Some results on Euler class groups

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Abstract: Let $A$ be a regular domain of dimension $d$ containing an infinite field and let $n$ be an integer with $2n \geq d + 3$. For a stably free $A$-module $P$ of rank $n$, we prove that (i) $P$ has a unimodular element if and only if the euler class of $P$ is zero in $E^n(A)$ and (ii) we define Whitney class homomorphism $w(P) : E^n(A) \to E^{n+s}(A)$, where $E^n(A)$ denotes the $n$th Euler class group of $A$ for $s \geq 1$.

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1 Introduction

Let $A$ be a commutative Noetherian ring of dimension $d$. For any $1 \leq s \leq d$, abelian group called the Euler class group $E^n(A)$ of $A$ is defined in [8] and given any projective $A$-module $P$ of rank $n < d$, a Whitney class homomorphism $w(P) : E^{d-n} \to E^d(A)$ is defined. Further it is proved that if $P$ has a unimodular element, then $w(P)$ is the zero map. Assume that $A$ is a regular domain of dimension $d$ containing an infinite field $k$. For a positive integer $n$ with $2n \geq d + 3$, we prove the following results:

(i) For a stably free $A$-module $P$ of rank $n$, we will associate an element $e(P)$ of $E^n(A)$ and prove that $e(P) = 0$ in $E^n(A)$ if and only if $P$ splits off a free summand of rank one (i.e. $P \oplus A$ for some projective $A$-module $Q$ of rank $n-1$). When $P \oplus A \cong A^{n+1}$, this result is due to Bhatwadekar and Raja Sridharan [4].

(ii) An element $(J, w_J)$ is zero in $E^n(A[T])$ if and only if $J$ is generated by $n$ elements and $w_J$ is a global orientation of $J$. This result is also proved by Das and Raja [6], Theorem 3.1), but their proof is different from ours. When $2n > d + 3$, this result follows from [8] for any Noetherian ring $A$. Hence the regularity of the ring is used only in the case $2n = d + 3$.

(iii) Given a stably free $A$-module $Q$ of rank $n$, we define a Whitney class homomorphism $w(Q) : E^n(A) \to E^{n+s}(A)$. Further, we prove that if $Q$ has a unimodular element, then $w(Q)$ is the zero map. When $n + s = d$, these results are proved in [8] for arbitrary projective module $Q$ over any Noetherian ring $A$.

It will be ideal to define the Whitney class homomorphism for all projective $A$-module $Q$ of rank $n$. For this first we need to define the euler class of $Q$ in $E^n(A)$ which is not known.

2 Euler class groups

All the rings considered are commutative Noetherian and all the modules are finitely generated. For a ring $A$ of dimension $d \geq 2$ and $1 \leq n \leq d$, the $n$th Euler class group of $A$, denoted by $E^n(A)$ is defined in [8] as follows:
Let $E_n(A)$ denote the group generated by $n \times n$ elementary matrices over $A$ and let $F = A^n$. A local orientation is a pair $(I, w)$, where $I$ is an ideal of $A$ of height $n$ and $w$ is an equivalence class of surjective homomorphisms from $F/IF$ to $I/I^2$. The equivalence is defined by $E_n(A/I)$-maps.

Let $L^n(A)$ denote the set of all pairs $(I, w)$, where $I$ is an ideal of height $n$ such that Spec $(A/I)$ is connected and $w : F/IF \to I/I^2$ is a local orientation. Similarly, let $L^n_0(A)$ denote the set of all ideals $I$ of height $n$ such that Spec $(A/I)$ is connected and there is a surjective homomorphism from $F/IF$ to $I/I^2$.

Let $G^n(A)$ denote the free abelian group generated by $L^n(A)$ and let $G^n_0(A)$ denote the free abelian group generated by $L^n_0(A)$.

Suppose $I$ is an ideal of height $n$ and $w : F/IF \to I/I^2$ is a local orientation. By ([3], Lemma 4.1), there is a unique decomposition $I = \cap I_i$, such that $I_i$’s are pairwise comaximal ideals of height $n$ and Spec $(A/I_i)$ is connected. Then $w$ naturally induces local orientations $w_i : F/I_iF \to I_i/I_i^2$. Denote $(I, w) = \sum (I_i, w_i) \in G^n(A)$. Similarly we denote $(I) = \sum (I_i) \in G^n_0(A)$.

We say a local orientation $w : F/IF \to I/I^2$ is global if $w$ can be lifted to a surjection $\Omega : F \to I$. Let $H^n(A)$ be the subgroup of $G^n(A)$ generated by global orientations. Also let $H^n_0(A)$ be the subgroup of $G^n_0(A)$ generated by $(I)$ such that $I$ is a surjective image of $F$.

The Euler class group of codimension $n$ cycles is defined as $E^n(A) = G^n(A)/H^n(A)$ and the weak Euler class group of codimension $n$ cycles is defined as $E^n_0(A) = G^n_0(A)/H^n_0(A)$.

The following result is proved in ([4], Corollary 2.4) in the case $P$ is free. Same proof works in this case, hence we omit the proof.

**Lemma 2.1** Let $A$ be a ring of dimension $d$ and let $n$ be an integer such that $2n \geq d + 1$. Let $I$ be an ideal of $A$ of height $n$. Let $P$ be a projective $A$-module of rank $n$. Suppose $\phi : P \to I/I^2$ is a surjection. Then, we can find a lift $\Phi' : P \to I$ of $\phi$ such that $\Phi'(P) = I \cap I'$, where $I'$ is an ideal of height $\geq n$ and comaximal with $I$.

Further, given any ideal $K$ of $A$ of height $\geq d - n + 1$, we can choose $I'$ to be comaximal with $K$.

Using ([2], (2), 4.11, 5.7) and following the proof of ([1], Proposition 3.3), we can prove the following result. Hence we omit the proof.

