Simple Model for (3+2) Neutrino Oscillations

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Abstract

We formulate a set of naturalness criteria for sterile neutrinos ($\nu'$) to be light, needed for reconciling the LSND neutrino anomaly with the other neutrino data. A light sterile neutrino becomes as natural as the light active neutrinos if it carries quantum numbers of a chiral gauge symmetry broken at the TeV scale. The simplest such theory is shown to be an $SU(2)$ gauge theory with the $\nu'$ transforming as a spin 3/2 multiplet. We develop this model and show that it leads naturally to the phenomenologically viable (3+2) neutrino oscillation scheme. We also present next-to-minimal models for light sterile neutrinos based on a chiral $U(1)$ gauge symmetry.

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1 Introduction

Recent solar [1], atmospheric [2], and reactor [3] neutrino oscillation experiments have significantly improved our knowledge about neutrino masses and mixing angles. In particular, the solar and the atmospheric neutrino data are very well described in a three-neutrino oscillation scenario where the mass squared splittings are respectively \( \Delta m_{\odot}^2 \simeq 7.5 \times 10^{-5} \text{eV}^2 \) and \( \Delta m_{\text{atm}}^2 \simeq 2.0 \times 10^{-3} \text{eV}^2 \) [4]. On the other hand, the \( \nu_{\mu} - \nu_e \) oscillation signal reported by the Liquid Scintillator Neutrino Detector (LSND) experiment at Los Alamos [5], which will soon be tested by the ongoing MiniBooNE [6] experiment at Fermilab, would require a third neutrino mass squared splitting \( \Delta m_{\text{LSND}}^2 \gtrsim 10^{-1} \text{eV}^2 \), which is impossible to implement in a three-neutrino oscillation scheme. Instead, one possibility to accommodate all the neutrino data is to add one or more light sterile neutrinos with masses of the order \( \sim 1 \text{eV} \), which would provide additional mass splittings. Although four-neutrino mass models with a single sterile neutrino [7, 8] are strongly disfavored by present data [9], a combined analysis of the short-baseline experiments Bugey [10], CCFR [11], CDHS [12], CHOOZ [13], KARMEN [14], and LSND shows, that (3+2) neutrino mass schemes with two sterile neutrinos can yield a satisfactory description of current neutrino oscillation data including LSND [15]. Generally, in (3+n) neutrino mass schemes, where \( n \) denotes the number of sterile neutrinos, it seems [15] that the LSND signal still remains compatible with the other data sets even when \( n > 2 \).

While the seesaw mechanism [16] provides a simple understanding of the smallness of active neutrino masses, it does not explain why a sterile neutrino \( \nu' \) would be light. In fact, if the effective low-energy theory is the Standard Model (SM), then there is no reason why \( \nu' \) would not acquire a mass of the order of the Planck scale \( M_{\text{Pl}} \sim 10^{19} \text{GeV} \). Thus, in any \( (3+n) \) neutrino mass model it is important to explain the smallness of the sterile neutrino masses. In this paper, we wish to formulate a set of naturalness criteria for light sterile neutrinos which would be as compelling as the seesaw mechanism for active neutrinos. We suggest and develop the simplest models which satisfy these criteria.

It is useful to recall the main ingredients that make the seesaw mechanism successful. Here, the set of left-handed SM neutrinos \( \nu_e, \nu_\mu, \) and \( \nu_\tau \) is extended by introducing three right-handed neutrinos \( N_1, N_2, \) and \( N_3 \), which are singlets under the SM gauge group \( G_{\text{SM}} = SU(3)_c \times SU(2)_L \times U(1)_Y \). In the basis \( (\nu_e, \nu_\mu, \nu_\tau, N_1, N_2, N_3) \), the resulting \( 6 \times 6 \) neutrino mass matrix then reads

\[
M_\nu = \begin{pmatrix} 0 & m_D^T \\ m_D & M_R \end{pmatrix},
\]

where the entries \( 0, m_D, \) and \( M_R \) are \( 3 \times 3 \) matrices which are characterized by the gauge-structure and the Higgs-content of the theory. It is significant that, in Eq. (1), the entries in the upper-left \( 3 \times 3 \) sector are all vanishing. This is because the SM is a chiral gauge-theory and does not permit a bare mass term for the left-handed neutrinos. In addition, there are no Higgs triplet fields, which could have directly coupled to \( \nu_l \). Furthermore, the matrix elements of \( m_D \) are of the order of the electroweak scale \( \sim 10^2 \text{GeV} \) and protected from becoming too large by electroweak gauge invariance. In contrast to this, the mass matrix \( M_R \) has unprotected entries of the order \( M_{\text{Pl}} \) or of order the \( B - L \) breaking scale \( M_{B-L} \sim 10^{15} \text{GeV} \). As a result, we obtain an effective \( 3 \times 3 \) neutrino mass matrix \( \tilde{M}_{\text{eff}} = -m_D^T M_R^{-1} m_D^T \), which leads to small neutrino masses of the order \( \sim 10^{-2} \text{eV} \).
By analogy with the seesaw mechanism for active neutrinos, we propose the following
criteria for a light sterile neutrino $\nu'$ (with mass of order $1\text{ eV}$) to be natural:

1. $\nu'$ must transform as a chiral representation of a “sterile” gauge symmetry $G'$ which is
   broken at the TeV scale.

