State Complexity of Permutation and Related Decision Problems on Alphabetical Pattern Constraints

Stefan Hoffmann

Informatikwissenschaften, FB IV, Universität Trier, Universitätsring 15, 54296 Trier, Germany, hoffmanns@informatik.uni-trier.de

Abstract. We investigate the state complexity of the permutation operation, or the commutative closure, on Alphabetical Pattern Constraints (APC). This class corresponds to level 3/2 of the Straubing-Thérien hierarchy and includes the finite, the piecewise-testable, or \( J \)-trivial, and the \( R \)-trivial and \( L \)-trivial languages. We give a sharp state complexity bound expressed in terms of the longest strings in the unary projection languages of an associated finite language. This bound is already sharp for the subclass of finite languages. Additionally, for two subclasses, we give sharp bounds expressed in terms of the size of a recognizing input automaton and the size of the alphabet. Lastly, we investigate the inclusion and universality problem on APCs up to permutational equivalence. These two problems are known to be \( \text{PSPACE} \)-complete on APCs in general, even for fixed alphabets. However, we show them to be decidable in polynomial time for fixed alphabets if we only want to solve them up to permutational equivalence.

Keywords: state complexity · finite automata · alphabetic pattern constraint language · commutative closure · inclusion problem

1 Introduction

In regular model checking [1], a set of initial configurations is modeled as a regular language and the actions of the system are modeled as a rewriting relation. For example, suppose we have an arbitrary number of processes that are connected linearly and need access to a common resource, but only one at a time and in order, starting from the first processor. Then, the state of a given processor could be modeled by \( \Sigma = \{0, 1\} \), where 1 means the processor has access to the resource, and 0 otherwise. The set of initial configurations is then the regular languages \( 10^* \), where a specific initial configuration is determined by the number of processors involved. The transition relation is given by the rule \( 10 \rightarrow 01 \) and the set of reachable configurations is the language \( 0^*10^* \). The bad configurations are given by the language \( (0 + 1)^*1(0 + 1)^*1(0 + 1)^* \), and we see that intersection of this set with the reachable configurations is empty.

The computation of the set of reachable configurations is the closure of the set of initial configurations under the rewriting relation. However, in this generality,
the framework is Turing-complete and hence restrictions have to be imposed. In [2] the class of Alphabetical Pattern Constraints (APC) was introduced as a class to describe initial and bad configurations, given by forbidden patterns, that is closed under semi-commutations. The constructions in [2] rely on an inductive transformation of an APC expression into another APC expression for the closure. Here, our constructions will give a more direct and efficient procedure for the full commutative closure and will also yield deterministic automata, which we then use to devise polynomial time decision procedures for the inclusion and universality problem up to permutational equivalence.

The state complexity of a regular language \( L \) is the minimal number of states needed in a deterministic automaton recognizing \( L \). Investigating the state complexity of the result of a regularity-preserving operation on regular languages, depending on the state complexity of the regular input languages, was first initiated in [15] and systematically started in [31]. As the number of states of a recognizing automaton could be interpreted as the memory required to describe the recognized language and is directly related to the runtime of algorithms employing regular languages, obtaining state complexity bounds is a natural question with applications in verification, natural language processing or software engineering [8].

In general, the permutation operation is not regularity-preserving. But it is regularity-preserving on finite languages, APCs and on group languages [2,9,11]. The state complexity on group languages was studied in [11], but it is not known if the derived bounds are tight. The state complexity of the permutation operation on finite languages was first investigated in [5,18]. However, sharp bounds were only obtained for subclasses and it is unknown if the general bound stated in [5,18] is sharp. Surely, every finite language is an APC.

The dot-depth hierarchy [6] is an infinitely increasing hierarchy whose union is the class of star-free languages. This hierarchy was motivated by alternately increasing the combinatorial and sequential complexity of languages and corresponding recognizing devices [4,20]. Later, the more fundamental Straubing-Thérien hierarchy was introduced [21,27,28]. Here, we start with \( \{ \emptyset, \Sigma^* \} \) at level zero and, alternately, build (1) the half-levels: finite unions of marked products of the form \( L_0a_1L_1a_2\cdots a_kL_k \) with \( k \geq 0 \), \( a_1, \ldots, a_k \in \Sigma \) and \( L_1, \ldots, L_k \) from the previous full-level or, (2) the full levels: the Boolean closure of the previous half-level. More formally, set \( \mathcal{L}(0) = \{ \emptyset, \Sigma^* \} \) and for \( n \geq 0 \), level \( \mathcal{L}(n + \frac{1}{2}) \) consists of all finite unions of languages \( L_0a_1L_1a_2\cdots a_kL_k \) with \( k \geq 0 \), \( L_0, \ldots, L_k \in \mathcal{L}(n) \) and \( a_1, \ldots, a_k \in \Sigma \), and level \( \mathcal{L}(n+1) \) consists of all finite Boolean combinations of languages from level \( \mathcal{L}(n + \frac{1}{2}) \). Every star-free language is contained in some level of this hierarchy, which is also infinitely increasing. The different levels could also be characterized logically by the quantifier alternation of first order sentences [29].

The membership problem and related decision and separation problems with respect to the levels of both hierarchies, and their connection to logic, have sparked much interest [14,21,29]. The APCs precisely correspond to the languages of level \( 3/2 \) in the Straubing-Thérien hierarchy [21,14].
Green’s relations are five equivalence relations, named \( \mathcal{H}, \mathcal{R}, \mathcal{L}, \mathcal{J} \) and \( \mathcal{D} \), that characterize the elements of a semigroup in terms of the principal ideals they generate \([10]\). By the notion of the syntactic monoid \([17,23]\), these relations entered into formal language theory and proved to be useful in the classification of formal languages \([7,13,19]\). For example, it turned out that the \( \mathcal{J} \)-trivial, or piecewise-testable languages, are precisely the languages of level one in the Straubing-Thérien hierarchy \([19,26]\). The \( \mathcal{H} \)-trivial languages are precisely the star-free languages \([24]\). Also, the \( \mathcal{R} \)-trivial and the \( \mathcal{L} \)-trivial languages are properly contained in level 3/2 of the Straubing-Thérien hierarchy, i.e., are APCs \([28,14]\).

2 Preliminaries and Definitions

We assume the reader to have some basic knowledge of automata and complexity theory. For all unexplained notions, as, for example, regular expressions, the Nerode equivalence relation and more formal definitions of \( \text{PSPACE} \), the class of problems solvable with polynomially bounded space, and \( P \), the class of problems solvable in polynomial time, we refer the reader to \([12]\).

For an alphabet (finite nonempty set) \( \Sigma \), denote by \( \Sigma^* \) the set of all finite words over the alphabet \( \Sigma \) including the empty word \( \varepsilon \). If \( u \in \Sigma^* \) and \( a \in \Sigma \), by \(|u|\) we denote the length of \( u \) and by \(|u|_a\) the number of occurrences of the symbol \( a \) in \( u \). A language over \( \Sigma \) is any subset of \( \Sigma^* \). Let \( L \subseteq \Sigma^* \). We set \( \text{Pref}(L) = \{ u \in \Sigma^* \mid \exists v \in \Sigma^* : uv \in L \} \). A word \( u \in \Sigma^* \) is a prefix of a word \( v \in \Sigma^* \), if \( u \in \text{Pref}(|v|) \). For \( a \in \Sigma \), the one-letter projection language is \( \pi_a(L) = \{ a^{[u]_a} : u \in L \} \) and, for \( u \in \Sigma^* \), we set \( \pi_a(u) = a^{[u]_a} \).

For a natural number \( n \geq 0 \), we set \([n] = \{ 0, \ldots, n-1 \}\). For a finite subset \( A \) of natural numbers, by \( \max A \) and \( \min A \) we denote the maximal and minimal element in \( A \) with respect to the usual order, where we set \( \max \emptyset = \min \emptyset = 0 \).

