A simple global representation for second-order normal forms of Hamiltonian systems relative to periodic flows

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Abstract

We study the determination of the second-order normal form for perturbed Hamiltonians \( H_\epsilon = H_0 + \epsilon H_1 + \epsilon^2 H_2 \), relative to the periodic flow of the unperturbed Hamiltonian \( H_0 \). The formalism presented here is global, and can be easily implemented in any computer algebra system. We illustrate it by means of two examples: the Hénon–Heiles and the elastic pendulum Hamiltonians.

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1. Introduction

In this paper we discuss some computational aspects of the normal form theory for Hamiltonian systems on general phase spaces, that is, Poisson manifolds. According to Deprit [9], a perturbed vector field

\[
A = A_0 + \epsilon A_1 + \frac{\epsilon^2}{2} A_2 + \cdots + \frac{\epsilon^k}{k!} A_k + O(\epsilon^{k+1})
\]
on a manifold \( M \), is said to be in normal form of order \( k \) relative to \( A_0 \) if \([A_0, A_i] = 0\) for \( i \in \{1, \ldots, k\} \). In the context of perturbation theory, the normalization problem is formulated as follows: to find a (formal or smooth) transformation which brings a perturbed dynamical system to a normal form up to a given order. The construction of a normalization transformation, in the framework of the Lie transform method [8, 12, 14, 16], is related to the solvability of a set of linear non-homogeneous equations, called the homological equations. If the homological equations admit global solutions, defined on the whole \( M \), we speak of a global normalization, which essentially depends on the properties of the unperturbed dynamics.

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Here we are interested in the global normalization of a perturbed Hamiltonian dynamics relative to periodic Hamiltonian flows. In this case, a result due to Cushman [6] states that if $A$ is a Hamiltonian vector field on a symplectic manifold, and the flow of the unperturbed vector field $A_0$ is periodic, then the true dynamics admits a global Deprit normalization to arbitrary order. The corresponding normal forms can be determined by a recursive procedure (the so-called Deprit diagram) involving the resolution of the homological equations at each step.

In this paper, we extend Cushman’s result to the Poisson case and derive an alternative coordinate-free representation for the second-order normal form, involving only three intrinsic operations: two averaging operators associated to the $S^1$-action, and the Poisson bracket. We give a direct derivation of this representation based on a period–energy argument [11] for Hamiltonian systems, and some properties of the periodic averaging on manifolds [3, 6, 19].

This formalism allows us to get an efficient symbolic implementation for some models related to polynomial perturbations of the harmonic oscillator with 1 : 1 resonance. In particular, we prove in section 5, paying particular attention to making explicit the steps in which the results developed so far are applied. The final section is devoted to a couple of examples which show that the results obtained by this method coincide with those based on the use of Lie derivatives.

2. Vector fields with periodic flow

Throughout the paper, we set $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We collect here some results regarding the flow $\text{Fl}_X$ of a vector field $X$ on an arbitrary manifold $M$, in the case when $\text{Fl}_X$ is periodic. Although these results are general, they will be later applied to the case of a Hamiltonian vector field on a Poisson manifold $(M, P)$.

Let $X \in \mathcal{X}(M)$ be a complete vector field whose flow is periodic with period function $T \in C^\infty(M)$, $T > 0$, that is, for any $p \in M$,

$$\text{Fl}_X^{-T}(p) = \text{Fl}_X^T(p). \quad (1)$$

Then, $X$ determines an $S^1$-action $S^1 \times M \rightarrow M$ given by $(t, p) \mapsto \text{Fl}_X^{t/\omega(p)}(p)$, where $\omega := 2\pi/T > 0$ is the frequency function, and $t \in S^1$. Thus, the $S^1$-action is periodic, with constant period $2\pi$.

The generator $\mathcal{Y}$ of this $S^1$-action can be readily computed:

$$\mathcal{Y}(p) = \frac{d}{dt} |_{t=0} \text{Fl}_X^{t/\omega(p)}(p) = \frac{1}{\omega(p)} \frac{d}{ds} |_{s=0} \text{Fl}_X^s(p) = \frac{1}{\omega(p)} X(p),$$

so $\mathcal{Y} = \frac{1}{\omega} X$. Notice from (1) that $T(p) > 0$ is the period of the integral curve of $X$ passing through $p \in M$ at $t = 0$, $c_p : \mathbb{R} \rightarrow M$ (which is such that $c(0) = p$ and $\dot{c}_p(0) = X(p)$). In
other words, \(c_p(0) = p = c_p(T(p))\). Also, each point on the image of the integral curve \(c_p\), gives the same value for the period \(T(p) = T(c_p(t))\), for all \(t \in \mathbb{R}\). In terms of the flow of \(X\), that means

\[
((\text{Fl}_X^t)^*T)(p) = T(\text{Fl}_X^t(p)) = T(p), \text{ for all } p \in M.
\]