2. There must exist no Higgs field which couples directly to $\nu'$.

Note that we require $G'$ to be a gauge symmetry, rather than a global symmetry, since only
gauge symmetries will survive quantum gravity corrections. In our constructions, we will
supplement the above criteria by the requirement of a minimal Higgs sector: a single Higgs
field breaks $G'$ and provides simultaneously sterile neutrino masses, analogous to the SM
Higgs doublet.

To illustrate the basic idea, let us consider the simplified case of one generation with one
active neutrino flavor $\nu$ and one sterile neutrino $\nu'$ [(1+1) model]. Following our criteria,
we extend the SM gauge symmetry to $G_{SM} \times G'$, with the $\nu'$ transforming chirally under
$G'$. All SM particles carry zero $G'$ charges. Next, we introduce two right-handed neutrinos
$N$ and $N'$, which are singlets under the total gauge group $G_{SM} \times G'$. In analogy with
the electroweak symmetry breaking in the SM, we assume that $G'$ is spontaneously broken
around the TeV scale by a suitable Higgs field $\Phi$ which has no direct Yukawa coupling of
the type $\nu' \nu' \Phi$. To keep the situation simple, we furthermore take $\Phi$ to be a singlet under
$G_{SM}$. In the basis $(\nu, \nu', N, N')$, the total $4 \times 4$ neutrino mass matrix takes then the form

$$ M_{\nu} = \begin{pmatrix}
0 & 0 & m_D & m_D'' \\
0 & 0 & m_D'' & m_D' \\
m_D & m_D' & M_R & M_R'' \\
m_D' & m_D'' & M_R & M_R''
\end{pmatrix}. \tag{2} $$

We hence observe, that the general principles which lead to the usual seesaw mechanism,
have also in this case dictated the canonical structure of $M_{\nu}$ in Eq. (1). Particularly, in
Eq. (2), the vanishing of the mass terms in the upper-left $2 \times 2$-block results from the chiral
nature of the $G_{SM} \times G'$ gauge theory and the absence of specific Higgs representations.
Moreover, $m_D$, $m_D'$, $m_D''$, and $m_D''$ are of the order $\sim 10^2\text{ GeV}$, since they are protected
by gauge invariance under $G_{SM}$ and $G'$ up to the TeV scale, where both $G_{SM}$ and $G'$ are
spontaneously broken. The entries $M_R$, $M_R'$, and $M_R''$ on the other hand, are unprotected
by $G_{SM} \times G'$ and thus of the order $\sim M_{B-L}$. At low energies, this will therefore give an
effective $2 \times 2$ neutrino mass matrix, which yields small masses in the (sub-)eV-range for
both the active and the sterile neutrinos. The generalization of this sterile neutrino seesaw
mechanism to a $(3+n)$ mass scheme is straightforward with $m_D$ becoming a $3 \times n$ matrix
and $M_R$ becoming an $n \times n$ matrix in Eq. (1). Notice that in the special case when $G'$ is
identified with a copy of $G_{SM}$, we arrive at the well-known scenario for “mirror” neutrinos
[17]. Alternative ways of realizing light sterile neutrinos have been suggested in Ref. [18].

In this paper, we construct the simplest neutrino mass model consistent with our criteria
for a light $\nu'$. As it turns out, the simplest model yields the phenomenologically viable
scenario of $(3+2)$ neutrino oscillations [15]. Here, we require invariance under the product
group $G_{SM} \times G'$, where $G'$ is a chiral anomaly-free continuous gauge symmetry. This implies,
in particular, that no extra discrete symmetry is imposed. The simplest example of this kind is found to be when $G' = SU(2)$, with the sterile neutrinos $\Psi$ in the spin 3/2 representation. A single spin 3/2 Higgs field $\Phi$ can spontaneously break this symmetry at the TeV scale without supplying large (TeV scale) masses to $\Psi$. In this setup, we calculate the most general neutrino mass matrix $M_\nu$ by explicitly minimizing the scalar potential for $\Phi$. The minimum of the potential preserves a $Z_3$ subgroup of the sterile isospin symmetry. The isospin $\pm 3/2$ components of $\Psi$ are neutral under this $Z_3$, while the $\pm 1/2$ components have charges $\pm 1$. Thus, only $\Psi_{\pm 3/2}$ will mix with the active neutrinos, yielding a (3+2) oscillation scheme. We also present the next simplest examples based on a chiral $U(1)$ gauge theory. Cancellation of chiral anomalies requires the existence of at least five – more naturally six – Weyl spinors, making these examples the second simplest.

