Quantum corrections to classical solutions via generalized zeta-function

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A general algebraic method of quantum corrections evaluations is presented. Quantum corrections to a few classical solutions of Landau-Ginzburg model (phi-in-quadro) are calculated in arbitrary dimensions. The Green function for heat equation with soliton potential is constructed by Darboux transformation. The generalized zeta-function is used to evaluate the functional integral and corrections to mass in quasiclassical approximation. Some natural generalizations for matrix equations are discussed in conclusion.

1 Introduction

In the papers of V.Konoplich [1] quantum corrections to a few classical solutions by means of Riemann zeta-function are calculated in dimensions $d > 1$. Most interesting of them are the corrections to the kink - the separatrix solution of field $\phi^4$ model. The method of [1] is rather complicated and it is desired to simplify it, that is the main target of this note with some nontrivial details missed in [1]. The suggested approach open new possibilities; for example it allows to show the way to calculate the quantum corrections to matrix models of similar structure, Q-balls [2] and periodic solutions of the models. The last problem is posed in the review [3].

We mainly concentrate on a technique of evaluation of the quantum corrections to static one-dimensional solutions of the d-dimensional Landau-Ginzburg model. Starting from a sketch of the method, in this introduction, we also give the detailed description of the solutions. The zeta-function is introduced in the next section, the last section is devoted to the evaluation of the diagonal of the Schrödinger operator kernel.
The nonlinear Klein-Gordon equation in the case of static one-dimensional solutions is reduced to

$$\phi'' - V'(\phi) = 0, \phi = \phi(x), x \in \mathbb{R}. \quad (1.1)$$

Suppose the potential $V(\phi)$ is twice continuously differentiable; it guarantees existence and uniqueness of the equations’ (1.1) Cauchy problem solution. The first integral of (1.1) is given by

$$E = \frac{1}{2}(\phi')^2 - V(\phi), \quad (1.2)$$

where $E$ is the integration constant. The equation (1.2) is ordinary first-order differential equation with separated variables. As the phase method shows, the solutions of this equations belong to the following families: constant, periodic, separatrix and the so-called “passing” one [3].

The approximate quantum corrections to the solutions of the equation (1.1) are obtained if the Feynmann functional integral by trajectories is evaluated by the continual stationary phase method. It gives the following relation

$$\exp\left[-\frac{S_{qu}}{\hbar}\right] \simeq \frac{A}{\sqrt{D}}, \quad (1.3)$$

where $S_{qu}$ denotes quantum action, corresponding the potential $V(\phi)$, $A$ - some quantity determined by the vacuum state at $V(\phi) = 0$, and $\det D$ is the determinant of the operator

$$D = -\partial^2_x - \Delta_y + V''(\phi(x)). \quad (1.4)$$

The variable $y \in \mathbb{R}^{d-1}$ stands for the transverse variables on which the solution $\phi(x)$ does not depend. The operator $D$ appears while the evaluation of the second variational derivative of the quantum action functional (which enter the Feynmann trajectory integral) is provided. For the vacuum action $S_{vac}$ the relation of the form (1.3) is valid if $S_{qu}$ is changed to $S_{vac}$ and $D$ is placed by the ”vacuum state” operator $D_0 = -\partial^2_x - \Delta_y$. Then, the quantum correction

$$\Delta S_{qu} = S_{qu} - S_{vac}, \quad (1.5)$$

is obtained by the mentioned twice use of the formula (1.3) as

$$\Delta S_{qu} = \frac{\hbar}{2} \ln\left(\frac{\det D}{\det D_0}\right). \quad (1.6)$$

Hence, the problem of determination of the quantum correction is reduced to one of evaluation of the determinants od $D$ and $D_0$. The methodic of the evaluation will be presented in the following section.