**Proposition 2.2** Let $A$ be a regular domain of dimension $d$ containing an infinite field and let $n$ be an integer such that $2n \geq d + 3$. Let $P = Q \oplus A$ be a projective $A$-module of rank $n$. Let $J, J_1, J_2$ be ideals of $A[T]$ of height $n$ such that $J$ is comaximal with $J_1$ and $J_2$. Assume that there exist surjections

$$\alpha : P[T] \to J \cap J_1, \ \beta : P[T] \to J \cap J_2$$

with $\alpha \otimes A[T]/J = \beta \otimes A[T]/J$. Suppose that there exists an ideal $J_3 \subset A[T]$ of height $n$ such that $J_3$ is comaximal with $J, J_1, J_2$ and there exists a surjection $\gamma : P[T] \to J_3 \cap J_1$ with $\alpha \otimes A[T]/J_1 = \gamma \otimes A[T]/J_1$.

Then there exists a surjection $\delta : P[T] \to J_3 \cap J_2$ with $\delta \otimes A[T]/J_3 = \gamma \otimes A[T]/J_3$ and $\delta \otimes A[T]/J_2 = \beta \otimes A[T]/J_2$.

If we replace $A[T]$ be any Noetherian ring $B$ of dimension $d$ and $P[T]$ by any projective $B$-module $\tilde{P} = Q \oplus B$ of rank $n$, then using ([2], Theorems 3.7 and 5.6) and following the proof of ([1], Proposition 3.3), we can prove ([2], in this case also.
Using (2, 4.11, 5.7), and following the proof of ([4], Theorem 4.2), we can prove the following result. This result is also proved in ([6], Theorem 3.1). Note that regularity of the ring is used only when $2n = d + 3$. When $2n > d + 3$, [2.3] holds for any ring $A$ by ([4], Theorem 4.2).

**Theorem 2.3** Let $A$ be a regular ring of dimension $d \geq 3$ containing an infinite field and let $n$ be an integer such that $2n \geq d + 3$. Assume that the image of $(J, w_J)$ is zero in $E^n(A[T])$, where $J \subset A[T]$ is an ideal of height $n$ and $w_J : (A[T]/J)^n \rightarrow J/J^2$ is an equivalence class of surjections. Then $J$ is generated by $n$ elements and $w_J$ can be lifted to a surjection $\theta : A[T]^n \rightarrow J$.

### 2.1 Euler class of Stably free modules

Let $A$ be a regular ring of dimension $d \geq 3$ containing an infinite field and let $n$ be an integer such that $2n \geq d + 3$. In ([4], a map from $\text{Um}_{n+1}(A)$ to $E^n(A)$ is defined and it is proved that, if $P$ is a projective $A$-module of rank $n$ defined by the unimodular element $[a_0, \ldots, a_n]$, then $P$ has a unimodular element if and only if the image of $[a_0, \ldots, a_n]$ in $E^n(A)$ is zero ([4], Theorem 5.4). Note that $P \oplus A \simeq A^{n+1}$.

For $r \geq 1$, let $\text{Um}_{r,n+r}(A)$ be the set of all $r \times (n+r)$ matrices $\sigma$ in $M_{r,n+r}(A)$ which has a right inverse, i.e., there exists $\tau \in M_{n+r,r}$ such that $\sigma \tau$ is the $r \times r$ identity matrix. For any element $\sigma \in \text{Um}_{r,n+r}(A)$, we have an exact sequence

$$0 \rightarrow A^r \xrightarrow{\sigma} A^{n+r} \rightarrow P \rightarrow 0,$$

where $\sigma(v) = \nu \sigma$ for $v \in A^r$ and $P$ is a stably free projective $A$-module of rank $n$. Hence, every element of $\text{Um}_{r,n+r}(A)$ corresponds to a stably free projective $A$-module of rank $n$ and conversely, any stably free projective $A$-module $P$ of rank $n$ will give rise to an element of $\text{Um}_{r,n+r}(A)$ for some $r$. We will define a map from $\text{Um}_{r,n+r}(A)$ to $E^n(A)$ which is a natural generalization of the map $\text{Um}_{n+1}(A) \rightarrow E^n(A)$ defined in [4].

Let $\sigma$ be an element of $\text{Um}_{r,n+r}(A)$.

$$\sigma = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n+r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,n+r} \end{bmatrix}$$

Let $e_1, \ldots, e_{n+r}$ be the standard basis of $A^{n+r}$ and let

$$P = A^{n+r}/(\sum_{i=1}^{n+r} a_{1,i} e_i, \ldots, \sum_{i=1}^{n+r} a_{r,i} e_i)A.$$

Let $p_1, \ldots, p_{n+r}$ be the images of $e_1, \ldots, e_{n+r}$ respectively in $P$. Then

$$P = \sum_{i=1}^{n+r} A p_i \text{ with relations } \sum_{i=1}^{n+r} a_{1,i} p_i = 0, \ldots, \sum_{i=1}^{n+r} a_{r,i} p_i = 0.$$

To the triple $(P, (p_1, \ldots, p_{n+r}), \sigma)$, we associate an element $e(P, (p_1, \ldots, p_{n+r}), \sigma)$ of $E^n(A)$ as follows:

Let $\lambda : P \rightarrow J$ be a surjection, where $J \subset A$ is an ideal of height $n$. Since $P \oplus A^r = A^{n+r}$ and $\dim A/J \leq d - n \leq n - 3$, by ([1], P/JP is a free $A/J$-module of rank $n$. Since $J/J^2$ is a surjective image of $P/JP$, $J/J^2$ is generated by $n$ elements.
Let "bar" denote reduce modulo \( J \). By Bass result (\( \mathbb{H} \)), there exists \( \Theta \in E_{n+r}(A/J) \) such that
\[
[\overline{a}_{1,1}, \ldots, \overline{a}_{1,n+r}] \Theta = [1, 0, \ldots, 0].
\]
Let \( \sigma \Theta = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\overline{b}_{2,1} & \overline{b}_{2,2} & \cdots & \overline{b}_{2,n+r} \\
\vdots & \vdots & & \vdots \\
\overline{b}_{r,1} & \overline{b}_{r,2} & \cdots & \overline{b}_{r,n+r}
\end{bmatrix}.
\]

