The Skyrmion limit of 
the Nambu–Jona-Lasinio soliton

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ABSTRACT

The special role of the isoscalar mesons ($\sigma$ and $\omega$) in the NJL soliton is discussed. Stable soliton solutions are obtained when the most general ansatz compatible with vanishing grand spin is assumed. These solutions are compared to soliton solutions of a purely pseudoscalar Skyrme type model which is related to the NJL model by a gradient expansion and the limit of infinitely heavy (axial-) vector mesons.

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Introduction

The theory of strong interactions, Quantum Chromo Dynamics (QCD), reduces for a large number of colors $N_c$ to an effective theory of weakly interacting meson fields$^1$. Furthermore baryons emerge as soliton solutions in this effective meson theory$^2$. On the other hand, simplified models with quark degrees of freedom can also possess solitonic solutions, as for example the Nambu–Jona-Lasinio (NJL) model$^3$. Selfconsistent soliton solutions have been found$^4$-$^6$ where the calculations had been restricted to the chiral field. It has then been observed that a straightforward extension of this model to allow for space dependent scalar fields leads to a collapse of the soliton$^7, 8$. Simultaneously the NJL model containing the chiral as well as (axial-) vector meson fields was shown to possess stable soliton solutions$^9$-$^13$. In case the scalar degrees of freedom are incorporated in the latter type of model, the repulsive character of the $\omega$ meson is supposed to prevent the soliton from collapsing.

The aim of the present investigations is twofold. First, we will discuss the special role of the isoscalar meson fields related to the above mentioned collapse. With all meson fields included in the soliton calculations the valence quarks are strongly bound and they indeed join the Dirac sea. Therefore the topological current, which arises in leading order in the gradient expansion of the vacuum part of the baryon current, becomes a suitable representation of the full baryon current. This fact supports the Skyrmion picture of baryons. For these investigations we are going to employ a static energy functional which has recently been motivated by studying the relevant analytic properties under Wick rotation$^14$. For the inclusion of all mesons similar calculations have previously been performed using a somewhat different definition of the energy functional$^15$. As the Minkowski energy functional is ambiguous if the $\omega$ meson is treated non-perturbatively$^14$ it is, of course, interesting to study the question of stability assuming an alternative definition for the energy functional. Combining the results of the two approaches should provide a general picture of the situation.

On the other hand it is known that a Skyrme type model can be derived from the bosonized NJL model$^16$. This suggests to compare the self-consistent soliton solution of the NJL model to the soliton solution of the corresponding Skyrme model. Such a comparison constitutes the second issue of this letter.

Description of the model

As starting point we assume the bosonized version of the two-flavor NJL model$^{17}$:

\begin{align}
\mathcal{A} &= \mathcal{A}_F + \mathcal{A}_m, \\
\mathcal{A}_F &= \text{Tr} \log(i\hat{D}) = \text{Tr} \log \left(i\partial + V + \gamma_5 A - (P_R \Sigma + P_L \Sigma^\dagger)\right), \\
\mathcal{A}_m &= \int d^4x \left( -\frac{1}{4g_1} \text{tr}(\Sigma^\dagger \Sigma - \hat{m}_0 (\Sigma + \Sigma^\dagger) + \hat{m}_0^2) + \frac{1}{4g_2} \text{tr}(V^\mu V_\mu + A^\mu A_\mu) \right).
\end{align}

Here $V^\mu = \sum_{a=0}^3 V^a_{\mu} \tau^a / 2$ and $A^\mu = \sum_{a=0}^3 A^a_{\mu} \tau^a / 2$ denote the vector and axial vector meson fields. The matrices $\tau^a / 2$ denote the generators of the flavor group ($\tau^0 = 1$). $P_{R,L} = (1 \pm \gamma_5) / 2$ are the projectors on right– and left–handed quark fields, respectively. The complex field $\Sigma$ describes the scalar and pseudoscalar meson fields $S_{ij} = \sum_{a=0}^3 S^a \tau^a_{ij} / 2$ and $P_{ij} = \sum_{a=0}^3 P^a \tau^a_{ij} / 2$:

\begin{equation}
\Sigma = S + iP = \Phi U
\end{equation}

where we have made use of a polar decomposition. This defines the chiral radius $\Phi$ as well as the chiral field $U = \exp(i\Theta)$ with $\Theta$ being the chiral angle. The current quark
mass matrix \( \hat{m}_0 = \text{diag}(m_0^u, m_0^d) \) only appears in the mesonic part of the action, \( \mathcal{A}_m \). The Schwinger–Dyson (or gap) equation relates \( \hat{m}_0 \) to the constituent quark mass matrix \( \hat{m} = \text{diag}(m^u, m^d) \). For the ongoing discussion we will adopt the isospin limit \( m_0^u = m_0^d =: m_0 \) which also implies \( m^u = m^d =: m \). As the NJL model is not renormalizable it needs regularization. This introduces one more parameter, the cut–off \( \Lambda \). Hence the model contains four parameters: \( m_0, g_1, g_2 \) and \( \Lambda \). As ingredients from the meson sector we use the pion decay constant \( f_\pi = 93\text{MeV} \) and the masses of the pion and the \( \rho^- \)–meson, \( m_\pi = 135\text{MeV} \) and \( m_\rho = 770\text{MeV} \), respectively. Then the Schwinger–Dyson equation allows one to choose the constituent quark mass \( m \) as the only free parameter and express \( m_0, g_1, g_2 \) and \( \Lambda \) in terms of it. We employ Schwinger’s proper time description\,[18] which has the desired feature of preserving gauge symmetry. Then the fermion determinant, \( \mathcal{A}_F \), may be decomposed into real (\( \mathcal{A}_R \)) and imaginary (\( \mathcal{A}_I \)) parts. Only \( \mathcal{A}_R \) is divergent. The proper time procedure is applicable to \( \mathcal{A}_R \) since \( \hbar \mathcal{D}_E \mathcal{P}_E \) is positive definite

\[
\mathcal{A}_R = \frac{1}{2} \text{Tr} \log \left( \mathcal{D}_E \mathcal{P}_E \right) \rightarrow -\frac{1}{2} \int_1^{\infty} \frac{ds}{s} \exp \left( -s \mathcal{D}_E \mathcal{P}_E \right).
\]

(5)

Although the imaginary part is UV finite, the proper time regularization may be applied as well\,[12, 13]. Note, however, that regularizing \( \mathcal{A}_I \) or not yields qualitatively different models and different results may occur.

The definition of the Euclidean Dirac one-particle Hamiltonian \( h \) via

\[
i\beta \mathcal{D}_E = -\partial_t - h
\]

(6)

is useful in the context of studying soliton solutions since one has \([\partial_t, h] = 0 \) for static fields. In the limit of large Euclidean times, \( T \to \infty \), the temporal part of the trace may be carried out straightforwardly by calculating Gaussian integrals. To evaluate the trace over the remaining degrees of freedom we require the eigenvalues of \( h \). As \( h \) is non–Hermitean, we have to distinguish between left and right eigenstates

\[
h|\Psi_\nu\rangle = \epsilon_\nu|\Psi_\nu\rangle \quad \langle \bar{\Psi}_\nu|h = \epsilon_\nu^*\langle \bar{\Psi}_\nu| \quad \text{i.e. \( h^\dagger|\Psi_\nu\rangle = \epsilon_\nu^*|\Psi_\nu\rangle \)}
\]

(7)

with the normalization condition \( \langle \bar{\Psi}_\mu|\Psi_\nu\rangle = \delta_{\mu\nu} \). The functional trace may then be expressed as sums over the one–particle energies \( \epsilon_\mu \). The resulting expression motivates the following definition for the Minkowski energy functional\,[14]:

\[
E[\varphi] = E^R_{\text{val}} + E^I_{\text{val}} + E^R_{\text{vac}} + E^I_{\text{vac}} + E_m - E^R_{\text{vac}}[\varphi_{\text{vac}}]
\]