As \(T\) is constant along the orbits of \(X\), its Lie derivative with respect to \(X\) vanishes:

\[
\mathcal{L}_X T = \frac{d}{dt} \bigg|_{t=0} (\text{Fl}_X^t)^* T = 0.
\]

Now, from \(T \omega = 2\pi\), we get

\[
0 = \mathcal{L}_X (T \omega) = (\mathcal{L}_X T) \omega + T \mathcal{L}_X \omega = \mathcal{L}_X \omega.
\]

But \(T > 0\), so this implies that \(\omega\) is a first integral (or invariant) of \(X\),

\[
\mathcal{L}_X \omega = 0.
\]

**Definition 2.1.** A smooth function \(f \in C^\infty(M)\) is said to be an \(S^1\)-invariant if it is invariant under the flow of the generator \(\mathbb{Y} = \frac{1}{\omega} X\), that is,

\[
\mathcal{L}_\mathbb{Y} f = 0.
\]

Clearly, this is equivalent to the condition \((\text{Fl}_\mathbb{Y}^t)^* f = f\), for all \(t \in [0, 2\pi]\). Notice that, by (2), the frequency function is also an invariant of the \(S^1\)-action,

\[
\mathcal{L}_\mathbb{Y} \omega = \frac{1}{\omega} \mathcal{L}_X \omega = 0.
\]

### 3. Averaging operators

Given a vector field \(X \in \mathcal{X}(M)\) with periodic flow, the associated \(S^1\)-action can be used to define two averaging operators, which we will denote by \(\langle \cdot \rangle\) and \(\mathcal{S}\). In this section, \(M\) will be an arbitrary manifold.

For any tensor field \(R \in \Gamma T^r_s(M)\) (\(r\)-covariant, \(s\)-contravariant), the average of \(R\) with respect to the \(S^1\)-action on \(M\) induced by \(X\) is the tensor field (of the same type as \(R\)) defined by

\[
\langle R \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_\mathbb{Y}^t)^* R \, dt.
\]

The properties of the flow [1] guarantee that \(\langle R \rangle\) is well-defined as a differentiable tensor field. Also, note that if \(R \in \Gamma T^r_s(M)\), and \(X_1, \ldots, X_r, \in \mathcal{X}(M), \alpha_1, \ldots, \alpha_s \in \Omega^\alpha(M)\) are arbitrary, then for every \(p \in M\), \(t \mapsto (\text{Fl}_\mathbb{Y}^t)^* R(X_1, \ldots, X_r, \alpha_1, \ldots, \alpha_s)(p)\) is a real differentiable function on the compact \([0, 2\pi]\), hence integrable. We will use this definition mainly applied to the case of functions \(f \in C^\infty(M)\) ((0, 0)-tensors) and vector fields \(Y \in \mathcal{X}(M)\) ((0, 1)-tensors).

The other averaging operator that will be important in what follows is the \(\mathcal{S}\) operator,

\[
\mathcal{S} : \Gamma T^r_s(M) \rightarrow \Gamma T^r_s(M).
\]

It is given by

\[
\mathcal{S}(R) := \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(\text{Fl}_\mathbb{Y}^t)^* R \, dt.
\]

Note that both \(\langle \cdot \rangle\) and \(\mathcal{S}\) are \(\mathbb{R}\)-linear operators. Other properties are listed below.

**Lemma 3.1.** For any complete vector field \(Y \in \mathcal{X}(M)\) (whose flow is not necessarily periodic) and smooth tensor field \(R \in \Gamma T^r_s(M)\), we have:

\[
\frac{d}{ds} \bigg|_{s=0} (\text{Fl}_Y^s)^* R = \frac{1}{2\pi} ((\text{Fl}_Y^{2\pi})^* R - R),
\]

where the averaging is taken with respect to the flow of \(Y\), that is, \(\langle R \rangle\) is given by

\[
\langle R \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_Y^t)^* R \, dt.
\]
Proof. Start from the identities (which follow directly from the definitions of flow and the Lie derivative):

\[(\text{Fl}_t)^* (\mathcal{L}_\gamma R) = \left. \frac{d}{dt} (\text{Fl}_t)^* R \right|_{t=0} = \left. \frac{d}{ds} (\text{Fl}_t)^* R \right|_{s=0} = (\text{Fl}_t)^* (\text{Fl}_t)^* R.
\]

Taking the integral with respect to \(t\) between 0 and 2\(\pi\) on both sides, we get, on the one hand:

\[
\frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_t)^* (\mathcal{L}_\gamma R) \, dt = \left. \frac{d}{ds} \right|_{s=0} \left( \int_0^{2\pi} (\text{Fl}_t)^* R \, dt \right) = \frac{d}{ds} \left( \int_0^{2\pi} (\text{Fl}_t)^* R \, dt \right) = \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_t)^* (\text{Fl}_t)^* R - R.
\]

and, on the other:

\[
\frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_t)^* (\mathcal{L}_\gamma R) \, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dt} (\text{Fl}_t)^* R \, dt = \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_t)^* (\text{Fl}_t)^* R - R.
\]

\[\Box\]

Proposition 1. For every \(R \in \Gamma T^1_s(M)\), the following properties hold.

