Critical behaviour of the random-field Ising model with long-range interactions in one dimension

Ivan Balog$^1$, Gilles Tarjus$^2$ and Matthieu Tissier$^2$

$^1$ Institute of Physics, PO Box 304, Bijenička cesta 46, HR-10001 Zagreb, Croatia
$^2$ LPTMC, CNRS-UMR 7600, Université Pierre et Marie Curie, boîte 121, 4 Pl. Jussieu, 75252 Paris cédex 05, France
E-mail: balog@ifs.hr, tarjus@lptmc.jussieu.fr and tissier@lptmc.jussieu.fr

Received 21 July 2014
Accepted for publication 31 August 2014
Published 9 October 2014

Online at stacks.iop.org/JSTAT/2014/P10017
doi:10.1088/1742-5468/2014/10/P10017

Abstract. We study the critical behaviour of the 1D random field Ising model (RFIM) with long-range interactions ($\propto r^{-(d+\sigma)}$) by the nonperturbative functional renormalization group. We find two distinct regimes of critical behaviour as a function of $\sigma$, separated by a critical value $\sigma_c$. What distinguishes these two regimes is the presence or not of a cusp-like nonanalyticity in the functional dependence of the renormalized cumulants of the random field at the fixed point. This change of behaviour can be associated with the characteristics of the large-scale avalanches present in the system at zero temperature. We propose some ways to check these predictions through lattice simulations. We also discuss the differences with the RFIM on the Dyson hierarchical lattice.

Keywords: classical phase transitions (theory), renormalisation group, disordered systems (theory), avalanches (theory)

ArXiv ePrint: 1407.4891
1. Introduction

The random field Ising model (RFIM) has been the focus of intense investigation as one of the paradigms of criticality in the presence of quenched disorder [1, 2]. It has found applications in physics and physical chemistry as well as in interdisciplinary fields such as biophysics and socio- and econo-physics. The model displays a phase transition, a paramagnetic-to-ferromagnetic one in the language of magnetic systems and the long-distance physics is dominated by disorder-induced sample-to-sample fluctuations rather than by thermal fluctuations. In the renormalization-group (RG) sense, the critical behaviour is then controlled by a fixed point at zero temperature and its universal properties can also be studied by investigating the model at zero temperature as a function of disorder strength.

One of the central issues arising in the RFIM was the so-called dimensional-reduction property, according to which the critical behaviour of the random system is the same as that of the pure system in a dimension reduced by 2. This was found at all orders of perturbation theory [3, 4] and was related to the presence of an underlying supersymmetry [5]. The property fails in a low dimension, in particular in $d = 3$ [6, 7] and a resolution of the problem was found within the framework of the nonperturbative functional RG (NP-FRG) [8–10]. Within the NP-FRG, the breakdown of dimensional reduction and the
associated spontaneous breaking of the underlying supersymmetry are attributed to the appearance of a strong enough nonanalytic dependence, a 'cusp', in the dimensionless renormalized cumulants of the random field at the fixed point.

What appears specific to random-field systems among the disordered models whose long–distance behaviour is controlled by a zero-temperature fixed point for which perturbation theory predicts the $d \to d - 2$ dimensional-reduction property, as, e.g., interfaces in a random environment [11–13], is the existence of two distinct regimes of the critical behaviour separated by a nontrivial value of the dimension [8,10], or of the number of components in the O(N) model [9], or else of the power-law exponent of the interactions for the long-range models [14]. This is actually what requires the RG treatment to be both functional and nonperturbative (hence, the NP-FRG).

It was shown in [15] that these two regimes are related to the large-scale properties of the ‘avalanches’, which are collective phenomena present at zero temperature. In equilibrium, such ‘static’ avalanches describe the discontinuous change in the ground state of the system at values of the external source that are sample-dependent. At the critical point, the avalanches always take place on all scales. However, whether or not they induce a cusp in the dimensionless renormalized cumulants of the random field at the fixed point depends on their scaling properties and more specifically on the fractal dimension $d_f$ of the largest typical avalanches at criticality compared to the scaling dimension of the total magnetization [15].

In the short-range RFIM the change in critical behaviour appears in a large, non-integer dimension $d \approx 5.1$ [8–10], which is therefore not accessible to lattice simulations. The interest in introducing long-range interactions is to reduce the dimensions where a phase transition can be observed and to provide an additional control parameter with the power-law exponent governing the spatial decay of the interactions. In [14], a 3D RFIM with both long-range interactions and long-range correlations of the random field was studied. The long–distance decays were chosen in such a way that the supersymmetry which is responsible for dimensional reduction is still present in the theory. It was then shown that the spontaneous breaking of the supersymmetry and the associated breakdown of dimensional reduction do take place in this $3-d$ model and that two regimes of critical behaviour are present and separated by a nontrivial value of the power-law exponents [14].

In this work, we further reduce the dimension of interest to $d = 1$ by considering the RFIM with long-range interactions but short-range correlations of the disorder. This model should be even more accessible to lattice simulations (see for instance the recent studies in [16,17]). This should allow an independent check of the scenario derived from the NP-FRG.

In the model the interactions decay with distance as $r^{-(d+\sigma)}$ with $\sigma > 0$. In some sense, varying $\sigma$ has a similar effect to that of changing the spatial dimension in the short-range case. For $\sigma \leq \sigma_G = 1/3$, the critical behaviour is governed by a Gaussian fixed point and the exponents are therefore the classical (mean-field) ones in the presence of long-range interactions: $\sigma_G$ is the analogue of an upper critical dimension. On the other hand, heuristic arguments predict that no phase transition takes place for $\sigma \geq \sigma_M = 1/2$ [17,18]: $\sigma_M$ is then the analogue of a lower critical dimension. The interesting range is therefore $1/3 \leq \sigma < 1/2$. (Rigorous results prove the existence of a phase transition for $2 - (\ln 3/\ln 2) \approx 0.4150\cdots < \sigma < 1/2$ [19] and it is even more likely that a transition exists for smaller values of $\sigma$.)
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

We investigate through the NP-FRG whether this 1D long-range RFIM displays, as the other random-field models studied so far, two regimes of critical behaviour separated by a nontrivial value of $\sigma$. In this model, there is of course no $d \to d - 2$ dimensional-reduction property and no associated supersymmetry. The two regimes should therefore be characterized by the presence or not of a cusp in the functional dependence of the (dimensionless) renormalized cumulants of the random field at the fixed point.

We do find two distinct regimes separated by the critical value $\sigma_c \approx 0.379$: a ‘cuspless’ one below $\sigma_c$ and a ‘cuspy’ one above. We calculate the critical exponents, which do not show any significant change of behaviour around $\sigma_c$. Of greater significance is the variation of the fractal dimension $d_f$ of the largest critical avalanches and we discuss how to assess the validity of the predictions in computer simulations. We also discuss the RFIM on a Dyson hierarchical model with the same parameter $\sigma$. Based on the results given in the literature [20, 21], we conclude that this model displays a unique cuspless regime over the whole range of $\sigma$, with the avalanche-induced cuspy contributions always being subdominant at the fixed point.

2. Model and NP-FRG formalism

We study the 1D random-field Ising model with power-law decaying long-range ferromagnetic interactions [3, 18]. It is described by the following Hamiltonian:

$$H = - \sum_{ij} J_{ij} s_i s_j - \sum_i h_i s_i$$  \hspace{1cm} (1)

where $s_i = \pm 1$ are Ising spins placed on the vertices of a lattice, with a ferromagnetic pairwise interaction decaying as

$$J_{ij} \propto |x_i - x_j|^{-(d+\sigma)}$$  \hspace{1cm} (2)

at long distance. The exponent $\sigma > 0$ characterizes the long-range power-law decay of the interaction and $d$ is the spatial dimensionality. In the present case $d = 1$ the disorder is introduced by the random fields $h_i$, which are independently distributed with a Gaussian distribution of width $\Delta_B$ around a zero mean:

$$P(h_i) = \frac{1}{\sqrt{2\pi \Delta_B^2}} e^{-\frac{h_i^2}{2\Delta_B^2}}.$$  \hspace{1cm} (3)

To make use of the NP-FRG formalism and investigate the long–distance properties of the model near its critical point, it is more convenient to reformulate the model in the field-theory setting. By using standard manipulations, the Hamiltonian in equation (1) is replaced by the ‘bare action’ for a scalar field $\varphi$:

$$S[\varphi; h] = S_B[\varphi] - \int_x h(x)\varphi(x)$$

$$S_B = \int_x \left[ \frac{\nu}{2!} \varphi(x)^2 + \frac{u}{4!} \varphi(x)^4 \right] + \frac{1}{2} \int_{x,y} J(|x - y|)\varphi(x)\varphi(y)$$  \hspace{1cm} (4)

doi:10.1088/1742-5468/2014/10/P10017
where \( \int_x = \int d^d x \) and \( h(x) \) is the continuous version of the random magnetic field with 
\( h(x) = 0 \) and \( \overline{h(x)h(y)} = \Delta_B \delta^{(d)}(x - y) \) (where as usual the overline denotes the average
over the quenched disorder). The interaction \( \mathcal{J}(x) \) goes as \( |x|^{-(d+\sigma)} \) at a large distance
and, accordingly, its Fourier transform behaves as

\[
\mathcal{J}(p) = \mathcal{J}[p]^{\sigma} + O(p^2)
\]

when \( p \to 0 \).

