Geodesic metric spaces with unique blow-up almost everywhere: properties and examples

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In this report we deal with metric spaces that at almost every point admit a tangent metric space. These spaces are in some sense generalizations of Riemannian manifolds. We will see that, at least at the level of the tangents, there is some resemblance of a differentiable structure and of (sub)Riemannian geometry. I will present some results and give examples.

Let \( X = (X, d_X) \) and \( Y = (Y, d_Y) \) be metric spaces. Fix \( x_0 \in X \) and \( y_0 \in Y \). If there exists \( \lambda_j \to \infty \) such that, in the Gromov-Hausdorff convergence,

\[
(X, \lambda_j d_X, x_0) \to (Y, d_Y, y_0), \quad \text{as } j \to \infty,
\]

then \((Y, y_0)\) is called a tangent (or a weak tangent, or a blow-up) of \( X \) at \( x_0 \).

Some remarks are due. Fixed \( x_0 \in X \), there might be more than one tangent. Moreover, in general there might not exist any tangent. However, if the distance is doubling, then, by the work of Gromov [Gro81], then tangents exists. Namely, for any sequence \( \lambda_j \to \infty \), there exists a subsequence \( \lambda_{j_k} \to \infty \) such that \((X, \lambda_{j_k} d_X, x_0)\) converges as \( k \to \infty \). A tangent is well defined up to pointed isometry. Thus we define the set of all tangents of \( X \) at \( x_0 \) as

\[
\text{Tan}(X, x_0) := \{ \text{tangents of } X \text{ at } x_0 \}/\text{pointed isometric equivalence}.
\]

We consider two questions: how big is \( \text{Tan}(X, x_0) \)? what happens when the tangent is unique? The rough answer that we will give are the following. Under some ‘standard’ assumptions, if \((Y, y_0) \in \text{Tan}(X, x_0)\), then \((Y, y) \in \text{Tan}(X, x)\), for all \( y \in Y \). Moreover, in the case of unique tangents, such tangents are very special, however, not much can be said about the initial space \( X \).

**Definition and examples.** Let \((X_j, x_j), (Y, y)\) be pointed geodesic metric spaces. We write \((X_j, x_j) \to (Y, y)\) in the Gromov-Hausdorff convergence if, for all \( R > 0 \), we have \( d_{GH}(B(x_j, R), B(y, R)) \to 0 \). Here

\[
d_{GH}(A, B) := \inf \{ d^2_H(A', B') : Z \text{ metric space, } A', B' \subseteq Z, A^\text{isom} = A', B^\text{isom} = B' \},
\]

and \( d^2_H(\cdot, \cdot) \) is the Hausdorff distance in the space \( Z \).

**Example 1.** When \( \mathbb{R}^n \) is endowed with the Euclidean distance (or more generally a norm), we have \( \text{Tan}(\mathbb{R}^n, p) = \{ (\mathbb{R}^n, \| \cdot \|, 0) \} \), \( \forall p \in \mathbb{R}^n \).

**Example 2.** Let \((M, d)\) be a Riemannian manifold (or more generally a Finsler manifold), we have \( \text{Tan}(M, d, p) = \{ (\mathbb{R}^n, \| \cdot \|, 0) \} \), \( \forall p \in \mathbb{R}^n \).

**Definition 3** (Carnot group). Let \( g \) be a stratified Lie algebra, i.e., \( g = V_1 \oplus \cdots \oplus V_s \), with \([V_j, V_i] = V_{j+i}, \) for \( 1 \leq j \leq s \), where \( V_{s+1} = \{ 0 \} \). Let \( G \) be the simply-connected Lie group whose Lie algebra is \( g \). Fix \( \| \cdot \| \) on \( V_1 \). Define, for any \( x, y \in G \),

\[
d_{CC}(x, y) := \inf \left\{ \int_0^1 \| \dot{\gamma}(t) \| dt : \gamma \in C^\infty([0, 1]; G), \gamma(0) = x, \gamma(1) = y, \dot{\gamma} \in V_1 \right\}.
\]
The pair \((G, d_{CC})\) is called Carnot group.

In particular, any Carnot group \(G\) is a metric space homeomorphic to the Lie group \(G\). Moreover, by the work of Pansu and Gromov [Pan83], the Carnot groups are the blow-downs of left-invariant Riemannian/Finsler distances on \(G\). Namely, if \(\|\cdot\|\) is a norm on \(\text{Lie}(G)\) extending the one on \(V_1\) and \(d_{\|\cdot\|}\) is the corresponding Finsler distance, \((G, \lambda d_{\|\cdot\|}, 1) \xrightarrow{\lambda \to 0} (G, d_{CC}, 1)\).

