The existence of perfect codes in a family of generalized Fibonacci cubes

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Abstract

The Fibonacci cube of dimension $n$, denoted as $\Gamma_n$, is the subgraph of the $n$-cube $Q_n$ induced by vertices with no consecutive 1’s. In an article of 2016 Ashrafi and his co-authors proved the non-existence of perfect codes in $\Gamma_n$ for $n \geq 4$. As an open problem the authors suggest to consider the existence of perfect codes in generalization of Fibonacci cubes. The most direct generalization is the family $\Gamma_n(s)$ of subgraphs induced by strings without 1’s as a substring where $s \geq 2$ is a given integer. We prove the existence of a perfect code in $\Gamma_n(s)$ for $n = 2^p - 1$ and $s \geq 3.2^{p-2}$ for any integer $p \geq 2$.

Keywords: Error correcting codes, perfect code, Fibonacci cube.

AMS Subj. Class.: 94B50,0C69

1 Introduction and notations

Let $G$ be a connected graph. The open neighbourhood of a vertex $u$ is $N(u)$ the set of vertices adjacent to $u$. The closed neighbourhood of $u$ is $N[u] = N(u) \cup \{u\}$. The distance between two vertices noted $d_G(x, y)$, or $d(x, y)$ when the graph is unambiguous, is the length of the shortest path between $x$ and $y$. We have thus $N[u] = \{v \in V(G); d(u, v) \leq 1\}$.

A dominating set $D$ of $G$ is a set of vertices such that every vertex of $G$ belongs to the closed neighbourhood of at least one vertex of $D$. In [2], Biggs initiated the study of perfect codes in graphs a generalization of classical 1-error perfect correcting codes. A code $C$ in $G$ is a set of vertices $C$ such that for all pair of distinct vertices $c, c’$ of $C$ we have $N[c] \cap N[c’] = \emptyset$ or equivalently such that $d_G(c, c’) \geq 3$.

A perfect code of a graph $G$ is both a dominating set and a code. It is thus a set of vertices $C$ such that every vertex of $G$ belongs to the closed neighbourhood of exactly one vertex of $C$. A perfect code is some time called an efficient dominating set. The existence or non-existence of perfect codes have been considered for many graphs. See the introduction of [1] for some references.

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The vertex set of the $n$-cube $Q_n$ is the set $\mathbb{B}_n$ of binary strings of length $n$, two vertices being adjacent if they differ in precisely one position. Classical 1-error correcting codes and perfect codes are codes and perfect codes in the graph $Q_n$. The concatenation of strings $x$ and $y$ is noted $x||y$ or just $xy$ when there is no ambiguity. A string $f$ is a substring of a string $s$ if there exist strings $x$ and $y$, may be empty, such that $s = xfy$.

A Fibonacci string of length $n$ is a binary string $b = b_1 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words a Fibonacci string is a binary string without 11 as substring. The Fibonacci cube $\Gamma_n$ ($n \geq 1$) is the subgraph of $Q_n$ induced by the Fibonacci strings of length $n$. Fibonacci cubes were introduced as a model for interconnection networks \cite{3} and received a lot of attention afterwards. These graphs also found an application in theoretical chemistry. See the survey \cite{4} for more results and applications about Fibonacci cubes.

The sets $\{00\}$ and $\{010, 101\}$ are perfect codes in respectively $\Gamma_2$ and $\Gamma_3$. In a recent paper \cite{11} Ashrafi and his co-authors proved the non-existence of perfect codes in $\Gamma_n$ for $n \geq 4$. As an open problem the authors suggest to consider the existence of perfect codes in generalization of Fibonacci cubes. The most complete generalization proposed in \cite{5} is, for a given string $f$, to consider $\Gamma_n(f)$ the subgraph of $Q_n$ induced by strings that do not contain $f$ as substring. Since Fibonacci cubes are $\Gamma_n(11)$ the most immediate generalization \cite{6,7} is to consider $\Gamma_n(1^s)$ for a given integer $s$. We will prove the existence of perfect codes in $\Gamma_n(1^s)$ for an infinite family of parameters $(n, s)$.

It will be convenient to consider the binary strings of length $n$ as vectors of $\mathbb{F}^n$ the vector space of dimension $n$ over the field $F = \mathbb{Z}_2$ thus to associate to a string $x_1x_2 \ldots x_n$ the vector $\theta(x_1x_2 \ldots x_n) = (x_1, x_2, \ldots, x_n)$. The Hamming distance between two vectors $x, y \in \mathbb{F}^n$, $d(x, y)$ is the number of coordinates in which they differ. The parity function is the function from $\mathbb{F}^n$ to $\mathbb{Z}_2$ defined by $\pi(x) = \pi(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$. By the correspondence $\theta$ we can define the sum $x + y$, the Hamming distance $d(x, y)$ and the parity $\pi(x)$ of strings in $\mathbb{B}_n$. Note that Hamming distance is the usual graph distance in $Q_n$. The complement of a string $x \in \mathbb{B}_n$ is the string $\overline{x} = x + 1^n$.

