Multi-dimensional dissipation strategy within advection upstream splitting methods in hypersonic flows

Shu-sheng Chen¹,², Hua Yang², Fang-jie Cai³ and Zheng-hong Gao²,*

¹ AVIC The First Aircraft Institute, Xi’an 710089, China
² School of Aeronautics, Northwestern Polytechnical University, Xi’an 710072, China
³ China Academy of Launch Vehicle Technology, Beijing 100076, China

E-mail: zgao@nwpu.edu.cn

Abstract. The work develops the multi-dimensional dissipation strategy within advection upstream splitting methods (AUSM) in hypersonic flows based on the affordable shock-stable item by Chen et al. (J. Comput. Phys. 373 (2018) 662-672). This strategy is related to the shear velocity difference and the pressure-based sensing function. It strengthens the robustness of AUSM-family schemes against shock anomalies and meantime preserves shear layer. A series of numerical cases illuminate its potential capability for hypersonic flows, particularly in the extreme situations such as the large aspect ratio of cells, skewed non-orthogonal grid, and unstructured grid.

1. Introduction
With the comprehensive advantages of efficiency and accuracy, shock-capturing schemes have been widely applied to Computational Fluid Dynamics (CFD). Among them, Advection Upstream Splitting Methods (AUSM), including AUSM+ [1], AUSMUP [2] and simple low-dissipation AUSM (SLAU) [3,4], have excellent performance in simulating many flows, but they are still prone to shock anomalies in hypersonic flows under some situations [6,7,8,9], such as the large aspect ratio of cells, skewed non-orthogonal grid, and unstructured grid. In [6], the author proposed a new approach to combat carbuncle phenomenon. Its idea is to introduce an affordable shock-stable item consistent with shear viscosity for the approximate Riemann solvers. Its shock robustness and potential application for hypersonic flows have been demonstrated by several well-known cases. Thus, the objective of the current work is to further improve the technique of [6] and develop the multi-dimensional dissipation strategy for AUSM-family schemes, accomplishing a series of efficient and robust shock-stable schemes.

2. Governing equations and AUSM-family schemes
Euler equations. The governing equations for the ideal two-dimensional Euler system in the x-direction are written as:
\[
\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = 0
\]

\[
\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix}, \quad \mathbf{F}(\mathbf{Q}) = \begin{bmatrix} \rho U \\ \rho u U + p n_x \\ \rho v U + p n_y \\ \rho h U \end{bmatrix}
\]

where \( \rho, \mathbf{u} = (u, v)^T \), \( p, e \) and \( h \) are density, velocity, pressure, total specific energy and total enthalpy. The normal velocity \( \mathbf{U} = n_x u + n_y v \) associated with the unit normal vector \( \mathbf{n} = (n_x, n_y)^T \). To close the equations, the following relationship is established by the prefect law

\[
a = \left( \frac{\gamma p}{\rho} \right)^{1/2}, \quad e = \frac{p}{\gamma - 1} + \frac{\|\mathbf{u}\|^2}{2}, \quad h = e + \frac{p}{\rho}
\]

with the specific heat ratio \( \gamma = 1.4 \). With the numerical flux \( \mathbf{F}_{i+1/2} \), this hyperbolic system is then solved numerically via conservative method

\[
\mathbf{Q}_{i+1}^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x} \left[ \mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2} \right]
\]

AUSM-family schemes. The numerical flux calculated by AUSM-family schemes [3] is generally expressed as follows:

\[
\mathbf{F}_{i+1/2} = \frac{\dot{m} + |\dot{m}|}{2} \Phi_L + \frac{\dot{m} - |\dot{m}|}{2} \Phi_R + p_s \mathbf{N}
\]

\[
\Phi = \begin{bmatrix} 1 & u & v & h \end{bmatrix}^T, \quad \mathbf{N} = \begin{bmatrix} 0 & n_x & n_y & 0 \end{bmatrix}^T
\]

with mass flux \( \dot{m} \) and pressure flux \( p_s \). The superscript “L” and “R” signify the left and right states depending on reconstruction methods. In the current study, all-speed AUSM-family schemes (AUSMUP [2] and SLAU [3]) are presented.

3. Multi-dimensional dissipation strategy

The formula for the affordable shock-stable item in [6] by the author is as follows

\[
\mathbf{S} \mathbf{V} = -g \cdot \frac{1}{2} \mathbf{\tilde{a}} \Delta \mathbf{V} \cdot \mathbf{\varphi} \cdot (0, -n_x, n_y, \mathbf{V})^T
\]

with the transverse velocity \( \mathbf{V} = -n_x u + n_y v \) and the jump of the left and right states \( \Delta(\cdot) = (\cdot)_R - (\cdot)_L \). The superscript tilde “~” stands for Roe average. The pressure-based sensing function \( g \) and \( \varphi \) are given as

\[
g = \frac{1 + \cos\left( \frac{\pi h}{2} \right)}{2}, \quad h = \min_k (h_k), \quad h_k = \min \left( \frac{P_{lk}}{P_{lk}} \cdot \frac{P_{lk}}{P_{lk}} \right)
\]

\[
\varphi = \max \left( 1 - \frac{|\tilde{M}|}{\tilde{M}}, 1 - |M_L|, 1 - |M_R|, 0 \right)
\]

