Scalable Quantum Networks based on Few-Qubit Registers

Liang Jiang\textsuperscript{1}, Jacob M. Taylor\textsuperscript{1,2}, Anders S. Sørensen\textsuperscript{3}, Mikhail D. Lukin\textsuperscript{1}

\textsuperscript{1} Department of Physics, Harvard University, Cambridge, Massachusetts 02138
\textsuperscript{2} Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139 and
\textsuperscript{3} Quantop and The Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen \textdegree{}, Denmark

(Dated: April 1, 2022)

We describe and analyze a hybrid approach to scalable quantum computation based on an optically connected network of few-qubit quantum registers. We show that probabilistically connected few-qubit quantum registers suffice for deterministic, fault-tolerant quantum computation even when state preparation, measurement, and entanglement generation all have substantial errors. We discuss requirements for achieving fault-tolerant operation for two specific implementations of our approach.

The key challenge in experimental quantum information science is to identify isolated quantum mechanical systems with good coherence properties that can be manipulated and coupled together in a scalable fashion. Substantial progress has been made towards the physical implementation of few-qubit quantum registers using systems of coupled trapped ions \textsuperscript{[1, 2, 3, 4]}, superconducting islands \textsuperscript{[5, 6]}, solid-state qubits based on electronic spins in semiconductors \textsuperscript{[7]}, and color centers in diamond \textsuperscript{[8, 9, 10, 11]}. While the precise manipulation of large, multi-qubit systems still remains an outstanding challenge, approaches for connecting such few qubit registers into large scale circuits are currently being explored both theoretically \textsuperscript{[12, 13, 14, 15, 16, 17]} and experimentally \textsuperscript{[18, 19]}. Of specific importance are approaches which can yield fault-tolerant operations with minimal resources and realistic (high) error rates.

In Ref. \textsuperscript{[13]} a novel technique to scalable quantum computation was suggested, where high fidelity local operations can be used to correct low fidelity non-local operations, using techniques that are currently being explored for quantum communication \textsuperscript{[20, 21, 22]}. In this Letter, we present a hybrid approach, which requires only 5 (or fewer)-qubit registers with local deterministic coupling, while providing additional improvements over the earlier protocol \textsuperscript{[13]}: reduced measurement errors, higher fidelity, and more efficient entanglement purification. The small registers are connected by optical photons, which enables non-local coupling gates and reduces the requirement for fault tolerant quantum computation \textsuperscript{[24]}. Specifically, we analyze two physical systems where this approach is very effective. We consider an architecture where pairwise non-local entanglement can be created in parallel, as indicated in Fig. 1. This is achieved via simultaneous optical excitation of the selected register pairs followed by photon-detection in specific channel. We use a Markov chain analysis to estimate the overhead in time and operational errors, and discuss the feasibility of large scale, fault-tolerant quantum computation using this approach.

The present work is motivated by experimental advances in two specific physical systems. Recent experiments have demonstrated quantum registers composed of few trapped ions, which can support high-fidelity local operations \textsuperscript{[2, 3, 4]}. The ion qubits can couple to light efficiently \textsuperscript{[24]} and were recognized early for their potential in an optically coupled component \textsuperscript{[13, 14]}. Probabilistic entanglement of remote ion qubits mediated by photons has also been demonstrated \textsuperscript{[25, 26]}. At the same time, few-qubit quantum registers have been recently implemented in high-purity diamond samples \textsuperscript{[9, 10, 11]}. Here, quantum bits are encoded in individual nuclear spins, which are extraordinarily good quantum memories \textsuperscript{[11]} and can also be manipulated with high precision using techniques from NMR \textsuperscript{[27]}. The electronic spin associated with a nitrogen-vacancy (NV) color center enables addressing and polarization of nuclei, and entanglement generation between remote registers. While for systems of trapped ions there exist several approaches for coupling remote few-qubit registers (such as those based on moving the ions \textsuperscript{[28]}), for NV centers in diamond it is difficult to conceive a direct construction of large scale multi-qubit systems without major advances in fabrication technology. For the latter scenario the hybrid approach developed here is required. Furthermore the use of light has the major advantage that it allows for con-
gate-teleportation circuit between registers $R$. When the communication qubits (an exponentially decreasing chance of continued failure. In contrast, local unitary operations may fail frequently ($p_L \lesssim 10^{-4}$) when quantum control techniques for small quantum system are utilized [2, 27]. We now show that the most important sources of imperfections, such as imperfect initialization, measurement errors for individual qubits in each quantum register, and entanglement generation errors between registers, can be corrected with a modest increase in register size. We determine that with just three additional auxiliary qubits and high-fidelity local unitary operations, all these errors can be efficiently suppressed by bit-verification and entanglement purification [20, 21]. This provides an extension of Ref. [13] that mostly focused on suppressing errors from entanglement generation.