From here, the NP-FRG equations can be derived by following two routes. In the first
one, the system is considered directly at zero temperature and one builds a superfield
formalism that can account for the fact that the equilibrium properties are given by
the ground state of the model \([10, 14]\). In the second one, the system is considered at
finite temperature and one works with the Boltzmann weight \([8, 9]\). The main advantage
of the former is that it makes the underlying supersymmetry that is responsible for the
\( d \to d - 2 \) dimensional-reduction property explicit and allows one to study its spontaneous
breaking along the NP-FRG flow. The latter one is however quite simpler to present and
in the following we will follow that route. We stress that the two derivations lead to the
same exact NP-FRG equations for the zero-temperature fixed point controlling the critical
behaviour of the RFIM.

Due to the presence of quenched disorder the generating functional of the (connected)
correlation functions, \( \mathcal{W}_h[J] = \ln \int \mathcal{D} \varphi \exp(-S[\varphi; h] + \int_x J(x)\varphi(x)) \), is random and can then be characterized by its cumulants. The latter are conveniently studied by considering copies or replicas of the system (see e.g. \([8, 10]\)) which, unlike the standard replica trick \([22]\), are each coupled to a distinct external source. After averaging over the disorder, the resulting ‘multicopy’ generating functional \( \mathcal{W}[\{J_a\}] = \ln \prod_a \exp(\mathcal{W}_h[J_a]) \) is given by

\[
e^{\mathcal{W}[\{J_a\}]} = \int \prod_a \mathcal{D} \varphi_a \exp\left\{ \sum_a \left( -S_B[\varphi_a] + \int_x J_a(x)\varphi_a(x) \right) - \Delta_B \sum_{a,b} \int_x \varphi_a(x)\varphi_b(x) \right\}. \tag{6}
\]

The cumulants are generated by expanding in increasing number of unrestricted (or free)
sums over replicas:

\[
\mathcal{W}[\{J_a\}] = \sum_a \mathcal{W}_1[J_a] + \frac{1}{2} \sum_{a,b} \mathcal{W}_2[J_a, J_b] + \cdots, \tag{7}
\]

where \( \mathcal{W}_{k1}[J_a] = \overline{\mathcal{W}_{h,k}[J_a]} \) is the first cumulant, \( \mathcal{W}_{k2}[J_a, J_b] = \overline{\mathcal{W}_{h,k}[J_a]\mathcal{W}_{h,k}[J_b]} \) the second cumulant, etc.

The formulation of the NP-FRG proceeds by modifying the partition function in
equation (6) with the introduction of an ‘infrared regulator’ that suppresses the integration
over the modes with momentum less than some cutoff \( k \) and takes the form of a generalized ‘mass’ (quadratic) term added to the bare action \([23]\):

\[
\Delta S_k[\{\varphi_a\}] = \frac{1}{2} \sum_{a,b} \int_{x,y} \varphi_a(x)\mathcal{R}_{k,ab}(|x - y|)\varphi_b(y), \tag{8}
\]

where \( \mathcal{R}_{k,ab} = \overline{\mathcal{R}_k\delta_{ab}} + \overline{\mathcal{R}_k} \) with \( \overline{\mathcal{R}_k} \) and \( \overline{\mathcal{R}}_k \) two functions enforcing an infrared cut-off on the fluctuations \([10]\). Through this procedure, one defines the generating functional of the (connected) correlation functions at scale \( k \), \( \mathcal{W}_k[\{J_a\}] \).

doi:10.1088/1742-5468/2014/10/P10017
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

In the NP-FRG approach, the central quantity is the ‘effective average action’ $\Gamma_k$, which is the generating functional of the one-particle irreducible (1PI) correlation function or vertices and is obtained from $W_k[\{J_a\}]$ through a (modified) Legendre transform:

$$\Gamma_k[\{\phi_a\}] + \Delta S_k[\{\phi_a\}] = -W_k[\{J_a\}] + \sum_a \int_x J_a(x) \phi_a(x),$$

where the field $\phi_a = \delta W_k / \delta J_a(x)$ is the average of the physical field in copy $a$.

Similar to $W_k[\{J_a\}]$, $\Gamma_k[\{\phi_a\}]$ can be expanded in an increasing number of free replica sums,

$$\Gamma_k[\{\phi_a\}] = \sum_a \Gamma_{k1}[\phi_a] - \frac{1}{2} \sum_{a,b} \Gamma_{k2}[\phi_a, \phi_b] + \cdots,$$

where the $\Gamma_{kp}$’s are essentially the cumulants of the renormalized disorder [8, 10].

The $k$ dependence of the effective average action is governed by an exact renormalization-group (RG) equation [23, 24]:

$$\partial_k \Gamma_k[\{\phi_a\}] = \frac{1}{2} \text{Tr} \tilde{\partial}_{k} \ln (\Gamma_k^{(2)}[\{\phi_a\}] + \mathcal{R}_k),$$

where $t = \ln(k/\Lambda)$, $\tilde{\partial}_k$ is a symbolic notation indicating a derivative acting only on the $k$ dependence of the cut-off functions (i.e. $\tilde{\partial}_k \equiv \partial_k \tilde{R}_k / \partial_k \hat{R}_k + \partial_k \hat{R}_k / \partial_k \tilde{R}_k$) and $\Gamma_k^{(2)}$ is the second functional derivative of the effective average action with respect to the replica fields (Generically, superscripts indicate functional differentiation with respect to the field arguments.). Finally, the trace involves summing over copy indices and integrating over spatial coordinates.

The initial condition of the RG flow when $k = \Lambda$, the microscopic scale (e.g. the inverse of the lattice spacing), is provided by the bare action and when $k \to 0$ one recovers the effective action of the full theory with all fluctuations accounted for.

After expanding both sides of equation (11) in an increasing number of free replica sums, one obtains a hierarchy of exact RG flow equations for the cumulants of the renormalized disorder. For instance, the equations for the first two cumulants read

$$\partial_k \Gamma_{k1}[\phi_a] = \frac{1}{2} \tilde{\partial}_k \text{tr} \left\{ \ln \hat{G}_k[\phi_a] + \hat{G}_k[\phi_a] \left( I_{k2}^{(11)}[\phi_a, \phi_a] - \tilde{R}_k \right) \right\},$$

$$\partial_k \Gamma_{k2}[\phi_a, \phi_b] = \frac{1}{2} \tilde{\partial}_k \left\{ - \Gamma_{k3}^{(101)}[\phi_a, \phi_b, \phi_a] \hat{G}_k[\phi_a] + \Gamma_{k2}^{(20)}[\phi_a, \phi_b] \hat{G}_k[\phi_a, \phi_a] + \Gamma_{k2}^{(11)}[\phi_a, \phi_b] - \tilde{R}_k \right\} \hat{G}_k[\phi_a, \phi_b] + \text{perm}(a, b),$$

where $\text{perm}(a, b)$ denotes the terms obtained by permuting the two indices $a$ and $b$ and the trace now only involves integrating over the spatial coordinates and the ‘propagators’ $\hat{G}_k$ and $G_k$ are defined as

$$\hat{G}_{k;x_1x_2}[\phi_a] = \left( I_{k1}^{(2)}[\phi_a] + \tilde{R} \right)^{-1} \big|_{x_1, x_2}$$

$$\hat{G}_{k;x_1x_2}[\phi_a, \phi_b] = - \int_{x_3x_4} \hat{G}_{k;x_1x_3}[\phi_a] \left( \Gamma_{k2}^{(11)}[\phi_a, \phi_b] - \tilde{R}_k(\{x_3 - x_4\}) \right) \hat{G}_{k;x_4x_2}[\phi_b].$$

doi:10.1088/1742-5468/2014/10/P10017
Up to now the RG equations are exact but they represent an infinite hierarchy of coupled functional equations and require approximations to be solved.

