Images in Christmas baubles

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Abstract

We describe light-reflection properties of spherically curved mirrors, such as baubles on a Christmas tree. In particular, we study the position of the image which is formed somewhere beyond the surface of a spherical mirror, when an eye observes the image of a pointlike light source. The considered problem, originally posed by Abu Ali Hasan Ibn al-Haitham—alias Alhazen—more than a millennium ago, turned out to have the now well-known analytic solution of a biquadratic equation, being still of great relevance, e.g., for the aberration-free construction of telescopes. We do not attempt to perform an exhaustive survey of the rich historical and engineering literature on the subject, but develop a simple pedagogical approach to the issue, which we believe to be of continuing interest in view of its maltreatment in many high-school textbooks.

1. Introduction

The basic property of geometrical optics for reflection is the equality of the angles of incidence and reflection, given by the equation

\[ t = i. \] (1)

Here, we will demonstrate that equality (1) is not respected when analysing curved mirrors, in particular spherical mirrors, in the educational programmes for high-school and university students [1].

One of the corner stones of geometrical optics at the high-school level is the concept of an ideal lens, which reduces the lens to an optical plane, principal and auxiliary axes, and a focal plane with principal and auxiliary foci. The rules for image formation are clear: incident light rays parallel to an auxiliary axis are refracted in the lens such that the refracted light ray passes through the corresponding auxiliary focus in the case of a positive ideal lens, or seem to originate from the corresponding auxiliary focus for a negative ideal lens. No further rules are needed to construct images for any ideal lens or set of ideal lenses. This concept is not only useful, as it approximates well real lenses applied in optical instruments, but is also a practical application of Euclidean geometry.
Figure 1. The image of the light source is assumed at the intersection of the directions in which observers 2 and 3 are supposed to receive light rays originating from the light source and reflected by the mirror. The light ray for observer 2 seems to come from the direction of the supposed focus $F$ on the supposed principal axis of the mirror, since the incident light ray, from the light source to the mirror, is parallel to the supposed principal axis. Similarly, the light ray for observer 3 is reflected in the direction parallel to the supposed principal axis, since it is emitted by the light source in the direction of the supposed focus. In the right-hand figure, we show the incident and reflection angles with respect to the local normal direction at the various vertices.

However, ideal lenses should not be confused with ideal spherical mirrors. The ideal spherical mirror is a mirror in the form of a perfect sphere. No simple rules can be applied to construct its images. Nevertheless, almost all physics courses for high-school and undergraduate students contain figures similar to the one represented in figure 1(a), in order to demonstrate image formation in convex (or concave) spherical mirrors. In figure 1(b), we show the results of measuring the incident $i$ and reflection $t$ angles at each of the vertices, applying simple Euclidean geometry. Being rather obvious for observer 5, we actually find that none of the vertices respects relation (1). We must, therefore, conclude that figure 1(a) cannot be correct.

The latter conclusion is no surprise, of course. Since a perfect spherical mirror is spherically symmetric, any choice of principal axis is as good (or bad) as any other. Descartes’ formula for the position of an image does not imply the existence of a focus. It just provides a simple method to determine the place of the image for an observer positioned right behind the light source with respect to the centre of the sphere. For observers who are just a very small angle away from that direction, it constitutes a reasonable approximation. The method given by Descartes can thus perfectly well be applied to optical instruments, where long tubes guarantee that the angles involved are small. However, for the images observed in Christmas baubles, it is not a correct method to be applied.
In the following paragraphs, we shall show a simple method of how to determine the images seen by each of the observers in figure 1. We are well aware that similar demonstrations must have been presented long ago, but, unfortunately, we have not found any references to such work (see also [2]). Since, moreover, it seems to have been completely forgotten, we judged it useful to present our material to a wider public than just our students.

2. Reflection

Let us consider a pointlike light source and a spherical mirror. We assume that the eye lens is small enough to be considered pointlike as well. The relevant light rays must all be in the plane determined by the centre of the sphere, the position of the light source and the eye. Consequently, we may study the subject in two dimensions.

