\textbf{d-MINIMAL SURFACES IN THREE-DIMENSIONAL SINGULAR SEMI-EUCLIDEAN SPACE $\mathbb{R}^{0,2,1}$}

\textbf{YUICHIRO SATO}

\textbf{Abstract.} In this paper, we study surfaces in singular semi-Euclidean space $\mathbb{R}^{0,2,1}$ endowed with a degenerate metric. We define \textit{d}-minimal surfaces, and give a representation formula of Weierstrass type. Moreover, we prove that \textit{d}-minimal surfaces in $\mathbb{R}^{0,2,1}$ and spacelike flat zero mean curvature (ZMC) surfaces in four-dimensional Minkowski space $\mathbb{R}^{4}_1$ are in one-to-one correspondence.

1. Introduction

In this paper, we investigate surfaces in three-dimensional singular semi-Euclidean space with the signature $(0, 2, 1)$. The history of surface theory is very long, and the research has been studied. Minimal surfaces which attain stationary values for the volume functional of surfaces have many results of the research. In particular, they are characterized by having the mean curvature vector field which vanishes identically. Recently, Umehara and Yamada et al. ([19], [8] and [7] etc.) study the zero mean curvature surfaces in three-dimensional Minkowski space actively. For such surfaces, they showed that singularities appear generically, and relate to the topology of surfaces.

On the other hand, the author [15] classified ruled minimal surfaces in semi-Euclidean space. As a consequence, we obtained that certain surfaces are included in three-dimensional subspaces whose metrics are degenerate forms. Inspired by this fact, we study the singular differential geometry, i.e. allow to have degenerate metrics. In particular, we consider the surface theory. We introduce a degenerate metric $dx^2 + dy^2$ to three-dimensional vector space $\mathbb{R}^3$ with the coordinates $(x, y, z)$. We call the pair $(\mathbb{R}^3, dx^2 + dy^2)$ three-dimensional singular semi-Euclidean space with the signature $(0, 2, 1)$. It is denoted by $\mathbb{R}^{0,2,1}$. Let $M$ be a surface in $\mathbb{R}^{0,2,1}$. We assume that the induced metric of $M$ is non-degenerate. Actually, this degenerate geometry is equivalent to simply isotropic geometry which is one of the Cayley-Klein geometries. For isotropic geometry, the well-known reference is [13]. We reformulate in terms of the geometry using metrics and connections.

Here, we remark how to use the terms. First, in the canonical three-dimensional Euclidean space $\mathbb{R}^3$, surfaces whose mean curvature vanishes identically give stationary values for the volume functional. In a certain situation, its value is minimal, but not extreme in general. Historically, we call such surfaces \textit{minimal}.

\textit{2010 Mathematics Subject Classification.} Primary 53A10; Secondary 53A40.

\textit{Key words and phrases.} minimal surface, isotropic geometry, semi-Riemannian geometry.
Next, in three-dimensional Minkowski space $\mathbb{R}^3_1$, surfaces whose mean curvature vanishes identically change its name with respect to the cases of the induced metrics. When the induced metric is spacelike, i.e., Riemannian, we call such surfaces maximal. This means that, when we consider the volume functional analogically, such surfaces always give maximal values unlike the Euclidean case. On the other hand, when timelike, i.e., Lorentzian, we simply call such surfaces minimal. We should remark that timelike minimal surfaces give stationary values for the volume functional, but give neither minimal nor maximal values. When connected surfaces have the part of spacelike maximal surfaces and that of timelike minimal surfaces, we call such surfaces mixed type (17).

In four-dimensional Minkowski space $\mathbb{R}^4_1$, surfaces whose mean curvature vector field vanishes identically are more complicated. Therefore, in order to treat uniformly, we call all such surfaces zero mean curvature when the ambient space is $\mathbb{R}^4_1$. This is why we have to pay attention to the terminology.

In the section two, we recall the fundamental fact in semi-Riemannian geometry, and recall properties of non-degenerate submanifolds. In particular, we explain the singular semi-Euclidean space.

In the section three, this is the main section. We define non-degenerate surfaces in $\mathbb{R}^{0,2,1}$ and study their properties in detail. In particular, $d$-minimal surfaces which we define are analogue objects to classical minimal surfaces. They are called isotropic minimal surfaces in terms of simply isotropic geometry (13). In addition to, we show a representation formula of Weierstrass type for $d$-minimal surfaces (Theorem 15), and claim that $d$-minimal surfaces allow to have isolated singularities. As an application, we prove that $d$-minimal surfaces and spacelike flat zero mean curvature (ZMC) surfaces in four-dimensional Minkowski space are in one-to-one correspondence (Corollary 26). In particular, we see that there exist infinitely many spacelike flat ZMC surfaces in $\mathbb{R}^4_1$, which are not congruent each other.

Because of these consequences, we see that spacelike flat ZMC surfaces in $\mathbb{R}^4_1$ are contained in a three-dimensional subspace endowed with a degenerate induced metric. However, we remark that it is the known fact by [1] and [9]. And, local expressions are given by [1], however, we study global expressions such as the representation formula and having singularities.

From Table 1, we see that $d$-minimal surfaces in $\mathbb{R}^{0,2,1}$ have neutral properties between minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{R}^3_1$. Regarding singularities, they do not appear on minimal surfaces. However, on maximal surfaces, cuspidal edges, swallowtails and cuspidal crosscaps appear in generic case. Refer to [8] in detail. On the other hand, for $d$-minimal surfaces, they allow to have isolated singularities. In this paper, these singularities are not classified.

2. Preliminaries

In this section, we explain the fundamental properties for semi-Riemannian manifolds and their non-degenerate submanifolds.

2.1. Semi-Riemannian manifolds. Let $(M, g)$ be an $n$-dimensional semi-Riemannian manifold. For each $x \in M$ and a tangent vector $X \in T_x M$, we
call $X$

\begin{align*}
\text{spacelike} \iff g(X,X) > 0 \text{ or } X = 0, \\
\text{timelike} \iff g(X,X) < 0, \\
\text{lightlike (or null)} \iff g(X,X) = 0.
\end{align*}

These are called \textit{causal properties} of tangent vectors \((\text{III})\). As in the case of Riemannian manifolds, there exists uniquely a torsion-free, and metric connection $\nabla$ for a semi-Riemannian manifold. We call $\nabla$ the \textit{Levi-Civita connection} of \((M,g)\). Hereinafter, we consider that connections for semi-Riemannian manifolds are Levi-Civita connections.

We define the \textit{curvature tensor field} $R$ of a semi-Riemannian manifold \((M,g)\) as

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad (X,Y,Z \in \Gamma(TM)).$$

Next, for each $x \in M$, let $P$ be a two-dimensional non-degenerate subspace of the tangent vector space $T_x M$, and let \\{\(X,Y\)\} be a basis of $P$. Then, we define the \textit{sectional curvature} $K(P)$ of $P$ as

$$K(P) := \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where a subspace $P \subset T_x M$ is called \textit{non-degenerate} if the restriction on $P$ of $g$ is the non-degenerate form, and it called \textit{degenerate} if not so. In particular, when the dimension of $M$ is two, sectional curvatures are called \textit{Gaussian curvatures}. We denote the set consisting of smooth functions on $M$ by $C^\infty(M)$. For each $u \in C^\infty(M)$, we define the \textit{gradient vector field} $\text{grad}u$ of $u$ as

$$g(\text{grad}u, X) = du(X) \quad (\forall X \in \Gamma(TM)),$$

where $du$ denotes the exterior derivative of $u$. Next, for each $X \in \Gamma(TM)$, we define the \textit{divergence} $\text{div}X$ of $X$ as

$$\text{div}X := \text{tr}((X_1, X_2) \mapsto g(\nabla_{X_1} X, X_2)) \quad (X_1, X_2 \in \Gamma(TM)).$$

For each $u \in C^\infty(M)$, we define the \textit{Laplacian} $\Delta_g u$ of $u$ with respect to $g$ as

$$\Delta_g u := \text{div}(\text{grad}u).$$

When $\Delta_g u \equiv 0$, we call a function $u$ \textit{harmonic}.

When let \\{\(e_1, \cdots, e_n\)\} be a local orthonormal frame of \((M,g)\), the gradient vector field and the divergence respectively have the following local expressions

\begin{align*}
\text{grad}u &= \sum_{i=1}^n \epsilon_i du(e_i)e_i, \\
\text{div}X &= \sum_{i=1}^n \epsilon_i g(\nabla_{e_i} X, e_i),
\end{align*}

where $\epsilon_i = g(e_i, e_i) = \pm 1$. 

2.2. Non-degenerate submanifolds. Let $M$ be an $m$-dimensional manifold, and let $(N, \bar{g})$ be an $n$-dimensional semi-Riemannian manifold. We assume that a $C^\infty$-mapping $f : M \to N$ is an immersion. Then, we call $M$ an immersed submanifold in $N$. In particular, when $f$ is injective, and $M$ is homeomorphic to the image $f(M)$ as the subspace of $N$, $M$ is said to be an embedded submanifold in $N$.

We denote the induced metric $f^* \bar{g}$ on $M$ by $g$. For semi-Riemannian manifolds, we remark that $g$ is not always non-degenerate even if $f$ is an immersion. When the induced metric $g$ is non-degenerate, we call $(M, g)$ a non-degenerate submanifold, or a semi-Riemannian submanifold of $(N, \bar{g})$.

