Energy Scattering for a Klein-Gordon Equation with a Cubic Convolution

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Abstract

In this paper, we study the global well-posedness and scattering problem in the energy space for both focusing and defocusing the Klein-Gordon-Hartree equation in the spatial dimension \(d \geq 3\). The main difficulties are the absence of an interaction Morawetz-type estimate and of a Lorentz invariance which enable one to control the momentum. To compensate, we utilize the strategy derived from concentration compactness ideas, which was first introduced by Kenig and Merle \cite{K1} to the scattering problem. Furthermore, employing technique from \cite{S}, we consider a virial-type identity in the direction orthogonal to the momentum vector so as to control the momentum in the defocusing case. While in the focusing case, we show that the scattering holds when the initial data \((u_0, u_1)\) is radial, and the energy \(E(u_0, u_1) < E(\mathcal{W}, 0)\) and \(\|\nabla u_0\|_2^2 + \|u_0\|_2^2 < \|\nabla \mathcal{W}\|_2^2 + \|\mathcal{W}\|_2^2\), where \(\mathcal{W}\) is the ground state.

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1 Introduction

This paper is devoted to the study of the Cauchy problem of the Klein-Gordon-Hartree equation

\[
\begin{aligned}
\ddot{u} - \Delta u + u + f(u) &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad d \geq 3, \\
u(0, x) &= u_0(x), \quad u_t(0, x) = u_1,
\end{aligned}
\]

where \(f(u) = \mu(\|x\|^{-\gamma} \ast |u|^2)u, \ 2 < \gamma < \min(4, d), \ \mu = \pm 1\) with \(\mu = 1\) known as the defocusing case and \(\mu = -1\) as the focusing case. Here \(u\) is a real-valued function defined in \(\mathbb{R}^{d+1}\), the dot denotes the time derivative, \(\Delta\) is the Laplacian in \(\mathbb{R}^d\), \(|x|^{-\gamma}\) is called the potential, and \(\ast\) denotes the spatial convolution in \(\mathbb{R}^d\).
Formally, the solution $u$ of (1.1) conserves the energy

$$E(u(t), u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} (\|\dot{u}(t, x)\|^2 + \|\nabla u(t, x)\|^2 + \|u(t, x)\|^2) \, dx$$

$$+ \frac{\mu}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^{\gamma}} \, dxdy$$

$$\equiv E(u_0, u_1),$$

and the momentum

$$P(u)(t) = \int_{\mathbb{R}^d} u_t(t, x) \nabla u(t, x) \, dx = P(u)(0). \quad (1.2)$$

The scattering theory for the Klein-Gordon equation with $f(u) = \mu |u|^{p-1} u$ has been intensively studied in [4], [5], [9], [12], [34] and [36]. For $\mu = 1$ and

$$1 + \frac{4}{d} < p < 1 + \gamma d \frac{4}{d - 2}, \quad \gamma d = \begin{cases} 1, & 3 \leq d \leq 9; \\ \frac{d}{d + 1}, & d \geq 10. \end{cases} \quad (1.3)$$

Brenner [5] established the scattering results in the energy space, which does not contain all subcritical cases for $d \geq 10$. Thereafter, Ginibre and Velo [9] exploited the Birman-Solomjak space $\ell^m(L^p, I, B)$ in [2] and the delicate estimates to improve the results in [5], which covered all subcritical cases. Finally K. Nakanishi [34] obtained the scattering results for the critical case by the strategy of induction on energy [7] and a new Morawetz-type estimate. And recently, S. Ibrahim, N. Masmoudi and K. Nakanishi [12, 13] utilized the concentration compactness ideas to give the scattering threshold for the focusing nonlinear Klein-Gordon equation. Their method also works for the defocusing case.

On the other hand, the scattering theory for the Hartree equation

$$i\dot{u} = -\Delta u + \mu (|x|^{-\gamma} * |u|^2) u$$

has been also studied by many authors (see [11, 17, 23, 24, 25, 26, 27, 28]). For the subcritical defocusing case, Ginibre and Velo [11] derived the associated Morawetz inequality and extracted an useful Birman-Solomjak type estimate to obtain the asymptotic completeness in the energy space. Nakanishi [35] improved the results by a new Morawetz estimate which doesn’t depend on the nonlinearity. For the critical case, Miao, Xu and Zhao [23] took advantage of a new kind of the localized Morawetz estimate, which is also independent of the nonlinearity, to rule out the possibility of the energy concentration at origin and established the scattering results in the energy space for the radial data in dimension $d \geq 5$. We refer also to [24, 25, 26, 27, 28] for the general data and focusing case.

For the equation (1.1), using the ideas of Strauss [40], [41], Pecher [39] and Mochizuki [32] showed that if $d \geq 3$, $2 \leq \gamma < \min(d, 4)$, then global well-posedness and scattering results with small data hold in the energy space $H^1 \times L^2$. We refer also to Miao-Zhang [31] where the low regularity for the cubic convolution defocusing Klein-Gordon-Hartree equation is discussed. In this paper, we develop in the energy space a complete
scattering theory for \([1.1]\) with the subcritical nonlinearity under some suitable assumptions. Compared with the classical Klein-Gordon equation with the local nonlinearity \(f = |u|^{p-1}u\), the nonlinearity \(f(u) = \mu(|x|^{-\gamma} * |u|^2)u\) is nonlocal, which brings us many difficulties. The main difficulty is the absence of a Lorentz invariance which could be used to control the momentum efficiently. We will overcome this difficulty by considering a Virial-type identity in the direction orthogonal to the momentum vector following the technique in \([37]\) for the defocusing case. Unfortunately, the type of virial identity in \([37]\) can not work for the focusing case. Inspired by the method in \([12]\) and \([22]\), we get over this difficulty by some new variational framework and the profile decomposition under the restriction that the initial data is radial.

Now we state our first result

**Theorem 1.1.** Assume that \(\mu = 1\), \(d \geq 3\), \(2 < \gamma < \min\{d, 4\}\) and \((u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\). Then there exists a unique global solution \(u(t)\) of \((1.1)\) which scatters in the sense that there exist solutions \(v_{\pm}\) of the free Klein-Gordon equation

\[
\ddot{v} - \Delta v + v = 0
\]

with \((v_{\pm}(0), \dot{v}_{\pm}(0)) \in H^1 \times L^2\) such that

\[
\| (u(t), \dot{u}(t)) - (v_{\pm}(t), \dot{v}_{\pm}(t)) \|_{H^1 \times L^2} \to 0, \quad \text{as} \quad t \to \pm \infty.
\]

In the focusing case, one can not expect to establish a similar result as Theorem 1.1 without any other restriction. In fact, let \(W\) be an element of the ground states which satisfy the elliptic equation

\[-\Delta \psi + \psi = (|x|^{-\gamma} * |\psi|^2)\psi.\]

One may find that \(W\) is a non-scattering solution of \((1.1)\) with the finite energy \(E(W, 0)\). The existence of ground state \(W\) was proved by \([19]\). We will discuss the ground state in the Section 3.

Now we state our second result as follows

**Theorem 1.2.** Assume \(d \geq 3\), \(\mu = -1\), and \(2 < \gamma < \min\{d, 4\}\). Let \((u_0, u_1) \in H^1 \times L^2\) be radial with

\[E(u_0, u_1) < E(W, 0),\]

and \(u\) be the corresponding solution of \((1.1)\) with maximal interval of existence \(I = (-T_-(u_0, u_1), T_+(u_0, u_1))\).

(i) If \(\|\nabla u_0\|_2^2 + \|u_0\|_2^2 < \|\nabla W\|_2^2 + \|W\|_2^2\), then \(u\) is global and scatters.

(ii) If \(\|\nabla u_0\|_2^2 + \|u_0\|_2^2 > \|\nabla W\|_2^2 + \|W\|_2^2\), then \(u\) blows up both forward and backward in finite time, i.e. \(T_- (u_0, u_1), T_+ (u_0, u_1) < \infty\).

**Remark 1.1.** The case \(\|\nabla u_0\|_2^2 + \|u_0\|_2^2 = \|\nabla W\|_2^2 + \|W\|_2^2\) is not compatible with \(E(u_0, u_1) < E(W, 0)\) by Lemma 3.3 and Lemma 3.6 below. In this sense, we give a complete classification for solutions under the restriction \(E(u_0, u_1) < E(W, 0)\).
The outline for the proof of Theorem 1.1 and Theorem 1.2 first, we define the scattering size of a solution to (1.1) on a time interval \( I \subset \mathbb{R} \) by
\[
ST(I) = [W](I) \cap [K](I),
\]
where
\[
[W](I) = L^2(I; B^1_2(R^d)), \quad [K](I) = L^2(I; B^2_2(R^d)),
\]
and
\[
[W](I) = L^2(I; B^1_{2d/(d+1)}(\mathbb{R}^d)), \quad [K](I) = L^2(I; B^2_{2d/(d+2)}(\mathbb{R}^d)).
\]
Then it is easy to see (cf Proposition 2.2 below) that, scattering for the Klein-Gordon-Hartree equation is implied by the finiteness of the scattering size \( \|u\|_{ST(I)} \) with \( I = \mathbb{R} \).

Second, we define the function \( \Lambda \) by
\[
\Lambda(E) = \sup \{ \|u\|_{ST(\mathbb{R})} : E(u, u_t) \leq E \}
\]
where the supremum is taken over all nonlinear solutions of (1.1) with energy not greater than \( E \), and define
\[
E^+_\max = \sup \{ E : \Lambda(E) < +\infty \},
\]
\[
E^-\max = \sup \{ E : \Lambda(E) < +\infty, \|\nabla u_0\|^2_2 + \|u_0\|^2_2 < \|\nabla W\|^2_2 + \|W\|^2_2 \},
\]
and
\[
E_{\max} = \begin{cases} 
E^+_\max, & \text{if } \mu = 1, \\
E^-\max, & \text{if } \mu = -1.
\end{cases}
\]
In particular, by Lemma 3.6, \( E^-\max \) is equivalent to
\[
E^-\max = \sup \{ E : \Lambda(E) < +\infty, K_1(u_0) > 0 \}.
\]
Our goal next is to prove that \( E^+_\max = +\infty \) and \( E^-\max = E(W, 0) \). We argue by contradiction. We show that if \( E^+_\max < +\infty \) (or \( E^-\max < E(W, 0) \)), then there exists a nonlinear solution of (1.1) with energy be exactly \( E_{\max} \). Moreover, this solution satisfies some strong compactness properties. This is completed in Section 6 where we utilize the profile decomposition that was established in [12], and a strategy introduced by Kenig and Merle [15]. We consider a virial-type identity in the direction orthogonal to the momentum vector following the technique [37] in the defocusing case to obtain a contradiction. We refer to Section 7 for more details.

The paper is organized as follows. In Section 2, we deal with the local theory for the equation (1.1). In Section 3, we discuss the property of the ground state. In Section 4, we prove the blow up part of Theorem 1.2. In Section 5, we give the linear and nonlinear profile decomposition and show some properties of the profile. In Section 6, we extract a critical solution. Finally in Section 7, we preclude the critical solution, which completes the proof of Theorem 1.1 and Theorem 1.2.

We conclude the introduction by giving some notations which will be used throughout this paper. We always assume the spatial dimension \( d \geq 3 \) and let \( 2^* = \frac{2d}{d-2} \). For any \( r, 1 \leq r \leq \infty \), we denote by \( \| \cdot \|_r \) the norm in \( L^r = L^r(\mathbb{R}^d) \) and by \( r' \) the conjugate
The integral equation for the Cauchy problem (2.1) can be written as

\[
\begin{align*}
\ddot{u} - \Delta u + u + f(u) &= 0, \\
u(0) &= u_0, \quad \dot{u}(0) = u_1.
\end{align*}
\]

The integral equation for the Cauchy problem (2.1) can be written as

\[
u(t) = \dot{K}(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds,\]

or

\[
\begin{pmatrix}
\dot{u}(t) \\
\ddot{u}(t)
\end{pmatrix} = V_0(t)\begin{pmatrix}
u_0(x) \\
u_1(x)
\end{pmatrix} - \int_0^t V_0(t-s)\begin{pmatrix}
0 \\
f(u(s))
\end{pmatrix}ds,
\]

where

\[
K(t) = \frac{\sin(t\omega)}{\omega}, \quad V_0(t) = \begin{pmatrix}
\dot{K}(t), K(t) \\
\dot{K}(t), \dot{K}(t)
\end{pmatrix}, \quad \omega = (1 - \Delta)^{1/2}.
\]

Let \(U(t) = e^{it\omega}\), then

\[
\dot{K}(t) = \frac{U(t) + U(-t)}{2}, \quad K(t) = \frac{U(t) - U(-t)}{2i\omega}.
\]

We begin by recalling the definition of strong solution to the Cauchy problem.

\[\text{exponent defined by } \frac{1}{p} + \frac{1}{q} = 1. \text{ For any } s \in \mathbb{R}, \text{ we denote by } H^s(\mathbb{R}^d) \text{ the usual Sobolev space. Let } \psi \in \mathcal{S}(\mathbb{R}^d) \text{ be such that } \text{supp } \hat{\psi} \subseteq \{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \} \text{ and } \sum_{j \in \mathbb{Z}} \hat{\psi}(2^{-j}\xi) = 1 \text{ for } \xi \neq 0. \text{ Define } \psi_0 \text{ by } \hat{\psi}_0 = 1 - \sum_{j \geq 1} \hat{\psi}(2^{-j}\xi). \text{ Thus } \text{supp } \hat{\psi}_0 \subseteq \{ \xi : |\xi| \leq 2 \} \text{ and } \hat{\psi}_0 = 1 \text{ for } |\xi| \leq 1. \text{ We denote by } \Delta_j \text{ and } P_0 \text{ the convolution operators whose symbols are respectively given by } \hat{\psi}(\xi/2^j) \text{ and } \hat{\psi}_0(\xi). \text{ For } s \in \mathbb{R}, 1 \leq r \leq \infty, \text{ the inhomogeneous Besov space } B^s_{r,2}(\mathbb{R}^d) \text{ is defined by}
\]

\[
B^s_{r,2}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d), \| P_0 u \|_{L^r}^2 + \| 2^j \| \Delta_j u \|_{L^r}^2 \right\}_{j \in \mathbb{N}} < \infty.
\]

For details of Besov space, we refer to [1]. For any interval \(I \subset \mathbb{R}\) and any Banach space \(X\) we denote by \(\mathcal{C}(I; X)\) the space of strongly continuous functions from \(I\) to \(X\) and by \(L^q(I; X)\) the space of strongly measurable functions from \(I\) to \(X\) with \(\|u(\cdot); X\| \in L^q(I)\).

Given \(d\), we define, for \(2 \leq r \leq \infty\),

\[
\delta(r) = d\left( \frac{1}{2} - \frac{1}{r} \right).
\]

Sometimes abbreviate \(\delta(r), \delta(r_i)\) to \(\delta, \delta_i\) respectively. We denote by \(\langle \cdot, \cdot \rangle\) the scalar product in \(L^2\). We let \(L^p_u\) denote the weak \(L^p\) space.

2 Preliminaries

2.1 Strichartz estimate

In this section, we consider the Cauchy problem for the equation (1.1)

\[
\begin{align*}
\ddot{u} - \Delta u + u + f(u) &= 0, \\
u(0) &= u_0, \quad \dot{u}(0) = u_1.
\end{align*}
\]

The integral equation for the Cauchy problem (2.1) can be written as

\[
u(t) = \dot{K}(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds,
\]

or

\[
\begin{pmatrix}
\dot{u}(t) \\
\ddot{u}(t)
\end{pmatrix} = V_0(t)\begin{pmatrix}
u_0(x) \\
u_1(x)
\end{pmatrix} - \int_0^t V_0(t-s)\begin{pmatrix}
0 \\
f(u(s))
\end{pmatrix}ds,
\]

where

\[
K(t) = \frac{\sin(t\omega)}{\omega}, \quad V_0(t) = \begin{pmatrix}
\dot{K}(t), K(t) \\
\dot{K}(t), \dot{K}(t)
\end{pmatrix}, \quad \omega = (1 - \Delta)^{1/2}.
\]

Let \(U(t) = e^{it\omega}\), then

\[
\dot{K}(t) = \frac{U(t) + U(-t)}{2}, \quad K(t) = \frac{U(t) - U(-t)}{2i\omega}.
\]

We begin by recalling the definition of strong solution to the Cauchy problem.
Definition 2.1. We call $u$ a strong solution to (2.1) on a time interval $I$, if $u \in C(I; H^1) \cap C^1(I; L^2)$ and satisfies the Duhamel formula (2.2) in the sense of tempered distributions for every $t \in I$.

Now we recall the following dispersive estimate for the operator $U(t) = e^{it\omega}$.

Lemma 2.1 ([5], [9]). Let $2 \leq r \leq \infty$ and $0 \leq \theta \leq 1$. Then
\[
\|e^{it\omega}f\|_{B_{r,2}^\theta} \leq \mu(t)\|f\|_{B_{r',2}^\theta},
\]
where
\[
\mu(t) = C \min \left\{ |t|^{-(d-1-\theta)}(\frac{1}{r} - \frac{1}{r_1}), |t|^{-(d-1-\theta)}(\frac{1}{r} - \frac{1}{r_2}) \right\}.
\]

According to the above lemma, the abstract duality and interpolation argument (see [10], [14]), we have the following Strichartz estimates.

