A Geometric Procedure for Computing Differential Characteristics of Multi-phase Electrical Signals using Geometric Algebra

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Abstract. This paper presents exploratory investigations on the concept of generalized geometrical frequency in electrical systems with an arbitrary number of phases by using Geometric Algebra and Differential Geometry. By using the concept of Darboux bivector it is possible to find a bivector that encodes the invariant geometrical properties of a spatial curve named electrical curve. It is shown how the traditional concept of instantaneous frequency in power networks can be intimately linked to the Darboux bivector. Several examples are used to illustrate the findings of this work.

Keywords: Geometric Algebra · Geometric Electricity · Power systems · Geometric Frequency.

1 Introduction

Multi-phase power systems play a crucial role in modern society due to the tremendous increase in energy needs. Moreover, with the proliferation of new generation smart grids based on a decentralized paradigm and focused on renewable energies, it is of utmost importance to investigate new methodologies that can deal with the distortion and unbalance scenarios produced by nonlinear loads. In this regard, voltage and frequency control is essential to achieve adequate stability, so appropriate tools are necessary for a better understanding of transient phenomena that can potentially disturb the grid. Currently, the concept of instantaneous frequency, widely used in electrical power systems, presents some issues in its classical definition as the time derivative of the phase angle of a signal. Apparently, this definition only holds for a sine representation of such a signal but fails for other representations where harmonics or transients are included \cite{9}. Existing techniques based on the Fourier or Hilbert Transform, for example, are unable to deal with these problems and give rise to a number of paradoxes, described in \cite{10}.

Geometric Algebra (GA) and Differential Geometry (DG) are among the most promising tools recently proposed, as evidenced by recent works \cite{11,12}. For example, in \cite{13,14} a geometrical interpretation of the frequency for three-phase electric circuits is proposed. Through the analysis of the invariants of spatial curves, it is possible to find a direct relation with the generalized concept of frequency that can be of interest in electrical power systems. However, a generalization to an arbitrary number of phases has yet to be presented. In particular, the study of multiphase electrical machines or electrical power systems with a number of phases greater than three can benefit from the investigations presented in this paper.
This approach not only allows a generalization through the use of exterior algebra or geometric algebra but also provides a unifying mathematical framework.

In this paper, GA and DG is applied to multi-phase electrical systems comprising any number of electrical phases by characterizing the voltage as a vector describing a curve in \( n \)-dimensional space. We will refer to such curves as “electrical curves” (EC). A new procedure is proposed that allows to obtain a multivector representation of the geometrical angular frequency, known as Darboux Bivector.

## 2 Electrical Curves and Geometric Properties

The electric curve approach is an effort for the application of concepts related to spatial curves within electrical systems. More specifically, geometric invariants (e.g. curvature) of the curve can accurately describe properties of interest for the power community. For example, the instantaneous or average grid frequency in power systems can be linked easily to curvature properties. This approach lead us to the concept of “geometric frequency” as introduced by Milano [11]. For higher dimensional, i.e., multi-phase power systems, the procedure depends on the derivations of Hestenes [5] (chapter 6) where arc-length parameterization is assumed. Here we provide a procedure that expresses this idea using time-dependent formulation of the original arc-length parameterization formulation.

### 2.1 Electrical Curve Definition and Parameterization

In practical power systems applications, we are typically given a uniformly sampled multi-phase and periodic voltage (or current) signals \( v_i[k] \) with \( k \) the sample index and \( i = 1, 2, \ldots, n \), the electrical phase index, and \( n \) the total number of electrical phases. We are allowed to create a discrete vector signal \( v[k] \) that describes a curve in \( n \)-dimensional Euclidean space. We call this object an “electrical curve” (EC).

Because the method depends on differential geometric characteristics of curves, the first step in this procedure is to use a suitable interpolation or fitting method to obtain a time-dependent differentiable curve \( v(t) \) that closely approximates the sampled signal \( v[k] \). However, real-world signals can typically contain a fair amount of noise and artifacts because of the Analog to Digital Converter (ADC) quantization sampling process, transient phenomena, etc., that make this a hard task. Fortunately, many procedures exist in practice [8,3]. We will assume this step is already implemented using any suitable method.

