BV-Quantization
of a
Noncommutative Yang–Mills Theory Toy Model

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Abstract
We review the Batalin-Vilkovisky quantization procedure for Yang–Mills theory on a 2-point space.

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In this talk we give a short summary of [1], where we proposed the quantization of one of the simplest toy models for noncommutative gauge theories which is (zero dimensional) Yang–Mills theory on a 2-point space.

Noncommutative geometry constitutes one of the fascinating new concepts in current theoretical physics research with many promising applications [2, 3, 4, 5, 6].

We quantize the Yang–Mills theory on a 2-point space by applying the standard Batalin–Vilkovisky method [7, 8]. Somewhat surprisingly we find that despite of the model’s original simplicity the gauge structure reveals infinite reducibility and the gauge fixing is afflicted with the Gribov [9] problem.

The basic idea of noncommutative geometry is to replace the notion of differential manifolds and functions by specific noncommutative algebras of functions. Following [10] we define the Yang–Mills Theory on a 2-point space in terms of the algebra $A = C \oplus C$ which is represented by diagonal complex valued $2 \times 2$ matrices. The differential p-forms are constant, diagonal or offdiagonal $2 \times 2$ matrices, depending on whether $p$ is even or odd, respectively. A nilpotent derivation $d$ acting on $2 \times 2$ matrices is defined by

$$ d a = i \begin{pmatrix} a_{21} + a_{12} & a_{22} - a_{11} \\ a_{11} - a_{22} & a_{21} + a_{12} \end{pmatrix} \quad \text{where} \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{ij} \in C. $$

The anti-Hermitean 1-forms $A$ can be parametrized by

$$ A = \begin{pmatrix} 0 & i \phi \\ -i \bar{\phi} & 0 \end{pmatrix} $$

and constitute the gauge fields of the model; here $\phi \in C$ denotes a (constant) scalar field. The (rigid) gauge transformations of $A$ are defined by

$$ A^U = U^{-1} A U + U^{-1} d U $$

with $U$ being a unitary element of the algebra $A$. It is a constant, diagonal and unitary matrix which can be parametrized by the diagonal matrix $\varepsilon$

$$ U = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} = e^{i\varepsilon}, \quad \varepsilon = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}. $$
Due to the nonabelian form of the gauge transformations [3] the $U(1) \times U(1)$ gauge model shares many interesting features with the standard Yang–Mills theory, yet it has no physical space-time dependence and allows extremely simple calculations.

We define a scalar product for $2 \times 2$ matrices $a, b$ by $\langle a | b \rangle = \text{tr} \, a^\dagger \, b$ where $^\dagger$ denotes taking the Hermitian conjugate. The curvature $\mathcal{F}$ is defined as usual by $\mathcal{F} = dA + A \, A$ and for an action which is automatically invariant under the gauge transformations [3] one takes

$$S_{\text{inv}} = \frac{1}{2} \langle \mathcal{F} | \mathcal{F} \rangle = \left( (\phi + \bar{\phi}) + \phi \, \bar{\phi} \right)^2 . \quad (4)$$

To discuss infinitesimal (zero-stage) gauge transformations we introduce a diagonal infinitesimal (zero-stage) gauge parameter matrix $\varepsilon^0_e$ in terms of which $U \simeq 1 + \varepsilon^0_e$. The infinitesimal (zero-stage) gauge variation of $A$ derives as

$$\delta \varepsilon^0_e \, A = i R^0 \, \varepsilon^0_e \text{ where } R^0 = D; \quad (5)$$

here the (zero-stage) gauge generator $R^0$ is defined in terms of the covariant matrix derivative $D$, which acting on $\varepsilon^0_e$ is given by $D\varepsilon^0_e = d\varepsilon^0_e + [A, \varepsilon^0_e]$.

