Sobolev Inequalities for Functions on Graphs

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Abstract. The aim of this paper is to study the Sobolev inequalities when $0 < p < 1$. Sobolev spaces defined by the Russian mathematician Sorgi Sobolev. The need for these spaces that some results are found in these spaces rather than the spaces of continuous functions. Sobolev and some authors proved Sobolev inequality in the cases $p = 1$ and then $p > 1$. Here we prove Sobolev types inequalities in the case $0 < p < 1$. An infinite dimensional Sobolev inequality for expander is also proved. Also we introduce estimates for Banach-Mazur distance between the spaces $S_p(G)$ and $l_p^m$ using Cheeger constant.

Keywords. Sobolev space on a graph, isoperimtric constant of a graph, Cheeger constant, Banach-Mazur distance.

Introduction
Sobolev spaces are named after the Russian mathematician Sorgi Sobolev. Sobolev space is important since it contains all the derivatives that we need for the functions. So it is better than the space of continuous functions.

The Sobolev seminorm is important
(1) Spectral graph theory, see [1] (see, also, [2, Chapter 11]).
(2) $l_p$-embeddability in graph metric, see [3], [4], and [5].
(3) Discrete Sobolev space.

In our work we talk about discrete analogue of Sobolev spaces of smooth functions. We take $G$ a finite graph with $V_G$ and $E_G$, note the vertices and edges set, respectively. The degree of a vertex $v \in V_G$ is denoted by $d_v$.

The edge connected between two vertices $u$ and $v$ we write $u \sim v$ instead of $(u, v) \in E_G$. We delete the symbol $G$ in $E_G$, $V_G$ etc. As a text for graph theory we have [6].

Ostrovskii [7] in 2005, study the Sobolev spaces on graphs when $1 \leq p \leq \infty$ and defined "the discrete Sobolev seminorm" of $f$ when $E = E_G$ and $p > 1$ is defined by:

$$
\|f\| = \|f\|_{E,p} = \left\{ \begin{array}{ll}
\sum_{(u,v) \in E} |f(u) - f(v)|^p & \text{if } 1 \leq p < \infty \\
\max_{(u,v) \in E} |f(u) - f(v)| & \text{if } p = \infty.
\end{array} \right.
$$

The norm $\|f\|_{E,p} = 0$ when only the functions $f$ are constant functions with a connected graph $G$.

Then $\|f\|_{E,p}$ is a considered as a norm of functions define on graph of vertices $V_G$ which doesn’t contain constants. We will consider all functions satisfies $\sum_{v \in V} f(v)dv = 0$ and will be denoted by $S_p(G)$. 
Weighted Sobolev seminorms can be generalized on the most of the results presented below. Let the function \( \mu: E_G \to (0, \infty) \). The weighted Sobolev seminorms function

\[
\|f\| = \|f\|_{E,P,\mu} = \left\{ \begin{array}{ll}
\left( \sum_{(u,v) \in E} |f(u) - f(v)|^p \mu(u,v) \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\
\max(u,v) \in E |f(u) - f(v)| \mu(u,v) & \text{if } p = \infty
\end{array} \right.
\]

To show the reason of called the normed Sobolev space, remember that the Sobolev space (of order 1) on \( \mathbb{R}^n \) can be defined under the norm

\[
\|f\| = \left( \sum_{i=1}^{n} \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_i}(x) \right|^p dx \right)^{\frac{1}{p}}
\]

of the space of all smooth compactly supported functions on \( \mathbb{R}^n \): The notion is the following the difference \( f(u) - f(v) \) we can suppose it as the "partial derivative " of \( f \) at \( v \) directed to \( u \). Some proofs of known results on Sobolev spaces in \( \mathbb{R}^n \) we can also prove some well know theorem in Sobolev spaces to Sobolev spaces on graphs then this analogy is very producer.

In this paper we work on discrete analogues of Sobolev spaces of smooth functions, for \( p \leq 1 \). We suppose \( f: V_G \to \mathbb{R} \), and \( 0 < p < 1 \) the discrete Sobolev seminorm on \( E = E_G \) is

\[
\|f\| = \left( \sum_{(u,v) \in E} |f(u) - f(v)|^p \mu(u,v) \right)^{\frac{1}{p}}
\]

For \( 0 < p < 1 \). If \( G \) is connected graph. To make \( \|f\| \) a norm we must remove constant functions.