Assume now a time-dependent vector \( v(t) = \sum_{i=1}^{n} v_i(t) \sigma_i \) that describes a curve in \( n \)-dimensional Euclidean space defined on the interval \( t \in [t_0, t_1] \). Theoretically, we can express the curve by a reparameterization \( v(s) = v(t(s)) = \sum_{i=1}^{n} v_i(t(s)) \sigma_i \) using the arc-length variable \( s(t) = \int_{t_0}^{t} \|v'(\alpha)\| \, d\alpha \), where the relation \( s(t) \) and its functional inverse \( t(s) = s^{-1}(t) \) between parameters \( s \) and \( t \) are one-to-one. Note that this reparameterization \( s(t), v_i(t(s)) \) is not always possible to express in closed form in most cases. Therefore, in-depth new knowledge is required to overcome the exposed challenges and issues.

### 2.2 Time and Arc-Length Derivatives of Electrical Curves

The computation of derivatives for the EC is of paramount importance. They can be obtained in two different ways: with respect to parameter \( t \) or with respect to arc-length \( s \). The relationship between \( s \) and \( t \) is crucial in this regard.
To investigate this point, we can start by calculating the \( t \)-derivatives of the arc-length parameter \( s(t) \) using the vector \( \mathbf{v} \) and basic rules of vector differentiation \([7]\). The following illustrates the first four derivatives:

\[
\begin{align*}
    s'(t) &= \sqrt{\mathbf{v}' \cdot \mathbf{v}'} \\
    s''(t) &= \frac{1}{s'} \mathbf{v}' \cdot \mathbf{v}'' \\
    s'''(t) &= \frac{1}{s'} \left( \mathbf{v}'' \cdot \mathbf{v}'' + \mathbf{v}' \cdot \mathbf{v}''' - (s'')^2 \right) \\
    s''''(t) &= \frac{1}{s'} \left( 3\mathbf{v}''' \cdot \mathbf{v}'' + \mathbf{v}' \cdot \mathbf{v}'''' - 3s'' s''' \right)
\end{align*}
\] (1)

Having a suitable closed form time-dependent curve \( \mathbf{v}(t) = \sum_{i=1}^{n} v_i(t) \sigma_i \) of class \( C^p \) (i.e. differentiable up to \( p \) times in \( t \)), it is simple to find the \( t \)-derivatives of arbitrary degree \( 0 < m \leq p \) using:

\[
\begin{align*}
    \mathbf{v}'(t) &= \partial_t \mathbf{v}(t) = \sum_{i=1}^{n} v_i'(t) \sigma_i \\
    \mathbf{v}''(t) &= \partial_t^2 \mathbf{v}(t) = \sum_{i=1}^{n} v_i''(t) \sigma_i \\
    \vdots \\
    \mathbf{v}^{(m)}(t) &= \partial_t^m \mathbf{v}(t) = \sum_{i=1}^{n} [\partial_t^m v_i(t)] \sigma_i
\end{align*}
\] (2)

On the other hand, the derivatives of the curve with respect to arc-length parameter \( s \) can be obtained in a similar fashion:

\[
\begin{align*}
    \dot{\mathbf{v}}(s) &= \partial_s \mathbf{v}(s) = \sum_{i=1}^{n} \dot{v}_i(s) \sigma_i \\
    \ddot{\mathbf{v}}(s) &= \partial_s^2 \mathbf{v}(s) = \sum_{i=1}^{n} \ddot{v}_i(s) \sigma_i \\
    \dddot{\mathbf{v}}(s) &= \partial_s^3 \mathbf{v}(s) = \sum_{i=1}^{n} \dddot{v}_i(s) \sigma_i \\
    \vdots \\
    \mathbf{v}^{(m)}(s) &= \partial_s^m \mathbf{v}(s) = \sum_{i=1}^{n} [\partial_s^m v_i(s)] \sigma_i
\end{align*}
\] (3)

Interestingly, if one wants to express the above set of equations (3) in terms of the parameter \( t \), it is found that they are much more intricate. For example, using the chain rule, the first \( s \)-derivative in terms of \( t \) can be obtained for every \( v_i(s) \):

\(^1\) We use a dot for \( s \)-derivatives instead of a prime used in \( t \)-derivatives.
\[ \dot{v}_i(s(t)) = \partial_s v_i(s(t)) = \frac{d}{dt} \left( \frac{v_i'(s)}{s'} \right) = \frac{1}{s'} \left( s'' v_i' - s''' \right) \] (4)

