Abstract. We study the rationality of gamma factors associated to certain Hasse zeta functions. We show many explicit examples of rational gamma factors coming from products of $\text{GL}(n)$.

1. Introduction

Let $X = \text{GL}(n_1) \times \cdots \times \text{GL}(n_r)$ for $n_1, \ldots, n_r \geq 1$ and $q$ be a prime power. Then the congruence zeta function $\zeta_{X/\mathbb{F}_q}(s)$ associated with $X$ over $\mathbb{F}_q$ is defined by

$$
\zeta_{X/\mathbb{F}_q}(s) = \exp \left( \sum_{m=1}^{\infty} \frac{|X(\mathbb{F}_q^m)|}{m} q^{-ms} \right).
$$

The Hasse zeta function $\zeta_{X/\mathbb{Z}}(s)$ associated with $X$ over $\mathbb{Z}$ is defined by

$$
\zeta_{X/\mathbb{Z}}(s) = \prod_{p\text{-primes}} \zeta_{X/\mathbb{F}_p}(s).
$$

In this case it is not difficult to show that $\zeta_{X/\mathbb{Z}}(s)$ has a functional equation of the type

$$
\Gamma_{X/\mathbb{Z}}(s) \zeta_{X/\mathbb{Z}}(s) = (\Gamma_{X/\mathbb{Z}}(d-s) \zeta_{X/\mathbb{Z}}(d-s))(-1)^{n_1+\cdots+n_r},
$$

where $d = \sum_{j=1}^{r} n_j (3n_j - 1)/2 + 1$ and $\Gamma_{X/\mathbb{Z}}(s)$ is expressed in terms of a product/quotient of the gamma function. We call $\Gamma_{X/\mathbb{Z}}(s)$ the gamma factor for $\zeta_{X/\mathbb{Z}}(s)$.

In this paper, we study the rationality of the gamma factor $\Gamma_{X/\mathbb{Z}}(s)$. We prove the following result.

**Main Theorem.** $\Gamma_{X/\mathbb{Z}}(s)$ is a non-rational function if and only if $n_1 = \cdots = n_r = 1$.

This result will be shown in Section 4 using Theorem 1 for absolute zeta functions proved in Section 3. Then we study its analogues over algebraic integer rings in Section 5.

We refer to Manin [7] and Connes and Consani [2] for gamma factors of Hasse zeta functions.

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2. Absolute zeta functions

In this section we study expressions of gamma factors of some Hasse zeta functions as absolute zeta functions. Absolute zeta functions were studied by Soulé [8] and Connes and Consani [1].

First, we recall absolute zeta functions following Kurokawa and Tanaka [4, 5]. Let

\[ f(x) \in \frac{1}{(1 - x^{-\omega_1}) \cdots (1 - x^{-\omega_r})} \mathbb{Z}[x] \]

for \( \omega_1, \ldots, \omega_r > 0 \). Assume the absolute automorphy

\[ f\left(\frac{1}{x}\right) = C x^{-D} f(x) \quad (2.1) \]

with \( C = \pm 1 \) and \( D \in \mathbb{R} \). We construct the absolute zeta function \( \zeta_f(s) \) via

\[ \zeta_f(s) = \exp\left(\frac{\partial}{\partial w} Z_f(w, s) \bigg|_{w=0}\right) \]

with

\[ Z_f(w, s) = \frac{1}{\Gamma(w)} \int_1^\infty f(x) x^{-s-1} (\log x)^{w-1} dx. \]

We see that \( \zeta_f(s) \) is a meromorphic function as in Kurokawa and Tanaka [4, 5] (cf. Kurokawa and Ochiai [3]). We call \( \zeta_f(s) \) the absolute zeta function associated to the absolute automorphic form \( f \). Moreover, for \( f(x) \in \mathbb{Z}[x] \), the Hasse zeta function \( \zeta_{f/\mathbb{Z}}(s) \) is defined by

\[ \zeta_{f/\mathbb{Z}}(s) = \prod_{p: \text{primes}} \exp\left(\sum_{m=1}^\infty \frac{f(p^m)}{m} p^{-ms}\right). \]

Let \( f(x) \in \mathbb{Z}[x] \). Assume the absolute automorphy (2.1) for \( x > 0 \) with \( C = \pm 1 \) and \( D \in \mathbb{Z} \). Expand

\[ f(x) = \sum_k a(k) x^k \quad (2.2) \]

and put

\[ \hat{\zeta}_{f/\mathbb{Z}}(s) = \zeta_{f/\mathbb{Z}}(s) \Gamma_f(s) \]

with

\[ \Gamma_f(s) = \prod_k \Gamma_{\mathbb{R}}(s - k)^{a(k)} \quad (2.3) \]

for

\[ \Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right). \]

Then by the absolute automorphy (2.1) \( \hat{\zeta}_{f/\mathbb{Z}}(s) \) satisfies the functional equation (see Theorem 1(3) for detail). So we call \( \Gamma_f(s) \) the gamma factor for the Hasse zeta function \( \zeta_{f/\mathbb{Z}}(s) \). Concerning gamma factors of Hasse zeta functions we prove the following results.
Theorem 1. We have the following.

