The role of exceptional points in quantum systems

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Abstract. Exceptional points are known in the mathematical literature for many years. They are singular points at which (at least) two eigenvalues of an operator coalesce. In physics, they can be studied best when the Hamiltonian of the system is non-Hermitian. Although the points themselves can not be directly identified in physics, their strong influence onto the neighborhood can be traced. Here, the exceptional points are called mostly crossing points (of the eigenvalue trajectories) or branch points or double poles of the $S$ matrix. In the present paper, first the mathematical basic properties of the exceptional points are discussed. Then, their role in the description of real physical quantum systems is considered (after solving the corresponding equations exactly). The Hamiltonian of these systems is non-Hermitian due to their embedding into an environment (continuum of scattering wavefunctions). Outside the energy window coupled directly to the continuum, the Hamiltonian is Hermitian but with corrections originating from the principal value integral of the coupling term via the continuum. Most interesting value of the non-Hermitian quantum physics is the phase rigidity of the eigenfunctions which varies (as function of a control parameter) between 1 (for distant non-overlapping states) and 0 (at the exceptional point where the resonance states completely overlap). This variation allows the system to incorporate environmentally induced effects. In the very neighborhood of a crossing (exceptional) point, the system can be described well by a conventional nonlinear Schrödinger equation. Here, the entanglement of the different states is large. In the regime of overlapping resonances, many eigenvalue trajectories cross or avoid crossing, and spectroscopic redistribution processes occur in the whole system. As a result, a dynamical phase transition takes place to which all states of the system contribute: a few short-lived resonance states are aligned to the scattering states of the environment by trapping the other states. The trapped resonance states are long-lived, show chaotic features, and are described well by means of statistical ensembles. Due to the alignment of a few states with the states of the environment, observable values (e.g. the transmission through the system) are enhanced. The dynamical phase transition breaks the spectroscopic relation of the short-lived and long-lived resonance states to the original individual states of the system. These results hold also for $\mathcal{PT}$ symmetric systems. The dynamical phase transition characteristic of non-Hermitian quantum physics, allows us to understand some experimental results which remained puzzling in the framework of conventional Hermitian quantum physics. The effects caused by the exceptional (crossing) points in physical systems allow us to manipulate them for many different applications.

PACS numbers: 03.65.Ta, 03.65.Ca, 02.40.Xx, 05.30.Rt, 03.65.Xp
1. Introduction

Many years ago, Kato [1] introduced the notation \textit{exceptional points} for singularities appearing in the perturbation theory for linear operators. Consider a family of operators of the form

\[ T(\varsigma) = T(0) + \varsigma T' \]  

where \( \varsigma \) is a scalar parameter, \( T(0) \) is the unperturbed operator and \( \varsigma T' \) is the perturbation. Then the number of eigenvalues of \( T(\varsigma) \) is independent of \( \varsigma \) with the exception of some special values of \( \varsigma \) where (at least) two eigenvalues coalesce. These special values of \( \varsigma \) are the \textit{exceptional points}. An example is the operator

\[ T(\varsigma) = \begin{pmatrix} 1 & \varsigma \\ \varsigma & -1 \end{pmatrix} \]  

In this case, the two values \( \varsigma = \pm i \) give the same eigenvalue 0.

Operators of the type (2) appear in the description of physical systems, for example in the theory of open quantum systems [2]. In this case, they represent a \( 2 \times 2 \) Hamiltonian describing a two-level system with the unperturbed energies \( \epsilon_1 \) and \( \epsilon_2 \) and the interaction \( \omega \) between the two levels,

\[ H(\omega) = \begin{pmatrix} \epsilon_1 & \omega \\ \omega & \epsilon_2 \end{pmatrix} \]  

In an open quantum system, two states can interact directly (corresponding to a first-order term) as well as via an environment (second-order term) [2]. In the present paper, we consider the case that the direct interaction is contained in the energies \( \epsilon_k \) (\( k = 1, 2 \)). Then \( \omega \) contains exclusively the coupling of the states via the environment which, in the case of an open quantum system, consists of the continuum of scattering wavefunctions into which the system is embedded. This allows to study environmentally induced effects in open quantum systems in a very clear manner [2].

The eigenvalues of the operator \( H(\omega) \) are

\[ \epsilon_{1,2} = \frac{\epsilon_1 + \epsilon_2}{2} \pm Z ; \quad Z = \frac{1}{2} \sqrt{(\epsilon_1 - \epsilon_2)^2 + 4\omega^2} \]  

The two eigenvalue trajectories cross when \( Z = 0 \), i.e. when

\[ \frac{\epsilon_1 - \epsilon_2}{2\omega} = \pm i \]  

At these \textit{crossing points}, the two eigenvalues coalesce,

\[ \epsilon_1 = \epsilon_2 \equiv \epsilon_0 \]  

The crossing points may be called therefore exceptional points.

However, there are some essential differences between the exceptional points considered in the mathematical literature and the crossing points which appear in physical systems. The differences arise from the fact that the crossing points are points in the continuum of scattering wavefunctions (which represents the environment). They are therefore of measure 0 and can not be observed directly. However, they influence the behavior of the eigenvalue trajectories \( \epsilon_k(\alpha) \) (where \( \alpha \) is a certain parameter) in their neighborhood in a non-negligible manner. Thus, the most interesting features of the exceptional (crossing) points in physical systems are not the properties at the crossing points themselves. Much more interesting are their effects onto the eigenvalue trajectories \( \epsilon_k(\alpha) \) in a finite parameter range around the
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critical value $\alpha = \alpha_{cr}$ (at which two trajectories cross) and, above all, the behavior of the eigenvalue trajectories in approaching the crossing point, $\varepsilon_k(\alpha) \to \varepsilon_k(\alpha_{cr})$. The phenomenon of avoided level crossing is known in physical systems since many years [3]. It occurs not only for discrete states but also for narrow resonance states [4]. In the scattering theory, the crossing points cause double poles of the $S$ matrix. For details see [2].

According to their influence on many physical observables, exceptional points are considered under different aspects in the physical literature. The topological features are theoretically considered by means of a $2 \times 2$ system in, e.g., [5, 6, 7] and experimentally studied on a microwave cavity in [8, 9]. The results of recent theoretical studies can be found in [10, 11]. A topological transition in a non-Hermitian quantum walk is discussed in [12].

In many-level quantum systems, the exceptional points are called crossing points of eigenvalue trajectories, e.g. [13, 14, 2], or double poles of the $S$ matrix, e.g. [15, 16, 17, 18], or branch points, e.g. [19]. In most studies, the biorthogonality of the eigenfunctions of the Hamiltonian plays an important role and is considered explicitly. In [16, 17], laser-induced continuum structures in atoms are studied. They are of interest especially in the neighborhood of crossing points (double poles of the $S$ matrix). The high-order harmonic generation in a driven two-level atom is related to abrupt population transfers between states at the avoided level crossings [20]. Recently, the laser control of vibrational transfer based on exceptional points is studied in [21]. The influence of exceptional points on the photoionization cross section is investigated in [22]. Also in nuclear physics, exceptional points at low energy appear for realistic values of the coupling to the continuum [18]. The relation between exceptional points and the Petermann factor characterizing the enhancement in intrinsic laser line widths and spontaneous emission rates, is discussed in [23]. In [2], the phase rigidity of the eigenfunctions of the non-Hermitian Hamilton operator in approaching an exceptional point is related to environmentally induced effects in quantum systems. This relation holds also for $\mathcal{PT}$ symmetric systems [24]. The phase rigidity is shown to be anti-correlated with the transmission probability through quantum dots [25].

The relation between exceptional points on the one hand, and the phenomenon of resonance trapping and dynamical phase transitions, on the other hand, is studied theoretically in many papers, see the recent review [2]. It is proven experimentally in [26]. As a result of resonance trapping in $\mathcal{P}$ symmetric systems, bound states in the continuum may appear [16, 27]. The relation between avoided level crossings and bound states in the continuum (the last phenomenon is called mostly population trapping in atomic physics), is first obtained in [13]. In [28], the statistical properties of the trapped states are considered. In [29], the lifetimes of electromagnetic quasibound states in dielectric microresonators with fully chaotic ray dynamics are statistically analyzed. The necessary renormalization is linked to the formation of short-lived resonances, i.e. to the resonance trapping mechanism. According to [30], the resonance states of a many-body system at high level density are described well by a statistical ensemble containing the interaction between all particles (e.g. the Gaussian orthogonal ensemble), while those at low level density are described best by a combination of one-body problems (e.g. the shell model). Meanwhile, an exceptional point is observed directly in a chaotic optical microcavity [31]. The influence of exceptional points in quantum chemistry is studied some time ago [32]. Recently, quantum dynamical phase transitions are found experimentally and theoretically in the spin swapping operation
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Resonance coalescence in molecular photodissociation is studied in [36]. The visualization of an exceptional point in a $\mathcal{PT}$ symmetric directional coupler is demonstrated in [37]. In [38] it is shown that the nature of the transport through a molecular junction is determined by a dimensionless parameter which measures the degree of resonance overlap in the system. Experimental studies in quantum point contacts show the importance of detector backaction [39]. In a $\mathcal{PT}$-symmetric square well, bound states appear below a certain threshold of the degree of non-Hermiticity, while beyond the threshold the two lowest real energies are shown analytically to merge and disappear [40]. The phase lapses observed experimentally [41, 42] in the transmission through small quantum dots, can be explained qualitatively [43] by the dynamical phase transition occurring in the regime of overlapping resonances.

Exceptional points are found to play a role also in Bose-Einstein condensates of gases [44]. In the quantum motion of a Bose-Einstein condensate in an optical cavity, the Dicke-model phase transition is observed [45], what is nothing but the resonance trapping phenomenon [2]. The doorway states in nuclear reactions can be considered to be a manifestation of the Dicke model super-radiant mechanism [46, 47]. The appearance of exceptional points is studied recently even in classical systems: in cosmic structure formation, where the magnetorotational instability is known to play an important role [48]. Here, the mechanism of instability transfer between modes through a spectral exceptional point is identified, which allows to explain some data.

