Modeling the dynamics of the distribution of company orders at their cost

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Abstract. Sales volume is one of the most important characteristic of business efficiency. In a detailed analysis and planning of sales it is necessary to consider not only lumped, but also distributed indicators. For this purpose, it is proposed to use developed mathematical model of distribution of orders at their cost. The model constructed by stochastic dynamics methods and described by the initial-boundary value problem for a one-dimensional partial differential equation of parabolic type. The paper presents the model rationale and also numerical example made in Comsol Multiphysics.

1. Introduction
To reach success, each company needs to conduct a financial and economic analysis of its activities and make forecasts. The most important characteristics of the company’s effectiveness are the indicators of sales.

In order to estimate the total sales volume, the company’s data for the previous periods are being analyzed. And the future state is predicted using various methods. The most common are the methods of expert estimates, time series and correlation and regression analysis [1-3]. The more advanced methods are based on mathematical models of sales volume [4, 5], including popular models of market demand [6-8].

However, when detailed analysis is conducted, one should also take into account distributed indicators. Especially it is actually the case when the studied indicator does not have structure like normal distribution and replacing it with one average value can lead to erroneous calculations. The advantages of distributed sales indicators are that they can qualitatively and quantitatively characterize the structure of sales. The authors justify the mathematical model of distribution of company orders at their cost taking into account random factors in this article. When constructing this model, elements of the theory of random processes and stochastic differential equations are used, as well as the approach proposed by V T Erofeenko and I S Kozlovskaya to describe the distribution of families ensemble by monetary savings in the work [9].

2. Model formulation
Applying the continuum principle, the authors will assume that \( x = x(t) \) is continuously differentiated function, meaning the cost of the order paid by the customer for the company goods at the moment of time \( t \), i.e. the purchase price. At fixed time moments, the function’s value is like a point on the axis. And it’s movement along the axis (with the speed \( dx/dt \)) reflects the fact that over time cost of
purchases by customers changes. The authors will also assume that orders of a particular customer are characterized as deterministic or random. Then it is possible to describe the dynamics of customer orders by a stochastic differential equation:

\[
dx = F(x,t)dt + dX
\]  

(1)

where \( F(x,t) \) is customer speed of deterministic orders (type of function depends on the specific client), \( dX = X(t + dt) - X(t) \) is differential of stochastic Markov process \( X(t) \), meaning random customer’s orders. Probability density \( \rho(x) = \rho(y,t;x,s) \) of random process \( X(t) \) is defined by functions \( [10] \)

\[
e(y,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{x - \epsilon}^{x + \epsilon} (x - y)\rho(y,t;x,t + \Delta t)dx, \quad b(y,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{x - \epsilon}^{x + \epsilon} (x - y)^2 \rho(y,t;x,t + \Delta t)dx.
\]

Consider a group of \( N_0 \) such customers, that the dynamics of their orders is described by the same equation (1). Let one customer make only one order at a fixed point of time. At the initial moment of time \( N_0 \) points (orders) are distributed along the axis in some way. Over time, the points move along the axis. \( \Delta Q(x,t) \) is a number of points on the segment \([x, x + \Delta x]\) at the time moment \( t \). The authors introduce the density distribution function \( u(x,t) \) of customers at the cost of their orders:

\[
u(x,t) = \lim_{\Delta x \to 0} \frac{\Delta Q(x,t)}{\Delta x}
\]  

(2)

Then \( Q(t) = \int_{x_1}^{x_2} u(x,t)dx \) means the number of orders of amounts within \([x_1, x_2]\) at the time moment \( t \). The change of the points number in the time segment \([t_1, t_2]\) on an arbitrary segment \([x_1, x_2]\) would look like \( \Delta Q_{t_1t_2} = Q(t_2) - Q(t_1) = \int_{x_1}^{x_2} u(x,t_2)dx - \int_{x_1}^{x_2} u(x,t_1)dx = \int_{x_1}^{x_2} u(x,t) \int_{t_1}^{t_2} dt \) dx.

So

\[
\Delta Q_{t_1t_2} = \int_{x_1}^{x_2} \frac{\partial u}{\partial t} dt dx
\]  

(3)

On the other hand, note, the change number of orders \( \Delta Q_{t_1t_2} \) occurs due to deterministic orders or random orders, as well as orders by new customers from other groups. The authors write this fact in the form of a sum

\[
\Delta Q_{t_1t_2} = \Pi_1 + \Pi_2 + \Pi_3
\]  

(4)

where \( \Pi_1 \) is the number of deterministic orders of the amount from \( x_1 \) to \( x_2 \) in the time from \( t_1 \) till \( t_2 \); \( \Pi_2 \) is the number of random orders of the amount from \( x_1 \) to \( x_2 \) in the time from \( t_1 \) till \( t_2 \); \( \Pi_3 \) is the number of orders by new customers from other groups. Negative values \( \Pi_j \) mean a decrease the number of orders of the amount from the segment \([x_1, x_2]\).

