ON NONLOCAL VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES WITH FRACTIONAL GRADIENT

JOSÉ FRANCISCO RODRIGUES AND LISA SANTOS

Abstract. We extend classical results on variational inequalities with convex sets with gradient constraint to a new class of fractional partial differential equations in a bounded domain with constraint on the distributional Riesz fractional gradient, the \(\sigma\)-gradient \((0 < \sigma < 1)\). We establish continuous dependence results with respect to the data, including the threshold of the fractional \(\sigma\)-gradient. Using these properties we give new results on the existence to a class of quasi-variational variational inequalities with fractional gradient constraint via compactness and via contraction arguments. Using the approximation of the solutions with a family of quasilinear penalisation problems we show the existence of generalised Lagrange multipliers for the \(\sigma\)-gradient constrained problem, extending previous results for the classical gradient case, i.e., with \(\sigma = 1\).

1. Introduction

In a series of two interesting papers [13] and [14], Shieh and Spector have considered a new class of fractional partial differential equations. Instead of using the well-known fractional Laplacian, their starting concept is the distributional Riesz fractional gradient of order \(\sigma \in (0, 1)\), which will be called here the \(\sigma\)-gradient \(D^\sigma\), for brevity: for \(u \in L^p(\mathbb{R}^N)\), \(1 < p < \infty\), we set

\[
(D^\sigma u)_j = \frac{\partial^\sigma u}{\partial x_j^\sigma} = \frac{\partial}{\partial x_j} I_{1-\sigma} u, \quad 0 < \sigma < 1, \quad j = 1, \ldots, N,
\]

where \(\frac{\partial}{\partial x_j}\) is taken in the distributional sense, for every \(v \in C_0^\infty(\mathbb{R}^N)\),

\[
\left\langle \frac{\partial^\sigma u}{\partial x_j^\sigma}, v \right\rangle = -\left\langle I_{1-\sigma} u, \frac{\partial v}{\partial x_j} \right\rangle = - \int_{\mathbb{R}^N} (I_{1-\sigma} u) \frac{\partial v}{\partial x_j} \, dx,
\]

with \(I_\alpha\) denoting the Riesz potential of order \(\alpha\), \(0 < \alpha < 1\):

\[
I_\alpha u(x) = (I_\alpha * u)(x) = \gamma_{N,\alpha} \int_{\mathbb{R}^N} \frac{u(y)}{|x - y|^{N-\alpha}} \, dy, \quad \text{with } \gamma_{N,\alpha} = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\pi^{\frac{N}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}.
\]

As it was shown in [13], \(D^\sigma\) has nice properties for \(u \in \mathcal{C}_0^\infty(\mathbb{R}^N)\), namely

\[
D^\sigma u \equiv D(I_{1-\sigma} u) = I_{1-\sigma} * Du,
\]

\[
(-\Delta)^\sigma u = - \sum_{j=1}^N \frac{\partial^\sigma}{\partial x_j^\sigma} \frac{\partial^\sigma}{\partial x_j^\sigma} u,
\]
where the well-known fractional Laplacian may be given, for a suitable constant $C_{N,\sigma}$, by (see, for instance, [4]):

$$(-\Delta)^\sigma u \equiv C_{N,\sigma} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\sigma}} \, dy.$$  

It was also observed in [14] that the $\sigma$-gradient is an example of the non-local gradients considered in [9], which can be also given by

$$(1.4) \quad D^\sigma u(x) = R(-\Delta)^{\frac{\sigma}{2}} u(x) = (1 - \sigma - N)\gamma_{N,1-\sigma} \int_{\mathbb{R}^N} \frac{u(x) - u(y) \, x - y}{|x-y|^{N+\sigma} \, |x-y|} \, dy,$$

in terms of the vector-valued Riesz transform (see [15], with $\rho_N = \Gamma(N+\frac{1}{2})/\pi^{\frac{N+1}{2}}$):

$$Rf(x) = \rho_N \text{P.V.} \int_{\mathbb{R}^N} f(y) \frac{x - y}{|x-y|^{N+1}} \, dy.$$  

We observe that, from the properties of $D^\sigma$ and a result of [7] on the Riesz kernel as approximation of the identity as $\alpha \to 0$, the $\sigma$-gradient approaches the standard gradient as $\sigma \to 1$: if $Du \in L^p(\mathbb{R}^N)$, then $D^\sigma u \to Du$ in $L^p(\mathbb{R}^N)$.

Introducing the vector space of fractional differentiable functions as the closure of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{p,\sigma} = \|u\|_{L^p(\mathbb{R}^N)}^p + \|D^\sigma u\|_{(L^p(\mathbb{R}^N))^m}^p, \quad 0 < \sigma < 1, \quad p > 1,$$

by [13] Theorem 1.7] it is exactly the Bessel potential space $L^{\sigma,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N)$, $0 \leq s < \sigma$, where $W^{s,p}(\mathbb{R}^N)$ denotes the usual fractional Sobolev space. In [13] the solvability of the fractional partial differential equations with variable coefficients and Dirichlet data was treated in the case $p = 2$, as well as the minimization of the integral functionals of the $\sigma$-gradient with $p$-growth, leading to the solvability of a fractional $p$-Laplace equation of a novel type.

In this work we are concerned with the Hilbertian case $p = 2$ in a bounded domain $\Omega \subset \mathbb{R}^N$, with Lipschitz boundary, where the homogeneous Dirichlet problem for a general linear PDE with measurable coefficients is considered under an additional constraint on the $\sigma$-gradient. We shall consider all solutions in the usual Sobolev space

$$(1.5) \quad H_0^\sigma(\Omega), \quad \text{with norm } \|u\|_{H_0^\sigma(\Omega)} = \|D^\sigma u\|_{L^2(\Omega)^N}, \quad 0 < \sigma < 1,$$

which, by the Sobolev-Poincaré inequality, is equivalent to the usual Hilbertian norm induced from $L^{\sigma,2}(\mathbb{R}^N) = W^{\sigma,2}(\mathbb{R}^N) = H^\sigma(\mathbb{R}^N), 0 < \sigma < 1$ in the closure of the Cauchy sequences of functions in $C^\infty_0(\Omega)$ (see [13]).

For nonnegative functions $g \in L^\infty(\Omega)$, we consider the nonempty convex sets of the type

$$(1.6) \quad \mathbb{K}_g = \{ v \in H_0^\sigma(\Omega) : |D^\sigma v| \leq g \text{ a.e. in } \Omega \}.$$
Let $f \in L^1(\Omega)$ and $A : \Omega \to \mathbb{R}^{N \times N}$ be a measurable, bounded and positive definite matrix. We shall consider, in Section 2, the well-posedness of the variational inequality

(1.7) $u \in K^\sigma_g : \quad \int_\Omega AD^\sigma u \cdot D^\sigma (v - u) \geq \int_\Omega f(v - u), \quad \forall v \in K^\sigma_g.$

In particular, we obtain precise estimates for the continuous dependence of the solution $u$ with respect to $f$ and $g$, and so we extend well-known results for the classical case $\sigma = 1$ (see [12] and its references).

