The final fate of spherical inhomogeneous dust collapse II: Initial data and causal structure of singularity

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ABSTRACT

Further to results in [9], pointing out the role of initial density and velocity distributions towards determining the final outcome of spherical dust collapse, the causal structure of singularity is examined here in terms of evolution of the apparent horizon. We also bring out several related features which throw some useful light towards understanding the nature of this singularity, including the behaviour of geodesic families coming out and some aspects related to the stability of singularity.
I. Introduction

The spherical gravitational collapse of inhomogeneous dust has been analyzed in several recent papers [1-9]. We now have a reasonably good understanding of the role of initial data in deciding whether the central singularity is naked or covered [8,9]. Continuing the analysis of [9], in the present note, we discuss the relation between the initial data and the dynamics of the apparent horizon - this study improves our understanding of the connection between initial data and the nature of the final singularity.

In Section II we recall some essential results and discuss some aspects of initial data which provide certain useful insights into the structure of the singularity. In particular, we consider the stability of naked singularity under perturbation of initial data. The dynamics of evolving apparent horizon is discussed in Section III for different initial data and Section IV examines in some detail the behaviour of outgoing families of geodesics, when the singularity is visible, for different classes of initial data.

Our overall results in this paper are as follows. We argue that the strong naked singularities in dust collapse are non-generic, in a certain sense discussed below. We show that the Oppenheimer-Snyder solution is stable against perturbations leading to a strong naked singularity, but unstable against perturbations leading to a weak naked singularity. We study the dynamics of the apparent horizon and show that the evolution of the apparent horizon for initial data leading to a naked singularity is very different from the case when the initial conditions lead to a covered singularity. In particular, we provide an example to show that absence of the apparent horizon until singularity formation does not imply the singularity is naked. This has relevance to searches for naked singularities in numerical investigations of collapse. We show that the geodesic structure for the weak naked singularity is different from that for the strong naked singularity.

II. Initial data and nature of the singularity

Spherical dust collapse is described by the Tolman-Bondi model [10,11] in comoving coordinates \((t, r, \theta, \phi)\) which has metric of the form

\[
\text{ds}^2 = -d\text{t}^2 + \frac{R'^2}{1 + f(r)} dr^2 + R^2 d\Omega^2,
\]

and energy-momentum tensor of the form of a perfect fluid with equation of state \(p=0\), i.e., \(T^{ij} = \epsilon \delta_i^t \delta_j^t\). Here \(\epsilon\) and \(R\) are functions of \(r\) and \(t\), \(d\Omega^2\) is the metric on the two-sphere \(S^2\), and \(f(r)\) is an arbitrary function of \(r\) which we call the energy function. The Einstein equations are

\[
\dot{R}^2 = \frac{F(r)}{R} + f(r),
\]

and

\[
\epsilon(r,t) = \frac{F'(r)}{R^2 R'},
\]
where the dot and prime denote $\partial/\partial t$ and $\partial/\partial r$ respectively. $F(r)$ is another free function of integration, and it is equal to twice the mass inside the sphere of radius $r$.

As we are concerned with the situation of gravitational collapse, we take $\dot{R} < 0$. The evolution leads to the formation of a shell-focussing curvature singularity described by the curve $R(t,r) = 0$. The singularity at $r = 0$ is called the central singularity. The free functions $F(r)$ and $f(r)$ are determined by the initial density profile $\rho(r) = \epsilon(r,0)$ and the initial velocity profile $v(r) = \dot{R}(r,0)$ as follows. Let $t = 0$ be the initial epoch, and by virtue of the freedom of scaling let us choose $R(0, r) = r$. Then we get from (3),

$$F(r) = \int \rho(r)r^2 dr, \quad (4)$$

and from (2) and (4) that

$$f(r) = v^2(r) - \frac{1}{r} \int \rho(r)r^2 dr. \quad (5)$$

We assume that the initial density and velocity profiles can be expanded in a power series, near the center $r = 0$,

$$\rho(r) = \rho_0 + \rho_1 r + \frac{1}{2}\rho_2 r^2 + \frac{1}{6}\rho_3 r^3 + ... \quad (6)$$