(8)

where for simplicity we have generically labeled the meson fields by \( \varphi \). The contribution of the trivial vacuum has been subtracted whereby the configuration \( \varphi_{\text{vac}} \) corresponds to \( \Sigma = m \) while the (axial-) vector meson fields are set to zero. \( E[\varphi] \) receives contributions from the explicit occupation of the valence quark level\,[19, 20]

\[
E^R_{\text{val}} = N_C \sum_\nu \eta_\nu|\epsilon^R_\nu|, \quad E^I_{\text{val}} = N_C \sum_\nu \eta_\nu \text{sgn}(\epsilon^R_\nu)\epsilon^I_\nu,
\]

(9)

the polarized Dirac sea

\[
E^R_{\text{vac}} = \frac{N_C}{4\sqrt{\pi}} \sum_\nu |\epsilon^R_\nu| \Gamma\left(-\frac{1}{2}, \left(\epsilon^R_\nu/\Lambda\right)^2\right), \quad E^I_{\text{vac}} = \frac{-N_C}{2} \sum_\nu \epsilon^I_\nu \text{sgn}(\epsilon^R_\nu)
\]

(10)

and the mesonic part, \( E_m \), which stems from \( \mathcal{A}_m \). In eqn. (1) \( \eta_\nu \) denote the occupation numbers of the valence quark orbits. They are adjusted to provide unit baryon number:
1 = \sum \left( \eta_\mu - \frac{1}{2} \text{sgn}(\epsilon_\mu^R) \right)$. In eqn. (11) we have displayed the vacuum part of the energy for the case that the imaginary part is not regularized; employing the regularization for the imaginary part yields \cite{12, 13}

$$E^I_{\text{vac}} = \frac{-NC}{2\sqrt{\pi}} \sum \epsilon_\nu \text{sgn}(\epsilon_\nu^R) \Gamma\left(\frac{1}{2}, (\epsilon_\nu^R/\Lambda)^2\right).$$

(11)

It is important to note that the different treatments of real and imaginary parts under regularization destroys the analytical structure of the action in the time components of the (axial–) vector meson fields. As already indicated this prohibits a unique extraction of a Minkowski energy functional. In ref. \cite{14}, however, it has been demonstrated that the definition (8) is motivated from the regularized energy functional in Euclidean space. This motivation is based on the fact that the energy eigenvalue, \(\epsilon_\mu\), may well be approximated by a sum consisting of two functionals of the meson fields. One of these two functionals is almost independent of \(\omega\) while the other has only a linear dependence on \(\omega\), see tables 2.1 and 2.2 of ref. \cite{14}. Furthermore, the energy functional (8) possesses the correct behavior under global flavor singlet transformations and it also yields the current field identities. A unique Minkowski energy functional can only be obtained when expanding the fermion determinant in terms of the time components of the (axial–) vector meson fields while treating all other fields as non–perturbative background fields (cf. ref. \cite{14}).

Let us next construct the most general static Euclidean Dirac Hamiltonian in the grand spin zero sector. The grand spin operator is defined as the sum

$$G = l + \frac{\sigma}{2} + \frac{\tau}{2}$$

with \(l\) being the orbital angular momentum, \(\sigma/2\) the spin and \(\tau/2\) the isospin operators. For the chiral field the well–known hedgehog ansatz

$$U(r) = \exp\left(i\tau \cdot \hat{r}\Theta(r)\right)$$

(12)

satisfies the condition of vanishing grand spin while for the scalar field this can only be accommodated by a radial function

$$\Phi(r) = m\phi(r).$$

(13)

For the (axial–) vector meson fields we impose the grand spin symmetric ansätze:

$$V_\mu^0 = \omega(r)\delta_{\mu4}, \quad V_4^a = 0, \quad V_i^a = \epsilon^{aik}\hat{r}^k G(r),$$

$$A_\mu^0 = 0, \quad A_4^a = 0, \quad A_i^a = \hat{r}^i\hat{r}^a F(r) + \delta^a H(r),$$

(14)

where the indices \(a, i\) and \(k\) take the values 1, 2 and 3. Then the Euclidean Dirac Hamiltonian reads

$$h = \alpha \cdot p + i\omega(r) + m\phi(r)\beta(\cos\Theta(r) + i\gamma_5 \tau \cdot \hat{r}\sin\Theta(r))$$

$$+ \frac{1}{2}(\alpha \times \hat{r}) \cdot \tau G(r) + \frac{1}{2}(\sigma \cdot \hat{r})(\tau \cdot \hat{r}) F(r) + \frac{1}{2}(\sigma \cdot \tau) H(r).$$