Lemma 2.2 ([5],[9],[30]). Let $0 \leq \theta_i \leq 1$, $\rho_i \in \mathbb{R}$, $2 \leq q_i, r_i \leq \infty$, $i = 1, 2$. Assume that $(\theta_i, d, q_i, r_i) \neq (0, 3, 2, \infty)$ satisfy the following admissible conditions
\[
0 \leq \frac{2}{q_i} \leq \min \left\{ (d - 1 + \theta_i)(\frac{1}{2} - \frac{1}{r_i}), 1 \right\}, \quad i = 1, 2
\]
\[
\rho_1 + (d + \theta_1)(\frac{1}{2} - \frac{1}{r_1}) - \frac{1}{q_1} = \mu,
\]
\[
\rho_2 + (d + \theta_2)(\frac{1}{2} - \frac{1}{r_2}) - \frac{1}{q_2} = 1 - \mu.
\]

Then, for $f \in H^\mu$, we have
\[
\|U(\cdot)f\|_{L^{q_1}(\mathbb{R}; B_{r_1,2}^{\rho_1})} \leq C\|f\|_{H^\mu};
\]
\[
\|K * f\|_{L^{q_1}(I; B_{r_1,2}^{\rho_1})} \leq C\|f\|_{L^{q_2}(I; B_{r_2,2}^{-\rho_2})};
\]
\[
\|K_{R} * f\|_{L^{q_1}(I; B_{r_1,2}^{\rho_1})} \leq C\|f\|_{L^{q_2}(I; B_{r_2,2}^{-\rho_2})},
\]

where the subscript $R$ stands for retarded, and
\[
K * f = \int_\mathbb{R} K(t-s)f(u(s))ds,
\]
\[
K_{R} * f = \int_0^t K(t-s)f(u(s))ds.
\]

Remark 2.1. One can check that (2.5) and (2.7) hold for any $(q_i, r_i, \theta_i, \rho_i), i = 1, 2$ satisfying the condition (2.4), thus the choice of exponents (especially of $\theta$) is very flexible, which is significant for the estimate of the nonlinearity. In fact, for any $(q_1, r_1, \theta_1, \rho_1), (q, r, \theta, \rho)$ satisfying the first two conditions of (2.4), we let
\[
B := L^{q_1}(I; B_{r_1,2}^{\rho_1}) \cap L^{q}(I; B_{r,2}^{\rho}),
\]
where $H$.

Hence, without loss of generality, it suffices to consider the special case

where $U$.

It follows from (2.3) and the abstract $TT^*$ method that

$$
\| U * f \|_{L^q(I;B_{r,2}^{0})} \leq \| U * f \|_B \leq C \| f \|_{B^*} \leq C \| f \|_{L^q(I;B_{r,2}^{0})}.
$$

where $U * f = \int_{\mathbb{R}} U(t-s)f(u(s))ds$.

**Remark 2.2.** According to the above remark, we know that if

$$
f(u) = (V * |u|^2)u = ((V_1 + V_2) * |u|^2)u =: f_1(u) + f_2(u),
$$

where $V_1 \in L^p_1, V_2 \in L^p_2$, then we have

$$
\| U * f \|_B \leq \| f \|_{B^*} \leq \| f_1 \|_{L^q(I;B_{r,2}^{0})} + \| f_2 \|_{L^q(I;B_{r,2}^{0})}.
$$

Hence, without loss of generality, it suffices to consider the special case $V \in L^p$ throughout the paper.

In addition to the $ST$-norm defined in (1.6), we also need the following norm

$$
ST^*(I) = [W]^*(I) + [K]^*(I),
$$

where

$$
[W]^*(I) = L^{2(d+1)}_{d+3}(I;B_{r,2}^{0}((\mathbb{R}^d)), [K]^*(I) = L^{2(d+2)}_{d+4}(I;B_{r,2}^{0}((\mathbb{R}^d)).
$$

Now we give a nonlinear estimate which will be applied to show the small data scattering which is the first step to obtain the global time-space estimate that lead to the scattering.

**Lemma 2.3.** Let $2 < \gamma < \min\{d, 4\}$, then we have

$$
\left\| \left( |x|^{-\gamma} * |u|^2 \right) u \right\|_{ST^*(I)} + \left\| \left( |x|^{-\gamma} * (uv) \right) u \right\|_{ST^*(I)} \leq C \left\| v \right\|_{K(I)} \left\| u \right\|_{2(d-2)} 2(d+2)_{d+4}(I;L^2_{d+2}) + C \left\| u \right\|_{K(I)} \left\| u \right\|_{d+4} 2(d-2)_{d+4}(I;L^2_{d+2}) \left\| v \right\|_{K(I)} \left\| v \right\|_{d+4} 2(d-2)_{d+4}(I;L^2_{d+2}) (2.8)
$$

$$
+ C \left\| v \right\|_{W(I)} \left\| u \right\|_{2(d-3)} 2(d-3)_{d+3}(I;H^1_{d+2}) \left\| v \right\|_{W(I)} + C \left\| u \right\|_{W(I)} \left\| u \right\|_{d+3} 2(d-3)_{d+3}(I;H^1_{d+2}) \left\| v \right\|_{W(I)} \left\| v \right\|_{W(I)}.
$$

In particular,

$$
\left\| \left( |x|^{-\gamma} * |u|^2 \right) u \right\|_{ST^*(I)} \leq C \left\| u \right\|_{2(d-2)} 2(d+2)_{d+4}(I;L^2_{d+2}) + C \left\| u \right\|_{W(I)} \left\| u \right\|_{d+4} 2(d-2)_{d+4}(I;L^2_{d+2}) (2.9)
$$

In the particular,
Proof. We only need to prove the estimate \( \|(x)^{-\gamma} \ast |u|^2 v\|_{ST^\ast(I)}, \) since the estimate \( \|(x)^{-\gamma} \ast (uv)u\|_{ST^\ast(I)} \) is similar. From the fractional Leibnitz rule and the Hölder and the Young inequalities, we have

\[
\| (V \ast |u|^2)v \|_{L^q(I,B^{1/2}_{q,2})} \
\lesssim \|V\|_p \|v\|_{L^q(I,B^{1/2}_{q,2})} \|u\|_{L^k(I,L^2)} \|u\|_{L^k(I,L^2)} \|v\|_{L^k(I,L^2)}, \quad (2.10)
\]

where we have assumed for simplicity that \( V \in L^p \), and the exponents satisfy

\[
\begin{cases}
\frac{d}{p} = 2\delta(r) + 2\delta(s), \\
\frac{2}{q} + \frac{2}{k} = 1.
\end{cases} \quad (2.11)
\]

If we set \( V_1 = (x)^{-\gamma} \chi_{|x| \leq 1}, V_2 = (x)^{-\gamma} \chi_{|x| > 1} \), then

\[
\|((x)^{-\gamma} \ast |u|^2)v\|_{ST^\ast(I)} \\ \lesssim \|V_1\|_p \|v\|_{L^q(I,B^{1/2}_{q,2})} \|u\|_{L^k(I,L^2)} \|u\|_{L^k(I,L^2)} \|v\|_{L^k(I,L^2)}, \quad (2.12)
\]

Using Hölder’s inequality and the Sobolev embedding theorem, we get

\[
\|v\|_{L^k_2(I,L^2)} \lesssim \|v\|_{[K](I)} \|v\|_{L^k_\infty(I,L^2)}, \quad (2.13)
\]

Plugging (2.14) into (2.13), we obtain

\[
I_2 \lesssim \|v\|_{[K](I)} \|u\|_{[K]\ast(K)} \frac{2d-2}{d-1} \|u\|_{L^2_\infty L^2} + \|u\|_{L^2_\infty L^2} \|v\|_{L^k_\infty(I,L^2)} \|v\|_{L^k_\infty(I,L^2)}. \quad (2.15)
\]

On the other hand, for \( d \geq 4 \), since \( V_1 = (x)^{-\gamma} \chi_{|x| \leq 1} \in L^\frac{4}{d-4} \), if we take admissible pair \( q = r = \frac{2(d+2)}{d-1} \) and \( \delta(s_1) = \frac{1}{d+1} \) (then \( \delta(r) = \frac{d}{d+1}, k_1 = d+1 \)), then

\[
I_1 = \|((x)^{-\gamma} \ast |u|^2)v\|_{L^q(I,B^{1/2}_{q,2})} \\
\lesssim \|v\|_{[W](I)} \|u\|_{L^2_\infty(I,L^2)} \|u\|_{L^2_\infty(I,L^2)} \|v\|_{L^k_\infty(I,L^2)} \|v\|_{L^k_\infty(I,L^2)}. \quad (2.16)
\]

The Hölder inequality and the Sobolev embedding theorem yield that

\[
\|v\|_{L^k_1(I,L^2)} \lesssim \|v\|_{L^2_\infty(I,L^2)} \|v\|_{L^2_\infty(I,L^2)} \|v\|_{L^2_\infty(I,L^2)} \lesssim \|v\|_{H^s(I)} \|v\|_{[W](I)}. \quad (2.17)
\]
Proof. We apply the Banach fixed point argument to prove this lemma. First we define

\[ B = \{ u \in C(I; H^1) : \|u\|_{L^\infty(I;H^1)} \leq 2C\|\{(u_0, u_1)\}_{H^1 \times L^2}, \|u\|_{ST(I)} \leq 2C\delta \} \]

with the metric \( d(u, v) = \|u - v\|_{ST(I) \cap L^\infty(I;H^1)} \).

Plugging (2.17) into (2.16), we get

\[ I_1 \lesssim \|v\|_{W(I)}\|v\|_{L^{2d-3}_{\text{inf}} H^{1}_{x} L^2_{t}} \|u\|_{W(I)} \|u\|_{L^\infty_{t} H^1_x} \|v\|_{L^\infty_{t} H^1_x} \|v\|_{L^2_{t} W(I)}. \]  

(2.18)

For \( d = 3 \), since \( V_1 = \|x^{-\gamma} \chi_{|x| \leq 1} \in L^1 \), if we take admissible pair \( q = r = k_1 = 4 \) and \( \delta(r) = \delta(s_1) = \frac{3}{4} \), then

\[ I_1 = \|V_1 + |u|^2v\|_{L^4(I,B^2_4 \Omega)} \lesssim \|v\|_{W(I)}\|u\|_{L^4(I;L^4)} + \|u\|_{W(I)}\|u\|_{L^4(I;L^4)}\|v\|_{L^4(I;L^4)}. \]  

(2.19)

Combining (2.12), (2.15), (2.18) and (2.19), we obtain

\[ \left\| \|x^{-\gamma} + |u|^2v\|_{ST(I)} \right\|_{ST(I)} \lesssim \|v\|_{W(I)}\|u\|_{L^2(I;L^2)} + \|u\|_{W(I)}\|u\|_{L^2(I;L^2)}\|v\|_{L^2(I;L^2)} \]

\[ + \|v\|_{W(I)}\|u\|_{L^2(I;L^2)}\|v\|_{W(I)} + \|v\|_{W(I)}\|u\|_{L^2(I;L^2)}\|v\|_{W(I)} + \|v\|_{W(I)}\|u\|_{L^2(I;L^2)}\|v\|_{W(I)} \]  

(2.20)

which completes the proof of Lemma 2.3. \( \square \)

We can now state the local well-posedness for (1.1) with large initial data and small
data scattering in the energy space.

**Theorem 2.1.** Assume \( d \geq 3 \), \( 2 < \gamma < \min\{d, 4\} \) and \( (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \).

There exists a small constant \( \delta = \delta(E) \) such that if \( \|\{u_0, u_1\}\|_{H^1 \times L^2} \leq E \) and \( I \) is an interval such that

\[ \|K(t)u_0 + K(t)u_1\|_{ST(I)} \leq \delta, \]

then there exists a unique strong solution \( u \) to (1.1) in \( I \times \mathbb{R}^d \), with \( u \in C(I; H^1) \cap C^1(I; L^2) \) and

\[ \|u\|_{ST(I)} \leq 2C\delta. \]  

(2.21)

Let \( (-T_{\text{min}}, T_{\text{max}}) \) be the maximal time interval on which \( u \) is well-defined. Then, if \( T_{\text{max}} < +\infty \), then \( \lim_{t \to T_{\text{max}}} \|u(t)\|_{H^1} = +\infty \). Similarly, if \( T_{\text{min}} < +\infty \), then

\[ \lim_{t \to -T_{\text{min}}} \|u(t)\|_{H^1} = +\infty. \]

**Proof.** We apply the Banach fixed point argument to prove this lemma. First we define the map

\[ \Phi(u(t)) = K(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds \]  

(2.22)

on the complete metric space \( B \)

\[ B = \{ u \in C(I; H^1) : \|u\|_{L^\infty(I;H^1)} \leq 2C\|\{(u_0, u_1)\}_{H^1 \times L^2}, \|u\|_{ST(I)} \leq 2C\delta \} \]
It suffices to prove that the operator defined by the RHS of (2.22) is a contraction map on $B$ for $I$. In fact, if $u \in B$, then by Lemma 2.3 and (2.4), we have
\[
\|\Phi(u)\|_{ST(I)} \leq C\|\hat{K}(t)u_0 + K(t)u_1\|_{ST(I)} + C\|f(u)\|_{ST^*(I)}
\leq C\delta + C\|u\|_{\|K(t)\|_{L^\infty(I;H^1)}}^{1 + \frac{d}{2}} \|u\|_{L^\infty(I;H^1)}^{2(d - 2)\frac{d}{d - 3}} + C\|u\|_{\|W(t)\|_{H^1(I;H^1)}}^{1 + \frac{4(d - 3)}{d - 1}} \|u\|_{L^\infty(I;H^1)}^{2(d - 2)\frac{d}{d - 3}}.
\]
Hence, if we take $\delta$ sufficiently small such that
\[
C2\frac{d + 4}{d} \delta^2 (2C(\|u_0\|_{H^1} + \|u_1\|_{H^1})^2 \|u\|_{ST^*(I)}) \leq \frac{1}{2}
\]
and
\[
C(2\delta)^\frac{1}{d - 1} (2C(\|u_0\|_{H^1} + \|u_1\|_{H^1})^2 \|u\|_{ST^*(I)}) \leq \frac{1}{2},
\]
then $\|\Phi(u)\|_{ST(I)} \leq 2C\delta$.

Similarly, we get $\|\Phi(u)\|_{L^\infty(I;H^1)} \leq 2C(\|u_0\|_{H^1} + \|u_1\|_{H^1})$, and so $\Phi(u) \in B$.

On the other hand, for $\omega_1, \omega_2 \in B$, by Strichartz estimate, we obtain
\[
d(\Phi(\omega_1), \Phi(\omega_2)) \leq C\|(|x|^{-\gamma} * |\omega_1|^2)\omega_1 - (|x|^{-\gamma} * |\omega_2|^2)\omega_2\|_{ST^*(I)}
\leq C\|(|x|^{-\gamma} * |\omega_1|^2)(\omega_1 - \omega_2)\|_{ST^*(I)} + C\|(|x|^{-\gamma} * (\omega_1 - \omega_2)^2)\omega_2\|_{ST^*(I)}
+ 2C\|(|x|^{-\gamma} * (\omega_2(\omega_1 - \omega_2))\omega_2\|_{ST^*(I)}.
\]
By Lemma 2.3, the above quantity can be controlled by
\[
Cd(\omega_1, \omega_2) \sum_{i,j=1}^2 \left(\|\omega_i\|_{ST(I)} \|\omega_j\|_{L^\infty(\mathbb{R})} + \|\omega_i\|_{ST(I)} \|\omega_j\|_{L^\infty(\mathbb{R})} + \|\omega_i\|_{ST(I)} \|\omega_j\|_{L^\infty(H^1)} + \|\omega_i\|_{ST(I)} \|\omega_j\|_{L^\infty(H_+)} \right),
\]
which allows us to derive
\[
d(\Phi(\omega_1), \Phi(\omega_2)) \leq \frac{1}{2}d(\omega_1, \omega_2)
\]
by taking $\delta$ small. This completes the proof. \qed

In the defocusing case ($\mu = 1$), using the conservation of energy, the above solutions can be extended globally. This gives the following proposition which is the starting point of our investigation. As we will see in Corollary 3.3, the above solutions can also be extended globally for the focusing case ($\mu = -1$) under the restriction $E(u_0, u_1) < E(0, 0)$ and $\|\nabla u_0\|_2^2 + \|u_0\|_2^2 < \|\nabla W\|_2^2 + \|W\|_2^2$.

**Proposition 2.1.** Assume $d \geq 3$, $\mu = 1$ and $2 < \gamma < \min\{4, d\}$. For all initial data $(u_0, u_1) \in H^1 \times L^2$, there exists a unique globally defined nonlinear solution $u \in C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, L^2) \cap ST_{loc}(\mathbb{R})$. Besides, the evolution flow $(u(0), u_t(0)) \in H^1 \times L^2 \mapsto u$ is continuous for all compact time interval $I$.

The above Proposition only gives the local in time bounds, while the key point to understand better the behaviour of these solutions is to gain access to global in time bounds. Actually, as mentioned in the introduction, a global in time bound of the $ST$ norm of the solution is sufficient for scattering, this is the object of the following proposition.
Proposition 2.2. Let $d \geq 3$, $\mu = 1$, $2 < \gamma < \min\{4, d\}$ and $u \in C(\mathbb{R}, H^1(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d))$ be a global strong solution of (1.1) with initial data $(u_0, u_1) \in H^1 \times L^2$. Then we have if

$$\|u\|_{ST(\mathbb{R})} < \infty,$$

then $u$ scatters.

Moreover, in the focusing case: $\mu = -1$, the above result also holds under the restriction $E(u_0, u_1) < E(W, 0)$, and $\|\nabla u_0\|_2^2 + \|u_0\|_2^2 < \|\nabla W\|_2^2 + \|W\|_2^2$.

