Development and $L^2$-Analysis of a Single-Step Characteristics Finite Difference Scheme of Second Order in Time for Convection-Diffusion Problems

Hirofumi Notsu$^a$, Hongxing Rui$^b$ and Masahisa Tabata$^{c,*}$

$^a$Waseda Institute for Advanced Study, Waseda University, Tokyo 169-8555, Japan
$^b$School of Mathematics, Shandong University, Jinan 250100, China
$^c$Faculty of Science and Engineering, Waseda University, Tokyo 169-8555, Japan

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ABSTRACT

A new finite difference scheme based on the method of characteristics is presented for convection-diffusion problems. The scheme is of single-step and second order in time, and the matrix of the derived system of linear equations is symmetric. Since it is a finite difference scheme, we can get rid of numerical integration which may cause some instability in the characteristics finite element method. An optimal error estimate is proved in the framework of the discrete $L^2$-theory. Numerical results are shown to recognize the convergence order and advantages of the scheme.

Keywords: The method of characteristics, Finite difference method, Second order in time, Discrete $L^2$-analysis.

1. INTRODUCTION

Convection-diffusion equation describes phenomena including both convection and diffusion effects, and appears in various fields of natural sciences, e.g., heat transfer, weather prediction and atmospheric radioactivity propagation. It may also be treated as a simplified model of the system of the Navier-Stokes equations, which are representative equations in fluid dynamics. Although the convection-diffusion equation is linear, numerical difficulty caused by convection effect is still remained. Nowadays, to deal with convection-dominant problems several upwind type ideas have been developed for flow
problems, e.g., upwind methods [2, 8, 9, 19, 23], characteristics (-based) methods [1, 5–7, 11–16] and so on. We focus on the approximation based on the method of characteristics. The idea of the method is to consider the trajectory of the fluid particle and discretize the material derivative term along the trajectory. The method has such a common advantage that the resulting matrix is symmetric, which is especially useful when we employ implicit schemes for the benefit of a good stability.

The characteristics finite element method of first order in time has been well studied in [5, 11, 12]. As for the scheme of second order in time, a multi-step scheme has been considered in [6] while a single-step scheme has been developed in [15], where they have pointed out that the conventional Crank-Nicolson method is not sufficient and that an additional correction term is indispensable in order to obtain a real second order scheme. In this paper we apply their idea to the finite difference method, and present a new characteristics scheme with a proper additional correction term for convection-diffusion problems in 2D.

In general, the finite difference method has less flexibility in the shape of domains to be applied than the finite element method. The reason why we consider the finite difference method nevertheless is that it requires no numerical integration in the execution. Every characteristics scheme includes composite function terms. When we employ the finite element method, some numerical integration procedure is often required to compute the integration of the composite functions, since they are not polynomials in each element. In the papers [20, 21] they have remarked that much attention should be paid to the numerical integration, because a rough numerical integration formula may yield oscillating results caused by the non-smoothness of the composite function. In order to overcome such a problem a characteristics finite element scheme without numerical integration has been presented in [14], where a mass-lumping technique is used to P1 (piecewise linear) element and \( L^2 \)-theory is applied to establish the convergence. For the application to flow problems and higher order elements, \( L^2 \)-analysis is preferable. The present scheme naturally requires no numerical integration as it is finite difference one, and it is analyzed by the discrete \( L^2 \)-theory.

The scheme has such advantages that this is of second order in time and the resulting matrix is symmetric and positive definite. The extension to 3D problems is straightforward with the expense of a little complicated notation. The stability and convergence theorems are proved in the framework of the discrete \( L^2 \)-theory. The convergence order and low computation cost of the scheme are observed by numerical results.
Let \( m \) be a non-negative integer. We use the Sobolev spaces \( W^{1,\infty}(\Omega) \) and \( H^m(\Omega) \) as well as \( C^m(\Omega) \). For any normed space \( X \) with norm \( \|x\|_X \), we define the function space \( C^m([0, T]; X) \) consisting of \( X \)-valued functions in \( C^m([0, T]) \). We often omit \([0, T]\) if there is no confusion, e.g., we write \( C'(\Omega) \) in place of \( C'(\Omega([0, T]) \). We introduce function spaces \( Z_m \) and \( Z_m^\ast \),

\[
Z_m^\ast = \{ \phi \in H^j(\Omega); j = 0, \ldots, m, \|\phi\|_{Z_m} < +\infty \},
\]

\[
Z_m = \{ \phi \in C^j(\Omega); j = 0, \ldots, m, \|\phi\|_{Z_m^\ast} < +\infty \},
\]

where the norms \( \|\cdot\|_{Z_m} \) and \( \|\cdot\|_{Z_m^\ast} \) are defined by

\[
\|\phi\|_{Z_m} = \max_{j=0,\ldots,m} \|\phi\|_{H^j(\Omega)}, \quad \|\phi\|_{Z_m^\ast} = \max_{j=0,\ldots,m} \|\phi\|_{C^j(\Omega)}. \]

The partial derivative \( \partial \phi / \partial x_i \) of a function \( \phi \) is simply denoted by \( \partial_i \phi \). We often consider a continuous function in \( \overline{\Omega} \) as a function defined on lattice points in \( \overline{\Omega} \). \( \delta_{ij}(i, j = 1, 2) \) is the Kronecker delta, and \( Z^\alpha = \{ Z + \alpha \} \) for \( \alpha \in [0, 1] \). The abbreviations LHS and RHS mean left- and right-hand sides, respectively.

### 2. A CHARACTERISTICS FINITE DIFFERENCE SCHEME OF SECOND ORDER IN TIME

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain, \( \Gamma = \partial \Omega \) be the boundary of \( \Omega \) and \( T \) be a positive constant. We consider an initial boundary value problem; find \( \phi: \Omega \times (0, T) \rightarrow \mathbb{R} \) such that

\[
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - v \Delta \phi = f \quad \text{in} \quad \Omega \times (0, T),
\]

\[
\phi = 0 \quad \text{on} \quad \Gamma \times (0, T),
\]

\[
\phi = \phi^0 \quad \text{in} \quad \Omega, \text{ at } t = 0,
\]

where \( v \) is a positive constant less than a fixed \( v_0 > 0 \), and \( u: \Omega \times (0, T) \rightarrow \mathbb{R}^2 \), \( f: \Omega \times (0, T) \rightarrow \mathbb{R} \) and \( \phi^0: \Omega \rightarrow \mathbb{R} \) are given functions.

To begin with, we summarize conditions to be imposed on the functions \( u \), \( \phi^0 \), \( f \) and \( \phi \). Each condition is referred to simply by, e.g., \([H_{0,1}(u)]\) in place of Hypothesis 1 \([H_{0,1}(u)]\).
Hypothesis 1 \((u)\).

\[ \begin{align*}
[H_{0,1}(u)] & \ u \in C^0(C^1(\overline{\Omega})), \\
[H_{0,2}(u)] & \ u \in C^0(C^2(\overline{\Omega})), \\
[H_{1c}(u)] & \ u \in Z^1_C, \\
[H_{2c}(u)] & \ u \in Z^2_C, \\
[H_{t}(u)] & \ u = 0 \text{ on } \Gamma \times [0, T].
\end{align*} \]

Hypothesis 2 \((\phi^0)\).

\[ \begin{align*}
[H_{0,0}(\phi^0)] & \ \phi^0 \in C^0(\overline{\Omega}) \text{ and } \phi^0 = 0 \text{ on } \Gamma.
\end{align*} \]

Hypothesis 3 \((f)\).

\[ \begin{align*}
[H_{0,0}(f)] & \ f \in C^0(C^0(\overline{\Omega})), \\
[H_{2c}(f)] & \ f \in Z^2_C.
\end{align*} \]

Hypothesis 4 \((\phi)\).

\[ \begin{align*}
[H_{0,1}(\phi)] & \ \phi \in C^0(C^1(\overline{\Omega})), \\
[H_{0,2}(\phi)] & \ \phi \in C^0(C^2(\overline{\Omega})), \\
[H_{1c}(\phi)] & \ \phi \in \mathbb{Z}^1_C, \\
[H_{2c}(\phi)] & \ \phi \in \mathbb{Z}^2_C, \\
[H_{1c}(\Delta \phi)] & \ \Delta \phi \in \mathbb{Z}^3_C.
\end{align*} \]

For the sake of simplicity we consider a rectangle domain \(\Omega = (0, L_1) \times (0, L_2)\) for positive numbers \(L_1\) and \(L_2\). For \(i = 1\) and 2 let \(N_i\) be a positive integer and \(h_i = L_i / N_i\) be the mesh size of \(x_i\)-direction. We assume \(h_1 = h_2\) to simplify the notation. We set lattice points \(x_{i,j} = (ih, jh)^T\) for \(i, j \in \mathbb{Z} \cup \mathbb{Z}^{1/2}\), where the superscript “T” means the transposition.

Let \(\Delta t\) be a time increment and \(N_T = \lceil T/\Delta t \rceil\) be a total step number, where \(\lceil \alpha \rceil\) is the greatest integer that is less than or equal to \(\alpha \in \mathbb{R}\). We set \(t^n = n\Delta t\) for \(n \in \mathbb{Z} \cup \mathbb{Z}^{1/2}\), and \(\phi^n = \phi(\cdot, t^n)\) for any function \(\phi\) defined in \(\Omega \times (0, T)\). Let \(U_0^\infty\) and \(U_1^\infty\) be constants defined by

\[ \begin{align*}
U_0^\infty & = \max \left\{ \left| u(x,t) \right| ; x \in \overline{\Omega}, t \in [0,T] \right\}, \\
U_1^\infty & = \max \left\{ \left| \nabla u_j(x,t) \right| ; x \in \overline{\Omega}, t \in [0,T], j = 1,2 \right\},
\end{align*} \]

where, for a vector \(a \in \mathbb{R}^2\), \(|a|_\infty = \max\{|a_i| ; i = 1, 2\}\) and \(|a|_1 = \Sigma_{i=1}^2 |a_i|\). Before the presentation of the scheme we summarize conditions on \(\Delta t\).

Hypothesis 5 \((\Delta t)\). Let \(C_1\) be any positive constant independent of \(h\) and \(\Delta t\).

\[ \begin{align*}
[H_u(\Delta t)] & \ \Delta t < 1/\|u\|_{C^0(W^{1,\gamma} (\Omega))}, \\
[H_{wCFL}(\Delta t)] & \ \Delta t \leq C_1 h / U_0^\infty.
\end{align*} \]

Remark 1. (i) \([H_u(\Delta t)]\) guarantees that all upwind points to be used in the present scheme are in \(\overline{\Omega}\) (cf. Proposition 1). (ii) \([H_{wCFL}(\Delta t)]\) with \(C_1 = 1\) is identical with the CFL condition (cf. [12]). Since \(C_1 > 1\) can be chosen, we call \([H_{wCFL}(\Delta t)]\) “weak-CFL condition”, whose abbreviation is put in the subscript.

Let \(X : (0,T) \rightarrow \mathbb{R}^2\) be a solution of the ordinary differential equation

\[ \frac{dX}{dt} = u(X,t). \quad (2) \]
Then, for a smooth function \( \phi \) we can write

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \phi(X(t), t) = \frac{d}{dt} \phi(X(t), t),
\]

which is a basic idea of the method of characteristics. Let \( X(t; x, t^n) \) be the solution of (2) subject to an initial condition \( X(t^n) = x \). Approximating the value \( X(t^{n-1}; x, t^n) \) by the Euler method and the second order Runge-Kutta method, we define

\[
X_1^n(x) \equiv x - u^n(x) \Delta t, \quad X_2^n(x) \equiv x - u^{n-1/2} \left( x - u^n(x) \frac{\Delta t}{2} \right) \Delta t.
\]

**Remark 2.** Instead of the second order Runge-Kutta method we can also use the Heun method,

\[
X_2^n(x) \equiv x - \left\{ u^n(x) + u^{n-1} \left( x - u^n(x) \Delta t \right) \right\} \frac{\Delta t}{2}.
\]

The following result has been proved in [15, Proposition 1] for any bounded domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\).