Extending the result of [2] for the gradient ($\sigma = 1$) case, we prove in Section 3 the existence of generalised Lagrange multipliers for the $\sigma$-gradient constrained problem. More precisely, we show the existence of $(\lambda, u) \in L^\infty(\Omega)' \times Y^\sigma_\infty(\Omega)$ such that

(1.8a) $\langle \lambda D^\sigma u, D^\sigma v \rangle_{L^\infty(\Omega)^N} + \int_\Omega AD^\sigma u \cdot D^\sigma v = \int_\Omega fv, \quad \forall v \in Y^\sigma_\infty(\Omega),$

(1.8b) $|D^\sigma u| \leq g \text{ a.e. in } \Omega, \quad \lambda \geq 0 \text{ and } \lambda(|D^\sigma u| - g) = 0 \text{ in } L^\infty(\Omega)'$

and, moreover, $u$ solves (1.7).

Here, for each $\sigma$, we have set

(1.9) $Y^\sigma_\infty(\Omega) = \{ v \in H^\sigma_0(\Omega) : D^\sigma v \in L^\infty(\Omega)^N \}, \quad 0 < \sigma < 1,$

and

$\langle \lambda \alpha, \beta \rangle_{L^\infty(\Omega)^N \times L^\infty(\Omega)^N} = \langle \lambda, \alpha \cdot \beta \rangle_{L^\infty(\Omega)' \times L^\infty(\Omega)} \quad \forall \lambda \in L^\infty(\Omega)' \forall \alpha, \beta \in L^\infty(\Omega)^N.$

Finally, in the Section 4 we consider the solvability of solutions to quasi-variational inequalities corresponding to (1.7) when the threshold $g = G[u]$ and therefore also the convex set (1.6) depend on the solution $u \in K^\sigma_g[u]$. We give sufficient conditions on the nonlinear and nonlocal operator $v \mapsto G[v]$ to obtain the existence of at least one solution $u$ of (1.7) with $K^\sigma_g$ replaced by $K^\sigma_{G[u]}$, by compactness methods, as in [6] for the case $\sigma = 1$.

In a special case, when $G[u](x) = \Gamma(u) \varphi(x)$ is strictly positive and separates variables with a Lipschitz functional $\Gamma : L^2(\Omega) \to \mathbb{R}^+$, we adapt an idea of [5] (see also [12]) to obtain, by a contraction principle, the existence and uniqueness of the solution of the quasi-variational inequality under the “smallness” of the product of $f$ with the Lipschitz constant of $\Gamma$ and the inverse of its positive lower bound.

2. The variational inequality with $\sigma$-gradient constraint

For some $a_*, a^* > 0$, let $A = A(x) : \Omega \to \mathbb{R}^{N \times N}$ be a bounded and measurable matrix, not necessarily symmetric, such that, for a.e. $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$:

(2.1) $a_* |\xi|^2 \leq A(x) \xi \cdot \xi \leq a^* |\xi|^2.$
Fixed $\nu > 0$, we define
\[(2.2) \quad L^\infty_\nu(\Omega) = \{ v \in L^\infty(\Omega) : v(x) \geq \nu > 0 \text{ a.e. } x \in \Omega \}.
\]

For any $g \in L^\infty_\nu(\Omega)$ it is clear that the convex set $K^\sigma_g$ defined in (1.6) is non-empty, closed and, by Sobolev embeddings, we have, using the notation (1.9), for all $0 < \beta < \sigma$:
\[(2.3) \quad K^\sigma_g \subset \Upsilon^\sigma(\Omega) \subset C^{0,\beta}(\overline{\Omega}) \subset L^\infty(\Omega),
\]
where $C^{0,\beta}(\overline{\Omega})$ is the space of Hölder continuous function with exponent $\beta$. Indeed, we recall (see for instance [3]) the embedding for the fractional Sobolev spaces $0 < \sigma \leq 1$, $1 < p < \infty$:
\[(2.4a) \quad W^{\sigma,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for every } q \leq \frac{Np}{N-\sigma p}, \text{ if } \sigma p < N,
\]
\[(2.4b) \quad W^{\sigma,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for every } q < \infty, \text{ if } \sigma p = N,
\]
\[(2.4c) \quad W^{\sigma,p}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^{0,\beta}(\overline{\Omega}), \quad \text{for every } 0 < \beta \leq \sigma - \frac{N}{p}, \text{ if } \sigma p > N,
\]
with continuous embeddings, which are also compact if also $q < \frac{Np}{N-\sigma p}$ in (2.4a) and $\beta < \sigma - \frac{N}{p}$ in (2.4c). In particular, we have
\[(2.5) \quad H^\sigma_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \quad \text{and} \quad L^{2^*}(\Omega) \hookrightarrow H^{-\sigma}(\Omega) = (H^\sigma_0(\Omega))', \quad 0 < \sigma < 1,
\]
where we set $2^* = \frac{2N}{N-2\sigma}$ and $2^* = \frac{2N}{N+2\sigma}$ when $\sigma < \frac{N}{2}$, and if $N = 1$ we denote $2^* = q$, $2^* = q' = \frac{q}{q-1}$ when $\sigma = \frac{1}{2}$ and $2^* = \infty$, $2^* = 1$ when $\sigma > \frac{1}{2}$.

Here we are also assuming that $\Omega \subset \mathbb{R}^N$ is an open, bounded domain with Lipschitz boundary, and we may conclude (2.3) from (2.4a)-(2.4c) by using a bootstrap argument.

Therefore, in the right hand side of the variational inequality (1.7), for $g_i \in L^\infty(\Omega)$, $g_i \geq 0$, we can take $f_i \in L^1(\Omega)$, and the first two theorems give continuous dependence results with precise estimates for two different problems with $i = 1, 2$:
\[(2.15)_i \quad u_i \in K^\sigma_{g_i} : \int_{\Omega} AD^\sigma u_i \cdot D^\sigma (v-u_i) \geq \int_{\Omega} f_i (v-u_i), \quad \forall v \in K^\sigma_{g_i}.
\]

**Theorem 2.1.** Under the assumptions (2.1), for each $f_i \in L^1(\Omega)$ and each $g_i \in L^\infty(\Omega)$, $g_i \geq 0$, there exists a unique solution $u_i$ to (2.15)$_i$ such that
\[(2.16) \quad u_i \in K^\sigma_{g_i} \cap C^{0,\beta}(\overline{\Omega}), \quad \text{for all } 0 < \beta < \sigma.
\]

When $g_1 = g_2$, the solution map $L^1(\Omega) \ni f \mapsto u \in H^\sigma_0(\Omega)$ is $\frac{1}{2}$-Hölder continuous, i.e., for some $C_1 > 0$, we have
\[(2.17) \quad \| u_1 - u_2 \|_{H^\sigma_0(\Omega)} \leq C_1 \| f_1 - f_2 \|_{L^1(\Omega)}^{\frac{1}{2}}.
\]
Moreover, if in addition \( f_i \in L^{2^*}(\Omega) \), \( i = 1, 2 \), \( 2^* \) defined in (2.5) and \( g_1 = g_2 \), then \( L^{2^*}(\Omega) \ni f \mapsto u \in H_0^\sigma(\Omega) \) is Lipschitz continuous:

\[
\|u_1 - u_2\|_{H_0^\sigma(\Omega)} \leq C^\# \|f_1 - f_2\|_{L^{2^*}(\Omega)},
\]

for \( C^\# = C_*/a_* > 0 \), where \( C_* \) is the constant of the Sobolev embedding \( H_0^\sigma(\Omega) \hookrightarrow L^{2^*}(\Omega) \).

