TOPOLOGICAL ENTROPY AND AF SUBALGEBRAS OF GRAPH $C^*$-ALGEBRAS

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Abstract. Let $A_E$ be the canonical AF subalgebra of a graph $C^*$-algebra $C^*(E)$ associated with a locally finite directed graph $E$. For Brown-Voiculescu’s topological entropy $ht(\Phi_E)$ of the canonical completely positive map $\Phi_E$ on $C^*(E)$, $ht(\Phi_E) = ht(\Phi_E|A_E) = h_0(E) = h_b(E)$ is known to hold for a finite graph $E$, where $h_0(E)$ is the loop entropy of Gurevic and $h_b(E)$ is the block entropy of Salama. For an irreducible infinite graph $E$, the inequality $h_l(E) \leq ht(\Phi_E|A_E)$ has been known recently. It is shown in this paper that $ht(\Phi_E|A_E) \leq \max\{h_b(E), h_b(\overset{\uparrow}{E})\}$, where $\overset{\uparrow}{E}$ is the graph $E$ with the direction of the edges reversed. Some irreducible infinite graphs $E_p(p > 1)$ with $ht(\Phi_E|A_{E_p}) = \log p$ are also examined.

1. Introduction

Voiculescu [20] introduced a notion of topological entropy $ht(\alpha)$ for an automorphism $\alpha$ of a nuclear unital $C^*$-algebra $A$ to measure the growth of $\alpha^n$ as $n \to \infty$ using the fact that a nuclear $C^*$-algebra has the completely positive approximation property. The definition extends very well to automorphisms of exact $C^*$-algebras (as done by Brown in [4]) due to the deep result by Kirchberg [12] that exact $C^*$-algebras are nuclearly embeddable. But without effort one can define $ht(\Phi)$ even for a completely positive (cp) map on an exact $C^*$-algebra as described in [2]. Since a $C^*$-subalgebra of an exact $C^*$-algebra is always exact, if $\Phi : A \to A$ is a cp map on an exact $C^*$-algebra $A$ and $B$ is a $\Phi$-invariant $C^*$-subalgebra of $A$ then $ht(\Phi|B)$ can be defined and the monotonicity $ht(\Phi|B) \leq ht(\Phi)$ holds.

The topological entropy has been computed in several cases, for example, the equality $ht(\alpha * \beta) = \max\{ht(\alpha), ht(\beta)\}$ for the reduced free product automorphism $\alpha * \beta$ was proved in [14], when the free product is with amalgamation over a finite dimensional $C^*$-algebra. Also Dykema [9] showed that $ht(\alpha) = 0$ for certain classes of automorphisms $\alpha$ of reduced amalgamated free products of $C^*$-algebras, which turns out to extend Størmer’s result [19] that the Connes-Størmer entropy of the free shift automorphism of the II$_1$-factor $L(F_\infty)$ is zero.

In this paper we are concerned with the topological entropy of the shift type cp maps on $C^*$-algebras arising from directed graphs. A typical one is the canonical cp map $\Phi_A : \mathcal{O}_A \to \mathcal{O}_A$ of the Cuntz Krieger algebra $\mathcal{O}_A$ given by

$$\Phi_A(x) = \sum_{i=1}^{n} s_i x s_i^*,$$

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where $s_1, \ldots, s_n$ are the partial isometries that generate $\mathcal{O}_A$. The reason we call $\Phi_A$ shift type is that $\mathcal{O}_A$ contains a $\Phi_A$-invariant commutative $C^*$-subalgebra $\mathcal{D}_A$ which is isomorphic to $C(X_A)$ in such a way that the restriction $\Phi_A|_{\mathcal{D}_A}$ corresponds to the shift map $\sigma_{X_A}$ on the (compact) shift space $X_A$ associated with the incidence matrix $A$. The topological entropy of $\Phi_A$ is then computed (see $[5, 2, 17]$) as $ht(\Phi_A) = \log r(A)$ ($r(A)$ is the spectral radius of $A$). But $\log r(A) = h_{top}(X_A)$ is a well known fact, so that one can deduce by $[8]$ that $ht(\Phi_A) = h(t(\Phi_A|_{\mathcal{D}_A})$. On the other hand, $\mathcal{O}_A$ also contains another important $\Phi_A$-invariant $C^*$-subalgebra $\mathcal{A}_A$ which is an AF algebra with $\mathcal{D}_A \subset \mathcal{A}_A$. Thus by monotonicity of entropy, we have $ht(\Phi_A) = h(t(\Phi_A|_{\mathcal{A}_A}) = h(t(\Phi_A|_{\mathcal{D}_A})$.

The Cuntz-Krieger algebras $\mathcal{O}_A$ are now well understood as graph $C^*$-algebras $C^*(E) = C^*(s_e, p_v)$ associated with finite directed graphs $E$ and the cp map $\Phi_A$ of $\mathcal{O}_A$ is interpreted as the map $\Phi_E : C^*(E) \to C^*(E)$ given by $\Phi_E(x) = \sum_{e \in E} s_e x s_e^*$. Hence if $E$ is a finite graph (possibly with sinks) which contains an infinite path, it follows that $ht(\Phi_E) = h(t(\Phi_E|_{\mathcal{A}_E}) = h(t(\Phi_E|_{\mathcal{D}_E}) = \log r(A_E)$, where $A_E$ is the AF subalgebra of $C^*(E)$ corresponding to $\mathcal{A}_A$ in $\mathcal{O}_A$ and $A_E$ is the edge matrix of $E$ (see $[11]$).

