ON THE IMMERSED SUBMANIFOLDS IN THE UNIT SPHERE
WITH PARALLEL BLASCHKE TENSOR II

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Abstract. As is known, the Blaschke tensor $A$ (a symmetric covariant 2-tensor) is one of the fundamental Möbius invariants in the Möbius differential geometry of submanifolds in the unit sphere $S^n$, and the eigenvalues of $A$ are referred to as the Blaschke eigenvalues. In this paper, we continue our job for the study on the submanifolds in $S^n$ with parallel Blaschke tensors which we simply call Blaschke parallel submanifolds to find more examples and seek a complete classification finally. The main theorem of this paper is the classification of Blaschke parallel submanifolds in $S^n$ with exactly three distinct Blaschke eigenvalues. Before proving this classification we define, as usual, a new class of examples.

1. Introduction

Let $S^n(r)$ be the standard $n$-dimensional sphere in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ of radius $r$, and denote $S^n = S^n(1)$. Let $\mathbb{H}^n(c)$ be the $n$-dimensional hyperbolic space of constant curvature $c < 0$ defined by

$$\mathbb{H}^n(c) = \{y = (y_0, y_1) \in \mathbb{R}^{n+1}_1; \langle y, y \rangle_1 = \frac{1}{c}, y_0 > 0\},$$

where, for any integer $N \geq 2$, $\mathbb{R}^N_1 \equiv \mathbb{R}_1 \times \mathbb{R}^{N-1}$ is the $N$-dimensional Lorentzian space with the standard Lorentzian inner product $\langle \cdot, \cdot \rangle_1$ given by

$$\langle y, y' \rangle_1 = -y_0 y'_0 + y_1 \cdot y'_1, \quad y = (y_0, y_1), \quad y' = (y'_0, y'_1) \in \mathbb{R}^N_1,$$

in which the dot "\cdot" denotes the standard Euclidean inner product on $\mathbb{R}^{N-1}$. From now on, we simply write $\mathbb{H}^n$ for $\mathbb{H}^n(-1)$.

Denote by $S^+_n$ the hemisphere in $S^n$ whose first coordinate is positive. Then there are two conformal diffeomorphisms

$$\sigma : \mathbb{R}^n \to S^n \setminus \{(-1, 0)\} \quad \text{and} \quad \tau : \mathbb{H}^n \to S^+_n$$

defined as follows:

$$\sigma(u) = \left(\frac{1 - |u|^2}{1 + |u|^2},\frac{2u}{1 + |u|^2}\right), \quad u \in \mathbb{R}^n, \quad (1.1)$$

$$\tau(y) = \left(\frac{1}{y_0} y_1, y_0\right), \quad y = (y_0, y_1) \in \mathbb{H}^n \subset \mathbb{R}^{n+1}_1. \quad (1.2)$$

Let $x : M^m \to S^{m+p}$ be an immersed umbilic-free submanifold in $S^{m+p}$. Without loss of generality, we usually assume that $x$ is linearly full, that is, $x$ can not be contained in a hyperplane in $\mathbb{R}^{m+p+1}$. Then it is known that there are four fundamental Möbius invariants of $x$, in terms of the light-cone model established by C. P. Wang in 1998 (24) that are the Möbius metric $g$, the Blaschke tensor $A$, the Möbius second fundamental form $B$ and the Möbius form $C$. Since the pioneer work of Wang, there have been obtained

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many interesting results in the Möbius geometry of submanifolds including some important classification theorems of submanifolds with particular Möbius invariants, such as, the classification of surfaces with vanishing Möbius forms ([10]), that of Möbius isotropic submanifolds ([22]), that of hypersurfaces with constant Möbius sectional curvature ([4]), that of Möbius isoparametric hypersurfaces ([8], [6], [12], etc), and that of hypersurfaces with Blaschke tensors linearly dependent on the Möbius metrics and Möbius second fundamental forms ([9], which is later generalized by [18] and [3], respectively, in two different directions. Here we should remark that, after the classification of all Möbius parallel hypersurfaces in $S^{m+1}$, that is, hypersurfaces with parallel Möbius second fundamental forms ([5]), Zhai-Hu-Wang recently proved in [25] an interesting theorem which classifies all 2-codimensional Möbius parallel submanifolds in the unit sphere.

Clearly, it is much natural to study submanifolds in the unit sphere $S^n$ with particular Blaschke tensors. Note that a submanifold in $S^n$ with vanishing Blaschke tensor also has a vanishing Möbius form, and therefore is a special Möbius isotropic submanifold; any Möbius isotropic submanifold is necessarily of parallel Blaschke tensor. Furthermore, all Möbius parallel submanifolds also have vanishing Möbius forms and parallel Blaschke tensors ([25]). Thus a rather natural and interesting problem is to seek a classification of all the submanifolds with parallel Blaschke tensors which we shall call for simplicity Blaschke parallel submanifolds.

To this direction, the first step is indeed the study of hypersurfaces. In fact, the following theorem has been established:

**Theorem 1.1 ([19]).** Let $x : M^m \to S^{m+1}$, $m \geq 2$, be a Blaschke parallel hypersurface. Then the Möbius form of $x$ vanishes identically and $x$ is either Möbius parallel, or Möbius isotropic, or Möbius equivalent to one of the following examples which have exactly two distinct Blaschke eigenvalues:

1. one of the minimal hypersurfaces as indicated in Example 3.2 of [19];
2. one of the non-minimal hypersurfaces as indicated in Example 3.3 of [19].

As the second step, we have proved earlier the following classification:

**Theorem 1.2 ([17]).** Let $x : M^m \to S^{m+p}$ be a Blaschke parallel submanifold immersed in $S^{m+p}$ with vanishing Möbius form $C$. If $x$ has two distinct Blaschke eigenvalues, then it must be Möbius equivalent to one of the following four kinds of immersions:

1. a non-minimal and umbilic-free pseudo-parallel immersion $\tilde{x} : M \to S^{m+p}$ with parallel mean curvature and constant scalar curvature, which has two distinct principal curvatures in the direction of the mean curvature vector;
2. the image under $\sigma$ of a non-minimal and umbilic-free pseudo-parallel immersion $\tilde{x} : M \to \mathbb{R}^{m+p}$ with parallel mean curvature and constant scalar curvature, which has two distinct principal curvatures in the direction of the mean curvature vector;
3. the image under $\tau$ of a non-minimal and umbilic-free pseudo-parallel immersion $\tilde{x} : M \to \mathbb{H}^{m+p}$ with parallel mean curvature and constant scalar curvature, which has two distinct principal curvatures in the direction of the mean curvature vector;
4. a submanifold $LS(m_1, p_1, r, \mu)$ for some parameters $m_1, p_1, r, \mu$.

**Remark 1.1.** Submanifolds $LS(m_1, p_1, r, \mu)$ with multiple parameters $m_1, p_1, r, \mu$ were first defined in Example 3.2 of [17]. As in [17], we call a Riemannian submanifold pseudo-parallel if the inner product of its second fundamental form with the mean curvature vector is parallel. In particular, if the second fundamental form is itself parallel, then we simply call this submanifold (Euclidean) parallel.

In this paper, we continue our work on the classification of the Blaschke parallel submanifolds in $S^n$ with vanishing Möbius forms. Naturally, due to Theorems 1.1 and 1.2, the next step is to study those Blaschke parallel submanifolds with three distinct Blaschke eigenvalues. To do this, we first construct in Section 3 a new class of Blaschke parallel submanifolds denoted by $LS(m, p, r, \mu)$ with, as desired,
vanishing Möbius forms and exactly three distinct Blaschke eigenvalues. The idea of this construction originates from those hypersurface examples that were first introduced in [19] (see also [20]) and are the only non-Möbius isoparametric but Blaschke isoparametric hypersurfaces (cf. [21]) with two distinct Blaschke eigenvalues. Note that, due to [11], any Blaschke isoparametric hypersurfaces with more than two distinct Blaschke eigenvalues must be Möbius isoparametric, which is an affirmative solution of the problem originally raised in [21] (see also [13] and [14]). It should also be remarked that, by [12] and [23], the Möbius isoparametric hypersurfaces (cf. [8]) have been completely classified and thus the work in [11] actually finishes the classification of the Blaschke isoparametric hypersurfaces (see also the latest partial classification theorem in [7]). Besides, there have been some parallel results on space-like hypersurfaces in the de Sitter space $S^{n+1}$ (see [15] and the references therein). Combining all we know on this subject, it turns out that our new examples and the argument in this present paper will shed a new light on the completement of our final classification work which will be done in a forth-coming paper.

The main theorem of this paper is now stated as follows:

**Theorem 1.3.** Let \( x : M^m \to S^{m+p} \) be a Blaschke parallel submanifold immersed in \( S^{m+p} \) with vanishing Möbius form \( C \). If \( x \) has three distinct Blaschke eigenvalues, then it must be Möbius equivalent to one of the following four kinds of immersions:

1. a non-minimal and umbilic-free pseudo-parallel immersion \( \tilde{x} : M^m \to S^{m+p} \) with parallel mean curvature and constant scalar curvature, which has three distinct principal curvatures in the direction of the mean curvature vector;
2. the image under \( \sigma \) of a non-minimal and umbilic-free pseudo-parallel immersion \( \bar{x} : M^m \to \mathbb{R}^{m+p} \) with parallel mean curvature and constant scalar curvature, which has three distinct principal curvatures in the direction of the mean curvature vector;
3. the image under \( \tau \) of a non-minimal and umbilic-free pseudo-parallel immersion \( \bar{x} : M^m \to H^{m+p} \) with parallel mean curvature and constant scalar curvature, which has three distinct principal curvatures in the direction of the mean curvature vector;
4. a submanifold \( LS(m, p, r, \mu) \) given in Example 3.2 for some multiple parameters \( m, p, r, \mu \) satisfying \( m^3 r^2 \neq m^2 r^3 \).

**Remark 1.2.** Indeed, it is directly verified that each of the immersed submanifolds stated in Theorem 1.3 is Blaschke parallel with vanishing Möbius form and exactly three distinct Blaschke eigenvalues (see Section 3).

We also remark that the final classification theorem will be much like Theorem 1.3 with the corresponding examples \( LS(m, p, r, \mu) \) being extended to the general case.

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## 2. Preliminaries

Let \( x : M^m \to S^{m+p} \) be an immersed umbilic-free submanifold. Denote by \( h \) the second fundamental form of \( x \) and \( H = \frac{1}{m} \text{tr} h \) the mean curvature vector field. Define

\[
\rho = \left( \frac{m}{m-1} \left( |h|^2 - m|H|^2 \right) \right)^{\frac{1}{4}}, \quad Y = \rho(1, x).
\]  

Then \( Y : M^m \to \mathbb{R}^{m+p+2} \) is an immersion of \( M^m \) into the Lorentzian space \( \mathbb{R}^{m+p+2} \) and is called the canonical lift (or the Möbius position vector) of \( x \). The function \( \rho \) given by (2.1) may be called the Möbius factor of the immersion \( x \). Denote

\[
C_+^{m+p+1} = \{ y = (y_0, y_1) \in \mathbb{R}_1 \times \mathbb{R}^{m+p+1} : \langle y, y \rangle_1 = 0, \ y_0 > 0 \}\]
and let $O(m+p+1, 1)$ be the Lorentzian group of all elements in $GL(m+p+2; \mathbb{R})$ preserving the standard Lorentzian inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{m+p+2}$. There is a subgroup $O^+(m+p+1, 1)$ of $O(m+p+1, 1)$ that is given by
\begin{equation}
O^+(m+p+1, 1) = \left\{ T \in O(m+p+1, 1); \ T(C^m_{m+p+1}) \subset C^m_{m+p+1} \right\}.
\end{equation}

The following theorem is well known.

