The partially alternating ternary sum in an associative dialgebra

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Abstract

The alternating ternary sum in an associative algebra,

\[ abc - acb - bac + bca + cab - cba, \]

gives rise to the partially alternating ternary sum in an associative dialgebra with products \( \dlane \) and \( \drane \) by making the argument \( a \) the center of each term:

\[ a \dlane b \dlane c - a \dlane c \dlane b - a \drane c + b \drane c \dlane a - a \drane b - b \drane a. \]

We use computer algebra to determine the polynomial identities in degree \( \leq 9 \) satisfied by this new trilinear operation. In degrees 3 and 5, we obtain

\[ [a, b, c] + [a, c, b] \equiv 0, \quad [a, [b, c, d], e] + [a, [c, b, d], e] \equiv 0; \]

these identities define a new variety of partially alternating ternary algebras. We show that there is a 49-dimensional space of multilinear identities in degree 7, and we find equivalent nonlinear identities. We use the representation theory of the symmetric group to show that there are no new identities in degree 9.

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1. Introduction

The Lie bracket \( [a, b] = ab - ba \) gives rise to two distinct ternary operations: the Lie triple product \( [[a, b], c] = abc - bac + cab + cba \), and the alternating ternary sum (ATS) \( [a, b, c] = abc - acb - bac + bca + cab - cba \), also called the ternary commutator [4] or ‘ternutator’ [19]. Apart from the obvious skew-symmetry in degree 3, the simplest non-trivial identities for the ATS have degree 7 and were found by Bremner [4] using computer algebra. Subsequent work of Bremner and Hentzel [7] showed that there are no new identities for the ATS in degree 9. (An identity is ‘new’ if it is not a consequence of identities in lower degree.) Recently, Curtright et al [15] have generalized Bremner’s identity to all odd \( n \). The ATS is also related to the \( n = 3 \) case of \( n \)-Lie algebras introduced by Filippov [20]: in the recent construction by Bremner and Elgendy [6] of universal associative enveloping algebras.
of Filippov algebras, the alternating $n$-ary product in the Filippov algebra is mapped to the alternating $n$-ary sum in the associative algebra. Ternary and $n$-ary algebras, and the closely related topic of multidimensional matrices and determinants, have been investigated since pioneering works in the middle of the 19th century. We mention in particular the papers by Cayley [12, 13] and Sylvester [37], and the relatively recent book by Gelfand, Kapranov and Zelevinsky [22] on algebraic geometry and homological algebra related to multidimensional determinants. The ATS appears in physics in the context of Nambu’s formulation of quantum mechanics [34]. He introduced a multilinear $n$-bracket (now called the Nambu $n$-bracket) which becomes the ATS for $n = 3$. This theory has been developed by Takhtajan [38], Gautheron [21] and Curtright and Zachos [16], among many others. Further applications of ternary and $n$-ary structures in physics have been studied by Vainerman and Kerner [39], Abramov et al [1], Kerner [28], Plyushchay and Rausch de Traubenberg [35], Bazunova et al [3] and Campoamor-Stursberg and Rausch de Traubenberg [10, 11]. In another direction, the works of Hanlon and Wachs [26] and Goze et al [23] introduce the strong homotopy $n$-Lie algebras, which permit a natural study of Maurer–Cartan computations for Nambu–Filippov algebras. The importance of polynomial identities for ternary algebras is also apparent from the work of Goze and Remm [24] on operads and homology of free $n$-ary algebras; in the study of deformation theory in physics, one has to compute cohomology based on the structure of the free algebra. However, the most important recent developments in physics related to $n$-ary algebras are the works of Bagger and Lambert [2] and Gustavsson [25], which aim at a world-volume theory of multiple M2-branes. For a very recent comprehensive survey of this entire area, from both the physical and mathematical points of view, see [18].

Motivated by the importance of the ATS in theoretical physics, as well as the recent works of Bremner and Peresi [5, 9] on polynomial identities satisfied by the quasi-Jordan product in an associative dialgebra, this paper will focus on the partially alternating ternary sum (PATS) in an associative dialgebra. We use computational linear algebra and the representation theory of the symmetric group to determine the polynomial identities in degree $\leq 9$ satisfied by this new trilinear operation.

2. Preliminaries on dialgebras

Unless otherwise stated, the base field $F$ is the field $\mathbb{Q}$ of rational numbers.

2.1. Dialgebras and Leibniz algebras

Dialgebras were introduced by Loday [30–32] to provide a natural setting for Leibniz algebras, a ‘noncommutative’ generalization of Lie algebras.

**Definition 2.1 ([17, 30]).** A Leibniz algebra is a vector space $L$ together with a bilinear map $L \times L \to L$, denoted as $(a, b) \mapsto [a, b]$ and called the Leibniz bracket, satisfying the Leibniz identity which says that right multiplications are derivations:

$$[[a, b], c] \equiv [[a, c], b] + [a, [b, c]].$$

If $[a, a] \equiv 0$, then the Leibniz identity is the Jacobi identity and $L$ is a Lie algebra.

Every associative algebra becomes a Lie algebra if the associative product is replaced by the Lie bracket. Loday introduced the notion of dialgebra which gives, by a similar procedure, a Leibniz algebra: one replaces $ab$ and $ba$ by two distinct operations, so that the resulting bracket is not necessarily skew-symmetric.
Definition 2.2 ([31]). An (associative) dialgebra is a vector space $A$ together with two bilinear maps $A \times A \to A$, denoted as $a \triangleright b$ and $a \vdash b$ and called the left and right products, satisfying the following identities:

$$(a \triangleright b) \triangleright c \equiv a \triangleright (b \triangleright c),$$

$$(a \vdash b) \vdash c \equiv a \vdash (b \vdash c),$$

$$(a \vdash b) \triangleright c \equiv a \vdash (b \triangleright c),$$

$$(a \triangleright b) \vdash c \equiv (a \vdash b) \vdash c.$$ 

From a dialgebra we construct a Leibniz product by $a \triangleright b - b \vdash a$.

2.2. Free dialgebras

Definition 2.3 ([32]). A (dialgebra) monomial on a set $X$ is a product $x = a_1 a_2 \cdots a_n$ where $a_1, \ldots, a_n \in X$ with some placement of parentheses and some choice of operations. The center of $x$ is defined inductively: if $n = 1$, then $c(x) = x$; if $n \geq 2$, then $x = y \vdash z$ and we set $c(y \vdash z) = c(y)$ or $c(y \triangleright z) = c(z)$.

A monomial is determined by the order of its factors and the position of its center.

Lemma 2.4 ([32]). If $x = a_1 a_2 \cdots a_n$ is a monomial with $c(x) = a_i$, then

$$x = (a_1 \vdash \cdots \vdash a_{i-1}) \vdash a_i \vdash (a_{i+1} \vdash \cdots \vdash a_n).$$

Definition 2.5. The right-hand side of the last equation is the normal form of $x$ and is abbreviated by the hat notation $a_1 \cdots a_{i-1} \hat{a}_i a_{i+1} \cdots a_n$.