If $E$ is infinite but locally finite then the map $\Phi_E$ is known to be a contractive cp map, and furthermore if $E$ is irreducible and $A_E$ is the canonical AF subalgebra of $C^*(E)$, the inequality $h_t(E) \leq h(t(\Phi_E|_{\mathcal{A}_E})$ is known to hold $[11]$. The purpose of the present paper is then to give an upper bound for $h(t(\Phi_E|_{\mathcal{A}_E})$ and we actually prove the following (see Theorem 3.9)

$$ht(\Phi_E|_{\mathcal{A}_E}) \leq \max\{h_b(E), h_b(l^1(E))\}.$$  

In particular, for an irreducible infinite graph $E_\rho$ constructed in $[18]$ so that $h_t(E_\rho) = h_b(E_\rho) = p > 1$, we have $ht(\Phi_{E_\rho}|_{\mathcal{A}_{E_\rho}}) = \log p$.

We believe that the result would be helpful to compute the entropy $ht(\Phi_E)$ of $\Phi_E$ on the whole graph $C^*$-algebra $C^*(E)$.

2. Preliminaries

2.1. Graph $C^*$-algebras. A (directed) graph is a quadruple $E = (E^0, E^1, r, s)$ of the vertex set $E^0$, the edge set $E^1$, and the range, source maps $r, s : E^1 \to E^0$. A family $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ of mutually orthogonal projections $p_v$ and partial isometries $s_e$ is called a Cuntz-Krieger $E$-family if the following relations hold:

\begin{align*}
\Delta^* s_e = p_v, & \quad s_e \Delta s_e^* \leq p_{s(e)}, \\
\Delta p_v = \sum_{s(e) = v} s_e \Delta s_e^*, & \quad \text{if } 0 < |s^{-1}(v)| < \infty.
\end{align*}

The graph $C^*$-algebra $C^*(E)$ is then defined to be a $C^*$-algebra generated by a universal Cuntz-Krieger $E$-family (see $[15, 8]$). We call $E$ locally finite if each vertex receives and emits only finitely many edges. Throughout this paper we consider only locally finite graphs, and adopt the notations in $[15]$. If a finite path $\alpha \in E^*$ of length $|\alpha| > 0$ is a return path, that is, $s(\alpha) = r(\alpha)$, then $\alpha$ is called a loop at $v = s(\alpha)$. A graph $E$ is said to be irreducible if for any two vertices $v, w$ there is a finite path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. It is known that if $E$ is irreducible and every loop has an exit then $C^*(E)$ is simple.

2.2. Topological entropy of cp maps. Let $A$ be a $C^*$-algebra, $\pi : A \to B(H)$ a faithful $*$-representation, and $Pf(A)$ be the set of all finite subsets of $A$. For
\[ \omega \in Pf(A) \text{ and } \delta > 0, \text{ put} \]
\[ CPA(\pi, A) = \{(\phi, \psi, B) \mid \phi : A \to B, \psi : B \to B(H) \text{ are contractive cp maps} \]
\[ \text{ and dim } B < \infty \}, \]
\[ rcp(\pi, \omega, \delta) = \inf \{ \text{rank}(B) \mid (\phi, \psi, B) \in CPA(\pi, A), \| \psi \circ \phi(x) - \pi(x) \| < \delta, \]
\[ \text{ for all } x \in \omega \}, \]
\[ \text{where rank}(B) \text{ denotes the dimension of a maximal abelian subalgebra of } B. \]
\[ \text{Since the cp } \delta\text{-rank } rcp(\pi, \omega, \delta) \text{ is independent of the choice of } \pi \text{ (see [4], [2]) and graph } C^*\text{-algebras } C^*(E) \text{ are nuclear we may write } rcp(\omega, \delta) \text{ for } rcp(\pi, \omega, \delta) \]
\[ \text{assuming that } C^*(E) \subset B(H) \text{ for a Hilbert space } H. \]

**Definition 2.1.** ([4], [2]) Let \( A \subset B(H) \) be a \( C^* \)-algebra and \( \Phi : A \to A \) be a cp map. Put
\[ ht(\Phi, \omega, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log (rcp(\bigcup_{i=0}^{n-1} \Phi^i(\omega), \delta)) \]
\[ ht(\Phi, \omega) = \sup_{\delta > 0} ht(\Phi, \omega, \delta). \]

Then \( ht(\Phi) := \sup_{\omega \in Pf(A)} ht(\Phi, \omega) \) is called the topological entropy of \( \Phi. \)

**Remark 2.2.** We refer the reader to [2] and [4] for the following useful properties and their proofs. Let \( A \) be an exact \( C^* \)-algebra and \( \Phi : A \to A \) be a cp map.

(a) If \( \theta : A \to B \) is a \( C^* \)-isomorphism then \( ht(\Phi) = ht(\theta \Phi \theta^{-1}). \)

(b) Let \( \tilde{A} \) be the unital \( C^* \)-algebra obtained by adjoining a unit and \( \tilde{\Phi} : \tilde{A} \to \tilde{A} \)
\[ \text{be the extension of } \Phi. \]
\[ \text{Then } ht(\tilde{\Phi}) = ht(\Phi). \]

(c) If \( A_0 \subset A \) is a \( \Phi \)-invariant \( C^* \)-subalgebra of \( A, \)
\[ ht(\Phi|_{A_0}) \leq ht(\Phi). \]

We will use the following Arveson’s extension theorem several times.

