Polyhedral Clinching Auctions for Two-sided Markets

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Abstract

In this paper, we present a new model and mechanism for auctions in two-sided markets of buyers and sellers, with budget constraints imposed on buyers. Our mechanism is viewed as a two-sided extension of the polyhedral clinching auction by Goel et al., and enjoys various nice properties, such as incentive compatibility of buyers, individual rationality, pareto optimality, strong budget balance. Our framework is built on polymatroid theory, and hence is applicable to a wide variety of models that include multiunit auctions, matching markets and reservation exchange markets.

1 Introduction

Mechanism design for auctions in two-sided markets is a challenging and urgent issue, especially for rapidly growing fields of internet advertisement. In ad-exchange platforms, the owners of websites want to get revenue by selling their ad slots, and the advertisers want to purchase ad slots. Auction is an efficient way of mediating them, allocating ad slots, and determining payments and revenues, where the underlying market is two-sided in principle. Similar situations arise from stock exchanges and spectrum license reallocation; see e.g., [2, 5]. Despite its potential applications, auction theory for two-sided markets is currently far from dealing with such real-world markets. The main difficulty is that the auctioneer has to consider incentives of buyers and sellers, both possibly strategic, and is confronted with nonexistence theorems of desirable mechanisms achieving both accuracy and efficiency, even in the simplest case of bilateral trade; see [15].

In this paper, we address auctions for two-sided markets, aiming to overcome such difficulties and provide a reasonable and implementable framework. To capture realistic models mentioned above, we deal with budget constraints on buyers. The presence of budgets drastically changes the situation in which the traditional auction theory is not applicable. Our investigation is thus based on two recent seminal works on auction theory of budgeted one-sided markets:

(i) Dobzinski et al. [4] presented the first effective framework for budget-constrained markets. Generalizing the celebrated clinching framework by Ausubel [1], they proposed an incentive compatible, individual rational, and pareto optimal mechanism, called “Adaptive Clinching Auction”, for markets in which the budget information is public to the auctioneer. This work triggers subsequent works dealing with more complicated settings [3, 6, 10, 11].

(ii) Goel et al. [11] utilized polymatroid theory to generalize the above result for a broader class of auction models including previously studied budgeted settings as well as new models for contemporary auctions such as Adwords Auctions. Here a polymatroid is a polytope associated

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with a monotone submodular function, and can represent the space of feasible transactions under several natural constraints. They presented a polymatroid-oriented clinching mechanism, “Polyhedral Clinching Auction,” for markets under polymatroidal environments. This mechanism enjoys incentive compatibility, individual rationality, and pareto optimality, and can be implemented via efficient submodular optimization algorithms developed in the literature of combinatorial optimization [8, 16].

The goal of this paper is to extend this line of research to reasonable two-sided settings.

Our contribution. We present a model and mechanism for auctions in two-sided markets. Our market is modeled as a bipartite graph of buyers and sellers, with transacting divisible and homogeneous goods through the links. Each buyer wants goods under a limited budget. Each seller constrains transactions of his goods by a monotone submodular function on the set of edges linked to him. Namely, possible transactions are restricted to the corresponding polymatroid. In the auction, each buyer reports his bid and budget to the auctioneer, and each seller reports his reserved price. In our model, the reserved price is assumed to be identical with his true valuation. The utilities are quasi-linear (within budget) on their valuations and payments/revenues. The goal of this auction is to determine transactions of goods, payments of buyers, and revenues of sellers, with which all participants are satisfied. In the case of single seller, this model coincides with that of Goel et al [11]. For this model, we present a mechanism that satisfies the incentive compatibility of buyers, individual rationality, pareto optimality and strong budget balance.

Our mechanism is a generalization of the polyhedral clinching auction by Goel et al., and is also built on the clinching framework and polymatroid theory. As price clocks increase, each buyer clinches a maximal amount of goods not affecting other buyers. Each buyer transacts with multiple sellers, and hence conducts a multidimensional clinching. We prove in Theorem 3.1 that feasible transactions for the clinching forms a polymatroid, which we call the clinching polytope, and moreover the corresponding submodular function can be computed in polynomial time. Thus our mechanism is implementable. We also reveal in Theorem 3.4 that the allocation of buyers obtained by our mechanism is the same as that by the original polyhedral clinching auction applied to the reduced one-side market, obtained by aggregating all sellers to one seller. This means that our mechanism achieves the same performance as the original one under a more complex setting of multiple sellers.

Our framework is applicable to a wide variety of auction models in two-sided markets, thanks to the strong expressive power of polymatroids. Examples include two-sided extensions of multiunit auctions [4] and matching markets [6] (for divisible goods), and reservation exchange markets [9]. We demonstrate how our framework is applied to auctions for display advertisements.

Related work. Double auction is the simplest auction for two-sided markets, where buyers and sellers have unit demand and unit supply, respectively. The famous Myerson-Satterthwaite impossibility theorem [15] says that there is no mechanism which simultaneously satisfies incentive compatibility (IC), individual rationality (IR), pareto optimality (PO), and budget balance (BB). McAfee [13] proposed a mechanism which satisfies (IC),(IR), and (BB). Recently, Colini-Baldeschi et al. [2] proposed a mechanism which satisfies (IC),(IR), and (BB), and achieves an $O(1)$-approximation to the maximum social welfare.

Goel et al. [9] considered a two-sided market model, called a reservation exchange market, for internet advertisement. They formulated several axioms of mechanisms for this model, and presented an (implementable) mechanism satisfying (IC) for buyers, (IR), maximum social welfare, and a certain fairness for sellers, called the envy-freeness. This mechanism is also based on the
clinching framework, and sacrifices (IC) for sellers to avoid the impossibility theorem. Their setting is non-budgeted.

Freeman et al. [7] formulated the problem of wagering as an auction in a special two-sided market, and presented a mechanism, called “Double Clinching Auctions”, satisfying (IC), (IR), and (BB). They verified by computer simulations that the mechanism shows near-pareto optimality. This mechanism is regarded as the first generalization of the clinching framework to budgeted two-sided settings, though it is specialized to wagering.

Our results in this paper provide the first generic framework for auctions in budgeted two-sided markets.

Organization of this paper. The rest of this paper is organized as follows. In Section 2, we introduce our model and present the main result. In Section 3, we present and analyze our mechanism. In Section 4, we give proofs.

Notation. Let \( \mathbb{R}_+ \) denote the set of nonnegative real numbers, and let \( \mathbb{R}_+^E \) denote the set of all functions from a set \( E \) to \( \mathbb{R}_+ \). For \( f \in \mathbb{R}_+^E \), we often denote \( f(e) \) by \( f_e \), and write as \( f = (f_e)_{e \in E} \).

Let us recall theory of polymatroids and submodular functions; see [8, 16]. A monotone submodular function on set \( E \) is a function \( f \) satisfying:

\[
\begin{align*}
  f(\emptyset) &= 0, \\
  f(S) &\leq f(T) \quad (S, T \subseteq E, S \subseteq T), \\
  f(S \cup \{e\}) - f(S) &\geq f(T \cup \{e\}) - f(T) \quad (S, T \subseteq E, S \subseteq T, e \in E \setminus T),
\end{align*}
\]

where the third inequality is equivalent to

\[
  f(S) + f(T) \geq f(S \cap T) + f(S \cup T) \quad (S, T \subseteq E).
\]

The polymatroid \( P := P(f) \) associated with monotone submodular function \( f \) is defined by

\[
P := \{ x \in \mathbb{R}_+^E | \sum_{e \in F} x_e \leq f(F) \quad (F \subseteq E) \},
\]

and the base polytope of \( f \) is defined by

\[
B := \{ x \in P | \sum_{e \in E} x_e = f(E) \},
\]

which is equal to the set of all maximal points in \( P \). A point in \( B \) is obtained by the greedy algorithm [8] in polynomial time, provided the value of \( f \) for each subset \( F \subseteq E \) can be computed in polynomial time.

2 Main result

We consider a two-sided market consisting of \( n \) buyers and \( m \) sellers. Our two-sided market is modeled as a bipartite graph \((N, M, E)\) of disjoint sets \( N \) and \( M \) of nodes and edge set \( E \subseteq N \times M \), where \( N \) and \( M \) represent a set of buyers and sellers, respectively, and buyer \( i \in N \) and seller \( j \in M \) are adjacent if and only if \( i \) wants the goods of seller \( j \). An edge \((i, j) \in E\) is denoted by \( ij \).

For buyer \( i \) (resp. seller \( j \)), let \( E_i \) (resp. \( E_j \)) denote the set of edges incident to \( i \) (resp. \( j \)). In the market, the goods are divisible and homogeneous. Each buyer \( i \) has three nonnegative real
numbers \( v_i, v'_i, B_i \in \mathbb{R}_+ \), where \( v_i \) and \( v'_i \) are his valuation and bid, respectively, for per unit of goods sold by buyers in \( E_i \), and \( B_i \) is his budget. Each buyer \( i \) acts strategically for maximizing his utility \( u_i \) (defined later), and hence his true valuation \( v_i \) is not necessarily equal to bid \( v'_i \). In this market, each buyer \( i \) reports \( v'_i \) and \( B_i \) to the auctioneer. Each seller \( j \) also has a valuation \( \rho_j \in \mathbb{R}_+ \) for the unit of his goods, and reports \( \rho_j \) to the auctioneer as the reserved price of his goods, the lowest price that he admits for the goods. In particular, he is assumed to be truthful (to avoid the impossibility theorem as in [9]). He also has a monotone submodular function \( f_j \) on \( E_j \), which constrains transactions of goods through \( E_j \). The value \( f_j(F) \) for \( F \subseteq E_j \) means the maximum possible amount of goods transacted through edge subset \( F \). In particular, \( f_j(E_j) \) is interpreted as his stock of goods. Note that these assumptions on sellers are characteristic of our model.