(15)

With the ansätze (12) - (14) the mesonic part of the energy is given by

$$E_m = 4\pi \int dr r^2 \frac{m^2}{2m_0} \left[ m(\phi^2(r) - 1) + 2m_0\phi(r)(1 - \cos \Theta(r)) \right]$$

$$+ \frac{m^2}{2m_0^2} \left[ G^2(r) + \frac{1}{2} F^2(r) + F(r) H(r) + \frac{3}{2} H^2(r) - 2\omega^2(r) \right].$$

(16)
Here $g_V$ is the universal vector coupling constant, $g_V = \left(\frac{1}{8\pi^2} \Gamma(0, m^2_{\pi^2})\right)^{-1/2}$ which is related to the coupling constant $g_2$ via the $\rho$-meson mass $g^2_V = 4g_2m^2_{\pi^2}[17]$.

The equations of motion for the meson fields are derived by extremizing the static energy functional [8]: $\delta E[\varphi]/\delta \varphi = 0$. In addition to the equations of motion listed in ref. [13] we obtain for the scalar field

$$\phi(r) = \frac{m_0}{m} \cos \Theta(r) - \frac{m_0 N_c}{m^2 f^2_\pi} \text{tr} \int \frac{d\Omega}{4\pi} \left( \cos \Theta(r) + i\gamma_5 \tau \cdot \hat{r} \sin \Theta(r) \right) \rho(r, r). \quad (17)$$

with the scalar density matrix $\rho(x, y)$ (which is bilinear in the eigenfunctions of $h$) being defined in ref. [13]. The formal structure of the other equations of motion is not affected by the presence of the scalar field $\phi$.

In order to diagonalize $h$ and to solve the equations of motion $\delta E[\varphi]/\delta \varphi = 0$ we discretize the Hamiltonian (15) in a suitable basis using a spherical cavity of finite radius $D$. Typical values are $D = 4\ldots 6\text{fm}$. For details about the numerical method see ref. [21]. The numerical solution to $\delta E/\delta \varphi = 0$ on the whole range $0 \leq r \leq D$ is plagued by finite size effects[13]. In order to avoid these effects we substitute the exact solution to $\delta E/\delta \varphi = 0$ by the phenomenologically motivated large distance behavior of the meson fields for $r \geq r_m (D/4 \leq r_m \leq D/2)$. For the chiral angle $\Theta$ this can be extracted from the solution of the free (P-wave) Klein-Gordon equation:

$$\Theta(r) \rightrightarrows \tilde{\Theta}(r) \sim e^{-m_\pi r} \left( m_\pi + \frac{1}{r} \right). \quad (18)$$

For the other fields we make use of their relation to the chiral angle in a local approximation yielding the asymptotic behavior\footnote{In order to accommodate the vacuum values of the fields at $r = D$ we furthermore multiply appropriate factors to the RHS of eqns. (18) and (19): $\varphi \rightarrow \varphi \tanh(a(1 - \frac{r}{D}))$ with $a \approx 10 - 15$.}

$$G(r) \sim \tilde{\Theta}^2(r),$$
$$\omega(r) \sim \tilde{\Theta}'(r) \tilde{\Theta}^2(r),$$
$$\phi(r) - 1 \sim \tilde{\Theta}^2(r) + ae^{-2mr},$$
$$F(r), H(r) \sim \tilde{\Theta}'(r). \quad (19)$$

The constants of proportionality, which are omitted in eqns. (18,19), are fixed by making contact with the exact solution to $\delta E/\delta \varphi = 0$ at $r = r_m$. For the scalar field we fix the additional constant $a$ by requiring the derivative of $\phi(r)$ to be continuous.

The role of the isoscalar meson fields

In some recent studies of the NJL model [4, 8] it has been shown that abandoning the chiral circle condition $S^2 + P^2 = m^2$ leads to the collapse of the soliton configuration if no additional mechanism to preserve the stability is incorporated (e.g. including a forth-order term in the scalar meson field [22, 23] or constraining the regularized baryon number [24]). It has already been demonstrated that in an alternative definition of the (ambiguous) non-perturbative energy functional the incorporation of the $\omega$ meson provides a further stabilisation mechanism[15]. This is intuitively clear because the $\omega$ meson is of repulsive character.

