QuDi: A Classical Simulation Platform for Qubit-Qudit Hybrid Quantum Systems

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In the recent years, numerous research advancements have extended the limit of classical simulation of quantum algorithms. Although, most of the state-of-the-art classical simulators are only limited to binary quantum systems, which restrict the classical simulation of higher-dimensional quantum computing systems. Through recent developments in higher-dimensional quantum computing systems, it is realized that implementing qudits improves the overall performance of a quantum algorithm by increasing memory space and reducing the asymptotic complexity of a quantum circuit. Hence, in this article, we introduce QuDi: a state-of-the-art user-friendly python-based higher-dimensional quantum computing simulator. QuDi offers multi-valued logic operations by utilizing generalized quantum gates with an abstraction so that any naive user can simulate qudit systems with ease as compared to the existing ones. We simulate various benchmark quantum circuits in QuDi and show the considerable speedup in simulation time as compared to the other simulators without loss in precision. Finally, QuDi provides a full qubit-qudit hybrid quantum simulator package with quantum circuit templates of well-known quantum algorithms for fast prototyping and simulation. The complete code and packages of QuDi is available at https://github.com/LegacYFTw/QuDi so that other platforms can incorporate it as a classical simulation option for qubit-qudit hybrid systems to their platforms.

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1 INTRODUCTION

A significant progress has been made in quantum computing recently due to its asymptotic advantage over classical computing [15, 27, 28]. Moreover, with the introduction of the mathematical notion of a qudit in [26], the boundaries of quantum computing are extended beyond the binary state space. Qudits are states in a $d$-dimensional Hilbert space where $d > 2$, thus allowing a much larger state space to store and process information as well as simultaneous control operations [6, 7, 12, 14, 21, 30, 34]. It has been shown that usage of qudits leads to circuit complexity...
reduction as well as enhanced efficiency of some quantum algorithms. For practical demonstrations, qudits have been realized on several different hardware, including photonic quantum systems [17], ion-trap systems [22], topological quantum systems [5, 9, 10], superconducting systems [23], nuclear magnetic resonance systems [13, 18], continuous spin systems [1, 3] and molecular magnets [25].

In order to continue innovation in the multi-valued logic space for quantum computing, an efficient, easy to use quantum computing simulator with the support of qudits is the need of the hour. Given the size of the unitaries in the qudit space, one also needs to consider the computational costs incurred while simulating such large complex systems, which presents a significant challenge even in the binary state-space simulators. The inception of this simulator was in the wake (or lack thereof) of accessible simulators that were capable of simulating multi-valued logic in an user-friendly manner. This meant that a lot of research time and effort was previously spent in the construction of matrices and checking their compatibility, dimensions and kronecker products manually before their output could be deciphered from a huge 1D-array [4, 11, 19]. As an example, let say a generalized gate is imposed on five qudits in a 4-dimensional quantum system. For simulation purpose, the matrix of that generalized gate of \(4^5 \times 4^5\) i.e., 1024 \(\times\) 1024 needs to be prepared manually, which apparently makes the simulation time-consuming and error-prone.

Our proposed simulator, named QuDiet, claims to solve all that by providing suitable abstractions that shy the user away from behemoth calculations and focus on purely the logic building and quantum phenomenology in the higher dimensional space. QuDiet does this thanks to its simple, yet effective, lean architecture that could be used to debug implementations quickly and provide outputs without unnecessary computational or memory overhead. It also provides the users with the flexibility of adding gates accordingly to a quantum circuit with only a few commands.

The main contributions of this work are as follows:

- The first of its kind proposal for a simulator based on higher-dimensional state space, multi-valued logic, utilizing generalized quantum gates.
- Using sparse matrices and related algorithms at the core of all quantum operations to unlock potential speed-up.
- Using GPU acceleration and efficient memory maps to process large matrices with considerable speedup.
- Benchmarking multiple quantum circuits in qudit systems and showing overall simulation time for the different backends for the first time to the best of our knowledge.
- A full package with quantum circuit templates for fast prototyping and simulation.

The structure of this article is as follows. Section. 2 describes the higher-dimensional quantum circuit and its classical simulation. Section. 3 proposes the higher-dimensional quantum simulator, QuDiet. Section. 4 analysis the efficiency of the proposed simulator with the help of benchmark circuits. Future scope of the proposed simulator is outlined in Section 5. Section. 6 captures our conclusions.

2 PRELIMINARIES

In this section, firstly, we discuss about qudits and generalized quantum gates. Later we put some light on classical simulation of a higher-dimensional quantum circuit.

2.1 Higher-dimensional quantum circuits

Any quantum algorithm can be expressed or visualized in the form of a quantum circuit. Commonly for binary quantum systems, logical qubits and quantum gates comprise these quantum circuits [2]. The number of gates present in a circuit is called gate count and the number of qubits present in a
circuit is known as qubit cost. In this work, we mainly deal with qudits and generalized quantum gates since our simulator is based on higher-dimensional quantum computing.

2.1.1 Qudits. A logical qudit that encodes a quantum algorithm’s input/output in $d$-ary or multi-valued quantum systems is often termed as data qudit. Another sort of qudit used to store temporary findings is the ancilla qudit. The unit of quantum information in $d$-dimensional quantum systems is qudit. In the $d$ dimensional Hilbert space $H_d$, qudit states can be substantiated as a vector.

The vector space is defined by the span of orthonormal basis vectors $\{ |0\rangle, |1\rangle, |2\rangle, \ldots |d-1\rangle \}$. In qudit systems, the general form of quantum state can be stated as

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle + \cdots + \alpha_{d-1} |d-1\rangle$$

where $|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_{d-1}|^2 = 1$ and $\alpha_0, \alpha_1, \ldots, \alpha_{d-1} \in \mathbb{C}^d$.