**Proposition 1.** Suppose \([H_{0,1}(u)], [H_{1}(u)] \) and \([H_u(\Delta t)]\). Then, it holds that

\[
X_1^n(\Omega) = X_2^n(\Omega) = \Omega.
\]

For a pair \((\alpha, \beta) \in \{(0,0), \left( \frac{1}{2}, 0 \right), \left( 0, \frac{1}{2} \right) \}\) we define sets of lattice points,

\[
\Omega_h^{(\alpha, \beta)} = \left\{ x_{i,j} \in \Omega; \ (i,j) \in \mathbb{Z}^\alpha \times \mathbb{Z}^\beta \right\}, \\
\bar{\Omega}_h^{(\alpha, \beta)} = \left\{ x_{i,j} \in \bar{\Omega}; \ (i,j) \in \mathbb{Z}^\alpha \times \mathbb{Z}^\beta \right\}, \\
\Gamma_h^{(\alpha, \beta)} = \Omega_h^{(\alpha, \beta)} \setminus \Omega_h^{(\alpha, \beta)}, \\
\Omega_h^{(1,0)} = \Omega_h^{(0,0)} \cup \left\{ x_{i,j}; (i,j) \in \{-1/2, N_1 + 1/2\} \times \{0, \ldots, N_2\} \right\}, \\
\Omega_h^{(0,1)} = \Omega_h^{(0,0)} \cup \left\{ x_{i,j}; (i,j) \in \{0, \ldots, N_1\} \times \{-1/2, N_2 + 1/2\} \right\},
\]

\[\Omega_h = \Omega_h^{(0,0)}, \Omega_h = \Omega_h^{(0,0)} \text{ and } \Gamma_h = \Gamma_h^{(0,0)}, \text{ and function spaces,}\]
\[ V^{(\alpha, \beta)}_h = \left\{ v_h : \Omega^{(\alpha, \beta)}_h \rightarrow \mathbb{R} \right\}, \quad V^{(\alpha, \beta)}_{h0} = \left\{ v_h \in V^{(\alpha, \beta)}_h : v_h |_{\Gamma^{(\alpha, \beta)}_0} = 0 \right\}, \]

\[ V^{(\alpha, \beta)}_{0h} = \left\{ v_h : \Omega^{(\alpha, \beta)}_h \rightarrow \mathbb{R} \right\}. \]

\[ V_h = V^{(0,0)}_h, \quad V_{h0} = V^{(0,0)}_{h0} \quad \text{and} \quad V_{0h} = V^{(0,0)}_{0h}. \] The space \( V_{h0} \) includes the essential boundary condition (1b).

For \((\alpha, \beta) \in \{(0,0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right)\}\) we define a bilinear interpolation operator \( \Pi^{(\alpha, \beta)}_h : V^{(\alpha, \beta)}_h \rightarrow C^0(\bar{\Omega}) \) by

\[ \left( \Pi^{(\alpha, \beta)}_h v_h \right)(x) = \sum_{(i,j) \in \Lambda^{(\alpha, \beta)}_h(x)} \tilde{v}_h(x_{i,j}) \phi_{i,j}(x), \]

where \( \phi_{i,j} : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a bilinear (basis) function with a support \( l = i \pm \frac{1}{2}, m = j \pm \frac{1}{2} \).

\( K_{l,m} \) is a closed rectangle,

\[ K_{l,m} = \left[ \left( l - \frac{1}{2} \right) h, \left( l + \frac{1}{2} \right) h \right] \times \left[ \left( m - \frac{1}{2} \right) h, \left( m + \frac{1}{2} \right) h \right], \]

\( \Lambda^{(\alpha, \beta)}_h(x) \subset \mathbb{Z}^\alpha \times \mathbb{Z}^\beta \) is a set of neighboring four lattice points of \( x \in \bar{\Omega} \), and

\[
\begin{align*}
\begin{cases}
v_h(x_{i,j}) = x_{i,j} & \text{for } x_{i,j} \in \bar{\Omega}_h \cup \overset{\frac{1}{2}}{\bar{\Omega}_h}^{(0,0)} \cup \overset{\frac{1}{2}}{\bar{\Omega}_h}^{(0,\frac{1}{2})} \\
2v_h(x_{1/2,j}) - v_h(x_{3/2,j}) & \text{for } x_{i,j} \in \overset{\frac{1}{2}}{\bar{\Omega}_h}^{(0,0)}, i = -1/2, \\
2v_h(x_{N_1-1/2,j}) - v_h(x_{N_1-3/2,j}) & \text{for } x_{i,j} \in \overset{\frac{1}{2}}{\bar{\Omega}_h}^{(0,0)}, i = N_1 + 1/2, \\
2v_h(x_{i,1/2}) - v_h(x_{i,3/2}) & \text{for } x_{i,j} \in \overset{\frac{1}{2}}{\bar{\Omega}_h}^{(0,\frac{1}{2})}, j = -1/2, \\
2v_h(x_{i,N_2-1/2}) - v_h(x_{i,N_2-3/2}) & \text{for } x_{i,j} \in \overset{\frac{1}{2}}{\bar{\Omega}_h}^{(0,\frac{1}{2})}, j = N_2 + 1/2.
\end{cases}
\end{align*}
\]
Π_{h(0,0)} is denoted simply by Π_h.

We use the symbol ° to represent the composition of functions, e.g.,

\[
(\phi \circ X^n_i)(x) \equiv \phi\left(X^n_i(x)\right).
\]

Let \( e_i \equiv (\delta_{i1}, \delta_{i2})^T \) \((i = 1, 2)\) be unit vectors and \( T^i_a(h) \) be a translation operator,

\[
\left\{T^i_a(h)\right\}(x) \equiv v(x + a e_i).
\]

We often omit \( h \) to write simply \( T^i_a \). For a discrete function \( v_h \) and an integer \( n \) \((= 1, \cdots, N_T)\) we set finite difference operators, for \( i = 1, 2 \),

\[
(3a)
\]

\[
(3b)
\]

\[
(3c)
\]

\[
(3d)
\]

\( \Delta t \) is often omitted and the operators are written as, e.g., \( \tilde{\nabla}^{(n)}_h \).

**Remark 3.** (i) For \( i = 1 \) and \( 2 \) we can write

\[
\nabla_{hi} = \frac{T^{i}_{1/2} - T^{i}_{-1/2}}{h}, \text{ i.e., } \left(\nabla_{hi} v_h\right)(x) = \frac{1}{h} \left\{v_h\left(x + \frac{h}{2} e_i\right) - v_h\left(x - \frac{h}{2} e_i\right)\right\},
\]

\[
\tilde{\nabla}^{(n)}_{h1} v_h = \left(\Pi_h^{(1,0)} \nabla_{h1} v_h\right) \circ X^n_1, \quad \tilde{\nabla}^{(n)}_{h2} v_h = \left(\Pi_h^{(0,1)} \nabla_{h2} v_h\right) \circ X^n_1.
\]
We note that, for $\nu \in V_h$, $\nabla h_1 \nu \in V_h$, $\nabla h_2 \nu \in V_h$, and $\nabla (2h)_1 \nabla (2h)_2 \nu$, $\Delta_h \nu$ and $\tilde{\Delta}_h^{(n)} \nu \in V_{0h}$.

A characteristics finite difference scheme of second order in time for (1) is to find $\{\phi_h^n\}_{n=0}^{N_T}$ such that, for $n = 1, \cdots, N_T$,

\begin{align}
\mathcal{A}_h^{n-1/2} \phi_h &= \frac{1}{\Delta t} \left( f^n + f^{n-1} \circ X_1^n \right) \quad \text{on } \Omega_h, \quad (4a) \\
\phi_h^0 &= \phi^0 \quad \text{on } \Omega_h, \quad (4b)
\end{align}

where $\mathcal{A}_h^{n-1/2}$ is a finite difference operator defined by

\begin{align}
\mathcal{A}_h^{n-1/2} \phi_h &= \phi_h^n - \frac{\Pi_h \phi_h^{n-1}}{\Delta t} X_2^n - \frac{\nu}{\Delta t} \frac{1}{2} \left( \Delta_h \phi_h^n + \tilde{\Delta}_h^{(n)} \phi_h^{n-1} \right) \\
&\quad - \frac{\nu \Delta t}{2} \left\{ \sum_{i=1}^{2} \left( \partial_i u_i^n \right) \Delta_n + \left( \partial_2 u_1^n + \partial_1 u_2^n \right) \nabla (2h)_1 \nabla (2h)_2 \right\} \phi_h^{n-1}. \quad (5)
\end{align}

\textbf{Remark 4.} The third term of $\mathcal{A}_h^{n-1/2} \phi_h$,

\begin{align}
- \frac{\nu \Delta t}{2} \left\{ \sum_{i=1}^{2} \left( \partial_i u_i^n \right) \Delta_n + \left( \partial_2 u_1^n + \partial_1 u_2^n \right) \nabla (2h)_1 \nabla (2h)_2 \right\} \phi_h^{n-1},
\end{align}

is an additional correction term in order to obtain a real second order scheme in time, whose significance will be shown in Lemma A.11.

We also consider a scheme corresponding to (4) for general initial values and right-hand sides. Let $a_h \in V_{0h}$ and $\{\mathcal{F}_h^{n-1/2}\}_{n=1}^{N_T} \subset V_{0h}$ be given. A general scheme is to find $\{\phi_h^n\}_{n=0}^{N_T} \subset V_{0h}$ such that, for $n = 1, \cdots, N_T$,

\begin{align}
\mathcal{A}_h^{n-1/2} \phi_h &= \mathcal{F}_h^{n-1/2} \quad \text{on } \Omega_h, \quad (6a)
\end{align}
\[ \phi^0 = a_h \quad \text{on } \bar{\Omega}_h. \] (6b)

Then, \( \phi_h = \{ \phi^n_h \}_{n=0}^{N_r} \) is called the solution of scheme (6) with \( \left( a_h, \{ \mathcal{F}^{-n/2}_h \}_{n=1}^{N_r} \right) \).

Obviously, the solution \( \phi_h = \{ \phi^n_h \}_{n=0}^{N_r} \) of scheme (4) is the solution of scheme (6) with \( \left( \phi^0, \left\{ \frac{1}{2} \left( f^n + f^{n-1} \circ X^n_h \right) \right\}_{n=1}^{N_r} \right) \).