Lemma 2.6 ([32]). The set of monomials $a_1 \cdots a_{i-1} \hat{a}_i a_{i+1} \cdots a_n$ in normal form with $1 \leq i \leq n$ and $a_1, \ldots, a_n \in X$ forms a basis of the free dialgebra on $X$.

2.3. Identities for algebras and identities for dialgebras

Kolesnikov [29] and Pozhidaev [36] recently introduced an algorithm for passing from identities of algebras to identities of dialgebras. This algorithm applies not only to associative algebras and dialgebras, but also to identities in which the monomials involve a certain placement of parentheses, in other words, to identities in general nonassociative algebras and dialgebras.

Let $I$ be a multilinear algebra identity in the variables $a_1, \ldots, a_n$. For each $i = 1, \ldots, n$ we convert $I$ into a multilinear dialgebra identity by making $a_i$ the center of each monomial; we do not change the placement of parentheses. In this way, one algebra identity in degree $n$ produces $n$ dialgebra identities. For example, associativity $(ab)c - a(bc)$ gives rise to $(\hat{a}b)c - a(\hat{b}c)$, $(ab)c - a(\hat{b}c)$, $(ab)c - a(\hat{b}c)$: associativity of the left product, inner associativity and associativity of the right product.

The same algorithm can be used to convert a multilinear algebra operation into a set of multilinear dialgebra operations. For example, the Lie bracket $ab - ba$ gives rise to the left and right Leibniz products $a \triangleright b - b \vdash a$ and $a \vdash b - b \triangleright a$, respectively. We use this method to obtain ternary dialgebra operations from ternary algebra operations.

3. Degree 3

The Kolesnikov–Pozhidaev algorithm produces the dialgebra version of the ATS.

Definition 3.1. The PATS is this trilinear operation in an associative dialgebra:

$$[a, b, c] = \hat{a}bc - \hat{a}cb - b\hat{a}c + b\hat{c}a + c\hat{a}b - c\hat{b}a.$$
Table 1. Transpose of the expansion matrix $E$ in degree 3.

\[
\begin{bmatrix}
1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 \\
-1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & -1 & 1 \\
\cdot & \cdot & 1 & -1 & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & -1 \\
\cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & 1 & -1 \\
\cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & -1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\end{bmatrix}
\]

Table 2. Row canonical form of $E$ and canonical basis of its nullspace.

\[
\begin{bmatrix}
1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]

It is easy to see that the PATS is an alternating function of its second and third arguments; proposition 4.3 shows that it becomes completely alternating (CA) in the second and third arguments of a nested monomial. The PATS is obtained from the ATS by making $a$ the center of each monomial; the alternating property of the ATS implies that making $b$ or $c$ the center gives equivalent operations.

Definition 3.2. A ternary algebra is a vector space $T$ together with a trilinear map $T \times T \times T \to T$ denoted by $(a, b, c)$. In a ternary algebra we define the polynomial $P(a, b, c) = (a, b, c) + (a, c, b)$.

Proposition 3.3. Every multilinear polynomial identity in degree $\leq 3$ satisfied by the PATS is a consequence of $P(a, b, c) \equiv 0$.

Proof. It is clear from the definition of the PATS that $[a, b, c] = [a, c, b]$, and this shows that the PATS satisfies $P(a, b, c) \equiv 0$. It remains to show the converse: if the PATS satisfies an identity $I(a, b, c) \equiv 0$, then $I(a, b, c) \equiv 0$ follows from $P(a, b, c) \equiv 0$. Consider the general trilinear identity of degree 3:

$$x_1[a, b, c] + x_2[a, c, b] + x_3[b, a, c] + x_4[b, c, a] + x_5[c, a, b] + x_6[c, b, a] = 0.$$

Each of the six ternary monomials expands using the PATS into a linear combination of six dialgebra monomials. Altogether we obtain 18 dialgebra monomials:

$\hat{a}bc, \hat{a}cb, \hat{b}ac, \hat{b}ca, \hat{c}ab, \hat{c}ba, \hat{a}\hat{b}c, \hat{a}\hat{c}b, \hat{b}\hat{c}a, \hat{c}\hat{b}a, \hat{b}\hat{a}c, \hat{c}\hat{a}b, \hat{a}\hat{c}b, \hat{b}\hat{a}c, \hat{c}\hat{b}a, \hat{b}\hat{c}a, \hat{c}\hat{b}a$.

Let $E$ be the $18 \times 6$ expansion matrix whose $(i, j)$ entry is the coefficient of the $i$th dialgebra monomial in the expansion of the $j$th ternary monomial (table 1). The coefficient vectors of the identities in degree 3 satisfied by the PATS are the vectors in the nullspace of $E$. We compute the row canonical form of $E$ and extract the canonical basis of the nullspace, see table 2. The basis vectors are the coefficient vectors of three permutations of $P(a, b, c)$, namely $[a, b, c] + [a, c, b], [b, a, c] + [b, c, a]$ and $[c, a, b] + [c, b, a]$.

4. Degree 5

For a ternary operation, there are three association types in degree 5: $((a, b, c), d, e)$, $(a, (b, c, d), e)$ and $(a, b, (c, d, e))$. 
Lemma 4.1. If a ternary operation satisfies \( P(a, b, c) \equiv 0 \), then every multilinear monomial in degree 5 equals one of the following 90 monomials:

\[
(a^o, b^p, c^r, d^s, e^t) \quad \text{for} \quad b^p < c^r < d^s < e^t,
\]

\[
(a^o, b^p, c^r, d^s, e^t) \quad \text{for} \quad c^r < d^s.
\]

Here \( \sigma \) is a permutation of \( \{a, b, c, d, e\} \) and \( \leq \) denotes alphabetical precedence; that is, if \( x \) and \( y \) are two elements of \( \{a, b, c, d, e\} \), then we say that \( x < y \) if and only if \( x \) precedes \( y \) in alphabetical order.

Proof. Since \( P(a, b, c) \equiv 0 \) implies \( (a, (b, c, d), e) \equiv -((a, c, b), d, e) \), we can ignore the third association type. Applying \( P(a, b, c) \equiv 0 \) to the first and second types gives

\[
((a, b, c), d, e) \equiv -((a, c, b), d, e) \equiv -((a, c, b), e, d) \equiv ((a, c, b), e, d),
\]

\[
(a, (b, c, d), e) \equiv -((a, b, d), c, e).
\]

Hence there are \( 5!/4 = 30 \) monomials in the first type and \( 5!/2 = 60 \) in the second. \( \square \)

Definition 4.2. In a ternary algebra, we define the polynomial \( Q(a, b, c, d, e) = (a, (b, c, d), e) + (a, (c, b, d), e) \).

Proposition 4.3. Every multilinear polynomial identity in degree \( \leq 5 \) satisfied by the PATS is a consequence of \( P(a, b, c) \equiv 0 \) and \( Q(a, b, c, d, e) \equiv 0 \).