**Arveson Extension Theorem** (see [4]) Let \( A \) be a unital \( C^* \)-algebra, \( S \subset A \)
\[ \text{a unital subspace with } S = S^*, \text{ and } \phi : S \to B \text{ be a contractive cp map where} \]
\[ B = B(H) \text{ or dim}(B) < \infty. \]
\[ \text{Then } \phi \text{ extends to a cp map } \phi : A \to B. \] If \( S \) is a \( C^* \)-subalgebra of \( A \) then we obtain a unital cp extension of \( \phi \) even when \( S \) does not contain the unit of \( A. \)

If \( E \) is a locally finite graph, the map \( \Phi_E : C^*(E) \to C^*(E), \)
\[ \Phi_E(x) = \sum_{e \in E^1} s_e x s_e^*, \]
\[ \text{is well defined, contractive, and completely positive [11]. For a finite graph } E, \text{ the}
\[ \text{topological entropy } ht(\Phi_E) \text{ has been obtained as follows (see [2], [5], [17], or [11].} \]

**Theorem 2.3.** Let \( E \) be a finite graph possibly with sinks and \( A_E \) be the edge matrix of \( E. \)
\[ \text{If } E \text{ contains an infinite path then} \]
\[ ht(\Phi_E) = \log r(A_E), \]
\[ \text{where } r(A_E) \text{ is the spectral radius of } A_E. \]

By \( h_{top}(X) \) we denote the topological entropy of a compact space \( (X, T) \) together with a continuous map \( T : X \to X \) (for definition, see [16] Definition 4.1.1 or [13] p.23] ). Let \( E \) be a locally finite infinite graph and \( X_E \) the locally compact shift
space of (one-sided) infinite paths with the one point compactification $\bar{X}_E$. Then the first identity in the following theorem is shown for the doubly infinite path space of $E$ by Gurevich \[11\]. See Definition 3.1 for $h_b(E)$.

**Theorem 2.4 (\[11\] Theorem 4.4).** Let $E$ be a locally finite irreducible infinite graph. Then

$$h_{top}(\bar{X}_E) = \sup_{E'} h_b(E') \leq ht(\Phi_E),$$

where the supremum is taken over all the finite (irreducible) subgraphs of $E$.

### 3. Main results

Throughout this section $E$ will denote a locally finite infinite graph unless stated otherwise. For a path $\alpha \in E^*$, let $\alpha^0$ be the set of vertices lying on $\alpha = \alpha_1 \cdots \alpha_n$, that is, $\alpha^0 = \{s(\alpha_1), r(\alpha_1), \ldots, r(\alpha_n)\}$. For a fixed vertex $v$ we consider the following subsets of finite paths $E^n_\alpha$ of length $n$.

(i) $E^n_\alpha(v) = \{\alpha \in E^n \mid v \in \alpha^0\}$,

(ii) $E^n_s(v) = \{\alpha \in E^n \mid s(\alpha) = v\}$,

(iii) $E^n_s(v^*) = \{\alpha \in E^n_s(v) \mid r(\alpha_i) \neq v, \ 1 \leq i \leq n\}$,

(iv) $E^n_t(v) = \{\alpha \in E^n \mid \alpha \text{ is a loop at } v\}$.

Similar we can think of $E^n_s(v)$ and $E^n_s(v^*)$.

**Definition 3.1.** Let $E$ be an irreducible graph and let $v \in E^0$.

(a) $h_l(E) := \limsup_n \frac{1}{n} \log |E^n_s(v)|$ is called the loop entropy of $E$.

(b) $h_b(E) := \limsup_n \frac{1}{n} \log |E^n_s(v)|$ is called the block entropy of $E$.

Note that both entropies $h_l(E)$ and $h_b(E)$ are independent of the choice of a vertex $v$ \[13\]. If $E^t$ denotes the graph $E$ with the direction of all edges reversed, then clearly $h_l(E) = h_l(E^t)$ while $h_b(E) \neq h_b(E^t)$ in general as we will see in Example 3.3.

We will use the following notation for the infinite series with coefficients from (i)-(iv) above.

(i) $E(v, z) := \sum |E^n_s(v)|z^n$,

(ii) $E_s(v, z) := \sum |E^n_s(v)|z^n$,

(iii) $E_s(v^*, z) := \sum |E^n_s(v^*)|z^n$,

(iv) $E_t(v, z) := \sum |E^n_t(v)|z^n$.

We denote the radius of convergence of the series $E_s(v^*, z)$ by $R_{E_s}$. Thus

$$R_{E_s}^{-1} = \limsup_{n \to \infty} |E^n_s(v^*)|^{1/n}.$$  

Similarly $R_{E_t}$ denotes the radius of convergence of $E_t(v^*, z) := \sum |E^n_t(v^*)|z^n$.

**Proposition 3.2 (\[13\]).** If $E$ is an irreducible graph, then

$$h_b(E) = \max\{\log (R_{E_s}^{-1}), h_l(E)\}.$$  

Note that if $E$ is irreducible then $h_b(E) = \limsup \frac{1}{n} \log |E^n_s(v)|$ and so from the above proposition we have

$$h_b(E) = \max\{\log (R_{E_s}^{-1}), h_l(E)\}.$$
The following example shows that \( h_b(E) \neq h_b(\ell E) \) in general.

**Example 3.3.** For each pair of positive real numbers \( 1 < p \leq q \), Salama \[18\] constructed an irreducible infinite graph \( E_{p,q} \) with

\[
h_l(E_{p,q}) = \log p \quad \text{and} \quad h_b(E_{p,q}) = \log q.
\]

For example, the following graph \( E := E_{2,8} \) satisfies \( h_l(E) = \log 2 \) and \( h_b(E) = \log 8 \). There are 8 edges from the vertex \( n \) to the vertex \( n + 1 \) for each \( n \geq 0 \).