Hereinafter, when we describe submanifolds, unless otherwise noted, we consider immersed, non-degenerate submanifolds. Then, for each $x \in M$, a normal vector space $T_x^\perp M$ is defined as

$$T_x^\perp M := \{v \in T_{f(x)}N \mid \bar{g}(df_x(w), v) = 0, \forall w \in T_x M\}.$$  

We obtain a vector bundle $T^\perp M = \bigcup_{x \in M} T_x^\perp M$ of rank $(n - m)$ over $M$. This is called a normal bundle of $M$. By this, for each $x \in M$, we have the orthogonal direct sum decomposition

$$T_{f(x)} N = T_x M \perp T_x^\perp M,$$

where $\perp$ stands for the orthogonal direct sum. In particular, we see that, as the orthogonal direct sum of vector bundles, it holds

$$f^* T N = T M \perp T^\perp M,$$

where $f^* T N$ is the pull-back bundle over $M$ by $f$. We denote the Levi-Civita connection of $(N, \bar{g})$ and that of $(M, g)$ by $\bar{\nabla}$ and $\nabla$ respectively. And, we define the set $\Gamma(T^\perp M)$ as the whole of smooth sections of the normal bundle $T^\perp M$. This section is said to be a normal vector field particularly.

For each $X, Y \in \Gamma(T M), \xi \in \Gamma(T^\perp M)$, by using the orthogonal direct sum decomposition given above, we have

\begin{align*}
\bar{\nabla}_X Y & = \nabla_X Y + h(X, Y), \\
\bar{\nabla}_X \xi & = -A_\xi X + \nabla^\perp_X \xi,
\end{align*}

where $h, A_\xi$ and $\nabla^\perp$ are called the second fundamental form, the shape operator with respect to $\xi$ and the normal connection on $M$ respectively. We call the formula (1) and (2) Gauss formula and Weingarten formula of $M$ respectively.

2.3. Singular semi-Euclidean spaces. We define the $n$-dimensional singular semi-Euclidean space with the signature $(p, q, r)$ as

$$\mathbb{R}^{p,q,r} := \left(\mathbb{R}^n, \langle \cdot, \cdot \rangle = -\sum_{i=1}^p dx_i^2 + \sum_{j=p+1}^{p+q} dx_j^2 + \sum_{k=p+q+1}^n 0 dx_k^2 \right),$$

where $n = p + q + r$ and $(x_1, \ldots, x_n)$ expresses the canonical coordinates on $\mathbb{R}^n$. We remark the following statement:

- When $r = 0$, $\mathbb{R}^{p,q,0}$ is called semi-Euclidean space having index $p$, and we denote it by $\mathbb{R}^p_n$.
- When $p = r = 0$, $\mathbb{R}^{0,n,0} = \mathbb{R}^n_0$ is nothing but Euclidean space $\mathbb{R}^n$. 

We remark that $r \geq 1$ if and only if the metric $(\cdot, \cdot)$ is degenerate. In isotropic geometry, the notation $\mathbb{R}^{0,n−1,1}$ is also known as the simply isotropic $n$-space $\mathbb{I}^n$ (\textbf{13}). From now on, we state fundamental objects for a semi-Euclidean space and its non-degenerate submanifolds.

For $n$-dimensional semi-Euclidean space $\mathbb{R}^n_p$ having index $p$ ($0 \leq p \leq q$), we assume that the semi-Euclidean metric is given by

$$\langle \cdot, \cdot \rangle_p := -\sum_{i=1}^p dx_i^2 + \sum_{j=p+1}^n dx_j^2,$$

where $(x_1, \cdots, x_n)$ is the canonical coordinates of $\mathbb{R}^n_p$. A non-zero vector $v$ in $\mathbb{R}^n_p$ is called spacelike, timelike and lightlike if it satisfies $\langle v, v \rangle_p > 0, \langle v, v \rangle_p < 0$ and $\langle v, v \rangle_p = 0$ respectively.

The $n$-dimensional semi-Euclidean space $\mathbb{R}^n_1$ having index one is said to be the $n$-dimensional Minkowski space. Moreover, the four-dimensional Minkowski space is closely related to the physics as the flat spacetime model.

Let $M$ be an $m$-dimensional non-degenerate submanifold in $\mathbb{R}^n_p$. We denote the Levi-Civita connections for $\mathbb{R}^n_p$ and $M$ by $\nabla$ and $\bar{\nabla}$ respectively. And, let $X,Y,Z,W$ and $\xi,\eta$ be tangent vector fields and normal vector fields on $M$ respectively.

\textbf{Gauss equation, Codazzi equation and Ricci equation of $M$ are given by the following}

\begin{align}
(3) \quad \langle R(X,Y)Z,W \rangle_p &= \langle h(Y,Z), h(X,W) \rangle_p - \langle h(X,Z), h(Y,W) \rangle_p, \\
(4) \quad \langle \nabla_X h(Y,Z) \rangle &= \langle \nabla_Y h(X,Z) \rangle, \\
(5) \quad \langle R^\perp(X,Y)\xi,\eta \rangle_p &= \langle [A_\xi, A_\eta]X,Y \rangle_p,
\end{align}

where $R$ and $R^\perp$ are curvature tensor fields with respect to connections $\nabla$ and $\bar{\nabla}$ respectively, and $\nabla_X h$ is the covariant derivative of the second fundamental form $h$ for the tangent vector field $X$, i.e. it is defined by

$$\nabla_X h(Y,Z) = \nabla_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

Moreover, the normal bundle $T^\perp M$ of $M$ is called flat if $R^\perp \equiv 0$.

Let $\{e_1, \cdots, e_m\}$ be a local orthonormal frame of the tangent bundle $TM$, and let $\{e_{m+1}, \cdots, e_n\}$ be a local orthonormal frame of the normal bundle $T^\perp M$. In addition to, setting $e_A := \langle e_A, e_A \rangle_p = \pm 1$, we use the following range of indices:

$$1 \leq A, B, C, \cdots \leq n, \quad 1 \leq i, j, k, \cdots \leq m, \quad m + 1 \leq \alpha, \beta, \gamma, \cdots \leq n.$$

We denote the connection form of $\nabla$ associated $\{e_1, \cdots, e_m\}$ by $\{\omega^i_j\}$, and we denote the connection form of $\nabla^\perp$ associated $\{e_{m+1}, \cdots, e_n\}$ by $\{\omega^\perp_{\beta}\}$. Then, from Gauss formula \textbf{(1)} and Weingarten formula \textbf{(2)}, we have

\begin{align}
\nabla_{e_k} e_i &= \sum_{j=1}^m \epsilon_j \omega^j_i(e_k)e_j + \sum_{\alpha=m+1}^n \epsilon_\alpha h^\alpha_{ik}e_\alpha, \\
\nabla_{e_k} e_\beta &= -\sum_{j=1}^m \epsilon_j h^\beta_{kj}e_j + \sum_{\alpha=m+1}^n \epsilon_\alpha \omega^\perp_{\beta}(e_k)e_\alpha,
\end{align}
where $h_{ij}^\alpha$ are coefficients of the second fundamental form. Moreover, we see that the mean curvature vector field $\vec{H}$ of $M$ is expressed by

$$\vec{H} = \frac{1}{m} \sum_{\alpha=m+1}^n \epsilon_\alpha \text{tr} A\alpha e_\alpha,$$

where $\text{tr} A\alpha$ is the trace of the shape operator $A_{e_\alpha}$ with respect to $e_\alpha$, i.e. $\text{tr} A\alpha = \sum_{i=1}^n \epsilon_i h_{ii}^\alpha$.

3. $d$-minimal surfaces in singular semi-Euclidean space

In this section, we consider three-dimensional singular semi-Euclidean space with the signature $(0, 2, 1)$. We define $\mathbb{R}^{0,2,1}$ as

$$\mathbb{R}^{0,2,1} := (\mathbb{R}^3, (\cdot, \cdot) = dx^2 + dy^2),$$

where let $(x, y, z)$ be the canonical coordinates. And, we study surfaces in $\mathbb{R}^{0,2,1}$.

3.1. Preparations. Let $M$ be a two-dimensional manifold, let $f: M \to \mathbb{R}^{0,2,1}$ be a $C^\infty$-immersion and let $g$ be the induced metric by $f$. We assume that the metric $g$ is a positive definite symmetric bilinear form. And, we call $f$ a non-degenerate immersion or a non-degenerate surface. Then, for each $x \in M$, a normal vector space $T_x^\perp M$ is defined by

$$T_x^\perp M := \{ \xi \in \mathbb{R}^3 \mid (df_x(v), \xi) = 0, \forall v \in T_x M\} = \text{span}_\mathbb{R}\{(0, 0, 1)\},$$

and we have a vector bundle of rank one over $M$

$$T^\perp M = \bigcup_{x \in M} T_x^\perp M.$$

Therefore, we obtain an orthogonal direct sum decomposition

$$T_f(x)^3 = T_x M \perp T_x^\perp M$$

for each $x \in M$. In particular, we see, as a vector bundle decomposition,

$$f^* T\mathbb{R}^3 = TM \perp T^\perp M,$$

where $TM$ is the tangent bundle over $M$ and $f^* T\mathbb{R}^3$ is the pull-back bundle by $f$ over $M$.