**Proof.** Suppose that \( f_i \in L^{2^*}(\Omega) \subset H^{-\sigma}(\Omega) \). Since the assumption (2.1) implies that \( A \) defines a continuous bilinear and coercive form over \( H_0^\sigma(\Omega) \), the existence and uniqueness of the solution \( u_i \in K_1^\sigma \) to (2.15); is an immediate consequence of the Stampacchia Theorem (see, for instance, [11, p. 95]), and (2.16) follows from (2.3).

With our notation (1.5), the estimate (2.18) follows easily from (2.15); with \( g_1 = g_2 \) and \( v = u_j \) (\( i, j = 1, 2 \), \( i \neq j \)) from

\[
a_*\|\overline{u}\|_{H_0^\sigma(\Omega)}^2 \leq \int_\Omega AD^\sigma \overline{u} \cdot D^\sigma \overline{u} \leq \|\overline{f}\|_{L^{2^*}(\Omega)}\|\overline{u}\|_{L^{2^*}(\Omega)} \leq C_*\|\overline{f}\|_{L^{2^*}(\Omega)}\|\overline{u}\|_{H_0^\sigma(\Omega)},
\]

where we have set \( \overline{u} = u_1 - u_2 \) and \( \overline{f} = f_1 - f_2 \).

By (2.3), letting \( \kappa \) be such that

\[
\|v\|_{L^\infty(\Omega)} \leq \kappa, \quad \forall v \in K_{g_1}^\sigma,
\]

we may easily conclude the estimate (2.17) with \( C_1 = \sqrt{2\kappa/a_*} \) for \( f_1, f_2 \in L^{2^*}(\Omega) \subset L^1(\Omega) \) from (1.5); and

\[
a_*\|\overline{u}\|_{H_0^\sigma(\Omega)}^2 \leq \|\overline{f}\|_{L^1(\Omega)}\|\overline{u}\|_{L^\infty(\Omega)} \leq 2\kappa\|\overline{f}\|_{L^1(\Omega)}.
\]

Finally, the solvability of (2.15); for \( f_i \) only in \( L^1(\Omega) \) can be easily obtained by taking an approximating sequence of \( f^n_i \in L^{2^*}(\Omega) \) such that \( f^n_i \to f_i \) in \( L^1(\Omega) \) and using (2.17) for that (Cauchy) sequence. The proof is complete. \( \Box \)

**Remark 2.1.** As in [13] it is possible to extend the variational inequality with \( \sigma \)-gradient to arbitrary open domains \( \Omega \subset \mathbb{R}^N \) with a generalised Dirichlet data \( \varphi \in H^\sigma(\mathbb{R}^N) \) such that \( I_{1-\sigma} \ast \varphi \) is well-defined and \( D^\sigma \varphi \in L^\infty(\mathbb{R}^N) \). This would require in the definition (1.6) of \( K_g^\sigma \) to replace \( H_0^\sigma(\Omega) \) by the space

\[
H_\varphi^\sigma = \{ v \in H^\sigma(\mathbb{R}^N) : v = \varphi \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}
\]

and, in addition, technical compatibility assumptions on \( \varphi \) and \( g \) to guarantee that the new \( K_g^\sigma \) \( \neq \emptyset \).

**Remark 2.2.** It is well-known that if, in addition, \( A \) is symmetric, i.e. \( A = A^T \), the variational inequality (1.7) corresponds (and is equivalent) to the optimisation problem (see, for instance, [11])

\[
u \in K_g^\sigma : \quad \mathcal{J}(u) \leq \mathcal{J}(v), \quad \forall v \in K_g^\sigma,
\]
where $\mathcal{J} : \mathbb{K}^\sigma_g \to \mathbb{R}$ is the convex functional

$$\mathcal{J}(v) = \frac{1}{2} \int_{\Omega} AD^\sigma v \cdot D^\sigma v - \int_{\Omega} f v.$$ 

**Theorem 2.2.** Under the framework of the previous theorem, when $f_1 = f_2 \in L^1(\Omega)$, the solution map

$$L^\infty_\nu(\Omega) \ni g \mapsto u \in H^\sigma_0(\Omega)$$

is also $\frac{1}{2}$-Hölder continuous, i.e., there exists $C_\nu > 0$ such that

$$\|u_1 - u_2\|_{H^\sigma_0(\Omega)} \leq C_\nu \|g_1 - g_2\|_{L^\infty(\Omega)}^{\frac{1}{2}}.$$ 

**Proof.** Let $\eta = \|g_1 - g_2\|_{L^\infty(\Omega)}$ and, for $i, j = 1, 2$, $i \neq j$, notice that

$$u_{ij} = \frac{\nu}{\nu + \eta} u_i \in \mathbb{K}^\sigma_g,$$

if $u_i$ denotes the unique solution of $(2.15)_i$ to $g_i$ and $f_i$.

Denote by $\kappa = \max_{i=1,2}\{\|g_i\|_{L^\infty(\Omega)}, \|u_i\|_{L^\infty(\Omega)}\}$ and observe that for $i = 1, 2$,

$$|u_i - u_{ij}| + |D^\sigma(u_i - u_{ij})| \leq \frac{\eta}{\nu + \eta} (|u_i| + |D^\sigma u_i|) \leq 2\kappa \frac{\eta}{\nu}.$$ 

Hence, letting $v = u_{ij}$ in $(2.15)_j$ and using (2.1) we get

$$a_\nu \|u_1 - u_2\|_{H^\sigma_0(\Omega)}^2 \leq \int_{\Omega} AD^\sigma(u_1 - u_2) \cdot D^\sigma(u_1 - u_2)$$

$$\leq \int_{\Omega} AD^\sigma u_1 \cdot D^\sigma(u_2 - u_2) + \int_{\Omega} AD^\sigma u_2 \cdot D^\sigma(u_1 - u_1) + \int_{\Omega} f((u_1 - u_{12}) + (u_2 - u_{21}))$$

$$\leq 2\kappa \frac{\eta}{\nu}(M\|g_1\|_{L^1(\Omega)} + M\|g_2\|_{L^1(\Omega)} + 2\|f\|_{L^1(\Omega)}) = C_\nu^2\|g_1 - g_2\|_{L^\infty(\Omega)},$$

with $C_\nu = \sqrt{2\kappa(M\|g_1\|_{L^1(\Omega)} + M\|g_2\|_{L^1(\Omega)} + 2\|f\|_{L^1(\Omega)})/a_\nu > 0}$, where $M = \|A\|_{L^\infty(\Omega)N^2}$ which yields (2.20). \qed

**Remark 2.3.** Using the trick of the above proof, if $g_n \to g$ in $L^\infty(\Omega)$ for a sequence $g_n \in L^\infty_\nu(\Omega)$, it is clear that, for any $w \in \mathbb{K}^\sigma_g$ we can choose $w_n \in \mathbb{K}^\sigma_{g_n}$ such that $w_n \to w$ in $H^\sigma_0(\Omega)$. On the other hand, also for any sequence $w_n \to w$ in $H^\sigma_0(\Omega)$-weak, with each $w_n \in \mathbb{K}^\sigma_{g_n}$, $g_n \to g$ in $L^\infty(\Omega)$ implies that also $w \in \mathbb{K}^\sigma_g$. These two conditions determine that if $g_n \to g$ in $L^\infty_\nu(\Omega)$ then the respective convex sets $\mathbb{K}^\sigma_{g_n}$ converge in the Mosco sense to $\mathbb{K}^\sigma_g$. An open question is to extend this convergence to the case $0 < \sigma < 1$, by dropping the strict positivity condition on $g_n$ and $g$, as in [1] for $\sigma = 1$. 