Under this setting, the goal is to design a mechanism determining a reasonable allocation. An allocation \( A \) of the auction is a triple \( A := (w, p, r) \) of a transaction vector \( w = (w_{ij})_{ij \in E} \), a payment vector \( p = (p_i)_{i \in N} \), and a revenue vector \( r = (r_j)_{j \in M} \), where \( w_{ij} \) is the amount of transactions of goods between buyer \( i \) and seller \( j \), \( p_i \) is the payment of buyer \( i \), and \( r_j \) is the revenue of seller \( j \). For each \( j \in M \), the restriction \( w|_{E_j} \) of the transaction vector \( w \) to \( E_j \subseteq E \) must belong to the polymatroid \( P_j \):

\[
    w|_{E_j} \in P_j \quad (j \in M).
\]

Also the payment \( p_i \) of buyer \( i \) must be within his budget \( B_i \):

\[
    p_i \leq B_i \quad (i \in N).
\]

A mechanism \( M \) is a function that gives an allocation \( A = (w, p, r) \) from public information \( I := ((N, M, E), \{v'_i\}_{i \in N}, \{B_i\}_{i \in N}, \{\rho_j\}_{j \in M}, \{f_j\}_{j \in M}) \) that the auctioneer can access. The true valuation \( v_i \) of buyer \( i \) is private information that only \( i \) can access. We regard \( I \) and \( \{v_i\}_{i \in N} \) as the input of our model.

Next we define the utilities of buyers and sellers. For an allocation \( A = (w, p, r) \), the utility \( u_i(A) \) of buyer \( i \) is defined by:

\[
    u_i(A) := \begin{cases} 
        v_i \sum_{ij \in E_i} w_{ij} - p_i & \text{if } p_i \leq B_i, \\
        -\infty & \text{otherwise.}
    \end{cases}
\]

Intuitively, this is the sum of surplus for their goods if the payment is within his budget, and \(-\infty\) otherwise. The utility of seller \( j \) is defined by:

\[
    u_j(A) := r_j + \rho_j(f_j(E_j)) - \sum_{ij \in E_j} w_{ij}. \tag{1}
\]

This is the sum of revenue and the total valuation of his remaining goods. In this model, we consider the following properties of mechanism \( M \).

(ICb) Incentive Compatibility of buyers: For every input \( I_i, \{v_i\}_{i \in N} \), it holds

\[
    u_i(M(I)) \leq u_i(M(I_i)) \quad (i \in N),
\]

where \( I_i \) is obtained from \( I \) by replacing bid \( v'_i \) of buyer \( i \) with his true valuation \( v_i \). This means that it is the best strategy for buyers to report their true valuation.

(IRb) Individual Rationality of buyers: For each buyer \( i \), there is a bid \( v'_i \) such that \( i \) always obtains nonnegative utility. If (ICb) holds, then (IRb) is written as

\[
    u_i(M(I_i)) \geq 0 \quad (i \in N).
\]
(IRs) **Individual Rationality** of sellers: The utility of each seller $j$ after the auction is at least the utility $\rho_j f_j(E_j)$ at the beginning:

$$u_j(M(I)) \geq \rho_j f_j(E_j) \quad (j \in M).$$

By (1), (IRs) is equivalent to

$$r_j \geq \rho_j \sum_{ij \in E_j} w_{ij}. \quad (2)$$

(SBB) **Strong Budget Balance**: All payments of buyers are directly given to sellers:

$$\sum_{i \in N} p_i = \sum_{j \in M} r_j.$$

(PO) **Pareto Optimality**: There is no allocation $A := (w, p, r)$ which satisfies $\sum_{i \in N} p_i \geq \sum_{j \in M} r_j$ and the following three conditions:

$$u_i(M(I^*)) \leq u_i(A) \quad (i \in N),$$

$$u_j(M(I^*)) \leq u_j(A) \quad (j \in M),$$

and at least one of the inequalities holds strictly, where $I^*$ is obtained from $I$ by replacing $\{v'_i\}_{i \in N}$ with $\{v_i\}_{i \in N}$. Namely, there is no other allocation superior to that given by $M$ in all aspects, provided all buyers report their true valuation.

They are desirable properties that mechanism should have. The main result is:

**Theorem 2.1.** There exists a mechanism which satisfies all of (ICb),(IRb),(IRs), (SBB), and (PO).

The detail of our mechanism is explained in the next section.

**Remark 1.** Maximizing the **social welfare**, the sum of utilities of all participants, is usually set as the goal in traditional auction theory. However, in budgeted settings, it is shown in [4] that the maximum social welfare and incentive compatibility of buyers cannot be achieved simultaneously. As in the previous mechanisms [3, 4, 6, 10, 11], we take priority over incentive compatibility.

**Application to display ad auction of multiple sellers and slots**  Here we consider following auction for display advertisements (ads) between advertisers and owners of websites. Each owner $j$ wants to sell ad slots in his website. Each advertiser $i$ wants to purchase the slots. Namely, the owners are sellers, and advertisers are buyers, where buyer $i$ is linked to seller $j$ if $i$ is interested in the slots of the website of $j$. The market is modeled as bipartite graph $(N, M, E)$ as above.

Each seller has $s_j$ ad slots. Each slot $l \in \{1, 2, \ldots, s_j\}$ has a barometer $\alpha^l_j \in \mathbb{R}^+$ for the quality of ads, which is measured by the expected number of views of $l$ over a certain period. The **view-impression** is a unit of this barometer; namely, slot $l$ has $\alpha^l_j$ view-impressions. In the market, buyers purchase view-impressions from sellers, and have bids and valuations for the unit view-impression.

After the auction, suppose that buyer $i$ obtains $w_{ij}$ view-impressions from $j$. Seller (owner) $j$ assigns ads of buyers (linked to $j$) to his slots so that the same ads cannot be displayed in distinct slots at the same time. An ad-slotting is naturally represented by $(S, \psi)$ for a subset $S$ of buyers and an injection $\psi$ from $S$ to the set of his slots $\{1, 2, \ldots, s_j\}$. (The number of slots is at least $|S|$
by adding dummy slots of 0 view-impressions.) Seller $j$ displays ads of $i$ according to a probability distribution $p$ on the set of all ad-slottings satisfying

$$w_{ij} = \sum_{(S, \psi)} p(S, \psi) \alpha^j_{\psi(i)}.$$  

(3)

Namely the expected number of view-impressions of ads $i$ in the website of $j$ is equal to $w_{ij}$. The existence of such a $p$ constrains transactions $w_{ij}$, and is equivalent to the condition that $\sum_{i \in S} w_{ij}$ is at most the sum of $|S|$ highest $\alpha^j_i$ for each subset $S$ of buyers linked to $j$. This condition can be written by a monotone submodular function, and seller $j$ restricts his transactions $w_{ij}$ by the corresponding polymatroid, to conduct the above way of ad-slotting after the auction. Now the market falls into our model, and our mechanism is applicable, where buyers can have budget limits.

This market is a modification of Adwords auctions in [11] for display ad auctions with multiple websites, where the way of ad-slotting according to (3) is based on their idea. Also this market is a special reservation exchange market in the sense of [9]. Our framework is applicable to general reservation exchange market with budget constraints and polymatroid constraints.

3 Polyhedral Clinching Auctions in Two-sided Markets

3.1 Mechanism

Here we describe our mechanism for Theorem 2.1. First we make the following preprocessing on the market. Buyers and sellers are numbered as $N = \{1, 2, ..., n\}$ and $M = \{1, 2, ..., m\}$. For each seller $j \in M$, add to $N$ a virtual buyer $n+j$ corresponding to $j$, and add to $E$ a new edge connecting $n+j$ and $j$; buyer $n+j$ transacts only with $j$. The virtual buyer $n+j$ has sufficiently large budget $B_{n+j} = \infty$, and reports the valuation $\rho_j$ of $j$ as bid $v'_{n+j}$. The valuation $v_{n+j}$ is set as $v_{n+j} := v'_{n+j}$ (though we do not use $v_{n+j}$). Submodular function $f_j$ is extended by

$$f_j(F) := \begin{cases} f_j(E_j) & \text{if } (n+j)j \in F, \\ f_j(F) & \text{otherwise.} \end{cases}$$  

(4)

Namely the goods purchased by buyer $n+j$ is interpreted as the unsold goods of $j$. The utility of seller $j$ (in the original market) is the sum of the revenue of seller $j$ and the utility of buyer $n+j$ after the auction; then we need not to take $\rho_j$ into account by this preprocessing.

After the preprocessing, we apply Algorithm 1. The meaning of variables $c_i, \xi_{ij}, w_{ij}, p_i, r_j, d_i, l$ and fixed parameter $\varepsilon > 0$ is explained as follows:

- $c_i$ is the price clock of buyer $i$, which is used as the transaction cost for per unit of goods at this moment. It starts at 0 and increases by $\varepsilon$ in each step.

- $\xi_{ij}$ is the increase of transactions between buyer $i$ and seller $j$ in the current iteration.

