Specify a randomized algorithm that, given a very large graph or network, extracts a random subgraph. What can we learn about the input graph from a single subsample? We derive laws of large numbers for the sampler output, by relating randomized subsampling to distributional invariance: Assuming an invariance holds is tantamount to assuming the sample has been generated by a specific algorithm. That in turn yields a notion of ergodicity. Sampling algorithms induce model classes—graphon models, sparse generalizations of exchangeable graphs, and random multigraphs with exchangeable edges can all be obtained in this manner, and we specialize our results to a number of examples. One class of sampling algorithms emerges as special: Roughly speaking, those defined as limits of random transformations drawn uniformly from certain sequences of groups. Some known pathologies of network models based on graphons are explained as a form of selection bias.

1. Introduction. Consider a large graph or network, and invent a randomized algorithm that generates a subgraph. The algorithm can be understood as a model of an experimental design—a protocol used to collect data in a survey, or to sample data from a network—or as an actual program extracting data from a database. Use the algorithm to extract a sample graph, small relative to input size. What information can be obtained from a single such sample? Certainly, that should depend on the algorithm. We approach the problem starting from a simple observation: Fix a sequence $y_n$ with $n$ entries. Generate a random sequence $X_k$ by sampling $k$ elements of $y_n$, uniformly and independently with replacement. Then $X_k$ is exchangeable, and it remains so under a suitable distributional limit $n \to \infty$ in input size. Similarly, one can generate an exchangeable graph (a random graph whose law is invariant under permutations of the vertex set) by sampling vertices of an input graph independently, and extracting the induced subgraph. The resulting class of random graphs is equivalent to graphon models [14, 15, 22]. Exchangeability is an example of distributional symmetry, that is, invariance of a distribution under a class of transformations [36]. Thus, a randomized algorithm (independent selection of elements) induces a symmetry principle (exchangeability of elements), and when applied to graphs, it also induces a model class (graphon models). The purpose of this work is to show how the interplay of these properties answers what can be learned from a single subgraph, both for the example above and for other algorithms.

1.1. Overview. Start with a large graph $y$. The graph may be unadorned, or a “network” in which each edge and vertex is associated with some mark or observed value. We always assume that the “initial subgraph of size $n$”, denoted $y|_n$, is unambiguously defined. This may be the induced subgraph on the first $n$ vertices (if vertices are enumerated), the subgraph incident to the first $n$ edges (if edges are enumerated), the neighborhood of size $n$ around a fixed root, et cetera; details follow in Section 2. Now invent a randomized al-
algorithm that generates a graph of size $k < n$ from the input $y|_n$, and denote this random graph $S_{n \to k}(y)$. For example:

**Algorithm 1.**

i.) Select $k$ vertices of $y|_n$ independently and uniformly without replacement.

ii.) Extract the induced subgraph $S_{n \to k}(y|_n)$ of $y|_n$ on these vertices.

iii.) Label the vertices of $S_{n \to k}(y|_n)$ by $1, \ldots, k$ in order of appearance.

We assume $y$ is so large that it is modeled as infinite, and hence ask for a limit in input size: Is there a random variable $S_{\infty}(y)$ that can be regarded as a sample of infinite size from an infinite input graph $y$? That is, a variable such that, for each output size $k$, the restriction $S_{\infty}(y)|_k$ is the distributional limit in input size $n$,

$$S_{n \to k}(y) \xrightarrow{d} S_{\infty}(y)|_k \quad \text{as} \quad n \to \infty.$$  

Necessary and sufficient conditions are given in Theorem 1. By sampling and passing to the limit, some information about $y$ is typically lost—$y$ cannot be reconstructed precisely from the sample output, which is of some importance to our purposes; see Remark 2.

The main problem we consider is inference from a single realization: By a model on $X$, we mean a set $\mathcal{P}$ of probability measures on $X$. Observed is a single graph $X_k$ of size $k$. We choose a sampling algorithm as a modeling assumption on how the data was generated, and take its limit as above. For a given set $\mathcal{Y}$ of input graphs, the algorithm induces a model $\mathcal{P} := \{P_y | y \in \mathcal{Y}\}$ where $P_y := L(S_{\infty}(y))$.

The observed graph $X_k$ is modeled as a sample $X_k = S_k(y)$ of size $n$ from an infinite input graph. Even as $n \to \infty$, this means that only a single draw from the model distribution is available. When $k$ is finite, this is further constrained to a partial realization.

