Stability analysis of mathematical model (sirb) in the spread of cholera with vaccination and disinfection

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Abstract. Cholera is a disease as a kind of acute diarrhea caused by bacteria V. Cholerae. The spread of cholera can be modeled in the form of nonlinear differential equation systems with 4 variables SIRB. This paper aims to research to analyze the local stability of the equilibrium point of the dynamical population in the spread of cholera by the Routh-Hurwitz stability criterion and bifurcation method. Next Generation Matrix (NGM) method is used to get the basic reproductive numbers ($R_0$) to find the local stability at the equilibrium point of the. The disease-free equilibrium point is locally asymptotically stable if $R_0 < 1$, while the endemic equilibrium point is locally asymptotically stable if $R_0 > 1$. The results of numerical simulations obtained $R_0 = 0.87$ indicated that the disease-free equilibrium point is locally asymptotically stable. In endemic condition ($R_0 > 1$) show that increasing the rate of vaccination and disinfection can reduce the population of susceptible, infected and bacteria of V. Cholerae.

1. Introduction

Vibrio Cholerae is a bacteria that causes cholera [1], which is characterized by diarrhea, vomit, and dehydration [2]. Without proper treatment, it can cause death [3]. The transmission of cholera usually through water or food that is contaminated by bacteria [4]. The emergence of cholera is often undetected, so appropriate prevention needs to be done.

Indonesia, China, and India are an endemic district of Cholera. In 1961 Cholera spread in 23 countries and the source of transmission came from Sulawesi [5], then spread to Europe and Japan in 1970 [6]. In Indonesia, there were 6882 cases from September 1994 to November/December 1999. Mathematical modeling can be used to study problems in the real world by constructing a mathematical model that according to the problems. Several papers have discussed the mathematics model in the spread of cholera, among others by looking at the effects of vaccination and demographic movements [7]. Further, Rahmi, et.al divided the population of Vibrio Cholerae into two types, they are hyper infectious bacteria and less infectious bacteria [8].

In this paper, discuss a mathematical model in the spread of cholera that been constructed by [9]. In previous paper analyzed the effect of the transmission rate (from human to human and from environment to human) in the dynamical population, while in this paper analyze the effect of vaccination and disinfection in the dynamical population. The population divide into two, namely the human population ($N$) and V. Cholerae population ($B$). The population of humans is divided into three subpopulations,
they are susceptible (S), infected (I), and recovered (R), then this model considers the vaccination and disinfection to control the spread of cholera.

2. Construction of model
The assumptions used in the process of modeling the spread of cholera are as follows, the values of birth rate and death rate are the same, equal to $\mu$. There are two ways of transmission, from human to human and environment to human. Vaccination and disinfection are used to control cholera. A recovered individual becomes invulnerable.

Susceptible individuals increase due to the natural birth rate $\mu$. It reduced due to interactions between the susceptible individual and V. Cholerae with a concentration in the environment $k$, the transmission rate is $\beta_i$, and the interactions with an infected individual, the transmission rate is $\beta_h$. It also reduced due to the natural death rate $\mu$ and vaccination rate $v$.

Infected individuals increase due to interactions between susceptible individuals and V. Cholerae with a concentration in the environment $k$, the transmission rate is $\beta_i$, and the interactions with infected individuals, the transmission rate is $\beta_h$. Infected individuals reduced due to the recovered rate from its self as $\gamma$ and natural death rate $\mu$. It influences the population of V. Cholerae $\xi I$ but does not affect the number of it.

Recovered individual increase due to recovered rate from infected individuals as $\gamma$ and vaccination rate from susceptible individuals as $v$. Then it reduces due to the natural death rate $\mu$. The population of V. Cholerae increase due to contribution rate of human that infected as $\xi$. It reduces due to the death rate from its self ($\delta$) and disinfection rate ($c$).

Obtained the system of differential equation [9],

$$\frac{dS}{dt} = \mu N - \beta_i S \frac{B}{k + B} - \beta_h SI - \mu S - vS,$$

$$\frac{dI}{dt} = \beta_i S \frac{B}{k + B} + \beta_h SI - \gamma I - \mu I,$$

$$\frac{dR}{dt} = \gamma I - \mu R + vS,$$

$$\frac{dB}{dt} = \xi I - \delta B - cB,$$

where $S$, $I$, $R$, and $B$ are susceptible, infected, recovery, and bacteria V. Cholerae, because variable $R$ only appears in equation (4), so the system can be reduced to:

$$\begin{align*}
\frac{dS}{dt} &= \mu N - \beta_i S \frac{B}{k + B} - \beta_h SI - \mu S - vS = f_1 \\
\frac{dI}{dt} &= \beta_i S \frac{B}{k + B} + \beta_h SI - \gamma I - \mu I = f_2 \\
\frac{dB}{dt} &= \xi I - \delta B - cB = f_3
\end{align*}$$