2.1.2 Generalized Quantum Gates. In this section, a brief discussion on generalized qudit gates is exhibited. The generalization can be described as discrete quantum states of any arity in this way. Unitary qudit gates are applied to the qudits to evolve the quantum states in a quantum algorithm. It is required to take into account one-qudit generalized gates such as NOT gate ($X_d$), Phase-shift gate ($Z_d$), Hadamard gate ($F_d$), two-qudit generalized CNOT gate ($C_{X,d}^d$) and generalized multi-controlled Toffoli gate ($C_{X, d}^n$) for logic synthesis of quantum algorithms in $d$-dimensional quantum systems. For better understanding, these gates are described in detail:

**Generalized NOT Gate:** $X_d^a$, the generalized NOT can be defined as $X_d^a \cdot |x\rangle = |(x + a) \mod d\rangle$, where $1 \leq a \leq d - 1$. For visualization of the $X_d^a$ gate, we have used a ‘rectangle’ ($\Box$). ‘X$_d^a$’ in the ‘rectangle’ box represents the generalized NOT.

**Generalized Phase-Shift Gate**

$Z_d$ is the generalized phase-shift gate represented by a $(d \times d)$ matrix as is follows, with $\omega = e^{2\pi i / d}$ henceforth:

$$Z_d = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \omega & 0 & \cdots & 0 \\
0 & 0 & \omega^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega^{d-1}
\end{pmatrix}$$

We have used $Z_d$ in the ‘rectangle’ ($\Box$) box to represent the generalized phase-shift gate.

**Generalized Hadamard Gate:** The superposition of the input basis states is produced via the generalized quantum Fourier transform, also known as the generalized Hadamard gate, $F_d$. The generalized quantum Fourier transform or generalized Hadamard gate, produces the superposition of the input basis states. We have used $F_d$ in the ‘rectangle’ ($\Box$) box to represent the generalized Hadamard gate. The $(d \times d)$ matrix representation of it is as shown below:
Generalized CNOT Gate: In a binary quantum system, a controlled NOT (CNOT) gate can achieve quantum entanglement, which is an unrivalled property of quantum mechanics. For $d$-dimensional quantum systems, the binary two-qubit CNOT gate is generalised to the INCREMENT gate:

\[ \text{INCREMENT} |x\rangle |y\rangle = |x\rangle |(x + a) \mod d\rangle \]

where 1 ≤ $a$ ≤ $d - 1$. In schematic design of the generalized CNOT gate, $C_{X,d}$, we have used a 'Black dot' (●) to represent the control, and a 'rectangle' (⊗) to represent the target. 'X$^a_d$' in the target box represents the increment operator.

The $(d^2 \times d^2)$ matrix representation of the generalized CNOT $C_{X,d}$ gate is as follows:

\[
C_{X,d} = \begin{pmatrix}
I_d & 0_d & 0_d & \ldots & 0_d \\
0_d & I_d & 0_d & \ldots & 0_d \\
0_d & 0_d & I_d & \ldots & 0_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0_d & 0_d & 0_d & \ldots & X^a_d \\
\end{pmatrix}
\]

where $I_d$ and $0_d$ are both $d \times d$ matrices as shown below:

\[
I_d = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\end{pmatrix}
\]

and,

\[
0_d = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

Generalized Multi-controlled Toffoli Gate: We expand the generalized CNOT or INCREMENT further to work over $n$ qudits as a generalized Multi-controlled Toffoli Gate or $n$-qudit Toffoli gate $C^n_{X,d}$. For $C^n_{X,d}$, the target qudit is increased by $a$ (mod $d$) only when all $n - 1$ control qudits have the value $d - 1$, where 1 ≤ $a$ ≤ $d - 1$. In schematic design of the generalized Multi-controlled Toffoli Gate, $C^n_{X,d}$, we have used 'Black dots' (●) to represent all the control qudits, and a 'rectangle' (⊗) to represent the target. 'X$^a_d$' in the target box represents the increment operator. The $(d^n \times d^n)$ matrix representation of generalized Multi-controlled Toffoli (MCT) gate is as follows:

\[
C^n_{X,d} = \begin{pmatrix}
I_d & 0_d & 0_d & \ldots & 0_d \\
0_d & I_d & 0_d & \ldots & 0_d \\
0_d & 0_d & I_d & \ldots & 0_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0_d & 0_d & 0_d & \ldots & X^a_d \\
\end{pmatrix}
\]

For the sake of simplicity, we decompose generalized Multi-controlled Toffoli gate into set of generalized CNOT gates in rest of the article [20, 29]. It is also exhibited that this decomposition of Toffoli gate can be logarithmic in depth as compared to linear depth while using conventional approach of using generalized $T$ gate. This depth reduction is also useful for implementing different
algorithms in quantum computing. A generalized Toffoli decomposition in a $d$-ary system using $|d\rangle$ state is shown in Figure 1.

\[
\begin{align*}
|q_0\rangle & \quad \equiv \quad |q_0\rangle \\
|q_1\rangle & \quad \equiv \quad |q_1\rangle \\
|q_2\rangle & \quad \equiv \quad X^{d+1} \\
\end{align*}
\]

Fig. 1. Generalized Toffoli in $d$-ary quantum systems — the control qudits in red circles activate on $|d-1\rangle$ and those in the blue dotted-circles activate on $|d\rangle$.

2.2 Classical simulation of a higher-dimensional quantum circuit

This section highlights how a program that runs on a classical computer can resemble the evolution of a quantum computer. Before jumping on that let us discuss about the challenges that the current qubit-only classical simulators are facing to simulate qudit systems or higher-dimensional quantum circuits.

- For the state-of-the-art qubit-only simulators [4, 11, 31, 32, 35], the simulators need to act on an $n$ qubit state with a $2^n \times 2^n$ matrix. The dimension of the matrices of the unitary gates are also quite straightforward as it is only based on qubit systems [24]. Due to the engineering challenge of maintaining the dimension of the matrices automatically for the qubit-qudit hybrid systems, the current simulators are unable to provide a solution to simulate the higher-dimensional quantum circuits efficiently, which is addressed in this paper.

- The other challenge is that it requires a significant amount of memory to store the higher-dimensional quantum state vectors and to perform matrix multiplication to simulate the generalized quantum gates. Hence, the current simulators reach long simulation times and memory limitations very quickly due to its conventional memory management. In this paper, we also address this issue by designing unitary matrix simulator with various backends to simulate qudit systems effectively.

Before explaining our proposed simulator more elaborately, we would like to discuss about the technicalities of the classical simulation of a higher-dimensional quantum circuit.