3. Nonperturbative ansatz

To truncate the exact hierarchy of functional RG equations, a systematic nonperturbative approximation scheme has been proposed and successfully applied to the short-range RFIM [8,10] and the 3-d RFIM with both long-range interactions and long-range random-field correlations [14]. It consists of formulating an ansatz for the effective average action that relies on truncating the expansion in cumulants and approximating the spatial dependence of the fields through a truncated expansion in gradients (or in fractional Laplacians).

We have adapted this approximation scheme to the present 1D long-range model. The main specificities of this model compared to the random-field systems studied before through the NP-FRG are that

(a) the underlying supersymmetry that is responsible for the \( d \to d-2 \) dimensional-reduction property, i.e. the super-rotational invariance, is not present (and dimensional reduction is of course not an issue in \( d = 1 \)) and

(b) the small momentum dependence of the 2-point 1-copy 1PI vertex function acquires anomalous terms.

An efficient ansatz that can capture the long-distance physics including the influence of rare events such as avalanches is then

\[
\Gamma_{k1}[\phi_a] = \int_x \left\{ U_k(\phi_a(x)) + \frac{1}{2} J_k(\phi_a(x)) \phi_a(x) (-\partial_x^2)_{\!\!\!\!\!\!\!\!\cdot}^{1/2} \phi_a(x) + \frac{1}{2} Z_k(\phi_a(x)) \phi_a(x) (-\partial_x^2)_{\!\!\!\!\!\!\!\!\cdot}^{1+2\sigma} \phi_a(x) \right\}, \\
\Gamma_{k2}[\phi_a, \phi_b] = \int_x V_k(\phi_a(x), \phi_b(x)), \\
\Gamma_{kp\geq3} = 0
\]  

where \((-\partial_x^2)^{\alpha}\), with \(\alpha\) a real number, denotes a fractional Laplacian: its Fourier transform generates a \((p^2)^{\alpha}\) term and for \(\alpha = 1\) it reduces to the usual Laplacian. The first of the fractional-Laplacian terms, \((-\partial_x^2)^{\frac{1}{2}}\), directly stems from the long-range interaction (see equation (5)). The second fractional-Laplacian term, \((-\partial_x^2)^{1+2\sigma}\), is specific to the present 1D long-range case. As will be shown later, this term is generated under renormalization and for the range of \(\sigma\) under consideration, with, \(\sigma < \frac{1}{2}\), it dominates at long-distance the conventional \((-\partial_x^2)^{\frac{1}{2}}\) term (in other words, \(|p|^{1+2\sigma}\) is dominant in the infrared compared to \(p^2\)).
When expressed at the level of the 2-point 1PI vertices, the above ansatz leads to

\[ I_{k; x_1 x_2}^{(2)}[\phi_a] = \left\{ U_k''(\phi_a(x_1)) + \lambda_k(\phi_a(x_1))(-\partial_{x_1}^2)^{\frac{\lambda}{2}} + \frac{1}{2} \lambda_k(\phi_a(x_1))(-\partial_{x_2}^2)^{\frac{\lambda}{2}} \phi_a(x_1) \right. \]
\[ + Y_k(\phi_a(x_1))(-\partial_{x_1}^2)^{\frac{1+2\sigma}{2}} + \frac{1}{2} Y_k(\phi_a(x_1))(-\partial_{x_1}^2)^{\frac{1+2\sigma}{2}} \phi_a(x_1) \left. \right\} \delta(x_1 - x_2), \]  
\[ I_{k; x_1 x_2}^{(11)}[\phi_a, \phi_b] = \Delta_k(\phi_a(x_1), \phi_b(x_1)) \delta(x_1 - x_2), \]  

where \( \lambda_k(\phi_a) = \partial_{\phi_a}[\mathcal{J}_k(\phi_a)\phi_a] \), \( Y_k(\phi_a) = \partial_{\phi_a}[\mathcal{Z}_k(\phi_a)\phi_a] \) and \( \Delta_k(\phi_a, \phi_b) = V_k^{(11)}(\phi_a, \phi_b) \).

For the present 1-d long-range model, we make a further simplifying step which is to set the cut-off function \( \tilde{R}_k \) to zero. The fluctuations are still suppressed in the infrared by the cut-off function \( \tilde{R}_k \) [8, 9]. The role of \( \tilde{R}_k \) is mainly to ensure that super-rotational invariance is not explicitly broken at the level of the regulator. As there is no such supersymmetry in the present case (see above), it is not crucial to keep \( \tilde{R}_k \) and we find it is more convenient to drop it.

The flow of the functions \( U_k'' \), \( \lambda_k \), \( Y_k \) and \( \Delta_k \) appearing in equations (19) and (20) can be obtained by inserting the ansatz in the exact RG flow equations in the equations (12) and (13), then considering uniform configurations of the replica fields and working in Fourier space. For instance, the flow equation for the 1-copy 2-point 1PI vertex becomes, in a graphical representation,

\[ \partial_t \Gamma_{k_1}^{(2)}(p; \phi) = -\frac{1}{2} \partial_t \int_q \left( \begin{array}{c}
q
\phi
p
\end{array} \right) + \begin{array}{c}
q+p
\phi
2x
p
\end{array} + \begin{array}{c}
q+p
\phi
2x
p
\end{array}, \]  

where lines denote the propagator \( \hat{G}_k \), dots the one-copy 1PI vertices and dots linked by dotted lines the two-copy 1PI vertices. The internal momentum is denoted by \( q \) and the operator \( \partial_t \) now acts only on \( \hat{G}_k \) through its dependence on \( \tilde{R}_k \); \( \partial_t \hat{G}_k(q; \phi) = -\hat{G}_k(q; \phi)^2 \partial_t \tilde{R}_k(q) \). To derive the above equation, we have used the fact that the expressions for the propagators \( \hat{G}_k \) and \( \hat{G}_k \) in equations (14) and (15) can be simplified for uniform fields, with

\[ \hat{G}_k(p; \phi_a, \phi_b) = \hat{G}_k(p; \phi_a) \hat{G}_k(p; \phi_b) \Delta_k(\phi_a, \phi_b) \]  

and

\[ \hat{G}_k(p; \phi) = \frac{1}{\lambda_k(\phi)|p|^{\sigma} + Y_k(\phi)|p|^{1+2\sigma} + \tilde{R}_k(p) + U_k''(\phi)}. \]  

The RG flow of \( U_k'' \) is then obtained from equation (21) when \( p = 0 \) and those of \( \lambda_k \) and \( Y_k \) are obtained by expanding the right-hand side of equation (21) in small \( p \) and identifying the anomalous dependence in \( |p|^{\sigma} \) and \( |p|^{1+2\sigma} \). A similar procedure with the 2-copy 2-point 1PI vertex for \( p = 0 \) allows one to derive the flow of \( \Delta_k(\phi_a, \phi_b) \).

\[ \text{doi:10.1088/1742-5468/2014/10/P10017} \]
We find that the flow equation for \(\lambda_k(\phi)\) is such that if \(\lambda_k\) is independent of the field \(\phi\) in the initial condition (see equation (5)), it does not flow and remains equal to its bare value for all RG times. So, without a loss of generality, we can set \(\lambda_k = 1\). On the other hand, the flow equations for \(U_k^p(\phi)\) and \(\Delta_k(\phi, \phi)\) are given in appendix A.

Finally, the derivation of the flow equation for the function \(Y_k(\phi)\) needs some special care and demonstrates why the small-momentum dependence is described by a term \(\propto |p|^{1+2\sigma}\) (in the range \(\sigma < -\frac{1}{2}\)). The key point is that even if one starts the RG flow with an initial condition where such a term is absent, it is generated along the flow.

