Abstract. Let $G$ be a split semisimple linear algebraic group over a field $k_0$. Let $E$ be a $G$-torsor over a field extension $k$ of $k_0$. Let $h$ be an algebraic oriented cohomology theory in the sense of Levine-Morel. Consider a twisted form $E/B$ of the variety of Borel subgroups $G/B$ over $k$.

Following the Kostant-Kumar results on equivariant cohomology of flag varieties we establish an isomorphism between the Grothendieck groups of the $h$-motivic subcategory generated by $E/B$ and the category of finitely generated projective modules of certain Hecke-type algebra $H$ which depends on the root datum of $G$, on the torsor $E$ and on the formal group law of the theory $h$.

In particular, taking $h$ to be the Chow groups with finite coefficients $\mathbb{F}_p$ and $E$ to be a generic $G$-torsor we prove that all indecomposable submodules of an affine nil-Hecke algebra $H$ of $G$ with coefficients in $\mathbb{F}_p$ are isomorphic to each other and correspond to the (non-graded) generalized Rost-Voevodsky motive for $(G, p)$.

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1. Introduction

Let $G$ be a split semisimple linear algebraic group over a field $k_0$ and let $E$ be a $G$-torsor over a field extension $k$ of $k_0$. Consider a twisted form $E/B$ of the variety of Borel subgroups $G/B$ of $G$ over $k$. Observe that $E/B$ is a smooth projective variety over $k$ that in general has no rational points. For example, for $G = PGL_p$ and a non-split $E$, $E/B$ is a variety of complete flags of ideals in a central simple division algebra of a prime degree $p$ over $k$.

Following [14, §64] consider the category of graded Chow motives $CM(k, \mathbb{F}_p)$ of smooth projective varieties over $k$ with finite coefficients $\mathbb{F}_p$. According to [25, Theorem 5.17] the motive $[E/B]$ of $E/B$ splits as a direct sum of Tate twists of some indecomposable motive $R$, a generalization of the Rost-Voevodsky motive, i.e.,

$$[E/B] \simeq \bigoplus_{i \in I} R(i).$$

Hence, if $\langle [E/B] \rangle$ denotes a pseudo-abelian subcategory generated by the motive $[E/B]$, i.e., a minimal pseudo-abelian category containing $[E/B]$, then

$$\langle [E/B] \rangle = \langle R(i) \rangle_{i \in I}.$$

Observe that in the non-graded case (in the category of motives $CM_k(k, \mathbb{F}_p)$ of [14, §64]) all Tate twists become isomorphic and we have $\langle [E/B]_s \rangle = \langle R_s \rangle$, where $[E/B]_s$ and $R_s$ denote the respective non-graded motives.

The motive $R$ has several remarkable properties (see [25, §5]). If $p$ is not a torsion prime of $G$, then $R$ coincides with the motive of a point, so $\langle [E/B] \rangle$ is generated by Tate twists $\mathbb{F}_p(i)$, $i = 0, \ldots, \dim G/B$. While being indecomposable over $k$, the motive $R$ becomes isomorphic to a direct sum of Tate twists over a splitting field $\bar{k}$ of $E$ (as $\bar{k}$ one can always take an algebraic closure of $k$ or a function field of $E/B$). Moreover, the Poincaré polynomial of $R$ over $\bar{k}$ is given by an explicit polynomial. For example, if $G$ is an exceptional group of type $F_4$ and $p = 3$, then $R|_{\bar{k}} \simeq \mathbb{F}_3 \oplus \mathbb{F}_3(4) \oplus \mathbb{F}_3(8)$ for a non-split $E$.

Only very few facts are known concerning the subcategory $\langle [E/B] \rangle$ of Chow motives with integer coefficients. An integer version of the motive $R$ was introduced and discussed in [31]; in [5], [10], [27] it was shown that $\langle [E/B] \rangle$ is not Krull-Schmidt (the uniqueness of a direct sum decomposition fails).

In the present paper we consider the category of $h$-motives with coefficients in a commutative ring $R = h(k)$, where $h$ is any algebraic oriented cohomology theory over $k$ in the sense of Levine-Morel [24], e.g., Chow ring with integer or finite coefficients, $K$-theory, algebraic cobordism $\Omega$ with coefficients in the Lazard ring. Let $\langle [E/B] \rangle_h$ (resp. $\langle [E/B]_s \rangle_h$) denote its pseudo-abelian subcategory generated by the (resp. non-graded) $h$-motive of $E/B$. Our main result (Theorem 8.1) establishes isomorphisms between the Grothendieck groups

$$K_0(\langle [E/B] \rangle_h) \simeq K_0(\mathbb{D}_{F}^{(0)})$$

and

$$K_0(\langle [E/B]_s \rangle_h) \simeq K_0(\mathbb{D}_{F}).$$

of the category $\langle [E/B] \rangle_h$ (resp. $\langle [E/B]_s \rangle_h$) and the category of finitely generated projective modules over a certain $R$-algebra $\mathbb{D}_{F}^{(0)}$ (resp. $\mathbb{D}_{F}$). More precisely, the algebra $\mathbb{D}_{F}^{(0)}$ is the degree 0 component of the $R$-algebra $\mathbb{D}_{F}$ defined using the formal push-pull operators (see Definition 7.5); it depends on the root datum of $G$, on the formal group law $F$ of the theory $h$ and on the subring of rational cycles in $h(G/B)$.
If $E$ is a generic $G$-torsor, then $\overline{D}_F$ can be replaced by the formal affine Demazure algebra $D_F$. The theory of such algebras and formal push-pull operators has been recently developed in [6], [19], [7], [8], [9] motivated by Bernstein-Gelfand-Gelfand [1], Demazure [11], [12], Bressler-Evens [2], [3], Kostant-Kumar [22], [21], Brion [4], Totaro [29] and Edidin-Graham [13]. The key properties of $D_F$ are

- It is a free module over the $T$-equivariant oriented cohomology ring $S = \mathcal{H}_{T}(k)$ of a point, where $T$ is a split maximal torus in $G$ [7].
- Its $S$-dual $D_F^S = Hom_S(D_F, S)$ is isomorphic to the $T$-equivariant oriented cohomology ring $\mathcal{H}_{T}(G/B)$ of $G/B$ [9].
- Its structure (generators and relations) is very close to those of the affine Hecke algebra [19].

For example, if $\mathcal{H}(-) = CH(-; \mathbb{F}_p)$ is the Chow ring with finite coefficients, then $D_F^S \simeq CH_T(G/B; \mathbb{F}_p)$ is the $T$-equivariant Chow ring and $D_F = H_{nil,p}$ is the affine nil-Hecke algebra over $\mathbb{F}_p$ (in the notation of Ginzburg [17, §12]) which is a free module of rank $|W|$ over the polynomial ring $\mathcal{S} = \mathbb{F}_p[x_1, \ldots, x_n]$, where $n$ is the rank of $G$ and $W$ is the Weyl group.

For generic $E$ the isomorphisms (1) then turn into (see Corollary 8.4)

$$K_0(\langle R(i) \rangle_{i \in I}) \simeq K_0(H^{0}_{nil,p}) \quad \text{and} \quad K_0(\langle R_+ \rangle) \simeq K_0(H_{nil,p}),$$

where the Tate twists $R(i)$ correspond to indecomposable $H^{0}_{nil,p}$-submodules. Moreover, there is a ring isomorphism

$$H_{nil,p} \simeq \text{Mat}_I \otimes \mathbb{R}(\text{End}(P_*)),$$

where $P_*$ is the projective $H_{nil,p}$-module corresponding to $R_+$ and $r$ is the $p$-part of the product of $p$-exceptional degrees of the group $G$.

The latter isomorphism specialized to $G = SL_n$ and $\mathcal{H} = CH$ gives [26, 3.1.16] and [23, Prop. 3.5]. Indeed, in this case $E$ is split, $r = 1$ and $\mathcal{S}$ is a free $S^W$-module with $\mathcal{H}(G/B) \simeq \mathbb{R} \otimes S^W \mathcal{S}$. Then by Lemma 7.3 one obtains that

$$H_{nil,p} \simeq S^W \otimes \mathbb{R} \text{Mat}_n(\mathcal{R}) \simeq \text{Mat}_n(S^W).$$

In the paper we restrict ourselves to varieties $E/B$ of Borel subgroups only. However, by [5] $B$ can be replaced by any special parabolic subgroup $P$ without affecting the isomorphism (1) for non-graded motives. For instance, for $G = PGL_n$, $\mathcal{H} = CH(-; \mathbb{Z})$ and $E$ corresponding to a generic central division algebra $A$ of degree $n$ we get

$$K_0(\langle [SB(A)]_+ \rangle) \simeq K_0(H_{nil,\mathbb{Z}}),$$

where $SB(A)$ is the Severi-Brauer variety of $A$ and $H_{nil,\mathbb{Z}}$ is the affine nil-Hecke algebra for $PGL_n$ with integer coefficients.

The paper is organized as follows. In section 2 we recall definitions and basic facts concerning Borel-Moore homology $h$ and the respective category of $h$-motives. We state a version of the Künneth isomorphism for cellular spaces. In the next section we generalize it to the equivariant setting. In section 4 we introduce the convolution product on the equivariant cohomology of products and study its properties. In the next section we identify the equivariant cohomology of $G$ with respect to the convolution product with the endomorphism ring of $T$-equivariant cohomology of $G/B$ and then in section 6 with the formal affine Demazure algebra. In section 7 we introduce the notion of a rational algebra of push-pull operators $\overline{D}_F$.
and identify it with the subring of rational endomorphisms. In the last section we prove isomorphisms (1) and provide applications and examples.

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2. Oriented (co-)homology

We recall definitions of an algebraic oriented Borel-Moore homology and of the respective category of correspondences. We also recall a version of the Künneth isomorphism for cellular spaces (Lemmas 2.4 and 2.5).

