Simple Causes of Complexity in Hedonic Games

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Abstract

Hedonic games provide a natural model of coalition formation among self-interested agents. The associated problem of finding stable outcomes in such games has been extensively studied. In this paper, we identify simple conditions on expressivity of hedonic games that are sufficient for the problem of checking whether a given game admits a stable outcome to be computationally hard. Somewhat surprisingly, these conditions are very mild and intuitive. Our results apply to a wide range of stability concepts (core stability, individual stability, Nash stability, etc.) and to many known formalisms for hedonic games (additively separable games, games with \(W\)-preferences, fractional hedonic games, etc.), and unify and extend known results for these formalisms. They also have broader applicability: for several classes of hedonic games whose computational complexity has not been explored in prior work, we show that our framework immediately implies a number of hardness results for them.

1 Introduction

Hedonic games [Drèze and Greenberg, 1980] [Banerjee et al., 2001] [Bogomolnaia and Jackson, 2002] provide an elegant and versatile model of coalition formation among strategic agents. In such games, each agent has preferences over coalitions (subsets of players) that she can be a part of, and an outcome of the game is a partition of agents into coalitions. Clearly, the quality of an outcome depends on how well it reflects the agents’ preferences. In particular, it is desirable to have outcomes that are stable, i.e., do not offer the agents an opportunity to profitably deviate. Many different concepts of stability have been proposed in the hedonic games literature (see Section 2 for a brief summary, and [Aziz and Savani, 2015] for an in-depth discussion), and for each of them a natural computational question is whether a given game admits an outcome that is stable in that sense.

The complexity of this question depends on how the game is represented: while every hedonic game can be described by explicitly listing each agent’s preference relation over all coalitions that may contain her, in recent years there has been a considerable amount of research on succinct representation formalisms for hedonic games, i.e., ones where a game description size scales polynomially with the number of agents \(n\). Typically, such formalisms are not universally expressive, but capture important classes of hedonic games. For instance, if the utility that an agent assigns to a coalition is given by the sum/average/minimum/maximum of the utilities she assigns to individual members of that coalition, the entire game can be described by \(n(n-1)\) numbers (such games are known as, respectively, additively separable games [Bogomolnaia and Jackson, 2002], fractional hedonic games [Aziz et al., 2014], and games with \(W\)- and \(B\)-preferences [Cechlárová and Hajduková, 2003, 2004b]). There are also representation formalisms that are universally expressive (and hence

| Formalism                          | SNS | SCR | CR | NS | IS |
|-----------------------------------|-----|-----|----|----|----|
| IRCL of length \(\leq 3\)         | NP-h| NP-c| NP-c| NP-c|    |
| IRCL of length \(\leq 9\)         | NP-h| NP-c| NP-c| NP-c| NP-c|
| Hedonic Coalition Nets            | NP-h| NP-h| NP-h| NP-c| NP-c|
| Stable Marriages (SMI) (no ties)  | (P) | NP-c| (P) | NP-c| NP-c|
| \(W\)-preferences (no ties)       | NP-h| NP-c| NP-c| NP-c| NP-c|
| \(W\)-preferences (no ties)       | (P) | NP-c| (P) | NP-c| NP-c|
| B- & \(W\)-hedonic games         | NP-h| NP-h| NP-h| NP-c| NP-c|
| Additively separable              | NP-h| NP-h| NP-h| NP-c| NP-c|
| Fractional hedonic games          | NP-h| NP-h| NP-h| NP-c| NP-c|
| Social FHGs                       | NP-h| NP-h| (+) | (+) |     |
| Median                            | NP-h| NP-h|     |     |     |
| Midrange (\(\frac{1}{2}B + \frac{1}{2}W\)) | NP-h| NP-h| NP-c| NP-c| NP-c|
| 3-Approval                         | NP-h| NP-h| NP-c| NP-c| NP-c|
| 4-Approval                         | NP-h| NP-h| NP-c| NP-c| NP-c|

Table 1: Some of the hardness results implied by our framework for the problem of identifying hedonic games with stable outcomes. Gray entries are results that have not appeared in the literature before. (P) indicates known polynomial-time algorithms, (+) means that a stable outcome always exists. See Section 6 for details.
exponentially verbose in the worst case), but provide succinct descriptions of hedonic games that have certain structural properties; examples include Individually Rational Coalition Lists [Ballester, 2004] and Hedonic Coalition Nets [Elkind and Wooldridge, 2009]. The complexity of stability-related problems under these and other representations for hedonic games has been investigated by a number of researchers (see [Woegeringer, 2013a] for a survey); with a few exceptions, checking whether a game admits a stable outcome turns out to be computationally hard.

In this paper, we unify and extend several known hardness results for this family of problems in order to uncover common causes of complexity of stability-related questions in hedonic games. In their simplest form, our results imply that if in a given representation formalism, agents are able to rank coalitions of size two in any way they wish, and if agents are to some extent averse to the presence of enemies, then the problem of checking whether a game admits a stable outcome is NP-hard. The precise meaning of being averse to enemies depends on the stability concept in question. We also introduce intuitively appealing conditions on how agents rank coalitions of size three, which turn out to entail NP-hardness even if the underlying preferences are strict. Our approach enables us to automatically derive new hardness results for hedonic games: instead of coming up with a hardness reduction, one can simply check whether the representation in question satisfies the relevant conditions on enemy-aversion and coalitions of size two or three. By doing so, we answer several questions that were left open by prior work, and substantially contribute to the understanding of computational complexity of somewhat less explored solution concepts: to the best of our knowledge, we are the first to obtain NP-hardness results for strong Nash stability (SNS), strict strong Nash stability (SSNS), and strong individual stability (SIS).

To provide further evidence of the power of our approach, we also consider several classes of hedonic games whose complexity has not been investigated before, and derive NP-hardness results for them using our methodology. Perhaps the most interesting of them is the class of median games, proposed by [Hajduković, 2006], where each agent assigns a utility to every other agent, and her utility for a coalition is the utility she assigns to the median agent in that coalition.

The complexity results implied by our analysis are summarized in Table 1. However, we believe that the sufficient conditions for hardness identified in our work are at least as important as the specific new results we have established. Indeed, these conditions indicate which additional constraints should be placed on a representation formalism to avoid the complexity trap, and may guide researchers towards identifying formalisms that adequately describe their application scenario, yet admit efficient algorithms for finding stable outcomes.

2 Preliminaries

Given a finite set of agents $N = \{1, \ldots, n\}$, a hedonic game is a pair $G = (N, (\succ_i)_{i \in N})$, where $\succ_i$ is a complete and transitive preference relation over $N_i = \{S \subseteq N : i \in S\}$. We write $S \succ_T T$ when $S \succ_T T$, but $T \not\succ_T S$. A class $C$ of hedonic games is any collection of hedonic games. We say that a class $C$ is polynomially representable if there exists a polynomial $p(x)$ and a poly-time algorithm $A$ such that each $(N, (\succ_i)_{i \in N}) \in C$ can be represented by a binary string of length at most $p(|N|)$, and, given this string, an agent $i \in N$, and a pair of coalitions $S, T \in N_i$, algorithm $A$ can decide whether $S \succ_T T$. For example, additively separable hedonic games mentioned in Section 1 form a polynomially representable class.

An outcome of a hedonic game is a partition $\pi$ of $N$ into disjoint coalitions. We write $\pi(i)$ for the coalition of $\pi$ that contains $i$. For partitions $\pi$ and $\pi'$, we write $\pi \succeq \pi'$ to mean $\pi(i) \succeq \pi'(i)$.

We are mainly interested in the stability of a given partition $\pi$ of $N$. We will consider seven stability concepts for hedonic games: two that are based on individual deviations, and five that are based on group deviations. The former group comprises Nash stability (NS) and individual stability (IS). A partition $\pi$ is NS if no player can benefit from moving to another (possibly empty) coalition $S$ in $\pi$, i.e., $\pi(i) \succeq S \cup \{i\}$ for all $S \in \pi \cup \{\emptyset\}$. Partition $\pi$ satisfies IS if no player can make such a beneficial move without making an agent in $S$ worse off, i.e., for each $S \in \pi \cup \{\emptyset\}$ it holds that $\pi(i) \succeq S \cup \{i\}$ or $S \succ \pi(i) \cup \{i\}$ for some $j \in S$.

The classic solution concept for group deviations is the core (CR). We say that a non-empty coalition $S$ CR-blocks $\pi$ if $S \succ \pi(i)$ for all $i \in S$; it CR-blocks $\pi$ if $S \succ \pi(i)$ for all $i \in S$ and, moreover, $S \succ \pi(i)$ for some $i \in S$. If no coalition CR-blocks $\pi$, it is in the core (CR); if no coalition CR-blocks it, it is in the strict core (SCR).\[\]

Karakaya [2011] introduced strong Nash stability (SNS), and Aziz and Brandl [2012] introduced the derived notions of strict strong Nash stability (SSNS) and strong individual stability (SIS). These solution concepts deal with deviations where the deviators do not necessarily form a single coalition. Given two partitions $\pi, \pi'$, we say that a coalition $H \subseteq N$ can reach $\pi'$ from $\pi$ if for all $i, j \notin H$ we have $\pi(i) = \pi'(j)$ if and only if $\pi'(i) = \pi'(j)$. Coalition $H$ SSNS-blocks $\pi$ if it can reach some $\pi'$ with $\pi'(i) \succeq \pi$ for all $i \in H$ and $\pi'(j) \succeq \pi$ for some $i \in H$. If $H$ can reach some $\pi'$ with $\pi'(i) \succeq \pi$ for all $i \in H$ then $H$ is said to SSNS-block $\pi$. If $H$ SSNS-blocks $\pi$ by reaching $\pi'$ and, moreover, for each $i \in H$ and each $j \in \pi'(i)$ we have $\pi'(j) \succeq \pi$, then $H$ is said to SIS-block $\pi$. A partition $\pi$ is $\alpha$-stable (where $\alpha \in \{SNS, SSNS, SIS\}$) if no coalition $\alpha$-blocks it. Intuitively, SNS-blocking coalitions allow groups of agents to swap places with each other. For SIS-blocking coalitions, agents joined by a deviator must consent to the changes.

The diagram above shows implication relationships among these concepts. A partition that is SSNS-stable is also stable under every other solution concept considered here. A coalition $S \ni i$ is individually rational (IR) for $i$ if $S \ni \{i\}$. A partition $\pi$ is said to be IR if $\pi(i)$ is IR for all $i \in N$.

3 Properties of Preferences

Our hardness results require a given class $C$ of hedonic games to be expressible enough to include hard instances. To this end,
Weakly\{which\} strictly of graph theory when talking about hedonic games, and, i\{all\} \{strict\} selection is if each \{donic game\} \{coalitions that contain too many enemies.\} key reason why they admit easiness results. and in view of our results in Section 4 this is a\{games. Notably these classes of games fail to be triangle-
W\{matching problems and the (structurally similar)\} outcomes in hedonic games are known, mainly confined to \{the agents in\} \{enemies of\} i. In what follows, it will not matter how \{orders \}E\{i\}—only its restriction on F\{i\} will be of interest.

We now describe a series of properties that relate i\{s\} preferences \{\geq\} i\{over the coalitions in N\} to her preferences \{\geq\} i\{over the agents in N\}. These properties express various ways in which \{\geq\} can be said to \{extend\} \{\geq\}. The numerical examples in brackets aim to illustrate the intuition behind these properties.

\textbf{Consistent on pairs.} For all \{j, k \in F_i \cup \{i\}\} it holds that \{(i, j) \geq_i \{i, k\}\} iff \{j \geq_i k\}.