To simulate a higher-dimensional quantum circuit, we first need to specify the dimension of each qudit so that generalized quantum gates or quantum operations can act on a sequence of qudits effectively [11]. This can be done through a method, which returns a tuple of integers corresponding to the required dimension of each qudit it operates on, as an instance $(2, 3, 4)$ means an object that acts on a qubit, a qutrit, and a ququad. To apply a generalized gate to some qudits, the dimensions of the qudits must match the dimensions it works on. For example, for a single qubit gate, its unitary is a $2 \times 2$ matrix, whereas for a single qutrit gate its unitary is a $3 \times 3$ matrix. A two qutrit gate will have a unitary that is a $9 \times 9$ matrix ($3 \times 3 = 9$) and a qubit-ququad gate will have a unitary that is an $8 \times 8$ matrix ($2 \times 4 = 8$). The size of the matrices involved in defining mixtures and channels follow the same pattern.

After simulating higher-dimensional quantum circuit by considering the dimension of qudits and generalized gates appropriately, the size of the resultant state is determined by the product of the dimensions of the qudits being simulated. For example, the state vector output after simulating a circuit on a qubit, a qutrit, and a ququad will have $2 \times 3 \times 4 = 24$ elements. Since, circuits on qudits are always assumed to start in the computational basis state $|0\rangle$, and all the computational
basis states of a qudit are assumed to be $|0\rangle, |1\rangle, \ldots, |d - 1\rangle$. Measurements of qudits are assumed to be in the computational basis and for each qudit return an integer corresponding to these basis states. Thus measurement results for each qudit are assumed to run from $|0\rangle$ to $|d\rangle$ like $|0\rangle$ to $|1\rangle$ for qubit systems.

3 QUDEIT: A QUBIT-QUDEIT HYBRID QUANTUM SIMULATOR

The lean architecture of QuDiet has been laid out briefly in the subsequent subsections. Before that the flow on the user end can be summed in Figure 2. This figure shows the high-level description of QuDiet to understand the general features of the proposed simulator. First, the quantum algorithm is expressed as a quantum circuit with the help of either QuDiet’s QASM specification or simple python console. Next, this needs to be compiled to a specific quantum gate set of QuDiet. Finally, the quantum circuit is simulated with QuDiet to get the final outcome of the given quantum algorithm.

![Fig. 2. A user-level compilation flow of the QuDiet, where the input is a quantum circuit and the output is the probable quantum states of the input circuit.](image)

Now that we have a high level understanding of the compiler, let’s take a deep dive and look at the internals in the following subsections.

3.1 High level architecture

At its core, QuDiet is managed by two integral parts: The Moment and the OperatorFlow objects. These two, however, are simple vector objects that behave like a stack, during the compiler’s operation. This is portrayed in Figure 3. A circuit, in essence, is an OperatorFlow object at its heart, running on a Backend. Once a QuantumCircuit object has been instantiated, in the background an OperatorFlow object is also created, which consists of a single Moment object. This Moment object can carry two types of objects: A InitState, which is the representation of an initial state of a quantum circuit or an QuantumGate, an abstract class, inherited by all quantum gates that are implementable by the simulator.

Once the user has implemented a quantum circuit, using the available commands, the measure_all() function is invoked, thereby pushing a Moment carrying measurement gates to all quantum registers. The measurement gate is a symbolic gate that tells the compiler that a program routine has ended and can be executed. The execution occurs when the QuantumCircuit.run() method is invoked, using the circuit’s specified Backend object which has several interfaces for plug and play operability.

3.2 The Quantum Circuit

The quantum circuit is represented by the QuantumCircuit class in the simulator. Whenever a new quantum circuit is invoked, a QuantumCircuit object is instantiated. This QuantumCircuit object takes the following arguments for instantiation:

- q_regs: The dimensions of the quantum registers is represented as a heterogeneous list of integer dimensions. In other words, the dimension of the quantum ‘wire’, in order, or as a
Fig. 3. High level overview of the **QuDiet** quantum simulator. This figure shows the different parts of the quantum simulator, namely the InitState objects, with the subscripts representing the dimensions of the respective qudits, and the QuantumGate objects with the $C_{X_2}^3$ gate being a generalized version of the CNOT gate with the target acting on a qutrit whereas the control being a qubit. The QuantumGate objects are enclosed using the Moment object and the Moment objects are enclosed within the OperatorFlow object.

- **tuple** represents a homogeneous register of fixed length, with the first qudit acting as the Least Significant Bit (LSB) and the last qudit in the register acting as the Most Significant Bit (MSB).
- **cregs**: The length of the classical register (optional)
- **name**: A string that represents the name of the quantum circuit (optional)
- **init_states**: Represents the configuration of the initial states of the register. This is represented by an array of the same length as that of qregs. If none is provided, the registers get automatically initialized to $|0\rangle$'s.
- **backend**: This represents the Backend on which the quantum circuit is to be executed. There are four Backend objects to choose from, and are elaborated in subsection 3.7. The default backend to be used is the SparseBackend
- **debug**: A flag argument that forms the base of a debugger engine, implemented in a simple manner in this release cycle and shall be expanded upon in future versions for easy debugging and callback functions.

For example, in order to make a circuit with 3 qudits of dimensions 4,5,3 respectively, with initial states being $|0\rangle$, $|3\rangle$ and $|2\rangle$, that calculates using the CUDASparseBackend we just need the following lines of python3 code:

```python
qreg_dims = [4,5,3]
init_states = [0,3,2]
backend = CUDASparseBackend
qc = QuantumCircuit(qregs=qreg_dims, init_states=init_states, backend=backend)
```

This shows how easily we can simulate different dimensional qudits with this simulator. Let’s now take a closer look at these initial states, or more generally, the quantum states that they represent and how does **QuDiet** handle them.

### 3.3 Representation of quantum states

Any quantum simulator is incomplete without their interpretation of quantum states. This preliminary version of **QuDiet** assumes that quantum states as state vectors. This comes with a caveat that **QuDiet** can only deal with states, as represented by an array or a vector list. This will of course be...
improved upon in future releases where we hope to incorporate density matrices, tensor-network and ZX/ZH calculus based representations. But we shall stick to the notion that these state vectors would be represented by:

\[
|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle + \cdots + \alpha_{d-1} |d-1\rangle
\]

Therefore, a register of qudits \(|\psi_1\psi_2 \ldots \psi_n\rangle\) would now need to be represented as:

\[
|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle = \left(\begin{array}{c}
\alpha_{10} \\
\alpha_{11} \\
\alpha_{12} \\
\vdots \\
\alpha_{1d-1}
\end{array}\right) \otimes \cdots \otimes \left(\begin{array}{c}
\alpha_{n0} \\
\alpha_{n1} \\
\alpha_{n2} \\
\vdots \\
\alpha_{nd-1}
\end{array}\right)
\]

Unlike qubit-only quantum simulators, this presents an engineering challenge: An efficient method of storing these matrices and their computations in memory. On observation, we note that the number these quantum states are, in essence very sparse matrices with non-zero elements scattered around them. Therefore, it seemed natural to use the sparse array format for storing these state vectors.