3. MAIN RESULTS

For a set \( S_h \) of lattice points and functions \( \nu_h \) and \( w_h \) in a function space \( \{ \nu_h : S_h \to \mathbb{R} \} \) we define an inner product by

\[ (\nu_h, w_h)_{S_h} \equiv h^2 \sum_{x \in S_h} \nu_h(x) w_h(x). \]

Let \( (\alpha, \beta) \in \{ (0,0), \left( \frac{1}{2}, 0 \right), \left( 0, \frac{1}{2} \right) \} \) be a pair of numbers. We define norms and seminorms, for a scalar-valued function \( \nu_h \), a vector-valued function \( w_h = (w_{h1}, w_{h2})^T \) and a set of scalar-valued functions \( \phi_h = \{ \phi^n_h \}_{n=0}^{N_r} \),

\[ \| \nu_h \|_{L^2(\Omega_h)} = \left\{ (\nu_h, \nu_h)_{\Omega_h} \right\}^{1/2}, \| \nu_h \|_{L^2(\Omega_h)} = \left\{ \| \nu_h \|_{L^2(\Omega_h)}^2 + \| \nu_h \|_{L^2(\Omega_h)}^2 \right\}^{1/2}. \]

\[ \| \phi_h \|_{k^2} = \max_{n=0, \ldots, N_r} \| \phi^n_h \|_{L^2(\Omega_h)} = \max_{n=0, \ldots, N_r} \| \phi^n_h \|_{L^2(\Omega_h)} \]

\[ \| \phi_h \|_{k^2} = \left\{ \Delta t \sum_{n=1}^{N_r} \| \phi^n_h \|_{L^2(\Omega_h)}^2 \right\}^{1/2}, \| \phi_h \|_{k^2} = \left\{ \frac{\Delta t \sum_{n=0}^{N_r} \| \phi^n_h \|_{L^2(\Omega_h)}^2}{\sum_{n=0}^{N_r} \phi^n_h \phi^{n-1}_h} \right\}^{1/2}. \]

\[ \| \phi_h \|_{k^2(\Omega_h')} = \left\{ \Delta t \sum_{n=1}^{N_r} \left( \nabla_h \phi^n_h + \nabla_h (\phi^{n-1}_h \cdot X^n_h) \right)^2 \right\}^{1/2}. \]
By deleting \( (\overline{\cdot}) \) from the above notations, norms \( \| f (\Omega_\nu) \| \) and \( \| f (\Omega_\nu)^{\frac{1}{2}} \| \) and a seminorm \( |\cdot|'_{\Omega_\nu} \) are similarly defined. We define the norm \( \| f (\Omega_\nu) \| \) for \( \mathcal{F}_h = \{ \mathcal{F}_h^n \}_{n=1}^N \) by

\[
\| \mathcal{F}_h \| _{\mathcal{F}_h} = \left\{ \sum_{n=1}^N \| \mathcal{F}_h^n \| _{\mathcal{F}_h}^{\frac{1}{2}} \right\}^{\frac{1}{2}}.
\]

**Theorem 1** (stability). Suppose \( [H_{\nu,2}(u)], [H_{\nu}(u)], [H_{\nu}(\Delta t)] \) and \( [H_{tCFL}(\Delta t)] \). Let \( a_h \in \mathbb{V}_{h0} \) be given. Let \( \phi_h = \{ \phi_h^n \}_{n=0}^N \subset \mathbb{V}_{h0} \) be the solution of scheme (6). Then, there exists a positive constant \( c = c \left( \| u \|_{C^2(\overline{\Omega})}, v_0, T \right) \), independent of \( \nu, h \) and \( \Delta t \), such that

\[
\| \phi_h \|_{\mathcal{F}_h} + \sqrt{\nu \Delta t} |\phi_h|_{\mathcal{F}_h} + \sqrt{v} |\phi_h|_{\mathcal{F}_h} \leq c \left( \| a_h \|_{\mathbb{V}_{h0}} + \sqrt{\nu \Delta t} |a_h|_{\mathbb{V}_{h0}} + \| \mathcal{F}_h \| _{\mathcal{F}_h} \right). \quad (7)
\]

**Corollary 1.** Suppose \( [H_{\nu,2}(u)], [H_{t}(u)], [H_{\nu,0}(f)], [H_u(\Delta t)] \) and \( [H_{tCFL}(\Delta t)] \). Let \( \phi_h = \{ \phi_h^n \}_{n=0}^N \subset \mathbb{V}_{h0} \) be the solution of scheme (4). Then, there exists a positive constant \( c = c \left( \| u \|_{C^2(\overline{\Omega})}, v_0, T \right) \), independent of \( \nu, h \) and \( \Delta t \), such that

\[
LHS \ of \ (7) \leq c \left( \| \phi_0 \|_{\mathcal{F}_h} + \sqrt{\nu \Delta t} |\phi_0|_{\mathbb{V}_{h0}} + |f|_{\mathcal{F}_h} \right). \quad (8)
\]

**Theorem 2** (error estimate). Suppose \( [H_{2C}(\phi)] \) and \( [H_{2C}(\Delta \phi)] \) for the solution \( \phi \) of (1). Suppose \( [H_{2C}(u)], [H_{t}(u)], [H_u(\Delta t)] \) and \( [H_{tCFL}(\Delta t)] \). Let \( \phi_h = \{ \phi_h^n \}_{n=0}^N \subset \mathbb{V}_{h0} \) be the solution of scheme (4). Then, there exists a positive constant \( c = c \left( \| u \|_{L^2}, v_0, T \right) \), independent of \( \nu, h \) and \( \Delta t \), such that
Corollary 2. Suppose \([H_{1,0}(\phi)]\) and \([H_{0,2}(\phi)]\) instead of \([H_{3,C}(\phi)]\) and \([H_{2,C}(\Delta \phi)]\) in the assumptions of Theorem 2. Then, it holds that

\[
\|\phi - \phi_h\|_{r_{(t^\prime)}} + \sqrt{\nu} \|\phi - \phi_h\|_{r_{(h^\prime)}} \leq c \left(\Delta t^2 + h\right) \left(\|\phi\|_{Z_c^r} + \|\Delta \phi\|_{Z_c^r}\right).
\]

(9)

Corollary 3. RHS of (9) can be replaced by \(c(\Delta t^2 + h)\left(\|\phi\|_{Z^r} + \|\Delta \phi\|_{Z^r}\right)\).

Remark 5. Theorems 1 and 2 and Corollary 1 ensure that the estimates (7)–(9) hold uniformly in \(\nu\) even when \(\nu\) tends to 0.

Remark 6. Since the relation \([H_{WCFL}(\Delta t)]\) is assumed, RHS of (9) can be written as \(c\left(\|\phi\|_{Z^r}^c \|\Delta \phi\|_{Z^r}^c\right)\), and \(h \downarrow 0\) in (10) is equivalent to the condition that \(h\) and \(\Delta t \downarrow 0\) under that relation.

Throughout the paper, we use \(c\) with or without subscript to denote the generic positive constant independent of \(h\) and \(\Delta t\), which may take different values at different places, e.g., \(c(A)\) means a constant depending on \(A\). We prepare positive constants,

\[
c_0 = c_0 \left(\|\mu\|_{C^0(\bar{\Omega})}\right), \quad c_1 = c_1 \left(\|\mu\|_{C^0(\bar{\Omega})}\right), \quad c_2 = c_2 \left(\|\mu\|_{C^0(\bar{\Omega})}\right),
\]

\[
c_3 = c_3 \left(\|\mu\|_{Z^r_c}\right), \quad c_4 = c_4 \left(\|\mu\|_{Z^r_c}\right).
\]

and sometimes add “‘(prime)” to the constants, e.g., \(c'_0\).

4. PROOF OF THEOREM 1

For a vector \(w \in \mathbb{R}^2\), a mesh size \(h\) and a time increment \(\Delta t\) we define a “proportional weight” of the \(w\)-upwind point of a lattice point \(x_{i,j}\) with respect to a lattice point \(x_{l,m}\) by

\[
c_{i,j}^{l,m}(w; \Delta t, h) \equiv \phi_{l,m}(x_{i,j} - w\Delta t),
\]

(11)

whose properties are summarized in Lemma A.1 of Appendix A.1.
Lemma 1. Suppose $[H_{0,1}(u)], [H_{u}(\Delta t)], [H_{u}(\Delta t)]$ and $[H_{u}(\Delta t)]$. Then, for any function $v_h \in V_h$, $n = 1, \cdots, N_T$ and $k = 1$ and 2, it holds that
\[
\left\| (\Pi_h v_h) \circ X^n_h \right\|_{L^2(\Omega_h)} \leq (1 + c_1 \Delta t) \left\| v_h \right\|_{L^2(\Omega_h)},
\]
(12a)
\[
\left\{ \left( \Pi_h^{(0,1)} v_h \right) X_1^n \left( \Omega_h \right)^{1/2} + \left( \Pi_h^{(0,1)} v_h \right) X_1^n \left( \Omega_h \right)^{1/2} \right\}^{1/2} \leq (1 + c_1 \Delta t) \left\| v_h \right\|_{L^2(\Omega_h)}.
\]
(12b)

Proof. We prove only (12a) with $k = 1$. Let $C_1$ be the constant in $[H_{wCFL}(\Delta t)]$, $n (\leq N_T)$ be a positive integer and $x_{i,j} \in \Omega_h$ be a lattice point. Since we have $X_1^n (x_{i,j}) \in \Omega$ by $[H_{u}(\Delta t)]$, it holds that, from Lemma A.1 (iv) with $w = u^n(x_{i,j})$,
\[
(\Pi_h v_h) \circ X_1^n (x_{i,j}) = \sum_{x_{i,m} \in \Omega_h} c_{i,j}^{l,m} (u^n(x_{i,j})) v_h(x_{l,m}),
\]
(13)
where $c_{i,j}^{l,m} = c_{i,j}^{l,m} (: \Delta t, h)$. Using the properties of $\left\{ c_{i,j}^{l,m} (w) \right\}_{i,j,l,m}$ in Lemma A.1 and the Schwarz inequality, we have
\[
(\text{LHS of (12a)})^2 = h^2 \sum_{x_{i,j} \in \Omega_h} \left\{ \sum_{x_{i,m} \in \Omega_h} c_{i,j}^{l,m} (u^n(x_{i,j})) v_h(x_{l,m}) \right\}^2 \leq h^2 \sum_{x_{i,j} \in \Omega_h} \left\{ \sum_{x_{i,m} \in \Omega_h} c_{i,j}^{l,m} (u^n(x_{i,j})) \sum_{x_{i,m} \in \Omega_h} c_{i,j}^{l,m} (u^n(x_{i,j})) v_h(x_{l,m}) \right\}^2 \leq h^2 \sum_{x_{i,j} \in \Omega_h} \left\{ \sum_{x_{i,m} \in \Omega_h} c_{i,j}^{l,m} (u^n(x_{i,j})) v_h(x_{l,m}) \right\}^2 \leq h^2 \sum_{x_{i,j} \in \Omega_h} \left\{ \sum_{x_{i,m} \in \Omega_h} c_{i,j}^{l,m} (u^n(x_{i,j})) - 1 \right\} \leq h^2 \sum_{x_{i,j} \in \Omega_h} v_h(x_{l,m})^2 + h^2 \sum_{x_{i,j} \in \Omega_h} \sum_{x_{i,m} \in \Omega_h} g_{i,j}^{l,m},
\]
(14)
where $\xi_{i,j}^{l,m} \equiv \left| c_{i,j}^{l,m}(u^n(x_{i,j}))-c_{i,j}^{l,m}(u^n(x_{l,m})) \right|$. Let $\Xi_l^{l,m}, \Xi_0^{l,m}$ and $\Xi_1^{l,m}$ be sets of lattice points,

$$
\Xi_l^{l,m} \equiv \{ x_{i,j} \in \bar{\Omega}_h; \xi_{i,j}^{l,m} \neq 0 \}, \quad \Xi_0^{l,m} \equiv \{ x_{i,j} \in \bar{\Omega}_h; c_{i,j}^{l,m}(u^n(x_{i,j})) \neq 0 \},
$$

$$
\Xi_1^{l,m} \equiv \{ x_{i,j} \in \Omega_h; c_{i,j}^{l,m}(u^n(x_{l,m})) \neq 0 \},
$$

and $\tilde{C}_1 \equiv [C_1]+1\text{ and } N_\Xi \equiv 2(2\tilde{C}_1+1)^2$ be integers. We note that, from the inequality $\|\Xi_l^{l,m}\| \leq (2\tilde{C}_1+1)^2 (p+1)$, it holds that

$$
\|\Xi_l^{l,m}\| \leq \|\Xi_0^{l,m}\| + \|\Xi_1^{l,m}\| \leq N_\Xi. \tag{15}
$$

Therefore, from Lemma A.2 and (15) the sum $\sum_{x_{i,j} \in \Omega_h} \xi_{i,j}^{l,m}$ is estimated as

$$
\sum_{x_{i,j} \in \Omega_h} \xi_{i,j}^{l,m} = \sum_{x_{i,j} \in \Xi_l^{l,m}} \xi_{i,j}^{l,m} \leq \sum_{x_{i,j} \in \Xi_0^{l,m} \cup \Xi_{1}^{l,m}} \xi_{i,j}^{l,m} \leq \sum_{x_{i,j} \in \Xi_0^{l,m}} \xi_{i,j}^{l,m} + \sum_{x_{i,j} \in \Xi_1^{l,m}} \xi_{i,j}^{l,m}
$$

$$
\leq (\|\Xi_0^{l,m}\| + \|\Xi_1^{l,m}\|)2U_1^{\infty}(C_1+1)\Delta t \leq 2N_\Xi U_1^{\infty}(C_1+1)\Delta t. \tag{16}
$$

Combining (16) with (14), we get (12a) for $c_1 = N_\Xi U_1^{\infty}(C_1+1)$. \hfill \square

In the next lemma we present discrete formulae of integration by parts. The proofs are omitted, as they are not difficult.