To obtain the flow equation for \(Y_k\) one needs to isolate all of the terms contributing to the order \(|p|^{1+2\sigma}\) when the right-hand side of equation (21) is expanded in small \(p\). There are two types of terms:

(a) The vertex terms—obtained by collecting the \(|p|^{1+2\sigma}\) dependence from the 1PI vertices and dropping the \(p\) dependence of the propagators. These terms are proportional to \(Y_k\) and its derivatives.

(b) The anomalous terms—produced by the singular momentum dependence in \(|p|^\sigma\) present in the propagators appearing in the 1-loop integrals. This point is further explained for a toy model in the appendix B. These terms generate a contribution to the effective average action \(\propto |p|^{1+2\sigma}\) even if it is not present in the initial condition of the flow.

The expression for the flow equation of the function \(Y_k\) can therefore be written as the sum of a vertex contribution \(\beta_{Y,ve}\) and an anomalous \(\beta_{Y,an}\), \(\partial_t Y_k(\phi) = \beta_{Y,ve}(\phi) + \beta_{Y,an}(\phi)\). The derivation of \(\beta_{Y,ve}\) is relatively straightforward and is not detailed whereas that of \(\beta_{Y,an}\) is sketched in the appendix C. We provide here the final expressions:

\[
\beta_{Y,ve}(\phi) = \int_0^{+\infty} \frac{dk}{\pi} \Delta_k(\phi, \phi)^2 \left\{ \left( \Delta_k^{(10)}(\phi, \phi) + \Delta_k^{(01)}(\phi, \phi) \right) \times Y'_k(\phi) + \Delta_k(\phi, \phi) \left( Y''_k(\phi) - 3 \hat{G}_k(q; \phi) Y_k''(\phi) |U_k''(\phi) + q^{1+2\sigma} Y_k'(\phi)| \right) \right\}, \quad (24)
\]

\[
\beta_{Y,an}(\phi) = \frac{2^{2\sigma} \sigma^2 \Gamma\left(\frac{1}{2} - \sigma\right)}{\sqrt{\pi} \Gamma(2 - \sigma)} \partial_t \hat{R}_k(0) \hat{G}_k(0; \phi) U_k''(\phi) \left( 5 \hat{G}_k(0; \phi) U_k''(\phi) \Delta_k(\phi, \phi) - 2 [\Delta_k^{(10)}(\phi, \phi) + \Delta_k^{(01)}(\phi, \phi)] \right). \quad (25)
\]

Note that the expression of \(\beta_{Y,an}(\phi)\) is nonzero even when \(Y_k = 0\).

### 4. Fixed-point equations

To describe the long-distance physics near the critical point and search for fixed points of the NP-FRG equations we first need to cast the latter in a dimensionless form. As the critical physics is related to a ‘zero-temperature’ fixed point, one needs to introduce a renormalized temperature \(T_k\) and an associated critical exponent \(\theta > 0\), such that \(T_k \propto k^\theta\) [8, 9]. The renormalized cumulants then scale as \(\Gamma_{k1} \sim T_k^{-1}\), \(\Gamma_{k2} \sim T_k^{-2}\) and the
replica fields as $\phi_a \sim k^{(d-2+\eta)/2} T_k^{-1/2}$. Since $\lambda_k$, the term in $|p|^\sigma$ in the 2-point 1PI vertex is not renormalized (see above), one immediately derives that the anomalous dimension $\eta$ is always given by $\eta = 2 - \sigma$.

One can thus introduce dimensionless quantities as follows (recall that $d = 1$):

$$
\phi_a = k^{(1-\sigma)/2} T_k^{-1/2} \varphi_a \sim k^{-\frac{3+\eta}{2}},
$$

$$
U_k''(\phi) = k^\sigma u_k''(\varphi),
$$

$$
Y_k(\phi) = k^{-(1+\sigma)} y_k(\varphi),
$$

$$
\Delta_k(\phi_a, \phi_b) = k^{-\sigma} T_k^{-1} \delta_k(\varphi_a, \varphi_b) \sim k^{-(2\eta-\bar{\eta})},
$$

where the additional anomalous dimension $\bar{\eta}$ is related to the temperature exponent $\theta$ and to $\eta$ through $\theta = 2 + \eta - \bar{\eta}$.

The cut-off function $R_k$ can also be put in a dimensionless form, $\tilde{R}_k(q) = k^\sigma s(q^2/k^2)$ and we have used $s(x^2) = (a + bx^2 + cx^4) e^{-x^2}$ with the parameters $a, b, c$ optimized through stability considerations and varied to provide error bars on the results [25–27].

We thus have to solve three coupled dimensionless flow equations, which can be symbolically written as

$$
\partial_t u_k''(\varphi) = \tilde{\beta}_u''(\varphi),
$$

$$
\partial_t y_k(\varphi) = \tilde{\beta}_y(\varphi),
$$

$$
\partial_t \delta_k(\varphi_a, \varphi_b) = \tilde{\beta}_d(\varphi_a, \varphi_b).
$$

The (running) exponent $\bar{\eta}_k$ can be calculated from the flow equation for $\delta_k$ by imposing that $\partial_t \delta_k = 0$ for a given arbitrary value of the fields. This directly gives an expression for $2\eta - \bar{\eta}_k$. The result is not sensitive to the fields value and we have chosen the point $\varphi_a = \varphi_b = 0$: then, without loss of generality, we impose $\delta_k(0, 0) = \delta_k \equiv 1$.

We have solved numerically the fixed-point equations, which are obtained by setting to zero the left-hand sides of equations (27) for $\frac{1}{2} \leq \sigma < \frac{1}{2}$. To do so, we have discretized the fields on a grid and used a variation of the Newton–Raphson method [28].

5. Results

For illustration, we first display in figure 1 the fixed-point function $y^*(\varphi)$ for a range of $\sigma$ near $\sigma_G = 1/3$. The system at $\sigma_G = 1/3$ is equivalent to a system at the upper critical dimension: the fixed point is Gaussian (in particular, $y^* = 0$) and the critical exponents for $\sigma < \sigma_G$ are the classical (mean-field) ones for a long-range system [18] (see also below). Above $\sigma_G$ the Gaussian fixed point is unstable due to the $\phi^4$ term in the potential $U_k(\phi)$ and another fixed point emerges.

For $1/3 < \sigma < 1/2$, we find two different regimes for the critical behaviour, separated by a critical value $\sigma_c \approx 0.379$. What distinguishes these regimes is the presence or the absence of a nonanalyticity taking the form of a linear cusp in the (dimensionless) renormalized second cumulant of the random field at the fixed point, $\delta^*(\varphi_a, \varphi_b)$, when $\varphi_b \to \varphi_a$: a cusp is present when $\sigma > \sigma_c$ but is absent when $\sigma < \sigma_c$. The existence of two such regimes has been found in the short-range RFIM [8–10], in the RFO(N)M [9].
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

Figure 1. Fixed-point solutions for the function $y^*(\varphi)$ for several values of $\sigma$ close to $\sigma_G = 1/3$. The system at $\sigma_G = 1/3$ is equivalent to a system at the upper critical dimension: the fixed point is Gaussian (in particular, $y^* = 0$). Another fixed point emerges for $\sigma > \sigma_G$.

and in the long-range RFIM in $d = 3$ as well [14]. In these cases, the cuspless fixed point is associated with a critical behaviour satisfying the dimensional-reduction property whereas the presence of a cusp breaks the dimensional reduction and the underlying supersymmetry. In the present 1D model, there is no dimensional reduction and no underlying supersymmetry whatsoever. The presence of a cusp in the dimensionless cumulant of the disorder has an effect on the values of the critical exponents, but not as dramatic as in the cases involving dimensional reduction and its breakdown.

This nonanalytical dependence in the dimensionless fields is more conveniently studied by introducing

$$\varphi = \frac{\varphi_a + \varphi_b}{2},$$

$$\delta \varphi = \frac{\varphi_b - \varphi_a}{2}.$$

The cumulant $\delta^*$ is an even function of $\varphi$ and $\delta \varphi$ separately. The presence of a cusp then means that

$$\delta^*(\varphi, \delta \varphi) = \delta_0^*(\varphi) + \delta_1^*(\varphi) |\delta \varphi| + \frac{1}{2} \delta_2^*(\varphi) \delta \varphi^2 + \cdots$$

with $\delta_1^*(\varphi) \neq 0$. We show in figure 2 $\delta_1^*(\varphi = 0)$ versus $\sigma$. It is equal to zero below $\sigma_c \approx 0.379$, indicating that the fixed point is cuspless and it becomes nonzero above $\sigma_c$, signalling a cuspy fixed point.

