Comment on “Summing one-loop graphs at multi-particle threshold”

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Abstract

The propagator of a virtual $\phi$-field with emission of $n$ on-mass-shell particles all being exactly at rest is calculated at the tree-level in $\lambda\phi^4$ theory by directly solving recursion equations for the sum of Feynman graphs. It is shown that the generating function for these propagators is equivalent to a Fourier transform of the recently found Green’s function within the background-field technique for summing graphs at threshold suggested by Lowell Brown. Also the derivation of the result that the tree-level on-mass-shell scattering amplitudes of the processes $2 \rightarrow n$ are exactly vanishing at threshold for $n > 4$ is thus given in the more conventional Feynman diagram technique.
The technique recently suggested by Lowell Brown\cite{1} for summing the tree graphs in $\lambda\phi^4$ theory for production at the threshold of $n$ on-mass-shell scalar $\phi$-bosons by a highly virtual $\phi$ field was extended\cite{2} to calculation of the loop corrections to the same process. The latter calculation is based on deriving the two-point Green function of the quantum field $\phi$ in the background of the complex field of the classical solution to the field equations considered by Brown. As a by-product of the calculation\cite{2} it was found that the sum of tree graphs for the processes $2 \to n$ in which all particles are on the mass shell are exactly vanishing at the threshold for any $n$ greater than 4 in the $\lambda\phi^4$ theory with unbroken symmetry. It was subsequently shown\cite{3} that in the theory with spontaneous symmetry breaking (i.e. with negative $m^2$) the same behavior of the tree-level threshold amplitudes holds for any $n$ larger than 2. Given that the method used in Refs.\cite{1, 3} is not entirely conventional it is worthwhile to present a derivation of the same results in more standard terms. It is the purpose of this comment to derive the Green function equivalent to the one obtained in Ref.\cite{2} as well as the aforementioned surprising threshold behavior of the on-mass-shell scattering amplitudes using standard Feynman diagram technique. Simultaneously this will substantiate the claim\cite{2} that the technique based on recursion relations\cite{4} for the sums of Feynman graphs is equivalent to Brown’s method, though perhaps less elegant. For simplicity we concentrate on the case of unbroken $\lambda\phi^4$ theory in which the Lagrangian reads as

$$
\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 .
$$

The calculations for the case of the theory with broken symmetry contain no principal differences from this somewhat simpler in terms of notation case.

Our purpose here is to calculate the propagator $D_n(p)$ of the field $\phi$ with the tree-level emission of $n$ particles all being at rest and $p$ being the final four-momentum in the propagator after the emission, see Fig.1. The incoming four-momentum in the propagator is fixed:

$$p_1 = p + nq ,$$

where $q$ is the four-momentum of each of the final particles. In the rest frame of the produced on-mass-shell particles one has $q_0 = m$, $q = 0$. For the first two values of $n$ the propagator is well known:
\[ D_0(p) = \frac{i}{p^2 - 1} , \quad D_2(p) = \frac{i \lambda}{(p^2 - 1)((p + 2q)^2 - 1)} \]  

(hereafter we set \( m \) equal to one, and it can be also noticed that \( n \) is always even due to the unbroken symmetry under the reflection \( \phi \rightarrow -\phi \)).

The propagators \( D_n(p) \) are related by recursion relations analogous to the ones considered in Ref.\[4\], which graphically are shown on Fig.2 and algebraically can be written as

\[ D_n(p) = \frac{3\lambda}{((p + nq)^2 - 1)} \sum_{n_1,n_2,n_3} \delta_{n_1+n_2+n_3,n} \frac{n!}{n_1!n_2!n_3!} a(n_1)a(n_2)D_{n_3}(p) , \]

where \( a(N) \) are the amplitudes of threshold production of \( N \) particles, which were found\[4,5,6\] to be given by:

\[ a(N) = \langle N | \phi(0)|0 \rangle = N! \left( \frac{\lambda}{8} \right)^{N-1} \]

where \( N \) and thus also \( n_1 \) and \( n_2 \) in eq.(4) are necessarily odd. Substituting this expression for the amplitudes \( a(N) \) into eq.(4) and introducing instead of \( D_n(p) \) the normalized propagator \( d_n(p) \):

\[ D_n(p) = i n! \left( \frac{\lambda}{8} \right)^{n/2} d_n(p) \]

one can rewrite the equation (4) in the form

\[ ((p + nq)^2 - 1)d_n = 24 \sum_{n_3(even)} \frac{n - n_3}{2} d_{n_3}(p) . \]

This equation can be solved by the technique of generating function\[7\], i.e. by introducing the function

\[ g_p(x) = \sum_{n=0}^{\infty} x^n d_n(p) . \]

Equation (7) can then be readily verified to be the \( n \)-th term of the expansion in \( x \) of the differential equation for the generating function

\[ \left[ 2x \frac{d^2}{dx^2} + (2(p \cdot q) + 1)x \frac{d}{dx} + p^2 - 1 - \frac{24x^2}{(1 - x^2)^2} \right] g_p(x) = 1 , \]
where the inhomogeneous term on the right hand side is fixed by the normalization of $D_0(p)$. To simplify the subsequent formulas we introduce the notations: $\epsilon = (p \cdot q)$ and $\omega^2 = (p \cdot q)^2 - p^2 + 1$. In the rest frame of the produced on-shell particles $\epsilon = p_0$ and $\omega^2 = p^2 + 1$. In the kinematics of the process $2 \to n$ i.e. when the initial and the final momenta in the propagator $D_n(p)$ correspond to incoming particles the value of $\epsilon$ is negative. The mass shell for the particle with the momentum $p$ corresponds to $\omega = -\epsilon$ and for the particle with the initial momentum $p + nq$: $\omega = n + \epsilon$. Thus $\epsilon = -n/2$ when both incoming particles are on the mass shell.