We take the centre of the circle, which has radius $R$, at the origin of the coordinate system. In figure 2, we show a light ray emitted by a light source located at $(x_s, y_s)$ and reflected by the spherically curved mirror at the vertex $(x, y)$.

Since we are essentially left with a two-dimensional problem, we may perform the geometrical calculations by using complex numbers. Thus, let us define the complex numbers $r_s$ (object location) and $r$ (vertex location), i.e.,

$$r_s = x_s + iy_s$$

and

$$r = x + iy = R e^{i\beta}.$$  

The vector $r_s - r$ points from the vertex to the light source (object) along the incident light ray. Furthermore, the angle $i = t \equiv \varphi$ (see figure 2) is defined as the angle between $r_s - r$ and $r$. Consequently,

$$\frac{r}{r_s - r} = \left| \frac{r}{r_s - r} e^{i\varphi} \right| \quad \text{or} \quad e^{i\varphi} = \left| \frac{r_s - r}{r} \right| \left| \frac{r}{r_s - r} \right|.$$  

When by using equation (3) we rotate the vector $r_s - r$ over an angle $2\varphi$, i.e.,

$$(r_s - r) e^{2i\varphi} = (r_s - r) \left( \left| \frac{r_s - r}{r} \right| \frac{r}{r_s - r} \right)^2 = \frac{r_s^* - r^*}{r^*} r,$$  

we obtain a vector in the direction of the reflected light ray. Using now equation (4), we deduce the following expression for the straight line which coincides with the reflected light ray ($\lambda$ real):

$$s(\lambda) = r + \lambda \frac{r_s^* - r^*}{r^*} r.$$  

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**Figure 2.** A light ray emitted by the light source at $(x_s, y_s)$ is reflected by the spherical mirror. The incident angle $i$ and the reflection angle $t$ at the surface of the mirror are equal: $i = t \equiv \varphi$. The vector connecting the coordinate origin to the vertex at $(x, y)$ makes an angle $\beta$ with the horizontal axis.
Figure 3. Light rays which are reflected in nearby directions, intersect at nearly the same location.

### 3. The image

The small bundle of light rays which strikes the eye lens has a very small area of convergence, as can be understood from figure 3. This small area is where the eye supposes the light rays are stemming from, hence the image of the light source.

By varying the angle $\beta$ in figure 2, we obtain different trajectories for light rays originating from the same light source at $r_s$. We are here interested in the intersection of the straight lines that coincide with the reflected rays of two different suchlike trajectories. We indicate their vertices by $r$ and $r' = r + \Delta r$, respectively. We define the image of the light source, for observers in the direction characterized by $r$, as the point of intersection of the two trajectories, in the limit of vanishing $|\Delta r|$. In the appendix, it is shown how one obtains, thus, the position $r_i$ of the image of the light source. From equation (A.12), one easily deduces the expression

$$
 r_i = r - \frac{3|\Delta r|^2 - r_s^* r_*^* \frac{2}{r} - 2r_s r^*}{4|\Delta r|^2 - 3r_s r^* - 3r_s^* r + 2|\Delta r|^2}.
$$

(6)

The denominator in this fraction is real. Consequently, it is an easy task to extract the $x$ and $y$ components of the image position. For a light source on the $x$ axis, one gets

$$
\begin{align*}
(x_i) = \frac{\left(\frac{R}{2}x^2 \cos(\beta) \left[1 + 2 \sin^2(\beta)\right] - R x_i\right)}{R^2 + 2x^2 - 3R x_i \cos(\beta)}, \\
y_i = \frac{2R x_i^2 \sin^2(\beta)}{R^2 + 2x^2 - 3R x_i \cos(\beta)}.
\end{align*}
$$

(7)

In figure 4, we show some pictures. We indicate the positions of the light source, the eye and the centre of the spherical mirror. Three different light rays are reflected on the surface of the sphere and then reach the eye. We assume the location of the image to lie where these rays intersect one another. A small dot labelled by $i$ indicates the position of the image for the central direction, which follows from equation (7). The image in the upper-left picture of figure 4 can approximately be reconstructed from Descartes’ formula, but the other three images come obviously at different locations.