For $\sigma < \sigma_c$, the cuspless fixed point and its vicinity are described by a renormalized second cumulant that behaves as

$$\delta_k(\varphi, \delta \varphi) = \delta_{k,0}(\varphi) + \frac{1}{2} \delta_{k,2}(\varphi) \delta \varphi^2 + \cdots$$

at small $\delta \varphi$. When inserting equation (30) into the RG flow equations, equations (27), it is easily realized that one obtains a closed system of three coupled equations for $u_k^*(\varphi)$.
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

Figure 2. Fixed-point value of the amplitude of the cusp, $\delta_1^{*}(\varphi = 0)$, versus $\sigma$. Below $\sigma_c \approx 0.379$ (denoted by the dotted vertical line), the fixed-point solution is cuspless around $\delta \varphi = 0$; a cusp appears and its magnitude grows further above $\sigma_c$.

Below $\sigma_c \approx 0.379$ (denoted by the dotted vertical line), the fixed-point solution is cuspless around $\delta \varphi = 0$; a cusp appears and its magnitude grows further above $\sigma_c$. A flow equation for $\delta_{k,2}(\varphi)$ is further obtained with a beta function that depends on $u_k''(\varphi)$, $y_k(\varphi)$, $\delta_{k,0}(\varphi)$ and $\delta_{k,2}(\varphi)$ only. The derivation is straightforward but cumbersome and the resulting equation is not shown here (see [15, 29] for a similar derivation). The function $\delta_{k,2}(\varphi)$ blows up at a finite scale $k_L$ for $\sigma > \sigma_c$ and the fixed-point function is then given by equation (29). On the other hand it reaches a finite fixed-point function for $\sigma < \sigma_c$. This allows us to find a precise estimate of $\sigma_c = 0.379 \pm 0.001$ within the present approximation.

Before discussing in more detail the significance and the physics of the two regimes, it is instructive to characterize how the fixed point changes from cuspless to cuspy by considering the stability of the cuspless fixed point with respect to a cuspy perturbation, i.e. a perturbation $\propto |\delta \varphi|$ at small $\delta \varphi$. To compute the eigenvalue $\lambda$ associated with such a cuspy perturbation, one then has to consider the vicinity of the fixed point with $\delta_k(\varphi, \delta \varphi) \simeq \delta^* \varphi \delta \varphi + k^2 f_1(\varphi, \delta \varphi)$ with $f_1(\varphi, \delta \varphi) \simeq |\delta \varphi| f_1(\varphi)$ when $\delta \varphi \to 0$. By linearizing the flow equation for $\delta_k$ around $\delta^*$ and expanding around $\delta \varphi = 0$ one easily derives the eigenvalue equation for $f_1(\varphi)$, which depends on $u''^*(\varphi)$, $y^*(\varphi)$, $\delta^*_{0}(\varphi)$ and $\delta^*_2(\varphi)$ only. The details are similar to those given in the supplementary material of [15] and the resulting equation is not reproduced here.

We show in figure 3 the evolution of $\lambda$ with $\sigma$ and $\lambda$. It is positive for $\sigma < \sigma_c$, thereby indicating that the cuspy perturbation is irrelevant. For $\sigma = \sigma_G = 1/3$, i.e. around the Gaussian fixed point, $\lambda$ can be computed exactly and is equal to $1/6$. The eigenvalue then decreases as $\sigma$ increases. The 1-loop perturbative result in $\sigma = \sigma_G + \epsilon$, which is reproduced by the present nonperturbative ansatz, is found to be $\lambda = 1/6 - \epsilon$ (see appendix D). When $\sigma = \sigma_c^-$, we obtain a very small yet strictly positive value, $\lambda_- = 0.0011 \pm 0.0001$. (This is a robust result as the value is always found to be strictly positive when varying the parameters of the cutoff function.) Above $\sigma_c$, the fixed point is cuspy and $\lambda$ is defined as

doi:10.1088/1742-5468/2014/10/P10017
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

Figure 3. The eigenvalue $\lambda$ associated with a cuspy perturbation versus $\sigma$. The dashed line is the calculation around the cuspless fixed point for $\sigma \leq \sigma_c \approx 0.379$: $\lambda$ decreases from $1/6$ when $\sigma = \sigma_G = 1/3$ to a very small but nonzero value in $\sigma_c$; above $\sigma_c$, the fixed point is characterized by a cusp and $\lambda$ is taken by definition to be strictly zero.

Consequently, $\lambda$ is discontinuous in $\sigma_c$, which, as explained in [29], is the signature that the cuspy fixed emerges continuously from the cuspless one through a boundary-layer mechanism.

We now turn to the values of the critical exponents. In addition to $\eta$, which is fixed to $2 - \sigma$ due to the long-range nature of the interactions, the critical behaviour of the system is characterized by the correlation-length exponent $\nu$ and the exponent $2\eta - \bar{\eta}$ that describes the scaling behaviour of the second renormalized cumulant of the random field. From these exponents, one can deduce the other critical exponents through scaling relations (which are exactly satisfied by the NP-FRG).

The results for $1/\nu$ and $2\eta - \bar{\eta}$ versus $\sigma$ are displayed in figures 4 and 5, respectively. Both $1/\nu$ and $2\eta - \bar{\eta}$ have a non-monotonic behaviour with $\sigma^3$ but this is not related to the change of regime at $\sigma_c$. At and around $\sigma_c$ the variation of the two exponents is smooth, as expected from the boundary-layer mechanism [29]. It is noteworthy that the value of $2\eta - \bar{\eta}$ is small over the whole range of $\sigma$, being at most of the order of .01.

As already mentioned, the present NP-FRG equations reproduce exactly the known 1-loop perturbative results for $\sigma = \sigma_G + \epsilon$: $1/\nu = 1/3 + O(\epsilon^2)$ and $2\eta - \bar{\eta} = O(\epsilon^2)$ (see appendix D). They are not exact at order $\epsilon^2$ but the coefficients of the $\epsilon^2$ terms are found to be close to the exact ones $[4, 18]$: $-10.61$ for $1/\nu$ and $1.498$ for $2\eta - \bar{\eta}$ compared to the exact $-10.8984 \cdots$ and $1.32498 \cdots$. The NP-FRG and the perturbative result at order $\epsilon^2$ stay close up to $\sigma \sim 0.4$ for $1/\nu$ but strongly differ even in the vicinity of $\sigma_G = 1/3$, which seems to indicate in this case large higher-order corrections. None of this however

3 The upturn in $1/\nu$ observed around $\sigma \approx 0.44$ is robust within the present approximation scheme and is also found in the model in $d = 3$ and 4 (unpublished). For larger values of $\sigma$ which we cannot access, $1/\nu$ should however reach a maximum and eventually decrease to reach zero when $\sigma = 1/2$. 

doi:10.1088/1742-5468/2014/10/P10017
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

**Figure 4.** Inverse of the correlation-length exponent $\nu$ versus $\sigma$ as obtained from the present NP-FRG theory (full line). The dashed line is the perturbative result at order $\epsilon^2$ around $\sigma_G = 1/3$ [4, 18]. The symbols with the error bars are results from the lattice simulations: [16] (circle), [17] (diamond). The dotted vertical line marks the value $\sigma_c \approx 0.379$ separating the cuspless and the cuspy regimes. Note that $1/\nu$ should eventually decrease to 0 when $\sigma$ reaches 1/2.

**Figure 5.** $2\eta - \bar{\eta}$ versus $\sigma$ as obtained from the present NP-FRG theory (full line). The dashed line is the perturbative result to order $\epsilon^2$ around $\sigma_G = 1/3$ [18]. The dotted vertical line marks the value $\sigma_c \approx 0.379$ separating the cuspless and the cuspy regimes. For $\sigma = 1/2$, $2\eta - \bar{\eta}$ should go to 0.

appears to be related to the change of regime at $\sigma_c$. Note finally that the result for $1/\nu$ at $\sigma = 0.4$, i.e. in the cuspy regime, is in good agreement with the value recently obtained from the numerical ground-state determination by Dewenter and Hartmann [16] (a little less so with the result of [17]).

doi:10.1088/1742-5468/2014/10/P10017
We have been able to follow the critical fixed point up to $\sigma \approx 0.46$, after which numerical instabilities proliferate. The reason is that $\sigma$ approaches the value of $1/2$, which is the analogue of a lower critical dimension below which there is no transition. It is then expected that the low-momentum dependence of the propagators is no longer well described by only taking into account the lowest-order terms of the derivative expansion as was done in the ansatz used in this work.