Proposition 1. We get an isomorphism as vector bundle

$$T^\perp M \cong M \times \mathbb{R}.$$ 

Proof. We can take $\xi = (0, 0, 1) \in \Gamma(T^\perp M)$ as a non-vanishing global section. \qed

Remark 2. For three-dimensional singular semi-Euclidean space with the signature $(p, q, r)$, where $p + q + r = 3$, $r \geq 1$, $p \leq q$, we can define non-degenerate surfaces when $r = 1$, i.e.

$$(p, q, r) = (0, 2, 1), (1, 1, 1).$$

When $r \geq 2$, the metric induced on surfaces is degenerate. We remark that $\mathbb{R}^{1,1,1}$ is equivalent to the pseudo-isotropic 3-space $\mathbb{I}^3_1$ (Refer to [10], [17] and [14]). And, as a notation, we define

$$|v| := \sqrt{(v, v)} = \sqrt{v_1^2 + v_2^2}$$
for a vector \( v = (v_1, v_2, v_3) \in \mathbb{R}^{0,2,1} \).

Next, we recall affine differential geometry ([10]). Let \((\mathbb{R}^{n+1}, d)\) be \((n+1)\)-dimensional Euclidean space with the canonical connection \( d \) and \( M \) be an \( n \)-dimensional manifold. A \( C^\infty \)-immersion \( f : M \to \mathbb{R}^{n+1} \) is an affine immersion if for any \( x \in M \) there exists a neighborhood \( U \) at \( x \) and a vector field \( \xi \) on \( U \) over \( \mathbb{R}^{n+1} \) such that

\[
T_{f(y)}\mathbb{R}^{n+1} = T_yM \oplus \mathbb{R} \xi_y \quad (\forall y \in U),
\]

where \( \oplus \) stands for the direct sum. In particular, when there exists \( \xi \) globally on \( M \), it is called a transversally vector field on \( M \). Then, a torsion-free connection \( \nabla \) is induced on \( M \), and it satisfies

\[
d_XY = \nabla_XY + h(X,Y)\xi
\]

for any \( X, Y \in \Gamma(TM) \). This implies that \( h \) is a \((0,2)\)-type symmetric tensor field over \( M \), and we call \( h \) an affine fundamental form (with respect to \( \xi \)). In affine differential geometry, we often assume that \( h \) is non-degenerate. Moreover, let \( f : M \to \mathbb{R}^{n+1} \) be an affine immersion and let \( \xi \) be its transversally vector field. we call \( \xi \) equiaffine when

\[
\forall X \in \Gamma(TM), \quad d_X\xi \in \Gamma(TM).
\]

Then, \( f \) is called an equiaffine immersion.

In terms of affine differential geometry, we see the following proposition.

**Proposition 3.** Let \( M \) be a two-dimensional manifold. A non-degenerate immersion \( f : M \to \mathbb{R}^{0,2,1} \) is an equiaffine immersion whose transversally vector field over \( M \) is \( \xi \equiv (0,0,1) \).

**Proof.** By using the orthogonal direct sum \( f^*T\mathbb{R}^3 = TM \perp T^\perp M \), and \( d_X\xi = 0 \) for all \( X \in \Gamma(TM) \), the proof is completed. \( \square \)

Hereinafter, let \( \xi \) be the constant vector field \( \xi = (0,0,1) \) and let \( d \) be the canonical connection as a linear connection, i.e. for all \( X, Y \in \Gamma(T\mathbb{R}^{0,2,1}) \), identifying \( Y \) with the vector-valued function \( Y = (Y_1,Y_2,Y_3) \),

\[
d_XY := dX(Y) = (X(Y_1), X(Y_2), X(Y_3)).
\]

Then, the connection \( d \) is torsion-free and preserves the degenerate metric \((\cdot,\cdot)\). Thus, the connection \( d \) plays the role of the Levi-Civita connection.

We define the automorphism group \( \text{Aut}(\mathbb{R}^{0,2,1}, d) \) with respect to \( \mathbb{R}^{0,2,1} \) and \( d \) as

\[
\text{Aut}(\mathbb{R}^{0,2,1}, d) := \{ A \in \text{Diff}(\mathbb{R}^3) \mid A^*d = d, \; A^*(\cdot,\cdot) = (\cdot,\cdot) \}
\]

\[
= O(0,2,1) \times \mathbb{R}^3,
\]

where \( \text{Diff}(\mathbb{R}^3) \) is the diffeomorphism group of \( \mathbb{R}^3 \) and

\[
O(0,2,1) := \left\{ \begin{pmatrix} T & 0 \\ a & b & c \end{pmatrix} \bigg| \begin{array}{l} a, b, c \in \mathbb{R}, \ c \neq 0, \ T \in O(2) \end{array} \right\}.
\]

We call \( \text{Aut}(\mathbb{R}^{0,2,1}, d) \) an affine isometry group. In particular, \( \text{Aut}(\mathbb{R}^{0,2,1}, d) \) is a seven-dimensional Lie group. From the view of Cayley-Klein geometry,
this automorphism group is nothing but the simply isotropic rigid motion group \( [16] \).

By using the decomposition \( f^*T\mathbb{R}^3 = TM \perp T^\perp M \), for each \( X,Y \in \Gamma(TM) \), \( \alpha \xi \in \Gamma(T^\perp M) \) (\( \alpha \in C^\infty(M) \)), we have

\[
\begin{align*}
d_XY &= \nabla_XY + h(X,Y) \xi, \\
d_X(\alpha \xi) &= X(\alpha) \xi.
\end{align*}
\]

Then, we see that the connection \( \nabla \) is the Levi-Civita connection with respect to the induced metric \( g \) on \( M \). And, we call the given affine fundamental form \( h \) a second fundamental form of the non-degenerate immersion \( f \).

For all \( X,Y,Z \in \Gamma(TM) \), since the connection \( d \) is flat, we obtain

\[
0 = d^R(X,Y)Z = \nabla^R(X,Y)Z + \{(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)\} \xi,
\]

where \( d^R \) and \( \nabla^R \) are the curvature tensor fields for \( d \) and \( \nabla \) respectively, and we define \( (\nabla_X h)(Y,Z) := X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \). Therefore, we get

\[
\begin{align*}
\nabla^R &\equiv 0, \\
(\nabla_X h)(Y,Z) &= (\nabla_Y h)(X,Z).
\end{align*}
\]

The formula (9) implies that the non-degenerate surface is always flat, and we call the formula (10) Gauss-Codazzi equation of the non-degenerate surface. These formulas (9) and (10) were obtained by Sachs in [13].

Let \( f : M \to \mathbb{R}^{0,2,1} \) be a non-degenerate immersion. Then, the image of \( f \) is locally expressed by the form of a graph surface \( \{(u,v,F(u,v)) \in \mathbb{R}^{0,2,1} \mid (u,v) \in U \} \), where \( F \) is a smooth function on an open subset \( U \subset \mathbb{R}^2 \).

Next, we define some classes for non-degenerate surfaces.

(i) \( d \)-totally geodesic surface \( \iff \) the second fundamental form \( h \equiv 0 \),

(ii) \( d \)-totally umbilical surface \( \iff \exists \lambda \in C^\infty(M) \mathrm{ s.t. } h = \lambda g \),

(iii) \( d \)-minimal surface \( \iff \mathcal{H} := \frac{1}{2} \mathrm{tr}_g h = \frac{1}{2} g^{ij} h_{ij} = 0 \),

where \( g^{ij} \) is the components of the inverse matrix of \( (g_{ij})_{1 \leq i,j \leq 2} \) and \( h_{ij} \) is the coefficients of the second fundamental form \( h \). We call \( \mathcal{H} \) the mean curvature of the non-degenerate surface. For (ii), we remark that (ii) is equivalent to (i) when \( \lambda = 0 \).

**Proposition 4.** Let \( M \) be a two-dimensional manifold, and let \( f : M \to \mathbb{R}^{0,2,1} \) be connected, not \( d \)-totally geodesic and \( d \)-totally umbilical surface, that is, there exists a function \( \lambda \in C^\infty(M) \) such that \( h = \lambda g \) and \( \lambda \neq 0 \). Then, \( \lambda \) is a constant function, and the image of \( f \) is an open subset of a paraboloid of revolution

\[
\left\{(u,v, \frac{\lambda}{2}(u^2 + v^2) + Au + Bv + C) \in \mathbb{R}^3 \mid (u,v) \in \mathbb{R}^2 \right\},
\]

where \( A, B, C \in \mathbb{R} \) are constant. In particular, it is, up to affine isometry, an open subset of

\[
\left\{(u,v,u^2 + v^2) \in \mathbb{R}^3 \mid (u,v) \in \mathbb{R}^2 \right\}.
\]
Proof. Since non-degenerate surfaces satisfy Gauss-codazzi equation (10), the function $\lambda$ is a constant. Let $g$ be the induced metric by $f$ and let $h$ be its second fundamental form. From the assumption, there exists a non-zero constant number $\lambda \in \mathbb{R}$ such that $h = \lambda g$. Since $f$ is the non-degenerate immersion, for each point of $M$, there exists a coordinate neighborhood $\{U; (u,v)\}$ such that

$$f(u,v) = (u,v,\varphi(u,v)) \in \mathbb{R}^{0,2,1},$$

where $\varphi$ is a $C^\infty$-function on $U$. Then, we get

$$h_{11} = \varphi_{uu}, \ h_{12} = \varphi_{uv}, \ h_{22} = \varphi_{vv}.$$ 

Therefore, since we have

$$\varphi_{uu} = \lambda g_{11} = \lambda, \ \varphi_{uv} = \lambda g_{12} = 0, \ \varphi_{vv} = \lambda g_{22} = \lambda,$$

there exist constant numbers $A, B, C \in \mathbb{R}$ such that

$$\varphi(u,v) = \frac{\lambda}{2}(u^2 + v^2) + Au + Bv + C.$$

Finally, gluing these pieces of surface in the whole of $M$, we obtain the consequence. □

Here, we define a relative Gaussian curvature $K$ which is introduced in [13] as

$$K := \frac{\det h}{\det g} \in C^\infty(M).$$

This quantity expresses the shape of the non-degenerate surface when we look from the ambient space $\mathbb{R}^3$. However, the canonical Gaussian curvature, i.e. the sectional curvature of two-dimensional Riemannian manifolds with respect to the induced metric, identically vanishes.

