We discuss the transformation of the QCD temporal-gauge Hamiltonian to a representation in which it can be expressed as a functional of gauge-invariant quark and gluon fields. We show how this objective can be realized by implementing the non-Abelian Gauss's law, and by using the mathematical apparatus developed for that purpose to also construct gauge-invariant quark and gluon fields. We demonstrate that, in the transformed QCD Hamiltonian, the interactions of pure-gauge components of the gauge field with color-current densities are replaced by nonlocal interactions connecting quark color-charge densities to each other and to 'glue'-color. We discuss the nonperturbative evaluation of these nonlocal interactions, which are non-Abelian analogs of the Coulomb interaction in QED, and we explore their implications for QCD in the low-energy regime.

1 Introduction

The problem that my students and I addressed in the investigation I am discussing in this report, deals with the role of the non-Abelian Gauss’s law in generating a nonlocal interaction between gauge-invariant color-bearing particle states — in particular, quark and multi-quark states. Our conjecture that this nonlocal interaction might play an important, perhaps a decisive, role in color confinement, is based on an analogy with Abelian gauge theories such as QED. In covariant-gauge QED, the spinor (electron) field $\psi$ is a gauge-dependent quantity. It interacts with 'pure gauge' components of the gauge field $A^\mu$, and that interaction is responsible for generating — perturbatively, order by order — the parts of the S-matrix elements that correspond to the Coulomb interaction between charges. On the other hand, when covariant-gauge QED is transformed to a representation in which Gauss’s law is implemented and $\psi$ becomes a gauge-invariant charged field, the same nonlocal Coulomb interaction that is seen in the Coulomb gauge appears explicitly in that case, even though the covariant-gauge condition continues to apply. Quite generally, when only gauge-invariant fields are used in constructing the QED Hamiltonian, the interactions between charged fields and pure-gauge components of gauge fields vanish, and a nonlocal interaction between charge densities — the Coulomb interaction — appears in their stead. In gauge-invariant QED, the only interaction besides the Coulomb interaction is between the transverse (gauge-invariant) part of the gauge field and the current density;
this interaction takes the form $-\int j_i(r)A_{Ti}(r)\,dr$, where $A_{Ti}$ is the transverse (gauge-invariant) part of the gauge field, and $j_i(r) = e\psi^\dagger(r)\partial_i\psi(r)$. Since the current density has a $v/c$ dependence in the nonrelativistic limit, the Coulomb interaction is, by far, the most important electrodynamic force in the low-energy regime. That fact provides a strong incentive for formulating QCD in terms of gauge-invariant fields, since then, in QCD too, the interactions between the 'pure gauge' components of the gauge field and the color-current density must necessarily be eliminated and replaced by a nonlocal interaction between color-charge densities. We have obtained such a nonlocal interaction and, in the remainder of this paper, will discuss how it comes about and how it can facilitate our study of the low-energy regime of QCD.

2 Implementing the Non-Abelian Gauss's Law

The strategy of using only gauge-invariant fields in formulating gauge theories, in QCD as well as in QED, faces major technical hurdles. The most daunting is the implementation of the non-Abelian Gauss's law. The Gauss's law operator for QCD is

$$\hat{G}^a(r) = \partial_i\Pi^a_i(r) + gf^{abc}A^b_i(r)\Pi^c_i(r) + j^a_0(r),$$

(1)

where $\Pi^a_i$ is the negative chromoelectric field and, also, the momentum conjugate to $A^a_i(r)$; $j^a_0(r)$ is the color-charge density $j^a_0(r) = g\psi^\dagger(r)\overleftrightarrow{\partial}\psi(r)$, and $f^{abc}$ represents the SU(3) structure constants. The Abelian Gauss's law operator for QED is

$$\hat{G}(r) = \partial_i\Pi_i(r) + j_0(r),$$

(2)

where $j_0$ is the electric charge density $j_0(r) = g\psi^\dagger(r)\psi(r)$. In QED, $\hat{G}(r)$ and $\partial_i\Pi_i(r)$ are unitarily equivalent, which makes it far easier to implement Gauss's law in QED than in QCD. We note, for example, that $\partial_i\Pi_i(r) |\xi\rangle = 0$, or an equivalent equation, is easy to solve. Moreover, it is relatively straightforward to explicitly construct a unitary transformation that relates $\hat{G}$ and $\partial_i\Pi_i(r)$.