\textbf{Monotone on triangles (\{7+6 > 7+5\}).} If \{j, j', k' \in F_i\} are such that \{j \geq_i j' \geq_i k' \}, then \{i, j, k\} \succ_i \{i, j', k'\}.

\textbf{Triangle-appreciating (\{7+5 > 7\}).} Two almost equally good friends together are preferable to the better friend alone: If \{j, k, \ell \in F_i\} are ranked \{j >_i k >_i \ell\} and they are immediate successors under \geq_i, then \{i, j, \ell\} \succ_i \{i, j\}.

Only few polynomial-time algorithms for finding stable outcomes in hedonic games are known, mainly confined to matching problems and the (structurally similar) W-hedonic games. Notably these classes of games fail to be triangle-appreciating, and in view of our results in Section 4 this is a key reason why they admit easiness results.

The following properties express that agents do not like coalitions that contain too many enemies.

\textbf{(a-b)-toxic.} If \{|S \cap F_i| = a\} but \{|S \cap E_i| \geq b\} then \{i \geq_i S\}.

\textbf{Strictly (a-b)-toxic.} Same as above with \{i \succ_i S\}.

\textbf{Weakly (a-b)-toxic.} Same as above with \{i, j \succ_i S\} for all \{j \in F_i\}.

\textbf{Intolerant in triangles.} If \{E_i' \subset E_i\} is non-empty and \{j, k \in F_i\} are distinct then \{i, j, k\} \succ_i \{i, j, k\} \cup E_i'.

We write ‘(strictly/weakly) \{(a_1-b_1, \ldots, a_m-b_m)\}-toxic’ for preferences that are (strictly/weakly) \{(a_i-b_i)\}-toxic for \{t = 1, \ldots, m\}.

Given a collection \{(\geq_i)\}_{i \in N} of orderings, we say that a hedonic game \{N, (\geq_i)_{i \in N}\} satisfies one of the properties above if each \geq_i satisfies it with respect to \geq_i. We say that the collection is \emph{strict} if each \geq_i is antisymmetric, so \{j \not\geq_i k\} implies \{j \ngeq_i k\}. The collection is \emph{mutual} if \{j \in F_i\} if and only if \{i \in F_j\} for all \{i, j\}. For a mutual collection of orderings, we may consider the \emph{friendship graph} with vertex set \{N\}, where an (unweighted) edge connects mutual friends. We will use standard terminology of graph theory when talking about hedonic games, and, in particular, speak of cliques, trees, and cycles of agents.

\section{Hardness Results}

Let \(C\) be a polynomially representable class of hedonic games. For every stability concept \(\alpha\) defined in Section 4 we will consider the following decision problem associated with \(C\).

\textbf{\(\alpha\)-existence for \(C\)}

\textbf{Instance:} Game \(\langle N, (\geq_i)_{i \in N}\rangle\) from \(C\) in its binary encoding.

\textbf{Question:} Is there an \(\alpha\)-stable partition \(\pi\) of \(N\)?

To avoid difficulties with binary representations that are very short, we will assume that the binary encoding of \(\langle N, (\geq_i)_{i \in N}\rangle\) lists the names of agents in \(N\), and hence contains at least \{|N|\} bits. Furthermore, when in the following theorems we assume that \(C\) contains various hedonic games \(\langle N, (\geq_i)_{i \in N}\rangle\) derived from orderings \((\geq_i)_{i \in N}\), we require that such games (i.e., their binary descriptions) can be constructed in time polynomial in \{|N|\}; this property is necessary for our hardness reductions to work in polynomial time and is satisfied by all classes of hedonic games considered in this paper.

Our first result has mild assumptions and applies to a large number of classes \(C\).

\textbf{Theorem 1.} CR-existence for \(C\) is \(NP\)-hard if for all \(N\) and every mutual collection of orderings \((\geq_i)_{i \in N}\) in which each agent has at most \(3\) friends, there is a game \(\langle N, (\geq_i)_{i \in N}\rangle \in C\) that is consistent on pairs, \{0-1\}-toxic and \{weakly \{1-1, 2-2\}\}-toxic with respect to \((\geq_i)_{i \in N}\).

\textbf{Remark 1.} Under the same set of conditions SIS-existence for \(C\) is also \(NP\)-hard; we obtain a hardness result for SNS-existence for \(C\) by strengthening weak \{1-1\}-toxicity to \{1-1\}-toxicity.

Effectively, Theorem 1 says that if agents are allowed to rank pairs as they wish, and if they do not have to like everyone, then finding a core-stable outcome is hard.

The assumptions are chosen so as to guarantee that a game like the pentagon displayed on the right has empty core. In this game, each agent has exactly two friends, the clockwise successor being preferred to the clockwise predecessor. All other agents are enemies. It can be checked that in agents’ preferences satisfy weak \{1-1-2-2\}-toxicity then this game has empty core. We use the 9-player version of this game as a gadget in our hardness reductions (see Figure 1).

A similar result holds for solution concepts based on individual deviations.

\textbf{Theorem 2.} NS- and IS-existence for \(C\) are \(NP\)-complete if for all \(N\) and every strict and mutual collection of orderings \((\geq_i)_{i \in N}\) in which each agent has at most \(3\) friends, there is a game \(\langle N, (\geq_i)_{i \in N}\rangle \in C\) that is consistent on pairs and strictly \{0-1-1-1-2-5\}-toxic with respect to \((\geq_i)_{i \in N}\).

In the case of NS-existence, the theorem remains true even if the orderings \((\geq_i)_{i \in N}\) are strict and bipartite (but not mutual), i.e. the friendship graph is bipartite. Thus, its conclusion also applies to NS-existence for the stable marriage problem with unacceptabilities. For the case with ties allowed, this result is also obtained by Aziz [2013].

The reduction establishing Theorem 1 makes essential use of indifferences in the underlying orderings \((\geq_i)_{i \in N}\) (this is also the reason why it does not go through for the strict core).
Theorem 3. CR- and SCR-EXISTENCE FOR $\mathcal{C}$ are NP-hard if for all $N$ and every collection of strict and mutual orderings $(\succeq_i)_{i \in \mathbb{N}}$ in which each agent has at most 4 friends, there is a game $(\mathcal{N},(\succeq_i)_{i \in \mathbb{N}}) \in \mathcal{C}$ that is consistent on pairs, triangle-approximating, monotone on triangles, $\{0,1\}$-toxic, weakly $\{1,2,2,3,3\}$-toxic, and intolerant in triangles with respect to $(\succeq_i)_{i \in \mathbb{N}}$.

Remark II. The same result holds for SIS-EXISTENCE FOR $\mathcal{C}$. It applies to SNS-EXISTENCE FOR $\mathcal{C}$ if we add $\{1,1\}$-toxicity and weak $\{2,1\}$-toxicity. It applies to SSNS-EXISTENCE FOR $\mathcal{C}$ if we add strict $\{0,1,1,1\}$-toxicity and weak $\{2,1\}$-toxicity.

5 The Reductions

The proofs of our results are by reduction from a restricted version of 3SAT. The reduction behind Theorem 1 is inspired by an argument of Ronn [1990] showing that STABLEROOMMATES with ties is NP-complete. Theorem 3 introduces triangles into this reduction to allow strict preferences.

We sketch the proof of Theorem 1 but omit proofs of the other claims due to space constraints. The omitted arguments are similar to the one given, but more complicated due to SNS-like stability concepts imposing little structure. Full proofs are given in the appendices A, B, and C.

Proof of Theorem 1 (sketch). We reduce from $(3, B2)$-SAT, which is 3SAT restricted to formulas in which each clause contains exactly 3 literals, and each variable occurs exactly twice positively and twice negatively [Berman et al., 2003].

Given an instance formula $\varphi$ with variable set $X$ and clause set $C$, we construct the following agent set $N$:

$$\bigcup_{c \in C} \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \cup \bigcup_{x \in X} \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$$

The four occurrences (two positive and two negative ones) of a variable $x \in X$ are called $x_1, x_2, x_3, x_4$, respectively. For a clause $c = \ell_1 \lor \ell_2 \lor \ell_3$, we write $c(\ell_i) = 1$, $c(\ell_i) = 0$ for $i = 1, 2, 3$.

Construct orderings $(\succeq_i)_{i \in \mathbb{N}}$ as follows:

$$\begin{align*}
\pi_1: & \ x_4 \succ x_2 \succ c(\pi_1) \\
\pi_2: & \ x_8 \succ x_7 \succ x_5 \\
\pi_3: & \ x_1 \succ x_8 \succ x_9 \\
\pi_4: & \ x_2 \succ x_7 \succ x_6
\end{align*}$$

For each agent $i$ we have only listed $i$'s friends $F_i$, each friend being strictly better than $i$. Any agent not mentioned in $i$'s list is an enemy, i.e., an element of $E_i$. Figure 1(a) illustrates the orderings $(\succeq_i)_{i \in \mathbb{N}}$. Note that no agent has more than 3 friends, and that these orderings are mutual.

By the assumptions of Theorem 1 there is a poly-time many-one reduction that takes a formula $\varphi$ as input and outputs the binary encoding of a game $G = (\mathcal{N}, (\succeq_i)_{i \in \mathbb{N}}) \in \mathcal{C}$ that is consistent on pairs, $\{0,1\}$-toxic and weakly $\{1,2,2,3,3\}$-toxic with respect to the $(\succeq_i)_{i \in \mathbb{N}}$ given above. We show that $\varphi$ is satisfiable if and only if $G$ admits a CR-stable partition.

Let $\mathcal{A}$ be a satisfying assignment of $\varphi$. Take the partition

$$\pi = \{\{\ell, c(\ell)\} : \ell \text{ a true variable occurrence}\}$$

$$\bigcup \{\{x_i, \pi_1\}, \{x_2, \pi_2\} : x \in X \text{ set true in } \mathcal{A}\}$$

$$\bigcup \{\{x_i, \pi_3\}, \{x_2, \pi_4\} : x \in X \text{ set false in } \mathcal{A}\}$$

$$\bigcup \{\{x_i, x_9\}, \{x_2, x_6\} : x \in X\}$$

$$\bigcup \{\{c_i, c_{i+1}, \ldots, c_j\} : c \in C\}.$$
even if the list of each agent includes at most 3 entries, each of which is a pair. A similar result is shown by Deineko and Woeginger [2013]. They prove that CR-EXISTENCE for IRCL is hard even for lists of length 2, with entries being coalitions of size 3. Theorem 5 applies if we allow lists up to length 9, which can encode a triangle-appreciating game where agents have up to 4 friends.

Hedonic Coalition Nets. Elkind and Wooldridge [2009] study a rule-based representation for hedonic games in which agents’ preferences are described by weighted boolean formulas. It can be shown that polynomial size nets are sufficient to describe, for any collection of orderings \( \{x_i\}_{i \in N} \), a game satisfying all our conditions, implying hardness of \( \alpha \)-EXISTENCE for all \( \alpha \) considered in this work. This is perhaps not surprising: while Elkind and Wooldridge only establish the hardness of CR-EXISTENCE in their work, they show that one can compile an IRCL representation into a hedonic coalition net representation with at most polynomial overhead. Because our hardness results hold even if each player is only allowed 3 or 4 friends, we can say in addition that \( \alpha \)-EXISTENCE for hedonic coalition nets remains hard even if we restrict each player’s preferences to be described by at most 4 or 5 formulas, and even if the weights of these formulas are given in unary.