From now on, we remove the \( t \) symbol to indicate dependency on time in \( s \) and \( v \) for simplicity. The second and third derivatives follow:

\[ \ddot{v}_i(s) = \partial_s^2 v_i(s) = \partial_s \dot{v}_i(s) = \frac{1}{s'} \frac{d}{dt} \left( \frac{v_i'(s)}{s'} \right) = \frac{1}{(s')^3} \left( s' s'' v_i' - s''' \right) \]

(5)

The relation between \( t \)-derivatives and \( s \)-derivatives of degree \( k \geq 2 \) in \( v_i \) is algebraically complicated, not at all as simple as the first derivative where \( \dot{v}(s) = \frac{dv}{ds} \). To illustrate this complex relation between \( t \)-derivatives and \( s \)-derivatives, the first 4 \( s \)-derivatives \( \dot{v}(s), \ddot{v}(s), \dddot{v}(s), \ddddot{v}(s) \) in terms of \( t \)-derivatives are presented:

\[ \dot{v}(s) = \frac{1}{s'} v' \]

\[ \ddot{v}(s) = \frac{1}{(s')^3} [s' v'' - s''' v'] \]

\[ \dddot{v}(s) = \frac{1}{(s')^3} \left[ (s'^2) v''' - 3 s' s'' v'' - \left( 3 s^2 - 3 (s')^2 \right) v' \right] \]

(6)

\[ \ddddot{v}(s) = \frac{1}{(s')^7} \left[ (s')^3 v'''' - 6 (s')^2 s'' v''' - \left( 4 (s')^2 s''' - 15 s' (s'')^2 \right) v'' + \left( 10 s' s'' s''' - 15 (s'')^3 - (s')^2 s'''' \right) v' \right] \]

Additionally, the following relations hold:

\[ ||\dddot{v}|| = 1 \] (7)

\[ ||\dddddot{v}|| = \frac{1}{(s')^2} \sqrt{||v''||^2 - 2 s' v' \cdot v'' + (s'')^2} \] (8)

\[ \dot{v} \cdot \dddot{v} = 0 \] (9)

The quantity \( ||\dddot{v}|| \) is important as it’s the base for computing the first curvature coefficient \( \kappa_1 \) of the curve \( v \) which has important implications for the geometrical frequency as illustrated later on.

### 2.3 Local Orthogonal Frames in Electrical Curves

The next step in the proposed procedure involves the application of an orthogonalization method, such as the Gram-Schmidt process, to the \( s \)-derivative vectors \( \dot{v}, \ddot{v} \) and so on. For a symbolic expression of the \( s \)-derivative vectors, this is simple to compute using either the Classical Gram-Schmidt
(CGS) [2], or the Geometric Algebra-based Gram-Schmidt (GAGS) [6]. For practical numerical computations, however, care must be taken when applying the CGS/GAGS as they are highly unstable numerically. A much more numerically stable alternative is the Modified Gram-Schmidt (MGS) procedure [2,1], which we found giving much better results when orthogonalizing higher (i.e. degree 3 or higher) s-derivatives. In any case, we essentially compute at each instant of time a local orthogonal frame \{u_1, u_2, \ldots, u_m\}, m \leq p from the set of p local s-derivative vectors \{\dot{v}, \ddot{v}, \ldots, \partial^p v\}. We can also normalize the frame \{u_1, u_2, \ldots, u_m\} to get a fully orthonormal local frame \{e_1, e_2, \ldots, e_m\}.

According to the chain rule presented in (4), the arc-length derivatives of the arc-length frame can be readily computed:

\[ \dot{e}_i(s) = \frac{1}{s'} e'_i \]  

(10)

It is also interesting to find the explicit relation between vectors \(e_1, e_2\) and vectors \(\dot{v}, \ddot{v}, v', v''\). This will be useful later for expressing the “grid angular velocity” blade, on which the concept of geometric frequency is based. Applying the GAGS process to \(\dot{v}, \ddot{v}\), we can write:

\[
\begin{align*}
\dot{u}_1 &= \dot{v} = \frac{1}{s'} v' \\
\dot{e}_1 &= \frac{u_1}{\|u_1\|} = \dot{v} = \frac{v'}{s'} \implies v' = s' e_1
\end{align*}
\]

(11)

