Estimating the number of communities in networks by spectral methods

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Abstract

Community detection is a fundamental problem in network analysis with many methods available to estimate communities. Most of these methods assume that the number of communities is known, which is often not the case in practice. We propose a simple and very fast method for estimating the number of communities based on the spectral properties of certain graph operators, such as the non-backtracking matrix and the Bethe Hessian matrix. We show that the method performs well under several models and a wide range of parameters, and is guaranteed to be consistent under several asymptotic regimes. We compare the new method to several existing methods for estimating the number of communities and show that it is both more accurate and more computationally efficient.

1 Introduction

The problem of clustering similar objects into groups is a fundamental problem in data analysis. In network analysis, it is known as community detection [26, 2, 8, 3]. Given a network, which consists of a set of nodes and a set of edges between them, the goal of community detection is to cluster the nodes into groups (communities) so that nodes in the same community share a similar connectivity pattern.

One of the simplest ways of modeling a community structure is the stochastic block model (SBM) [16]. Given the number of communities $K$, $n$ node labels $c_i$ are drawn independently from a multinomial distribution with parameter $\pi = (\pi_1, \ldots, \pi_K)$. The edges between pairs of nodes $(i, j)$ are then drawn independently from a Bernoulli distribution with parameter $P_{c_i c_j}$ and collected in the $n \times n$ adjacency matrix $A$, with $A_{ij} = 1$ if nodes $i$ and $j$ are connected by an edge, and 0 otherwise. A limitation of the stochastic block model is that all nodes in the same communities are equivalent and follow the same degree distribution, whereas many real networks contain a small number of high-degree nodes, the so-called hubs. To address this limitation, the degree-corrected stochastic block model (DCSBM) [17] assigns a degree parameter $\theta_i$ to each node $i$, and edges between nodes are drawn independently with probabilities $\theta_i \theta_j P_{c_i c_j}$. The community detection task is to recover the labels $c_i$, given the adjacency matrix $A$.

A large number of methods have been proposed for finding the underlying community structure [22, 25, 2, 8, 28, 11, 3, 18, 32, 23, 29]. Most of these methods require the number of communities $K$ as input, but in practice $K$ is often unknown. To address this problem, a few likelihood-based methods have been proposed to estimate $K$ [13, 19, 27, 30, 33], under either the SBM or the DCSBM. These methods use BIC-type criteria for choosing the number of communities from a set of possible values, which requires computing the likelihood, done using either MCMC or the variational method, which are both computationally very challenging for large networks. A different approach based on the distribution of leading eigenvalues of an appropriately scaled version of the adjacency matrix was
proposed by [7, 20]. Under the SBM, distributions of the leading eigenvalues converge to the Tracy-Widom distribution; this fact is used to determine \( K \) through a sequence of hypothesis tests. Since the rate of convergence is slow for relatively sparse networks, a bootstrap correction procedure was employed, which also leads to a high computational cost. A cross-validation approach was proposed by [12], which requires estimating communities on many random network splits, and was shown to be consistent under the SBM and the DCSBM.

To the best of our knowledge, all existing methods are either restricted to a specific model or computationally intensive. In this paper we propose a fast and reliable method that uses spectral properties of either the Bethe Hessian or the non-backtracking matrices. Under a simple SBM in the sparse regime, these matrices have been used to recover the community structure [18, 29, 10]; it was also observed that the informative eigenvalues (i.e., those corresponding to eigenvectors which encode the community structure) of these matrices are well separated from the bulk. We will show that the number of “informative” (to be defined explicitly below) eigenvalues of these matrices directly estimates the number of communities, and the estimate performs well under different network models and over a wide range of parameters, outperforming existing methods that are designed specifically for finding \( K \) under either SBM or DCSBM. This method is also very computationally efficient, since all it requires is computing a few leading eigenvalues of just one typically sparse matrix.

2 Preliminaries

Recall \( A \) is the \( n \times n \) symmetric adjacency matrix. Let \( d_i = \sum_{j=1}^{n} A_{ij} \) be the degree of node \( i \). Treating \( A \) as a random matrix, we denote by \( \bar{A} \) the expectation of \( A \), and by \( \lambda_n \) the average of expected node degrees, \( \lambda_n = \frac{1}{n} \sum_{i=1}^{n} \bar{d}_i \). For a symmetric matrix \( X \), let \( p_k(X) \) the \( k \)-th largest eigenvalue of \( X \). We say that a property holds with high probability if the probability that it occurs tends to one as \( n \to \infty \). Next, we recall the definitions of the non-backtracking and the Bethe Hessian matrices which we will use to estimate the number of communities.