The reason that we see a ‘sharp’ image of the pointlike light source is due to the smallness of our eye lens. More accurately, it stems from the ratio of the diameter of the pupil of the eye and the curvature radius $R$ of the spherical mirror. Hence, perfectly sharp images are only to be expected for an infinite radius of curvature, which just represents a flat mirror. Nevertheless, for large spherical mirrors, the resolution of the virtual image observed by the eye is good enough to be considered sharp. However, its location depends on the position of the eye.
For a light source on the $x$ axis ($y_s = 0$) and for $\beta = 0$, we obtain for the image location

$$x_i = R \frac{x_s}{2x_s - R} \quad \text{and} \quad y_i = 0.$$  \hfill (8)

Moreover, one then finds

$$\frac{1}{x_s} + \frac{1}{x_i} = \frac{1}{x_s} + \frac{2x_s - R}{Rx_s} = \frac{2}{R},$$  \hfill (9)

which is Descartes’ formula for spherical mirrors.

4. The angle $\beta$

The question is how to determine the angle $\beta$ in the case where the positions of the light source $(x_s, y_s)$ and the eye $(x_0, y_0)$ are given.

Let us define the complex number $r_o$ for the coordinates of the eye lens, i.e.,

$$r_o = x_o + iy_o.$$  \hfill (10)

From figure 2, and in analogy with equation (3), we obtain

$$r_o - r = |r_o - r| e^{i(\beta + \phi)} \quad \text{and} \quad \frac{r_o - r}{r} = \frac{r_o - r}{r} e^{i\phi}.$$  \hfill (11)

Combining equations (3) and (11) gives

$$|r_s - r| \frac{r}{r_s - r} = e^{i\phi} = \frac{r}{r_o - r} \frac{r_o - r}{r}.$$  \hfill (12)
Figure 5. Two different situations for the initial positions of the two billiard balls. In the left-hand picture, we have an example where there are only two solutions to equation (15). The right-hand picture shows an example where, by just selecting a different position of the red ball, we find four solutions.

or

$$\frac{(r_s - r)(r_o - r)}{r^2} = \left| \frac{(r_s - r)(r_o - r)}{r^2} \right|.$$  \hspace{1cm} (13)

The right-hand side of equation (13) is real, whence for the imaginary part of the left-hand side we conclude

$$\frac{(r_s - r)(r_o - r)}{r^2} - \left( \frac{(r_s - r)(r_o - r)}{r^2} \right)^* = 0,$$

which is equivalent to the fourth-order equation

$$\frac{r_s^* r_o^*}{R^2} (e^{i\beta})^4 - \frac{r_s^*}{R} + \frac{r_o^*}{R} (e^{i\beta})^3 + \frac{r_s + r_o}{R} (e^{i\beta}) - \frac{r_s r_o}{R^2} = 0.$$  \hspace{1cm} (15)

4.1. The solutions to equation (15)

The biquadratic equation (15) is in the literature known as the billiard problem of al-Hasan, named after the Arab scientist Abu Ali al-Hasan ibn al-Haytham (965–1040 AD) [3, 4]. It corresponds to the following problem, which had already been formulated in 150 AD by the Greek scientist Ptolemy: ‘Find, for arbitrary initial positions of the red ball and the white ball, the point on the edge of a circular billiard table at which the white ball must be aimed in order to carom once off the edge and collide head-on with the red ball’.

From the definitions of $r_s, r_o, r$ and the subsequent definition of $\beta$ in equation (2), it may be clear that reflection in the interior of the circle representing the cross section of the spherical mirror can be studied in exactly the same way as reflection in the exterior, by considering $r - r_s$ instead of $r_s - r$ and $r - r_o$ instead of $r_o - r$, which leads to exactly the same equation (15).

In general, there are four solutions $z_1, z_2, z_3$ and $z_4$ to the equation

$$\frac{r_s^* r_o^*}{R^2} z^4 - \frac{r_s^*}{R} z^3 + \frac{r_s + r_o}{R} z - \frac{r_s r_o}{R^2} = 0.$$

However, when $|z| \neq 1$, the solution to equation (16) is not a physical solution to equation (15). Although we have not studied the situation exhaustively, we find that generally there are two
Figure 6. The locations of the various images as seen by each of the five observers introduced in figure 1. We also indicate the angles of incidence and reflection, in order to make sure that they are equal.