2 The generalized Riemann zeta-function, preliminaries.

The generalized zeta-function appears in many problems of quantum mechanics and quantum field theories which use the Lagrangian $L = (\partial \phi)^2/2 - V(\phi)$ and it is necessary to calculate a Feynmann functional integral in quasiclassical approximation.
The scheme is following. Let \( \{\lambda_n\} = S \) be a set of all eigenvalues of a linear operator \( L \), then, logarithm of the operator determinant is represented by the formal sum by this set

\[
\ln(\det L) = \sum_{\lambda_n \in S} \ln \lambda_n. \tag{2.7}
\]

Let us next define the generalized Riemann zeta-function \( \zeta_L(s) \) by the equality

\[
\zeta_L(s) = \sum_{\lambda_n \in S} \lambda_n^{-s}. \tag{2.8}
\]

This definition should be interpreted as analytic continuation from the half plane \( \Re s > \sigma \) in which the sum in (2.8) converge. Differentiating the relation (2.8) with respect to \( s \) at the point \( s = 0 \) yields

\[
\ln(\det L) = \zeta'_L(0). \tag{2.9}
\]

The generalized function (2.8) admits the representation via the diagonal \( g_D \) of a Green function of the operator \( \partial_t + L \). The representation is obtained as follows.

Let \( r \in \mathbb{R} \) be the set of independent variables of the operator \( L \); particularly, the operator \( D \) of (1.4) depends on \( r = (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1} \), then

\[
(\partial_t + L)g(t, r, r_0) = \delta(r - r_0) \tag{2.10}
\]

and

\[
g(t, r, r_0) = 0, \quad t < 0.
\]

There is a representation in terms of the formal sum

\[
g_D(t, r, r_0) = \sum_n \exp\left[-\lambda_n t\right] \psi_n(r) \psi^*_n(r_0), \tag{2.11}
\]

where the normalized eigenfunctions \( \psi_n(r) \) correspond to eigenvalues \( \lambda_n \) of the operator \( D \). Let us introduce the function

\[
\gamma_D(t) = \int dr g_D(t, r, r) = \sum_n \exp\left[-\lambda_n t\right], \tag{2.12}
\]

that follows from (2.11) and normalization. The Mellin transformation of (2.12) yields in

\[
\zeta_D(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \gamma_D(t) dt, \tag{2.13}
\]

where the \( \Gamma(s) \) is the Euler Gamma-function.

The generalized zeta-function, defined by the relations (2.12, 2.13), will be referred as the zeta-function of the operator \( D \).

From the relation (2.12) for the function \( \gamma_D(t) \) it follows most important property of multiplicity: if the operator \( D \) is a sum of two differential operators \( D = D_1 + D_2 \), which depend on different variables, the following equality holds

\[
\gamma_D(t) = \gamma_{D_1}(t) \gamma_{D_2}(t). \tag{2.14}
\]
We will need the value of the function \( \gamma_D(t) \) for the vacuum state, when the operator \( D = D_0 = -\Delta \) is equal to the d-dimensional Laplacian. In this case the formal sum in the r.h.s of \((2.12)\) goes to d-dimensional Poisson integral

\[
\gamma_D(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}} d\mathbf{k} \exp(-|\mathbf{k}|^2 t) = (4\pi t)^{-d/2}. \tag{2.15}
\]

The basic relation \((1.6)\) points to a necessity of evaluation of the determinants of the operators

\[
D = D_0 + u(x), \quad D_0 = -\partial_x^2 - \Delta_y + \lambda \tag{2.16}
\]

where \( \lambda \) is a positive number and \( x \in \mathbb{R} \) is one of variables, while \( y \in \mathbb{R}_{d-1} \) is a set of other variables. The operator \( \Delta_y \) is the Laplacian in \( d-1 \) dimensions, \( u(x) \) is one-dimensional potential that is defined by the condition

\[
V''(\phi_0(x)) = \lambda + u(x), \tag{2.17}
\]

where \( \phi_0(x) \) is the classical static solution of the equation of motion.

A quantum correction to the action in one-loop approximation for the classical solution \( \phi(x) \) is calculated via zeta-function by the formula

\[
\Delta \epsilon = -\zeta_D'(0)/2, \tag{2.18}
\]

where

\[
\zeta_D(s) = M^{2s} \int_0^\infty \gamma(t) t^{s-1} dt / \Gamma(s); \tag{2.19}
\]

here \( \Gamma(s) \) is the Euler gamma function and \( M \) is a mass scale.