The aim of the present paper is to give a consistent and unifying representation of the role of singular (exceptional) points in quantum systems. The constraints originating from the physical boundary conditions are taken into account. Among others, it will be shown that exceptional points influence not only the resonance states, but also the discrete states of the system. Here, they cause the avoided level crossing phenomenon known since many years [3], as well as effective forces used in almost all numerical calculations. Most interesting is the regime of overlapping resonances where the meaning of exceptional points for physical processes and their impact on the dynamics of the system can be controlled. Here, symmetry breaking caused by exceptional points plays an important role.

The paper is organized in the following manner. In sections 2 and 3 the eigenvalues and eigenfunctions of a $2 \times 2$ Hamilton operator of the type (3) are considered. Here, the basic properties of the exceptional (crossing) points are sketched. At these singular points, level repulsion passes into width bifurcation. The eigenfunctions $\phi_k$ of the non-Hermitian Hamilton operator $H$ are biorthogonal leading to some freedom for their normalization (since $\langle \phi_k^* | \phi_l \rangle$ is not necessarily a real number). We choose $\langle \phi_k^* | \phi_l \rangle = \delta_{kl}$ in order to describe the transition from overlapping to non-overlapping resonance states in a smooth manner (the last ones are normalized as $\langle \phi_k^* | \phi_l \rangle = \delta_{kl}$ according to conventional quantum theory). As a consequence, the phases of the eigenfunctions of $H$ are not rigid in approaching an exceptional (crossing) point. This mathematical result is surely the most interesting one of non-Hermitian quantum physics. It allows the system to incorporate environmentally induced effects (feedback from the coupling to the environment). In the neighborhood of the crossing points, the system is described well by a conventional nonlinear Schrödinger equation.

Section 4 shows that the basic results of the $2 \times 2$ problem survive when realistic systems with many levels are considered. The eigenvalues of the Hamiltonian are complex or real, according to the boundary conditions. In the first case, the eigenstates are resonant (with, usually, a finite lifetime) while they are discrete (corresponding
to an infinitely long lifetime) in the second case. The coupling of the states via
the continuum becomes important in the regime of overlapping resonances. For the
discrete states, it introduces effective forces. Section 5 gives the solution $\Psi^E_c$ of the
Schrödinger equation in the total function space, including discrete and scattering
states. The Hamilton operator of the whole system is Hermitian. The solution $\Psi^E_c$
is found by using a projection operator formalism. The two subspaces correspond to
system (localized in space) and environment (extended in space). The solution $\Psi^E_{c,\text{int}}$
inside the localized part of the system can be represented in a set of biorthogonal
wavefunctions. Hence, the phases of the $\Psi^E_{c,\text{int}}$ are not rigid such that an alignment
of some of them with the channel wave functions of the environment is possible also
in the many-level case. This alignment occurs by trapping other resonance states,
i.e. by width bifurcation. In section 6 the $S$ matrix is given by using the $\Psi^E_c$. Most
interesting are the double poles of the $S$ matrix appearing at the crossing points. Here,
the line shape of the resonance shows nonlinear effects.

The entanglement of the states is considered in section 7. It is most interesting in
the regime of overlapping resonances where many true and avoided level crossings
occur. In section 8 the avoided level crossing phenomenon is traced, by means
of a control parameter, from resonance states in the overlapping regime to narrow
resonance states and finally to discrete states. The relation to quantum chaos is
discussed. In section 9 the interplay between system and environment is discussed. It
is shown that the entanglement of the states via the continuum occurring in the regime
of overlapping resonances of a many-level system, is nothing but a dynamical phase
transition. At and in the neighborhood of the crossing points, the resonance states
lose their individual spectroscopic features under the influence of the environment.
The aligned states cause some transparency of the system while the trapped states are
described best by a statistical ensemble, e.g. by the Gaussian orthogonal ensemble. In
section 10 some experimental results are sketched which are puzzling in conventional
Hermitian quantum physics, but may be explained (at least qualitatively) by means of
dynamical phase transitions, i.e. by considering the exceptional points characteristic
of non-Hermitian quantum physics. The results are summarized in the last section.

2. The eigenvalues of a non-Hermitian $2 \times 2$ Hamilton operator

We consider the Hamiltonian (3) with the unperturbed energies $\epsilon_i$ ($i = 1, 2$) of the two
states and the interaction $\omega$ between them. The interaction $\omega$ contains exclusively the
coupling of the states via the environment, which consists of the continuum of decay
channels into which the states are embedded. The interaction $\omega$ is therefore a second-
order interaction term. The two eigenvalues $\epsilon_k$ ($k = 1, 2$) of (3) are given in (4). The
Hamiltonian may be Hermitian or non-Hermitian.

For a Hermitian operator, the unperturbed energies $\epsilon_i$ of the states are real. The
interaction $\omega$ being the principal value integral of the coupling term via the continuum,
is also real (2). Accordingly, the two eigenvalue trajectories $\epsilon_i(\alpha) = \epsilon_i(\alpha)$ (where $\epsilon_i(\alpha)$
is real) cannot cross (for $\omega \neq 0$) when traced as a function of a certain parameter $\alpha$, see
(4). Instead, they avoid crossing. This phenomenon is very well known for about 70
years (3). The fictive crossing point is called diabolic point. The topological structure
of this point is characterized by the Berry phase (49) which is studied theoretically
and experimentally in many papers.

The situation is another one for a non-Hermitian operator. In such a case, the
unperturbed energies $\epsilon_i$ are usually complex. Also the interaction $\omega$ is complex,
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in general, since it contains the principal value integral as well as the residuum of the coupling term describing the interaction of the two states via the environment (continuum of scattering wavefunctions) [2]. The states can decay, in general, and the two eigenvalues of (3) can be written as

$$\varepsilon_{1,2} = \varepsilon_{1,2} - \frac{i}{2} \gamma_{1,2} \quad \text{(with } \gamma_{1,2} \geq 0)$$

(7)

The widths \( \gamma_i \) are proportional to the inverse lifetimes \( \tau_i^{-1} \) of the states, \( i = 1, 2 \). The two eigenvalue trajectories \( \varepsilon_i(\alpha) \) may cross according to (4) and (6). The crossing point is an exceptional point in agreement with the definition given in [1], see (1) and (2). The topological phase of the exceptional point is twice the Berry phase [2, 7]. This theoretical result is proven experimentally by means of a microwave cavity [8].

According to the eigenvalue equation (4) \( Z \) is complex, usually. \( \text{Re}(Z) \) causes repulsion of the levels in energy. This result corresponds to the avoided level crossing phenomenon known for discrete states since many years [3]. It is the dominant part also in the case when the resonance states are narrow (long-lived), i.e. when the interaction \( |\omega| \) of the states via the continuum of scattering wavefunctions is small. The value \( \text{Im}(Z) \) has another physical meaning. It is the dominant part when \( |\omega| \) is large what is the case, above all, when the two resonances overlap. According to (4), \( \text{Im}(Z) \) is related to a bifurcation of the widths of the levels.

Due to width bifurcation, resonance states with long lifetime may appear together with short-lived states. The time scales characterizing these two different types of states, may differ strongly from one another. It is possible even that the widths of some states vanish, i.e. that \( \gamma_i = 0 \) for some states. These states with vanishing width are called, usually, bound states in the continuum [13]. Examples are studied in calculations for laser-induced continuum structures in atoms [10] as well as for the transmission through quantum dots [27]. In these calculations, resonance states with zero width appear at realistic parameter values. Tracing their appearance as a function of a parameter, one can see that they are nothing but special resonance states. The only hint in the cross section to such a state is the (elastic) scattering phase shift which passes into a jump by \( \pi \) at the energy of the state, see Fig. 5 in [27] for an example. These bound states in the continuum coexist with short-lived states. In [50, 51], the bound states in the continuum are called spectral singularities.

Hence, the real and imaginary parts of the complex eigenvalues (7) of the Hamiltonian (3) have a physical meaning in a quantum system in which the localized states of the system are embedded in an extended continuum of scattering states. The real parts \( \varepsilon_i \) stand for the positions in energy of the (almost) localized states while the imaginary parts \( \gamma_i \) give the widths (inverse lifetimes) of these states. It is \( \gamma_i > 0 \) when the decay of the states is not forbidden by any selection rule (and when the states are inside the energy window coupled to the continuum). The decay is an irreversible process [2]. Only at strong coupling to the continuum [corresponding to \( \text{Im}(Z) \gg \text{Re}(Z) \) in (1)], discrete states may appear due to width bifurcation also inside the energy window coupled to the continuum.

Starting with the papers [52, 53] by Bender et al., it has been shown that a wide class of \( \mathcal{PT} \) symmetric non-Hermitian Hamiltonians provides entirely real spectra. In order to realize complex \( \mathcal{PT} \) symmetric structures, the formal equivalence of the quantum mechanical Schrödinger equation to the optical wave equation in \( \mathcal{PT} \) symmetric optical lattices can be exploited by involving symmetric index guiding and an antisymmetric gain/loss profile [54, 55, 56, 57]. Meanwhile, experimental
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studies are performed. The results given in [58] have confirmed the expectations and have, furthermore, demonstrated the onset of passive $\mathcal{PT}$ symmetry breaking within the context of optics. This phase transition was found to lead to a loss induced optical transparency in specially designed pseudo-Hermitian potentials. In [59], the wave propagation in an active $\mathcal{PT}$ symmetric coupled wave guide system is studied. Both spontaneous $\mathcal{PT}$ symmetry breaking and power oscillations violating left-right symmetry are observed. Moreover, the relation of the relative phases of the eigenstates of the system to their distance from the level crossing (exceptional) point is obtained. Approaching this point, the phase transition occurs. In [60], the Floquet-Bloch modes in $\mathcal{PT}$ symmetric optical lattices are examined in detail.