Further, there exists \( \Theta_1 \in E_{n+r}(A/J) \) such that \( \sigma \Theta \Theta_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \overline{b}_{2,2} & \cdots & \overline{b}_{2,n+r} \\
\vdots & \vdots & & \vdots \\
0 & \overline{b}_{r,2} & \cdots & \overline{b}_{r,n+r}
\end{bmatrix}.
\]

It is clear that the first row of the elementary matrix \((\Theta \Theta_1)^{-1}\) is \([\overline{a}_{1,1}, \ldots, \overline{a}_{1,n+r}]\) and the matrix
\[\sigma_1 = \begin{bmatrix}
\overline{b}_{2,2} & \cdots & \overline{b}_{2,n+r} \\
\vdots & \vdots & \vdots \\
\overline{b}_{r,2} & \cdots & \overline{b}_{r,n+r}
\end{bmatrix}\]
belongs to \( U_{m(r-1),(n+r-1)}(A/J) \). Hence, by induction on \( r \), there exists \( \Theta_2 \in E_{n+r-1}(A/J) \) such that the first \( r-1 \) rows of \( \Theta_2 \) are \( \sigma_1 \). Hence \( \sigma \) can be completed to an elementary matrix \( \Delta \in E_{n+r}(A/J) \) (i.e. \( \sigma \) is the first \( r \) rows of an elementary matrix \( \Delta \in E_{n+r}(A/J) \)).

Since \( \sum_{i=1}^{n+r} a_{1,i} p_i = 0, \ldots, \sum_{i=1}^{n+r} a_{r,i} p_i = 0 \), we get
\[
\Delta[p_{1}^t, \ldots, p_{n+r}]^t = [0, \ldots, 0, q_{1}, \ldots, q_{n}]^t,
\]
where \( t \) stands for transpose.

Thus \([q_{1}, \ldots, q_{n}]\) is a basis of the free module \( P/JP \). Let \( w_j \) be given by the set of generators \( \lambda(q_{1}), \ldots, \lambda(q_{n}) \) of \( J/J^2 \), i.e \( w_j : (A/J)^n \rightarrow J/J^2 \) given by \( w_j(e_i) = \lambda(q_{i}) \) for \( i = 1, \ldots, n \).

We define \( e(P, (p_1, \ldots, p_{n+r}), \sigma) = (J, w_j) \in E^n(A) \). We need to show that \( e(P, (p_1, \ldots, p_{n+r}), \sigma) \) is independent of the choice of the elementary completion of \( \sigma \).

**Lemma 2.4** Suppose \( \Gamma \in E_{n+r}(A/J) \) is chosen so that its first \( r \) rows are \( \sigma \). Let \( \Gamma[p_{1}^t, \ldots, p_{n+r}]^t = [0, \ldots, 0, q_{1}, \ldots, q_{n}]^t \). Then there exists \( \Psi \in E_{n}(A/J) \) such that \( \Psi[p_{1}^t, \ldots, p_{n}]^t = [q_{1}, \ldots, q_{n}]^t \).

**Proof** The matrix \( \Gamma \Delta^{-1} \in E_{n+r}(A/J) \) is such that its first \( r \) rows are
\[
\begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & \cdots & 0
\end{bmatrix}.
\]
Therefore, there exists \( \Psi \in SL_n(A/J) \cap E_{n+r}(A/J) \) such that \( \Psi[p_{1}^t, \ldots, p_{n}]^t = [q_{1}, \ldots, q_{n}]^t \). Since \( n > \dim A/J + 1 \), by (\( \mathbb{H} \), Theorem 3.2), \( \Psi \in E_{n}(A/J) \).

The remaining arguments needed to show that \( e(P, (p_1, \ldots, p_{n+r}), \sigma) \) is a well defined element of \( E^n(A) \) is same as in (\( \mathbb{H} \), p. 152-153), hence we omit it. Therefore we have a well defined map \( U_{m(r,n+r)}(A) \xrightarrow{\epsilon} E^n(A) \).

The following result can be proved by following the proof of (\( \mathbb{A} \), Theorem 5.4). Hence we omit the proof.

**Theorem 2.5** Let \( A \) be a regular ring of dimension \( d \) containing an infinite field \( k \) and let \( n \) be an integer such that \( 2n \geq d+3 \). Let \( P \) be a stably free \( A \)-module of rank \( n \) defined by \( \sigma \in U_{m(r,n+r)}(A) \). Then \( P \) has a unimodular element if and only if \( e(P) = e(\sigma) = 0 \) in \( E^n(A) \).
2.2 Whitney class homomorphism

Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field $k$ and let $Q$ be a stably free $A$-module of rank $n$ with $2n \geq d + 3$. In [Lus99], we proved that $e(Q) = 0$ in $E^n(A)$ if and only if $Q$ has a unimodular element. Using this result we will establish a Whitney class homomorphism of stably free modules. When $n + s = d$, then [Lus99] is proved in ([8], Theorem 3.1) for any projective $A$-module $Q$. Our proof is a simple adaptation of their proof.

Theorem 2.6 Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field $k$. Suppose $Q$ is a stably free $A$-module of rank $n$ defined by $s \in \text{Um}_{r,n+r}(A)$. Then there exists a homomorphism $w(Q) : E^n(A) \to E^{n+s}(A)$ for every integer $s \geq 1$ with $2n + s \geq d + 3$.

Proof Write $F = A^n$ and $F' = A^n$. Let $I$ be an ideal of height $s$ and $w : F'/IF' \to I/I^2$ be an equivalence class of surjective homomorphisms, where the equivalence is defined by $E_s(A/I) = E(F'/IF')$ maps. To each such pair $(I, w)$, we will associate an element $w(Q) \cap (I, w) \in E^{n+s}(A)$.