The function \( \gamma(t) \) in the Mellin integral \((2.19)\) is expressed via the Green functions difference \( G(x,y,t;x_0,y_0,t_0) \) and \( G_0(x,y,t;x_0,y_0,t_0) \) of \( \partial_t + D \) and \( \partial_t + D_0 \) in the following way; let

\[
g(x,t) = G(x,y,t;x_0,y_0,0) - G_0(x,y,t;x_0,y_0,0),
\]

due to the translational invariance along \( y \) of operators \( D \) and \( D_0 \) the function \( g \) does not depend on \( y \); the contraction of \( G_0 \) is necessary for deleting of ultraviolet divergence. Then the following formula is valid

\[
\gamma(t) = \int_{-\infty}^{\infty} g(x,t) dx. \tag{2.20}
\]

In the case of static solutions of the \( \phi^4 \) model the potential and the Lagrangian are determined by the formulas

\[
V(\varphi) = g \varphi^4/4 - m^2 \varphi^2/2,
\]

\[
L = -(\varphi')^2 - g \varphi^4/4 + m^2 \varphi^2/2,
\]

therefore the equation of motion has the form

\[
\varphi''(x) + m^2 \varphi - g \varphi^3 = 0. \tag{2.21}
\]

Its separatrix solution is the kink

\[
\varphi_0 = m \tanh(mx/\sqrt{2})/\sqrt{g}. \tag{2.22}
\]
After the substitution of (2.22) into (2.17) we obtain the following potential \( u(x) \):

\[
u(x) = -\frac{6b^2}{ch^2(bx)},
\]

with the meaning of the constant \( b = m/\sqrt{2} \). As a result the two-level reflectionless potential of one-dimensional Schrödinger equation \(-\partial_x^2 + u(x)\) appears. Eigenvalues and the normalized eigenfunctions of which are correspondingly (its numeration is chosen from above to lowercase).

\[
\lambda_1 = -b^2, \quad \psi_1(x) = \sqrt{3b/2} \sinh(bx)/\cosh^2(bx);
\]
\[
\lambda_2 = -4b^2, \quad \psi_2(x) = \sqrt{3b/2} \cosh(bx).
\]

In the next section (sec.3) we suppose for generality that the potential \( u(x) \) from (2.17) has the form of \( n \)-level reflectionless potential

\[
u(x) = -\frac{n(n+1)b^2}{\cosh^2(bx)}.
\]

(2.24)

with eigenvalues \( \lambda_m = -m^2b^2, \ m = 1, ..., n \). We would note for further generalization that this function may be considered as degenerate limit of \( n \)-gap Lame potential of Hill equation (see conclusion). Such potentials correspond to higher solitonic models.

The kink case corresponds \( n = 2, \lambda = n^2b^2 \). The quantum correction to its action will be calculated in sec.4 from general formulas of sec.3.

### 3 Zeta-function representation and its derivative at zero point.

Formula (2.20) for \( \gamma(t) \) may be simplified if one expresses the integrand via the Green function of one-dimensional Schrödinger operator with nonzero and zero potentials.

**Proposition 1.** Let \( G^{(1)}(x, t; x_0, t_0) \) and \( G^{(1)}_0(x, t; x_0, t_0) \) are Green functions of one dimensional evolution operators \( \partial - \partial_x^2 + u(x) \) and \( \partial - \partial_x^2 \). And let

\[
e(x, t) = G^{(1)}(x, y, t; x_0, y_0, t_0) - G^{(1)}_0(x, y, t; x_0, y_0, t_0),\]

then the following representation for \( g(x, t) \) from (2.20) is valid:

\[
g(x, t) = (4\pi t)^{(1-d)/2} \exp[-\lambda t] e(x, t).\]

(3.25)

**Proof.** The Green functions \( G(x, y, t; x_0, y_0, t_0) \) and \( G_0(x, y, t; x_0, y_0, t_0) \) are easily expressed via \( G^{(1)}(x, t; x_0, t_0) \) and \( G^{(1)}_0(x, t; x_0, t_0) \) by means of Fourier transformations by \( y \); putting in these expressions \( x_0 = x, y_0 = y, t_0 = t \) we get

\[
g(x, t) = \exp[-\lambda t] e(x, t) \int_{\mathbb{R}^{d-1}} \exp[-|k|^2 t] dk/(2\pi)^{d-1}\]

(here \( k \) is the vector Fourier transform parameter). The known Poisson integral (2.15) of power \( d-1 \) appears; using its value one goes to (3.25).