Proof. We just prove that $u$ scatters at $+\infty$, the proof for the scattering at $-\infty$ is similar. Using Duhaml’s formula, the solution with initial data $(u(0), \dot{u}(0)) = (u_0, u_1) \in H^1 \times L^2$ of (1.1) can be written as

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \int_0^t V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds, \quad (2.23)$$

where $V_0$ is defined by (2.23). Denote the scattering data $(u_0^+, u_1^+)$ by

$$\left( \begin{array}{c} u_0^+ \\ u_1^+ \end{array} \right) = \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) - \int_0^\infty V_0(-s) \begin{pmatrix} 0 \\ \mu(|x|^{-2} + |u|^2)u(s) \end{pmatrix} ds.$$

Then, by Strichartz estimate (2.27) and (2.20), we can obtain

$$\left\| \begin{pmatrix} u \\ \dot{u} \end{pmatrix} - V_0(t) \begin{pmatrix} u_0^+ \\ u_1^+ \end{pmatrix} \right\|_{H^1 \times L^2} = \left\| \int_0^\infty V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds \right\|_{H^1 \times L^2} \leq \lesssim \left\| (|x|^{-2} + |u|^2)u \right\|_{ST^*(t, \infty)}$$

$$\lesssim \|u\|_{1+\frac{4}{d-2}} \|u\|_{L^\infty_t L^2_x}^{2(d-2)} + \|u\|_{1+\frac{4}{d-1}} \|u\|_{L^\infty_t H^1_x}^{2(d-3)} \rightarrow 0, \quad \text{as} \quad t \rightarrow \infty.$$

Thus $u$ scatters. Here we used the fact that: $\|u\|_{L^\infty_t(\mathbb{R}; H^1)}$ can be controlled by $E(u_0, u_1)$ from the conservation of the energy in the defocusing case, and $\|u\|_{L^\infty_t(\mathbb{R}; H^1)}$ can also be controlled by $E(u_0, u_1)$ for the focusing case by Lemma 3.1 below.

2.2 Perturbation lemma

We here record the short and long time perturbations as in [7]. Roughly speaking, the stability proposition says that, if initial data are close enough and the perturbation term is small in some sense, then the solutions will be close.

Lemma 2.4 (Short-time perturbations). Let $I$ be a time interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^d$ which is a near solution to (1.1) in the sense that

$$\left( \square + 1 \right) \tilde{u} = -f(\tilde{u}) + e \quad (2.24)$$

for some function $e$. Assume that

$$\|\tilde{u}\|_{L^\infty_t H^1_x(I \times \mathbb{R}^d)} + \|\partial_t \tilde{u}\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} \leq E$$

Then $u$ is controlled by $E > 0$.
for some constant $E > 0$. Let $t_0 \in I$, and let $(u(t_0), u_t(t_0))$ be close to $(\tilde{u}(t_0), \tilde{u}_t(t_0))$ in the sense that
\begin{equation}
\| (u(t_0) - \tilde{u}(t_0), u_t(t_0) - \tilde{u}_t(t_0)) \|_{H^1 \times L^2} \leq E' \tag{2.25}
\end{equation}
and assume also that we have smallness conditions
\begin{align}
\| \hat{K}(t - t_0)(u(t_0) - \tilde{u}(t_0)) + K(t - t_0)(u_t(t_0) - \tilde{u}_t(t_0)) \|_{ST(I)} & \leq \epsilon, \tag{2.26} \\
\| \tilde{u} \|_{ST(I)} & \leq \epsilon, \quad \| \epsilon \|_{ST^*(I)} \leq \epsilon \tag{2.27}
\end{align}
for some $0 < \epsilon < \epsilon_0$, where $\epsilon_0 = \epsilon_0(E) > 0$ is a small enough constant.

We conclude that there exists a solution $u$ to (1.1) on $I \times \mathbb{R}^d$ with the specified initial data $(u(t_0), u_t(t_0))$ at $t_0$, and furthermore
\begin{align}
\| u - \tilde{u} \|_{ST(I)} & \lesssim \epsilon, \\
\| (\Box + 1)(u - \tilde{u}) \|_{ST^*(I)} & \lesssim \epsilon, \tag{2.28}
\end{align}
where $\Box := \partial_t - \Delta$.

Proof. First we claim that
\begin{equation}
\| \tilde{u} \|_{L^{k_i}(I; L^{s_i})} \leq \epsilon_i, \quad (i = 1, 2),
\end{equation}
where $\epsilon_i$ is a small constant depending on $\epsilon_0$ and $E, k_i, s_i$ defined in (2.13), (2.16) and (2.19). In fact, by (2.14) and (2.17), we obtain
\begin{align}
\| \tilde{u} \|_{L^{k_i}(I; L^{s_i})} & \leq \| \tilde{u} \|_{ST(I)} \| \tilde{u} \|_{L^\infty(I; H^1)} \\
& \lesssim \epsilon_0 E^{1-\delta_i} \equiv \epsilon_i.
\end{align}
This concludes the claim. Similarly,
\begin{equation}
\| \hat{K}(t - t_0)(u(t_0) - \tilde{u}(t_0)) + K(t - t_0)(u_t(t_0) - \tilde{u}_t(t_0)) \|_{L^{k_i}(I; L^{s_i})} \leq \epsilon_i, \quad (i = 1, 2).
\end{equation}

Let $w := u - \tilde{u}$ and $Y(I) = ST(I) \cap L^{k_1}(I; L^{s_1}) \cap L^{k_2}(I; L^{s_2})$, then by Lemma 2.2 we have
\begin{equation}
\| w \|_{Y(I)} \lesssim \epsilon + \| (\Box + 1)w \|_{ST^*(I)} =: \epsilon + S(I), \tag{2.29}
\end{equation}
where
\begin{equation}
(\Box + 1)w = -f(u) + f(\tilde{u}) - \epsilon. \tag{2.30}
\end{equation}
Hence, by Lemma 2.3 and (2.10), we have
\begin{align}
S(I) & \lesssim \| f(u) - f(\tilde{u}) \|_{ST^*(I)} + \| \epsilon \|_{ST^*(I)} \\
& \lesssim \| (|x|^{-\gamma} \ast |\tilde{u}|^2)w \|_{ST^*(I)} + \| (|x|^{-\gamma} \ast |w|^2)\tilde{u} \|_{ST^*(I)} + \| (|x|^{-\gamma} \ast |w|^2)w \|_{ST^*(I)} \\
& \quad + \| (|x|^{-\gamma} \ast (w\tilde{u}))w \|_{ST^*(I)} + \| (|x|^{-\gamma} \ast (w\tilde{u}))\tilde{u} \|_{ST^*(I)} + \epsilon \\
& \lesssim \sum_{l=1}^{3} \| w \|_{Y(I)}^l \| \tilde{u} \|_{Y(I)}^{3-l} + \epsilon \\
& \lesssim \sum_{l=1}^{3} (S(I) + \epsilon)^l (\epsilon + \epsilon_1 + \epsilon_2)^{3-l} + \epsilon.
\end{align}
Making use of a standard continuity argument, it follows that \( S(I) \lesssim \epsilon \) and then \( \|w\|_{Y(I)} \lesssim \epsilon \). This, together with \([2.27]\), implies the first estimate in \([2.28]\). This completes the proof.

Lemma 2.5 (Long-time perturbations). Let \( I \) be a time interval, and let \( \tilde{u} \) be a function on \( I \times \mathbb{R}^n \) which is a solution to \([2.24]\) such that

\[
\|\tilde{u}\|_{ST(I)} \leq M,
\]

\[
\|\tilde{u}\|_{L^\infty_t H^1_x(I \times \mathbb{R}^d)} + \|\partial_t \tilde{u}\|_{L^2_t L^2_x(I \times \mathbb{R}^d)} \leq E,
\]

for some constant \( M, E > 0 \). Let \( t_0 \in I \), and let \((u(t_0), u_t(t_0))\) be close to \((\tilde{u}(t_0), \tilde{u}_t(t_0))\) in the sense that

\[
\|(u(t_0) - \tilde{u}(t_0), u_t(t_0) - \tilde{u}_t(t_0))\|_{H^1_t L^2_x} \leq E',
\]

and assume also that we have smallness conditions

\[
\|\dot{K}(t - t_0)(u(t_0) - \tilde{u}(t_0)) + K(t - t_0)(u_t(t_0) - \tilde{u}_t(t_0))\|_{ST(I)} + \|\varepsilon\|_{ST'(I)} \leq \epsilon
\]

for some small \( 0 < \epsilon < \epsilon_1 \), where \( \epsilon_1 = \epsilon_1(M, E) > 0 \). We conclude that there exists a solution \( u(t) \) to \([1.1]\) on \( I \times \mathbb{R}^d \) with specific initial data \((u(t_0), u_t(t_0))\) at \( t_0 \), and furthermore

\[
\|u - \tilde{u}\|_{ST(I)} \leq C(M, E)\epsilon,
\]

\[
\|u\|_{ST(I)} \leq C(M, E, E')
\]

Proof. Since \( \|\tilde{u}\|_{ST(I)} \leq M \), we may subdivide \( I \) into \( N = C(M, \varepsilon_0) \) intervals \( I_j = [t_j, t_{j+1}] \) such that

\[
\|\tilde{u}\|_{ST(I_j)} \leq \varepsilon_0, \quad 1 \leq j \leq C(M, \varepsilon_0),
\]

where \( \varepsilon_0 = \varepsilon_0(E) > 0 \) is the same as Lemma 2.4.

Next we can use inductively the short-time perturbations lemma for \( j = 0, 1, \cdots, N \) to get

\[
\|u - \tilde{u}\|_{ST(I_j)} \leq C(j)\epsilon,
\]

\[
\|(\Box + 1)(u - \tilde{u})\|_{ST'(I_j)} \leq C(j)\epsilon
\]

and then we obtain

\[
\|(u(t_{j+1}) - \tilde{u}(t_{j+1}), \dot{u}(t_{j+1}) - \dot{\tilde{u}}(t_{j+1}))\|_{H^1_t L^2_x} \leq \|(u(t_j) - \tilde{u}(t_j), \dot{u}(t_j) - \dot{\tilde{u}}(t_j))\|_{H^1_t L^2_x} + C(j)\epsilon,
\]

thus the claim follows by the standard argument.

\[\square\]

3 Variational characterizations

In this section, we discuss some properties of ground states, and some preliminary lemmas for the study in the focusing case. The idea is similar to S. Ibrahim, N. Masmoudi, K. Nakanishi [12] and C. Miao, Y. Wu [22].
First, by the symmetry
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{x \cdot (x - y)}{|x - y|^{d+2}} \phi(x)^2 \phi(y)^2 \, dx \, dy = -\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y \cdot (x - y)}{|x - y|^{d+2}} \phi(x)^2 \phi(y)^2 \, dx \, dy,
\]
and a direct computation we have the following identities:

**Lemma 3.1.** Assume \( \phi \in S(\mathbb{R}^d) \), then
\[
-\int_{\mathbb{R}^d} \Delta \phi x \cdot \nabla \phi \, dx = -\frac{d-2}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 \, dx,
\]
\[
\int_{\mathbb{R}^d} \phi x \cdot \nabla \phi \, dx = -\frac{d}{2} \int_{\mathbb{R}^d} |\phi|^2 \, dx,
\]
\[
\int_{\mathbb{R}^d} (|x|^{-\gamma} * |\phi|^2) \phi x \cdot \nabla \phi \, dx = \left( -\frac{d}{2} + \frac{\gamma}{4} \right) \int_{\mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x - y|^{\gamma}} \, dx \, dy.
\]

**Lemma 3.2.** Assume that \( 2 < \gamma < \min\{4, d\} \). Let \( \phi \) be the \( H^1(\mathbb{R}^d) \) solution of the following equation
\[
-\Delta \phi + \phi = (|x|^{-\gamma} * |\phi|^2) \phi,
\]
then the following identity holds:
\[
K_1(\phi) \triangleq \int_{\mathbb{R}^d} (|\nabla \phi|^2 + |\phi|^2) \, dx - \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x - y|^{\gamma}} \, dx \, dy = 0.
\]
\[
K_2(\phi) \triangleq \int_{\mathbb{R}^d} |\nabla \phi|^2 \, dx - \frac{\gamma}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x - y|^{\gamma}} \, dx \, dy = 0.
\]

**Proof.** \( K_1(\phi) = 0 \) can be obtained by multiplying (3.1) both sides by \( \phi \) and integrating. By Lemma 3.1 \( K_2(\phi) = 0 \) is obtained by multiplying (3.1) both sides by \( x \cdot \nabla \phi + \frac{d}{2} \phi \), and integrating.

Let the static energy \( J \) be defined by
\[
J(\phi) = \frac{1}{2} \int_{\mathbb{R}^d} [|\nabla \phi|^2 + |\phi|^2] \, dx - \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x - y|^{\gamma}} \, dx \, dy.
\]
Let \( \Omega \triangleq \{ \phi \in H^1(\mathbb{R}^d) \setminus \{0\} : \phi \text{ solves (3.1)} \} \), then the ground state set of the elliptic equation (3.1) is defined as
\[
\Lambda \triangleq \{ \varphi \in \Omega : J(\varphi) \leq J(\phi), \forall \phi \in \Omega \}.
\]

The existence of the ground state was shown in [19] where it had been shown that the ground state a radial, rapidly decaying function. And let \( \mathcal{W} \) be an element of the ground state set. For our purpose, we will give two characterizations of the ground state based on the functional \( K_1, K_2 \), which will be important to describe the structures of the dichotomy of blow up and scattering associated to the nonlinear Klein-Gordon-Hartree equation.
Lemma 3.3. Let $\Omega_1 = \{ \phi \in H^1(\mathbb{R}^d) : \phi \not\equiv 0, K_1(\phi) = 0 \}$, and

$$
\Lambda_1 \triangleq \{ \varphi \in \Omega_1 : J(\varphi) \leq J(\phi), \forall \phi \in \Omega_1 \},
$$

$$
m \triangleq \inf \{ J(\phi) : \phi \in H^1(\mathbb{R}^d), \phi \not\equiv 0, K_1(\phi) = 0 \}
$$

then $\Lambda_1 \not= \emptyset$, and moreover, $m = J(\mathcal{W})$, i.e. $m$ is attained by the ground state $\mathcal{W}$.

Before proving Lemma 3.3 we introduce some notations. First we decompose $K_j(\phi)$ $(j = 1, 2)$ into the quadratic and nonlinear parts:

$$
\begin{cases}
K_j(\phi) = K_j^Q(\phi) + K_j^N(\phi), & j = 1, 2, \\
K_1^Q(\phi) = \| \phi \|_{L^2}^2 + \| \phi \|_{H^1(\mathbb{R}^d)}^2, & K_2^Q(\phi) = \| \nabla \phi \|_2^2, \\
K_1^N(\phi) = -\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x-y|^\gamma} \, dx \, dy, & K_2^N(\phi) = \frac{1}{4} K_1^N(\phi).
\end{cases}
$$

(3.4)

Remark 3.1. If $\phi \in H^1(\mathbb{R}^d)$, then $K_j^Q(e^\lambda \phi) = e^{2\lambda}(\| \phi \|_{L^2}^2 + \| \phi \|_{H^1}^2) \rightarrow 0$, as $\lambda \rightarrow -\infty$.

Lemma 3.4. Assume $2 < \gamma < \min\{4, d\}$, then for any bounded sequence $\{ \phi_n \} \subset H^1(\mathbb{R}^d) \setminus \{ 0 \}$ such that $K_j^Q(\phi_n) \rightarrow 0$, we have for large $n$,

$$
K_1(\phi_n) > 0.
$$

Proof. Using the Hölder and generalized Young inequality, we have

$$
|K_1^N(\phi)| = \| (|x|^{-\gamma} * |\phi|^2)|^2 \|_{L^1} \leq \| |x|^{-\gamma} \|_{L^\infty} \| \phi \|_{L^2}^4 \| x \|_{L^\frac{d}{\gamma}}^\frac{4}{d},
$$

(3.5)

here $p = \frac{2d}{2d-\gamma}, 1 + \frac{1}{p} = \frac{\gamma}{d} + \frac{1}{p}$, and by the Hölder inequality and Sobolev imbedding theorem, one has

$$
\| \phi \|_{L^2} \leq \| \phi \|_{L^2}^{1-\frac{\gamma}{d}} \| \phi \|_{L^\frac{2d}{\gamma}}^{\frac{\gamma}{d}} \leq \| \phi \|_{L^2}^{1-\frac{\gamma}{d}} \| \phi \|_{H^\frac{\gamma}{d}}^\frac{\gamma}{d}.
$$

(3.6)

Plugging (3.6) into (3.5), we obtain

$$
|K_1^N(\phi)| \leq \| \phi \|_{L^2}^4 |x|_{L^\frac{d}{\gamma}}^\frac{4}{d} \| \phi \|_{H^\frac{\gamma}{d}}^\frac{\gamma}{d} \leq K_1^Q(\phi)^2.
$$

(3.7)

Since $\phi_n \in H^1(\mathbb{R}^d) \setminus \{ 0 \}$ satisfies $K_j^Q(\phi_n) \rightarrow 0$, (3.7) and $K_1(\phi_n) = K_1^Q(\phi_n) + K_1^N(\phi_n)$, we get for large $n$,

$$
K_1(\phi_n) > 0.
$$

□

Lemma 3.5. If we set $H_1(\phi) \triangleq -\frac{1}{4} K_1^N(\phi) = \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x-y|^\gamma} \, dx \, dy$, and

$$
m_1 = \inf \{ H_1(\phi) : \phi \in H^1(\mathbb{R}^d), \phi \not\equiv 0, K_1(\phi) \leq 0 \},
$$

(3.8)

then $m_1 = m$. 