3. Equilibrium point of the model
The equilibrium point is a condition where there are no changes in each population over time. System (5) can be written:
\[
\mu N - \left( \beta_+ S \frac{B}{k+B} + \beta_+ SI - \mu S - \nu S \right) = 0, \quad (6)
\]
\[
\beta_+ S \frac{B}{k+B} + \beta_+ SI - (\gamma + \mu) I = 0, \quad (7)
\]
\[
\xi I - \delta B - cB = 0. \quad (8)
\]

Disease-free equilibrium (DFE) point \( E_0 = (S_0, I_0, B_0) \) is a condition when there is no disease and bacteria in populations \( (I_0 = B_0 = 0) \), by substituting \( I_0 = B_0 = 0 \) into equation (6) was obtained \( E_0 \) as follows:
\[
E_0 = (S_0, I_0, B_0) = \left( \frac{\mu N}{(\mu + \nu)}, 0, 0 \right).
\]

To look for basic reproductive number (\( \mathcal{R}_0 \)) is used NGM method, \( \mathcal{R}_0 \) is a parameter used to know the effect of the spread of cholera in population, it was obtained
\[
\mathcal{R}_0 = \frac{\beta_+ \mu N}{(\mu + \nu)(\gamma + \mu)} + \frac{\beta_+ \mu N \xi}{(\mu + \nu)(\gamma + \mu)(\delta + c)k}.
\]

The endemic equilibrium point \( E^* = (S^*, I^*, B^*) \) is a condition where there are diseases and bacteria in populations. By solving the equations (3.6) – (3.8) was obtained \( E^* \) as follows:
\[
S^* = \frac{k(\delta + c) + \xi I^*}{\beta_+ \xi} + \frac{\xi I^*}{\beta_+ (k(\delta + c) + \xi I^*)},
\]
\[
B^* = \frac{\xi I^*}{(\delta + c)}, \quad \text{and}
\]
\[
m_1 I^* + m_2 I^* + m_3 = 0, \quad \text{where}
\]
\[
m_1 = \left[ \beta_+ (\gamma + \mu) \xi \right]
\]
\[
m_2 = \left[ \beta_+ (\gamma + \mu) k(\delta + c) + ((\mu + \nu)(\gamma + \mu) \xi) + (\beta_+ (\gamma + \mu) \xi) - (\beta_+ \mu N \xi) \right]
\]
\[
m_3 = \left[ ((\mu + \nu)(\gamma + \mu) k(\delta + c)) - (\beta_+ \mu N k(\delta + c)) - (\beta_+ \mu N \xi) \right]
\]

To state the population of \( I^* \) having at least one positive root, so \( I_1^* I_2^* < 0 \) if and only if
\[
D = m_2^2 - 4m_1m_3 > 0 \quad \text{and} \quad \frac{m_1}{m_3} < 0 \quad \text{by inequality analysis} \quad m_3 < 0 \quad \text{and} \quad m_1 > 0.
\]
\[
m_1 = \left[ \beta_+ (\gamma + \mu) \xi \right] \quad \text{so} \quad m_1 > 0
\]
\[
m_2 = ((\mu + \nu)(\gamma + \mu) k(\delta + c)) - (\beta_+ \mu N k(\delta + c)) - (\beta_+ \mu N \xi) < 0
\]
\[
\Leftrightarrow (\mu + \nu)(\gamma + \mu) k(\delta + c)) - (\beta_+ \mu N k(\delta + c)) - (\beta_+ \mu N \xi) < 0
\]
\[
\Leftrightarrow 1 < \frac{\beta_+ \mu N}{(\mu + \nu)(\gamma + \mu)} + \frac{\beta_+ \mu N \xi}{(\mu + \nu)(\gamma + \mu) k(\delta + c)}
\]
\[
\Leftrightarrow \mathcal{R}_0 > 1
\]

So the existence \( I^* \geq 0 \) is guaranteed by \( \mathcal{R}_0 > 1 \).
4. Stability analysis

Stability analysis is carried out to obtain the behavior of the equilibrium points. Stability of the DFE point can be stated by theorem;

