Sampling by Quaternion Reproducing Kernel Hilbert Space Embedding

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Abstract

The Whittaker-Shannon-Kotel’nikov (WSK) sampling theorem provides a reconstruction formula for the bandlimited signals. In this paper, a novel kind of the WSK sampling theorem is established by using the theory of quaternion reproducing kernel Hilbert spaces. This generalization is employed to obtain the novel sampling formulas for the bandlimited quaternion-valued signals. A special case of our result is to show that the 2D generalized prolate spheroidal wave signals obtained by Slepian can be used to achieve a sampling series of cube-bandlimited signals. The solutions of energy concentration problems in Quaternion Fourier transform are also investigated.

Keywords: Spectral theorem, prolate spheroidal wave function, quaternion reproducing-kernel Hilbert spaces, quaternion Fourier transform, Whittaker-Shannon-Kotel’nikov sampling formulae.

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1 Introduction

The sampling theory, as one of the basic and fascinating topics in engineering sciences, is crucial for reconstructing the continuous signals from the information collected at a series of discrete points without aliasing because it bridges the continuous physical signals and the discrete domain. After the celebrated Whittaker Shannon Kotel’nikov (WSK) sampling theorem established, there have been numerous proposals in the literature to generalize the classical WSK sampling expansions in various areas. The goal of this paper is to use the theory of quaternion reproducing kernel Hilbert method to obtain a generalization of WSK sampling for a general class of bandlimited quaternion-valued signals.

On the other hand, special functions [1] such as Hermite and Laguerre functions have played an important role in classical analysis and mathematical physics. In a series of papers, Slepian et al. [2, 3, 4, 5] extensively investigated the remarkable properties of the prolate spheroidal wave signals (PSWFs) which are a class of special functions. For fixed $\tau$ and $\sigma$, the PSWFs of degree $n$ denoted by $\varphi_n$ constitute an orthogonal basis of the space of $\sigma$-bandlimited signals with finite energy, that is, for continuous finite energy signals whose Fourier transforms have

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support in \([-\sigma, \sigma]\). They are also maximally concentrated on the interval \([-\tau, \tau]\) and depend on parameters \(\tau\) and \(\sigma\). PSWFs are characterized as the eigenfunctions of an integral operator with kernel arising from the sinc functions \(\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}\):

\[
\frac{\sigma}{\pi} \int_{-\tau}^{\tau} \varphi_n(x) \text{sinc}\left(\frac{\sigma}{\pi}(y - x)\right) dx = \mu_n \varphi_n(y), \quad |y| \leq \tau. \tag{1.1}
\]

It has been shown that (1.1) has solutions in \(L^2([-\tau, \tau])\) only for a discrete set of real positive values of \(\mu_n\), say \(\mu_1 > \mu_2 > \ldots\) and that \(\lim_{n \to \infty} \mu_n = 0\). The corresponding solutions, or eigenfunctions, \(\varphi_1(y), \varphi_2(y), \ldots\) can be chosen to be real and orthogonal on \((-\tau, \tau)\).

The variational problem that led to (1.1) only requires that equation to hold for \(|y| \leq \tau\). With \(\varphi_n(x)\) on the left-hand side of (1.1) gives for \(|x| \leq \tau\), however, the left is well defined for all \(y\). We use this to extend the range of definition of the \(\varphi_n\)'s and so define

\[
\varphi_n(y) := \frac{\sigma}{\pi \mu_n} \int_{-\tau}^{\tau} \varphi_n(x) \text{sinc}\left(\frac{\sigma}{\pi}(y - x)\right) dx, \quad |y| \geq \tau.
\]

The eigenfunctions \(\varphi_n\) are now defined for all \(y\). This leads to a dual orthogonality

\[
\int_{-\tau}^{\tau} \varphi_n(x) \varphi_m(x) dx = \mu_n \delta_{mn}, \tag{1.2}
\]

\[
\int_{-\infty}^{\infty} \varphi_n(x) \varphi_m(x) dx = \delta_{mn}. \tag{1.3}
\]

In [6], Zayed proved that there are other systems of functions possess similar properties to those of Prolate Spheroidal Wave Functions (PSWFs). Moumni and Zayed [7] then extended the results to the higher dimension and derived a novel sampling formula for general class of bandlimited functions. This sampling formula [7] is a generalization of Walter and Shen’s result [8] on sampling with the PSWFs. In the present paper, we study the Prolate Spheroidal Quaternion Wave Signals (PSQWSs), which refine and extend the PSWFs. The PSQWSs are ideally suited to study certain questions regarding the relationship between quaternionic signals and their Fourier transforms. We illustrate how to apply the PSQWSs for the quaternionic Fourier transform to analyze Slepian’s energy concentration problem and sampling theory. We address all the above issues and explore some basic facts of the arising quaternionic function theory.

Quaternion and Quaternionic Fourier transform have already shown advantages over complex and classical Fourier transform within color image processing, computer graphics, and robotics communities, for their modelling of rotation, orientation, and cross-information between multichannel data, see for instance, [9, 10, 11, 12]. We are motivated to develop a counterpart of the PSWFs in quaternion algebra. We apply the theory of reproducing-kernel Hilbert spaces and compact normal operators on Hilbert spaces, which are used in [7] to the Quaternion algebra. Due to the non-commutative property of the quaternion ring, there are some differences in the proof between the quaternions and complex cases.

The paper is organized as follows. In the next section, we collect some basic concepts in quaternion analysis. In Section 3, we derive a spectral theorem for compact normal operators of quaternionic Hilbert spaces. In Section 4, we introduce the PSQWSs and some of their properties. Moreover, these signals are used to obtain two sampling formulae of bandlimited...
quaternion-valued signals. In Section 5, we present an example which demonstrate our results. More importantly, the maximum-energy-problem (Slepian’s energy concentration problem) is also investigated.

2 Preliminaries

2.1 Quaternion Algebra

Throughout the paper, let\( \mathbb{H} := \{ q = q_0 + iq_1 + jq_2 + kq_3 | q_0, q_1, q_2, q_3 \in \mathbb{R} \} \), be the Hamiltonian skew field of quaternions, where the elements \( i, j, k \) obey the Hamilton’s multiplication rules:
\[ i^2 = j^2 = k^2 = ijk = -1. \]

For every quaternion \( q := q_0 + q_i + q_j + q_k \), the scalar and vector parts of \( q \), are defined as \( \text{Sc}(q) := q_0 \) and \( \text{Vec}(q) := q_i + q_j + q_k \), respectively. If \( q = \text{Vec}(q) \), then \( q \) is called pure imaginary quaternion. The quaternion conjugate is defined by \( q := q_0 - q_i - q_j - q_k \), and the norm \( |q| \) of \( q \) is defined as \( |q|^2 := qq = q_0^2 + q_1^2 + q_2^2 + q_3^2 \). Then we have
\[ \overline{q} = q, \quad \overline{p + q} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q} \overline{p}, \quad |pq| = |p||q|, \quad \forall p, q \in \mathbb{H}. \]

Using the conjugate and norm of \( q \), one can define the inverse of \( q \in \mathbb{H}\setminus\{0\} \) as \( q^{-1} := \overline{q}/|q|^2 \).

The quaternion has subsets \( \mathbb{C}_\mu := \{ a + b\mu : a, b \in \mathbb{R}, |\mu| = 1, \mu = \text{Vec}(\mu) \} \). For each fixed unit pure imaginary quaternion \( \mu \), \( \mathbb{C}_\mu \) is isomorphic to the complex plane.

2.2 Quaternion Module and Quaternionic Hilbert Space

In order to state our results, we shall need some further notations. The left quaternion module is similar with the right quaternion module, except that the quaternion ring acts on the left. The right quaternion module version of all these facts can be found e.g. in (\([13, 14]\)).