We have already observed the sampler output $S_{\infty}(y)$ is exchangeable for certain algorithms, or more generally invariant under some family of transformations of $X$. A fundamental principle of ergodic theory is that invariance under a suitable family of transformations $T$ defines a set $\text{Erg}(T)$ of distinguished probability measures, the $T$-ergodic measures. Section 4 reviews the relevant concepts. The utility of invariance to our problem is that ergodic measures of suitable transformation families are distinguishable by a single realization: If a model is chosen as a subset $\mathcal{P}$ of these $T$-ergodic measures, and a single draw $X \sim P$ from some (unknown) element $P$ of $\mathcal{P}$ is observed, the distribution $P$ is unambiguously determined by $X$. The following consequence is made precise in Section 5:

*If the limiting sampler can be shown to have a suitable invariance property, then inference from a single realization is possible in principle.*

We make the provision in principle as the realization $S_{\infty}(y)$ only determines $P_y$ abstractly—for an actual algorithm, there is no obvious way to derive $P_y$ from an observed graph. We hence ask under what conditions the expectations

$$\mathbb{E}[f(S_{\infty}(y))] = P_y(f) \quad \text{for some } f \in L_1(P_y)$$

can be computed or approximated given an observed realization. (Here and throughout, we use the notation $P(f) = \int f dP$.) Theorem 5 provides the following answer: If $T$ is
specifically a group satisfying certain properties, one can choose a specific sequence of finite subsets \( A_k \) of \( T \) and define

\[
F^k_x := \frac{1}{|A_k|} \sum_{t \in A_k} \delta_{t(x)} \quad \text{hence} \quad F^k_x(f) = \frac{1}{|A_k|} \sum_{t \in A_k} f(t(x)) .
\]

The sequence \( (F^k_x) \) can be thought of as an empirical measure, and satisfies a law of large numbers: If a sequence \( (f_k) \) of functions converges almost everywhere to a function \( f \), then

\[
F^k_{S_\infty(y)}(f_k) \xrightarrow{k \to \infty} E[f(S_\infty(y))] \tag{1}
\]

under suitable conditions on the sampler. Theorem 5 formulates such convergence as a law of large numbers for symmetric random variables, and subsumes several known results on exchangeable structures, graphons, etc.

The graph \( S_\infty(y) \) in (1) is typically infinite. To work with a finite sample \( S_k(y) \), we formulate additional conditions on \( T \) that let transformations act on finite structures, which leads to a class of groups we call prefix actions. We then define sampling algorithms by randomly applying a transformation: Fix a prefix action \( T \), draw a random transformation \( \Phi_n \) uniformly from those elements of \( T \) that affect only a subgraph of size \( n \), and define

\[
S_{n \to k}(y) = \Phi_n(y|_n)|_k .
\]

In words, randomly transform \( y|_n \) using \( \Phi_n \) and then truncate at output size \( k < n \). These algorithms turn out to be of particular interest: They induce various known models—graphons, edge-exchangeable graphs, and others—and generically satisfy \( T \)-invariance and other non-trivial properties; see Theorem 9. The law of large numbers strengthens to

\[
F^k_{S_k(y)}(f_k) \xrightarrow{k \to \infty} E[f(y)] . \tag{2}
\]

In contrast to (1), the approximation is now a function of a finite sample of size \( k \), and the right-hand side a functional of the input graph \( y \), rather than of \( S_\infty(y) \). See Corollary 10.

With the general results in place, we consider specific algorithms. In some cases, the algorithm induces a known class of random graphs as its family \( \{P_y|y \in Y\} \) of possible output distributions; see Table 1. Section 8 concerns Algorithm 1, exchangeable graphs, and graphon models. We consider modifications of Algorithm 1, and show how known misspecification problems that arise when graphons are used to model network data can be explained as a form of selection bias. Section 9 relates well-known properties of exchangeable

| sec. | sampling scheme | symmetry | induced model class |
|------|----------------|---------|-------------------|
| 8.1  | \( k \) independent vertices | vertices exchangeable | graphon models [14] |
| 8.4  | select vertices by coin flips + delete isolated vertices | underlying point process exchangeable | generalized graphons [16, 48] |
| 10.2 | \( k \) independent edges (input multigraph) | edges exchangeable | “edge-exchangeable graphs” [21, 18] |
| 11.2 | neighborhood of random vertex | involution invariance | certain local weak limits [8] |

Table 1
sequences and partitions to algorithms sampling from fixed sequences and partitions. That serves as preparation for Section 10, on algorithms that select a random sequence of edges, and report the subgraph incident to those edges. If the input graph \( y \) is simple, a property of \( y \) can be estimated from the sample output if and only if it is a function of the degree sequence of \( y \). If \( y \) is a multigraph and the limiting relative multiplicities of its edges sum to 1, the algorithm generates edge-exchangeable graphs in the sense of [21, 18, 33]. The two cases differ considerably: For simple input graphs, the sample output is completely determined by vertex degrees, for multigraphs by edge multiplicities. If a sampling algorithm explores a graph by following edges—as many actual experimental designs do, see e.g. [40]—the stochastic dependence between edges tends to become more complicated, and understanding symmetries of such algorithms is much harder. Section 11 puts some previously known properties of methods algorithms that sample neighborhoods in the context of this work.

