COSMOLOGICAL DISTRIBUTION FUNCTIONS

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Abstract

The evolution of probability distribution functions (PDFs) of continuous density, velocity and velocity derivatives (deformation tensor) fields in the theory of cosmological gravitational instability are considered. We show that in the Newtonian theory the dynamical equations cannot be reduced to the closed set of Lagrangian equations. Since continuous fields from galaxy surveys need sufficiently large smoothing which exceeds the scale of nonlinearity, one can use the Zel’dovich approximation to describe the mildly non-linear matter evolution, which allows the closed set of Lagrangian equations. The closed kinetic equation for the joint PDF of cosmological continuous fields is derived in this approximation. The analytical theory of the cosmological PDFs with arbitrary (including Gaussian) initial statistics is developed, based on the solution on the kinetic equation.

For Gaussian initial fluctuations, the PDFs are parametrized by only linear \textit{rms} fluctuations $\sigma$ on given filtering scale. Density PDF $P(\rho, t)$ and PDF $M(\lambda_1, \lambda_2, \lambda_3; t)$ of eigenvalues of the deformation tensor field in the Eulerian space evolve very rapidly in non-linear regime. On the contrary, velocity PDF $Q(\vec{v}, t)$ remains invariant under non-linear evolution. For small $\sigma$ the Edgworth series is suggested to reconstruct $P(\rho, t)$. Density PDF $P(\rho, t)$ is close to the \textit{log-normal} PDF for the CDM model at moderate $\sigma$, but differs from that in general case.

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1 Introduction

One of the standard methods to study the statistical property of the non-linear Large Scale Structure (LSS) of the universe is the two-point correlation function \( \langle y(\vec{r}_1)y(\vec{r}_2) \rangle = \xi_y^{(2)}(\vec{r}_1, \vec{r}_2) \) and further, n–point functions \( \xi_y^{(n)}(\vec{r}_1, ..., \vec{r}_n) \) of the point-like statistical ensemble of galaxies, \( y \) may be the density, velocity components or other fields. The full statistics of the ensemble is described by the n–point PDFs \( P(y_1, ..., y_n) \) which are connected with \( \xi_y^{(n)} \) \([3]\). It is insufficient to know a few lower order correlation functions of galaxy distribution to define the full statistics of non-linear matter distribution in the universe. In this contribution we will consider the simplest one–point PDFs of the continuous non-linear cosmic fields. On the statistics of cosmological point processes see \([3]\).

The theoretical development of the subject begun with the calculations of the lowest moments \( \langle \delta^{(n)} \rangle \) of the density fields in the perturbation theory \([37], [18]\), in the Zel’dovich approximation \([19]\), for different cosmologies \([10]\), for filtered field \([22]\), in numerical simulations for non-linear regime \([12]\).

Recently even more attention has been drawn to the PDFs which can probe the statistics of primordial fluctuations. The standard CDM model with the scale free Gaussian initial fluctuations is hardly compatible with the set of observations. One way (fortunately not the only) is to switch the model to start with non-Gaussian primordial inhomogeneities. In the cosmic inflation scenario in principle, there is some room to introduce non-Gaussian fluctuations \([27]\), although it needs fine tuning in parameters, as a rule. Non-Gaussian perturbations also arise in other scenarios, where progenitors of inhomogeneities are topological defects, such as textures \([12]\), cosmic strings \([11]\), or explosions \([31]\).

The phenomenological attempts to design the density PDF at non-linear stage were made from different arguments. In \([10]\) the formula for \( P(\rho) \) from “thermodynamical” treatment of gravitational system was suggested, which evokes some scepticism. The log-normal mapping in order to mimick the non-linear density evolution was argued in \([13]\). Unfortunately the log-normal model does not work \([14]\), although the features of the log-normal distribution have been recognized in the density PDF \([21], [13], [29]\). An elegant analytical method has been developed \([1]\) to summarize the series of moments for the small density fluctuations \( \delta \) in the approximation when \( \langle \delta^{(n)} \rangle \approx S_n \sigma^{2(n-1)} \).

In \([20], [29]\) we calculated \( P(\rho) \) and \( F(\vec{v}) \) analytically in the “truncated” Zel’dovich approximation (i.e. with initial smoothing), and found a good agreement with the results of N-body simulations (with final smoothing). Recently two different methods from papers \([1]\) and \([29]\) were compared resulting in a good mutual agreement \([2]\).

