Dynamical description of quantum computing: generic nonlocality of quantum noise

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We develop dynamical non-Markovian description of quantum computing in weak coupling limit, in lowest order approximation. We show that the long-range memory of quantum reservoir (such as the $1/t^4$ one exhibited by electromagnetic vacuum) produces strong interrelation between structure of noise and quantum algorithm, implying nonlocal attacks of noise. This shows that the implicit assumption of quantum error correction theory – independence of noise and self-dynamics – fails in long time regimes. We also use our approach to present pure decoherence and decoherence accompanied by dissipation in terms of spectral density of reservoir. The so-called dynamical decoupling method is discussed in this context. Finally, we propose minimal decoherence model, in which the only source of decoherence is vacuum. We optimize fidelity of quantum information processing under the trade-off between speed of gate and strength of decoherence.

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I. INTRODUCTION

In spite of many remarkable results on decoherence in open quantum systems \textsuperscript{1} the problem still remains open involving some conceptual and interpretational difficulties in the description of dynamical quantum effect. Recently the problem become crucial in connection with the idea of quantum computers \textsuperscript{2}. The latter stimulated a huge theoretical and experimental effort to control evolution of quantum systems. This progress allows to hope that theoretical obstacles to build quantum computer (QC) can be overcome. To deal with unavoidable decoherence, the quantum error correcting codes have been designed \textsuperscript{3,22} resulting in the theory of fault-tolerant (FT) quantum computation \textsuperscript{23}. In this theory it is tacitly assumed that decoherence acts independently of the structure of the controlled self-evolution of QC (the one including quantum algorithm and the scheme of error correction). The self-evolution only propagates the errors. In fact, decoherence depends on the kind of dynamics of the system of interest, and this should be taken into account.

The main purpose of this paper is to provide more complete general analysis of interrelations between decoherence and controlled evolution, basing on well established theory of open systems \textsuperscript{15}. We develop the dynamical, Hamiltonian description of decohering quantum computer, deriving and analysing non-Markovian master equation. We show that the long-range quantum memory in conjunction with self-dynamics of quantum computer implies highly nonlocal structure of noise. This has remarkable implications on quantum error correction concept. The key idea of the latter is that the noise is local, so that the quantum information can be hidden in multi-qubit entanglement. Once we do not couple the qubits through quantum gates, the noise is local indeed. However, to protect the information against the noise, we need to produce multi-qubit entanglement. Due to memory, the reservoir will then “see” the evolution as highly nonlocal, and after some time it will act nonlocally itself. We argue that the long-range quantum memory is unavoidable for fundamental reasons: it is exhibited by interaction with vacuum, which can be never removed. It turns out that the memory is relevant even for quantum optics regime, despite the fact that the deviation from exponential decay is unobservable.

In Refs. \textsuperscript{11} it was noted that FT method may not work, if time or space correlations do not decay exponentially. What we show here, is that correlations in time cause the noise to follow the evolution of the system that is to protect against the noise. It follows that FT scheme should be revisited to take into account the memory effect. There is a hope that it is possible, as the collective noise attacks due to memory are of special type: they can be viewed as a result of propagation backward in time of single-qubit errors.

One should mention, that the long quantum memory is exhibited also by phonon environment which characterized stronger coupling than electromagnetic vacuum. This means that within the program of solid state quantum computation the memory effects can become practically relevant.

Other programs to avoid decoherence use dynamical decoupling \textsuperscript{3}, and decoherence free subspaces \textsuperscript{11}. We discuss the first approach in the context of real physical environment and conclude that it is hard to find conditions for which the method could be useful. The decoherence free subspaces approach bases on opposite paradigm to the one of quantum error correction - it exploits symmetries of collective interaction with environment. Even though our approach is fully general, in this paper we will discuss the independent interaction case.

We also introduce a minimal decoherence model of evolution, within the framework of non-Markovian approach. The motivation behind the model is to try to take into account only fundamental obstacles for building quantum computer. That is we remove any decoherence mechanism that could be, in principle, avoided at some tech-
ology. In the model the only environment is the vacuum, and there is no internal, natural self-evolution (the only evolution of the system is constituted by quantum gates). It follows that there is some room to optimize the process of quantum computation within this model. Subsequently, we obtain time-energy trade-off due to that fact that to perform a given gate one needs a constant amount of “action” that can be split in different ways into time and energy. Higher energy implies quick gate, but simultaneously, enhances decoherence. In result it is possible to keep high fidelity at the expense of worse scaling. For sequence of $n$ gates, the physical time needed to keep high fidelity scales as $n^{3/2}$.

This paper is organized as follows. In Sect. II we make some preliminary remarks. Then we develop Hamiltonian approach in Sect. II. We obtain the transition map (called here error map). Then we pass to the Markovian limit for time-independent Hamiltonian and indicate that in non-Markovian stage the reservoir recognizes structure of levels of the systems. We discuss possible types of spectral densities causing decoherence and the problem of high frequency cut-offs. Subsequently we discuss the dynamical decoupling method. In Sect. III we present two examples - QC in memoryless reservoir, and QC driven by kicked dynamics. We discuss error path interference effect. In Sect. IV C memory causing by interaction with vacuum is derived. Then we pass to the main result of the paper (Sect IV) deriving and discussing the master equation for decohering QC. We relate the results to quantum error correction concept. In Sect. V we postulate minimal decoherence model, and provide formula for fidelity within the model. An example of fidelity for single qubit rotation is presented and the time-energy trade-off is discussed.

II. PRELIMINARIES

A. Fidelity in interaction picture

We consider the general case of QC interacting with environment $E$. The initial state of QC is given by $\psi_{t_0}$. The dynamics of QC without decoherence, is given by unitary evolution of quantum algorithm

$$\psi_{t_0} \rightarrow \psi_t = U(t, t_0)\psi_{t_0}$$

(1)

We will call algorithm the total controlled evolution, i.e. the computation algorithm itself plus the error correction procedures. Due to interaction with environment, the initial state will evolve into mixture

$$\psi_{t_0} \rightarrow \varrho_t = \tilde{\Lambda}(t, t_0)\varrho_{t_0}$$

(2)

with $\varrho_{t_0} = \psi_{t_0}$. The fidelity of decohered computation is given by

$$F_t = \langle \psi_t | \varrho_t | \psi_t \rangle.$$  

(3)

It is convenient to use interaction picture. Putting

$$\Lambda(t, t_0) = \tilde{U}(t_0, t)\hat{\Lambda}(t, t_0)$$

(4)

we obtain

$$F_t = \langle \psi_{t_0} | \Lambda(t, t_0)(|\psi_{t_0}\rangle|\psi_{t_0}\rangle)|\psi_{t_0}\rangle.$$  

(5)

Here we used the notation

$$\tilde{U}(\varrho) = U\varrho U^\dagger,$$

(6)

where $U$ is unitary operation.