Proof. We order the multilinear ternary monomials of lemma 4.1 first by association type and then by lexicographical (lex) order of the permutation; that is, if \( x \) and \( y \) are two permutations of \( abcd \), we say that \( x \leq y \) if and only if \( x \) comes before \( y \) in dictionary order. Each ternary monomial expands into a sum of 36 dialgebra monomials. For the first two association types we have

\[
[[a, b, c], d, e] = \widehat{abcde} - \widehat{abcd} - \widehat{acbd} + \widehat{abde} + \widehat{bcde} + \widehat{cabd} - \widehat{cabe} + \widehat{ebcd} + \widehat{bcaed} - \widehat{bcade} - \widehat{cbed} + \widehat{ebad} + \widehat{bdeca} - \widehat{debca} + \widehat{debc}.
\]

Altogether there are \( 5 \times 5! = 600 \) dialgebra monomials; we order them first by the position of the center and then lexicographically:

\[
\widehat{abce}, \widehat{acde}, \widehat{abdc}, \ldots, \widehat{bced}, \ldots, \widehat{cabd}, \ldots, \widehat{abdc}, \widehat{abcd}, \widehat{abde}, \ldots, \widehat{bacd}, \ldots, \widehat{bcde}, \ldots, \widehat{cabd}, \ldots, \widehat{dabc}, \ldots, \widehat{eabd}, \ldots, \widehat{eabc}, \ldots, \widehat{abcd}, \ldots.
\]

We used a Maple program to compute the expansions of the 90 ternary monomials of lemma 4.1 as linear combinations of the 600 dialgebra monomials. We then created a matrix \( E \), called the expansion matrix, of size \( 600 \times 90 \) and set the \((i, j)\) entry \( E_{ij} \) equal
to the coefficient of the $i$th dialgebra monomial in the expansion of the $j$th ternary monomial. The nullspace of this matrix contains the coefficient vectors of the polynomial identities of degree 5 satisfied by the PATS. We used the Maple package \texttt{LinearAlgebra} to manipulate this matrix: \texttt{Rank} returns the value 50 and so the nullspace has dimension $90 - 50 = 40$; \texttt{ReducedRowEchelonForm} computes the row canonical form which has only 50 nonzero rows and finally \texttt{DeleteRow} removes the 550 zero rows at the bottom of the matrix. The result is a $50 \times 90$ matrix; considering this matrix as the coefficient matrix of a homogeneous linear system in the variables $\{x_1, x_2, \ldots, x_{90}\}$, we must determine a basis for the solution set of the system (that is, the nullspace of the matrix). Our Maple program found that the leading 1s of the rows of this matrix occur in the columns with the following 50 indices:

$$L = \{1-33, 36, 43, 44, 45, 48, 55, 56, 57, 60, 67, 68, 69, 72, 79, 80, 81, 84\}.$$  

Our program took the complement of this subset of the column indices $\{1, 2, \ldots, 90\}$ to obtain the 40 indices corresponding to the free variables in the solution of the homogeneous linear system:

$$N = \{34, 35, 37-42, 46, 47, 49-54, 58, 59, 61-66, 70, 71, 73-78, 82, 83, 85-90\}.$$  

We wrote another Maple procedure to obtain the canonical basis for the nullspace of the matrix using the classical algorithm: for each $j \in N$, we initialize the free variables by setting $x_j = 1$ and $x_k = 0$ for $k \in N \setminus \{j\}$ and then use the linear relations in the 50 rows of the matrix to solve for the leading variables $x_\ell$ with $\ell \in L$. The output of this procedure is a set of 40 coefficient vectors of length 90 which form a basis of the nullspace of the expansion matrix $E$.

Applying these coefficient vectors to the 90 ternary monomials, we obtain the following identities, 20 of each form:

$$\begin{align*}
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] &\equiv 0, \\
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] &\equiv [a^\sigma, [b^\sigma, d^\sigma, c^\sigma], e^\sigma] \\
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] &\equiv [a^\sigma, [d^\sigma, b^\sigma, c^\sigma], e^\sigma] \\
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] &\equiv [a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] \\
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] &\equiv [a^\sigma, [b^\sigma, d^\sigma, c^\sigma], e^\sigma] \\
[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] &\equiv [a^\sigma, [d^\sigma, b^\sigma, c^\sigma], e^\sigma].
\end{align*}$$

Every identity of the first form is equivalent to $Q(a, b, c, d, e) \equiv 0$. We have

$$[a^\sigma, [b^\sigma, c^\sigma, d^\sigma], e^\sigma] \equiv [a^\sigma, [b^\sigma, d^\sigma, c^\sigma], e^\sigma] \equiv [a^\sigma, [d^\sigma, b^\sigma, c^\sigma], e^\sigma] \equiv 0,$$

and so every identity of the second form follows from $P$ and $Q$. \hfill \Box

**Corollary 4.4.** If a ternary algebra satisfies $P(a, b, c) \equiv 0$ and $Q(a, b, c, d, e) \equiv 0$, then every monomial $(a, b, c, d, e)$ is an alternating function of $b, c, d$ and every monomial $(a, b, c, d, e)$ is an alternating function of $c, d, e$.

### 5. Partially alternating ternary algebras

Let $A$ be an (associative) dialgebra with operations $\cdot$ and $\cdot$. On the underlying vector space of $A$ we introduce a new ternary operation $\cdot$ defined by the PATS in $A$:

$$[a, b, c] = a \cdot b \cdot c - a \cdot c \cdot b - b \cdot a \cdot c + b \cdot c \cdot a + c \cdot a \cdot b - c \cdot b \cdot a.$$

We denote the resulting ternary algebra by $A^{(\cdot)}$. Propositions 3.3 and 4.3 allow us to conclude that this new ternary operation satisfies the identities $P(a, b, c) \equiv 0$ and $Q(a, b, c, d, e) \equiv 0$, and so $A^{(\cdot)}$ is a partially alternating ternary algebra (PATA) in the sense of the following definition.

**Definition 5.1.** A completely alternating ternary algebra (CATA) is one with a completely alternating (CA) product: $(a^\sigma, b^\sigma, c^\sigma) = \epsilon(\sigma)(a, b, c)$ for all permutations $\sigma$ where $\epsilon$ is the sign. A PATA is one with a partially alternating (PA) product: the product satisfies $P(a, b, c) \equiv 0$ and $Q(a, b, c, d, e) \equiv 0$. 6
Algebras with a CA product appear naturally in the computation of the duals of quadratic operads, see [33], section 1.12.

Lemma 5.2. Let \((\cdot, \cdot, \cdot)\) be a PA ternary product and let \(s = a_1 \ldots a_n\) be a monomial with some placement of (ternary) parentheses. Then the second and third arguments of any submonomial \((x, y, z)\) of \(s\) are CA.

Proof. We proceed by induction on \(n\). For \(n \leq 3\) the claim is obvious. Set \(n > 3\) and assume that the result holds for any monomial of degree \(< n\). We have the factorization \(s = (t, u, v)\); then the second or third argument of a submonomial of \(s\) is either (i) \(u\) or \(v\), or (ii) the second or third argument of a submonomial of \(t\), \(u\) or \(v\). In case (ii), the claim follows from the inductive hypothesis. In case (i), it suffices to prove the claim for the second argument \(u\), since \(P \equiv 0\) implies \(s = -(t, v, u)\) and so the claim also holds for \(v\). To show the claim for \(u\), it is enough to note that \(P \equiv 0\) (respectively \(Q \equiv 0\)) implies that \(u\) alternates in its second and third arguments (respectively its first and second arguments). Since these two transpositions generate the symmetric group \(S_3\), \(u\) is CA.

