistently composed as the sum of these two separate dynamical pieces.

Proof of the Proposition: The following Hamiltonian \( h \) (and conserved energy) will yield the HVBK equations in the frame of the normal fluid upon using this Lie-Poisson bracket

\[
h = \int d^{3}x \left[ -\frac{1}{2} \rho \mathbf{v}_{n}^{2} + (\mathbf{M} - \rho \mathbf{A}) \cdot \mathbf{v}_{n} + \varepsilon_{0}(\rho, S, \mathbf{v}_s - \mathbf{v}_n, \omega) \right] .
\]

The variational derivatives of the Hamiltonian \( h \) are computed in this reference frame by using the thermodynamic first law (6). Namely,

\[
\delta h = \int d^{3}x \left[ \left( \mu - \frac{1}{2} \mathbf{v}_{n}^{2} - \mathbf{A} \cdot \mathbf{v}_{n} \right) \delta \rho + T \delta S + \mathbf{v}_{n} \cdot \delta \mathbf{M} + (\mathbf{p} + \text{curl} \lambda) \cdot \delta \mathbf{u} 
- (\mathbf{p} + \text{curl} \lambda + \rho \mathbf{v}_{n}) \cdot \delta \mathbf{A} + (\mathbf{M} - \mathbf{p} - \rho \mathbf{v}_{n} - \rho \mathbf{A}) \cdot \delta \mathbf{v}_{n} \right].
\]

Here we have used the velocity split \( \delta \mathbf{v}_{s} = \delta \mathbf{u} - \delta \mathbf{A} \) and assumed the boundary condition \( \mathbf{n} \cdot \omega \times \lambda = 0 \) when integrating by parts. This boundary condition is satisfied identically, since \( \lambda = \lambda \hat{\omega} \) in the HVBK theory. Upon substituting these variational derivatives into the Lie-Poisson bracket, Corollary #1 yields the following equations expressed in the normal fluid reference frame,

\[
\partial_{t} S = \{ S, h \} = -\text{div}(S \mathbf{v}_{n}) ,
\]

\[
\partial_{t} n = \{ n, h \} = -\text{div}(\rho \mathbf{v}_{n} + \mathbf{p} + \text{curl} \lambda) ,
\]

\[
\partial_{t} \rho = \{ \rho, h \} = -\text{div}(\rho \mathbf{v}_{n} + \mathbf{p} + \text{curl} \lambda) ,
\]

(Hence, the condition \( n = \rho \) is preserved.)

\[
\partial_{t} \mathbf{u} = \{ \mathbf{u}, h \} = -\nabla \left( \mu - \frac{1}{2} \mathbf{v}_{n}^{2} + \mathbf{v}_{n} \cdot \mathbf{v}_{s} \right) + \mathbf{v}_{n} \times \text{curl} \mathbf{u} ,
\]

(Hence, \( \mathbf{v}_{t} = \mathbf{v}_{n} \) is preserved.)

\[
\partial_{t} \mathbf{A} = \{ \mathbf{A}, h \} = n^{-1}(\rho \mathbf{v}_{n} + \mathbf{p} + \text{curl} \lambda) \times \text{curl} \mathbf{A} ,
\]

(Hence, \( \mathbf{v}_{t} = \mathbf{v}_{n} \) when \( n = \rho \) is used.)

\[
\partial_{t} (M_{j} - n A_{j}) = \{ M_{j} - n A_{j}, h \} = -\partial_{k} T^{k}_{j}
\]
Remarks:

(1.) Preservation of the condition \( n = \rho \) by these equations allows the introduction of the momentum-carrying field \( A \) as an independent degree of freedom without introducing additional material inertia, provided the dynamically preserved condition \( n = \rho \) holds initially. This is reminiscent of the preservation of Gauss’s Law by the continuity equation for mass conservation in a fluid plasma.