Fix a smooth scheme $S$ over a field $k$. Let $Sch_S$ denote the category of finite type quasi-projective separated $S$-schemes and let $Sm_S$ denote its full subcategory consisting of smooth quasi-projective $S$-schemes.

Following [24, Def. 5.1.3] consider an oriented graded Borel-Moore homology theory $\mathbf{h}_*$ defined on some admissible [24, (1.1)] subcategory $\mathcal{V}$ of $Sch_S$. So that there are pull-backs $f^*: \mathbf{h}_*(X) \to \mathbf{h}_*(Y)$ for l.c.i. morphisms $f: Y \to X$ in $\mathcal{V}$ of relative dimension $d$ and push-forwards $f_*: \mathbf{h}_*(Y) \to \mathbf{h}_*(X)$ for projective morphisms $f: X \to Y$ in $\mathcal{V}$. According to [24, Prop. 5.2.1] the Borel-Moore homology $\mathbf{h}_*$ restricted to $Sm_S$ defines an algebraic oriented cohomology theory $\mathbf{h}^*$ (with values in the category of graded commutative rings with unit) in the sense of [24, Def. 1.1.2] by

$$\mathbf{h}^{dim_S}X^{-\bullet}(X) := \mathbf{h}_*(X), \quad X \in Sm_S.$$ 

If the (co-)dimension is clear from the context we will write simply $\mathbf{h}(X)$.

Following [14, §63] and [31, §2] we define the category of $h$-correspondences $h-MR(S)$ over $S$. The objects are pairs $([X \to Y], (\alpha))$, where $[X \to S]$ is an isomorphism class of a smooth projective map $X \to S$ and $\alpha \in \mathbb{Z}$. The morphisms are defined by

$$Hom\mathbb{h}^hMR(S)(([X \to Y], (\alpha)), ([X \to Y], (\beta))) := \bigoplus_l Hom_{l-j}([X \to S], [X \to Y]),$$

taken over all connected components $Y_l$ of $Y$, where

$$Hom\mathbb{h}_*([X \to S], [X \to Y]) := h_{dim} Y_l \times_S X.$$  

The composition of morphisms is given by the correspondence product. Namely, if $p_i: X_1 \times_S X_2 \times_S X_3 \to X_j \times_S X_j$ denotes the projection obtained by removing the $i$-th coordinate, then given $\alpha \in h(X_1 \times_S X_2)$ and $\beta \in h(X_2 \times_S X_3)$ we set

$$\beta \circ \alpha := (p_2)_* (p_1^*(\beta) \cdot p_3^*(\alpha)) \in h(X_1 \times_S X_3).$$

The idempotent completion of $h-MR(S)$ denoted by $h-M(S)$ is called the category of $h$-motives. We simply write $[X]$ for the respective class in $h-M(S)$.

We also consider the non-graded version of $h-MR(S)$ and of $h-M(S)$ denoted by $h-MR_*(S)$ and $h-M_*(S)$ respectively, were the objects are given by isomorphisms classes $[X \to S]$ of smooth projective maps and the morphisms are defined by

$$Hom\mathbb{h}^hMR_*(S)([X \to Y], [X \to Y]) := h(Y \times_S X).$$

Definition 2.1. (cf. [24, (CD)]) Let $X$ be smooth projective over $S$. Suppose that there is a filtration by proper closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

such that
each irreducible component $X_{ij}$ of $X_i \setminus X_{i-1}$ is a locally trivial affine fibration over $S$ of rank $d_{ij}$, and

- the closure of $X_{ij}$ in $X$ admits a resolution of singularities $\overline{X}_{ij} \to \overline{X}_{ij}$ over $S$; we set $g_{ij} : \overline{X}_{ij} \to \overline{X}_{ij} \to X$ and, therefore, $(g_{ij})_*(1_{\overline{X}_{ij}}) \in h_{d_{ij}}(X)$.

We call such $X$ (together with the filtration) a cellular space over $S$.

**Definition 2.2.** We say that the theory $h$ satisfies the cellular decomposition (CD) property if given a cellular space $X$ over $S$ the respective elements $(g_{ij})_*(1_{\overline{X}_{ij}})$ form a $h(S)$-basis of $h(X)$.

**Example 2.3.** The property (CD) holds for any oriented Borel-Moore homology $h$ over a field $k$ of characteristic 0.

Indeed, the same reasoning as in [14, Thm. 66.2] shows that for every $Z \in Sm_S$ there is an isomorphism

$$\sum (g_{ij})_*(1) \times id_Z : \bigoplus_{ij} CH_{d_{ij}}(Z) \to CH_*(Z \times_S X).$$

By the Yoneda lemma (cf. [14, Lemma 63.9]) the latter induces an isomorphism in the category $CM_*(S)$ (cf. [14, Cor. 66.4]).

Following [30, §2] consider the specialization functor $\Omega \cdot M(S) \to CM_*(S)$, $[f : Y \to X] \mapsto f_*(1_Y)$. It is surjective on the classes of objects and morphisms. Moreover, for every $X$ the kernel of

$$\Omega_{\dim S} X(X \times_S X) \to CH_{\dim S} X(X \times_S X)$$

is $\Omega_{\geq 1}(k) \cdot \Omega_*(X \times_S X)$ by [24, Rem. 4.5.6]. Hence for every $y$ in this kernel

$$y^{\dim S+1} \in \Omega_{\dim S} X(X \times_S X) \cap (\Omega_{\geq (\dim S+1)}(k) \cdot \Omega_*(X \times_S X)).$$

So $y = 0$ since $\Omega_{<0}(Y) = 0$. Therefore, the kernel of

$$\text{End}_{\Omega \cdot M(S)}([X], i) \to \text{End}_{CM_*(S)}([X], i)$$

consists of nilpotents.

Finally, by [30, Lemma 2.1] the isomorphism $\sum_{ij} (g_{ij})_*(1)$ in $CM_*(S)$ can be lifted to an isomorphism in the category $\Omega \cdot M(S)$. Specializing it via $\Omega \to h$ we obtain the desired isomorphism.

**Lemma 2.4.** Assume that $h$ satisfies the property (CD). Let $X$ be a cellular space over $S$. Then there is an isomorphism in $h \cdot M(S)$

$$\sum_{ij} (g_{ij})_*(1_{\overline{X}_{ij}}) : \bigoplus_{ij} ([S], d_{ij}) \to [X],$$

where $(g_{ij})_*(1_{\overline{X}_{ij}}) \in h_{d_{ij}}(X) = Hom_{h \cdot M(S)}(([S], d_{ij}), [X])$.

**Proof.** Transversal base change implies that there is an isomorphism

$$\sum (g_{ij})_*(1) \times id_Z : \bigoplus_{ij} h_{-d_{ij}}(Z \times_S S) \to h_*(Z \times_S X)$$

for any $Z$ smooth projective over $S$. So by the Yoneda lemma (cf. [14, Lemma 63.9]) it induces an isomorphism in $h \cdot M(S)$ (cf. [14, Cor. 66.4]).
Lemma 2.5. Assume that $\mathfrak{h}$ satisfies the property (CD). Let $X$ be a cellular space over $S$. The pairing $\langle \cdot, \cdot \rangle: \mathfrak{h}(X) \otimes_{\mathfrak{h}(S)} \mathfrak{h}(X) \to \mathfrak{h}(S)$ given by $(a, b) = p_*(ab)$ is non-degenerate and the map

$$f: (\mathfrak{h}(X \times_S X), \circ) \to \text{End}_{\mathfrak{h}(S)}(\mathfrak{h}(X))$$

given by $a \mapsto f_a$, $f_a(x) = (p_2)_*(p_1^*(x) \cdot a)$ is an $\mathfrak{h}(S)$-linear isomorphism of graded rings. In particular, it gives an $\mathfrak{h}(S)$-linear isomorphism

$$(\mathfrak{h}_{\dim_S} X (X \times_S X), \circ) \simeq \text{End}_{\mathfrak{h}_M(S)}(X).$$

Observe that the endomorphism ring of $\mathfrak{h}(S)$-linear operators $\text{End}_{\mathfrak{h}(S)}(\mathfrak{h}(X))$ is a graded ring. Its $n$-th graded component consists of operators increasing the codimension by $n$. By definition the subring of degree-0 operators (preserving the codimension) coincides with $\text{End}_{\mathfrak{h}_M(S)}(X)$.

Proof. By the previous lemma there is an isomorphism

$$\bigoplus_{i,j} \mathfrak{h}(S) = \bigoplus_{k=-\infty}^{\infty} \text{Hom}([[S], k], \oplus_{ij}([S], d_{ij})) \xrightarrow{\cong} \bigoplus_{k=-\infty}^{\infty} \text{Hom}(([[S], k], [S]) = \mathfrak{h}(X),$$

where each component is given by $x \mapsto x \cdot (g_{ij})_*(1)$. Let $\sum_{i,j} a_{i,j}: [X] \to \oplus_{ij}([S], d_{ij})$ be the inverse isomorphism in $\mathfrak{h}(M(S))$. Observe that

$$a_{i,j} \in \text{Hom}([X], ([S], d_{ij})) = \mathfrak{h}_{\dim(X/S)-d_{ij}}(X).$$

Since $a_{i,j} \circ (g_{ij})_*(1) = p_*(a_{i,j} \cdot (g_{ij})_*(1)) = \delta_{i,j}$, the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate.

The pairing $\langle \cdot, \cdot \rangle$ gives an isomorphism $\mathfrak{h}(X) \to \text{Hom}_{\mathfrak{h}(S)}(\mathfrak{h}(X), \mathfrak{h}(S))$ and, hence, an isomorphism $\text{End}_{\mathfrak{h}(S)}(\mathfrak{h}(X)) \xrightarrow{\cong} \mathfrak{h}(X) \otimes_{\mathfrak{h}(S)} \mathfrak{h}(X)$. Consider the composition

$$\rho: \mathfrak{h}(X \times_S X) \xrightarrow{f} \text{End}_{\mathfrak{h}(S)}(\mathfrak{h}(X)) \xrightarrow{\cong} \mathfrak{h}(X) \otimes_{\mathfrak{h}(S)} \mathfrak{h}(X)$$

and a map $\pi: \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X \times_S X)$ given by $\pi(a \otimes b) = p_1^*(a) \cdot p_2^*(b)$.