2 A simple chiral $SU(2)$ model

The existence of a chiral gauge symmetry $G'$, broken at the TeV scale, plays a crucial rôle in our criteria for realizing naturally a light sterile neutrino. The vanishing of the axial vector anomalies and the mixed gauge-gravitational anomalies [19] sets non-trivial constraints on such a theory. We are naturally led to the choice $G' = SU(2)$, where these anomalies automatically vanish for any representation. Furthermore, $SU(2)$ admits chiral representations, i.e., fermionic representations for which mass terms are forbidden by gauge invariance. Chiral $U(1)$ theories, while also interesting, are not the simplest as they require at least five spin 1/2 Weyl fermions for non-trivial anomaly cancellation. These next-to-minimal models are discussed in the next section.

Consider an $SU(2)$ gauge theory with one fermion field $\Psi$ in the spin $j$ representation. The spin $j$ representation of $SU(2)$ yields for $j = 0, 1, 2, \ldots$ (bosonic case) a unitary and for $j = 1/2, 3/2, 5/2, \ldots$ (fermionic case) a projective unitary representation (with essential cocycle) of $SO(3)$. Although the axial vector anomalies and the mixed gauge-gravitational anomalies are zero for any $j$, the spin 1/2 representation of $SU(2)$ is plagued with a global Witten anomaly [20]. The spin 1 representation will not suit our needs as it is vectorial. The global $SU(2)$ anomaly vanishes, however, when $\Psi$ transforms under the spin 3/2 representation, which has an even quadratic index. In this case, $\Psi$ also cannot have an explicit mass term. Therefore, $SU(2)$ with a single fermion $\Psi$ in the spin 3/2 representation is the simplest anomaly-free chiral gauge theory. $SU(2)$ with spin 3/2 matter fields has been studied in the context of dynamical supersymmetry breaking in Ref. [21]. Non-Abelian chiral gauge theories are necessary ingredients for dynamical supersymmetry breaking and have been analyzed extensively [22].

The gauge symmetry of our model is $G_{SM} \times SU(2)$. We will assume here that the $SU(2)$ symmetry is spontaneously broken by the vacuum expectation value (VEV) of a single Higgs field $\Phi$ at the TeV scale. Like the fermion $\Psi$, we put $\Phi$ into the spin 3/2 representation of the $SU(2)$ symmetry. In component form, one can write the $SU(2)$ spin 3/2 representations $\Psi$ and $\Phi$ as $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ and $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, where $\psi_1, \psi_2, \psi_3$, and $\psi_4$ denote 2-component Weyl spinors and $\phi_1, \phi_2, \phi_3$, and $\phi_4$ are complex-valued scalar fields. Here, we

1Recall that the spin $j$ representation of $SU(2)$ is defined on the space of polynomial functions on $\mathbb{C}^2$
take all SM particles to be singlets under the $SU(2)$ symmetry, while $\Psi$ and $\Phi$, on the other hand, are sterile with respect to $G_{SM}$. In addition, we assume seven right-handed neutrinos $N_\alpha$ ($\alpha = 1, \ldots, 7$), which are total singlets under $G_{SM} \times SU(2)$. For $n$ light $\nu'$ fields we will assume a total of $n + 3$ superheavy fields $N_\alpha$. As a result of the product group-structure and the fermionic charge assignment, this model is automatically free of all anomalies.

The renormalizable Lagrangian relevant for neutrino masses is given by

$$\mathcal{L}_Y = a_{i\alpha} \ell_i H N_\alpha + b_{\alpha} \Psi \Phi^* N_\alpha + c_{\alpha} \Psi \Phi N_\alpha + M_{\alpha\beta} N_\alpha N_\beta + \text{h.c.},$$

(3)

where $H$ is the SM Higgs doublet, $\ell_i (i = e, \mu, \tau)$ denotes the SM lepton doublets, $a_{i\alpha}, b_{\alpha}$, and $c_\alpha$ are Yukawa couplings of order unity and $M_{\alpha\beta} (\alpha, \beta = 1, \ldots, 7)$ are of order $10^{14} - 10^{16}$ GeV. Note that Eq. (3) leads to a mass matrix structure as given in Eq. (2). The effective dimension-five Lagrangian for neutrino masses is obtained after integrating out the $N_\alpha$ fields:

$$\mathcal{L}_{\text{eff}} = \frac{H}{\Lambda} \ell_i \left[ Y_{1i} (\psi_1 \phi_1^* + \psi_2 \phi_2^* + \psi_3 \phi_3^* + \psi_4 \phi_4^*) + Y_{2i} (\psi_1 \phi_4 - \psi_2 \phi_3 + \psi_3 \phi_2 - \psi_4 \phi_1) \right] + \frac{Y_3}{\Lambda} (\psi_1 \phi_1^* + \psi_2 \phi_2^* + \psi_3 \phi_3^* + \psi_4 \phi_4^*)^2 + \frac{Y_4}{\Lambda} (\psi_1 \phi_4 - \psi_2 \phi_3 + \psi_3 \phi_2 - \psi_4 \phi_1)^2 + \frac{Y_5}{\Lambda} (\psi_1 \phi_4 - \psi_2 \phi_3 + \psi_3 \phi_2 - \psi_4 \phi_1) (\psi_1 \phi_1^* + \psi_2 \phi_2^* + \psi_3 \phi_3^* + \psi_4 \phi_4^*) + \frac{Y_{ij}}{\Lambda} H^2 \ell_i \ell_j + \text{h.c.},$$

