On Graded Lie Algebras of Characteristic Three
With Classical Reductive Null Component

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0 Introduction

\textbf{Abstract}

We consider finite-dimensional irreducible transitive graded Lie algebras $L = \sum_{i=-q}^{r} L_i$ over algebraically closed fields of characteristic three. We assume that the null component $L_0$ is classical and reductive. The adjoint representation of $L$ on itself induces a representation of the commutator subalgebra $L'_{0}$ of the null component on the minus-one component $L_{-1}$. We show that if the depth $q$ of $L$ is greater than one, then this representation must be restricted.

Over algebraically closed fields $F$ of characteristic $p > 0$, the classification of the finite-dimensional simple Lie algebras relies on the classification of the finite-dimensional irreducible transitive graded Lie algebras $L = \bigoplus_{i=-q}^{r} L_i$ of depth $q \geq 1$ with classical reductive null component $L_0$. We recall some of the progress that has been made in the classification of such Lie algebras $L$. In the case in which $L_{-1}$ is not only irreducible but also restricted as an $L_0$-module, such Lie algebras are described by the Recognition Theorem of Kac [10] for $p > 5$. (See also [3].) In [1] it is shown that for $p > 5$, $L_{-1}$ is necessarily a restricted $L'_0$-module. (The assertion is also true for $p = 5$. [3]) When $p = 3$, the situation is more complicated. In characteristic three, there are series of simple graded Lie algebras which satisfy the conditions of Kac’s Recognition Theorem, but which are neither classical Lie algebras nor Lie algebras of Cartan.

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type. (See [6], [12], [14].) Moreover, for \( q = 1 \), examples exist in which \( L_{-1} \) is not a restricted \( L_0' \)-module. All simple depth-one graded Lie algebras over algebraically closed fields of characteristic three with non-restricted \( L_0' \)-module \( L_{-1} \) were determined in [4]. (In [11], the authors classified all simple depth-one graded Lie algebras over algebraically closed fields of characteristic three in which \( L_{-1} \) is a restricted \( L_0' \)-module.) In [2], two-graded (i.e., depth-two, graded) Lie algebras were examined, and it was proved that when \( p = 3 \) and \( q = 2 \), the \( L_0' \)-module \( L_{-1} \) must be restricted. For \( q = 3 \), the corresponding statement was proved in [7]. It was conjectured in [2] that a non-restricted \( L_0' \)-module \( L_{-1} \) can exist only in Lie algebras of depth one. The present paper completes the proof of that conjecture. One has, of course, to exclude the case of \( L(2 : n, \omega) \) with the reverse gradation; however, since the one-component of \( L(2 : n, \omega) \) with its original gradation is abelian, this example does not satisfy condition (D) of the Main Theorem below (for depth greater than one). One has also to exclude the sum of a degenerate Lie algebra (in the sense of Theorem 1.3 below) and a (simple, classical) Lie algebra which resides in the null component and acts non-restrictedly on the minus-one component. We note, as in [2], that because there are only finitely many irreducible restricted modules for the derived algebra of a classical reductive Lie algebra, what needs to be considered in classifying graded Lie algebras over algebraically closed fields of characteristic three is reduced. In this paper, we prove the following theorem, which we will henceforth refer to as the “Main Theorem.”

**Theorem 0.1 (Main Theorem)** Let \( L = L_{-q} \oplus L_{-q+1} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r, q > 1, r > 0, \) be a finite-dimensional graded Lie algebra over an algebraically closed field \( F \) of characteristic \( p = 3 \) such that

(A) \( L_0 \) is classical reductive;

(B) \( L_{-1} \) is an irreducible \( L_0 \)-module (i.e., \( L \) is irreducible);

(C) for all \( j \geq 0 \), if \( x \in L_j \) and \( [x, L_{-1}] = (0) \), then \( x = 0 \) (i.e., \( L \) is transitive);

(D) \( L_{-i} = [L_{-i+1}, L_{-1}] \) for all \( i > 1 \); and

(E) \( L_{-2} \not\subseteq M(L) \), where \( M(L) \) is the largest ideal of \( L \) contained in the sum of the negative gradation spaces. (See Theorem 1.3 below.)

Then \( L_{-1} \) is a restricted module for \( L_0' \) under the adjoint action, except when \( L = L^1 + I_0 \), where \( L^1 \) is degenerate (in the sense of Theorem 1.3 below) and where the representation (induced by the adjoint representation of \( L \)) of \( I_0 \), a summand of the null component of \( L \), on \( L_{-1} \) is not restricted.

To help to motivate hypothesis (E) above, we offer the following

**Example 0.2** For characteristic \( p \geq 3 \), consider the irreducible transitive graded Lie algebra

\[
R \overset{\text{def}}{=} \mathcal{O}(2 : (1, 1)) \oplus H(2 : (1, 1)) = \bigoplus_{i=-2-2(p-1)}^{2p-5} R_i,
\]
where $R_i = H(2 : (1, 1))_i$ for $i \geq -1$, and $R_i = O(2 : (1, 1))_{i+2p}$ for $-2p = -2 - 2(p - 1) \leq i \leq -2$. Here, the divided-power algebra $O(2 : (1, 1))$ is an abelian ideal of $R$, and $H(2 : (1, 1))$ has its usual Lie algebra bracket operation and its usual action on $O(2 : (1, 1))$, except that $[D_{x_1}, D_{x_2}] = x_1^{p-1}x_2^{p-1} \in R_{-2}$. (See Theorem 1.3 below.) Then $R/M(R) \cong R/O(2 : (1, 1))$ has depth one. In general, if we consider the free Lie algebra generated by the local part of any depth-one graded Lie algebra $L$, and take a co-finite-dimensional subideal $C$ of the maximal ideal $D$ in the negative part (See [5].), then $M(L \oplus D/C) = D/C$, and $(L \oplus D/C)/M(L \oplus D/C) \cong L$ has depth one.

To illustrate the degenerate case in the conclusion of the Main Theorem, we offer the following

**Example 0.3** Consider a graded Lie algebra

$$L = \bigoplus_{i=-q}^{1} L_i$$

where

$$L_1 = \langle \partial_j, j = 1, \ldots, n \rangle$$

$$L_0 = S + T$$

where $S$ and $T$ are classical, and

$$L_{-i} = S \otimes \langle x_1^{k_1} \ldots x_n^{k_n}, k_1 + \ldots + k_n = i \rangle,$$

and where $T$ is a subalgebra of $\mathfrak{gl}(n) = \langle x_i\partial_j, i, j = 1, \ldots, n \rangle = W(n : 1)_0$ with a non-restricted action on $\langle x_1, \ldots, x_n \rangle$; i.e., the representation of $T$ in $\langle x_1, \ldots, x_n \rangle$ is a non-restricted representation of $T$. (See (ii) of Theorem 1.3.) More succinctly, $L = S \otimes A(n : 1) + 1 \otimes T + 1 \otimes \langle \partial_i, i = 1, \ldots, n \rangle$. The minimal example corresponds to $T = \mathfrak{sl}(2)$ and $n = 3$, so that, therefore, $q = 6$.

We have noted that the Main Theorem has been proved for $q = 2$ in [2] and for $q = 3$ in [7]. When we refer to the Main Theorem to substantiate certain claims below, it will be for the cases already proved.

We conclude this section with a sketch of the plan of the rest of the work. In Section 1, we establish terminology and notation, and gather some previously known results. We continue to gather previous results at the beginning of Section 2; here, however, the results are of a more technical nature, and we immediately apply them to showing that under quite natural assumptions, the $L_0$-module $L_{-2}$ is irreducible. We conclude Section 2 by establishing other technical lemmas that we’ll use later on. In Sections 3, 4, and 5, we prove the Main Theorem under the “natural” assumptions just referred to, for the cases $q \geq 6$, $q = 4$, and $q = 5$, respectively. In Section 6, we complete the proof of the Main Theorem in its full generality.
1 Preliminaries

In this section, we recall definitions and introduce notation, after which we gather results from the literature that we will use later in the work.

Recall that if one takes a $\mathbb{Z}$-form (Chevalley basis) of a complex simple Lie algebra and reduces the scalars modulo $p$, one obtains a Lie algebra over $I/(p)$. If $F$ is any field of characteristic $p$, then, by tensoring the Lie algebra we obtained over $I/(p)$ by $F$, we obtain a Lie algebra over $F$. In characteristic $p$, any such Lie algebra so obtained is referred to as classical, even those with root systems $E_6$, $E_7$, $E_8$, $F_4$, and $G_2$. This process may result in a Lie algebra with a non-zero center; such a Lie algebra is still referred to as classical, as is the quotient of such a Lie algebra by its center. For example, the Lie algebras $\mathfrak{g}(k)$ and $\mathfrak{pg}(k)$ are both considered to be classical Lie algebras. Thus, a classical reductive Lie algebra $A$ may have a nontrivial center $Z(A)$ as well.

A classical reductive Lie algebra $\mathfrak{g}$ is the sum of commuting ideals $\mathfrak{g}_j$ which are classical Lie algebras, and an at-most-one-dimensional center $Z(\mathfrak{g})$:

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_k + Z(\mathfrak{g}) \quad (1.1)$$

For any classical Lie algebra $\mathfrak{g}_j$, the derived algebra $\mathfrak{g}_j'$ is a restricted Lie algebra with a natural $p$-structure such that $e_\alpha'^{[p]} = 0$, and $h_i'^{[p]} = h_i$ for any Chevalley basis $\{e_\alpha, h_i | \alpha \in R, i = 1, \ldots, \text{rank}(\mathfrak{g}_j')\}$ of $\mathfrak{g}_j'$, where $R$ is the root system of the corresponding complex simple Lie algebra. For a classical reductive Lie algebra $\mathfrak{g}$, we will consider only that $p$-structure on $\mathfrak{g}' = \mathfrak{g}_1' + \cdots + \mathfrak{g}_k'$ whose restriction to each classical summand is the natural $p$-structure on that summand.

Let $\pi : L \rightarrow \mathfrak{gl}(n)$ be a finite-dimensional irreducible representation of a restricted Lie algebra $L$. The character of $\pi$ is the linear functional $\chi$ on $L$ such that $\chi(y)I = \pi(y)p - \pi(y^{[p]})$ for all $y \in L$. The representation $\pi$ is restricted when the character $\chi$ equals 0.

**Lemma 1.2** (See [7] Lemma 1.) Assume that $L$ is a graded Lie algebra satisfying conditions (A)-(D) of the Main Theorem. If $\chi$ is the character of $L_0'$ on $L_{-1}$, then $L_0'$ has character $-j\chi$ on $L_j$ for all $j$.

The following theorem of Weisfeiler [15] plays a fundamental rôle in the study of graded Lie algebras. In what follows, we will sometimes refer to Theorem [1.3] as “Weisfeiler’s Theorem.”

**Theorem 1.3 (Weisfeiler’s Theorem)** Let $L = L_{-q} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r$ be a graded Lie algebra such that conditions (B)-(D) of the Main Theorem hold. Let $M(L)$ denote the largest ideal of $L$ contained in $L_{-q} \oplus \cdots \oplus L_{-1}$. Then
(i) $L/M(L)$ is semisimple and contains a unique minimal ideal $I = S \otimes O(n : 1)$, where $S$ is a simple Lie algebra, $n$ is a non-negative integer, and $O(n : 1) = F[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^p)$. The ideal $I$ is graded and $I_1 = (L/M(L))_1$ for all $i < 0$.

(ii) If $I_1 = (0)$, then for some $k$, $1 \leq k \leq n$, the algebra $O(n : 1)$ is graded by setting $\deg(x_i) = -1$ for $1 \leq i \leq k$ and $\deg(x_i) = 0$ for $k < i \leq n$. Then $I_1 = S \otimes O(n : 1)$ for all $i$, $L_2 = (0)$, $I_0 = [L_{-1}, L_1]$, and $L_1 \subseteq \{D \in 1 \otimes \text{Der}O(n : 1) | \deg(D) = 1\}$.

(iii) If $I_1 \neq (0)$, then $S$ is graded and $I_i = S_i \otimes O(n : 1)$ for all $i$. Moreover, $(0) \neq [L_{-1}, L_1] \subseteq I_0$.

Set

$$L_{<0} \overset{\text{def}}{=} \bigoplus_{i=-q}^{-1} L_i,$$

and

$$L_{>0} \overset{\text{def}}{=} \bigoplus_{i=1}^{r} L_i.$$

In [8], the following theorem is proved.

**Theorem 1.4** Let $L = \oplus_{i \in \mathbb{Z}} L_i$ be a non-degenerate graded Lie algebra over an algebraically closed field $F$ of characteristic $p > 2$ satisfying conditions (A)-(D) of the Main Theorem. If $[[L_{-1}, V], V] = 0$ for some proper $L_0$-submodule $V \subset L_1$, $\dim V > 1$, then $\dim L = \infty$.

By Lemma 1.2, the representation of $L'_0$ on $L_1$ is restricted when and only when the representation of $L'_0$ on $L_{-1}$ is restricted. Since no non-restricted representation of $L'_0$ can have dimension one, we have the following corollary.

**Corollary 1.5** Let $L$ be as in the above theorem, and suppose that the representation of $L'_0$ on $L_{-1}$ is not restricted. If $[[L_{-1}, V], V] = 0$ for some proper $L_0$-submodule $V \subset L_1$, then $L$ is an infinite-dimensional Lie algebra.

We will make use of the following results from [2]. For definitions of the Lie algebras $L(e), M, H(2 : n_2, \omega)$, and $CH(2 : n, \omega)$ mentioned in the conclusion of Proposition 1.6 below, see, for example, Section 2 of [2]. When we make use of certain properties of these Lie algebras in later sections, we will explicitly state the properties we need.

**Proposition 1.6** (See Lemma 2.12 of [2].) Let $L = L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_r$ be a graded Lie algebra satisfying conditions (A), (B), and (C) of the Main Theorem, and suppose that $L_1 \neq 0$. If $L_{-1}$ is a nonrestricted $L_0$-module, then either $L$ is isomorphic to one of the Lie algebras $L(e)$ or $M$, or $L$ is a Hamiltonian Lie algebra such that $H(2 : n, \omega) \subseteq L \subseteq CH(2 : n, \omega)$, where $n = (1, n_2)$, $\omega = (\exp x^{(3)})dx \wedge dy$, and the grading is of type $(0, 1)$. 

Corollary 1.7 (See Corollary 2.13 of [2].) Under the assumptions of Proposition 1.6, \( L_0 \cong \mathfrak{sl}(2) \), \( L_1 \) is an irreducible three-dimensional \( L_0 \)-module, and \([L_1, L_1] = 0\). In addition, \([L_{-1}, L_1] \cong \mathfrak{sl}(2)\) if and only if \( L \) is a Hamiltonian Lie algebra; otherwise, \([L_{-1}, L_1] \cong \mathfrak{gl}(2)\).

Lemma 1.8 (See Lemma 2.14 of [2].) Let \( L = L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_q \) be one of the Lie algebras \( L(\varepsilon) \), \( M \), or \( H(2 : n, \omega) \) with \( n = (1, n_2) \), let \( \chi \) be the nonzero character of the \( L_0 \)-module \( L_{-1} \), and let \( V \) be an \( L \)-module such that \( l^3 \cdot V = 0 \) for all \( l \in L_{-1} \cup L_1 \). Suppose that \( W \) is an irreducible \( L_0 \)-submodule of \( V \) with character \( \chi_W = \zeta \chi \), \( \zeta \in \mathbf{F}^\times \), and suppose that \( L_1 \cdot W = 0 \). Then \( L_{2} \cdot W \neq 0 \). Similarly, if \( L_{-1} \cdot W = 0 \), then \( L_{-1} \cdot W \neq 0 \).

In what follows, all Lie algebras will be finite-dimensional over an algebraically closed field \( \mathbf{F} \) of characteristic \( p = 3 \). The commutator ideal \([L, L] \) of a Lie algebra \( L \) will be denoted by \( L' \), and the \( i \)-th commutator \((\mathrm{ad} X)^{i-1}X \) of any set \( X \) will be written as \( X^i \). The annihilator of an \( L_0 \)-module \( M \subseteq L \) in an \( L_0 \)-module \( N \subseteq L \) will be denoted by \( \mathrm{Ann}_N M \).

2 Properties of irreducible transitive graded Lie algebras

This section contains technical lemmas and a proof that under hypotheses which we list here, the representation of the null component on the minus-two component must be irreducible.

We begin this section by recalling a few results from [1]. Let \( L, M(L), I, S = \sum_{i=-q}^s S_i \) be as in Weisfeiler’s Theorem (Theorem 1.3). Throughout this section, we make the following two blanket assumptions:

(i) \( M(L) = 0 \)

(ii) \( I = S \).

In this regard, please see [13] (2.4.6)], and note that the Lie algebra \( S + L_0 \) satisfies (B) of the Main Theorem.

Lemma 2.1 (See [1], Lemma 6.] For any \( x \) in \( L \backslash L_{-q}, [L_{-1}, x] \neq 0 \).

Lemma 2.2 (See [1], Lemma 7.] \( S_j = (\mathrm{ad} S_{-1})^{s-j} S_s \) for all \( j, -q \leq j \leq s \). If \( q(t-1) < i \leq s \), then \( (\mathrm{ad} S_{-q})^j S = 0 \) if and only if \( (\mathrm{ad} S_{-q})^j S_i = 0 \) for some \( i, q(t-1) \leq i \leq s \).

Lemma 2.3 (See [1], Lemma 8.] \( [L_{-q}, L_i] \neq 0 \) for all \( i = 0, \ldots, r \). In addition, \( S_j = (\mathrm{ad} L_{-1})^{r-j} L_r \) for all \( j, -q \leq j \leq r - 1 \), so that \( s = r - 1 \) or \( r \). \( S_s \) is an irreducible \( S_0 \)-module.

Lemma 2.4 (See [1], Lemma 9.] \( S_{-q} \) is an irreducible \( S_0 \)-module. In particular, \( L_{-q} \) is an irreducible \( L_0 \)-module.
Lemma 2.5 (See [1, Lemma 10].) \( \text{Ann} L_0 L_i \cap \text{Ann} L_0 V_{i+1} = 0 \) for all \( i = -q, \ldots, r-1 \), where \( V_{i+1} \) is any non-zero \( L_0 \)-submodule of \( L_{i+1} \).

Lemma 2.6 (See [1, Lemma 11].) \( \text{Ann} L_i L_q \cap \text{Ann} L_i V_{q+1} = 0 \) for all \( i = 0, \ldots, r \), where \( V_{q+1} \) is any non-zero \( L_0 \)-submodule of \( L_{q+1} \).

Lemma 2.7 (See [1, Lemma 12].) \( \text{Ann} L_{q-1} L_{q+1} = 0 \).

Lemma 2.8 (See [1, Lemma 13].) If \( r \geq q \), then \( L_{-q+i} = [L_{-q}, L_i] \) for \( i = 0, 1, \ldots, q-1 \).

Lemma 2.9 (See [1, Lemma 14].) Let \( U, V \) be \( L_0 \)-submodules of \( L \) such that \( [U, V] \subseteq L_0 \) and \( [U, [U, V]] = 0 \). Then \( \text{ad} [u, v]u \in U, v \in V \) is weakly closed (in the sense of [2, p.31]); consequently, if \( \text{ad} [u, v]^2 M = 0, u \in U, v \in V \), for some \( i > 1 \), and \( L_0 \)-module \( M \), then \( \text{ad} M [U, V] \) is “associative nilpotent.” (See Theorem II.2.1 of [2].)

Lemma 2.10 (See [1, Lemma 15].) If \( r \geq q \), then \( (\text{ad} L_{-q})^2 L \neq (0) \).

Corollary 2.11 If \( s \geq q \), then \( [S_{-q}, [S_{-q}, S_q]] \neq (0) \). In particular, if \( s \geq q \), then \( [L_{-q}, [L_{-q}, L_q]] \neq (0) \).

Proof By Lemma 2.10 applied to the Lie algebra \( S + L_0 \), \( (\text{ad} S_{-q})^2 S \neq 0 \). Thus, in view of Lemma 2.2 it cannot be that \( (\text{ad} S_{-q})^2 S_q = 0 \). \( \square \)

Lemma 2.12 (See [1, Lemma 16].) Let \( V \) be an \( L_0 \)-submodule of \( L_{-q+i} \) for some \( i \), where \( 0 < i \leq \frac{q}{2} \), and suppose that \( [V, L_{q-i-1}] = 0 = [V, [V, L_{q-i}]] \). Suppose further that \( L_{-q+i-1} \) is an irreducible \( L_0 \)-submodule of \( L \), and that \( [L_{-q+i-1}, L_{q-i}] \neq 0 \) (so that it equals \( L_{-1} \)). Then \( V = 0 \).

Lemma 2.13 (See [1, Lemma 17].) Suppose that \( V \) is an irreducible \( L_0 \)-submodule of \( L_{-q+i} \) for some \( i \), where \( 0 < i < \frac{q-1}{2} \), such that \( [V, L_{q-i-1}] \neq 0 \) (so that it equals \( L_{-1} \)). Then \( L_{-q+i} \) is an irreducible \( L_0 \)-module; i.e., it equals \( V \).

From our observations at the beginning of this section, we have that \( S \subseteq L \subseteq \text{Der} S \) where \( S = S_{-q} \oplus S_{-q+1} \oplus \cdots \oplus S_{-1} \oplus S_0 \oplus S_1 \oplus \cdots \oplus S_s \) is a simple Lie algebra with \( S_i = L_i \) for \( i < 0 \). Since \( S_1 \) is an \( L_0 \)-submodule of \( L_1 \), it follows that if \( L_1 \) is an irreducible \( L_0 \)-module, then \( S_1 = L_1 \). If, in addition, \( L \) is generated by its local part \( L_{-1} \oplus L_0 \oplus L_1 \), then for \( i \geq 1 \), we have

\[
L_i = L_i^1 = S_i^1 \subseteq S_i \subseteq L_i
\]

so that \( S_i = L_i \) for \( i > 0 \), and \( L \) could differ from \( S \) only in the null component. In particular, \( s \) would equal \( r \). (See Lemma 2.3)

In the lemmas that follow, we will consider graded Lie algebras \( L = L_{-q} \oplus L_{-q+1} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_s \) satisfying assumptions (i) and (ii) below. Other assumptions will be noted in the statements of the results for which we use them. Note, for example, that, as noted in the paragraph above, assumption (iii)
follows from assumptions (iv) and (v) (and assumptions (i) and (ii), of course). Also, assumption (viii) can be assumed whenever the previous assumptions are true, since if they hold, we can reverse the gradation, and have that all of the hypotheses of the Main Theorem continue to be true for the reversed gradation. Indeed, (iv) is the “reverse” of assumption (B) above, (vi) is the “reverse” of (C), and (v) is the “reverse” of (D). Of course, by the transitivity (C) of $L$, there can be no ideals of $L$ in the positive part of $L$, so (E) holds in the “reverse” direction. In addition, by Lemma 2.12, the representation of $L_0$ on $L_{-1}$ is restricted if and only if the representation of $L_0'$ on $L_1$ is restricted.

(i) $L$ satisfies conditions (A)-(D) of the Main Theorem.

(ii) $L \subseteq \text{Der}S$ where $S = S_{-q} \oplus S_{-q+1} \oplus S_{-1} \oplus S_0 \oplus S_1 \oplus \cdots \oplus S_s$ is a simple graded Lie algebra.

(iii) $L_i = S_i$, $i \neq 0$.

(iv) (See the discussion before Lemma 2.13.) $L_1 = S_1$ is an irreducible $L_0$-module.

(v) $L_{i+1} = [L_i, L_1]$ for $i > 0$.

(vi) If $x$ is a non-zero element in $L_{-i}$ for some $i \geq 0$, then $[L_1, x] \neq (0)$.

(vii) The character $\chi$ of $L_0'$ on $L_{-1}$ is non-zero.

(viii) $r \geq q$.

**Lemma 2.14** If assumptions (iv) and (v) hold, and $S_1 \neq 0$, then $\text{Ann}_{L_0} L_1 = 0$.

**Proof** Suppose, on the contrary, that $A_0 \overset{\text{def}}{=} \text{Ann}_{L_0} L_1 \neq 0$. Then (as in Lemma 18)

$$[L_{-1}, L_1] = [[L_{-1}, A_0], L_1] = [[L_{-1}, L_1], A_0] \subset A_0,$$

so that by transitivity (C),

$$0 \neq [S_{-1}, L_1] \subset A_0 \cap S_0 \subset \text{Ann} S_0 S_1.$$

Since for $i > 0$, we have by assumptions (iv) and (v) that $S_i = S_i^{1}$, and since

$J \overset{\text{def}}{=} S_{-q} \oplus S_{-q+1} \oplus \cdots \oplus S_{-1} + A_0 \cap S_0$ is invariant under $\text{ad}S_1$, $-q \leq i \leq s$, it follows, from our assumption that $S_1 \neq 0$, that $J$ is a proper ideal of the simple Lie algebra $S$; i.e., we have obtained a contradiction. Thus, we must conclude that $\text{Ann}_{L_0} L_1 = 0$. □

**Lemma 2.15** If assumption (vi) holds, then $[V_{-2}, L_1] = L_{-1}$ for any non-zero $L_0$-submodule $V_{-2}$ of $L_{-2}$.

**Proof** This lemma follows from assumptions (vi) and (B). □

**Lemma 2.16** If assumption (iv) holds, then $\text{Ann}_{L_i} L_{-q} = 0$ for all $i > 0$. 8
Proof Consider first the case in which $i = 1$. If $\text{Ann}_{L_1} L_q \neq 0$, then, since we are assuming (iv) that $L_1$ is an irreducible $L_0$-module, we would have $\text{Ann}_{L_1} L_q = L_1$. But then

$$[L_q, L_1] = [L_q, \text{Ann}_{L_1} L_q] = 0,$$

to contradict Lemma 2.3. Consequently, $\text{Ann}_{L_1} L_q = 0$. Now, if $Q_i \text{ def } = \text{Ann}_{L_1} L_q \neq 0$ for some $i > 1$, then by transitivity (C), we would have

$$0 \neq (\text{ad } L_{-1})^{i-1} Q_i \subset \text{Ann}_{L_i} L_q,$$

to contradict what we just showed. Thus, $\text{Ann}_{L_i} L_q = 0$ for all $i > 0$, which is what we wanted to show. ∎

Lemma 2.17 If assumptions (vi) and (viii) hold, then $L_q = [L_{q-1}, L_{-i}], 0 \leq i \leq q$.

