Optimal Renormalization-Group Improvement of Two Radiatively-Broken Gauge Theories

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Abstract

In the absence of a tree-level scalar-field mass, renormalization-group (RG) methods permit the explicit summation of leading-logarithm contributions to all orders of the perturbative series for the effective-potential functions utilized in radiative symmetry breaking. For scalar-field electrodynamics, such a summation of leading logarithm contributions leads to upper bounds on the magnitudes of both gauge and scalar-field coupling constants, and suggests the possibility of an additional phase of spontaneous symmetry breaking characterized by a scalar-field mass comparable to that of the theory’s gauge boson. For radiatively-broken electroweak symmetry, the all-orders summation of leading logarithm terms involving the dominant three couplings (quartic scalar-field, $t$-quark Yukawa, and QCD) contributing to standard-model radiative corrections leads to an RG-improved potential characterized by a 216 GeV Higgs boson mass. Upon incorporation of electroweak gauge couplants we find that the predicted Higgs mass increases to 218 GeV. The potential is also characterized by a quartic scalar-field coupling over five times larger than that anticipated for an equivalent Higgs mass obtained via conventional spontaneous symmetry breaking, leading to a concomitant enhancement of processes (such as $W^+W^- \to ZZ$) sensitive to this coupling. Moreover, if the QCD coupling constant is taken to be sufficiently strong, the tree potential’s local minimum at $\phi = 0$ is shown to be restored for the summation of leading logarithm corrections. Thus if QCD exhibits a two-phase structure similar to that of $N = 1$ supersymmetric Yang-Mills theory, the weaker asymptotically-free phase of QCD may be selected by the large logarithm behaviour of the RG-improved effective potential for radiatively broken electroweak symmetry.

1 Introduction: Radiatively Broken Abelian Gauge Symmetry

In their 1973 paper\cite{1}, S. Coleman and E. Weinberg demonstrated that spontaneous symmetry breaking occurs within gauge theories in which scalar fields are initially massless. This approach to symmetry breaking has considerable predictive power and led to perhaps the first definitive prediction for the magnitude of the Higgs boson mass. Although this prediction has since proved to be incorrect (the mass of the top-quark was unknown at the time of their paper), Coleman and Weinberg’s work also demonstrated the nontrivial role radiative corrections play in determining observable consequences of gauge theories, with eventual applications to cosmology and empirical standard model physics.

The simplest example of calculable radiatively induced symmetry breaking considered by Coleman and Weinberg is that of massless scalar electrodynamics, in which an initially massless complex scalar field (or alternatively, its two constituent real-field components) is minimally coupled to an unbroken Abelian gauge theory.

The effective potential of this massless scalar electrodynamics is generated from the tree-level potential

$$V = \lambda (\phi_1^2 + \phi_2^2)^2 / 24$$

by the scalar field self-interaction and the interaction Lagrangian involving real scalar $\phi_1, \phi_2$, and gauge fields $A_\mu$:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_1 - e A_\mu \phi_2)^2 + \frac{1}{2} (\partial_\mu \phi_2 + e A_\mu \phi_1)^2 - V.$$
The effective potential for this theory is calculated in Landau gauge in Ref. [1],

$$V_{\text{eff}} = \phi^4 \left[ \frac{\lambda}{24} + \left( \frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2} \right) \left( \log \frac{\phi^2}{\mu^2} + k \right) + \mathcal{O}(\lambda^3, e^6) \right], \quad (1.3)$$

and in arbitrary covariant gauge in Appendix A. The absence of an explicit scalar field mass term precludes the need for a cosmological constant term in $V_{\text{eff}}$. The renormalization constant $k$ is determined by the (choice-of-scheme) definition of the quartic scalar interaction constant $\lambda$ as the fourth derivative of the effective potential with respect to the classical field $\phi_1c$ (or $\phi_2c$, $\phi_1c^2 + \phi_2c^2 \equiv \phi_c^2$) when evaluated at the renormalization mass $\mu$:

$$\frac{d^4V}{d\phi_1^4}\bigg|_{\phi^2 = \mu^2} \equiv \lambda \quad (1.4)$$

in which case

$$k = -\frac{25}{6}. \quad (1.5)$$

The condition that $dV_{\text{eff}}/d\phi_c = 0$ at the vacuum expectation value $\langle \phi \rangle$, in conjunction with the assumption that $\lambda$ and $e^4$ are of equivalent order, leads to the constraint

$$\lambda = 33e^4/8\pi^2. \quad (1.6)$$

Given this constraint, one finds that

$$V_{\text{eff}} = \frac{3e^4}{64\pi^2} \phi_c^4 \left[ \log \frac{\phi_c^2}{\langle \phi \rangle^2} - \frac{1}{2} \right] + \mathcal{O}(e^6). \quad (1.7)$$

The scalar field and gauge field mass terms [the RG-invariance of the former is discussed in Appendix B] are respectively given by

$$m_\phi^2 = \frac{d^4V}{d\phi_1^4} \langle \phi \rangle, \quad m_A^2 = e^2 \langle \phi \rangle^2, \quad (1.8)$$

and one finds from Eq. (1.7) that [1]

$$\frac{m_\phi^2}{m_A^2} = 3e^2/8\pi^2. \quad (1.9)$$

The result (1.5) is obtained entirely via the leading logarithmic contribution (1.3) to the scalar field self-coupling and by assuming that $\lambda^2$ can be neglected compared to $e^4$. At this juncture, if this latter assumption were not true, then Eq. (1.4) would no longer be true, and $\lambda$ could be sufficiently large to render order $\lambda^3$-and-higher contributions to Eq. (1.3) too important to neglect. In Section V of Ref. [1], renormalization group methods were utilized to show that the range of validity of the above results [particularly Eq. (1.9)] could be extended to arbitrary but still small values of $\lambda$ and $e$. The form of renormalization group improvement employed in Ref. [1] is the introduction of running coupling constants in the effective potential; it is then argued that $\lambda$ can be moved from any (small) value to an $\mathcal{O}(e^4)$ value via a change in renormalization mass $\mu$ for which $e$ retains a small value.

In the following sections we employ a more “optimal” form of renormalization-group (RG) improvement in which the leading logarithms of the effective potential for massless scalar electrodynamics are explicitly summed to all orders in perturbation theory. Indeed, we show in the next section that the logarithmic contribution to the effective potential (1.3) is just the first term of such a summation of leading logarithms, and that one can obtain this term directly from the renormalization-group equation (RGE), rather than via the explicit calculation and summation of one loop diagrams (as in Ref. [1]). Our methodological point of view (as articulated in general terms by Maxwell [2]) is to incorporate all information about higher order contributions to the effective potential that is accessible via RG methods. Such all-orders summations of logarithms, which are compared to more conventional forms of RG-improvement in refs. [3] [4], have been previously applied to the effective potential of $\phi^4$ scalar field theory [5] [6] [7] [8], various effective actions [9], as well as to correlation functions, decay rates and cross-sections for a number of perturbatively-calculated processes [10].

In the present paper, we re-examine both the massless scalar electrodynamics (MSED) and the radiatively-broken $SU(2) \times U(1)$ electroweak symmetry considered in Ref. 1 after inclusion of all RG-accessible information about higher order terms within the effective potentials of these theories. In Section 2, we demonstrate how the RG-equation for the effective potential of MSED leads to recursion relations which serve to determine all leading logarithm contributions.
to the perturbative effective potential series, despite the presence of two perturbative couplings, $\lambda/4\pi^2$ and $\alpha/\pi(=e^2/4\pi^2)$, in the theory. We restructure the theory into a power series in the latter (presumably small) gauge coupling by obtaining closed-form expressions in Section 3 for the summation-of-leading-logarithm contributions to all orders of the self-interaction coupling $\lambda/4\pi^2$ involving a given power of $\alpha/\pi$. This procedure reduces a double-summation over powers of both couplings to a summation over powers of a single (presumably known) gauge coupling $\alpha/\pi$, a summation in which the series coefficients are obtained by solving successive first-order differential equations in the unknown scalar-field self-interaction coupling $\lambda/4\pi^2$.

In Section 4, we incorporate into the effective potential this entire summation-of-leading-logarithm series, as opposed to just the first two terms of this series (leading and one-loop) present in Eq. (1.1). By applying the same set of finite renormalization conditions delineated above to the full leading-logarithm potential, we are able to obtain results that are not limited by assumptions of either couplant being perturbatively small. Surprisingly, we find a nonlinear relation between the two couplings of the theory that places upper bounds on the magnitudes of both couplants. Moreover, the radiative symmetry breaking epitomized by Eq. (1.9) is seen to correspond to only the weaker of two possible phases for the scalar-field self-interaction couplant. In the stronger phase, the scalar field mass is substantially larger, comparable in magnitude to the mass that symmetry-breaking generates for the Abelian gauge field.

This latter result provides motivation for a similar re-appraisal of the Standard Model for electroweak physics with a single (initially-) massless complex scalar field doublet, since the Higgs boson mass is already known to be larger than the broken-symmetry $W$ and $Z$ gauge-boson masses for $SU(2) \times U(1)$ gauge theory. In Section 5 we argue that the dominant three interaction coupling parameters for this theory are the scalar-field self-interaction coupling $\lambda/4\pi^2$, the $t$-quark Yukawa-interaction coupling $g_t^2/4\pi^2$, and the QCD gauge-interaction coupling $\alpha_s/\pi$, even though this latter couplant does not contribute to leading-logarithms within the scalar-field effective potential until two-loop order. With this simplification, the RGE for the effective potential leads to recursion relations from which leading-logarithm contributions may be extracted to arbitrary order in all three couplants, including, of course, the one-loop Yukawa and scalar-field couplant contributions obtained in Ref. [1].

In Section 6 we reorganize perturbative leading-logarithm contributions to the effective potential as a power series in the Yukawa couplant $g_t^2/4\pi^2$, whose series coefficients are closed-form all-orders functions of the remaining two couplants. These functions are determined from a set of successive partial differential equations. The closed form solutions for the first three of these series coefficients are obtained through explicit solution of such partial differential equations. [The solution of the fourth series coefficient is presented in Appendix C.]

It is shown in Section 7 that any prediction of the Higgs boson mass is sensitive to terms with at most four powers of the logarithm appearing in the effective potential's perturbative series. In anticipation of this result, Section 6 lists every such term arising from contributions of the three dominant couplants to the leading-logarithms of that series. In Section 7, these results are utilized to obtain a prediction of 216 GeV for the Higgs boson mass, a result that follows from a nonlinear relationship between the scalar-field self-interaction coupling and the known QCD gauge- and $t$-quark Yukawa couplants. Moreover, the predicted scalar-field couplant is seen to be several times larger than the scalar couplant corresponding to an equivalent Higgs boson mass obtained from conventional spontaneous symmetry-breaking. Thus, given the discovery of a Higgs boson with mass at or near 216 GeV, a clear signal for radiative symmetry breaking (as opposed to conventional spontaneous symmetry breaking) would be the amplification of processes such as the $W^+W^- \rightarrow ZZ$ scattering cross-section which exhibit significant sensitivity to the scalar-field self-interaction couplant.

These results are discussed further in Section 8. Residual renormalization-scale dependence of the one-loop effective potential is shown to be substantially larger than that obtained when leading logarithm contributions are summed. Moreover, this minimal residual scale dependence, if indicative of unknown subsequent-to-leading-logarithm corrections, buttresses the case for the 216 GeV prediction for the Higgs boson mass to be meaningful.

The summation of leading logarithms, however, is of greatest value in ascertaining large-logarithm properties of the effective potential, properties corresponding to that potential's large-field and zero-field limits. In Section 8, these two limits are examined separately. Large-field contributions to each power of the Yukawa couplant in the summation-of-leading-logarithms series are shown to grow singular when $\phi \sim 22(\phi)$. Since term-by-term singularities in a series are not necessarily singularities in the function represented by the series, this ultraviolet singularity represents a bound on the domain of the series obtained in Section 6, as opposed to a fundamental property of the effective potential itself. Moreover, it is shown in Section 8 that every term in the summation-of-leading-logarithms series diverges positively as $\phi$ approaches this singularity near $22(\phi)$ from below. Such a result is consistent with this series representation (or truncations thereof) being bounded from below over its entire domain of applicability. The boundedness of the potential prior to an $O(5 \text{ TeV})$ singularity is shown to be confirmed by analysis of a closed-form exact solution for the summation of leading logarithms that is obtainable in the limit where QCD is turned off (i.e. the QCD couplant is set to zero). The correspondence between the explicit leading-logarithm summation and its
equivalent method-of-characteristics solution is also demonstrated in Section 8 for the case of radiatively broken electroweak symmetry breaking, which facilitates estimation of the contributions of sub-dominant electroweak gauge couplants to radiative symmetry breaking. Such contributions are shown to raise the extracted value of the Higgs boson mass from 216 GeV to 218 GeV.

We conclude Section 8 by examining this summation-of-leading-logarithms series in the zero-field limit. Convergence of this series is demonstrated when the QCD couplant is sufficiently strong \((\alpha_s \geq 0.4)\). The series summation for this case is shown to exhibit a local minimum at \(\phi = 0\), suggestive of electroweak symmetry restoration when the QCD gauge couplant is sufficiently large. The applicability of this result in distinguishing between possible coexisting strong and asymptotically-free phases of QCD is briefly discussed.

## 2 RG-Equation for the Effective Potential of MSED

The effective potential of massless scalar electrodynamics may be expressed in terms of a perturbative series \(S = \frac{\lambda}{4\pi^2} + \mathcal{O}\left(\lambda^2, e^4, e^2\lambda\right)\) such that

\[
V_{\text{eff}} = \frac{\pi^2 \phi^4}{6} S(\lambda, e^2, L) \tag{2.1}
\]

where

\[
\phi^2 \equiv \phi_1^2 + \phi_2^2 \tag{2.2}
\]

\[
L \equiv \log(\phi^2/\mu^2). \tag{2.3}
\]

The statement that \(V_{\text{eff}}\) is independent of the renormalization mass scale \(\mu\) (i.e., that \(\mu \, dV_{\text{eff}}/d\mu = 0\)) implies that \(\lambda, e^2\) and \(\phi^2\) are all implicit functions of \(\mu\) such that

\[
\left\{ \frac{-2 + 2\gamma}{L} + \beta_\gamma \frac{\partial}{\partial e^2} + \beta_\lambda \frac{\partial}{\partial \lambda} + 4\gamma \right\} S(\lambda, e^2, L) = 0 \tag{2.4}
\]

where the chain rule coefficients in Eq. (2.4) are just the RG functions \[1\]

\[
\gamma \equiv \frac{\mu}{\phi_1} \frac{d\phi_1}{d\mu} \quad \left( \equiv \frac{\mu}{\phi_2} \frac{d\phi_2}{d\mu} \right) = \frac{\mu}{2\phi^2} \frac{d\phi^2}{d\mu} = \frac{3e^2}{16\pi^2} + \mathcal{O}(e^4, \lambda e^2, \lambda^2) \tag{2.5}
\]

\[
\beta_e \equiv \mu \frac{d e^2}{d\mu} = \frac{e^4}{24\pi^2} + \mathcal{O}(e^6, \lambda e^4, \lambda^2 e^2, \lambda^3) \tag{2.6}
\]

\[
\beta_\lambda \equiv \mu \frac{d\lambda}{d\mu} = \frac{5}{24\pi^2} \lambda^2 - \frac{3}{4\pi^2} \lambda e^2 + \frac{9}{4\pi^2} e^4 + \mathcal{O}(\lambda^3, e^2 \lambda^2, e^4 \lambda, e^6). \tag{2.7}
\]

In Eqs. (2.4), (2.5) and (2.6), we have listed only the one-loop contributions to \(\lambda, \beta_e\) and \(\beta_\lambda\). These contributions are sufficient in themselves to determine leading-logarithm contributions to the series \(S(\lambda, e^2, L)\) to all orders of perturbation theory. Such leading logarithm contributions to a given order necessarily involve a power of the logarithm \(L\) that is always one less than the aggregate power of the coupling constants \(e^2\) and \(\lambda\). The all-orders series of leading logarithm contributions may be represented in terms of the couplant parameters

\[
x(\mu) \equiv e^2/4\pi^2 \tag{2.8}
\]

\[
y(\mu) \equiv \lambda/4\pi^2 \tag{2.9}
\]

as follows:

\[
S_{LL} = \sum_{n=1}^{\infty} \left( R_{n,n-1} \ y^n L^{n-1} + \sum_{k=0}^{\infty} T_{n,k} \ x^n y^k L^{n+k-1} \right). \tag{2.10}
\]

The only \textit{ab initio} known coefficients of this series are \(R_{1,0} = 1\) and \(T_{1,0} = 0\), as required to obtain correspondence between Eq. (2.10) and the tree-order \(\lambda \phi^4/24\) contribution to \(V_{\text{eff}}\) (2.1):

\[
S_{LL} = \frac{\lambda}{4\pi^2} + \mathcal{O}\left(\lambda^2, e^4, \lambda e^2\right) = y + \mathcal{O}\left(y^2, x^2, x y\right); \tag{2.11}
\]

\textit{i.e.}, \(T_{1,0}\) must vanish, as there is no \(e^2 \phi^4\) tree level contribution to the potential. The other coefficients in Eq. (2.10) may be extracted by considering only those contributions to the RGE (2.1) which either lower the power of the
logarithm $L$ by one, or which raise the aggregate power of the couplings $x$ or $y$ by one. Such terms are entirely known from the one-loop RG functions (2.5), (2.6), and (2.7), and are seen to lead via Eqs. (2.8) and (2.9) to the following RGE for determining leading-logarithm coefficients:

$$
-2 \frac{\partial}{\partial L} + \left( \frac{5}{6} y^2 - 3xy + 9x^2 \right) \frac{\partial}{\partial y} + \frac{x^2}{6} \frac{\partial}{\partial x} + 3x \right) S_{LL}(x, y, L) = 0. \tag{2.12}
$$

