Development of optimal design theory for series multiple tuned mass dampers considering stroke and multiple structural modes

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Abstract. A tuned mass damper (TMD) system generates structural control forces through large motions of mass units. Therefore, it may not be functional if the stroke capacity of its spring and damper components are insufficient. This paper focuses on a novel mass-damper system, the series multiple tuned mass damper (SMTMD) system, which consists of multiple interconnected TMDs, of which only the first is connected to the primary structure. The main purpose of this paper is to compare the control effectiveness and TMD stroke of SMTMDs with those of a conventional TMD device. In addition, the ability of the studied SMTMD to suppress multiple structural modes is also investigated. First, the optimal design theory for an SMTMD installed on an arbitrary floor of a multi-storey building is developed. To optimize the SMTMD parameters, two performance indices are established by combining multiple modal responses of the primary structure. The developed theory is demonstrated analytically by using a three-story building. The results show that the SMTMD with a higher number of TMD units places lower demands on the TMD stroke and is more adaptive in controlling multiple structural modes of the primary structure.

1. Introduction
A tuned mass damper (TMD) system is one type of passive vibration control system. The idea of TMD can date back to 1909 and the first application for civil engineering structures was implemented in the 1970s [1]. It has being a popular device for skyscrapers to mitigate excess vibration induced by wind and earthquake. A TMD is a single-degree-of-freedom (SDOF) system, generating vibration attenuation force through the movement of its mass with respect to the primary structure. Wang et al. [2] compared the stroke (displacement relative to the installed floor) of TMD mass and the roof displacement of a three-story building excited by a white-noise base excitation and found that the ratio of root-mean-square values of TMD stroke to the roof displacement is 5.4 when the mass of TMD is 1% of the mass of the primary building. This means that the spring and damping components of a TMD need to provide enough stroke capacity to satisfy such a high stroke demand. To reduce the demand of TMD stroke, Wang et al. [2] developed a two-stage parameter optimization theory that considers the reduction of the TMD stroke. They found that when stroke reduction is required, the optimal TMD has a similar frequency ratio but a higher damping ratio compared with the conventionally optimal TMD; however, this sacrifices the effectiveness in reducing the dynamic
response of the primary building. In addition, a TMD is an SDOF system that can control only a single-mode structural response. Therefore, a multiple-TMD (MTMD) system for controlling multiple structural modes has also been researched [3]. Nonetheless, the stroke remains a concern for such multiple parallel TMD systems.

In this study, a series multiple tuned mass damper (SMTMD) system for building structures was investigated, emphasizing its stroke and multi-mode control performance. Past research has mainly focused on either two-degree-of-freedom (2DOF) SMTMDs or SDOF primary structures [4–6]. This study considered a multi-DOF SMTMD system, installed on an arbitrary floor of a multistory building, to develop an optimal design theory. To optimize the SMTMD parameters, two performance indices were established by combining multiple modal responses of the primary structure. The developed theory was demonstrated numerically by using a three-story building. The results are included in this paper.

2. Development of optimal design theory

This section presents the theory for the design of an SMTMD applied for building response control. The theory begins with the establishment of equations of motion, followed by the transformation of coordinates of the primary structure from physical to modal domains. Next, the performance indices are defined by combining the modal responses of the structure. These indices are the objective functions to be minimized to determine the optimal parameters of the SMTMD.

2.1. Dynamic equations of the system

Consider a coupled system consisting of a \( q \)-DOF SMTMD subsystem and an \( n \)-DOF building subsystem. The SMTMD subsystem is installed on the \( l \)th-floor of the building, as shown in Figure 1. Conventionally, it is more convenient to represent these two subsystems according to the displacement with respect to their own bases. Express the building system as the mass matrix \( M_p \), damping matrix \( C_p \), and stiffness matrix \( K_p \), and the SMTMD system as \( M_s \), \( C_s \), and \( K_s \). The coupled equations...
of motion for base excitation of the \( N \)-DOF \( (N = q + n) \) composite SMTMD-primary systems are written as follows:

\[
\mathbf{M} \ddot{\mathbf{x}}(t) + \mathbf{C} \dot{\mathbf{x}}(t) + \mathbf{K} \mathbf{x}(t) = \mathbf{r} \ddot{\mathbf{g}}(t),
\]