Figure 7. Images of an extended object, as seen in a Christmas bauble from different angles.

solutions to equation (15), but sometimes there are four (see also [5]), depending on the complex parameters \( r_s \) and \( r_o \). The different cases, corresponding to \( r_s = 0, r_o = 0, r_s = r_o = 0, |r_s| = R, |r_o| = R \) and \( |r_s| = |r_o| = R \), must be studied separately. In figure 5, we show two examples for a circular billiard.
5. Deformation

Now that we have solved the problem of finding the vertex on the curved mirror where the light ray passing from the object to the eye is reflected, we may construct the images for the situation shown in figure 1. The result is depicted in figure 6.

Furthermore, we may now construct the images of extended objects and study their deformation. In figure 7, we give a few simple examples.

6. Conclusions

The complete resolution of the billiard problem (15), posed by al-Hasan in the context of what nowadays is called geometric optics, had to wait for many centuries. First, it was necessary to derive an analytic form for the solutions to biquadratic equations like equation (16). Apparently found by Ferrari, it was published for the first time in Cardano’s ‘Ars Magna’ in 1545. The first to subsequently solve the billiard problem of al-Hasan was Christian Huygens.

The problem would probably have ended up in the history books, since ‘once solved, forever solved’. However, for optical equipment, where the angles of reflection $\varphi$ are small, Descartes formulated an approximation, amounting to basically linear equations. Anything that has larger reflection angles is considered an aberration in his philosophy. Ever since, a whole generation of physicists emerged completely unaware of al-Hasan’s billiard problem, who systematically applied Descartes’ approximation also to large angles $\varphi$, thus arriving at totally wrong conclusions. Therefore, we should highlight al-Hasan’s billiard problem and reintroduce it into our optics courses, so that new generations can fully enjoy the perfectly sharp images in Christmas baubles, without feeling uneasy for having been told that such wonderful images should be considered aberrations.

For undergraduate students the full subject can be treated using some simple computer programming. One can either choose a point on the spherical mirror and construct the image by the use of formulae (7) and from the direction of the reflected light ray, or one can select the position of the eye pupil and determine the vertex by means of equation (15). However, for youngsters at high schools, we recommend one only deal with it qualitatively, using examples such as figures 3 and 6.

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Appendix. The point of convergence for adjacent reflected light rays

We consider here two reflected light rays, characterized by two different vertices $r$ and $r'$. In accordance with formula (5), we may represent the two straight lines coinciding with the two reflected rays by $(\lambda, \lambda’$ real)

$$s(\lambda) = r + \lambda \frac{r^s - r^r}{r^r} r$$

and

$$s'(\lambda') = r' + \lambda' \frac{r'^s - r'^r}{r'^r} r'.$$

(A.1)

The intersection of the two trajectories follows from

$$s(\lambda) = s'(\lambda'),$$

(A.2)
which, since $\lambda$ and $\lambda'$ are real, actually corresponds to two equations, one for the real part and one for the imaginary part, namely,

$$s(\lambda) + s^*(\lambda) = s'(\lambda') + s'^*(\lambda').$$  \hfill (A.3)

The system of equations (A.3) can be solved for $\lambda$ and $\lambda'$.

In order to avoid lengthy formulae, we define

$$A = \frac{r^2 - r^* r}{r^*} \quad \text{and} \quad A' = \frac{r^2 - r'^* r'}{r'^*}.$$  \hfill (A.4)

Using definitions (A.4) and formula (A.1), we obtain

$$s(\lambda) \pm s^*(\lambda) = r \pm r^* + \lambda(A \pm A^*), \quad \text{and} \quad s'(\lambda') \pm s'^*(\lambda') = r' \pm r'^* + \lambda'(A' \pm A'^*).$$  \hfill (A.5)