Thus, the formal equivalence of the optical wave equation in $\mathcal{PT}$ symmetric optical lattices to the quantum mechanical Schrödinger equation allows us to study the properties of quantum systems the states of which can not only decay due to their coupling to the environment, but may also be formed out of the environment due to this coupling. In optics, these two possibilities are called loss and gain. The theory contains both possibilities. This fact makes the study of $\mathcal{PT}$ symmetric optical lattices a very attractive one.

In $\mathcal{PT}$ symmetric optical lattices, the eigenvalues are

$$\varepsilon_{1,2} = e_{1,2} \pm \frac{i}{2} \gamma_{1,2}$$

(with $\gamma_{1,2} \geq 0$ and $e_1 = e_2$) (8)

in difference to (7). Due to $\mathcal{PT}$ symmetry, all eigenvalues $\varepsilon_i = e_i$ may be real (corresponding to $\gamma_i = 0$) when $\text{Re}(Z) \gg \text{Im}(Z)$ in (4), i.e. at low coupling of the states to the continuum. Under these conditions, the optical wave equation describes a reversible process. However, the $\mathcal{PT}$ symmetry breaks at $\text{Im}(Z) \gg \text{Re}(Z)$ and then $\gamma_{1,2} \neq 0$.

It follows immediately that the $\mathcal{PT}$ symmetric models can not be mapped onto models of open quantum systems [24], although formally such a mapping seems to be possible by adding a constant imaginary energy shift to the eigenvalues. Both models differ fundamentally from one another when applied to the description of physical systems. It is this difference between the two models which will allow us to receive interesting information on quantum systems by studying not only open quantum systems (which exist in nature) but also $\mathcal{PT}$ symmetric systems (which are formally equivalent to them).

3. The eigenfunctions of a non-Hermitian $2 \times 2$ Hamilton operator

The eigenfunctions of the non-Hermitian Hamilton operator $H$ are biorthogonal,

$$\langle \phi^*_k | \phi_l \rangle = \delta_{k,l} .$$

From these equations follows

$$\langle \phi_k | \phi_k \rangle = A_k \geq 1$$

and

$$\langle \phi_k | \phi_{l \neq k} \rangle = -\langle \phi_{l \neq k} | \phi_k \rangle \equiv B^l_k ; \quad |B^l_k| \geq 0 .$$

At the crossing point

$$A^{(\text{cr})}_k \rightarrow \infty \quad \text{and} \quad |B^l_k|^{(\text{cr})} \rightarrow \infty ,$$

for details see [2].
The relation between the eigenfunctions $\phi_1$ and $\phi_2$ of the operator (3) at the crossing point is

$$\phi_1^{cr} \rightarrow \pm i \phi_2^{cr} \quad \phi_2^{cr} \rightarrow \mp i \phi_1^{cr}$$

(13)

according to analytical [4, 61] as well as numerical studies [17]. The two eigenfunctions are linearly dependent of one another at the crossing point such that the number of eigenfunctions of $H$ is reduced at this point. This result shows once more that the crossing point is an exceptional point in the sense defined by Kato [1].

In an experimental study on a microwave cavity [8], the topological structure of the exceptional point and its surrounding is studied by encircling it and tracing the relative amplitudes of the wavefunctions (field distributions inside the cavity). As a result, the wavefunctions including their phases are restored after four surroundings.

The authors [8] interpreted the experimental data by two theoretical assumptions: (i) the two wavefunctions coalesce into one at the exceptional point, $\phi_1^{cr} \leftrightarrow \phi_2^{cr}$, and (ii) only one of the wavefunctions picks up a phase of $\pi$ (a sign change) when encircling the critical point. With these two assumptions, the wavefunctions are restored after four surroundings as found experimentally.

The experimental result can be explained, however, without any additional assumptions by using the relations (13)

1. cycle : $\varepsilon_{1,2} \rightarrow \varepsilon_{2,1}$, $\phi_{1,2} \rightarrow \pm i \phi_{2,1}$
2. cycle : $\varepsilon_{2,1} \rightarrow \varepsilon_{1,2}$, $\pm i \phi_{2,1} \rightarrow - \phi_{1,2}$
3. cycle : $\varepsilon_{1,2} \rightarrow \varepsilon_{2,1}$, $- \phi_{1,2} \rightarrow \mp i \phi_{2,1}$
4. cycle : $\varepsilon_{2,1} \rightarrow \varepsilon_{1,2}$, $\mp i \phi_{2,1} \rightarrow \phi_{1,2}$

(14)

As can be seen, the eigenvalues are restored after two surroundings and the eigenfunctions are restored after four surroundings, in full agreement with the experimental result.

In any case, $|\phi_1^{cr}| = |\phi_2^{cr}|$ at the crossing point in agreement with the statement that the number of eigenstates is reduced at the exceptional point. The topological phase is twice the Berry phase, in accordance with the enlarged function space in open quantum systems.

Theoretical studies [61] have shown that associated vectors $\phi_i^{cr,a}$ defined by the Jordan relations, appear at the crossing points. The corresponding equations are

$$(H - \varepsilon_0) \phi_i^{cr,a} = 0$$

$$\phi_i^{cr,a} = \phi_i^{cr}.$$  

(15)

The existence of two states in the very neighborhood of the exceptional point has been seen in a numerical calculation for the elastic scattering of a proton on a light nucleus [18]. The elastic scattering phase shifts jump always by $2\pi$ (and not by $\pi$ as for a single resonance state).

Furthermore, the phases of the wavefunctions jump by $\pi/4$ at the crossing point (when traced as a function of a parameter) due to the biorthogonality [9] of the eigenfunctions of the non-Hermitian Hamiltonian $H$, see also [11]. This result has been proven in many numerical studies, see [2].

Let us now consider the consequences of the biorthogonality relations [9] and [10] for the two borderline cases characteristic of neighboring resonance states.

(i) The two levels are distant from one another. Then the eigenfunctions are (almost) orthogonal

$$\langle \phi_k^* | \phi_k \rangle \approx \langle \phi_k | \phi_k \rangle = A_k \approx 1.$$  

(16)
(ii) The two levels cross. Then the two eigenfunctions are linearly dependent according to (13) and
\[
\langle \phi_k | \phi_k \rangle = A_k \rightarrow \infty .
\] (17) according to (12). The two relations (16) and (17) show that the phases of the two eigenfunctions relative to one another change when the crossing point is approached. This can be expressed quantitatively by defining the phase rigidity \( r_k \) of the eigenfunctions \( \phi_k \),
\[
r_k \equiv \frac{\langle \phi_k^* | \phi_k \rangle}{\langle \phi_k | \phi_k \rangle} = A_k^{-1} .
\] (18)
According to (16) and (17) holds
\[
1 \geq r_k \geq 0 .
\] (19)
The non-rigidity \( r_k \) of the phases of the eigenfunctions of \( H \) follows also from the fact that \( \langle \phi_k^* | \phi_k \rangle \) is a complex number (in difference to the norm \( \langle \phi_k | \phi_k \rangle \) which is a real number) such that the normalization condition (9) can be fulfilled only by the additional postulation \( \text{Im} \langle \phi_k^* | \phi_k \rangle = 0 \) (what corresponds to a rotation [2]).

The variation of \( r_k \) according to (19) in approaching the crossing point of two eigenvalue trajectories is proven experimentally by means of a study on a microwave cavity [9]. As a result of the experimental study, the phase difference between two modes is \( \pi \) at large distance and decreases to \( \pi/2 \) at the crossing point. The authors of [9] interpreted the experimental data by assuming (i) that the singular point is a chiral state (in spite of the phase jump occurring at the crossing point, when traced as a function of a certain parameter), (ii) that the number of states is reduced from 2 to 1 at the crossing point (in spite of the existence of the associate vector (15)) and (iii) that a single point in the continuum can be identified (although it is of measure zero). The authors are unable to explain the large parameter range in which the phase difference decreases in approaching the crossing point.

Considering the phase rigidity \( r_k \) in the regime of the two overlapping resonance states, no additional assumptions are required for the explanation of the experimental results given in [9], since the phase rigidity (being a quantitative measure for the degree of resonance overlapping) varies smoothly in a comparably large parameter range. It can therefore be concluded that the experimental results [9] prove the statement that the phases of the eigenfunctions of the non-Hermitian Hamilton operator \( H \) are not rigid in approaching the crossing point, but vary according to (16) to (19).

According to (3), the Schrödinger equation with the unperturbed Hamilton operator \( H_0 \) and a source term arising from the interaction \( \omega \) with another state via the continuum of scattering states reads [4]
\[
(H_0 - \epsilon_n) | \phi_n \rangle = - \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} | \phi_n \rangle \equiv W | \phi_n \rangle
= \sum_{k=1,2} \langle \phi_k | W | \phi_n \rangle \sum_{l=1,2} \langle \phi_k | \phi_l \rangle | \phi_l \rangle
= \sum_{k=1,2} \langle \phi_k | W | \phi_n \rangle \{ A_k | \phi_k \rangle + \sum_{l \neq k} B^l_k | \phi_l \rangle \} .
\] (20)
Here \( \langle \phi_k | \phi_k \rangle \equiv A_k \geq 1 \) according to (10) and \( \langle \phi_k | \phi_{l \neq k} \rangle = - \langle \phi_{l \neq k} | \phi_k \rangle \equiv B^l_k \). \( |B^l_k| \geq 0 \) according to (11). The \( A_k \) and \( B^l_k \) characterize the degree of resonance overlapping.
In the regime of overlapping resonances, \( A_k > 0, |B_k| > 0 \), and equation (20) is nonlinear. The most important part of the nonlinear contributions is contained in

\[
(H_0 - \epsilon_n) \left| \phi_n \right> = \left< \phi_n \left| W \right| \phi_n \right> |\phi_n|^2 |\phi_n\rangle
\]

which is a nonlinear Schrödinger equation. According to (20), the nonlinear Schrödinger equation (21) goes over smoothly into a linear Schrödinger equation when departing from the exceptional point, i.e. in its neighborhood.

