Sturm Liouville Equations in the frame of fractional operators with Mittag-Leffler kernels and their discrete versions

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Abstract.
Very recently, some authors have studied new types of fractional derivatives whose kernels are non-singular. In this article, we study Sturm-Liouville Equations (SLEs) in the frame of fractional operators with Mittag-Leffler kernels. We formulate some Fractional Sturm-Liouville Problems (FSLPs) with the differential part containing the left and right sided derivatives. We investigate the self-adjointness, eigenvalue and eigenfunction properties of the corresponding Fractional Sturm-Liouville Operators (FSLOs) by using fractional integration by parts formulas. The nabla discrete version of our results are also established.

Keywords. Fractional Sturm-Liouville problem, ABR and ABC fractional derivatives, ABR and ABC fractional differences, Mittag-Leffler kernel.

1 Introduction and Preliminaries

Fractional calculus has been studied in the last two decades or so. It has been used effectively in the modelling of many problems in various fields of science and engineering. It has reflected successfully the description of the properties of non-local complex systems \cite{1,5}. On the other hand, the discrete fractional calculus was of interest among several mathematicians \cite{6,20} and has been developing rapidly. For the sake of finding more fractional operators with different kernels, recently some authors have introduced and studied new non-local derivatives with non-singular kernels and have applied them successfully to some real world problems \cite{21,26}. The extension to higher order fractional operators and their Lyapunov type inequalities have been investigated in \cite{?,?}. The proposed kernels are non-singular such as those with Mittag-Leffler kernels. The approach in defining such operators is different from the one of classical fractional operators which is through an iterative process of either the usual integration or differentiation. The idea behind this is to define the fractional derivatives first by imposing a non-singular kernel depending on the degree $\alpha$ so that as $\alpha \to 1$ the usual derivative is obtained and then by applying a Laplace transform method to find their corresponding fractional integrals. What makes those fractional derivatives with Mittag-Leffler kernels more interesting is that their corresponding fractional integrals contain Riemann-Liouville fractional integrals as a part of their structure. The advantage of such operators is that they enable numerical analysts to develop more efficient algorithms in solving fractional dynamical systems by concentrating only on the coefficients of the differential equations rather than worrying about the singularity of the kernels in the case of the classical fractional.
operators [29,31]. Later, the discrete counterparts of these fractional operators were introduced, studied, and their monotonicity properties were analyzed [38,39,40,41,42,43,44,45,46,47,48,49].

The SLEs, which were investigated a long time ago, have many applications in various areas of science, engineering, and mathematics [38,39]. However, its formulation in the frame of classical fractional calculus has started very recently [40,41]. The classical Sturm-Liouville problem (SLP) for a linear differential equation of second order is a boundary value problem (BVP) of the form:

$$-\frac{d}{dt} \left( p(t) \frac{dx}{dt} \right) + q(t)x(t) = \lambda r(t)x(t), \quad t \in [a, b],$$

$$c_1x(a) + c_2x'(a) = 0,$$

$$d_1x(b) + d_2x'(b) = 0,$$

where \( p, p', q, r \) are continuous functions on the interval \([a, b]\) such as \( p(t) > 0, r(t) > 0 \) on \([a, b]\). The differential equation can be written in the form

$$L(x) = \lambda r(t)x,$$

where \( L(x) = -[p(t)x']' + q(t)x \). A \( \lambda \) for which the above BVP has a nontrivial solution is called an eigenvalue, and the corresponding solution, an eigenfunction.

Motivated by what we have mentioned above, we introduce and analyze fractional SLEs in the frame of fractional operators with Mittag-Leffler kernels and their discrete counterparts. The corresponding fractional operator \( \tilde{L} \) is introduced so that it contains left and right sided fractional operators with Mittag-Leffler kernels which makes possible to apply the suitable integration by parts formulas presented in [24,33].

This article is organized as follows: In the rest of this section, we recall some basic concepts concerning the classical fractional calculus, classical nabla discrete fractional calculus, fractional operators with Mittag-Leffler kernels, and discrete fractional operators with discrete Mittag-Leffler kernels. In section 2, we state the main results which is divided into two parts. The first part is devoted to SLEs in the frame of fractional operators with Mittag-Leffler kernels and the second part to SLEs in the frame of nabla discrete fractional operators with discrete Mittag-Leffler kernels. Finally, in section 3, we present an open problem for the higher order discrete fractional SLE of order \( \alpha \in (1, \frac{3}{2}) \).

Below, we first recall some basic concepts from classical fractional calculus.

**Definition 1.** (2) The Mittag-Leffler function of one parameter is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \text{Re}(\alpha) > 0,$$

and the one with two parameters \( \alpha \) and \( \beta \) by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0,$$

where \( E_{\alpha,1}(z) = E_\alpha(z) \).
Definition 2. ([3]) The generalized Mittag-Leffler function of three parameters is defined by

\[ E_{\alpha,\beta}^\rho(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha, \beta, \rho \in \mathbb{C}, \ Re(\alpha) > 0, \ Re(\beta) > 0, \ Re(\rho) > 0, \]

where \((\rho)_k = \frac{\Gamma(\rho+k)}{\Gamma(\rho)}\).

Notice that \((1)_k = k!\) so \(E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z)\).

- The left fractional integral of order \(\alpha > 0\) starting at \(a\) has the following form
  \[
  \left( a \int^t f(s) ds \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds.
  \]

- The right fractional integral of order \(\alpha > 0\) ending at \(b\) is defined by
  \[
  \left( I_b^\alpha f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds.
  \]

- The left Riemann-Liouville fractional derivative of order \(0 < \alpha < 1\) starting at \(a\) is given by
  \[
  \left( a D^\alpha f \right)(t) = \frac{d}{dt} \left( a t^{1-\alpha} f \right)(t).
  \]

- The right Riemann-Liouville fractional derivative of order \(0 < \alpha < 1\) ending at \(b\) has the form
  \[
  \left( D_b^\alpha f \right)(t) = -\frac{d}{dt} \left( I_b^{1-\alpha} f \right)(t).
  \]

Definition 3. ([21]) Let \(f \in H^1(a,b), \ a < b, \ \alpha \in [0,1]\). Then the left Caputo fractional derivative with Mittag-Leffler kernel is defined by

\[
 a^\text{ABC} D^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_{\alpha} \left( \frac{-\alpha}{1-\alpha} (t-x)\alpha \right) dx,
\]

and the left Riemann-Liouville one by

\[
 a^\text{ABR} D^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x) E_{\alpha} \left( \frac{-\alpha}{1-\alpha} (t-x)\alpha \right) dx,
\]

where \(B(\alpha) > 0\) is a normalization function with \(B(0) = B(1) = 1\). In addition, the associated fractional integral is defined by

\[
 a^\text{AB} \int^\alpha f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} a^\alpha f(t).
\]
If $f$ is defined on the interval $[a,b]$, then the action of the $Q$–operator is defined as $(Qf)(t) = f(a + b - t)$. From classical fractional calculus, it is known that $(aI^\alpha Qf)(t) = Q(A^\alpha f)(t)$ and $(aD^\alpha Qf)(t) = Q(D^\alpha f)(t)$. In [24], by making use of the $Q$–operator, the authors defined the right versions of the $ABR$ and $ABC$ fractional derivatives and their corresponding integral as follows:

