On the Zero-freeness of Tall Multirate Linear Systems

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Abstract—In this paper, tall discrete-time linear systems with multirate outputs are studied. In particular, we focus on their
zeros. In systems and control literature zeros of multirate systems are defined as those of their corresponding time-invariant blocked
systems. Hence, the zeros of tall blocked systems resulting from blocking of linear systems with multirate outputs are mainly
explored in this work. We specifically investigate zeros of tall blocked systems formed by blocking tall multirate linear systems
with generic parameter matrices. It is demonstrated that tall blocked systems generically have no finite nonzero zeros; however,
they may have zeros at the origin or at infinity depending on the choice of blocking delay and the input, state and output
dimensions.

I. INTRODUCTION

Multirate linear systems have been studied in different subdisciplines, such as sampled-data control [27], signal processing [28] and econometric modeling [8] for some decades. Especially, with recent theoretical advances in the field of econometric modeling (see e.g. [15]), multirate linear systems analysis has found more potential applications in ‘mixed frequency’ data analysis; mixed frequency data refers to the fact that in econometric modeling, it is common to have some data which are collected monthly, while other data may be obtained quarterly or even annually [26], [8] (in most advanced countries, the number of such time series generally easily exceeds 100). The authors of the present paper have also become interested in multirate linear systems analysis while studying generalized dynamic factor models (GDFMs) [15], which are a recent tool in the field of econometric modeling. In GDFMs, linear dynamic systems driven by white noise are used to model measured high-dimensional time series, and virtually always, such systems have a much larger number of outputs than inputs [24], [12] i.e. the systems are tall (if not very tall). Typical research questions include how such models can be identified, and how they can be used for near-term forecasting.

Tall and very linear multi-rate systems have not been studied in great depth. This paper however does try to formulate some general properties of tall multi-rate systems. Consequently, we do not focus on a particular application problem but rather on a bigger framework which is the system theoretical issues associated with such systems

As first attempts to understand the properties of tall multirate linear systems, the authors of [12] and [13] have considered just the single-rate scenario and shown that the underlying model is generically zero-free. This has the key consequence that identification of the model from measured output data (assuming a white noise input) becomes far simpler than for a normal system, as the system parameters can be identified through linear calculations from the observed data, using a set of equations known as the Yule-Walker equations [17]. A corresponding demonstration till now has been lacking for the multirate case, and the central task of this paper is to address that shortcoming. Specifically, we show that tall multirate linear systems are generically zero-free, apart possibly for zeros at infinity or zero.

While our prime motivation has been to demonstrate a property which implies, as noted above, substantial simplification in the identification or modelling task, we comment that the result may have separate importance from a control design perspective; zeros which are unstable or stable but close to a stability boundary can provide obstructions to the existence of inverses of linear systems and more generally, the design of high performance controllers. The results of this paper suggest that, when one is dealing with a generic system, the controller design may then be easier if one can add extra sensors to make the system have more outputs than inputs, and thereby suppress occurrence of any zeros at all, apart possibly from zeros at zero or infinity.

There exists a large number of works in systems and control literature dealing with multirate linear systems; for example, one can refer to [4], [27], [5], [9], [22], [10], [11] and references listed therein. In order to deal with this type of system, a technique termed blocking or lifting has been developed in systems and control [4] and signal processing [28]. In systems and control, blocking has been largely used to transform linear discrete-time periodic systems to linear time-invariant (LTI) systems, so that analysis and design of the former can be done using the well-developed tools in LTI systems. In particular, in [4] and [3] the notions of poles and zeros of LTI systems have been extended to linear periodic systems. Moreover, the authors of [16] and [5] have defined zeros of multirate linear systems as those of their corresponding blocked systems. However, to the best of our knowledge there are few works on zeros of multirate systems. Among such works we should mention [5], [16], [4], [33], [6].

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References [5] and [16] have explored zeros of blocked systems obtained from blocking of linear periodic systems. The results show that the blocked system has a finite zero if it is obtained from a LTI unblocked system, and the latter has a finite zero, which is a form of sufficiency condition. References [33] and [6] have used different approaches but they have obtained largely similar results. The results in those references show that a tall blocked system has a zero if and only if its associated LTI unblocked system has a zero. Later, in [32] the authors have obtained more general results by relaxing the assumptions made in [33] and [6] on the normal rank and the structure of the transfer function matrices. While references [33], [6] and [32] have mainly considered LTI unblocked systems, as opposed to multirate systems, in [31] zeros of a class of unblocked multirate linear systems have been explored. It has been shown that the tall blocked systems obtained from blocking of multirate systems with generic parameter matrices have no finite nonzero zeros. Finally, some of the results in [33] and [31] are reviewed in [2].

The main objective of this paper is to investigate zeros of tall blocked systems resulting from blocking of a multirate linear system with a generic choice of parameter matrices appearing in a state-variable description of the system. The results of this study reveal what kind of zeros tall blocked systems have for almost all choices of parameter matrices. Note that there are already some results in the literature dealing with zeros of unblocked tall LTI systems with generic parameter matrices [1], [13], [30] and [19]. However, there has been a gap in the literature regarding the study of blocked systems formed by blocking of multirate linear system; this process result in a time-invariant system with relations among the entries of the state-variable matrices of the blocked system, i.e. so that the state-variable matrices are not fully generic. As mentioned earlier, reference [31] has partially addressed this problem, showing that tall blocked systems generically have no finite nonzero zeros. However, zeros at the origin and zeros at infinity have not been completely studied in [31]. Moreover, [31] has mainly focused on situations where the system matrix associated with the blocked system attains full-column normal rank; perhaps surprisingly, this is a significant restriction, and in this paper, there is no such restriction at all.

More precisely, in this paper, we provide for the first time a complete analysis of zeros for tall blocked systems obtained by blocking multirate linear systems with generic parameter matrices. In particular, as far as zeros at infinity and the origin are concerned, the results of this paper go far beyond the scope of [31]. Here, we show that in general blocked systems may have zeros at the origin or infinity depending on the delay associated with blocking and the dimension of the input, state and output vectors. Moreover, regarding finite nonzero zeros, the current paper improves some deficiencies in results of [31] and shows that tall blocked systems generically have no finite nonzero zeros.

Since the analysis of zeros for tall blocked systems is quite involved, we consider three cases separately, that is, 1) finite nonzero system zeros; 2) system zeros at infinity; and 3) system zeros at zero. The next section of the paper is focused on zeros of tall blocked systems associated with finite nonzero zeros. It is explicitly established that tall blocked systems generically have no finite nonzero zeros. As a byproduct in this section, we also establish results on the generic rank of a system matrix resulting from blocking a multi-rate system. Following this, in Section III zeros of tall blocked systems are examined at \( Z = 0 \) and \( Z = \infty \). It is shown when tall blocked systems can have a zero at \( Z = 0 \) or \( Z = \infty \) and when they are zero-free at those aforementioned points. Finally, Section IV offers concluding remarks.

II. BLOCKED SYSTEMS WITH GENERIC PARAMETERS-FINITE NONZERO ZEROS

In this section, first the formulation of the problem under study is introduced. Then attention is given to the analysis of zeros for tall blocked systems with generic parameters, considering in this section finite nonzero zeros only. In the next section, infinite zeros and zeros at the origin are explored.

The dynamics of an underlying system operating at the highest sample rate are defined by

\[
x(k+1) = Ax(k) + Bu(k) \\
y(k) = Cx(k) + Du(k),
\]

where \( x(k) \in \mathbb{R}^n \) is the state, \( y(k) \in \mathbb{R}^p \) the output, and \( u(k) \in \mathbb{R}^m \) the input. For this system, \( y(k) \) exists for all \( k \), and, separately, can be measured at every time \( k \). However, we are also interested in the situation where though \( y(k) \) exists for all \( k \), not every entry is measured for all \( k \). In particular, we consider the case where \( y(k) \) has components that are observed at different rates. For simplicity, in this paper we consider a case where outputs are provided at two rates which we refer to as the fast rate and the slow rate.

Without loss of generality we decompose \( y(k) \) as \( y(k) = [y^f(k)^T \ y^s(k)^T]^T \) where the fast part \( y^f(k) \in \mathbb{R}^{p_1} \) is observed at all \( k \), and the slow part \( y^s(k) \in \mathbb{R}^{p_2} \) is observed at \( k = 0, N, 2N, \ldots \) also \( p_1 > 0, p_2 > 0 \) and \( p_1 + p_2 = p \). Accordingly, we decompose \( C \) and \( D \) as

\[
C = \begin{bmatrix} C^f \\ C^s \end{bmatrix}, \quad D = \begin{bmatrix} D^f \\ D^s \end{bmatrix}.
\]

Thus, the multirate linear system corresponding to what is measured has the following dynamics:

\[
x(k+1) = Ax(k) + Bu(k) \quad k = 0, 1, 2, \ldots \\
y^f(k) = C^f x(k) + D^f u(k) \quad k = 0, 1, 2, \ldots \\
y^s(k) = C^s x(k) + D^s u(k) \quad k = 0, N, 2N, \ldots
\]

We have actually \( N \) distinct alternative ways to block the system, depending on how the fast signals are grouped with the slow signals. Even though these \( N \) different systems share some common poles, their zeros are not identical in the whole complex plane (see [4], pages 173-179).
We index these systems with an integer \( \tau \in \{1, 2, \ldots, N\} \), and define

\[
U_\tau(k) \triangleq \begin{bmatrix}
    u(k + \tau) \\
    u(k + \tau + 1) \\
    \vdots \\
    u(k + \tau + N - 1)
\end{bmatrix},
\]

\[
Y_\tau(k) \triangleq \begin{bmatrix}
    y^f(k + \tau) \\
    y^f(k + \tau + 1) \\
    \vdots \\
    y^f(k + \tau + N - 1)
\end{bmatrix},
\]

\[
x_\tau(k) \triangleq x(k + \tau),
\]

where \( k = 0, N, 2N, \ldots \).

