Strong Feller properties and uniqueness of sticky reflected distorted Brownian motion

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Abstract

For a specified class of drift functions we construct a strong Feller transition semigroup for sticky reflected distorted Brownian motion on \( E := [0, \infty)^n, n \in \mathbb{N} \), in order to improve the "quasi everywhere" statements of [FGV14] to "everywhere" statements. In particular, the relations of the underlying Dirichlet form to random time changes and Girsanov transformations are presented. Moreover, we prove uniqueness of weak solutions to the corresponding stochastic differential equation.

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1 Introduction

In [FGV14] the authors constructed via Dirichlet form techniques a reflected distorted Brownian motion in \( E := [0, \infty)^n, n \in \mathbb{N} \), with sticky boundary behavior which solves the system of stochastic differential equations

\[
\begin{align*}
 \text{d}X^j_t &= \mathbb{1}_{(0, \infty)}(X^j_t) \left( \sqrt{2} \text{d}B^j_t + \partial_j \ln(\varrho)(X^j_t) \, \text{d}t \right) + \frac{1}{\beta} \mathbb{1}_{(0)}(X^j_t) \, \text{d}t, \quad j \in I, \\
\text{d}^{0,j}_t &= \mathbb{1}_{(0, \infty)}(X^j_t) \left( \sqrt{2} \text{d}B^j_t + \partial_j \ln(\varrho)(X^j_t) \, \text{d}t \right) + \text{d}^{0,j}_t,
\end{align*}
\]

or equivalently

weakly for quasi every starting point with respect to the underlying Dirichlet form. Here \( I := \{1, \ldots, n\} \), \( \beta \) is a real and positive constant and \( (B^j_t)_{t \geq 0} \) are one dimensional independent standard Brownian motions, \( j \in I \). \( \varrho \) is a continuously differentiable density on \( E \) such that for all \( B \subset I \), \( \varrho \) is...
almost everywhere positive on $E_+(B)$ with respect to the Lebesgue measure and for all $\emptyset \neq B \subset I$, $\sqrt{\phi} E_+(B)$ is in the Sobolev space of weakly differentiable functions on $E_+(B)$, square integrable together with its derivative, where $E_+(B) := \{x \in E \mid x_i > 0 \text{ for all } i \in B, \; x_i = 0 \text{ for all } i \in I \setminus B\}$. $\phi$ continuously differentiable on $E$ implies that the drift part $(\partial_j \ln(\phi))_{j \in I}$ is continuous on $\{\phi > 0\}$. Moreover, $\ell_{t}^{0,j}$ is the central local time of the solution to (1.1), i.e., it holds almost surely

$$\ell_{t}^{0,j} = \frac{1}{\beta} \int_{0}^{t} 1_{\{0\}}(X_{s}^{j}) \, ds = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{0}^{t} 1_{(-\varepsilon,\varepsilon)}(X_{s}^{j}) \, d(X_{s}^{j})_s.$$ 

A solution to the associated martingale problem is even provided under the weaker assumptions that $\phi$ is almost everywhere positive, integrable on each set $E_+(B)$ with respect to the Lebesgue measure and that the respective Hamza condition is fulfilled. This kind of stochastic differential equation is strongly related to the sticky Brownian motion on the half-line $[0, \infty)$ (which is occasionally also called Brownian motion with delayed reflection or slowly reflecting Brownian motion). In [EP12] the authors study Brownian motion on $[0, \infty)$ with sticky boundary behavior and provide existence and uniqueness of solutions to the SDE system

$$\begin{cases} 
\mathrm{d}X_t = \frac{\beta}{2} \ell_{t}^{0,+}(X) + 1_{(0,\infty)}(X_t) \, dB_t \\
1_{\{0\}} \, dt = \frac{1}{\mu} \ell_{t}^{0,+}(X), 
\end{cases}$$

for reflecting Brownian motion $X$ in $[0, \infty)$ sticky at 0, where $X := (X_t)_{t \geq 0}$ starts at $x \in [0, \infty)$, $\mu \in (0, \infty)$ is a given constant, $\ell_{t}^{0,+}(X)$ is the right local time of $X$ at 0 and $B := (B_t)_{t \geq 0}$ is the standard Brownian motion. In particular, H.-J. Engelbert and G. Peskir show that the system (1.3) has a jointly unique weak solution and moreover, they prove that the system (1.3) has no strong solution, thus verifying Skorokhod’s conjecture of the non-existence of a strong solution in this case. For an outline of the historical evolution in the study of sticky Brownian motion we refer to the references given in [EP12] and also to [KPS10].

The results of [FGV14] apply to the so-called wetting model (also refered to as the Ginzburg-Landau $\nabla \phi$ interface model with entropic repulsion and pinning). More precisely, in a finite volume $\Lambda \subset \mathbb{Z}^d$, $d \in \mathbb{N}$, the scalar field $\phi_t := (\phi_t(x))_{x \in \Lambda}$, $t \geq 0$, is described by the stochastic differential equations

$$d\phi_t(x) = -1_{(0,\infty)}(\phi_t(x)) \sum_{y \in \Lambda} \frac{V'(\phi_t(x) - \phi_t(y))}{|x-y|=1} dt + 1_{(0,\infty)}(\phi_t(x)) \sqrt{2d} B_t(x) + d\ell_0^\phi(x), \quad x \in \Lambda, \quad (1.4)$$

subject to the conditions:

$$\phi_t(x) \geq 0, \quad \ell_0^\phi(x) \text{ is non-decreasing with respect to } t, \quad \ell_0^\phi(x) = 0,$$

$$\int_{0}^{\infty} \phi_t(x) \, d\ell_0^\phi(x) = 0,$$

$$\beta \ell_0^\phi(x) = \int_{0}^{t} 1_{\{0\}}(\phi_s(x)) \, ds \quad \text{for fixed } \beta > 0,$$

where $\ell_0^\phi(x)$ denotes the local time of $\phi_t(x)$ at 0. Here $|\cdot|$ denotes the norm induced by the euclidean scalar product on $\mathbb{R}^d$, $V \in C^2(\mathbb{R})$ is a symmetric, strictly convex potential and $\{(B_t(x))_{t \geq 0} \mid x \in \Lambda\}$
are independent standard Brownian motions. In dimension $d=2$ this model describes the wetting of a solid surface by a fluid. More details on interface models are presented in e.g. [Gia02], [Fun05]. In [Fun05, Sect. 15.1] J.D. Deuschel and T. Funaki investigated (1.4) and gave reference to classical solution techniques as developed e.g. in [IW89]. The methods provided therein require more restrictive assumptions on the drift part as in our situation (e.g. the drift is assumed to be bounded and Lipschitz continuous), moreover, do not apply directly (the geometry and the behavior on the boundary differs). First steps in the direction of applying [IW89] are discussed in [Fun05] by J.-D. Deuschel and T. Funaki.

As far as we know the only reference that applies to the system of stochastic differential equations (1.4) is [Gra88]. By means of a suitable choice of the coefficients the system of equations given by [Gra88, (II.1)] coincides with (1.4), but amongst others the drift part is also assumed to be Lipschitz continuous and boundend. For this reason, it is not possible to apply the results of [Gra88] to the setting investigated by J.-D. Deuschel and T. Funaki, since the potential $V$ naturally causes an unbounded drift.

In view of the results provided in [EP12], the construction of a weak solution as given in [FGV14] is the only reasonable one. However, the construction via Dirichlet form techniques has the well-known disadvantage that the constructed process solves the underlying stochastic differential equation only for quasi-every starting point with respect to the Dirichlet form. Hence, in the present paper we investigate properties of the corresponding semigroup in order to strengthen the results of [FGV14], i.e., we construct a solution to (1.1) and show an ergodicity theorem for every starting point in the state space $E$ under the assumptions on the density given in [Fun05]. Moreover, we establish connections between our Dirichlet form construction procedure and classical probabilistic methods. Using this relations, we additionally prove uniqueness of weak solutions to (1.1).