$$v(r) = v_1 r + \frac{1}{2}v_2 r^2 + \frac{1}{6}v_3 r^3 + ... \quad (7)$$

We assume the density to decrease outwards from the center, hence the first non-vanishing derivative of density has negative sign. The terms $\rho_n$ and $v_n$ denote the $n$th derivative at the center, of the respective quantity. The inclusion of the $\rho_1$ term in (6) means there is a central cusp in the initial density, and such a term should be dropped to avoid this feature. It is included here for completeness, and results have been worked out both with and without this term. In (7), the first term in the expansion is of order $r$ - the center is taken to be at rest, because of spherical symmetry. From (4) and (6) it follows that the mass $F(r)$ near the center is given by

$$F(r) = F_0 r^3 + F_1 r^4 + F_2 r^5 + F_3 r^6 + ... \quad (8)$$

where $F_n = \rho_n/(n + 3)n!$. Using (5), (7) and (8) it is seen that the energy function $f(r)$ is given near the center by

$$f(r) = f_2 r^2 + f_3 r^3 + f_4 r^4 + f_5 r^5 + ... \quad (9)$$

where the coefficients $f_n$ are determined by the coefficients in (7) and (8). Note that $f(r)$ cannot have a term lower than the quadratic one near the center - this is a consequence of
the center being at rest, and of the fact that $F(0)$, the mass at the center, is zero. Thus $f(r)$ must necessarily vanish at $r = 0$. Further, we assume $f_2 \neq 0$, as its vanishing is only possible by a fine tuning of the density and velocity profiles.

It can be shown that the shell-focussing singularity at $r > 0$ is covered by the event horizon, but the central singularity at $r = 0$ is locally naked for some initial data, and covered for other initial data. (In this paper, when we say the singularity is naked, we mean it is locally naked. We will not be concerned with the issue of global nakedness which has been discussed elsewhere [7,9]). Also, the results on the curvature strength of the naked singularity, as to whether it is strong or weak (this is a technical criterion quantifying the rate of curvature growth in the limit of approach to the singularity along the outgoing future directed nonspacelike trajectories; see e.g. ref.[7] for definitions), have been worked out. For the marginally bound case, $f = 0$, these results are as follows [3,6,9]:

(i) If $\rho_1 < 0$ the singularity is naked and weak.

(ii) If $\rho_1 = 0, \rho_2 < 0$ the singularity is naked and weak.

(iii) If $\rho_1 = \rho_2 = 0, \rho_3 < 0$ the singularity is naked if $\xi = F_3/F_0^{5/2} = \sqrt{3}\rho_3/4\rho_0^{5/2}$ is less than the critical value $\xi_c = -25.9904$, and covered if $\xi > \xi_c$. Further, the naked singularity is a strong curvature singularity.

(iv) If $\rho_1 = \rho_2 = \rho_3 = 0$ the singularity is covered.

For the non-marginally bound case $f \neq 0$ the results are as follows. We define the quantity $Q_q$ as

$$Q_q = \frac{3}{2} \left(1 - \frac{f_2}{2F_0}\right) \left(G(-f_2/F_0) \left(\frac{F_4}{F_0} - \frac{3f_{q+2}}{2f_2}\right) \left(1 + \frac{f_2}{2F_0}\right) + \frac{f_{q+2}}{f_2} - \frac{F_q}{F_0}\right).$$  (10)

Here $q = 1$ if at least one of $F_1$ and $f_3$ are non-zero, $q = 2$ if $F_1 = f_3 = 0$ and at least one of $F_2$ and $f_4$ are non-zero, and $q = 3$ if $F_1, F_2, f_3, f_4$ are zero, and at least one of $F_3$ and $f_5$ are non-zero. If $q = 1$, the singularity is naked for $Q_1 > 0$ and if naked it is weak. If $q = 2$ the singularity is naked for $Q_2 > 0$ and if naked it is weak. If $q = 3$ the singularity is naked if $\xi = -2Q_3/F_0^{3/2}$ is less than the above mentioned critical value $\xi_c$, and covered if $\xi$ exceeds $\xi_c$. Further the naked singularity is strong. If all of $F_1, F_2, F_3, f_3, f_4$ and $f_5$ are zero, the singularity is covered.