(a) \(R\) is invariant under the flow of \(\gamma\) (that is, \(S^1\)-invariant) if and only if \(\langle R \rangle = R\).
(b) \(\mathcal{L}_\gamma (R) = 0\).
(c) If \(g \in C^\infty(M)\) is \(S^1\)-invariant, then \(\langle g R \rangle = g \langle R \rangle\).
(d) The averaging operator commutes with tensor contractions whenever one of the tensors is \(S^1\)-invariant, that is, if \(S \in \Gamma T^1_s(M)\) is \(S^1\)-invariant and \(C_i^k\) is any contraction, then

\[\langle C_i^k (R \otimes S) \rangle = \langle C_i^k \rangle \langle R \otimes S \rangle.\]

Proof.

(a) If \(R\) is invariant under the flow of \(\gamma\), then \((\text{Fl}_t)^* R = R\), for all \(t \in [0, 2\pi]\), and from this it is immediate that \(\langle R \rangle = R\). Reciprocally, if \(\langle R \rangle = R\) we may apply the preceding lemma to obtain:

\[
\left. \frac{d}{dt} \right|_{t=0} (\text{Fl}_t)^* R = \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_t)^* R - R,
\]

and from the fact that the flow of \(\gamma\) is \(2\pi\)-periodic,

\[\mathcal{L}_\gamma R = \left. \frac{d}{dt} \right|_{t=0} (\text{Fl}_t)^* R = 0.\]

(b) From the properties of the Lie derivative and the definition of \(\langle R \rangle\):

\[
(\text{Fl}_t)^* (\mathcal{L}_\gamma (\langle R \rangle)) = \left. \frac{d}{dt} \right|_{t=0} (\text{Fl}_t)^* (\langle R \rangle) = \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_{t+s})^* R \, ds = \left. \frac{d}{dt} \right|_{t=0} \int_t^{t+2\pi} (\text{Fl}_t)^* R \, du.
\]

Now, because \(\text{Fl}_t^\gamma\) is \(2\pi\)-periodic:

\[
(\text{Fl}_t)^* (\mathcal{L}_\gamma (\langle R \rangle)) = \left. \frac{d}{dt} \right|_{t=0} \int_0^{2\pi} (\text{Fl}_t)^* R \, du = 0,
\]

so, as \(\text{Fl}_t^\gamma\) is a diffeomorphism, \(\mathcal{L}_\gamma (\langle R \rangle) = 0\).

(c) It is a straightforward computation.

(d) It is just a consequence of the commutativity between the pull-back and the tensor contractions, and the functorial property \((\text{Fl}_t)^* (R \otimes S) = (\text{Fl}_t)^* R \otimes (\text{Fl}_t)^* S\). \[\Box\]

Remark 1. In particular, from (1) we get that if \(Y \in \mathcal{X}(M)\) and \(\alpha \in \Omega(M)\) is \(S^1\)-invariant, then \(i_{[Y,\alpha]} = i_Y \alpha\).
Proposition 2. For any \( R \in \Gamma T^*_r(M) \) and \( g \in C^\infty(M) \) \( S^1 \)-invariant, the following hold:

(a) \( S(g R) = g S(R) \).
(b) \( (\mathcal{L}_\gamma \circ S)(R) = R - \langle R \rangle \).

Proof.

(a) A straightforward computation.

(b) With an obvious change of variable, we have:
\[
(\text{Fl}_t^\gamma)^* S(R) = \frac{1}{2\pi} \int_0^{2\pi} (t - \pi) (\text{Fl}_t^\gamma)^* R \, dt = \frac{1}{2\pi} \int_s^{s+2\pi} (u - s - \pi) (\text{Fl}_t^\gamma)^* R \, du.
\]

Differentiating both sides of this identity with respect to the parameter \( s \), and taking into account the \( 2\pi \)-periodicity of the flow \( \text{Fl}_t^\gamma \), it results in:
\[
\frac{d}{ds} (\text{Fl}_t^\gamma)^* S(R) = (\text{Fl}_t^\gamma)^* (R - \langle R \rangle). \tag{3}
\]

The statement follows by recalling that \( \text{Fl}_t^\gamma \) is a diffeomorphism, and the identity (see [1]):
\[
\frac{d}{ds} (\text{Fl}_t^\gamma)^* S(R) = (\text{Fl}_t^\gamma)^* (\mathcal{L}_\gamma S(R)). \tag{4}
\]

Finally, let us give some useful properties involving the averaging operators.