where

\[
\mathbf{M} = \begin{bmatrix} \mathbf{M}_p + \mathbf{Q}^T \mathbf{M}_s \mathbf{Q} & \mathbf{Q}^T \mathbf{M}_s \\ \mathbf{M}_s \mathbf{Q} & \mathbf{M}_s \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_s \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_s \end{bmatrix},
\]

\[
\mathbf{x} = \begin{bmatrix} \mathbf{x}_p \\ \mathbf{x}_s \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}_p \\ \mathbf{r}_s \end{bmatrix},
\]

\[
\mathbf{M}_p = \mathrm{diag} \{ m_{p_1}, m_{p_2}, \ldots, m_{p_q} \}, \quad \mathbf{M}_s = \mathrm{diag} \{ m_{s_1}, m_{s_2}, \ldots, m_{s_q} \},
\]

\[
\mathbf{C}_p = \begin{bmatrix} 0 & -c_{p_2} + c_{p_1} & \cdots & 0 \\ -c_{p_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & -c_{p_2} + c_{p_1} & \cdots & 0 \\ -c_{p_2} + c_{p_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & -c_{p_2} + c_{p_1} & \cdots & 0 \end{bmatrix}, \quad \mathbf{K}_p = \begin{bmatrix} k_{p_1} + k_{p_2} & 0 & \cdots & 0 \\ -k_{p_2} & k_{p_2} + k_{p_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -k_{p_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ -k_{p_2} & k_{p_2} + k_{p_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & -k_{p_2} & \cdots & 0 \end{bmatrix},
\]

\[
\mathbf{C}_s = \begin{bmatrix} 0 & -c_{s_2} + c_{s_1} & \cdots & 0 \\ -c_{s_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & -c_{s_2} + c_{s_1} & \cdots & 0 \\ -c_{s_2} + c_{s_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & -c_{s_2} + c_{s_1} & \cdots & 0 \end{bmatrix}, \quad \mathbf{K}_s = \begin{bmatrix} k_{s_1} + k_{s_2} & 0 & \cdots & 0 \\ -k_{s_2} & k_{s_2} + k_{s_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -k_{s_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ -k_{s_2} & k_{s_2} + k_{s_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & -k_{s_2} & \cdots & 0 \end{bmatrix},
\]

In Eq. (2), \( \mathbf{x}_p \) is the displacement vector, with respect to the ground, of the primary building; \( \mathbf{x}_s \) is the displacement vector, with respect to the \( l \)th floor, of the SMTMD; \( \mathbf{r} \) is the excitation influence vector; \( \mathbf{Q} \) is the \( q \times n \) pseudostatic influence matrix, expressed as follows:

\[
\mathbf{Q} = \begin{bmatrix} 0 & \cdots & 1^{(l)} & \cdots & 0 \\ 0 & \cdots & 1^{(l)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1^{(l)} & \cdots & 0 \end{bmatrix}_{q \times n},
\]

where the superscript \( (l) \) means the entry 1 is located at the \( l \)th column of \( \mathbf{Q} \). Eq. (2) shows that the equations of motion of the two subsystems are coupled only in the mass matrix. Considering that the relative displacements (strokes) between the mass units of the SMTMD, defined as \( \mathbf{v}_s \), are more crucial than the relative displacements, we propose a preliminary coordinate transformation:

\[
\mathbf{x}_s = \mathbf{L} \mathbf{v}_s,
\]

where \( \mathbf{L} \) is the \( q \times q \) unit lower triangular matrix, expressed as follows:

\[
\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{q \times q}.\]
Substituting Eq. (4) into Eq. (1) enables deriving a new equation of motion:

\[ M_r \ddot{x}_r(t) + C_r \dot{x}_r(t) + K_r x_r(t) = r_x g(t), \tag{6} \]

where

\[
M_r = \begin{bmatrix} M_p + Q^T M_s Q & 0 \\ Q^T M_r L & M_s L \end{bmatrix}, \quad C_r = \begin{bmatrix} C_p & 0 \\ 0 & C_s L \end{bmatrix}, \quad K_r = \begin{bmatrix} K_p & 0 \\ 0 & K_s L \end{bmatrix}, \quad x_r = \begin{bmatrix} x_p \\ v_s \end{bmatrix}. \tag{7}
\]

