Cohomological rigidity for Fano Bott manifolds

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Abstract
In the present paper, we characterize Fano Bott manifolds up to diffeomorphism in terms of three operations on matrix. More precisely, we prove that given two Fano Bott manifolds \( X \) and \( X' \), the following conditions are equivalent:

1. the upper triangular matrix associated to \( X \) can be transformed into that of \( X' \) by those three operations;
2. \( X \) and \( X' \) are diffeomorphic;
3. the integral cohomology rings of \( X \) and \( X' \) are isomorphic as graded rings.

As a consequence, we affirmatively answer the cohomological rigidity problem for Fano Bott manifolds.

Keywords Cohomological rigidity · Toric Fano manifold · Bott manifold

Mathematics Subject Classification Primary 57R19; Secondary 14M25 · 14J45 · 57S15

1 Introduction

1.1 Cohomological rigidity problem

A toric variety is a normal complex algebraic variety with an algebraic action of a \( \mathbb{C}^* \)-torus which has an open dense orbit. In what follows, we call a compact smooth toric variety a toric manifold.

A fundamental fact on toric varieties claims that each toric variety one-to-one corresponds to a combinatorial object, called a fan. Moreover, the compactness and the smoothness of toric varieties can be interpreted in terms of the associated fans. It is well known that the
set of toric manifolds up to algebraic varieties one-to-one corresponds to the set of complete nonsingular fans up to unimodular equivalence. Moreover, it is also known by Batyrev [2] that toric varieties are isomorphic as algebraic varieties if and only if there is a bijection between primitive collections which preserves their associated primitive relations (see Sect. 2.2 for the details of them). In particular, the classification of toric manifolds as algebraic varieties is completely done by using combinatorial objects.

However, the classification of toric manifolds as smooth manifolds is not established yet. Inspired by a classification of a certain class of Bott manifolds up to diffeomorphism in [14], the following naive problem was proposed:

Problem 1.1 (cf. [15], Cohomological rigidity problem for toric manifolds) Are two toric manifolds diffeomorphic if their integral cohomology rings are isomorphic as graded rings?

We denote the integral cohomology ring $H^*(X; \mathbb{Z})$ of a toric manifold $X$ by $H^*(X)$. In what follows, we omit “integral” of integral cohomology.

We say that a family $\mathcal{F}$ of toric manifolds is cohomologically rigid if any two toric manifolds in $\mathcal{F}$ whose cohomology rings are isomorphic as graded rings are diffeomorphic.

No counterexample to the cohomological rigidity problem for toric manifolds is known. Towards the solution of Problem 1.1, many results have been obtained, all of which affirmatively answer the problem. Many results are related to Bott manifolds [4–10, 14, 15]. Those will be explained below.

1.2 Bott manifolds and toric Fano manifolds

The main object of the present paper is Bott manifolds. Note that if $B$ is a toric manifold and $E$ is a Whitney sum of complex line bundles over $B$, then the projectivization $P(E)$ of $E$ is again a toric manifold. Starting with $B$ as a point and repeating this construction, say $d$ times, we obtain a sequence of toric manifolds as follows:

$$B_d \xrightarrow{\pi_d} B_{d-1} \xrightarrow{\pi_{d-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{ \text{a point} \},$$

where the fiber of $\pi_i : B_i \to B_{i-1}$ for $i = 1, \ldots, d$ is a complex projective space $\mathbb{CP}^{n_i}$. This sequence is called a generalized Bott tower of height $d$, and we call $B_d$ a $d$-stage generalized Bott manifold. We say that $B_d$ is a $d$-stage Bott manifold (omitted “generalized”) when $n_i = 1$ for every $i$.

Bott manifolds are very well-studied objects in the area of toric topology. In fact, there are many results on cohomological rigidity problem for (generalized) Bott manifolds. The following theorems are a part of the known results on cohomological rigidity problem for toric manifolds concerning with (generalized) Bott manifolds.

Theorem 1.2 [10, Theorem 1.1] Let $X$ be a $d$-stage generalized Bott manifold. If $H^*(X)$ is isomorphic to $H^*(\prod_{i=1}^d \mathbb{CP}^{n_i})$ as graded rings, then every fibration is topologically trivial; in particular, $X$ is diffeomorphic to $\prod_{i=1}^d \mathbb{CP}^{n_i}$.

Theorem 1.3 [4, 10] The following results are known:

- 2-stage generalized Bott manifolds are cohomologically rigid [10, Theorem 1.3];
- 3-stage Bott manifolds are cohomologically rigid [10, Theorem 1.3];
- 4-stage Bott manifolds are cohomologically rigid [4, Theorem 3.3].

Note that Theorem 1.3 is still open for $d$-stage Bott manifolds with $d \geq 5$.

The following is one of the most important contributions to cohomological rigidity problem for Bott manifolds:
Theorem 1.4 [7, Theorem 1.1] Any graded ring isomorphism between the cohomology rings of two Bott manifolds preserves their Pontrjagin classes.

Many researchers also investigate toric Fano manifolds. Here, we say that a nonsingular projective variety is Fano if its anticanonical divisor is ample. Note that there are only finitely many toric Fano $d$-folds for each fixed $d$. The classification problem of toric Fano $d$-folds is of particular interest in the study of toric Fano manifolds. The classification of toric Fano 4-folds was accomplished by Batyrev [2] and Sato [18]. The key tool for this classification is primitive collections and primitive relations (see Sect. 2.2). After their classification, Øbro [16] succeeded in constructing an algorithm, called SFP algorithm, which produces the complete list of smooth Fano $d$-polytopes for a given positive integer $d$. For small $d$’s, the database of all smooth Fano $d$-polytopes is available in the following web page:

$$\text{http://www.grdb.co.uk/forms/toricsmooth}$$  \hspace{1cm} (1.1)

The classification problem of toric Fano manifolds was solved in some sense. By using those classifications, the authors and Masuda proved the following:

Theorem 1.5 [12, Theorem 1.1] Toric Fano $d$-folds with $d \leq 4$ are cohomologically rigid except for possibly two toric Fano 4-folds.

Note that the exceptional toric Fano 4-folds have ID 50 and ID 57 in the above web page. Furthermore, the following has been also proved in [5] by Cho, Lee, Masuda and Park:

Theorem 1.6 [5, Theorem 1.2] For two Fano Bott manifolds $X$ and $X'$, if there is an isomorphism between their cohomology rings which preserves the first Chern class, then $X$ and $X'$ are isomorphic as algebraic varieties.

In [8, 13], cohomological rigidity problem for real Bott manifolds is investigated. Here, we call a manifold $X$ a real Bott manifold if there is a sequence of $\mathbb{R}P^1$ bundles such that for each $j = 1, \ldots, d$, $B_j \to B_{j-1}$ is the projective bundle of the Whitney sum of a real line bundle and the trivial real line bundle over $B_{j-1}$, where $B_0$ is a point. It is well known that any $d$-stage real Bott manifold is determined by a certain upper triangular matrix with its entry in $\mathbb{Z}/2 = \{0, 1\}$, called a Bott matrix. Moreover, in [8, Section 3], three operations for Bott matrices are introduced. We say that two Bott matrices are Bott equivalent if those can be transformed by those three matrix operations.

The following theorem gives a complete characterization of real Bott manifolds in terms of Bott matrices and answers the cohomological rigidity problem for real Bott manifolds in some certain sense.

Theorem 1.7 ([8, Theorem 1.1], Classification of real Bott manifolds) Let $A$ and $B$ be Bott matrices and let $X(A)$ and $X(B)$ be the associated real Bott manifolds to $A$ and $B$, respectively. Then the following three conditions are equivalent:

1. $A$ and $B$ are Bott equivalent;
2. $X(A)$ and $X(B)$ are affinely diffeomorphic;
3. $H^*(X(A)) \otimes \mathbb{Z}/2$ and $H^*(X(B)) \otimes \mathbb{Z}/2$ are isomorphic as graded rings.

Note that the equivalence of (2) and (3) was originally proved in [13, Theorem 1.1].

Our motivation to organize the present paper is to obtain an analogue of Theorem 1.7 for Fano Bott manifolds.
1.3 Main Result

The main result of the present paper is the following:

**Theorem 1.8** (Main Result). *Let X and X' be Fano Bott manifolds and let \( A(X) \) and \( A(X') \) be the upper triangular matrices associated to X and X', respectively. Then the following three conditions are equivalent:

1. \( A(X) \) and \( A(X') \) are Fano Bott equivalent;
2. \( X \) and \( X' \) are diffeomorphic;
3. \( H^*(X) \) and \( H^*(X') \) are isomorphic as graded rings.*

See Sect. 5 for the precise definition of \( A(X) \) and Fano Bott equivalence. The key idea of the proof of Theorem 1.8 relies on the identification of Fano Bott manifolds with rooted signed forests (see Sect. 4).

**Remark 1.9** By the proof of Theorem 1.8, we see that the three conditions (1)–(3) are equivalent to the following fourth condition:

(4) the signed rooted forests \( T_X \) and \( T_{X'} \) have equivalent signs by exchanging the signs assigned to the edges adjacent to the roots if necessary.

See Sect. 6.

As an immediate corollary of Theorem 1.8, we conclude the following:

**Corollary 1.10** *Fano Bott manifolds are cohomologically rigid.*

Note that in the case \( d \leq 4 \), the cohomological rigidity for \( d \)-stage Bott manifolds is already known by [4, Theorem 3.3] as well as [12, Theorem 1.1].

