ON SMOOTHNESS OF THE ELEMENTS OF SOME INTEGRABLE TEICHMÜLLER SPACES

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Abstract. In this paper we focus on the integrable Teichmüller spaces $T^p$ ($p > 0$) which are subspaces of the symmetric subspace of the universal Teichmüller space. We prove that any element of $T^p$ for $0 < p \leq 1$, is a $C^1$-diffeomorphism.

1. Introduction

The universal Teichmüller space $T$ is the space of quasisymmetric homeomorphisms of the unit circle $S^1$ fixing 1, $i$, and $-1$. A mapping $f : S^1 \to S^1$ is said to be quasisymmetric if there exists $M > 0$ such that

$$\forall \theta \in \mathbb{R}, \forall t > 0, \quad \frac{1}{M} \leq \frac{|f(e^{i(\theta+t)}) - f(e^{i\theta})|}{|f(e^{i\theta}) - f(e^{i(\theta-t)})|} \leq M.$$ 

Due to a well-known result by Ahlfors and Beurling [3] one can give an equivalent description of $T$. More precisely, the universal Teichmüller space can be defined as the set of Teichmüller equivalence classes of quasiconformal mappings of the unit disc $D$ fixing 1, $i$, and $-1$ where two such mappings are Teichmüller equivalent if they coincide on $S^1$. A mapping $F : D \to F(D)$, where $D \subset \mathbb{C}$ is a domain, is called quasiconformal (or q.c. for short) if it is an orientation-preserving homeomorphism and if its distributional derivatives $\partial_x F$ and $\partial_y F$ can be represented by locally square integrable functions (also denoted by $\partial_x F$ and $\partial_y F$) on $D$ such that

$$\left\| \frac{\partial_x F}{\partial_y F} \right\|_{\infty} = \sup_{z \in D} \left| \frac{\partial_x F(z)}{\partial_y F(z)} \right| < 1.$$ 

We also recall that for $z = x + iy$, $\partial_x = \frac{1}{2}(\partial_x + i\partial_y)$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. Furthermore, if $F$ is a quasiconformal mapping, the function $\mu_F = \frac{\partial_x F}{\partial_y F}$, defined a.e., is called the Beltrami coefficient associated with $F$. By the measurable Riemann mapping theorem, if a measurable function $\mu$ on $D$ is such that $\|\mu\|_{\infty} < 1$, then it is the Beltrami coefficient of some quasiconformal mapping, which we will denote here by $F^\mu$.

Let us now introduce an important subspace of $T$, namely, the symmetric Teichmüller space denoted here by $T_s$. Following a terminology introduced by Gardiner and Sullivan [14], it is the space of symmetric homeomorphism of $S^1$ fixing 1, $i$, and $-1$. One recalls that $f : S^1 \to S^1$ is symmetric if it is an orientation-preserving

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homeomorphism of $S^1$ such that
\begin{equation}
\frac{f(e^{i(t+i)}) - f(e^{it})}{f(e^{it}) - f(e^{i(-t)})} \to 1,
\end{equation}
with respect to the uniform convergence on $\mathbb{R}$. As for the universal Teichmüller space one has an equivalent description of such a space that involves quasiconformal mappings. Indeed, Gardiner and Sullivan proved (see Theorem 2.1 in [14]) that $T_1$ corresponds to the space of Teichmüller equivalent classes of quasiconformal mappings of $\mathbb{D}$ fixing $1, i$, and $-1$ admitting a representative which is \textit{asymptotically conformal} on $S^1$. Let us recall that a quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ is said to be asymptotically conformal on $S^1$ if for every $\epsilon > 0$, there exists a compact subset $K_\epsilon$ of $\mathbb{D}$ such that for any $z \in \mathbb{D} \setminus K_\epsilon$, $|\mu_F(z)| < \epsilon$.