Proof By Lemmas 2.3 and 2.4, $[L_0, L_q] = L_q$, so the lemma is true for $i = 0$ and $i = q$. For $i = 1$, we use Lemmas 2.1 and 2.4. Now note that for $1 \leq i \leq q - 1$, we have

$$[(\text{ad } L_1)^i L_q, (\text{ad } L_1)^{q-i} L_q] = [(\text{ad } L_1)^i L_q, [L_1, \text{ad } L_1)^{q-(i+1)} L_q] = [(\text{ad } L_1)^{i+1} L_q, (\text{ad } L_1)^{q-(i+1)} L_q],$$

so that in view of (vi), (B), and Lemmas 2.1 and 2.4

$$L_q = [(\text{ad } L_1)^i L_q, (\text{ad } L_1)^{q-i} L_q] = [(\text{ad } L_1)^i L_q, (\text{ad } L_1)^{q-i} L_q], 1 \leq i \leq q.$$

Then

$$L_q = [(\text{ad } L_1)^i L_q, (\text{ad } L_1)^{q-i} L_q] \subseteq [L_{q-1}, L_{-i}] \subseteq L_q, 1 \leq i \leq q.$$  

□

Lemma 2.18 Let $V$ be any (non-zero) irreducible $L_0$-submodule of $L_{q+i}$, where $0 \leq i < \frac{q-1}{2}$. If assumptions (v) and (viii) hold, then $[V, L_{q-(i+1)}] \neq 0$, so that $[V, L_{q-(i+1)}] = L_{-1}$. Moreover, $L_{q+i}$ is an irreducible $L_0$-module.

Proof If $[V, L_{q-(i+1)}] = 0$, then, since the positive gradation spaces are assumed (v) to be generated by $L_1$,

$$[V, L_{q-i}] = [V, [L_1, L_{q-(i+1)}]] = [(V, L_1), L_{q-(i+1)}] \subseteq [L_{q+(i+1)}, L_{q-(i+1)}].$$

Consequently, since $i$ is assumed to be less than $\frac{q-1}{2}$, so that $2i + 1 < q$,
\[ [V, [V, L_{q-i}]] \subseteq [V, [L_{q+(i+1)}, L_{q-(i+1)}]] \]
\[ = [L_{q+(i+1)}, [V, L_{q-(i+1)}]] \]
\[ = [L_{q+(i+1)}, 0] = 0. \]

Now, \( L_q = S_q \) is an irreducible \( L_0 \)-module by Lemma 2.4 so we can assume by induction on \( i \) that \( L_{q+i-1} \) is an irreducible \( L_0 \)-module. Also, since \( [L_q, L_{q-1}] = L_1 \) by Lemma 2.3 and (B), we can assume by induction that \( [L_{q+i-1}, L_{q-1}] = L_1 \). But then Lemma 2.12 would imply that \( V = 0 \), contrary to assumption. Thus, \( [V, L_{q-(i+1)}] \) is a non-zero \( L_0 \)-submodule of \( L_1 \), so that by irreducibility (B), \( [V, L_{q-(i+1)}] = L_{-1} \). The last assertion follows from Lemma 2.13.

**Lemma 2.19** Suppose that assumptions (vi) and (viii) hold. Then for any \( i, 1 \leq i \leq q \), we have and \( [L_q, [L_i, L_i]] = L_{-q} \); in particular, \( [L_i, L_i] \neq 0 \).

**Proof** The lemma will follow from Lemma 2.4 once we show that \( [L_q, [L_i, L_i]] \neq 0 \). For \( i = q \), the lemma follows from Corollary 2.11. Let \( 1 \leq i < q \). Then by Lemma 2.17 we have \( L_q = [L_{q+i}, L_{-i}] \), and by Lemma 2.8 we have \( L_{q+i} = [L_q, L_i] \). Then we have

\[ L_q = [L_{q+i}, L_{-i}] = [L_{q+i}, L_{-i}] = [L_q, [L_i, L_i]], \]

so that \( [L_q, [L_i, L_i]] \neq 0 \), as required.

**Lemma 2.20** Suppose that assumptions (iv), (v), (and therefore, (iii)), (vi), and (viii) hold and that \( 0 < i < \frac{q-3}{2} \). Then \( [L_{q+i}, L_{q-i+1}] = L_1 \).

**Proof** Suppose that \( [L_{q+i}, L_{q-i+1}] = 0 \). Then, since by (iii) and Lemma 2.3 \( L_j = S_j = [S_{j+1}, S_{-1}] = [L_{j+1}, L_{-1}] \subseteq L_j \) for all \( j, 0 < j < r \), we have

\[ [L_{q+i}, L_{q-i}] = [L_{q+i}, [L_{q-i+1}, L_{-1}]] \]
\[ = [L_{q-i+1}, [L_{q+i}, L_{-1}]] \]
\[ = [L_{q-i+1}, L_{q+i-1}], \]

so that (since \( i < \frac{q-3}{2} \) implies that \( 2i + 3 < q \), so that a fortiori \( 2i - 1 < q \))

\[ [L_{q+i}, [L_{q+i}, L_{q-i}]] = [L_{q+i}, [L_{q-i+1}, L_{q+i-1}]] \]
\[ = [L_{q+i-1}, [L_{q+i}, L_{q-i+1}]] \]
\[ = [L_{q+i-1}, 0] = 0. \]

Let \( v \in L_{q+i} \) and \( u \in L_{q-i} \). Then (since \( 2i + 1 < 2i + 3 < q \), so that \( [v, L_{q+i+1}] = 0 \))

\[ 2(ad[v, u])^2L_{q+i+1} = (ad v)^2(ad u)^2L_{q+i+1} \]
\[ \subseteq (ad v)^2L_{q-i+1} \]
\[ \subseteq (ad v)[L_{q+i}, L_{q-i+1}] = (ad v)0 = 0. \]
Consequently, \( \text{ad}_{L_{q+i+1}}[L_{q+i}, L_{q-i}] \) is a nilpotent set of linear transformations by Lemma \( \ref{lem:2.8} \). Since we are assuming that \( i < \frac{q^2}{2} \), we have \( i + 1 < \frac{q^2 - 1}{2} \), so we can apply Lemma \( \ref{lem:2.18} \) to conclude that \( L_{q+i+1} \) is an irreducible \( L_0 \)-module. It follows that \( \text{ad}_{L_{q+i+1}}[L_{q+i}, L_{q-i}] \) annihilates \( L_{q+i+1} \). Thus (since, again, \( 2i + 1 < 2i + 3 < q \))

\[
0 = [[L_{q+i}, L_{q-i}], L_{q+i+1}] = [L_{q+i}, [L_{q-i}, L_{q+i+1}]].
\]

If \( [L_{q-i}, L_{q+i+1}] \neq 0 \), then, since \( L_1 \) is assumed (iv) to be irreducible, \( [L_{q-i}, L_{q+i+1}] \) would have to equal \( L_1 \), and the above-displayed formula would imply a lack of \( \{-1\}\)-transitivity (vi) of \( L \) in its negative part. It follows that \( [L_{q-i}, L_{q+i+1}] = 0 \). Then, in view of our initial assumption that \( [L_{q+i}, L_{q-i+1}] = 0 \), we would have

\[
0 = [[L_{q+i}, L_1], L_{q-i}] = [[L_{q+i}, L_{q-i}], L_1],
\]

to contradict Lemma \( \ref{lem:2.14} \) in view of Lemma \( \ref{lem:2.19} \). Thus, it must be that \( [L_{q+i}, L_{q-i+1}] \neq 0 \), so that by the assumed irreducibility of \( L_1 \), \( [L_{q+i}, L_{q-i+1}] = L_1 \), as required. \( \square \)

**Lemma 2.21** Let \( q > 5 \), and suppose that assumptions (iv), (v) (and therefore (iii)), (vi), and (viii) hold. If \( q \) is even, then \( L_{q-2}^2 = L_{-q} \), while if \( q \) is odd, then \( L_{q-2}^{q-1} = L_{q+1} \).

**Proof** We have by Lemma \( \ref{lem:2.8} \) that \( L_{-2} = [L_{-q}, L_{q-2}] \) and by Lemma \( \ref{lem:2.18} \) (since \( q > 3 \)) that \( L_{-1} = [L_{-q+1}, L_{q-2}] \). Thus, for any \( j, 1 < j < q-1 \), we have by \( \{-1\}\)-transitivity (Lemma \( \ref{lem:2.1} \)) that

\[
0 \neq [V_{-j}, L_{-1}] = [V_{-j}, [L_{-q+1}, L_{q-2}]] = [L_{-q+1}, [V_{-j}, L_{q-2}]].
\]

where \( V_{-j} \) is any non-zero \( L_0 \)-submodule of \( L_{-j} \). Consequently, \( [V_{-j}, L_{q-2}] \neq 0 \). Then by Lemma \( \ref{lem:2.10} \) when \( j < q-2 \), or, when \( j = q-2 \), by Lemma \( \ref{lem:2.10} \)

\[
0 \neq [L_{-q}, [V_{-j}, L_{q-2}]] = [V_{-j}, [L_{-q}, L_{q-2}]] = [V_{-j}, L_{-2}]
\]

by Lemma \( \ref{lem:2.8} \). If, for \( j = 1, 2, \ldots, \left\lfloor \frac{q}{2} \right\rfloor - 1 \), we successively let \( V_{-2j} \overset{\text{def}}{=} L_{-2} \), we can conclude that \( L_{-2}^2 \neq 0 \) if \( q \) is even, and \( L_{-2}^{q-1} \neq 0 \) if \( q \) is odd. Then, by Lemmas \( \ref{lem:2.4} \) and \( \ref{lem:2.18} \) respectively, \( L_{-2}^2 = L_{-q} \) if \( q \) is even, or \( L_{-2}^{q-1} = L_{-q+1} \) if \( q \) is odd. \( \square \)

**Lemma 2.22** Let \( q > 5 \), and suppose that assumptions (iv), (v) (and therefore (iii)), (vi) and (viii) hold. Then \( L_{-2} \) is an irreducible \( L_0 \)-module.
Proof Let $V_{-2}$ be any irreducible $L_0$-submodule of $L_{-2}$. Since $[L_{-q+1}, [V_{-2}, L_q]] = [V_{-2}, L_{-q+1}, L_q] = [V_{-2}, L_{-1}] = L_{-1}$ by Lemmas 2.20 and 2.15, it follows that for any $j, 0 < j < \frac{q}{2}$ (i.e., $0 < j \leq \frac{q-1}{2}$) for which $V_{-2}^j \neq 0$, we have by transitivity (Lemma 2.1) that

$$[L_{-q+1}, [V_{-2}^j, [V_{-2}, L_q]]] = [V_{-2}^j, [V_{-2}, [L_{-q+1}, L_q]]] = [V_{-2}^j, L_{-1}] \neq 0,$$

so we conclude that $[V_{-2}^j, [V_{-2}, L_q]] \neq 0$ and therefore that $(\text{ad} V_{-2})^j L_q \neq 0$. Thus, so long as $2(j + 1) < q$ (i.e., $j < \frac{q}{2} - 1$), we have by Lemma 2.16 that

$$0 \neq [L_q, [V_{-2}^j, [V_{-2}, L_q]]] = [V_{-2}^j, [V_{-2}, [L_q, L_q]]] \subseteq [V_{-2}^j, V_{-2}].$$

Thus, $V_{-2}^j \neq 0$ for all $j, 0 < j \leq \frac{q-1}{2}$, and $(\text{ad} V_{-2})^j L_q \neq 0$ for all $j, 0 < j \leq \frac{q-1}{2}$.

If $q$ is odd, then, since $q > 5$, we have by Lemma 2.18 that $V_{-2}^{\frac{q-1}{2}} = L_{-q+1}$, while if $q$ is even, we have $V_{-2}^{\frac{q}{2}-1} = L_{-q+2}$.

In the case of odd $q$, we have, by the irreducibility (B) of $L$, that $L_{-1} = (\text{ad} V_{-2})^{\frac{q}{2}} L_q$, so that

$$L_{-2} = [L_{-1}, L_{-1}] = [L_{-1}, (\text{ad} V_{-2})^{\frac{q}{2}} L_q] \subseteq [V_{-2}, [L_q, L_q]] + [V_{-2}, L_0] \subseteq V_{-2}.$$

Thus, when $q$ is odd, we see that $L_{-2}$ is irreducible. In the case of even $q$, we have by Lemma 2.18 (since $q > 5$) that $L_{-1} = [L_{-q+2}, L_{q-3}] = [V_{-2}^{\frac{q}{2}-1}, L_{q-3}] \subseteq (\text{ad} V_{-2})^{\frac{q}{2}-1} L_{q-3} \subseteq L_{-1}$. By (D) and transitivity (Lemma 2.1),

$$L_{-q+1} = [L_{-1}, L_{-q+2}] = [(\text{ad} V_{-2})^{\frac{q}{2}-1} L_{q-3}, L_{q+2}] \subseteq (\text{ad} V_{-2})^{\frac{q}{2}-1} L_{q-3} \subseteq L_{-q+1},$$

so that (See also Lemma 2.18) $(\text{ad} V_{-2})^{q/2} L_{q-3} = L_{-q+1}$. Then, by Lemma 2.17,

$$L_{-q} = [L_{-1}, L_{-q+1}] = [L_{-1}, (\text{ad} V_{-2})^{q-2} L_{q-3}] \subseteq [V_{-2}, L_{-q+2}] + [(\text{ad} V_{-2})^{q-2} L_{-1}, L_{q-3}] \subseteq [V_{-2}, V_{-2}^{\frac{q}{2}-1}] + [0, L_{q-3}] \subseteq V_{-2}^{\frac{q}{2}}.$$
Now, by Lemma 2.16 and irreducibility (B), we have
\[ L^{-1} = [L^{-q}, L_{q-1}] = [V^{-\frac{q}{2}}, L_{q-1}] \subseteq (\text{ad} V_{-2})_{\frac{q}{2}} L_{q-1} \subseteq L^{-1}. \]

Consequently, we have
\[ L^{-2} = [L^{-1}, L^{-1}] = [L^{-1}, (\text{ad} V_{-2})_{\frac{q}{2}} L_{q-1}] = [(\text{ad} V_{-2})_{\frac{q}{2}} L_{q+1}, L_{q-1}] + [V, L_0] \subseteq [0, L_{q-1}] + V_{-2} \subseteq V_{-2} \]

as required. □

Now suppose that (vii) holds. Note that by (D) of the Main Theorem, \( L^{-2} = [L^{-1}, L^{-1}] \); that is, \( L^{-2} \) is spanned by brackets of elements of the three-dimensional \( L_0 \)-module \( L^{-1} \). Consequently, \( L^{-2} \) is at most three-dimensional.

On the other hand, by Lemma 1.2, the character \( \chi \) is non-zero on \( L^{-2} \), so \( L^{-2} \) is not a restricted \( L_0 \)-module, so its dimension is, in fact, three, as are all irreducible non-restricted \( \mathfrak{sl}(2) \)-modules in characteristic three. Thus (Compare Lemma 2.22),

Lemma 2.23 If assumptions (i) and (vii) hold, then \( L^{-2} \) is an irreducible three-dimensional non-restricted \( L_0 \)-module.

Lemma 2.24 If \( q > 2 \) and assumptions (vi) and (vii) hold, then \( \text{Ann}_{L_1} L^{-2} = 0. \)

Proof Set \( A_1 = \text{Ann}_{L_1} L^{-2} \), and suppose that \( A_1 \neq 0 \). Since
\[ [L^{-2}, [L^{-q+1}, A_1]] = [L^{-q+1}, [L^{-2}, A_1]] = 0, \]
we have
\[
0 = [L^{-2}, [L^{-q+1}, A_1]] \\
\supseteq [[L^{-3}, L_1], [L^{-q+1}, A_1]] = [L^{-3}, [L_1, [L^{-q+1}, A_1]]] \\
\supseteq [[L^{-4}, L_1], [L_1, [L^{-q+1}, A_1]]] = [L^{-4}, [L_1, [L_1, [L^{-q+1}, A_1]]]] \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \cdots \\
\supseteq [L^{-q+1}, (\text{ad} L_1)^{q-3} [L^{-q+1}, A_1]].
\]

Now, if \( [L^{-q+1}, A_1] \neq 0 \), then by (vi) and irreducibility (B), we would have \( (\text{ad} L_1)^{q-3} [L^{-q+1}, A_1] = L^{-1} \), so that \( [L^{-q+1}, L^{-1}] = 0 \), to contradict transitivity (Lemma 2.1). Thus, we must have
\[ [L^{-q+1}, A_1] = 0. \quad (2.25) \]

Now, since \( [L^{-q}, [A_1, A_1]] \subseteq [L^{-q+1}, A_1] = 0 \), and, clearly, \( [L^{-q+1}, [A_1, A_1]] = 0 \), it follows from Lemma 2.0 that \( [A_1, A_1] = 0 \). Then \( L^+ \) def \( L_{-q} \oplus \)
\[ \cdots \oplus L_{-1} \oplus L_0 \oplus A_1)/(L_{-q} \oplus \cdots \oplus L_{-2}) \] is a depth-one Lie algebra which satisfies conditions (A) through (C) of the Main Theorem. Consequently, by Proposition 1.6 \((L^1)'\) is one of the Lie algebras enumerated in the hypothesis of Lemma 1.8. If we set \(V = L_{-q} \oplus L_{-q+1}\), then the \((L^1)'\)-module \(V\) satisfies the hypotheses of Lemma 1.8. Set \(W = L_{-q}\). Since \([A_1, L_{-q+1}] = 0\), we must conclude that if \(\zeta\) is as in Lemma 1.8, then \(\zeta = q - 1\); furthermore, since \([L_{-1}, [L_{-1}, L_{-q+1}] \subseteq L_{-q} = 0\), we must have that \(\zeta = q - 1 = 0 \mod 3\). However, if we then set \(W = L_{-q}\), we have \([L_{-1}, L_{-q}] = 0\) and, by (2.25),

\[ [A_1, [A_1, L_{-q}] \subseteq [L_{-q+1}, A_1] = 0 \]

so we must, by similar reasoning, conclude that \(\zeta = q = 0 \mod 3\). Since both \(q - 1\) and \(q\) cannot be equivalent to zero modulo three, we have arrived at a contradiction. We therefore conclude that \(\text{Ann}_{L_1}L_{-2} = A_1 = 0\), as required.

\[ \square \]

Lemma 2.26 If conditions (iv), (vi) and (vii) hold, then

\[ \text{Ann}_{L_1}L_{-2} = 0. \]

**Proof** If \(q = 2\), this lemma follows from Lemma 2.16. If \(q > 2\), it follows from Lemma 2.24. \(\square\)

Lemma 2.27 If \(M_1\) is a non-zero \(L_0\)-submodule of \(L_1\) such that \(\text{Ann}_{L_0}M_1 \neq 0\), then \([L_{-1}, M_1], M_1] = 0\), and \([M_1, M_1] = 0\).

**Proof** Set \(X \overset{\text{def}}{=} \text{Ann}_{L_0}M_1 \neq 0\), and suppose that \(X \neq 0\). Then by transitivity (C) and irreducibility (B), \([L_{-1}, X] = L_{-1}\), so

\[ [L_{-1}, M_1] = [[L_{-1}, M_1], M_1] = [[L_{-1}, M_1], X] \subseteq X. \]

Thus, \([L_{-1}, [M_1, M_1]] \subseteq [L_{-1}, M_1], M_1] \subseteq [X, M_1] = 0\), so, by transitivity (C), \([M_1, M_1] = 0\). \(\square\)

Lemma 2.28 We may assume that \((S_\omega =) [[L_{-1}, L_1], L_r] = L_r.\)

**Proof** Suppose \([[L_{-1}, L_1], L_r] \neq L_r\). We distinguish two cases:

(i) \([[L_{-1}, L_1], L_r] = 0.\)

(ii) \([[L_{-1}, L_1], L_r] \neq 0.\)

(i) Suppose first that \([[L_{-1}, L_1], L_r] \) were equal to zero. Then we would have

\[ 0 = [[L_{-1}, L_1], L_r] = [L_1, [L_{-1}, L_r]] \]

so

\[ S \overset{\text{def}}{=} \sum_{i \geq 0} (\text{ad } L_{-1})^i [L_{-1}, L_r] \subseteq S \]

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would be an ideal in \(S\) entailing equality by the definition of \(S\); in particular, we would have \([L_{-1}, L_r] = S_s\).

If, in addition, \([L_{-1}, L_1], [L_{-1}, L_r]\) were also equal to zero, we could repeat the argument and get that

\[
\mathcal{S} \overset{\text{def}}{=} \sum_{i \geq 0} (\text{ad} L_{-1})^i [L_{-1}, [L_{-1}, L_r]] \subseteq S
\]

would be a proper ideal of \(S\), to contradict the simplicity of \(S\). We conclude that \([L_{-1}, L_1], [L_{-1}, L_r] \neq 0\), and, in the case that \([L_{-1}, L_1], L_r = 0\), that \([L_{-1}, L_r] = S_s\), which is an irreducible \(L_0\)-module by Lemma 2.3.

Then,

\[
[[L_{-1}, L_1], [L_r, L_{-1}]] = [[L_{-1}, L_1], S_s] = S_s = [L_{-1}, L_r]
\]

Hence, if in the case that \([L_{-1}, L_1], L_r = 0\), we replace \(L_r\) with \([L_{-1}, L_r]\), the lemma follows.

(ii) Now suppose that \([L_{-1}, L_1], L_r \neq 0\). Since

\[
0 \neq [[L_{-1}, L_1], L_r] = [[L_{-1}, L_r], L_1] \subseteq [S_{r-1}, L_1] \subseteq S,
\]

it would follow that \(\sum_{i \geq 0} (\text{ad} L_{-1})^i [[L_{-1}, L_1], L_r]\) would be an ideal of \(S\) and hence all of \(S\). In particular, \(s\) would equal \(r\), and \([L_{-1}, L_1], L_r\) would equal \(S_s\). Thus, \([L_{-1}, L_1], L_r\) is an irreducible \(L_0\)-module by Lemma 2.3. Consequently, if we replace \(L\) by the Lie algebra generated by \(L_{-1}, L_0, L_1\), and \([L_{-1}, L_1], L_r\) (\(= [S_{s-1}, L_1] = S_s\)), then the highest gradation space will be of the form \([L_{-1}, L_1], L_r\), as required. \(\square\)

**Lemma 2.29** Let \(L\) be as in the statement of the Main Theorem, and suppose that \(L_2 \neq 0\), that \([L_{-2}, L_1] = 0 = [L_{-2}, L_2]\), and that assumption (vii) holds. Let \(\tilde{L}\) be the Lie subalgebra of \(L\) generated by \(L_{-1}, L_0, L_1\), and \(L_2\). If \(M(\tilde{L})\) is as in Weisfeiler’s Theorem (Theorem 1.3), then \(\tilde{L}/M(\tilde{L})\) is Hamiltonian, and we have \([L_{-1}, L_1] \cong \mathfrak{sl}(2)\).

**Proof** Let \(\tilde{L}\) be the Lie subalgebra of \(L\) generated by \(L_{-1}, L_0, L_1\), and \(L_2\). Since \([L_{-2}, L_1] = 0 = [L_{-2}, L_2]\), we have \(M(\tilde{L}) = L_{-q} \oplus \cdots \oplus L_{-2} = M(\tilde{L})\). But the depth \(\tilde{L}/M(\tilde{L})\) is then one, so by Proposition 1.6, \(\tilde{L}/M(\tilde{L})\) is either Hamiltonian (i.e., between \(H(2 : \mathfrak{h}, \omega)\) and \(CH(2 : \mathfrak{h}, \omega)\)) or is isomorphic to a Lie algebra of type \(L(\epsilon)\) or \(M\). However, the height of the latter two Lie algebras is one, and, since \(L_2 \neq 0\), the height of \(\tilde{L}/M(\tilde{L})\) is at least two. Thus, \(\tilde{L}/M(\tilde{L})\) must be Hamiltonian, so \([L_{-1}, L_1] \cong \mathfrak{sl}(2)\). It now follows from Corollary 1.7 that \(\tilde{L}/M(\tilde{L})\) is Hamiltonian, as well. \(\square\)

**Lemma 2.30** Let \(L\) be as in the statement of the Main Theorem, and suppose that \(L/M(\tilde{L})\) (as above) is isomorphic to \(L(\epsilon)\) or \(M\). Then \(\text{Ann}_{\tilde{L}} L_{-2} = 0\) and in the proof of the Main Theorem, where we assume that \([L_{-2}, L_1] = 0\), we may assume that \(L_{-2}\) is an irreducible \(L_0\)-module.
Proof Suppose first that \( \text{Ann}_{L_2} L_{-2} \neq 0 \). Then, as in the proof of the previous lemma, we may consider the Lie algebra \( \tilde{L} \) generated by \( L_{-1}, L_0, L_1, \) and \( \text{Ann}_{L_2} L_{-2} \). Then \( \tilde{L}/M(\tilde{L}) \) has depth one and height greater than one, but null component isomorphic to \( \mathfrak{gl}(2) \), to contradict Corollary 1.7. Thus, \( \text{Ann}_{L_2} L_{-2} = 0 \), so that if \( M_2 \) is any irreducible \( L_0 \)-module of \( L_2 \), then \([L_{-2}, M_2] \neq 0\). Replacing \( L \) by the Lie algebra generated by \( L_{-1}, L_0, L_1, \) and \( M_2 \), we complete the proof of the lemma. \( \square \)

**Lemma 2.31** If \([L_{-2}, L_1] = 0\) and \([L_{-2}, L_2] \neq 0\) and (vii) holds, then we may assume in the proof of the Main Theorem that \([L_{-2}, L_2] = [L_{-1}, L_1]\).