4. The many-level system

We consider now a conventional quantum system with \( N \) discrete states. The wavefunctions \( \Phi_k^B \) of the states of this system are eigenfunctions of a Hermitian Hamilton operator \( H_B = H_0^B + V \) which is assumed to contain the direct interaction \( V \) between the different states. Such a system is localized in space. We assume that this system is, in a certain energy window, embedded into the extended continuum of scattering wavefunctions \( \xi_{E}^c \). The energy window is defined by the two threshold energies \( E_{\text{thr}}^l \) and \( E_{\text{thr}}^h \) determining the conductance band of, e.g., a quantum dot. In nuclei, \( E_{\text{thr}}^h \rightarrow \infty \). In this manner, an open quantum system is defined. In the following, we use this definition.

The mathematical description of this system meets the problem that the two wavefunctions \( \Phi_k^B \) and \( \xi_{E}^c \) are of different type. They are normalized differently,

\[
\left< \Phi_i^B | \Phi_j^B \right> = \delta_{i,j}
\]

and

\[
\left< \xi_c^E | \xi_{c'}^E \right> = \delta(E - E') \delta_{c,c'}
\]

where \( c \) stands for a certain decay channel and \( E \) is the energy of the system. For the channel wavefunctions, the shortened notation \( \xi_{E}^c \) is used here (see [2]). Also the boundary conditions are different for the two types of wavefunctions.

From the mathematical point of view, this problem can be overcome best in the following manner [62]. It is convenient to separate the total function space into two subspaces, one of which (the \( Q \) subspace) contains the \( \Phi_k^B \) while the other one (the \( P \) subspace) consists of the \( \xi_{E}^c \). In the two subspaces, the corresponding Schrödinger equation (including the boundary conditions) can be solved by using the well-known standard methods. With \( P + Q = 1 \), the two subspaces (system and environment) are well defined. The solution in the total function space can then be obtained by combining the solutions obtained in the two subspaces, see section 5.

In the energy window coupled directly to the continuum of scattering wavefunctions, the discrete states (with infinite lifetime) of the \( Q \) subspace pass into resonance states (with finite lifetime) due to their embedding into the \( P \) subspace. Beyond the energy window, the discrete states remain discrete. Thus, also the boundary conditions between the two subspaces play an important role in considering the many-level system.

In the open quantum system, the states of the \( Q \) subspace can interact via the common environment, i.e. via the states of the \( P \) subspace. Hence, the Hamilton operator consists of a first-order and a second-order interaction term,

\[
H_{\text{eff}} = H_B + V_{BC} \frac{1}{E^+ - H_C} V_{CB}
\]

(24)
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with

\[
\text{Re}\{\langle \Phi_i^B | H_{\text{eff}} | \Phi_j^B \rangle\} = \langle \Phi_i^B | H_B | \Phi_j^B \rangle + \frac{1}{2\pi} \sum_c P \int_{E_{\text{thr}}^c}^{E_{\text{thr}}^c+t} \frac{\hat{\gamma}_i^c \hat{\gamma}_j^c}{E - E'} \, dE' \tag{25}
\]

\[
\text{Im}\{\langle \Phi_i^B | H_{\text{eff}} | \Phi_j^B \rangle\} = -\frac{1}{2} \sum_c \hat{\gamma}_i^c \hat{\gamma}_j^c . \tag{26}
\]

Here, \( P \) denotes the principal value integral and

\[
\hat{\gamma}_k^c = \sqrt{2\pi} \langle \xi_{E_c}^c | V | \Phi_k^B \rangle \tag{27}
\]

is the coupling matrix element between the wavefunctions of the two subspaces. The direct (first-order) interaction \( V \) is included in \( H_B \) and its eigenfunctions \( \Phi_k^B \).

In conventional quantum mechanics, the effective Hamilton operator \( H_{\text{eff}} \) is assumed to be Hermitian, i.e. the matrix elements \( \text{Re}\{\langle \Phi_i^B | H_{\text{eff}} | \Phi_j^B \rangle\} \) are considered to correspond to effective forces. The non-Hermitian part is not all considered, i.e. \( \text{Im}\{\langle \Phi_i^B | H_{\text{eff}} | \Phi_j^B \rangle\} = 0 \) is assumed.

Here, we are looking for the exact solution of the problem. We calculate not only \( \text{Im}\{\langle \Phi_i^B | H_{\text{eff}} | \Phi_j^B \rangle\} \), but also \( \text{Re}\{\langle \Phi_i^B | H_{\text{eff}} | \Phi_j^B \rangle\} \), and that by including the principal value integral and without any statistical assumptions. The Schrödinger equation reads

\[
(H_{\text{eff}} - z_k) \Phi_k = 0 \tag{28}
\]

with the eigenvalues \( z_k \) and eigenfunctions \( \Phi_k \) of \( H_{\text{eff}} \). In detail:

(i) The states inside the energy window are coupled directly to the environment such that the effective Hamilton operator \( H_{\text{eff}} \) is non-Hermitian, i.e. the principal value integral in (25) as well as the residuum (26) have to be calculated. The eigenvalues are complex,

\[
z_k = E_k - \frac{i}{2} \Gamma_k \tag{29}
\]

in general, and the eigenfunctions \( \Phi_k \) are complex and biorthogonal,

\[
\langle \Phi_i^* | \Phi_j \rangle = \delta_{i,j} , \tag{30}
\]

compare (9). The coupling matrix elements between the \( \Phi_k \) and the \( \xi_{E_c}^c \) are

\[
\gamma_k^c = \sqrt{2\pi} \langle \xi_{E_c}^c | V | \Phi_k \rangle \tag{31}
\]

in analogy to (27).

(ii) The states outside the energy window are not coupled directly to the environment such that the effective Hamiltonian \( H_{\text{eff}} \) is Hermitian at the energy of the states, i.e. only the principal value integral in (25) has to be calculated. At the energy of the states, the eigenvalues \( z_k = E_k \) are real, i.e. \( \Gamma_k = 0 \), and the \( \Phi_k \) are orthogonal in the standard manner,

\[
\langle \Phi_i | \Phi_j \rangle = \delta_{i,j} . \tag{32}
\]

The coupling matrix elements (31) between the \( \Phi_k \) and the \( \xi_{E_c}^c \) vanish at the energy of the state. They are, however, different from zero at energies inside the window coupled directly to the environment and contribute to the principle value integral.
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Thus, the non-Hermitian Hamilton operator $H_{\text{eff}}$ of the open system provides, according to the boundary conditions, resonance or discrete states. The method for numerical calculations is given in [63, 64] for nuclei, in [65] for atoms, in [66, 67] for quantum dots.

The individual states of the many-level system depend on parameters in a different manner according to their different spectroscopic properties. They may therefore cross or avoid crossing as discussed in sections 2 and 3. The most interesting effects appear in the very neighborhood of the crossing points where the contributions of all the other states to the crossing phenomenon need not to be considered. Hence, the exceptional points defined in (2) and (3) play an important role also in the many-level system. For example, the avoided crossing phenomenon of discrete and narrow resonance states is well known, see section 8.

Most interesting is the population transfer related to an exceptional point. The population transfer takes place not only at the crossing point itself but also in its neighborhood, i.e. at the critical point of an avoided level crossing. In realistic systems, population transfer may be induced by means of lasers. In [20], the connection between high-order harmonic generation and the periodic level crossings is investigated in detail. The knowledge of the physical mechanism allows one to manipulate the adiabatic states and consequently the harmonic spectra. The results can be extended to a broader parameter range, as, for instance, those characteristic of solid-state systems in strong fields. Another example studied recently, is the laser control of vibrational transfer occurring at and in the neighborhood of exceptional points [21] where the resonances exchange their labels. It is possible therefore to control, by means of a laser, the vibrational transfer of the undissociated molecules from one field-free state to another.

5. The solution of the Schrödinger equation in the total function space

The question arises now whether or not the properties discussed in the foregoing sections survive when solving the many-level problem in the total function space. In order to find an answer to this question let us sketch the Feshbach projection operator formalism [62] that allows a unified description of structure and reaction, for details see [2]. The structure is determined by the spectroscopic properties of the system ($Q$ subspace) while the reaction is induced by the environment of scattering wavefunctions ($P$ subspace).

The Schrödinger equation in the total function space reads

$$(H_{\text{full}} - E) \Psi_E^c = 0$$

where

$$H_{\text{full}} \equiv H_{QQ} + H_{QP} + H_{PQ} + H_{PP}$$

is Hermitian and $H_{QQ} \equiv QHQ$, $H_{QP} \equiv QHP$ and so on. The two projection operators $Q$ and $P$ are defined by

$$(H_E - E^E_E) \Phi_k^H = 0 \quad \rightarrow \quad Q = \sum_k |\Phi_k^B\rangle\langle\Phi_k^B|$$

$$(H_E - E^E_c) \xi_E^c = 0 \quad \rightarrow \quad P = \sum_c \int_{E_c}^{E_c'} dE |\xi_E^E\rangle\langle\xi_E^E|$$
and \( P + Q = 1 \) is assumed, see section 1. The operator \( H_B = H_0 + V \) contains the interaction \( V \) between the different basic states while the coupling between the discrete and scattering states is given by (27). Thus, \( \Psi^E \) contains (by definition) everything and \( H^\text{full} \) is Hermitian. The solution of the Schrödinger equation in the total function space reads [2]

\[
\Psi^E_c = \xi^E_c + \sum_{k,l=1}^N \langle \Phi^B_k | H_{QP} | \xi^E_c \rangle \frac{1}{E - H_{\text{eff}}} \langle \Phi^B_k | | \Phi^B_l \rangle
\]

where \( H_{\text{eff}} \) is given by (24) and

\[
\omega_k^0 = G^{(+)}_P H_{QP} \cdot \Phi^B_k; \quad G^{(+)}_P = P(E - H_{PP})^{-1} P.
\]

After diagonalizing \( H_{\text{eff}} \) (see equations (29) and (30)), the solution (37) reads

\[
\Psi^E_c = \xi^E_c + \sum_{k=1}^N \Omega_k \cdot \langle \Phi^*_k | H_{QP} | \xi^E_c \rangle \frac{1}{E - z_k}.
\]

Here

\[
\Omega_k = (1 + G^{(+)}_P H_{QP}) \Phi_k \equiv (1 + \omega_k) \Phi_k
\]

is the wavefunction of the resonance state \( k \). The tail of the resonance wavefunction is determined by \( \omega_k \), i.e. by a value analogue to (38). The solution (39) is exact in relation to the assumption \( P + Q = 1 \).

