The Volume Operator
for Singly Polarized Gravity Waves
with Planar or Cylindrical Symmetry

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Abstract
A previous paper constructed a kinematic basis for spin networks with planar or cylindrical symmetry and arbitrary polarization. This paper imposes a constraint which limits the gravitational wave to a single polarization. The spectrum of the constraint contains a physically reasonable number of zero eigenvalues, and the zero eigenvectors can be constructed explicitly. Commutation of the constraint with the Hamiltonian is expected to lead to a further constraint. This new constraint is not investigated in this paper, but I argue it will be non-local, relating states at two or more neighboring vertices.

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I Introduction
In a previous paper, I constructed a kinematic basis for a spin network with planar or cylindrical symmetry and studied the eigenfunctions and eigenvalues of the volume operator[1]. In that paper, the gravitational wave could have either polarization. In this paper I study the volume operator for a wave limited to a single polarization.

In classical gravity, and in quantum gravity based on local field theory, the assumption of a single polarization greatly simplifies the

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discussion. If both polarizations are present the matrix of metric components (or triad components, if one is using connection triad variables) has a 1x1 subblock and a 2x2 subblock after gauge fixing. The 1x1 subblock contains the longitudinal component of the metric or triad (r component in the cylindrical case; z component in the planar case); the 2x2 subblock contains components transverse to the direction of the wave. In the single polarization case, the matrix simplifies further, becoming diagonal. In terms of ADM metric connection variables, the off-diagonal element \( g_{xy} \) (or \( g_{\theta \phi} \))=0.

In the connection triad variables used in this paper, the equivalent statement is

\[
\text{Tr}(\tilde{E}_x \tilde{E}_y) = 0, \tag{1}
\]

or the analogous statement for cylindrical coordinates.

In local field theory, when the constraint eq. (1) is commuted with the Hamiltonian, one obtains a further constraint which states that the conjugate ADM momentum \( \pi_{xy} \) also vanishes. In triad connection language,

\[
0 = Tr(\tilde{E}_x \tilde{E}_a)Tr(\tilde{E}_y A_a) + (x \leftrightarrow y) - \Gamma^{xy};
\]

\[
\Gamma^{xy} = -iTr((\tilde{E}_x \partial_z \tilde{E}_y)\tilde{E}_z). \tag{2}
\]

The \( \Gamma \) term subtracts out the spin connection part of A. Imposing these constraints causes a degree of freedom and its conjugate momentum to disappear entirely from the calculation, which simplifies considerably.

In the spin network case, it is straightforward to impose eq. (1). This constraint resembles the volume operator, in that it is constructed entirely from triads and is derivative free. If one makes the simplest assumption that only the field strengths (or other operators with derivatives) need to be non-local, then one can take the operator of eq. (1) to be local. That is, the operator acts at a single vertex, merely reshuffling the holonomic basis states at that vertex. The constraint is no harder to handle than the volume operator, which is also local. (All \( \tilde{E} \) operators should be integrated over an area, but this is straightforward and I suppress the area integrations.)

However, in a spin network approach, the constraint eq. (2) is unacceptable. Connections A are not diffeomorphism invariant unless integrated over an edge and included in a holonomy. It is also
much easier to construct a field theory based on holonomies than one based on A’s, because holonomies may be described by a compact manifold, the Euler angles \[2\].

Perhaps one could propose a spin network generalization of eq. (2) that is local. (Promote the A operator to a holonomy which acts at the same vertex as the \(\tilde{E}\) operators in eq. (2)) However, I will give two arguments that the holonomy (and perhaps some of the \(\tilde{E}\) as well) will have to be non-local. That is, the holonomy which regularizes the A in eq. (2) will include segments joining two or more neighboring vertices.

(First argument.) Since the constraint eq. (2) is a momentum, the time derivative of the original constraint, the natural way to obtain (the spin network version of) eq. (2) is to commute eq. (1) with the spin network Hamiltonian. It is not clear at present what to use for that Hamiltonian; but it must be non-local, in order for propagation to occur. Commuting even a local object with a non-local object is very likely to lead to a non-local object.

(Second argument). Consider the Killing vectors which define planar symmetry. (I am concentrating on the planar case. The discussion for the cylindrical case is virtually identical \[3\].) The space possesses two spacelike Killing vectors which commute, and one can choose coordinates so that these two vectors become \(\partial/\partial x\) and \(\partial/\partial y\). This planar symmetry by itself does not imply one polarization only. Given only the existence of the two Killing vectors, all three transverse components of the metric, \(g_{xx}, g_{yy}, g_{xy}\) or all four transverse triads can be non-zero; this is enough to permit two polarizations.