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Proof. It is trivial to prove that \( m_1 \leq m \) because \( J(\phi) = H_1(\phi) \) if \( K_1(\phi) = 0 \). So it suffices to show \( m \leq m_1 \). Take \( \phi \in H^1 \) such that \( K_1(\phi) < 0 \).

It follows from Remark 3.1 and Lemma 3.3 that
\[
\exists \lambda_1 < 0, \text{ s.t. } K_1(e^{\lambda_1} \phi) > 0.
\]
Combining this with \( K_1(\phi) < 0 \), we deduce that
\[
\exists \lambda \in (\lambda_1, 0), \text{ s.t. } K_1(e^{\lambda} \phi) = 0, \quad J(e^{\lambda} \phi) = H_1(e^{\lambda} \phi) = e^{4\lambda} H_1(\phi) \leq H_1(\phi), \quad (3.9)
\]
and so \( m \leq m_1 \). This completes the proof of Lemma 3.5.

The proof of Lemma 3.3

Step 1: We claim that \( \Lambda_1 \neq \emptyset \). Indeed, let \( \phi_n \in H^1 \) be a minimizing sequence for (3.8), i.e.
\[
K_1(\phi_n) \leq 0, \quad \phi_n \neq 0, \quad H_1(\phi_n) \searrow m.
\]
Let \( \phi_n^* \) be the Schwartz symmetrization of \( \phi_n \), i.e. the radial decreasing rearrangement (see [18]). Since \( \|\nabla \phi_n^*\|_2 \leq \|\nabla \phi_n\|_2, \|\phi_n^*\|_2 = \|\phi_n\|_2 \) and
\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi_n(x)^2 \phi_n(y)^2}{|x - y|^{7}} \, dx \, dy \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\phi_n^*(x)|^2 |\phi_n^*(y)|^2}{|x - y|^{7}} \, dx \, dy,
\]
(by the General rearrangement inequality, see page 93 Theorem 3.8 in [18]), therefore we have
\[
\phi_n^* \neq 0, \quad K_1(\phi_n^*) \leq K_1(\phi_n) \leq 0, \quad H_1(\phi_n^*) \geq H_1(\phi_n).
\]
If we choose \( \lambda_n \) such that \( e^{4\lambda_n} = \frac{H_3(\phi_n)}{H_1(\phi_n)} \), then
\[
\phi_n^* \neq 0, \quad K_1(e^{\lambda_n} \phi_n^*) \leq 0, \quad H_1(\phi_n) = H_1(e^{\lambda_n} \phi_n^*) \searrow m.
\]
Then by (3.9), we may replace it by symmetric \( \varphi_n \in H^1 \) such that
\[
\varphi_n \neq 0, \quad K_1(\varphi_n) = 0, \quad J(\varphi_n) = H_1(\varphi_n) \to m. \quad (3.10)
\]
Notice that \( J(\varphi_n) = \frac{1}{4}(\|\nabla \varphi_n\|_2^2 + \|\varphi_n\|_2^2) \to m \), we know that \( \{\varphi_n\} \) is bounded in \( H^1 \).
And so up to subsequence, it converges to some \( \varphi \) weakly in \( H^1 \). By the radial symmetry, it also converges strongly in \( L^p \) for all \( 2 < p < 2^* = \frac{2d}{d-2} \). By the Hölder and Young inequality, we deduce that
\[
\|(x^{-\gamma} \varphi_n)^2 - (|x|^{-\gamma} |\varphi|^2)^2\|_{L^1} \leq \|\varphi_n - \varphi\|_p \|\varphi_n + \varphi\| \left(\|\varphi_n\|_p^2 + \|\varphi\|_p^2\right),
\]
where \( 2 < p = \frac{4d}{2d-\gamma} < 2^* \), and so the nonlinear parts \( K_1^N(\varphi_n) \) converges. And by the Fatou Lemma, we have
\[
K_1(\varphi) \leq \lim_{n \to \infty} K_1(\varphi_n) \leq 0, \quad \text{and} \quad H_1(\varphi) \leq \lim_{n \to \infty} H_1(\varphi_n) \leq m.
\]
If \( \varphi = 0 \), then \( K_1(\varphi_n) = 0 \) implies that \( K_1^Q(\varphi_n) = -K_1^N(\varphi_n) \to -K_1^N(\varphi) = 0 \) by \( \varphi = 0 \), and by Lemma 3.4 we have \( K_1(\varphi_n) > 0 \) for large \( n \), which contradicts with \( K_1(\varphi_n) = 0 \). Hence \( \varphi \neq 0 \).
By (3.9), we may replace \( \varphi \) by \( e^\lambda \varphi \), so that \( K_1(\varphi) = 0, J(\varphi) = H(\varphi) \leq m \) and \( \varphi \neq 0 \). Then \( \varphi \) is a minimizer and \( m = H(\varphi) > 0 \). Hence \( \Lambda_1 \neq \emptyset \).

**Step 2:** \( m = J(W) \). That is, we need to prove \( J(\varphi) = J(W) \), where \( \varphi \) is attained in Step 1. Since \( \varphi \) is a minimizer for (3.3), there exists a Lagrange multiplier \( \eta \in \mathbb{R} \) such that

\[
J'(\varphi) = \eta K_1'(\varphi).
\]

Here \( J'(\varphi)(\phi) = \frac{d}{d\lambda} J(\varphi + \lambda \phi) \big|_{\lambda=0} \) and also \( J'(\varphi)(\varphi) = K_1(\varphi) = 0 \). On the other hand, a direct computation gives

\[
K_1'(\varphi)(\varphi) = 2\left( ||\nabla \varphi||_2^2 + ||\varphi||_2^2 \right) - 4 \int_{\mathbb{R}^d} \varphi(x)^2 \varphi(y)^2 \frac{d\varphi}{|x-y|^\gamma} dxdy
\]

\[
= -2 \int_{\mathbb{R}^d} \varphi(x)^2 \varphi(y)^2 \frac{d\varphi}{|x-y|^\gamma} dxdy < 0,
\]

and so \( \eta = 0 \). Hence \( J'(\varphi) = 0 \), namely \( J'(\varphi)(\phi) = 0, \forall \phi \in H^1 \):

\[
J'(\varphi)(\phi) = \int_{\mathbb{R}^d} (-\Delta \varphi + \varphi - (|x|^{-\gamma} * |\varphi|^2)\varphi) \phi dx = 0, \forall \phi \in H^1.
\]

So \( \varphi \) satisfies the elliptic equation: \(-\Delta \varphi + \varphi - (|x|^{-\gamma} * |\varphi|^2)\varphi = 0 \). Therefore \( J(\varphi) \geq J(W) \). On the other hand, it is trivial that \( J(\varphi) \leq J(W) \) by \( K_1(W) = 0 \). Hence \( J(\varphi) = J(W) \), which concludes the proof of Lemma 3.3.

The following lemma gives an equivalent description of the functional \( K_1(u) \) under \( E(u_0, u_1) < E(W, 0) \).

**Lemma 3.6.** Assume \( \phi \in H^1 \) such that \( J(\phi) < J(W) \), then

1. \( K_1(\phi) > 0 \iff ||\nabla \phi||_2^2 + ||\phi||_2^2 < ||\nabla W||_2^2 + ||W||_2^2 \);
2. \( K_1(\phi) < 0 \iff ||\nabla \phi||_2^2 + ||\phi||_2^2 > ||\nabla W||_2^2 + ||W||_2^2 \).

**Proof.** We only prove (1), since (2) can be obtained by the similar way. It is easy to see that

\[
J(W) = \frac{1}{4} \int_{\mathbb{R}^d} (||\nabla W||^2 + ||W||^2) dx,
\]

and

\[
J(\phi) - \frac{1}{4} K_1(\phi) = \frac{1}{4} \int_{\mathbb{R}^d} (||\nabla \phi||^2 + ||\phi||^2) dx.
\]

Thus, if \( J(\phi) < J(W) \) and \( K_1(\phi) > 0 \), then

\[
||\nabla \phi||_2^2 + ||\phi||_2^2 < ||\nabla W||_2^2 + ||W||_2^2.
\]

Next we prove the reverse statement. Since

\[
K_1(e^{\lambda} \phi) = e^{2\lambda} \left( \int_{\mathbb{R}^d} (||\nabla \phi||^2 + ||\phi||^2) dx - e^{2\lambda} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x-y|^\gamma} dxdy \right).
\]
If we take \( \lambda \in \mathbb{R} \) such that
\[
e^{2\lambda} = \frac{\int_{\mathbb{R}^d} \|\nabla \phi\|^2 + |\phi|^2 \, dx}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x-y|^\gamma} \, dx \, dy},
\]
then \( K_1(e^\lambda \phi) = 0 \). By (3.15), we know that
\[K_1(\phi) > 0 \iff \lambda > 0.\]

On the other hand, from \( K_1(e^\lambda \phi) = 0 \) and Lemma 3.3 one has
\[J(e^\lambda \phi) \geq J(W).\]

While by (3.15) and the assumption in Lemma 3.6 we obtain
\[
J(e^\lambda \phi) = e^{2\lambda} \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 + |\phi|^2 \, dx - \frac{e^{2\lambda}}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\phi(x)^2 \phi(y)^2}{|x-y|^\gamma} \, dx \, dy \right\}
\[= \frac{e^{2\lambda}}{4} \int_{\mathbb{R}^d} |\nabla \phi|^2 + |\phi|^2 \, dx \]
\[< \frac{e^{2\lambda}}{4} \int_{\mathbb{R}^d} |\nabla W|^2 + |W|^2 \, dx \]
\[= e^{2\lambda} J(W).\]

Combining this with (3.16), one gives that \( \lambda > 0 \), and so \( K_1(\phi) > 0 \), which concludes the proof.

**Remark 3.2.** By the similar argument as above, one may find that if \( E(u_0, u_1) < E(W, 0) \), then \( K_1(u) \cdot K_2(u) > 0 \).

As a consequence of Lemma 3.3 we deduce that the sign of \( K_1(u) \) is invariance along the flow of (1.1) under the restriction of \( E(u, \dot{u}) < E(W, 0) \):

**Corollary 3.1.** Let
\[
\mathcal{A}^+ = \{(u, \dot{u}) \in H^1 \times L^2(\mathbb{R}^d) : E(u, \dot{u}) < E(W, 0), K_1(u) > 0\},
\]
\[
\mathcal{A}^- = \{(u, \dot{u}) \in H^1 \times L^2(\mathbb{R}^d) : E(u, \dot{u}) < E(W, 0), K_1(u) < 0\},
\]
then \( \mathcal{A}^+ \) and \( \mathcal{A}^- \) are invariant under the flow of (1.1). That is, if \( (u_0, u_1) \in \mathcal{A}^\pm \), then \( (u(t), \dot{u}(t)) \in \mathcal{A}^\pm \), for any \( t \in (-T_-(u_0, u_1), T_+(u_0, u_1)) \).

**Proof.** Since \( J(u) < E(u, \dot{u}), J(W) = E(W, 0) \) and \( E(u(t), \dot{u}(t)) = E(u_0, u_1) \), we deduce that if given \( E(u_0, u_1) < E(W, 0) \), then \( J(u) < J(W) \), for any \( t \in (-T_-(u_0, u_1), T_+(u_0, u_1)) \).

On the other hand, if there exists \( t \in (-T_-(u_0, u_1), T_+(u_0, u_1)) \) such that \( K_1(u(t)) = 0 \), then by Lemma 3.3 we get \( J(u(t)) \geq J(W) \), which contradicts with \( J(u) < J(W) \).
By Lemma 3.6, one may replace $A^\pm$ by

$$A^+ = \{(u, ̇u) \in H^1 \times L^2 : E(u, ̇u) < E(W, 0), \|\nabla u\|_2^2 + \|u\|_2^2 < \|\nabla W\|_2^2 + \|W\|_2^2\};$$
$$A^- = \{(u, ̇u) \in H^1 \times L^2 : E(u, ̇u) < E(W, 0), \|\nabla u\|_2^2 + \|u\|_2^2 > \|\nabla W\|_2^2 + \|W\|_2^2\}.$$

Now, we need a stronger result than Corollary 3.1, which is important in the viral analysis. Let

$$A_{\delta, \delta}^+ \triangleq \{(u, ̇u) \in H^1 \times L^2 : E(u, ̇u) < (1 - \delta)E(W, 0), K_1(u) > \bar{\delta}\max\{K_1^Q(u), -K_1^N(u)\},$$
$$K_2(u) > \bar{\delta}\max\{K_2^Q(u), -K_2^N(u)\}\};$$

**Proposition 3.1.** Assume $(\phi_0, \phi_1) \in H^1 \times L^2$ and $(\phi_0, \phi_1) \in A_{\delta, \delta}^+$, then there exists some $\bar{\delta} \in (0, 1)$ independent of $(\phi_0, \phi_1)$ such that $(\phi_0, \phi_1) \in A_{\delta, \delta}^+$.

**Proof.** The proof is the same as [22]. For convenience, we give a full proof. We only need to prove

$$K_1(\phi_0) > \bar{\delta}\max\{\|\nabla \phi_0\|_2^2 + \|\phi_0\|_2^2, -K_1^N(\phi_0)\},$$

since the other is given by the similar way. From $(\phi_0, \phi_1) \in A_{\delta, \delta}^+$, we know that $K_1(\phi_0) > 0$, so it suffices to prove that

$$K_1(\phi_0) > \bar{\delta}(\|\nabla \phi_0\|_2^2 + \|\phi_0\|_2^2). \quad (3.18)$$

We suppose for contradiction that there exists a sequence $\{(\varphi_0^n, \varphi_1^n) \in A_{\delta, \delta}^+\}_n$, and a sequence $\{\bar{\delta}_n\}_n$ satisfying $\bar{\delta}_n \to 0$, as $n \to \infty$, and

$$J(\varphi_0^n) < (1 - \delta)J(W), 0 < K_1(\varphi_0^n) \leq \bar{\delta}_n(\|\nabla \varphi_0^n\|_2^2 + \|\varphi_0^n\|_2^2),$$

which implies that

$$|K_1^N(\varphi_0^n)| < K_1^Q(\varphi_0^n) \leq \frac{1}{1 - \bar{\delta}_n}|K_1^N(\varphi_0^n)|.$$  

Thus there exists $\lambda_n$ such that

$$K_1(e^{\lambda_n} \varphi_0^n) = 0,$$

and $1 < e^{2\lambda_n} \leq \frac{1}{1 - \bar{\delta}_n}$. So by Lemma 3.3 we get $J(e^{\lambda_n} \varphi) \geq J(W)$, which implies that

$$\frac{e^{2\lambda_n}}{2}K_1^Q(\varphi_0^n) + \frac{e^{4\lambda_n}}{4}K_1^N(\varphi_0^n) \geq J(W). \quad (3.19)$$

On the other hand, since $(\varphi_0^n, \varphi_1^n) \in A_{\delta, \delta}^+$, $J(\varphi_0^n) < E(\varphi_0^n, \varphi_1^n) < (1 - \delta)E(W, 0) = (1 - \delta)J(W)$, which means that

$$\frac{1}{2}K_1^Q(\varphi_0^n) + \frac{1}{4}K_1^N(\varphi_0^n) < (1 - \delta)J(W). \quad (3.20)$$

Combining (3.19) and (3.20), one gives

$$0 \leq -\frac{e^{2\lambda_n} - 1}{4}K_1^N(\varphi_0^n) < (1 - \delta - e^{-2\lambda_n})J(W).$$

But this can not happen for large $n$, since $e^{2\lambda_n} \to 1$ as $n \to \infty$. \qed
Combining the energy conservation law, Corollary 3.1 with the above proposition, we obtain the following result.

**Corollary 3.2.** Assume \((u_0, u_1) \in A^+\), then there exist some \(\delta, \bar{\delta} \in (0, 1)\) depending on \((u_0, u_1)\), such that the corresponding solution \((u(t), \dot{u}(t)) \in A^+_{\delta, \bar{\delta}}\) for any \(t \in (-T_-(u_0, u_1), T_+(u_0, u_1))\).

It is easy to observe that the free energy and the nonlinear energy are equivalent in the set \(K_1 > 0\).

**Lemma 3.7.** Assume \(2 < \gamma < \min\{4, d\}\), then for any \((u_0, u_1) \in H^1 \times L^2\), we have

\[
K_1(u_0) > 0 \implies \begin{cases} J(u_0) \leq \frac{1}{2}K_1^Q(u_0) \leq 2J(u_0) \\ E(u_0, u_1) \leq E_0(u_0, u_1) \leq 2E(u_0, u_1) \end{cases}
\] (3.21)

where \(E_0(u_0, u_1) = \frac{1}{2}K_1^Q(u_0) + \frac{1}{2}\|u_1\|_2^2 \), and \(E(u_0, u_1) = E_0(u_0, u_1) + \frac{1}{4}K_1^N(u_0)\).

**Proof.** First, we recall that

\[
\begin{align*}
K_1(u_0) &= K_1^Q(u_0) + K_1^N(u_0), \\
J(u_0) &= \frac{1}{2}K_1^Q(u_0) + \frac{1}{4}K_1^N(u_0).
\end{align*}
\]

Hence, by direct computation, we get \(K_1(u_0) > 0\), which gives \(J(u_0) \leq \frac{1}{2}K_1^Q(u_0) \leq 2J(u_0)\). Similarly, we obtain \(E(u_0, u_1) \leq E_0(u_0, u_1) \leq 2E(u_0, u_1)\) under the condition \(K_1(u_0) > 0\).

