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The $Z$-Dirac and massive Laplacian operators in the $Z$-invariant Ising model. (English) Zbl 1469.82007 Electron. J. Probab. 26, Paper No. 53, 86 p. (2021).

Summary: Consider an elliptic parameter $k$; we introduce a family of $Z^u$-Dirac operators $(K(u))_{u \in \mathbb{C}},$ relate them to the $Z$-massive Laplacian of C. Boutillier et al. [Invent. Math. 208, No. 1, 109–189 (2017; Zbl 1372.82016)], and extend to the full $Z$-invariant case the results of R. Kenyon [Invent. Math. 150, No. 2, 409–439 (2002; Zbl 1038.58037)] on discrete holomorphic and harmonic functions, which correspond to the case $k = 0.$ We prove through combinatorial identities, how and why the $Z^u$-Dirac and $Z$-massive Laplacian operators appear in the $Z$-invariant Ising model, considering the case of infinite and finite isoradial graphs. More precisely, consider the dimer model on the Fisher graph $G^F$ arising from a $Z$-invariant Ising model. We express coefficients of the inverse Fisher Kasteleyn operator as a function of the inverse $Z^u$-Dirac operator and also as a function of the $Z$-massive Green function; in particular this proves a (massive) random walk representation of important observables of the Ising model. We prove that the squared partition function of the Ising model is equal, up to a constant, to the determinant of the $Z$-massive Laplacian operator with specific boundary conditions, the latter being the partition function of rooted spanning forests. To show these results, we relate the inverse Fisher Kasteleyn operator and that of the dimer model on the bipartite graph $G^Q$ arising from the XOR-Ising model, and we prove matrix identities between the Kasteleyn matrix of $G^Q$ and the $Z^u$-Dirac operator, that allow to reach inverse matrices as well as determinants.

MSC:

82B20 Lattice systems (Ising, dimer, Potts, etc.) and systems on graphs arising in equilibrium statistical mechanics
82B23 Exactly solvable models; Bethe ansatz
05A19 Combinatorial identities, bijective combinatorics
33E05 Elliptic functions and integrals

Keywords:
dimer model; discrete massive harmonic and holomorphic functions; Ising model; massive Laplacian and Dirac operators; spanning forests and spanning trees; $Z$-invariance

Full Text: DOI arXiv

References:

[1] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, National Bureau of Standards Applied Mathematics Series, vol. 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. · Zbl 0171.38503
[2] H. Au-Yang and J. H. H. Perk, Critical correlations in a $Z$-invariant inhomogeneous Ising model, Physica A 144 (1987), 44-104.
[3] H. Au-Yang and J. H. H. Perk, Correlation functions and susceptibility in the $Z$-invariant Ising model, MathPhys Odyssey 2001: Integrable Models and Beyond (Birkhäuser Boston M. Kashiwara and T. Miwa, eds., ed.), 2002, pp. 23-48.
[4] R. J. Baxter, Solvable eight-vertex model on an arbitrary planar lattice, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 289 (1978), no. 1359, 315-346.
[5] R. J. Baxter, Free-fermion, checkerboard and $Z$-invariant lattice models in statistical mechanics, Proc. Roy. Soc. London Ser. A 404 (1986), no. 1826, 1-33.
[6] R. J. Baxter, Exactly solved models in statistical mechanics, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989, Reprint of the 1982 original.
[7] V. Belfara and H. Duminil-Copin, Smirnov’s fermionic observable away from criticality, Ann. Probab. 40 (2012), no. 6, 2667-2689. · Zbl 1260.08031
[8] C. Boutillier, D. Cimasoni, and B. de Tilière, Elliptic dimers on minimal graphs and genus 1 harnack curves, 2007.14699 (2020), 71 p.
[9] C. Boutiller and B. de Tilière, “The critical Z-invariant Ising model via dimers: the periodic case,” Probab. Theory Related Fields 147 (2010), 379-413. · Zbl 1195.82011