In classical general relativity, and in local quantum field theory treatments of gravity, one must demand hypersurface orthogonality of the Killing vectors in order to get one polarization. Hypersurface orthogonality can be shown to imply that \(g_{xy}\) and its canonical momentum vanish \[3\].

Hypersurface orthogonality is an intrinsically non-local idea. If a Killing vector field such as \(\partial/\partial x\) exists, then in each plane \(z = \text{constant} = z_1\) there exist one-dimensional integral curves with the Killing vector as tangent vector. The plane at \(z = z_2\) possesses similar integral curves, but no relation is implied between the two sets of curves at \(z_1\) and \(z_2\). The additional assumption of hypersurface orthogonality creates such a relation. The two sets of integral
curves must be normal to a common set of hyperplanes. The integral curves can be described as curves of increasing \( x \), and the hyperplanes as hyperplanes of constant \( x \).

Without hypersurface orthogonality, the integral curves of \( \partial/\partial x \) may twist in any manner as one goes from one value of \( z \) to the next. With hypersurface orthogonality, this twisting is strongly constrained.

In the spin network context, going from one value of \( z \) to the next is equivalent to going from one vertex to the next. Hypersurface orthogonality implies a relation between structures at two different vertices, necessarily a non-local relation. (End of second argument).

If eq. (2) must be promoted to a non-local constraint, then within a spin network context, the one-polarization case is harder than the two-polarization case. In this paper I carry the discussion of the one-polarization case as far as I can, given that I have a kinematic basis but no Hamiltonian, therefore no way to construct eq. (2). I impose eq. (1) on the states of the kinematic basis, but not eq. (2). I then ask which states of the two-polarization case are ruled out by the constraint.

For orientation, it is helpful to keep in mind an example from quantum mechanics, the two dimensional simple harmonic oscillator with Hamiltonian

\[
H = -\left(\hbar^2/2m\right)[\partial^2/\partial x^2 + \partial^2/\partial y^2] + (k/2)(x^2 + y^2).
\] (3)

This problem is separable, with eigenfunctions \( |n_xn_y> \) labelled by occupation numbers. Suppose I wish to impose the constraint that the oscillator possesses only a single ”polarization”, say it oscillates only along \( x \). I can impose the constraint on the classical theory first, then quantize (constraint first). Alternatively, I can follow Dirac and quantize first, then impose the constraint (quantize first) [4]. For reasons to be explored in the next section, I will choose to follow the second, Dirac procedure and quantize first.

Suppose I apply the Dirac procedure to the oscillator problem. I know the answer I want: the constraints should rule out all states except those of the form \( |n_x; n_y = 0> \). However, if I require physical states to obey

\[
y |\geq p_y |\geq 0,
\] (4)

I find these requirements are too strong. From the theory of coherent states, there is no ket which satisfies even one of these equations
Recall that both coordinate and momentum contain creation operators, which make it difficult for these operators to completely destroy any state.)

I can try the milder constraint,

\[ <| y | > = 0, \]

which is perhaps enough to guarantee a satisfactory classical limit. However, whereas the previous constraint was too strong, this constraint is too weak. For example, it is satisfied by any state \(| >\) which has definite parity, including states with \(n_y\) non-zero. To get the answer I want, I must impose at least eq. (5) and

\[ <| y^2 | > = 0. \]

Eq. (6) gives the energy a satisfactory classical limit. I conclude a straightforward application of the constraint to a ket, the analog of eq. (4), may not work; I may have to impose averaged constraints, eqs. (5) and (6).

For another example, this time from field theory, consider quantization of the electromagnetic field in Lorentz gauge. One must impose both of the constraints

\[ 0 = <| \partial \cdot A | >; \]
\[ 0 = <| (\partial \cdot A)^2 | >, \]

in order to get the correct classical limit, including the correct classical energy [6]. Again,

\[ 0 = \partial \cdot A | > \]

is too strong.

These examples determine much of the discussion in section two. First I discuss the relative merits of constraint first vs. quantize first approaches, and I opt for the latter. I then impose the analog of eq. (4),

\[ \text{Tr}(\hat{E}^x \hat{E}^y) | > = 0. \]

In the oscillator and Lorentz gauge examples, constraints of this type have no solutions. However, in the spin network case I was able to find normalizable states satisfying eq. (4). The kernel of the
constraint is non-trivial. As a check, I impose the constraints in the average sense:

\[ 0 = \langle \text{Tr} (\tilde{E}^x \tilde{E}^y) \rangle > \]
\[ = \langle \text{Tr} (\tilde{E}^x \tilde{E}^y)^2 \rangle > . \tag{10} \]

I find that the states which satisfy these constraints are the same as the states which satisfy eq. (9).