A nondeterministic finite automaton (NFA) is given by \( A = (\Sigma, Q, \delta, q_0, F) \), where \( \Sigma \) is an input alphabet, \( Q \) a finite set of states, \( \delta : Q \times \Sigma \rightarrow 2^Q \) the transition function, having a set of states as image, \( q_0 \) the initial state and \( F \subseteq Q \) the set of final states. If, for any \( q \in Q \) and \( a \in \Sigma \), we have \( |\delta(q, a)| \leq 1 \), then \( A \) is called a partial deterministic finite automaton (PDFA). If \( A \) is a PDFA, then the transition function is often written as a partial function \( Q \times \Sigma \rightarrow Q \). In the usual way, the transition function \( \delta \) can be extended to the domain \( Q \times \Sigma^* \). The language recognized by \( A \) is \( L(A) = \{ u \in \Sigma^* \mid \delta(q_0, u) \cap F \neq \emptyset \} \). The finite simple language associated with \( A \) is \( L^{\text{simple}}(A) = \{ w \in \Sigma^* \mid w \text{ labels a simple accepting path in } A \} \), where a path is simple if no state occurs more than once along the path, i.e., the states we end up after each prefix (along the path) are distinct for distinct prefixes, and a path is accepting if it starts at the initial state of \( A \) and ends in a final state. The language \( L^{\text{simple}}(A) \) is the set of all words in \( L(A) \) that label paths with no loops\(^1\).

\(^1\) The length of the longest word in \( L^{\text{simple}}(A) \) is called the depth in \([16]\).
Lemma 1. Let \( a \in \Sigma \) and \( n = \max\{|u|_a \mid u \in L^{\text{simple}}(A)\} + 1 \). Then, for any \( w \in \Sigma^* \) with \( |w|_a \geq n \), we have: \( w \in \text{perm}(L) \iff wa \in \text{perm}(L) \).

The state complexity of a regular language is the smallest number of states in any PDFA recognizing the language.

Let \( A = (\Sigma, Q, \delta, q_0, F) \). A state \( q \in Q \) is said to be reachable from a state \( p \in Q \), if there exists \( u \in \Sigma^* \) such that \( q \in \delta(p, u) \).

An automaton is called partially ordered, if the reachability relation is a partial order. Equivalently, if the only loops are self-loops. Partially ordered automata are also known as weakly acyclic automata [22].

The shuffle operation of two languages \( U, V \subseteq \Sigma^* \) is defined by

\[
U \shuffle V = \{ w \in \Sigma^* \mid w = x_1y_1x_2y_2 \cdots x_ny_n \text{ for some words } x_1, \ldots, x_n, y_1, \ldots, y_n \in \Sigma^* \text{ such that } x_1x_2 \cdots x_n \in U \text{ and } y_1y_2 \cdots y_n \in V \}.
\]

and \( u \shuffle v = \{ u \} \shuffle \{ v \} \) for \( u, v \in \Sigma^* \). For languages \( L_1, \ldots, L_n \subseteq \Sigma^* \), we set \( \bigcup_{i=1}^n L_i = L_1 \shuffle \cdots \shuffle L_n \). Let \( L \subseteq \Sigma^* \). If \( L = \bigcup_{u \in \Sigma} \{ [u]_a \mid u \in L \} \), then we call it a strict shuffle language.

Example 1. Let \( \Sigma = \{a, b\} \).

1. If \( u \in \Sigma^* \), then \( \text{perm}(u) \) is a strict shuffle language.
2. The language \( \{ u \in \{a, b\}^* \mid |u|_a = 1 \text{ and } 2 \leq |u| \leq n \} \) is a strict shuffle language.
3. \( \text{perm}\{aaabb, ab\} \) is not a strict shuffle language.
4. \( \text{perm}\{aaabb, abbb, aaab, ab\} \) is a strict shuffle language.

The permutation operation, or commutative closure, on a language is the set of words that we get when permuting the letters of the words from the language. Formally, for \( L \subseteq \Sigma^* \), we set \( \text{perm}(L) = \{ u \in \Sigma^* \mid \exists v \in L \forall a \in \Sigma : |u|_a = |v|_a \} \).

For example, \( \text{perm}\{abb\} = \{abb, bab, bba\} \). For \( u \in \Sigma^* \), we also write \( \text{perm}(u) \) for \( \text{perm}\{u\} \). A language \( L \subseteq \Sigma^* \) is called commutative, if \( \text{perm}(L) = L \). Note that for strict shuffle languages \( L \subseteq \Sigma^* \) we have \( \text{perm}(L) = L \).

An Alphatical Pattern Constraint (APC) is an expression \( p_1 + \cdots + p_n \), where each \( p_i \) is of the form \( \Sigma_0^* a_1 \Sigma_1^* \cdots a_n \Sigma_n^* \) with \( \Sigma_0, \ldots, \Sigma_n \subseteq \Sigma \) and \( a_1, \ldots, a_n \in \Sigma \). In the following, we will not distinguish between the expression and the language it denotes, and taking the liberty to denote “+” by the union symbol as well. Hence, an APC is a finite union of languages of the form \( \Sigma_0^* a_1 \Sigma_1^* \cdots a_n \Sigma_n^* \) as above. Equivalently, as concatenation distributes over union, it is the closure of the subsets \( \Gamma^* \), \( \Gamma \subseteq \Sigma \), and \( \{a\} \) for \( a \in \Sigma \) under concatenation and finite union. The APCs are precisely the languages recognized by partially ordered NFAs [1425].

\[2 \text{ The assumption } |w|_a \geq n \text{ is needed. For example, consider } L = a^{n-1}b^* \].

\[3 \text{ With the shorthand } \Gamma^* = (a_1 + \cdots + a_n)^* \text{ for } \Gamma = \{a_1, \ldots, a_n\} \subseteq \Sigma \].

\[4 \text{ Note that, for } \Sigma_0, \Sigma_1 \subseteq \Sigma \text{ nonempty, we have } \Sigma_0^* \Sigma_1^* = \Sigma_0^* \cup \bigcup_{\Sigma_1} \Sigma_0^* a \Sigma_1^* \].
3 State Complexity Bound of Permutation on APCs

The APCs are closed under the permutation operation. However, they are not closed under complementation. For example, the complement of $\Sigma^*ab\Sigma^* \cup b\Sigma^* \cup \Sigma^*a$ over $\Sigma = \{a, b\}$ is $(ab)^*$. As perm($(ab)^* = \{u \in \Sigma^* | |u|_a = |u|_b\}$ is not regular, it is not an APC. This also shows that level 3/2 of the Straubing-Thérien hierarchy is the lowest level in which the permutation of any language is regular.

Remark 1. Let $L = \bigcup_{i=1}^{n_0} r_{0}^{(i)} a_{1}^{(i)} \cdots a_{n_i}^{(i)} \Sigma_{n_i}^{(i)}$. Then, perm$(L) = \bigcup_{i=1}^{n_0} \text{perm}(a_{1}^{(i)} \cdots a_{n_i}^{(i)} \Gamma^{(i)}) = \bigcup_{i=1}^{n_0} \text{perm}(a_{1}^{(i)} \cdots a_{n_i}^{(i)}) \cup \Gamma^{(i)}$.

Hence, the permutational closure, as a finite union of languages of the form $\Gamma^{*} a_{1} \Gamma^{*} \cdots a_{n} \Gamma^{*}$, is itself an APC.

Theorem 2. Let $L$ be an APC recognized by a partially ordered NFA $A$. Then, perm$(L)$ is recognizable by a PDFA that uses at most (where we set max $\mathcal{O} = 0$)

$$\prod_{a \in \Sigma} (\max\{|u|_a : u \in L^{\text{simple}}(A)\} + 1)$$

many states and this bound is sharp even for finite languages.

Proof. Suppose we have $k$ symbols and $\Sigma = \{a_1, \ldots, a_k\}$. Set $n_j = \max\{|u|_{a_j} : u \in L^{\text{simple}}(A)\} + 1$ for $j \in \{1, \ldots, k\}$. Construct $B = (\Sigma, Q, \delta, q_0, F)$ with $Q = [n_1 + 1] \times \cdots \times [n_k + 1]$ and

$$\delta((s_1, \ldots, s_k), a_j) = \begin{cases} (s_1, \ldots, s_{j-1}, s_j + 1, s_{j+1}, \ldots, s_k) & \text{if } s_j < n_j; \\ (s_1, \ldots, s_k) & \text{if } s_j = n_j \text{ and } a_1^{s_1} \cdots a_k^{s_k} a_j \in \text{perm}(\text{Pref}(L)). \end{cases}$$

Also $q_0 = (0, \ldots, 0)$ and $F = \{q(w, u) \mid w \in L \text{ and } \forall j \in \{1, \ldots, k\} : |w|_{a_j} \leq n_j\}$.

Claim: We have $L(B) = \text{perm}(L)$.