We denoted the corresponding weighted Sobolev seminorm as

\[
\|f\| = \left( \sum_{(u,v) \in E} |f(u) - f(v)|^p \mu(u,v) \right)^{\frac{1}{p}}
\]

for \( 0 < p < 1 \). \( \mu: E \to (0,1) \).

We define the norm on the Sobolev space (of order 1) on \( \mathbb{R}^n \) as

\[
\|f\| = \left( \sum_{i=1}^{n} \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_i}(x) \right|^p dx \right)^{\frac{1}{p}} \quad (1.1)
\]

**Sobolev Type Inequalities**

Suppose \( G \) be a finite graph. We suppose \( X \subseteq V_G \) then the volume of a subset \( X \) is \( \text{vol}(X) = \sum_{v \in X} dv \) and the \( L^p \)-norm \( 0 < p < 1 \) on a discrete Sobolev space is

\[
\|f\|_{p,\mu} = \left( \sum_{v \in V_G} |f(v)|^p \mu(v) \right)^{\frac{1}{p}} 
\]

Let \( X, Y \subseteq V_G \), we note the complement \( V_G \setminus X \) by \( X^c \). The number of edges in \( E_G \) with one end vertex in \( X \) and the other end vertex in \( Y \), denote by \( e_G(X,Y) \).

The graph \( G \) has "isoperimetric dimension" \( \lambda \) with "isoperimetric constant" \( C_\lambda > 0 \) if

\[
e_G(x,x^c) \geq C_\lambda (\text{min\{vol}(x, \text{vol}(x^c))^{\frac{\lambda-1}{\lambda}}
\]

with \( C_\lambda \) does not depend on \( X \) and for every proper subset \( X \) of \( V_G \). [7]

We suppose \( m = m(f) \) be median of \( f \), then the following two inequalities satisfying with a real number.

\[
\text{vol}(\{v: f(v) > m\}) \leq \frac{\text{vol}(V_G)}{2}
\]

and

\[
\text{vol}(\{v: f(v) < m\}) \leq \frac{\text{vol}(V_G)}{2}
\]

Let us now introduce our first result for Sobolev type inequality :

**Theorem 2.4.** Let \( G \) be a graph and suppose \( m = m(f) \) be a median of \( f: V_G \to \mathbb{R} \) then

\[
\|f - m(f)\|_{L^q(V_G)} \leq c(p) \|f\|_{E,p}
\]

**Proof.**

assume \( 0 < q < p < 1 \) then

\[
\|f - m(f)\|_{L^q(V_G)} \leq \left( \sum_{v \in V_G} |f(v) - m(f(v))|^q \right)^{\frac{1}{q}}
\]
\( \leq (\sum_{v \in V_G}|f(v) - m(f(v))|^p)^{\frac{1}{p}}. \)

From the definition of the median, there exists \( c > 0, \)
\[
\|f - m(f)\|_q(v) \leq c(p) (\sum_{(u,v) \in E_G}|f(u) - f(v)|^p)^{\frac{1}{p}} = c(p) \|f\|_{E,p}.
\]

**Lemma 2.5.** [8] If \( 0 < p < q < \infty, \)
\[
(\sum_{v \in V_G}|f(v)|^p)^{\frac{1}{p}} \leq c(p) (\sum_{v \in V_G}|f(v)|^q)^{\frac{1}{q}}.
\]

**Lemma 2.6.** [7] \( \int_0^\infty(|f(v)|^p)^{\frac{1}{p}} dv \leq c(p) \int_0^\infty|f(v)|^q dv. \)

**Theorem 2.7.** If \( G \) is a connected graph with isoperimetric dimension \( \lambda \) and \( f \in L_p(G) \), then
\[
\|f\|_{E,p} \geq c(p,\lambda)\|f - m(f)\|_{\Lambda-1,v}^\lambda,
\]
where \( m \) is the median of \( f \), and \( c(p,\lambda) \) is appositive constant depends on \( p \) and \( \lambda \)

**Proof.**

Suppose \( I(v) = f(v) - m \)

