Solutions of Fractional Verhulst Model by Modified Analytical and Numerical Approaches

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Abstract In this chapter, we are interested in the fractional release of the Verhulst model according to Caputo’s sense, which is popular in applying environmental, biological, chemical and social studies describing the population growth model. Such a model, which is sometimes called logistic growth model related to systems in which the rate of change depends on their previous memory. In the light of this, three advanced numerical and analytical algorithms are presented to obtain approximate solutions for different classes of logistical growth problems, including reproducing kernel algorithm, fractional residual series algorithm and successive substitutions algorithm. The first technique relies on the reproducing property that characterises a specific function of building a complete orthogonal system at desired Hilbert spaces. The RPS technique relies on residual error function and generalised Taylor series to reduce residual errors and generate a converging power series, while the last technique converts the fractional logistic model to Volterra integral equation based on Riemann-Liouville integral operator. To demonstrate consistency with the theoretical framework, some realistic applications are tested to show the accuracy and efficiency of the proposed schemes. Numerical results are displayed in tables and figures for different fractional orders to illustrate the effect of the fractional parameter on population growth behaviour. The results confirm that the proposed schemes are very convenient, effective and do not require long-term calculations.
1 Introduction

Mathematical modelling is an important branch of mathematics to deal with various aspects of real applications that occur in engineering, science and social studies. There are many types of models, in particular, the mathematical model concerned with population dynamics, which is called the logistic growth model, first proposed by Pierre Verhulst in 1838 [1]. So, it is sometimes called the Verhulst model. Anyhow, the continuous form of the non-linear logistical equation proposed at that time was as follows:

\[ \frac{dW}{dt} = CW \left( 1 - \frac{W}{k} \right), \]

where \( W(t) \) is population at time \( t \), \( C > 0 \) is Malthusian parameter concerned with the growth rate and \( k \) describes the carrying capacity.

Thus, if we set \( \omega(t) = \frac{W}{k} \), the following standard logistic differential equation can be converted as

\[ \frac{d\omega}{dt} = \delta \omega(1 - \omega), \tag{1} \]

which has the following exact solution

\[ \omega(t) = \frac{\omega_0}{\omega_0 + (1 - \omega_0)e^{-\delta t}}, \tag{2} \]

whereas \( \omega_0 = \omega(0) \) is related to the initial population.

Logistic differential equation has many applications such as modelling of population growth of tumours in medicine [2], modelling of social dynamics of replacement technologies by Fisher and Pry [3] and the adaptability of society to innovation. On the other hand, the generalisation of the classical concept of the derivatives of the integer-order system to derivatives of arbitrary order is nowadays a growing interest area of mathematics to study various natural phenomena. Recent advancement on fractional operators has demonstrated that generalised fractional models are much better compared to classic integer-order models. This feature is due to a variety of choices of the order that often lead to better results. The classical derivative describes the instantaneous rate of change of a given function, while the fractional parameter can be interpreted as a memory indication of the function variance considering previous moments. These benefits encouraged researchers to use fractional operators when studying complex physical phenomena, especially dealing with memory processes or viscoelastic materials [4–7]. To read more about fractional calculus applications in rheology modelling, fluid mechanics, biology, entropy theory, etc., the reader can refer to [8–11]. In fact, there are many definitions of fractional derivatives and integrals. Some of these are Riemann-Liouville, Caputo, Grunwald-Letinkov, Weyl, Riesz, Feller, Caputo-Fabrizio and Atangana-Baleanu [12–15]. In this chapter, the
fractional derivative $D^\alpha$ is considered in the Caputo sense which has the property that the derivative of any constant is zero and hence the initial conditions of the fractional differential equation takes on the classical form similar to those of integer order.

The fractional form of the logistic differential equation can be obtained by replacing the first-order derivative in Eq. (1) with a fractional derivative $D^\alpha$, $0 < \alpha \leq 1$. Consequently, we consider the fractional logistic differential equation (FLDE) of the following form:

$$D^\alpha \omega(t) = \delta \omega(t)(1 - \omega(t)), \quad t > 0, \quad \delta > 0,$$

subject to the initial condition

$$\omega(0) = \omega_0, \quad \omega_0 > 0.$$

The stability, existence and uniqueness of the FLDE have been studied in [14]. For $\alpha = 1$, the FLDE in (3) reduces to the classic LDE in (1). Recently, West in [15] has provided the function

$$\omega(t) = \sum_{k=0}^{\infty} \frac{\omega_0 - 1}{\omega_0} E_\alpha(-k\delta t^\alpha), \quad t \geq 0,$$

as an exact solution to the FLDEs (3) and (4). Here, $E_\alpha(z)$ is the Mittag-Leffler function that is defined as a power series given by $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$. But in [16], Area et al. proved that the function in (5) is the solution of (3) if and only if the power is first order, so it is not the exact solution of arbitrary fractional order. Indeed, most fractional derivatives have a non-local property. So, the exact solutions of fractional differential equations are not easy to obtain. This is one of the reasons that why fractional calculus is more popular and a rich field for research. However, despite recent major efforts on the topic of fractional calculus, there are no methods in literature to produce exact solutions of non-linear fractional differential equations till now. So, approximate and numerical methods are needed. Several numerical techniques have been applied to solve fractional logistic problem in Caputo sense, including homotopy perturbation method spectral Laguerre collocation method, variational iteration method, operational matrices of Bernstein polynomials and finite difference method [17–23]. Different types of numerical method can be found in [24–31].

In this chapter, we employ three different methods to obtain approximate solutions for the FLDE and to study the effect of a fractional operator to the curves of population growth of species. The first method is the reproducing kernel Hilbert space (RKHS) method [32–35]. This method has many advantages; first, it is global in nature due to its ability to solve different mathematical and physical problems, especially non-linear problems; second, it is accurate and its numerical results can be obtained easily; third, it doesn’t require discretisation of the variables, and it is not affected by round off errors of computations. Reproducing kernel theory has important applications in mathematics, image processing, machine learning, finance and
probability [36–40]. Recently, a lot of research work has been devoted to the applications of RKHS method for wide classes of stochastic and deterministic problems involving operator equations, differential equations and integro-differential equations. To see these applications, the reader can refer to [41–46] and the references therein.

The second proposed method in this chapter is the fractional residual power series (FRPS) method in which the solution is assumed to have a generalised fractional Taylor series representation. This technique has been first developed to approximate numerical solutions for certain class of differential equations under uncertainty. Later, the generalised Lane-Emden equation has been investigated numerically by utilising the FRPS method. Also, the method was applied successfully in solving composite and non-composite fractional differential equations, fractional boundary value problems, stiff systems and time-fractional Fokker–Planck models [47–52]. Further, [53] asserts that the FRPS method is easy and powerful to construct power series solution for strongly linear and non-linear equations without terms of perturbation, discretisation and linearisation. Unlike the classical power series method, the FRPS method distinguishes itself in several important aspects such as it does not require making a comparison between the coefficients of corresponding terms and a recursion relation is not needed. It provides a direct way to ensure the rate of convergence for series solution by minimising the residual error-related [54]. In fact, it is just a repetition of Caputo derivative to obtain the values of unknown coefficients of desired fractional series solution by solving sequence of algebraic equations with choosing a fit initial data. The gained series solution and all fractional derivatives are valid for all mesh points of the domain of interest. The third technique that we apply in this chapter for solving FLDE depends on a property of Riemann-Liouville integral and Caputo derivative. We just apply the Riemann-Liouville integral to both sides of the FLDE then use a successive substitution (SS) to solve the resulting integral equation starting by the initial data. The simplicity and accuracy of this iterative scheme appears in the numerical and graphical results.

This chapter is outlined in five sections. In Sect. 2, some basics and needed properties related to fractional calculus and generalised fractional power series are given. Section 3 consists out of three subsections. Each subsection is a brief description of one of the proposed techniques with the main definitions, theorems and algorithms to construct analytic and approximate solutions of the FLDE. In Sect. 4, two examples are carried out to illustrate the reliability and the simplicity of the proposed methods and to display how the fractional operator affects the behaviour of the solution of the population growth curves. Finally, a brief conclusion is presented in Sect. 5.

2 Preliminaries of Fractional Calculus

This section is devoted to introducing some necessary definitions and mathematical preliminaries of fractional calculus, especially those related to Caputo operator, and basics of the generalised fractional power series. For more details, the reader can
refer to [8, 11, 35, 42] for more details about fractional derivatives. Throughout this chapter, we deal with the following spaces:

- \( L_p[a, b] = \left\{ \omega : [a, b] \to \mathbb{R} : \int_a^b |\omega(x)|^p dx < \infty, \right\} \), \( 1 \leq p < \infty; \)
- \( C[a, b] = \{ \omega : [a, b] \to \mathbb{R} : \omega \) is continuous on \([a, b])\};
- \( AC[a, b] = \{ \omega : [a, b] \to \mathbb{R} : \omega \) is absolutely continuous on \([a, b])\};
- \( AC^n[a, b] = \{ \omega : [a, b] \to \mathbb{R} : \omega \) is \( n \) time absolutely continuous on\([a, b])\}.