In the theory of Dirichlet forms it is a common approach to use results of the regularity theory of elliptic partial differential equations in order to deduce that the associated resolvent and semigroup admit a certain regularity and thereby, it is possible to construct a pointwise solution to the underlying martingale problem or stochastic differential equation for an explicitly known set of starting points under very weak assumptions on the density $\rho$. For example, this has recently been realized in case of distorted Brownian motion on $\mathbb{R}^d$, $d \in \mathbb{N}$, in [AKR03], in case of absorbing distorted Brownian motion on $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, in [BGS13] and in case of reflecting distorted Brownian motion on $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, under some smoothness condition on the boundary $\partial \Omega$ in [FG07] and [BG14]. However, in the present setting which involves multiple measures on the boundary of the state space $E$ the elliptic regularity theory is not yet investigated and from our present point of view the required results are hard to prove. For this reason, we choose the more probabilistic approach of random time changes and Girsanov transformations.

Our paper is organized as follows. In Section 2 we state our main results. In Section 3 we recall some facts about sticky Brownian motion and present the connections of the Dirichlet form constructed in [FGV14] to classical methods from probability theory. In particular, we establish relations to random time changes and Girsanov transformations. In Section 4 a Feller transition semigroup is constructed. Moreover, this semigroup is used to construct a pointwise solution to (1.1) and the corresponding Dirichlet form is identified. Finally, we prove uniqueness of weak solutions to (1.1) in Section 5.
2 Main results

Let \( \varrho : E \to (0, \infty) \) be defined by a potential \( H \) with nearest neighbor pair interaction, i.e., \( \varrho = \exp(-H) \) and \( H \) is given by

\[
H(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i,j \in \{0, \ldots, n+1\}, |i-j|=1} V(x_i - x_j),
\]

where \( x_0 := x_{n+1} := 0 \) and \( V : \mathbb{R} \to [-b, \infty), b \in [0, \infty) \), fulfills the conditions of [Fum05, (2.2)]:

(i) \( V \in C^2(\mathbb{R}) \),

(ii) \( V \) is symmetric, i.e., \( V(r) = V(-r) \) for all \( r \in \mathbb{R} \),

(iii) \( V \) is strictly convex, i.e., \( c_- \leq V''(r) \leq c_+ \) for all \( r \in \mathbb{R} \) and some constants \( c_-, c_+ > 0 \).

Denote by \( \phi := \sqrt{\varrho} \) the square root of \( \varrho \). In the following we denote by \( dx_i \) the one dimensional Lebesgue measure and by \( \delta^i_0 \) the Dirac measure in 0, where \( i = 1, \ldots, n \) gives reference to the component of \( x = (x_1, \ldots, x_n) \in E \). Define the product measure \( d\mu_n := \prod_{i=0}^n (dx_i + \beta \delta^i_0) \) on \((E, \mathcal{B}(E))\).

Under the above assumptions on \( \varrho \) it holds:

**Theorem 2.1.** There exists a conservative diffusion process \( \mathbb{M}^\varrho = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_0^x)_{x \in E}) \) on \( E \) with strong Feller transition function \( (p_t)_{t \geq 0} \), i.e., \( p_t(\mathcal{B}_0(E)) \subset C_b(E) \), such that the associated Dirichlet form is given by the closure of the symmetric bilinear form \((\mathcal{E}^\varrho, \mathcal{D})\) on \( L^2(E; g d\mu_n) \), where

\[
\mathcal{E}^\varrho(f,g) := \sum_{B \in \mathcal{B} \setminus \{1, \ldots, n\}} \mathcal{E}_B(f,g) = \int_E \sum_{i=1}^n 1_{\{x_i \neq 0\}} \partial_i f \partial_i g \ g d\mu_n \quad \text{for } f, g \in \mathcal{D} := C^1_c(E)
\]

with

\[
\mathcal{E}_B(f,g) := \int_E \sum_{i \in B} \partial_i f \partial_i g \ g d\lambda_B^{n,\beta},
\]

where \( d\lambda_B^{n,\beta} := \beta^{n-\#B} \prod_{j \in B} dx_j \prod_{j \in B^c} \delta^j_0 \). In particular, \( (p_t)_{t \geq 0} \) fulfills the absolute continuity condition [FOTTT1, (4.2.9)].

Let \( \mathcal{V}'(i, x) \) for \( i = 1, \ldots, n \) and \( x \in E \) be given by

\[
\mathcal{V}'(i, x) := \frac{1}{2} \sum_{j \in \{0, \ldots, n+1\}, |i-j|=1} V'(x_i - x_j).
\]

**Theorem 2.2.** Let \( \mathbb{M}^\varrho \) be the diffusion process of Theorem 2.1. It holds for each \( i = 1, \ldots, n \)

\[
X_t^i = X_0^i + \sqrt{2} \int_0^t 1_{(0, \infty)}(X_s^i) dB_s^i - \int_0^t 1_{(0, \infty)}(X_s^i) \mathcal{V}'(i, X_s) ds + \frac{1}{\beta} \int_0^t 1_{(0, \infty)}(X_s^i) ds \quad (2.2)
\]
\textbf{3 Sticky Brownian motion and Dirichlet form transformations}

\section{Sticky Brownian motion on the halfline}

Define the Dirichlet form

\begin{equation}
\mathbb{P}_x^\delta \text{-a.s. for every } x \in E, \text{ where } (B^i_t)_{t \geq 0}, \ i = 1, \ldots, n, \text{ are independent standard Brownian motions. Moreover, it holds}
\lim_{t \to \infty} \frac{1}{t} \int_0^t F(X_s)ds = \frac{\int_E F \, d\mu}{\int_E \, d\mu}
\end{equation}

\textbf{Remark 2.3.} (i) Note that the drift function $\mathbb{V}'(i, x)$ is not necessarily bounded, but $\mathbb{V}'(i, x) \geq 0$ for all $i = 1, \ldots, n$ and $x \in E$.

(ii) Let $\Gamma \subset \partial E$ such that $\int_\Gamma \, d\mu > 0$. Then it follows by (2.3) that

\begin{equation}
\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{1}_\Gamma(X_s)ds = \frac{\int_\Gamma \, d\mu}{\int_E \, d\mu} > 0
\end{equation}

\textbf{P}^\delta_x \text{-a.s. for every } x \in E. \text{ This confirms the sticky behavior of the process on the boundary.}

\textbf{Theorem 2.4.} The solution to (2.2) is unique in law.

\section{Sticky Brownian motion and Dirichlet form transformations}

\subsection{Sticky Brownian motion on the halfline}

Define the Dirichlet form $(\hat{\mathbb{E}}, D(\hat{\mathbb{E}}))$ as the closure of

$$
\hat{\mathbb{E}}(f, g) := \int_{[0, \infty)} f'(x)g'(x)dx, \ f, g \in C^1_c([0, \infty)),
$$

on $L^2([0, \infty); dx)$. It is well-known that reflecting Brownian motion is associated to $(\hat{\mathbb{E}}, D(\hat{\mathbb{E}}))$ and $D(\hat{\mathbb{E}}) = H^{1,2}((0, \infty))$ is the Sobolev space of order one.

Let $(\hat{B}_t)_{t \geq 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $\hat{X}_t := |x + \sqrt{2}\hat{B}_t|$, $t \geq 0$, yields reflecting Brownian motion on $[0, \infty)$ starting at $x \in [0, \infty)$ and by Tanaka’s formula

$$
\hat{X}_t = x + \sqrt{2}\hat{B}_t + L_{t}^{0+}, \ t \geq 0,
$$

where $\hat{B}_t := \int_0^t \text{sgn}(x + \sqrt{2}\hat{B}_s)d\hat{B}_s$, $t \geq 0$, is a standard Brownian motion and $(L_{t}^{0+})_{t \geq 0}$ is the right local time in 0, i.e.,

$$
L_{t}^{0+} = \lim_{\epsilon \to 0} \int_0^t 1_{[0, \epsilon)}(\hat{X}_s)ds
$$

in probability. The Dirichlet form associated to $(\hat{X}_t)_{t \geq 0}$ is $(\hat{\mathbb{E}}, D(\hat{\mathbb{E}}))$ and $(L_{t}^{0+})_{t \geq 0}$ is an additive functional which is in Revuz correspondence with the Dirac measure $\delta_0$ in 0. Consider the additive functional $A_t := t + \beta L_{t}^{0+}, \ t \geq 0$, for some real constant $\beta > 0$. Note that $A_0 = 0$ and $A_t \to \infty$ a.s. as $t \to \infty$. Then sticky Brownian motion on $[0, \infty)$ is usually constructed by a random time change using the inverse $\tau(t)$ of $A_t$. More precisely, $X_t := \hat{X}_{\tau(t)}$ (starting in $x$) solves the stochastic differential equation

$$
dX_t = 1_{(0,\infty)}(X_t)\sqrt{2}dB_t + \frac{1}{\beta} 1_{\{0\}}(X_t)dt,
$$

(3.2)
where \((B_t)_{t \geq 0}\) is a standard Brownian motion. For details on Feller’s Brownian motions and in particular, sticky Brownian motion and its transition semigroup, see e.g. \([\text{EP12}], \text{KPS10}, \text{GS72}\) or \([\text{Kni81}]\).