**Theorem 2.1.** (24) Two submanifolds $x, \tilde{x}: M^m \to \mathbb{S}^{m+p}$ with Möbius position vectors $Y, \tilde{Y}$, respectively, are Möbius equivalent if and only if there is a $\langle \cdot, \cdot \rangle$ and let
\begin{equation}
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\end{equation}
the orthogonal complement of $R$ vector field $\kappa$ where $\delta$ denotes the normalized scalar curvature of the Möbius metric $g$. By Theorem 2.1, the induced metric $\Phi$ preserves the inner products as well as the connections on $\mathbb{R}^{m+p+2}$ with respect to the Lorentzian product $\langle \cdot, \cdot \rangle_1$ is a Möbius invariant Riemannian metric (cf. [1], [2], [24]), and is called the Möbius metric of $x$. Using the vector-valued function $\Phi$ and the Laplacian $\Delta$ of the metric $g$, one can define another important vector-valued function $N : M^m \to \mathbb{R}^{m+p+2}$, called the Möbius biposition vector, by
\begin{equation}
N = -\frac{1}{m} \Delta Y - \frac{1}{2m^2} \langle \Delta Y, \Delta Y \rangle_1 Y.
\end{equation}

Then it is verified that the Möbius position vector $Y$ and the Möbius biposition vector $N$ satisfy the following identities [24]:
\begin{align}
\langle \Delta Y, Y \rangle_1 &= -m, \quad \langle \Delta Y, dY \rangle_1 = 0, \quad \langle \Delta Y, \Delta Y \rangle_1 = 1 + m^2 \kappa, \\
\langle Y, Y \rangle_1 &= \langle N, N \rangle_1 = 0, \quad \langle Y, N \rangle_1 = 1,
\end{align}
where $\kappa$ denotes the normalized scalar curvature of the Möbius metric $g$.

Let $V \to M^m$ be the vector subbundle of the trivial Lorentzian bundle $M^m \times \mathbb{R}^{m+p+2}$, defined to be the orthogonal complement of $RY \oplus RN \oplus Y_*(TM^m)$ with respect to the Lorentzian product $\langle \cdot, \cdot \rangle_1$. Then $V$ is called the Möbius normal bundle of the immersion $x$. Clearly, we have the following vector bundle decomposition:
\begin{equation}
M^m \times \mathbb{R}^{m+p+2} = \mathbb{R}Y \oplus RN \oplus Y_*(TM^m) \oplus V.
\end{equation}

Denote by $T^\perp M^m$ the normal bundle of the immersion $x : M^m \to \mathbb{S}^{m+1}$. Then the mean curvature vector field $H$ of $x$ defines a bundle isomorphism $\Phi : T^\perp M^m \to V$ by
\begin{equation}
\Phi(e) = (H \cdot e, (H \cdot e)x + e) \quad \text{for any } e \in T^\perp M^m.
\end{equation}

It is known that $\Phi$ preserves the inner products as well as the connections on $T^\perp M^m$ and $V$ ([24]).

To simplify notations, we make the following conventions on the ranges of indices used frequently in this paper:
\begin{equation}
1 \leq i, j, k, \cdots \leq m, \quad m + 1 \leq \alpha, \beta, \gamma, \cdots \leq m + p.
\end{equation}

For a local orthonormal frame field $\{e_i\}$ for the induced metric $dx \cdot dx$ with the dual $\{\theta^i\}$ and for an orthonormal normal frame field $\{e_\alpha\}$ of $x$, we set
\begin{equation}
e_i = \rho^{-1} e_i, \quad \omega^i = \rho \theta^i, \quad E_\alpha = \Phi(e_\alpha).
\end{equation}

Then $\{E_\alpha\}$ is a local orthonormal frame field on $M^m$ with respect to the Möbius metric $g$, $\{\omega^i\}$ is the dual of $\{e_i\}$, and $\{E_\alpha\}$ is a local orthonormal frame field of the Möbius normal bundle $V \to M$. Clearly, $\{Y, N, Y_i := Y_*(E_i), E_\alpha\}$ is a moving frame of $\mathbb{R}^{m+p+2}$ along $M^m$. If the basic Möbius invariants $A, B$ and $C$ are respectively written as
\begin{equation}
A = \sum A_{ij} \omega^i \omega^j, \quad B = \sum B_{ij}^\alpha \omega^i \omega^j E_\alpha, \quad C = \sum C_{i}^\alpha \omega^i E_\alpha,
\end{equation}
then we have the following equations of motion ([24]):
\begin{align}
dY &= \sum Y_i \omega^i, \quad dN = \sum A_{ij} \omega^i Y_j + C_{i}^\alpha \omega^i E_\alpha, \\
dY_i &= - \sum A_{ij} \omega^j Y - \omega^j N + \sum \omega^j Y_j + \sum B_{ij}^\alpha \omega^j E_\alpha, \\
dE_\alpha &= - \sum C_{i}^\alpha \omega^j Y - \sum B_{ij}^\alpha \omega^j Y_i + \sum \omega^j E_\beta.
\end{align}
where $\omega^j$ are the Levi-Civita connection forms of the Möbius metric $g$ and $\omega^\beta_\alpha$ are the (Möbius) normal connection forms of $x$. Furthermore, by a direct computation one can find the following local expressions (2.4):

$$A_{ij} = -\rho^{-2} \left( \operatorname{Hess}_{ij}(\log \rho) - \epsilon_i(\log \rho)\epsilon_j(\log \rho) - \sum H^\alpha h_{ij}^\alpha \right)$$

$$- \frac{1}{2} \rho^{-2} \left( |d \log \rho|^2 - 1 + |H|^2 \right) \delta_{ij},$$

(2.14)

$$B^\alpha_{ij} = \rho^{-1} \left( h_{ij}^\alpha - H^\alpha \delta_{ij} \right),$$

(2.15)

$$C^\alpha_i = -\rho^{-2} \left( H_{ij}^\alpha + \sum (h_{ij}^\alpha - H^\alpha \delta_{ij}) \epsilon_j(\log \rho) \right),$$

(2.16)

in which the subscript “$i$” denotes the covariant derivative with respect to the induced metric $dx \cdot dx$ and in the direction $e_i$.

Denote, respectively, by $R_{ijkl}$, $R^\alpha_{ij\beta\gamma}$ and $C^\alpha_{ij}$ the components of the Möbius Riemannian curvature tensor and the curvature operator of the Möbius normal bundle with respect to the tangent frame field $\{E_i\}$ and the Möbius normal frame field $\{E_\alpha\}$. Then we have (2.4):

$$\operatorname{tr} A = \frac{1}{2m} (1 + m^2 \kappa), \quad \operatorname{tr} B = \sum B^\alpha_{ij} E_\alpha = 0, \quad |B|^2 = \sum (B^\alpha_{ij})^2 = \frac{m - 1}{m}.$$  

(2.17)

$$R_{ijkl} = \sum (B^\alpha_{il} B^\alpha_{jk} - B^\alpha_{ik} B^\alpha_{jl}) + A_{il} \delta_{jk} - A_{ik} \delta_{jl} + A_{jk} \delta_{il} - A_{jl} \delta_{ik},$$

(2.18)

$$R_{ij...}^\alpha = \sum (B^\beta_{jk} B^\beta_{ik} - B^\beta_{ik} B^\beta_{jk}).$$

(2.19)

We should remark that both equations (2.18) and (2.19) have the opposite sign from those in [-24] due to the different notations of the Riemannian curvature tensor. Furthermore, let $A_{ijk}$, $B^\alpha_{ij}$ and $C^\alpha_{ij}$ denote, respectively, the components with respect to the frame fields $\{E_i\}$ and $\{E_\alpha\}$ of the covariant derivatives of $A$, $B$ and $C$, then the following Ricci identities hold [-24]:

$$A_{ijk} - A_{ikj} = \sum (B^\alpha_{il} C^\alpha_{jk} - B^\alpha_{ik} C^\alpha_{lj}),$$

(2.20)

$$B^\alpha_{ij...} - B^\alpha_{ij...} = \delta_{ij} C^\alpha_{lj} - \delta_{lj} C^\alpha_{ij},$$

(2.21)

$$C^\alpha_{ij} - C^\alpha_{ji} = \sum (B^\beta_{ik} A_{kj} - B^\beta_{kj} A_{ik}).$$

(2.22)

Denote by $R_{ij}$ the components of the Ricci curvature. Then by taking trace in (2.18) and (2.19), one obtains

$$R_{ij} = -\sum B^\alpha_{ik} B^\alpha_{ij} + \delta_{ij} \operatorname{tr} A + (m - 2) A_{ij},$$

(2.23)

$$(m - 1) C^\alpha_i = -\sum B^\alpha_{ij}.$$  

(2.24)

Moreover, for the higher order covariant derivatives $B^\alpha_{ij...kl}$, we have the following Ricci identities:

$$B^\alpha_{ij...kl} - B^\alpha_{ij...kl} = \sum B^\alpha_{iq...kl} + \sum B^\alpha_{ij...qkl} + \cdots - \sum B^\beta_{ij...} R^\alpha_{ijkl}. $$

(2.25)

By (2.17), (2.23) and (2.24), if $m \geq 3$, then the Blaschke tensor $A$ and the Möbius form $C$ are determined by the Möbius metric $g$, Möbius second fundamental form $B$ and the (Möbius) normal connection of $x$. Thus the following theorem holds:

**Theorem 2.2** (cf. [-24]). Two submanifolds $x : M^m \to S^{m+p}$ and $\tilde{x} : \tilde{M}^m \to \tilde{S}^{m+p}$, $m \geq 3$, are Möbius equivalent if and only if they have the same Möbius metrics, the same Möbius second fundamental forms and the same (Möbius) normal connections.
3. THE NEW EXAMPLES

Before proving the main theorem, we need to find more examples of Blaschke parallel submanifolds in the unit sphere $S^{m+p}$ as many as possible with parallel Blaschke tensors and with three distinct Blaschke eigenvalues. We note that, by Zhai-Hu-Wang (25), all Möbius parallel submanifolds in $S^{m+p}$ are necessarily Blaschke parallel ones. This kind of examples are listed in [25]. In this section we define a new class of Blaschke parallel examples which are in general not Möbius parallel.

**Example 3.1.** The following three classes of submanifolds have been studied in [17] (cf. [25]).

1. The umbilic-free pseudo-parallel submanifolds $\tilde{x} : M^m \to S^{m+p}$ with parallel mean curvature $\tilde{H}$ and constant scalar curvature $\tilde{S}$.

2. The composition $\tilde{x} = \sigma \circ \bar{x}$ where $\bar{x} : M^m \to \mathbb{R}^{m+p}$ is an umbilic-free pseudo-parallel submanifolds with parallel mean curvature $\bar{H}$ and constant scalar curvature $\bar{S}$.

3. The composition $\tilde{x} := \tau \circ \bar{x}$ where $\bar{x} : M^m \to \mathbb{H}^{m+p}$ is an umbilic-free pseudo-parallel submanifold with parallel mean curvature $\bar{H}$ and constant scalar curvature $\bar{S}$.