**Theorem 1.** Point $E_0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$.

**Proof:**

System (3.5) is a nonlinear system, so it must first be linearized by forming Jacobian Matrix around $E_0$ , was obtained:

$$
J(E_0) = \begin{bmatrix}
-(\mu + v) & -\frac{\mu N \beta_h}{\mu + v} & -\frac{\beta_t \mu N}{(\mu + v)k} \\
0 & \frac{\mu N \beta_h}{\mu + v} - \gamma - \mu & \frac{\beta_t \mu N}{(\mu + v)k} \\
0 & \xi & -(\delta + c)
\end{bmatrix}
$$

The characteristic equation of the Jacobian $J(E_0)$ is

$$(\lambda + \mu + v)[\lambda^2 + \lambda(\delta + c - p + \gamma + \mu) + (\gamma + \mu)(\delta + c) - p(\delta + c) - \xi q] = 0$$

(9)

where $p = \frac{\mu N \beta_h}{\mu + v}$ and $q = \frac{\beta_t \mu N}{(\mu + v)k}$. From equation (9) we obtain eigenvalue are $\lambda_1 = -(\mu + v)$ and polynomial equation $P(\lambda) = a_0 \lambda^2 + a_1 \lambda + a_2 = 0$, where

$$
a_0 = 1 \\
a_1 = \delta + c - p + \gamma + \mu \\
a_2 = (\gamma + \mu)(\delta + c) - p(\delta + c) - \xi q
$$

Based on Routh-Hurwitz criteria, $E_0$ locally asymptotically stable if $a_1 > 0$, $a_2 > 0$, and $a_1 a_2 > 0$, these conditions are met if $\mathcal{R}_0 < 1$.

Further, the stability of the $E^*$ point can be state by theorem;

**Theorem 2.** Point $E^*$ is locally asymptotically stable if $\mathcal{R}_0 > 1$.

**Proof:**

Theory of Manifold Center [10] used to carry out the stability of $E^*$ in $\mathcal{R}_0 > 1$. The parameter bifurcation from $\mathcal{R}_0 = 1$ is $\beta_h$, where $\beta_h = \beta_h^*$ , was obtained:

$$
\beta_h = \frac{(\mu + v)(\gamma + \mu)(\delta + c)k - \beta_t \mu N \xi}{\mu N(\delta + c)k}
$$

Matrix Jacobian around DFE point when $\beta_h = \beta_h^*$ is :

$$
J(E_0, \beta_h^*) = \begin{bmatrix}
-(\mu + v) & -\frac{(\mu + v)(\gamma + \mu)(\delta + c)k - \beta_t \mu N \xi}{(\mu + v)(\delta + c)k} & -\frac{\beta_t \mu N}{(\mu + v)k} \\
0 & \frac{(\mu + v)(\gamma + \mu)(\delta + c)k - \beta_t \mu N \xi}{(\mu + v)(\delta + c)k} - \gamma - \mu & \frac{\beta_t \mu N}{(\mu + v)k} \\
0 & \xi & -(\delta + c)
\end{bmatrix}
$$

It has simple eigenvalue as $\lambda_2 = 0$. Then obtain right eigen vector as $w$ and left eigen vector as $v$ that corresponding to the eigenvalue $\lambda_2 = 0$. 


The right eigenvector denoted by \( \mathbf{w} = [w_1, w_2, w_3]^T \) satisfied \( J(E_0, \beta_i^*) \mathbf{w} = 0 \)
\[
(-\mu - v)w_1 \left( -\frac{(\mu + v)(\gamma + \mu)(\delta + c)k - \beta_i \mu N \xi}{(\mu + v)(\delta + c)k} \right) w_2 \left( -\frac{\beta_i \mu N}{(\mu + v)k} \right) w_3 = 0
\]

taken \( w_2 = 1 \), so obtained \( w_3 = \frac{\xi}{(\delta + c)} \) and \( w_1 = -\frac{(\gamma + \mu)}{(\mu + v)} \). The right eigenvector is
\[
\mathbf{w} = \begin{bmatrix}
\frac{(\gamma + \mu)}{(\mu + v)} \\
1 \\
\frac{\xi}{(\delta + c)}
\end{bmatrix}
\]