2.1 The non-backtracking matrix

Let \( m \) be the number of edges in the undirected network. To construct the non-backtracking matrix \( B \), we represent the edge between node \( i \) and node \( j \) by two directed edges, one from \( i \) to \( j \) and the other from \( j \) to \( i \). The \( 2m \times 2m \) matrix \( B \), indexed by these directed edges, is defined by

\[
B_{i \to j, k \to l} = \begin{cases} 
1 & \text{if } j = k \text{ and } i \neq l \\
0 & \text{otherwise}. 
\end{cases}
\]

It is well-known [4, 18] that the spectrum of \( B \) consists of \( \pm 1 \) and eigenvalues of an \( 2n \times 2n \) matrix

\[
\tilde{B} = \begin{pmatrix}
0_n & D - I_n \\
-I_n & A
\end{pmatrix}.
\]

Here \( 0_n \) is the \( n \times n \) matrix of all zeros, \( I_n \) is the \( n \times n \) identity matrix, and \( D = \text{diag}(d_i) \) is \( n \times n \) diagonal matrix with degrees \( d_i \) on the diagonal. It was observed in [18] that if a network has \( K \) communities then the first \( K \) largest eigenvalues in magnitude of \( \tilde{B} \) are real-valued and well separated from the bulk, which is contained in a circle of radius \( \|\tilde{B}\|^{1/2} \). We will refer to these \( K \) eigenvalues as informative eigenvalues of \( \tilde{B} \). It was also shown in [18] that the spectral norm of the non-backtracking matrix is approximated by

\[
\tilde{d} = \left( \sum_{i=1}^{n} d_i \right)^{-1} \left( \sum_{i=1}^{n} d_i^2 \right) - 1.
\]

(1)

For the special case of the SBM, [10] proved that the leading eigenvalues of \( \tilde{B} \) concentrate around non-zero eigenvalues of \( \bar{A} \) and the bulk is contained in a circle of radius \( \|\tilde{B}\|^{1/2} \), and used the corresponding leading eigenvectors to recover the community labels.

2.2 The Bethe Hessian matrix

The Bethe Hessian matrix is defined as

\[
H(r) = (r^2 - 1)I - rA + D,
\]

(2)
where \( r \in \mathbb{R} \) is a parameter. In graph theory, the determinant of \( H(r) \) is the Ihara-Bass formula for the graph zeta function. It vanishes if \( r \) is an eigenvalue of the non-backtracking matrix \([15, 5, 4]\). Saade et al \([29]\) used the Bethe Hessian for community detection. Under the SBM, they argued that the best choice of \( r \) is \( |r_c| = \sqrt{\lambda_n} \), depending on whether the network is assortative or disassortative; for a more general network, their choice of \( r \) is \( |r_c| = \|B\|^{1/2} \). For assortative sparse networks with \( K \) communities and bounded \( \lambda_n \), they showed that the \( K \) eigenvalues of \( H(r_c) \) whose corresponding eigenvectors encode the community structure are negative, while the bulk of \( H(r_c) \) are positive. Thus, the number of negative eigenvalues of \( H(r_c) \) corresponds to the number of communities. We will refer to these negative eigenvalues of \( H(r_c) \) as informative eigenvalues.

3 Spectral estimates of the number of communities

The spectral properties of the non-backtracking and the Bethe Hessian matrices lead to natural estimates of the number of communities, but they have not been previously considered specifically for this purpose. We now propose two methods (one for each matrix) to determine the number of communities \( K \).

3.1 Estimating \( K \) from the non-backtracking matrix

Under the SBM, the informative eigenvalues of the non-backtracking matrix are real-valued and separated from the bulk of radius \( \|B\|^{1/2} \) \([10]\). Therefore we can estimate \( K \) by counting the number of real eigenvalues of \( B \) that are at least \( \|B\|^{1/2} \). We denote this method by NB (for non-backtracking). As shown by numerical results in Section 5, this estimate of \( K \) also works under the DCSBM. When the network is balanced (communities have similar sizes and edge densities), NB performs well; however, the accuracy of NB drops if the communities are unbalanced in either size or edge density. Computationally, since \( \tilde{B} \) is not symmetric, computing the eigenvalues of \( \tilde{B} \) is more demanding for large networks. Thus we focus instead on the Bethe Hessian matrix, which is symmetric.