1.2. Related work.\(^1\) Related previous work largely falls into two categories: One concerns random graphs, exchangeability, graph limits, and related topics. This work is mostly theoretical, and intersects probability, combinatorics, and mathematical statistics [14, 15, 22, 36, 5, 10, 27, 39, 45]. A question closely related to the problem considered here—what probabilistic symmetries aside from exchangeability of vertices are applicable to networks analysis problems—was posed in [45]. One possible solution, due to Caron and Fox [19], is to require exchangeability of an underlying point process. This idea can be used to generalize graph limits to sparse graphs [48, 16]. Another answer are random multigraphs whose edges, rather than vertices, are exchangeable [21, 18, 33]. These are exchangeable partitions, in the sense of Kingman [38], of the upper triagonal of the set \( \mathbb{N}^2 \). The second related category of work covers experimental design in networks, and constitutes a substantial literature, see [40] for references. This literature tends to be more applied, although theoretical results have been obtained [e.g. 41]. The two bodies of work are largely distinct, with a few notable exceptions, such as the results on identifiability problems in [20].

The specific problem considered here—the relationship between sampling and symmetry—seems largely unexplored, but Aldous reasons about exchangeability in terms of uniform sampling in [2], and, in joint work with Lyons [1], extends the work of Benjamini and Schramm [8] from a symmetry perspective. (Kallenberg [34] and other authors use the term sampling differently, for the explicit generation of a draw from a suitable representation of a distribution.) More closely related from a technical perspective than experimental design in networks are two ideas popular in combinatorics. One is property testing, which samples uniform parts of a large random structure \( X \), and then asks with what probability \( X \) is in a certain class; see [4]. The second is used to define convergence of discrete structures: Start with a set \( \mathcal{X} \) of such structures, and equip it with some initial metric (so convergence in distribution is defined). Invent a randomized algorithm that generates a “substructure” \( S(x) \) of a fixed structure \( x \in \mathcal{X} \), and call a sequence \( (x_n) \) convergent if the laws \( \mathcal{L}(S(x_n)) \) converge weakly. The idea is exemplified by [8], but seems to date back further, and is integral to the construction of graph limits: The topology on dense graphs defined in this manner by Algorithm 1 is metrizable, by the “cut metric” of Frieze and Kannan [14, 15].

\(^1\)The ideas proposed here are used explicitly in forthcoming work of Veitch and Roy [49] and Borgs, Chayes, Cohn, and Veitch [17], both already available as preprints. Cf. Section 8.4.
2. Spaces of graphs and discrete structures. Informally, we consider a space $X$ of infinite structures, spaces $X_n$ of finite substructures of size $n$, and a map $x \mapsto x|_n$ that takes a structure of size $\geq n$ to its initial substructure of size $n$. For example, if $X$ (resp. $X_n$) consists of labeled graphs vertex set $\mathbb{N}$ (resp. $[n]$), then $\bullet|_n$ may map to the induced subgraph on the first $n$ vertices. More formally, these objects are defined as follows: Let $X_n$, for $n \in \mathbb{N}$, be countable sets. Require that for each pair $m \leq n$, there is a surjective map

$$(3) \quad \bullet|_n : X_n \to X_m \quad \text{such that} \quad x|_m|_k = x|_k \quad \text{if} \quad x \in X_n \quad \text{and} \quad k \leq m \leq n.$$ 

We write $x_m \preceq x_n$ whenever $x_n|_m = x_m$. In words, $x_m$ is a substructure of $x_n$. An infinite sequence

$$(4) \quad x := (x_1, x_2, \ldots) \quad \text{with} \quad x_n \in X_n \quad \text{and} \quad x_n \preceq x_{n+1} \quad \text{for all} \quad n \in \mathbb{N}$$

can then be regarded as a structure of infinite size. The set of all such sequences is denoted $X$. The maps $\bullet|_n$ can be extended to $X$ by defining $x|_n = x_n$ if $x = (x_1, x_2, \ldots)$ as above.