In order to achieve this, QuDiet makes use of scipy’s Compressed Sparse Column array implementation. We shall be improving on this format to reduce latency of Sparse Matrix Vector multiplication (SpMV) and General Matrix Vector multiplication (GeMV). This is elaborated in the Section V. However we have added support for numpy matrices for small circuits.

This technique is also used when representing quantum operators or quantum gates as we shall see in the following subsection.

### 3.4 Representation of quantum gates

One of the key features of QuDiet is it’s ability to construct generalized gates and operators for quantum computing using multi-valued logic automatically. This is a standout feature and there does not exist a simulator in known literature that allows the construction of single and multi-qudit quantum gates and operators with the ease as that of QuDiet.

The QuDiet contains a limited gate set. These are as follows, and their descriptions are available in the Section 2:

1. NOT Gate (XGate)
2. Phase-Shift Gate (ZGate)
3. Hadamard Gate (HGate)
4. CNOT Gate (CXGate)
5. User Defined Gate (QuantumGate)
6. Measurement Gate
7. Identity Gate (IGate)

Each of these quantum gates take into account two things: The qudit register it is acting on, and the dimension of the qudit register. Using a user-defined acting register, these quantum gates are able to take into account the dimension of the acting register, and dynamically construct a gate
unitary at runtime. The XGate and the CXGate have an added functionality, that an arbitrary shift can be induced, as long as the shift has a value less than the dimension of the qudit register. The Measurement Gate in **QuDi**et is a special gate in that, it has no unitary associated with it. Up until this version, the Measurement Gate merely acts as flag that signifies the end of a quantum circuit. Any gate post measurement will be ignored by the simulator.

Once the unitary of a generic quantum gate has been created, it then utilizes the same sparse matrix format to store the data. Contrary to quantum states, however, quantum gates utilize scipy’s Compressed Sparsed Row matrix implementation. This format will of course, be improved upon to not only facilitate SpMV and GeMV but also SpGeMM or Sparse Matrix-Matrix multiplication and GeMM, or General Matrix-Matrix multiplication [16]. However we have added support for numpy matrices for small circuits.

The backends have been engineered to provide minimal speedup using naive algorithms, and has CUDA support for GPU executions. These backends are elaborated in subsection 3.7.

Therefore for demonstration, if we were to use the same quantum circuit as previously, in order to add a Hadamard gate acting on the first qudit and a CNOT gate with a shift of plus 2, acting on the first and the third qudit, we simply invoke the following lines of python3 code:

```python
qc.h(0)
qc.cx((0,2), plus=2)
```

**QuDi**et also decomposes Toffoli gates (Let’s say, `qc.Toffoli((0,1,2), plus=1)`) into suitably mapped higher order CXGate objects. This is given as follows:

```python
qc.cx((0,1), plus=1)
qc.cx((1,2), plus=1)
qc.cx((0,1), plus=2)
```

As stated before, these form the basis of a Moment object, which we shall elaborate on next.

### 3.5 The Moment

The Moment object is a forms an abstraction between the QuantumGate and the OperatorFlow objects, in that it maintains the orientation of the quantum gates that are acting on the respective qudits and the OperatorFlow maintains the sequence of execution of the quantum operations. The Moment object is inherently an array of length equal to the breadth of the quantum circuit. The primary job of the Moment object is to maintain the position of an acting register on the intended qudit so that errors in evaluating Kronecker products are avoided when the quantum circuit is executed.

There can be a single Moment object containing a list of InitState objects spanning across all the qudits in the quantum register and is initialized and pushed into the OperatorFlow object’s stack whenever a quantum circuit is initialized. The OperatorFlow object can hold an arbitrary number of Moment objects, containing quantum gates across the breadth of the quantum register.

As per the **QuDi**et, whenever the user invokes a quantum gate onto a quantum circuit, the QuantumGate is pushed into the specific register corresponding to the index in the Moment object. All other corresponding registers or indices in the Moment object shall contain the IdentityGate object.

Something to note here is that, the OperatorFlow and the Moment data structures only store the data when it is pushed. No kronecker product or matrix multiplication operation would be done until the Measurement gates would be pushed and the user invokes the `run()` method from the quantum circuit object.

**QuDi**et also performs preliminary optimizations at the logic level whenever a gate is pushed. Whenever a new gate is pushed into the OperatorFlow stack, the gate is enclosed in a Moment
object, while ensuring that an index in the Moment array corresponds to the acting qudit as specified by the user. All the other registers have an Identity gate acting on them. When pushing a Moment object containing a quantum gate into the OperatorFlow stack, the simulator checks if any other immediate predecessor Moment has an Identity gate in them at the same position, if so, the Identity Gate is swapped out for the currently incoming quantum gate. This is done until it reaches an InitState object or it finds a QuantumGate object in any of its immediate predecessors.

### 3.6 The Operator Flow Stack

The OperatorFlow object is at the heart of the QuDiet: it maintains the order of execution of QuantumGate objects nestled inside the Moment objects. The OperatorFlow inherently maintains a vector list which acts like a stack during execution.

In order to run a quantum circuit, a `measure_all()` is called. This places MeasurementGate objects, across all quantum registers, thereby raising flag variables inside a Moment object. Any Moment containing quantum gates pushed after the Moment object shall be discarded, post the `measure_all()` command. In order to execute the said circuit, the `run()` is invoked, thereby outputting the statevectors of the quantum states that have non-negative probabilities.