Here we focus on some interesting infinite dimensional subspaces of $T$, the $p$-\textit{integrable Teichmüller spaces}, which we define for each $p > 0$ as the set
\[
T^p = \{ f \in T \mid \exists F : \mathbb{D} \to \mathbb{D}, \text{q.c. such that } F_{|z^1} = f \text{ and } \mu_F \in L^p(\mathbb{D}, \sigma) \},
\]
where $\sigma$ is the hyperbolic measure on $\mathbb{D}$, that is, for any $z = x + iy \in \mathbb{D}$, $d\sigma(z) = (1 - |z|^2)^{-2} dx dy$. It is elementary to observe from such a definition that if $q > p > 0$, then $T^p \subset T^q$. The spaces $T^p$, $p \geq 2$, were first introduced by Guo [15] through an equivalent description involving univalent functions. At about the same time, Cui [9] studied the case $p = 2$ and gave a few important characterizations of the elements of $T^2$. In particular, he proved that the Beltrami coefficient associated with the Douady–Earle extension (see [10]) of any element of $T^2$ belongs to $L^2(\mathbb{D}, \sigma)$. Later on, Takhtajan and Teo [22] introduced a Hilbert manifold structure on the universal Teichmüller space that makes the space $T^2$ the connected component of the identity mapping $id_{S^1}$. With respect to such a structure, they proved that the so-called \textit{Weil–Petersson metric} is a Riemannian metric on $T$. Following Takhtajan and Teo’s work, the space $T^2$ is now referred to as the \textit{Weil–Petersson Teichmüller space}. For further results on $T^2$ we refer to [20]. Let us point out that one can obtain $T^2 \subset T_1$ by combining [9] Theorem 2 and Lemma 2 and [12] Theorem 4, see [13] Section 3 for a more detailed explanation. One can also mention the paper [21] by Tang where in particular, Cui’s result concerning the Douady–Earle extension is extended to all spaces $T^p$ with $p \geq 2$. Recently, the second author of this paper proved in [6] that $T^2 \subset T_1$ using an approach based on module techniques and the so-called \textit{Teichmüller’s Modulsatz} (see [24] §4), and later on using a different method she proved that for any $p > 0$, $T^p \subset T_1$ (see [5]).

In this paper we only deal with $T^p$ for $0 < p \leq 1$ and we give a proof of the following result:

\textbf{Theorem 1.} \textit{Let $p \leq 1$. Then, any element of $T^p$ is a $C^1$-diffeomorphism.}

The strategy of the proof takes advantage of an approach used by the second author of this paper and J. A. Jenkins [8], modified to the case of the unit disc. We first use the \textit{Teichmüller–Wittich–Bellinskii} to show that each element of $T^1$ has a non-vanishing derivative at each point of $S^1$. Then, we use properties of the reduced module of a simply-connected domain to show that the derivatives of the elements of $T^1$ are continuous. As mentioned earlier, since $T^p \subset T^1$ for $0 < p \leq 1$, it follows immediately that for $0 < p \leq 1$, any element in $T^p$ is continuously differentiable with non-vanishing derivative.
2. Background

In this section we recall some classic notions from geometric function theory. Such notions are most notably and thoroughly investigated in Teichmüller’s Habilitationsschrift (Habilitation Thesis) [23].

2.1. Module of a doubly-connected domain. Let $D$ be a (non-degenerate) doubly-connected domain of the extended complex plane, that is, the complement of $D$ is an union of two disjoint simply-connected domains, each bounded by a Jordan curve. It is well known (see [17], [23]) that there exist a biholomorphic function that maps $D$ onto an annulus of inner radius $r_1$ and outer radius $r_2$ for some $0 < r_2 < r_1 < \infty$. The module $\text{Mod}(D)$ of $D$ is $\ln \left( \frac{r_2}{r_1} \right)$. It is a conformal invariant, namely, if $\Psi : D \to \Psi(D)$ is a biholomorphic function, then $\text{Mod}(D) = \text{Mod}(\Psi(D))$.