II Dirac Quantization of the One-Polarization Case

I can impose the constraint on the classical theory first, then quantize the single-polarization classical theory (constrain first); or I can quantize first, then impose the constraint as a condition on the quantized states (quantize first; Dirac quantization). If I constrain first, it is natural to follow Bojowald and introduce fields which have the constraint built in [7].

\[ \tilde{E}^x_A S_A = \tilde{E}^x \left[ \cos(\alpha + \beta) S_1 + \sin(\alpha + \beta) S_2 \right]; \]
\[ \tilde{E}^y_A S_A = \tilde{E}^y \left[ -\sin(\alpha + \beta) S_1 + \cos(\alpha + \beta) S_2 \right]; \]
\[ A^x_A S_A = A^x \left[ \cos(\beta) S_1 + \sin(\beta) S_2 \right]; \]
\[ A^y_A S_A = A^y \left[ -\sin(\beta) S_1 + \cos(\beta) S_2 \right]. \tag{11} \]

The first two lines, for example, relate the four ”Cartesian” fields (\(\hat{E}^x_A\) and \(\hat{E}^y_A\)) to three ”polar” fields (the magnitudes \(E^x, E^y\), and the angle \(\alpha + \beta\)). One degree of freedom is lost, but this is acceptable: the new polar fields have the constraint \(\text{Tr}(\tilde{E}^x \tilde{E}^y) = 0\) built in. From eq. (11), the A’s obey a similar constraint,

\[ \text{Tr} A^x A^y = 0, \tag{12} \]

which again removes one degree of freedom, and is easier to implement than the constraint of eq. (2). Bojowald carries out the canonical transformation from Cartesian to polar fields and finds the following pairs of canonical variables [7].

\[ (E^x \cos(\alpha) = P^x, A^x); \]
\[ (E^y \cos(\alpha) = P^y, A^y); \]
\[ (\sin \alpha (A^x P^x + A^y P^y) = P^\beta, \beta); \]
\[ (\hat{E}^z_A, A^z). \tag{13} \]

\[ \frac{A^z}{E^z} = \beta, \]
$P^\beta$ turns out to be the Gauss Constraint. The angle $\alpha$ is not an independent degree of freedom, but is given by
\[
\tan \alpha = P^\beta/(A_x P^x + A_y P^y).
\]
These are classical field theory definitions. It is not clear how to order the operators in $\tan \alpha$ after quantizing, or what the spin network definitions should be.

The lack of precise definitions is already a problem at the kinematic level, since the formula for the volume operator in polar coordinates involves $\alpha$.
\[
V^2 = \tilde{E}_z^2 E^x E^y = \tilde{E}_z^2 P^x P^y/(\cos \alpha)^2.
\]
(14)

Since I was unable to supply any convincing spin network definitions of $\alpha$ or $V^2$, I determined to avoid polar coordinates as much as possible. Every time I attempted a ”constrain first” quantization, however, I found myself introducing the polar variables, explicitly or implicitly. A constrain first quantization which avoids polar fields may be possible; but I was unable to find it.

For the rest of this paper I will use the quantize first approach. This has the advantage that the first steps (gauge fixing, quantization, kinematic basis and dot product) have been worked out already in the previous paper; and the polar fields need not be used at any stage.

I now impose the constraint, in its unaveraged form, eq. (9). I take the ket to be a linear combination of states in the kinematic basis. Note that the trace in eq. (9), is very similar in structure to the 2x2 transverse subblock of the (square of the) volume operator:
\[
(V_2)^2 = \epsilon_{ZAB} \tilde{E}_A^x \tilde{E}_B^y
= i(\tilde{E}_+^x \tilde{E}_-^y - \tilde{E}_-^x \tilde{E}_+^y);
\]
\[
E^{xy} := 2Tr(\tilde{E}^x \tilde{E}^y)
= \tilde{E}_+^x \tilde{E}_-^y + \tilde{E}_-^x \tilde{E}_+^y.
\]
(15)
on the second lines I have used the linear combinations with simple $U(1)$ transformation properties,
\[
\tilde{E}_\pm^x = (\tilde{E}_x^x \pm i\tilde{E}_y^x)/\sqrt{2}.
\]
The two expansions in eq. (15) are identical, except for the extra i’s and the extra minus sign in $(V_2)^2$. Therefore I can get the action of the constraint from a simple modification of the action of the $(V_2)^2$ operator.