**Remark 3.1.** It is shown in [17] that all the examples $\tilde{x} : M^m \to S^{m+p}$ given in (1), (2) and (3) above are Blaschke parallel with vanishing Möbius form. Furthermore, $\tilde{x}$ in (1) has three distinct Blaschke eigenvalues if and only if it is not minimal and has three distinct principal curvatures in the direction of the mean curvature vector $\tilde{H}$, while $\tilde{x}$ in (2) (resp. in (3)) has three distinct Blaschke eigenvalues if and only if the corresponding $\bar{x} : M^m \to \mathbb{R}^{m+p}$ (resp. $\bar{x} : M^m \to \mathbb{H}^{m+p}$) is not minimal and has three distinct principal curvatures in the direction of the mean curvature vector $\bar{H}$. Note that $\tilde{x}$ is Möbius isotropic, or equivalently, $\tilde{x}$ has only one distinct Blaschke eigenvalue, if and only $\tilde{x}$ in (1), or $\tilde{x}$ in (2) or in (3) is minimal (22). In addition, it is not hard to see that (25) a submanifold $\tilde{x}$ is Möbius parallel if and only if $\tilde{x}$ in (1), or $\tilde{x}$ in (2) or in (3) is (Euclidean) parallel.

**Example 3.2.** Submanifolds $LS(m, p, r, \mu)$.

We start with a multiple parameter data $(m, p, r, \mu)$ where

$$ m := (m_1, m_2, m_3), \quad p := (p_1, p_2, p_3), \quad r := (r_1, r_2, r_3), \quad \mu := (\mu_1, \mu_2, \mu_3) $$

with $m_1, m_2, m_3$ and $p_1, p_2, p_3$ being integers satisfying

$$ m_1, m_2, m_3 \geq 1, \quad p_1, p_2, p_3 \geq 0; $$

and with $r_1, r_2, r_3$ and $\mu_1, \mu_2, \mu_3$ being real numbers satisfying

$$ r_1, r_2, r_3 > 0, \quad r_1^2 = r_2^2 + r_3^2, \quad \mu_1, \mu_2, \mu_3 \geq 0, \quad \mu_1 + \mu_2 + \mu_3 = 1. $$

Denote

$$ m := m_1 + m_2 + m_3, \quad p := p_1 + p_2 + p_3 + 1. $$

Since

$$ \det \begin{bmatrix} m + m_1 & m_2 & m_3 \\ m_1 & m + m_2 & m_3 \\ m_2 & m_2 & m + m_3 \end{bmatrix} = 2m^3 \neq 0, $$

there exist real numbers $\lambda_1, \lambda_2, \lambda_3$ that are uniquely determined by

$$ \begin{cases} (m + m_1)\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = -\frac{m_1}{r_1^2}, \\
m_1\lambda_1 + (m + m_2)\lambda_2 + m_3\lambda_3 = \frac{r_2}{r_3}, \\
m_1\lambda_1 + m_2\lambda_2 + (m + m_3)\lambda_3 = \frac{m_3}{r_3}. \end{cases} \quad (3.1) $$

Let $B_1^0, B_2^0, B_3^0$ be real numbers defined by

$$ m_1B_1^0 + m_2B_2^0 + m_3B_3^0 = 0, \quad B_a^0B_b^0 = -(\lambda_a + \lambda_b), \quad a \neq b \quad (3.2) $$
for a permutation \(\leq\). Then the following lemma can be shown by a direct computation using (3.1), (3.2), (3.3) and (3.4):

\[
(a, b)'' \leq m \quad \text{is an even permutation of } 1, 2, 3.
\]

Note that, by (3.2)

\[
2\lambda_a = (\lambda_a + \lambda_{a'}) + (\lambda_a + \lambda_{a''}) - (\lambda_a' + \lambda_{a''}) = -B_a^0 B_a^0 - B_a^0 B_{a''} + B_a^0 B_{a''},
\]

for a permutation \(a, a', a''\) of 1, 2, 3.

Then the following lemma can be shown by a direct computation using (3.1), (3.2), (3.3) and (3.4):

**Lemma 3.1.** It holds that

\[
2\lambda_1 + (B_1^0)^2 = -\frac{1}{r_1^2}, \quad 2\lambda_2 + (B_2^0)^2 = \frac{1}{r_2^2}, \quad 2\lambda_3 + (B_3^0)^2 = \frac{1}{r_3^2},
\]

\[
- \frac{m_1 - 1}{r_1^2} + (B_1^0)^2 = (m + m_1 - 2)\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = (m - 2)\lambda_1 + \sum_a m_a\lambda_a,
\]

\[
- \frac{m_2 - 1}{r_2^2} + (B_2^0)^2 = m_1\lambda_1 + (m + m_2 - 2)\lambda_2 + m_3\lambda_3 = (m - 2)\lambda_2 + \sum_a m_a\lambda_a,
\]

\[
- \frac{m_3 - 1}{r_3^2} + (B_3^0)^2 = m_1\lambda_1 + m_2\lambda_2 + (m + m_3 - 2)\lambda_3 = (m - 2)\lambda_3 + \sum_a m_a\lambda_a,
\]

\[
\begin{align*}
&= \frac{m_1(m_1 - 1)}{r_1^2} + \frac{m_2(m_2 - 1)}{r_2^2} + \frac{m_3(m_3 - 1)}{r_3^2} \\
&= (2m_1(m_1 - 1) - (m + m_1))\lambda_1 + (2m_2(m - 1) - (m + m_2))\lambda_2 \\
&+ (2m_3(m - 1) - (m + m_3))\lambda_3,
\end{align*}
\]

\[
m_1(B_1^0)^2 + m_2(B_2^0)^2 + m_3(B_3^0)^2 = (m + m_1)\lambda_1 + (m + m_2)\lambda_2 + (m + m_3)\lambda_3,
\]

\[
- r_1^2 B_1^0 + r_2^2 B_2^0 + r_3^2 B_3^0 = 0,
\]

\[
- r_1^2 (B_1^0)^2 + r_2^2 (B_2^0)^2 + r_3^2 (B_3^0)^2 = -\lambda_1 r_1^2 + \lambda_2 r_2^2 + \lambda_3 r_3^2 = 1.
\]

Let

\[
\tilde{y} = (\tilde{y}_0, \tilde{y}_1) : M_1 \rightarrow \mathbb{H}^{m_1+p_1} \left( -\frac{1}{r^2} \right) \subset \mathbb{R}^{m_1+p_1+1}
\]

be an immersed minimal submanifold of dimension \(m_1\) with constant scalar curvature

\[
\tilde{S}_1 = - \frac{m_1(m_1 - 1)}{r_1^2} + \mu_1 \left( m_1(B_1^0)^2 + m_2(B_2^0)^2 + m_3(B_3^0)^2 - \frac{m - 1}{m} \right),
\]

and

\[
\tilde{y}_2 : M_2 \rightarrow \mathbb{S}^{m_2+p_2}(r_2) \subset \mathbb{R}^{m_2+p_2+1}, \quad \tilde{y}_3 : M_3 \rightarrow \mathbb{S}^{m_3+p_3}(r_3) \subset \mathbb{R}^{m_3+p_3+1}
\]

be two immersed minimal submanifolds of dimensions \(m_2, m_3\) with constant scalar curvatures

\[
\tilde{S}_2 = \frac{m_2(m_2 - 1)}{r_2^2} + \mu_2 \left( m_1(B_1^0)^2 + m_2(B_2^0)^2 + m_3(B_3^0)^2 - \frac{m - 1}{m} \right),
\]

\[
\tilde{S}_3 = \frac{m_3(m_3 - 1)}{r_3^2} + \mu_3 \left( m_1(B_1^0)^2 + m_2(B_2^0)^2 + m_3(B_3^0)^2 - \frac{m - 1}{m} \right),
\]

where \(1 \leq a, b \leq 3\). Clearly, \(B_1^0, B_2^0, B_3^0\) are unique up to a common sign. In fact we have

\[
(B_a^0)^2 = \frac{1}{m_a}((m_{a'} + m_{a''})\lambda_a + m_{a'}\lambda_{a'} + m_{a''}\lambda_{a''})
\]

where \(a, a', a''\) is an even permutation of 1, 2, 3.
respectively. Then by (3.9) and (3.10)
\[
\tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 = -\frac{m_1(m_1-1)}{r_1^2} + \frac{m_2(m_2-1)}{r_2^2} + \frac{m_3(m_3-1)}{r_3^2} - \frac{m-1}{m} + m_1(B_1^0)^2 + m_2(B_2^0)^2 + m_3(B_3^0)^2 = 2(m-1) \sum_a m_a \lambda_a - \frac{m-1}{m}. \tag{3.17}
\]

Set
\[
\tilde{M}^m = M_1 \times M_2 \times M_3, \quad \tilde{Y} = (\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3). \tag{3.18}
\]
Then \( \tilde{Y} : \tilde{M}^m \to \mathbb{R}^{m+p+2} \) is an immersion satisfying \( \langle \tilde{Y}, \tilde{Y} \rangle_1 = 0 \) with the induced Riemannian metric
\[
g = \langle d\tilde{Y}, d\tilde{Y} \rangle_1 = -d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2 + d\tilde{y}_3^2.
\]
Thus
\[
(\tilde{M}^m, g) = (M_1, \langle d\tilde{y}, d\tilde{y} \rangle_1) \times (M_2, d\tilde{y}_2^2) \times (M_3, d\tilde{y}_3^2) \tag{3.19}
\]
as Riemannian manifolds. Define
\[
\tilde{x}_1 = \frac{\tilde{y}_1}{\tilde{y}_0}, \quad \tilde{x}_2 = \frac{\tilde{y}_2}{\tilde{y}_0}, \quad \tilde{x}_3 = \frac{\tilde{y}_3}{\tilde{y}_0}, \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3). \tag{3.20}
\]
Then \( \tilde{x}^2 = 1 \) and thus \( \tilde{x} : \tilde{M}^m \to \mathbb{S}^{m+p} \) is an immersed submanifold which we denote simply by \( \text{LS}(\mathbb{S}^{m+p}) \). Since
\[
d\tilde{x} = -\frac{d\tilde{y}_0}{\tilde{y}_0}(\tilde{y}_1, \tilde{\tilde{y}}_2, \tilde{\tilde{y}}_3) + \frac{1}{\tilde{y}_0}(d\tilde{y}_1, d\tilde{y}_2, d\tilde{y}_3), \tag{3.21}
\]
the induced metric \( \tilde{g} = d\tilde{x} \cdot d\tilde{x} \) on \( \tilde{M}^m \) is related to \( g \) by
\[
\tilde{g} = \tilde{y}_0^{-2}(-d\tilde{y}_0^2 + d\tilde{y}_1^2 + d\tilde{y}_2^2 + d\tilde{y}_3^2) = \tilde{y}_0^{-2} g. \tag{3.22}
\]
Denote
\[
\bar{E}_{\alpha 0} := -(B_1^0 \tilde{y}_0, B_2^0 \tilde{y}_1, B_3^0 \tilde{y}_2, B_3^0 \tilde{y}_3), \tag{3.23}
\]
and let
\[
\{ \bar{E}_\alpha; m + 1 \leq \alpha \leq m + p_1 \},
\{ \bar{E}_\alpha; m + p_1 + 1 \leq \alpha \leq m + p_1 + p_2 \},
\{ \bar{E}_\alpha; m + p_1 + p_2 + 1 \leq \alpha \leq m + p_1 + p_2 + p_3 \}
\]
be orthonormal normal frame fields of \( \tilde{y}, \tilde{\tilde{y}}_2, \tilde{\tilde{y}}_3 \), respectively, with
\[
\bar{E}_\alpha = (\bar{E}_{\alpha 0}, \bar{E}_{\alpha 1}) \in \mathbb{R}^1 \times \mathbb{R}^{m_1+p_1} \equiv \mathbb{R}^{m_1+p_1+1}, \text{ for } \alpha = m + 1, \cdots, m + p_1.
\]
Define
\[
\hat{e}_\alpha = (\bar{E}_{\alpha 1}, 0, 0) - \bar{E}_{\alpha 0} \tilde{x} \in \mathbb{R}^{m_1+p_1} \times \mathbb{R}^{m_2+p_2+1} \times \mathbb{R}^{m_3+p_3+1} \equiv \mathbb{R}^{m+p+1},
\]
for \( \alpha = m + 1, \cdots, m + p_1 \); \( \hat{e}_\alpha = (0, \bar{E}_\alpha, 0) \in \mathbb{R}^{m_1+p_1} \times \mathbb{R}^{m_2+p_2+1} \times \mathbb{R}^{m_3+p_3+1} \equiv \mathbb{R}^{m+p+1}, \)
for \( \alpha = m + p_1 + 1, \cdots, m + p_1 + p_2 \); \( \hat{e}_\alpha = (0, 0, \bar{E}_\alpha) \in \mathbb{R}^{m_1+p_1} \times \mathbb{R}^{m_2+p_2+1} \times \mathbb{R}^{m_3+p_3+1} \equiv \mathbb{R}^{m+p+1}, \)
for \( \alpha = m + p_1 + p_2 + 1, \cdots, m + p_1 + p_2 + p_3 \); \( \hat{e}_{\alpha 0} = -(B_1^0 \tilde{y}_1, B_2^0 \tilde{y}_2, B_3^0 \tilde{y}_3) + B_1^0 \tilde{y}_0 \tilde{x} \). \tag{3.27}