**Definition 2.1 (Quaternion module)** Let \( H \) be a left quaternion module, that is, \( H \) consists of an abelian group with a left scalar multiplication \( (q, u) \mapsto qu \) from \( \mathbb{H} \times H \) into \( H \), such that for all \( u, v \in H \) and \( p, q \in \mathbb{H} \)
\[ (p + q)u = pu + qu, \quad p(u + v) = pu + qu, \quad (pq)u = p(qu). \]

**Definition 2.2 (Quaternionic pre-Hilbert space)** A left quaternion module \( H \) is called quaternionic pre-Hilbert space if there exists a quaternion-valued function (inner product) \( (\cdot, \cdot) : H \times H \to \mathbb{H} \) with the following properties:

1) \((u, v) = (v, u)\);

2) \((pu + qv, w) = p(u, w) + q(v, w)\);

3) \((u, u) \in \mathbb{R}^+ \) and \((u, u) = 0 \) if and only if \( u = 0 \);
where \( p, q \in \mathbb{H} \) and \( u, v, w \in H \).

For each \( u \in H \), putting \( \|u\|^2 = (u, u) \), the Cauchy-Schwarz inequality and triangular inequality (see \([13]\)) hold as \( |(u, v)|^2 \leq (u, u)(v, v) \) and \( \|u + v\| \leq \|u\| + \|v\| \). The quaternionic pre-Hilbert space \( H \) is said to be a quaternionic Hilbert space if it is complete under the norm \( \| \cdot \| \). In what follows, by the notation \( H \), we mean a (left) quaternionic Hilbert space.

For each \( A \subset H \), define \( A^\perp := \{ u \in H \mid (u, v) = 0, \forall v \in A \} \) and \( \text{Span}(A) = \left\{ \sum_{k=1}^n q_k u_k : q_k \in \mathbb{H}, u_k \in A, n \geq 1 \right\} \). Define \( U(H) := \{ u \in H, \|u\| = 1 \} \). A is called an orthonormal set in \( H \) if \( A \subset U(H) \) and for any \( u, v \in A \), \( (u, v) = 0 \) for \( u \neq v \).

**Theorem 2.3** Let \( E \) be an orthonormal set in \( H \). Then the following statements are equivalent.

1) \( E \) is a maximal orthonormal set (i.e. if \( E' \) is an orthonomal set such that \( E \subset E' \), then \( E' = E \)).

2) \( E \) is total in \( H \), that is, \( \text{Span}(E) = H \).

3) \( E^\perp = \{0\} \).

4) \( u = \sum_{z \in E} (u, z) z \) holds for every \( u \in H \);

5) \( (u, v) = \sum_{z \in E} (u, z)(z, v) \) holds for all \( u, v \in H \).

6) \( \|u\| = \sum_{z \in E} |(u, z)|^2 \) holds for every \( u \in H \).

**Theorem 2.4** Let \( A \) be a left \( \mathbb{H} \)-linear subspace in \( H \). Then the following assertions hold.

1) \( A^\perp \) is a left \( \mathbb{H} \)-linear closed subspace of \( H \).

2) If \( A \) is closed, then \( A = A^\perp \perp \) and \( H = A \oplus A^\perp \), every \( u \in H \) admits a unique decomposition \( u = u_1 + u_2 \) with \( u_1 \in A \) and \( u_2 \in A^\perp \).

3) If \( A \) is closed, calling \( P_A(u) = u_1, u \in H \), we obtain that \( P_A \) is a projection operator in \( H \). Moreover \( A = \text{Range}(P_A), A^\perp = \text{Null}(P_A) \).

A left \( \mathbb{H} \)-linear operator is a map \( T : H \to H \) such that

\[
T(pu + qv) = pT(u) + qT(v)
\]

if \( p, q \in \mathbb{H} \) and \( u, v \in H \). Such an operator is called bounded if there exists a constant \( c \geq 0 \) such that for all \( u \in H \),

\[
\|Tu\| \leq c\|u\|.
\]

As in the complex case, the norm of a bounded \( \mathbb{H} \)-linear operator \( T \) is defined by

\[
\|T\| = \sup \{ \|Tu\| : \|u\| \leq 1 \}.
\]

The set of bounded left \( \mathbb{H} \)-linear operators is denoted by \( \mathcal{B}(H) \).
Proposition 2.5 Equip $B(H)$ with the metric $\text{Dist}(T_1, T_2) = \|T_1 - T_2\|$. Then $B(H)$ is a complete metric space.

For every $T \in B(H)$, the Riesz representation theorem, as proposed in [13], guarantees that there exists a unique operator $T^* \in B(H)$, which is called the adjoint of $T$, such that for all $u, v \in H$, $(Tu, v) = (u, T^*v)$.

Definition 2.6 Like the complex case, an operator $T \in B(H)$ is said to be
1) self-adjoint if $T = T^*$;
2) positive if $T$ is self-adjoint and $(Tu, u) \geq 0$ for every $u \in H$;
3) normal if $TT^* = T^*T$;
4) compact if for every bounded set $B$ of $H$, $\overline{T(B)}$ is a compact set of $H$.

The set of all compact operators on $H$ is denoted by $B_0(H)$. Clearly, $B_0(H) \subset B(H)$. Indeed, $B_0(H)$ is a closed subset of $B(H)$ (see e.g. [15]) just like the complex case.

3 A Spectral Theorem for Compact Normal Operators in Quaternionic Hilbert Space

Over the years, the spectral properties of bounded operators in quaternionic Hilbert space have been studied (see e.g. [15, 16, 17]). Ghiloni, Moretti and Perotti presented a spectral theorem for compact normal operators in right quaternionic Hilbert space in [17]. However, they did not sort the eigenvalues by norm in descending order. In this part, we use different approaches from those used in [17] to derive a spectral theorem for compact normal operators. More importantly, the eigenvalues are sorted by norm in descending order.

As in the complex case, define the eigenvalue $q$ of $T \in B(H)$ by

$$Tu = qu, \quad u \in H \setminus \{0\}.$$ 

However, as mentioned in [14], the eigenspace of $q$ cannot be a left $\mathbb{H}$-linear subspace. If $\lambda \neq 0$, then $\lambda u$ is an eigenvector of $\lambda q^{-1}$ rather than $q$. The eigenvalues of compact normal operators are not necessary to be real. It is hard to consider the spectral property of compact normal operators, because there is a big difference between the complex case and the quaternionic case to consider this problem.

Therefore, we need to consider the entire similarity orbit $\theta(q)$ of $q$ (see [16]):

$$\theta(q) := \{\lambda q \lambda^{-1} : \lambda \in \mathbb{H} \setminus \{0\}\} = \{\lambda q \lambda^{-1} : \lambda \in \mathbb{H}, |\lambda| = 1\}.$$ 

If $q \in \mathbb{R}$, then $\theta(q)$ contains only one element. In all other cases, $\theta(q)$ contains infinitely many elements. But the following lemma indicate that only two of those are complex.

Lemma 3.1 ([16]) If $q \in \mathbb{H}$ is nonreal, then there is a nonreal $\lambda \in \mathbb{C}_\mu$ such that $\theta(q) \cap \mathbb{C}_\mu = \{\lambda, \overline{\lambda}\}$. In particular, if $\lambda \in \mathbb{C}_\mu$, then $\theta(\lambda) \cap \mathbb{C}_\mu = \{\lambda, \overline{\lambda}\}$. 

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Lemma 3.2 If $$\theta(q_1) \cap \theta(q_2) \neq \emptyset$$, then $$\theta(q_1) = \theta(q_2)$$.

For each $$\theta(q)$$, define the corresponding characteristic set as

$$H_q := \{ u \in H : Tu = pu, p \in \theta(q) \}.$$  

Clearly, $$H_q$$ is a left $$\mathbb{H}$$-linear subspace of $$H$$ when $$q \in \mathbb{R}$$. As $$H_q$$ is not always a left $$\mathbb{H}$$-linear subspace of $$H$$, we call $$H_q$$ an eigenspace only when $$H_q$$ is a left $$\mathbb{H}$$-linear subspace of $$H$$.

Given the similarity between complex Hilbert space and quaternion Hilbert space, some proofs of the following results are omitted and the proofs of complex version can be found in [18, 19]. We only present the proofs of those which deserve an explicit proof.