Example 5.3. First, consider the monomial \((a, (b, c, d), e)\) in a PATA. To show that this is a CA function of \(b, c, d\), it suffices to show that it alternates in \(b, c\) and in \(c, d\), since these two transpositions generate the symmetric group on \(b, c, d\). If we apply the identity \(P \equiv 0\) to the inner triple, we obtain \((a, (b, c, d), e) = -(a, (b, d, c), e)\), showing that the monomial alternates in \(c, d\). If we apply the identity \(Q \equiv 0\) to the monomial, we obtain \((a, (b, c, d), e) = -(a, (c, b, d), e)\), showing that the monomial alternates in \(b, c\). Second, consider the monomial \((a, b, (c, d, e))\) in a PATA. If we apply the identity \(P \equiv 0\) to the outer triple, we transpose the second argument \(b\) and the third argument \((c, d, e)\) and change the sign, obtaining \((a, (b, (c, d, e))) = -(a, (c, d, e), b)\). If we apply the previous case to \((a, (c, d, e), b)\), we see that it alternates in \(c, d, e\), and hence so does the monomial \((a, b, (c, d, e))\).

A ternary operation has monomials only in odd degrees. We generate the association types for a PA ternary product inductively by degree; we simultaneously generate the CA types and the PA types. For both CA and PA types, the basis of the induction is the trivial type \(a\) in degree \(1\). Suppose that we have already generated ordered lists of the CA and PA types of degree \(\leq n\). To generate the CA and PA association types of degree \(n\), we consider all partitions of \(n = i + j + k\) into three odd parts, and then perform the following algorithm.

(i) If \(i > j > k\), then for all CA types \(t, u, v\) of degrees \(i, j, k\) respectively, we include \((t, u, v)\) as a new CA type in degree \(n\).
(ii) If \(i = j > k\), then for all CA types \(t, u, v\) of degrees \(i, i, k\) where \(t\) precedes \(u\) in the types of degree \(i\), we include \((t, u, v)\) as a CA type in degree \(n\).
(iii) If \(i > j = k\), then for all CA types \(t, u, v\) of degrees \(i, j, j\) where \(u\) precedes \(v\) in the types of degree \(j\), we include \((t, u, v)\) as a CA type in degree \(n\).
(iv) If \(i = j = k\), then for all CA types \(t, u, v\) of degree \(i\) where \(t\) precedes \(u\) and \(u\) precedes \(v\), we include \((t, u, v)\) as a CA type in degree \(n\).
(v) If \(j > k\), then for all PA types \(t\) of degree \(i\) and all CA types \(u, v\) of degrees \(j, k\) respectively, we include \((t, u, v)\) as a new PA type in degree \(n\).
(vi) If \(j = k\), then for all PA types \(t\) of degree \(i\) and all CA types \(u, v\) of degree \(j\) where \(u\) precedes \(v\), we include \((t, u, v)\) as a PA type in degree \(n\).

The Maple program implementing this algorithm appears in figure 1; we use the symbol 0 as a placeholder to indicate the occurrence of a variable. The output of this program is presented in standard mathematical notation in table 3.
\[
\begin{array}{c}
\text{maxdeg} := 9: \ \text{CA} := \text{table}(): \ \text{CA}\{1\} := \{0\}: \ \text{PA} := \text{table}(): \ \text{PA}\{1\} := \{\}\; \text{for n from 3 to maxdeg by 2 do}
\text{CA}\{n\} := \{\}; \ \text{PA}\{n\} := \{\}; \ \\
\text{for i to n−2 by 2 do for j to n−i−1 by 2 do}
\text{k} := n−i−j: \ \\
\text{if i > j and j > k then}
\text{for t in CA}\{i\} do \text{for u in CA}\{j\} do \text{for v in CA}\{k\} do
\text{CA}\{n\} := \{ \text{op(CA}\{n\}), \{t,u,v\}\} od od od fi:
\text{if i = j and j > k then}
\text{for p to nops(CA}\{i\} do for q to nops(CA}\{j\} do
\text{for v in CA}\{k\} do
\text{CA}\{n\} := \{ \text{op(CA}\{n\}), \{CA}\{i\}[p],CA}\{j\}[q],v\}\} od od od fi:
\text{if i > j and j = k then}
\text{for t in CA}\{i\} do
\text{for p to nops(CA}\{i\} do for q to nops(CA}\{i\} do
\text{for v in CA}\{k\} do
\text{CA}\{n\} := \{ \text{op(CA}\{n\}), \{t,CA}\{i\}[p],CA}\{j\}[q],CA}\{k\}[r]\}\} od od od fi:
\text{if j > k then}
\text{for t in PA}\{i\} do \text{for u in CA}\{j\} do \text{for v in CA}\{k\} do
\text{PA}\{n\} := \{ \text{op(PA}\{n\}), \{t,u,v\}\} od od od fi:
\text{if j = k then}
\text{for t in PA}\{i\} do \text{for u in CA}\{j\} do \text{for v in CA}\{k\} do
\text{PA}\{n\} := \{ \text{op(PA}\{n\}), \{t,CA}\{j\}[q],CA}\{k\}[r]\}\} od od od fi:
\end{array}
\]

**Figure 1.** Maple program to generate CA and PA types.

**Table 3.** Association types for ternary products.

| a   | (a, b, c) | ((a, b, c), (d, e), f, g) | (((a, b, c), d, e), f, g), h, i) |
|-----|-----------|--------------------------|----------------------------------|
|     | (a, (b, c, d), e) | ((a, (b, c, d), e), f, g) | (((a, b, c), d, e), f, g), h, i) |
|     | (a, b, c, d, e) | ((a, b, c, d, e), f, g) | (((a, b, c, d), e, f), g), h, i) |
|     | (a, (b, c, d), e, f) | ((a, (b, c, d), e, f), g) | (((a, b, c, d), e, f), g), h, i) |
|     | a         | ((a, b, c, d), (e, f), g) | (((a, b, c, d), (e, f), g), h, i) |

**Completely alternating types**

| a   | (a, b, c) | ((a, b, c), (d, e), f, g) | (((a, b, c), d, e), f, g), h, i) |
|-----|-----------|--------------------------|----------------------------------|
|     | (a, (b, c, d), e) | ((a, (b, c, d), e), f, g) | (((a, b, c, d), e, f), g), h, i) |
|     | (a, b, c, d, e) | ((a, b, c, d, e), f, g) | (((a, b, c, d), e, f), g), h, i) |
|     | (a, (b, c, d), e, f) | ((a, (b, c, d), e, f), g) | (((a, b, c, d), e, f), g), h, i) |
|     | (a, b, c, d, e, f) | ((a, b, c, d, e, f)) | (((a, b, c, d), (e, f), g), h, i) |
|     | (a, (b, c, d), e, f) | ((a, (b, c, d), e, f)) | (((a, b, c, d), (e, f), g), h, i) |
|     | (a, b, c, d, e, f) | ((a, b, c, d, e, f)) | (((a, b, c, d), (e, f), g), h, i) |
|     | a         | ((a, b, c, d), (e, f), g) | (((a, b, c, d), (e, f), g), h, i) |

