Singularities and Characteristic Classes for Differentiable Maps

Toru Ohmoto
Hokkaido University
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Eu gostaria de agradecer os organizadores por me convidar esta conferência maravilhosa!
This mini-course is about
What’s about?

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... about the polynomial named in honor of him
What’s about?

- Alg. equation over \( \mathbb{C} \) (\( \rightsquigarrow \mathcal{K} \)-classification)

\[
P(x) = x^d + a_1 x^{d-1} + \cdots + a_d = 0, \quad \#_{\text{vir sol.}} = d
\]

taking account of multiplicities \( e = 1 + \mu \) (nondeg. sol. \( \leftrightarrow \mu = 0 \))
- Function $y = P(x)$ (\(\sim\) $A$-classification)

$$f : M \rightarrow N \quad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\})$$
What’s about?

- Function $y = P(x)$ (≈ $\mathcal{A}$-classification)

$$f : M \to N \quad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\})$$

$$\#_{\text{vir}} \text{ crit. pt} =$$
What's about?

- Function \( y = P(x) \) (\( \sim \) A-classification)

\[ f : M \to N \quad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}) \]

\[ \#_{\text{vir crit. pt}} = \int_M \mu(f, x) \, d\chi \]
What’s about?

- Function $y = P(x)$ (\(\sim\) $A$-classification)

$$f : M \rightarrow N \quad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\})$$

$$\#_{vir \text{ crit. pt}} = \int_{M} \mu(f, x) \, d\chi = 2d - 2$$
What’s about ?

- Function $y = P(x)$  \(\implies\) $A$-classification

\[ f : M \to N \quad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}) \]

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\#_{vir \text{ crit. pt}} = \int_M \mu(f, x) \, d\chi = 2d - 2 = \deg f \cdot \chi(N) - \chi(M)
\]
What’s about?

- Function $y = P(x)$ (\(\sim\) $A$-classification)

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  \[\#_{vir} \text{ crit. pt} = \int_M \mu(f, x) \, d\chi = 2d - 2 = \deg f \cdot \chi(N) - \chi(M)\]

  \[= c_1(TN) \cap f_*[M] - c_1(TM) \cap [M]\]
What's about?

- Function $y = P(x)$ (related to $A$-classification)

$$f : M \to N \quad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\})$$

$$\#_{\text{vir crit. pt}} = \int_M \mu(f, x) \, d\chi = 2d - 2 = \deg f \cdot \chi(N) - \chi(M)$$

$$= c_1(TN) \cap f_*[M] - c_1(TM) \cap [M]$$

$$= c_1(f^*TN - TM) \cap [M]$$
What's about?

• Function $y = P(x)$ ($\rightsquigarrow \mathcal{A}$-classification)

$$f : M \rightarrow N \quad (M = N = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\})$$

\[
\begin{align*}
\#_{\text{vir crit. pt}} &= \int_M \mu(f, x) \, d\chi = 2d - 2 = \deg f \cdot \chi(N) - \chi(M) \\
&= c_1(TN) \cap f^*[M] - c_1(TM) \cap [M] \\
&= c_1(f^*TN - TM) \cap [M] \\
&= \text{Thom polynomial of } A_1 \text{ for } f
\end{align*}
\]
What’s about?

I will talk about a generalization of this picture, in particular,

*hunting invariants of map-germs by localizing ‘higher $Tp$’*

**Contents**

- Preliminary: very basics
- Thom polynomials for singularities of maps
- Thom polynomials for multi-singularities of maps
- Higher Thom polynomials associated to CSM class
- Computing numerical invariants: Bezout type theorems
- $Tp$ for real singularities and Vassiliev type invariants

We works in the complex holomorphic context throughout. To be elementary and self-contained as much as possible.
First we recall a few basic notions about stable singularities of maps:
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\[ \mathcal{O}(m, n) := \{ f : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0 \text{ holomorphic} \} \]
Classification of map-germs: Equivalence

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- **A-classification**
  Classifies map-germs up to isomorphisms of source and target
  
  \[ A = \text{Diff}(\mathbb{C}^m, 0) \times \text{Diff}(\mathbb{C}^n, 0) \] acts on \( \mathcal{O}(m, n) \) by
  \[ (\sigma, \tau).f := \tau \circ f \circ \sigma^{-1} \]
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- **\( K \)-classification**
  
  Classifies the zero locus \( f^{-1}(0) \) as a scheme (i.e., defining ideal) up to the isomorphisms of source.
  