**Lemma 2** (summation by parts). For $v_h$ and $w_h \in V_{h0}$ we have

$$
-(\tilde{\Delta}_{h,1}^{(n)} v_h, w_h)_{\Omega_h} = \left( \nabla_{h1}^{(n)} v_h, \nabla_{h1} w_h \right)_{\Omega_h}^{<0},
$$

$$
-(\tilde{\Delta}_{h,2}^{(n)} v_h, w_h)_{\Omega_h} = \left( \nabla_{h2}^{(n)} v_h, \nabla_{h2} w_h \right)_{\Omega_h}^{<0}, \tag{17a}
$$

$$
-(\nabla_{(2h)} v_{(2h)} w_h)_{\Omega_h} = \left( \nabla_{(2h)} v_{(2h)}, \nabla_{(2h)} w_h \right)_{\Omega_h},
$$

$$
= \left( \nabla_{(2h)} v_{(2h)}, \nabla_{(2h)} w_h \right)_{\Omega_h}. \tag{17b}
$$
Now we prove the stability theorem and its corollary.

**Proof of Theorem 1.** Multiplying both sides of (6a) by $h^2 \phi_h^n$ and summing up for all $x \in \Omega_h$, we have

$$\left(\mathcal{A}_h^{n-1/2} \phi_h^n, \phi_h^n\right)_{\Omega_h} = \left(\mathcal{F}_h^{n-1/2} \phi_h^n\right)_{\Omega_h}.$$  \hspace{1cm} (18)

The definition of $\mathcal{A}_h^{n-1/2}$ leads to

\[
\text{LHS of (18)} = \left(\phi_h^n - \left(\Pi_h \phi_h^{n-1}\right) \circ X_{2h}^n\right)_{\Omega_h} - \frac{v}{2} \left(\Delta_h \phi_h^n + \Delta_h^{(n)} \phi_h^{n-1} \cdot \phi_h^n\right)_{\Omega_h} \\
= -\frac{v\Delta t}{2} \left(\sum_{i=1}^{2} \left(\partial_i u_i^n\right) \Delta x_{ij} + \left(\partial_2 u_1^n + \partial_1 u_2^n\right) \nabla (2h)_{1} \nabla (2h)_{2}\right) \phi_h^{n-1} \cdot \phi_h^n_{\Omega_h} \\
= I_1 + I_2 + I_3.
\]

Let $\overline{D}_t$ be the backward difference operator $\overline{D}_t \phi^n \equiv \left(\phi^n - \phi^{n-1}\right) / \Delta t$. Lemmas 1 and 2 imply the estimates, from identities $(a - b)a = (a^2 - b^2)/2 + (a - b)^2/2$ and $(a + b)a = (a^2 + b^2)/2 + (a + b)^2/2$,

\[
I_1 \geq \overline{D}_t \left(\frac{1}{2} \left\|\phi_h^n\right\|_{L^2(\Omega_h)}^2\right) - c_1 \left\|\phi_h^{n-1}\right\|_{L^2(\Omega_h)}^2 + \frac{1}{2\Delta t} \left\|\phi_h^n - \left(\Pi_h \phi_h^{n-1}\right) \circ X_{2h}^n\right\|_{L^2(\Omega_h)}^2, \hspace{1cm} (19a)
\]

\[
I_2 \geq \overline{D}_t \left(\frac{v\Delta t}{4} \left\|\phi_h^n\right\|_{H^1(\Omega_h)}^2\right) - c_1 v\Delta t \left\|\phi_h^{n-1}\right\|_{H^1(\Omega_h)}^2 \\
+ v \left\|\nabla \phi_h^n + \nabla (n) \phi_h^{n-1}\right\|_{L^2(\Omega_h \times \partial^1 (\Omega_h))}. \hspace{1cm} (19b)
\]

\[
I_3 \leq v\Delta t \left(\frac{1}{2\delta_0} \left\|\phi_h^{n-1}\right\|_{L^2(\Omega_h)}^2 + c_2 \left\|\phi_h^n\right\|_{L^2(\Omega_h)}^2\right) \\
\leq c_2 v\Delta t \left\{\left\|\phi_h^{n-1}\right\|_{L^2(\Omega_h)}^2 + \delta_0 \left(\left\|\phi_h^n\right\|_{L^2(\Omega_h)}^2 + \left\|\phi_h^n\right\|_{L^2(\Omega_h)}^2\right)\right\}, \hspace{1cm} (19c)
\]

for any positive number $\delta_0$. Here we have used the following inequalities to obtain (19c), for $v_h \in V_{h0}$.
\[
\left( \nabla_{(2h)1} v_h, \nabla_{(2h)1} v_h \right)_{\Omega_h} \leq \| \nabla_{h1} v_h \|^2_{L^2(\Omega_h^{(1/2)})},
\]
\[
\left( \nabla_{(2h)2} v_h, \nabla_{(2h)2} v_h \right)_{\Omega_h} \leq \| \nabla_{h2} v_h \|^2_{L^2(\Omega_h^{(1/2)})}.
\]

It is obvious that
\[
\text{RHS of (18)} = \left( \mathcal{F}^{-n/2}_h, \phi^n_h \right)_{\Omega_h} \leq \delta_0 \left\| \phi^n_h \right\|^2_{L^2(\Omega_h)} + \frac{1}{4\delta_0} \left\| \mathcal{F}^{-n/2}_h \right\|^2_{L^2(\Omega_h)}. \tag{20}
\]

Combining the inequalities (19) and (20) with (18), we have
\[
\delta_0 \left\| \phi^n_h \right\|^2_{L^2(\Omega_h)} + \frac{1}{4\delta_0} \left\| \mathcal{F}^{-n/2}_h \right\|^2_{L^2(\Omega_h)} \leq \delta_0 \left\| \phi^n_h \right\|^2_{L^2(\Omega_h)} + \frac{1}{2\Delta t} \left\| \phi^n_h - \left( \Pi_h \phi^{n-1}_h \right) X^n_2 \right\|^2_{L^2(\Omega_h)}
\]
\[
+ \left\| \nabla \phi^n_h + \nabla \phi^{n-1}_h \right\|^2_{L^2(\Omega_h)} \right) + \frac{1}{2\Delta t} \left\| \phi^n_h - \left( \Pi_h \phi^{n-1}_h \right) X^n_2 \right\|^2_{L^2(\Omega_h)}
\]
\[
+ \left( 1 + \nu_0 \Delta t \right) \left\| \phi^n_h \right\|^2_{L^2(\Omega_h)} + \nu \Delta t \left\| \phi^n_h \right\|^2_{L^2(\Omega_h)} \right) + \delta_0 \left( 1 + \frac{1}{2\delta_0} \right) \left\| \phi^n_{n-1} \right\|^2_{L^2(\Omega_h)}
\]
\[
+ \frac{1}{4\delta_0} \left\| \mathcal{F}^{-n/2}_h \right\|^2_{L^2(\Omega_h)}. \tag{21}
\]

Applying the discrete Gronwall inequality (cf. [22]) to (21) with a proper \( \delta_0 \), we get (7). \( \square \)

\textbf{Proof of Corollary 1.} Since \( \phi_h \) is nothing but the solution of scheme (6) with
\[
\left\{ \phi^0, \left\{ \frac{1}{2} \left( f^n + f^{n-1} \circ X^n_1 \right) \right\}_{n=1}^{N_T} \right\},
\]
it holds that
\[
\text{LHS of (8)} \leq c \left( \left\| \phi^0 \right\|_{L^2(\Omega_h)} + \sqrt{\nu \Delta t} \left\| \phi^0 \right\|_{h^1(\Omega_h)} + \left\| \mathcal{F}_h \right\|_{L^2(\Omega_h)} \right)
\]
for \( \mathcal{F}^{-n/2}_h = \frac{1}{2} \left( f^n + f^{n-1} \circ X^n_1 \right) \) \((n = 1, \ldots, N_T)\). From Lemma 1 we have
\[ \| \mathcal{F}_h^{n-1/2} \|_{C(\Omega)} \leq \frac{1}{2} \left\{ \| f^n \|_{C(\Omega)} + (1 + c_1 \Delta t) \| f^{n-1} \|_{C(\Omega)} \right\}, \]

which implies

\[ \| \mathcal{F}_h \|_{L^\infty(T')} \leq c_1 \| f \|_{L^\infty(T')} \].

5. PROOF OF THEOREM 2

For functions \( u \in C^0 \left( C^0 \left( \overline{\Omega} \right) \right) \) and \( \phi \in C^1 \left( C^0 \left( \overline{\Omega} \right) \right) \cap C^0 \left( C^2 \left( \overline{\Omega} \right) \right) \) we define an operator \( \mathcal{A}^{n-1/2} \) and a function \( Y^n_1(x) \) by

\[ \mathcal{A}^{n-1/2} \phi = \frac{\partial \phi^{n-1/2}}{\partial t} + u^{n-1/2} \cdot \nabla \phi^{n-1/2} - v \Delta \phi^{n-1/2}, \quad Y^n_1(x) = \frac{x + X^n_1(x)}{2}. \]

We evaluate scheme (4) at a point \( P^{n-1/2}(x) \equiv \left( Y^n_1(x), t^{n-1/2} \right) \).