Stable Roommates. The reduction behind Theorem 1 is a modified version of Ronn’s construction showing that CR-EXISTENCE for SRT, the stable roommate problem with ties, is NP-complete [Ronn, 1990]. It is thus no surprise that the class of stable roommate problems, considered as hedonic games in which sets with 3 or more members are unacceptable, fulfills the conditions of Theorem 1 (but note that this formulation corresponds to SRTI, not SRT). Indeed, CR-EXISTENCE for SRTI remains hard even if the preference list of each agent has length at most 3, and by Theorem 2 this is also true of NS- and IS-EXISTENCE. Now, consider a generalization of STABLE-ROOMMATES where rooms have capacity 1, 2, or 3, and rooms with capacity 3 are generally preferred because they are cheaper per person. Then it can be checked that the conditions of Theorems 2 and 3 are satisfied, giving hardness of \( \alpha \)-EXISTENCE for all \( \alpha \) for this model.

Stable Marriages. A version of Theorem 2 implies that NS-EXISTENCE for SMI, the stable marriage problem with incomplete lists, is NP-complete. This extends the NP-completeness result for SMTI obtained by [Aziz, 2013]. Aziz also notes that it is possible to embed SMTI into other classes \( C \) of hedonic games, and thus to deduce hardness of NS-EXISTENCE for \( C \); this observation provides an (alternative) method of deriving hardness results for several classes of hedonic games.

\( W \)-preferences. Cechlárová and Hajduková [2004b] consider hedonic games where each agent first ranks all other agents and then compares coalitions based on their worst member under this ranking. Clearly, the game so obtained is consistent on pairs and strictly \( \{k-1\} \)-toxic for all \( k \). It follows that (with ties allowed) CR-EXISTENCE is NP-hard by Theorem 1 (a result first obtained by [Cechlárová and Hajduková, 2004b]), and that NS- and IS-EXISTENCE are NP-complete by Theorem 2, first shown by Aziz et al. [2012]. NS- and IS-EXISTENCE are hard even if preferences are strict; the latter result was previously unknown.

\( WB \)-preferences. Noting that agents in \( W \)-hedonic games are extremely pessimistic, Cechlárová and Hajduková [2004a] propose a compromise: Agents still rank coalitions according to their worst member, but break ties in favor of the coalition with better best member. Again the game obtained is consis-
tent on pairs and strictly \{k-1\}-toxic for all \(k\), so CR-, NS-, and IS-EXISTENCE are hard.

**W- and B-hedonic games.** In these two classes of games [Aziz et al., 2012, 2013], agents rank coalitions according to their worst or best member, but coalitions containing an enemy are not individually rational. As for \(\mathcal{V}\)-preferences, we see that CR-, NS-, and IS-EXISTENCE are hard.

In all of the following classes of games, agents first assign cardinal utilities \(v_i(j) \in \mathbb{R}\) to all agents in \(N = \{1, \ldots, n\}\), and then lift these utilities to coalitions (e.g., by computing the sum or average of the utilities of coalition members).

The following method of constructing integer-valued functions \(v : N \to \mathbb{R}\) from orderings \((\geq_i)_{i \in N}\) will be used repeatedly: given \(x, y \in \mathbb{Z}\), we set \(v_i(i) = 0, v_i(j) = x\) for \(j \in E_i\) and let \(y \leq v_i(j) \leq y + n\) for \(j \in F_i\) so that for each \(j, k \in F_i\) we have \(v_i(j) \geq v_i(k)\) iff \(j \geq k\) (this is accomplished by assigning utility \(y + k + 1\) to friends at the \(k\)-th ‘preference level’). We refer to such utilities as \([x,y]\)-utilities.

**Additively Separable Games (ASGs).** In these games, preferences are given by \(S \triangleright_{\geq_i} T\) iff \(\sum_{j \in S} v_i(j) \geq \sum_{j \in T} v_i(j)\). This class of games satisfies all our theorems, so \(\alpha\)-EXISTENCE is hard for all \(\alpha\) we consider. Indeed, given \(N = \{1, \ldots, n\}\) and \((\geq_i)_{i \in N}\), we consider the ASG with \([- (n^2 + 2n), 4]\)-utilities. Then a coalition containing an enemy of \(i\) is not individually rational for \(i\), so this game is strictly \{k-1\}-toxic for all \(k\), and it is obviously consistent on pairs, triangle-appreciating and monotone on triangles. \(\alpha\)-EXISTENCE remains hard even if players are allowed at most 3 or 4 friends (depending on \(\alpha\)), so for ASGs, it remains hard even if \(v_i(j)\) is positive for at most 3 or 4 agents \(j\). This improves on the reduction in [Sung and Dimitrov, 2010], where agents have up to 11 friends.

**Fractional Hedonic Games (FHGs).** This class of games was recently proposed by Aziz et al. [2014]. Preferences are given by \(S \triangleright_{\geq_i} T\) iff \(1/|S| \sum_{j \in S} v_i(j) \geq 1/|T| \sum_{j \in T} v_i(j)\). Brandl et al. [2015] have shown hardness of CR-, NS-, and IS-EXISTENCE. We recover these results and complement them by showing hardness of SSNS-, SNS-, SIS- and SCR-EXISTENCE; all these results hold even if the underlying preferences are strict. FHGs with \([- (n^2 + 5n), 5]\)-utilities satisfy all of our properties; choosing \(y = 5\) ensures triangle-appreciation.

**Social FHGs.** An FHG is social if agents’ utilities for each other are non-negative. Theorem 3 applies to the class of social FHGs. Indeed, given \((\geq_i)_{i \in N}\) we can construct a social FHG with \([0, 7n]\)-utilities. Toxicity follows from \(v_i(j) \geq 7n\) for \(j \in E_i\), and other properties can be checked as for FHGs. To ensure that our framework applies to social FHGs, we carefully crafted our constructions to only require weak toxicity whenever possible.

The next five classes of hedonic games are based on fairly intuitive ways of deriving utilities for coalitions from utilities for individual players; however, to the best of our knowledge we are the first to consider the computational complexity of stability-related problems for these games (median games have been suggested by [Hajduková, 2006] as an interesting topic; the other four classes appear to be entirely new).

**Median Games.** Agents evaluate coalitions according to their median value, which in odd-size coalitions is the middle element, and in even-size coalitions is the mean of the middle two elements. Median games with \([0, 5]\)-utilities satisfy Theorem 5. Notice that in this construction \(v_i(j)\) are non-negative, so hardness holds even for ‘social median games’ with non-negative underlying utilities. There are various other ways of defining median games. In particular, we can use a purely ordinal version by taking the worse of the middle two players in even-sized coalitions, satisfying Theorem 1 if agents take either the ordinal or cardinal median of the coalition \(S \setminus \{i\}\) then both Theorems 1 and 2 apply.

**Geometric Mean Games.** In these games agents evaluate coalitions according to the geometric mean \(\sqrt[n]{\prod_{j \in S} v_i(j)}\) of member utilities. We obtain the same hardness results as for FHGs by taking logs.

**Nash Product Games.** This is the class of games that are ‘multiplicatively separable’; agents evaluate coalitions according to \(\prod_{j \in S} v_i(j)\). As far as hardness is concerned these games behave identically to additively separable games, again by taking logs.

**Midrange \((\frac{1}{3} + \frac{1}{3} X)\).** In this case, agents evaluate a coalition by averaging the maximum and minimum utility in it. With \([-3n, 1]\)-utilities, these games are strictly \{k-1\}-toxic for all \(k\) and consistent on pairs, so Theorems 1 and 2 apply.

**r-Approval.** Starting with cardinal utilities, sum the (up to) \(r\) highest elements of a coalition. If \(r \geq 3\), then games with \([-6rn, 4]\)-utilities satisfy the conditions of Theorems 1 and 2. If \(r \geq 4\), they satisfy the conditions of Theorem 3.

7 Conclusions

We have developed a framework that enables us to prove NP-hardness of \(\alpha\)-EXISTENCE for \(C\) for many choices of \(\alpha\) and \(C\). Our results show that problems in this family tend to be hard even for representation formalisms with very limited expressivity, and, moreover, are unlikely to admit an efficient parametrized algorithm for many natural choices of parameter (such as length and coalition size in the IRCL representation or number of formulas per agent in the hedonic coalition nets representation). However, they also indicate which features of hedonic games may lead to tractability of stability-related problems. In particular, restricting the number of different ‘preference intensities’ (e.g., the range of \(v_i(j)\) in ASGs, FHGs, and median games) rules out consistency on pairs, so one may hope for easiness results when this number is small.

While we focused on the problem of checking whether a stable partition exists, another important stability-related problem is checking whether a specific partition is stable. This problem is in \(P\) for AS and NS for all classes of hedonic games considered here, simply because the number of possible deviations is polynomially bounded; however, for notions of stability that are based on group deviations it is often coNP-complete. It would be interesting to extend our framework to handle this problem as well.
Since verifying stability is often hard, $\alpha$-EXISTENCE FOR $C$ is usually not known to be in NP for stability notions based on group deviations. Thus most of our hardness results do not have a tight complexity upper bound. For all representation formalisms we consider, these problems are in $\Sigma^p_2$, and CR-EXISTENCE FOR ASGs is known to be complete for this complexity class [Woeginger, 2013b]. A natural open question is whether our framework can be extended from NP-hardness proofs to $\Sigma^p_2$-hardness proofs.

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References
[Aziz and Brandl, 2012] Haris Aziz and Florian Brandl. Existence of Stability in Hedonic Coalition Formation Games. AAMAS ’12, pages 763–770, 2012.

[Aziz and Savani, 2015] Haris Aziz and Rahul Savani. Hedonic games. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, Handbook of Computational Social Choice, chapter 15. Cambridge University Press, 2015.

[Aziz et al., 2012] Haris Aziz, Paul Harrenstein, and Evangelia Pyrga. Individual-based Stability in Hedonic Games Depending on the Best or Worst Players. AAMAS ’12, pages 1311–1312, 2012.

[Aziz et al., 2013] Haris Aziz, Felix Brandt, and Paul Harrenstein. Pareto optimality in coalition formation. Games and Economic Behavior, 82:562–581, 2013.

[Aziz et al., 2014] Haris Aziz, Felix Brandt, and Paul Harrenstein. Fractional Hedonic Games. AAMAS ’14, pages 5–12, 2014.

[Aziz, 2013] Haris Aziz. Stable marriage and roommate problems with individual-based stability. AAMAS ’13, pages 287–294, 2013.

[Ballester, 2004] Coralio Ballester. NP-completeness in hedonic games. Games and Economic Behavior, 49(1):1–30, October 2004.

[Banerjee et al., 2001] Suryapratin Banerjee, Hideo Konishi, and Tayfun Sönnmez. Core in a simple coalition formation game. Social Choice and Welfare, 18(1):135–153, January 2001.

[Berman et al., 2003] Piotr Berman, Marek Karpinski, and Alexander D. Scott. Approximation Hardness of Short Symmetric Instances of MAX-3SAT. Technical Report 049, 2003.

[Bogomolnaia and Jackson, 2002] Anna Bogomolnaia and Matthew O. Jackson. The Stability of Hedonic Coalition Structures. Games and Economic Behavior, 38(2):201–230, February 2002.

[Brandl et al., 2015] Florian Brandl, Felix Brandt, and Martin Strobel. Fractional Hedonic Games: Individual and Group Stability. AAMAS ’15, pages 1219–1227, 2015.

[Cechlárová and Hajduková, 2003] Katarína Cechlárová and Jana Hajduková. Computational complexity of stable partitions with B-preferences. International Journal of Game Theory, 31(3):353–364, 2003.

[Cechlárová and Hajduková, 2004a] Katarína Cechlárová and Jana Hajduková. Stability of partitions under WB-preferences and BW-preferences. International Journal of Information Technology & Decision Making, 03(04):605–618, December 2004.