And for the second vector we have:

\[
\begin{align*}
u_2 &= \ddot{v} = \frac{1}{(s')^2} (v'' - s'' e_1) \\
e_2 &= \frac{u_2}{\|u_2\|} = \frac{v'' - s'' e_1}{(s')^2 \|\ddot{v}\|} = \implies v'' = s'' e_1 + (s')^2 \|\ddot{v}\| e_2 = s'' e_1 + \sqrt{(v'')^2 - 2 \frac{s''}{s'} (v' \cdot v'') + (s'')^2} e_2
\end{align*}
\]

(12)

2.4 Frénet-Serret Curvature Coefficients of an Electrical Curve

The Frénet-Serret equations were formulated for three dimensions by Jean Frédéric Frénet and Joseph Alfred Serret and generalized to higher dimensions by Camille Jordan in the XIX century. They describe some dynamic properties of moving objects along curves in space by establishing a relationship between an orthonormal frame and its derivatives that also moves along the curve. The coefficients of these equations are known as curvature coefficients \(\kappa_i, i = 1, 2, \ldots, n - 1\) with \(n\) the number of dimensions. They satisfy the Frénet equations [5]:

\[
\begin{align*}
\dot{e}_1 &= \kappa_1 e_2 \\
\dot{e}_2 &= -\kappa_1 e_1 + \kappa_2 e_3 \\
\dot{e}_i &= -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1} \\
\dot{e}_n &= -\kappa_{n-1} e_{n-1}
\end{align*}
\]

(13)
We can re-write these equations as:

\[
\begin{align*}
\dot{e}_1 &= \kappa_1 e_2 \\
\dot{e}_2 + \kappa_1 e_1 &= \kappa_2 e_3 \\
\dot{e}_i + \kappa_{i-1} e_{i-1} &= \kappa_i e_{i+1} \\
\dot{e}_n &= -\kappa_{n-1} e_{n-1}
\end{align*}
\]  

(14)

This directly leads to the solutions:

\[
\begin{align*}
\kappa_1 &= \dot{e}_1 \cdot e_2 \\
\kappa_i &= (\dot{e}_i + \kappa_{i-1} e_{i-1}) \cdot e_{i+1} = \dot{e}_i \cdot e_{i+1}
\end{align*}
\]  

(15)

Another simpler alternative for computing \(\kappa_i\) is to use the relation from [4]:

\[
\kappa_i = \frac{\|u_{i+1}\|}{\|u_i\|}
\]

(16)

Here, the vectors \(u_i\) are defined as

\[
u_i = \partial^j_s v - \sum_{j=1}^{i-1} \frac{\partial^j_s v \cdot u_j}{u_j \cdot u_j} u_j
\]

(17)

and are computed using the CGS or MGS process, not the GAGS as before. However, in this work, a slightly different expression will be used for practical applications in power systems where the frequency is ultimately dependent on the time variable \(t\) (instead of the arc-length variable \(s\)) through the voltage \(v(t)\). A scaled version of \(\kappa_i\) will be used known as “scaled curvature coefficient”, \(k_i\):

\[
k_i (t) = s' \frac{\|u_{i+1}\|}{\|u_i\|} = s' \kappa_i
\]

(18)

These scaled curvature coefficients depend explicitly on the time variable \(t\).

3 The Darboux Blades

The original Darboux bivector \(\Omega_H\) is described in [5]. Note that we use the subscript \(H\) to highlight the Hestenes definition. It contains a summary of the local differential geometric information of the electrical curve. In this work, a slight modification of this original definition is used, where we flip the order of multiplication. The rationale behind this decision is to fit the practical definitions of frequency in power systems and the recently proposed geometric frequency for particular cases, such as sinusoidal and balanced conditions. We can use either of the following relations as the definition for the new Darboux bivector \(\Omega\):
\[ \Omega = \frac{1}{2} s' \sum_{i=1}^{n} e_i \wedge \dot{e}_i \]  
\[ = \sum_{i=1}^{n-1} k_i e_i \wedge e_{i+1} \]  
\[ = s' \sum_{i=1}^{n-1} \frac{u_i \wedge u_{i+1}}{\|u_i\|^2} \]  
\[ = s' \sum_{i=1}^{n-1} u_i^{-1} \wedge u_{i+1} \]  
\[ (19) \]

For the special case of \( s' = 1 \), then \( \Omega = -\Omega_H \). Using this definition for the Darboux bivector, the following important relation holds true:

\[ \dot{e}_i = e_i \mid \Omega = -\Omega \mid e_i \]  
\[ (23) \]