In \([\text{FOT11}]\) Chapter 6] and \([\text{CF11}]\) Chapter 5] is presented how a random time change by an additive functional affects the underlying Dirichlet form. Let \(\mu\) denote the Revuz measure corresponding to \((A_t)_{t \geq 0}\). Clearly, \(d\mu = dx + \beta \delta_0\). In particular, \(\mu\) has full support \([0, \infty)\). Thus, the Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) associated to \((X_t)_{t \geq 0}\) has the representation

\[
\mathcal{E}(f, g) = \hat{\mathcal{E}}(f, g) = D(\mathcal{E}) = D(\hat{\mathcal{E}}) \cap L^2([0, \infty); d\mu).
\]

In particular, \(D(\mathcal{E}) = H^{1, 2}([0, \infty)) \cap L^2([0, \infty); d\mu) = H^{1, 2}(\mathbb{R})\) by Sobolev embedding. Moreover, \(C^1_c([0, \infty))\) is dense in \(D(\mathcal{E})\) by \([\text{CF11}]\) Theorem 5.2.8(i) and thus, it is a special standard core of \((\mathcal{E}, D(\mathcal{E}))\). Hence, the closure of

\[
\mathcal{E}(f, g) = \int_{[0, \infty)} f'(x) g'(x) dx = \int_{[0, \infty)} \frac{x}{(1 + x^2)^{3/2}} f'(x) g'(x) dx, \quad f, g \in C^1_c([0, \infty)), \quad \text{(3.3)}
\]

on \(L^2([0, \infty); d\mu)\) is the Dirichlet form associated to \((X_t)_{t \geq 0}\).

Remark 3.1. Note that our notion for the solution to the equations \((3.1)\) and \((3.2)\) as reflecting Brownian motion and sticky reflecting Brownian motion on \([0, \infty)\) respectively differs by the factor \(\sqrt{2}\) from classical literature in view of the underlying SDE \((1.1)\). If \((Y_t^\gamma)_{t \geq 0}\) solves

\[
dY_t^\gamma = \mathbb{1}_{[0, \infty)}(Y_t^\gamma) dB_t + \frac{1}{\gamma} \mathbb{1}_{[0, \gamma)}(Y_t^\gamma) dt \quad \text{for } \gamma > 0,
\]

we obtain the solution to \((3.2)\) by setting \(X_t := \sqrt{2} Y_t^\sqrt{2\beta}\). This identity is useful in order to derive the resolvent density and transition density for the solution to \((3.2)\).

Let \(F\) be a locally compact separable metric space and denote by \(C_0(F) := \{f \in C(F) | \forall \epsilon > 0 \exists K \subset F \text{ compact} : |f(x)| < \epsilon \forall x \in F \setminus K\}\) the space of continuous functions on \(F\) vanishing at infinity. We can specify the resolvent and transition semigroup of sticky Brownian motion on \([0, \infty)\). \([\text{KPS10}]\) Corollary 3.10, Corollary 3.11 state the following (see also \([\text{Kni81}]\) Section 6.1):

Theorem 3.2. The transition function of sticky Brownian motion on \([0, \infty)\) yields a Feller semigroup on \(C_0([0, \infty))\), i.e., \(p_t(C_0([0, \infty))) \subset C_0([0, \infty))\) and \(\lim_{t \to 0} \|p_t f - f\|_\infty = 0\) for each \(f \in C_0([0, \infty))\). For \(\lambda > 0, x, y \in [0, \infty)\), the resolvent kernel \(r^\beta_\lambda(x, dy)\) of the Brownian motion with sticky origin (i.e., the solution to \((3.2)\)) is given by

\[
r^\beta_\lambda(x, dy) = \frac{r^\lambda_D(x, \sqrt{2} y)}{\sqrt{2}} dy + \frac{1}{2(\sqrt{2} \lambda + \beta \lambda)} (2e^{-\sqrt{2} \lambda(x+y)} dy + \sqrt{2} \beta |2 \beta - \sqrt{2} \lambda| y \delta_0(dy)), \quad \text{(3.4)}
\]

where \(r^\lambda_D(x, y) = \frac{1}{\sqrt{2} \lambda} (e^{-\sqrt{2} \lambda |x-y|} - e^{-\sqrt{2} \lambda (x+y)})\) is the resolvent density of Brownian motion with Dirichlet boundary conditions.

Furthermore, by the inverse Laplace transform it follows that, for \(t > 0\), the transition kernel \(p^\beta(t, x, dy)\) of the Brownian motion with sticky origin is given by

\[
p^\beta(t, x, dy) = \frac{p^D(t, x, \sqrt{2} y)}{\sqrt{2}} dy + 2 g_0, \sqrt{2} \beta (t, x + \sqrt{2} y) dy + \sqrt{2} \beta g_0, \sqrt{2} \beta (t, x) \delta_0(dy), \quad \text{(3.5)}
\]
where \( p^D(t,x,y) = p(t,x,y) - p(t,x,-y) \) is the transition density for Brownian motion with Dirichlet boundary conditions, \( p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \) and

\[
g_{0,\gamma}(t,x) = \frac{1}{\gamma} \exp\left(\frac{2x}{\gamma} + \frac{2t}{\gamma^2}\right) \operatorname{erfc}\left(\frac{x}{\sqrt{2t}} + \frac{\sqrt{2t}}{\gamma}\right), \quad \text{for } \gamma > 0, \ t > 0, \ x \geq 0,
\]

with the complementary errorfunction \( \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} \, dz, \ x \in \mathbb{R} \).

**Remark 3.3.** Note that (3.4) implies that \( p^\beta(t,x,\cdot) \) is absolutely continuous with respect to the measure \( d\mu = dx + \beta \delta_0 \) for each \( x \in [0,\infty) \), \( t > 0 \). Therefore, the so-called absolute continuity condition \([\text{FOT11}, (4.2.9)]\) is fulfilled. In the following we see that the transition semigroup possesses even stronger properties.

Thus, with \( p^\beta(t,x,dy) \) as above and \( p^\beta_t \), \( t > 0 \), the transition semigroup of sticky Brownian motion starting in \( x \in [0,\infty) \), it holds

\[
\mathbb{E}_x(f(X_t)) = p^\beta_t f(x) = \int_{[0,\infty)} f(y) \, p^\beta(t,x,dy)
\]

for each \( f \in C_0([0,\infty)) \). Furthermore, the resolvent \( r^\beta_\lambda \) is given by

\[
\mathbb{E}_x\left(\int_0^\infty e^{-\lambda s} f(X_s) \, ds\right) = \int_0^\infty e^{-\lambda s} p^\beta_s f(x) \, ds = r^\beta_\lambda f(x) = \int_{[0,\infty)} f(y) \, r^\beta_\lambda(x,dy).
\]

The proof of Theorem 3.2 is based on the so-called first passage time formular (see \([\text{Kni81}, (6.4)]\)). Let \( \lambda > 0 \) and define \( A^\beta := \lambda - (r^\beta_\lambda)^{-1} \) on \( \mathcal{D} := r^\beta_\lambda(C_0([0,\infty))) \) (which is independent of \( \lambda \)). By \([\text{Kni81}, \text{Theorem 6.2, Theorem 6.4}]\) it holds that

\[
A^\beta f = f'', \quad f \in \mathcal{D} = \{ f \in C_0([0,\infty)) \cap C^2([0,\infty)) \mid f'' \in C_0([0,\infty)) \} \quad \text{and} \quad \beta f''(0) = f'(0). \quad (3.6)
\]

The condition \( \beta f''(0) = f'(0) \) for \( f \in C^2([0,\infty)) \) is called Wentzell boundary condition.