[Cechlárová and Hajduková, 2004b] Katarína Cechlárová and Jana Hajduková. Stable partitions with W-preferences. Discrete Applied Mathematics, 138(3):333–347, 2004.

[Deineko and Woeginger, 2013] Vladimir G Deineko and Gerhard J Woeginger. Two hardness results for core stability in hedonic coalition formation games. Discrete Applied Mathematics, 161(13):1837–1842, 2013.

[Drèze and Greenberg, 1980] Jacques H Drèze and Joseph Greenberg. Hedonic Coalitions: Optimality and Stability. Econometrica, 48(4):987–1003, May 1980.

[Elkind and Wooldridge, 2009] Edith Elkind and Michael Wooldridge. Hedonic Coalition Nets. AAMAS ’09, pages 417–424, 2009.

[Hajduková, 2006] Jana Hajduková. Coalition Formation Games: A Survey. International Game Theory Review (IGTR), 08(04):613–641, 2006.

[Karakaya, 2011] Mehmet Karakaya. Hedonic coalition formation games: A new stability notion. Mathematical Social Sciences, 61(3):157–165, May 2011.

[Ronn, 1990] Eytan Ronn. NP-complete stable matching problems. Journal of Algorithms, 11(2):285–304, June 1990.

[Sung and Dimitrov, 2010] Shao-Chin Sung and Dinko Dimitrov. Computational complexity in additive hedonic games. European Journal of Operational Research, 203(3):635–639, June 2010.

[Woeginger, 2013a] Gerhard J Woeginger. Core stability in hedonic coalition formation. In SOFSEM 2013: Theory and Practice of Computer Science, pages 33–50. Springer, 2013.

[Woeginger, 2013b] Gerhard J. Woeginger. A hardness result for core stability in additive hedonic games. Mathematical Social Sciences, 65(2):101–104, March 2013.
A Proof of Theorem 1

Theorem. SIS- and CR-EXISTENCE (SNS-EXISTENCE) for $\mathcal{C}$ is NP-hard if for each $N$ and every collection of orderings $(\geq_i)_{i \in \mathbb{N}}$ there is a game $(\mathcal{N}, (\geq_i)_{i \in \mathbb{N}}) \in \mathcal{C}$ that is consistent on pairs, $\{0\text{-}1\}$-toxic ( $\{0\text{-}1,1\text{-}1\}$-toxic) and weakly $\{1\text{-}1,2\text{-}2\}$-toxic with respect to $(\geq_i)_{i \in \mathbb{N}}$.

We will prove both statements in the theorem together, with the proof for SNS-EXISTENCE added in $()$-brackets. Proof by reduction from (3,B2)-SAT (each clause contains exactly 3 literals, each variable occurs exactly twice positively and twice negatively).

Given a formula $\varphi$ we denote

(a) for any variable $x$ its four occurrences by $x_1, x_2, x_3, x_4$, 
(b) for any variable occurrence $c, c(\ell)$ is the clause that $\ell$ occurs in.

Given an instance $\varphi$ of (3,B2)-SAT, take the following set $N$ of agents, with 9 agents per clause and 10 agents per variable.

$N = \{c_1, \ldots, c_9 \mid c \in \text{Clauses}(\varphi)\} \cup \{x_1, x_2, x_3, x_4, x'_a, x_b, x'_b, x''_a, x''_b \mid x \in \text{Var}(\varphi)\}$.

For a clause $c = \ell_1 \lor \ell_2 \lor \ell_3$, where $\ell_i$ is a variable occurrence, we will connect $c_1$ with $\ell_1$, $c_4$ with $\ell_2$, and $c_7$ with $\ell_3$.

For $c_1, c_4, c_7$, we call the variable occurrence connected with them ‘its literal’. For a variable occurrence $\ell$, we will call the $c_i$ player connected with it ‘its clause player’ and denote it by $c(\ell)$. We generate the following orderings, which only show the friends of each player. The last entry of player $i$’s list is strictly better than $i$ who is strictly better than all players not mentioned:

$c_1 : \ell_1 > c_2 > c_9$
$c_4 : \ell_2 > c_5 > c_3$
$c_7 : \ell_3 > c_8 > c_6$
$c_i : c_{i+1} > c_{i-1}$ for $i \neq 1, 4, 7$ with subscripts mod $9$
$x_1 : x_a > x_2 > c(x_1)$
$x_3 : x_a > x_2 > c(x_3)$
$x_5 : x_b > x_1 > c(x_2)$
$x_7 : x_b > x_1 > c(x_2)$
$x_9 : x_1 \sim x_3 > x'_a$
$x'_a : x_a > x''_a$
$x'_a : x_a > x''_a$
$x_b : x_2 > x'_b > x'_b$
$x'_b : x_b > x'_b$
$x''_b : x'_b$

Notice the following facts:

- No player has more than 3 friends.
- The orderings are mutual: $j$ is $i$’s friend if and only if $i$ is $j$’s friend.
- The friendship graph has girth $6$. In particular, it is triangle-free.

By the assumptions of the theorem, we can in polynomial time find a hedonic game which is consistent on pairs, $\{0\text{-}1\}$-toxic ($\{0\text{-}1,1\text{-}1\}$-toxic), and weakly $\{1\text{-}1,2\text{-}2\}$-toxic with respect to the orderings given. We show that $\varphi$ is satisfiable if and only if this game admits a core- and SIS-stable (SNS-stable) partition.

**Satisfiable $\Rightarrow$ Stable.**

Suppose $A$ is a satisfying assignment for $\varphi$. Construct the following partition $\pi$ of $N$, which consists only of pairs and singletons:

- Match true variable occurrences with their clause player.
- If $x$ is false then $\{x_1, x_a\}, \{x_2, x_b\} \in \pi$.
- If $x$ is true then $\{x_1, x_a\}, \{x_2, x_b\} \in \pi$.
- $\{x'_a, x''_a\} \in \pi$ and $\{x'_b, x''_b\} \in \pi$.
- For each clause, match the non-matched $c$-players in some stable way:
  - If exactly 1 player is matched with a variable occurrence, say $c_1$ is matched, then we take $\{c_2, c_3\}, \{c_4, c_5\}, \{c_6, c_7\}, \{c_8, c_9\} \in \pi$.
  - If exactly 2 players are matched with a variable occurrence, say $c_1$ and $c_4$ are matched, then $\{c_2, c_3\}, \{c_5\}, \{c_6, c_7\}, \{c_8, c_9\} \in \pi$.
  - If all 3 players are matched with a variable occurrence, then $\{c_2, c_3\}, \{c_5, c_6\}, \{c_8, c_9\} \in \pi$.

Each of these three cases is illustrated in the drawing above.

Suppose that $\pi \xrightarrow{H} \pi'$ for some $H \subseteq N$ with $\pi' \succeq_i \pi$ for all $i \in H$. We will prove that this is not an SIS-deviation (not an SNS-deviation). Note that if there are no SIS-deviations, then $\pi$ is also core-stable.

Some terminology and observations:

- an agent is matched if it is in a pair in $\pi$, and lonely if it is in a singleton in $\pi$ (these terms always refer to $\pi$ and never to $\pi'$).
- an agent is a deviator if it is in $H$, and a non-deviator otherwise.
- no 2 enemies are together in $\pi$.
- any 2 lonely players have distance at least $5$, which in the following will mean that at most 1 player in a coalition considered below can be lonely.
By definition of $\pi \xrightarrow{H} \pi'$, no two non-deviators can be in the same coalition in $\pi'$ if they weren’t together in $\pi$ already. This fact will be used often and not particularly stressed in the following argument.

**Lemma 1.** No matched deviator $i$ has exactly 2 friends in $\pi'(i)$.

**Proof.** Suppose $i$ is a matched deviator and ends up in a coalition $S \in \pi'$ which includes exactly 2 friends of $i$. Since $i$ prefers $\pi'$ to $\pi$, by weak $\{2-2\}$-toxicity, $S$ includes at most 1 enemy of $i$, so $|S|$ is either 3 or 4.

If $|S|$ is 3, say $S = \{i, j, k\}$, then since the game is triangle-free and friendship is mutual, each of $j$ and $k$ have 1 friend (namely $i$) and 1 enemy in $S$. Since $j$ and $k$ are enemies, they are not together in $\pi$ and hence they cannot be both non-deviators (and still end up in the same coalition in $\pi'$). Say $j$ is a deviator. Then $j$ cannot be matched by weak $\{1-1\}$-toxicity ($j$ has an enemy in $S$). So $j$ is lonely and hence $k$ is not lonely since lonely players are far apart. Now $k$ is made worse off by the deviation (since $k$ is not lonely and by weak $1$-1-toxicity), so the deviation is not SIS. (With $\{1-1\}$-toxicity, $j$ is not made happier by the deviation, so not an SNS deviation.)

\[
\begin{array}{ccc}
\text{j} & \xrightarrow{\mu} & \text{i} \\
\downarrow & & \downarrow \\
\ell & & k
\end{array}
\]

Suppose $|S|$ is 4, say $S = \{i, j, k, \ell\}$ where $\ell$ is an enemy of $i$. Suppose first that $\ell$ had no friends in $S$. Then $\ell$ is not better off under $\pi'$ by $\{0-1\}$-toxicity and thus is a non-deviator. Since $j$ and $k$ were not previously together with $\ell$, they must both be deviators. If either of them was matched, they’d now be unhappy by weak $\{1-1\}$-toxicity. Since they are deviators, they cannot be unhappy, and hence both were lonely. But then $i$ is friends with 2 lonely players, contradiction. Hence $\ell$ has a friend in $S$. Since there are no 4-cycles in the game, $\ell$ cannot be friends with both $j$ and $k$, and thus must be friends with exactly 1 of $j$ and $k$, say $k$. So $j$ and $\ell$ are enemies. Since they are not together under $\pi$, at least 1 of them must be a deviator, say $j$. Since there are enemies in $S$, $j$ must have been lonely, and hence $\ell$ is not lonely. Now $\ell$ is made worse off by the deviation (since $\ell$ is not lonely), so the deviation is not SIS. (With $\{1-1\}$-toxicity, $j$ is not made happier by the deviation, so not SNS.)

**Lemma 2.** No matched deviator $i$ has exactly 3 friends in $\pi'(i)$.

**Proof.** Suppose $i$ is a matched deviator and ends up in a coalition $S \in \pi'$ which includes exactly 3 friends of $i$, called $j$, $k$, $\ell$ who are (necessarily) pairwise enemies. Thus at least 2 of the 3 friends, say $j$ and $k$, must deviate, and thus at least 1 of these 3 friends is a matched deviator, say $j$. By weak $\{1-1, 2-2\}$-toxicity, $j$ must have at least 3 friends in $S$, including $i$. Apart from $i$, 1 more friend of $j$ must deviate, called $m$. Now one of $m$ and $k$ must be matched, so is a matched deviator, and hence must have 3 friends in $S$. Hence $S$ contains at least 3 players with 3 friends in $S$. Also $|S| \geq 5$, and hence $S$ cannot include matched deviators who only have 2 or fewer friends in $S$, by toxicity. Call a matched player in $S$ with at most 2 friends in $S$ sad. Then $S$ consists of lonely players, players with 3 friends in $S$, and at most 2 sad players (who are non-deviators). If there are 2 sad players, they must be together in $\pi$.