Of special interest is the scattering wavefunction inside the localized part of the system. According to (39), it can be represented in a set \( \{ \Phi_k \} \) of biorthogonal wavefunctions,

\[
|\Psi^E_{c, \text{int}}\rangle = \sum_k c_k^E |\Phi_k\rangle; \quad \langle \Psi^E_{c, \text{int}} | = \sum_k c_k^E \langle \Phi_k^* |
\]

with the coefficients

\[
c_k^E = \langle \Phi_k^* | H_{QP} | \xi^E_c \rangle \frac{1}{E - z_k} \equiv \frac{\gamma_k^E}{\sqrt{2\pi}} \frac{1}{E - z_k}
\]

which depend on energy. The coefficients \( \gamma_k^E \) are defined in (31). Due to this representation, the phases of the wavefunctions \( \Psi^E_{c, \text{int}} \) are not rigid. In analogy to (18), the phase rigidity \( \rho \) of the \( \Psi^E_{c, \text{int}} \) with \( 1 \geq \rho \geq 0 \) can be defined [2]. It is possible therefore that some wavefunctions align with the channel wavefunctions \( \xi^E_c \) \((c = 1, \ldots, C)\). This alignment occurs by trapping other resonance states [2]: finally, all but the aligned resonance states are more or less decoupled (trapped) from the continuum of scattering wavefunctions [2].

The resonance trapping phenomenon is proven experimentally [26]. The alignment of resonance states to the scattering states of the environment is a collective phenomenon to which all resonance states in a large energy region contribute, see section 9.

6. The \( S \) matrix

The \( S \) matrix is given in the following manner [15, 2]

\[
S_{cc'} = \delta_{cc'} - \int \frac{\langle \chi^E_{c'} | V | \Psi^E_c \rangle}{E - E'} dE' = \delta_{cc'} - \mathcal{P} \int \frac{\langle \chi^E_{c'} | V | \Psi^E_c \rangle}{E - E'} dE' - 2i\pi \langle \chi^E_{c'} | V | \Psi^E_c \rangle
\]

\[
\equiv \delta_{cc'} - S_{cc'}^{(1)} - S_{cc'}^{(2)}.
\]
where the $\chi^E_c$ are the unperturbed scattering wavefunctions and $\Psi^E_c$ is given in (39).

The $S$ matrix consists of two parts, one of which

$$
S^{(1)}_{cc'} = \mathcal{P} \int \frac{\langle \chi^E_c | V | \Psi^E_{c'} \rangle}{E - E'} dE' + 2i\pi \langle \chi^E_c | V | \xi^E_c \rangle
$$

(44)

depends smoothly on energy, and the other one

$$
S^{(2)}_{cc'} = i \sqrt{2\pi N} \sum_{k=1}^{N} \langle \chi^E_{c'} | V | \Omega_k \rangle \cdot \frac{\gamma^c_k}{E - z_k}
$$

(45)

is the resonance term with the eigenvalues $z_k$ defined in (29), and the coupling coefficients $\gamma^c_k$ defined in (31). If the resonance states are excited via the continuum of decay channels it holds

$$
\xi^E_c = (1 + G^{(+)}_P \cdot V) \chi^E_c
$$

(46)

and therefore

$$
\langle \chi^E_{c'} | V | \Omega_k \rangle = \langle \xi^E_{c'} | V | \Phi_k \rangle = \frac{1}{\sqrt{2\pi}} \frac{\gamma^c_k}{\gamma^c_k}.
$$

(47)

Using this relation, the resonance part of the $S$ matrix passes into the familiar expression

$$
S^{(2)}_{cc'} = i \sum_{k=1}^{N} \frac{\gamma^c_k \gamma^c_{k'}}{E - z_k}
$$

(48)

However, there are some differences to the conventional expression of the $S$ matrix: the coupling vectors $\gamma^c_k$ are calculated by means of the eigenfunctions $\Phi_k$ of $H_{\text{eff}}$ according to (31), the $z_k$ are eigenvalues of $H_{\text{eff}}$, see (29), and the $\gamma^c_k$ as well as the $z_k$ are energy dependent functions since $H_{\text{eff}}$ depends explicitly on energy according to (24). Further, the $S$ matrix is always unitary.

In the standard theory, the spectroscopic information is obtained from the poles of the $S$ matrix. According to (15), this procedure is equivalent to $E = z_k$, i.e., the spectroscopic properties are obtained from the eigenvalues $z_k(E)$ when $E$ is continued into the complex plane. Hence, the double poles of the $S$ matrix contain the information on the exceptional points. In the case with two resonance states coupled to one channel $c$, the $S$ matrix at the crossing point reads [4]

$$
S = 1 - i \sum_{k=1}^{2} \frac{\gamma^c_k \gamma^c_{k'}}{E - z_k}
$$

$$
= 1 - 2i \frac{\Gamma_d}{E - E_d + \frac{i}{2} \Gamma_d} - \frac{\Gamma_d^2}{(E - E_d + \frac{i}{2} \Gamma_d)^2}
$$

(49)

where $E_1 = E_2 \equiv E_d$, $\Gamma_1 = \Gamma_2 \equiv \Gamma_d$ and (29) is used. This expression shows a non-linear behavior around the crossing point. At the crossing point, the cross section vanishes due to interferences [71]. The interference minimum is however washed out in the neighborhood of the double pole [72]. In any case, the resonance observed is broader than a Breit-Wigner resonance according to (49). Different numerical studies are performed for realistic systems: for atoms in [16] [17] [73] and for nuclei in [18]. In all cases, the dependence of, e.g., the cross section on a certain parameter is nonlinear in the neighborhood of crossing points [18] [16] [17].

As a result, it can be stated that the effects induced by the exceptional points survive when the problem in the total function space is considered and observables are calculated.
7. The entanglement of states

The eigenfunctions $\Phi_i$ of the non-Hermitian Hamiltonian $H_{\text{eff}}$ can be represented in relation to different sets of basic wavefunctions:

(i) Representation of the $\Phi_i$ in the $\{\Phi_0^n\}$,

$$\Phi_i = \sum_{j=1}^N b_{ij} \Phi_0^j ; \quad b_{ij} = \langle \Phi_0^j | \Phi_i \rangle ,$$

(50)

where the $\Phi_0^j$ are eigenfunctions of the non-Hermitian unperturbed operator $H_{\text{eff}}^0$ (with vanishing non-diagonal matrix elements),

$$(H_{\text{eff}}^0 - z^0_i) |\Phi_0^i\rangle = 0 .$$

(51)

(ii) Representation of the $\Phi_i$ in the $\{\Phi_B^n\}$

$$\Phi_i = \sum_{j=1}^N a_{ij} \Phi_B^j ; \quad a_{ij} = \langle \Phi_B^j | \Phi_i \rangle$$

(52)

where the $\Phi_B^j$ are eigenfunctions of the Hermitian operator $H_B = H_{\text{eff}}^0 + V$,

$$(H_B - E^B_i) |\Phi_B^i\rangle = 0 .$$

(53)

When the $\Phi_i$ and the wavefunctions of the basic set describe localized non-overlapping states, the representations (50) and (52) are linear and well defined. In such a case, the number of states described by the $\Phi_i$ is equal to the number of states described by the $\Phi_0^i$ and $\Phi_B^i$, respectively.

The situation is more complicated when the mixing of the wavefunctions $\Phi_i$ in the overlapping regime is considered. The reason is the fact that the number of states is reduced at the exceptional points. Here and at the critical points of avoided level crossings, respectively, width bifurcation starts. As a result, some of the states become delocalized (short-lived) and do no longer contribute to the number of localized (long-lived) states. Hence, the number $N_{\text{loc}}$ of narrow (localized) resonance states described by the $\Phi_i$ may be different from the number $N$ of basic states described by the $\Phi_0^i$ and $\Phi_B^i$, respectively. In spite of $N_{\text{loc}} \neq N$, a representation of the wavefunctions $\Phi_i$ in the set $\{\Phi_0^n\}$ of $N$ wavefunctions is formally possible. However, the spectroscopic linear relation between the long-lived localized eigenstates and the basic localized resonance states (or the basic localized discrete states) is lost. For the coefficients $b_{ij}$ holds

$$\delta_{i,j} = \langle \Phi_i^* | \Phi_j \rangle = \sum_{k,l=1}^N b_{ik}b_{jl} \langle \Phi_k^0 | \Phi_l^0 \rangle = \sum_{k=1}^N b_{ik}b_{jk} .$$

(54)

Numerical studies for the nuclear reaction $^{15}\text{F} + p$ have shown the strong energy dependence of the coefficients $b_{ij}$ around the critical point of avoided and true crossings of eigenvalue trajectories [15]. They show the exchange of the two resonance states at the critical point of the avoided level crossing (characterized by level repulsion); they become infinitely large at the crossing point, and depend resonance-like on energy at the crossing point of the energy trajectories after width bifurcation. This picture agrees with that obtained for laser-induced continuum structures in atoms [16, 17], and also with that received from a study of the $2 \times 2$ toy model [4].

