ON A DISCRETE SELF-ORGANIZED-CRITICALITY FINITE TIME RESULT

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Abstract. In this paper we deal with theoretical and numerical aspects of some nonlinear problems related to sandpile models. We introduce a purely discrete model for infinitely many particles interacting according to a toppling process on a uniform two-dimensional grid and prove the convergence of the solutions to a differential initial value problem.

1. Introduction. At the turn of the eighties and nineties, a general idea surfaced in the physics literature: certain nonlinear dissipative systems drive themselves autonomously to a critical state acting as a finite-time attractor of the dynamics. This property is referred to as self-organized-criticality, SOC.

At the same time, in the early nineties, particle models for avalanches and sand piles provided the numerical evidence supporting the SOC finite-time property. The mechanism underlying the dynamics was described as a toppling process involving the particles of the pile, and the pile was supposed to consist only of a finite number of particles. We refer to the early papers by P.Bak, C. Tang and K. Wiesenfeld [2] and by D. Dahr [17] and M.Creutz [16], as well as to the papers by P. Bántay, I.M. Jánosi [3] and by J.S. Carlson and G.R. Swindle [14] who reformulated the automata models as nonlinear difference equations. In particular in [14], given a critical state \( z^c \) and an initial supercritical height \( z^0 \geq z^0 \) on a uniform coordinate grid of points \( P = ij \) of the plane, the authors write the implicit equations

\[
z^{t+1}_{ij} - z^t_{ij} = -\Delta_{ij,k\ell} H(z^{t+1}_{k\ell} - z^c_{k\ell})
\]

where the matrix \( \Delta_{ij,k\ell} \) is given by

\[
\begin{cases}
\Delta_{ij,k\ell} = 4, & \text{if } k\ell = ij \\
\Delta_{ij,k\ell} = -1, & \text{if } k\ell = \text{one of the four nearest sites of } ij \\
\Delta_{ij,k\ell} = 0, & \text{otherwise}
\end{cases}
\] (1.1)

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and $H$ is the Heaviside function

$$
\begin{align*}
H(r) &= 1, \quad \text{for } r \geq 0 \\
H(r) &= 0, \quad \text{for } r < 0.
\end{align*}
$$

(1.2)

In these particle models, the toppling of 4 particles at the site $ij$ occurs only if the height of the pile at $ij$ and at each of the four receiving sites $k\ell$ remains also supercritical.

A related sand pile problem was formulated in 2010 by V. Barbu in [6], see also [7], as a multivalued nonlinear diffusion equation in the plane. In this model, given a critical configuration $h_c(x)$, where $x$ belongs to a bounded open domain $\Omega$ of $\mathbb{R}^2$, and set $\zeta(x,t) = h(x,t) - h_c(x)$, where $h(x,t)$ represents the height of the pile at $x$ at the time $0 < t \leq T$, $T > 0$, the problem is formulated as

$$
\begin{align*}
\frac{\partial \zeta}{\partial t} - \Delta \eta &= 0, \quad \eta \in H(\zeta) \\
\eta &\in L^2(0,T;H^1_0(\Omega)), \\
\zeta(x,0) &= \zeta_0(x),
\end{align*}
$$

(1.3)

where $\Delta u = u_{xx} + u_{yy}$ is the two-dimensional Laplace operator in $\Omega$, $H$ denotes the Heaviside graph

$$
H(s) = \begin{cases} 
0 & \text{for } s < 0 \\
1 & \text{for } s > 0 \\
[0,1] & \text{for } s = 0.
\end{cases}
$$

(1.4)

For this problem Barbu proves that if $\zeta_0 \in L^2(\Omega)$ and $\zeta_0 \geq 0$, then there exists a unique solution

$$
\zeta(x,t) \in L^2(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;H^{-1}(\Omega))
$$

and $\zeta \geq 0$. Moreover, by further assuming that $\zeta_0 \in L^\infty(\Omega)$, there exists a finite $T^*$ such that $\zeta(x,t) = 0$ for all $t \geq T^*$. Above, $H^1_0(\Omega)$ is the usual Sobolev space of functions vanishing on $\partial \Omega$ (Dirichlet boundary condition), $H^{-1}(\Omega)$ is the dual of $H^1_0(\Omega)$ and $W^{1,2}(0,T;H^{-1}(\Omega))$ is the space of functions $u(t,.)$ with values in $H^{-1}(\Omega)$ which are in $L^2(0,T)$ with time-derivative in $H^{-1}(\Omega)$. A stochastic version of this model was developed by V. Barbu, G. Da Prato, Röker [8] and, later, by Gess [19]. A problem left open by Barbu, see [6], and co-authors is whether the initial value problem can be approximated by discrete, finite-difference particle equations. In this regard, some remarks on the numerical solutions of the PDE model of [6] can be found in that paper.

The models mentioned so far are of variational nature. Related sand pile theories, based on the notion of angle of repose and differential equations, were also developed by various authors, see e.g. [1], [9] and [18] but they do not play a role in the present paper.

It should be noticed that between the particle models and the mathematical models based on partial differential equations there is no obvious direct connection. In fact, sandpiles and toppling processes are exemplary in general physical terms of large particle systems with short-range and fast-time dynamics and are intrinsically discrete in nature, contrary to the initial value problems mentioned before, which have a continuous nature and contain no toppling process overtly visible. There is thus an unexplored field between the two kinds of models.

A purely discrete model involving infinitely many particles was presented in [22]. This paper is based on the general principle that particle interactions occurring at shorter and shorter range – as in sand piles and avalanches – must be recorded at faster
and faster observation times. This principle is implemented by applying a suitable synchronization of the time and space variables. The main achievement of this work is to obtain the SOC finite-time property in a fully discrete setting, where the limit from finitely many particles to infinitely many particle is carried out. The relevance of this result is that it shows that the finite-time property of self-organized-criticality systems has its roots -- as it is in the physics literature -- in particle theory and not in partial differential equations. In [22], however, no connection with PDEs is carried out.

The present paper gives such a connection and provides a bridge between particles and PDEs, between discrete models and models in the continuum, and in this sense it is an additional contribution to the sand pile models mentioned so far.

More precisely, in this paper we present a discrete sand pile model based on suitable finite-difference equations in time and space, which involves a sequence of piles composed of a finite number of particles of increasing cardinality distributed on planar uniform grids of decreasing mesh size, and we prove that the solutions of the finite-difference equations have a limit, though in a weak sense, when the number of particles tends to infinity. Moreover, the limit function is the unique solution of a nonlinear PDE.

The limit from finite-differences to PDEs is carried out in coordinate Sobolev spaces

$$X = H^1(\Omega)_1 \times H^1(\Omega)_2,$$

where

$$H^1(\Omega)_i = \{ v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega) \}, \quad i = 1, 2,$$

and is based on the external approximation of solutions of PDEs in terms of suitable projection operators $p_h^0$, $p_h^1$ and restriction operators $r_h$ for the Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$ on the domain $\Omega$.

This limit is in turn related to the theory of so-called $M$--convergence of subspaces of a Hilbert space introduced in [20], [21]. $M$--convergence properties are summarized in our paper [23].

The plan of the paper is the following. In Section 2 we construct the external approximation of the Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$ and collect the main results connecting Sobolev space approximation and $M$--convergence. In Section 3 we show the existence and uniqueness of the solutions to the discrete finite-difference problems of our SOC model. In Section 4 we prove the self-organized-criticality finite-time property in the discrete setting. In Section 5 we prove the convergence of the discrete solutions to the solution of the differential initial value problem.

2. Projection and restriction operators in Sobolev spaces. We now specify the domain $\Omega$ to be the coordinate open square of side $2L$, $L > 0$ being a scaling parameter, centered at the origin:

$$\Omega = (-L, L) \times (-L, L).$$

We consider the coordinate Sobolev space

$$X \equiv H^1(\Omega)_1 \times H^1(\Omega)_2$$

of all pairs $u = (u_1, u_2)$ with $u_1 \in H^1(\Omega)_1$ and $u_2 \in H^1(\Omega)_2$, where

$$H^1(\Omega)_1 = \{ v \in L^2(\Omega) : \frac{\partial v}{\partial x_1} \in L^2(\Omega) \}$$
and
\[ H^1(\Omega)_2 = \{ v \in L^2(\Omega) : \frac{\partial v}{\partial x_2} \in L^2(\Omega) \} . \]
The space \( H^1(\Omega)_i \) is a Hilbert space with the inner product
\[ (u,v)_{H^1(\Omega)_i} = \int_\Omega (uv + \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i}) dx, \quad i = 1,2 \]
and the product space \( X \) is a Hilbert space with the product norm
\[ ||u||_X = (||u_1||^2_{L^2(\Omega)} + ||\frac{\partial u_1}{\partial x_1}||^2_{L^2(\Omega)} + ||u_2||^2_{L^2(\Omega)} + ||\frac{\partial u_2}{\partial x_2}||^2_{L^2(\Omega)})^{\frac{1}{2}} . \]
The finite-dimensional approximations of the spaces introduced so far are constructed in \( \Omega \) with the help of regular coordinate discrete grids \( G_m, m \in \mathbb{N} \), of increasing cardinality as \( m \to +\infty \). For every \( m \) we set
\[ h = h_m = \frac{2L}{2^m} \]
and define the discrete grid \( G_m \) in \( \Omega \), the boundary grid \( \partial G_m \) in \( \partial \Omega \) and the grid \( \overline{G}_m \) in \( \bar{\Omega} \) – all of mesh size \( h \) – as follows:
\[ G_m := \left\{ \left( \frac{2L}{2^m}, q \frac{2L}{2^m} \right) : (q_1, q_2) \in \mathbb{Z}^2, |q_1| \leq 2^{m-1} - 1, |q_2| \leq 2^{m-1} - 1 \right\} \]
\[ \partial G_m := \left\{ \left( \frac{2L}{2^m}, q \frac{2L}{2^m} \right) : (q_1, q_2) \in \mathbb{Z}^2, |q_1| = 2^{m-1}, |q_2| \leq 2^{m-1} - 1 \right\} \]
\[ \cup \left\{ \left( \frac{2L}{2^m}, q \frac{2L}{2^m} \right) : (q_1, q_2) \in \mathbb{Z}^2, |q_1| \leq 2^{m-1} - 1, |q_2| = 2^{m-1} \right\} \]
\[ \cup \left\{ \left( \frac{2L}{2^m}, q \frac{2L}{2^m} \right) : (q_1, q_2) \in \mathbb{Z}^2, |q_1| = 2^{m-1}, |q_2| = 2^{m-1} \right\} \]
\[ \overline{G}_m := G_m \cup \partial G_m . \]
\( G_m \) is composed of \( (2^{m-1} - 1) \times (2^{m-1} - 1) \) points in \( \Omega \) while \( \partial G_m \) is composed of \( 2^{m-1} - 1 \) points on \( \partial \Omega \). Moreover, the grid \( G_m \) decomposes the square \( \Omega \) into coordinate square elements of side \( h \).