Now if a player of type $x_a$ is in $S$, then either $x_a$ is sad or $x_a' \in S$ and $x_a'$ is sad. Similarly for $x_b$. Further, if a player of type $c_1/(c_2/c_3)$ is in $S$, then either he is sad, or a neighbour of it is in $\pi$ and sad. Therefore at most 1 player of type $x_a/(x_b/c_1)/c_2/c_3$ can be in $S$. Suppose a literal player $x_1$ is in $S$. Then either $x_1$ is sad, or else both $c(x_1) \in S$ and $x_a \in S$, but we previously said that there is at most 1 of these players in $S$, so this is impossible. Thus all literal players in $S$ must be sad, and so there is at most 1 literal player in $S$. Hence there are at most 2 degree-3 players in $S$, contradicting our observation before that there are at least 3.

Now suppose $i$ is a matched deviator (with $\{i, j\} \in \pi$). Then by the lemmas, $i$ must end up in a coalition $S \in \pi'$ which includes exactly 1 friend of $i$. By weak $\{1-1\}$-toxicity, $S$ must be a pair, $S = \{i, k\}$, say. Since $\pi' \Rightarrow \pi$ because $i$ is a deviator, we must have $k > j$ by consistency on pairs. Now by inspection it is seen that in $\pi$ there is no edge $\{i, j\}$ that is strictly better than $\pi$ for both $i$ and $j$. Hence $k$ cannot be strictly better off in $\{i, k\}$ than $k$ is in $\pi$. So $k$ is not a deviator, and hence is either lonely or matched to a deviator.

We now go through each type $i$ of matched player and show that there is no edge $\{i, j\}$ satisfying these conditions: any preferred edge (if any) involves a partner who is matched to a non-deviator.

- $x_a'$ and $x_a''$, $x_a'$ and $x_b$, and false variable occurrences are in a favourite edge, so none of them are deviators.
- $x_a'$ and $x_a''$ have a strictly preferred edge with $x_a$ and $x_b$ respectively, but these are matched to false variable occurrences who we know are non-deviators.
- True variable occurrences: all preferred edges are partitioned with a non-deviating player.
- Those $c_1$, $c_4$, or $c_3$ players matched to their literal player are in their favourite edge so not deviating.
- Those $c_1$ players together with $c_i+1$ are either in their favourite edge or (if they have a literal friend) that friend is false, so together with a non-deviator.
- Those $c_1$ players together with $c_i-1$ such that $c_i+1$ is in a pair where both members are confirmed non-deviators are themselves then clearly non-deviators. Such $c_i$ players exist: namely those preceding a $c$-player matched with their literal. If we repeat this observation, we find that all matched $c$-players are non-deviators.

Hence (together with the lemmas above) no matched player is a deviator.

Now consider a lonely player $i$. If $i$ deviates, then $i$ ends up in a coalition $S \in \pi'$ consisting of lonely players and possibly 2 players that are in a pair in $\pi$ (because no matched players are deviators). If $S$ consists entirely of lonely players, then by $\{0-1\}$-toxicity, $i$ is not better off, so won’t deviate. Hence $S$ also contains 2 players in a pair. At least 1 of these 2 is enemies with $i$ and is thus worse off by weak $\{1-1\}$-toxicity,
so this is not an SIS-deviation. \((\text{With \{1-1\}-toxicity, } i \text{ is not better off in } S, \text{ so not an SNS-deviation.})\)

Thus we conclude that no player is a deviator. Hence \(\pi\) is SIS-stable (SNS-stable).

**Stable \(\Rightarrow\) Satisfiable.**

Suppose \(\pi\) is a core-stable partition of the game. We show that \(\varphi\) is then satisfiable.

**Lemma 3.** The 9 players of a clause cannot all be together in the same coalition in \(\pi\).

**Proof.** If they were, then \(\{c_2, c_3\}\) would block by weak \(2\{2\}\)-toxicity.

**Lemma 4.** For any given clause, at least 1 of its players must be together with their literal player in \(\pi\).

**Proof.** Suppose not. Let \(T = \{c_1, \ldots, c_9\}\). Suppose 3 or more agents from \(T\) are together in \(S \in \pi\). Take an agent \(c_i \in S\) such that \(c_i - 1 \notin S\) (this exists since \(T \notin \pi\)). By weak \(\{1,1\}\)-toxicity, \(c_i\) prefers any edge to \(S\). Under \(\pi\), \(c_i - 1\) can have at most one friend (because \(c_i\) is committed to \(S \not\equiv c_i - 1\) and we assumed that \(c_i - 1\) is not together with its literal player). Thus under \(\pi\), \(c_i - 1\) is not better off than \(\{c_i - 2, c_i - 1\}\) (again by weak \(\{1,1\}\)-toxicity). It follows that \(c_{i - 1}, c_i\) is blocking \(\pi\), a contradiction. So at most two agents from \(T\) are in the same coalition in \(\pi\). Since \(|T|\) is odd, there is a \(c_i\) not together with any other agent from \(T\) in \(\pi\). By our assumption that \(c_i\) has no literal friends, this \(c_i\) has no friends in \(\pi(c_i)\), so is not better off than being alone by \(0\{1\}\)-toxicity. On the other hand, \(c_i - 1\) is not better off than \(\{c_i - 2, c_i - 1\}\). It follows that \(\{c_i - 2, c_i - 1\}\) is blocking \(\pi\).

**Lemma 5.** No \(c_i\) can have 3 friends in \(\pi\). A \(c_i\) together with its literal has at most 1 enemy in \(\pi\).

**Proof.** Suppose \(c_i\) does have 3 friends in \(\pi\).

In particular, \(c_i\) is together with \(c_i - 1\). If \(c_i\) is also together with \(c_i - 2\), then each of \(c_i - 1\) and \(c_i - 2\) have at most 2 friends in \(\pi\) (their degree is 2), but at least 2 enemies (the 2 other friends of \(c_i\)). Hence \(\{c_i - 1, c_i - 2\}\) blocks by weak \(\{1, 1, 2\}\)-toxicity. Hence \(c_i - 2\) is in a different coalition from \(c_i - 1\); thus \(c_i - 2\) has at most 1 friend in \(\pi\). On the other hand \(c_i - 1\) has 1 friend but at least 2 enemies in \(\pi\). Hence by weak \(\{1, 1\}\)-toxicity and consistency on pairs, \(\{c_i - 1, c_i - 2\}\) blocks. So \(c_i\) cannot have 3 friends in \(\pi\).

Now to the second claim. We know \(c_i\) has at most 2 friends in \(\pi\). Suppose \(c_i \in S \in \pi\), where \(c_i\) has 2 enemies in \(S\), and its literal player is part of \(S\). By weak \(\{1, 1, 2\}\)-toxicity, \(c_i\) prefers any edge to \(S\). If \(c_i - 1 \not\equiv S\), then \(\{c_i, c_i - 1\}\) blocks. If \(c_i - 1 \in S\) but \(c_i - 2 \not\equiv S\), then \(\{c_i, c_i - 2\}\) blocks. If both \(c_i - 1 \in S\) and \(c_i - 2 \in S\) then \(|S| \geq 5\) (\(S\) includes \(c_i\), the literal, \(c_i - 1\), and two enemies of \(c_i\)), so that \(\{c_i - 1, c_i - 2\}\) blocks since they have at most 2 friends and at least 2 enemies.

**Lemma 6.** \(x_1\) and \(x_1\) cannot both be together with their clause player.

**Proof.** Suppose they are. Now if \(x_a\) was together with both \(x_1\) and \(x_1\) in \(\pi\), this would mean that the associated clause players have more than 1 enemy which is impossible. So \(x_a\) has at most 2 friends. But if \(x_a\) had friends \(x_a\) and \(x_1\) then the clause player of \(x_1\) would have enemies \(x_a\) and \(x_a\), which is impossible. So \(x_a\) has either no or 1 friend in \(\pi\).

Suppose \(x_a\) has as friend either \(x_1\) or \(x_1\) in \(\pi\). Then by weak \(\{1, 1\}\)-toxicity (since the clause player is an enemy), \(\{x_a, x_a\}\) blocks.

Hence either \(x_a\) has no friends in \(\pi\), or its only friend is \(x_a\).

Now if either \(x_1\) or \(x_1\) has only 1 friend in \(\pi\), then \(\{x_a, x_1\}\) or \(\{x_a, x_1\}\) blocks. So both \(x_1\) and \(x_1\) have 2 friends in \(\pi\), so \(x_1\) is together with \(x_1\) and \(x_1\) is together with \(x_1\). Since the clause players of \(x_1\) and \(x_1\) can have at most 1 enemy in \(\pi\), it follows that \(x_1\) and \(x_1\) have only 1 friend in \(\pi\) (namely \(x_1\) and \(x_1\) respectively). It follows that \(x_a\) has at most 1 friend, namely possibly \(x_a\). Hence by weak \(\{1, 1\}\)-toxicity, \(\{x_a, x_a\}\) blocks.

**Lemma 7.** \(x_1\) and \(x_1\) cannot both be together with their clause player.

**Proof.** Suppose they were. Since the clause players can have at most 1 enemy, \(x_1\) and \(x_1\) cannot be in the same coalition in \(\pi\). Now if \(x_1\) and \(x_1\) both have only 1 friend (their clause player) in \(\pi\), then \(\{x_1, x_1\}\) blocks by consistency on pairs. Otherwise, at least 1 of them, say \(x_1\), has 2 friends in \(\pi\), hence is together with \(x_a\). Since \(x_1\)’s clause player can have at most 1 enemy, \(x_a\) is not in the same coalition. But then \(\{x_a, x_a\}\) blocks by weak \(\{1, 1\}\)-toxicity.

Define a propositional assignment \(A\) that sets literals that are in a coalition with their clause player true. By the last two lemmas, this is well-defined. By the lemma before, each clause has at least 1 literal that is set true by \(A\). Hence \(A\) satisfies \(\varphi\).

**B Proof of Theorem 2**

**B.1 Bipartite Case**

We call a collection of orderings \((\geq i)\) \(i \in N\) bipartite if the friendship graph is bipartite, i.e., there is a partition \((N_1, N_2)\) of the agent set \(N\) such that for each \(i \in N_1\) we have \(F_i \subseteq N_2\) and for each \(i \in N_2\) we have \(F_i \subseteq N_1\). The following theorem gives a hardness result even for bipartite preferences, and so it applies for example to the stable marriage case. This result works for Nash stability but not for individual stability (and it cannot, for individual stability is poly-time solvable for stable marriages). The next section will consider the non-bipartite case which also applies to individual stability.

**Theorem.** NS-Existence for \(C\) is NP-complete if for all \(N\) and every strict bipartite collection of orderings \((\geq i)\) \(i \in N\) there is a game \(\langle N, (\varpi_i)_{i \in N}\rangle \in C\) that is consistent on pairs and strictly \(0\{1, 1, 2\}\)-toxic with respect to \((\geq i)\) \(i \in N\).

The problem is in NP since a Nash-stable partition is a certificate: we can in polynomial time check for each player \(i\)
whether he wishes to deviate (this follows from the definition of a polynomially representable class).

We reduce from (3,B2)-SAT (each clause contains exactly 3 literals, each variable occurs exactly twice positively and twice negatively).

Given a formula \( \varphi \) we denote

(a) for any variable \( x \) its four occurrences by \( x_1, x_2, \overline{x}_1, \overline{x}_2 \),

(b) for any variable occurrence \( \ell \), \( c(\ell) \) is the clause that \( \ell \) occurs in.