6. Discussion

The existence of two different regimes in the critical behaviour of the RFIM can be attributed to the role of ‘avalanches’, which are collective phenomena present at zero temperature. In equilibrium, such ‘static’ avalanches describe the discontinuous change in the ground state of the system at values of the external source that are sample-dependent (for an illustration see the figures in [30,31]). At the critical point, the avalanches take place on all scales and their size distribution follows a power law with a nontrivial exponent $\tau$.

It was shown in [15] for the RFIM (and in [32,33] for interfaces in a random environment) that the presence of avalanches necessarily induces a cusp in the functional dependence of the renormalized cumulants of the disorder and in properly chosen correlation functions. Actually, such a cuspy dependence is already present in the $d = 0$ RFIM [15]. As the critical behaviour is affected by a cusp in the dimensionless quantities [8–10], the question then is whether the amplitude of the cusp is an irrelevant contribution at the fixed point, or not.

In physical terms, the two regimes therefore correspond to two different situations where the scale of the largest typical avalanches at criticality, which in a system of linear size $L$ goes as $L^{d_f}$ with $d_f \leq d$, is equal to the scale of the total magnetization, i.e. $L^{d-(d-4+\bar{\eta})/2}$, or is subdominant. This is precisely characterized by the eigenvalue $\lambda$ calculated in the previous section. In the regime where the fixed point is cuspless, $d_f = d - (d - 4 + \bar{\eta})/2 - \lambda$ with $\lambda > 0$ whereas $d_f = d - (d - 4 + \bar{\eta})/2$ in the regime where the fixed point is cuspy.

The interest in the 1D long-range RFIM is mainly because it can be studied by computer simulations for large system sizes. Efficient algorithms exist to determine the ground state in a polynomial time and one can further reduce the computational cost by considering a diluted (Levy) lattice which is expected to be in the same universality class [16,17]. Sizes up to $L = 256,000$ in [17] and $L = 524,288$ in [16] have thus been investigated.

As discussed previously, the critical exponents characterizing the leading scaling behaviour do not show any significant change of dependence with $\sigma$ around the critical value $\sigma_c$ (they are predicted to be continuous in $\sigma_c$). To distinguish the two regimes it seems preferable to investigate the avalanche distribution at $T = 0$ and try to extract the fractal dimension $d_f$ of the largest avalanches. This can be obtained through a finite-size scaling of the distribution of the avalanches or of their first moments. For instance, the ratio of the second to the first moment of the avalanche size, $\langle S^2 \rangle / \langle S \rangle$, should scale as $L^{d_f}$ where $L$ is large enough. Having extracted $d_f$ and $2\eta - \bar{\eta}$, one can obtain the eigenvalue...
\[ \lambda = \frac{d + 4 - \bar{\eta}}{2} - d_f = \frac{1}{2} + \sigma + \frac{2\eta - \bar{\eta}}{2} - d_f \quad (31) \]

where \(2\eta - \bar{\eta}\) is anyhow expected to be very small. If \(\lambda > 0\), one is in the cuspless regime and if \(\lambda = 0\) it is in the cuspy one. By studying values of \(\sigma\) near \(\sigma_G = 1/3\) where \(\lambda\) should be \(\approx 1/6\) and values of \(\sigma \gtrsim 0.4\) where \(\lambda = 0\) it should be possible to distinguish the two regimes in computer simulations.

In addition, one could also consider the corrections to scaling. In the vicinity of the critical value \(\sigma_c \approx 0.379\), these corrections should be dominated by the lowest (positive) eigenvalue in the \(Z_2\)-symmetric subspace of perturbations around the fixed point. For \(0.372 \lesssim \sigma \lesssim \sigma_c\), this eigenvalue is equal to \(\lambda\) and is therefore expected to be very small (with \(\lambda^- \approx 0.0011\)). Right above \(\sigma_c\), the smallest irrelevant eigenvalue is also associated with a cuspy perturbation and it emerges from a value \(\lambda_c^+ \lesssim \lambda_c^-\) at \(\sigma_c^+\) [29]. (Note that this irrelevant eigenvalue is different than \(\lambda\) computed from equation (31) which is equal to zero in this region of \(\sigma\).) The exponent describing the main correction to scaling should therefore display a minimum as a function of \(\sigma\) with a value near zero for \(\sigma = \sigma_c \approx 0.379\).

Finally, we conclude by discussing the RFIM on the Dyson hierarchical lattice. This lattice mimics the behaviour of the 1D chain with long-range interactions specified by the same parameter \(\sigma\). Its critical behaviour was studied by Rodgers and Bray [20] and more recently by Parisi and Rocchi [34]; its avalanche distribution was characterized in detail by Monthus and Garel [21].

It was shown in [20] that in this case the temperature exponent is always given by \(\theta = \sigma\), which implies, together with the result \(\eta = 2 - \sigma\), that \(2\eta - \bar{\eta} = 0\). In addition, it was found in [21] that the fractal dimension of the largest avalanches at criticality is \(d_f = 2\sigma\). These two results, \(2\eta - \bar{\eta} = 0\) and \(d_f = 2\sigma\), when inserted into the equation (31) predict that \(\lambda = 1/2 - \sigma\). As a consequence, \(\lambda\) decreases from 1/6 to 0 as \(\sigma\) increases from 1/3 to 1/2 and is always strictly positive, except at the analogue of the lower critical dimension. The RFIM on the Dyson hierarchical level therefore appears to display a unique regime over the whole range \(1/3 \leq \sigma < 1/2\): avalanches are present at all scales but their scaling dimension is never large enough to induce a cuspy fixed point. This is a notable difference with the long-range RFIM on the standard 1D chain. This finding is rather surprising. It is known that the binary-tree structure of the Dyson hierarchical lattice may lead to critical exponents that differ from those of the 1D power law model (for the same value of \(\sigma\)). In the context of disordered systems, this has been obtained for instance for the hierarchical Edwards–Anderson spin-glass model [35]. However, the difference found in the present case is more severe as it concerns the overall fixed-point scenario. This indicates that the qualitative equivalence between the large-scale behaviour of 1D long-range models and that of their Dyson-lattice counterparts should not be taken for granted.

To summarize: The NP-FRG predicts that the critical behaviour of the RFIM generically shows two distinct regimes separated by a nontrivial critical value of the dimension, the number of components or the power-law exponent of the long-range interactions, when present. These regimes are associated with properties of the large-scale avalanches at zero temperature. We suggest that investigating the avalanche characteristics in the 1D long-range RFIM through numerical simulations should provide a direct check of the prediction.
Appendix A. The flow equations for $U_k''(\phi)$ and $\Delta_k(\phi_a, \phi_b)$

Starting from equation (21) and setting the external momentum to 0 one easily obtains the flow equation for $U_k''(\phi)$, $\partial U_k''(\phi) = \beta_{U''}(\phi)$, with

\[
\beta_{U''} = -\frac{1}{2} \int_0^\infty \frac{dq}{\pi} \partial_t \hat{R}(q) \hat{G}_k(q; \phi)^2 \left\{ \frac{\Delta_k^{(20)}(\phi, \phi)}{\Delta_k^{(10)}(\phi, \phi)} + 2 \Delta_k^{(11)}(\phi, \phi) + 2 \hat{G}_k(q; \phi) \right. \\
\times \left[ -2[\Delta_k^{(10)}(\phi, \phi) + \Delta_k^{(01)}(\phi, \phi)][U_k''(\phi) + q^{1+2\sigma} Y_k''(\phi)] - \Delta_k(\phi, \phi) \right] \\
\times \left( U_k^{(4)}(\phi) + q^{1+2\sigma} Y_k''(\phi) - 3 \hat{G}_k(q; \phi)[U_k''(\phi) + q^{1+2\sigma} Y_k''(\phi)]^2 \right) \right\}.
\] (A.1)

