A three-dimensional limit equilibrium method for analysing the basal upheaval stability of deep slender foundation pits

Chenghua Wang 1*, Yajie Li 1
1 School of civil engineering, Tianjin University, Tianjin, 300354, China*
*Corresponding author’s e-mail: Chwang@tju.edu.cn

Abstract. Deep slender foundation pits, which are of small plane dimensions and deep retaining structures, have been increasingly used in engineering practice. Spatial effects of this type of foundation pits are obvious, but the existing stability analysis methods are not suitable for such deep foundation pits. Therefore, three-dimensional basal upheaval stability analysis methods considering the spatial effect are badly needed. On basis of former tests and numerical solutions, a new model of upheaval failure for deep slender foundation pits was suggested by application of the limit equilibrium technique for entire foundation pits, considering the influences of spatial effects of the retaining structures. A global factor of safety against upheaval instability was defined and the applicability of the method was demonstrated with some parametric analyses.

1. Introduction
A deep slender foundation pit is generally referred as a pit that with smaller plain dimensions with respect to a great depth of the pit. With the continuing development in municipal engineering, deep slender foundation pits are becoming more and more popular and hence their safety and economy are hence very much concerned. As the spatial effects of this kind of foundation pits are very obvious, three-dimensional effects must be taken into the design of the retaining structures for the pits. The basal upheaval failure, which is referred as a very large swelling or upheaval deformation developed in the bottom of a deep pit while companied by significant sacking behind the retaining wall of the pit with the excavation process, is one of the main issues in design works of the foundation pits, but the methods for analysing basal upheaval failure used nowadays are just for plain strain cases and inadequate in case that the foundation pit is deep enough and hence spatial effects are too strong to be neglected. Therefore, it is very urgent to study the analysis methods for evaluating basal upheaval stability of deep slender pits.

The basal upheaval stability is affected by many factors and the basal failure mechanism is very complicated, hence attracted many attentions among researchers. Nowadays, there three types of methods, i.e. limit equilibrium methods, limit analysis methods and numerical methods for analysing the upheaval stability problem of foundation pits. The limit equilibrium methods though suffered with the assumptions in sliding surfaces and are considered that they rarely yield rigorous solutions, but they are mostly traditional and relatively simple in principles and easy to use. Limit analysis methods are quite complicated in analysing stress fields in complex geological and geometry conditions, which make them faces many difficulties in application in practical engineering. Though there are some researches in three-dimensional analysis of basal upheaval stability, spatial effects in mechanical aspect had been not included [1] [2] or the spatial effects were considered not suitable with cases where the retaining structure were hollow slenderized [3] [4]. Therefore, it is crucial to solve the stability problems in deep slenderer foundation pit. On the basis of former analytical solutions, a new model of upheaval failure for deep hollow slender foundation pit was suggested with consideration of corner effects in global
retaining structure, and a factor of safety against upheaval instability in corresponding was defined and the applicability was demonstrated with some parametric analyses.

2. Three-dimensional basal upheaval failure model

For a deep slenderer foundation pit with dimensions smaller in plain but relatively larger in depth as shown in Figures 1 and 2, where \( a \) is the length of the pit, \( b \) the width and \( D \) the embedded depth for retaining structure, while \( H \) is the total depth of excavation. The thickness of the retaining structure is \( t \), while \( q \) represents the surcharge loading on ground surface behind the retaining walls. As there are very strong spatial effects, according to recent results of model experiments and numerical simulations [5], a three-dimensional slip surface combining plenary and partial ellipse surfaces for describing the basal upheaval failure of deep slender foundation pits.

With reference of the failure models of shallow foundations, the possible failure case as shown in Figure 2, where lower part retaining structure which penetrate into base soil body was shown. For the cases in deep hollow slender pits, the most possible failure pattern is shown in Figure 2, which shows the form of failure profile around the pit, with two slip surfaces contact under the centre of bottom, due to the smallness of the plenary dimensions.