Proposition 3. For all \( R \in \Gamma T^*_r(M) \), the operators \( \mathcal{L}_\gamma \), \( \langle \cdot \rangle \), and \( S \) satisfy the relations:

(a) \( \langle \mathcal{L}_\gamma R \rangle = \mathcal{L}_\gamma \langle R \rangle = 0 \).
(b) \( \langle S(R) \rangle = S(\langle R \rangle) = 0 \).
(c) \( \langle d\alpha \rangle = d(\langle \alpha \rangle) \), for all \( \alpha \in \Omega(M) \).

Proof. Straightforward computations, making use of proposition 2 and the fact that \( d \) commutes with pull-backs.

4. The Hamiltonian case

Let \((M, P)\) be an \( m \)-dimensional Poisson manifold, where \( P \in \Gamma \Lambda^2 TM \) is a Poisson bivector determining a bracket \( \{f, g\} = P(df, dg) \), for all \( f, g \in C^\infty(M) \). For every \( f \), its Hamiltonian vector field \( X_f \in \mathcal{X}(M) \) is given by \( X_f(g) := \{f, g\} \), for any \( g \in C^\infty(M) \), equivalently,
\[
X_f = i_f P. \tag{5}
\]

At any point the distribution spanned by the Hamiltonian vector fields is involutive, as a consequence of Jacobi’s identity for the Poisson bracket \( \{\cdot, \cdot\} \). Thus, these Hamiltonian vector fields give rise to a foliation whose leaves turn out to be symplectic manifolds (see [21]). On each leaf \( S \), the restriction \( P|_S \) is a non-degenerate Poisson bivector field which determines a symplectic structure \( \sigma_S \) through
\[
\sigma_S(X_f, X_g) := \{f, g\}. \tag{6}
\]

Indeed, by the splitting theorem due to Weinstein ([21]), the local structure of \((M, P)\) can be described as follows: for any \( p \in M \) there exists a chart \((U, \phi)\) of \( M \) around \( p \) such that, if \[ q_1, \ldots, q_k, p_1, \ldots, p_{l-k}, y_1, \ldots, y_l \] are the coordinates of \( \phi: U \to \mathbb{R}^m \) \((2k + l = m)\), then
\[
P|_U = \sum_{i=1}^k \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j=1}^l \psi_{ij}(y_1, \ldots, y_l) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}. \tag{7}
\]
Proposition 4. Let $X$ be a vector field on the symplectic manifold $(S, \sigma)$ whose flow is periodic with period function $T \in C^\infty(M)$, $T > 0$ (and frequency $\omega = 2\pi/T$). If $X$ is the Hamiltonian vector field of a certain function $f \in C^\infty(M)$ (that is, $i_X\sigma = -df$), then:

$$d\omega \wedge df = 0 = dT \wedge df.$$  \hfill (5)

Proof. By hypothesis, we have,

$$L_X\sigma = i_Xd\omega + d(i_X\sigma) = -d^2f = 0.$$  

On the other hand, using the generator $\Upsilon = X/\omega$ of the $S^1$-action induced by $X$:

$$L_X\sigma = \omega L_{\Upsilon}\sigma - \frac{1}{\omega}d\omega \wedge df.$$  

Recalling that $\omega, f$ are first integrals of $X$, and hence $S^1$-invariants, applying the averaging operator $\langle \cdot \rangle$ to the last identity, taking into account that $\langle L_{\Upsilon}\sigma \rangle = 0$ (proposition 3 (a)), and the commutativity between $d$ and $\langle \cdot \rangle$ (proposition 3 (c)), we get:

$$0 = \langle \omega L_{\Upsilon}\sigma \rangle - \left\{ \frac{1}{\omega}d\omega \wedge df \right\} = \omega \langle L_{\Upsilon}\sigma \rangle - \frac{1}{\omega}d\omega \wedge df = -\frac{1}{\omega}d\omega \wedge df.$$  

\hfill □

Remark 2. Notice that, in terms of Hamiltonian vector fields, we can write the energy–period relation (5) as follows,

$$X_\omega \wedge X_f = 0.$$  \hfill (6)

Also, in the course of the proof we have seen that, if $\Upsilon = \frac{1}{\sigma}X$ is the generator of the $S^1$-action induced by $X$:

$$0 = L_X\sigma = \omega L_{\Upsilon}\sigma - \frac{1}{\omega}d\omega \wedge df,$$

so from (5) we get the following consequence.

Corollary 1. The symplectic form $\sigma$ is $S^1$-invariant, $L_{\Upsilon}\sigma = 0$. In particular, $\langle \sigma \rangle = \sigma$.