**Corollary 3.3.** Let \(\mu = -1\), \(2 < \gamma < \min\{d, 4\}\) and \((u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\). Assume that \(E(u_0, u_1) < E(W, 0)\) and \(K_1(u_0) > 0\), then the solution of (1.1) is global.

**Proof.** It is a consequence of Corollary 3.1 energy conservation, Lemma 3.7 and the standard blow-up criterion, see Theorem 2.1.

The next lemma gives a upper bound on \(K_1\) in the set \(K_1 < 0\), which will be important for the blow up.

**Lemma 3.8.** Suppose \(2 < \gamma < \min\{4, d\}\), \(\phi \in H^1\), \(J(\phi) < J(W)\), and \(K_1(\phi) < 0\), then

\[
K_1(\phi) < -2(J(W) - J(\phi)).
\]

**Proof.** Let \(j(\lambda) = J(e^\lambda \phi)\), then by a directive computation, we have

\[
j'(\lambda) = K_1(e^\lambda \phi), \quad j''(\lambda) = 2K_1^Q(e^\lambda \phi) + 4K_1^N(e^\lambda \phi),
\]

and so \(j'(0) = K_1(\phi), \quad j''(\lambda) < 2j'(\lambda)\). If we choose \(\lambda_0 \in \mathbb{R}\) such that

\[
e^{2\lambda_0} = \frac{K_1^Q(\phi)}{-K_1^N(\phi)}.
\]

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then \( j'(\lambda_0) = K_1(e^{\lambda_0} \phi) = 0 \), and \( \lambda_0 < 0 \) by \( K_1(\phi) < 0 \). Therefore, we have

\[
K_1(\phi) = j'(0) - j'(\lambda_0) = \int_{\lambda_0}^{0} j''(\lambda) d\lambda < 2 \int_{\lambda_0}^{0} j'(\lambda) d\lambda = 2(j(0) - j(\lambda_0)).
\] (3.22)

By Lemma 3.3 and \( K_1(e^{\lambda_0} \phi) = 0 \), we get \( j(\lambda_0) = J(e^{\lambda_0} \phi) \geq J(W) \), and so

\[
(3.22) \leq 2(J(\phi) - J(W)).
\]

This completes the proof. \( \square \)

**Corollary 3.4.** Suppose \( 2 < \gamma < \min\{4, d\} \), \( (u_0, u_1) \in H^1 \times L^2 \), \( E(u_0, u_1) < E(W, 0) \) and \( K_1(u_0) < 0 \), then

\[
K_1(u) \leq -2(E(W, 0) - E(u_0, u_1)), \text{ for any } t \in (-T_-(u_0, u_1), T_+(u_0, u_1)).
\]

**Proof.** From Corollary 3.1 we have

\[
K_1(u) < 0, \text{ for } t \in (-T_-(u_0, u_1), T_+(u_0, u_1)).
\]

And so by Lemma 3.8 we obtain

\[
K_1(u) \leq -2(E(W, 0) - E(u_0, u_1)), \text{ for } t \in (-T_-(u_0, u_1), T_+(u_0, u_1)),
\]

where we use the fact that \( J(u) \leq E(u, \dot{u}) = E(u_0, u_1) \), and \( J(W) = E(W, 0) \). \( \square \)

## 4 Blow up

In this section we prove the blow-up part of Theorem 1.2. The idea is essentially due to Payne-Sattinger [35], but we give a complete proof for convenience.

By contradiction we assume that the solution \( u \) exists for all \( t > 0 \). The proof for \( t < 0 \) is the same.

Denote \( y(t) = \int |u|^2 dx, \) then we have \( y'(t) = 2 \int u\dot{u} dx, \) and

\[
y''(t) = 2\|\dot{u}\|^2_2 - 2K_1(u) = 6\|\dot{u}\|^2_2 + 2K_1^Q(u) - 8E(u_0, u_1).
\] (4.1)

It follows from Corollary 3.4 that there exists \( \delta > 0 \) such that \( K_1(u) < -\delta \), and so \( y''(t) > 2\delta > 0 \). Thus

\[
y''(t) > 2\delta > 0.
\]

Then by the lower bound on \( y''(t) \), there exists \( t_0 > 0 \) such that \( y'(t_0) > 0 \), and hence \( y'(t) > 0 \) for \( t > t_0 \). By Lemma 3.6 and \( K_1(u) < 0 \), we get \( K_1^Q(u) > K_1^Q(W) \), and so

\[
E(u, \dot{u}) < E(W, 0) = J(W) = \frac{1}{4} K_1^Q(W) < \frac{1}{4} K_1^Q(u).
\]

Therefore, using Cauchy-Schwarz inequality and (4.1), we obtain for any \( t > t_0 \)

\[
y''(t) \geq 6\|\dot{u}\|^2_2 \geq \frac{3}{2} \frac{y^2}{y}.
\]
So that, for \( t > t_0 \),
\[
\frac{y''(t)}{y'(t)} \geq \frac{3}{2} \frac{y'(t)}{y(t)}.
\]

Hence for \( t > t_0 \),
\[
y(t)^{-\frac{1}{2}} \leq y(t_0)^{-\frac{1}{2}} - \frac{1}{2} \frac{y'(t_0)}{y(t_0)^{\frac{3}{2}}} (t - t_0).
\]

Therefore \( T_+ \leq t_0 + \frac{2y'(t_0)}{y(t_0)^{\frac{3}{2}}} \), which contradicts with \( T_+ = +\infty \).

### 5 Profile decomposition

In this section, we first recall the linear profile decomposition of the sequence of \( H^1 \)-bounded solutions of \((1.1)\) which was established in [12]. And then we utilize it to show the orthogonal analysis for the nonlinear energy and the nonlinear profile decomposition which will be used to construct the critical element and obtain its compactness properties. In order to do it, we now recall some notations in [12].

With any real-valued function \( u(t, x) \), we associate the complex-valued function \( \vec{u}(t, x) \) by
\[
\vec{u} = \langle \nabla \rangle u - i\dot{u}, \quad u = \Re\langle \nabla \rangle^{-1} \vec{u}.
\]

Then the free and nonlinear Klein-Gordon equations are given by
\[
\begin{cases}
(\Box + 1)u = 0 \iff (i\partial_t + \langle \nabla \rangle) \vec{u} = 0, \\
(\Box + 1)u = f(u) \iff (i\partial_t + \langle \nabla \rangle) \vec{u} = f(\langle \nabla \rangle^{-1} \vec{u}),
\end{cases}
\]
and the energy are written as
\[
\tilde{E}(\vec{u}) = E(u, \dot{u}) = \frac{1}{2} \int_{\mathbb{R}^d} (|\vec{u}|^2 + |\nabla u|^2 + |u|^2) \, dx + \frac{\mu}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^\gamma} \, dx dy.
\]

We denote the set of Fourier multipliers on
\[
\mathcal{MC} = \{ \mu = \mathcal{F}^{-1} \tilde{\mu} \mathcal{F} | \tilde{\mu} \in C(\mathbb{R}^d), \exists \lim_{|x| \to \infty} \tilde{\mu}(x) \in \mathbb{R} \}.
\]

#### 5.1 Linear profile decomposition

First, we state the linear profile decomposition( which was established in [12]) as follows

**Lemma 5.1.** Let \( \vec{v}_n \) be a sequence of free Klein-Gordon solutions with uniformly bounded \( L^2 \) norm. Then after replacing it with some subsequence, there exist \( K \in \{0, 1, 2, \ldots, \infty\} \) and, for each integer \( j \in [0, k) \), \( \varphi^j \in L^2(\mathbb{R}^d) \) and \( \{(t_n^j, x_n^j)\}_{n \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{R}^d \) satisfying the following. Define \( \vec{v}_n^j \) and \( \vec{w}_n^k \) for each \( j < k \leq K \) by
\[
\vec{v}_n(t, x) = \sum_{j=0}^{k-1} \vec{v}_n^j(t, x) + \vec{w}_n^k(t, x), \quad \vec{v}_n^j(t, x) = e^{i\langle \nabla \rangle (t - t_n^j)} \varphi^j(x - x_n^j),
\]
Lemma 5.2. Let \( \mu \in \mathcal{MC} \), any \( l < j < k \leq K \) and any \( t \in \mathbb{R} \),
\[
\lim_{n \to \infty} \left( \frac{\mu \nu_n(t)}{L^2} \right)^k_{L^2} = 0 = \lim_{n \to \infty} \left( \frac{\mu \nu_n(t)}{L^2} \right)^k_{L^2},
\]
and for any \( \mu \in \mathcal{MC} \), any \( l < j < k \leq K \) and any \( t \in \mathbb{R} \),
\[
\lim_{n \to \infty} \left| t_n^k - t_n^j \right| + \left| x_n^k - x_n^j \right| = +\infty.
\]

Remark 5.1. We call \( \nu_n \) the free concentrating wave. From (5.5), we have the following asymptotic orthogonality
\[
\lim_{n \to +\infty} \left( \| \mu \nu_n(t) \|_{L^2} - \sum_{j=1}^{k-1} \| \mu \nu_n(t) \|_{L^2} - \| \mu \nu_n(t) \|_{L^2} \right) = 0.
\]

Next we begin with the orthogonal analysis for the nonlinear energy.

Lemma 5.2. Let \( \nu_n \) be a sequence of free Klein-Gordon solutions satisfying \( \nu_n(0) \in L^2_x \).
Let \( \nu_n = \sum_{j=0}^{k-1} \nu_j^k + \nu_k^k \) be the linear profile decomposition given by Lemma 5.1. Then if \( \mu = 1 \) (defocusing) and \( \lim_{n \to \infty} \hat{E}(\nu_n(0)) < +\infty \), then we have \( \nu_n(0) \in L^2_x \) for large \( n \), and
\[
\lim_{k \to K, n \to \infty} \left| \hat{E}(\nu_n(0)) - \sum_{j=0}^{k-1} \hat{E}(\nu_j^k(0)) - \hat{E}(\nu_k^k(0)) \right| = 0.
\]
Moreover we have for all \( j < k \)
\[
0 \leq \lim_{n \to \infty} \hat{E}(\nu_j^k(0)) \leq \lim_{n \to \infty} \hat{E}(\nu_n(0)) \leq \lim_{n \to \infty} \hat{E}(\nu_n(0)),
\]
where the last inequality becomes equality only if \( K = 1 \) and \( \omega_1^k \to 0 \) in \( L^\infty_t L^2_x \).

Furthermore, if \( \mu = -1 \) (focusing), \( (\nu_n(0), \nu_n(0)) \in \mathcal{A}^+ \) and \( \lim_{n \to \infty} \hat{E}(\nu_n(0)) < E(W, 0) \), then we have \( \nu_n(0) \in L^2_x \) for large \( n \) and all \( j < K \), \( (\nu_n^j(0), \nu_n^j(0)) \in \mathcal{A}^+ \), and (5.8), (5.9) also holds true.

Proof. By Sobolev imbedding theorem and (5.1), we have
\[
\lim_{k \to K} \lim_{n \to \infty} \| \omega_{n}^k \|_{L^{\frac{d}{d-\gamma}}_{x}} \leq \lim_{k \to K} \lim_{n \to \infty} \| \omega_{n}^k \|_{L^{\theta}} \| \omega_{n}^k \|_{L^{\frac{1}{1-\theta}}} = 0,
\]
where \( \theta = \frac{d - \gamma}{2} \in (0, 1) \), and \( \omega_{n}^k = \Re \langle \nabla \rangle^{-1} \omega_{i}^k \). This implies that, if there exists \( u_i = \omega_{n}^k \) (\( i = 1, 2, 3, 4 \)), then by the Hölder and general Young inequality, we obtain
\[
\lim_{k \to K} \lim_{n \to \infty} \| |x|^{-\gamma} \ast (u_1 u_2) (u_3 u_4) \|_{L^1_x} \leq \lim_{k \to K} \lim_{n \to \infty} \prod_{i=1}^{4} \| u_i \|_{L^{\frac{d}{d-\gamma}}_{x}} = 0.
\]
This together with (5.5) reduces us to prove
\[
\lim_{k \to K} \lim_{n \to \infty} \left\{ \| (|x|^{-\gamma} * v_n - \omega_n^k) |v_n - \omega_n^k|^2 \|_{L_1^k} - \sum_{j=0}^{k-1} \| (|x|^{-\gamma} * v_n^j) |v_n^j|^2 \|_{L_1^k} \right\} = 0.
\]

For this purpose, we discuss in two cases
\[
\begin{cases}
\text{case 1: } \exists \ j, \ |t_n^j| \to \infty \text{ as } n \to \infty, \\
\text{case 2: } |t_n^j| \text{ is uniformly bounded in } n, j.
\end{cases}
\]

For the first case, by the decay of \( e^{it(\nabla)} \) in \( S \to L^p_x \) uniform w.r.t \( n \) and the Sobolev embedding \( \dot{H}^s \subset L^p \), we have
\[
\| v_n^j \|_{L_x^{2s,\gamma}} = \| \Re \langle \nabla \rangle^{-1} e^{i(\nabla)(t-t_n^j)} \varphi^j(x-x_n^j) \|_{L_x^{2s,\gamma}} \to 0, \text{ as } n \to \infty.
\]

Thus by the linear profile decomposition, the Hölder and generalized Young inequality, we have
\[
\left\| (|x|^{-\gamma} * (v_n^{k_1} v_n^{k_2}))(v_n^{k_3} v_n^{k_4}) \right\|_{L_1^k} \lesssim \prod_{i=1}^{4} \| v_n^{k_i} \|_{L_x^{2s,\gamma}} \to 0, \text{ as } n \to \infty,
\]
for some \( k_i = j \).

Next we consider the second case. Since
\[
\left\| (|x|^{-\gamma} * \sum_{j=0}^{k-1} v_n^j)^2 \|_{L_1^k} = \sum_{j_1, j_2, j_3, j_4 < k} \left\| (|x|^{-\gamma} * (v_n^{j_1} v_n^{j_2}))(v_n^{j_3} v_n^{j_4}) \right\|_{L_1^k},
\]
we only need to prove that
\[
\left\| (|x|^{-\gamma} * (v_n^{j_1} v_n^{j_2}))(v_n^{j_3} v_n^{j_4}) \right\|_{L_1^k} \to 0, \text{ as } n \to \infty,
\]
provided that \( t_n^i \) is bounded for any \( i = 1, 2, 3, 4 \), and at least two of \( j_1, j_2, j_3, j_4 \) are different. Moreover, by (5.6), we know that
\[
|x_n^{j_i} - x_n^{j_{i'}}| \to \infty, \text{ as } n \to \infty, \text{ for } j_i \neq j_i'.
\]

To prove (5.10), we should split it into the following two cases:
\[
\begin{cases}
\text{Case 1: } j_1 \neq j_2 \text{ or } j_3 \neq j_4, \\
\text{Case 2: } j_1 = j_2, j_3 = j_4, j_1 \neq j_3.
\end{cases}
\]

Keep in mind that \( \vec{v}_n^j(t, x) = e^{i(\nabla)(t-t_n^j)} \varphi^j(x-x_n^j) \). Without loss of generality, we may assume that \( \varphi^j(i = 1, 2, 3, 4) \) have compact support in \( \{x \in \mathbb{R}^d : |x| \leq R\} \).

**Case 1:** \( j_1 \neq j_2 \) or \( j_3 \neq j_4 \),
\[
\text{LHS of (5.10)} \lesssim \| v_n^{j_1} v_n^{j_2} \|_p \| v_n^{j_3} v_n^{j_4} \|_p \to 0, \text{ as } n \to \infty,
\]

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where we use the orthogonal condition (5.11) and \( p = \frac{2d}{2d - \gamma} \).

**Case 2:** \( j_1 = j_2, j_3 = j_4, j_1 \neq j_3 \).

The LHS of (5.10) is

\[
\begin{align*}
\text{LHS of (5.10)} &= \| (|x|^{-\gamma} * |v_{n_1}^{j_1}|^2)|v_{n_1}^{j_3}|^2 \|_{L^2_x} \\
&\leq \| (V_1 * |v_{n_1}^{j_1}|^2)|v_{n_1}^{j_3}|^2 \|_{L^2_x} + \| (V_2 * |v_{n_1}^{j_1}|^2)|v_{n_1}^{j_3}|^2 \|_{L^2_x} \\
&\triangleq I_1 + I_2.
\end{align*}
\]

where \( V_1 = |x|^{-\gamma} \chi_{|x| \leq LR}, V_2 = |x|^{-\gamma} \chi_{|x| > LR} \) for large \( L \gg 1 \). By supp \( \varphi^{j_1} \subset \{ x \in \mathbb{R}^d : |x| \leq R \} \), the Hölder inequality and Young inequality, we get

\[
I_1 = \| (V_1 * |e^{-it\tilde{\Gamma} \varphi} \varphi^{j_1} (\cdot - x_{n_1}^{j_1})|^2)|e^{-it\tilde{\Gamma} \varphi} \varphi^{j_3} (x - x_{n_1}^{j_3})|^2 \|_{L^2_x} \\
= \| [V_1 * |e^{-it\tilde{\Gamma} \varphi} \varphi^{j_1} (\cdot)|^2] \chi_{|x| \leq R} (x) |e^{-it\tilde{\Gamma} \varphi} \varphi^{j_3} (x - (x_{n_1}^{j_3} - x_{n_1}^{j_1}))|^2 \|_{L^2_x} \\
\leq \| V_1 * |e^{-it\tilde{\Gamma} \varphi} \varphi^{j_1} (\cdot)|^2 \|_{L^{2 \gamma}_{t,x}} \| \chi_{|x| \leq R} (x) e^{-it\tilde{\Gamma} \varphi} \varphi^{j_3} (x - (x_{n_1}^{j_3} - x_{n_1}^{j_1}))|^2 \|_{L^{2 \gamma}_{t,x}} \\
\lesssim \| V_1 \|_{L^{\frac{2 \gamma}{\gamma - 2 \gamma}}_{t,x}} \| v_{n_1}^{j_1} \|_{L^{2 \gamma}_{t,x}} \| v_{n_1}^{j_3} \|_{L^{2 \gamma}_{t,x}} \\
&\to 0, \quad \text{as} \quad n \to \infty.
\]

(5.12)

For \( I_2 \), by the Hölder and Young inequality, one infers that

\[
I_2 = \| (V_2 * |v_{n_1}^{j_1}|^2)|v_{n_1}^{j_3}|^2 \|_{L^2_x} \\
\leq \| V_2 * |v_{n_1}^{j_1}|^2 \|_{L^\infty} \| v_{n_1}^{j_3} \|_{L^2_x} \\
\lesssim (LR)^{-\gamma} \| v_{n_1}^{j_1} \|_{L^2_x} \| v_{n_1}^{j_3} \|_{L^2_x} \\
&\to 0, \quad \text{as} \quad L \to \infty.
\]

(5.13)

Combining (5.12) with (5.13), we obtain (5.10), which concludes the proof of (5.8).