First we can find an ideal $\overline{I} \subset A$ of height $n + s$ and a surjective homomorphism $\psi : Q/\overline{I}Q \to \overline{I}/I$ (this is just the existence of a generic surjection of $Q/\overline{I}Q$). Let $\psi \circ A/\overline{I} = \psi$. Then $\psi : Q/\overline{I}Q \to \overline{I}/(I + \overline{I}^2)$ is a surjection.

Since $\dim A/\overline{I} \leq d - (n + s) \leq n - 3$, $Q/\overline{I}Q$ is a free $A/\overline{I}$-module, by Bass result ([1]). Let “bar” denotes reduction modulo $\overline{I}$, then $\overline{T} \in \text{Um}_{r,n+r}(A)$ can be completed to an elementary matrix $\Theta \in E_{n+r}(\overline{A})$. This gives a well defined basis $[\overline{t}_1, \ldots, \overline{t}_n]$ for $Q$ which does not depends on the elementary completions of $\overline{T}$ (in the sense that any two basis of $\overline{T}$ obtained this way will be connected by an element of $E_{n+r}(\overline{A})$).

Let $\gamma : F/\overline{I}F \to Q/\overline{I}Q$ be the isomorphism given by $\gamma(\overline{t}_i) = \overline{t}_i$ for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the standard basis of the free module $F$. Let $\beta = \overline{\psi} \gamma : F/\overline{I}F \to \overline{I}/(I + \overline{I}^2)$ be a surjection and let $\beta' : F/F' \to \overline{I}/\overline{I}^2$ a lift of $\beta$.

Further, $w : F'/IF' \to I/I^2$ induces a surjection $\overline{w} : F'/\overline{I}F' \to (I + \overline{I}^2)/\overline{I}^2$. Composing $\overline{w}$ with the natural inclusion $(I + \overline{I}^2)/\overline{I}^2 \subset \overline{I}/\overline{I}^2$, we get a map $w' : F'/\overline{I}F' \to \overline{I}/\overline{I}^2$.

Combining $w'$ and $\beta'$, it is easy to see that we get a surjective homomorphism

$$\Delta = \beta' \oplus w' : F/\overline{I}F \oplus F'/\overline{I}F' = (F \oplus F')/\overline{I}(F \oplus F') = \overline{I}/\overline{I}^2$$

(surjectivity follows by considering the exact sequence $0 \to (I + \overline{I}^2)/\overline{I}^2 \subset \overline{I}/\overline{I}^2 \to 0$). We have $(\overline{I}, \Delta)$ a local orientation of $\overline{I}$. We will show that the image of $(\overline{I}, \Delta)$ in $E^{n+s}(A)$ is independent of choices of $\psi$, the lift $\beta'$ and the representative of $w$ in the equivalence class.

Step 1. First we show that for a fixed $\psi$, $(\overline{I}, \Delta)$ in $E^{n+s}$ is independent of the lift $\beta'$ and the representative of $w$.

(a) Suppose $w, w_1 : F'/IF' \to I/I^2$ are two equivalent local orientations of $I$. Then $w_1 = w_0 \epsilon$ for some $\epsilon \in E(F'/IF')$. Using the canonical homomorphisms $E(F'/IF') \to E(F'/\overline{I}F') \to E((F \oplus F')/\overline{I}(F \oplus F'))$, we get that $w_1 = w \epsilon_1$ for some $\epsilon_1 \in E((F \oplus F')/\overline{I}(F \oplus F'))$.

Let $\Delta_1$ be the local orientation of $\overline{I}$ obtained by using $\beta'$ and $w_1$. Then $\Delta_1 = \Delta \epsilon_1$. Hence $(\overline{I}, \Delta) = (\overline{I}, \Delta_1)$ in $E^{n+s}(A)$. 

5
(b) Let $\beta'' : F/IF \to \bar{I}/\bar{I}^2$ be another lift of $\beta$. Then $\phi = \beta'' - \beta'' : F/IF \to (I + \bar{I}^2)/\bar{I}^2$. Since $\bar{w}_1 : F'/IF' \to (I + \bar{I}^2)/\bar{I}^2$ is a surjection, there exists $g : F/IF \to F'/IF'$ such that $\bar{w}_1 g = \phi$.

Let $\varepsilon_2 = (\begin{smallmatrix} 3 \varepsilon \\ 0 \end{smallmatrix}) \in E((F \oplus F')/(F \oplus F'))$. Then $(\beta'' + w') \varepsilon_2 = (\beta' + w')$. Therefore, if $\Delta_2 = \beta'' + w'$, then $\Delta_2 \varepsilon_2 = \Delta_1 = \Delta \varepsilon_1$.

This completes the proof of the claim in step 1.

**Step 2.** Now we will show that $(\bar{I}, \Delta) \in E^{n+s}(A)$ is independent of $\psi$ also (i.e. it depends only on $(I, w)$).

Recall that $w : F'/IF' \to I/I^2$ is a surjection. It is easy to see that we can lift $w$ to a surjection $\Omega : F' \to I \cap K$, where $K + I = A$ and $K$ is an ideal of height $s$ (or $K = A$).

We can find an ideal $\bar{K} \subset A$ of height $\geq n + s$ and a surjective homomorphism $\psi' : Q/KQ \to \bar{K}/K$. Let $\psi' \otimes A/\bar{K} = \bar{\psi'}$. Then $\bar{\psi} : Q/\bar{K}Q \to \bar{K}/(K + \bar{K}^2)$ is a surjection.

Again, since $\dim A/\bar{K} \leq n - 3$, $Q/\bar{K}Q$ is a free $A/\bar{K}$-module. If "bar" denotes reduction modulo $\bar{K}$, then $\bar{\sigma} \in \text{Um}_r(A/\bar{K})$ can be completed to an elementary matrix which gives a basis $\bar{p}_1, \ldots, \bar{p}_n$ for $Q/\bar{K}Q$. Let $\gamma' : F'/KF \to Q/KQ$ be the isomorphism given by $\gamma'(\bar{p}_i) = \bar{p}_i$. Let $\eta = \bar{\psi} \gamma' : F/\bar{K}F \to \bar{K}/(I + \bar{K}^2)$ be a surjection and let $\eta' : F/\bar{K}F \to \bar{K}/\bar{K}^2$ be a lift of $\eta$.