**Proposition 5** ([13], Definition 8.11). Let $M$ be a two-dimensional manifold, and let $f : M \to \mathbb{R}^{0,2,1}$ be a non-degenerate immersion. Let $K$ be its relative Gaussian curvature. Then, in a sense of surface theory in the canonical Euclidean space $\mathbb{R}^3$, we have

$$K(x) > 0 \iff x : \text{elliptic point},$$

$$K(x) < 0 \iff x : \text{hyperbolic point},$$

$$K(x) = 0 \iff x : \text{parabolic point},$$

for each $x \in M$. However, when we consider $f$ as an immersion to Euclidean space $\mathbb{R}^3$, the canonical Gaussian curvature do not correspond with the relative Gaussian curvature in general.

**Remark 6.** We consider the sign of the relative Gaussian curvature for some surfaces. First, for $d$-totally geodesic surfaces, since we have $h = 0$ by definition, it holds

$$K = \frac{\det h}{\det g} \equiv 0.$$ 

Next, for $d$-totally umbilical surfaces, we have, by definition, there exists a constant number $\lambda \in \mathbb{R}$ such that $h = \lambda g$. We assume $\lambda \neq 0$. Then, we obtain

$$K = \frac{\det h}{\det g} = \frac{\lambda^2 \det g}{\det g} = \lambda^2 > 0,$$
that is, all points are elliptic. Finally, for $d$-minimal surfaces, we make use of isothermal coordinates, that is, we choose the coordinates in which the coefficients of the induced metric hold
\[ g_{11} = g_{22} > 0, \quad g_{12} = 0. \]
Then, since the mean curvature identically vanishes, we have
\[ 2\mathcal{H} = \text{tr}_g h = \frac{g_{22}h_{11} + g_{11}h_{22}}{g_{11}g_{22}} = \frac{h_{11} + h_{22}}{g_{11}} \equiv 0. \]
Moreover, by using $h_{22} = -h_{11}$, we obtain
\[ K = \frac{\det h}{\det g} = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22}} = -\frac{h_{11}^2 + h_{12}^2}{g_{11}^2} \leq 0, \]
that is, almost all points are hyperbolic.

Here, we give some descriptions for curves in $\mathbb{R}^{0,2,1}$. For a connected open interval $I \subset \mathbb{R}$, let $c$ be a $C^\infty$-map $c : I \to \mathbb{R}^{0,2,1}$. We call $c$ a curve in $\mathbb{R}^{0,2,1}$. Moreover, we call $c$ a regular curve if it holds
\[ \forall t \in I, \quad c'(t) \neq 0. \]
Next, let $\pi$ be the projection to $xy$-plane, i.e.
\[ \pi : \mathbb{R}^{0,2,1} \ni (x,y,z) \mapsto (x,y) \in \mathbb{R}^2. \]
And, we call a parameter $s$ of a curve $c = c(s)$ arc-length if it holds
\[ |c'(s)| \equiv 1. \]
Then, we obtain the following propositions.

**Proposition 7.** Let $c = c(t)$ ($t \in I$) be a regular curve in $\mathbb{R}^{0,2,1}$. The following are equivalent:
(i) The curve $c = c(t)$ admits an arc-length parameter.
(ii) For all $t \in I$, it holds $|c'(t)| > 0$.
(iii) The mapping $\pi \circ c$ is regular as a planar curve in $\mathbb{R}^2$.

**Proof.** Easy calculations. $\square$

We call a regular curve $c = c(t)$ ($t \in I$) in $\mathbb{R}^{0,2,1}$ null if it holds
\[ |c'(t)| \equiv 0. \]

**Proposition 8.** A regular curve $c : I \to \mathbb{R}^{0,2,1}$ is null if and only if it is a spacial line which is parallel with the $z$-axis.

**Proof.** Easy calculations. $\square$

**Proposition 9.** For any connected surfaces in $\mathbb{R}^{0,2,1}$,
(0) $d$-totally geodesic surfaces in $\mathbb{R}^{0,2,1}$ are non-degenerate planes only (\cite{13}, Theorem 9.4).
(1) a graph surface in $\mathbb{R}^{0,2,1}$
\[ \{(u,v,f(u,v)) \in \mathbb{R}^{0,2,1} \mid (u,v) \in U \subset \mathbb{R}^2\} \]
is $d$-minimal if and only if $f$ is a harmonic function on $U$ (\cite{13}, Eq. (9.31)).
(2) non-planar, ruled $d$-minimal surfaces in $\mathbb{R}^{0,2,1}$ are locally, up to affine isometry, open subset of
(a) \( f(u, v) = (v \cos u, v \sin u, u) \) (refer to Figure 1),
(b) \( f(u, v) = (u, v, uv) \) (refer to Figure 2),
where \((u, v) \in \mathbb{R}^2\) (15, Theorem 6).

(3) non-planar, \( d \)-minimal rotational surfaces in \( \mathbb{R}^{0,2,1} \) are locally, up to
affine isometry, open subset of
\[
\begin{align*}
f(u, v) &= (e^u \cos v, e^u \sin v, u) \\
&= (\gamma(s, t) = (t \cos s, t \sin s, s)) \in \mathbb{R}^{0,2,1}.
\end{align*}
\]
(refer to Figure 3), where the rotational surfaces mean the rotation
group, which acts on the \( xy \)-plane, \( SO(2) \)-invariant surfaces.

**Proof.** (0) and (1) are proved by easy calculations.

In case of (2), we apply the method of classification described by [15].
Since we have the fact that the induced metrics of non-degenerate
immersions are positive definite, ruled \( d \)-minimal surfaces of cylinder type are
planes only. Thus, we have only to investigate the case of non-cylinder type.
Let curves \( \gamma(s) \) and \( x(s) \) be a direction curve and a base curve of the given
ruled surface respectively. Since we consider the case of non-cylinder, the
direction curve \( \gamma \) is regular. When \( |\gamma'| \neq 0 \), we can take the arc-length pa-
rameter of \( \gamma \) from Proposition 7. Then, we may set \( \eta := |\gamma'|^2 = (\gamma', \gamma') \equiv 1 \)
or 0 in generic.

When \( \eta \equiv 1 \), we see that the direction curve is \( \gamma(s) = \cos s e_1 + \sin s e_2 \),
where vectors \( e_1, e_2 \in \mathbb{R}^{0,2,1} \) satisfy \( |e_1| = |e_2| = 1 \), \( (e_1, e_2) = 0 \). Therefore,
it holds that \( e_1, e_2 \) are tangent vectors. If we assume that \( (\gamma'(s), x'(s)) \equiv 0 \),
it holds \( (x'(s), x'(s)) \equiv 0 \). Thus, regarding the Table 2 in [15], the case of (i)
does not exist. Since the case of (iii) is reduced to the case of (ii), we have
only to consider the case of (ii). For the case of (ii), by an affine isometry,
we have
\[
f(s, t) = (t \cos s, t \sin s, s).
\]

Next, when \( \eta \equiv 0 \), we see that \( \gamma' \) is a normal vector. Thus, it holds
\( (\gamma'(s), x'(s)) \equiv 0 \). Therefore, we have only to consider the case of (v). Then,
by an affine isometry, we have
\[
f(s, t) = (s, t, st).
\]