If each point in $X$ is an infinite graph, two natural ways to measure size of subgraphs is by counting vertices, or by counting edges. Since the notion of size determines the definition of the restriction map $x \mapsto x|_n$, the two lead to different types of almost discrete spaces:

(i) Counting vertices. Choose $X$ as the set of graphs a given type (e.g. simple and undirected) with vertex set $\mathbb{N}$, and $X_n$ as the analogous set of graphs with vertex set $[n]$. The restriction map $x \mapsto x|_n$ extracts the induced subgraph on the first $n$ vertices, i.e. $x|_n$ is the graph with vertex set $[n]$ that contains those edges $(i, j)$ of $x$ with $i, j \leq n$. Graph size is the cardinality of the vertex set.

(ii) Counting edges. A graph $x$ with vertices in $\mathbb{N}$ is represented as a sequence of edges, $x = ((i_1, j_1), (i_2, j_2), \ldots)$, where $i_k, j_k \in \mathbb{N}$. Each set $X_n \subset (\mathbb{N}^2)^n$ consists of all graphs with $n$ edges, $x_n = ((i_1, j_1), \ldots, (i_n, j_n))$, and vertex set $\{i_k, j_k | k \leq n\} \subset \mathbb{N}$. The restriction map

$$(5) \quad x \mapsto x|_n := ((i_1, j_1), \ldots, (i_n, j_n))$$

extracts the first $n$ edges, and graphs size is cardinality of the edge set.

To define probabilities on $X$ requires a notion of measurability, and hence a topology. We endow each countable set $X_n$ with its canonical, discrete topology, and $X$ with the smallest topology that makes all maps $x \mapsto x|_n$ continuous. A topological space constructed in this manner is called procountable. Any procountable space admits a canonical “prefix metric”

$$(6) \quad d(x, x') := \inf_{n \in \mathbb{N}} \{2^{-n} | x|_n = x'|_n \},$$

which is indeed an ultrametric. The ultrametric space $(X, d)$ is complete. An almost discrete space $(X, d)$ is a procountable space that is separable, and hence Polish. Throughout, all sets of infinite graphs are almost discrete spaces, or subsets of such spaces. If every set $X_n$ is finite, $X$ is called profinite (or Boolean, or a Stone space) [28]. A space is profinite iff it is almost discrete and compact. A random element $X$ of $X$ is defined up to almost sure equivalence by a sequence $X_1, X_2, \ldots$ satisfying (4) almost surely. A probability measure $P$ on $X$
Our first main result shows the necessary condition that $B$ can be collected into a vector, exist for all $x$ for all $y$. For that to be the case, it is certainly necessary that the limits variable $S_n |\{\text{an} \}$. This representation can be refined to represent a measure on topological subspaces of $X$. Since each sampling step augments the previously sampled graph, we require these random variables to cohere accordingly, as

$$S_{n \to k} : Y_n \times [0,1]^k \to X_k,$$

which we will often read as an $X_k$-valued random variable $S_{n \to k}(y_n)$, parametrized by $y_n$. Since each sampling step augments the previously sampled graph, we require these random variables to cohere accordingly, as

$$S_{n \to k}(y_n) \preceq S_{n \to k+1}(y_n) \quad \text{almost surely.}$$

It suffices to require that $S_{n \to k}(y)$ exists for $n$ sufficiently large: For each $y$ and $k$, there is an $n(k)$ such that $S_{n \to k}(y)$ is defined for all $n \geq n(k)$. A sampling algorithm is a family $(S_{n \to k})_{k \in \mathbb{N}, n \geq n(k)}$ as in (7) that satisfies (8).

To explain sampling from an infinite graph $y \in \mathcal{Y}$, we ask whether there is a limiting variable $S_k(y)$ such that convergence $S_{n \to k}(y|_n) \to S_k(y)$ holds in distribution as $n \to \infty$. For that to be the case, it is certainly necessary that the limits

$$t_{x_k}(y) := \lim_{n \to \infty} \mathbb{P}_{\text{alg}} \{ S_{n \to k}(y|_n) = x_k \}$$

exist for all $x \in X, k \in \mathbb{N}$. We call the limit $t_{x_k}(y)$ a prefix density. These prefix densities can be collected into a vector,

$$t(y) := (t_{x_k}(y))_{k \in \mathbb{N}, x_k \in X_k} \quad \text{which is a measurable map} \quad t : \mathcal{Y} \to [0,1]^{\cup_k X_k}.$$ 

Our first main result shows the necessary condition that $t$ exists is indeed sufficient:

**Theorem 1.** Let $\mathcal{Y}$ be an almost discrete space, and $\mathcal{Y}$ any subset, equipped with the restriction $\mathcal{B}(\mathcal{Y})$ of the Borel sets of $\mathcal{Y}$. Let $S = (S_{n \to k})$ be a sampling algorithm $\mathcal{Y} \to X$. If the prefix densities $t(y)$ exist on $\mathcal{Y}$, there exists a jointly measurable function

$$S_\infty : \mathcal{Y} \times [0,1] \to X \quad \text{satisfying} \quad S_{n \to k}(y|_n, U) \xrightarrow{d} S_\infty(y, U)|_k \quad \text{as } n \to \infty$$

for all $y \in \mathcal{Y}$ and $k \in \mathbb{N}$.
There is hence a random variable $S_\infty(y)$, with values in $X$, which can be interpreted as an infinite or “asymptotically large” sample from an infinite input graph $y$. Each restriction $S_k(y) = S_\infty(y)|_k$ represents a sample of size $k$ from $y$. If repeated application of sampling preserves the output distribution, i.e. if

$$S_{m \to k}(S_m(y)) \overset{d}{=} S_k(y) \quad \text{and} \quad S_{m \to k}(S_{n \to m}(y)) \overset{d}{=} S_{n \to k}(y) \quad \text{whenever} \quad k \leq m \leq n,$$

we call the algorithm idempotent.

**Remark 2.** The limit in output size $k$ is an inverse limit: A growing graph is assembled as in (4), and all information in $S_k$ can be recovered from $S_\infty$. In contrast, the limit in input size is distributional, so the input graph $y$ can typically not be reconstructed, even from an infinite sample $S_\infty(y)$. The limit of Algorithm 1, for example, will output an empty graph if $y$ has a finite number of edges, or indeed if the number of edges in $y|_n$ grows sub-quadratically in $n$. We regard $S$ as a measurement of properties of a “population” (see [40] for a discussion of populations in network problems, and [45] for graph limits as populations underlying exchangeable graph data). An infinitely large sample $S_\infty(y)$ makes asymptotic statements valid, in the sense that any effect of finite sample size can be made arbitrarily small, but does not exhaust the population.

4. **Background: Invariance and symmetry.** We use the term *invariance* to describe preservation of probability distributions under a family of transformations; we also call an invariance a *symmetry* if this family is specifically a group. Let $X$ be a standard Borel space, and $\mathcal{T}$ a family of measurable (but not necessarily invertible) transformations $X \to X$. A random element $X$ of $X$ is $\mathcal{T}$-invariant if its law remains invariant under every element of $\mathcal{T}$,

$$(9) \quad t(X) \overset{d}{=} X \quad \text{for all} \quad t \in \mathcal{T}.$$  

Analogously, a probability measure $P$ is $\mathcal{T}$-invariant if the image measure $t(P) = P \circ t^{-1}$ satisfies $t(P) = P$ for all $t \in \mathcal{T}$. We denote the set of all $\mathcal{T}$-invariant probability measures on $X$ by $\text{Inv}(\mathcal{T})$. It is trivially convex, though possibly empty if the family $\mathcal{T}$ is “too large”.

4.1. **Ergodicity.** Inference from a single instance relies on the concept of ergodicity. A Borel set $A \subset X$ is invariant if $t(A) = A$ for all $t \in \mathcal{T}$, and almost invariant if

$$P(A \triangle t^{-1}A) = 0 \quad \text{for all} \quad t \in \mathcal{T} \quad \text{and} \quad P \in \text{Inv}. $$

We denote the system of all invariant sets $\sigma(\mathcal{T})$, and that of all almost invariant sets $\overline{\sigma}(\mathcal{T})$. Both are $\sigma$-algebras. Recall that a probability $P$ is trivial on a $\sigma$-algebra $\sigma$ if $P(A) \in \{0, 1\}$ for all $A \in \sigma$. For a probability measure $P$ on $X$, we define:

$$P \text{ is } \mathcal{T}\text{-ergodic} \iff P \text{ is } \mathcal{T}\text{-invariant, and trivial on } \overline{\sigma}(\mathcal{T}).$$

The set of all ergodic measures is denoted $\text{Erg}(\mathcal{T})$. 
4.2. Groups and symmetries. We reserve the term symmetry for invariance under transformation families $T$ that form a group. A useful concept in this context is the notion of a group action: For example, if $X$ is a space of graphs, a group of permutations may act on a graph $x \in X$ by permuting its vertices, by permuting its edges, by permuting certain subgraphs, etc. Such different effects of one and the same group can be formalized as maps $T(\phi, x)$ that explain how a permutation $\phi$ affects the graph $x$. Formally, let $G$ be a group, with unit element $e$. An action of $G$ on $X$ is a map $T: G \times X \to X$, customarily denoted $T_\phi(x) = T(\phi, x)$, with the properties

(i) $T_e(x) = x$ for all $x \in X$ and
(ii) $T_\phi \circ T_{\phi'} = T_{\phi \phi'}$ for all $\phi, \phi' \in G$.