[10] C. Boutiller and B. de Tilière, “The critical Z-invariant Ising model via dimers: locality property,” Comm. Math. Phys. 301 (2011), no. 2, 473-516. · Zbl 1245.05027

[11] C. Boutiller, B. de Tilière, and K. Raschel, “The Z-invariant massive Laplacian on isoradial graphs,” Invent. Math. 208 (2017), no. 1, 109-189. · Zbl 1372.82016

[12] C. Boutiller, B. de Tilière, and K. Raschel, “The Z-invariant Ising model via dimers,” Probab. Theory Related Fields. 174 (2019), no. 1-2, 235-305. · Zbl 1419.82008

[13] C. Boutiller and B. de Tilière, “Height representation of XOR-Ising loops via bipartite dimers,” Electron. J. Probab. 19 (2014), no. 80, 1-33. · Zbl 1302.82014

[14] R. Burton and R. Pemantle, “Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances,” Ann. Probab. 21 (1993), no. 3, 1329-1371. · Zbl 0785.60007

[15] D. Chelkak, “Planar Ising model at criticality: state-of-the-art and perspectives,” Proceedings of the ICM 2018 3 (2019), 2789-2816.

[16] D. Chelkak, D. Cimasoni, and A. Kassel, “Revisiting the combinatorics of the 2D-Ising model,” Ann. Inst. Henri Poincaré, D 4 (2017), no. 3, 309-385. · Zbl 1380.82017

[17] D. Chelkak and S. Smirnov, “Discrete complex analysis on isoradial graphs,” Adv. in Math. 228 (2011), no. 3, 1590-1630. · Zbl 1253.82010

[18] D. Chelkak and S. Smirnov, “Universality in the 2D Ising model and conformal invariance of fermionic observables,” Inv. Math. 189 (2012), 515-580. · Zbl 1257.82020

[19] D. Chelkak and Y. Wan, “On the convergence of massive loop-erased random walks to massive SLE(2) curves,” arXiv preprint 1903.08045 (2019).

[20] S. Chhita, “The height fluctuations of an off-critical dimer model on the square grid,” J. Stat. Phys. 148 (2012), no. 1, 67-88. · Zbl 1253.82010

[21] D. Cimasoni, “Kac-Ward operators, kasteleyn operators, and s-holomorphicity on arbitrary surface graphs,” Ann. Inst. Henri Poincaré, D 2 (2015), 115-168. · Zbl 1334.82008

[22] D. Cimasoni and H. Dumitriu-Copin, “The critical temperature for the Ising model on planar doubly periodic graphs,” Electron. J. Probab. 18 (2013), no. 44, 1-18.

[23] H. Cohn, R. Kenyon, and J. Propp, “A variational principle for domino tilings,” J. Amer. Math. Soc. 14 (2001), no. 2, 297-346 (electronic). · Zbl 1037.82016

[24] R. G. Cooke, “Infinite matrices and sequence spaces,” Dover books on mathematics, 2014.

[25] R. Costa-Santos, “Geometrical aspects of the Z-invariant Ising model,” The European Physical Journal B 53 (2006), no. 1, 85-90.

[26] N. G. de Bruijn, “Algebraic theory of Penrose’s non-periodic tilings of the plane. I,” Indagationes Mathematicae (Proceedings) 84 (1981), no. 1, 39-52. · Zbl 0457.05021

[27] N. G. de Bruijn, “Algebraic theory of Penrose’s non-periodic tilings of the plane. II,” Indagationes Mathematicae (Proceedings) 84 (1981), no. 1, 53-66. · Zbl 0457.05022

[28] B. de Tilière, “Quadril-tilings of the plane,” Probab. Theory Related Fields 137 (2007), no. 3-4, 487-518. · Zbl 1109.05032

[29] B. de Tilière, “From cycle rooted spanning forests to the critical Ising model: an explicit construction,” Comm. Math. Phys. 319 (2013), no. 1, 69-110. · Zbl 1269.82016

[30] B. de Tilière, “Critical Ising model and spanning trees partition functions,” Ann. Inst. Henri Poincaré Probab. Stat. 52 (2016), no. 3, 1382-1405. · Zbl 1354.82013

[31] V. S. Dotsenko and V. S. Dotsenko, “Critical behaviour of the phase transition in the 2d ising model with impurities,” Adv. in Math. 32 (1983), no. 2, 129-172.

