SPLINE-INTERPOLATION SOLUTION OF 3D DIRICHLET PROBLEM FOR ONE CLASS OF SOLIDS

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Abstract. We present the spline-interpolation approximate solution of the Dirichlet problem for the Laplace equation in the bodies of revolution, cones and cylinders. Our method is based on reduction of the 3D problem to the sequence of 2D Dirichlet problems. The main advantage of the spline-interpolation solution of the 3D Dirichlet problem is its continuity in the whole domain up to the boundary even for the case of the linear spline.

1. Introduction

We present the new — spline-interpolation — approximate solution of the Dirichlet problem for the Laplace equation which has wide applications to electric and seismic prospecting [1, 3, 4, 5, 6, 9]. This method is applicable for the cylinders, cones and the bodies of revolution. There exist many methods of approximate solution of this problem. The classic approximate method is the finite-element one. The spline-interpolation method is the analog of finite-element method where the discrete boundary nodes are the closed boundary curves and the cells are the layers between two parallel planes. Our method is based on reduction of the 3D problem to the sequence of 2D Dirichlet problems. Note that one can apply the functions of complex variable in the case of 2D Dirichlet problem. There also exists another spline technique of solution, but the one presented here is substantially different from the one given, for example, in the book [2]. The main advantage of the spline-interpolation solution of the 3D Dirichlet problem is its continuity in the whole domain up to the boundary even for the case of the linear spline. The piece-wise analytic form of the spline-interpolation solution is also convenient for the applications.

2. Formulation of the problem

Let $M$ be a body of revolution with the smooth generatrix $x = \Phi(h)$ in $XYH$ space. Let us denote by $S$ the boundary of $M$.

The classic internal Dirichlet problem for the Laplace equation

(1) $\Delta_3 u = 0$,

is the following: given the function $U_0$ on $S$ it should be found the function $u$ such that it satisfies equation (1) everywhere in $M$ and takes the given

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boundary values \( U_0 \), that is

\[ u \big|_S = U_0. \]

Here

\[ \Delta_3 = \partial_x^2 + \partial_y^2 + \partial_h^2. \]

We call by the spline-interpolation solution of the problem the function which satisfies equation (1) almost everywhere in \( M \) and takes the given continuous boundary values at a finite number of the curves being the sections of \( S \) by the planes \( h = h_j \). So we do not solve the initial boundary value problem but reduce it to the problem with the data given at some finite set of sections of \( S \).

For the case when the domain \( M \) has the upper and the lower ends \( D_A \) and \( D_B \) on the planes \( h = A \) and \( h = B \) we assume that the given boundary values at these ends are polyharmonic functions. The spline-interpolation solution takes these given values at the ends. Note that the requirement of poly-harmonicity is not too restrictive because every continuous function given on the plane can be approximated with any accuracy and any smoothness by a polyharmonic function.

It is well-known [8] that the external problem can be reduced to the internal one by Kelvin transform [7]. Let \( \mathbb{R}^3 \setminus M \) be the unbounded domain in which we must solve equation (1). We assume that the unit ball \( B_1(0) \) is a subset of \( M \). Then the inversion of \( \mathbb{R}^3 \setminus M \) with respect to the unit sphere belongs to \( B_1(0) \). It is well-known, that \( u(x, y, h) \) is the solution of equation (1) inside the ball \( B_1(0) \) if and only if the function \( v(x, y, h) = u(x/r^2, y/r^2, h/r^2)/r \), \( r = \sqrt{x^2 + y^2 + h^2} \) is the solution of (1) outside the ball \( B_1(0) \) which vanishes at infinity (\( O(1/r) \)). So we present and demonstrate the spline-interpolation solution of the internal problem for the body of revolution and only then discuss the case of the external problem and the problem for cylinders or cones.

3. Analysis

We construct the spline-interpolation solution for every fragment of \( M \) which is the layer between two planes \( h = h_j, h = h_{j+1} \). The form of spline-interpolation solution is polynomial with respect to \( h \). If we put the solution in the form

\[ u(x, y, h) = \sum_{k=0}^{p} h^k u_k(x, y), \]

in equation (1) and equate the coefficients with the same powers of \( h \) we get the following relations:

\[(k + 2)(k + 1)u_{k+2} + \Delta_2 u_k = 0, \quad k = 0, \ldots, p - 2,\]

and

\[\Delta_2 u_k = 0, \quad k = p, p - 1,\]

where

\[ \Delta_2 = \partial_x^2 + \partial_y^2. \]
Now the solution \( u(x, y, h) \) satisfies equation (11) in the layer and \( u(x, y, h) \) is a \((\frac{3}{2}+1)\)-harmonic function for any fixed \( h \). The main problem now is to satisfy the boundary condition and to glue continuously the solutions constructed for the adjacent layers. Such spline-interpolation solution is named here the continuous solution. The derivative \( u_h(x, y, h) \) of the continuous solution is bounded but discontinuous at the points of \( M \) on the cuts \( h = h_j \). We also show how to minimize these discontinuity.

4. THE CONTINUOUS SOLUTION. THE CASE OF A DOMAIN WITHOUT ENDS

Assume that the body of revolution \( M \) lies between the planes \( h = A \) and \( h = B \), the vertices \( P_A \) and \( P_B \) being the common points of the domain and the corresponding planes. Assume that the boundary values are given at \( m \) circles \( C_j \), \( j = 1, ..., m \), which are the sections of \( S \) by the planes \( h = h_j \), \( j = 1, ..., m \), in the form \( f_j : C_j \rightarrow \mathbb{R} \) and also at the points \( P_A \) and \( P_B \).

We begin the construction of the spline-interpolation solution within the layer which contains one of the vertices, namely \( P_A \), so it is the layer \( h \in [A, h_1] \).