By definition, we have

$$f p_1^*(a) p_2^*(b)(x) = (p_2)_*(p_1^*(x) p_1^*(a) p_2^*(b)) = (x, a)b.$$

Hence, $\rho(\pi(a \otimes b)) = a \otimes b$ and the map $\rho$ is surjective. By the property (CD) for $X \times_S X \to X$, $\mathfrak{h}(X \times_S X)$ is a free $\mathfrak{h}(X)$-module of rank $rk_{\mathfrak{h}(S)}(\mathfrak{h}(X))$. Thus, $\rho$ is a surjective homomorphism between free modules of the same rank, hence, it is an isomorphism. \hfill $\Box$

Let $C$ be any pseudo-abelian category. For an object $X \in C$ consider a subcategory $\langle X \rangle$ generated by $X$, i.e., the smallest pseudo-abelian subcategory of $C$ that contains $X$.

Lemma 2.6. The category $\langle X \rangle$ is equivalent to the category of finitely generated projective $End_C(\langle X \rangle)$-modules.

Proof. Denote $End_C(\langle X \rangle)$ by $R$. Every element $Y$ of $\langle X \rangle$ is isomorphic to the image $p(X^{\oplus n})$ of some idempotent $p \in End_C(\langle X \rangle) = Mat_n(R)$ and $\text{Hom}_C(X, p(X^{\oplus n})) = p(R^n)$. Note that

$$\text{Hom}_C(p(X^{\oplus n}), p'(X^{\oplus n'})) = p' \text{Hom}_C(X^{\oplus n}, X^{\oplus n'}) p = \text{Hom}_R(p(R^n), (p'R^{n'})).$$

Then the functor $Y \mapsto \text{Hom}_C(X, Y)$ establishes an equivalence between $\langle X \rangle$ and the category of finitely generated projective right $R$-modules. \hfill $\Box$
Corollary 2.7. The category \( \langle [E/B]_h \rangle \) (resp. \( \langle [E/B]_h \rangle^* \)) is equivalent to the category of finitely generated projective modules over the endomorphism ring of the (resp. non-graded) \( h \)-motive of \( E/B \).

3. The equivariant Künneth isomorphism

In the present section we introduce an equivariant Borel-Moore homology following [7, §2] and [18]. We provide an equivariant analogue of the Künneth isomorphism (Lemma 3.7).

Let \( G \) be a smooth group scheme over \( S \). Consider an admissible subcategory \( \mathcal{V}^G \) of the category of \( G \)-varieties \( X \in \text{Sch}_S \) with \( G \)-equivariant morphisms. By a \( G \)-equivariant oriented (graded) Borel-Moore homology theory we will call an additive functor \( h^G_\bullet \) from \( \mathcal{V}^G \) to graded abelian groups such that

1. There are pull-backs for l.c.i. maps and push-forwards for projective maps that satisfy

\[
\text{(TS) (l.c.i. base change)} \quad \text{For a Cartesian square } \begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\text{ where } f \text{ (hence } f' \text{) is l.c.i. and } g \text{ (hence } g' \text{) is projective, we have } f^* g_* = g'_*(f')^*.
\]

\[
\text{(Loc) (localization)} \quad \text{If } U \subset X \text{ is an open } G \text{-equivariant embedding with } Z = X \setminus U, \text{ then there is a right exact sequence:}
\]

\[
h^G_\bullet(Z) \to h^G_\bullet(X) \to h^G_\bullet(U) \to 0.
\]

2. The functor \( h^G_\bullet \) restricted to \( \text{Sm}_S \) defines a graded \( G \)-equivariant oriented cohomology theory \( h^G_\bullet \) in the sense of [9] (we refer to [9, §2, A1-9] for the precise definition) by

\[
h^G_{\dim S} X^{-\bullet}(X) := h^G_\bullet(X), \quad X \in \text{Sm}_S.
\]

In addition to the axioms of [9, §2] we require that \( h^G \) satisfies the following stronger version of the homotopy invariance axiom:

\[
\text{(HI) (extended homotopy invariance)} \quad \text{Let } p: Y \to X \text{ be a } G \text{-equivariant torsor of a vector bundle of rank } r \text{ over } X, \text{ then the pull-back induced by projection}
\]

\[
p^*: h^G_\bullet(X) \to h^G_\bullet(Y)
\]

is an isomorphism.

If a variety is smooth we will always use the cohomology notation.

Example 3.1. Given a linear algebraic group \( G \) over a field \( k \) of characteristic zero an example of such \( G \)-equivariant Borel-Moore homology theory \( h^G_\bullet \) was constructed in [18] as follows.

Consider a system of \( G \)-representations \( V_i \) and its open subsets \( U_i \subseteq V_i \) such that

- \( G \) acts freely on \( U_i \) and the quotient \( U_i/G \) exists as a scheme over \( k \),
- \( V_{i+1} = V_i \oplus W_i \) for some representation \( W_i \),
- \( U_i \subseteq U_i \oplus W_i \subseteq U_{i+1} \), and \( U_i \oplus W_i \to U_{i+1} \) is an open inclusion, and
- \( \text{codim}(V_i \setminus U_i) \) strictly increases.
Such a system is called a good system of representations of $G$.

Let $X \in Sch_k$ be a $G$-variety. Following [18, §3 and §5] the inverse limit induced by pull-backs

$$
\lim_{i} h^\bullet \cdot \dim G + \dim U_i (X \times^G U_i), \quad X \times^G U_i = (X \times_k U_i)/G,
$$

does not depend on the choice of the system $(V_i, U_i)$ and, hence, defines the $G$-equivariant oriented homology group $h^*_G(X)$.

In the present paper we will extensively use the following property (cf. [9, §2, A6]) of an equivariant theory

(Tor) Let $X \to X/G$ be a $G$-torsor over $S$ and a $G'$-equivariant map for some group scheme $G'$ over $S$. Then there is an isomorphism

$$
\mathbb{E}_{G \times G'}^\bullet(X) \xrightarrow{\sim} \mathbb{E}_{G'}^\bullet(X/G),
$$

that is natural with respect to the maps of pairs

$$(\phi, \gamma): (X, G \times G') \to (X_1, G_1 \times G'_1), \quad \phi(x \cdot (g, g')) = \phi(x) \cdot \gamma(g, g').$$

Observe that the theory of Example 3.1 satisfies this property by [18, Prop. 27].

We have the following equivariant analogues of Definitions 2.1 and 2.2

**Definition 3.2.** Let $X \in V^G$. Suppose that there is a filtration by $G$-equivariant proper closed subschemes

$$
\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n = X
$$
such that

- each irreducible component $X_{ij}$ of $X_i \setminus X_{i-1}$ is a $G$-equivariant (locally trivial) affine fibration over $S$ of rank $d_{ij}$, and
- the closure of $X_{ij}$ in $X$ admits a $G$-equivariant resolution of singularities

$$
g_{ij}: \bar{X}_{ij} \to \bar{X}_{ij} \text{ over } S.
$$

We call such $X$ (together with the filtration) a $G$-equivariant cellular space over $S$.

**Definition 3.3.** We say that the equivariant theory $h^G$ satisfies the cellular decomposition (CD) property if given a $G$-equivariant cellular space X over S the respective elements $(g_{ij}), (1_{X_{ij}})$ form a $h^G(S)$-basis of $h^G(X)$.

**Lemma 3.4.** Suppose a morphism $f: X \to Y$ in $Sm_k$ factors as $f: X \to L \to Y$ where $p: L \to X$ is a vector bundle, $z: X \to L$ is a zero section and $j$ is an open embedding.

Then for every projective map $a: Y' \to Y$ and $X' = X \times_Y Y'$ the following diagram of pull-back and push-forward maps commutes (we omit the grading)

$$
\begin{array}{c}
\text{h}(X') \xrightarrow{\alpha^*} \text{h}(X) \\
\text{f}^* \downarrow \quad \downarrow \text{f}^* \\
\text{h}(Y') \xrightarrow{\alpha^*} \text{h}(Y)
\end{array}
$$

*Proof.* Observe that the map $f': X' \to Y'$ factors as $X' \to L \times_Y Y' \to Y'$ where $z'$ is the zero section of the vector bundle $p': L' = L \times_Y Y' \to X'$ and $j'$ is an open embedding. Let $b$ denote the canonical map $L' \to L$. Since $j$ and $j'$ are flat,
we have \( j^*a_* = b_*j^* \) by the l.c.i. base change for oriented theories. Note that by the homotopy invariance \( z^* = (p^*)^{-1} \) and \( z'^* = (p'^*)^{-1} \). Since \( p \) and \( p' \) are flat, \( p^*a'_* = b_*p'^* \). Then \( z^* b_* = a'_*z'^* \) and

\[
f^* a_* = z^* j^* a_* = z*b_* j^* = a'_* z'^* j^* = a'_* f'^*.
\]

\( \square \)

**Remark 3.5.** If \((V_i, U_i)\) is a good system of representations of Example 3.1, then for any \( G \)-variety \( X \) the connecting maps \( X \times^G U_i \rightarrow X \times^G U_{i+1} \) factor as in Lemma 3.4, i.e., we have \( X \times^G U_i \rightarrow X \times^G (U_i \oplus W_i) \rightarrow X \times^G U_{i+1} \).