**Definition 4.** ([24]) Let $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$. Then the right Caputo fractional derivative with Mittag-Leffler kernel is defined by

$$\text{ABC} D^\alpha_b f(t) = -\frac{B(\alpha)}{1 - \alpha} \int_t^b f'(x) E_\alpha \left( \frac{-\alpha}{1 - \alpha} (x - t)^\alpha \right) \, dx,$$

and the right Riemann-Liouville one by

$$\text{ABR} D^\alpha_b f(t) = -\frac{B(\alpha)}{1 - \alpha} \frac{d}{dt} \int_t^b f(x) E_\alpha \left( \frac{-\alpha}{1 - \alpha} (x - t)^\alpha \right) \, dx.$$

In addition, the corresponding fractional integral is defined by

$$\text{AB} I^\alpha_b f(t) = \frac{1 - \alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} I^\alpha_b f(t).$$

The following function spaces were introduced in [24] in order to present an integration by parts formula for $ABR$ fractional derivatives. For $p \geq 1$ and $\alpha > 0$,

$$_a\text{AB} I^\alpha (L_p) = \{ f : f = _a\text{AB} I^\alpha \varphi, \quad \varphi \in L_p(a,b) \},$$

and

$$_b\text{AB} I^\alpha (L_p) = \{ f : f = _b\text{AB} I^\alpha \phi, \quad \phi \in L_p(a,b) \}.$$

In [21][24], it was shown that the left and right fractional operators $_a\text{AB} D^\alpha$ and $_b\text{AB} D^\alpha$ and their associated fractional integrals $_a\text{AB} I^\alpha$ and $_b\text{AB} I^\alpha$ satisfy

$$_a\text{AB} D^\alpha_a\text{AB} I^\alpha f(t) = f(t), \quad _b\text{AB} D^\alpha_b\text{AB} I^\alpha f(t) = f(t),$$

and also

$$_a\text{AB} I^\alpha_a\text{AB} D^\alpha f(t) = f(t), \quad _b\text{AB} I^\alpha_b\text{AB} D^\alpha f(t) = f(t). \quad (1)$$

Hence, from (1), it follows that the function spaces $a\text{AB} I^\alpha(L_p)$ and $b\text{AB} I^\alpha(L_p)$ are nonempty.

**Theorem 1.** ([24]) (Integration by parts formula for $ABR$ fractional derivatives)

Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).

- If $\varphi(x) \in L_p(a,b)$ and $\psi(x) \in L_q(a,b)$, then

$$\int_a^b \varphi(x)\ _a\text{AB} I^\alpha \psi(x) \, dx = \int_a^b \psi(x)\ _a\text{AB} I^\alpha \varphi(x) \, dx.$$
• If \( f(x) \in {}^AB I^\alpha_b(L_p) \) and \( g(x) \in {}^AB I^\alpha_a(L_q) \), then

\[
\int_a^b f(x) \ {}^ABR D^\alpha g(x) dx = \int_a^b g(x) \ {}^ABR D^\alpha f(x) dx.
\]

From [21], we recall the following relation between the left \( ABR \) and \( ABC \) fractional derivatives as

\[
0^{ABC} D^\alpha f(t) = 0^{ABR} D^\alpha f(t) - \frac{B(\alpha)}{1 - \alpha} f(0) E^\alpha_\alpha \left( -\frac{t^\alpha}{1 - \alpha} \right). \tag{2}
\]

Right version of (2) was proved in [24] by making use of the \( Q \)-operator as follows:

\[
{}^{ABC} D^\alpha_0 f(t) = {}^{ABR} D^\alpha_0 f(t) - \frac{B(\alpha)}{1 - \alpha} f(b) E^\alpha_\alpha \left( -\frac{b^\alpha}{1 - \alpha} \right). \tag{3}
\]

From [42], recall the left generalized fractional integral operator as

\[
E^\alpha_{\alpha,\beta,\omega,a} f(x) = \int_a^x (x - t)^{\beta - 1} E^\alpha_{\alpha,\beta}(\omega(x - t)^\alpha) f(t) dt, \quad x > a. \tag{4}
\]

Analogously, the right generalized fractional integral operator can be defined by

\[
E^\alpha_{\alpha,\beta,\omega,b} f(x) = \int_x^b (t - x)^{\beta - 1} E^\alpha_{\alpha,\beta}(\omega(t - x)^\alpha) f(t) dt, \quad x < b \tag{5}
\]
(see also [24]).

**Remark 1.** By means of (4) and (5), the \( ABR \) and \( ABC \) fractional derivatives can be expressed as

\[
\begin{align*}
{}^ABR D^\alpha f(t) &= \frac{B(\alpha)}{1 - \alpha} \frac{d}{dt} E^{1,\alpha,\frac{\omega}{\alpha}}_{\alpha,1,\frac{\omega}{\alpha},a} f(t), \\
{}^ABR D^\alpha_b f(t) &= -\frac{B(\alpha)}{1 - \alpha} \frac{d}{dt} E^{1,\alpha,\frac{\omega}{\alpha}}_{\alpha,1,\frac{\omega}{\alpha},b} f(t), \\
{}^{ABC} D^\alpha f(t) &= \frac{B(\alpha)}{1 - \alpha} E^{1,\alpha,\frac{\omega}{\alpha}}_{\alpha,1,\frac{\omega}{\alpha},a} f'(t), \\
{}^{ABC} D^\alpha_0 f(t) &= -\frac{B(\alpha)}{1 - \alpha} E^{1,\alpha,\frac{\omega}{\alpha}}_{\alpha,1,\frac{\omega}{\alpha},b} f'(t).
\end{align*}
\]

**Proposition 1.** (Integration by parts formula for ABC fractional derivatives)
Let \( f, g \in H^1(a,b) \) and \( 0 < \alpha < 1 \). Then we have

\[
\begin{align*}
&\int_a^b g(t) \ {}^{ABC} D^\alpha f(t) dt = \int_a^b f(t) \ {}^{ABR} D^\alpha_b g(t) dt + \frac{B(\alpha)}{1 - \alpha} f(t) \ E^{1,\alpha,\frac{\omega}{\alpha}}_{\alpha,1,\frac{\omega}{\alpha},b} g(t)|_a^b, \\
&\int_a^b f(t) \ {}^{ABC} D^\alpha_b f(t) dt = \int_a^b f(t) \ {}^{ABR} D^\alpha_a g(t) dt - \frac{B(\alpha)}{1 - \alpha} f(t) \ E^{1,\alpha,\frac{\omega}{\alpha}}_{\alpha,1,\frac{\omega}{\alpha},a} g(t)|_a^b.
\end{align*}
\]
The proof of Proposition [1] was presented in [24] by making use of the relations (2) and (3) and the ABR integration by parts formula in Theorem [1]. Below, we present an alternative proof for Proposition [1] by using an integration by parts formula for the generalized fractional integral operators defined in (4) and (5) and the ordinary integration by parts.