Define \( \Pi_1 \). Consider an arbitrary customer who has made a deterministic order for an amount in the neighbourhood of the point \( x_1 \). According to equation (1) point \( x(t) \) moves along the axis \( Ox \) with speed \( F(x_1,t) \). For a period of time \( \Delta t \) it will pass the way \( \Delta S = F(x_1,t)\Delta t \). Then the number of points that fall on the segment \([x_1, x_2]\) through point \( x_1 \) for time \( \Delta t \) is defined by the expression\( M(x_1) = u(x_1,t)\Delta S = u(x_1,t)F(x_1,t)\Delta t \).
Similarly for the point \( x_2 \) we get: \( M(x_2) = -u(x_2,t)F(x_2,t)\Delta t \). The sign "−" means that with a positive \( F(x_2,t) \) there will be a decrease the number of orders of the amount of the segment \([x_1, x_2]\).

Therefore, the number of points that falls on the segment \([x_1, x_2]\) for the time \( \Delta t \) is defined by the expression: \( \Delta M = -\int_{x_1}^{x_2} [u(x_2,t)F(x_2,t) - u(x_1,t)F(x_1,t)] dt \)

Summing up \( \Delta M_i \) over all elementary intervals \( \Delta t_i \), divided into the time segment \([t_1, t_2] \), the authors get the integral sum [9]:

\[
\sum_i \Delta M_i = -\int_{x_1}^{x_2} [u(x_2,t)F(x_2,t) - u(x_1,t)F(x_1,t)] dt \rightarrow -\int_{t_1}^{t_2} [u(x_2,t)F(x_2,t) - u(x_1,t)F(x_1,t)] dt
\]

Thus, the authors come to the formula:

\[
\Pi_1 = -\int_{t_1}^{t_2} \frac{\partial}{\partial x}(uF(x,t)) dx dt
\] (5)

Define \( \Pi_2 \). Consider two time points \( t \) and \( t + \Delta t \). Let’s calculate the number of points that will move due to random orders on the segment \([x_1, x_2]\) from the array \( \Omega(t) = (-\infty < x < x_1) \cup (x_2 < x < \infty) \) at the point time \( t + \Delta t \) starting the moment \( t \). To succeed the authors will divide array \( \Omega(t) \) to elementary segments of length \( \Delta y_i \). At the moment of time \( t \) on the elementary segment \( \Delta y_i \) are \( u(y_i,t)\Delta y_i \) points. Through the time span \( \Delta t \) these points are distributed along the whole axis \( Ox(-\infty < x < \infty) \), with probability density \( \rho(y,t; x, t + \Delta t) \) at moment of time \( t + \Delta t \).

Integral \( \int_{y_i}^{y_2} \rho(y,t; x, t + \Delta t) dy \) means the probability that a point from \( y_i \) falls on a segment \([x_1, x_2]\) at the moment of time \( t + \Delta t \). It means that \( \Delta L_i = \int_{y_i}^{y_2} \rho(y,t; x, t + \Delta t) dy \rho(y_i, t)\Delta y_i \) is the number of points that would move from segment \( \Delta y_i \) to segment \([x_1, x_2]\).

Summing up the values \( \Delta L_i \) over all segments \( \Delta y_i \) of array \( \Omega(t) \), the authors get the integral sum:

\[
\sum_i \Delta L_i = \sum_i \int_{y_i}^{y_2} \rho(y,t; x, t + \Delta t) dy \rho(y_i, t)\Delta y_i \rightarrow \int_{y_i}^{y_2} \int_{\Delta y_i \rightarrow \Omega(t)} \rho(y,t; x, t + \Delta t) dy \Delta y_i
\]

Value \( I_1 \) means amount of points, which falls from the array \( \Omega(t) \) to the segment \([x_1, x_2]\) at the moment \( \Delta t \).

Similarly, the number of points \( I_2 \), which goes from the segment \([x_1, x_2]\) to the array \( \Omega(t) \):

\[
I_2 = \int_{x_1}^{x_2} \int_{\Omega(t)} \rho(y,t; x, t + \Delta t) dy dx
\]

As a result, the increase of random orders over time \( \Delta t \) on the segment \([x_1, x_2]\) is defined by the value:
\[ \Delta I = I_1 - I_2 = \int_{x_1}^{x_2} \int_{\Omega_0} u(y, t) \rho(y, t; x, t + \Delta t) dy \, dx - \int_{x_1}^{x_2} \int_{\Omega_0} u(y, t) \rho(y, t; x, t + \Delta t) dy \, dx \]