**Partially alternating types**

| a   | (a, b, c) | ((a, b, c), (d, e), f, g) | (((a, b, c), d, e), f, g), h, i) |
|-----|-----------|--------------------------|----------------------------------|
|     | (a, (b, c, d), e) | ((a, (b, c, d), e), f, g) | (((a, b, c, d), e, f), g), h, i) |
|     | (a, b, c, d, e) | ((a, b, c, d, e), f, g) | (((a, b, c, d), e, f), g), h, i) |
|     | (a, (b, c, d), e, f) | ((a, (b, c, d), e, f), g) | (((a, b, c, d), e, f), g), h, i) |
|     | (a, b, c, d, e, f) | ((a, b, c, d, e, f)) | (((a, b, c, d), (e, f), g), h, i) |
|     | (a, (b, c, d), e, f) | ((a, (b, c, d), e, f)) | (((a, b, c, d), (e, f), g), h, i) |
|     | (a, b, c, d, e, f) | ((a, b, c, d, e, f)) | (((a, b, c, d), (e, f), g), h, i) |
Input:
A completely or partially alternating ternary association type $x$; a Boolean variable $\text{flag}$ which is true for CA and false for PA.

Procedure:
If $\deg(x) > 1$ then write $x = (x_1, x_2, x_3)$:
If $\text{flag} = \text{true}$ then
  countsymmetry$(x_1, \text{true})$;
  countsymmetry$(x_2, \text{true})$;
  countsymmetry$(x_3, \text{true})$;
If all three of $x_1, x_2, x_3$ have the same degree and association type then set $d \leftarrow 6d$.
If only two of $x_1, x_2, x_3$ have the same degree and association type then set $d \leftarrow 2d$.
else
  countsymmetry$(x_1, \text{false})$;
  countsymmetry$(x_2, \text{true})$;
  countsymmetry$(x_3, \text{true})$;
If $x_2, x_3$ have the same degree and association type then set $d \leftarrow 2d$.

Figure 2. Recursive procedure countsymmetry$(x, \text{flag})$.

To enumerate the multilinear monomials in a CA or PA association type in degree $n$, we need to determine the number $d$ of skew-symmetries of the association type; the number of monomials is then $n! / d$. To compute $d$ we use the recursive procedure of figure 2; before calling the procedure we set $d \leftarrow 1$.

6. Degree 7

From now on we usually omit the commas in all ternary monomials.

Lemma 6.1. In a PATA, every multilinear monomial in degree 7 equals one of the following 1960 monomials:

1. \(((a^\sigma b^\sigma c^\sigma )d^\sigma e^\sigma )f^\sigma g^\sigma \) (\(b^\sigma < c^\sigma , d^\sigma < e^\sigma , f^\sigma < g^\sigma\))
2. \(((a^\sigma b^\sigma c^\sigma d^\sigma )e^\sigma f^\sigma g^\sigma \) (\(b^\sigma < c^\sigma < d^\sigma , f^\sigma < g^\sigma\))
3. \(((a^\sigma b^\sigma c^\sigma d^\sigma )e^\sigma f^\sigma g^\sigma \) (\(b^\sigma < c^\sigma , d^\sigma < e^\sigma < f^\sigma\))
4. \((a^\sigma (b^\sigma c^\sigma d^\sigma )e^\sigma f^\sigma g^\sigma \) (\(b^\sigma < c^\sigma < d^\sigma , e^\sigma < f^\sigma\))
5. \((a^\sigma (b^\sigma c^\sigma d^\sigma )e^\sigma f^\sigma g^\sigma \) (\(b^\sigma < c^\sigma < d^\sigma , e^\sigma < f^\sigma < g^\sigma , b^\sigma < e^\sigma\)).

Here $\sigma$ is a permutation of \{a, b, c, d, e, f, g\} and $<$ denotes alphabetical precedence.

Proof. In degree 7 there are 12 association types for a ternary operation, but in a PATA these reduce to 5:

\begin{align*}
  ((abc)de)fg &= \text{type (1)} , \\
  ((ab(cde))f)g &= -(a((cde)b)fg) , \\
  ((abc)(def)g) &= \text{type (3)} , \\
  (a(bcdef)) &= -(a((cdef)b)g) , \\
  (a((def)c))g &= -(a((def)c)g) , \\
  (ab(cdef)) &= -(a((def)c)g) .
\end{align*}
To enumerate the multilinear monomials in each type, we count the skew-symmetries:

\[
((abc)de)fg) \quad \text{alternates in } b, c \text{ and } d, e \text{ and } f, g \quad \frac{7!}{8} = 630
\]

\[
((a(bcd)e)fg) \quad \text{alternates in } b, c, d \text{ and } f, g \quad \frac{7!}{12} = 420
\]

\[
((abc)(def)g) \quad \text{alternates in } b, c \text{ and } d, e, f \quad \frac{7!}{12} = 420
\]

\[
(a((bcd)ef)g) \quad \text{alternates in } b, c, d \text{ and } e, f \quad \frac{7!}{12} = 420
\]

\[
(a(bcd)(efg)) \quad \text{alternates in } b, c, d \text{ and } e, f, g \text{ and } bcd, efg \quad \frac{7!}{72} = 70.
\]

We order these monomials first by association type and then lexicographically. □

**Definition 6.2.** In a PATA we consider the following polynomials of degree 7:

\[
R(a, b, c, d, e, f, g) = \frac{1}{12} \sum_{\sigma \in S_6} \epsilon(\sigma) ((ab^\sigma c^\sigma d^\sigma e^\sigma) f^\sigma g^\sigma)
\]

\[- \frac{1}{12} \sum_{\sigma \in S_6} \epsilon(\sigma) ((ab^\sigma c^\sigma)(d^\sigma e^\sigma f^\sigma) g^\sigma),
\]

\[
S(a, b, c, d, e, f, g) = \frac{1}{4} \sum_{\sigma \in S_5} \epsilon(\sigma) (((ab^\sigma c^\sigma) d^\sigma g) e^\sigma f^\sigma)
\]

\[- \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma) ((ab^\sigma c^\sigma d^\sigma) e^\sigma) f^\sigma g) + \frac{1}{4} \sum_{\sigma \in S_5} \epsilon(\sigma) ((ab^\sigma c^\sigma g) d^\sigma) e^\sigma f^\sigma
\]

\[+ \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma) ((ab^\sigma c^\sigma)(d^\sigma e^\sigma f^\sigma) g) - \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma)((ab^\sigma c^\sigma d^\sigma) e^\sigma) f^\sigma g).
\]

Some of the permutations \(\sigma\) in \(R\) and \(S\) produce monomials which are not ‘straightened’: identities \(P\) and \(Q\) need to be applied to convert such a monomial into \((\pm)\) an equivalent monomial in which the permutation \(\sigma'\) satisfies \(\sigma' < \sigma\) in lex order. This produces repetitions; to cancel the resulting coefficients, we use appropriate fractions. (This ‘straightening’ algorithm is described in detail in subsection 8.2.) After this process, each monomial has coefficient \(\pm 1\), and each polynomial has 120 terms.