(4)

where $Y_{ij}, Y_{1i}, Y_{2i}, Y_3, Y_4,$ and $Y_5 (i, j = e, \mu, \tau)$ are dimensionless couplings related to $a_{i\alpha}$ and $b_{\alpha}$ and $\Lambda \sim M_{ij}$. Here, the couplings $Y_{1i}$ and $Y_{2i}$, for example, arise respectively from the terms $\sim a_{i\alpha} b_{\alpha}$ and $\sim a_{i\alpha} c_\alpha$ in Eq. (3). The most general dimension-five neutrino mass operators which arise by integrating out arbitrary fermion representations (i.e., by integrating out $SU(2)$ spin $j = 1, 2, 3$ fermions in addition to the $j = 0$ states $N_\alpha$) are given in Appendix A. These mass terms however, will not alter our general results here.

Following Appendix B where the most general scalar potential for $\Phi$ has been minimized, we can assume a VEV of the form $\langle \Phi \rangle = (v_1, 0, 0, v_4)$, with $v_1$ and $v_4$ as given in Eqs. (17) and $v_1, v_4 \sim 10^2$ GeV. Since $\langle \Phi \rangle$ breaks $SU(2)$ completely, the component-fields of $\Psi$ will finally appear as four sterile neutrinos $(\nu'_1, \nu'_2, \nu'_3, \nu'_4) \equiv (\psi_1, \psi_4, \psi_2, \psi_3)$ in the low-energy theory (note in the definition the permutation of indices).

Integrating out the right-handed neutrinos $N_\alpha$, the sterile neutrino seesaw mechanism leads to five light neutrinos with finite masses in the (sub-) eV-range and two massless neutrinos. The massless states are $\nu'_3$ and $\nu'_4$ which decouple from $\nu_e, \nu_\mu, \nu_\tau, \nu'_1$, and $\nu'_2$ (this is actually independent of the total number of right handed neutrinos $N_\alpha$). The vacuum respects an unbroken $Z_3$ symmetry, which is a subgroup of $I_3$, under which $\nu'_3$ and $\nu'_4$ have charges $\pm 1$ while the other fermionic fields are all neutral. This $Z_3$ symmetry forbids the mixing of $\nu'_3$ and $\nu'_4$ with the other neutrinos. These states will acquire (sub-) eV masses once the effective Lagrangian $\mathcal{L}'_{\text{eff}}$ in Eq. (14) is taken into account. The resulting non-vanishing $5 \times 5$ effective neutrino mass matrix can be written in the basis $(\nu_e, \nu_\mu, \nu_\tau, \nu'_1, \nu'_2)$ as

$$M_{\text{eff}} = \begin{pmatrix} M_{\nu} & M'_{\nu} \\ M'_{\nu} & M''_{\nu} \end{pmatrix},$$

(5)

that are homogeneous of degree $2j$, which is a complex representation space.
where $\mathcal{M}_\nu$ is an arbitrary $3 \times 3$ matrix with entries of the order $\sim 10^{-2}$ eV, while $\mathcal{M}_\nu'$ is given by the $3 \times 2$ matrix

$$\mathcal{M}_\nu' = \frac{\langle H \rangle}{\Lambda} \begin{pmatrix} Y_{1\alpha}v_1^* + Y_{2\alpha}v_4 & Y_{1\alpha}v_4^* - Y_{2\alpha}v_1 \\ Y_{1\mu}v_1^* + Y_{2\mu}v_4 & Y_{1\mu}v_4^* - Y_{2\mu}v_1 \\ Y_{1\tau}v_1^* + Y_{2\tau}v_4 & Y_{1\tau}v_4^* - Y_{2\tau}v_1 \end{pmatrix}, \quad (6)$$

and the $2 \times 2$ matrix $\mathcal{M}_\nu''$ reads

$$\mathcal{M}_\nu'' = \frac{Y_3}{\Lambda} \begin{pmatrix} v_1^2 & v_1^*v_4^* \\ v_4^* & v_4^2 \end{pmatrix} - \frac{Y_4}{\Lambda} \begin{pmatrix} v_1^2 & v_1^*v_4^* \\ v_4^* & v_4^2 \end{pmatrix} + \frac{Y_5}{\Lambda} \left( \frac{1}{2}(v_4^2 - |v_1|^2) - |v_1|^2 \right). \quad (7)$$

It is therefore seen that the effective interactions in Eq. (4) which generate the matrix $\mathcal{M}_\nu'$ introduce a non-zero mixing of $\nu_1'$ and $\nu_2'$ with the active neutrinos. Although the inclusion of the effective operators $\mathcal{L}'_{\text{eff}}$ in Eq. (14) lifts the zero neutrino masses to small values of the order $\Lambda^{-1}v_1v_2 \sim 1$ eV, the fields $\nu_1'$ and $\nu_2'$ will still remain decoupled from the rest of the neutrinos, owing to the unbroken $Z_3$ symmetry. In total, the model therefore gives in any case a (3+2) neutrino mass scheme for sterile neutrino oscillations.

