Outline

• Fresnel vs. Fraunhofer diffraction
• Talbot effect
• Fresnel zones & plates
• Fraunhofer diffraction
• 2D Fourier Transforms
Fresnel vs. Fraunhofer diffraction

Fresnel: occurs when either S or P are close enough to the aperture that wavefront curvature is not negligible

Fraunhofer: both incident and diffracted waves may be considered to be planar (i.e. both S and P are far from the aperture)
Fresnel vs. Fraunhofer criterion

\[ \Delta \approx \frac{h^2}{2r'} \approx \frac{h^2}{2p} \]

**view from source:**

\[ \Delta \approx \frac{h^2}{2r} \approx \frac{h^2}{2q} \]

**view from point of interest:**

**near field** ≡

\[ \Delta > \lambda \]

\[ \frac{1}{2} \left( \frac{1}{p} + \frac{1}{q} \right) h^2 > \lambda \]

where \( d \) represents \( p \) or \( q \) (=distance from source or point to aperture)

\[ d < \frac{A}{\lambda} \]

\( A \) is aperture area
Fraunhofer diffraction occurs when:

\[ F = \frac{h^2}{d\lambda} \ll 1 \]

Fresnel diffraction occurs when:

\[ F = \frac{h^2}{d\lambda} \geq 1 \]

where \( h \) = aperture or slit size
\( \lambda \) = wavelength
\( d \) = distance from the aperture (\( p \) or \( q \))
From Fresnel to Fraunhofer diffraction

Incident plane wave

\[ F >> 1 \]

\[ F << 1 \]
Fresnel diffraction from infinite array of slits: Talbot effect

- one of the few Fresnel diffraction problems that can be solved analytically
- beam pattern alternates between two different fringe patterns
Talbot “carpet”
Fresnel zones (180° phase difference)

Fresnel’s approach to diffraction from circular apertures

Zone spacing = \( \lambda/2 \):

- \( r_1 = r_0 + \lambda/2 \)
- \( r_2 = r_0 + \lambda \)
- \( r_3 = r_0 + 3\lambda/2 \)
- \( \vdots \)
- \( r_n = r_0 + n\lambda/2 \)

These are called the Fresnel zones

(Note: all zones have equal areas)
Adding up light from the zones

as we draw a phasor diagram where each zone is subdivided into 15 subzones

\[ a_1 = A_1 \]

- obliquity factor shortens successive phasors
- circles do not close, but spiral inwards
- amplitude \( a_1 = A_1 \): resultant of subzones in 1st half-period zone
- composite amplitude at \( P \) from \( n \) half-period zones:

\[
A_n = a_1 + a_2 e^{i\pi} + a_3 e^{i2\pi} + a_4 e^{i3\pi} + \ldots + a_n e^{i(n-1)\pi}
\]

\[
A_n = a_1 - a_2 + a_3 - a_4 + \ldots a_n
\]
Some interesting implications of Fresnel zones

A circular aperture is matched in size with the first Fresnel zone:

*What is amplitude of the wavefront at P?*

\[ A_P = a_1 \]

Now open the aperture wider to also admit zone 2:

\[ A_P \sim 0! \]

Now remove aperture, allowing all zones to contribute:

\[ A_P = \frac{1}{2} a_1 !!! \]

(To find intensity – square the amplitudes, i.e. it’s only \(\frac{1}{4}\) of the 1st zone!)
Some interesting implications of Fresnel zones

A circular disk is matched in size with the first Fresnel zone:

What is amplitude of the wavefront at $P$?

- all zones except the first contribute
- first contributing zone is the second

$$A_P = \frac{1}{2} a_2$$

Irradiance at center of shadow nearly the same as without the disk present!

How absurd!

Siméon Denis Poisson (1781-1840)
Poisson/Arago spot

Francois Arago (1786-1853)
The Fresnel zone plate

$A_n = a_1 - a_2 + a_3 - a_4 + \ldots a_n$

If the 2nd, 4th, 6th, etc. zones are blocked, then:

$A_{16} = a_1 + a_3 + a_5 + a_7 + a_9 + a_{11} + a_{13} + a_{15}$

Amplitude at $P$ is 16 times the amplitude of $a_1/2$

Irradiance at $P$ is $(16)^2$ times! (a.k.a. focusing)
An alternative to blocking zones

Phases of adjacent Fresnel zones changed by $\pi$
Fresnel lighthouse lens

other applications:    overhead projectors
                      automobile headlights
                      solar collectors
                      traffic lights
Back to Fraunhofer diffraction

- Typical arrangement (or use laser as a source of plane waves)
- Plane waves in, plane waves out