We are assuming in the following a separation of error probabilities: any internal, unitary operation of the register fails with low probability, $p_L$, while all operations connecting the communication qubit to the outside world (initialization, measurement, and entanglement generation) fail with error probabilities that can be several orders of magnitude higher. For specificity, we set these error probabilities to $p_I$, $p_M$, and $1 - F$, respectively. In terms of these quantities the error probability in the non-local C-NOT gate circuit is of order $p_{CNOT} \sim (1 - F) + 2p_L + 2p_M$. We now show how this fidelity can be greatly increased.

Robust measurement can be implemented by bit-verification: a majority vote among the measurement outcomes (Fig. 2a), following a sequence of C-NOT operations between the auxiliary/storage qubit and the communication qubit. This also allows robust initialization by measurement. High-fidelity robust entanglement generation is achieved via entanglement purification [13, 20, 21] (Fig. 2bc), in which lower fidelity entanglement between the communication qubits is used to purify entanglement between the auxiliary qubits, which can then be used for the remote C-NOT operation. To make the most efficient use of physical qubits, we introduce a new two-level entanglement pumping scheme. Our circuit (Fig. 2b) uses raw Bell pairs to repeatedly purify (“pump”) against bit-errors, then the bit-purified Bell pairs are used to pump against phase-errors (Fig. 2c).

Entanglement pumping, like entanglement generation, is probabilistic; however, failures are detected. Still, in computation, where each logical gate should be completed within the allocated time (clock cycle), failed entanglement pumping can lead to gate failure. To demonstrate the feasibility of our approach for quantum computation, we next analyze the time required for robust initialization, measurement and entanglement generation, and show that the failure probability for these procedures can be made sufficiently small with reasonable time over-

FIG. 2: (color online). Circuits for robust operations. (a) Robust measurement of the auxiliary/storage qubit, $a/s$, based on majority vote from $2m+1$ outcomes of the communication qubit, $c$. Robust measurement is denoted by the box shown in the upper left corner. (b)(c) Using entanglement pumping to create high fidelity entangled pairs between two registers $R^i$ and $R^j$. If the two outcomes are the same, it is a successful step of pumping; otherwise generate new pairs and restart the pumping operation from the beginning. The two circuits are for the first level pumping and the second level pumping, purifying bit- and phase-errors, respectively.

necting qubits over long distances, which reduces the requirement for fault-tolerant quantum computation [23].

We define a quantum register as a few-qubit device that contains one communication qubit, with a photonic interface; one storage qubit, with very good coherence times; and several auxiliary qubits, used for purification and error correction (described below).

The simplest quantum register requires only two qubits: one for storage and the other for communication. Entanglement between two remote registers may be generated using probabilistic approaches from quantum communication [22] and references therein). In general, such entanglement generation produces a Bell state of the communication qubits from different registers, conditioned on certain measurement outcomes. If state generation fails, it can be re-attempted until success, with an exponentially decreasing chance of continued failure. When the communication qubits ($c^1$ and $c^2$) are prepared in the Bell state, we can immediately perform the remote C-NOT gate on the storage qubits ($s^1$ and $s^2$) using the gate-teleportation circuit between registers $R^1$ and $R^2$. This can be accomplished [14, 31, 32] via a sequence of local C-NOTs within each register, followed by measurement of two communication qubits and subsequent local rotations. Since arbitrary rotations on a single qubit can be performed within a register, the C-NOT operation between different quantum registers is in principle sufficient for universal quantum computation. Similar approaches are also known for deterministic generation of graph states [33] – an essential resource for one-way quantum computation [34].