That unitary transformation enables us to transform solutions of $\partial_i\Pi_i(r) |\xi\rangle = 0$ into solutions of $\hat{G}(r) |\hat{\xi}\rangle = 0$, thereby implementing Gauss's law. In QCD, however, $\hat{G}^a$ and $\partial_i\Pi^a_i$ cannot be unitarily equivalent, because their commutator algebras differ. Whereas $[\partial_i\Pi^a_i(r), \partial_i\Pi^b_i(r')] = 0$,

$$[\hat{G}^a(r), \hat{G}^b(r')] = igf^{abc}\hat{G}^c(r)\delta(r - r').$$

Although $\partial_i\Pi^a_i(r) |\xi\rangle = 0$ is just as easy to solve as its Abelian counterpart, we cannot make use of that solution to implement the non-Abelian Gauss’s law

$$\hat{G}^a(r) |\hat{\Psi}\rangle = 0.$$  

(3)
Similarly, and which obeys the operator differential equation Eq.(4) also contains the chain of SU(3) structure constants necessary to find an explicit solution of Eq.(4). We provided such a solution in our work but, for lack of space, cannot repeat it here.

That fact requires us to use a fundamentally different strategy for implementing Gauss’s law in QCD.

We have constructed states that satisfy Gauss’s law in QCD and Yang-Mills theory, by solving an operator-valued differential equation. In that work, we defined an operator-valued quantity \( \widetilde{A}_i(r) \), which we will call the resolvent gauge field, and which obeys the operator differential equation

\[
i \int dr' \left[ \partial_i \Pi^\alpha(r), \widetilde{A}_j^\alpha(r') \right] V_j^\gamma(r') + ig f^{a\beta\gamma} A_i^a(r) \int dr' \left[ \Pi^\alpha(r), \widetilde{A}_j^\alpha(r') \right] V_j^\gamma(r')
\]

\[
= -g f^{a\beta\gamma} A_i^a(r) V_i^\gamma(r) + \sum_{\eta=1}^{n+1} B(\eta) \frac{f^{a\beta\gamma}}{\eta(\eta+1)} A_i^a(r) \frac{\partial}{\partial \eta} \left( \mathcal{M}^\alpha_{(\eta)}(r) \partial_i V_i^\gamma(r) \right)
\]

\[- g f^{a\beta\gamma} A_i^a(r) \sum_{\eta=0}^{n} \sum_{t=1} (-1)^t t^{+d} \frac{B(\eta)}{\eta(!)(t+1)!} f^{\alpha\gamma \lambda} A_i^a(r) \frac{\partial}{\partial \eta} \left( \mathcal{R}_{(\eta)}^\beta(\gamma)(r) \mathcal{M}^\alpha_{(\eta)}(r) \partial_i V_i^\gamma(r) \right)
\]

where \( \mathcal{R}_{(\eta)}^\beta(\gamma)(r) \) is the product of functions of the longitudinal gauge field

\[
\mathcal{R}_{(\eta)}^\beta(\gamma)(r) = \prod_{m=1}^{\eta} \lambda^\alpha[m](r) \quad \text{with} \quad \lambda^\alpha(r) = \left[ \frac{\partial}{\partial \eta} A_i^\gamma(r) \right].
\]

Similarly, \( \mathcal{M}^\alpha_{(\eta)}(r) \) is a functional of the resolvent gauge field

\[
\mathcal{M}^\alpha_{(\eta)}(r) = \prod_{m=1}^{\eta} \widetilde{\lambda}^\alpha[m](r) \quad \text{with} \quad \widetilde{\lambda}^\alpha(r) = \frac{\partial}{\partial \eta} \widetilde{A}_i^\gamma(r).
\]

Eq.(6) also contains the chain of SU(3) structure constants

\[
f^{\alpha\beta\gamma}_{(\eta)} = f^{[\alpha][\beta][\gamma]} f^{b[\alpha][\beta][\gamma]} f^{b[\alpha][\beta][\gamma]} \ldots f^{b[\alpha][\beta][\gamma]} f^{[\alpha][\beta][\gamma]},
\]

the Bernoulli numbers \( B(\eta) \), and \( V_i^\gamma(r) \), which represents an arbitrary vector field in the adjoint representation of SU(3). As shown in our earlier work, Eq.(6) is a reformulation of the non-Abelian Gauss’s law as a requirement on the resolvent gauge field \( \widetilde{A}_i(r) \). In order to implement Gauss’s law, it is also necessary to find an explicit solution of Eq.(6). We provided such a solution in our work but, for lack of space, cannot repeat it here.
3 Construction of the Gauge-Invariant Fields