  \( K \subset \text{Diff}(\mathbb{C}^m \times \mathbb{C}^n, 0) \), preserving fibers \( \star \times \mathbb{C}^n \) and \( \mathbb{C}^m \times 0 \), acts on \( \mathcal{O}(m,n) \) measuring the tangency of graph \( y = f(x) \) and \( y = 0 \)
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- **A \subset K**
  Thus, orbits \( A.f \subset K.f \)
Classification of map-germs: Infinitesimal stability

- \( f = (x^3 + yx, y) \) and \( g = (x^3, y) \) in \( O(2, 2) \) are \( K \)-equivalent but not \( A \)-equivalent. \( A.f \neq K.f \)
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- $f = (x^3 + yx, y)$ and $g = (x^3, y)$ in $O(2, 2)$ are $\mathcal{K}$-equivalent but not $\mathcal{A}$-equivalent. $\mathcal{A}.f \neq \mathcal{K}.f$

- The $\mathcal{A}$-class of $f = (x^3 + yx, y)$ is called a cusp or $A_2$-singularity. The discriminant (s = singular value curves on the plane) looks as
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- \( f : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0 \) is a stable germ if taking any small perturbation of any representative \( f : U \to \mathbb{C}^n \), still the same singularity remains at some point nearby 0. The above cusp singularity is stable.
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- (J. Mather IV) If $f$ is a stable germ, $A.f = \{\text{Stable germs}\} \cap K.f$
Classification of map-germs: Jet-extension

Given a map \( f : M \rightarrow N \), we may think of it as

\[
\text{a family of mono-germs } f : M, x \rightarrow N, f(x) \\
\text{parameterized by the source space } M.
\]

(cf. a family of \textit{multi-germs} parametrized by the target \( N \))
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$$J(TM, TN)$$

$$j f$$

$$\uparrow$$

$$M \xrightarrow{(id, f)} M \times N$$
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\[
\begin{array}{c}
J(TM, TN) \\
\downarrow jf \\
M \xrightarrow{(id,f)} M \times N
\end{array}
\]

$f: M, x \rightarrow N, y$ is stable

$\iff jf: M \rightarrow J(TM, TN)$ is transverse to the $A$-orbit at $x$.

$\iff jf: M \rightarrow J(TM, TN)$ is transverse to the $K$-orbit at $x$ (Mather)
Notation: For a $\mathcal{K}$ (or $A$)-orbit $\eta$ in $\mathcal{O}(m, n)$, define

$$\eta(f) := \{ x \in M \mid \text{the germ } f \text{ at } x \text{ is of type } \eta \} = jf^{-1}(\eta(M, N))$$

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\[
\begin{array}{ccc}
M & \xrightarrow{(id,f)} & M \times N \\
\downarrow & & \downarrow \\
J(TM, TN) & \xrightarrow{jf} & \\
\end{array}
\]

Of our particular interest is

\[
\text{Dual } [\eta(f)] \in H^*(M)
\]

If \( \text{codim } \eta = \text{dim } M \) and \( M \) compact, this gives \( \sharp \) \( \eta \)-singular pts.
Classification of map-germs: Jet-extension

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**Of our particular interest** is

$$\text{Dual } [\eta(f)] \in H^*(M)$$

If codim $\eta = \dim M$ and $M$ compact, this gives $\sharp \eta$-singular pts.

"counting $\eta$-singular points = describing this cohomology class"
Recall a basic notion in topology:

A **vector bundle** $p : E \to M$ is a locally trivial fibration with fiber $\mathbb{C}^n$ and structure group $GL_n$.