In case of (3), we explain the meaning of \( SO(2) \)-invariant firstly. It is
well-known that
\[
SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in M_2(\mathbb{R}) \bigg| \theta \in \mathbb{R} \right\}.
\]
We realize \( SO(2) \) as a subgroup of \( \text{Aut}(\mathbb{R}^{0,2,1}, d) \) as below.
\[
H := \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Aut}(\mathbb{R}^{0,2,1}, d) \bigg| \theta \in \mathbb{R} \right\}.
\]
Then, the group \( H \) is isomorphic to \( SO(2) \) as a Lie group. We simply denote
\( H \) as \( SO(2) \). A surface is said to \( SO(2) \)-invariant if it is invariant under the
action of this group. Such surfaces are locally parametrized by
\[
f(u, v) = (x(u) \cos v, x(u) \sin v, y(u)) \in \mathbb{R}^{0,2,1},
\]
where \( x, y \) are real variable functions satisfying \( x > 0 \), \( (x')^2 + (y')^2 = 1 \). Then, we have
\[
f_u = (x' \cos v, x' \sin v, y'), \quad f_v = (-x \sin v, x \cos v, 0).
\]
Thus, we compute
\[
g_{11} = (x')^2, \quad g_{12} = 0, \quad g_{22} = x^2.
\]
The non-degeneracy implies \( x' \neq 0 \). Moreover, since we compute
\[
f_{uu} = (x'' \cos v, x'' \sin v, y'') = \frac{x''}{x'} f_u + \left( -\frac{x''}{x'} y' + y'' \right) \xi,
\]
\[
f_{uv} = (-x' \sin v, x' \cos v, 0) = \frac{x'}{x} f_v,
\]
\[
f_{vv} = (-x \cos v, -x \sin v, 0) = -\frac{x}{x'} f_u + \frac{x}{x'} y' \xi,
\]
the coefficients of second fundamental form \( h \) hold
\[
h_{11} = -\frac{x''}{x} y' + y'', \quad h_{12} = 0, \quad h_{22} = \frac{x}{x'} y'.
\]
Therefore, we compute that the mean curvature of \( SO(2) \)-invariant \( d \)-minimal surfaces is
\[
2\mathcal{H} = g^{ij} h_{ij} = \frac{1}{(x')^3} \left[ -y'' \frac{y'}{x} + x' y'' + \frac{y'}{x} \right] = 0.
\]
Since \( x' \neq 0 \), by the coordinate transformation, we can represent \( y \) as a function with respect to \( x \). Then, the equation (11) is equal to the following equation
\[
\frac{d^2 y}{dx^2} = -\frac{1}{x} \frac{dy}{dx}.
\]
By solving the ordinary differential equation, we have
\[
y(x) = C_1 \log x + C_2 \quad (C_1, C_2 \in \mathbb{R} : \text{constants}).
\]
Again, when we replace the parameter \( x \) with \( x(w) = e^w \), we get \( y(w) = C_1 w + C_2 \). In particular, if \( C_1 = 0 \), then it is a plane. So, if it is not a plane, by an affine isometry, we obtain
\[
f(u, v) = (e^u \cos v, e^u \sin v, u).
\]
The proof is completed. \( \square \)
Remark 10. We recall that non-degenerate surfaces are locally expressed by graph surfaces. However, (a) of Proposition 9 is an example which cannot be entirely expressed as a graph surface.

We consider the canonical connection $d$ as a linear connection for $\mathbb{R}^{0,2,1}$. This connection $d$ is a torsion-free connection which is parallel with respect to the degenerate metric $(\cdot, \cdot)$, i.e. $d$ plays the role of Levi-Civita connection. However, since the metric is degenerate, connections having such properties are not unique. For example, let $\lambda \in \mathbb{R}$ be a real parameter, and we define a tensor field $L_\lambda \in \Gamma(S^2T^*\mathbb{R}^3)$ as

$$L_\lambda(X,Y) := \lambda \sum_{i,j} X_i Y_j,$$

where the set $\Gamma(S^2T^*\mathbb{R}^3)$ expresses the whole of $(0,2)$-type symmetric tensor fields over $\mathbb{R}^3$ and $X,Y$ are vector fields over $\mathbb{R}^3$, and we regard $X$ and $Y$ respectively as vector-valued functions $X = (X_1, X_2, X_3), Y = (Y_1, Y_2, Y_3)$.

Then, when we put $d^\lambda := d + L_\lambda \xi$, $d^\lambda$ is a flat connection over $\mathbb{R}^{0,2,1}$ which has the same properties of Levi-Civita connections.

When we consider $d^\lambda$-totally geodesic surfaces defined as the case of $d$, non-trivial examples appear, i.e. there exist examples which are not planes (refer to fig. 4).

As an example satisfying $h^\lambda \equiv 0$ except for planes, we find, for instance,

$$F(u, v) = \frac{1}{\lambda} \log |\lambda u + 1| - u - v,$$

where $u < -\frac{1}{\lambda}, u > -\frac{1}{\lambda}$.

As a remark, let $\nabla$ be a torsion-free, metric connection on $\mathbb{R}^{0,2,1}$, if all non-degenerate plane are $\nabla$-totally geodesic, then it holds $\nabla = d$, i.e. the case of simply isotropic geometry.
3.2. Representation formula of Weierstrass type for $d$-minimal surfaces. Let $f : M \to \mathbb{R}^{0,2,1}$ be a non-degenerate immersion. When we set $f = (f_1, f_2, f_3)$, we define Laplacian $\Delta_g f$ of $f$ with respect to the induced metric $g$ as Laplacians of each coordinate functions $f_i$ ($i = 1, 2, 3$), i.e.

$$\Delta_g f := (\Delta_{g} f_1, \Delta_{g} f_2, \Delta_{g} f_3).$$

**Proposition 11.** Let $H$ be the mean curvature of a non-degenerate immersion $f$. Then, $2H\xi \in \Gamma(T^\perp M)$ is equal to Laplacian $\Delta_g f$ of $f$ with respect to the induced metric $g$. In particular, the non-degenerate surface is a $d$-minimal if and only if coordinate functions of $f$ are all harmonic with respect to $g$.

**Proof.** Since $f$ is non-degenerate, there exists a coordinate neighborhood $U$ of $M$ such that the local expression of $f$ is

$$f(u, v) = (u, v, F(u, v)) \in \mathbb{R}^{0,2,1},$$

where $F$ is a function on $U$. By using this coordinate, we get

$$2H\xi = (0, 0, F_{uu} + F_{vv}) = \Delta_g f.$$

The proof is completed. $\square$

In case of graph surfaces, Proposition 11 is equivalent to the formula (8) in [12].

Next, we prepare some simple lemmas.

**Lemma 12.** For a real two variable function $f(u, v)$, we define a complex function $F(w)$ with respect to the complex variable $w = u + iv$ as

$$F(w) := \frac{\partial f}{\partial u}(u, v) - i\frac{\partial f}{\partial v}(u, v).$$

Then, $F$ is a holomorphic function if and only if $f(u, v)$ is a harmonic function.

**Proof.** The Cauchy-Riemann’s equations imply. $\square$

**Lemma 13.** In $\mathbb{R}^{0,2,1}$, we consider a surface given by

$$f(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^{0,2,1}.$$
We define a complex function \( \varphi, \psi \) with respect to the complex variable \( w = u + iv \) as

\[
\varphi(w) := \frac{\partial x}{\partial u}(u, v) - i \frac{\partial x}{\partial v}(u, v), \quad \psi(w) := \frac{\partial y}{\partial u}(u, v) - i \frac{\partial y}{\partial v}(u, v).
\]

Then, the coordinates \((u, v)\) is isothermal if and only if it holds

\[
\varphi^2 + \psi^2 \equiv 0.
\]

**Proof.** By direct calculations, we have

\[
\varphi^2 + \psi^2 = |f_u|^2 - |f_v|^2 - 2i(f_u, f_v).
\]

This completes the proof. \(\square\)

**Theorem 14.** Let \( U \) be an open subset of \( uv\)-plane. In \( \mathbb{R}^{0.2,1} \), let \( f \) be an immersion on \( U \) which is parametrized by \( f(u, v) = (x(u, v), y(u, v), z(u, v)) \). We assume that \((u, v)\) is the isothermal coordinates and \( f \) is \( d\)-minimal. Then, complex functions \( \varphi_1, \varphi_2, \varphi_3 \) with respect to the complex variable \( w = u + iv \) defined by

\[
\varphi_1(w) = \frac{\partial x}{\partial u} - i \frac{\partial x}{\partial v}, \quad \varphi_2(w) = \frac{\partial y}{\partial u} - i \frac{\partial y}{\partial v}, \quad \varphi_3(w) = \frac{\partial z}{\partial u} - i \frac{\partial z}{\partial v}
\]

are all holomorphic, and it holds

\[
|\varphi_1|^2 + |\varphi_2|^2 > 0, \quad \varphi_1^2 + \varphi_2^2 = 0.
\]

Moreover, it holds

\[
(f_u, f_u) = (f_v, f_v) = \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2).
\]

Conversely, let \( U \) be a simply-connected domain on \( \mathbb{C} \), and we assume that holomorphic functions \( \varphi_1(w), \varphi_2(w), \varphi_3(w) \) satisfy the formula (13). Then, when we set \( w = u + iv \in U \), there exists a \( d\)-minimal surface satisfying the formula (12) such that, for the parametrized expression \( f(u, v) = (x(u, v), y(u, v), z(u, v)) \), the coordinates \((u, v)\) are isothermal.