3. Existence of Lagrange multipliers

In this section we prove the existence of solution of the problem (1.8a)-(1.8b).

For \( \varepsilon \in (0, 1) \) and denoting \( \hat{k}_\varepsilon = \hat{k}_\varepsilon(D^\sigma u^\varepsilon) = k_\varepsilon(|D^\sigma u^\varepsilon| - g) \) for simplicity, we define a family of approximated quasi-linear problems

\[
\int_\Omega \left( \hat{k}_\varepsilon(D^\sigma u^\varepsilon)D^\sigma u^\varepsilon + AD^\sigma u^\varepsilon \right) \cdot D^\sigma v = \int_\Omega f v \quad \forall v \in H^s_0(\Omega)
\]

where \( k_\varepsilon : \mathbb{R} \to \mathbb{R} \) is defined by

\[
k_\varepsilon(s) = 0 \text{ for } s < 0, \quad k_\varepsilon(s) = e^{\frac{s}{\varepsilon}} - 1 \text{ for } 0 \leq s \leq \frac{1}{\varepsilon}, \quad k_\varepsilon(s) = e^{1 - \varepsilon s} - 1 \text{ for } s > \frac{1}{\varepsilon}.
\]

Proposition 3.1. Suppose that \( g \in L^\infty_\nu(\Omega) \), \( f \in L^2#(\Omega) \) and \( A : \Omega \to \mathbb{R}^{N \times N} \) is a measurable, bounded and positive definite matrix. Then the quasi-linear problem (3.1) has a unique solution \( u^\varepsilon \in H^s_0(\Omega) \).

\[\square\]

Proof. The operator \( B_\varepsilon : H^s_0(\Omega) \to H^{-\sigma}(\Omega) \) defined by

\[
\langle B_\varepsilon v, w \rangle = \int_\Omega \left( \hat{k}_\varepsilon(D^\sigma v)D^\sigma v + AD^\sigma v \right) \cdot D^\sigma w
\]

is bounded, strongly monotone, coercive and hemicontinuous, so problem (3.1) has a unique solution (see, for instance, [8]).

Lemma 3.1. If \( g \in L^\infty_\nu(\Omega) \), \( f \in L^2#(\Omega) \), \( A : \Omega \to \mathbb{R}^{N \times N} \) is a measurable, bounded and positive definite matrix and \( 1 \leq q < \infty \), there exist positive constants \( C \) and \( C_q \) such that, for \( 0 < \varepsilon < 1 \), setting \( \hat{k}_\varepsilon = k_\varepsilon(|D^\sigma u^\varepsilon| - g) \), the solution \( u^\varepsilon \) of the approximated problem (3.1) satisfies

\[
\begin{align*}
(3.2a) & \quad \|\hat{k}_\varepsilon|D^\sigma u^\varepsilon|^2\|_{L^1(\Omega)} \leq C, \\
(3.2b) & \quad \|\hat{k}_\varepsilon\|_{L^1(\Omega)} \leq C, \\
(3.2c) & \quad \|\hat{k}_\varepsilon D^\sigma u^\varepsilon\|_{(L^\infty(\Omega))^N'} \leq C, \\
(3.2d) & \quad \|\hat{k}_\varepsilon\|_{L^\infty(\Omega)^N'} \leq C, \\
(3.2e) & \quad \|D^\sigma u^\varepsilon\|_{L^q(\Omega)^N} \leq C_q.
\end{align*}
\]

Proof. Using \( u^\varepsilon \) as test function in (3.1), we get

\[
\int_\Omega \left( \hat{k}_\varepsilon + a_* \right)|D^\sigma u^\varepsilon|^2 \leq \int_\Omega \hat{k}_\varepsilon|D^\sigma u^\varepsilon|^2 + AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon
\]

\[
= \int_\Omega f u^\varepsilon \leq \frac{C^2}{2a_*} \|f\|_{L^2#(\Omega)}^2 + \frac{a_*}{2} \|D^\sigma u^\varepsilon\|_{L^2(\Omega)^N}^2
\]
since \( A\xi \cdot \xi \geq a_* |\xi|^2 \) for any \( \xi \in \mathbb{R}^N \) by the assumptions on \( A \). But \( \hat{k}_\varepsilon \geq 0 \) and so
\[
\frac{a_*}{2} \int_{\Omega} |D^\sigma u^\varepsilon|^2 \leq \frac{C_{\#}^2}{2a_*} \| f \|_{L^2(\Omega)}^2,
\]
concluding then (3.2a).

Observing that the function \( \varphi_\varepsilon = \hat{k}_\varepsilon (t^2 - g^2) + g^2 \hat{k}_\varepsilon \geq \nu^2 \hat{k}_\varepsilon \) and using (3.2a), there exists a positive constant \( C \) independent of \( \varepsilon \) such that
\[
\nu^2 \int_{\Omega} \hat{k}_\varepsilon \leq C.
\]
This implies the uniform boundedness of \( \hat{k}_\varepsilon \) in \( L^1(\Omega) \) and also in \( L^\infty(\Omega) \), i.e., (3.2b) and (3.2d) respectively.

To prove (3.2c), it is enough to notice that, for \( \beta \in L^\infty(\Omega)^N \),
\[
\| \hat{k}_\varepsilon D^\sigma u^\varepsilon \|_{(L^\infty(\Omega)^N)'} = \sup_{\beta \in L^\infty(\Omega)^N} \int_{\Omega} \hat{k}_\varepsilon D^\sigma u^\varepsilon \cdot \beta \leq \left( \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \hat{k}_\varepsilon |\beta|^2 \right)^{\frac{1}{2}} \leq C \| \beta \|_{L^\infty(\Omega)^N}.
\]
Because for \( t - g > 0 \) we have \( \hat{k}_\varepsilon (t - g) \geq \frac{1}{m!} (t - g)^m \), for any \( m \in \mathbb{N} \), then using (3.2b) we conclude (3.2c). (for details see, for instance [10]).

\[\Box\]

**Proposition 3.2.** For \( g \in L^\infty(\Omega) \), \( f \in L^2(\Omega) \) and \( A : \Omega \to \mathbb{R}^{N \times N} \) a measurable, bounded and positive definite matrix, the family \( \{ u^\varepsilon \}_\varepsilon \) of solutions of the approximated problems (3.1) converges weakly in \( H^\sigma_0(\Omega) \) to the solution of the variational inequality (1.7).