Let \( \phi \) be the solution of (1), \( \phi_h^{N_T} \subset V_{h0} \) be the solution of (4) and \( e_h^{N_T} \subset V_{h0} \) be a function set defined by

\[ e^n_h(x) = \phi^n_h(x) - \phi^n(x) \quad (x \in \overline{\Omega}_h). \tag{22} \]

From (4) and the fact that

\[ \mathcal{A}^{n-1/2} \phi = f^{n-1/2} \quad \text{in } \Omega, \]

we have, for \( n = 1, \ldots, N_T \),

\[ \mathcal{A}^{n-1/2} e_h^n = R^n_f + R^n_{\mathcal{A}}, \tag{23} \]

where

\[ R^n_f \equiv \frac{1}{2} \left( f^n + f^{n-1} \circ X^n_1 \right) - f^{n-1/2} \circ Y^n_1, \tag{24a} \]

\[ R^n_{\mathcal{A}} \equiv \left( \mathcal{A}^{n-1/2} \phi \right) \circ Y^n_1 - \mathcal{A}^{n-1/2} \phi \equiv \sum_{i=1}^{4} R^n_i + v \sum_{i=5}^{8} R^n_i, \tag{24b} \]
In order to prove Theorem 2 we prepare two lemmas, which give estimates of 

\begin{align}
R_1^n &\equiv \frac{D\phi}{Dt} n^{-1/2} o Y_1^n - \frac{D\phi}{Dt} n^{-1/2} \left(X(t^{n-1/2};\cdot, t^n)\right), \\
R_2^n &\equiv \frac{D\phi}{Dt} n^{-1/2} \left(X(t^{n-1/2};\cdot, t^n)\right) - \frac{\phi^n - \phi^n \left(X(t^{n-1};\cdot, t^n)\right)}{\Delta t}, \\
R_3^n &\equiv \frac{\phi^{n-1} o X_2^n - \phi^{n-1} \left(X(t^{n-1};\cdot, t^n)\right)}{\Delta t}, \\
R_4^n &\equiv \frac{\left(\Pi h \phi^n\right) o X_2^n - \phi^{n-1} o X_2^n}{\Delta t}, \\
R_5^n &\equiv \frac{1}{2} (\Delta_h - \Delta) \phi^n, \\
R_6^n &\equiv \frac{1}{2} \left\{ \Delta^n (\phi^{n-1} - \nabla \cdot \left(\nabla \phi^{n-1} o X_1^n\right)\right\}, \\
R_7^n &\equiv \frac{1}{2} \left\{ \nabla \cdot \left(\nabla \phi^{n-1} o X_1^n - \Delta \phi^{n-1} o X_1^n\right)\right\} + \frac{\Delta t}{2} \sum_{j=1}^{2} \left(\partial_1 u_1^n + \partial_2 u_2^n\right) \nabla_{(2h)} \nabla_{(2h)2} \phi^{n-1}, \\
R_8^n &\equiv \frac{1}{2} \left(\Delta \phi^n + \Delta \phi^{n-1} o X_1^n\right) - \Delta \phi^{n-1/2} o Y_1^n.
\end{align}

In order to prove Theorem 2 we prepare two lemmas, which give estimates of 

\[ \|R_f\|_{L^2(I')} \] and \[ \|R_{\phi}\|_{L^2(I')} \].

**Lemma 3.** Suppose \([H_0(u)], [H_1(u)], [H_2c(f)]\) and \([H_u(\Delta t)]\). Then, there exists a positive constant \(M_f\) such that 

\[ \|R_f\|_{L^2(I')} \leq c \Delta^2 \|f\|_{L^2}, \]

where \(M_f\) satisfies 

\[ M_f \leq c_1 \|f\|_{L^2}, c_1' \|f\|_{L^2}. \]
Proof. Let $g$ and $F$ be functions defined by

$$g(x, t) = f\left(x - \left(t^n - t\right)u^n(x), t\right), \quad (x, t) \in \Omega \times \left[t^{n-1}, t^n\right],$$
$$F(s) = F(s; x, t^n) = g\left(x, t^{n-1/2} + s\right).$$

Then $F \in C^2[-\Delta t/2, \Delta t/2]$ and by Lemma A.4 we have, for $\theta \in (-1/2, 1/2)$,

$$R^\theta_f(x) = \Gamma_1\left(F(\cdot; x, t^n); \Delta t\right) = \frac{\Delta t^2}{8} F''(\theta \Delta t),$$

$$\left|R^\theta_f(x)\right| = \frac{\Delta t^2}{8} \left[\frac{\partial}{\partial t} + u^n(x) \cdot \nabla\right]^2 f\left[x - \left(\frac{1}{2} - \theta\right)u^n(x) \Delta t, t^{n-1/2} + \theta \Delta t\right] \leq c_1 \Delta t^2 \|f\|_{L^2}^2,$$

which implies (25a) with the first inequality of (25b).

Now we prove the second inequality of (25b). At first we prepare an estimate more precise than (26). From (A.3c) and the relation

$$f^1_{-1} F''\left(\frac{\Delta t}{2} s; x, t^n\right)^2 ds = f^1_{-1} \frac{\partial^2 g}{\partial t^2}\left(x, t^{n-1/2} + \frac{\Delta t}{2} s\right)^2 ds = \frac{\Delta t}{2} \int_{t^{n-1}}^{t^n} \frac{\partial^2 g}{\partial t^2}(x, t)^2 dt,$$

we have

$$\|R_f\|_{L^2} \leq \frac{\Delta t^2}{8} \left\{\frac{2}{\Delta t} \int_{t^{n-1}}^{t^n} \frac{\partial^2 g}{\partial t^2}(\cdot, t)^2 dt\right\}^{1/2}$$

$$= \frac{\Delta t^2}{8} \left\{\sum_{n=1}^{N_t} \Delta t \sum_{x \in \Omega} h_x h_2 \frac{2}{\Delta t} \int_{t^{n-1}}^{t^n} \frac{\partial^2 g}{\partial t^2}(x, t)^2 dt\right\}^{1/2}$$

$$= \sqrt{2} \frac{\Delta t^2}{8} \left\{\int_{0}^{T} \left\|\frac{\partial^2 g}{\partial t^2}(\cdot, t)\right\|_{L^2(\Omega)}^2 dt\right\}^{1/2} = \sqrt{2} \frac{\Delta t^2}{8} \left\|\frac{\partial^2 g}{\partial t^2}\right\|_{L^2(0, T; L^2(\Omega))}.$$

Since any sequence of Riemann sums $\left\{\left\|\frac{\partial^2 g}{\partial t^2}(\cdot, t)\right\|_{L^2(\Omega)}\right\}_{h \downarrow 0}$ converges to $\left\|\frac{\partial^2 g}{\partial t^2}(\cdot, t)\right\|_{L^2(\Omega)}$, there exists a constant $h_* = h_*(g) > 0$ such that, for any $h \leq h_*$,
Transforming the variable \( x \) into \( y = x - (t^n - t)u^n(x) \) and evaluating the Jacobian by \( 1 + c_1 \Delta t \), we have

\[
\left\| \frac{\partial^2 g}{\partial t^2} \right\|_{L^2(0,T;L^2(\Omega))} \leq 2 \left\| \frac{\partial^2 g}{\partial t^2} \right\|_{L^2(0,T;L^2(\Omega))}.
\]

which completes the proof.

Lemma 4 (truncation error of \( s^i_h \)). Suppose \([H_{2C}(u)], [H_\Gamma(u)], [H_{3C}(\phi)], [H_{2C}(\Delta \phi)] \) and \([H_\mu(\Delta t)]\). Then, there exists a positive constant \( M_{\text{sl}} \) such that

\[
\left\| R_{s} \right\|_{L^2(i^i)} \leq c_1 \left( \Delta t^2 + h \right) M_{\text{sl}},
\]  

(27a)

where \( M_{\text{sl}} \) satisfies

\[
M_{\text{sl}} \leq c_4 \left( \left\| \phi \right\|_{Z^2} + \left\| \Delta \phi \right\|_{Z^2} \right), c_4 \left( \left\| \phi \right\|_{Z^2} + \left\| \Delta \phi \right\|_{Z^2} \right).
\]

(27b)

Proof. Let \( M_i, i = 1, \cdots, 8 \), be constants in Lemmas A.5–A.12. We set \( M_{\text{sl}} = \sum_{i=1}^{4} M_i + \nu \sum_{i=5}^{8} M_i \). From (24b) and Lemmas A.5–A.12 we have

\[
\left\| R_{s} \right\|_{L^2(i^i)} \leq \sum_{i=1}^{4} \left\| R_i \right\|_{L^2(i^i)} + \nu \sum_{i=5}^{8} \left\| R_i \right\|_{L^2(i^i)} \leq c_1 \left( \Delta t^2 + h \right) M_{\text{sl}},
\]

which leads to (27a). Estimates (27b) follow from Lemmas A.5–A.12. 

Now we prove the error estimates.
Proof of Theorem 2. Let \( e_h^n \in V_{h0} \), \( R^n_j \) and \( R^n_{x\delta} \) be functions defined by (22), (24a) and (24b), respectively. Then, (23) implies that \( e_h = \left\{ e_h^n \right\}_{n=0}^{N_f} \subset V_{h0} \) is the solution of scheme (6) with \( \left( 0, \left\{ R^n_j + R^n_{x\delta} \right\}_{n=1}^{N_f} \right) \). Applying Theorem 1 for \( e_h \), we have, from Lemmas 3 and 4,

\[
\text{LHS of (9)} \leq c \left\| R_j + R_{x\delta} \right\|_{L^2(I_f)} \leq c \left( \left\| R_j \right\|_{L^2(I_f)} + \left\| R_{x\delta} \right\|_{L^2(I_f)} \right) \leq c \left( \Delta t^2 + h \right) M,
\]

where

\[
c = c \left( \left\| u \right\|_{C^0(C^1(\Omega))}, V_0, T \right),
\]

\[
M \equiv M_f + M_{x\delta} \leq c \left( \left\| u \right\|_{Z^c_z}, V_0 \right) \left( \left\| \phi \right\|_{Z^c_z} + \left\| \Delta \phi \right\|_{Z^c_z} \right).
\]

Therefore the inequality (9) holds for a constant \( c = c \left( \left\| u \right\|_{Z^c_z}, V_0, T \right) \) independent of \( \nu, h \) and \( \Delta t \).

Proof of Corollary 2. Let \( \varepsilon > 0 \) be any fixed number. It holds that

\[
\left\| \phi - \phi_h \right\|_X \leq \left\| \phi - \phi^\delta \right\|_X + \left\| \phi^\delta - \phi_h \right\|_X + \left\| \phi_h - \phi \right\|_X,
\]

where \( \left\| \cdot \right\|_X = \left\| \cdot \right\|_{L^2(I_f)} + \left\| \sqrt{V} \cdot \right\|_{L^2(I_f')} \), \( \delta > 0 \) is a (small) number, \( \phi^\delta \) is a mollification of \( \phi [4] \),

\[
\phi_h^\delta = \left\{ \phi_h^\delta, n \right\}_{n=0}^{N_f} \text{ is the solution of scheme (6) with } \left( \phi^\delta, 0, \left\{ \frac{1}{2} \left( f^\delta, n + f^\delta, n-1 \circ X^h \right) \right\}_{n=1}^{N_f} \right),
\]

\[
\phi^\delta, 0 = \phi^\delta (\cdot, 0) \in C^0(\Omega), f^\delta, n = f^\delta (\cdot, n\Delta t) \text{ and } f^\delta = D\phi^\delta / Dt - \nu \Delta \phi^\delta \in C^0(C^0(\Omega)).
\]

There exists a \( \delta_1 > 0 \), independent of \( h \), such that, for \( \delta \leq \delta_1 \),

\[
\left\| \phi - \phi^\delta \right\|_X \leq c \left\| \phi - \phi^\delta \right\|_{C^0(C^1(\Omega))} < \frac{\varepsilon}{3},
\]

(30a)
Let us consider \(\|\phi_h^\delta - \phi_h\|_X\). Since \(\phi_h\) is the solution of scheme (6) with 
\[
\left\{ \phi^0, \left\{ \frac{1}{2} \left( f^n + f^{n-1} \circ x_1^n \right) \right\}_{n=1}^{N_t} \right\},
\]
there exists a \(\delta_2 > 0\), independent of \(h\), such that, for \(\delta \leq \delta_2\),
\[
\|\phi_h^\delta - \phi_h\|_X \leq c \left( \|\phi^0\|_{C^1(\overline{\Omega})} + \sqrt{\nu} \Delta t \|\phi^0\|_{C^1(\overline{\Omega})} + \|f^\delta - f\|_{L^2(I^2)} \right) 
\leq c \left( \|\phi^0\|_{C^1(\overline{\Omega})} + \sqrt{\nu} \Delta t \|\phi^0\|_{C^1(\overline{\Omega})} + \|f^\delta - f\|_{C^0(C^1(\overline{\Omega}))} \right) < \frac{\varepsilon}{3},
\]
(30b)
from Theorem 1 (stability), \([H_{1,0}(\phi)]\) and \([H_{0,2}(\phi)]\). Now we fix \(\delta = \min \{\delta_1, \delta_2\}\). Then, there exists a constant \(h_* = h_* (\phi^\delta) > 0\) such that, for \(h \leq h_*\),
\[
\|\phi^\delta - \phi_h^\delta\|_X \leq c \left( \Delta t^2 + h \right) \left( \|\phi^\delta\|_{L^2}^2 + \|\Delta \phi^\delta\|_{L^2}^2 \right) < \frac{\varepsilon}{3},
\]
(30c)
from Theorem 2 (error estimate) and \([H_{wCFL}(\Delta t)]\). Combining (30) with (29),
we obtain
\[
\|\phi - \phi_h\|_X < \varepsilon,
\]
which implies (10).