**Lemma 1.** Let \( \alpha > 0, \ p \geq 1, \ q \geq 1 \), and \( \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha \) (\( p \neq 1 \) and \( q \neq 1 \) in case \( \frac{1}{p} + \frac{1}{q} = 1 + \alpha \)). If \( \varphi(x) \in L_p(a, b) \) and \( \psi(x) \in L_q(a, b) \), then

\[
\int_a^b \varphi(t) \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, a}^{p,q} \psi(t) dt = \int_a^b \psi(t) \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, b}^{p,q} \varphi(t) dt.
\]

**Proof.** The proof follows from the definition of the generalized fractional integral operators and interchanging the order of integration.

Now, we present the alternative proof of Proposition [1].

**Proof.** Using Remark [1] Lemma [1] and the ordinary integration by parts, we get

\[
\int_a^b g(t) \, ^{ABC}D^\alpha f(t) dt = \frac{B(\alpha)}{1 - \alpha} \int_a^b g(t) \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, a} \varphi(t) f(t) dt
\]

\[
= \frac{B(\alpha)}{1 - \alpha} \int_a^b f' \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, b} \varphi(t) g(t) dt
\]

\[
= \frac{B(\alpha)}{1 - \alpha} f \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, b} \varphi(t) g(t)_{b} - \frac{B(\alpha)}{1 - \alpha} f \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, b} \varphi(t) g(t)_{a}
\]

\[
= \frac{B(\alpha)}{1 - \alpha} f \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, b} \varphi(t) g(t)
\]

\[
+ \int_a^b f \, ^{ABC}D^\alpha g(t) dt.
\]

Similarly, the proof of the second part is as follows:

\[
\int_a^b g(t) \, ^{ABC}D^\alpha f(t) dt = -\frac{B(\alpha)}{1 - \alpha} \int_a^b g(t) \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, b} \varphi(t) f(t) dt
\]

\[
= -\frac{B(\alpha)}{1 - \alpha} \int_a^b f' \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, a} \varphi(t) g(t) dt
\]

\[
= -\frac{B(\alpha)}{1 - \alpha} f \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, a} \varphi(t) g(t)_{a}
\]

\[
+ \frac{B(\alpha)}{1 - \alpha} f \ E_{\alpha,1, \frac{\alpha}{1 - \alpha}, a} \varphi(t) g(t)_{b}
\]

\[
+ \int_a^b f \, ^{ABC}D^\alpha g(t) dt.
\]
Now, we recall some notations and basic definitions concerning the classical nabla discrete fractional calculus. For more details, we refer the reader to [6–9, 20] and the references cited therein.

The functions we consider will be defined on sets of the form

\[ \mathbb{N}_a = \{a, a + 1, a + 2, \ldots\}, \quad \mathbb{N}_b = \{\ldots, b - 2, b - 1, b\}, \]

where \(a, b \in \mathbb{R}\), or a set of the form

\[ \mathbb{N}_{a,b} = \{a, a + 1, a + 2, \ldots, b\}, \]

where \(b - a\) is a positive integer.

**Definition 5 (6,20).** (i) For a natural number \(m\) and \(t \in \mathbb{R}\), the \(m\) rising (ascending) factorial of \(t\) is defined by

\[
\text{r}_m^t = \prod_{k=0}^{m-1} (t+k), \quad \text{r}_0^t = 1.
\]

(ii) For any real number \(\alpha\), the (generalized) rising function is defined by

\[
\text{r}_\alpha^t = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}, \quad 0^0 = 0.
\]

**Definition 6 (8,9).** For a function \(f : \mathbb{N}_a \to \mathbb{R}\), the nabla left fractional sum of order \(\alpha > 0\) (starting from \(a\)) is given by

\[
a \nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\alpha-1} f(s), \quad t \in \mathbb{N}_{a+1}.
\]

The nabla right fractional sum of order \(\alpha > 0\) (ending at \(b\)) for \(f : \mathbb{N}_b \to \mathbb{R}\) is defined by

\[
\nabla^{-\alpha}_b f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (s - \rho(t))^{\alpha-1} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s) - t)^{\alpha-1} f(s), \quad t \in b-1\mathbb{N}.
\]

**Definition 7 (8,9).** For a function \(f : \mathbb{N}_a \to \mathbb{R}\), the nabla left Riemann-Liouville fractional difference of order \(0 < \alpha < 1\) (starting from \(a\)) is defined by

\[
a \nabla^{-\alpha} f(t) = \nabla \text{r}^{-\alpha}_1 f(t) = \nabla \left[ \frac{1}{\Gamma(1-\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{-\alpha} f(s) \right], \quad t \in \mathbb{N}_{a+1},
\]
and for \( f : b\mathbb{N} \to \mathbb{R} \), the nabla right Riemann-Liouville fractional difference of order \( 0 < \alpha < 1 \) (ending at \( b \)) by

\[
\nabla_b^\alpha f(t) = \nabla_\Delta b^{\alpha-1} f(t) = -\Delta \left[ \frac{1}{\Gamma(1-\alpha)} \sum_{s=t}^{b-1} (s-\rho(t))^{-\alpha} f(s) \right], \quad t \in b-1\mathbb{N}.
\]

In the above, \( \rho \) and \( \sigma \) are the backward and forward jump operators, respectively.

**Definition 8** ([8, 9]). For a function \( f : \mathbb{N}_a \to \mathbb{R} \), the nabla left Caputo fractional difference of order \( 0 < \alpha < 1 \) (starting from \( a \)) is defined by

\[
(C_a \nabla^\alpha f)(t) = a \nabla^{-1+\alpha} \nabla f(t), \quad t \in \mathbb{N}_{a+1},
\]

and for \( f : b\mathbb{N} \to \mathbb{R} \), the nabla right Caputo fractional difference of order \( 0 < \alpha < 1 \) (ending at \( b \)) by

\[
(C_b \nabla^\alpha f)(t) = b \nabla^{-1+\alpha} \nabla f(t), \quad t \in b-1\mathbb{N}.
\]

**Definition 9.** (Nabla Discrete Mittag-Leffler functions) ([8–10]) For \( \lambda \in \mathbb{R}, \ |\lambda| < 1 \), and \( \alpha, \beta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), the nabla discrete Mittag-Leffler function is defined by

\[
E_{\alpha,\beta}^\lambda(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\lambda \alpha + \beta - 1}}{\Gamma(\alpha k + \beta)}.
\]

For \( \beta = 1 \), it is written that

\[
E_{\alpha,1}^\lambda(\lambda, z) = E_{\alpha,1}^\lambda(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\lambda \alpha}}{\Gamma(\alpha k + 1)}.
\]

**Definition 10.** ([33]) The nabla discrete generalized Mittag-Leffler function of three parameters \( \alpha, \beta, \) and \( \rho \) is defined by

\[
E_{\alpha,\beta}^\rho(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k (\rho)_k \frac{z^{\lambda \alpha + \beta - 1}}{k! \Gamma(\alpha k + \beta)}.
\]

Notice that \( E_{\alpha,\beta}^1(\lambda, z) = E_{\alpha,\beta}(\lambda, z) \).