As already remarked in the preceding section regularizing the imaginary part of the action or not leads to quite different models. Taking a finite cut-off for the imaginary part
Table 1: The soliton energy $E$ as well as its Dirac sea and mesonic contributions $E_{\text{vac}}$ and $E_m$ for different values of the constituent quark mass $m$. Also shown is the energy of the ‘dived’ level $\epsilon_{\text{val}}$ and the axial charge $g_A$.

| $m$ (MeV) | 350   | 400   | 500   |
|-----------|-------|-------|-------|
| $E$ (MeV) | 1125  | 1091  | 1022  |
| $E_{\text{vac}}^R$ (MeV) | 2271  | 1337  | 883   |
| $E_{\text{vac}}^I$ (MeV)  | 177   | 206   | 223   |
| $E_m$ (MeV) | -1323 | -451  | -83   |
| $\epsilon_{\text{val}}^R/m$ | -0.28 | -0.51 | -0.72 |
| $\epsilon_{\text{val}}^I/m$ | 0.15  | 0.15  | 0.12  |
| $g_A$ | 0.41  | 0.36  | 0.32  |

(11) also yields a regularized baryon number. As a matter of fact this regularized baryon number density vanishes for the field configuration associated with the above mentioned collapse[24] indicating that the collapse is due to the transition from the baryon number $N_B = 1$ sector to the $N_B = 0$ sector. Since the baryon number density represents the source of the $\omega$ field nothing prevents the $\omega$ field from being zero when the imaginary part of the action is regularized and the scalar field is allowed to be space dependent. Consequently, the collapse also appears if all vector mesons are included and $A_I$ is regularized. The ongoing explorations will therefore be constrained to the case when $A_I$ is not regularized. Then a non-vanishing source for the $\omega$ field is present and the stability of the soliton depends on the strength of the $\omega$ field. It has been noted that the numerical calculations will pretend to a “pseudo-stable solution” if too small a basis for diagonalizing $h$ in momentum space is adopted[8]. By varying the size of this basis we have ensured that our solutions are not subject to this “pseudo-stability”. At low constituent quark masses, $m$, the trivial minimum with $N_c$ free and unbound valence quark orbits occupied is energetically favored compared to the soliton configuration. Actually a lower bound exists for $m$ below which the equation of motion are solved by this trivial configuration only. Numerically we obtain for this bound approximately 330MeV which it is somewhat larger ($\sim 410$MeV) when neither $\rho$ nor $a_1$ fields are present. It should be stressed that this instability is completely different in character from the above mentioned collapse. In case of the collapse the valence quark is strongly bound and its energy eigenvalue tends to $-m$.

For stable solitons the valence quark is strongly bound ($\epsilon_{\text{val}}^R \lesssim -m/4$) and the chiral radius $\phi$ lies near the chiral circle $\phi(r) = 1$ (cf. fig. 1). Then the scalar-isoscalar meson field has only minor influence on the energy and on the other meson fields. In table 1 the soliton energy $E$, its Dirac sea and mesonic contributions, $E_{\text{vac}}$ and $E_m$, as well as the energy of the valence quark level ($\epsilon_{\text{val}}^R,I$) are shown for different values of the constituent quark mass $m$. Also displayed is the axial charge $g_A$ obtained directly from the axial vector field[13], $g_A = - (2\pi/g_2) \int dr r^2 \frac{1}{3} (H(r) + F(r))$. The relative contributions from the Dirac sea and the mesonic part of the scalar-isoscalar meson field depends on the choice of the point $r_m$ where the tail for the meson field is fitted. Nevertheless the total energy and the valence quark energy of the soliton are stable in a wide range $D/4 < r_m < D/2$. To be definite, we choose $r_m$ such that $\partial_r \phi(r)|_{r_m} = 0$ which yields $r_m \approx 0.35 \cdots 0.375D$.