It is also well known (see [17], [23]) that the module is superadditive. More precisely, if $D_1$ and $D_2$ are two disjoint doubly-connected subdomains of a doubly-connected domain $D_3$, where each separates some $z_0 \in \mathbb{C}$ from $\infty$, then

$$\text{Mod}(D_1) + \text{Mod}(D_2) \leq \text{Mod}(D_3).$$

In saying that a doubly-connected domain separates $z_0$ from $\infty$, we mean that one component of its complement contains $z_0$ in its interior while the other component contains $\infty$.

Let us now recall two inequalities that will be used in the proof of the main result. For $0 < r_2 < r_1$ and $\zeta \in \mathbb{C}$ we set $A_{\zeta, r_2, r_1} = \{ z \mid r_2 < |z - \zeta| < r_1 \}$. Let $F : A_{\zeta, r_2, r_1} \to F(A_{\zeta, r_2, r_1})$ be a quasiconformal mapping. Then setting $z = \zeta + re^{i\theta}$, $r_2 < r < r_1$ we have

$$\text{Mod}(F(A_{\zeta, r_2, r_1})) \leq \frac{1}{2\pi} \int_{A_{\zeta, r_2, r_1}} \frac{1 + \mu_F(z)}{1 - \mu_F(z)} \cdot \frac{dxdy}{|z - \zeta|^2} \leq \text{Mod}(F(A_{\zeta, r_2, r_1})).$$

These estimates could be obtained following Teichmüller’s approach based on the length-area method in [23] §6.3, where he arrived at weaker versions of (3) and (4).—Estimates equivalent to (3) and (4)—some proved under more general assumptions and different methods—can be found in [15], [16], [4] and others.

2.2. Reduced module of a simply-connected domain. Let $\Omega$ be a simply-connected domain of the complex plane different from $\mathbb{C}$. Let $\zeta \in \Omega$. For $r > 0$, let $D(\zeta, r)$ denote the disc of radius $r$ centered at $\zeta$ and let $0 < r_2 < r_1$ be small enough so that $D(\zeta, r_1) \subset \Omega$. From [22] follows

$$\text{Mod}(\Omega \setminus D(\zeta, r_1)) + \ln \left( \frac{r_1}{r_2} \right) \leq \text{Mod}(\Omega \setminus D(\zeta, r_2)),$$

and therefore

$$\text{Mod}(\Omega \setminus D(\zeta, r_1)) + \ln (r_1) \leq \text{Mod}(\Omega \setminus D(\zeta, r_2)) + \ln (r_2).$$

One defines the reduced module $M^\text{red}(\Omega, \zeta)$ of $\Omega$ at $\zeta$ as $\lim_{r \to 0} \text{Mod}(\Omega \setminus D(\zeta, r)) + \ln (r)$. Using, for example, Koebe distortion theorem one can show that this limit is finite and $M^\text{red}(\Omega, \zeta) = \ln (|\Psi(0)|)$, where $\Psi : \mathbb{D} \to \Omega$ is a biholomorphic function.
mapping 0 onto \( \zeta \). A detailed proof can be found in [23, §1.6]. From here it follows directly that \( \zeta \mapsto M^\text{red}(\Omega, \zeta) \) is continuous.

Before concluding this subsection let us add one more property of the reduced module that we will use later.

If \( F : \mathbb{C} \to \mathbb{C} \) is a homeomorphism then, for any \( r > 0 \), the function \( \zeta \mapsto M^\text{red}(F(D(\zeta, r)), F(\zeta)) \) is continuous. Indeed, if \( \zeta_n \to \zeta \), then by applying a sequence of biholomorphic functions \( z \mapsto F(z + \zeta_n - \zeta) - F(\zeta_n) + F(\zeta), z \in D(\zeta, r) \), one obtains a sequence of domains \( D_n \), which are all images of \( D(\zeta, r) \). Since \( F(z) \) is a homeomorphism it follows that \( D_n \to F(D(\zeta, r)) \) (with respect to the topology induced by the Hausdorff distance on the set of subsets of \( \mathbb{C} \)). Consider the sequence of biholomorphic functions \( \Psi_n : \mathbb{D} \to D_n \) mapping 0 onto \( F(\zeta) \), normalized by \( \Psi'_n(0) > 0 \). Then for any \( n \), \( \ln (\Psi'_n(0)) = M^\text{red}(D_n, F(\zeta)) = M^\text{red}(F(D(\zeta_n, r)), F(\zeta_n)) \) since a translation does not change the reduced module. Furthermore, the sequence of functions \( \Psi_n \) forms a normal family and thus, up to a subsequence, \( \Psi_n \) converges uniformly (on any compact subset of \( \mathbb{D} \)) to a biholomorphic function \( \Psi_\infty : \mathbb{D} \to F(D(\zeta, r)) \) mapping 0 onto \( \zeta \). This implies