Proof of the Claim: By Lemma 1 for any $w \in \Sigma^*$ with $|w|_{a_j} \geq n_j$, we have

$$w \in \text{perm}(L) \Leftrightarrow w a_j \in \text{perm}(L).$$

Let $w \in \text{perm}(L)$. Then $a_1^{\min\{n_1, |w|_{a_1}\}} \cdots a_k^{\min\{n_k, |w|_{a_k}\}} \in \text{perm}(L)$. Hence,

$$\delta(q_0, a_1^{\min\{n_1, |w|_{a_1}\}} \cdots a_k^{\min\{n_k, |w|_{a_k}\}}) \in F.$$

Furthermore, if $|w|_{a_j} \geq n_j$, then for

$$q = (\min\{n_1, |w|_{a_1}\}, \ldots, \min\{n_k, |w|_{a_k}\}),$$

we have $\delta(q, a_j) = q$ for any $j \in \{1, \ldots, k\}$ such that $|w|_{a_j} \geq n_j$. So,

$$\delta(q_0, w) = \delta(q_0, a_1^{\min\{n_1, |w|_{a_1}\}} \cdots a_k^{\min\{n_k, |w|_{a_k}\}}) \in F.$$
Conversely, suppose $\delta(q, w) \in F$. If $|w|_{a_j} > n_j$ with $j \in \{1, \ldots, k\}$, then, for the state $q = (\min\{n_1, |w|_{a_1}\}, \ldots, \min\{n_k, |w|_{a_k}\})$, we have $\delta(q, |w|_{a_j}) = q$ and so $\text{perm}(a_1^{\min\{n_1, |w|_{a_1}\}} \cdots a_k^{\min\{n_k, |w|_{a_k}\}}) = \text{perm}(L)$ by the above definition of the transition function $\delta$. Hence, the letter $a_j$ could be appended $n_j - |w|_{a_j}$ many times and $B$ stays in the same state, for every such letter with $|w|_{a_j} > n_j$. So, we find $\text{perm}(a_1^{\min\{n_1, |w|_{a_1}\}} \cdots a_k^{\min\{n_k, |w|_{a_k}\}}) \subseteq \text{perm}(L)$, which is equivalent to $w \in \text{perm}(L)$.

That the bound is sharp is shown in Remark 2.

Lemma 3. Let $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ be a partially ordered NFA. If any NFA for $\text{L}_\text{simple}(\mathcal{A})$ needs at least $n$ states, then $|Q| \geq n$. A similar statement holds true for PDFAs.

Let $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ be a partially ordered NFA. As $\text{L}_\text{simple}(\mathcal{A})$ is finite, every path from the start state to a final state in any recognizing automaton has no loops. Hence, the length of a longest string in $\text{L}_\text{simple}(\mathcal{A})$ is a lower bound for the number of states of any NFA recognizing $\text{L}_\text{simple}(\mathcal{A})$. Surely, for $a \in \Sigma$ and $u \in \text{L}_\text{simple}(\mathcal{A})$, the number $|u|_a$ is a lower bound for the length of the longest string in $\text{L}_\text{simple}(\mathcal{A})$. So, combining with Lemma 3 we have $\max\{|u|_a : u \in \text{L}_\text{simple}(\mathcal{A})\} \leq |Q|$ for $a \in \Sigma$. This yields the next corollary to Theorem 2.

Corollary 4. Let $L$ be an APC recognized by a partially ordered NFA with $n$ states. Then, $\text{perm}(L)$ is recognizable by a PDFA with at most $n^{\lceil |\Sigma| \rceil}$ many states.

We have formulated Theorem 2 and the above corollary in terms of partially ordered NFAs recognizing a given APC. However, APC expressions and partially ordered NFAs are closely connected, for example, see Lemma 9 in Section 6. Hence a corresponding statement could be made for APC expressions, where $\text{L}_\text{simple}(\mathcal{A})$ corresponds to the set of words resulting if we delete all subexpressions $\Sigma_i^*$ in the parts of the unions.

4 When $\text{perm}(\text{L}_\text{simple}(\mathcal{A}))$ is a Strict Shuffle Language

Here, we investigate a class of languages for which we can devise a sharp bound expressed in the size of the input NFA. The bound is formulated with the number of states and the size of the alphabet of the input automaton. As the bound is sharp for a subclass of languages, it also yields a lower bound for the general case.

For finite strict shuffle languages, we can derive the following lower bound for the size of recognizing NFAs, which we will need in the proof of Theorem 6.

Lemma 5. Let $L \subseteq \Sigma^*$ be finite. If $\text{perm}(L)$ is a strict shuffle language, then any NFA recognizing $L$ needs at least $(\sum_{a \in \Sigma} \max\{|u| : u \in \pi_a(L)\}) + 1$ many states.

Next, we state the main result of this section.
Theorem 6. Let $L$ be an APC language recognized by a partially ordered NFA $A$ with $n$ states such that $\text{perm}(L^{\text{simple}}(A))$ is a strict shuffle language. Then, $\text{perm}(L)$ is recognizable by a PDFA with at most
\[
\left\lfloor \frac{n - 1}{|\Sigma|} + 1 \right\rfloor^{|\Sigma|}
\]
many states and this bound is sharp even for finite languages.

Proof. By Lemma 5, any automaton for $L^{\text{simple}}(A)$ needs at least $(\Sigma \subseteq \Sigma \max\{|u|_a : u \in \pi_a(L)\}) + 1$ many states. So, by Lemma 3 we have $0 \leq (\Sigma \subseteq \Sigma \max\{|u|_a : u \in \pi_a(L)\}) + 1 \leq n$. The value $\prod_{a \subseteq \Sigma}(\max\{|u|_a : u \in L^{\text{simple}}(A)\}) + 1$ from Theorem 2 with the constraint $0 \leq (\Sigma \subseteq \Sigma \max\{|u|_a : u \in \pi_a(L)\}) + 1 \leq n$ is maximized if $\max\{|u|_a : u \in L^{\text{simple}}(A)\}$ equals $(n - 1)/|\Sigma|$ for every $a \subseteq \Sigma$, which gives the claim. That the bound is sharp is shown in Remark 2.

Note that for a single word $u \in \Sigma^*$, we have $\text{perm}(u) = \prod_{a \subseteq \Sigma} \pi_a(u)$, i.e., the commutative closure is a strict shuffle language. Hence, we get the next corollary from Theorem 6 which is also sharp, as shown by Remark 2.

Corollary 7. Let $L = \Sigma^* a_1 \Sigma^* a_2 \cdots a_m \Sigma^*$. Then, $\text{perm}(L)$ is recognizable by a PDFA with at most $\left\lfloor \frac{m}{|\Sigma|} + 1 \right\rfloor^{|\Sigma|}$ many states. In particular, the commutative closure of a single word $u$ could be recognized by a PDFA with at most $\left\lfloor \frac{|u|}{|\Sigma|} + 1 \right\rfloor^{|\Sigma|}$ many states and this bound is sharp.

Proof. The NFA $A$ with state set $Q = \{q_0, q_1, \ldots, q_m\}$, transition function $\delta(q_i, a) = \{q_i : a \subseteq \Sigma_i\} \cup \{q_{i+1} : i < m$ and $a = a_{i+1}\}$ for $i \in \{0, \ldots, m\}$ and $a \subseteq \Sigma$, start state $q_0$ and final state set $\{q_m\}$ recognizes $L$. We have $L^{\text{simple}}(A) = \{a_1 a_2 \cdots a_m\}$ and $\text{perm}(L^{\text{simple}}(A))$ is a strict shuffle language. Note that, by Lemma 3 and Lemma 5, $A$ has the least possible number of states. Then, Theorem 6 gives the claim and the bound is sharp by Remark 2.

Remark 2. Suppose $\Sigma = \{a_1, \ldots, a_k\}$. Let $m > 0$ and $u = a_1^{n_1} \cdots a_k^{n_k}$. Then, any PDFA recognizing $\text{perm}(u)$ needs at least $(m + 1)^k$ many states. For let $0 \leq m_1 \leq \cdots \leq m_k$, such that there exists $j \in \{1, \ldots, k\}$ with $m_j < m_j$. Then, choose $r_i$ for each $i \in \{1, \ldots, k\}$ such that $n_i + r_i = m$. Set $w = a_1^{n_1} \cdots a_k^{n_k} a_1^{r_1} \cdots a_k^{r_k}$. As $|w|_{a_i} = m$ for any $i \in \{1, \ldots, k\}$, we find $w \in \text{perm}(u)$. However, for $w' = a_1^{m_1} \cdots a_k^{m_k} a_1^{r_1} \cdots a_k^{r_k}$ we have $|w'|_{a_j} < m$, so that $w' \notin \text{perm}(u)$. So, $w$ and $w'$ represent different Nerode right-congruence classes [12] for the language $\text{perm}(u)$, which yields the lower bound for the number of states of any recognizing automaton.