First recall the equation for the action of $(V_2)^2$. Write an eigenfunction of this operator as a linear combination of states in the kinematic basis:

$$|\vec{c}(\lambda; L_x, L_y, F)\rangle = \sum \Sigma_D c(D; \lambda, L_x, L_y, F) Y_{L_x m_x} Y_{L_y m_y}.$$  \hspace{1cm} (16)

$\lambda$ is the eigenvalue, $\vec{c}$ is the eigenvector with components $\{c(D; \lambda, L_x, L_y, F)\}$, the Y’s are the spherical harmonics constructed in [1], and

$$F = (m_x + m_y)/2;$$
$$D = (m_x - m_y)/2. \hspace{1cm} (17)$$

The eigenfunction contains a sum over $D$ only; the volume operator does not change $F$. From the previous paper, $(V_2)^2$ acting on this state leads to the following recurrence relation for the c’s.

$$2\lambda c(D; \lambda) = ig_- c(D - 1; \lambda) - ig_+ c(D + 1; \lambda);$$
$$g_-(D) = \sqrt{(L_x - F - D + 1)(L_x + F + D)};$$
$$g_+(D) = \sqrt{(L_y + F - D + 1)(L_y - F + D)}; \hspace{1cm} (18)$$

(For simplicity I have suppressed some of the arguments of the c’s.) To obtain the action of $E_{xy}$, I use eq. (15). I drop the i’s and the minus sign in eq. (18). Also, I replace the eigenvalue $\lambda$ by zero, since $E_{xy} \psi = 0$.

$$0 = g_-(D - 1) + g_+(D + 1). \hspace{1cm} (19)$$

I denote the expansion coefficients by $e$, rather than $c$, because the c’s (the volume eigenfunctions) in general will not satisfy eq. (19).

The following lemma is straightforward to prove.

Lemma. If the eigenvector $\vec{c}(\lambda)$ with components $\{c(D; \lambda)\}$ is a solution to eq. (18), then the eigenvectors

$$\vec{e}(\pm \lambda) = \{(\mp i)^D c(D; \lambda)\} \hspace{1cm} (20)$$

are eigenvectors of eq. (19) with eigenvalues $\pm \lambda$, i.e.

$$\pm 2\lambda e(D; \pm \lambda) = g_- e(D - 1; \pm \lambda) + g_+ e(D + 1; \pm \lambda). \hspace{1cm} (21)$$
This lemma implies that \((V_2)^2\) and the constraint operator have exactly the same eigenvalue spectrum (although their eigenvectors are not the same). (For example, in the previous paper I proved that the non-zero eigenvalues of \((V_2)^2\) occur in pairs \((\lambda, -\lambda)\); and the lemma indicates a similar pairing for the eigenvalues of the constraint operator.)

From the previous paper, \((V_2)^2\) has zero eigenvalues \([1]\). Hence the constraint operator has zero eigenvalues, and the constraint equation eq. \((9)\) has non-trivial solutions.

From the discussion in the introduction, I should also investigate the averaged constraints, eq. \((10)\). It is easy to see that there are a large number of solutions to the first, linear constraint: the combinations \(|\vec{e}(\lambda)\rangle \pm |\vec{e}(-\lambda)\rangle\) are sent into the orthogonal combinations \(|\vec{e}(\lambda)\rangle \mp |\vec{e}(-\lambda)\rangle\) by the constraint. (As at eq. \((16)\), \(|\vec{e}(\lambda)\rangle\) is shorthand for the eigenstate of the constraint operator \(\Sigma D e(D; \lambda) Y_{L_x m_x} Y_{L_y m_y}\).

However, the second, quadratic constraint in eq. \((10)\) is more restrictive. Let the quadratic constraint act on an arbitrary state

\[
|> = \sum_\lambda a(\lambda) |\vec{e}(\lambda)\rangle .
\]

The expectation value of the quadratic constraint is then

\[
\Sigma_\lambda \lambda^2 |a(\lambda)|^2 .
\]

This is not zero unless all of the \(\lambda\) are zero. The averaged constraints, eq. \((10)\), give the same result as the unaveraged constraint, eq. \((9)\).

In the oscillator example discussed in the introduction, the imposition of a constraint lowers the dimension of the Hilbert space significantly. The unconstrained oscillator states are labeled by two integers \((n_x, n_y)\), and there are \((\text{countable infinity})^2\) states in the Hilbert space. After the constraint restricts the states to \(n_y = 0\) only, the number of admissible states drops to \((\text{countable infinity})^1\). Roughly speaking, this is the square root of the original number of states.