3.2 Estimating \( K \) from the Bethe Hessian matrix

The number of communities corresponds to the number of negative eigenvalues of \( H(r) \); the challenge is in choosing an appropriate value of \( r \).

It was argued in \([29]\) that when \( r = \|B\|^{1/2} \), the informative eigenvalues of \( H(r) \) are negative, while the bulk are positive; by \([18]\), \( \|B\| \) can be approximated by \( \tilde{d} \) from \([1]\). Following these results, we first choose \( r \) to be \( r_m = \tilde{d}^{1/2} \) and denote the corresponding method by BHm. Simulations show that using \( r = r_m \) and \( r = \|B\|^{1/2} \) produce similar results; we choose \( r = r_m \) because computing \( r_m \) is less demanding than computing \( \|B\|^{1/2} \).

Another choice of \( r \) is \( r_a = \sqrt{(d_1 + \cdots + d_n)/n} \), which was proposed in \([29]\) for recovering the community structure under the SBM; we denote the corresponding method by BHa. We have found that when the network is balanced, NB, BHm, and BHa perform similarly; when the network is unbalanced, BHa produces better results.

Both BHm and BHa tend to underestimate the number of communities, especially when the network is unbalanced. In that setting, some informative eigenvalues of \( H(r) \) become positive, although they may still be far from the bulk. Based on this observation, we correct BHm and BHa by also using positive eigenvalues of \( H(r) \) that are much close to zero than to the bulk. Namely, we sort eigenvalues of \( H(r) \) in non-increasing order \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n \), and estimate \( K \) by

\[
K = \max \{ k : t\rho_{n-k+1} \leq \rho_{n-k} \}, \tag{3}
\]

where \( t > 0 \) is a tuning parameter. Note that if \( \rho_{n-k+1} < 0 \) then \( K \geq k_0 \) because \( \rho_{n-k_0+1} \leq \rho_{n-k_0} \), therefore the number of negative eigenvalues of \( H(r) \) is always upper bounded by \( K \). Heuristically, if the bulk follows the semi-circular law and \( \rho_{n-k} \geq 0 \) is given, then the probability that \( 0 \leq \rho_{n-k+1} \leq \rho_{n-k}/t \) is less than \( 1/t \). When \( 1/t \) is sufficiently small, we may suspect that \( \rho_{n-k+1} \) is an informative eigenvalue. In practice we find that \( t \in [4, 6] \) works well; we will set \( t = 5 \) for
all computations in this paper. Simulations show that \( \hat{K} \) performs well, especially for unbalanced networks. The resulting methods are denoted by BHmc and BHac, respectively. We will also use BH to refer to all the methods that use the Bethe Hessian matrix.

4 Consistency

The consistency of the non-backtracking matrix based method (NB) for estimating the number of communities in the sparse regime under the stochastic block model follows directly from Theorem 4 in [10]. We state this consistency result here for completeness. The proof given in [10] is combinatorial in nature and this approach unfortunately does not extend to any other regimes or the Bethe-Hessian matrix.

**Theorem 4.1** (Consistency in the sparse regime). Consider a stochastic block model with \( \pi = (\pi_1, ..., \pi_K) \) and \( P = (P_{ik}) = \frac{1}{n} P^{(0)} \) for some fixed \( K \times K \) symmetric matrix \( P^{(0)} \). Assume that \( \text{diag}(\pi P)^r \) has positive entries for some positive integer \( r \). Further, assume that \( \mathbb{E} d_i = \lambda_n > 1 \) for all \( i \), and all \( K \) non-zero eigenvalues of \( P \) are greater than \( \sqrt{\lambda_n} \). Then with probability tending to one as \( n \to \infty \), the number of real eigenvalues of \( \hat{B} \) that are at least \( \|\hat{B}\|^{1/2} \) is equal to \( K \).

To better understand the condition on the eigenvalues of \( P \), consider the simple model \( G(n, \frac{a}{n}, \frac{b}{n}) \). This model assumes that there are two communities of equal sizes and nodes are connected with probability \( a/n \) if they are in the same community, and \( b/n \) otherwise. Since the two non-zero eigenvalues of \( P \) are \((a+b)/2 \) and \((a-b)/2 \), the condition on eigenvalues of \( P \) is \((a-b)^2 > 2(a+b)\).