Now, we review some main concepts concerning the nabla discrete fractional differences with discrete Mittag-Leffler kernels following the notations in [33].

**Definition 11.** ([33]) Assume \( f : \mathbb{N}_a \to \mathbb{R} \) and \( \alpha \in (0, 1/2) \). Then the nabla discrete left Caputo fractional difference in the sense of Atangana and Baleanu is defined by

\[
\nabla^\alpha f(t) = B(\alpha) \sum_{s=a+1}^{t} \nabla f(s) E_{\alpha,1}^\lambda \left( \frac{-\alpha}{1-\alpha}, t - \rho(s) \right), \quad t \in \mathbb{N}_{a+1},
\]
Definition 12. Assume by right Riemann-Liouville fractional difference with discrete Mittag-Leffler kernel is defined and ABR

\[ \text{ABR}^{\alpha} f(t) = \frac{B(\alpha)}{1 - \alpha} \Delta_a \sum_{s=a+1}^{t} f(s) \mathcal{E}_{\alpha} \left( \frac{-\alpha}{1 - \alpha}, t - \rho(s) \right), \quad t \in \mathbb{N}_{a+1}, \]

where \( B(\alpha) > 0 \) is a normalization function with \( B(0) = B(1) = 1 \). In addition, the associated fractional sum is defined by

\[ \text{AB}^{\alpha} f(t) = \frac{1 - \alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \Delta_a f(t), \quad t \in \mathbb{N}_{a+1}. \]

Similar to the continuous case, for a function \( f \) defined on \( \mathbb{N}_{a,b} \), the action of the \( Q \)-operator is defined as \( (Qf)(t) = f(a + b - t) \). From classical discrete fractional calculus, it is known that \( (\alpha \nabla^{-\alpha} Qf)(t) = Q(\nabla_{b}^{\alpha} f)(t) \) and \( (\alpha \nabla^{\alpha} Qf)(t) = Q(\nabla_{b}^{\alpha} f)(t) \).

In [33], by making use of the \( Q \)-operator, the authors defined the right versions of the \( ABR \) and \( ABC \) nabla fractional differences and their corresponding sum as follows:

**Definition 12.** (33) Assume \( f : b \mathbb{N} \to \mathbb{R} \) and \( \alpha \in (0, 1/2) \). Then the nabla discrete right Riemann-Liouville fractional difference with discrete Mittag-Leffler kernel is defined by

\[ \text{ABR}^{\alpha}_{b} f(t) = -\frac{B(\alpha)}{1 - \alpha} \Delta_{b} \sum_{s=t}^{b-1} f(s) \mathcal{E}_{\alpha} \left( \frac{-\alpha}{1 - \alpha}, s - \rho(t) \right), \quad t \in b_{-1} \mathbb{N}, \]

and the right Caputo one by

\[ \text{ABC}^{\alpha}_{b} f(t) = -\frac{B(\alpha)}{1 - \alpha} \Delta_{b} \sum_{s=t}^{b-1} f(s) \mathcal{E}_{\alpha} \left( \frac{-\alpha}{1 - \alpha}, s - \rho(t) \right), \quad t \in b_{-1} \mathbb{N}. \]

In addition, the associated fractional sum is defined by

\[ \text{AB}^{\alpha}_{b} f(t) = \frac{1 - \alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \Delta_{b} f(t), \quad t \in b_{-1} \mathbb{N}. \]

In [33], it was shown that the left and right fractional difference operators \( \text{ABR}^{\alpha}_{a} \) and \( \text{ABR}^{\alpha}_{b} \) and their associated fractional sums \( \text{AB}^{\alpha}_{a} \) and \( \text{AB}^{\alpha}_{b} \) satisfy

\[ \text{ABR}^{\alpha}_{a} \text{AB}^{\alpha}_{a} f(t) = f(t), \quad \text{ABR}^{\alpha}_{b} \text{AB}^{\alpha}_{b} f(t) = f(t), \]

and also

\[ \text{AB}^{\alpha}_{a} \text{AB}^{\alpha}_{a} f(t) = f(t), \quad \text{AB}^{\alpha}_{b} \text{AB}^{\alpha}_{b} f(t) = f(t). \]

From [33], we also recall the following relation between the left \( ABR \) and \( ABC \) nabla fractional differences as

\[ \text{ABC}^{\alpha}_{a} f(t) = \text{ABR}^{\alpha}_{a} f(t) - f(a) \frac{B(\alpha)}{1 - \alpha} \mathcal{E}_{\alpha} \left( \frac{-\alpha}{1 - \alpha}, t - a \right). \tag{6} \]

Right version of (6) was proved in [33] by making use of the \( Q \)-operator as follows:

\[ \text{ABC}^{\alpha}_{b} f(t) = \text{ABR}^{\alpha}_{b} f(t) - f(b) \frac{B(\alpha)}{1 - \alpha} \mathcal{E}_{\alpha} \left( \frac{-\alpha}{1 - \alpha}, b - t \right). \tag{7} \]
Theorem 2. (33) (Integration by parts formula for ABR fractional sums)
Assume \(f, g: \mathbb{N}_{a,b} \to \mathbb{R}\) and \(\alpha \in (0, 1/2)\). Then we have
\[
\sum_{s=a+1}^{b-1} g(s) ABR_{a}^{\alpha} f(s) = \sum_{s=a+1}^{b-1} f(s) ABR_{b}^{\alpha} g(s).
\]

Theorem 3. (33) (Integration by parts formula for ABR fractional differences)
Assume \(f, g: \mathbb{N}_{a,b} \to \mathbb{R}\) and \(\alpha \in (0, 1/2)\). Then we have
\[
\sum_{s=a+1}^{b-1} f(s) ABR_{a}^{\alpha} g(s) = \sum_{s=a+1}^{b-1} g(s) ABR_{b}^{\alpha} f(s).
\]

Before presenting an integration by parts formula for the left ABC fractional differences, we first recall the discrete versions of the left and right generalized fractional integral operators given in (4) and (5).

