Estimation of Weibull parameters from parameters of initial distribution of flaw size

C Wakabayashi¹, K Yasuda¹ and T Shiota¹

¹Department of Metallurgy and Ceramics Science, Graduate School of Science and Engineering, Tokyo Institute of Technology, 2-12-1-S7-14, Ookayama, Meguro-ku, Tokyo, Japan

E-mail: wakabayashi.c.aa@m.titech.ac.jp

Abstract. The distribution of the largest flaw size is derived from the initial distribution of flaw size based on extreme value statistics, and also the distribution of fracture origin size is given by transforming Weibull distribution by fracture mechanical relation. These two distributions are equivalent under uniaxial loading. By using this relation, their parameters are related each other and Weibull parameters are estimated from the parameters of the initial distribution of flaw size and the number of links.

1. Introduction
In ceramics processing, flaws such as pores, large grains and cracks are generated. And they lead to material fracture. The flaws’ shape and size are not uniform and the fracture strength shows wide scattering. Therefore, in order to use ceramics as structural components, reliability analysis is essential. For this purpose, fracture strength test is conducted with many specimens. And it is assumed that the fracture strength follows Weibull distribution and Weibull parameters are estimated. However, this method takes long time and low cost-efficiency, so that more economical way including non-destructive inspection is in need of reliability analysis.

The authors derived the exact distribution of fracture strength from the initial distribution of flaw size via the distribution of the largest flaw size. [1-3] Parameters of the distribution of the largest flaw size are parameters of the initial distribution of flaw size and the number of links. On the other hand, Matsuo et al. [4] obtained the distribution of fracture origin size by transforming Weibull distribution by fracture mechanical relation. These two distributions are equivalent. Therefore, the Weibull parameters are related to the parameters of the initial distribution. By using only the parameters of the initial distribution of flaw size, the distribution of fracture strength can be estimated theoretically. Due to this method, the low cost efficiency of the current reliability analysis method is solved.

2. Theory
2.1. Derivation of the exact distribution of fracture strength based on the initial distribution of flaw-size
In extreme value statistics, there are two kinds of the initial distribution. The difference is convergence of the probability. The probability of exponential type rapidly decreases when the random valuable increases. On the other hand, Cauchy type’s convergence is comparatively slower. In general, the
number of small flaw is far larger than the number of large flaw in brittle materials. Therefore, the
initial distribution of flaw size should be exponential type distribution. Gamma distribution, one of
exponential type distribution, is widely used for its flexible shape change by their parameters. In this
paper, gamma distribution is used as the initial distribution of flaw size $\Phi_i$. The probability density
function of gamma distribution is given as follows,

$$\phi_i(x) = \frac{1}{\Gamma(\nu)} \alpha^{\nu} x^{\nu-1} \exp(-\alpha x)$$  \hspace{1cm} (1)

, where $x$ is flaw size, $\alpha$ and $\nu$ are scale and shape parameters, respectively. The validation of gamma
distribution for the initial distribution of flaw size is demonstrated by applying flaw size data to Eq. (1).
These data were reported by Zhang et al. [5] and Abe et al. [6]. Hereafter, Zhang’s specimens are
denoted by AL1 and Abe’s are AL2, AL3 and AL4, respectively. In Table 1, characteristics of
specimens are shown. The parameters of gamma distribution are estimated by the most likelihood
method and shown in Table 2. In Fig.1, the flaw size data and gamma distribution of AL1 are shown. The
correlation coefficient R is 0.999 in the data. From its quite high correlation in Fig.1, gamma
distribution can be used as the initial distribution of flaw size. For other specimen, AL2, AL3 and AL4,
almost the same tendencies are obtained.

Table 1 Mechanical properties of alumina
ceramics\cite{4,5}.

| Sample | Average Strength $\bar{\sigma}$ / MPa | Shape Parameter of Weibull Distribution $m$ / 1 | Fracture Toughness $K_{IC}$ / MPa$\sqrt{m}$ |
|--------|-----------------------------------|-----------------------------------------------|-----------------------------------------------|
| AL1    | 360                               | 16.2                                          | 4.00                                          |
| AL2    | 260                               | 28                                            | 4.31                                          |
| AL3    | 315                               | 25                                            | 3.90                                          |
| AL4    | 359                               | 33                                            | 4.41                                          |

Table 2 Best estimators of $\hat{\alpha}$ and $\hat{\nu}$ of
gamma distribution of flaw size in alumina ceramics.

| Sample | $\hat{\alpha}$ / $\mu m^{-1}$ | $\hat{\nu}$ |
|--------|-------------------------------|------------|
| AL1    | 0.667                         | 14.2       |
| AL2    | 0.206                         | 5.34       |
| AL3    | 0.400                         | 9.17       |
| AL4    | 0.489                         | 11.3       |

The distribution of the largest flaw size is obtained from the initial distribution of flaw size based on
extreme value statistics.

$$\Phi_n(x) = \left( \int_0^x \frac{1}{\Gamma(\nu)} \alpha^{\nu} x^{\nu-1} \exp(-\alpha x')dx' \right)^n$$  \hspace{1cm} (2)
where $x$ is the largest flaw size, $\alpha$ and $\nu$ are scale and shape parameters of the initial distribution of flaw size and $n$ is the number of links in the weakest link theory. In Fig. 2, the change in the distribution of the largest flaw size of AL1 is shown. The distribution of the largest flaw size is equal to the distribution of fracture origin size under uniaxial loading. Therefore, the distribution of fracture strength is obtained by transformation of the distribution of the largest flaw size. For this transformation, the following relation in fracture mechanics is used.

\[
\sigma = K_{IC} \sqrt{\frac{\pi}{2c}} \quad (3)
\]

where $c$ is fracture origin size and $K_{IC}$ is fracture toughness and $\sigma$ is fracture strength. By using Eq. (3), the distribution of the largest flaw size is transformed to the distribution of fracture strength $F_n$.

\[
F_n(\sigma) = 1 - \Phi_n \left( \frac{\pi K_{IC}^2}{2\sigma^2} \right) \quad (4)
\]

In order to determine the number of links, the distribution of fracture strength $F_n$ is drawn in Fig.3, that is, in Weibull plot (x-axis is logarithm of fracture strength, and y-axis is double logarithm of $1/F_n$). By comparing with the Weibull parameters estimated from fracture strength data, the number of links is determined as $10^6$.