Notice that, under the hypothesis of proposition 4, if $g \in C^\infty(S)$ is $S^1$-invariant, then its Hamiltonian vector field $X_g \in \mathfrak{X}(S)$ is also $S^1$-invariant. Indeed, recalling that $d$ commutes with the averaging (proposition 3 (c)), remark 1, and the preceding corollary, we get:

$$i_{X_g}\sigma = -d(g) = -(dg) = i_{X_g}\sigma = i_{\Upsilon}\sigma.$$  

Hence, by the non-degeneracy of $\sigma$, $\langle X_g \rangle = X_{\langle g \rangle}$. Now, if $g$ is $S^1$-invariant, $\langle g \rangle = g$, and so $\langle X_g \rangle = X_g$.

As a consequence, for any $S^1$-invariant $g \in C^\infty(S)$, we have

$$L_{\Upsilon}X_g = [X_\Upsilon, X_g] = 0.$$
Now, suppose that we are given a function $H \in C^\infty(M)$ on the Poisson manifold $(M, P)$ such that its Hamiltonian vector field $X_H \in \mathcal{X}(M)$ has periodic flow (with the frequency function $\omega \in C^\infty(M)$, $\omega > 0$). Let $\Upsilon = \frac{1}{\omega}X_H$ be the generator of the associated $S^1$-action. From the results above we know that $M$ is foliated by symplectic leaves $S$ in such a way that $P|_{\delta}$ is equivalent to a symplectic form $\sigma_\delta$ (recall (4)), and these are invariant under Hamiltonian flows. Thus:

$$0 = \mathcal{L}_{X_H}P = \mathcal{L}_{\omega \Upsilon} P = \omega \mathcal{L}_\Upsilon P - \frac{\omega \Upsilon}{\omega} \wedge i_{\mathcal{L}_\Upsilon P} = \omega \mathcal{L}_\Upsilon P + \frac{1}{\omega}X_H \wedge X_\omega,$$

where we have used the formula $\mathcal{L}_fXA = f \mathcal{L}_X A - X \wedge i_A f$ (valid for any function $f \in C^\infty(M)$, vector field $X \in \mathcal{X}(M)$ and multivector field $A \in \Gamma(\Lambda TM)$; see [18, p 358]), as well as (3) and the fact that $\omega > 0$. From this identity and the energy–period relation (6), we deduce that $P$ is $S^1$-invariant, $\mathcal{L}_\Upsilon P = 0$.

Moreover, if $g \in C^\infty(M)$ is $S^1$-invariant, the flow of its Hamiltonian vector field $X_g$ leaves the integral submanifolds $S$ invariant and, as we have seen, on each of them it satisfies $\mathcal{L}_\Upsilon X_g = 0$, so this is also true on $M$. In other words, the flows of $\Upsilon$ and $X_g$ commute on $M$. The following result exploits this fact.

**Proposition 5.** Let $(M, P)$ be a Poisson manifold, and $H \in C^\infty(M)$ such that its Hamiltonian vector field $X_H \in \mathcal{X}(M)$ has periodic flow. If $f, g \in C^\infty(M)$ and $g$ is $S^1$-invariant, then:

1. **(a)** if $\omega$ is the frequency function of $X_H$, then, $X_H \wedge X_\omega = 0$;
2. **(b)** $[H, g] = 0$;
3. **(c)** $\langle [f, g] \rangle = \langle [f], g \rangle$.

**Proof.** Item (a) follows easily from the above considerations, while (b) is proved by a straightforward computation. Item (c) is a direct consequence of the $S^1$-invariance of $g$ and the fact that the flows of $\Upsilon$ and $X_g$ commute. \(\Box\)

5. The main result

Let $H_\varepsilon = H_0 + \varepsilon H_1 + \frac{1}{2} \varepsilon^2 H_2 + O(\varepsilon^3)$ be an $\varepsilon$-dependent Hamiltonian function which describes a perturbed Hamiltonian system on a Poisson manifold $(M, P)$, with associated bracket $\{ \cdot, \cdot \}$. We will denote by $X_{H_\varepsilon} = X_{H_0} + \varepsilon X_{H_1} + \frac{1}{2} \varepsilon^2 X_{H_2} + O(\varepsilon^3)$ the corresponding Hamiltonian vector field. Recall that the perturbed Hamiltonian vector field $X_{H_\varepsilon}$ is in (Deprit) normal form relative to $X_{H_0}$ of order $k$ in $\varepsilon$ if

$$[X_{H_\varepsilon}, X_{H_\varepsilon}] = 0, \text{ for all } i \in \{1, 2, \ldots, k\}. \quad (7)$$

In terms of Hamiltonian functions, (7) is satisfied whenever

$$[H_{\varepsilon}, H_i] = 0, \text{ for all } i \in \{1, 2, \ldots, k\}.$$

Usually, one can bring the Hamiltonian to a normal form by means of near-to-identity transformations. Let us recall some definitions and basic properties.