Note that in the interaction picture the quantum algorithm does not compute, but only “interacts” with errors, causing them to spread more and more. The map $\Lambda(t, t_0)$ describes the net effect of decoherence. As we will see, in FT model the decoherence is “decoupled” from algorithm, so that the total decoherence amounts to the faults caused by environment, which are then propagated by the algorithm. In Hamiltonian approach, the interaction with environment will be “entangled” with algorithm, due to quantum memory, and no such simple description will apply.

B. General evolution

Through the paper we will deal with equation

$$\frac{d}{dt}\sigma_t = \mathcal{L}_t\sigma_t + L\sigma_t$$

(7)

where $\mathcal{L}_t$ is some time dependent operator (it will describe self-evolution), while $L$ is time independent operator (interaction), $\sigma_t$ is vector. Let the free evolution of $\sigma$ be given by the operator $\Gamma(t, t_0)$

$$\Gamma(t, t_0)\sigma_{t_0} = \sigma_t.$$  

(8)

The operator satisfies

$$\frac{d}{dt}\Gamma(t, t_0) = \mathcal{L}_t\Gamma(t, t_0).$$

(9)

The total evolution denoted by $\hat{\Lambda}(t, t_0)$ satisfies

$$\frac{d}{dt}\hat{\Lambda}(t, t_0) = \mathcal{L}_t\hat{\Lambda}(t, t_0) + \hat{L}\hat{\Lambda}(t, t_0).$$

(10)

Then the evolution in interaction picture given by $\Lambda(t, t_0) = \Gamma(t_0, t)\hat{\Lambda}(t, t_0)$ we have the following formal expansion

$$\Lambda(t, t_0) = \mathbb{I} + \sum_{m=1}^{\infty} \int_{t_0}^{t} dt_m \ldots \int_{t_0}^{t_0} dt_1 \times$$

$$\Gamma(t_0, t_m)\Gamma(t_m, t_{m-1})L \ldots \Gamma(t_1, t_0)$$

(11)

Full evolution is given by

$$\hat{\Lambda}(t, t_0) = \Gamma(t, t_0) + \sum_{m=1}^{\infty} \int_{t_0}^{t} dt_m \ldots \int_{t_0}^{t_2} dt_1 \times$$

$$\Gamma(t, t_m)\Gamma(t_m, t_{m-1})L \ldots \Gamma(t_1, t_0).$$

(12)
III. EVOLUTION OF QC IN SECOND ORDER APPROXIMATION

In this section we derive reduced dynamics of QC interacting with environment R. Consequently, in eq. \( \rho \) the operator \( L_i \) will be sum of free Hamiltonians of the QC and R, the operator \( L \) – the interaction Hamiltonian, and \( \alpha \) – the wave function of the total QC + R system.

A. Non-Markovian Born approximation

The Hamiltonian of the total system is of the form
\[
H = H_{QC} + H_R + \lambda H_{int}.
\]
Here \( H_{QC} \) is time-dependent Hamiltonian of the computer, \( H_R \) is time-independent self-Hamiltonian of the environment, \( \lambda \) is coupling constant, which is assumed to be small (\( \lambda \ll 1 \)). In the following we will remove \( \lambda \) from formulas by incorporating it into interaction Hamiltonian \( H_{int} \). The latter is of the form
\[
H_{int} = \sum_{\alpha} S_{\alpha} \otimes R_{\alpha}
\]
where \( S_{\alpha}, R_{\alpha} \) are self-adjoint operators. The resulting evolution of the total system is described by unitary transformation \( U(t, s) \) such that
\[
U(t, s) \varrho(s) = \varrho(t).
\]
We use the notation:
\[
\hat{H}(\varrho) = [H, \varrho].
\]
The self-evolution of QC is given by
\[
U_C(t, s) = T e^{-i \int_s^t H_{C}(u) du}.
\]
where \( T \) is time ordering operator. Finally, the free evolution of the total system is
\[
\hat{U}_0(t, s) = \hat{U}_C(t, s) \otimes e^{-i \hat{H}_R(t-s)}.
\]
The initial state of the total system is given by
\[
\varrho(s) = \varrho_C(s) \otimes \omega_R,
\]
where \( \varrho_C \) is the initial state of quantum computer, while \( \omega_R \) is stationary state of environment. Without loss of generality, one can assume that
\[
\text{Tr} R_{\alpha} \omega_R = 0.
\]
Besides, the state \( \omega_R \) commutes with dynamics of environment
\[
[\omega_R, H_R] = 0
\]
We consider lowest order approximation of the evolution, obtaining from equation (11)
\[
\hat{U}(t, s) \simeq \hat{U}_0(t, s) - i \int_s^t du \hat{U}_0(t, u) \hat{H}_{int} \hat{U}_0(u, s) -
\]
\[
\int_s^t du \int_s^t dw \hat{U}_0(t, u) \hat{H}_{int} \hat{U}_0(u, w) \hat{H}_{int} \hat{U}_0(w, s).
\]
Then the evolution of the reduced density matrix reads as \( \varrho_C(t) = \text{Tr}_R(\varrho(t)) \) is given by
\[
\varrho_C(t) = \hat{U}_0(t, s) \varrho_C(s) - \text{Tr}_R \left\{ \int_s^t du \int_s^t dw \hat{U}_0(t, u) \times \hat{H}_{int} \hat{U}_0(u, w) \hat{H}_{int} \hat{U}_0(w, s) \right\}.
\]
Subsequently, we find that \( \varrho_C(t) \) can be written in the following compact form
\[
\varrho_C(t) = \hat{U}_C(t, s) \left( \varrho_C(s) - \frac{1}{2} [A(t, s) \varrho_C(s) - \varrho_C(s) A(t, s)] + \hat{\Phi}(t, s) \varrho_C(s) - i [h(t, s), \varrho_C(s)] \right)
\]
The completely positive superoperator \( \hat{\Phi}(t, s) \) is given by the following formula
\[
\hat{\Phi}(t, s) \varrho_C(s) = \sum_{\alpha \beta} \int_s^t du \int_s^t dw \langle R_{\beta} R_{\alpha}(u-w) \rangle \omega_R \times
\]
\[
S_{\beta}(u, s) \varrho_C(s) S_{\beta}(w, s)
\]
where \( S_{\alpha}(u, s) = \hat{U}_C^{-1}(u, s) S_{\alpha} \), \( R_{\alpha}(t) = e^{i H_R t} R_{\alpha} e^{-i H_R t} \) are the versions of \( S_{\alpha} \) and \( R_{\alpha} \) evolving in Heisenberg picture according to free evolution; \( \langle R_{\beta} R_{\beta}(t) \rangle \omega_R = \text{Tr}(\omega_R R_{\beta} R_{\beta}(t)) \) is the autocorrelation function of the environment. The operator \( A(t, s) \) is given by
\[
A(t, s) = A^\dagger(t, s) = \hat{\Phi}^*(t, s) \| \]
where \( \hat{\Phi}^* \) is dual map to \( \hat{\Phi} \). Explicitly
\[
A(t, s) = \sum_{\alpha \beta} \int_s^t du \int_s^t dw \langle R_{\beta} R_{\alpha}(u-w) \rangle \omega_R S_{\alpha}(u, s) S_{\beta}(w, s).
\]
Finally, \( h(t, s) \) describes the Hamiltonian contribution to the dynamics due to the interaction with environment (Lamb shift, collective Lamb shift, etc.). In the following we put \( h(s, t) \equiv 0 \) by applying renormalization procedure. We add to the Hamiltonian \( H_{QC}(t) \) the appropriate counter-terms which cancel the contribution \( h(t, s) \) in
a given order of perturbation calculus. Therefore in the following $H_{QC}(t)$ is the full physical Hamiltonian containing all relevant terms. In this way, when passing to Markovian approximation for the reservoir at the thermal equilibrium we obtain the Gibbs state corresponding to full Hamiltonian, as it should be.