We will first recall some basic results about perfect codes in $Q_n$. Since $Q_n$ is a regular graph of degree $n$ the existence of a perfect code of cardinality $|C|$ implies $|C|(n + 1) = 2^n$ thus a necessary condition of existence is that $n + 1$ is a power of 2 thus that $n = 2^p - 1$ for some integer $p$.

For any integer $p$ Hamming \cite{8} constructed, a linear subspace of $\mathbb{F}^{2^p - 1}$ which is a perfect code. It is easy to prove that all linear perfect codes are Hamming codes.

In 1961 Vasilev \cite{9}, and later many authors, see \cite{10,11} for a survey, constructed perfect codes which are not linear codes. Let us recall Vasilev’s construction of perfect codes.

**Theorem 1.1** \cite{9} Let $C_r$ be a perfect code of $Q_r$. Let $f$ be a function from $C_r$ to $\mathbb{Z}_2$ and $\pi$ be the parity function. Then the set $C_{2r+1} = \{x||\pi(x) + f(c)||x + c; x, c \in C_r\}$
is a perfect code of $Q_{2r+1}$

We recall also the proof of Theorem 1.1 in such a way our article will be self contained.

**Proof.** First notice that $|C_{2r+1}| = 2^r |C_r| = 2^r \frac{2^r}{r+1} = \frac{2^{2r+1}}{2r+2}$. Thus is is sufficient to prove that the distance between to different elements of $C_{2r+1}$ is at least 3.

Consider $d(x||z||c) = d(x, c, x' + z + c') = d_1 + d_2 + d_3$ where $d_1 = d(x, x')$, $d_2 = d(x, f(c), x' + f(c'))$ and $d_3 = d(x + c, x' + c')$.

If $d_1 = 0$ then $x = x'$ thus $d_3 = d(c, c') \geq 3$.

If $d_1 = 1$ and $c = c'$ then $d_2 = d_3 = 1$

If $d_1 = 1$ and $c \neq c'$ then $d_3 \geq 2$ otherwise $d(c, c') \leq 2$

If $d_1 = 2$ then $d_3 \neq 0$ otherwise $d(c, c') = 2$

Thus $d = d_1 + d_2 + d_3 \geq 3$. \hfill \Box

If $f(c) = 0$ for any $c \in C_r$ we obtain the classical inductive construction of Hamming codes with $C_1 = \{0\}$ as basis.

In the next section we will use this construction starting from the Hamming code in $Q_r$ as $C_r$ and a function $f$ chosen in such way that the strings of the constructed code $C_{2r+1}$ has not a too big number of consecutive 1’s.

## 2 Main Result

**Lemma 2.1.** Let $m$ be an integer. Let $A_0$ be the set of strings $A_0 = \{0^{m+1}y : y \in \mathbb{B}_m\}$. For $i \in \{1, \ldots, m\}$ let $A_i = \{z10^{m+1}y : z \in \mathbb{B}_{i-1}, y \in \mathbb{B}_{m-i}\}$. Then the sets $A_i$ are disjoint and any string of $B_{2m+1}$ containing $0^{m+1}$ as substring belongs to a $A_i$.

**Proof.** Let $x$ be a string of $B_{2m+1}$ containing $0^{m+1}$ as substring and $i$ be the minimum integer such that $x_ix_{i+1}x_{i+2} \ldots x_{i+m+1} = 0^{m+1}$. Then $i = 0$, and $x$ belongs to $A_0$, or $m \geq i \geq 1$. In this case $x_i = 1$ thus $x \in A_i$. Assume $x \in A_i \cap A_j$ with $m \geq j > i \geq 0$ then $x_j = 1$ thus $j \geq i + m + 2 > m$ a contradiction.

**Theorem 2.2.** Let $n = 2^p - 1$ where $p \geq 2$ and let $s = 3.2^{p-2}$. There exists a perfect code $C$ in $Q_n$ such that no elements of $C$ contains $1^s$ as substring.

**Proof.** Let $m = 2^{p-2} - 1$ thus $2m + 1 = 2^{p-1} - 1$ and $s = 3m + 3$. Let $C_{2m+1}$ be a perfect code in $Q_{2m+1}$. Let $f$ be the function from $B_{2m+1}$ to $\mathbb{Z}_2$ defined by