**Example 4.** If \((G, d_{CC})\) is a Carnot group, then \(\text{Tan}(G, d_{CC}, 1) = \{(G, d_{CC}, 1)\}\). Indeed, for all \(\lambda > 0\), there is a group homomorphism \(\delta_\lambda : G \to G\) such that \((\delta_\lambda)_*|_{V_1}\) is the multiplication by \(\lambda\). Consequently, \((\delta_\lambda)_*d_{CC} = \lambda d_{CC}\). QED

**Results.** Our main theorem is the following.

**Theorem 5 ([LD11]).** Let \((X, d)\) be a geodesic metric space. Let \(\mu\) be a doubling measure. Assume that, for \(\mu\)-almost every \(x \in X\), the set \(\text{Tan}(X, x)\) contains only one element. Then, for \(\mu\)-almost every \(x \in X\), the element in \(\text{Tan}(X, x)\) is a Carnot group.

**Example 6** (SubRiemannian manifolds). Let \(M\) be a Riemannian manifold (or more generally Finsler). Let \(\Delta \subseteq TM\) be a smooth sub-bundle. Let \(X^1(\Delta)\) be the vector fields tangent to \(\Delta\). By induction, define \(X^{k+1}(\Delta) := X^k(\Delta) + [X^1(\Delta), X^k(\Delta)]\). Assume that there exists \(s \in \mathbb{N}\) such that \(X^s(\Delta) = TM\) and that, for all \(k\), the function \(p \mapsto \dim X^k(\Delta)(p)\) is constant. Define, for any \(x, y \in M\),

\[
d_{CC}(x, y) := \inf \{\text{Length}(\gamma) \mid \gamma \in C^\infty([0, 1]; M), \gamma(0) = x, \gamma(1) = y, \dot{\gamma} \in \Delta\}.
\]

Then \((M, d_{CC})\) is called an (equiregular) subFinsler manifold. In such a case, by a theorem of Mitchell, see [Mit85, MM95],

\[
\text{Tan}(M, d_{CC}, p) = \{(G, d_{CC}, 1)\}, \quad \forall p \in M,
\]

with \((G, d_{CC})\) a Carnot group, which might depend on \(p\).

Theorem 5 is proved using the following general property.

**Theorem 7 ([LD11]).** Let \((X, \mu, d)\) be a doubling-measured metric space. Then, for \(\mu\)-almost every \(x \in X\), if \((Y, y) \in \text{Tan}(X, x)\), then \((Y, y') \in \text{Tan}(X, x)\), for all \(y' \in Y\).

If \# \text{Tan}(X, x_0) = 1, then \((Y, y_0) = (Y, y)\), for all \(y \in Y\). In other words, the isometry group \(\text{Isom}(Y)\) acts on \(Y\) transitively. Thus we use the following.

**Theorem 8** (Gleason-Montgomery-Zippin, [MZ74]). Let \(Y\) be a metric space that is complete, proper, connected, and locally connected. Assume that the isometry group \(\text{Isom}(Y)\) of \(Y\) acts transitively on \(Y\). Then \(\text{Isom}(Y)\) is a Lie group with finitely many connected components.

Regarding the conclusion of the proof of Theorem 8, since moreover \(Y\) is geodesic, being \(X\) so, then \(Y\) is a subFinsler manifold, by [Bert88]. From Mitchell’s Theorem and the fact that \(\{Y\} = \text{Tan}(Y, y)\), \(Y\) is a Carnot group. QED
**Comments and more examples.** There are other settings in which the tangents are (almost everywhere) unique. The snow flake metrics \((\mathbb{R}, \|\cdot\|^{\alpha})\) with \(\alpha \in (0,1)\) are such examples. Some examples on which the tangents are Euclidean spaces are the Reifenberg vanishing flat metric spaces, which have been considered in [CC97, DT99]. Alexandrov spaces have Euclidean tangents almost everywhere, [BGP92].

However, even in the subRiemannian setting, the tangents are not local model for the space. Indeed, there are subRiemannian manifolds with a different tangent at each point, [Var81]. In fact, there exists a nilpotent Lie group equipped with left invariant sub-Riemannian metric that is not locally biLipschitz equivalent to its tangent, see [LDOW11]. Such last fact can be seen as the local counterpart of a result by Shalom, which states that there exist two finitely generated nilpotent groups \(\Gamma\) and \(\Lambda\) that have the same blow-down space, but they are not quasi-isometric equivalent, see [Sha04].

Another pathological example from [HH00] is the following. For any \(n > 1\), there exists a geodesic space \(X\) supporting a doubling measure \(\mu\) such that at \(\mu\)-almost all point of \(X\) the tangent is \(\mathbb{R}^n\), but \(X\) has no manifold points.

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