Now we show that

\[
\log \left( R_{E^*}^{-1} \right) \leq h_l(E),
\]

which then implies \( h_b(\ell E) = h_l(E) \) by the above proposition (hence \( h_b(\ell E) \neq h_b(E) \)). For a fixed vertex 0 we have

\[
R_{E^*}^{-1} = \limsup_{n \to \infty} |E_{\ell}^n(0^*)|^{1/n}
\]

\[
= \limsup_{n \to \infty} \{|\alpha \in E_{\ell}^n(0) \mid s(\alpha_i) \neq 0, \text{ for } 1 \leq i \leq n\}|^{1/n}.
\]

With \( n_{k+1} = 4k + 1 (k \geq 0) \),

\[
|E_{\ell}^{n_{k+1}-1}(0^*)| = |E_{\ell}^{4k}(0^*)| = 1 + 8^{k-1} + 8^{k-4} + 8^{k-7} + \ldots,
\]

and a computation gives

\[
\limsup_{k \to \infty} |E_{\ell}^{4k}(0^*)|^{1/4} = 8^{1/4}.
\]

But it is not hard to see that

\[
\limsup_{n \to \infty} |E_{\ell}^n(0^*)|^{1/n} = \limsup_{k \to \infty} |E_{\ell}^{4k}(0^*)|^{1/4},
\]

hence \( \log(R_{E^*}^{-1}) = \log 8^{1/4} < \log 2 = h_l(E) \).

**Lemma 3.4.** If \( E \) is an irreducible graph then the value

\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v)|
\]

is independent of the choice of a vertex \( v \).

**Proof.** Let \( v, w \) be two vertices of \( E \). Then there exist two paths \( \mu \in E^k, \nu \in E^m \) with \( s(\mu) = r(\nu) = v, s(\nu) = r(\mu) = w \) because \( E \) is irreducible. We assume that \( \mu \) and \( \nu \) have the smallest length, respectively. If \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_n \in E^n(v) \) then with \( i_0 = \min\{i \mid s(\alpha_i) = v\} \) write \( \alpha = \alpha' \alpha'' \), where \( \alpha' = \alpha_1 \cdots \alpha_{i_0-1} \) and \( \alpha'' = \alpha_{i_0} \cdots \alpha_n \) (if \( i_0 = 0, \alpha = \alpha'' \)). Then the map

\[
E^n(v) \to E^{n+k+m}(w), \quad \alpha = \alpha' \alpha'' \mapsto \alpha' \mu \alpha''
\]
is injective, hence \(|E^n(v)| \leq |E^{n+k+m}(w)|\) for each \(n\). Therefore
\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v)| \leq \limsup_{n \to \infty} \frac{1}{n} \log |E^{n+k+m}(w)| \leq \limsup_{n \to \infty} \frac{1}{n} \log |E^n(w)|.
\]

\[\square\]

**Proposition 3.5.** Let \(E\) be an irreducible graph and \(v_0 \in E^0\).
(a) If \(E\) is finite, then
\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v_0)| = \limsup_{n \to \infty} \frac{1}{n} \log |E^n|.
\]
In particular, \(h_l(E) = h_b(E) = h_b(\mathcal{E})\).
(b) If \(E\) is infinite, then
\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n(v_0)| = \max \{h_b(E), h_b(\mathcal{E})\}.
\]

**Proof.** (a) Let \(E^0 = \{v_0, v_1, \ldots, v_{k-1}\}\). Since \(E\) is irreducible there exist finite paths \(\{\mu_i, r_i \mid 0 \leq i \leq k - 1\}\) such that \(s(\mu_i) = r(v_i) = v_0, r(\mu_i) = v_i = s(v_i)\).
Suppose \(|\mu_i| = m_i, |r_j| = l_j\). If \(\alpha \in E^n\) is a path with \(s(\alpha) = v_i, r(\alpha) = v_j\) then \(\mu_i, \alpha \in E_{l_i}^{n+m_i+l_j}(v_0)\) is a loop at \(v_0\). The map \(\alpha \mapsto \mu_i, \alpha \in E^n\) is not necessarily injective, but there exist at most \(k_0\) paths in \(E^n\) that have the same image in \(E_{l_i}^{n+m_i+l_j}(v_0)\) under the map, where \(k_0 = \max_{i,j} \{m_i + l_j\}\). Hence we have
\[
|E^n| \leq k_0 \cdot \bigcup_{0 \leq i, j \leq k-1} E_{l_i}^{n+m_i+l_j}(v_0) \leq k_0 k^2 \max_{i,j} |E_{l_i}^{n+m_i+l_j}(v_0)|.
\]

On the other hand, for each \(n\), there exists a \(k_n \in \{0, \ldots, k_0\}\) such that
\[
|E_{l_i}^{n+k_n}(v_0)| = \max_{i,j} |E_{l_i}^{n+m_i+l_j}(v_0)|.
\]
Then \(|E^n| \leq k_0 k^2 |E_{l_i}^{n+k_n}(v_0)|\) and it follows that
\[
\limsup_{n \to \infty} \frac{1}{n} \log |E^n| \leq \limsup_{n \to \infty} \frac{1}{n} \log |E^n(v_0)|.
\]

(b) Note first that
\[
|E^n(v)| = |\bigcup_{k=0}^n \{\alpha \beta \mid \alpha \in E_{r}^{k}(v^\ast), \beta \in E_{s}^{n-k}(v_0)\}| = \sum_{k=0}^n |E_{r}^{k}(v^\ast)||E_{s}^{n-k}(v)| = \sum_{k=0}^n \left( (tE)^k_s(v^\ast) \right) |E_{s}^{n-k}(v)|.
\]
Then
\[
E(v, z) = \sum_{n} \left( \sum_{k=0}^n \left( (tE)^k_s(v^\ast) \right) |E_{s}^{n-k}(v)| \right) z^n = \left( \sum_{n} \left( (tE)^n_s(v^\ast) \right) z^n \right) \left( \sum_{n} |E_s^n(v)| z^n \right) = \left( (tE)_s(v^\ast, z) \right) \cdot E_s(v, z),
\]
so that the radius of convergence \(R_E\) of \(E(v, z)\) is equal to \(\min \{R_{(tE)\ast}, R_{E_s}\}\). Thus
\[
R_E^{-1} = \max \{ R_{(tE)\ast}^{-1}, R_{E_s}^{-1} \}.
\]
Lemma 3.6. Let \( R^{-1}_E \) be an irreducible infinite graph and let \( D_E \) be the commutative \( C^* \)-subalgebra of \( C^*(E) \) generated by the projections \( \{ p_\alpha = s_\alpha s_\alpha^* \mid \alpha \in E^* \} \). Then \( D_E = \text{span}\{p_\alpha \mid \alpha \in E^*\} \) and the map

