Triplon Supersolid of Cold Atoms in a Ladder-Shaped Optical Lattice

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We investigate the ground state phase diagram of the cold atoms in a ladder-shaped optical lattice by using stochastic series expansion quantum Monte Carlo method. We demonstrate how triplon supersolid could emerge, in which the triplon superfluid can coexist with the long-range triplon solid. In the triplon superfluid the quasi-condensation appears in the form of power-law decaying phase correlations. We suggest that a quasi-supersolid phase could be charted in the phase diagram, in which both superfluid and solid orders are quasi-long range ones. We also discuss how this phenomenon can be realized and detected in real experiment.

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\textbf{Introduction.}—With the advantage of cleanness and controllability, the cold atoms system in optical lattice has been a powerful platform to simulate a variety of quantum system. Many works in experimental and theoretical aspects have been made in the past few decades \cite{1,2}. Various types of optical lattices with interesting geometric structure, such as triangular, kagome and honeycomb \cite{2,3}, could be realized by adjusting the potential formed by lasers. Further more, the interactions between atoms can also be tuned by Feshbach resonance \cite{2}. With the aid of ultracold gas in optical lattice, the quantum phase transition from superfluid to Mott insulator is also observed \cite{1}. Beyond that, we ask whether some fascinating quantum states, such as supersolids (SS) which is not observed in real materials so far, could be implemented by using cold atoms in some new kinds of optical lattices, such as a ladder-shaped optical lattice.

Featured by the coexistence of both diagonal and off-diagonal long-range order, SS is an intriguing exotic quantum system. Many works in experimental and theoretical fields have been made in the past few decades \cite{13–21}. In another route, lattice SS have been designed \cite{23–30}. Recently, the cold atoms in isolated double-well have been realized in experiment \cite{2,3}. Could we go beyond those experiments to design a new system and realize a novel SS phase in optical lattices?

In this Letter, we investigate SS phase by extending the isolated double-well \cite{2,3} to a spin dependent ladder-shaped optical lattice, as illustrated in Fig. 1. It’s can be viewed as double-well lattice that are coupled along y direction. Comparing with real compounds, the advantage of this system are the cleanness and highly controllability of the interactions which allow us extend previous studies of ladder system \cite{31,32} to the parameters region where the triplon SS can emerge. This novel phase can be observed in ladder-shaped optical lattice by varying the lattice depth, the zeeman field and the interaction strength via the Feshbach resonance. When the interactions between cold atoms is larger than its kinetic enough i.e., $U \gg t$, the system enter Mott insulator region. The dynamics of atoms just involves spin freedom of degree. By adjusting the zeeman field and the hopping of cold atoms with different spin, the triplon SF is induced by the power-law decaying phase correlations, while the triplon solid emerges as the result of triplon interactions. In a proper parameters region, triplon SF and solid appear simultaneously to form the triplon SS.

The cold atoms in a ladder-shaped optical lattice.—A spin dependent ladder-shaped optical potential is constructed as Fig. 1. We assume that only two of the internal ground states participate in the dynamics, which we call spin up and down, $| \uparrow \rangle$ and $| \downarrow \rangle$. By second-order perturbation theory for coupling of the ground state levels to the excited atomic states, the potential can be obtained as, $U_{\sigma} = V_{z_{1},\sigma} \cos^{2}(k_{z}x) + V_{z_{2},\sigma} \cos^{2}(2k_{z}x) + V_{y,\sigma} \cos^{2}(k_{y}y)$, where $\sigma = \uparrow, \downarrow$ represents the two states of cold atoms and $V_{z_{1},\sigma}$ and $V_{z_{2},\sigma}$ are the barrier heights of the two standing waves along the x direction, $V_{y,\sigma}$ is the one along the y direction. In experiment a harmonic potential with frequency $\omega_{z}$ should be added in z direction which restricts the moving of atoms along it. To simulate a decoupled two-leg ladder potential, $4V_{z_{2},\sigma} > V_{z_{1},\sigma}$ is required and the ratio of potential heights between intra-ladder to inter-ladder along x direction should satisfies $\Delta V = (4V_{z_{2},\sigma} - V_{z_{1},\sigma})^{2}/(4V_{z_{2},\sigma} + V_{z_{1},\sigma})^{2} \ll 1$. The widths of the two barriers along rung direction and rail direction are $a = \alpha/k_{z}$ and $b = \pi/k_{y}$, respectively, with $\alpha = \arccos(V_{z_{1},\sigma}/4V_{z_{2},\sigma})$. With the considerations, then by using harmonic approximation in each bottom of trap,
FIG. 1: (Color online) A ladder-shaped optical potential is formed by three standing wave lasers indicated as a pair of counter yellow arrows $V_{s,1}, V_{s,2}, V_{s,3}$. Two $V_{s,1}, V_{s,2}$ of them along $x$ direction and another one $V_{s,3}$ along $y$ direction, where $V_{s,2}$ (short yellow arrows) has twice the period of $V_{s,1}$ (long yellow arrows). The cold atoms with two internal states denoted by spin up $|\uparrow\rangle$ (red up arrows) and spin down $|\downarrow\rangle$ (blue down arrows) are trapped by this potential. An antiferromagnetic (AF) configuration with staggered up-and-down moments on the ladder is showed. The white lines in the potential contour lines represent the ladder. The effective strength of interactions is $J_\perp$ along $x$ direction, $J_\parallel^U, J_\parallel^L$ along $y$ direction. $h$ is zeeman field.