- $w_{ij}$ is the total amount of transaction between buyer $i$ and seller $j$. Transaction vector $(w_{ij})_{ij \in E}$ must belong to the polymatroid

$$P := \bigoplus_{j \in M} P_j = \{ w \in \mathbb{R}_+^E \mid w|_{E_j} \in P_j \ (j \in M) \}.$$  

- $p_i$ is the payment of buyer $i$.  

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Algorithm 1 Polyhedral Clinching Auction for Two-Sided Market

1: \( p_i := 0, \) \( c_i := 0, \) \( d_i := \infty \) (\( i \in N \)) and \( l := 1. \)
2: \( r_j := 0 \) (\( j \in M \)) and \( w_{ij} := 0 \) (\( ij \in E \)).
3: while \( d_i \neq 0 \) for some \( i \in N \) do
4: for \( i = 1, 2, \ldots, n + m \) do
5: Clinch a maximal increase (\( \xi_{ij} \)) \( (ij \in E_i) \) not affecting other buyers.
6: \( w_{ij} := w_{ij} + \xi_{ij} \) (\( ij \in E_i \)).
7: \( p_i := p_i + c_i \sum_{ij \in E_i} \xi_{ij} \) (\( i \in N \)).
8: \( d_i := \begin{cases} 0 & \text{if } c_i < v'_i, \\ \frac{(B_i - p_i)}{c_i} & \text{otherwise}, \end{cases} \) (\( i \in N \)).
9: end for
10: \( r_j := r_j + \sum_{ij \in E_j} c_i \xi_{ij} \) (\( j \in M \)).
11: \( c_l := c_l + \varepsilon. \)
12: \( l := l + 1 \mod m + n. \)
13: \( d_l := \begin{cases} 0 & \text{otherwise}. \end{cases} \)
14: end while

- \( r_j \) is the revenue of seller \( j. \)
- \( d_i \) is the demand of buyer \( i, \) which is interpreted as the maximum amount of transactions that buyer \( i \) can get in the future.
- \( l \) is the buyer whose price clock is increased in the next iteration.

Algorithm 1 terminates when \( d_i = 0 \) for all buyer \( i, \) and outputs \( (w, p, r) \) at this moment. The allocation \( (w, p, r) \) obtained by Algorithm 1 includes that for virtual buyers. Thus we transform this allocation into the allocation without virtual buyers in the following way. For each \( j, \) we regard \( w_{(n+j)j} \) as the remaining goods of seller \( j, \) and update the revenue \( r_j \) by

\[
    r_j := r_j - p_{n+j}.
\]

After the update, the triple \( (w, p, r) \) for non-virtual buyers and sellers is regarded as the allocation in the original market, and as the output of our mechanism.

We explain the details of “not affecting other buyers” in line 5. For transaction vector \( w \) and demand vector \( d := (d_i)_{i \in N}, \) define the remnant supply polytope \( P_{w,d} \) by

\[
    P_{w,d} := \{ x \in \mathbb{R}^E_+ \mid w + x \in P, \sum_{ij \in E_i} x_{ij} \leq d_i \ (i \in N) \}. \tag{5}
\]

In addition, for \( \xi := (\xi_{ij})_{ij \in E_i} \in \mathbb{R}^E_+ \), define the remnant supply polytope \( P_{w,d}^i(\xi) \) of remaining buyers \( N \setminus \{i\} \) by

\[
    P_{w,d}^i(\xi) := \{ u \in \mathbb{R}^{N \setminus \{i\}}_+ \mid \exists x \in P_{w,d}, x|_{E_i} = \xi, \sum_{k \in E_k} x_{kj} = u_k \ (k \in N \setminus \{i\}) \}. \tag{6}
\]

The first polytope \( P_{w,d} \subseteq P \) represents the feasible increases of transactions under \( w \) and \( d, \) and the second polytope \( P_{w,d}^i(\xi) \subseteq \mathbb{R}^{N \setminus \{i\}}_+ \) represents the possible amounts of goods that buyers except
i can get in the future, provided i got $\xi \in \mathbb{R}^{E_i}$ in this iteration. Then the condition for clinching $\xi_i := (\xi_{ij})_{ij \in E_i}$, not to affect other buyers is naturally written as $P^{i}_{w,d}(\xi) = P^{i}_{w,d}(0)$. This motivates to define the polytope $P^{i}_{w,d}$ by

$$P^{i}_{w,d} := \{\xi \in \mathbb{R}^{E_i} | P^{i}_{w,d}(\xi) = P^{i}_{w,d}(0)\},$$

which we call the clinching polytope of buyer $i$. A clinching vector $(\xi_{ij})_{ij \in E_i}$ in line 5 is chosen as a maximal vector in the clinching polytope $P^{i}_{w,d}$. Such a clinching can be done in polynomial time by the following theorem, which implies that our mechanism is implementable in practice.

**Theorem 3.1.** The clinching polytope $P^{i}_{w,d}$ is a polymatroid, and the value of the corresponding submodular function can be computed in polynomial time. In particular, a maximal vector of $P^{i}_{w,d}$ can be computed in polynomial time by the greedy algorithm.

The proof of Theorem 3.1 is given in Section 4.2.

### 3.2 Analysis

Here we analyze our mechanism for proving Theorem 2.1. First we show (SBB) for our mechanism. In line 8 and line 10 of Algorithm 1 the total payment of buyers (including virtual buyers) is directly given to sellers in each iteration. Also both the total payment and the total revenue are decreasing by the total payment of virtual buyers in the transformation of the allocation. Thus we have:

**Lemma 3.2.** Our mechanism satisfies (SBB).

For the individual rationality of sellers (IRs), we will prove in Section 4.4 that each seller $j$ and non-virtual buyer $i$ transact only when the price clock $c_i$ is at least valuation $\rho_j$ and virtual buyer $n + j$ vanishes, i.e., $d_{n+j} = 0$. Consequently we obtain (IRs); see also (2).

**Proposition 3.3.** Our mechanism satisfies (IRs).

Next we consider other properties (ICb),(IRb), and (PO) in Theorem 2.1. For proving these properties, we utilize the framework by Goel et al. [11] on one-sided markets under polymatroidal environments. From our two-sided market (including virtual buyers), we construct a one-sided market consisting of $N$ and one seller (= auctioneer) to which their framework is applicable. The polymatroidal environment on $N$ is defined by the following polytope $\tilde{P} \subseteq \mathbb{R}_+^N$:

$$\tilde{P} := \{y \in \mathbb{R}_+^N | \exists w \in P, y_i = \sum_{ij \in E_i} w_{ij} \ (i \in N)\}.$$

Then $\tilde{P}$ is a polymatroid, which immediately follows from a network induction of a polymatroid [14]. An allocation of this market is a pair $(y,p)$ of $y,p \in \mathbb{R}_+^N$, where $y_i$ and $p_i$ is the transaction and payment, respectively, of buyer $i$ to the seller. The transaction vector $y$ must belong to the polymatroid $\tilde{P}$. For each buyer $i$, payment $p_i$ must be within his budget $B_i$. The utility of buyer $i$ is defined in (13) by replacing $\sum_{ij \in E_i} w_{ij}$ with $y_i$. Now the original polyhedral clinching auction [11] is applicable to this one-sided market, and is given in Algorithm 2. The variables $p_i, c_i, d_i, l$ and the parameter $\varepsilon$ are the same as in Algorithm 1. The meaning of other variables are:

- $y_i$ is the amount of goods of buyer $i$. 

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Algorithm 2 Polyhedral Clinching Auction [11] for One-sided Market.

1: \( y_i := 0 \), \( p_i := 0 \), \( c_i := 0 \), \( d_i := \infty \) \((i \in N)\) and \( l := 1 \)
2: while \( d \neq 0 \) do
3: Clinch a maximal increase \((\zeta_i)_{i \in N}\) not affecting other buyers.
4: \( y_i := y_i + \zeta_i \), \( p_i := p_i + c_i \zeta_i \) \((i \in N)\)
5: \( c_l := c_l + \epsilon \)
6: \( d_i := \begin{cases} (B_i - p_i)/c_i & \text{if } c_i < v'_i, \\ 0 & \text{otherwise,} \end{cases} \) \((i \in N)\).
7: \( l := l + 1 \mod m + n \)
8: end while

- \( \zeta_i \) is the amount of goods that buyer \( i \) clinches in the current iteration.

Again we explain the details of “not affecting other buyers” in line 3 according to [11]. For transaction vector \( y \) and demand vector \( d \), define the remnant supply polytope \( \tilde{P}_{y,d} \) by

\[
\tilde{P}_{y,d} := \{ z \in \mathbb{R}^N_+ \mid y + z \in \tilde{P}, y_i \leq d_i \ (i \in N) \}.
\]

Also, for \( \zeta \in \mathbb{R}_+ \), define the remnant supply polytope of remaining buyers \( N \setminus \{i\} \) by

\[
\tilde{P}_{y,d}^i(\zeta) := \{ \sigma \in \mathbb{R}^{N\setminus\{i\}}_+ \mid \exists z \in \tilde{P}_{y,d}, z_i = \zeta, z_k = \sigma_k \ (k \in N \setminus \{i\}) \}.
\]

These polytopes correspond to the polytopes (5) and (6) in Algorithm 1, respectively. For each \( i \), clinching \( \zeta_i \) in line 3 is chosen as the maximum \( \zeta \) for which \( \tilde{P}_{y,d}^i(\zeta) = \tilde{P}_{y,d}^i(0) \).