Then, by Lemma 3.4 \( \{ \hat{e}_\alpha, \hat{e}_{\alpha 0}; m + 1 \leq \alpha \leq m + p - 1 \} \) is an orthonormal normal frame field of \( \text{LS}(\mathbb{S}^{m+p}) \).
respectively. Then

\[ \{E_i : 1 \leq i \leq m_1\}, \quad \{E_i : m_1 + 1 \leq i \leq m_1 + m_2\}, \quad \{E_i : m_1 + m_2 + 1 \leq i \leq m\} \]

be local orthonormal frame fields for

\[ (M_1, (d\tilde{y}, d\tilde{y})_1), \quad (M_2, d\tilde{y}_2^2), \quad (M_3, d\tilde{y}_3^2), \]

respectively. Then \(\{\tilde{e}_i : 1 \leq i \leq m\}\) is a local orthonormal frame field for \((M^m, \tilde{g})\). Thus for \(\alpha = m + 1, \ldots, m + p_1\),

\[
\begin{aligned}
\tilde{h}_o^a &= \tilde{h}_o^a(\tilde{e}_i, \tilde{e}_j) = \tilde{g}^o_{ij}\tilde{h}_o^a(E_i, E_j) = \tilde{y}_0\tilde{h}_o^a(E_i, E_j) + \tilde{E}_{\alpha 0} g(E_i, E_j) \\
&= \tilde{y}_0\tilde{h}_o^a_{ij} + \tilde{E}_{\alpha 0}\delta_{ij}, \quad \text{when } 1 \leq i, j \leq m_1, \\
\tilde{h}_o^a &= \tilde{E}_{\alpha 0}\delta_{ij}, \quad \text{otherwise;}
\end{aligned}
\]

while

\[
\begin{aligned}
\tilde{h}_o^a &= \tilde{h}_o^a(\tilde{e}_i, \tilde{e}_j) = \tilde{g}^o_{ij}\tilde{h}_o^a(E_i, E_j) = \tilde{y}_0\tilde{h}_o^a(E_i, E_j) = \tilde{y}_0\tilde{h}_o^a_{ij}, \\
&= \tilde{y}_0\tilde{h}_o^a_{ij} + \tilde{E}_{\alpha 0}\delta_{ij}, \quad \text{when } m_1 + 1 \leq i, j \leq m_1 + m_2, \\
\tilde{h}_o^a &= 0, \quad \text{otherwise.}
\end{aligned}
\]
for \( \alpha = m + p_1 + 1, \cdots , m + p_1 + p_2 \), and
\[
\begin{cases}
\bar{h}_{ij}^\alpha = \bar{h}^{\alpha}(\bar{e}_i, \bar{e}_j) = \bar{y}_0 \bar{h}^{\alpha}(E_i, E_j) = \bar{y}_0 \bar{h}_{ij}^\alpha, \\
\text{when } m_1 + m_2 + 1 \leq i, j \leq m,
\end{cases}
\]
\[\bar{h}_{ij}^\alpha = 0, \quad \text{otherwise} \tag{3.38}\]
for \( \alpha = m + p_1 + p_2 + 1, \cdots , m + p - 1 \). Furthermore
\[
\bar{h}_{ij}^{\alpha 0} = \bar{h}^{\alpha 0}(\bar{e}_i, \bar{e}_j) = \bar{y}_0 \bar{h}^{\alpha 0}(E_i, E_j) = \begin{cases}
\bar{y}_0 (B_2^0 - B_1^0) \delta_{ij}, & \text{for } m_1 + 1 \leq i, j \leq m_1 + m_2, \\
\bar{y}_0 (B_3^0 - B_1^0) \delta_{ij}, & \text{for } m_1 + m_2 + 1 \leq i, j \leq m, \\
0, & \text{otherwise} \tag{3.39}
\end{cases}
\]
Since \( \bar{y}, \bar{y}_2 \) and \( \bar{y}_3 \) are minimal, the mean curvature
\[
\bar{H} = \frac{1}{m} \left( \sum_{\alpha=m+1}^{m+p-1} \sum_{i=1}^{m} \bar{h}_{ii}^\alpha \bar{e}_\alpha + \sum_{i=1}^{m} \bar{h}_{ii}^{\alpha 0} \bar{e}_{\alpha 0} \right)
\]
of \( \text{LS}(m, p, r, \mu) \) is given by
\[
\bar{H}^\alpha = \frac{1}{m} \sum_{i=1}^{m} \bar{h}_{ii}^\alpha = \frac{\bar{y}_0}{m} \sum_{i=1}^{m} \bar{h}_{ii}^\alpha + \bar{E}_{\alpha 0} = \bar{E}_{\alpha 0}, \quad \text{for } m + 1 \leq \alpha \leq m + p_1; \tag{3.40}
\]
\[
\bar{H}^\alpha = \frac{1}{m} \sum_{i=1}^{m} \bar{h}_{ii}^\alpha = \frac{\bar{y}_0}{m} \sum_{i=m_1+1}^{m_1+m_2} \bar{h}_{ii}^\alpha = 0, \quad \text{for } m + p_1 + 1 \leq \alpha \leq m + p_1 + p_2; \tag{3.41}
\]
\[
\bar{H}^\alpha = \frac{1}{m} \sum_{i=1}^{m} \bar{h}_{ii}^\alpha = \frac{\bar{y}_0}{m} \sum_{i=m_1+m_2+1}^{m} \bar{h}_{ii}^\alpha = 0, \quad \text{for } m + p_1 + p_2 + 1 \leq \alpha \leq m + p - 1; \tag{3.42}
\]
\[
\bar{H}^{\alpha 0} = \frac{1}{m} \sum_{i=1}^{m} \bar{h}_{ii}^{\alpha 0} = \frac{\bar{y}_0}{m} \left( m_2(B_2^0 - B_1^0) + m_3(B_3^0 - B_1^0) \right) = -\bar{y}_0 B_1^0. \tag{3.43}
\]
From \( 3.22, 3.23, 3.24, 3.25, 3.26, 3.27, 3.28, 3.29, 3.30, 3.31 \) and the Gauss equations of \( \bar{y}, \bar{y}_2 \) and \( \bar{y}_3 \), we find
\[
\bar{h}^2 = \bar{y}_0^2 \sum_{\alpha=m+1}^{m+p_1} \sum_{i,j=1}^{m} \bar{h}_{ii}^{\alpha 0} + m \sum_{\alpha=m+1}^{m+p_1} (\bar{E}_{\alpha 0})^2 + \bar{y}_0^2 \sum_{\alpha=m+1}^{m+p_1} \sum_{i,j=m+1}^{m+p_1} (\bar{h}_{ii}^\alpha)^2 \\
+ \bar{y}_0^2 \sum_{\alpha=m+p_1+1}^{m+p-1} \sum_{i,j=m+p_1+1}^{m+p-1} (\bar{h}_{ii}^\alpha)^2 + \bar{y}_0^2 (m_2(B_2^0 - B_1^0)^2 + m_3(B_3^0 - B_1^0)^2)
\]
\[= \frac{m-1}{m} \bar{y}_0^2 + m \sum_{\alpha=m+1}^{m+p_1} (\bar{E}_{\alpha 0})^2 + m \bar{y}_0^2 (B_1^0)^2, \tag{3.44}
\]
\[
|\bar{H}|^2 = \sum_{\alpha=m+1}^{m+p_1} (\bar{H}^\alpha)^2 + \sum_{\alpha=m+p_1+1}^{m+p-1} (\bar{H}^\alpha)^2 + \sum_{\alpha=m+p_1+1}^{m+p-1} (\bar{H}^{\alpha 0})^2 + (\bar{H}^{\alpha 0})^2
\]
\[= m \sum_{\alpha=m+1}^{m+p_1} (\bar{E}_{\alpha 0})^2 + m \bar{y}_0^2 (B_1^0)^2. \tag{3.45}
\]
It then follows that
\[
|\bar{h}|^2 - m|\bar{H}|^2 = \frac{m-1}{m} \bar{y}_0^2 > 0,
\]
implicating that \( \bar{x} \) is umbilic-free, and the M"obius factor \( \bar{\rho} = \bar{y}_0 \). So \( \bar{Y} \) is the M"obius position of \( \text{LS}(m, p, r, \mu) \). Consequently, the M"obius metric of \( \text{LS}(m, p, r, \mu) \) is nothing but \( \langle d\bar{Y}, d\bar{Y} \rangle_1 = g \). Furthermore, if we denote
by \( \{ \omega^i \} \) the local coframe field on \( M^m \) dual to \( \{ E_i \} \), then the Möbius second fundamental form

\[
\tilde{B} = \sum_{\alpha=m+1}^{m+p} B^\alpha \Phi(\tilde{e}_\alpha) = \sum_{\alpha=m+1}^{m+p} \tilde{B}_{ij}^\alpha \omega^i \omega^j \Phi(\tilde{e}_\alpha)
\]

of \( \text{LS}(m, p, r, \mu) \) is given by

\[
\tilde{B}^\alpha = \tilde{\rho}^{-1} \sum_{i,j=1}^{m_1} (\tilde{h}^\alpha_{ij} - \tilde{H}^\alpha_{ij}) \omega^i \omega^j = \sum_{i,j=1}^{m_1} \tilde{h}^\alpha_{ij} \omega^i \omega^j,
\]

for \( \alpha = m + 1, \ldots, m + p_1 \); \hspace{1cm} (3.46)

\[
\tilde{B}^\alpha = \tilde{\rho}^{-1} \sum_{i,j=1}^{m_1+m_2} (\tilde{h}^\alpha_{ij} - \tilde{H}^\alpha_{ij}) \omega^i \omega^j = \sum_{i,j=1}^{m_1+m_2} \tilde{h}^\alpha_{ij} \omega^i \omega^j,
\]

for \( \alpha = m + p_1 + 1, \ldots, m + p_1 + p_2 \); \hspace{1cm} (3.47)

\[
\tilde{B}^\alpha = \tilde{\rho}^{-1} \sum_{i,j=1}^{m_1+m_2+m_3} (\tilde{h}^\alpha_{ij} - \tilde{H}^\alpha_{ij}) \omega^i \omega^j = \sum_{i,j=1}^{m_1+m_2+m_3} \tilde{h}^\alpha_{ij} \omega^i \omega^j,
\]

for \( \alpha = m + p_1 + p_2 + 1, \ldots, m + p - 1 \); \hspace{1cm} (3.48)

\[
\tilde{B}^{\alpha_0} = B^\alpha_1 \sum_{i=1}^{m_1} (\omega^i)^2 + B^\alpha_2 \sum_{i=m_1+1}^{m_1+m_2} (\omega^i)^2 + B^\alpha_3 \sum_{i=m_1+m_2+1}^{m} (\omega^i)^2;
\]

or, equivalently

\[
\tilde{B}^\alpha_{ij} = \begin{cases} 
\tilde{h}^\alpha_{ij}, & \text{if } m + 1 \leq \alpha \leq m + p_1, \ 1 \leq i, j \leq m_1, \\
\text{otherwise}.
\end{cases}
\]