If $G$ is equipped with a topology, and with the corresponding Borel sets, $T$ is a measurable action if it is jointly measurable in both arguments. Any measurable action $T$ on $X$ defines a family $\mathcal{T} := T(G) := \{T_\phi | \phi \in G\}$ of transformations $X \to X$. Clearly, each element of $\mathcal{T}$ is a bimeasurable bijection, and $\mathcal{T}$ is again a group. The orbit of an element $x$ of $X$ under a group action is the set $T_G(x) := \{T_\phi(x), \phi \in G\}$. The orbits of $T$ form a partition of $X$ into disjoint sets. If there exists a Polish topology on $G$ that makes $T$ measurable, each orbit is a measurable set [7, §2.3].

4.3. Characterization of ergodic components. The main relevance of ergodicity to our purposes is that, informally, the elements of a model $P$ chosen as a subset $P \subset \text{Erg}(\mathcal{T})$ can be distinguished from one another by means of a single realization, provided that $\mathcal{T}$ is not too complex. In other words, if a random variable $X$ is assumed to be distributed according to some distribution in $P$, then a single draw from $X$ determines this distribution unambiguously within $P$. That is a consequence of the ergodic decomposition theorem, whose various shapes and guises are part of mathematical folklore. To give a reasonably general statement, we have to formalize that $\mathcal{T}$ be “not too complex”: Call $\mathcal{T}$ separable if it is jointly measurable in both arguments. Any measurable action $T$ on $X$ defines a family $\mathcal{T} := T(G) := \{T_\phi | \phi \in G\}$ of transformations $X \to X$. Clearly, each element of $\mathcal{T}$ is a bimeasurable bijection, and $\mathcal{T}$ is again a group. The orbit of an element $x$ of $X$ under a group action is the set $T_G(x) := \{T_\phi(x), \phi \in G\}$. The orbits of $T$ form a partition of $X$ into disjoint sets. If there exists a Polish topology on $G$ that makes $T$ measurable, each orbit is a measurable set [7, §2.3].

(10) $\text{INV}(\mathcal{T}_0) = \text{INV}(\mathcal{T})$.

If so, we call $\mathcal{T}_0$ a separating subset. Criteria for verifying separability are reviewed in Appendix A. If $\mathcal{T}_0$ is separating, both the ergodic probability measures and the almost invariant sets defined by the two families coincide,

(11) $\text{ERG}(\mathcal{T}_0) = \text{ERG}(\mathcal{T})$ and $\text{\sigma}(\mathcal{T}_0) = \text{\sigma}(\mathcal{T})$.

The following form of the decomposition theorem is amalgamated from [25, 29, 43]:

Theorem 3 (Folklore). Let $\mathcal{T}$ be a separable family of measurable transformations of a standard Borel space $X$. Then the $\mathcal{T}$-ergodic measures are precisely the extreme points of $\text{INV}(\mathcal{T})$, and for every pair $P \neq P'$ of ergodic measures, there is a set $A \in \text{\sigma}(\mathcal{T})$ such that $P(A) = 1$ and $P'(A) = 0$. A random element $X$ of $X$ is $\mathcal{T}$-invariant if and only if there is a random probability measure $\xi$ on $X$ such that

(12) $\xi \in \text{ERG}(\mathcal{T})$ and $P[X \in \cdot | \xi] = \xi(\cdot)$ almost surely. If so, the law of $\xi$ is uniquely determined by the law of $X$. 

The decomposition theorem can be understood as a generalization of the representation theorems of de Finetti, of Aldous and Hoover, and similar results: In expectation, the almost sure identity (12) takes the weaker but more familiar form

\[ P(\cdot) = \int_{\text{Erg}(T)} \nu(\cdot) \mu_\xi(d\nu) \quad \text{where } \mu_\xi := \mathcal{L}(\xi) . \]

In de Finetti's theorem, the ergodic measures are the laws of i.i.d. sequences; in the Aldous-Hoover theorem applied to simple, undirected, exchangeable graphs, they are those distributions represented by graphons; etc. The generality of Theorem 3 comes at a price: The theorems of de Finetti, Kingman, and Aldous-Hoover provide constructive representations of (12): There is a collection \((U_i)\) of independent, uniform random variables on \([0,1]\), and a class of measurable mappings \(H\), such that each ergodic random element \(X\) can be represented as \(X \overset{d}{=} h((U_i))\) for some \(h \in H\). The representation is non-trivial in that each finite substructure \(X|_k\) can be represented analogously by a finite subset of the collection \((U_i)\). Kallenberg [36] calls such a representation a \textbf{coding}. Existence of a coding can be much harder to establish than (12), and not all invariances seem to admit codings.