**Definition 3.4.** Let \( F \) be a locally compact separable metric space. A transition semigroup \( p_t \), \( t > 0 \), of an \( F \)-valued Markov process is said to have the **Feller property** if \( p_t(C_0(F)) \subset C_0(F) \) and \( \lim_{t \downarrow 0} \| p_t f - f \|_\infty = 0 \) for each \( f \in C_0(F) \). Furthermore, it is called **strong Feller** if \( p_t(B_b(F)) \subset C_b(F) \) for each \( t > 0 \). If the transition semigroup has both Feller and strong Feller property, we say that it possesses the **doubly Feller property**.

We can also deduce the following:

**Proposition 3.5.** The transition semigroup \( (p^\beta_t)_{t>0} \) of sticky Brownian motion on \([0,\infty)\) has the doubly Feller property.

**Proof.** In consideration of Theorem 3.2 it rests to show that \( p_t(B_b([0,\infty))) \subset C_b([0,\infty)) \). Let \( f \in B_b([0,\infty)) \) and \( t > 0 \). It is well-known that

\[
\frac{e^{-x^2}}{x + \sqrt{x^2 + 2}} < \operatorname{erfc}(x) \leq \frac{e^{-x^2}}{x + \sqrt{x^2 + 1}}
\]

for each \( x \geq 0 \) (see \([\text{AS64}, 7.1.13]\)). Let \( x \in [0,\infty) \) and \( (x_n)_{n \in \mathbb{N}} \) a sequence in \([0,\infty)\) such that \( x_n \to x \) as \( n \to \infty \). Then \( G_n(y) := f(y)g_{0,x_n}(t,x_n + \frac{y}{\sqrt{2}}) \) converges for each fixed \( y \in [0,\infty) \) to
G(y) := f(y)g_{0,\sqrt{2\beta}}(t, x + \frac{y}{\sqrt{2}}) as n \to \infty by continuity of g_{0,\sqrt{2\beta}} in the second variable. Moreover, for each y \in [0, \infty) holds

\[|G_n(y)| \leq \|f\|_\infty K_1 \exp(\frac{\sqrt{2}x_n + y}{\beta}) \operatorname{erfc}(\frac{x_n}{\sqrt{2t}} + \frac{y}{2\sqrt{t}})
\leq \|f\|_\infty K_2 \exp(\frac{y}{\beta}) \operatorname{erfc}(\frac{y}{2\sqrt{t}})
\leq \|f\|_\infty K_3 \exp(\frac{y}{\beta}) \exp(-\frac{y^2}{4t}) =: H(y)\]

for suitable constants K_1, K_2 and K_3. Note that the function H is integrable with respect to the Lebesgue measure on [0, \infty). Thus, dominated convergence yields

\[\int_{[0, \infty)} G_n(y)dy \to \int_{[0, \infty)} G(y)dy\]

and by this, we can conclude that p_\beta^n f is continuous and bounded. \hfill \Box

**Remark 3.6.** Denote by \( (T_t^\beta)_{t \geq 0} \) the \( L^2([0, \infty); d\mu) \)-semigroup of \( (E, D(\mathcal{E})) \) defined in (3.3). Then, by the previous considerations, for all \( f \in B_b([0, \infty)) \cap L^2([0, \infty); d\mu) \) it holds that \( p_\beta^n f \) is a \( \mu \)-version of \( T_t^\beta f \). Note also that the \( L^2([0, \infty); d\mu) \)-generator \( (L, D(L)) \) is given by

\[Lf(x) = 1_{(0, \infty)}(x)f''(x) + 1_{\{0\}}(x)\frac{1}{\beta}f'(x) \quad \text{for } f \in D(L) = H^{2,2}((0, \infty)),\]

where \( H^{2,2}((0, \infty)) \) denotes the Sobolev space of order two. This can be shown using integration by parts, the fact that \( D(\mathcal{E}) = H^{1,2}((0, \infty)) \) and the definition of the space \( H^{2,2}((0, \infty)) \). For \( f \in C_c^0([0, \infty)) \subset D(L) \) such that the Wentzell boundary condition \( \beta f''(0) = f'(0) \) is fulfilled, it holds \( Lf = f'' \) similar to the generator of the \( C_0([0, \infty)) \)-semigroup given in (3.6). However, in the \( L^2 \)-setting the boundary behavior is rather described by the measure \( \mu \) instead of the domain of the generator.

Next we will construct the Dirichlet form corresponding to \( n \) independent sticky Brownian motions on \([0, \infty), n \in \mathbb{N}\). In [BH91, Chapter V, Section 2.1] it is shown how to construct finite tensor products of Dirichlet spaces. Moreover, the corresponding semigroup of the product Dirichlet form has an explicit representation. In our setting this construction yields the semigroup of an \( n \)-dimensional process on \( E = [0, \infty)^n, n \in \mathbb{N} \), such that the components are independent sticky Brownian motions on \([0, \infty)\). In particular, this approach justifies the choice of the Dirichlet form structure used in [FGV14].

Let \( (\mathcal{E}_i, D(\mathcal{E}_i)), i = 1, \ldots, n, \) be \( n \) copies of the Dirichlet form in (3.3). Note that each such form is defined on the space \( L^2([0, \infty); d\mu) \). In accordance with [BH91, Definition 2.1.1] we define the product Dirichlet form \( (\mathcal{E}^n, D(\mathcal{E}^n)) \) on \( L^2([0, \infty)^n; d\mu_n) \) with \( d\mu_n = \prod_{i=1}^n (dx_i + \beta \delta_0) \) by

\[\mathcal{E}^n(f, g) := \sum_{i=1}^n \int_{[0, \infty)^{n-1}} \mathcal{E}_i(f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n), g(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)) \prod_{j \neq i} (dx_j + \beta \delta_0)\]

(3.7)

for \( f, g \in D(\mathcal{E}^n) \), where

\[D(\mathcal{E}^n) := \{ f \in L^2([0, \infty)^n; d\mu_n) \} \text{ for each } i = 1, \ldots, n \text{ and for } \prod_{j \neq i} (dx_j + \beta \delta_0) - a.e.\]
First, we prove the following:

**Lemma 3.7.** \( C^1([0, \infty)^n) \) is dense in \( D(\mathcal{E}) \).

**Proof.** Note that \( C^1([0, \infty)^n) \subset D(\mathcal{E}) \) by definition of \( D(\mathcal{E}) \).

W.l.o.g. let \( n = 2 \). By [BH91, Proposition 2.1.3 b)] \( D(\mathcal{E}_1) \otimes D(\mathcal{E}_2) \) is dense in \( D(\mathcal{E}^2) \). We show that \( C^1([0, \infty)^n) \otimes C^1([0, \infty)^2) \subset C^1([0, \infty)^2) \) is dense in \( D(\mathcal{E}_1) \otimes D(\mathcal{E}_2) \). Then the assertion follows by a diagonal sequence argument. So let \( h \in D(\mathcal{E}_1) \otimes D(\mathcal{E}_2) \) such that \( h(x_1, x_2) = f(x_1)g(x_2) \) for \( \mu_2 \)-a.e. \( (x_1, x_2) \in [0, \infty)^2 \), \( f \in D(\mathcal{E}_1) \) and \( g \in D(\mathcal{E}_2) \). Choose sequences \((f_k)_{k \in \mathbb{N}}\) and \((g_k)_{k \in \mathbb{N}} \) in \( C^1([0, \infty)) \) such that \( f_k \rightarrow f \) in \( D(\mathcal{E}_1) \) and \( g_k \rightarrow g \) in \( D(\mathcal{E}_2) \) as \( k \rightarrow \infty \) and define, for \( k \in \mathbb{N} \), \( h_k(x_1, x_2) := f_k(x_1)g_k(x_2) \), \( x_1, x_2 \in [0, \infty) \). Then it follows immediately by assumption and the product structure that \( h_k \rightarrow h \) as \( k \rightarrow \infty \) in \( L^2([0, \infty)^2; d\mu) \). Moreover, for \( k, l \in \mathbb{N} \)