Let \( I^+(v) = \max\{I(v), 0\} \) and \( I^-(v) = \max\{-I(v), 0\} \)
\[
(\sum_{v \in V_G}|I^+(v)|^\lambda d\nu)^{\frac{1}{\lambda}} \leq c(p,\lambda) \sum_{v \in V_G}\|\nabla(I^+)(v)\|_p
\]

Let us start by proving the inequality

Let
\[
I^+(v) = \int_0^\infty \chi \{z: I(z) > t\}(v) dt
\]

Then
\[
(\sum_{v \in V_G}|I^+(v)|^\lambda d\nu)^{\frac{1}{\lambda}} \leq \int_0^\infty (\sum_{v \in V_G}\chi \{z: I^+(z) > t\}(v)|^\lambda d\nu)^{\frac{1}{\lambda}} dt
\]

Suppose \( z_t = \{z: I^+(z) > t\} \). For \( t > 0 \), we have the inequality
\[
vol z_t \leq vol z_{t^*} \quad (since \ 0 \ is \ the \ median \ of \ I(v)).
\]

Then
\[
vol \{z: I^+(z) > t\}\)^{\frac{1}{\lambda}} \leq c(p,\lambda) e_G(z_t,z_{t^*})
\]

By the definition of the isoperimetric constant, we have
\[
e_G(z_t,z_{t^*}) = \sum_{t = t} 1 \quad (t \in I^+(V_G))
\]

Then, by Lemma 2.6, when \( g = 1 \), we get
\[
\int_0^\infty (c(p,\lambda) \sum_{t = t} 1) dt = c(p,\lambda) \sum_{v \in V_G}\|\nabla(I^+)(v)\|_p
\]

We have \( I(v) = I^+(v) - I^-(v) \)
\[
(\sum_{v \in V_G}|I(v)|^\lambda d\nu)^{\frac{1}{\lambda}} \leq (\sum_{v \in V_G}|I^+(v)|^\lambda d\nu)^{\frac{1}{\lambda}} + (\sum_{v \in V_G}|I^-(v)|^\lambda d\nu)^{\frac{1}{\lambda}}
\]
\[
\leq c(p,\lambda) \sum_{v \in V_G}\|\nabla(I^+)(v)\| + c(p,\lambda) \sum_{v \in V_G}\|\nabla(I^-)(v)\|
\]
\[
= c(p,\lambda) \sum_{u \in V_G}\|\nabla I(v)\|
\]
\[
= c(p,\lambda) (\sum_{u \in V_G}|I(u) - I(v)|^p)^{\frac{1}{p}}.
\]
Sobolev Inequalities For 0 < p < 1

The first type of Sobolev inequalities was introduced in [9] for the continuous case. In the discrete case Sobolev inequalities introduce in [5]. We introduce Sobolev inequalities type for any function defined on graph belongs to \( L_p(V_G) \) spaces for p < 1.

**Theorem 3.1.** If \( G \) be an arbitrary graph, With isoperimetric constant \( c_\lambda \), \( f \in L_q \), \( 0 < q < 1 \). Then

\[
\left( \sum_{v \in V_G} |f(v)|^\gamma dv \right)^{1/\gamma} \leq c(\lambda, c_\lambda, p) \| f \|_{E, q}^{\gamma-1} \left( \sum_{u \sim v} |f(u) - f(v)|^\gamma \right)^{1/(\gamma+1)}
\]

**Proof.**

Let

\[
\left( \sum_{v \in V_G} |f(v)|^\gamma dv \right)^{1/\gamma} \leq c(\lambda, c_\lambda, p) \left( \sum_{u \sim v} |f(u)|^{\lambda-1} + |f(v)|^{\lambda-1} \right)^{\gamma/(\gamma+1)} \left( \sum_{u \sim v} |f(u) - f(v)|^\gamma \right)^{1/(\gamma+1)}
\]

For \( 0 < p < 1 \) using the inequality \( |a + b|^p < |a|^p + |b|^p \) we get

\[
\left( \sum_{v \in V_G} |f(v)|^\gamma dv \right)^{1/\gamma} \leq c(\lambda, c_\lambda, p) \left( \sum_{u \sim v} |f(u)|^{\lambda-1} + |f(v)|^{\lambda-1} \right)^{\gamma/(\gamma+1)} \left( \sum_{u \sim v} |f(u) - f(v)|^\gamma \right)^{1/(\gamma+1)}
\]

Expanders Sobolev Inequality The Infinite Dimensional Case.