Note that this equation is sufficient in itself to determine the one-loop effective potential. If we substitute the series (2.10) into Eq. (2.12), we find that the aggregate coefficient of $y^2$ vanishes provided

$$
-2R_{2,1} + \frac{5}{6} R_{1,0} = 0. \tag{2.13}
$$

Since $R_{1,0} = 1$, as argued above, we see that $R_{2,1} = 5/12$. The aggregate coefficient of $xy$ in Eq. (2.12) vanishes provided $T_{1,1} = 0$, which explains the absence of a $\lambda e^2 \log \phi^2$ “cross term” in the one-loop effective potential. Similarly, the aggregate coefficient of $x^2$ in Eq. (2.12) vanishes provided

$$
-2T_{2,0} + 9R_{1,0} + \frac{19}{6} T_{1,0} = 0. \tag{2.14}
$$

Since $T_{1,0} = 0$ and $R_{1,0} = 1$, we see that $T_{2,0} = 9/2$. Using the series (2.10) to obtain only the (one-loop) contributions linear in the (leading) logarithm $L$, we find that

$$
V_{LL} = \frac{\pi^2}{6} \phi^4 S_{LL} = \frac{\pi^2}{6} \phi^4 \left( R_{1,0}y + T_{1,0}x + R_{2,1}y^2 + T_{1,1}xy + T_{2,0}x^2 + O(L^2) \right)
$$

$$
= \frac{\pi^2}{6} \phi^4 \left( y + \frac{5}{12} y^2 + \frac{9}{2} x^2 + O(L^2) \right)
$$

$$
= \frac{\lambda^4}{24} \left( \frac{5\lambda^2}{1152\pi^2} + \frac{3\epsilon^2}{64\pi^2} \right) \left( \phi^4 \log \left( \frac{\phi^2}{\mu^2} \right) \right) + O \left( \log^2 \frac{\phi^2}{\mu^2} \right). \tag{2.15}
$$

The result (2.15) is, of course, the same as Coleman and Weinberg’s (11) direct one-loop calculation quoted in Eq. (1.3). The remaining term in Eq. (1.3) is just the finite $\frac{25}{6} \left( \frac{5\lambda^2}{1152\pi^2} + \frac{3\epsilon^2}{64\pi^2} \right) \phi^4$ counterterm required by the $d^4V/d\phi^4|_\mu = \lambda$ renormalization condition, as discussed in the previous section. Of course, any RG approach, such as that leading to Eq. (2.15), ultimately relies on Feynman diagrammatic calculations of the RG functions (2.5), (2.6) and (2.7). It is nevertheless reassuring that these one-loop RG functions lead, via RG methods, to the same one-loop effective potential as one obtains explicitly from Feynman diagrams with external scalar field legs.\footnote{An early analysis of radiative spontaneous symmetry breaking using the RG equation also appears in Ref. [10].}

However, the result (2.15) does not include all information available from the leading-logarithm RGE (2.12). If we substitute Eq. (2.10) into Eq. (2.12), we obtain recursion relations which determine all coefficients $R_{n,n-1}$ and $T_{n,k}$ in the series (2.10) for leading-logarithm contributions to $V_{eff}$. For arbitrary power $p \geq 2$, we find that the aggregate coefficient of $y^p L^{p-2}$ vanishes in Eq. (2.12) provided

$$
-2(p-1)R_{p,p-1} + \frac{5}{6} (p-1) R_{p-1,p-2} = 0. \tag{2.16}
$$

Since $R_{1,0} = 1$, we see from this constraint that $R_{p,p-1} = \left( \frac{5}{6} \right)^{p-1}$. Similarly, we find that the aggregate coefficients of $x^p L^{p-1}$ and $x^2 y^p L^p$ vanish provided

$$
-2pT_{1,p} + \frac{5}{6} (p-1) T_{1,p-1} - 3p R_{p,p-1} + 3R_{p,p-1} = 0, \quad p \geq 1 \tag{2.17}
$$

$$
-2(p+1)T_{2,p} + \frac{5}{6} (p-1) T_{2,p-1} - 3pT_{1,p} + 9(p+1) R_{p+1,p} + \frac{1}{6} T_{1,p} + 3T_{1,p} = 0, \quad p \geq 1. \tag{2.18}
$$

Since all coefficients $R_{p,p-1}$ are known from Eq. (2.16), Eq. (2.17) is sufficient to determine all coefficients $T_{1,k}$; note that the result $T_{1,1} = 0$ follows directly from Eq. (2.17) with $p = 1$. Similarly Eqs. (2.18) and (2.19) are sufficient to determine all coefficients $T_{2,k}$ in the series (2.10). Subsequent coefficients of terms degree-3-and-higher in $x$ are obtained by demanding that the aggregate coefficient of $x^p y^p L^{p-2}$ vanish:

$$
-2(p+n-1)T_{n,p} + \frac{5}{6} (p-1) T_{n,p-1} - 3pT_{n-1,p} + 9(p+1) T_{n-2,p+1} + \frac{n+17}{6} T_{n-1,p} = 0, \quad n \geq 3, \quad p \geq 1 \tag{2.19}
$$

$$
-2(n-1)T_{n,0} + 9T_{n-2,1} + \frac{n+17}{6} T_{n-1,0} = 0, \quad n > 2. \tag{2.20}
$$
Thus, one could in principle use the above set of recursion relations to determine the *entire* leading-logarithm series (2.10), as opposed to its one-loop projection (2.15). However, we find it most useful to restructure the series (2.10) into a series that is perturbative in the couplant $x = e^2/4\pi^2$ but which includes summation over all powers of $y = \lambda/4\pi^2$,

$$S_{LL} = yS_0(yL) + xS_1(yL) + x^2LS_2(yL) + \ldots = yS_0(yL) + \sum_{j=1}^{\infty} x^j L^{j-1} S_j(yL),$$

(2.21)

since $x$, the electromagnetic couplant $\alpha/\pi$, is anticipated to be perturbatively small. If we equate Eqs. (2.21) and (2.10), we find that

$$S_0(yL) = \sum_{n=1}^{\infty} R_{n,n-1}(yL)^{n-1}$$

(2.22)

$$S_j(yL) = \sum_{k=0}^{\infty} T_{j,k}(yL)^k.$$  

(2.23)

In the next section we will utilize the recursion relations (2.16) – (2.20) to obtain closed-form expressions for the summations $S_0, S_1, S_2$ and $S_3$. Although $S_k$ with $k > 3$ can also be determined from these recursion relations, we will be able to show (Section 4) that the couplant $x$ is constrained by minimization of $V_{eff}$ to be small, in which case contributions $x^k L^{k-1} S_k(yL)$ to Eq. (2.21) with $k \geq 3$ can be safely disregarded. Note also from the organization of the series (2.21) that no *a priori* assumptions are required concerning the magnitude of the couplant $y$, since all-orders $y$-dependence resides in the closed-form summations obtained for $S_j$ in the next section.

### 3 Summations of Leading Logarithms in MSED

The series (2.22) for $S_0(yL)$ is just a geometric series, since $R_{p+1,p}/R_{p,p-1} = 5/12$ by the recursion relation (2.16). Since $R_{1,0} = 1$, we find easily that

$$S_0(yL) = \frac{1}{1 - \frac{5}{12} yL} \equiv \frac{1}{w}.$$  

(3.1)

We will find it convenient to make use of the variable $w = 1 - \frac{5}{12} yL$ to parametrise subsequent summations, and will henceforth denote by $S_k[w]$ their functional dependence on this variable (i.e., $S_0[w] = 1/w$).

To find the series $S_1(u)$, where $u = yL$, we multiply each term of the recursion relation (2.17) by $u^{p-1}$ and then sum from $p = 1$ to $\infty$:

$$-2 \sum_{p=1}^{\infty} p T_{1,p} u^{p-1} + \frac{5}{6} \sum_{p=1}^{\infty} (p-1) T_{1,p-1} u^{p-1} - 3 \sum_{p=1}^{\infty} (p-1) R_{p,p-1} u^{p-1} = 0.$$  

(3.2)

We note from the expressions (2.22) and (2.23) that

$$S_0(u) = \sum_{p=1}^{\infty} R_{p,p-1} u^{p-1},$$

(3.3)

$$S_1(u) = \sum_{p=1}^{\infty} T_{1,p-1} u^{p-1}.$$  

(3.4)

Consequently Eq. (3.2) is just the first order differential equation

$$-2 \left(1 - \frac{5}{12} u \right) \frac{dS_1}{du} - 3u \frac{dS_0}{du} = 0.$$  

(3.5)

We change variables to $w = 1 - \frac{5}{12} u$ and, noting from Eq. (3.1) that $S_0 = 1/w$, we find that

$$\frac{dS_1}{dw} = \frac{18}{5} (w^{-3} - w^{-2}),$$

(3.6)
with an initial condition obtained from Eq. (3.4) in the $u \to 0$ limit:

$$\lim_{w \to 1} S_1[w] = \lim_{u \to 0} S_1(u) = T_{1,0} = 0. \quad (3.7)$$

The solution to this differential equation is

$$S_1[w] = -\frac{9}{5} \frac{(w - 1)^2}{w^2} \quad (3.8)$$

where $w = 1 - \frac{5}{12} y L$.

A differential equation for the series

$$S_2(u) = \sum_{k=0}^{\infty} T_{2,k} u^k \quad (3.9)$$

can be obtained by multiplying the recursion relation \((2.18)\) by $u^p$ and then summing from $p = 1$ to infinity:

$$-2w \frac{dS_2}{du} - 2(S_2 - T_{2,0}) = \frac{5}{6} u \frac{dS_2}{du} - 3u \frac{dS_1}{du} + 9w \frac{dS_0}{du} + 9(S_0 - 1) + \frac{19}{6} S_1 = 0. \quad (3.10)$$

The constant terms in Eq. \((3.10)\) cancel; $T_{2,0} = 9/2$, as obtained from Eq. \((2.20)\). If we make the change of variable $w = 1 - \frac{5}{12} u$, we find that

$$\frac{dS_2}{dw} + \frac{1}{w(w - 1)} S_2 = -\frac{3}{2w} \frac{dS_1}{dw} + \frac{19}{12 w(w - 1)} S_1 + \frac{9}{2w} \frac{dS_0}{dw} + \frac{9}{2w(w - 1)} S_0 \quad (3.11)$$

with initial condition

$$\lim_{w \to 1} S_2[w] = \lim_{u \to 0} S_2(u) = T_{2,0} = \frac{9}{2}. \quad (3.12)$$

Substituting the solutions \((3.1)\) and \((3.8)\) for $S_0$ and $S_1$ into the right hand side of Eq. \((3.1)\), one finds that

$$S_2[w] = -\frac{1}{20 w^3} \left[ -20 w^3 - 77 w^2 + 34 w - 27 \right] . \quad (3.13)$$

A similar analysis of the recursion relations \((2.18)\) and \((2.20)\) leads to the differential equations

$$\frac{dS_k}{dw} + \frac{k - 1}{w(w - 1)} S_k = -\frac{3}{2w} \frac{dS_{k-1}}{dw} + \frac{17 + k}{12 w(w - 1)} S_{k-1} - \frac{15}{8 w(w - 1)} \frac{dS_{k-2}}{dw} \equiv f_k[w] \quad (3.14)$$

where $w = 1 - 5u/12$ and where $S_k(u)$ is defined by Eq. \((2.22)\). The solution to Eq. \((3.13)\) is uniquely determined by the requirement that $S_k$ not be singular at $w = 1$, since $\lim_{w \to 1} S_k = \lim_{u \to 0} S_k = T_{k,0}$. Consequently we find from Eq. \((3.13)\) that

$$S_k = \frac{w^{k-1}}{(w - 1)^{k-1}} \int_1^w dr \frac{(r - 1)^{k-1}}{r^{k-1}} f_k[r], \quad (3.15)$$

where the function $f_k$ is defined to be the inhomogeneous driving term in Eq. \((3.14)\) obtained from knowledge of $S_{k-1}[w]$ and $S_{k-2}[w]$. One finds, for example, that

$$S_3[w] = \frac{1}{240 w^4} \left[ 580 w^4 + 760 w^3 - 323 w^2 + 126 w - 243 \right] \quad (3.16)$$

and that $\lim_{w \to 1} S_3 = T_{3,0} = \frac{15}{4}$, consistent with Eq. \((2.20)\) [note that $T_{1,1} = 0$ and $T_{2,0} = 9/2$].

### 4 Analysis of the Leading-Logarithm MSED Effective Potential

The RGE \((2.12)\) was shown in Section 2 to determine all coefficients of the leading-logarithm series $S_{LL}$ for the effective potential \((2.4)\) of MSED. Using the results of Section 3, this effective potential may be expressed as follows:

$$V_{eff}^{LL} = \frac{\pi^2 \phi^4}{6} \left[ yS_0[w] + \sum_{n=1}^{\infty} x^n L^{n-1} S_n[w] \right] + K\phi^4 \quad (4.1)$$
where the logarithm \( L \) and the couplings \( x \) and \( y \) \((w = 1 - 5yL/12)\) are respectively defined in Eqs. (2.23), (2.8) and (2.9), and where the series \( S_k[w] \) are given explicitly by Eqs. (3.11), (3.8), (3.13) and, for \( k \geq 3 \), by Eq. (3.15). Note that Eq. (4.1) includes a finite \( K\phi^4 \) counterterm. Such a counterterm is also present in the “unimproved” one-loop effective potential \((4.3)\), as discussed in Section 1, but the value of this counterterm will be shifted as a result of the leading logarithm contributions to Eq. (4.1) past one-loop order.

Let us first consider the leading three contributions to Eq. (4.1):

\[
V_{eff}^{LL} = \frac{\pi^2}{6} [yS_0[w] + xS_1[w] + x^2LS_2[w]] + K\phi^4 + O(x^3).
\]  

(4.2)

We show below that \( x \) is constrained by this potential to be small, providing an \textit{a posteriori} justification for truncation of the series past terms quadratic in \( x \). [The quadratic term in Eq. (4.2) contains the \((3e^4/64\pi^2)\phi^4L\) term occurring within the one-loop expression \((4.3)\).] As before, the finite \( \phi^4 \) counterterm is determined by application of the renormalization condition \((1.4)\) onto the effective potential \((4.2)\). We then find that

\[
\frac{6K}{\pi^2} = - \left( \frac{125y^2}{72} + \frac{75y^3}{4} \right) - \frac{2625y^3 + 1875y^4 + 625y^5}{1296}
\]

\[
+ \frac{x(175y^2 + 250y^3 + 125y^4)}{48} - x^2 \left( \frac{9450y + 10275y^2 + 5275y^3}{432} \right)
\]

(4.3)

Note that the degree two terms in Eq. (4.3) lead to precisely the same finite counterterm as in Eq. (1.3):

\[
K\phi^4 = - \frac{\pi^2}{6} \left( \frac{125y^2}{72} + \frac{75y^3}{4} \right) \phi^4 + \ldots = - \frac{25}{6} \left( \frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2} \right) \phi^4 + \ldots
\]

(4.4)

If we substitute Eq. (4.3) into Eq. (4.2), the minimization condition at \( \mu^2 = \langle \phi \rangle^2 \), \( \text{i.e., at } L = 0 \) becomes

\[
0 = \frac{dV_{eff}^{LL}}{d\phi} \bigg|_\mu = \pi^2 \left( \frac{1296y - 1980y^2 - 2625y^3 - 1875y^4 - 625y^5}{1944} \right)
\]

\[
+ x \left( \frac{175y^2 + 250y^3 + 125y^4}{72} \right) - x^2 \left( \frac{7128 + 9450y + 10275y^2 + 5275y^3}{648} \right)
\]

(4.5)

Eq. (4.5) is a quadratic equation in \( x \) whose solution yields \( x \) as a function of \( y \) \( \text{when } \mu = \langle \phi \rangle \). The \text{(positive)} solutions to this equation are plotted in Fig. 1. We note the following features:

1) For each value of \( x \), there are \textit{two} allowed values of \( y \). In other words, for a given value of \( e^2 \) there are both a “strong-\( \lambda \)” and “weak-\( \lambda \)” phase of the spontaneously broken theory.

2) The solution space of Eq. (4.5) places upper bounds on both \( x \) and \( y \), with a fairly small numerical bound on \( x \). Thus the spontaneously broken theory is truly perturbative in \( x \), \text{i.e., in } \( e^2 \), consistent with truncation of Eq. (4.1) past its degree-2 terms in \( x \).

3) When \( y \leq 0.1 \), well into the weak-\( \lambda \) phase, the solution curve in Fig. 1 reduces to \( y = 33x^2/2 \), consistent with the constraint \((1.6)\) for the “unimproved” one-loop effective potential.