The map $\Omega : F' \to I \cap K$ induces a surjection $\Omega \otimes A/K = \Omega' : F'/KF' \to K/K^2$ which in turn induces a surjection $\Omega' \otimes A/\bar{K} = w'' : F'/\bar{K}F' \to (K + \bar{K}^2)/\bar{K}^2$. Since $(K + \bar{K}^2) \subset \bar{K}$, we get a map $w'' : F'/\bar{K}F' \to \bar{K}/\bar{K}^2$.

Combining $w''$ and $\eta'$, we get a surjection $\Delta' = \eta' \oplus w'' : (F \oplus F')/\bar{K}(F \oplus F') \to \bar{K}/\bar{K}^2$.

**Claim.** $(\bar{I}, \Delta) + (\bar{K}, \Delta') = 0$ in $E^{n+s}(A)$.

Since $I + K = A$, we get $\bar{I} + \bar{K} = A$. Further, we get a surjection $\Psi = \psi \oplus \psi' : Q/(I \cap K)Q \simeq Q/IQ \oplus Q/KQ \to \bar{I}/\bar{I} \oplus \bar{K}/K \simeq (\bar{I} \cap \bar{K})/(I \cap K)$.

Let $\bar{\Psi} : Q \to \bar{I} \cap \bar{K}$ be a lift of $\Psi$ such that the following holds:

(i) $\bar{\Psi} \otimes A/\bar{I} = \bar{\psi}$, where $\bar{\psi} : Q/\bar{I}Q \to \bar{I}/(I + \bar{I}^2)$ is a surjection and

(ii) $\bar{\Psi} \otimes A/\bar{K} = \bar{\psi}'$, where $\bar{\psi}' : Q/\bar{K}Q \to \bar{K}/(K + \bar{K}^2)$ is a surjection.

Let $\bar{\Psi}_1 : Q/\bar{I}Q \to \bar{I}/\bar{I}^2$ be a lift of $\bar{\psi} \otimes A/\bar{I}$ and let $\bar{\Psi}_2 : Q/\bar{K}Q \to \bar{K}/\bar{K}^2$ be a lift of $\bar{\psi}' \otimes A/\bar{K}$. Then $\bar{\Psi}_1$ and $\bar{\Psi}_2$ induces a map $\bar{\Psi}_3 : Q/(\bar{I} \cap \bar{K})Q \to (\bar{I} \cap \bar{K})/(I \cap K)^2$.

Since $\beta = \bar{\psi} \gamma = (\bar{\psi} \otimes A/\bar{I}) \gamma$ and $\beta' : F/IF \to \bar{I}/\bar{I}^2$ is a lift of $\beta$, we get that $\alpha_1 = \beta' \gamma^{-1} - \bar{\Psi}_1$ is a map from $Q/IQ$ to $(I + \bar{I}^2)/\bar{I}^2 \subset \bar{I}/\bar{I}^2$. Similarly, $\alpha_2 = \eta'(\gamma')^{-1} - \bar{\Psi}_2$ is a map from $Q/\bar{K}Q$ to $(K + \bar{K}^2)/\bar{K}^2 \subset \bar{K}/\bar{K}^2$.

Since $\bar{w} : F'/IF' \to (I + \bar{I}^2)/\bar{I}^2$ is a surjection, we can find $g_1 : Q/\bar{I}Q \to F'/\bar{I}F'$ such that $\bar{w}_1 g_1 = \alpha_1$. Similarly, we can find $g_2 : Q/\bar{K}Q \to F'/\bar{K}F'$ such that $w'' g_2 = \alpha_2$ (here $w'' = \Omega' \otimes A/\bar{K}$).

Let $g$ be given by $g_1, g_2$ and $\tilde{\gamma}$ be given by $\gamma, \gamma'$. Then

(a) $\left( \begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix} \right)$ is an isomorphism from $(F \oplus F')/(\bar{I} \cap \bar{K})(F \oplus F')$ to $(Q \oplus F')/(\bar{I} \cap \bar{K})(F \oplus F')$ and
Write $\Gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Since $\Psi$ is a lift of $\Phi$, $\Psi$ is a surjection from $Q/(I \cap K)Q$ to $(\tilde{I} \cap \tilde{K})/(I \cap K)$ and $\Omega : F' \to I \cap K$ is a surjection, we get that $\Psi \oplus \Omega : Q \oplus F' \to \tilde{I} \cap \tilde{K}$ is a surjection.

Let $\Theta = \begin{pmatrix} \Psi \oplus \Omega \end{pmatrix} \otimes A/(\tilde{I} \cap \tilde{K})$. Then $\Theta : (Q \oplus F')/(\tilde{I} \cap \tilde{K})(Q \oplus F') \to (\tilde{I} \cap \tilde{K})/(\tilde{I} \cap \tilde{K}) \cdot I, \tilde{K}$. Let $(\delta, \delta') : (F \oplus F')/(\tilde{I} \cap \tilde{K})(F \oplus F') \to (\tilde{I} \cap \tilde{K})/(\tilde{I} \cap \tilde{K}) \cdot I, \tilde{K}$ be the surjection induced from $\delta, \delta'$. We claim that $(\delta, \delta') = \Theta \Gamma$. (This follows by checking on $V(\tilde{I})$ and $V(\tilde{K})$ separately, but we give a direct proof below.)