Here we put

\[
\begin{align*}
  u^1(x, y, h) &= u_0^1(x, y) + (h - h_1)u_1^1, \\
  u^1(x, y, h) &= u_0^1(x, y) + (h - h_1)u_1^1,
\end{align*}
\]

where \( u_0^1(x, y) \) is a harmonic function in the domain which is the section of \( M \) by the plane \( h = h_1 \) and \( u_1^1 \) is a constant. Evidently this spline satisfies equation (11). The function \( u_0^1(x, y) \) can be restored via the boundary values \( f_1 \). We find the constant \( u_1^1 \) using the relation

\[
u_1^1 = \frac{u(P_A) - u_0^1(x_A, y_A)}{A - h_1},\]

where \( u(P_A) \) is the given value at the vertex \( P_A \), the parameters \( x_A \) and \( y_A \) are the abscissa and the ordinate of the point \( P_A \), respectively.

In the next layer \( h \in [h_1, h_2] \) the spline has the form

\[
u^2(x, y, h) = u_0^2(x, y) + (h - h_1)u_1^2(x, y),
\]

where both of the unknown functions \( u_0^2(x, y) \) and \( u_1^2(x, y) \) are harmonic in the union of the projections of the sections of \( M \) by the planes \( h = h_1, h = h_2 \) onto the plane \( XOY \). Evidently this spline satisfies equation (11).

We put \( u^2_0(x, y) = u_0^1(x, y) \). So the boundary conditions at the curve \( C_1 \) are met. Now we find the boundary values of the function \( u_1^2 \) which are equal to

\[
f_2 - \frac{u_0^2(C_2)}{h_2 - h_1},
\]

and restore the harmonic in the section of \( M \) by the plane \( h = h_2 \) function \( u_1^2(x, y) \) via its boundary values.

In the layer \( h \in [h_2, h_3] \) we put

\[
u^3(x, y, h) = u_0^3(x, y) + (h - h_2)u_1^3(x, y),
\]

where the harmonic function \( u_0^3(x, y) \) can be restored by setting \( u_0^3(x, y) = u_0^2(x, y) + (h_2 - h_1)u_1^2(x, y) \), and the harmonic function \( u_1^3(x, y) \) can be restored.
via the boundary values

\[
\frac{f_3 - u_0^3(C_3)}{h_3 - h_2}.
\]

We move from one layer to the next one successively and restore the functions \(u^k_0(x, y), u^k_1(x, y), k = 1, \ldots, m\) in the whole domain \(M\). The construction for the last layer is similar to that for the first one.

Note that the harmonic on one cut of \(M\) by \(h = h_k\) function must be continuous and harmonic at the neighbor cuts. It holds true if the boundary data on the curves \(C_j, j = 1, \ldots, m\), is given in the form of trigonometric polynomials.

The constructed spline-interpolation solution is harmonic at the points of the common plane of two adjacent layers. The corresponding function is restored uniquely via its boundary values, so this solution is continuous over the axis \(O\). It is also continuous in any plane parallel to \(XOY\) up to the boundary of \(M\).

The derivatives of the constructed solution with respect to \(x\) and to \(y\) are continuous in \(M\) at every level \(h = \text{const}\), but the first derivative with respect to \(h\) is discontinuous at the points of \(M\) at the levels \(h = h_j, j = 1, \ldots, m\). We show that this discontinuity tends to zero when \(m \to \infty\) and \(\max_j |h_{j+1} - h_j| \to 0\) if the boundary data of the corresponding classic Dirichlet problem is a smooth function and the boundary surface of \(M\) is smooth.

Suppose that the boundary data of the classic Dirichlet problem is the function \(f(\theta, h), \theta \in [0, 2\pi], h \in [A, B]\), which has the continuous derivative \(f_h(\theta, h)\), which belongs to H"older class \(H(\alpha)\) with respect to \(h\) uniformly for \(\theta \in [0, 2\pi]\). So there exist the constants \(L > 0\) and \(\alpha \in (0, 1]\) such that \(|f_h'(\theta, h') - f_h'(\theta, h'')| \leq L|h' - h''|^\alpha\) for any \(\theta \in [0, 2\pi]\). Let the generatrix \(x = \Phi(h)\) of the boundary surface of \(M\) be smooth: \(\Phi'(h)\) belongs to \(H(\alpha)\).

We assume that the distance between any two adjacent planes \(h = h_j\) equals \(t = (B - A)/(m + 1)\). Consider two layers in \(M\) with the common plane \(h = 0\) and the splines constructed for these layers:

\[
u^j(x, y, h) = u_0(x, y) + u^j_1(x, y)h, \quad h \in [-t, 0],
\]

and

\[
u^{j+1}(x, y, h) = u_0(x, y) + u^{j+1}_1(x, y)h, \quad h \in [0, t].
\]

We have

\[
u^j(\Phi(-t) \cos \theta, \Phi(-t) \sin \theta, -t) = f(\theta, -t),
\]

\[
u^j(\Phi(0) \cos \theta, \Phi(0) \sin \theta, 0) = u^{j+1}(\Phi(0) \cos \theta, \Phi(0) \sin \theta, 0) = f(\theta, 0),
\]

\[
u^{j+1}(\Phi(t) \cos \theta, \Phi(t) \sin \theta, t) = f(\theta, t),
\]

so

\[
u_0(\Phi(0) \cos \theta, \Phi(0) \sin \theta, 0) = f(\theta, 0),
\]

\[-tu^j_1(\Phi(-t) \cos \theta, \Phi(-t) \sin \theta) = [f(\theta, -t) - f(\theta, 0)] +
\]

\[+\left[u_0(\Phi(0) \cos \theta, \Phi(0) \sin \theta, 0) - u_0(\Phi(-t) \cos \theta, \Phi(-t) \sin \theta)\right],
\]

\[tu^{j+1}_1(\Phi(t) \cos \theta, \Phi(t) \sin \theta) = [f(\theta, t) - f(\theta, 0)] +
\]