For the Bethe Hessian, no formal results have been previously established that can be applied directly. We will show that both BHm and BHa methods produce consistent estimator of \( K = \text{rank}(\hat{A}) \) in the dense regime when \( \lambda_n \) grows linearly in \( n \), under the inhomogeneous Erdos-Renyi model with edge probability matrix \( \hat{A} \) (see [9]), which includes as a special case the stochastic block model with \( K \) communities. The inhomogeneous Erdos-Renyi model assumes that edges are drawn independently between nodes \( i \) and \( j \) with probabilities \( \hat{A}_{ij} \). Let 

\[
    d_0 = \min \mathbb{E} d_i, \quad d = \max_{i,j} n \hat{A}_{ij}.
\]

It is clear that \( d_0 \leq \lambda_n \leq d \). For the simple model \( G(n, \frac{a}{n}, \frac{b}{n}) \) we have \( d_0 = \lambda_n = d = (a+b)/2 \).

**Theorem 4.2** (Consistency in the dense regime). Consider an inhomogeneous Erdos-Renyi model with \( \text{rank}(\hat{A}) = K \) such that

\[
    \rho_K(\hat{A}) \geq 5d/\sqrt{d_0}, \quad \text{and} \quad d_0 \geq (1+\varepsilon)d(1-d/n)
\]

for some constant \( \varepsilon > 0 \). Then with high probability, the Bethe Hessian \( H(\rho) \) with \( \rho = \rho_m \) or \( \rho = \rho_a \) has exactly \( K \) negative eigenvalues.

**Proof.** Let us first rewrite the Bethe Hessian as

\[
    H(\rho) = (\rho^2 - 1)I - \rho(A - \hat{A}) + D - \rho\hat{A} =: \hat{H}(\rho) - \rho\hat{A}.
\]

We will show that eigenvalues of \( \hat{H}(\rho) \) are non-negative and are of smaller order than non-zero eigenvalues of \( \rho\hat{A} \). This in turn implies that \( K \) eigenvalues of \( H(\rho) \) are negative while the rest are positive.

To bound \( A - \hat{A} \), we use the concentration result in [31]: with high probability,

\[
    \|A - \hat{A}\| \leq 2\sqrt{d(1-d/n)} + C_0 n^{1/4} \log n, \tag{4}
\]

for some constant \( C_0 > 0 \). To bound the node degrees, we use the standard Bernstein’s inequality: there exists a constant \( C_1 > 0 \) such that, with high probability,

\[
    \|D - \mathbb{E} D\| \leq C_1 \sqrt{d \log n}, \quad |\rho^2 - \lambda_n| \leq C_1 \sqrt{d \log n}. \tag{5}
\]

For square matrices \( X, Y \) we use \( X \succeq Y \) to signify that \( X - Y \) is semi-positive definite. Since \( \mathbb{E} D \succeq d_0 I \), from [4, 5], and the assumption that \( d_0 \geq (1+\varepsilon)d(1-d/n) \), we obtain

\[
    \hat{H}(\rho) \succeq \left[ d_0 + \lambda_n - 2\sqrt{\lambda_n d(1-d/n)} + o(d) \right] I \succeq 0. \tag{6}
\]
For a subspace $U \subseteq \mathbb{R}^n$, we denote by $\dim(U)$ the dimension of $U$, and by $U^\perp$ the orthogonal complement of $U$. Let $\text{col}(A)$ be the column space of $A$. Using the Courant min-max principle (see e.g. [6] Corollary III.1.2) and (6), we have

$$
\rho_{n-K}(H(r)) = \max_{\dim(U) = n-K} \min_{x \in U, \|x\| = 1} \langle H(r)x, x \rangle \geq \min_{x \in \text{col}(A)^\perp, \|x\| = 1} \langle H(r)x, x \rangle \geq 0.
$$

Therefore the $n-K$ largest eigenvalues of $H(r)$ are non-negative.

It remains to show that the $K$ smallest eigenvalues of $H(r)$ are negative. From (4), (5), and the triangle inequality, we obtain

$$
\|H(r)\| \leq \lambda_n + d + 2d\sqrt{1 - d/n} + o(d) < 4d. \quad (7)
$$

On the other hand, from (5) and the assumption $\rho_K(A) \geq 5d/\sqrt{d_n}$, we have

$$
\rho_K(r\tilde{A}) \geq \left[1 + o(1)\right] \lambda_n^{1/2} \rho_K(\tilde{A}) \geq 4d. \quad (8)
$$

Combining (7), (8), and using the Courant min-max principle again implies that $K$ smallest eigenvalues of $H(r)$ are negative, which completes the proof. \hfill \square

More work is needed on the case of “intermediate” rate of $\lambda_n$ not covered by either of the theorems, which will require fundamentally different approaches. This is a topic for future work.