In practice, the qubit measurement, initialization, and entanglement generation can be fairly noisy with error probabilities as high as a few percent, due to practical limitations such as finite collection efficiency and poor interferometric stability. As a result the corresponding error probability in non-local gate circuit will also be very high. In contrast, local unitary operations may fail frequently ($p_L \lesssim 10^{-4}$) when quantum control techniques for small quantum system are utilized [2, 27]. We now show that the most important sources of imperfections, such as imperfect initialization, measurement errors for individual qubits in each quantum register, and entanglement generation errors between registers, can be corrected with a modest increase in register size. We determine that with just three additional auxiliary qubits and high-fidelity local unitary operations, all these errors can be efficiently suppressed by bit-verification and entanglement purification [20, 21]. This provides an extension of Ref. [13] that mostly focused on suppressing errors from entanglement generation.

We are assuming in the following a separation of error probabilities: any internal, unitary operation of the register fails with low probability, $p_L$, while all operations connecting the communication qubit to the outside world (initialization, measurement, and entanglement generation) fail with error probabilities that can be several orders of magnitude higher. For specificity, we set these error probabilities to $p_I$, $p_M$, and $1 - F$, respectively. In terms of these quantities the error probability in the non-local C-NOT gate circuit is of order $p_{CNOT} \sim (1 - F) + 2p_L + 2p_M$. We now show how this fidelity can be greatly increased.

Robust measurement can be implemented by bit-verification: a majority vote among the measurement outcomes (Fig. 2a), following a sequence of C-NOT operations between the auxiliary/storage qubit and the communication qubit. This also allows robust initialization by measurement. High-fidelity robust entanglement generation is achieved via entanglement purification [13, 20, 21] (Fig. 2bc), in which lower fidelity entanglement between the communication qubits is used to purify entanglement between the auxiliary qubits, which can then be used for the remote C-NOT operation. To make the most efficient use of physical qubits, we introduce a new two-level entanglement pumping scheme. Our circuit (Fig. 2b) uses raw Bell pairs to repeatedly purify (“pump”) against bit-errors, then the bit-purified Bell pairs are used to pump against phase-errors (Fig. 2c).

Entanglement pumping, like entanglement generation, is probabilistic; however, failures are detected. Still, in computation, where each logical gate should be completed within the allocated time (clock cycle), failed entanglement pumping can lead to gate failure. To demonstrate the feasibility of our approach for quantum computation, we next analyze the time required for robust initialization, measurement and entanglement generation, and show that the failure probability for these procedures can be made sufficiently small with reasonable time over-
head.

The measurement circuit shown in Fig. 2a, yields the correct result based on majority vote from $2m + 1$ consecutive readouts (bit-verification). Since the evolution of the system (C-NOT gate) commutes with the measured observable (Z operator) of the auxiliary/storage qubit, it is a quantum non-demolition (QND) measurement, which can be repeated many times. The error probability for majority vote measurement scheme is:

$$
\varepsilon_M \approx \left( \frac{2m + 1}{m + 1} \right) (p_l + p_M)^{m+1} + \frac{2m + 1}{2} p_L. 
$$

Suppose $p_l = p_M = 5\%$, we can achieve $\varepsilon_M \approx 8 \times 10^{-4}$ by choosing $m^* = 6$ for $p_L = 10^{-4}$, or even $\varepsilon_M \approx 12 \times 10^{-6}$ for $m^* = 10$ and $p_L = 10^{-6}$. Recently, measurement with very high fidelity ($\varepsilon_M$ as low as $6 \times 10^{-4}$) has been demonstrated in the ion-trap system [33], using similar ideas as above. The time for robust measurement is

$$
t_M = (2m + 1) (t_I + t_L + t_M),
$$

where $t_I$, $t_L$, and $t_M$ are times for initialization, local unitary gate, and measurement, respectively.