the system can be described by Hubbard model as, 

$$
H = \sum_{i,j,\sigma} \left(-t_{\mu,\sigma} c_{i,\sigma}^\dagger c_{j,\sigma} + h.c\right) + U_{\uparrow\downarrow} \sum_i n_{i,\uparrow} n_{i,\downarrow} 
$$

$$
+ \sum_i U_{\sigma} n_{i,\sigma}(n_{i,\sigma} - 1) - \tilde{h} \sum_i (n_{i,\uparrow} - n_{i,\downarrow}),
$$

where a zeeman term $\tilde{h}$ is also introduced. $c_{i,\sigma}^\dagger (c_{i,\sigma})$ is the creation (annihilation) operator of boson or fermion in site $i$ with spin $\sigma$. The coefficients $t_{\mu,\sigma}$, with $\mu = \perp, \parallel$, represent intraladder hopping along rung, rail direction, respectively. $U_{\uparrow\downarrow}, U_{\sigma}$ are the on-site interaction between atoms. We can work out those interaction parameters as a function of $V_{s,1,\sigma}, V_{s,2,\sigma}, V_{s,3,\sigma}, a_s, E_x, E_y, E_{gr}$, where $a_s$ is the scattering lengths and $E_{x,y,gr}$ is $k^2 q^2 x, y, 2m$ are the atomic recoil energy along $x, y$ direction respectively.

Considering $t_{\mu} \ll U_{\parallel}, U_{\perp}$ and $1/2$ filling, by a second order perturbation theory, this system will be described by an effective Hamiltonian with spin freedom of degree only, 

$$
\tilde{H} = \sum_{i,j} \left[J_{\mu}^0 (S_i^\mu S_j^\mu \pm J_{\mu}^x (S_i^x S_j^x + S_i^y S_j^y)) - h \sum_i S_i^z\right],
$$

where $\mathbf{S} = \frac{1}{2} \sigma \mathbf{c}, \sigma$ is pauli matrix and $\mathbf{c} = (c_\uparrow, c_\downarrow)$. The upper sign before $J_{\mu}^0$ is for fermionic atoms and down sign for bosonic one. The interaction coefficients for bosons can be figured out as, 

$$
J_{\mu}^0 = \frac{4 t_{\mu}^2 + U_{\parallel}}{4 t_{\parallel}^2 - U_{\parallel}^2} J_{\mu}^x = \frac{4 t_{\perp}^2 + U_{\perp}}{4 t_{\perp}^2 - U_{\perp}^2}, h = 2\tilde{h} + \frac{6 t_{\perp}^2}{U_{\perp}} - \frac{6 t_{\parallel}^2}{U_{\parallel}}.
$$

For fermion, we need to omit the last two term in $J_{\mu}^x$.

Since the lattice is bipartite, without loss of generality, we focus on the region with positive $J_{\perp}^x$ of the spin XXZ model $\tilde{H}$ and hereafter we take $J_{\perp}^x = J_{\parallel}^x = J_{\parallel}^y = 1$ as energy unit. The QMC simulation [34] is performed on a ladder with length $L$ and temperature $T = 6/L$. A mean-field (MF) theory is also employed for comparison. The ground-state phase diagram is illustrated in Fig. 4.