Let $M$ be a manifold, $N \subset M$ be a non-empty open domain, and $\delta > 0$. A smooth mapping $\Phi : (-\delta, \delta) \times N \to M$ is said to be a near-to-identity transformation if, for each $\varepsilon \in (-\delta, \delta)$, the map $\Phi_\varepsilon : N \to M$ given by

$$\Phi_\varepsilon(x) = \Phi(\varepsilon, x)$$

is such that it is a diffeomorphism onto its image and, moreover, $\Phi_0 = \text{id}_M$. 


These transformations have the following important property: whenever we have a time-dependent vector field $A_\varepsilon$ on $M$, and a near-to-identity transformation $\Phi_\varepsilon$, the pull-back $\Phi_\varepsilon^*A_\varepsilon$ is again an $\varepsilon$-dependent vector field on $\mathcal{N}$, and it is such that,

$$\Phi_\varepsilon^*A_\varepsilon|_{\varepsilon=0} = A_0.$$ 

In other words, thinking of $A_\varepsilon$ as a perturbed vector field, near-to-identity transformations preserve the unperturbed part.

Actually, we will construct the required transformations out from the flow of a perturbed vector field. The following properties say that we can do so on each open domain with compact closure.

**Proposition 6.** Let $F: \mathbb{R} \times M \to M$ be a smooth mapping, sending $(\varepsilon, x)$ to $F_\varepsilon(x) = F(\varepsilon, x)$, such that $F_0 = \text{id}_M$. Then, for any open domain with compact closure $N \subset M$, there exists a $\delta > 0$ such that, for each $\varepsilon \in (-\delta, \delta)$, the restriction $F_\varepsilon|_N$ is a diffeomorphism onto its image.

**Proof.** It is an immediate consequence of the fact that the closure $\overline{N}$ can be covered by a finite number of open neighborhoods, such that the implicit function theorem applies to them. □

**Proposition 7.** Let $A_\varepsilon = A_0 + \varepsilon R_\varepsilon$ be a smooth vector field on a manifold $M$. Assume that the unperturbed vector field $A_0$ is complete on $M$. Then, for any open domain $N \subset M$, with compact closure, and any constant $\delta > 0$, there exists another constant $L > 0$ such that the flow $F_\varepsilon|_N$ of $A_\varepsilon$, is well-defined on $N$ for any $t \in [0, L/\varepsilon]$ and each $\varepsilon \in (0, \delta]$.

**Proof.** If $X, Y$ are vector fields on the manifold $M$, their flows are related by

$$F_t^Y \circ F_\varepsilon^X = F_{\varepsilon t}^Y,$$  

where $P_t$ is the time-dependent vector field given by $P_t = -X + (F_t^Y)^*Y$. Now, let

$$(F_{\varepsilon t}^Y)^*A_\varepsilon - A_0 = \varepsilon R_t(\varepsilon),$$

where $R_t(\varepsilon) = (F_{\varepsilon t}^Y)^*R_\varepsilon$ depends smoothly on $t$ and $\varepsilon$, and fix a $\delta > 0$. By the flow-box theorem and the compactness of the closure $\overline{N}$, there exists an $L > 0$ such that the flow of $R_t(\varepsilon)$ is well-defined on $N$ for any $t \in [0, L]$. Applying (8) to $X = A_0, Y = A_\varepsilon$, and $P_t = R_t(\varepsilon)$, we get

$$F_{\varepsilon t}^Y = F_0^X \circ F_{\varepsilon t}^{R_t(\varepsilon)},$$

and, since $F_0^X$ is well-defined for all $t \in \mathbb{R}$, the statement follows. □

**Definition 5.1.** We say that the system described by a vector field of the form $A_\varepsilon = A_0 + \varepsilon R_\varepsilon$, where $A_0$ has complete flow, admits a global normalization of order $k$ if, for each open domain $N \subset M$ with compact closure, there exist a $\delta > 0$ and a near-to-identity transformation $F: (-\delta, \delta) \times N \to M$, which brings $A_\varepsilon$ to a normal form of order $k$.

**Theorem 5.2.** Suppose that the flow of $X_{H_0}$ is periodic with frequency function $\omega \in C^\infty(M)$, $\omega > 0$. Then, the perturbed Hamiltonian system admits a global normalization of arbitrary order $k$. In particular, the second-order normal form can be expressed as:

$$H_\varepsilon \circ \Phi_\varepsilon = H_0 + \varepsilon \{H_1\} + \frac{\varepsilon^2}{2} \left(\{\{H_1\}, H_1\} + O(\varepsilon^3)\right).$$  

(9)
Proof. If the Hamiltonian vector field $X_{H_0}$ has periodic flow, the existence of the near-identity canonical transformation $\Phi_\varepsilon$ follows from the above propositions (see also [3, 6, 16, 17]). Here we give a explicit formula for it.