**Proof.** Since \( f \) is \( d\)-minimal, each coordinate functions are harmonic from Proposition 11. Thus, by using Lemma 12 each \( \varphi_i \) are holomorphic. And, since \((u, v)\) is isothermal coordinates, it holds \( \varphi_1^2 + \varphi_2^2 \equiv 0 \) from Lemma 13. Next, since we compute

\[
|\varphi_1|^2 + |\varphi_2|^2 = x_u^2 + y_u^2 + x_v^2 + y_v^2 = |f_u|^2 + |f_v|^2 = 2|f_u|^2 = 2|f_v|^2 > 0,
\]

the former of the claim holds. For the latter, we assume that holomorphic functions \( \varphi_1, \varphi_2, \varphi_3 \) on a simply-connected domain \( U \) satisfy the formula (13). We fix a point \( w_0 \in U \) and define a real function \( x = x(u, v) \) as

\[
x(u, v) := \text{Re} \int_{w_0}^{w} \varphi_1(w) dw \quad (w = u + iv \in U).
\]

This is well-defined since \( U \) is simply-connected. When we act on this formula by the differential operator

\[
\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} = 2 \frac{\partial}{\partial w},
\]

we have

\[
\frac{\partial x}{\partial u} - i \frac{\partial x}{\partial v} = 2 \frac{\partial}{\partial w} \text{Re} \int_{w_0}^{w} \varphi_1(w) dw = \varphi_1(w).
\]
As above, when we define \( y = y(u,v) \) and \( z = z(u,v) \), we have
\[
\frac{\partial y}{\partial u} - i \frac{\partial y}{\partial v} = \varphi_2(w), \quad \frac{\partial z}{\partial u} - i \frac{\partial z}{\partial v} = \varphi_3(w).
\]
From Lemma 12 again, we see that \( x(u,v), y(u,v), z(u,v) \) are harmonic functions on \( U \). Next, we prove that the mapping \( f = f(u,v) \) gives a surface, i.e. two-dimensional manifold. For the purpose of that, we prove that the Jacobi matrix
\[
\begin{pmatrix}
x_u & y_u & z_u \\
x_v & y_v & z_v
\end{pmatrix}
\]
is rank two for any point \( w \in U \). We prove by using contradiction, i.e. we assume that there is a point \( w' \in U \) such that the rank of its Jacobi matrix is less than one. Since we have
\[
0 < |\varphi_1|^2 + |\varphi_2|^2 = (x_u)^2 + (x_v)^2 + (y_u)^2 + (y_v)^2,
\]
at the point \( w' \), we see that either of column vectors
\[
\begin{pmatrix}
x_u \\
x_v
\end{pmatrix}, \begin{pmatrix}
y_u \\
y_v
\end{pmatrix}
\]
is not the zero vector. So, we suppose that the former is not the zero vector. From the assumption of contradiction, since we may set \( \exists \lambda \in \mathbb{R} \) s.t. \( \varphi_2 = \lambda \varphi_1 \), we compute
\[
\varphi_1(w')^2 + \varphi_2(w')^2 = (1 + \lambda^2)\varphi_1(w')^2 \neq 0.
\]
at \( w' \). This contradicts the formula (13). Thus, since \( f \) is a \( C^\infty \)-immersion, \( f(u,v) = (x(u,v), y(u,v), z(u,v)) \) gives a surface in \( \mathbb{R}^{0,2,1} \), and \( (u,v) \in U \) is the isothermal coordinates from the condition (14). In particular, \( f \) is a \( d \)-minimal surface satisfying the formula (12).

**Theorem 15** (Weierstrass-type representation formula for \( d \)-minimal surfaces). Let \( U \subset \mathbb{C} \) be a simply-connected domain, and let \( F,G \) be holomorphic functions on \( U \), where \( F \) does not have zero points on \( U \). Then, a mapping
\[
f(u,v) = \text{Re} \int_w (F, -iF, G) dw \quad (w := u + iv \in U)
\]
gives a \( d \)-minimal surface in \( \mathbb{R}^{0,2,1} \), and the coordinates \( (u,v) \in U \) are isothermal. Moreover, it holds
\[
(f_u, f_u) = (f_v, f_v) = |F|^2.
\]
Conversely, a \( d \)-minimal surface in \( \mathbb{R}^{0,2,1} \) locally have the expression as above.

**Proof.** For the former of the claim, when we set \( \varphi_1 := F, \varphi_2 := -iF, \varphi_3 := G \), it immediately holds from Theorem 14. For the latter of the claim, given a \( d \)-minimal surface, it is locally considered on a simply-connected domain. From Theorem 14 again, we have the parametrized expression
\[
f(u,v) = \text{Re} \int_w (\varphi_1, \varphi_2, \varphi_3) dw.
\]
Since it satisfies
\[ |\varphi_1|^2 + |\varphi_2|^2 > 0, \quad \varphi_1^2 + \varphi_2^2 = 0, \]
setting \( F := \varphi_1, G := \varphi_3 \), we obtain the expression which we want. \( \square \)

Zero points of \( G \) correspond with singularities of \( d \)-minimal surfaces. For example, we see cross-caps on \( d \)-minimal surfaces. We remark that there exist some singularities not only cross-caps. We state in detail in the next section.

At the end of this section, for Weierstrass type expression formula for \( d \)-minimal surfaces
\[
f(u, v) = \text{Re} \int_w (F, -iF, G)dw \quad (w := u + iv \in U),
\]
the function \( F \) expresses the induced metric, i.e. it holds \((f_u, f_u) = (f_v, f_v) = |F|^2\). On the other hand, the function \( G \) is concerned with the second fundamental form \( h \) by the following proposition.

**Proposition 16.** Under the situation stated above, it holds
\[
h = \left\{ (\text{Re}G)_u - \frac{|F|_u}{|F|} (\text{Re}G) - \frac{|F|_v}{|F|} (\text{Im}G) \right\} (du^2 - dv^2) + \left\{ (\text{Re}G)_v - \frac{|F|_v}{|F|} (\text{Re}G) + \frac{|F|_u}{|F|} (\text{Im}G) \right\} (2dudv).
\]
In particular,
\[
\det h = - \left( |G|_u^2 + |G|_v^2 \right) - \frac{|G|^2}{|F|} \left( |F|_u^2 + |F|_v^2 \right) + 2 \left| \frac{G}{F} \right| \left( |F|_uG|_u + |F|_vG|_v \right).
\]

**Proof.** By direct calculations. \( \square \)

**Remark 17.** The pair \((F, G)\) is called a Weierstrass data. And, for any \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \),
\[
f_\theta(s, t) = \cos \theta \left( \text{Re} \int (F, -iF, G)dw \right) + \sin \theta \left( \text{Im} \int (F, -iF, G)dw \right)
\]
is a \( d \)-minimal surface in \( \mathbb{R}^{0.2.1} \) and this gives an isometric deformation.

In fact, for holomorphic functions \( F, G \), it follows
\[
\text{Re} \int_{w_0}^w (-iF, -F, -iG)dw = \text{Im} \int_{w_0}^w (F, -iF, G)dw.
\]
Thus, \( d \)-minimal surfaces defined by the Weierstrass data \((-iF, -iG)\) correspond with the imaginary part of the formulas defined by the Weierstrass data \((F, G)\). For \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \), when we consider the \( d \)-minimal surface whose Weierstrass data is \((e^{-i\theta}F, e^{-i\theta}G)\), the given immersion is called an associated family and, when we denote \( f_\theta \), we have the \( S^1 \)-family of mappings. Moreover, we see
\[
f_\theta(u, v) = \text{Re} \int_{w_0}^w (e^{-i\theta}F, -ie^{-i\theta}F, e^{-i\theta}G)dw = \cos \theta \left( \text{Re} \int_{w_0}^w (F, -iF, G)dw \right) + \sin \theta \left( \text{Im} \int_{w_0}^w (F, -iF, G)dw \right),
\]
In particular, when $\theta = 0, \frac{\pi}{2}$, they correspond with the $d$-minimal surfaces given by the real part and imaginary part from $(F,G)$ respectively. Moreover, for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, since the induced metric of $f_{\theta}$ satisfies
\[
((f_{\theta})_u, (f_{\theta})_u) = ((f_{\theta})_v, (f_{\theta})_v) = |e^{-i\theta}F|^2 = |F|^2, \quad ((f_{\theta})_u, (f_{\theta})_v) = 0,
\]
it gives an isometric deformation between $f = f_0$ and $f_{\theta}$. We call $f_{\frac{\pi}{2}}$ a conjugate surface of $f_0$.

**Example.**

(0) When $(F,G) = (\alpha, \beta)$ ($\alpha, \beta \in \mathbb{C}, \alpha \neq 0$), a non-degenerate plane appears.

(1) When $(F,G) = (z, 1)$, we have
\[
f_0(u, v) = \left( \frac{1}{2}(u^2 - v^2), uv, u \right), \quad f_{\frac{\pi}{2}}(u, v) = \left( uv, -\frac{1}{2}(u^2 - v^2), v \right).
\]
These are surfaces which have the self-intersection and both give singularities called as cross-caps at $(u,v) = (0,0)$ (refer to Figure 5).

(2) When $(F,G) = (e^z, 1)$, we have
\[
f_0(u, v) = (e^u \cos v, e^u \sin v, u), \quad f_{\frac{\pi}{2}}(u, v) = (e^u \sin v, -e^u \cos v, v).
\]
$f_0$ is the $d$-minimal rotational surface given by Proposition 9 (3), and $f_{\frac{\pi}{2}}$ is the elliptic helicoid of the second kind (refer to Figure 1, Figure 3).

(3) When $(F,G) = (1, z)$, we have
\[
f_0(u, v) = \left( u, v, \frac{1}{2}(u^2 - v^2) \right), \quad f_{\frac{\pi}{2}}(u, v) = (u, -v, uv).
\]
These both are minimal hyperbolic paraboloids (refer to Figure 2).

**Remark 18.** The above Weierstrass-type representation formula is essentially known in [1]. However, the formula stated in [1] does not give singularities on surfaces. In this sense, Theorem 15 is more complete. On the other hand, We can see isotropic minimal surfaces which have isolated singularities in [12].