2.2. Component-mode synthesis

Component-mode synthesis \cite{7} reduces the order of the full nodal dynamic equations by using limited modal equations of motions of subsystems. Here, a transformation of coordinates, defined using component-mode synthesis, is adopted:

\[ x_r = \Phi_r \eta \quad \text{where} \quad \Phi_r = \begin{bmatrix} \Phi_p & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} \eta_p \\ v_s \end{bmatrix}. \tag{8} \]

In Eq. (8), \( I \) is a \( q \times q \) unity matrix, \( \eta_p \) is the modal displacement vector of the primary building of order \( n_c \), and \( \Phi_p \) is the mode shape matrix of the primary building, which is normalized so that

\[
\Phi_p^T M_p \Phi_p = \begin{bmatrix} \Phi_p^T M_p \Phi_p \\ \Phi_p^T \Phi_p \end{bmatrix} = \Phi_p^T M_{p,\text{eff}}, \tag{9}
\]

where \( \Phi_p \) indicates the \( j \)th mode-shape vector that is extracted from the \( j \)th column of the matrix \( \Phi_p \), and \( M_{p,\text{eff}} \) is the \( j \)th modal effective mass of the primary building. The criterion for choosing the order \( n_c \) of the mode shape \( \Phi_p \) for the coordinate transformation can be based on the quantity of \( M_{p,\text{eff}} \), for example, \( n_c \) satisfying \( \sum_{j=1}^{n_c} \sum_{p=1}^{n_p} m_{p,\text{eff}}^j > 95\% \sum_{p=1}^{n_p} m_{p,\text{eff}} \). Therefore, the dimensions of \( \Phi_p \) and \( \Phi_s \) are \( n \times n_c \) and \( (n+q) \times (n_c+q) \), respectively, where \( n_c \leq n \) is the considered number of building modes.

Using Eq. (8) and pre-multiplying with \( \Phi_p^T \) transforms Eq. (6) into a set of differential equations of order \( n_c + q \), lower than the order of the original systems of equations, which can be written as

\[ M_r^* \ddot{\eta}(t) + C_r^* \dot{\eta}(t) + K_r^* \eta(t) = \Gamma_r^* \ddot{g}(t), \tag{10} \]

where

\[
M_r^* = \begin{bmatrix} I_p + \Phi_p^T Q^T M_s Q \Phi_p & \Phi_p^T Q^T M_s L \\ Q \Phi_p^T M_p \Phi_p & Q \Phi_p^T \Phi_p \end{bmatrix}, \quad C_r^* = \begin{bmatrix} \Xi_p & 0 \\ 0 & C_s L / M_s \end{bmatrix}, \tag{11a, 11b}
\]

\[
K_r^* = \begin{bmatrix} \Omega_p^2 & 0 \\ 0 & K_s L / M_s \end{bmatrix}, \quad \Gamma_r^* = \begin{bmatrix} \Phi_p^T M_p \Phi_p & \Phi_p^T Q^T M_s \Phi_p \\ \Phi_p^T M_p \Phi_p & \Phi_p^T \Phi_p \end{bmatrix} = \begin{bmatrix} \Gamma_p^* \\ \Gamma_s^* \end{bmatrix}, \tag{11c, 11d}
\]

where \( \Xi_p = \text{diag} \{ 2 \xi_p, \omega_p \} \) and \( \Omega_p^2 = \text{diag} \{ \omega_p^2 \} \) are \( n_c \times n_c \) diagonal matrices with the modal parameters of the primary building.

2.3. Transfer functions and mean-square responses

To address the SMTMD design problem in the frequency domain, we took the Fourier transform \cite{8} of two sides of Eq. (10) and obtained
where \( \omega \) is the frequency of the excitation and \( i = \sqrt{-1} \) is the imaginary unit. Define \( H_{\eta,\bar{\eta}}(\omega) = \eta_p(\omega) / \bar{\eta}_p(\omega) \) and \( H_{v,x}(\omega) = v_s(\omega) / \bar{x}_g(\omega) \) as the transfer functions from the input \( \bar{x}_g(\omega) \) to the modal displacements of the building and the strokes of the SMTMD, respectively, and let

\[
(-\omega^2M^*_r + i\omega C_r + K^*_r)\eta(\omega) = \Gamma_r \bar{x}_g(\omega),
\]