\[+\left[u_0(\Phi(0) \cos \theta, \Phi(0) \sin \theta, 0) - u_0(\Phi(t) \cos \theta, \Phi(t) \sin \theta)\right].
\]
We apply Lagrange formula and have
\[ u_1^j(\Phi(-t) \cos \theta, \Phi(-t) \sin \theta) = f_j'(\theta, \eta(\theta)t) - [u_{0x}^j(\Phi(\mu(\theta)t) \cos \theta, \Phi(\mu(\theta)t) \sin \theta) \cos \theta + \\
+ u_{0y}^j(\Phi(\mu(\theta)t) \cos \theta, \Phi(\mu(\theta)t) \sin \theta) \sin \theta]\Phi'(\mu(\theta)t), \quad \eta(\theta), \mu(\theta) \in (-1, 0), \]
\[ u_{1+}^j(\Phi(t) \cos \theta, \Phi(t) \sin \theta) = f_j'(\theta, \xi(\theta)t) - [u_{0x}^j(\Phi(\mu(\theta)t) \cos \theta, \Phi(\mu(\theta)t) \sin \theta) \cos \theta + \\
+ u_{0y}^j(\Phi(\mu(\theta)t) \cos \theta, \Phi(\mu(\theta)t) \sin \theta) \sin \theta]\Phi'(\nu(\theta)t), \quad \xi(\theta), \nu(\theta) \in (0, 1). \]

Note that the functions \( u_{0x}^j(x, y) \) and \( u_{0y}^j \) are differentiable in the union of the disks with the radii \( \Phi(-t) \), \( \Phi(0) \) and \( \Phi(t) \).

Now
\[ u_1^j(\Phi(0) \cos \theta, \Phi(0) \sin \theta) = u_1^j(\Phi(-t) \cos \theta, \Phi(-t) \sin \theta) + \\
[u_1^j(\Phi(0) \cos \theta, \Phi(0) \sin \theta) - u_1^j(\Phi(-t) \cos \theta, \Phi(-t) \sin \theta)], \]
\[ u_{1+}^j(\Phi(0) \cos \theta, \Phi(0) \sin \theta) = u_{1+}^j(\Phi(t) \cos \theta, \Phi(t) \sin \theta) + \\
[u_1^{j+1}(\Phi(0) \cos \theta, \Phi(0) \sin \theta) - u_{1+}^{j+1}(\Phi(t) \cos \theta, \Phi(t) \sin \theta)]. \]

Finally we have according to supposition
\[ |u_{1+}^{j+1}(\Phi(0) \cos \theta, \Phi(0) \sin \theta) - u_1^j(\Phi(0) \cos \theta, \Phi(0) \sin \theta)| \leq Kt^\alpha. \]

The function \( u_{1+}^{j+1}(x, y) - u_1^j(x, y) \) is harmonic in the disk \( x^2 + y^2 \leq \Phi(0) \), so \( |u_{1+}^{j+1}(x, y) - u_1^j(x, y)| \leq Kt^\alpha \) everywhere in this disk, and we can see that the discontinuity of the function \( u_h(x, y, h) \) vanishes when the distance \( t \) between any two adjacent planes \( h = h_j \) tends to zero.

We have the representation
\[ u_{1+}^{j+1}(\Phi(0) \cos \theta, \Phi(0) \sin \theta) - u_1^j(\Phi(0) \cos \theta, \Phi(0) \sin \theta) = t\phi(\theta, t) \]
where \( \phi(\theta, t) \) is bounded when \( t \) tends to zero if the functions \( \Phi''(h) \) and \( f_{hh}(\theta, h) \) are continuous.

Note that we can construct the solution beginning from the lowest point as well as from the highest point. If \( u_i(x, y, h) \) and \( u_f(x, y, h) \) are such solutions then \( u_i(x, y, h) + (1 - \alpha)u_f(x, y, h) \) is also the solution for any \( \alpha \in (0, 1) \).

5. The continuous solution. The case of a domain with the ends

Here the domain \( M \) is situated between the planes \( h = A \) and \( h = B \), the end \( D_A \) is located on the plane \( h = A \) and the end \( D_B \) is located on the plane \( h = B \), the curves \( C_j \) are located on the planes \( h = h_j, j = 1, ..., m, \). The boundary data on the ends has the form \( U_A(x, y), (x, y) \in E_A \) and \( U_B(x, y), (x, y) \in E_B \). The boundary values at the points of the curves \( C_j \) are given by the functions \( f(\theta, h_j) \).

The form of the solution depends on the order of harmonicity of the functions \( U_A(x, y) \) and \( U_B(x, y) \). Assume that the highest order of harmonicity of these functions is equal to \( (n + 1) \).

We construct the spline for the layer \( h \in [A, h_1] \) in the form
\[ u_1^j(x, y, h) = u_1^j(x, y) + (h - A)u_1^j(x, y) + \sum_{k=1}^{n} u_{2k}^j(x, y)(h - A)^{2k}, \]
where \( u_0^1(x, y) = U_A(x, y) \). The spline satisfies equation (11) if and only if

\[
u_{2k}^1(x, y) = (-1)^{\frac{k}{2}!} \Delta^k u_0^1(x, y), k = 1, ..., n,
\]

where \( \Delta \) is \( k \) times applied Laplace operator \( \Delta_2 = \partial_x^2 + \partial_y^2 \). Hence we find all coefficients with the even powers of \( h \) if we satisfy equation (11) in the layer \( h \in [A, h_1] \). The only odd coefficient \( u_1^1(x, y) \) must satisfy the equation

\[
\Delta u_1^1(x, y) = 0.
\]

Therefore we have to restore the function \( u_1^1(x, y) \) which is harmonic in the union of the projections of the sections of \( M \) by the planes \( h = A, h = h_1 \) onto \( XOY \). We do it using the boundary condition

\[
u_1^1(x, y)|_{C_1} = f_1 - u_0^1(C_1) - \sum_{k=1}^n u_{2k}^1(C_1)
\]

For the next layer \( h \in [h_1, h_2] \) we take the spline in the form

\[
u^2(x, y, h) = u_0^2(x, y) + (h - h_1)u_1^2(x, y) + \sum_{k=1}^n u_{2k}^2(x, y)(h - h_1)^{2k},
\]

where

\[
u_0^2(x, y) = u_1^1(x, y) + (h_1 - A)u_1^1(x, y) + \sum_{k=1}^n u_{2k}^1(x, y)(h_1 - A)^{2k},
\]

and

\[
u_{2k}^2(x, y) = (-1)^{\frac{k}{2}!} \Delta^k u_0^2(x, y), k = 1, ..., n.
\]

The harmonic function \( u_1^2(x, y) \) can be restored via the boundary data at the curve \( C_2 \).