1.4 Structure of the paper

An overview of the present paper is as follows. In Sect. 2, we recall the theory of toric varieties (e.g. primitive collections and primitive relations) and the description of cohomology rings of toric manifolds. We also introduce the invariants (s.v.e. and maximal basis number) on cohomology rings. We also give a sufficient condition (Lemma 2.7) for two Bott manifolds to be diffeomorphic. In Sect. 3, we introduce the upper triangular matrix \( A(X) \) arising from a Fano Bott manifold \( X \) and describe the cohomology ring \( H^*(X) \) of \( X \) in terms of \( A(X) \). We also discuss s.v.e. and maximal basis number of \( H^*(X) \) (Lemma 3.3). In Sect. 4, we associate signed rooted forests \( T_X \) from Fano Bott manifolds \( X \) and we prove a key proposition for the proof of Theorem 1.8 (Proposition 4.5). In Sect. 5, we introduce three operations on the upper triangular matrices associated to Fano Bott manifolds and the notion of Fano Bott equivalence. We also see that those operations correspond to certain operations on the signs of some edges of the signed rooted forest (Proposition 5.7). Finally, in Sect. 6, after preparing some more lemmas, we give a proof of Theorem 1.8.

2 Preliminaries

In this section, we recall the well-known description of the cohomology rings of toric manifolds. For our discussions, we recall the notions, *primitive collections* and *primitive relations*, which represent the toric variety in terms of linear relations of primitive ray vectors of the
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associated fan of a toric variety. We also introduce the invariants of cohomology rings to distinguish cohomology rings. At last, we give a sufficient condition for two toric manifolds to be diffeomorphic (Lemma 2.7).

2.1 Complete nonsingular fans and toric manifolds

Please consult e.g. [11] or [17] for the introduction to toric varieties and the associated fans.

First, we recall the notion of complete nonsingular fans and their underlying simplicial complexes. A fan of dimension $d$ is a collection $\Sigma$ of rational polyhedral pointed cones in $\mathbb{R}^d$ such that

i) each face of a cone $\sigma$ in $\Sigma$ also belongs to $\Sigma$, and
ii) the intersection of two cones in $\Sigma$ is also a face of each of those cones.

It is well known that the set of toric varieties of complex dimension $d$ up to algebraic isomorphism one-to-one corresponds to the set of fans of dimension $d$ up to unimodular equivalence. For a toric variety $X$, let $F_X$ be the corresponding fan. Similarly, for a fan $\Sigma$, let $X_{\Sigma}$ be the corresponding toric variety. It is known that a toric variety $X$ is compact if and only if $F_X$ is complete, i.e., $\bigcup_{\sigma \in F_X} \sigma = \mathbb{R}^d$, and $X$ is smooth if and only if $F_X$ is nonsingular, i.e., the set of primitive ray generators of every maximal cone in $F_X$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. Hence, the set of toric manifolds (namely, compact smooth toric varieties) one-to-one corresponds to the set of complete nonsingular fans.

For a fan $\Sigma$, let $\Sigma_1$ be the set of primitive ray generators of 1-dimensional cones in $\Sigma$. Given a complete nonsingular fan $\Sigma$, since nonsingular fans are simplicial, we see that $\Sigma$ has a structure of an abstract simplicial complex. Let $\Sigma_1 = \{v_1, \ldots, v_m\}$. We define $K(\Sigma) = \{I \subset [m] : \text{cone}(v_i : i \in I) \in \Sigma\}$, where $[m] = \{1, \ldots, m\}$. We call $K(\Sigma)$ the underlying simplicial complex of $\Sigma$.

2.2 Primitive collections and primitive relations

Next, we recall what primitive collections and primitive relations are.

Definition 2.1 (Primitive collection, [1, Definition 2.6]) For a complete nonsingular fan $\Sigma$, we call a nonempty subset $P = \{x_1, \ldots, x_k\} \subset \Sigma_1$ of $\Sigma_1$ a primitive collection of $\Sigma$ if for each generator $x_i \in P$ the elements of $P\setminus\{x_i\}$ generate a $(k - 1)$-dimensional cone in $\Sigma$, while $P$ does not generate any $k$-dimensional cone in $\Sigma$. In other words, $P$ is a minimal non-face of $K(\Sigma)$. Let $PC(\Sigma)$ be the set of all primitive collections of $\Sigma$.

Definition 2.2 (Primitive relation, [1, Definition 2.8]) For a complete nonsingular fan $\Sigma$, let $P = \{x_1, \ldots, x_k\}$ be a primitive collection of $\Sigma$. Let $\sigma$ be the cone in $\Sigma$ of the smallest dimension containing $x_1 + \cdots + x_k$ and let $y_1, \ldots, y_m \in \Sigma_1$ be the minimal system of primitive ray generators of $\sigma$. Then there exists a unique linear combination $n_1y_1 + \cdots + n_my_m$ with positive integer coefficients $n_i$ which is equal to $x_1 + \cdots + x_k$. We call the linear relation

\[ x_1 + \cdots + x_k = n_1y_1 + \cdots + n_my_m \] (2.1)

the primitive relation associated with $P$. 

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Given a primitive collection $\mathcal{P}$ with its associated primitive relation $(2.1)$, let $\deg(\mathcal{P}) = k - (n_1 + \cdots + n_m)$, which we call the degree of $\mathcal{P}$.

We can verify whether the toric manifold $X$ is Fano or not by the degrees of primitive relations of $\Sigma_X$.

Proposition 2.3 [2, Proposition 2.3.6] A toric manifold $X$ is Fano if and only if we have $\deg(\mathcal{P}) > 0$ for every primitive collection $\mathcal{P}$ of $\Sigma_X$.

Given two complete nonsingular fans $\Sigma$ and $\Sigma'$, we say that $\text{PC}(\Sigma)$ and $\text{PC}(\Sigma')$ are isomorphic if there is a bijection between $\Sigma_1$ and $\Sigma'_1$ which induces a bijection between not only $\text{PC}(\Sigma)$ and $\text{PC}(\Sigma')$ but also their primitive relations.

Proposition 2.4 [2, Proposition 2.1.8 and Theorem 2.2.4] Two toric Fano manifolds $X$ and $X'$ are isomorphic as algebraic varieties if and only if $\text{PC}(\Sigma_X)$ and $\text{PC}(\Sigma_{X'})$ are isomorphic.

2.3 Cohomology rings of toric manifolds and their invariants

Next, we recall the description of the cohomology rings of toric manifolds.

Proposition 2.5 [3, Theorem 5.3.1] Let $X$ be a toric manifold of complex dimension $d$, let $\Sigma = \Sigma_X$ and let $\Sigma_1 = \{v_1, \ldots, v_m\} \subset \mathbb{R}^d$. Then the cohomology ring of $X$ can be described as follows:

$$H^*(X) \cong \mathbb{Z}[x_1, \ldots, x_m]/(I_X + J_X),$$

where

$$I_X = \left( \prod_{i \in F} x_i : F \subset [m], \{v_i : i \in F\} \in \text{PC}(\Sigma_X) \right)$$

and $J_X = \left( \sum_{j=1}^{m} v_{ij}^j x_j : i = 1, \ldots, d \right)$

and $v_{ij}$ denotes the $i$-th entry of $v_j \in \mathbb{R}^d$.

In order to distinguish cohomology rings up to isomorphism, we prepare the invariants on cohomology rings.

Definition 2.6 (s.v.e., maximal basis number) Let $X$ be a toric manifold. A nonzero linear form in $H^*(X)$ (i.e., an element of $H^2(X)$) is said to be s.v.e. (square vanishing element) if it is primitive and its second power vanishes in $H^*(X)$. Note that this notion was already introduced in [6].

Let $V$ be the set of all s.v.e. of the cohomology ring $H^*(X)$ of $X$. Define

$$B = \{ S \subset V : S \text{ is a part of a } \mathbb{Z}\text{-basis of } H^*(X) \}. $$

Then there exists $S_{\text{max}} \in B$ such that $|S| \leq |S_{\text{max}}|$ for any $S \in B$. We call a set $S_{\text{max}}$ a maximal basis of s.v.e. of $H^*(X)$ and $|S_{\text{max}}|$ a maximal basis number of $H^*(X)$.

Note that the number of s.v.e. and the maximal basis number are invariants of $H^*(X)$. We refer the readers to [12, Examples 2.7 and 2.9] for examples of s.v.e and maximal basis numbers of $H^*(X)$ and how to compute them.
2.4 Diffeomorphism lemma

Finally, we recall the key lemma for the proof of Theorem 1.8. The following lemma directly follows from [12, Lemma 2.3].

Lemma 2.7 (cf. [12, Lemma 2.3]) Let $X, X'$ be $d$-stage Fano Bott manifolds and let $\Sigma, \Sigma'$ be the associated complete nonsingular fans, respectively. Let $\Sigma_1 = \{v_1, \ldots, v_{2d}\}$ and let $\Sigma'_1 = \{v'_1, \ldots, v'_{2d}\}$. If $v_i = \pm v'_i$ for each $i = 1, \ldots, 2d$ by reordering $v_i$'s if necessary such that $K(\Sigma)$ is unchanged, then $X$ and $X'$ are diffeomorphic.

3 Fano Bott manifolds and their cohomology rings

In this section, we introduce upper triangular matrices $(n_{ij})$ associated to Fano Bott manifolds. The cohomology rings of Fano Bott manifolds can be described by using such matrices (see (3.3)). We discuss the s.v.e. of their cohomology rings (Lemma 3.3). We refer the reader to [3, Section 7.8] for the introduction to Bott manifolds.

3.1 Upper triangular matrices associated to Fano Bott manifolds

Let $X$ be a $d$-stage Fano Bott manifold and let $\Sigma = \Sigma_X$. Then $|\Sigma_1| = 2d$. Let $\Sigma_1 = \{v_i^\pm : i = 1, \ldots, d\}$. In general, the underlying simplicial complex of the associated fan of any Bott manifold is the boundary complex of the cross-polytope of dimension $d$. Thus, the primitive collections look like $\{v_i^+, v_i^-\}$ for each $i$. Moreover, by Proposition 2.3, the associated primitive relations of $PC(\Sigma)$ look as follows:

$$v_1^+ + v_1^- = v_1^{\sigma(1)},$$
$$v_2^+ + v_2^- = v_2^{\sigma(2)},$$
$$\vdots$$
$$v_d^+ + v_d^- = v_d^{\sigma(d)},$$

where $\varphi$ is a map $\varphi : [d] \to [d + 1]\backslash\{1\}$ satisfying that there is $k_i$ with $\varphi^{k_i}(i) = d + 1$ for each $i \in [d]$ and $\sigma$ is a map $\sigma : [d] \to \{\pm\}$. Note that the condition “there is $k_i$ with $\varphi^{k_i}(i) = d + 1$ for each $i \in [d]$” is derived from the linear independence of $v_1^{\epsilon_1}, \ldots, v_d^{\epsilon_d}$ for any choice of $\epsilon_i \in \{\pm\}$. Up to the equivalence of $PC(\Sigma)$, we may assume that $\varphi$ satisfies that $i < \varphi(i) \leq d + 1$ for each $i$.