Now we turn to prove that \( (v_{n_0}^j(0), \dot{v}_{n_0}^j(0)) \in A^+ \) for large \( n \) and all \( j < K \) in the focusing case. In fact, by (5.5), we have

\[
\lim_{k \to K} \lim_{n \to \infty} \left| K^Q (v_n(0)) - \sum_{j < k} K^Q_1 (v_n^j(0)) - K^Q_1 (w_k^j(0)) \right| = 0,
\]

where \( K^Q_1 (\phi) = \| \phi \|_2^2 + \| \nabla \phi \|_2^2 \). It follows from \( v_n(0) \in A^+ \) and Lemma 3.6 that

\[
K^Q_1 (v_n^j(0)) < K^Q_1 (W),
\]

and so \( (v_{n_0}^j(0), \dot{v}_{n_0}^j(0)) \in A^+ \) for large \( n \). Lemma 3.7 shows that the last inequality in (5.9) becomes equality only if \( K = 1 \) and \( \omega_n^1 \to 0 \) in \( L^\infty_t L^2_x \).

5.2 Nonlinear profile decomposition

After the linear profile decomposition of a sequence of initial data in the last subsection, we now show the nonlinear profile decomposition of a sequence of the solutions of (1.1) with the same initial data in the energy space \( H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \).
First we construct a nonlinear profile corresponding to a free concentrating wave. Let \( \vec{v}_n \) be a free concentrating wave for a sequence \((t_n, x_n)\),

\[
\begin{align*}
(i\partial_t + \langle\nabla\rangle)\vec{v}_n &= 0, \\
\vec{v}_n(t_n) &= \phi(x - x_n), \quad \phi(x) \in L^2,
\end{align*}
\]
and let \( u_n \) be the nonlinear solution with the same initial data

\[
\begin{align*}
(i\partial_t + \langle\nabla\rangle)\vec{u}_n &= f(u_n), \\
\vec{u}_n(0) &= \vec{v}_n(0).
\end{align*}
\]

Next we define

\[
\vec{V}_n(t, x) = \vec{v}_n(t + t_n, x + x_n), \quad \vec{U}_n(t, x) = \vec{u}_n(t + t_n, x + x_n).
\]

Then they satisfy the rescaled equations

\[
\begin{align*}
\vec{V}_n(t, x) &= e^{it\langle\nabla\rangle}\phi(x), \\
\vec{U}_n(t) &= \vec{V}_n(t) - i \int_{-t_n}^t e^{i(t-s)\langle\nabla\rangle} f(\Re\langle\nabla\rangle^{-1}\vec{U}_n)ds.
\end{align*}
\]

Extracting a subsequence, we may assume convergence

\[
t_n \to t_\infty \in [-\infty, \infty].
\]

Thus the limit equations are given by

\[
\begin{align*}
\vec{V}_\infty(t, x) &= e^{it\langle\nabla\rangle}\phi(x), \\
\vec{U}_\infty(t) &= \vec{V}_\infty(t) - i \int_{-t_\infty}^t e^{i(t-s)\langle\nabla\rangle} f(\Re\langle\nabla\rangle^{-1}\vec{U}_\infty)ds.
\end{align*}
\]

The unique existence of a local solution \( \vec{U}_\infty \) around \( t = t_\infty \) is known in all cases, including \( t = \pm \infty \) which corresponds to the existence of the wave operators, by using the standard iteration with the Strichartz estimate.

**Definition 5.1.** The nonlinear concentrating wave \( \vec{u}_{(n)} \) associated with \( \vec{v}_n \) is defined by

\[
\vec{u}_{(n)}(t, x) = \vec{U}_\infty(t - t_n, x - x_n).
\]

**Remark 5.2.**

(i) \( u_{(n)} \) solves (1.1).

(ii) By definition, we have \( \|\vec{u}_{(n)}(0) - \vec{u}_{(n)}(0)\|_{L^2_x} \to 0 \), as \( n \to \infty \). In fact

\[
\begin{align*}
\|\vec{u}_{(n)}(0) - \vec{u}_{(n)}(0)\|_{L^2_x} &= \|\vec{v}_n(0) - \vec{u}_{(n)}(0)\|_{L^2_x} \\
&= \|\vec{V}_n(-t_n) - \vec{U}_\infty(-t_n)\|_{L^2_x} \\
&= \|\vec{U}_\infty(-t_\infty) - \vec{U}_\infty(-t_n)\|_{L^2_x} \\
&\to 0, \quad \text{as} \quad n \to \infty.
\end{align*}
\]

Let \( u_n \) be a sequence of (local) solutions of (1.1) around \( t = 0 \), and let \( v_n \) be the sequence of the free solutions with the same initial data. We consider the linear profile decomposition of \( \{\vec{v}_n\} \) given by Lemma 5.1

\[
\vec{v}_n = \sum_{j=0}^{k-1} \vec{v}_{n}^j + \vec{\omega}_n^{k}, \quad \vec{v}_n = e^{i\langle\nabla\rangle(t-t_n^j)} \varphi^j(x - x_n^j).
\]
Definition 5.2. (Nonlinear profile decomposition) Let \( \{ \vec{v}_n^j \}_{n \in \mathbb{N}} \) be the free concentrating wave, and \( \{ \vec{w}_n^j \}_{n \in \mathbb{N}} \) be the sequence of the nonlinear concentrating wave associated with \( \{ \vec{v}_n^j \}_{n \in \mathbb{N}} \). Then we define the nonlinear profile decomposition of \( u_n \) by

\[
\vec{u}_n^{< k} := \sum_{j=0}^{k-1} \vec{w}_n^j.
\]

We are going to prove that \( \vec{u}_n^{< k} + \vec{\omega}_n^{k} \) is a good approximation for \( \vec{u}_n \). And the following two lemmas derive from Lemma 5.1 and the perturbation lemma. The first lemma concerns the orthogonality in the Strichartz norms.

Lemma 5.3. Suppose that in the nonlinear profile decomposition (5.16), we have

\[
\| \Re(\nabla)^{-1} \vec{U}_n^j \|_{ST(\mathbb{R})} + \| \vec{U}_n^j \|_{L_t^\infty L_x^2(\mathbb{R})} < \infty, \forall j < k.
\]

Then, for any finite interval \( I, j < k \), we have

\[
\lim_{n \to \infty} \| u_n^j \|_{ST(I)} \lesssim \| \Re(\nabla)^{-1} \vec{U}_n^j \|_{ST(\mathbb{R})},
\]

\[
\lim_{n \to \infty} \| u_n^{< k} \|_{ST(I)} \lesssim \lim_{n \to \infty} \sum_{j=0}^{k-1} \| u_n^j \|_{ST(\mathbb{R})},
\]

where the implicit constants do not depend on \( I \) or \( j \). We also have

\[
\lim_{n \to \infty} \left( \| (|x|^{-\gamma} * |u_n^{< k})^2| u_n^{< k} - \sum_{j=0}^{k-1} (|x|^{-\gamma} * |u_n^j|^2) u_n^j \|_{ST^*(I)} \right) = 0.
\]

Proof. It is easy to get (5.18) from the definition of \( u_n^j \). Now we prove (5.19). Let \( \chi(t, x) \in C^\infty(\mathbb{R}^n+1) \) satisfy \( \chi(t, x) = 1 \) for \( |(t, x)| \leq 1 \) and \( \chi(t, x) = 0 \) for \( |(t, x)| \geq 2 \), and

\[
\chi_R(t, x) = \chi(\frac{t}{R}, \frac{x}{R}).
\]

Define \( u_n^j(R) \) for \( R \gg 1 \) by

\[
u_n^j(t, x) = \chi_R(t, x) u_n^j(t, x), \quad u_n^{< k} = \sum_{j=0}^{k-1} u_n^j(R).
\]

Then we have

\[
\| u^{< k} - u^{< k}(R) \|_{ST(I)} \lesssim \sum_{j=0}^{k-1} \| (1 - \chi_R) \Re(\nabla)^{-1} \vec{U}_n^j \|_{ST(\mathbb{R})} \to 0, \quad \text{as } R \to \infty,
\]

thus we may replace \( u^{< k} \) by \( u^{< k}(R) \). The homogenous Besov norm in ST is equivalent to

\[
\| u_n^{< k}(R) \|_{L^q(I; B^q_{q,2})} \cong \| u_n^{< k}(R) \|_{L^q(I; \tilde{B}^\frac{q}{2}_{q,2})} + \| u_n^{< k}(R) \|_{L^q_t L^2_x}
\]

\[
\cong \left( \int_I \left( \int_{\mathbb{R}^d} \frac{\| u_n^{< k}(x-y) - u_n^{< k}(x) \|^2 L_y^q \, dy}{|y|^q} \right)^\frac{1}{q} \right)^{\frac{1}{p}} + \| u_n^{< k}(R) \|_{L^q_t L^2_x},
\]

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By the orthogonality (5.6), we get for large $n$

$$|u_{(n),R}^<(x - y) - u_{(n),R}^<(x)| = \left\{ \sum_{j=0}^{k-1} |u_{(n),R}^j(x - y) - u_{(n),R}^j(x)|^2 \right\}^{\frac{1}{2}},$$

and so

$$\tag{5.21} \left( \int_I \left( \int_{\mathbb{R}^d} \left\| \sum_{j=0}^{k-1} |u_{(n),R}^j(x - y) - u_{(n),R}^j(x)|^2 \right\|_{L^q} \frac{dy}{|y|^d} \right)^{\frac{q}{2}} dy \right)^{\frac{1}{q}} + \|u_{(n),R}^<\|_{L^q_{s,x}}$$

By Minkowski inequality, the above quantity can be controlled by

$$\tag{5.23} \left( \int_{\mathbb{R}^d} \left( \int_I \left\| \sum_{j=0}^{k-1} |u_{(n),R}^j(x - y) - u_{(n),R}^j(x)|^2 \right\|_{L^q} \frac{dy}{|y|^d} \right)^{\frac{q}{2}} dy \right)^{\frac{1}{q}} + \|u_{(n),R}^<\|_{L^q_{s,x}}$$

This implies that

$$\|u_{(n),R}^<\|_{L^q_{s(I,B_{\frac{1}{q},2})}} \lesssim \left( \sum_{j=0}^{k-1} \|u_{(n),R}^j\|_{L^q_{s(I,B_{\frac{1}{q},2})}}^2 \right)^{\frac{1}{2}}.$$ 

Thus we obtain (5.19).

Finally, we turn to prove (5.20). After the smooth cut-off, we have for large $n$

$$\tag{5.22} \left| \sum_{j=0}^{k-1} u_{(n),R}^j \right|^2 = \sum_{j=0}^{k-1} |u_{(n),R}^j|^2.$$

Hence by the triangle inequality and (5.22), we obtain for large $n$

$$\|(|x|^{-\gamma} \ast |u_{(n),R}^<|^2) u_{(n),R}^< - \sum_{j<k} (|x|^{-\gamma} \ast |u_{(n),R}^j|^2) u_{(n),R}^j\|_{ST^*(I)}$$

$$\leq \|(|x|^{-\gamma} \ast |u_{(n),R}^<|^2) u_{(n),R}^< - (|x|^{-\gamma} \ast |u_{(n),R}^<|^2) u_{(n),R}^<\|_{ST^*(I)}$$

$$+ \sum_{j=0}^{k-1} \|(|x|^{-\gamma} \ast |u_{(n),R}^j|^2) u_{(n),R}^j - (|x|^{-\gamma} \ast |u_{(n),R}^j|^2) u_{(n),R}^j\|_{ST^*(I)}$$

$$+ \sum_{0 \leq j \neq l < k} \|(|x|^{-\gamma} \ast |u_{(n),R}^j|^2) u_{(n),R}^l\|_{ST^*(I)}.$$

therefore it suffices to prove

$$\|(|x|^{-\gamma} \ast |u_{(n),R}^<|^2) u_{(n),R}^< - (|x|^{-\gamma} \ast |u_{(n),R}^<|^2) u_{(n),R}^<\|_{ST^*(I)} \to 0,$$

and

$$\|(|x|^{-\gamma} \ast |u_{(n),R}^j|^2) u_{(n),R}^l\|_{ST^*(I)} \to 0, \quad j \neq l.$$

(5.24)
For (5.23), using the triangle inequality, we have

\[
\text{LHS of (5.23)} \lesssim \left\| \left( |x|^{-\gamma} \ast |u_n^k|^2 \right)(u_n^k - u_n^k, R) \right\|_{ST^\ast(I)} \\
+ \left\| \left( |x|^{-\gamma} \ast |u_n^k|^2 - |u_n^k, R|^2 \right) u_n^k, R \right\|_{ST^\ast(I)} \\
\triangleq I_1 + I_2.
\]

For \( I_1 \), by (2.20), we get

\[
I_1 \lesssim \left\| v \right\|_{[K](I)} \left\| u_n^k \right\|_{[K](I)} \left\| u_n^k \right\|_{L^2(I)}^{2(d-2)} + \left\| u_n^k \right\|_{[K](I)} \left\| u_n^k \right\|_{L^2(I)}^{2(d-3)} \left\| v \right\|_{L^1(I)} \left\| v \right\|_{L^2(I)}^{2(d-3)} \\
+ \left\| v \right\|_{[W](I)} \left\| u_n^k \right\|_{[W](I)} \left\| u_n^k \right\|_{[W](I)} \left\| u_n^k \right\|_{[W](I)} \left\| u_n^k \right\|_{[W](I)} \left\| u_n^k \right\|_{[W](I)} \left\| v \right\|_{L^2(I)}^{2(d-3)} \\
\rightarrow 0, \quad \text{as} \quad R \to \infty,
\]

(5.25)

where \( v = u_n^k - u_n^k, R \). Similarly, \( I_2 \to 0 \) as \( R \to \infty \).

Now we turn to prove (5.24). By the triangle inequality, we get

\[
\text{LHS of (5.24)} \lesssim (V_1 * |u_n^j, R|^2) u_n^j, R \right\|_{[K]^\ast(I)} + (V_2 * |u_n^j, R|^2) u_n^j, R \right\|_{[K]^\ast(I)} \\
\triangleq I_3 + I_4,
\]

(5.26)

where \( V_1 = |x|^{-\gamma} \chi_{|x| \leq R} \), \( V_2 = |x|^{-\gamma} \chi_{|x| \geq R} \).

For \( I_3 \), by the compact support of the function \( u_n^j, R \) and (2.16), we obtain

\[
I_3 = \left\| (V_1 * |u_n^j, R(x)|^2) \chi_{|x| \leq R} u_n^j, R(x - (x_n^j - x_n^j)) \right\|_{[W]^\ast(I)} \\
\lesssim \left\| \chi_{|x| \leq R} u_n^j, R(x - (x_n^j - x_n^j)) \right\|_{[W]^\ast(I)} \left\| u_n^j, R(x) \right\|_{L^2(I)}^{2(d-2)} \left\| u_n^j, R(x) \right\|_{L^2(I)}^{2(d-3)} \left\| v \right\|_{L^1(I)} \left\| v \right\|_{L^2(I)}^{2(d-3)} \\
+ \left\| u_n^j, R(x) \right\|_{L^2(I)} \left\| u_n^j, R(x) \right\|_{L^2(I)} \left\| \chi_{|x| \leq R} u_n^j, R(x - (x_n^j - x_n^j)) \right\|_{L^1(I)}^{d-2} \\
\times \left\| \chi_{|x| \leq R} u_n^j, R(x - (x_n^j - x_n^j)) \right\|_{L^1(I)}^{d-2} \\
\rightarrow 0 \quad \text{as} \quad n \to \infty,
\]

(5.27)

For \( I_4 \), by the same argument as (2.15), and \( \left\| V_2 \right\|_{L^d} = R^{(1-\frac{d}{2})} \), we have

\[
I_4 = \left\| (V_2 * |u_n^j, R|^2) u_n^j, R \right\|_{[K]^\ast(I)} \\
\lesssim R^{(1-\frac{d}{2})} \left\{ \left\| u_n^j, R \right\|_{[K]^\ast(I)} \left\| u_n^j, R \right\|_{L^2(I)}^{d-2} \right\} \\
\rightarrow 0 \quad \text{as} \quad R \to \infty.
\]

(5.28)

This concludes the proof.
After this preliminaries, we now show that \( \vec{u}_{(n)}^{<k} + \vec{w}_{n}^{k} \) is a good approximation for \( \vec{u}_{n} \) provided that each nonlinear profile has finite global Strichartz norm.