All the splines for the next layers are constructed similarly. Only the last spline where we must satisfy both the given data on the section and on the end \( D_B \) is different.

We take the last spline for the layer \( h \in [h_m, B] \) in the form

\[
u^m(x, y, h) = u_0^m(x, y) + (h - h_m)u_1^m(x, y) + \sum_{k=1}^n u_{2k}^m(x, y)(h - h_m)^{2k} + \sum_{k=1}^n u_{2k+1}^m(x, y)(h - h_m)^{2k+1},
\]

where

\[
u_0^m(x, y) = u_0^{m-1}(x, y) + (h_m - h_{m-1})u_1^{m-1}(x, y) + \sum_{k=1}^n u_{2k}^{m-1}(x, y)(h_m - h_{m-1})^{2k},
\]

\[
u_{2k}^m(x, y) = (-1)^{\frac{k}{2}!} \Delta^k u_0^m(x, y).
\]

So we glue continuously the last spline with the previous one at the points of the common section and provide equality of the coefficients with the even
powers of $h$ in (1). Now we must provide equality of the coefficients with the odd powers of $h$ in (1), that is why we put

$$u_{2k+1}^m(x, y) = \frac{(-1)^k}{(2k+1)!} \Delta_2^k u_1^m(x, y), \quad k = 1, ..., n.$$  

Now equation (1) is satisfied in the layer $[h_m, B]$ and we must satisfy the boundary data at the points of $D_B$. We write this boundary condition in the form

$$\sum_{k=0}^n \frac{(-1)^k}{(2k+1)!} \Delta_2^k u_1^m(x, y)(B - h_m)^{2k+1} = U_B(x, y) - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} \Delta_2^k u_1^m(x, y)(B - h_m)^{2k+1},$$

where the right side of the relation is known $(n+1)$-harmonic function, we denote it by $F_B(x, y)$. Our aim is to restore the $(n+1)$-harmonic function $u_1^m(x, y)$ with the help of the identity

$$\sum_{k=0}^n \frac{(-1)^k}{(2k+1)!} \Delta_2^k u_1^m(x, y)(B - h_m)^{2k+1} = F_B(x, y).$$  

Let us apply the operator $\Delta_2$ to both sides of (2), now harmonicity of the function

$$u_{2n+1}^m(x, y) = \frac{(-1)^n}{(2n+1)!} \Delta_2^n u_1^m(x, y)$$

implies

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!} \Delta_2^{k+1} u_1^m(x, y)(B - h_m)^{2k+1} = \Delta_2 F_B.$$

When we apply the operator $\Delta_2$ to both sides of the last relation, the function $u_{2n-1}^m$ disappears from the left part. We continue application of the operators $\Delta_2^k$ till $k = n$. We obtain $n$ additional equalities from equality (2). After we apply the operator $\Delta_2^n$ to (2) we have

$$(B - h_m) \Delta_2^n u_1^m = (B - h_m)^{2n+1} \frac{(2n+1)!}{(-1)^n} u_{2n+1}^m = \Delta_2^n F_B,$$

and it is the expression of the harmonic function $u_{2n+1}^m(x, y)$.

When we apply $\Delta_2^{n-1}$ we have:

$$(B - h_m) \Delta_2^{n-1} u_1^m + (B - h_m)^3 \Delta_2^{n-1} u_1^m =$$

$$= (B - h_m)^{2n-1} \frac{(2n-1)!}{(-1)^n} u_{2n-1}^m - (B - h_m)^3 \frac{1}{6} \Delta_2^{n-1} u_1^m =$$

$$= (h_m - B)^{2n-1} \frac{(2n-1)!}{(-1)^n} u_{2n-1}^m - (B - h_m)^3 \frac{2n+1)!}{6(-1)^n} u_{2n+1}^m = \Delta_2^{n-1} F_B.$$

After we put the known expression of $u_{2n+1}^m$ into the last relation we get the biharmonic function $u_{2n-1}^m$. We continue our inverse movement over the additional equalities and restore all functions $u_k^m(x, y)$, beginning with $u_{2n+1}^m(x, y)$ up to $u_1^m(x, y)$. We get the last function directly from equality (2). When we
restore all functions $u_{mn}^{k} (x, y)$, $k = 1, \ldots, n$, we have the last spline in the layer $h \in [h_m, B]$.

The solution of the problem for a cylinder can be reduced to the solution for the circular cylinder due to the additional conform mapping.

Note that as at the previous section we can construct the solution beginning from the lowest end as well as from the highest end. If $u_l(x, y, h)$ and $u_h(x, y, h)$ are such solutions then $\alpha u_l(x, y, h) + (1 - \alpha) u_h(x, y, h)$ is also the solution for any $\alpha \in (0, 1)$.