Definition 13. (33)
• The discrete (left) generalized fractional integral operator is defined by
\[
E_{\alpha, 1/\alpha, a}^{\alpha, 1/\alpha, a+} \varphi(t) = \sum_{s=a+1}^{t} (t - \rho(s))^{\alpha-1} E_{\alpha, \alpha}^{\alpha, 1/\alpha, a+} \varphi(s), \quad t \in \mathbb{N}_{a}.
\]

• The discrete (right) generalized fractional integral operator is defined by
\[
E_{\alpha, 1/\alpha, b}^{\alpha, 1/\alpha, b-} \varphi(t) = \sum_{s=t}^{b-1} (s - \rho(t))^{\alpha-1} E_{\alpha, \alpha}^{\alpha, 1/\alpha, b-} \varphi(s), \quad t \in \mathbb{N}_{b}.
\]

Remark 2. By means of Definition 13, the ABR and ABC fractional differences can be expressed as:
\[
ABR_{a}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \nabla E_{\alpha, 1/\alpha, a}^{\alpha, 1/\alpha, a+} f(t),
\]
\[
ABR_{b}^{\alpha} f(t) = -\frac{B(\alpha)}{1-\alpha} \nabla E_{\alpha, 1/\alpha, b}^{\alpha, 1/\alpha, b-} f(t),
\]
\[
ABC_{a}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} E_{\alpha, 1/\alpha, a}^{\alpha, 1/\alpha, a+} \nabla f(t),
\]
\[
ABC_{b}^{\alpha} f(t) = -\frac{B(\alpha)}{1-\alpha} E_{\alpha, 1/\alpha, b}^{\alpha, 1/\alpha, b-} \nabla f(t).
\]

Lemma 2. Assume \(f, g: \mathbb{N}_{a,b} \to \mathbb{R}\) and \(\alpha \in (0, 1/2)\). Then we have
\[
\sum_{s=a+1}^{b-1} f(s) E_{\alpha, 1/\alpha, a}^{\alpha, 1/\alpha, a+} g(s) = \sum_{s=a+1}^{b-1} g(s) E_{\alpha, 1/\alpha, b}^{\alpha, 1/\alpha, b-} f(s).
\]
Proof. The proof follows from the definition of the generalized fractional sums and interchanging the order of summations.

In [33], the authors presented integration by parts formulas for the ABC nabla fractional differences by using Theorem 3, (6) and (7). In what follows, we will present an integration by parts formula for the left ABC fractional differences by making use of Remark 2, Lemma 2, and the integration by parts in the ordinary difference calculus.

**Theorem 4.** (Integration by parts formula for left ABC fractional differences)

Assume $f, g : \mathbb{N}_{a,b} \to \mathbb{R}$ and $\alpha \in (0, 1/2)$. Then we have

$$\sum_{s=a+1}^{b-1} f(s) \frac{ABC}{a} \nabla^\alpha g(s) = \sum_{s=a+1}^{b-1} g(s-1) \frac{ABR}{b} \nabla^\alpha f(s-1) + g(t) \frac{B(\alpha)}{1-\alpha} \frac{E_{1-\alpha}^{1\alpha,1\alpha}}{1-\alpha} f(t)^{b-1}.$$  

In the above, it is easy to see that $\frac{E_{1-\alpha}^{1\alpha,1\alpha}}{1-\alpha} b - f(b-1) = (1-\alpha)f(b-1)$.

**Remark 3.** The integration by parts formula in Theorem 4 can now be stated as follows:

$$\sum_{s=a+1}^{b} f(s) \frac{ABC}{a} \nabla^\alpha g(s) = \sum_{s=a+1}^{b} g(s-1) \frac{ABR}{b+1} \nabla^\alpha f(s-1) + g(t) \frac{B(\alpha)}{1-\alpha} \frac{E_{1-\alpha}^{1\alpha,1\alpha}}{1-\alpha} f(t)^{b}.$$  

### 2 Main Results

Denoting the SL operator as

$$L_1 x(t) = \frac{ABR}{a} D^\alpha (p(t)) \frac{ABR}{b} D^\alpha x(t) + q(t)x(t),$$

consider the fractional SLE

$$\frac{ABR}{a} D^\alpha (p(t)) \frac{ABR}{b} D^\alpha x(t) + q(t)x(t) = \lambda r(t)x(t), \quad t \in (a,b),$$  

where $\alpha \in (0, 1)$, $p(t) > 0$, $r(t) > 0 \forall t \in [a,b]$, $p, q, r$ are real valued continuous functions on the interval $[a,b]$.

**Theorem 5.** The fractional SL operator $L_1$ is self-adjoint with respect to the inner product

$$<u, v> = \int_{a}^{b} \overline{u(t)} v(t) \, dt.$$  

Proof. We have

$$v(t)L_1 u(t) = v(t) \frac{ABR}{a} D^\alpha (p(t)) \frac{ABR}{b} D^\alpha x(t) + q(t)x(t)v(t)$$

$$\overline{v(t)L_1 v(t)} = \overline{v(t)} \frac{ABR}{a} D^\alpha (p(t)) \frac{ABR}{b} D^\alpha x(t) + q(t)x(t)v(t).$$

Subtracting (10) from (9), we have
\[ v(t) \overline{L_1 u(t)} - \overline{v(t)} L_1 v(t) = v(t) ABR D^\alpha(p(t) ABR D_b^\alpha \overline{u(t)}) - \overline{u(t)} ABR D^\alpha(p(t) ABR D_b^\alpha v(t)). \]

Integrating from \(a\) to \(b\), we get
\[
\int_a^b \left( v(t) \overline{L_1 u(t)} - \overline{v(t)} L_1 v(t) \right) dt = \int_a^b \left( v(t) ABR D^\alpha(p(t) ABR D_b^\alpha \overline{u(t)}) - \overline{u(t)} ABR D^\alpha(p(t) ABR D_b^\alpha v(t)) \right) dt.
\]

Now, by applying the integration by parts formula in Theorem 1, we obtain
\[
\int_a^b \left( v(t) \overline{L_1 u(t)} - \overline{v(t)} L_1 v(t) \right) dt = \int_a^b p(t) ABR D_b^\alpha \overline{u(t)} ABR D_b^\alpha v(t) dt - \int_a^b p(t) ABR D_b^\alpha v(t) ABR D_b^\alpha \overline{u(t)} dt = 0.
\]

Hence, \( < L_1 u, v > = < u, L_1 v > \). That is, \( L_1 \) is self-adjoint.

**Theorem 6.** The eigenvalues of the SLE (8) are real.

**Proof.** Assume that \( \lambda \) is the eigenvalue for (8) corresponding to eigenfunction \( x \). Then \( x \) and its complex conjugate \( \overline{x} \) satisfy
\[
L_1 x(t) = \lambda r(t) x(t),
\]
and
\[
L_1 \overline{x}(t) = \overline{\lambda} r(t) \overline{x}(t),
\]
respectively. We multiply (11) by \( \overline{x}(t) \) and (12) by \( x(t) \), respectively, and subtract to obtain
\[
(\overline{\lambda} - \lambda) r(t) x(t) \overline{x}(t) = x(t) L_1 \overline{x}(t) - \overline{x}(t) L_1 x(t).
\]

Integrating from \(a\) to \(b\), and by using the fact that \( L_1 \) is self-adjoint, we get
\[
(\overline{\lambda} - \lambda) \int_a^b r(t) |x(t)|^2 dt = 0,
\]
and since \( x \) is a nontrivial solution and \( r(t) > 0 \), we conclude that \( \lambda = \overline{\lambda} \).