**Example 3.6.** Let \( h^G \) be the equivariant theory of Example 3.1. Then the property (CD) holds for \( h^G \).

Indeed, consider a good system of representations \( \{(V_i, U_i)\}_i \) for \( X \). The sub-varieties \( X_i \times^G U_j, i = 0 \ldots n \) form a cellular filtration on \( X \times^G U_j \) over \( S \times^G U_j \). Note that \( \overline{X_i} \times^G U_j \) is a resolution of singularities of \( X_i \times^G U_j \). By (CD) for \( h \) the set \( \{(f_i \times^G id_{U_j})_*(1)\}_i \) forms a basis of \( h(X \times^G U_j) \) as a \( h(S \times^G U_j) \)-module. By Lemma 3.4 the following diagram commutes:

\[
\begin{array}{ccc}
  h(\overline{X_i} \times^G U_{j+1}) \ar[d]^{i^*_j} & \ar[l]_{(g_{i,j+1})_*} & h(X \times^G U_{j+1}) \ar[d]^{i^*_j} \\
  h(\overline{X_i} \times^G U_j) & \ar[l]_{(g_{i,j})_*} & h(X \times^G U_j)
\end{array}
\]

So \( i^*_m((f_i \times^G id_{U_{j+1}})_*(1)) = (f_i \times^G id_{U_j})_*(1) \), which implies that the elements \( f_*(1) = \lim_j((f_i \times^G id_{U_j})_*(1)) \) form a basis of \( h^G(X) \) over \( h^G(S) \).

As for usual oriented theories we then obtain

**Lemma 3.7.** Assume that \( h^G \) satisfies the property (CD). Let \( X \) be a \( G \)-equivariant cellular space over \( S \). Then the pairing \( \langle \cdot, \cdot \rangle: h^G(X) \otimes_{h^G(S)} h^G(X) \rightarrow h^G(S) \) given by \( (a, b) = p_*(ab) \) is non-degenerate and the map

\[ f: (h^G(X \times^S X), \circ) \rightarrow \text{End}_{h^G(S)} h^G(X) \]

given by \( a \mapsto f_a, f_a(x) = (p_2)_*(p_1^*(x) \cdot a) \) is an \( h^G(S) \)-linear isomorphism of rings. In particular, there is an \( h^G(S) \)-linear isomorphism

\[ (h^G_{\text{dim}_S X} X \times^S X, \circ) \rightarrow \text{End}_{h^G \text{-}M(S)} h^G(X), \]

where \( h^G \text{-}M(S) \) is the respective category of \( G \)-equivariant motives.

4. THE CONVOLUTION PRODUCT

In the present section we introduce the convolution product on the equivariant Borel-Moore homology (Definition 4.3) of the product \( G \times^G \times \ldots \times^G \). We relate this product to the usual correspondence product for the associated torsors (Lemma 4.6) and study its behaviour under the base change (diagram (6)).

Let \( G \) be a smooth algebraic group over \( k \) and let \( E \) be a \( G \)-torsor over \( k \) (\( G \) acts on the right). By definition there is an isomorphism \( \rho: E \times_k G \rightarrow E \times_k E \) given on points by \( (e, g) \mapsto (e, eg) \). For each \( i \geq 0 \) it induces an isomorphism

\[ \rho_i: E \times_k G^i \rightarrow E^{i+1}, \quad (e, g_1, g_2, \ldots, g_i) \mapsto (e, eg_1, eg_2, \ldots, eg_i). \]

Consider the composition

\[ \gamma_i: E^{i+1} \xrightarrow{\rho_{i-1}} E \times_k G^i = E \times_k G^i \xrightarrow{pr} G^i. \]
The coordinate-wise right $G^{i+1}$-action on $E^{i+1}$ induces an action on $E \times_k G^i$ and, hence, on $G^i$. For instance, on points it is given by

\[(3) \quad (e, g_1, \ldots, g_i) \cdot (h_1, \ldots, h_{i+1}) = (eh_1h_1^{-1}g_1h_2, \ldots, h_i^{-1}g_i h_{i+1}).\]

Consider projections $p_j: E^{i+1} \to E^i$ obtained by removing the $j$-th coordinate and the respective $G^i$-action on $E^i$. For each $i \geq 1$, $1 \leq j \leq i+1$ there is a commutative diagram of $G^i$-equivariant maps

\[
\begin{array}{ccc}
E^{i+1} & \xrightarrow{\gamma_i} & G^i \\
\downarrow{p_j} & & \downarrow{\pi_j} \\
E^i & \xrightarrow{\gamma_{i-1}} & G^{i-1}
\end{array}
\]

where $\pi_1(g_1, \ldots, g_i) = (g_1^{-1}g_2, \ldots, g_1^{-1}g_i)$ and $\pi_j(g_1, \ldots, g_i) = (g_1, \ldots, g_{j-1}, g_i)$ for $j > 1$.

**Example 4.1.** For $i = 1$ it gives a commutative diagram of $G$-equivariant maps

\[
\begin{array}{ccc}
E \times_k E & \xrightarrow{\gamma_1} & G \\
\downarrow{p_j} & & \downarrow{\pi_j} \\
E & \xrightarrow{\gamma_0} & \text{Spec } k
\end{array}
\]

where $\gamma_0, \pi_1, \pi_2$ are the structure maps, $p_1, p_2$ are the corresponding projections and $\gamma_1(e, eg) = g$. Moreover, if $E$ is trivial, then $\gamma_1 = \pi_1: G \times_k G \to G$, $(g_1, g_2) \mapsto g_1^{-1}g_2$.

Let $H$ be an algebraic subgroup of $G$ such that $G/H$ is a smooth variety over $k$. We can view $G^i$ as an $H$-torsor over $G^i/H$, where $H$ acts on $G^i$ via the $j$-th coordinate of $G^{i+1}$. By definition, the $H^i$-equivariant map $\pi_j$ factors as

\[\pi_j: G^i \xrightarrow{\pi_j} G^i/H \xrightarrow{\pi_j} G^{i-1},\]

where the second map $\pi_j$ is a fibration with a fibre $G/H$.

**Example 4.2.** The map $\pi_1$ factors through the quotient maps modulo the diagonal action

\[\pi_1: G^i \xrightarrow{\pi_1} G^i/\Delta(H) \xrightarrow{\pi_1} G^i/\Delta(G) = G^{i-1},\]

which are equivariant with respect to the usual coordinate-wise $H^i$-action.

Consider an equivariant Borel-Moore homology theory $h$. For every $1 \leq j \leq i+1$ consider the action of the $j$-th copy of $H$ on $G^i$. The property (Tor) gives an isomorphism

\[(5) \quad h_{H^j}(G^i/H) \xrightarrow{\cong} h_{H^{i+1}}(G^i),\]

where $H^{i+1}$ acts on $G^i$ as in (3). Unless explicitly mentioned we will always identify these two rings.

Set $S = h_H(G^0) = h_H(k)$ and set the convolution product on $S$ to be the usual intersection product.

**Definition 4.3.** Assume that $G/H$ is a smooth projective variety over $k$. We define the $S$-linear convolution product $'*'$ on $h_H(G^{i-1})$, $i \geq 2$ to be the composite

\[
h_{H^i}(G^i) \otimes h_{H^i}(G^{i-1}) \xrightarrow{\pi_{i-1}^* \otimes \pi_{i+1}^*} h_{H^{i+1}}(G^i) \otimes h_{H^{i+1}}(G^{i-1}) \xrightarrow{'}
\]
The central object of the present paper is the convolution ring \((h_{H^2}(G), \circ)\), i.e., the case \(i = 2\). In the next sections we will show that \((h_{H^2}(G), \circ)\) (where \(B\) is a Borel subgroup of a semisimple split \(G\)) can be identified with the formal affine Demazure algebra.

**Example 4.4.** In the case \(i = 3\) the convolution ring \((h_{H^3}(G^2), \circ)\) is isomorphic to \(h_{\Delta(H)}((G/H)^2)\) with respect to the usual correspondence product. Indeed, the maps \(\pi_i: G^3 \to G^2\), \(i = 2, 3, 4\) induce \(\Delta(H)\)-equivariant projections \((G/H)^3 \to (G/H)^2\). The isomorphism then follows by (Tor).

Observe that if \(G/H\) is an \(H\)-equivariant cellular space and \(h_H\) satisfies (CD), then by Lemma 3.7 there is an \(S\)-linear ring isomorphism

\[
(h_{H^3}(G^2), \circ) \cong \text{End}_S h_H(G/H).
\]

**Lemma 4.5.** For \(i \geq 1\) the map \(\pi_1\) induces an injective ring homomorphism with respect to the convolution products

\[
(h_{H^i}(G^{i-1}), \circ) \xrightarrow{\pi_1^*} (h_{H^{i+1}}(G^i), \circ).
\]

**Proof.** For \(i = 1\) it follows from the fact that the convolution product on \(h_{H^2}(G)\) is \(S\)-linear.

For \(i \geq 2\) for each \(i - 1 \leq j \leq i + 1\) we have \(\pi_j \circ \pi_1 = \pi_1 \circ \pi_{j+1}\). Since push-forwards commute with flat pull-backs by (TS), there are commutative diagrams in equivariant cohomology

\[
\begin{array}{ccc}
h_{H^{i+1}}(G^i) & \xrightarrow{\pi_1^*} & h_{H^{i+2}}(G^{i+1}) \\
\downarrow \pi_j & & \downarrow \pi_{j+1}^* \\
h_{H^i}(G^{i-1}) & \xrightarrow{\pi_1^*} & (h_{H^{i+1}}(G^i))
\end{array}
\]

Finally, there is a \(H^1\)-equivariant section of the map \(\bar{\pi}_1: G^i/\Delta(H) \to G^{i-1}\) given by \((g_1, \ldots, g_{i-1}) \mapsto (1, g_1, \ldots, g_{i-1})\), so \(\bar{\pi}_1^*\) is injective. \(\square\)

**Lemma 4.6.** The map \(\gamma_1\) induces a ring homomorphism

\[
(h_{H^2}(G), \circ) \xrightarrow{\gamma_1^*} (h_{H^2}(E^2), \circ) \xrightarrow{\cong} (h((E/H)^2), \circ),
\]

where the last ring is viewed with respect to the correspondence product (2).