For the last two spaces, a function \( \omega : [a, b] \to \mathbb{R} \) is called absolutely continuous, if \( \forall \epsilon > 0, \) there is \( \delta > 0 \) such that for any finite set of disjoint intervals \( \{[x_i, y_i], i = 1, 2, \ldots, k\} \) in \([a, b]\) with \( \sum_{i=1}^k |y_i - x_i| < \delta \) then \( \sum_{i=1}^k |\omega(y_i) - \omega(x_i)| < \epsilon \).

Among many definitions for the integrals and derivatives of fractional order, the most famous of them are the definitions of Riemann-Liouville fractional integral and Caputo fractional derivative which have some privacy. In this section, we give these definitions and some of their properties.

**Definition 1** The Riemann-Liouville fractional integral of order \( \alpha > 0 \) over the interval \([a, b]\) for a function \( \omega \in L_1[a, b] \) is defined by

\[
(J^\alpha_{a+}\omega)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\omega(z)}{(t-z)^{1-\alpha}} d\xi, \quad t > a.
\]

For \( \alpha = 0, J^0_{a+} \) is the identity operator.

**Definition 2** The Riemann-Liouville fractional derivative of order \( \alpha \) for a function \( \omega \in AC^n[a, b], n \in \mathbb{N}, t > a, n - 1 < \alpha \leq n, \) is defined by

\[
(D^\alpha_{a+}\omega)(t) = \left( \frac{d}{dt} \right)^n (J^{n-\alpha}_{a+}\omega)(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t \frac{g(\xi)}{(t-\xi)^{\alpha-n+1}} d\xi.
\]

In particular, if \( 0 < \alpha < 1, \) then \((D^\alpha_{a+}\omega)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{\omega(t)}{(t-\xi)^\alpha} d\xi, \quad t > a.\)

**Definition 3** The Caputo fractional derivative of order \( \alpha > 0 \) for a function \( \omega \in AC^n[a, b], n \in \mathbb{N}, \) is defined by

\[
(CD^\alpha_{a+}\omega)(t) = \left( J^{n-\alpha}_{a+}D^n \omega \right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{\omega^{(n)}(\xi)}{(t-\xi)^{\alpha-n+1}} d\xi, \quad t > a, n - 1 < \alpha \leq n.
\]

Specifically, for \( 0 < \alpha < 1, \) and \( \omega(t) \in AC[a, b], \) we have \((CD^\alpha_{a+}\omega)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{\omega(\xi)}{(t-\xi)^\alpha} d\xi.\)

Some properties of the Caputo fractional derivative and Riemann-Liouville integral are given in the following theorem [11]:

**Theorem 1** Let \( \omega \in AC^n[a, b] \) and \( \alpha > 0. \) Then,
\[ (J_{a+}^\alpha C_{a+}^\alpha \omega)(t) = \omega(t) - \sum_{k=0}^{n-1} \frac{\omega^{(k)}(a)}{k!} (t - a)^k. \]

In particular, if \( 0 < \alpha \leq 1 \) and \( f \in AC[a, b] \) or \( \omega \) is continuous on \([a, b] \), then
\[ (J_{a+}^\alpha C_{a+}^\alpha \omega)(t) = \omega(t) - \omega(a). \]

The Caputo fractional derivative can be easily computed for any function \( \omega(t) \) through the formula
\[ C_{a+}^\alpha \omega(t) = \sum_{n=0}^{\infty} \frac{\omega^{(n)}(\xi)}{\Gamma(n + 1 - \alpha)} (t - \xi)^{n\alpha}, \quad 0 \leq \xi < t, \alpha > 0. \]

The next theorem shows the relations between Caputo and Riemann-Liouville fractional derivatives.

**Theorem 2** Let \( \alpha > 0 \) and \( n - 1 < \alpha \leq n \) for \( n \in \mathbb{N} \). Then
\[ (C_{a+}^\alpha D_{a+}^\alpha \omega)(t) = \left(D_{a+}^\alpha \left[\omega(x) - \sum_{k=0}^{n-1} \frac{\omega^{(k)}(a)}{k!} (x - a)^k\right]\right)(t).\]

However, since the Caputo fractional derivative will be only used in this chapter with \( a = t_0 = 0 \), then the symbol \( D_{a+}^\alpha \) will be instead of \( C_{a+}^\alpha \).

**Definition 4** A power series (PS) expansion at \( t = t_0 \) of the following form
\[ \sum_{m=0}^{\infty} a_m(t - t_0)^{m\alpha} = a_0 + a_1(t - t_0)^\alpha + a_1(t - t_0)^{2\alpha} + \cdots, \]
for \(-1 < \beta \leq n, n \in \mathbb{N} \) and \( t \leq t_0 \), is called the fractional power series (FPS) \([48]\).

**Theorem 3** There are only three possibilities for the FPS \( \sum_{m=0}^{\infty} a_m(t - t_0)^{m\alpha} \). They are as follows:

1. The series converges only for \( t = t_0 \). That is, the radius of convergence equals zero.
2. The series converges for all \( t \geq t_0 \). That is, the radius of convergence equals infinity.
3. The series converges for \( t \in [t_0, t_0 + R) \), for some positive real number \( R \) and diverges for \( t > t_0 + R \). Here, \( R \) is the radius of convergence for the FPS.

**Theorem 4** Suppose that \( \omega(t) \) has FPS representation at \( t = t_0 \) of the form
\[ \omega(t) = \sum_{m=0}^{\infty} c_m(t - t_0)^{m\alpha}. \]
If \( \omega(t) \in C[t_0, t_0 + R] \), and \( D^{\alpha_0} \omega(t) \in C(t_0, t_0 + R) \), for \( m = 0, 1, 2, \ldots \), then coefficients \( c_m \) will be in the form \( c_m = \frac{T_r^{\alpha_0 \omega(t_0)}}{T(m)!} \), where \( D^{\alpha_0} = D^\alpha \cdot D^\alpha \cdots D^\alpha \) (\( m \)-times) \[49\].

### 3 Description of FRPS, RKHS and SS Approaches

In this section, we give a brief description for the three proposed techniques that we will apply to solve fractional LDE. For each method, we present some necessary definitions and prove some convergence results. We summarise each proposed procedure by an algorithm.

#### 3.1 The Reproducing Kernel Hilbert Space Method

**Definition 5** Let \( S \) be any nonempty abstract set. Then a function \( K : S \times S \to \mathbb{C} \) is called a reproducing kernel of the Hilbert space \( H \) if and only if

(a) \( \forall t \in S, K(\cdot, t) \in \mathcal{H}, \)

(b) \( \forall t \in S, \forall \omega \in \mathcal{H}, (\omega(t), K(\cdot, t)) = \omega(t). \)

The condition in (b) is called “the reproducing property” because it indicates that the value of the function \( \varphi \) at the point \( t \) is reproduced by the inner product of \( \omega \) with \( K(\cdot, t) \) \[34\]. The function \( K \) is called the reproducing kernel function of \( H \).

This function possesses some important properties such as being unique, conjugate symmetric and positive-definite. A Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS).

**Definition 6** The space of functions \( W^2_1[a, b] \) is defined as

\[
W^2_1[a, b] = \{ \omega : [a, b] \to \mathbb{R} : \omega \in AC[a, b], o' \in L^2[a, b] \}.
\]

The inner product and norm are given, respectively, by \( \langle \omega_1, \omega_2 \rangle_{W^2_1} = \int_a^b (\omega_1(t) \omega_2(t) + \omega_1'(t) \omega_2'(t)) dt \) and \( \| \omega_1 \|_{W^2_1} = \sqrt{\langle \omega_1(t), \omega_1(t) \rangle_{W^2_1}}, \omega_1, \omega_2 \in W^2_1[a, b] \) \[33\].