**Proof.** The uniform boundedness of \( \{ u^\varepsilon \}_\varepsilon \) in \( H^\sigma_0(\Omega) \) implies that, at least for a subsequence,
\[
(3.3) \quad u^\varepsilon \xrightarrow{\varepsilon \to 0} u \quad \text{in} \quad H^\sigma_0(\Omega).
\]
For \( v \in K^\sigma_g \) we have, since \( \hat{k}_\varepsilon > 0 \) when \( |D^\sigma u^\varepsilon| > g \geq |D^\sigma v| \),
\[
\hat{k}_\varepsilon D^\sigma u^\varepsilon \cdot D^\sigma (v - u^\varepsilon) \leq \hat{k}_\varepsilon |D^\sigma u^\varepsilon| (|D^\sigma v| - |D^\sigma u^\varepsilon|) \leq 0
\]
and so, testing the first equation of (3.1) with \( v - u^\varepsilon \), we get
\[
\int_{\Omega} AD^\sigma u^\varepsilon \cdot D^\sigma (v - u^\varepsilon) \geq \int_{\Omega} f (v - u^\varepsilon).
\]
But
\[
\int_{\Omega} AD^\sigma u^\varepsilon \cdot D^\sigma (v - u^\varepsilon) = \int_{\Omega} AD^\sigma (u^\varepsilon - v) \cdot D^\sigma (v - u^\varepsilon) + \int_{\Omega} AD^\sigma v \cdot D^\sigma (v - u^\varepsilon) \leq \int_{\Omega} AD^\sigma v \cdot D^\sigma (v - u^\varepsilon)
\]
So, utilizing the weak convergence $u^\varepsilon \rightharpoonup u$ in $H^\sigma_0(\Omega)$,
\[
\int_\Omega AD^\sigma v \cdot D^\sigma (v - u) \geq \int_\Omega f(v - u).
\]
Let $w \in K^\sigma_g$ and setting $v = u + \theta(w - u)$, then $v \in K^\sigma_g$ for any $\theta \in (0, 1]$ and we get
\[
\theta \int_\Omega AD^\sigma (u + \theta(w - u)) \cdot D^\sigma (w - u) \geq \theta \int_\Omega f(w - u).
\]
Dividing this inequality by $\theta$ and letting $\theta \to 0$, we obtain (1.7). The proof is concluded if we show that $u \in K^\sigma_g$. Indeed we split $\Omega$ in three subsets
\[
U_\varepsilon = \{|D^\sigma u^\varepsilon| - g \leq \sqrt{\varepsilon}\}, \quad V_\varepsilon = \{\sqrt{\varepsilon} \leq |D^\sigma u^\varepsilon| - g \leq \frac{1}{\varepsilon}\}, \quad W_\varepsilon = \{|D^\sigma u^\varepsilon| - g > \frac{1}{\varepsilon}\}
\]
and, following the steps in [10], we conclude that
\[
\int_\Omega (|D^\sigma u| - g)^+ \leq \lim_{\varepsilon \to 0} \int_\Omega (|D^\sigma u^\varepsilon| - g) \vee 0 \wedge \frac{1}{\varepsilon}
= \lim_{\varepsilon \to 0} \left( \int_{U_\varepsilon} (|D^\sigma u^\varepsilon| - g) \vee 0 + \int_{V_\varepsilon} (|D^\sigma u^\varepsilon| - g) + \int_{W_\varepsilon} \frac{1}{\varepsilon} \right)
\leq \lim_{\varepsilon \to 0} \left( \sqrt{\varepsilon} |\Omega| + ||D^\sigma u^\varepsilon| - g|_{L^2(\Omega)} |V_\varepsilon|^{\frac{1}{2}} + \int_{W_\varepsilon} \frac{1}{\varepsilon} \right) \xrightarrow{\varepsilon \to 0} 0,
\]
because
\[
|V_\varepsilon| \leq \int_{V_\varepsilon} \frac{\tilde{k}_\varepsilon + 1}{\varepsilon^{\frac{1}{2}}} \leq Ce^{\sqrt{\varepsilon}} \xrightarrow{\varepsilon \to 0} 0 \quad \text{and} \quad \int_{W_\varepsilon} \frac{1}{\varepsilon} = \frac{1}{\varepsilon} \int_{W_\varepsilon} \frac{\tilde{k}_\varepsilon + 1}{\varepsilon^{\frac{1}{2}}} \leq \frac{C}{\varepsilon} e^{-\frac{1}{\varepsilon^{2}}} \xrightarrow{\varepsilon \to 0} 0.
\]
So $|D^\sigma u| \leq g$ a.e. in $\Omega$, which means that $u \in K^\sigma_g$.

The uniqueness of solution of the variational inequality (1.7) implies that the whole sequence $\{u^\varepsilon\}_\varepsilon$ converges to $u$ in $H^\sigma_0(\Omega)$.

\[\square\]

**Theorem 3.1.** If $g \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $A : \Omega \to \mathbb{R}^{N \times N}$ is a measurable, bounded and positive definite matrix, then problem (1.8a)-(1.8b) has a solution
\[
(\lambda, u) \in L^\infty(\Omega)' \times K^\sigma_\infty(\Omega).
\]

**Proof.** By estimates (3.2c) and (3.2d) and the Banach-Alaoglu-Bourbaki theorem we have, at least for a subsequence,
\[
\tilde{k}_\varepsilon D^\sigma u^\varepsilon \xrightarrow{\varepsilon \to 0} \Lambda \text{ weak in } (L^\infty(\Omega)^N)'
\]
and
\[
\tilde{k}_\varepsilon \xrightarrow{\varepsilon \to 0} \lambda \text{ weak in } L^\infty(\Omega)'.
\]

For $v \in H^\sigma_0(\Omega)$, since
\[ (\cdot) (3.4) \quad \int_\Omega \left( \tilde{k}_\varepsilon D^\sigma u^\varepsilon + AD^\sigma u^\varepsilon \right) \cdot D^\sigma v = \int_\Omega f v,
\]
we obtain, letting $\varepsilon \to 0$ with $v \in \mathcal{V}_\infty^\sigma(\Omega)$,

$$\langle \Lambda, D^\sigma v \rangle + \int_{\Omega} A D^\sigma u \cdot D^\sigma v = \int_{\Omega} f v.$$  

Taking $v = u^\varepsilon$ in (3.4) we get

$$\int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 + \int_{\Omega} A D^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon = \int_{\Omega} f u^\varepsilon$$

Observe first that

$$\int_{\Omega} A D^\sigma (u^\varepsilon - u) \cdot D^\sigma u^\varepsilon = \int_{\Omega} A D^\sigma (u^\varepsilon - u) \cdot D^\sigma (u^\varepsilon - u)$$

$$+ \int_{\Omega} A D^\sigma (u^\varepsilon - u) \cdot D^\sigma u \geq \int_{\Omega} A D^\sigma (u^\varepsilon - u) \cdot D^\sigma u$$

and therefore

$$\int_{\Omega} A D^\sigma u \cdot D^\sigma u \leq \lim_{\varepsilon \to 0} \int_{\Omega} A D^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon.$$ 

So, from (3.6) and (3.5) with $v = u$,

$$\lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 + \int_{\Omega} A D^\sigma u \cdot D^\sigma u \leq \lim_{\varepsilon \to 0} \left( \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 + \int_{\Omega} A D^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon \right)$$

$$= \int_{\Omega} f u = \langle \Lambda, D^\sigma u \rangle + \int_{\Omega} A D^\sigma u \cdot D^\sigma u$$

and then

$$\lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 \leq \langle \Lambda, D^\sigma u \rangle.$$ 

Using $\hat{k}_\varepsilon(|D^\sigma u^\varepsilon|^2 - g^2) \geq 0$, we obtain

$$\langle \Lambda, D^\sigma u \rangle \geq \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 \geq \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon g^2 = \langle \lambda, g^2 \rangle \geq \langle \lambda, |D^\sigma u|^2 \rangle.$$ 