In \([36]\) further approximation was made within the Zel’dovich theory to calculate \( P(\rho) \) after the final smoothing. Unfortunately the assumptions made are valid for small fraction of the Lagrangian volume and result does not fit the numerical calculations. The count-in-cell statistics which is tightly related to the \( P(\rho) \) was calculated numerically in \([5]\). In \([44]\) \( P(\rho) \) from a range of numerical simulations with different statistics was plotted.

From observational side, the skewness measured from the IRAS QDOT \([39]\) is in the range \( s/\sigma = 1 - 2 \), as expected from the Gaussian initial statistics. However, the IRAS galaxies are underrepresented in clusters, which correspond to high density tail of \( P(\rho) \), contributing to the skewness. This has little effect on \( P(\rho) \) itself because clusters occupy a small fraction of volume. The density and velociy PDF as deduced from 1.9 Jy IRAS survey and POTENT analysis is
consistent with Gaussian initial statistics \[^{[29]}\], the count-in-cell analysis of 1.2 Jy IRAS data also consistent with this hypothesis \[^{[9]}\], although in all cases the larger sample volume is needed for more quantitative conclusion.

2 Basic Non-Local Equations

At the epoch of the LSS formation, most of the mass is in the form of dark matter of relic origin (for instance, the Cold Dark Matter), with no pressure; nonlinear LSS originated by gravitational instability of small initial fluctuations. Let \( \vec{x}, \vec{v} = a \frac{dx}{dt}, \rho(t, \vec{x}) \) and \( \phi(t, \vec{x}) \) be, respectively, the comoving coordinates, peculiar velocity and density of dark matter (neglecting baryons), and peculiar Newtonian gravitational potential. In the Newtonian theory, the motion of dark matter before the particle orbits cross obeys a nonlinear system of equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a} \nabla_x (\rho \vec{v}) &= 0, \\
\frac{\partial \vec{v}}{\partial t} + \frac{1}{a} (\vec{v} \nabla_x) \vec{v} + H\vec{v} &= -\frac{1}{a} \nabla_x \phi,
\end{align*}
\]

where \( a(t) \) is a scalar factor of an expanding Universe, \( H = \dot{a}/a \) and \( \bar{\rho} \) is a mean density. These equations are valid in the single stream regime, and admit the evident generalization in the regions of multiple streams.

For simplicity we assume that Inflation produces a flat universe \( \Omega = 1, \) and \( \Lambda = 0. \) The growing mode of adiabatic perturbations \( D(t) \) in the Einstein-de Sitter universe is \( D(t) = a(t) \propto t^{2/3}. \)

Let us perform further analyses of the basic equations. It is convenient to use the growing solution \( D(t) = a(t) \) as a new time variable instead of \( t \) (e.g. \[^{[20]}\], \[^{[26]}\]) and introduce a comoving velocity \( \vec{u} = \frac{dx}{da} = \vec{v}/a\dot{a} \) in respect with this time variable. Then eq. (2) has the form

\[
\frac{\partial \vec{u}}{\partial a} + (\vec{u} \nabla_x) \vec{u} = (a\dot{a})^{-2} \left( 3a^2 \ddot{a} \vec{u} - \nabla_x \phi \right)
\]

Let \( \Phi \) be velocity potential so \( \vec{u} = \nabla_x \Phi. \) We also observe that the combination \( A = (\frac{3}{2} H a^3)^{-1} = -(3\ddot{a}a^2)^{-1} \) does not depend on time in the matter-dominated Einstein-de Sitter Universe. Then from eq. (4) we find a general relation between the velocity potential and the Newtonian gravitational potential \[^{[25]}\]

\[
\frac{\partial \Phi}{\partial a} + \frac{1}{2} (\nabla_x \Phi)^2 = \frac{3}{2a} \left( \Phi + A\phi \right).
\]

This equation is often referred to as the Bernoulli equation. Substituting density \( \rho \) from the right hand side of eq. (3) into eq. (4), we find the second equation linking scalar potentials \( \phi \) and \( \Phi \)

\[
\nabla_{x_i} \left[ \frac{\partial}{\partial a} (a \nabla_{x_i} \phi) + \left( A^{-1} + a \nabla_{x_i}^2 \phi \right) \nabla_{x_i} \Phi \right] = 0.
\]
The structure of this equation is $\nabla_x \cdot \vec{\Sigma} = 0$, where the vector $\vec{\Sigma}(\vec{x})$ is the expression in the square brackets in eq. (6). Using the Helmholtz theorem, we find

$$\vec{\Sigma}(\vec{x}) = \nabla_x \times \left[ \frac{1}{4\pi} \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} \nabla_{x'} \times \vec{\Sigma}(\vec{x}') \right]. \quad (7)$$