\[
\mathcal{E}^2(h_k - h_l) = \int_{[0, \infty)} \mathcal{E}_1((h_k - h_l)(\cdot, x_2))(dx_2 + \beta \delta^i_0) + \int_{[0, \infty)} \mathcal{E}_2((h_k - h_l)(x_1, \cdot))(dx_1 + \beta \delta^i_0)
\leq \mathcal{E}_1(f_k - f_l) \| g_k \|_{L^2([0, \infty); dx + \beta \delta^i_0)} + \mathcal{E}_1(f_l - f_k) \| g_k - g_l \|_{L^2([0, \infty); dx + \beta \delta^i_0)}
+ \mathcal{E}_2(g_k - g_l) \| f_k \|_{L^2([0, \infty); dx + \beta \delta^i_0)} + \mathcal{E}_2(g_l - g_k) \| f_k - f_l \|_{L^2([0, \infty); dx + \beta \delta^i_0)}.
\]

Hence, \( \mathcal{E}^2(h_k - h_l) \rightarrow 0 \) as \( k, l \rightarrow \infty \) and thus, \( h_k \rightarrow h \) as \( k \rightarrow \infty \) in \( D(\mathcal{E}^2) \). \( \Box \)

Let \( f, g \in C^1([0, \infty)^n) \). Then for each \( i = 1, \ldots, n \) and fixed \((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in [0, \infty)^{n-1}\) we have

\[
\mathcal{E}_i(f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n), g(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)) = \int_{[0, \infty)} \partial_i f(x_1, \ldots, x_n) \partial_i g(x_1, \ldots, x_n) \, dx_i.
\]

Set \( \{j \neq i\} := \{1, \ldots, i, i + 1, \ldots, n\} \). If \( A \) is a subset of some set \( I \), we denote by \( A^c \) the set \( I \setminus A \). Due to the identity

\[
\prod_{j \neq i} (dx_j + \beta \delta^i_0) = \sum_{A \subset \{j \neq i\}} \beta^{#A^c} \prod_{j \in A} dx_j \prod_{j \in A^c} \delta^i_j
\]

we get by rearranging the terms that

\[
\mathcal{E}^n(f, g) = \sum_{\emptyset \neq B \subset \{1, \ldots, n\}} \mathcal{E}_B(f, g)
\]

with

\[
\mathcal{E}_B(f, g) := \int_{[0, \infty)^n} \sum_{i \in B} \partial_i f \partial_i g \, d\lambda_B^{n, \beta},
\]

where \( d\lambda_B^{n, \beta} := \beta^{n-#B} \prod_{j \in B} dx_j \prod_{j \in B^c} \delta^j_0 \). In other words, \((\mathcal{E}^n, D(\mathcal{E}^n))\) defined in (3.7) coincides with the form defined in [FGV14 (2.3)] disregarding that in our present setting the density function \( g \) is identically one. Moreover, (3.7) can also be rewritten in the form

\[
\mathcal{E}^n(f, g) = \int_E \sum_{i = 1}^n \mathbb{1}_{\{x_i \neq 0\}} \partial_i f \partial_i g \, d\mu_n \quad \text{for } f, g \in C^1_c(E).
\]
From the present point of view \((\mathcal{E}^n, D(\mathcal{E}^n))\), defined as in (3.7), is the sum of \(n\) subforms and each such form for \(i = 1, \ldots, n\) describes the dynamics of the process on \([0, \infty)^n\) for all configurations where the \(i\)-th component is not pinned to zero. In contrast, the forms \(\mathcal{E}_B, \emptyset \neq B \subset \{1, \ldots, n\}\) describe the dynamics of the process for all configurations where the components specified by \(B\) are non-zero.

By a minor generalization of the results in [FGV14] we get the following lemma:

**Lemma 3.8.** The Dirichlet form \((\mathcal{E}^n, D(\mathcal{E}^n))\) on \(L^2([0, \infty)^n; d\mu_n), n \in \mathbb{N}\), is conservative, strongly local, regular and symmetric.

Let \(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in [0, \infty)^n, n \in \mathbb{N}\). Then the transition kernel \(p_t^{\beta, n}(x, dy)\) of \(n\) independent sticky Brownian motions on \([0, \infty)\) is given by

\[
p_t^{\beta, n}(x, dy) = \prod_{i=1}^n p_t^{\beta}(x_i, dy_i).
\]

Thus, for \(f \in C_0([0, \infty)^n)\) we have

\[
p_t^{\beta, n}f(x) = \int_{[0, \infty)^n} f(y_1, \ldots, y_n) \prod_{i=1}^n p_t^{\beta}(x_i, dy_i).
\]

By Theorem 3.2 we have an explicit representation of \(p_t^{\beta, n}(x, dy)\) and by the same arguments as in Proposition 3.3 the doubly Feller property holds also for \(p_t^{\beta, n}\):

**Proposition 3.9.** The transition semigroup \((p_t^{\beta, n})_{t > 0}\) of \(n\) independent sticky Brownian motions on \([0, \infty)\) has the doubly Feller property.

Let \((T_t^i)_{t \geq 0}\) be the \(L^2([0, \infty); d\mu)\)-semigroup of the forms \((\mathcal{E}_i, D(\mathcal{E}_i)), i = 1, \ldots, n\). Set for \(f \in L^2([0, \infty)^n; d\mu_n), i = 1, \ldots, n,\) and \(\mu_n\)-a.e. \((x_1, \ldots, x_n) \in [0, \infty)^n\)

\[
\hat{T}_t^{\beta,i} f(x_1, \ldots, n) := T_t^i f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, n)(x_i).
\]

and

\[
T_t^{\beta,n} f = \hat{T}_t^{\beta,1} \cdots \hat{T}_t^{\beta,n} f.
\]

By [BH91] Proposition 2.1.3 a) \((T_t^{\beta,i})_{t \geq 0}\) is the \(L^2([0, \infty)^n; d\mu_n)\)-semigroup associated to the form \((\mathcal{E}^n, D(\mathcal{E}^n))\) defined in (3.7) and the order of the \(\hat{T}_t^{\beta,i}, i = 1, \ldots, n,\) is arbitrary.

Let \(f \in B_b([0, \infty)^n) \cap L^2([0, \infty)^n; d\mu_n).\) Then we have for \(\mu_n\)-a.e. \(x = (x_1, \ldots, x_n) \in [0, \infty)^n\)

\[
\hat{T}_t^{\beta,n} f(x_1, \ldots, x_n) = T_t^n f(x_1, \ldots, x_{n-1}, \cdot)(x_n) = \int_{[0, \infty)} f(x_1, \ldots, x_{n-1}, y_n) p_t^{\beta}(x_n, dy_n)
\]

and similarly

\[
\hat{T}_t^{\beta,n-1} \hat{T}_t^{\beta,n} f(x_1, \ldots, x_n) = \int_{[0, \infty)} \int_{[0, \infty)} f(x_1, \ldots, x_{n-2}, y_{n-1}, y_n) p_t^{\beta}(x_{n-1}, dy_{n-1}) p_t^{\beta}(x_n, dy_n) p_t^{\beta}(x_{n-1}, dy_{n-1}).
\]

**Remark 3.10.** Proceeding successively as in (3.8) and (3.9) yields that \(p_t^{\beta, n} f\) is a \(\mu_n\)-version of \(T_t^{\beta,n} f\) for every \(f \in B_b([0, \infty)^n) \cap L^2([0, \infty)^n; d\mu_n).\)

In particular, the Dirichlet form corresponding to \(n\) independent sticky Brownian motions on \([0, \infty)\) is indeed given by the form \((\mathcal{E}^n, D(\mathcal{E}^n))\) defined in (3.7).
3.2 Girsanov transformations

We summarize some results on Girsanov transformations of a Markov process and the associated Dirichlet form. The statements can be found in [Ebe96] and [FOT11, Chapter 6]. In some cases we do not state the results in full generality, since for our purposes it is sufficient to simplify the assumptions.