In what follows, we will always assume (3.2).

In general, $d$-stage Bott manifolds are determined by a collection of integers $(n_{ij})_{1 \leq i < j \leq d}$ arranged in an upper triangular matrix form (see [3, Section 7.8]). In the case of Fano Bott manifolds, we can construct this upper triangular matrix $(n_{ij})$ from the primitive relations as follows. Since we have $v_i^- = -v_i^+ + v_{\varphi(i)}^{\sigma(i)}$ for each $i$, by the condition (3.2), we have

$$v_i^+ + v_i^- = -v_i^{\sigma(i)} - v_i^{\varphi^2(i)} - \cdots - v_i^{\varphi^{k_i-1}(i)} + v_i^{\varphi^{k_i}(i)}.$$

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Thus, we can rewrite the primitive relations (3.1) as follows:

\[
\begin{align*}
\mathbf{v}^+ + \mathbf{v}^- &= n^+_{12} \mathbf{v}^+_2 + n^+_{13} \mathbf{v}^+_3 + n^+_{14} \mathbf{v}^+_4 + \cdots + n^+_{1d} \mathbf{v}^+_d, \\
\mathbf{v}^+ + \mathbf{v}^- &= n^+_{23} \mathbf{v}^+_3 + n^+_{24} \mathbf{v}^+_4 + \cdots + n^+_{2d} \mathbf{v}^+_d, \\
\mathbf{v}^+ + \mathbf{v}^- &= n^+_{34} \mathbf{v}^+_4 + \cdots + n^+_{3d} \mathbf{v}^+_d, \\
\vdots \\
\mathbf{v}^+ + \mathbf{v}^- &= n^+_{d-1,d} \mathbf{v}^+_d, \\
\mathbf{v}^+ + \mathbf{v}^- &= 0,
\end{align*}
\]

where \( n_{ij} \in \{0, \pm 1\} \).

By [19], Fano Bott manifolds are characterized in terms of \((n_{ij})_{1 \leq i < j \leq d}\).

**Theorem 3.1** [19, Theorem 8] Let \( X \) be a \( d \)-stage Bott manifold and let \((n_{ij})_{1 \leq i < j \leq d}\) be the associated upper triangular matrix. Then \( X \) is Fano if and only if for any \( p = 1, \ldots, d - 1 \), one of the following conditions holds:

1. \( n_{p,p+1} = \cdots = n_{pd} = 0 \),
2. \( n_{pr} = \begin{cases} 
1 & (r = q) \\
0 & (r \neq q) 
\end{cases} \) for \( r = p + 1, \ldots, d \),
3. \( n_{pr} = \begin{cases} 
-1 & (r = q + 1, \ldots, d) \\
q_r & (r = q + 1, \ldots, d) 
\end{cases} \).

Let \( R^{d \times d} \) denote the set of all \( d \times d \) matrices whose entries are in the set \( R \).

**Definition 3.2** Given a \( d \)-stage Fano Bott manifold \( X \), let

\[ A(X) = (n_{ij})_{1 \leq i, j \leq d} \in \{0, \pm 1\}^{d \times d} \]

be the upper triangular matrix whose upper triangle part is equal to \( n_{ij} \) defined as above and whose remaining parts are all 0. Moreover, let

\[ \mathcal{FB}(d) = \{ A(X) \in \{0, \pm 1\}^{d \times d} : X \text{ is a } d \text{-stage Fano Bott manifold} \} \]

Namely, \( \mathcal{FB}(d) \) consists of all upper triangular matrices satisfying the condition in Theorem 3.1.

### 3.2 Cohomology rings of Fano Bott manifolds and their s.v.e.

For a \( d \)-stage Fano Bott manifold \( X \), by using the upper triangular matrix \((n_{ij})\) associated to \( X \), the cohomology ring \( H^*(X) \) can be written as follows (see, e.g., [5, (2.5)]):

\[
H^*(X) \cong \mathbb{Z}[x_1, \ldots, x_d]/\mathcal{I}, \quad \text{where} \\
\mathcal{I} = (f_i : i = 1, \ldots, d) \quad \text{and} \\
f_i = x_i^2 - (n_{i1}x_1 + \cdots + n_{i-1,i}x_{i-1})x_i \quad \text{for } i = 1, \ldots, d.
\]
For the analysis of s.v.e. of $H^*(X)$, we perform the following computations. Let $a_i \in \mathbb{Z}$. Then we have

$$(a_1 x_1 + \cdots + a_d x_d)^2 = \sum_{i=1}^{d} a_i^2 x_i^2 + \sum_{1 \leq i < j \leq d} 2a_i a_j x_i x_j$$

$$= \sum_{i=1}^{d} a_i^2 f_i + \sum_{j=1}^{d} a_j^2 (n_1 j x_1 + \cdots + n_{j-1} j x_{j-1}) x_j$$

$$+ \sum_{1 \leq i < j \leq d} 2a_i a_j x_i x_j$$

$$= \sum_{i=1}^{d} a_i^2 f_i + \sum_{1 \leq i < j \leq d} a_j (a_j n_{ij} + 2a_i) x_i x_j.$$ 

Hence, we obtain the following:

$$(a_1 x_1 + \cdots + a_d x_d)^2 = 0 \text{ in } H^*(X)$$

$$\iff \text{“}a_2 = 0\text{” or “}n_{12} a_2 = -2a_1\text{”, and “}a_3 = 0\text{” or “}n_{13} a_3 = -2a_1 \text{ and } n_{23} a_3 = -2a_2\text{”, and “}a_d = 0\text{” or “}n_{1d} a_d = -2a_1 \text{ and } \cdots \text{ and } n_{d-1,d} a_d = -2a_{d-1}\text{”}.$$

Let $S(H^*(X))$ be the set of all s.v.e. of $H^*(X)$. Then we see the following:

**Lemma 3.3** Work with the same notation as above. Then $S(H^*(X))$ can be divided into three disjoint subsets \{$(g_i), (g'_i) \text{ and } (h_j)$\} with some $I, J \subset [d]$ satisfying the following:

1. \{$(g_i)_{i\in I'} \cup (g'_i)_{i\in I' \setminus I} \cup (h_j)_{j\in J}$\} is a maximal basis of $S(H^*(X))$ for any (possibly empty) $I' \subset I$;
2. $H^*(X)/(g_i) \cong H^*(X)/(g'_i)$ for each $i$.

**Proof (The first step):** Let $g = a_1 x_1 + \cdots + a_d x_d \in S(H^*(X))$. First, we determine what kind of $g$ appears.

The case $a_p \neq 0$ and $a_i = 0$ for any $i$ with $i \neq p$: Then $g = x_p$ and $g^2 = x_p^2 = 0$.

The case $a_p \neq 0, a_q \neq 0$ and $a_i = 0$ for any $i$ with $i \neq p, q$: Let $p < q$. Then $g^2 = 0$ implies that $n_{pq} a_q = -2a_p \neq 0$. Thus, $n_{pq} = \pm 1$, i.e., $a_q = \pm 2a_p$. Moreover,

$$n_{iq} a_q = -2a_i = 0 \quad (1 \leq i \neq p < q) \Rightarrow n_{iq} = 0 \quad (1 \leq i \neq p < q).$$

Hence, we may assume that $n_{pq} = 1$, i.e., $a_q = -2a_p$. Namely, we have $(x_p - 2x_q)^2 = 0$. Moreover, in this case, we have

$$n_{iq} a_q = -2a_i = 0 \quad (1 \leq i < p) \Rightarrow n_{ip} = 0 \quad (1 \leq i < p),$$

so we obtain that $x_p^2 = 0$.

The case $a_p \neq 0, a_q \neq 0$ and $a_r \neq 0$ with $p < q < r$: Then it follows from (3.4) that

$$n_{pq} a_q = -2a_p \neq 0, \quad n_{pr} a_r = -2a_p \neq 0 \text{ and } n_{qr} a_r = -2a_q \neq 0.$$

Thus, $n_{pq}, n_{pr}, n_{qr} \in \{\pm 1\}$. However, in this case, we have $\begin{cases} a_r = \pm a_q \\ a_r = \pm 2a_q, \end{cases}$ implying that

$$a_q = a_r = 0,$$

a contradiction.

Therefore, we see that $S(H^*(X)) \subset \{x_p : 1 \leq p \leq d\} \cup \{x_p - 2x_q : 1 \leq p < q \leq d\}$. 

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(The second step): Let \((x_{p_1} - 2x_{q_1})^2 = 0\) and \((x_{p_2} - 2x_{q_2})^2 = 0\). We claim that \(\{p_1, q_1\} \neq \{p_2, q_2\}\) implies \(\{p_1, q_1\} \cap \{p_2, q_2\} = \emptyset\). By the above discussions, we may assume that \(n_{p_1q_1} = 1\) and \(n_{p_2q_2} = 1\), and we also have

\[
\begin{align*}
n_{i}q_1 &= 0 \quad (1 \leq i < q_1, i \neq p_1), \quad n_{i}p_1 &= 0 \quad (1 \leq i < p_1), \\
n_{i}q_2 &= 0 \quad (1 \leq i < q_2, i \neq p_2), \quad n_{i}p_2 &= 0 \quad (1 \leq i < p_2).
\end{align*}
\]

Then \(p_1 = q_2\) and \(p_2 = q_1\) never happen. If \(q_1 = q_2\), then \(p_1 = p_2\). If \(p_1 = p_2\), Theorem 3.1 (2) implies that \(q_1 = q_2\) since we assume \(n_{p_1q_1} = n_{p_2q_2} = 1\). Hence, \(\{p_1, q_1\} \cap \{p_2, q_2\} = \emptyset\) holds if \(\{p_1, q_1\} \neq \{p_2, q_2\}\).