**Lemma 5.4.** Assume \( \mu = 1 \). Let \( u_{n} \) be a sequence of local solutions of (1.1) around \( t = 0 \) satisfying \( \lim_{n \to \infty} E(u_{n}, \dot{u}_{n}) < +\infty \). Suppose that in its nonlinear profile decomposition (5.16), every nonlinear profile \( \vec{U}_{\infty}^{j} \) has finite global Strichartz and energy norms, i.e.

\[
\| \Re(\nabla)^{-1} \vec{U}_{\infty}^{j} \|_{ST(R)} + \| \vec{U}_{\infty}^{j} \|_{L_{t}^{\infty}L_{x}^{2}(R)} < \infty.
\]

(5.29)

Then \( u_{n} \) is bounded for large \( n \) in the Strichartz and the energy norms, i.e.

\[
\lim_{n \to \infty} \| u_{n} \|_{ST(R)} + \| \vec{u}_{n} \|_{L_{t}^{\infty}L_{x}^{2}(R)} < +\infty.
\]

(5.30)

Moreover, assume \( \mu = -1 \) and let \( u_{n} \) be a sequence of local solutions of (1.1) around \( t = 0 \) in \( A^{+} \) satisfying \( \lim_{n \to \infty} E(u_{n}, \dot{u}_{n}) < E(W, 0) \). Then the above results also hold true.

**Proof.** We only need to verify the condition of Lemma 2.5. For this purpose, we always use the fact that \( u_{(n)}^{<k} + w_{n}^{k} \) satisfies that

\[
(\partial_{tt} - 1 - \Delta)(u_{(n)}^{<k} + w_{n}^{k}) = -f(u_{(n)}^{<k} + w_{n}^{k}) + (f(u_{(n)}^{<k}) + (f(u_{(n)}^{<k}) - f(u_{(n)}^{<k})) + (f(u_{(n)}^{<k}) - \sum_{j=0}^{k-1} f(u_{(n)}^{j})).
\]

First, by the definition of the nonlinear concentrating wave \( u_{(n)}^{j} \) and Remark 5.2 we have

\[
\| (\vec{u}_{(n)}^{<k}(0) + \vec{w}_{n}^{k}(0)) - \vec{u}_{n}(0) \|_{L_{x}^{2}} \leq \sum_{j=0}^{k-1} \| \vec{u}_{(n)}^{j}(0) - \vec{u}_{n}(0) \|_{L_{x}^{2}} \to 0,
\]

as \( n \to +\infty \).

Next, by the linear profile decomposition in Lemma 5.1 we get

\[
\| \vec{u}_{n}(0) \|_{L_{x}^{2}}^{2} = \| \vec{v}_{n}(0) \|_{L_{x}^{2}}^{2} \geq \sum_{j=0}^{k-1} \| \vec{v}_{n}^{j}(0) \|_{L_{x}^{2}}^{2} + o_{n}(1) = \sum_{j=0}^{k-1} \| \vec{u}_{(n)}^{j}(0) \|_{L_{x}^{2}}^{2} + o_{n}(1),
\]

(5.31)

Hence except for a finite set \( J \subset \mathbb{N} \), the energy of \( u_{(n)}^{j} \) with \( j \notin J \) is smaller than the iteration threshold (the small data scattering in Lemma 2.1), and so

\[
\| u_{(n)}^{j} \|_{ST(R)} \lesssim \| \vec{u}_{(n)}^{j}(0) \|_{L_{x}^{2}}, \quad j \notin J.
\]

This together (5.18), (5.19), (5.29) and (5.31) yields that for any finite interval \( I \)

\[
\sup_{k} \lim_{n \to \infty} \| u_{(n)}^{<k} \|_{ST(I)} \lesssim \sum_{j \in J} \| u_{(n)}^{j} \|_{ST(R)} + \sum_{j \notin J} \| u_{(n)}^{j} \|_{ST(R)} \lesssim \sum_{j \in J} \| \Re(\nabla)^{-1} \vec{U}_{\infty}^{j} \|_{ST(R)} + \lim_{n \to \infty} \| \vec{u}_{n}(0) \|_{L_{x}^{2}} < +\infty.
\]

(5.32)
This together with the Strichartz estimate for $w^n_k$ implies that

$$\sup_{k} \lim_{n \to \infty} \| u^{<k}_n + w_n^k \|_{ST(I)} < +\infty.$$ 

At last, by Lemma 5.16 and Lemma 5.3, we have

$$\| f(u^{<k}_n + w_n^k) - f(u^{<k}_n) \|_{ST^*(I)} \to 0,$$

and

$$\| f(u^{<k}_n) - \sum_{j=0}^{k-1} f(w^j_n) \|_{ST^*(I)} \to 0,$$

as $n \to +\infty$. Therefore, by Lemma 2.5 we can obtain the desired result.

\[ \square \]

### 6 Concentration Compactness

By the profile decomposition in the previous section and the stability theory, we argue in this section that if the scattering result does not hold, then there must exist a minimal energy solution with some good compactness properties. This is the object of the following proposition.

**Proposition 6.1.** Let $\mu = 1$. Suppose that $E^+_{\max} < +\infty$. Then there exists a global solution $u_c$ of (1.1) satisfying

$$E(u_c) = E_{\max}, \quad \| u_c \|_{ST(R)} = +\infty. \quad (6.1)$$

Moreover, there exists $c(t) : \mathbb{R} \to \mathbb{R}^d$, such that $K = \{(u_c, \dot{u}_c)(t, x - c(t)) \mid t \in \mathbb{R}^+\}$ is precompact in $H^1 \times L^2$. Besides, one can assume that $c(t)$ is $C^1$ and satisfies

$$|\dot{c}(t)| \lesssim u_c(1) \quad (6.2)$$

uniformly in $t$.

Furthermore, let $\mu = -1$, and suppose that $E^-_{\max} < E(W, 0)$. Then the above results also hold true and $(u_c, \dot{u}_c) \in A^+$. 

**Proof.** By the definition of $E_{\max}$, we can choose a sequence $\{u_n(t)\}$ such that

$$E(u_n, \dot{u}_n) \to E^+_{\max}, \text{ and } \| u_n \|_{ST(R)} \to \infty, \text{ as } n \to \infty, \quad (defocusing) \quad (6.3)$$

or

$$\left\{ \begin{array}{l}
E(u_n, \dot{u}_n) \to E^-_{\max}, \text{ and } \| u_n \|_{ST(R)} \to \infty, \text{ as } n \to \infty, \\
\| u_n \|_2^2 + \| \nabla u_n \|_2^2 < \| W \|_2^2 + \| \nabla W \|_2^2,
\end{array} \right. \quad (focusing) \quad (6.4)$$

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then \( u_n \) is global by Proposition 2.1 in the defocusing case and by Corollary 3.3 in the focusing case. Now we consider the linear and nonlinear profile decompositions of \( u_n \), using Lemma 5.1.

\[
e^{i(t-n)} \tilde{u}_n(0) = \sum_{j=0}^{k-1} \tilde{v}_n^j + \tilde{w}_n^k, \quad \tilde{v}_n^j = e^{i(t-n)}(x-x_n^j),
\]

\[
u_{(n)}^{c} = \sum_{j=0}^{k-1} u_j(0), \quad \tilde{w}_n^j(t, x) = \tilde{U}_n^j(t-n, x-x_n^j), \quad \|\tilde{v}_n^j(0) - \tilde{w}_n^j(0)\|_{L^2_x} \to 0, \text{ as } n \to \infty.
\]

Lemma 5.4 precludes that all the nonlinear profiles \( \tilde{U}_n^j \) have finite global Strichartz norm. On the other hand, every solution of (1.1) with energy less than \( E_{max} \) has global finite Strichartz norm by the definition of \( E_{max} \). Hence by (5.8), we deduce that there is only one profile, i.e. \( K = 1 \), and so for large \( n \)

\[
\tilde{E}(u_n^0) = E_{max}, \quad \|U_n^0\|_{ST(\mathbb{R})} = \infty, \quad \lim_{n \to \infty} \|\tilde{w}_n\|_{L^\infty_t L_x^2} = 0.
\]

Hence we obtain (6.1) for \( u_c = U_0^0 \), and also there exist a sequence \((t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d \) and \( \phi \in L^2(\mathbb{R}^d) \) such that along some subsequence,

\[
\|\tilde{u}_n(0, x) - e^{-i(t_n-n)} \phi(x-x_n)\|_{L^2_x} \to 0, \quad n \to \infty.
\]

Moreover, since (1.1) is symmetric in \( t \), we may assume that

\[
\|u_c\|_{ST(0, +\infty)} = +\infty.
\]

Now we need only find \( c(t) \) satisfying the right properties. The proof of [37] can be adapted verbatim, but we give a sketch for the sake of completeness. For \((u, v, y) \in H^1 \times L^2 \times \mathbb{R}^d \), we define

\[
E_0(u, v, y, R) = \int_{|x-y| \leq R} (|v(x)|^2 + |\nabla u(x)|^2 + |u(x)|^2) dx,
\]

\[
\lambda(u, v, R) = \sup_{y \in \mathbb{R}^d} E_0(u, v, y, R),
\]

\[
\rho(u, v, \delta) = \inf \{ R : \lambda(u, v, R) > (1 - \delta)E_0(u, v) \}.
\]

We claim that for all fixed \( \delta > 0 \), \( \rho(u_c(t), \dot{u}_c(t), \delta) \) remains bounded. In fact, if this were not true, there would exist a sequence of times \( \{t_n\} \) such that we have for all \( n \) and all \( y \), that \( E_0(u_c(t_n), \dot{u}_c(t_n), y, n) \leq (1 - \delta)E_0(u_c(t_n), u_c(t_n)) \). But the sequence \( \{(u_n(t), \dot{u}_n(t)) = (u_c(t+t_n), \dot{u}_c(t+t_n))\} \) satisfies the hypothesis (6.3) (or (6.4)), hence by (6.7), there exists a sequence \((y_n, t_n) \in \mathbb{R} \times \mathbb{R}^d \) and \( \phi \in L^2(\mathbb{R}^d) \) such that, up to a subsequence,

\[
\|\tilde{u}_n(0, x) - e^{-i(t_n-n)} \phi(x-y_n)\|_{L^2_x} \to 0, \quad n \to \infty.
\]

And so

\[
\|\tilde{u}_c(t_n, x) - e^{-i(t_n-n)} \phi(x-y_n)\|_{L^2_x} \to 0, \quad n \to \infty.
\]
Now we claim that $\{t'_n\}$ is bounded. Indeed, if $t'_n \to -\infty$, then by triangle inequality and Strichartz estimate, we have
\[
\| \langle \nabla \rangle^{-1} e^{it \langle \nabla \rangle} \tilde{u}_c(t_n) \|_{ST(0, \infty)} \leq \| \langle \nabla \rangle^{-1} e^{it \langle \nabla \rangle} (\tilde{u}_c(t_n) - e^{-it'_n \langle \nabla \rangle} \phi(x - x'_n)) \|_{ST(0, \infty)} + \| \langle \nabla \rangle^{-1} e^{i(t-t'_n) \langle \nabla \rangle} \phi(x - x'_n) \|_{ST(0, \infty)} \lesssim \| \tilde{u}_c(t_n, x) - e^{-it'_n \langle \nabla \rangle} \phi(x - x'_n) \|_{L^2_x} + \| \langle \nabla \rangle^{-1} e^{it \langle \nabla \rangle} \phi \|_{ST(-t'_n, \infty)} \to 0, \text{ as } n \to \infty,
\]
so that we can solve (1.1) of $u_c$ for $t > t_n$ with large $n$ globally by iteration with small Strichartz norms, contradicting with (6.8).

If $t'_n \to +\infty$, by the similar argument, we get
\[
\| \langle \nabla \rangle^{-1} e^{it \langle \nabla \rangle} \tilde{u}_c(t_n) \|_{ST(-\infty, 0)} = \| \langle \nabla \rangle^{-1} e^{it \langle \nabla \rangle} \phi \|_{ST(-\infty, -t'_n)} + o(1) \to 0, \text{ as } n \to \infty.
\]
And so by Lemma 2.1 we can solve (2.1) of $u$ for $t < t_n$ for large $n$ with
\[
\| u_c(t) \|_{ST(-\infty, t_n)} \lesssim \| \langle \nabla \rangle^{-1} e^{i(t-t_n) \langle \nabla \rangle} \tilde{u}_c(t_n) \|_{ST(-\infty, t_n)} = \| \langle \nabla \rangle^{-1} e^{it \langle \nabla \rangle} \tilde{u}_c(t_n) \|_{ST(-\infty, 0)}.
\]
This implies $u_c = 0$ by taking the limit, which contradicts with (6.8). Therefore $t'_n$ is bounded, which means that $\{t'_n\}$ is precompact.

And so by (6.10), there exists $(w_0, w_1) \in H^1 \times L^2$, up to a subsequence,
\[
\| (u_c(t_n, x + Y_n), \dot{u}_c(t_n, x + Y_n)) - (w_0, w_1) \|_{H^1 \times L^2} \to 0, \text{ as } n \to \infty. \tag{6.11}
\]
Consequently,
\[
E_0(w_0, w_1, y, R) = \lim_{n \to +\infty} E_0(u_c(t_n, x + Y_n), \dot{u}_c(t_n, x + Y_n), y, R) \leq (1 - \delta)E_0(w_0, w_1)
\]
for all $R$. Thus $E_0(w_0, w_1) \leq (1 - \delta)E_0(w_0, w_1)$, which contradicts the fact that $E(w_0, w_1) = \lim_{n \to +\infty} E(u_c(t_n), \dot{u}_c(t_n)) = E_{max}$. Consequently, there exists an decreasing function $R$ such that $\rho(u_c(t), \dot{u}_c(t), \delta) < R(\delta)$ for all $t \geq 0$.

A similar proof shows that there exists $\kappa(\delta) > 0$ such that for all $t \geq 0$,
\[
\lambda(u_c(t), \dot{u}_c(t), R(\delta)) > \kappa(\delta). \tag{6.12}
\]
We choose $\delta$ to be small such that $\delta < \frac{1}{24}$ and $\sqrt{\delta} < \frac{\kappa(\delta)}{S\lambda_{max}}$, and let $c(t)$ be such that
\[
\lambda(u_c(t), \dot{u}_c(t), R(\delta)) = E_0(u_c(t), \dot{u}_c(t), -c(t), R(\delta)).
\]
We claim that the set $K = \{(u_c, \dot{u}_c)(t, x - c(t)) \mid t \in \mathbb{R}^+\}$ is precompact in $H^1 \times L^2$. Suppose it were not true, then there would exist $\varepsilon > 0$ and a sequence of times $t_i$ such that
\[
E_0(u_c(t_i, x - c(t_i)) - u_c(t_j, x - c(t_j)), \dot{u}_c(t_i, x - c(t_i)) - \dot{u}_c(t_j, x - c(t_j))) > \varepsilon \tag{6.13}
\]
for all $i \neq j$. Using the same argument as (6.11), we obtain that there exists a sequence $\{Y_k\}_k$ and $(w_0, w_1) \in H^1 \times L^2$ such that, up to a subsequence,
\[
(U(t_i), \dot{U}(t_i)) = (u_c(t_i, x - (y(t_i) - Y_i)), \dot{u}_c(t_i, x - (y(t_i) - Y_i))) \to (w_0, w_1) \text{ in } H^1 \times L^2,
\]
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as $i \to \infty$. In particular, $(U(t_i), U_t(t_i))$ is a Cauchy sequence. Let $i_0$ be such that for all $j \geq i_0$, there holds that

$$E_0(U(t_i) - U(t_j), U_t(t_i) - U_t(t_j)) < \frac{\kappa(\delta)}{4}, \quad (6.14)$$

and suppose that there exists a subsequence such that $|Y_k - Y_{i_0}| \to +\infty$ as $k \to +\infty$. Then, for $|Y_k - Y_{i_0}| > 2R(\delta)$, we have

$$E_0(U(t_j) - U(t_{i_0}), U_t(t_j) - U_t(t_{i_0}))$$

$$= -2 \int_{\mathbb{R}^d} \left( U(t_{i_0})U_t(t_j) + \nabla U(t_j) \nabla U(t_{i_0}) + U(t_j)U(t_{i_0}) \right)$$

$$\geq 2\kappa(\delta) - 2 \int_{|x - Y_j| < R(\delta)} \left( U(t_{i_0})U_t(t_j) + \nabla U(t_j) \nabla U(t_{i_0}) + U(t_j)U(t_{i_0}) \right)$$

$$\geq 2\kappa(\delta) - 8\sqrt{\delta E_{\max}} \geq \kappa(\delta),$$

this contradicts (6.14), where we use the fact that $E_0(u_c, \dot{u}_c) \leq 2E(u_c, \dot{u}_c)$ in the third inequality. Thus, the sequence $\{ Y_k \}_{k}$ remains bounded. Therefore, up to a subsequence, we can assume that $Y_k \to Y_s$. This fact implies that

$$(U(t_n, x + Y_s), U_t(t_n, x + Y_s)) = (u_c(t_n, \cdot - Y(t_n) - (Y_n - Y_s)), \dot{u}_c(t_n, \cdot - Y(t_n) - (Y_n - Y_s)))$$

is a Cauchy sequence, which contradicts (6.13). And so the set $K = \{(u_c, \dot{u}_c)(t, x - c(t)) \mid t \in \mathbb{R}^+\}$ is precompact in $H^1 \times L^2$.