Considering the following two aspects: (1) To further improve the compatibility of the affordable shock-stable item \( \mathbf{S} \mathbf{V} \) with the pressure flux \( p_s \) in AUSM-family schemes; (2) To reduce some
unnecessary computation cost by using existing variables in AUSM-family schemes. Then, Roe-averaged variables in Eq. (5) can be approximated through the relationship:

$$\frac{1}{2} \rho a \approx \frac{\gamma}{4} \frac{p_L + p_R}{a_{u/2}}, \quad \tilde{V} = \frac{V_L + V_R}{2}$$

(7)

Here, $a_{u/2}$ is the numerical sound speed of AUSM-family schemes.

In addition, the pressure-based sensing function $g$ can effectively detect shock waves and restore shear layers. And $\phi$ becomes smaller with closer to shock wave, ranging from 0 to 1. Obviously, there is a possible inconsistency between $g$ and $\phi$ at the shock waves, which reduces shock robustness. To avoid this contradiction, we can take the maximum value of $\phi$, namely

$$\phi' = \max(\phi) = 1$$

Then, combining Eq. (5) and Eq. (7)-(8), we acquire the improved item:

$$SVM = -g \cdot \frac{\gamma}{4} \frac{p_L + p_R}{a_{u/2}} \cdot \Delta V \cdot \left(0, -n_y, n_x, \frac{V_L + V_R}{2}\right)^T$$

(9)

This strategy can be considered as the multi-dimensional dissipation strategy, since it only works in multidimensional situations but not in one dimension. It tends to zero in the most computational domains, and works near the shock regions. It can be very easy to apply within AUSMUP and SLAU, resulting in the improved shock-stable versions, called AUSMUP+SVM and SLAU+SVM. In addition, there is no reason that this approach cannot be applied to other all-speed schemes, such as SLAU2 [4], LDFSS [10], etc. Then, AUSMUP+SVM and SLAU+SVM schemes are expressed

$$F_{u/2}^{AUSMUP+SVM} (Q_L, Q_R) = F_{u/2}^{AUSMUP} (Q_L, Q_R) + SVM$$

(10)

$$F_{u/2}^{SLAU+SVM} (Q_L, Q_R) = F_{u/2}^{SLAU} (Q_L, Q_R) + SVM$$

(11)

4. Numerical results

4.1. Structured grids: Hypersonic inviscid flow over a cylinder

**Figure. 1 Uniform orthogonal grid**

Hypersonic inviscid flow over a cylinder at Mach=20 is carried out to illuminate its shock stability with structured grids. Two sets of structured grids are used: Grid 1 (Figure. 1) is a uniform orthogonal grid, containing 715 (circumferential) × 41 (wall normal) grid points with the large aspect ratio of cells,
while Grid 2 (Figure. 2) is a severely skewed non-orthogonal grid with 239 (circumferential) × 121 (wall normal) grid points. For both computations, the exterior boundary employs the farfield boundary conditions and the wall uses the slip boundary conditions. All simulations are done up to 20000 time iterations by first-order accurate method and implicit LU-SGS with CFL=5. Three types of flux schemes are compared, including the original AUSM-type schemes (AUSMUP and SLAU, first type), the schemes with the original shock-stable item [6] (AUSMUP+SV and SLAU+SV, second type) and the schemes with the current multi-dimensional dissipation strategy (AUSMUP+SVM and SLAU+SVM, third type).

The pressure contours calculated by each scheme using grid 1 are shown in Figure. 3. The first type of schemes suffers from shock instability on uniform orthogonal grid, while the second and the third types of schemes get clear and smooth pressure distributions without the appearance of shock anomalies. Such results demonstrate that both the original and the improved shock-stable item can effectively enhance shock robustness and suppress carbuncle phenomenon.

Figure. 4 provides pressure contours computed by each scheme using grid 2. Under this circumstance, the first type of schemes still meets with severe shock instability. Different from the previous computation, the second type has shock anomalies, which illustrates that the original shock-stable item has defects of insufficient robustness. In comparison, the third type has higher shock instability, and the result is clear and smooth. This indicates that the current multi-dimensional dissipation strategy raises shock robustness and can better cope with extreme conditions.
The convergence histories are exhibited in Figure 5. Whether using a uniform orthogonal grid with large aspect ratio or skewed non-orthogonal grid, the residual of the first type of schemes can hardly converge while the third type reaches superior convergence histories. As for the second type, it converges well with the uniform orthogonal grid but gets poor convergence histories with the skewed non-orthogonal grid.