The right one is called **the trivial bundle**. How can we measure 'non-trivial gluing' in the left?
Chern class of vector bundles: Definition

Recall a basic notion in topology:

Take a section $s : M \to E$ and observe its intersection with $Z$, that leads us the definition of the top Chern class of $E$

$$c_n(E) := s^* \text{Dual } [Z] = \text{Dual } [s^{-1}(Z)] \in H^{2n}(M; \mathbb{Z})$$

For the above picture, $c_n(Left) \neq 0$ and $c_n(Right) = 0$
The Chern class of complex vector bundles is uniquely characterized as the assignment

\[ \text{vector bdle } E \rightarrow M \sim \Rightarrow c_i(E) \in H^{2i}(M; \mathbb{Z}), \quad (i = 0, 1, 2, \cdots) \]

satisfying the following axioms:
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satisfying the following axioms:

- \( c_0(E) = 1 \) and \( c_i(E) = 0 \) \( (i > n = \text{rank } E) \), i.e.,

\[
c(E) := \sum_{i \geq 0} c_i(E) = 1 + c_1(E) + \cdots + c_n(E) : \text{total Chern class}
\]
Chern class of vector bundles: Definition

Theorem 2.1 (or Definition: see Milnor’s or Hirzebruch’s textbooks.)

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\[ \text{vector bdle } E \rightarrow M \leadsto c_i(E) \in H^{2i}(M; \mathbb{Z}), \quad (i = 0, 1, 2, \cdots) \]

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  \[ c(E) := \sum_{i \geq 0} c_i(E) = 1 + c_1(E) + \cdots + c_n(E) \quad : \text{total Chern class} \]
- \( c(f^*E) = f^*c(E) \) for the pullback via \( f : M' \rightarrow M \) : naturality
- \( c(E \oplus F) = c(E) \cdot c(F') \) : Whitney sum formula
- \( c_1(O_{\mathbb{P}^1}(1)) \) equals the divisor class \( a \in H^2(\mathbb{P}^1) \) : normalization
Trivial bundle: \( c_1(\varepsilon^1) = 0 \), hence for the trivial \( n \)-bundle,
\[
c(\varepsilon^n) = c(\bigoplus \varepsilon^1) = 1.
\]
Chern class of vector bundles: Remark

- **Trivial bundle**: $c_1(\epsilon^1) = 0$, hence for the trivial $n$-bundle, $c(\epsilon^n) = c(\oplus \epsilon^1) = 1$.

- **Tensor product** of line bundles $\ell_1, \ell_2$ over $M$:

  $$c_1(\ell_1 \otimes \ell_2) = c_1(\ell_1) + c_1(\ell_2) \quad \text{(additive group law)}$$
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  \]

- The Chern class of a complex manifold \( M \) means \( c(TM) \) of the tangent bundle. The top Chern class is the **Euler characteristic**:

  \[
  c_n(TM) \sim [M] = \chi(M) \cdot [pt] \in H_0(M)
  \]

  That is **the Poincaré-Hopf theorem** : for a vector field \( v : M \to TM \)

  \[
  c_n(TM) = \sum \text{Ind}(v, p)^{\text{P.H.}} \equiv \chi(M)
  \]
Difference Chern class: To measure the difference between two vector bundles \( E \) and \( F \) over the same base space, we define by using formal expansion

\[
\frac{1}{1+A} = 1 - A + A^2 - A^3 + \cdots
\]

\[
c(F - E) := \frac{1 + c_1(F') + c_2(F') + \cdots}{1 + c_1(E) + c_2(E) + \cdots}
\]
Difference Chern class: To measure the difference between two vector bundles $E$ and $F$ over the same base space, we define by using formal expansion $\frac{1}{1+A} = 1 - A + A^2 - A^3 + \cdots$

$$c(F - E) := \frac{1 + c_1(F') + c_2(F') + \cdots}{1 + c_1(E') + c_2(E') + \cdots}$$

Obviously,

- If $F = E \oplus E'$, then $c(F - E) = c(E')$ by Whitney sum formula.

- For line bundles, $c(\ell' - \ell) = \frac{1+b}{1+a} = (1 + b)(1 - a + a^2 - \cdots)$ where $a = c_1(\ell)$ and $b = c_1(\ell')$
Now, return back to our setting:

Let $\eta \subset J(m, n)$ be a $K$-orbit. Given a stable map $f : M \to N$,

$$
\begin{array}{c}
J(TM, TN) & \overset{\eta(M, N)}{\longrightarrow} \\
\downarrow jjf & \downarrow \\
\eta(f) & \longrightarrow M & \longrightarrow M \times N
\end{array}
$$

How to describe $\text{Dual } [\eta(f)] \in H^*(M)$
Theorem 3.1 (Thom ('57), Damon ('72) etc)