Then the blocked system \( \sum_\tau \) is defined by

\[
x_{\sum_\tau}(k + N) = A_\tau x_\tau(k) + B_\tau U_\tau(k)
\]

\[
Y_{\sum_\tau}(k) = C_\tau x_\tau(k) + D_\tau U_\tau(k).
\]

where

\[
A_\tau \triangleq A^N,
\]

\[
B_\tau \triangleq \begin{bmatrix}
    A^{N-1}B & A^{N-2}B & \ldots & AB & B
\end{bmatrix},
\]

\[
C_\tau \triangleq \begin{bmatrix}
    C^f & A^f C^f & \ldots & A^{(N-1)f} C^f & A^{(N-\tau)f} C^s T
\end{bmatrix}^T,
\]

\[
D_\tau \triangleq \begin{bmatrix}
    D^f & 0 & \ldots & 0 \\
    C^f B & D^f & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    C^f A^{N-2} B & C^f A^{N-3} B & \ldots & D^f
\end{bmatrix}.
\]

where \( D^*_\tau = [C^s A^{N-\tau-1} B \ldots C^s B \ D^s \ 0 \ldots 0] \) for \( \tau < N \) with \( \tau - 1 \) zero blocks of size \( p_2 \times m \), and when \( \tau = N \), it is given by \( D^*_\tau = [D^s \ 0 \ldots 0] \) where there are \( N - 1 \) zero blocks of size \( p_2 \times m \).

Reference [4] defines zeros of (2) at time \( \tau \) as zeros of its corresponding blocked system \( \sum_\tau \). Hence, in the rest of this section we focus on zeros of the blocked system \( \sum_\tau \forall \tau \in \{1, 2, \ldots, N\} \).

For completeness, we recall the following standard definition [19].

**Definition 2.1:** The finite zeros of the system \( \sum_\tau \) are defined to be the finite values of \( Z \) for which the rank of the following system matrix falls below its normal rank

\[
M_\tau(Z) = \begin{bmatrix}
    Z I - A_\tau & -B_\tau \\
    C_\tau & D_\tau
\end{bmatrix}.
\]

Further, \( V_\tau(Z) = C_\tau (Z I - A_\tau)^{-1} B_\tau + D_\tau, \tau \in \{1, 2, \ldots, N\} \), is said to have an infinite zero when \( n + \text{rank}(D_\tau), \tau \in \{1, 2, \ldots, N\} \), is less than the normal rank of \( M_\tau(Z) \), \( \tau \in \{1, 2, \ldots, N\} \), or equivalently the rank of \( D_\tau \), \( \tau \in \{1, 2, \ldots, N\} \), is less than the normal rank of \( V_\tau(Z) \), \( \tau \in \{1, 2, \ldots, N\} \).

\(^2\text{Zeros of the transfer function defined from (4) are identical with those defined here, provided the quadruple \{A_\tau, B_\tau, C_\tau, D_\tau\} is minimal.}\)

We also provide the following definition for the geometric multiplicity of a zero:

**Definition 2.2:** The geometric multiplicity of a finite zero \( Z_0 \in C \) is normal rank of \( M_\tau(Z_0) \)-rank \( (M_\tau(Z_0)) \). Moreover, the geometric multiplicity of a zero at infinity is normal rank of \( M_\tau(Z) - n - \text{rank} (D_\tau) \).

In this paper we use the term multiplicity to refer to the geometric multiplicity.

We treat zeros of \( \sum_\tau \forall \tau \in \{1, 2, \ldots, N\} \), under a genericity assumption on the matrices of the unblocked system and a tallness assumption. Given that \( p_1, p_2 > 0 \), it proves convenient to consider a partition of the set of possible values of \( p_1 \) and \( p_2 \) defining tallness of the blocked transfer function into two subsets, as follows:

1. \( p_1 > m \).
2. \( p_1 \leq m, Np_1 + p_2 > Nm \).

The first case is common, perhaps even overwhelmingly common, in econometric modeling but the second case is important from a theoretical point of view, and possibly in other applications. Our results are able to cover both cases, but separate treatment is required.

**A. Case \( p_1 > m \)**

According to Definition 2.1, the normal rank for the system matrix of \( \sum_\tau \forall \tau \in \{1, 2, \ldots, N\} \), plays an important role in the analysis of its zeros; thus, we state the following straightforward and preliminary result for the normal rank of \( \sum_\tau \forall \tau \in \{1, 2, \ldots, N\} \).

**Lemma 2.1:** For generic choice of the matrices \( \{A, B, C^s, C^f, D^f, D^s\} \), \( p_1 \geq m \), the system matrix of \( \sum_\tau \forall \tau \in \{1, 2, \ldots, N\} \), has normal rank of \( n + Nm \).

**Proof:** In a generic setting and with \( p_1 \geq m \), the matrix \( D^f \) is of full-column rank. So, due to the structure of \( D_\tau \forall \tau \in \{1, 2, \ldots, N\} \), one can easily conclude that \( D_\tau \forall \tau \in \{1, 2, \ldots, N\} \), is of full-column rank as well. Furthermore,

\[
M_\tau(Z) = \begin{bmatrix}
    Z I - A_\tau & -B_\tau \\
    C_\tau & D_\tau
\end{bmatrix} = \begin{bmatrix}
    I & 0 \\
    C_\tau(Z I - A_\tau)^{-1} I & 0 & C_\tau(Z I - A_\tau)^{-1} B_\tau + D_\tau
\end{bmatrix}
\]

Now observe that \( M_\tau(Z) \) has \( n + Nm \) columns so, \( n + Nm \geq \text{normal rank}(M_\tau(Z)) = \text{normal rank}(Z I - A_\tau) + \text{normal rank}(C_\tau(Z I - A_\tau)^{-1} B_\tau + D_\tau) \geq n + \text{rank}\{\lim_{Z \to \infty}[C_\tau(Z I - A_\tau)^{-1} B_\tau + D_\tau]\} = n + \text{rank}(D_\tau) = n + Nm \). Hence, the normal rank of \( M_\tau(Z) \) equals the number of its columns.

In the situation where \( p_1 > m \), obtaining a result on the absence of infinite nonzero zeros is now rather trivial, since the blocked system contains a subsystem obtained by deleting some outputs which is provably zero-free.

**Theorem 2.1:** For a generic choice of the matrices \( \{A, B, C^s, C^f, D^f, D^s\} \), \( p_1 > m \), the system matrix of \( \sum_\tau \forall \tau \in \{1, 2, \ldots, N\} \), has full-column rank for all finite nonzero \( Z \).

**Proof:** Define a system matrix \( M^f(Z) \) by deleting those rows of \( M_\tau(Z), \tau \in \{1, 2, \ldots, N\} \), which contain any entries
of $C^s$. Thus $M^f(Z)$ is a system matrix associated with a blocked version of the original system with slow outputs completely discarded, i.e. of a time-invariant and not just periodic system. With $p_1 > m$, it was shown in [33] that $M^f(Z)$ is generically of full-column rank for all finite nonzero $Z$. Then it is immediate that $M_r(Z), \tau \in \{1, 2, \ldots, N\}$, will be of full-column rank for all finite nonzero $Z$.

### B. Case $p_1 \leq m, Np_1 + p_2 > Nm$

In the previous subsection the case $p_1 > m$ was treated where only considering the fast outputs alone generically leads to a zero-free blocked system, and the zero-free property is not disturbed by the presence of the further slow outputs. A different way in which the blocked system will be tall arises when $p_1 \leq m$ and $Np_1 + p_2 > Nm$. The main result of this subsection is to show that $\sum_{\tau} \forall \tau \in \{1, 2, \ldots, N\}$ with $p_1 \leq m, Np_1 + p_2 > Nm$ is again generically zero-free. This case is harder to treat; in the conference paper [31], we treated the case under a restrictive assumption, namely that the system matrix of the blocked system had full column rank, and we shall drop this assumption here. That the system matrix of the blocked system may indeed have less than full column rank, so the extension is warranted, is exhibited in the following example.

**Example 2.1:** Consider a tall multi-rate system with $n = 1$, $m = 3, N = 2, p_1 = 1, p_2 = 5$. Let the parameter matrices for the multirate system be $A = a, B = [b_1 b_2 b_3], C^f = c^f, C^s = [c_1^f c_2^f c_3^f c_4^f]^T, D^f = [d_{11}^f d_{12}^f d_{13}^f]$ and

$$D^s = \begin{bmatrix} d_{11}^s & d_{12}^s & d_{13}^s \\ \vdots & \vdots & \vdots \\ d_{51}^s & d_{52}^s & d_{53}^s \end{bmatrix}.$$  

All the scalar parameters are generic. We consider $\tau = 1$ and write the associated system matrix as

$$M_1(Z) = \begin{bmatrix} Z - a_b^2 & -ab_1 & -ab_2 & -ab_3 & -b_1 & -b_2 & -b_3 \\ c^f & d_{1}^f & d_{2}^f & d_{3}^f & 0 & 0 & 0 \\ c^f a & c^f b_1 & c^f b_2 & c^f b_3 & d_{1}^f & d_{2}^f & d_{3}^f \\ c^f a b_1 & c^f a b_2 & c^f a b_3 & d_{1}^f & d_{2}^f & d_{3}^f \\ c^f a b_1 & c^f a b_2 & c^f a b_3 & d_{1}^f & d_{2}^f & d_{3}^f \\ c^f a b_1 & c^f a b_2 & c^f a b_3 & d_{1}^f & d_{2}^f & d_{3}^f \end{bmatrix}.$$ 

It is obvious that first the two rows are (generically) linearly independent. Now, consider rows from 3 to 8; they can be written as a product of matrices $G \bar{F}$, with

$$G \triangleq \begin{bmatrix} c^f & c^f & c^f & d_{1}^f & d_{2}^f & d_{3}^f \\ c_1^f & c_1^f & c_1^f & d_{11}^f & d_{12}^f & d_{13}^f \\ c_2^f & c_2^f & c_2^f & d_{21}^f & d_{22}^f & d_{23}^f \\ c_3^f & c_3^f & c_3^f & d_{31}^f & d_{32}^f & d_{33}^f \\ c_4^f & c_4^f & c_4^f & d_{41}^f & d_{42}^f & d_{43}^f \\ c_5^f & c_5^f & c_5^f & d_{51}^f & d_{52}^f & d_{53}^f \end{bmatrix}$$ 

and $\bar{F} \triangleq \text{diag}(a, b_1, b_2, b_3, I_3)$. The matrix $G$ has rank at most 4; hence, with generic parameter matrices the normal rank of $M_1(Z)$ equals 6 and thus $M_1(Z)$ cannot attain full-column normal rank.