One can pass to the frequency domain, putting

$$ (R_{\alpha}R_{\beta}(t))_{\omega_R} = \int_{-\infty}^{+\infty} R_{\alpha\beta}(\omega)e^{-i\omega t}d\omega, \quad (28) $$

$$ Y_\alpha(\omega) = \int_s^t S_\alpha(u,s)e^{-i\omega u}du. \quad (29) $$

The function $R_{\alpha\beta}$ is called spectral density. We then can write $\hat{\Phi}$ as

$$ \hat{\Phi}(t,s)g_C(s) = \sum_{\alpha,\beta} \int_{-\infty}^{+\infty} d\omega R_{\alpha\beta}(\omega)Y_\beta(\omega)g_C(s)Y_\alpha^*(\omega) \quad (30) $$

where we do not write explicitly the dependence on $t$ and $s$ of the operators $Y_\alpha$. Denoting $\hat{\Phi}(t,0) = \hat{\Phi}_t$, the evolution in interaction picture can be thus written (for $s = 0$) as

$$ g_C(t) = \hat{U}_0(t,0)\left[ g_0 - \frac{1}{2}\{\hat{\Phi}_t^\dagger,\bar{g}_0\} + \hat{\Phi}_t\bar{g}_0 \right], \quad (31) $$

where $g_0 = g_C(0)$. The above equation closely resembles the form of the generator of the Markovian semi-group \cite{12,13}. In next section we will exhibit Markovian approximation of this formula.

One notes that the crucial element is here the transition map $\hat{\Phi}$ (we will call it error map). Operators $S_\alpha$ are errors that can occur during interaction with reservoir (cf. \cite{13}). In particular, the map contains propagation of errors which is nothing but evolution of $S_\alpha$ in interaction picture. The overall error is proportional to $\|\hat{\Phi}\|$, where $\|\cdot\|$ is some norm (see \cite{13}).

B. Markovian limit for time independent Hamiltonian

Let us put $t = \frac{\tau}{2}$, $s = -\frac{\tau}{2}$, and

$$ H_C = \sum_j \epsilon_j|j\rangle\langle j| \quad (32) $$

where $\{|j\rangle\}$ is the orthonormal basis in the Hilbert space of QC. To find the meaning of the completely positive map $\hat{\Phi}_t$ we will find the behaviour of evolution for long time. We have

$$ S_\alpha(u,s) = \sum_{\omega_k = \epsilon_j - \epsilon_j'} S_\alpha(\omega_k)e^{i\omega_k u}e^{-\frac{1}{2}\omega_k \tau} \quad (33) $$

with

$$ S_\alpha(\omega_k) = \sum_{j,j',\epsilon_j - \epsilon_j' = \omega_k} |j\rangle\langle j|S_\alpha(\omega_k)^*|j'\rangle\langle j'|. \quad (34) $$

Then the formula (30) reads as

$$ \hat{\Phi}(s,t)g_C(s) = \pi \sum_{\alpha,\beta} \int d\omega R_{\alpha\beta}(\omega)\delta_{\tau}(\omega - \omega_k) \quad (2) $$

$$ S_{\beta}(\omega_k)g_C(s)S_{\alpha}^d(\omega_k) \tau $$

$$ + \pi^2 \sum_{\alpha,\beta; \omega_k \neq \omega_l} \int d\omega R_{\alpha\beta}(\omega)\delta_{\tau}(\omega - \omega_k)\delta_{\tau}(\omega - \omega_l) \quad (35) $$

Here we have used two models of Dirac delta function with width $\frac{1}{\pi}$

$$ \delta_{\tau}(x) = \frac{\sin \pi x}{\pi x}, \quad \delta_{\tau}(x) = \frac{\sin^2 \pi x}{\pi^2 x^2} \quad (36) $$

Approximately we have $(\delta_{\tau}(x))^2 \simeq \pi \tau \delta_{\tau}(x)$. The second, “non-resonant” term of eq. (35) will vanish in the limit of large $\tau$, as the overlap between two deltas will decrease for large $\tau$. (This is an alternative form of a well known rotating-wave approximation). More precisely, one requires $\tau \gg \frac{1}{\Delta \omega}$, where

$$ \Delta \omega = \min\{|\epsilon_j - \epsilon_j'|; \epsilon_j \neq \epsilon_j'\} $$

Finally, we obtain

$$ \hat{\Phi}(t,s)g_C(s) = \tau\pi \sum_{\alpha,\beta; \omega_k} R_{\alpha\beta}(\omega_k)S_{\beta}(\omega_k)g_C(s)S_{\alpha}^d(\omega_k) \quad (37) $$

Note that the map depends on $t$ and $s$ only through the factor $\tau = t - s$. Thus $\frac{1}{\tau}\hat{\Phi}$ is exactly the transition map for the quantum dynamical semi-group in the weak coupling limit \cite{14,15}

$$ \hat{g}_C = Lg_C \quad (38) $$

with

$$ Lg_C = -i[H_C, g_C] - \frac{1}{2\tau}\{\hat{\Phi}^\dagger(\bar{l}), g_C\} + \frac{1}{\tau}\hat{\Phi}g_C. \quad (39) $$

One can see \cite{16} that we can distinguish the initial, non-Markovian stage, when reservoir is “learning” the structure of the Hamiltonian of the system. This requires time $\tau \gg \frac{1}{\Delta \omega}$. Once the structure is recognized, we have Markovian stage, during which the system is being
relaxed towards the Gibbs state of thermal equilibrium 
\[ Z e^{-\beta H} \] determined by \( H \).

Suppose now that we have two qubit system, where the components are coupled to one another: e.g. the self evolution is to produce some two qubit gate. Then, in Markovian stage the system relaxes to compound Gibbs state. Thus, the noise, after recognizing that the Hamiltonian is compound, becomes compound itself \[ \text{[17]} \]. We will analyse it in more detail in Sect. \[ \text{[\text{V}]} \].