There exists a unique polynomial \( tp(\eta) \in \mathbb{Z}[c_1, c_2, \cdots] \) in abstract Chern classes so that

- homogeneous in degree \( = \text{codim} \eta \) \( (\deg c_i = 2i) \)
- it depends only on \( \eta \subset J(*, * + k) \),
- for any generic map \( f : M \to N \) of map-codim. \( \dim N - \dim M = k \), the polynomial evaluated by \( c_i = c_i(f) := c_i(f^*TN - TM) \) expresses the singular locus of type \( \eta \):

\[
    tp(\eta)(f) = \text{Dual} \left[ \overline{\eta(f)} \right] \in H^{2 \text{codim} \eta}(M)
\]

We call \( tp(\eta) \) the Thom polynomial of stable singularity type \( \eta \)
Thom polynomials of stable singularities

Example 3.2 (Thom ('56): Case of map codimension $k = 0$)

Thom polynomials of stable singularities $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ are

$$tp(A_0) = 1, \quad tp(A_1) = c_1, \quad tp(A_2) = c_1^2 + c_2$$

| type       | normal form            |
|------------|------------------------|
| $A_0$ (regular) | $(x, y) \mapsto (x, y)$ |
| $A_1$ (fold)   | $(x, y) \mapsto (x^2, y)$ |
| $A_2$ (cusp)   | $(x, y) \mapsto (x^3 + xy, y)$ |
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More examples of stable singularities $\mathbb{C}^n, 0 \to \mathbb{C}^n, 0$,

\[
\begin{align*}
tp(A_3) &= c_1^3 + 3c_1c_2 + 2c_3, \\
tp(A_4) &= c_1^4 + 6c_1^2c_2 + 2c_2^2 + 9c_1c_3 + 6c_4, \\
tp(I_{22}) &= c_2^2 - c_1c_3, \cdots
\end{align*}
\]
Localization formula

Let’s compute $tp(A_2)$ by the restriction method due to Richard Rimanyi. Since $\text{codim } A_2 = 2$, the Thom polynomial has the form

$$tp(A_2) = Ac_1^2 + Bc_2$$

and we want to determine the unknowns $A, B$. 
Localization formula

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The key point is that the normal forms of stable germs admit a natural torus action $\mathbb{C}^* = \mathbb{C} - \{0\}$:

$$\begin{align*}
(x, y) &\xrightarrow{A_2} (x^3 + yx, y) \\
\rho_0 &= \alpha \oplus \alpha^2 \\
\rho_1 &= \alpha^3 \oplus \alpha^2 \\
\alpha &\in \mathbb{C}^*
\end{align*}$$
Localization formula

Let us think of $\alpha$ as the gluing map for the canonical line bundle
$\ell = O_{\mathbb{P}^N}(1)$ over $\mathbb{P}^N$ ($N \gg 0$).
Localization formula

Let us think of $\alpha$ as the gluing map for the canonical line bundle $\ell = O_{\mathbb{P}^N}(1)$ over $\mathbb{P}^N$ ($N \gg 0$). Define two vector bundles of rank 2

$$E_0 := \ell \oplus \ell^\otimes 2,$$
$$E_1 := \ell^\otimes 3 \oplus \ell^\otimes 2$$
Localization formula

Let us think of $\alpha$ as the gluing map for the canonical line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)$ over $\mathbb{P}^N$ ($N \gg 0$). Define two vector bundles of rank 2

$$E_0 := \mathcal{L} \oplus \mathcal{L}^{\otimes 2}, \quad E_1 := \mathcal{L}^{\otimes 3} \oplus \mathcal{L}^{\otimes 2}$$

That is, take $\{U_i\}$ of the base giving a local trivialization of $\mathcal{L}$; glueing maps $g_{ij} : U_i \cap U_j \to GL_2$ for $E_0$ and $E_1$ are of the form

$$U_i \cap U_j \xrightarrow{\alpha} \mathbb{C}^* \xrightarrow{\rho} GL_2, \quad \rho_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}, \quad \rho_1 = \begin{bmatrix} \alpha^3 & 0 \\ 0 & \alpha^2 \end{bmatrix},$$

respectively.
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$$E_0 := \ell \oplus \ell \otimes^2, \quad E_1 := \ell \otimes^3 \oplus \ell \otimes^2$$