Something similar happens in the present case. Consider the set of kinematic basis states \(\{Y_{L_x m_x} Y_{L_y m_y}; \forall m_x, m_y\}\). Before imposition of the constraint, there are

\[
(2L_x + 1)(2L_y + 1)
\]
unconstrained states
in this set. After imposition of the constraint, the set contains only those linear combinations which are eigenstates of the constraint with eigenvector zero. From the lemma, the volume and constraint operators have exactly the same number of zero eigenvalues. The necessary condition for the volume operator to have a zero was worked out in the previous paper [1]. The eigenvectors are sums of the form eq. (16): sums over D with F held fixed. There will be a zero eigenvector iff the number of D’s in the sum is odd. Therefore I can compute the number of zero eigenvectors by constructing the following rectangular lattice of points. Make the x-axis \(m_x\) and y-axis \(m_y\); draw one point with coordinates \((m_x, m_y)\), for each allowed pair of m’s, \((2L_x + 1)(2L_y + 1)\) points in all.

In this lattice, diagonals at 135 degrees are lines of constant \(F = (m_x + m_y)/2\), while diagonals at 45 degrees are lines of constant \(D = (m_x - m_y)/2\). The number of zeros is the number of 135 degree diagonals which contain an odd number of points (corresponding to an odd number of D’s in the sum over D with F held fixed). After some experimenting with specific examples, one arrives at the general formula for the number of diagonals with an odd number of points = number of states surviving the constraint.

\[2 \max(L_x, L_y) + 1\] constrained states

We have gone from order \((2L)^2\) states to order \((2L)^1\) states. This is roughly a square root, similar to the oscillator example and therefore not unreasonable.

In the previous paper I computed the zero eigenvectors of the volume operator [1]. Given that result, and the lemma, it is possible to compute the zero eigenvectors of the constraint. From the previous paper, the zero eigenvectors of the volume operator have components

\[
c(D; \lambda = 0) = N \sqrt{f(D-1)/f(D)};
\]

\[
f(D) = \left( \frac{L_y - F + D}{2} \right)! \left( \frac{L_y + F - D}{2} \right)! \times \left( \frac{L_x - F - D}{2} \right)! \left( \frac{L_x + F + D}{2} \right)!, \tag{22}
\]

for \(D = \max D, \max D - 2, \max D - 4, \ldots, \min D;\) and \(c(D) = 0\) otherwise. From the lemma, eq. (20), the components of the
corresponding zero eigenvector of the constraint are just
\[ e(D; \lambda = 0) = (i)^D c(D; \lambda = 0). \]
(The two states \( e(\pm; \lambda = 0) \) in eq. (20) give the same zero eigenvector, except perhaps for an overall minus sign, because \( c(D; \lambda = 0) \) is zero for every other value of \( D \).

These zero eigenvectors illustrate a point made earlier, that volume and constraint operators have (the same eigenvalues but) very different eigenvectors. Zero constraint eigenvalue does not imply zero volume eigenvalue. An eigenvector of the constraint operator with zero eigenvalue is a linear combination of eigenvectors of the volume operator, with eigenvalues which are in general different from zero. Although the volume operator and constraint have the same eigenvalue spectrum, they do not commute.

I return to the point made in the introduction: in classical or quantum field theory, the single polarization case is simpler; whereas for spin networks, the single polarization case is harder. For example, experience with field theory might suggest using eigenfunctions of the constraint as a basis; however, these eigenfunctions are not eigenfunctions of the volume, and the spin network Hamiltonian would be hard to evaluate in that basis.

There is a further complication. The constraint I have been studying in this paper is a local constraint. The Hamiltonian acting on an eigenfunction of this local constraint will produce another local constraint eigenfunction only if the state satisfies the spin network version of eq. (2). As argued in the introduction, that constraint is likely to be non-local. Given the problems with choosing a basis and the existence of a non-local constraint, it would not be surprising if solutions for a general polarization were obtained first, before solutions for a single polarization.

There may be ways around these complications, however. In the classical limit, the spin vectors associated with the \( A_x \) and \( A_y \) holonomies will have sharp expectation values with minimal uncertainties. The local constraint implies that the two spin vectors are perpendicular; while the volume eigenvalue is given by the cross product of the two spin vectors. It is not possible for the classical state to be simultaneously an exact eigenfunction of both the volume operator and the local constraint; but in the classical limit it should be possible to find states which are eigenfunctions of both
constraints to very good approximation. Further, the non-local con-
straint, when combined with the Hamiltonian, may simplify the
Hamiltonian. It is too early to tell. Much work remains to be
done.
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