We now make several observations here regarding the nature of this singularity, which arises as the final state of inhomogeneous gravitational collapse of pressureless dust, and its relationship with the initial data. In our view, this brings out and throws some useful light on certain important aspects related to this singularity which are not widely known and which should help us understand better the structure of singularity.

To begin with, we should comment on the series expansions in Eqns. (6) to (9). In their work, Christodoulou [2] and Newman [3] assumed that the density and metric functions are smooth ($C^\infty$). It then follows that a power series expansion near the center for the density
\( \rho(r) \) and for the energy function \( f(r) \) can have only even powers of \( r \). This is because if odd powers of \( r \) are present, then in a Cartesian coordinate system set up at the center, some of the Cartesian derivatives of odd-powered terms are not defined - this makes the function non-smooth. For instance, if in (6), the linear term is absent, and the cubic term present, the density function is \( C^2 \), but not \( C^\infty \). Further, since the spherical coordinate system is singular at the origin, strictly one must set up Cartesian coordinates to describe quantities near the center.

As is evident from Eqns. (6) to (9), we are not restricting ourselves to smooth functions. It could be asked, does a physical density function have to be smooth [12], where by physical we mean ‘occurring in a real system’. In our view, the answer is in the negative. As a justification and for illustration, we consider the case of a spherically symmetric star in Newtonian gravity, described by a polytropic equation of state. If the star is in hydrostatic equilibrium, the density distribution is described by the Lane-Emden equation [13], and the substitution of a power-series expansion for the density into the equation indeed implies that all odd-powered terms drop out and the density function is necessarily smooth. However, if one is considering gravitational collapse of the star, the Lane-Emden equation is replaced by the dynamical equation for the evolution of the radius \( R \) of a fluid element,

\[
\ddot{R} = -4\pi R^2 \int \rho R^2 dR - \frac{1}{\rho} \frac{dp}{dR},
\]

where \( p = K \rho^{1+1/n} \) describes the polytrope. Setting \( \ddot{R} = 0 \) reduces this equation to the Lane-Emden equation. While considering collapse, initial conditions must be set using the above dynamical equation and not using the Lane-Emden equation. Thus when a power-series expansion for the initial density, having odd as well as even powers, is substituted in this dynamical equation, the odd powers do not drop out. Instead, this equation determines the initial acceleration, given an initial density profile. A situation analogous to this Newtonian case holds in general relativity as well. It should also be emphasized that the restriction to even powers arising from the Lane-Emden equation is not applicable to dust collapse as such, there being no pressure, and hence no equilibrium.

We conclude that a priori, physical density functions need not necessarily be smooth functions. (See also, for instance, the discussion in [14] of observations suggesting possible cusps in globular cluster cores). In certain equilibrium cases, the field equations imply that they have to be smooth, but this is not true in general. Also, there need not be a restriction on the initial velocity distribution to be smooth. In our view, physical quantities should at the most be required to be \( C^2 \), so as to ensure solvability of the field equations. For instance, the self-similar density profile, which is often of physical interest (see e.g. [15], [16]), is a \( C^2 \) function in the case of dust [16].

All the same, it is important to note that the issue of whether physical quantities should be smooth or not does not have a bearing on our overall conclusion regarding censorship in dust collapse. As we explain below, the occurrence of a strong naked singularity does not necessarily require the density to be non-smooth; it is sufficient that the density is smooth, and the energy function \( f(r) \) is \( C^4 \), (the velocity \( v(r) \) is then \( C^2 \)). However, the strong naked
singularities are non-generic in either case, in the sense described below. The generic naked singularity is weak and hence it is \textit{genericity}, as opposed to \textit{smoothness}, which limits the importance of strong naked singularities. In other words, even if one allows non-smoothness in the initial data, any resulting strong naked singularities are non-generic. Perhaps it will also be useful to mention that when one considers equations of state other than dust, strong naked singularities do arise from smooth initial data [16].