![Figure 1. The plan layout of a foundation pit](image1)

![Figure 2. The failure profiles in ground of a pit.](image2)

In Figure 2, the two slip surfaces contact under the centre of bottom ground, without cross over each other. From comprehensive analysis of the results form model experiments and numerical simulations [5], the authors suggest that the failure pattern in Figure 2 is adequate for describing the basal upheaval failure of deep slender foundation pits.

In this model, the range of failure zone outside the pit is shown in Figure 1 with outer dash line, as also shown is the rectangular ABCD pit plan with retaining sheet pile structure with length \( a \) and width \( b \). In the bottom the pit, the failure slip lines were assumed as in Figure 3(a) for the length to width ratio \( a/b>1 \), and as Figure 3(b) while \( a/b=1 \).

![Figure 3. Plane diagrams of slip lines in a pit: (a) \( a/b>1 \); (b) \( a/b=1 \)](image3)

3. Geometry of slip surfaces

3.1. Spatial shape of failure model

As mentioned above, an ellipse slip surface is adopted to describe the failure zone below the bottom of a pit. As shown in Figure 4, for a rectangular shape pit, taking the asymmetrical and three-dimensional view of the failure zone, for convenience the thickness is neglected in the Figure. In Figure 4(a), \( a, b, H, D \) are the dimensions stated earlier, while \( d \) denotes the width of failure soil mass outside the pit.
Figures 4(a) and (b), the digits 1 to 8 indicate the numbers of individual part of soil mass failed and a surcharge $q$ is assumed exerted on the top surfaces of mass number 1, 2, and 3 with the width $d$. The corresponding two-dimensional slip lines are also given in Figures 2 and 3.

![Figure 4](image)

Figure 4. A sketch of three-dimensional sliding surface of a pit: (a) dimension notations; (b) an inner view; (c) an outer view

3.2 Mathematical description of slip surfaces.

In order to analyse three-dimensional failure model for the basal upheaval stability of a deep slender foundation pit, taking the global retaining structure for the pit rather than its individual sections as the objective to exam, a coordination system is shown in Figure 5. Based on the structure and coordination in Figure 5, a composite circumferential axis (CCA) is defined as shown as the dark rectangular lines in Figure 5. The CCA is located at bottom level of the pit, and consisted of the four segmental axial lines, i.e., AB, CD, AC and BD, as it can be seen in Figures 5(a) and (b).

![Figure 5](image)

Figure 5. The coordination and composite circumferential axis in plan: (a) AB axis; (b) AC axis

For convenience, with reference of Figure 9, the origin of coordinates for axial line segments AB and CD is chosen at corner point A, with $y$ axis is in the normal direction of line segment AB, while the origin of coordinates for axial line segments AC and BD is also chosen at corner point but with $y$ axis is in the normal direction of line segment AC. The overall slip surface will be defined with reference on the setup of coordination system and the total factor of safety for stability of the pit against basal upheaval can be obtained from the calculation of partial factors of safety about each segmental axis.

For the failure model, the slip surface can be easily expressed mathematically with the setup of the coordinate system. The mathematical equations for description slip surfaces are summarized in Table 1 for axis AB and AC, with the notation of the ordered number of surfaces in brackets before the equations.

It is assumed that the Mohr-Coulomb's law is obeyed when slip failure occurred in foundation soils, and the internal friction angle $\phi$ in determination of the range of failure and corresponding slip surfaces to get the parameter $m$ in Table 1, and is given by equation (1).