As $u$ is recognizable by a minimal NFA $A$ with $k \cdot m + 1$ many states, $|u|_{a_i} = m$ for any $i \in \{1, \ldots, k\}$ and $L^{\text{simple}}(A) = L(A)$, as $L(A)$ is finite, the bounds of Theorem 6 and of Corollary 7 are all met by this example.

More precisely, if $\Sigma = \{a_1, \ldots, a_k\}$, we seek to maximize the function $f(x_1, \ldots, x_n) = \prod_{i=1}^n (x_i + 1)$ due to the constraint $0 \leq \sum_{i=1}^k x_i \leq n - 1$, which happens for $x_1 = \ldots = x_k = \max(1, 1 - \frac{1}{k})$ with maximum value $(\frac{1}{k} + 1)^k.$
5 State Complexity on General Chain Automata

A general chain automaton \( \mathcal{A} = (\Sigma, Q, \delta, q_0, F) \) is a NFA such that the state set is totally ordered, i.e., we can assume \( Q = \{0, \ldots, n-1\} \) with the usual order and \( q_0 = 0 \) and \( F = \{n-1\} \) and, for any \( q \in Q \setminus \{n-1\} \) and \( a \in \Sigma \), we have \( \delta(q, a) \subseteq \{q, q+1\} \). If \( \mathcal{A} \) is a general chain automaton, then \( L_{\text{simple}}(\mathcal{A}) \subseteq \Sigma^{n-1} \).

These automata, with no self-loops allowed\(^6\), were introduced in \( [5] \) under the name chain automata. The sharp bound we will give is essentially an adaption of the bound derived in \( [5] \). Note that we only have a result for binary alphabets.

**Proposition 8.** Let \( \Sigma = \{a, b\} \) and \( \mathcal{A} \) be a general chain automaton with \( n \) states. Then, \( \text{perm}(L(\mathcal{A})) \) is recognizable by a PDFA with at most \( \frac{n^2 + n + 1}{3} \) many states and this bound is sharp even on finite languages.

**Proof (sketch).** Let the set of states of \( \mathcal{A} \) be \( \{0, \ldots, n-1\} \), where 0 is the start state and \( n-1 \) is the only final state. Set \( \Gamma = \{x \in \Sigma \mid \exists q \in Q : q \in \delta(q, x)\} \), the symbols which label self-loops. Note that \( L(\mathcal{A}) \) is finite if and only if \( \Gamma = \emptyset \).

For \( 0 \leq h \leq n-2 \), the transitions only go from \( h \) to \( h+1 \) or we have a self-loop from \( h \) to \( h \). We have three possibilities for outgoing transitions from a state \( 0 \leq h \leq n-2 \) that are not self-loops:

1. \( \{h+1\} \subseteq \delta(h, a) \) and \( \delta(h, b) \cap \{h+1\} = \emptyset \) (a-transition);
2. \( \{h+1\} \subseteq \delta(h, b) \) and \( \delta(h, a) \cap \{h+1\} = \emptyset \) (b-transition);
3. \( \{h+1\} \subseteq \delta(h, a) \cap \delta(h, b) \) (a&b-transition).

The order of the different types of transitions (a, b, or a&b) of \( \mathcal{A} \) does not affect the language \( \text{perm}(L(\mathcal{A})) \). A similar reasoning applies to the self-loops. Hence, without loss of generality, we can assume that \( \mathcal{A} \) has first a (possibly empty) sequence of a-transitions, followed by a (possibly empty) sequence of b-transitions, followed by a (possibly empty) sequence of a&b-transitions and only self-loops with labels from the (possibly empty) subset \( \Gamma \subseteq \Sigma \) at the final state.

Thus, we can assume that \( L(\mathcal{A}) = a^i b^j (a+b)^k \ast \) for some non-negative integers \( i, j, k \) such that \( i + j + k = n - 1 \). By modifying a construction from \( [5] \), we can construct a PDFA for \( \text{perm}(L(\mathcal{A})) \) with \( f(i, j, k) = (i + 1) \cdot (j + 1) + k \cdot j + k \cdot i + k \) many states. In order to get an upper bound for the state complexity of \( \text{perm}(L(\mathcal{A})) \) as a function of the size of \( \mathcal{A} \), we determine for which values of \( i, j, k \), where \( i + j + k = n - 1 \), the function \( f(i, j, k) \) has a maximal value. The function \( f \) is maximized if \( ij + kj + ki \) is maximal, thus if \( i = j = k = \frac{n-1}{3} \).

More generally,

\[
\max_{i+j+k=n-1} f(i, j, k) = \begin{cases} \frac{n^2 + n + 1}{3} & \text{if } n \equiv 1 \pmod{3}; \\ \frac{n^2 + n - 2}{3} & \text{otherwise}. \end{cases}
\]

In \( [5] \) Lemma 4.2, as every chain automaton is a general chain automaton recognizing a finite language, it was shown that for \( n \equiv 1 \pmod{3} \) there exists a language recognized by a chain automaton with \( n \) states such that any automaton for the commutative closure needs at least \( \frac{n^2 + n - 1}{3} \) many states. \( \square \)

\(^6\) This is no restriction when we have no self-loops.
6 Complexity Results

Here, we consider the alphabet to be fixed in advance and not part of the input.

In model checking, when the specification and the implementation could be represented by finite automata, the inclusion problem arises naturally \[1,30\]. In this problem, we are given two automata and ask if the recognized language of the first is contained in the recognized language of the second automaton.

In \[2\] it was shown that the universality problem, i.e., deciding if a given APC\(^7\) denotes \(\Sigma^*\), is PSPACE-complete, even for fixed binary alphabets. This implies PSPACE-completeness of the inclusion problem.

Here, we show the somewhat surprising result that the above decision problems are polynomial time solvable modulo permutational equivalence, i.e., if we ask the same questions for the commutative closure of the input languages, see Theorem \([11]\) and Corollary \([12]\).

This result is not as artificial as it might seem. For example, consider the introductory example from regular model checking in Section \([1]\). Here, the set of reachable configurations \(0^*10^*\) is closed under the commutative closure, as well as the set of bad configurations \((0 + 1)^*1(0 + 1)^*1(0 + 1)^*\) and its complement. More specifically, these sets are commutative languages and the original decision problem is equivalent to the same decision problem modulo permutational equivalence.

At the heart of this result lies the fact that the PDFA constructed in the proof of Theorem \([2]\) could be constructed, for a fixed alphabet, in polynomial time. This will be shown in Proposition \([10]\). But before this result, let us first state that, with respect to polynomial time, it makes no difference if the input is given as an APC expression or a partially ordered NFA.

**Lemma 9.** For a given partially ordered NFA \(A\) an APC expression of \(L(A)\) could be computed in \(P\) and for every APC expression a partially ordered NFA is computable in \(P\). This result also holds for variable input alphabets.

So, we are ready to derive that from a given partially ordered NFA, a PDFA recognizing the commutative closure could be computed in \(P\).

**Proposition 10.** Given a partially ordered NFA \(A\) with \(n\) states, the recognizing PDFA for \(\text{perm}(L(A))\) from Theorem \([2]\) could be constructed in polynomial time for a fixed alphabet. More precisely in time \(O(n|\Sigma|^2)\).

*Proof (sketch).* This is only a rough and intuitive outline of the procedure.

Let \(\Sigma = \{a_1, \ldots, a_k\}\) and \(A = (\Sigma, S, \mu, s_0, E)\) be a partially ordered NFA. We outline a polynomial time algorithm to compute \(B = (\Sigma, Q, \delta, q_0, F)\) as defined in the proof of Theorem \([2]\) We can assume that \(s_0\) is minimal for the partial order of \(A\) and every maximal state is final. Set \(n_a = \max\{|u|_a : u \in L\text{simple}(A)\}\) for \(a \in \Sigma\)

The state set, and hence the numbers \(n_a\), could be computed by a dynamic programming scheme starting at the maximal final states and ending...
at the start state. For each letter \( a \in \Sigma \), we store at every state \( q \) the number \( \max\{|u|_a \mid \delta(q, u) \in F \) and no loops are entered by \( u \) in \( A \)\), i.e., the longest unary projection string for that letter when starting at this state, ending at a final state and traversing no self-loops.\(^8\) For a final maximal state, those numbers are initialized to zero and for every other state, they are computable from the predecessor states. For the start state, the last state in this procedure, these are precisely the numbers \( n_a \), from which \( Q \) is easily constructible.