After some vector algebra, we end up with the formula

$$\frac{\partial}{\partial a} (a \nabla_x \phi) + A^{-1} \nabla_x \Phi = \frac{a}{4\pi} \nabla_x^k \nabla_x^l \left[ \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} (\nabla_{x'_k} \Phi) (\nabla_{x'_l}^2 \phi) \right]. \quad (8)$$

Thus we obtain two equations (3), (8) for two fields $\Phi$ and $\phi$. In the Newtonian theory we can construct the traceless symmetric tensors $\sigma_{ij}^N = \nabla_{x_i} \nabla_{x_j} \Phi - \frac{1}{3} \delta_{ij} \nabla^2 \Phi$ and $E_{ij}^N = \nabla_{x_i} \nabla_{x_j} \phi - \frac{1}{3} \delta_{ij} \nabla^2 \phi$. Tensor $E_{ij}^N$ is the gravitational tidal field, and corresponds to the Newtonian limit of the electric part of the Weyl tensor.

Taking derivatives $\nabla_{x_i} \nabla_{x_j}$ from eq. (4), we can get the equation for the Lagrangian time derivative along the trajectory $\frac{d}{da} \sigma_{ij}^N$ plus other local terms only. Taking derivative $\nabla_{x_j}$ of eq. (8), we obtain equation containing the term $\frac{d}{da} E_{ij}^N$ plus other local terms in the left hand side, and non-local integral term $\nabla_{x_j} \nabla_{x_k} \nabla_{x_i} \left[ \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} (\nabla_{x'_k} \Phi) (\nabla_{x'_l}^2 \phi) \right]$ in the right hand side. Apparently, this non-local term cannot be reduced to the combination of the local terms. It prevents us from obtaining a closed set of Lagrangian equations in the Newtonian theory. However, in general relativity there is a closed set of Lagrangian equations for the Weyl tensor and $\sigma_{ij}$ [17]. In general relativity 00 component of the Einstein Eqs. (which contains the Poisson equation as the Newtonian limit) plays the role of constraint equation, which is automatically resolved for arbitrary time $t > t_0$ if it is resolved once at the initial hypersurface $t = t_0$ and evolution equations are satisfied. Recently this fact drew much attention in connection with possible application to the LSS dynamics beyond the linear theory [33], [14], [4]. However its implication for the Newtonian theory is unclear, because in the Newtonian limit the Poisson equation has to be resolved for each moment $t$.

In this contribution I will consider the general method of following the time evolution of PDFs for the dynamical systems which obey the closed set of Lagrangian equations. In the case of Newtonian gravity, we will use an approximation where basic equations are truncated and might be reduced to the closed set of equations containing full time derivatives only.

### 3 Truncated Zel’dovich Approximation

The dynamics of the system which obeys the multiple stream generalization of the basic equations is complicated and requires the N-body simulations. For interesting cosmological models such as the CDM scenario, the structure formation looks like complicated hierarchical pancaking and clustering from very small to large cosmological scales. However the gravitational clustering at sufficiently large scales $R$ can be considered in the quasilinear theory in a single stream regime ignoring small scale details. For this goal let us apply the Zel’dovich approximation for the smoothed initial...
gravitational potential filtered by the window function \( W(R) \) with filtering scale \( R \):

\[
\phi(\vec{x}; R) = \frac{1}{2\pi^2} \int_0^\infty d^3\phi(\vec{x}') W(|\vec{x} - \vec{x}'|; R).
\]  

(9)

We will call this approach the truncated Zel’dovich approximation, which was used for different purposes in papers [26], [28], [35], [14], [29].

The mean space separation between galaxies is about \( R_0 \sim 5h^{-1}\text{Mpc} \). To generate the continuous field from galaxies, one has to smooth the survey with the filter exceeding the minimal scale \( R_0 \). At these scales the density contrast \( \delta \rho/\rho \) less than unity and quasilinear truncated Zel’dovich approximation can be applied. The approximated Zel’dovich solution [43] of the basic equations describes the gradient mapping of the Lagrangian particles space \( \vec{q} \) into physical Eulerian space \( \vec{x} \).

The velocity field is defined by the gradient of the initial velocity potential \( \Phi(\vec{q}) \). In this approximation the follow of particles density field evolves as \( \rho(\vec{x}) = \bar{\rho}(\vec{q}) |\frac{\partial \vec{q}}{\partial \vec{x}}|^{-1} \). From mathematical point of view the truncated Zel’dovich approximation means just the Zel’dovich approximation which is applied to the truncated initial potential \( \Phi(\vec{q}) \) to ensure being in the single stream quasilinear regime. In this Section the approximation is formulated in terms of an equation for the gravitational potential and its derivatives corresponding to the tidal force.