On the other hand, since the Möbius metric \( g \) is the direct product of \( \langle d\tilde{y}, d\tilde{y} \rangle_1, d\tilde{y}_2 \cdot d\tilde{y}_2 \) and \( d\tilde{y}_3 \cdot d\tilde{y}_3 \), one finds by the minimality and the Gauss equations of \( \tilde{y}, \tilde{y}_2 \) and \( \tilde{y}_3 \) that the Ricci tensor of \( g \) is given as follows:

\[
R_{ij} = -\frac{m_1 - 1}{r_1^2} \delta_{ij} - \sum_{\alpha_1}^{m_1} \sum_{k=1}^{m_1} \tilde{h}^\alpha_{ik} \tilde{h}^\alpha_{kj}, \quad \text{if } 1 \leq i, j \leq m_1,
\]

\[
R_{ij} = \frac{m_2 - 1}{r_2^2} \delta_{ij} - \sum_{\alpha_2}^{m_2} \sum_{k=m_1+1}^{m_2} \tilde{h}^\alpha_{ik} \tilde{h}^\alpha_{kj}, \quad \text{if } m_1 + 1 \leq i, j \leq m_1 + m_2,
\]

\[
R_{ij} = \frac{m_3 - 1}{r_3^2} \delta_{ij} - \sum_{\alpha_3}^{m_3} \sum_{k=m_1+m_2+1}^{m} \tilde{h}^\alpha_{ik} \tilde{h}^\alpha_{kj}, \quad \text{if } m_1 + m_2 + 1 \leq i, j \leq m,
\]

\[
R_{ij} = 0, \quad \text{otherwise},
\]

where

\[
m + 1 \leq \alpha_1 \leq m + p_1, \quad m + p_1 + 1 \leq \alpha_2 \leq m + p_1 + p_2, \quad m + p_1 + p_2 + 1 \leq \alpha_3 \leq m + p - 1.
\]

On the other hand, by the definitions of \( \tilde{y}, \tilde{y}_2 \) and \( \tilde{y}_3 \), the trace of \( A \) is given by

\[
\text{tr } A = \frac{1}{2m} (1 + m^2 \lambda) = \sum_{\alpha} m_\alpha \lambda_\alpha.
\]
Since \( m \geq 3 \), it follows by (3.20), (3.22) and (3.30)–(3.35) that the Blaschke tensor of \( \text{LS}(m, p, r, \mu) \) is given by \( A = \sum A_{ij} \omega_i \omega_j \) where, for \( 1 \leq i, j \leq m_1 \),
\[
A_{ij} = \frac{1}{m - 2} \left( -\frac{m_1 - 1}{r^2} + (B_1^0)^2 - \sum a \lambda_a \right) \delta_{ij} = \lambda_1 \delta_{ij}.
\]

Similarly,
\[
A_{ij} = \lambda_2 \delta_{ij}, \quad \text{for} \quad m_1 + 1 \leq i, j \leq m_1 + m_2,
\]
\[
A_{ij} = \lambda_3 \delta_{ij}, \quad \text{for} \quad m_1 + m_2 + 1 \leq i, j \leq m,
\]
\[
A_{ij} = 0, \quad \text{otherwise}.
\]

Therefore, \( A \) has constant eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \). It follows that \( \text{LS}(m, p, r, \mu) \) is Blaschke parallel since \( \omega_i^j = 0 \) for \( A_{ii} \neq A_{jj} \).

**Proposition 3.2.** For each of the submanifolds \( \text{LS}(m, p, r, \mu) \) defined in Example 3.2, we have

1. The Möbius form \( C \) vanishes identically;
2. The Blaschke eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) are distinct if and only if \( m_3 r_3^2 \neq m_2 r_3^2 \);
3. The Möbius second fundamental form \( B \) is parallel if and only if

\[
\tilde{y} : M_1 \to \mathbb{H}^{m_1 + p_1} \left( -\frac{1}{r_1^2} \right), \quad \tilde{y}_2 : M_2 \to S^{m_2 + p_2}(r_2) \quad \text{and} \quad \tilde{y}_3 : M_3 \to S^{m_3 + p_3}(r_3)
\]

are all parallel as Riemannian submanifolds. Furthermore, if it is the case, then \( \tilde{y}(M_1) \) is isometric to the totally geodesic hyperbolic space \( \mathbb{H}^{m_1} \left( -\frac{1}{r_1^2} \right) \) and \( \tilde{y} \) can be taken as the standard embedding of \( \mathbb{H}^{m_1} \left( -\frac{1}{r_1^2} \right) \) in \( \mathbb{H}^{m_1 + p_1} \left( -\frac{1}{r_1^2} \right) \).

**Proof.** The proof of (1) and (3) is omitted here since it is similar to that of Proposition 3.1 in [17]; the conclusion (2) is direct from (3.1).

**Remark 3.2.** It is not hard to show that, if \( m_3 r_3^2 = m_2 r_3^2 \), then \( \lambda_2 = \lambda_3 \). In this case, \( \text{LS}(m, p, r, \mu) \) has two distinct Blaschke eigenvalues and is included as one special case of Example 3.1 or Example 3.2 in [17].

4. **Proof of the main theorem**

Let \( x : M^m \to S^{m+p} \) be an umbilic-free submanifold in \( S^{m+p} \) satisfying all the conditions in the main theorem, and \( \lambda_1, \lambda_2, \lambda_3 \) be the three distinct Blaschke eigenvalues of \( x \). Since the Möbius form \( C \equiv 0 \) and the Blaschke tensor \( A \) is parallel, \((M, g)\) is isometric to a direct product of three Riemannian manifolds \((M_1, g^{(1)})\), \((M_2, g^{(2)})\) and \((M_3, g^{(3)})\) with
\[
m_1 := \dim M_1, \quad m_2 := \dim M_2, \quad m_3 := \dim M_3
\]
such that, under the orthonormal frame field \( \{E_i\} \) of \((M^m, g)\) satisfying
\[
E_1, \ldots, E_{m_1} \in TM_1, \quad E_{m_1+1}, \ldots, E_{m_1+m_2} \in TM_2, \quad E_{m_1+m_2+1}, \ldots, E_m \in TM_3,
\]
the components \( A_{ij} \) of \( A \) with respect to \( \{E_i\} \) are diagonalized as follows:
\[
A_{i_1j_1} = \lambda_1 \delta_{i_1j_1}, \quad A_{i_2j_2} = \lambda_2 \delta_{i_2j_2}, \quad A_{i_3j_3} = \lambda_3 \delta_{i_3j_3}, \quad A_{i_4j_4} = A_{i_5j_5} = A_{i_6j_6} = 0, \quad (4.1)
\]
where and from now on we agree with
\[
1 \leq i_1, j_1, k_1, \ldots \leq m_1, \quad m_1 + 1 \leq i_2, j_2, k_2, \ldots \leq m_1 + m_2, \quad m_1 + m_2 + 1 \leq i_3, j_3, k_3, \ldots \leq m.
\]
Furthermore, as done in Section 2, write \( B = \sum B^\alpha_{ij} \omega^j E_a \) for some Möbius normal frame field \( \{ E_a \} \). Then, by \( C \equiv 0 \) and (2.22), the corresponding components \( B^\alpha_{ij} \) satisfy
\[
B^\alpha_{i_1 i_2} = B^\alpha_{i_1 i_3} = B^\alpha_{i_2 i_3} \equiv 0, \quad \text{for all } \alpha, i_1, i_2, i_3. \tag{4.2}
\]
In general, we have

**Lemma 4.1.** It holds that
\[
B^\alpha_{i_1 \ldots k} \equiv 0, \tag{4.3}
\]
if there exist two of the indices \( i, j, \ldots, k \) assuming the forms \( i_a, i_b \) with \( a \neq b \), where \( i j \ldots k \) denotes a multiple index of order no less than 2.

**Proof.** Due to (4.2) and the method of induction, it suffices to prove that if (4.3) holds then
\[
B^\alpha_{i_1 \ldots k l} \equiv 0 \tag{4.4}
\]
for indices \( i, j, \ldots, k, l \) in which there exist two assuming the forms \( i_a, i_b \) with \( a \neq b \).

In fact, we only need to consider the following two cases:

(i) There exist two of the indices \( i, j, \ldots, k \) which assume the forms \( i_a, i_b \) with \( a \neq b \).

In this case, we use (4.3) and \( \omega^j_b = 0 \) to find
\[
B^\alpha_{i_1 j \ldots k l} \omega^l = dB^\alpha_{i_1 j \ldots k} - \sum B^\alpha_{i_1 j \ldots k} \omega^j - \sum \sum B^\alpha_{i_1 j \ldots k} \omega^j - \sum \sum \sum B^\alpha_{i_1 j \ldots k} \omega^j + \sum B^\beta_{i_1 j \ldots k} \omega^\beta \equiv 0.
\]
So (4.3) is true.

(ii) Either \( 1 \leq i, j, \ldots, k \leq m_1 \), or \( m_1 + 1 \leq i, j, \ldots, k \leq m_1 + m_2 \), or \( m_1 + m_2 + 1 \leq i, j, \ldots, k \leq m \).

Without loss of generality, we assume the first. Then it must be that \( l = j_a \) for \( a = 2 \) or \( a = 3 \). Note that by (2.19) and (4.2),
\[
R^\perp_{i_a j i_j j_a} = \sum_q (B^\alpha_{j_a q} B^\alpha_{i_j q} - B^\alpha_{i_j q} B^\alpha_{j_a q}) \equiv 0, \quad \forall i_1, j_a, a = 2, 3. \tag{4.5}
\]
This together with Case (i), the Ricci identities (2.25) and the fact that \( R_{i_1 j_a i_j} \equiv 0 \) shows that
\[
B^\alpha_{i_1 j \ldots k j_a} \equiv \sum B^\alpha_{i_1 j \ldots k j_a} + \sum B^\alpha_{i_1 j \ldots k j_a} + \cdots - \sum B^\beta_{i_1 j \ldots k j_a} \equiv 0.
\]
\( \square \)

**Lemma 4.2.** It holds that, for all \( i_a, j_b, k_a, \ldots, l_a, i_a, j_b, \ldots, k_b \) and \( 1 \leq a \neq b \leq 3 \),
\[
\sum B^\alpha_{i_a j_a} B^\alpha_{i_b j_b} = -(\lambda_a + \lambda_b) \delta_{i_a j_a} \delta_{i_b j_b}, \tag{4.6}
\]
\[
\sum B^\alpha_{i_a j_a k_a} B^\alpha_{i_b j_b} = 0. \tag{4.7}
\]
More generally,
\[
B^\alpha_{i_a j_a k_a \ldots l_a} B^\alpha_{i_b j_b k_b \ldots l_b} = 0, \tag{4.8}
\]
where \( i_a j_a k_a \ldots l_a \) is a multiple index of order no less than 3.