The computation of the transition function and the final state set is more involved. Note that for states \( s_1, \ldots, s_k \in Q \) with \( s_i < n_{a_i} \) for \( i \in \{1, \ldots, k\} \) the transition function is easily computable. The only difficulty is to determine which “boundary” states should be labeled by self-loops. We do this by constructing an auxiliary automaton \( A' \) out of \( A \) by “unfolding” the self-loops into paths of length \( |S| + 1 \). The automaton \( A' \) then has no loops anymore. Now, we label the states of this auxiliary automaton with those states from \( Q \) that are reachable in \( B \) by words that go from the start state to the state under consideration of \( A' \). If such a word passes an unfolded path completely, then, as they are sufficiently long, we know that it must traverse a self-loop in \( A \) labeled by the same letter \( a \) as the unfolded path. In this case, for every “boundary” state of \( Q \) in the labeling of the target state of the word in \( A' \) we add a self-loop for the letter \( a \) to that state from \( Q \) in \( B \).

Finally, a state from \( Q \) is declared to be final if and only if it appears in a label of a final state of \( A' \).

This procedure indeed computes \( B \) and could be made to run in the stated time bound.

With Proposition \(10\) we derive that, given two APCs, the inclusion problem modulo permutational equivalence is solvable in polynomial time.

**Theorem 11.** Fix an alphabet \( \Sigma \). Then, the following problem is in \( \mathcal{P} \):
Input: Two APC expressions \( L_1, L_2 \) over \( \Sigma^* \).
Question: Is \( \text{perm}(L_1) \subseteq \text{perm}(L_2) \)?

Given an APC, the universality problem modulo permutational equivalence is solvable in polynomial time, as it is reducible to the corresponding inclusion problem up to permutational equivalence.

**Corollary 12.** Fix an alphabet \( \Sigma \). Then, the following problem is in \( \mathcal{P} \):
Input: An APC expression \( L \) over \( \Sigma^* \).
Question: Is \( \text{perm}(L) = \Sigma^* \)?

As for commutative languages \( L \subseteq \Sigma^* \) we have \( \text{perm}(L) = L \), we get the next corollary. This generalizes a corresponding reduction of complexity for unary alphabets \(14\).

**Corollary 13.** Fix an alphabet \( \Sigma \). Given an APC describing a commutative language, the universality problem is in \( \mathcal{P} \). Also, given two APCs describing commutative languages, the inclusion problem is solvable in polynomial time.

\(^8\) So, essentially we are working in the automaton that results if we delete all self-loops, which gives a recognizing automaton for \( L_{\text{simple}}(A) \) for partially ordered NFAs \( A \).
7 Conclusion

We have given a sharp upper bound for the number of states needed in a deterministic recognizing automata for the commutative closure of APCs. Additionally, we have shown that the recognizing automaton could be computed in polynomial time for fixed alphabets. Using this result, we have shown that the inclusion and universality problem modulo permutational equivalence are solvable in polynomial time for a fixed input alphabet. This contrasts with the general inclusion and universality problem for APCs. Both are \textsc{PSPACE}-complete even for binary alphabets [14]. For two subclasses of the APC languages, we have given sharp bounds for the commutative closure expressed in the size of the input automata. In the case that the language is given by a general chain automaton, the result was only established for binary alphabets. The case for larger alphabets is still open.

Acknowledgement. I thank the anonymous reviewers for careful reading, noticing a reoccurring typo in the proof of Theorem 1 that was luckily spotted and fixed and helping me identifying some unclear formulations throughout the text.

References

1. Abdulla, P.A., Jonsson, B., Nilsson, M., Saksena, M.: A survey of regular model checking. In: Gardner, P., Yoshida, N. (eds.) CONCUR 2004 - Concurrency Theory, 15th International Conference, London, UK, August 31 - September 3, 2004, Proceedings. LNCS, vol. 3170, pp. 35–48. Springer (2004)
2. Bouajjani, A., Muscholl, A., Touili, T.: Permutation rewriting and algorithmic verification. Inf. Comput. 205(2), 199–224 (2007)
3. Brzozowski, J.A., Fitch, F.E.: Languages of \(\mathcal{R}\)-trivial monoids. Journal of Computer and System Sciences 20(1), 32–49 (Feb 1980)
4. Brzozowski, J.A.: Hierarchies of aperiodic languages. RAIRO Theor. Informatics Appl. 10(2), 33–49 (1976)
5. Cho, D., Goc, D., Han, Y., Ko, S., Palioudakis, A., Salomaa, K.: State complexity of permutation on finite languages over a binary alphabet. Theor. Comput. Sci. 682, 67–78 (2017)
6. Cohen, R.S., Brzozowski, J.A.: Dot-depth of star-free events. J. Comput. Syst. Sci. 5(1), 1–16 (1971)
7. Colcombet, T.: Green's relations and their use in automata theory. In: Dediu, A., Inenaga, S., Martín-Vide, C. (eds.) Language and Automata Theory and Applications - 5th International Conference, LATA 2011, Tarragona, Spain, May 26-31, 2011. Proceedings. LNCS, vol. 6638, pp. 1–21. Springer (2011)
8. Gao, Y., Moreira, N., Reis, R., Yu, S.: A survey on operational state complexity. Journal of Automata, Languages and Combinatorics 21(4), 251–310 (2017)
9. Gómez, A.C., Guiaiana, G., Pin, J.: Regular languages and partial commutations. Inf. Comput. 230, 76–96 (2013)
10. Green, J.A.: On the structure of semigroups. Annals of Mathematics (second series). 54, 163–172 (1951)
11. Hoffmann, S.: State complexity bounds for the commutative closure of group languages. In: Jirásková, G., Pighizzini, G. (eds.) Descriptional Complexity of Formal Systems - 22nd International Conference, DCFS 2020, Vienna, Austria, August 24-26, 2020, Proceedings. LNCS, vol. 12442, pp. 64–77. Springer (2020)
12. Hopcroft, J.E., Ullman, J.D.: Introduction to Automata Theory, Languages, and Computation. Addison-Wesley Publishing Company (1979)
13. J.E. Pin, J.E.: Varieties Of Formal Languages. Plenum Publishing Co. (1986)
14. Krötzsch, M., Masopust, T., Thomazo, M.: Complexity of universality and related problems for partially ordered nfas. Inf. Comput. 255, 177–192 (2017)
15. Maslov, A.N.: Estimates of the number of states of finite automata. Dokl. Akad. Nauk SSSR 194(6), 1266–1268 (1970)
16. Masopust, T., Krötzsch, M.: Partially ordered automata and piecewise testability. CoRR abs/1907.13115 (2019), http://arxiv.org/abs/1907.13115
17. McNaughton, R., Papert, S.A.: Counter-Free Automata (M.I.T. Research Monograph No. 65). The MIT Press (1971)
18. Palioudakis, A., Cho, D., Goc, D., Han, Y., Ko, S., Salomaa, K.: The state complexity of permutations on finite languages over binary alphabets. In: Shallit, J.O., Okhotin, A. (eds.) Descriptive Complexity of Formal Systems - 17th International Workshop, DCFS 2015, Waterloo, ON, Canada, June 25-27, 2015. Proceedings. Lecture Notes in Comp. Science, vol. 9118, pp. 220–230. Springer (2015)
19. Pin, J.: Syntactic semigroups. In: Rozenberg, G., Salomaa, A. (eds.) Handbook of Formal Languages, Volume 1, pp. 679–746. Springer (1997)
20. Pin, J.: The dot-depth hierarchy, 45 years later. In: Konsttinidis, S., Moreira, N., Reis, R., Shallit, J.O. (eds.) The Role of Theory in Computer Science - Essays Dedicated to Janusz Brzozowski. pp. 177–202. World Scientific (2017)
21. Place, T., Zeitoun, M.: Generic results for concatenation hierarchies. Theory Comput. Syst. 63(4), 849–901 (2019)
22. Ryzhikov, A.: Synchronization problems in automata without non-trivial cycles. Theor. Comput. Sci. 787, 77–88 (2019)
23. Schützenberger, M.P.: On an application of semi groups methods to some problems in coding. IRE Trans. Inf. Theory 2(3), 47–60 (1956)
24. Schützenberger, M.P.: On finite monoids having only trivial subgroups. Inf. Control. 8(2), 190–194 (1965)
25. Schwentick, T., Thérien, D., Vollmer, H.: Partially-ordered two-way automata: A new characterization of DA. In: Kuich, W., Rozenberg, G., Salomaa, A. (eds.) Developments in Language Theory, 5th International Conference, DLT 2001, Vienna, Austria, July 16-21, 2001, Revised Papers. Lecture Notes in Computer Science, vol. 2295, pp. 239–250. Springer (2001)
26. Simon, I.: Piecewise testable events. In: Barkhage, H. (ed.) Automata Theory and Formal Languages, 2nd GI Conference, Kaiserslautern, May 20-23, 1975. Lecture Notes in Computer Science, vol. 33, pp. 214–222. Springer (1975)
27. Straubing, H.: A generalization of the schützenberger product of finite monoids. Theor. Comput. Sci. 13, 137–150 (1981)
28. Thérien, D.: Classification of finite monoids: The language approach. Theor. Comput. Sci. 14, 195–208 (1981)
29. Thomas, W.: Classifying regular events in symbolic logic. J. Comput. Syst. Sci. 25(3), 360–376 (1982)
30. Vardi, M.Y.: An automata-theoretic approach to linear temporal logic. In: Moller, F., Birtwistle, G.M. (eds.) Logics for Concurrency - Structure versus Automata (8th Banff Higher Order Workshop, Banff, Canada, August 27 - September 3, 1995, Proceedings). LNCS, vol. 1043, pp. 238–266. Springer (1995)
31. Yu, S., Zhuang, Q., Salomaa, K.: The state complexities of some basic operations on regular languages. Theoretical Computer Science 125(2), 315–328 (Mar 1994)
A. Proofs for Section 2 (Preliminaries and Definitions)