\[ w : D_E \to C_0(X_E), \quad w(p_\alpha) = \chi_{[\alpha]}, \]

is a \( C^* \)-isomorphism [11]. Here \( \chi_{[\alpha]} \) is the characteristic function on the cylinder set \([\alpha] = \{ \beta \in X_E \mid \beta = \alpha \beta \} \) which is both open and closed. Furthermore, from the proof of Theorem 2.4 we know that \( h_{top}(X_E) = h_{top}(\Phi_E|D_E) \). Put

\[ A_E := \text{span}\{ s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, |\alpha| = |\beta| \}. \]

Then \( A_E \) is a \( \Phi_E \)-invariant AF \( C^* \)-subalgebra of \( C^*(E) \) with \( D_E \subset A_E \), hence \( h_{top}(\Phi_E|D_E) \leq h_{top}(\Phi_E|A_E) \).

Also, for each \( v \in E^0 \), \( A_E \) contains a \( \Phi_E \)-invariant AF subalgebra \( A_E(v) \),

\[ A_E(v) := \text{span}\{ s_\alpha s_\beta^* \mid r(\alpha) = r(\beta) = v, |\alpha| = |\beta| \}. \]

Lemma 3.6. Let \( v \) be a vertex of an irreducible graph \( E \) with at least two vertices and let \( n \geq 1 \). Then the elements in the set

\[ \omega(n, v) = \{ s_\alpha s_\beta^* \mid r(\alpha) = r(\beta) = v, |\alpha| = |\beta| \leq n \} \]

are linearly independent.

Proof. We prove the assertion by induction on \( n \). For \( n = 1 \), suppose

\[ x = \sum_{\substack{e, f \in E^1 \\atop r(e) = r(f) = v}} \lambda_{e,f} s_es_f^* + \lambda_0 p_v = 0. \]

If \( e_0 \) and \( f_0 \) are edges with \( r(e_0) = r(f_0) = v \) and either \( s(e_0) \neq v \) or \( s(f_0) \neq v \) then \( s_{e_0}^* p_v s_{f_0} = 0 \), hence

\[ 0 = s_{e_0}^* s_{f_0} = \lambda_{e_0 f_0} (s_{e_0}^* s_{e_0})(s_{f_0}^* s_{f_0}) = \lambda_{e_0 f_0} p_v, \]

thus \( \lambda_{e_0 f_0} = 0 \). Similarly, \( \lambda_{e,f} = 0 \) if \( e \) and \( f \) are loops at \( v \) and \( e \neq f \). Then \( x \) becomes

\[ x = \sum_{e \in E^1_v} \lambda_{e,v} s_es_e^* + \lambda_0 p_v = 0. \]

By irreducibility of \( E \) and the assumption that \( |E^0| > 1 \), there exists an edge \( f \) with \( s(f) = v, r(f) \neq v \). Then \( sfs_f^*x = \lambda_0 sfs_f^* = 0 \), so that \( \lambda_0 = 0 \) and we have \( x = \sum_{e \in E^1_v} \lambda_{e,v} s_es_e^* = 0 \). Since the projections \( \{ s_es_e^* \mid e \in E^1_v \} \) are mutually orthogonal, it follows that \( \lambda_{e,v} = 0 \) for each \( e \in E^1_v \).
Now suppose that the assertion is true for $n - 1$. If

$$x = \sum_{\lambda, \beta: |\lambda| = |\beta| \leq n, r(\lambda) = r(\beta) = v} \lambda_{\alpha, \beta} s_{\alpha} s_{\beta}^* = 0, \quad \lambda_{\alpha, \beta} \in \mathbb{C},$$

then for an edge $e \in E^1$ we have

$$0 = s_e^* x s_e = \sum_{\alpha = \alpha', \beta = \beta'} \lambda_{\alpha, \beta} s_{\alpha} s_{\beta}^* s_e = \sum_{|\alpha'| = |\beta'| \leq n - 1} \lambda_{(\alpha\gamma'), (\beta\gamma')} s_{\alpha'} (s_{\beta'})^*.$$

Note that the elements $s_{\alpha'} (s_{\beta'})^*$ appearing in the sum are distinct. Thus by induction hypothesis, one sees that $\lambda_{(\alpha\gamma'), (\beta\gamma')} = 0$. But the edge $e$ was arbitrary, and so we conclude that the coefficients $\lambda_{\alpha, \beta}$ are all zero.

**Proposition 3.7.** (cf. [4, Proposition 2.6]) Let $\Phi: A \to A$ be a contractive cp map of an exact $C^*$-algebra $A$. If $\{\omega_\lambda\}_{\lambda \in A}$ is a net (partially ordered by inclusion) of finite subsets in $A$ such that the linear span of $\bigcup_{\lambda \in \mathbb{Z}^+} \Phi^j(\omega_\lambda)$ is dense in $A$ then

$$ht(\Phi) = \sup_{\lambda} ht(\Phi, \omega_\lambda).$$

**Theorem 3.8.** Let $E$ be an irreducible infinite graph. Then for each $v \in E^0$,

$$ht(\Phi_E|_{A_E(v)}) \leq h_b(tE).$$

**Proof.** Let $A_n(v)$ be the $C^*$-subalgebra of $A_E(v)$ generated by $\omega(n, v)$. Then from

$$s_{\alpha} s_{\beta}^* \cdot s_{\mu} s_{\nu}^* = \begin{cases} s_{\alpha'\mu'} s_{\nu}', & \text{if } \mu = \beta \mu', \\ s_{\alpha} s_{\nu', \mu'}, & \text{if } \beta = \mu \beta', \\ 0, & \text{otherwise}, \end{cases}$$

we see that $A_n(v) = \text{span}(\omega(n, v))$ is finite dimensional.