4.4. Definitions of exchangeability. The term \textit{exchangeability} generically refers to invariance under an action of either the finitary symmetric group \(S_F\), or of the infinite symmetric group \(S_N\) of all bijections of \(\mathbb{N}\). For the purposes of Theorem 3, both definitions are typically equivalent: The group \(S_N\) inherits its natural topology from the product space \(\mathbb{N}^\mathbb{N}\), which makes the subgroup \(S_F\) a dense subset. If \(T\) is a continuous action of \(S_N\) on a metrizable space, the image \(T(S_F)\) hence lies dense in \(T(S_N)\) in pointwise convergence, which in turn implies \(T(S_F)\) is a separating subset for \(T(S_N)\) (see Appendix A.2).

In terms of their orbits, the two actions differ drastically: If \(X = \{0,1\}^\infty\), for example, and \(T\) permutes sequence indices, each orbit of \(S_F\) is countable. Not so for \(S_N\): Let \(A \subset X\) be the set of sequences containing both an infinite number of 0s and of 1s. For any two \(x, x' \in A\), there exists a bijection \(\phi \in S_N\) with \(x' = T_\phi(x)\). The set \(A\) thus constitutes a single, uncountable orbit of \(S_N\), which is complemented by a countable number of countable orbits. That illustrates the role of almost invariant sets: By de Finetti’s theorem, the ergodic measures are factorial Bernoulli laws. For all Bernoulli parameters \(p \in (0,1)\), these concentrate on \(A\), and \(A\) does not subdivide further into strictly invariant sets. In other words, \(\sigma(S_N)\) does not provide sufficient resolution to guarantee mutual singularity in Theorem 3, but the almost invariant sets \(\tilde{\sigma}(S_N)\) do. Vershik [50] gives a detailed account. Unlike Theorem 3, more explicit results like the law of large numbers in Section 6 rely on the orbit structure, and must be formulated in terms of \(S_F\).

5. Sampling and symmetry. We now consider the fundamental problem of drawing conclusions from a single observed instance in the context of sampling. For now, we assume the entire, infinite output graph \(S_\infty(y)\) is available. Consider a sampling algorithm \(S\), with input set \(Y \subset \mathcal{Y}\) and output space \(X\), defined as in Section 3, whose prefix densities \(t(y)\) exist for all \(y \in \mathcal{Y}\). We generically denote its output distributions

\[ P_y := \mathcal{L}(S_\infty(y)) . \]
Suppose a model $P$ is chosen as a subset of $\{P_y | y \in Y\}$. Can two elements $P_y, P_{y'} \in P$ be distinguished from another given a single sample $S_\infty(y)$? That can be guaranteed only if
\begin{equation}
P_y(A) = 1 \quad \text{and} \quad P_{y'}(A) = 0 \quad \text{for some Borel set } A \subset X.
\end{equation}
To decide more generally which distribution in $\{P_y | y \in Y\}$ (and hence which input graph $y$) accounts for $S_\infty(y)$, we define
\begin{equation}
\Sigma := \bigcap_{y \in Y} \Sigma_y \quad \text{where} \quad \Sigma_y := \{ A \in \mathcal{B}(X) | P_y(A) \in \{0, 1\} \}.
\end{equation}
Then $\Sigma$ is a $\sigma$-algebra. From (13), we conclude:

*Determining the input graph based on a single realization of $S_\infty$ is possible if the output laws $P_y$ are pairwise distinct on $\Sigma$.*