To obtain the expression for the flow equation of $\Delta_k$, $\partial \Delta_k(\phi_a, \phi_b) = \beta_{\Delta}(\phi_a, \phi_b)$, we derive equation (13) with respect to $\phi_a$ and $\phi_b$, insert the ansatz for the effective average action and express the output in terms of the Fourier transformed quantities. The flow of $\Delta_k$ is then given by setting the external momentum $p$ to zero, which leads to the following graphical expression:

\[
\partial_t \Delta_k(\phi_a, \phi_b) = -\frac{1}{2} \partial_t \int_q \left( \begin{array}{ccc}
4x & b & +4x \\
+4x & a & +4x \\
+2x & b & +2x \\
+2x & a & +2x \\
\end{array} \right) \left( \begin{array}{ccc}
4x & b & +4x \\
+4x & a & +4x \\
+2x & b & +2x \\
+2x & a & +2x \\
\end{array} \right) \left( \begin{array}{ccc}
4x & b & +4x \\
+4x & a & +4x \\
+2x & b & +2x \\
+2x & a & +2x \\
\end{array} \right) \right). \quad \text{(A.2)}
\]

Explicitly written, the expression for $\beta_{\Delta}$ reads

\[
\beta_{\Delta} = -\int_0^\infty \frac{dq}{\pi} \partial_t \hat{R}(q) \left\{ -\hat{G}_k(q; \phi_a)^3 \left[ \Delta_k(\phi_a, \phi_a) \Delta_k^{(20)}(\phi_a, \phi_b) + \Delta_k^{(10)}(\phi_a, \phi_b) \right] \\
\times \left( \Delta_k^{(01)}(\phi_a, \phi_a) + \Delta_k^{(10)}(\phi_a, \phi_a) - 3[U_k^{(3)}(\phi_a) + q^{1+2\sigma} Y_k''(\phi_a)] \right) \\
\times \left[ \Delta_k(\phi_a, \phi_b) \gG_k(q; \phi_a) \right] - \left[ \Delta_k(\phi_a, \phi_b) \Delta_k^{(11)}(\phi_a, \phi_b) + \Delta_k^{(01)}(\phi_a, \phi_b) \right] \\
\times \left( \Delta_k^{(10)}(\phi_a, \phi_b) - 2[U_k^{(3)}(\phi_a) + q^{1+2\sigma} Y_k''(\phi_a)] \Delta_k(\phi_a, \phi_b) \hat{G}_k(q; \phi_a) \right) \\
\times \left( \Delta_k^{(01)}(\phi_a, \phi_b) - [U_k^{(3)}(\phi_b) + q^{1+2\sigma} Y_k''(\phi_b)] \Delta_k(\phi_a, \phi_b) \hat{G}_k(q; \phi_a) \right) \\
\times \left( \Delta_k^{(01)}(\phi_a, \phi_b) - [U_k^{(3)}(\phi_b) + q^{1+2\sigma} Y_k''(\phi_b)] \Delta_k(\phi_a, \phi_b) \hat{G}_k(q; \phi_a) \right) \\
\times \left( \Delta_k^{(01)}(\phi_a, \phi_b) - [U_k^{(3)}(\phi_b) + q^{1+2\sigma} Y_k''(\phi_b)] \Delta_k(\phi_a, \phi_b) \hat{G}_k(q; \phi_a) \right) \right\}. \quad \text{(A.3)}
\]
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

Appendix B. Toy model integral

We consider the following 1D integral

\[
I(p) = \int_{-\infty}^{\infty} dq \frac{1}{|q + p|^\sigma + f((q + p)^2)}
\]

with \(1/3 < \sigma < 1/2\), in the vicinity of \(p = 0\). (As \(I(p) = I(-p)\), we restrict ourselves to \(p \geq 0\).) The function \(f(q^2)\) is chosen such that \(f(0) \neq 0\) and that its large \(q^2\) behaviour guarantees the convergence of the integral. As a result \(I(p = 0)\) is finite and we look for the leading \(p\)-dependence when \(p \to 0\). A naive expansion in \(p\) predicts a \(p^2\) dependence, but this turns out to be wrong. To obtain the correct behaviour, we first rewrite \(I(p) - I(0)\) as

\[
I(p) - I(0) = \int_{-\infty}^{\infty} dq \frac{1}{|q + p|^\sigma + f((q + p)^2)}
\]

where, for convenience, we have expressed \(I(p = 0)\) in a symmetrized form by changing the integration variable to \(q \pm \frac{p}{2}\).

We now introduce the variable \(x = \sqrt{q^2 + (p^2/4)}\), so that

\[
|q \pm \frac{p}{2}|^\sigma + f \left( |q \pm \frac{p}{2}|^2 \right) = f(x^2) + |x|^\sigma \pm \frac{qp}{|x|^{2-\sigma}} \left( \frac{\sigma}{2} + |x|^{2-\sigma} f'(x^2) \right)
\]

where the ellipses denote terms with higher-order powers in \(p\) (at fixed \(x\)).

After inserting equation (B.3) in equation (B.2) and changing the variable from \(q\) to \(z = q/p\), we obtain

\[
I(p) - I(0) = -\frac{p^{1+2\sigma}}{2} \int_{-\infty}^{\infty} dz \frac{z^2}{(z^2 + \frac{1}{4})^{2-\sigma}} \left[ \frac{\sigma^2 + O(p^{2-\sigma})}{f(0)^4 + O(p^\sigma)} \right]
\]

so that

\[
\lim_{p \to 0} \frac{I(p) - I(0)}{p^{1+2\sigma}} = -\frac{\sigma^2}{2f(0)^4} \int_{-\infty}^{\infty} dz \frac{z^2}{(z^2 + \frac{1}{4})^{2-\sigma}}.
\]

where the integral over \(z\) converges in the range of \(\sigma\) considered.

Appendix C. Derivation of \(\beta_{Y,an}\)

Consider the diagrams shown in figure C1. They enter in the right-hand side of the flow equation of \(F^{(2)}(p; \phi)\) as

\[
I_{an}(p) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dq \partial_t R_k(q) \hat{G}_k(p + q; \phi) \times U''''(\phi) \left( -2[\Delta^{(10)}(\phi, \phi) + \Delta^{(01)}(\phi, \phi)] + U''''(\phi) \Delta_k(\phi, \phi) \right) \hat{G}_k(q; \phi) + \hat{G}_k(q + p; \phi) \right)
\]

with \(\hat{G}_k(q; \phi)\) given by equation (23).

doi:10.1088/1742-5468/2014/10/P10017

18
immediately obtains the equation (25).

After inserting the above expressions in (C.1) and performing the integral over \( u \), where one expects \( \epsilon = \sigma - 1/3 \), expand in small \( p \) and finally change the integration variable from \( q \) to \( z = q/p \). The outcome at the leading order in \( p \) is then

\[
\begin{align*}
I_{an,1}(p) - I_{an,1}(0) &= -p^{1+2\sigma} \partial_t \hat{R}_k(0) \hat{G}_k(0; \phi) \int_{-\infty}^{+\infty} dz \frac{\sigma^2 z^2}{(z^2 + \frac{1}{4})^{2-\sigma}}, \\
I_{an,2}(p) - I_{an,2}(0) &= -p^{1+2\sigma} \partial_t \hat{R}_k(0) \hat{G}_k(0; \phi) \int_{-\infty}^{+\infty} dz \frac{3\sigma^2 z^2}{(2(z^2 + \frac{1}{4})^{2-\sigma}} \\
I_{an,3}(p) - I_{an,3}(0) &= -p^{1+2\sigma} \partial_t \hat{R}_k(0) \hat{G}_k(0; \phi) \int_{-\infty}^{+\infty} dz \frac{2\sigma^2 z^2}{(z^2 + \frac{1}{4})^{2-\sigma}}.
\end{align*}
\]  

(C.2)

After inserting the above expressions in (C.1) and performing the integral over \( z \), one immediately obtains the equation (25).