Comparison with the Skyrme model

Since the energy of the valence quark orbit is negative the baryon number can well
be approximated by the topological current. For that reason the NJL soliton strongly supports Witten’s conjecture that baryons may be described as solitons within purely mesonic models. We will now study the explicit connection to the Skyrme model. In a first step we assume a gradient type expansion for the pseudoscalar field. Secondly, we adopt the static limit for all other mesons, i.e. the inverse propagators are approximated by the corresponding mass terms. This allows one to integrate out these mesons yielding an extended Skyrme lagrangian:

\[ \mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_{SB}, \]

\[ \mathcal{L}_2 = -\frac{f^2}{4} \text{tr} L^\mu L^\mu, \]

\[ \mathcal{L}_4 = \frac{\cos \chi}{32 e^2} \text{tr} [L^\mu, L^\nu] [L^\mu, L^\nu] + \frac{\sin \chi}{24 e^2} (\text{tr} (L^\mu L^\nu))^2, \]

\[ \mathcal{L}_6 = -\frac{e^6}{2} B^\mu B^\mu, \]

\[ \mathcal{L}_{SB} = \frac{f^2 m^2}{4} \text{tr} \left(U + U^\dagger - 2\right). \]  

(20)

Here \( L^\mu = U^\dagger \partial_\mu U \) and \( B^\mu = (1/24\pi^2) e^\mu_\nu_\rho \text{tr} L^\nu L^\rho L^\lambda \) denote the left Maurer-Cartan form and the topological current, respectively. In the static limit the parameters \( e, e_6 \) and \( \chi \) are obtained to be

\[ \frac{1}{e^2} = \sqrt{\left(\frac{2a - 1}{g_\nu a}\right)^4 + \left(\frac{\sqrt{3} f_\pi}{4m}\right)^4} \quad \text{with} \quad a = 1 + \frac{m^2}{6m^2}, \]

\[ e_6 = \frac{N_c g_\nu}{2m_\rho}, \]

\[ \tan \chi = \left(\frac{\sqrt{3} f_\pi}{4m} \frac{g_\nu a}{|2a - 1|}\right)^2. \]  

(21)

For a given chiral field \( U \) the soliton energy or mass obtained from the Lagrangian \( \mathcal{L} \) is given by

\[ M = \int d^3r \left[ \frac{f^2}{2} \left( \Theta^2(r) + \frac{2 \sin^2(\Theta(r))}{r^2} \right) + \frac{1}{e^2} \sin^2(\Theta(r)) \left( \frac{\Theta^2(r)}{r^2} + \frac{\sin^2(\Theta(r))}{2r^2} \right) \cos \chi \right. \]

\[ - \frac{2}{3e^2} \left( \Theta^2(r) + \frac{2 \sin^2(\Theta(r))}{r^2} \right) \sin \chi + \frac{e^2}{8\pi^4} \Theta^2(r) \frac{\sin^4(\Theta(r))}{r^4} + m^2 f^2_\pi \left( 1 - \cos(\Theta(r)) \right) \bigg] \].

(22)

If we perform slow rotations \( A(t) \in SU(2) \) on the static soliton \( U(t) = A(t) U A^\dagger(t) \) we can derive the following Hamiltonian in the space of collective angular degrees of freedom

\[ H = M + \frac{J^2}{2\lambda} \]  

(23)

with \( \lambda \) being the moment of inertia given by

\[ \lambda = \frac{2}{3} \int d^3r \sin^2(\Theta(r)) \left[ \frac{f^2}{e^2} + \frac{\cos \chi}{e^2} \left( \frac{\Theta^2(r)}{r^2} + \frac{\sin^2(\Theta(r))}{r^2} \right) \right. \]

\[ - \frac{2 \sin \chi}{3e^2} \left( \frac{\Theta^2(r)}{r^2} + \frac{2 \sin^2(\Theta(r))}{r^2} \right) + \frac{e^2}{4\pi^4} \Theta^2(r) \frac{\sin^4(\Theta(r))}{r^4} \bigg]. \]  

(24)

\(^a\)In the NJL model the mass of the scalar meson is given by \( 2m \).