Theorem 3.4. Suppose that \( \epsilon \) is the same in both Algorithms. At the end of algorithm, \((y,p)\) in Algorithm 2 is equal to \(((\sum_{ij \in E_i} w_{ij})_{i \in N}, p)\) in Algorithm 1.

The proof of Theorem 3.4 is given in Section 4.3. Goel et al. [11] proved that Algorithm 2 satisfies (ICb) and (IRb). Consequently we have:

Corollary 3.5. Our mechanism satisfies (ICb) and (IRb).

Goel et al. [11] proved (PO) for Algorithm 2. We combine Theorem 3.4 with their proof method, and careful considerations on utilities of sellers, and prove (PO) for our mechanism:

Theorem 3.6. Our mechanism satisfies (PO).

The proof is given in Section 4.5.

Remark 2. Our model and mechanism are naturally adapted to the case where utility \( u_i \) of each buyer \( i \) is quasi-linear on a convex region defined by a concave non-decreasing function \( \phi_i : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \phi_i(0) = 0 \):

\[
u_i(A) := \begin{cases} v_i \sum_{ij \in E_i} w_{ij} - p_i & \text{if } p_i \leq \phi_i(\sum_{ij \in E_i} w_{ij}), \\ -\infty & \text{otherwise.} \end{cases}
\]

Such a concave budget constraint was considered by Goel et al. [10] for one-sided markets. They showed that the polyhedral clinching auction can be generalized to this setting. By using their idea, our two-sided version can also be adapted to have the same conclusion in Theorem 2.1 where each buyer reports the region \( \{(p_i, z_i) \mid p_i \leq \phi(z_i)\} \) to the auctioneer, and the budget feasibility is replaced by \( p_i \leq \phi(\sum_{ij \in E_i} w_{ij}) \).
Remark 3. In our mechanism, feasible increases in clinching form a polymatroid, and hence a maximal increase can be easily obtained by the greedy algorithm. However there is a degree of freedom of choosing a maximal increase. The price of goods is increasing, and each seller obtains (actual) revenues from non-virtual buyers after his virtual buyer vanishes. Hence sellers obtain higher revenues when their goods are sold in later iterations. This means that the point selection in each clinching is directly related to sellers’ revenues. Formalizing an appropriate fairness concept for sellers and designing a fair clinching rule deserve challenging future work.

4 Proofs

In this section, we use the following abbreviated notation. For \( \phi \in \mathbb{R}^S \) and \( X \subseteq S \), let

\[
\phi(X) := \sum_{x \in X} \phi(x).
\]

In the market \((N, M, E)\), for node subset \( Z \), let \( E_Z \) denote the set of edges incident to nodes in \( Z \). For edge subset \( F \), let \( N_F \) denote the set of buyers incident to \( F \).

In the following, we often use minimum-value functions of form

\[
G(X) = \min_Y H(X,Y).
\]

By a minimizer \( Y \) of \( G(X) \) we mean such \( Y \) that satisfies \( G(X) = H(X,Y) \).

4.1 Polymatroidal network flow

In the proof, we utilize polymatroidal network flow model by Lawler and Martel [12]. A polymatroidal network is a directed network \((V,E)\) with source \( s \) and sink \( t \) such that each node \( v \) has polymatroids \( P^+_v \) and \( P^-_v \) defined on the sets \( \delta^+_v \) and \( \delta^-_v \) of edges leaving \( v \) and entering \( v \), respectively.

A flow is a function \( \varphi : E \to \mathbb{R}_+ \) satisfying

\[
\varphi(\delta^+ v) = \varphi(\delta^- v) \quad (v \in V \setminus \{s, t\}),
\]

\[
\varphi|_{\delta^+ v} \in P^+_v,
\]

\[
\varphi|_{\delta^- v} \in P^-_v.
\]

Let \( f^+_v \) and \( f^-_v \) be monotone submodular functions corresponding to \( P^+_v \) and \( P^-_v \), respectively. In the case where the network has edge-capacity \( c : E \to \mathbb{R}_+ \), the polymatroid representing capacity-constraint is given by submodular function

\[
F \mapsto c(F) \quad (F \subseteq \delta^+_v \text{ (or } \delta^-_v)).
\]

The flow-value of a flow \( \varphi \) is defined as \( \varphi(\delta^+_s) - \varphi(\delta^-_s) \) (= \( \varphi(\delta^-_t) - \varphi(\delta^+_t) \)). The following is a generalization of the max-flow min-cut theorem for polymatroidal networks, and is also a version of the polymatroid intersection theorem.

Theorem 4.1 (Lawler and Martel [12]). The maximum value of a flow is equal to

\[
\min_{U, A, B} \left\{ \sum_{v \in V \setminus U} f^-_v (\delta^-_v \cap A) + \sum_{v \in U} f^+_v (\delta^+_v \cap B) \right\},
\]

where \( U \) ranges over all node subsets with \( s \in U \neq t \) and \( \{A, B\} \) ranges over all partitions of edges leaving \( U \).
4.2 Proof of Theorem 3.1

Let $d$ be the demand vector, and $w$ the transaction vector in line 4 in the current iteration. We start to study polytopes $P_{w,d}$, $P_{w,d}^i(\xi)$, and $P_{w,d}^i$ from viewpoint of polymatroidal network flow. For each seller $j$, define polytope $P_{j,w}$ by

$$P_{j,w} := \{ x \in \mathbb{R}_+^E \mid w|_{E_j} + x \in P_j \}.$$  

Since $P_{j,w}$ is a contraction of $P_j$ (see; e.g. Fujishige [8 Section 3.1]), $P_{j,w}$ is a polymatroid, and the corresponding submodular function $f_{j,w} : 2^{E_j} \to \mathbb{R}_+$ is given by

$$f_{j,w}(F) := \min_{F' \supseteq F} \{ f_j(F') - w(F') \} \quad (F \subseteq E_j).$$

Let us construct a polymatroidal network from the market $(N,M,E)$, demand $d$, and polymatroids $P_{j,w}$ for each $j \in M$. Each edge in $E$ is directed from $N$ to $M$. For each buyer $i$, consider copy $i'$ and edge $i'i$. The set of the copies of $X \subseteq N$ is denoted by $X'$. Add source node $s$ and edge $si'$ for each $i' \in N'$. Also, add sink node $t$ and edge $jt$ for each $j \in M$. Edge-capacity $c$ is defined by

$$c(e) := \begin{cases} d_i & \text{if } e = i'i \text{ for } i \in N, \\ \infty & \text{otherwise}. \end{cases}$$

(9)

Two polymatroids of each node in $N \cup N' \cup M \setminus \{s,t\}$ is defined as follows. For $j \in M$, the polymatroid on $\delta_j^- = E_j$ is defined as $P_{j,w}$, and the other polymatroid on $\delta_j^+$ is defined by capacity as in [9]. Both two polymatroids of other nodes are defined by capacity [9]. Let $\mathcal{N}$ denote the resulting polymatroidal network.

A flow $\varphi$ in $\mathcal{N}$ is an edge-weight satisfying the flow-conservation law, the capacity constraint, and polymatroid constraint $\varphi|_{E_j} \in P_{j,w} (j \in M)$. Now the remnant supply polytope $P_{w,d}$ is written by using flows in the network $\mathcal{N}$:

$$P_{w,d} := \{ x \in \mathbb{R}_+^E \mid \exists \varphi : \text{flow in } \mathcal{N}, x = \varphi|_E \}.$$  

In the proof, the following function $f_{w,d} : 2^E \to \mathbb{R}_+$ plays a key role

$$f_{w,d}(F) := \max_{x \in P_{w,d}} x(F) = \max_{\varphi: \text{flow in } \mathcal{N}} \varphi(F) \quad (F \subseteq E).$$

(10)

We also define $f, f_w : 2^E \to \mathbb{R}_+$ by

$$f(F) := \sum_{j \in M} f_j(E_j \cap F) \quad (F \subseteq E),$$

$$f_w(F) := \min_{F' \supseteq F} \{ f(F') - w(F') \} = \sum_{j \in M} f_{j,w}(E_j \cap F) \quad (F \subseteq E).$$

Note that both $f$ and $f_w$ is monotone submodular. In particular, $f$ corresponds to the polymatroid $P$ (by a theorem in McDiarmid [14]). Then we obtain the following:

**Theorem 4.2.** For $F \subseteq E$, it holds

$$f_{w,d}(F) = \min_{X \subseteq N_F} \{ f_w(E_X \cap F) + d(N_F \setminus X) \},$$

(11)

and $f_{w,d}(F)$ can be computed in polynomial time, provided the value oracle of each $f_j$ is available.
leaving edge 5.5]. There are several polynomial time algorithms for this problem; see Fujishige [8, Section

Lemma 4.3, the value \( f_{w,d}(F) \) is equal to \( \mathfrak{s} \). Observe that subset \( U \) attaining the minimum can take a form of \( \{s\} \cup N^c \cup X \) for \( X \subseteq N_F \). Then the set \( \delta_{U}^+ \) of edges leaving \( U \) is equal to \( \{i' | i \in N_F \setminus X \} \cup E_X \). In a partition \( \{A,B\} \) of \( \delta_U^+ \) attaining minimum, edge \( i' \) contributes \( d_i \) in both cases of \( i' \in \mathcal{A} \) and \( i' \in \mathcal{B} \). Each edge \( e \) in \( F \) must be in \( A \) (by \( c(e) = \infty \)), and each edge \( e \) in \( E_X \setminus F \) can be in \( B \) (by \( c(e) = 0 \)). Thus we have \( \mathfrak{s} \).