In the next part of this subsection, we first characterise the normal rank of $M_r(z)$; following that, we turn to the question of zero existence.

**Proposition 2.1:** Consider the system $\sum_{\tau} \forall \tau \in \{1, 2, \ldots, N\}$, with $p_1 \leq m, Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then

1) if $n \leq (N - \tau)(m - p_1)$, the matrix $D_\tau$ has rank equal to $(N - 1)p_1 + m + n$;

2) if $n > (N - \tau)(m - p_1)$, the matrix $D_\tau$ has rank equal to $(\tau - 1)p_1 + (N - \tau + 1)m$.

**Proof:** Refer to the appendix for a proof.

Now the general result on the normal rank of $M_r(z)$ is as follows:

**Theorem 2.2:** Consider the system $\sum_{\tau} \forall \tau \in \{1, 2, \ldots, N\}$, with $p_1 < m, Np_1 + p_2 > Nm$ and generic values of the defining matrices $\{A, B, C^f, C^s, D^f, D^s\}$. Then the normal rank of the system matrix $M_r(Z)$ is equal to:

1) $(N - 1)p_1 + m + 2n,$ if $n < (N - 1)(m - p_1);

2) $n + Nm$, if $n \geq (N - 1)(m - p_1)$.

**Proof:** The proof is provided in the appendix.

We return to the main task of studying the zeros of the blocked system. For this purpose, we first review briefly properties of the Kronecker canonical form of a matrix pencil. The system matrix of $\sum_{\tau} \forall \tau \in \{1, 2, \ldots, N\}$ is actually a matrix pencil, and the Kronecker canonical form turns out to be a very useful tool to obtain insight into the zeros of $\tilde{L}$ and the structure of the kernels associated with those zeros.

The main theorem on the Kronecker canonical form of a matrix pencil is obtained from [29].

**Theorem 2.3:** [29] Consider a matrix pencil $zR + S$. Then under the equivalence defined using pre- and postmultiplication by nonsingular constant matrices $\bar{P}$ and $\bar{Q}$, there is a canonical quasi-diagonal form:

$$\bar{P}(zR + S)\bar{Q} = \text{diag}[\tilde{L}_{\mu_1}, \ldots, \tilde{L}_{\mu_r}, \tilde{L}_{\eta_1}, \ldots, \tilde{L}_{\eta_s}, tzN-I, zI-K]$$

where:

1) $L_\mu$ is the $\mu \times (\mu + 1)$ bidiagonal pencil

$$\begin{bmatrix} z & -1 & 0 & \ldots & 0 & 0 \\ 0 & z & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & z & -1 \end{bmatrix}$$

2) $\tilde{L}_\mu$ is the $(\mu + 1) \times \mu$ transposed bidiagonal pencil

$$\begin{bmatrix} -1 & 0 & \ldots & 0 & 0 \\ z & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & z & -1 \\ 0 & 0 & \ldots & 0 & z \end{bmatrix}$$

3) $N$ is a nilpotent Jordan matrix.

4) $K$ is in Jordan canonical form.

Furthermore, the possibility that $\mu = 0$ exists. The associated $L_0$ is deemed to have a column but not a row and $\tilde{L}_0$ is deemed to have a row but not a column, see [29].
The following corollary can be directly derived easily from
the above theorem and provides detail about the vectors in
the null space of the Kronecker canonical form. Because the
matrices \( P \) and \( Q \) are nonsingular, it is trivial to translate
these properties back to an arbitrary matrix pencil, including
a system matrix.

Corollary 2.1: With the same hypothesis as Theorem 2.3
and with \( \Lambda(K) \) denoting the set of eigenvalues of \( K \), the
following hold:

1) For all \( z \notin \Lambda(K) \), the kernel of the Kronecker canonical
form has dimension equal to the number of matrices \( L_\mu \)
appearing in the form; likewise the co-kernel dimension is
determined by the number of matrices \( \bar{L}_\mu \).

2) The vector \([1 \ z \ z^2 \ldots z^\mu]^T\) is the generator of the kernel
of \( L_\mu \), a set of vectors
\[
[0 \ldots 0 \ 1 \ 0 \ 0 \ldots 0 \ 0 \ldots 0 \ 0 \ldots 0]^T
\]
are generators for the kernel of the whole canonical form which depend continuously on \( z \), provided that
\( z \notin \Lambda(K) \); when \( z \in \Lambda(K) \), the vectors form a subset
of a set of generators.

3) When \( z \in \Lambda(K) \) equals an eigenvalue of \( K \), the dimen-
sion of the kernel jumps by the geometric multiplicity of
that eigenvalue, the rank of the pencil drops below the
normal rank by that geometric multiplicity, and there
is an additional vector or vectors in the kernel apart
from those defined in point 2, which are of the form
\([0 \ 0 \ldots v^T]^T\), where \( v \) is an eigenvector of \( K \). Such a vector
is orthogonal to all vectors in the kernel which are a linear combination of the generators listed in
the previous point.

4) Let \( \lambda_0 \in \Lambda(K) \) the associated kernel of the matrix
pencil can be generated by two types of vectors: those
which are the limit of the generators defined by adding
extra zeros to vectors such as \([1 \ \lambda_0 \lambda_0^2 \ldots \lambda_0^\mu]^T\) (these
being the limits of the generators when \( z \neq \lambda_0 \) but
continuously approaches \( \lambda_0 \)), and those obtained by
adjointing zeros to the eigenvector(s) corresponding to
\( \lambda_0 \), the latter set being orthogonal to the former set.

In the rest of this subsection, we explore zeros of \( M_r(Z) \)
\( \forall r \in \{1, 2, \ldots, N\} \). To achieve this, we first focus on the
particular case of \( M_1(Z) \). Later, we introduce the main result
for zeros of \( M_r(Z) \) \( \forall r \in \{1, 2, \ldots, N\} \).

We begin by studying a square matrix generated from
certain rows of \( M_1(Z) \); these are the rows remaining after
excluding certain output variables from consideration. To
this end, we argue first that the first \( n + Np_1 \) rows of \( M_1(Z) \)
are linearly independent. For the submatrix formed by
these rows is the system matrix of the blocked system obtained
by blocking the fast system defined by \([A, B, C^f, D^f]\), and
accordingly has full-row normal rank, since the unblocked
system is generic and square or fat under the condition
\( p_1 \leq m \). Now define the square submatrix of \( M_1(Z) \):

\[
N(Z) \triangleq \begin{bmatrix}
ZI - A_1 & -B_1 \\
C_1 & D_1
\end{bmatrix},
\]

such that normal rank \( (N(Z)) = \) normal rank \( (M_1(Z)) \),
by including the first \( n + Np_1 \) rows of \( M_1(Z) \) and followed by
appropriate other rows of \( M_1(Z) \) to meet the normal rank and
squareness requirements. Note that there exists a permutation
matrix \( P \) such that

\[
PM_1(Z) = \begin{bmatrix}
N(Z) \\
C_2 & D_2
\end{bmatrix},
\]

where \( C_2 \) and \( D_2 \) capture those rows of \( C_1 \) and \( D_1 \) that are
not included in \( C_1 \) and \( D_1 \), respectively.

The zero properties of \( N(Z) \) are studied in the following
proposition (we will build on them to obtain the zero properties
of \( M_1(Z) \)).

Proposition 2.2: Let the matrix \( N(Z) \) be the submatrix of
\( M_1(Z) \) formed via the procedure described. Then for generic
values of the matrices \( A, B, \) etc. with \( p_1 \leq m \) and \( Np_1 + p_2 > Nm \),
for any finite \( Z_0 \) for which the matrix \( N(Z_0) \) has less
rank than its normal rank, its rank is one less than its normal
rank.

Proof: We distinguish two cases, \( p_1 = m \), \( p_1 = m \). In
case \( p_1 = m \), then \( N(Z) \) is the system matrix for the system
obtained by blocking the original system with slow outputs
discarded. As such, the blocked system zeros are precisely
the \( N \)-th powers of the unblocked system zeros \( \{33\} \). For
generic coefficient matrices, the unblocked system will have
\( n \) distinct zeros; then the blocked system will have the same
property. Further, the unblocked system will generically have a
nonsingular direct feedthrough matrix, as will then the blocked
system, so that \( D_1 \) can be assumed to be nonsingular. It follows
then that the zeros of the system with system matrix \( N(Z) \) are
identical with the eigenvalues of \( A_1 - B_1D_1^{-1}C_1 \), which are
then distinct, and since this matrix is \( n \times n \), the eigenvector
associated with each zero will be uniquely defined to within a
scaling constant. It follows easily that there is a unique vector
(to within scaling) in the kernel of \( N(Z_0) \) where \( Z_0 \) is the
zero of the blocked system.

We turn therefore to the case \( p_1 < m \). We study the co-
kernel of \( N(Z_0) \). Let \( Z_1, Z_2, \ldots \), be a sequence of complex
numbers such that \( (a) \ Z_i \to Z_0 \) and \( (b) \ \text{rank}(N(Z_i)) \) equals
the normal rank of \( N(Z) \). From what has been described
earlier using the Kronecker canonical form, we know that the
sequence of co-kernels of \( N(Z_i) \) converges, say to \( \hat{K} \),
with any vector in this limit also in the co-kernel of
\( N(Z_0) \). In addition, since \( N(Z_0) \) has lower rank than the normal rank
of \( N(Z) \), the co-kernel, call it \( \hat{K} \), will be strictly greater than \( \hat{K} \).
Suppose its dimension is at least two more than that of \( \hat{K} \).
We shall show this situation is nongeneric.