Equation (9) is solved by introducing new variable $t$ according to $x = i e^t$ and then seeking the solution in the form

$$g_p(x(t)) = e^{-\epsilon t} y_p(t) .$$

In terms of the function $y_p(t)$ the equation (9) reads as

$$\left[ \frac{d^2}{dt^2} - \omega^2 + \frac{6}{(\cosh t)^2} \right] y_p(t) = e^{\epsilon t} .$$

The differential operator in the homogeneous part of this equation is the same as in eq.(17) of Ref.[2]. Therefore one can use the result for the Green function from there and write the solution to the equation (9) in the form

$$g_p(x(t)) = -e^{-\epsilon t} \frac{W}{W} \left[ f_1(t) \int_{-\infty}^t e^{\epsilon s} f_2(s) \, ds + f_2(t) \int_t^\infty e^{\epsilon s} f_1(s) \, ds \right] ,$$

where

$$f_1(t) = \frac{2 - 3 \omega + \omega^2 - 8 u^2 + 2 \omega^2 u^2 + 2 u^4 + 3 \omega u^4 + \omega^2 u^4}{w^\omega (1 + u^2)^2}$$

and

$$f_2(t) = \frac{u^w (2 + 3 \omega + \omega^2 - 8 u^2 + 2 \omega^2 u^2 + 2 u^4 - 3 \omega u^4 + \omega^2 u^4)}{(1 + u^2)^2}$$

are the two solutions of the homogeneous equation (11) written in terms of $u(t) = -ix(t) = e^t$ and $W$ is the Wronskian of these two solutions:

$$W = f_1(t)f_2'(t) - f_1'(t)f_2(t) = 2 \omega (\omega^2 - 1)(\omega^2 - 4) .$$

The usual ambiguity in the solution of a second-order differential equation, which amounts to the freedom of adding arbitrary linear combination of the two solutions
of the homogeneous equation, is fixed in the equation (12) by the requirement that for arbitrary $\omega$ the function $g_p(x)$ should have expansion in ascending integer (in fact integer even) powers of $x$.

Equation (12) demonstrates that the generating function $g_p(x)$ for the propagators $d_n(p)$ is related to the Green function $G_{\omega}(t_1, t_2)$ of Ref.[2] by the Fourier transform in time, which proves the statement [2] about the equivalence of the two techniques at least at this level of the perturbation theory in $\lambda$.

From the equation one can also readily see the nullification of the on-mass-shell threshold amplitudes for the scattering amplitudes $2 \to n$. Indeed, this amplitude is given by the residue of the propagator $D_n(p)$ at the double pole, when both the final momentum $p$ and the initial $p + nq$ are on the mass shell and $p_0 = \epsilon$ is negative. Therefore one can set $\epsilon$ at its on-mass-shell value, $\epsilon = -n/2$, and look for the double pole at $\omega = n/2$. (Thus the difference $\omega - n/2$ can be used as a measure of how far the incoming particles are off the mass shell.) The integration in eq.(12) however produces only single poles at $\omega = \pm (\epsilon + 2k)$ with integer $k$, and the missing single pole can come only from a zero of the Wronskian (15). Thus the tree-level amplitude of the on-mass-shell scattering $2 \to n$ is non-vanishing only for $\omega = 1$ and $\omega = 2$, i.e. for $n = 2$ and $n = 4$.

I believe that the presented here calculation based on the recursion relation (4) for the propagators $D_n(p)$ somewhat clarifies the relation between this approach and the technique suggested by Brown and also can be helpful for an interpretation of the further calculations within Brown’s technique.

As a final remark it can be noticed that the nullification of the on-mass-shell threshold amplitudes arises due to the special value, namely 6, of the coefficient of $(\cosh t)^{-2}$ in the differential operator in the equation (11). It is well known that when this coefficient is equal to $N(N + 1)$ with positive integer $N$ the operator has $N$ eigenvalues: $\omega = 1, 2, \ldots, N$ and the solutions of the homogeneous equation at arbitrary $\omega$ have the form of rational functions of $u(t)$ times $u^{\pm \omega}$. This coefficient is theory-dependent, e.g. in a similar analysis in the Sine-Gordon theory the coefficient of $(\cosh t)^{-2}$ in the analog of eq.(11) would be equal to 2, so that $N = 1$. One can readily write a theory with several bosonic fields where this coefficient can take arbitrary value depending on the coupling constants. From a simple generalization of the present analysis one thus concludes that every time this coefficient takes the value $N(N + 1)$ the tree-level amplitudes of the scattering $2 \to n$ should vanish at the
threshold for \( n > 2N \), though the nullification can also occur at smaller \( n \) for other reasons\( ^3 \).

I am thankful to Peter Arnold, whose questions and remarks have stimulated writing this comment. This work is supported in part by the DOE grant DE-AC02-83ER40105.

References

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Figure captions

Fig. 1. The propagator \( D_n(p) \) with emission of \( n \) on-mass-shell particles all being at rest. The circle represents the sum of all tree graphs.

Fig. 2. The recursion equation (4) for the propagators \( D_n(p) \). The circles correspond to the sums of all tree graphs.
\[ n \]
\[ p + nq \]
\[ p \]

Figure 1

\[ n \]
\[ n_1 \]
\[ n_2 \]
\[ n_3 \]
\[ p \]
\[ n \]
\[ p \]

\[ = \sum \]

Figure 2