(The third step): Now, we consider the following subsets of \(S(H^*(X))\):

\[
\begin{align*}
\{g_i\}_i &= \{x_p : (x_p - 2x_q)^2 = 0 \text{ for some } q\}, \quad \{h_j\}_j = \{x_p : x_p^2 = 0\}\setminus \{g_i\}, \\
\{g'_i\}_i &= \{x_p - 2x_q : (x_p - 2x_q)^2 = 0\}.
\end{align*}
\]

Since \(x_p \in \{g_i\}_i\) if and only if \(x_p - 2x_q \in \{g'_i\}_i\) (see the first step), we can simultaneously index both of the sets \(\{g_i\}\) and \(\{g'_i\}\) by \(I \subseteq [d]\). Moreover, it follows from the second step that the condition (1) holds.

We claim that the condition (2) also holds. More precisely, we prove the ring isomorphism \(H^*(X)/(x_p) \cong H^*(X)/(x_p - 2x_q)\). We may assume that \(n_{pq} = 1\). By Theorem 3.1 (2), we have \(n_{pi} = 0 \quad (p < i \neq q)\), and by \(n_{iq} = 0 \quad (1 \leq i < q, i \neq p)\) and \(n_{ip} = 0 \quad (1 \leq i < p)\), we see that a generator of \(I\) in which \(x_p\) appears is either \(x_p^2\) or \(x_q(x_q - x_p)\). Therefore, the ring homomorphism \(F : H^*(X) \to H^*(X)\) defined by

\[
F(x_i) = x_i \quad (i \neq p), \quad F(x_p) = x_p - 2x_q
\]

induces an isomorphism between \(H^*(X)/(x_p)\) and \(H^*(X)/(x_p - 2x_q)\).

\(\square\)

Remark 3.4 For a Fano Bott manifold, a maximal basis of \(S(H^*(X))\) is not unique as claimed in the condition (1) of Lemma 3.3. However, the condition (2) of Lemma 3.3 says that for any maximal basis \(S_{\text{max}}\) of \(S(H^*(X))\), \(\{H^*(X)/(f) : f \in S_{\text{max}}\}\) is unique up to ring isomorphisms. Therefore, when we consider the family of quotient rings by s.v.e., we may only consider s.v.e. of the form \(x_p\), namely, we do not need to treat s.v.e. of the form \(x_p - 2x_q\).

4 Correspondence between Fano Bott manifolds and signed rooted forests

In this section, we introduce the notion of signed rooted forests. We construct the signed rooted forest \(T_{\varphi, \sigma}\) from a map \(\varphi : [d] \to [d+1] \setminus \{1\}\) with (3.2) and a map \(\sigma : [d] \to \{\pm\}\). Namely, we can construct a signed rooted forest \(T_X\) from a Fano Bott manifold \(X\). By using this, we establish a correspondence between Fano Bott manifolds and signed rooted forests \(T_{\varphi, \sigma}\). We also observe that an operation on a signed rooted forest, called leaf-cutting by \(v_{\alpha}\), corresponds to taking a quotient by a certain element \(x_{\alpha}\).

We call \(T = (V, E, v_0)\) a rooted tree on the vertex set \(V\) with its root \(v_0\) and the edge set \(E\) if \((V, E)\) is a tree, which is a connected graph having no cycle, and a vertex \(v_0 \in V\), called a root, is fixed. A rooted forest is a disjoint union of rooted trees. We recall some notations on a rooted forest \(T\):

- A vertex \(v\) (resp. \(v')\) in \(T\) is called a child (resp. parent) of a vertex \(v'\) (resp. \(v\)) if \(\{v', v\}\) is an edge of a connected component \(T_0\) of \(T\) and \(v'\) is closer to the root of \(T_0\) than \(v\).
• A descendant of a vertex \( v \) in \( T \) means any vertex which is either a child of \( v \) or recursively a descendant of \( v \).

• We call a vertex \( v \) in \( T \) a leaf if no child is adjacent to \( v \).

We call a rooted forest signed if \( + \) or \( - \) is assigned to each of its edges. Given an edge \( e \), we denote the sign of \( e \) by \( \text{sign}(e) \).

Let \( T \) and \( T' \) be two rooted forests. We say that \( T \) and \( T' \) are isomorphic as rooted forests if there is a bijection \( f : V(T) \rightarrow V(T') \) between the sets of vertices which induces a bijection between the roots and between the sets of edges, and we call \( f \) an isomorphism as signed rooted forests if \( f \) also preserves all signs of the edges.

**Definition 4.1** (Signed rooted forests associated to Fano Bott manifolds) Let \( \varphi : [d] \rightarrow [d + 1]\backslash\{1\} \) be a map with (3.2) and let \( \sigma : [d] \rightarrow \{\pm\} \). We define the signed rooted forest \( T_{\varphi,\sigma} \) from \( \varphi \) and \( \sigma \) as follows:

\[
V(T_{\varphi,\sigma}) = \{v_1, \ldots, v_d\},
\]

\[
E(T_{\varphi,\sigma}) = \{(v_{\varphi(i)}, v_i) : 1 \leq i \leq d, \ \varphi(i) \neq d + 1\} \text{ with } \text{sign}((v_{\varphi(i)}, v_i)) = \sigma(i) \text{ for each } i,
\]

the roots are \( v_i \)'s with \( \varphi(i) = d + 1 \).

As mentioned in Sect. 3, we can associate the above maps \( \varphi \) and \( \sigma \) from Fano Bott manifolds, so we can construct \( T_{\varphi,\sigma} \) from Fano Bott manifolds. Let \( T_X = T_{\varphi,\sigma} \) be such signed rooted forest.

**Remark 4.2** Given a Fano Bott manifold \( X \), we see that the signed rooted forest \( T_{\varphi,\sigma} \) associated to \( X \) constructed in the above way is well-defined. In fact, for two Fano Bott manifolds \( X \) and \( X' \) and their associated signed rooted forests \( T_X \) and \( T_{X'} \), it is straightforward to check that the equivalence between \( \text{PC}(\Sigma X) \) and \( \text{PC}(\Sigma X') \), which is a bijection between \( (\Sigma X)_1 \) and \( (\Sigma X')_1 \), directly implies the isomorphism between two corresponding signed rooted forests \( T_X \) and \( T_{X'} \), which is a bijection between \( V(T_X) \) and \( V(T_{X'}) \).

On the other hand, given a signed rooted forest \( T \), we can reconstruct the primitive relations associated to \( T \), so we can associate a Fano Bott manifold from \( T \). Let \( X_T \) be the Fano Bott manifold associated to a signed rooted forest \( T \).

**Example 4.3** (1) Let us consider the Fano Bott manifold \( X \) associated to the following primitive relations:

\[
\begin{align*}
v_1^+ + v_1^- &= v_2^+, \\
v_2^+ + v_2^- &= v_5^-, \\
v_3^+ + v_3^- &= v_4^+, \\
v_4^+ + v_4^- &= v_5^+, \\
v_5^+ + v_5^- &= 0.
\end{align*}
\]

Then the associated signed rooted forest \( T_X \) looks as follows.
(2) Let us consider the Fano Bott manifold $X$ associated to the following primitive relations:

\[
\begin{align*}
 v_1^+ + v_1^- &= v_3^+ , \\
v_2^+ + v_2^- &= v_3^- , \\
v_3^+ + v_3^- &= 0 , \\
v_4^+ + v_4^- &= v_5^+ , \\
v_5^+ + v_5^- &= 0 . \\
\end{align*}
\]

Then the associated signed rooted forest $T_X$ looks as follows.

\[
\begin{align*}
 &\begin{array}{c}
v_3 \\
v_1 \end{array} & \begin{array}{c}
v_5 \\
v_2 \end{array} & \begin{array}{c}
v_4 \\
v_3 \end{array}
\end{align*}
\]

Remark 4.4 Let $X$ be a $d$-stage Fano Bott manifold and let $A(X) = (n_{ij})$ be the upper triangular matrix associated to $X$. Then Theorem 3.1 claims that $A(X)$ is completely determined by the left-most nonzero entry of each row. Moreover, the left-most nonzero entry of each row, which is 1 or $-1$, one-to-one corresponds to the signed edge of the associated signed rooted forest $T_X$. Namely, we see that

\[
\begin{align*}
n_{i,i+1} = \cdots = n_{i,j-1} = 0, \ n_{ij} = 1 \ (\text{resp. } n_{ij} = -1) \\
\iff \ \{v_j, v_i\} \in E(T_X) \text{ and } \text{sign}(\{v_j, v_i\}) = + \ (\text{resp. } \text{sign}(\{v_j, v_i\}) = -) \ (4.1)
\end{align*}
\]

Furthermore, we can read off all entries of $(n_{ij})$ from $T_X$ as follows: for $1 \leq i < j \leq d$, let

\[
n_{ij} = \begin{cases} 
1 , & \text{if } \{v_i, v_j\} \in E(T_X) \text{ and } \text{sign}(\{v_i, v_j\}) = + , \\
1 , & \text{if there is an upward path } (v_{i_0}, \ldots, v_{i_k}) \text{ from } v_i = v_{i_0} \text{ to } v_j = v_{i_k} \text{ such that} \\
\text{sign}(\{v_{i_{\ell-1}}, v_{i_\ell}\}) = - \text{ for } \ell = 1, \ldots, k - 1 \text{ and } \text{sign}(\{v_{i_{k-1}}, v_{i_k}\}) = + , \\
-1 , & \text{if there is an upward path from } v_i \text{ to } v_j \text{ all of whose edges have the sign } - , \\
0 , & \text{otherwise},
\end{cases}
\]

where we call a path $(v_{i_0}, v_{i_1}, \ldots, v_{i_k})$ in the signed rooted forest upward if $v_j$ is a child of $v_{i_{j+1}}$ for all $j = 1, \ldots, k - 1$. For Example 4.3 (1), we see that

\[
n_{12} = n_{35} = 1, \ n_{25} = n_{34} = n_{45} = -1, \ n_{ij} = 0 \text{ otherwise}.
\]
Let $T_1, \ldots, T_k$ be the connected components of a signed rooted forest $T$. It then follows from the above construction that the upper triangular matrix associated to $T$ becomes a direct sum of those of $T_1, \ldots, T_k$.