The presence of a strong-\( \lambda \) phase in the radiatively broken theory suggests the possibility of a large scalar-field mass solution to the spontaneously broken theory. The only available scale within the spontaneously broken theory for assessing the scalar-field mass \( m_\phi \) is the gauge boson mass \( m_A = e\langle \phi \rangle \). Using the effective potential \((4.2)\) \text{[with} counterterm coefficient \( K \) given by Eq. (4.3)], we find using Eqs. (1.8) that

\[
\frac{m_\phi^2}{m_A^2} = \left( \frac{y}{2x} - \frac{27x}{4} \right) - \frac{1}{2592x} \left[ 1620y^2 + 2475y^3 + 1875y^4 + 625y^5 \right]
\]

\[- x(4455y^2 + 6750y^3 + 3375y^4) + x^2(26730y + 30825y^2 + 15825y^3) \]

(4.6)

where \( x \) is the positive solution to the quadratic equation \((4.5)\). In the weak-\( \lambda \) phase where \( y \approx 33x^2/2 \), the leading contribution to Eq. (4.6) is just

\[
\frac{m_\phi^2}{m_A^2} = \left( \frac{3x}{2} \right) + [O(x^3)],
\]

(4.7)

consistent with Eq. (1.9) for the unimproved one-loop effective potential. In Fig. 2 we plot the solution \((4.6)\) as a function of \( y \) \( \approx \lambda/4\pi^2 \). The plot shows progressive deviation of the ratio from its one-loop effective-potential value \((4.7)\) as \( y \) increases in magnitude.\(^2\) As \( y \) approaches its upper bound near 0.4, the ratio grows infinite, corresponding

\(^2\)Note that \( y = 33x^2/2 \) in the one-loop potential, in which case \( m_\phi^2/m_A^2 = \sqrt{3y/22} \) by Eq. (4.7).
Figure 1: The scalar self-interaction couplant $y$ as a double-valued function of the gauge couplant $x$ of $\mathbb{MSED}$. The solid curve is the solution to the quadratic equation (4.5). The dotted curve is obtained by incorporating all terms within $V_{eff}^{LL}$ that can contribute to this double-valued relation, as discussed in the text.

to the decoupled radiatively-broken massless scalar-field theory one would obtain in the limit $e \to 0$, $m_A = e \langle \phi \rangle \to 0$. The two-phase nature of the spontaneous symmetry breaking is illustrated in Fig. 3, where $x$ (instead of $y$) is used as the independent variable in plotting the mass ratio (4.6). The figure shows that for a given value of $x$ in the allowed domain of the constraint (4.5), there are two values of the mass ratio $m_\phi^2/m_A^2$, respectively corresponding to the two allowed values of $y$ (or of $\lambda$) evident in Fig. 1. The weak-$\lambda$ phase is seen to yield a mass ratio quite close to Coleman and Weinberg’s original one-loop prediction (1.9) for almost the entire allowed domain in $x$. However, the strong-$\lambda$ phase yields a scalar-boson mass $m_\phi > 0.45m_A$ that is comparable to or even larger [subject to subsequent-to-leading-log corrections] than the gauge boson mass.

Note that the results described above are not contingent upon (or an artifact of) the truncation of the series within Eq. (4.1) to terms of degree-2 or less in $x$; we have performed such a truncation to provide insight [via Eq. (4.5)] into how the double valued structure of $x(y)$ occurs. In Section 7 it is shown that only those series terms degree-4 and less in $L$ contribute to the information we extract from the effective potential in Figs. 1, 3 a consequence following from renormalization conditions [such as Eq. (1.4)] that involve $L = 0$ values of at most four derivatives of the effective potential. The only such terms omitted from Eq. (4.2) are

$$x^3L^2S_3 = \frac{15}{4}x^3L^2 - \frac{5}{6}x^3yL^3 - \frac{9415}{12}x^3y^2L^4 + O(L^5), \quad (4.8)$$

$$x^4L^3S_4 = 5x^4L^3 + \frac{2015}{3}x^4yL^4 + O(L^5), \quad (4.9)$$

$$x^5L^4S_5 = \frac{65}{48}x^5L^4 + O(L^5), \quad (4.10)$$

results which can be obtained via successive solutions of Eq. (4.14). The effect of including these new terms within $V_{eff}^{LL}$ is displayed in the additional dashed curves displayed in Figs. 1 and 3. Specifically, we find from Fig. 1 that the upper bound on the couplant $x$ decreases from 0.078 to 0.073, and that the mass ratio curve of Fig. 3 is “pulled in” accordingly. However, the two-phase structure described above remains evident in these dashed curves, which are now inclusive of all contributing leading-logarithm effects.
Figure 2: The ratio of the squares of the scalar-field ($\phi$) and gauge-field ($A$) masses for MSED as a function of the scalar couplant $y$. The solid curve is obtained from Eq. (4.6), with $x(y)$ given by the solution to Eq. (4.5). The dotted curve displays corresponding results from the Coleman-Weinberg relations (1.9) and (1.6).

5 The RGE for Radiative Electroweak Symmetry Breaking

In the absence of an explicit scalar-field mass term, the one-loop (1L) effective potential for $SU(2) \times U(1)$ gauge theory is given by [1, 11]

$$V_{\text{eff}}^{(1L)} = \frac{\lambda \phi^4}{4} + \phi^4 \left[ \frac{12 \lambda^2 - 3 g_t^2}{64 \pi^2} + \frac{3(3g_t^2 + 2g_2^2 + g_4^4)}{1024 \pi^2} \right] \left( \log \frac{\phi^2}{\mu^2} - \frac{25}{6} \right) \tag{5.1}$$

where the $-25/6$ constant is chosen to ensure that

$$\frac{d^2 V_{\text{eff}}^{(1L)}}{d\phi^2} \Bigg|_{\mu} = \frac{d^2 V_{\text{tree}}}{d\phi^2} = 6\lambda. \tag{5.2}$$

There are four distinct coupling constants appearing in Eq. (5.1), the $SU(2)$ coupling constant $g_2$, the $U(1)$ coupling constant $g_1$, and the quartic scalar-field self-interaction coupling constant $\lambda$. Three of these are known in terms of the electromagnetic coupling $e$, the weak angle $\theta_w$, and the masses of the $t$-quark and $W$-boson:

$$g_2^2 \equiv \frac{e^2}{\sin^2 \theta_w} \approx 0.436 \tag{5.3}$$
$$g_1^2 \equiv \frac{e^2}{\cos^2 \theta_w} \approx 0.127 \tag{5.4}$$
$$g_t = m_t \sqrt{2}/\langle \phi \rangle = e \frac{m_t}{(\sqrt{2} m_W \sin \theta_w)} \approx 1.00 \tag{5.5}$$

Prior to the discovery of the $t$-quark, contributions of known-quark Yukawa couplings $g_q \leq m_q/175$ GeV to Eq. (5.1) could be safely ignored relative to the contributions of gauge coupling constants, permitting an analysis very similar to that for massless scalar electrodynamics. By assuming $\lambda$ to be order $g_2^4$, one then obtains a scalar-boson mass ($m_\phi$) whose magnitude is perturbatively suppressed [11]:

$$\frac{m_\phi^2}{m_W^2} = \frac{4}{g_2^2 \langle \phi \rangle^2} \frac{d^2 V_{\text{eff}}^{(1L)}}{d\phi^2} \Bigg|_{\phi = 0} = \frac{3\alpha(2 + \sec^4 \theta_w)}{8\pi \sin^2 \theta_w}, \tag{5.6}$$

a ratio corresponding to a scalar field mass of order 10 GeV. As in scalar electrodynamics, such an approach is self-consistent; optimization of the potential (5.1) leads to an $\mathcal{O}(g_2^2)$ value for $\lambda$, so corrections past one-loop order are seen not to alter appreciably Eq. (5.6).
Figure 3: The dependence of the ratio of squares of the scalar field and gauge-field masses for MSED on the gauge couplant x. The solid curve is obtained from Eq. (4.6) with y(x) given implicitly by Eq. (4.5). The dotted curve, as in Figure 1, incorporates all additional contributing terms to $V_{\text{LL}}^{\text{eff}}$.

Of course, with the discovery of the t-quark, the neglect of Yukawa couplings is no longer justifiable. Indeed, the t-quark’s Yukawa coupling-constant contributions to Eq. (5.1) are large compared to those of SU(2) × U(1) gauge coupling constants. If $g_t = 1$, one finds that the $V_{\text{eff}}^{(1L)}((\phi)) = 0$ minimization condition implies that $\lambda \sim 3.6 \frac{g_t^2}{\lambda} \text{ and } m_\phi \sim 350 \text{ GeV}$. This result is questionable, however, because the value for $\lambda$ may be too large to justify the neglect of higher-loop contributions.

This failure to obtain a clear prediction for a conformally-invariant electro-weak-symmetry potential suggests the utility of extending the one-loop effective potential (5.1) to include summation over all leading logarithms, as already considered in massless scalar electrodynamics. To obtain such a sum, we first note that the RGE for the potential may be expressed as follows:

$$0 = \mu \frac{d}{d\mu} V[\lambda(\mu), g_t(\mu), g_3(\mu), \phi^2(\mu), \mu]$$

$$= \left( \mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_t \frac{\partial}{\partial g_t} + \beta_3 \frac{\partial}{\partial g_3} - 2 \gamma \phi^2 \frac{\partial}{\partial \phi^2} \right) V(\lambda, g_t, g_3, \phi^2, \mu),$$

(5.7)

where, to one-loop order in $\lambda, g_t, \text{ and the QCD coupling-constant } g_3$ [11, 12],

$$\beta_\lambda \equiv \mu \frac{d\lambda}{d\mu} = \frac{48\lambda g_t^2}{64\pi^2} + \frac{12 \lambda^2}{8\pi^2} - \frac{3 g_t^4}{8\pi^2} + \mathcal{O}(\lambda^k g_3^{6-2k})$$

(5.8)

$$\beta_t \equiv \mu \frac{dg_t}{d\mu} = -\frac{7 g_t^3}{16\pi^2} + \mathcal{O}(g_t^5 g_3^{5-2k})$$

(5.9)

$$\beta_3 \equiv \mu \frac{dg_3}{d\mu} = -\frac{7 g_3^3}{16\pi^2} + \mathcal{O}(g_3^5)$$

(5.10)

$$\gamma \equiv -\mu \frac{d\phi}{d\mu} = \frac{3 g_t^2}{16\pi^2} + \mathcal{O}(\lambda^k g_3^{4-2k}).$$

(5.11)

Since the contribution of the Yukawa coupling-constant $g_t$ to Eq. (5.8) dominates the contributions of the much smaller $SU(2) \times U(1)$ gauge coupling-constants [as evident in Eqs. (5.3)–(5.5)], we work in the approximation in which $g$ and $g'$ are equal to zero. However, the contribution of the QCD coupling constant to the Yukawa $\beta$-function (5.9) cannot be neglected, since $g_3^2$ is the largest known coupling constant contributing to the effective potential [$g_3^2 \approx 4\pi\alpha_s(M_z) \approx 1.50$].

To assess the full leading-logarithm (LL) contribution to the electroweak effective potential in the absence of an
explicit scalar-field mass term, we utilize the couplant parameters $x$, $y$, $z$ defined at $\mu = \langle \phi \rangle = 2^{-1/4}G_F^{-1/2} \equiv v$:

\begin{align}
  x &\equiv g_t^2(v)/4\pi^2 \ (\equiv 0.0253) \\
  y &\equiv \lambda/4\pi^2 \\
  z &\equiv g_3^2(v)/4\pi^2 \ (\equiv 0.0329)
\end{align}

(5.12) (5.13) (5.14)

with corresponding one-loop RG-functions derived from Eqs. (5.8)–(5.11):

\begin{align}
  \frac{dx}{d\mu} &= 9x^2 - 4xz \\
  \frac{dy}{d\mu} &= 6y^2 + 3yx - \frac{3}{2}x^2 \\
  \frac{dz}{d\mu} &= -\frac{7}{2}z^2 \\
  \gamma &= \frac{3x}{4}
\end{align}

(5.15) (5.16) (5.17) (5.18)

[The value (5.14) is obtained from $\alpha_s(M_z) \approx 0.12$ via evolution of $\alpha_s$ from $M_z$ to $v$.] As in Section 2, we write the summation-of-leading-logarithms effective potential for radiative broken electroweak symmetry (RBEWS) in the form

\[
V_{LL} = \pi^2 \phi^4 S_{LL} = \pi^2 \phi^4 \left\{ \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\infty} y^k \sum_{\ell=0}^{\infty} z^\ell C_{n,k,\ell} L^{n+k+\ell-1} \right\}, \quad (C_{0,0,0} = 0)
\]

(5.19)

where the series $S_{LL}$ is the sum of all contributions involving a power of the logarithm $L \equiv \log(\phi^2/\mu^2)$ that is only one degree lower than the aggregate power of the couplants $\{x, y, z\}$. We keep only those terms in the RGE (5.7) that either lower the power of $L$ or raise the aggregate power of couplants by one,

\[
\left[ -2 \frac{\partial}{\partial L} + \left( \frac{9}{4} x^2 - 4xz \right) \frac{\partial}{\partial x} + \left( 6y^2 + 3yx - \frac{3}{2}x^2 \right) \frac{\partial}{\partial y} - \frac{7}{2}z^2 \frac{\partial}{\partial z} - 3x \right] S_{LL}(x, y, z, L) = 0,
\]

(5.20)

since such terms (which arise entirely from one-loop RG functions) are sufficient in themselves to determine all coefficients $C_{n,k,\ell}$ within $S_{LL}$. Note that the leading contributions to $S_{LL}$ [Eq. (5.19)] are

\[
S_{LL} = (C_{1,0,0} x + C_{0,1,0} y + C_{0,0,1} z) + (C_{0,2,0} y^2 + C_{2,0,0} x^2 + C_{0,0,2} z^2 + C_{1,1,0} xy + C_{1,0,1} xz + C_{0,1,1} yz) L + \ldots
\]

(5.21)

The leading coefficients $C_{1,0,0} = 0$, $C_{0,1,0} = 1$, $C_{0,0,1} = 0$ are known from the $\lambda \phi^4/4$ tree potential. If we substitute Eq. (5.21) with these values into Eq. (5.20), we find that the aggregate term independent of $L$ is just

\[
-2 \left( C_{0,2,0} y^2 + C_{2,0,0} x^2 + C_{0,0,2} z^2 + C_{1,1,0} xy + C_{1,0,1} xz + C_{0,1,1} yz \right) + 6y^2 - \frac{3}{2}x^2 = 0,
\]

(5.22)

in which case $C_{0,2,0} = 3$, $C_{2,0,0} = -\frac{3}{4}$, and the remaining degree-2 coefficients within Eq. (5.20) are zero:

\[
S_{LL} = y + 3y^2 L - \frac{3}{4} x^2 L + \ldots = \frac{\lambda}{4\pi^2} + \left( \frac{3\lambda^2}{16\pi^4} - \frac{3g_t^4}{64\pi^4} \right) \log \left( \frac{\phi^2}{\mu^2} \right) + \ldots
\]

(5.23)

The one-loop $O(\lambda^2, g_t^4)$ diagrammatic contributions to the effective potential (5.10) for RBEWS are easily obtained upon substitution of Eq. (5.22) into Eq. (5.19); RG-invariance and the one-loop RG-functions (5.8)–(5.11) are sufficient in themselves to determine $O(\lambda^2, g_t^4)$ contributions to the one-loop effective potential (5.10) obtained directly from diagrams in Ref. [1].

### 6 Summations of Leading Logarithms in RBEWS

The leading-logarithm summation $S_{LL}$ defined by Eq. (5.14) may be expressed as a series expansion in the Yukawa couplant $x$ [Eq. (5.12)]:

\[
S_{LL} = yF_0(w, \zeta) + \sum_{n=1}^{\infty} x^n L^{-1} F_n(w, \zeta),
\]

(6.1)
where we find it convenient to utilize the new variables

$$\zeta \equiv zL, \quad w \equiv 1 - 3yL.$$ \hspace{1cm} (6.2)

The coefficient functions $F_k$ appearing in Eq. (6.1) are the $O(x^k)$ contributions to the leading log series $S_{LL}$. Comparison to Eq. (5.19) shows these functions to be themselves summations over the new variables $\zeta$ and $w$:

$$F_n(w, \zeta) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} C_{n,\ell,k} \left( \frac{1 - w}{3} \right)^{\ell} \zeta^k.$$ \hspace{1cm} (6.3)

Upon substituting Eq. (6.1) into the RGE (5.20), we obtain the following recursive set of partial differential equations for $F_k(w, \zeta)$:

$$\zeta \left( 1 + \frac{7}{4} \zeta \right) \frac{\partial}{\partial \zeta} F_0(w, \zeta) = (1 - w) \left[ \frac{w}{\partial w} + 1 \right] F_0(w, \zeta),$$ \hspace{1cm} (6.4)

$$\zeta \left( 1 + \frac{7}{4} \zeta \right) \frac{\partial}{\partial \zeta} + 2\zeta + (w - 1)w \frac{\partial}{\partial w} F_1(w, \zeta) = \frac{(w - 1)^2}{2} \frac{\partial}{\partial w} F_0(w, \zeta),$$ \hspace{1cm} (6.5)

$$\zeta \left( 1 + \frac{7}{4} \zeta \right) \frac{\partial}{\partial \zeta} + (w - 1)w \frac{\partial}{\partial w} + (1 + 4\zeta) F_2(w, \zeta)$$

$$= \left[ \frac{3}{2} \frac{w - 1}{\partial w} - \frac{3}{8} \right] F_1(w, \zeta) - \frac{3}{4} \left[ (w - 1) \frac{\partial}{\partial w} + 1 \right] F_0(w, \zeta),$$ \hspace{1cm} (6.6)

$$\zeta \left( 1 + \frac{7}{4} \zeta \right) \frac{\partial}{\partial \zeta} + (w - 1)w \frac{\partial}{\partial w} + (k - 1 + 2k\zeta) F_k(w, \zeta)$$

$$= \left[ \frac{3(3k - 7)}{8} + \frac{3}{2} (w - 1) \frac{\partial}{\partial w} \right] F_{k-1}(w, \zeta) + \frac{9}{2} \frac{\partial F_{k-2}}{\partial w}(w, \zeta), \quad (k \geq 3).$$ \hspace{1cm} (6.7)

The functions $F_k(w, \zeta)$ are themselves summations of all leading logarithms analogous to the functions $S_k[w]$ of Section 2. Their successive solutions are obtained below.