**Proof** By (6.14),

\[
\mathfrak{sl}(2) \subseteq [L_{-1}, L_1] \subseteq \mathfrak{gl}(2)
\]

By Lemma 2.23 \( L_{-2} \) is an irreducible, non-restricted \( L'_0 \)-module of dimension three. Let \( B(L_{-2}) \) be as in Section 3 of [1], and let \( M(B(L_{-2})) \) be as in Weisfeiler’s Theorem (Theorem 1.3). We focus on

\[
X \overset{\text{def}}{=} B(L_{-2})/M(B(L_{-2})),
\]

whose depth is no greater than half of that of \( L \). If the depth of \( X \) is greater than one, then we can apply the Main Theorem to conclude that the representation of \( L'_0 \) on the minus-one component (namely, \( L_{-2} \)) of \( X \) is restricted, so that (See Lemma 1.2) the representation of \( L'_0 \) on \( L_{-1} \) is also restricted, to contradict assumption (vii). We can therefore assume that the depth of \( X \) is one, so that by Proposition 1.9 \( X \) is isomorphic either to a Hamiltonian Lie algebra, or to \( M \), or to \( L(\epsilon) \) for some \( \epsilon \). Then, by (6.14) again,

\[
\mathfrak{sl}(2) \subseteq [L_{-2}, L_2] \subseteq \mathfrak{gl}(2)
\]

Since \([L_{-2}, L_2] \subseteq [L_{-1}, L_1]\) by (D), we are done unless \([L_{-2}, L_2] \cong \mathfrak{sl}(2)\), and \([L_{-1}, L_1] \cong \mathfrak{gl}(2)\). In that case, we have by Corollary 1.7 that \( X \) is Hamiltonian. Let \( e, f, \) and \( h \) be the usual basis of \( L'_0 \). Then, according to (1.4) of [1], \((-\text{ad}_{L_{-1}} h + 1)^2 + -\text{ad}_{L_{-1}} e + \text{ad}_{L_{-1}} f\) acts as the identity on the minus-one component of \( X \), namely, \( L_{-2} \). On the other hand, if \( \tilde{L} \) is as in Lemma 2.29, then, again by Corollary 1.7, either \( \tilde{L}/M(\tilde{L}) \cong L(\epsilon) \), or \( \tilde{L}/M(\tilde{L}) \cong M \). If \( \tilde{L}/M(\tilde{L}) \cong L(\epsilon) \), then \((-\text{ad}_{L_{-1}} h + 1)^2 + -\text{ad}_{L_{-1}} e + \text{ad}_{L_{-1}} f\) acts as multiplication by the scalar \((\epsilon^3 + 1)/(\epsilon + 1)(\epsilon - 1)^2\). If this were to be consistent with its action on the minus-one component \( (L_{-2}) \) of \( X \), then we would have

\[
1 = \frac{(\epsilon^3 + 1)}{(\epsilon + 1)(\epsilon - 1)^2} = \frac{(\epsilon + 1)(\epsilon^2 - \epsilon + 1)}{(\epsilon + 1)(\epsilon - 1)^2}
\]

from which it would follow that \(\epsilon^2 - \epsilon + 1 = (\epsilon - 1)^2\) or \(\epsilon = 0\). However, if \( \epsilon = 0 \), it follows from (1.8) of [1], that the representation of the null component on the
minus-one component of \(L(e)\) would be a \(p\)-representation, contrary to assumption. Thus, we can conclude that \([L_{-2}, L_2] = [L_{-1}, L_1]\), when \(\bar{L}/M(\bar{L}) \cong L(e)\). Finally, if \(\bar{L}/M(\bar{L}) \cong M\), then \((-\text{ad}_{L_{-1}}, h+1)^2 + -\text{ad}_{L_{-1}}e + \text{ad}_{L_{-1}}f\) acts as zero on \(L_{-2} = [L_{-1}, L_{-1}]\), which, again, is incompatible with its value on the minus-one component \(L_{-2}\) of the Hamiltonian Lie algebra \(X\), as we just observed. We can therefore infer that \([L_{-2}, L_2] = [L_{-1}, L_1]\) in the case where \(\bar{L}/M(\bar{L}) \cong M\), also, to complete the proof of the Lemma. \(\square\)

3 The Main Theorem under additional assumptions

In this and the following two sections, we assume that assumptions ((i) and (ii), of course), (iv), (v) (and therefore (iii)), (vi), and (viii) of the previous section hold, so that, in particular, by Lemma 2.22, \(L_{-2}\) is an irreducible \(L_0\)-module. We begin this section by forming the irreducible, transitive Lie algebra \(B(L_{-2})\). (See, for example, Section 3 of [1].) Indeed, consider the subalgebra

\[
E = E_{-\ell} \oplus \cdots \oplus E_0 \oplus \cdots \oplus E_{\ell}\]

of \(L\) consisting of the gradation spaces \(E_i = L_{-i}^\perp\) for \(i < 0\), and \(E_i = L_{2i}\) for \(i \geq 0\). Set \(T_0 = \text{Ann}_E E_{-1} = \text{Ann}_L L_{-2}\), and for \(i = 1, 2, \ldots\), let

\[
T_i = \{x \in E_i \mid [x, E_{-1}] \subseteq T_{i-1}\}.
\]

Then \(T = T_0 \oplus T_1 \oplus \cdots \oplus T_\ell\) is an ideal of \(E\), and the factor algebra

\[
G = E/T = G_{-\ell} \oplus \cdots \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus \cdots \oplus G_{\ell}
\]

is a transitive graded Lie algebra (See [1] Lemma 3.). Thus, the Lie algebra \(B(L_{-2}) \overset{\text{def}}{=} G\) satisfies conditions (A)-(D) of the Main Theorem. (It is shown in, for example, [3] that the process of forming \(B(L_{-2})\) preserves condition (A).)

**Lemma 3.1** If conditions (i) through (viii) of Section 2 hold, and \(q \geq 6\), then \([[L_{-2}, L_2], L_2] = 0\).

**Proof** By Lemma 2.22, \(L_{-2}\) is irreducible as an \(L_0\)-module. We consider \(B(L_{-2})\). If the height of \(B(L_{-2})\) is greater than one, Lemma 3.1 follows from Corollary 1.5 applied to \(B(L_{-2})\). Thus, we may assume that the height of \(B(L_{-2})\) is one.

To prove Lemma 3.1 we assume that \([[L_{-2}, L_2], L_2] = 0\) and derive a contradiction. Since we are assuming that the height of \(B(L_{-2})\) is one, we must have that \(L_4 \subseteq T\); that is, that \([L_{-2}, [L_{-2}, L_2]] = 0\). Therefore, in view of our assumption that \([[L_{-2}, L_2], L_2] = 0\) (so that \([L_{-2}, [L_{-2}, L_4]], L_2] = 0\) we have

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0 = [[L_{-2}, [L_{-2}, L_4]], L_2] = [[L_{-2}, L_2], [L_{-2}, [L_{-2}, L_4]]],
so [L_{-2}, [L_{-2}, L_4]] is abelian, and is therefore contained in the center of L_0.
But we have seen that any non-zero element of the center acts via the adjoint representation as a non-zero scalar on L_{-1} and, consequently, on L_{-2}, so we can conclude that

\[ [L_{-2}, [L_{-2}, L_4]] = 0 \]  \hspace{1cm} (3.2)

Since by hypothesis \([[[L_{-2}, L_2], L_2] = 0, we have

\[ [L_{-2}, [L_2, L_2]] \subseteq [[[L_{-2}, L_2], L_2] = 0. \]  \hspace{1cm} (3.3)

Let \(M_4\) be any \(L_0\)-submodule of \(L_4\) such that

\[ [L_{-2}, M_4] = 0 \]  \hspace{1cm} (3.4)

We will endeavor to show that \(M_4 = 0\). If \(q\) is even, then by Lemma 2.21, \(L_{-q} = L_{-2}\), so \([L_{-q}, M_4] = 0\). Then by Lemma 2.16, \(M_4 = 0\). If \(q\) is odd, then by Lemma 2.21, \(L_{-q+1} = L_{-2}\). Then \([L_{-q+1}, M_4] = 0\), so

\[ 0 = [L_{-5}, [L_{-q+1}, M_4]] = [L_{-q+1}, [L_{-5}, M_4]] \]

so if \([L_{-5}, M_4] \neq 0\), then by the irreducibility of \(L\) we would have \([L_{-5}, M_4] = L_{-1}\), so that \([L_{-q+1}, L_{-1}] = 0\), to contradict Lemma 2.1. Thus,

\[ [L_{-5}, M_4] = 0 \]  \hspace{1cm} (3.5)

Then by (3.4) and (3.5),

\[ [L_{-2}, [L_{-3}, M_4]] \subseteq [L_{-5}, M_4] + [L_{-3}, [L_{-2}, M_4]] \subseteq 0 + [L_{-3}, 0] = 0 \]

so that

\[ [L_{-3}, M_4] = 0 \]  \hspace{1cm} (3.6)

since otherwise, by (iv) we would have \([L_{-3}, M_4] = L_1\), and (vi) would be violated. Thus, by the definition of \(M_4\), we have by (D) and (3.4) that

\[ 0 = [L_{-3}, M_4] = [[L_{-2}, L_{-1}], M_4] = [L_{-2}, [L_{-1}, M_4]] \]

Consequently, by Lemma 2.21 again, \([L_{-q+1}, [L_{-1}, M_4]] = 0\). Then \(0 = [L_{-q+1}, [L_{-1}, M_4]]\), so by the irreducibility (B) of \(L\) and Lemma 2.1 again, we must have

\[ 0 = [L_{-4}, [L_{-1}, M_4]] = [L_{-1}, [L_{-4}, M_4]] \]

since by (3.5), \([L_{-5}, M_4] = 0\). Thus, by the transitivity (C) of \(L\),
Recapitulating, we have, by (3.4), (3.5), (3.6), and (3.7) that

\[ [L_{-i}, M_4] = 0 \text{ for } i = 2, 3, 4, 5 \]  

(3.8)

Then

\[ 0 = [L_{-1}, 0] = [L_{-1}, [L_{-3}, M_4]] = [L_{-3}, [L_{-1}, M_4]] \]

so

\[ 0 = [[L_{-1}, L_{-2}], [L_{-1}, M_4]] = [L_{-2}, [L_{-1}, [L_{-1}, M_4]]] \]  

(3.9)

(because by (3.8) and the Jacobi Identity, \([L_{-2}, [L_{-1}, M_4]] = 0\)). Now, again by (3.8) and the Jacobi Identity,

\[ [L_{-3}, [L_{-1}, M_4]] = 0 \]

so, by (3.8) and the Jacobi Identity, and by (3.9),

\[ 0 = [[L_{-1}, L_{-3}], [L_{-1}, [L_{-1}, M_4]]] = [L_{-3}, [L_{-1}, [L_{-1}, M_4]]] \]

However, as above, in view of (iv), this contradicts (vi) if \([L_{-1}, [L_{-1}, M_4]] \neq 0\). Consequently,

\[ [L_{-1}, [L_{-1}, M_4]] = 0, \]

so that, by transitivity,

\[ M_4 = 0 \]  

(3.10)

In view of (3.10), we can set \(M_4\) equal to \([L_2, L_2]\), and conclude that \(L_2\) is abelian.

If \(q\) is even, then by Lemma 2.21, \(L_{-q} = \frac{L_{-q}}{2}\) = \(\frac{L_{-1}}{2}\), so by (3.2), \([L_{-q}, L_4] \subseteq (\text{ad} L_{-2})^r L_4 = 0\). Then by Lemma 2.3, \(L_4 = 0\) (and \(r \leq 3\)).

If \(q\) is odd, then, since \(q \geq 6\), we must have \(q \geq 7\), so by Lemma 2.21, \(L_{-q+1} = \frac{L_{-q+1}}{2}\). Then again by (3.2), \([L_{-q+1}, L_4] = 0\), so we can argue as in the proof of (3.3) to show that \([L_{-5}, L_4] = 0\). Similarly, since by (3.2) again, its bracket with \(L_{-q+1}\) is zero, \([L_{-3}, [L_{-2}, L_4]] = 0\) must equal zero, since otherwise it would by irreducibility (B) equal \(L_{-1}\), and transitivity (Lemma 2.1) would be violated. Since again by (3.2) \([L_{-2}, [L_{-2}, L_4]] = 0\), we have by (D) that

\[ 0 = [L_{-3}, [L_{-2}, L_4]] \]

\[ = [[L_{-1}, L_{-2}], [L_{-2}, L_4]] \]

\[ = [[L_{-1}, [L_{-2}, L_4]], L_{-2}] \]
Consequently, if \([L_{-1}, [L_{-2}, L_4]]\) were not zero, it would, by (iv), equal \(L_1\), and (vi) would be violated. We can therefore conclude that \([L_{-1}, [L_{-2}, L_4]] = 0\), so that, by the transitivity of \(L\), \([L_{-2}, L_4] = 0\). We can now set \(M_4 = L_4\), where \(M_4\) is as in the argument above, and conclude that \(L_4 = 0\). Thus, \(r \leq 3\). But, according to (viii), we have \(6 \leq q \leq r\). This contradiction completes the proof of the lemma. \(\Box\)

**Lemma 3.11** Let \(L\) be as in the statement of the Main Theorem, and suppose that assumptions (i) - (viii) hold. If \(q \geq 6\), then \(B(L_{-2})\) is an irreducible, transitive graded Lie algebra of height greater than zero and depth greater than 1, and \(B(L_{-2})^{-2} \not\subseteq M(B(L_{-2}))\). Consequently, since the depth of \(B(L_{-2})\) is no greater than half of the depth of \(L\), we can, using induction, apply the Main Theorem to conclude that the character of the representation of \(B(L_{-2})^{-2}\) on \(B(L_{-2})^{-1}\) is equal to zero. Then the character of \(L'_{0} = B(L_{-2})_{0}^{+}\) on \(L_{-1}\) is also zero.

**Proof** By Lemma 2.22 or Lemma 2.23 \(L_{-2}\) is an irreducible \(L_{0}\)-module. To show that the height of \(B(L_{-2})\) is positive, we begin by proving that since \(q \geq 6\), we have \([L_{-q+1}, L_2] \neq 0\). Indeed, we have by Lemma 2.20 and (vi) that

\[
[L_{-q+1}, L_2] \supseteq [L_{-q+1}, [L_{-q+2}, L_q]] = [L_{-q+2}, [L_{-q+1}, L_q]] = [L_{-q+2}, L_1] \neq 0,
\]

as required, so that if \(q\) is odd, we have by Lemma 2.21 that

\[
0 \neq [L_{-q+1}, L_2] \subseteq (\text{ad } L_{-2}) \Leftrightarrow L_2.
\]

On the other hand, if \(q\) is even, we have by Lemma 2.24 and Lemma 2.21 that

\[
0 \neq [L_{-q}, L_2] \subseteq (\text{ad } L_{-2}) \Leftrightarrow L_2.
\]

In either case,

\[
[L_{-2}, [L_{-2}, L_2]] \neq 0, \quad (3.12)
\]

so \([L_{-2}, L_2] \not\subseteq T_0\). Thus, \(T_1 \neq L_2\), so \(G_1 \neq (0)\), and the height of \(B(L_{-2})\) is positive.

We must now verify hypothesis (E) of the Main Theorem for the Lie algebra \(B(L_{-2})\); that is, we must show that \([L_{-2}, L_{-2}]\) is not contained in \(M(B(L_{-2}))\). Thus, suppose that \([L_{-2}, L_{-2}]\) is contained in \(M(B(L_{-2}))\); i.e., that

\[
[L_2, [L_{-2}, L_{-2}]] = 0 \quad (3.13)
\]

We will arrive at a contradiction by successively considering the two cases:

---

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1) even $q$, and
2) odd $q$.

1) Suppose first that $q$ is even. Then by Lemma 2.21, $L_q = L_{\frac{q}{2}}$

$$L_{\frac{q}{2}} = \text{ad}(L_{\frac{q}{2}}) = (\text{ad}(L_{\frac{q}{2}})^2)[L_{\frac{q}{2}}, L_{\frac{q}{2}}],$$

so that $L_{\frac{q}{2}} \in M(B(L_{\frac{q}{2}}))$, which would be a contradiction, for example, (3.12) above. Thus, $q$ cannot be even.

2) Next suppose that $q$ is odd. Then by Lemma 2.21 again, $L_{\frac{q+1}{2}} = \text{ad}(L_{\frac{q+1}{2}})^2)[L_{\frac{q+1}{2}}, L_{\frac{q+1}{2}}]$, so that $L_{\frac{q+1}{2}} \in M(B(L_{\frac{q+1}{2}}))$. Now, by Lemma 2.18, $[L_{\frac{q+1}{2}}, L_{\frac{q-3}{2}}] = L_{-1}$. If $[L_{\frac{q+1}{2}}, L_{\frac{q-3}{2}}]$ were equal to zero, then we would have

$$0 = [L_{\frac{q+1}{2}}, 0] = [L_{\frac{q+1}{2}}, [L_{\frac{q+1}{2}}, L_{\frac{q-3}{2}}]] = [L_{\frac{q+1}{2}}, L_{\frac{q+1}{2}}, L_{\frac{q-3}{2}}] = [L_{\frac{q+1}{2}}, L_{-1}]$$

to contradict transitivity (Lemma 2.1). We therefore conclude that

$$[L_{\frac{q+1}{2}}, L_{\frac{q-3}{2}}] \neq 0$$

Since by Lemma 2.22, $L_{-2}$ is an irreducible $L_0$–module, it follows that $L_{-2} = [L_{\frac{q+1}{2}}, L_{\frac{q-3}{2}}] \subseteq M(B(L_{\frac{q+1}{2}}))$. But we saw at the conclusion of 1) above that $L_{-2}$ cannot be contained in $M(B(L_{\frac{q+1}{2}}))$. This second contradiction shows that $B(L_{\frac{q+1}{2}}) = [L_{-2}, L_{-2}]$ is in fact not contained in $M(B(L_{\frac{q+1}{2}}))$, no matter what the parity of $q$ is.

Consequently, we can conclude that $B(L_{\frac{q+1}{2}})$ satisfies hypothesis (E), and therefore all of the hypotheses of the Main Theorem. Since the depth of $B(L_{\frac{q+1}{2}})$ is greater than one, but less than or equal to $\frac{q}{2}$, we can now apply to conclude that the character $\chi$ of $B(L_{\frac{q+1}{2}})$ on $B(L_{\frac{q+1}{2}})$ is zero. Then $\frac{1}{2}\chi$, which is the character of $L_{\frac{q}{2}}$ on $L_{\frac{q}{2}}$, must be zero as well, and Lemma 3.11 is proved. □

We now address the depth-four and depth-five cases individually.

4 The depth-four case

Suppose

$$L = L_{-4} \oplus L_{-3} \oplus \cdots \oplus L_r$$

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satisfies conditions (i) through (viii) of Section 2. By Lemma 2.15

\[ [L_2, L_1] = L_{-1}. \]

Now suppose that \([L_2, L_2] = 0\). Then by (D),

\[ [L_3, L_1] = [L_3, [L_2, L_1]] = [L_2, [L_3, L_1]] \subseteq [L_2, L_2] = 0, \]

to contradict \([-1]\)–transitivity (Lemma 2.1). Thus, it follows that

\[ [L_2, L_2] \neq 0. \]

It now follows from Lemma 2.4 that \([L_2, L_2] = L_{-4}\). Furthermore, from Lemma 2.3 we have \([L_4, L_2] \neq 0\). Then

\[ 0 \neq [L_4, L_2] = ([L_2, L_2], L_2] \subseteq [L_2, [L_2, L_2]] \quad (4.1) \]

Now let \(V_2\) be any irreducible \(L_0\)–submodule of \(L_2\). If \([V_2, L_3] = 0\), then by Lemma 2.3 and irreducibility (B), \(0 = [L_4, [V_2, L_3]] = [V_2, [L_4, L_3]] = [V_2, L_{-1}]\) to contradict transitivity (Lemma 2.1). Thus, we can assume that \([V_2, L_3] = L_1\), since we are assuming that \(L_1\) is \(L_0\)–irreducible (iv). Then by Lemma 2.15 \(L_{-1} = [V_2, [V_2, L_3]]\). Then we have by condition (D) of the Main Theorem that

\[
L_2 = [L_{-1}, L_{-1}]
= [L_{-1}, [V_2, [V_2, L_3]]]
\subseteq [[L_{-1}, V_2], [V_2, L_3]] + [V_2, L_0]
= [V_2, [[L_{-1}, V_2], L_3]] + [V_2, L_0]
\subseteq [V_2, L_0]
\subseteq V_2
\]

so that \(L_2\) is an irreducible \(L_0\)–module.

Thus, in the depth-two irreducible, transitive graded Lie algebra \(B(L_2)\), we have by (4.1) that \(B(L_2)_{-1} \neq 0\). Furthermore, it follows again from (4.1) above that

\[ B(L_2)_{-2} = L_{-4} = [L_{-2}, L_{-2}] \not\subseteq M(B(L_2)), \]

so that hypothesis (E) of the Main Theorem is satisfied for \(B(L_2)\), as are the other hypotheses of the Main Theorem. Then the Main Theorem (proved for the case \(q = 2\) in [2]) applies to show that the representation of \(B(L_2)_{0}\) on \(B(L_2)_{-1}\) is restricted. Consequently, the character \(\chi\) of \(L_0\) on \(L_{-2}\) is zero, as must be the character \(\frac{1}{2}\chi\) of \(L_0\) on \(L_{-1}\).

5 The depth-five case

Suppose
\[ L = L_{-5} \oplus L_{-4} \oplus \cdots \oplus L_r \]

satisfies conditions (i) through (viii) of Section 2. Since \( L \) is transitive (C), \([L_{-1}, L_5] \neq 0\). Hence, by Lemma 2.16 \([L_{-5}, [L_{-1}, L_5]] \neq 0\), so that, by irreducibility (B), \([L_{-5}, [L_{-1}, L_5]] = L_{-1}\).

Now suppose that \([L_{-4}, L_5] = 0\). If also \([L_{-3}, [L_{-3}, L_5]] = 0\), then we would have

\[
0 = [L_{-1}, [L_{-3}, [L_{-3}, L_5]]] = [L_{-3}, [L_{-1}, L_5]].
\]

However, since \([L_{-5}, [L_{-1}, L_5]] = L_{-1}\), we have by transitivity (Lemma 2.1) that

\[
0 \neq [L_{-1}, L_{-3}] = [L_{-3}, [L_{-5}, [L_{-1}, L_5]]] = [L_{-5}, [L_{-3}, [L_{-1}, L_5]]]
\]

so that \([L_{-3}, [L_{-1}, L_5]] \neq 0\). Since \(L_1\) is assumed (iv) to be irreducible, we must have \([L_{-3}, [L_{-1}, L_5]] = L_1\). Then by \(\{1\}\)–transitivity (vi),

\[
0 \neq [L_{-3}, L_1] = [L_{-3}, [L_{-1}, L_5]],
\]

counter to what was derived above. Thus, we can assume that \([L_{-3}, [L_{-3}, L_5]] \neq 0\). But then, by the irreducibility (B) of \(L\), we must have \([L_{-3}, [L_{-3}, L_5]] = L_{-1}\). Then, by \(\{-1\}\)–transitivity (Lemma 2.1),

\[
0 \neq [L_{-4}, L_{-1}] = [L_{-4}, [L_{-3}, L_5]] = [L_{-3}, [L_{-4}, L_5]],
\]

so that \([L_{-4}, L_5] \neq 0\). Since we are assuming (iv) that \(L_1\) is irreducible, it follows that

\[
[L_{-4}, L_5] = L_1. \quad (5.1)
\]

Now let \(V_{-2}\) be any non-zero \(L_0\)–submodule of \(L_{-2}\). Then we have by \((5.1)\) and \(\{1\}\)–transitivity (vi) that

\[
0 \neq [V_{-2}, L_1] = [V_{-2}, [L_{-4}, L_5]] = [[L_{-4}, [V_{-2}, L_5]],
\]

so

\[
[L_{-4}, [V_{-2}, L_5]] = L_{-1},
\]

by the irreducibility (B) of \(L\). Then, by transitivity (Lemma 2.1),

\[
0 \neq [V_{-2}, L_{-1}] = [V_{-2}, [L_{-4}, [V_{-2}, L_5]]] = [L_{-4}, [V_{-2}, [V_{-2}, L_5]],
\]

so that \([V_{-2}, [V_{-2}, L_5]] \neq 0\). Thus, by the assumed irreducibility (iv) of \(L_1\), we must have \([V_{-2}, [V_{-2}, L_5]] = L_1\). Then, as above, by the \(\{1\}\)–transitivity (vi) of \(L\), we have
0 \neq [V_{-2}, L_1] = [V_{-2}, [V_{-2}, [V_{-2}, L_5]]],

so that

\[ [V_{-2}, [V_{-2}, [V_{-2}, L_5]]] = L_{-1}, \tag{5.2} \]

by the irreducibility (B) of \( L \). Then, since the negative gradation spaces are generated (D) by \( L_{-1} \), we have

\[ L_{-2} = [L_{-1}, L_{-1}] = [L_{-1}, [V_{-2}, [V_{-2}, L_5]]] \subseteq [V_{-2}, L_0] \subseteq V_{-2}, \]

so that \( L_{-2} \) is an irreducible \( L_0 \)-module.