Let $\alpha_3 : Q/(\tilde{I} \cap \tilde{K})Q \to (\tilde{I} \cap \tilde{K})/(\tilde{I} \cap \tilde{K}) \cdot I, \tilde{K}$ be the map induced from $\alpha_1, \alpha_2$ and let $\tau : F/(\tilde{I} \cap \tilde{K}) \to (\tilde{I} \cap \tilde{K})/(\tilde{I} \cap \tilde{K}) \cdot I, \tilde{K}$ be the map induced from $\beta, \eta$. Then we have $\alpha_3 = \tau \gamma^{-1} - \Psi_3$. Let $\Omega : F'/(\tilde{I} \cap \tilde{K})F' \to (\tilde{I} \cap \tilde{K})/(\tilde{I} \cap \tilde{K}) \cdot I, \tilde{K}$ be the map induced from $\bar{w}, w'$. Then we have $\Omega g = \alpha_3$.

Let $\Theta = \Theta(0, y) = (\tilde{I}, \tilde{K})/(0, 0)$ and $\Theta \Gamma(x, 0) = \Theta(\gamma(x), \gamma(x)) = \bar{\Psi}_3 \gamma(x) + \bar{\Omega} \gamma(x) = \bar{\Psi}_3 \gamma(x) + \tau \gamma^{-1} \gamma(x) - \bar{\Psi}_3 \gamma(x) = \tau(x) = (\delta, \delta')(0, x)$.

This proves that $(\delta, \delta') = \Theta \Gamma$. By [2.4], Theorem 4.2, we get that $\tilde{I}(\tilde{I}) + (\tilde{K}, \tilde{K}) = 0$ in $E^{n+1}(A)$. Since $(\tilde{K}, \tilde{K})$ depends only on $(I, w)$, it follows that $(\tilde{I}, \tilde{K})$ is independent of the choice of $\psi$. This establishes the claim in step 2.

If $(I, w)$ is a global orientation, then we can take $K = A$ in the above proof and it will follow that $(\tilde{I}, \tilde{K})$ is also a global orientation.

Thus the association $(I, w) \mapsto (\tilde{I}, \tilde{K}) \in E^{n+1}(A)$ defines a homomorphism $\phi(Q) : G^s(A) \to E^{n+1}(A)$, where $(I, w)$ are the free generators of $G^s(A)$. Further $\phi(Q)$ factors through a homomorphism $w_0(Q) : E^s(A) \to E^{n+1}(A)$ sending $(I, w)$ to $E^s(A)$ to $(\tilde{I}, \tilde{K}) \in E^{n+1}(A)$. This completes the proof of the theorem.

**Corollary 2.7** Let $A$ be a regular domain of dimension $d \geq 2$ containing an infinite field. Suppose $Q$ is a stably free $A$-module of rank $n$. Then there exists a homomorphism $w_0(Q) : E^0_0(A) \to E^{n+1}_0(A)$ for every integer $s \geq 1$ with $2n + s \geq d + 3$.

**Proof** The proof is similar to that of [2.4] and we give an outline. Write $F = A^n$ and $F' = A^s$.

Suppose $(I)$ is a generator of $G^0_0(A)$. Here $I$ is an ideal of height $s$, $\text{Spec}(A/I)$ is connected and there is a surjection from $F'/IF'$ to $I/I^2$. There is a surjection $\psi : Q/IQ \to \tilde{I}, I$, where $\tilde{I}$ is an ideal of height $\geq n + s$. For such a generator $(I)$, we associate $(\tilde{I}) \in E^{n+1}_0(A)$.

For well-definedness, fix a local orientation $w : F'/IF' \to I/I^2$ and a surjective lift $\Omega : F' \to I \cap K$ of $w$, where $K$ is an ideal of height $\geq s$ and $K + I = A$. Let $\psi' : Q/KQ \to \tilde{K}/K$ be a surjection, where $\tilde{K}$ is an ideal of height $\geq n + s$. As in [2.0], there exists a surjection from $F \oplus F' \to \tilde{I} \cap \tilde{K}$. This shows that $(\tilde{I}) + (\tilde{K}) = 0$ in $E^{n+1}_0(A)$ and so $(\tilde{I}) \in E^{n+1}_0(A)$ is independent of the choice of $\psi$.

The association $(I) \mapsto (\tilde{I}) \in E^{n+1}_0(A)$ extends to a homomorphism $\phi_0 : G^0_0(A) \to E^{n+1}_0(A)$.

If $(I)$ is global (i.e. $I$ is a surjective homomorphism of $F'$), then taking $K = A$ in the above argument, we can prove that $(\tilde{I})$ is also global. So $\phi_0$ factors through a homomorphism $w_0(Q) : E^0_0(A) \to E^{n+1}_0(A)$.

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**Remark**
Definition 2.8 The homomorphism \( w(Q) \) in theorem \( \text{[2.6]} \) will be called the Whitney class homomorphism. The image of \((I, w) \in E^n(A)\) under \( w(Q) \) will be denoted by \( \Phi(I, w) \).

Similarly, the homomorphism \( w_0(Q) \) in \( \text{[2.7]} \) will be called the weak Whitney class homomorphism. The image of \((I) \in E^n_0(A)\) under \( w_0(Q) \) will be denoted by \( \Phi_0(I) \).

The proof of the following result is same as \( \text{[8], Corollary 3.4} \), hence we omit it.

Corollary 2.9 Let \( A \) be a regular domain of dimension \( d \geq 2 \) containing an infinite field. Suppose \( Q \) is a stably free \( A \)-module of rank \( n \). For every integer \( s \geq 1 \) with \( 2n + s \geq d + 3 \), we have

\[
\Phi_0(Q) \xi^s = \xi^{n+s} \Phi(Q) \quad \text{and} \quad C^n(Q^*) \eta^s = \eta^{n+s} \Phi(Q),
\]

where \( i ) \xi^r : E^n(A) \to E^n_0(A) \) is a natural surjection obtained by forgetting the orientation,

(ii) \( \eta^r : E^n_0(A) \to CH^n(A) \) is a natural homomorphism, sending \( (I) \) to \([A/I] \). Here \( CH^n(A) \) denotes the Chow group of cycles of codimension \( r \) in \( \text{Spec} (A) \) and

(iii) \( C^n(Q^*) \) denote the top Chern class homomorphism \( \text{[7]} \).