6. Example

Let the domain $M$ be the half-ball bounded by the plane $XOY$ from below and by the sphere $x^2 + y^2 + h^2 = 1$ from above. We construct the simplest spline-interpolation solution with the given data on the base $h = 0$, on the level $h = 1/2$ of the spherical surface and at the vertex at the level $h = 1$. We suppose that the data on the base is a biharmonic function on the variables $x$ and $y$:

$$u(x, y, 0) = \Re\left(\sum_{k=0}^{N} (a_0^k - ib_0^k) (x + iy)^k\right) +$$

$$+ (x^2 + y^2) \Re\left(\sum_{k=0}^{N} (a_1^k - ib_1^k) (x + iy)^k\right).$$

The data on the curve $\{x^2 + y^2 = \frac{3}{4}, h = \frac{1}{2}\}$ has the form

$$u\left(\frac{\sqrt{3}}{2} \cos \phi, \frac{\sqrt{3}}{2} \sin \phi, \frac{1}{2}\right) = \frac{a_0}{2} + \sum_{k=0}^{N} (a_k \cos \phi + b_k \sin \phi).$$

The data at the vertex is $u(0, 0, 1) = u_0$.

We search for the solution for both layers $h \in [0, \frac{1}{2}]$ and $h \in [\frac{1}{2}, 1]$ in the form of the polynomial of the second power over $h$.

In the layer $h \in [0, 1/2]$ the spline has the form

$$u(x, y, h) = u_0^1(x, y) + hu_1^1(x, y) + h^2u_2^1(x, y),$$

where

$$u_0^1(x, y) = \Re\left(\sum_{k=0}^{N} (a_0^k - ib_0^k) (x + iy)^k\right) +$$

$$+ (x^2 + y^2) \Re\left(\sum_{k=0}^{N} (a_1^k - ib_1^k) (x + iy)^k\right),$$
SPLINE-INTERPOLATION SOLUTION OF 3D DIRICHLET PROBLEM FOR ONE CLASS OF SOLIDS

\[ u_1^1(x, y) = 2\left(\frac{a_0}{2} + \text{Re}(\sum_{k=1}^{N} (a_k - ib_k)(\frac{2}{\sqrt{3}})^k(x + iy)^k) - \right. \]
\[ \text{Re}(\sum_{k=0}^{N} (a_k^0 - ib_k^0)(x + iy)^k) - \frac{a_1^1}{4} + \]
\[ \left. \frac{1}{4}\text{Re}(\sum_{k=1}^{N} (a_k^1 - ib_k^1)(2k - 1)(x + iy)^k)\right], \]
\[ u_1^2(x, y) = -2[a_0^1 + \text{Re}(\sum_{k=1}^{N} (a_k^1 - ib_k^1)(k + 1)(x + iy)^k)]. \]

In the layer \( h \in [1/2, 1] \) the spline has the form
\[ u(x, y, h) = u_0^2(x, y) + (h - \frac{1}{2})u_1^2(x, y) + (h - \frac{1}{2})^2u_2^2(x, y), \]
where
\[ u_0^2(x, y) = \frac{a_0}{2} + \text{Re}(\sum_{k=1}^{N} (a_k - ib_k)(\frac{2}{\sqrt{3}})^k(x + iy)^k) - \]
\[ -\frac{3}{4}\text{Re}(\sum_{k=0}^{N} (a_k^1 - ib_k^1)(x + iy)^k) + (x^2 + y^2)\text{Re}(\sum_{k=0}^{N} (a_k^1 - ib_k^1)(x + iy)^k), \]
\[ u_1^2(x, y) = 2u_0 + \frac{5}{2}a_0 - a_0, \]
\[ u_2^2(x, y) = -2[a_0^1 + \text{Re}(\sum_{k=1}^{N} (a_k^1 - ib_k^1)(k + 1)(x + iy)^k)]. \]

7. APPROXIMATION ESTIMATE OF THE CONTINUOUS SOLUTION

Assume that \( \tilde{u} \) is the difference between the exact solution of Dirichlet problem and the spline-interpolation solution presented in the previous sections. This function vanishes at the points of \( C_j \), at the points of the ends or at the top points. The function \( \tilde{u}(x, y, h) \) has continuous derivatives \( \tilde{u}_x(x, y, h) \) and \( \tilde{u}_y'(x, y, h) \) at any intersection of the plane \( h = \text{const} \) with \( M \). So we have the estimate
\[ \left| \tilde{u}(x, y, h) \right| \leq \hat{u}(\hat{h}) + l \max_{(x, y, h) \in M} \sqrt{\tilde{u}_x^2(x, y, h) + \tilde{u}_y^2(x, y, h)}, \]
where \( \hat{u}(\hat{h}) \) is the maximum of the module of the boundary difference between the classic solution and the constructed approximate solution on the level \( h = \hat{h} \), \( l \) is the distance from the point \( (x, y) \) to the boundary of the intersection of \( M \) with the plane \( h = \hat{h} \). Evidently, \( \hat{u}(\hat{h}) = 0 \) for \( \hat{h} = h_j, j = 1, 2, ..., m \). The values of \( \hat{u}(\hat{h}) \) tend to zero when we increase the number of \( m \) and decrease the value \( \max_j \left| h_{j+1} - h_j \right| \) if the corresponding classic Dirichlet problem has smooth boundary values.
The estimate shows that the continuous solution is rather accurate in the neighbourhood of the boundary $S$ of the body $M$.

8. Construction of the smoothing solution

Here we construct the continuous solution which minimizes the difference between the derivatives with respect to $h$ of the adjacent splines at the common disk $E_j$. We consider a body of revolution without the ends or with one end. We suppose that the function $\Phi''(h)$ is continuous and the boundary data of the corresponding classic Dirichlet problem $f(\theta, h)$ has the form

$$\sum_{k=0}^{l} C_k(t) \cos k\theta + D_k \sin k\theta$$

where the functions $C_k(t)$ and $D_k(t)$ are at least 3-differentiable, $h \in [A, B]$.

The spline for the layer $h \in [h_j, h_{j+1}]$ has the following form:

$$u^{j+1}(x, y, h) = \sum_{k=0}^{2N+1} u^{j+1}_k(x, y) h^k,$$

where

$$u^{j+1}_k(x, y) = (-1)^p \Delta^p u^{j+1}_0(x, y)/(2p)!,$$

and the norm of the difference $u^{j+1}(x, y, h_j) - u^j(x, y, h_j)$ in the space $L_2$ is close to minimal.