1. \( \zeta_{f/Z}(s) = \prod_k \zeta_Z(s - k)^{a(k)} \), \hspace{1cm} (2.4)

where \( \zeta_Z(s) \) is the Riemann zeta function.

2. \( \hat{\zeta}_{f/Z}(s) = \prod_k \hat{\zeta}_Z(s - k)^{a(k)} \),

where \( \hat{\zeta}_Z(s) \) is the completed Riemann zeta function defined as \( \hat{\zeta}_Z(s) = \zeta_Z(s) \Gamma_R(s) \).

3. \( \hat{\zeta}_{f/Z}(s) \) satisfies the functional equation

\[ \hat{\zeta}_{f/Z}(D + 1 - s) = \hat{\zeta}_{f/Z}(s). \]

4. Let

\( f_\infty(x) = \frac{f(x)}{1 - x^{-2}}. \)

Then

\( f_\infty\left(\frac{1}{x}\right) = -Cx^{-(D+2)} f_\infty(x) \)

and

\[ \Gamma_f(s) = \zeta_{f_\infty}(s)(4\pi)^{f(1)/2}(2\pi)^{-f(1)/2} (2\pi)^{-f(1)/2}. \]

5. \( \Gamma_f(s) \) is a rational function (in \( s \)) if and only if \( f(\pm 1) = 0. \)

3. Proof of Theorem 1

1. Let

\[ \zeta_{f/P}(s) = \exp\left(\sum_{m=1}^\infty \frac{f(p^m)}{m} p^{-ms}\right). \]

By (2.2),

\[ \zeta_{f/P}(s) = \exp\left(\sum_{m=1}^\infty \frac{1}{m} \left(\sum_k a(k) p^{mk}\right) p^{-ms}\right) = \prod_k \left(1 - p^{k-s}\right)^{-a(k)}. \]

Multiplying these over the prime numbers \( p \), we obtain (1).

2. By (2.3) and (2.4) we obtain

\[ \hat{\zeta}_{f/Z}(s) = \prod_k \zeta_Z(s - k)^{a(k)} \Gamma_R(s - k)^{a(k)} = \prod_k \hat{\zeta}_Z(s - k)^{a(k)}. \]
(3) From the absolute automorphy (2.1) we see that
\[ \sum_k a(k)x^{-k} = C \sum_k a(k)x^{-(D-k)} \]
\[ = C \sum_k a(D-k)x^{-k}. \]
Hence using \( C = \pm 1 \) we get \( a(D-k) = Ca(k) \). Now, the expression (3.1) gives
\[ \hat{\zeta}_{f/\mathbb{Z}}(D + 1 - s)^C = \prod_k \hat{\zeta}_Z(D - k + 1 - s)^{Ca(k)} \]
\[ = \prod_k \hat{\zeta}_Z((D - k) + 1 - s)^{(D-k)} \]
\[ = \prod_k \hat{\zeta}_Z(k + 1 - s)^{a(k)}. \]
Then, from the functional equation \( \hat{\zeta}_Z(1 - s) = \hat{\zeta}_Z(s) \), we see \( \hat{\zeta}_Z(k + 1 - s) = \hat{\zeta}_Z(s - k) \).
Thus we obtain \( \hat{\zeta}_{f/\mathbb{Z}}(D + 1 - s)^C = \hat{\zeta}_{f/\mathbb{Z}}(s) \) as claimed.