We introduce the following simplified notation for the generic point of the grid \( \overline{G}_m \):
\[ (q_1 \frac{2L}{2^m}, q_2 \frac{2L}{2^m}, ) = (j, k)h. \]
At every point \((jh, kh) \in \overline{G}_m \) we define the cross-like region \( I \cup J \), where
\[ I := \{ x = (x_1, x_2) : (j - \frac{1}{2})h \leq x_1 \leq (j + 1)h, \ (k - \frac{1}{2})h \leq x_2 \leq (k + \frac{1}{2})h \} \]
\[ J := \{ x = (x_1, x_2) : (j - \frac{1}{2})h \leq x_1 \leq (j + 1)h, \ (k - \frac{1}{2})h \leq x_2 \leq (k + 1)h \} . \]
\( I \) is a horizontal strip of base \( 2h \) and height \( h \), \( J \) is a vertical strip of base \( h \) and height \( 2h \). The central square element of this cross-like region:
\[ I \cap J = \{ x = (x_1, x_2) : (j - \frac{1}{2})h \leq x_1 \leq (j + 1)h, \ (k - \frac{1}{2})h \leq x_2 \leq (k + 1)h \} \]
is called the central square element of the cross. The characteristic function of this set is
\[ \theta_{jk}(x) := \chi(x_{1} - j) \chi(x_{2} - k) \quad (2.1) \]
where \( \chi(s), s \in \mathbb{R} \) is the characteristic function of the real interval \( [-\frac{1}{2}, \frac{1}{2}] \) \( \subset \mathbb{R} \). We introduce also the pair of functions (tent functions):
\[ \varphi_{jk}^{1}(x) := \varphi(x_{1} - j) \chi(x_{2} - k), \quad \varphi_{jk}^{2}(x) := \chi(x_{1} - j) \varphi(x_{2} - k), \quad (2.2) \]
where \( \varphi(s) = \chi * \chi(s) \) is the convolution of \( \chi \) with itself. It is easy to check that
\[ \text{supp} \varphi_{jk}^{1}(x) = I, \quad \text{supp} \varphi_{jk}^{2}(x) = J. \]
We fix \( m \in \mathbb{N} \), hence also \( h = h_{m} = 2^{-m} 2L \). We define the Hilbert space of elements
\[ L_{h}^{2}(G_{m}) := \{ v^{h} : v^{h} = \{ v^{h}_{j,k} \}_{(jh,kh) \in \Omega_{m}}, \quad v^{h}_{j,k} \in \mathbb{R} \} \]
with inner product
\[ (v^{h}, u^{h})_{h} := h^{2} \sum_{j, k = -2^{m-1}}^{2^{m-1}} v^{h}_{j,k} u^{h}_{j,k} \]
and the Hilbert space of functions
\[ V_{h}(\Omega) := \{ v_{h} : v_{h}(x) = \sum_{j, k = -2^{m-1}}^{2^{m-1}} v_{j,k} \theta_{jk}(x), \quad x \in \Omega, \quad v_{j,k} \in \mathbb{R} \} \]
with inner product
\[ (v_{h}, w_{h}) := \int_{\Omega} v_{h}(x) w_{h}(x) \, dx. \]
We also consider also the subspace
\[ L_{h,0}^{2}(G_{m}) := \{ v^{h} : v^{h} = \{ v^{h}_{j,k} \}_{(jh,kh) \in \Omega_{m}}, \quad v^{h}_{j,k} = 0 \quad \forall (jh, kh) \in \partial G_{m} \} \]
with the induced inner product from \( L_{h}^{2}(G_{m}) \).
We now define the projection operator
\[ p_{h}^{0} : L_{h}^{2}(G_{m}) \rightarrow V_{h}(\Omega) \quad (2.3) \]
by setting
\[ v_{h} = p_{h}^{0} v^{h} := \sum_{j, k = -2^{m-1}}^{2^{m-1}} v^{h}_{j,k} \theta_{jk}(x) \]
and the restriction operator
\[ r_{h} : L^{2}(\Omega) \rightarrow L_{h}^{2}(G_{m}) \quad (2.4) \]
by setting
\[ r_{h} v := v^{h} \equiv \{ v^{h}_{j,k} \}_{(jh,kh) \in \Omega_{m}} \]
where
\[ v^{h}_{j,k} = h^{-2} \int_{\Omega} v(x) \theta_{jk}(x) \, dx. \]
The Sobolev space \( H^{1}(\Omega) \) can be identified with diagonal of the product space \( X \) introduced before. With the help of the tent functions \( \varphi_{jk}^{1}(x) \) and \( \varphi_{jk}^{2}(x) \), we construct the following finite dimensional subspaces \( X_{h} \) of \( X \):
\[ X_{h} = \left\{ (v^{1}(x)_{|\Omega}, v^{2}(x)_{|\Omega}) \right\} \]
Proposition 2.1.

Lemma 2 and to formula (5.14) of [23].

1 and Lemma 2 below, for the proof of which we refer to Proposition 2.1, Lemma 1, with all terms $v$ where by $I$

we have:

**Lemma 1.**

The operators $p^1_h$, $p^2_h$ and $r_h$ will be used in the estimates of the model presented in the following sections.

It is easy to see that

$$\frac{1}{4} (|v^h|, I)_h \leq \|p^0_h v^h\|_{L^1(\Omega)} \leq (|v^h|, I)_h, \quad \forall v^h \in L^2_h(G_m),$$

where by $I$ we denote the unit element of $L^2(G_m)$, that is, the element $v^h$ of $L^2(G_m)$ with all terms $v^h_{j,k} \equiv 1$. In the article [23] we have proved Proposition 2.1, Lemma 1 and Lemma 2 below, for the proof of which we refer to Proposition 2.1, Lemma 1, Lemma 2 and to formula (5.14) of [23].

**Proposition 2.1.**

$$\frac{1}{2} \|v^h\|_{L^2(G_m)} \leq \|p^0_h v^h\|_{L^2(\Omega)} \leq \|v^h\|_{L^2(G_m)}, \quad \forall v^h \in L^2_h(G_m)$$

$$|r_h v^h|_{L^2(G_m)} \leq \|v^h\|_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega),$$

$$|v - p^0_h r_h v|_{L^2(\Omega)} \to 0 \quad \text{as} \quad h \to 0, \quad \forall v \in L^2(\Omega),$$

$$\|p^1_h v^h\|_{L^2(\Omega) \times L^2(\Omega)} \leq 2 \sqrt{2} \|v^h\|_{L^2(G_m)} \quad \forall v \in L^2(\Omega),$$

where $\|p^1_h v^h\|_{L^2(\Omega) \times L^2(\Omega)} = \left( \|v^1\|_{L^2(\Omega)}^2 + \|v^2\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$

**Lemma 1.** We have:

(b) for every sequence $v_h \in X_h$ and every subsequence $v_{h_k}$ of $v_h$ weakly converging to $u$ in $X$ as $k \to +\infty$, $u$ belongs to $H^1(\Omega)$;

(c) for every sequence $v_h \in X_{h,0}$ and every subsequence $v_{h_k}$ of $v_h$ weakly converging to $u$ in $X$ as $k \to +\infty$, $u$ belongs to $H^1_0(\Omega)$.

**Lemma 2.** We have:

(a) for every $u \in H^1(\Omega)$, the elements $u_h = p^1_h r_h u$ belong to $X_h$ and converge strongly to $u$ in $X$;

(aa) if $u \in H^1_0(\Omega)$, then there exist elements in $X_{h,0}$ strongly converging to $u$ in $X$. 

where

$$v^1(x) = \sum_{j,k=-2^{m-1}}^{2^{m-1}} v^1_{j,k} \varphi_{jk}(x), \quad v^2(x) = \sum_{j,k=-2^{m-1}}^{2^{m-1}} v^2_{j,k} \varphi_{jk}(x),$$

the coefficients $v^1_{j,k}$ being variable in $\mathbb{R}$. By taking coefficients $v^1_{j,k} = 0$ in the previous sums whenever the point $(jh, kh)$ is a boundary point $(jh, kh) \in \partial \Omega$ of $\Omega$, we get the subspace $X_{h,0} \subset X_h$:

$$X_{h,0} = \{(v^1(x), v^2(x)) \in X_h : v^1(x) = v^2(x) = 0 \quad \forall x \in \partial \Omega\}.$$ 

(2.5)

The spaces $X_h$ and $X_{h,0}$ are external approximations of the Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$. For every $h$, we introduce the projection operator

$$p^1_h : L^2_h(G_m) \to X_h$$

by setting

$$p^1_h(v^h) = (v^1(x)|_\Omega, v^2(x)|_\Omega)$$

where

$$v^1(x) = \sum_{j,k=-2^{m-1}}^{2^{m-1}} v^1_{j,k} \varphi_{jk}(x), \quad v^2(x) = \sum_{j,k=-2^{m-1}}^{2^{m-1}} v^2_{j,k} \varphi_{jk}(x).$$

The operators $p^0_h$, $p^1_h$ and $r_h$ will be used in the estimates of the model presented in the following sections.
3. Existence and uniqueness of the discrete solution. We now come back to the discrete SOC model described in the Introduction. We first give the following rigorous definition of the discrete solution of the sand piles model stated in (1.3).

**Definition 1.** For a given \( \zeta_{h,0} \in V_h(\Omega) \), we say that \( \zeta_h \) is the discrete (variational) solution of problem (1.3) (with initial condition \( \zeta_0 = \zeta_{h,0} \)), if \( \zeta_h \) belongs to \( W^{1,2}(0,T; V_h(\Omega)) \), \( \zeta_h(x,0) = \zeta_{h,0}(x) \), and there exists \( \eta_h \in L^2(0,T; X_{h,0}) \), \( \eta_h(x,t) = p^1_h(\Upsilon^h) \), \( \Upsilon^h(t) = (\Upsilon^h(t))_{j,k} \in L^2_{h,0}(G_m) \), \( p^0_h(\Upsilon^h(t)) \in H(\zeta_h(x,t)) \) a.e. in \( \Omega \times (0,T) \), and

\[
\begin{aligned}
\int_0^T \int_{\Omega} \frac{\partial \zeta_h}{\partial t} \Phi_h^0 \, dx \, dt + & \int_0^T \int_{\Omega} \left( \frac{\partial \Phi^1_h}{\partial x_1} \frac{\partial \zeta_h}{\partial x_1} + \frac{\partial \Phi^2_h}{\partial x_2} \frac{\partial \zeta_h}{\partial x_2} \right) \, dx \, dt = 0,
\end{aligned}
\]

\( \forall \Phi_h \in L^2(0,T; X_{0,h}) \).

Here

\[
\eta_h(x,t) = (\eta^1_h(x,t), \eta^2_h(x,t)) = \left( \sum_{j,k=-2^{m-1}}^{2^{m-1}} (\Upsilon^h(t))_{j,k} \varphi^1_{jk}(x), \sum_{j,k=-2^{m-1}}^{2^{m-1}} (\Upsilon^h(t))_{j,k} \varphi^2_{jk}(x) \right),
\]

and for all \( \Phi_h \in L^2(0,T; X_{0,h}) \), we have with \( \Phi^h(t) = (\Phi^h(t))_{j,k} \in L^2_{h,0}(G_m) \):

\[
\Phi_h(x,t) = p^1_h(\Upsilon^h(t)) = (\Phi^1_h(x,t), \Phi^2_h(x,t)) = \left( \sum_{j,k=-2^{m-1}}^{2^{m-1}} (\Phi^h(t))_{j,k} \varphi^1_{jk}(x), \sum_{j,k=-2^{m-1}}^{2^{m-1}} (\Phi^h(t))_{j,k} \varphi^2_{jk}(x) \right).
\]

We define

\[
\Phi^0_h = p^0_h(\Upsilon^h(t)) = \sum_{j,k=-2^{m-1}}^{2^{m-1}} (\Phi^h(t))_{j,k} \theta_{jk}(x).
\]

The following theorem holds:

**Theorem 3.1.** Assume \( \zeta_{h,0}(x) = p^0_h(Z^h), Z^h \in L^2_{h,0}(G_m), \zeta_{h,0} \geq 0 \). Then, there exists a unique discrete solution \( \zeta_h \) of our sand piles model (1.3) according to Definition 1. Moreover

\[
0 \leq \zeta_h(x,t) \leq M(h),
\]

where \( M(h) \) denotes the maximum value of the entries of \( Z^h \), i.e.,

\[
M(h) = \max_{j,k} (Z^h)_{j,k}.
\]

The proof of Theorem 3.1 will be carried on in various steps, as follows.

**Step 1.** Here for any fixed \( m \in \mathbb{N} \) and \( h = \frac{2}{2^m} \), we introduce a suitable maximal monotone operator \( A_h \) in \( L^2_{h,0}(G_m) \). For definitions and properties of maximal monotone operators see for example [10].