Given a Fano Bott manifold $X$, we observe the following:

- By definition of $T_{\varphi,\sigma}$, we see that $v_\alpha$ is a leaf of $T_{\varphi,\sigma}$ if and only if $\alpha \notin \varphi([d])$. Thus, for a leaf $v_\alpha$ of $T_{\varphi,\sigma}$, we obtain that $n_{1\alpha} = 0$ for each $i = 1, \ldots, \alpha - 1$, i.e., $x_{\alpha}^2 = 0$ in $H^*(X)$.
- On the other hand, if $x_{\alpha}^2 = 0$ in $H^*(X)$, then $n_{1\alpha} = \cdots = n_{\alpha-1,\alpha} = 0$ by (3.4). Thus, we obtain that $\alpha \notin \varphi([d])$, i.e., $v_\alpha$ is a leaf of $T_X$.

Therefore, the set of leaves of $T_X$ one-to-one corresponds to $\{x_i : x_i^2 = 0 \in H^*(X)\}$.

Now, we consider the quotient ring $H^*(X)/(x_\alpha)$ where $x_\alpha^2 = 0$ in $H^*(X)$. Then we have $n_{1\alpha} = 0$ for each $i = 1, \ldots, \alpha - 1$. By $H^*(X)/(x_\alpha) \cong \mathbb{Z}[x_1, \ldots, x_d]/(I_X + (x_\alpha))$, where $I_X = I_X + J_X$ in Proposition 2.5, we see that $H^*(X)/(x_\alpha) \cong \mathbb{Z}[\hat{x}_\alpha, \ldots, x_d]/\overline{I}_X$, where $\overline{I}_X$ is the image of $I_X$ by the natural projection $\mathbb{Z}[x_1, \ldots, x_d] \to \mathbb{Z}[\hat{x}_\alpha, \ldots, x_d]$. Then $\mathbb{Z}[x_1, \ldots, \hat{x}_\alpha, \ldots, x_d]/\overline{I}_X$ is isomorphic to the cohomology ring of a Fano Bott manifold $\overline{X}$ whose primitive relations are

\[
v_{1}^+ + v_{1}^- = v_{\varphi(1)},
\]

\[
v_{\alpha-1}^+ + v_{\alpha-1}^- = v_{\varphi(\alpha-1)},
\]

\[
v_{\alpha+1}^+ + v_{\alpha+1}^- = v_{\varphi(\alpha+1)},
\]

\[
v_{d}^+ + v_{d}^- = v_{\varphi(d)},
\]

where $\varphi : [d] \setminus \{\alpha\} \to [d+1] \setminus \{1, \alpha\}$ (resp. $\overline{\varphi} : [d] \setminus \{\alpha\} \to \{\pm\}$) is defined by $\varphi(i) = \varphi(i)$ (resp. $\overline{\varphi}(i) = \sigma(i)$). This implies that $T_X$ is isomorphic to $T_X \setminus v_\alpha$ as signed rooted forests.

Therefore, when $x_\alpha^2 = 0$ or $(x_\alpha - 2x_\beta)^2 = 0$ in $H^*(X)$, taking the quotient $H^*(X)/(x_\alpha)$ (which is isomorphic to $H^*(X)/(x_\alpha - 2x_\beta)$) is equivalent to a leaf-cutting by $v_\alpha$ (see Remark 3.4). Moreover, Lemma 3.3 says that for two Fano Bott manifolds $X$ and $X'$, $H^*(X) \cong H^*(X')$ implies $\{H^*(X)/(x_i) : x_i^2 = 0\} \cong \{H^*(X')/(x_i') : x_i'^2 = 0\}$. These mean that the number of leaves of $T_X$ is equal to the maximal basis number of $H^*(X)$.

The following proposition will play a crucial role in the proof of Theorem 1.8.

**Proposition 4.5** Let $X$ and $X'$ be two Fano Bott manifolds. Assume that $H^*(X) \cong H^*(X')$ and let $F : H^*(X) \to H^*(X')$ be a ring isomorphism.

1. $F$ induces an isomorphism between $T_X$ and $T_{X'}$ as rooted forests.
2. Let $f : T_X \to T_{X'}$ be an isomorphism induced by $F$. Take a rooted subtree $T_0 \subset T_X$ whose root is the same as that of $T_X$. Then $H^*(X_{T_0}) \cong H^*(X_{f(T_0)})$.

**Proof** (1) Since $\{x_\alpha \in H^*(X) : x_\alpha^2 = 0\}$ one-to-one corresponds to the leaves of $T_X$, and since we may assume that $F$ sends $\{x_\alpha \in H^*(X) : x_\alpha^2 = 0\}$ to $\{x_\alpha' \in H^*(X') : x_\alpha'^2 = 0\}$ (see the above discussion), we obtain a bijection $f$ between the leaves of $T_X$ and those of $T_{X'}$ induced by $F$, as discussed above.

\[ \square \]
Let \( v_1, \ldots, v_k \) be the leaves of \( T_X \) and let \( v_i' = f(v_i) \) \((i = 1, \ldots, k)\). Remark that taking a quotient by \( x_1 \) is equivalent to taking a quotient by \( x_i' \). We prove the assertion by induction on \( k \).

Let \( k = 1 \). Then \( F \) induces the correspondence \( v_1 \) and \( v_1' \). Consider \( \overline{F} : H^*(X)/(x_1) \to H^*(X')/(x_1') \). Since \( \overline{F} \) is also an isomorphism and \( H^*(X)/(x_1) \) (resp. \( H^*(X')/(x_1') \)) corresponds to a rooted tree \( T_X \setminus v_1 \) (resp. \( T_X' \setminus v_1' \)) which contains only one leaf, we also obtain the correspondence between their leaves, which are the parents of \( v_1 \) and \( v_1' \), respectively. By repeating this procedure, we obtain a bijection of all vertices of \( T_X \) and \( T_X' \) which induces an isomorphism between \( T_X \) and \( T_X' \).

Let \( k > 1 \). Then \( F \) gives a bijection between the leaves of \( T_X \) and \( T_X' \). Take one leaf \( v_1 \) and consider \( \overline{F} : H^*(X)/(x_1) \to H^*(X')/(x_1') \). Note that \( \overline{F} \) gives a bijection between the leaves of \( T_X \setminus v_1 \) and \( T_X' \setminus v_1' \). Since \( \overline{F} \) is induced from \( F \), we see that \( \overline{F} \) preserves the bijectionity between \( \{v_2, \ldots, v_k\} \) and \( \{v_2', \ldots, v_k'\} \). Note that the number of leaves of \( T_X \setminus v_1 \) is either the same as that of \( T_X \) or minus one.

- When the number of leaves of \( T_X \setminus v_1 \) decreases from that of \( T_X \), by the hypothesis of induction, we obtain an isomorphism \( g : T_X \setminus v_1 \to T_X' \setminus v_1' \). Let \( v \) (resp. \( v' \)) be the parent of \( v_1 \) (resp. \( v_1' \)). Once we can see that

\[
g(v) = v',
\]

by combining the correspondence between \( v_1 \) and \( v_1' \), we obtain an isomorphism \( T_X \) and \( T_X' \) induced by \( F \).

- When the number of leaves of \( T_X \setminus v_1 \) stays the same, we see that new leaves \( v_{l''} \) and \( v_{r''} \) appear both in \( T_X \setminus v_1 \) and \( T_X' \setminus v_1' \), respectively. Since \( \overline{F} \) still gives a bijection between \( \{v_2, \ldots, v_k\} \) and \( \{v_2', \ldots, v_k'\} \), \( v_{l''} \) should correspond to \( v_{l''} \). Consider \( H^*(X)/(x_1, x_{l''}) \) and \( H^*(X')/(x_1', x_{l'''}) \). By the same discussion, we can repeat this procedure until the number of leaves decreases. Hence, the assertion follows.

Our remaining task is to show (4.2). Note that there is another leaf, say \( u_{l_2} \), which is a descendant of \( v \). When \( u_{l_2} \) is a child of \( v \), by the hypothesis of induction, we obtain an isomorphism \( h : T_X \setminus u_{l_2} \to T_X' \setminus u_{l_2}' \). Since \( h \) still gives a bijection between \( \{v_1, v_3, \ldots, v_k\} \) and \( \{v_1', v_3', \ldots, v_k'\} \), we obtain that \( h(v) = v' \). Even if \( u_{l_2} \) is not a child of \( v \), by removing the leaves until the number of leaves decreases, we obtain the same conclusion. Hence, we see that \( h(v) = g(v) = v' \).

(2) Let \( f : T_X \to T_X' \) be an isomorphism constructed in the above way. For any rooted subtree \( T_0 \subset T_X \) whose root is the same as that of \( T_X \), since

\[
H^*(X_{T_0}) \cong H^*(X)/(x_{\alpha} : v_{\alpha} \in V(T_X \setminus T_0))
\]

and \( f \) sends \( V(T_X \setminus T_0) \) to \( V(T_X' \setminus f(T_0)) \) by construction of \( f \), we obtain that \( H^*(X_{T_0}) \cong H^*(X_{f(T_0)}) \), as required. 

\( \square \)
5 Three operations on matrices and Fano Bott equivalence

In this section, we introduce the equivalence relation in $\mathcal{FB}(d)$ (see Definition 3.2), which we call Fano Bott equivalence. The notion of Fano Bott equivalence is derived from Bott equivalence defined in [8].