Let $h_j = 0$, $h_{j-1} = -t$, $h_{j+1} = t$, $\Phi(h_j) = 1$, $\Phi(h_{j+1}) = 1 + tb(t)$, where $b(t)$ is bounded when $t$ tends to zero. Gluing of the neighbor splines at the unit disk yields the equation $u^{j+1}_2(x, y) = u^{j}_2(x, y)$.

After we consider the boundary values $u^{j+1}(x, y, t)$ and $u^j(x, y, -t)$ and apply Lagrange formula we obtain in the same way as for the liner splines the representation

$$\sum_{p=0}^{N} [u^{j+1}_{2p+1}(\cos \theta, \sin \theta) - u^{j+1}_{2p+1}(\cos \theta, \sin \theta)] t^{2p+1} = t^2 \phi(\theta, t),$$

where $\phi(\theta, t)$ is bounded when $t$ tends to zero.

Consider the function

$$u(x, y, h) = \sum_{p=0}^{N} [u^{j+1}_{2p+1}(\cos \theta, \sin \theta) - u^{j+1}_{2p+1}(\cos \theta, \sin \theta)] h^{2p+1} =$$

$$\sum_{p=0}^{N} u_{2p+1}(x, y) h^{2p+1}$$

in the intersection of $M$ with the layer $h \in [0, t]$.
Let us introduce the polar coordinates and take the coefficient \( u_1(x, y) = u_1(r, \theta) \) in the form of the following expansions

\[
 u_1(r, \theta) = \sum_{p=0}^{N-1} \sum_{k=0}^{l} (\alpha_k^{2p} \cos k\theta + \beta_k^{2p} \sin k\theta) r^{k+2p}.
\]

The last relation provides us with the expansions of the coefficients \( u_{2p+1}(r, \theta) \), \( p = 1, \ldots, N \). Due to representation (3) we can take the boundary data on the level \( h = t \) in the form

\[
 u(\Phi(t), \theta) = t^2 \sum_{k=0}^{l} (\lambda_k \cos k\theta + \mu_k \sin k\theta) + t^3 \psi(t, \theta).
\]

Therefore we have the relation

\[
 t^l \sum_{k=0}^{l} (\alpha_k^0 \cos k\theta + \beta_k^0 \sin k\theta)(1 + tb(t))^k + \sum_{k=0}^{l} (\alpha_k^2 \cos k\theta + \beta_k^2 \sin k\theta)(1 + tb(t))^{k+2} + \ldots
\]

\[
 + \left[ \sum_{k=0}^{l} (\alpha_k^2 N \cos k\theta + \beta_k^2 N \sin k\theta)(1 + tb(t))^{k+2N} \right] - \frac{t^3}{3!} \sum_{k=0}^{l} 4(k + 1)(\alpha_k^2 \cos k\theta + \beta_k^2 \sin k\theta)(1 + tb(t))^{k+2N} + \ldots + (-1)^N \frac{t^{2N+1}}{(2N + 1)!} \sum_{k=0}^{l} 4N(k + N)! (\alpha_k^2 N \cos k\theta + \beta_k^2 N \sin k\theta)(1 + tb(t))^k =
\]

\[
 = t^2 \sum_{k=0}^{l} (\lambda_k \cos k\theta + \mu_k \sin k\theta) + t^3 \psi(t, \theta).
\]

The last relation yields the following representation of the coefficients \( \alpha_k^0, \beta_k^0 \) through the coefficients \( \alpha_k^2, \beta_k^2 \), \( k = 0, \ldots, l, j = 2, \ldots, N \),

\( \alpha_k^0 = t\lambda_k - \alpha_k^2 - \alpha_k^4 - \ldots - \alpha_k^{2N} + i \sum_{j=1}^{N} c_k^j \alpha_k^j + t^2 \phi_k(t), \)

(4)

\( \beta_k^0 = t\mu_k - \beta_k^2 - \beta_k^4 - \ldots - \beta_k^{2N} + i \sum_{j=1}^{N} d_k^j \beta_k^j + t^2 \psi_k(t), \)

(5)

where \( \phi_k(t) \) and \( \psi_k(t) \) are bounded when \( t \) tends to zero.

Now we must choose the coefficients \( \alpha_k^2, \beta_k^2 \) so that the value \( \|u_1\|_{L_2(E)} \) becomes rather small. Here

\[
 \|u_1\|_{L_2(E)}^2 = 2\pi \int_{0}^{1} \left[ \alpha_0^0 + \alpha_0^2 r^2 + \ldots + \alpha_0^{2N} r^{2N} \right] r^2 dr + \pi \sum_{k=1}^{l} \int_{0}^{1} (\alpha_k^0 + \alpha_k^2 r^{2+k} + \ldots + \alpha_k^{2N} r^{2N+k})^2 dr.
\]
Lemma 1. The minimum of the quadratic form
\[ \int_0^1 [a_0 + a_1 t^2 + \ldots + a_N t^{2N}]^2 rdr \]
with the condition \( \sum_{k=0}^N a_k = 1 \) is equal to \( \frac{1}{2(N+1)^2} \).

Proof. We introduce the variable \( t = r^2 \) and represent the expression
\[ a_0 + a_1 t + \ldots + a_N t^N = \sum_{k=0}^N b_k \hat{P}_k(t). \]
This representation is unique and gives us the linear relations between the coefficients \( a_k \) and \( b_k, k = 0, \ldots, N \). Now the given condition takes the form
\[ \sum_{k=0}^N a_k = \sum_{k=0}^N b_k = 1. \]
and the given quadratic form is equal to
\[ \frac{1}{2} \int_0^1 (\sum_{k=0}^N b_k \hat{P}_k(t))^2 dt = \frac{1}{2} \sum_{k=0}^N \frac{b_k^2}{2k+1}. \]

We apply Lagrange method and find the values \( \tilde{b}_k = \frac{2k+1}{(N+1)^2} \), which provide minimum of the corresponding quadratic form with the given condition. So the minimum of quadratic form equals to
\[ \frac{1}{2} \sum_{k=0}^N \frac{\tilde{b}_k^2}{2k+1} = \frac{1}{2(N+1)^2}. \]

We denote by \( \tilde{a}_k^N, k = 0, \ldots, N \) the values of the coefficients which provide minimum of the quadratic form given in the statement of Lemma 1.