The following result is about vanishing of Whitney class homomorphism. When \( n + s = d \), it is proved in \( \text{[8], Theorem 3.5} \) for arbitrary projective module \( Q \) and our proof is an adaptation of \( \text{[8]} \). We will follow the proof of \( \text{[2.6]} \) with necessary modifications.

Theorem 2.10 Let \( A \) be a regular domain of dimension \( d \geq 2 \) containing an infinite field. Suppose \( Q \) is a stably free \( A \)-module of rank \( n \) defined by \( \sigma \in \text{Um}_{n,n+r}(A) \). Let \( s \geq 1 \) be an integer with \( 2n + s \geq d + 3 \).

Write \( F = A^n \) and \( F' = A^s \). Let \( I \) be an ideal of height \( s \) and let \( w : F'/IF' \to I/I^2 \) be a surjection. If \( Q/IQ = P_0 \oplus A/I, \) then \( w(Q) \cap (I, w) = 0 \) in \( E^{n+s}(A) \).

In particular, if \( Q = P \oplus A \), then the homomorphism \( w(Q) : E^n(A) \to E^{n+s}(A) \) is identically zero. Similar statements hold for \( w_0(Q) \).

Proof Step 1. We can find an ideal \( \tilde{I} \subset A \) of height \( n + s \) and a surjective homomorphism \( \psi : Q/IQ \to \tilde{I}/I \). Let \( \tilde{\psi} = \psi \otimes A/\tilde{I} : Q/\tilde{I} \to \tilde{I}/(I + \tilde{I}^2) \).

Let \( \Omega : F' \to I \) be a lift of \( w \) and let \( \overline{\nu} = w \otimes A/\tilde{I} : F'/IF' \to I/I\tilde{I} \). Composing \( \overline{\nu} \) with the natural map \( I/I\tilde{I} \to \tilde{I}/I\tilde{I} \), we get a map \( w' : F'/IF' \to \tilde{I}/I\tilde{I}^2 \).

Since \( Q/IQ = P_0 \oplus A/I, \) we can write \( \psi = (\theta, \overline{\nu}) \) for some \( \theta \in P_0^* \). We may assume that \( \psi(P_0) = \tilde{J}/I, \) for some ideal \( \tilde{J} \subset A \) of height \( n + s - 1 \). Note that \( \tilde{I} = (\tilde{J}, a) \).

Since \( \dim A/\tilde{J} = d - (n + s - 1) \leq n - 2 \) and \( P_0/IP_0 \) is stably free \( A/\tilde{I} \)-module of rank \( n - 1 \), \( P_0/JP_0 \) is free. If “prime” denotes reduction modulo \( \tilde{J} \), then \( \sigma' \) can be completed to an elementary matrix in \( E_{n+s}(A/\tilde{J}) \). This gives a canonical basis of \( P_0/JP_0, \) say \( q_1', \ldots, q_{n-1}' \). Let \( \gamma' : (A/\tilde{J})^{n-1} \to P_0/JP_0 \) be the isomorphism given by \( [q_1', \ldots, q_{n-1}'] \).

Let \( \gamma : F/IF = (A/\tilde{I})^n \to Q/IQ = P_0/JP_0 \) be the isomorphism given by \( (\gamma', 1) \), i.e. \( \gamma = [\overline{\nu}, \ldots, q_{n-1}] \). Let \( \beta = \overline{\psi} \gamma : F/IF \to \tilde{I}/(I + \tilde{I}^2) \) and let \( \beta' : F/IF \to \tilde{I}/I\tilde{I}^2 \) be a lift of \( \beta \).

As in the proof of \( \text{[2.6]} \), combining \( w' \) and \( \beta' \), we get a surjection \( \Delta = \beta' \circ w' : (F \oplus F')/\tilde{I}(F \oplus F') \to \tilde{I}/I\tilde{I}^2 \) and \( (\tilde{I}, \Delta) = (Q/IQ \cap (I, w) \cap \Delta = 0 \) in \( E^{n+s}(A) \).
Step 2. In this step, we will prove the claim. The surjection \( \theta : P_0 \rightarrow \tilde{J}/I \) induces a surjection \( \overline{\theta} = \theta \otimes A/\overline{J} : P_0/\overline{J}P_0 \rightarrow \tilde{J}/(I + \tilde{J}^2) \). Let \( \zeta = \overline{\theta}^* : (A/\overline{J})^{n-1} \rightarrow \tilde{J}/(I + \tilde{J}^2) \) and let \( \zeta' : (A/\overline{J})^{n-1} \rightarrow \tilde{J}/\tilde{J}^2 \) be a lift of \( \zeta \).

If \( \overline{\zeta} \) denotes the composition of \( \zeta' \otimes A/\overline{J} : (A/\overline{J})^{n-1} \rightarrow \tilde{J}/\tilde{J}^2 \) with natural maps \( \tilde{J}/\tilde{J}^2 \rightarrow \tilde{J}/\tilde{J}^2 \rightarrow \tilde{J}/\tilde{J}^2 \), we get that \( (\overline{\zeta}, \overline{\pi}) \) is a lift of \( \beta : F/IF \rightarrow \tilde{J}/(I + \tilde{J}^2) \). Since \( w(Q) \cap (I, w) \) is independent of \( \tilde{\beta} \), we may assume that \( \tilde{\beta} = (\overline{\zeta}, \overline{\pi}) \).

If \( \delta : A^{n-1} \rightarrow J \) is a lift of \( \zeta' \), then \( (\delta, a, \Omega) : F \oplus F' \rightarrow \tilde{J} \) is a lift of \( (\beta', w') \). If \( \tilde{J} \) is the image of \( (\delta, \Omega) \), then \( \tilde{J} = \tilde{J}' + \tilde{J}^2 \). (To see this, let \( y \in \tilde{J} \), then there exists \( x \in A^{n-1} \) such that \( \delta(x) - y = y_1 + z \) for some \( y_1 \in I \) and \( z \in \tilde{J}^2 \). Choose \( x \in F' \) such that \( y_1 - \Omega(x_1) = z_1 \in I^2 \subset \tilde{J}^2 \). Therefore \( \delta(x) - \Omega(x_1) = y \) modulo \( \tilde{J}^2 \).