Under the hood, during this time, all circuits prior to the Moment containing the MeasurementGate objects are called to be executed in reverse order. This means that the OperatorFlow object will first ‘pop’ out that Moment and evaluate the Kronecker product depending on the type of Backend, inside the Moment and store it in a variable. Then it will advance onto the second-to-last Moment, evaluate the Kronecker product, and then store it in a separate variable. Now once these two Kronecker products have been evaluated, QuDiet will perform a SpGEMM or a GEMM operation, depending on the Backend selected for the circuit. When the matrix products have been evaluated, the two variable storing the Kronecker product is freed from memory. This continues until the Moment containing the InitState objects are reached where the last operation is and SpMV or a GEMV, depending on the Backend chosen.

### 3.7 Acceleration and the Backends

In QuDiet, acceleration can be achieved in two different ways. One is through a GPU, i.e., hardware acceleration, and the other is by using sparse matrices, i.e., algorithmic or software acceleration. These accelerations are delivered through different Backends, like CudaBackend. To access the GPU as a host for hardware acceleration, we use CuPy, which is a GPU equivalent for NumPy and SciPy. In terms of software acceleration, SciPy is used instead of NumPy, because of its useful interface for Sparse matrices. The very basis of QuDiet, lies in the availability and choice of Backends. For example, nearly dense matrices have showed no acceleration when run on the SparseBackend. On the contrary, it is a better choice to use CudaBackend instead and ignore the sparse like Backends when dealing with nearly dense matrices. A more detail discussion on Backends is carried out in next.

QuDiet is a model-level library, providing high-level building blocks for developing quantum circuit algorithms. The low-level operations such as dot product, kronecker product, etc. These low-level operations are interfaced through the class Backend. This enables one to use the best backend option based on the type of the circuit and the hardware accessible.

Right now, the backends accessible for use are NumpyBackend, SparseBackend, CudaBackend and CudaSparseBackend.

#### 3.7.1 NumpyBackend

NumpyBackend is the default backend used when no backend type is explicitly defined. This interfaces to the basic numpy operations, without any external optimization.
3.7.2 SparseBackend. SparseBackend interfaces to scipy’s sparse module instead of interfacing to numpy’s ndarray. It stores only the nonzero elements of the matrix and reduce the computation time by eliminating operations on zero elements. In cases, this can reduce the matrix size exponentially, showing an overall speedup in execution runtime and memory compression.

This speedup directly depends on the nature of the circuit, where the worst scenario is the arrays being mostly dense. In that case, the SparseBackend will perform nearly like a NumpyBackend.

3.7.3 CUDABackend. CudaBackend interfaces with CuPy’s Numpy Routine, using numpy-like dense matrices, accessing the GPUs with the purpose of runtime speedup only.

3.7.4 CUDASparseBackend. CUDASparseBackend interfaces with CuPy’s Scipy Routine, using sparse representation of the matrices and then accessing the GPUs with the purpose of runtime speedup.

The CUDASparseBackend optimizes the operations using sparse matrices, followed by hardware (GPU) optimization.

3.8 Output and Interpretability of results

Output Representation of a quantum circuit is still a less explored domain, especially when dealing with large circuits. QuDiet additionally provides some small contributions in terms of Output Representation and Interpretability for the sake of better research experience. QuDiet comes with two types of output representation, OUTPUTTYPE and two types of output method, OUTPUTMETHOD. OUTPUTTYPE is the way of output representation, which has two types, print and state. The OUTPUTTYPE.STATE provides the raw output state as a binary array. Whereas the OUTPUTTYPE.PRINT provides the output state as a ket string. On the contrary, OUTPUTMETHOD dictates whether the output would provide the probability distribution or the amplitude of the quantum states. By default, a quantum circuit returns the final ket representation along with the distribution probability, which looks like {\(|111010000000\rangle : 1.0\)} for an instance.

3.9 Example Workflow

Let us now, take an example to understand the flow of work in QuDiet. The circuit to be executed is shown in Figure 4.

![Figure 4. A quantum circuit to be simulated using QuDiet](image)

In order to do this, we must first create the quantum circuit by specifying the dimensions of the circuit lines in the quantum circuit, this is done by the following lines of python code:

```python
qreg_dims = [2, 3, 3]
init_states = [0, 0, 0]
backend = SparseBackend
qc = QuantumCircuit(qregs=qreg_dims, init_states=init_states, backend=backend)
```
These lines of code do the following:

1. The `InitState` objects are created for each of the circuit lines. Each of these `InitState` objects have the following matrices:

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

for the initiated circuit lines respectively.

2. The `Moment` object is initialized and then the `InitState` objects are pushed into the `Moment`. This object is then pushed into the `OperatorFlow` object's stack.

Now that the circuit has been initialized with the desired states, we can now add in the required gates. These gates are invoked by calling the respective methods of the circuit, which then creates the respective objects of the quantum gates. These quantum gates, at the time of their creation, take into account the dimensions of the acting register, among other factors, to construct the correct unitary automatically and push it into the `Moment` object, which is then pushed into the `OperatorFlow` object. This is done by the following lines of code:

```python
qc.h(0)
qc.cx((0,1), plus=2)
```

These lines of code does the following:

1. The first line of the above snippet detects the dimension of the acting register. Since the dimension of the acting register is 2, it shall construct a $2 \times 2$ unitary suitably as follows:

\[
\begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{pmatrix}
\]

Once done, it will assign it to the instance variable of the `HGate` object.

2. Before pushing the `HGate` object to the `Moment` list, `QuDiet` will create Identity gates tailored to the dimensions of the qudit register and push it into the `Moment` object.

3. Next, `QuDiet` will create the `CXGate` object using a generated unitary and then place the Identity gates before pushing it into the `Moment` object.

The state of the `OperatorFlow` object and its internals are given as follows in Figure 5.