Proof of Corollary 3. Since (28) can be replaced by
\[
M \leq c \left( \|u\|_{L^2}, v_0 \right) \left( \|\phi\|_{L^2} + \|\Delta \phi\|_{L^2} \right)
\]
in virtue of Lemmas 3 and 4 in the proof of Theorem 2, we obtain the result.

6. NUMERICAL RESULTS

Example 1 (rotating Gaussian hill). In the problem (1) we set
\[
\Omega = (0,1)^2, \quad T = 2\pi, \quad u = \left( -(x_2 - 0.5), x_1 - 0.5 \right)^T, \quad f = 0,
\]
and three values of \(v\),
\[
v = 5 \times 10^{-4}, 10^{-3}, 2 \times 10^{-3}.
\]
The initial function $\phi^0$ is given so that the exact solution is

$$
\phi(x_1, x_2, t) = \frac{\sigma}{\sigma + 4vt} \exp \left\{ -\frac{(\bar{x}_1(t) - x_{1,c})^2 + (\bar{x}_2(t) - x_{2,c})^2}{\sigma + 4vt} \right\},
$$

(31)

where

$$
(\bar{x}_1(t), \bar{x}_2(t)) \equiv \left( (x_1 - 0.5) \cos t + (x_2 - 0.5) \sin t, -(x_1 - 0.5) \sin t + (x_2 - 0.5) \cos t \right),
$$

$$
(x_{1,c}, x_{2,c}) \equiv (0.25, 0), \quad \sigma \equiv 0.01.
$$

\textbf{Example 2.} In the problem (1) we set

$$
\Omega = (0, \pi)^2, \quad T = 2\pi, \quad u = (1 + \sin t)\left( -\sin^2 x_1 \sin (2x_2), \sin^2 x_2 \sin (2x_1) \right)^T,
$$

and three values of $\nu$,

$$
\nu = 10^{-6}, 10^{-3}, 1.
$$

The functions $\phi^0$ and $f$ are given so that the exact solution is

$$
\phi(x_1, x_2, t) = (1 + \sin t) \sin (2x_1) \sin (2x_2).
$$

(32)

Examples 1 and 2 are solved by the present scheme (4) with division numbers $N_1 = N_2 = 64, 128, 256$ and 512. In Example 1, $U_0 = 1/2$. We choose $\Delta t = 10h$, which satisfies $[H_{wCFL}(\Delta t)]$ with $C_1 = 5$. Since the function (31) does not satisfy (1b) exactly, we impose the inhomogeneous Dirichlet boundary condition derived from (31). The left of Fig. 1 shows the graph of $Err \equiv \|\phi - \phi_h\|_{L^1(T)} / \|\phi\|_{L^1(T)}$ versus $\Delta t$ in logarithmic scale for all $\nu$. As mentioned in Remark 6, the theoretical convergence order of the present scheme under $[H_{wCFL}(\Delta t)]$ is $O(\Delta t^2 + h) = O(h)$, and it is $O(\Delta t)$ if $\Delta t = ch$. In the left of Fig. 1 we can see $Err$ with $\Delta t = 10h$ (white symbols) is almost of first order in $\Delta t$ for all $\nu$. The results are consistent with Theorem 2. Next, violating the condition $[H_{wCFL}(\Delta t)]$, we set $\Delta t = 2\sqrt{h}$. Even for this choice, the computations have been performed stably.
for all cases. The black symbols in the left of Fig. 1 show those results, where the convergence order seems to be almost $O(\Delta t^2)$. The results suggest that $[H_{wCFL}(\Delta t)]$ may be weakened in Theorems 1 and 2.

In Example 2 conditions (1b) and $[H_t(u)]$ hold and $U_0 = 2$. We choose $\Delta t = 2h$ and $\sqrt{h}/2$. The former relation satisfies $[H_{wCFL}(\Delta t)]$ with $C_1 = 4$. Although the latter violates $[H_{wCFL}(\Delta t)]$, the computation have been performed stably for all cases. The right of Fig. 1 shows the graph of $Err$ versus $\Delta t$ in logarithmic scale for all $\nu$. Since the results with black symbols for $\nu = 10^{-6}$ and $10^{-3}$ are approximately same, we cannot distinguish them in the graph. We can observe similar results to ones in Example 1, i.e., $Err$ with $\Delta t = 2h$ and $\sqrt{h}/2$ is almost of first and second order in $\Delta t$ for all $\nu$, respectively.

**Remark 7.** For the computation of $\left[ \Pi_h \phi_h^{n-1} \right] \odot X_2^n(x)$ in scheme (4), we have to find a pair $(i, j) \in \mathbb{Z}^{1/2} \times \mathbb{Z}^{1/2}$ such that $X_2^n(x) \in K_{i,j}$. For $y = X_1^n(x)$ it is written easily as
while it costs much more to find an element where $X^n_2(x)$ belongs in unstructured meshes.

Here we compare the present scheme with other implicit finite difference (FD) and finite element (FE) schemes. First, we roughly summarize features of the numerical schemes, which are classified into three types, characteristics FD, characteristics FE and others, cf. Table 1. In general, the convection term causes non-symmetric matrices and a solver for the non-symmetric sparse systems of linear equations is employed. The idea of the method of characteristics, however, makes schemes symmetric. Therefore, matrices of characteristics FD and characteristics FE are symmetric and the advantage reduces the computation cost of solving the system of linear equations. In the case of characteristics FE we need to compute integrals of the composite functions of the form

$$
\frac{1}{\Delta t} \int_\Omega \phi_h^{n-1} \circ X^n_1(x) \psi_h(x) dx,
$$

where $\phi_h^{n-1}$ and $\psi_h$ are FE functions. Since the composite function $\phi_h^{n-1} \circ X^n_1$ is not smooth enough, it is difficult to exactly compute the integral (34). Hence, for the integration we usually employ a quadrature formula.

Five schemes are picked up from the groups of schemes in Table 1, characteristics FE (C-FE) [12], central FD (FD1), P1-Galerkin FE (FE1), upwind FD (FD2) [18] and P1-SUPG-FE (FE2) schemes;

| Scheme          | Matrix | Quadrature |
|-----------------|--------|------------|
| char. FD        | sym.   | free       |
| char. FE        | sym.   | required*  |
| others          | non-sym.| free      |

Table 1. Features of numerical schemes (char.: characteristics;*: in the case of lumped-mass type the quadrature is free [14].)
FE2 is derived by applying the backward Euler method for time integration and P1-FE space for space discretization to [23, eq. (15)]. For C-FE a quadrature formula of degree two is employed.

Example 1 with \( \nu = 10^{-3} \) is solved by the six schemes including the present one. We use square meshes with \( N_1 = N_2 = 64, 128, 256 \) and 512, and set \( \Delta t = 10h \).

For the symmetric and non-symmetric matrices we employ CG (conjugate gradient) and CR (conjugate residual) methods (cf. [3, 17]), respectively. D-ILU preconditioner (cf. [3]) is applied for both solvers. The error for FE schemes is defined by

\[
\| \phi_h - \prod_h^{p1} \phi \|_{L^\infty (L^2(\Omega))} / \| \prod_h^{p1} \phi \|_{L^\infty (L^2(\Omega))},
\]

where \( \phi_h \) and \( \phi \) are the P1-FE and the exact solutions, respectively, and \( \prod_h^{p1} \) is the P1-interpolation operator.

The left of Fig. 2 shows a comparison of computation times, where DOF means “degrees of freedom”. We can observe that for every scheme each value of \( h = 1/512, 1/256 \) and 1/128 is roughly 10 times larger than that of \( h = 1/256, 1/128 \) and 1/64, respectively, that the values of FE2 are the largest, and that values of C-FD are the smallest. Since main part of the computation time is spent in solving the system of linear equations, the symmetric positive definite
matrices of C-FE and C-FD reduce considerably the computation cost (memory and time). As the graphs only show the total computation times, we add some detail of the contents. While the computation time of C-FE for solving the system of linear equations is shorter (about 0.6) than that of FE1 and FE2, C-FE requires the additional computation of the integrals (34), which yields computation times nearly equivalent to FE1. On the other hand, C-FD needs few time for the computation of the value \( \frac{1}{\Delta t}(\prod_k \phi_h^{t-1}) \circ X_2^n(x) \) (cf. (33)). This is the reason why the computation time of C-FD is the shortest of the six schemes. The present scheme has not only a property of second order accuracy in time but also a significant advantage of the low computation cost.

Finally, we show a comparison of errors in the right of Fig. 2, which exhibits error versus \( h \) in logarithmic scale. Under the relation \( \Delta t = 10h \) in Example 1 with \( v = 10^{-3} \), C-FD: scheme (4), C-FE: characteristics FE, FD1: central FD, FE1: P1-Galerkin FE, FD2: upwind FD and FE2: P1-SUPG-FE.

The computation was performed on an Intel Xeon processor X5690 (3.46 GHz).
7. CONCLUSIONS
We have presented a new characteristics finite difference scheme for convection-diffusion problems, which is of second order in $\Delta t$ and symmetric, and have analyzed it in the framework of the discrete $L^2$-theory. The finite difference scheme corresponds to the characteristics finite element scheme of second order in time in [15]. In the case of characteristics finite element methods we need to pay attention to numerical integration of composite functions. However, in the case of characteristics finite difference methods we do not need it. For scheme (4) we have proved that the scheme is stable and convergent in the discrete $L^2$-norm under the condition $U_0 \Delta t \leq ch$, and that the convergence order is $O(\Delta t^2 + h)$. The convergence order and the rather low computation cost have been recognized in the numerical results. It is possible to extend the present scheme to a higher order one with respect to $h$. It will be shown in a forthcoming paper [10].

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APPENDIX

A.1. Tools for the Proof of Lemma 1

Lemma A.1. (Properties of the “proportional weight”). Let \( w \in \mathbb{R}^2 \) be a constant vector, \( h \) be a mesh size and \( \Delta t \) be a time increment. The proportional weights \( \{ c_{i,j}^{l,m}(w; \Delta t, h) \}_{i,j,l,m \in \mathbb{Z}} \) defined by (11) have the following properties.

(i) \( c_{i,j}^{l,m}(w) \geq 0 \) (i, j, l, m \( \in \mathbb{Z} \)).