We will introduce 9 players for each variable and 1 player for each clause.

\[ N = \{ x_{\text{stalker}}, v_{\text{main}}, x_{\text{pos}}, x_{\text{neg}}, x_{\text{garbage}} \mid x \in \text{Var}(\varphi) \} \]
\[ \cup \{ x_1, x_2, \overline{x}_1, \overline{x}_2 \mid x \in \text{Var}(\varphi) \} \]
\[ \cup \{ c \mid c \in \text{Clauses}(\varphi) \} \]

We take the following strict orderings, which only show the friends of each player. The last entry of player \( i \)'s list is strictly better than \( i \) who is strictly better than all players not mentioned.

\[ x_{\text{stalker}} : \quad x_{\text{main}} \]
\[ x_1 : \quad c(x_1) > x_{\text{pos}} > x_{\text{main}} \]
\[ x_2 : \quad c(x_2) > x_{\text{pos}} > x_{\text{garbage}} \]
\[ \overline{x}_1 : \quad c(\overline{x}_1) > x_{\text{neg}} > x_{\text{main}} \]
\[ \overline{x}_2 : \quad c(\overline{x}_2) > x_{\text{neg}} > x_{\text{garbage}} \]
\[ c : \quad \text{its three variable occurrences in any order} \]
\[ x_{\text{main}} : \quad x_1 > \overline{x}_1 \]
\[ x_{\text{pos}} : \quad x_1 > x_2 \]
\[ x_{\text{neg}} : \quad \overline{x}_1 > \overline{x}_2 \]
\[ x_{\text{garbage}} : \quad x_2 > \overline{x}_2. \]

Notice that no one has more than 3 friends, and that these orderings are bipartite: all friends of a red player are blue, all friends of a blue player are red.

Let \( G \) be any hedonic game that is consistent on pairs and strictly \( 0\{1,1,1,2,5\} \)-toxic with respect to these orderings. This game has a Nash stable outcome if and only if \( \varphi \) is satisfiable.

We call a variable occurrence \( \ell \) matched if it is true\((c)\) for its clause \( c \), and unmatched otherwise.

We now describe a partition \( \pi \) of \( N \) that is Nash stable in \( G \).

- \( \{ x_{\text{stalker}} \} \) is always alone.
- \( \{ c, \text{true}(c) \} \) forms a love marriage.
- Suppose now that \( x \) is true in \( A \).
  - \( \{ x_{\text{main}}, \overline{x}_1 \} \).
  - \( \{ x_{\text{neg}}, \overline{x}_2 \} \).
  - If \( x_1 \) is matched and \( x_2 \) is matched, then \( \{ x_{\text{pos}} \} \) and \( \{ x_{\text{garbage}} \} \).
  - If \( x_1 \) is matched and \( x_2 \) is unmatched, then \( \{ x_2, x_{\text{pos}} \} \) and \( \{ x_{\text{garbage}} \} \).
  - If \( x_1 \) is unmatched and \( x_2 \) is matched, then \( \{ x_1, x_{\text{pos}} \} \) and \( \{ x_{\text{garbage}} \} \).
  - If \( x_1 \) is unmatched and \( x_2 \) is unmatched, then \( \{ x_1, x_{\text{pos}} \} \) and \( \{ x_2, x_{\text{garbage}} \} \).
- Suppose now that \( x \) is false in \( A \). (everything is symmetric to the positive case)
  - \( \{ x_{\text{main}}, x_1 \} \).
  - \( \{ x_{\text{neg}}, x_2 \} \).
  - If \( \overline{x}_1 \) is matched and \( \overline{x}_2 \) is matched, then \( \{ x_{\text{neg}} \} \) and \( \{ x_{\text{garbage}} \} \).
  - If \( \overline{x}_1 \) is matched and \( \overline{x}_2 \) is unmatched, then \( \{ \overline{x}_2, x_{\text{neg}} \} \) and \( \{ x_{\text{garbage}} \} \).
  - If \( \overline{x}_1 \) is unmatched and \( \overline{x}_2 \) is matched, then \( \{ \overline{x}_1, x_{\text{neg}} \} \) and \( \{ x_{\text{garbage}} \} \).
  - If \( \overline{x}_1 \) is unmatched and \( \overline{x}_2 \) is unmatched, then \( \{ \overline{x}_1, x_{\text{neg}} \} \) and \( \{ \overline{x}_2, x_{\text{garbage}} \} \).

The \( \pi \) as above is Nash stable (note all coalitions have size at most 2, so it is a marriage matching). It is easily seen to be individually rational since no one is together with an enemy and consistency on pairs holds. Notice that because the underlying preferences are bipartite they are triangle-free which means that no player ever wants to join a pair of players because at least one of them is an enemy and strict \{0-1,1-1\}-toxicity holds. So any possible Nash deviation would involve a player joining a singleton coalition. The only players that are possibly single in \( \pi \) are \( x_{\text{pos}}, x_{\text{neg}}, x_{\text{garbage}}, x_{\text{stalker}} \).

- No one is friends with \( x_{\text{stalker}} \) so no one can benefit by joining him.
- Player \( x_{\text{pos}} \) is single only if both \( x_1 \) and \( x_2 \) are matched, and because they prefer their clause to \( x_{\text{pos}} \), they do not benefit by joining \( x_{\text{pos}} \). No one else is friends with \( x_{\text{pos}} \), so they do not join either.
- Similarly for \( x_{\text{neg}} \).
- Player \( x_{\text{garbage}} \) is single only in situations where his friends \( x_2 \) and \( \overline{x}_2 \) are together with either their clauses or the \( x_{\text{pos}}, x_{\text{neg}} \) players. Both \( x_2 \) and \( \overline{x}_2 \) prefer this situation to joining \( x_{\text{garbage}} \).

Hence no deviations are possible and thus \( \pi \) is stable.
NS-stable ⇒ satisfiable

Suppose π is NS-stable in G.

Lemma 8. All coalitions in π have size 1 or 2.

Proof. Remember that no player has more than 3 friends. Let S ∈ π with |S| ≥ 3. Then by toxicity and individual rationality of π, each player has exactly 3 friends in S, or 2 friends in S but at most 4 enemies in S. So each member of S has ‘degree’ at least 2, and thus S contains a cycle 1. All cycles in G have length 8 or more, thus |S| ≥ 8. If some member i of S had exactly 2 friends in S, then i would have 5 enemies in contradiction to individual rationality by strict (2-5)-toxicity. Hence every i ∈ S has exactly 3 friends in S. The only players who have 3 friends are variable occurrences x1, x2, x3, x4 and clauses c. If some variable occurrence were in S then so would be its friends of types xmain, xpos, xneg, xgarbage who themselves cannot have 3 friends. Hence S can only contain clause players; but no two clause players are friends.

Thus every player is either alone or together with exactly 1 friend. Consider xmain. If he is alone, then xstalker will want to join him. Since π is Nash stable, this cannot happen. So xmain is together with a friend, which is either x1 or x2.

Define the following propositional assignment A:

A(x) = true ⇐⇒ {xmain, x1} ∈ π
A(x) = false ⇐⇒ {xmain, x1} ∉ π

By what we said above, this is well-defined. We will show that A satisfies φ.

Let c be a clause. If c is alone in π then one of its literals joins c (and is welcome to do so). Hence c is not alone and thus together with one of its literals, say ℓ. We show that ℓ is true under A.

Suppose not and it is false. Of the two false literal occurrences of a variable, the first is in a pair with xmain by definition of A. Thus the second occurrence (ℓ) is the one together with the clause; for concreteness suppose the situation is {xmain, x1} ∈ π and {x2, c} ∈ π. It then follows that {xpos} ∈ π because both friends of xpos are otherwise engaged. But then x1 wants to join xpos (and is welcome to do so). This is a contradiction to π being stable. Hence ℓ is true. Therefore each clause contains a true literal under A and hence A satisfies φ. Thus φ is satisfiable.

B.2 Non-bipartite case

Theorem. NS- and IS-existence for C are NP-complete if for all N and every strict collection of orderings (≥1)i∈N there is a game (N, (πi)i∈N) ∈ C that is consistent on pairs, {0-1}-toxic, weakly {1-2, 2-3}-toxic {1-1}-toxic and weakly [2-1]-toxic, intolerant in triangles, triangle-appreciating, and monotone on triangles with respect to (≥1)i∈N.

1We can use graph theory terminology by referring to a graph on N with an edge between mutual friends. Note that since xstalker has only 1 friend, we must have xstalker ∈ S, so we may pretend that all friendships are mutual so that all edges are indeed undirected.

2Bipartiteness excludes cycles of odd length, so we need only check that there are no cycles of length 4 or 6. There are none. A shortest cycle is xmain → x1 → x3pos → x2 → x3garbage → x2 → xneg → x1 → xmain or cycles like c → x1 → x3pos → x2 → c → x2 → y2 → ypos → y1 → c.

If we do not insist on mutual preferences that also holds for IS. This is done by adding by a 9-gon. The argument that an NS-stable partition exists will be very similar to before, using triangle-freeness. Since the girth of the game continues to be at least 8, and since we have been careful that most deviations considered are actually IS-deviations, the remainder of the proof needs few adjustments.

In more detail:

\[ N = \{ x_1, \ldots, x_9, x_{\text{stalker}}, x_{\text{main}}, x_{\text{pos}}, x_{\text{neg}}, x_{\text{garbage}} \mid x \in \text{Var}(\varphi) \} \]

\[ \cup \{ x_1, x_2, x_3, x_2 \mid x \in \text{Var}(\varphi) \} \]

\[ \cup \{ c \mid c \in \text{Clauses}(\varphi) \} \]

We take the following strict orderings:

\[ x_{\text{main}} : x_{\text{stalker}} > x_1 > x_3 \]

\[ x_{\text{stalker}} : x_{\text{main}} > x_1 \]

\[ x_1 : x_{\text{stalker}} > x_1 > x_4 \]

\[ x_1 : x_{\text{stalker}} > x_1 > x_4 \]

\[ x' : x'^{i+1} > x'^{i-1} \text{ with superscripts mod 9 (i = 2, \ldots, 9)} \]

and everyone else as before. Given an assignment, the partition π generated is the same as before, but instead of \{x_{\text{stalker}}, x'\} ∈ π, we now take \{x_{\text{stalker}}, x'\}, \{x^2, x^3\}, \{x^2, x^3\}, \{x^4, x^5\}, \{x^6, x^9\} ∈ π. Checking that this π is NS proceeds as before.

Suppose the game has an IS partition π. Similar to before, all coalitions in π have size 1 or 2. We can then see that we must have \{x_{\text{stalker}}, x'\} ∈ π. Otherwise, there must be some single \{x'\} ∈ π and then \{x'^{i+1}\} will want to join \{x'\} and is welcome to do so. But now x_{\text{main}} cannot be alone, else x_{\text{stalker}} will want to join him and is welcome to do so. Hence x_{\text{main}} is together with a friend. The rest of the argument can now proceed as before.

C Proof of Theorem 3

Theorem. (SN-), (SSN-), SIS-, CR- and SCR-existence for C are NP-hard if for all N and every collection of strict orderings (≥1)i∈N there is a game (N, (πi)i∈N) ∈ C that is consistent on pairs, {0-1}-toxic, weakly {1-2, 2-3}-toxic {1-1}-toxic and weakly {2-1}-toxic, intolerant in triangles, triangle-appreciating, and monotone on triangles with respect to (≥1)i∈N.

We will prove all three statements in the theorem together. Proof by reduction from (3,B2)-SAT.