**Theorem 5** The space \( W^2_1[a, b] \) is a complete RKHS with the reproducing kernel function \( T_r(s) \) such that

\[
T_r(s) = \frac{1}{2\sinh(b - a)}[\cosh(t + s - b - a) + \cosh(|t - s| - b + a)].
\]

**Definition 7** The space of real functions \( W^2_2[a, b] \) is defined as follows:
\[ W^2[a, b] = \{ \omega : \omega, \omega' \in AC[a, b], \omega'' \in L_2[a, b], \omega(a) = 0 \}. \]

The inner product and the norm are, respectively, given by \( \langle \omega_1, \omega_2 \rangle_{W^2} = \omega_1(a)\omega_2(a) + \int_a^b \omega_1'(t)\omega_2'(t)dt \) and \( \| \omega_1 \|_{W^2} = \sqrt{\langle \omega_1(t), \omega_1(t) \rangle_{W^2}}, \omega_1, \omega_2 \in W^2[a, b] \) \cite{[34]}

**Theorem 6** The space \( W^2[a, b] \) is a RKHS and its reproducing kernel function \( K_t(s) \) has the form

\[
K_t(s) = \begin{cases} 
\frac{1}{6}(s-a)(2a^2 - s^2 + 3t(2 + s) - a(6 + 3s + t)), & s \leq t, \\
\frac{1}{6}(t-a)(2a^2 - t^2 + 3s(2 + t) - a(6 + 3s + t)), & s > t. 
\end{cases}
\]

Now, let us consider the FLDE in \( (3) \) with the initial condition \( (4) \). To solve this equation using the RKHSM, we first homogenise the initial condition using the substitution \( P(t) = \omega(t) - \omega_0 \) to get

\[ D^\alpha P(t) + D^\alpha \omega_0 = \delta(P(t) + \omega_0)(1 - P(t) - \omega_0). \]

But \( D^\alpha \omega_0 = 0 \), so Eqs. \( (3) \) and \( (4) \) becomes

\[ D^\alpha P(t) = \delta(P(t) + \omega_0)(1 - P(t) - \omega_0), P(0) = 0. \] \quad (7)

Now, by defining the differential operator \( L : W^2[a, b] \rightarrow W^1_2[a, b] \) such that \( LP(t) = D^\alpha P(t) \), the Eq. \( (7) \) can be rewritten as \( LP(t) = \delta(P(t) + \omega_0)(1 - P(t) - \omega_0), \quad t > 0. \)

**Theorem 7** The operator \( L \) is a bounded linear from \( W^2[a, b] \) to \( W^1_2[a, b] \) such that \( LP(t) = D^\alpha P(t) \).

**Proof** The linearity of the operator \( L \) results from the linearity of the Caputo derivative, so it is clear. For boundedness, we need to find \( M > 0 \) such that \( \| LP \|_{W^2_2} \leq M \| P \|_{W^2_2} \) \( \forall P \in W^2_2[a, b] \). To do this, we have

\[
\| LP \|^2_{W^2_2} = \langle LP, LP \rangle_{W^2_2} = \int_a^b \left\{ \left[ (LP)(t) \right]^2 + \left[ (LP)'(t) \right]^2 \right\} dt.
\]

By the reproducing property of \( K_t(s) \), we can write \( P(t) = \langle P(.), K_t(.) \rangle_{W^2_2} \) and \( (D^\alpha P)(t) = \langle P(.), (D^\alpha K_t)(.) \rangle_{W^2_2} \). Using Schwarz Inequality and the fact that \( D^\alpha K_t(s) \) is continuous and uniformly bounded in \( s \) and \( t \), we get \( |(LP)(t)| = |(D^\alpha P)(t)| \leq \| P \|_{W^2_2} \| (D^\alpha K_t) \|_{W^2_2} = M_1 \| P \|_{W^2_2} \) and \( |(LP)'(t)| = |\frac{d}{dt}(D^\alpha P)(t)| = \left| \left[ \langle P(.), \frac{d}{dt}(D^\alpha K_t)(.) \rangle_{W^2_2} \right] \right| \leq \| P \|_{W^2_2} \| \frac{d}{dt}(D^\alpha K_t) \|_{W^2_2} = M_2 \| P \|_{W^2_2} \)

Therefore, \( \| LP \|^2_{W^2_2} \leq \left( (M_1)^2 + (M_2)^2 \right) (b - a) \| P \|^2_{W^2_2} = \| LP \|_{W^2_2} \leq M \| P \|_{W^2_2} \).
where $M_{ir} = \sqrt{(M^2_1 + M^2_2)(b - a)}$. Hence, $L$ is bounded.

Now, we construct an orthogonal function system of the space $W^2_2[a, b]$ by considering the countable dense set $\{t_i\}_{i=1}^{\infty}$ of the interval $[a, b]$, and let $\varphi_i(t) = T_i(t)$ and $\psi_i(t) = L^* \varphi_i(t)$, where $L^*$ is the adjoint operator of $L$. In terms of the properties of the reproducing kernel $T_i(.)$, we have $\langle P(t), \psi_i(t) \rangle_{W^2_2} = \langle P(t), L^* \varphi_i(t) \rangle_{W^2_2} = \langle LP(t), \varphi_i(t) \rangle_{W^2_2} = LP(t_i), i = 1, 2, \ldots$. Using the Gram-Schmidt orthogonalisation process on $\{\psi_i(t)\}_{i=1}^{\infty}$, we can form the orthonormal function system $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ of $W^2_2[a, b]$ by defining $\overline{\psi}_i(t) = \sum_{l=1}^{i} \beta_{il} \psi_l(t), i = 1, 2, 3, \ldots$, where $\beta_{il}$ are the orthogonalisation coefficients which are given by:

$$
\beta_{i1} = \frac{1}{\|\psi_1\|_{W^2_2}},
$$

$$
\beta_{il} = \frac{1}{\sqrt{\|\psi_l\|_{W^2_2}^2 - \sum_{p=1}^{i-1} \langle \psi_l(t), \overline{\psi}_p(t) \rangle_{W^2_2}^2}},
$$

and

$$
\beta_{il} = \frac{-\sum_{p=1}^{i-1} \langle \psi_l(t), \overline{\psi}_p(t) \rangle_{W^2_2}}{\sqrt{\|\psi_l\|_{W^2_2}^2 - \sum_{p=1}^{i-1} \langle \psi_l(t), \overline{\psi}_p(t) \rangle_{W^2_2}^2}},
$$

for $i > 1$.

**Theorem 8** If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a, b]$ and assuming that the solution of Eq. (7) is unique, then it has the form:

$$
P(t) = \delta \sum_{i=1}^{\infty} \sum_{l=1}^{i} \beta_{il} (P(t_l) + \omega_0)(1 - P(t_l) - \omega_0)\overline{\psi}_l(t).
$$

The $n$-term approximate solution $P^n(t)$ of $P(t)$ is given by the finite sum:

$$
P^n(t) = \delta \sum_{i=1}^{n} \sum_{l=1}^{i} \beta_{il} (P(t_l) + \omega_0)(1 - P(t_l) - \omega_0)\overline{\psi}_l(t).
$$

**Proof** According to the orthonormal basis $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ of $W^2_2[a, b]$, we have

$$
P(t) = \sum_{i=1}^{\infty} \langle P(t), \overline{\psi}_i(t) \rangle_{W^2_2} \overline{\psi}_i(t)
$$

$$
= \sum_{i=1}^{\infty} \sum_{l=1}^{i} \beta_{il} \langle P(t), L^* \varphi_l(t) \rangle_{W^2_2} \overline{\psi}_l(t)
$$

$$
= \sum_{i=1}^{\infty} \sum_{l=1}^{i} \beta_{il} \langle \delta(P(t) + \omega_0)(1 - P(t) - \omega_0), \varphi_l(t) \rangle_{W^2_2} \overline{\psi}_l(t)
$$
\[
= \delta \sum_{i=1}^{\infty} \sum_{l=1}^{i} \beta_{il}(P(t_l) + \omega_0)(1 - P(t_l) - \omega_0)\overline{\psi}_i(t).
\]

**Theorem 9** For any function \(\omega(t)\) in \(W_2^2[a, b]\), the approximate series \(\omega^n(t)\) and \(\frac{d\omega^n}{dt}\) are uniformly convergent to \(\omega(t)\) and \(\omega'(t)\), respectively.

According the previous discussion, the approximate solution of (3) and (4) is

\[
\omega^n(t) = \delta \sum_{i=1}^{n} \sum_{l=1}^{i} \beta_{il}(P(t_l) + \omega_0)(1 - P(t_l) - \omega_0)\overline{\psi}_i(t) + \omega_0.
\]

It is worth to mention that we can define another operator in order to solve the non-linear FLDE by applying the Riemann-Liouville fractional integral \(J_0^\alpha\) to the two sides of Eq. (7). So \(J_0^\alpha[D^\alpha P](t) = J_0^\alpha[\delta(P(t) + \omega_0)(1 - P(t) - \omega_0)]\) which are equivalent to

\[
P(t) = \frac{\delta}{\Gamma(\alpha)} \int_0^t (P(s) + \omega_0)(1 - P(s) - \omega_0)\frac{ds}{(t - s)^{1-\beta}}, \quad t > 0.
\] (8)

Define the operator \(I : W_2^2[a, b] \rightarrow W_2^1[a, b]\) by \(IP(t) = P(t)\). Obviously, \(I\) is a bounded linear operator. Thus, (7) can be rewritten as \(IP(t) = \delta(P(t) + \omega_0)(1 - P(t) - \omega_0)\). Applying the RKHS method with the operator \(I\), we can obtain approximate solution of (3) and (4) of the form

\[
P^n(t) = \frac{\delta}{\Gamma(\alpha)} \sum_{i=1}^{n} \sum_{l=1}^{i} \beta_{il}^n(P(t_l) + \omega_0)(1 - P(t_l) - \omega_0)\overline{\psi}_i(t),
\]

which converges to the analytic solution

\[
P^n(t) = \frac{\delta}{\Gamma(\alpha)} \sum_{i=1}^{\infty} \sum_{l=1}^{i} \beta_{il}^n(P(t_l) + \omega_0)(1 - P(t_l) - \omega_0)\overline{\psi}_i(t).
\]

**Algorithm 1** To approximate the solution of the LFDE (3) and (4), we do the following steps:

**Step 1:** Put \(P(t) = \omega(t) - \omega_0\) to homogenise the initial condition.

**Step 2:** Define the reproducing function over \([a, b]\) as

If \(s \leq t\), set \(K_i(s) = \frac{1}{6}(s - a)\left(2a^2 - s^2 + 3t(2 + s) - a(6 + 3t + s)\right)\);

else set \(K_i(s) = \frac{1}{6}(t - a)\left(2a^2 - t^2 + 3s(2 + t) - a(6 + 3s + t)\right)\).