By the above modification, \( f_{w,d}(F) \) can be computed by solving a maximum polymatroidal flow problem. There are several polynomial time algorithms for this problem; see Fujishige [8, Section 5.5]. \( \square \)

Note that \( f_{w,d} \) is not necessarily submodular. Let \( \xi \in \mathbb{R}_+^E \). Next we consider the remnant supply polytope of remaining buyers except \( i \)

\[
P_{w,d}^i(\xi) := \{ u \in \mathbb{R}_+^{N\setminus\{i\}} | \exists x \in P_{w,d}, x|_{E_i} = \xi, \sum_{k \in E_k} x_{kj} = u_k (k \in N \setminus \{i\}) \}.
\]

Define a set function \( h^i_\xi \) on \( N \setminus \{i\} \) by

\[
h^i_\xi(Y) = \min_{G \subseteq E_Y} \{ f_{w,d}(E_Y \cup G) - \xi(G) \} \quad (Y \subseteq N \setminus \{i\}).
\]

**Theorem 4.3.** \( h^i_\xi \) is monotone submodular, and \( P_{w,d}^i(\xi) \) coincides with the polymatroid associated with \( h^i_\xi \).

**Proof.** We utilize polymatroidal network \( \mathcal{N} \). Then \( u \in P_{w,d}^i(\xi) \) if and only if there exists a flow \( \varphi \) in \( \mathcal{N} \) such that \( \varphi(ij) = \xi(ij) \) for \( ij \in E_i \), \( \varphi(i'j) = \xi(E_i) \), and \( \varphi(sk') = u_k \) for \( k \in N \setminus \{i\} \). By modifying \( \mathcal{N} \), we provide an equivalent condition as follows. Replace \( ij \in E_i \) by \( sj \) with capacity \( \xi_{ij} \), change capacity of \( sk' \) to \( u_k \) for \( k \in N \setminus \{i\} \), delete nodes \( i' \) and \( i \). By \( \xi(E_i) \leq d_i \), the above condition is equivalent to: the maximum flow value is equal to \( \xi(E_i) + u(N \setminus \{i\}) \), or equivalently, \( U = \{s\} \) and \( \{A,B\} = (\emptyset, \delta_{U}^+) \) attain the minimum of \( \mathfrak{s} \). In the minimum of \( \mathfrak{s} \), it suffices to consider \( U \) of form \( U = \{s\} \cup Y' \cup Z \) for \( Z \subseteq Y \subseteq N \). Then the set \( \delta_{U}^+ \) of edges leaving \( U \) is equal to \( E_i \cup \{sk' | k \in N \setminus \{Y \cup \{i\}\}\} \cup \{kk' | k \in Y \setminus Z\} \subseteq E_Z \). In any partition \( \{A,B\} \) of \( \delta_{U}^+ \), edge \( sk' \) contributes \( u_k \), and \( kk' \) contributes \( d_k \). It must hold \( E_Z \subseteq A \). Thus it suffices to consider a partition \( \{G, E_i \setminus G\} \subseteq E_i \). From this and Theorem 4.1 \( u \in P_{w,d} \) if and only if

\[
\xi(E_i) + u(N \setminus \{i\}) \leq f_{w}(E_Z \cup G) + \xi(E_i \setminus G) + u((N \setminus \{i\}) \setminus Y) + d(Y \setminus Z) \\
(G \subseteq E_i, Z \subseteq Y \subseteq N \setminus \{i\}).
\]

This is equivalent to

\[
u(Y) \leq \min_{Z \subseteq \mathcal{Y}, G \subseteq E_i} \{ f_{w}(E_Z \cup G) + d(Y \setminus Z) - \xi(G) \} \quad (Y \subseteq N \setminus \{i\}). \tag{12}
\]

Define \( \tilde{h}^i_\xi : 2^{N\setminus\{i\}} \rightarrow \mathbb{R}_+ \) by the right hand side of (12). It suffices to show that \( \tilde{h}^i_\xi \) is monotone submodular, and equals to \( h^i_\xi \). Here \( h^i_\xi \) is viewed as a network induction of \( f_{w} \), and is necessarily monotone submodular; one can also see this fact directly by taking minimizers of \( h^i_\xi(X) \) and \( \tilde{h}^i_\xi(Y) \).

Finally we show \( h^i_\xi(Y) = \tilde{h}^i_\xi(Y) \). Here \( h^i_\xi \) is also written as

\[
h^i_\xi(Y) = \min_{G \subseteq E_i} \left\{ \min_{Z \subseteq N_{E_Y \cup G}} \{ f_{w}(E_Z \cap (E_Y \cup G)) + d(N_{E_Y \cup G} \setminus Z) \} - \xi(G) \right\}
\]

\[
= \min_{Z \subseteq Y \cup \{i\}, G \subseteq E_i} \{ f_{w}(E_Z \setminus (E_i \setminus G)) + d(N_{E_Y \cup G} \setminus Z') - \xi(G) \}, \tag{13}
\]

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where we use the fact that if $G = \emptyset$, then $Z'$ and $Z' \setminus \{i\}$ have the same value in (13). We consider the minimum of (13) over all $Z' \subseteq Y$ and over all $Z' = Z \cup \{i\}$ with $Z \subseteq Y$. The first minimum $\alpha$ is equal to

$$\min_{Z' \subseteq Y, \ G \subseteq E_i} \{ f_w(E'_Z) + d(N_{E_Y \cup G} \setminus Z) - \xi(G) \}$$

$$= \min_{Z' \subseteq Y, \ G \subseteq E_i} \{ f_w(E'_Z) + d(Y \setminus Z) + d(N_G) - \xi(G) \}$$

$$= \min_{Z' \subseteq Y} \{ f_w(E'_Z) + d(Y \setminus Z) \},$$

where $G = \emptyset$ attains the minimum since $d_i \geq \xi(G)$. The second minimum $\beta$ is equal to

$$\min_{Z \subseteq Y, \ G \subseteq E_i} \{ f_w(E_{Z \cup \{i\} \setminus (E_i \setminus G)}) + d(N_{E_Y \cup G} \setminus (Z \cup \{i\})) - \xi(G) \}$$

$$= \min_{Z \subseteq Y, \ G \subseteq E_i} \{ f_w(E_{Z \cup G}) + d(Y \setminus Z) - \xi(G) \}$$

$$= \tilde{h}_\xi^i(Y)$$

Observe that $\alpha \geq \beta$. Thus $h_\xi^i(Y) = \min\{\alpha, \beta\} = \beta = \tilde{h}_\xi^i(Y)$ as required. \qed

Finally we consider the clinching polytope $P_{w,d}^i$ of buyer $i$. Define a set function $h_{w,d}^i$ on $E_i$ by

$$h_{w,d}^i(F) := f_{w,d}(E_{N \setminus \{i\}} \cup F) - f_{w,d}(E_{N \setminus \{i\}}) \quad (F \subseteq E_i).$$

**Theorem 4.4.** $h_{w,d}^i$ is monotone submodular, and $P_{w,d}^i$ coincides with the polymatroid associated with $h_{w,d}^i$.

Theorem 3.1 is obtained as an immediate corollary of Theorem 4.2 and Theorem 4.4. For the proof of Theorem 4.4, we show the following lemma.

**Lemma 4.5.** (i) Let $F \subseteq E_i$. The function $X \mapsto f_{w,d}(E_X \setminus (E_i \setminus F))$ is monotone submodular on $N$.

(ii) Let $X \subseteq N \setminus \{i\}$. The function $G \mapsto f_{w,d}(E_X \cup G)$ is monotone submodular on $E_i$.

**Proof.** The monotonicity of both functions in (i) and (ii) is immediate from the monotonicity of $f_{w,d}$ in Theorem 4.2. Thus we show the submodularity.

(i) The submodularity can directly be shown by using formula (11) and taking minimizers of $f_{w,d}(E_{X \setminus (E_i \setminus F)})$ and $f_{w,d}(E_{Y \setminus (E_i \setminus F)})$. We use a flow interpretation. Consider $N$, and remove all edges in $E \setminus (E_X \setminus (E_i \setminus F))$. Then $f_{w,d}(E_X \setminus (E_i \setminus F))$ is equal to the maximum value of a flow $X'$ to $t$. Thus, $X \rightarrow f_{w,d}(E_X \setminus (E_i \setminus F))$ is a network induction of $f_w$, and submodular.

(ii) Let $F, G \subseteq E_i$. It suffices to consider the case where $F \subseteq G$ and $G \subseteq F$. In particular, both $F$ and $G$ are nonempty. Thus $N_{E_X \cup F} = N_{E_X \cup G} = X \cup \{i\}$.

Suppose that $Y, Z \subseteq X \cup \{i\}$ are minimizers of $f_{w,d}(E_X \cup F)$ and $f_{w,d}(E_X \cup G)$, respectively. Namely $f_{w,d}(E_X \cup F) = f_w(E_Y \cap (E_X \cup F)) + d((X \cup \{i\}) \setminus Y)$ and $f_{w,d}(E_X \cup G) = f_w(E_Z \cap (E_X \cup G)) + d((X \cup \{i\}) \setminus Z)$. 