(2.) The curl of the dynamical equation for the field \( A \) implies the vortex line velocity 
\[
\mathbf{v}_\ell = -\frac{1}{n} \frac{\delta h}{\delta A} = \mathbf{v} + \rho^{-1} \text{curl} \lambda, \quad \text{where} \quad \mathbf{v}_n = \mathbf{v} + \rho^{-1} \mathbf{p} = \rho^{-1} \mathbf{J}.
\]  
(62)
The velocity \( \mathbf{v} \) is the mass averaged velocity. The vortex slip velocity \( s \) corresponding to the vortex line velocity \( \mathbf{v}_\ell \) is the basis for the phenomenological reactive and mutual friction forces \( f \) and Rayleigh dissipation function \( R \) in the HVBK system. Namely, 
\[
s = \mathbf{v}_\ell - \mathbf{v}_n = \rho^{-1}(\mathbf{p} + \text{curl}\lambda), \quad \text{with} \quad \lambda = \lambda \hat{\omega}.
\]  
(63)
As expected, this expression agrees with BK [1961] at zero temperature. Note that the renormalized HVBK equations introduce no new parameters.

(3.) The corresponding equation for \( \mathbf{v}_s = \mathbf{u} - \mathbf{A} \) is then obtained as
\[
\partial_t \mathbf{v}_s = -\nabla(\mu - \frac{1}{2} |\mathbf{v}_s - \mathbf{v}_n|^2) + \mathbf{v}_\ell \times \omega, \quad \text{where} \quad \mathbf{v}_\ell = \mathbf{v} + \rho^{-1} \text{curl} \lambda \hat{\omega}.
\]  
(64)
This may be expressed equivalently in manifestly Galilean invariant form as 
\[
\partial_t \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = -\nabla(\mu - \frac{1}{2} |\mathbf{v}_s - \mathbf{v}_n|^2) + \mathbf{f}', \quad \text{where} \quad \mathbf{f}' = (\mathbf{v}_\ell - \mathbf{v}_s) \times \omega.
\]  
(65)
The term \( \mathbf{f}' \) is the Hamiltonian contribution to the reactive vortex force. This contribution would vanish if the vortex lines moved with the superfluid velocity.

(4.) The stress tensor \( T^k_j = \pi^k_j + \tau^k_j \) for total momentum conservation is given by summing 
\[
\pi^k_j = (\rho_s v_{sj} v_s^k + \rho_n v_{nj} v_n^k) + P \delta^k_j, \quad \tau^k_j = \partial_t \epsilon_{klm}(v_{sj} \lambda_m) - \lambda_j \omega^k + \omega \cdot \lambda \delta^k_j.
\]  
(66)
The divergence of \( \tau^k_j \) defines the vortex line tension \( T \) as 
\[
\partial_k \tau^k_j = -\omega \cdot \nabla \lambda + \nabla(\omega \cdot \lambda) = -\rho_s T + \nabla(\omega \cdot \lambda),
\]  
(67)
in the stress tensor \( \pi^k_j \) the pressure \( P \) is defined by the Euler relation, 
\[
P = -\varepsilon_0 + \mu \rho + TS,
\]  
(68)
so that in the normal-fluid frame the pressure satisfies 
\[
dP = \rho d\mu + SdT - \mathbf{p} \cdot d(\mathbf{v}_s - \mathbf{v}_n) - \lambda \cdot d\omega.
\]  
(69)
The stress tensor \( T^k_j = \pi^k_j + \tau^k_j \) may be derived by using Corollary #2 for the Hamiltonian formulation.
Implications of the HVBK vortex dynamics. The new Hamiltonian formulation of the renormalized HVBK equations presented in the Proposition provides a formula for the slip velocity of a vortex line element in a turbulent superfluid at finite temperature. Namely, for the Hamiltonian $h$ in equation (61), one finds

$$s = v_\ell - v_n = \rho^{-1}(p + \text{curl}\, \lambda).$$  \hspace{1cm} (70)

This formula for $v_\ell$ recovers the HVBK expression in BK [1961] at zero temperature. Otherwise, it provides an extension to finite temperature of the HVBK vortex force

$$f = (v_L - v_n) \times \omega, \quad \text{with} \quad v_L = v_n - \rho_s (\alpha s + \beta \hat{\omega} \times s),$$  \hspace{1cm} (71)

where the renormalized vortex slip velocity is given by

$$s = v_\ell - v_n = \frac{\rho_s}{\rho} (v_\ell^0 - v_n), \quad \text{for constant} \ \rho_s.$$  \hspace{1cm} (72)

The corresponding heating rate $R$ is given by

$$R = (\mathbf{J} - \rho v_n + \text{curl}\, \lambda) \cdot \omega \times (v_L - v_n) = \rho \rho_s \beta \omega |s \times \hat{\omega}|^2,$$  \hspace{1cm} (73)

which is positive, as it must be.

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