### 3 Simple chiral $U(1)$ models

In Sec. 2 we have analyzed a simple gauge extension of $G_{SM}$ to $G_{SM} \times SU(2)$. It is instructive to compare this model with a similar setup, where $SU(2)$ is replaced by a sterile $U(1)$ gauge symmetry to give the total gauge group $G_{SM} \times U(1)$. Let us therefore consider now $N$ Weyl spinors $\Psi_{n_i} (i = 1, \ldots, N)$, where $\Psi_{n_i}$ carries the charge $n_i$ under the $U(1)$ gauge group. In this model, the anomaly cancellation conditions read $\sum_{i=1}^N n_i = 0$ (mixed gauge-gravitational anomaly) and $\sum_{i=1}^N n_i^3 = 0$ (cubic gauge anomaly). It is easy to see, that for $N \leq 4$ these conditions can only be fulfilled if the theory is vector-like, i.e., the $U(1)$ model must contain at least five fermions to be chiral. Motivated by charge quantization, we shall require all charges $n_i$ to be rational numbers, in which case they can be taken to be integers. Before discussing the case of $N = 5$ fermions, let us first consider simple chiral $U(1)$ models with $N = 6$. For this case, we find the following anomaly-free charge assignments:

- Model (a) : $2 \times \{5\} + 1 \times \{-3\} + 1 \times \{-2\} + 1 \times \{1\} + 1 \times \{-6\}$, \quad (8a)
- Model (b) : $2 \times \{4\} + 3 \times \{-1\} + 1 \times \{-5\}$. \quad (8b)

Here, Model (a), e.g., has two Weyl fermions with $U(1)$ charge 5 and one state each with charge $-3$, $-2$, 1, and $-6$. For Model (a), we minimally extend the Higgs sector by adding a single scalar singlet field $\Phi$ with $U(1)$ charge $-5$. From the charge assignment in Eq. (8a) we then obtain the effective interaction Lagrangian for the neutrinos

$$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda} \ell_i H \Psi_5^0 \Phi + \frac{1}{\Lambda} \ell_j H H + \frac{1}{\Lambda} \Psi_5^0 \Psi_5^0 \Phi \Phi + \Psi_{-3}^0 \Phi^* + \Psi_1^0 \Phi^* + \text{h.c.}, \quad (9)$$

where $i = e, \mu, \tau$ and $\alpha, \beta = 1, 2$ and the Yukawa couplings have not been explicitly displayed.

Similar to the $SU(2)$ model in Sec. 2 we suppose that $\Phi$ acquires its VEV at the TeV scale.
Hence, $\Psi_1, \Psi_{-2}, \Psi_{-3}$, and $\Psi_{-6}$ will decouple below the TeV scale and we are left at low energies with a (3+2) model which is similar to the $SU(2)$ model.

For Model (b), a minimal extension of the Higgs sector by a scalar $\Phi$ with charge $-4$ leads to the effective neutrino mass Lagrangian

$$L_{\text{eff}} = \frac{1}{\Lambda} \bar{\ell}_i H \Psi_\alpha^\alpha \Phi + \frac{1}{\Lambda} \bar{\ell}_i \ell_j HH + \frac{1}{\Lambda} \bar{\Psi}_\alpha^\alpha \Psi_\beta^\beta \Phi \Phi + \text{h.c.},$$

where $\alpha, \beta = 1, 2$. This gives essentially a (3+2) model with four additional extremely light neutrinos (the fields with charges $-1$ and $-5$) which decouple from the active neutrinos. When $\Phi$, instead, carries the charge $+1$ we have the effective Lagrangian

$$L_{\text{eff}} = \frac{1}{\Lambda} \bar{\ell}_i H \Psi_\alpha^{-1} \Phi + \frac{1}{\Lambda} \bar{\ell}_i \ell_j HH + \frac{1}{\Lambda} \bar{\Psi}_\alpha^{-1} \Psi_\gamma^{-1} \Phi \Phi + \Psi_\gamma \Psi_{-5} \Phi + \text{h.c.},$$

where $\alpha, \beta = 1, 2, 3$ and $\gamma = 1, 2$. These operators give rise to a (3+3) scheme with one additional massless neutrino (a linear combination of $\Psi_4^\gamma$) and two heavy neutrinos ($\Psi_{-5}$ and one linear combination of $\Psi_4^\gamma$) which all decouple.

Let us now consider the case of $N = 5$ fermions. In Diophantine analysis it has been shown that every integer $n \neq \pm 4 \pmod{9}$ can be expressed as a sum of the cubes of four integers. The integers $n = \pm 8 \pmod{18}$, for example, can be written as

$$(k - 5)^3 + (-k + 14)^3 + (3k - 30)^3 + (-3k + 29)^3 = 18k + 8 \quad (k \in \mathbb{Z}).$$

Choosing in Eq. (12) the value $k = 28$, we arrive at the integer solution $(n_1, n_2, n_3, n_4, n_5) \equiv (23, -14, 54, -55, -8)$ of the cubic anomaly cancellation condition. Note that none of the charges is vector-like. Simultaneously, this solution also gives a zero mixed gauge-gravitational anomaly. As a result, the simplest anomaly-free chiral $U(1)$ theory with only rational charges is given by

$$\text{Model (c): } 1 \times \{23\} + 1 \times \{-14\} + 1 \times \{54\} + 1 \times \{-55\} + 1 \times \{-8\}.$$  (13)

In comparison with the $N = 6$ models (a) and (b) in Eqs. (8), however, the charges in Eq. (13) involve rather large numbers, which makes this model less attractive.