**Step 3:** To obtain orthogonal function system \(\psi_{ij}(t)\) for \(i = 1, 2, \ldots, n\), do the following:

Set \(t_i = a + \frac{i - 1}{n - 1}(b - a)\) so that \(\psi_i(t) = D^\alpha[K_i(s)]_{s=t_i}\);
Step 4: For $i = 1, \ldots, n$ and $l = 1, 2, \ldots, i - 1$, set:

$$\beta_{11} = \frac{1}{\|\psi_1\|_W^2},$$

$$\beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|_W^2 - \sum_{p=1}^{i-1} \langle \psi_i(t), \overline{\psi_p(t)} \rangle^2_W}},$$

and

$$\beta_{il} = \frac{-\sum_{p=1}^{i-1} \langle \psi_i(t), \overline{\psi_p(t)} \rangle^2_W \beta_{pl}}{\sqrt{\|\psi_i\|_W^2 - \sum_{p=1}^{i-1} \langle \psi_i(t), \overline{\psi_p(t)} \rangle^2_W}},$$

for $i > l$.

Step 5: For $i = 1, \ldots, n$, $\overline{\psi_i(t)} = \sum_{l=1}^{i} \beta_{il} \psi_i(t)$ to get the orthonormal function system $\overline{\psi_i(t)}$.

Step 6: Set $\omega^0(t) = \delta \sum_{i=1}^{n} \sum_{l=1}^{i} \beta_{il}(P(t_l) + \omega_0)(1 - P(t_l) - \omega_0)\overline{\psi_i(t)} + \omega_0$ to obtain the approximate RKHS solution for the FLDE.

### 3.2 The Fractional Residual Power Series Method

In this subsection, we describe the main steps needed to implement the FRPS algorithm in order to solve the continuous logistic equation in the fractional sense by expanding FPS and utilising repeated fractional differentiation of the truncated residual functions. This analysis aims to extend the application of fractional Taylor series framework to get an accurate analytic series solution of fractional system (3) and (4). To perform this, let the fractional logistic equation (3) and (4) has the solution form about $t = 0$:

$$\omega(t) = \sum_{n=0}^{\infty} a_n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \quad (9)$$

Thus, if we use the initial data given by (4), $D_0^\alpha \omega(0) = \omega_0$, as initial truncated series of $\omega(t)$, so the FPS solution of Eq. (3) can be written by

$$\omega(t) = \omega_0 + \sum_{n=1}^{\infty} a_n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \quad (10)$$

Therefore, the $j$-th truncated series solution of $\omega(t)$ is given by

$$\omega_j(t) = \omega_0 + \sum_{n=1}^{j} a_n \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}. \quad (11)$$
According to the FRPS approach, we define the $j$-th residual function, $\text{Res}_j^\omega(t)$, for the proposed logistic model as follows

$$\text{Res}_j^\omega(t) = D_0^\alpha \omega_j(t) - \delta \omega_j(t)(1 - \omega_j(t)), \quad j = 1, 2, 3, \ldots, \quad (12)$$

whereas the residual function, $\text{Res}_\omega(t)$, can be defined by

$$\text{Res}_\omega(t) = \lim_{j \to \infty} \text{Res}_j^\omega(t) = D_0^\alpha \omega(t) - \delta \omega(t)(1 - \omega(t)).$$

In this point, we noted that $\text{Res}_\omega(t) = 0$ for all $t \geq 0$, which leads to $D_0^k \text{Res}_\omega(0) = D_0^k \text{Res}_\omega(0) = 0$, for all $k = 1, 2, \ldots, j$. Consequently, the following fractional relations assist us to determine the unknown coefficients, $a_n$, $n = 1, 2, \ldots, j$, of Eq. (10)

$$D_0^{(j-1)\alpha} \omega(0) = 0, \quad j = 1, 2, 3, \ldots \quad (13)$$

To show the iteration concept of the FRPS technique to find out $a_1$, follow the steps:

- Let $\omega_1(t) = \omega_0 + a_1 \frac{t^\alpha}{\Gamma(\alpha + 1)}$;
- Substitute in Eq. (12) at $j = 1$ to get that

$$\text{Res}_1^\omega(t) = \delta(\omega_0 - 1) \omega_0 + a_1 \left( 1 + \frac{\delta(2\omega_0 - 1)}{\Gamma(\alpha + 1)} t^\alpha \right) + \frac{\delta a_1^2}{\Gamma^2(\alpha + 1)} t^{2\alpha}. \quad (14)$$

Set $\text{Res}_1^\omega(0) = 0$, which yields

$$a_1 = \delta \omega_0(1 - \omega_0). \quad (15)$$

Thus, we get the first FRPS approximation for Eqs. (3) and (4) as

$$\omega_1(t) = p_0 \left( 1 + \frac{\delta(1 - \omega_0)}{\Gamma(\alpha + 1)} t^\alpha \right).$$

In the same way, to find out $a_2$, follow the steps:

- Put the second truncated series

$$\omega_2(t) = \omega_0 \left( 1 + \frac{\delta(1 - \omega_0)}{\Gamma(\alpha + 1)} t^\alpha \right) + a_2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)};$$
• Substitute it in \(\text{Res}_\omega^2(t)\) such as
\[
\text{Res}_\omega^2(t) = D_0^\alpha \omega_2(t) - \delta \omega_2(t)(1 - \omega_2(t)) = \delta \omega_0(1 - \omega_0) \\
\left(1 + \left(1 + \frac{\delta(2\omega_0 - 1)}{\Gamma(\alpha + 1)} t^\alpha + a_2 \frac{2\delta}{\Gamma(\alpha + 1) \Gamma(2\alpha + 1)} t^{2\alpha}\right)\right) + \frac{\delta^2 \omega_0^2(1 - \omega_0)^2}{\Gamma^2(\alpha + 1)} t^{2\alpha} \\
+ a_2 \left(\frac{1}{\Gamma(\alpha + 1)} t^\alpha + \frac{\delta(2\omega_0 - 1)}{\Gamma(\alpha + 1)} t^{2\alpha}\right) + a_2^2 \frac{\delta}{\Gamma^2(2\alpha + 1)} t^{4\alpha}. \tag{16}
\]

• Apply the operator \(D_0^\alpha\) on both sides of Eq. 16 to get
\[
D_0^\alpha \text{Res}_\omega^2(t) = \frac{\lambda \delta^3 \Gamma(2\alpha + 1) \omega_0^2(1 - \omega_0)^2}{\Gamma^3(\alpha + 1)} t^\alpha + \delta^2 \omega_0(1 - \omega_0) \\
\left(2\omega_0 - 1 + a_2 \frac{2\delta}{\Gamma(3\alpha + 1)} \right) t^{2\alpha} \\
+ a_2 \left(1 + \frac{\delta(2\omega_0 - 1)}{\Gamma(\alpha + 1)} t^\alpha + \frac{\delta \Gamma(4\alpha + 1) a_2}{\Gamma^2(2\alpha + 1) \Gamma(3\alpha + 1)} t^{2\alpha}\right). \tag{17}
\]

• Use the fact \(D_0^\alpha \text{Res}_\omega^2(0) = 0\) in the above equation, to obtain
\[
a_2 = -\delta^2 \omega_0(1 - \omega_0)(2\omega_0 - 1).
\]

• Rewrite the second FRPS approximation as
\[
\omega_2(t) = \omega_0 \left(1 - \delta \frac{(\omega_0 - 1)}{\Gamma(\alpha + 1)} t^\alpha + \delta^2 \frac{(\omega_0 - 1)(2\omega_0 - 1)}{\Gamma(2\alpha + 1)} t^{2\alpha}\right).
\]