As pointed out above, in the marginally bound case, the strong singularity arises from an initial density profile involving a cubic term, which is not smooth, but $C^2$. (We wish to recall that when we call a function of $r$ non-smooth we mean that some of the Cartesian derivatives are ill-defined; all the derivatives with respect to $r$ are well-defined though). However, an important question in this connection is the following: Is it true that a strong curvature naked singularity must necessarily involve and arise from a non-smooth density profile? The answer to this question is no. In fact, as can be deduced from the above, in the non-marginal case, one can get a strong naked singularity from density functions that are smooth, and a metric that is $C^4$. Consider, for instance, the case $\rho(r) = \rho_0 = \text{const.}$, or a smooth $\rho(r)$ with $\rho_2 = 0$, and $f_3 = f_4 = 0$, $f_5 < 0$. This initial data again leads to a strong naked singularity. Hence, non-smoothness of the density is not a pre-requisite for the strong naked singularity to occur.

Next, let us consider the stability of the Oppenheimer-Snyder (O-S) dust collapse solution [17] (as a special case of models considered here), under perturbation of initial data, in the marginally bound collapse. As we know, the O-S model describes homogeneous dust collapse leading to the formation of a black hole. If the O-S initial data is perturbed by switching on an infinitesimal negative $\rho_1$ or $\rho_2$ terms, we get a naked singularity instead. The black hole is not stable to small perturbations. On the other hand, if the O-S black hole scenario is perturbed by switching on a small $\rho_3$, it continues to be a black hole. Only large enough perturbations from homogeneity at the level of the third derivative convert the black hole to a naked singularity. If we make the plausible proposition that the O-S black hole must be stable to small perturbations in density, we can conclude that the weak naked singularity is not a genuine naked singularity. On the other hand, from this point of view, the strong naked singularity is a genuine naked singularity. This inference is strengthened by the likely possibility that spacetime may be extendible through a weak enough naked singularity, but not through a strong naked singularity.

We now comment on the possible measure of initial data which leads to a strong naked singularity in the case of inhomogeneous dust collapse. This argument is due to Reza Tavakol. Consider first the marginally bound case. Only the three derivatives $\rho_1$, $\rho_2$ and $\rho_3$ play a role in deciding whether or not the singularity is naked. Hence consider a three-dimensional space of initial data, labelled by these three derivatives. A generic point in this space will have all the three derivatives non-zero. For physical reasons we restrict to the half-space $\rho_1 < 0$. Since the first non-vanishing derivative is the relevant one, it follows that the generic singularity is weak and naked. If we set $\rho_1 = 0$ the generic naked singularity continues to be weak and naked. The strong naked singularity arises from part of the half-line $\rho_1 = \rho_2 = 0$, $\rho_3 < 0$, and hence from initial data that is of measure zero in this three-space; (see also [18]). A similar
consideration holds for the non-marginal case - the relevant space is now six dimensional, consisting of the first three derivatives of $\rho(r)$ and of $f(r)$. It can be concluded that the strong naked singularity arises from an initial data set of measure zero. Perhaps it should be emphasized that this result (strong naked singularity arises from data of zero measure) is specific to dust collapse, and may not hold for more general forms of matter. In fact, in the gravitational collapse of imploding radiation described by the Vaidya space-time, an initial data set of non-zero measure leads to a strong naked singularity [19]. In this model no initial data lead to a weak naked singularity.

The preceding two arguments together suggest that the generic naked singularity in dust collapse is weak, which we are proposing is not a genuine singularity. Should this be taken to mean that dust collapse is consistent with the cosmic censorship hypothesis? According to us the answer is no. Even if the singularity is weak (the quantity $R_{ij}V^iV^j$ grows as $1/k$, rather than as $1/k^2$ for the case of a strong singularity, where $k = 0$ at the center) and spacetime is possibly extendible, one will still see regions of arbitrarily high curvatures without any bound via the outgoing nonspacelike geodesics which start from near the center. As we see it, while the above arguments provide a good pointer to a possible direction of a cosmic censorship statement, their rigorous mathematical formulation and proof might turn out to be a difficult task to achieve as has been the case for the attempts so far. This is mainly because of the formidable difficulties in formulating a suitable stability analysis in general relativity and the related issues such as the complexity caused by the non-linearity of Einstein equations etc. What is needed, in our view, is some form of physical formulation and justification of the cosmic censorship principle [20].