After rearrangement of integrals and transformations, we obtain:

\[ \Delta I = \int_{x_1}^{x_2} \int_{-\infty}^{\infty} u(y, t) \rho(y, t; x, t + \Delta t) dy \, dx - u(x, t) \]

Using the formula [4]

\[ \int_{-\infty}^{\infty} u(y) \rho(y, t; x, t + \Delta t) dy - u(x) = \left[ -\frac{\partial}{\partial x} (c(x, t) u(x)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (b(x, t) u(x)) \right] \Delta t + o(\Delta t) \]

The authors come to the ratio:

\[ \Delta I = \int_{x_1}^{x_2} \left[ -\frac{\partial}{\partial x} (cu) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (bu) \right] dx \Delta t. \]

Summing up these integrals over all elementary intervals \( \Delta t', \) which are the parts of divided interval \([t_1, t_2],\) the authors get the integral sum:

\[ \sum_i \Delta I_i = \sum_i \int_{x_{i-1}}^{x_i} \left[ -\frac{\partial}{\partial x} (c(x, t') u(x, t')) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (b(x, t') u(x, t')) \right] dx \Delta t' \xrightarrow{\Delta t' \to 0} \int_{x_1}^{x_2} \left[ -\frac{\partial}{\partial x} (c(x, t) u(x, t)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (b(x, t) u(x, t)) \right] dx \Delta t \]

Define \( \Pi_3 \). An auxiliary function is introduced \( f(x, t) \) that is the number of orders by customers from other groups for a single time interval in the neighbourhoods \( x \) and \( t. \) It's obvious that

\[ \Pi_3 = \int_{t_1}^{t_2} f(x, t) dx \, dt \]

(7)

Substituting (4), (5), (6), (7) in the balance equation (3), the authors obtain the integral identity

\[ \int_{t_1}^{t_2} \frac{\partial u}{\partial t} dx \, dt = \int_{t_1}^{t_2} \left[ \int_{x_1}^{x_2} -\frac{\partial}{\partial x} (c u) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (bu) \right] dx \Delta t + \int_{t_1}^{t_2} f \, dx \, dt \]

Omitting integrals by virtue of the average theorem and arbitrariness of integration intervals \([t_1, t_2],\) \([x_1, x_2],\) the authors obtain a parabolic partial differential equation

\[ \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} (c + F) u + \frac{1}{2} \frac{\partial^2}{\partial x^2} (bu) + f \]

(8)

The equation (8) would be called the equation of the distribution of orders of company customers group at their cost. Adding to the parabolic equation (8) the initial and boundary conditions of the form

\[ u(x, 0) = u_{\text{start}}(x), x \in [L_1, L_2] \]

(9)

\[ u(L_1, t) = u_{\text{left}}, u(L_2, t) = u_{\text{right}}, t \geq 0, \]

(10)

The authors get an initial-boundary problem, that modeling the dynamics of the distribution of company’s orders at their cost.

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3. Numerical example
The authors would consider the case when all customers of the company belong to the same group. Then the function $f(x,t)$ would mean the number of orders by new customers. The authors give an illustrative example of model calculation. Define the modeling area $x \in [l;100]$ conditional monetary units, $t \in [0;36]$ months. Let the company has data on the characteristic distribution of orders at their cost for the previous period in the segment $[l;100]$. Thus, the function $u_{\text{start}}(x)$ is defined and its graph is shown in Figure 1.

Note that in direct problems of mathematical physics, researchers seek to find explicitly or approximate functions that describe physical phenomena or economic processes. The environment properties, i.e. from a mathematical point of view the coefficients of the equation are assumed to be known. However, often environment properties are unknown functions. This means that it is necessary to formulate and solve inverse problems in which it is required to find the coefficients of the equations.

Suppose that the random component is described by the Wiener process $(c = 0, \ b = 1)$. Thus two of the three unknown coefficients of the model are fixed. Then based on the known initial distribution $u_{\text{start}}(x)$ the third coefficient (the function $F$) have been calculated as a solving of the inverse problem with using special tools and optimization algorithms of Comsol Multiphysics. The graph of the function $F(x)$ is shown in Figure 2.

The obtained parameters have been substituted into the model and the distribution of orders at the cost over the next 3 years has been calculated. The results are shown in Figure 3. As it can be seen from the
graph, the number of orders worth about 50 conventional money units will increase. This trend will lead to a unimodality of the sales structure in the stationary case, which is illustrated in Figure 4.

4. Conclusion
The paper presents the mathematical model of the distribution of company orders at their cost. It is formulated with applying stochastic dynamics methods and described by an initial-boundary value problem for a one-dimensional partial differential equation of parabolic type. The example of a numerical implementation of the model was carried out using Comsol Multiphysics. Such calculations of this distributed mathematical model can provide more detailed sales analysis, in contrast to lumped models of sales volume.

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