**Proof.** This lemma mainly comes from the Möbius Gauss equation (2.18) and the parallel assumption of the Blaschke tensor \( A \). In fact, since \( a \neq b \), (4.6) is given by (2.18), (4.1), (4.2) and that \( R_{i_a i_b j_a j_b} \equiv 0 \); (4.7) is given by (2.18), (4.3), \( R_{i_a i_b j_a j_b} \equiv 0 \) and the parallel of \( A \); Finally, (4.8) can be shown by the method of induction using (4.7) and Lemma 4.1. \( \square \)
As the corollary of (4.6), we have for \(a \neq b\)
\[
\sum_\alpha B^\alpha_{i_a j_a} (B^\alpha_{b k_b} - B^\alpha_{j_b j_b}) = 0, \quad (4.9)
\]
\[
\sum_\alpha B^\alpha_{i_a j_a} B^\alpha_{k_b k_b} = 0, \text{ if } i_a \neq j_a. \quad (4.10)
\]

Define
\[
V_a = \text{Span} \left\{ \sum_\alpha B^\alpha_{i_a j_a \cdots k_a} E_a \right\}, \quad a = 1, 2, 3; \quad (4.11)
\]
\[
V_{a0} = V_a \cap (V_{a'} + V_{a''})^\perp, \quad \text{so that } V_{a0} \perp V_{b0} \text{ for } a \neq b, \quad (4.12)
\]

where, as mentioned earlier, \(a, a', a''\) is an even permutation of 1, 2, 3.

Let \(V'_{a0} (a = 1, 2, 3)\) be the orthogonal complement of \(V_{a0}\) in \(V_a\) and denote
\[
V_0 := V'_{10} + V'_{20} + V'_{30}.
\]

**Lemma 4.3.** It holds that \(1 \leq \text{dim } V_0 \leq 2\).

**Proof.** For any \(i, j\), we denote by \(B^V_{i j} V_{a0}\) the \(V_{a0}\)-component of \(B_{ij}\), \(a = 1, 2, 3\). Then it follows from (4.9) and (4.10) that
\[
B^V_{i a j a} = 0, \quad B^V_{i a j a} = B^V_{i a j a}, \quad \text{for any } i_a, j_a, i_a \neq j_a. \quad (4.13)
\]

So that
\[
V'_{a0} = \text{Span} \{ B^V_{i a j a} \}, \quad \text{for each fixed } i_a, a = 1, 2, 3. \quad (4.14)
\]

In particular, \(\text{dim } V'_{a0} \leq 1, a = 1, 2, 3\).

On the other hand, by the second equation in (2.17), we have
\[
\sum_{i_1} B_{i_1 i_1} + \sum_{i_2} B_{i_2 i_2} + \sum_{i_3} B_{i_3 i_3} = 0. \quad (4.15)
\]

But, for any \(a\) and \(i_a\),
\[
B_{i_a i_a} = B^V_{i a i a} + B^V_{i_a i_a} = B^V_{i a i a} + B^V_{i_a i_a},
\]
and \(V_{a0} \perp V_{b0}\) for \(a \neq b\), so that (4.13) reduces to
\[
\sum_{i_a} B^V_{i a i a} = 0, \quad a = 1, 2, 3, \quad \sum_{i_1} B^V_{i_1 i_1} + \sum_{i_2} B^V_{i_2 i_2} + \sum_{i_3} B^V_{i_3 i_3} = 0. \quad (4.16)
\]

The second equality in (4.16) together with (4.13) shows that, for fixed \(i_1, i_2\) and \(i_3\),
\[
m_1 B^V_{i_1 i_1} + m_2 B^V_{i_2 i_2} + m_3 B^V_{i_3 i_3} = 0 \quad (4.17)
\]
which with (4.14) proves that \(\text{dim } V_0 \leq 2\).

Finally, if \(\text{dim } V_0 = 0\), then for any \(a \neq b\), \(B_{i_a i_a} \perp B_{i_b i_b}\) which with (4.6) implies that \(\lambda_a + \lambda_b = 0\), contradicting the assumption that \(\lambda_1, \lambda_2, \lambda_3\) are distinct. \(\Box\)

Clearly by definition, \(V_0, V_{10}, V_{20}\) and \(V_{30}\) are orthogonal to each other. Denote \(\iota := \text{dim } V_0\). Then we can properly choose an orthonormal normal frame field \(\{ E_\alpha \}\) such that
\[
E_{\alpha \nu} \in V_0, \quad \nu = 1, \cdots, \iota; \quad E_{\alpha \alpha}, E_{\beta \alpha}, E_{\gamma \alpha}, \cdots \in V_{a0}, \quad \text{for } a = 1, 2, 3. \quad (4.18)
\]

**Lemma 4.4.** \(V_0, V_{10}, V_{20}\) and \(V_{30}\) are parallel in the Möbius normal bundles \(V\). In particular, they are all of constant dimension.

**Proof.** Let \(\xi_a\) and \(\xi_0\) be sections of \(V\) such that \(\xi_a \in V_{a0}\) \((a = 1, 2, 3)\) and \(\xi_0 \in V_0\). Then, by the definition of the subspaces \(V_0, V_{10}, V_{20}\) and \(V_{30}\), \(\xi_a\) (resp. \(\xi_0\)) is a linear combination of \(B^V_{i_a j_a \cdots k_a}\) (resp.
of $B_{i_1i_2}^{V_0}$, $B_{i_2i_3}^{V_0}$ and $B_{i_3i_1}^{V_0}$ for some $i_1, i_2, i_3$). Thus, by (4.18) and (4.19), it is not hard to conclude that

$$\langle D^\perp \xi_\alpha, \xi_\alpha \rangle_1 = \langle D^\perp \xi_\alpha, \xi_\alpha \rangle_1 = 0, \quad b \neq a.$$  \hfill (4.19)

In fact, we take, for example, $a = 1$ and $\xi_\alpha = m_1 B_{11}^{V_0}$. Write $B_{i_1i_2} = \sum_\alpha B_{i_1i_2}^\alpha E_\alpha$. Since by (4.18),

$$m_1 B_{11}^{V_0} = (B_{11}^{V_0} - B_{22}^{V_0}) + \cdots + (B_{11}^{V_0} - B_{m_1m_1}^{V_0}) = \sum_\alpha ((B_{11}^\alpha - B_{22}^\alpha) + \cdots + (B_{m_1m_1}^\alpha)) E_\alpha,$$

we have

$$D^\perp \xi_\alpha = m_1 D^\perp B_{11}^{V_0} = \sum_\alpha ((d B_{11}^\alpha - d B_{22}^\alpha) + \cdots + (d B_{m_1m_1}^\alpha)) E_\alpha$$

$$+ \sum_{i, \alpha, \beta} ((B_{11}^\alpha - B_{22}^\alpha) + \cdots + (B_{m_1m_1}^\alpha)) \omega_\beta^i E_\alpha$$

$$= \sum_{i, \alpha} ((B_{11}^\alpha - B_{22}^\alpha) + \cdots + (B_{m_1m_1}^\alpha)) \omega_i^i E_\alpha$$

$$+ 2 \sum_{i, \alpha} (B_{11}^\alpha \omega_1^i - B_{22}^\alpha \omega_2^i) E_\alpha + \cdots + 2 \sum_{i, \alpha} (B_{m_1m_1}^\alpha \omega_1^i - B_{m_1m_1}^\alpha \omega_2^i) E_\alpha$$

$$+ \sum_{i_1, \alpha_1} ((B_{11}^\alpha - B_{22}^\alpha) + \cdots + (B_{m_1m_1}^\alpha)) \omega_i^{i_1} E_{\alpha_1}$$

$$+ 2 \sum_{i_1, \alpha_1} (B_{11}^\alpha \omega_1^{i_1} - B_{22}^\alpha \omega_2^{i_1}) E_{\alpha_1} + \cdots + 2 \sum_{i_1, \alpha_1} (B_{m_1m_1}^\alpha \omega_1^{i_1} - B_{m_1m_1}^\alpha \omega_2^{i_1}) E_{\alpha_1} \in V_1$$

implying that (4.19) holds in this case. Other cases can be similarly but more easily considered. Now, from (4.19) directly follows Lemma 4.3.

**Remark 4.1.** The conclusion that $\dim V_0$ is constant along $M^m$ can also be directly proved as follows:

For some fixed $i_1, i_2, i_3$, we have by (4.17)

$$\sum_a m_a \langle B_{i_1i_2}^{V_0}, B_{i_3i_3}^{V_0} \rangle_1 = 0, \quad b = 1, 2, 3$$

(4.20)

which together with (4.10) shows that $\langle B_{i_1i_2}^{V_0}, B_{i_3i_3}^{V_0} \rangle_1$ are constant for any $a, b$. It then follows from the Lagrangian identity that, for $a \neq b$,

$$(B_{i_1i_2}^{V_0} \times B_{i_3i_3}^{V_0})^2 = \langle B_{i_1i_2}^{V_0}, B_{i_3i_3}^{V_0} \rangle_1 \langle B_{i_1i_2}^{V_0}, B_{i_3i_3}^{V_0} \rangle_1 - (B_{i_1i_2}^{V_0}, B_{i_3i_3}^{V_0})^2 = \text{const}.$$  \hfill (4.21)

On the other hand, by (4.17), $B_{i_1i_3}^{V_0}$ is parallel to $B_{i_2i_2}^{V_0}$ if and only if $B_{i_1i_2}^{V_0}$, $B_{i_2i_3}^{V_0}$, $B_{i_3i_1}^{V_0}$ are parallel to each other, that is, $\dim V_0 = 1$. The remark is proved.

Hence there are only two cases that need to be considered:

Case 1. $\dim V_0 = 2$. In this case, we can find two special indices $\alpha_0$ and $\alpha_0'$ such that

$$B_{i_1i_2}^{V_0} = B_{i_1i_2}^{V_0} E_{\alpha_0}, \quad B_{i_2i_3}^{V_0} = B_{i_2i_3}^{V_0} E_{\alpha_0} + B_{i_2i_3}^{V_0} E_{\alpha_0'}, \quad \text{for any } i, \quad a = 2, 3.$$  \hfill (4.21)

with

$$B_{i_1i_2}^{V_0} \neq 0.$$  \hfill (4.22)

**Lemma 4.5.** $B_1^0$, $B_2^0$, $B_3^0$, $B_4^0$, $B_5^0$ are constants.

**Proof.** First, (2.17) and (4.21) give that

$$m_1 B_1^0 + m_2 B_2^0 + m_3 B_3^0 = m_2 B_2^0 + m_3 B_3^0 = 0.$$  \hfill (4.23)
On the other hand, by \((4.19), (4.20)\) and \((4.21)\), we find
\[
\begin{align*}
  m_1(B_1^0)^2 &= m_2(\lambda_1 + \lambda_2) + m_3(\lambda_1 + \lambda_3) = \text{const}, \\
  m_2((B_2^0)^2 + (B_2^0)^2) &= m_1(\lambda_1 + \lambda_2) + m_3(\lambda_2 + \lambda_3) = \text{const}, \\
  m_3((B_3^0)^2 + (B_3^0)^2) &= m_1(\lambda_1 + \lambda_3) + m_2(\lambda_2 + \lambda_3) = \text{const}.
\end{align*}
\]
This with \((4.6)\) and \((4.23)\) easily shows that \(B_1^0, B_2^0, B_3^0, B_3^0, B_3^0\) are all constants.