**Corollary.** The following trace formulae takes place:

\[
\gamma(t) = (4\pi t)^{(1-d)/2} \exp[-\lambda t] \gamma_0(t),\]

(3.26)
\[ \gamma_0(t) = \int_{-\infty}^{\infty} e(x,t)dx \]

**Remark 1.** Formula (3.26) is named as trace formula because \( \gamma_0(t) \) is the difference of traces of reciprocals of operators \( \partial - \partial_x^2 + u(x) \) and \( \partial - \partial_x^2 \). This sense of \( \gamma_0(t) \) is reserved for possible generalizations of the problem under consideration (see conclusion).

Now we go to the determination of the explicit form of the function \( \gamma_0(t) \). From the results of [4] it is possible to extract the following

**Proposition 2.** Let the integral

\[ \hat{e}(x,t) = \int_{0}^{\infty} \exp[-pt]dt \]  

be the Laplace transformation of \( e(x,t) \), then

\[ \hat{e}(x,p) = \frac{1}{\sqrt{p}} \sum_{\nu=1}^{n} \frac{\psi_\nu(x)\nu b/(p - \nu^2b^2)}{d - 1 - s} \Gamma(s + 1 - d/2, 1, 3/2; \nu^2b^2/\lambda) \]  

where \( \psi_\nu(x) \) is a normalized eigen function that correspond to the eigenvalue \( \lambda_\nu = -\nu^2b^2, \nu = 1, \ldots, n \), of the one- dimensional operator with the potential (2.24).

From (3.26)-(3.28) we obtain the Laplace transform of the function \( \gamma_0(t) \)

\[ \int_{-\infty}^{\infty} \hat{e}(x,p)dx = \frac{1}{\sqrt{p}} \sum_{\nu=1}^{n} \nu b/(p - \nu^2b^2), \]

and by the table of inverse Laplace transformations [8] one finds

\[ \gamma_0(t) = \sum_{\nu=1}^{n} \exp[\nu^2b^2t]Erf(\nu b \sqrt{t}). \]  

Now one can derive the basic result (probably absent in publications) of the section.

**Theorem.** Zeta-function \( \zeta_D(s) \) has the following representations:

\[ \zeta_D(s) = \frac{4\Gamma(s + 1 - d/2)}{(4\pi)^{d/2}\Gamma(s)} \sum_{\nu=1}^{n} (\nu b)^{d-1}(M/\nu b)^{2s} \int_{0}^{1} (\tau^2 - 1 + \lambda/\nu^2b^2)^{d/2-1-s}d\tau; \]

\[ \zeta_D(s) = b(\frac{\lambda}{4\pi})^{2-1}(\frac{M^2}{\lambda})^{s} \frac{\Gamma(s + 1 - 1/2)}{\pi\Gamma(s)} \sum_{\nu=1}^{n} \nu F(s + 1 - d/2, 1, 3/2; \frac{\nu^2b^2}{\lambda}). \]

**Proof.** The integral that define error function may be transformed by the simple change of variables to the form

\[ Erf(z) = 2z \int_{0}^{1} \exp[-z^2\tau^2]d\tau/\sqrt{\pi}. \]

Substituting this representation to the formulae (3.29) and further in (3.26), (2.20) and (2.19), by means of analytical prolongation in \( s \), one arrives at the expression (3.30). The integral in it is transformed via Hypergeometrical function [9] that gives (3.31). This expression is cited from esthetic point; further only the formulae (3.30) is used.
4 Quantum correction to the kink and periodic solutions mass in the 1, 2, 3, 4 dimensions

Now we go to the zeta-function derivative evaluation. The meaning of $\zeta'_D(0)$ should be calculated in different ways for $d=1$, even $d > 2$, and odd ones. For the beginning we formulate and prove the following useful intermediate result.