Lemma 1. Let \( a \in \Sigma \) and \( n = \max \{|u|_a \mid u \in L_{\text{simple}}(A)\} + 1 \). Then, for any \( w \in \Sigma^* \) with \( |w|_a \geq n \), we have: \( w \in \text{perm}(L) \iff wa \in \text{perm}(L) \).

Proof. If \( w \in \text{perm}(L) \) is such that \( |w|_a \geq n \), then in \( A \) we have to enter a loop labeled by \( a_j \) when reading \( w \), as after removing the loops every accepting path has at most \( n_j - 1 \) transitions labeled by the letter \( a_j \). So, we can read in \( a_j \) one additional time and find \( wa_j \in \text{perm}(L) \). Conversely, if \( wa_j \in \text{perm}(L) \), with the same reasoning, we find that we have to traverse a loop labeled by \( a_j \) at least once, which could be left out and so \( w \in \text{perm}(L) \).

B. Proofs for Section 3 (State Complexity Bound of Permutation on APCs)

Lemma 3. Let \( A = (\Sigma, Q, \delta, q_0, F) \) be a partially ordered NFA. If any NFA for \( L_{\text{simple}}(A) \) needs at least \( n \) states, then \(|Q| \geq n\). A similar statement holds true for PDFA.

Proof. By deleting all self-loops in \( A \), we find an automaton for \( L_{\text{simple}}(A) \) with \(|Q| \) many states. Similarly if \( A \) is a PDFA. \( \Box \)

C. Proofs for Section 4 (When \( \text{perm}(L_{\text{simple}}(A)) \) is a Strict Shuffle Language)

Lemma 5. Let \( L \subseteq \Sigma^* \) be finite. If \( \text{perm}(L) \) is a strict shuffle language, then any NFA recognizing \( L \) needs at least \( \sum_{a \in \Sigma} \max\{|u| : u \in \pi_a(L)\} + 1 \) many states.

Proof. Suppose \( \Sigma = \{a_1, \ldots, a_k\} \). Set \( m_i = \max\{|u| : u \in \pi_{a_i}(L)\} \) for \( i \in \{1, \ldots, k\} \) and \( w = a_1^{m_1} \cdots a_k^{m_k} \). Then \( w \in \pi_{a_1}(L) \cup \ldots \cup \pi_{a_k}(L) = \text{perm}(L) \). Hence, we find \( u \in L \) with \( u \in \text{perm}(w) \).

If \( v \in L \) is arbitrary, then \( |v| = \sum_{i=1}^{k} |v|_{a_i} \leq \sum_{i=1}^{k} m_i = |u| \). So, \( |u| = \max\{|w'| : w' \in L\} \). For finite languages, as we could not have any loops on a path from the start state to some final state, the length of the longest string in the language plus one is a lower bound for the number of states for any recognizing NFA. As \(|u| \) is a longest string in \( L \), this gives the claim. \( \Box \)

D. Proofs for Section 5 (State Complexity on General Chain Automata)

Proposition 8. Let \( \Sigma = \{a, b\} \) and \( A \) be a general chain automaton with \( n \) states. Then, \( \text{perm}(L(A)) \) is recognizable by a PDFA with at most \( \frac{n^2 + n + 1}{3} \) many states and this bound is sharp even on finite languages.

The assumption \( |w|_a \geq n \) is needed. For example, consider \( L = a^{n-1}b^* \).
Proof. Let the set of states of $A$ be $\{0, \ldots, n - 1\}$, where 0 is the start state and $n - 1$ is the only final state. Set $\Gamma = \{x \in \Sigma \mid \exists q \in Q : q \in \delta(q, x)\}$, the symbols which label self-loops. Note that $L(A)$ is finite if and only if $\Gamma = \emptyset$. For $0 \leq h \leq n - 2$, the transitions only go from $h$ to $h + 1$ or we have a self-loop from $h$ to $h$. We have three possibilities for outgoing transitions from a state $0 \leq h \leq n - 2$ that are not self-loops:

1. $\{h + 1\} \subseteq \delta(h, a)$ and $\delta(h, b) \cap \{h + 1\} = \emptyset$ (a-transition);
2. $\{h + 1\} \subseteq \delta(h, b)$ and $\delta(h, a) \cap \{h + 1\} = \emptyset$ (b-transition);
3. $\{h + 1\} \subseteq \delta(h, a) \cap \delta(h, b)$ (a&b-transition).

The order of the different types of transitions ($a$, $b$, or $a\cdot b$) of $A$ does not affect the language $\text{perm}(L(A))$. A similar reasoning applies to the self-loops. Hence, without loss of generality, we can assume that $A$ has first a (possibly empty) sequence of $a$-transitions, followed by a (possibly empty) sequence of $b$-transitions, followed by a (possibly empty) sequence of $a\cdot b$-transitions and only self-loops with labels from the (possibly empty) subset $\Gamma \subseteq \Sigma$ at the final state. Thus, we can assume that $L(A) = a^i b^j (a + b)^k \Gamma^*$ for some non-negative integers $i, j, k$ such that $i + j + k = n - 1$.

Now, the language $\text{perm}(L(A))$ is recognized by the PDFA $B_{i,j,k} = (\{(a, b), \gamma, q_0, F_B\} \ gamma where $Q = \{(r, s) \mid 0 \leq h \leq i + k, 0 \leq j < l\} \cup \{(r, s) \mid 0 \leq j \leq i + k, 0 \leq r < i\} \cup \{z_0, z_1, \ldots, z_k\}$,

$F_B = \{z_k\}$, $q_0 = (0, 0)$ and the transitions are defined by setting $\gamma(r, s, a) = \begin{cases} (r + 1, s) & \text{if } r < i - 2 \text{ or } (r < i + k \text{ and } s < j); \\ z_{s-j} & \text{if } r = i - 1 \text{ and } s \geq j; \\ (r, s) & \text{if } r + 1 > i + k \text{ and } a \in \Gamma; \\ \text{undefined} & \text{if } r + 1 > i + k \text{ and } a \notin \Gamma; \end{cases}$

$\gamma(r, s, b) = \begin{cases} (r, s + 1) & \text{if } s < j - 2 \text{ or } (s < j + k \text{ and } r < i); \\ z_{s-i} & \text{if } s = j - 1 \text{ and } r \geq i; \\ (r, s) & \text{if } s + 1 > j + k \text{ and } b \in \Gamma; \\ \text{undefined} & \text{if } s + 1 > j + k \text{ and } b \notin \Gamma; \end{cases}$

and, for $x \in \Sigma$,

$\gamma(z_l, x) = \begin{cases} z_{l+1} & \text{if } l \leq i < k; \\ z_k & \text{if } l = k \text{ and } x \in \Gamma; \\ \text{undefined} & \text{if } l = k \text{ and } x \notin \Gamma. \end{cases}$

A computation of $B_{i,j,k}$ reaches a state of the form $(r, s)$ after encountering $r$ occurrences of $a$ if $r < i + k$ and $s < j$ or at least $i + k$ occurrences of $a$ if $r = i + k$ and $s < j$ and $s$ occurrences of $b$ if $s < j + k$ and $r < i$ or at least $j + k$ occurrences of $b$ if $s = j + k$ and $r < i$. A state $z_l$, $0 \leq l \leq k$, is reached after encountering at least $l$ occurrences of $a$ and at least $j$ occurrences of $b$, where the input processed thus far has length at least $i + j + k$. Thus, $B_{i,j,k}$ reaches the accepting state $z_k$ exactly on input of length at least $i + j + k$ that have at least $i$ occurrences of $a$ and at least $j$ occurrences of $b$.