Lemma 3.3 $$T \in \mathcal{B}(H)$$ is a compact operator if and only if for every bounded sequence $$\{u_n\} \subset H$$, the sequence $$\{Tu_n\}$$ has a convergent subsequence.

Proposition 3.4 Let $$T \in \mathcal{B}(H)$$ be a normal operator. Then the following assertion hold.

1) $$\|Tu\| = \|T^*u\|$$ for every $$u \in H$$;

2) $$Tu = qu$$ if and only if $$T^*u = \overline{q}u$$, where $$u \in H$$ and $$q \in \mathbb{H};$$

3) if $$q_1$$ and $$q_2$$ are eigenvalues of $$T$$ and $$\theta(q_1) \neq \theta(q_2)$$, then $$H_{q_1} \perp H_{q_2}.$$

Proof. 2 It can be easily seen from $$(Tu - qu, Tu - qu) = (T^*u - \overline{q}u, T^*u - \overline{q}u)$$. 3 If $$Tu = \lambda_1 u$$ ($$\lambda_1 \in \theta(q_1)$$) and $$Tv = \lambda_2 v$$ ($$\lambda_2 \in \theta(q_2)$$), we have $$T^*u = \overline{\lambda_1}u$$ and $$T^*v = \overline{\lambda_2}v$$ by 2.

Therefore

$$\lambda_1(u,v) = (Tu,v) = (u,T^*v) = (u,\overline{\lambda_2}v) = (u,v)\lambda_2.$$  

By Lemma 3.2, $$\theta(q_1) \cap \theta(q_2) = \emptyset$$, thus $$(u,v) = 0$$.  \qed

Proposition 3.5 Let $$T \in \mathcal{B}(H)$$ be a compact self-adjoint operator. Then the following assertion hold.

1) $$\|T\| = \sup \{ |\lambda| : \lambda \in V(T) \}$$, where $$V(T) = \{(Tu,u) : u \in U(H)\};$$

2) every eigenvalue of $$T$$ is real;

3) $$T$$ has an eigenvalue of absolute value $$\|T\|$$. Moreover, if $$T$$ is positive, then $$\|T\|$$ is an eigenvalue of $$T$$.

Since every eigenvalue of compact self-adjoint operator is real. So it is easy to obtain the following theorem.

Theorem 3.6 Let $$H$$ be a left quaternionic Hilbert space and $$T \in \mathcal{B}(H)$$ be a compact self-adjoint operator. Then there is a (possibly finite) sequence $$\{\lambda_k\}$$ of real numbers satisfying $$|\lambda_1| \geq |\lambda_2| \geq \cdots$$ and a sequence $$\{\xi_k\} \subset U(H)$$ such that

1) $$\{\lambda_k\}$$ are eigenvalues of $$T$$ and $$\lambda_k \to 0$$ if $$\{\lambda_k\}$$ is infinite;
2) \( \mathcal{T}\xi_k = \lambda_k\xi_k \) for every \( k \). For every \( u \in H \), we have
\[
\mathcal{T}u = \sum_{k=1}^{\infty} (u, \xi_k)\lambda_k \xi_k;
\]

3) if \( 0 \) is not an eigenvalue of \( \mathcal{T} \), then \( \{\xi_k\} \) is an orthonormal basis of \( H \);

4) \( \dim H_{\lambda_k} < \infty \) for \( \lambda_k \neq 0 \).

**Proof.** If \( \mathcal{T} = 0 \), we are done. Suppose that \( \mathcal{T} \neq 0 \), then \( \|\mathcal{T}\| \neq 0 \). By Proposition 3.5, there exists a pair \( (\lambda_1, \xi_1) \in \mathbb{R} \times U(H) \) satisfying \( |\lambda_1| = \|\mathcal{T}\| \) such that \( \mathcal{T}\xi_1 = \lambda_1\xi_1 \). Let \( W_1 = \text{Span}\{\xi_1\} \), \( V_1 = \{u \in H : u \perp \xi_1\} \), then \( W_1, V_1 \) are left \( \mathbb{H} \)-linear closed subspace of \( H \) and \( H = W_1 \oplus V_1 \). Moreover, \( \mathcal{T}(V_1) \subset V_1 \), since if \( v \in V_1 \), we have
\[
(\mathcal{T}v, \xi_1) = (v, \mathcal{T}^*\xi_1) = (v, \sum_{k=1}^{\infty} (V, \xi_k)\lambda_k \xi_1) = (v, \xi_1)\lambda_1 = 0.
\]

\( V_1 \) is still a left quaternionic Hilbert space. Define \( \mathcal{T}_1 = \mathcal{T}|_{V_1} \), then \( \mathcal{T}_1 \) a compact self-join operator on \( V_1 \). If \( \mathcal{T}_1 = 0 \), then for any \( u \in H \), we have \( u = w_1 + v_1 \), where \( w_1 \in W_1, v_1 \in V_1 \). Thus
\[
\mathcal{T}u = \mathcal{T}w_1 + \mathcal{T}v_1 = \mathcal{T}(w_1, \xi_1)\xi_1 + 0 = \mathcal{T}(u, \xi_1)\xi_1 = (u, \xi_1)\lambda_1 \xi_1.
\]

If \( \mathcal{T}_1 \neq 0 \), then there exists a pair \( (\lambda_2, \xi_2) \in \mathbb{R} \times U(V_1) \) satisfying \( |\lambda_2| = \|\mathcal{T}_1\| \) such that \( \mathcal{T}\xi_2 = \lambda_2\xi_2 \). Let \( W_2 = \text{Span}\{\xi_1, \xi_2\}, V_2 = \{u \in H : u \perp W_2\} \), then \( W_2, V_2 \) are left \( \mathbb{H} \)-linear closed subspace of \( H \) and \( H = W_2 \oplus V_2 \). Furthermore, \( \mathcal{T}(V_2) \subset V_2 \). Let \( \mathcal{T}_2 = \mathcal{T}_1|_{V_2} = 0 \), then for any \( u \in H \), we have
\[
\mathcal{T}u = (u, \xi_1)\lambda_1 \xi_1 + (u, \xi_2)\lambda_2 \xi_2.
\]

If \( \mathcal{T}_2 \neq 0 \), continue the above procedure. If there is a \( n \in \mathbb{N} \) such that \( \mathcal{T}_n = 0 \), then
\[
\mathcal{T}u = \sum_{k=1}^{n} (u, \xi_k)\lambda_k \xi_k.
\]

Otherwise, \( \lambda_k \) is infinite, and \( |\lambda_1| \geq |\lambda_2| \geq \cdots \). From the definitions of \( W_n \) and \( V_n \), we have \( \xi_k \perp \xi_l \) for \( k \neq l \). Furthermore, \( \lambda_k \to 0 \) as \( k \to \infty \). If not, there is a subsequence \( \{\lambda_{s_m}\} \) such that \( |\lambda_{s_m}| \geq \delta > 0 \). Then
\[
\|\mathcal{T}\xi_{s_m} - \mathcal{T}\xi_{s_n}\|^2 = \|\mathcal{T}\xi_{s_m}\|^2 + \|\mathcal{T}\xi_{s_n}\|^2 = |\lambda_{s_m}|^2 + |\lambda_{s_n}|^2 \geq 2\delta^2 > 0,
\]

which is contradict with the compactness of \( \mathcal{T} \). Let \( W_\infty = \text{Span}\{\xi_k : k \geq 1\} \) and \( V_\infty = \{u \in H : u \perp W_\infty\} \subset V_k \) \( (k = 1, 2, \cdots) \). If \( u \in V_\infty \), then
\[
\|(\mathcal{T}u, u)\| = \|T_k u, v\| \leq \|T_k\|\|u\|^2 = |\lambda_k|\|u\|^2 \to 0,
\]
as \( k \to \infty \). Therefore, \( (\mathcal{T}u, u) = 0 \) for every \( u \in V_\infty \). Given \( u \in U(V_\infty) \), let \( v \in U(V_\infty) \) be such that \( \mathcal{T}u = \|\mathcal{T}u\|v \). Then \( (\mathcal{T}u, v) = (u, \mathcal{T}v) = \|\mathcal{T}u\| \) and thus
\[
\|\mathcal{T}u\| = (\mathcal{T}u, v) = \frac{1}{4} [(\mathcal{T}(u+v), u+v) - (\mathcal{T}(u-v), u-v)] = 0.
\]
Hence $T|_{V_\infty} = 0$. Therefore, for every $u \in H$, let $u = w_\infty + v_\infty$, where $w_\infty \in W_\infty$ and $v_\infty \in V_\infty$, we have
\[
Tu = T w_\infty + T v_\infty = T \left( \sum_{k=1}^{\infty} (w_\infty, \xi_k) \xi_k \right) + 0
\]
\[
= \sum_{k=1}^{\infty} (w_\infty, \xi_k) T \xi_k = \sum_{k=1}^{\infty} (w_\infty, \xi_k) \lambda_k \xi_k
\]
\[
= \sum_{k=1}^{\infty} (u, \xi_k) \lambda_k \xi_k.
\]
The third equality holds for the boundness of $T$. If $0$ is not an eigenvalue of $T$, then $H = W_\infty$ and $\{\xi_k\}$ is an orthonormal basis of $H$. $\dim H_{\lambda_k} < \infty$ for $\lambda_k \neq 0$ as $T$ is compact. \qed