**The triplon superfluidity.**—Let’s start from the isotropic case, i.e., $J_{\parallel}^y = J_{\parallel}^x = J_\parallel$, which has been extensively studied in the context of real materials [35]. The four eigenstates of $j_{th}$ decoupled rung provide very helpful basis: the singlet state, $\ket{s} = \frac{1}{2\sqrt{2}} (\ket{\uparrow\downarrow}_{j} - \ket{\downarrow\uparrow}_{j})$, and three triplet states, $\ket{\uparrow\uparrow}_{j} = \ket{\uparrow\uparrow}_{j}, \ket{\downarrow\downarrow}_{j} = \frac{1}{2\sqrt{2}} (\ket{\uparrow\downarrow}_{j} + \ket{\downarrow\uparrow}_{j}), \ket{\downarrow\downarrow}_{j} = \ket{\downarrow\downarrow}_{j}$. If no (or weak) magnetic field is present, the ladder is in a gapped SD phase, whose ground state is roughly a chain of singlets. While the system will be in a FP phase in the strong magnetic field limit. The SD and FP phases can be readily expressed by MF wave functions, $\psi_{SD} = \prod_j \ket{s}_{j}$ and $\psi_{FP} = \prod_j \ket{\uparrow\uparrow}_{j}$, respectively. If moderate field is applied, a gapless triplon SF phase emerges, which is induced by the quasi-condensation of triplons when the energy level of $\ket{\uparrow\uparrow}_{j}$ is down to the vicinity of state $\ket{s}$. And the upper $\ket{\uparrow\uparrow}_{j}$ and $\ket{\downarrow\downarrow}_{j}$ bands could be integrated out from the renormalization point of view. One can introduce triplon bosonic operators $\hat{b}^\dagger_{j} = \frac{1}{\sqrt{2}} (S_{j,1}^+ - S_{j,1}^-)$ that fulfill $\hat{b}^\dagger_{j} \ket{s}_{j} = \ket{\uparrow\uparrow}_{j}$. Then the system could be mapped to a 1D hardcore bose-Hubbard model. It can be exactly solved by Bethe Ansatz or approximately by Bosonization method, gives the Luttinger liquid (LL) nature [36]. In this regime, the state $\ket{\uparrow\uparrow}_{j}$ undergoes a quasi-condensation characterized by a power-law decreasing phase correlations $\langle \hat{b}^\dagger_{j} \hat{b}_{j+\pi} \rangle \sim r^{-1/2K_{\pi}}$, where $K$ is the LL parameter. Although true condensation is absent in 1D systems, the triplon superfluidity is still possible due to the power-law decreasing phase correl-
SD to AF is of second order, which is well signified by the isotropic case (see Fig. 3(d) and (e)). The staggered magnetization \( m_s \) and (e) its corresponding staggered magnetic susceptibility \( \chi_s \) for \( J_1^s = 0.35 \) and \( h = 0.5 \).

The exact solutions of \( \langle |\psi_{TS}\rangle \rangle = \prod_j (u_A |s\rangle_2 + (-1)^j v_A |t_+\rangle_2) \otimes (u_B |s\rangle_2 + (-1)^j v_B |t_+\rangle_2) \) with variational parameters \( u_A \neq u_B \) and \( v_A \neq v_B \). The TS order breaks the translational symmetry of the ladder along the rail. To detect the TS order, we use the trilpion structure factor \( S(\pi) = \sum \langle |\tilde{n}_{s}^t| \rangle \langle \tilde{n}_{s}^t \rangle \rangle \). True TS order means a finite diagonal order parameter squared, \( \langle |\psi_{TS}\rangle \rangle ^2 = \frac{1}{2} S(\pi) \). The true TS order is a consequence of the interplay between the anisotropy and the strong field. While in the SF regime, there may exist a weak triplon solid order that arises from the power-law decaying correlators \( |\tilde{n}_{s}^t| \) in anisotropic case \( J_1^s = 4.0 \). The dotted lines have slope value 1 which facilitates to discern the occurrence of true TS order.