Now, by Lemma 2.17, \( L_{-5} = [L_{-2}, L_{-3}] \). Consequently, in view of (D) and (5.2) above

\[ L_{-5} = [L_{-2}, L_{-3}] = [L_{-2}, [L_{-1}, L_{-2}]] = [L_{-2}, [L_{-2}, [L_{-2}, L_5]]]. \]

Then by \( \{1\} \)-transitivity (vi), we have

\[ 0 \neq [L_1, L_{-5}] = [L_1, [L_{-2}, [L_{-2}, [L_{-2}, [L_{-2}, L_5]]]]] \subseteq [L_{-2}, [L_{-2}, L_0]], \]

so that \( [L_{-2}, L_{-2}] \neq 0 \). Furthermore, since the negative gradation spaces are generated (D) by \( L_{-1} \), we have by (5.2) above that

\[ L_{-4} = [L_{-1}, L_{-3}] = [L_{-1}, [L_{-1}, L_{-2}]] = [L_{-1}, [L_{-2}, [L_{-2}, [L_{-2}, L_5]]]] \subseteq [L_{-2}, L_{-2}], \]

so

\[ L_{-4} = [L_{-2}, L_{-2}]. \tag{5.3} \]

Now suppose that \( [L_{-4}, L_2] = 0 \), and suppose further that \( [L_{-3}, L_2] = 0 \).

Then we would have by (C) and (D) that

\[ 0 = [L_{-4}, L_2] = [[L_{-3}, L_{-1}], L_2] = [L_{-3}, [L_{-1}, L_2]] = [L_{-3}, L_1] \]

by the assumed irreducibility (iv) of \( L_1 \), to contradict \( \{1\} \)-transitivity (vi). Thus, \( [L_{-3}, L_2] \neq 0 \), so that by the irreducibility (B) of \( L \), \( [L_{-3}, L_2] = L_{-1} \). Then

\[ [[L_{-3}, [L_{-4}, L_2]] = [[L_{-4}, [L_{-3}, L_2]] = [L_{-4}, L_{-1}] \neq 0, \]

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by \{-1\}-transitivity (Lemma 2.1). Thus, it must be true that
\[
[L_{-4}, L_2] \neq 0,
\] (5.4)
so that
\[
0 \neq [L_{-4}, L_2] = [[L_{-2}, L_{-2}], L_2] \subseteq [L_{-2}, [L_{-2}, L_2]].
\] (5.5)

By construction and Lemma 2.22 or Lemma 2.23, \(B(L_{-2})\) is a transitive, irreducible depth-two Lie algebra. By (5.5) above, \(B(L_{-2})_1 \neq 0\). By (5.3) and (5.4) above, \(B(L_{-2})\) satisfies hypothesis (E) of the Main Theorem. Therefore, as in the depth-four case above, it follows from the Main Theorem (proved for the case \(q = 2\) in [2]) that the character of \(B(L_{-2})_0\) on \(B(L_{-2})_{-1} = L_{-2}\) is zero. Consequently, the character of \(L_{-1}\) on \(L_{-1}\) is zero, as well, and \(L_{-1}\) is a restricted \(L_{-1}\)-module.

6 Conclusion of the proof of the Main Theorem

Let \(L\) be as in the statement of the Main Theorem, and let \(S = \sum_{i=-q}^* S_i\) be as in Weisfeiler’s Theorem (Theorem 1.3). If \([S_{\geq 0}, L_{-2}]\) were equal to zero, then \(L_{-q} \oplus \cdots \oplus L_{-2} = \sum_{i \geq 0} (\text{ad}(L_{-1}))^i L_{-2}\) would be an ideal of the simple Lie algebra \(S\). Consequently, it must be that \([S_{\geq 0}, L_{-2}] \neq 0\). Let \(j > 0\) be minimal such that
\[
[L_{-2}, S_j] \neq 0.
\] (6.1)

We wish to show that \(j = 1\). Suppose not.

We begin by establishing a few basic properties. By the Jacobi Identity, if \(1 \leq k \leq j - 1\) and \(2 \leq i \leq k + 1\), then (since in the sum that follows, \(0 \leq \kappa \leq i - 2 \leq k - 1 \leq j - 2\), so that \(j - 1 \geq k \geq k - \kappa \geq 1\); here \(\kappa\) is the number of \((\text{ad} L_{-1})s\) that act on \(S_k\) before it brackets with \(L_{-2}\) and annihilates it)
\[
[L_{-i}, S_k] = [(\text{ad}(L_{-1}))^{i-2} L_{-2}, S_k]
\leq \sum_{0 \leq \kappa \leq i-2} (\text{ad}(L_{-1}))^{(i-2-\kappa)} [L_{-2}, S_{k-\kappa}]
= 0;
\]
i.e.,
\[
[L_{-i}, S_k] = 0, 2 \leq i \leq k + 1, 1 \leq k \leq j - 1.
\] (6.2)

In particular, we have (letting \(i = k\), \(i = k + 1\), and \(k = j - 1\), respectively)
\[
[L_{-k}, S_k] = 0, 2 \leq k \leq j - 1,
\] (6.3)
\[
[L_{-k-1}, S_k] = 0, 1 \leq k \leq j - 1,\] (6.4)
\[
[L_{-i}, S_{j-1}] = 0, 2 \leq i \leq j.
\] (6.5)
We will now show that \([L_{-i}, S_j] \neq 0, 1 \leq i \leq j + 1\). Since \(L\) is irreducible (B), it will follow that
\[
[L_{-j-1}, S_j] = L_{-1}. \tag{6.6}
\]
Note that (since we are assuming that \([L_{-2}, S_j] \neq 0\) and \([L_{-2}, S_j'] = 0\) for \(1 \leq j' \leq j - 1\)), we have by (D), by (6.5), by transitivity (C), and by induction, that
\[
[L_{-i}, S_j] = [[L_{-i+1}, L_{-1}], S_j] = [[L_{-i+1}, S_j], L_{-1}] \neq 0, 2 < i \leq j + 1.
\]
Thus, we have
\[
[L_{-1}, S_j] \neq 0, 1 \leq i \leq j + 1. \tag{6.7}
\]
We will now show that
\[
\text{Ann}_{L_j} L_{-j} = \text{Ann}_{L_j} L_{-2}. \tag{6.8}
\]
Thus, suppose first that \(T_j\) is an \(L_0\)-submodule of \(L_j\) such that \([L_{-j}, T_j] = 0\). Then, in view of (D) and (6.5), we have
\[
0 = [L_{-j}, T_j] = [[L_{-j+1}, L_{-1}], T_j] = [[L_{-j+1}, T_j], L_{-1}] \tag{6.9}
\]
so that by transitivity (C), we have \([L_{-j+1}, T_j] = 0\). Then we may replace \(j\) in the above formula by, successively, \(j - 1, j - 2\), etc., to arrive at \([L_{-2}, T_j] = 0\). On the other hand, if we rather define \(T_j = \text{Ann}_{L_j} L_{-2}\), then, again in view of (6.5), we may bracket the equation \([L_{-2}, T_j] = 0\) with \(L_{-1}\) again and again to conclude that
\[
[L_{-3}, T_j] = 0, [L_{-4}, T_j] = 0, \ldots, [L_{-j}, T_j] = 0,
\]
as required.

**Lemma 6.10** If \(j\) is as above, then \([L_{-j-1}, S_{j+1}] \neq 0\).

**Proof** Suppose that \([L_{-j-1}, S_{j+1}] = 0\). Then, since by (6.6) \([L_{-j-1}, S_j] = L_{-1}\), we have
\[
[[S_j, S_{j+1}], L_{-j-1}] = [[L_{-j-1}, S_j], S_{j+1}] = [L_{-1}, S_{j+1}] = S_j \neq 0.
\]
Then
\[
[[S_j, S_{j+1}], L_{-j-1}] = [[[L_{-j-1}, S_j], S_{j+1}], S_{j+1}] = [[[L_{-1}, S_{j+1}], S_{j+1}], S_{j+1}] = [S_j, S_{j+1}] \neq 0.
\]
Continuing in this manner, we get homogeneous non-zero \(L_0\)-submodules of arbitrarily high gradation degree. This of course cannot happen in a finite-dimensional Lie algebra, so it must be that \([L_{-j-1}, S_{j+1}] \neq 0\). \(\square\)
Lemma 6.11 If \( j \geq 2 \), and \( k \) is the smallest integer greater than two for which \( [L_{-k}, S_r] \neq 0 \), then \( k = j \).

Proof Since by (D), \( L_{<0} \) is generated by \( L_{-1} \), we have by (6.7) and (6.3) and (iv) that

\[
(0 \neq) [L_{-j}, S_j] = [[L_{-1}, L_{-j+1}], S_j] = [L_{-1}, [L_{-j+1}, S_j]] = [L_{-1}, L_1]
\]

so \( [L_{-j}, S_r] \neq 0 \), since by Lemma 2.28 we have

\[
0 \neq [[L_{-j}, L_1], S_r] = [[L_{-j}, S_j], S_r] = [[L_{-j}, S_r], S_j]
\]

Consequently,

\[
k \leq j. \quad (6.12)
\]

On the other hand, by definition of \( k \), \( [L_{-k}, S_r] \neq 0 \), and \( [L_{-i}, S_r] = 0 \) for \( k - 1 \geq i \geq 2 \). It follows that if \( [L_{-k}, S_{k-1}] = 0 \) and \( [L_{-k}, S_k] = 0 \), then

\[
U \overset{\text{def}}{=} \sum_{i \geq 0} (\text{ad} L_{-1})^i[L_{-k}, S_r]
\]

would be a proper ideal of \( S \), to contradict the simplicity of \( S \). Thus, it must be that either \( [L_{-k}, S_k] \neq 0 \) or \( [L_{-k}, S_{k-1}] \neq 0 \).

Now, by (6.3) and (6.7), \( j \geq 2 \) is minimal such that \( [L_{-j}, S_j] \neq 0 \), so if \( [L_{-k}, S_k] \neq 0 \), then \( j \leq k \), so, since we have observed earlier that \( k \leq j \), we must have \( j = k \). If, on the other hand, \( [L_{-k}, S_k] = 0 \), then, as we noted above, we must have \( [L_{-k}, S_{k-1}] \neq 0 \). But by (6.4) and (6.7), \( j + 1 \) is the smallest \( i \geq 2 \) such that \( [L_{-i}, S_{i-1}] \neq 0 \), so we must have \( j + 1 \leq k \), so \( j \leq k - 1 \), to contradict (6.12). Thus, we have, in any case that

\[
j = k
\]

as required. □

Define \( H \) to be the Lie algebra generated by \( L_{-1} \oplus L_0 \oplus S_1 \oplus \cdots \oplus S_{j-1} \). Since \( [L_{-2}, S_1 \oplus \cdots \oplus S_{j-1}] = 0 \), it follows from (6.2) that if \( M(H) \) is as in the statement of Weisfeiler’s Theorem (Theorem 1.3), then \( M(H) = L_{-q} \oplus \cdots \oplus L_{-2} \), and \( H/M(H) \) is a depth-one graded Lie algebra which inherits transitivity (C) and irreducibility (B) from \( L \). From Proposition 1.6 we conclude that \( H/M(H) \) is either between \( H(2 : (1, 1), \omega) \) and \( CH(2 : (1, 1), \omega) \), or is equal to \( L(\epsilon) \) or \( M \). In each of these cases, \( S_i \) is an irreducible abelian \( L_0 \)-module. Thus, we can from now on assume (See Corollary 1.7) that assumption (iv) of Section 2 holds and

\[
[L_1, L_1] = 0 \quad (6.13)
\]

and, as a Lie algebra, \( [L_{-1}, L_1] \) lies between \( \mathfrak{sl}(2) \) and \( \mathfrak{gl}(2) \):
of the following three-dimensional 

\[ \mathfrak{sl}(2) \subseteq [L_{-1}, L_1] \subseteq \mathfrak{gl}(2) \] (6.14)

Now, by Lemma 2.29, \( j \leq 2 \) in the non-Hamiltonian cases. Thus, suppose that we are in the Hamiltonian case, and suppose, for a contradiction, that \( j > 2 \). For this (Hamiltonian, \( j > 2 \)) case, we will assume without loss of generality, that \( L \) is generated by \( L_{<0} \oplus L_0 \oplus S_1 \oplus \cdots \oplus S_{j-1} \oplus S_j \). By definition of \( j \) and the Jacobi Identity, \( [S_i, [L_{-2}, L_{-2}]] = 0, 1 \leq i \leq j - 1 \). Furthermore, since \( j \) is assumed to be greater than two, we have \([S_j, [L_{-2}, L_{-2}]] \subseteq [L_{-2}, S_{j-2}] = 0\). Thus, again by the Jacobi Identity, \([L_{>0}, [L_{-2}, L_{-2}]] = 0\), so, if \([L_{-2}, L_{-2}] \) were not equal to zero then \( \sum_{i \geq 0} (\text{ad } L_{-1})^i [L_{-2}, L_{-2}] \) would be a proper ideal of \( S \), to contradict the simplicity of \( S \). Thus, we may assume in this case that

\[ [L_{-2}, L_{-2}] = 0 \] (6.15)

In addition, \([L_{-2}, S_{j-2}] = 0\) also entails that

\[ [L_{-2}, [L_{-2}, L_i]] = 0, 0 \leq i \leq 2j - 1 \] (6.16)

**Lemma 6.17** If \( j > 2 \), then

\[ [L_{-2}, [L_{-2}, L_{j+2}]] = 0 \]

**Proof.** Since we are assuming that \( j > 2 \), it follows that \( 2j > j + 2 \), so the lemma follows from (6.16). \( \square \)

We adopt the notation of \[ \text{[4].} \]

**Lemma 6.18** If \( j > 2 \), and \( j \not\equiv 0 \pmod{3} \), then \([L_{-2}, [L_2, L_j]] \neq 0\).

**Proof:** Suppose not. Then

\[ 0 = [L_{-2}, [L_2, L_j]] = [L_2, [L_{-2}, L_j]] \]

since we are assuming that \( j > 2 \). But \( 0 \neq [L_{-2}, L_j] \subseteq H_{j-2} \) (See the discussion preceding (6.13)), and \( H_{j-2} \) is \( L_0 \)-irreducible. Consequently, we have \([H_2, H_{j-2}] = 0\). But \( \{ y^{(3)}, xy^{(j-1)} \} = y^{(2)}y^{(j-1)} \equiv jy^{(j+1)} \neq 0 \) (since \( j \not\equiv 0 \pmod{3} \)). \( \square \)

**Lemma 6.19** If \( M \) is an \( L_0 \)-module such that \([L_{-2}, M] = 0 \) and \([L_{-2}, [1, M]] = 0\), then \((\text{ad } 1)^3 \) is an \( L_0 \)-homomorphism of \( M \) into \((\text{ad } 1)^3 M\).

**Proof.** Since \( \{1, xy\} = 0 = \{1, x(2)y\} \), we have \((\text{ad } 1)^3 [L_0, M] = [(\text{ad } 1)^3 L_0, M] + [L_0, (\text{ad } 1)^3 M] = [(\text{ad } 1)^2 x(2), M] + [L_0, (\text{ad } 1)^3 M] \subseteq [1, [L_{-2}, M]] + [[1, M], L_{-2}] = 0 \). \( \square \)

By transitivity (C), \( (S_j \subseteq L_j \) must be contained in the nine-dimensional \( L_0 \)-module \( L_{-1}^* \otimes S_{j-1} \). In what follows, we will make use of this implication of (C) without further comment. Note also that if the gradation degree of any of the following three-dimensional \( L_0 \)-modules

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is not equivalent to zero modulo three, then the $L_0$–module is $L_0$–irreducible.

We consider separately the cases where $j$ is equivalent to one, two, or zero modulo three. We will show that in each case, if $j > 2$, then $L$ can be replaced with a subalgebra in which the smallest integer $i \geq 2$ such that $L_{i-2}$ has nonzero bracket with the $i^{th}$ gradation space of the replacement is greater than $j$.

When $j \equiv 1 \pmod{3}$, $L^*_{-1} \otimes S_{j-1}$ (which, by transitivity (C) contains $L_j$) contains the two $L_0$–submodules

\[
X_{1,j,1} \overset{\text{def}}{=} <1^* \otimes x^{(2)}y^{(j)}, 2 \cdot x^* \otimes x^{(2)}y^{(j)} + 1^* \otimes xy^{(j)},
(x^{(2)})^* \otimes x^{(2)}y^{(j)} + x^* \otimes xy^{(j)} + 1^* \otimes y^{(j)}> \]

and

\[
X_{1,j,2} \overset{\text{def}}{=} <x^* \otimes x^{(2)}y^{(j)}, 2 \cdot (x^{(2)})^* \otimes x^{(2)}y^{(j)} + x^* \otimes xy^{(j)},
1^* \otimes x^{(2)}y^{(j)} + (x^{(2)})^* \otimes xy^{(j)} + x^* \otimes y^{(j)}> \]

Now, $[L_{-2}, X_{1,j,2}] = 0$. Denote by $\Xi$ the Lie algebra generated by $L_{\leq 0} \oplus L_1 \oplus \cdots \oplus L_{j-1}$ and $X_{1,j,2}$. Then $\Xi/M(\Xi)$ (like $H/M(H)$ above) must be Hamiltonian, so $X_{1,j,2}$ must contain $\langle y^{(j+1)}, x^{(2)}y^{(j+1)} \rangle$. Then the analysis of $L^*_{-1} \otimes S_{j-1}$ when $j \equiv 2 \pmod{3}$ below would apply to $L^*_{-1} \otimes X_{1,j,2}$.

Now,

\[
L_j \subseteq L^*_{-1} \otimes S_{j-1} = X_{1,j,1} \oplus X_{1,j,3}, \quad (6.20)
\]

where modulo its submodule $X_{1,j,3}$, the indecomposable $L_0$–module $X_{1,j,3}$ is spanned by $(x^{(2)})^* \otimes x^{(2)}y^{(j)}, 2 \cdot 1^* \otimes x^{(2)}y^{(j)} + (x^{(2)})^* \otimes xy^{(j)}$, and $1^* \otimes xy^{(j)} + x^* \otimes x^{(2)}y^{(j)} + (x^{(2)})^* \otimes y^{(j)}$.

Then we have

\[
L_{j+1} \subseteq L^*_{-1} \otimes L_j \subseteq L^*_{-1} \otimes (X_{1,j,1} \oplus X_{1,j,3}) = L^*_{-1} \otimes X_{1,j,1} \oplus L^*_{-1} \otimes X_{1,j,3}
\]

Note that if $X_{1,j,2} \cap L_j = 0$, then $X_{1,j,3} \cap L_j = 0$.

We define

\[
a \overset{\text{def}}{=} 1^* \otimes x^{(2)}y^{(j)} \\
b \overset{\text{def}}{=} 2 \cdot x^* \otimes x^{(2)}y^{(j)} + 1^* \otimes xy^{(j)} \\
c \overset{\text{def}}{=} (x^{(2)})^* \otimes x^{(2)}y^{(j)} + x^* \otimes xy^{(j)} + 1^* \otimes y^{(j)}.
\]

Then $L^*_{-1} \otimes X_{1,j,1}$ contains the following submodules:
\[ X_{1,j+1,1} \stackrel{\text{def}}{=} \langle 1^* \otimes c + (x^{(2)})^* \otimes a + x^* \otimes b, \\
2 \cdot x^* \otimes a + 1^* \otimes b, 1^* \otimes a \rangle \]

\[ X_{1,j+1,2} \stackrel{\text{def}}{=} \langle 1^* \otimes a + (x^{(2)})^* \otimes b + x^* \otimes c, \\
2 \cdot (x^{(2)})^* \otimes a + x^* \otimes b, x^* \otimes a \rangle \]

Set

\[
\begin{align*}
\alpha & \stackrel{\text{def}}{=} 1^* \otimes a \\
\beta & \stackrel{\text{def}}{=} 2 \cdot x^* \otimes a + 1^* \otimes b \\
\gamma & \stackrel{\text{def}}{=} 1^* \otimes c + (x^{(2)})^* \otimes a + x^* \otimes b \\
\delta & \stackrel{\text{def}}{=} x^* \otimes a \\
\epsilon & \stackrel{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes a + x^* \otimes b \\
\zeta & \stackrel{\text{def}}{=} 1^* \otimes a + (x^{(2)})^* \otimes b + x^* \otimes c
\end{align*}
\]

Then \( L_{*1}^* \otimes X_{1,j,1} = X_{1,j+1,3} + X_{1,j+1,1} \), where \( X_{1,j+1,3} \) is an indecomposable \( L_0 \)-module with irreducible \( L_0 \)-submodule \( X_{1,j+1,2} \). Modulo \( X_{1,j+1,2} \), \( X_{1,j+1,3} \) is spanned by \((x^{(2)})^* \otimes a, 2 \cdot 1^* \otimes a + (x^{(2)})^* \otimes b, \) and \( 1^* \otimes b + x^* \otimes a + (x^{(2)})^* \otimes c \). Here, also, if \( X_{1,j+1,2} \cap L_{j+1} = 0 \), then \( X_{1,j+1,3} \cap L_{j+1} = 0 \).

Note that \( L_{*1}^* \otimes X_{1,j+1,1} \) contains the following \( L_0 \)-submodules:

\[ X_{1,j+2,1} \stackrel{\text{def}}{=} \langle (x^{(2)})^* \otimes a + x^* \otimes \beta + 1^* \otimes \gamma, \\
2 \cdot x^* \otimes a + 1^* \otimes \beta, 1^* \otimes \alpha \rangle \]

and

\[ X_{1,j+2,2} \stackrel{\text{def}}{=} \langle 1^* \otimes a + (x^{(2)})^* \otimes \beta + x^* \otimes \gamma, \\
2 \cdot (x^{(2)})^* \otimes a + x^* \otimes \beta, x^* \otimes \alpha \rangle \]

and that \( L_{*1}^* \otimes X_{1,j+1,1} = X_{1,j+2,1} + X_{1,j+2,3} \), where, modulo \( X_{1,j+2,2} \), \( X_{1,j+2,3} \) is spanned by \((x^{(2)})^* \otimes a, 2 \cdot 1^* \otimes a + (x^{(2)})^* \otimes \beta, \) and \( x^* \otimes a + 1^* \otimes \beta + (x^{(2)})^* \otimes \gamma \). Here we have \((\text{ad } [x^{(2)}, 1]) \cdot ((x^{(2)})^* \otimes a) = [a, 1] = 2a, \) and \((\text{ad } [1, x]) \cdot a = (\text{ad } [1, x]) \cdot (1^* \otimes x^{(2)} y(j)) = [x^{(2)} y(j), x] = x^{(2)} y(j-1) \neq 0 \), so by Lemma 0.17.
Similarly, $(\text{ad} [x^{(2)}, 1])(2 \cdot 1^* \otimes \alpha + (x^{(2)})^* \otimes \beta) = 2 \cdot [x^{(2)}, \alpha] + [\beta, 1] = 2b$, and $(\text{ad} [1, x]) \cdot b = [xy^{(j)}, x] + 2 \cdot [1, x^{(2)}y^{(j)}] = xy^{(j-1)} \neq 0$, and $(\text{ad} [1, x])^2 (x^* \otimes \alpha + 1^* \otimes \beta + (x^{(2)})^* \otimes \gamma) = 2 \cdot (\text{ad} [1, x]) \cdot a \neq 0$, as we just saw. Thus,

\[
X_{1, j+2, 3} \cap L_{j+2} \subseteq X_{1, j+2, 2}. \tag{6.21}
\]

Focusing on $X_{1, j+2, 2}$, we see that $(\text{ad} [1, x]) \cdot (x^* \otimes \alpha) = [1, \alpha] = a$, and, from above, $(\text{ad} [1, x]) \cdot a = x^{(2)}y^{(j-1)}$, so Lemma 6.17 shows that

\[
X_{1, j+2, 2} \cap L_{j+2} = 0. \tag{6.22}
\]

It follows from (6.21) that

\[
X_{1, j+2, 3} \cap L_{j+2} = 0. \tag{6.23}
\]

Consequently, we have

\[
L_{s-1}^{*} \otimes X_{1, j+1, 1} \cap L_{j+2} \subseteq X_{1, j+2, 1}. \tag{6.24}
\]

Note that

\[
[L_{-2}, X_{1, j+2, 1}] = 0 \tag{6.25}
\]

(Note also the following: Suppose, for example, that $L_{j+2} \cap X_{1, j+2, 1} \neq 0$. Since $[L_{-2}, X_{1, j+1, 1}] = 0$ and $[L_{-2}, X_{1, j+2, 1}] = 0$, it follows from Lemma 6.19 that $(\text{ad} 1)^3$ is an $L_0$–isomorphism from $X_{1, j+2, 1}$ to $S_{j-1}$:

\[
(\text{ad} 1)^3 \cdot (1^* \otimes \alpha) = (\text{ad} 1)^2 \alpha = (\text{ad} 1) \alpha = x^{(2)}y^{(j)}
\]

\[
(\text{ad} 1)^3 \cdot (2 \cdot x^* \otimes \alpha + 1^* \otimes \beta) = (\text{ad} 1)^2 \beta = (\text{ad} 1) b = xy^{(j)}
\]

\[
(\text{ad} 1)^3 \cdot ((x^{(2)})^* \otimes \alpha + x^* \otimes \beta + 1^* \otimes \gamma) = (\text{ad} 1)^2 \gamma = (\text{ad} 1) c = y^{(j)}
\]

Since $X_{1, j+2, 1} \cong S_{j-1}$ as an $L_0$–module, and since the construction of the “$X$” modules depends only on the $L_0$–module properties of $S_{j-1}$ and $L_{s-1}$, and $L_{-1}$ [and its bracket with itself, $L_{-2}$] only interact with the first factor of the tensors, we can repeat the process with $X_{1, j+2, 1}$ [Compare $X_{2, j+2, 2}$ in the $j = 2$ case below.] in place of $S_{j-1}$ and continue to repeat it until we arrive at the highest gradation space. Since $(\text{ad} [1, x]) \cdot (1^* \otimes x^{(2)}y^{(j)}) = x^{(2)}y^{(j-1)} \neq 0$, it follows that $[L_{-2}, X_{1, j, 1}] \neq 0$. Consequently, when we arrive at the highest gradation space, $L_{-2}$ will have non-zero bracket with either $L_{r-2}$, or $L_{r-1}$, or $L_r$. Then by Lemma 6.11 we would have

\[
j = k \leq 2.
\]

We note that $L_{s-1}^{*} \otimes X_{1, j+1, 2}$ contains the following submodules, and is contained in their sum:
\[ X_{1, j+2, 4} \overset{\text{def}}{=} (1^* \otimes \delta + (x^{(2)})^* \otimes \epsilon + x^* \otimes \zeta) + (2 \cdot (x^{(2)})^* \otimes \delta + x^* \otimes \epsilon, x^* \otimes \delta) \]

\[ X_{1, j+2, 5} \overset{\text{def}}{=} (1^* \otimes \delta + (x^{(2)})^* \otimes \epsilon + x^* \otimes \zeta) + (2 \cdot x^* \otimes \epsilon + 1^* \otimes \zeta, 1^* \otimes \epsilon) \]

\[ X_{1, j+2, 6} \overset{\text{def}}{=} (1^* \otimes \delta + ((x^{(2)})^* \otimes \delta + x^* \otimes \epsilon + 1^* \otimes \zeta, 1^* \otimes \epsilon + 2 \cdot x^* \otimes \delta) \]

\[ X_{1, j+2, 7} \overset{\text{def}}{=} ((x^{(2)})^* \otimes \epsilon + x^* \otimes \zeta) + ((x^{(2)})^* \otimes \delta + x^* \otimes \epsilon + 1^* \otimes \zeta, 1^* \otimes \epsilon + 2 \cdot x^* \otimes \delta) \]

Here \( X_{1, j+2, 4} \) and \( X_{1, j+2, 5} \) are a “bouquet” with the common one-dimensional submodule \( (1^* \otimes \delta + (x^{(2)})^* \otimes \epsilon + x^* \otimes \zeta) \). The irreducible submodules of \( X_{1, j+2, 6} \) and \( X_{1, j+2, 7} \) are the respectively indicated two-dimensional subspaces.