**Theorem 3.7** Let $H$ be a left quaternionic Hilbert space and $T \in B(H)$ be a compact positive (self-join) operator. Then there is a (possibly finite) sequence $\{\mu_k\}$ of nonnegative real numbers satisfying $\mu_1 > \mu_2 > \cdots$ such that

1) $\{\mu_k\}$ are eigenvalues of $T$ and $\mu_k \to 0$ if $\{\mu_k\}$ is infinite;

2) if $H_k = \{u \in H : Tu = pu, p \in \theta(\mu_k)\}$, then $H_k \perp H_l$ for $k \neq l$ and $\dim H_k < \infty$ for $\mu_k \neq 0$.

An exposition of the spectral theory of normal matrices with quaternion entries was presented in [16] by Farenick and Pidkowich. They also obtained a spectral theorem for compact operators in $n$-dimensional quaternionic Hilbert spaces by establishing relations between compact operators and quaternion normal matrices.

**Theorem 3.8** Assume that $H$ is an $n$-dimensional quaternionic Hilbert space. Then an operator $T : H \to H$ is normal if and only if there exists an orthonormal set $E = \{\xi_1, \xi_2, \ldots, \xi_n\} \subset U(H)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}_1^+$ such that:

1) $T \xi_k = \lambda_k \xi_k$;

2) $Tu = \sum_{k=1}^{n} (u, \xi_k) \lambda_k \xi_k$ for any $u \in H$.

**Remark 3.9** Theorem 3.8 is just the left version of Farenick and Pidkowich’s result in [16]. Indeed, $\lambda_k$ is not necessary to be restricted in $\mathbb{C}_1^+$, it can be any elements of $\theta(\lambda_k)$.

**Theorem 3.10** If $n$ is a positive integer and $H_0$ is an $n$-dimensional subspace of a quaternionic Hilbert space $H$, then

1) every one-to-one $\mathbb{H}$-linear mapping of $\mathbb{H}^n$ onto $H_0$ is an isomorphism, and

2) $H_0$ is closed, that is, $H_0$ is complete.
From Theorem 3.7, we see that every eigenspace belonging to the non-zero eigenvalues of a compact positive operator is finite-dimensional. Noticing that every normal operator in finite-dimensional quaternionic Hilbert spaces can be diagonalized, we obtain the following spectral theorem for compact normal operator in quaternionic Hilbert spaces.

**Theorem 3.11** Let $H$ be a (left) quaternionic Hilbert space and $T \in B(H)$ be a compact normal operator. Then there is a (possibly finite) sequence $\{\lambda_k\}$ of quaternions satisfying $|\lambda_1| \geq |\lambda_2| \geq \cdots$ and a sequence $\{\xi_k\} \in U(H)$ such that

1) $\{\lambda_k\}$ are eigenvalues of $T$ and $|\lambda_k| \to 0$ if $\{\lambda_k\}$ is infinite;
2) $T \xi_k = \lambda_k \xi_k$ and $K \xi_k = |\lambda_k|^2 \xi_k$ for every $k$, where $K = TT^*$. For every $u \in H$, we have

$$T u = \sum_{k=1}^{\infty} (u, \xi_k) \lambda_k \xi_k \quad \text{and} \quad Ku = \sum_{k=1}^{\infty} (u, \xi_k) |\lambda_k|^2 \xi_k;$$

3) if 0 is not an eigenvalue of $T$, then $\{\xi_k\}$ is an orthonormal basis of $H$.

**Proof.** Let $H_1, H_2, \ldots$ be the eigenspaces belonging to the non-zero eigenvalues $\mu_1 > \mu_2 > \cdots$ of the compact positive operator $K = TT^* = T^*T$, and let $H_0 = \text{Ker} K$. Since $KT = TK$, each $H_n(n \geq 1)$ is invariant under $T$, i.e. $T(H_n) \subset H_n$. For $u \in H_0$ we have

$$\|Tu\|^2 = (Tu, Tu) = (T^*Tu, u) = (Kx, x) = 0$$

and so $T|H_0 = 0$. Furthermore, for $n \geq 1$ the restriction of $T$ to $H_n$ is normal; as $H_n$ is finite-dimensional, we know from Theorem 3.8 that $H_n$ has an orthonormal basis consisting of eigenvectors of $T$. Denote the dimension of $H_n(n \geq 1)$ by $l_n$ and let $l_0 = 0$, $s_n = \sum_{k=0}^{n} l_n$, then we can find $\lambda_{s_n-1+1}, \lambda_{s_n-1+2}, \ldots, \lambda_{s_n} \in \mathbb{H}$ and $\{\xi_{s_n-1+1}, \xi_{s_n-1+2}, \ldots, \xi_{s_n}\} \subset H_n$ such that

1) $T \xi_k = \lambda_k \xi_k$ for every $s_{n-1} < k \leq s_n$;
2) $K \xi_k = |\lambda_k|^2 \xi_k$ for every $s_{n-1} < k \leq s_n$;
3) $|\lambda_k|^2 \in \theta(\mu_n) = \{\mu_n\}$, i.e. $|\lambda_k|^2 = \mu_n$ for every $s_{n-1} < k \leq s_n$.

Taking the union of these bases, together with an orthonormal basis of $H_0$, we obtain an orthonormal basis consisting of eigenvectors of $T$. □

**Remark 3.12** Unlike the complex cases, $\lambda_k$ can be replaced any element in $\theta(\lambda_k)$ of Theorem 3.11.

## 4 Prolate Spheroidal Quaternion Wave Signals

Recently, the existence and uniqueness conditions of quaternion reproducing kernel Hilbert spaces (QRKHS) were established by Tabar and Mandic in [20]. They also introduced the quaternion version of positive definiteness kernel and Moore-Aronszajn theorem.
To study the integral transforms in quaternionic Hilbert spaces, we follow similar techniques to those used in [21] by Saitoh. Let $X$ be an arbitrary set and $\mathfrak{F}(X)$ a $\mathbb{H}$-linear space composed of all $\mathbb{H}$-valued signals on $X$. Let $H$ be a (possibly finite-dimensional) quaternionic Hilbert space with inner product $(\cdot, \cdot)_H$, and $E : X \to H$ be a vector-valued function from $X$ into $H$. Consider the left $\mathbb{H}$-linear mapping $T$ from $H$ into $\mathfrak{F}(X)$ defined by

$$f(x) = (TF)(x) = (F, E(x))_H,$$

where $TF = f$, $F \in H$, $f \in \mathfrak{F}(X)$.

Let $\mathcal{H}$ and $N$ denote the range and null space of $T$. Then $N$ is a closed subspace of $H$. Denote $M = N^\perp$, then $H = M \oplus N$.