**Theorem 7.** The eigenfunctions, corresponding to distinct eigenvalues of the SLE (8) are orthogonal with respect to the weight function \( r \) on \([a,b]\) that is
\[
< x_{\lambda_1}, x_{\lambda_2} > = \int_a^b r(t) x_{\lambda_1}(t) x_{\lambda_2}(t) dt = 0, \quad \lambda_1 \neq \lambda_2,
\]
when the functions \( x_{\lambda_i} \) correspond to eigenvalues \( \lambda_i, i = 1, 2 \).
Proof. Let $\lambda_1$ and $\lambda_2$ be two distinct eigenvalues of (8) corresponding to the eigenfunctions $x_{\lambda_1}$ and $x_{\lambda_2}$, respectively. Then we have

$$a^\alpha D_t^\alpha (p(t) a^\alpha D_t^\alpha x_{\lambda_1}(t)) + q(t)x_{\lambda_1}(t) = \lambda_1 r(t)x_{\lambda_1}(t) \quad (13)$$

$$a^\alpha D_t^\alpha (p(t) a^\alpha D_t^\alpha x_{\lambda_2}(t)) + q(t)x_{\lambda_2}(t) = \lambda_2 r(t)x_{\lambda_2}(t). \quad (14)$$

We multiply (13) and (14) by $x_{\lambda_2}(t)$ and $x_{\lambda_1}(t)$, respectively, and subtract the results to obtain

$$(\lambda_1 - \lambda_2) r(t)x_{\lambda_1}(t)x_{\lambda_2}(t) = x_{\lambda_2}(t) a^\alpha D_t^\alpha (p(t) a^\alpha D_t^\alpha x_{\lambda_1}(t)) - x_{\lambda_1}(t) a^\alpha D_t^\alpha (p(t) a^\alpha D_t^\alpha x_{\lambda_2}(t)).$$

Now, integrating from $a$ to $b$, and using the integration by parts formula in Theorem 1 to the right side of the equation, we get

$$\int_a^b r(t)x_{\lambda_1}(t)x_{\lambda_2}(t) dt = 0.$$ 

Since $\lambda_1 \neq \lambda_2$, it follows that

$$\int_a^b r(t)x_{\lambda_1}(t)x_{\lambda_2}(t) dt = 0,$$

which completes the proof. \qed

Now, consider the nabla discrete fractional SL operator

$$L_2 x(t) = a^\alpha \nabla_t^\alpha (p(t) a^\alpha \nabla_t^\alpha x(t)) + q(t)x(t),$$

and the corresponding SLE

$$a^\alpha \nabla_t^\alpha (p(t) a^\alpha \nabla_t^\alpha x(t)) + q(t)x(t) = \lambda r(t)x(t), \quad t \in \mathbb{N}_{a+1,b-1}, \quad (15)$$

where $\alpha \in (0, 1/2)$, $p(t) > 0$, $r(t) > 0 \forall t \in \mathbb{N}_{a,b}$, $p, q, r$ are real valued functions on $\mathbb{N}_{a,b}$.

**Theorem 8.** The discrete fractional SL operator $L_2$ is self-adjoint with respect to the inner product

$$< u, v > = \sum_{t=a+1}^{b-1} u(t)v(t).$$

Proof. The proof is similar to that of Theorem 5. However, it follows by making use of the discrete fractional integration by parts in Theorem 3. The details are left to the reader. \qed

**Theorem 9.** The eigenvalues of the SLE (15) are real.

Proof. The proof is similar to that of Theorem 6. The details are left to the reader. \qed
Theorem 10. The eigenfunctions, corresponding to distinct eigenvalues of the SLE (15) are orthogonal with respect to the weight function \( r \) on \( \mathbb{N}_{a,b} \) that is

\[
<x_{\lambda_1}, x_{\lambda_2}> = \sum_{t=a+1}^{b-1} r(t) x_{\lambda_1}(t) x_{\lambda_2}(t) = 0, \quad \lambda_1 \neq \lambda_2,
\]

when the functions \( x_{\lambda_i} \) correspond to eigenvalues \( \lambda_i, i = 1, 2 \).

Proof. The proof is similar to that of Theorem 7. However, it follows by making use of the discrete fractional integration by parts in Theorem 3. The details are left to the reader.

Denoting the SL operator as

\[
\mathcal{C} L_1 x(t) = ABC D^\alpha_a (p(t) ABR D^\alpha_b x(t)) + q(t) x(t),
\]

consider the \( ABC \) type fractional SLE

\[
ABC D^\alpha_a (p(t) ABR D^\alpha_b x(t)) + q(t) x(t) = \lambda r(t) x(t), \quad t \in (a, b),
\]

where \( \alpha \in (0, 1), p(t) \neq 0, r(t) > 0 \forall t \in [a, b], p, q, r \) are real valued continuous functions on the interval \( [a, b] \), and the boundary conditions:

\[
c_1 \ E_{1,1,-\alpha}^{1,\alpha,1-\alpha} x(a) + c_2 ABR D^\alpha_b x(a) = 0, \quad (17)
\]

\[
d_1 \ E_{1,1,-\alpha}^{1,\alpha,1-\alpha} x(b) + d_2 ABR D^\alpha_b x(b) = 0, \quad (18)
\]

where \( c_1^2 + c_2^2 \neq 0 \) and \( d_1^2 + d_2^2 \neq 0 \).

Theorem 11. The eigenvalues of the SLP (16)-(18) are real.

Proof. By the help of the integration by parts formula in Proposition 1, we observe that

\[
\int_a^b u(t) C L_1 v(t) dt = \int_a^b q(t) u(t) v(t) dt + \\
\int_a^b p(t) ABR D^\alpha_b v(t) ABR D^\alpha_b u(t) dt + \frac{B(\alpha)}{1-\alpha} p(t) \ E_{1,1,-\alpha}^{1,\alpha,1-\alpha} u(t) ABR D^\alpha_b v(t)_{\alpha}^b.
\]

Assume that \( \lambda \) is the eigenvalue for (16)-(18) corresponding to eigenfunction \( x \). Then, \( x \) and its complex conjugate \( \bar{x} \) satisfy

\[
C L_1 x(t) = \lambda r(t) x(t), \quad (20)
\]

\[
c_1 \ E_{1,1,-\alpha}^{1,\alpha,1-\alpha} x(a) + c_2 ABR D^\alpha_b x(a) = 0, \quad (21)
\]

\[
d_1 \ E_{1,1,-\alpha}^{1,\alpha,1-\alpha} x(b) + d_2 ABR D^\alpha_b x(b) = 0, \quad (22)
\]

\[
\text{Proof. The proof is similar to that of Theorem 7. However, it follows by making use of the discrete fractional integration by parts in Theorem 3. The details are left to the reader.}
\]
Finally, by making use of the boundary conditions (21), (22), (24), and (25), we obtain

\[ C L_1 \overline{\alpha}(t) = \overline{\alpha} r(t) \overline{\alpha}(t), \]

\[ c_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - \overline{\alpha}(a) + c_2 ABR D_{p}^{\alpha} \overline{\alpha}(a) = 0, \]

\[ d_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b + d_2 ABR D_{p}^{\alpha} \overline{\alpha}(b) = 0, \]

with \( c_1^2 + c_2^2 \neq 0 \) and \( d_1^2 + d_2^2 \neq 0 \). We multiply (20) by \( \overline{\alpha}(t) \) and (23) by \( x(t) \), respectively, and subtract to obtain

\[ (\overline{\lambda} - \lambda) r(t) x(t) \overline{\alpha}(t) = x(t) C L_1 \overline{\alpha}(t) - \overline{\alpha}(t) C L_1 x(t). \]