Note that

\[ [L_{-1}, X_{1, j+2, 4}] = 0. \tag{6.26} \]

Focusing on \( X_{1, j+2, 5} \), we have

\[ [[x^{(2)}, 1], [[1, x], 1^* \otimes \epsilon]] = [[x^{(2)}, 1], 2b] = 2 \cdot [x^{(2)}, xy^{(j)}] = 2x^{(2)}y^{(j-1)} \neq 0 \]

so that by Lemma 6.17,

\[ X_{1, j+2, 5} \cap L_{j+2} = 0. \tag{6.27} \]

We focus next on \( X_{1, j+2, 6} \) and \( X_{1, j+2, 7} \). Here we have \( 1^* \otimes \epsilon + 2 \cdot x^* \otimes \delta \in X_{1, j+2, 6} \cap X_{1, j+2, 7} \). Observe that

\[ [[1, x], 2 \cdot 1^* \otimes \epsilon + x^* \otimes \delta] = 2 \cdot [\epsilon, x] + [1, \delta] = [x, \epsilon] + 0 = b \]

and (as above)

\[ [[x^{(2)}, 1], b] = 2 \cdot [xy^{(j)}, x^{(2)}] = x^{(2)}y^{(j-1)} \neq 0 \]

so by Lemma 6.17 again,

\[ X_{1, j+2, 6} \cap L_{j+2} = 0 \text{ and } X_{1, j+2, 7} \cap L_{j+2} = 0. \tag{6.28} \]

Consequently, in view of (6.27) and (6.28), we have

\[ L_{-1}^* \otimes X_{1, j+1, 2} \cap L_{j+2} \subseteq X_{1, j+2, 4} \tag{6.29} \]

Define
follows from Lemma 6.17 that (modulo \( \eta + \cdot 2 \)) and from Lemma 6.17 that (modulo \( L \)) it follows from Lemma 6.17 that (modulo \( L \)) is the indicated one-dimensional subspace, the irreducible \( X \) \( \theta + 1 \) \( L \)

\[ \eta \overset{\text{def}}{=} (x^{(2)})^* \otimes a \]

\[ \theta \overset{\text{def}}{=} 2 \cdot 1^* \otimes a + (x^{(2)})^* \otimes b \]

\[ \iota \overset{\text{def}}{=} 1^* \otimes b + x^* \otimes a + (x^{(2)})^* \otimes c \]

Modulo \( L_{-1} \otimes X_{1,j+1,2}, L_{-1} \otimes X_{1,j+1,3} \) is spanned by

\[ X_{1,j+2,8} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \eta + x^* \otimes \theta + 1^* \otimes \iota > + < 2 \cdot x^* \otimes \eta + 1^* \otimes \theta, 1^* \otimes \eta > \]

\[ X_{1,j+2,9} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \iota + x^* \otimes \eta + 1^* \otimes \theta, 2 \cdot x^* \otimes \iota + 1^* \otimes \eta > + < 1^* \otimes \iota > \]

and

\[ X_{1,j+2,10} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \theta + x^* \otimes \iota + 1^* \otimes \eta, 2 \cdot (x^{(2)})^* \otimes \eta + x^* \otimes \theta, x^* \otimes \eta > \]

Note that (modulo \( L_{-1} \otimes X_{1,j+1,2} \)), the irreducible \( L_0 \)-submodule of \( X_{1,j+2,8} \) is the indicated one-dimensional subspace, the irreducible \( L_0 \)-submodule of \( X_{1,j+2,9} \) is the indicated two-dimensional subspace, and \( X_{1,j+2,10} \) is an irreducible \( L_0 \)-module.

We focus first on \( X_{1,j+2,8} \) and note that since \( (\text{ad } [1, x]) \cdot ((x^{(2)})^* \otimes \eta + x^* \otimes \theta + 1^* \otimes \iota) = [1, \theta] + [\iota, x] \equiv a \), and from above \( (\text{ad } [1, x]) \cdot a = x^{(2)} y^{(j-1)} \neq 0 \), it follows from Lemma 6.17 that (modulo \( L_{-1} \otimes X_{1,j+1,2} \)),

\[ X_{1,j+2,8} \cap L_{j+2} \equiv 0 \quad (6.30) \]

We focus next on \( X_{1,j+2,9} \) and note that since \( (\text{ad } [1, x]) \cdot (2 \cdot x^* \otimes \iota + 1^* \otimes \eta) = 2 \cdot [1, \iota] + [\eta, x] = 2b \), and, from above, \( (\text{ad } [1, x]) \cdot b = x y^{(j-1)} \neq 0 \), it follows from Lemma 6.17 that (modulo \( L_{-1} \otimes X_{1,j+1,2} \)),

\[ X_{1,j+2,9} \cap L_{j+2} \equiv 0 \quad (6.31) \]

Finally, we focus on \( X_{1,j+2,10} \) and note that since \( (\text{ad } [1, x]) \cdot (2 \cdot (x^{(2)})^* \otimes \eta + x^* \otimes \theta) = [1, \theta] = 2a \), and from above \( (\text{ad } [1, x]) \cdot a = x^{(2)} y^{(j-1)} \neq 0 \), it follows from Lemma 6.17 that (modulo \( L_{-1} \otimes X_{1,j+1,2} \)),

\[ X_{1,j+2,10} \cap L_{j+2} \equiv 0 \quad (6.32) \]

From (6.30), (6.31), and (6.32), we can conclude that

\[ L_{j+2} \cap (L_{-1} \otimes X_{1,j+1,3}) \subseteq L_{j+2} \cap (L_{-1} \otimes X_{1,j+1,2}) \quad (6.33) \]

We now turn our attention to \( X_{1,j,2} \). Define
Then $L_{-1} \otimes X_{1,1,4}$ is the sum of the following $L_0$-submodules:

$$
X_{1,j+1,4} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes d + x^* \otimes e + 1^* \otimes f, 2 \cdot x^* \otimes d + 1^* \otimes e \rangle + \langle 1^* \otimes d \rangle
$$

$$
X_{1,j+1,5} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes e + x^* \otimes f + 2 \cdot 1^* \otimes d \rangle + \langle 2 \cdot x^* \otimes e + 1^* \otimes f, 1^* \otimes e \rangle
$$

$$
X_{1,j+1,6} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes e + x^* \otimes f + 1^* \otimes d \rangle + \langle x^* \otimes e + 2 \cdot (x^{(2)})^* \otimes d, x^* \otimes d \rangle
$$

$$
X_{1,j+1,7} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes f + 2 \cdot x^* \otimes d + 1^* \otimes e, x^* \otimes f + 2 \cdot (x^{(2)})^* \otimes e, x^* \otimes e \rangle
$$

Note that

$$
[L_{-2}, X_{1,j+1,6}] = 0.
$$

Then, since $X_{1,j,2}$ also has zero bracket with $L_{-2}$, we have by Lemma 6.19 that $X_{1,j+1,6} \cong S_{j-2}$.

Now, since $(\text{ad} [1, x]) \cdot (1^* \otimes d) = [d, x] = 2 \cdot x^* \otimes d + 1^* \otimes e) = 2 \cdot [1, d] + [e, x] = 2xy(j) \neq 0$, and $(\text{ad} [1, x]) \cdot (2 \cdot x^* \otimes d + 1^* \otimes e) = 2 \cdot [1, d] + [e, x] = 2xy(j) \neq 0$, if $L_{j+1} \cap X_{1,j+1,4} \neq 0$, then $[L_{-2}, L_{j+1} \cap X_{1,j+1,4}] \neq 0$, and we can replace $L$ with the Lie algebra generated by $L_{\leq 0} \oplus L_1 \oplus \cdots \oplus L_{j-1} \oplus (L_j \cap X_{1,j,2}) \oplus (L_{j+1} \cap X_{1,j+1,4})$, and we will have a Lie subalgebra of $L$ such that the minimal $i \geq 2$, such that $L_{-2}$ has non-zero bracket with the $i\text{th}$ gradation space of the subalgebra, is $j + 1$. Thus, we may assume in what follows that

$$
L_{j+1} \cap X_{1,j+1,4} = 0.
$$

Furthermore, since $(\text{ad} [1, x]) \cdot (1^* \otimes e) = [e, x] = 2 \cdot x^* \otimes e + 2 \cdot 1^* \otimes d) = [1, f] + 2 \cdot [d, x] = 2x^{(2)}y^{(j)} \neq 0$, if $L_{j+1} \cap X_{1,j+1,5} \neq 0$, then $[L_{-2}, L_{j+1} \cap X_{1,j+1,5}] \neq 0$, and we can replace $L$ with the Lie algebra generated by $L_{\leq 0} \oplus L_1 \oplus \cdots \oplus L_{j-1} \oplus (L_j \cap X_{1,j,2}) \oplus (L_{j+1} \cap X_{1,j+1,5})$, and we will have a Lie subalgebra of $L$ such that the minimal $i \geq 2$, such that $L_{-2}$ has non-zero bracket with the $i\text{th}$ gradation space of the subalgebra, is $j + 1$. Thus, we may assume in what follows that
\[ L_{j+1} \cap X_{1,j+1,5} = 0. \]  

(6.35)

In addition, since \((\text{ad} [x, x^{(2)}]) (x^* \otimes e) = [e, x^{(2)}] = 2[x^{(2)}, e] = x^{(2)} y^{(j)} \neq 0, \)
and \((\text{ad} [1, x]) (2(x^{(2)})^* \otimes e + x^* \otimes f) = [1, f] = x^{(2)} y^{(j)} \neq 0, \) if \(L_{j+1} \cap X_{1,j+1,7} \neq 0, \) then \([L_{-2}, L_{j+1} \cap X_{1,j+1,7}] \neq 0, \) and we can replace \(L \) with the Lie algebra generated by \(L_{\leq 0} \oplus L_{1} \oplus \cdots \oplus L_{j-1} \oplus (L_{j} \cap \bigoplus_{1,j+2} X_{1,j,2}) \oplus (L_{j+1} \cap X_{1,j+1,7}), \) and we will have a Lie subalgebra of \(L \) such that the minimal \(i \geq 2, \) such that \(L_{-2} \) has non-zero bracket with the \(i^{\text{th}} \) gradation space of the subalgebra, is \(j+1. \) Thus, we may assume in what follows that

\[ L_{j+1} \cap X_{1,j+1,7} = 0. \]  

(6.36)

Define

\[ \kappa \overset{\text{def}}{=} x^* \otimes d \]

\[ \lambda \overset{\text{def}}{=} x^* \otimes e + 2 \cdot (x^{(2)})^* \otimes d \]

\[ \mu \overset{\text{def}}{=} (x^{(2)})^* \otimes e + x^* \otimes f + 1^* \otimes d \]

Then \(L_{-1}^* \otimes X_{1,j+1,6} \) is the sum of the following \(L_0\)-modules:

\[ X_{1,j+2,11} \overset{\text{def}}{=} <1^* \otimes \mu + (x^{(2)})^* \otimes \kappa + x^* \otimes \lambda > + 2 \cdot (x^{(2)})^* \otimes \mu + x^* \otimes \kappa, x^* \otimes \mu > \]

\[ X_{1,j+2,12} \overset{\text{def}}{=} <1^* \otimes \kappa + (x^{(2)})^* \otimes \lambda + x^* \otimes \mu, 2 \cdot (x^{(2)})^* \otimes \kappa + x^* \otimes \lambda, x^* \otimes \kappa > \]

\[ X_{1,j+2,13} \overset{\text{def}}{=} <(x^{(2)})^* \otimes \mu + x^* \otimes \kappa + 1^* \otimes \lambda, 2 \cdot x^* \otimes \mu + 1^* \otimes \kappa > + <1^* \otimes \mu > \]

Note that

\[ [L_{-2}, X_{1,j+2,12}] = 0 \]  

(6.37)

Now, since \((\text{ad} [1, x]) \cdot (x^* \otimes \mu) = [1, \mu] = d, \) and \((\text{ad} [1, x]) \cdot ((x^{(2)})^* \otimes \kappa + x^* \otimes \lambda + 1^* \otimes \mu) = [\mu, x] + [1, \lambda] = 2f \neq 0, \) if \(L_{j+2} \cap X_{1,j+2,11} \neq 0, \) then \([L_{-2}, L_{j+2} \cap X_{1,j+2,11}] \neq 0, \) and we can replace \(L \) with the Lie algebra generated by \(L_{\leq 0} \oplus L_{1} \oplus \cdots \oplus L_{j-1} \oplus (L_{j} \cap \bigoplus_{1,j+2} X_{1,j,2}) \oplus (L_{j+1} \cap X_{1,j+1,7}) \oplus (L_{j+2} \cap X_{1,j+2,11}), \) and we will have a Lie subalgebra of \(L \) such that the minimal \(i \geq 2, \) such that \(L_{-2} \) has non-zero bracket with the \(i^{\text{th}} \) gradation space of the subalgebra, is \(j+2. \) Thus, we may assume in what follows that

\[ L_{j+2} \cap X_{1,j+2,11} = 0. \]  

(6.38)

Also, since \((\text{ad} [1, x]) \cdot (1^* \otimes \mu) = [\mu, x] = 2f, \) and \((\text{ad} [1, x]) \cdot (2 \cdot x^* \otimes \mu + 1^* \otimes \kappa) = 2 \cdot [1, \mu] + [\kappa, x] = 2d + 2d \neq 0, \) if \(L_{j+2} \cap X_{1,j+2,13} \neq 0, \) then

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and we will have a Lie subalgebra of $L$ by $L/L_i$. Thus, we may assume in what follows that $L$ contains a subalgebra such that the minimal $i \geq 2$, such that $L_{-2}$ has non-zero bracket with the $i^\text{th}$ gradation space of the subalgebra, is $j+2$.

Thus, we may assume in what follows that

$$L_{j+2} \cap X_{1,j+2,13} = 0. \quad (6.39)$$

We next turn our attention to $L_{-1}^* \otimes X_{1,j,3}$. (See (6.20)) Define

$$g \overset{\text{def}}{=} (x^{(2)})^* \otimes x^{(2)} y^{(j)}, \quad h \overset{\text{def}}{=} 2 \cdot (1^* \otimes x^{(2)} y^{(j)} + (x^{(2)})^* \otimes x y^{(j)})$$

$$i \overset{\text{def}}{=} 1^* \otimes x y^{(j)} + x^* \otimes x^{(2)} y^{(j)} + (x^{(2)})^* \otimes y^{(j)}$$

Then modulo $L_{-1}^* \otimes X_{1,j,2}$, $L_{-1} \otimes X_{1,j,3}$ is spanned by

$$X_{1,j+1,8} \overset{\text{def}}{=} < (x^{(2)})^* \otimes g + x^* \otimes h + 1^* \otimes i > + < 2 \cdot x^* \otimes g + 1^* \otimes h, 1^* \otimes g >$$

$$X_{1,j+1,9} \overset{\text{def}}{=} < (x^{(2)})^* \otimes i + 2 \cdot x^* \otimes g + 2 \cdot 1^* \otimes h, 2 \cdot x^* \otimes i + 2 \cdot 1^* \otimes g > + < 1^* \otimes i >$$

and

$$X_{1,j+1,10} \overset{\text{def}}{=} < (x^{(2)})^* \otimes h + x^* \otimes i + 1^* \otimes g > + < 2 \cdot (x^{(2)})^* \otimes g + x^* \otimes h, x^* \otimes g >$$

Looking first at $X_{1,j+1,8}$, we observe that if $X_{1,j+1,8} \cap L_{j+1} \neq 0$, then $1^* \otimes g \in L_{j+1}$, which must then contain $(\text{ad} x^{(2)} y) \cdot (1^* \otimes g) = 2 \cdot (1^* \otimes d) \in X_{1,j+1,4}$, so that $X_{1,j+1,4} \subset L_{j+1}$, from which we have inferred above that $L$ contains a subalgebra such that the smallest $i \geq 2$ such that $L_{-1}$ has non-zero bracket with the $i^\text{th}$ component of the subalgebra is greater than $j$, as required. Thus, we may assume in what follows that modulo $X_{1,j,2}$,

$$L_{j+1} \cap X_{1,j+1,8} \equiv 0. \quad (6.40)$$

Looking next at $X_{1,j+1,9}$, we define

$$\nu \overset{\text{def}}{=} 1^* \otimes i$$

$$\xi \overset{\text{def}}{=} 2 \cdot x^* \otimes i + 2 \cdot 1^* \otimes g$$

$$\pi \overset{\text{def}}{=} (x^{(2)})^* \otimes i + 2 \cdot x^* \otimes g + 2 \cdot 1^* \otimes h.$$
\( X_{1,j+2,14} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \nu + x^* \otimes \xi + 1^* \otimes \pi, 2 \cdot x^* \otimes \nu + 1^* \otimes \xi > + < 1^* \otimes \nu > \)

\( X_{1,j+2,15} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \xi + x^* \otimes \pi + 2 \cdot 1^* \otimes \nu > + < 2 \cdot (x^{(2)})^* \otimes \nu + x^* \otimes \xi, 1^* \otimes \xi > \)

and

\( X_{1,j+2,16} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \xi + x^* \otimes \pi + 1^* \otimes \nu > + < 2 \cdot (x^{(2)})^* \otimes \nu + x^* \otimes \xi, x^* \otimes \nu > \)

Note that (modulo \( L_{-1} \otimes (L_{-1} \otimes X_{1,j,2}) \))

\[
[L_{-2}, X_{1,j+2,14}] \equiv 0 \quad (6.41)
\]

Focusing next on \( X_{1,j+2,15} \), we note that (ad \([1, x]) \cdot (1^* \otimes \xi) = [\xi, x] = 2 \cdot [x, \xi] = i, \text{ and (ad} [1, x]) \cdot i = [xy^{(j)}, x] + [1, x^{(2)}y^{(j)}] = xy^{(j-1)} \neq 0, \text{ it follows from Lemma } 6.17 \text{ that (modulo } L_{-1} \otimes (L_{-1} \otimes X_{1,j,2})\))

\[
X_{1,j+2,15} \cap L_{j+2} \equiv 0. \quad (6.42)
\]

Looking finally at \( X_{1,j+1,10} \), we define

\[
\rho \overset{\text{def}}{=} x^* \otimes g \\
\sigma \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes g + x^* \otimes h \\
\tau \overset{\text{def}}{=} (x^{(2)})^* \otimes h + x^* \otimes i + 1^* \otimes g.
\]

Then (modulo \( L_{-1} \otimes (L_{-1} \otimes X_{1,j,2}) \)) \( L_{-1} \otimes X_{1,j+1,10} \) is equal to the sum of the following \( L_0 \)-modules:

\[
X_{1,j+2,17} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \rho + x^* \otimes \sigma + 1^* \otimes \tau, 2 \cdot x^* \otimes \rho + 1^* \otimes \sigma, 1^* \otimes \rho >
\]

\[
X_{1,j+2,18} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \sigma + x^* \otimes \tau + 1^* \otimes \rho, 2 \cdot (x^{(2)})^* \otimes \rho + x^* \otimes \sigma > + < x^* \otimes \rho >
\]

and
\[X_{1,j+2,19} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \tau + x^* \otimes \rho + 1^* \otimes \sigma, 2 \cdot (x^{(2)})^* \otimes \sigma + x^* \otimes \tau, x^* \otimes \sigma \rangle\]

Note that (modulo \(L^*_1 \otimes (L^*_1 \otimes X_{1,j,2})\))

\[\{ -2, X_{1,j+2,18} \} = 0. \quad (6.44)\]

Focusing next on \(X_{1,j+2,17}\), we note that \([(\text{ad}[1,x]) \cdot (1^* \otimes \rho) = [\rho, x] = 2 \cdot [x, \rho] = 2g, \text{and} \quad (\text{ad}[x, x^{(2)}]) \cdot g = [x, x^{(2)} y^{(j)}] = 2 \cdot x^{(2)} y^{(j-1)} \neq 0\)), it follows from Lemma 6.17 that (modulo \(L^*_1 \otimes (L^*_1 \otimes X_{1,j,2})\))

\[X_{1,j+2,17} \cap L_{j+2} = 0. \quad (6.45)\]

Focusing finally on \(X_{1,j+2,19}\), we note that \([(\text{ad}[x, x^{(2)}]) \cdot (x^* \otimes \sigma) = [\sigma, x^{(2)}] = g, \text{and} \quad (\text{ad}[x, x^{(2)}]) \cdot g = 0, \text{so} \quad x^* \otimes \sigma \not\in L_{j+2}. \text{Similarly}, \quad (\text{ad}[x, x^{(2)}]) \cdot (2 \cdot (x^{(2)})^* \otimes \sigma + x^* \otimes \tau) = 2 \cdot [x, \sigma] + [\tau, x^{(2)}]) = 2 \cdot x^{(2)} y^{(j)} \neq 0, \text{so} \quad (x^{(2)})^* \otimes \sigma + x^* \otimes \tau \not\in L_{j+2}. \text{Finally,} \]

\[(\text{ad}[1, x]) \cdot ((x^{(2)})^* \otimes \tau + x^* \otimes \rho + 1^* \otimes \sigma) = [1, \rho] + [\sigma, x] = 0 + 2 \cdot [x, \sigma] = 2h, \text{and} \quad (\text{ad}[1, x]) \cdot h = 2 \cdot x^{(2)} y^{(j)} \neq 0, \text{so} \quad (x^{(2)})^* \otimes \tau + x^* \otimes \rho + 1^* \otimes \sigma \not\in L_{j+2}. \text{It therefore follows from Lemma 6.17 that (modulo \(L^*_1 \otimes (L^*_1 \otimes X_{1,j,2})\))}

\[X_{1,j+2,19} \cap L_{j+2} = 0. \quad (6.46)\]

Now, by \((6.34), (6.35), (6.36), \text{and} (6.40)\), we may assume that \(X_{1,j+1,4}, X_{1,j+1,5}, X_{1,j+1,7}\), and \(X_{1,j+1,8}\) all have zero intersection with \(L_{j+1}\). (See also \((6.24), (6.29), \text{and} (6.33)\).) Consequently, we can infer from \((6.22), (6.23), (6.27), (6.28), (6.30), (6.31), (6.32), (6.33), (6.38), (6.39), (6.42), (6.43), \text{and} (6.46)\) that

\[L_{j+2} = L_{j+2} \cap L^*_1 \otimes (L^*_1 \otimes S_{j-1}) \subseteq X_{1,j+2,1} \cup X_{1,j+2,4} + X_{1,j+2,12} + X_{1,j+2,14} + X_{1,j+2,18} \]

It then follows from \((6.29), (6.29), (6.37), (6.41), \text{and} (6.42)\) that \([L_{-2}, L_{j+2}] = 0\), to contradict Lemma 6.18.

Thus, if \(j \equiv 1 \pmod{3}\), then \(L\) contains a subalgebra such that if \(i \geq 2\) is minimal such that \(L_{-2}\) has non-zero bracket with the \(i\)th gradation space of that subalgebra, then \(i\) is greater than \(j\).

If \(j \equiv 2 \pmod{3}\), then \(L_{j} \subseteq (L_{-1})^* \otimes S_{j-1}\), which equals the sum of the \(L_0\)-submodules

\[X_{2,j,1} \overset{\text{def}}{=} \langle 1^* \otimes xy^{(j)}, 2 \cdot x^* \otimes xy^{(j)} + 1^* \otimes y^{(j)}, (x^{(2)})^* \otimes xy^{(j)} + x^* \otimes y^{(j)} + 2 \cdot 1^* \otimes x^{(2)} y^{(j)} \rangle\]

and
\[ X_{2,j,2} \overset{\mathrm{def}}{=} \begin{cases} x^* \otimes x^{(2)} y^{(j)}, & 2 \cdot (x^{(2)})^* \otimes x^{(2)} y^{(j)} + x^* \otimes x y^{(j)}, \\ 1^* \otimes x^{(2)} y^{(j)} + (x^{(2)})^* \otimes x y^{(j)} + x^* \otimes y^{(j)} > & \end{cases} \]

and

\[ X_{2,j,3} \overset{\mathrm{def}}{=} \begin{cases} x^* \otimes x y^{(j)}, & 2 \cdot (x^{(2)})^* \otimes x y^{(j)} + x^* \otimes y^{(j)}, \\ 2 \cdot x^* \otimes x^{(2)} y^{(j)} + (x^{(2)})^* \otimes y^{(j)} + 1^* \otimes x y^{(j)} > & \end{cases} \]

Since \([L_{-2}, X_{2,j,2}] = 0\), it follows from the definition of \(j\) that either \(X_{2,j,1} \cap L_j \neq 0\) or \(X_{2,j,3} \cap L_j \neq 0\).