**Theorem 4.1** $(\mathcal{H}, (\cdot, \cdot)_\mathcal{H})$ is a quaternionic Hilbert space that is isometric to $(M, (\cdot, \cdot)_H)$, where

$$(f, g)_\mathcal{H} = (TF, TG)_\mathcal{H} = (P_M F, P_M G)_H. \quad (4.1)$$

Moreover, $(\mathcal{H}, (\cdot, \cdot)_\mathcal{H})$ is a quaternion reproducing kernel Hilbert space with the kernel $S(x, y)$ defined as

$$S(x, y) = (E(y), E(x))_H.$$ 

**Proof.** The proof of this theorem is similar with its complex version, but for the sake of the convenience we present the proof here. It is easy to see that $T$ is a bijection from $M$ into $H$. It implies that for every $f \in \mathcal{H}$, there exists a unique element $F^M_f \in M$ such that $TF^M_f = f$. We can show that $\|F^M_f\|^2_H = \inf\{\|F\|^2_H : f = TF\}$. It is obviously that $\|F^M_f\|^2_H \geq \inf\{\|F\|^2_H : f = TF\}$. On the other hand, for every $F \in \{F \in H : TF = f\}$, it can be decomposed as $F = F^M_f + F'$, where $F' \in N$. Then

$$\|F\|^2_H = (F^M_f + F', F^M_f + F')_H$$

$$= \|F^M_f\|^2_H + (F', F^M_f)_H + (F^M_f, F')_H + \|F'\|^2_H$$

$$\geq \|F^M_f\|^2_H.$$

Thus $\inf\{\|F\|^2_H : f = TF\} \geq \|F^M_f\|^2_H$. Hence $\|f\|^2_H = \|F^M_f\|^2_H = \inf\{\|F\|^2_H : f = TF\}$ and $T |_M$ is an isometry between $(M, (\cdot, \cdot)_H)$ and $(\mathcal{H}, (\cdot, \cdot)_\mathcal{H})$.

Next, we prove that for every $x \in X, E(x) \in M$. Noticing that when $F \in N$, then for every $x \in X$ we have

$$f(x) = (F, E(x))_H = 0.$$

That means for every $x \in X, E(x) \in N^\perp = M$ and $P_M E(x) = E(x)$. Since $S(x, y) = (E(y), E(x))_H = T(E(y))(x)$, thus

$$(f, S(\cdot, y))_\mathcal{H} = (TF, T(E(y))_\mathcal{H} = (P_M F, P_M E(y))_H$$

$$= (F^M_f, E(y))_H = TF^M_f(y) = f(y)$$

which completes the proof. \qed
For \( \mathbb{H} \)-valued signals \( f, g : D \to \mathbb{H} \) where \( D \) is a compact connected subset of \( \mathbb{R}^d \), we can define the \( \mathbb{H} \)-valued inner product \( (f, g) = \int_D f(\omega)\overline{g(\omega)}d\omega \). The left \( \mathbb{H} \)-linear quaternionic Hilbert space \( L^2(D, \mathbb{H}) \) consists of all \( \mathbb{H} \)-valued signals which are square-integrable on \( D \):

\[
L^2(D, \mathbb{H}) = \left\{ f | f : D \to \mathbb{H}, \| f \| := \int_D |f(\omega)|^2d\omega < \infty \right\}.
\]

Now we apply Theorem 4.1 to a specific case. Let \( H = L^2(D, \mathbb{H}) \) and \( X \) be an open, connected subset of \( \mathbb{R}^d \) containing \( D \). The \( \mathbb{H} \)-valued function \( E(\omega, x) \) on \( D \times X \) satisfying \( E(\omega, x) \in L^2(D, \mathbb{H}) \) for any \( x \in X \). In the next, we consider the integral transform of \( F \in L^2(D, \mathbb{H}) \),

\[
f(x) = (TF)(x) = \int_D F(\omega)|E(\omega, x)d\omega.
\]

**Theorem 4.2** Suppose that \( E(\omega, x) \) is square-integrable on \( D \times D \). Then \( T \in B_0(H) \) where \( H = L^2(D, \mathbb{H}) \) and \( T \) is given by (4.2).

**Proof.** Clearly, \( T \) is linear. Since

\[
\int_D |f(x)|^2dx = \int_D \int_D F(\omega)|E(\omega, x)d\omega|^2 dx \\
\leq \int_D \left( \int_D |F(\omega)|^2 d\omega \int_D |E(\omega, x)|^2 d\omega \right) dx \\
\leq \int_D |F(\omega)|^2 d\omega \int_{D^2} |E(\omega, x)|^2 d\omega dx < \infty.
\]

Therefore \( f(x) \in H \). Since \( \|TF\|^2 \leq \alpha^2 \|F\|^2 \), where

\[
\alpha = \left( \int_{D^2} |E(\omega, x)|^2 d\omega dx \right)^{\frac{1}{2}},
\]

then \( T \) is bounded.

Suppose \( \{e_n(\omega)\} \) is an orthonormal basis of \( H \). It is easily checked that \( \overline{e_n(x)}e_m(\omega) \) is an orthonormal basis of \( L^2(D^2, \mathbb{H}) \). Thus \( E(\omega, x) = \sum_{m,n} c_{mn}e_n(x)e_m(\omega) \) where

\[
c_{mn} = \int_{D^2} E(\omega, x)e_n(x)e_m(\omega)d\omega dx \\
= \int_{D^2} E(\omega, x)\overline{e_m(\omega)}e_n(x)d\omega dx.
\]

From Parseval’s identity, we have \( \sum_{m,n} |c_{mn}|^2 = \alpha^2 < \infty \). For each \( e_m(\omega) \in H \), \( g_m(x) = \overline{(Te_m)(x)} \in H \), then

\[
g_m(x) = \sum_n d_{mn}e_n(x),
\]

where

\[
d_{mn} = \int_D g_m(x)e_n(x)dx \\
= \int_D \left( \int_D e_m(\omega)\overline{E(\omega, x)}d\omega \right) e_n(x)dx \\
= \int_{D^2} E(\omega, x)e_m(\omega)e_n(x)d\omega dx = c_{mn}.
\]
Since \( \sum_{m,n} |c_{mn}|^2 \) converges, then
\[
\lim_{m \to \infty} \|Te_m\|^2 = \lim_{m \to \infty} \|g_m\|^2 = \lim_{m \to \infty} \sum_{n=1}^{\infty} |c_{mn}|^2 = 0.
\]

In fact, for the same reason, we have
\[
\lim_{k \to \infty} \sum_{m>k} \|Te_m\|^2 = \lim_{k \to \infty} \sum_{m>k} \sum_{n=1}^{\infty} |c_{mn}|^2 = 0.
\]

For every \( x \in D \), \( E(\omega, x) \in H \), thus
\[
E(\omega, x) = \sum_{n=1}^{\infty} (E(\cdot, x), e_n(\cdot))e_n(\omega) = \sum_{n=1}^{\infty} Te_n(x)e_n(\omega).
\]

Truncate the kernel \( E(\omega, x) \) by
\[
E_k(\omega, x) = \sum_{n=1}^{k} Te_n(x)e_n(\omega) = \sum_{n=1}^{k} g_n(x)e_n(\omega),
\]

These give the finite-rank operators
\[
(\mathcal{T}_k F)(x) = \int_{D} F(\omega)\overline{E_k(\omega, x)}d\omega, \quad \text{for every} \quad F(\omega) \in H.
\]

We claim that \( \{\mathcal{T}_k\} \) tend to \( \mathcal{T} \) in operator norm. Indeed, for every \( F(\omega) = \sum b_n e_n(\omega) \) in \( H \), we have
\[
((\mathcal{T} - \mathcal{T}_k) F)(x) = (F, E(\cdot, x)) - (F, E_k(\cdot, x))
= \sum_{n=1}^{\infty} (F, e_n(\cdot))e_n(\omega) - (F, \sum_{n=1}^{k} Te_n(x)e_n)
= \sum_{n=1}^{\infty} (F, e_n)Te_n(x) - \sum_{n=1}^{k} (F, e_n)Te_n(x)
= \sum_{n>k} b_n Te_n(x).
\]

Therefore
\[
\| (\mathcal{T} - \mathcal{T}_k) F \| = \left\| \sum_{n>k} b_n Te_n \right\|
\leq \sum_{n>k} |b_n| \| Te_n \|
\leq \left( \sum_{n>k} |b_n|^2 \right)^{\frac{1}{2}} \left( \sum_{n>k} \| Te_n \|^2 \right)^{\frac{1}{2}}
\leq \| F \| \left( \sum_{n>k} \| Te_n \|^2 \right)^{\frac{1}{2}}.
\]

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As \( \lim_{k \to \infty} \sum_{n > k} \|T e_n\|^2 = 0 \), we have \( \lim_{k \to \infty} T_k = T \).