Select two vectors \( w_1, w_2 \) which are in \( \hat{K} \) and which are
orthogonal to \( \hat{K} \). Then it is evident that there are two vectors
call them \( v_1, v_2 \), constructed from linear combinations of
\( w_1, w_2 \), which belong to \( \hat{K} \), which are still orthogonal to \( \hat{K} \),
and for which some pair \( r < s \) have 1 and 0 in the \( r \)-th entry
and 0 and 1 in the \( s \)-th entry respectively. Choose \( v_1, v_2 \) so
that firstly, \( s \) is maximal, and secondly, for that \( s \) then \( r \)
is maximal. It is not difficult to see that this means that \( v_1 \) has
zero entries beyond the \( r \)-th and \( v_2 \) has zero entries beyond the
\( s \)-th.

Now again we must consider two cases. Suppose firstly that
\( s \) obeys \( n + Np_1 + 1 \leq s \leq n + Nm \); in forming the product
\( v_2^T N(Z_0) \), the \( s \)-th entry of \( v_2 \) will be multiplying entries of
$N(Z_0)$ defined using $C^*, A, B, D^*$. Consider an entry in the $s$-th row of $N(Z_0)$ and in the last $m$ columns. Such an entry is an entry of $D^*$, and is independent of all other entries in $N(Z_0)$. Suppose this entry of $D^*$ is continuously perturbed by a small amount. Then clearly $v_1$ remains in the co-kernel of $N(Z_0)$ but $v_2$ cannot.

The particular values of $Z$ for which $N(Z)$ has rank less than its normal rank, i.e. the zeros of $N(Z)$, will depend continuously on the perturbation.

Accordingly, with a small enough perturbation, those not equal before perturbation to $Z_0$ will never change to $Z_0$, and it is therefore guaranteed that with a small enough nonzero perturbation, the co-kernel of $N(Z_0)$ is reduced by one in dimension, though never to zero. If the original (before perturbation) co-kernel $\bar{K}$ had dimension greater than two in excess of the dimension of $K$, and the excess after perturbation is still greater than one, the argument can be repeated. Eventually, the co-kernel of $N(Z_0)$ will have an excess dimension over $K$ of 1, i.e. $N(Z_0)$ will have rank one less than the normal rank of $N(Z)$.

Now suppose that $s$ obeys $s \leq n + Np_1$. Then the last $N(m - p_1)$ entries of each of $v_1, v_2$ are zero. Remove these entries to define two linearly independent vectors $\tilde{v}_1, \tilde{v}_2$ of length $n + Np_1$, which evidently satisfy

$$
\begin{bmatrix}
ZI_n - A^N & -A^{N-1}B & \cdots & -B \\
C^f & D^f & 0 \\
\vdots & \vdots & \vdots & \vdots \\
C^fA^{N-1} & C^fA^{N-2}B & \cdots & D^f
\end{bmatrix}
\begin{bmatrix}
\tilde{v}_1^T \\
\tilde{v}_2^T
\end{bmatrix} = 0, \quad i = 1, 2.
$$

(14)

The above equation contains a fat system matrix, corresponding to a blocked version of a fat time-invariant unblocked system. It can be concluded easily form the results provided in \[53\] that for generic values of the underlying matrices, there can be no $Z_0$ for which an equation such as \[14\] can even hold for a single nonzero $\tilde{v}_i$, let alone two linearly independent ones. This ends the proof.

The result of the previous proposition, although restricted to $\tau = 1$, enables us to establish the main result of this section, applicable for any $\tau$. Before we state the main theorem we need to recall the following lemma from \[6\] and \[9\].

**Lemma 2.2:** The pair $(A, B)$ is reachable if and only if the pair $(A_\tau, B_\tau)$, $\forall \tau \in \{1, 2, \ldots, N\}$ is reachable.

**Theorem 2.4:** Consider the system $\sum_{\tau} Z_\tau$, $\forall \tau \in \{1, 2, \ldots, N\}$, with $p_1 \leq m$, and $Np_1 + p_2 > Nm$. Then for generic values of the defining matrices $\{A, B, C^f, D^f, C^*, D^*\}$ the system matrix $M_\tau(Z)$, $\forall \tau \in \{1, 2, \ldots, N\}$, has rank equal to its normal rank for all finite nonzero values of $Z_0$, and accordingly $\sum_{\tau}$ has no finite nonzero zero.

**Proof:** We first focus on the case $\tau = 1$. Now, apart from the $p_2 - N(m - p_1)$ rows of the $C^*, D^*$ which do not enter the matrix $N(Z)$ defined by \[12\], choose generic values for the defining matrices, so that the conclusions of the preceding proposition are valid.

Let $Z_a, Z_b, \ldots$ be the finite set of $Z$ for which $N(Z)$ has less rank than its normal rank (the set may have less than $n$ elements, but never has more), and let $w_a, w_b, \ldots$ be vectors which are in the corresponding kernels (not co-kernels) and orthogonal to the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \ldots$. Now, due to the facts that $M_1(Z)$ and $N(Z)$ have the same normal rank and relation \[13\] holds, it follows that for generic $Z$, the kernels of $M_1(Z)$ and $N(Z)$ are identical (and may be both empty). Hence one can conclude that the subspace in the kernel obtained from the limit of the kernel of $N(Z)$ as $Z \to Z_a, Z_b, \ldots$ etc. coincides with the subspace in the kernel obtained from the limit of the kernel of $M_1(Z)$ as $Z \to Z_a, Z_b, \ldots$.

Now, to obtain a contradiction, we suppose that the system matrix $M_1(Z_0)$ is such that, for $Z_0 \neq 0$, $M_1(Z_0)$ has rank less than its normal rank, i.e. the dimension of its kernel increases. Since the kernel of $M_1(Z_0)$ is a subspace of the kernel of $N(Z_0)$, $Z_0$ must coincide with one of the values of $Z_a, Z_b, \ldots$, and the rank of $M_1(Z_0)$ must be only one less than its normal rank; moreover, there must exist an associated nonzero $\tilde{w}_1$ unique up to a scalar multiplier, in the kernel of $M_1(Z_0)$ which is orthogonal to the limit of the kernel of $M_1(Z)$ as $Z \to Z_0$.

Then $\tilde{w}_1$ is necessarily in the kernel of $N(Z_0)$, orthogonal to the limit of the kernel of $N(Z)$ as $Z \to Z_0$ and thus $\tilde{w}_1$ in fact must coincide to within a nonzero multiplier with one of the vectors $w_a, w_b, \ldots$.

Write this $\tilde{w}_1$ as $w_1 = [x_1^{T} u_1^{T} u_2^{T} \ldots w_N^{T}]^{T}$ and suppose the input sequence $u(i) = u_1$ is applied for $i = 1, 2, \ldots, N$ to the original system, starting in initial state $x_1$ at time 1. Let $y^{f}(1), y^{f}(2), \ldots$ denote the corresponding fast outputs and $y^{s}(N)$ the slow output at time $N$. Break this up into two subvectors, $y^{s1}(N), y^{s2}(N)$, where $y^{s1}(N)$ is associated with those rows of $C^s, D^s$ which are included in $C_1, D_1$ (see \[12\]) and $y^{s2}(N)$ is related with the remaining rows of $C^s$ and $D^s$. We have $N(Z_0)w_1 = [Z_0I_n - A^N - A^{N-1}B - A^{N-2}B \cdots - B]$

$$
\begin{bmatrix}
C^f & D^f & 0 & \cdots & 0 \\
C^fA & C^fB & D^f & \cdots & 0 \\
C^fA^{N-1} & C^fA^{N-2}B & C^fA^{N-3}B & \cdots & D^f \\
C^fA^{N-1} & C^fA^{N-2}B & C^fA^{N-3}B & \cdots & D^s
\end{bmatrix}
\begin{bmatrix}
Z_0x_1 - x(N + 1) \\
y^{f}(1) \\
y^{f}(2) \\
y^{s1}(N) \\
y^{s2}(N)
\end{bmatrix} = 0.
$$

(15)

Now it must be true that $x_1 \neq 0$. For otherwise, we would have $N(Z)w_1 = 0$ for all $Z$, which would violate assumptions. Since also $Z_0 \neq 0$, there must hold $x(N + 1) \neq 0$. Hence there cannot hold both $x(N) = 0$ and $u(N) = 0$. Consequently, we can always find $C^{s2}, D^{s2}$ such that $y^{s2}(N) = C^{s2}x(N) + D^{s2}u(N) \neq 0$, i.e. the slow output value is necessarily nonzero, no matter whether $w_1 = w_a, w_b$, etc. Equivalently, the equation $[C_2, D_2]w_1 = 0$ cannot hold. Hence, if $M_1(Z)$ defines a system with a finite zero and
it is nonzero, this is a nongeneric situation. Hence, \( M_1(Z) \) generically has rank equal to its normal rank for all finite nonzero \( Z \). It now remains to show that this property carries over to all \( M_\tau(Z) \), \( \tau \in \{2, 3, \ldots, N\} \). First, note that the pair \((A, B)\) is generically reachable; then by Lemma 2 the pair \((A_\tau, B_\tau), \forall \tau \in \{1, 2, \ldots, N\}\), is also reachable. Consider \( Z_\tau \in \mathbb{C} - \{0, \infty\} \); if \( Z_\tau \) does not coincide with any eigenvalue of \( A_\tau \), then

\[
\text{rank} (M_\tau(Z_\tau)) = n + \text{rank} (V_\tau(Z_\tau)). \tag{16}
\]

Hence, using the result of Proposition A.1 (see the appendix), it is immediate that \( \text{rank} (M_\tau(Z_\tau)) = \text{rank} (M_{\tau+1}(Z_\tau)) \). If \( Z_\tau \) does coincide with an eigenvalue of \( A_\tau \), then \( \text{rank} (V_\tau(Z_\tau)) \) is ill-defined. However, since zeros of \( M_\tau(Z) \), \( \tau \in \{1, 2, \ldots, N\} \), are invariant under state feedback and the pair \((A_\tau, B_\tau)\) is reachable, one can easily find a state feedback to shift that eigenvalue \([34]\) and then \([16]\) is a well-defined equation and \( \text{rank} (M_\tau(Z_\tau)) = \text{rank} (M_{\tau+1}(Z_\tau)) \). Thus, we can conclude that all \( M_\tau(Z) \), \( \tau \in \{1, 2, \ldots, N\} \) generically have no finite nonzero zeros. This ends the proof.