Let \( G \) be a graph and the definition of "cheeger constant" \( h_G \) is

\[
h_G = \min_{x \in V_G} \frac{\varphi_G(x, x)}{\min(\text{vol}_x, \text{vol}_x)}
\]

We note that the relation between the definition of the isoperimetric constant and the Cheeger constant is \( h_G = \lim_{d \to \infty} c_\lambda \). The expanders is the families of graphs with Cheeger constants uniformly bounded below by some non-zero constants. The theory of expanders have many applications and many connections with different branches of mathematic see for example [2], [10]. The following theorem can be considered as a version of a theorem proved in [11, problem 30]. F.R.K. Chung and P. Tetali proved a same version in [12].

**Theorem 4.1.** If \( G \) is a graph with the Cheeger constant \( h_G \), then

\[
\left( \sum_{u \sim v} |f(u) - f(v)|^q dv \right)^{1/q} = \| f \|_{E, p} \geq c(h_G, p, \alpha) \| f(v) - m \|_{L_p} = \left( \sum_v |f(v) - m|^p dv \right)^{1/p}
\]

With \( m = m(f) \) be a median of \( f \), where \( 0 < p < 1 \).

**Proof.**

Suppose \( c_\lambda = q > 1 \) in Theorem 3.1 to get

\[
\left( \sum_{v \in V_G} |f(v)|^q dv \right)^{\gamma/(\gamma+1)} \leq c(\lambda, c_\lambda, p, \alpha) \| f \|_{E, q}^{\gamma-1} \left( \sum_{u \sim v} |f(u) - f(v)|^q \right)^{1/(\gamma+1)} \]

If \( p < 1 \) then

\[
\left( \sum_{v \in V_G} |f(v)|^p dv \right)^{\gamma/(\gamma+1)} \leq c(\lambda, c_\lambda, p, \alpha) \| f \|_{E, p}^{\gamma-1} \left( \sum_{u \sim v} |f(u) - f(v)|^p \right)^{1/(\gamma+1)}
\]

By taking the limit for \( c_\lambda \to \infty \) we get

\[
\left( \sum_{v \in V_G} |f(v)|^p dv \right)^{\gamma/(\gamma+1)} \leq c(h_G, p, \alpha) \| f \|_{E, q}^{\gamma-1} \left( \sum_{u \sim v} |f(u) - f(v)|^p \right)^{1/(\gamma+1)}
\]

As a small modification for lemma (2) in [5]. We get the following lemma

**Lemma 4.2.** Let \( G \) be a graph with expander \( \phi \), then for the reals \( x_1, x_2, ..., x_n \) and \( 0 < p < 1 \), we have

\[
\sum_{(i,j) \in E} |x_i - x_j|^p \geq \phi \sum_{i \in E} |x_i - m|^p
\]

Where \( m \) is the median of \( x_i \) s' \( , i = 1, 2, ..., n \).
Theorem 4.3. If $G$ be an arbitrary graph with expander constant $\phi$. Let $f : V_G \to \mathbb{R}$ and suppose $m = m(f)$ be a median of $f$. Then
\[
(\sum_{u,v \in E} |f(u) - f(v)|^p)^{\frac{1}{p}} \geq c(p, \phi) (\sum_{v \in V_G} |f(u) - m|^p)^{\frac{1}{p}}
\]
Proof.
Using Lemma 4.2 we get the desired