Now we put \( \alpha_{2p}^0 = t \lambda_0 \tilde{a}_k^N, p = 1, \ldots, N \). So the first summand from (6) according to (4) with \( k = 0 \) has the following representation
\[ 2\pi \int_0^1 \left[ \alpha_0^0 + \alpha_0^2 r^2 + \ldots + \alpha_0^{2N} r^{2N} \right]^2 rdr = 2\pi \int_0^1 \left\{ t \lambda_0 \left[ 1 - (\tilde{a}_1^N + \ldots + \tilde{a}_N^N) + \tilde{a}_1^N r^2 + \ldots + \tilde{a}_N^N r^{2N} \right] + t^2 R_0(r, t, N) \right\}^2 dr = \frac{\pi t^2 \lambda_0^2}{(N+1)^2} \left[ 1 + t \bar{R}_0(t, N) \right]. \]

We need to find minima of some other quadratic forms but we have to introduce some polynomial bases different from Legendre polynomials in order to find the exact value of these minima. So we give here only the estimates of them.
Lemma 2. The minimum of the quadratic form
\[ \int_{0}^{1} [a_0 + r^k(a_1 r^2 + \ldots + a_N r^{2N})]^2 dr \]
with the condition \( \sum_{k=0}^{N} a_k = 1 \) is less than \( \frac{1}{2N^2} \).

Proof. We have
\[ \int_{0}^{1} [a_0 + r^k(a_1 r^2 + \ldots + a_N r^{2N})]^2 dr = \frac{a_0}{2} + 2a_0 \int_{0}^{1} r^{k+1}(a_1 r^2 + \ldots + a_N r^{2N}) dr + \]
\[ + \int_{0}^{1} r^{2k+4}(a_1 + \ldots + a_N r^{2N-2})^2 dr. \]

Let us put \( a_0 = 0 \). Now
\[ \int_{0}^{1} [a_0 + r^k(a_1 r^2 + \ldots + a_N r^{2N})]^2 dr = \int_{0}^{1} r^{2k+4}(a_1 + \ldots + a_N r^{2N-2})^2 dr \leq \]
\[ \int_{0}^{1} (a_1 + \ldots + a_N r^{2N-2})^2 dr. \]

The minimum of the quadratic form \( \frac{1}{2N^2} \) according to Lemma 1 is equal to \( \frac{1}{2N^2} \). This quadratic form achieves the minimal value when \( a_k = \tilde{a}_{k-1}, k = 1, \ldots, N \). Lemma 2 is proved.\( \square \)

Now we put
\[ \alpha_k^0 = \lambda_k t_{k-1}^{N-1}, \]
\[ \beta_k^0 = \mu_k t_{k-1}^{N-1}, \quad p = 1, \ldots, N, k = 1, \ldots, l, \]
and obtain according to (4) and (5) the estimates of the following summands from (6):
\[ \pi \int_{0}^{1} [\alpha_k^0 + \alpha_k^2 r^{2k+2} + \ldots + \alpha_k^{2N} r^{2N+k}]^2 dr \leq \frac{\pi \mu_k^2 t^2}{2N^2} (1 + \tilde{R}_k(t, N)), \]
\[ \pi \int_{0}^{1} [\beta_k^0 + \beta_k^2 r^{2k+2} + \ldots + \beta_k^{2N} r^{2N+k}]^2 dr \leq \frac{\pi \mu_k^2 t^2}{2N^2} (1 + \tilde{Q}_k(t, N)). \]

Finally
\[ \|u_1\|_{L_2(E)}^2 \leq \frac{A t^2}{N^2} + t^3 \Phi(t, N), \]
where \( A \) is constant and \( \Phi(t, N) \) is bounded for the fixed \( N \) when \( t \) tends to zero and therefore
\[ \|u_k^{j+1} - u_k^j\|_{L_2(E_j)}^2 \leq \frac{C t^2}{N^2} + t^3 \Phi(t, N). \]

(7)
9. **Approximation estimate for the smoothing solution**

In this section we present integral approximation estimate of the constructed solution. We suppose that the distance \(|h_{j+1} - h_j|\) is unique for all \(j = 1, \ldots, m\), and equals to \(t = (B - A)/(m + 1)\).

Assume that \(\hat{u}\) is the difference between the exact solution of Dirichlet problem and the smoothing spline-interpolation solution. Let us first consider the equality which holds true due to Gauss-Ostrogradsky theorem

\[
\int \int_S \hat{u}(u_h + \hat{u}_h + \hat{u}_y dy dx + \sum_{j=1}^m \int \int_{E_j} \hat{u}(u_h(x, y, h_j) - u_h^{j+1}(x, y, h_j)) dx dy =
\]

\[
= \int \int \int_M |\text{grad}\hat{u}(x, y, h)|^2 dx dy dh,
\]

where \(u^j(x, y, h)\) is the spline at the layer \(h \in [h_{j-1}, h_j]\) and \(E_j\) is the disk which is the intersection of \(M\) with the plane \(h = h_j\).