III. BLOCKED SYSTEMS WITH GENERIC PARAMETERS-ZEROS AT THE ORIGIN AND INFINITY

In the previous section zeros of tall blocked systems with generic parameters for the choice of finite nonzero zeros were studied. In this section zeros of the latter systems are investigated for choices of zeros at zero and infinity. As in the previous section, it is convenient to break up our examination of tall systems into separate cases based on the relation between \( p_1 \) and \( m \).

We first state the following result which, perhaps surprisingly, relates zeros of the system \( \sum_{\tau} \) at infinity to zeros of the system \( \sum_{N+1} \) at the origin and conversely.

**Lemma 3.1:** Consider the family of systems \( \sum_{\tau} \forall \tau \in \{1, 2, \ldots, N\} \), where the defining matrices \( \{A, B, C^f, D^f, C^s, D^s\} \) assume generic values. Then the following fact holds: \( \sum_{\tau} \) has \( \kappa \) zeros at \( Z = 0 \) and \( \mu \) zeros at \( Z = \infty \) if and only if \( \sum_{N+1} \) has \( \mu \) zeros at \( Z = 0 \) and \( \kappa \) zeros at \( Z = \infty \).

**Proof:** Consider a reverse-time description of the system \([2]\), namely

\[
x(k-1) = A^{-1}x(k) - A^{-1}Bu(k-1), \quad k = 1, 2, \ldots
\]

\[
y^f(k-1) = C^f x(k-1) + D^f u(k-1), \quad k = 1, 2, \ldots
\]

\[
y^s(k-1) = C^s x(k-1) + D^s u(k-1), \quad k = 1, N+1, \ldots
\]

and define the following matrices

\[
\tilde{A} \triangleq A^{-1}, \quad \tilde{B} \triangleq -A^{-1}B
\]

\[
\tilde{C}^f \triangleq C^f A^{-1}, \quad \tilde{D}^f \triangleq D^f - C^f A^{-1}B
\]

\[
\tilde{C}^s \triangleq C^s A^{-1}, \quad \tilde{D}^s \triangleq D^s - C^s A^{-1}B
\]

which are still in a generic setting since the genericity of \( \{A, B, C^f, D^f, C^s, D^s\} \) is assumed. Note that the matrix \( A^{-1} \) is well-defined, since \( A \) is generically full rank. Recall the blocking procedure introduced in \([3]\) for a given value of \( \tau \); we can obtain the blocked time-invariant system associated with the system \([17]\) (again a reverse-time system) as

\[
x_\tau(k - N) = A_\tau x_\tau(k) + B_\tau u_\tau(k - N)
\]

\[
Y_\tau(k - N) = C_\tau x_\tau(k) + D_\tau u_\tau(k - N),
\]

where \( k = N, 2N, \ldots \), and

\[
\tilde{A}_\tau \triangleq \tilde{A}^N, \quad \tilde{B}_\tau \triangleq \begin{bmatrix} \tilde{B} & \tilde{A} \tilde{B} & \ldots & \tilde{A}^{N-2} \tilde{B} & \tilde{A}^{N-1} \tilde{B} \end{bmatrix},
\]

\[
\tilde{C}_\tau \triangleq \begin{bmatrix} \tilde{C}^f \tilde{A}^f & \ldots & \tilde{C}^f \tilde{A}^{N-3} \tilde{B} & \tilde{C}^f \tilde{A}^{N-2} \tilde{B} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \tilde{D}^f & \tilde{C}^f \tilde{B} \\
0 & \ldots & 0 & \tilde{D}^f \end{bmatrix}.
\]

In the latter expression, when \( \tau > 1 \) the matrix \( \tilde{D}_\tau^s \) is equal to \([0 \ldots 0 \tilde{D}^s \ldots \tilde{C}^s \tilde{A}^{N-2} \tilde{B}] \), with \( N - \tau \) zero blocks of size \( p_2 \times m \), while, when \( \tau = 1 \), it becomes \([0 \ldots 0 \tilde{D}^s]\). Now let us introduce the \( N \)-step backward operator \( \zeta \), such that \( \zeta(x)(k) = x(k - N) \). Then the transfer function \( V_\tau(\zeta) \triangleq \tilde{C}_\tau(\zeta I - \tilde{A}_\tau)^{-1} \tilde{B}_\tau + \tilde{D}_\tau \) is associated with the blocked system \([19]\). It can be easily checked through simple computations that this transfer function is connected to the transfer function \( V_\tau(Z) \) associated with the system \( \sum_{\tau} \) at the points zero and infinity through the equalities

\[
\tilde{V}_\tau(0) = \lim_{Z \to \infty} V_\tau(Z) \quad \lim_{\zeta \to \infty} \tilde{V}_\tau(\zeta) = V_\tau(0). \tag{21}
\]

Define the system matrix associated with the system \([19]\) as

\[
\tilde{M}_\tau(\zeta) \triangleq \begin{bmatrix} \zeta I - \tilde{A}_\tau & -\tilde{B}_\tau \\
\tilde{C}_\tau & \tilde{D}_\tau \end{bmatrix}. \tag{22}
\]

For our purpose in this paper, we define the following equalities

\[
\text{rank} (\lim_{Z \to \infty} M_\tau(Z)) \triangleq n + \text{rank} (D_\tau)
\]

\[
\text{rank} (\lim_{\zeta \to \infty} \tilde{M}_\tau(\zeta)) \triangleq n + \text{rank} (D_\tau). \tag{23}
\]

Then using the equation \([21]\) one can write

\[
\text{rank} (\lim_{Z \to \infty} M_\tau(Z)) = \text{rank} (\tilde{M}_\tau(0))
\]

\[
\text{rank} (\lim_{\zeta \to \infty} \tilde{M}_\tau(\zeta)) = \text{rank} (M_\tau(0)). \tag{24}
\]

Again, note that the above equalities are well-defined since, due to the genericity assumption of the matrix \( A \), the matrices \( A_\tau \) and \( \tilde{A}_\tau \) do not have any eigenvalues at the origin. Now, by comparing \([7]\) and \([20]\), one can verify that there exist permutation matrices \( Q_1 \) and \( Q_2 \) such that \( Q_1 M_\tau(\zeta)Q_2 = \Psi_\tau(\zeta) \) and \( \Psi_\tau(\zeta) \) is exactly \( M_{N-\tau+1}(Z) \) when \( A, B, C^f, D^f, D^s, \zeta \) are replaced by \( A, B, C^f, D^f, C^s, D^s, Z \), accordingly. Since the parameter matrices \( A, B, C, D \) assume generic values, we have the following equalities

\[
\text{rank} (\lim_{\zeta \to \infty} \tilde{M}_\tau(\zeta)) = \text{rank} (\lim_{Z \to \infty} M_{N-\tau+1}(Z))
\]

\[
\text{rank} (\tilde{M}_\tau(0)) = \text{rank} (M_{N-\tau+1}(0)). \tag{25}
\]
Then, by combining equations (24) and (25) we obtain
\[
\text{rank}(\lim_{Z \to \infty} M_r(Z)) = \text{rank}(M_{N-\tau+1}(0))
\] (26)
and
\[
\text{rank}(M_r(0)) = \text{rank}(\lim_{Z \to \infty} M_{N-\tau+1}(Z)).
\] (27)
Thus, by using equations (26), (27) and the fact that the normal rank of \(M_r(Z)\) does not depend on \(\tau\) (see Proposition A.1 in Appendix), the conclusion of the lemma readily follows. ■

A. Case \(p_1 > m\)

**Theorem 3.1:** For a generic choice of the matrices \(\{A, B, C^s, C^f, D^f, D^s\}\), \(p_1 > m\), the system matrix of \(\sum_{s} \forall\tau \in \{1, 2, \ldots, N\}\) has full-column rank at \(Z = 0\) and \(Z = \infty\), and accordingly \(\sum_{s}\) has no zero at \(Z = 0\) and \(Z = \infty\).

**Proof:** We first consider the zeros at \(Z = 0\). It was shown in (53) that \(M^f(0)\), where the system matrix \(M^f(0)\) can be formed by deleting rows of \(M_1(0)\) which are related to \(C^s\) and \(D^s\), has full-column rank at \(Z = 0\) for generic parameter matrices \(A, B, E\), etc. Then it is immediate that \(M_r(0) \forall\tau \in \{1, 2, \ldots, N\}\) has full-column rank, implying that the system \(\sum_{s}\) has no zero at \(Z = 0\). Next, consider zeros at infinity. Using Lemma 3.1 it follows that \(M_r(Z) \forall\tau \in \{1, 2, \ldots, N\}\) is full-column rank. Hence, \(\sum_{s}\) has no zeros at infinity. ■

B. Case \(p_1 \leq m, Np_1 + p_2 > Nm\)

As in the previous subsection, we study zeros of tall blocked systems at infinity and the origin. We shall start with the former. According to Definition 2.1 the rank of matrix \(D_r\) plays a crucial role in the determination of the zeros at infinity. We now use the result of Proposition 2.1 to determine the multiplicity of zeros at infinity.

**Theorem 3.2:** Consider the system \(\sum_{s} \forall\tau \in \{1, 2, \ldots, N\}\), with \(p_1 \leq m\) and \(Np_1 + p_2 > Nm\). Assume that the defining matrices \(\{A, B, C^f, D^f, C^s, D^s\}\) take generic values. Then \(M_r(Z)\) has zeros at \(Z = \infty\) with multiplicity equal to:

1) \(0\) if \(n \leq (N - \tau)(m - p_1)\);
2) \(n - (N - \tau)(m - p_1)\) if \((N - \tau)(m - p_1) < n \leq (N - 1)(m - p_1)\);
3) \((\tau - 1)(m - p_1)\) if \(n > (N - 1)(m - p_1)\).