We also have

$$0 \leq \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma (u^\varepsilon - u)|^2 = \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 - 2 \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon D^\sigma u^\varepsilon \cdot D^\sigma u + \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u|^2$$

$$\leq \langle \Lambda, D^\sigma u \rangle - 2 \langle \Lambda, D^\sigma u \rangle + \langle \lambda, |D^\sigma u|^2 \rangle$$

$$= -\langle \Lambda, D^\sigma u \rangle + \langle \lambda, |D^\sigma u|^2 \rangle,$$

and therefore we conclude

$$\langle \Lambda, D^\sigma u \rangle = \langle \lambda, |D^\sigma u|^2 \rangle \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma (u^\varepsilon - u)|^2 = 0.$$
Given \( v \in K_g \), we have

\[
(3.8) \quad \lim_{\varepsilon \to 0} \left| \int_{\Omega} \hat{k}_\varepsilon D^\sigma (u^\varepsilon - u) \cdot D^\sigma v \right| \leq \lim_{\varepsilon \to 0} \left( \int_{\Omega} \hat{k}_\varepsilon |D^\sigma (u^\varepsilon - u)|^2 \right)^{\frac{1}{2}} \| \hat{k}_\varepsilon \|_{L^1(\Omega)}^\frac{1}{2} \| D^\sigma v \|_{L^\infty(\Omega)} = 0,
\]

because, by estimate (3.2b), \( \hat{k}_\varepsilon \) is uniformly bounded in \( L^1(\Omega) \). So, for any \( v \in K_g \),

\[
\int_{\Omega} f v = \lim_{\varepsilon \to 0} \int_{\Omega} (\hat{k}_\varepsilon + A) D^\sigma u^\varepsilon \cdot D^\sigma v = \lim_{\varepsilon \to 0} \left( \int_{\Omega} (\hat{k}_\varepsilon + A) D^\sigma (u^\varepsilon - u) \cdot D^\sigma v \right)
+ \lim_{\varepsilon \to 0} \int_{\Omega} (\hat{k}_\varepsilon + A) D^\sigma u \cdot D^\sigma v
= \langle \lambda D^\sigma u, D^\sigma v \rangle + \int_{\Omega} A D^\sigma u \cdot D^\sigma v,
\]

concluding the proof of (1.8a).

Since \( \int_{\Omega} \hat{k}_\varepsilon v \geq 0 \) for all \( v \in L^\infty(\Omega) \) such that \( v \geq 0 \) then, for such \( v \), we also have \( \langle \lambda, v \rangle \geq 0 \), which means that \( \lambda \geq 0 \).

For \( v \in L^\infty(\Omega) \) set \( v^+ = \max\{v, 0\} \), \( v^- = (-v)^+ \). Since \( \hat{k}_\varepsilon (|D^\sigma u^\varepsilon|^2 - g^2) \geq 0 \) then

\[
\langle \lambda, g^2 v^\pm \rangle \leq \lim_{\varepsilon \to 0} \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 v^\pm
= \lim_{\varepsilon \to 0} \left( \int_{\Omega} \hat{k}_\varepsilon D^\sigma (u^\varepsilon - u)^2 v^\pm - 2 \int_{\Omega} \hat{k}_\varepsilon D^\sigma (u^\varepsilon - u) \cdot D^\sigma uv^\pm + \int_{\Omega} \hat{k}_\varepsilon |D^\sigma u|^2 v^\pm \right)
= \langle \lambda, |D^\sigma u|^2 v^\pm \rangle , \quad \text{using (3.8)},
\]

concluding that

\[
\langle \lambda, (|D^\sigma u|^2 - g^2) v^\pm \rangle \geq 0.
\]

The fact that \( \hat{k}_\varepsilon \geq 0 \) and \( u \in K_g^\sigma \) imply \( \hat{k}_\varepsilon (|D^\sigma u|^2 - g^2) v^\pm \leq 0 \) and, therefore, integrating and letting \( \varepsilon \to 0 \), \( \langle \lambda, (|D^\sigma u|^2 - g^2) v^\pm \rangle \leq 0 \), and so

\[
\langle \lambda, (|D^\sigma u|^2 - g^2) v \rangle = 0.
\]

Writing \( v = \frac{w}{|D^\sigma u| + g} \), for any \( w \in L^\infty(\Omega) \), we conclude (1.8b).

\[\square\]

4. The quasi-variational inequality with \( \sigma \)-gradient constraint

In this section we consider a map \( G \) such that

\[
(4.1) \quad G : L^{2^*}(\Omega) \to L^\infty(\Omega)
\]

is a continuous and bounded operator, where \( 2^* \) is the Sobolev exponent as in (2.5) for \( 0 < \sigma < 1 \).

We set

\[
(4.2) \quad K^\sigma_{G[u]} = \{ v \in H^\sigma_0(\Omega) : |D^\sigma v| \leq G[u] \text{ a.e. in } \Omega \}
\]
and we shall consider the quasi-variational inequality

\[
(4.3) \quad u \in K^\sigma_{G[u]} : \quad \int_{\Omega} AD^\sigma u \cdot D^\sigma (v - u) \geq \int_{\Omega} f(v - u), \quad \forall v \in K^\sigma_{G[u]}. \tag{4.3}
\]

Generalising a compactness argument of [6] where quasi-variational inequalities of this type were considered for the gradient case \(\sigma = 1\), we may give a general existence theorem.

**Theorem 4.1.** Under the assumptions \((2.1)\), for continuous and bounded operators \(G\) satisfying \((4.1)\) and for any \(f \in L^{2^*}(\Omega)\), with \(2^*\) as in \((2.5)\), there exists at least one solution for the quasi-variational inequality \((4.3)\).

**Proof.** Let \(u = S(f,g)\) be the unique solution of the variational inequality \((1.7)\) with \(g = G[w]\) for any \(w \in L^2(\Omega)\). If \(C_\sigma > 0\) denotes the Sobolev constant as in Theorem \(2.1\), since \(f_2 = 0\) corresponds always to the solution \(u_2 = 0\), we have the a priori estimate

\[
(4.4) \quad \|u\|_{L^{2^*}(\Omega)} \leq C_\sigma \|u\|_{H^\sigma_0(\Omega)} \leq \frac{C_\sigma}{\alpha} \|f\|_{L^{2^*}(\Omega)} \equiv c_f,
\]

independently of \(g \in L^{2^*}_\nu(\Omega)\).

Set \(B_{c_f} = \{ v \in L^2(\Omega) : \|v\|_{L^{2^*}(\Omega)} \leq c_f \}\) and define the nonlinear map \(T = S \circ G : L^2(\Omega) \ni w \mapsto u \in L^2(\Omega)\) where \(u = S(f,G[w]) \in K^\sigma_{G[w]} \cap C^{0,\beta}(\Omega),\) \(0 < \beta < \sigma\) by \((2.16)\).

Clearly, \((4.4)\) implies \(T(B_{c_f}) \subset B_{c_f}\) and, by the continuity of \(G\) and Theorem \(2.2\), \(T\) is also a continuous map. On the other hand, \(G\) is bounded, i.e. transforms bounded sets in \(L^2(\Omega)\) into bounded sets of \(L^\infty(\Omega)\) and \(S \circ T\) is also a bounded operator. Therefore, by \((2.16)\), \(T(B_{c_f})\) is also a bounded set of \(C^{0,\beta}(\Omega)\). Since the embedding \(C^{0,\beta}(\Omega) \hookrightarrow L^2(\Omega)\) is compact, the Schauder fixed point theorem guarantees the existence of \(u = Tu\), which solves \((4.3)\).