**Proposition 3.** When $d=1$, $\lambda > n^2b^2$ and when $d = 2N-1$, $N = 2, 3, \ldots$, $\lambda \geq n^2b^2$ the following equality is valid

$$
\zeta'_D(0) = 4(4\pi)^{1/2-N}\Gamma(3/2 - N) \sum_{\nu=1}^{n} (\nu b)^{2(N-1)} R_N(\lambda/\nu^2b^2),
$$

(4.32)

where,

$$
R_N(z) = \int (\tau^2 - 1 + z)^{N-3/2}d\tau.
$$

(4.33)

When $d = 1$, $\lambda \geq n^2b^2$,

$$
\zeta'_D(0) = \frac{2}{\sqrt{\pi}} \sum_{\nu=1}^{n-1} R_1(n^2/\nu^2) - \frac{d}{ds} \left| \frac{\Gamma(s + 1/2)}{\Gamma(s + 1)} \left( \frac{M}{\nu^2b^2} \right)^2 \right|_{s=0}.
$$

(4.34)

When $d = 2N$, $N = 1, 2, \ldots$, $\lambda \geq n^2b^2$,

$$
\zeta'_D(0) = \frac{4(-1)^N}{(4\pi)^N(N-1)!} \sum_{\nu=1}^{n} (\nu b)^{2N-1} ((\gamma_N + 2 \ln(\frac{M}{\gamma b})) P_{N-1}\left( \frac{\lambda}{\nu^2b^2} \right) - J_{N-1}\left( \frac{\lambda}{\nu^2b^2} \right),
$$

(4.35)

where, for $N = 0, 1, 2, \ldots$,

$$
P_N(z) = \int_{0}^{1} (\tau^2 - 1 + z)^N d\tau, \quad J_N(z) = \int_{0}^{1} (\tau^2 - 1 + z)^N \ln(\tau^2 - 1 + z)d\tau,
$$

(4.36)

and

$$
\gamma_N = \sum_{j=1}^{N-1} 1/j.
$$

Proof. At $d=2N$ the formulae (3.30) converts in the following one:

$$
\zeta'_D(s) = 4(4\pi)^{1/2-N}\frac{\Gamma(s + 3/2 - N)}{\Gamma(s)} \sum_{\nu=1}^{n} (\nu b)^{2(N-1)} (\frac{M}{\nu b})^s \int_{0}^{1} (\tau^2 - 1 + \lambda/\nu^2b^2)^{N-s-3/2}d\tau
$$

(4.37)

In the cases $N \geq 2, \lambda \geq n^2b^2$, and $N = 1, \lambda > n^2b^2$, the function in the right-hand side is analytical in the vicinity of the point $s = 0$ and $\lim_{s \to 0} \zeta'_D(s) = 0$. Therefore $\zeta'_D(0) = \lim_{s \to 0} \zeta'_D(s)/s$. As $\lim_{s \to 0} \frac{1}{\Gamma(s)} = 1$, from (4.37) one immediately obtain (4.32).

The case $d = 1$, $\lambda = n^2b^2$ is singular because the integral in the last term of (4.37) diverges at $s = 0$ (when $\nu = n$ it degenerates into the integral $\int_{0}^{1} \tau^{-1-2s}d\tau$). However, due to $1/\Gamma(s) \simeq s$ when $s \to 0$, the term with $\nu = n$ is continued analytically till $s = 0$ but now $\lim_{s \to 0} \zeta'_D(s) \neq 0$.

Really, evaluating the integral in this term inside the region of convergence $\Re s <$, one gets $\int_{0}^{1} \tau^{-1-2s}d\tau = -1/2s$. Thus the last term in (4.37) in the case of $d=1, \lambda = n^2b^2$ yields in
analytical (in the vicinity of $s = 0$) function $\frac{\Gamma(s+1/2)}{\sqrt{\pi s+1}} \left( \frac{M}{n} \right)$. Separating further in r.h.s. of (4.37) the last term and calculating derivatives of the rest terms in $s = 0$ in the same manner, one arrives at (4.34).