Similarly, we have
• Let \(\omega_2(t) = \omega_0 + \sum_{n=1}^{3} a_n \frac{t^{\omega_n}}{\Gamma(\omega_n + 1)}\); 
• Substitute it into the residual function \(\text{Res}_\omega^3(t)\) such that
\[
\text{Res}_\omega^3(t) = D_0^\alpha \omega_3(t) - \delta \omega_3(t)(1 - \omega_3(t)).
\]

• Apply \(D_0^{2\alpha}\) to \(\text{Res}_\omega^3(t)\).
• Solve \(D_0^{2\alpha} \text{Res}_\omega^3(t)\big|_{t=0} = 0\) for \(a_3\) to get
\[ a_3 = \delta^3 \omega_0 (1 - \omega_0) \left( \frac{\Gamma^2(\alpha + 1) - \omega_0 (1 - \omega_0) (4 \Gamma^2(\alpha + 1) + \Gamma(2\alpha + 1))}{\Gamma^2(\alpha + 1)} \right). \]

- The third FRPS approximation is

\[ \omega_3(t) = \omega_0 (1 - \delta (\omega_0 - 1) t^\alpha + \delta^2 (\omega_0 - 1)(2\omega_0 - 1) t^{2\alpha} \]

\[ - \delta^3 (\omega_0 - 1)(\Gamma^2(\alpha + 1) + \omega_0 (\omega_0 - 1)(4 \Gamma^2(\alpha + 1) + \Gamma(2\alpha + 1))) \frac{t^{3\alpha}}{\Gamma^2(\alpha + 1) \Gamma(3\alpha + 1)} \]

Further, when the same routine is repeated as above up to the arbitrary order \( k \), the coefficients \( a_n, n = 4, 5, 6, \ldots, k \), can be obtained. Anyhow, the next algorithm summarises the procedure in the previous discussion to find the unknown coefficients of the fractional power series that represents the solution of the FLDE.

**Algorithm 2** To determine the required coefficients of \( \omega(t) \) in (11) which represents the analytic solution of (3) and (4), do the following steps:

**Step 1:** Consider the initial condition \( \omega_0 = \omega(0) \), which is the zeroth FPS approximate solution of \( \omega(t) \).

**Step 2:** Substitute \( \omega_j(t) = \omega_0 + \sum_{n=1}^{j} a_n t^{n\alpha} \) into the \( j^{th} \) residual function \( Res^j_\omega(t) \).

**Step 3:** Compute \( D_i^{(j-1)\alpha} Res_i^j_\omega(t) \) for \( j = 1, 2, \ldots, k \).

**Step 4:** Solve the resulting fractional equation \( D_i^{(j-1)\alpha} Res_i^j_\omega(0) = 0 \) for \( a_j \).

**Step 5:** Substitute the obtained coefficients, for \( j = 1, 2, \ldots, k \), back into Eq. (7) which yields the required analytic solution.

**Lemma 1** Suppose that \( \omega(t) \in C[0, R], R > 0, D_0^{i\alpha} \omega(t) \in C(0, R), \) and \( 0 < \alpha \leq 1 \). Then for any \( j \in \mathbb{N} \), we have

\[ \left( J_0^{i\alpha} D_0^{i\alpha} \omega \right)(t) - \left( J_0^{(j+1)\alpha} D_0^{(j+1)\alpha} \omega \right)(t) = \frac{D_0^{i\alpha} \omega(0) t^{i\alpha}}{\Gamma((j+1)\alpha + 1)}. \]

**Theorem 10** Let \( \omega(t) \) has the FPS in (7) with radius of convergence \( R > 0 \), and suppose that \( \omega(t) \in C[0, R], D_0^{i\alpha} \omega(t) \in C(0, R) \) for \( j = 0, 1, 2, \ldots, N + 1 \). Then, \( \omega(t) = \omega_N(t) + R_N(\xi) \), where \( \omega_N(t) = \sum_{j=0}^{N} D_0^{i\alpha} \omega(0) t^{i\alpha} \) and \( R_N(\xi) = \frac{D_0^{(N+1)\alpha} \omega(\xi)}{\Gamma((N+1)\alpha + 1)} t^{(N+1)\alpha} \), for some \( \xi \in (0, t) \).

In view of the previous theorem, we can consider the formula of \( \omega_N(t) \) as an approximation of \( \omega(t) \), and \( R_N(\xi) \) as the truncation (remainder) error that results from approximating \( \omega(t) \) by \( \omega_N(t) \). Also, the upper bound of the error can be predicted by
\[ |R_N(\xi)| \leq \left| \text{Sup}_{t \in [0,R]} \frac{M_t^{(N+1)\alpha}}{\Gamma((N + 1)\alpha + 1)} \right|, \]

provided that \[ |D_0^{(N+1)\alpha} \omega(\xi)| < M \text{ on } [0, R). \]

### 3.3 The Successive Substitutions Technique

In this subsection, we provide a simple iterative scheme to solve system (3) and (4). To do this, by applying the Riemann-Liouville operator \( J_0^{\alpha} \) to both sides of Eq. (3), the following Volterra integral equation can be obtained

\[ \omega(t) = \omega_0 + \frac{\delta}{\Gamma(\alpha)} \int_0^t \omega(\xi)(1 - \omega(\xi))(t - \xi)^{\alpha-1} d\xi. \]

In order to approximate the solution of Volterra integral equation, and consequently the solution of the FLDE, we may apply a successive substitution (SS) technique as follows.

Let \( \omega_0(t) = \omega_0 = \omega(0) \),

\[ \omega_{n+1}(t) = \omega_0 + \frac{\delta}{\Gamma(\alpha)} \int_0^t \omega_n(\xi)(1 - \omega_n(\xi))(t - \xi)^{\alpha-1} d\xi, \ n = 0, 1, 2, \ldots \]

Hence, the first approximation is \( \omega_1(t) = \omega_0 \left( 1 - \frac{t^\alpha(-1+\omega_0)}{\Gamma(\alpha+1)} \right) \), and the second approximation is

\[ \omega_2(t) = \omega_0(\omega_0 - 1) \left( 1 - \frac{t^\alpha \delta}{\Gamma[1+\alpha]} - \frac{4^\alpha t^{3\alpha} \delta^3 \Gamma\left[\frac{1}{2} + \alpha\right]}{\sqrt{\pi} \Gamma[1+\alpha] \Gamma[1+3\alpha]} \left( \frac{\eta - 1}{\eta} \right) \right). \]

Continuing this process, we may obtain the exact solution as \( \omega(t) = \lim_{n \to \infty} \omega_n(t) \).

**Algorithm 3** To determine the nth SS approximate solution \( \omega^n(t) \) for Eq. (3) and (4), do the following two steps:

**Step 1**: Consider the initial condition \( \omega_0 = \omega(0) \) as the zeroth SS approximate solution of \( \omega(t) \).

**Step 2**: For \( k = 1, 2, \ldots, n \), substitute

\[ \omega_k(t) = \omega_0 + \frac{\delta}{\Gamma(\alpha)} \int_0^t \omega_{k-1}(\xi)(1 - \omega_{k-1}(\xi))(t - \xi)^{\alpha-1} d\xi. \]
4 Numerical Applications

In this section, we compute approximate solution for the FLDE using three numerical approaches: the RKHS, the FRPS and the SS methods. Using these techniques, we study two examples of the FLDE. For each example, we apply the proposed methods for different values of the fractional derivative. To show the efficiency and accuracy of these methods, we compare the exact with the approximate solutions in the case of integer-order derivative and compute absolute and relative errors. Moreover, we compare the numerical solutions for different orders of the Caputo derivative in graphs and tables. We observe that all the three techniques are convenient methodologies for controlling the convergence of the solution and the results are found in good agreement. The graphical results show that the variety of choice of fractional orders leads to a variety in predicted curves for the population growth, which of course, leads to better results. All computations are performed using Mathematica 10.

Example 1  Consider the following non-linear FLDE:

\[ D_0^\alpha \omega(t) = \frac{1}{3} \omega(t)(1 - \omega(t)), \quad t \in [0, 1], \quad 0 < \alpha \leq 1, \]  \tag{18} 

subject to the initial condition

\[ \omega(0) = \frac{1}{3}. \]  \tag{19} 

The exact solution of the above IVP is \( \omega(t) = \frac{e^{t/3}}{2 + e^{t/3}}. \)

Following Algorithm 1, we have to homogenise the IC to get

\[ D_0^\alpha P(t) = \frac{1}{3} \left( P(t) + \frac{1}{3} \right) \left( \frac{2}{3} - P(t) \right), \quad P(0) = 0, \quad t \in [0, 1], \quad 0 < \alpha \leq 1. \]

Taking \( n = 25 \), Table 1 shows the accuracy of this technique. While, the approximate solution of the FLDE for different values of \( \alpha \) is given in Table 2. On the other hand, following the procedures outlined in this chapter for the FRPS method, we take six iterations and summarise some of the results in Tables 3 and 4.