**Example 4.1.** Consider the operator \(G : L^2(\Omega) \to L^\infty(\Omega)\) defined as follows:

\[
(4.5) \quad G[u](x) = F(x,w(x)),
\]

where \(F : \Omega \times \mathbb{R} \to \mathbb{R}\) is a function bounded in \(x \in \Omega\) and continuous in \(w \in \mathbb{R}\), uniformly in \(x \in \Omega\), satisfying, for some \(\nu > 0\),

\[
(4.6) \quad 0 < \nu \leq F(x,w) \leq \varphi(|w|) \quad \text{a.e. } x \in \Omega,
\]

and for some monotone increasing function \(\varphi\). We may choose

\[
(4.7) \quad w(x) = \int_{\Omega} \vartheta(x,y)u(y) \, dy,
\]

where we give \(\vartheta \in L^\infty(\Omega_x, L^{2^*}(\Omega_y))\). For \(u_n \to u\) in \(L^2(\Omega)\), from the estimate

\[
\sup_{x \in \Omega} |w_n(x) - w(x)| = \sup_{x \in \Omega} \left| \int_{\Omega} \vartheta(x,y)(u_n(y)-u(y)) \, dy \right| \leq \sup_{x \in \Omega} \|\vartheta(x,\cdot)\|_{L^{2^*}(\Omega)} \|u_n - u\|_{L^2(\Omega)}
\]
and by the uniform continuity of $F$, we have

$$
\|G[u_n] - G[u]\|_{L^\infty(\Omega)} = \|F(w_n) - F(w)\|_{L^\infty(\Omega)} \to 0,
$$

implying the continuity of $G$.

The boundedness of $G$ is a consequence of (4.6) and therefore $G$ satisfies the assumptions of Theorem 4.1.

**Example 4.2.** Consider now the operator $G : H^\sigma_0(\Omega) \to L^\infty_\nu(\Omega)$ given also by (4.5) with $F$ under the same assumptions as in the previous example, but now with

$$
(4.8)
$$

where $\Theta \in C(\overline{\Omega} \times; L^2(\Omega)^N)$. Now $G$ is not only bounded but also completely continuous, since $\Phi : H^\sigma_0(\Omega) \to C(\overline{\Omega})$ is also completely continuous. Indeed, if $u_n \rightharpoonup u$ in $H^\sigma_0(\Omega)$-weak, then $w_n = \Phi(u_n) \rightharpoonup \Phi(u) = w$ in $C(\overline{\Omega})$, because $\{D^\sigma u_n\}_n$, being bounded in $L^2(\Omega)^N$ implies $\{w_n\}_n$ uniformly bounded in $C(\overline{\Omega})$,

$$
|w_n(x)| \leq \|\Theta(x,\cdot)\|_{L^2(\Omega)^N} \|D^\sigma u_n\|_{L^2(\Omega)^N}, \quad \forall x \in \overline{\Omega}
$$

and also equicontinuous in $\overline{\Omega}$ by

$$
|w_n(x) - w_n(z)| \leq C\|\Theta(x,\cdot) - \Theta(z,\cdot)\|_{L^2(\Omega)^N}.
$$

But $G$ is not defined in the whole $L^2(\Omega)$ and therefore we cannot apply Theorem 4.1 to solve (4.3). Nevertheless, the solvability of (4.3) in this example is an immediate consequence of the following theorem.

**Theorem 4.2.** Assume (2.1) and let $f \in L^2(\Omega)$ as previously. If the nonlinear and nonlocal operator $G$ satisfies

$$
(4.9)
$$

then there exists a solution $u$ to the quasi-variational inequality (4.3).

**Proof.** Due to the estimate (4.4) and the assumption (4.9), the proof is analogous by applying the Schauder fixed point theorem to the nonlinear completely continuous map

$$
T = S \circ G : H^\sigma_0(\Omega) \ni w \mapsto u = S(f, G[w]) \in H^\sigma_0(\Omega).
$$

□

**Example 4.3.** By restricting the domain of $G$ and using the same type of Carathéodory function $F$ as in Example 4.1, we can introduce the superposition operator

$$
(4.10)
$$

where $u \in C(\overline{\Omega})$, $x \in \Omega$. 
In order to guarantee that $G : C(\overline{\Omega}) \to L^\infty_\nu(\Omega)$ is a continuous and bounded operator in an appropriate space to obtain a fixed point, we need to require that the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded function in $x \in \Omega$ in each compact for the variable $u$, continuous in $u \in \mathbb{R}$ uniformly in $x \in \Omega$, and satisfying (4.6), where the monotone increasing function $\varphi$ satisfies

\begin{equation}
0 < \nu \leq \varphi(t) \leq C_0 + C_1 t^{2^*/p}, \quad t \in \mathbb{R},
\end{equation}

for some $p > \frac{N}{\sigma}$ and $2^*$ the Sobolev exponent as in (2.5).

This situation is covered by the next theorem, since the assumption (4.11) implies the condition (4.13) below.

**Theorem 4.3.** Assume (2.1), let $f \in L^2(\Omega)$ and the functional $G$ be such that

\begin{equation}
G : C^0(\overline{\Omega}) \to L^\infty_\nu(\Omega)
\end{equation}

is a continuous operator and satisfying, for some positive monotone increasing function $\eta$,

\begin{equation}
\|G[w]\|_{L^p(\Omega)} \leq \eta(\|w\|_{L^{2^*}(\Omega)})
\end{equation}

for some $p > \frac{N}{\sigma}$ and $2^*$ the Sobolev exponent of $H^\sigma_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$. Then there exists a solution of the quasi-variational inequality (4.3).

**Proof.** As before, we set $T = S \circ G : C^0(\overline{\Omega}) \to H^\sigma_0(\Omega)$, where $u = S(f, G[w])$, for $w \in C^0(\overline{\Omega})$ solves (1.7) with $g = G[w]$. In order to apply the Leray-Schauder principle, we set

\[ \mathcal{S} = \{w \in C^0(\overline{\Omega}) : w = \theta T w, \theta \in [0, 1]\} \]

and we show that $\mathcal{S}$ is a priori bounded. For any $w \in \mathcal{S}$, $u = Tw$ solves (1.7) with $g = G[w]$. Hence, by (2.4c) and the assumption (4.13) we have, noting that $w = \theta u$,

\[ \|w\|_{C^0(\overline{\Omega})} \leq C_\sigma \|D^\sigma w\|_{L^p(\Omega)^N} \leq C_\sigma \theta \|G[w]\|_{L^p(\Omega)^N} \leq C_\sigma \eta(\|w\|_{L^{2^*}(\Omega)}) \leq C_\sigma \eta(c_f), \]

by the a priori estimate (4.4).

Since, by (2.3), $T(C^0(\overline{\Omega})) \hookrightarrow C^{0, \beta}(\overline{\Omega}) \hookrightarrow C^0(\overline{\Omega})$ and this last embedding is compact, we may conclude that $T$ is a completely continuous mapping into a closed ball of $C^0(\overline{\Omega})$ and its fixed point $u = Tu$ solves (4.3). \[\square\]

It is clear that in general we cannot expect the uniqueness of solution to quasi-variational inequalities of the type (4.3). However, the Lipschitz continuity of the solution map $f \mapsto u$ to the variational inequality (1.7), given by Theorem 2.1 allows us to obtain, via the strict contraction Banach fixed point principle, a uniqueness result in a special case of “small”
and controlled variations of the convex sets for the quasi-variational situation with separation of variables in the nonlocal constraint $G$.