Inserting the result (A.5) into the two relations (A.3) gives the two equations

$$(A \pm A^*)\lambda - (A' \pm A'^*)\lambda' = -r \mp r^* + r' \pm r'^*,$$  \hfill (A.6)

which can be cast in matrix form and are readily solved by

$$\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix} = \begin{pmatrix} A + A^* & -A' - A'^* \\ A - A^* & -A' + A'^* \end{pmatrix}^{-1} \begin{pmatrix} -r - r^* + r' + r'^* \\ -r + r^* + r' - r'^* \end{pmatrix}$$

$$= \begin{pmatrix} A(r^* - r'^*) - A'^*(r - r') \\ A(r - r'^*) - A^*(r - r') \end{pmatrix}.$$

\hfill (A.7)

Next, we take $r' = r + \Delta r$ and expand the various terms of equation (A.7) to first order in $\Delta r$ and $\Delta r^*$. In the limit $|\Delta r| \downarrow 0$, one has moreover

$$\frac{\Delta r^*}{\Delta r} \rightarrow -\frac{r^*}{r}.$$  \hfill (A.8)

This can most easily be understood, when one, using equation (2), defines

$$r + \Delta r = R e^{i(\beta + \Delta \beta)} \approx R e^{i\beta} (1 + i \Delta \beta),$$

hence, to lowest order in $\Delta \beta$, one finds $\Delta r = i R e^{i\beta} \Delta \beta$. Consequently, since $R$, $\beta$ and $\Delta \beta$ are real,

$$\frac{\Delta r^*}{\Delta r} \rightarrow -\frac{R e^{-i\beta}}{R e^{i\beta}} = -\frac{r^*}{r}.$$  \hfill (A.9)

Using, moreover, the definitions (A.4), we determine, in the limit $|\Delta r| \downarrow 0$,

$$A' = \frac{A^* - r^* - \Delta r^*}{r^* + \Delta r^*} (r + \Delta r) \approx A \left( 1 + \frac{\Delta r}{r} - \frac{\Delta r^*}{r^*} \right) - \frac{r}{r^*} \Delta r^*$$

$$\approx A + \frac{\Delta r}{r} \left( A - \frac{r}{r^*} \frac{\Delta r^*}{\Delta r} - \frac{r}{r^*} \frac{\Delta r^*}{\Delta r} \right) \rightarrow A + \frac{\Delta r}{r} (2A + r),$$

and similarly,

$$AA'^* - A^* A' \rightarrow \frac{\Delta r}{r} (-4|A|^2 - Ar^* - A^* r),$$  \hfill (A.9)

$$A'(r^* - r'^*) - A'^*(r - r') \rightarrow \frac{\Delta r}{r} (Ar^* + A^* r).$$  \hfill (A.10)

On substitution of the results (A.9) and (A.10) into expressions (A.7), we obtain for the parameter $\lambda$ at the point of intersection, in the limit of vanishing $|\Delta r|$, the result

$$\lambda = -\frac{Ar^* + A^* r}{4|A|^2 + Ar^* + A^* r}.$$  \hfill (A.11)
Finally, using equation (A.11), we determine the point of intersection, \( r_i \), of the two straight lines (A.1) in the limit of vanishing \(|\Delta r|\):

\[
    r_i = r + \lambda A = \frac{4|A|^2 r + (Ar^* + A^* r)(r - A)}{4|A|^2 + Ar^* + A^* r}.
\]  

(A.12)

The variable \( \lambda \) indicates how deep the ‘image’ is below the surface of the spherical mirror, measured from the mirror surface, along the direction of reflection, to the point where the small bundle of reflected rays seems to emerge. Alternatively, for practical purposes, one could determine this ‘depth’ from

\[
    \frac{1}{|\lambda A|} = \frac{2}{R \cos(\varphi)} + \frac{1}{|r_i - r|}.
\]

One measures with a ruler, the lengths of \(|\vec{r}_i - \vec{r}|\) and \(R \cos(\varphi)\), the latter equals half the distance from the vertex to the other intersection of the mirror and the line which coincides with the reflected light ray, and performs the above calculation. For small angles \(\varphi\), one obtains the image location of Descartes, whereas for larger angles the location of the image is different. For \(\varphi = 90^\circ\), one obtains zero depth.

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