### 6.1 Solution for $F_0(w, \zeta)$

Consider first Eq. (6.4) when $\zeta$ is fixed at zero:

$$w \frac{d}{dw} F_0(w, 0) + F_0(w, 0) = 0$$ \hspace{1cm} (6.8)

We know that the lead term in the series $S_{LL}$ is $y \left[ C_{0,1,0} = 1 \right.$ and $C_{1,0,0} = C_{0,0,1} = 0$ in Eq. (5.11), in which case we see from Eqs. (6.1) and (6.2) that $F_0(1, 0) = 1$ to ensure that $S_{LL} \rightarrow y$. With this initial condition, the solution to Eq. (6.8) is $F(w, 0) = 1/w$. Now suppose we substitute $F_0(w, \zeta) = G(\zeta)K(w)$ into Eq. (6.4). This separation of variables implies that

$$\frac{\zeta \left( 1 + \frac{7}{4} \zeta \right) G'(\zeta)}{G(\zeta)} = (1 - w) \left[ \frac{wK'(w)}{K(w)} + 1 \right]$$ \hspace{1cm} (6.9)

We see that when $w = 1$, $G'(\zeta) = 0$, in which case $G$ is constant. Since $G(\zeta)$ is independent of $w \left[ i.e., \right.$ since the left-hand side of Eq. (6.9) is independent of $w], G$ is constant and

$$F_0(w, \zeta) = F_0(w, 0) = 1/w.$$ \hspace{1cm} (6.10)

Indeed this result is not unexpected. $F_0$ is the sum of all leading logarithm contributions to the effective potential that do not involve the Yukawa coupling (i.e. the couplant $x$). Since scalar fields do not couple directly to $SU(3)$ gluons, the only graphs with external scalar field legs that contain the strong interaction coupling constant necessarily contain internal quark lines, which can then emit/absorb virtual gluons, and such quark lines cannot occur unless a Yukawa interaction is present to couple scalar field lines to quark-antiquark pairs. For the case where the Yukawa couplant $x$ does not occur, namely, the coefficient $F_0(w, \zeta)$ in the series $S_{LL}$, there are no $SU(3)$ fermions coupled to the $F_0$ subset of leading log graphs, which necessarily implies that $F_0$ is independent of the $SU(3)$ couplant $z$. Since $\zeta \equiv zL$ is the source of any $z$ dependence within $F_0(w, \zeta), F_0(w, \zeta)$ clearly must be independent of $\zeta$, as is evident in Eq. (6.10).
6.2 Solution for $F_1(w, \zeta)$

If we substitute the solution (6.10) for $F_0(w, \zeta)$ into Eq. (6.5), we find the right-hand side of this equation is just equal to $\frac{1}{4} [(w-1)/w]^2$. Since $[(w-1)/w]^2$ is an eigenstate of the operator $(w-1)w(\partial/\partial w)$ appearing on the left-hand side of Eq. (6.5), one can choose

$$F_1(w, \zeta) = \frac{1}{4} M(\zeta) [(w-1)/w]^2$$

(6.11)

and find that $[(w-1)/w]^2$ factors out of Eq. (6.5) to yield the first-order ordinary differential equation

$$\left(1 + \frac{7}{4} \zeta\right) M'(\zeta) + 2 \left(\frac{\zeta+1}{\zeta} M(\zeta) = \frac{2}{\zeta}\right.$$

(6.12)

The integrating factor for this equation,

$$g(\zeta) = \exp \left[\frac{8}{7} \int \frac{(\zeta+1)}{\zeta(\zeta+4/7)} d\zeta\right] = \zeta^2 \left(\zeta + \frac{4}{7}\right)^{-6/7},$$

(6.13)

vanishes at $\zeta = 0$. Thus, the requirement that $M(\zeta)$ be singular at $\zeta = 0$ [i.e., not be singular in the limit the QCD couplant vanishes] uniquely specifies the solution

$$M(\zeta) = \frac{8}{7g(\zeta)} \int_0^\zeta dr g(r) r^{-1} (r + 4/7)^{-1} = \frac{8}{\zeta} + \frac{16}{3\zeta^2} \left[1 - \left(1 + \frac{7}{4} \frac{\zeta}{1} \right)^{6/7}\right] = 1 - \frac{2}{3} \zeta + \frac{5}{8} \zeta^2 + \ldots $$

(6.14)

$F_1(w, \zeta)$ is then obtained by explicit substitution of the final line of Eq. (6.14) into Eq. (6.11).

6.3 Solution for $F_2(w, \zeta)$

To obtain a solution to Eq. (6.6), we wish to exploit the identity

$$w(w-1) \frac{\partial}{\partial w} \left[ H(\zeta) \left(\frac{w-1}{w}\right)^k \right] = k H(\zeta) \left(\frac{w-1}{w}\right)^k$$

(6.15)

by expressing the $w$-dependence of the right-hand side of Eq. (6.6) as a series in the form $\sum_{k=0}^{\infty} A_k(\zeta) [(w-1)/w]^k$. From the solution (6.10) for $F_0$, we see that

$$\left[\frac{w-1}{w} + 1\right] \frac{1}{w} = 1 - 2 \left(\frac{w-1}{w}\right) + \left(\frac{w-1}{w}\right)^2.$$  

(6.16)

Moreover, we see from Eq. (6.11) that

$$(w-1) \frac{\partial}{\partial w} F_1(w, \zeta) = \frac{1}{2} M(\zeta) \left[ \left(\frac{w-1}{w}\right)^3 - \left(\frac{w-1}{w}\right)^3\right],$$

(6.17)

in which case the right-hand side of Eq. (6.6) may be expressed as

$$-\frac{3}{4} + \frac{3}{4} \left(\frac{w-1}{w}\right) + \left(\frac{21}{32} M(\zeta) - \frac{3}{4} \left(\frac{w-1}{w}\right)^2 - \frac{3}{4} M(\zeta) \frac{w-1}{w}\right)^3.$$  

(6.18)

To solve for $F_2$, we utilize the property (6.15) within Eq. (6.6) by writing

$$F_2(w, \zeta) = H_0(\zeta) + H_1(\zeta) \left(\frac{w-1}{w}\right) + H_2(\zeta) \left(\frac{w-1}{w}\right)^2 + H_3(\zeta) \left(\frac{w-1}{w}\right)^3.$$  

(6.19)

We substitute Eq. (6.19) into Eq. (6.6) and equate powers of $[(w-1)/w]^k$ to find that

$$\zeta \left(1 + \frac{7}{4} \zeta\right) H_0'(\zeta) + (1 + 4\zeta) H_0(\zeta) = -3/4,$$  

(6.20)

$$\zeta \left(1 + \frac{7}{4} \zeta\right) H_0'(\zeta) + (2 + 4\zeta) H_1(\zeta) = 3/2,$$  

(6.21)

$$\zeta \left(1 + \frac{7}{4} \zeta\right) H_2'(\zeta) + (3 + 4\zeta) H_2(\zeta) = \frac{21}{32} M(\zeta) - \frac{3}{4},$$  

(6.22)

$$\zeta \left(1 + \frac{7}{4} \zeta\right) H_3'(\zeta) + (4 + 4\zeta) H_3(\zeta) = -\frac{3}{4} M(\zeta),$$  

(6.23)
where $M(\zeta)$ is given by Eq. (6.13). The requirement that \( \{H_0(\zeta), H_1(\zeta), H_2(\zeta), H_3(\zeta)\} \) not be singular in the $\zeta = 0$ limit in which QCD is turned off uniquely specifies the following solutions:

\[
H_0(\zeta) = -\frac{3}{7\zeta} (\zeta + \frac{4}{7})^{9/7} \int_0^\zeta \left( r + \frac{4}{7} \right)^{2/7} dr = \frac{1 + \frac{2\zeta}{7}}{3\zeta} - 1 = -\frac{3}{4} + \frac{3}{2} \zeta - \frac{23}{8} \zeta^2 + \ldots, \tag{6.24}
\]

\[
H_1(\zeta) = \frac{6}{7\zeta^2} (\zeta + \frac{4}{7})^{2/7} \int_0^\zeta dr \left( r + \frac{4}{7} \right)^{-5/7} = \frac{2\zeta - 4 \left[ 1 - \left( 1 + \frac{7\zeta}{4} \right)^{-2/7} \right]}{3\zeta^2} = \frac{3}{4} - \zeta + \frac{23}{16} \zeta^2 + \ldots, \tag{6.25}
\]

\[
H_2(\zeta) = \frac{8(\zeta + 4/7)^{5/7}}{7\zeta^3} \int_0^\zeta dr \left( r + \frac{4}{7} \right)^{2/7} \left[ \frac{21}{32} M(r) - \frac{3}{4} \right] = \frac{1}{\zeta^3} \left[ \frac{20}{3} + \frac{71}{6} \zeta - \frac{\zeta^2}{3} + \frac{22}{3} \left( 1 + \frac{7\zeta}{4} \right)^{5/7} - 14 \left( 1 + \frac{7\zeta}{4} \right)^{6/7} \right] = -\frac{1}{32} - \frac{5}{64} \zeta + \frac{11}{64} \zeta^2 + \ldots, \tag{6.26}
\]

\[
H_3(\zeta) = \frac{4(\zeta + 4/7)^{12/7}}{7\zeta^4} \int_0^\zeta dr \left( r + \frac{4}{7} \right)^{19/7} \left( -\frac{3M(r)}{4} \right) = \frac{1}{\zeta^4} \left[ -\frac{16}{3} - \frac{16\zeta - 12\zeta^2 + \frac{32}{21} \left( 1 + \frac{7\zeta}{4} \right)^{6/7}}{3} - \frac{16}{3} \left( 1 + \frac{7\zeta}{4} \right)^{12/7} + \frac{64}{7} \left( 1 + \frac{7\zeta}{4} \right)^{13/7} \right] \tag{6.27}
\]

\[
= -\frac{3}{16} + \frac{1}{4} \zeta - \frac{61}{192} \zeta^2 + \ldots.
\]

Thus \( F_2(w, \zeta) \) is obtained explicitly by substitution of Eqs. (6.24) - (6.27) into Eq. (6.19).

### 6.4 \( \mathcal{O}(x^3) \) Contribution to \( S_{LL} \)

We now consider the \( k = 3 \) case of Eq. (6.27). We substitute into this equation’s right-hand side the solutions (6.11) and (6.19) for \( F_1(w, \zeta) \) and \( F_2(w, \zeta) \), and we then expand this side in terms of powers of \([ (w - 1)/w ]^k \) to obtain the partial differential equation

\[
\left[ \zeta \left( 1 + \frac{7\zeta}{4} \right) \frac{\partial}{\partial \zeta} + (w - 1)w \frac{\partial}{\partial w} + 2(1 + 3\zeta) \right] F_3(w, \zeta)
\]

\[
= \frac{3}{4} H_0(\zeta) + \frac{9}{8} \left( 2H_1(\zeta) + M(\zeta) \right) \left[ \frac{w - 1}{w} \right] + \frac{3}{4} \left( 5H_2(\zeta) - 2H_1(\zeta) - 3M(\zeta) \right) \left[ \frac{w - 1}{w} \right]^2 \tag{6.28}
\]

\[
+ \frac{3}{8} \left( 14H_3(\zeta) - 8H_2(\zeta) + 3M(\zeta) \right) \left[ \frac{w - 1}{w} \right]^3 - \frac{9}{2} H_3(\zeta) \left[ \frac{w - 1}{w} \right]^4.
\]

As before, we express \( F_3 \) in powers of \([ (w - 1)/w ] \),

\[
F_3(w, \zeta) = \sum_{k=0}^4 N_k(\zeta) \left[ \frac{w - 1}{w} \right]^k, \tag{6.29}
\]

to obtain the following first-order differential equations for \( N_k(\zeta) \):

\[
\zeta \left( 1 + \frac{7\zeta}{4} \right) N_0(\zeta) + 2(1 + 3\zeta) N_0(\zeta) = \frac{3}{4} H_0(\zeta), \tag{6.30}
\]

\[
\zeta \left( 1 + \frac{7\zeta}{4} \right) N_1(\zeta) + 3(1 + 2\zeta) N_1(\zeta) = \frac{9}{4} H_1(\zeta) + \frac{9}{8} M(\zeta), \tag{6.31}
\]

\[
\zeta \left( 1 + \frac{7\zeta}{4} \right) N_2(\zeta) + 2(2 + 3\zeta) N_2(\zeta) = \frac{3}{4} \left[ 5H_2(\zeta) - 2H_1(\zeta) - 3M(\zeta) \right], \tag{6.32}
\]

\[
\zeta \left( 1 + \frac{7\zeta}{4} \right) N_3(\zeta) + (5 + 6\zeta) N_3(\zeta) = \frac{3}{8} \left[ 14H_3(\zeta) - 8H_2(\zeta) + 3M(\zeta) \right], \tag{6.33}
\]

\[
\zeta \left( 1 + \frac{7\zeta}{4} \right) N_4(\zeta) + 6(1 + \zeta) N_4(\zeta) = -\frac{9}{2} H_3(\zeta). \tag{6.34}
\]
These can all be solved exactly using the requirement that $N_k(\zeta)$ is nonsingular at $\zeta = 0$, and their solutions are tabulated in Appendix C. However, our analysis of the effective potential in the next section will prove to be sensitive only up to $O(L^4)$ terms in the leading logarithm series $S_{LL}$ [Eqs. (5.10) or (5.1)]. Thus it is sufficient for us here to generate series solutions for $F_3$ such that the term $x^3L^2F_3(w, \zeta)$ within $S_{LL}$ is specified up to (and including) terms of order $L^4$. Since $\zeta = zL$, we thus need to know $N_0(\zeta)$ only up to terms quadratic in $\zeta$. Moreover, since $w - 1 = -3yL$, we see from Eq. (6.29) that we need to know $N_1(\zeta)$ only to terms linear in $\zeta$, and $N_2(\zeta)$ only to its constant term $N_2(0)$. The functions $N_3(\zeta)$ and $N_4(\zeta)$ will not participate at all in the analysis of the next section, since they correspond to contributions of order $L^5$ and higher to the series $S_{LL}$.

From Eqs. (C.32) and (C.33) of Appendix C, we find that

\begin{equation}
N_0(\zeta) = -\frac{9}{32} + \frac{15}{16} \zeta - \frac{603}{256} \zeta^2 + \ldots
\end{equation}

\begin{equation}
N_1(\zeta) = \frac{15}{16} - \frac{69}{32} \zeta + \ldots
\end{equation}

\begin{equation}
N_2(\zeta) = -\frac{447}{512} + O(\zeta)
\end{equation}

We substitute these results into Eq. (6.29), and make use of Eq. (6.2) to express $F_3$’s contribution to the leading-logarithm series $S_{LL}$ in terms of the couplants $x$, $y$, and $z$:

\begin{equation}
x^3L^2F_3 = \left( -\frac{9x^3}{32} \right) L^2 + \frac{15x^3}{16} (z - 3y)L^3 + \left( \frac{x^3}{32} \left[ 207yz - \frac{603}{8} z^2 - \frac{8343}{16} y^2 \right] \right) L^4 + O(L^5).
\end{equation}

### 6.5 $O(x^4)$ and $O(x^5)$ Contributions to $S_{LL}$

By substituting expressions (6.29) and (6.1) for $F_3$ and $F_2$ into the $k = 4$ version of Eq. (6.7), we find that

\[
\left[ \zeta \left( 1 + \frac{7}{4} \zeta \right) \frac{\partial}{\partial \zeta} + (3 + 8 \zeta) + w(w - 1) \frac{\partial}{\partial w} \right] F_4(w, \zeta) = \left[ \frac{15}{8} N_0(\zeta) + \frac{9}{4} H_1(\zeta) \right] + \left[ \frac{w - 1}{w} \right] \left[ \frac{27}{8} N_1(\zeta) - \frac{9}{2} H_1(\zeta) + \frac{9}{2} H_2(\zeta) \right] \\
+ \left[ \frac{w - 1}{w} \right]^2 \left[ \frac{39}{8} N_2(\zeta) - \frac{3}{2} N_1(\zeta) + \frac{9}{4} H_1(\zeta) - 9H_2(\zeta) + \frac{27}{4} H_3(\zeta) \right] \\
+ \left[ \frac{w - 1}{w} \right]^3 \left[ \frac{51}{8} N_3(\zeta) - 3N_2(\zeta) + \frac{9}{2} H_2(\zeta) - \frac{27}{2} H_3(\zeta) \right] \\
+ \left[ \frac{w - 1}{w} \right]^4 \left[ \frac{63}{8} N_4(\zeta) - \frac{9}{2} N_3(\zeta) + \frac{27}{4} H_3(\zeta) \right] + \left[ \frac{w - 1}{w} \right]^5 \left[ -6N_4(\zeta) \right].
\]

As before, we define

\begin{equation}
F_4(w, \zeta) = \sum_{k=0}^{5} P_k(\zeta) \left( \frac{w - 1}{w} \right)^k
\end{equation}

so as to generate first order differential equations for $P_0, P_1, \ldots, P_5$:

\begin{equation}
\zeta \left( 1 + \frac{7}{4} \zeta \right) P'_0(\zeta) + (3 + 8 \zeta) P_0(\zeta) = \frac{15}{8} N_0(\zeta) + \frac{9}{4} H_1(\zeta),
\end{equation}

\begin{equation}
\zeta \left( 1 + \frac{7}{4} \zeta \right) P'_1(\zeta) + 4(1 + 2 \zeta) P_1(\zeta) = \frac{27}{8} N_1(\zeta) - \frac{9}{2} H_1(\zeta) + \frac{9}{2} H_2(\zeta),
\end{equation}

\textit{etc.} In Appendix C we list solutions for $\{N_0(\zeta), \ldots, N_4(\zeta)\}$ obtained by solving Eqs. (6.30) – (6.34) with the requirement that $N_k(\zeta)$ be finite at $\zeta = 0$. Consequently, one could solve differential equations such as (6.41) and (6.42) for $P_k(\zeta)$ by imposing a similar requirement of finiteness at $\zeta = 0$. However, as noted above, our extraction of a scalar-field mass presented in the next section is sensitive only to terms up to degree-4 in $L$ within the series (6.1) or (5.1) for $S_{LL}$; the analysis will be insensitive to terms of $O(L^5)$. Since $[(w - 1)/w] = -3yL + O(L^2)$ and $\zeta = zL$, we see that the only $O(x^4)$ contributions $x^4L^2F_3(w, \zeta)$ to $S_{LL}$ relevant to the analysis in the section that
follows arise from knowing \( P_0(\zeta) \) to its term linear in \( \zeta \), and from knowing \( P_1(\zeta) \)'s constant lead term \( P_1(0) \). By substituting the final series in Eq. (6.25) for \( H_1(\zeta) \) and the series \( (6.35) \) for \( N_0(\zeta) \) into Eq. (6.41), we find that

\[
P_0(\zeta) = \frac{99}{256} - \frac{459\zeta}{512} + \mathcal{O}(\zeta^2).
\]