The gauge field, and the resolvent gauge field, play a crucial role in determining the gauge-invariant spinor (quark) and gauge (gluon) fields. As we pointed out in earlier work, the Gauss’s law operator $\hat{G}^a(r)$, given in Eq. (1), and the ‘pure glue’ Gauss’s law operator $G^a(r) = \partial_i \Pi^i_a(r) + g f^{abc} A^i_b(r) \Pi^i_c(r)$, are unitarily equivalent, so that $G^a$ may be taken to represent $\hat{G}^a(r)$ in a different, unitarily equivalent representation. We refer to the representation in which $\hat{G}^a(r)$ is the Gauss’s law operator, the $C$ representation; and the representation in which $G^a$ represents the entire Gauss’s law operator, with the color-charge density $j^a_0(r)$ included (though implicitly only), the $N$ representation. The unitary equivalence is expressed as

$$\hat{G}^a(r) = U_C G^a(r) U_C^{-1},$$

where $U_C$ is a functional of the gauge field and the resolvent gauge field, as well as the color-charge density: $U_C = e^{C_0} e^{\bar{C}}$ with $C_0$ and $\bar{C}$ given by

$$C_0 = i \int dr \lambda^a(r) j^a_0(r), \quad \text{and} \quad \bar{C} = i \int dr \overline{\lambda}^a(r) j^a_0(r).$$

Since the quark field $\psi$ trivially commutes with $G^a(r)$, $\psi$ is manifestly gauge-invariant in the $N$ representation. The unitary operator $U_C$ transforms the quark field $\psi$ so that the gauge-invariant spinor field in the $C$ representation is

$$\psi_{GI}(r) = U_C \psi(r) U_C^{-1} = V_C(r) \psi(r),$$

where

$$V_C(r) = \exp \left( -ig\overline{\lambda}^a(r) \lambda^a \right) \exp \left( -ig\lambda^a(r) \overline{\lambda}^a \right).$$

and $\lambda^b$ represents the Gell-Mann SU(3) matrices. And, since $V_C(r)$ is a unitary operator, we can also readily see that the corresponding gauge-invariant gauge (gluon) field $A^b_{GI,i}(r)$ is given by

$$[ A^b_{GI,i}(r) \lambda^a ] = V_C(r) [ A^b_i(r) \lambda^a ] V_C^{-1}(r) + \frac{1}{g} V_C(r) \partial_i V_C^{-1}(r),$$

or, equivalently,

$$A^b_{GI,i}(r) = A^b_{T,i}(r) + [ \delta_{ij} - \frac{\partial_i}{\partial^2} ] A^b_{j}(r).$$

We can expand $\psi_{GI}(r)$ and $A^b_{GI,i}(r)$ to arbitrary order in $g$, and, in that way, we have verified that our gauge-invariant quark and gluon fields agree with the perturbative calculations of Lavelle, McMullan, et al., to the highest order to which their perturbative calculations were available.
4 The Gauge-Invariant QCD Hamiltonian

We can construct the gauge-invariant QCD Hamiltonian by systematically transforming the temporal-gauge QCD Hamiltonian from the conventional \( C \) representation to the \( N \) representation. In that representation, \( \psi(\mathbf{r}) \) represents the gauge-invariant quark field, and \( j_0^a(\mathbf{r}) \) the gauge-invariant color-charge density. In the \( N \) representation, \( j_0^a(\mathbf{r}) \) therefore implicitly includes ‘glue-color’ as well as the color of the bare quarks.

The QCD Hamiltonian that results from the transformation to the \( N \) representation is

\[
\tilde{H} = \int d\mathbf{r} \left[ \frac{1}{2} \Pi_i^a(\mathbf{r}) \Pi_i^a(\mathbf{r}) + \frac{1}{4} F_{ij}^a(\mathbf{r}) F_{ij}^a(\mathbf{r}) + \psi^\dagger(\mathbf{r}) (\beta m - i \alpha_i \partial_i) \psi(\mathbf{r}) \right] + \tilde{H}'.
\] (14)

\( \tilde{H}' \) describes interactions involving the gauge-invariant quark field. The parts of \( \tilde{H}' \) relevant to the dynamics of quarks and gluons can be expressed as

\[
\tilde{H}' = \tilde{H}_{j-A} + \tilde{H}_{LR}.
\] (15)

\( \tilde{H}_{j-A} \) describes the interaction of the gauge-invariant gauge field with the gauge-invariant quark color-current density, and is given by

\[
\tilde{H}_{j-A} = -g \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \alpha_i \frac{\lambda^b}{2} \psi(\mathbf{r}) A_{Gl}^b(\mathbf{r});
\] (16)