Now, integrate over the interval \([a, b]\) and apply (19) with \( u(t) = x(t) \) and \( v(t) = \overline{\alpha}(t) \) and vice versa to obtain

\[ (\overline{\lambda} - \lambda) \int_{a}^{b} r(t) |x(t)|^2 dt = \]

\[ \frac{B(\alpha)}{1 - \alpha} p(b) \left[ E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x(b) ABR D_{p}^{\alpha} \overline{\alpha}(b) - E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x(b) ABR D_{p}^{\alpha} x(b) \right] + \]

\[ \frac{B(\alpha)}{1 - \alpha} p(a) \left[ E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x(a) ABR D_{p}^{\alpha} \overline{\alpha}(a) - E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x(a) ABR D_{p}^{\alpha} x(a) \right]. \]

Finally, by making use of the boundary conditions (21), (22), (24), and (25), we obtain

\[ (\overline{\lambda} - \lambda) \int_{a}^{b} r(t) |x(t)|^2 dt = 0. \]

Because \( x \) is a nontrivial solution and \( r(t) > 0 \), we conclude that \( \lambda = \overline{\lambda} \). □

**Theorem 12.** The eigenfunctions, corresponding to distinct eigenvalues of the SLP (16)-(18) are orthogonal with respect to the weight function \( r \) on \([a, b]\) that is

\[ < x_{\lambda_1}, x_{\lambda_2} > = \int_{a}^{b} r(t) x_{\lambda_1}(t) x_{\lambda_2}(t) dt = 0, \quad \lambda_1 \neq \lambda_2, \]

when the functions \( x_{\lambda_i} \) correspond to eigenvalues \( \lambda_i, \ i = 1, 2 \).

**Proof.** Assume that \( x_{\lambda_1} \) and \( x_{\lambda_2} \) are eigenfunctions corresponding to two distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively. Then we have

\[ C L_1 x_{\lambda_1}(t) = \lambda_1 r(t) x_{\lambda_1}(t), \]

\[ c_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x_{\lambda_1}(a) + c_2 ABR D_{p}^{\alpha} x_{\lambda_1}(a) = 0, \]

\[ d_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x_{\lambda_1}(b) + d_2 ABR D_{p}^{\alpha} x_{\lambda_1}(b) = 0, \]

and

\[ C L_1 x_{\lambda_2}(t) = \lambda_2 r(t) x_{\lambda_2}(t), \]

\[ d_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x_{\lambda_2}(b) + d_2 ABR D_{p}^{\alpha} x_{\lambda_2}(b) = 0, \]

\[ d_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x_{\lambda_2}(b) + d_2 ABR D_{p}^{\alpha} x_{\lambda_2}(b) = 0, \]

\[ c_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x_{\lambda_2}(a) + c_2 ABR D_{p}^{\alpha} x_{\lambda_2}(a) = 0, \]

\[ c_1^1 E_{\alpha,1, \frac{\alpha}{1-\alpha}}^{1} b - x_{\lambda_2}(a) + c_2 ABR D_{p}^{\alpha} x_{\lambda_2}(a) = 0. \]
\[ c_1 \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 b^{-}x_{\lambda_2}(a) + c_2 {^{ABR}}D^\alpha b x_{\lambda_2}(a) = 0, \quad (30) \]
\[ d_1 \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 b^{-}x_{\lambda_2}(b) + d_2 {^{ABR}}D^\alpha b x_{\lambda_2}(b) = 0. \quad (31) \]

We multiply (26) by \( x_{\lambda_2}(t) \) and (29) by \( x_{\lambda_1}(t) \), respectively, and subtract to obtain
\[ x_{\lambda_2}(t) {^C}L_1 x_{\lambda_1}(t) - x_{\lambda_1}(t) {^C}L_1 x_{\lambda_2}(t) = (\lambda_1 - \lambda_2) r(t)x_{\lambda_1}(t)x_{\lambda_2}(t). \]
Now, we integrate over the interval \([a,b]\) and apply (19) with \( u(t) = x_{\lambda_2}(t) \) and \( v(t) = x_{\lambda_1}(t) \) and vice versa to obtain
\[ (\lambda_1 - \lambda_2) \int_a^b r(t)x_{\lambda_1}(t)x_{\lambda_2}(t) dt = \]
\[ \frac{B(\alpha)}{1-\alpha} p(b) \left[ \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 b^{-}x_{\lambda_2}(b) {^{ABR}}D^\alpha b x_{\lambda_1}(b) - \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 b^{-}x_{\lambda_1}(b) {^{ABR}}D^\alpha b x_{\lambda_2}(b) \right] + \]
\[ \frac{B(\alpha)}{1-\alpha} p(a) \left[ \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 b^{-}x_{\lambda_2}(a) {^{ABR}}D^\alpha b x_{\lambda_1}(a) - \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 b^{-}x_{\lambda_1}(a) {^{ABR}}D^\alpha b x_{\lambda_2}(a) \right]. \]
Finally, by using the boundary conditions (27), (28), (30), and (31), we conclude that
\[ (\lambda_1 - \lambda_2) \int_a^b r(t)x_{\lambda_1}(t)x_{\lambda_2}(t) dt = 0, \]
and hence, as \( \lambda_1 \neq \lambda_2, < x_{\lambda_1}, x_{\lambda_2} >= 0. \]

**Remark 4.** The integration by parts formula in the second part of Proposition 1 suggests the following SLE:
\[ {^{ABC}}D^\alpha_0 (p(t) {^a}_{^{ABR}}D^\alpha t x(t)) + q(t)x(t) = \lambda r(t)x(t), \quad t \in (a, b), \quad (32) \]
with the boundary conditions:
\[ c_1 \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 a^{+} x(a) + c_2 {^a}_{^{ABR}}D^\alpha a x(a) = 0, \quad (33) \]
\[ d_1 \text{E}_{\alpha,1,\frac{\alpha}{1-\alpha}}^1 a^{+} x(b) + d_2 {^a}_{^{ABR}}D^\alpha a x(b) = 0, \quad (34) \]
where \( c_1^2 + c_2^2 \neq 0 \) and \( d_1^2 + d_2^2 \neq 0 \). Similar properties can also be proved for such a SLP (32)-(34) as proved for the SLP (17)-(18).