Let us first focus on \(X_{2,j,1}\). Define

\[
\begin{align*}
    a & \overset{\mathrm{def}}{=} 1^* \otimes x y^{(j)} \\
    b & \overset{\mathrm{def}}{=} 2 \cdot x^* \otimes x y^{(j)} + 1^* \otimes y^{(j)} \quad \text{and} \\
    c & \overset{\mathrm{def}}{=} (x^{(2)})^* \otimes x y^{(j)} + x^* \otimes y^{(j)} + 2 \cdot 1^* \otimes x^{(2)} y^{(j)}
\end{align*}
\]

Then \(L_{-1}^* \otimes X_{2,j,1}\) is the sum of the following \(L_0\)-submodules:

\[
X_{2,j+1,1} \overset{\mathrm{def}}{=} \langle (x^{(2)})^* \otimes a + x^* \otimes b + 1^* \otimes c \rangle + \langle 2 \cdot x^* \otimes a + 1^* \otimes b, 1^* \otimes a \rangle
\]

\[
X_{2,j+1,2} \overset{\mathrm{def}}{=} \langle (x^{(2)})^* \otimes c + x^* \otimes a + 1^* \otimes b, 2 \cdot x^* \otimes c + 1^* \otimes a \rangle + \langle 1^* \otimes c \rangle
\]

\[
X_{2,j+1,3} \overset{\mathrm{def}}{=} \langle (x^{(2)})^* \otimes b + x^* \otimes c + 1^* \otimes a, 2 \cdot (x^{(2)})^* \otimes a + x^* \otimes b, x^* \otimes a \rangle
\]

We focus first on \(X_{2,j+1,1}\) and set

\[
\begin{align*}
    \alpha & \overset{\mathrm{def}}{=} 1^* \otimes a \\
    \beta & \overset{\mathrm{def}}{=} 2 \cdot x^* \otimes a + 1^* \otimes b \\
    \gamma & \overset{\mathrm{def}}{=} (x^{(2)})^* \otimes a + x^* \otimes b + 1^* \otimes c
\end{align*}
\]

Now \(L_{-1}^* \otimes X_{2,j+1,1}\) is the sum of the following \(L_0\)-submodules:

\[
X_{2,j+2,1} \overset{\mathrm{def}}{=} \langle (x^{(2)})^* \otimes \gamma, 2 \cdot x^* \otimes \gamma, 1^* \otimes \gamma \rangle
\]

\[
X_{2,j+2,2} \overset{\mathrm{def}}{=} \langle (x^{(2)})^* \otimes \alpha + x^* \otimes \beta + 1^* \otimes \gamma, 2 \cdot x^* \otimes \alpha + 1^* \otimes \beta, 1^* \otimes \alpha \rangle
\]


\[ X_{2, j+2, 3} \overset{\text{def}}{=} \langle (x(2))^* \otimes \beta + x^* \otimes \gamma + 1^* \otimes \alpha, x^* \otimes \beta + 2 \cdot (x(2))^* \otimes \alpha, x^* \otimes \alpha \rangle \]

Focusing on \( X_{2, j+2, 1} \), we have \([1, x] \cdot (1^* \otimes \gamma) = [\gamma, x] = 2 \cdot [x, \gamma] = 2b \), and \((\text{ad } 1) \cdot b = (\text{ad } 1, x) \cdot (2 \cdot x^* \otimes xy(j) + 1^* \otimes y(j)) = 2 \cdot [1, xy(j)] + [y(j), x] = y(j-1) \neq 0\), so by Lemma 6.17 again,

\[ X_{2, j+2, 1} \cap L_{j+2} = 0. \quad (6.47) \]

Focusing on \( X_{2, j+2, 3} \), we have \((\text{ad } 1) \cdot (x^* \otimes \alpha) = [1, \alpha] = a\), and \((\text{ad } 1) = (\text{ad } 1) \cdot (1^* \otimes xy(j)) = xy(j), x = xy(j-1) \neq 0\), so by Lemma 6.17 again,

\[ X_{2, j+2, 3} \cap L_{j+2} = 0. \quad (6.48) \]

It follows that if \( X_{2, j+1, 1} \subset S_{j+1} \), then \( X_{2, j+2, 2} \subset L_{j+2} \). Note that

\[ [L_{-2}, X_{2, j+2, 2}] = 0. \quad (6.49) \]

(Note here also that since \([L_{-2}, X_{2, j+2, 2}] = 0 \) and \([L_{-2}, X_{2, j+1, 1}] = 0 \), if \( L_{j+2} \cap X_{2, j+2, 2} \neq 0 \), then we have by Lemma 6.19 that \((\text{ad } 1)^3 \) is an isomorphism from \( L_{j+2} \cap X_{2, j+2, 2} \) to \( S_{j-1} \):

\[
(\text{ad } 1)^3 \cdot (1^* \otimes \alpha) = (\text{ad } 1)^2 \alpha = (\text{ad } 1) a = xy(j) \\
(\text{ad } 1)^3 \cdot (2 \cdot x^* \otimes \alpha + 1^* \otimes \beta) = (\text{ad } 1)^2 \beta = y(j) \\
(\text{ad } 1)^3 \cdot ((x(2))^* \otimes \alpha + x^* \otimes \beta + 1^* \otimes \gamma) = (\text{ad } 1)^2 \gamma = (\text{ad } 1) c = 2x(2)y(j)
\]

Since \( X_{2, j+2, 2} \overset{\text{def}}{=} S_{j-1} \) as an \( L_0 \)-module, and since the construction of the “\( X \)” modules depends only on the \( L_0 \)-module properties of \( S_{j-1} \) and \( L_{j-1} \) [and its bracket with itself \( L_{-2} \)] only interact with the first factor of the tensors, we can repeat the process with \( X_{2, j+2, 2} \) [Compare \( X_{1, j+2, 1} \) in the \( j \equiv 1 \) case above.] in place of \( S_{j-1} \) and continue to repeat it until we arrive at the highest gradation space. Since \((\text{ad } 1) \cdot (1^* \otimes xy(j)) = xy(j-1) \neq 0\), it follows that \([L_{-2}, X_{2, j+1, 1}] \neq 0\). Consequently, when we arrive at the highest gradation space, \( L_{-2} \) will have non-zero bracket with either \( L_{r-2} \), or \( L_{r-1} \), or \( L_r \). [A similar argument works for \( X_{2, j+2, 1} \) in place of \( X_{2, j+2, 2} \), and \( X_{2, j+1, 9} \) in place of \( X_{2, j+1, 1} \).] Then by Lemma 6.11 we would have

\[ j = k \leq 2. \]

If \( X_{2, j+1, 2} \cap L_{j+1} \neq 0 \), then \( \tilde{X}_{2, j+1, 2} \cap L_{j+1} \neq 0 \), where

\[ \tilde{X}_{2, j+1, 2} \overset{\text{def}}{=} \langle (x(2))^* \otimes c + x^* \otimes a + 1^* \otimes b, 2 \cdot x^* \otimes c + 1^* \otimes a \rangle. \]

\( \tilde{X}_{2, j+1, 2} \) is the irreducible \( L_0 \)-submodule of the indecomposable \( L_0 \)-module \( X_{2, j+1, 2} \). Set
\[ \delta \overset{\text{def}}{=} (x^{(2)})^* \otimes c + x^* \otimes a + 1^* \otimes b \text{ and} \]
\[ \epsilon \overset{\text{def}}{=} 2 \cdot x^* \otimes c + 1^* \otimes a \]

Now, \( L_{-1}^* \otimes \tilde{X}_{2,j+1,2} \) is the sum of the following irreducible (because \( -(j+2) \omega \not\equiv 0 \pmod{3} \)) \( L_0 \)-submodules

\[ X_{2,j+2,4} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \epsilon + x^* \otimes \delta, 2 \cdot x^* \otimes \epsilon + 1^* \otimes \delta, 1^* \otimes \epsilon \rangle \]

and

\[ X_{2,j+2,5} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \delta + 1^* \otimes \epsilon, 2 \cdot (x^{(2)})^* \otimes \epsilon + x^* \otimes \delta, x^* \otimes \epsilon \rangle \]

If, in addition, we set

\[ \zeta \overset{\text{def}}{=} 1^* \otimes c \]

then

\[ L_{-1}^* \otimes X_{2,j+1,2} = L_{-1}^* \otimes \tilde{X}_{2,j+1,2} + X_{2,j+2,6} = X_{2,j+2,4} + X_{2,j+2,5} + X_{2,j+2,6} \]

where

\[ X_{2,j+2,6} \overset{\text{def}}{=} <1^* \otimes \zeta> + <2 \cdot x^* \otimes \zeta + 1^* \otimes \epsilon, (x^{(2)})^* \otimes \zeta + x^* \otimes \epsilon + 1^* \otimes \delta> \]

Note that

\[ [L_{-2}, X_{2,j+2,6}] = 0. \tag{6.50} \]

Consider \( X_{2,j+2,4} \). We have

\[ (\text{ad} [1, x]) \cdot (1^* \otimes \epsilon) = [\epsilon, x] = 2 \cdot [x, \epsilon] = c \]

and

\[ (\text{ad} [1, x]) \cdot x = 2 \cdot x^{(2)} y^{(j)} \cdot [1, y^{(j)}] = 2x^{(2)} y^{(j-1)} - x^{(2)} y^{(j-1)} - x^{(2)} y^{(j-1)} \neq 0 \]

so by Lemma 6.17

\[ X_{2,j+2,4} \cap L_{j+2} = 0. \tag{6.51} \]

Next consider \( X_{2,j+2,5} \). Here we have

\[ (\text{ad} [1, x]) \cdot (x^* \otimes \epsilon) = [1, \epsilon] = a \]

and

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\[(\text{ad } [1, x]) \cdot a = (\text{ad } [1, x]) \cdot (1^* \otimes xy^{(j)}) = [xy^{(j)}, x] = xy^{(j-1)} \neq 0\]

so by Lemma 6.17 again,

\[X_{2,j+2,5} \cap L_{j+2} = 0. \quad (6.52)\]

It follows that

\[\tilde{X}_{2,j+1,2} \cap L_{j+1} = 0. \quad (6.53)\]

We focus next on \(X_{2,j+1,3}\). We set

\[\eta \overset{\text{def}}{=} x^* \otimes a, \quad \theta \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes a + x^* \otimes b, \quad \iota \overset{\text{def}}{=} (x^{(2)})^* \otimes b + x^* \otimes c + 1^* \otimes a\]

Then \(L_{-1}^* \otimes X_{2,j+1,3}\) contains the following two \(L_0\)–submodules:

\[X_{2,j+2,7} \overset{\text{def}}{=} ((x^{(2)})^* \otimes \eta + x^* \otimes \theta + 1^* \otimes \iota, 2 \cdot x^* \otimes \eta + 1^* \otimes \theta, 1^* \otimes \eta)\]

and

\[X_{2,j+2,8} \overset{\text{def}}{=} ((x^{(2)})^* \otimes \eta + x^* \otimes \theta + 1^* \otimes \iota + 2 \cdot (x^{(2)})^* \otimes \eta + x^* \otimes \theta, x^* \otimes \eta)\]

Furthermore, \(L_{-1}^* \otimes X_{2,j+1,3} = X_{2,j+2,7} + X_{2,j+2,9}\), where, modulo it’s irreducible submodule \(X_{2,j+2,8}, X_{2,j+2,9}\) is spanned by \((x^{(2)})^* \otimes \eta, (x^{(2)})^* \otimes \theta + 2 \cdot 1^* \otimes \eta\), and \((x^{(2)})^* \otimes \iota + x^* \otimes \eta + 1^* \otimes \theta\). Now,

\[
\begin{align*}
(\text{ad } [x, x^{(2)}]) \cdot ((x^{(2)})^* \otimes \eta) &= [x, \eta] = a \\
(\text{ad } [x^{(2)}, x]) \cdot ((x^{(2)})^* \otimes \theta + 2 \cdot 1^* \otimes \eta) &= [\theta, x] = 2b \\
(\text{ad } [1, x]) \cdot ((x^{(2)})^* \otimes \iota + x^* \otimes \eta + 1^* \otimes \theta) &= [1, \eta] + [\theta, x] = 2b
\end{align*}
\]

so, since from above we know that \((\text{ad } [1, x]) \cdot a\) and \((\text{ad } [1, x]) \cdot b\) are both non-zero, it follows from Lemma 6.17 that

\[L_{j+2} \cap X_{2,j+2,9} \subseteq X_{2,j+2,8}, \quad (6.54)\]

Note that

\[\left[ L_{-2}, X_{2,j+2,8} \right] = 0. \quad (6.55)\]

Focusing on \(X_{2,j+2,7}\) we have
so, since, again, from above we know that \((\text{ad} \,[1, \, x]) \cdot a\) is not zero, it follows from Lemma 6.17 that

\[
X_{2,j+2, \, 17} \cap L_{j+2} = 0. \quad (6.56)
\]

We focus next on \(X_{2,j, \, 3}\). Set

\[
d \overset{\text{def}}{=} x^* \otimes xy^{(j)} \\
e \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes xy^{(j)} + x^* \otimes y^{(j)} \text{ and} \\
f \overset{\text{def}}{=} (x^{(2)})^* \otimes y^{(j)} + 2 \cdot x^* \otimes (x^{(2)})y^{(j)} + 1^* \otimes xy^{(j)}
\]

Then \(L_{j+1}^* \otimes X_{2,j, \, 3}\) contains the following submodules:

\[
X_{2,j+1, \, 4} \overset{\text{def}}{=} ((x^{(2)})^* \otimes e + x^* \otimes f + 1^* \otimes d, 2 \cdot (x^{(2)})^* \otimes d + x^* \otimes e) + \langle x^* \otimes d \rangle
\]

\[
X_{2,j+1, \, 5} \overset{\text{def}}{=} ((x^{(2)})^* \otimes d + x^* \otimes e + 1^* \otimes f, 2 \cdot x^* \otimes d + 1^* \otimes e, 1^* \otimes d)
\]

Now, \(L_{j+1}^* \otimes X_{2,j, \, 3} = X_{2,j+1, \, 6} + X_{2,j+1, \, 5}\), where, modulo \(X_{2,j+1, \, 4}\), \(X_{2,j+1, \, 6}\) is spanned by \((1^* \otimes e + x^* \otimes d + (x^{(2)})^* \otimes f)\) and \((2 \cdot (x^{(2)})^* \otimes e + x^* \otimes f, x^* \otimes e)\).

Focusing first on \(X_{2,j+1, \, 4}\), we set

\[
\kappa \overset{\text{def}}{=} x^* \otimes d \\
\lambda \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes d + x^* \otimes e \\
\mu \overset{\text{def}}{=} (x^{(2)})^* \otimes e + x^* \otimes f + 1^* \otimes d
\]

Then \(L_{j+1}^* \otimes X_{2,j+1, \, 4}\) is the sum of the following three \(L_0\)-submodules:

\[
X_{2,j+2, \, 10} \overset{\text{def}}{=} ((x^{(2)})^* \otimes \lambda + x^* \otimes \mu) + (2 \cdot x^* \otimes \lambda + 1^* \otimes \mu, 1^* \otimes \lambda)
\]

and

\[
X_{2,j+2, \, 11} \overset{\text{def}}{=} ((x^{(2)})^* \otimes \kappa + x^* \otimes \lambda + 1^* \otimes \mu, 2 \cdot x^* \otimes \kappa + 1^* \otimes \lambda) + (1^* \otimes \kappa)
\]

and

\[
X_{2,j+2, \, 12} \overset{\text{def}}{=} (1^* \otimes \lambda + (x^{(2)})^* \otimes \mu, 2 \cdot (x^{(2)})^* \otimes \lambda + x^* \otimes \mu, x^* \otimes \lambda)
\]

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The irreducible $L_0$–submodules are the one-, two-, and three-dimensional subspaces noted respectively above.

In the case of $X_{2,j+2,10}$, we have $(\text{ad} [1, x]) · ((x^{(2)})^* ⊗ \lambda + x^* ⊗ \mu) = [1, \mu] = d,$ and $(\text{ad} [x, x^{(2)}]) · d = [xy^{(j)}, x^{(2)}] = 2x^{(2)}y^{(j-1)} \neq 0$, so by Lemma 6.17

$$X_{2,j+2,10} \cap L_{j+2} = 0.$$  

(6.57)

In the case of $X_{2,j+2,11}$, we have $(\text{ad} [1, x]) · (2 · x^* ⊗ \kappa + 1^* ⊗ \lambda) = 2 · [1, \kappa] + [\lambda, x] = 2e$, and $2 · (\text{ad} [1, x]) · e = (\text{ad} [1, x]) · ((x^{(2)})^* ⊗ xy^{(j)} + 2 · x^* ⊗ (x^{(2)})^*) = 2 · [1, y^{(j)}] = x^{(2)}y^{(j-1)} \neq 0$, so by Lemma 6.17

$$X_{2,j+2,11} \cap L_{j+2} = 0.$$  

(6.58)

In the case of $X_{2,j+2,12}$, we have $(\text{ad} [x, x^{(2)}]) · (x^* ⊗ \lambda) = [\lambda, (x^{2})] \equiv d$, and, as above, $(\text{ad} [x, x^{(2)}]) · d = 2x^{(2)}y^{(j-1)} \neq 0$, so by Lemma 6.17

$$X_{2,j+2,12} \cap L_{j+2} = 0.$$  

(6.59)

Thus,

$$X_{2,j+1,4} \cap L_{j+1} = 0.$$  

(6.60)

It follows that since $X_{2,j+1,4}$ is the irreducible $L_0$–submodule of the indecomposable $L_0$–module $X_{2,j+1,6}$, we must also have

$$X_{2,j+1,6} \cap L_{j+1} = 0.$$  

(6.61)

Focusing next on $X_{2,j+1,5}$, we set

$$\nu \defeq 1^* \otimes d$$

$$\xi \defeq 2 · x^* \otimes d + 1^* \otimes e$$

$$\pi \defeq (x^{(2)})^* \otimes d + x^* \otimes e + 1^* \otimes f$$

Then $L^{*}_{-1} \otimes X_{2,j+1,5}$ contains the following $L_0$–submodules:

$$X_{2,j+2,13} \defeq ((x^{(2)})^* \otimes \nu + x^* \otimes \xi + 1^* \otimes \pi, 2 · x^* \otimes \nu + 1^* \otimes \xi, 1^* \otimes \nu)$$

and

$$X_{2,j+2,14} \defeq ((x^{(2)})^* \otimes \xi + x^* \otimes \pi + 1^* \otimes \nu, 2 · (x^{(2)})^* \otimes \nu + x^* \otimes \xi) + (x^* \otimes \nu)$$

Furthermore, $L^{*}_{-1} \otimes X_{2,j+1,5} = X_{2,j+2,13} + X_{2,j+2,15}$, where modulo $X_{2,j+2,14}$, $X_{2,j+2,15}$ is spanned by $(x^{(2)})^* \otimes \nu, x^* \otimes \nu + 1^* \otimes \xi + (x^{(2)})^* \otimes \pi,$ and $2 · 1^* \otimes \nu + (x^{(2)})^* \otimes \xi$. We have
\[
\begin{align*}
(\text{ad} [x^{(2)}, 1]) \cdot ((x^{(2)})^* \otimes \nu) &= 2d \\
(\text{ad} [1, x]) \cdot (x^* \otimes \nu + 1^* \otimes \xi + (x^{(2)})^* \otimes \pi) &= 2d \\
(\text{ad} [x, x^{(2)}]) \cdot (2 \cdot 1^* \otimes \nu + (x^{(2)})^* \otimes \xi) &= 2d
\end{align*}
\]

so, since from above \((\text{ad} [x, x^{(2)}]) \cdot d = 2x^{(2)} y^{(j-1)} \neq 0\), it follows from Lemma 6.17 that

\[
X_{2, j+2, 15} \cap L_{j+2} \subseteq X_{2, j+2, 14}
\]

(6.62)

Focusing on \(X_{2, j+2, 14}\), we have \((\text{ad} [1, x]) \cdot (2 \cdot (x^{(2)})^* \otimes \nu + x^* \otimes \xi) = [1, \xi] = e\), and \((\text{ad} [1, x]) \cdot e = [1, y^{(j)}] = 2x^{(2)} y^{(j-1)} \neq 0\), so by Lemma 6.17

\[
X_{2, j+2, 14} \cap L_{j+2} = 0.
\]

(6.63)

It follows from (6.62) and (6.63) that

\[
X_{2, j+2, 15} \cap L_{j+2} = 0.
\]

(6.64)

Note that

\[
[L_{-2}, X_{2, j+2, 13}] = 0.
\]

(6.65)

It remains to consider \(X_{2, j, 2}\). Set

\[
g \overset{\text{def}}{=} x^* \otimes x^{(2)} y^{(j)}
\]

\[
h \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes x^{(2)} y^{(j)} + x^* \otimes x y^{(j)}
\]

\[
i \overset{\text{def}}{=} 1^* \otimes x^{(2)} y^{(j)} + (x^{(2)})^* \otimes x y^{(j)} + x^* \otimes y^{(j)}
\]

Then \(L_{-1}^* \otimes X_{2, j, 2}\) is the sum of the following \(L_0\)-modules:

\[
X_{2, j+1, 7} \overset{\text{def}}{=} < (x^{(2)})^* \otimes g + x^* \otimes h + 1^* \otimes i > + < 2 \cdot x^* \otimes g + 1^* \otimes h, 1^* \otimes g >
\]

\[
X_{2, j+1, 8} \overset{\text{def}}{=} < (x^{(2)})^* \otimes i + x^* \otimes g + 1^* \otimes h, 2 \cdot x^* \otimes i + 1^* \otimes g > + < 1^* \otimes i >
\]

\[
X_{2, j+1, 9} \overset{\text{def}}{=} < (x^{(2)})^* \otimes h + x^* \otimes i + 1^* \otimes g, 2 \cdot (x^{(2)})^* \otimes g + x^* \otimes h, x^* \otimes g >
\]

We focus first on \(X_{2, j+1, 7}\). Since \((\text{ad} [1, x]) \cdot (1^* \otimes g) = [g, x] = 2x^{(2)} y^{(j)} \neq 0\), and \((\text{ad} [1, x]) \cdot ((x^{(2)})^* \otimes g + x^* \otimes h + 1^* \otimes i) = 2y^{(j)} \neq 0\), if \(L_{j+1} \cap X_{2, j+1, 7} \neq 0\), then \([L_{-2}, L_{j+1} \cap X_{1, j+1, 7}] \neq 0\), and we can replace \(L\) with the Lie algebra generated by \(L_{\leq 0} \oplus L_1 \oplus \cdots \oplus L_{j-1} \oplus (L_j \cap X_{2, j, 2}) \oplus (L_{j+1} \cap X_{2, j+1, 7})\), and we will have a Lie subalgebra of \(L\) such that the minimal \(i \geq 2\) such that \(L_{-2}\) has
non-zero bracket with the \(i\)th gradation space of this subalgebra is \(j + 1\). Thus, we may assume in what follows that
\[ L_{j+1} \cap X_{2, j+1, \tau} = 0. \] (6.66)

We focus next on \(X_{2, j+1, 8}\). Since \((\text{ad} [1, x]) \cdot (1^* \otimes i) = [i, x] = 2g^{(j)} \neq 0\), and \((\text{ad} [1, x]) \cdot (2 \cdot x^* \otimes i + 1^* \otimes g) = x^{(2)}g^{(j)} \neq 0\), if \(L_{j+1} \cap X_{2, j+1, 8} \neq 0\), then \([L_{-2}, L_{j+1} \cap X_{1, j+1, 8}] \neq 0\), and we can replace \(L\) with the Lie algebra generated by \(L_{\leq 0} \oplus L_{1} \oplus \cdots \oplus L_{j-1} \oplus (L_j \cap X_{2, j, 2}) \oplus (L_{j+1} \cap X_{2, j+1, 8})\), and we will have a Lie subalgebra of \(L\) such that the minimal \(i \geq 2\) such that \(L_{-2}\) has non-zero bracket with the \(i\)th gradation space of this subalgebra is \(j + 1\). Thus, we may assume in what follows that
\[ L_{j+1} \cap X_{2, j+1, 8} = 0. \] (6.67)

Looking next at \(X_{2, j+1, 9}\), we set
\[
\pi \overset{\text{def}}{=} x^* \otimes g \\
\rho \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes g + x^* \otimes h \\
\sigma \overset{\text{def}}{=} (x^{(2)})^* \otimes h + x^* \otimes i + 1^* \otimes g
\]

Note that
\[ [L_{-2}, X_{2, j+1, 9}] = 0, \]
(so that by Lemma 6.19 \(X_{2, j+1, 9} \cong S_{j-2}\)). Note also that \(L_{-1} \otimes X_{2, j+1, 9}\) is the sum of the following \(L_0\)-modules:
\[
X_{2, j+2, 16} \overset{\text{def}}{=} <(x^{(2)})^* \otimes \sigma, 2 \cdot x^* \otimes \sigma > + < 1^* \otimes \sigma >
\]
\[
X_{2, j+2, 17} \overset{\text{def}}{=} <(x^{(2)})^* \otimes \pi + x^* \otimes \rho + 1^* \otimes \sigma, 2 \cdot x^* \otimes \pi + 1^* \otimes \rho, 1^* \otimes \pi >
\]
\[
X_{2, j+2, 18} \overset{\text{def}}{=} <(x^{(2)})^* \otimes \rho + x^* \otimes \sigma + 1^* \otimes \pi, 2 \cdot (x^{(2)})^* \otimes \pi + x^* \otimes \rho > + < x^* \otimes \pi >
\]

Note that
\[ [L_{-2}, X_{2, j+2, 18}] = 0 \] (6.68)

Also, since \((\text{ad} [1, x]) \cdot (1^* \otimes \sigma) = [\sigma, x] = 2i\), and \((\text{ad} [1, x]) \cdot (2 \cdot x^* \otimes \sigma) = 2 \cdot [1, \sigma] = 2g \neq 0\), if \(L_{j+2} \cap X_{2, j+2, 16} \neq 0\), then \([L_{-2}, L_{j+2} \cap X_{2, j+2, 16}] \neq 0\), and we can replace \(L\) with the Lie algebra generated by \(L_{\leq 0} \oplus L_{1} \oplus \cdots \oplus L_{j-1} \oplus (L_j \cap X_{2, j, 2}) \oplus (L_{j+1} \cap X_{2, j+1, 9}) + (L_{j+2} \cap X_{2, j+2, 16})\), and we will have a Lie subalgebra of \(L\) such that the minimal \(i \geq 2\), such that \(L_{-2}\) has non-zero bracket
with the $i^{th}$ gradation space of the subalgebra, is $j + 2$. Thus, we may assume in what follows that

$$L_{j+2} \cap X_{2,j+2,16} = 0.$$  \hspace{1cm} (6.69)

Also, since $(\text{ad}[1, x]) \cdot (1^* \otimes \pi) = [\pi, x] = 2g$, and $(\text{ad}[1, x]) \cdot (2 \cdot x^* \otimes \pi + 1^* \otimes \rho) = 2 \cdot [1, \pi] + [\rho, x] = 2h \neq 0$, if $L_{j+2} \cap X_{2,j+2,17} \neq 0$, then $[L_{j-2}, L_{j+2} \cap X_{2,j+2,17}] \neq 0$, and we can replace $L$ with the Lie algebra generated by $L_{\leq 0} \oplus L_1 \oplus \cdots \oplus L_{j-1} \oplus (L_j \cap X_{2,j,2}) \oplus (L_{j+1} \cap X_{2,j+1,9}) \oplus (L_{j+2} \cap X_{2,j+2,17})$, and we will have a Lie subalgebra of $L$ such that the minimal $i \geq 2$, such that $L_{-2}$ has non-zero bracket with the $i^{th}$ gradation space of the subalgebra, is $j + 2$. Thus, we may assume in what follows that

$$L_{j+2} \cap X_{2,j+2,17} = 0.$$  \hspace{1cm} (6.70)

Now, we may assume from $(6.53)$, $(6.60)$, $(6.51)$, $(6.61)$, and $(6.67)$, that $X_{2,j+1,2}$, $X_{2,j+1,4}$, $X_{2,j+1,6}$, $X_{2,j+1,7}$, and $X_{2,j+1,8}$ all have zero intersection with $L_{j+1}$. It therefore follows from $(6.47)$, $(6.48)$, $(6.50)$, $(6.52)$, $(6.53)$, $(6.57)$, $(6.58)$, $(6.59)$, $(6.63) \text{ (and (6.62)), (6.64), (6.69), and (6.70)}$ that

$$L_{j+2} = L_{j+2} \cap (L_{-1}^* \otimes (L_{-1}^* \oplus (L_{-1}^* \otimes S_{j-1}))) \subseteq \sum X_{2,j+2,2} + X_{2,j+2,6} + X_{2,j+2,8} + X_{2,j+2,13} + X_{2,j+2,18}$$

It then follows from $(6.49)$, $(6.50)$, $(6.55)$, $(6.65)$, and $(6.68)$, that $[L_{-2}, L_{j+2}] = 0$, to contradict Lemma $6.18$. Thus, if $j \equiv 2 \pmod{3}$, then $L$ contains a subalgebra such that if $i \geq 2$ is minimal such that $L_{-2}$ has non-zero bracket with the $i^{th}$ gradation space of that subalgebra, then $i$ is greater than $j$.