The normal form of $A_2$, $(x, y) \mapsto (x^3 + yx, y)$, is invariant under the action, thus we can glue the map on $U_i$’s together.
Localization formula

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The normal form of $A_2$, $(x, y) \mapsto (x^3 + yx, y)$, is invariant under the action, thus we can glue the map on $U_i$'s together. This defines a stable map $f_{A_2}: E_0 \to E_1$ between the total spaces

![Diagram](attachment:image.png)

$A_2$-singularity locus $A_2(f_{A_2}) = \text{the zero section of } E_0$. 
Localization formula

$E_0$  $E_1$  Base
Compute the Chern classes. Put $a = c_1(\ell)$ and then

$$H^*(\mathbb{P}^N) = \mathbb{Z}[a]/(a^{N+1}), \quad N \gg 0$$

Note that $H^*(E_0) = H^*(E_1) = H^*(\mathbb{P}^N)$ via the pullback $p_0^*$ and $p_1^*$.

$$c(E_0) = c(\ell \oplus \ell \otimes 2) = (1 + a)(1 + 2a),$$
$$c(E_1) = c(\ell \otimes 3 \oplus \ell \otimes 2) = (1 + 3a)(1 + 2a)$$
Compute the Chern classes. Put $a = c_1(\ell)$ and then

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$$c(E_1) = c(\ell \otimes 3 \oplus \ell \otimes 2) = (1 + 3a)(1 + 2a)$$

$$c(f_{A_2}) = c(f^*TE_1 - TE_0) = c(p_1^*E_1 - p_0^*E_0) = \frac{(1+3a)(1+2a)}{(1+a)(1+2a)} = \frac{1+3a}{1+a}$$

$$= 1 + 2a - 2a^2 + 2a^3 - \cdots$$

Thus we have $c_1(f_{A_2}) = 2a$, $c_2(f_{A_2}) = -2a^2$, ... etc.
Apply the Thom polynomial theorem to this map $f_{A_2} : E_0 \rightarrow E_1$,
\[
    tp(A_2)(f_{A_2}) = \text{Dual } [A_2(f_{A_2})]
\]
Localization formula

Apply the Thom polynomial theorem to this map $f_{A_2} : E_0 \to E_1$,

$$tp(A_2)(f_{A_2}) = \text{Dual } [\overline{A_2}(f_{A_2})]$$

Substitute $c_2(E_0) = 2a^2$, $c_1(f_{A_2}) = 2a$, $c_2(f_{A_2}) = -2a^2$.

$$tp(A_2)(f_{A_2}) = Ac_1^2 + Bc_2$$
$$= A(2a)^2 + B(-2a^2) = (4A - 2B)a^2$$

Dual [$\overline{A_2}(f_{A_2})$] = Dual [Zero] = $c_2(E_0) = 2a^2$
Localization formula

Apply the Thom polynomial theorem to this map $f_{A_2} : E_0 \to E_1$,

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$$tp(A_2)(f_{A_2}) = Ac_1^2 + Bc_2$$
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Dual $[\overline{A}_2(f_{A_2})] = \text{Dual} [\text{Zero}] = c_2(E_0) = 2a^2$

Thus we get

$$2A - B = 1$$
Localization formula

Do the same thing for other singularities:

\[(x, y) \xrightarrow{A_1} (x^2, y) \quad \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}^* \]

\[\rho_0 = \alpha \oplus \beta \quad \rho_1 = \alpha^2 \oplus \beta\]

We obtain a stable map \( f_{A_1} : E_0 \to E_1 \); It has only \( A_1 \)-singularities, so the \( A_2 \)-singularity locus \( A_2(f_{A_1}) \) is empty. Thus, Tp Theorem says that

\[tp(A_2)(f_{A_1}) = \text{Dual } [\emptyset] = 0\]

Since \( c(f_{A_1}) = \frac{(1+2a)(1+b)}{(1+a)(1+b)} = 1 + a - a^2 + \cdots \), one obtains

\[A - B = 0\]
Localization formula

Do the same thing for other singularities:

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Since \(c(f_{A_1}) = \frac{(1+2a)(1+b)}{(1+a)(1+b)} = 1 + a - a^2 + \cdots\), one obtains

\[A - B = 0\]