A gauge symmetry is called irreducible if the (zero stage) gauge generator $R^0$ does not possess any zero mode. It is amusing to note that the Yang–Mills theory on the 2-point space reveals an infinitely reducible gauge symmetry: We observe that $D \, d$ is vanishing on arbitrary offdiagonal matrices. Thus there exists a zero mode $\varepsilon^1_e$ for the (zero-stage) gauge generator $R^0$, such that

$$R^0 \, \varepsilon^1_e = 0 \text{ where } \varepsilon^1_e = R^1 \, \varepsilon^1_o \text{ with } R^1 = d. \quad (6)$$

Here $\varepsilon^1_o$ denotes an offdiagonal, infinitesimal (first-stage) gauge parameter matrix and $R^1$ the corresponding (first-stage) gauge generator. As a matter of fact an infinite tower of (higher-stage) gauge generators $R^s$, $s = 1, 2, 3, \cdots$ with never ending gauge invariances for gauge invariances is arising: We define $R^s = d$ for $s = 1, 2, 3, \cdots$ so that for each gauge generator there exists an additional zero mode

$$R^1 \, \varepsilon^2_o = 0, \text{ where } \varepsilon^2_o = R^2 \, \varepsilon^2_e$$
\[ \mathbf{R}^2 \varepsilon_e^3 = 0, \quad \text{where} \quad \varepsilon_e^3 = \mathbf{R}^3 \varepsilon_o^3 \]
\[ \cdots \cdots \cdots \quad \cdots \cdots \cdots \]

due to the nilpotency \( d^2 = 0 \).

Now we straightforwardly apply the usual field theory BV-path integral quantization scheme \([4, 8]\) to the 2-point model: In addition to the original gauge field \( A \equiv C_{-1} \) we introduce ghost fields \( C_s^k, \infty \geq s \geq -1, s \geq k \geq -1 \) with \( k \) odd, as well as auxiliary ghost fields \( \tilde{C}_s^k, \infty \geq s \geq 0, s \geq k \geq 0 \) with \( k \) even. Furthermore we add Lagrange multiplier fields \( \pi_s^k, \infty \geq s \geq 1, s \geq k \geq 1 \) with \( k \) odd and \( \tilde{\pi}_s^k, \infty \geq s \geq 0, s \geq k \geq 0 \) with \( k \) even; finally we introduce antifields \( C_s^{k*}, \tilde{C}_s^{k*} \). The BV-action obtains as
\[
S_{BV} = S_{inv} + S_{aux} - \langle C_{-1}^{-1} | DC_0^{-1} \rangle - \sum_{s=1,3,5,...}^\infty \langle C_s^{-1} | d C_{s+1}^{-1} \rangle + \sum_{s=0,2,4,...}^\infty i \langle C_s^{-1} | d C_{s+1}^{-1} \rangle, \tag{8}
\]

where we denote by \( S_{aux} \) the auxiliary field action
\[
S_{aux} = \sum_{s=0,2,4,...}^\infty \sum_{k=0,2,4,...}^\infty \langle \pi_s^k | \tilde{C}_s^{k*} \rangle + \sum_{k=1,3,5,...}^\infty \sum_{s=0,2,4,...}^\infty \langle C_s^{k*} | \pi_s^k \rangle. \tag{9}
\]

Gauge fixing conditions similar to the usual Feynman gauge are implemented by introducing the gauge fixing fermion \( \Psi = \Psi_\delta + \Psi_\pi \)
\[
\Psi_\delta = \sum_{s=0,2,4,...}^\infty \sum_{k=0,2,4,...}^\infty \left( -\langle \tilde{C}_s^k | \delta C_{s-1}^k \rangle + \langle \delta C_{s+1}^k | C_{s+2}^k \rangle \right) + i \langle \tilde{C}_{s+1}^k | \delta C_s^{k-1} \rangle + i \langle \delta C_s^k | C_{s+1}^{k+1} \rangle,
\]
\[
\Psi_\pi = \frac{1}{2} \sum_{s=0,2,4,...}^\infty \sum_{k=0,2,4,...}^\infty \left( \langle \tilde{C}_s^k | \tilde{\pi}_{s+1}^{k+1} \rangle + \langle \tilde{\pi}_s^k | C_{s+1}^{k+1} \rangle \right) + i \langle \tilde{C}_{s+1}^k | \tilde{\pi}_{s+1}^{k+1} \rangle + i \langle \tilde{\pi}_{s+1}^k | C_{s+1}^{k+1} \rangle + \frac{1}{2} \sum_{k=1,3,5,...}^\infty \langle \tilde{C}_s^k | \tilde{\pi}_s^k \rangle. \tag{10}
\]