For practical usage we found that it is advisable to separate the Darboux bivector into several 2-blades \( \Omega_i \), which we will call the Darboux Blades (DBs), as follows:

\[ \Omega = \frac{1}{2} \sum_{i=1}^{n-1} \Omega_i \]

\[ \Omega_1 = k_1 e_1 \wedge e_2 \rightarrow \|\Omega_1\| = |k_1| \]

\[ \Omega_i = k_i e_{i-1} \wedge e_i + k_i e_i \wedge e_{i+1} = s' u_i^{-1} \wedge u_{i+1} + s' u_{i-1}^{-1} \wedge u_i \]

Note that \( \Omega_i \) are always 2-blades (i.e. represent planes in \( n \)-dimensions), while the Darboux bivector \( \Omega \) is generally a bivector, not a 2-blade (except in 3-dimensions where all bivectors are 2-blades).

The first DB \( \Omega_1 \), called the grid angular velocity blade, has special relevance for our analysis, satisfying the following relations:

\[ \dot{\theta} = \dot{e}_i \mid \Omega = \frac{1}{s'} \sqrt{\|\dot{\theta}\|^2 - 2 s'' (\dot{\theta} \cdot \dot{\theta}) + (s'')^2} e_1 \wedge e_2 \]

\[ \dot{\theta} \mid (\Omega - \Omega_1) = \sum_{i=2}^{n-1} \Omega_i = 0 \]
The difference $\Omega - \Omega_1$ is proportional to the bivector $B$ of relation (3.8) in [5], which always satisfies $\dot{v} \mid B = 0$. When the curve $v(t)$ is a planar one (such as in the case for 3-phase line-to-line voltages signal), all curvature coefficients are zero except $\kappa_1$. In this specific case, we have $\Omega - \Omega_1 = 0$, and the two quantities are equivalent $\Omega = \Omega_1$.

4 Example Signals

A number of theoretical examples are now presented to validate the proposed method. Sinusoidal and non-sinusoidal multi-phase systems with an arbitrary number of phases are studied. The goal is to obtain a geometric representation of the generalized frequency grid by using the concept of DB.

4.1 Multi-phase Balanced Sinusoidal Signal

Assume we have a balanced multi-phase sinusoidal electrical signal. It means that the amplitude is the same for all phases and the phase angle among them is $\frac{2\pi m}{n}$ with $m = 0, 1, \ldots, n-1$ and $n$ the number of phases.

$$v(t) = V \sum_{m=0}^{n-1} \cos \left( \omega t - \frac{2\pi m}{n} \right) \sigma_{m+1} = V \sum_{m=0}^{n-1} \left[ \cos \left( 2\pi \frac{m}{n} \right) \cos (\omega t) + \sin \left( 2\pi \frac{m}{n} \right) \sin (\omega t) \right] \sigma_{m+1}$$

$$= \cos (\omega t) V \sum_{m=0}^{n-1} \cos \left( 2\pi \frac{m}{n} \right) \sigma_{m+1} + \sin (\omega t) V \sum_{m=0}^{n-1} \sin \left( 2\pi \frac{m}{n} \right) \sigma_{m+1}$$

$$= \cos (\omega t) a + \sin (\omega t) b$$

(24)