The fourth FPS for \( \alpha = 1, \alpha = 0.8, \alpha = 0.5 \) are listed below:

\[ \omega_{4,\alpha=1}(t) = \frac{1}{3} + 0.0740741 t + 0.00411523 t^2 - 0.0000635066 t^3; \]

\[ \omega_{4,\alpha=0.8}(t) = \frac{1}{3} + 0.0795312 t^{0.8} + 0.00575707 t^{1.6} - 0.000147395 t^{3.2}; \]
Table 1  Absolute and relative errors for Example 1 for $\alpha = 1$ using RKHS method

| $t$  | RKHS solution | Absolute error | Relative error |
|------|---------------|----------------|---------------|
| 0.0  | 0.33333333    | 0.0            | 0.0           |
| 0.1  | 0.340781423   | $6.55734649 \times 10^{-9}$ | $1.92420887 \times 10^{-8}$ |
| 0.2  | 0.348308985   | $1.31846654 \times 10^{-8}$ | $3.78533586 \times 10^{-8}$ |
| 0.3  | 0.355913052   | $1.96469139 \times 10^{-8}$ | $5.52014396 \times 10^{-8}$ |
| 0.4  | 0.363590507   | $2.59394049 \times 10^{-8}$ | $7.13423552 \times 10^{-8}$ |
| 0.5  | 0.371338094   | $3.20587129 \times 10^{-8}$ | $8.6329421 \times 10^{-8}$ |
| 0.6  | 0.379152415   | $3.80026985 \times 10^{-8}$ | $1.00230654 \times 10^{-7}$ |
| 0.7  | 0.387029942   | $4.37705229 \times 10^{-8}$ | $1.13093363 \times 10^{-7}$ |
| 0.8  | 0.394967015   | $4.93626510 \times 10^{-8}$ | $1.24979157 \times 10^{-7}$ |
| 0.9  | 0.402959856   | $5.47808413 \times 10^{-8}$ | $1.35946132 \times 10^{-7}$ |
| 1.0  | 0.411004569   | $6.00281270 \times 10^{-8}$ | $1.46052192 \times 10^{-7}$ |

Table 2  RKHS solutions of Example 1 for different values of fractional derivative

| $t$  | $\alpha = 0.9$ | $\alpha = 0.8$ | $\alpha = 0.7$ | $\alpha = 0.6$ | $\alpha = 0.5$ | $\alpha = 0.4$ |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.0  | 0.333333      | 0.333333      | 0.333333      | 0.333333      | 0.333333      | 0.333333      |
| 0.1  | 0.342354      | 0.344504      | 0.347041      | 0.349995      | 0.353982      | 0.357204      |
| 0.2  | 0.351427      | 0.355279      | 0.359757      | 0.364896      | 0.370713      | 0.377189      |
| 0.3  | 0.359926      | 0.364564      | 0.369790      | 0.375611      | 0.382003      | 0.388904      |
| 0.4  | 0.367993      | 0.372816      | 0.378034      | 0.383607      | 0.389459      | 0.395473      |
| 0.5  | 0.375912      | 0.380739      | 0.385766      | 0.390909      | 0.396053      | 0.401041      |
| 0.6  | 0.383753      | 0.388472      | 0.393218      | 0.397879      | 0.402311      | 0.406341      |
| 0.7  | 0.391531      | 0.396018      | 0.400374      | 0.404473      | 0.408164      | 0.411274      |
| 0.8  | 0.399254      | 0.403388      | 0.407251      | 0.410711      | 0.413620      | 0.415818      |
| 0.9  | 0.406929      | 0.410611      | 0.413893      | 0.416646      | 0.418731      | 0.420003      |
| 1.0  | 0.414579      | 0.417755      | 0.420409      | 0.422419      | 0.423656      | 0.423993      |

$$\omega_{4,\alpha=0.5}(t) = \frac{1}{3} + 0.0835836 t^{0.5} + 0.00823045 t - 0.00106387 t^{1.5} - 0.000383401 t^2.$$  

For the last technique, the FLDE can be solved using SS method with $\omega_0 = \frac{1}{3}$.  

The first two SS iterations are

$$\omega_1(t) = \frac{1}{3} + \frac{2t^\alpha}{27\Gamma(\alpha + 1)} \quad \text{and} \quad \omega_2(t) = \frac{1}{3} - \frac{8t^{3\alpha} \Gamma(2\alpha)}{(\Gamma(\alpha + 1))^2 \Gamma(3\alpha)} + \frac{54}{6561\alpha^2 \Gamma(\alpha)} \left( 9 + \frac{4^{\alpha} \sqrt{\pi} t^{2\alpha}}{\Gamma(\alpha + \frac{1}{2})} \right).$$

The fourth iteration for $\alpha = 1, \alpha = 0.8, \alpha = 0.5$ are listed below:
Table 3  Absolute and relative errors for Example 1 for \(\alpha = 1\) by FRPS method

| \(t\)  | FRPS solution | Absolute error | Relative error |
|------|--------------|----------------|----------------|
| 0.0  | 0.3333333333 | 0.0            | 0.0            |
| 0.1  | 0.340781429  | 1.27675647 \times 10^{-15} | 3.74655323 \times 10^{-15} |
| 0.2  | 0.348308998  | 1.48991929 \times 10^{-13} | 4.27759122 \times 10^{-13} |
| 0.3  | 0.355913071  | 2.37349029 \times 10^{-12} | 6.66873595 \times 10^{-12} |
| 0.4  | 0.363590533  | 1.64749880 \times 10^{-12} | 4.531192797 \times 10^{-11} |
| 0.5  | 0.37138125   | 7.22676918 \times 10^{-11} | 1.946142527 \times 10^{-10} |
| 0.6  | 0.379152452  | 2.36192787 \times 10^{-10} | 6.229493852 \times 10^{-10} |
| 0.7  | 0.387029984  | 6.27320917 \times 10^{-10} | 1.620858696 \times 10^{-9} |
| 0.8  | 0.394967063  | 1.424094787 \times 10^{-9} | 3.605603894 \times 10^{-9} |
| 0.9  | 0.402959908  | 2.849543213 \times 10^{-9} | 7.071530277 \times 10^{-9} |
| 1.0  | 0.411004623  | 5.119068058 \times 10^{-9} | 1.245501315 \times 10^{-8} |

Table 4  FRPS solutions of Example 1 for different values of the fractional derivative

| \(t\)  | \(\alpha = 0.9\) | \(\alpha = 0.8\) | \(\alpha = 0.7\) | \(\alpha = 0.6\) | \(\alpha = 0.5\) | \(\alpha = 0.4\) |
|------|------------------|------------------|------------------|------------------|------------------|------------------|
| 0.0  | 0.3333333333     | 0.3333333333     | 0.3333333333     | 0.3333333333     | 0.3333333333     | 0.3333333333     |
| 0.1  | 0.3431061        | 0.34607989       | 0.34985592       | 0.35461228       | 0.36050372       | 0.36788773       |
| 0.2  | 0.3516901        | 0.35570238       | 0.36042250       | 0.36592033       | 0.37224872       | 0.37942738       |
| 0.3  | 0.3599338        | 0.36448306       | 0.36958364       | 0.37522532       | 0.38137402       | 0.38794792       |
| 0.4  | 0.3679885        | 0.37278715       | 0.37795795       | 0.38344549       | 0.38915855       | 0.39496007       |
| 0.5  | 0.3759200        | 0.38076197       | 0.38580186       | 0.39094948       | 0.39608036       | 0.40103094       |
| 0.6  | 0.3837683        | 0.38849262       | 0.39325249       | 0.39793077       | 0.40238497       | 0.40644127       |
| 0.7  | 0.3915414        | 0.39603018       | 0.40039364       | 0.40450624       | 0.40821844       | 0.41135893       |
| 0.8  | 0.3992664        | 0.40340854       | 0.40728107       | 0.41075307       | 0.41367612       | 0.41588750       |
| 0.9  | 0.4069478        | 0.41065145       | 0.41395408       | 0.41672549       | 0.41882433       | 0.42010209       |
| 1.0  | 0.4145917        | 0.41777615       | 0.42044161       | 0.42246332       | 0.42371145       | 0.42405671       |

\[
\omega_{a,\alpha=1}(t) = \frac{1}{3} + \frac{2t}{27} + \frac{t^2}{243} - \frac{t^3}{2187} - \frac{5t^4}{78732} + \frac{t^5}{531441} + \frac{41t^6}{47829690} - \frac{4519905705}{584t^{10}} - \frac{162716605380}{349t^{11}} + \frac{16475056294725}{688t^{13}} - \frac{26689511974545}{281041395309195885} + \frac{27239367761220627}{256t^{15}}.
\]
\[
\omega_{4, \alpha=0.8}(t) = \frac{1}{3} + 0.0795312 t^{0.8} + 0.00575707 t^{1.6} - 0.000704323 t^{2.4} \\
- 0.000147395 t^{3.2} + 2.7352 \times 10^{-6} t^4 + 3.57138 \times 10^{-6} t^{4.8} \\
- 3.92022 \times 10^{-10} t^{5.6} - 4.3661 \times 10^{-8} t^{6.4} - 3.6982 \times 10^{-10} t^{7.2} \\
+ 4.32763 \times 10^{-10} t^8 + 2.36072 \times 10^{-12} t^{8.8} - 3.09023 \times 10^{-12} t^{9.6} \\
+ 5.81878 \times 10^{-14} t^{10.4} + 8.82091 \times 10^{-15} t^{11.2} \\
- 3.24929 \times 10^{-16} t^{12}.
\]