**Proof:** Denote by \(\sigma\) the multiplicity of zeros at infinity. Then, by Definition 2.1 we have \(\sigma = \text{normal rank} M_r(Z) - n - \text{rank} D_r\). Consider the following cases:

1) \(n \leq (N - \tau)(m - p_1)\). From Theorem 2.2 we have that normal rank \(M_r(Z) = (N - 1)p_1 + m + 2n\), while Proposition 2.1 yields that rank \(D_r = (N - 1)p_1 + m + n\). Then we easily conclude that \(\sigma = 0\).

2) \((N - \tau)(m - p_1) < n \leq (N - 1)(m - p_1)\). From Theorem 2.2 we still have that normal rank \(M_r(Z) = (N - 1)p_1 + m + 2n\), while now Proposition 2.1 yields rank \(D_r = (\tau - 1)p_1 + (N - \tau + 1)m\). Hence, in this case we obtain \(\sigma = n - (N - \tau)(m - p_1)\).

3) \(n > (N - 1)(m - p_1)\). In this case, from Theorem 2.2 we have that the system matrix is full-column normal rank, namely \(n + Nm\), while, according to Proposition 2.1 the rank of \(D_r\) is still \((\tau - 1)p_1 + (N - \tau - 1)m\). Then we can conclude that \(\sigma = (\tau - 1)(m - p_1)\).

The following corollary studies zeros at the origin.

**Corollary 3.1:** Consider the system \(\sum_{s} \forall\tau \in \{1, \ldots, N\}\), with \(p_1 \leq m\) and \(Np_1 + p_2 > Nm\). Assume that the defining matrices \(\{A, B, C^f, D^f, C^s, D^s\}\) take generic values. Then \(M_r(Z)\) has zeros at \(Z = 0\) with multiplicity equal to:

1) \(0\) if \(n \leq (\tau - 1)(m - p_1)\);
2) \(n - (\tau - 1)(m - p_1)\) if \((\tau - 1)(m - p_1) < n \leq (N - 1)(m - p_1)\);
3) \((N - \tau)(m - p_1)\) if \(n > (N - 1)(m - p_1)\).

**Proof:** Pick \(\tau\) in the set \(\{1, 2, \ldots, N\}\) and consider the following situations:

1) \(n \leq (N - \tau)(m - p_1)\). In this case, from Theorem 3.2 one can see that the system \(\sum_{s}\) has no zeros at infinity. Then, recalling Lemma 3.1 we also have that \(\sum_{s - 1}\) has no zeros at \(Z = 0\). Then, by defining \(\tau = N - \tau + 1\) and substituting in the inequality \(n \leq (N - \tau)(m - p_1)\), one can easily obtain that, when \(n \leq (\tau - 1)(m - p_1)\), the system \(\sum_{s - 1}\) has no zeros at \(Z = 0\).

2) \((N - \tau)(m - p_1) < n \leq (N - 1)(m - p_1)\). In this case, \(\sum_{s}\) has \(n - (N - \tau)(m - p_1)\) zeros at infinity. Using the same arguments employed for the previous case, we can conclude that, when \((\tau - 1)(m - p_1) < n \leq (N - 1)(m - p_1)\), \(\sum_{s}\) has \(n - (\tau - 1)(m - p_1)\) zeros at \(Z = 0\).

3) \(n > (N - 1)(m - p_1)\). Again, since \(\sum_{s}\) has \((\tau - 1)(m - p_1)\) zeros at infinity, we have that \(\sum_{s}\) has \((N - \tau)(m - p_1)\) zeros at the origin.

**Remark 1:** The above results reveal that, assuming \(A, B, \) etc. generic with \(p_1 \leq m\) and \(Np_1 + p_2 > Nm\), when \(\tau = 1\) all zeros are at the origin and no zero at infinity. Conversely, when \(\tau = N\) all zeros are at infinity and there are no zeros at the origin. Furthermore, when \(\tau = 1\) there is always at least one zero at the origin, while when \(\tau = N\) there is always at least one zero at infinity (unless one considers a system with no dynamics, i.e. a system with \(n = 0\)).

**Remark 2:** When \(p_1 = m\), the conditions given in Theorem 3.2 and the subsequent Corollary on the presence of zeros at \(Z = 0\) and \(Z = \infty\) shrink to empty sets. Then, it follows that \(\sum_{s}\) has neither zeros at the origin nor at infinity.

**Remark 3:** In some special cases depending on the state, input and output dimensions, \(\sum_{s}\) may have zeros at the origin or at infinity for some values of \(\tau\) but be completely zero-free for other values of \(\tau\). For example, consider \(\sum_{s}\) for particular choice of \(n = 5, m = 5, p_1 = 3, p_2 = 24\) and \(N = 8\) which has zeros for all values of \(\tau\), except for \(\tau = 4, 5\). In these particular cases, the system \(\sum_{s}\) is totally zero-free. This can be easily checked by using Theorem 3.2 and the subsequent Corollary.

Various theorems have been introduced in this paper regarding zeros of the system \(\sum_{s}\) given a generic underlying multirate system. Accordingly, we summarize results obtained in this paper in the table below.
TABLE I
Showing the Results Obtained in This Paper.

| Zero Region | \( p_1 \geq m \) | \( p_1 < m \) | \( N_{p_1 + p_2} > Nm \) |
|-------------|-----------------|-----------------|-----------------|
| Finite nonzero zeros | No | No |
| Zeros at zero | No | Zeros can be at these points |
| Zeros at infinity | No | No |

IV. CONCLUSIONS

Zeros of tall discrete-time multirate linear systems were addressed in this paper, with the zeros of multirate linear systems being defined as those of their corresponding blocked systems. The system matrix of tall blocked systems was investigated for generic choice of parameter matrices. It was specifically shown that tall blocked systems generically have no finite nonzero zeros. However, we showed that there are situations in which these systems present zeros at \( Z = 0 \) or \( Z = \infty \) or both. Such situations can be characterized in terms of the relevant integer parameters (input, state, and output dimensions and ratio of sampling rates. As part of the investigation, we also identified the generic rank assumed by the system matrix of a blocked system and the transfer function of that system. As part of our future work, we intend to generalize the results of this paper. In particular, we are interested in a general case where there are two output streams, one available every \( \omega \) time instants and the other every \( \tau \) time instants, with \( \omega \) and \( \tau \) coprime integers.

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APPENDIX

A. Proof of Proposition 27

We first need to introduce the following lemma.

**Lemma A.1:** Consider a generic pair of matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \). Then, given \( \nu \in \mathbb{N} \), the matrix

\[
C = \begin{bmatrix} B & AB & \cdots & A^{\nu-1}B \end{bmatrix}
\]

is always full rank, i.e. its rank is equal to:

\[
\sum_{k=0}^{\nu-1} \text{rank}(A^k) = n - 1.
\]
1) its number of rows, \( n \), if \( n \leq \nu m \),
2) its number of columns, \( \nu m \), if \( n > \nu m \).

Proof: Since the case \( m \geq n \) is straightforward, we focus on the case \( n > m \). The statement can be proven by finding a pair \((A, B)\) such that the matrix \( C \) attains full rank, since it means that this happens for any generic pair of such matrices. Accordingly, choose the following matrices

\[
A = \begin{bmatrix}
0_{m \times (n-m)} & I_m \\
I_{n-m} & 0_{(n-m) \times m}
\end{bmatrix},
\]

\[B = \begin{bmatrix}
I_m \\
0_{(n-m) \times m}
\end{bmatrix},
\]

(29)

regarding which we point out the following properties.

1) The matrix \( A \) acts as a circular left-shift
transform matrix through \( m \) positions and can be written in terms of the canonical basis of \( \mathbb{R}^n \), as \( A = [e_{m+1} \ldots e_n \ e_1 \ldots e_m] \).

Then, if for example \( n > 3m + 1 \), one has

\[
A^2 = [e_{2m+1} \ldots e_n \ e_1 \ldots e_{2m}],
\]

\[A^3 = [e_{3m+1} \ldots e_n \ e_1 \ldots e_{3m}].
\]

2) The matrix \( B \) selects the first \( m \) columns of any matrix which premultiplies it. Furthermore, the columns of \( B \) correspond to \( e_1, \ldots, e_m \).

Based on these considerations, we have then

\[
B = \begin{bmatrix}
e_1 \ldots e_m
\end{bmatrix}
\]

\[AB = \begin{bmatrix}
e_{m+1} \ldots e_{2m}
\end{bmatrix}
\]

\[A^2 B = \begin{bmatrix}
e_{2m+1} \ldots e_{3m}
\end{bmatrix}
\]

\[\vdots
\]

\[A^{\nu-1} B = \begin{bmatrix}
e_{(\nu-1)m+1} \ldots e_{\nu m}
\end{bmatrix},
\]

where for simplicity we have adopted the notation \( e_{(k+1)m+i} = e_i \), \( i = 1, \ldots, n, k \in \mathbb{N} \). Then, it is easy to conclude that:

1) if \( n \leq \nu m \), all the vectors of the canonical basis of \( \mathbb{R}^n \) enter in the matrix \( C \) at least once, and thus \( C \) is full row rank;
2) if \( n > \nu m \), there are \( \nu m \) distinct vectors of the canonical basis of \( \mathbb{R}^n \) entering in the matrix \( C \) and thus \( C \) is full column rank.