with

$$a = V \sum_{m=0}^{n-1} \cos \left( 2\pi \frac{m}{n} \right) \sigma_{m+1}$$

$$b = V \sum_{m=0}^{n-1} \sin \left( 2\pi \frac{m}{n} \right) \sigma_{m+1}$$

In this case the two $n$-dimensional vectors $a, b$ are orthogonal with $\|a\|^2 = \|b\|^2 = \frac{n}{2} V^2$. This signal describes a perfect circular curve in the plane spanned by the $n$-dimensional orthogonal vectors $a, b$ inside the larger signal space defined by orthonormal basis vectors $\{\sigma_i\}_{i=1}^n$. We can express all relevant quantities using vectors $a$ and $b$ as follows:
\[ v' = -\omega (\sin(\omega t) a - \cos(\omega t) b) \]
\[ v'' = -\omega^2 (\cos(\omega t) a + \sin(\omega t) b) \]
\[ \|v'\|^2 = \frac{\omega^2}{2} nV^2 \]
\[ s' = \|v'\| = \frac{\omega}{\sqrt{2}} \sqrt{nV} \]
\[ u_1 = \dot{v} = -\frac{\sqrt{2}}{\sqrt{nV}} [\sin(\omega t) a - \cos(\omega t) b] \]
\[ u_2 = \ddot{v} = -\frac{2}{nV^2} [\cos(\omega t) a + \sin(\omega t) b] \]
\[ e_1 = \frac{u_1}{\|u_1\|} = -\frac{\sqrt{2}}{\sqrt{nV}} [\sin(\omega t) a - \cos(\omega t) b] \]
\[ e_2 = \frac{u_2}{\|u_2\|} = -\frac{\sqrt{2}}{\sqrt{nV}} [\cos(\omega t) a + \sin(\omega t) b] \]
\[ k_1 = s' \frac{\|u_2\|}{\|u_1\|} = \omega \]
\[ \Omega_1 = k_1 e_1 \wedge e_2 = \omega e_1 \wedge e_2 = 2\omega \left[ \frac{1}{nV^2} \right] a \wedge b \]

Note the constant nature of \( \|\Omega_1\| = \omega \) for this signal, which confirms the curve being a perfect circle.

### 4.2 Multi-phase Unbalanced Sinusoidal Signal

Assume we have a general (possibly unbalanced) multi-phase sinusoidal electrical signal. This now means that the amplitude can be different among phases and phase angle is not regularly spaced by \( \frac{2\pi m}{n} \). In this case the voltage vector is:

\[ v(t) = \sum_{m=1}^{n} V_m \cos(\omega t - \varphi_m) \sigma_m = \sum_{m=1}^{n} [V_m \cos(\varphi_m) \cos(\omega t) + V_m \sin(\varphi_m) \sin(\omega t)] \sigma_m \]
\[ = \cos(\omega t) \left[ \sum_{m=1}^{n} V_m \cos(\varphi_m) \sigma_m \right] + \sin(\omega t) \left[ \sum_{m=1}^{n} V_m \sin(\varphi_m) \sigma_m \right] \]
\[ = \cos(\omega t) a + \sin(\omega t) b \]

with

\[ a = \sum_{m=1}^{n} V_m \cos(\varphi_m) \sigma_m \]
\[ b = \sum_{m=1}^{n} V_m \sin(\varphi_m) \sigma_m \]
This signal describes an ellipse in the plane spanned by the \(n\)-dimensional vectors \(a, b\) inside the signal space defined by orthonormal basis vectors \({\sigma_i}\)\(^n\)\(_{i=1}\). We can express all relevant quantities using vectors \(a\) and \(b\) as follows:

\[
\begin{align*}
\mathbf{v}' &= -\omega (\sin (\omega t) \mathbf{a} - \cos (\omega t) \mathbf{b}) \\
\mathbf{v}'' &= -\omega^2 (\cos (\omega t) \mathbf{a} + \sin (\omega t) \mathbf{b}) \\
\|\mathbf{v}'\|^2 &= \frac{\omega^2}{2} g^2 \\
s' &= \|\mathbf{v}'\| = \frac{\omega}{\sqrt{2} g} \\
g &= \sqrt{(b^2 - a^2) \cos (2\omega t) - 2 (a \cdot b) \sin (2\omega t) + (b^2 + a^2)} \\
\mathbf{u}_1 &= \hat{v} = -\frac{\sqrt{2}}{g} [\sin (\omega t) \mathbf{a} - \cos (\omega t) \mathbf{b}] \\
\mathbf{u}_2 &= \hat{\mathbf{v}} = -\frac{4}{g^4} \left[ (b^2 \cos (\omega t) - (a \cdot b) \sin (\omega t)) \mathbf{a} + (a^2 \sin (\omega t) - (a \cdot b) \cos (\omega t)) \mathbf{b} \right] \\
\mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = -\frac{\sqrt{2}}{g} [\sin (\omega t) \mathbf{a} - \cos (\omega t) \mathbf{b}] \\
\mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = -\frac{\sqrt{2}}{g} \left[ \frac{b^2 \cos (\omega t) - (a \cdot b) \sin (\omega t)}{\sqrt{a^2 b^2 - (a \cdot b)^2}} \mathbf{a} + \frac{a^2 \sin (\omega t) - (a \cdot b) \cos (\omega t)}{\sqrt{a^2 b^2 - (a \cdot b)^2}} \mathbf{b} \right] \\
\mathbf{k}_1 &= s' \frac{\|\mathbf{u}_2\|}{\|\mathbf{u}_1\|} = \left( \frac{2}{g^2} \sqrt{a^2 b^2 - (a \cdot b)^2} \right) \omega \\
\Omega_1 &= \mathbf{k}_1 \mathbf{e}_1 \wedge \mathbf{e}_2 = \left( \frac{2}{g^2} \sqrt{a^2 b^2 - (a \cdot b)^2} \right) \omega \mathbf{e}_1 \wedge \mathbf{e}_2 \\
&= 2 \omega \left[ \frac{1}{g^2} - (a \cdot b) \frac{\sin (2\omega t)}{g^4} \right] \mathbf{a} \wedge \mathbf{b}
\end{align*}
\]