(12)

where \( \omega \) is the frequency of the excitation and \( i = \sqrt{-1} \) is the imaginary unit. Define \( H_{\eta,\bar{\eta}}(\omega) = \eta_p(\omega)/\bar{\eta}_p(\omega) \) and \( H_{v,x}(\omega) = v_s(\omega)/\bar{x}_g(\omega) \) as the transfer functions from the input \( \bar{x}_g(\omega) \) to the modal displacements of the building and the strokes of the SMTMD, respectively, and let

\[
(-\omega^2M^*_r + i\omega C_r + K^*_r)^{-1} = \begin{bmatrix} H_{pp} & H_{ps} \\ H_{sp} & H_{ss} \end{bmatrix}.
\]

(13)

Then,

\[
\begin{bmatrix} H_{\eta_p,\bar{\eta}_p}(\omega) \\ H_{v_s,\bar{x}_g}(\omega) \end{bmatrix} = \begin{bmatrix} H_{pp} & H_{ps} \\ H_{sp} & H_{ss} \end{bmatrix} \begin{bmatrix} \Gamma^*_p \\ \Gamma^*_s \end{bmatrix} = \begin{bmatrix} H_{pp} \Gamma^*_p + H_{ps}r_s \\ H_{sp} \Gamma^*_s + H_{ss}r_s \end{bmatrix}.
\]

(14)

It is shown that \( \eta_p(\omega) = H_{\eta,\bar{\eta}}(\omega) \bar{\eta}_p(\omega) \) and \( v_s(\omega) = H_{v,x}(\omega) \bar{x}_g(\omega) \) are functions of \( \omega \). Because \( \bar{x}_g(\omega) \) is wideband, the mean-square (MS) values of the \( j \)th modal displacement \( \eta_j(\omega) \), \( j \)th modal acceleration \( \dot{\eta}_j(\omega) \), and stroke of the \( k \)th SMTMD unit \( \bar{v}_k(\omega) \) caused by the external excitation \( \bar{x}_g(\omega) \) are calculated to represent the modal responses of the primary building and the SMTMD using the following equations, which can be found in books of random vibration [8]

\[
E[\eta^2_j(\omega)] = \int_{-\infty}^{\infty} |H_{\eta_j,\bar{x}_g}(\omega)|^2 S_{\bar{x}_g}(\omega) d\omega
\]

(15)

\[
E[\dot{\eta}^2_j(\omega)] = \int_{-\infty}^{\infty} \omega^2 |H_{\dot{\eta}_j,\bar{x}_g}(\omega)|^2 S_{\bar{x}_g}(\omega) d\omega
\]

(16)

\[
E[\bar{v}^2_k(\omega)] = \int_{-\infty}^{\infty} |H_{\bar{v}_k,\bar{x}_g}(\omega)|^2 S_{\bar{x}_g}(\omega) d\omega.
\]

(17)

where \( S_{\bar{x}_g}(\omega) \) is the power spectral density of \( \bar{x}_g(\omega) \).

2.4. Performance index

Using Eqs. (15)–(17) established two performance indices, \( J_d \) and \( J_a \), to measure the control effectiveness of the SMTMD in structural displacement and acceleration as follows:

\[
J_d = \sum_{j=1}^{n} \beta_j R_{\eta_j}, \quad J_a = \sum_{j=1}^{n} \beta_j R_{\dot{\eta}_j},
\]

(18, 19)

where

\[
R_{\eta_j} = \frac{E[\eta^2_j(\omega)]_{\text{w/o SMTMD}}}{E[\eta^2_j(\omega)]_{\text{w/SMTMD}}}, \quad R_{\dot{\eta}_j} = \frac{E[\dot{\eta}^2_j(\omega)]_{\text{w/o SMTMD}}}{E[\dot{\eta}^2_j(\omega)]_{\text{w/SMTMD}}}.
\]

(20, 21)

Eqs. (20) and (21) represent the ratios of the MS values of \( j \)th modal displacement and acceleration, respectively, of the primary building with and without the SMTMD. \( J_d, J_a < 1 \) indicates the SMTMD is effective. The constant coefficient \( \beta_j \) indicates the weighting factor of the \( j \)th modal response ratio of the primary building among the considered modes, where \( \sum_{j=1}^{n} \beta_j = 1 \). \( \beta_1 = \beta_2 = \ldots = \beta_{n_e} = 1/n_e \) was used in this study.