In the Zel’dovich approximation \( \phi = -A^{-1}\Phi \), and then the dynamical equation for the velocity potential is reduced to the “shortened” eq. (4) without the right hand side:

\[
\frac{\partial \Phi}{\partial a} + \frac{1}{2}(\nabla_x \Phi)^2 = 0.
\]  

(10)

We introduce the tensor of the velocity derivatives \( S_{ij}(x, t) = \nabla_x v_j \) in the Eulerian space; for the potential motion it is reduced to \( S_{ij} = \nabla_x \nabla_x \Phi \). Let \( \lambda_i \) be its eigenvalues. The field of the \( S_{ij}(t) \)-tensor evolves in time, its initial value (in the Lagrangian space) coincides with the Lagrangian deformation tensor \( D_{ij} = \nabla_q \nabla_q \Phi_0 \). We will call the \( S_{ij} \)-tensor as the Eulerian deformation tensor.

It is convenient to use three invariants of the \( S_{ij} \)-tensor:

\[
J_1 = \lambda_1 + \lambda_2 + \lambda_3; \quad J_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3; \quad J_3 = \lambda_1 \lambda_2 \lambda_3.
\]  

(11)

Taking the derivatives of \( \Phi \) in respect with \( \nabla_x \), we get from eq. (10) the following set of equations. First equation is

\[
\frac{D\vec{u}}{Da} = \frac{\partial \vec{u}}{\partial a} + (\vec{u} \nabla_x)\vec{u} = 0,
\]  

(12)

which is the truncated Euler equation (cf. eq. [4]) describing the free streaming of particles in a comoving coordinates ( \( \frac{D}{Da} \) is the lagrangian derivative along the trajectory). Further derivatives of this equation give us the dynamical equations for the invariants of the deformation tensor field

\[
\frac{DJ_1}{Da} + J_1^2 - 2J_2 = 0,
\]  

(13)

\[
\frac{DJ_2}{Da} + J_1 J_2 - 3J_3 = 0,
\]  

(14)

\[
\frac{DJ_3}{Da} + J_1 J_2 = 0.
\]  

(15)
Also, we can rewrite the continuity equation in new variables. Let \( \varrho = a^3 \rho \) be the comoving density, then from eq. (1) we have

\[
\frac{D \varrho}{Da} + \varrho J_1 = 0. \tag{16}
\]

The advantage of the set of equations (12)-(16) is that it is the closed system of equations which describes the dynamics of the cosmological potential density, velocity and deformation tensor fields \( \Phi, \varrho, \vec{u}, J_1, J_2, J_3 \) as function of time \( a(t) \) and the Eulerian position \( \vec{x} \) in the Zel’dovich approximation. In the next Section, using the system (12)-(16), we will derive and solve the kinetic equation for the joint distribution function of these fields.

Let us introduce the principal radii of curvature \( R_i = \lambda_i^{-1} \) of the three dimensional hypersurfaces of the potential \( \Phi \). Then \( J_i \) can be expressed through \( R_i \)-th. The solutions of eq. (13)-(15) in terms of \( R_i \) are extremely simple:

\[
R_i = R_{0i} + a, \tag{17}
\]

the principal radii of curvature of the hypersurface of the potential in the Zeldovich approximation linearly increase with “time” \( a(t) \). It is nothing but the Huygens principle of geometrical optics (note the optical-mechanical similarity between the ray propagations and particle trajectories in the Zel’dovich approximation [41]). Using (11), (17) we obtain the time evolution of the fields from their initial values \( (J_{01}, J_{02}, J_{03}) \) along with the characteristic:

\[
J_1 = \frac{J_{01} - 2aJ_{02} + 3a^2J_{03}}{1 - aJ_{01} + a^2J_{02} - a^3J_{03}}, \tag{18}
\]

\[
J_2 = \frac{J_{02} - 3aJ_{03}}{1 - aJ_{01} + a^2J_{02} - a^3J_{03}}, \tag{19}
\]

\[
J_3 = \frac{J_{03}}{1 - aJ_{01} + a^2J_{02} - a^3J_{03}}, \tag{20}
\]

\[
\varrho = \frac{\varrho_0}{1 - aJ_{01} + a^2J_{02} - a^3J_{03}}, \tag{21}
\]

and \( \Phi(\vec{q}, a) = \Phi_0(\vec{q}), \quad \vec{u}(\vec{q}, a) = \vec{u}_0(\vec{q}) \). From the set of eqs. (18)-(21) we can obtain the reverse formulas

\[
J_{01} = \frac{J_1 + 2aJ_2 + 3a^2J_3}{1 + aJ_1 + a^2J_2 + a^3J_3}, \tag{22}
\]

\[
J_{02} = \frac{J_2 + 3aJ_3}{1 + aJ_1 + a^2J_2 + a^3J_3}, \tag{23}
\]

\[
J_{03} = \frac{J_3}{1 + aJ_1 + a^2J_2 + a^3J_3}, \tag{24}
\]

\[
\varrho_0 = \frac{\varrho}{1 + aJ_1 + a^2J_2 + a^3J_3}. \tag{25}
\]