**Proof.** By (TS) the diagram (4) gives rise to commutative diagrams in cohomology

\[
\begin{array}{ccc}
h_{H^2}(G^2) & \xrightarrow{\gamma_1^*} & h_{H^2}(E^3) \\
\downarrow (\bar{\pi}_2) & & \downarrow (\bar{\pi}_2^* \\
h_{H^2}(G) & \xrightarrow{\gamma_1^*} & h_{H^2}(E^2)
\end{array}
\]

The last isomorphism follows by (Tor). \(\square\)
Let \( \bar{k} \) denote the splitting field of a \( G \)-torsor \( E \) so that \( G_{\bar{k}} = E_{\bar{k}} \). Since the base change preserves the convolution product, combining Lemmas 4.5 and 4.6 we obtain two commutative diagrams of convolution (correspondence) rings

\[
\begin{array}{ccc}
\gamma_1^* : h_{H^2}(G) & \xrightarrow{pr^*} & h_{H^2}(E) \\
\downarrow \text{res}_{k/k} & \text{res}_{k/k} & \downarrow \text{res}_{k/k} \\
\gamma_1^* : h_{H^2}(G_{\bar{k}}) & \xrightarrow{\bar{p}_1^*} & h_{H^2}(G_{\bar{k}}^2) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\gamma_0^* : h_H(k) & \xrightarrow{\rho_0opr^*} & h_H(E) \\
\downarrow \text{res}_{k/k} & \text{res}_{k/k} & \downarrow \text{res}_{k/k} \\
\gamma_0^* : h_H(\bar{k}) & \xrightarrow{q^*opr_1^*} & h_H(G_{\bar{k}}) \\
\end{array}
\]

where \( \text{res}_{k/k} \) is the base change map. Combining these two diagrams we obtain a commutative diagram of convolution rings

\[
\begin{array}{ccc}
h_H(E) \otimes_S h_{H^2}(G) & \xrightarrow{(p_1^*, \gamma_1^*)} & h_{H^2}(E) \\
\downarrow \text{res}_{k/k} & \text{res}_{k/k} & \downarrow \text{res}_{k/k} \\
h_H(G_{\bar{k}}) \otimes_S h_{H^2}(G_{\bar{k}}) & \xrightarrow{(\bar{p}_1^*, \gamma_1^*)} & h_{H^2}(G_{\bar{k}}^2),
\end{array}
\]

where the left convolution rings are \( h_H(E) \)- and \( h_H(G_{\bar{k}}) \)-linear.

5. The Subring of Push-Pull Operators

In the present section we prove that if \( H \) is the Borel subgroup of a split semisimple linear algebraic group, then the convolution ring \( h_{H^2}(G) \) of Definition 4.3 can be identified with the subring of push-pull operators (Corollary 5.3). Our arguments are essentially based on the Bruhat decomposition of \( G \) stated using the \( G \)-orbits on the product \( G/H \times_k G/H \) and the resolution of singularities (8).

As before assume that \( G/H \) is a smooth projective variety over \( k \). In the notation of the previous section consider the \( H^2 \)-equivariant maps of Example 4.2.

\[
\pi_1 : G^2 \xrightarrow{g} G^2/\Delta(H) \xrightarrow{\pi_1} G^2/\Delta(G) = G, \quad (g_1, g_2) \mapsto g_1^{-1} g_2.
\]

Since \( G^2 \) is a \( \Delta(G) \)-torsor over \( G \) (\( \Delta(H) \)-torsor over \( G^2/\Delta(H) \)), by the property (Tor) the induced \( \Delta(G) \times H^2 \)-equivariant pull-backs on cohomology coincide with the forgetful maps

\[
\begin{array}{ccc}
\gamma_1^* : h_{H^2}(G) & \xrightarrow{\pi_1^*} & h_{H^2}(G^2) \\
\downarrow \cong & \downarrow \cong & \downarrow \cong \\
\gamma_1^* : h_{H^2}((G/H)^2) & \xrightarrow{q^*} & h((G/H)^2)
\end{array}
\]

Moreover, by Lemma 4.5 it is a commutative diagram of convolution rings.
Let $G$ be a split semisimple linear algebraic group over $k$ and let $h$ be an equivariant theory that satisfies property (CD). We fix a Borel subgroup $B$ of $G$ containing a split maximal torus $T$. By Bruhat decomposition (e.g. [28])

$$G = \Pi_{w \in W} B \dot{w} B, \quad \dot{w} \in N_T,$$

is the disjoint union of $B^2$-orbits of $G$, where $W = N_T/T$ is the Weyl group and $N_T$ is the normalizer of $T$ in $G$. Projecting this decomposition onto $X = G/B$ gives a $B$-equivariant cellular filtration on $X$ by closures $\overline{X}_w$ of affine spaces $X_w = B \dot{w} B/B$ of dimension $l(w)$ (the length of $w$). The preimage $\pi_1^{-1}(B \dot{w} B)$ is a $\Delta(G)$-orbit in $G^2$ (here $H = B$). Let $\mathcal{O}_w$ denote its image via $G^2 \to X^2$ and let $\overline{\mathcal{O}}_w$ denote its closure. Observe that both $\mathcal{O}_w$ and $\overline{\mathcal{O}}_w$ are $\Delta(G)$-invariant in $X^2$.

By properties of the Bruhat decomposition (see [28, §1]) it follows that the projection $\mathcal{O}_w \to X^2 \to X$ is a torsor of a vector bundle over $X$ with fibre $X_w$. Indeed, the transition functions are affine since they are given by the action of $B$ on the left on $B \dot{w} B/B$ that is by $T$ acting on the product of the respective root subgroups $\prod_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha \cdot \mathfrak{p}_{\alpha})$. Let $\mathfrak{p}_{\alpha}$ via the multiplication $t \cdot x = \alpha(t)x$, $t \in T$, $x \in \mathfrak{g}_\alpha$. Then by the property (CD) the cohomology $h(X)$ and $h(X)$ with filtration given by the closures $\overline{\mathcal{O}}_w$.

Assume that for each $w \in W$ we are given a $G$-equivariant resolution of singularities $\mathcal{O}_w \to \overline{\mathcal{O}}_w$. Let $[\mathcal{O}_w]_G$ denote the relative class in $h_G(X \setminus l(w))(X^2)$. Then by the property (CD) the cohomology $h_G(X^2)$ (resp. $h_B(X^2)$) is a free module over $h_B(X)$ (resp. over $h_B(X)$ and $h(X)$) with basis $\{[\mathcal{O}_w]_G\}_{w \in W}$ (resp. $\{[\mathcal{O}_w]_B\}_{w \in W}$). Hence, the forgetful maps of (7) send $[\mathcal{O}_w]_G \mapsto [\mathcal{O}_w]_B \mapsto [\overline{\mathcal{O}}_w]$ and change the coefficients by $-\otimes h_G(X) h_B(X)$ and $-\otimes h_B(X) h(X)$ respectively, where the map $S = h_G(X) \to h_B(X) \to h(X)$ is the classical characteristic map.

We now construct such $G$-equivariant resolutions as follows. For the $i$-th simple reflection $s_i$ we denote $X_{s_i}$ (resp. $\mathcal{O}_{s_i}$) simply by $X_i$ (resp. by $\mathcal{O}_i$). Let $P_i$ be the minimal parabolic subgroup corresponding to a simple root $\alpha_i$ and let $q_i : X \to G/P_i$ denote the respective quotient map.

**Lemma 5.1.** We have $\overline{\mathcal{O}}_i = X \times_{G/P_i} X$ and, in particular, $\overline{\mathcal{O}}_i$ is smooth.

**Proof:** We have $(g_1 B, g_2 B) \in X \times_{G/P_i} X$, $g_1, g_2 \in G$ if and only if $g_1 P_i = g_2 P_i$, so $g_2 = g_1 h$ for some $h \in P_i$. Since $P_i = B \cup B s_i B$, it means that either $g_2 B = g_1 B$ or $g_2 B = g_1 B s_i B$, so $(g_1 B, g_2 B) \in \mathcal{O}_{s_i} \cup \Delta X = \overline{\mathcal{O}}_i$. \hfill $\square$

For any $w \in W$ we choose a reduced decomposition $w = s_{i_1} s_{i_2} \ldots s_{i_l}$ and set $I_w = (i_1, i_2, \ldots, i_l)$. Consider a variety

$$\overline{\mathcal{O}}_{I_w} = X \times_{G/P_{i_1}} X \times_{G/P_{i_2}} \ldots \times_{G/P_{i_l}} X.$$

The projection on the first and the last factor $pr : \overline{\mathcal{O}}_{I_w} \to X \times_k X$ gives a $G$-equivariant resolution of singularities of $\overline{\mathcal{O}}_w$.

**Theorem 5.2.** For $H = B$ or $1$, the image of $[\overline{\mathcal{O}}_{I_w}]_H \in h_H(X \times_k X)$ under the Künneth isomorphism

$$(h_H(X \times_k X), \circ) \overset{\cong}{\to} \text{End}_{h_H(k)}(h_H(X))$$
is the composition of push-pull operators $q_{i_1}^* q_{i_2} \circ \ldots \circ q_{i_n}^* q_{i_1}^*$. 

**Proof.** By definition the image of $[\mathcal{O}_{I_u}]_H$ is the $h_H(k)$-linear operator 

$$h_H^*(X) \xrightarrow{pr_{i+1}} h_H(X \times_k X) \xrightarrow{[\mathcal{O}_{I^L_u}]} h_H(X \times_k X) \xrightarrow{pr_{i+2}} h_H^{*-(l(u))}(X).$$

By the projection formula and (TS) it can be also written as 

$$h_H^*(X) \xrightarrow{pr_{i+1}} h_H(\mathcal{O}_{I_u}) \xrightarrow{pr_{i+2}} h_H^{*-(l(u))}(X),$$

where $pr_j$ denotes the projection on the $j$-th coordinate (recall that $p_j$ denotes the projection obtained by removing the $j$-th coordinate).