2.5. Parameter optimization
The aforementioned performance indices are functions of the parameters of the primary building and the SMTMD. It is assumed that the parameters of the primary building are known, and the total mass of the SMTMD $m_{s_i}$ is a designated value.

2.5.1 Unconstrained optimization

With necessary constraints, $m_{s_i}, c_{s_i}, k_{s_i} > 0$ and $\sum_{k=1}^{q}m_{s_i} = m_{st}$, the optimal mass $(m_{s_i})_{opt}$, damping $(c_{s_i})_{opt}$, and stiffness $(k_{s_i})_{opt}$ of the SMTMD units $(k = 1, 2, \ldots, q)$ that minimize the performance index $J$ ($J_d$ or $J_u$) can be obtained by solving the following $3q$ equations:

\[
\begin{align*}
\frac{\partial J}{\partial m_{s_1}} &= 0, \quad \frac{\partial J}{\partial m_{s_2}} = 0, \ldots, \quad \frac{\partial J}{\partial m_{s_q}} = 0 \\
\frac{\partial J}{\partial c_{s_1}} &= 0, \quad \frac{\partial J}{\partial c_{s_2}} = 0, \ldots, \quad \frac{\partial J}{\partial c_{s_q}} = 0 \\
\frac{\partial J}{\partial k_{s_1}} &= 0, \quad \frac{\partial J}{\partial k_{s_2}} = 0, \ldots, \quad \frac{\partial J}{\partial k_{s_q}} = 0
\end{align*}
\] (22)

2.5.2 Conditional optimization

Calculating Eq. (22) is time consuming because numerous variables must be optimized. Alternatively, we reduce the problem scale by letting the masses of the SMTMD units being linearly distributed between $m_{s_1}$ and $m_{s_q}$. By doing this, the $q$ unknown mass variables can then be represented by the single unknown variable, the mass distribution factor, which is defined as $\delta m = (m_{s_q} - m_{s_1})/m_{s_1}$. Moreover, the uniform unit damping and stiffness coefficients, namely, $c_{s_1} = c_{s_2} = \ldots = c_{s_q}$ and $k_{s_1} = k_{s_2} = \ldots = k_{s_q}$, are more beneficial for the practice. Therefore, the number of variables to be optimized is reduced from $3q$ to three, namely, $(\delta m)_{opt}$, $(c_{s_i})_{opt}$, and $(k_{s_i})_{opt}$, which can be obtained by solving the following three equations:

\[
\frac{\partial J}{\partial (\delta m)} = 0, \quad \frac{\partial J}{\partial c_{s_1}} = 0, \quad \frac{\partial J}{\partial k_{s_1}} = 0.
\] (23)

The optimal mass of the $k$th SMTMD unit can then be obtained using

\[
(m_{s_k})_{opt} = m_{st} \left[ \frac{1}{q} + \left( k - 1 \right) \frac{1}{q - 1} \right] (\delta m)_{opt},
\] (24)

where $\delta m$ must satisfy $-2/q < \delta m < 2/q$ to ensure $m_{s_i} > 0$. Figure 2 depicts the SMTMD mass distribution in three $\delta m$ situations. Note that for the $q=1$ situation, the SMTMD becomes a single TMD; thus, Eq. (24) is not applied and using $\delta m$ is unnecessary. In this study, the optimization process was numerically performed using the MATLAB software toolbox, rather than by solving the aforementioned partial difference equations analytically.

Figure 2. Configurations of three types of SMTMD mass distribution.
### Table 1. Dynamic properties of the three-story building frame

| Modal parameters (Identified) | Physical parameters (Calculated) |
|------------------------------|----------------------------------|
| \(\omega_{p,j}\) (Hz)        | \(M_p\) (kg)                     |
| \{1.08 3.24 5.03\}           | \[
| \(\xi_{p,j}\) (%)            | \(C_p\) (kN·sec/m)               |
| \{2.19 0.20 0.16\}           | 0.652 0.233 0.295                |
| \(\Phi_p\)                   | \(K_p\) (kN/m)                   |
| \[
| \{0.503 −0.295 0.199\}       | 3487 −1918 126                   |
| \{0.924 −0.128 −0.214\}      | −1918 3815 −1801                 |
| \{1.255 0.255 0.114\}        | 126 −1801 1525                   |