While the left eigenvector denotes by \( \mathbf{v} = [v_1, v_2, v_3] \) satisfied \( \mathbf{v}^T J(E_0, \beta_i^*) = 0 \)
\[
\left\{ \left( -\frac{(\mu + v)(\gamma + \mu)(\delta + c)k - \beta_i \mu N \xi}{(\mu + v)(\delta + c)k} \right) v_1 + \left( -\frac{\beta_i \mu N}{(\mu + v)k} \right) v_2 + \xi v_3 \right\}^T = 0
\]

obtained \( v_1 = 0 \), \( v_2 = \frac{(\mu + v)(\delta + c)k}{\beta_i \mu N} v_3 \), and \( v_3 = v_3 \), then look for \( \mathbf{v} \) that satisfy \( \mathbf{v} \cdot \mathbf{w} = 1 \)
\[
\begin{bmatrix}
0 \\
\frac{(\mu + v)(\delta + c)k}{\beta_i \mu N} v_3 \\
\frac{(\gamma + \mu)}{(\mu + v)} \\
1 \\
\frac{\xi}{(\delta + c)}
\end{bmatrix} = 1
\]

\[
\frac{(\mu + v)(\delta + c)k}{\beta_i \mu N} v_3 + \frac{\xi}{(\delta + c)} v_3 = 1
\]

obtained \( v_3 = \frac{\beta_i \mu N(\delta + c)}{(\mu + v)(\delta + c)k + \beta_i \mu N \xi} \), so the left eigenvector is
\[
\mathbf{v} = \begin{bmatrix}
0 \\
\frac{(\mu + v)k}{(\mu + v)k + \beta_i \mu N \xi} \\
\frac{\beta_i \mu N(\delta + c)}{(\mu + v)(\delta + c)k + \beta_i \mu N \xi}
\end{bmatrix}
\]
Suppose that $S = y_1$, $I = y_2$, and $B = y_3$. Derivative partial levels two of the system equation (3.5) in free equilibrium disease case are:

$$
\frac{\partial^2 f_1}{\partial y_1 \partial y_2} = \frac{\partial^2 f_2}{\partial y_1 \partial y_3} = \beta_b, \quad \frac{\partial^2 f_2}{\partial y_2 \partial y_1} = \frac{\partial^2 f_3}{\partial y_2 \partial y_3} = \frac{\beta_i k}{(k + y_3)^2}, \quad \frac{\partial^2 f_3}{\partial y_3 \partial y_1} = -\frac{2\beta_y k}{(k + y_3)^2}, \quad \frac{\partial^2 f_3}{\partial y_3 \partial \beta_h} = y_2,
$$

and $\frac{\partial^2 f_2}{\partial y_2 \partial \beta_h} = y_1$

Then find the parameter $a$ and $b$, where:

$$a = \sum_{k,i,j=1}^{n} w_k w_i \frac{\partial^2 f_k}{\partial y_i \partial y_j} (0,0) \quad \text{and} \quad b = \sum_{k,i,j=1}^{n} w_k w_i \frac{\partial^2 f_k}{\partial y_i \partial \beta_h} (0,0)
$$

obtained

$$a = \frac{k \left[ \beta_h (\gamma + \mu)(\delta + c)^2 (k + y_3)^3 + \beta_v (\gamma + \mu)(\delta + c)(k + y_3) \xi k^2 + 2\beta_v (\mu + v) \xi^2 k y_1 \right]}{[(\mu + v)k + \beta_v \mu N \xi]} < 0
$$

and $b = \frac{(\mu + v) k y_1}{(\mu + v) k + \beta_v \mu N \xi} > 0$. Based on the theory of Manifold Center [10] if $a < 0$ and $b > 0$,

then case 4 applies to the system (3.5) and $E^*$ the point is locally asymptotically stable if $R_0 > 1$. [4]

5. Numerical simulation

Parameters used for numerical simulation based on the parameter from Rahmi, et al [8] and Sun, et al [9] are as follows, natural birth ($\mu$) = $0.0066$ /day and death rate ($\mu$) = $0.0066$ /day, concentration of Vibrio Cholerae in environment ($k$) = 500 cell/ml, human population ($N$) = $1.36 \times 10^9$ individuals, transmission rate from environment to human ($\beta_v$) = $62669 \times 10^{-6}$ /day, transmission rate from human to human ($\beta_h$) = $53508 \times 10^{-9}$ /day, recovered rate ($\gamma$) = 0.2 /day, Rate of human contribution to Vibrio Cholerae ($\xi$) = 10 cell/ml/day, death rate of V. Cholerae ($\delta$) = 0.033 /day.

In disease-free conditions $v = 0.5$ and $c = 0.01$ are taken [8], obtained $R_0 = 0.8723273720$, and $E_0 = (49181,0,0)$. It shows that the stability DFE point of cholera is reached when the susceptible individuals are 49181. the graphic is:
In the initial situation, the number of susceptible individuals was 100000 people and then decline until at $t > 30$ stable approaching 49181 people. In an infected individual with an initial number of 1000 people decline until at $t > 30$ stable approaching 0 people. Whereas in bacteria $V. Cholera$ the initial amount of 100 cells/ml has fluctuated until $t > 130$ is stable close to 0.