By the property (TS) we obtain a commutative diagram 

$$
\begin{array}{cccc}
  h_H(X) & \xrightarrow{pr^*_i} & h_H(\mathcal{O}_{I_i}) & \xrightarrow{pr_{i+1}} h_H(\mathcal{O}_{I_{i-1}}) & \ldots \\
  \downarrow q_{i+1} & & \downarrow pr_{i+1} & & \downarrow pr_{i+2} \\
  h_H(G/P_{i+1}) & \xrightarrow{q_i} & h_H(X) & \xrightarrow{pr^*_i} h_H(\mathcal{O}_{I_{i-1}}) & \ldots \\
  \downarrow q_{i+1} & & \downarrow pr_{i+1} & & \downarrow pr_{i+2} \\
  h_H(G/P_{i-1}) & \xrightarrow{q_i} & h_H(X) & \xrightarrow{pr^*_i} h_H(\mathcal{O}_{I_{i-2}}) & \ldots \\
  \downarrow q_{i+2} & & \downarrow pr_{i+1} & & \downarrow pr_{i+2} \\
  \vdots & & \vdots & & \vdots \\
  \ldots & & \ldots & & \ldots \\
  h_H(X) & & & & \\
\end{array}
$$

where $pr_{ijk\ldots}$ denote the projection on the $i$-th, $j$-th, $k$-th, ..., coordinates. The result then follows since the top horizontal row gives $pr_{i+1}^*$ and the right vertical column gives $pr_{1+}^*$. \[\square\]

Observe that the theorem cannot be stated for $H = G$ as $X$ is not a $G$-equivariant cellular space so we can not use the Künneth isomorphism of Lemma 3.7.

Combining Diagram (7) and Theorem 5.2 we obtain

**Corollary 5.3.** There is a commutative diagram of convolution rings

$$
\begin{array}{cccc}
  h_{B^2(G)}^* \xrightarrow{\pi_{B^2(G)}} h_{B^2(G)} X^2 \xrightarrow{\sim} h_B(X^2) \xrightarrow{\sim} \text{End}_S(h_B(X)) \\
  \downarrow q^* & & \downarrow & & \downarrow \\
  h_{B^2(G)}^* \xrightarrow{\sim} h(X^2) \xrightarrow{\sim} \text{End}_R(h(x)) \\
\end{array}
$$

where the image of $(h_{B^2(G)}, \circ)$ in $\text{End}_S(h_B(X))$ is the subring generated by the push-pull operators $q_i^* q_{i+1}$ (of degree $(-1)$) and the image of the forgetful map $S \leftarrow h_{B^2(G)}^*(X) \rightarrow h_{B^2(G)}^*(X)$ (of degrees $\bullet$) and the last vertical arrow is induced by the augmentation map $S \rightarrow R = h(k)$.

6. Self-duality of the algebra of push-pull operators

In the present section we identify the convolution ring $h_{B^2(G)}$ with the formal affine Demazure algebra $D_F$ of [19] and show that it is self-dual with respect to the convolution product (Theorem 6.2). Our arguments are based on the results of [19], [7], [8] and, especially, [9]. We use the notation of [9].
Recall that algebraic oriented cohomology theories $\mathfrak{h}$ correspond (up to universality) to one-dimensional commutative formal group laws $F(u, v)$: the formal group law corresponds to $\mathfrak{h}$ by means of the Quillen formula expressing the first characteristic classes

$$c_1^k(L_1 \otimes L_2) = F(c_1^k(L_1), c_1^k(L_2))$$

and the respective cohomology theory $\mathfrak{h}$ is defined from $F$ by tensoring with the algebraic cobordism

$$\mathfrak{h}(-) = \Omega(-) \otimes_{\Omega(k)} \mathbb{R},$$

where $\Omega(k) \to \mathbb{R}$ defines $F$ by specializing the coefficients in the Lazard ring (see [9, §2] for details). For example, the additive formal group law correspond to Chow groups and the periodic multiplicative law corresponds to $K$-theory.

By [9, Thm. 3.3] the completed $B$-equivariant coefficient ring $S = \mathfrak{h}_B(k)$ can be identified with the formal group algebra $\mathbb{R}[[T^*]]$, where $T^*$ is the group of characters of a split maximal torus $T \subset B$ and $F$ is the respective formal group law.

Following [9, §5] (we assume that $S$ satisfies regularity condition [9, 5.1]) consider the localized algebra $Q = \mathbb{R}[[T^*]]\mathbb{R}[\frac{1}{T^*}]$ (where $\alpha$ runs through all simple roots) and the smash products $Q_W = Q \otimes_{\mathbb{R}} \mathbb{R}[W]$ and $S_W = S \otimes_{\mathbb{R}} \mathbb{R}[W]$ with the multiplication given by

$$q \delta_w \cdot q' \delta_w' = q(wq') \delta_{ww'}$$

for $q, q' \in Q$ (respectively $S$) and $w, w' \in W$ (the Weyl group). Consider the duals $Q^*_W = Hom_Q(Q_W, Q)$ and $S^*_W = Hom_S(S_W, S)$. By definition $Q^*_W$ and $S^*_W$ can be identified with the ring of functions $Hom(W, Q)$ and $Hom(W, S)$ respectively.

As in [19, Def. 6.2, 6.3] for each simple root $\alpha_i$ of the root system for $G$ define the push-pull element

$$Y_i = (1 + \delta_i) \frac{1}{x^{-i}} \in Q_W.$$

Define the formal affine Demazure algebra $D_F$ as the subalgebra of $Q_W$ generated by multiplications by $S$ and the elements $Y_i$.

By [7, Thm. 7.9] (see also [19, Thm. 5.14]) the $R$-algebra $D_F$ satisfies the following (complete) set of relations: for $i, j = 1 \ldots rk(G)$ and $u \in S$

- $Y_i^2 = \kappa_i Y_i$,
- $Y_i u = s_i(u) Y_i + \Delta_{-i}(u)$, where $\Delta_{-i}(u) = \frac{u - s_i(u)}{x^{-i}}$,
- $(Y_i Y_j)^{m_{ij}} - (Y_j Y_i)^{m_{ij}} = \sum_{I_w} c_{I_w} Y_{I_w}$, where the sum is taken over all reduced expressions $I_w$ of elements $w$ of the subgroup $\langle s_i, s_j \rangle \subseteq W$, and the coefficients $c_{I_w}$ are given by the formulas of [19, Prop. 5.8]

**Example 6.1.** If $F$ corresponds to Chow groups, then $D_F = H_{nil}$ is the affine nil-Hecke algebra over $\mathbb{Z}$ in the notation of [17]. If $F$ corresponds to $K$-theory, then $D_F$ is the 0-affine Hecke algebra over $\mathbb{Z}$ ($q \to 0$ in the affine Hecke algebra). If $F$ corresponds to the generic hyperbolic formal group law of [8, §9], then by [8, Prop. 9.2] the constant part of $D_F$ is isomorphic to the localized classical Iwahori-Hecke algebra.

Let $D_F^* = Hom_S(D_F, S)$ denote its dual. Observe that the main result of [9] (Thm. 8.2 loc.cit.) says that $D_F^*$ is isomorphic to the $R$-algebra $\mathfrak{h}_F(X)$. We then obtain the following generalization of [17, Prop. 12.8]
Theorem 6.2. Let \( G \) be a split semisimple linear algebraic group over a field \( k \) and let \( \mathfrak{h} \) be an equivariant theory that satisfies property (CD).

Then the convolution algebra \((\mathfrak{h}_{B^2}(G), \circ)\) is isomorphic (as an \( \mathbb{R} \)-algebra) to the formal affine Demazure algebra \( \mathcal{D}_F \). So there is an \( \mathbb{R} \)-algebra isomorphism

\[
(\mathcal{D}_F, \circ) \simeq (\mathcal{D}_F, \cdot)
\]

Proof. By Corollary 5.3 the ring \((\mathfrak{h}_{B^2}(G), \circ) \simeq (\mathfrak{h}_B(X), \circ)\) is isomorphic to the subalgebra of \( \text{End}_Q(\mathfrak{h}_B(X)) \) generated by the image of the forgetful map \( \mathfrak{h}_G(X) \to \mathfrak{h}_B(X) \) and push-pull operators \( q_i^* q_* \). Since the map \( B \to B/T \) is an affine fibration, the natural map \( \mathfrak{h}_B(X) \to h_T(X) \) is an isomorphism. Hence we may identify \( S \) with \( h_T(k) \) and \( \text{End}_Q(\mathfrak{h}_B(X)) \) with \( \text{End}_Q(h_T(X)) \). Observe that these identifications preserve push-pull operators. The inclusion of \( T \)-fixed point set \( W \to X \) gives an embedding \( h_T(X) \to h_T(W) = S_W \subseteq Q_W \). By \cite[Corollary 8.7]{9} there is the following commutative diagram

\[
\begin{array}{ccc}
\mathfrak{h}_T(X) & \longrightarrow & S_W \\
\downarrow q_i^* q_* & & \downarrow & A_i \\
h_T(X) & \longrightarrow & S_W \\
\end{array}
\]

where the Hecke operator \( A_i \) is given by

\[ A_i(f)(x) = f(x \cdot Y_i) \quad \text{for} \quad x \in Q_W, f \in Q_W. \]

Moreover, the forgetful map

\[ S \cong \mathfrak{h}_G(X) \to \mathfrak{h}_T(X) = \bigoplus_{w \in W} S \]

is given by the formula \( s \mapsto (w \cdot s)_{w \in W} \) for any \( s \in S \). Then the multiplication in \( h_T(X) = S_W \) by the image of any element in \( s \in \mathfrak{h}_G(X) \) induces a right multiplication by \( s \) in \( Q_W \). Since \( Q_W \) is a free \( Q \)-module of finite rank, the natural map \( \iota: Q_W \to \text{End}_Q(Q_W) \) given by \( \iota(x)(f)(y) = f(yx) \) is an inclusion. Note that every \( A_i \) lies in the image of \( \iota \). Then by diagram (9) the image of \( \mathfrak{h}_{B^2}(G) \) is isomorphic to a subalgebra of \( Q_W \) generated by \( S \) and \( Y_i \) which is \( \mathcal{D}_F \). \qed

7. The rational algebra of push-pull operators

In the present section we introduce the rational algebra of push-pull operators \( \mathcal{D}_F \) (Definition 7.5) and show that it can be identified with the subring of rational endomorphisms of \( G/B \) (Theorem 7.6).