(4) The absolute automorphy of \( f_\infty(x) \) is easily seen:
\[ f_\infty\left(\frac{1}{x}\right) = \frac{f(1/x)}{1 - x^2} \]
\[ = -x^{-2} \frac{Cx^{-D}f(x)}{1 - x^{-2}} \]
\[ = -Cx^{-(D+2)}f_\infty(x). \]
Next, we calculate \( \Gamma_f(s) \) as
\[ \Gamma_f(s) = \prod_k \Gamma_Z(s - k)^{a(k)} \]
\[ = \prod_k \left( \pi^{-(s-k)/2} \Gamma\left(\frac{s-k}{2}\right) \right)^{a(k)} \]
\[ = \pi^{-(f(1)/2)s} \pi^{f'(1)/2} \prod_k \Gamma\left(\frac{s-k}{2}\right)^{a(k)}. \]
where we have used the following facts:
\[ f(1) = \sum_k a(k), \quad f'(1) = \sum_k ka(k). \]
Finally, we calculate \( \zeta_{f_\infty}(s) \). Since
\[ f_\infty(x) = \sum_k a(k) \frac{x^k}{1 - x^{-2}}, \]
we have
\[ Z_{f\infty}(w, s) = \frac{1}{\Gamma(w)} \int_{1}^{\infty} \left( \sum_k a(k) \frac{x^k}{1 - x^{-2}} \right) x^{-s-1} (\log x)^{w-1} dx \]
\[ = \sum_k a(k) \zeta_1(w, s - k, (2)). \]
Here
\[
\zeta_1(s, x, (\omega)) = \sum_{n=0}^{\infty} (n\omega + x)^{-s} = \omega^{-s} \sum_{n=0}^{\infty} \left(n + \frac{x}{\omega}\right)^{-s} = \omega^{-s} \zeta(s - 1, x/\omega)
\]
and \(\zeta(s, x)\) is the Hurwitz zeta function. By the formula of Lerch [6] we have
\[
\frac{\partial}{\partial s} \zeta_1(s, x, (\omega)) \bigg|_{s=0} = -\log(\omega) \left(\frac{1}{2} - \frac{x}{\omega}\right) + \log \frac{\Gamma(x/\omega)}{\sqrt{2\pi}}.
\]
Hence we get
\[
\zeta_{f_\infty}(s) = \exp \left( \frac{\partial}{\partial w} Z_{f_\infty}(w, s) \bigg|_{w=0} \right) = \prod_k \left( \frac{\Gamma((s-k)/2) 2^{(s-k)/2-1/2}}{\sqrt{2\pi}} \right)^{a(k)} = (4\pi)^{-f(1)/2} 2^{f(1)/2} 2^{-f'(1)/2} \prod_k \Gamma \left( \frac{s-k}{2} \right)^{a(k)}.
\]
Comparing equations (3.2) and (3.3), we obtain (4).

(5) First, suppose \(f(\pm 1) = 0\). Then, we know that \(f_\infty(x) \in \mathbb{Z}[x]\). Hence \(\zeta_{f_\infty}(s)\) is a rational function. Thus
\[
\Gamma_f(s) = \zeta_{f_\infty}(s) (2\pi)^{f'(1)/2}
\]
is a rational function.

Next, suppose that \(\Gamma_f(s)\) is a rational function. Then, it is obvious that
\[
\lim_{s \to +\infty} \frac{\log \Gamma_f(s)}{s \log s} = 0.
\]
On the other hand, applying the Stirling formula
\[
\lim_{s \to +\infty} \frac{\log \Gamma(s)}{s \log s} = 1
\]
to (3.2), we find
\[
\lim_{s \to +\infty} \frac{\log \Gamma_f(s)}{s \log s} = \frac{1}{2} \sum_k a(k) = \frac{1}{2} f(1).
\]
Hence, we get \(f(1) = 0\). Thus
\[
\zeta_{f_\infty}(s) = \Gamma_f(s) (2\pi)^{-f'(1)/2}
\]
is a rational function. Hence, by the rationality criterion (see Kurokawa and Tanaka [4, Theorem 1]) we see that \(f_\infty(x) \in \mathbb{Z}[x]\). Hence, \(f(-1) = 0\) also. This completes the proof of Theorem 1.
4. Proof of Main Theorem

For prime powers \( q \) put
\[
f(q) = |X(\mathbb{F}_q)|,
\]
where \( X = \text{GL}(n_1) \times \cdots \times \text{GL}(n_r) \); and
\[
f(q) = f_1(q) \cdots f_r(q)
\]
with
\[
f_j(q) = |\text{GL}(n_j, \mathbb{F}_q)|.
\]

It is well known that each \( f_j(x) \in \mathbb{Z}[x] \), so \( f(x) \in \mathbb{Z}[x] \). Actually,
\[
|\text{GL}(n, \mathbb{F}_q)| = q^{n(n-1)/2}(q-1)(q^2-1)\cdots(q^n-1) \in \mathbb{Z}[q].
\]

Now, applying Theorem 1 to this \( f(x) \) with \( C = (-1)^{n_1+\cdots+n_r} \) and \( D = \sum_{j=1}^r n_j(3n_j-1)/2 \), we know that \( \Gamma_{X/\mathbb{Z}}(s) = \Gamma_f(s) \) and that
\[
\Gamma_{X/\mathbb{Z}}(s) \text{ is a rational function } \iff f(\pm 1) = 0.
\]

Looking at the vanishing of \( f(\pm 1) = f_1(\pm 1) \cdots f_r(\pm 1) \) we see that
\[
\Gamma_{X/\mathbb{Z}}(s) \text{ is a rational function } \iff \text{some } n_j \geq 2.
\]

Thus,
\[
\Gamma_{X/\mathbb{Z}}(s) \text{ is a non-rational function } \iff n_1 = \cdots = n_r = 1.
\]

This completes the proof of the Main Theorem.