The operator \( A_h \) is defined to be the operator \( A_h = \triangle_h H \), where \( H \) is the maximal monotone graph in \( \mathbb{R} \) defined in (1.4) of the Introduction and \( \triangle_h \) is the discrete Laplacian on \( L^2_h(G_m) \), which for any \( u^h \in L^2_{h,0}(G_m) \) and every \( j,k \) has the entry

\[
\{\triangle_h u^h\}_{j,k} = \frac{1}{h^2} \{4u_{j,k} - u_{j-1,k} - u_{j+1,k} - u_{j,k-1} - u_{j,k+1}\}.
\]
The domain of the operator $A_h$ is

$$D(A_h) = \{ v^h \in L^2_{h,0}(G_m) : \exists \eta^h \in L^2_{h,0}(G_m), \eta^h \in H(v^h) \} = L^2_{h,0}(G_m).$$

Then:

$$\{ A_h u^h \}_{j,k} = \frac{1}{h^2} \{ 4\eta_{j,k} - \eta_{j-1,k} - \eta_{j+1,k} - \eta_{j,k-1} - \eta_{j,k+1} \} \text{ where } \eta_{j,k} \in H(u_{j,k}). \quad (3.5)$$

In [22] it has been proved that $A_h = \triangle_h H$ is a maximal monotone operator in $L^2_{h,0}(G_m)$.

From now on we will denote by $E_h(\cdot, \cdot)$ the energy form associated with the operator $\triangle_h$, i.e.,

$$E_h(u^h, v^h) = h^2 \sum_{j,k=-2m-1}^{2m-1} \left( D^+_1 u^h_{j,k} D^+_1 v^h_{j,k} + D^+_2 u^h_{j,k} D^+_2 v^h_{j,k} \right), \quad (3.6)$$

where $D^+_1 u^h_{j,k} = \frac{1}{h} (u^h_{j+1,k} - u^h_{j,k})$ and $D^+_2 u^h_{j,k} = \frac{1}{h} (u^h_{j,k+1} - u^h_{j,k}).$

For the form $E_h(u^h, v^h)$ the following coerciveness estimate holds:

$$E_h(u^h, u^h) \geq C_P(u^h, u^h) \forall u^h \in L^2_{h,0}(G_m), \quad (3.7)$$

with a constant $C_P$ that does not depend on $h$, (see [22], (2.26) and (2.24)).

By using formulas (2.10) and (2.16) in [22] (pp. 2413-14), we find that for every $v^h, u^h \in L^2_{h,0}(G_m)$ we have

$$\left\{ \begin{array}{l}
(\triangle_h u^h, v^h)_h = h^2 \sum_{j,k=-2m-1}^{2m-1} (\triangle_h u^h)_{j,k} v^h_{j,k} = \\
h^2 \sum_{j,k=-2m-1}^{2m-1} D^+_1 u^h_{j,k} D^+_1 v^h_{j,k} + D^+_2 u^h_{j,k} D^+_2 v^h_{j,k},
\end{array} \right. \quad (3.8)$$

$$(D^+_1 u^h_{j,k} = \frac{1}{h} (u^h_{j+1,k} - u^h_{j,k}) \text{ and } D^+_2 u^h_{j,k} = \frac{1}{h} (u^h_{j,k+1} - u^h_{j,k})).$$

By using formulas (2.27) - (2.35) in [22] (pp. 2416, 2417), it can be shown that

$$R(I + A_h) = L^2_{h,0}(G_m)$$

and that for every $u^h, v^h \in D(A_h)$, $\eta^h \in A_h u^h, \zeta^h \in A_h v^h$

$$\eta^h - \zeta^h, u^h - v^h \geq 0.$$

From these properties it follows that the operator $A_h = \triangle_h H$, is a maximal monotone operator in $L^2_{h,0}(G_m)$ (see [10]).

**Step 2.** The summary of this step is as follows. For any $T > 0$ we introduce problem (3.9), which is a discrete problem in space and time and which involves the operator $A_h$. Since – as seen in Step 1 – the operator $A_h$ is maximal monotone, for fixed $h$ and $i$ the discrete problem (3.9) admits a unique solution $U^i_{h, \varepsilon}$ in the space $L^2_{h,0}(G_m)$. Then we shall prove that for every $i = 1, \ldots, n$, and every $\varepsilon > 0$, the solution $U^i_{h, \varepsilon}$ of (3.9) is nonnegative and bounded by the initial datum, as stated in (3.12), (3.13) and (3.14) below.

Here are the details of the proof of the properties stated in the summary of Step 2.

For any $\varepsilon > 0$ and $T > 0$ we construct an $\varepsilon$– discretization for the problem (1.3).
We put $t_0 = 0 < t_1 < \cdots < t_n = T$, with $t_i - t_{i-1} = \varepsilon = \frac{T}{n}$. Let $m \in \mathbb{N}$ and $h = \frac{2L}{2m}$.

For every $i = 1, \ldots, n$, we consider the problem:

$$
\begin{align*}
    &U^h_{i,\varepsilon} = U^h_{i-1,\varepsilon} - \varepsilon \Delta_h (\Upsilon^h_{i,\varepsilon}) \\
    &\Upsilon^h_{i,\varepsilon} \in \mathbf{H}(U^h_{i,\varepsilon}) \\
    &\Upsilon^h_{i,\varepsilon} \in L^2_{h,0}(G_m) \\
    &U^h_{0,\varepsilon} = Z^h
\end{align*}
$$

(3.9)

where we have assigned on $\Omega_m$ the initial condition:

$$
Z^h = \{Z^h_{j,k} : (j, k, h) \in \overline{G}_m\}, \quad Z^h \in L^2_{h,0}(G_m), \quad Z^h \geq 0.
$$

(3.10)

Problem (3.9) has for every $i$ a unique solution $U^h_{i,\varepsilon}$ because, as seen in Step 1, the operator $A_h = \Delta_h \mathbf{H}$ is a maximal monotone operator in $L^2_{h,0}(G_m)$.

Therefore, at any point $P_{j,k} = (jh, kh)$ of the grid $G_m$, we have solved the problem:

$$
\begin{align*}
    &\{(U^h_{i,\varepsilon})_{j,k} = (U^h_{i-1,\varepsilon})_{j,k} - \frac{\varepsilon}{h^2} \{4(\Upsilon^h_{i,\varepsilon})_{j,k} - (\Upsilon^h_{i,\varepsilon})_{j-1,k} - (\Upsilon^h_{i,\varepsilon})_{j+1,k} - (\Upsilon^h_{i,\varepsilon})_{j,k-1} - (\Upsilon^h_{i,\varepsilon})_{j,k+1}\} \\
    &\{(\Upsilon^h_{i,\varepsilon})_{j,k} \in \mathbf{H}(U^h_{i,\varepsilon})_{j,k}\} \\
    &U^h_{0,\varepsilon} = Z^h_{j,k} = 0 \quad \text{for} \quad (j, k, h) \in \partial G_m
\end{align*}
$$

(3.11)

We now prove a few inequalities satisfied by the solutions $(U^h_{i,\varepsilon})_{j,k}$. The first inequality is the following:

$$
(U^h_{i,\varepsilon})_{j,k} \geq 0.
$$

(3.12)

For the proof we refer to [22], p. 2421-2422.

The second inequality is:

$$
(U^h_{i,\varepsilon})_{j,k} \leq \max_{j,k} (U^h_{i-1,\varepsilon})_{j,k}.
$$

(3.13)

The proof is as follows. Let $M_{i-1,\varepsilon} = M^h_{i-1,\varepsilon} = \max_{j,k} (U^h_{i-1,\varepsilon})_{j,k}$. We note that $(U^h_{i,\varepsilon})_{j,k} \geq 0$ for every $i$ implies that also $M_{i-1,\varepsilon} \geq 0$ for every $i$; moreover, as $U^h_{i,\varepsilon} \in L^2_{h,0}(G_m)$ then $(U^h_{i,\varepsilon} - M_{i-1,\varepsilon})^+ \in L^2_{h,0}(G_m)$. Now we subtract the quantity $M_{i-1,\varepsilon}$ to both the sides of the first equation in (3.11), we multiply by $((U^h_{i,\varepsilon} - M_{i-1,\varepsilon})^+)(j, k)$, we sum on the indices and we obtain,

$$
\begin{align*}
    &\sum_{j, k = -2^{-m-1}}^{2^{-m-1}} ((U^h_{i,\varepsilon})_{j,k} - M_{i-1,\varepsilon})^+ \leq \frac{\varepsilon \Delta_h}{h^2} \{4(\Upsilon^h_{i,\varepsilon})_{j,k} - (\Upsilon^h_{i,\varepsilon})_{j-1,k} - (\Upsilon^h_{i,\varepsilon})_{j+1,k} - (\Upsilon^h_{i,\varepsilon})_{j,k-1} - (\Upsilon^h_{i,\varepsilon})_{j,k+1}\} ((U^h_{i,\varepsilon})_{j,k} - M_{i-1,\varepsilon})^+ + \\
    &\sum_{j, k = -2^{-m-1}}^{2^{-m-1}} \frac{\varepsilon \Delta_h}{h^2} (-\Upsilon^h_{i,\varepsilon})_{j+1,k} - (\Upsilon^h_{i,\varepsilon})_{j-1,k} - (\Upsilon^h_{i,\varepsilon})_{j,k-1} - (\Upsilon^h_{i,\varepsilon})_{j,k+1} ((U^h_{i,\varepsilon})_{j,k} - M_{i-1,\varepsilon})^+ = \\
    &\sum_{j, k = -2^{-m-1}}^{2^{-m-1}} ((U^h_{i,\varepsilon})_{j,k} - M_{i-1,\varepsilon})^+ ((U^h_{i-1,\varepsilon})_{j,k} - M_{i-1,\varepsilon}).
\end{align*}
$$
We note that the second term on the left hand side of the previous identity is non-negative (see [22] p. 2423), and the term at the right hand side is non-positive. Therefore
\[
\sum_{j,k=2^{m-1}}^{2^{m-1}} \left( (U_{i,\varepsilon}^h)_{j,k} - M_{i-1,\varepsilon}^+ \right)^2 \leq 0,
\]
what proves estimate (3.13).

The third inequality is
\[
(U_{i,\varepsilon}^h)_{j,k} \leq M(h),
\]
where the quantity \( M(h) = \max_{j,k} (Z_j^h)_{j,k} \) has already been defined in (3.3). This inequality follows from the previous inequality (3.13) by an iteration over the index \( i \).

**Step 3.** The summary of Step 3 is as follows. We use the matrices \( U_{i,\varepsilon}^h \) and \( Y_{i,\varepsilon}^h \) to construct suitable step-functions in time with values in \( L_{h,0}^2(G_m) \), denoted by \( U_{\varepsilon}^h(t) \) and \( Y_{\varepsilon}^h(t) \), see (3.15) and (3.16) below. Moreover we will prove some estimates for \( U_{\varepsilon}^h(t) \) and \( Y_{\varepsilon}^h(t) \), see the estimates (3.22), (3.23) and (3.24) below.