We now use robust measurement and entanglement generation to perform entanglement pumping. Suppose the raw Bell pairs have initial fidelity $F = \langle \Phi^+ \rangle \langle \Phi^+ \rangle$ due to depolarizing error. We apply two-level entanglement pumping. The first level has $n_b$ steps of bit-error pumping using raw Bell pairs (Fig. 2a) to produce a bit-error-purified entangled pair. The second level uses these bit-error-purified pairs for $n_p$ steps of phase-error pumping (Fig. 2b).

For successful purification, the infidelity of the purified pair, $\varepsilon_E^{(n_b,n_p)}$, depends on both the control parameters $(n_b, n_p)$ and the imperfection parameters $(F, p_L, \varepsilon_M)$. For depolarizing error, we find

$$
\varepsilon_E^{(n_b,n_p)} \approx \frac{3 + 2n_b p_L}{4} + \frac{4 + 2(n_b + n_p)}{3} (1 - F) \varepsilon_M + (n_p + 1) \left( \frac{2 (1 - F)}{3} \right)^{n_b + 1} + \left( \frac{n_b + 1}{3} (1 - F) \right)^{n_b + 1}
$$

to the leading order of $p_L$ and $\varepsilon_M$. The dependence on the initial infidelity $1 - F$ is exponentially suppressed at the cost of a linear increase of error from local operations $p_L$ and robust measurement $\varepsilon_M$. Measurement-related errors are suppressed by the prefactor $1 - F$, since measurement error does not cause infidelity unless combined with other errors. In the limit of ideal operations ($p_L, \varepsilon_M \rightarrow 0$), the infidelity $\varepsilon_E^{(n_b,n_p)}$ can be arbitrarily close to zero [33]. On the other hand, if we use the standard entanglement pumping scheme [20, 21] (that alternates purification of bit and phase errors within each pumping level), the reduced infidelity from two-level pumping is always larger than $(1 - F)^2 / 9$. Therefore, for very small $p_L$ and $\varepsilon_M$, the new pumping scheme is crucial to minimize the number of qubits per register.

The overall success probability can be defined as the joint probability that all successive steps succeed. We use the model of finite-state Markov chain [34] to directly calculate the failure probability of $(n_b, n_p)$-two-level entanglement pumping using $N_{tot}$ raw Bell pairs, denoted as $\varepsilon_E^{(n_b,n_p)} (N_{tot})$. See Ref. [20] for detailed analysis.

For given $F$, $p_L$, and $\varepsilon_M$, the purified pair has minimum infidelity $\Delta_{min} = \varepsilon_E^{(n_b,n_p)}$, obtained by the optimal choice of the control parameters $(n_b^*, n_p^*)$. Then, we calculate the typical value for $N_{tot}$, by requiring the failure probability and the minimum infidelity to be equal, $\varepsilon_E^{(n_b,n_p)} (N_{tot}) = \Delta_{min}$. The total error probability is

$$
\varepsilon_E \approx \varepsilon_E^{10^3,10^2} (N_{tot}) + \Delta_{min} = 2 \Delta_{min}.
$$

We remark that a faster and less resource intensive approach may be used if the unpurified Bell pair is dominated by dephasing error. And one-level pumping may be sufficient (i.e. no bit-error purification, $n_b = 0$). The total time for robust entanglement generation $t_E$ is

$$
t_E \approx \langle N_{tot} \rangle \times (t_I + t_L + t_M),
$$

where $t_E$ is the average generation time of the unpurified Bell pair.

Figure 3 shows the contours of $\varepsilon_E$ and $N_{tot}$ with respect to the imperfection parameters $p_L$ and $1 - F$. We assume $p_l = p_M = 5\%$ for the plot. The choice of $p_L$ and $p_M$ ($< 10\%$) has marginal effect to the contours, since they only modifies $\varepsilon_M$ marginally. For initial fidelity $F_0 > 0.95$, the contours of $\varepsilon_E$ are almost vertical;
that is $\varepsilon_E$ is mostly limited by $p_L$ with an overhead factor of about 10. The contours of $N_{\text{bar}}$ indicate that the entanglement pumping needs about tens or hundreds of raw Bell pairs to ensure a very high success probability.