**Theorem 6.3.** Every multilinear polynomial identity of degree \(\leq 7\) satisfied by the PATS is a consequence of \(P \equiv 0\), \(Q \equiv 0\), \(R \equiv 0\) and \(S \equiv 0\).

**Proof.** The strategy of the proof is essentially the same as for proposition 4.3, but now the matrix is much larger. By lemma 6.1 we know that there are 1960 distinct multilinear ternary monomials in degree 7; as before, we order them first by association type, and within each type by lexicographical order of the permutation of the variables. There are \(7 \times 7! = 35280\) dialgebra monomials in degree 7; as before, we order them first by the position of the center and then by lexicographical order of the permutation. The expansion matrix \(E\) has 35280 rows and 1960 columns; the \((i, j)\) entry \(E_{ij}\) is the coefficient of the \(i\)th dialgebra monomial in the expansion using the PATS of the \(j\)th ternary monomial. It is not practical to use rational arithmetic with such a large matrix, so we use the Maple package LinearAlgebra[Modular].

The 1960-dimensional vector space spanned by the multilinear ternary monomials is a representation of the symmetric group \(S_7\) which acts by permuting the variables. For any positive integer \(n\), the group algebra \(FS_n\) of the symmetric group \(S_n\) is semisimple over any field \(F\) of characteristic 0 or \(p > n\). It therefore decomposes into the direct sum of full matrix
algebras according to the classical theory of Young tableaux. In particular, $FS_n$ has a basis consisting of elements satisfying the matrix unit relations $e^{(1)}_{rs} e^{(m)}_{uv} = \delta_{tw} \delta_{su} e^{(m)}_{rv}$; see Clifton [14] for a concise summary of this theory. Hence the structure of $FS_n$ is essentially the same over the rational field $\mathbb{Q}$ as over the finite field $\mathbb{F}_p$ with $p$ elements for $p > n$. Since elementary row operations on matrices can be expressed in terms of multiplication by elementary matrices, it follows that we can compute ranks in $\mathbb{Q}S_n$ by computing in $\mathbb{F}_pS_n$. We used $p = 101$; this is the smallest prime greater than 100, and makes it easy to recognize the modular forms of small rational numbers (for example, $\frac{1}{2} \equiv 51$, $\frac{1}{3} \equiv 34$).

We wrote a Maple program to compute the expansions of the 1960 ternary monomials as linear combinations of the 35 280 dialgebra monomials. We used the command `Create` to allocate memory for the expansion matrix $E$ of size 35 280 × 1960, and initialized the entries of this matrix with the coefficients of the expansions (reduced modulo 101). The procedure `RowReduce` computed the row canonical form of $E$ and also returned its rank, 1911. It follows that the nullspace of $E$ has dimension 1960 − 1911 = 49. The nullspace of this reduced matrix consists of the coefficient vectors of the linear combinations of the ternary monomials that collapse to zero when they are expanded into the free dialgebra; that is, the nullspace consists of the multilinear polynomial identities in degree 7 for the PATS which are not consequences of $P \equiv 0$ and $Q \equiv 0$. The procedure `Basis` computes the canonical basis of the nullspace, which was defined in the proof of proposition 4.3. This basis consists of 49 vectors of length 1960 over the field $\mathbb{F}_{101}$; we used Maple to sort these vectors by increasing number of nonzero components (that is, increasing number of terms in the corresponding identities). The corresponding list of polynomial identities in degree 7 satisfied by the PATS can be described briefly as follows.

The first 7 identities: 120 terms; coefficients $\pm 1$; association types 2, 3.
The next 28: 120 terms; coefficients $\pm 1$; association types 1, 2, 3, 4.
The next 7: 120 terms; coefficients $\pm 1$; association types 1, 2, 3, 4, 5.
The last 7: 180 terms; coefficients $\pm 1, \pm 2$; association types 1, 2, 3, 4.

The identities $R$ and $S$ of definition 6.2 correspond respectively to the first identity in group 1 and the first identity in group 3. (In other words, their coefficient vectors are numbers 1 and 36 in the sorted basis of the nullspace.)

We wrote another Maple procedure to apply every permutation of the seven variables to these two identities, and to compute a basis for the subspace spanned by these 2 × 7! = 10 080 vectors in the 1960-dimensional space spanned by the multilinear ternary monomials. Using these results, we verified that identities $R$ and $S$ generate the entire nullspace: every identity in the nullspace is a linear combination of permutations of $R$ and $S$. Moreover, $R$ (respectively, $S$) generates a subspace of dimension 7 (respectively, 42). Hence, the 49-dimensional nullspace is the direct sum of these two subspaces, and so $R$ and $S$ are independent.

\[ \square \]

7. Representation theory of the symmetric group

To study nonlinear identities, and identities of higher degree, we use the representation theory of the symmetric group $S_n$; our main reference is James and Kerber [27].

**Definition 7.1.** A partition of $n$ is a tuple $\lambda = (n_1, \ldots, n_\ell)$ with $n = n_1 + \cdots + n_\ell$ and $n_1 \geq \cdots \geq n_\ell \geq 1$. The frame $[\lambda]$ consists of $n$ boxes in $\ell$ left-justified rows with $n_i$ boxes in row $i$. A tableau is a bijection between $\{1, \ldots, n\}$ and the boxes of $[\lambda]$. In a standard tableau the numbers increase from left to right and from top to bottom.
The irreducible representations of $S_n$ correspond to the partitions of $n$. The dimension $d_\lambda$ associated with $\lambda$ is the number of standard tableaux with frame $[\lambda]$. If $F = \mathbb{Q}$ or $F = \mathbb{F}_p$ for $p > n$, then the group algebra $FS_n$ decomposes into an orthogonal direct sum of two-sided ideals isomorphic to simple matrix algebras:

$$FS_n \approx \bigoplus_\lambda M_{d_\lambda}(F).$$

Given a permutation $\pi$ and a partition $\lambda$, we need to compute the projection of $\pi$ onto $M_{d_\lambda}(F)$: the matrix for $\pi$ in the representation $\lambda$. A simple algorithm for this was found by Clifton [14]. We fix an ordering of the standard tableaux $T_1, \ldots, T_d$ ($d = d_\lambda$) and construct a matrix $R^\lambda_\pi$ as follows: apply $\pi$ to $T_j$, obtaining a (possibly non-standard) tableau $\pi T_j$. If there exist two numbers in the same column of $T_i$ and the same row of $\pi T_j$, then $(R^\lambda_\pi)_{ij} = 0$. Otherwise, $(R^\lambda_\pi)_{ij}$ is the sign of the permutation of $T_i$ which leaves the columns invariant as sets and moves the numbers to the correct rows of $\pi T_j$. The matrix $R^\lambda_\pi$ for the identity permutation may not be the identity matrix but is invertible. An explicit algorithm for computing $R^\lambda_\pi$ is given by Bremner and Peresi [9]; that reference also contains a more detailed discussion of the application of representation theory to polynomial identities.