Note that the tensor and inner products are not evaluated at the time of creation. In order to begin the process of execution, we invoke the following lines of code:

```python
qc.measure_all()  
qc.run()
```

These will add measurement operators, interpreted here as flag variables, to signify the end of all operations within a circuit. The circuit is then executed with the `run()` command, which begins calculating the tensor and the inner products using the backend specified. The final output is as follows:

```
Build elapsed: 0.0001919269561767578s  
Execution elapsed: 0.0010845661163330078s
```

```
[ {\'\mid000\>\': 0.7071067811865475}, {\'\mid120\>\': 0.7071067811865475} ]
```

The final simulation result comes with loading-time and execution-time of the given circuit as shown in the above example. We also obtain the final output quantum states as $|000\>$ and $|120\>$ with amplitude $0.7071067811865475$ for the example circuit.
4 EXPERIMENTS AND DISCUSSION

Apart from python console, QuDiet offers another form of circuit specification i.e., QuDiet’s QASM. With the increasing usage of quantum circuit description, QASM (Quantum Assembly Language) [33] was introduced. Through QuDiet’s QASM, one can declare the qubits or qudits and can describe the operations (gates) on those qubits or qudits to be run on QuDiet. For ease of understanding, a sample QASM program on QuDiet is presented as following:

```
.qudit 3
qudit x0 (2)
qudit x1 (3)
qudit x2 (3)
.begin
X x0
H x0
Z x0
X x1
X X2 2
CX x0 x1
CX x1 x1 2
.end
```

In this QASM program, we declare 3 qudits, one is qubit (x0), one is qutrit (x1) and last one is qutrit (x2). NOT, Hadamard and phase gate are applied on qubit x0. Then a generalized NOT gate with +1 is applied on qutrit x1 followed by a generalized NOT gate with +2 is applied on qutrit x2. A generalized CNOT with +1 is on x0 and x1 and a generalized CNOT with +2 is on x1 and x2. The most widely used QASM, i.e., OpenQASM [8] can be converted to the QuDiet’s QASM form with the help of inbuilt lexer that is available in QuDiet to make it more user-friendly.

**Benchmarking with state-of-the-art simulators.** We have taken 21 benchmark circuits as an initial benchmarking, ranging from 3 qubit-qutrit to 7 qubit-qutrit in the form of QASM from [36] to verify our proposed simulator. To simulate all the 21 circuits, Toffoli gate is decomposed with intermediate qutrits as discussed earlier to get the algorithmic advantage. The simulation results are shown in Table 1. The complete simulation time is based on three different parameters, (i)
We run these circuits with two backends, Numpy and Sparse. The maximum run-time (loading-time + execution-time) of these circuits is 0.3 seconds, which is akin to [11], albeit the total simulation-time is much lower since the prepossessing-time is much higher for [11] as gates are needed to be defined manually based on dimensions. The exact prepossessing-time of [11] can never be determined since it is manual and very complicated to be defined mathematically. In our case, QuDiet being fully automatic, the prepossessing-time is negligible.

We further simulate more larger circuits on our proposed simulator. The results are shown in Table 2. These 17 medium-sized circuits are also taken from [36]. It is exhibited through numerical simulation that these circuits are well executable with Sparse-cuda backend due to its dense nature as compared to other backends. It can also be noted that if these 38 qubit-only circuits without decomposition from Table 1 and 2 are simulated on QuDiet, the performance time is same as [4, 11].

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Table 2. Loading and Execution Times (in milliseconds) of some benchmark circuits on Sparse-cuda Backend

| circuit             | (width, depth) | loading-time | execution-time |
|---------------------|----------------|--------------|----------------|
| hwb4_49_tof         | (5, 51)        | 2.58         | 760.01         |
| one-two-three-v0_97_tof | (5, 56)      | 575.1        | 963.46         |
| mod8-10_177_tof     | (6, 89)        | 568.71       | 1656.48        |
| mod5adder_127_tof   | (6, 106)       | 462.74       | 696.1          |
| sf_274_tof          | (6, 148)       | 5.91         | 1717.76        |
| ham7_104_tof        | (7, 73)        | 3.18         | 2196.92        |
| C17_204_tof         | (7, 106)       | 4.4          | 5135.62        |
| majority_239_tof    | (7, 149)       | 5.9          | 5966.78        |
| sym6_145_tof        | (7, 945)       | 24.09        | 43371.93       |
| f2_232_tof          | (8, 293)       | 9.59         | 19534.77       |
| con1_216_tof        | (9, 227)       | 10.2         | 25159.1        |
| mini_alu_305_tof    | (10, 39)       | 784.24       | 884254.49      |
| sys6-v0_111_tof     | (10, 46)       | 7.4          | 2060.27        |
| wim_266_tof         | (11, 219)      | 7.87         | 997068.12      |
| dc1_220_tof         | (11, 435)      | 13.96        | 2329661.45     |
| 0410184_169_tof     | (14, 60)       | 6.58         | 337.46         |

We have employed the multiplication of $3 \times 2$ as an example to illustrate our simulator’s efficiency in designing a quantum multiplier with intermediate qutrit. In Figure 6(a), in light of the preceding example, a multiplier circuit has been provided in accordance with [36], in which all the qubits are initialized with $|0\rangle$. In this circuit, the first four qubits ($q_0 - q_3$) are the input qubits, where the first two qubits ($q_0$ and $q_1$) represent the number 3 by applying two NOT gates on them and the other two qubits ($q_2$ and $q_3$) represent the number 2 by applying NOT gate on qubit $q_2$. Subsequently, using Toffoli gates, we conduct a multiply operation on these qubits and store the result in ancilla qubits ($q_4 - q_7$). Now, using CNOT gates on an ancilla qubit $q_8$, we execute addition. Lastly, to obtain the resultant output of $3 \times 2$, we need to measure the qubits ($q_4$, $q_7$ and $q_8$). Additionally, each of the Toffoli gates shown in Fig. 6(a) are realized with the help of the intermediate qutrit method as shown in Fig. 6(b) to achieve asymptotic advancement of the circuit. Our numerical simulation on QuDiet also yields $3 \times 2 = 6$ appropriately. Our simulation results further show that if we use Numpy backend, then the loading-time is 34.631 (ms) and execution-time is 136.532 (ms). For Sparse backend, the loading-time is 17.157 (ms) and the execution-time is 1.865 (s) for the multiplier of $3 \times 2$ circuit, whose width is 9 and depth is 15.

Fig. 6. (a) Quantum Multiplier Circuit for the multiplication of $3 \times 2$; (b) Quantum Multiplier with Intermediate Qutrit for the multiplication of $3 \times 2$. 
4.1 Simulation of Some Well-known Quantum Algorithms

We simulate some well-known quantum algorithms like, Grover’s algorithm, Simon algorithm, Berstein-Vazirani algorithm etc on QuDiет considering different backends and amalgamate them as a package for faster prototyping. The result of the simulation is shown in Table 3. The trade-off of loading-time and execution-time between different backends for various quantum algorithms is exhibited in Table 3. As a concluding remark, using the QuDiет simulator, we also simulate a comparatively larger circuit of 20 qutrits on depth 80 quite comprehensibly. The simulation has been performed on a local computer with processor Intel(R) Core(TM) i5-6300U CPU 2.40 GHz 2.50 GHz, RAM 8.00 GB, and 64-bit windows operating system.