$$m = \tan(45° - \frac{\phi}{2})$$

(1)
Table 1. Mathematical equation describing the slip surfaces

| Axis AB | Equation | Domain | Axis AC | Equation | Domain |
|---------|----------|--------|---------|----------|--------|
| 1       | \( y = d \) | \( 0 \leq x \leq a, \) \( 0 \leq z \leq H \) | 3       | \( y = d \) | \( -b \leq x \leq 0, \) \( 0 \leq z \leq H \) |
| 2       | \( x^2 + y^2 = d^2 \) \( -d \leq x \leq 0, \) \( 0 \leq y \leq (d^2 - x^2)^{1/2}, \) \( 0 \leq z \leq H \) | 2       | \( x^2 + y^2 = d^2 \) \( 0 \leq x \leq d, \) \( 0 \leq y \leq (d^2 - x^2)^{1/2}, \) \( 0 \leq z \leq H \) |
| 6       | \( m_y + z + D = 0 \) \( 0 \leq x \leq a, \) \( -b \leq y \leq 0, \) \( -D \leq z \leq -D - m_y \) | 4       | \( \frac{2}{d^2} \frac{y^2}{d^2} + \frac{z^2}{D^2} = 1 \) \( -D(1 - \frac{2}{d^2})^{1/2} \leq z \leq 0, \) \( 0 \leq y \leq d \) |
| 7       | \( \frac{2}{d^2} \frac{y^2}{d^2} + \frac{z^2}{D^2} = 1 \) \( 0 \leq x \leq a, \) \( -D(1 - \frac{2}{d^2})^{1/2} \leq z \leq 0, \) \( 0 \leq y \leq d \) | 5       | \( m_y + z + D = 0 \) \( -b \leq x \leq 0, \) \( -b \leq y \leq 0, \) \( -D \leq z \leq -D - m_y \) |
| 8       | \( \frac{2}{d^2} \frac{y^2}{d^2} + \frac{z^2}{D^2} = 1 \) \( -D(1 - \frac{2}{d^2})^{1/2} \leq z \leq 0, \) \( 0 \leq y \leq \sqrt{d^2 - x^2} \) | 8       | \( \frac{2}{d^2} \frac{y^2}{d^2} + \frac{z^2}{D^2} = 1 \) \( -D(1 - \frac{2}{d^2})^{1/2} \leq z \leq 0, \) \( 0 \leq y \leq \sqrt{d^2 - x^2} \) |

4. Definition of factor of safety against basal upheaval

4.1. The definition of global factor of safety

The global factor of safety against the instability of basal upheaval is defined as the ratio of anti-slide moment over the slide moment rotating about the composite circumferential axis (CCA). The sliding moment is induced by the gravity of soil mass and the surcharge loading exerted on the ground surface outside the pit, while the anti-sliding moment is from the action of self-weight of soil mass inside the pit and the resistance of soil friction on all parts of the sliding surface including the surfaces both outside and inside the pit. The corner part is divided into two parts with reference of the diagonal line in Figure 3, and the moments can be calculated respectively about the two rotating axes individually. In order to reflect the influence of all parts in two directions, the overall stability factor can be calculated with a weighted method, i.e. taking the ratio of the length of a side in one direction over the total length in two directions as the weight function for the factor of safety about the axis, as given by equation (2), after the partial factors of safety K1 and K2 of rotation about axis AB and axis AC are obtained, respectively.

\[
K_s = \frac{M_r}{M_s} = \frac{a}{a+b} K_1 + \frac{b}{a+b} K_2
\]  
(2)

Where, \( M_r \) is the global anti-sliding moment, and \( M_s \) is the global sliding moment; the partial factors of safety K1 and K2 can be obtained from equations (3) and (4), respectively.

For the partial factor K1 about the axis AB is defined in limit equilibrium analyses as follow:

\[
K_1 = \frac{M_6 + M_{R1} + 2M_{R2,AB} + M_{R6} + M_{R5} + 2M_{R8,AB}}{M_Q + 2M_{Q2,AB} + M_1 + 2M_{2,AB} + M_4 + 2M_{8,AB}}
\]  
(3)

Where, the M6, MR1, MR2AB, MR6, MR7 and MR8AB are the anti-sliding moments form parts of soil mass number 6, 1, 2, 6, 7, 8, respectively. The Mq1, Mq2AB, M1, M2AB, M7 and M8AB are the sliding moments form parts of soil mass of 1, 2, 1, 2, 7, and 8, respectively.

While the partial factor K2 about the axis AC is defined in limit equilibrium analyses as follows:

\[
K_2 = \frac{M_4 + M_{R3} + 2M_{R2,AC} + M_{R5} + M_{R4} + 2M_{R8,AC}}{M_Q + 2M_{Q2,AC} + M_3 + 2M_{2,AC} + M_4 + 2M_{8,AC}}
\]  
(4)
Where the $M_5$, $MR_3$, $MR_2AC$, $MR_5$, $MR_4$ and $MR_8AC$ are the anti-sliding moments form part of 5, 3, 2, 5, 4, and 8, respectively. The $Mq_3$, $Mq_2AC$, $M_3$, $M_2AC$, $M_4$ and $M_8AC$ are the sliding moments form part of 1, 2, 1, 2, 7, and 8, respectively.