We introduce the clock cycle time $t_C = \bar{t}_E + 2 t_L + \bar{t}_M \approx \bar{t}_E$ and the effective error probability $\gamma = \varepsilon_E + 2 p_L + 2 \varepsilon_M$ for general coupling gate between two registers, which can be compared with a similar approach as the remote C-NOT gate [22]. We now provide an estimate of clock cycle time based on realistic parameters. The time for optical initialization/measurement is $t_I = t_M \approx \frac{\ln p_{\text{EF}}}{\ln(1 - \eta)} \bar{\varepsilon}$, with photon collection/detection efficiency $\eta$, vacuum radiative lifetime $\tau$, and the Purcell factor $C$ for cavity-enhanced radiative decay. We assume that entanglement is generated based on detection of two photons [29], which takes time $t_E \approx (t_I + \tau/C)/\eta^2$. If the bit-errors are efficiently suppressed by the intrinsic purification of the entanglement generation scheme, one-level pumping is sufficient; otherwise two-level pumping is needed. Suppose the parameters are $(t_L, \tau, \eta, C) = (0.1 \mu s, 10 ns, 0.2, 10)$ [38, 39, 40] and $(1 - F, p_L, p_M, p_L, \varepsilon_M) = (5\%, 5\%, 5\%, 10^{-3}, 12 \times 10^{-6})$. For depolarizing errors, two-level pumping can achieve $(t_C, \gamma) = (997 \mu s, 4.5 \times 10^{-5})$. If all bit-errors are suppressed by the intrinsic purification of the coincidence scheme, one-level pumping is sufficient and $(t_C, \gamma) = (140 \mu s, 3.4 \times 10^{-5})$. Finally, $t_C$ should be much shorter than the memory time of the storage qubit, $t_{\text{mem}}$. This is indeed the case for both trapped ions (where $t_{\text{mem}} \sim 10$ s has been demonstrated [41, 42]) and proximal nuclear spins of NV centers (where $t_{\text{mem}}$ approaches 1 s [11, 30].

This approach yields gates between quantum registers to implement arbitrary quantum circuits. Errors can be further suppressed by using quantum error correction. For example, as shown in Fig. 3, $(p_L, F) = (10^{-4}, 0.95)$ can yield $\gamma \leq 2 \times 10^{-3}$, well below the 1% threshold for fault tolerant computation for approaches such as the $C_4/C_6$ code [43] or 2D toric codes [43]: $(p_L, F) = (10^{-6}, 0.95)$ can achieve $\gamma \leq 5 \times 10^{-5}$, which allows efficient codes such as the BCH [127,43,13] code to be used without concatenation. Following Ref. [45], we estimate 10 registers per logical qubit to be necessary for a calculation involving $10^4$ logical qubits and $10^6$ logical gates.

In conclusion, we have analyzed a hybrid approach to fault-tolerant quantum computation with optically coupled few-qubit quantum registers. We further note that it is possible to facilitate fault-tolerant quantum computation with special operations from the hybrid approach such as partial Bell measurement [36] or with systematic optimization using dynamic programming [40].

The authors wish to thank Gurudev Dutt, Lily Childress, Paola Cappellaro, Phillip Hemmer, and Charles Marcus. This work is supported by NSF, DTo, ARO-MURI, the Packard Foundations, Pappalardo Fellowship, and the Danish Natural Science Research Council.