III. Initial data and dynamics of the apparent horizon.

The behaviour and evolution of the apparent horizon in the course of gravitational collapse gives insight into the causal structure near the singularity. In this Section we work out some aspects of the evolution of the apparent horizon for the marginally bound case, $f = 0$ and examine the relation between initial data and the formation or otherwise of a naked singularity. This helps us understand why some initial data lead to a naked singularity, while the other would produce a black hole as the end product of collapse. From Eqn. (2), using the scaling $R(0, r) = r$ at the initial epoch $t = 0$, we find the solution to be

$$R^{3/2} = r^{3/2} - \frac{3}{2} \sqrt{F} t.$$

The singularity curve $R(t_s(r), r) = 0$ is given by $t_s(r) = 2r^{3/2}/3\sqrt{F}$. The apparent horizon, which is the boundary of the trapped region, is given by the curve $R(t_{ah}(r), r) = F(r)$, and using this relation, we find from (11) that

$$t_{ah}(r) = \frac{2r^{3/2}}{3\sqrt{F}} - \frac{2}{3} F(r).$$
Evidently, dust shells with \( r > 0 \) become trapped before they become singular, whereas at \( r = 0 \) the singularity and apparent horizon form simultaneously. We are interested in the properties of the function \( t_{ah}(r) \). Although the function \( t_{s}(r) \) is monotonically increasing, \( t_{ah}(r) \) is not necessarily monotonic.

Let us first work out its behaviour for the Oppenheimer-Snyder model. Now, we have \( F(r) = F_{0}r^{3} \) and hence

\[
t_{ah}(r) = \frac{2}{3\sqrt{F_{0}}} - \frac{2F_{0}}{3} r^{3}.
\]

Clearly, the boundary \( r_{b} \) of the star gets trapped first, and the apparent horizon moves inwards, the center being the last point to get trapped. The event horizon begins to form a finite time before the boundary gets trapped. (See Fig. 1(i)). Note that for the homogeneous case, \( t_{ah}(r) < t_{ah}(0) \) near the origin, signifying that the neighborhood of the center gets trapped before the center. Since \( t_{ah}(0) = t_{s}(0) \) the neighboring regions get trapped before the center becomes singular, and this helps understand why the central singularity is not naked.

Consider now the case of inhomogeneous collapse. The global evolution of the apparent horizon will depend on the nature of \( F(r) \) throughout the star, but near \( r = 0 \) it will be determined by \( F(r) \) near the center. If at \( r = 0 \) the first non-vanishing derivative of the density is the \( n \)th one, then using (8) and (12) we get to leading order

\[
t_{ah}(r) = \frac{2}{3\sqrt{F_{0}}} - \frac{F_{n}r^{n}}{3F_{0}^{3/2}} - \frac{2}{3}F_{0}r^{3}.
\]

This equation helps us understand the evolution of the apparent horizon for an inhomogeneous density profile and the following conclusions can be drawn. If the first non-vanishing derivative of the density at the center is either \( \rho_{1} \) or \( \rho_{2} \), the so called weak naked singularity occurs and in that case \( t_{ah}(r) > t_{ah}(0) = t_{s}(0) \). That is, the center becomes singular before its neighborhood gets trapped and the first point of singularity coincides with the first point in time of the apparent horizon. The qualitative behaviour of the apparent horizon formation is as shown in Fig. 1(ii). The dynamical evolution of apparent horizon now is totally different from the homogeneous case, and helps understand why in this case the singularity is not naked.

Next, if the first non-vanishing derivative is \( \rho_{3} \), we get \( t_{ah}(r) > t_{ah}(0) \) if the parameter \( \xi \) defined earlier satisfies \( \xi < -2 \) and \( t_{ah}(r) < t_{ah}(0) \) if \( \xi > -2 \). Recall that the singularity is naked for \( \xi < -25.9904 \) and covered if \( \xi \) is greater than this number. This means there is the range \(-25.9904 < \xi < -2\) in which the singularity is not naked, even though the center gets trapped before its neighborhood. Put differently, the apparent horizon is absent until the formation of the singularity, but the singularity is not naked. This is a counterexample to the criterion used by Shapiro and Teukolsky in their numerical studies [21], where they suggest that if the apparent horizon is absent until singularity formation, the singularity is
naked. As is clear from this consideration, the condition that the center get trapped before its neighboring region is necessary but not sufficient for nakedness.