The cardinality of $Q$ is

$$f(i, j, k) = (i + 1) \cdot (j + 1) + k \cdot j + k \cdot i + k$$

The automaton is essentially the same as given in [CGH+17] for chain automata except for adding certain self-loops for symbols in $\Gamma$. The proof is also very similar.
In [HU79] it is only described for deterministic automata, but it actually works for the characterization of APCs, mentioned in Section 2, as the closure of a PL given by an APC, which yields the claim for \( t \) computed in dered NF A for an expression of the form \( \Sigma \) and ultimately for

\[
\begin{align*}
\max_{i+j+k-n-1} f(i, j, k) &= \begin{cases} 
\frac{n^2 + n + 1}{3} & \text{if } n \equiv 1 \pmod{3}; \\
\frac{n^2}{3} & \text{otherwise.}
\end{cases}
\end{align*}
\]

In [CGH+17, Lemma 4.2], as every chain automaton is a general chain automaton recognizing a finite language, it was shown that for \( n \equiv 1 \pmod{3} \) there exists a language recognized by a chain automaton with \( n \) states such that any automaton for the commutative closure needs at least \( \frac{n^2 - n + 1}{3} \) many states.

### E Proofs for Section 6 (Complexity Results)

**Lemma 9** For a given partially ordered NFA \( \mathcal{A} \) an APC expression of \( L(\mathcal{A}) \) could be computed in \( P \) and for every APC expression a partially ordered NFA is computable in \( P \). This result also holds for variable input alphabets.

**Proof.** Kleene’s algorithm [HU79] for the conversion of any NFA to a regular expression could be used, arguing that it yields an APC when applied to a partially ordered NFA. However, we also give a more direct approach. Let \( \mathcal{A} = (\Sigma, Q, \delta, q_0, F) \). As in the proof of Proposition 10, we can assume \( q_0 \) is a minimal state with respect to the partial order of the automaton. Compute a topological ordering \( Q = \{q_0, q_1, \ldots, q_n\} \) for \( i \in \{0, \ldots, n\} \), let \( L_i = L(\langle \Sigma, Q, \delta, q_i, F \rangle) \) be the language when \( \mathcal{A} \) is started at state \( q_i \). For any other \( i \in \{0, \ldots, n\} \) set \( \Sigma_i = \{a \in \Sigma \mid q_i \in \delta(q_i, a)\} \). We have

\[
L_i = \bigcup_{a \in \Sigma, q_j < q_i} \Sigma_i^{\ast} \cdot \{a\} \cdot L_j.
\]

This equation is actually valid for any NFA. However, as \( \mathcal{A} \) is partially ordered with \( q_0 < q_1 < \ldots < q_n \), in the above equation we have, for \( q_j \in \delta(q_i, a) \setminus \{q_i\} \) with \( q_j \in Q \) and \( a \in \Sigma \), that \( i < j \). So, the index works as an induction parameter, where the base cases are the maximal indices respectively states. For the maximal states \( q_j \) we must have \( L_j = \Sigma_j^{\ast} \); in particular \( L_n = \Sigma_n^{\ast} \). Hence, we can assume inductively that the \( L_j \) are given by an APC, which yields the claim for \( L_i \) and ultimately for \( L_0 = L(\mathcal{A}) \) (recall the characterization of APCs, mentioned in Section 2 as the closure of \( \Gamma^{\ast} \), \( \Gamma \subseteq \Sigma \), and \( \{a\} \), \( a \in \Sigma \), under concatenation and finite union). It is clear that the above recursive computation could be performed in polynomial time (in fact, the topological ordering does not need to be computed and was only used for notational convenience). Also, note that by omitting \( \Sigma_n^{\ast} \) in the above equation, the language \( \mathcal{L}_{\text{simple}}(\mathcal{A}) \) could be computed by the resulting recursion.

Conversely, suppose we have an APC expression \( L \) over \( \Sigma^{\ast} \). Now, a partially ordered NFA for an expression of the form \( \Sigma_0^{\ast} a_1 \Sigma_1^{\ast} \cdots a_n \Sigma_n^{\ast} \) is a single path labelled by \( a_0 \cdots a_n \) and with self-loops labelled by the symbols in \( \Sigma_i \) at the \( i \)-th state in this path.

**Note.** In [HU79] it is only described for deterministic automata, but it actually works for NFAs the same way.
A finite union of such languages corresponds to an NFA that branches in the initial state to a path corresponding to each part of the form \( \Sigma^* a_1 \Sigma^* \cdots a_n \Sigma^* \) in the union.

Lastly, we see that the presented algorithms also run in polynomial time when the alphabet is allowed to be part of the input. \( \square \)

**Proposition 10.** Given a partially ordered NFA \( A \) with \( n \) states, the recognizing PDFA for \( \text{perm}(L(A)) \) from Theorem 2 could be constructed in polynomial time for a fixed alphabet. More precisely in time \( O(n^{13 \Sigma^3 + 2}) \).

**Proof.** Let \( \Sigma = \{ a_1, \ldots, a_k \} \) and \( A = (\Sigma, S, \mu, s_0, E) \) be the input NFA. We refer to the proof of Theorem 2 for the definition of \( B = (\Sigma, Q, \delta, q_0, F) \) and further notation and show that the defining parameters of the automaton \( B = (\Sigma, Q, \delta, q_0, F) \) could be computed in polynomial time by giving algorithms running in polynomial time.

1. The state set \( Q \): We use that the states are partially ordered by the reachability relation. If a state is maximal for this order and not final, we can ignore this state without altering the recognized language. So, we assume that every maximal state is final. Then, for every state we compute a vector \((i_1, \ldots, i_k)\) that stores for any \( j \in \{1, \ldots, k\} \) in the entry \( i_j \) the maximum of the number of times the letter \( a_j \) appears in the words that go from this state to a maximal final state without repeating a state, i.e., not entering a loop. We compute these vectors according to the following rules:

   (a) For any maximal final state, set it equal to \((0, \ldots, 0)\).

   (b) For any \( p \in Q \) that is not maximal compute \((i_1, \ldots, i_k)\) according to the following scheme. Inductively, we can assume that the vectors for states strictly larger in the partial order are already computed. We use the vectors at the direct successors (i.e., those reachable by a single letter) to compute the vector for the state under consideration. For every state \( q \in Q \) reachable by at least one letter from \( p \), let \( \Sigma_q \subseteq \Sigma \) be the set of letters from which we can reach it, i.e., \( \Sigma_q = \{ a \in \Sigma \mid \delta(p, a) = q \} \). By choice of \( q \), we have \( \Sigma_q \neq \emptyset \). Let \((i_1, \ldots, i_k)\) be the vector corresponding to \( q \). Then, let \((i_1', \ldots, i_k')\) be the vector defined by

   \[
   i'_j = \begin{cases} 
   i_j + 1 & \text{if } a_j \in \Sigma_q; \\
   i_j & \text{otherwise}.
   \end{cases}
   \]

Let \( S_q \subseteq \mathbb{N}_0^k \) be the set of all these vectors for each state \( q \in Q \) as above. Then, the vector of the state \( p \in Q \) is

\[
(m_1, \ldots, m_k)
\]

where \( m_j = \max\{i_j \mid (i_1, \ldots, i_k) \in S_q \} \) for \( j \in \{1, \ldots, k\} \).