In this signal we have \(\|\Omega_1\| = \text{h}(t) \omega\), where the constant angular frequency of the grid \(\omega\) is scaled by the periodic time dependent factor:

\[
\text{h}(t) = \frac{2 \sqrt{a^2 b^2 - (a \cdot b)^2}}{(b^2 - a^2) \cos (2\omega t) - 2 (a \cdot b) \sin (2\omega t) + (b^2 + a^2)}
\]

Which has unit average value \(\bar{h} = \frac{1}{T} \int_0^T \text{h}(t) \, dt = 1\) where \(T = \frac{\pi}{\omega}\) is the time of a single cycle of \(\text{h}(t)\), which is half the time of a single cycle of the signal \(\mathbf{v}(t)\). This clearly indicates that by averaging \(\Omega = \Omega_1\) on one half cycle of this signal, followed by taking the norm of the resulting bivector, we again get the grid frequency \(\omega\) as intuitively expected.
4.3 Multi-phase Balanced Harmonic Signal

Finally, assume we have the following harmonic electrical signal:

\[ v(t) = \sqrt{2} \left[ 200 \sin(\omega t) + 20 \sin(2\omega t) - 30 \sin(7\omega t) \right] \sigma_1 + \sqrt{2} \left[ 200 \sin(\omega t - \frac{2\pi}{3}) + 20 \sin\left(2 \left(\omega t - \frac{2\pi}{3}\right)\right) - 30 \sin\left(7 \left(\omega t - \frac{2\pi}{3}\right)\right) \right] \sigma_2 + \sqrt{2} \left[ 200 \sin(\omega t + \frac{2\pi}{3}) + 20 \sin\left(2 \left(\omega t + \frac{2\pi}{3}\right)\right) - 30 \sin\left(7 \left(\omega t + \frac{2\pi}{3}\right)\right) \right] \sigma_3 \]

This electrical signal traces a planar symmetric curve in the plane orthogonal to vector \((1, 1, 1)\).

The expression for the first DB is:

\[ \Omega_1(t) = \frac{5\omega}{\sqrt{3}} \left( \frac{16 \cos(3\omega t) + 672 \cos(6\omega t) + 84 \cos(9\omega t) - 691}{-160 \cos(3\omega t) + 840 \cos(6\omega t) + 168 \cos(9\omega t) - 857} \right) (\sigma_{1,2} - \sigma_{1,3} + \sigma_{2,3}) \]

The average DB and its norm are given by:

\[ \bar{\Omega}_1 = \frac{1}{T} \int_0^T \Omega_1(t) \, dt = \sqrt{3}\omega (\sigma_{1,2} - \sigma_{1,3} + \sigma_{2,3}) \quad T = \frac{2\pi}{\omega} \]

\[ \|\bar{\Omega}_1\| = 3\omega \]

The norm of the average angular velocity blade \(\|\bar{\Omega}_1\|\) is proportional to the grid nominal angular frequency \(\omega\).

5 Conclusion

This paper has presented the generalized concept of geometrical angular frequency applied to multi-phase systems with arbitrary number of phases, extending previous works where linear algebra were used to define the concept of geometric frequency. Geometric Algebra and Differential Geometry have been used to represent vectors in \(n\)-dimensional spaces and to compute the geometric invariants associated to the generated spatial curves. Voltage signals have been used to create such vectors and to compute the Darboux Bivector, which encodes the specific differential geometric properties in the curves known as electrical curves. The method can be conveniently employed for a variety of engineering problems such as voltage stability, frequency control, to mention a few.

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