Similarly, we find from the lead term of the final series contributions within Eqs. (6.25), (6.26) and (6.36) to the right-hand side of Eq. (6.42) that

\[
P_1(\zeta) = -\frac{45}{512} + \mathcal{O}(\zeta).
\]

We can substitute these results into Eq. (6.40) to find the aggregate \( \mathcal{O}(x^4) \) contribution to \( S_{LL} \) to \( \mathcal{O}(L^4) \):

\[
x^4 L^3 F_4(w, \zeta) = \left( \frac{99}{256} x^4 \right) L^3 + x^4 \left[ -\frac{459}{512} z + \frac{135}{512} y \right] L^4 + \mathcal{O}(L^5).
\]

Finally we see that the only contribution to \( S_{LL} \) from its \( \mathcal{O}(x^5) \) term \( x^5 L^4 F_5(w, \zeta) \) that is degree-4 in \( L \) is just \( x^5 L^4 F_5(1, 0) \). This constant term is found from constant-term contributions to the \( k = 5 \) version of Eq. (6.47), which satisfy the algebraic relation

\[
8 F_5(1, 0) = 6 P_0(0) + \frac{9}{2} N_1(0).
\]

Using Eqs. (6.36) and (6.43), we find that \( F_5(1, 0) = -837/1024 \). This result, in conjunction with Eqs. (6.48) and (6.43), yields all \( \mathcal{O}(x^3) \) and higher contributions to \( S_{LL} \) that enter into the extraction of the Higgs mass in the section that follows:

\[
S_{LL} = \frac{y}{w} + x M(\zeta) \left( \frac{w - 1}{w} \right)^2 + x^2 L \left[ \sum_{k=0}^{3} H_k(\zeta) \left( \frac{w - 1}{w} \right)^k \right] + \left\{ -\frac{9 x^3}{32} \right\} L^2 + \left( \frac{15 x^3 (z - 3 y)}{16} + \frac{99}{256} x^4 \right) L^4
\]

\[
+ \left\{ \frac{x^3}{32} \right\} \left( 207 y z - \frac{603}{8} z^2 - \frac{8343}{16} y^2 \right) + \frac{x^4}{512} \left( 135 y - 459 z \right) + \frac{837}{1024} x^5 \right\} + \mathcal{O}(L^5),
\]

where \( M(\zeta) \) and \( H_k(\zeta) \) are given by Eqs. (6.44) and (6.24) - (6.27).

## 7 Extraction of \( m_\phi \) from the RBEWS Summation-of-Leading-Logarithms Effective Potential

We have seen in the previous section that the RGE \( (6.20) \) can be used to determine all coefficients of the leading logarithm series \( S_{LL} \) for the effective potential \( (6.19) \). The result \( (6.47) \) for this series includes summations of logarithms to all orders in the scalar self-interaction couplant \( y \) and QCD gauge-couplant \( z \), and to quadratic terms in the \( t \)-quark Yukawa couplant \( x \). However, the result \( (6.47) \) also includes contributions to coefficients in the series \( (5.19) \) of the leading four powers of \( L \) that are valid to all orders of \( \{x, y, z\} \). Such terms are sufficient in themselves for a prediction of the scalar-field mass to all-orders in \( \{x, y, z\} \) for the summation-of-leading-logarithms series \( (6.47) \). One can expand this series in powers of \( L = \log(\phi^2/\mu^2) \) and choose \( \mu = \langle \phi \rangle \), the vacuum expectation value of the scalar field:

\[
S_{LL} = A + B \log(\phi^2/\langle \phi \rangle^2) + C \log^2(\phi^2/\langle \phi \rangle^2) + D \log^3(\phi^2/\langle \phi \rangle^2) + E \log^4(\phi^2/\langle \phi \rangle^2) + \ldots,
\]

where, from Eq. (6.47),

\[
B = 3y^2 - \frac{3}{4} x^2,
\]

\[
C = 9 y^3 + \frac{9}{4} x y^2 - \frac{9}{4} x^2 y + \frac{3}{2} x^2 z - \frac{9}{32} x^3,
\]

\[
D = 27 y^4 + \frac{27}{2} x y^3 - \frac{3}{2} x^2 y z + 3 x^2 y z - \frac{225}{32} x^2 y^2 - \frac{23}{8} x^2 z^2 + \frac{15}{16} x^3 z - \frac{45}{16} x^3 y + \frac{99}{256} x^4,
\]

\[
E = 81 y^5 + \frac{243}{4} x y^4 - 9 x y z^3 + \frac{45}{32} x y^2 z^2 - \frac{69}{16} x^2 y z^2 - \frac{135}{8} x^2 y^3 + \frac{531}{64} x^2 y z^2 + \frac{345}{64} x^2 z^3 + \frac{603}{256} x^3 z^2 + \frac{207}{32} x^3 y z - \frac{8343}{512} x^3 y^2 - \frac{459}{512} x^4 z + \frac{135}{512} x^4 y + \frac{837}{1024} x^5.
\]
The coefficient $A$ will include a finite counterterm $K$,

$$A = y + K,$$

so that the definition for the summation-of-leading-logarithms potential,

$$V_{LL} = \pi^2 \phi^4 S_{LL},$$

is inclusive of this counterterm. We can expand this potential about $\langle \phi \rangle$ to obtain

$$V_{LL} = \pi^2 \langle \phi \rangle^4 \left[ A + (4A + 2B) \left( \frac{\phi - \langle \phi \rangle}{\langle \phi \rangle} \right) + (6A + 7B + 4C) \left( \frac{\phi - \langle \phi \rangle}{\langle \phi \rangle} \right)^2 
+ \left( 4A + \frac{26}{3}B + 12C + 8D \right) \left( \frac{\phi - \langle \phi \rangle}{\langle \phi \rangle} \right)^3 + \left( A + \frac{25}{6}B + \frac{35}{3}C + 20D + 16E \right) \left( \frac{\phi - \langle \phi \rangle}{\langle \phi \rangle} \right)^4 + \ldots \right].$$

The condition that $\langle \phi \rangle$ is indeed an extremum of this potential implies that the coefficient of $(\phi - \langle \phi \rangle)/\langle \phi \rangle$ must vanish,

$$A = -B/2,$$

serving to remove entirely the finite counterterm $K$ from the series. The condition that the $\phi^4$ vertex extracted from the effective potential coincides with its tree-level value, i.e., that

$$\frac{d^4 V_{LL}}{d\phi^4} \bigg|_{\langle \phi \rangle} = \frac{d^4}{d\phi^4} \left( \frac{\lambda \phi^4}{4} \right) = 24\pi^2 y,$$

implies that the coefficient of $(\phi - \langle \phi \rangle)^4/\langle \phi \rangle^4$ in the series is equal to $y$:

$$y = \frac{11}{3}B + \frac{35}{3}C + 20D + 16E.$$  

This is a degree-5 equation for $y$, as is evident from Eqs. (7.12)–(7.15), since the Standard-Model values for the couplings $x$ and $z$ are known [Eqs. (5.12) and (5.14)]. Once $y$ is determined, one can obtain the scalar-field mass $m_\phi$ explicitly from the coefficient of $(\phi - \langle \phi \rangle)^2/\langle \phi \rangle^2$ in Eq. (7.8):

$$m_\phi^2 = \frac{d^2 V}{d\phi^2} \bigg|_{\langle \phi \rangle} = 8\pi^2 \langle \phi \rangle^2 (B + C).$$

Indeed, the procedure delineated above for extracting $m_\phi$ within RBEWS is mathematically equivalent to that of Section 4 for MSED. We have employed the series expansion only to illustrate the insensitivity of this procedure to terms $O(L^5)$ and higher within the series for $S_{LL}$.

In the case at hand, we find three real solutions to the constraint:

$$y_1 = 0.0538, \quad y_2 = -0.278, \quad y_3 = -0.00143,$$

given the substitution of known numerical values into $B$, $C$, $D$ and $E$ [Eqs. (5.12)–(5.15)]. Only the values and correspond to $\langle \phi \rangle$ being a local minimum of the potential; the value yields a negative value for $d^2 V/d\phi^2$ at $\langle \phi \rangle$. If we substitute the value or the value into Eqs. (7.12) and (7.13), and then substitute the resulting values of $B$ and $C$ into Eq. (7.16), we find corresponding values for the scalar field mass:

$$m_{\phi_1} = \sqrt{8\pi^2 \langle \phi \rangle^2 (B(y_1) + C(y_1))} = 216 \text{ GeV},$$

$$m_{\phi_2} = \sqrt{8\pi^2 \langle \phi \rangle^2 (B(y_2) + C(y_2))} = 453 \text{ GeV}.$$  

In obtaining these values, we have utilized the known value for the vacuum expectation value $\langle \phi \rangle = 246 \text{ GeV} \equiv v$ consistent with the magnitude of the Fermi constant and the $SU(2)$ gauge coupling constant $g_2$ characterizing broken electroweak symmetry in the Standard Model.
It is instructive to relate these results to the results of Section 5 for the one-loop effective potential of Coleman and Weinberg. The results \( \lambda \cong 3.6 \) (i.e., \( y = 0.093 \)), \( m_\phi \cong 350 \text{ GeV} \) obtained in that section can be recovered from Eqs. (7.10), (7.11) and (7.12) simply by setting \( C, D \) and \( E \) equal to zero while retaining the value of \( B \) in Eq. (7.12) — i.e., by ignoring all terms subsequent to the term linear in the logarithm in the series (7.11). It is evident that the mass prediction (7.11) relies on an unacceptably large and negative value for the couplant \( y \) [Eq. (7.14)]. The prediction (7.10), however, is not only phenomenologically reasonable, but is also contingent upon a determination (7.13) of the scalar self-interaction couplant \( y \) that is more in line with the known magnitudes (4) and (5) of the couplants \( x \) and \( z \) than the value \( y = 0.093 \) following from the purely one-loop treatment of Section 5. The contributions of \( y \) alone to the \( \beta \)-function (7.10) correspond to the \( \beta \)-function of an \( O(4) \)-symmetric scalar field theory, which has been calculated to four subleading orders in the scalar-field self-interaction coupling (13). Using these results we find that

\[
\lim_{\gamma \to 0} \frac{\gamma}{\gamma} \sim y^2 \left[ 1 - \frac{3}{2} y + \frac{195}{16} y^2 - 132.9 y^3 + \ldots \right].
\]

If \( y = 0.093 \), the right-hand side of (7.18) evaluated term by term is \( 10^{-2}[5.2 - 1.6 + 1.4 - 1.9 + 3.1 + \ldots] \), whose increasing magnitudes are indicative of a failure to converge. However, if \( y = 0.0538 \), terms on the right-hand side of Eq. (7.18), which are now \( 10^{-3}[17.5 - 3.04 + 1.58 - 1.22 + 1.16 + \ldots] \), continue to decrease monotonically, though very slowly. Similarly, the anomalous dimension of the scalar field (5.11) is seen from (7.11) is seen from (7.12) is seen from (7.11) is seen from (7.11) is seen from (7.11) to be proportional to the series (14)

\[
\lim_{z \to 0} \frac{\gamma}{\gamma} = 6y^2 - \frac{39}{2} y^3 + 187.85 y^4 - 2698.3 y^5 + 47975 y^6 + \ldots.
\]

If \( y = 0.0930 \), the series in square brackets ceases to decrease after its third term, \( [1 - 0.140 + 0.105 - 0.107 + \ldots] \), while the \( y = 0.0538 \) version of this same series, \( [1 - 0.0807 + 0.0353 - 0.0207 + \ldots] \), continues to decrease. Thus, the aggregate effect of summing leading logarithms appears to bring Section 5’s \( m_\phi \cong 350 \text{ GeV} \) one-loop estimate, obtained via a problematical determination of the couplant \( y \), down to \( 216 \text{ GeV} \) via a determination of \( y \) considerably closer in magnitude to those of the known QCD couplant \( z \) (\( \equiv \alpha_s(\langle \phi \rangle)/\pi \)) and the t-quark Yukawa couplant \( x \) (\( \equiv g_t^2(\langle \phi \rangle)/4\pi \)). This is demonstrably more consistent with the convergence of RG-functions when subsequent-to-leading logarithms are taken into consideration.

Of course, any future observation of a Higgs boson mass at or near \( 216 \text{ GeV} \) is not in itself proof of radiative electroweak symmetry breaking. This prediction is subject to unknown corrections from subsequent-to-leading logarithms within the perturbative series for the effective potential, though in the next section we present arguments that such corrections are likely to be small. However, the value of the quartic scalar couplant corresponding to an \( O(200 \text{ GeV}) \) Higgs boson is five times larger in RBWSS than in conventional spontaneous symmetry breaking. For the latter case, in which a negative mass term is explicitly present in the tree-level potential prior to spontaneous symmetry breaking, the scalar field interaction couplant is predicted to be \( y = (\lambda/4\pi^2) = m_\phi^2/(8\pi^2\langle \phi \rangle^2) \), less than one fifth the value obtained in Eq. (7.10) for radiative symmetry breaking if \( m_\phi = 216 \text{ GeV} \). Present indirect standard-model bounds on the Higgs boson mass, which come from the \( \log(m_\phi) \) dependence of \( m_t, M_W, M_{20} \) and \( \Gamma_{20} \) (13), are insensitive to the quartic scalar-field self-interaction coupling \( \lambda \). If electroweak symmetry breaking is indeed radiative, processes such as the \( W^+W^- \to ZZ \) scattering cross-section which are sensitive to \( \lambda \) should be greatly enhanced relative to standard model expectations. In short, if an \( O(200 \text{ GeV}) \) Higgs is discovered, a “smoking gun” indication of symmetry breaking along the lines proposed in this work would be a factor of 30 enhancement of \( \sigma(W_L^0W_L^0 \to Z_0^0Z_0^0) \) relative to conventional standard-model expectations.

8 Discussion

8.1 Perturbative Consistency and Residual Scale Dependence

The result \( m_\phi = 216 \text{ GeV} \) is not contingent on any fine-tuning of the known-interaction couplants \( x \) and \( z \). Indeed, if we let \( x \) and \( z \) go to zero, the scalar field mass increases only slightly from \( 216 \text{ GeV} \) to \( 221 \text{ GeV} \), with the value for \( y \) determined via Eq. (7.11) correspondingly increasing from 0.0538 to 0.0541. At this juncture, however, we cannot know if next-to-leading-logarithm contributions to the effective potential serve (or fail) to destabilize this summation-of-leading-logarithms solution. The effect of summing such nonleading-logarithm contributions is clearly an area for further investigation. What is encouraging, however, is the fact that the value of \( y \) obtained from summing leading logarithms appears to be not very different than the known magnitudes of the dominant contributing couplants \( x \) and \( z \). Logarithms subsequent to leading are accompanied by additional powers of the couplants \( \{x, y, z\} \). Thus, next-to-leading-logarithm contributions to \( B, C, D \) and \( E \) are respectively degree 3, 4, 5 and 6 in the couplants.
\{x, y, z\}, one degree higher than the leading-logarithm contributions listed in Eqs. (7.2)–(7.5). If \{x, y, z\} are all comparably “small,” one might expect such subsequent contributions to be perturbatively suppressed.

Such perturbative consistency is supported by an examination of the residual renormalization-scale dependence of the leading logarithm effective potential. To see this, we first compare the residual scale dependence occurring ...

1) \ldots \text{within}

\[ V_{1L} = \pi^2 \phi^2(\mu) \left[ y(\mu) + \left( 3y^2(\mu) - \frac{3}{4}x^2(\mu) \right) \log \left( \frac{\phi^2(\mu)}{\mu^2} \right) \right], \quad (8.1) \]

the leading-logarithm contribution to the one-loop potential [Fig. 4, and ... \]

2) \ldots \text{within}

\[ V_{LL}^{(2)} = \pi^2 \phi^2(\mu) \left[ y(\mu) + x(\mu)F_1[w(\mu), \zeta(\mu)] + x^2(\mu)L(\mu)F_2[w(\mu), \zeta(\mu)] \right], \quad (8.2) \]

the summation-of-leading logarithms series [Fig. 5] truncated after \(O(x^2)\), where

\[ L(\mu) = \log \left( \frac{\phi^2(\mu)}{\mu^2} \right), \quad w(\mu) = 1 - 3y(\mu)L(\mu), \quad \zeta(\mu) = z(\mu)L(\mu). \quad (8.3) \]

Eq. (8.1) is just the one-loop potential (5.1) without its finite \(K\phi^4\) counterterm, which (since it is more than one degree higher in couplants than in the logarithm) ultimately generates summations of subsequent-to-leading logarithms upon incorporation of two-and-higher-loop order RG-functions within the RG equation (5.7). The finite counterterm is similarly excluded from Eq. (8.2).

Figure 4: The residual renormalization-scale dependence of Eq. (8.1), the leading-logarithm contribution to the one-loop effective potential, with couplant and field values evolving from \(\mu = v\) as indicated in the text. The top, middle and bottom curves at the right boundary of the figure correspond respectively to \(\mu = 2v\), \(\mu = v\) and \(\mu = v/2\).