\[
M^\text{red}(F(D(\zeta, r)), F(\zeta)) = \lim_{n \to \infty} \ln (\Psi'_n(0)) = \lim_{n \to \infty} M^\text{red}(F(D(\zeta_n, r)), F(\zeta_n)) ,
\]

and thus we have continuity.

2.3. **Teichmüller–Wittich–Bellinskiǐ theorem.** First, let us recall that a mapping \( F : \mathbb{C} \to \mathbb{C} \) is said to be *conformal* at \( z_0 \) if \( \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} \) exists and is different from 0. Following [17, Chapter V, Theorem 6.1] the well-known Teichmüller–Wittich–Bellinskiǐ theorem can be stated as follows:

**Theorem 2.** Let \( D \) be a domain of the complex plane and let \( z_0 \in D \). Let \( F : D \to F(D) \) be a quasiconformal mapping. If there exists a neighborhood \( U \) of \( z_0 \) contained in \( D \) such that

\[
\iint_U \frac{|\mu_F(z)|}{|z - z_0|^2} \, dx \, dy < \infty;
\]

then \( F \) is conformal at \( z = z_0 \).

The history of this theorem and its extensions is rather long and we may refer the curious reader to some of the following papers [2, 13, 11, 7, 16, 4, 19] and to [1].

3. **Proof of the main result**

Let \( f \in T^1 \). By definition, there exists a quasiconformal extension \( F \) of \( f \) to the closed unit disc such that

\[
\iint_D |\mu_F(z)| \, d\sigma(z) < \infty. \tag{5}
\]

Let \( \tilde{\mu} \) be a function defined on the extended complex plane which coincides with \( \mu \) on \( \mathbb{D} \) and which is identically 0 outside the disc. Let \( F^{\tilde{\mu}} \) be the unique quasiconformal mapping of the complex plane with Beltrami coefficient \( \tilde{\mu} \) that fixes 1, \( i \), and \( -i \). Therefore, we have \( F^{\tilde{\mu}}|_\mathbb{D} = F \) and \( F^{\tilde{\mu}}|_{\mathbb{S}} = f \).
Claim 1. The quasiconformal mapping $F\tilde{\mu}$ is conformal at any point of $S^1$. Therefore, $f$ is a diffeomorphism of $S^1$.

We apply Theorem 2 to derive the conformality of $F\tilde{\mu}$.

Proof of Claim 1. Let $\zeta_0 \in S^1$. Because of (5), one can find a compact subset $K$ of $D$ such that

$$\int\int_{D \setminus K} |\mu_F(z)| \, d\sigma(z) < 1.$$  

Let $r > 0$ be such that $D \setminus D(\zeta_0, r) \subset D \setminus K$. One first observes that

$$\forall z \in D(\zeta_0, r) \cap D, \left(1 - |z|^2\right) \leq (1 - |z|^2)^2 \cdot (1 + |z|^2)^2$$

$$\leq |\zeta_0 - z|^2 \cdot (1 + |z|^2)^2$$

$$< 4 \cdot |\zeta_0 - z|^2,$$

and therefore

$$\forall z \in D(\zeta_0, r) \cap D, \frac{1}{|z - \zeta_0|^2} < 4 \cdot \frac{1}{(1 - |z|^2)^2}.$$  