2.2 The distribution of fracture origin size

In former section, it is shown that the distribution of the largest flaw size is the distribution of fracture origin size. Matsuo et al. [4] obtained the distribution of fracture origin size $H(x)$ from transforming Weibull distribution by fracture mechanics. Matsuo et al. assumed Weibull distribution $F(\sigma)$ as follows,

\[
F(\sigma) = 1 - \exp \left\{ - \left( \frac{\sigma}{\sigma_0} \right)^m \right\} \quad (5)
\]

where $m$ and $\sigma_0$ are shape and scale parameters of Weibull distribution and $K_{IC}$ is fracture toughness. By using Eq. (3) in fracture mechanics, $H(x)$ is given as below,
This is the distribution of fracture origin size under uniaxial loading. It is originally equal to the distribution of the largest flaw size.

2.3 Method of estimation of Weibull parameters from the initial distribution of flaw size

By using the relation between the distribution of the largest flaw size \( \Phi_n(x) \) and the distribution of fracture origin size \( H(x) \), Weibull parameters are estimated from parameters of the initial distribution of flaw size. The parameters (\( \alpha \) and \( \nu \)) in Eq. (2) are known, and the probability for any flaw size \( x \) can be calculated. On the other hand, the distribution of fracture origin size \( H(x) \) has two parameters (\( m \) and \( \sigma_0 \)). Therefore, if there are two relations including these four parameters, the parameters (\( m \) and \( \sigma_0 \)) are determined uniquely. For this purpose, 50 percentile and 25 percentile of Eq. (2) are calculated and it is supposed that Eq.(6) gives the same probabilities. Practically, next equations are solved.

\[
\Phi_n(x) = \left( \int_0^x \frac{1}{\Gamma(\nu)} \alpha' x^{\nu-1} \exp(-\alpha' x') dx' \right)^n = 0.5 \tag{7}
\]

\[
\Phi_n(x) = \left( \int_0^x \frac{1}{\Gamma(\nu)} \alpha' x^{\nu-1} \exp(-\alpha' x') dx' \right)^n = 0.25 \tag{8}
\]

As the number of links, \( 10^6 \) is used to calculate 50 percentile and 25 percentile of Eq. (7) and (8) [3]. Therefore, \( x \) satisfies

\[
H(x) = \exp \left\{ -\left( \frac{\pi}{2} \frac{K_{IC}}{\sigma_0} \right) x^{\frac{m}{2}} \right\} = 0.5
\]

\[
\Rightarrow x = \left[ \frac{\ln 2}{\nu \left( \frac{\pi}{2} \frac{K_{IC}}{\sigma_0} \right)^\frac{2}{m}} \right]^\frac{m}{2}
\]

and
Finally, two relations about $m$ and $\sigma_0$ are obtained, and $m$ and $\sigma_0$ are determined by solving Eq. (9) and Eq. (10) as simultaneous equations.

$$H(x) = \exp\left\{ -\left(\frac{\pi}{2} \frac{K_{IC}}{\sigma_0}\right) x^{\frac{m}{2}} \right\} = 0.25$$

$$\Rightarrow x = \frac{\frac{2 \ln 2}{V\left(\frac{\pi}{2} \frac{K_{IC}}{\sigma_0}\right)^{\frac{m}{2}}}}{\frac{2}{m}}$$

(10).

$2.4$ Results and discussion

In Table 3, the estimators of Weibull parameters are shown.

| Sample | $\hat{m}$ | $\hat{\sigma}_0$ |
|--------|----------|------------------|
| AL1    | 26.0     | 211              |
| AL2    | 22.3     | 372              |
| AL3    | 24.2     | 258              |
| AL4    | 24.8     | 262              |

The experimental Weibull parameters in Table 1 and the estimates of Weibull parameters in Table 3 have a good agreement in the order of magnitude. Therefore, it is shown that this theory is valid. By this theory, fracture strength can be estimated only from the parameters of the initial distribution of flaw size and the number of links.

$3$. Conclusion

In this paper, the relation between the distribution of the largest flaw size and the distribution of fracture origin size is discussed. By this relation, Weibull parameters are estimated from parameters of the initial distribution of flaw size and the number of links. This method is useful to reliability analysis of brittle materials much easier.

References

[1] Wakabayashi C, Matsuo Y, Yasuda K and Shiota T 2008 $JAME$, A74 162
[2] Wakabayashi C, Yasuda K and Shiota T $J. Ceram. Soc. Japan$, 117 162
[3] Wakabayashi C, Yasuda K and Shiota T 2009 $J. Solid Mechanics and Mat. Eng.$ 3 2
[4] Masuo Y, Kitakami K and Kimura S 1989 $JSME$, A55 1
[5] Zhang Y, Inoue M, Uchida N and Uematsu K 1999 $J. Mater. Res.$ 14 3370
[6] Abe H, Naito M, Hotta T, Shinohara N and Uematsu K 2003 $J. Am, Ceram. Soc.$ 86 1019