5 Numerical results

Here we empirically compare the accuracy in estimating the number of communities using the non-backtracking matrix (NB), and all the versions based on the Bethe Hessian matrix (BHm, BHmc, BHa, and BHac), described in Sections 4.1 and 4.2. We compare them with two other methods proposed specifically for estimating the number of communities in networks: the network cross-validation method (NCV) proposed by [12] and a likelihood-based BIC-type method (VLH, for variational likelihood) proposed by [33]. We use NCVbm and NCVdc to denote the versions of the NCV method specifically designed for the SBM and the DCSBM, respectively; VLH is only designed to work under the SBM, so it is not included in the DCSBM comparisons. To make comparisons with VLH computationally feasible, instead of using the variational method to estimate the posterior of the community labels as done in [33], we estimate the node labels by the pseudo-likelihood method proposed by [3] and then compute the posterior following [33]. In small-scale simulations where both approaches are computationally feasible (results omitted) we found that substituting pseudo-likelihood for the variational method has very little effect on the estimate of $K$.

The tuning parameter of VLH is set to one (following [33]). We do not include the method of [7] in these comparisons due to its high computational cost.

5.1 Synthetic networks

To generate test networks, we fix the label vector $c \in \{1, \ldots, K\}^n$ so that $c_i = k$ if $n\pi_{k-1} + 1 \leq i < n\pi_k$, where $\pi_0 = 0$. The label matrix $Z \in \mathbb{R}^{n \times K}$ encodes $c$ by representing each node with a row of $K$ elements, exactly one of which is equal to 1 and the rest are equal to 0: $Z_{ik} = 1(c_i = k)$. Let $\bar{P}$ be an $K \times K$ matrix with diagonal $w = (w_1, \ldots, w_K)$ and off-diagonal entries $\beta$, and $M = ZPZ^T$. Under the stochastic block model, we generate $A$ according to an edge probability matrix $\bar{A} = \bar{E}A$ proportional to $M$; the average degree $\lambda_n$ is controlled by appropriately rescaling $M$. The parameter $w$ controls the relative edge densities within communities, and $\beta$ controls the out-in probability ratio. Smaller values of $\beta$ and larger values of $\lambda_n$ make the problem easier.

For the DCSBM, we generate the degree parameters $\theta_i$ from a distribution that takes two values, $P(\theta = 1) = 1 - \gamma$ and $P(\theta = 0.2) = \gamma$. Parameter $\gamma$ controls the fraction of “hubs”, the high-degree nodes in the network, and setting $\gamma = 0$ gives back the regular SBM. Given $\theta = (\theta_1, \ldots, \theta_n)$, the edges are generated independently with probabilities $A = \bar{E}A$ proportional to $\text{diag}(\theta)M\text{diag}(\theta)$, where $\text{diag}(\theta)$ is a diagonal matrix with $\theta_i$’s on the diagonal.

The number of nodes is set to $n = 1200$, the out-in probability ratio $\beta = 0.2$, and we vary the average degree $\lambda_n$, weights $w$, and community sizes. We consider three different values for the number of communities, $K = 2, 4$, and 6. For each setting, we generate 200 replications of the
network and record the accuracy, defined as the fraction of the times a method correctly estimates the true number of communities $K$. The methods NCV and VLH require a pre-specified set of $K$ values to choose from; we use the set $\{1, 2, \ldots, 8\}$ for synthetic networks and $\{1, 2, \ldots, 15\}$ for real-world networks.

We start by varying the average degree $\lambda_n$, which controls the overall difficulty of the problem, and keeping all community sizes equal. Figure 1 shows the performance of all methods when all edge density weights are also equal, $w_i = 1$ for all $1 \leq i \leq K$; in Figure 2 $w = (1, 2)$ for $K = 2$, $w = (1, 1, 2, 3)$ for $K = 4$, and $w = (1, 1, 1, 1, 2, 3)$ for $K = 6$, resulting in communities with varying edge density. In all figures, the top row corresponds to the SBM ($\gamma = 0$) and the bottom row to the DCSBM ($\gamma = 0.9$, which means that 10% of nodes are hubs).

In general, we see that when everything is balanced (Figure 1), all spectral methods perform fairly similarly and outperform both cross-validation (NCV) and the BIC-type criterion (VLH). Also, for larger $K$ and especially under DCSBM, we can see that the corrected versions are slightly better than the uncorrected ones, and the best Bethe Hessian based methods are better than the non-backtracking estimator.