\[
\omega_{4, \alpha=0.5}(t) = \frac{1}{3} + 0.0835836 t^{0.5} + 0.00823045 t - 0.0010687 t^{1.5} \\
- 0.000383401 t^2 - 6.94706 \times 10^{-6} t^{2.5} + 0.000019413 t^3 \\
+ 4.15567 \times 10^{-8} t^{3.5} - 4.41534 \times 10^{-7} t^4 - 1.40437 \times 10^{-8} t^{4.5} \\
+ 8.61155 \times 10^{-9} t^5 + 2.03093 \times 10^{-10} t^{5.5} - 1.25157 \times 10^{-10} t^6 \\
+ 2.47645 \times 10^{-12} t^{6.5} + 6.9536 \times 10^{-13} t^7 - 3.3294 \times 10^{-14} t^{7.5}.
\]

The errors when applying this iterative method are shown in Table 5, while numerical values are given in Table 6. For these results, we take six successive substitutions.

To notice if there is difference between results from these three procedures, we graphed the approximate solutions in the same plane in Fig. 1. Also, we graph absolute errors that result from applying each of these numerical procedures in Fig. 2. Obviously, the solution behaviour indicates that an increase of the fractional parameter changes the nature of the solution. Also, from the present results, we can conclude that all three proposed algorithms are convenient to solve FFDEs and in good agreement.

**Table 5** Absolute and relative errors for Example 1 for \( \alpha = 1 \) using SSM

| t   | SS solution | Absolute error | Relative error |
|-----|-------------|---------------|---------------|
| 0.0 | 0.333333333 | 0.0           | 0.0           |
| 0.1 | 0.340781429 | 0.0           | 0.0           |
| 0.2 | 0.348308998 | 2.220446049 \times 10^{-16} | 6.374931627 \times 10^{-16} |
| 0.3 | 0.355913071 | 3.885780586 \times 10^{-15} | 1.09177993 \times 10^{-14} |
| 0.4 | 0.363590533 | 2.46495115 \times 10^{-14} | 6.778765923 \times 10^{-14} |
| 0.5 | 0.371338125 | 9.908740495 \times 10^{-14} | 2.668387599 \times 10^{-14} |
| 0.6 | 0.379152453 | 2.988165271 \times 10^{-13} | 7.881170876 \times 10^{-13} |
| 0.7 | 0.387029985 | 7.346900865 \times 10^{-13} | 1.898276915 \times 10^{-12} |
| 0.8 | 0.394970664 | 1.553090989 \times 10^{-12} | 3.932203790 \times 10^{-12} |
| 0.9 | 0.402959911 | 2.919331443 \times 10^{-12} | 7.244719294 \times 10^{-12} |
| 1.0 | 0.411004629 | 4.992062319 \times 10^{-12} | 1.214600023 \times 10^{-11} |
### Table 6  SS solutions of Example 1 for different values of the fractional derivative

| $t$ | $\alpha = 0.9$ | $\alpha = 0.8$ | $\alpha = 0.7$ | $\alpha = 0.6$ | $\alpha = 0.5$ | $\alpha = 0.4$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.0 | 0.333333       | 0.333333       | 0.333333       | 0.333333       | 0.333333       | 0.333333       |
| 0.1 | 0.343106       | 0.3460798      | 0.3498559      | 0.3546122      | 0.3605503      | 0.3678867      |
| 0.2 | 0.351690       | 0.3557023      | 0.3604225      | 0.3659203      | 0.3722487      | 0.3794268      |
| 0.3 | 0.359933       | 0.3644850      | 0.3695836      | 0.3752253      | 0.3813740      | 0.3879473      |
| 0.4 | 0.367988       | 0.3727871      | 0.3779579      | 0.3834455      | 0.3891586      | 0.3949600      |
| 0.5 | 0.375920       | 0.3807619      | 0.3858018      | 0.3909495      | 0.3960804      | 0.4010302      |
| 0.6 | 0.383763       | 0.3884926      | 0.3932525      | 0.3979308      | 0.4023852      | 0.4064412      |
| 0.7 | 0.391541       | 0.3960301      | 0.4003936      | 0.4045063      | 0.4082188      | 0.4113584      |
| 0.8 | 0.399266       | 0.4034085      | 0.4072811      | 0.4107533      | 0.4136767      | 0.4158884      |
| 0.9 | 0.406947       | 0.4106515      | 0.4139542      | 0.4167258      | 0.4188252      | 0.4201042      |
| 1.0 | 0.414591       | 0.4177762      | 0.4204418      | 0.4224639      | 0.4237127      | 0.4240585      |

### Fig. 1  A comparison between the RKHS, FRPS and SS solutions of Example 1 for different orders fractional derivatives: … RKHS $\omega^n(t)$, FRPS $\omega^n(t)$, … SS $\omega^n(t)$

### Example 2  Consider the following non-linear FLDE:

$$D^\alpha_0\omega(t) = \frac{1}{2} \omega(t)(1 - \omega(t)), \quad t \in [0, 1], \quad 0 < \alpha \leq 1,$$

(20)
subject to the initial condition
\[ \omega(0) = \frac{1}{5}. \]  

The exact solution of the above IVP is \( \omega(t) = e^{t/2}. \)

Following the Algorithm 1, we have to homogenise the IC to get
\[ D_\alpha^0 P(t) = \frac{1}{2} \left( P(t) + \frac{1}{5} \right) \left( \frac{4}{5} - P(t) \right), \quad P(0) = 0, \quad t \in [0, 1], \quad 0 < \alpha \leq 1. \]

Taking \( n = 25, \) Table 7 shows the accuracy of this technique. For different values of \( \alpha, \) the approximate solution of the FLDE is given in Table 8.

On the other hand, following the procedures outlined in this chapter for the FRPS method, we take six iterations and summarise some of the results in Tables 9 and 10. For the last technique, the FLDE can be solved using SS method with \( \omega_0 = \frac{1}{5}. \) The first three SS iterations are

\[ \omega_1(t) = \frac{1}{5} + \frac{2t^\alpha}{25\Gamma(\alpha + 1)}, \]
\[ \omega_2(t) = \frac{1}{5} + \frac{t^\alpha}{625} \left( \frac{50}{\Gamma(\alpha + 1)} + \frac{154 - \alpha \sqrt{\pi} t^\alpha}{\Gamma(\alpha + 1)\Gamma(\alpha + \frac{1}{2})} - \frac{4t^{2\alpha} \Gamma[2\alpha]}{(\Gamma(\alpha + 1))^2 \Gamma(3\alpha + 1)} \right), \]
\[ \omega_3(t) = \frac{1}{5} + \frac{t^\alpha}{32812500\Gamma(\alpha)} \left( \frac{-23625\pmb{\Gamma}(3\alpha)}{\alpha^2 \Gamma(2\alpha)\Gamma(4\alpha)} + 875 \left( \frac{3000}{\alpha} - \frac{52^3 + 2a_t^2 \Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}a_t^2 \Gamma(3\alpha)} \right) \right). \]
Table 7 Absolute and relative errors for Example 2 for $\alpha = 1$ using RKHS method

| $t$  | RKHS solution | Absolute Error | Relative Error |
|------|---------------|----------------|---------------|
| 0.1  | 0.208120076   | $3.379174099 \times 10^{-8}$ | $1.623665343 \times 10^{-7}$ |
| 0.2  | 0.216480619   | $6.994193039 \times 10^{-8}$ | $3.230862332 \times 10^{-7}$ |
| 0.3  | 0.225081557   | $1.065140960 \times 10^{-7}$ | $4.732242243 \times 10^{-7}$ |
| 0.4  | 0.233922198   | $1.433683575 \times 10^{-7}$ | $6.12886908 \times 10^{-7}$ |
| 0.5  | 0.243001189   | $1.803603639 \times 10^{-7}$ | $7.422195346 \times 10^{-7}$ |
| 0.6  | 0.252316499   | $2.173427363 \times 10^{-7}$ | $8.613885731 \times 10^{-7}$ |
| 0.7  | 0.261865377   | $2.541666377 \times 10^{-7}$ | $9.705994472 \times 10^{-7}$ |
| 0.8  | 0.271644341   | $2.906834533 \times 10^{-7}$ | $1.070087236 \times 10^{-6}$ |
| 0.9  | 0.281649145   | $3.267465498 \times 10^{-7}$ | $1.160117744 \times 10^{-6}$ |
| 1.0  | 0.291874770   | $3.62130897 \times 10^{-7}$  | $1.240986467 \times 10^{-6}$ |