Let $\mathcal{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in F})$ be a $\mu$-symmetric strong Markov process with state space $F \subset \mathbb{R}^n$, $n \in \mathbb{N}$, continuous sample paths and infinite lifetime, where $\mu$ is a positive Radon measure on $(F, \mathcal{B}(F))$ with full support. We suppose that the process is canonical, i.e., $\Omega = C([0, \infty), F)$ and $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \geq 0$. Moreover, assume that its Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(F; \mu)$ is regular, strongly local, conservative and that it possesses a square field operator $\Gamma$. Moreover, we can choose as sample space $\Omega = C([0, \infty), F)$. In the sequel, we denote by $(p_t)_{t \geq 0}$ the transition semigroup of $\mathcal{M}$, i.e., for $f \in \mathcal{B}_b(F)$ holds

$$p_t f(x) := \mathbb{E}_x(f(X_t)),$$

and we suppose that the transition density $p_t(x, \cdot)$, $x \in F$, $t > 0$, possesses the absolute continuity condition [FOT11, (4.2.9)].

A function $f$ is said to be in $D(\mathcal{E})_{\text{loc}}$ if for any relatively compact open set $G$ there exists a function $g \in D(\mathcal{E})$ such that $f = g$ $\mu$-a.e. on $G$. Fix some $\phi \in D(\mathcal{E})_{\text{loc}}$ such that $\phi > 0$ $\mu$-a.e. and $\phi$ is bounded. Define $\varrho := \phi^2$ and the symmetric bilinear form $(\mathcal{E}^\varrho, D^\varrho)$ on $L^2(F; \varrho \mu)$ by

$$D^\varrho := D(\mathcal{E})$$

$$\mathcal{E}^\varrho(f, g) := \int_F \Gamma(f, g) \varrho \mu$$

which is well-defined by the boundedness of $\phi$. Suppose that there exists a dense subspace $\mathcal{D}$ of $D(\mathcal{E})$ that is closed under composition with Lipschitz continuous functions $h \in C^\infty_b(\mathbb{R}^d)$, $d \in \mathbb{N}$, vanishing at the origin. Furthermore, let $\varphi_k \in C^1(\mathbb{R})$ such that $0 \leq \varphi_k \leq 1$, $\varphi_k = 1$ on $[-k, k]$ and $\text{supp}(\varphi_k) \subset [-k - 2, k + 2]$, $k \in \mathbb{N}$. Denote by $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ with respect to the norm induced by $\mathcal{E}^\varrho_1$. We assume that $\phi$ fulfills

$$\varphi_k(\ln \phi) \in \overline{\mathcal{D}} \quad \text{for all} \quad k \in \mathbb{N}.$$  \hspace{1cm} (3.11)

By [Ebe96, Theorem 1.1, Theorem 1.2] we can conclude the following:

Lemma 3.11. The symmetric bilinear form $(\mathcal{E}^\varrho, D(\mathcal{E}))$ is densely defined and closable on $L^2(F; \varrho \mu)$ and its closure $(\mathcal{E}^{\varrho}, D(\mathcal{E}^{\varrho}))$ is a strongly local Dirichlet form. Assume that the condition (3.11) holds true. Then $(\mathcal{E}^{\varrho}, D(\mathcal{E}^{\varrho})) = (\mathcal{E}^{\varrho}, \overline{\mathcal{D}})$, i.e., $\mathcal{D}$ is a dense subset of $D(\mathcal{E}^{\varrho})$.

Due to [FOT11, Theorem 5.5.1] it is possible to give a Fukushima decomposition of the process $\mathcal{M}$ of the form

$$\ln \phi(X_t) - \ln \phi(X_0) = M_t^{\ln \phi} + N_t^{\ln \phi} \quad \mathbb{P}_x \text{- a.s. for each} \quad x \in F,$$  \hspace{1cm} (3.12)

where $M_t^{\ln \phi}$ is a martingale additive functional and $N_t^{\ln \phi}$ is a continuous additive functional. The function $\ln \phi$ is possibly unbounded. In this case, the decomposition (3.12) requires some localization argument (see e.g. [FOT11, (6.3.19)]). Define the positive multiplicative functional $(Z_t)_{t \geq 0}$ by

$$Z_t = \exp(M_t^{\ln \phi} - \frac{1}{2}(M_t^{\ln \phi})^2),$$  \hspace{1cm} (3.13)
Furthermore, let \((\tilde{p}_t)_{t>0}\) be defined by
\[
\tilde{p}_tf(x) := E_x(Z_tf(X_t))
\]
for \(f \in \mathcal{B}_b(F)\).

By [FOT11] Section 6.3 \((\tilde{p}_t)_{t>0}\) is a transition function and there exists a corresponding \(\mu\)-symmetric right process \(\tilde{M}^e = (\Omega, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in F})\). Moreover, the Dirichlet form of \(\tilde{M}^e\) is given by \((\mathcal{E}^e, D(\mathcal{E}^e))\). We say that the process \(\tilde{M}^e\) is a Girsanov transformation of \(\tilde{M}\) by the multiplicativy functional \((Z_t)_{t \geq 0}\).

### 4 Construction of the Feller transition semigroup

In [Chu85] criteria are given under which the doubly Feller property is preserved under the transformation by a multiplicativy functional \((Z_t)_{t \geq 0}\). This concept is extended in [CK08]. It is shown that the conditions on \((Z_t)_{t \geq 0}\) can be weakened. Moreover, the setting is applied to Feynman-Kac and Girsanov transformations. In particular, precise conditions on the Revuz measure of the underlying additive functionals are given. We quote a result of [CK08] concerning the preservation of the doubly feller property under Girsanov transformations. Since we deal with strong Markov processes with continuous sample paths, we restrict the results to this setting instead of stating them in full generality.

Let \(\tilde{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in F})\) be again a \(\mu\)-symmetric strong Markov process with state space \(F \subset \mathbb{R}^n, n \in \mathbb{N}\), continuous sample paths and infinite lifetime, where \(\mu\) is a positive Radon measure on \((F, \mathcal{B}(F))\) with full support. As before, denote by \((p_t)_{t>0}\) the transition semigroup of \(\tilde{M}\). Assume that \((p_t)_{t>0}\) possesses the doubly Feller property.

Let \(r_\lambda(x, y), \lambda > 0, x, y \in F\), be the resolvent kernel of \(\tilde{M}\), i.e., the resolvent \((r_\lambda)_{\lambda>0}\) of \(\tilde{M}\) is given by
\[
r_\lambda f(x) = \int_F f(y)r_\lambda(x, y)d\mu(y)
\]
for \(f \in \mathcal{B}_b(F), \lambda > 0\) and \(x \in F\). For a Borel measure \(\nu\) on \(\mathcal{B}(F)\) we define the \(\lambda\)-potential of \(\nu\) by
\[
R_\lambda \nu(x) := \int_F r_\lambda(x, y)d\nu(y), \lambda > 0.
\]

Let \(B\) be a non-empty open subset of \(F\) and denote by \(B_{\Delta_B} := B \cup \{\Delta_B\}\) the one-point compactification of \(B\). Define \((X_t^B)_{t \geq 0}\) by
\[
X_t^B := \begin{cases} 
X_t & \text{if } t < \tau_B \\
\Delta_B & \text{if } t \geq \tau_B
\end{cases}
\]

where \(\tau_B := \inf\{t > 0 | X_t \notin B\}\). The transition semigroup of \((X_t^B)_{t \geq 0}\) is given by
\[
p_t^B(x, A) = \mathbb{P}_x(X_t \in A, \ t < \tau_B)
\]

and
\[
p_t^B(x, \{\Delta_B\}) := 1 - p_t^B(x, B), \ p_t^B(\Delta_B, \{\Delta_B\}) := 1,
\]

for \(x \in B, A \in \mathcal{B}(B)\). A function \(f \in \mathcal{B}_b(F)\) is extended to \(\Delta_B\) by setting \(f(\Delta_B) = 0\). For functions of this form, the transition semigroup of \((X_t^B)_{t \geq 0}\) reads
\[
p_t^B f(x) = E_x(f(X_t)1_{\{t < \tau_B\}}).
\]
The set $B$ is called regular if for each $x \in F \backslash B$, we have $\mathbb{P}_x(\tau_B = 0) = 1$.