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Therefore the function $G$ where the third equality follows from the submodularity (Lemma 4.5 (i)) of $P_{N}$ on $f$.

Case 2: one of $Y$ and $Z$ does not contain $i$, say $i \not\in Z$. Since $G \neq \emptyset$, we have

$$f_{w,d}(E_X \cup F) + f_{w,d}(E_X \cup G) = f_{w,d}(E_X \cup F) + f_{w,d}(E_X) + d(N_{F \cup G})$$

By the monotonicity of $f_{w,d}$ and (11), we obtain

$$f_{w,d}(E_X \cup F) + f_{w,d}(E_X \cup G) = f_{w,d}(E_X \cup F) + f_{w,d}(E_X) + d(N_{F \cup G}) \geq f_{w,d}(E_X \cup (F \cap G)) + f_{w,d}(E_X) + d(N_{F \cup G})$$

Therefore the function $G \mapsto f_{w,d}(E_X \cup G)$ is submodular.

**Proof of Theorem 4.4.** By Theorem 4.3, we have $P_{w,d}(\xi) = P_{w,d}(0)$ if and only if $h_{\xi}^{i}(X) = h_{0}^{i}(X)$ for each $X \subseteq N \setminus \{i\}$. By the monotonicity of $f_{w,d}$, the latter condition is equivalent to

$$\min_{F \subseteq E_{i}} \{f_{w,d}(E_X \cup F) - \xi(F)\} = f_{w,d}(E_X) \quad (X \subseteq N \setminus \{i\}).$$

Thus $\xi \in \mathbb{R}^{E_{i}}$ belongs to $P_{w,d}$ if and only if for each $F \subseteq E_{i}$, it holds

$$\xi(F) \leq f_{w,d}(E_X \cup F) - f_{w,d}(E_X) \quad (X \subseteq N \setminus \{i\})$$

where the third equality follows from the submodularity (Lemma 4.5 (i)) of

$$X \mapsto f_{w,d}(E_X \setminus (E_{i} \setminus F)) = \begin{cases} f_{w,d}(E_X \setminus (E_{i} \setminus F)) & \text{if } i \in X, \\ f_{w,d}(E_X) & \text{otherwise}, \end{cases}$$

on $N$. By Lemma 4.3 (ii), $h_{w,d}^{i}$ is monotone submodular on $E_{i}$. This concludes that the clinching polytope $P_{w,d}^{i}$ is the polymatroid associated with $h_{w,d}^{i}$, as required.
By [14] in the proof, we obtain the following corollary.

**Corollary 4.6.** Suppose that buyer \( i \) clinches \( \xi \in \mathbb{R}_{+}^{E_i} \) of goods. For each \( F \subseteq E_i \) it holds

\[
\xi(F) \leq f_{w,d}(E_X \cup F) - f_{w,d}(E_X) \quad (X \subseteq N \setminus \{i\}).
\]

Also we can obtain the following.

**Corollary 4.7.** Buyer \( i \) gets \( \xi(E_i) = f_{w,d}(E) - f_{w,d}(E \setminus E_i) \) amount of goods.

### 4.3 Proof of Theorem 3.4

Here we give a proof of Theorem 3.4. It is known in [11] that \( \tilde{P} \) is a polymatroid, and the corresponding submodular function \( g \) is

\[
g(X) = f(E_X) \quad (X \subseteq N).
\]

Suppose that transaction vector \( w \in \mathbb{R}^E \) and demand vector \( d \in \mathbb{R}_+^N \) are given. Then \( \tilde{P}_{w,d} \) is a polymatroid, and the corresponding submodular function \( g_{w,d} \) is given by

\[
g_{w,d}(X) = \min \{ \min_{Z \subseteq X} \min_{Z' \supseteq Z} \{ g(Z') - w(E_{Z'}) \} + d(X \setminus Z) \}.
\]

The clinch \( \zeta_i \) in Algorithm [2] is given as follows:

**Theorem 4.8** (Goel et al. [11]). It holds \( \zeta_i = g_{w,d}(N) - g_{w,d}(N \setminus \{i\}) \) for each \( i \in N \).

The goal of this section is to prove following:

**Theorem 4.9.** It holds \( f_{w,d}(E_X) = g_{w,d}(X) \) for each \( X \subseteq N \).

**Proposition 4.10.** The value \( f_{w,d}(E) - f_{w,d}(E \setminus E_i) \) does not change after the clinch of buyer \( k \in N \setminus \{i\} \).

In particular, the total amount \( \xi(E_i) \) of the clinch of each buyer \( i \) is determined by \( w \) and \( d \) at the beginning of For Loop in line 4 (i.e., independent of the ordering of buyers). By Theorems 4.8 and 4.9, we obtain Theorem 3.4, and conclude that in Algorithm [1] each buyer obtains the same total amount of goods as in Algorithm [2].

To prove Theorem 4.9, we focus on the minimizer of \( \tilde{g}_{w,d}(X) \), where \( \tilde{g}_{w,d} : 2^N \rightarrow \mathbb{R}_+ \) is defined by

\[
\tilde{g}_{w,d}(X) := f_{w,d}(E_X) = \min_{Z \subseteq X} \{ f_w(E_Z) + d(X \setminus Z) \} \quad (X \subseteq N).
\]

Then the following lemma holds.

**Lemma 4.11.** For \( Z \subseteq N \), the following conditions are equivalent.

1. \( Z \) is a (unique) minimal minimizer of \( \tilde{g}_{w,d}(Z) \).
2. \( Z \) is a minimal minimizer of \( \tilde{g}_{w,d}(X) \) for some \( X \) with \( Z \subseteq X \subseteq N \).

**Proof.** It suffices to show that (ii) implies (i). Suppose that \( Z \) is a minimal minimizer of \( \tilde{g}_{w,d}(X) \), but \( Z \) is not a minimal minimizer of \( \tilde{g}_{w,d}(Z) \). Then there exists a set \( Z' \subseteq Z \) such that

\[
f_w(E_{Z'}) + d(Z \setminus Z') \leq f_w(E_Z).
\]

Then

\[
f_w(E_Z) + d(X \setminus Z) \geq f_w(E_{Z'}) + d(Z \setminus Z') + d(X \setminus Z) = f_w(E_{Z'}) + d(X \setminus Z'),
\]

which contradicts the minimality of \( Z \). \( \square \)
Let $\mathcal{U}$ be the set of subsets $Z \subseteq N$ satisfying the conditions in Lemma 4.11. Then we obtain the following proposition.

**Proposition 4.12.** For $X \in \mathcal{U}$, it holds

$$\tilde{g}_{w,d}(X) = f(E_X) - w(E_X).$$

**Proof.** Since $X \in \mathcal{U}$, we have

$$\tilde{g}_{w,d}(X) = f_w(E_X) := \min_{F \supseteq E_X} \{ f(F) - w(F) \}.$$ 

At the beginning of the auction, it holds $\mathcal{U} = 2^N$, and (15) holds because of $w_{ij} = 0$ ($ij \in E$) and the monotonicity of $f_j$ ($j \in M$). We are going to show that subsets in $\mathcal{U}$ vanish only when $d_i$ is recalculated in line 13. First we show that $\mathcal{U}$ does not change, and (15) holds throughout the For loop (line 4-line 9). In an iteration, for $X \in \mathcal{U}$, suppose that (15) holds in the line 4. Let $d$ be the current demand vector, and $w$ the current transaction vector. Then for each proper subset $Z \subset X$, it holds

$$f_w(E_X) < f_w(E_Z) + d(X \setminus Z).$$

Suppose that buyer $i \in X$ clinches $\xi \in \mathbb{R}_{+}^{E_i}$ of goods. Recall that $\tilde{g}_{w,d}$ is represented by

$$\tilde{g}_{w,d}(X) := f_{w,d}(E_X) = \min_{Z \subseteq X} \min_{F \supseteq E_Z} \{ f(F) - w(F) \} + d(X \setminus Z) \quad (X \subseteq N).$$

For each $F \supseteq E_X$, $f(F) - w(F)$ decreases by $\xi(E_i)$ after $w$ is recalculated in line 6. Then the equality

$$f_w(E_X) = f(E_X) - w(E_X).$$

is keeping after line 6. Also the demand $d_i$ decreases by $\xi(E_i)$ after $d_i$ is recalculated in line 8. Note that nontrivial clinch $\xi \neq 0$ occurs only if $d_i > 0$ and $c_i < v'_i$. Thus, $\min_{F \supseteq E_Z} \{ f(F) - w(F) \} + d(X \setminus Z)$ decreases by $\xi(E_i)$ for each $Z \subseteq X$ (regardless of whether $Z$ contains $i$ or not). Therefore, $X$ is still a minimal minimizer of $\tilde{g}_{w,d}(X)$; $X \in \mathcal{U}$. By this fact together with (17), we obtain (15), as required.