Given an instance φ of (3,B2)-SAT, take the following set N of agents, with 9 agents per clause and 8 agents per variable.

\[ N = \{ c_1, \ldots, c_9 \mid c \in \text{Clauses}(\varphi) \} \]

\[ \cup \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \mid x \in \text{Var}(\varphi) \} \]

For a clause c = ℓ_1 ∨ ℓ_2 ∨ ℓ_3, where ℓ_i is a variable occurrence, we will connect c with ℓ_1, ℓ_2, and ℓ_3 with c with ℓ_3. For c_1, c_4, c_7, we call the variable occurrence connected with them ‘its literal’. For a variable occurrence ℓ, we will call the ℓ player connected with it ‘its clause player’ and denote it by i(ℓ). We generate the following orderings, which only show the friends of each player. The last entry of player i(ℓ)’s list is
strictly better than \( i \) who is strictly better than all players not mentioned.

\[
\begin{align*}
c_1 : \ell_1 > c_2 > c_9 \\
c_4 : \ell_2 > c_5 > c_3 \\
c_7 : \ell_3 > c_8 > c_6 \\
c_i : c_{i+1} > c_{i-1} & \quad \text{for } i \neq 1, 4, 7 \text{ with subscripts mod } 9 \\
x_1 : x_a > x_b > x_2 > c(x_1) \\
x_1 : x_b > x_2 > c(x_1) \\
x_2 : x_c > x_d > x_1 > c(x_2) \\
x_2 : x_d > x_c > x_1 > c(x_2) \\
x_a : x_b > x_1 > x \in (x_1 \cup x_2) \\
x_c : x_d > x_2 > x \in (x_1 \cup x_2) \\
x_d : x_c > x_2 > x \in (x_1 \cup x_2)
\end{align*}
\]

These orderings only show the friends of each player. The last entry of player \( i \)'s list is strictly better than \( i \) who is strictly better than all players not mentioned. Notice the following facts:

- No player has more than 4 friends.
- The orderings are mutual: \( j \) is \( i \)'s friend if and only if \( i \) is \( j \)'s friend.
- The friendship graph has no chordless 4-cycles.

By the assumptions of the theorem, we can in polynomial time find a hedonic game which satisfies the relevant conditions as in the theorem statement. We show that \( \varphi \) is satisfiable if and only if this game admits a core- and SIS-stable (SNS-stable) \( \langle \text{SNS-stable} \rangle \) partition.

**Satisfiable \( \Rightarrow \) Stable.**

Suppose \( \mathcal{A} \) is a satisfying assignment for \( \varphi \). Construct the following partition \( \pi \) of \( N \), which consists of singletons, pairs, and triangles:

- Match true variable occurrences with their clause player.
- If \( x_1 \) is false then \( \{x_1, x_a, x_b\} \in \pi \). If \( \pi_1 \) is false then \( \{x_1, x_a, x_b\} \in \pi \).
- If \( x_2 \) is false then \( \{x_2, x_c, x_d\} \in \pi \). If \( \pi_2 \) is false then \( \{x_2, x_c, x_d\} \in \pi \).
- For each clause, match the non-matched \( c \)-players in the same way as before.

Suppose that \( \pi \xrightarrow{H} \pi' \) for some \( H \subseteq N \) with \( \pi' \supset \pi \langle \langle \pi' \supset j \pi \rangle \rangle \) for all \( i \in H \). We will prove that this is not an SIS-deviation \( \langle \text{not an SNS-deviation} \rangle \langle \text{not an SSNS-deviation} \rangle \). Note that if there are no SIS-deviations, then \( \pi \) is also core-stable.

Some terminology and observations:

- an agent is lonely if it is in a singleton in \( \pi \) and matched otherwise (these terms always refer to \( \pi \) and never to \( \pi' \)).
- We also use ‘matched’ for agents in triangles.
- an agent is a deviator if it is in \( H \), and a non-deviator otherwise.
- no 2 enemies are matched in \( \pi \).
- any 2 lonely players have distance 5, which in the following will mean that at most 1 player in a coalition considered below can be lonely.

**Lemma 9.** No matched deviator \( i \) has exactly 2 friends \( j \) and \( k \) in \( \pi' \) where \( j \) and \( k \) are enemies.

**Proof.** Essentially identical to Lemma 11. Replace “because there are no 4-cycles” by “because there are no chordless 4-cycles”, and (modify remarks about SNS to also include the SSNS case).

**Lemma 10.** No matched deviator \( i \) has exactly 2 friends \( j \) and \( k \) in \( \pi' \) where \( j \) and \( k \) are friends.

**Proof.** Suppose not. By weak \( \langle 2-2 \rangle \)-toxicity, \( i \)'s coalition \( S \) in \( \pi' \) has size either 3 or 4. (By weak \( \langle 2-1 \rangle \)-toxicity, \( i \)'s coalition \( S \) in \( \pi' \) has size 3.)

If \( |S| = 3 \), then \( S \) must be a triangle of the form \( \{x_1, x_a, x_b\} \) where \( x_1 \) is true. Now one of \( x_a \) and \( x_b \) is made worse off in this triangle compared to the triangle \( \{x_1, x_a, x_b\} \) which is part of \( \pi \) (by monotonicity on triangles), in this case \( x_b \). Thus \( x_b \) does not deviate. Since \( \pi(x_1) = \pi(x_b) \) but \( \pi'(x_1) \neq \pi'(x_b) \), \( x_1 \) must deviate. But \( x_1 \) is now strictly worse off by triangle-appreciation and monotonicity on triangles, a contradiction.
Suppose \(|S| = 4\), with \(S = \{i, j, k, ℓ\}\) with \(i, j, k, ℓ\) forming a triangle, and \(i\) and \(ℓ\) being enemies. For this case we only have to worry about SIS-deviations. Suppose first that \(ℓ\) is friends with both \(j\) and \(k\). Now in \(π\), either \(\{i, j, k\}\) formed a coalition, in which case \(S\) is worse for \(i\) by intolerance in triangles, or \(\{j, k, ℓ\}\) formed a coalition. In the latter case, \(ℓ\) is made worse off by \(i\) joining (using intolerance in triangles), so that the deviation is not SIS.

Hence \(ℓ\) is not friends with both \(j\) and \(k\). Let’s condition on whether \(\{i, j, k\} \in π\) or not. If \(\{i, j, k\} \in π\), then \(i\) is worse off in \(S\) by intolerance in triangles. So suppose \(\{i, j, k\} \notin π\). If \(ℓ\) has no friends in \(S\), then 1 of \(i, j, k\) is member of a triangle in \(π\) that is better than \(\{i, j, k\}\), and thus is worse off in \(S\) by intolerance in triangles, so no SIS. Similarly if \(ℓ\) has exactly 1 friend in \(S\), say \(j\) (it cannot be \(i\) who has exactly 2 friends), then \(j\) must be a literal player, and one of \(i\) and \(k\) was better off in \(π\) (where they were in a better triangle than \(\{i, j, k, ℓ\}\), and without enemies), so this is not an SIS-deviation.

**Lemma 11.** No deviator has 3 or 4 friends in \(π\).

**Proof.** Suppose \(i \in S \in π\) is a deviator where \(S\) includes 3 or 4 friends. We condition on the type of \(i\).

1. **i is a clause player:** Say \(i\) has name \(c_i\). \(c_i\)’s 3 friends are pairwise enemies, so at least 2 of them deviate, including another clause player \(j \in \{c_{i+1}, c_{i−1}\}\). If \(j\) is matched then \(j\) is worse off since there are 2 enemies in \(S\) and \(j\) is of degree 2, contradiction. So \(j\) is single, and hence \(j = c_{i−1}\). Then the other clause friend \(c_{i+1}\) of \(c_i\) is matched and now worse off so not an SIS deviation. With \(\{x\\} 1-1\)-toxicity, \(c_{i−1}\) is not made better off unless \(c_{i−1}\) has 2 friends (\(c_i\) and \(c_{i−2}\)) in \(S\). Since \(c_{i+1}\) does not deviate, \(c_{i−2}\) must deviate. But \(c_{i−2}\) is matched and has enemies \(c_i\) and \(c_{i+1}\), so must have 3 friends which is impossible since \(c_{i−2}\) has degree 2, contradiction.

2. **i is a player of type \(x_a, x_b, x_c, \) or \(x_d\):** Say \(x_a\), and \(π(x_a) = \{x_a, x_b, x_1\}\). By assumption, \(x_a\) has 3 friends in \(S\), so both \(x_1\) and \(x_1\) are in \(S\).

   - By intolerance in triangles, this cannot be an SIS deviation unless \(x_1\) has at least 3 friends in \(S\), so there is an extra friend \(j \in S\):

   ![Diagram](attachment:triangle_diagram.png)

   - (By weak \{2-1\}-toxicity, this cannot be an SNS or SSNS deviation unless 1 of \(x_1\) or \(x_1\) has at least 3 friends in \(S\) (since one of them deviates). Say \(x_1\) has extra friend \(j \in S\), also giving us the picture above.)

   Now both \(x_1\) and \(j\) have at least 2 enemies in \(S\), and one of them must deviate, hence have at least 3 friends in \(S\). If this is \(j\), then \(|S| \geq 7\) and \(x_a\) is unhappier by weak \{3-3\}-toxicity, contradiction. So \(j\) doesn’t deviate and \(x_1\) has an extra friend in \(S\), say \(k\). Since \(j\) doesn’t deviate, \(k\) does deviate. Since \(x_1\) has no lonely friends, \(k\) is matched. Hence \(k\) has 3 friends in \(S\). Then \(|S| \geq 7\) and \(x_a\) is unhappier by weak \{3-3\}-toxicity, contradiction.

   i is a literal player, say \(x_1\): Suppose first that \(S\) includes \(x_a\) and \(c(x_1)\), so that there are 3 friends of \(x_1\) in \(S\) that are pairwise enemy, and hence 2 friends who must be deviating. Of those 2, at most 1 is of the \(x_a\) or \(x_3\) kind; such a player must have 3 friends in \(S\) by intolerance in triangles. All other types must have 3 friends in \(S\) by toxicity. The 2 deviating friends of \(x_1\) thus together contribute at least 3 extra friends to \(S\), and hence \(|S| \geq 7\). Then at least 1 of the deviating friends of \(x_1\) has 3 friends in \(S\) but also 3 enemies, and is thus worse off in \(S\), a contradiction.

   Suppose otherwise that \(S\) includes exactly 3 friends of \(x_1\), including \(x_a\) and \(x_b\), and also \(j \in \{x_2, x(c_1)\}\). Suppose \(x_a\) and \(x_b\) do not deviate. Then everyone else (except possibly \(x_1\)) in \(S\) must deviate; in particular \(j\) must deviate and hence have 3 friends in \(S\) (which means 2 extra agents in \(S\)), who each must also deviate, which means 1 extra agent for each of the 2 extra agents in \(S\). Hence \(|S| \geq 7\) and \(x_1\) is worse off in \(S\) by weak \{3-3\}-toxicity, a contradiction. Otherwise, at least 1 of \(x_a\) and \(x_b\) is deviating. Now unless \(x_1\) is in \(S\), by intolerance in triangles one of \(x_a\) or \(x_b\) is worse off, preventing this from being an SIS deviation. Otherwise by weak \{2-1\}-toxicity, the deviator from \(x_a\) and \(x_b\) needs 3 friends. Either way we conclude \(x_1\) is in \(S\). Now at least 1 of \(x_1\) and \(j\) must be deviating, and thus must have 3 friends in \(S\). If \(j\) is deviating then \(|S| \geq 7\). If \(x_1\) is deviating, then either its extra friend or \(j\) is deviating, bringing 3-3, so that \(x_1\) is worse off by weak \{3-3\}-toxicity.

Now suppose \(i\) is a matched deviator. Then by the lemmas, \(i\) must end up in a coalition \(S \in π\) which includes exactly 1 friend of \(i\). By weak \{1-1\}-toxicity, \(S\) must be a pair, \(S = \{i, k\}\), say.