**Lemma 7.2 ([14]).** The matrix representing $\pi$ in the partition $\lambda$ equals $(R^\lambda_{\pi})^{-1} R^\lambda_\pi$.

Any polynomial identity (not necessarily multilinear or even homogeneous) of degree $\leq n$ over a field $F$ of characteristic 0 or $p > n$ is equivalent to a finite set of multilinear identities; see [40], chapter 1. We consider a multilinear identity $I(x_1, \ldots, x_n)$ of degree $n$ and collect the terms with the same association type: $I = I_1 + \cdots + I_k$. The monomials in each $I_k$ differ only by a permutation of $x_1, \ldots, x_n$; hence, $I_k$ is an element of $FS_n$ and $I$ is an element of $(FS_n)^t$. If $U \subseteq (FS_n)^t$ is the span of the $n!$ identities obtained by applying all permutations in $S_n$ to $I$, then $U$ is a representation of $S_n$, and so $U$ is the direct sum of components corresponding to the irreducible representations of $S_n$. This breaks down a large computational problem into smaller pieces. We fix a partition $\lambda$ with the associated dimension $d = d_\lambda$. To determine the $\lambda$-component of $U$, we construct a $d \times dt$ matrix $M_\lambda$ consisting of $t$ blocks of size $d \times d$; in block $j$ we put the representation matrix for the terms of $I_j$.

**Definition 7.3.** The rank of $M_\lambda$ is the rank of the identity $I$ in the partition $\lambda$.

We modify this procedure to determine the nullspace of the expansion matrix $E$ for the PATS. In degree $n$, let $t = t_n$ be the number of association types for a partially alternating ternary product. Consider the monomial in ternary association type $i$ with the identity permutation of the variables, and let $E^i$ be its expansion using the PATS. We have $E^i = E^i_1 + \cdots + E^i_n$ where $E^i_j$ contains the dialgebra monomials with center in position $j$. We construct a $td \times (n+t)td$ matrix $X_\lambda$ with $t$ rows and $n+t$ columns of $d \times d$ blocks (table 4). On the right side, in block $(i, n+i)$ for $1 \leq i \leq t$, we put $-I_d$ (identity matrix); the other blocks of the right side are zero. On the left side, in block $(i, j)$ for $1 \leq i \leq t$ and $1 \leq j \leq n$, we put $p_{ij}(E^i_j)$, the representation matrix of $E^i_j$. The matrix $X_\lambda$ is the representation matrix for the components in the partition $\lambda$ of the expansions of the ternary association types in degree $n$. We compute the row canonical form of $X_\lambda$, and distinguish the upper (respectively lower) part containing the rows with leading ones in the left (respectively right) side. The rows of the lower right part represent polynomial identities satisfied by the PATS as a result of dependence relations among the dialgebra expansions of the ternary association types.

**Definition 7.4.** The number of (nonzero) rows in the lower right part of the row canonical form of $X_\lambda$ is the rank of identities satisfied by the PATS in the partition $\lambda$. 


Table 4. Representation matrix $X_\lambda$ in the partition $\lambda$.

| $\rho_\lambda(E^1_1)$ | $\rho_\lambda(E^1_2)$ | ... | $\rho_\lambda(E^1_{n-1})$ | $\rho_\lambda(E^1_n)$ | $-I_d$ | $O$ | $O$ | $O$ |
| $\rho_\lambda(E^2_1)$ | $\rho_\lambda(E^2_2)$ | ... | $\rho_\lambda(E^2_{n-1})$ | $\rho_\lambda(E^2_n)$ | $O$ | $-I_d$ | $O$ | $O$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| $\rho_\lambda(E^t_1)$ | $\rho_\lambda(E^t_2)$ | ... | $\rho_\lambda(E^t_{n-1})$ | $\rho_\lambda(E^t_n)$ | $O$ | $O$ | $O$ | $-I_d$ |

8. Degree 7: nonlinear identities

In this section we find nonlinear identities for the PATS in degree 7 which are shorter than the 120-term multilinear identities $R$ and $S$ of definition 6.2.

Definition 8.1. A polynomial identity is nonlinear if it is homogeneous of degree $n$ and there is a partition $(n_1, \ldots, n_\ell)$ of $n$ with some $n_i \geq 2$ (equivalently $\ell < n$) such that the variables in each monomial are a permutation of $a_1, \ldots, a_{n_1}$, $a_2, \ldots, a_{n_2}, \ldots, a_\ell, \ldots, a_{n_\ell}$.

The representation theory of the symmetric group tells us which partitions to use in our search for shorter nonlinear identities. We can apply the techniques discussed in this section to any degree, in particular to degrees 3 and 5; but since the multilinear identities $P$ and $Q$ that we have already found have only two terms each, there is nothing to be gained. The general theory of linearization and delinearization of polynomial identities for nonassociative algebras is discussed in detail in [40], chapter 1. In particular, since we consider ternary structures which do not have identity elements, the processes of linearization and delinearization preserve the degree of the identities.

8.1. Application of representation theory

In section 6, we found the inequivalent multilinear monomials in each ternary association type. That method was based on monomials and used the skew-symmetries implied by $P \equiv 0$ and $Q \equiv 0$ to reduce the number of monomials in each type. In contrast, the representation theory method is based on association types and expresses the skew-symmetries as multilinear polynomial identities.

Lemma 8.2. In a PATA, every skew-symmetry of the five association types in degree 7 is a consequence of the 15 identities in table 5.

Proof. This follows by applying the identities $P \equiv 0$ and $Q \equiv 0$. □

Let $\lambda$ be a partition of $n = 7$ with associated irreducible representation of dimension $d = d_\lambda$. There are 15 skew-symmetry identities and five association types, requiring a $15d \times 5d$ matrix $M_\lambda$. In each skew-symmetry, the first term has the identity permutation with the representation matrix $I_d$, and the second term has a permutation $\pi$ of order 2 with the representation matrix $(R^{\pi}_d)^{-1} R^{\pi}_d$ from lemma 7.2. The $d \times d$ block in position $(i, j)$ contains the sum of these matrices, where $i$ and $j$ are the index number and the association type of the skew-symmetry, respectively. The rank of $M_\lambda$ is ‘symrank’ in table 6. For each $\lambda$, we
construct the representation matrix $X_\lambda$ (table 4) and compute the rank of its lower right part (definition 7.4); this is ‘exprank’ in table 6. Column ‘newrank’ is the difference between ‘symrank’ and ‘exprank’: this is the rank of the new identities in degree 7 for the partition $\lambda$, that is, the identities which are non-trivial consequences of the skew-symmetries of the association types. We check the results by summing, over all representations, the product of ‘newrank’ and ‘dimension’: $1 \times 15 + 1 \times 14 + 3 \times 6 + 2 \times 1 = 49$. This is the dimension of the nullspace of the expansion matrix from section 6; column ‘newrank’ gives the decomposition of the nullspace into irreducible representations.