Let $(M_t)_{t \geq 0}$ be a continuous locally square integrable martingale additive functional and denote by $\mu_{(M)}$ the Revuz measure of $((M)_t)_{t \geq 0}$. Furthermore, the transition semigroup $(\tilde{p}^B_t)_{t \geq 0}$ is given by

$$\tilde{p}^B_t f(x) := \mathbb{E}_x(\langle Z_t f(X_t) \mathbb{1}_{\{t < \tau_B\}} \rangle),$$

where $Z_t := \exp(M_t - \frac{1}{2} \langle M_t \rangle_t), t \geq 0$ and corresponds to the process obtained from $M^\varnothing$ (see Section 3.2) killed when leaving $B$. In the special case $B = F$ this definition reduces to the transition semigroup of $M^\varnothing$.

**Definition 4.1.** A Borel measure $\nu$ on $B(F)$ is said to be of

- (i) **Kato class** if $\lim_{\lambda \to \infty} \sup_{x \in F} R_\lambda \nu(x) = 0$,

- (ii) **extended Kato class** if $\lim_{\lambda \to \infty} \sup_{x \in F} R_\lambda \nu(x) < 1$,

- (iii) **local Kato class** if $1_K \nu$ is of Kato class for every compact set $K \subset F$.

**Theorem 4.2.** Assume that $\frac{1}{2} \mu_{(M)}$ is a positive Radon measure of local and extended Kato class and let $B$ be a regular open subset of $F$. Then $(\tilde{p}^B_t)_{t \geq 0}$ has the doubly Feller property. Moreover, $(Z_t)_{t \geq 0}$ is a martingale and

$$\lim_{t \to 0} \sup_{x \in D} \mathbb{E}_x(|Z_t - 1| \mathbb{1}_{\{t < \tau_B\}}) = 0$$

for any relatively compact open set $D \subset B$, and

$$\sup_{0 \leq s \leq t} \sup_{x \in B} \mathbb{E}_x(Z^p_s \mathbb{1}_{\{s < \tau_B\}}) < \infty$$

for some $p > 1$ and each $t > 0$.

**Proof.** See [CK08, Theorem 3.3].

Consider again the $n$ independent sticky Brownian motions on $[0, \infty)$ discussed in Section 3.1 with transition function $(\tilde{p}^{\beta,n}_t)_{t \geq 0}$ and Dirichlet form $(\mathcal{E}^n, D(\mathcal{E}^n))$ on $L^2(E; d\mu_n)$. In the following, we introduce a density function $\varrho = \phi^2$. Under suitable conditions on $\phi$ it is possible to perform a Girsanov transformation such that the transition semigroup of the transformed process $M^\varrho$ still possesses the strong Feller property (or even the doubly Feller property). By the preceding section the transformed Dirichlet form is of the form considered in [FGV14]. In this way, we are able to strengthen the results in [FGV14].

**Remark 4.3.** For functions $\phi$ such that the conditions of Theorem 4.2 are fulfilled for $(Z_t)_{t \geq 0}$ as in Section 3.1 and $B = E$, we immediately get that the transition function has the doubly Feller property and the process $M^\varrho$ solves $1_n$ for every starting point in $E$. Unfortunately, we are also interested in densities $\varrho$ such that the corresponding Revuz measure is not of extended Kato class. Such potentials are of particular interest for the application to the so-called wetting model in the theory of stochastic interface models. For this reason, we construct a strong Feller transition semigroup for a larger class of densities using Theorem 4.2 and an approximation argument. A direct application fails, since the Kato condition on $\mu_{(M)}$ ensures that the drift caused by the Girsanov transformation does not "explode". However, this criterion does only take into account the variation of the drift, but not its direction, which is of particular importance in our setting.

Let $\phi \in C^2(E)$ such that $\phi$ is bounded and $\phi > 0$ everywhere. Then $\phi \in D(\mathcal{E}^n)_{loc}$ and the energy measure $\mu_{(ln,\phi)} = \mu_{(M_{ln,\phi})}$ is given by

$$d\mu_{(ln,\phi)}(x) = 2 \sum_{i=1}^{n} (\partial_i \ln \phi(x))^2 \, dx \prod_{j \neq i} (dx_j + \beta \delta_0^j) = 2 \sum_{i=1}^{n} \mathbb{1}_{(0,\infty)}(x_i) \, (\partial_i \ln \phi(x))^2 \, d\mu_n$$

(4.1)
and thus, by Revuz correspondence we see that

$$\langle M^{[\ln \phi]} \rangle_t = 2 \sum_{i=1}^n \int_0^t (\partial_i \ln \phi(X_s))^2 \mathbb{1}_{(0,\infty)}(X_s^i) ds. \quad (4.2)$$

By this we can deduce that $(M^{[\ln \phi]}_t)_{t \geq 0}$ has the representation

$$M^{[\ln \phi]}_t = \sqrt{2} \sum_{i=1}^n \int_0^t \partial_i \ln \phi(X_s) \mathbb{1}_{(0,\infty)}(X_s^i) dB^i_s \quad (4.3)$$

which also follows by Ito’s formula.

**Example 4.4.** Let $n = 1$ and $\phi(x) := \exp(-\frac{1}{2}x^2)$. In this case, $(\ln \phi)'(x) = -x$. Hence, we expect that the process $M^\phi$ has the representation

$$dX_t = \sqrt{2} \mathbb{1}_{(0,\infty)}(X_t) dB_t - 2X_t \mathbb{1}_{(0,\infty)}(X_t) dt + \frac{1}{\beta} \mathbb{1}_{\{0\}}(X_t) dt.$$ 

Note that the additional drift term is always non-positive, since $X_t \in [0, \infty)$ for all $t > 0$ and thus, it attracts the process to 0. However, the logarithmic derivative of $\phi$ is unbounded and the energy measure is even not of extended Kato class. Indeed,

$$R_{\lambda \mu([\ln \phi])}(x) = \int_{[0,\infty)} r^\beta_{\lambda}(x,y)d\mu([\ln \phi])(y)$$

$$= 2 \int_{[0,\infty)} \left( \frac{1}{\sqrt{2\lambda}}(e^{-\sqrt{2\lambda}x-y} - e^{-\sqrt{2\lambda}(x+y)}) + \frac{1}{\sqrt{2\lambda + \beta \lambda}} 2e^{-\sqrt{2\lambda}(x+y)} \right) y^2 dy$$

is unbounded in $x$ for each fixed $\lambda > 0$, since

$$\int_{[0,\infty)} \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}x-y} y^2 dy = \frac{1}{\lambda} x^2 - \frac{1}{2\lambda^2} e^{-\sqrt{2\lambda}x} + \frac{1}{\lambda^2} \to \infty \text{ as } x \to \infty,$$

whereas the remaining terms converge to 0 as $x \to \infty$. Thus, it is not possible to apply Theorem 4.2 to this specific choice of $\phi$.

**Example 4.5.** Assume additionally that $\nabla \ln \phi$ is bounded. Then $\frac{1}{2} \mu([\ln \phi]) = \frac{1}{2} \mu(M^{[\ln \phi]})$ is of local and extended Kato class.

In the following, we assume for simplicity that $\phi$ is given by a potential $H$ with nearest neighbor pair interaction, i.e., $\phi = \sqrt{\varrho}$ and $\varrho = \exp(-H)$, where $H$ is defined by

$$H(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i,j \in \{0, \ldots, n+1\} \setminus \{i-j=1\}} V(x_i - x_j), \quad (4.4)$$

where $x_0 := x_{n+1} := 0$ and $V : \mathbb{R} \to [-b, \infty)$, $b \in [0, \infty)$, fulfills the conditions of [Fun05, (2.2)]:

(i) $V \in C^2(\mathbb{R})$,

(ii) $V$ is symmetric, i.e., $V(r) = V(-r)$ for all $r \in \mathbb{R}$,

(iii) $V$ is strictly convex, i.e., $c_- \leq V''(r) \leq c_+$ for all $r \in \mathbb{R}$ and some constants $c_-, c_+ > 0$. 