[32] J. Dubédat, “Exact bosonization of the Ising model,” 1112.4399 (2011).

[33] R. J. Duffin, “Potential theory on a rhombic lattice,” J. Combinatorial Theory 5 (1968), 258-272. · Zbl 0247.31003

[34] C. Fan and F. Y. Wu, “General lattice model of phase transitions,” Phys. Rev. B 2 (1970), 723-735.

[35] M. E. Fisher, “On the dimer solution of planar Ising models,” J. Math, Phys. 7 (1966), 1776-1781.

[36] V. V. Fock, “Inverse spectral problem for gk integrable system,” arXiv preprint 1903.08045 (2019).

[37] A. B. Goncharov and R. Kenyon, “Dimers and cluster integrable systems,” 46 (2013), no. 5, 747-813.

[38] L. P. Kadanoff and A. C. Brown, “Correlation functions on the critical lines of the Baxter and Ashkin-Teller models,” Ann. Phys. 121 (1979), no. 1, 85-90.

[39] L. P. Kadanoff and H. Ceva, “Determination of an operator algebra for the two-dimensional Ising model,” Phys. Rev. B 3 (1971), 3918-3939.

[40] L. P. Kadanoff and F. J. Wegner, “Some critical properties of the eight-vertex model,” Phys. Rev. B 4 (1971), 3989-3993.

[41] P. W. Kasteleyn, “The statistics of dimers on a lattice: I. the number of dimer arrangements on a quadratic lattice,” Physica 27 (1961), 1209-1225. · Zbl 1253.82010

[42] P. W. Kasteleyn, “Graph theory and crystal physics,” Graph Theory and Theoretical Physics, Academic Press, London,
S. Sheffield, \textit{Random surfaces}, Astérisque (2005), no. 304, vi+175.

S. Smirnov, \textit{Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model}, Ann. Math. Zbl 1062.05045

B. Nienhuis and H. J. F. Knops, \textit{Spinor exponents for the two-dimensional Potts model}, Phys. Rev. B. 32 (1985), and the

R. Kenyon, \textit{Determinantal spanning forests on planar graphs}, Ann. Probab. 47 (2019), no. 2, 952-988. - Zbl 1416.82010

R. Kenyon, A. Okounkov, and S. Sheffield, \textit{Dimers and amoebae}, Ann. of Math. (2) 163 (2006), no. 3, 1019-1056.

R. Kenyon and J.-M. Schlenker, \textit{Rhombic embeddings of planar quad-graphs}, Trans. Amer. Math. Soc. 357 (2005), no. 9, 3443-3458 (electronic). - Zbl 1062.05056

R. W. Kenyon, J. G. Propp, and D. B. Wilson, \textit{Trees and matchings}, Electron. J. Combin 7 (2000), no. 1, R25. - Zbl 0939.05066

G. Kirchhoff, \textit{Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird}, Annalen der Physik 148 (1847), 497-508.

H. A. Kramers and G. H. Wannier, \textit{Statistics of the two-dimensional ferromagnet. Part I}, Phys. Rev. 60 (1941), no. 3, 252-262. - Zbl 0027.28505

H. A. Kramers and G. H. Wannier, \textit{Statistics of the two-dimensional ferromagnet. Part II}, Phys. Rev. 60 (1941), no. 3, 263-276. - Zbl 0027.28506

G. Kuperberg, \textit{An exploration of the permanent-determinant method}, Electron. J. Combin. 5 (1998), no. 1, R46. - Zbl 0906.05055

D. F. Lawden, \textit{Elliptic functions and applications}, Applied Mathematical Sciences, vol. 80, Springer-Verlag, New York, 1989.