As in Step 2, for arbitrary \( T > 0 \) and \( n \in \mathbb{N} \) we put \( \varepsilon = T/n \) and select the points \( t_0 = 0 < t_1 < ... < t_n = T \). For any \( i = 1, ..., n \) and \( m \in \mathbb{N} \), we put \( h = 2L/2^m \) and we define \( U^h_{\varepsilon}(t) \) and \( Y_{\varepsilon}^h(t) \) for \( 0 \leq t < T \) by:
\[
U_{\varepsilon}^h(t) = \begin{cases} U_{0,\varepsilon}^h & \text{for } t \in [0, t_1), \\ U_{i,\varepsilon}^h & \text{for } t \in [t_i, t_{i+1}), \ i = 1,...n, \end{cases}
\]
and
\[
Y_{\varepsilon}^h(t) = \begin{cases} 0 & \text{for } t \in [0, t_1), \\ Y_{i,\varepsilon}^h & \text{for } t \in [t_i, t_{i+1}) \ i = 1,...n. \end{cases}
\]

We now go back to (3.9). We make the inner product in \( L_{h}^2(G_m) \) of both terms of the first equation of (3.9) with \( Y_{i,\varepsilon}^h \) and, by taking (3.8) into account, for any \( i = 1,..., n \) we obtain:
\[
\begin{align*}
&(U_{i,\varepsilon}^h, Y_{i,\varepsilon}^h)_{h} + \\
&\varepsilon h^2 \sum_{j,k=2^{m-1}}^{2^{m-1}} \left( D_1^+(Y_{i,\varepsilon}^h)_{j,k} D_1^+(Y_{i,\varepsilon}^h)_{j,k} + D_2^+(Y_{i,\varepsilon}^h)_{j,k} D_2^+(Y_{i,\varepsilon}^h)_{j,k} \right) = (3.17)
\end{align*}
\]
In this equation, as in (3.9), we have \( (U_{i,\varepsilon}^h)_{j,k} \geq 0 \) and \( Y_{i,\varepsilon}^h \in H(U_{i,\varepsilon}^h) \), therefore, in view of the properties (1.4) of \( H \) as a graph, \( Y_{i,\varepsilon}^h \) has non-zero entries only at nodes \( j,k \) where \( (U_{i,\varepsilon}^h)_{j,k} \geq 0 \), and their value is either \( Y_{i,\varepsilon}^h = 1 \) or \( 0 \leq Y_{i,\varepsilon}^h \leq 1 \). Since \( (U_{i-1,\varepsilon}^h)_{j,k} \geq 0 \) at all nodes \( j,k \), if \( I \) is defined to have unit entries at all nodes \( j,k \), then \( I \in L_{h}^2(G_m) \) and we get \( (U_{\varepsilon}^h, Y_{i,\varepsilon}^h)_{h} = (U_{i,\varepsilon}^h, I)_{h} \) and \( (U_{i-1,\varepsilon}^h, Y_{i,\varepsilon}^h)_{h} \leq (U_{i-1,\varepsilon}^h, I)_{h} \). Therefore, the equation (3.17) gives the inequality:
\[
\begin{align*}
&(U_{i,\varepsilon}^h, I)_{h} + \\
&\varepsilon h^2 \sum_{j,k=2^{m-1}}^{2^{m-1}} \left( D_1^+(Y_{i,\varepsilon}^h)_{j,k} D_1^+(Y_{i,\varepsilon}^h)_{j,k} + D_2^+(Y_{i,\varepsilon}^h)_{j,k} D_2^+(Y_{i,\varepsilon}^h)_{j,k} \right) \leq (3.18)
\end{align*}
\]
We now prove that the following integrated inequality holds for every $t > 0$:

\[
\begin{align*}
(U_\varepsilon^h(t), I)_h + & \int_0^t \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left(D_1^+(\Upsilon^h_\varepsilon)_{j,k}D_1^+(\Upsilon^h_\varepsilon)_{j,k} + D_2^+(\Upsilon^h_\varepsilon)_{j,k}D_2^+(\Upsilon^h_\varepsilon)_{j,k}\right) \, ds \\
& \leq (Z^h, I)_h.
\end{align*}
\]

(3.19)

We first consider the left-open intervals $[t_0 = 0, t_1)$ and $[t_1, t_2)$. We recall from (3.9) that at every nodes $j,k$ of $\mathcal{G}_m$ there is an initial condition for $i = 0$ given by $Z^h$, moreover $t_1 - t_0 = t_2 - t_1 = \varepsilon$. When we integrate the inequality (3.18) first from $t_0$ to $t$ with $t_0 \leq t < t_1$ then from $t_1$ to $t$ with $t_1 \leq t < t_2$, we get

\[
\begin{align*}
(U_\varepsilon^h(t), I)_h + & \int_0^t \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left(D_1^+(\Upsilon^h_\varepsilon)_{j,k}D_1^+(\Upsilon^h_\varepsilon)_{j,k} + D_2^+(\Upsilon^h_\varepsilon)_{j,k}D_2^+(\Upsilon^h_\varepsilon)_{j,k}\right) \, ds \\
& \leq (Z^h, I)_h.
\end{align*}
\]

(3.20)

In order to prove (3.19) for all intervals $[t_i, t_{i+1})$, we now proceed by induction over the index $i$. Suppose that for $t \in [t_i, t_{i+1})$ we have proved that

\[
(U_\varepsilon^h(t), I)_h + \int_0^t \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left(D_1^+(\Upsilon^h_\varepsilon)_{j,k}D_1^+(\Upsilon^h_\varepsilon)_{j,k} + D_2^+(\Upsilon^h_\varepsilon)_{j,k}D_2^+(\Upsilon^h_\varepsilon)_{j,k}\right) \, ds = (U_{i+1,\varepsilon}^h, I)_h + \int_0^t \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left(D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k} + D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}\right) \, ds + \varepsilon \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left(D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k} + D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}\right) h^2 \, ds \leq (U_{i+1,\varepsilon}^h, I)_h + \int_0^t \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left(D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k} + D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}\right) \, ds \\
& \leq (Z^h, I)_h,
\end{align*}
\]

(3.21)

where we have taken into account that

\[
\sum_{j,k=-2^{m-1}}^{2^{m-1}} \left(D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_1^+(\Upsilon^h_{i+1,\varepsilon})_{j,k} + D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}D_2^+(\Upsilon^h_{i+1,\varepsilon})_{j,k}\right) h^2 \geq 0,
\]

as it follows from (3.6) and (3.7). Thus the proof of estimate (3.19) has been completed.

We proceed with Step 3. By (3.14) and (3.15), we obtain that

\[
\sup_{t} \max_{j,k}(U_\varepsilon^h(t))_{j,k} \leq M(h).
\]

(3.22)

We now recall that the two terms on the left hand side of inequality (3.19) are non-negative (see (3.6), (3.7) and (3.12)). Therefore, by (3.14), (3.15), (3.19) and by taking
into account the definition of the inner product in $L^2_h(G_m)$, we obtain

$$\sup_t |U^h_\varepsilon(t)|_h \leq \sqrt{M(h)(Z^h, I)_h}.$$  

(3.23)

Finally, by (3.6), (3.7), and (3.19) we get the last estimate we had to prove:

$$C_P \int_0^t (Y^h_\varepsilon, Y^h_\varepsilon)_h ds \leq \int_0^t E_h(Y^h_\varepsilon, Y^h_\varepsilon) ds \leq (Z^h, I)_h.$$  

(3.24)

This conclude the proof of Step 3.

**Step 4.** In this step we apply the Crandall-Liggett theorem (see Theorem II and Formula (2.5) in [15]) in order to identify the weak-limit (as $\varepsilon \to 0$) of the sequence $U^h_\varepsilon$ with the strong solution of the evolution problem (3.31) in the space $L^2(0,T;L^2_h(G_m))$. Moreover, we shall prove some important estimates, namely the estimates (3.27),(3.28), (3.29), (3.30) and (3.33). Since the estimate (3.22) is uniform in $\varepsilon$, there exists a subsequence in $\varepsilon$ of the sequence $U^h_\varepsilon$ still denoted by $U^h_\varepsilon$, and a function $U^h \in L^\infty((0,T) \times G_m)$ such that

$$U^h_\varepsilon \to U^h$$  

(3.25)

in the weakly-star-convergence in $L^\infty(G_m \times (0,T))$ and in $L^\infty(0,T;L^2_h(G_m))$ as $\varepsilon \to 0$.

Analogously, by possible extracting a further subsequence, we find from estimate (3.24) that there exists a function $\Upsilon^h \in L^2(0,T;L^2_h(G_m))$ such that

$$\Upsilon^h_\varepsilon \to \Upsilon^h$$  

(3.26)

weakly in $L^2(0,T;L^2_h(G_m))$.

The limit functions $U^h$ and $\Upsilon^h$ inherit the estimates satisfied by the functions $U^h_\varepsilon$ and $\Upsilon^h_\varepsilon$. More precisely, from (3.12), (3.22), (3.23) and the convergence property (3.25) we get

$$\{U^h(t)\}_{j,k} \geq 0 \text{ a.e. } t \in (0,T),$$  

(3.27)

as well as

$$\sup_{j,k} \max_{t \in (0,T)} |U^h(t)| \leq M(h) \text{ a.e. } t \in (0,T),$$  

(3.28)

and

$$\sup_t |U^h(t)|_h \leq \sqrt{M(h)(Z^h, I)_h} \text{ a.e. } t \in (0,T).$$  

(3.29)

Moreover, from estimate (3.24) we obtain

$$C_P \int_0^t (\Upsilon^h, \Upsilon^h)_h ds \leq \int_0^t E_h(\Upsilon^h, \Upsilon^h) ds \leq (Z^h, I)_h.$$  

(3.30)

As the operator $A_h = \triangle_h H$ is a maximal monotone operator in $L^2_{h,0}(G_m)$, by Theorem 3.1 in [10] for any initial datum $Z^h$ in the domain $D(A_h)$ there exists a unique strong solution $u^h$ of the problem

$$\begin{cases}
\frac{\partial u^h}{\partial t} + A_h u^h = 0 \\
u^h(0) = Z^h.
\end{cases}$$  

(3.31)

According to the Crandall-Liggett theorem (see Theorem II and Formula (2.5) in [15]) $u^h(t) = \lim_{n \to \infty} (I + \frac{1}{n} A_h)^{-n} Z^h = \lim_{\varepsilon \to 0} U^h_\varepsilon(t)$ (strongly in $L^2_h(G_m)$ and uniformly in $t$) and hence

$$U^h = u^h.$$  

(3.32)
Moreover \( \frac{\partial U_h}{\partial t} \in L^2(0, T; L^2_h(G_m)) \) and (see Theorem 3.1 in [10])

\[
\| \frac{\partial U_h}{\partial t} \|_{L^\infty(0, T; L^2_h(G_m))} \leq |A^0(Z^h)|_h \tag{3.33}
\]

where for every element \( u^h \in D(A_h) \) we denote by \( A^0_h u^h \) the element of \( A_h u^h \) having minimal norm. In particular (see (3.32) and (3.31))

\[
U^h(0) = Z^h. \tag{3.34}
\]

The operator \( H \), being maximal monotone in \( L^2(0, T; L^2_h(G_m)) \times L^2(0, T; L^2_{h,0}(G_m)) \) (see [10] Examples 2.1.3 and 2.3.3), is closed in strong \( \times \) weak topology (see Corollary 2.4 in [5] or Proposition 2.5 in [10]). As \( U^h \) strongly converge to \( U^h \) (in \( L^2(0, T; L^2_h(G_m)) \)) and \( \Upsilon^h \) weakly converge to \( \Upsilon^h \) (in \( L^2(0, T; L^2_{h,0}(G_m)) \)) then \( \Upsilon^h \in H(U^h) \) a.e. in \( G_m \times (0, T) \). The proof of Step 4 is completed.

**Step 5.** In this step we make use of the projection operators introduced in Section 2 (see (2.6)), we construct the tent-functions associated with the elements of the space \( L^2_{h,0}(G_m) \) and we focus our attention on the relation between the inner product in the Hilbert space \( X_{h,0} \) and the energy form \( E_h(\cdot, \cdot) \) on \( L^2_{h,0}(G_m) \) (see (3.6) and (3.36)).

More precisely, let \( v_h = p_h^1 v^h, \ u_h = p_h^1 u^h \), where \( v^h, \ u^h \in L^2_{h,0}(G_m) \) then we have \( v_h, u_h \in X_{h,0}, \ v_h = (v^1_h, v^2_h), \ u_h = (u^1_h, u^2_h) \) and

\[
\frac{\partial v^1_h}{\partial x_1} = \sum_{j,k=-m^{-1}}^{2m^{-1}-1} v_{j+1,k} - v_{j,k} h \chi_j(x_1) \chi_k(x_2)
\]

\[
\frac{\partial u^1_h}{\partial x_1} = \sum_{j,k=-m^{-1}}^{2m^{-1}-1} u_{j+1,k} - u_{j,k} h \chi_j(x_1) \chi_k(x_2)
\]

where \( \chi_k(x_2) = \chi(x_2) - k \) is the characteristic function of the interval \([k - 1/2)h, (k + 1/2)h]\) and \( \chi_j(x_1) \) is the characteristic function of the interval \([j h, (j + 1)h]\).