Appendix D. Recovering the perturbative result at order \( \epsilon = \sigma - 1/3 \) from the NP-FRG

We studied the NP-FRG flow equations expressed in dimensionless quantities in the vicinity of the boundary value to classical (long-range) behaviour, \( \sigma_l = \frac{1}{3} \). The fixed point should be close to the Gaussian fixed point characterized by \( u_k'' = y_k = 0 \) and \( \delta_k(\varphi, \delta \varphi) = 1 \) (recall that the long-range kinetic part is not renormalized so that \( \eta = 2 - \sigma \)). For \( \sigma = 1/3 + \epsilon \), we expand the functions in powers of the fields,

\[
\begin{align*}
u_k''(\varphi) &= u_{k,2} + \frac{1}{2} u_{k,4} \varphi^2 + O(\varphi^4), \\
y_k(\varphi) &= \frac{1}{2} y_{k,2} \varphi^2 + O(\varphi^4), \\
\delta_k(\varphi, \delta \varphi) &= 1 + \frac{1}{2} (\delta_{k,20} \varphi^2 + \delta_{k,02} \delta \varphi^2) + O(\varphi^4, \delta \varphi^4, \varphi^2 \delta \varphi^2),
\end{align*}
\]  

(D.1)

where one expects \( u_{k,2}, u_{k,4}, y_{k,2}, \delta_{k,20}, \delta_{k,02} \) to be at least of \( O(\epsilon) \) around the fixed point.
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

After inserting the above expressions in the beta function $\tilde{\beta}_{\alpha'}(\varphi)$ obtained from equation (A.1), one finds the following flow equation for the coupling constant $u_{k,4}$:

$$\partial_t u_{k,4} = -3\epsilon u_{k,4} + 36u_{k,4}^2 I_4(u_{k,2}) + O(\epsilon^2)$$  \hspace{1cm} (D.2)

where

$$I_n(u_2) = \frac{1}{2} \int_0^\infty dq \frac{\tilde{\partial} s(q^2) \tilde{p}_0(q)^n}{\pi}$$  \hspace{1cm} (D.3)

with $\tilde{p}_0(q) = [q^\sigma + s(q^2) + u_2]^{-1}$ and $\tilde{\partial} s(q^2) = \sigma s(q^2) - 2q^2 s'(q^2)$. Equation (D.2) admits a nontrivial fixed-point solution $u_4^* = \epsilon/[12I_4(0)] + O(\epsilon^2)$.

Similarly, the flow of $u_{k,2}$ reads

$$\partial_t u_{k,2} = -\sigma u_{k,2} - 4u_{k,4}^2 I_3(u_{k,2}) + O(\epsilon^2),$$  \hspace{1cm} (D.4)

which leads to $u_2^* = -\epsilon I_3(0)/[3\sigma I_4(0)] + O(\epsilon^2)$.

On the other hand, from the beta function for $\delta_k$ and the condition $\partial_t \delta_k(0,0) = 0$, one derives that

$$2\eta - \bar{\eta} = O(\epsilon^2).$$  \hspace{1cm} (D.5)

To derive the $\epsilon$ dependence of the exponent $1/\nu$, one begins from the flow equation for $u_{k,2}$ and perturbs the fixed point by a small constant $k^{-1/\nu} \delta u_2$. By using that $I_3(u_2^* + \delta u_2) \approx I_3(u_2^*) - 3I_4(u_2^*) \delta u_2$ one finds the following linearized equation:

$$-\frac{1}{\nu} \delta u_2 = (-\sigma + \epsilon) \delta u_2 + O(\epsilon^2),$$  \hspace{1cm} (D.6)

which implies, as $\sigma = 1/3 + \epsilon$,

$$\frac{1}{\nu} = \frac{1}{3} + O(\epsilon^2).$$  \hspace{1cm} (D.7)

The above expression for $1/\nu$ is identical to the perturbative result [4].

Finally, we can also derive the expression for the eigenvalue $\lambda$. In the vicinity of the Gaussian fixed point we look for an eigenfunction $f_\lambda(\varphi)$ of the form $f_\lambda(\varphi) = f_{\lambda,0} + (1/2)f_{\lambda,2}\varphi^2 + \cdots$. We find that $f_{\lambda,2} = O(\epsilon^2)$ and the eigenvalue equation then reads

$$\lambda f_{\lambda,0} = (\frac{1}{6} - \epsilon) f_{\lambda,0} + O(\epsilon^2),$$  \hspace{1cm} (D.8)

which leads to

$$\lambda = \frac{1}{6} - \epsilon + O(\epsilon^2).$$  \hspace{1cm} (D.9)

This is the result quoted in the main text.

References

[1] Imry Y and Ma S K 1975 Phys. Rev. Lett. 35 1399
[2] Nattermann T 1998 Spin Glasses and Random Fields (Singapore: World Scientific) p 277
[3] Grinstein G 1976 Phys. Rev. Lett. 37 944

doi:10.1088/1742-5468/2014/10/P10017
Critical behaviour of the random-field Ising model with long-range interactions in one dimension

[4] Young A P 1977 J. Phys. C: Solid State Phys. 10 L257
[5] Parisi G and Sourlas N 1979 Phys. Rev. Lett. 43 744
[6] Imbrie J Z 1984 Phys. Rev. Lett. 53 1747
[7] Brichmont J and Kupiainen A 1987 Phys. Rev. Lett. 59 1829
[8] Tarjus G and Tissier M 2004 Phys. Rev. Lett. 93 267008
Tarjus G and Tissier M 2008 Phys. Rev. B 78 024203
[9] Tissier M and Tarjus G 2006 Phys. Rev. Lett. 96 087202
Tissier M and Tarjus G 2008 Phys. Rev. B 78 024204
[10] Tissier M and Tarjus G 2011 Phys. Rev. Lett. 107 041601
Tissier M and Tarjus G 2012 Phys. Rev. B 85 104202
Tissier M and Tarjus G 2012 Phys. Rev. B 85 104203
[11] Fisher D S 1986 Phys. Rev. Lett. 56 1964
Narayan O and Fisher D S 1992 Phys. Rev. B 46 11520
[12] Le Doussal P, Wiese K J and Chauve P 2002 Phys. Rev. B 66 174201
Le Doussal P, Wiese K J and Chauve P 2004 Phys. Rev. E 69 026112
[13] Le Doussal P and Wiese K J 2009 Phys. Rev. E 79 051106
Le Doussal P and Wiese K J 2012 Phys. Rev. E 85 061102
[14] Baczyn M, Tissier M, Tarjus G and Sakamoto Y 2013 Phys. Rev. B 88 014204
[15] Tarjus G, Baczyn M and Tissier M 2013 Phys. Rev. Lett. 110 135703
[16] Dewenter T and Hartmann A K 2014 Phys. Rev. B 90 014207
[17] Leuzzi L and Parisi G 2013 Phys. Rev. B 88 224204
[18] Bray A J 1986 J. Phys. C: Solid State Phys. 19 6225
[19] Cassandro M, Orlandi E and Picco P 2009 Commun. Math. Phys. 288 731
[20] Rodgers G J and Bray A J 1988 J. Phys. A: Math. Gen. 21 2177
[21] Monthus C and Garel T 2011 J. Stat. Mech. P07010
[22] Mezard M, Parisi G and Virasoro M A 1987 Spin Glass Theory and Beyond (Singapore: World Scientific)
[23] Berges J, Tétardis N and Wetterich C 2002 Phys. Rep. 363 223
[24] Wetterich C 1993 Phys. Lett. B 301 90
[25] Litim D F 2000 Phys. Lett. B 486 128
[26] Litim D F 2002 Nucl. Phys. B 651 90
[27] Canet L, Delamotte B, Mouhanna D and Vidal J 2003 Phys. Rev. D 67 065004
[28] Pawlowski J M 2007 Ann. Phys. 322 2831
[29] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 Numerical Recipes vol 1 (Cambridge: Cambridge University Press) (section 9.4)
[30] Baczyk M, Tarjus G, Tissier M and Balog I 2014 J. Stat. Mech. P06010
[31] Wu Y and Machta J 2005 Phys. Rev. Lett. 95 137208
Wu Y and Machta J 2006 Phys. Rev. B 74 064418
[32] Liu Y and Dahmen K A 2007 Phys. Rev. E 76 031106
[33] Balents L, Bouchaud J-P and Mezard M 1996 J. Physique I 6 1007
Middleton A A, Le Doussal P and Wiese K J 2007 Phys. Rev. Lett. 98 155701
Le Doussal P, Middleton A A and Wiese K J 2009 Phys. Rev. E 79 050101
[34] Parisi G and Rocchi J 2014 Phys. Rev. B 90 024203
[35] Castellana M and Parisi G 2010 Phys. Rev. E 82 040105
Castellana M and Parisi G 2011 Phys. Rev. E 83 041134

doi:10.1088/1742-5468/2014/10/P10017