### Table 2. MS responses of the three-story building under the white noise of unit intensity

| MS modal displacement (m²) | MS modal acceleration (m/ sec²)² |
|---------------------------|----------------------------------|
| \(E[\eta^2_{p_1}]\)       | \(E[\eta^2_{p_2}]\)              |
| 0.1188                    | 0.0804                           |
| \(E[\eta^2_{p_3}]\)       | 0.0040                           |
| 288                       | 13831                            |
| \(E[\dot{\eta}^2_{p_1}]\) | \(E[\dot{\eta}^2_{p_2}]\)       |
| \(E[\dot{\eta}^2_{p_3}]\) | 3973                             |

### 3. Numerical Study

This section presents a study to demonstrate the proposed SMTMD design theory.

#### 3.1. Details of the primary building and SMTMD

The dynamic properties of a full-scale three-story experimental building frame weighing 18 t and measuring 9 m high were used in this study as presented in Table 1, which were identified based on shaking-table test data. The frequency and damping ratio of the first mode are 1.08 Hz and 2.2%, respectively, whereas the damping ratios of the second and third modes are quite small. The MS modal displacements and modal accelerations of the structure under the white noise of unit intensity \(S_{\eta_0}(\omega) = 1\) calculated using Eqs. (15) and (16) are shown in Table 2. The second modal displacement is comparable with that of the first mode, whereas the second and third modal accelerations are considerably greater than that of the first mode. It was expected that a multiple-mode control is required for this structure. The mass of the SMTMD was set to 2% of the primary building (18 tons), or 360 kg. In this study, \(J_d\) was used as the performance index with \(\beta_j = 1/n_c\) where \(j = 1 \sim n_c\). Situations involving various numbers of SMTMD units were discussed (\(q\) ranges from 1 to 5). A single mode or multiple modes of the primary building were considered (\(n_c = 1, 2, \text{or } 3\)).

#### 3.2. Optimal parameters of the SMTMD

The optimal mass distribution factor \((\delta m)_{\text{opt}}\), damping coefficient \((c_{\text{opt}})\), and stiffness coefficient \((k_{\text{opt}})\) under different combinations of \(n_c\) and \(q\) are listed in Table 3. The modal properties of the SMTMD system are presented in Table 4. When only the first mode of the primary building is considered (\(n_c = 1\)), \((\delta m)_{\text{opt}}\) is always close to the upper or lower bound of \(\delta m\) regardless of the number of SMTMD units. In other words, either side of the SMTMD unit has near-zero mass. Therefore, the \(q\)-DOF SMTMD becomes similar to a \((q - 1)\) DOF system. This is because with a lower DOF, the first modal effective mass of the SMTMD system is greater, resulting in greater tuning mass.

For the \(q = 2\) situation, \((\delta m)_{\text{opt}}\) is close to the upper bound of \(\delta m\) regardless of the number of considered modes: almost all the SMTMD mass is concentrated to the second mass unit. In other words, the 2DOF SMTMD is similar to a single TMD. As shown in Table 4, either \(\xi_{s,1}\) or \(\xi_{s,2}\) is
greater than 100%, meaning that one of the vibration modes of the SMTMD cannot be oscillated. With increasing \( q \), the value of \( (\ddot{\phi})_{opt} \) appears to change toward zero (uniform mass distribution) and the three modes of the SMTMD are gradually tuned to the respective three modes of the building, even when \( n_c=1 \). It is believed that the SMTMD with higher \( q \) is more flexible in adjusting the first three modal frequencies to the target values.

### Table 3. Optimal parameters of the SMTMD under various values of \( q \) and \( n_c \)

| Optimal parameters of the SMTMD (up to the third mode) |
|-------------------------------------------------------|
| Number of SMTMD units, \( q \) | 1 | 2 | 3 | 4 | 5 |
|--------------------------------|---|---|---|---|---|
| Optimal mass distribution, \( (\ddot{m})_{opt} \)          | \( n_c=1 \) | 0 | ~2/\( q \) | ~2/\( q \) | ~2/\( q \) | ~2/\( q \) |
|                                             | \( n_c=2 \) | 0 | ~2/\( q \) | 0.080 | 0.028 | 0.004 |
|                                             | \( n_c=3 \) | 0 | ~2/\( q \) | 0.254 | 0.046 | 0.014 |
| Optimal damping coefficient, \( (c_{jp})_{opt}, (kN\cdot sec/m) \) | \( n_c=1 \) | 416 | 831 | 1023 | 1238 | 631 |
|                                             | \( n_c=2 \) | 1557 | 3108 | 409 | 464 | 532 |
|                                             | \( n_c=3 \) | 5087 | 10159 | 42.2 | 411 | 410 |
| Optimal stiffness coefficient, \( (k_{jp})_{opt}, (kN/m) \) | \( n_c=1 \) | 15214 | 30428 | 37821 | 46044 | 24312 |
|                                             | \( n_c=2 \) | 16014 | 31955 | 28215 | 33779 | 39459 |
|                                             | \( n_c=3 \) | 25251 | 50439 | 29190 | 34548 | 40333 |