Fig. 1(i): The standard picture of spherical homogeneous dust collapse, the Oppenheimer-Snyder model. The shaded portion is the trapped region. Note that first the boundary of the star gets trapped, then the trapped region expands into the star, towards the center. The center is the last point to get trapped. The curvature singularity forms simultaneously all over the star. Fig. 1(ii): The corresponding space-time diagram for inhomogeneous dust-collapse leading to a locally naked singularity. The apparent horizon curve is as given by Eqn. (16), with $\xi < -2$. The shaded portion is the trapped region. Note that unlike in the Oppenheimer-Snyder model, the center gets trapped first, and the trapped region moves out towards the boundary of the star. Different shells become singular at different times, the center becoming singular first.

If the first three derivatives are zero, then $n \geq 4$ and it follows from (14) that $t_{ah}(r) < t_{ah}(0)$ - the center gets trapped later than its neighborhood, and hence the central singularity is not naked; the qualitative picture being again same as Fig. 1a. We see that there is a qualitative change in the behaviour of the apparent horizon near the center as the density profile (6) is made more and more homogeneous by setting successive derivatives to zero. This provides a physical picture for the relation between the initial data and the nature of the singularity, as to whether it is naked or covered.

As is clear from (12), the global behaviour of the function $t_{ah}(r)$ can be worked out only if we know $F(r)$ throughout the matter cloud. To understand the overall evolution of the apparent horizon within the cloud, we consider below an illustrative example of a typical initial density profile where the density is inhomogeneous, decreasing away from the center,
and we work out the evolution of the apparent horizon through the star. Consider the initial profile

$$\rho(r) = \rho_0 \left(1 - \frac{r^3}{r_b^3}\right)$$  \hspace{1cm} (15)

where \(\rho_0\) is the central density and \(r_b\) is the boundary of the cloud. Recall that we are using the scaling \(R = r\) at the initial epoch, so this is the physical initial density function. We can compute the resulting mass-function \(F(r)\) using (4) and the apparent horizon curve using (12). We get

$$t_{ah}(r) = \frac{2}{\sqrt{3\rho_0}} \left(1 - \frac{r^3}{2r_b^3}\right)^{-1/2} - \frac{2\rho_0}{9} \left(r^3 - \frac{r^6}{2r_b^3}\right).$$  \hspace{1cm} (16)

It can be verified that for \(\xi < -2\) this is a monotonically increasing function. That is, the center is the first point to get trapped, and shells with larger and larger initial radii get trapped at later and later times. Again, this behaviour should be contrasted with that of the trapped surface in the Oppenheimer-Snyder model (see Fig. 1(ii)). If \(\xi > -2\) the above \(t_{ah}(r)\) starts to decrease near the center, but has a turning point at some location in the interior of the star (other than the center). This means some point other than the center or the boundary is the first one to get trapped, and as time progresses, the trapped region moves outwards as well as inwards. Figs. 2(i) and 2(ii) show \(t_{ah}(r)\) as given by (16), for various representative values of \(\xi\). In Fig. 2(i) we have set \(\rho_0 = 1\) and varied \(r_b\), while in Fig. 2(ii) we have set \(r_b = 1\) and varied \(\rho_0\).
Figure 2: A plot of the apparent horizon curves described by Eqn. (16), for various values of $\xi$. For $\xi < -2$ the center is the first point to get trapped, whereas for $\xi > -2$ some surface in the interior of the star is the first one to get trapped, and the trapped region moves both inwards and outwards. Fig. 2(i) is obtained by setting $\rho_0 = 1$ and varying $r_b$, whereas in Fig. 2(ii) we have $r_b = 1$ and $\rho_0$ has been varied.