[1] J. I. Cirac and P. Zoller, Physics Today 57, 38 (2004).
[2] D. Leibfried, et al., Nature (London) 422, 412 (2003).
[3] H. Haffner, et al., Nature (London) 438, 643 (2005).
[4] R. Reichle, et al., Nature (London) 443, 838 (2006).
[5] T. Yamamoto, et al., Nature (London) 421, 343 (2003).
[6] A. Wallraff, et al., Nature (London) 431, 162 (2004).
[7] J. R. Petta, et al., Science 309, 2180 (2005).
[8] J. Wrachtrup, et al., Opt. Spectros. 91, 429 (2001).
[9] F. Jelezko, et al., Phys. Rev. Lett. 93, 130501 (2004).
[10] L. Childress, et al., Science 314, 281 (2006).
[11] M. V. G. Dutt, et al., Science 316, 1312 (2007).
[12] J. I. Cirac, et al., Phys. Rev. Lett. 78, 3221 (1997).
[13] W. Dür and H. J. Briegel, Phys. Rev. Lett. 90, 067901 (2003).
[14] L. M. Duan, et al., Quantum Inf. Comput. 4, 165 (2004).
[15] Y. L. Lim, et al., Phys. Rev. A 73, 012304 (2006).
[16] D. K. L. Oi, et al., Phys. Rev. A 74, 052313 (2006).
[17] R. van Meter, et al., quant-ph/0607160 (2006).
[18] T. Legero, et al., Phys. Rev. Lett. 93, 070503 (2004).
[19] K. M. Birnbaum, et al., Nature (London) 436, 87 (2005).
[20] H.-J. Briegel, et al., Phys. Rev. Lett. 81, 5932 (1998).
[21] W. Dür, et al., Phys. Rev. A 59, 169 (1999).
[22] L. Childress, et al., Phys. Rev. A 72, 052330 (2005), Phys. Rev. Lett. 96, 070504 (2006).
[23] K. M. Svore, et al., Phys. Rev. A 72, 022317 (2005).
[24] B. B. Blinov, et al., Nature (London) 428, 153 (2004).
[25] D. L. Moehring, et al., Nature (London) 449, 68 (2007).
[26] P. Maunz, et al., Nature Phys. 3, 538 (2007).
[27] L. M. K. Vandersypen and I. L. Chuang, Rev. Mod. Phys. 76, 1037 (2004).
[28] D. Kielpinski, et al., Nature (London) 417, 709 (2002).
[29] L. M. Duan and H. J. Kimble, Phys. Rev. Lett. 90, 253601 (2003). C. Simon and W. T. M. Irvine, Phys. Rev. Lett. 91, 110405 (2003).
[30] C. Kim, et al., Ieee Journal of Selected Topics in Quantum Electronics 13, 322 (2007).
[31] D. Gottesman and I. L. Chuang, Nature (London) 402, 390 (1999). A. Sorensen and K. Mølmer, Phys. Rev. A 58, 2745 (1998). X. Zhou, D. W. Leung, and I. L. Chuang, Phys. Rev. A 62, 052316 (2000).
[32] J. Eisert, et al., Phys. Rev. A 62, 052317 (2000).
[33] S. C. Benjamin, et al., N. J. Phys. 8, 141 (2006).
[34] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[35] D. B. Hume, T. Rosenband and D. J. Wineland, Phys. Rev. Lett. 99, 120502 (2007).
[36] L. Jiang, et al., arXiv: 0709.4539 (2007).
[37] S. P. Meyn and R. L. Tweedie, Markov Chains and Stochastic Stability (Springer-Verlag, New York, 1993).
[38] J. J. García-Ripoll, P. Zoller, and J. I. Cirac, Phys. Rev. Lett. 91, 157901 (2003).
[39] M. Keller, et al., Nature (London) 431, 1075 (2004).
[40] T. Steinmetz, et al., App. Phys. Lett. 89, 111110 (2006).
[41] C. Langer, et al., Phys. Rev. Lett. 95, 060502 (2005).
[42] H. Haffner, et al., Appl. Phys. B 81, 151 (2005).
[43] E. Knill, Nature (London) 434, 39 (2005).
[44] R. Raussendorf and J. Harrington, Phys. Rev. Lett. 98, 190504 (2007).
[45] A. M. Steane, Phys. Rev. A 68, 042322 (2003).
[46] L. Jiang, et al., Proc. Natl. Acad. Sci. U. S. A. 104, 17291 (2007).