It only remains to prove (6.2). By the precompactness of $K$, and the continuity of the flow, there exists $s_0 > 0$ such that for every solution $u$ of (1.1) with initial data $(w_0, w_1) \in K$, there holds that

$$E_0(u_c(s), \dot{u}_c(s), 0, 2R(\delta)) \geq (1 - \delta)E_0(u_c(0), \dot{u}_c(0), 0, R(\delta))$$

$$E_0(u_c(s), \dot{u}_c(s)) \leq (1 - \delta)^{-1}E_0(u_c(0), \dot{u}_c(0))$$

for every time $s$ such that $|s| \leq s_0$. In particular,

$$E_0(u_c(s), \dot{u}_c(s), 0, 2R(\delta)) \geq (1 - \delta)^3E_0(u_c(s), \dot{u}_c(s)).$$

This implies that $E_0(u_c(s), \dot{u}_c(s), 0, R(\delta)) \geq (1 - \delta)^3E_0(u_c(s), \dot{u}_c(s))$ when $|Y| > 3R(\delta)$. Consequently, for all $t \geq 0$, for all sufficiently small $s \leq s_0$, there holds that $|c(t) - c(t + s)| \leq 6R(\delta)$. Now, let $t_j = js_0$ for $j \in \mathbb{N}$, and let $\tilde{c}(t)$ be a smooth function such that $\tilde{c}(t_j) = c(t_j)$, and $|\tilde{c}'(t)| \leq 8R(\delta)s_0^{-1}$. Then $|c(t) - \tilde{c}(t)| \leq 14R(\delta) \lesssim u_c 1$, hence $\{(u_c(t, x - \tilde{c}(t)), \dot{u}_c(t, x - \tilde{c}(t)) \mid t \in \mathbb{R}^+\}$ also is precompact in $H^1 \times L^2$ and replacing $c(t)$ by $\tilde{c}(t)$, we obtain (6.2). This concludes the proof. \qed

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Remark 6.1. In particular, for the radial data as in [15, 16, 29], we have

\[ c \]

Using the precompactness of \( K \), there holds that

\[ \text{E}(u(t,x)) \leq \text{E}(u(t,y)) \]

then for any \( \eta > 0 \), there exists \( R(\eta) > 0 \) such that

\[ E_{R(\eta),c(t)} \leq \eta E(u, \dot{u}), \text{ for any } t > 0. \]

Corollary 6.2. Let \( u \) be a nonlinear strong solution of (1.1) such that the set \( K \) defined in Proposition [2.1] is precompact in \( H^1 \times L^2 \), and \( E(u, \dot{u}) \neq 0 \). Then there exists a constant \( \beta = \beta(\tau) > 0 \) such that, for all time \( t \), there holds that

\[ \int_t^{t+\tau} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^\gamma + 2} |u(s, x)|^2 |u(s, y)|^2 \text{d}x \text{d}y \text{d}s \geq \beta, \quad (6.15) \]

\[ \int_t^{t+\tau} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^\gamma} |u(s, x)|^2 |u(s, y)|^2 \text{d}x \text{d}y \text{d}s \geq \beta. \quad (6.16) \]

In particular, there holds that \( \int_0^\tau \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^\gamma + 2} |u(t, x)|^2 |u(t, y)|^2 \text{d}x \text{d}y \text{d}t \geq t. \)

Proof. If (6.15) were not true, then there exists \( \tau > 0 \) and a sequence \( \{t_k\}_k \) such that

\[ \int_t^{t+\tau} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^\gamma + 2} |u(t_k, x)|^2 |u(t_k, y)|^2 \text{d}x \text{d}y \text{d}s < \frac{1}{k}. \quad (6.17) \]

Using the precompactness of \( K \), we can extract a subsequence and assume that \( (u(t_k, \cdot - c(t_k)), u(t_k, \cdot - c(t_k))) \rightarrow (U_0, U_1) \) in \( H^1 \times L^2 \). Let \( U \) be the solution of (1.1) with initial data \( (U_0, U_1) \), then \( E(U, \dot{U}) = E(u, \dot{u}) \neq 0 \). Meanwhile, by (6.17) we have

\[ \int_0^\tau \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^\gamma + 2} |U(t, x)|^2 |U(t, y)|^2 \text{d}x \text{d}y \text{d}t = 0. \]

Consequently, we have \( U(t) = 0 \) a.e. for all \( t \in (0, \tau) \), which contradicts with \( E(U_0, U_1) = E(U, \dot{U}) = E(u, \dot{u}) \neq 0 \). Hence, we finish the proof.

7 Extinction of the critical element

In this section, we prove that the critical solution constructed in Section 6 does not exist, thus ensuring that \( E_{\text{max}} = +\infty \), and \( E_{\text{max}} = E(W, 0) \). This implies Theorem 1.1 and Theorem 1.2.
7.1 The defocusing case: $\mu = 1$.

**Proposition 7.1.** Assume $d \geq 3$, $\mu = 1$, and $2 < \gamma < \min\{4, d\}$, then $E_{\max}^+ = +\infty$.

**Proof.** We use a Virial-type estimate in a direction orthogonal to the Momentum vector. Up to relabeling the coordinates, we might assume that $\text{Mom}(u)$ is parallel to the first coordinate. Thus we have

$$\int_{\mathbb{R}^d} u_t(t, x) \partial_j u(t, x) dx = 0, \quad \forall j \geq 2. \quad (7.1)$$

Let $\phi_R(x) = \phi(x/R)$ where $\phi(x)$ is a nonnegative smooth radial function such that $\text{supp}\phi \subseteq B(0, 2)$ and $\phi \equiv 1$ in $B(0, 1)$. We define the Virial action

$$I(t) = \int_{\mathbb{R}^d} z_2 \phi_R(z) \partial_2 u(t, x) u_t(t, x) dx,$$

where $z = x - c(t)$ and $z_2$ denotes the second component of $z \in \mathbb{R}^d$. Integrating by parts we get by (7.1)

$$\partial_t I(t) = \int_{\mathbb{R}^d} \partial_t(z_2 \phi_R(z)) \partial_2 u(t, x) u_t(t, x) dx + \frac{1}{2} \int_{\mathbb{R}^d} z_2 \phi_R(z) \partial_2 (u_t(t, x))^2 dx$$

$$+ \int_{\mathbb{R}^d} z_2 \phi_R(z) \partial_2 u(t, x) (\Delta u - u - (|x|^{-\gamma} * |u|^2)u) dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} (-|u_t|^2 + |u|^2 + |\nabla u|^2 + (|x|^{-\gamma} * |u|^2)|u|^2) dx - \int_{\mathbb{R}^d} \partial_2 u^2 dx$$

$$+ \frac{\gamma}{2} \int_{\mathbb{R}^d} z_2 \phi_R(z) |u|^2 (\frac{x_2}{|x|^\gamma + 2} * |u|^2)) dx$$

$$+ \int_{|z| \geq R} \mathcal{O}_1(u) dx,$$

where

$$\mathcal{O}_1(u) = \frac{1}{2} \frac{z_2}{R} \phi_R - (1 - \phi_R(x))\left[ -|u_t|^2 + |u|^2 + |\nabla u|^2 + (|x|^{-\gamma} * |u|^2)|u|^2 \right]$$

$$- (c'(t) \cdot \nabla \phi_R) \frac{z_2}{R} \partial_2 u u_t - \frac{c_2(t)}{R} (1 - \phi_R(z)) \partial_2 u u_t - (\nabla \phi_R \cdot \nabla u) z_2 \partial_2 u,$$

is bounded by a constant multiple of $(|u|^2 + |\nabla u|^2 + |\partial_t u|^2)$ and is supported on the set $|z| \geq R$. Besides, we define the equi-partition of energy action

$$J(t) = \int_{\mathbb{R}^d} \phi_R(z) u(t, x) u_t(t, x) dx.$$

Then

$$\partial_t J(t) = \int_{\mathbb{R}^d} \left(|u_t|^2 - |u|^2 - |\nabla u|^2 - (|x|^{-\gamma} * |u|^2)|u|^2\right) dx + \int_{|z| \geq R} \mathcal{O}_2(u) dx, \quad (7.2)$$

where

$$\mathcal{O}_2(u) = (1 - \phi_R(z))\left[ |u_t|^2 - |u|^2 - |\nabla u|^2 - (|x|^{-\gamma} * |u|^2)|u|^2 \right] + (c'(t) \cdot \nabla \phi_R) \frac{u_t}{R} - \frac{u}{R} \nabla \phi_R \cdot \nabla u,$$

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has the same properties as $O_1(u)$.

Considering $A(t) = I(t) + \frac{1}{2}J(t)$, we get

$$|A(t)| \lesssim RE(u, \dot{u}), \text{ for all time } t,$$

and

$$\partial_t A(t) = -\int_{\mathbb{R}^d} |\partial_2 u|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_R(x - c(t))(x_2 - c_2(t)) \frac{x_2 - y_2}{|x - y|^{\gamma+2}} |u(t, x)|^2 |u(t, y)|^2 dxdy$$

$$- \int_{|z| \geq R} (O_1(u) + \frac{1}{2}O_2(u))dx.$$

And so by symmetrization, $\partial_t A(t)$ can be rewritten as

$$-\partial_t A(t) = \int_{\mathbb{R}^d} |\partial_2 u|^2 dx + \frac{\gamma}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_R(z) \frac{|x_2 - y_2|^2}{|x - y|^{\gamma+2}} |u(t, x)|^2 |u(t, y)|^2 dxdy$$

$$+ \frac{\gamma}{4}I_2 + \int_{|z| \geq R} (O_1(u) + O_2(u))dx,$$

where

$$I_2 = \int_{\mathbb{R}^d \times \mathbb{R}^d} [(x_2 - c_2(t))\phi_R(x - c(t)) - (y_2 - c_2(t))\phi_R(y - c(t)) - (x_2 - y_2)]$$

$$\times \frac{x_2 - y_2}{|x - y|^{\gamma+2}} |u(t, x)|^2 |u(t, y)|^2 dxdy.$$

We will show that $I_2$ constitute only a small fraction of $E(u, u_t)$. First, by Corollary 6.2 we know that if $R$ is sufficient large depending on $u$ and $\eta$, then

$$E_{R,c(t)}(u, u_t) \leq \eta E(u, u_t).$$

Let $\chi$ denote a smooth cutoff to the region $|x - c(t)| \geq \frac{R}{2}$ such that $\nabla \chi$ is bounded by $R^{-1}$ and supported where $|x - c(t)| \sim R$. In the region where $|x - c(t)| \sim |y - c(t)|$, we have

$$|x - c(t)| \sim |y - c(t)| \gtrsim R,$$

since otherwise $I_2$ vanish. Moreover, note that

$$|(x_2 - c_2(t))\phi(x - c(t)) - (y_2 - c_2(t))\phi(y - c(t))| \lesssim |x - y|,$$

we use Hölder inequality to control the contribution to $I_2$ from this regime by

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\chi u(t, x)|^2 |\chi u(t, y)|^2}{|x - y|^\gamma} dxdy \lesssim \|\chi u\|_2^{4-\gamma} \|\nabla (\chi u)\|_2^\gamma \lesssim \eta^2.$$

In the region where $|x - c(t)| \ll |y - c(t)|$, we use the fact that

$$|x - c(t)| \ll |y - c(t)| \sim |x - y| \text{ and } |y - c(t)| \gtrsim R$$

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to estimate the contribution from this regime by
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^{\gamma + 2}} |u(t, x)|^2 |u(t, y)|^2 dxdy \lesssim \eta.
\]
The last line follows from the same computation as the first case. Finally, since the remaining region \(|y - c(t)| \ll |x - c(t)|\) can be estimated in the same way, we conclude that
\[
I_2 \lesssim \eta.
\]

Chosen \(\eta\) sufficiently small depending on \(u\) and \(R\) sufficiently large depending on \(u\) and \(\eta\), we obtain
\[
-\partial_t A(t) \geq \frac{4}{\gamma} \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_R(z) \frac{|x_2 - y_2|^2}{|x - y|^{\gamma + 2}} |u(t, x)|^2 |u(t, y)|^2 dxdy - \eta E(u, u_t). \tag{7.5}
\]
If \(E^+_{\max} < \infty\), integrating \((7.5)\) from \(0\) to \(T > 0\) and using Corollary \(6.2\) we get that there exists \(\alpha = \alpha(1, u) > 0\) such that
\[
\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^{\gamma + 2}} |u(s, x)|^2 |u(s, y)|^2 dxdyds \geq \alpha T,
\]
for all \(T > 1\). Thus \(-A(t) \gtrsim T\) for large \(T\), which contradicts with \((7.3)\). Hence we have \(E^+_{\max} = +\infty\), this concludes the proof of Proposition \(7.1\). \(\square\)

7.2 The focusing case: \(\mu = -1\)

Proposition 7.2. Assume \(d \geq 3\), \(\mu = -1\), and \(2 < \gamma < \min\{4, d\}\), then \(E^-_{\max} = E(W, 0)\).

Proof. Let \(\phi_R(x) = \phi(x/R)\) where \(\phi(x)\) is a nonnegative smooth radial function such that \(\text{supp} \phi \subseteq B(0, 2)\) and \(\phi = 1\) in \(B(0, 1)\). We define the Virial action
\[
I(t) = \int_{\mathbb{R}^d} \phi_R(x) x \cdot \nabla u(t, x) u_t(t, x) dx.
\]
Using \((1.1)\), and integrating by parts we get
\[
\partial_t I(t) = \frac{1}{2} \int_{\mathbb{R}^d} \phi_R(x) x \cdot \nabla (u_t(x, t))^2 dx + \int_{\mathbb{R}^d} \phi_R(x) x \cdot \nabla u(\Delta u - u + (|x|^{-\gamma} * |u|^2) u) dx
\]
\[
= \frac{d}{2} \int_{\mathbb{R}^d} (-|u_t|^2 + |u|^2 + |\nabla u|^2 - (|x|^{-\gamma} * |u|^2)|u|^2) dx - \int_{\mathbb{R}^d} |\nabla u|^2 dx \tag{7.6}
\]
\[
+ \frac{\gamma}{2} \int_{\mathbb{R}^d} \phi_R(x) |u|^2 x \cdot \bigg( \frac{x}{|x|^{\gamma + 2}} * |u|^2 \bigg) dx + \int_{|x| > R} \mathcal{O}_1(u) dx \tag{7.7}
\]
where
\[
\mathcal{O}_1(u) = \bigg[ \frac{1}{2} x \cdot \nabla \phi - \frac{d}{2} (1 - \phi_R(x)) \bigg] \bigg[ -|u_t|^2 + |u|^2 + |\nabla u|^2 - (|x|^{-\gamma} * |u|^2)|u|^2 \bigg]
\]
\[
+ \frac{1}{2} \sum_{j=1}^d \frac{x_j}{R} \partial_j \phi_R |\partial_j u|^2 - (\nabla \phi_R \cdot \nabla u) \frac{x}{R} \cdot \nabla u,
\]
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is bounded by a constant multiple of $|u|^2 + |\nabla u|^2 + |\partial_t u|^2$ and is supported on the set $|x| \geq R$. Besides, we define the equi-repartition of energy action

$$J(t) = \int_{\mathbb{R}^d} \phi_R(x) u(t, x)u_t(t, x)dx.$$ 

Then

$$\partial_t J(t) = \int_{\mathbb{R}^d} (|u_t|^2 - |u|^2 - |\nabla u|^2 + (|x|^{-\gamma} * |u|^2)) dx + \int_{|x| > R} \mathcal{O}_2(u)dx,$$  

(7.8)

where

$$\mathcal{O}_2(u) = (1 - \phi_R(x)) [ |u_t|^2 - |u|^2 - |\nabla u|^2 - (|x|^{-\gamma} * |u|^2)] + \frac{u}{R} \nabla \phi_R \cdot \nabla u,$$

has the same properties as $\mathcal{O}_1$. Considering $A(t) = I(t) + \frac{1}{2}J(t)$, by the definition of $I(t)$ and $J(t)$, we get that

$$|A(t)| \lesssim RE(u, u),$$  

(7.9)

for all time $t$ and

$$\partial_t A(t) = - \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\gamma}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_R(x) \frac{x \cdot (x - y)}{|x - y|^2} |u(t, x)|^2 |u(t, y)|^2 dxdy$$

$$- \int_{|x| > R} (\mathcal{O}_1(u) + \mathcal{O}_2(u))dx,$$

And so by symmetrization, $\partial_t A(t)$ can be rewritten as

$$-\partial_t A(t) = \int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_R(x) \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^\gamma} dxdy$$

$$+ \int_{|x| > R} (\mathcal{O}_1(u) + \mathcal{O}_2(u))dx$$

$$= K_2(u) + \int_{|x| > R} (\mathcal{O}_1(u) + \mathcal{O}_2(u))dx,$$

Using Proposition 3.1 and Remark 6.1 we obtain

$$-\partial_t A(t) \geq \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^\gamma} dxdy - \eta E(u, u_t).$$  

(7.10)

By Corollary 6.2 we get that there exists $\alpha = \alpha(1, u) > 0$ such that

$$\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(s, x)|^2 |u(s, y)|^2}{|x - y|^\gamma} dxdyds \geq \alpha T,$$

for all $T > 1$. Integrating the inequality (7.10) from 0 to $T > 0$ and taking $\eta$ small, we obtain $-A(t) \gtrsim T$ for large $T$, which contradicts with (7.9).

\[ \square \]

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