Proposition 4.4. If $a = a(f) = \frac{1}{|V_G|} \sum_{v \in V_G} f(v) dv$. Then for $0 < p < 1$
\[
\frac{1}{|V_G|} \|f - b\|_p(v) \leq \|f - a\|_p(v) \leq (\sum_{v \in V_G} d(v))^\frac{1}{p} \|f - b\|_p(v)
\]
Proof.
\[
\|f - a\|_p(v) = \left\| f - \frac{1}{|V_G|} \sum_{v \in V_G} f(v) dv \right\|_p(v) = (\sum_{u,v} |f(u) - t(f)|^p d(u)d(v))^{\frac{1}{p}}
\]
Where $t(f) = \frac{1}{|V_G|} f(v)$
Then $\|f - a\|_p(v) \leq (\sum_{v \in V_G} d(v))^\frac{1}{p} (\sum_{u,v} |f(u) - b|^p d(u)d(v))^{\frac{1}{p}}$
Hence the second inequality is proved.
\[
\|f - a\|_p(v) = \left\| f - \frac{1}{|V_G|} \sum_{v \in V_G} f(v) dv \right\|_p(v) = (\sum_{u,v} |f(u) - \frac{1}{|V_G|} f(v)|^p d(u)d(v))^{\frac{1}{p}}
\]
\[
> \frac{1}{|V_G|} (\sum_{u,v} |f(u) - f(v)|^p d(u)d(v))^{\frac{1}{p}}
\]
\[
> \frac{1}{|V_G|} (\sum_{u \in V_G} |f(u) - t(f)|^p d(u))^{\frac{1}{p}}
\]
\[
= \frac{1}{|V_G|} \|f - b\|_{L_p}(v)
\]

"The Banach-Mazur distance" between $S_p(G)$ and $l_p^m$ in terms of Cheeger constant.
Let $G$ be a connected graph by $l_p^m(G)$. We denote the 1-codimensional subspace of $L_p(V_G)$ given by
\[
\sum_{v \in V_G} f(v) dv = 0. \text{ We know that}
\]
d$(l_p^0(G), l_p^m) \leq 9 \text{ for } 1 \leq p < \infty$. \[7\]
With $m = |V_G| - 1$
Suppose $l_p : S_p(G) \to l_p^0(G)$ the identity mapping.
The inequality (5-1) satisfying that in order to estimate $d(S_p(G), l_p^m)$ from above it is enough to estimate the norms $\|l_p\|$ and $\|l_p^{-1}\|$ in the corresponding spaces of operators.
\[
\|f\|_{E,p} = \left( \sum_{u \sim v} |f(u) - f(v)|^p \right)^{\frac{1}{p}} \leq 2 \frac{1}{p-1} (\sum_{u \sim v} (|f(u)|^p + |f(v)|^p))^{\frac{1}{p}}
\]
\[
= 2 \frac{1}{p-1} \left( \sum_{v \in V_G} |f(v)|^p dv \right)^\frac{1}{p} \leq 2 \frac{1}{p-1} \left( \sum_{v \in V_G} \frac{d(v)}{h_G} \text{ for } 0 < p < 1. \right)
\]
(5-2)

Theorem 5.3. $d(S_p(G), l_p^0(V_G)) \leq c(p) \frac{\sum_{v \in V_G} d(v)}{h_G}$ for $0 < p < 1$.

Proof.
l_p^0(V_G) is the space of all functions $f : V_G \to \mathbb{R}$, satisfying $\sum_{v \in V_G} f(v) dv = 0. \text{ In } [7] \text{ we have}$
d$(l_p^0(G), l_q^0) \leq 9 \text{ for } 1 \leq p < \infty. \text{ So for } 0 < q < 1, \text{ we get}$
d$(l_p^0(G), l_p^m) \leq d(l_q^0(G), l_p^m) \leq 9 \text{ for } 0 < q < 1.$ \[5-4\]
Where $m = |V_G| - 1$.
Define $l_p : S_p(G) \to l_p^0(V_G)$, the identity map. From (5-1), to estimate $d(S_p(G), l_p^m)$ it is enough to estimate the norms $\|l_p\|$ and $\|l_p^{-1}\|$. 

Then let us estimate $\|I^{-1}_p\|$. In fact (5-2). Using Theorem 4.1 and proposition 4.4 we get $\|I_p\| \leq c(p) \frac{\sum_{v \in V_G} dv}{h_G}$.

From (15) we obtain
$$d(S_p(G), l^0_p(V_0)) \leq c(p) \frac{\sum_{v \in V_G} dv}{h_G}.$$ 

**Conclusion**

Sobolev types inequalities, Infinite dimensional Sobolev inequality for expander, estimates for Banach -Mazur distances between $S_p(G)$ and $l^0_p$ can be proved in the case $0 < p < 1$.

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