(ii) For any fixed integers \( i \) and \( j \) there are at most four non-zero values in \( \{ c_{i,j}^{l,m}(w) \}_{l,m \in \mathbb{Z}} \), and it holds that

\[
\sum_{x_{i,j} \in \Omega_i} c_{i,j}^{l,m}(w) = \sum_{x_{i,j} \in \Omega_i} c_{i,j}^{l,m}(w) = \sum_{l,m \in \mathbb{Z}} c_{i,j}^{l,m}(w) = 1.
\]

(iii) For any fixed integers \( l \) and \( m \) there are at most four non-zero values in \( \{ c_{i,j}^{l,m}(w) \}_{i,j \in \mathbb{Z}} \), and it holds that

\[
\sum_{x_{i,j} \in \Omega_i} c_{i,j}^{l,m}(w) = \sum_{x_{i,j} \in \Omega_i} c_{i,j}^{l,m}(w) = \sum_{l,m \in \mathbb{Z}} c_{i,j}^{l,m}(w) = 1.
\]

(iv) Assume \( v_h \in V_h \), \( i \) and \( j \in \mathbb{Z} \), and \( x_{i,j} - w \Delta t \in \Omega_i \). Then, it holds that

\[
\left( \sum_{(l,m) \in \Lambda_{i,j}} c_{i,j}^{l,m}(w) v_h(x_{i,m}) \right)
\]

Proof. Since the support of \( \phi_{l,m} \) is equal to \( \bigcup_{\alpha=\pm} K_{\alpha,\pm} \), the above results follow immediately from the definition (11). \( \square \)
Lemma A.2. Let \( w \in C^1(\Omega) \) be a velocity satisfying \( w \big|_\Gamma = 0 \) and \( W_0 \) and \( W_1 \) be positive constants defined by

\[
W_0 \equiv \max \{ |w(x)|_{\infty} : x \in \Omega \}, \quad W_1 \equiv \max \{ |\nabla w_j(x)|_{1} : x \in \Omega, j = 1, 2 \}.
\]

Let \( C_1 \) be any positive constant independent of \( h \) and \( \Delta t \). Assume \( \Delta t \) satisfies inequalities \( \Delta t < 1/\|w\|_{W^{1,\infty}(\Omega)} \) and \( \Delta t \leq C_1 h/W_0 \). Suppose \( x_{i,j} \) and \( x_{i,m} \in \Omega \) and \( x_{i,j} - w(x_{i,j}) \Delta t \in \text{supp}(\phi_{i,m}) \). Then, it holds that

\[
\left| c_{i,j}^{l,m} \left( w(x_{i,j}) \right) - c_{i,j}^{l,m} \left( w(x_{i,m}) \right) \right| \leq 2W_1 \Delta t (C_1 + 1), \tag{A.1}
\]

where \( c_{i,j}^{l,m} = c_{i,j}^{l,m} (; \Delta t, h) \).

Proof. From the Taylor formula we have

\[
\text{LHS of (A.1)} = \left| \phi_{i,m} \left( x_{i,j} - w(x_{i,j}) \Delta t \right) - \phi_{i,m} \left( x_{i,j} - w(x_{i,m}) \Delta t \right) \right|
\]

\[
\leq \frac{\Delta t}{h} \left| w_1 \left( x_{i,j} \right) - w_1 \left( x_{i,m} \right) \right| + \left| w_2 \left( x_{i,j} \right) - w_2 \left( x_{i,m} \right) \right|
\]

\[
\leq 2W_1 \frac{\Delta t}{h} \left( W_0 \Delta t + h \right) \quad \text{(by } x_{i,j} - w(x_{i,j}) \Delta t \in \text{supp}(\phi_{i,m})\text{)}
\]

\[
\leq 2W_1 \Delta t (C_1 + 1) \quad \text{(by } \Delta t \leq C_1 h/W_0\text{)}.
\]

\[\square\]

A.2. Tools for the Estimate of Truncation Errors

Let \( \hat{I} \) be the identity operator, \( \hat{\partial}_k \equiv \partial/\partial \hat{x}_k, \hat{\nabla} \equiv (\hat{\partial}_1, \hat{\partial}_2)^T, \hat{\epsilon}_k \equiv (\delta_{k1}, \delta_{k2})^T (k = 1, 2), \hat{x}_{ij} \equiv i\hat{e}_1 + j\hat{e}_2, \hat{\Lambda} \equiv \{(0, 0), (1, 0), (0, 1), (1, 1)\}, \)

\[
\hat{\phi}_{0,0}(\hat{x}) \equiv (1 - \hat{x}_1)(1 - \hat{x}_2), \quad \hat{\phi}_{1,0}(\hat{x}) \equiv \hat{x}_1(1 - \hat{x}_2), \quad \hat{\phi}_{0,1}(\hat{x}) \equiv (1 - \hat{x}_1)\hat{x}_2, \quad \hat{\phi}_{1,1}(\hat{x}) \equiv \hat{x}_1\hat{x}_2,
\]

and

\[
\left( \hat{\Pi} \hat{f} \right)(\hat{x}) \equiv \sum_{(i,j) \in \hat{\Lambda}} \hat{f}(\hat{x}_{ij}) \hat{\phi}_{i,j}(\hat{x}) \quad \left( \hat{f} \in C^0([0,1]^2) \right).
\]
Lemma A.3. Let $\hat{f} \in C^2([0, 1]^2)$ be a function and $\hat{x} \in [0, 1]^2$ be any point. Then, it holds that

$$\left( \hat{\Pi} - \hat{f} \right) \hat{f} (\hat{x}) = \sum_{(i,j) \in \Lambda} \hat{T}(\hat{x}; i, j) \hat{\phi}_{i,j}(\hat{x}), \quad \text{(A.2)}$$

where

$$\hat{T}(\hat{x}; i, j) \equiv \int_0^1 ds_1 \int_0^{s_1} \left[ \left\{ \hat{x}_{i,j} - \hat{x} \right\} \hat{V} \right]^2 \hat{f}(\hat{x} + \hat{s}_2 (\hat{x}_{i,j} - \hat{x})) ds_2.$$

Proof. The result follows from the identities, $\sum_{(i,j) \in \Lambda} \hat{\phi}_{i,j}(\hat{x}) = 1$ and $\sum_{(i,j) \in \hat{\Lambda}} \hat{x}_{i,j} \hat{\phi}_{i,j}(\hat{x}) = \hat{x}$.

By the scaling argument we obtain

Corollary A.1. Let $(\alpha, \beta) \in \{(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2})\}$ be a fixed pair, $(l, m) \in \mathbb{Z}^{\alpha+1/2} \times \mathbb{Z}^{\beta+1/2}$ be another pair, $v \in C^2(K_{l,m})$ be a function and $x \in K_{l,m}$ be any point. Then, it holds that

$$\left( \Pi_h^{(\alpha, \beta)} - I \right) v(x) = \sum_{(i,j) \in \Lambda^{(\alpha, \beta)}(x)} T(x; i, j) \phi_{i,j}(x),$$

where

$$T(x; i, j) \equiv \int_0^1 ds_1 \int_0^{s_1} \left[ \left\{ (x_{i,j} - x) \cdot \nabla \right\}^2 v \right](x + s_2 (x_{i,j} - x)) ds_2.$$

Moreover, it holds that

$$\left\| \left( \Pi_h^{(\alpha, \beta)} - I \right) v(x) \right\| \leq c h^2 \| v \|_{C^2(K_{l,m})}.$$

Lemma A.4. (i) Let $f \in C^2([-1, 1])$ and $F \in C^2([-\delta, \delta])$ be functions and $\delta$ be a positive number. Then, it holds that

$$\frac{1}{2} \{ f(1) + f(-1) \} - f(0) = \frac{1}{2} \int_0^1 ds_1 \int_{-s_1}^{s_1} f''(s_2) ds_2, \quad \text{(A.3a)}$$
\[ \Gamma_i(F; \delta) \equiv \frac{1}{2} \left\{ F\left(\frac{\delta}{2}\right) + F\left(-\frac{\delta}{2}\right) \right\} - F(0) = \frac{\delta^2}{8} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} F'' \left(\frac{\delta}{2} s_2 \right) ds_2 \] (A.3b)

(ii) Let \( F = F(\cdot; x, t^n) \in C^2[-\delta/2, \delta/2] \) be a function for \( x \in \bar{\Omega}_n \) and \( n = 1, \ldots, N_T \) and \( r^n_1: \bar{\Omega}_n \to \mathbb{R} \) be a function defined by

\[ r^n_1(x) \equiv \Gamma_1(F(\cdot; x, t^n); \delta). \]

Then, it holds that

\[ \left\| r^n_1 \right\|_{F'(F)} \leq \frac{\delta^2}{8} \left\{ \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} F'' \left(\frac{\delta}{2} s_2 \right) ds_2 \right\}^{1/2} \] (A.3c)

**Proof.** We prove only (A.3c) as proofs of (A.3a) and (A.3b) are easy. From (A.3b) and the Schwarz inequality we have

\[
 r^n_1(x)^2 \leq \left\{ \frac{\delta^2}{8} \right\} \left\{ \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} F'' \left(\frac{\delta}{2} s_2 \right) ds_2 \right\} \left\{ \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} F'' \left(\frac{\delta}{2} s_2 \right) ds_2 \right\} \\
 = \left( \frac{\delta^2}{8} \right)^2 \int_{-\delta}^{\delta} F'' \left(\frac{\delta}{2} s_2 \right) ds_2, 
\]

which implies (A.3c). \( \Box \)

**A.3. Estimates of the Truncation Error**

Here we evaluate each term \( R_i \) (\( i = 1, \ldots, 8 \)) of the truncation error \( R_{i,j} \) in (24c)–(24j). In the following Lemmas A.5–A.12 we prove only inequalities corresponding to the first one of (27b). The remaining proofs are similar to that of Lemma 3. The lemmas are used in Lemma 4.

**Lemma A.5.** Suppose \([H_i(u)], [H_i(\phi)], [H_1(\phi)]\) and \([H_i(\Delta t)]\). Then, there exists a positive constant \( M_1 \) such that

\[ \left\| R_1 \right\|_{F'(F)} \leq c \Delta t^2 M_1, \] (A.4a)
where $M_1$ satisfies

$$M_1 \leq c_3 \| \nabla \phi \|_{L^1_c}, \ c'_3 \| \nabla \phi \|_{L^3_c}.$$  \hspace{1cm} (A.4b)

**Proof.** Let $X(t) = X(t; x, t^n)$. There exist constants $\theta_1$ and $\theta_2$ $(0, 1)$ such that

$$R^n_1(x) = \frac{D\phi^{n-1/2}}{Dt} \circ Y^n_1(x) - \frac{D\phi^{n-1/2}}{Dt} \left( X \left( t^{n-1/2} \right) \right) = \left\{ Y^n_1(x) - X \left( t^{n-1/2} \right) \right\}$$

$$\nabla \frac{D\phi^{n-1/2}}{Dt} \left( \theta_1 Y^n_1(x) + (1 - \theta_1) X \left( t^{n-1/2} \right) \right), Y^n_1(x) - X \left( t^{n-1/2} \right)$$

$$= \left\{ X \left( t^n \right) - X \left( t^n \right) \Delta t \right\} - X \left( t^n - \frac{\Delta t}{2} \right) = -\frac{\Delta t^2}{8} X \left( t^n - \frac{\theta_2 \Delta t}{2} \right) \right.$$}

which implies the result. \hfill \Box

**Lemma A.6.** Suppose $[H_{2C}(u)]$, $[H_{1}(u)]$ and $[H_{3C}(\phi)]$. Then, there exists a positive constant $M_2$ such that

$$\| R^n_2 \|_{L^2_c(t^n)} \leq c \Delta t^2 M_2,$$  \hspace{1cm} (A.6a)

where $M_2$ satisfies

$$M_2 \leq c_4 \| \phi \|_{L^1_c}, \ c'_4 \| \phi \|_{L^3_c}.$$  \hspace{1cm} (A.6b)

**Proof.** Let $F$ be a function defined by

$$F(s) \equiv \phi \left( X \left( t^{n-1/2} + s \right), t^{n-1/2} + s \right),$$

where $X(t) = X(t; x, t^n)$. Then, there exists a $\theta \in (-1/2, 1/2)$ such that

$$R^n_2(x) = \frac{D\phi^{n-1/2}}{Dt} \left( X \left( t^{n-1/2} \right) \right) - \frac{\phi^n - \phi^{n-1}}{\Delta t} \left( X \left( t^{n-1} \right) \right)$$

$$= F'(0) - \frac{F(\Delta t/2) - F(-\Delta t/2)}{\Delta t} = -\frac{\Delta t^2}{24} F'''(\theta \Delta t)$$

$$= -\frac{\Delta t^2}{24} \frac{D^3 \phi}{Dt^3} \left( X \left( t^{n-1/2} + \theta \Delta t \right), t^{n-1/2} + \theta \Delta t \right),$$

which implies the result. \hfill \Box
Lemma A.7. Suppose \( [H_2 C(u)], [H_1(u)], [H_{0,1}(\phi)] \) and \( [H_u(\Delta t)] \). Then, there exists a positive constant \( M_3 \) such that

\[
\| R_3^n \|_{L^c(t^n)} \leq c \Delta t^2 M_3, \tag{A.7a}
\]

where \( M_3 \) satisfies

\[
M_3 \leq c_4 \| \phi \|_{C^0(C^1(\Omega))}, \quad c_4' \| \phi \|_{L^c(H^1(\Omega))}. \tag{A.7b}
\]