4.2. Odd-even decoupling
The odd-even decoupling test problem [11] is simulated to reveal the potential ability of multi-dimensional dissipation strategy against carbuncle instability. The initial condition is given as

\[ (\rho, u, v, p)_L = (1.6, 0.1), (\rho, u, v, p)_R = (5.25, 0.353, 0, 40.64) \]  (12)

The initial shock is located at \( x = 8 \). The computational domain is \([0,10] \times [0,1]\) with \(200(X) \times 80(Y)\) grid cells. The centerline grid is disturbed by small numerical error \( \pm 10^{-4} \). All simulations are done by the explicit 3rd TVD Runge-Kutta scheme with CFL=0.4 and first-order accurate method.

Figure 6 provides the density contours computed by different schemes. The odd-even decoupling test problem can trigger numerical catastrophes through the grid perturbation. AUSMUP and SLAU schemes meet with severe shock oscillations under the current large aspect ratio mesh. On the contrary, the improved AUSMUP+SVM and SLAU+SVM schemes exhibit the clean profile along the shock. The improved schemes enhance shock robustness and restrain shock anomalies.

![Figure 6 Density contours of odd-even decoupling](image)

### 4.3. Unstructured grids: Hypersonic inviscid flow over a cylinder

![Figure 7 The unstructured mesh of a cylinder](image)

![Figure 8 Convergence histories](image)

This case generalizes the application of unstructured grids under hypersonic flows and investigates shock stability of multi-dimensional dissipation strategy with unstructured grids. Hypersonic inviscid
flow over a cylinder at Mach=10 is conducted. A great of numerical experiments [9] indicate that both the all-speed AUSMUP and SLAU schemes suffer from shock anomalies with unstructured grids. As shown in Figure. 7, the computational mesh contains 3710 grid points, with farfield boundary conditions at exterior boundaries and slip boundary conditions at the wall. All simulations are done up to 20000 time iterations by first-order accurate method and implicit FGMRES technique with CFL=1.5. Figure. 8 plots the convergence histories of different schemes by using unstructured grid. The improved schemes reach the excellent convergence histories, yet the original schemes can hardly converge. Figure. 9 presents the density contours computed by each scheme with unstructured grid. Shock anomalies occur in all original schemes, but the improved schemes get clear and smooth flow fields. This proves that the multi-dimensional dissipation strategy can be adapted to hypersonic flows on unstructured grids and suppress the occurrence of shock anomalies.

5. Conclusions
The paper has developed the multi-dimensional dissipation strategy within AUSM-family schemes in hypersonic flows. This strategy is a simple and efficient way against shock anomalies. It is related to the sear velocity difference and the pressure-based sensing function. For hypersonic inviscid flow over a cylinder, the multi-dimensional dissipation strategy raises shock robustness and can better cope with extreme conditions, including the large aspect ratio of cells, skewed non-orthogonal grid, and unstructured grid. The odd-even decoupling test problem demonstrates that this strategy exhibits the clean profile along the shock. All in all, it strengthens the robustness of AUSM-family schemes against shock anomalies and deserves further researches for complex flows.

6. Acknowledgements
This research was supported by the Space Science and Technology Fund Project of China (No. 2020-HT-XG-14).

References
[1] M.S. Liou, A sequel to AUSM: AUSM+, J. Comput. Phys. 129 (1996) 364-382.
[2] M.S. Liou, A sequel to AUSM, Part II: AUSM+-up for all speeds, J. Comput. Phys. 214 (2006) 137-170.
[3] E. Shima, K. Kitamura, Parameter-free simple low-dissipation AUSM-family scheme for all speeds, AIAA J. 49 (2011) 1693-1709.
[4] K. Kitamura, E. Shima, Towards shock-stable and accurate hypersonic heating computations: A new pressure flux for AUSM-family schemes, J. Comput. Phys. 245 (2013) 62-83.
[5] A.V. Rodionov, Artificial viscosity in Godunov-type schemes to cure the carbuncle phenomenon, J. Comput. Phys. 345 (2017) 308-329.
[6] S.-S. Chen, C. Yan, B.-Xi. Lin, L.-Y. Liu, J. Yu, Affordable shock-stable item for Godunov-type schemes against carbuncle phenomenon, J. Comput. Phys. 373 (2018) 662-672.
[7] K. Chakravarthy, D. Chakraborty, Modified SLAU2 scheme with enhanced shock stability, Comput. Fluids 100 (2014) 176-184.
[8] S.-S. Chen, F.-J. Cai, H.-C. Xue, N. Wang, C. Yan, An improved AUSM-family scheme with robustness and accuracy for all Mach number flows, Appl. Math. Model. 77 (2020) 1065-1081.
[9] F. Zhang, J. Liu, B. Chen, W. Zhong, A robust low-dissipation AUSM-family scheme for numerical shock stability on unstructured grids, Int. J. Numer. Methods Fluids 84 (2016) 135-151.
[10] J.R. Edwards, Towards unified CFD simulation of real fluid flows, AIAA Paper 2001-2524, 2001.
[11] J. Quirk, A contribution to the great Riemann solver debate, Int. J. Numer. Methods Fluids 18 (1994) 555-574.