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In particular, \( \kappa := \int_{\mathbb{R}} \exp(-V(r))dr < \infty \), \( V \) is convex, \( V' (0) = 0 \) and \( V' \) is non-decreasing. Then \( \partial_i \ln \phi (x) \) has the representation
\[
\partial_i \ln \phi (x) = -\nabla' (i, x) := -\frac{1}{2} \sum_{j \in \{0, \ldots, n+1\} \setminus \{i-j\}} \nabla'' (x_i - x_j) \quad (=-\frac{1}{2} \nabla' (x_i - x_{i-1}) - \frac{1}{2} \nabla'' (x_i - x_{i+1}))
\]
for \( i = 1, \ldots, n \) and moreover,\[
\partial_i^2 \ln \phi (x) = -\nabla'' (i, x) := -\frac{1}{2} \sum_{j \in \{0, \ldots, n+1\} \setminus \{i-j\}} \nabla''' (x_i - x_j) \quad (=-\frac{1}{2} \nabla'' (x_i - x_{i-1}) - \frac{1}{2} \nabla''' (x_i - x_{i+1}))
\]
Using (4.3), (4.2) and Ito’s formula we see that
\[
M_t^\ln \phi [\nabla'] - \frac{1}{2} (M_t^\ln \phi )_t = \sum_{i=1}^{n} \int_0^t \partial_i \ln \phi (X_s) 1_{(0, \infty)}(X_s^i) dB_s^i - \frac{1}{2} \sum_{i=1}^{n} \int_0^t (\partial_i \ln \phi (X_s))^2 1_{(0, \infty)} (X_s^i) ds
\]
\[
= H(X_0) - H(X_t) + \frac{1}{\beta} \sum_{i=1}^{n} \int_0^t \nabla' (i, X_s) 1_{(0)} (X_s^i) ds
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \int_0^t (\nabla'' (i, X_s) - \nabla' (i, X_s)^2) 1_{(0, \infty)} (X_s^i) ds
\]
\[
\leq H(x) + nb + \frac{1}{2} nc \cdot t
\]
\( \mathbb{P}_x \)-a.s. for each \( x \in E \), since \( \nabla' (i, x) = \frac{1}{2} (\nabla' (-x_{i-1}) + \nabla' (-x_{i+1})) \leq 0 \) if \( x_i = 0 \).

Let \( k > 0 \) and \( K := [0, k]^n \) as well as \( \tau_K := \inf \{ t > 0 : X_t \notin K \} \). Choose a bounded, positive function \( \phi_k \in C^2 (E) \) such that \( \phi_k = \phi \) on \( K \) and \( \nabla \ln \phi_k \) is bounded. We define the exponential functional \( (Z_t^k)_{t \geq 0} \) by
\[
Z_t^k := \exp (M_t^\ln \phi_k - \frac{1}{2} (M_t^\ln \phi_k)_t).
\]

**Theorem 4.6.** Let \( \rho = \phi^2 \) be given by (4.4) and \( Z_t = \exp (M_t^\ln \phi - \frac{1}{2} (M_t^\ln \phi)_t) \), \( t \geq 0 \). Then the transition function \( (p_t)_{t \geq 0} \) defined by \( p_t f (x) = \mathbb{E}_x (Z_t f (X_t)) \) for \( f \in \mathcal{B}_b (E) \) and \( x \in E \) which corresponds to the strong Markov process \( \mathbb{M}^\rho \) has the strong Feller property.

**Proof.** Let \( k > 0 \) and \( K := [0, k]^n \). \( K \) is regular, i.e., \( \mathbb{P}_x (\tau_K = 0) = 1 \) for each \( x \in E \setminus K \). We define the transition function \( (p_t^k)_{t \geq 0} \) similar as \( (p_t)_{t \geq 0} \) by \( p_t^k f (x) := \mathbb{E}_x (Z_t^k f (X_t) 1_{t < \tau_K}) \). By the assumptions on \( \phi_k \), Example 4.5 and Theorem 4.2 \( (p_t^k)_{t \geq 0} \) has the doubly Feller property for each \( k > 0 \). Let \( f \in \mathcal{B}_b (E) \) and choose a constant \( C (f) < \infty \) such that \( f (x) \leq C (f) \) for all \( x \in E \). Clearly, \( p_t f \in \mathcal{B}_b (E) \). Hence, it suffices to show that \( p_t f \) is continuous. We have for \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \)
\[
|p_t f (x) - p_t^k f (x)| = |\mathbb{E}_x (Z_t f (X_t)) - \mathbb{E}_x (Z_t^k f (X_t) 1_{t < \tau_K})|
\]
\[
= |\mathbb{E}_x (Z_t f (X_t) 1_{t \geq \tau_K})|
\]
\[
\leq C (f) (\mathbb{E}_x ((Z_t)^p))^\frac{1}{p} (\mathbb{P}_x (\tau_K \leq t))^{\frac{1}{q}}
\]
\[
\leq C (f) \exp (H(x) + nb + \frac{1}{2} nc \cdot t) (\mathbb{P}_x (\tau_K \leq t))^{\frac{1}{q}}
\]
Uniqueness of weak solutions

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by (1.5). Define $C_t := \max_{i=1,\ldots,n} \max_{0 \leq s \leq t} X_i^s$ for $t \geq 0$ and let $D := [0,d]^n$ with $d > 0$ fixed. Then for $x \in D$ and $k > d$

$$
\mathbb{P}_x(\tau_K \leq t) \leq \mathbb{P}_0(C_t \geq k - d) \leq n \sqrt{\frac{t}{2\pi k - d}} \exp\left(-\frac{d^2}{2t}\right) =: C(k) \to 0 \text{ as } k \to \infty
$$
due to [KS98, p.96,(8.3')], since $C_t \leq \max_{i=1,\ldots,n} \max_{0 \leq s \leq t} |B_i^s|$ almost surely with respect to $\mathbb{P}_0$. Thus, we have

$$
\|p_t f - p_k t f\|_{\infty,d} := \sup_{x \in D} |p_t f(x) - p_k t f(x)| \leq C(f) \exp(H_{\max}(d) + nb + \frac{1}{2}nc_+ t) C(k)^\frac{1}{q} \to 0 \text{ as } k \to \infty,
$$

where $H_{\max}(d) := \max_{x \in D} H(x) < \infty$. Hence, $p_t f$ is continuous on $D$ for each $d > 0$ and so $p_t f \in C_b(E)$.

Proof of Theorem 2.2 By Section 3.2 there exists a strong Markov process $\mathbb{M}^\alpha$ with transition semigroup $(p_t)_{t \geq 0}$ and the Dirichlet form associated to $\mathbb{M}^\alpha$ is given by the closure of $(\mathcal{E}^\alpha, \mathcal{D})$ on $L^2(E; d\mu_\alpha)$. The strong Feller property is shown in Theorem 4.6 and the last statement holds by [FOT11, Exercise 4.2.1].

Proof of Theorem 2.3 The statement follows by the results proven in [FGV14, Corollary 4.18, Theorem 5.6] considering that the absolute continuity condition [FOT11, (4.2.9)] is fulfilled.

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Theorem 5.1. Let $\varrho = \varphi^2$ be given as in Theorem (4.6). Then the solution to (2.2) is unique in law.

Proof. By [GS72, §24, Theorem 1, Corollary 1] the one dimensional sticky Brownian motion on $[0,\infty)$ is unique in law. Thus, the same holds true for $n$ independent sticky Brownian motions for each $n \in \mathbb{N}$. Finally, we can conclude that the solution to (2.2) is unique in law due to [IW89, Chapter IV, Theorem 4.2], since its law is constructed by a Girsanov transformation.

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