Since every finite-rank operator is compact, and \( \mathcal{B}_0(H) \) is a closed subset of \( \mathcal{B}(H) \), thus \( \mathcal{T} \) is compact.

\[ \square \]

**Remark 4.3** By Theorem 4.1 and Theorem 4.2, we see that \( \mathcal{T} \) can be regarded an operator of both \( \mathcal{B}(H, \mathcal{H}) \) and \( \mathcal{B}(H) \).

**Theorem 4.4** Let \( E(\omega, x) \) be a \( \mathbb{H} \)-valued continuous function defined on \( X^2 \subset \mathbb{R}^{2d} \) satisfying the following conditions:

1) \( E(\omega, x) = E(x, \omega) \) for every \( (\omega, x) \in X^2 \).
2) \( A = \{E(\omega, x)\}_{x \in X} \subset H = L^2(D, \mathbb{H}) \) and \( A^\perp = \{0\} \).
3) \( S(x, y) = \int_D E(\omega, y) \overline{E(\omega, x)} d\omega \in \mathbb{R} \) for every \( (x, y) \in X^2 \);
4) \( \int_D E(\omega, y) \overline{E(\omega, x)} d\omega = \int_D E(\omega, x) E(\omega, y) d\omega \) for every \( (x, y) \in X^2 \).

Then we can find a countably infinite set of \( \mathbb{H} \)-valued signals \( \{\phi_n(x)\}_{n=1}^\infty \) called Prolate Spheroidal Quaternion Wave Signals(PSQWSs) and a set of \( \mathbb{H} \)-valued numbers \( |\lambda_1| \geq |\lambda_2| \geq \cdots \) with the following properties:

1) The \( \phi_n(x) \) \( (n = 1, 2, \cdots) \) are \( D \)-bandlimited in the sense of transformation \( T \), orthonormal and complete in \( \mathcal{H} \), where \( \mathcal{H} \) is a quaternionic Hilbert space with the inner product defined by (4.1):

\[ (\phi_m, \phi_n)_\mathcal{H} = \delta_{mn}. \]

2) The \( \phi_n(x) \) \( (n = 1, 2, \cdots) \) are orthogonal and complete in \( L^2(D, \mathbb{H}) \):

\[ \int_D \phi_m(\omega) \overline{\phi_n(\omega)} d\omega = \mu_n \delta_{mn}, \quad \text{with} \quad \mu_n = |\lambda_n|^2. \]

3) For every \( x \in X \), we have

\[ \int_D \phi_n(\omega) \overline{E(\omega, x)} d\omega = \lambda_n \phi_n(x), \quad (4.3) \]

\[ \int_D \phi_n(\omega) S(x, y) d\omega = \mu_n \phi_n(x). \quad (4.4) \]

4) The \( \phi_n(x) \) \( (n = 1, 2, \cdots) \) are uniformly continuous on \( X_1 \), where \( X_1 \) being any compact subset of \( X \); and, hence \( \phi_n(x) \) is continuous on \( X \).

5) \( E(\omega, x) \) and \( S(x, y) \) can be expanded by \( \phi_n \):

\[ E(\omega, x) = \sum_{n=1}^\infty \phi_n(x) \lambda_n^{-1} \phi_n(\omega) \quad (\omega, x) \in D \times X \quad (4.5) \]

and

\[ S(x, y) = \sum_{n=1}^\infty \phi_n(y) \phi_n(x) \quad (x, y) \in X^2. \quad (4.6) \]
For any fixed \( x \in X \), the series (4.5) converges in the norm. The series (4.6) converges absolutely on \( X \times X \). Furthermore, if \( E(\omega, x) \) is bounded on \( D \times X \), then the series (4.6) converges uniformly on \( X_1 \times X_2 \), where \( X_1, X_2 \) being any compact subsets of \( X \).

6) If there exists a sequence of points \( \{x_n\} \subset X \) such that \( \{E(\omega, x_n)\} \) is an orthonormal basis of \( L^2(D, \mathbb{H}) \), then for any \( f \in \mathcal{H} \),

\[
f(x) = \sum_{n=1}^{\infty} f(x_n) S(x, x_n) \tag{4.7}
\]

and

\[
f(x) = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} f(x_m) \overline{\phi_n(x_m)} \right) \phi_n(x). \tag{4.8}
\]

7) The signals \( \{\phi_n\} \) satisfy the discrete orthogonality relation

\[
\sum_{n=1}^{\infty} \overline{\phi_n(x)} \phi_n(x_m) = \delta_{ml};
\]

and

\[
\int_D S(x, x) dx = \sum_{n=1}^{\infty} \mu_n.
\]

8) For every \( f(x) \in \mathcal{H} \) with \( \|f\|_H \neq 0 \), we form the ratio

\[
\beta_f = \frac{\|f(x)\|_H^2}{\|f(x)\|_H^2} \tag{4.9}
\]

If we put \( \tilde{\beta} = \sup \{\beta_f : f \in \mathcal{H}, \|f\|_H \neq 0\} \), then \( \tilde{\beta} = \mu_1 \).

**Proof.** Since \( S(x, y) = \int_{D} E(\omega, x) \overline{E(\omega, y)} d\omega = S(y, x) \), and \( S \) is real, then \( S(x, y) = S(y, x) \).

We consider the integral transform \( T \) defined by (4.2). Since \( E(\omega, x) \) is continuous on compact set \( D \times D \), by Theorem 4.2, we have \( T \in B_0(H) \) when \( x \) is restricted to \( D \). We now prove that \( T \) is normal. Consider its adjoint operator \( T^* \). Indeed,

\[
T^* F(x) = \int_D F(\omega) E(x, \omega) d\omega,
\]

since

\[
(F_1, T^* F_2)_H = \int_D F_1(\omega) \overline{\left( \int_D F_2(y) E(\omega, y) dy \right)} d\omega = \int_D \int_D F_1(\omega) E(\omega, y) \overline{F_2(y)} dy d\omega = \int_D TF_1(y) \overline{F_2(y)} dy = (TF_1, F_2)_H.
\]

Therefore

\[
T(T^* F)(x) = T \left( \int_D F(\omega) E(y, \omega) d\omega \right) (x) = \int_D F(\omega) E(y, \omega) \overline{E(y, x)} dy d\omega = \int_D F(\omega) S(x, \omega) d\omega.
\]

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and

\[ T^*(TF)(x) = T^* \left( \int_D F(\omega)E(\omega, y) d\omega \right)(x) = \int_D F(\omega)E(\omega, y)E(x, \omega) d\omega dy = \int_D F(y)S(x, y) dy. \]

Hence, \( T^*T = TT^* \). Let \( T^*T = TT^* = K \), then

\[ g(x) = KF(x) = \int_D F(y)S(x, y) dy, \quad x \in D. \] (4.10)

Thus, by Theorem 3.11, there is a sequence \( \{\lambda_n\} \) of quaternions satisfying \(|\lambda_1| \geq |\lambda_2| \geq \cdots \) and a sequence \( \{\Phi_n\}_{n=1}^\infty \in U(H) \), such that

\[ \int_D \Phi_n(\omega)\overline{E(\omega, x)} d\omega = \lambda_n \Phi_n(x), \quad x \in D \] (4.11)

and

\[ \int_D \Phi_n(y)S(x, y) dy = |\lambda_n|^2 \Phi_n(x) = \mu_n \Phi_n(x), \quad x \in D. \] (4.12)

Since \( A^\perp = 0 \), which implies the null space of \( T \) is \( \{0\} \) and \( \lambda_n \neq 0 \) for every \( n \). Hence, \( \{\Phi_n\}_{n=1}^\infty \) is an orthonormal basis of \( H \).