In both Equations (1) and (3), the subscript $q$ denotes the cause of moment is surcharge $q$, and $R$ soil weight, while the subscript $AB$ or $AC$ indicate the axis $AB$ or $AC$ for a moment.

4.2. Sliding moments

The equations for calculating the sliding moments due to surcharge and the gravity of failure soil mass is listed in Table 2.

| Part No. | Equations of sliding moments | Axis | Causes |
|----------|-----------------------------|------|--------|
| 1        | $M_{q1} = \frac{1}{2} q ad^2$ | AB   | Surcharge |
| 2        | $M_{q2,AB} = M_{q2,AC} = \frac{\sqrt{2}}{6} q d^3$ | AC   | Surcharge |
| 3        | $M_{q3} = \frac{1}{2} q bd^2$ | AC   | Surcharge |
| 1        | $M_1 = \frac{1}{2} \gamma H ad^2$ | AB   | Weight |
| 2        | $M_{2,AB} = M_{2,AC} = \frac{\sqrt{2}}{6} d^3 \gamma H$ | AB, AC | Weight |
| 3        | $M_3 = \frac{1}{2} \gamma Hbd^2$ | AC   | Weight |
| 4        | $M_4 = \frac{1}{3} \gamma Db d^2$ | AC   | Weight |
| 7        | $M_7 = \frac{1}{3} \gamma Da d^2$ | AB   | Weight |
| 8        | $M_{8,AB} = M_{8,AC} = \frac{\sqrt{2}}{32} \pi \gamma D d^3$ | AB, AC | Weight |

4.3. Anti-sliding moments

The equations for calculating the anti-sliding moments due to the strength and the weight of failure soil mass is listed in Table 3.

| Part No. | Equations of anti-sliding moments | Axis | Causes |
|----------|----------------------------------|------|--------|
| 5        | $M_5 = \gamma \left( \frac{b^3 m}{96} - \frac{D b^3}{24} \right)$ | AC   | Weight |
| 6        | $M_6 = \gamma \left( \frac{b^3 D}{12} + \frac{ab^3 m}{24} - \frac{b^3 m}{32} - \frac{a D b^5}{8} \right)$ | AB   | Weight |
| 1        | $M_{H1} = d a H \left( m^2 \tan \varphi \left( q + \frac{\gamma H}{2} \right) + c \right)$ | AB   | Strength |
| 2        | $M_{H2,AB} = \frac{\sqrt{2}}{2} d^2 H \left( m^2 \tan \varphi \left( q + \frac{\gamma H}{2} \right) + c \right)$ | AB, AC | Strength |
| 3        | $M_{H3} = d b H \left( m^2 \tan \varphi \left( q + \frac{\gamma H}{2} \right) + c \right)$ | AC   | Strength |
| 4        | $M_{H4} = \int_0^d \left( (q + \gamma (H - z))(1 + m^4 \frac{D^4}{d^4} \frac{\gamma^3}{2}) \cdot \frac{z^2 d^4 + \gamma^4 D^3}{z^2 d^4 + \gamma^4 D^3} \tan \varphi + c \right) \left( \frac{D z}{z^2} \right) dy$ | AC   | Strength |
5. Some parametric analyses

In order to show the validity and basic function of the method, a simple case is used for some parametric analyses. A set of basic parameters, i.e., the influencing factors for the case are listed in Table 4. In making the parametric analysis of a single parameter, take the parameter vary within a range while others remain unchanged. The influences of the soil friction angle, unit weight and the width of the pit on the factor of safety against basal upheaval instability are analysed herein.

5.1 The influence of soil strength parameters

Taking the internal friction angle φ of soil as a variant changing from 5° to 30°, for each value of soil cohesion, while remain other parameters unchanged, the variation of factor of safety with the increase in soil is shown in Figure 6. for each value of soil cohesion, the tendency of the curves is similar, shown increase with the increase in internal friction angle φ. The higher is the cohesion, the greater the factor of safety.