It follows

$$\int\int_{D(\zeta_0, r) \setminus D(\zeta_0, r) \cap D} |\mu_F(z)| \, d\sigma(z) < 4 \cdot \int\int_{D(\zeta_0, r) \cap D} |\mu_F(z)| \, d\sigma(z)$$

$$< 4.$$  

We deduce, by Theorem 2, that $F\tilde{\mu}$ is conformal at $z = \zeta_0$ which proves that $f$ is differentiable at $\zeta_0$ and $|f'(\zeta_0)| > 0$. Since this is true for any $\zeta_0 \in S^1$, we deduce that $f$ is a diffeomorphism of $S^1$. 

The following two additional results will be needed in the proof of the continuity of $f'$ on $S^1$.

Claim 2. Let $\epsilon > 0$. Then, there exists $r_\epsilon > 0$ such that

$$\forall \zeta \in S^1, \forall 0 < \rho_2 < \rho_1 \leq r_\epsilon, \left| \text{Mod} \left( F\tilde{\mu} (A_{\zeta, \rho_2, \rho_1}) \right) - \ln \left( \frac{\rho_1}{\rho_2} \right) \right| < \epsilon.$$  

Claim 3. Let $\zeta \in S^1$ and $r > 0$. Then,

$$\lim_{\rho \to 0} \text{Mod} \left( F\tilde{\mu} (A_{\zeta, \rho, r}) \right) + \ln (|f'(\zeta)|) \rho = \text{Mod} \left( F\tilde{\mu} (D(\zeta, r)), f(\zeta) \right).$$  

Proof of Claim 3. Let $\zeta \in S^1$ and $0 < \rho_2 < \rho_1$. One the one hand, by applying (3) one gets

$$\text{Mod} \left( F\tilde{\mu} (A_{\zeta, \rho_2, \rho_1}) \right) - \ln \left( \frac{\rho_1}{\rho_2} \right) \leq \frac{1}{2\pi} \int\int_{A_{\zeta, \rho_2, \rho_1}} \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \cdot \frac{dxdy}{|z - \zeta|^2} - \ln \left( \frac{\rho_1}{\rho_2} \right)$$

$$= \frac{1}{2\pi} \int\int_{A_{\zeta, \rho_2, \rho_1}} \left( \frac{1 + |\mu(z)|}{1 - |\mu(z)|} - 1 \right) \cdot \frac{dxdy}{|z - \zeta|^2}$$

$$\leq \frac{1}{\pi (1 - ||\mu_F||_\infty)} \int\int_{A_{\zeta, \rho_2, \rho_1}} |\mu_F(z)| \cdot \frac{dxdy}{|z - \zeta|^2}.$$  

□
On the other hand since
\[ \int_0^{2\pi} \frac{1 + |\bar{\mu}(z)|}{1 - |\bar{\mu}(z)|} d\theta \geq 2\pi, \]
by means of (4) one obtains
\[ \text{Mod} \left( F^{\tilde{\nu}}(A_{\zeta, r_2, r_1}) \right) - \ln \left( \frac{\rho_1}{\rho_2} \right) \geq 2\pi \int_{\rho_2}^{\rho_1} \frac{1}{2\pi} \frac{1}{\frac{1 + |\bar{\mu}(z)|}{1 - |\bar{\mu}(z)|}} \cdot \frac{dr}{r} - \ln \left( \frac{\rho_1}{\rho_2} \right) \]
\[ = \int_{\rho_2}^{\rho_1} 2\pi - \frac{2\pi - 2\pi}{\frac{1 + |\bar{\mu}(z)|}{1 - |\bar{\mu}(z)|}} \cdot \frac{dr}{r} \]
\[ = \int_{\rho_2}^{\rho_1} \frac{2\pi - 2\pi}{\frac{1 + |\bar{\mu}(z)|}{1 - |\bar{\mu}(z)|}} \cdot \frac{dr}{r} \]
\[ \geq -\frac{1}{\pi} \iint_{A_{\zeta, r_2, r_1} \cap \mathbb{D}} \frac{1}{\mu_F(z)} \cdot \frac{dxdy}{|z - \zeta|^2} \]
\[ \geq -\frac{1}{\pi (1 - \|\mu_F\|_\infty)} \iint_{A_{\zeta, r_2, r_1} \cap \mathbb{D}} |\mu_F(z)| \cdot \frac{dxdy}{|z - \zeta|^2}. \]
(9)