- $f(0^{m+1}y) = 1$ for $y \in \mathbb{B}_m$
- $f(10^{m+1}y) = 0$ for $y \in \mathbb{B}_{m-1}$
- $f(z10^{m+1}y) = \pi(z)$ for $z \in \mathbb{B}_{i-1}$ and $y \in \mathbb{B}_{m-i}$ for $i = 2$ to $m$.
- $f = 0$ otherwise.
Note that from the previous lemma the function is well defined. Let \( C \) be the perfect code obtained from Vasilev’s construction from \( C_{2m+1} \) and \( f \). Assume there exists a string \( d \) in \( C \) with \( 1^{3m+3} \) as substring. Therefore \( d \) is obtained from \( x = d_1d_2 \ldots d_{2m+1} \) and \( c \in C_{2m+1} \). Since \( n = 4m + 3 \) note first that \( d_{m+1}d_{m+2} \ldots d_{3m+3} = 1^{2m+2} \). Let \( i \) be the minimum integer such that \( d_id_{i+1} \ldots d_{3m+i+2} = 1^{3m+3} \). We consider 3 cases

- \( i = 1 \) then \( x = d_1d_2 \ldots d_{2m+1} = 1^{2m+1} \) and \( d_{2m+2}d_{2m+3} \ldots d_{3m+3} = 1^{m+2} \). Since \( c + x = 1^{m+1}d_{3m+4}d_{3m+5} \ldots d_{4m+3} \) we have \( c = 0^{m+1}y \) for some \( y \in B_{m} \). Thus \( f(c) = 1 \) and since \( \pi(x) = 1 \) we obtain \( d_{2m+2} = f(c) + \pi(x) = 0 \) a contradiction.

- \( i = 2 \) then \( x = 01^{2m} \) and \( d_{2m+2}d_{2m+3} \ldots d_{3m+4} = 1^{m+3} \). Since \( c + x = 1^{m+2}d_{3m+5}d_{3m+6} \ldots d_{4m+3} \) we have \( c = 10^{m+1}y \) for some \( y \in B_{m-1} \). Thus \( f(c) = 0 \) and since \( \pi(x) = 0 \) we obtain \( d_{2m+2} = f(c) + \pi(x) = 0 \) a contradiction.

- \( i \geq 3 \) then \( x = z01^{2m-i+2} \) for \( z \in B_{i-2} \) and \( d_{2m+2}d_{2m+3} \ldots d_{3m+2+i} = 1^{m+i+1} \). Since \( c + x = 1^{n+i}d_{3m+i+3}d_{3m+i+4} \ldots d_{4m+3} \) we have \( c = z10^{m+1}y \) for some \( y \in B_{m-i+1} \). Thus \( f(c) = \pi(z) \). Since \( \pi(x) = \pi(z) + \pi(1^{2m-i+2}) \) and \( \pi(z) + \pi(z) = \pi(1^{i-2}) \) we obtain \( d_{2m+2} = f(c) + \pi(x) = \pi(1^{2m}) = 0 \) a contradiction.

Therefore there exists no string \( d \) in \( C \) with \( 1^{3m+3} \) as substring. \( \square \)

**Corollary 2.3** Let \( n = 2^p - 1 \) where \( p \geq 2 \) and let \( s \geq 3.2^{p-2} \). There exists a perfect code in \( \Gamma_n(1^s) \).

**Proof.** Indeed let \( C \) be a perfect code in \( Q_n \) such that no element of \( C \) contains \( 1^{3.2^{p-2}} \) as substring. The strings of \( C \) are in \( V(\Gamma_n(1^s)) \). Let \( x \) be a vertex of \( V(\Gamma_n(1^s)) \). If \( x \notin C \) then \( x \) is adjacent in \( Q_n \) to a vertex \( c \) in \( C \). Note that \( x \) and \( c \) are also adjacent in \( \Gamma_n(1^s) \) thus \( C \) is a dominating set of \( \Gamma_n(1^s) \). If \( c \) and \( c' \) are two strings of \( C \) then \( d_{\Gamma_n(1^s)}(c,c') \geq d_{Q_n}(c,c') \geq 3 \). Therefore \( C \) is a perfect code in \( \Gamma_n(1^s) \).

**3 Concluding remark and open problems**

Whenever \( n = 2^p - 1 \) it will be interesting to determine the minimum \( s \) such that there exists a perfect code in \( \Gamma_n(1^s) \).

Corollary 2.3 is not always the best result possible. For example for \( n = 7 \) the code \( C_7 \) obtained in Vasilev’s construction starting from \( C_3 = \{000,111\} \) with \( f(000) = f(111) = 1 \) is a perfect code in \( \Gamma_n(1^5) \). Indeed

- \( 11111ab \) or \( 0011111 \) cannot be in \( C_7 \) since the \( P(111) + 1 = P(001) + 1 = 0 \)

- \( 011111a \) cannot be in \( C_7 \) since the possible codewords beginning with \( 011 \) are \( 0111011 \) and \( 0111100 \).
Note that that all strings of this code are obtained from strings in the Hamming code of length 7 by a translation of 0001000. This simple idea can be generalized but is less efficient than our result in the general case.

We propose also the following conjecture:

**Conjecture 3.1** For $n \geq 3$ and $s \geq 1$ if $C$ is a perfect code in $\Gamma_n(1^s)$ then $n = 2^p - 1$ for some integer $p$ and furthermore $C$ is a perfect code in $Q_n$.

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