According to these results, the scenario in the regime of overlapping resonances is the following. At a (true or avoided) crossing point of the trajectories of two
resonance states, one of the resonance states starts to align to a scattering state of the environment and loses its localization while the other one remains localized but loses also its spectroscopic relation to the basic localized states. In a many-level system, this scenario repeats hierarchically [2] [68]. Finally, the number of localized long-lived states is reduced and, moreover, the surviving narrow resonance states – although (almost) localized – have lost their spectroscopic relation to the basic individual localized states. They are strongly entangled. This scenario is called dynamical phase transition.

Dynamical phase transitions are observed in different experimental studies. Some of them will be sketched in section 10. Here, it will be underlined only that – according to the above discussion – dynamical phase transitions can be traced back to the existence of exceptional points in quantum systems and to the nonlinearities caused by them. Moreover, the \( \omega \) in [3] and [20] contain solely the coupling of the states via the environment. The dynamical phase transition is therefore an environmentally induced phenomenon.

8. The avoided crossing phenomenon of discrete and narrow resonance states

According to section 4 two states may interact via the environment of scattering wavefunctions even if their energy is outside the energy window coupled directly to the environment of scattering wavefunctions. Numerical calculations have shown that the avoided crossings of discrete states can be traced back, indeed, to the crossing points of resonance states by varying one or two parameters [4]. The model Hamiltonian used in the calculations is

\[
H = \begin{pmatrix}
\epsilon_k^0(a) & 0 \\
0 & \epsilon_k^0(a)
\end{pmatrix} - \begin{pmatrix}
\frac{i}{2} \gamma_1^0 & \omega \\
\omega & \frac{i}{2} \gamma_2^0
\end{pmatrix}
\]

with the notation \( \epsilon_k \equiv \epsilon_k^0 - \frac{1}{\lambda k} \). The calculations are performed as a function of the parameter \( a \) for different \( \gamma_1^0 \) (and fixed ratio \( \gamma_1^0/\gamma_2^0 \) and fixed \( \omega \)) such that – with decreasing \( \gamma_1^0 \) – the levels (i) have always different widths and cross freely in energy at the value \( a = a^{cr} \), (ii) cross in energy and width at \( a = a^{cr} \) when \( \gamma_1^0 = \gamma_1^{cr} \), (iii) avoid crossing in energy when the widths cross at \( a = a^{cr} \) and (iv) pass into discrete states (with vanishing width) and avoid crossing at \( a = a^{cr} \) (in a similar manner as the narrow resonance states with \( \gamma_1^0 < \gamma_1^{cr} \) do).

Most interesting results are obtained for the coefficients \( |b_{ij}|^2 \) defined in (50). At the critical value \( a^{cr} \), it is \( \delta \equiv |b_{i,j}|^2 - |b_{i,j \neq i}|^2 = 1 \) when the two levels cross freely in energy (for \( \gamma_1^0 > \gamma_1^{cr} \)) according to the fact that the two states exist at different time scales and are therefore well defined. Approaching the crossing point (at \( \gamma_1^0 = \gamma_1^{cr} \)), \( |b_{i,j}|^2 \) and \( |b_{i,j \neq i}|^2 \) increase up to \( \infty \) and \( \delta \to 0 \) what is achieved due to (12). When the two states avoid crossing (for \( \gamma_1^0 < \gamma_1^{cr} \)), \( |b_{i,j}|^2 \) and \( |b_{i,j \neq i}|^2 \) decrease, however \( \delta = 0 \) remains at the critical point \( a^{cr} \). This result is an expression for the fact that the two states are exchanged at the critical point. Also for discrete states (\( \gamma_1^0 = 0 \)), \( \delta = 0 \) at the critical point \( a^{cr} \) of avoided level crossing. Here, \( |b_{i,j}|^2 = |b_{i,j \neq i}|^2 = 1/2 \).

It is interesting to see that \( |b_{i,j}| \neq 1 \) and \( |b_{i,j \neq i}| \neq 0 \) in a comparably large parameter range around the critical point. This range is the larger the smaller the widths of the states are (when traced as a function of the parameter \( a \) as in the above example). The range shrinks to one point (the crossing point) when the two states cross, i.e. when the two states overlap completely. The largest range occurs for discrete states although the discrete states do not overlap. This means that also
discrete states are mixed via the continuum in a finite range $\Delta a$ of the parameter $a$ and that this mixing is caused by the existence of the exceptional point.

In any case, resonance states as well as discrete states are mixed in a finite range $\Delta a$ of the parameter $a$ around the critical point $a = a^{cr}$ of an avoided crossing. At high level density where many neighboring levels avoid crossing, the ranges $\Delta a$ corresponding to a non-vanishing mixing of two states in each case, may overlap. As a consequence, the eigenfunctions $\Phi_i$ of $H_{eff}$ that describe localized states, lose their spectroscopic relation to the wavefunctions $\Phi_0^i$ of the basic individual states in a certain range of the parameter $a$ which is determined by the sum of the overlapping $\Delta a$. This statement holds true also for discrete states. It agrees fully with the conclusion drawn in section 7 on the dynamical phase transition that occurs in the regime with many avoided level crossings.

Thus, the strong mixing of the wavefunctions at high level density expressed by the lost of their spectroscopic relation to the individual basic resonance states is, on the one hand, a hint to the existence of exceptional points in the continuum of scattering wavefunctions. On the other hand, it points to the dynamical phase transition related to the many avoided crossings of localized states and the nonlinear effects caused by them.

The strongly mixed trapped (localized) states can be described best by statistical methods, e.g. by the Gaussian orthogonal ensemble. An example is the nuclear data ensemble analyzed by Bohigas et al. [69] many years ago. It caused many studies on quantum chaos. The states of the Gaussian orthogonal ensemble are different from those of a two-body random ensemble. They do not decay according to an exponential law [70]. This points to the fact that they differ from individual resonance states in spite of their small widths preventing them from overlapping (level repulsion is one of the characteristics of the Gaussian orthogonal ensemble). Indeed, the long-lived chaotic states in nuclei coexist with a broad single-particle resonance by which they are overlapped. For details see [30]. In [28] a shot noise analysis of the states of a microwave billiard is performed. In the long-time scale, the system shows features characteristic of quantum chaos while the system is regular in the short-time scale. In a recent study, the statistical properties of lifetimes of electromagnetic quasibound states in dielectric microresonators with fully chaotic ray dynamics are discussed [29]. The results show regular short-lived resonances and many long-lived resonances. The level statistics of the last ones is very well described by a random-matrix model, provided that two effective parameters are appropriately renormalized. This renormalization is linked by the authors [29] to the formation of the short-lived resonances, i.e. to the resonance-trapping phenomenon.

All these studies show, on the one hand, a correlation between exceptional points and quantum chaos. On the other hand, they show that quantum chaotic states are different from individual resonance states which characterize the system at low level density. In other words, quantum chaotic states are the result of a dynamical phase transition induced by the environment (including boundary conditions). They coexist with a few (aligned) short-lived states.

9. The interplay between system and environment

Some years ago, the question has been studied [74] whether or not the resonance trapping phenomenon is related to some type of phase transition. The study is
performed by using the toy model

\[ H_{\text{toy}}^{\text{eff}} = H_0 + i\alpha V V^+ \]  

(56)

in the one-channel case and with the assumption that (almost) all exceptional points accumulate in one point \([7]\). It has been found that resonance trapping may be understood, in this case, as a second-order phase transition. The calculations are performed for a linear chain consisting of a finite number \(m = 2n + 1\) of states. The state in the center of the spectrum traps the other ones and becomes a collective state in a global sense: it contains components of almost all basic states of the system, also of those which are not overlapped by it. The normalized width \(\Gamma_0/m\) of this state can be considered as the order parameter: it increases linearly as a function of \(\alpha\), and the first derivative of \(\Gamma_0/m\) jumps at the critical value \(\alpha = \alpha^{\text{cr}}\). The two phases differ by the number of localized states. In the case considered, this number is \(m\) at \(\alpha < \alpha^{\text{cr}}\), and \(m-1\) at \(\alpha > \alpha^{\text{cr}}\).

Much more interesting is the realistic case with the Hamiltonian (24). Here, trapping of resonance states occurs in the regime of overlapping resonances hierarchically, i.e. one by one \([68]\). The crossing points do not accumulate in one point, but are distributed over a certain range of the parameter. In this case, a dynamical phase transition takes place in a finite parameter range inside the regime of overlapping resonances \([2]\). Also in this case, almost all resonance states are involved in the phase transition and, furthermore, the number \(N\) of localized states is reduced. That means, the \(Q\) subspace splits into two parts under the influence of the environment. One part contains the few short-lived states which are (more or less) aligned to the scattering states of the environment, while the other part contains the trapped, long-lived and well localized states. Both time scales are well separated from one another.

An example are the short-lived whispering gallery modes in a microwave cavity with convex boundary which coexist with many long-lived states \([28, 75, 76]\). Another example is known in nuclear physics: the short-lived single-particle resonances are responsible for the fast direct reaction part, while the long-lived ones cause the slow compound-nucleus reaction part. In the Feshbach unified theory of nuclear reactions \([62]\), the direct reaction part is described exactly while the compound-nucleus reaction part is described by means of statistical ensembles. Similar representations are used in other fields of physics at high level density.

Interesting is the enhancement of observable values in the parameter range in which the phase transition takes place. The enhancement is a direct consequence of the alignment that occurs in such a manner that the aligned state fits best to the environment, i.e. that the corresponding \(\gamma_k\) is maximal. An example is studied theoretically in \([25]\). Here, an anticorrelation between the conductance \(|t|^2\) of a quantum dot and the phase rigidity \(|\rho|^2\) is found. The alignment is basic for the solution of the brachistochrone problem in quantum mechanics \([2, 77]\).

In any case, the regime at low level density (or small coupling via the continuum) differs from the regime at high level density (corresponding to strong coupling via the continuum). At small coupling via the continuum, the resonance states show individual spectroscopic features which are lost at large coupling. Here, many narrow (trapped) resonances are superimposed on broad (aligned) resonances. The trapped resonance states show chaotic features (section 8).