There are four representations where ‘newrank’ is positive: numbers 11, 13, 14, 15. This suggests that a slight modification of the techniques of section 6 will produce nonlinear identities in these partitions: identities in which the variables in each term are a permutation of $\{a, a, a, b, c, d, e\}$, $\{a, a, b, b, c, d, e\}$ or $\{a, a, b, c, d, e, f\}$. (We omit representation 15 since it corresponds to the multilinear case.)

8.2. Straightening algorithm

We need an algorithm to convert a monomial (multilinear or nonlinear) to its ‘straightened’ form with respect to $P \equiv 0$ and $Q \equiv 0$. We apply $P$ and $Q$ to convert a monomial with a given permutation of the variables into ($\pm$) a monomial with a different permutation.
which lexicographically precedes the original permutation. We use the recursive procedures completestraighten (CS) and partialstraighten (PS); for both the input is a monomial $x$. If the straightened form of $x$ is 0, then both return 0. If $\deg(x) = 1$, then both return $x$. If $\deg(x) > 1$, then we write $x = (x_1, x_2, x_3)$ and proceed as follows.

- CS recursively computes $CS(x_1)$, $CS(x_2)$, $CS(x_3)$. If any of them is 0, then CS returns 0. If two or more are equal, then CS returns 0. Otherwise, CS puts them in the correct order using strictlyprecedes (see below).
- PS recursively computes $PS(x_1)$, $CS(x_2)$, $CS(x_3)$. If any of them is 0, then PS returns 0. If $CS(x_2) = CS(x_3)$, then PS returns 0. Otherwise, PS puts $CS(x_2)$ and $CS(x_3)$ in the correct order using strictlyprecedes.

Procedure strictlyprecedes compares monomials $x$ and $y$. If $\deg(x) \neq \deg(y)$, then it returns true if $\deg(x) < \deg(y)$, and false if $\deg(x) > \deg(y)$. If $\deg(x) = \deg(y)$, then:

- if both $x$ and $y$ have degree 1, it uses the total order on the generators;
- if both have degree > 1, then $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$; it finds the least $i$ with $x_i \neq y_i$, and recursively calls strictlyprecedes($x_i$, $y_i$).

In other words, first compare the degrees, if the degrees are equal then compare the association types, and if the types are equal then compare the permutations.

**Theorem 8.3.** There is one identity for the partition 31111; it has 60 terms:

$$I_{31111}^{(1)} = \frac{1}{4} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a a^e b^e c^e d^e e^e] - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a [a^e b^e c^e] d^e e^e]$$

$$- \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a [a^e b^e] [c^e d^e e^e]] - \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a [b^e c^e d^e] e^e]] = 0.$$

**Proof.** We first generate all 840 permutations of $a$, $a$, $a$, $b$, $c$, $d$, $e$. For each association type, we apply the type to each permutation, find the straightened form of the resulting monomial and retain only those monomials which equal their own straightened forms. We sort the remaining monomials in each type by lex order of the permutation. For the partition 31111, the five association types contain respectively $60 + 34 + 34 + 3 = 165$ monomials.

The expansion matrix $E$ has 165 columns and 7 rows. For $j = 1, \ldots, 165$ we store in column $j$ the PATS expansion of the ternary monomial $j$. The rank is 164, and so the nullspace has dimension 1. Hence, up to a scalar multiple, there is exactly one identity for the PATS with these variables; this identity has 60 terms with coefficients $\pm 1$ in association types 1–4.

**Theorem 8.4.** There are two identities for partition 22111; both have 60 terms:

$$I_{22111}^{(1)} = \frac{1}{4} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a a^e b^e c^e d^e e^e] - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a [a^e b^e c^e] b d^e e^e]$$

$$- \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a [a^e b^e] [c^e d^e e^e]] b - \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma) [[a [a^e b^e c^e] b d^e] e^e]] = 0,$$

and $I_{22111}^{(2)}$ which is obtained by interchanging $a$ and $b$.

**Proof.** Similar to the proof of theorem 8.3.
Theorem 8.5. There are 12 identities for partition 211111; only 5 have 60 terms:

\[ I^{(1)}_{211111} = \frac{1}{3} \sum_{\sigma \in S_5} \epsilon(\sigma) \left[ [ba^\sigma c^\sigma] e^\sigma f^\sigma \right] - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma) \left[ [b[a^\sigma c^\sigma d^\sigma] e^\sigma f^\sigma] \right] \]

\[ - \frac{1}{12} \sum_{\sigma \in S_5} \epsilon(\sigma) \left[ [ba^\sigma c^\sigma] [d^\sigma e^\sigma f^\sigma] a \right] - \frac{1}{6} \sum_{\sigma \in S_5} \epsilon(\sigma) \left[ [a^\sigma c^\sigma d^\sigma] ae^\sigma f^\sigma \right] \equiv 0, \]

and \( I^{(2)}_{211111}, \ldots, I^{(5)}_{211111} \) obtained by interchanging \( b \) with \( c, d, e, f \), respectively.

Proof. Similar to the proof of theorem 8.3. \( \square \)

9. Degree 9

In degree 9 there are 12 association types for a PATA (table 3). We compute the following matrix ranks using the representation theory of the symmetric group:

- symrank: the rank of the skew-symmetry identities of the association types.
- symlifrank: the rank of the skew-symmetries combined with the consequences in degree 9 of the identities \( R \) and \( S \) in degree 7 from definition 6.2, that is,

\[ T \left( (a, c, d, e, f, g) \right), T \left( a, (b, c, d, e, f, g) \right), \ldots, T \left( a, b, c, d, e, f, (g, h) \right), \]

\[ T \left( a, b, c, d, e, f, g, h \right), T \left( (a, b, c, d, e, f, g), i \right), (h, T \left( a, b, c, d, e, f, g \right), i \right), (h, i, T \left( a, b, c, d, e, f, g \right) \right), \]

where \( T = R \) and \( T = S \).
- exprank: the rank of the lower right part of the expansion matrix (definition 7.4).

For every partition ‘symlifrank’ equals ‘exprank’; there are no new identities.

10. Degrees \( \geq 11 \)

Our computational techniques apply in principle to higher degrees, but the size of the matrices involved makes these calculations impractical at present, even using modular arithmetic and the representation theory of the symmetric group.

11. Conclusion

Trilinear operations in an associative algebra have recently been classified by Bremner and Peresi [8]: there are six isolated operations (the alternating, symmetric and cyclic sums, the cyclic commutator, the weakly commutative and anticommutative operations), and four infinite families (the Lie, Jordan and anti-Jordan families, and a fourth family which seems unrelated to Lie and Jordan structures). The Kolesnikov–Pozhidaev algorithm can be applied to all these operations: we choose one of the three arguments and make it the center of each monomial. Unlike the alternating sum, which corresponds to the one-dimensional sign representation of the symmetric group \( S_3 \), most of the other algebra operations will produce essentially different dialgebra operations for each choice of the central argument. The polynomial identities satisfied by these new dialgebra operations will define varieties of ternary algebras with great potential for applications in pure mathematics and theoretical physics.
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