### Table 4. Modal parameters of the optimal SMTMD system (up to the third mode)

| Modal parameters of the optimal SMTMD system (up to the third mode) |
|-------------------------------------------------------|
| Number of SMTMD units, \( q \) | 1 | 2 | 3 | 4 | 5 |
|--------------------------------|---|---|---|---|---|
| Modal frequency ratios, \( \omega_{s,1}/\omega_{p,1} \) | \( n_c=1 \) | 0.958 | 0.958 | 1.191 | 1.174 | 0.815 |
|                                             | \( n_c=2 \) | 0.983 | 0.983 | 0.952 | 0.955 | 0.956 |
|                                             | \( n_c=3 \) | 1.234 | 0.732 | 0.265 | 0.997 | 0.968 |
| Modal damping ratios, \( \xi_{s,1} \) | \( n_c=1 \) | 8.88 | 8.88 | >100 | 32.1 | 21.5 |
|                                             | \( n_c=2 \) | 32.4 | 32.4 | >100 | >100 | 32.0 |
|                                             | \( n_c=3 \) | >100 | >100 | >100 | >100 | >100 |

3.3. **Structural response performance**

The MS \( j^{th} \) modal displacement \( E[\eta_{p,j}^2] \) of the primary building without \( (q = 0) \) and with \( (q > 0) \) SMTMD are shown in Figure 3 (left column). The corresponding MS \( j^{th} \) modal accelerations \( E[\dddot{\eta}_{p,j}^2] \) are also plotted (right column). The ratios of MS modal displacement \( R_{\eta_{p,j}} \) and MS modal acceleration \( R_{\dddot{\eta}_{p,j}} \) of the primary building with the SMTMD to those without the SMTMD are shown in Figure 4.

Note that the three modal responses \( R_{\eta_{p,j}} \) and \( R_{\dddot{\eta}_{p,j}} \) (\( j = 1, 2, \) and \( 3 \)) of the primary building can always be obtained once the SMTMD parameters are determined for any set of \( n_c \) and \( q \). It is seen that if only the first structural mode is considered \( (n_c=1) \), the single TMD performs higher control efficiency than the SMTMD; however, the difference in controlled response between \( q = 1 \) and \( q = 5 \) is less than 5% for both \( E[\eta_{p,j}^2] \) and \( E[\dddot{\eta}_{p,j}^2] \). With lower \( q \) and increasing \( n_c \), the effectiveness of the SMTMD decreases for the first mode but increases for higher modes, because a lower \( q \) indicates a lower number of the tunable mode. With a higher \( q \), the ability to control three structural modes is greater. In addition, the SMTMD can reduce both \( E[\eta_{p,j}^2] \) and \( E[\dddot{\eta}_{p,j}^2] \), although it was designed to optimize the displacement performance \( J_d \).
Figure 3. MS values of modal displacement and acceleration of the primary structure with and without the SMTMD.