Let \( \epsilon > 0 \). Still because of (3) there exists a compact set \( K_\epsilon \) of \( \mathbb{D} \) such that
\[ \int_{\mathbb{D} \setminus K_\epsilon} |\mu_F(z)| d\sigma(z) < \frac{\pi (1 - \|\mu_F\|_\infty)}{4} \epsilon. \]
(10)

Let \( r_\epsilon > 0 \) be the distance between \( \mathbb{S}^1 \) and \( K_\epsilon \). Thus, for any \( 0 < \rho_2 < \rho_1 \leq r_\epsilon \) one obtains by combining (3), (9), (7) and (10)
\[ \forall \zeta \in \mathbb{S}^1, \quad -\epsilon < \text{Mod} \left( F^{\tilde{\nu}}(A_{\zeta, r_2, r_1}) \right) - \ln \left( \frac{\rho_1}{\rho_2} \right) < \epsilon, \]
and therefore Claim 2 follows. \( \square \)

Proof of Claim 3. Let \( \zeta \in \mathbb{S}^1 \) and let \( r > 0 \). For any \( 0 < \rho < r \), let
\[ m(\rho) = \min_{|z - \zeta| = \rho} |F^{\tilde{\nu}}(z) - f(\zeta)| \quad \text{and} \quad M(\rho) = \max_{|z - \zeta| = \rho} |F^{\tilde{\nu}}(z) - f(\zeta)|. \]

Since \( F^{\tilde{\nu}} \) is conformal at \( \zeta \) one has
\[ \lim_{\rho \to 0} \frac{|f'(\zeta)| \rho}{M(\rho)} = \lim_{\rho \to 0} \frac{|f'(\zeta)| \rho}{m(\rho)} = 1. \]
(11)

Furthermore, it is evident that
\[ \text{Mod} \left( F^{\tilde{\nu}}(D(f(\zeta), r)) \setminus D(\zeta, M(\rho)) \right) \leq \text{Mod} \left( F^{\tilde{\nu}}(A_{\zeta, r, r}) \right) \]
\[ \leq \text{Mod} \left( F^{\tilde{\nu}}(D(\zeta, r)) \setminus D(f(\zeta), m(\rho)) \right). \]

Therefore, by adding \( \ln (|f'(\zeta)| \rho) \), using (11), and letting \( \rho \to 0 \) it follows that
\[ \lim_{\rho \to 0} \text{Mod} \left( F^{\tilde{\nu}}(A_{\zeta, r, r}) \right) + \ln (|f'(\zeta)| \rho) = M^{\text{red}} \left( F^{\tilde{\nu}}(D(\zeta, r)), f(\zeta) \right), \]
which proves Claim 3. \( \square \)
We have now all the ingredients necessary to complete the proof of our main
Theorem 1.

Let \( \zeta_0 \in S^1 \). Let \( \epsilon > 0 \). Let \( r_\xi > 0 \) be as in Claim 2. By the continuity of the
reduced module discussed earlier one can find a \( \delta_\xi > 0 \) such that if \( \zeta \in S^1 \) and
\( |\zeta - \zeta_0| < \delta_\xi \), then

\[
|M(\xi)_{r_\xi} - M_{r_\xi}| < \frac{\epsilon}{3}.
\]