Table 8 RKHS solutions of Example 2 for different values of the fractional derivative

| $t$   | $\alpha = 0.95$ | $\alpha = 0.85$ | $\alpha = 0.75$ | $\alpha = 0.65$ | $\alpha = 0.55$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.0   | 0.2             | 0.2             | 0.2             | 0.2             | 0.2             |
| 0.1   | 0.2085909388    | 0.210651316     | 0.213077939     | 0.2158996236    | 0.2191301069    |
| 0.2   | 0.2176968227    | 0.2215403565    | 0.2260325928    | 0.231221992     | 0.2371380454    |
| 0.3   | 0.2270357312    | 0.232098817     | 0.2377706961    | 0.244317699     | 0.2517120038    |
| 0.4   | 0.236403584     | 0.242061520     | 0.2483834620    | 0.2554405964    | 0.2632591967    |
| 0.5   | 0.2457805209    | 0.2517473876    | 0.2582547508    | 0.2652963638    | 0.272801322     |
| 0.6   | 0.2551847215    | 0.2612933947    | 0.2677531712    | 0.2744944401    | 0.2814094655    |
| 0.7   | 0.2647676723    | 0.2708877764    | 0.2771755006    | 0.2835098158    | 0.289720914     |
| 0.8   | 0.2745550464    | 0.2805544582    | 0.2865470405    | 0.2923758538    | 0.2978301946    |
| 0.9   | 0.2845305358    | 0.2902745947    | 0.2958409331    | 0.3010472160    | 0.3056651410    |
| 1.0   | 0.2946087218    | 0.300022144     | 0.3050864641    | 0.3096053875    | 0.313396499     |

The errors when applying this iterative method are shown in Table 11, while a sample of the numerical results are given in Table 12. For these results, we take six successive substitutions. To notice if there is difference between results from these
Table 9 Absolute and relative errors for Example 2 for $\alpha = 1$ using FRPS method

| $t$  | FRPS Solution | Absolute error | Relative error |
|------|---------------|----------------|----------------|
| 0.0  | 0.2           | 0.0            | 0.0            |
| 0.1  | 0.208120110   | 8.00748356 x 10^{-14} | 3.847529 x 10^{-13} |
| 0.2  | 0.216480689   | 1.03367869 x 10^{-11} | 4.774923 x 10^{-11} |
| 0.3  | 0.225081664   | 1.77954123 x 10^{-10} | 7.906202 x 10^{-10} |
| 0.4  | 0.233922340   | 1.34282263 x 10^{-9}  | 5.7404633 x 10^{-9}  |
| 0.5  | 0.243001363   | 6.44730971 x 10^{-9}  | 2.6531989 x 10^{-8}  |
| 0.6  | 0.252316693   | 2.32531752 x 10^{-8}  | 9.2158679 x 10^{-8}  |
| 0.7  | 0.261865563   | 6.88311416 x 10^{-8}  | 2.6284908 x 10^{-7}  |
| 0.8  | 0.271644455   | 1.76294589 x 10^{-7}  | 6.4898977 x 10^{-7}  |
| 0.9  | 0.281649067   | 4.04247487 x 10^{-7}  | 1.4352858 x 10^{-6}  |
| 1.0  | 0.291874283   | 8.49407245 x 10^{-7}  | 2.9101733 x 10^{-6}  |

Table 10 FRPS solutions of Example 2 for different values of the fractional derivative

| $t$  | $\alpha = 0.95$ | $\alpha = 0.85$ | $\alpha = 0.75$ | $\alpha = 0.65$ | $\alpha = 0.55$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.0  | 0.2             | 0.2             | 0.2             | 0.2             | 0.2             |
| 0.1  | 0.209326063     | 0.2122614428    | 0.2160541797    | 0.2209413786    | 0.2272259965    |
| 0.2  | 0.218315534     | 0.2225536329    | 0.2276652747    | 0.2338068760    | 0.2411521445    |
| 0.3  | 0.227349931     | 0.2324254862    | 0.2382910547    | 0.2450298545    | 0.2527108790    |
| 0.4  | 0.236498213     | 0.2421252989    | 0.2484162179    | 0.2553899593    | 0.2630302820    |
| 0.5  | 0.245789092     | 0.2517561978    | 0.2582379505    | 0.2651976372    | 0.2725500299    |
| 0.6  | 0.255236644     | 0.2613715314    | 0.2678584462    | 0.2746131007    | 0.2814954330    |
| 0.7  | 0.26487780      | 0.2710017612    | 0.2773374643    | 0.2837308534    | 0.2899996493    |
| 0.8  | 0.274625211     | 0.2806649490    | 0.2867124049    | 0.2926136189    | 0.2981486946    |
| 0.9  | 0.284568847     | 0.2903716588    | 0.2960075485    | 0.3012954313    | 0.3060016177    |
| 1.0  | 0.294676544     | 0.3001275373    | 0.3052388626    | 0.3098116011    | 0.3136007800    |

Table 11 Absolute and relative errors for Example 2 for $\alpha = 1$ using SSM

| $t$  | SS solution $\omega^n(t)$ | Absolute error | Relative error |
|------|---------------------------|----------------|----------------|
| 0.1  | 0.208120110               | 1.026956298 x 10^{-15} | 4.93444049 x 10^{-15} |
| 0.3  | 0.2250816644              | 1.944999717 x 10^{-12} | 8.64130679 x 10^{-12} |
| 0.5  | 0.2430013702              | 5.729930419 x 10^{-11} | 2.35798276 x 10^{-10} |
| 0.7  | 0.2618656315              | 4.906398066 x 10^{-10} | 1.873631919 x 10^{-9} |
| 0.9  | 0.2816494695              | 2.276855093 x 10^{-9}  | 8.084002712 x 10^{-9}  |


Table 12  SS solutions of Example 2 for different values of the fractional derivative

| t  | $\alpha = 0.95$ | $\alpha = 0.85$ | $\alpha = 0.75$ | $\alpha = 0.65$ | $\alpha = 0.55$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.1 | 0.2093260630    | 0.2122614428    | 0.2160541797    | 0.2209413780    | 0.2272259993    |
| 0.2 | 0.2183155346    | 0.2225536331    | 0.2276652760    | 0.2338068718    | 0.2411521873    |
| 0.3 | 0.2273499315    | 0.2324254885    | 0.2382910663    | 0.2450298550    | 0.2527110946    |
| 0.4 | 0.2364982168    | 0.2421253119    | 0.2484162714    | 0.2553900182    | 0.2630309643    |
| 0.5 | 0.2457891050    | 0.2517562474    | 0.2582381266    | 0.2651979050    | 0.2725517020    |
| 0.6 | 0.2552366879    | 0.2613716801    | 0.2678589132    | 0.2746138898    | 0.2814989163    |
| 0.7 | 0.2648479033    | 0.2710021379    | 0.2773385312    | 0.2837327154    | 0.2900061343    |
| 0.8 | 0.2746255118    | 0.2806657922    | 0.2867145888    | 0.2926151761    | 0.2981598119    |
| 0.9 | 0.2845695109    | 0.2903733762    | 0.2960116590    | 0.3013025036    | 0.3060195091    |

three procedures, we graphed the approximate solutions in the same plane in Fig. 3, and the absolute error curves for each method are shown in Fig. 4.

Fig. 3  A comparison between the RKHS, FRPS and SS solutions of Example 2 for different orders fractional derivatives: … RKHS $\omega^\alpha(t)$, __FRPS $\omega^\alpha(t)$, … SS $\omega^\alpha(t)$. 

\[ \text{Fig. 3} \quad \text{A comparison between the RKHS, FRPS and SS solutions of Example 2 for different orders fractional derivatives: … RKHS $\omega^\alpha(t)$, __FRPS $\omega^\alpha(t)$, … SS $\omega^\alpha(t)$.} \]
5 Conclusion

In this work, we applied three analytic and numeric schemes in order to obtain approximate solutions for the non-linear FLDE, which are the RKHS, the FRPS and the SS methods. The fractional derivative was described in the Caputo sense. Two examples were given to show the efficiency of the proposed methods. By comparing our results with the exact solution for integer order derivative, we observe that the proposed methods yield accurate approximations. To see the effects of the fractional derivative on the logistic curve, we solved the same FLDE for different values of the fractional order using these three procedures. The results showed that the curves of FLDE approach the curve of LDE as the fractional order approaches the integer order. Obviously, the solution behaviour indicates that an increase of the fractional parameter changes the nature of the solution. Also, from the present results, we can conclude that all three proposed algorithms are convenient to solve FFDEs and in good agreement.

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