Table 3. Loading and Execution Times (in milliseconds) of some supremacy circuits on various Backend

| circuit (backend)       | (width, depth) | loading-time | execution-time |
|-------------------------|----------------|--------------|----------------|
| Grover_n2 (sparse)     | (2, 9)         | 2.55         | 25.76          |
| Grover_n2 (cuda)       | (2, 9)         | 1.16         | 248.16         |
| Grover_n2 (sparse-cuda)| (2, 9)         | 1.19         | 44.05          |
| Simon_n6 (sparse)     | (6, 13)        | 5.39         | 92.94          |
| Simon_n6 (cuda)       | (6, 13)        | 1.05         | 30.7           |
| Simon_n6 (sparse-cuda)| (6, 13)        | 2.11         | 183.22         |
| Seca_n11 (cuda)       | (11, 69)       | 180.8        | 13536.19       |
| Seca_n11 (sparse-cuda)| (11, 69)       | 524.42       | 12008.1        |
| SAT_n11 (cuda)        | (11, 139)      | 5.9          | 16861.99       |
| SAT_n11 (sparse-cuda)| (11, 139)      | 8.62         | 12229.81       |
| BV_n14 (cuda)         | (14, 16)       | 2.25         | 75112.75       |
| BV_n14 (sparse-cuda)  | (14, 16)       | 4.65         | 932159.35      |

5 FUTURE SCOPE

In this section, we provide a glimpse of the future version of QuDiет. In future, we can use a multi-threaded C++ core that uses AVX instructions, which can achieve state-of-the-art performance in matrix-vector multiplication of quantum states. The C++ core, which uses a similar syntax of numpy.dot can be integrated into our simulator whenever a matrix-vector multiplication is needed. Formation of a DAG interfaces for better optimization workflows can be incorporated in the future. Large-scale simulations on heterogeneous HPC clusters via MPI can also be looked into. Memorization and lookup tables for fast simulation can be a good case-study for future version of QuDiет. Simulation with noise models shall give a more broader picture of qudit simulation. Interfaces for different quantum hardware topologies shall need to be taken care in the future version of QuDiет to reduce the gap between physical and logical qudit simulation.

6 CONCLUSION

In this paper, we introduced QuDiет, a hybrid qubit-qudit simulator that can classically simulate any finite-dimensional quantum system. It is exhibited that the QuDiет offers user-friendly environment to simulate qudit systems with an abstraction. QuDiет is efficient since generalized gates can be easily used without spending much time to defining them manually during simulation. We also showed considerable speed-up in simulation time for benchmark circuits. Finally, we simulated some well-known quantum algorithms in qudit setting to analysis the performance of our proposed simulator QuDiет. Furthermore, other available platforms can integrate QuDiет as a classical
simulation option for higher-dimensional quantum systems to their platforms, since QuDiet is an open-source python-based simulator.

REFERENCES

[1] M. R. A. Adcock, P. Høyer, and B. C. Sanders. 2016. Quantum computation with coherent spin states and the close Hadamard problem. Quantum Information Processing 15, 4 (Jan 2016), 1361–1386. https://doi.org/10.1007/s11128-015-1229-0

[2] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter. 1995. Elementary gates for quantum computation. Physical Review A 52 (Nov 1995), 3457–3467. Issue 5. https://doi.org/10.1103/PhysRevA.52.3457

[3] S. D. Bartlett, H. de Guise, and B. C. Sanders. 2002. Quantum encodings in spin systems and harmonic oscillators. Physical Review A 65, 5 (May 2002). https://doi.org/10.1103/physreva.65.052316

[4] Luciano Bello, Jim Challenger, Andrew Cross, Ismael Faro, Jay Gambetta, Juan Gomez, Ali J. Abhari, Paco Martin, Diego Moreda, Jesus Perez, Erick Winston, and Chris Wood. 2021. Qiskit: An Open-source Framework for Quantum Computing. https://doi.org/10.5281/zenodo.2573505

[5] Alex Bocharov, Shawn X. Cui, Martin Roetteler, and Krysta M. Svore. 2015. Improved Quantum Ternary Arithmetics. arXiv:1512.03824 [quant-ph]

[6] Alex Bocharov, Martin Roetteler, and Krysta M. Svore. 2017. Factoring with qutrits: Shor’s algorithm on ternary and metaplectic quantum architectures. Physical Review A 96, 1 (Jul 2017). https://doi.org/10.1103/physreva.96.012306

[7] Ye Cao, Shi-Guo Peng, Chao Zheng, and Gui Long. 2011. Quantum Fourier Transform and Phase Estimation in dit System. Communications in Theoretical Physics 55 (05 2011), 790–794. https://doi.org/10.1088/0253-6102/55/5/11

[8] Andrew W. Cross, Lev S. Bishop, John A. Smolin, and Jay M. Gambetta. 2017. Open Quantum Assembly Language. https://doi.org/10.48550/ARXIV.1707.03429

[9] S. X. Cui and Z. Wang. 2015. Universal quantum computation with weakly integral anyons. Quantum Information Processing 14, 8 (May 2015), 2687–2727. https://doi.org/10.1007/s11128-015-1016-y

[10] S. X. Cui and Z. Wang. 2015. Universal quantum computation with metaplectic anyons. J. Math. Phys. 56, 3 (Mar 2015), 032202. https://doi.org/10.1063/1.4914941

[11] Cirq Developers. 2022. Cirq. https://doi.org/10.5281/zenodo.6599601 See full list of authors on Github: https://github.com/quantumlib/Cirq/graphs/contributors.