Additionally, we can directly get the solution of eq. (11)

\[
\Phi(a, \vec{x}) = \Phi_0(\vec{q}) + \frac{(\vec{x} - \vec{q})^2}{2a}. \tag{26}
\]

The solution (26) describes the deformations of 3D-hypersurface of the potential \( \Phi \), and is valid until the formation of folds of \( \Phi \)-hypersurface, which corresponds to caustics (pancakes). Taking
the derivatives of \( \Phi \) in respect with \( \nabla_x \) and \( \nabla_q \), from (26) we get the usual form of the Zel’dovich approximation [43]: 
\[
\vec{x} = \vec{q} + a \nabla_q \Phi_0(\vec{q}).
\]

4 Kinetic equation for the PDFs

In the Zel’dovich approximation, we have derived the closed system of the dynamical equations (12)–(16), and found its solutions which describes the evolution of the set of the fields \( \Phi, \rho, \vec{u}, J_1, J_2, J_3 \) (note that, for instance, the set of fields \( \Phi, \rho, \vec{u} \) does not obey the closed system of equations). For the closed system of dynamical equations one can directly derive a so-called “kinetic” equation for the joint PDF. In the context of the two dimensional geometric-optics problem kinetic equation was derived in [32], [38].

Here we derive the one-point joint PDF \( W(\Phi, \rho, \vec{u}, J_1, J_2, J_3) \) of the fields \( \Phi, \rho, \vec{u}, J_1, J_2, J_3 \) in the Zel’dovich approximation. Let us introduce an arbitrary function of these fields \( f(\Phi, \rho, \vec{u}, J_1, J_2, J_3) \). Taking the full time derivative \( \frac{df}{da} \) and using the set of equations (12)–(16), we get

\[
\frac{\partial f}{\partial a} + (\nabla_x)(\vec{u}f) - J_1f + \rho J_1 \frac{\partial f}{\partial \rho} + (J_1^2 - 2J_2) \frac{\partial f}{\partial J_1} + (J_1J_2 - 3J_3) \frac{\partial f}{\partial J_2} + J_1J_3 \frac{\partial f}{\partial J_3} = 0.
\]

(27)

Convolving this expression with \( W(\Phi, \rho, \vec{u}, J_1, J_2, J_3) \), and using the fact that the function \( f \) is an arbitrary one, we obtain the sought kinetic equation for the joint PDF:

\[
\frac{DW}{Da} - J_1W - J_1 \frac{\partial}{\partial \rho} (\rho W) - \frac{\partial}{\partial J_1} (J_1^2 - 2J_2)W - \frac{\partial}{\partial J_2} (J_1J_2 - 3J_3)W - \frac{\partial}{\partial J_3} (J_1J_3W) = 0.
\]

(28)

Let \( W_0(\Phi_0, \rho_0, \vec{u}_0, J_{01}, J_{02}, J_{03}) \) be an initial joint PDF of the initial fields \( (\Phi_0, \rho_0, \vec{u}_0, J_{01}, J_{02}, J_{03}) \) defined in the Lagrangian space.

Thus, eq. (28) together with the given initial condition, describes the time evolution of the statistics of the field in the problem. Eq. (28) admits a simple analytical solution which can be obtained applying the method of characteristics. Using the time evolution (18)–(21) of the fields along of characteristics, we finally obtain the solution of the kinetic equation

\[
W(\Phi, \rho, \vec{u}, J_1, J_2, J_3; a) = (1 + aJ_1 + a^2J_2 + a^3J_3)^{-6}W_0(\Phi_0, \rho_0, \vec{u}_0, J_{01}, J_{02}, J_{03}),
\]

(29)

where we have to substitute the expressions (22)–(25) as the arguments of the function \( W_0 \).

Formula (29) is the main result of the paper. It describes the time evolution of the joint PDF of the cosmological potential, density, velocity and its derivatives fields in the Zel’dovich approximation, from the given (Gaussian or non-Gaussian) initial distribution function \( W_0 \).