C. Decoherence and dissipation in Markovian regime.

The properties of the decoherence in Markovian limit can be read from the formula \[ \text{[57]} \]. We will now analyse the formula to get first intuition about decoherence. First of all, one can single out pure decoherence which is not connected with energy exchange with environment. The populations of energy levels are kept, but the phases subject randomization. The relevant term is

\[ \pi \tau \sum_{\alpha, \beta} R_{\alpha \beta}(0) S_{\beta}(0) \rho C S_{\alpha}^\dagger(0). \]  

Indeed \( S_{\alpha}(0) \) commutes with \( H_C \). Hence this term does not lead to transitions between energy levels. Thus the term \( R_{\alpha \beta}(0) \) stands for the strength of pure dephasing. On the contrary the other terms of the sum correspond to the decoherence accompanied by energy exchange, because the operators \( S_{\alpha}(\omega_k) \) either do not exist (if \( S_{\alpha} \) commute with \( H_C \)) or describe transitions between energy levels. Note that if the spectral density \( R_{\alpha \beta} \) vanishes for \( \omega = 0 \), there is no pure decoherence: the decoherence is always accompanied by dissipation. Assume, for example, that a system does not have self-Hamiltonian (e.g. spin of free electron). Then there is no dissipation. If in addition \( R_{\alpha \beta}(0) = 0 \) then there no decoherence at all.

D. Important examples of reservoirs

We note that an important characteristic of the interaction with environment is the shape of the spectral density. A general property of the heat bath at the temperature \( T \) is the following

\[ R_{\alpha \beta}(-\omega) = e^{-\frac{\omega}{T}} R_{\alpha \beta}(\omega) \]  

In particular, at zero temperature, only nonnegative frequencies are relevant. Let us now present three important examples of environments.

a. Linear coupling to bosonic field Operators \( R_{\alpha} \) are given by

\[ R_{\alpha} = a(\phi_{\alpha}) + a^\dagger(\phi_{\alpha}) \]  

Where \( a^\dagger(\phi_{\alpha}) \), \( a(\phi_{\alpha}) \) are creation and annihilation operators for fields \( \phi_{\alpha} \), respectively. The following form of diagonal elements of \( R_{\alpha \beta} \) can be obtained

\[ R_{\alpha \alpha}(\omega) = \int d\mathbf{k} \delta(\Omega(k) - \omega)(n(k) + 1)|\phi_{\alpha}(k)|^2 \]  

where \( \Omega(k) \) is the energy of the boson (\( \Omega(k) = \frac{k^2}{2m} \) or \( \Omega = \hbar k \)), \( n(k) = \frac{\hbar k}{\epsilon_c} \). Typically \( R_{\alpha \alpha}(\omega) \) grows like \( |\omega|^d \) for small \( \omega \) and then rapidly falls down for \( |\omega| > \omega_c \) (\( \omega_c \) is a cut-off frequency). For electromagnetic interactions in dipol approximation we have \( R_{\alpha \alpha}(\omega) \approx \omega^3 \) for small \( \omega \). Similarly for some cases of phonon interaction. Cut-off parameter \( \omega_c \) is model-dependent (see, e.g., \[ \text{[18]} \]) and characterizes the range of validity of model but does not mean that the frequencies \( \omega > \omega_c \) are not influenced by noise. For example, the electromagnetic interaction can be described by the dipol approximation for \( \frac{\hbar k}{\epsilon_c} = \frac{\hbar c}{2\epsilon} \) where \( r_0 = \frac{\hbar c}{\epsilon} \) is classical electron radius \[ \text{[19]} \]. For larger frequencies different model involving non-bounded (scattering) states of electrons should be taken into account. The other cut-off for free electrons \( \omega_c \approx 2mc^2 \) means only that for larger frequencies the pair production should be taken into account. In solid state \( \omega_c \) is a Debye frequency. For higher frequencies the coupling to phonons makes no sense and we have to consider local interactions with the neighbouring atoms or ions and electrons.

b. Interaction with a dilute gas. We use the model of free bosonic or fermionic gas in the low density approximation

\[ R_{\alpha} = a^\dagger(\phi_{\alpha})a(\phi_{\alpha}) - \langle a^\dagger(\phi_{\alpha})a(\phi_{\alpha}) \rangle \]  

The spectral density is then of the form

\[ R_{\alpha \alpha}(\omega) = \int \int d\mathbf{k} d\mathbf{l} n(k)|\phi_{\alpha}(k)|^2|\phi_{\alpha}(l)|^2 \delta(\Omega(k) - \Omega(l) - \omega) \]  

Typically \( R_{\alpha \alpha}(0) > 0 \) and, again \( R_{\alpha \alpha}(\omega) \) falls down for \( |\omega| > \omega_c \) — a model dependent cut-off frequency.

c. Fluctuations of molecular field. Influence of a local fluctuating field can be described by a classical noise \( F_{\alpha}(t) \). For example for a coloured noise model we have

\[ \langle F_{\alpha} F_{\alpha}(t) \rangle = D e^{-\frac{\omega}{T}} \]  

Then the spectral density is Lorentzian

\[ R_{\alpha \alpha}(\omega) = \frac{D}{\omega^2 + \frac{\omega_c^2}{\tau_c^2}}. \]  

Such a density is relevant for relaxation mechanism in pulsed NMR experiments.
E. Discussion of dynamical decoupling method

From eq. (27) we see that one can gamble with decoherence by changing spectrum of the Hamiltonian trying to choose frequencies $\omega_k$ for which the spectral density is small. In Ref. [11] the method of dynamical decoupling was exhibited. It bases on adding periodic, rapidly alternating term to the self-Hamiltonian of QC which averages out the interaction part, leaving some room for controlled evolution of QC. The method is supported by an elegant group-theoretic framework. Here we would like to determine what types of reservoirs allow to apply such method. By use of a toy model we will argue, that, unfortunately, typical reservoirs presented in sec. 3 do not allow for dynamical decoupling as proposed in [11]. Consider decoherence in QC with “bang-bang” term only, given by

$$H(t) = H \cos \Omega t$$

where $H$ is time independent, $\Omega$ is large frequency. We need to derive $Y_\alpha(\omega)$. Using notation of eq. (24) we have

$$S_\alpha(t, 0) = \sum_{\omega_k} S_\alpha(\omega_k)e^{it\frac{\omega_k}{2}\sin \Omega t}$$

Expanding the time dependent term into Taylor series and applying Fourier transform we obtain

$$Y_\alpha(\omega) = \sum_{\omega_k} S_\alpha(\omega_k) \left[ c_0 \delta(\omega) + c_1 \frac{\omega_k}{\Omega} (\delta(\omega - \Omega) + \delta(\omega + \Omega)) + \ldots, \right]$$

where $\delta$ stands for $\delta^{(1)}_\omega$ (see eq. (30)), $c_{0,1}$ are constants and we omitted higher harmonics. Putting $Y_\alpha$ into formula (23) we see that to obtain low decoherence, we need $R_{\alpha\beta}$ (i) to be small for $\omega \approx 0$ and (ii) to have cut-off frequency satisfying $\omega_c \leq \Omega$. Thus one would need bell-shaped spectral density. The bosonic field reservoirs satisfy (i), but, as we have already mentioned, do not have physical cut-offs. On the other hand collisional or the coloured noise have cut-offs, yet allow for pure decoherence due to nonzero $R_{\alpha\beta}(0)$. The conditions for dynamical decoupling can be in principle met in QED cavity. Then however, the system loses its fundamental simplicity: the quantum computer must now include modes of cavity, while the reservoir become the atoms building the cavity.