Table 4. Parameters for parametric analyses

| Influencing factor                  | Value |
|-------------------------------------|-------|
| Depth of pit H (m)                  | 10.0  |
| Embedded depth of wall D (m)        | 8.0   |
| Length of pit a (m)                 | 5.0   |
| Width of pit b (m)                  | 3.0   |
| Surcharge loading q (kPa)           | 10.0  |
| Unit weight of soil γ (kNm⁻³)       | 18.0  |
| Cohesion of soil c (kPa)            | 10.0  |
| Internal friction angle of soil φ  (°) | 10.0  |

Figure 6. The influences of soil cohesion and friction angle on safety factor
5.2 The influence of soil unit weight

The influence of unit weight $\gamma$ of soil on the factor of safety is studied by varying the unit weight from 15.0 kNm$^{-3}$ to 20.0 kNm$^{-3}$, while remain all other parameters unchanged. Figure 7 shows a curve of the relation between factor of safety and the unit weight of soil. The curve indicates that the factor of safety decrease with the increase of the unit weight of soil. From the curve factor of safety changed quite slowly with the unit weight, as the soil mass gravity contribute to both sliding moment and anti-sliding moment.

![Figure 7. Relationship between unit weight and safety factor](image1)

![Figure 8. Relationship between pit width and safety factor](image2)

5.3 The influence of pit dimensions

There two aspect of pit dimensions that can influence the basal upheaval stability behaviour of a pit depth of the pit and the plane dimension, i.e., the length and the width of the pit. For a pit of given depth and length, the pit shape can be analysed in terms of its width. By changing the pit width $b$ from 2 m to 5m, with respect to the definite pit length $a = 5$ m, while again maintain the other factors unchanged, the influence of pit dimension on the factor of safety is representatively given in Figure 8. As can be seen from Figure 8, the factor of safety against pit upheaval failure increases gently with the increase of the pit width, as the shape of the pit become more and more squared rather than narrow ditch shaped and exhibiting more spatial effects.

6. Conclusions

A three-dimensional model for the basal upheaval failure of deep slender foundation pit was suggested based on earlier experimental and numerical simulations, in which slip failure in soil mass was assumed take place and the failure soil mass rotate about the composite circumferential rotation axis of the pit. The corresponding spatial slip surface of upheaval failure in soil mass was mathematical described.

The global factor of safety against the basal upheaval failure of deep slender foundation pit was defined as according to the failure model and can be calculated with the weighted partial factors of safety in two directions that are perpendicular to each other.

The parametric analyses of the influence of several factors including cohesion, internal friction angle, and unit weight of foundation soil and dimensions of foundation pit are briefly made. The result of the analyses indicates that the general tendency in the variation of the safety with the changes of the influencing factors is in accordance with engineering experience, especially in spatial effects.

Acknowledgments

The authors wish to express their grateful thanks to the National Natural Science Foundation of China who financially supported this work (Grant No. 51478313).

References

[1] Zhang, Z., Feng, H., Liu, G.B. A Three dimensional limit equilibrium method for basal stability analysis. Journal of Shenyang Jianzhu University, 2012, 28(1): 37-43. (in Chinese)
[2] Wang, H.X. Stability safety factor of foundation pit anti uplift considering two-dimensional and three-dimensional size effects. Journal of Geotechnical Engineering, 2013, 35(11):2144-2152. (in Chinese)
[3] Zhao, F. Stability analysis of anti-uplift of circular foundation pit based on limit analysis upper limit method. Chongqing: Chongqing University. 2013. (in Chinese)

[4] Cai, F., Ugai, K., Hagiwara T. Base stability of circular excavations in soft clay. Journal of Geotechnical & Geoenvironmental Engineering, 2002, 128(8):702-706.

[5] Li, Y. A model experimental study and numerical simulations of the upheaval vs. overall stability of deep slender foundation pits, A thesis submitted to Tianjin University in partial fulfillment of the requirement for a master’s degree of science. Tianjin: Tianjin University, 2017. (in Chinese)