## 4 Discussion

There are several experimental signatures of our models for naturally light sterile neutrinos. Generally speaking, the most striking consequences will be in the neutrino sector with very little effect elsewhere.

First, a confirmation of the LSND neutrino anomaly by MiniBooNE will clearly give credence to this class of models. Second, since a (3+2) neutrino mass scheme requires $U_{e5} \simeq 0.07$ \cite{15}, the model can be tested in the future by $\bar{\nu}_e$ (or $\nu_e$) disappearance experiments. Moreover, with a fifth neutrino mass eigenvalue $m_5$ in the range $m_5 \sim 4 - 6$ eV, the effective

This is a subject which is mainly concerned with the discussion of the rational or integer solutions of a polynomial equation $f(n_1, n_2, \ldots, n_N) = 0$ with integer coefficients.
Majorana mass in neutrinoless double $\beta$-decay $|\langle m \rangle|$, receives a contribution of the order $\sim 0.02$ eV, which has a good chance to be tested in next generation neutrinoless double $\beta$-decay experiments like GENIUS, EXO, MAJORANA, and MOON, which will have a sensitivity for $|\langle m \rangle| \sim 0.01$ eV.

Due to the non-zero mixing of $H$ and $\Phi$, the SM Higgs will have invisible decay modes such as $H \rightarrow \Phi \Phi$ and $H \rightarrow W'W'$, if these decays are kinematically allowed. This can be tested at LHC or a future linear collider.

Clearly, the requirement $N_\nu < 4$ on the total number of neutrino species $N_\nu$ from $^4$He abundance in standard big bang nucleosynthesis (BBN) is violated, since in all our schemes $\nu'$ will thermalize. However, there are suggestions that a primordial lepton asymmetry will weaken this bound. Similarly, the neutrino mass limit $\sum m_\nu < 0.7 - 1.0$ eV (@95\% C.L.) from recent cosmological data may also be avoided for a suitable primordial $\nu_e$ chemical potential. Our viewpoint here is, that if the (3+2) neutrino oscillation scheme is indeed confirmed by MiniBooNE, one will have to revise the standard BBN paradigm.

Finally, it has been suggested that a sterile neutrino in the 1–20 keV range with very small mixing ($\sin^2 \theta \sim 10^{-11} - 10^{-7}$ for $\nu' - \nu_e$ mixing) with the active neutrinos can serve as a possible dark matter candidate and may be responsible for the observed pulsar velocities exceeding $\sim 500$ km/sec. Our models are readily adaptable to such a scenario.

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**A Effective mass operators**

Apart from the mass terms in Eq. (4), there exists in general a second type of effective dimension-five neutrino mass operators, which arise by integrating out arbitrary fermionic $SU(2)$ representations. The most general Lagrangian of these interactions reads

$$\mathcal{L}'_{\text{eff}} = \frac{Y_6}{\Lambda} \left[ 2 (\psi_1 \psi_4 - \frac{1}{3} \psi_2 \psi_3) (\phi_1 \phi_4 - \frac{1}{3} \phi_2 \phi_3) - \frac{4}{3} \left( \frac{1}{\sqrt{3}} \phi^2 - \psi_1 \psi_3 \right) \left( \frac{1}{\sqrt{3}} \phi^2 - \phi_2 \phi_4 \right) \right]$$

$$- \frac{4}{3} \left( \frac{1}{\sqrt{3}} \phi^2 - \psi_2 \psi_4 \right) \left( \frac{1}{\sqrt{3}} \phi^2 - \phi_1 \phi_3 \right) + \frac{Y_7}{\Lambda} \left[ \frac{4}{3} \left( \frac{1}{\sqrt{3}} \phi^2 - \psi_1 \psi_3 \right) \left( \frac{1}{\sqrt{3}} \phi^2 - \phi_1 \phi_3 \right) \right]$$

$$+ 2 (\psi_1 \psi_4 - \frac{1}{3} \psi_2 \psi_3) (\phi^*_1 \phi^*_4 - \frac{1}{3} \phi^*_2 \phi^*_3) + \frac{4}{3} \left( \frac{1}{\sqrt{3}} \phi^2 - \psi_2 \psi_4 \right) \left( \frac{1}{\sqrt{3}} \phi^2 - \phi_2 \phi'_4 \right) \right]$$

$$+ \frac{Y_8}{\Lambda} \left[ \frac{2}{3} \left( \frac{1}{\sqrt{3}} \phi^2 - \psi_1 \psi_3 \right) \left( \phi^*_1 \phi^*_2 + \frac{2}{3} \phi^*_2 \phi^*_3 + \phi^*_3 \phi^*_4 \right) + (|\phi_1|^2 + \frac{1}{3} |\phi_2|^2 - \frac{1}{3} |\phi_3|^2 - |\phi_4|^2) \right]$$