Let $\Phi_\varepsilon$ be the flow of the perturbed vector field $Z_\varepsilon = Z_0 + \varepsilon Z_1$ where $Z_0$ and $Z_1$ are the Hamiltonian vector field of the functions $G_0 = \frac{1}{\omega}S(H_1)$ and $G_1 = \frac{1}{\omega}S(H_2 + \{S(\frac{1}{\omega}H_1), H_1 + \langle H_1 \rangle\})$, respectively. Using the Lie transform method [6, 8, 12, 14], the second-order development of $H_\varepsilon \circ \Phi_\varepsilon$ is given by:

$$H_\varepsilon \circ \Phi_\varepsilon = H_0 + \varepsilon (\mathcal{L}_{Z_0}H_0 + H_1) + \frac{\varepsilon^2}{2} \left( \mathcal{L}_{Z_0}^2 H_0 + 2 \mathcal{L}_{Z_0}H_1 + \mathcal{L}_{Z_0}H_0 + H_2 \right) + O(\varepsilon^3). \quad (10)$$

Now, we apply the results of the preceding sections to put this Hamiltonian in the form (9). To this end, we derive the following relations. First, by a direct computation we have

$$\mathcal{L}_{Z_0}H_1 = \mathcal{L}_{X_{G_0}} H_1 = \left\{ \frac{1}{\omega} S(H_1), H_1 \right\}.$$

Then, it follows from proposition 2 that:

$$\mathcal{L}_{Z_0}H_0 = -\mathcal{L}_{X_{G_0}} \left\{ \frac{1}{\omega} S(H_1) \right\} = \langle H_1 \rangle - H_1,$$

and

$$\mathcal{L}_{Z_0}^2 H_0 = \mathcal{L}_{X_{G_0}} \left( \langle H_1 \rangle - H_1 \right) = \left\{ \frac{1}{\omega} S(H_1), \langle H_1 \rangle - H_1 \right\}.$$

Finally, again by proposition 2, proposition 3 (b), and proposition 5 (c), we get

$$\mathcal{L}_{Z_0}H_0 = -\mathcal{L}_{X_{G_0}} S \left( H_2 + \left\{ \frac{1}{\omega} H_1, H_1 + \langle H_1 \rangle \right\} \right)$$

$$= \langle H_2 \rangle + \left\{ \left\{ \frac{1}{\omega} H_1, H_1 \right\} - \left( H_2 + \left\{ \frac{1}{\omega} H_1, H_1 + \langle H_1 \rangle \right\} \right) \right\}.$$

Substituting these identities into (10), we obtain the normal form (9).

$\Box$

6. Examples

In this section we illustrate the computation of the normal form of two particular Hamiltonians on $\mathbb{R}^2$ endowed with the canonical symplectic form, $\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_1$ (and the corresponding canonical Poisson bracket). If we have a system admitting an $S^1$-action, described by a perturbed Hamiltonian $H = H_0 + \varepsilon H_1$, and such that the Hamiltonian vector field of $H_0$, $X_{H_0}$ has periodic flow with frequency $\omega$ then, as shown in theorem 5.2, its second-order normal form is given by:

$$H_0 + \varepsilon \langle H_1 \rangle + \frac{\varepsilon^2}{2} \left( \left\{ \left\{ \frac{1}{\omega} H_1, H_1 \right\} \right\} \right).$$

Example 1 (Hénon–Heiles Hamiltonian). This example is taken from [7]. The Hamiltonian is

$$H = H_0 + \varepsilon H_1 = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} (q_1^2 + q_2^2) + \varepsilon \left( \frac{q_1^3}{3} - q_1 q_2^2 \right)$$

(note that the perturbation term is a homogeneous polynomial of degree 3). The frequency function for the flow of $X_{H_0}$ is readily found to be constant, $\omega = 1$, and, after carrying on the computations indicated in theorem 5.2, the second-order normal form is found to be:

\[
\begin{align*}
&\frac{p_2^2}{2} + \frac{q_2^2}{2} + \frac{q_1^2}{2} - \frac{\varepsilon^2}{48} (5q_2^4 + (10q_1^2 + 10p_2^2 - 18p_1^2)q_1^2) + 56p_1 p_2 q_1 q_2 + 5q_1^2 + (10p_1^2 - 18p_2^2)q_1^2 + 5q_2^4 + 10p_1^2 p_2^2 + 5p_1^4). \\
\end{align*}
\]
It is usual to express the normal form in terms of the Hopf variables $w_1, w_2, w_3, w_4$, as a previous step to carry on the reduction of symmetry process (see [6, 7]). For the case in which $H_0$ is the Hamiltonian of the 2D-harmonic oscillator, these variables form a system of functionally independent generators of the algebra of first integrals of $H_0$, and are defined as $w_1 = 2(q_1 q_2 + p_1 p_2), w_2 = 2(q_1 p_2 - q_2 p_1), w_3 = q_1^2 + p_1^2 - q_2^2 - p_2^2, w_4 = q_1^2 + q_2^2 + p_1^2 + p_2^2$.