The causal nature of the apparent horizon near the center can be worked out by writing the induced metric on the apparent horizon. It is also convenient to use the mass-function $F(r)$ as a coordinate, instead of $r$. Using (1), (11) and (12) the induced metric is shown to be

$$ds^2 = \left( 2 \frac{\partial t_s(r)}{\partial F(r)} - \frac{1}{3} \right) dF^2(r). \quad (17)$$

It can then be shown that if the first non-vanishing derivative of the density at the center is either $\rho_1$ or $\rho_2$, the apparent horizon is spacelike, and as we know, the singularity is naked. If the first non-vanishing derivative is $\rho_3$, the apparent horizon is spacelike for $\xi < -0.5$ and past-timelike for $\xi > -0.5$. That is, if the singularity is naked the horizon is spacelike, and if it is covered, the horizon is spacelike or past-timelike. If the first non-vanishing derivative is the fourth or higher, the apparent horizon is past-timelike, and the singularity is covered.

The event horizon must be outside the apparent horizon covering the trapped region. Hence in the homogeneous case it must start forming before the first point of the singularity.
forms at the center. When the initial density is inhomogeneous and the resulting singularity covered, the event horizon will again start forming before the central singularity occurs. In the case of inhomogeneous collapse leading to a naked singularity, the event horizon may begin to form prior to or simultaneously with the central singularity, depending on whether the singularity is only locally naked, or globally naked as well.

IV. Naked singularity and the nature of geodesics

In this Section we discuss the nature of null geodesics families emerging from the naked singularity, for marginally bound collapse. In particular, we wish to highlight the structural difference between the cases of weak and strong naked singularities. In order to avoid some repetition of earlier results, we refer to [7] for the derivation of the geodesic equation in this case. Here, $\alpha = 1 + 2q/3$, where the first non-vanishing derivative in the Taylor expansion (6) for the density is the $q$th one. The central singularity is the point $R = 0, u = 0$. In terms of these variables, the geodesic equation is the Eqn. (25) of [7]:

$$\frac{dR}{du} = \left(1 - \sqrt{\frac{\Lambda}{X}}\right) \frac{H(X, u)}{\alpha} \equiv U(X, u).$$

Here, $\Lambda(r) = F(r)/r^\alpha$, $X = R/r^\alpha$. The function $H(X, u)$ is defined in [7] - we will need its form only in the limit $r \to 0$, that limiting expression is reproduced below. The singularity turns out to be naked if the equation $V(X_0) = 0$ admits one or more positive real roots $X_0$,

$$V(X) = U(X, 0) - X = \left(1 - \sqrt{\frac{\Lambda_0}{X}}\right) \frac{H(X, 0)}{\alpha} - X.$$  

We have denoted $\Lambda(0)$ as $\Lambda_0$. The quantity $H(X, 0)$ is given in [9] as $H(X, 0) = X + \Theta_0/\sqrt{X} \equiv H_0$ where $\Theta_0 = -qF_q/3F_0$, and $F_q$ is defined in Eqn. (8).

We are interested in finding out the nature of geodesics for the weak and strong naked singularities. For this purpose, given a positive root to the equation $V(X_0) = 0$, we integrate Eqn. (18) as follows. We can write this equation as

$$\frac{dX}{du} = \frac{1}{u} \left(\frac{dR}{du} - X\right) = \frac{U(X, u) - X}{u}. $$

The geodesics will be given as solutions $X = X(u)$ of this equation. Given a root $X_0$ of $V(X_0) = 0$, we can write $V(X) = (X - X_0)(h_0 - 1) + h(X)$ where $h_0$ is a constant defined by $h_0 = (dU/dX)_{X=X_0}$ and the function $h(X)$ contains higher order terms in $(X - X_0)$, that
is, \( h(X_0) = (dh/dX)_{X=X_0} = 0 \). The constant \( h_0 \) can be evaluated using (19) and is seen to be
\[
h_0 = \frac{\Lambda_0^{1/2} H_0}{2\alpha X_0^{3/2}} + \frac{1}{\alpha} \left( 1 - \sqrt{\frac{\Lambda_0}{X_0}} \right) \left( 1 - \frac{\Theta_0}{2X_0^{3/2}} \right). \tag{21}
\]
Eqn. (20) can be written
\[
\frac{dX}{du} - (X - X_0) \frac{(h_0 - 1)}{u} = \frac{S}{u} \tag{22}
\]
where we have defined \( S(X, u) = U(X, u) - U(X, 0) + h(X) \), and \( S(X_0, 0) = 0 \). This equation can be integrated to get the solution
\[
X - X_0 = Du^{h_0 - 1} + u^{h_0 - 1} \int S u^{-h_0} du, \tag{23}
\]
\( D \) being a constant of integration that labels different geodesics. Note that the last term in this equation always vanishes as \( X \to X_0, \ u \to 0 \), irrespective of the value of \( h_0 \). The first term, \( Du^{h_0 - 1} \), vanishes in this limit if \( h_0 > 1 \), goes to a constant for \( h_0 = 1 \) and diverges for \( h_0 < 1 \).