**Lemma 4.6.** For the Möbius normal frame field \(\{E_\alpha\} \equiv \{E_\alpha, E_\alpha, E_\alpha\}\) chosen above, the Möbius normal connection forms \(\omega^\beta_\alpha\) satisfy
\[
\omega^\beta_\alpha = \omega^\beta_\alpha = \omega^\beta_\alpha = 0, \quad a \neq b.
\]

**Proof.** Due to Lemma 4.4, \((4.18)\) and \((4.21)\), we only need to show that \(\omega^\alpha_\alpha = 0\). In fact, by \((4.8)\), we know that \(\sum B^\alpha_\alpha \omega^\beta = 0\) which with Lemma 4.5 and \((4.21)\) shows that
\[
0 = dB^\alpha_\alpha - B^\alpha_j \omega_j^i - B^\alpha_i \omega_i^j + B^\alpha_0 \omega_0^j + B^\alpha_1 \omega_1^j = B^\alpha_0 \omega_0^j,
\]
where we have used \(\omega^\alpha_\alpha = 0, \ a = 1, 2, 3\), which are directly obtained by Lemma 4.4. Thus \(\omega^\alpha_\alpha = 0\). □

Let \(Y\) and \(N\) be the Möbius position vector and the Möbius biposition vector of \(x\), respectively. As done earlier by many authors (for example, \([22, 9, 18, 25]\) and the very recent paper \([17]\), we define another vector-valued function
\[
c := N + \lambda Y + \mu B_\alpha = \mu_0 E_\alpha
\]
for some constants \(\lambda, \mu_0\) and \(\mu_0\) to be determined. Then by using \((4.11)\) and \((4.13)\) we find
\[
dc = \sum_{a,j,i} (\lambda_a + \lambda - \mu_0 B^0_a - \mu_0 B^0_a) \omega_i^a Y_i
\]
with \(B^0_i = 0\). Note that, by \((4.23)\)
\[
\begin{vmatrix}
  1 & B^0_1 & 0 \\
  1 & B^0_2 & B^0_2 \\
  1 & B^0_3 & B^0_3
\end{vmatrix} = B^0_i (B^0_i - B^0_3) + B^0_3 (B^0_2 - B^0_2) = \frac{1}{m_3} B^0_1 B^0_2 (m_1 + m_2 + m_3) = \frac{m}{m_3} B^0_1 B^0_2 \neq 0,
\]
the system of linear equations
\[
\begin{align*}
  \lambda_1 + \lambda - \mu_0 B^0_1 &= 0, \\
  \lambda_2 + \lambda - \mu_0 B^0_2 - \mu_0 B^0_2 &= 0, \\
  \lambda_3 + \lambda - \mu_0 B^0_3 - \mu_0 B^0_3 &= 0,
\end{align*}
\]
for \(\lambda, \mu_0, \mu_0\) has a unique solution as
\[
\begin{align*}
  \lambda &= -\frac{m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3}{m}, \\
  \mu_0 &= -\frac{1}{m B^0_1} \left( \sum m_\alpha \lambda_\alpha - m \lambda_1 \right), \\
  \mu_0' &= -\frac{1}{m B^0_1 B^0_2} \left( (m_1 B^0_1 + (m_2 + m_3) B^0_2) \lambda_1 + ((m_1 + m_3) B^0_1 + m_2 B^0_2) \lambda_2 + m_3 (B^0_1 - B^0_3) \lambda_3 \right).
\end{align*}
\]
Thus the following lemma is proved:

**Lemma 4.7.** Let \(\lambda, \mu_0\) and \(\mu_0'\) be given by \((4.27)\). Then the vector-valued function \(c\) defined by \((4.25)\) is constant on \(M^m\) and
\[
\langle c, c \rangle = 2\lambda + \mu_0^2 + \mu_0'^2, \quad \langle c, Y \rangle = 1.
\]
Next we have to consider the following three subcases:

Subcase (1): \( c \) is time-like; Subcase (2): \( c \) is light-like; Subcase (3): \( c \) is space-like.

Since the argument that follows here is standard and same as that of [17] (see Case 1 there), we omit the detail of it and only state the corresponding conclusions:

**Proposition 4.8.** Let \( x : M^m \to \mathbb{S}^{m+p} \) be as in the main theorem (Theorem 1.3). If \( \dim V_0 = 2 \) and \( c \) is the constant vector given by (4.25), then

(1) \( c \) is time-like and \( x \) is Möbius equivalent to a non-minimal and umbilic-free pseudo-parallel immersion \( \bar{x} : M^m \to \mathbb{S}^{m+p} \) with parallel mean curvature and constant scalar curvature, which has three distinct principal curvatures in the direction of the mean curvature vector;

(2) \( c \) is light-like and \( x \) is Möbius equivalent to the image under \( \sigma \) of a non-minimal and umbilic-free pseudo-parallel immersion \( \bar{x} : M^m \to \mathbb{S}^{m+p} \) with parallel mean curvature and constant scalar curvature, which has three distinct principal curvatures in the direction of the mean curvature vector;

(3) \( c \) is space-like and \( x \) is Möbius equivalent to the image under \( \tau \) of a non-minimal and umbilic-free pseudo-parallel immersion \( \bar{x} : M^m \to \mathbb{S}^{m+p} \) with parallel mean curvature and constant scalar curvature, which has three distinct principal curvatures in the direction of the mean curvature vector.

Case 2. \( \dim V_0 = 1 \).

In this case, there is an index \( a_0 \) such that \( V_0 = \mathbb{R}E_{a_0} \). Thus we can write

\[
\left< B^V_{a_0} \right> = \left< B^0_{a_0} E_{a_0} \right>, \quad \text{for each } i_a, \quad a = 1, 2, 3. \tag{4.25}
\]

It follows that

\[
m_1 B^0_{a_1} + m_2 B^0_{a_2} + m_3 B^0_{a_3} = 0, \quad B^0_{a_1} B^0_{a_0} = -(\lambda_a + \lambda_b), \quad \text{for } a \neq b \tag{4.26}
\]

which implies

\[
(B^0_{a})^2 = \frac{1}{m_a} \left( (m_{a'} + m_{a''}) \lambda_a + m_{a'} \lambda_{a'} + m_{a''} \lambda_{a''} \right), \quad a = 1, 2, 3 \tag{4.27}
\]

where \( a, a', a'' \) is an even permutation of 1, 2, 3.

Furthermore, Lemma [14] implies in the present case that

\[
\omega_{\alpha_0}^a = \omega_{\alpha_0}^{a_0} = 0, \quad \text{for all } a \text{ and } b \neq a. \tag{4.28}
\]

Define

\[
z_a = N + \lambda_a Y - B^0_{a} E_{a_0}, \quad a = 1, 2, 3. \tag{4.29}
\]

Then, by (2.11), (2.13) and (4.32), we find that

\[
dz_a = \sum_{i,j} A_{ij} \omega^i Y_j + \lambda_a \sum_i \omega^i Y_j + B^0_a \sum_{i,j} B^0_{ij} \omega^i Y_j = (2\lambda_a + (B^0_a)^2) \sum_i \omega^i Y_j, \quad a = 1, 2, 3. \tag{4.30}
\]

Thus \( z_a \) is constant on \( M_a \) for \( b \neq a \).

Using (2.12), (2.13), (4.32) and (4.34), the following lemma is easily proved:

**Lemma 4.9.** The subbundles \( \mathbb{R}z_a, Y_a(T M_a), \mathbb{R}E_{a_0}, V_{a_0}, a = 1, 2, 3, \) are mutually orthogonal, and the Möbius normal connection on the Möbius normal bundle \( V \) is the direct sum of its restrictions on \( \mathbb{R}E_{a_0}, V_{a_0}, a = 1, 2, 3. \) Moreover,

\[
\mathbb{R}z_a \oplus Y_a(T M_a) \oplus V_{a_0}, \quad a = 1, 2, 3
\]

are orthogonal to each other in \( \mathbb{R}^{m+p+2}_1 \) and are constant on \( M_a, a = 1, 2, 3, \) respectively.

Subcase (i): One of \( 2\lambda_1 + (B^0_{a_1})^2, 2\lambda_2 + (B^0_{a_2})^2 \) and \( 2\lambda_3 + (B^0_{a_3})^2 \) vanishes.

Without loss of generality, we assume \( 2\lambda_1 + (B^0_{a_1})^2 = 0 \). Then by (4.34), \( dz_1 \equiv 0 \) and thus \( z_1 = c \) is a constant vector on \( M^m \). Furthermore,

\[
\langle c, c \rangle = 2\lambda_1 + (B^0_{a_1})^2 = 0, \quad \langle c, Y \rangle = 1.
\]
Therefore the according argument in [17] (see Subcase (ii) of Case 1 there) applies to the present case and proves the following conclusion:

**Proposition 4.10.** Let \( x : M^n \to S^{m+p} \) be as in the main theorem (Theorem 1.3). If \( \dim V_0 = 1 \) and there exists some \( a, 1 \leq a \leq 3 \), such that \( 2\lambda_a + (B^0_a)^2 = 0 \), then \( x \) is Möbius equivalent to

1. The image under \( \sigma \) of a non-minimal and umbilic-free pseudo-parallel immersion \( \tilde{x} : M \to \mathbb{R}^{m+p} \) with parallel mean curvature and constant scalar curvature, which has three distinct principal curvatures in the direction of the mean curvature vector.

Subcase (ii): \( 2\lambda_a + (B^0_a)^2 \neq 0, a = 1, 2, 3 \).

In this subcase, by using (4.30) and (4.31) the following lemma can be easily proved by a direct computation:

**Lemma 4.11.** The three constants \( B^0_a, a = 1, 2, 3, \) have the properties that

\[
\sum_a \frac{1}{2\lambda_a + (B^0_a)^2} = \sum_a \frac{B^0_a}{2\lambda_a + (B^0_a)^2} = 0, \quad \sum_a \frac{\lambda_a}{2\lambda_a + (B^0_a)^2} = 1. \tag{4.35}
\]

**Remark 4.2.** Note that \( \mathbb{R}E_{a0}, \bigoplus_a (Y_x(TM_a) \oplus V_{a0}) \) are space-like, and

\[
\langle z_a, z_a \rangle = 2\lambda_a + (B^0_a)^2 \neq 0, \quad a = 1, 2, 3. \tag{4.36}
\]

It follows that there exists one and only one index \( a \) such that

\[
\langle z_a, z_a \rangle < 0, \quad \text{or equivalently,} \quad 2\lambda_a + (B^0_a)^2 < 0.
\]

With no loss of generality we assume that

\[
r_1 := -\frac{1}{2\lambda_1 + (B^0_1)^2}, \quad r_2 := \frac{1}{2\lambda_2 + (B^0_2)^2}, \quad r_3 := \frac{1}{2\lambda_3 + (B^0_3)^2}.
\]

for positive numbers \( r_1, r_2, r_3 \). Then by (4.30), (4.31) and (4.35) we have

\[
r_1^2 = r_2^2 + r_3^2, \quad m_3r_2^2 \neq m_2r_3^2. \tag{4.38}
\]

Now from the Möbius second fundamental form \( B \), we define for each \( a \)

\[
^{(a)}B = \sum_s B^{(a)}_{iajs} \omega^i \omega^j E_{s a}.
\]

Then \( ^{(a)}B \) is a \( V_{a0} \)-valued symmetric 2-form on \( M_a \) with components \( \mathcal{B}^{(a)}_{iajs} = B^{(a)}_{iajs} \).

Let \( ^{(a)}B_{iajs} \) be the components of the covariant derivatives of \( ^{(a)}B \) with the induced connection on \( V_{a0} \).