When $j \equiv 0 \pmod{3}$, there is only one irreducible three-dimensional $L_0$-submodule $Q$ of $(L_{-1})^* \otimes S_{j-1}$. It is spanned by

$$x^* \otimes x^{(2)} y^{(j)}$$
$$x^{(2)} y^{(j)} + 2 \cdot x^* \otimes x y^{(j)}$$
$$1^* \otimes x^{(2)} y^{(j)} + (x^{(2)})^* \otimes x y^{(j)} + x^* \otimes y^{(j)}$$

and is the only $L_0$-submodule of $(L_{-1})^* \otimes S_{j-1}$ which has zero bracket with $L_{-2}$. Note that

$$\{y^{(2)}, y^{(j)}\} = x^{(2)} y^{(j+1)} \neq 0$$

so, in fact, $Q \subseteq L_j$. Then, as with the discussion of $\Xi$ in the $j \equiv 1$ case, $Q$ must equal $H_j$ and the analysis of $L_{-1}^* \otimes Q$ would be the same as that of $L_{-1}^* \otimes S_{j-1}$ when $j \equiv 1 \pmod{3}$.

In view of the definition of $j$, $L_j$ must have non-zero intersection with either the three-dimensional indecomposable $L_0$-submodule of $(L_{-1})^* \otimes S_{j-1}$ generated by

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\( a \overset{\text{def}}{=} 1^* \otimes y^{(j)} \)

which contains the two-dimensional irreducible \( L_0 \)-module \( X_{0,j,1} \) spanned by

\[
\begin{align*}
  b & \overset{\text{def}}{=} 2 \cdot x^* \otimes y^{(j)} + 1^* \otimes x^{(2)} y^{(j)} \text{ and } \\
  c & \overset{\text{def}}{=} (x^{(2)})^* \otimes y^{(j)} + x^* \otimes x^{(2)} y^{(j)} + 1^* \otimes x y^{(j)} .
\end{align*}
\]

or one of the “bouquet” of \( L_0 \)-submodules (a linear combination of which appears in the indecomposable module above) generated by the extreme vectors

\( x^* \otimes y^{(j)} \) and \( 1^* \otimes x^{(2)} y^{(j)} \),

respectively, namely,

\[
\begin{align*}
  x^* \otimes y^{(j)} \\
  2 \cdot (x^{(2)})^* \otimes y^{(j)} + x^* \otimes x^{(2)} y^{(j)} \\
  1^* \otimes y^{(j)} + (x^{(2)})^* \otimes x^{(2)} y^{(j)} + x^* \otimes x y^{(j)}
\end{align*}
\]

and

\[
\begin{align*}
  1^* \otimes x^{(2)} y^{(j)} \\
  2 \cdot x^* \otimes x^{(2)} y^{(j)} + 1^* \otimes x y^{(j)} \\
  1^* \otimes y^{(j)} + (x^{(2)})^* \otimes x^{(2)} y^{(j)} + x^* \otimes x y^{(j)}
\end{align*}
\]

the third vector in each of the above indecomposable modules spanning the (same) (irreducible) (one-dimensional) trivial submodule, namely,

\[
X_{0,j,2} \overset{\text{def}}{=} 1^* \otimes y^{(j)} + (x^{(2)})^* \otimes x^{(2)} y^{(j)} + x^* \otimes x y^{(j)}
\]

(Note that the difference of the “bouquet” submodules is the irreducible submodule \( L_{0,j,1} \) of the previous indecomposable submodule.)

Consider \( L_{-1} \otimes L_{0,j,1} \), which is the sum of the following two \( L_0 \)-submodules:

\[
X_{0,j+1,1} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes b + x^* \otimes c > + 2 \cdot x^* \otimes b + 1^* \otimes c, 1^* \otimes b \rangle
\]

and

\[
X_{0,j+1,2} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes c + 1^* \otimes b, 2 \cdot (x^{(2)})^* \otimes b + x^* \otimes c, x^* \otimes b \rangle
\]

We look first at \( X_{0,j+1,1} \). Set

\[
\alpha \overset{\text{def}}{=} 1^* \otimes b
\]
\[
\beta \overset{\text{def}}{=} 2 \cdot x^* \otimes b + 1^* \otimes c \\
\gamma \overset{\text{def}}{=} (x^{(2)})^* \otimes b + x^* \otimes c
\]

Then \( L_{-1}^* \otimes X_{0,j+1,1} \) is the sum of the \( L_0 \)-submodules

\[
\begin{align*}
X_{0,j+2,1} & \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \gamma + 2 \cdot x^* \otimes \alpha + 2 \cdot 1^* \otimes \beta, 2 \cdot x^* \otimes \gamma + 2 \cdot 1^* \otimes \alpha, 1^* \otimes \gamma \rangle \\
X_{0,j+2,2} & \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \alpha + x^* \otimes \beta + 1^* \otimes \gamma, 2 \cdot x^* \otimes \alpha + 1^* \otimes \beta, 1^* \otimes \alpha \rangle \\
X_{0,j+2,3} & \overset{\text{def}}{=} \langle 2 \cdot (x^{(2)})^* \otimes \alpha + 2 \cdot x^* \otimes \beta + 1^* \otimes \gamma, 2 \cdot x^* \otimes \alpha + 2 \cdot (x^{(2)})^* \otimes \gamma, x^* \otimes \gamma \rangle
\end{align*}
\]

Focusing first on \( X_{0,j+2,1} \), we have \((\text{ad} [1, x]) \cdot (1^* \otimes \gamma) = [\gamma, x] = 2 \cdot [x, \gamma] = 2c\), and \((\text{ad} [1, x]) \cdot c = [xy^{(j)}(j), x] + [1, x^{(2)}y^{(j)}] = xy^{(j-1)} \neq 0\), so by Lemma 6.17,

\[
X_{0,j+2,1} \cap L_{j+2} = 0. \tag{6.71}
\]

Note that

\[
[L_{-2} \cdot X_{0,j+2,2}] = 0. \tag{6.72}
\]

Focusing finally on \( X_{0,j+2,3} \), we have \((\text{ad} [x, x^{(2)}]) \cdot (x^* \otimes \gamma) = [\gamma, x^{(2)}] = 2 \cdot [x^{(2)}, \gamma] = 2b\), and \((\text{ad} [x, x^{(2)}]) \cdot b = 2 \cdot [y^{(j)}, x^{(2)}] = 2xy^{(j-1)} \neq 0\), so by Lemma 6.17,

\[
X_{0,j+2,3} \cap L_{j+2} = 0. \tag{6.73}
\]

Now consider \( X_{0,j+1,2} \). Set

\[
\begin{align*}
\delta & \overset{\text{def}}{=} x^* \otimes b \\
\epsilon & \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes b + x^* \otimes c \\
\zeta & \overset{\text{def}}{=} (x^{(2)})^* \otimes c + 1^* \otimes b
\end{align*}
\]

Then \( L_{-1}^* \otimes X_{0,j+1,2} \) contains the following \( L_0 \)-submodules:

\[
\begin{align*}
X_{0,j+2,4} & \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \delta + x^* \otimes \epsilon + 1^* \otimes \zeta, 1^* \otimes \epsilon + 2 \cdot x^* \otimes \delta, 1^* \otimes \delta \rangle \\
X_{0,j+2,5} & \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \epsilon + x^* \otimes \zeta + 1^* \otimes \delta, x^* \otimes \epsilon + 2 \cdot (x^{(2)})^* \otimes \delta, x^* \otimes \delta \rangle
\end{align*}
\]

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Then \( L_{-1}^* \otimes X_{0,j+1,2} = X_{0,j+2,4} + X_{0,j+2,6} \), where \( X_{0,j+2,6} \supset X_{0,j+2,5} \), and \( X_{0,j+2,6} \) is spanned over \( X_{0,j+2,5} \) by \((x^{(2)})^* \otimes \zeta + x^* \otimes \delta + 1^* \otimes \epsilon, 2 \cdot 1^* \otimes \delta + (x^{(2)})^* \otimes \epsilon, \) and \((x^{(2)})^* \otimes \delta \). We have

\[
\text{(ad} [x^{(2)}, 1]) \cdot ((x^{(2)})^* \otimes \zeta + x^* \otimes \delta + 1^* \otimes \epsilon) = [\zeta, 1] + [x^{(2)}, \epsilon] = 2b + 2b = b \neq 0
\]

and from above \((\text{ad} [x, x^{(2)})] \cdot b \neq 0\). Similarly, \((\text{ad} [1, x]) \cdot 2 \cdot 1^* \otimes \delta + (x^{(2)})^* \otimes \epsilon = 2 \cdot [\delta, x] = b = (\text{ad} [x, x^{(2)})] \cdot ((x^{(2)})^* \otimes \delta)\), and, as we showed, \((\text{ad} [x, x^{(2)})] \cdot b \neq 0\).

It therefore follows from Lemma 6.17 that

\[
L_{j+2} \cap X_{0,j+2,6} \subseteq X_{0,j+2,5}
\]  \hspace{1cm} (6.74)

Note that

\[
[L_{-2}, X_{0,j+2,5}] = 0.
\]  \hspace{1cm} (6.75)

Focusing on \( X_{0,j+2,4} \), we have \((\text{ad} [1, x]) \cdot (1^* \otimes \delta) = [\delta, x] = 2 \cdot [x, \delta] = 2b\), and, as above, \((\text{ad} [x, x^{(2)})] \cdot b \neq 0\), so by Lemma 6.17

\[
X_{0,j+2,4} \cap L_{j+2} = 0.
\]  \hspace{1cm} (6.76)

Now consider \( X_{0,j,3} \overset{\text{def}}{=} X_{0,j,1} + <1^* \otimes y^{(j)}>\). Recall that

\[
a = 1^* \otimes y^{(j)}.
\]

Then \( L_{-1}^* \otimes X_{0,j,3} = L_{-1}^* \otimes X_{0,j,1} + X_{0,j+1,3} \), where

\[
X_{0,j+1,3} \overset{\text{def}}{=} <(x^{(2)})^* \otimes a + x^* \otimes b + 1^* \otimes c, 2 \cdot x^* \otimes a + 1^* \otimes b > + <1^* \otimes a>
\]

Set

\[
\begin{align*}
\eta & \overset{\text{def}}{=} 1^* \otimes a \\
\theta & \overset{\text{def}}{=} 2 \cdot x^* \otimes a + 1^* \otimes b \\
\lambda & \overset{\text{def}}{=} (x^{(2)})^* \otimes a + x^* \otimes b + 1^* \otimes c
\end{align*}
\]

Then \( L_{-1}^* \otimes X_{0,j+1,3} \) is the sum of the following \( L_0 \)-submodules:

\[
X_{0,j+2,7} \overset{\text{def}}{=} <(x^{(2)})^* \otimes \theta + x^* \otimes \lambda + 2 \cdot 1^* \otimes \eta > + <2 \cdot x^* \otimes \theta + 1^* \otimes \lambda, 1^* \otimes \theta >
\]

\[
X_{0,j+2,8} \overset{\text{def}}{=} <(x^{(2)})^* \otimes \eta + x^* \otimes \theta + 1^* \otimes \lambda, 2 \cdot x^* \otimes \eta + 1^* \otimes \theta > + <1^* \otimes \eta >
\]

\[
X_{0,j+2,9} \overset{\text{def}}{=} <(x^{(2)})^* \otimes \theta + x^* \otimes \lambda + 1^* \otimes \eta > + <2 \cdot (x^{(2)})^* \otimes \eta + x^* \otimes \theta, x^* \otimes \eta >
\]

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Note that both
\[ [L_{-2}, X_{0,j+1.3}] = 0 \text{ and } [L_{-2}, X_{0,j+2.8}] = 0. \]  
(6.77)

Consequently, we can apply Lemma 6.19 to conclude that \( X_{0,j+2.8} \cong S_{j-1} \).

Focusing on \( X_{0,j+2.7} \), we have \( (\operatorname{ad} [1, x]) \cdot (1^* \otimes \theta) = [\theta, x] = a, \) and \((\operatorname{ad} [1, x]) \cdot a = [y^{(j)}, x] = y^{(j-1)} \neq 0\), so by Lemma 6.17
\[ X_{0,j+2.7} \cap L_{j+2} = 0. \]  
(6.78)

Focusing on \( X_{0,j+2.9} \), we have \((\operatorname{ad} [1, x]) \cdot (x^* \otimes \eta) = [1, \eta] = a, \) and, as above, \((\operatorname{ad} [1, x]) \cdot a \neq 0\), so by Lemma 6.17
\[ X_{0,j+2.9} \cap L_{j+2} = 0. \]  
(6.79)

We now consider
\[ X_{0,j,2} = \langle d \rangle, \]
where \( d \overset{\text{def}}{=} 1^* \otimes y^{(j)} + (x^{(2)})^* \otimes x^{(2)}y^{(j)} + x^* \otimes xy^{(j)} \). Set \( 1 \overset{\text{def}}{=} 1^* \otimes d, \overline{x} \overset{\text{def}}{=} x^* \otimes d, \) and \((x^{(2)})^* \overset{\text{def}}{=} (x^{(2)})^* \otimes d \). We denote the span of these three elements by \( X_{0,j+1.4} \). Then \( L_{-1}^* \otimes X_{0,j+1.4} \) is the sum of the following \( L_0 \)-submodules:
\[ X_{0,j+2.10} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes 1^* + 2 \cdot 2^* \otimes x^* + 1^* \otimes (x^{(2)})^*, \ 2 \cdot x^* \otimes 1^* + 2 \cdot 1^* \otimes x^*, \ 1^* \otimes 1^* \rangle \]
and
\[ X_{0,j+2.11} \overset{\text{def}}{=} \langle (x^{(2)})^* \otimes \overline{x}^* + 2 \cdot x^* \otimes (x^{(2)})^* + 1^* \otimes 1^* \rangle \]
and
\[ X_{0,j+2.12} \overset{\text{def}}{=} \langle 1^* \otimes \overline{x}^* + 2 \cdot (x^{(2)})^* \otimes (x^{(2)})^* + x^* \otimes 1^*, \ 2 \cdot (x^{(2)})^* \otimes x^* + 2 \cdot x^* \otimes (x^{(2)})^*, \ x^* \otimes x^* \rangle \]

Note that
\[ [L_{-2}, X_{0,j+2.10}] = 0 = [L_{-2}, X_{0,j+2.12}]. \]  
(6.80)

Focusing on \( X_{0,j+2.11} \), we have \( (\operatorname{ad} [1, x]) \cdot (1^* \otimes \overline{x}^*) = [\overline{x}^*, x] = 2 \cdot [x, \overline{x}^*] = 2 \cdot d, \) and \((\operatorname{ad} [1, x]) \cdot d = [y^{(j)}, x] + [1, xy^{(j)}] = y^{(j-1)} \neq 0\), so by Lemma 6.17
\[ X_{0,j+2.11} \cap L_{j+2} = 0. \]  
(6.81)

Set
\[ X_{0,j,4} \overset{\text{def}}{=} X_{0,j,2} + < x^* \otimes y^{(j)}, \ 2 \cdot (x^{(2)})^* \otimes y^{(j)} \rangle + x^* \otimes x^{(2)}y^{(j)} > \]
and

\[ X_{0,j,5} \overset{\text{def}}{=} X_{0,j,2+} < 1^* \otimes x^{(2)} y^{(j)}, 2 \cdot x^* \otimes x^{(2)} y^{(j)} + 1^* \otimes x y^{(j)} > \]

Set

\[ e \overset{\text{def}}{=} x^* \otimes y^{(j)} \]
\[ f \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes x \otimes y^{(j)} + x^* \otimes x^{(2)} y^{(j)} \]
\[ g \overset{\text{def}}{=} 1^* \otimes x^{(2)} y^{(j)} \]
\[ h \overset{\text{def}}{=} 2 \cdot x^* \otimes x^{(2)} y^{(j)} + 1^* \otimes x y^{(j)} \]

Then

\[ L_{-1}^* \otimes X_{0,j,4} = L_{-1} \otimes X_{0,j,2} + X_{0,j+1,5} + X_{0,j+1,6} \]

and

\[ L_{-1}^* \otimes X_{0,j,5} = L_{-1} \otimes X_{0,j,2} + X_{0,j+1,7} + X_{0,j+1,8} \]

where

\[ X_{0,j+1,5} \overset{\text{def}}{=} < (x^{(2)})^* \otimes e + x^* \otimes f + 1^* \otimes d > + < 2 \cdot x^* \otimes e + 1^* \otimes f, 1^* \otimes e > \]

\[ X_{0,j+1,6} \overset{\text{def}}{=} < (x^{(2)})^* \otimes f + x^* \otimes d + 1^* \otimes e, 2 \cdot (x^{(2)})^* \otimes e + x^* \otimes f, x^* \otimes e > \]

\[ X_{0,j+1,7} \overset{\text{def}}{=} < (x^{(2)})^* \otimes g + x^* \otimes h + 1^* \otimes d > + < 2 \cdot x^* \otimes g + 1^* \otimes h, 1^* \otimes g > \]

and

\[ X_{0,j+1,8} \overset{\text{def}}{=} < (x^{(2)})^* \otimes h + x^* \otimes d + 1^* \otimes g, 2 \cdot (x^{(2)})^* \otimes g + x^* \otimes h, x^* \otimes g > \]

Set

\[ \kappa \overset{\text{def}}{=} 1^* \otimes e \]
\[ \lambda \overset{\text{def}}{=} 2 \cdot x^* \otimes e + 1^* \otimes f \]
\[ \mu \overset{\text{def}}{=} (x^{(2)})^* \otimes e + x^* \otimes f + 1^* \otimes d \]
Then $L_{-1}^* \otimes X_{0,j+1,5}$ is the sum of the following $L_0$–submodules:

\[
X_{0,j+2,13} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \mu + 2 \cdot x^* \otimes \kappa + 2 \cdot 1^* \otimes \lambda, 2 \cdot x^* \otimes \mu + 2 \cdot 1^* \otimes \kappa > + < 1^* \otimes \mu >
\]

\[
X_{0,j+2,14} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \kappa + x^* \otimes \lambda + 1^* \otimes \mu > + < 2 \cdot x^* \otimes \kappa + 1^* \otimes \lambda, 1^* \otimes \kappa >
\]

\[
X_{0,j+2,15} \overset{\text{def}}{=} < 2 \cdot (x^{(2)})^* \otimes \kappa + 2 \cdot x^* \otimes \lambda + 1^* \otimes \mu > + < 2 \cdot (x^{(2)})^* \otimes \mu + 2 \cdot x^* \otimes \kappa, x^* \otimes \mu >
\]

Note that

\[
[X_{-2}, X_{0,j+2,14}] = 0 \quad \text{(6.82)}
\]

Focusing on $X_{0,j+2,13}$, we have $(\text{ad} [1, x]) \cdot (1^* \otimes \mu) = [\mu, x] = 2f$, and

\[
(\text{ad} [x, x^{(2)}]) \cdot f = 2 \cdot [x, y(j)] + [x^{(2)}y(j), x^{(2)}] = y(j-1) \neq 0,
\]

so by Lemma 6.17, $X_{0,j+2,13} \triangle L_{j+2} = 0$. \hfill (6.83)

Focusing on $X_{0,j+2,15}$, we have $(\text{ad} [1, x]) \cdot (x^* \otimes \mu) = [1, \mu] = d$, and, from above, $(\text{ad} [1, x]) \cdot d \neq 0$, so by Lemma 6.17, $X_{0,j+2,15} \triangle L_{j+2} = 0$. \hfill (6.84)

Set

\[
\nu \overset{\text{def}}{=} x^* \otimes e
\]

\[
\xi \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes e + x^* \otimes f
\]

\[
\pi \overset{\text{def}}{=} (x^{(2)})^* \otimes f + x^* \otimes d + 1^* \otimes e
\]

Then $L_{-1}^* \otimes X_{0,j+1,6}$ contains the following $L_0$–submodules:

\[
X_{0,j+2,16} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \nu + x^* \otimes \xi + 1^* \otimes \pi, 2 \cdot x^* \otimes \nu + 1^* \otimes \xi, 1^* \otimes \nu >
\]

\[
X_{0,j+2,17} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \xi + x^* \otimes \pi + 1^* \otimes \nu, 2 \cdot (x^{(2)})^* \otimes \nu + x^* \otimes \xi > + < x^* \otimes \nu >
\]

Now, $L_{-1}^* \otimes X_{0,j+1,6} = X_{0,j+2,16} + X_{0,j+2,18}$, where, modulo $X_{0,j+2,17}$, $X_{0,j+2,18}$ is spanned by $\langle 1^* \otimes \xi + 2 \cdot x^* \otimes \nu + (x^{(2)})^* \otimes \pi \rangle$ and $\langle 2 \cdot (x^{(2)})^* \otimes \xi + x^* \otimes \pi, x^* \otimes \xi \rangle$. Since $(\text{ad} [x, x^{(2)}]) \cdot (x^* \otimes \xi) = [\xi, x^{(2)}] = e$, and $(\text{ad} [1, x]) \cdot e = [1, y(j)] = 2x^{(2)}y(j-1) \neq 0$; and $(\text{ad} [1, x]) \cdot (2 \cdot (x^{(2)})^* \otimes \xi + x^* \otimes \pi) = [1, \pi] = e$;

and $(\text{ad} [1, x]) \cdot (1^* \otimes \xi + 2 \cdot x^* \otimes \nu + (x^{(2)})^* \otimes \pi) = [\xi, x^{(2)}] + 2 \cdot [1, \nu] = 2f$ and, from above, $(\text{ad} [x, x^{(2)}]) \cdot f = y(j-1) \neq 0$, it follows from Lemma 6.17 that
\[ L_{j+2} \cap X_{0,j+2,18} \subseteq X_{0,j+2,17} \]  \hspace{1cm} (6.85)

Note that
\[ [L_{-2}, X_{0,j+2,17}] = 0 \]  \hspace{1cm} (6.86)

Focusing on \( X_{0,j+2,16} \), we have \((\text{ad } [1, x]) \cdot (1^* \otimes \nu) = [\nu, x] = 2e\), and as we saw above \((\text{ad } [1, x]) \cdot e = 2x(2)y^{(j-1)} \neq 0\), so by Lemma 6.17,
\[ X_{0,j+2,16} \cap L_{j+2} = 0. \]  \hspace{1cm} (6.87)

Set
\[ \rho \overset{\text{def}}{=} 1^* \otimes g \]
\[ \sigma \overset{\text{def}}{=} 2 \cdot x^* \otimes g + 1^* \otimes h \]
\[ \tau \overset{\text{def}}{=} (x^{(2)})^* \otimes g + x^* \otimes h + 1^* \otimes d \]