Combine it with \(2A - B = 1\), gets \(A = B = 1\), i.e., \(tp(A_2) = c_1^2 + c_2\)
Remark 3.3

- Rimanyi’s restriction method works well for simple orbits in classification up to the lowest codimension of moduli strata of orbits. In fact the restriction of $tp$ to an orbit is the Atiyah-Bott localization for torus action (the origin is a fixed point).
Remark 3.3

- Rimanyi’s restriction method works well for simple orbits in classification up to the lowest codimension of moduli strata of orbits. In fact the restriction of $t_p$ to an orbit is the Atiyah-Bott localization for torus action (the origin is a fixed point).

- The universal map $f_\eta : E_0 \to E_1$ is a key ingredient in Thom-Pontrjagin-Szücs construction of classifying space of singular maps.
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- Rimanyi’s restriction method works well for simple orbits in classification up to the lowest codimension of moduli strata of orbits. In fact the restriction of \( t_p \) to an orbit is the Atiyah-Bott localization for torus action (the origin is a fixed point).

- The universal map \( f_\eta : E_0 \to E_1 \) is a key ingredient in Thom-Pontrjagin-Szücs construction of classifying space of singular maps.

- Why the difference Chern classes \( c_i(f) = c_i(f^*TN - TM) \) arise? It is that the \( \mathcal{K} \)-equivalence admits a stabilization of dimensions: the embedding \( J(m, n) \to J(m + r, n + r) \), \( j f(0) \to j(f \times \text{id}_r)(0) \), is transverse to any \( \mathcal{K} \)-orbits (not true for \( \mathcal{A} \)-orbits).
Tp for $\mathcal{A}$-finite singularities

What’s then about Tp for unstable but $\mathcal{A}$-finite singularities of maps?
What’s then about $T_p$ for unstable but $A$-finite singularities of maps?

It makes sense.

But such a $T_p$ is no longer a polynomial in $c_i(f)$ in general and it’s for families of maps: a proper setting should be as follows:
Consider the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p_0} & & \downarrow{p_1} \\
B & & B
\end{array} \]

where \( X, Y, B \) are complex manifolds, \( p_0 : X \to B \) and \( p_1 : Y \to B \) are submersions of constant relative dimension, say \( \dim = 2 \).
Consider the diagram

\[
\begin{array}{c}
X \\
p_0 \\
\downarrow \\
B \\
p_1 \\
\uparrow \\
Y
\end{array}
\]

where \( X, Y, B \) are complex manifolds, \( p_0 : X \to B \) and \( p_1 : Y \to B \) are submersions of constant relative dimension, say \( \dim = 2 \).

For each \( x \in X \), a map-germ of \( f \) restricted to the fiber is defined:

\[
f\big|_{p_0^{-1}(p_0(x))} : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0 \quad \text{(centered at } x \text{ and } f(x))
\]
Consider the diagram

\[
\begin{array}{ccc}
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\]

Given an \( \mathcal{A} \)-finite singularity type \( \eta \), the **singularity locus** \( \eta(f) \subset X \) and the **bifurcation locus** \( B_{\eta}(f) = p_0(\eta(f)) \subset B \) are defined.
Theorem 4.1

Let \( \eta \) be an \( \mathcal{A} \)-finite singularity type. For generic maps \( f : X \to Y \), Dual \( [\bar{\eta}(f)] \in H^*(X) \) is expressed by a universal polynomial \( tp^A(\eta) \) in the Chern class \( c_i = c_i(T_{X/B}) \) and \( c_j = c_j(T_{Y/B}) \) of relative tangent bundles. Dual \( [\bar{B}_\eta(f)] \in H^*(B) \) is also expressed by the pushforward \( p_0*tp^A(\eta) \).

\[
\begin{array}{ccc}
\bar{\eta}(f) & \xhookrightarrow{} & X \\
\downarrow p_0 & & \downarrow p_0 \\
\bar{B}_\eta(f) & \xhookrightarrow{} & B \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( jf \)} \\
\downarrow \\
J(T_{X/B}, f^*T_{Y/B})
\end{array}
\]
Theorem 4.1

Let \( \eta \) be an \( A \)-finite singularity type. For generic maps \( f : X \to Y \),
Dual \( [\overline{\eta}(f)] \in H^*(X) \) is expressed by a universal polynomial \( t p^A(\eta) \) in the
Chern class \( c_i = c_i(T_{X/B}) \) and \( c_j = c_j(T_{Y/B}) \) of relative tangent bundles.
Dual \( [\overline{B_\eta}(f)] \in H^*(B) \) is also expressed by the pushforward \( p_0*tp^A(\eta) \).