Since $\{\omega(n, v)\}_{n}$ is an increasing sequence of finite subsets in $A_E(v)$ such that the linear span of $\cup_n \omega(n, v)$ is dense in $A_E(v)$, by Proposition 3.7 it suffices to show that

$$ht(\Phi_E, \omega(n, v)) \leq h_b(tE), \quad n \in \mathbb{N}.$$

Set $E^*_k(v) := \cup_{k \geq 0} E^*_k(v)$ and $r(n) := \cup_{k=0}^n E^*_k(v)$. Fix $n_0 \in \mathbb{N}$, and define a map $\phi: \omega(n_0, v) \to M_{r(n_0)}$ by

$$\phi(s_{\alpha} s_{\beta}^*) = \sum_{|\alpha\gamma| \leq n_0, \gamma \in E^*_k(v)} e_{(\alpha\gamma)(\beta\gamma)};$$

where $\{e_{\mu\nu}\}$ is the standard matrix units of of the matrix algebra $M_{r(n_0)}$. Since the elements in $\omega(n_0, v)$ are linearly independent by Lemma 3.6, one can extend the map to the linear map $\phi: A_{n_0}(v) \to M_{r(n_0)}$. Now we show that $\phi$ is in fact a $*$-isomorphism. To prove that it is a $*$-homomorphism, we only need to see that $\phi((s_{\alpha} s_{\beta}^*)(s_{\mu} s_{\nu}^*)) = \phi(s_{\alpha} s_{\beta}^*) \phi(s_{\mu} s_{\nu}^*)$. 

If $\beta = \mu \beta'$, then $s_\alpha s_\beta^* s_{\mu} s_{\nu} = s_\alpha (s_{\mu} s_{\nu})^*$ and

$$
\phi(s_\alpha s_\beta^*)\phi(s_{\mu} s_{\nu})^* = \sum_{|\alpha\gamma| \leq n_0} e_{(\alpha\gamma)}(\mu \beta' \gamma) \cdot \sum_{|\mu\delta| \leq n_0} e_{(\mu\delta)}(\nu \delta) = \sum_{|\alpha\gamma| \leq n_0} e_{(\alpha\gamma)}(\nu \beta' \gamma) = \phi(s_\alpha (s_{\mu} s_{\nu})^*) = \phi(s_\alpha s_\beta^* s_{\mu} s_{\nu}^*).
$$

If $\mu = \beta \mu'$, a similar proof works. Otherwise, $\phi((s_\alpha s_\beta^*) (s_{\mu} s_{\nu})^*) = 0 = \phi(s_\alpha s_\beta^*) \phi(s_{\mu} s_{\nu})$. In order to show that $\phi$ is injective, let $\phi(\sum_{\alpha, \beta} \lambda_{\alpha \beta} s_\alpha s_\beta^*) = 0$. Then

$$
\sum_{\alpha, \beta} \lambda_{\alpha \beta} \phi(s_\alpha s_\beta^*) = \sum_{\alpha, \beta} \lambda_{\alpha \beta} \left( \sum_{|\alpha\gamma| \leq n_0} e_{(\alpha\gamma)}(\beta \gamma) \right) = 0.
$$

But the vectors, $\sum_{|\gamma| \leq n_0} e_{(\alpha\gamma)}(\beta \gamma)$, $(r(\alpha) = r(\beta) = v, |\alpha| = |\beta| \leq n_0)$, are linearly independent in $M_{r(n_0)}$. In fact, if $A := \sum_{\alpha, \beta} \lambda_{\alpha \beta} (\sum_{|\alpha\gamma| \leq n_0} e_{(\alpha\gamma)}(\beta \gamma)) = 0$, then $e_{vv} A e_{vv} = \lambda_{vv} e_{vv} = 0$, that is, $\lambda_{vv} = 0$, and for any $\alpha, \beta \in E_f(v)$, $e_{\alpha} A e_{\beta} = \lambda_{\alpha \beta} e_{\alpha \beta} = 0$, hence $\lambda_{\alpha \beta} = 0$. Repeating the process one has $\lambda_{\alpha \beta} = 0$ for any $\alpha, \beta \in \cup_{k=0}^{n_0} E_k(v)$. Therefore $\sum_{\alpha, \beta} \lambda_{\alpha \beta} s_\alpha s_\beta^* = 0$, and the map $\phi$ is injective. The surjectivity of $\phi$ follows from $\dim(A_{n_0}(v)) = r(n_0)^2$. We simply write $\phi$ for $\phi : A_{n_0+1}(v) \to M_{r(n_0+1)}$ $(l \geq 0)$, and $\phi$ for its contractive cp extension to $A_E(v)$ that exists by Arveson's extension theorem.

For each $n \in \mathbb{N}$ and $0 \leq l \leq n - 1$, note that

$$
\cup_{i=0}^{n-1} \Phi_E(\omega(n_0, v)) \subseteq \text{span}(\omega(n_0 + n - 1, v)).
$$

Then the element

$$(\widetilde{\phi}, \psi := \phi^{-1}, M_{r(n_0+n-1)}) \in CPA(id, A_E(v))$$

satisfies $\psi \circ \widetilde{\phi} \mid_{\omega(n_0+n-1, v)} = id_{\omega(n_0+n-1, v)}$. Thus for each $\delta > 0$

$$ rcp(id, \omega(n_0 + n - 1, v), \delta) ≤ r(n_0 + n - 1),
$$

and so

$$
ht(\Phi_E|_{A_E(v)}, \omega(n_0, v), \delta) ≤ \limsup_{n \to \infty} \frac{1}{n} \log(r(n_0 + n - 1))
$$

$$
= \limsup_{n \to \infty} \frac{1}{n} \log(r(n))
$$

$$
= \limsup_{n \to \infty} \frac{1}{n} \log \left( \cup_{k=0}^{n} E_k(v) \right)
$$

$$
= h_{l}(tE).
$$

For the last equality, note that if $k \leq n$ then $|E_k(v)| ≤ |E_n(v)|$, hence $|\cup_{k=0}^{n} E_k(v)| \leq (n + 1) \cdot |E_n(v)|$. $\square$