In even dimensions $d = 2N$, the formula (3.30) takes the form

$$\zeta_D(s) = \frac{4\Gamma(s+1-N)}{(4\pi)^N \Gamma(s)} \sum_{\nu=1}^n (\nu b)^{2N-1} \left( M/\nu b \right)^{2s} \int_0^1 (\tau^2 - 1 + \frac{\lambda}{\nu^2 b^2})^{N-1-s} d\tau; \quad (4.38)$$

Differentiating it in $s = 0$ with account of the equality

$$\frac{d}{ds} \ln \frac{\Gamma(s+1-N)}{\Gamma(s)} \bigg|_{s=0} = \gamma_N$$

one obtains (4.35). The statement 3 gives an intermediate result; the final form derivation needs the calculation of integrals in (4.33,4.36).

Let us start with

$$R_1(z) = \int (\tau^2 - 1 + z)^{-1/2} d\tau = \frac{1}{2} \ln \frac{\sqrt{z+1}}{\sqrt{z-1}}, \quad z > 1.$$  

Therefore $\zeta(0) = \ln \prod_{\nu=1}^n \frac{\sqrt{\lambda+\nu b}}{\sqrt{\lambda-\nu b}}, \quad d=1, \lambda > n^2 b^2$.

It is easy to check (see e.g. [9])

$$\frac{d}{ds} \ln \frac{\Gamma(s+1/2)}{\Gamma(s+1)} \bigg|_{s=0} = -\ln 4;$$

using this one finds from (4.34) for $d=1, \lambda = n^2 b^2$:

$$\zeta_D(0) = \ln \left( \frac{4C^{n-1} n^2 b^2}{M^2} \right).$$

The values of $\zeta_D(0)$ are calculated by the recurrences

$$R_{N+1}(z) = \frac{2N-1}{N} (z-1) R_N + \frac{z^{N-1/2}}{2N}, \quad N = 1, 2, \ldots,$$

$$P_{N+1}(z) = \frac{2N+1}{2N+3} (z-1) P_N + \frac{z^{N+1}}{2N+3}, \quad N = 0, 1, 2, \ldots,$$

$$J_{N+1}(z) = \frac{2N-1}{N} (z-1) J_N + \frac{1}{2} N z^{N-1/2}, \quad N = 0, 1, 2, \ldots;$$

that start from

$$R_1, \quad P_0 = 1, \quad J_0(z) = 2\sqrt{z-1} \text{arcsin} \frac{1}{\sqrt{z}};$$

(obtained by integrating by parts in integrals of (4.33) and (4.36)). After all transformations, the following expressions for $d = 1.2.3.4$ are collected at the next

**Statement 4.** At $d = 1$,

$$\zeta'_D(0) = \ln \prod_{\nu=1}^n \frac{\sqrt{\lambda+\nu b}}{\sqrt{\lambda-\nu b}}$$
if $\lambda > n^2 b^2$; otherwise,

$$\zeta_D'(0) = \ln \left( \frac{4 C_{2n-1}^n n^2 b^2}{M^2} \right),$$

\hspace{1cm} (4.39)

if $\lambda = n^2 b^2$.

At $d=2$

$$\zeta_D'(0) = \frac{n(n+1)b}{2\pi} \left( \ln \frac{M^2}{\lambda} - \frac{2}{\pi} \sum_{\nu=1}^n \sqrt{\frac{\lambda - \nu^2 b^2}{\lambda + \nu^2 b^2}} \arcsin \frac{\nu b}{\sqrt{\lambda + \nu^2 b^2}} \right), \quad \lambda > n^2 b^2$$

$$\zeta_D'(0) = \frac{n(n+1)b}{\pi} \left( \ln \frac{M}{mb} - \frac{2b}{\pi} \sum_{\nu=1}^n \sqrt{\frac{n^2 b^2 - \nu^2}{n^2 b^2 - \lambda}} \arcsin \frac{\nu}{n} \right), \quad \lambda = n^2 b^2$$

At $d=3$

$$\zeta_D'(0) = -\frac{n(n+1)b \sqrt{\lambda}}{4\pi} + \sum_{\nu=1}^n \left( \lambda - \nu^2 b^2 \right) \ln \frac{\sqrt{\lambda + \nu b}}{\sqrt{\lambda - \nu b}} \left( \frac{\arcsin \frac{\nu b}{\sqrt{\lambda + \nu^2 b^2}}}{\sqrt{\lambda - \nu^2 b^2}} \right), \quad \lambda > n^2 b^2$$

$$\zeta_D'(0) = -\frac{b^2}{4\pi} \left( n^2 (n+1)b + \sum_{\nu=1}^n \left( n^2 - \nu^2 \right) \frac{\arcsin \frac{\nu}{n}}{\sqrt{\lambda - \nu^2 b^2}} \right), \quad \lambda = n^2 b^2$$