The second type of solid order is the alternative arrangement of singlet state \( |s\rangle \) and triplet state \( |t_+\rangle \) on the rung along the rail of the ladder. It is the TS order represented by \( |\psi_{TS}\rangle \rangle = \prod_j (u_A |s\rangle_2 + (-1)^j v_A |t_+\rangle_2) \otimes (u_B |s\rangle_2 + (-1)^j v_B |t_+\rangle_2) \) with variational parameters \( u_A \neq u_B \) and \( v_A \neq v_B \). The TS order breaks the translational symmetry of the ladder along the rail. To detect the TS order, we use the trilpion structure factor \( S(\pi) = \sum \langle |\tilde{n}_{s}^t| \rangle \langle \tilde{n}_{s}^t \rangle \rangle \). True TS order means a finite diagonal order parameter squared, \( \langle |\psi_{TS}\rangle \rangle ^2 = \frac{1}{2} S(\pi) \). The true TS order is a consequence of the interplay between the anisotropy and the strong field. While in the SF regime, there may exist a weak triplon solid order that arises from the power-law decaying correlators \( |\tilde{n}_{s}^t| \) in anisotropic case \( J_1^s = 4.0 \). The dotted lines have slope value 1 which facilitates to discern the occurrence of true TS order.

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prevail. Before entering into the AF phase, we find the system gains a considerable small fraction of $n_0 \sim 0.13$, which plays the role of impurities in the SS phase [26]. This may be the reason why SS can not emerge in general 1D hard-core boson model or XXZ model but in this system. It is worthwhile to note that the simple mean-field description for the superfluidity in 1D is not quite right. One can combine all the components in a wave function as $|\psi\rangle = \prod_j (|u\rangle_j + (-1)^j |v_L\rangle_j + (-1)^j |v_T\rangle_j)$. But this assumption would lead to a true condensation and, thus, totally failed to capture the quasi-condensation nature. So the situation here is quite similar to the 1D spin 1 chain with uniaxial exchange and single-ion anisotropies, where the LL behavior also plays an important role in providing the 1D superfluidity [28].

It is important to preclude the possibility of phase separation. This can be done by measuring $m_z$ of the small area of size (say $l = 8(12)$) throughout the whole lattice ($L = 88$). If no phase separation occurs, we should observe only a single peak in the histograms of $m_z$ [28]. This is indeed the case as shown in Fig. 6.

**Experimental proposal.**—The system discussed in this paper may be realized in a Mott insulating phase of cold bosonic or fermionic atoms in a ladder-shaped optical lattice if the atoms possess two internal states that play the role of the spin (such as $^{87}$Rb in $|\uparrow\rangle \equiv |F = 1, m_F = 1\rangle$, $|\downarrow\rangle \equiv |F = 1, m_F = -1\rangle$). Let’s estimate the typical energy scales. For Rb atoms with a lattice constant $\pi/k_B \approx 2\pi/k_B \approx 426$ nm and about $10^5$ atoms in a Bose-Einstein condensate, we can choose $t^2/(\hbar U) \sim 0.1$ kHz (corresponding to a time scale of 10 ms) with a conservative choice of $U \sim 2$ kHz and $(t/U)^2 \sim 1/20$. These energy scales are clearly compatible with current experiments [1] and make the system in a Mott insulating area.

In experiment, the density of condensates in momentum space $\langle \hat{n}_\pi \rangle$ can be measured by noise correlations which can be linked to spin-spin correlations [11–13]. We can use Bragg scattering of light, which gives rise to the spin structure factor, to detect $S(\pi)$ [44]. An alternative technique for imaging spin states in optical lattices has been put forward [45]. Thus, the SS discussed in this paper can be detected in experiment.

**Conclusions.**—In summary, we propose a scheme to realize triplon SS of cold atoms in a ladder-shaped optical lattice. There is a q-SS phase emerging as a precursor of the true SS phase and TS phases. The most importance is that this triplon SS can be detected in recent experimental technology. These results are beyond both previous experimental researches in isolated double-well and previous studies of ladder system. It can be used as a guide for future experimental studies of supersolid and quasi-supersolid.

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[1] M. Greiner et al., Nature (London) 415, 39 (2002).
[2] S. Trotzky et al., Science 319, 295 (2008).
[3] S. Fölling et al., Nature (London) 448, 1029 (2007).
[4] Y. A. Chen, S. Nascimbène, M. Aidelsburger, M. Atala, S. Trotzky, and I. Bloch, Phys. Rev. Lett. 107, 210405 (2011).
[5] J. Sebby-Strabley, M. Anderlini, P. S. Jessen, and J. V. Porto, Phys. Rev. A 73, 033605 (2006).
[6] L. M. Duan, E. Demler and M. D. Lukin, Phys. Rev. Lett. 91, 090402 (2003).
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