We denote, for $R > 0$,

$$B_R = \{ v \in H_0^\sigma(\Omega) : \|v\|_{H_0^\sigma(\Omega)} \leq R \}.$$

**Theorem 4.4.** Let $f \in L^2#(\Omega)$, $\varphi \in L^\infty(\Omega)$ and

$$G[u](x) = \varphi(x)\Gamma(u), \quad x \in \Omega,$$

where $\Gamma : H_0^\sigma(\Omega) \to \mathbb{R}^+$ is a functional satisfying

i) $0 < \eta(R) \leq \Gamma(u) \leq E(R)$, $\forall u \in B_R$,

ii) $|\Gamma(u_1) - \Gamma(u_2)| \leq \gamma(R)\|u_1 - u_2\|_{H_0^\sigma(\Omega)}$, $\forall u_1, u_2 \in B_R$,

for sufficiently large $R \in \mathbb{R}^+$, with $\eta, E$ and $\gamma$ being monotone increasing positive functions of $R$.

Then the quasi-variational inequality (4.3) has a unique solution, provided

$$2C\#\frac{\gamma(R_f)}{\eta(R_f)} \|f\|_{L^2#(\Omega)} < 1,$$

where $R_f \equiv C\#\|f\|_{L^2#(\Omega)}$ with $C\# = C_*/a_*$ and $C_*$ is the constant of the Sobolev embedding as in (4.4).

**Proof.** Let $S : B_R \ni v \mapsto u \in H_0^\sigma(\Omega)$ be the solution map with $u = S(f, G[v])$ being the unique solution of the variational inequality (1.7) with $g = G[v]$.

The a priori estimate (4.4) implies $S(B_{R_f}) \subset B_{R_f}$.

Given $v_i \in B_{R_f}$, let $u_i = S(v_i) = S(f, \varphi \Gamma(v_i))$, $i = 1, 2$, and choose $\mu = \frac{\Gamma(v_2)}{\Gamma(v_1)} > 1$, without loss of generality.

Setting $g = \varphi \Gamma(v_1)$, we have $\mu g = \varphi \Gamma(v_2)$ and

$$S(\mu f, \mu g) = \mu S(f, g),$$

$$\mu - 1 = \frac{\Gamma(v_2) - \Gamma(v_1)}{\Gamma(v_1)} \leq \frac{\gamma(R_f)}{\eta(R_f)} \|v_1 - v_2\|_{\sigma}$$

by recalling the assumptions i) and ii) and denoting $\|w\|_{\sigma} = \|w\|_{H_0^\sigma(\Omega)}$ for simplicity.

Consequently, using (4.4) and (2.18) with $f_1 = f$ and $f_2 = \mu f$, we have

$$\|S(v_1) - S(v_2)\|_{\sigma} \leq \|S(f, g) - S(\mu f, \mu g)\|_{\sigma} + \|S(\mu f, \mu g) - S(f, \mu g)\|_{\sigma}$$

$$\leq (\mu - 1)\|u_1\|_{\sigma} + (\mu - 1)C\#\|f\|_{L^2#(\Omega)}$$

$$\leq 2C\#(\mu - 1)\|f\|_{L^2#(\Omega)}$$

$$\leq 2C\#\frac{\gamma(R_f)}{\eta(R_f)}\|v_1 - v_2\|_{\sigma}\|f\|_{L^2#(\Omega)}$$

and the conclusion of the theorem follows immediately. \qed
**Example 4.4.** We can take $\Gamma$ of the form

$$\Gamma(u) = \int_{\Omega} e(y, u(y), D^\sigma u(y)) \, dy, \quad u \in H^\sigma_0(\Omega),$$

with $e : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [\eta, \infty)$, for some $\eta > 0$, under a local Lipschitz condition of the type

$$|e(y, v, \xi) - e(y, w, \zeta)| \leq \gamma(R)(|v - w| + |\xi - \eta|)$$

for $|v|$, $|w|$, $|\xi|$ and $|\zeta|$ less or equal to $R$.

**Remark 4.1.** Assumptions i) and ii) have been used in Appendix B of [5] under the implicit assumptions of smallness of the term $f$, and in [12] in a simplified and more precise form in the case of gradient type (i.e. $\sigma = 1$) and for a class of general operators of $p$-Laplacian type.

**Remark 4.2.** The existence of solution of the quasi-variational inequality (4.3) is obtained in this section by finding a fixed point of the map $w \mapsto S(f, G[w]) = u$, under suitable assumptions. But when $u = S(f, G[w])$ is the solution of (1.7) then there exists $\lambda \in L^\infty(\Omega)'$ such that $(u, \lambda)$ solves problem $\text{(1.8a)}$-$(\text{1.8b})$ with data $(f, G[w])$. In particular, when $u$ is a fixed point $u = S(f, G[u])$ it solves the quasi-variational inequality, and we immediately get existence of a solution $(\lambda, u)$ of problem $\text{(1.8a)}$-$(\text{1.8b})$ for the quasi-variational case.

**Acknowledgements**

The authors would like to thank to the Reviewers for their helpful comments.

The research of J. F. Rodrigues was partially done under the framework of the project PTDC/MAT-PUR/28686/2017 at CMAFcIO/ULisboa and L. Santos was partially supported by the Centre of Mathematics the University of Minho through the Strategic Project PEst UID/MAT/00013/2013.

**References**

[1] Azevedo, A. and Santos, L.: *Convergence of convex sets with gradient constraint*, Journal of Convex Analysis 11 (2004) 285–301.

[2] Azevedo, A. and Santos, L.: *Lagrange multipliers and transport densities*, J. Math. Pures Appl. 108 (2017) 592–611.

[3] Demengel, F. and Demengel, G., Functional spaces for the theory of elliptic partial differential equations, *Springer, London, EDP Science, Les Ulis*, 2012.

[4] Di Nezza, E., Palatucci, G. and Valdinoci, E., *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012) 521–573.

[5] Hintermüller, M. and Rautenberg, C., *A sequential minimization technique for elliptic quasi-variational inequalities with gradient constraints*, SIAM J. Optim. 22 (2013) 1224–1257.
[6] Kunze, M. and Rodrigues, J.-F.: An elliptic quasi-variational inequality with gradient constraints and some of its applications, Math. Methods Appl. Sci. 23 (2000) 897–908.

[7] Kurokawa, T., On the Riesz and Bessel kernels as approximations of the identity, Sci. Rep. Kagoshima Univ. 30 (1981) 31–45.

[8] Lions, J.-L., Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier-Villars, Paris 1969.

[9] Mengesha, T. and Spector, D., Localization of nonlocal gradients in various topologies, Calc. Var. Partial Differential Equations 52 (2015) 253–279.

[10] Miranda, F., Rodrigues, J.-F. and Santos, L.: On a p-curl system arising in electromagnetism, Discrete Contin. Dyn. Syst Ser. S 5 (2012) 605–629.

[11] Rodrigues, J.-F., Obstacle problems in mathematical physics. North-Holland Mathematics Studies 134, 1987.

[12] Rodrigues, J.-F. and Santos, L., Variational and quasi-variational inequalities with gradient type constraints in Topics in Applied Analysis and Optimisation, Proceedings of the CIM-WIAS Workshop, CIM-Springer Series, 2019.

[13] Shieh, T. and Spector, D., On a new class of fractional partial differential equation, Adv. Calc. Var. 8 (2015) 321–336.

[14] Shieh, T. and Spector, D., On a new class of fractional partial differential equations II, Adv. Calc. Var. 11 (2018) 289–307.

[15] Stein, E., Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, 1970.

CMAFcIO – Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa P-1749-016 Lisboa, Portugal

Email address: jfrodrigues@ciencias.ulisboa.pt

CMAT and Departamento de Matemática, Escola de Ciências, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal

Email address: lisa@math.uminho.pt