In endemic conditions, variations in the value of $v$ are taken, they are 0.09, 0.3, and 0.4, then variations values of $c$ are 0.5, 0.7, and 0.9. these values result $R_0 > 1$, then it is used to see the effect of the vaccination and disinfection rate on dynamical populations of cholera spread. Obtained the graphics are:

From Figure 2 – Figure 4, In the initial situation the number of susceptible individuals is 100000, in the infected individual are 1000 people, and in bacteria $V. Cholera$ the initial are 100 cells/ml. From figure 2 the susceptible individual is increased until $t > 70$ is stable approaching 248.340 people. In infected individual decline until $t > 30$ stable approaching 10 people. while in bacteria $V. Cholera$ has fluctuated until $t > 130$ is stable close to 2.280 cells/ml. From figure 3 the susceptible individual is decline until $t > 10$ is stable approaching 7.452 people. In infected individual decline until $t > 10$ stable approaching 0 people. while in bacteria $V. Cholera$ has fluctuated until $t > 130$ is stable close to 57 cells/ml. From figure 4 the susceptible individual is decline until $t > 10$ is stable approaching 5.589 people. In infected individual decline until $t > 10$ stable approaching 0 people. while in bacteria $V. Cholera$ has fluctuated until $t > 130$ is stable close to 44 cells/ml.

Table 1. Total populations in 140th day with various values of $v$

| Simulations | $v$  | Susceptible Individuals (person) | Infected Individuals (person) | $V. Cholerae$ (cells/ml) | $R_0$ |
|-------------|------|----------------------------------|-------------------------------|--------------------------|-------|
| 1           | 0.09 | 248.340                          | 10                            | 2.280                    | 4.86  |
| 2           | 0.3  | 7.452                            | 0                             | 57                       | 1.46  |
| 3           | 0.4  | 5.589                            | 0                             | 44                       | 1.09  |
From table 1 appears that increase of vaccination and disinfection rate can reduce the population

**Figure 5.** \(c=0.5\) and \(v=0.02\)  
**Figure 6.** \(c=0.7\) and \(v=0.02\)  
**Figure 7.** \(c=0.9\) and \(v=0.02\)

From Figure 5 – Figure 7, In the initial situation the number of susceptible individuals is 100000, in the infected individual are 1000 people, and in bacteria \(V.\ Cholera\) the initial are 100 cells/ml. From figure 5 the susceptible individual increases until \(t > 180\) is stable approaching 105.450 people. In infected individual decline until \(t > 10\) stable approaching 10 people, while in bacteria \(V.\ Cholera\) has fluctuated until \(t > 10\) is stable close to 210 cells/ml.

From figure 6 the susceptible individual increases until \(t > 180\) is stable approaching 105.270 people. In infected individual decline until \(at > 10\) stable approaching 10 people, while in bacteria \(V.\ Cholera\) has fluctuated until \(t > 10\) is stable close to 130 cells/ml.

From figure 7 the susceptible individual increases until \(t > 180\) is stable approaching 104.930 people. In infected individual decline until \(at > 10\) stable approaching 10 people, while in bacteria \(V.\ Cholera\) has fluctuated until \(t > 130\) is stable close to 80 cells/ml.

**Table 2.** Total populations in 140th day with various values of \(c\)

| Simulations | \(c\) | Susceptible Individuals (person) | Infected Individuals (person) | \(V.\ Cholerae\) (cell/ml) | \(R_0\) |
|-------------|------|----------------------------------|------------------------------|--------------------------|-------|
| 1           | 0.5  | 105.450                          | 10                           | 210                      | 2.65  |
| 2           | 0.7  | 105.270                          | 10                           | 130                      | 2.17  |
| 3           | 0.9  | 104.930                          | 10                           | 80                       | 1.91  |

From table 2, it appears that the increase of the disinfection rate can reduce the populations.

6. Conclusion

This paper has proposed a mathematical model in the form of first-order non-linear differential equation systems with 4 variables \(SIRB\). The disease-free equilibrium point is locally asymptotically stable if \(R_0 < 1\), while the endemic equilibrium point is locally asymptotically stable if \(R_0 > 1\). The results of the numerical simulations obtained \(R_0 = 0.87\) indicated that the DFE point is locally asymptotically stable. In endemic conditions (\(R_0 > 1\)) show that increasing the rate of vaccination and disinfection can reduce the population of susceptible, infected, and bacteria of \(V.\ Cholerae\).
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