5 Evolution of the PDFs from Gaussian initial fluctuations
5.1 General formalism for Gaussian initial statistics

In this Section we study the statistics of the continuous cosmological fields evolving from the initial Gaussian fluctuations, which is apparently the most attractive model of primordial perturbation. On the other hand, in this case we can make more advanced predictions for the statistics of the matter distribution and motion. For the sake of simplicity, we consider the joint PDF of the cosmological density, velocity and deformation tensor. For the Gaussian initial conditions we write the initial joint PDF as

$$W_0(\rho_0, \vec{u}_0, J_{01}, J_{02}, J_{03}) = Q_0(\vec{u}_0) \cdot \delta(\rho_0 - \bar{\rho}) \cdot G_0(J_{01}, J_{02}, J_{03}).$$  \hspace{1cm} (30)

The first factor is the Gaussian velocity distribution function. The velocity dependence is factorized because there is no correlation between $\nabla_q \Phi_0$ and other fields involved in (30) for the Gaussian fluctuations. The second factor is the initial density distribution function, which corresponds to the perfectly homogeneous density distribution $\rho_0 = \bar{\rho}$. This is just the formal limit of the Gaussian density distribution with $\sigma \to 0$. The third factor is the joint distribution function of the invariant of the initial deformation tensor. Substituting the form (30) into (29), we get the time evolution of the joint PDF from Gaussian initial fluctuations.

5.2 Evolution of the velocity PDF from Gaussian initial fluctuations

The initial PDF of the velocity $\vec{u}(x, b)$ for the Gaussian field is

$$Q_0(\vec{u})d^3u = \frac{1}{(2\pi \sigma_u^2)^{3/2}} \exp \left[ -\frac{\vec{u}^2}{2 \sigma_u^2} \right], \quad \sigma_u^2 = \langle |\nabla_q \Phi_0|^2 \rangle, \hspace{1cm} (31)$$

where $\sigma_u = \sigma_u(R)$ is the initial dispersion of the velocity $\vec{u}$, which depends on the filtering scale $R$.

From the joint PDF (29) for the Gaussian initial condition (30) we see that the velocity dependence remains to be factorized for an arbitrary moment of time $a(t)$. Integrating the joint PDF over all the arguments, except velocity, we get the Eulerian velocity PDF:

$$Q(\vec{u}, b) = Q_0(\vec{u}_0) = \frac{1}{(2\pi \sigma_u^2)^{1/2}} \exp \left[ -\frac{\vec{u}^2}{2 \sigma_u^2} \right]. \hspace{1cm} (32)$$

Thus we find that the Eulerian velocity PDF $Q(\vec{u})$ is time-invariant under the Zel’dovich approximation, in accordance with paper [29], where this result was obtained by a different method, and supported by N-body simulations beyond the Zel’dovich approximation. The time-invariance means that the Eulerian velocity PDF remains Gaussian with the dispersion defined by the linear theory. The distribution of the velocity field $\vec{u}$ is isotropic. Using eq. (32), we can obtain the PDF of the physical velocity $\vec{v} = a(t) \frac{d\vec{u}}{dt} = a\dot{a}\vec{u}$: it is Gaussian with the dispersion $\sigma_v(R)$ defined by the linear theory and the smoothing filter.
5.3 Evolution of the density PDF from Gaussian initial fluctuations

To calculate the Eulerian density PDF $P(\varrho, a)$, we have to integrate the joint PDF given by eq. (30) over all of its arguments except density

$$P(\varrho, a) = \int d^3u\; dJ_1\; dJ_2\; dJ_3\; W(\varrho, \bar{u}, J_1, J_2, J_3; a) .$$  (33)

To take the integral (33), we need the joint distribution function of the invariants of the Lagrangian deformation tensor $G_0(J_{01}, J_{02}, J_{03})$. The joint distribution function of its Lagrangian (initial) eigenvalues $\lambda_{0i}$ is given by

$$M(\lambda_{01}, \lambda_{02}, \lambda_{03}) = \frac{5^{5/2}}{8\pi\sigma_{in}^6} (\lambda_{01} - \lambda_{02})(\lambda_{01} - \lambda_{03})(\lambda_{02} - \lambda_{03}) \exp \left[ -\frac{1}{\sigma_{in}^2} \left( 3J_{01}^2 - \frac{15}{2} J_{02} \right) \right] ,$$  (34)

where $J_{01}, J_{02}$ are expressed through $\lambda_{0i}$, and $\sigma_{in} = \sigma_{in}(R)$ is the initial variance of $\varrho$, which depends on the filtering scale $R$. Then the joint distribution function of the invariants $J_i$ is

$$G(J_{01}, J_{02}, J_{03}) = \frac{5^{5/2}}{8\pi\sigma_{in}^6} \exp \left[ -\frac{1}{\sigma_{in}^2} \left( 3J_{01}^2 - \frac{15}{2} J_{02} \right) \right] .$$  (35)