Then \( L_{-1}^* \otimes X_{0,j+1,7} \) is the sum of the following \( L_0 \)-submodules:
\[ X_{0,j+2,19} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \tau + 2 \cdot x^* \otimes \rho + 2 \cdot 1^* \otimes \sigma, 2 \cdot x^* \otimes \tau + 2 \cdot 1^* \otimes \rho > + < 1^* \otimes \tau > \]
\[ X_{0,j+2,20} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \rho + x^* \otimes \sigma + 1^* \otimes \tau > + < 2 \cdot x^* \otimes \rho + 1^* \otimes \sigma, 1^* \otimes \rho > \]
\[ X_{0,j+2,21} \overset{\text{def}}{=} < 2 \cdot (x^{(2)})^* \otimes \rho + 2 \cdot x^* \otimes \sigma + 1^* \otimes \tau > + < 2 \cdot (x^{(2)})^* \otimes \tau + 2 \cdot x^* \otimes \rho, x^* \otimes \tau > \]

Note that
\[ [L_{-2}, X_{0,j+2,20}] = 0 \]  \hspace{1cm} (6.88)

Focusing on \( X_{0,j+2,19} \), we have \((\text{ad } [1, x]) \cdot (2 \cdot x^* \otimes \tau + 2 \cdot 1^* \otimes \rho) = 2 \cdot [1, \tau] + 2 \cdot [\rho, x] = 2d \neq 0\), and, as above, \((\text{ad } [1, x]) \cdot d = y^{(j-1)} \neq 0\), so by Lemma 6.17,
\[ X_{0,j+2,19} \cap L_{j+2} = 0. \]  \hspace{1cm} (6.89)

Focusing on \( X_{0,j+2,21} \), we have \((\text{ad } [1, x]) \cdot (x^* \otimes \tau) = [1, \tau] = d\), and, as above, \((\text{ad } [1, x]) \cdot d = y^{(j-1)} \neq 0\), so by Lemma 6.17,
\[ X_{0,j+2,21} \cap L_{j+2} = 0. \]  \hspace{1cm} (6.90)

Set
\[ \upsilon \overset{\text{def}}{=} x^* \otimes g \]
\[ \phi \overset{\text{def}}{=} 2 \cdot (x^{(2)})^* \otimes g + x^* \otimes h \]
\[ \chi \overset{\text{def}}{=} (x^{(2)})^* \otimes h + x^* \otimes g + 1^* \otimes g \]

Then \( L^- \otimes X_{0,j+1,8} \) contains the following \( L_0 \)-submodules:

\[ X_{0,j+2,22} \overset{\text{def}}{=} < (x^{(2)})^* \otimes v + x^* \otimes \phi + 1^* \otimes \chi, 2 \cdot x^* \otimes v + 1^* \otimes \phi, 1^* \otimes v > \]

\[ X_{0,j+2,23} \overset{\text{def}}{=} < (x^{(2)})^* \otimes \phi + x^* \otimes \chi + 1^* \otimes v, 2 \cdot (x^{(2)})^* \otimes v + x^* \otimes \phi > + < x^* \otimes v > \]

Note that

\[ [L_{-2}, X_{0,j+2,23}] = 0 \quad (6.91) \]

Focusing on \( X_{0,j+2,22} \), we have \((\text{ad}[1, x]) \cdot (1^* \otimes v) = [v, x] = 2g\), and \((\text{ad}[1, x]) \cdot g = [x^{(2)}y^{(j)}, x] = x^{(2)}y^{(j-1)} \neq 0\), so by Lemma 6.17

\[ X_{0,j+2,22} \cap L_{j+2} = 0. \quad (6.92) \]

Now, \( L^- \otimes X_{0,j+1,8} = X_{0,j+2,22} + X_{0,j+2,24} \), where, modulo \( L_{j+2} \), \( X_{0,j+2,24} \) is spanned by \((1^* \otimes \phi + 2 \cdot x^* \otimes v + (x^{(2)})^* \otimes \chi)\) and \((2 \cdot (x^{(2)})^* \otimes \phi + x^* \otimes \chi, x^* \otimes \phi)\). Since \((\text{ad}[x, x^{(2)}]) \cdot (\phi \otimes x^{(2)}) = g\), and, from above, \((\text{ad}[1, x]) \cdot g = x^{(2)}y^{(j)} \neq 0\); and \((\text{ad}[x, x^{(2)}]) \cdot (2 \cdot (x^{(2)})^* \otimes \phi + x^* \otimes \chi) = 2 \cdot [x, \phi] + [\chi, x^{(2)}] \equiv h\), and \((\text{ad}[x^{(2)}, 1]) \cdot h = (\text{ad}[x^{(2)}, 1]) \cdot (2 \cdot x^* \otimes x^{(2)}y^{(j)} + 1^* \otimes xy^{(j)}) = [x^{(2)}, xy^{(j)}] = x^{(2)}y^{(j-1)} \neq 0\); and \((\text{ad}[1, x]) \cdot (1^* \otimes \phi + 2 \cdot x^* \otimes v + (x^{(2)})^* \otimes \chi) = [\phi, x] + 2 \cdot 1^* \otimes [1, v] = 2h\); it follows from Lemma 6.17 that

\[ L_{j+2} \cap X_{0,j+2,24} \subseteq X_{0,j+2,23} \quad (6.93) \]

In view of \( \{6.71\}, \{6.73\}, \{6.76\}, \{6.74\}, \{6.78\}, \{6.79\}, \{6.80\}, \{6.81\}, \{6.83\}, \{6.84\}, \{6.87\}, \{6.85\}, \{6.89\}, \{6.90\}, \{6.92\}, \{6.93\}, \) it follows from \( \{6.72\}, \{6.77\}, \{6.80\}, \{6.82\}, \{6.86\}, \{6.88\}, \) and \( \{6.91\} \) that

\[ [L_{-2}, L_{j+2}] = 0 \]

It remains to consider \( X_{0,j+2,2}, X_{0,j+2,5}, X_{0,j+2,8}, X_{0,j+2,10}, X_{0,j+2,12}, X_{0,j+2,14}, X_{0,j+2,17}, X_{0,j+2,20}, \) and \( X_{0,j+2,23} \). (We have shown that we can assume that all of the other \( \text{"}X_{0,j+2}"\)’s can be assumed to be absent from \( L_{j+2} \).) Since \( L \) is closed under the bracket operation, it follows that \( L \) contains \([L_3, L_{j-1}] = [H_3, H_{j-1}] = H_{j+2}\) (which is an irreducible \( L_0 \)-module), which is contained in \( L_{j+2} \).

Therefore, we can assume that \( H_{j+2} \) equals one of the following \( L_0 \)-modules:

\( X_{0,j+2,2}, X_{0,j+2,5}, X_{0,j+2,8}, X_{0,j+2,10}, X_{0,j+2,12}, X_{0,j+2,14}, X_{0,j+2,17}, X_{0,j+2,20}, \) or \( X_{0,j+2,23} \), since all are three dimensional. However, in the various cases, we would have
These contradictions enable us to conclude the following: Let \( r' = r \) if there have been no replacements of \( L \), or let \( r' \) be the height of the most recent replacement of \( L \) otherwise. Then, in either case, if \( j \geq 2 \) is minimal such that \( \{ L_{-2}, L_j \} \neq 0 \), and \( j < r' - 2 \), we can find a subalgebra of \( L \) such that the
smallest $i \geq 2$, such that $L_{-2}$ has non-zero bracket with the $i^{th}$ gradation space of that subalgebra, is greater than $j$. Thus, by induction, $L$ contains a subalgebra such that the smallest $i \geq 2$ such that $L_{-2}$ has non-zero bracket with the $i^{th}$ gradation space of the subalgebra, is at least $r' - 2$. Thus, $(j =) k \leq 2$, and we can now conclude that in any case $j \leq 2$.

To show that $j = 1$, we will, for a contradiction, assume that $j = 2$. We begin by using an inductive argument from [13, Lemma 2.14] to show that the centralizer of $S_s$ in $L_{<0}$ is zero. Denote by $Z$ the centralizer of $S_s$ in $L$. Then $Z = \oplus Z_i$ is a homogeneous subspace of $L$. Since $S_s$ is stable under $\text{ad} L_{\geq 0}$, $Z$ is, as well. The component $Z_{-1}$ is an $L_0$–submodule of $L_{-1}$, and $Z_{-1}$ by definition has zero bracket with $S_s$. Thus, by (B) and (C),

$$Z_{-1} = 0 \quad (6.94)$$

We will show that (See [13, Lemma 2.14].)

$$Z_i = 0 \quad (6.95)$$

for all $i < 0$, proceeding by (downward) induction on $i$. Assume that $i < -1$, and that

$$Z_{\ell} = 0, \ i < \ell < 0. \quad (6.96)$$

By analogy with the previous argument, we here (where $j = 2$) assume that $L$ is generated by $L_{<0} + L_0 + S_1 + S_2$. If we show that $[S_1, Z_i] = [S_2, Z_i] = 0$, then $[L_{>0}, Z_i] = 0$, so that

$$\sum_{i \geq 0} (\text{ad} L_{-1})^i Z_i$$

would be an ideal of $L$ properly contained in $S$, the simple ideal of $L$, to contradict the simplicity of $S$. Consequently, to verify that $Z_i = 0$, we need only show that $[S_1, Z_i] = [S_2, Z_i] = 0$.

By (6.5), we have

$$[Z_{-2}, S_1] = 0$$

and by (6.96) that $[S_1, Z_i] \subseteq Z_{i+1} = 0$. Similarly, $[S_2, Z_i] \subseteq Z_{2+i} = 0$ when $i < -2$. If $i = -2$ and $[S_2, Z_{-2}] \neq 0$, then since by Lemma 2.23, $L_{-2}$ is an irreducible $L_0$–module, we would have by Lemma 2.31 that $0 \neq [S_2, Z_{-2}] = [S_2, L_{-2}] = [S_1, L_{-2}] = L_0$. (See Corollary 1.7.) But then we would have $[L_0, S_s] = 0$, to contradict [13, Lemma 2.13]. (See also Lemma 2.28.) This contradiction shows that $[Z_{-2}, S_2] = 0$, and completes the verification of (6.95).

Since we are assuming that $j = 2$, we have

$$[L_{-2}, S_2] \neq 0 \quad (6.97)$$

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and

\[ [L_{-2}, S_1] = 0 \] (6.98)

We will first address the case in which \( q > r \), and then deal with the case in which \( r \geq q \). Thus, assume first that \( q > r \).

Since (as we observed at the beginning of this section) \( S_1 \) is an irreducible \( L_0 \)-module, it follows from (6.95) that

\[ [L_{-r+1}, S_r] = S_1 \]

Then by (6.98),

\[ [[L_{-r+1}, L_{-2}], S_r] = [L_{-r+1}, [L_{-2}, S_r]] \]

If both sides of the equation were non-zero, then by irreducibility (B) they must both equal \( L_{-1} \). Then we would have, in view of Lemma 2.28 and (6.95) and (6.98), that, if \( r > 2 \),

\[
0 \neq [[L_{-1}, S_1], S_r] = [[[L_{-r+1}, [L_{-2}, S_r]], S_1], S_r] \\
= [[[L_{-r+1}, S_1], [L_{-2}, S_r]], S_r] \\
= [[L_{-r+1}, S_r], [L_{-2}, S_r]] \\
\subseteq [[S_1, S_1], [L_{-2}, S_r]]
\]

which would imply that \([S_1, S_1] \neq 0\), contrary to (6.13). If, on the other hand, \( r = 2 \), then the third line above becomes “\( \subseteq [[L_{-r+1}, S_r], [L_{-2}, S_r]] + [[L_{-r+1}, S_r], S_1]\)”, which leads to a similar contradiction. Thus, we can conclude that

\[ [L_{-r+1}, [L_{-2}, S_r]] = 0 \] (6.99)

Now note that under the present assumptions, \( r \) cannot equal four. Indeed, by Lemma 2.28 \([L_{-1}, S_1], S_r] = [[L_{-1}, S_r], S_1]\) can be assumed to be non-zero and

\[ L_r \supseteq [S_1, S_{r-1}] \supseteq [S_1, [L_{-1}, S_r]] = [[L_{-1}, S_1], S_r] = S_r \] (6.100)

If \( r \) were four, we would have by (6.100), (6.98), and (6.13) that

\[ [L_{-2}, S_r] = [L_{-2}, S_4] = [L_{-2}, [S_1, S_3]] = [S_1, S_1] = 0 \]

to contradict (6.95). Similarly, (6.13) and (6.100) contradict one another when \( r = 2 \). Lastly, \( r \) cannot equal three, either; indeed, by (6.98), \([[L_{-2}, L_{-2}], L_3] \subseteq [L_{-2}, L_1] = 0\), so if \( r \) were three, then by (6.96), \([L_{-2}, L_{-2}] = 0\). Then (replacing, if necessary, \( L_2 \) with one of its irreducible \( L_0 \)-submodules which has non-zero bracket with \( L_{-2} \), which we may do, since \( j \) is assumed to be two and)

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setting $L = L_{-2} \oplus L_0 \oplus L_2$ (See Lemma 1.2 and Proposition 1.6) and $W = L_1$ in the latter option of Lemma 1.8, we arrive at a contradiction. Since $r \geq j = 2$, it follows that

$r > 4$

If we now bracket (6.99) by $S_{r-2}$, we obtain

$0 = [[L_{-r+1}, [L_{-2}, S_r]], S_{r-2}] = [[L_{-r+1}, S_{r-2}], [L_{-2}, S_r]]$

which (since in view of our assumption that $j = k = 2$, we have $[L_{-2}, S_r] \neq 0$) would imply that

$$[L_{-r+1}, S_{r-2}] = 0$$

(6.101)

since otherwise it would equal $L_{-1}$ by irreducibility (B), and transitivity (C) would be violated.

If we next bracket (6.99) by $S_{r-1}$, we get

$$[[L_{-r+1}, S_{r-1}], [L_{-2}, S_r]] = 0$$

(6.102)

Now suppose that

$$[L_{-r}, S_{r-2}] = 0$$

(6.103)

Then we would have

$$0 = [[L_{-r}, S_{r-2}], S_{r-1}] = [[L_{-r}, S_{r-1}], S_{r-2}]$$

which by (B) and (C) would imply that

$$[L_{-r}, S_{r-1}] = 0$$

(6.104)

which in turn would (in view of (D)) imply that

$$[L_{-r-1}, S_{r-1}] = [[L_{-r}, L_{-1}], S_{r-1}] \subseteq [L_{-r}, S_{r-2}]$$

the right-hand side of which we have assumed to be zero. But then we would have $[L_{-r-1}, S_{r-1}] = 0$, so

$$0 = [0, S_r] = [L_{-r-1}, S_{r-1}], S_r] = [[L_{-r-1}, S_r], S_{r-1}] = [L_{-1}, S_{r-1}]$$

by (6.95) and irreducibility (B), to contradict transitivity (C). We conclude that

$$[L_{-r-1}, S_{r-1}, S_r] \neq 0$$

(6.105)

to imply that (See (6.104).)

$$[L_{-r}, S_{r-1}] \neq 0$$

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so that by irreducibility (B),
\[ [L_{-r}, S_{r-1}] = L_{-1} \]  
and to further imply that (See (6.103).)
\[ [L_{-r}, S_{r-2}] \neq 0 \]  
Then, since by Lemma 2.23 \( L_{-2} \) is an irreducible \( L_0 \)-module, it follows that
\[ [L_{-r}, S_{r-2}] = L_{-2} \]  
and, in view of (6.105), that also
\[ [L_{-r-1}, S_{r-2}] = L_{-2} \]  
If \( [L_{-r}, [L_{-2}, S_r]] = 0 \), we would have by (6.106) that
\[
0 = [0, S_{r-1}]
= [[[L_{-r}, [L_{-2}, S_r]], S_{r-1}]
= [[L_{-r}, S_{r-1}], [L_{-2}, S_r]]
= [L_{-1}, [L_{-2}, S_r]]
\]
to contradict transitivity (C), since we are assuming that \((k =) j = 2\). Consequently,
\[ [L_{-r}, [L_{-2}, S_r]] \neq 0 \]
so that, as above
\[ [L_{-r}, [L_{-2}, S_r]] = L_{-2} \]  
By (6.109),
\[ [L_{-2}, S_2] = [[L_{-r-1}, S_{r-1}], S_2] = [[L_{-r-1}, S_2], S_{r-1}] \subseteq [L_{r-1}, L_{r-1}] \]  
(6.111)
Now, if \( q > r + 1 \), by (6.95) and Lemma 2.23 we have \([L_{-r-2}, S_r] = L_{-2}\), so
\[ [L_{-2}, S_2] \subseteq [[L_{-r-2}, S_r], S_2] = [[L_{-r-2}, S_2], S_r] \subseteq [L_{r-1}, S_r]. \]  
(6.112)
If, on the other hand, \( q = r + 1 \), then, since by (6.95) and irreducibility (B)
\([L_{-r-1}, S_r] = L_{-1}\), we have
\[ [L_{-1}, S_1] = [[L_{-r-1}, S_r], L_1] \subseteq [L_{r-1}, S_r] \subseteq [L_{-1}, S_1] \]
since by (D), \([L_{-i}, S_i] \subseteq [L_{-1}, S_1] \) for all \( i, 1 \leq i \leq \min\{q, r\} \), so that in particular, \([L_{-2}, S_2] \subseteq [L_{-1}, S_1] = [L_{-r}, S_r]\), so that (6.112) holds when \( q =
Thus, we have by (6.111) and (6.112), respectively, (since we are assuming that $j = 2$)

$$0 \neq [L_{-2}, S_2] \subseteq [L_{-r+1}, S_{r-1}] \subseteq [L_{-1}, S_1]$$

By Lemma 2.31, we may assume that $[L_{-2}, S_2] = [L_{-1}, S_1]$ and thus conclude that

$$[L_{-r+1}, S_{r-1}] = [L_{-r}, S_r] \quad (6.113)$$

Then we have from (6.99), (6.110) and (6.114) that (See also (6.102).)

$$0 = [0, S_{r-1}]$$

Furthermore, we would then have by induction that

$$[L_{-i}, M_{-q+i}] = 0, \text{ for all } i \geq 2 \quad (6.117)$$

Now, if $[[L_{-i}, S_{q-1}], M_{-q+i}]$ were not equal to zero, then it would have to equal $L_{-1}$ by the irreducibility (B) of $L$, and we would have, by Lemmas 2.1 and 2.4 and (6.117) that for $1 < i < q - 1$,

$$L_{-q} = [L_{-q+1}, [L_{-1}, S_{q-1}], M_{-q+i}] \subseteq [L_{-1}, M_{-q+i}] = 0$$

This contradiction shows that

$$j = 1 \quad (6.115)$$

Let us now assume that $q \leq r$. For $i \geq 1$, set

$$M_{-q+i} \overset{\text{def}}{=} (\text{ad } S_1)^i L_{-q} \quad (6.116)$$

Then we would have, by (6.98), that $[L_{-2}, M_{-q+i}] = 0$ for all $i$. Furthermore, we would then have by induction that

$$[L_{-i}, M_{-q+i}] = 0, \text{ for all } i \geq 2 \quad (6.117)$$

This final contradiction enables us to conclude that if $q > r$,

$$j = 1 \quad (6.115)$$
\[ [L_{-1}, S_{q-1}], M_{-q+i}] = 0, \quad 2 \leq i < q - 1 \]

so that for \( i < \frac{q}{2} \), we have by (6.117) that

\[
0 = [L_{-q+i}, 0] = [L_{-q+i}, [L_{-i}, S_{q-1}], M_{-q+i}] = [[L_{-q+i}, [L_{-i}, S_{q-1}]], M_{-q+i}] \]

Since for \( i = \frac{q}{2} \), we have \([L_{-q+i}, M_{-q+i}][L_{-q+i}, M_{-q+i}] = 0\), also by (6.117), the above-displayed calculation is valid for \( i \leq \frac{q}{2} \). Consequently, if \([L_{-q+i}, [L_{-i}, S_{q-1}]]) were not zero, then by the irreducibility (B) of \( L \), it would equal \( L_{-1} \), so that by Lemma 2.1, \( M_{-q+i} \) would equal zero for \( i \leq \frac{q}{2} \). In [2] and [7], we proved the Main Theorem for all cases less than or equal to three. Consequently, we may assume that \( q \geq 4 \), so that \( M_{-q+i} \) would equal zero for all \( i \leq 2 \). However, when \( i = 1 \), it follows from Lemma 2.8 that \( L_{-q+1} = [L_{-q}, S_1] = M_{-q+1} \), so that \( M_{-q+2} = [L_{-q}, S_1] = [L_{-q+1}, S_1] \). Moreover, it follows from Lemma 1.8 (with \( W = M_{-q+1} \) or \( W = L_{-q} \)) that \( M_{-q+2} \neq 0 \). We conclude that \([L_{-q+i}, [L_{-i}, S_{q-1}]] = 0 \) for \( i \leq 2 \). In particular,

\[ [[L_{-q+2}, L_{-2}], S_{q-1}] = 0 \quad (6.118) \]

Now, if \([L_{-q+2}, L_{-2}] \) were not equal to zero, it would, by Lemma 2.3, equal \( L_{-q} \), and we would have \([L_{-q}, S_{q-1}] = 0 \), to contradict Lemma 2.3. Thus, by Lemma 2.8, we must have

\[
0 = [L_{-q+2}, L_{-2}] = [[L_{-q}, S_2], L_{-2}] = [L_{-q}, [L_{-2}, S_2]]
\]

By Lemma 2.3, \([L_{-2}, S_2] = [L_{-1}, S_1] \), so we have, in view of Lemmas 2.8,

\[
0 = [L_{-q}, [L_{-2}, S_2]] = [L_{-q}, [L_{-1}, S_1]] = [L_{-1}, [L_{-q}, S_1]] = [L_{-1}, L_{-q+1}]
\]

to contradict Lemma 2.1. This contradiction shows that there, too, (6.115) must be true.

Let \( \tilde{L} \) be as in the statement of Lemma 2.24 and let \( V_1 \) be any irreducible \( L_0 \)-submodule of \( S_1 \). Because \( \tilde{L}_{>0} \) is generated by \( S_1 \), we have \( \{1\} \)-transitivity (vi) in the negative part of \( \tilde{L}/M(\tilde{L}) \), and since by (6.115) \([L_{-2}, S_1] \neq 0 \), we can apply Lemma 2.24 to \( \tilde{L}/M(\tilde{L}) \) to conclude that

\[ [L_{-2}, V_1] \neq 0 \quad (6.119) \]

Consequently, if \( \tilde{L} \) is the Lie algebra generated by \( L_{-1} \oplus L_0 \oplus V_1 \), then the depth \( \tilde{q} \) of \( \tilde{L}/M(\tilde{L}) \) (Again, see Theorem 1.3) is (also) greater than one. Let \( \tilde{r} \) denote the height of \( \tilde{L}/M(\tilde{L}) \).
Case I: \( \tilde{q} < \tilde{r} \). Suppose first that the depth \( \tilde{q} \) is less than \( \tilde{r} \). If \( \tilde{q} \) is less than \( q \), then we can apply the Main Theorem to \( \tilde{L}/M(\tilde{L}) \) to (inductively) conclude that the representation of \( \tilde{L}_0 = L_0 \) on \( \tilde{L}_{-1} = L_{-1} \) is restricted, and see that the Main Theorem is true in this case. If \( \tilde{q} = q \), the Main Theorem follows from [2, \ref{7}], Sections 4 and 5, and Lemma \ref{5.11}.

Case II: \( \tilde{q} \geq \tilde{r} \). From now on, then, we will assume that \( \tilde{r} \) is less than or equal to \( \tilde{q} \).

Case IIA: \( \tilde{q} \geq \tilde{r} > 1 \). Since \( \tilde{r} > 1 \), it follows from the definition of \( \tilde{L} \) that \([V_1, V_1] \neq 0\), so that by Lemma \ref{2.27}, \( L_0 V_1 = 0 \). Clearly, \( B(V_1) \) (See Section 3.) satisfies the conditions of the Main Theorem. (Condition (E), for example, follows from the transitivity (C) of \( L \), which shows that actually \( M(B(V_1)) = 0 \).) Now, \( q \) is assumed to be greater than one, so we have

\[
0 \neq [L_{-1}, L_{-1}] = [(B(V_1))_1, (B(V_1))_1]
\]

so \( B(V_1) \) is not degenerate. Consequently, (since the depth \( \tilde{r} \) of \( B(V_1) \) is less than or equal to \( \tilde{q} \) which is less than or equal to \( q \)) we can, as in Case I, apply the Main Theorem to conclude that the representation of \( L'_0 \) on \( V_1 \) is restricted, and that the representation of \( L'_0 \) on \( B(V_1)_1 = L_{-1} \) is restricted, as well (since \( L_{-1} = B(V_1)_1 \subseteq \Hom(V_1, L_0) \); see also Lemma \ref{1.2}).

Case IIB: \( \tilde{q} \geq \tilde{r} = 1 \). Since \( \tilde{r} = 1 \),

\[
[V_1, V_1] = 0.
\]

By Corollary \ref{1.30} either \( L \) is degenerate and \( r = 1 \), as in the last sentence of the Main Theorem, or \( L \) and \( \tilde{L}/M(\tilde{L}) \) are not degenerate, in which case we can apply Proposition \ref{1.6} to \( B((\tilde{L}/M(\tilde{L}))_1) = B(V_1) \) to conclude that \( B(V_1) \) is isomorphic to \( L(e) \) or \( M \), or is Hamiltonian (i.e., between \( H(2 : n, \omega) \) and \( CH(2 : n, \omega) \)). But in those cases, the one-component \( (B(V_1))_1 = L_{-1} \) is abelian; i.e., \([L_{-1}, L_{-1}] \subseteq M(\tilde{L}) \), so that by (D), \( L_{-2} = [L_{-1}, L_{-1}] \subseteq M(\tilde{L}) \), so \([L_{-2}, V_1] = 0 \), to contradict \ref{6.119}.

The proof of the Main Theorem is now complete.

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