The dynamical phase transitions are surely the most interesting feature of non-Hermitian quantum physics. They are environmentally induced, see section 7 and \([24]\). Mathematically, this phenomenon is directly related to the existence of exceptional
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points, i.e. to the coupling matrix elements $\omega$ in (3), to the phase rigidity $r_k$ of the
eigenfunctions and to the nonlinear terms in the Schrödinger equation, see (20) and (21). In detail:

(i) The phases of the eigenfunctions of the non-Hermitian Hamilton operator are not
rigid in approaching the exceptional point: $r_k < 1$ in the regime of overlapping
resonances.

(ii) Due to $r_k < 1$, some resonance states may align with the scattering states of the
environment while other ones decouple from the environment (width bifurcation).

(iii) The short-lived aligned resonance states lose, to a great deal, their localization
and make the system (almost) transparent.

(iv) The long-lived trapped resonance states are well localized and show chaotic
features.

(v) The spectroscopic relation between the localized states at low level density
(without resonance overlapping) and those at high level density (with overlapping
short-lived and long-lived resonances) is lost.

The appearance of dynamical phase transitions can explain some puzzles that
are observed experimentally and could not explained theoretically in the framework
of conventional Hermitian quantum theory. Some experimental results of such a type
will be sketched in the following section 10, together with experimental results which
prove directly the dynamical phase transition.

10. Dynamical phase transitions in experimental data

10.1. Experimental verification of the resonance trapping phenomenon

About 10 years ago, the first direct experimental verification of the counterintuitive
resonance trapping phenomenon is presented [26]. The experiment is based on
the equivalence of the electromagnetic spectrum for flat cavities to the quantum
mechanical spectrum of the corresponding system. This equivalence holds also when
the system is opened by coupling the discrete states of the cavity to an attached
waveguide. In the experiment [26], a microwave Sinai cavity with an attached
waveguide with variable slit width was used.

As a result of this experimental study, agreement with theory is observed: the
widths of all resonance states first increase with increasing coupling strength to the
channels (continuum of scattering wavefunctions) but finally decrease again for most
of the states. Thus, the dynamical phase transition has been directly traced in this
experiment.

10.2. Spectra of light and heavy nuclei

It is a well-known result of nuclear physics studies during many years that the
resonance states in light nuclei are different from those in heavy nuclei. In light nuclei,
resonance states appear mostly at low excitation energy of the nucleus, where the level
density is small. The lifetimes of the resonance states are often near to the limit for
single-particle (or alpha) decay. All resonance states show individual spectroscopic
features.

The situation in heavy nuclei is completely different. The first (elastic) threshold
for particle decay is at about 8 MeV excitation energy of the nucleus where the level
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20 density is extremely high. In a small energy region above this threshold, the so-called neutron (compound nucleus) resonances are identified. They are extremely long-lived corresponding to decay widths of the order of eV and show chaotic features [69], see section 8. Much less discussed in literature are the so-called single-particle resonances in heavy nuclei the widths of which are of the order of magnitude of MeV. Their energy is mostly just below the elastic decay threshold and their width at energies above the threshold (see section 5 for the energy dependence of the widths) is much larger than the widths of the long-lived states. In the cross section, they appear as a smooth background for the very narrow neutron resonances. The time scales of these two different types of resonance states are well separated from one another: up to 10^6 neutron resonances are overlapped by one single-particle resonance.

In medium-mass nuclei, the first (elastic) decay threshold is at a comparably low excitation energy of the nucleus where the level density is still relatively low. These nuclei are characterized by overlapping resonances with different lifetimes. They are described well by the \emph{doorway picture} according to which doorway states coexist with long-lived compound nucleus resonance states. The doorway states being comparably short-lived, are coupled directly to the decay channels \emph{and} to the narrow compound nucleus resonance states. The narrow resonance states, however, are assumed to be coupled to the continuum \emph{only} via the doorway states. This model gives a good description of medium-mass nuclei, see [30]. In [47], the doorway picture is related to the Dicke model super-radiant mechanism which is nothing but the mechanism of resonance trapping (see [46] and section 5).

In [18], exceptional points are identified in nuclei under realistic conditions. This allows us to consider nuclei at low and high level density as quantum systems, respectively, below and above a dynamical phase transition. The phase transition itself can not be controlled today by means of a parameter since the strong nuclear forces do not allow a manipulation of nuclei. In [30], the resonance states at high level density are shown to be trapped states, i.e. states originating from a dynamical phase transition. They are described well by a statistical ensemble containing the interaction between all particles (e.g. by the Gaussian orthogonal ensemble), and not by a two-body ensemble. Beyond a critical value, the widths of these states decrease with increasing coupling strength between system and environment (continuum of scattering wave functions) [78]. The states of the Gaussian orthogonal ensemble decay according to a power law [70].

10.3. Phase lapses

In experiments [41, 42] on Aharonov-Bohm rings containing a quantum dot in one arm, both the phase and the magnitude of the transmission amplitude \( T = |T| e^{i\beta} \) of the dot can be extracted. The obtained results caused much discussion since they do not fit into the standard understanding of the transmission process. As a function of the plunger gate voltage \( V_g \), a series of well-separated transmission peaks of rather similar width and height has been observed in many-electron dots and, according to expectations, the transmission phases \( \beta(V_g) \) increase continuously by \( \pi \) across every resonance. In contrast to expectations, however, \( \beta \) always jumps sharply downwards by \( \pi \) in each valley between any two successive peaks. These jumps called phase lapses, were observed in a large succession of valleys for every many-electron dot studied. Only in few-electron dots, the expected so-called mesoscopic behavior is observed, i.e. the phases are sensitive to details of the dot configuration. The problem is considered
theoretically, in the framework of conventional Hermitian quantum physics, in many papers over many years, however without solving it.

In [43], the phase lapses observed experimentally at high level density are related to the trapped resonance states resulting from the dynamical phase transition. In accordance to this picture, only the resonance states at low level density show individual spectroscopic features. At high level density, the observed resonances arise from trapped states. They show level repulsion (see section 8) and have almost no spectroscopic relation to the open decay channels such that phase lapses appear. It follows further, that any theoretical study on the basis of conventional Hermitian quantum physics is unable to explain the experimental results convincingly. In other words: the experimentally observed phase lapses can be considered to be a proof for the dynamical phase transitions occurring in mesoscopic systems.

10.4. Dephasing at very low temperature

Comparing the basic ingredients of the theory of open quantum systems with the experimental results on dephasing at very low temperature, it should firstly be stated that the concept dephasing is used differently in different papers. Here, we consider the phase coherence time $\tau_\phi$ characterizing dephasing at very low temperature. In the following, a very short discussion of the results obtained experimentally will be given. The discussion is qualitatively by using the results obtained in different recent studies. It avoids to comment the many controversial discussions that exist in the literature to this question.

In the proceedings of a recent conference, the experimental progress on the saturation problem in metallic quantum wires is reviewed [79]. As a conclusion of this analysis, based on all presently available measurements of the phase coherence time $\tau_\phi$ in very clean metallic wires, it is hard to conceive that the apparent saturation of $\tau_\phi$ is solely due to the presence of an extremely small amount of magnetic impurities.

The absolute value of $\tau_\phi$ (and not just its temperature dependence) is studied in [80]. It is found that the electron dwell time $\tau_d$ is the central parameter governing the saturation of phase coherence at low temperature. The condition for the occurrence of saturation is found to be $\tau_\phi^{sat} \approx \tau_d$ where $\tau_\phi^{sat}$ is the saturated coherence time. This simple behavior holds over the three orders of magnitude covered by the available data in the literature. According to the authors, $\tau_\phi$ is found to be intrinsic to the physics of the quantum dots, and not due to the coherence time of the electrons themselves. Furthermore it is found [80] that $\tau_\phi$ is strongly influenced by the population of the second electronic subband in the quantum well.

According [81], one consensus has been reached by several groups, saying that the responsible electron dephasing processes in highly disordered and weakly disordered metals might be dissimilar. That means, while one mechanism is responsible for dephasing in weakly disordered metals, another mechanism may be relevant for the saturation (or very weak temperature dependence) of $\tau_\phi$ found in highly disordered alloys. According to the authors of [81], the intriguing electron dephasing is very unlikely due to magnetic scattering. It may originate from specific dynamical structure defects in the samples.

Experimental data from many different publications for $\tau_\phi^0$ obtained in metallic samples with different diffusion coefficients, are collected in [82]. The conclusion is that low temperature saturation of $\tau_\phi$ is universally caused by electron-electron interactions. The authors found seemingly contradicting dependencies of $\tau_\phi^0$ on the
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diffusion coefficient \( D \) in weakly and strongly disordered conductors. While the trend \textit{less disorder} – \textit{less decoherence} for sufficiently clean conductors is quite obvious, the opposite trend \textit{more disorder} – \textit{less decoherence} in strongly disordered structures is unexpected.

All these statements obtained from the results of many experimental studies fit qualitatively into the expectations received by considering the quantum dot as an open quantum system. First of all, the saturation of \( \tau_0 \) appears in a natural manner since most states of an open quantum dot have a finite lifetime at zero temperature. The value of the lifetime can be obtained from the imaginary part of the complex eigenvalue \( z_\lambda \) of the non-Hermitian Hamilton operator \( H_{\text{eff}} \) [i.e. from \( \text{Im}(z_\lambda) \)]. It expresses the time the electron stays in the quantum dot. This time is called usually dwell time. Thus, the result obtained in [80] supports the description of the quantum dot as an open quantum system.