There is another aspect of the discussed method, that has not been investigated so far. Namely, the time dependent Hamiltonian is obtained by use of external fields that can be described classically (e.g. coherent light of the laser beam). However, rapidly alternating field will get entangled with the controlled system (this is called quantum back-reaction) so it must be treated quantum mechanically. The resulting disturbance was evaluated to be weaker than the “regular” (say collisional) decoherence [24]. However, it seems that for the bang-bang control, the considered effect can become relevant.

An interesting version of dynamical decoupling method was considered in Ref. [25]. Namely, the starting point was modulate the coupling constant in interaction picture rather than self Hamiltonian. As a result, in Schrödinger picture, the self-Hamiltonian was of the form $\omega H_C \sin \omega t$. Thus in addition to modulation, the level differences of the Hamiltonian were rescaled. Then the above analysis does not apply. However, one can show that in the cases where the above method works, more elementary strategies can protect the system against the reservoir. We will discuss it elsewhere.

IV. EXAMPLES

We will present here two examples. The first one will serve us to introduce fault map describing the action of noise with propagation excluded. In the second one we consider kicked dynamics of algorithm, and illustrate the difference between quantum and classical memory.

A. Memoryless reservoir: error map and fault map

Memoryless reservoir has the white-noise spectral density. Its auto-correlation function is $R_{\alpha\beta}(s - t) = R_{\alpha\beta}^0 \delta(s - t)$. Let us, for simplicity, assume independent interaction $R_{\alpha\beta}(\omega) = R_{\alpha} \delta_{\alpha\beta}$. The resulting error map is given by

$$\Phi_t(\varrho) = \sum_{\alpha} R_{\alpha}^0 \int_s^t du S_\alpha(u, s) \varrho S_\alpha(u, s).$$

Suppose that $S_\alpha$ are one-qubit operators. According to notation of Ref. [3] we will call them faults - these are errors caused by the reservoir, and should be distinguished from the errors resulting from propagation of faults by algorithm (we will denote it by $U(t, s)$ in this section). The propagation of the faults is described by the fact that the error map involves $S_\alpha$ in Heisenberg picture. Note, that the overall error is caused solely by one qubit faults and their propagation, averaged in time. Such evolution can be described by the following master equation

$$\frac{d}{dt} \varrho_t = i[H_t, \varrho_t] + L\varrho_t$$

with

$$L = -\frac{1}{2} \{\hat{\Phi}(\varrho), \varrho\} + \hat{\Phi}(\varrho)$$

for $\hat{\Phi}(\varrho) = \sum_{\alpha} S_\alpha \varrho S_\alpha$. Indeed, let equation (52) $\sigma_\alpha$ be density matrix of QC, $L_t = -iH_t$ - where $H_t$ is Hamiltonian of algorithm and $L$ - the above generator. Applying the formula (12) in first order approximation we obtain
\[ \varrho_t = \hat{U}(t, t_0)\left[ \varrho_{t_0} - \frac{1}{2}\{A, \varrho_{t_0}\} + \sum_\alpha \int_s^t du \, S_\alpha(u, s) \varrho S_\alpha(u, s) \right] = \hat{U}(t, t_0)\left[ \varrho - \frac{1}{2}(\Phi_t(\{I\}, \varrho) + \Phi_t(\varrho)) \right]. \] (54)

Even though the memoryless case is not especially interesting from our point of view, the very form of the master equation (53) is particularly useful in our context, as the faults themselves and the propagation are separated from one another. The propagation is described by the Hamiltonian term, while the faults are described by the map \( \Phi \), which we will call fault map. We will derive such master equation in general case in next section. Here, let us only mention that the “fault part” of the master equation is a sum of one-qubit operators. This means that environment acts locally, and the reason for occurring multi-qubit errors is solely due to propagation.

**B. Decoherence under kicked dynamics of QC: Error path interference effect**

We will assume that the gates are performed quickly, so that they can be generated by Hamiltonian with time dependence given by delta function. This will not work for electrodynamic or phonon vacuum reservoir, as due to the behaviour \( \tilde{\omega}^3 \), any rapid changes in self-Hamiltonian cause large or infinite contribution. Thus in this case one needs to work with Gaussian pulses. On the other hand, the kicked dynamics can be used for Lorentzian spectral density.

Thus we divide computation time \( t \) into \( N \) pieces of length \( \tau \). The Hamiltonian is given by

\[ H(t) = \sum_{j=1}^N \delta(t - (j - 1)\tau) h_j \] (55)

where \( h_j \) generates \( j \)-th step of computation. The unitary evolution is given by

\[ U(t) = \prod_{j=1}^N U(j, j - 1) \] (56)

with \( U(j, j - 1) = e^{-i\theta_j \tau} \) is the \( j \)-th step of computation (the group of gates performed at time \( j\tau \), \( \theta \) is step function. Consequently

\[ U_\tau = \begin{cases} U(1, 0) & t \in (0, \tau) \\ U(2, 1)U(1, 0) & t \in (\tau, 2\tau) \\ \vdots \\ U(N, N - 1)U(N - 1, N - 2)\ldots U(1, 0) & t \in ((N - 1)\tau, N\tau) \end{cases} \] (57)

Substituting such dynamics into the error map of eq (25), we obtain

\[ \Phi^H = \sum_\alpha \sum_{j,k=1}^N r_{jk}^{\alpha} S_{\alpha,j}^k S_{\alpha}^k \] (58)

where \( S_{\alpha,j}^k = U_{j}^\dagger S_{\alpha} U_j \). For simplicity we assumed that \( R_{\alpha,\beta} = R_{\alpha} \delta_{\alpha,\beta} \). The coefficients \( r_{jk}^{\alpha} \) are given by

\[ r_{jk}^{\alpha} = \int_{(k-1)\tau}^{k\tau} \frac{dk'}{\tau} \int_{(j-1)\tau}^{j\tau} dv R_{\alpha\alpha}(u - v). \] (59)

For memoryless case we obtain discrete counterpart of

\[ \Phi^{ph} = \tau \sum_\alpha \sum_{k=1}^N S_{\alpha,k}^j S_{\alpha}^k, \] (60)

In both formulas we see propagation of error, this time, in discrete manner: \( S_{\alpha,j}^k \) denotes error resulting in propagation of fault \( S_{\alpha,\tau} \) within the interval \((0, j\tau)\). The difference is that in equation (60) we have interference due to quantum memory. This effect of interference of error can remarkably modify the process of error propagation.