Suppose that $i \in N \setminus X$ clinches $\xi \in \mathbb{R}_{+}^{E_i}$ of goods. By Corollary 4.6 it holds that

$$\xi(F) \leq h_{w,d}(F) \leq f_{w,d}(E_Z \cup F) - f_{w,d}(E_Z) \leq f_w(E_Z \cup F) - f_w(E_Z)$$

for each $F \subseteq E_i, Z \subseteq N$. In this case, demand $d_k$ for each $k \in X$ is unchanged in line 8. After the recalculation of $w$ and $d$, $\tilde{g}_{w,d}(X)$ is written as

$$\min_{Z \subseteq X} \min_{F \supseteq E_Z} \{ f(F) - w(F) - \xi(E_i \cap F) \} + d(X \setminus Z).$$

Since it holds

$$\min_{G \subseteq E_i} \{ f_w(E_Z \cup G) - \xi(G) \} = \min_{G \subseteq E_i} \min_{G' \supseteq E_Z \cup G} \{ f(G') - w(G') \} - \xi(G)$$

$$= \min_{F \supseteq E_Z} \min_{F' \subseteq E_i} \{ f(F) - w(F) - \xi((E_i \cap F) \setminus F') \} = \min_{F \supseteq E_Z} \{ f(F) - w(F) - \xi(E_i \cap F) \},$$

we obtain

$$\min_{F \supseteq E_Z} \{ f(F) - w(F) - \xi(E_i \cap F) \} = \min_{G \subseteq E_i} \{ f_w(E_Z \cup G) - \xi(G) \} = f_w(E_Z),$$

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where the second equality is obtained by (18). Thus, (19) is equal to
\[ \min_{Z \subseteq X} \{ f_w(E_Z) + d(X \setminus Z) \}. \]

By (16), \( X \) is still a minimal minimizer of \( \tilde{g}_{w,d}(X) \); \( X \in \mathcal{U} \). This indicates that \( \tilde{g}_{w,d}(X) \) does not change, and (15) still holds after the clinch of buyer \( i \in N \setminus X \). Therefore, the transactions of buyers do not change the set \( \mathcal{U} \), and it still holds (15) in each iteration.

Finally we show that no sets are added to \( \mathcal{U} \) by the calculation of \( d_l \) in line 13. It is sufficient for us to consider the case of \( l \in X \). Here for each \( X \notin \mathcal{U} \) which contains \( l \), there exists a set \( Z \subseteq X \) such that
\[ f_w(E_Z) + d(X \setminus Z) \leq f_w(E_X). \]

Since \( f_w \) does not depend on \( d \), and \( d \) is non-increasing, this inequality still holds after the recalculation of \( d_l \). Therefore it still holds that \( X \notin \mathcal{U} \). \( \square \)

**Proof of Proposition 4.10.** Suppose that \( k \) clinches \( \xi_k \in \mathbb{R}_+ \) amount of goods in an iteration. We prove that both \( f_{w,d}(E) \) and \( f_{w,d}(E \setminus E_i) \) decrease by \( \xi_k(E_k) \). Suppose that \( Z \) is a minimal minimizer of \( f_{w,d}(E) \) in line 4. Then it holds
\[ f_w(E_Z) + d(N \setminus Z) \leq f_w(E_Y) + d(N \setminus Y) \quad (Y \subseteq N). \] (20)

By Proposition 4.11, we obtain \( Z \in \mathcal{U} \). Thus by Proposition 4.12 it holds
\[ f_{w,d}(E) = f_w(E_Z) + d(N \setminus Z) = f(E_Z) - w(E_Z) + d(N \setminus Z). \]

After the recalculation in line 9, \( f(E_Z) - w(E_Z) + d(N \setminus Z) \) decreases by \( \xi_k(E_k) \) regardless of whether \( Z \) contains \( k \) or not (since \( k \in N \)).

Let \( Z' \) be a minimal minimizer of \( f_{w,d}(E) \) in line 9. As seen in the proof of Proposition 4.12, the family \( \mathcal{U} \) is monotone non-increasing throughout Algorithm 1 and hence it holds that \( Z' \in \mathcal{U} \) in line 4. Therefore the value \( f_w(E_{Z'}) + d(N \setminus Z') = f(E_{Z'}) - w(E_{Z'}) + d(N \setminus Z') \) also decreases by \( \xi_k(E_k) \) after the recalculation. Thus we obtain \( f_w(E_Z) + d(N \setminus Z) = f_w(E_{Z'}) + d(N \setminus Z') \) in the For Loop. We can conclude that \( f_{w,d}(E_N) \) decreases by \( \xi_k(E_k) \) through the clinch of \( \xi_k(E_k) \) amount of goods by buyer \( k \). The same argument is true for \( f_{w,d}(E \setminus E_i) \) (since \( k \in N \setminus \{i\} \)). \( \square \)

**Proof of Theorem 4.9.** By Theorem 4.2
\[ f_{w,d}(E_X) = \tilde{g}_{w,d}(X) = \min_{Z \subseteq X} \{ f_w(E_Z) + d(X \setminus Z) \} \]
\[ = \min_{Z \subseteq X} \{ \min_{F \supseteq E_Z} \{ f(F) - w(F) \} + d(X \setminus Z) \} \]
\[ \leq \min_{Z \subseteq X} \{ \min_{Z' \supseteq Z} \{ f(E_{Z'}) - w(E_{Z'}) \} + d(X \setminus Z) \} \]
\[ = \min_{Z \subseteq X} \{ \min_{Z' \supseteq Z} \{ g(Z') - w(E_{Z'}) \} + d(X \setminus Z) \} = \tilde{g}_{w,d}(X). \]

Let \( Z \subseteq X \) be a minimal minimizer of \( \tilde{g}_{w,d}(X) \). Then by Lemma 4.11, it holds \( Z \in \mathcal{U} \). Thus by Proposition 4.12
\[ \tilde{g}_{w,d}(Z) = f_w(E_Z) = \min_{E \supseteq E_Z} \{ f(F) - w(F) \} = f(E_Z) - w(E_Z). \]

Since it holds
\[ f_w(E_Z) = \min_{E \supseteq E_Z} \{ f(F) - w(F) \} \leq \min_{Z' \supseteq Z} \{ g(Z') - w(E_{Z'}) \} \leq f(E_Z) - w(E_Z), \]

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we obtain equalities
\[ f_w(E_Z) = \min_{F \subseteq E_Z} \{ f(F) - w(F) \} = \min_{Z' \supseteq Z} \{ g(Z') - w(E_{Z'}) \} = f(E_Z) - w(E_Z). \]

It holds
\[ f_{w,d}(E_X) = g_{w,d}(X) = f_w(E_Z) + d(X \setminus Z) = \min_{Z' \supseteq Z} \{ g(Z') - w(E_{Z'}) \} + d(X \setminus Z) \geq g_{w,d}(X). \]

Therefore it holds \( f_{w,d}(E_X) = g_{w,d}(X) \). \( \square \)

4.4 Proof of Proposition [3.3]

Lemma 4.13. For each seller \( j \), if \( c_{n+j} < \rho_j \), then \( w_{ij} = 0 \) \((i \in N_j \setminus \{n+j\})\).

Proof. We consider the transaction of seller \( j \) with buyer \( i \in N_j \setminus \{n+j\} \) when \( c_{n+j} < \rho_j \) and \( w_{ij} = 0 \) \((i \in N_j \setminus \{n+j\})\). By Corollary 4.6, the transaction \( \xi \in \mathbb{R}^{E_i}_+ \) of buyer \( i \in N_j \setminus \{n+j\} \) through edge subset \( F \subseteq E_i \) satisfies
\[ \xi(F) \leq f_{w,d}(E_X \cup F) - f_{w,d}(E_X) \quad (X \subseteq N \setminus \{i\}). \]

Letting \( X = \{n+j\} \) and \( F = \{ij\} \), we obtain
\[ \xi_{ij} \leq f_{w,d}\left(\{(n+j)j\} \cup \{ij\}\right) - f_{w,d}\left(\{(n+j)j\}\right). \]

By the formula (11) of \( f_{w,d} \), modification (4) of \( f_j \), and \( d_{n+j} = \infty \),
\[ f_{w,d}\left(\{(n+j)j\} \cup \{ij\}\right) = \min\{f_w(\{(n+j)j\} \cup \{ij\}), f_w(\{(n+j)j\}) + d_i\} = \min\{f_j(E_j) - w_{(n+j)j}, f_j(E_j) - w_{(n+j)j} + d_i\} = f_j(E_j) - w_{(n+j)j}, \]
\[ f_{w,d}\left(\{(n+j)j\}\right) = \min\{f_w(\{(n+j)j\})\} = f_j(E_j) - w_{(n+j)j}, \]
and thus we obtain \( \xi_{ij} = 0 \). \( \square \)

Using this lemma, we prove Proposition [3.3].

Proof of Proposition [3.3]. Lemma 4.13 indicates that seller \( j \) does not transact with buyers other than \( n+j \) until \( c_{n+j} = \rho_j \). Therefore, after transforming the allocation of Algorithm [1] into the allocation without virtual buyers, it holds
\[ r_j = \sum_{ij \in E_j} c_i w_{ij} \geq c_{n+j} w(E_j) \geq \rho_j w(E_j) \quad (j \in M), \]
where \( c_i \geq c_{n+j} \) always holds for non-virtual buyer \( i \) in Algorithm [1]. Thus we obtain
\[ u_j(M(I)) = r_j + \rho_j(f_j(E_j) - w(E_j)) \geq \rho_j f_j(E_j). \]
\[ \square \]
4.5 Proof of Theorem 3.6

We denote the set of non-virtual buyers by \( N^* \) and the set of virtual buyers by \( V \). Namely, \( N = N^* \cup V \). By (ICb) (Corollary 3.5), we may assume that buyers report their true valuations; \( w_i' = w_i \) (\( i \in N^* \)). Let \( \mathcal{A} = (w, p, r) \) be the allocation obtained by Algorithm 1 (including the allocation of virtual buyers). By Theorem 3.4, we can utilize the following proposition obtained in the proof of Lemma 3.8 by Goel et al. [11]. A subset \( X \subseteq N \) is said to be 
\( \text{tight} \) with respect to \( w \) if \( w(X) = f(E_X)(= g(X)) \).