Remark 4.2

The case of rel. dim. 1: Kazarian-Lando for the study of Hurwitz numbers.
Tp for $A$-finite singularities

$A$-classification of $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ (Rieger-Ruas, Arnold-Platonova)

| type            | codim | miniversal unfolding                  |
|-----------------|-------|---------------------------------------|
| lips (beaks)    | 3     | $(x^3 + xy^2 + ax, y)$                |
| swallowtail     | 3     | $(x^4 + xy + ax^2, y)$                |
| goose           | 4     | $(x^3 + xy^3 + axy + bx, y)$          |
| gull            | 4     | $(x^4 + xy^2 + x^5 + axy + bx, y)$    |
| butterfly       | 4     | $(x^5 + xy + x^7 + ax^3 + bx^2, y)$   |
| $I_{2,2}^{1,1}$ (dertoid) | 4     | $(x^2 + y^3 + ay, y^2 + x^3 + bx)$   |

Lips
Tp for $A$-finite singularities

**Example 4.3 (Ohm)**

Tp for $A$-classification of map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ is defined as

$$tp^A(\eta) \in \mathbb{Z}[c_1, c_2, c'_1, c'_2]$$

where $c_i, c'_i$ are Chern classes of relative tangent bundles:

| Type     | Formula                                                                 |
|----------|--------------------------------------------------------------------------|
| lips/beaks | $-2c_1^3 + 5c_1^2c'_1 - 4c_1c'_2 - c_1c_2 + c_2c'_1 + c'_3$               |
| swallowtail | $-6c_1^3 + 11c_1^2c'_1 - 6c_1c'_2 + 7c_1c_2 - 5c_1c'_2 - 5c'_1c_2 + 3c'_1c'_2 + c'_3$ |
| goose     | $8c_1^4 - 24c_1^3c'_1 + 26c_1^2c'_2 - 12c_1c'_3 + 2c'_4$ + $4c_1^2c_2 - 6c_1c'_1c_2 + 2c'_1c_2$ |
| gull      | $6c_1^4 - 17c_1^3c'_1 + 17c_1^2c'_2 - 7c_1c'_3 + c'_4$ $-c_1^2c_2 + 5c_1^2c'_2 + c_1c'_1c_2 - 7c_1c'_1c'_2 + 2c'_1c'_2 - c_2^2 + c'_2$ |
| butterfly | $24c_1^4 - 50c_1^3c'_1 - 46c_1^2c'_2 - 10c_1c'_3 + c'_4 - 46c_1^2c_2 + 6c_1^2c'_2$ + $60c_1c'_1c_2 - 20c_1c'_1c'_2 - 20c'_1c_2 + 6c'_1c'_2 + 3c_2^2 - 3c'_2$ |
| $I_{2,2}^{1,1}$ | $c_2^2 - c_1c_2c'_1 + c_2c'_2 + c_1c'_2 - 2c_2c'_2 - c_1c'_1c'_2 + c'_2$ |
Today’s summary

Definition of Thom polynomials of stable singularities: That is a universal expression in terms of $c_i = c_i(f) (T_N - T_M)$ s.t. $tp(f) = \text{Dual} \left[ (f) \right] \in H(M)$

Torus action and computation of $T_p$
Today’s summary

- Definition of Thom polynomials of stable singularities: That is a universal expression in terms of $c_i = c_i(f^*TN - TM)$ s.t.

$$tp(\eta)(f) = \text{Dual } [\overline{\eta(f)}] \in H^*(M)$$
Today’s summary

- Definition of Thom polynomials of stable singularities: That is a universal expression in terms of $c_i = c_i(f^*TN - TM)$ s.t.
  \[ tp(\eta)(f) = \text{Dual} \left[ \eta(f) \right] \in H^*(M) \]

- Torus action and computation of $Tp$
Até amanhã. Tchau!

ではまた明日！