The difference is similar to that between separable and entangled mixed states. We can view the error map as (subnormalized) density matrix and the Kraus operators as the pure components of ensemble giving rise to the density matrix. In general, the noise process can then behave as classical-like or as quantum stochastic process with time correlations of the form

\[ \Phi_{cl} = \sum_{j_1 \ldots j_m} \rho(j_1 \ldots j_m) S(t_{j_m}) \ldots S(t_{j_1}) \varrho S(t_{j_1})^\dagger \ldots S(t_{j_m})^\dagger \] (61)

\[ \Phi_{qm} = \sum_{j_1 \ldots j_m, i_1 \ldots i_m} C(j_1 \ldots j_m, i_1 \ldots i_m) S(t_{j_m}) \ldots S(t_{j_1}) \varrho S(t_{i_1})^\dagger \ldots S(t_{i_m})^\dagger, \]

respectively. In the first case we have always mixture of product Kraus operators, in the second one - we can have a map for which such representation may be impossible. Indeed, in equation (60) the errors occurring in different times do not interfere with each other, they are added as in classical probability calculus.

However it seems that the crucial point is not the difference between quantum and classical memory, but its range. In Sect. we will exhibit remarkable connection between the range of the memory and occurring multi-qubit faults: the interference caused by long range quantum memory will give raise to nonlocal noise.

**C. Memory of electromagnetic vacuum reservoir**

Here we will calculate and discuss the memory caused by vacuum in free space. This is important point in our paper, as the vacuum is unavoidable, we cannot remove it. Of course we can put the QC into cavity, but then, we
can consider the QC plus cavity as the total system, that is again in free space. We then prefer the latter setup, as it is more fundamental.

Consider the two-level atom and electromagnetic field with dipol interaction

\[ H = \frac{1}{2} \omega_0 \sigma_3 + \sigma_1 \otimes dE \]  

(62)

The spectral density of reservoir in temperature \( T \) is given by

\[
R(\omega) = \frac{8 \pi}{3} d \omega^3 \left( 1 + \frac{1}{e^{\hbar \omega/kT} - 1} \right) \text{ for } \omega > 0
\]

\[
R(\omega) = \frac{8 \pi}{3} d |\omega|^3 \left( \frac{1}{e^{\hbar \omega/kT} - 1} \right) \text{ for } \omega \leq 0
\]

(63)

We must now calculate autocorrelation function in zero temperature. Taking inverse Fourier transform we obtain

\[ R(t) = \frac{1}{(t + i \epsilon)^4} \]

(64)

We see that the memory scales as power of \( t \) rather than exponential function. Then there is no characteristic time. This is due to the fact, that Coulomb interaction has no characteristic range. Thus the range of vacuum memory is infinite. Nevertheless, in quantum optics there is a notion of time of memory, which is the inverse of \( \omega_0 \) - the transition frequency for the atom, and one can work with Markovian master equation, within times much longer than \( 1/\omega_0 \).

This does not mean that there exists some characteristic time scale for vacuum reservoir. Rather, in the interaction picture, the autocorrelation function acquires oscillating term \( e^{i \omega_0 t} \) which effectively makes the memory much weaker. For optical transitions, the memory has then almost no observable consequences. One could think that the memory problem is not relevant in optical implementations of quantum computation. However, it is not the case. For two atoms, there are degenerated levels, so that if one wants to drive gate coupling these levels, there is no high frequency that would diminish the memory effect: the frequency driving the gate and the frequency of the transition are the same. Then the time of memory and time of controlled evolution are of the same order. The resulting non-Markovian effects will be most relevant for many atoms (as needed in quantum computation schemes), as the number of degenerate levels scales exponentially with number of atoms, so that it will be hard to perform computation without use of the gates suffering from memory effects.

V. QUANTUM LONG RANGE MEMORY IMPLIES NONLOCAL STRUCTURE OF NOISE

\[ \dot{\rho} = -i[H_{QC}(t), \rho] + L_t \rho. \]  

(65)

we will now derive such equation in second order approximation. To this end, we differentiate the equation \( [24] \) obtaining

\[
\dot{\rho}(t) = \dot{U}_C(t, s) \rho(s) + \dot{U}_C(t, s) \left[ -\frac{1}{2} \{ \hat{\Phi} \hat{\Phi}^*, \rho(s) \} + \hat{\Phi} \rho(s) \right] + \\
\dot{U}_C(t, s) \left[ -\frac{1}{2} \{ \hat{\Phi} \hat{\Phi}^*, \rho(s) \} + \hat{\Phi} \rho(s) \right].
\]

(66)

We make first step of approximation by removing the second term. Differentiating \( \dot{\Phi}(t, s) \) and putting \( s = 0 \) we obtain

\[
\dot{\rho} = -i[H_{QC}(t), \rho] - \frac{1}{2} \{ \hat{\Phi} \hat{\Phi}^*, \rho \} + \hat{\Phi} \rho
\]

(67)

where \( \hat{\Phi}_t = \hat{\Phi}(t, 0) \) is given by

\[
\hat{\Phi}(t, s)(\cdot) = \sum_{\alpha \beta} (Y_{\alpha \beta}(\cdot) S_\alpha + S_\beta(\cdot) Y_{\alpha \beta}^* )
\]

(68)

with

\[
Y_{\alpha \beta} = \int_s^t du R_{\alpha \beta}(u - t) S_\beta(u, t)
\]

(69)

On the right hand side of the equation we have density matrices evolving according to self-Hamiltonian of the system. The second step of approximation is then to replace them with the ones subjected to full evolution. (Better justification of such integral differential master equation can be obtained within cumulant expansion method, see e.g. \([21]\)).

Now, the decoherence is essentially characterized by the map \( \hat{\Upsilon}_t \) which we will call “fault map” in analogy to the error map \( \hat{\Phi}_t \). The latter one characterizes error, i.e. the net effect of the “attacks” of noise (faults) and their propagation. The present map \( \hat{\Upsilon}_t \) describes solely the action of noise. The propagation will be due to the Hamiltonian term in the above master equation.

Let us now discuss how the form of \( \hat{\Upsilon}_t \) depends on memory. The crucial term is, of course, \( \hat{\Upsilon}_{\alpha \beta} \). In the case of no memory (and, for simplicity, independent decoherence), we obtain \( \hat{\Upsilon}_{\alpha \beta} = \delta_{\alpha \beta} S_\alpha \) which is compatible with Sect. \( \[\text{VIA}\] \). Assuming that each \( \alpha \) denotes different qubit, we obtain that decoherence is local, and multi-qubit errors are solely due to propagation. We can then write the generator in the form

\[ L_t = \sum \alpha L_{1, \alpha}^t. \]  

(70)
where $L^\alpha_t$ operates only on $\alpha$th qubit. If we admit exponentially decaying memory (e.g. coloured noise) with characteristic time $\tau_R$ comparable with the time of performing single computational step, but short in comparison with time between gates, there will be some time for reservoir to learn something about the structure of two qubit Hamiltonians, so that the noise will be a sum of two qubit superoperators

$$L_t = \sum_{\alpha\beta} L^\alpha_{t,\beta}.$$  \hfill (71)

In both cases decoherence is local. The possible faults are, roughly speaking, symbolized by $L^\alpha_{t,\alpha}$, $L^\alpha_{t,\beta}$ and they couple at most two qubits (in general, they involve so many qubits, as the used gates do).