**Definition 5.1** (Three operations on $\mathcal{FB}(d)$) For $A \in \mathbb{Z}^{d \times d}$ and $i \in [d]$, let $A_i$ (resp. $A^i$) denote the $i$-th column (resp. row) vector of $A$. Let $e_i$ (resp. $e^i$) denotes the $i$-th unit column (resp. row) vector. Given a permutation $\pi$ on $[d]$, we define a permutation matrix $P$ by setting $P_j = e_{\pi(j)}$ for each $j$.

(Op1): For a permutation matrix $P$ corresponding to a permutation $\pi$ on $[d]$, we define a map $\Phi_P : \mathbb{Z}^{d \times d} \rightarrow \mathbb{Z}^{d \times d}$ by $\Phi_P(A) := PAP^{-1}$ for $A \in \mathbb{Z}^{d \times d}$. Namely, we see that $A_j^i = (\Phi_P(A))_{\pi(j)}^i$.

(Op2±): For $k \in [d]$, we define a map $\Phi_k^\pm : \mathbb{Z}^{d \times d} \rightarrow \mathbb{Z}^{d \times d}$ as follows: for $A \in \mathbb{Z}^{d \times d}$, we multiply $-1$ to the $k$-th column and add the $k$-th column times $(k, j)$-entry to the $j$-th column for each $j \in [d] \setminus \{k\}$. Namely, we see that

$$(\Phi_k^\pm(A))_{j}^i = \begin{cases} -A_k^i, & \text{if } j = k, \\ A_j^i, & \text{if } A_k^i \neq 0, \text{ otherwise.} \end{cases}$$

Notice that $\Phi_k^\pm$ is nothing but a kind of unimodular transformation.

(Op3±): Given $A \in \mathbb{Z}^{d \times d}$, assume that $A^\ell = 0$ and $A^k = \pm e^\ell$ for some $k$ and $\ell$. Then we define $\Phi_{k,\ell}^\pm(A)$ as follows: we multiply $-1$ to $A^k_\ell$, and multiply $A_k^i$ to $A^i_\ell$ if $A_k^i \neq 0$ for $i \in [d] \setminus \{\ell\}$, and the other entries stay the same. Namely, we see that

$$(\Phi_{k,\ell}^\pm(A))_{i}^\ell = \begin{cases} -A_k^\ell, & \text{if } i = k, \\ A_k^i, & \text{if } A_k^i \neq 0, \text{ and } (\Phi_{k,\ell}^\pm(A))_{j}^i = A_j^i \text{ for } j \neq \ell. \end{cases}$$

**Definition 5.2** (Fano Bott equivalence) Given two matrices $A$ and $A'$ in $\mathcal{FB}(d)$, we say that $A$ and $A'$ are Fano Bott equivalent if $A$ can be transformed into $A'$ through a sequence of the three operations (Op1), (Op2±) and (Op3±).

**Proposition 5.3** Let $A \in \mathcal{FB}(d)$. Then $\Phi_k^\pm(A) \in \mathcal{FB}(d)$ for any $k \in [d]$, and $\Phi_{k,\ell}^\pm(A) \in \mathcal{FB}(d)$ for any $k, \ell \in [d]$ satisfying $A^\ell = 0$ and $A^k = \pm e^\ell$.

**Proof** Given $A \in \mathcal{FB}(d)$, we prove that $\Phi_k^\pm(A)$ and $\Phi_{k,\ell}^\pm(A)$ satisfy the conditions in Theorem 3.1.

For each $p \in [d]$, it follows from Theorem 3.1 that we have $A^p = 0$, or $A^p = e^q$ for some $q$ with $p < q \leq d$, or $A^p = -e^q + A^q$ for some $q$ with $p < q \leq d$.

Let $A^p = 0$. Then we easily see that $(\Phi_k^\pm(A))^p = (\Phi_{k,\ell}^\pm(A))^p = 0$.

Let $A^p = e^q$ for some $q$ with $p < q \leq d$.

- We see that $(\Phi_k^\pm(A))^p = e^q$ if $k \neq q$ and $(\Phi_k^\pm(A))^p = -e^q + A^q$ if $k = q$.

- Note that $(\Phi_{k,\ell}^\pm(A))^\ell = 0$. We see that $(\Phi_{k,\ell}^\pm(A))^p = e^q$ if $q \neq \ell$. When $q = \ell$, if $p \neq k$ then $A_k^0 = 0$. If $p = k$, then $(\Phi_{k,\ell}^\pm(A))^p = -e^q$, so the assertion holds.

Let $A^p = -e^q + A^q$ for some $q$ with $p < q \leq d$. Then $A^p = A_j^q$ holds for any $j \neq q$. 

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We see that \((\Phi^\pm_k(A))^p = -e^q + (\Phi^\pm_k(A))^q\) if \(k \neq q\) and \((\Phi^\pm_k(A))^p = e^q\) if \(k = q\).

If \(q = \ell\), then \(A^q = A^\ell = 0\). Thus, \((\Phi^\pm_k,\ell(A))^p = -e^q\) (resp. \(e^q\)) when \(p \neq k\) (resp. \(p = k\)). If \(q \neq \ell\), then we see that \((\Phi^\pm_k,\ell(A))^p = -e^q + (\Phi^\pm_k,\ell(A))^q\). 

\[ \square \]

Remark that (Op1) does not necessarily preserve \(FB(d)\), while this corresponds to an isomorphism of the associated signed rooted forests.

**Example 5.4** Let us consider \(A = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\). Then it is straightforward to check that \(A \in FB(6)\). Moreover, we see the following:

\[
\Phi^\pm_3(A) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Phi^\pm_5(A) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\Phi^\pm_{3,6}(A) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Phi^\pm_{5,6}(A) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Let \(X\) be a \(d\)-stage Fano Bott manifolds. We discuss the relationship between \(A(X)\) and \(T_X\). As mentioned above, (Op1) on \(A(X)\) corresponds to an isomorphism of \(T_X\). More precisely, given a permutation matrix \(P\) associated to a permutation \(\pi\) on \([d]\), \(\Phi_P\) corresponds to a bijection on \(V(T_X)\) associated to \(\pi\). For clarifying the relationship between (Op2\(\pm\)) and (Op3\(\pm\)) and the operations on \(T_X\), we introduce a new notion and operation on the signed rooted forest \(T_X\).

**Definition 5.5** (Equivalent signs) Let \(T\) and \(T'\) be signed rooted forests. We say that \(T\) and \(T'\) have equivalent signs if \(T\) and \(T'\) are isomorphic as rooted forests and the signs of the edges of \(T\) can be transformed, up to isomorphism as rooted forests, into the signs of the edges of \(T'\) by some replacements of all the signs of the edges \(\{v_i, v_j\}\) for all \(j \in \varphi^{-1}(i)\) at the same time. When this is the case, we write \(T \sim T'^\prime\).

It directly follows from the construction of the signed rooted forests from primitive relations that if \(T \sim T'\) then the corresponding primitive relations are equivalent.

**Example 5.6** Let us consider the following three signed rooted forests \(T_1, T_2, T_3\) (but actually, trees).

We see that \(T_1 \sim T_2\). In fact, by exchanging the signs of the edges \(\{v_5, v_1\}, \{v_5, v_2\}, \{v_5, v_4\}\) and exchanging the vertices \(v_1\) and \(v_2\) by a graph isomorphism, we see that \(T_1\) can be transformed into \(T_2\).

On the other hand, \(T_1\) and \(T_3\) are not equivalent. In fact, the signs of the edges \(\{v_5, v_1\}\) and \(\{v_5, v_2\}\) in \(T_1\) should be different, while those in \(T_3\) should be the same.
The following proposition will be a crucial part of the proof of Theorem 1.8.

**Proposition 5.7** Let $X$ be a $d$-stage Bott manifold. Let $A = (a_{ij}) \in FB(d)$ be the upper triangular matrix associated to $X$. Let $T = T_X$ with $V(T) = \{v_1, \ldots, v_d\}$ and let $\varphi$ be the map with (3.2).

1. For $k \in [d]$, the operation $\Phi^\pm_k$ of (Op$^\pm_2$) corresponds to the simultaneous change of the signs of the edges $\{v_k, v_j\}$ of $T$ for all $j \in \varphi^{-1}(k)$.

2. For $k, \ell \in [d]$, if $A^\ell = 0$ and $A^k = \pm e^\ell$, then $v_\ell$ is a root of $T$ and $v_k$ is its child. Moreover, the operation $\Phi^\pm_{k, \ell}$ of (Op$^\pm_3$) corresponds to the change of the sign of the edge $\{v_\ell, v_k\}$ of $T$.

**Proof** (1) By (4.1), it suffices to show that for each $i = 1, \ldots, d - 1$, the left-most nonzero entry of $(\Phi^\pm_k(A))^i$ stays the same as that of $A^i$ when it is not in the $k$-th column, while the left-most nonzero entry of $(\Phi^\pm_k(A))^i$ changes from that of $A^i$ when it is in the $k$-th column. This directly follows from the definition of the operation $\Phi^\pm_k$. (2) By our assumption, we know that the left-most nonzero entry of $A^k$ is in the $\ell$-th column. Similar to (1), it suffices to show that for each $i$, the the left-most nonzero entry of $(\Phi^\pm_{k, \ell}(A))^i$ stays the same as that of $A^i$ when $i \neq k$, while the left-most nonzero entry of $(\Phi^\pm_{k, \ell}(A))^k$ changes from that of $A^k$. This also directly follows from the definition of the operation $\Phi^\pm_{k, \ell}$. □

**Remark 5.8** From Proposition 5.7 (1) together with Proposition 2.4 and Remark 4.2, we see that given two Fano Bott manifolds, the following three conditions are equivalent:

- $X$ and $X'$ are isomorphic as varieties;
- $PC(\Sigma_X)$ and $PC(\Sigma_{X'})$ are equivalent;
- $A(X)$ can be transformed into $A(X')$ through a sequence of the two operations (Op1) and (Op2$^\pm$).