Z. Li, \textit{Critical temperature of periodic Ising models}, Comm. Math. Phys. 315 (2012), 337-381 (English). - Zbl 1259.82024

M. Lis, \textit{The fermionic observables in the ising model and the inverse kac-ward operator}, Annales Henri Poincaré 15 (2014), no. 10, 1945-1965. - Zbl 1305.82017

M. Lis, \textit{Phase transition free regions in the Ising model via the Kac-Ward operator}, Comm. Math. Phys. (2014), 1-16 (English).

N. Makarov and S. Smirnov, \textit{Off-critical lattice models and massive sles}, XVIIth International Congress on Mathematical Physics, 2010, pp. 362-371. - Zbl 1205.82055

C. Mercat, \textit{Discrete Riemann surfaces and the Ising model}, Comm. Math. Phys. 218 (2001), no. 1, 177-216. - Zbl 1043.82005

R. J. Messikh, \textit{The surface tension near criticality of the 2d-Ising model}, math/0610636 (2006).

C. Nash and D. O’Connor, \textit{Dimer geometry, amoeba and a vortex dimer model}, J. Phys. A: Math. Theor. 50 (2017).

B. Nienhuis, \textit{Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas}, J. Statist. Phys. 34 (1984), no. 5-6, 731-761. - Zbl 0595.76071

B. Nienhuis and H. J. F. Knops, \textit{Spinor exponents for the two-dimensional Potts model}, Phys. Rev. B. 32 (1985), 1872-1875.

L. Onsager, \textit{Crystal statistics. I. A two-dimensional model with an order-disorder transition}, Phys. Rev. 65 (1944), no. 3-4, 117-149. - Zbl 0060.40001

J. H. H. Perk and H. Au-Yang, \textit{Yang Baxter equations}, Encyclopedia of Mathematical Physics (Jean-Pierre Françoise, Gregory L. Naber, and Tsou Sheung Tsun, eds.), Academic Press, Oxford, 2006, pp. 465-473.

S. Sheffield, \textit{Random surfaces}, Astérisque (2005), no. 304, vi+175.

S. Smirnov, \textit{Towards conformal invariance of 2D lattice models}, Proceedings of the ICCM, Madrid, vol. 2, 2006, pp. 1421-1452. - Zbl 1112.82014

S. Smirnov, \textit{Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model}, Ann. Math. 172 (2010), no. 2, 1435-1467. - Zbl 1200.82011

S. Smirnov, \textit{Discrete complex analysis and probability}, Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II-IV: Invited Lectures, World Scientific, 2010, pp. 595-621. - Zbl 1251.30049

W. Sun, \textit{Toroidal dimer model and temperley’s bijection}, 1603.00690 (2016).

H. N. V. Temperley, \textit{In Combinatorics: Proceedings of the British combinatorial conference 1973}, (1974), 202-204.

H. N. V. Temperley and M. E. Fisher, \textit{Dimer problem in statistical mechanics-an exact result}, Philosophical Magazine 6 (1961), no. 68, 1061-1063. - Zbl 0126.25102

W. T. Tutte, \textit{The dissection of equilateral triangles into equilateral triangles}, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 44, Cambridge Univ Press, 1948, pp. 463-482. - Zbl 0030.40903

D. B. Wilson, \textit{XOR-Ising loops and the Gaussian free field}, 1102.3782 (2011).
[73] F. W. Wu, *Ising model with four-spin interactions*, Phys. Rev. B 4 (1971), 2312-2314.

[76] F. Y. Wu and K. Y. Lin, *Staggered ice-rule vertex model - The Pfaffian solution*, Phys. Rev. B 12 (1975), 419-428.

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