Suppose now, that the reservoir has long range memory. Still, in each step of computation, there are two-qubit gates. The environment learns about their structure a bit. The effect accumulates due to memory, so that after long time, the noise is strongly nonlocal: the generator $L_t$ involves multiqubit operators as it keeps the record of all the history of self-evolution (i.e. algorithm). The decoherence becomes as “entangled” as the quantum algorithm is entangling. Note, that the effect does not depend on the time of performing gates. For short gates, the reservoir has short time to distinguish the temporary levels (driving the gate), but they are easily distinguishable, as the energy differences are high. If instead the energy differences are low, then the time must be longer. This is the result of time-energy uncertainty principle. Thus, the relevant parameter is here “action” i.e. product of time of gate, and strength of applied Hamiltonian (energy). To perform the gate, the needed amount of action must be always of order of 1. Thus the general structure of noise does not depend very much on the way the gates are performed. In Sect. $\mathcal{V}$ we will see that the strength of the decoherence does depend on it, so that we have the mechanism of possible optimization.

**B. Quantum memory and error correction**

Let us now discuss the implication of the above results on quantum error correction (QEC) method \cite{22}. The basic idea of QEC rests on two assumptions: (i) the environment acts locally (ii) the environment acts independently of self-evolution of the system. If the above assumptions are satisfied, it is indeed possible to protect the quantum information against decoherence by encoding it into highly entangled multiqubit states. In literature the assumption (i) was made explicitly, while the assumption (ii) was tacit. Thanks to assumption (i) such nonlocally encoded information will not be affected by decoherence (acting locally). The assumption (ii) ensures that the self-evolution will not modify the local structure of decoherence. As we have argued, the latter assumption is never satisfied. If there is short range memory, the dependence on self-evolution is irrelevant, and QEC still applies. However, the interaction with vacuum, which is unavoidable, introduces long-range quantum memory, which causes environment to be rather malevolent: it traces self-evolution, and the more entangled the latter is, the more nonlocal the faults become caused by noise. On the other hand, the self evolution must be entangling, since it is to hide the quantum information just into entangled states. Thus, we have something a bit analogous to the Lenz rule: the structure of environment attack is proportional to the structure of the evolution that is to protect against it. It seems to be a rather pessimistic result, as it implies, that the very idea of active protection against environment is fundamentally inadequate, as far as long quantum computation is concerned.

However, the conclusion need not be so pessimistic. First of all, decoherence caused by vacuum is small in comparison with other sources of decoherence, e.g. dephasing due to collisions (pressure decoherence), coloured noise or thermal noise. It seems that QEC (and hence FT method) can be used to fight with the latter sources of decoherence, within the time window in which the vacuum-memory effect can be neglected. Strictly speaking other types of reservoirs also have long tails in memory, however, the dominant part decays exponentially. Thus FT could be certainly performed within the time regime satisfying the following conditions: (i) the memory is exponentially decaying, (ii) the memory majorize vacuum memory. This regime may prove sufficient to perform some quantum computational tasks.

There is also another point to address \cite{22}. The claim that lack of memory is sufficient for FT does not mean that it is in general impossible to improve FT against some kind of memory. As a matter of fact, the mechanism producing multi-qubit faults we described has some features that may be used to defend FT scheme. Namely, as seen from formula (69), the multi-qubit fault at some place $\alpha$ arise from backward propagation (with decreasing strength) of single qubit fault $S_\alpha$. Now since FT method is able to fight with propagation forward, it cannot be excluded that it will be robust against faults correlated in such a way. However it goes beyond the scope of this paper.

**VI. MINIMAL DECOHERENCE MODEL FOR QUANTUM CONTROLLED SYSTEMS**

In this section we will postulate minimal decoherence model. We then derive formula for fidelity of quantum computation within the model. We shall consider a class of controlled open systems for which $H_C(t) \neq 0$ only during computation of subsequent gates. The spectral density $R_{\alpha\beta}$ satisfies
\[
\int_{-\epsilon}^{\epsilon} \frac{R_{\alpha\beta}(\omega)}{\omega^2} d\omega < \infty \quad \text{for some } \epsilon > 0. \tag{72}
\]

This excludes reservoirs with any pure decoherence effects, such as gas or colored noise. Our motivation is that we would like to investigate only fundamental obstacles to build quantum computers. The pure decoherence is not such a one, as in principle one could isolate the quantum computer from any influence of the above kind (even though it might require technology that will never be achieved). Since obtaining arbitrarily low temperature is also a matter of technology, we can work with the only environment being vacuum. We removed time-independent part of self-Hamiltonian, as it will only cause additional dissipation.

Example of the system could be n well-separated spins-\(\frac{1}{2}\) which interact with an electromagnetic field and possibly with phonons. The controlled Hamiltonian \(H_C(t)\) can be realized by switching external magnetic fields and suitable interaction between spins. Another example is atomic degenerate metastable level, with Stark of Zee-man splitting used for controlling. Of course it may be impossible to keep possibility of controlling interactions (a cornerstone of a quantum computer) and remove the influence of pure decoherence. However, as said, we assume the most optimistic case, and keep only the obstacle that cannot be removed for fundamental reasons. By removing time-independent self-Hamiltonian, we obtain that there is no decoherence on qubits that are not active. Thus decoherence will be caused by the vacuum solely due to gate operations that require to split energy levels during the time of performance operation. In the case of independent interactions of qubits with vacuum, this temporal difference of energy levels is made of use by the vacuum which forces the upper level to relax by spontaneous emission. In the case of collective interaction (Dicke limit) this is not the case, where one has decoherence-free subspaces [3].

### A. Fidelity in minimal decoherence model

We put \(g_C(s) = \langle \psi | \psi \rangle\) and \(s = -\infty, t = +\infty\). The fidelity of computation is given by

\[
F = 1 - \delta = \langle U_C(t, s)\psi|g_C(t)|U_C(t, s)\psi\rangle \tag{73}
\]

We will denote \(S_\alpha(t) = S_\alpha(-\infty, t) = U_C^{-1}(t, -\infty)S_\alpha\). To avoid infinite contribution from the interaction, we will assume that the interaction is adiabatically switched off at infinity (the standard method in quantum field theory [23]). Thus put into formula (73) the following operators

\[
Y_\alpha(\omega) = \int_{-\infty}^{\infty} \hat{S}_\alpha(t) e^{-i\sigma^+|t|} e^{-i\omega t} dt. \tag{74}
\]

Performing partial integration and then passing to the limit \(\epsilon \to 0\) we obtain

\[
Y_\alpha(\omega) = \frac{1}{i\omega} \int_{-\infty}^{\infty} dt e^{-i\sigma^+|t|} \left( \frac{d}{dt} S_\alpha(t) \right) = \frac{1}{\omega} X_\alpha(\omega). \tag{75}
\]

We then obtain the following formula for error \(\delta\)

\[
\delta = \sum_{\alpha, \beta} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} R_{\alpha\beta}(\omega) \left\{ \langle \psi | X_\alpha^\dagger(\omega) X_\beta(\omega) | \psi \rangle - \langle \psi | X_\alpha^\dagger(\omega) | \psi \rangle \langle \psi | X_\beta(\omega) | \psi \rangle \right\}. \tag{76}
\]