Additionally, we have to integrate (33) over the allowed region in the $(J_{01}, J_{02}, J_{03})$-space, for which all three eigenvalues $\lambda_{0i}$ are real. To find that region is a non-trivial task, the basic idea is outlined in [29]. After that, taking integral over the velocity in eq. (33), we get

$$P(\varrho, a) = \int \frac{dJ_1 dJ_2 dJ_3}{(1 + aJ_1 + a^2J_2 + a^3J_3)} \delta(\varrho_0 - \bar{\varrho})G_0(J_{01}, J_{02}, J_{03}) .$$  (36)

We have to substitute $G_0$ from (33) and arguments $\varrho_0, J_{01}, J_{02}, J_{03}$ from (22)–(25) in this formula. After some tedious algebra we can reduce the integral (36) to the simpler one-dimensional integral which has to be performed numerically

$$P(\varrho, a) = \frac{9 \cdot 5^{3/2} \bar{\varrho}^3}{4\pi N_s \bar{\varrho} \sigma^4} \int_{3\bar{\varrho}^3}^{\infty} ds\; e^{-(s-3)/2\sigma^2} \left( 1 + e^{-6s/\sigma^2} \right) \left( e^{-\beta_1^2/2\sigma^2} + e^{-\beta_2^2/2\sigma^2} - e^{-\beta_3^2/2\sigma^2} \right) ,$$  (37)

where the only parameter $\sigma(t) = a(t)\sigma_{in}$ is the standard deviation of the density fluctuations $\varrho/\bar{\varrho}$ in the linear theory, and $N_s$ is the mean number of streams, $N_s = 1$ in the single stream regime. The expression (37) was derived by the different method earlier in [26], [29].

For $\sigma \ll 1$ the expression (37) is reduced to the Gaussian distribution $P(\varrho) = (2\pi\sigma^2)^{-1/2}\exp \left( (\varrho - \bar{\varrho})^2/2\sigma^2 \right) .$

For $\sigma \to 0$ we formally get $P(\varrho) \to \delta(\varrho - \bar{\varrho})$, in accordance with (30). The density PDF $P(\varrho, a)$ calculated from (37) is plotted in [26], [29] for different values of parameter $\sigma$. The matter is evacuated from the underdense regions with $\varrho < \bar{\varrho}$, resulting in voids which expand in time and tend to occupy a larger fraction of volume, as well as formation of anisotropic collapsed dense pancakes which tend to occupy a smaller fraction of volume. The positive density contrast can reach
any large value while the negative density contrast is restricted by \( \rho \geq 0 \). Hence the probability function \( P(\rho, a(t)) \), meaning the fraction of volume with a given value of density, is expected to be very non-Gaussian even in the quasilinear stage. At large densities \( \rho \gg \bar{\rho} \) PDF from (37) has the pancakes induced asymptota \( \propto \rho^{-3} \), which affects such values as skewness, \( S_3^Z = 4 \) in this approximation instead of the actual value \( S_3 \approx \frac{34}{7} - (n + 3) \). However, the final smoothing (or the adhesion) regularizes high-density asymptota of \( P(\rho) \), see \cite{29}, \cite{2}. Formula (37) very well describes actual \( P(\rho) \) in the practically interesting region \( 0 < \rho < \text{several } \bar{\rho} \) for \( \sigma \leq 1/2 \).

5.4 Evolution of the deformation tensor PDF from Gaussian initial fluctuations

The density and velocity PDFs were considered earlier in the literature. Let us consider the new statistics: PDF of the Eulerian deformation tensor. The time evolution of the deformation tensor gives us alternative approach to the LSS \cite{5}. To calculate its PDF, we have to integrate the joint PDF over all the arguments except \( J_i \)-th:

\[
G(J_1, J_2, J_3; a) = \int d^3 u \, \rho \, W(\rho, \bar{u}, J_1, J_2, J_3; a) \ .
\]