We note that for any \( \ell = -m^{-1}, \ldots, 2m^{-1} - 1 \) (and for any choice of real numbers \( A_j \))

\[
\int_{\Omega}^{(\ell+1)h} \sum_{j=-m^{-1}}^{2m^{-1}-1} A_j \chi^*_j(x_1) dx_1 = h A_{\ell}
\]

hence

\[
\int_{\Omega} \frac{\partial u^1_h}{\partial x_1} \frac{\partial v^1_h}{\partial x_1} dx = h^{-2} \int_{\Omega} \sum_{j,k=-m^{-1}}^{2m^{-1}-1} (v_{j+1,k} - v_{j,k}) (u_{j+1,k} - u_{j,k}) \chi_j^*(x_1) \chi_k(x_2) dx =
\]

\[
h^{-2} \int_{-L}^{L} dx_2 \sum_{\ell=-m^{-1}}^{2m^{-1}-1} \int_{(\ell+1)h}^{2m^{-1}} dx_1 \sum_{j,k=-m^{-1}}^{2m^{-1}-1} (v_{j+1,k} - v_{j,k}) (u_{j+1,k} - u_{j,k}) \chi_j^*(x_1) \chi_k(x_2) dx_1 =
\]

\[
h^{-1} \int_{-L}^{L} dx_2 \sum_{k=-m^{-1}}^{2m^{-1}-1} \sum_{j=-m^{-1}}^{2m^{-1}-1} (v_{j+1,k} - v_{j,k}) (u_{j+1,k} - u_{j,k}) \chi_k(x_2) =
\]

\[
h^{-1} \sum_{r=-m^{-1}+1}^{2m^{-1}-1} \sum_{j=-m^{-1}}^{2m^{-1}-1} \sum_{k=-m^{-1}}^{2m^{-1}-1} (v_{j+1,k} - v_{j,k}) (u_{j+1,k} - u_{j,k}) \chi_k(x_2) dx_2 +
\]
\[ h^{-1} \int_{(2m-1)h}^{(2m-1+1/2)h} \sum_{j=-2m-1}^{2m-1-1} \sum_{k=-2m-1}^{2m-1} (v_{j+1,k} - v_{j,k})(u_{j+1,k} - u_{j,k}) \chi_k(x_2) d x_2 + \]

\[ h^{-1} \int_{(2m-1+1/2)h}^{(2m-1)h} \sum_{j=-2m-1}^{2m-1-1} \sum_{k=-2m-1}^{2m-1} (v_{j+1,k} - v_{j,k})(u_{j+1,k} - u_{j,k}) \chi_k(x_2) d x_2. \]

Again we note that, for any \( r = -2^{m-1} + 1, \ldots, 2^{m-1} - 1 \) (and for any choice of real numbers \( A_k \))

\[ \int_{(r+1/2)h}^{(r-1/2)h} \sum_{k=-2m-1}^{2m-1} A_k \chi_k(x_2) d x_2 = h A_r, \]

moreover

\[ \int_{(2m-1)h}^{(2m-1+1/2)h} \sum_{k=-2m-1}^{2m-1} A_k \chi_k(x_2) d x_2 = \frac{h}{2} A_{2m-1} \]

and

\[ \int_{(-2m-1)h}^{(-2m-1+1/2)h} \sum_{k=-2m-1}^{2m-1} A_k \chi_k(x_2) d x_2 = \frac{h}{2} A_{-2m-1}. \]

In a similar way we evaluate the second components:

\[ \frac{\partial v_h^2}{\partial x_2} = \sum_{j,k=-2m-1}^{2m-1-1} v_{j,k+1} - v_{j,k} \chi_j(x_1) \chi_k^*(x_2) \]

\[ \frac{\partial u_h^2}{\partial x_2} = \sum_{j,k=-2m-1}^{2m-1-1} u_{j,k+1} - u_{j,k} \chi_j(x_1) \chi_k^*(x_2) \]

where \( \chi_j^*(x_2) \) is the characteristic function of the interval \([kh, (k+1)h]\) and \( \chi_j(x_1) = \chi(\frac{x_1}{h} - j) \) is the characteristic function of the interval \([j-1/2)h, (j+1/2)h)\).

Finally, taking into account that the entries of the elements \( v^h, u^h \in L^2_{h,0}(G_m) \) vanish on \( \partial G_m \), we obtain

\[ \int_G \left( \frac{\partial u_h^1}{\partial x_2} \frac{\partial v_h^1}{\partial x_1} + \frac{\partial u_h^2}{\partial x_2} \frac{\partial v_h^2}{\partial x_2} \right) dx = \]

\[ \sum_{j,k=-2m-1}^{2m-1-1} \left( (v_{j+1,k} - v_{j,k})(u_{j+1,k} - u_{j,k}) + (v_{j,k+1} - v_{j,k})(u_{j,k+1} - u_{j,k}) \right) = \]

\[ \sum_{j,k=-2m-1}^{2m-1-1} \left( D_1^+ u_{j,k}^h D_1^+ v_{j,k}^h + D_2^+ u_{j,k}^h D_2^+ v_{j,k}^h \right) h^2 = E_h(u^h, v^h), \]

where \( D_1^+ u_{j,k}^h = \frac{1}{h} (u_{j+1,k}^h - u_{j,k}^h) \) and \( D_2^+ u_{j,k}^h = \frac{1}{h} (u_{j,k+1}^h - u_{j,k}^h) \).

Combining (3.8) and (3.35) we obtain

\[ \int_G \left( \frac{\partial u_h^1}{\partial x_2} \frac{\partial v_h^1}{\partial x_1} + \frac{\partial u_h^2}{\partial x_2} \frac{\partial v_h^2}{\partial x_2} \right) dx = \]

\[ \sum_{j,k=-2m-1}^{2m-1-1} \left( D_1^+ u_{j,k}^h D_1^+ v_{j,k}^h + D_2^+ u_{j,k}^h D_2^+ v_{j,k}^h \right) h^2 = \left( \Delta_h u^h, v^h \right) \]

\[ \sum_{j,k=-2m-1}^{2m-1-1} \left( \Delta_h u^h \right)_{j,k} v_{j,k}^h h^2. \]

This conclude the proof of Step 5.

We are finally ready to prove Theorem 3.1.
Proof. We recall that \( \zeta_{h,0}(x) = p_0^h(Z^h) \) is the element in the discrete space \( L_{h,0}^2(G_m) \) by using the projections operators introduced in Section 2 (see (2.3) and (2.6)). Let \( U^h \) be the element defined in (3.25) we set

\[
\zeta_h(x, t) = p_0^h(U^h(t)) = \sum_{j,k=-2^{m-1}}^{2^{m-1}} (U^h(t))_{j,k} \theta_{j,k}(x),
\]

hence we have

\[
\frac{\partial \zeta_h(x, t)}{\partial t} = \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left( \frac{\partial U^h(t)}{\partial t} \right)_{j,k} \theta_{j,k}(x).
\]

From estimates (3.33), (3.38) and (2.8) we also deduce

\[
\left\| \frac{\partial \zeta_h}{\partial t} \right\|_{L^2(\Omega \times (0,T))} \leq T \left\| \frac{\partial \zeta_h}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} = T \left\| p_0^h \left( \frac{\partial U^h}{\partial t} \right) \right\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{T} \left\| p_0^h \right\|_{H^1(\Omega)} = \sqrt{T} \left\| A_0^h \right\|_{L^2(\Omega)} \leq 4T \left\| A_0^h \right\|_{L^2(\Omega)}.
\]

hence

\[
\left\| \frac{\partial \zeta_h}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq \left\| A_0^h \right\|_{L^2(\Omega)}. \tag{3.39}
\]

and

\[
\left\| \frac{\partial \zeta_h}{\partial t} \right\|_{L^2(\Omega \times (0,T))} \leq 2\sqrt{T} \left\| p_0^h \right\|_{L^2(\Omega)} \leq 4T \left\| A_0^h \right\|_{L^2(\Omega)}. \tag{3.40}
\]

In particular, the function \( \zeta_h \) belongs to the space \( W^{1,2}(0, T; V^h(\Omega)) \), and (see (3.34)) \( \zeta_h(0, x) = \zeta_{h,0}(x) \) as required in Definition 1.

By the definition of the function \( \zeta_h \) (see (3.37)) and estimates (3.27), (3.28) we deduce that \( \zeta_h \) is nonnegative, \( \zeta_h \in L^\infty(\Omega \times (0,T)) \) and

\[
\left\| \zeta_h \right\|_{L^\infty(\Omega \times (0,T))} \leq M(h), \tag{3.41}
\]

in particular estimates (3.2) are established.

Moreover, again by the construction, estimate (3.29), (2.8) and (2.7) we deduce that \( \zeta_h \in L^\infty(0, T; V^h(\Omega)) \) and

\[
\sup_t \left\| \zeta_h(t) \right\|_{L^2(\Omega)} \leq \sqrt{M(h)\left\| \zeta_{h,0} \right\|_{L^1(\Omega)}}. \tag{3.42}
\]

Let \( \Upsilon^h \) be the element defined in (3.26) we put

\[
\eta_h(x, t) = p_1^h(\Upsilon^h(t)) = (\eta_1^h(x, t), \eta_2^h(x, t)), \tag{3.43}
\]

where

\[
p_1^h(\Upsilon^h(t)) = \left( \sum_{j,k=-2^{m-1}}^{2^{m-1}} (\Upsilon^h(t))_{j,k} \varphi_{j,k}(x), \sum_{j,k=-2^{m-1}}^{2^{m-1}} (\Upsilon^h(t))_{j,k} \varphi_{j,k}(x) \right).
\]

The element \( \eta_h \) belongs to the space \( L^2(0, T; X_{0,h}) \) moreover as \( \Upsilon^h(t) \in H(U^h(t)) \) a.e in \( G_m \times (0,T) \) then \( p_1^h(\Upsilon^h) \in H(\zeta_h(x, t)) \) a.e in \( \Omega \times (0,T) \) as required in Definition 1.

Moreover we deduce from estimate (3.30), having in mind (2.3), (2.7), (3.6) and (3.35), that

\[
\int_0^t \int_\Omega \left( \frac{\partial \eta_1^h}{\partial x_1} \right) \left( \frac{\partial \eta_2^h}{\partial x_1} \right) + \left( \frac{\partial \eta_1^h}{\partial x_2} \right) \left( \frac{\partial \eta_2^h}{\partial x_2} \right) dxds \leq 4\left\| \zeta_{h,0} \right\|_{L^1(\Omega)}.
\tag{3.44}
\]
From (3.31) and (3.32) taking into account (3.36), we deduce formula (3.1). This concludes the proof of Theorem 3.1. }

4. SOC. In this section we show the Self Organized Criticality (SOC) properties.

**Theorem 4.1.** Assume $\zeta_{h,0} = p_h^0(Z^h)$, $Z^h \in L^2_{h,0}(G_m)$, $\zeta_{h,0} \geq 0$ then $\exists T^* = T^*(h)$ such that $\zeta_h(x,t) = 0 \forall t \geq T^*$ where $\zeta_h(x,t)$ is the discrete solution of problem (1.3). Moreover if $||\zeta_h(0)||_{L^1(\Omega)} \cap L^\infty(\Omega)$ is uniformly bounded with respect to $h$ then $T^*$ is independent of $h$. More precisely

$$T^* \leq C||\zeta_{h,0}||_{L^1(\Omega)}^{1/2}||\zeta_{h,0}||_{L^\infty(\Omega)}^{1/2}$$

(4.1)

where the constant $C$ is independent of $h$.