5. Gamma factors of Hasse zeta functions for \( \text{GL}(n_1) \times \cdots \times \text{GL}(n_r) \) over algebraic integer rings

In this section we explain briefly an extension of the Main Theorem and Theorem 1 to the case of algebraic number fields briefly. Let \( f(x) \in \mathbb{Z}[x] \), \( K \) be an algebraic number field and \( \mathcal{O}_K \) be its integer ring. Then the Hasse zeta function \( \zeta_{f/\mathcal{O}_K}(s) \) associated with \( f \) over \( \mathcal{O}_K \) is defined by
\[
\zeta_{f/\mathcal{O}_K}(s) = \prod_{\mathcal{P}} \zeta_{f/(\mathcal{O}_K/\mathcal{P})}(s),
\]
where \( \mathcal{P} \) runs over the maximal ideals of \( \mathcal{O}_K \) and
\[
\zeta_{f/(\mathcal{O}_K/\mathcal{P})}(s) = \exp\left(\sum_{m=1}^{\infty} \frac{f(N(\mathcal{P})^m)}{m} N(\mathcal{P})^{-ms}\right)
\]
with \( N(\mathcal{P}) = |\mathcal{O}_K/\mathcal{P}| \). In the same manner as in the proof of Theorem 1 (1) we see that
\[
\zeta_{f/\mathcal{O}_K}(s) = \prod_{k} \zeta_K(s-k)^{a(k)},
\]
where \( f(x) = \sum_k a(k)x^k \) and \( \zeta_K(s) \) is the Dedekind zeta function of \( K \). Below we always assume the absolute automorphy (2.1) for \( f \). We put
\[
\Gamma_{f/O_K}(s) = \prod_k (|\Delta_K|^{(s-k)/2}\Gamma_{\mathbb{R}}(s-k)\Gamma_{\mathbb{C}}(s-k)^2)^{a(k)},
\]
where \( \Delta_K \) is the discriminant of \( K \), \( r_1 \) (respectively \( r_2 \)) is the number of real (respectively complex) places for \( K \) and \( \Gamma_{\mathbb{R}}(s) = 2(2\pi)^{-s}\Gamma(s) \). Then the absolute automorphy and the functional equation \( \widehat{\zeta_K}(s) = \zeta_K(1-s) \), where \( \widehat{\zeta_K}(s) = |\Delta_K|^{s/2}\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s)^2\zeta_K(s) \), yield
\[
\widehat{\zeta_{f/O_K}}(D+1-s)^{C} = \widehat{\zeta_{f/O_K}}(s),
\]
where \( \widehat{\zeta_{f/O_K}}(s) = \Gamma_{f/O_K}(s)\zeta_{f/O_K}(s) \). We call \( \Gamma_{f/O_K}(s) \) the gamma factor for \( \zeta_{f/O_K}(s) \).

Taking the above observation into account, we can show the following claim, which can be regarded as an extension of Theorem 1.

**Theorem 2.** \( \Gamma_{f/O_K}(s) \) is a rational function (in \( s \)) if and only if
\[
\begin{align*}
f(1) = 0 & \quad \text{if } K \text{ is a totally imaginary field}, \\
f(\pm 1) = 0 & \quad \text{if } K \text{ is not a totally imaginary field}.
\end{align*}
\]

**Sketch of proof.** By the Stirling formula we have
\[
\lim_{s \to +\infty} \frac{\log(\prod_k \Gamma_{\mathbb{R}}(s-k)^{a(k)})}{s \log s} = f(1)/2,
\]
\[
\lim_{s \to +\infty} \frac{\log(\prod_k \Gamma_{\mathbb{C}}(s-k)^{a(k)})}{s \log s} = f(1).
\]
This gives
\[
\lim_{s \to +\infty} \frac{\log \Gamma_{f/O_K}(s)}{s \log s} = \left( \frac{r_1}{2} + r_2 \right) f(1).
\]  
(5.1)