![Diagram of Fraunhofer diffraction](image)
Fraunhofer diffraction

\[ g_{\text{out}}(x', y'; z) = \frac{1}{i\lambda z} \exp \left\{ i2\pi \frac{z}{\lambda} \right\} \iint g_{\text{in}}(x, y) \exp \left\{ i\pi \frac{(x' - x)^2 + (y' - y)^2}{\lambda z} \right\} \, dx \, dy \]

\[ g_{\text{out}}(x', y'; z) = \frac{1}{i\lambda z} \exp \left\{ i2\pi \frac{z}{\lambda} \right\} \iint g_{\text{in}}(x, y) \exp \left\{ i\pi \frac{x'^2 + x^2 - 2xx' + y'^2 + y^2 - 2yy'}{\lambda z} \right\} \, dx \, dy \]

\[ \approx \exp \left\{ i2\pi \frac{z}{\lambda} + i\pi \frac{x'^2 + y'^2}{\lambda z} \right\} \iint g_{\text{in}}(x, y) \exp \left\{ -i2\pi \frac{xx' + yy'}{\lambda z} \right\} \, dx \, dy \]

\[ |...| = 1 \]

\[ g_{\text{out}}(x', y'; z) \approx \exp \left\{ i2\pi \frac{z}{\lambda} + i\pi \frac{x'^2 + y'^2}{\lambda z} \right\} \iint g_{\text{in}}(x, y) \exp \left\{ -i2\pi \left( ux' + uy' \right) \right\} \, dx \, dy \]
Fraunhofer diffraction $\propto$ Fourier Transform: Rectangular aperture

\[ g_{in}(x, y) = \text{rect} \left( \frac{x}{x_0} \right) \text{rect} \left( \frac{y}{y_0} \right) \]

\[ G_{in}(u, v) = x_0 y_0 \text{sinc} \left( x_0 u \right) \text{sinc} \left( y_0 v \right) \]

\[ g_{out}(x', y'; z \to \infty) \propto \text{sinc} \left( \frac{x_0 x'}{\lambda z} \right) \text{sinc} \left( \frac{y_0 y'}{\lambda z} \right) \]

Free space propagation by $l \to \infty$
Circular aperture

\[ g_{in}(x, y) = \text{circ} \left( \frac{\sqrt{x^2 + y^2}}{r_0} \right) \]

\[ G_{in}(u, v) = r_0^2 \text{jinc} \left( r_0 \sqrt{u^2 + v^2} \right) \]

\[ \equiv r_0 J_1 \left( \frac{2\pi \sqrt{u^2 + v^2}}{\sqrt{u^2 + v^2}} \right) g_{out}(x', y'; z \to \infty) \propto \text{jinc} \left( \frac{2\pi r_0 \sqrt{x'^2 + y'^2}}{\lambda z} \right) \]

Input field

Far-field

Airy pattern

free space propagation by

\[ l \to \infty \]
Fourier transform pair

\[ G(\nu) = \int_{-\infty}^{+\infty} g(t) \exp \{ -i2\pi \nu t \} \, dt. \]

1D

\[ g(t) = \int_{-\infty}^{+\infty} G(\nu) \exp \{ i2\pi \nu t \} \, d\nu. \]

\[ G(u, v) = \iint_{-\infty}^{+\infty} g(x, y) \exp \{ -i2\pi (ux + vy) \} \, dx \, dy. \]

2D

\[ g(x, y) = \iint_{-\infty}^{+\infty} G(u, v) \exp \{ i2\pi (ux + vy) \} \, dudv. \]
Spatial domain ↔ (angular) frequency domain

\[ \cos \left( 2\pi \frac{x}{\Lambda} \right) \]

\[ \frac{1}{2} \delta \left( u + \frac{1}{\Lambda} \right) \delta (v) + \frac{1}{2} \delta \left( u - \frac{1}{\Lambda} \right) \delta (v) \]

Space domain

Frequency (Fourier) domain
Tilted grating

\[
\cos \left( 2\pi \frac{x \sin \theta - y \cos \theta}{\Lambda} \right)
\]

\[
\frac{1}{2} \delta \left( u + \frac{\sin \theta}{\Lambda} \right) \delta \left( v - \frac{\cos \theta}{\Lambda} \right) + \frac{1}{2} \delta \left( u - \frac{\sin \theta}{\Lambda} \right) \delta \left( v + \frac{\cos \theta}{\Lambda} \right)
\]

Space domain

Frequency (Fourier) domain
Linear superposition

\[ a_1 \cos \left( \frac{2\pi x}{\Lambda_1} \right) + a_2 \cos \left( \frac{2\pi x}{\Lambda_2} \right) \]

\[ \frac{a_1}{2} \delta \left( u + \frac{1}{\Lambda_1} \right) \delta (v) + \frac{a_2}{2} \delta \left( u + \frac{1}{\Lambda_2} \right) \delta (v) + \]

\[ \frac{a_1}{2} \delta \left( u - \frac{1}{\Lambda_1} \right) \delta (v) + \frac{a_2}{2} \delta \left( u - \frac{1}{\Lambda_2} \right) \delta (v) \]
Scaling

\[ \mathcal{F} \left\{ g \left( \frac{x}{a}, \frac{y}{b} \right) \right\} = |ab| G(au, bv) \]
Shift theorem

\[ \mathcal{F} \{ g(x - a, y - b) \} = \exp \{ i2\pi(au + bv) \} \ G(u, v) \]
The convolution theorem

\[ \mathcal{F}\{f \ast g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\} \quad \text{or} \quad \mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} \ast \mathcal{F}\{g\} \]
Links/references

http://edu.tnw.utwente.nl/inlopt/overhead_sheets/Herek2010/week7/13.Fresnel%20diffraction.ppt

http://ocw.mit.edu/courses/mechanical-engineering/2-71-optics-spring-2009/video-lectures/lecture-17-fraunhofer-diffraction-fourier-transforms-and-theorems/MIT2_71S09_lec17.pdf