From Theorem 4.1, we see that \( T \) is an isometry between \( (H, (\cdot, \cdot)_H) \) and \( (\mathcal{H}, (\cdot, \cdot)_\mathcal{H}) \). Set

\[ \phi_n(x) = (T \Phi_n)(x), \quad x \in X. \]

Then \( \{\phi_n\}_{n=1}^\infty \) is an orthonormal basis of \( \mathcal{H} \) as \( \{\Phi_n\}_{n=1}^\infty \) is an orthonormal basis of \( H \). Moreover, \( \phi_n(x) = \lambda_n \Phi_n(x) \) for every \( x \in D \) from (4.11). Thus \( \phi_n(x) \) is an extension of \( \lambda_n \Phi_n \).

Furthermore,

\[ (\phi_m, \phi_n)_\mathcal{H} = (\Phi_m, \Phi_n)_H = \delta_{mn}, \]

and

\[ \int_D \phi_m(\omega)\overline{\phi_n(\omega)} d\omega = \int_D \lambda_n \Phi_m(\omega)\overline{\lambda_n \Phi_m(\omega)} d\omega = \lambda_n (\Phi_m, \Phi_n)_H \overline{\lambda_n} = \mu_n \delta_{mn}. \]

If we regard \( T \) as an operator of \( B(H, \mathcal{H}) \), then \( \mathcal{K} = TT^* \) is also an operator of \( B(H, \mathcal{H}) \).

Thus \( x \in D \) of (4.10) can be replaced by \( x \in X \). Since \( T^*\Phi_n = \overline{\lambda_n} \Phi_n \), then

\[ \overline{\lambda_n} T \Phi_n = T (\overline{\lambda_n} \Phi_n) = TT^* \Phi_n = K \Phi_n. \]

Hence, \( \phi_n(x) = \overline{\lambda_n}^{-1} K \Phi_n(x) \). From the definition of \( T \) and \( K \), we easily find

\[ \lambda_n \phi_n(x) = \lambda_n T \Phi_n(x) = \lambda_n \int_D \Phi_n(\omega)\overline{E(\omega, x)} d\omega = \int_D \lambda_n \Phi_n(\omega)\overline{E(\omega, x)} d\omega = \int_D \phi_n(\omega)\overline{E(\omega, x)} d\omega \]

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and 

\[
\begin{align*}
\mu_n \phi_n(x) & = \lambda_n \lambda_n^{-1} K \Phi_n(x) \\
& = \int_D \lambda_n \Phi_n(y) S(x, y) dy \\
& = \int_D \phi_n(y) S(x, y) dy.
\end{align*}
\]

Then we obtain (4.3) and (4.4).

For given \( F \in H \), \( f(x) = T F(x) \), we have

\[
|f(x) - f(y)| = \left| \int_D F(\omega) \left[ E(\omega, x) - E(\omega, y) \right] d\omega \right| 
\leq \|F\|_H \left[ \int_D |E(\omega, x) - E(\omega, y)|^2 d\omega \right]^\frac{1}{2}
\]

Since \( E(\omega, x) \) is continuous on compact set \( D \times X_1 \), then \( E(\omega, x) \) is uniformly continuous on \( D \times X_1 \). Thus \( \forall \varepsilon > 0, \exists \delta > 0 \), when \( x, y \in X_1 \) and \( d(x, y) < \delta \),

\[
|E(\omega, x) - E(\omega, y)| < \frac{\varepsilon}{\|F\|_H \cdot (\rho(D))^{\frac{1}{2}}}
\]

holds for every \( \omega \in D \), where \( \rho(D) \) is the Lebesgue measure of \( D \). Therefore

\[
|f(x) - f(y)| < \varepsilon.
\]

Hence, \( f(x) \) is uniformly continuous on \( X_1 \).

Since \( \{ \Phi_n \}_{n=1}^\infty \) is an orthonormal basis of \( H \), thus

\[
\begin{align*}
E(\omega, x) & = \sum_{n=1}^\infty (E(\cdot, x), \Phi_n(\cdot))_H \Phi_n(\omega) \\
& = \sum_{n=1}^\infty (\Phi_n(\cdot), E(\cdot, x))_H \Phi_n(\omega) \\
& = \sum_{n=1}^\infty \phi_n(x) \lambda_n^{-1} \phi_n(\omega),
\end{align*}
\]

\[
\begin{align*}
S(x, y) & = (E(\cdot, y), E(\cdot, x))_H \\
& = \sum_{n=1}^\infty (E(\cdot, y), \Phi_n(\cdot))_H (\Phi_n(\cdot), E(\cdot, x))_H \\
& = \sum_{n=1}^\infty \phi_n(y) \phi_n(x).
\end{align*}
\]

If \( E(\omega, x) \) is continuous and bounded on \( D \times X \), suppose that \( |E(\omega, x)| < M_1 \) for every \( (\omega, x) \in D \times X \). Then

\[
0 < S(x, x) = \sum_{n=1}^\infty |\phi_n(x)|^2 \\
= (E(\cdot, x), E(\cdot, x))_H \\
= \int_D |E(\omega, x)|^2 d\omega < M_1^2 \cdot (\rho(D)) < \infty.
\]
We conclude that the convergence of the theorem, the convergence of \( (\sum_{n=1}^{\infty} |\phi_n(y)|^2 ) \) to \( S(y, y) \) on \( X \) must be uniform. The dependence upon \( y \) of \( N(y, \varepsilon) \) is thus actually extrinsic, from which fact the desired uniform (and absolute) convergence of the series to \( S(x, y) \) on \( X \times X \) immediately follows.

Since \( \{ E_n(\omega) = E(\omega, x_n) \} \) is an orthonormal basis of \( H \), we have \( F(\omega) = \sum_{n=1}^{\infty} (F, E_n)_H E_n(\omega) \). Set \( F_m(\omega) = \sum_{n \leq m} (F, E_n)_H E_n(\omega) \) then \( F_m \) converges to \( F \) in the norm as \( m \to \infty \). Note that \( f(x_n) = (F, E_n)_H \) and \( S(x, x_n) = (E_n, E(\cdot, x)_H \), therefore

\[
\mathcal{T} F_m(x) = \sum_{n \leq m} (F, E_n)_H (E_n, E(\cdot, x)_H
\]

\[
= \sum_{n \leq m} f(x_n) S(x, x_n).
\]

We conclude that \( \mathcal{T} F_m(x) \) converges uniformly to \( \mathcal{T} F(x) \) on \( X \), that is, (4.7) converges uniformly on \( X \), since

\[
\begin{align*}
|\mathcal{T} F(x) - \mathcal{T} F_m(x)|^2 &= \left| \int_D [F(\omega) - F_m(\omega)] \overline{E(\omega, x)} d\omega \right|^2 \\
&\leq \int_D |F(\omega) - F_m(\omega)|^2 d\omega \cdot \int_D |E(\omega, x)|^2 d\omega \\
&\leq \|F - F_m\|^2_H \cdot M^2 \cdot \varrho(D).
\end{align*}
\]

Since \( \{ E_n(\omega) \} \) is an orthonormal basis of \( H \), then

\[
f(x) = (F, E_x)_H
\]

\[
= \sum_{n=1}^{\infty} (F, \Phi_n)_H (\Phi_n, E_x)_H
\]

\[
= \sum_{n=1}^{\infty} (F, \Phi_n)_H \phi_n(x)
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} (F, E_m)_H (E_m, \Phi_n)_H \right) \phi_n(x)
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} f(x_m) \phi_n(x_m) \right) \phi_n(x)
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} f(x_m) \phi_n(x_m) \right) \phi_n(x).
\]
On the one hand, \( S(x_m, x_t) = \sum_{n=1}^\infty \phi_n(x_t)\phi_n(x_m) \); on the other hand, \( S(x_m, x_t) = (E_t, E_m)_H = \delta_{ml} \). Therefore, 
\[
\sum_{n=1}^\infty \phi_n(x_t)\phi_n(x_m) = \delta_{ml}.
\]