In the proof of the previous theorem we will use the following discrete Sobolev inequality.

**Lemma 3.**

$$\left( h^2 \sum_{j,k=-2^{m-1}}^{2^{m-1}} |v^h_{j,k}|^4 \right)^{1/4} \leq C_S \left( h^2 \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left( (D^+_1 v_{j,k})^2 + (D^+_2 v_{j,k})^2 \right) \right)^{1/4}$$

(4.2)

for any $m \in \mathbb{N}$, $h = \frac{2^m}{m}$ and $v^h \in L^2_{h,0}(G_m)$ where $0 < C_S = C_S(L,m)$ is bounded as $m \to +\infty$.

**Proof.** In the previous notation we have for $V^h \in L^2_{h,0}(G_m)$

$$V^h_{j,k} = h \sum_{l=-2^{m-1}}^{j-1} D^+_1 V^h_{l,k} = h \sum_{r=-2^{m-1}}^{k-1} D^+_2 V^h_{j,r}.$$  

(4.3)

Hence

$$|V^h_{j,k}|^2 \leq h^2 \sum_{l=-2^{m-1}}^{2^{m-1}} |D^+_1 V^h_{l,k}| \sum_{r=-2^{m-1}}^{2^{m-1}} |D^+_2 V^h_{j,r}|,$$

(4.4)

then

$$h^2 \sum_{j,k=-2^{m-1}}^{2^{m-1}} |V^h_{j,k}|^2 \leq h^4 \sum_{l,-2^{m-1}}^{2^{m-1}} |D^+_1 V^h_{l,k}| \sum_{r,-2^{m-1}}^{2^{m-1}} |D^+_2 V^h_{j,r}|,$$

(4.5)

and also

$$\left( h^2 \sum_{j,k=-2^{m-1}}^{2^{m-1}} |(V^h)_{j,k}|^2 \right)^{1/2} \leq \frac{h^2}{2} \left( \sum_{l,-2^{m-1}}^{2^{m-1}} |D^+_1 V^h_{l,k}| + \sum_{i,-2^{m-1}}^{2^{m-1}} |D^+_2 V^h_{i,k}| \right).$$

(4.6)

Now we put $V^h_{j,k} = (v^h)_{j,k}$ and we note that

$$\begin{cases}
|D^+_1 (v^h)_{j,k}^2| = \frac{1}{h^2} |(v^h_{j+1,k})^2 - (v^h_{j,k})^2| = \frac{1}{h^2} |v^h_{j+1,k} - v^h_{j,k}||v^h_{j+1,k} + v^h_{j,k}| \leq |D^+_1 v^h_{j,k}| \left( |v^h_{j,k}| + |v^h_{j+1,k}| \right) \leq |D^+_1 v^h_{j,k}| \left( 2|v^h_{j,k}| + h|D^+_1 v^h_{j,k}| \right),
\end{cases}$$

(4.7)

and analogously

$$|D^+_2 (v^h)_{j,k}^2| \leq |D^+_2 v^h_{j,r}| \left( 2|v^h_{j,r}| + h|D^+_2 v^h_{j,r}| \right).$$

(4.8)
Combining estimates (4.6) (4.7) and (4.8) we deduce

\[
\left( \frac{h^2}{2} \sum_{l,k=-2^{m-1}}^{2^{m-1}} |v_{l,k}^h|^4 \right) \leq \left( \frac{h^2}{2} \sum_{l,k=-2^{m-1}}^{2^{m-1}} |v_{l,k}^h|^4 \right)^{\frac{1}{2}} \cdot \left( \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_1^+ v_{j,k}^h \right| (2|v_{j,k}^h| + h|D_1^+ v_{j,k}^h|) \right).
\]

By using the Hölder inequality with \( q = 4 \) (and hence \( q' = \frac{4}{3} \)) and putting \( h^2 = \sqrt{\frac{4}{3}} h^3 \) we obtain

\[
\frac{h^2}{2} \sum_{l,k=-2^{m-1}}^{2^{m-1}} 2|D_1^+ v_{l,k}^h||v_{l,k}^h| \leq \left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} |v_{j,k}^h|^4 \right)^{\frac{1}{4}} \cdot \left( \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_1^+ v_{j,k}^h \right| \right)^{\frac{3}{4}}.
\]

Analogously

\[
\frac{h^2}{2} \sum_{l,k=-2^{m-1}}^{2^{m-1}} 2|D_2^+ v_{l,k}^h||v_{l,k}^h| \leq \left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} |v_{j,k}^h|^4 \right)^{\frac{1}{4}} \cdot \left( \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_2^+ v_{j,k}^h \right| \right)^{\frac{3}{4}}.
\]

Combining (4.9), (4.11) and (4.12) we obtain

\[
\left\{ \begin{array}{l}
\left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} |v_{j,k}^h|^4 \right)^{\frac{1}{2}} \leq \left( \frac{4}{3} \left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} |v_{j,k}^h|^4 \right)^{\frac{1}{4}} \right)^{\frac{3}{2}} + \\
\left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_1^+ v_{j,k}^h \right|^2 \right)^{\frac{1}{2}} + \left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_2^+ v_{j,k}^h \right|^2 \right)^{\frac{1}{2}}
\end{array} \right.
\]

Now we evaluate the second term in (4.13) using the Hölder inequality with \( q = \frac{3}{2} \) (and hence \( q' = 3 \)) we obtain

\[
\left\{ \begin{array}{l}
\left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_1^+ v_{j,k}^h \right|^4 \right)^{\frac{1}{4}} \leq \left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_1^+ v_{j,k}^h \right|^2 \right)^{\frac{3}{4}} \times (2^m + 1)^{\frac{1}{2}}
\end{array} \right.
\]

Analogously

\[
\left\{ \begin{array}{l}
\left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_2^+ v_{j,k}^h \right|^4 \right)^{\frac{1}{4}} \leq \left( \frac{h^2}{2} \sum_{j,k=-2^{m-1}}^{2^{m-1}} \left| D_2^+ v_{j,k}^h \right|^2 \right)^{\frac{3}{4}} \times (2^m + 1)
\end{array} \right.
\]
and combining (4.13), (4.14) and (4.15)
\[
\left\{ \begin{array}{l}
\left( h^2 \sum_{j,k=-2m-1}^{2m-1} |v_{j,k}^h|^4 \right)^{\frac{1}{4}} \leq \\
2 \left\{ h^3 (2^m + 1) \left( \sum_{j,k=-2m-1}^{2m-1} |D_1^+ v_{j,k}^h|^2 + \sum_{j,k=-2m-1}^{2m-1} |D_2^+ v_{j,k}^h|^2 \right) + \\
\frac{h^3}{2} \left( \sum_{j,k=-2m-1}^{2m-1} |D_1^+ v_{j,k}^h|^2 + \sum_{j,k=-2m-1}^{2m-1} |D_2^+ v_{j,k}^h|^2 \right) \right\}^{\frac{1}{2}}
\end{array} \right.
\]
(4.16)

Finally, recalling that $h = \frac{2L}{2m}$,
\[
\left\{ \begin{array}{l}
\left( h^2 \sum_{j,k=-2m-1}^{2m-1} |v_{j,k}^h|^4 \right)^{\frac{1}{4}} \leq \\
\frac{\sqrt{2} (2^m + 1 + \frac{1}{2} \frac{2L}{2m}) \left\{ h^2 \sum_{j,k=-2m-1}^{2m-1} \left( (D_1^+ v_{j,k}^h)^2 + (D_2^+ v_{j,k}^h)^2 \right) \right\}^{\frac{1}{2}}}
\end{array} \right.
\]
(4.17)

hence estimate (4.2) is proved with
\[
C_S = C_S(L, M) = \sqrt{2 (2^m + 1 + \frac{1}{2} \frac{2L}{2m})} \leq \sqrt{7L} \ \forall m \in \mathbb{N}.
\]
(4.18)

We prove now Theorem 4.1

**Proof.** As in the previous section, we put
\[
U_{i,\varepsilon,h}(x) = \sum_{j,k=-2m-1}^{2m-1} (U_{i,\varepsilon}^h)_{j,k} \theta_{j,k}(x)
\]
(4.19)

and
\[
\left\{ \begin{array}{l}
\Upsilon_{i,\varepsilon,h}(x) = (\Upsilon_{i,\varepsilon}^h(x), \Upsilon_{i,\varepsilon}^2,h(x)) = \\
(\sum_{j,k=-2m-1}^{2m-1} (\Upsilon_{i,\varepsilon}^h)_{j,k} \varphi_{j,k}^1(x), \sum_{j,k=-2m-1}^{2m-1} (\Upsilon_{i,\varepsilon}^h)_{j,k} \varphi_{j,k}^2(x))
\end{array} \right.
\]
(4.20)

then we deduce from (3.18) for any $i = 1, ..., n$
\[
\int_{\Omega} U_{i,\varepsilon,h} \, dx + \varepsilon \int_{\Omega} \left( \frac{\partial \Upsilon_{i,\varepsilon,h}}{\partial x_1} \right)^2 + \left( \frac{\partial \Upsilon_{i,\varepsilon,h}}{\partial x_2} \right)^2 \, dx \leq \int_{\Omega} U_{i-1,\varepsilon,h} \, dx.
\]
(4.21)

We use the discrete Sobolev inequality (4.2), we choose in (4.2) $\psi^h = \Upsilon_{i,\varepsilon}^h$ and we derive:
\[
\left( h^2 \sum_{j,k=-2m-1}^{2m-1} |(\Upsilon_{i,\varepsilon}^h)_{j,k}|^4 \right)^{\frac{1}{4}} \leq C_S \left\{ h^2 \sum_{j,k=-2m-1}^{2m-1} \left( (D_1^+ (\Upsilon_{i,\varepsilon}^h)_{j,k})^2 + (D_2^+ (\Upsilon_{i,\varepsilon}^h)_{j,k})^2 \right) \right\}^{\frac{1}{2}}.
\]
(4.22)

We note that, using the definition of the operator $p_0^h$ (2.3) and estimate in the right hand side of (2.8),
\[
\int_{\Omega} (p_0^h (\Upsilon_{i,\varepsilon}^h))^4 \, dx \leq 2 \sum_{j,k=-2m-1}^{2m-1} |(\Upsilon_{i,\varepsilon}^h)_{j,k}|^4
\]
and then from (4.22) and (3.35) we derive
\[
\left( \int_{\Omega} \rho_{h}^{0} \left( \Psi_{i,\varepsilon}^{h} \right)^{4} dx \right)^{\frac{1}{2}} \leq C_{S}^{2} \int_{\Omega} \left( \frac{\partial \Psi_{1}^{h}}{\partial x_{1}} \right)^{2} + \left( \frac{\partial \Psi_{2}^{h}}{\partial x_{2}} \right)^{2} dx. \tag{4.23}
\]
As \( \Psi_{i,\varepsilon}^{h} \in H(U_{l,\varepsilon}^{h}) \) we have
\[
\left( \int_{\Omega} \left| \rho_{h}^{0} \left( \Psi_{i,\varepsilon}^{h} \right) \right|^{4} dx \right)^{\frac{1}{2}} \geq \left| \{ x \in \Omega : U_{i,\varepsilon}^{h} > 0 \} \right|^{\frac{1}{2}} \tag{4.24}
\]
where by \(|E|\) we denote the 2-dimensional Lebesgue measure of the set \( E \).