Let \( \zeta \in S^1 \) be such that \( |\zeta - \zeta_0| < \delta_\xi \). By Claim 2, there exist \( r_{\zeta_0,1}, r_{\zeta,1} < r_\xi \),
such that for any \( \rho \leq r_{\zeta_0,1} \)
\[
\left| \text{Mod} \left( \nu \left( A_{\zeta_0,1} \right) \right) + \ln \left( |f'(\zeta)| \right) - M(\xi)_{r_\xi} \right| < \frac{\epsilon}{3},
\]
and for any \( \rho \leq r_{\zeta,1} \)
\[
\left| \text{Mod} \left( \nu \left( A_{\zeta,1} \right) \right) + \ln \left( |f'(\zeta)| \right) - M(\xi)_{r_\xi} \right| < \frac{\epsilon}{3}.
\]

Thus, from the triangle inequality, Claim 2, and Inequalities (12), (13), and (14), we obtain
\[
\ln \left( |f'(\zeta)| \right) - \ln \left( |f'(\zeta_0)| \right) = \ln \left( |f'(\zeta)| \right) - \ln \left( |r_{\zeta,1}| \right) - \ln \left( |r_{\zeta_0,1}| \right) + \ln \left( |r_{\zeta_0,1}| \right) 
\leq \ln \left( |f'(\zeta)| \right) + \text{Mod} \left( \nu \left( A_{\zeta,1} \right) \right) - M(\xi)_{r_\xi} 
+ \ln \left( |f'(\zeta_0)| \right) + \text{Mod} \left( \nu \left( A_{\zeta_0,1} \right) \right) - M(\xi)_{r_\xi} 
+ M(\xi)_{r_\xi} - M(\xi)_{r_\xi} 
+ \text{Mod} \left( \nu \left( A_{\zeta,1} \right) \right) - \ln \left( |r_{\zeta,1}| \right) + \text{Mod} \left( \nu \left( A_{\zeta_0,1} \right) \right) + \ln \left( |r_{\zeta_0,1}| \right) 
\leq \frac{\epsilon}{3} + \text{Mod} \left( \nu \left( A_{\zeta,1} \right) \right) + \ln \left( |r_{\zeta,1}| \right) 
+ \text{Mod} \left( \nu \left( A_{\zeta_0,1} \right) \right) - \ln \left( |r_{\zeta_0,1}| \right) 
\leq \epsilon.
\]

This shows the continuity of \( |f'| \) at any \( \zeta_0 \in S^1 \), thus \( f' \) is continuously differen-
tiable on \( S^1 \) and since the derivative is never 0, any element \( f \in T^1 \) is a
\( C^1 \)-diffeomorphism on \( S^1 \). Since \( T^p \subset T^1 \) (\( p \leq 1 \)) we have shown that Theorem 1 holds.

Since every differentiable quasisymmetric function \( f \) on \( S^1 \) is symmetric in the
sense of (1), the following already known property follows from Theorem 1.

**Corollary 3.** Let \( 0 < p \leq 1 \). Then, \( T^p \subset T_s \).

Let us point out that although \( T^1 \subset T_s \), the quasiconformal extension \( F \) of
\( f \) we were working with may not necessarily be asymptotically conformal on \( S^1 \) and Claim 2 is not obvious. However, for \( p \geq 2 \), if one specifically employs the
Douady–Earle extension, then Claim 2 holds. It seems natural to ask:

**Question 1.** Let \( f \in T^p \) (with \( 0 < p \leq 2 \)). Is there a quasiconformal asympto-
tically conformal extension \( F \) of \( f \) to the closed unit disc for which \( \mu_F \in L^p(D, \sigma) \)?

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**ON A SMOOTHNESS RESULT**
Furthermore, since we obtain smoothness properties for the elements of $T^p$ (for $p \leq 1$), we suggest that one can show higher and higher order of smoothness for $p < 1$, as $p$ gets smaller and smaller. If this is the case we would like to find sharp results on how the order of smoothness depends on $p$, a question that seems to be similar to finding a characterization of $T^p$ using Sobolev spaces for $p \geq 2$. In addition, we pose the following question:

**Question 2.** What is $\bigcap_{p>0} T^p$?

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