### 3.3. Applications.

**Theorem 19.** Let $(M, g)$ be a connected, two-dimensional complete Riemannian manifold, and let $f : (M, g) \to \mathbb{R}^{0,2,1}$ be an isometric immersion. Then, $(M, g)$ is isometric to the canonical two-dimensional Euclidean space $\mathbb{R}^2$, and the image of $f$ corresponds with an entire graph
\[
\{(u,v,F(u,v)) \in \mathbb{R}^{0,2,1} \mid (u,v) \in \mathbb{R}^2\},
\]
where $F$ is a $C^\infty$-function on $\mathbb{R}^2$.

**Proof.** We define $C^\infty$-functions $\alpha, \beta, \gamma$ on $M$ as
\[
f(x) = (\alpha(x), \beta(x), \gamma(x)) \quad (x \in M).
\]
We assume that $\mathbb{R}^2$ is the canonical Euclidean space which treats $(u,v)$ as the coordinates, and define a $C^\infty$-map $f_0 : (M, g) \to \mathbb{R}^2$ as
\[
f_0(x) := (\alpha(x), \beta(x)) \quad (x \in M).
$f_0$ is an isometric immersion. We prove that $f_0$ is the isometric diffeomorphism. We remark that $\dim M = \dim \mathbb{R}^2 = 2$ and, from the inverse function theorem, $f_0$ is the locally diffeomorphism. Thus, in order to prove that $f_0$ is the isometric diffeomorphism, it is sufficient to prove that $f_0$ is bijective.

For the surjectivity, since $f_0$ is the locally homeomorphism, $f_0$ is the open mapping. Thus, $\text{Im}f_0$ is an open subset of $\mathbb{R}^2$. Next, since isometric mappings preserve the geodesic completeness, from Hopf-Rinow’s theorem, $(\text{Im} f_0, d_{\mathbb{R}^2} + dv^2) \subset \mathbb{R}^2$ is complete, where we consider $\text{Im} f_0$ as the metric subspace of $\mathbb{R}^2$ naturally. Thus, $\text{Im} f_0$ is a closed subset of $\mathbb{R}^2$. Therefore, since $\text{Im} f_0$ is the open and closed subset of $\mathbb{R}^2$, it holds $\text{Im} f_0 = \mathbb{R}^2$, i.e. $f_0 : M \to \mathbb{R}^2$ is surjective.

For the injectivity, we denote the Riemannian distance with respect to the metric $g$ by $d_M$. For arbitrary points $x, y \in M$ which are distinct, since $(M, g)$ is complete, there exist a short geodesic $\delta : [0, 1] \to M$ such that $\delta(0) = x, \delta(1) = y$. Moreover, since $f_0$ is isometric, $f_0 \circ \delta : [0, 1] \to \mathbb{R}^2$ is a geodesic in $\mathbb{R}^2$ which connects $f_0(x)$ and $f_0(y)$. For a curve $c$, when we denote the length of $c$ by $L(c)$, we see

$$0 < d_M(x, y) = L(\delta) = L(f_0 \circ \delta) = |f_0(x) - f_0(y)|_{\mathbb{R}^2}.$$ This implies $f_0(x) \neq f_0(y)$, i.e. $f_0 : M \to \mathbb{R}^2$ is injective. As a remark, we use the fact that geodesics in $\mathbb{R}^2$ are straight lines for the last equation above.

In summary, since we obtain that $f_0 : M \to \mathbb{R}^2$ is a locally isometric diffeomorphism and bijection, it is an isometric diffeomorphism, that is, $(M, g)$ is isometric to the canonical two-dimensional Euclidean space $\mathbb{R}^2$. We denote the inverse of $f_0$ by $\phi : \mathbb{R}^2 \to M$. For any $(u, v) \in \mathbb{R}^2$, we have

$$f(\phi(u, v)) = (\alpha(\phi(u, v)), \beta(\phi(u, v)), \gamma(\phi(u, v))) = ((f_0 \circ \phi)(u, v), (\gamma \circ \phi)(u, v)) = (u, v, F(u, v)),$$

where $F := \gamma \circ \phi$ is a $C^\infty$-function on $\mathbb{R}^2$. Therefore, the image of $f$ is the entire graph expressed by a function $F$ on $\mathbb{R}^2$. $\square$

**Corollary 20.** Let $f : M^2 \to \mathbb{R}^{0.2.1}$ be a connected, complete $d$-minimal surface. Then, the image of $f$ corresponds with the entire graph

$$\{(u, v, \psi(u, v)) \in \mathbb{R}^{0.2.1} \mid (u, v) \in \mathbb{R}^2\},$$

where $\psi$ is a harmonic function on $\mathbb{R}^2$.

**Proof.** From Proposition 2 (2), it follows immediately. $\square$

**Corollary 21.** Let $M$ be a connected, compact two-dimensional manifold, i.e. a connected closed surface. Then, there exist no non-degenerate immersion $f : M \to \mathbb{R}^{0.2.1}$.

**Proof.** We prove the corollary by contradiction. We assume that there exist a non-degenerate immersion $f : M \to \mathbb{R}^{0.2.1}$. When we denote the induced metric by $g$, $(M, g)$ is a connected, compact Riemannian manifold. In particular, it is complete. From Theorem 19 as we have a homeomorphism $M \cong \mathbb{R}^2$, this contradicts the compactness of $M$. $\square$
Let $f: M \to \mathbb{R}^{0,2,1}$ be a non-degenerate immersion, and let $h$ be its second fundamental form. Then, we recall that the Gauss-Codazzi equation of the non-degenerate immersion is given by the formula (10), i.e.

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \quad (X, Y, Z \in \Gamma(TM)).$$

By using the flat local coordinates $(u, v)$, the formula (10) is equivalent to

$$(h_{11})_v = (h_{12})_u, \quad (h_{22})_u = (h_{12})_v,$$

where $h_{ij}$ are coefficients of $h$.

**Theorem 22** (The fundamental theorem of non-degenerate surfaces, [13], Theorem 8.8). Let $U \subset \mathbb{R}^2$ be a simply-connected domain, and let $(u, v)$ be coordinates on $U$. And, let $h_{11}, h_{12}, h_{22}$ be $C^\infty$-functions on $U$. Then, there exist, up to affine isometry, a non-degenerate immersion whose the induced metric and the second fundamental form are

$$du^2 + dv^2, \quad h_{11} du^2 + 2h_{12} dudv + h_{22} dv^2$$

respectively if and only if the functions $h_{ij}$ satisfy the Gauss-Codazzi equation (14) of the non-degenerate surface.

From now on, we consider four-dimensional Minkowski space $\mathbb{R}^4_1$ which equips with the Lorentzian metric

$$\langle \cdot, \cdot \rangle_1 := -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where $(x_1, x_2, x_3, x_4)$ is the canonical coordinates of $\mathbb{R}^4$. We deal with spacelike surfaces only, i.e. we require that the induced metric of surfaces is positive definite.

A surface $M$ is called **zero mean curvature** if it holds $\vec{H} \equiv 0$, where $\vec{H}$ is the mean curvature vector field of $M$, and a surface $M$ is called **flat** if it holds $K \equiv 0$, where $K$ is the Gaussian curvature of $M$. We abbreviate zero mean curvature to ZMC.

On the other hand, $\mathbb{R}^{0,2,1}$ is isometrically embedded in $\mathbb{R}^4_1$ by the natural way. In fact, the following mapping

$$(15) \quad \iota: \mathbb{R}^{0,2,1} \ni (x, y, z) \mapsto (z, x, y, z) \in \mathbb{R}^4_1$$

is an isometric embedding.

**Remark 23.** We give one of the motivations of studying flat and zero mean curvature surfaces. We firstly remark that flat minimal submanifolds in $n$-dimensional Euclidean space $\mathbb{R}^n$ and spacelike flat ZMC surfaces in three dimensional Minkowski space $\mathbb{R}^3_1$ are totally geodesic. On the other hand, there exists timelike flat ZMC surfaces in $\mathbb{R}^3_1$ ([15]). Thus, we are interested in the case of spacelike surfaces.

Next, spacelike flat ZMC surfaces in four-dimensional Minkowski space $\mathbb{R}^4_1$ are not always planes. In particular, we also remark that spacelike flat ZMC surfaces in four-dimensional semi-Euclidean space $\mathbb{R}^4_2$ equipped with the neutral metric are totally geodesic again.

**Theorem 24.** Let $f: M^2 \to \mathbb{R}^4_1$ be an immersion which gives a non-totally geodesic, connected spacelike flat ZMC surface, and let $h$ be the second fundamental form of $M$. We define a subset $E$ of $M$ as

$$E := \{ x \in M \mid h_x = 0 \}.$$
Then, it holds the following assertions:

1. $M \setminus E$ is an open dense subset of $M$, and it is connected.
2. The normal bundle of $M$ is flat, i.e. the normal curvature $R \perp \equiv 0$.
3. $M$ is, by an isometry of $\mathbb{R}^4_1$, immersed in $\mathbb{R}^{0,2,1}_0 \subset \mathbb{R}^4_1$, and it is a $d$-minimal surface.

Proof. The claim (1) is easily proved that $E$ is a closed subset of $M$. Since
$M$ is not totally geodesic, we obtain $E = \emptyset$. Moreover, we see that the set
$E$ do not have accumulation points, that is, the set $E$ is a discrete subset of
$M$ which is made of isolated points. Thus, $M \setminus E$ is an open dense subset
of $M$. And, since $M$ is connected and $E$ is discrete, it is proved for $M \setminus E$
to be connected.