Since \( \tilde{J} = \tilde{J}' + \tilde{J}^2 \), we can find \( e \in \tilde{J}^2 \) such that \( (1 - e)\tilde{J} \subset \tilde{J}' \) and \( \tilde{J} = (\tilde{J}', e) \). Therefore by (9), Lemma 1), \( \tilde{J} = (\tilde{J}', a) = (\tilde{J}', b) \), where \( b = e + (1 - e)a \). Thus \( (\delta, b, \Omega) : F \oplus F' \rightarrow \tilde{J} \) is a surjection which is a lift of \( (\beta', w') \). This proves that \( (\tilde{I}, \Delta) = 0 \) in \( E^{n+s}(A) \). This completes the proof.

2.3 Remark on some results of Yang

We start this section by describing some results of Yang [11].

(1) Let \( R \) be a Noetherian commutative ring of dimension \( d \) and let \( n \) be an integer with \( 2n \geq d + 3 \). Let \( l \) be an ideal of \( R \) and let \( \rho : R \rightarrow \overline{R} = R/l \) be the natural surjection. Yang [11] defines a group homomorphism \( E(\rho) : E^n(R; R) \rightarrow E^n(\overline{R}; \overline{R}) \), called the restriction map of Euler class group, as \( E(\rho)(I, w_I) = (\overline{I}, w_{I/l}) \), where \( (I, w_I) = (I', w_{I'}) \) in \( E^n(R; R) \) with height of \( \overline{I} + I \geq n \) in \( \overline{R} \).

(2) Further, let \( A \) be a Noetherian commutative ring of dimension \( s \) with \( 2n \geq s + 3 \). Assume there exists a ring homomorphism \( \phi : R \rightarrow A \) such that for any local orientation \( (I, w_I) \in E^n(R; R) \), height of \( \phi(I) \) is \( \geq n \). Then Yang defines a group homomorphism \( E(\phi) : E^n(R; R) \rightarrow E^n(A; A) \), called the extension map of Euler class group, as \( E(\phi)(I, w_I) = (\phi(I), w_{\phi(I)}) \).

(3) Let \( D(R, l) \) denotes the double of \( R \) along \( l \), then

\[
0 \rightarrow l \rightarrow D(R, l) \xrightarrow{p_l} R \rightarrow 0
\]

is a split exact sequence. The relative Euler class group of \( R \) and \( l \) is defined as

\[
E^n(R, l; R) = \ker(E(p_l)) : E^n(D(R, l); D(R, l)) \rightarrow E^n(R; R),
\]

where \( E(p_l) \) is the restriction map.

(4) (Homology sequence) Let \( p_2 \) denote the second projection from \( D(R, l) \rightarrow R \). Then the following Homology sequence of Euler class group is exact.

\[
E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l).
\]

(5) (Excision theorem) Further assume that there exists a splitting of \( \rho : R \rightarrow R/l \) (i.e., a ring homomorphism \( \beta : R/l \rightarrow R \) such that \( \rho \beta = id \)) satisfying the condition that for any local orientation \( (J, w_J) \in E^n(R/l; R/l) \), height of \( \beta(J) \) is \( \geq n \). Then we have the following exact sequence, called the Excision sequence of Euler class group.

\[
0 \rightarrow E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l) \rightarrow 0.
\]
Note that the existence of a splitting of ρ is sufficient for the injectivity of Homology sequence.

(6) As a consequence of above results, if 2n ≥ d + 4, then we have the following split short exact sequence:

$$0 \rightarrow E^n(R[T], (T); R[T]) \xrightarrow{E(p_2)} E^n(R[T]; R[T]) \xrightarrow{E(\rho)} E^n(R; R) \rightarrow 0.$$  

Further if \(R\) is a regular affine domain essentially of finite type over an infinite perfect field, then it is proved that \(E(\rho) : E^n(R[T]; R[T]) \rightarrow E^n(R; R)\) is an isomorphism. Note that Das and Raja (\cite{6}, Theorem 3.8) also proved this isomorphism for \(2n \geq d + 3\).

Using (\cite{2}, 4.11, 5.7) and following the proof in (\cite{11}), we get the following stronger results in case of polynomial ring over \(R\).

**Theorem 2.11** Assume that \(R\) is a regular domain of dimension \(d\) containing an infinite field and let \(n\) be an integer with \(2n \geq d + 3\). Then we have the following results:

(i) (Homology sequence) Let \(p_2\) denote the second projection from \(D(R[T], l) \rightarrow R[T]\), where \(l\) is an ideal of \(R[T]\). Then we have the following Homology exact sequence of Euler class group:

$$E^n(R[T], l; R[T]) \xrightarrow{E(p_2)} E^n(R[T]; R[T]) \xrightarrow{E(\rho)} E^n(R[T]/l; R[T]/l).$$

(ii) (Excision theorem) Further assume that there exists a splitting \(\beta : R[T] \rightarrow R[T]/l\) satisfying the condition that for any local orientation \((J, w, r) \in E^n(R[T]/l; R[T]/l)\), height of \(\beta(J)\) is \(\geq n\). Then we have the following Excision exact sequence of Euler class group:

$$0 \rightarrow E^n(R[T], l; R[T]) \xrightarrow{E(p_2)} E^n(R[T]; R[T]) \xrightarrow{E(\rho)} E^n(R[T]/l; R[T]/l) \rightarrow 0.$$  

In particular, when \(l = (T)\), then we have the following split short exact sequence:

$$0 \rightarrow E^n(R[T], (T); R[T]) \xrightarrow{E(p_2)} E^n(R[T]; R[T]) \xrightarrow{E(\rho)} E^n(R; R) \rightarrow 0.$$  

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