Let \( M(h) \) as in (3.3) hence, by using the inequalities \( 0 \leq U_{i,\varepsilon}^{h} \leq M(h) \) (see (3.12) and (3.14)) we derive from (4.24)
\[
\left( \int_{\Omega} \left| \rho_{h}^{0} \left( \Psi_{i,\varepsilon}^{h} \right) \right|^{4} dx \right)^{\frac{1}{2}} \geq M(h)^{-\frac{1}{2}} \left( \int_{\Omega} U_{i,\varepsilon}^{h} dx \right)^{\frac{1}{2}}. \tag{4.25}
\]
Combining (4.21), (4.23) and (4.25) we obtain
\[
\int_{\Omega} U_{i,\varepsilon}^{h} dx + \varepsilon C_{S}^{-2} M(h)^{-\frac{1}{2}} \left( \int_{\Omega} U_{i,\varepsilon}^{h} dx \right)^{\frac{1}{2}} \leq \int_{\Omega} U_{i-1,\varepsilon}^{h} dx. \tag{4.26}
\]
Hence, for any positive integer \( l < i \) we have
\[
\int_{\Omega} U_{i,\varepsilon}^{h} dx + \varepsilon C_{S}^{-2} M(h)^{-\frac{1}{2}} \sum_{r=i}^{i+l} \left( \int_{\Omega} U_{r,\varepsilon}^{h} dx \right)^{\frac{1}{2}} \leq \int_{\Omega} U_{i,\varepsilon}^{h} dx. \tag{4.27}
\]
According to (3.15) we put
\[
U_{\varepsilon,h}(t) = \begin{cases} 
\zeta_{0,h} & \text{for } t \in [0, t_{1}) \\
U_{i,\varepsilon,h} & \text{for } t \in [t_{i}, t_{i+1}).
\end{cases} \tag{4.28}
\]
As \( \varepsilon \to 0 \), according to the Crandall-Liggett theorem (see Theorem II and Formula (2.5) in [15]), the functions \( U_{\varepsilon,h} \) converge strongly in \( C(0,T;L^{2}(\Omega)) \) to the function \( \zeta_{h} \). Keeping in mind (4.28) for all \( 0 \leq s < t \) we obtain, passing to the limit as \( \varepsilon \to 0 \)
\[
\int_{\Omega} \zeta_{h}(x,t) dx + C_{S}^{-2} M(h)^{-\frac{1}{2}} \int_{s}^{t} \left( \int_{\Omega} \zeta_{h}(x,\tau) dx \right)^{\frac{1}{2}} d\tau \leq \int_{\Omega} \zeta_{h}(x,s) dx. \tag{4.29}
\]
We set
\[
\beta(t) = \int_{\Omega} \zeta_{h}(x,t) dx \geq 0
\]
and we derive from (4.29)
\[
\beta(t) + C_{S}^{-2} M(h)^{-\frac{1}{2}} \int_{s}^{t} (\beta(\tau))^{\frac{1}{2}} d\tau \leq \beta(s) \quad \forall \ 0 \leq s \leq t. \tag{4.30}
\]
As the function \( \beta \) is absolutely continuous we deduce from (4.30)
\[
2 \frac{d}{dt} \left( \beta(t) \right)^{\frac{1}{2}} + C_{S}^{-2} M(h)^{-\frac{1}{2}} \leq 0
\]
and therefore, taking into account (3.3) and recalling that
\[
\zeta_{h,0} = \rho_{h}^{0}(Z^{h}) = \sum_{j,k=-2^{m-1}}^{2^{m-1}} (Z^{h})_{j,k} \theta_{j,k}(x),
\]
Then the discrete solutions \( \zeta \) weakly converge, in the estimates established in Section 3)

\[
\beta(t) = 0 \quad \forall t > T^* = \frac{2\sqrt{\beta(0)}}{C_S M(h)^{\frac{1}{2}}} \leq 2C^2 ||\zeta_{h,0}||_{L^1(\Omega)}^{\frac{1}{2}} ||\zeta_{h,0}||_{L^\infty(\Omega)}^{\frac{1}{2}}.
\]

By the uniform boundedness of the initial datum there exist two numbers \( M_\infty \) and \( M^* \) independent of \( h \) such that
\[
||\zeta_{h,0}||_{L^\infty(\Omega)} = \max_{j,k} (Z^h)_{j,k} = M(h) \leq M_\infty, \text{ and } ||\zeta_{h,0}||_{L^1(\Omega)} \leq M^*.
\]

The proof of (4.1) is now complete. \( \square \)

5. **Convergence.** In this section we deal with the convergence of the discrete solution as the discretization parameter \( h \) tends to 0.

**Theorem 5.1.** Let \( \zeta_0 \in C_0(\Omega) \), \( \zeta_0 \geq 0 \) we choose as initial datum \( \zeta_{h,0} = p^0_{h}(r_h \zeta_0) \). Then the discrete solutions \( \zeta_h \) given in Theorem 3.1 weakly converge as \( h \to 0 \), in \( L^2(0,T;L^2(\Omega)) \) to the solution \( \zeta \) of problem (1.3). Moreover the functions \( \eta_h \), (see Definition 1) weakly converge, in \( L^2(0,T;X) \), to the function \( \eta \) in problem (1.3).

We give an idea of the proof of Theorem 5.1. For fixed \( \varepsilon > 0 \) and \( h = \frac{2M}{\varepsilon^2} \) let \( U^h_{\varepsilon}(t) \) denote the function defined in formula (3.15), which is a step-function in time with value in \( L^2_{h,0}(G_m) \). We use the projection operator \( p^0_h \) and we construct the step-function in time and space \( \zeta^h_\varepsilon(x,t) \) that belongs to the space \( L^2(0,T;V_h(\Omega)) \). \( \zeta^h_\varepsilon(x,t) = p^0_h(U^h_{\varepsilon}(t)) \).

Similarly let \( \Upsilon^h_{\varepsilon}(t) \) denote the function defined in formula (3.16) which is a step-function in time with value in \( L^2_{h,0}(G_m) \). We use the projection operator \( p^1_h \) and we construct the element \( \Upsilon^h_{\varepsilon}(x,t) \in L^2(0,T;X_{h,0}) \). \( \Upsilon^h_{\varepsilon}(x,t) = p^1_h(\Upsilon^h_{\varepsilon}(t)) \) whose components are step-functions in time and tent functions in space.

By the estimates established in Section 3 we deduce that we can pass in the limit for \( \varepsilon \to 0 \) (and \( h \) fixed) as well as for \( h \to 0 \) (and \( \varepsilon \) fixed). Now the functions \( \zeta^h_{\varepsilon}(x,t) \) as \( \varepsilon \to 0 \) (and \( h \) fixed) converge to the discrete solution \( \zeta_h \) of the sand pile model (1.3) (see Definition 1 and Theorem 3.1) and we denote by \( \zeta^* \) the limit for \( h \to 0 \) of the sequence \( \zeta_h \) (that exists again by the estimates established in Section 3).

If instead \( \varepsilon \) is fixed and \( h \) tends to 0 then the functions \( \zeta^h_{\varepsilon}(x,t) \) converge to \( U^h_{\varepsilon} \) that is the Crandall- Liggett approximation of the solution of the sand pile model (1.3) and the functions \( U^h_{\varepsilon} \) converge for \( \varepsilon \to 0 \) to the unique solution \( \zeta \) of (1.3). Finally we conclude that if we show that the convergence (for \( \varepsilon \to 0 \) of the sequence \( \zeta^h_{\varepsilon} \) is uniform in \( h \) then the diagram commutes and we conclude that \( \zeta^* = \zeta \) and the proof of Theorem 5.1 will be complete.

We can summarize the described setting by the following diagram:

\[ \begin{array}{ccc}
\zeta^h_{\varepsilon} & \xrightarrow{\varepsilon \to 0 \ (\text{uniformly})} & \zeta_h \\
\downarrow h \to 0 & & \downarrow h \to 0 \ (\text{subseq}) \\
U^h_{\varepsilon} & \xrightarrow{\varepsilon \to 0} & \zeta^* \\
\end{array} \]

\( (\zeta^* = \zeta) \)
Proof. We begin by noticing that as the initial datum $\zeta_0$ has compact support in $\Omega$ then (for $h$ sufficiently small) $Z^h = \eta h_0$ belongs to the space $L^2_{h,0}(G_m)$. Moreover we can choose in (4.31)

$$M_\infty = \max_{x \in \Omega} \zeta_0(x) \quad \text{and} \quad M^* = 4M_\infty L^2. \quad (5.1)$$

By (2.10) in Section 2 we have

$$\zeta_{h,0} = p^0_h \eta h_0 \rightarrow \zeta_0 \text{ strongly in } L^2(\Omega). \quad (5.2)$$

Analogously for any function $\Phi \in C^1_0(\Omega)$ we set

$$\Phi_h = p^1_h \eta h \Phi = p^1_h \Phi^h = (\Phi^1_h, \Phi^2_h) = \left( \sum_{j,k=\pm 2^{m-1}} (\Phi^h)_{j,k} \varphi^1_{j,k}, \sum_{j,k=\pm 2^{m-1}} (\Phi^h)_{j,k} \varphi^2_{j,k} \right), \quad (5.3)$$

where $(\Phi^h)_{j,k} = \int_{\Omega} \Phi_{j,k}^h dx$ and, for $h$ sufficiently small, $(\Phi^h)_{j,k} \in L^2_{h,0}(G_m)$ and then $\Phi_h \in X_{h,0}$. By condition (a) in Lemma 2 the sequence $\Phi_h$ strongly converges in $X$ towards $\Phi$.

According to the notation of Section 3 we denote by

$$\Upsilon_{i,\epsilon,h} = p^1_h (\Upsilon^h_{i,\epsilon}) = \left( \Upsilon^1_{i,\epsilon,h}(x), \Upsilon^2_{i,\epsilon,h}(x) \right) = \left( \sum_{j,k=\pm 2^{m-1}} (\Upsilon^h_{i,\epsilon})_{j,k} \varphi^1_{j,k}(x), \sum_{j,k=\pm 2^{m-1}} (\Upsilon^h_{i,\epsilon})_{j,k} \varphi^2_{j,k}(x) \right)$$

Similarly we denote by $U_{i,\epsilon,h} = p^0_h \Upsilon^h_{i,\epsilon} = \sum_{j,k=\pm 2^{m-1}} (U^h_{i,\epsilon})_{j,k} \theta_{j,k}(x)$, moreover for every element $\Phi_h \in X_{0,h}$ let us denote by $\Phi^0_h = \Phi^0_h(\Phi^h) = \sum_{j,k=\pm 2^{m-1}} (\Phi^h)_{j,k} \theta_{j,k}(x)$.