Then, as the consequence of (1.12), (1.32) and Lemma 1.19, we have

\[
^{(a)}B_{iajs} = B^{(a)}_{iajs}. \tag{4.39}
\]

Since \( B^{(a)}_{iajs} = 0 \) for \( b \neq a \), the vanishing of the Möbius form \( C \) together with (2.18), (2.19), (2.21), (4.1), (4.2) and (4.39) proves the following lemma:

**Lemma 4.12.** The Riemannian manifold \( (M_a, g^{(a)}) \) and the vector bundle valued symmetric tensor \( ^{(a)}B \) satisfies the Gauss equation, Codazzi equation and Ricci equation for submanifolds in a space form of constant curvature \( 2\lambda_a + (B^0_a)^2 \). Namely

\[
R_{iajsl}^{(a)} = \sum_{\beta} (B_{iajs}^{(a)})_{\beta} - (B_{iajs}^{(a)})_{\beta} = \sum_{\alpha} (B_{iajs}^{(a)})_{\beta} - (B_{iajs}^{(a)})_{\beta}, \quad \delta_{iajsl}^{(a)} = \sum_{\alpha} (B_{iajs}^{(a)})_{\beta} - (B_{iajs}^{(a)})_{\beta}, \tag{4.40}
\]

\[
R_{iajs}^{(a)} = \sum_{\alpha} (B_{iajs}^{(a)})_{\beta} = \sum_{\alpha} (B_{iajs}^{(a)})_{\beta}, \quad R_{iajs}^{(a)} = \sum_{\alpha} (B_{iajs}^{(a)})_{\beta} - (B_{iajs}^{(a)})_{\beta}. \tag{4.41}
\]
By Lemma 4.12, there exist an isometric immersion
\[ \tilde{y} = (\tilde{y}_0, \tilde{y}_1) : (M_1, g^{(1)}) \to H^{m_1+p_1} - \frac{1}{r_1^2} \subset \mathbb{R}^{m_1+p_1+1} \]
with \( B \) as its second fundamental form, and two isometric immersions
\[ \tilde{y}_a : (M_a, g^{(a)}) \to S^{m_a+p_a} (r_a) \subset \mathbb{R}^{m_a+p_a+1}, \quad a = 2, 3 \]
with \( B \) as their second fundamental forms, respectively.

Note that \( B^\alpha_{i_a,j_a} \equiv 0 \) for \( b \neq a \). It follows from (2.17) that both \( \tilde{y} \) and \( \tilde{y}_a, a = 2, 3 \), are minimal immersions. Furthermore, if denote by \( \tilde{S}_a \) the scalar curvatures of \( M_a \), then by (4.37), (4.40) and the minimality, we have
\[ \tilde{S}_1 = -\frac{m_1(m_1-1)}{r_1^2} - \sum (B^\alpha_{i_1,j_1})^2, \quad \tilde{S}_a = \frac{m_a(m_a-1)}{r_a^2} - \sum (B^\alpha_{i_a,j_a})^2, \quad a = 2, 3 \] (4.42)
showing that
\[ \tilde{S}_1 + \frac{m_1(m_1-1)}{r_1^2} = -\sum (B^\alpha_{i_1,j_1})^2 \leq 0, \]
\[ \tilde{S}_a - \frac{m_a(m_a-1)}{r_a^2} = -\sum (B^\alpha_{i_a,j_a})^2 \leq 0, \quad a = 2, 3. \] (4.43)
(4.44)

On the other hand, by (2.17),
\[ \sum_{a,i,j} (B^\alpha_{i_a,j_a})^2 = \sum (B^\alpha_{i_j})^2 - \sum m_a (B^\alpha_a) = \frac{m-1}{m} - \sum m_a (B^\alpha_a)^2 = \text{const.} \] (4.45)
Thus by (4.42) and (4.45),
\[ \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 = -\frac{m_1(m_1-1)}{r_1^2} + \frac{m_2(m_2-1)}{r_2^2} + \frac{m_3(m_3-1)}{r_3^2} - \frac{m-1}{m} + m_1 (B^0_1)^2 + m_2 (B^0_2)^2 + m_3 (B^0_3)^2 = \text{const.} \] (4.46)

Since \( \tilde{S}_a \)'s are functions defined on \( M_a \)'s, respectively, it follows that all \( \tilde{S}_a \)'s are constant on \( M^m \) and, by (4.43), (4.41), we can write
\[ \tilde{S}_1 = -\frac{m_1(m_1-1)}{r_1^2} + \mu_1 \left( m_1 (B^0_1)^2 + m_2 (B^0_2)^2 + m_3 (B^0_3)^2 - \frac{m-1}{m} \right) \] (4.47)
\[ \tilde{S}_a = \frac{m_a(m_a-1)}{r_a^2} + \mu_a \left( m_a (B^0_a)^2 + m_2 (B^0_2)^2 + m_3 (B^0_3)^2 - \frac{m-1}{m} \right), \quad a = 2, 3 \] (4.48)
for some positive constants \( \mu_1, \mu_2, \mu_3 \) satisfying \( \mu_1 + \mu_2 + \mu_3 = 1 \).

Now let \( \text{LS}(m, p, r, \mu) \) be one of the submanifolds in Example 3.2 defined by \( \tilde{y}, \tilde{y}_2 \) and \( \tilde{y}_3 \). Then it is not hard to see that \( \text{LS}(m, p, r, \mu) \) has the same Möbius metric \( g \) and the same Möbius second fundamental form \( B \) as those of \( x \). Furthermore, by choosing the normal frame field \( \{ \tilde{e}_\alpha \} \) as given in (3.24)–(3.27) where, in the present case,
\[ \tilde{E}_\alpha = E_\alpha, \quad m+1 \leq \alpha \leq m+p, \]
we compute directly:
\[
\tilde{\omega}_\alpha^\beta = d\hat{e}_\alpha \cdot \hat{e}_\beta = (dE_\alpha, E_\beta)_1 = \begin{cases}
\omega_\alpha^\beta, & \text{for either } m + 1 \leq \alpha, \beta \leq m + p_1, \\
0, & \text{otherwise},
\end{cases}
\]
\[
\text{for } m + p_1 + 1 \leq \alpha, \beta \leq m + p_1 + p_2,
\]
\[
\text{or } m + p_1 + p_2 + 1 \leq \alpha, \beta \leq m + p;
\]

implying that \( x \) and \( \text{LS}(m, p, r, \mu) \) have the same Möbius normal connection. Therefore, by Theorem 2.2, \( x \) is Möbius equivalent to \( \text{LS}(m, p, r, \mu) \). So we have proved the following proposition:

**Proposition 4.13.** Let \( x: M^m \to S^{m+p} \) be as in the main theorem (Theorem 1.3). If \( \dim V_0 = 1 \) and \( 2\lambda_a + (B_0^a)^2 \neq 0, a = 1, 2, 3 \), then \( x \) is Möbius equivalent to

(4) a submanifold \( \text{LS}(m, p, r, \mu) \) given in Example 3.2 for some multiple parameters \( m, p, r, \mu \) satisfying \( m_3r_2^2 \neq m_2r_3^2 \).

The proof of the main theorem (Theorem 1.3).

As discussed earlier in this section, there are only the following two cases with additional subcases that need to be considered:

(1) \( \dim V_0 = 2 \).

(2) \( \dim V_0 = 1 \):

Subcase (i), one of \( 2\lambda_a + (B_0^a)^2 \) \((a = 1, 2, 3) \) vanishes;

Subcase (ii), \( 2\lambda_a + (B_0^a)^2 \neq 0, a = 1, 2, 3 \).

Thus the main theorem follows directly from Propositions 4.8, 4.10 and 4.13.

**References**

[1] W. Blaschke, Vorlesungen über Differentialgeometrie, Vol. 3, Springer Berlin, 1929.

[2] B. Y. Chen, Total mean curvature and submanifolds of finite type, Ser. Pure Math. 1, World Scientific Publishing, Singapore, 1984.

[3] Q.-M. Cheng, X. X. Li and X. R. Qi, A classification of hypersurfaces with parallel para-Blaschke tensor in \( S^{m+1} \), Int. J. Math., 21(2010), 297–316.

[4] Z. Guo, T. Z. Li, L. M. Lin, X. Ma and C. P. Wang, Classification of hypersurfaces with constant Möbius curvature in \( S^{m+1} \), Math. Z., to appear.

[5] Z. J. Hu and H. Z. Li, Classification of hypersurfaces with parallel Möbius second fundamental form in \( S^{n+1} \), Sci. China, Ser. A, 47(2004), 417–430.

[6] Z.J. Hu, H. Li and D.Y. Li, Möbius isoparametric hypersurfaces with three distinct principal curvatures, Pacific J. Math., 232(2007), 289–311.

[7] Z. J. Hu, X. X. Li and S. J. Jie, On the Blaschke isoparametric hypersurfaces in the unit sphere with three distinct Blaschke eigenvalues. Sci China Math, 54(2011), 10: 2171-2194, doi: 10.1007/s11425-011-4291-9

[8] H. Z. Li, H. L. Liu, C. P. Wang and G. S. Zhao, Möbius isoparametric hypersurfaces in \( S^{n+1} \) with two distinct principal curvatures, Acta Math. Sin. (Eng. Ser.) 18(2002), 437–446.

[9] H. Z. Li and C. P. Wang, Möbius geometry of hypersurfaces with constant mean curvature and scalar curvature, Manuscripta Math. 112(2003), 1–13.

[10] H. Z. Li and C. P. Wang, Surfaces with vanishing Möbius form in \( S^n \), Acta Math. Sin. (Eng. Ser.) 19(2003), 671–678.

[11] T. Z. Li, C. P. Wang, A note on Blaschke isoparametric hypersurfaces, Int. J. Math. 25(2014), 1450117 [9 pages] DOI: 10.1142/S0129167X14501171.

[12] T. Z. Li, J. Qing and C. P. Wang, Möbius and Laguerre geometry of Dupin Hypersurfaces, arXiv [math.DG]:1503.02979, 2015.

[13] X. X. Li and Y. J. Peng, Blaschke isoparametric hypersurfaces in the unit sphere \( S^6 \), Scientia Sinica, Mathematica (in Chinese), 40(2010), 827–928.

[14] X. X. Li and Y. J. Peng, Classification of the Blaschke isoparametric hypersurfaces with three distinct Blaschke Eigenvalues, Results. Math., 58(2010), 145–172.

[15] X. X. Li and H. R. Song, Regular space-like hypersurfaces in \( S^{m+1}_1 \) with parallel Blaschke tensors, arXiv [math. DG]:1511.02979, 2015.
[16] X. X. Li and H. R. Song, Regular space-like hypersurfaces in $S^{m+1}_1$ with parallel para-Blaschke tensors, arXiv [math. DG]: 1511.03261, 2015.
[17] X. X. Li and H. R. Song, On the immersed submanifolds in the unit sphere with parallel Blaschke tensor, arXiv [math. DG]: 1511.02560, 2015.
[18] X. X. Li and F. Y. Zhang, A Möbius characterization of submanifolds in real space forms with parallel mean curvature and constant scalar curvature, Manuscripta Math. 117(2005), 135–152.
[19] X. X. Li and F. Y. Zhang, A classification of immersed hypersurfaces in spheres with parallel Blaschke tensors, Tohoku Math. J., 58(2006), 581–597.
[20] X. X. Li and F. Y. Zhang, Immersed hypersurfaces in the unit sphere $S^{m+1}$ with constant Blaschke eigenvalues, Acta Math. Sinica, English Series, 23(2007), 533–548.
[21] X. X. Li and F. Y. Zhang, On the Blaschke isoparametric hypersurfaces in the unit sphere, Acta Math. Sinica, English Series, 25(2009), 657–678.
[22] H. L. Liu, C. P. Wang and G. S. Zhao, Möbius isotropic submanifolds in $S^n$, Tohoku Math. J. (2) 53(2001), 553–569.
[23] L. A. Rodrigues and K. Tenenblat, A characterization of Moebius isoparametric hypersurfaces of the sphere, Monatsh. Math., 158(2009), 321-327.
[24] C. P. Wang, Möbius geometry of submanifolds in $S^n$, Manuscripta Math. 96(1998), 517–534.
[25] S. J. Zhai, Z. J. Hu and C. P. Wang, On submanifolds with parallel Möbius second fundamental form in the unit sphere, Int. J. Math., 25(2014), DOI: 10.1142/S0129167X14500621.

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