Denoting the SL operator as
\[ {^d}C{^L_1} x(t) = {^{ABC}}D^\alpha_a (p(t) {^a}_{^{ABR}}D^\alpha \nabla b x(t)) + q(t)x(t), \]
consider the following nabla discrete \( {^{ABC}} \) type SLE:
\[ {^{ABC}}\nabla^\alpha (p(t) {^{ABR}}\nabla b^\alpha x(t)) + q(t)x(t) = \lambda r(t)x(t), \quad t \in \mathbb{N}_{a+1, b-1}, \quad (35) \]
where $\alpha \in (0, 1/2)$, $p(t) \neq 0$, $r(t) > 0 \forall t \in \mathbb{N}_{a,b-1}$, $p, q, r$ are real valued functions on $\mathbb{N}_{a,b-1}$, and the boundary conditions:

\begin{align}
&c_1 \mathbb{E}_{\alpha,1,1-\alpha,b}^1(\cdot) x(a) + c_2 ABR\nabla_b^\alpha x(a) = 0, \tag{36} \\
&d_1 \mathbb{E}_{\alpha,1,1-\alpha,b}^1(\cdot) x(b-1) + d_2 ABR\nabla_b^\alpha x(b-1) = 0, \tag{37}
\end{align}

where $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$.

Now, we make use of the integration by parts formula in Theorem 4 to prove the corresponding properties for the SLP (35)-(37). For such a discrete fractional SLP, we use the following inner product

\[ <f,g> = \sum_{t=a+1}^{b-1} r(t)f(t)g(t). \] (38)

Remark 5. If we want to use the integration by parts formula stated in Remark 3, then we consider the following nabla discrete fractional SLE

\[ dC L x(t) = ABC \nabla^\alpha(p(t) ABR\nabla_{b+1}^\alpha x(t)) + q(t)x(t) = \lambda r(t)x(t), \quad t \in \mathbb{N}_{a+1,b}, \]

where $\alpha \in (0, 1/2)$, $p(t) \neq 0$, $r(t) > 0 \forall t \in \mathbb{N}_{a,b}$, $p, q, r$ are real valued functions on $\mathbb{N}_{a,b}$, and the boundary conditions:

\begin{align}
&c_1 \mathbb{E}_{\alpha,1,1-\alpha,b+1}^1(\cdot) x(a) + c_2 ABR\nabla_{b+1}^\alpha x(a) = 0, \\
&d_1 \mathbb{E}_{\alpha,1,1-\alpha,b+1}^1(\cdot) x(b) + d_2 ABR\nabla_{b+1}^\alpha x(b) = 0,
\end{align}

where $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$, together with the inner product

\[ <f,g> = \sum_{t=a+1}^{b} r(t)f(t)g(t). \]

Theorem 13. The eigenvalues of the nabla discrete fractional SLP (35)-(37) are real.

Proof. The proof is similar to that of Theorem 11. However, it follows by making use of the discrete fractional integration by parts in Theorem 4. The details are left to the reader. \qed

Theorem 14. The eigenfunctions, corresponding to distinct eigenvalues of the SLP (35)-(37) are orthogonal with respect to the weight function $r$ on $[a,b]$ that is

\[ <x_{\lambda_1}, x_{\lambda_2}> = \sum_{t=a+1}^{b-1} r(t)x_{\lambda_1}(t)x_{\lambda_2}(t) = 0, \quad \lambda_1 \neq \lambda_2, \]

when the functions $x_{\lambda_i}$ correspond to eigenvalues $\lambda_i$, $i = 1, 2$. 

Proof. The proof is similar to that of Theorem [12]. However, it follows by making use of the discrete fractional integration by parts in Theorem [4] and through the use of the inner product defined in (38). The details are left to the reader. □

Remark 6. From the proofs of Theorem [11] and Theorem [13], it follows that the operators $C_{L1}$ and its discrete version $dC_{L1}$ are self-adjoint. Similarly, from Remark [5], we see that the discrete operator $dC_{L2}$ is also self-adjoint.

3 The higher order discrete fractional SLE and an open problem

In the previous section, the values of $\alpha$ in the discrete SLE are taken in the interval $(0, \frac{1}{2})$ in order to guarantee the convergence of the discrete Mittag-Leffler kernel used to define the ABR and ABC fractional differences. Therefore, the ordinary difference SLE can not be obtained as $\alpha \to 1^-$. For this purpose, we shall recommend for a discrete fractional SLE with the values of $\alpha \in (1, \frac{3}{2})$ so that the ordinary difference case can be obtained as $\alpha \to 1^+$. For the main concepts regarding to higher order fractional calculus with discrete Mittag-Leffler kernel, we refer to [?]. For higher order fractional operators with non-singular Mittag-Leffler kernels, we refer the reader to [?], where a Lyapunov type inequality was formulated for a BVP of order $2 < \alpha < 3$ and the ordinary Lyapunov inequality was obtained as $\alpha \to 2^+$. Using Definition 2.1. in [?] and Definition [13] Lemma [2] and Remark [2] in our paper that if $\alpha \in (1, 2)$, $\beta = \alpha - 1 \in (0, 1)$, $\lambda_\beta = -\frac{\beta}{1-\beta} = -\frac{\alpha-1}{2-\alpha}$ and $|\lambda_\beta| < 1$ if $\alpha \in (1, \frac{3}{2})$, then the following four results hold:

\[
\begin{align*}
\text{ABC}_{a} \nabla^\alpha f(t) &= \text{ABC}_{a} \nabla^\beta \nabla f(t) = \frac{B(\alpha - 1)}{2 - \alpha} \nabla \text{E}^{\frac{1}{\beta-1, \lambda_\beta, a}} \nabla^2 f(t), \\
\text{ABR}_{a} \nabla^\alpha f(t) &= \text{ABR}_{a} \nabla^\beta \nabla f(t) = \frac{B(\alpha - 1)}{2 - \alpha} \nabla \text{E}^{\frac{1}{\beta-1, \lambda_\beta, a}} \nabla^2 f(t), \\
\text{ABC}_{b} \nabla^\alpha f(t) &= \text{ABC}_{b} \nabla^\beta (-\Delta f)(t) = \frac{B(\alpha - 1)}{2 - \alpha} \Delta \text{E}^{\frac{1}{\beta-1, \lambda_\beta, b^-}} \Delta^2 f(t), \\
\text{ABR}_{b} \nabla^\alpha f(t) &= \text{ABR}_{b} \nabla^\beta (-\Delta f)(t) = \frac{B(\alpha - 1)}{2 - \alpha} \Delta \text{E}^{\frac{1}{\beta-1, \lambda_\beta, b^-}} \Delta f(t).
\end{align*}
\]

Remark 7. Notice that the ordinary SLE can be obtained in the non-discrete case as $\alpha \to 1^-$. In fact, since non-discrete Mittag-Leffler kernels do not have a convergence problem, the ABR and ABC fractional operators become well-defined for any $\alpha \in (0, 1]$. For that reason, we present the following open problem in the discrete case.

Open problem: Depending on the above discussion, can we present integration by parts formulas for the ABR and ABC fractional differences of order $\alpha \in (1, \frac{3}{2})$ which will be used to prove the self-adjointness, eigenvalue, and eigenfunction properties of suitable fractional difference SLEs with proper boundary conditions?
4 Acknowledgements

This study was supported by The Scientific and Technological Research Council of Turkey while the first author visiting the University of Nebraska-Lincoln. The second author would like to thank Prince Sultan University for funding this work through the research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

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