For (2), see Corollary 1.2 in [1]. The claim (3) is proved by using Proposition 3.5 in [1]. □

Remark 25. The set $E$ is a discrete subset of $M$ consisted of isolated points.
As an example which satisfies $E \neq \emptyset$, when we define a $C^\infty$-immersion $f : \mathbb{R}^2 \to \mathbb{R}^{0,2,1}_0 \subset \mathbb{R}^4_1$
as $f(u,v) := (u^3 - 3uv^2, u, v, u^3 - 3uv^2)$, it is a spacelike flat ZMC surface which satisfies $h = 0$ at the origin $(0,0)$ only.

Let $f : M \to \mathbb{R}^{0,2,1}_0$ be a $d$-minimal surface. Then, by the isometric embedding $\iota$ given by (15), we see that $M$ is a spacelike flat ZMC surface in
$\mathbb{R}^4_1$. $M$ is a spacelike flat surface since $\iota$ is an isometric embedding. To show
that $M$ is ZMC, we directly calculate the mean curvature vector field of $M$.
By using a harmonic function $\varphi$, since we can locally expressed by
$f(u,v) = (u,v,\varphi(u,v))$,
from the composition of $\iota$, we have
$(\iota \circ f)(u,v) = (\varphi(u,v), u, v, \varphi(u,v))$.
Thus, we compute that the mean curvature vector field $\vec{H}$ is
$2\vec{H} = (\iota \circ f)_{uu} + (\iota \circ f)_{vv} = (\varphi_{uu} + \varphi_{vv})(1,0,0,1) \equiv 0$.
Therefore, we obtain the following corollary.

Corollary 26. Let $X$ be the set of the isometric classes of spacelike flat
ZMC surfaces in $\mathbb{R}^4_1$, and let $Y$ be the set of equivalence classes of $d$-minimal surfaces in $\mathbb{R}^{0,2,1}_0$ by a subgroup
$K := \begin{cases} 
\begin{pmatrix} T & 0 \\
0 & 0 \end{pmatrix}, & c \neq 0, \ T \in O(2) \end{cases} \times \mathbb{R}^3 \subset \text{Aut}(\mathbb{R}^{0,2,1}_0, d)$.
Then, except for planes, we have that $X$ and $Y$ are in one-to-one correspondence.

Proof. It is obvious as long as we remark that this subgroup $K$ corresponds
to the subgroup of isometric group of $\mathbb{R}^4_1$ which preserves the degenerate
subspace $\mathbb{R}^{0,2,1}_0 \subset \mathbb{R}^4_1$. □
Regarding minimal surfaces in $\mathbb{R}^3$, maximal surfaces in $\mathbb{R}^3_1$ and $d$-minimal surfaces in $\mathbb{R}^{0,2,1}$, we have

$$\{\text{minimal surfaces, maximal surfaces and } d\text{-minimal surfaces}\} \subset \{\text{spacelike ZMC surfaces in } \mathbb{R}^4_1\}.$$ 

In fact, for the spaces $\mathbb{R}^3$ and $\mathbb{R}^3_1$, there exist isometric embeddings defined by

- $\mathbb{R}^3 \ni (x, y, z) \mapsto (0, x, y, z) \in \mathbb{R}^4_1$,
- $\mathbb{R}^3_1 \ni (x, y, z) \mapsto (x, y, z, 0) \in \mathbb{R}^4_1$,

respectively. Therefore, we see that there quite fruitfully exist ZMC surfaces in $\mathbb{R}^4_1$.

In general, singularity points appear in $d$-minimal surfaces. Refer to the figures from 5 to 10 as such examples. From the Whitney’s criterion, a cross-cap appears in Figure 5, and from the Saji’s criterion ([14]), a $D_{5}$-type singularity appears in Figure 9. Other singularities have been not identified and classified.

At the end of this paper, we give a table which compares properties among each surfaces. We assume the connectedness of surfaces:

|                    | min. | max. | $d$-min. |
|--------------------|------|------|----------|
| Compact            | $\notin$ | $\notin$ | $\notin$ (Cor. [21]) |
| Entire graph       | Planes only | Planes only | $\exists$ (Prop. [9]) |
| Singularity        | $\notin$ | $\exists$ ([8]) | $\exists$ |
| Complete           | $\exists$ | Planes only | $\exists$ (Thm. [19]) |
| Gaussian curvature | $\leq 0$ | $\geq 0$ | $\equiv 0$ |

Table 1.

where, in terms of singularity, the symbol $\exists$ expresses that singularities appear, and in terms of otherwise, $\exists$ expresses that there exist such surfaces which are not planes. In addition to, the abbreviations min., max. and $d$-min. are minimal surfaces in $\mathbb{R}^3$, maximal surfaces in $\mathbb{R}^3_1$ and $d$-minimal surfaces in $\mathbb{R}^{0,2,1}$ respectively.

Acknowledgment

The author would like to express his deepest gratitude to his advisor, Professor Takashi Sakai. The author is also very grateful to Shintaro Akamine and Luis Carlos Barbosa da Silva for their comments and valuable advices.
• Examples of isolated singularities whose the rank of Jacobi matrix is one for $C^\infty$-mappings $f$ giving $d$-minimal surfaces.

\begin{figure}[h]
\centering
\subfloat{\includegraphics[width=0.4\textwidth]{figure5.png}}
\subfloat{\includegraphics[width=0.4\textwidth]{figure6.png}}
\caption{$(F,G) = (z, 1)$.}
\end{figure}

\begin{figure}[h]
\centering
\subfloat{\includegraphics[width=0.4\textwidth]{figure7.png}}
\subfloat{\includegraphics[width=0.4\textwidth]{figure8.png}}
\caption{$(F,G) = (z^3, 1)$.}
\end{figure}

\begin{figure}[h]
\centering
\subfloat{\includegraphics[width=0.4\textwidth]{figure9.png}}
\subfloat{\includegraphics[width=0.4\textwidth]{figure10.png}}
\caption{$(F,G) = (z^2, z)$.}
\end{figure}

• Examples of isolated singularities whose the rank of Jacobi matrix is zero for $C^\infty$-mappings $f$ giving $d$-minimal surfaces.

\begin{figure}[h]
\centering
\subfloat{\includegraphics[width=0.4\textwidth]{figure11.png}}
\subfloat{\includegraphics[width=0.4\textwidth]{figure12.png}}
\caption{$(F,G) = (z, z^2)$.}
\end{figure}
References

[1] L. J. Alías, B. Palmer, *Curvature properties of zero mean curvature surfaces in four-dimensional Lorentzian space forms*, Math. Proc. Camb. Phil. Soc. **124** (1998), 315–327.

[2] H. Anciaux, *Minimal submanifolds in pseudo-Riemannian geometry*, World Scientific (2011).

[3] M. E. Aydin, *Classification results on surfaces in the isotropic 3-space*, AKU J. Sci. Eng. **16** (2016), 239–246.

[4] M. E. Aydin, *Constant curvature surfaces in a pseudo-isotropic space*, Tamkang J. Math. **49** (2018), 221–233.

[5] A. Bejancu, K. L. Duggal, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Kluwer Academic Publishers (1996).

[6] B. Y. Chen, *Black holes, marginally trapped surfaces and quasi-minimal surfaces*, Tamkang J. Math. **40** (2009), 313–341.

[7] S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Entire zero-mean curvature graphs of mixed type in Lorentz-Minkowski 3-space*, Quarterly J. Math. **67** (2016), 801–837.

[8] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, Math. Z. **259** (2008), 827–848.

[9] X. Ma, C. P. Wang and P. Wang, *Global geometry and topology of spacelike stationary surfaces in the 4-dimensional Lorentz space*, Adv. Math. **249** (2013), 311–347.

[10] K. Nomizu, T. Sasaki, *Affine differential geometry*, Cambridge University Press (1994).

[11] M. O’Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, London (1983).

[12] H. Pottmann, P. Grohs and N. J. Mitra, *Laguerre minimal surfaces, isotropic geometry and linear elasticity*, Adv. Comput. Math., **31** (2009), 391–419.

[13] H. Sachs, *Isotrope geometrie des raumes*, Vieweg, Braunschewig/Wiesbaden (1990).

[14] K. Saji, *Criteria for $D_4$ singularities of wave fronts*, Tohoku Math. J. **63** no.1 (2011), 137–147.

[15] Y. Sato, *On the classification of ruled minimal surfaces in pseudo-Euclidean space*, arXiv:1705.03187.

[16] L. C. B. da Silva, *Rotation minimizing frames and spherical curves in simply isotropic and pseudo-isotropic 3-spaces*, arXiv:1707.06321.

[17] L. C. B. da Silva, *The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces*, arXiv:1801.01187.

[18] O. C. Stoica, *On singular semi-Riemannian manifolds*, Int. J. Geom. Methods Mod. Phys., **11**, (2014), no. 5, 1450041.

[19] M. Umehara, K. Yamada, *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J. **35** (2006), 13–40.

Department of Mathematical Sciences, Tokyo Metropolitan University, Minami-Osawa 1-1, Hachioji, Tokyo, 192-0397, Japan.

E-mail address: satou-yuuichirou@ed.tmu.ac.jp