Finally, for the state \( s_0 \), which could be assumed to be minimal, for otherwise all predecessor states could be ignored without altering the recognized language, we have \( m_j = \max\{|u|_{a_j} \mid u \in L^\text{simple}(A)\} \). Recall that in the proof of Theorem 2, these number were denoted by \( n_j \), i.e., the vector \((m_1, \ldots, m_k)\) for the state \( s_0 \) is precisely the vector \((n_1, \ldots, n_k)\) introduced in the proof of Theorem 2. Having these numbers, the state set \( Q \) could be easily constructed, as it is only a cartesian product \([n_1 + 1] \times \ldots \times [n_k + 1] \).

Note that, in this computation, possible final states that are not maximal are handled like ordinary non-maximal states in the second step above.
2. The transition function and the final state set \( F \).

Recall from the proof of Theorem 2 that \( Q = \{n_1 + 1\} \times \ldots \times \{n_k + 1\} \) with \( n_j = \max\{|u|_{a_j} \mid u \in L_{\text{simple}}(A)\} + 1 \). The transition function among states \((s_1, \ldots, s_k) \in Q\) with \( s_j < n_j, j \in \{1, \ldots, k\}\), when an \( a_j \in \Sigma \) is read is easily computable - simply increase \( s_j \) by one. Hence, the only difficulty is to determine if a self-loop should be added in case \( s_j = n_j \). Intuitively, we “unfold” the self-loops in \( A \) into long enough paths of length \(|s|\), such that we can detect if an input word has to pass a self-loop, i.e., we read along these “unfolded” paths which after a certain point, as they have length \(|S|\), enforce a loop in \( A \). We do this by maintaining for each state of the unfolded automaton a list of states of \( B \) that are reachable by words that go into this state of the unfolded automaton.

As before, we can assume that from every state of \( A \), a final state is reachable and that \( s_0 \) is the minimal element in the partial order induced by the reachability relation.

First, we construct an automaton \( A' = (\Sigma, S', \mu', s'_0, E') \) by “unfolding” the self-loops. The result \( A' \) will have no loops. For a state \( s \in S \), set \( \Sigma_s = \{a \in \Sigma \mid s \in \mu(s, a)\} \). If \( \Sigma_s \neq \emptyset \), do the following: (1) remove the self-loops, (2) set \( s_1 = s \) and add the new states \( \{s_2, \ldots, s_{|S|+1}\} \) and the following transitions

\[
\{(s_{i-1}, a, s_i) \mid a \in \Sigma_s, i \in \{2, \ldots, |S| + 1\}\} \cup
\{(s, b, t) \mid b \in \Sigma, t \in \mu(s, b) \setminus \{s\}, i \in \{2, \ldots, |S| + 1\}\}.
\]

For every such state \( s \), we add \(|S|\) new states, hence \(|S'| \in O(|S|^2)\). Also, set \( E' = E \cup \{s_i \mid s \in E, i \in \{2, \ldots, |S| + 1\}\} \) and \( s'_0 = s_0 \).

In the following computation, we will only refer to the automaton \( A' \). First, topologically sort the states of \( A' \). Then, for any state \( s \in S' \), we compute a subset \( T_s \subseteq Q \) of states in \( B \) that are reachable by words that go from the start state to \( s \) in \( A' \). Note that, by construction of \( A' \), this is only a finite set of words. Also note that, as a subset is associated with every state, we will construct at most \(|S'| \) many subsets during the procedure, i.e., we have no blow-up here and the final algorithm will run in polynomial time.

Initially, to every state in \( S' \) the empty set is assigned. Then, proceed in the following way:

(a) Begin with the start state \( s'_0 \) and associate the subset \( T_{s_0} = \{0, \ldots, 0\} \).

(b) For the current state \( s \in S' \) under consideration, do the following. For any direct successor state of \( s \), i.e., those reachable by a single symbol, if \( a_j, j \in \{1, \ldots, k\} \) is the letter leading to this successor state, update the assigned subset of this successors state by adding to it the subset from the current state, but with every vector inside increased by the vector that has zero everywhere but at the \( j \)-th position, where the entry is one, except for the states inside this set that equal \( n_j \) at the \( j \)-th position. In this case, do nothing and add a self-loop labelled by \( a_j \) to the state in \( Q \) under consideration (here, that we can reach a final state is important, so that the word read up so far is a prefix of some word in \( L \), see the definition of the transition function in the proof of Theorem 2 for comparison, as this is a defining condition to add self-loops). More precisely, if \( t \in S' \) is a successor state to \( s \in S' \) for the letter \( a_j \), i.e., \( t = \delta(s, a_j) \), then, add to \( T_t \) the sets

\[
\{(i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_k) \mid (i_1, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_k) \in T_s \text{ and } i_j < n_j\}
\]

Permutation on Alphabetical Pattern Constraints 17
and
\[
\{(i_1, \ldots, i_{j-1}, n_j, i_{j+1}, \ldots, i_k) \mid (i_1, \ldots, i_{j-1}, n_j, i_{j+1}, \ldots, i_k) \in T_n\}.
\]
Also, in case the following holds true
\[
\{(i_1, \ldots, i_{j-1}, n_j, i_{j+1}, \ldots, i_k) \mid (i_1, \ldots, i_{j-1}, n_j, i_{j+1}, \ldots, i_k) \in T_n\} \neq \emptyset,
\]
add the self-loop
\[
\mu((i_1, \ldots, i_{j-1}, n_j, i_{j+1}, \ldots, i_k), a_j) = (i_1, \ldots, i_{j-1}, n_j, i_{j+1}, \ldots, i_k)
\]
to \(B\).

c) After we have done the previous step for every direct successor state, move on to the next state in the topological order.
In total, in the above procedure, only a polynomial number of steps are performed. So, also the subsets involved in the computation do not grow too much.
Finally, for every final state, collect the associated subsets of \(Q\) and these states are the final states of \(B\). More precisely, \((i_1, \ldots, i_k) \in F\) if and only if there exists \(s \in E^*\) such that \((i_1, \ldots, i_k) \in T_s\).

We have \(|Q| \leq n^{||\Sigma||}\), and in the above procedure of unfolding and traversing the at most \(n^2\) states of \(A\), we have to perform, for each letter at most \(|Q|\) many operations, as this is the maximal size of the associated sets. So, we have given a polynomial time algorithm to compute \(B\) in time \(O(|\Sigma||Q|^{||\Sigma||+2}) = O(|Q|^{||\Sigma||+2})\).

\textbf{Theorem 11.} Fix an alphabet \(\Sigma\). Then, the following problem is in \(P\):
\begin{itemize}
  \item \textbf{Input:} Two APC expressions \(L_1, L_2\) over \(\Sigma^*\).
  \item \textbf{Question:} Is \(\text{perm}(L_1) \subseteq \text{perm}(L_2)\) ?
\end{itemize}

\textbf{Proof.} First, we construct two NFAs \(A_1\) and \(A_2\) for \(L_1\) and \(L_2\), which could be done in \(P\) by Lemma \(9\). Then, we compute PDFAs \(B_1\) and \(B_2\) for their respective commutative closures, which could be done in \(P\) by Proposition \(10\). Now, for deterministic automata the inclusion problem is solvable in \(P\). More precisely, we have: \(L(B_1) \subseteq L(B_2) \iff L(B_1) \cap L(B_2) = \emptyset\). Then, on deterministic automata the Boolean operations are performable in polynomial time with the product automaton construction [HU79] and switching of final and non-final states. Also, the non-emptiness problem, i.e., deciding if the recognized language is non-empty, for automata (even NFAs) is NL-complete [HK11], hence also in \(P\). So, we can perform the above emptiness check in \(P\).

\textbf{References for the Appendix}

CGH\textsuperscript{+}17. Da-Jung Cho, Daniel Goc, Yo-Sub Han, Sang-Ki Ko, Alexandros Palioudakis, and Kai Salomaa. State complexity of permutation on finite languages over a binary alphabet. \textit{Theor. Comput. Sci.}, 682:67–78, 2017.

HK11. Markus Holzer and Martin Kutrib. Descriptive and computational complexity of finite automata - A survey. \textit{Inf. Comput.}, 209(3):456–470, 2011.

HU79. John E. Hopcroft and Jeff D. Ullman. \textit{Introduction to Automata Theory, Languages, and Computation}. Addison-Wesley Publishing Company, 1979.