We can now prove Proposition 2.1. For the sake of brevity, we treat only the case \( n \geq m \), since the case \( n < m \) is virtually the same. Fix \( \tau \) and first assume \( n \leq (N-\tau)(m-p_1) \). We consider a particular system, defined by the matrices

\[
A = \begin{bmatrix}
0_{(m-p_1) \times (n-m-p_1)} & I_{m-p_1} \\
I_{n-m-p_1} & 0_{(n-m-p_1) \times (m-p_1)}
\end{bmatrix}
\]

\[B = \begin{bmatrix}
I_{m-p_1} \\
0_{(n-m-p_1) \times m}
\end{bmatrix},
\]

\[C^f = 0_{p_1 \times n}
\]

\[D^f = \begin{bmatrix}
0_{p_1 \times (m-p_1)} & I_{p_1}
\end{bmatrix},
\]

(30)

\[C^s = \begin{bmatrix}
I_n \\
I_{(p_2-p_1) \times n}
\end{bmatrix},
\]

\[D^s = \begin{bmatrix}
I_{m-p_1} & 0_{n \times m} \\
0_{(m-p_1) \times p_1} & I_{(p_2+p_1-n-m) \times m}
\end{bmatrix}
\]

Note that, under the working assumptions, the dimensions of the various matrices involved in the construction of such system are consistent. In particular, since \( n \leq (N-\tau)(m-p_1) \) and, by assumption of tallness, \( p_2 > N(m-p_1) \), one has

\[
n + m \leq (N-\tau)(m-p_1) + m \leq (N-1)(m-p_1) + m \\
= p_1 + N(m-p_1) < p_1 + p_2
\]

and so \( p_2 + p_1 - n - m > 0 \). Below, we adopt the notation that, if a submatrix has zero rows or columns, then it does not appear in the relative matrix. Before writing \( D_T \) explicitly, we focus on the submatrix

\[
\begin{bmatrix}
C^s A^{N-\tau-1} B & C^s A^{N-\tau-2} B & \ldots & C^s B
\end{bmatrix},
\]

which enters in the block row associated with the slow dynamics of the blocked system. Due to the structure of \( C^s \), a first rewriting yields

\[
\begin{bmatrix}
A^{N-\tau-1} B & A^{N-\tau-2} B & \ldots & B
\end{bmatrix}.
\]

(31)

Now, we point out the following properties of \( A \) and \( B \).

1) The matrix \( A \) acts as a circular left-shift operator matrix through \( m-p_1 \) positions. Furthermore, the columns of \( A \) are orthogonal.
2) The matrix \( B \) selects the first \( m-p_1 \) columns of any matrix which premultiplies it. The other \( p_1 \) columns of the resulting matrix are set to zero. Furthermore, the nonzero columns of \( B \) correspond to \( e_1, \ldots, e_{m-p_1} \).

Based on these considerations, we have then

\[
B = \begin{bmatrix}
e_1 \ldots e_{m-p_1} \ 0_{n \times p_1}
\end{bmatrix}
\]

\[AB = \begin{bmatrix}
e_{m-p_1+1} \ldots e_{(m-p_1)+n} \ 0_{n \times p_1}
\end{bmatrix}
\]

\[A^2 B = \begin{bmatrix}
e_{2(m-p_1)+1} \ldots e_{3(m-p_1)} \ 0_{n \times p_1}
\end{bmatrix}
\]

\[\vdots
\]

\[A^{N-\tau-1} B = \begin{bmatrix}
e_{(N-\tau)(m-p_1)+1} \ldots e_{(N-\tau)(m-p_1)+n} \ 0_{n \times p_1}
\end{bmatrix}
\]

where for simplicity we have adopted the notation \( e_{(k+1)(m-p_1)+i} = e_i \), \( i = 1, \ldots, n, k \in \mathbb{N} \). Thus, since we assumed \( n \leq (N-\tau)(m-p_1) \), the above matrix has rank equal to \( n \). Defining

\[
E_i = [e_{(i-1)(m-p_1)+1} \ldots e_{(i)(m-p_1)}],
\]

we can write \( D_T = \)

\[
\begin{bmatrix}
0_{p_1 \times (m-p_1)} & I_{p_1} \\
\vdots
\end{bmatrix},
\]

\[0_{p_1 \times (m-p_1)} & I_{p_1} \]

\[0_{p_1 \times (m-p_1)} & I_{p_1}
\]

\[\vdots
\]

\[\begin{bmatrix}
0_{p_1 \times (m-p_1)} & I_{p_1} \\
\vdots
\end{bmatrix}.
\]

(32)
This expression reveals that the rank of $D_r$ can be calculated by summing the ranks of each nonzero submatrix entering it. More precisely, we have $N$ identity matrices of size $p_1$ and one identity matrix of size $m - p_1$, plus the $E_1$'s which provide $n$ linearly independent columns in total. Hence, for this choice of parameter matrices and $n \leq (N - \tau)(m - p_1)$ we have rank $D_r = (N - 1)p_1 + m + n$. We conclude that, for generic choice of parameter matrices, under these assumptions, rank $D_r \geq (N - 1)p_1 + m + n$.

Now, still assuming $n \leq (N - \tau)(m - p_1)$ we seek an upper bound for the generic rank of $D_r$ and show that indeed it coincides with the lower bound just found. For this, assume generic parameter matrices and introduce the matrix $D_r \triangleq \left[ \begin{array}{c|c|c}
D^f & 0 & \ldots \\
\vdots & \ddots & \vdots \\
D^f & \cdots & 0 \\
C^{f} A^{N-\tau-1} B & \cdots & D^f \\
C^{s} A^{N-\tau-1} B & \cdots & D^s \\
C^{f} A^{N-2} B & \cdots & C^{f} A^{\tau-2} B \\
C^{s} A^{N-\tau-1} B & \cdots & C^{s} A^{\tau-3} B \\
D^f & 0 & \ldots & 0 \\
D^s & \cdots & D^f \\
\end{array} \right]$

\[
\begin{bmatrix}
\Delta_1 & 0 \\
\ast & \Delta_2
\end{bmatrix}
\] (33)

which, being just a row permutation of $D_r$, has the same rank. Hence, from now on we shall refer to the rank of $D_r$. The presence of the fat matrix $D^f$ on the block diagonal of $\Delta_2 \in \mathbb{R}^{(\tau-1)p_1 \times (\tau-1)m}$ ensures that $\Delta_2$ is full row rank, namely rank $(\Delta_2) = (\tau - 1)p_1$. This implies that the matrix indicated as “*” does not influence the rank of $D_r$. Thus rank $(D_r) = \text{rank}(\Delta_1) + \text{rank}(\Delta_2)$ and so we focus on $\Delta_1$. We define

$\Delta_a \triangleq \left[ \begin{array}{c|c|c}
D^f & 0 & \ldots \\
\vdots & \ddots & \vdots \\
C^{f} B & D^f & 0 \\
C^{f} A^{N-\tau-2} B & \cdots & D^f \\
\end{array} \right]$

$\Delta_b \triangleq \left[ \begin{array}{c|c|c|c}
C^{f} A^{N-\tau-1} B & \cdots & C^{f} B & D^f \\
C^{s} A^{N-\tau-1} B & \cdots & C^{s} B & D^s \\
\end{array} \right]$

so that

$\Delta_1 = \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix}$

and rank $(\Delta_1) \leq \text{rank}(\Delta_a) + \text{rank}(\Delta_b)$. Note that, $\Delta_1$ is a tall matrix, since it includes the slow rate outputs whose dimension ensure tallness in the whole system. Hence, its maximum achievable rank is given by the number of its columns, namely $(N - \tau + 1)m$. Thus we can find a first upper bound for the rank of $D_r$, that is

\[ \text{rank}(D_r) \leq (N - \tau + 1)m + (\tau - 1)p_1 \tag{34} \]

and this will be used below. Meanwhile, we focus on the analysis of $\Delta_2$. It is well-known (see e.g. [33]) that, due to genericity of the matrix $D^f$, $\Delta_a$ is full row rank, namely $(N - \tau)p_1$. For $\Delta_b$, we consider the following factorization

$\Delta_b = \left[ \begin{array}{c|c|c}
C^{f} & D^f \\
C^{s} & D^s \\
\end{array} \right] \left[ \begin{array}{c|c|c|c}
A^{N-\tau-1} B & A^{N-\tau-2} B & \cdots & B & 0 \\
0 & 0 & \ldots & 0 & I_m \\
\end{array} \right]$

\[
\triangleq HR. \tag{35}
\]

Since by assumption $n \leq (N - \tau)(m - p_1)$, from Lemma A.1 one can see that the matrix $R$ is full row rank, namely $n + m$. Thus, the rank of $\Delta_b$ is determined by $H \in \mathbb{R}^{(p_1 + p_2)(n + m)}$. On the one hand, assumption of tallness of the blocked system ensures $p_2 > N(m - p_1)$; on the other hand, since $n \leq (N - \tau)(m - p_1)$, one has $n + m < p_2 + p_1$. Hence $\Delta_b$ is tall, and so generically rank $(\Delta_b) = n + m$ and rank $(\Delta_1) \leq \text{rank}(\Delta_a) + \text{rank}(\Delta_b) = (N - \tau)p_1 + n + m$, which in turn implies

\[ \text{rank}(D_r) = \text{rank}(\Delta_1) + \text{rank}(\Delta_2) \]

\[ \leq (N - \tau)p_1 + n + m + (\tau - 1)p_1 \]

\[ = (N - 1)p_1 + n + m, \]

which corresponds to the lower bound found previously.