$$\times \left( \psi_1 \psi_4 - \frac{1}{3} \psi_2 \psi_3 \right) = \frac{2}{3} \left( \frac{1}{\sqrt{3}} \phi^2 - \psi_3 \psi_4 \right) \left( \phi^*_1 \phi^*_2 + \frac{2}{3} \phi^*_2 \phi^*_3 + \phi^*_3 \phi^*_4 \right) \right] + \text{h.c.}, \quad (14)$$

where $Y_6, Y_7,$ and $Y_8$ denote Yukawa couplings of order unity. The most general effective neutrino mass operators are thus given by the sum $\mathcal{L}_{\text{eff}} + \mathcal{L}'_{\text{eff}}$. The gauge singlets in Eq. (14) can be determined from a Clebsh-Gordan table or by representing $\Psi$ as a totally symmetric
tensor \( \psi_{ijk} \), where \( i, j, k = 1, 2 \) and the (normalized) components are defined as \( \psi_{111} = \psi_1, \psi_{112} = \psi_{121} = \psi_{211} = \frac{1}{\sqrt{3}} \psi_2, \psi_{122} = \psi_{221} = \frac{1}{\sqrt{3}} \psi_3, \) and \( \psi_{222} = \psi_4 \) (correspondingly for \( \Phi \)). In this notation, the coupling \( \sim Y_7 \), \( \text{e.g.} \), can be obtained from the term \( \psi_{ijk} \phi^{ab} \phi^{cjk} \) (summation of indices understood).

## B Properties of the scalar potential

The most general renormalizable scalar potential of a \( SU(2) \) spin 3/2 Higgs representation \( \Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \) is given by

\[
V = -\mu^2 (|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2) + \lambda_1 (|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2)^2 + \lambda_2 \left( \left| \sqrt{\frac{2}{3}} \phi_1 \phi_3 - \frac{\sqrt{2}}{\sqrt{3}} \phi_2 \phi_3 \right|^2 + |\phi_1 \phi_4 - \frac{1}{\sqrt{3}} \phi_2 \phi_3|^2 + \left| \sqrt{\frac{2}{3}} \phi_2 \phi_4 - \frac{\sqrt{2}}{\sqrt{3}} \phi_3 \right|^2 \right) + \lambda_3 \left[ 2 \left( \sqrt{\frac{2}{3}} \phi_1 \phi_3 - \frac{\sqrt{2}}{\sqrt{3}} \phi_2 \phi_3 \right) \left( \sqrt{\frac{2}{3}} \phi_2 \phi_4 - \frac{\sqrt{2}}{\sqrt{3}} \phi_3 \phi_4 \right) - (\phi_1 \phi_4 - \frac{1}{\sqrt{3}} \phi_2 \phi_3)^2 \right] + \lambda_4 \left[ \phi_1^* \left( \phi_2^2 \phi_4 + \frac{2}{\sqrt{3}} \phi_2 \phi_3^3 - \phi_1 \phi_2 \phi_3 \right) + \phi_2^* \left( \phi_1 \phi_2 \phi_4 - \frac{2}{\sqrt{3}} \phi_1 \phi_3^2 + \frac{1}{\sqrt{3}} \phi_2 \phi_3 \right) + \phi_3^* \left( -\phi_1 \phi_3 \phi_4 - \frac{1}{\sqrt{3}} \phi_2 \phi_3^2 + \frac{2}{\sqrt{3}} \phi_2 \phi_4 \right) + \phi_4^* \left( \phi_2 \phi_3 \phi_4 - \frac{2}{\sqrt{3}} \phi_1 \phi_3^2 - \phi_1 \phi_4^2 \right) \right] + \text{h.c.},
\]
where the coefficients \( \mu, \lambda_1, \lambda_2, \) and \( \lambda_3 \) are real-valued\(^3\) and \( \lambda_4 = |\lambda_4| \cdot \exp(i \beta) \) with some arbitrary phase \( \beta \). Notice that the potential \( V \) possesses the following \( U(1) \) symmetry which is part of the \( SU(2) \) symmetry and allows to set one phase of the fields \( \phi_i \) always to zero:

\[
U(1) : \phi_1 \rightarrow e^{+i \varphi} \phi_1 , \quad \phi_2 \rightarrow e^{+i \varphi/3} \phi_2 , \quad \phi_3 \rightarrow e^{-i \varphi/3} \phi_3 , \quad \phi_4 \rightarrow e^{-i \varphi} \phi_4 .
\]