**Example 5.9** Let us consider the Bott Fano manifolds $X$ and $X'$ associated to the following primitive relations, respectively:

$$PC(\Sigma_X) = \begin{cases} v_1^+ + v_1^- = v_3^+, \\ v_2^+ + v_2^- = v_3^+, \\ v_3^+ + v_3^- = 0, \end{cases} \quad \text{and} \quad PC(\Sigma_{X'}) = \begin{cases} v_1^+ + v_1^- = v_3^+, \\ v_2^+ + v_2^- = v_3^+, \\ v_3^+ + v_3^- = 0. \end{cases}$$

Note that $X$ (resp. $X'$) corresponds to the smooth Fano 3-polytope with ID 11 (resp. ID 18) of the database (1.1). Then $A(X) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $A(X') = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Since $A(X)$ and
Here, we consider the entries of $H^-$ we identify. Let $X$ and $X'$ be the Fano Bott manifolds corresponding to $B$ and $B'$.

\[\Phi_1 \sim H^-(X) \cong H^-(X').\]

Once we can prove that $R \not\sim R'$, we see that both $A(X)$ and $A(X')$ contain only one nonzero column, we see that $A(X)$ cannot be transformed into $A(X')$ through a sequence of the operations (Op1) and (Op2$^\pm$). On the other hand, we have that $\Phi_2 \sim (A(X)) = A(X')$. Hence, we will see from Theorem 1.8 that $X$ and $X'$ are diffeomorphic 3-stage Fano Bott manifolds that are not isomorphic. This fact was already showed in [12, Section 4].

### 6 Proof of Theorem 1.8

The goal of this section is to complete our proof of Theorem 1.8. Proposition 6.3 is an essential part of the proof, which is the statement on the distinguishments of cohomology rings by the precise observations of the signed rooted forests associated to Fano Bott manifolds.

Here, we introduce a special kind of rooted trees:

**Definition 6.1** Let

\[
V = \{v_0, v_1, \ldots, v_p, w_1, w_2, \ldots, w_q\} (p \geq 1, q \geq 2),
\]

\[
E = \{(v_{i-1}, v_i) : i = 1, \ldots, p\} \cup \{(v_p, w_j) : j = 1, \ldots, q\}.
\]

We call the rooted tree $(V, E, v_0)$ a broom with $q$ leaves.

**Lemma 6.2** Let $B$ and $B'$ be brooms and assign the certain signs of the edges of $B$ and $B'$. Let $X$ and $X'$ be the Fano Bott manifolds corresponding to $B$ and $B'$, respectively. Assume that $H^*(X) \cong H^*(X')$. Then $B \sim B'$.

**Proof** Let $B$ and $B'$ be the same broom with $q$ leaves $(V, E, v_0)$ as in (6.1).

First, we prove the assertion in the case $q = 2$. Then there are only two possibilities $B$ and $B'$ of assignments of signs up to equivalence as follows:

\[
B : \text{sign}((v_{i-1}, v_i)) = + \text{ for } i = 1, \ldots, p, \quad \text{sign}((v_p, w_1)) = \text{sign}((v_p, w_2)) = +,
\]

\[
B' : \text{sign}((v_{i-1}, v_i)) = + \text{ for } i = 1, \ldots, p, \quad \text{sign}((v_p, w_1)) = +, \quad \text{sign}((v_p, w_2)) = -.
\]

Namely, $B \sim B'$. Let $R = H^*(X) \otimes \mathbb{Z}/2$ and $R' = H^*(X') \otimes \mathbb{Z}/2$. For the convenience to analyze $R$ and $R'$, we rename the vertices as follows:

\[
w_1, w_2, v_p, v_{p-1}, \ldots, v_0 \longrightarrow v_1, v_2, v_3, \ldots, v_{p+3}.
\]

Once we can prove that $R \not\cong R'$, we conclude $H^*(X) \not\cong H^*(X')$ as required.

Let $(n_{ij})$ (resp. $(n'_{ij})$) be the upper triangular matrix corresponding to $B$ (resp. $B'$). By Remark 4.4, we can compute them as follows:

\[
n_{13} = 1, \quad n_{i,i+1} = 1 \text{ for } i = 2, \ldots, p + 2, \quad n_{ij} = 0 \text{ otherwise},
\]

\[
n'_{23} = -1, \quad n'_{13} = n'_{24} = 1, n'_{i,i+1} = 1 \text{ for } i = 3, \ldots, p + 2, \quad n'_{ij} = 0 \text{ otherwise}.
\]

Here, we consider the entries of $(n_{ij})$ and $(n'_{ij})$ as the elements of $\mathbb{Z}/2$. More concretely, we identify $-1$ with 1. Consider the cut-rank discussed in [8, Section 8.5]. Let $D$ (resp. $D'$) denote the acyclic digraph corresponding to $(n_{ij}) \in (\mathbb{Z}/2)^{(p+3)\times(p+3)}$ (resp. $(n'_{ij}) \in (\mathbb{Z}/2)^{(p+3)\times(p+3)}$). In this case, we see that both $L_0(D)$ and $L_0(D')$ correspond to $\{1, 2\}$ and

\[
\rho_D(L_0(D)) = \text{rank} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} = 1, \quad \text{and}
\]

\[
\rho_{D'}(L_0(D')) = \text{rank} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \end{pmatrix} = 2.
\]

Therefore, we obtain that $R \not\cong R'$ by [8, Theorem 1.1 and Proposition 8.6 (i)].
Now, let us consider the general case. We assign the signs of the edges \( \{v_p, w_j\} \) for \( j = 1, \ldots, q \) as follows:

\[
B : |\{ j \in [q] : \text{sign}(\{v_p, w_j\}) = +\}| = a, \quad |\{ j \in [q] : \text{sign}(\{v_p, w_j\}) = -\}| = b,
\]

\[
B' : |\{ j \in [q] : \text{sign}(\{v_p, w_j\}) = +\}| = a', \quad |\{ j \in [q] : \text{sign}(\{v_p, w_j\}) = -\}| = b'.
\]

Here, we have \( a + b = a' + b' = q \). When \( B \approx B' \), we have \( \{a, b\} \neq \{a', b'\} \). Then, for any isomorphism \( f : V \rightarrow V \) between signed rooted trees \( B \) and \( B' \), we can find a pair of leaves \( w_i, w_j \) such that

\[
(\text{sign}(\{v_p, w_i\}), \text{sign}(\{v_p, w_j\})) \neq (\text{sign}(\{v_p, f(w_i)\}), \text{sign}(\{v_p, f(w_j)\})).
\]

Let us fix such an isomorphism \( f \) and such leaves \( w_i, w_j \). Let \( B_0 \) be the subtree of \( B \) induced by \( v_0, v_1, \ldots, v_p \) and \( w_i, w_j \). In particular, \( B_0 \) is a broom with \( 2 \) leaves. Then it follows from the above discussion that \( H^*(X_{B_0}) \nRightarrow H^*(X_{f(B_0)}) \). Therefore, Proposition 4.5 (2) implies that \( H^*(X) \nRightarrow H^*(X') \), as desired.

The following proposition plays the crucial role for the proof of Theorem 1.8.

**Proposition 6.3** Let \( X \) and \( X' \) be Fano Bott manifolds. Assume that \( H^*(X) \cong H^*(X') \). Then \( T_X \) and \( T_{X'} \) have equivalent signs by exchanging the signs assigned to the edges adjacent to the roots if necessary.

**Proof** By Proposition 4.5 (1), there is an isomorphism \( f : T_X \rightarrow T_{X'} \) as rooted forests.

Suppose, on the contrary, that \( T_X \napprox T_{X'} \). Then Definition 5.5 says that there is a vertex \( v \) of \( T_X \) such that the signs of the edges adjacent to \( v \) are not preserved by \( f \) even if we exchange all those signs at the same time. Since the equivalence of signs is considered up to isomorphism as rooted forests, we have to treat a certain part of children of \( v \) which can be transferred by an isomorphism as rooted forests. Let \( w^{(i)}_j \) \( (i = 1, \ldots, p, j = 1, \ldots, q_i) \) be all the children of \( v \) such that \( w^{(i)}_j \) can be transferred into \( w^{(i')}_{j'} \) by \( f \) if and only if \( i = i' \). Let \( v' = f(v) \), let \( w^{(i)}_j = f(w^{(i)}_j) \) for each \( i \) and \( j \), and let \( a_i = |\{ j \in [q_i] : \text{sign}(\{v, w^{(i)}_j\}) = +\}| \) (resp. \( a'_i = |\{ j \in [q_i] : \text{sign}(\{v', w^{(i)}_{j'}\}) = +\}| \)) and let \( b_i = q_i - a_i \) (resp. \( b'_i = q'_i - a'_i \)) for each \( i \). The assumption \( T \sim T' \) implies that there is \( I \subset [p] \) with

\[
\left\{ \sum_{i \in I} a_i, \sum_{i \in I} b_i \right\} \neq \left\{ \sum_{i \in I} a'_i, \sum_{i \in I} b'_i \right\},
\]

otherwise it is a contradiction to the choice of \( v \). Let us fix such \( I \).

Note that the length from the root \( v_0 \) of \( T_X \) to \( v \) is at least \( 1 \). Let \( v_0, v_1, \ldots, v_k = v \) be the vertices between \( v_0 \) and \( v \). Consider the broom \( B \) defined by \( v_0, v_1, \ldots, v_k \) and \( w^{(i)}_j \) for \( i \in I \) and \( j \in [q_i] \). Let \( B' = f(B) \). Note that \( B \napprox B' \). By Lemma 6.2, we see that \( H^*(X_B) \nRightarrow H^*(X_{B'}) \).

On the other hand, since \( B \) and \( B' \) are the rooted subtrees of \( T_X \) and \( T_{X'} \) whose roots are the same as that of \( T_X \) and \( T_{X'} \), respectively, we see that \( H^*(X_B) \cong H^*(X_{B'}) \) by Proposition 4.5 (2), a contradiction.