Carrying out this integral and switching from invariant \( J_i \)-th back to \( \lambda_i \)-th, we obtain the joint distribution of the Eulerian eigenvalues of the deformation tensor \( \frac{\partial u}{\partial x} \):

\[
M(\lambda_1, \lambda_2, \lambda_3; a) = \frac{5^{5/2} \pi^{27}}{8 \sigma^n^6} \left[ \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}{((1 + a\lambda_1)(1 + a\lambda_2)(1 + a\lambda_3))^5} \right] \exp \left[ -\frac{1}{\sigma_m^2} \left( 3J_{01}^2 - \frac{15}{2} J_{02} \right) \right] \ ,
\]

where \( J_{01} \) and \( J_{02} \) must be expressed through \( \lambda_i \) via eqs. (22)–(25). In the limit \( \sigma \to 0 \) this distribution is reduced to the initial Doroshkevich’s distribution (34) based on the Gaussian statistics. As \( \sigma = a(t)\sigma_m \) is growing, the Eulerian distribution \( M(\lambda_1, \lambda_2, \lambda_3; a) \) rapidly departs from the initial Lagrangian distribution, and might be an interesting discriminative test for the initial statistics.

6 Properties of the density PDF

6.1 PDF from the Edgeworth perturbation series

In this Section we consider some properties of the cosmological density PDF using other approaches besides the Zel’dovich approximation. In the case of the weakly non-linear dynamics (\( \delta < 1 \)) when slight departure from the initial Gaussian distribution is expected, one can use the general decomposition series around the gaussian PDF \cite{24}. Inspired by the paper \cite{30} on the weakly
non-linearities on statistical distributions in the theory of two-dimensional sea waves, we suggest similar decomposition for the cosmological density \( P(\delta) \) (in the three-dimensional case). The so-called Edgeworth form of the Cram-Charlier series for density distribution function reads as

\[
P(\delta) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left( \frac{\delta^2}{2\sigma^2} \right) \left[ 1 + \sigma \cdot \frac{S_3}{6} \cdot H_3 + \sigma^2 \cdot \left( \frac{S_4}{24} H_4 + \frac{S_3}{72} H_6 \right) + \ldots \right],
\]

(41)

where \( H_n(\frac{\delta}{\sigma}) \) is the Hermit polynomials. Numerical coefficients \( S_n \) can be calculated in the perturbation series [1]. We found [2] that a few iterations of the expansion (41) reproduce the peak of \( P(\delta) \) in the interval of \(|\delta| \leq 1/2\) around it relatively well for small parameter \( \sigma \leq 1/2 \). It fails to reproduce \( P(\delta) \) at larger \(|\delta|\) because the series (41) is an asymptotic expansion. We understand that similar decomposition was independently suggested in [23].

### 6.2 Mystery of the log-normal distribution

As it was noted in [13], [21], [23], the lognormal distribution

\[
P(\varrho) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp \left[ \frac{(\ln \varrho - \mu_l)^2}{2\sigma^2_l} \right] \cdot \frac{1}{\varrho}
\]

(42)

is a good approximation of the actual \( P(\varrho) \), \( \mu_l = \ln \mu - 0.5\sigma^2_l \), \( \sigma^2_l = \ln(1 + \sigma^2/\mu^2) \). However, this fit was checked for CDM model for moderate \( \sigma \sim 1/2 \) only. The question arises, if the log-normal distribution is a universal form of \( P(\varrho) \) due to the non-linear dynamics of the cosmological system, or it is just a convenient fit for particular cosmological models in some intermediate regime? We argue [2] that the log-normal PDF is not a universal form but is close to the actual \( P(\varrho) \) for some cosmological models (including CDM) for moderate \( \sigma \).

One can use the Edgeworth seria (41) for \( P(\varrho) \). From (42) we find the skewness of the log-normal distribution \( S_{3}^{(log)} = 3 + \sigma^2 \). On the other hand, the skewness of the filtered density field is \( S_3 = \frac{34}{7} - (n_{eff} + 3) \). To be the distributed lognormally, density field has to satisfy the following equation [2]:

\[
3 + \sigma^2 = \frac{34}{7} - (n_{eff} + 3).
\]

(43)

This is a necessary (but not sufficient) condition for the \( \sigma - n_{eff} \)-dependence to correspond to the perfect log-normal distribution. Clearly, it is not a general condition for the density field in arbitrary cosmological models. For instance, for CDM model a rough approximation (for interval \(-2 < n_{eff} < -1\)) is \( \sigma(n_{eff}) \approx 1.4 \cdot (-n_{eff} - 0.85) \). This \( \sigma(n_{eff}) \) dependence for CDM model is close to that of log-normal distribution given by eq. (43) for moderate \( \sigma \sim 0.4 - 0.6 \) only, and departs from log-normal formula for small and large \( \sigma \). To support this conclusion, the systematic comparison of \( P(\varrho) \) against the log-normal distribution for different \( \sigma(n_{eff}) \) is needed. Thus, some “lognormalish” features in the observed density distribution mean that realistic cosmological model is close to the CDM one.
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