Since \( S(x, x) = \sum_{n=1}^\infty |\phi_n(x)|^2 \) converges uniformly on \( D \), thus
\[
\int_D S(x, x)\,dx = \int_D \sum_{n=1}^\infty |\phi_n(x)|^2\,dx = \sum_{n=1}^\infty \int_D |\phi_n(x)|^2\,dx = \sum_{n=1}^\infty \mu_n.
\]

Suppose that \( f = TF \). From (4.1), we have
\[
\|f\|_H^2 = \|F\|_H^2 = \sum_{n=1}^\infty |a_n|^2,
\]
where \( a_n = (F, \Phi_n)_H \). As \( f(x) \in L^2(D, \mathbb{H}) \) when \( x \) is restricted in \( D \), it can be expanded into a series \( f(x) = \sum_{n=1}^\infty b_n \Phi_n(x) \) where
\[
b_n = \int_D f(x)\Phi_n(x)\,dx = \int_D \left( \int_D F(\omega)E(\omega, x)\,d\omega \right) \Phi_n(x)\,dx = \int_D F(\omega) \left( \int_D \Phi_n(x)E(\omega, x)\,dx \right)\,d\omega = \int_D F(\omega) \Phi_n(\omega)\,d\omega = (\int_D F(\omega)\Phi_n(\omega)\,d\omega) \lambda_n = a_n \lambda_n.
\]

Thus
\[
\|f\|_H^2 = \sum_{n=1}^\infty |a_n|^2 |\lambda_n|^2 = \sum_{n=1}^\infty |a_n|^2 \mu_n.
\]

Since \( \{\mu_n\} \) is monotonically decreasing and positive, Therefore
\[
\beta_f = \frac{\sum_{n=1}^\infty |a_n|^2 \mu_n}{\sum_{n=1}^\infty |a_n|^2} \leq \mu_1.
\]

Taking \( f(x) = \phi_1(x) \), we have \( \beta_{\phi_1} = \mu_1 \). Thus \( \tilde{\beta} = \mu_1 \).

5 Examples

Let \( \tau, \sigma > 0 \), \( D = [-\tau, \tau]^2 \), \( X = \mathbb{R}^2 \), \( E(\omega, x) = \frac{1}{2\pi} e^{-i\sigma x_1 \omega_1 / \tau} e^{-i\sigma x_2 \omega_2 / \tau} \). Then
\[
S(x, y) = \frac{\sin \sigma(x_1 - y_1) \sin \sigma(x_2 - y_2)}{\sigma(x_1 - y_1) \sigma(x_2 - y_2)}.
\]
We consider the following finite modified inverse quaternion Fourier transform

\[
f(x_1, x_2) = TF(x) = \frac{1}{2\pi} \int_{\mathbb{D}} F(\omega_1, \omega_2) e^{i\sigma x_2 \omega_2/\tau} e^{i\sigma x_1 \omega_1/\tau} d\omega_1 d\omega_2
\]

where \(F(\omega_1, \omega_2) \in L^2(\mathbb{D}, \mathbb{H})\). It is easy to see that \(f(x_1, x_2)\) is \(\sigma\)-bandlimited in QFT sense by variable substitution. Moreover, all the conditions of Theorem 4.4 are satisfied. Then we can find a countably infinite set of \(\mathbb{H}\)-valued signals \(\{\phi_n(x)\}_{n=1}^\infty\) called Prolate Spheroidal Quaternion Wave Signals (PSQWSs) and a set of \(\mathbb{H}\)-valued numbers \(|\lambda_1| \geq |\lambda_2| \geq \cdots\) such that for every \((x_1, x_2) \in \mathbb{R}^2\),

\[
\frac{\lambda_n}{2\pi} \phi_n(x_1, x_2) = \frac{1}{2\pi} \int_{[-\tau, \tau]^2} \phi_n(\omega_1, \omega_2) e^{i\sigma x_2 \omega_2/\tau} e^{i\sigma x_1 \omega_1/\tau} d\omega_1 d\omega_2
\]

and

\[
\int_{[-\tau, \tau]^2} \phi_n(y_1, y_2) \sin \sigma(x_1 - y_1) \sin \sigma(x_2 - y_2) \sigma(x_1 - y_1) \sigma(x_2 - y_2)^2 dy_1 dy_2 = |\lambda_n|^2 \phi_n(x_1, x_2).
\]

Since \(\{E(\omega, x_{n_1} n_2) = \frac{1}{2\pi} e^{-i\omega_1 \sigma n_1} e^{-i\omega_2 \sigma n_2}\}_{(n_1, n_2) \in \mathbb{Z}^2}\) is an orthonormal basis of \(L^2(\mathbb{D}, \mathbb{H})\), then for any \(f \in \mathcal{H}\), \(f(x_1, x_2)\) equals to

\[
\sum_{n_1, n_2} f(\frac{\pi}{\sigma} n_1, \frac{\pi}{\sigma} n_2) \sin(\sigma x_1 - n_1 \pi) \sin(\sigma x_2 - n_2 \pi) \sigma(x_1 - n_1 \pi) \sigma(x_2 - n_2 \pi)
\]

and

\[
\sum_{n=1}^\infty \left( \sum_{m_1, m_2} f(\frac{\pi}{\sigma} m_1, \frac{\pi}{\sigma} m_2) \phi_n(\frac{\pi}{\sigma} m_1, \frac{\pi}{\sigma} m_2) \right) \phi_n(x_1, x_2).
\]

From Plancherel theorem, we have \(f \in L^2(\mathbb{R}^2, \mathbb{H})\) and

\[
\int_{\mathbb{R}^2} f(x_1, x_2)g(x_1, x_2) dx_1 dx_2 = \frac{\pi^2}{\sigma^2} \int_{[-\tau, \tau]^2} F(\omega_1, \omega_2) \overline{G(\omega_1, \omega_2)} d\omega_1 d\omega_2.
\]

Notice that the inner product of \(\mathcal{H}\) is defined by

\[
(f, g)_\mathcal{H} = (F, G)_\mathcal{H}.
\]

Thus

\[
(f, g)_\mathcal{H} = \frac{\sigma^2}{\pi^2} \int_{\mathbb{R}^2} f(x_1, x_2) \overline{g(x_1, x_2)} dx_1 dx_2
\]

and \((\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})\) is a subspace of \(L^2(\mathbb{R}^2, \mathbb{H})\). By Theorem 4.1, we have \(f(x_1, x_2)\) equals to

\[
\int_{\mathbb{R}^2} f(y_1, y_2) \sin \sigma(x_1 - y_1) \sin \sigma(x_2 - y_2) \sigma(x_1 - y_1) \sigma(x_2 - y_2) dy_1 dy_2
\]
for any \( f(x_1, x_2) \in \mathcal{H} \). Furthermore,

\[
\sup \left\{ \frac{\int_{[-\tau, \tau]^2} |f(x_1, x_2)|^2 dx_1 dx_2}{\int_{\mathbb{R}^2} |f(x_1, x_2)|^2 dx_1 dx_2} \right\} = \frac{\sigma^2 \lambda_1^2}{\pi^2}.
\]

The extrema is reached if \( f(x_1, x_2) = T \Phi_1(x_1, x_2) = \phi_1(x_1, x_2) \).

**Remark 5.1** Suppose that \( \{\varphi_n(x)\} \) and \( \{\alpha_n\} \) are classical PSWFs and corresponding eigenvalues with parameters \( \tau, \sigma \) respectively. Let \( \phi_{mn}(x_1, x_2) = \varphi_m(x_1)\varphi_n(x_2) \) and \( \lambda_{mn} = \frac{\sigma^2}{\pi^2} \alpha_m \alpha_n \). Then \( \{\phi_{mn}(x_1, x_2), \lambda_{mn}\} \) satisfy (5.1) and (5.2). Moreover, \( \phi_{mn}(x_1, x_2) \) are orthogonal and complete in \( L^2(D, \mathbb{H}) \) and \( \lambda_{00} \) is the maximum of eigenvalues of (5.1).

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