To express \( \Gamma_{f/O_K}(s) \) in terms of absolute zeta functions, we introduce \( f_{\text{real}}(x) \) and \( f_{\text{cpx}}(x) \) by
\[
f_{\text{real}}(x) = \frac{f(x)}{1 - x^{-2}}, \quad f_{\text{cpx}}(x) = \frac{f(x)}{1 - x^{-1}}.
\]
After a routine calculation using Lerch’s formula, we have
\[
\zeta_{f_{\text{real}}}(s) = 2^{(s/2-1)f(1)-f'(1)/2}\pi^{-f(1)/2} \prod_k \Gamma_{\mathbb{R}}(s-k)^{a(k)},
\]
\[
\zeta_{f_{\text{cpx}}}(s) = (2\pi)^{-f(1)/2} \prod_k \Gamma_{\mathbb{C}}(s-k)^{a(k)}.
\]
This gives
\[
\prod_k \Gamma_{\mathbb{R}}(s-k)^{a(k)} = 2^{f(1)/2}(2\pi)^{-f(1)/2}\zeta_{f_{\text{real}}}(s),
\]
\[
\prod_k \Gamma_{\mathbb{C}}(s-k)^{a(k)} = 2^{f(1)}(2\pi)^{-f(1)/2 + f'(1)/2}\zeta_{f_{\text{cpx}}}(s).
\]
We consider the case \( r_1 = 0 \). Firstly, we assume \( f(1) = 0 \). Then \( \Gamma_{f/\mathcal{O}_K}(s) \) is written as
\[
\Gamma_{f/\mathcal{O}_K}(s) = |\Delta_K|^{-f'(1)/2} (2\pi)^{r_2/2} f'(1) \xi_{f_{\text{cpx}}}(s)^{r_2}.
\]
Since \( f_{\text{cpx}}(x) \) is a polynomial with integer coefficients, \( \xi_{f_{\text{cpx}}}(s) \) is a rational function. This implies \( \Gamma_{f/\mathcal{O}_K}(s) \) is a rational function. Secondly we assume that \( \Gamma_{f/\mathcal{O}_K}(s) \) is a rational function. Then (5.1) says \( f(1) = 0 \). These complete the proof of the first claim.

We consider the case \( r_1 \geq 1 \). Firstly we assume \( f(1) = f(-1) = 0 \). Then
\[
\Gamma_{f/\mathcal{O}_K}(s) = |\Delta_K|^{-f'(1)/2} (2\pi)^{(r_1/2+r_1)} f'(1) \xi_{f_{\text{real}}}(s)^{r_1} \xi_{f_{\text{cpx}}}(s)^{r_2}.
\]
(5.2)
Since both \( f_{\text{real}}(x) \) and \( f_{\text{cpx}}(x) \) are polynomials with integer coefficients, we see that \( \Gamma_{f/\mathcal{O}_K}(s) \) is a rational function. Secondly we assume that \( \Gamma_{f/\mathcal{O}_K}(s) \) is a rational function. Then (5.1) implies \( f(1) = 0 \). This gives (5.2) and that \( \xi_{f_{\text{cpx}}}(s) \) is a rational function. In consequence \( \xi_{f_{\text{real}}}(s)^{r_1} \) is a rational function. This implies that \( f_{\text{real}}(x) \) is a polynomial, which gives \( f(-1) = 0 \).

Let \( X = \text{GL}(n_1) \times \cdots \times \text{GL}(n_r) \). The Hasse zeta function associated with \( X \) over \( \mathcal{O}_K \) is defined by
\[
\xi_{X/\mathcal{O}_K}(s) = \prod_{\mathfrak{p}} \xi_{X/(\mathcal{O}_K/\mathfrak{p})}(s).
\]
We specialize \( f \) in Theorem 2 in the same manner as in Section 4. Then \( \xi_{X/\mathcal{O}_K}(s) \) has a functional equation of the type
\[
\Gamma_{X/\mathcal{O}_K}(s) \xi_{X/\mathcal{O}_K}(s) = (\Gamma_{X/\mathcal{O}_K}(d-s) \xi_{X/\mathcal{O}_K}(d-s))(-1)^{n_1+\cdots+n_r},
\]
where \( d = \sum_{j=1}^{r} n_j (3n_j-1)/2 + 1 \) and we obtain the following.

**Corollary.** \( \Gamma_{X/\mathcal{O}_K}(s) \) is a rational function if and only if
\[
\begin{cases} 
\text{any} & \text{if } K \text{ is a totally imaginary field,} \\
(n_1, \ldots, n_r) \neq (1, \ldots, 1) & \text{if } K \text{ is not a totally imaginary field.}
\end{cases}
\]

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