At $d=4$

$$\zeta_D'(0) = \frac{1}{3\pi^2} \left\{ \frac{n(n+1)b}{8} \left( n(n+1)b^2 - 3\lambda \right) \ln \frac{M^2}{\lambda} + \sum_{\nu=1}^n \left( \lambda - \nu^2 b^2 \right)^{3/2} \frac{\arcsin \frac{\nu b}{\sqrt{\lambda + \nu^2 b^2}}}{\sqrt{\lambda - \nu^2 b^2}} \right\}, \quad \lambda > n^2 b^2$$

$$\zeta_D'(0) = -\frac{b^4}{3\pi^2} \left\{ \frac{n^2(n+1)(13n^2 - 8)}{24} \ln \frac{M}{mb} - \sum_{\nu=1}^{n-1} \left( n^2 - \nu^2 \right)^{3/2} \frac{\arcsin \frac{\nu}{n}}{\sqrt{\lambda - \nu^2 b^2}} \right\}, \quad \lambda = n^2 b^2.$$

These formulas (by other method and notations) at $d = 2, 3, 4$ were derived in [1].

In the case of the kink (see the sec. 2). $n=2$, $b= m/\sqrt{2}, \lambda = 4b^2 = 2m^2$, basing on the case $\lambda = n^2 b^2$, one obtains the quantum correction to the mass $\Delta \epsilon = \zeta_D'(0)$

$$d = 1: \quad \zeta_D'(0) = -2 \ln \frac{M}{2\sqrt{6}m},$$

\hspace{1cm} (4.43)

$$d = 2: \quad \zeta_D'(0) = -\frac{3\sqrt{2}m}{\pi} \left( 1 + \ln \frac{M}{\sqrt{2}m} \right)$$

\hspace{1cm} (4.44)

$$d = 3: \quad \zeta_D'(0) = -\frac{3m^2}{8} \left( \ln 3 + 4 \right)$$

\hspace{1cm} (4.45)

$$d = 4: \quad \zeta_D'(0) = \frac{m^3}{8} \left( \frac{1}{4\sqrt{6}} - \frac{3}{2\sqrt{2}\pi} \left( 1 + \ln \frac{M}{2\sqrt{6}m} \right) \right)$$

\hspace{1cm} (4.46)

These expressions reproduce the formulas for kink mass correction from [1] for $d=2,3,4$.

5 Conclusion

Let us note that all results related to the scalar nonlinear Klein-Gordon equation may be applied directly to many-component model $\phi(x) \in S^\infty$ with account of $SO(m)$ symmetry. The equation \hspace{1cm} (1.1) takes the form

$$-\phi'' + V'(|\phi|) \frac{\phi}{||\phi||} = 0.$$ 

\hspace{1cm} (5.47)

The scalar operator $D$ is defined by \hspace{1cm} (1.4) goes to the matrix one

$$D = [-\partial^2_x - \Delta_y + V'(|\phi|) \frac{\phi}{||\phi||} I_m + [V''(|\phi|) - V'(|\phi|) \frac{\phi \otimes \phi}{||\phi||^3}].$$

\hspace{1cm} (5.48)
The technique developed in this paper is transported to quantum corrections for periodic static solutions of $\phi^4$ model:

$$\phi_0(x) = \frac{km}{1+k^2} \sqrt{2} \frac{\text{sn}(\frac{mx}{\sqrt{1+k^2}}; k)}{g}, \quad 0 < k \leq 1; \quad (5.49)$$

where $k$ is a module of the elliptic function. When $k = 1$ the formula (5.49) goes to one for kink (2.22). The substitution of (5.49) into (2.17) yields in two-gap Lame potential that is embedded in Darboux Transformations theory by chain representation [10] that give a possibility to derive the Green function analogue for this case. The results will be published elsewhere. Some recent papers open new field for applications [11].

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