The \( B^2 \)-equivariant isomorphism \( E \times_k G \to E \times_k E \), \( (e, g) \mapsto (e, eg) \) induces an isomorphism \( E \times B^2 G/B \to E/B \times_k E/B \). For all \( w \in W \) fix a reduced decomposition \( I_w = (i_1, \ldots, i_l) \) and the corresponding Bott-Samelson resolution \( X_{I_w} \to G/B \) of the Schubert cell. This map is \( B \)-equivariant, so it descends to a map \( Y_{I_w} = E \times^B X_{I_w} \to E \times^B G/B \).

Lemma 7.1. The classes \( [Y_{I_w}] \) form a basis of \( \mathfrak{h}(E/B \times_k E) \) over \( \mathfrak{h}(E/B) \), where the module structure is given by the pullback of the projection \( pr_1^*: \mathfrak{h}(E/B) \to \mathfrak{h}(E/B \times_k E/B) \).

Proof. Since \( B \) is special, \( G \)-torsor \( E \) splits over the function field of \( E/B \). Then by \cite[Lemma 3.3]{25} projection \( pr_1: E/B \times_k E/B \to E/B \) is a cellular fibration in
the sense of [25, Definition 3.1] so that \((E/B)^2\) is a cellular space over \(E/B\). Let \(\xi\) be the generic point of \(E/B\). The pullback of an open embedding

\[ j^*: h(E/B \times_k E/B) \rightarrow h(\xi \times_k E/B) \simeq h(G/B) \]

is surjective and any preimage of \(R\)-basis of \(h(G/B)\) gives a basis of \(h(E/B \times_k E/B)\). Thus it is sufficient to check that \(j^*\) sends \([Y_{I_w}]\) to a basis of \(h(\xi \times_k E/B)\). Let \(p: E \rightarrow E/B\) be the projection. Note that

\[ E \times_B X_{I_w} \times (E/B \times_k E/B) \xi \times_k E/B = p^{-1}(\xi) \times_B X_{I_w} = \xi \times X_{I_w}, \]

since \(p^{-1}(\xi) \rightarrow \xi\) is a trivial \(B\)-torsor. Thus \(j^*([Y_{I_w}]) = [\xi \times X_{I_w}]\) forms a basis of \(h(\xi \times E/B) = h(\xi \times G/B)\) over \(h(\xi) = R\).

Consider a \(B\)-equivariant map

\[ f: E \times_B G \rightarrow B \setminus G, \quad (e, g)B \mapsto Bg. \]

Let \(X'_{I_w} = (P_{i_1} \times \ldots \times P_{i_l})/B^l\) where \(B^l\)-action on \(P_{i_1} \times \ldots \times P_{i_l}\) is given by \((p_1, \ldots, p_l) \rightarrow (b_1 p_1 b_2, \ldots, b_l^{-1} p_l)\). Then \(X'_{I_w}\) gives the Bott-Samelson class for \(B \setminus G\).

**Lemma 7.2.** The composition \(h_B(B \setminus G) \xrightarrow{f^*} h_B(E \times_B G) \simeq h(E/B \times_k E/B)\) maps \([X'_{I_w}]_B\) to \([Y_{I_w}]\).

**Proof.** Consider the map \(P_{i_1} \times_B P_{i_2} \times \ldots \times B P_{i_l} \rightarrow G\) given by \((p_1, \ldots, p_l) \rightarrow p_1 \ldots p_l\). It is \(B\)-equivariant with respect to the left multiplication, so it descends to a map \(M_{I_w} = E \times_B P_{i_1} \times_B P_{i_2} \times_B \ldots \times_B P_{i_l} \rightarrow E \times_B G\). By construction we have an isomorphism

\[ M_{I_w} \simeq Y_{I_w} \times_{E \times h(B \setminus G)} (E \times_B G). \]

Then \([M_{I_w}]_B\) is mapped to \([Y_{I_w}]\) via the isomorphism \(h(E \times_B G/B) \rightarrow h_B(E \times B)\). Thus it is sufficient to check that \(f^*[X'_{I_w}]_B = [M_{I_w}]_B\), which follows from the fact that

\[ M_{I_w} = E \times_B (P_{i_1} \times \ldots \times P_{i_l}/B^{l-1}) \simeq (E \times_B G) \times_{B\times G} X'_{I_w}. \]

**Lemma 7.3.** (cf. [25, Corollary 3.4]) The composition

\[ (p_1^*, \gamma_1^*): h_B(E) \otimes_S h_B(\xi) \rightarrow h_B(E^2) \simeq h((E/B)^2) \]

of the diagram (6) (for \(H = B\)) is an isomorphism.

**Proof.** Consider the basis of \(h_B(\xi)\) over \(S\) given by the classes of Bott-Samelson resolutions \(\zeta_{I_w}\). Then by Lemma 7.2 \(\gamma_1^*(\zeta_{I_w})\) forms a basis of \(h_B(E^2)\) over \(h_B(E)\) induced by the respective cellular filtration.

Consider the restriction map \(h(E/B) \rightarrow h(E_\bar{k}/B) = h(X_\bar{k})\) on cohomology induced by the scalar extension \(\bar{k}/k\) (here \(\bar{k}\) is a splitting field of \(E\)). Let \(\overline{h}(X)\) denote its image.

**Corollary 7.4.** The image of the ring homomorphism

\[ \text{res}_{\bar{k}/k}: (h(E/B \times_k E/B), \circ) \rightarrow (h(X_\bar{k} \times_k X_\bar{k}), \circ). \]

is the subalgebra generated by the multiplication by the elements of \(\overline{h}(X)\) and the push-pull operators \(q_i^* q_i^*: h(X) \rightarrow h(G/P_i) \rightarrow h(X)\) for all simple roots \(\alpha_i\).

**Proof.** Follows by (6), Lemma 7.3 and Corollary 5.3. 

\[ \square \]
There is a natural action of $W$ on $\mathfrak{h}(X)$ that comes from the $W$-action on $E/T$. So we can endow $\mathfrak{h}(X) \otimes_S Q/W$ with a structure of an $R$-algebra.

**Definition 7.5.** Let $\mathbf{D}_F$ denote its subalgebra $\mathfrak{h}(X) \otimes_S D_F$. We call it the rational algebra of push-pull operators. By $\mathbf{D}_F^{(m)}$ we denote its degree $m$ homogeneous component assuming that all $Y_i$’s have degree $(-1)$ and elements of $\mathfrak{h}(X)$ (and of $S = \mathfrak{h}_F^*(k)$) have degree ‘$ullet$’.

Observe that if $E$ is split, then $\mathbf{D}_F = \mathfrak{h}(X) \otimes_S D_F$ does not coincide with $D_F$.

Set $N = \dim X$.

**Theorem 7.6.** Consider the restriction

$$\text{res}_{k/k}: \text{End}_{h,M(k)}([E/B]) \rightarrow \text{End}_{\bar{h},M(k)}([X_{\bar{k}}])$$

on endomorphism rings of the respective motives (i.e., preserving the grading of $h(X)$). Its image can be identified with $\mathbf{D}_F^{(0)}$ via the injective forgetful map

$$\phi: (\mathfrak{h}(X) \otimes_S h_G(X^2))(\mathcal{O}_{\bar{L}_n}) \rightarrow (\mathfrak{h}^N(X^2), \cdot)$$

Proof. By (7) both $h_G(X^2)$ and $\mathfrak{h}(X^2)$ are free modules over $h_G(X)$ and $\mathfrak{h}(X)$ with basis given by the classes $[\mathcal{O}_{\bar{L}_n}]_G$ and $[\mathcal{O}_{\bar{L}_n}]_{\bar{k}}$ respectively. The map $\phi$ sends $[\mathcal{O}_{\bar{L}_n}]_G \mapsto [\mathcal{O}_{\bar{L}_n}]_{\bar{k}}$ and leaves the coefficients invariant. The result follows by Corollary 7.4, Corollary 5.3 and Theorem 6.2.

We say that a (co-)homology theory $h$ satisfies the Dimension Axiom if (Dim) For any smooth variety $Y$ over $k$ we have $h^n(Y) = 0$ for all $n > \dim Y$.

**Example 7.7.** Any theory $h$ over a field $k$ of characteristic 0 obtained by specialization of coefficients of the Lazard ring (e.g. Chow groups, connective algebraic cobordism $\Omega$) satisfies (Dim). The graded $K$-theory $K_0(-)[\beta, \beta^{-1}]$ of [24, Example 1.1.5] does not satisfy (Dim).

Observe that the image of the characteristic map $c: S \rightarrow h(X)$ is contained in $\mathfrak{h}(X)$ (see [16, Thm. 4.5]). Consider both the induced map $c: D_F \rightarrow \mathbf{D}_F$ and the restriction map $\text{res}_{k/k}: (h_B(E) \otimes_S D_F) \rightarrow \mathbf{D}_F$. We will use the following substitute of the Rost nilpotence theorem.

**Lemma 7.8.** Assume that the theory $h$ satisfies (Dim), then the kernels of $c$ and $\text{res}_{k/k}$ are complete. In other words, there is a commutative diagram of maps of convolution rings

$$
\begin{array}{ccc}
\mathbf{D}_F & \overset{\gamma^1}{\longrightarrow} & h((E/B)^2) \\
\downarrow c & & \downarrow \text{res}_{k/k} \\
\mathbf{D}_F & \longrightarrow & \mathbf{D}_F
\end{array}
$$

with complete kernels (cf.[32, Ch.2, Lemma 2.2]).