Proof. Let \( X(t) = X(t; x, t^n) \). There exists a \( \theta_1 \in (0, 1) \) such that

\[
R_3^n (x) = \frac{\phi^{n-1} \circ X_2^n (x) - \phi^{n-1} \left( X \left( t^{n-1} \right) \right)}{\Delta t}
= \frac{1}{\Delta t} \left[ X_2^n (x) - X \left( t^{n-1} \right) \right] \cdot \nabla \phi^{n-1} \left( \theta_1 X_2^n (x) + (1 - \theta_1) X \left( t^{n-1} \right) \right).
\]

By (A.5) there exist \( \theta_2 \in (-1/2, 1/2), \theta_3 \) and \( \theta_4 \in (0, 1) \) such that

\[
X_2^n (x) - X \left( t^{n-1} \right) = \left\{ x - u^{n-1/2} \left( x - u^n (x) \Delta t/2 \right) \right\} - X \left( t^{n-1} \right)
= \left\{ X \left( t^n \right) - X \left( t^{n-1/2} \right) \Delta t - X \left( t^{n-1} \right) \right\}
+ \left\{ u^{n-1/2} \left( X \left( t^{n-1/2} \right) \right) - u^{n-1/2} \left( Y_1^n (x) \right) \right\} \Delta t
= \frac{\Delta t^3}{24} \left[ X'' \left( t^{n-1/2} + \theta_2 \Delta t \right) + X \left( t^{n-1/2} \right) - Y_1^n (x) \right] \cdot \nabla u^{n-1/2} \left( \theta_3 X \left( t^{n-1/2} \right) + (1 - \theta_3) Y_1^n (x) \right) \Delta t
= \frac{\Delta t^3}{8} \left[ \frac{1}{3} X'' \left( t^{n-1/2} + \theta_2 \Delta t \right) + X'' \left( t^n - \frac{\theta_4 \Delta t}{2} \right) \right] \cdot \nabla u^{n-1/2} \left( \theta_3 X \left( t^{n-1/2} \right) + (1 - \theta_3) Y_1^n (x) \right). \tag{A.9}
\]

(A.8) and (A.9) imply the result. \( \square \)

Lemma A.8. Suppose \( [H_{0,1}(u)], [H_1(u)], [H_{0,2}](\phi) \) and \( [H_u(\Delta t)] \). Then, there exists a positive constant \( M_4 \) such that
where $M_4$ satisfies

$$M_4 \leq c \|\phi\|_{C^0(C^1(\bar{\Omega}))}, \quad c' \|\phi\|_{L^2(H^1(\Omega))}.$$  

(A.10b)

Proof. Let $g^n(y) = [(\Pi_{h} - I)\phi^{n-1}](y)$ for $y \in \bar{\Omega}$ and $x \in \Omega_h$. From $g^n(x) = 0$, there exists a $\theta \in (0, 1)$ such that

$$R_4^n(x) = \frac{[\frac{1}{2}d^2 (\Pi_{h} - I)\phi^{n-1}]}{\Delta t} \cdot \frac{X_2^n(x)}{\Delta t} = \frac{g^n \circ X_2^n(x) - g^n(x)}{\Delta t}.$$

(A.11)

The result follows from (A.11), because $g^n$ has a property $|\nabla g^n(y)| \leq ch\|\phi^n\|_{C^2(\bar{\Omega})}$ ($y \in \Omega$), which is derived from Corollary A.1.

\[\Box\]

Lemma A.9. Suppose $[H_{0,3}(\phi)]$. Then, there exists a positive constant $M_5$ such that

$$\|R_5\|_{L^2(I)} \leq chM_5,$$

(A.12a)

where $M_5$ satisfies

$$M_5 \leq c \|\phi\|_{C^0(C^1(\bar{\Omega}))}, \quad c' \|\phi\|_{L^2(H^1(\Omega))}.$$  

(A.12b)

Proof. The result follows from the fact that $\Delta_t \phi^n$ has a first order approximation of $\Delta \phi^n$ in space for $\phi^n \in C^3(\bar{\Omega})$. $R_5^n$ has the second order accuracy if $\phi^n \in C^4(\bar{\Omega})$. \[\Box\]

Lemma A.10. Suppose $[H_{0,2}(u)], [H_1(u)], [H_{0,3}(\phi)] \text{ and } [H_u(\Delta t)]$. Then, there exists a positive constant $M_6$ such that

$$\|R_6\|_{L^2(I)} \leq c_0 hM_6,$$

(A.13a)

where $M_6$ satisfies

$$M_6 \leq c_2 \|\phi\|_{C^0(C^1(\bar{\Omega}))}, \quad c' \|\phi\|_{L^2(H^1(\Omega))}.$$  

(A.13b)
Proof. Let \( R^n_{6k} \) \((k = 1, 2)\) be functions defined by
\[
R^n_{6k} \equiv \frac{1}{2} \left\{ \nabla_{hk} \bar{V}^{(n)}_{hk} \phi^{n-1} - \partial_k \left( \partial_k \phi^{n-1} \circ X^n_1 \right) \right\} \quad (k = 1, 2).
\]
Then, we have \( R^n_6 = \sum_{k=1}^{2} R^n_{6k} \). We consider only \( k = 1 \). \( R^n_{61} \) can be written as
\[
R^n_{61} = \frac{1}{2} \nabla_{h_1} \left\{ \left( \Pi_h^{(1/2,0)} \phi^{n-1} \right) \circ X^n_1 \right\} - \nabla_{h_1} \left\{ \left( \Pi_h^{(1/2,0)} \partial_1 \phi^{n-1} \right) \circ X^n_1 \right\}
+ \frac{1}{2} \nabla_{h_1} \left\{ \left( \Pi_h^{(1/2,0)} \partial_1 \phi^{n-1} \right) \circ X^n_1 \right\} - \nabla_{h_1} \left\{ \left( \partial_1 \phi^{n-1} \right) \circ X^n_1 \right\}
+ \frac{1}{2} \nabla_{h_1} \left\{ \left( \partial_1 \phi^{n-1} \right) \circ X^n_1 \right\} - \partial_1 \left\{ \left( \partial_1 \phi^{n-1} \right) \circ X^n_1 \right\}
\equiv R^n_{611} + R^n_{612} + R^n_{613}.
\]

Let \( g^n_1 = \{ (\nabla_{h_1} - \partial_1) \phi^{n-1} \} \), \( g^n_2 = \{ (\Pi_h^{(1/2,0)} - I) \partial_1 \phi^{n-1} \} \) and \( g^n_3 = \{ \partial_1 \phi^{n-1} \circ X^n_1 \} \) be functions. Then, it holds that, for \( x \in \Omega_h \),
\[
|R^n_{611}(x)| = \frac{1}{2} \left\| \nabla_{h_1} \left\{ \left( \Pi_h^{(1/2,0)} g^n_1 \right) \circ X^n_1 \right\} (x) \right\|_{W^{1\nu}(\Omega)} \leq \frac{1}{2} \left\| \left( \Pi_h^{(1/2,0)} g^n_1 \right) \circ X^n_1 \right\|_{W^{1\nu}(\Omega)} \leq \frac{1}{2} \left\| X^n_1 \right\|_{C'([\Omega])} \left\| g^n_1 \right\|_{C'([\Omega])},
\]
\[
|R^n_{612}(x)| = \frac{1}{2} \left\| \nabla_{h_1} \left( g^n_2 \circ X^n_1 \right)(x) \right\|_{W^{1\nu}(\Omega)} \leq \frac{1}{2} \left\| g^n_2 \circ X^n_1 \right\|_{W^{1\nu}(\Omega)} \leq \frac{1}{2} \left\| X^n_1 \right\|_{C'([\Omega])} \left\| \partial_1 \phi^{n-1} \right\|_{C'([\Omega])},
\]
\[
|R^n_{613}(x)| = \frac{1}{2} \left\| \nabla_{h_1} \left( g^n_3 \circ X^n_1 \right)(x) \right\|_{W^{1\nu}(\Omega)} \leq \frac{1}{2} \left\| g^n_3 \circ X^n_1 \right\|_{W^{1\nu}(\Omega)} \leq \frac{1}{2} \left\| X^n_1 \right\|_{C'([\Omega])} \left\| \partial_1 \phi^{n-1} \right\|_{C'([\Omega])}.
\]
Combining properties \( \|X_i^n\|_{c'((\Omega))} \leq c_k (k = 1, 2) \) and (A.15) with (A.14), we obtain that \( R_{61}^n \) is of first order in space and consequently (A.13).

**Lemma A.11.** Suppose \([H_{0,1}(u)], [H_1(u)], [H_{0,3}(\phi)]\) and \([H_{\Delta t}]\). Then, there exists a positive constant \( M_7 \) such that

\[
\|R_7\|_{L^1(\Omega)} \leq c_1 \left( \Delta t^2 + h^2 \right) M_7, 
\]

where \( M_7 \) satisfies

\[
M_7 \leq c_1 \|\phi\|_{C^0(\Omega)} \cdot c_1 \|\phi\|_{L^2(\Omega)}. 
\]

**Proof.** The key identity for the proof is

\[
\nabla \left\{ \left( \nabla \phi^{n-1} \right) \circ X_1^n \right\} = \sum_{i=1}^{2} \partial_i \left\{ \left( \partial_i \phi^{n-1} \right) \circ X_1^n \right\} 
= \left( \nabla \phi^{n-1} \right) \circ X_1^n - \Delta t \sum_{i,j=1}^{2} \partial_i u_j^n \left( \partial_i \partial_j \phi^{n-1} \right) \circ X_1^n. 
\]

Substituting (A.17) into the RHS of (24i), we have

\[
R_7^n = \frac{\Delta t}{2} \sum_{i=1}^{2} \partial_i u_i^n \left( \Delta_{h,i} - \partial_i^2 \right) \phi^{n-1} + \frac{\Delta t}{2} \left( \partial_2 u_1^n + \partial_1 u_2^n \right) 
\times \left( \nabla (2h)_1 \nabla (2h)_2 - \partial_1 \partial_2 \right) \phi^{n-1} + \frac{\Delta t}{2} \epsilon^n \equiv R_7^n + R_7^n + R_7^n, 
\]

where \( \epsilon^n \equiv \sum_{i,j=1}^{2} \partial_i u_i^n \left( \partial_i \partial_j \phi^{n-1} \right) - \left( \partial_i \partial_j \phi^{n-1} \right) \circ X_1^n \). Combining the inequalities,
with (A.18), we obtain the result.

Lemma A.12. Suppose \([H_{0,1}(u)], [H_{1}(u)], [H_{2c}(\Delta \phi)]\) and \([H_u(\Delta t)]\). Then, there exists a positive constant \(M_8\) such that

\[
\|R_8\|_{\mathcal{V}(\Omega)} \leq c_1 \Delta t^2 M_8, \tag{A.19a}
\]

where \(M_8\) satisfies

\[
M_8 \leq c_1 \|\Delta \phi\|_{L^2}, \quad c_1' \|\Delta \phi\|_{L^2}. \tag{A.19b}
\]

Proof. The result follows by regarding \(\Delta \phi\) as \(f\) in Lemma 3. \(\square\)