As in the case of QED, \( \tilde{H}_{j-A} \) couples the gauge-invariant gauge field to the current density \( j_i^b(\mathbf{r}) = g \psi^\dagger(\mathbf{r}) \alpha_i \frac{\lambda^b}{2} \psi(\mathbf{r}) \), and can be expected to have a \( v/c \) dependence for non-relativistic quarks, leaving the nonlocal \( \tilde{H}_{LR} \) as the dominant term in the low-energy regime. \( \tilde{H}_{LR} \) is the nonlocal interaction

\[
\tilde{H}_{LR} = H_{g-Q} + H_{Q-Q}.
\] (17)

A useful formulation of \( H_{Q-Q} \) can be given as

\[
H_{Q-Q} = \frac{1}{2} \int d\mathbf{r} d\mathbf{x} j_0^b(\mathbf{r}) \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) j_0^a(\mathbf{x}).
\] (18)

where the Green’s function \( \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \) is represented as

\[
\mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) = \frac{\delta_{ab}}{4\pi|\mathbf{r} - \mathbf{x}|} + 2g f^{h_{(1)}}(1)\frac{dy}{4\pi|\mathbf{r} - \mathbf{y}|} A_{Gl}^{b_{(1)}}(\mathbf{y}) \partial_i \frac{1}{4\pi|\mathbf{y} - \mathbf{x}|} +
\]

\[
\ldots + (-1)^{(n-1)(n+1)} g^n f^{h_{(2)}}(2) \cdots f^{h_{(n-1)}}(n) \frac{dy_1}{4\pi|\mathbf{r} - \mathbf{y}_1|} A_{Gl}^{b_{(2)}}(\mathbf{y}_1) 
\]

\[
\partial_i \frac{dy_2}{4\pi|\mathbf{y}_1 - \mathbf{y}_2|} \cdots \frac{dy_n}{4\pi|\mathbf{y}_{(n-1)} - \mathbf{y}_n|} A_{Gl}^{b_{(n)}}(\mathbf{y}_n) \partial_i \frac{1}{4\pi|\mathbf{y}_n - \mathbf{x}|}.
\] (19)
We make the following observations about $F^{ba}(r, x)$: The initial term resembles the Coulomb interaction. The infinite series of further terms consists of chains through which the interaction is transmitted from one color-charge density to the other. Each chain contains a succession of ‘links’, which have the characteristic form

$$\text{link} = g f^{s(1) \delta s(2)} A_{\alpha j}^b(x) \delta_j \frac{1}{4\pi |x - y|}. \quad (20)$$

The ‘links’ are coupled through summations over the $s(n)$ indices and integrations over the spatial variables. All the quantities in $H_{Q-Q}$ are gauge-invariant — the color-charge density $j_0^a(x)$ as well as the gauge field $A_{\alpha j}^a(x)$. $H_{g-Q}$ — the other nonlocal interaction in $\tilde{H}_{LR}$ — couples quark to gauge-invariant gluon color. In $H_{g-Q}$ the quark color-charge density is coupled, through the same Green’s function $F^{ba}(r, x)$, to a gauge-invariant expression describing ‘glue’-color, given by

$$K_d^b(r) = g f^{d\sigma} \text{Tr} \left[ V_{d^{-1}}(r) \frac{\delta^\sigma}{\pi^2} V_c(r) \frac{\delta^\lambda}{\pi^2} A_{\alpha i}^c(r) \Pi_b^\lambda(r) \right].$$

A number of further observations about Eq. (19) have relevance for low-energy QCD dynamics. The first of these is based on a kind of ‘color multipole’ expansion of $\int F^{ba}(r, x) j^a_0(x) d\mathbf{x}$ — which appears as part of $H_{Q-Q}$ as well as $H_{g-Q}$ — about the point $x = x_0$:

$$\int d\mathbf{x} \left\{ F^{ba}(r, x_0) + X_i \partial_i F^{ba}(r, x_0) + \frac{1}{2} X_i X_j \partial_i \partial_j F^{ba}(r, x_0) + \cdots \right\} j^a_0(x) \quad (21)$$