For networks with equal size communities but different edge densities within communities (Figure 2), cross-validation performs poorly, but VLH relatively improves. For larger $K$ the spectral methods are also distinguishable, with all BH methods dominating NB, and corrected versions providing improvement. Overall, BHac is the best spectral method, comparable to VLH for the SBM, and best overall for DCSBM where VLH is not applicable.

Communities of different sizes present a challenge for community detection methods in general, and the presence of relatively small communities makes the problem of estimating $K$ difficult. To test the sensitivity of all the methods to this factor, we change the proportions of nodes falling into each community setting $\pi_1 = r/K$, $\pi_K = (2-r)/K$, and $\pi_i = 1/K$ for $2 \leq i \leq K-1$, and varying $r$ in the range $[0.2, 1]$. As $r$ increases, the community sizes become more similar, and are all equal when $r = 1$. Figure 3 shows the performance of all methods as a function of $r$. The top row corresponds to the SBM ($\gamma = 0$), the bottom row to the DCSBM ($\gamma = 0.9$), and the within-community edge density parameters $w_i = 1$ for all $1 \leq i \leq K$. Here we see that VLH is less sensitive to $r$ than the spectral methods, but unfortunately it is not available under the DCSBM. Cross-validation is still dominated by spectral methods except for very small values of $r$, where all methods perform poorly. The corrections still provide a slight improvement for Bethe Hessian based methods, although all spectral methods perform fairly similarly in this case.
5.2 Real world networks

Finally, we test the proposed methods on several popular network datasets. In the college football network [14], nodes represent 115 US college football teams, and edges represent the games played in 2000. Communities are the 12 conferences that the teams belong to. The political books network [24], compiled around 2004, consists of 105 books about US politics; an edge is “frequently purchased together” on Amazon. Communities are “conservative”, “liberal”, or “neutral”, labelled manually based on contents. The dolphin network [21] is a social network of 62 dolphins, with edges representing social interactions, and communities based on a split which happened after one
dolphin left the group. Similarly, the karate club network \cite{34} is a social network of 34 members of a karate club, with edges representing friendships, and communities based on a split following a dispute. Finally, the political blog network \cite{1}, collected around 2004, consists of blogs about US politics, with edges representing web links, and communities are manually assigned as “conservative” or “liberal”. For this dataset, as is commonly done in the literature, we only consider its largest connected component of 1222 nodes.

Table \ref{tab:community_numbers} shows the estimated number of communities in these networks. All spectral methods estimate the correct number of communities for dolphins and the karate club, and do a reasonable job for the college football and political books data. For political blogs, all methods but NCV and VLH estimate a much larger number of communities, suggesting the estimates correspond to smaller sub-communities with more uniform degree distributions that have been perviously detected by other authors. We also found that the VLH method was highly dependent on the tuning parameter, and the estimates of NCVbm and NCVdc varied noticeably from run to run due to their use of random partitions.

| Dataset       | NB  | BHm | BHmc | BHa | BHac | NCVbm | NCVdc | VLH | Truth |
|---------------|-----|-----|------|-----|------|-------|-------|-----|-------|
| College football | 10  | 10  | 10   | 10  | 14   | 13    | 9     | 12  |       |
| Political books | 3   | 3   | 4    | 4   | 4    | 2     | 6     | 3   |       |
| Dolphins      | 2   | 2   | 2    | 2   | 2    | 4     | 3     | 2   |       |
| Karate club   | 2   | 2   | 2    | 2   | 2    | 3     | 3     | 4   | 2     |
| Political blogs | 8   | 7   | 8    | 7   | 8    | 10    | 2     | 1   | 2     |

Table 1: Estimates of the number of communities in real-world networks.

6 Discussion

In summary, the numerical experiments suggest that the spectral methods provide extremely fast and reliable estimates of the number of communities $K$ for balanced networks, with the Bethe Hessian based method with the threshold choice $r_a$ and the correction described in \cite{3} the best choice in most scenarios. With communities of significantly different sizes, they tend to underestimate $K$ by combining small communities together, which seems to be an intrinsic limitation of spectral methods. This suggests that their estimates can be used as a lower bound on $K$ and a starting point for a more elaborate and computationally demanding likelihood-based method like VLH, in the same way that spectral clustering can be used to initialize a more sophisticated community detection method. Having a small set of plausible values of $K$ to focus on can significantly reduce the computational cost and improve the accuracy of estimating the number of communities.

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