Proof. If $c(x) = 0$ (resp. $\text{res}_{k/k}(x) = 0$) then $x = \sum_w a_w Y_w$ with $a_w \in S$, $c(a_w) = 0$ (resp. with $a_w \in h(E/B)$, $\text{res}_{k/k}(a_w) = 0$). Since $c$ (resp. $\text{res}_{k/k}$) commutes with push-pull operators on $S$ (resp. on $h(E/B)$), the product of $n$ such elements $x_1 \circ x_2 \circ \ldots \circ x_n$ corresponds to

$$(\sum_w a_w^1 Y_w) \ldots (\sum_w a_n^w Y_w) = \sum_w a_{w,n} Y_w, \quad a_{w,n} \in (\ker c)^n, \quad (\ker \text{res}_{k/k})^n$$
Since $\ker c$ is contained in the augmentation ideal, and $S$ is complete, then the series $\sum_n a_{w,n}$ converges in $S$ (resp. $\ker \text{res}_{\overline{k}/k}$ is contained in $h > 0$ $(E/B)$, then $a_{w,n} = 0$ for $n > \dim E/B$), thus $\ker c$ is complete and $\ker \text{res}_{\overline{k}/k}$ is nilpotent, hence complete.

**Lemma 7.9.** If $E$ is a generic $G$-torsor in the sense of [15, §3, p.108], then the map $\gamma_1^*$ and, hence, $c$, of the lemma 7.8 is surjective.

**Proof.** Observe that if $E$ is generic, then it admits a generically open $G$-equivariant embedding into $A^N_k$. So the projection $E \times_k G_i \to G_i$ in the definition of $\gamma_i$ factors through $A^N_k \times_k G_i$. By (Loc) and (HI) the induced pullback $\gamma_i^*$ is surjective. □

### 8. Applications and examples

Our main application is the following

**Theorem 8.1.** Let $G$ be a split semisimple linear algebraic group over a field $k_0$ and let $E$ be a $G$-torsor over a field extension $k$ of $k_0$. Let $h$ be an oriented cohomology theory over $k$ that satisfies both (CD) and (Dim) axioms. Let $\langle [E/B] \rangle_h$ (resp. $\langle [E/B]_\ast \rangle_h$) denote the pseudo-abelian subcategory generated by the (resp. non-graded) $h$-motive of $E/B$. Then

(i) There are isomorphisms between the Grothendieck groups $K_0(\langle [E/B] \rangle_h) \simeq K_0(\overline{D}_F^{(0)})$ and $K_0(\langle [E/B]_\ast \rangle_h) \simeq K_0(\overline{D}_F)$.

(ii) There is a 1-1 correspondence between direct sum decompositions of the $h$-motive $[E/B]$ and direct sum decompositions of the $\overline{D}_F^{(0)}$-module $\overline{D}_F^{(0)}$. Moreover, any two direct summands in the motivic decomposition of $E/B$ that are Tate twists of each other correspond to isomorphic $\overline{D}_F$-modules.

(iii) If $E$ is generic, then the algebra $\overline{D}_F$ in (i) and (ii) can be replaced by the algebra $D_F$.

**Proof.** Consider a graded endomorphism ring of the $h$-motive of $E/B$ $C_F = (\text{End}_{h-M(k)}([E/B]), \circ)$. Our main result (Theorem 7.6) together with Lemma 7.8 says that the restriction map gives a surjective ring homomorphism with complete kernel $\text{res}_{\overline{k}/k}: C_F \to \overline{D}_F$.

The part (i) then follows from [32, Ch. 2, Lemma 2.2].

The part (ii) follows from [25, §2 and Prop. 2.6] applied to the map $\text{res}_{\overline{k}/k}$ (an isomorphism between Tate twists in the non-graded category of motives corresponds to an isomorphism between idempotents of [25, §2.1]).

The part (iii) follows from Lemma 7.8 applied to the map $c$. □

**Example 8.2.** If $G$ is special, i.e. $G$ is simple simply-connected of type $A$ or $C$, then any $G$-torsor $E$ splits, and the respective non-graded subcategory $\langle [E/B]_\ast \rangle_h$ is generated by the motive of a point. So by (ii) and (iii) of the theorem all indecomposable direct summands of $\overline{D}_F$ (resp. $D_F$) are isomorphic to the $\overline{D}_F$-(resp. $D_F$-)module $S$ and

$$K_0(\overline{D}_F) \simeq K_0(D_F) \simeq \mathbb{Z}.$$
Lemma 8.3. If the coefficient ring $R$ is Artinian, then both $C_F$ and $D_F$ and, hence, the categories $(E/B)_0$ and $\text{Proj } D_F$ satisfy the Krull-Schmidt property (uniqueness of a direct sum decomposition).

Observe that in general the ring $D_F$ is not Krull-Schmidt (and not semi-simple).

Proof. If $R$ is Artinian, then both $D_F$ and $C_F$ are Artinian (as $D_F$ is finite dimensional over $R$). So they are both Noetherian which implies that the respective tautological modules $D_F$ and $C_F$ have finite length and, hence, the Krull-Schmidt property holds for both $D_F$ and $C_F$. \hfill $\square$

As a direct application of the main result of [25] one obtains the following characterization of modular representations of the (affine) nil-Hecke algebra ($F$ is an additive formal group law and $h(-) = CH(-; F_p)$).

Corollary 8.4. Let $G$ be a split semisimple linear algebraic group over a field $k_0$. Consider the affine nil-Hecke algebra $H_{nil,p}$ for $G$ with coefficients in $R = F_p$, $p$ is a prime. Then

$$K_0(H_{nil,p}) \simeq K_0((R_*)_p),$$

where $R_*$ is the generalized (non-graded) Rost-Voevodsky motive corresponding to a generic $G$-torsor $E$ and the prime $p$.

In particular, all indecomposable graded submodules of $H_{nil,p}$ are isomorphic to a graded indecomposable submodule $P_*$ corresponding to $R_*$. Moreover, they are free $S$-modules of rank $r$ that is equal to the $p$-part of the product of $p$-exceptional degrees of $G$ and we have

$$H_{nil,p} \simeq \text{Mat}_{W/F}(\text{End}(P_*)).$$

Proof. The $S$-rank coincides with the number of Tate motives in the decomposition of $R$ over a splitting field of $E$, that is $g_*(E) = \prod_{i=1}^{t} \frac{1}{1-t^{-1}}$ (in the notation of [25]) which is equal to the $p$-part $p^{\sum_{i=1}^{t} t_i}$ of $p$-exceptional degrees of $[20, \text{p.73}]$. The last statement follows from the fact that the ring of graded endomorphisms of $\mathcal{R}_*$ coincides with the ring of endomorphisms of $\mathcal{R}$. \hfill $\square$

We now switch to integer coefficients. Recall that in this case the Krull-Schmidt property usually fails.

Example 8.5. Consider the root system of type $A_1$. In this case $T^* = \mathbb{Z} \omega$ or $T^* = \mathbb{Z} \alpha$, $\alpha = 2\omega$ is the simple root and $\omega$ is the fundamental weight. The Weyl group $W = \{1, s\}$ acts by $s: \omega \mapsto -\omega$, where $s$ is the simple reflection. By definition, $S = R[[x]]_F$ (where $x = x_\omega$ or $x = x_\alpha$), $Q = S[\frac{1}{\tau}]$, $Q_W = \{q(x)\omega \mid q(x) \in Q, w \in W\}$ with

$$q(-x)\delta_w = s(q(x))\delta_w = \delta_s q(x),$$

where $-x$ is the formal inverse of $x$. Observe that $x_\alpha = x_{\omega+\omega} = F(x_\omega, x_\omega)$. The $R$-algebra $D_F$ is a free left $S$-submodule of rank 2 in $Q_W$ with basis

$$\{1, Y = \frac{1}{-x} + \frac{1}{x_\alpha} \delta_s\}.$$

It satisfies the relations

$$Y^2 = \kappa Y \text{ and } Y q(x) = q(-x)Y + \Delta(q(x)), $$

where $\kappa = \frac{1}{-x} + \frac{1}{x_\alpha}$ and $\Delta(q(x)) = \frac{q(x)-q(-x)}{-x}$. 

Let $p = a + bY$, where $a, b \in S$, be an idempotent in $D_F$, i.e., $p^2 = p$. Then we obtain in $S$

\[(10) \quad a^2 + b\Delta(a) = a \quad \text{and} \quad (a + s(a) + s(b)\kappa + \Delta(b))b = b.\]

In the case $h(-) = CH(-;\mathbb{Z})$ ($F$ is additive) we have $R = \mathbb{Z}, S = \mathbb{Z}[x], \kappa = 0, -Fx = -x, x_\alpha = 2x_\omega$ and the second equation turns into

\[(a + s(a) + \Delta(b))b = b.\]

If $x = x_\alpha$ ($G = PGL_2$), the polynomials $a + s(a)$ and $\Delta(b)$ are divisible by 2, so $b = 0$. Hence, $a = 0$ or 1 from the first equation. So $D_F$ is an indecomposable module over itself. By Theorem 8.1 this implies that both graded and non-graded motives $[E/B]$ and $[E/B]_\star$ of a generic conic are indecomposable.

If $x = x_\omega$ ($G = SL_2$) and $p$ is homogeneous, we get $a \in \mathbb{Z}, b = cx, c \in \mathbb{Z}$ and the system (10) has solutions only for $c = \pm 1$. Therefore, the algebra $D_F$ has only two indecomposable graded submodules which correspond to the idempotents $1 - x_\omega Y$ and $x_\omega Y$. In other words, the motives $[E/B]$ and $[E/B]_\star$ split into a direct sum of two indecomposable motives.

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