We note that, (for fixed $\epsilon > 0$ and $h = \frac{2\epsilon}{\sqrt{m}}$) by construction, the inner product in $L^2_{h,0}(G_m)$ of the functions $U_{i,\epsilon,h}$ and $\Phi^0_h$ can be expressed in terms of the components $i.e.$

$$\int_\Omega U_{i,\epsilon,h} \Phi^0_h \ dx = h^2 \sum_{j,k=\pm 2^{m-1}} (U^h_{i,\epsilon})_{j,k} (\Phi^h)_{j,k}$$

and, by taking (3.35) into account, also

$$\int_\Omega \left( \frac{\partial \Upsilon^1_{i,\epsilon,h}}{\partial x_1} \frac{\partial \Phi^1_h}{\partial x_1} + \frac{\partial \Upsilon^2_{i,\epsilon,h}}{\partial x_2} \frac{\partial \Phi^2_h}{\partial x_2} \right) \ dx = h^2 \sum_{j,k=\pm 2^{m-1}} \left( D^1_{i,\epsilon,h})_{j,k} D^1_{1}(\Phi^h)_{j,k} + D^2_{2}(\Upsilon^h_{i,\epsilon,h})_{j,k} D^2_{2}(\Phi^h)_{j,k} \right)$$

where for every element $\Phi_h \in X_{0,h}$ we put

$$\Phi_h(x) = p^1_h(\Phi^h) = (\Phi^1_h(x), \Phi^2_h(x)) = \left( \sum_{j,k=\pm 2^{m-1}} (\Phi^h)_{j,k} \varphi^1_{j,k}(x), \sum_{j,k=\pm 2^{m-1}} (\Phi^h)_{j,k} \varphi^2_{j,k}(x) \right).$$
We now go back to (3.9) and we deduce that, for any \( i = 1, \ldots, n \), the function \( U_{i,\varepsilon,h} \) is solution of the problem

\[
\begin{aligned}
&\int_{\Omega} U_{i,\varepsilon,h} \Phi_0^h \, dx + \\
&\varepsilon \int_{\Omega} \left( \frac{\partial U_{i,\varepsilon,h}}{\partial x_1} \frac{\partial \Phi_h}{\partial x_1} + \frac{\partial U_{i,\varepsilon,h}}{\partial x_2} \frac{\partial \Phi_h}{\partial x_2} \right) \, dx = \\
&\int_{\Omega} U_{i-1,\varepsilon,h} \Phi_0^h \, dx \quad \forall \quad \Phi_h \in X_{0,h} \\
&\int_{\Omega} U_{i,\varepsilon,h} \Phi_0^h \, dx + \Phi_h \in X_{0,h} \\
&\int_{\Omega} U_{0,\varepsilon,h} \Phi_0^h \, dx + \Phi_h \in X_{0,h} \\
&\int_{\Omega} U_{i,\varepsilon,h} \Phi_0^h \, dx + \Phi_h \in X_{0,h}.
\end{aligned}
\]  

(5.4)

From (3.14), (5.1), and the definition of the function \( \zeta_{h,0} \) we derive

\[ ||U_{i,\varepsilon,h}||_{L^\infty(\Omega)} \leq M_\infty \]

and in particular (up to pass to a subsequence), as the parameter \( h \) tends to 0, the functions \( U_{i,\varepsilon,h} \) converge weakly-star in \( L^\infty(\Omega) \) to a function \( U_{i,\varepsilon} \).

By construction the elements \( \Upsilon_{i,\varepsilon,h} \) belong to the space \( X_{h,0} \). From formulas (4.21), (3.14), (3.35), (4.20) and (5.1) (taking into account that \( U_{i,\varepsilon,h} \geq 0 \)) we derive that (for any fixed \( i = 1, \ldots, n \))

\[ \varepsilon E_h(\Upsilon_{i,\varepsilon,h}, \Upsilon_{i,\varepsilon,h}) = \varepsilon \int_\Omega \left( \left( \frac{\partial Y_{i,\varepsilon,h}}{\partial x_1} \right)^2 + \left( \frac{\partial Y_{i,\varepsilon,h}}{\partial x_2} \right)^2 \right) \, dx \leq 4M_\infty L^2, \]  

(5.5)

by estimate (3.7) we deduce

\[ C_P ||\Upsilon_{i,\varepsilon,h}||_{L^2(G_m)}^2 \leq E_h(\Upsilon_{i,\varepsilon,h}, \Upsilon_{i,\varepsilon,h}). \]  

(5.6)

Moreover by estimate (2.11) we deduce

\[ ||Y_{i,\varepsilon,h}||_{L^2(\Omega)} \leq 4|\Upsilon_{i,\varepsilon,h}|_{L^2(G_m)}^2 \]  

and

\[ ||Y_{i,\varepsilon,h}||_{L^2(\Omega)} \leq 4|\Upsilon_{i,\varepsilon,h}|_{L^2(G_m)}^2. \]  

(5.7)

Hence the elements \( \Upsilon_{i,\varepsilon,h} \) (up to pass to a subsequence) converge weakly in \( X \), as the parameter \( h \) tends to 0, to a function \( \Upsilon_{i,\varepsilon} \in H^1_0(\Omega) \). From estimates (5.5), (5.7) and using the Riesz-Fréchet-Kolmogorov criterium (see e.g. Teorema IV.26 in [11]) we derive that the functions \( p_h^0 \Upsilon_{i,\varepsilon,h} \) converge strongly in \( L^2(\Omega) \) to \( \Upsilon_{i,\varepsilon} \) and in particular

\[ ||p_h^0 \Upsilon_{i,\varepsilon,h}||_{L^2(\Omega)} \to ||\Upsilon_{i,\varepsilon}||_{L^2(\Omega)} \quad \text{as} \quad h \to 0. \]  

(5.8)

Now we pass to the limit in (5.4) as \( h \to 0 \) and we have

\[
\begin{aligned}
&\int_{\Omega} U_{i,\varepsilon} \Phi \, dx + \varepsilon \int_{\Omega} \left( \frac{\partial Y_{i,\varepsilon}}{\partial x_1} \frac{\partial \Phi}{\partial x_1} + \frac{\partial Y_{i,\varepsilon}}{\partial x_2} \frac{\partial \Phi}{\partial x_2} \right) \, dx = \\
&\int_{\Omega} U_{i-1,\varepsilon} \Phi \, dx + \Phi \in C^1_0(\Omega) \\
&U_{0,\varepsilon} = \zeta_0.
\end{aligned}
\]

(5.9)

Moreover as \( p_h^0(\Upsilon_{i,\varepsilon,h}) \in H(U_{i,\varepsilon,h}) \) then \( p_h^0 \Upsilon_{i,\varepsilon,h} \in H(U_{i,\varepsilon,h}) \). The operator \( H \), being maximal monotone in \( L^2(\Omega) \times L^2(\Omega) \), is closed in \( \text{weak} \times \text{strong} \) topology and as \( U_{i,\varepsilon,h} \) weakly converge to \( U_{i,\varepsilon} \) and \( p_h^0 \Upsilon_{i,\varepsilon,h} \) strongly converge to \( \Upsilon_{i,\varepsilon} \) then \( \Upsilon_{i,\varepsilon} \in H(U_{i,\varepsilon}) \) a.e in \( \Omega \) (see Corollary 2.4 In [5] or Proposition 2.5 in [10]). As a consequence \( U_{i+1,\varepsilon} \) is the Crandall-Liggett approximation of the solution \( \zeta \) of the sand pile model (1.3)
(it coincides with the function $y_{r_{1}+1}$ in Problem (2.7) in [7]); in particular the whole sequence $U_{r_{i},r_{1h}}$ converges (weakly-star in $L^{\infty}(\Omega)$, as the parameter $h$ tends to 0) to the function $U_{r_{1}}$ which is the Crandall-Liggett approximation of the solution $\zeta$.

We derive that the functions $\zeta_{h}(x,t) = p_{h}^{0}U_{h}(t)$ (see formula (3.15)), which are step-functions in time and space, converge (as $h \to 0$, weakly in $L^{2}(\Omega \times (0,T))$) to the function $U_{\varepsilon}$

$$U_{\varepsilon}(t) = \begin{cases} 
\zeta_{0} & \text{for } t \in [0,t_{1}) \\
U_{i} & \text{for } t \in [t_{i}, t_{i+1}).
\end{cases}$$

(5.10)

Furthermore the unique solution $\zeta$ of Problem (1.3) is the (strong) limit as $\varepsilon \to 0$ of the functions $U_{\varepsilon}$ defined in (5.10).

We inverse the order in the limit procedure: again from inequalities (3.41), (3.42) and (4.31) we derive that (up to pass to a subsequence) the functions $\eta$ converges weakly in $L^{2}(\Omega \times (0,T))$ and in $L^{\infty}(0,T;L^{2}(\Omega))$ to a function $\zeta^{*}$.

Analogously, we consider the functions $\eta_{i}$ defined in (3.43), and from inequalities (5.5), (5.7) and (5.6) we deduce that (up to pass to a subsequence) the functions $\eta_{i}$ converge weakly in $L^{2}(0,T;X)$ to a function $\eta \in L^{2}(0,T;H_{0}^{1}(\Omega))$ (see condition (c) in Lemma 1).

We assume now that there exists a constant $M^{**}$ independent of $h$ such that

$$|A_{h}^{0}r_{h}\zeta_{h,0}|_{h} \leq M^{**}$$

(5.11)

where, as before, we denote by $A_{h}^{0}u^{h}$ the element of $A_{h}u^{h}$ having minimal norm. We will remove later this assumption. We derive from (3.39) and (5.11)

$$\|\partial_{\zeta h}/\partial t\|_{L^{\infty}(0,T;L^{2}(\Omega))} \leq M^{**}$$

and also (up to pass to a subsequence) the time derivatives $\partial_{\zeta_{h}}/\partial t$ converge weakly-star in $L^{\infty}(0,T;L^{2}(\Omega))$ to $\partial_{\zeta^{*}}/\partial t$ (see the proof of Theorem 3.1 and the appendix in [10]).

Now for every function $\Phi \in L^{2}(0,T;C_{1}^{0}(\Omega))$ we set $\Phi_{h}^{0} = p_{h}^{0}r_{h}\Phi$ and we have $\Phi_{h}^{0} \in L^{2}(0,T;V_{h}(\Omega))$; similarly we set $\Phi_{h} = p_{h}^{0}r_{h}\Phi$ and we have $\Phi_{h} \in L^{2}(0,T;X_{h,0})$ (for $h$ sufficiently small). Moreover by (2.10) and Lemma 2 we derive

$$\Phi_{h}^{0} \rightarrow \Phi \text{ strongly in } L^{2}(0,T;L^{2}(\Omega)) \text{ and } \Phi_{h} \rightarrow \Phi \text{ strongly in } L^{2}(0,T;X).$$

(5.12)

Hence we can pass to the limit in (3.1) as $h \to 0$ and we have

$$\int_{0}^{T} \int_{\Omega} \partial_{\zeta h}^{*} \Phi \, dx \, dt + \int_{0}^{T} \int_{\Omega} \left( \frac{\partial \eta}{\partial x_{1}} \frac{\partial \Phi}{\partial x_{1}} + \frac{\partial \eta}{\partial x_{2}} \frac{\partial \Phi}{\partial x_{2}} \right) \, dx \, dt = 0,$$

(5.13)

and this yields $\zeta^{*} \in W^{1,2}(0,T;L^{2}(\Omega))$.

Coming back to the elements $U_{h}^{0}(t)$ in (3.15) and, using again estimates (3.33) and (5.11), we prove by arguing as in Theorem 2.1 of [9] that the elements $U_{h}^{0}(t)$ converge strongly in $L^{2}_{h,0}(G_{m})$ uniformly in $t$ and in $h$ to $U^{0}$ as $\varepsilon \to 0$.

We deduce that the functions $\zeta_{h}(t) = p_{h}^{0}U_{h}^{0}(t)$ converge strongly in $L^{2}(\Omega)$, uniformly in $t$ and in $h$, to $\zeta_{h}(t)$ as $\varepsilon \to 0$. Moreover the functions $\zeta_{h}$ converge weakly in $L^{2}(0,T;L^{2}(\Omega))$ to $\zeta^{*}$ as $h \to 0$. On the other hand, we showed before that the functions $p_{h}^{0}U_{h}^{0}(t)$ converge weakly in $L^{2}(\Omega)$ as $h \to 0$ to the function $U_{\varepsilon}(t)$ in (5.10) and the functions $U_{\varepsilon}(t)$ converge strongly in $L^{2}(\Omega)$ and as $\varepsilon \to 0$ to the (unique) solution $\zeta$ of Problem (1.3). As the convergence of $p_{h}^{0}U_{h}^{0}(t)$ (as $\varepsilon \to 0$) is uniform on $h$ we can change the order in the limit procedure and we deduce that $\zeta = \zeta^{*}$. 


Now remove condition (5.11). Firstly we note that if the initial datum \( \zeta_0 \) is non-negative and with a compact support in \( \Omega \) then we can choose in \( H(\zeta_0) \) a smooth function \( \mu \) (say \( \mu \in W^{2,\infty}(\Omega) \)). Recalling the definitions of \( A_h^0 \) and of \( A_h = \Delta_h H \) in order to prove (5.11) we have to show that the norm \( |\Delta_h r_h \mu|_h \) is bounded uniformly in \( h \) and this bound follows from the definition of \( \Delta_h \) in (3.4) and the smoothness of \( \mu \).

This concludes the proof of Theorem 5.1.

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