In Figs. 4 and 5 \(x(\mu), y(\mu), z(\mu)\) and \(\phi(\mu)\) evolve from initial values at \(\mu = v\) via the one-loop RG-functions (5.15)–(5.18). For the three couplants, these initial values are \(x(v) = 0.0253, y(v) = 0.0538,\) and \(z(v) = 0.0329\) [Eqs. (5.12), (5.14) and (7.13)]. The field’s initial value \(\phi(v)\) is an input parameter exhibited along the abscissae of both figures in units of the vacuum expectation value \(v\). We see from Figure 4 that \(V_{1L}\) varies substantially from \(\mu = v/2\) to \(\mu = 2v\), but that this variation all but vanishes (Figure 5) when leading logarithms are summed [Eq. (8.2)].
Figure 5: The residual renormalization-scale dependence of Eq. (8.2), the summation-of-leading-logarithms potential truncated after $O(x^2)$. Evolution of couplants and $\phi(\mu)$ is as in Fig. 4. The dotted and solid curves (which overlap almost completely) correspond respectively to $\mu = v/2$ and $\mu = 2v$; the $\mu = v$ curve, which falls between these two, has been omitted for visual clarity.

Indeed such diminution of residual scale dependence through summation of logarithms is observed within RG improvement of a wide spectrum of perturbative calculations [3]. However, if we assume such residual scale dependence to be indicative of next-order corrections, we can then expect only modest departures from the $m_\phi = 216$ GeV prediction obtained at $\mu = v$. The counterterm $K[x(\mu), y(\mu), z(\mu)]\phi^4$ is necessarily more than degree-two in couplants, and is degree zero in the logarithm $L$. Consequently this term can be partitioned into terms that contribute to the summation of successively subleading logarithms, as leading logarithm terms $[x, y, z]$ are only one degree lower in $L$ than in aggregate powers of the couplants $x$, $y$ and $z$. Thus the counterterm $K\phi^4$ does not enter subsequent terms via the leading-logarithm RGE (5.20). In Figure 6, we evaluate the summation-of-leading-logarithms effective potential augmented by the $K\phi^4$ counterterm, which was obtained above by requiring that the coefficient of $(\phi - \langle\phi\rangle)^4/\langle\phi\rangle^4$ in Eq. (7.8) equal $y$:

$$K = -\left(\frac{25}{6}B + \frac{35}{3}C + 20D + 16E\right),$$

for $\{B, C, D, E\}$ as given by Eqs. (7.2) – (7.5) with $x = 0.0253$, $y = 0.0538$ and $z = 0.0329$, as obtained earlier. Since $K\phi^4$ is not a term contributing to the sum of leading logarithms within the original perturbative series [5,19], we assume this term to be an RG-invariant contribution (in the leading-log sense) to the effective potential, whose residual $\mu$-dependence is assumed to reside entirely in the contribution to the summation of leading logarithms. We see from Figure 6 that this construction of the effective potential varies controllably over the range $v/2 \leq \mu \leq 2v$.\(^3\) Moreover, we find over this same range of $\mu$ that the value of $m_\phi$ extracted from this potential varies only from 208 GeV at $\mu = v/2$ to 217 GeV at $\mu = 2v$. If we assume such scale dependence to be indicative of subsequent subleading-logarithm corrections, we then can expect only modest departures from the $m_\phi = 216$ GeV prediction at $\mu = v$. Such scale uncertainties in $m_\phi$ dominate any uncertainties in $m_\phi$ deriving from the couplant

\(^3\)A similar construction based upon the one-loop contribution exhibits much larger variation in $m_\phi$ over this same range of $\mu$. 
values themselves (i.e., the error in $g_t(v)$ and $\alpha_s(M_Z)$), which affect $m_\phi$ only negligibly.

Figure 6: The residual renormalization-scale dependence of the effective potential obtained through augmentation of Eq. (8.2) with its appropriate $K\pi^2\phi^4$ counterterm [Eq. (8.4)]. As discussed in the text, renormalization scale-dependence of this counterterm (which is a sum of non-leading logarithm terms) is necessarily next-order in the RGE; consequently, this counterterm contribution has been assumed to be RG-invariant. The evolution of couplings and $\phi(\mu)$ from their values at $\mu = v$ is as described in the text, and corresponds to that of Fig. 7. As in Figs. 4 and 5 the horizontal axis is $\phi(v)/v$, and the vertical axis is the corresponding value of $V_{LL}^{\mu}\phi/v^4$ for the each curve’s choice of $\mu$. The top, middle and bottom curves respectively correspond to $\mu = v/2$ ($m_\phi = 208$ GeV), $\mu = v$ ($m_\phi = 216$ GeV), and $\mu = 2v$ ($m_\phi = 217$ GeV).

8.2 Large-Field Behaviour of the Effective Potential

The leading logarithm contribution to the effective potential is given by the series (6.1), with $F_0$ given explicitly by (6.10), $F_1$ given by Eqs. (6.11) and (6.14), $F_2$ given by Eqs. (6.19) and (6.24) – (6.27), and $F_3$ given by Eq. (6.29) and Eqs. (C.4) – (C.7) of Appendix C. The solutions to Eqs. (6.5) – (6.7) for $F_n(v, \zeta)$ are all of the general form

$$F_n(w, \zeta) = \sum_{k=0}^{n+1} f_{n,k}(\zeta) \left[ \frac{w-1}{w} \right]^k. \tag{8.5}$$

In this notation, we see from Sections 6.2, 6.3 and 6.4 that $M(\zeta)/4 = f_{1.2} (f_{1.0} = f_{1.1} = 0)$, $\{H_0, H_1, H_2, H_3\} = \{f_{2.0}, f_{2.1}, f_{2.2}, f_{2.3}\}$, and $\{N_0, N_1, N_2, N_3, N_4\} = \{f_{3.0}, f_{3.1}, f_{3.2}, f_{3.3}, f_{3.4}\}$. If we substitute Eq. (8.5) into Eq. (6.7) and make use of the identities

$$\frac{(w-1) w}{d w} \left[ \frac{w-1}{w} \right]^k = k \left[ \frac{w-1}{w} \right]^k, \tag{8.6}$$

$$\frac{(w-1)}{d w} \left[ \frac{w-1}{w} \right]^k = k \left[ \frac{w-1}{w} \right]^k - k \left[ \frac{w-1}{w} \right]^{k+1}, \tag{8.7}$$

$$\frac{d}{d w} \left[ \frac{w-1}{w} \right]^k = k \left( \frac{w-1}{w} \right)^{k-1} - 2 \left[ \frac{w-1}{w} \right]^k + \left[ \frac{w-1}{w} \right]^{k+1}, \tag{8.8}$$
we then obtain the following recursion relation for $f_{p,k}(\zeta)$ when $p \geq 3$:

$$
\left[ \frac{7\zeta^2}{2} \frac{d}{d\zeta} + 4p\zeta \right] f_{p,k} + \left[ 2\zeta \frac{d}{d\zeta} + 2(p-1) + 2k \right] f_{p,k} = \left[ \frac{9p - 21}{4} + 3k \right] f_{p-1,k-1} - 3(k-1) f_{p-1,k-1} - 2k f_{p-2,k+1} + (k+1) f_{p-2,k+1},
$$

(8.9)

where $f_{p,k} \equiv 0$ when $k < 0$ or $k > p + 1$, and where $f_{p,k}$ is analytic (finite) at $\zeta = 0$.

One of the motivations for summing leading logarithms is to ascertain the large logarithm behaviour of the effective potential, behaviour corresponding to the potential in either the large-field or zero-field limit. For the large-field case, one is not able to extrapolate past the $w = 0$ poles characterizing every $F_n(w, \zeta)$ in the series (6.1) [as evident from Eqs. (6.10), (6.11), (6.19), (6.29) and, generally speaking, (8.5)]. Of course, $w = 0$ corresponds to a Landau pole at $L = 1/3y$ [Eq. (6.2)], which implies singular behaviour of the summation-of-leading logarithms effective potential at $\phi = v \exp[1/6y] \approx 22.2v$ for our $y = 0.0538$ solution (7.13). In Fig. 7 we plot the effective potential of Figure 6 for $\mu = v$ over the range $1.5v < \phi < e^{1/6y}v$ to illustrate the singularity occurring at the latter bound. Such summation-of-logarithms singularities occur in other contexts and are not necessarily singularities of the function itself represented by the series. Artificial singularities in the series expression for the RG-invariant couplant are discussed in Ref. [9]; similar singularities in the perturbative electron-positron annihilation cross-section are discussed in refs. [4, 16]. Moreover, the terms $S_0$, $S_1$, $S_2$ and $S_3$ within the series for the effective potential of MSED, as considered in Section 3, also exhibit such an ultraviolet singularity at $1 - \frac{1}{2}yL = 0$, where $y$ is the scalar couplant of that theory.

Figure 7: The continuation of the $\mu = v$ curve of Figure 6 to large values of $\phi$. The ordinate is now logarithmic, as indicated.

Since $w = 1 - 3y\log(\phi^2/v^2)$, we see that as $\phi$ increases from its vacuum expectation value $v$ to $v \exp[1/6y]$, $w$ approaches zero from above. We can show from Eqs. (8.5) and (8.9), however, that as $w$ approaches zero from above, all $F_n$ diverge positively; i.e.

$$
\lim_{w \to 0^+} F_n(w, \zeta) \to +\infty,
$$

(8.10)

consistent with each term of the series $S_{LL}$ (6.1) within $V_{LL} = \pi^2 \phi^4 (S_{LL} + K)$ being bounded from below prior to the singularity. To see this, first note from Eq. (8.5) that as $w \to 0$, the asymptotic behaviour of each $F_n(w, \zeta)$ is
dominated by the coefficient \( f_{n,n+1} \):

\[
F_n(w, \zeta) \xrightarrow{w \to 0^+} (-1)^n f_{n,n+1}(\zeta)/w^{n+1}.
\]  

(8.11)

Note also that \( \zeta \) remains small in this limit: when \( w \to 0^+ \), \( L \to 1/3y \) and \( \zeta \to z/3y = 0.204 \ [z(v) = 0.0329 \) and \( y(v) = 0.0538 \). But if \( \zeta \) is small, we see from Eq. 8.9 that leading contributions to the power series for \( f_{n,n+1}(\zeta) \) satisfy the recursion relation

\[
f_{n,n+1}(0) = -\frac{3}{4} f_{n-1,n}(0).
\]  

(8.12)

Since \( f_{3,4}(0) = N_4(0) = 9/64 \) [Eq. (C.7)], and since \( \zeta \) is small, we see from Eqs. 8.11 and 8.12 that

\[
F_n(w, \zeta) \xrightarrow{w \to 0} (3n^{-1}/4^n)w^{-(n+1)}, \quad n \geq 3.
\]  

(8.13)

Thus, if \( n \geq 3 \), \( F_n \to +\infty \) as \( w \) approaches zero from above, i.e., as the field \( \phi \) increases to approach the singularity at \( L = 1/3y \) from below. This behaviour also characterizes \( F_n \) for \( n < 3 \),

\[
0 = 1/w, \quad F_1 \xrightarrow{w \to 0} M(0)/4w^2 = 1/4w^2, \quad F_2 \xrightarrow{w \to 0} -H_3(0)/w^3 = 3/16w^3,
\]  

(8.14)

as is evident from Eqs. (6.14) and (6.27). Consequently, we see that incorporation of arbitrarily many terms of the series \( \xi_n \) into the effective potential necessarily leads to a potential that diverges positively as \( \phi \) increases to approach the singularity at \( w = 0 \), corresponding to an effective potential which is bounded from below in the \( w > 0 \) region where the series \( \xi_n \) is meaningful. Moreover, it should be noted from the large \( y \) limit of Eq. 5.10 for the evolution of \( y(\mu) \) that \( y(\mu) = y(v)/[1 - 6y(v) \log(\mu/v)] \), an expression which exhibits a Landau pole at \( \mu = v \exp[1/6y(v)] \) in correspondence with the singularity discussed above. Consequently, one may argue (as in ref. 17) that the scale \( \Lambda \) for new physics must occur prior to this Landau pole, in which case \( \Lambda \leq 5.5 \text{ TeV} \) within the context of radiative (as opposed to conventional) spontaneous symmetry breaking.

### 8.3 Method of Characteristics and Electroweak-Couplant Corrections

An alternative RG approach to incorporating higher order effects is the method of characteristics, as opposed to use of the explicit forms for leading logarithms and secondary expansions, such as Eqs. (2.21) or (6.1). The closed-form solution to the RG equation (8.24) obtained by the method of characteristics is equivalent to the summation of leading logarithms and is given by

\[
V_{eff}^{L/2} = \pi^2 \tilde{g} \langle L/2 \rangle \tilde{\phi}^3 \langle L/2 \rangle = \pi^2 \tilde{g} \langle L/2 \rangle \phi^4 \exp \left[-3 \int_0^{L/2} \tilde{x}(t) \, dt \right].
\]  

(8.15)

where \( \tilde{x}(t), \tilde{y}(t), \tilde{z}(t), \) and \( \phi(t) \) are characteristic functions defined by the differential equations and initial conditions

\[
\frac{d\tilde{x}}{dt} = -\frac{7}{2} \tilde{z}^2, \quad \tilde{z}(0) = z; \tag{8.16}
\]

\[
\frac{d\tilde{y}}{dt} = \frac{9}{4} \tilde{x}^2 - 4\tilde{x}\tilde{z}, \quad \tilde{x}(0) = x; \tag{8.17}
\]

\[
\frac{d\tilde{\phi}}{dt} = \frac{3}{4} \tilde{x}\tilde{\phi}, \quad \tilde{\phi}(0) = \phi; \tag{8.18}
\]

\[
\frac{d\tilde{y}}{dt} = 6\tilde{y}^2 + 3\tilde{x}\tilde{y} - \frac{3}{2} \tilde{x}^2, \quad \tilde{y}(0) = y. \tag{8.19}
\]

Since \( f_{p,k} = 0 \) when \( k > p + 1 \), terms \( f_{n-1,n+1}, f_{n-2,n}, \) etc. appearing in Eq. 8.9 all vanish when \( p = n, k = n + 1 \).
in obvious correspondence with the one-loop RG functions defined by Eqs. (5.15) – (5.18). These differential equations lead to series solutions
\[
\tilde{\phi}(t) = \phi \exp \left[ -\frac{3}{4} \int_0^t \tilde{x}(s) ds \right] = \phi \left[ 1 - \frac{3}{4} xt + \left( -\frac{9}{16} x^2 + \frac{3}{2} xz \right) t^2 + \left( -\frac{45}{64} x^3 + \frac{9}{4} x^2 z - \frac{15}{4} xz^2 \right) t^3 + \cdots \right]
\]
\[
\tilde{y}(t) = y + \left( \frac{6y^2 - \frac{3}{2} x^2 + 3xy}{2} \right) t + \left( \frac{36y^3 - \frac{9}{8} yx^2 + 27xy^2 - \frac{45}{8} x^3 - 6xy z + 6x^2 z}{2} \right) t^2
\]
\[
+ \left( \frac{216y^4 + 18y^2 x^2 + 216y^3 x - \frac{337}{32} x^4 - \frac{477}{16} yx^3 - 48xy^2 - \frac{15}{2} yx^2 z + \frac{69}{2} x^3 z + 15x y z^2 - 23x^2 z^2}{2} \right) t^3
\]
\[
+ \left( \frac{1296y^5 + \frac{165}{4} yx^2 z^2 + \frac{891}{8} yx^3 z - 360y^3 x^3 z + \frac{675}{2} y^2 x^3 z + 1620y^4 x^4 + \frac{345}{4} y^3 x^3 z - \frac{513}{12} x^5}{2} \right) t^4 + \cdots
\]

which, when substituted into the intermediate expression in Eq. (8.15), lead to recovery of the series coefficients \( \{B, C, D, E\} \) [Eqs. (8.16) – (8.19)]. Indeed, the potential (8.15), as constructed from one-loop RG equations (8.16) – (8.19), provides the easiest means of estimating the effect of the subdominant electroweak gauge couplants \( r \equiv \frac{g_2^2}{4\pi^2} \) and \( s \equiv \frac{g^2}{4\pi^2} \) on the Section 7 predictions for radiatively broken electroweak symmetry. These couplants, which evolve from initial values \( g_2^2(v)/4\pi^2 = \tilde{r}(0) = 0.0109 \), \( g^2(v)/4\pi^2 = \tilde{s}(0) = 0.00324 \) [consistent with values (5.3) and (5.4) for \( g_2(M_Z) \) and \( g'(M_Z) \)] via the one-loop RG equations [19]
\[
\frac{d\tilde{r}}{dt} = -\frac{19}{12} \tilde{r}^2, \quad \frac{d\tilde{s}}{dt} = \frac{41}{12} \tilde{s}^2
\]

are seen to enter the potential (8.15) via the following additional electroweak (ew) contributions [11] to the right-hand sides of Eqs. (8.17) – (8.18):
\[
\Delta_{ew} \left( \frac{dx}{dt} \right) = -\frac{9}{8}\tilde{x} - \frac{17}{24}\tilde{x}s,
\]
\[
\Delta_{ew} \left( \frac{d\phi}{dt} \right) = -\phi \left[ -\frac{9}{16} \tilde{r} - \frac{3}{16} \tilde{s} \right],
\]
\[
\Delta_{ew} \left( \frac{dy}{dt} \right) = -\frac{9}{4}\tilde{y} - \frac{3}{4}\tilde{y}s + \frac{3}{32}s^2 + \frac{3}{16}\tilde{r}s + \frac{9}{32}\tilde{s}^2
\]

The above corrections allow incorporation of the contributions from the electroweak couplants \( r \) and \( s \) into the Eq. (7.1) series coefficients \( \{B, C, D, E\} \):
\[
\Delta_{ew} B = \frac{9}{64} r^2 + \frac{3}{32} rs + \frac{3}{64} s^2
\]
\[
\Delta_{ew} C = -\frac{33}{1024} r^3 + r^2 \left( \frac{43}{1024} s + \frac{27}{64} y - \frac{27}{256} x \right) + r \left( -\frac{27}{16} y^2 + s \left( -\frac{9}{128} x + \frac{9}{32} y \right) + \frac{127}{1024} s^2 \right)
\]
Given the “$v$-valued” couplings \(r, s, x, z, y\) \(\{0.0109, 0.00324, 0.0253, 0.0329\}\), we obtain a 1% increase in the predicted Higgs mass (to 218 GeV) within radiative electroweak symmetry breaking, as well as an 1% increase in the extracted value of the scalar field self-interaction couplant \(y\) (to \(y = 0.0545\)). We further note that the method of characteristics approach leading to Eq. (8.13) [and ultimately, to the series coefficients within Eq. (7.3)] is gauge parameter independent to the order we consider, since the specific one-loop RG functions entering the effective potential \(S_{15}\) are all gauge-parameter independent. However, the gauge independence of results from the effective potential \(I_{21}\) in general rests upon the implementation of Nielsen identities \(V_{20,22}\), whose applicability to radiatively broken scenarios is problematical \(V_{20,22}\).

