On definition of CI-quasigroup

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Abstract

Groupoid \((Q, \cdot)\) in which equality \((xy)Jx = y\) is true for all \(x, y \in Q\), where \(J\) is a map of the set \(Q\), is a CI-quasigroup.

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1 Introduction

Necessary definitions can be found in [4, 2, 6, 7].

Definition 1. Binary groupoid \((Q, \circ)\) is called a quasigroup if for any ordered pair \((a, b) \in Q^2\) there exist the unique solutions \(x, y \in Q\) to the equations \(x \circ a = b\) and \(a \circ y = b\) [2].

Definition 2. A quasigroup \((Q, \cdot)\) with an element \(1 \in Q\), such that \(1 \cdot x = x \cdot 1 = x\) for all \(x \in Q\), is called a loop.

We start from classical definition of Artzy [1].

Definition 3. Loop \((Q, \cdot)\) satisfying one of the equivalent identities \(x \cdot yJx = y\), \(xy \cdot Jx = y\), where \(J\) is a bijection of the set \(Q\) such that \(x \cdot Jx = 1\), is called a CI-loop.

In [1] it is proved that \(J\) is an automorphism of loop \((Q, \cdot)\).

Definition 4. Quasigroup \((Q, \cdot)\) with the identity \(xy \cdot Jx = y\), where \(J\) is a map of the set \(Q\), is called a CI-quasigroup [3].

Notice, in this case the map \(J\) is a permutation of the set \(Q\) [3]. In any CI-quasigroup the permutation \(J\) is unique [7, Lemma 2.25].

Definition 5. Groupoid \((Q, \cdot)\) with the identity

\[ xy \cdot J_r x = y, \]  

where \(J_r\) is a map of the set \(Q\) into itself, is called a left CI-groupoid.

Groupoid \((Q, \cdot)\) with the identity

\[ J_l x \cdot yx = y, \]  

where \(J_l\) is a map of the set \(Q\) into itself, is called a right CI-groupoid.

Groupoid \((Q, \cdot)\) with both identities [1] and (2) is called a CI-groupoid.
Definition 5 is given in [3]. A groupoid with the equations (1) and (2) is called a CI-groupoid in [5].

Any CI-groupoid is a quasigroup [3]. In CI-quasigroup the identities (1) and (2) are equivalent [3]. From the results of [3] it follows that any left CI-groupoid is a left quasigroup.

From the results of Keedwell and Shcherbacov (see, for example, [7, Proposition 3.28]) it follows that the left CI-groupoid in which the map \( J_r \) is bijective, is a CI-quasigroup. Any finite left CI-groupoid is a quasigroup [5].

2 Result

Lemma 1. Any left CI-groupoid is a left quasigroup [3].

Proof. We prove that in the left CI-groupoid \((Q, \cdot)\) the equation

\[ a \cdot x = b \]  

has the unique solution. From the equation (3) we have \( ax \cdot J_r a = b \cdot J_r a \). If we substitute last expression in (3), then we obtain the following equality:

\[ a \cdot b J_r a = b. \]  

Uniqueness. Suppose that there exist two solutions of equation (3), say, \( x_1 \) and \( x_2 \). Then \( ax_1 = ax_2 \), \( ax_1 \cdot J_r a = ax_2 \cdot J_r a \) and from equality (1) we obtain that \( x_1 = x_2 \).

Therefore any left translation \( L_x \) of groupoid \((Q, \cdot)\) is a bijective map.

Lemma 2. There exists a bijection between the set \( Q \) and the set \( R \), the map \( J_r \) is bijective and \( J_r Q = Q \).

Proof. We can rewrite the identity (1) in the following translation form:

\[ R_{J_r x} L_x = \varepsilon. \]  

From the equality (5) and Lemma 1 it follows that the map \( R_{J_r d} \) is a bijection of the set \( Q \) for any fixed element \( d \in Q \).

There exists a bijection between the set \( Q \) and the set \( L \) of all left translations of groupoid \((Q, \cdot)\). Namely \( x \leftrightarrow L_x, Q \leftrightarrow L \).

From the equality (5) we have that there exists a bijection between the set \( L \) and the set \( R \) of all translations (bijections) of the form \( R_{J_r x} \), namely, \( L_x \leftrightarrow R_{J_r x}, L \leftrightarrow R \).

Therefore there exists a bijection between the set \( Q \) and the set \( R \), the map \( J_r \) is bijective and \( J_r Q = Q \).

Theorem 1. Any left CI-groupoid \((Q, \cdot)\) is a CI-quasigroup.

Proof. Taking into consideration Lemma 1 we must only prove that in the left CI-groupoid \((Q, \cdot)\) the equation

\[ y \cdot a = b \]  

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has the unique solution. Using the language of translations we re-write equation (6) in the following form: \( R_a y = b \). By Lemma 2, the map \( R_a \) is a bijection and right translation \( R_a \) there exists for any \( a \in Q \). Then \( y = R_a^{-1} b \).

Therefore any left CI-groupoid \((Q, \cdot)\) is a CI-quasigroup.

Notice, Theorem 1 can be proved using Lemmas 1, 2 and Proposition 3.28 from [7].

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