Since $h_l(E) = \sup_{E \subseteq E'} h(X_{E'})$ is well known (see [13 Proposition 7.2.6]), we see from Theorem 2.4 and its proof that $h_l(E) \leq h(\Phi_E|_{A_E})$ holds. The following theorem gives an upper bound for $ht(\Phi_E|_{A_E})$. 
Theorem 3.9. Let $E$ be an irreducible infinite graph and $\mathcal{A}_E$ be the AF subalgebra of $C^*(E)$ generated by the partial isometries $\{s_\alpha s_\beta^* | \alpha, \beta \in E^*, |\alpha| = |\beta|\}$. Then

$$ht(\Phi_E|\mathcal{A}_E) \leq \max\{h_b(E^+), h_b(E)\}.$$  

Proof. Let $E^0 = \{v_1, v_2, \cdots\}$. For each $n_0 \in \mathbb{N}$ and $n_1 \in \mathbb{Z}^+$, put

$$\omega(n_0, n_1) := \left\{s_\alpha s_\beta^* | \alpha, \beta \in E^{n_1}, r(\alpha) = r(\beta) \in \{v_1, \cdots, v_{n_0}\}\right\},$$

$$\omega_{\Sigma}(n_0, n_1) := \left\{\sum s_\alpha s_\beta^* | s_\alpha, s_\beta^* \in \omega(n_0, n_1)\right\}.$$ 

Note that $\omega_{\Sigma}(n_0, n_1)$ is not the linear span of $\omega(n_0, n_1)$. Then $\{\omega_{\Sigma}(n_0, n_1) | n_0 \in \mathbb{N}, n_1 \in \mathbb{Z}^+\}$ is a net of finite subsets in $\mathcal{A}_E$ which is partially ordered by inclusion. In fact, given two finite sets $\omega_{\Sigma}(n_0, n_1), \omega_{\Sigma}(m_0, m_1) (n_1 \leq m_1)$, one may write each element $s_\alpha s_\beta^* \in \omega(n_0, n_1)$ as

$$s_\alpha s_\beta^* = s_\alpha \left(\sum_{|\mu| = m_1 - n_1} s_\mu s_\mu^* s_\beta^* = \sum s_{\alpha \mu} (s_{\beta \mu})^* \in \omega_{\Sigma}(m_2, m_1),$$

where $m_2 > \max\{n_0, m_0\}$ is an integer large enough so that $r(\alpha \mu) \in \{v_1, \cdots, v_{m_2}\}$ for any $\alpha \mu$ appearing in the last sum, then clearly $\omega_{\Sigma}(n_0, n_1) \cup \omega_{\Sigma}(m_0, m_1)$ is contained in $\omega_{\Sigma}(m_2, m_1)$.

Since the linear span of the set $\cup_{n_0, n_1} \phi_E^n(\omega_{\Sigma}(n_0, n_1))$ is dense in $\mathcal{A}_E$, by Proposition 3.7, we show that for each finite set $\omega_{\Sigma}(n_0, n_1)$,

$$ht(\Phi_E, \omega_{\Sigma}(n_0, n_1)) \leq \max\{h_b(E), h_b(E^+)\}.$$ 

If $s_\alpha s_\beta^* \in \omega(n_0, n_1)$, $r(\alpha) = r(\beta) = v$, then for $l \leq n - 1$,

$$\phi_E(s_\alpha s_\beta^*) = \sum_{|\mu| = l} s_{\mu \alpha} s_{\mu \beta}^* = \sum_{|\mu| = l} s_{\mu \alpha} \left(\sum_{|\nu| = n - l} s_{\nu} s_{\nu}^* s_{\nu}^* s_{\mu \beta}^* s_{\nu} = \sum_{|\mu \nu| = n + n_1} s_{\mu \nu} (s_{\beta \nu})^* \right),$$

because $p_v = \sum_{|\nu| = n - l} s_{\nu} s_{\nu}^* \nu$. Hence one sees that

$$\cup_{n=1}^{n_0-1} \Phi_E^l(\omega_{\Sigma}(n_0, n_1)) \subseteq \left\{\sum s_{\mu \nu} (s_{\beta \nu})^* | s_{\alpha} s_{\beta}^* \in \omega(n_0, n_1)\right\}.$$ 

Since the set $\{s_{\mu} s_{\mu}^* | \mu, \nu \in \cup_{i=1}^{n_0} E^{n_1+n}(v_i)\}$ forms a matrix units, it generates the $C^*$-subalgebra of $\mathcal{A}_E$ which is isomorphic to $M_{k_n}$, where $k_n = |\cup_{i=1}^{n_0} E^{n_1+n}(v_i)|$. Let

$$\rho_n : \text{span}\{s_{\alpha} s_{\beta}^* | \alpha, \beta \in \cup_{i=1}^{n_0} E^{n_1+n}(v_i)\} \to M_{k_n}$$

be a *-isomorphism with the inverse $\rho^{-1}$. Then by Arveson’s extension theorem $\rho$ extends to a contractive cp map $\tilde{\rho} : \mathcal{A}_E \to M_{k_n}$, so that we obtain an element $(\tilde{\rho}, \rho^{-1}, M_{k_n}) \in CPA(id, \mathcal{A}_E)$ such that $\|\rho^{-1} \circ \tilde{\rho}(x) - x\| = 0$ if

$$x \in \cup_{n=1}^{n_0-1} \Phi_E^l(\omega_{\Sigma}(n_0, n_1)) \subseteq \text{span}\{s_{\alpha} s_{\beta}^* | \alpha, \beta \in \cup_{i=1}^{n_0} E^{n_1+n}(v_i)\}.$$ 