Therefore, we conclude that \( T_X \napprox T_{X'} \), as required.

**Proof of Theorem 1.8** Note that \( (2) \Rightarrow (3) \) is trivial and \( (3) \Rightarrow (1) \) directly follows from Propositions 5.7 and 6.3.
In what follows, we prove (1) $\Rightarrow$ (2). Let $X$ and $X'$ be $d$-stage Fano Bott manifolds and let $T = T_X$ and $T' = T_{X'}$ be the associated signed rooted forests, respectively. By Proposition 5.7, the assumption of (1) says that $T \cong T'$ as (non-signed) rooted forests and $T$ and $T'$ have equivalent signs by exchanging the signs assigned to the edges adjacent to the roots. Namely, there is an isomorphism $f : T \to T'$ as rooted forests which preserves the signs of all edges of $T$ and $T'$ except for the edges adjacent to roots. We assume that $f$ is an “identity”, more precisely, $f(v_i) = v_i'$, where $V(T) = \{v_1, \ldots, v_d\}$ and $V(T') = \{v'_1, \ldots, v'_d\}$.

Let $\Sigma_1 = (\Sigma_X)_1 = \{v_i^\pm : i = 1, \ldots, d\}$ and let $\Sigma'_1 = (\Sigma_{X'})_1 = \{w_i^\pm : i = 1, \ldots, d\}$. Let $A := A(X) = (n_{ij})_{1 \leq i, j \leq d}$ (resp. $A' := A(X') = (n'_{ij})_{1 \leq i, j \leq d}$). As mentioned in Remark 4.4, we can read off $(n_{ij})$ (resp. $(n'_{ij})$) from $T$ (resp. $T'$). Since $v_1^+, \ldots, v_d^+$ (resp. $w_1^+, \ldots, w_d^+$) are linearly independent, we can take $v_i^+ = w_i^+ = e^i$ for each $i$. Thus, we see that the matrix $M$ (resp. $M'$) whose row vectors consist of $v_1^+, \ldots, v_d^+, v_1^-, \ldots, v_d^-$ (resp. $w_1^+, \ldots, w_d^+, w_1^-, \ldots, w_d^-$) looks as follows:

$$
M = \begin{pmatrix}
E \\
-1 n_{12} n_{13} \cdots n_{1d} \\
-1 n_{23} \cdots n_{2d} \\
-1 \cdots n_{3d} \\
\vdots \\
-1
\end{pmatrix}
$$

and

$$
M' = \begin{pmatrix}
E \\
-1 n'_{12} n'_{13} \cdots n'_{1d} \\
-1 n'_{23} \cdots n'_{2d} \\
-1 \cdots n'_{3d} \\
\vdots \\
-1
\end{pmatrix}.
$$

where $E \in \{0, 1\}^{d \times d}$ denotes the identity matrix. Namely, the top halves of $M$ and $M'$ are $E$ and the bottom half of $M$ (resp. $M'$) is $-E + A$ (resp. $-E + A'$).

Here, we see that the operation (Op1) is a unimodular transformation on $M$. In fact, (Op1) permutes the corresponding columns and the corresponding rows of the top half and the bottom half. Moreover, since (Op2±) is a unimodular transformation expressed by $T_k :=$ 

$$
\begin{pmatrix}
 e^1 \\
 \vdots \\
- e^k + A^k \\
 \vdots \\
 e^d
\end{pmatrix} \in \mathbb{Z}^{d \times d},
$$

we see that $E \cdot T_k = \begin{pmatrix}
 e^1 \\
 \vdots \\
- e^k + A^k \\
 \vdots \\
 e^d
\end{pmatrix}$ and

$$
(-E + A) \cdot T_k = - T_k + \Phi_k^\pm (A) = 
\begin{pmatrix}
 - e^1 + \Phi_k^\pm (A)^1 \\
 \vdots \\
 e^k \\
 \vdots \\
- e^d + \Phi_k^\pm (A)^d
\end{pmatrix}.
$$

Let $M''$ be the resulting matrix after applying a certain sequence of unimodular transformations corresponding to (Op1) and (Op2±) to $M$ and let $T''$ be the signed rooted forest associated to (the bottom half of) $M''$. Since $A$ and $A'$ are Fano Bott equivalent, we may assume that $M''$ can be transform into $M'$ by a certain sequence of transformations corresponding to (Op3±).
First, we assume that $T$ and $T'$ are trees, i.e., connected. Let $v_d$ be the unique root of $T$. By our assumption, we see that $M''$ and $M'$ agree except for the $d$-th columns. (See Proposition 5.7.) Let $v_{i_1}, \ldots, v_{i_k}$ be the children of the root $v_d$ such that the signs of $\{v_{d}, v_{i_j}\}$ and $\{v'_{d}, v'_{i_j}\}$ are different. Then, let us consider the subtrees $T_{i_j}$ of $T$ induced by all descendants of $v_{i_j}$ with its root $v_{i_j}$. By the construction of $(n_{ij})$ from $T$, we see that $n_{pq} \neq 0$ only if both $v_p$ and $v_q$ are contained in the same $T_{i_j}$ for some $j$. Now, apply the unimodular transformations $I_j = (e^{(j)}_{pq})_{1 \leq p, q \leq d} \in \mathbb{Z}^{d \times d}$ for $j = 1, \ldots, k$ to $M''$, where

$$e^{(j)}_{pp} = \begin{cases} -1 & \text{if } v_p \in V(T_{i_j}), \\ 1 & \text{otherwise}, \end{cases}$$

and $e^{(j)}_{pq} = 0$ if $p \neq q$. Then we see that the row vectors of $M'' \cdot (I_1 \cdots I_k)$ and $M'$ coincide up to sign. Hence, those satisfy the condition in Lemma 2.7, so we conclude that $X$ and $X'$ are diffeomorphic.

Even if $T$ and $T'$ have at least two connected components, we may apply the above procedure to each connected component.

**Example 6.4** Let us consider the following signed rooted forests $T$ and $T'$.

We see that $T \cong T'$ by the difference of signs of edges $\{v_7, v_3\}$ and $\{v_7, v_6\}$, but the corresponding Fano Bott manifolds are diffeomorphic since the remaining signs can be transformed. The following matrices are $M$, $M''$ and $M'$ appearing in the above proof and the unimodular transformations $I_1$ which corresponds to the subtree induced by the descendants of $v_6$ used in the proof:

$$M = \begin{pmatrix} -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

$$E = \begin{pmatrix} E \end{pmatrix}$$
We can transform $M$ into $M''$ by applying the permutation matrix corresponding to the transposition $(1, 2)$ and $T_6$ appearing in the above proof. Note that $T''$ above is the signed rooted tree corresponding to $M''$.

We can see that $M''I_1$ coincides with $M'$ up to signs of rows. In fact, the first, second, third and seventh rows of the top and bottom halves of $M''I_1$ are exactly equal to those of $M'$, while the fourth, fifth and sixth rows are equal up to signs.

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References

1. Batyrev, V.V.: On the classification of smooth projective toric varieties. Tohoku Math J. 43, 569–585 (1991)
2. Batyrev, V.V.: On the classification of toric Fano 4-folds. J. Math. Sci. (New York) 94, 1021–1050 (1999)
3. Buchstaber, V., Panov, T.: Toric Topology, Mathematical Surveys and Monographs, vol. 204. American Mathematical Society, Providence (2015)
4. Choi, S.: Classification of Bott manifolds up to dimension 8. Proc. Edinb. Math. Soc. (2) 58(3), 653–659 (2015)
5. Cho, Y., Lee, E., Masuda, M., Park, S.: Unique toric structure on a Fano Bott manifold. arXiv:2005.02740
6. Choi, S., Masuda, M.: Classification of $\mathbb{Q}$-trivial Bott manifolds. J. Symplectic Geom. 10(3), 447–461 (2012)
7. Choi, S., Masuda, M., Murai, S.: Invariance of Pontrjagin classes for Bott manifolds. Algebr. Geom. Topol. 15(2), 965–986 (2015)
8. Choi, S., Masuda, M., Oum, S.: Classification of real Bott manifolds and acyclic digraphs. Trans. Am. Math. Soc. 369(4), 2987–3011 (2017)
9. Choi, S., Masuda, M., Suh, D.Y.: Quasitoric manifolds over a product of simplices. Osaka J. Math. 47(1), 109–129 (2010)
10. Choi, S., Masuda, M., Suh, D.Y.: Topological classification of generalized Bott towers. Trans. Am. Math. Soc. 362(2), 1097–1112 (2010)
11. Fulton, W.: Introduction to Toric Varieties. Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton (1993)
12. Higashitani, A., Kurimoto, K., Masuda, M.: Cohomological rigidity for toric Fano manifolds of small dimension or large Picard number. Osaka J. Math. 59(1), 177–215 (2022)
13. Kamishima, Y., Masuda, M.: Cohomological rigidity of real Bott manifolds. Algebr. Geom. Topol. 9(4), 2479–2502 (2009)
14. Masuda, M., Panov, T.: Semi-free circle actions, Bott towers, and quasitoric manifolds. Mat. Sb. 199(8), 95–122 (2008)
15. Masuda, M., Suh, D.Y.: Classification Problems of Toric Manifolds via Topology, Contemporary Mathematics, vol. 460. American Mathematical Society, Providence (2008)
16. Øbro, M.: An algorithm for the classification of smooth Fano polytopes. arXiv:07040049v1
17. Oda, T.: Convex Bodies and Algebraic Geometry—An Introduction to the Theory of Toric Varieties. Ergeb. Math. Grenzgeb. (3), vol. 15. Springer, Berlin (1988)
18. Sato, H.: Towards the classification of higher-dimensional toric Fano varieties. Tohoku Math. J. 52, 383–413 (2000)
19. Suyama, Y.: Fano generalized Bott manifolds. Manuscr. Math. 20, 20 (2019)

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