Figure 4. Ratios of MS modal displacement and acceleration of the building between with and without the SMTMD.
To determine the best set of \( q \) and \( n_c \) resulting in the minimal structural response, the combined modal response ratios, \( R_{\eta_j} = \sum_{j=1}^{3} c_{d_j} R_{\eta_j} \) and \( R_{\eta_g} = \sum_{j=1}^{3} c_{a_j} R_{\eta_j} \), were used as the observed indices, where \( \sum_{j=1}^{3} c_{d_j} = 1 \), \( \sum_{j=1}^{3} c_{a_j} = 1 \), \( c_{d_j} : c_{d_2} : c_{d_3} = E[\eta_p^2] : E[\eta_p^2] : E[\eta_p^2] \), and \( c_{a_1} : c_{a_2} = E[\eta_p^2] : E[\eta_p^2] \), meaning that the modal responses were combined by using the ratios of the modal responses of the uncontrolled primary building as weighting factors. The definition of \( R_{\eta_j} \) and \( R_{\eta_g} \) are different from \( J_d \) and \( J_a \); the former ones are always the combination of the three modal responses of the primary building regardless of \( n_c = 1,2, \) or \( 3 \). Figure 5 plots \( R_{\eta_j} \) and \( R_{\eta_g} \) for various sets of \( q \) and \( n_c \) for this study case. It can be seen that the set of \( q = 5 \) and \( n_c = 2 \) resulted in the minimal \( R_{\eta_g} \), and the set of \( q = 3 \) and \( n_c = 3 \) resulted in the minimal \( R_{\eta_g} \).

![Figure 5. Ratios of MS modal displacement and acceleration of the building between with and without the SMTMD.](image)

### 3.4. TMD stroke performance.

Table 5 presents the maximal MS stroke of the TMD in the SMTMD of different \( n_c \) and \( q \). For a single TMD (\( q = 1 \)), the MS stroke is 1.067 for the optimal performance condition (\( n_c = 1 \)). This is nine multiplying the corresponding structural modal response \( E[\eta_p^2] \) (=0.119). Moreover, the 5-DOF SMTMD (\( q = 5 \)) exhibits superior performance in the first and second structural modes compared with a single TMD; the maximal MS stroke is 0.396, which is 37% of that of a single TMD.

| Considered Structural modes in design, \( n_c \) | Number of SMTMD unit, \( q \) |
|------------------------------------------------|-------------------------------|
| \( 1 \) (TMD) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) |
| \( 1 \) | \( 0.267 \) | \( 0.170 \) | \( 0.114 \) | \( 0.396 \) |
| \( 2 \) | \( 0.090 \) | \( 0.038 \) | \( 0.009 \) | \( 0.000 \) |
| \( 3 \) | \( 0.127 \) | \( 0.051 \) | \( 0.038 \) | \( 0.000 \) |

![Table 5. MS stroke of each unit in a SMTMD (top - the first unit; bottom - the last unit).](image)
3.5. Transfer functions of floor response

Figure 6 illustrates the transfer function amplitudes of displacement and acceleration of the building roof with and without the TMD/SMTMD. Without the SMTMD, both the first and second modes dominate the floor response (gray dashed lines). With a single TMD and \( n_c = 1 \), the first resonant peaks are significantly reduced, but the second peaks change little. On the other side, with the 5-DOF SMTMD (\( q = 5 \)), both the first and second resonant peaks are reduced significantly.

![Figure 6. Transfer function amplitudes of the building roof with and without the SMTMD.](image)

3.6. Time-history analysis

For comparison, Figure 7 presents a plot of the time-history displacement and acceleration on the roof of the three-story building with the TMD, with the SMTMD (\( q = 5 \) and \( n_c = 2 \)), and without any control under the scaled ground acceleration recorded at National Chung Hsing University (NCHU) during the 1999 Chi-Chi earthquake. The TMD and SMTMD demonstrate similar control performance for roof displacement, but the SMTMD more effectively reduces roof acceleration than the single TMD does. This is because the roof acceleration is mainly contributed by the first and second modes, and the SMTMD can control multiple modes. Regarding the TMD stroke, Figure 8 shows that the stroke of the single TMD is markedly larger than those of the first and fourth TMDs. This shows the benefits of the SMTMD according to the time-history phase.

![Figure 7. Time-history displacement and acceleration of the three-story building with and without control subjected to the scaled Chi-Chi earthquake ground motion at NCHU.](image)
4. Conclusions

In this study, an SMTMD design theory for response control of multistory buildings was developed. The theory was demonstrated using an experimental building as the primary building. The optimal SMTMD parameters and the control performance were discussed. The analytical results show that if a single structural mode is considered, a single TMD is more effective than an SMTMD; however, the difference is as small as 5%. If multiple structural modes are considered, an SMTMD with higher degrees of freedom demonstrates superior control performance. Simultaneously, the maximal TMD stroke in an SMTMD is smaller than that of a single TMD. Therefore, the SMTMD device can be an alternative solution when TMD stroke demand is greater than the stroke capacity of the TMD stiffness and damping units.

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