**Proposition 4.14** (Goel et al. [11]). There exist a chain \( \emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_t = N \) of 
\( \text{tight} \) sets and \( i_k \in X_k \setminus X_{k-1} \) for \( k = 1, 2, \ldots, t \) such that we have \( v_i \geq v_{i_k} \) for each \( k = 1, 2, \ldots, t \) and \( i \in X_k \).

**Proof of Theorem 3.6.** Suppose to the contrary that there exists an allocation (without virtual buyers) \( \mathcal{A}' = (w', p', r') \) with \( \sum_{i \in N^*} p_i' \geq \sum_{j \in M} r_j' \) such that \( \mathcal{A}' \) satisfies

\[
\begin{align*}
  v_i w(E_i) - p_i & \leq v_i w'(E_i) - p_i' \quad (i \in N^*), \\
  (r_j - p_{n+j}) + \rho_j w(n+j) & \leq r_j' + \rho_j(f_j(E_j) - w'(E_j)) \quad (j \in M),
\end{align*}
\]

and at least one of the inequalities holds strictly. Here \( \mathcal{A}' \) is extended to an allocation with virtual buyers by:

\[
\begin{align*}
  w'(n+j) & := f_j(E_j) - w'(E_j) \quad (j \in M), \\
  p_{n+j} & := 0 \quad (j \in M).
\end{align*}
\]

In the sequel, edges to virtual buyers are included in \( E \) and \( E_j \). Then we have

\[
w'(E_N) = \sum_{j \in M} f_j(E_j) = f(E_N). \tag{23}
\]

Then (21) and (22) are rewritten as

\[
\begin{align*}
  v_i(w(E_i) - w'(E_i)) & \leq p_i - p_i' \quad (i \in N^*), \\
  (r_j - r_j') - (p_{n+j} - p_{n+j}') & \leq \rho_j(w(n+j) - w'(n+j)) \quad (j \in M).
\end{align*}
\]

Consider a chain \( \emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_t = N \), and \( i_k \in X_k \setminus X_{k-1} \) in Proposition 4.14. By (23), it still holds that

\[
\begin{align*}
  f(E_{X_k}) & = w(E_{X_k}) \quad (k \in \{0, 1, 2, \ldots, t\}), \\
  f(E_{X_t}) & = f(E_{X_N}) = w'(E_{X_N}) = w'(E_{X_t}).
\end{align*}
\]

Also for each \( k \in \{1, 2, \ldots, t\} \), we define the following sets:

\[
\begin{align*}
  X_k^* & := X_k \cap N^*, \\
  Z_k & := X_k \setminus X_{k-1}, \\
  Z_k^* & := Z_k \cap N^*.
\end{align*}
\]

Adding (24) and (25), we prove, by induction, that

\[
\sum_{i \in X_k^*} (p_i - p_i') \geq \sum_{j : n+j \in X_k \cap V} ((r_j - r_j') - (p_{n+j} - p_{n+j}')) + v_{i_k}(w(E_{X_k}) - w'(E_{X_k})) \quad (k \in \{1, 2, \ldots, t\}). \tag{26}
\]
For each \( k \in \{1, 2, \ldots, t\} \)

\[
\sum_{i \in Z_k^*} (p_i - p_i') \geq \sum_{i \in Z_k^*} v_{ik}(w(E_i) - w'(E_i)) = v_{ik}(w(E_{Z_k^*}) - w'(E_{Z_k^*})),
\]

(27)

where the inequality follows from (24) and \( v_i \geq v_{ik} \ (i \in X_k) \). In the case of \( k = 1 \),

\[
\sum_{i \in X_1^*} (p_i - p_i') = \sum_{i \in Z_1^*} (p_i - p_i') \geq \sum_{i \in Z_1^*} v_{i_1}(w(E_i) - w'(E_i))
\]

\[
= \sum_{i \in X_1^*} v_{i_1}(w(E_i) - w'(E_i)) + v_{i_1}(w'(E_{X_1}) - w(E_{X_1}) - w'(E_{X_1}))
\]

\[
= v_{i_1}(w'(E_{X_1}) - w'(E_{X_1})) - w'(E_{X_1}) + v_{i_1}(w(E_{X_1}) - w'(E_{X_1}))
\]

\[
\geq \sum_{j:n+j \in X_{1}\cap V} ((r_j - r'_j) - (p_{n+j} - p'_{n+j})) + v_{i_1}(w(E_{X_1}) - w'(E_{X_1}))
\]

where the second inequality follows from (25). Thus (26) holds in \( k = 1 \). Suppose that (26) holds in \( k = s \). We have

\[
\sum_{i \in X_{s+1}^*} (p_i - p_i') = \sum_{i \in X_{s+1}^*} (p_i - p_i') + \sum_{i \in Z_{s+1}^*} (p_i - p_i')
\]

\[
+ v_{i_{s+1}}(w(E_{X_{s+1}}) - w'(E_{X_{s+1}})) + v_{i_{s+1}}(w'(E_{X_{s+1}}) - w(E_{X_{s+1}}))
\]

\[
\geq \sum_{j:n+j \in X_{s}\cap V} ((r_j - r'_j) - (p_{n+j} - p'_{n+j})) + v_{i_s}(w(E_{X_{s}}) - w'(E_{X_{s}}))
\]

\[
+ v_{i_{s+1}}(w'(E_{Z_{s+1}}) - w'(E_{Z_{s+1}})) + v_{i_{s+1}}(w(E_{X_{s+1}}) - w'(E_{X_{s+1}}))
\]

where the inequality follows from (26) for \( k = s \) and (27). Here we obtain

\[
v_{i_s}(w(E_{X_{s}}) - w'(E_{X_{s}})) + v_{i_{s+1}}(w(E_{Z_{s+1}}) - w'(E_{Z_{s+1}})) + v_{i_{s+1}}(w'(E_{X_{s+1}}) - w(E_{X_{s+1}}))
\]

\[
\geq v_{i_{s+1}}(w(E_{X_{s}}) - w'(E_{X_{s}})) + v_{i_{s+1}}(w'(E_{Z_{s+1}}) - w'(E_{Z_{s+1}})) + v_{i_{s+1}}(w'(E_{X_{s+1}}) - w'(E_{X_{s+1}}))
\]

\[
= v_{i_{s+1}}(w'(E_{Z_{s+1}\cap V}) - w(E_{Z_{s+1}\cap V}))
\]

where the inequality follows from \( v_i \geq v_{i_s} \). Therefore,

\[
\sum_{i \in X_{s+1}^*} (p_i - p_i') \geq \sum_{j:n+j \in X_{s}\cap V} ((r_j - r'_j) - (p_{n+j} - p'_{n+j})) + v_{i_{s+1}}(w(E_{X_{s+1}}) - w'(E_{X_{s+1}}))
\]

\[
+ v_{i_{s+1}}(w'(E_{Z_{s+1}\cap V}) - w(E_{Z_{s+1}\cap V}))
\]

\[
\geq \sum_{j:n+j \in X_{s}\cap V} ((r_j - r'_j) - (p_{n+j} - p'_{n+j}))
\]

\[
+ \sum_{j:n+j \in Z_{s+1}\cap V} ((r_j - r'_j) - (p_{n+j} - p'_{n+j})) + v_{i_{s+1}}(w(E_{X_{s+1}}) - w'(E_{X_{s+1}}))
\]

\[
= \sum_{j:n+j \in X_{s+1}\cap V} ((r_j - r'_j) - (p_{n+j} - p'_{n+j})) + v_{i_{s+1}}(w(E_{X_{s+1}}) - w'(E_{X_{s+1}}))
\]

where the second inequality follows from (25). Therefore, we obtain (26).
Since \( w(E_X) = w'(E_X) = f(E_X) \) for \( k = t \), we have

\[
\sum_{i \in X^*_t} (p_i - p'_i) \geq \sum_{j: n+j \in X_t \cap V} ((r_j - r'_j) - (p_{n+j} - p'_{n+j})).
\]

By (SBB), it holds

\[
\sum_{i \in X^*_t} p_i = \sum_{i \in N^*} p_i = \sum_{j \in M} (r_j - p_{n+j}) = \sum_{j: n+j \in X_t \cap V} (r_j - p_{n+j}).
\]

Substituting this to (28), we obtain

\[
\sum_{i \in N^*} p'_i = \sum_{i \in X^*_t} p'_i \leq \sum_{j: n+j \in X_t \cap V} (r'_j - p'_{n+j}) = \sum_{j \in M} (r'_j - p'_{n+j}).
\]

Since \( p'_{n+j} = 0 \) \( (j \in M) \),

\[
\sum_{i \in N^*} p'_i \leq \sum_{j \in M} r'_j. \tag{29}
\]

Since \( \sum_{i \in N^*} p'_i \geq \sum_{j \in M} r'_j \), the equality holds in (29). Here (29) was deduced by adding all inequalities in (24) and (25). Thus all inequalities in (24) and (25) hold in equality, which contradicts the assumption of \( \mathcal{A}' \) that at least one of inequalities is strict. \( \square \)

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