8.4 The \(z = 0\) Case

In principle, the coupled equations \((8.10) - (8.19)\) could be solved numerically to provide \(V_{eff}\) in \(8.13\). However, in the radiatively-broken theory, the initial condition for \(y\) needed for a numerical solution must be self-consistently determined from the shape of the effective potential itself, a formidable numerical problem. Since the summation of leading logarithms contains the analytic dependence on \(y\), such a summation facilitates the extraction of the self-consistent solution for \(y\) and the associated prediction of the Higgs mass. These issues for a radiatively-broken theory should be contrasted with standard spontaneous symmetry breaking, where the relation \(y = m_3^2/(8\pi^2 v^2)\) between \(y\) and the Higgs mass can be exploited for a numerical exploration of the effective potential in the Higgs-mass parameter space \(I_{17}\).
However, a closed form (method-of-characteristics) non-numerical solution to Eqs. (8.30)–(8.32) does exist in the $z=0$ limit. This closed form solution is of particular value in determining the scale for new physics. As already noted in Section 8.1, the singularity at $w = 0$ characterizing each term of the series (6.1), as evident in Eq. (8.33), need not represent a true singularity of the leading logarithm sum. To confirm the need for an $\mathcal{O}(5\text{TeV})$ scale for new physics within a radiative approach to the breakdown of electroweak symmetry, we utilize the explicit solutions to Eqs. (8.30)–(8.32) when the QCD couplant $z$ is equal to 0:

$$\ddot{t}(t) = \frac{x}{1 - \frac{4}{3}xt}, \quad (8.30)$$

$$\phi(t) = \phi \left[ 1 - \frac{9}{4}xt \right]^{1/3}, \quad (8.31)$$

$$\ddot{y}(t) = \ddot{x}(t) \left[ \frac{u_+ \left( \frac{u}{x} - u_- \right) - u_- \left( \frac{u}{x} - u_+ \right) \left( 1 - \frac{4}{3}xt \right)^{-\sqrt{65}/3}}{\frac{u}{x} - u_- \left( \frac{u}{x} - u_+ \right) \left( 1 - \frac{4}{3}xt \right)^{-\sqrt{65}/3}} \right], \quad (8.32)$$

where

$$u_+ = -1 \pm \sqrt{\frac{65}{16}}. \quad (8.33)$$

The $z = 0$ effective potential, including the arbitrary $K\phi^4$ counterterm, is found from Eqs. (8.30)–(8.32) to be

$$V = \pi^2\phi^4 \left\{ K + xR^{1/3} \left[ \frac{u_+ \left( \frac{u}{x} - u_- \right) - u_- \left( \frac{u}{x} - u_+ \right) \left( 1 - \frac{4}{3}xt \right)^{-\sqrt{65}/3}}{\frac{u}{x} - u_- \left( \frac{u}{x} - u_+ \right) \left( 1 - \frac{4}{3}xt \right)^{-\sqrt{65}/3}} \right] \right\}, \quad R \equiv 1 - \frac{9}{8}xL. \quad (8.34)$$

Given that $x = \frac{g_t^2}{4\pi^2} = 0.0253$ as before, we follow the procedure of the previous section by imposing the renormalization conditions

$$V'(v) = 0, \quad V^{(4)}(v) = 24\pi^2y, \quad (8.35)$$

at the electroweak vacuum expectation value $v = 2^{-1/4}G_\mu^{1/2} = 246\text{GeV}$, and find that $y = 0.05403$ and $K = -0.05817$. Upon incorporation of these numbers into Eq. (8.33), we obtain a potential that grows from its minimum at $\phi = v$ (i.e. $R = 1$) to $+\infty$ as $\phi$ approaches $16.5v \approx 4\text{TeV}$ from below, i.e. as

$$R \to \left[ \frac{\left( \frac{u}{x} - u_+ \right)}{\left( \frac{u}{x} - u_- \right)} \right]^{3/\sqrt{65}} = 0.8404 \quad (8.36)$$

from above. Thus, the $z = 0$ exact solution to the one-loop RG equation is seen to be a potential that is bounded from below over the domain for which it is valid. The singularity which terminates this domain is assumed, as before, to set the scale for new physics at a value not too different from the $5.5\text{TeV}$ singularity seen previously to characterize each term in the series expansion (6.1).

### 8.5 Zero-Field Limit of the Effective Potential

The zero-field limit of the effective potential $V_{LL} = \pi^2\phi^4(S_{LL} + K)$ corresponds to the large logarithm limit of the series (6.4) in which $L \to -\infty$ and $|(w - 1)/w|^k \to 1$. In this limit, the asymptotic behaviour of the leading contributions to the series (6.4) is seen from Eqs. (6.10), (6.11), (6.14), (6.19) and (6.24)–(6.27) to be

$$yF_0 = \frac{y}{w} \left. \frac{w}{|L|\to\infty} \right| - \frac{1}{3L}, \quad (8.37)$$

$$xF_1 = \frac{xM(zL)}{4} \left. \left( \frac{w - 1}{w} \right)^2 \right|_{|L|\to\infty} \frac{2}{L} \left( \frac{x}{z} \right), \quad (8.38)$$

$$x^2LF_2 = x^2L \sum_{k=0}^{3} H_k(zL) \left( \frac{w - 1}{w} \right)^k \left. \frac{3}{2L} \left( \frac{x}{z} \right)^2 \right|_{|L|\to\infty} - \frac{3}{2L} \left( \frac{x}{z} \right)^2. \quad (8.39)$$
For $n \geq 3$, the asymptotic behaviour of $x^n L^{-1} F_n(w, \zeta)$ can be extracted from Eq. (8.7) by noting from Eq. (8.6) that dependence of $F_n$ on the variable $w = 1 - 3yL$ disappears in the large logarithm limit

$$F_n \rightarrow \sum_{k=0}^{n+1} f_{n,k}(\zeta).$$

(8.40)

In the large $w$ limit, we see from Eqs. (8.37) and (8.38) that we can ignore all derivatives with respect to $w$ appearing on the right-hand side of Eq. (6.7). Moreover, in the large $L$ limit, the operators on the left hand side of Eq. (6.7) that are degree-1 in the variable $\zeta = zL$ dominate the remaining operators $|w(w - 1) d/dw$ is degree-zero via Eq. (8.6), leading to the following recursion relation for $F_n$ in the large-$|L|$ limit:

$$F_n = \frac{1}{2n} F_{n-1}, \quad (n \geq 3).$$

(8.41)

We see from Eq. (8.41) that $F_n$ is asymptotically one degree lower in $\zeta$ than $F_{n-1}$. Since the large-$|L|$ limit of $F_2$ is just $-\frac{3}{2} \zeta^{-2}$ [Eq. (8.39)], we see from Eq. (8.41) that $F_n$ in the large-$|L|$ limit is necessarily proportional to $\zeta^{-n}$,

$$\lim_{|L| \rightarrow \infty} F_n(w, \zeta) \equiv F_n(\zeta) = f_n \zeta^{-n},$$

(8.42)

such that

$$f_n = \frac{3(3n - 7)}{2n} f_{n-1}, \quad (n \geq 3),$$

(8.43)

with $f_2 = f_3 = -\frac{3}{2}$. Since $\zeta = zL$, it is evident from Eq. (8.5) that each term in the series $S_{LL}$ (6.1) goes like $1/L$ in the large-$|L|$ limit,

$$x^n L^{-1} F_n(w, \zeta) \rightarrow \frac{f_n}{L} \left(\frac{x}{z}\right)^n,$$

(8.44)

a structural result also upheld for $n = \{0, 1, 2\}$ in Eqs. (8.37) – (8.39). Thus truncations of the series $S_{LL}$ vanish term-by-term as $|L| \rightarrow \infty$ (i.e. as $\phi \rightarrow 0$ or $\infty$), a result that could hardly be anticipated from the form of $S_{LL}$ originally presented in Eq. (6.1).

Indeed, it is clear from Eq. (8.43) that the large-$|L|$ limit infinite series (6.1) for $S_{LL}$ can be summed explicitly if $|x/z|$ is sufficiently small. One easily sees from Eq. (8.43) that $|x/z| < 2/9$ for convergence of $S_{LL}$ in this limit. One can then make use of Eqs. (8.37) – (8.39) and (8.41) within Eq. (6.1) to obtain explicitly the large-$|L|$ limit of the effective potential itself, which is governed by the sum of its leading logarithm contributions:

$$V_{eff} \rightarrow \frac{\pi^2 \phi^4}{L} \left[ -\frac{1}{3} + 2 \left(\frac{x}{z}\right) - \frac{3}{2} \left(\frac{x}{z}\right)^2 + T \left(\frac{x}{z}\right) \right],$$

(8.45)

where

$$T(\rho) \equiv \sum_{n=3}^{\infty} \rho^n f_n.$$

(8.46)

The summation $T(\rho)$, as defined by Eq. (8.46), may be evaluated from the recursion relation (8.43), first by multiplying both sides of Eq. (8.43) by $2n\rho^{n-1}$, and then by summing from $n = 4$ to $\infty$:

$$2 \sum_{n=4}^{\infty} \rho^{n-1} f_n = 9\rho \sum_{n=4}^{\infty} (n - 1)\rho^{n-2} f_{n-1} - 12 \sum_{n=4}^{\infty} \rho^{n-1} f_{n-1}.$$

(8.47)

Eq. (8.47) is just a first order differential equation for $T(\rho)$, as defined by Eq. (8.46):

$$2 \left[ \frac{dT}{d\rho} - 3\rho^2 f_3 \right] = 9\rho \frac{dT}{d\rho} - 12T$$

(8.48)
with \( f_3 = -3/2 \) and with \( T(0) = 0 \). When the solution to this equation for \( 0 < \rho < 2/9 \),

\[
T(\rho) = \frac{1}{3} - 2\rho + \frac{3}{2} \rho^2 - \frac{1}{3} \left[ 1 - \frac{9\rho}{2} \right]^{4/3},
\]

is substituted into Eq. (8.49), one finds that

\[
V_{\text{eff}} \rightarrow - \frac{\pi^2 \phi^4}{3L} \left[ 1 - \frac{9x}{2z} \right]^{4/3}, \quad 0 \leq x/z \leq 2/9.
\] (8.50)

Since \( L = \log(\phi^2/v^2) \rightarrow -\infty \) as \( \phi \rightarrow 0^+ \), one immediately sees that \( V_{\text{eff}} \rightarrow 0^+ \) as \( \phi \rightarrow 0^+ \). Thus if leading logarithms dominate the zero-field limit, then \( \phi = 0 \) corresponds to a local minimum of the effective potential when \( 0 \leq x/z \leq 2/9 \), a most surprising result. In Appendix D, it is shown that Eq. (8.50) cannot be extended past the \( x/z < 2/9 \) radius of convergence for the series \( T(x/z) \). Thus, we come away knowing only that \( \phi = 0 \) is a local minimum of the effective potential when QCD is sufficiently strong (\( \alpha_s > 0.36 \)). On the basis of (8.50), we cannot say anything about \( \phi = 0 \) being (or not being) a local minimum if \( \alpha_s \) is below this bound, as is the case for the standard model parameters we have been using [\( z = 0.0329 = \alpha_s(v)/\pi \)].

Nevertheless, this result may prove indicative of electroweak symmetry restoration within a strong-phase context for QCD. Recent work based upon the ordering of Padé-approximant zeros and poles for the \( \chi^2 \) QCD \( \beta \)-function series [23] suggests that when \( n_f < 6 \), the QCD couplant may exhibit the same two-phase behaviour known to characterize the exact \( \beta \)-function for \( N = 1 \) supersymmetric Yang-Mills theory (SYM), in which coexisting strong-coupland and (asymptotically-free) weak-coupland phases evolve toward a common infrared attractive point [24]. Within such a context, one may envision a scenario in which the \( \phi = 0 \) minimum of preserved electroweak symmetry is upheld by an effective potential involving the strong phase of QCD, but in which a \( \phi = v \) minimum of radiatively-broken electroweak symmetry characterizes this same effective potential when QCD is in its asymptotically-free weak phase. Since the weak-phase minimum at \( \phi = v \) is deeper than the strong-phase minimum at \( \phi = 0 \) [\( V_{\text{eff}}(v) < 0 \) for \( z = 0.033 \), and \( V_{\text{eff}}(0) = 0 \) for all \( z \)], the weak-phase of QCD is seen to be energetically preferred. Thus, radiative electroweak symmetry breaking may also serve to select the asymptotically-free phase of QCD over any coexisting strong phase.

### Appendix A - The One-Loop Effective Action in Scalar QED

In this appendix, we will compute the one-loop effective action in arbitrary covariant gauge for massless scalar quantum electrodynamics up to second order in the derivative of the scalar field.

The classical Lagrangian for scalar QED with masses is

\[
\mathcal{L}_{\text{cl}} = \Delta \mathcal{L} - \frac{m^2}{2} (\phi_1^2 + \phi_2^2) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa^2}{2} (A_\mu + (1/\kappa) \partial_\mu S)^2, \quad (D_\mu = \partial_\mu - ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu) \quad (A.1)
\]

where \( \Delta \mathcal{L} \) is given by Eq. (1.2) in terms of the real scalar fields \( \phi_1 \) and \( \phi_2 \), which have been assigned equal masses in Eq. (A.1). \( A_\mu \) is a \( U(1) \) vector and \( S \) is a Stueckelberg scalar [24]. For calculational convenience, we work in Euclidean space \( [(\partial_\mu \phi_k)^2 \rightarrow -(\partial_\mu \phi_k)^2] \). If \( \Phi \equiv (\phi_1 + i\phi_2)/\sqrt{2} \), we note that \( \mathcal{L}_{\text{cl}} \) is invariant under the gauge transformation

\[
\Phi \rightarrow e^{i\Lambda} \Phi \quad (A.2)
\]

\[
A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \Lambda
\]

\[
S \rightarrow S - \frac{\kappa}{e} \Lambda
\] (A.3) (A.4)

We now split \( \phi_1 \) and \( \phi_2 \) into the sum of background fields \( f_1, f_2 \) and quantum fields \( h_1, h_2 \):

\[
\phi = f_1 + h_1, \quad \phi_2 = f_2 + h_2.
\] (A.5)

---

5 Strictly speaking the \( z = 0 \) potential also exhibits a local minimum at \( \phi = 0 \) \( [R = +\infty] \) immediately followed by a numerically tiny local maximum at a value \( \phi = v \exp(-2466) \) [corresponding to \( R \approx 141.4 \)] that is only infinitesimally separated from the \( \phi = 0 \) local minimum. Subject to the precision limitations of an actual plot, which are insensitive to such small separations in the values of \( \phi \), Eq. (8.34) is seen to exhibit an apparent maximum at \( \phi = 0 \), followed by its minimum at \( \phi = v \) and its subsequent positive approach to the singularity at \( \phi = 16.5v \).
The gauge fixing Lagrangian
\[ \mathcal{L}_{gf} = -\frac{1}{2\alpha} [\partial \cdot A + icA(f_1 + if_2)(h_1 - ih_2) + \alpha \kappa S] [\partial \cdot A - icA(f_1 - if_2)(h_1 + ih_2) + \alpha \kappa S] \]  
(A.6)
is now employed. Upon setting the mass parameters \( m^2 \) and \( \kappa^2 \) equal to zero (thereby decoupling \( S \)), the terms in \( \mathcal{L}_c + \mathcal{L}_{gf} \) that are bilinear in the quantum fields \( A_\mu, h_1 \) and \( h_2 \) are
\[ \mathcal{L}^{(2)} = -\frac{1}{2} V' M V, \]  
(A.7)
where \( V = (h_1, h_2, A_\mu) \), \( T_{\mu\nu} = \delta_{\mu\nu} - p_\mu p_\nu/p^2 \), \( L_{\mu\nu} = p_\mu p_\nu/p^2 \) \( (p = -i\partial) \).

The one loop effective action is given by \( \Gamma^{(1)} = \ln \det^{-1/2} \hat{M} \).

This matrix \( \hat{M} \) is then split into two parts, \( \hat{M} = H_0 + H_1 \) where \( H_0 \) consists of these contributions to \( \hat{M} \) that are independent of \( \partial_\lambda \phi_k \) and \( \partial_\lambda \partial_\sigma \phi_k \). One now regulates \( \Gamma^{(1)} \) in Eq. (A.9) using the \( \zeta \)-function
\[ \Gamma^{(1)} = \frac{1}{2} \frac{d}{ds} \right|_{s=0} \left[ (\mu^2)^s \zeta(s) \right], \]  
(A.10)
where
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{tr} e^{-\hat{M}t}. \]  
(A.11)

Now \( \text{tr} e^{-\hat{M}t} \) is expanded using the Schwinger formula \( \text{tr} e^{(H_0 + H_1)t} = \text{tr} \left\{ e^{-H_0 t} + \frac{(-t)}{1} e^{-H_0 t} H_1 + \frac{(-t)^2}{2} \int_0^1 du e^{-(1-u)H_0 t} H_1 e^{-uH_0 t} H_1 + \ldots \right\} \).