Hence

$$rcp(\cup_{n=1}^{n_0-1} \Phi_E^l(\omega_{\Sigma}(n_0, n_1)), \delta) \leq k_n$$

holds for any $\delta > 0$. Thus

$$ht(\Phi_E, \omega_{\Sigma}(n_0, n_1)) \leq \limsup_{n \to \infty} \frac{1}{n} \log(k_n).$$
On the other hand, the irreducibility of \( E \) implies that there is an \( N \) such that 
\[
|E^{n_0+n+N}(v_1)| \leq |E^{n_0+n+N}(v_1)| \quad \text{for } 1 \leq i \leq n_0.
\]
Hence \( k_n = \sum_{i=1}^{n_0} E^{n_0+n+N}(v_1) \leq n_0|E^{n_0+n+N}(v_1)| \). Therefore
\[
\limsup_{n \to \infty} \frac{1}{n} \log k_n \leq \limsup_{n \to \infty} \frac{1}{n} \log |E^{n}(v_1)|,
\]
and the assertion then follows from Proposition 3.5(b). \( \square \)

**Example 3.10.** Let \( E := E_{\{v_n\},\{\ell_n\}} \) be a Salama’s infinite irreducible graph (see [13]). We assume here that \( \ell_n + 1 \leq \ell_{n+1} \) for each \( n \). There are \( r_k \) edges from the vertex \( k \) to \( k+1 \), and there is only one path (of length \( l_k - l_{k-1} \)) from the vertex \( v_k \) to \( v_{k-1} \).

\[
\begin{array}{cccccccc}
E & v_1 & l_2 - l_1 & v_2 & l_3 - l_2 & v_3 & l_4 - l_3 & v_4 & \cdots \\
0 & r_1 & r_2 & r_3 & r_4 & \cdots & \cdots & r_5 \\
\end{array}
\]

Note that for each \( n \), 
\[
|E^n(0^*)| \leq |E^n(0^*)|,
\]
which then implies by Proposition 3.2 that 
\[
h_b(\mathcal{A}) \leq h_b(E).
\]

Thus from Theorem 3.9, we have 
\[
h_t(\Phi_E|\mathcal{A}_E) \leq h_b(E).
\]

In particular, if \( E_p := E_{p,p} \) (\( p > 1 \)) is an irreducible infinite graph of Salama satisfying \( h_t(E_p) = h_b(E_p) = \log p \), we have 
\[
h_t(\Phi_{E_p}|\mathcal{A}_{E_p}) = \log p.
\]

**References**

1. N. P. Brown, K. Dykema and D. Shlyakhtenko, *Topological entropy of free product automorphisms*, Acta Math. **189** (2002), 1–35.
2. F. P. Boca and P. Goldstein, *Topological entropy for the canonical endomorphism of Cuntz-Krieger algebras*, Bull. London Math. Soc. **32** (2000), 345–352.
3. T. Bates, D. Pask, I. Raeburn and W. Szymanski, *The C*-algebras of row-finite graphs*, New York J. Math. 6 (2000), 307–324.
4. N. P. Brown, *Topological entropy in exact C*-algebras*, Math. Ann. **314** (1999), 347–367.
5. M. Choda, *Endomorphisms of shift type (entropy for endomorphisms of Cuntz algebras)*, Operator Algebras and Quantum Field Theory (Rome, 1996), 469–475, International Press, Cambridge, MA.
6. J. Cuntz, *Simple C*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.
7. J. Cuntz and W. Krieger, *A class of C*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.
8. V. Deaconu, *Entropy estimates for some C*-endomorphisms*, Proc. Amer. Math. Soc. **127** (1999), no. 12, 3653–3658.
9. K. Dykema, *Topological entropy of some automorphisms of reduced amalgamated free product C*-algebras*, Ergodic Theory Dynam. Systems, **21** (2001), 1683–1693.
10. B. M. Gurevich, *Topological entropy of enumerable Markov chains*, Dokl. Akad. Nauk SSSR **187** (1969), 216–226. Soviet Math. Dokl. **10** (1969), 911–915.
[11] J. A Jeong and G. H Park, Topological entropy for the canonical completely positive maps of graph $C^*$-algebras, Bull. Austral. Math. Soc., to appear.
[12] E. Kirchberg, On subalgebras of the CAR-algebra, J. Funct. Anal. 129 (1995), no. 1, 35–63.
[13] B. P. Kitchens, Symbolic Dynamics, Springer 1998.
[14] A. Kumjian, Notes on $C^*$-algebras of graphs, Contemporary Math. 228, Operator Algebras and Operator Theory, 1998, AMS.
[15] A. Kumjian, D. Pask and I. Raeburn, Cuntz-Krieger algebras of directed graphs, Pacific J. Math. 184 (1998), 161–174.
[16] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press 1999.
[17] C. Pinzari, Y. Watatani and K. Yonetani, KMS states, entropy and the variational principle in full $C^*$-dynamical systems, Commun. Math. Phys. 213 (2000), 331-379.
[18] I. A. Salama, Topological entropy and recurrence of countable chains, Pacific J. Math. 134 (1988), 325–341.
[19] E. Størmer, Entropy of some automorphisms of the $II_1$ factor of the free group in infinite number of generators, Invent. Math. 110 (1992), 63–73.
[20] D. Voiculescu, Dynamical approximation entropies and topological entropy in operator algebras, Comm. Math. Phys. 170 (1995), 249–281.
[21] P. Walters, An introduction to ergodic theory, GTM 79, Springer 1982.

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