Also the more complicated result of different processes in weakly and strongly disordered systems is by no means in contradiction to the properties known for the eigenstates of open quantum systems. In some cases, \( \tau_0 \) depends only weakly on the electron diffusion constant \( D \): it is somewhat smaller when \( D \) is larger. That means, states with a large lifetime give only a small contribution to the diffusion – a result which is very well known. In other cases, the relation between \( \tau_0 \) and the diffusion constant \( D \) shows the opposite trend. Also in this case the states with a large lifetime give, of course, a small contribution to the diffusion. In contrast to the foregoing case, however, the main contribution to the diffusion arises obviously from short-lived states (according to the resonance trapping phenomenon). Finally, the short-lived states form some background for the long-lived resonance states. The diffusion constant is determined mainly by the contribution of the background states. Therefore, the diffusion constant \( D \) increases with increasing \( \tau_0 \) of the (long-lived) resonance states – a result being counterintuitive in the same manner as the resonance trapping effect. The last one is directly proven experimentally [26].

In this respect another experimental result obtained in [80] is interesting. It shows that, in the systems considered, the quantity \( \tau_0 \) is strongly influenced by the population of the second electronic subband in the quantum well. Obviously this means that the degree of overlapping of the states plays an important role for the lifetimes of the states – according to one of the basic properties of the eigenstates of \( H_{\text{eff}} \). Further experimental studies related to this question would be very useful.

As a result of this discussion: dephasing shows features that might be related to the non-rigidity of the phases of the wavefunctions of an open quantum system and to the dynamical phase transition occurring in the regime of overlapping resonances. Accordingly, the coherence time \( \tau_0 \) is intrinsic to the physics of the quantum dot, and not due to the coherence time of the electrons. This conclusion agrees qualitatively with that obtained from the experimental results. A quantitative description of the experimental data by using the theory of open quantum systems with a non-Hermitian Hamilton operator, is not performed up to now. It is, however, interesting to remark that a decoherence rate \( 1/\tau_0 \) appears also in the dynamics of a spin swapping operation where it is well defined (section 10.5).

10.5. Quantum dynamical phase transition in the spin swapping operation

A swapping gate in a two-spin system exchanges the degenerate states \(| \uparrow, \downarrow \rangle \) and \(| \downarrow, \uparrow \rangle \). Experimentally, this is achieved by turning on and off the spin-spin interaction
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That splits the energy levels and induces an oscillation with a natural frequency \( \omega \). An interaction \( \hbar/\tau_{SE} \) with an environment of neighboring spins degrades this oscillation within a decoherence time scale \( \tau_\phi \). The experimental frequency \( \omega \) is expected to be roughly proportional to \( b/\bar{\hbar} \) and the decoherence time \( \tau_\phi \) proportional to \( \tau_{SE} \). In [33], experimental data are presented that show drastic deviations in both \( \omega \) and \( \tau_\phi \) from this expectation. Beyond a critical interaction with the environment, the swapping freezes and the decoherence rate drops as \( 1/\tau_\phi \propto (b/\bar{\hbar})^2 \tau_{SE} \). That means, the relaxation decreases when the coupling to the environment increases. The transition between these two quantum dynamical phases occurs when \( \omega \propto \sqrt{(b/\bar{\hbar})^2 - (k/\tau_{SE})^2} \) becomes imaginary (where \( k \) depends only on the anisotropy of the system-environment interaction, \( 0 \leq k \leq 1 \)). The experimental results are interpreted by the authors as an environmentally induced quantum dynamical phase transition occurring in the spin swapping operation [33, 34].

Further theoretical studies within the Keldysh formalism showed that \( \tau_\phi \) is a non-trivial function of the system-environment interaction rate \( \tau_{SE} \); indeed: it is \( 1/\tau_\phi \propto 1/\tau_{SE} \) at low \( \tau_{SE} \) (according to the Fermi golden rule) but \( 1/\tau_\phi \propto \tau_{SE} \) at large \( \tau_{SE} \). This theoretical result is in (qualitative) agreement with the experimental results. In [35], the dynamical phase transition in the spin swapping operation is related to the existence of an exceptional point.

The dynamical phase transition observed experimentally in the spin swapping operation and described theoretically within the Keldysh formalism shows qualitatively the same features as the dynamical phase transitions discussed in the present paper on the basis of the resonance trapping phenomenon (width bifurcation).

10.6. Loss induced optical transparency in complex optical potentials

Recently, the prospect of realizing complex PT symmetric potentials within the framework of optics has been suggested [55, 56, 57]. It is based on the fact that the optical wave equation is formally equivalent to the quantum mechanical Schrödinger equation. One expects therefore that PT symmetric optical lattices show a behavior which is qualitatively similar to that discussed for open quantum systems in the present paper.

Experimental studies showed, indeed, a phase transition that leads to a loss induced optical transparency in specially designed non-Hermitian guiding potentials [58, 59]: the output transmission first decreases, attains a minimum and then increases with increasing loss. The phase transition is related, in these papers, to PT symmetry breaking. In a following theoretical paper [60], the Floquet-Bloch modes are investigated in PT symmetric complex periodic potentials. As a result, the modes are skewed (nonorthogonal) and nonreciprocal. That means, they show the same features as modes of an open quantum system under the influence of exceptional points. A detailed discussion of this analogy is given in [24]. The optical realization of relativistic non-Hermitian quantum mechanics is considered in [83]. Here, the PT symmetry breaking of the Dirac Hamiltonian is shown to be related to resonance narrowing what is nothing but resonance trapping.

The title of one of the papers published in Nature Physics [59] to this topic reads: Broken symmetry makes light work. It is exactly this property which characterizes the phase transition in complex optical potentials. However, the situation in open quantum systems is qualitatively the same: in the dynamical phase transition, the spectroscopic relation to the individual resonance states at low level density (including
all symmetries) is broken and the system becomes transparent, see e.g. section 9.

11. Summary

In the present paper, exceptional points are shown to be responsible for mainly two properties of quantum systems. Both condition one another.

First, the spectroscopy of discrete and resonance states is strongly influenced by exceptional points in their neighborhood. Both types of states are eigenstates of one and the same (non-Hermitian) Hamilton operator, but the boundary conditions differ from one another. The states are discrete (corresponding to an infinite long lifetime) when their energy is beyond the window coupled to the continuum of scattering wavefunctions. The states are resonant (corresponding, in general, to a finite lifetime) when their energy is inside the window coupled to the continuum of scattering wavefunctions. Accordingly, the singularities (crossing points) in the continuum influence not only the behavior of resonance states but also that of discrete states.

Discrete states are described well in the framework of conventional quantum mechanics as known for very many years, although it is necessary to introduce effective forces in the conventional theory (which arise, at least partly, from the principal value integral of the coupling term via the continuum). The Hamiltonian is Hermitian and \( A_k = 1 \), the phases of the eigenfunctions are rigid corresponding to \( r_k \equiv A_k^{-1} = 1 \), the discrete states avoid crossing and the topological phase of the diabolic point is the Berry phase. Due to \( A_k = 1 \), the Schrödinger equation is linear, but the levels are mixed (entangled) in the total parameter range of avoided level crossing. At the critical point, the mixing is maximal (1:1).

Resonance states are described well when the quantum theory is extended by including the environment of scattering wavefunctions. The Hamiltonian is, in general, non-Hermitian and \( A_k \geq 1 \), the phases of the eigenfunctions are, in general, not rigid corresponding to \( 0 \leq r_k \equiv A_k^{-1} \leq 1 \), the resonance states can cross in the continuum and the topological phase of the crossing point is twice the Berry phase. When \( 0 < r_k \equiv A_k^{-1} < 1 \) (regime of resonance overlapping and avoided level crossings), the Schrödinger equation is nonlinear and the levels are strongly mixed (entangled). The parameter range in which mixing appears, shrinks to one point when the levels cross, i.e. when \( r_k \equiv A_k^{-1} \to 0 \).

Secondly, dynamical phase transitions are caused by exceptional points. According to the results given in the present paper, a dynamical phase transition occurs in the regime of overlapping resonances. It is produced by width bifurcation, is environmentally induced and breaks spectroscopic symmetries characteristic of the system. It consists in the reduction of the number of localized states by alignment of a few resonance states to the (extended) scattering states. By this, it breaks the spectroscopic relation between states below and beyond the dynamical phase transition.

The two phases below and beyond the dynamical phase transition are characterized by the following properties. In one of the phases, the discrete and narrow resonance states have individual spectroscopic features. Here, the real parts (energies) of the eigenvalue trajectories avoid crossing while the imaginary ones (widths) can cross. In the other phase, the narrow resonance states are superimposed with a smooth background and the individual spectroscopic features of the states are lost. The narrow resonance states and, respectively, the corresponding discrete states show...
chaotic features. They do not cross in energy, but show level repulsion. The real parts (energies) of the eigenvalue trajectories of narrow resonance states can cross with those of the broad states since the narrow and broad states exist at well separated time scales. In the transition region, the different time scales corresponding to the short-lived and long-lived resonance states are formed, and the overlapping of the different resonance states is directly visible in the cross section. In this regime, the cross section is enhanced due to the (at least partial) alignment of some states with the scattering states of the environment.

It is interesting to see that the system behaves according to expectations only at low level density. After passing the transition region with overlapping resonances by further variation of the parameter, the behavior of the system becomes counterintuitive: the narrow resonance states decouple more or less from the continuum of scattering wavefunctions and the number of localized states decreases. According to the results represented in the present paper, the role of exceptional points in quantum physics can be seen best in the non-Hermitian quantum physics. Knowing the mathematical properties of the exceptional points it is possible, on the one hand, to explain (qualitatively) some experimental results which could not be understood in the framework of the conventional Hermitian quantum physics in spite of much effort. Numerical calculations for some realistic cases have to be performed in order to compare theory and experiment in detail. On the other hand, quantum systems can be manipulated systematically for applications. Another interesting topic of non-Hermitian quantum physics results from the formal equivalence of the optical wave equation in $PT$ symmetric optical lattices to the quantum mechanical Schrödinger equation. This equivalence allows to receive much new information on quantum systems.

In any case, further theoretical and experimental studies in the field of non-Hermitian quantum physics, including that of exceptional points, will broaden our understanding of quantum mechanics. Moreover, the results are expected to be of great value for applications.

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