Working separately with the independent term and the coefficient of $\epsilon^2$ in the expression above, we get:

$$
\frac{w_4}{2},
$$

and

$$
\frac{w_2^2}{48} \frac{(48 \lambda + 7)}{48} - \frac{w_4^2}{48} \frac{(48 \lambda + 5)}{48} + w_5^2 \lambda + w_6^2 \lambda.
$$

In the process of expressing the $q_i, p_i$ variables in terms of the $w_j$, a parameter $\lambda$ appears as a consequence of the fact that the corresponding system of equations is indeterminate. The formulas appearing in [7] are recovered by choosing the value 0 of the parameter:

$$
\frac{7 w_2}{48} = 5 w_4.
$$

Thus, the second-order normal form of the Hénon–Heiles system is

$$
H_e \circ \Phi_e = \frac{w_4}{2} + \frac{\epsilon^2}{48} \left(7 w_2^2 - 5 w_4^2\right) + O(\epsilon^3).
$$

**Example 2** (The elastic pendulum). Consider the case of the Hamiltonian of a elastic pendulum (see [4, 5, 10]):

$$
H(q_1, q_2, p_1, p_2) = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^2 + q_2^2}{2} - \frac{\epsilon}{2} q_1^2(1 + q_2),
$$

which is that of a perturbed system $H_0 + \epsilon H_1$, where

$$
H_0(q_1, q_2, p_1, p_2) = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^2 + q_2^2}{2},
$$

and

$$
H_1(q_1, q_2, p_1, p_2) = -\frac{q_1^2(1 + q_2)}{2}.
$$

Note that the perturbation term now is *not* homogeneous. The computation of the normal form in the original variables gives the result:

$$
\frac{p_2^2 + p_1^2}{2} + \frac{q_2^2 + q_1^2}{2} - \frac{\epsilon}{4} (q_1^2 + p_1^2) - \frac{\epsilon^2}{192} (20 q_1^2 - 4 p_1^2) q_2^2 + 48 p_1 p_2 q_1 q_2 + 5 q_1^4 + (-4 p_2^2 + 10 p_1^2 + 12) q_2^2 + 20 p_1^2 p_2^2 + 5 p_1^4 + 12 p_1^2).
$$

As before, we can express in terms of the Hopf variables the independent terms and the coefficient of $\epsilon$, getting:

$$
\frac{w_4}{2},
$$

and

$$
-\frac{w_4}{8} = -\frac{w_3}{8}.
$$

Note, however, that the coefficient of $\epsilon^2$ is *not* a homogeneous polynomial (of degree 4); there are two 2-degree terms: $(q_1^2 + p_1^2)/16$. Luckily, these terms can be easily expressed in terms of...
the variables $w_1, w_2, w_3, w_4$ (as $(q_1^2 + p_1^2)/16 = (w_1 + w_3)/32$) and then we can analyze the remainder, which is a polynomial of degree 4. Again, a parameter $\mu$ appears in the process:

$$-rac{w_2^2 (768 \mu + 25)}{768} + \frac{w_3^2 (256 \mu + 5)}{256} + \frac{w_4^2 (32 \mu + 1)}{32} + w_1^2 \mu - \frac{5 w_3 w_4}{384}.$$  

Let us take the simplest solution $\mu = 0$:

$$-rac{25 w_2^2}{768} - \frac{5 w_3 w_4}{384} + \frac{5 w_3^2}{256} + \frac{w_2^2}{32}.$$  

The remainder in the coefficient of $\epsilon^2$ is:

$$\frac{w_4}{32} + \frac{w_3}{32}.$$  

Thus, we get the second-order normal form of the elastic pendulum in the Hopf variables:

$$H_\epsilon \circ \Phi = \frac{w_4}{2} - \frac{\epsilon}{8} (w_4 + w_3) + \frac{\epsilon^2}{32} \left( w_4 + w_3 + w_2^2 - \frac{25 w_4^2}{24} - \frac{5 w_3 w_4}{12} + \frac{5 w_3^2}{8} \right) + O(\epsilon^3).$$

**Remark 3.** One of the advantages of representation (9) for the second-order normal form, is that it allows an easy implementation in any computer algebra system, as it does not involve the resolution of the homological equations. Indeed, the computations above were carried out with a package written in Maxima [15], available at http://galia.fc.uaslp.mx/~jvallejo/pdynamics.zip. It contains a detailed documentation illustrating its use with the preceding examples.

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