[12] Yao-Min Di and Hai Rui Wei. 2013. Synthesis of multivalued quantum logic circuits by elementary gates. Physical Review A 87, 1 (Jan 2013). https://doi.org/10.1103/physreva.87.012325

[13] S. Dogra, Arvind, and K. Dorai. 2014. Determining the parity of a permutation using an experimental NMR qutrit. Physics Letters A 378, 46 (Oct 2014), 3452–3456. https://doi.org/10.1016/j.physleta.2014.10.003

[14] Y. Fan. 2007. A Generalization of the Deutsch-Jozsa Algorithm to Multi-Valued Quantum Logic. In 37th International Symposium on Multiple-Valued Logic (ISMVL’07). IEEE Computer Society, Los Alamitos, CA, USA, 12. https://doi.org/10.1109/ISMVL.2007.3

[15] Edward Farhi and Sam Gutmann. 1998. Quantum computation and decision trees. Physical Review A 58, 2 (Aug 1998), 915–928. https://doi.org/10.1103/physreva.58.915

[16] Jianhua Gao, Weixing Ji, Zhaonian Tan, and Yueyan Zhao. 2020. A Systematic Survey of General Sparse Matrix-Matrix Multiplication. https://doi.org/10.48550/ARXIV.2002.11273

[17] X. Gao, M. Erhard, A. Zeilinger, and M. Krenn. 2020. Computer-Inspired Concept for High-Dimensional Multipartite Quantum Gates. Physical Review Letters 125, 5 (Jul 2020). https://doi.org/10.1103/physrevlett.125.050501

[18] Z. Gedik, I. A. Silva, B. Çakmak, G. Karpat, E. L. G. Vidoto, D. O. Soares-Pinto, E. R. deAzevedo, and F. F. Fanchini. 2015. Computational speed-up with a single qudit. Scientific Reports 5, 1 (Oct 2015). https://doi.org/10.1038/srep14671

[19] Andres Giraldo-Carvajal, Daniel A. Duque-Ramirez, and Jose A. Jaramillo-Villegas. 2021. QuantumSkynet: A High-Dimensional Quantum Computing Simulator. https://doi.org/10.48550/ARXIV.2106.15833

[20] Pranav Gokhale, Jonathan M. Baker, Casey Duckering, Natalie C. Brown, Kenneth R. Brown, and Frederic T. Chong. 2019. Asymptotic improvements to quantum circuits via qutrits. Proceedings of the 46th International Symposium on Computer Architecture (Jun 2019). https://doi.org/10.1145/3307650.3322253

[21] F. S. Khan and M. Perkowski. 2006. Synthesis of multi-qudit hybrid and d-valued quantum logic circuits by decomposition. Theoretical Computer Science 367, 3 (Dec 2006), 336–346. https://doi.org/10.1016/j.tcs.2006.09.006

[22] A. B. Klimov, R. Guzmán, J. C. Retamal, and C. Saavedra. 2003. Qutrit quantum computer with trapped ions. Physical Review A 67 (Jun 2003), 062313. Issue 6. https://doi.org/10.1103/PhysRevA.67.062313

[23] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. 2007. Charge-insensitive qubit design derived from the Cooper pair box. Phys. Rev. A 76 (Oct 2007), 042319. Issue 4. https://doi.org/10.1103/PhysRevA.76.042319
Turbasu Chatterjee, Arnav Das, Subhayu Kumar Bala, Amit Saha, Anupam Chattopadhyay, and Amlan Chakrabarti

[24] Ryan LaRose. 2019. Overview and Comparison of Gate Level Quantum Software Platforms. Quantum 3 (mar 2019), 130. https://doi.org/10.22331/q-2019-03-25-130

[25] M. N. Leuenberger and D. Loss. 2001. Quantum computing in molecular magnets. Nature 410, 6830 (Apr 2001), 789–793. https://doi.org/10.1038/35071024

[26] Ashok Muthukrishnan and C. R. Stroud. 2000. Multivalued logic gates for quantum computation. Physical Review A 62, 5 (Oct 2000). https://doi.org/10.1103/physreva.62.052309

[27] Michael A. Nielsen and Isaac L. Chuang. 2010. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press. https://doi.org/10.1017/CBO9780511976667

[28] J. Preskill. 2018. Quantum Computing in the NISQ era and beyond. Quantum 2 (Aug 2018), 79. https://doi.org/10.22331/q-2018-08-06-79

[29] Amit Saha, Ritajit Majumdar, Debasri Saha, Amlan Chakrabarti, and Susmita Sur-Kolay. 2022. Asymptotically improved circuit for a d-ary Grover’s algorithm with advanced decomposition of the n-qudit Toffoli gate. Phys. Rev. A 105 (Jun 2022), 062453. Issue 6. https://doi.org/10.1103/PhysRevA.105.062453

[30] Amit Saha, Sudhini Bikash Mandal, Debasri Saha, and Amlan Chakrabarti. 2021. One-Dimensional Lazy Quantum Walk in Ternary System. IEEE Transactions on Quantum Engineering 2 (2021), 1–12. https://doi.org/10.1109/TQE.2021.3074707

[31] Robert S. Smith, Michael J. Curtis, and William J. Zeng. 2017. A Practical Quantum Instruction Set Architecture. arXiv:1608.03355 [quant-ph]

[32] Damian S. Steiger, Thomas Häner, and Matthias Troyer. 2018. ProjectQ: an open source software framework for quantum computing. Quantum 2 (Jan. 2018), 49. https://doi.org/10.22331/q-2018-01-31-49

[33] K Svore, A Cross, A Aho, I Chuang, and I Markov. 2004. Toward a Software Architecture for Quantum Computing Design Tools. Proceedings of the 2nd International Workshop on Quantum Programming Languages (QPL) (2004), 145–162.

[34] Y. Wang, Z. Hu, B. C. Sanders, and S. Kais. 2020. Qudits and High-Dimensional Quantum Computing. Frontiers in Physics 8 (Nov 2020). https://doi.org/10.3389/fphy.2020.589504

[35] Dave Wecker and Krysta M. Svore. 2014. LIQUi|J}: A Software Design Architecture and Domain-Specific Language for Quantum Computing. https://doi.org/10.48550/ARXIV.1402.4467

[36] Robert Wille, Daniel Grobe, Lisa Teuber, Gerhard W. Dueck, and Rolf Drechsler. 2008. RevLib: An Online Resource for Reversible Functions and Reversible Circuits. In 38th International Symposium on Multiple Valued Logic (ismvl 2008), 220–225. https://doi.org/10.1109/ISIMVL.2008.43

, Vol. 1, No. 1, Article 1. Publication date: January 2016.