Thus, if \( h_0 > 1 \), a family of geodesics, labelled by the parameter \( D \), will terminate at the singularity with the root \( X_0 \) as their tangent. The situation is different for \( h_0 \leq 1 \). Consider first the range \( 0 < h_0 \leq 1 \). Now, as \( u \to 0 \), only one geodesic, labelled by \( D = 0 \), will terminate at the singularity with \( X_0 \) as tangent. There will however be a family of geodesics labelled by \( D \neq 0 \), for which \( X \to \infty \) as \( u \to 0 \). These correspond to geodesics having the \( R \)-axis as their tangent, rather than the root \( X_0 \) as tangent. If \( h_0 \leq 0 \), then by writing the solution (23) near the singularity as \( R - X_0 u = Du^{h_0} \) we see there will be no geodesics terminating at \( R = 0, u = 0 \).

We now work out the classification of null geodesics families for various initial data configurations - this classification will clearly depend on the value of \( h_0 \), which can be worked out from Eqn. (21). Consider first the density profile for which \( \rho_1 \neq 0 \). In this case we have \( q = 1, \alpha = 5/3, \Lambda_0 = 0 \). Using the result from [9] that \( V(X_0) = 0 \) has one positive root, given by \( X_0^{3/2} = \Theta_0/(\alpha - 1) \) in (21), we get \( h_0 = 2/5 \). For the case of the density profile \( \rho_1 = 0, \rho_2 \neq 0 \), similar considerations give \( h_0 = 1/7 \). Thus for both these profiles, which result in a weak naked singularity, we have \( 0 < h_0 < 1 \). Hence there will be one geodesic having the root \( X_0 \) as tangent, and an entire family having the \( R \)-axis as the limiting tangent.

The density profile \( \rho_1 = \rho_2 = 0, \rho_3 < 0 \) leads to a naked singularity if the parameter \( \xi \) defined earlier lies in a certain range, as described in Section 1. Further it is a strong curvature naked singularity. In this case, the expression for \( h_0 \) is not as trivial as we found above for the case of the weak naked singularity. We have \( q = 3, \alpha = 3 \) and with some effort it can be shown from (21) that
\[
h_0 = -\frac{1}{6\xi^4} \left( \xi + 2\xi^3 \right). \tag{24}
\]
Here $X_0 = F_0 x^2$, and we have used the result from [9] that $x$ satisfies the quartic equation

$$2x^4 + x^3 + \xi x - \xi = 0. \quad (25)$$

Eqn. (24) admits values for $h_0$ which are greater than unity. For instance, $x = 1.1, \xi = -42.6$ satisfies (25) and gives $h_0 = 4.5$ from (24). Further, we know from [9] that for this density profile, whenever the singularity is naked, there are two positive real roots. Since $h_0 - 1$ is the value of the derivative $dV/dX$ at $X = X_0$, clearly the derivative will be negative at one root and positive at the other. Hence one of the roots, say $X_1$, leads to $h_0 > 1$, and the other root, say $X_2$, gives $h_0 < 1$. As a result, there will be an entire family of outgoing geodesics with $X_1$ as the tangent, only one geodesic with $X_2$ as tangent, and a family of geodesics with the $R$-axis as tangent. This brings out the difference in structure of geodesic families coming out for both the cases of weak and strong naked singularities. For completeness we mention that the central naked singularity is a null singularity, since it occurs at one instant of time and at one point in space.

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