\(^d\)For notation see ref. \[25\]
Table 2: Comparison of the mass $M$, eq. (22), the masses of the nucleon ($N$) and Δ-resonance (25) for the self-consistent NJL model and the soliton of the extended Skyrme model (20). Shown are the results for two different values of the constituent quark mass $m$. The physical $\rho$ meson mass is adopted, $m_\rho = m^{ph}_\rho$.

| $m$ (MeV) | 400 | 500 |
|-----------|-----|-----|
| soliton   | eq. (20) | NJL | eq. (20) | NJL |
| $e$       | 4.65 | 5.41 |
| $e_6$ ($10^{-3}$/MeV) | 12.7 | 13.7 |
| $\chi$   | 0.221 | 0.191 |
| $M$ (MeV) | 1760 | 3519 | 1714 | 4124 |
| $M_N$ (MeV) | 1812 | 3643 | 1770 | 4261 |
| $M_\Delta$ (MeV) | 2023 | 4139 | 1995 | 4809 |
| $M_\Delta - M_N$ (MeV) | 210 | 496 | 225 | 548 |

The mass for the $N$ and $\Delta$ might be extracted from eq. (23):

$$M_N = M + \frac{3}{8\lambda}$$
$$M_\Delta = M + \frac{15}{8\lambda}.$$

(25)

For comparison we have calculated the chiral angle $\Theta(r)$ (cf. fig. 2): on one hand the self-consistent NJL soliton and on the other the extended Skyrmion obtained from the Euler-Lagrange equation associated with the Skyrme Lagrangian (20). We have then calculated the $N$ and $\Delta$ masses (25) for the NJL soliton chiral field as well as for the Skyrmion. As can be seen from fig. 2 using the physical $\rho$ meson mass $m_\rho = m^{ph}_\rho = 770$ MeV leads to quite different profiles for the chiral angle $\Theta(r)$. Accordingly, the soliton mass $M$ and therefore the $N$ and $\Delta$ masses are quite different in the static approximation (cf. tab. 2). Note that the soliton mass $M$ is significantly larger than the self-consistent NJL soliton energy (see table 1). This indicates that the physical $\rho$ meson mass is too low for the validity of the static limit. Indeed if we increase the $\rho$ meson mass successively the two chiral angles approach piece by piece (cf. fig. 2). For $m_\rho = 8m^{ph}_\rho$ the soliton mass $M$ obtained by substituting the self-consistent chiral angle of the NJL soliton into the static approximation and the Skyrmion are in very good agreement as can be seen from tab. 3. Furthermore, this soliton mass is very close to the corresponding self-consistent NJL soliton energy which is $E = 1162$ MeV ($E = 1026$ MeV) for $m_\rho = 8m^{ph}_\rho$ and $m = 400$ MeV ($m = 500$ MeV), respectively.

**Conclusions**

To summarize, we have shown that for an appropriate definition of the NJL model energy functional the inclusion of all meson fields in a way compatible with the hedgehog ansatz leads to stable soliton solutions in a wide range of constituent quark masses. We would like to put emphasis on the fact that the chiral radius deviates only mildly from its vacuum expectation value when all meson fields are included in the evaluation of the self-consistent soliton. This justifies the frequently adopted approximation $\phi(r) \equiv 1$.

Including all meson fields with vanishing grand spin the valence quarks are strongly bound and the baryon number is completely carried by the polarized Dirac sea. This indicates that baryons can be described as purely mesonic topological solitons like the
Table 3: Same as table 2 for $m_\rho = 8m_{ph}^{\rho}$

| $m$ (MeV) | 400  | 500  |
|------------|------|------|
| soliton    | eq. (20) NJL | eq. (20) NJL |
| $e$         | 5.17 | 6.46 |
| $e_6$ ($10^{-3}$/MeV) | 2.53 | 3.15 |
| $\chi$     | 0.274 | 0.274 |
| $M$ (MeV)  | 1160 | 1162 | 1008 | 1013 |
| $M_N$ (MeV) | 1326 | 1328 | 1235 | 1272 |
| $M_\Delta$ (MeV) | 1989 | 1991 | 2140 | 2307 |
| $M_\Delta - M_N$ (MeV) | 663  | 663  | 906  | 1035 |