In order to complete our proof, it remains to analyze the case $n \geq (N - \tau)(m - p_1)$. To do so, we first make an observation concerning the case $n = (N - \tau)(m - p_1)$, which was covered in the first part of the proof. In this particular case, rank $(D_r) = (N - 1)p_1 + n + m = (N - \tau + 1)m + (\tau - 1)p_1$, which corresponds to the upper bound on rank of $D_r$, given by (34). Now, the proof for the case $n > (N - \tau)(m - p_1)$ can be completed by showing that such an upper bound is attained by any generic tall system with $n = (N - \tau)(m - p_1) + q$, $q \in \mathbb{N}$. This can be verified by choosing the system

\[
A = \left[ \begin{array}{c|c|c}
0_{(m - p_1) \times ((N - \tau - 1)(m - p_1))} & I_{m - p_1} \\
0_{q \times ((N - \tau - 1)(m - p_1))} & 0_{q \times (m - p_1)} \\
\vdots & \ddots & \vdots \\
0_{(m - p_1) \times q} & \cdots & 0_{q \times q} \\
\end{array} \right]
\]

\[ B = \left[ \begin{array}{c|c|c}
I_{m - p_1} & 0_{(m - p_1) \times p_1} & 0_{(m - p_1) \times m} \\
0_{(m - p_1) \times p_1} & 0_{(m - p_1) \times (m - p_1)} \\
\end{array} \right] \tag{36}
\]

\[ C^f = 0_{p_1 \times n} \quad D^f = \left[ \begin{array}{c}
0_{p_1 \times (m - p_1)} \\
I_{p_1}
\end{array} \right] \]

\[ C^s = \left[ \begin{array}{c|c|c|c}
0_{(N - \tau)(m - p_1) \times m} & 0_{(N - \tau)(m - p_1) \times q} \\
0_{(m - p_1) \times (m - p_1)} & 0_{(m - p_1) \times m} \\
\end{array} \right] \]

\[ D^s = \left[ \begin{array}{c|c|c}
I_{m - p_1} & 0_{(m - p_1) \times p_1} & 0_{(m - p_1) \times m} \\
0_{(m - p_1) \times p_1} & 0_{(m - p_1) \times (m - p_1)} \\
\end{array} \right], \]

which generates a matrix $D_r$ equal to the one generated by the system (30), when $n = (N - \tau)(m - p_1)$. Since we have previously proven that, in that case, the rank is $(\tau - 1)p_1 + (N - \tau + 1)m$ (which is also the maximum rank achievable), then also for any $n > (N - \tau)(m - p_1)$ we have rank $(D_r) = (\tau - 1)p_1 + (N - \tau + 1)m$. This completes the proof.

B. Proof of Theorem 2.2

Before proving our result on the normal rank of $M_r(Z)$, we need to introduce three preliminary results. The following lemma is adopted from [3] and modified for our own purpose.

Lemma A.2: The transfer function $V_r(Z)$ associated with
the blocked system \( \mathbf{b} \) has the following property
\[
\mathbf{V}_{\tau+1}(Z) = \begin{bmatrix}
0 & \mathbb{I}_{p_2} & 0 & 0
\end{bmatrix} \begin{bmatrix}
\mathbf{V}(Z) & Z & I_m(N-1)
\end{bmatrix},
\]
where \( \tau \in \{1, 2, \ldots, N-1\} \).

**Proposition A.1:** The normal rank of the system matrix \( \mathbf{M}_\tau(Z) \) is same for every value of \( \tau \in \{1, 2, \ldots, N\} \).

**Proof:** Using the above lemma, one can easily conclude that the transfer function matrices \( \mathbf{V}_{\tau+1}(Z_0) \) and \( \mathbf{V}_\tau(Z_0) \) have the same rank provided that \( Z_0 \) does not belong to the finite set of poles of the \( \mathbf{V}_\tau(Z) \) (which is the same as that of \( \mathbf{V}_{\tau+1}(Z) \)) and \( Z_0 \notin \{0, \infty\} \). Hence, we can conclude that \( \mathbf{V}_{\tau+1}(Z) \) and \( \mathbf{V}_\tau(Z) \) have the same normal rank and so do their associated system matrices i.e. \( \mathbf{M}_{\tau+1}(Z) \) and \( \mathbf{M}_\tau(Z) \).

**Proposition A.2:** Consider the system \( \sum_1 \) (i.e. the blocked system obtained with \( \tau = 1 \)), with \( p_1 < m \), \( Np_1 + p_2 > Nm \) and generic values of the defining matrices \( \{A, B, C^f, C^s, D^f, D^s\} \). Then:

1. if \( n \leq (N-1)(m-p_1) \), the matrix \( \mathbf{D}_1 \) has rank equal to \( (N-1)p_1 + m + n \);
2. if \( n > (N-1)(m-p_1) \), the matrix \( \mathbf{D}_1 \) has full column rank, namely \( Nm \).

**Proof:** The proof follows easily from Proposition A.1 by letting \( \tau = 1 \).

We are now ready to prove Theorem 2.2. Here, we focus on the matrix \( \mathbf{M}_1(Z) \); every result on its normal rank can be easily extended to any value of \( \tau \in \{2, \ldots, N\} \) using Proposition A.1.

Consider the matrix \( \mathbf{D}_1 \) and define \( r \triangleq \text{rank}(\mathbf{D}_1) \); note that the condition of tallness of the system implies \( r \leq Nm \). Define the full row rank matrix \( \mathbf{D}_1 \in \mathbb{R}^{r \times Nm} \), obtained by discarding a proper number of linearly dependent rows of \( \mathbf{D}_1 \). Similarly, define \( \mathbf{C}_1 \) discarding the corresponding rows from \( \mathbf{C}_1 \). Without loss of generality assume \( \mathbf{A} \) diagonal. This hypothesis is not limiting; in fact, under a generic setting, \( \mathbf{A} \) has \( n \) distinct eigenvalues and so it is diagonalizable. If one considers a change of basis \( T \) such that \( T^{-1} \mathbf{A} T \) is diagonal, then the other parameter matrices \( T^{-1}B \) and \( \mathbf{C}^T \) are still in a generic setting. Define \( \mathbf{M}_1(Z) \) as follows
\[
\mathbf{M}_1(Z) = \begin{bmatrix}
Z - a_n^N & \vdots & \vdots & 0 & -b_n^T
\end{bmatrix},
\]
where the \( a_i \)'s represent the diagonal elements of \( \mathbf{A} \), \( b_n^T \) is the \( i \)-th row of \( \mathbf{B}_1 \) and \( \bar{c}_{1,1} \) is the \( i \)-th column of \( \mathbf{C}_1 \). Consider the submatrix \( [\bar{c}_{1,1} \ \mathbf{D}_1] \). Since \( \mathbf{D}_1 \) is full row rank, this matrix is full row rank. Consider the equation
\[
v^T [\bar{c}_{1,n} \ \mathbf{D}_1] = [Z - a_n^N \ -b_n^T],
\]
in which \( v \) and \( Z \) are yet to be specified and which can be rewritten as
\[
\begin{cases}
\begin{aligned}
v^T \bar{c}_{1,n} &= Z - a_n^N \\
v^T \bar{c}_{1,n} &= -b_n^T
\end{aligned}
\end{cases}
\]
which is clearly full row rank, namely \( r+1 \). Write the equation
\[
v^T [\bar{c}_{1,n} \ \mathbf{D}_1] = [Z - a_n^N \ -b_n^T],
\]
which in turn can be rewritten as
\[
\begin{cases}
\begin{aligned}
v^T [\bar{c}_{1,n} \ \mathbf{D}_1] &= Z - a_n^N \\
v^T [\bar{c}_{1,n} \ \mathbf{D}_1] &= -b_n^T
\end{aligned}
\end{cases}
\]
and hence normal rank \( \mathbf{M}_1(Z) \) is included; clearly the rank turns out to be \( r+n \). Since \( \mathbf{M}_1(Z) \) is a submatrix of \( \mathbf{M}_1(Z) \), the normal rank of \( \mathbf{M}_1(Z) \) is greater than or equal to \( r + n \). There are two cases in the theorem statement. Treating the second one first, suppose \( n \geq (N-1)(m-p_1) \). Recalling Proposition A.2, \( r = Nm \); hence normal rank (\( \mathbf{M}_1(Z) \)) = \( n + Nm \) and \( \mathbf{M}_1(Z) \) is full normal rank.

For the second case, suppose \( n < (N-1)(m-p_1) \). In this case, from Proposition A.2 we have \( r = (N-1)p_1 + m + n \), hence normal rank (\( \mathbf{M}_1(Z) \)) \( \geq \) normal rank (\( \mathbf{M}_1(Z) \)) = \( (N-1)p_1 + m + 2n \). Now, consider the submatrix formed by the first \( n + (N-1)p_1 \) rows of \( \mathbf{M}_1(Z) \). Such a submatrix is full normal rank, since it can be seen also as a submatrix of the system matrix
\[
\begin{bmatrix}
Z & -A^N & -A^{N-1}B & \cdots & -B \\
C^f & D^f & & & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
C^s A^{N-1} & C^s A^{N-2}B & \cdots & C^s B & D^s
\end{bmatrix}
\]
which is the system matrix of a blocked fat system with generic parameter matrices. From \( \mathbf{33} \), it is well-known that \( \mathbf{33} \) is full normal rank. Now consider the remaining rows of \( \mathbf{M}_1(Z) \), i.e. the matrix
\[
\mathbf{\Pi} = \begin{bmatrix}
\begin{array}{ccccc}
C^f A^{N-1} & C^f A^{N-2}B & \cdots & C^f B & D^f \\
C^s A^{N-1} & C^s A^{N-2}B & \cdots & C^s B & D^s
\end{array}
\end{bmatrix}
\]
which can be factorized as
\[
\mathbf{\Pi} = \begin{bmatrix}
C^f & D^f & \begin{array}{cc}
A^{N-1} & A^{N-2}B & \cdots & B & 0 \\
0 & 0 & \cdots & 0 & I_m
\end{array}
\end{bmatrix} \triangleq H \mathbf{R}.
\]
Since $A$ is full rank, then also $A^{N-1}$ is full rank and thus the matrix $\bar{R}$ is full row rank, namely $n + m$. Thus, the rank of $\Pi$ depends on the rank of $H$, which, for generic choice of matrices $C^s, D^s, C^f, D^f$, is equal to $\alpha \triangleq \min\{p_1 + p_2, m + n\}$. Then normal rank $(M_1(Z)) \leq n + (N - 1)p_1 + \alpha$. However, since for the condition of tallness $p_2 > N(m - p_1)$ and by assumption $n < (N - 1)(m - p_1)$, we have $n + m < (N - 1)(m - p_1) + m = N(m - p_1) + p_1 < p_2 + p_1$, and so $\alpha = n + m$. Hence normal rank $(M_1(Z)) \leq (N - 1)p_1 + m + 2n$. Combining this bound with the lower bound found previously, we conclude that normal rank $(M_1(Z)) = (N - 1)p_1 + m + 2n$. 