We now go through each type \(i\) of matched player and show that \(i\) does not deviate into a pair, and is thus not a deviator.

- By triangle-appreciation, \(x_a, x_b, x_c, \) and \(x_d\), and false variable occurrences are strictly better off than in any pair, so none of them are deviators.
- True variable occurrences: all preferred players are in triangles consisting of non-deviators.
- \(c_i\) players don’t deviate for the same reason as before.

Hence (together with the lemmas above) no matched player is a deviator.

Now consider a lonely player \(i\). If \(i\) deviates, then \(i\) ends up in a coalition \(S \in π\) consisting of lonely players and possibly 2 players that are in a pair in \(π\) or 3 players in a triangle in \(π\) (because no matched players are deviators). If \(S\) consists entirely of lonely players, then by \{strict\} \{0-1\}-toxicity, \(i\) is \{worse off\} not better off, so won’t deviate (SCR: no-one is strictly better off). Hence \(S\) also contains a pair or triangle. At least 1 of these 2 or 3 players is enemies with \(i\) and is thus worse off by weak \{1-1\}-toxicity or intolerance in triangles (since all lonely players are enemies to triangles), so this is not an SIS-deviation. (With \{strict\} \{1-1\}-toxicity, \(i\) is not better off in \(S\), so not an SNS-deviation \{SSNS-deviation\}.)

Thus we conclude that no player is a deviator. Hence \(π\) is SIS-stable \{SNS-stable\} \{SSNS-stable\}.
Stable $\Rightarrow$ Satisfiable.
Suppose $\pi$ is a core-stable partition of the game. We show that $\varphi$ is then satisfiable. The first 3 lemmas are proved exactly as before.

**Lemma 12.** The 9 players of a clause cannot all be together in the same coalition in $\pi$.

**Lemma 13.** For any given clause, at least 1 of its players must be together with their literal player in $\pi$.

**Lemma 14.** No $c_i$ can have 3 friends in $\pi$. A $c_i$ together with its literal has at most 1 enemy in $\pi$.

**Lemma 15.** $x_1$ and $\pi_1$ cannot both be together with their clause player.

**Proof.** Suppose they are. Note that if either $x_1$ or $\pi_1$ was together with either of $x_a$ or $x_b$ (cannot be together with both) then $\{x_a,x_b\}$ blocks. But then by triangle-appreciating and monotonicity $\{x_1,x_a,x_b\}$ blocks. $\Box$

**Lemma 16.** $x_1$ and $\pi_2$ cannot both be together with their clause player.

**Proof.** Suppose they are. Since the clause players can have at most 1 enemy, $x_1$ and $\pi_2$ cannot be in the same coalition in $\pi$. Also $x_1$ cannot be together with either $x_3$ or $x_4$ since then $\{x_3,x_4\}$ blocks, and similarly $\pi_2$ cannot be together with either $x_3$ or $x_4$ since then $\{x_3,x_4\}$ blocks. Hence $\{x_1,\pi_2\}$ blocks by consistency on pairs. $\Box$

Define a propositional assignment $A$ that sets literals that are in a coalition with their clause player true. By the last two lemmas, this is well-defined. By the lemma before, each clause has at least 1 literal that is set true by $A$. Hence $A$ satisfies $\varphi$.

**D Class Properties**
In this section, we check in more detail that the conditions of our theorems are satisfied for various classes. All of these are routine.

Throughout we may assume $N \neq \emptyset$ and so $n \geq 1$, because all conditions are satisfied vacuously by the empty hedonic game. It is useful to note that strict $\{3,1\}$-toxicity implies intolerance in triangles, so that we do not need to check this condition in all cases.

- **IRCL**
  Given $N$ and $(\geq i)_{i \in N}$ with friend sets $(F_i)_{i \in N}$, run the following algorithm producing IRCL lists.
  1. List $\{(j,k) \in F_i \times F_j : j \geq i, k \text{ or } j \sim_i k \text{ and } j \text{ comes earlier than } k \text{ in listing of } N\}$.
  2. Sort this list according to $(j,k) \gg (j',k')$ iff $j \geq i$ and $k \gg_i k'$. Break ties arbitrarily.
  3. Output this list with entries written as triangles $\{i,j,k\}$, any two entries separated by $\gg_i$.
  4. Output $F_i \cup \{i\}$, written as pairs $\{i,j\}$, with $\geq_i$ replaced by $\gg_i$ and $\sim_i$ replaced by $\sim_i$.
  5. End of output.

Clearly this algorithm terminates in polynomial time. The game described by the output is triangle-appreciating in all senses, because all friend-triangles come before all other coalitions. By step 4 the game is consistent on pairs. Because no coalition including an enemy is listed, they are not individually rational, so strict $\{k,1\}$-toxicity is satisfied.

- **Stable Roommates**
  This is the game produced by the IRCL-algorithm when we start it in step 4. So it is consistent on pairs, and strict $\{k,1\}$-toxicity is satisfied.

Neither of the triangle conditions is satisfied.

- **W-Games**
  **Consistency on pairs.** For agents $j,k \in F_i \cup \{i\}$, by definition $W_i(\{i,j\}) = j$ for both cases $j \neq i$ and $j = i$, and so $\{i,j\} \gg^i \{i,k\}$ iff $W_i(\{i,j\}) \geq W_i(\{i,k\})$ iff $j \geq^i k$.

  **Strict $\{k,1\}$-toxicity.** If $S$ contains an enemy $e \in E_i$, then $W_i(S) \leq i \leq i = W_i(\{i\})$, so $S \sim_i \{i\}$.
  We do not have triangle-appreciating. We do have monotonicity on triangles, but this is irrelevant.

- **WB-Games**
  The analysis is very similar to the case of W-games.

- **Additively Separable Games**
  Use $[-(n^2 + 2n), 4]$-utilities.

  **Consistency on pairs.** $\{i,j\} \gg^i \{i,k\}$ iff $v_i(j) \geq v_i(k)$ iff $j \geq^i k$.

  **Strict $\{k,1\}$-toxicity.** Suppose $S$ contains an enemy $e \in E_i$. Then $v_i(S) = \sum_{j \in S} v_i(j) \leq (n-2)(n+4) - (n^2 + 2n) = -8 < 0$, so $S \sim_i \{i\}$.

  **Triangle-appreciating.** If $j,k \in F_i$ distinct with $j \geq^i k$ then $v_i(\{i,j,k\}) = v_j(i) + v_k(i) > v_j(i) = v_i(\{i,j\})$ since $v_j(i) \geq 4 > 0$ in $[-(n^2 + 2n), 4]$-utilities.

  **Monotone on triangles.** Suppose $j,j',k,k' \in F_i$ are such that $j \gg^i j', k \gg_i k'$. Then $v_i(\{i,j,k\}) = v_j(i) + v_k(i) > v_i(j') + v_i(k') = v_i(\{i,j',k\})$.

- **Hedonic Coalition Nets**
  We essentially encode the additively separable game with $[-(n^2 + 2n), 4]$-utilities from above as a hedonic coalition net. Write $E_i = \{e'_1, \ldots, e'_k\}$. Then use the net $j \mapsto v_i(j)$, for friends $j \in F_i$, $e'_1 \lor \cdots \lor e'_k \mapsto -i$ $-(n^2 + 2n)$

As noted in the paper, $|F_i| \leq 4$ for all $i$, so the net above uses at most 5 formulas per agent. We can verify the properties exactly as we did for the Additively Separable Game above; toxicity goes through since in our check we only used the presence of a single enemy.

All weights we used in the net are of size polynomial in $n$. Since we always have an explicit list of $N$ as input to our algorithms, we have $n$ available in unary, so we are allowed to write the weights in unary.

- **Fractional Hedonic Games**
  Use $[-(n^2 + 5n), 7]$-utilities.
Consistent on Pairs. For \( j, k \in F_i \cup \{i\}, \{i, j\} \succ_i \{i, k\} \) iff \( v_i(j)/2 > v_i(k)/2 \) iff \( j \geq i, k \).

Strictly \( \{k\}-\)toxic. Suppose \( S \) contains an enemy \( e \in E_i \). Then \( v_i(S) = \frac{1}{|S|} \sum_{j \in S} v_i(j) \leq \sum_{j \in S} v_i(j) \leq (n - 2) \times (n + 7) - (n^2 + 5n) = -n + 7 < 0 \).

Triangle appreciating. Let \( j, k \in F_i \) be distinct with \( j \geq i, k \) and satisfying the closeness condition, which implies \( v_i(j) - v_i(k) \leq 2 \). Then \( v_i(\{i, j, k\}) = (v_i(j) + v_i(k))/3 \geq (2v_i(j) - 2)/3 = \frac{2}{3}v_i(j) - \frac{2}{3} > v_i(j)/2 = v_i(\{i, j\}) \) because \( v_i(j) \geq 5 \) by choice of utilities.

Monotone on triangles. Suppose \( j, j', k, k' \in F_i \) are such that \( j \geq i, j' > i, k > i, k' \). Then \( v_i(\{i, j, k\}) = (v_i(j) + v_i(k))/3 > (v_i(j') + v_i(k'))/3 = v_i(\{i, j', k'\}) \).

- Social FHGs
  - Use \([0, 7n]\)-utilities.
    - Note that because \( 7n \geq 7 \), all the `positive properties' hold as they did for straight FHGs. We only need to check the negative properties.
    - Weakly \( \{1, 1\}-\)toxic. A coalition \( S \) in which \( i \) only has enemies obtains value 0.
    - Weakly \( \{1, 1\}-\)toxic. Let \( S \) be \( \{1, 1\} \). Note that then \( |S| \geq 3 \). So \( v_i(S) = \frac{1}{|S|} \sum_{j \in S} v_i(j) = \frac{1}{8}v_i(1) + \frac{1}{8}v_i(1) = \frac{\frac{1}{7}n + \frac{1}{7}n}{2} = \frac{1}{7}n \) is the minimal utility obtained in pairs.
    - Weakly \( \{2, 2\}-\)toxic. Follows from \( \frac{3}{8}n < \frac{1}{7}n \).
    - Weakly \( \{3, 3\}-\)toxic. Follows from \( \frac{3}{8}n < \frac{1}{7}n \).

- Median Games
  - Use \([0, 5]\)-utilities.
    - Weakly \( \{1, 1\}-\)toxic. Let \( S \) be \( \{1, 1\} \). Then all utilities in \( S \) are 0, so it is evaluated at 0, so \( \{i\} \succ_i S \).
    - Weakly \( \{k, k\}-\)toxic. Let \( S \) be \( \{k, k\} \). Then the median value corresponds to a value or average of \( v_i(\text{enemy}) = 0 \) or \( v_i(\text{friend}) = \frac{n}{2} \) so is 0. On the other hand, in a pair with a friend, \( i \) obtains utility at least \( \frac{5}{2} > 0 \) by choice of utilities. Hence \( \{i\} \succ_i S \) for friends \( j \).

\( \ell \)-Approval. Assume \( \ell \geq 4 \), and use \([−6\ell n, 4]\)-utilities.

Consistent on pairs. A pair is valued with the utility of the partner.

Strictly \( \{k\}-\)toxic for \( k < \ell \). Suppose \( S \) is \( \{k\} \) with \( k < \ell \). Then \( v_i(S) = Y_1 + \cdots + Y_k + Y_{k+1} = Y_1 + \cdots + Y_k - 6\ell n \leq k(n + 4) - 6\ell n < \ell(n + 4) - 6\ell n = \ell(−5n + 4) < 0 \) since \( \ell \geq 0 \) and \( n \geq 1 \).

The checks of the remaining conditions are identical to the case of additively separable games, noting that there we never sum more than the first 3 entries.