skyrmion. This result has motivated the comparison between the solitons of the NJL model and a Skyrme type model. The latter has been related to the NJL model in the limit of large (axial-) vector meson masses and a gradient expansion for the pseudoscalar field. It has turned out that these two models possess different soliton solutions for the physical values of the vector meson mass ($m_{ph}^{\rho}$). However, as $m_\rho$ is increased these two solitons become similar in shape and size. Reasonable agreement is achieved for $m_\rho \geq 8m_{ph}^{\rho}$. This indicates that the kinetic term for the (axial-) vector mesons, here represented by the fermion determinant, carries important information about the meson fields. We have thus collected some support for Skyrme type models from a microscopic model for the quark flavor dynamics. One might argue that purely mesonic models with explicit (axial-) vector degrees of freedom might be more feasible for the computation of nucleon properties. However, as one is interested in the explicit quark structure of baryons it is unremitting to consider a microscopic model like the one of Nambu and Jona-Lasinio.
References

[1] G. ’t Hooft, Nucl. Phys. B72 (1974) 461.
[2] E. Witten, Nucl. Phys. B160 (1979) 57.
[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.
[4] H. Reinhardt and R. Wünsch, Phys. Lett. B215 (1988) 577.
[5] T. Meissner, F. Grümmer and K. Goeke, Phys. Lett. B227 (1989) 296.
[6] R. Alkofer, Phys. Lett. B236 (1990) 310.
[7] T. Watabe and H. Toki, Prog. Theor. Phys. 87 (1992) 651.
[8] P. Sieber, Th. Meißner, F. Grümmer and K. Goeke, Nucl. Phys. A547 (1992) 459.
[9] R. Alkofer and H. Reinhardt, Phys. Lett. B244 (1991) 461.
[10] R. Alkofer, H. Reinhardt, H. Weigel and U. Zückert, Phys. Rev. Lett. 69 (1992) 1874.
[11] C. Schüren, E. Ruiz Arriola and K. Goeke, Phys. Lett. B287 (1992) 283.
[12] R. Alkofer, H. Reinhardt, H. Weigel and U. Zückert, Phys. Lett. B298 (1993) 132.
[13] U. Zückert, R. Alkofer, H. Reinhardt and H. Weigel, Nucl. Phys. A570 (1994) 445.
[14] H. Weigel, U. Zückert, R. Alkofer and H. Reinhardt, “On the analytic properties of chiral solitons in the presence of the $\omega$ meson”, Tübingen University preprint, UNITU-THEP-13/1994, [help-ph/9407304], July 1994.
[15] E. Ruiz Arriola, F. Döring, C. Schüren and K. Goeke, J. Phys. G20 (1994) 399.
[16] H. Reinhardt and B.V. Dang, Nucl. Phys. A500 (1989) 563.
[17] D. Ebert and H. Reinhardt, Nucl. Phys. B271 (1986) 188.
[18] J. Schwinger, Phys. Rev. 82 (1951) 664.
[19] H. Reinhardt, Nucl. Phys. A503 (1989) 825.
[20] R. Alkofer, H. Reinhardt and H. Weigel, “Baryons as Chiral Solitons in the Nambu–Jona-Lasinio Model”, in preparation.
[21] U. Zücker, R. Alkofer and H. Weigel, “Self-consistent solution to a complex fermion determinant with space dependent fields”, UNITU-THEP-2/1994, Computer Physics Communications in press.
[22] C. Weiss, R. Alkofer and H. Weigel, Mod. Phys. Lett. A8 (1993) 79.
[23] Th. Meißner et al., Phys. Lett. B299 (1993) 183.
[24] J. Schlienz, H. Weigel, H. Reinhardt and R. Alkofer, Phys. Lett. B315 (1993) 6.
[25] D.E.L. Pottinger and E. Rathske, Phys. Rev. D33 (1986) 2448.
Figure captions

**Figure 1**
The chiral radius field $\phi(r)$ for $m = 400\text{MeV}$.

**Figure 2**
The chiral angle $\Theta(r)$ of the NJL model and the extended Skyrme model for $m = 400\text{MeV}$.
This figure "fig1-1.png" is available in "png" format from:

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This figure "fig2-1.png" is available in "png" format from:

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