The potential \( V \) has a local extremum of the form \( \langle \Phi \rangle = (v_1, 0, 0, v_4) \), where the complex entries \( v_1 \) and \( v_4 \) have a relative phase \( \alpha \), \( \text{i.e.} \), it is \( v_1 v_2 = |v_1 v_2| \cdot \exp(i \alpha) \). For simplicity, we may consider the limit \( |\lambda_4| \ll 1 \), in which case these quantities can be expressed to leading order as

\[
|v_1|^2 \simeq \frac{\mu^2}{4 \lambda_1 + \lambda_2 - 2 |\lambda_3|} \left( 1 \pm \frac{2 |\lambda_4|}{\lambda_2 - 2 |\lambda_3|} \cos(\beta) \right) ,
\]

\[
\frac{|v_4|}{|v_1|} \simeq 1 \pm \frac{2 |\lambda_4|}{\lambda_2 - 2 |\lambda_3|} \cos(\beta) ,
\]

\[
\alpha \simeq \frac{|\lambda_4|^2 \sin(2 \beta)}{\lambda_3 (\lambda_2 - 2 |\lambda_3|)} .
\]

Notice in Eq. (15) that each interaction involves either zero, two, or four of the fields \( \phi_2 \) and/or \( \phi_3 \). In the minimum \( (v_1, 0, 0, v_4) \), the mixing of \( \phi_2 \) and \( \phi_3 \) with \( \phi_1 \) and \( \phi_4 \) is hence zero. As a consequence, the mass matrix of \( \phi_2 \) and \( \phi_3 \) has one pair of zero eigenvalues which correspond to two (would-be) Nambu-Goldstone bosons and two degenerate non-zero mass-squared eigenvalues of the form

\[
m^2_{H^\pm} = \frac{2}{3} \left( |v_1|^2 + |v_4|^2 \right) \left( \lambda_2 + 6 \frac{|\lambda_4 v_1 v_4| \cos(\alpha)}{|v_4|^2 - |v_1|^2} \right) .
\]

\(^3\)The phase of \( \lambda_3 \) can always be removed by an appropriate phase-redefinition \( \Phi \rightarrow e^{i \varphi} \Phi \).
To calculate the remaining scalar masses, we consider the fluctuations \( \phi_1 = v_1 + \tilde{\phi}_1 \), and \( \phi_4 = v_4 + \tilde{\phi}_4 \) about the minimum \((v_1, 0, 0, v_2)\). The corresponding mass eigenstates \( G, A, H_1, \) and \( H_2 \) can be expressed as

\[
G = \frac{\sqrt{2} \text{Im}(v_1^* \tilde{\phi}_1 - v_4^* \tilde{\phi}_4)}{\sqrt{|v_1|^2 + |v_2|^2}}, \quad A = \frac{\sqrt{2} \text{Im}(v_4^* \tilde{\phi}_1 + v_1^* \tilde{\phi}_4)}{\sqrt{|v_1|^2 + |v_2|^2}}, \tag{19a}
\]

\[
H_1 = \frac{\sqrt{2} \text{Re}(v_1^* \tilde{\phi}_1 - v_4^* \tilde{\phi}_4)}{\sqrt{|v_1|^2 + |v_2|^2}}, \quad H_2 = \frac{\sqrt{2} \text{Re}(v_4^* \tilde{\phi}_1 + v_1^* \tilde{\phi}_4)}{\sqrt{|v_1|^2 + |v_2|^2}}. \tag{19b}
\]

The scalar \( G \) is a massless (would-be) Nambu-Goldstone boson which has zero mixing with the other fields. In the limit \( |\lambda_4| \ll 1 \), the \( 3 \times 3 \) mixing matrix of the fields \( A, H_1, \) and \( H_2 \) has the mass-squared eigenvalues

\[
m_{1,2}^2 \simeq (2\lambda_1 + \lambda_2)(|v_1|^2 + |v_2|^2)
\]

\[
\pm \sqrt{(2\lambda_1 + \lambda_2)^2(|v_1|^2 + |v_4|^2)^2 - 8\lambda_1 \lambda_2(|v_1|^2 - |v_4|^2)}, \tag{20a}
\]

\[
m_3^2 \simeq + \frac{|\lambda_4|}{|v_1| |v_4|}(|v_1|^2 + |v_2|^2)(|v_1|^2 - |v_4|^2) \cos(\beta). \tag{20b}
\]

In total we see, that for a range of parameters the extremum described in Eqs. \((14)\) will be a local minimum. In this minimum, the \( SU(2) \) gauge symmetry is completely broken, thereby leaving three (would-be) Nambu Golstone bosons, which must be eaten by the gauge bosons via the Higgs mechanism. The kinetic term of \( \Phi \) is obtained from the covariant derivative

\[
D_\mu \phi_{ijk} = \partial_\mu \phi_{ijk} - i \frac{g'_2}{2} \left[ (W'_\mu)^a_{il} \phi_{ajk} + (W'_\mu)^a_{jl} \phi_{iak} + (W'_\mu)^a_{lk} \phi_{ija} \right],
\]

where \( g'_2 \) is the gauge coupling and \((W'_\mu)^l_i \ (i, l = 1, 2)\) are the \( SU(2) \) gauge bosons. In the minimum \( \langle \Phi \rangle = (|v_1|, 0, 0, |v_4| \cdot e^{i\alpha}) \), the gauge boson masses are

\[
m_{W_3}^2 = \frac{9}{2} g'_2^2 (|v_1|^2 + |v_4|^2) = 3m_{W_3}^2. \tag{21}
\]

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