where $X_i = (x - x_0)_i$ and $\partial_i = \partial / \partial x_i$. When we perform the integration in Eq. (21), the first term contributes $F^{ba}(r, x_0) Q^a$, where $Q^a = \int d\mathbf{x} j_0^a(x)$ (the integrated ‘color charge’). Since the color charge is the generator of infinitesimal rotations in SU(3) space, it will annihilate any multiquark state vector in a singlet color configuration. Multiquark packets in a singlet color configuration therefore are immune to the initial term of the nonlocal $H_{Q-Q}$. Color-singlet configurations of quarks are only subject to the color multipole terms, which act as color analogs to the Van der Waals interaction. The scenario that this model suggests is that the leading term in $H_{Q-Q}$, namely $Q^b F^{ba}(r_0, x_0) Q^a$ for a quark color charge $Q^a$ at $r_0$ and another quark color charge $Q^b$ at $x_0$, contributes to the confinement of quarks and packets of quarks that are not in color-singlet configurations. Moreover, assuming that $F^{ba}(r, x_0)$ varies only gradually within a volume occupied by quark packets, the effect of the higher order color multipole forces on a packet of quarks in a color-singlet configuration becomes more significant as the packet increases in size. As small quark packets move through gluonic matter, they will experience only insignificant effects from the multipole contributions to $H_{Q-Q}$, since, as can be seen from
Eq. (21), the factors $X_i, X_j, \cdots, X_{i(1)} \cdots X_{i(n)}$, keep the higher order multipole terms from making significant contributions to $\int d\mathbf{x} F^{ba}(\mathbf{r}, \mathbf{x}) j_0^a(\mathbf{x})$ when they are integrated over small packets of quarks. As the size of the quark packets increases, the regions over which the multipoles are integrated also increases, and the effect of the multipole interactions on the color-singlet packets can become larger. This dependence on packet size of the final-state interactions experienced by color-singlet states — i.e. the increasing importance of final-state interactions as color-singlet packets grow in size — is in qualitative agreement with the characterizations of color transparency and color coherence given by Miller and by Jain, Pire and Ralston.  

Another observation pertains to the series given in Eq.(19). To fully achieve the purpose of this investigation, it is necessary to evaluate $F^{ba}(\mathbf{r}, \mathbf{x})$ nonperturbatively. The $n^{th}$ order term of $F^{ba}(\mathbf{r}, \mathbf{x})$ — i.e. the chain with $n$ links — has such a regular structure, that an explicit form can easily be written, without requiring knowledge of the lower order terms in the series. A nonperturbative evaluation of $F^{ba}(\mathbf{r}, \mathbf{x})$ is therefore far more accessible here than in the ‘standard’ gauge-dependent formulation in the $C$ representation, in which it is much more difficult to identify and isolate the interactions that make the dominant contributions in the low-energy regime.

In order to carry out a nonperturbative evaluation of $H_{Q-Q}$, either the $F^{ba}(\mathbf{r}, \mathbf{x})$ series has to be formally summed, or a more indirect method has to be found to achieve that objective. As of now, we have not yet addressed the evaluation of the gauge-invariant gauge field, which is an important component of $F^{ba}(\mathbf{r}, \mathbf{x})$. We have investigated an SU(2) model of QCD under the assumption that gluon correlations can be neglected, so that the expectation value $\langle 0 | A^a_{\mu i}(\mathbf{r}) | 0 \rangle$, in a state that implements the Gauss’s law $G^a(\mathbf{r}) | 0 \rangle = 0$, can be substituted for the operator-valued $A^a_{\mu i}(\mathbf{r})$ in Eq.(19). Furthermore, we made the ansatz that the transverse $\langle 0 | A^a_{\mu i}(\mathbf{r}) | 0 \rangle$ can be modeled by the manifestly transverse SU(2) ‘hedgehog’ function $\langle A^a_{\mu i}(\mathbf{r}) \rangle = \epsilon_{ij} j_i \phi(\mathbf{r})$. In this work, we showed how the SU(2) algebra enabled us to reduce the evaluation of $F^{ba}(\mathbf{r}, \mathbf{x})$ to the solution of a sixth-order differential equation, which circumvented the need for a perturbative evaluation of $F^{ba}(\mathbf{r}, \mathbf{x})$.

5 Discussion  

Abelian and non-Abelian gauge theories resemble each other in an important respect: The common thread that connects them is that the use of gauge-invariant fields eliminates interactions of pure-gauge components of gauge fields with charge and current densities, and replaces those interactions with nonlocal interactions among charge densities — the Coulomb interaction in QED, and
$\hat{H}_{LR}$, given in Eq. (17), in QCD. The structure of the Green’s function $F^{ba}(r, x)$ — a series of chains consisting of coupled links, in which the $n^{th}$ order chain is a product of $n$ links — suggests features closely associated with QCD: flux tubes, ‘string’-like structures tying colored objects to each other, etc. Further work will be required to determine whether these suggestive analogies are supported by detailed calculations.

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