By \( \delta \) we denote a nilpotent coderivative operator \( \delta a = i \left( \begin{array}{cc} a_{12} & -a_{21} \\ -a_{11} & a_{22} \end{array} \right) \)
where \( a = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right), \quad a_{ij} \in \mathbb{C} \). We eliminate the antifields by using the gauge fixing
fermion $\Psi$ via

$$\langle C_s^k \rangle = \frac{\partial \Psi}{\partial |C_s^k|}, \quad \langle \bar{C}_s^k \rangle = \frac{\partial \Psi}{\partial |\bar{C}_s^k|},$$  \hspace{1cm} (11)$$

so that the gauge fixed action $S_{\Psi}$ reads

$$S_{\Psi} = S_{inv} - i\langle C_0^0 | \delta D C_0^{-1} \rangle$$

$$- i \sum_{s=1,3,5,\ldots} \langle \bar{C}_{s+1}^0 | \delta d C_{s+1}^{-1} \rangle + \sum_{s=0,2,4,\ldots} \langle \bar{C}_{s+1}^0 | \delta d C_{s+1}^{-1} \rangle$$

$$+ \sum_{k=0,2,4,\ldots} \sum_{s=k+1, odd} \langle \bar{n}_k^s | \pi_{k+1}^s \rangle + \langle \bar{n}_s^k | (i \delta \bar{C}_{s-1} + dC_{s+1}^{k+1}) \rangle$$

$$+ \langle (i \delta \bar{C}_{s-1} - dC_{s+1}^{k+2}) | \pi_s^k \rangle)$$

$$+ \sum_{k=0,2,4,\ldots} \sum_{s=k+2, even} \langle \bar{n}_k^s | \pi_{k+1}^s \rangle + \langle \bar{n}_s^k | (-\delta C_{s-1} + i dC_{s+1}^{k+1}) \rangle$$

$$+ \langle (\delta \bar{C}_{s-1} + i dC_{s+1}^{k+2}) | \pi_s^k \rangle).$$  \hspace{1cm} (12)$$

We can now eliminate the Lagrange multiplier fields $\pi_s^k$ and $\bar{n}_s^k$ and arrive at

$$S_{\Psi} \longrightarrow S_{inv} + \frac{1}{2} \langle A | d\delta A \rangle - i \langle \bar{C}_0^0 | (\delta D + d\delta) C_0^{-1} \rangle$$

$$- i \sum_{s=1,3,5,\ldots} \langle \bar{C}_{s+1}^0 | (\delta d + d\delta) C_{s+1}^{-1} \rangle)$$

$$+ \sum_{s=0,2,4,\ldots} \langle \bar{C}_{s+1}^0 | (\delta d + d\delta) C_{s+1}^{-1} \rangle$$

$$- i \sum_{k=0,2,4,\ldots} \sum_{s=k+1, odd} \langle \bar{C}_{s+1}^{k+2} | (\delta d + d\delta) C_{s+1}^{k+1} \rangle$$

$$+ \sum_{k=0,2,4,\ldots} \sum_{s=k+2, even} \langle \bar{C}_{s+1}^{k+2} | (\delta d + d\delta) C_{s+1}^{k+1} \rangle$$

$$+ \frac{1}{2} \sum_{k=0,2,4,\ldots} \langle C_{k+1}^{k+1} | (\delta d + d\delta) C_{k+1}^{k+1} \rangle.$$  \hspace{1cm} (13)$$

All the higher-stage ghost contributions can be integrated away as $\delta d + d\delta = 4 \cdot 1$ and we simply obtain

$$S_{\Psi} \longrightarrow S_{inv} + \frac{1}{2} \langle A | d\delta A \rangle - i \langle \bar{C}_0^0 | (\delta D + d\delta) C_0^{-1} \rangle.$$  \hspace{1cm} (14)$$
We summarize that the zero dimensional Yang–Mills theory model on a 2-point space reveals infinite reducibility; after applying the standard BV-quantization procedure the action finally contains invertible quadratic parts for the gauge field, as well as for the ghost fields. A closer inspection [1] shows that the model suffers from a Gribov problem [9].

We expect that our present investigations will lead to a study of the renormalization effects at higher orders; it may also be possible to compare the perturbative calculations with explicit analytic integrations (for related attempts see [11]).

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