Upon diagonalizing the matrix \( \hat{M} \), we find the first contribution to Eq. (A.12) to be
\[ \text{tr} e^{-H_0 t} = \text{tr} e^{-p^2/2 + g_A \phi^2} \]  
(A.13)
where \( g_a = \lambda/2 + e^2 \alpha, \ g_b = \lambda/6 + e^2 \alpha, \ g_A = e^2 \), with \( \phi^2 \equiv \phi_1^2 + \phi_2^2 \). Since \( T_{\mu\nu} \) and \( L_{\mu\nu} \) constitute a complete set of orthogonal projection operators, Eq. (A.13) reduces to
\[ \text{tr} e^{-H_0 t} = \text{tr} \left\{ e^{-\left(p^2 + g_A \phi^2\right)t} + e^{-\left(p^2 + g_a \phi^2\right)t} \right\} \]  
(A.14)
The term linear in \( H_1 \) in Eq. (A.12) will not contribute to the one-loop effective action up to and including terms containing two derivatives of the background field. From the term in Eq. (A.12) that is quadratic in \( H_1 \) we find that
\[ \text{tr} e^{-H_0 t} H_1 e^{-H_0 t} H_1 \]  
\[ = \text{tr} \left\{ \left( 2g_a \phi x \cdot \partial \phi \right)e^{-\left(p^2 + g_a \phi^2\right)t} (2g_A \phi x \cdot \partial \phi) \right\} \]  
(A.15)
where $\phi x \cdot \partial \phi \equiv \sum_{k=1}^{2} \phi_{k}x_{\lambda}(\partial_{\lambda} \phi_{k})$. If the traces in Eq. (A.14) and (A.15) are computed in momentum space, we see upon using the identities

$$\text{tr } e^{-\frac{1}{2}t} = \int \frac{d^{4}p}{(2\pi)^{4}} e^{-\frac{1}{2}t} = \frac{1}{(4\pi t)^{2}}$$ \hspace{1cm} (A.16)

and

$$\text{tr } (f(p)x_{\mu}g(p)x_{\nu}) = \int \frac{d^{4}p}{(2\pi)^{4}} \left( i \frac{\partial}{\partial p_{\mu}} f(p) \right) \left( i \frac{\partial}{\partial p_{\nu}} g(p) \right)$$ \hspace{1cm} (A.17)

that

$$\text{tr } e^{-\frac{1}{2}t} = \frac{1}{(4\pi t)^{2}} \left( e^{-g_{0}\phi^{2}t} + e^{-g_{0}\phi^{2}t} + (3 + \alpha^{2})e^{-g_{A}\phi^{2}t} \right)$$ \hspace{1cm} (A.18)

and that

$$\int_{0}^{\infty} dt t^{-1} \text{tr } \left[ \frac{(-t)^{2}}{2} \int_{0}^{1} du e^{-(1-u)t} H_{1} e^{-uH_{0}t} H_{1} \right]$$

$$= 2\phi^{2}(\partial_{\mu}\phi)^{2} \int_{0}^{1} du \int \frac{d^{4}p}{(2\pi)^{4}} \left\{ -u(1-u)p^{2}\Gamma(s+4) \left( \frac{g_{0}^{2}}{(p^{2}+g_{0}\phi^{2})^{s+4}} + \frac{g_{0}^{2}}{(p^{2}+g_{0}\phi^{2})^{s+4}} + \frac{3g_{A}^{2}}{(p^{2}+g_{A}\phi^{2})^{s+4}} + \frac{g_{A}^{2}}{(p^{2}+g_{A}\phi^{2})^{s+4}} \right) \right\}. \hspace{1cm} (A.19)$$

The standard integral

$$\int \frac{d^{n}k}{(2\pi)^{n}} \frac{(k^{2})^{s}}{(k^{2} + M^{2})^{b}} = \frac{1}{(4\pi)^{n/2}} \left( M^{2} \right)^{n/2+\alpha-b} \frac{\Gamma(\alpha + \frac{n}{2})}{\Gamma(\frac{n}{2})} \frac{\Gamma(b - a - \frac{n}{2})}{\Gamma(b)} \hspace{1cm} (A.20)$$

in conjunction with Eqs. (A.15), (A.16) and (A.17) yields

$$16\pi^{2}\Gamma(s)\zeta(s) = (g_{0}\phi^{2})^{2-s} + (g_{0}\phi^{2})^{2-s} + (3 + \alpha^{2}) (g_{A}\phi^{2})^{2-s}$$

$$+ 2\phi^{2}(\partial_{\mu}\phi)^{2} \int_{0}^{1} du \left[ -2u(1-u)\Gamma(s+1) \left( g_{0}^{2}(g_{0}\phi^{2})^{-1-s} + g_{0}^{2}(g_{0}\phi^{2})^{-1-s} + 3g_{A}^{2}(g_{A}\phi^{2})^{-1-s} + \alpha g_{A}^{2}(g_{A}\phi^{2})^{-1-s} \right) \right. \hspace{1cm} (A.21)$$

$$\left. + \frac{3}{2}g_{A}^{2}\Gamma(s+1) \left( (g_{A}\phi^{2})^{-1-s} + \alpha (g_{A}\phi^{2})^{-1-s} - \frac{2}{1-u + \frac{2\alpha}{\alpha}} (g_{A}\phi^{2})^{-1-s} \right) \right].$$

leading via Eq. (A.10) to the following one-loop effective action:

$$32\pi^{2}\Gamma^{(1)} = \frac{1}{2} \left[ (g_{0}\phi^{2})^{2} \left( \frac{3}{2} - \ln \frac{g_{0}\phi^{2}}{\mu^{2}} \right) + (g_{0}\phi^{2})^{2} \left( \frac{3}{2} - \ln \frac{g_{0}\phi^{2}}{\mu^{2}} \right) + (3 + \alpha^{2}) (g_{A}\phi^{2})^{2} \left( \frac{3}{2} - \ln \frac{g_{A}\phi^{2}}{\mu^{2}} \right) \right.$$ \hspace{1cm} (A.22)

$$+ 2(\partial_{\mu}\phi)^{2} \left[ -\frac{1}{3} (g_{A} + g_{A} + (3 + \alpha)g_{A}) + \frac{3}{2} \left( g_{A} \left( \frac{\alpha - 2\alpha}{\alpha - 1} \right) \ln g_{A} \right) \right]$$

$$[\phi^{2} \equiv \phi_{1}^{2} + \phi_{2}^{2}, \ (\partial_{\mu}\phi)^{2} \equiv (\partial_{\mu}\phi_{1})^{2} + (\partial_{\mu}\phi_{2})^{2}].$$

Further contributions to $\Gamma^{(1)}$ that contain more derivatives of the background field can also be computed.

The one-loop (1L) Landau gauge contribution to the effective potential follows from the negative of the non-derivative terms in Eq. (A.22) with $\alpha$ taken to be equal to zero:

$$\Delta V_{eff}^{1L} = \frac{1}{64\pi^{2}} \left( g_{0}^{2} + g_{0}^{2} + 3g_{A}^{2} \right) \left( \phi_{1}^{2} + \phi_{2}^{2} \right)^{2} \left[ \log \left( \frac{\phi_{1}^{2} + \phi_{2}^{2}}{\mu^{2}} \right) - \frac{3}{2} g_{A}^{2} \log g_{A} + \frac{3}{2} \left( \frac{\phi_{1}^{2} + \phi_{2}^{2}}{\mu^{2}} + \text{constants} \right) \right]. \hspace{1cm} (A.23)$$

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a result consistent with Eq. (13). Since we have been using operator regularization [28], no explicit renormalization is required to excise divergences. Nevertheless, finite counterterms may be added to $\Gamma^{(1)}$ in Eq. (2.22) to accommodate renormalization conditions such as that of Eq. (1.4), which is why the constants in Eq. (A.23) remain unspecified; e.g. under the renormalization condition (1.4), the usual numerical factor $-25/6$ would replace “constants” in the final line of Eq. (A.23).

The derivative terms in Eq. (A.22) also contribute to the kinetic term within the Euclidean-space effective action

$$\Gamma \equiv -V_{\text{eff}} - \frac{1}{2} Z_\phi \left[ (\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 \right] + \ldots .$$  \hspace{1cm} (A.24)

Noting that $(\partial_\mu \phi_1)^2 \rightarrow - (\partial_\mu \phi_1)^2$ in the Euclidean space version of Eq. (1.2), we find from Eq. (A.22) that

$$Z_\phi = 1 + \frac{g_a + g_b}{24\pi^2} - \frac{g_{\lambda}}{8\pi^2} \left[ \frac{1}{2} + \alpha \left( \frac{7}{6} - \frac{3 \ln \alpha}{\alpha - 1} \right) \right],$$  \hspace{1cm} (A.25)

hence that

$$Z_\phi = 1 + \frac{\lambda}{36\pi^2} + \frac{\varepsilon^2}{16\pi^2} + O(\lambda^2, \varepsilon^4, \lambda \varepsilon^2)$$  \hspace{1cm} (A.26)
in Landau $(\alpha = 0)$ gauge.

**Appendix B: Scale-Invariant Scalar-Field Mass in MSED**

For massless scalar electrodynamics, the requirement that $V$ be independent of the value of the scale parameter $\mu$ leads to the renormalization group equation [11]

$$DV \equiv \left[ \mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_\varepsilon \frac{\partial}{\partial \varepsilon} + \gamma(\lambda, \varepsilon^2) \left( \phi \frac{\partial}{\partial \phi_1} + \phi_2 \frac{\partial}{\partial \phi_2} \right) \right] V \left[ \mu, \lambda, \varepsilon^2, \phi_k \right] = 0$$  \hspace{1cm} (B.1)

with RG functions defined as in Eqs. (2.5), (2.6) and (2.7). Taking derivatives of Eq. (B.1) with respect to scalar field components leads to

$$\frac{D + \gamma}{D \phi_k} \frac{\partial V}{\partial \phi_k} = 0$$  \hspace{1cm} (B.2)

$$\frac{D + 2\gamma}{D \phi_1 \partial \phi_k} \frac{\partial^2 V}{\partial \phi_1 \partial \phi_k} = 0$$  \hspace{1cm} (B.3)

These equations are upheld at all values of $\phi_k$, including the vacuum expectation value (vev) defined by $\partial V / \partial \phi_k \mid_{\phi_1=0, \phi_2=0} = 0$. Indeed this vev-defining equation ensures that the vev components $\phi_{10}, \phi_{20}$ are themselves of the functional form $\phi_{10} = \phi_{10} [\mu, \lambda(\mu), \varepsilon^2(\mu)]$ and $\phi_{20} = \phi_{20} [\mu, \lambda(\mu), \varepsilon^2(\mu)]$. Furthermore, since the anomalous field dimension associated with scalar field components $\phi_k$ satisfies Eq. (2.5), vev components necessarily satisfy the equation

$$\left( \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_\varepsilon \frac{\partial}{\partial \varepsilon} \right) \phi_{10} = + \gamma \phi_{10}.$$  \hspace{1cm} (B.4)

We also see from Eq. (A.24) that

$$(D + 2\gamma) Z_{\phi} = 0.$$  \hspace{1cm} (B.5)

We now define the scalar-field mass as the square-root of the nonzero eigenvalue of the matrix

$$m_{\phi}^2 \equiv \frac{D^2 V / D \phi_1 \partial \phi_k}{Z_{\phi}} \mid_{\phi_1=0, \phi_2=0}.$$  \hspace{1cm} (B.6)

This eigenvalue $(m_{\phi}^2)$ is a renormalization-group invariant. To see this, we consider

$$\mu \frac{dm_{\phi}^2}{d\mu} = \left[ \mu \left( \frac{\partial \phi_0}{\partial \mu} \frac{\partial}{\partial \mu} + \frac{\partial}{\partial \mu} \right) + \beta_\lambda \left( \frac{\partial \phi_0}{\partial \lambda} \frac{\partial}{\partial \phi_0} + \frac{\partial}{\partial \lambda} \right) + \beta_\varepsilon \left( \frac{\partial \phi_0}{\partial \varepsilon} \frac{\partial}{\partial \phi_0} + \frac{\partial}{\partial \varepsilon} \right) \right]$$

$$\times \frac{D^2 V [\mu, \lambda(\mu), \varepsilon^2(\mu)]}{Z_{\phi} [\mu, \lambda(\mu), \varepsilon^2(\mu)]}.$$  \hspace{1cm} (B.6)

One easily sees from Eqs. (B.3), (B.4) and (B.5) that $\mu dm_{\phi}^2 / d\mu = 0$. Note that $Z_{\phi}$ is only perturbatively removed from unity [Eq. (A.26)], justifying the operational use of the second derivative of the effective potential at the vev for calculating the scalar field mass to a given order of perturbation theory. We further note that the RG-invariance of $V'' / Z_{\phi}$ has also been used to determine a nonperturbative $\beta$-function within a toy $\phi^4$ model context [29].
Appendix C: Solutions for \( F_3(w, \zeta) \)

\( F_3(w, \zeta) \) is given by Eq. (6.29), with the individual \( N_k(\zeta) \) constituting solutions of Eqs. (6.30)–(6.34) that are not singular at \( \zeta = 0 \). Thus Eq. (8.50) is just the inhomogeneous linear first order differential equation

$$\frac{dN_0}{d\zeta} + \frac{8(1 + 3\zeta)}{\zeta(\zeta + 4)} N_0 = \frac{3H_0(\zeta)}{\zeta(\zeta + 4)}$$

(C.1)

with \( H_0(\zeta) \) given by Eq. (8.24). The integrating factor for Eq. (C.1) is

$$g(\zeta) = \zeta^2(\zeta + 4/7)^{10/7}.$$  

The only solution to Eq. (C.1) that is not singular as \( \zeta \to 0 \) is

$$N_0(\zeta) = \frac{3}{g(\zeta)} \int_0^\zeta dt g(t)H_0(t)/[t(7t + 4)] = \frac{\zeta^{-2}}{10} \left[ 10 \left(1 + \frac{7\zeta}{4}\right)^{-9/7} - 9 \left(1 + \frac{7\zeta}{4}\right)^{-10/7} - 1 \right].$$

(C.3)

If one expands Eq. (C.3) about \( \zeta = 0 \), one obtains Eq. (6.30).

Corresponding solutions of Eqs. (6.31)–(6.34) are listed below:

$$N_1(\zeta) = \zeta^{-3} \left[ \frac{21\zeta}{5} - \frac{8}{5} + 12 \left(1 + \frac{7\zeta}{4}\right)^{-2/7} - \frac{8}{3} \left(1 + \frac{7\zeta}{4}\right)^{6/7} - \frac{116}{15} \left(1 + \frac{7\zeta}{4}\right)^{-3/7} \right],$$

(C.4)

$$N_2(\zeta) = \zeta^{-4} \left[ -\frac{81}{10} \zeta^2 + \frac{2023}{30} \zeta + \frac{1273}{30} + \frac{64}{21} \left(1 + \frac{7\zeta}{4}\right)^{13/7} - \frac{831}{7} \left(1 + \frac{7\zeta}{4}\right)^{6/7} + \frac{738}{7} \left(1 + \frac{7\zeta}{4}\right)^{5/7} - \frac{16}{21} \left(1 + \frac{7\zeta}{4}\right)^{-2/7} - \frac{943}{30} \left(1 + \frac{7\zeta}{4}\right)^{4/7} \right],$$

(C.5)

$$N_3(\zeta) = \zeta^{-5} \left[ 4\zeta^3 - \frac{418}{3} \zeta^2 - \frac{524}{3} \zeta - \frac{160}{3} \left(1 + \frac{7\zeta}{4}\right)^{20/7} + \frac{7440}{49} \left(1 + \frac{7\zeta}{4}\right)^{13/7} + \frac{704}{49} \left(1 + \frac{7\zeta}{4}\right)^{6/7} - \frac{1136}{7} \left(1 + \frac{7\zeta}{4}\right)^{12/7} - \frac{176}{21} \left(1 + \frac{7\zeta}{4}\right)^{5/7} + \frac{176}{3} \left(1 + \frac{7\zeta}{4}\right)^{11/7} \right],$$

(C.6)

$$N_4(\zeta) = \zeta^{-6} \left[ 72\zeta^3 + 144\zeta^2 + 96\zeta + \frac{64}{3} - \frac{6}{7} \left(\frac{384}{7} \left(1 + \frac{7\zeta}{4}\right)^{20/7} + \frac{128}{7} \left(1 + \frac{7\zeta}{4}\right)^{13/7} + \frac{32}{21} \left(1 + \frac{7\zeta}{4}\right)^{6/7} - \frac{64}{3} \left(1 + \frac{7\zeta}{4}\right)^{19/7} - \frac{32}{3} \left(1 + \frac{7\zeta}{4}\right)^{12/7} + \frac{224}{9} \left(1 + \frac{7\zeta}{4}\right)^{18/7} \right] \right].$$

(C.7)

Appendix D: Ambiguity of Zero-Field Limit when \( x/z > 2/9 \)

One must be careful not to extend the result past the radius of convergence of the series \( T(\frac{x}{z}) \). Naively, the result would imply \( V_{eff} \to 0^+ \) regardless of \( x/z \), since \([1 - 9x/2z]^{4/3}\) is positive-definite. However, the solution for \( T(\rho) \) is referenced to the initial condition \( T(0) = 0 \). If one assumes one can extend Eq. (8.46) past its \( \rho = 2/9 \) radius of convergence, one must again solve Eq. (8.48) subject to the new initial condition \( T(2/9) = -1/27 \), as obtained from Eq. (8.39) when \( \rho = 2/9 \). The most general solution to the differential equation subject to this new initial condition is

$$T(\rho) = \frac{1}{3} - 2\rho + \frac{3}{2} \rho^2 + c(\rho - 2/9)^{4/3}, \quad (\rho > 2/9),$$

(D.1)

where the constant \( c \) is arbitrary. If one now substitutes Eq. (D.1) into Eq. (8.35), the sign of \( V_{eff} \) remains undetermined,

$$V_{eff} \to \frac{\pi^2 A c}{L} \left[ \frac{x}{z} - \frac{2}{9} \right]^{4/3}, \quad \frac{x}{z} > \frac{2}{9},$$

(D.2)

reflecting the ambiguity in attempting to extend the determination of a series past its known radius of convergence.
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