So

\[
\int \int \int_M |\text{grad}\hat{u}(x, y, h)|^2 dx dy dh \leq \max_S |\hat{u}| \int \int_S |\text{grad}\hat{u}| dS + \max_M |\hat{u}| \sum_{j=1}^m \int \int_{E_j} |(u^j_h(x, y, h_j) - u_h^{j+1}(x, y, h_j))| dx dy.
\]

The right-hand side of the last inequality can be made arbitrary small due to the approximation of the boundary data and the smoothing of the derivatives with respect to \(h\). Really

\[
\sum_{j=1}^m \int \int_{E_j} |(u^j_h(x, y, h_j) - u_h^{j+1}(x, y, h_j))| dx dy \leq \frac{Dm(B - A)}{N(m + 1)} + m \frac{(B - A)^2}{(m + 1)^2} \tilde{\Psi}(t, N),
\]

where \(D\) is constant and \(\tilde{\Psi}(t, N)\) is bounded when \(t\) tends to zero, due to (7) and to Hölder inequality. Let \(\max_M |\hat{u}| > 0\). For arbitrary small \(\varepsilon > 0\) we can choose \(N\) so large that the first summand in the right side of the previous inequality is less than \(\varepsilon/(3 \max_M |\hat{u}|)\). Now with the chosen number \(N\) we can choose the number \(m\) so large that the second summand in the right side of the previous inequality is less than \(\varepsilon/(3 \max_M |\hat{u}|)\).

Obviously the value of \(\max_S |\hat{u}| \int \int_S |\text{grad}\hat{u}| dS\) can be made less than \(\varepsilon/3\) for sufficiently large number \(m\).

Therefore \(\int \int_M |\text{grad}\hat{u}(x, y, h)|^2 dx dy dh < \varepsilon\) for large values of \(N\) and \(m\). Hence the Hölder inequality implies

\[
\int \int_M |\text{grad}\hat{u}(x, y, h)| dx dy dh \leq \sqrt{V(M)} \varepsilon,
\]

where \(V(M)\) is the volume of \(M\).
Now we prove the following estimate:

$$\int \int_{P_{\tilde{h}}} |\hat{u}(x,y,\tilde{h})| \, dx \, dy \leq Q\sqrt{\varepsilon}$$

for any $\tilde{h} \in [A,B]$, where $E_{\tilde{h}}$ is the section of $M$ by the plane $h = \tilde{h}$.

We prove this estimate at first for the case when $M$ is a convex body of revolution.

Let us fix $\tilde{h}$ and choose a point $s^* \in U$, where $U = \bigcup_{j=1}^{n} C_j \cup C_A \cup C_B$; $C_j$, $j = 1, \ldots, n$, are the curves with the given boundary data on $S$ and $C_A$, $C_B$ are the corresponding edges (or points). The cone $CH(P_{\tilde{h}}, s^*)$ which is the convex hull of $P_{\tilde{h}} \cup \{s^*\}$ is a subset of $M$ since $M$ is convex. Let us consider the set \{\gamma_{\tilde{h}}(x,y,s^*)\} of the straight lines, connecting the points $(x,y)$ of $P_{\tilde{h}}$ with the point $s^*$ and parametrized linearly by $h$:

$$\gamma_{\tilde{h}}(x,y,s^*) = (x(h), y(h), h), \quad h \in [\tilde{h}, h^*],$$

where $h^*$ is the $h$-coordinate of $s^*$. One can find the characteristic

$$K(\tilde{h}, s^*) = \max_{(x,y) \in P_{\tilde{h}}} \left| \frac{d\gamma_{\tilde{h}}(x,y,s^*)}{dh} \right|.$$

Now we find the characteristic $K(\tilde{h}) = \min_{s^* \in U} K(\tilde{h}, s^*)$ and the point $s(\tilde{h})$ which realises this minimum, the $h$-coordinate of the point $s(\tilde{h})$ being $h^*$. The values of the function $\hat{u}$ vanish on the set $U$, so for any point $p \in P_{\tilde{h}}$ we have

$$u(p) = \int_{\tilde{h}}^{h^*} \frac{d\hat{u}(x(h), y(h), h)}{dh} \, dh = \int_{\tilde{h}}^{h^*} (\text{grad}\hat{u}(x,y,h), \frac{d\gamma_{\tilde{h}}}{dh}) \, dh.$$

Hence the estimate of $\frac{d\gamma_{\tilde{h}}}{dh}$ implies

$$|\hat{u}(p)| \leq K(\tilde{h}) \int_{\tilde{h}}^{h^*} \|\text{grad}\hat{u}(x,y,h)\| \, dh.$$

Thus

$$\int \int_{P_{\tilde{h}}} |\hat{u}(x,y,\tilde{h})| \, dx \, dy = \int_{CH(P_{\tilde{h}}, s^*(\tilde{h}))} \int \int |(\text{grad}\hat{u}(x,y,h), \frac{d\gamma}{dh})| \, dx \, dy \, dh \leq K(\tilde{h}) \int_{M} \int |\text{grad}\hat{u}(x,y,h)| \, dx \, dy \, dh \leq K(\tilde{h}) \sqrt{V(M)} \varepsilon = Q\sqrt{\varepsilon}.$$

Here $Q = \sqrt{V(M)} \max_{h \in [A,B]} \{K(\tilde{h})\}$.

For another types of bodies we can obtain similar estimate if we divide any section $P_{\tilde{h}}$ into finite number of domains, construct the cone with the vertex from $U$ which lies in $M$ for every domain and take the integral of $|\hat{u}|$ over $P_{\tilde{h}}$ as the sum of the integrals over the introduced domains.
10. On the similar problems

As it was noted above the external problem for the body of revolution can be reduced to the internal one when the unbounded domain in which we solve the problem contains any ball in its exterior. We make the inversion with respect to the corresponding sphere and obtain the internal problem for the bounded domain. We construct the spline-interpolation solution of this internal problem and then make the inversion once more.

The most simple case of the external problem is the case of the problem for the unit ball $B_1(0)$ exterior. The points of the unit sphere are invariant under the inversion. So the boundary conditions remain the same.

We also can apply the spline-interpolation solution presented in this paper to the Dirichlet value problem for a cylinder or a cone with the section which is the known conformal map of the unit disk. Solution of 2D Dirichlet problem for the corresponding cuts can be reduced to solution of this problem in the unit disk.

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