### B. Quantum information processing in minimal decoherence regime

We consider the simplest quantum gate: rotation of single qubit by an angle \(\alpha \in [0, 2\pi]\), in the case of two-level atom coupled to electromagnetic field in zero temperature. The time dependent Hamiltonian will be of the form \(H_C(t) = \frac{1}{2} f(t) \sigma_z\), with

\[
\int_{-\infty}^{\infty} f(t) dt = \alpha, \quad \lim_{t \to \pm \infty} f(t) = 0. \tag{77}
\]

The interaction Hamiltonian is \(H_{int} = \sigma_x \otimes \phi\), and the spectral density, as discussed earlier, is

\[
R(\omega) = \begin{cases} 
R_0 \omega^3 & \text{for } \omega \geq 0 \\
0 & \text{for } \omega < 0 
\end{cases} \tag{78}
\]

The full unitary transformation is given by

\[
U_C = U_C(\infty, -\infty) = e^{-\frac{i}{2} \alpha \sigma_z}. \tag{79}
\]

The parameter \(\alpha\) is the amount of “action” needed to perform the transformation. Roughly speaking, \(\alpha\) is product of energy pumped into system during performing gate and the time of operation. In the case of non-rectangular pulse shape, the time is given by the width of pulse. Typically, if the energy is low, then accompanied dissipation is small, however time of the operation is long (while in quantum computation, one would prefer short time gates). Adding more energy, we obtain short time, but simultaneously, enhance decoherence, causing loss of fidelity. Let us now calculate the fidelity. We have

\[
X(\omega) = F_-(\omega) \sigma^- + F_+(\omega) \sigma^+, \tag{80}
\]

where

\[
F_\pm = \int_{-\infty}^{\infty} dt e^{-i\omega t} f(t) e^{\pm i\phi(t)} \tag{81}
\]

with \(\phi(t) = \int_{-\infty}^{t} f(u) du\). From the formula (76) one easily finds that any rapid variation of \(f(t)\) will result in
large error. Indeed, the changes produce long tails of the Fourier transforms \(F_\pm\), which integrated with \(R_0\omega\) for \(\omega \geq 0\) give poor fidelity. In order to minimize this effect, we choose Gaussian shape of the pulse

\[
f(t) = \frac{\alpha}{\sqrt{2\pi}t_C}e^{-\frac{1}{2}(t/t_1)^2}
\]

where \(t_1\) can be taken as the time of performing a single gate. Using approximate formula \(\phi(t) \simeq f(0)t + \frac{\alpha}{2}\) one finds

\[
|F_\pm(\omega)| = \alpha \exp\{-\frac{1}{2}(\omega t_C \pm \frac{1}{\sqrt{2\pi}})^2\}.
\]

Consequently the error can be estimated as follows

\[
\delta \simeq \langle (\psi|\sigma^+\sigma^-|\psi) - |(\psi|\sigma^+|\psi)|^2\rangle \alpha^2 R_0 t_1^{-2}
\]

Taking the average over possible input states \(\psi\) we get

\[
\delta_1 \sim R_0 \alpha^2 t_1^{-2}
\]

up to a constant factor.

We execute “algorithm” with action \(A\) in \(n\) steps (gates) i.e. \(\alpha = A/n = O(1)\). It turns out that the formula (84) is additive with respect to composition of gates. To see it consider \(n\) gaussian pulses of width \(t_1\) separated with time \(\tau = mt_1\), where \(m \gg 1\). This is smooth realization of kicked dynamics considered earlier. One finds

\[
X_\alpha(\omega) = \sum_{j=1}^{n} e^{-i\omega j\tau}U^\dagger(j\tau,0)Y_\alpha^j(\omega)
\]

where

\[
Y_\alpha^j(\omega) = \int_{-\tau/2}^{\tau/2} e^{-i\omega t} \frac{d}{dt} U^\dagger(t,j\tau)S_\alpha.
\]

Now, we have for example

\[
\langle \psi|X_\alpha^\dagger(\omega)X_\beta(\omega)|\psi\rangle = \sum_{j,j'} e^{-i\omega(j-j')} \langle U(j\tau,0)|\psi|Y_\alpha^j(\omega)Y_\beta^{j'}(\omega)U(j\tau,0)|\psi\rangle.
\]

Taking large separation between gates, we obtain that the terms involving \(j \neq j'\) are averaged out by rapid phase rotation. The remaining term is nothing but the sum of single gate terms.

Consequently, for \(n\) pulses with average error \(\delta_1\) the total error is given by

\[
\delta_n \sim n\delta_1
\]

Suppose now that we would like to keep the total error below some threshold \(\epsilon\) then we obtain the following estimation for the time of computation \(t_C = n(m+1)t_1\) (where \(t_1\) is the time of single pulse)

\[
t_C \geq \frac{1}{\sqrt{\epsilon}} R_0^2 (m+1)^{3/2}.
\]

In conclusion, it is possible to keep high fidelity, at the expense of worse scaling of the physical time \(t_C \sim n^{3/2}\) in comparison with “algorithmic” time \(t_{alg} \sim n\).

Finally, one should mention, that our result is not restricted to the simple one-qubit system. If one takes two qubits, and some two qubit gate, the reasoning will be similar. If the system is \(K\)-qubit, (still with, say, at most two-qubit gates), the only difference will be the need of scaling fidelity per one qubit as \(1/K\) to get high fidelity of the total final \(K\)-qubit state. This can be achieved by further slowing down the gates. Finally, to keep high fidelity of quantum computer in vacuum we need physical time to scale as \(t_C \sim n^{3/2}\sqrt{k}\) or \(t_C \sim n\sqrt{v}\) where \(v = nK\) is complexity of the problem (or volume of the algorithm) i.e. number of steps times number of qubits needed to run the algorithm.

VII. CONCLUDING REMARKS

We have developed dynamical description of decohering quantum computer. We have obtained dependence of the structure of decoherence on quantum memory of the reservoir. The nonlocal structure of decoherence implied by vacuum memory suggests, that long quantum computation at constant error rate is impossible. Also the high frequency “bang-bang” control is not useful for typical reservoirs. Instead, one should deal with short time quantum computation by use of low frequencies. This of course requires optimization of the kind we performed in the last section. Finally, we have designed the most optimistic minimal decoherence model, and provided fidelity formula, in first order approximation. Using it we have shown how the time-energy trade-off influences scaling of physical time of computation.

In general we have argued that in description of quantum computing the Markovian approach fails, as far as linear coupling to boson field is considered. The effects are relatively small for coupling to electromagnetic vacuum (though certainly much stronger than the non-observable non-exponential decay of radiating atom). Our results can have practical meaning for the systems interacting with phonons, like e.g. quantum dots (cf. [24]), where the coupling is much stronger. Then the memory effects can be of similar order as other sources of decoherence. There are also less fundamental but practical sources of memory such as e.g. heating mechanism in ion traps [27].

Finally, we believe that our non-Markovian dynamical description of quantum computing will have some meaning for future practically useful implementations of quantum computer involving large number of qubits and relatively long time of computation.
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