The sampling algorithm does not typically preserve all information provided by the input graph, due to the distributional limit defining $S_\infty$. Thus, demanding that all pairs $y \neq y'$ be distinguishable may be too strong a requirement. The $\sigma$-algebra $\Sigma$ defines a natural equivalence relation on $Y$,
\begin{equation}
y \equiv_S y' :\iff P_y(A) = P_{y'}(A) \text{ for all } A \in \Sigma.
\end{equation}
More colloquially, $y \equiv_S y'$ means $y$ and $y'$ cannot be distinguished given a single realization $S_\infty(y)$. We note $y \equiv_S y'$ does not generally imply $P_y = P_{y'}$: The measures may be distinct, but detecting that difference may require multiple realizations. We call the algorithm *resolvent* if
\begin{equation}
y \equiv_S y' \quad \text{implies} \quad P_y = P_{y'}.
\end{equation}
Let $\hat{y}$ denote the equivalence class of $y$. If $S$ is resolvent, we can define $P_{\hat{y}} := P_y$, and formulate the condition above as
\begin{equation}
P_{\hat{y}} \text{ and } P_{\hat{y}'} \text{ are mutually singular on } \Sigma \text{ whenever } \hat{y} \neq \hat{y}'.
\end{equation}
Establishing mutual singularity requires identifying a suitable system of sets $A$ in (13), which can be all but impossible: Since $X$ is Polish, each measure $P_y$ has a unique support (a smallest, closed set $F$ with $P_y(F) = 1$), but these closed support sets are *not* generally disjoint. To satisfy (13), $A$ is chosen more generally as measurable, but unlike the closed support, the measurable support of a measure is far from unique. One would hence have to identify a (possibly uncountable) system of not uniquely determined sets, each chosen just so that (13) holds pairwise.

If an invariance holds, Theorem 3 solves the problem. That motivates the following definition: A measurable action $T$ of a group $G$ on $X$ is a *symmetry* of the algorithm $S$ if all output distributions $P_y$ are $G$-ergodic. If $G$ is countable, that is equivalent to demanding
\begin{equation}
(i) \quad T_\phi(S_\infty(y)) \overset{d}{=} S_\infty(y) \quad \text{for all } y \in Y, \phi \in G \quad \text{and} \quad (ii) \quad \sigma(G) \subset \Sigma.
\end{equation}
If $G$ is uncountable, (ii) must be strengthened to $\sigma(G) \subset \Sigma$. Clearly, an algorithm that admits a separable symmetry is resolvent; thus, symmetry guarantees (15). We note mutual singularity could be deduced without requiring $T$ is a group action; this condition anticipates the law of large numbers in Section 6.
5.1. A remark: What can be said without symmetry. If we randomize the input graph by substituting a random element \( Y \) of \( \mathcal{Y} \) with law \( \nu \) for \( y \), the resulting output distribution is the mixture \( \int P_y \nu(dy) \). We define the set of all such laws as \( M := \{ \int P_y \nu(dy) \mid \nu \in \text{PM}(\mathcal{Y}) \} \), where \( \text{PM}(\mathcal{Y}) \) is the space of probability measures on \( \mathcal{Y} \). Clearly, \( M \) is convex, with the laws \( P_y \) as its extreme points. Without further assumptions, we can obtain the following result. It makes no appeal to invariance, and cannot be deduced from Theorem 3 above.

**Proposition 4.** Let \( S \) be a sampling algorithm with prefix densities. Then for every \( P \in M \), there exists a measurable subset \( Q_P \subset \text{PM}(\mathcal{X}) \) of probability measures such that all measures in \( Q_P \) are (i) mutually singular and (ii) 0–1 on \( \Sigma \). There exists a random probability measure \( \xi_P \) on \( \mathcal{X} \) such that \( \xi_P \in Q_P \) and \( P[X \in \cdot \mid \xi_P] = \xi_P \) almost surely.

A structure similar to Theorem 3 is clearly recognizable. That said, the result is too weak for our purposes: The set of \( Q_P \) of representing measures depends on \( P \), which means it cannot be used as a model, and the result does not establish a relationship between the measure \( P_y \) and the elements of \( Q_P \). Note it holds if, but not only if, \( P \in M \).

6. Symmetric laws of large numbers. Consider a similar setup as above: A random variable \( X \) takes values in a standard Borel space \( \mathcal{X} \), and its distribution \( P \) is invariant under a measurable action \( T \) of a group \( G \). Let \( f : \mathcal{X} \to \mathbb{R} \) be a function in \( L^1(\mathcal{X}) \). If \( T \) is separable, Theorem 3 shows that \( X \) is generated by drawing an instance of \( \xi \)—that is, by randomly selecting an ergodic measure—and then drawing \( X \mid \xi \sim \xi \). The expectation of \( f \) given the instance of \( \xi \) that generated \( X \) is

\[
\xi(f) = \mathbb{E}[f(X) \mid \xi] = \mathbb{E}[f(x) \mid \sigma(G)] \quad \text{a.s.}
\]

Again by Theorem 3, observing \( X \) completely determines the instance of \( \xi \). In principle, \( X \) hence completely determines \( \mathbb{E}[f(X) \mid \sigma(G)] \). These are all abstract quantities, however; is it possible to compute \( \mathbb{E}[f(X) \mid \sigma(G)] \) from a given instance of \( X \)?

If the group is finite, the elementary properties of conditional expectations imply

\[
\mathbb{E}[f(X) \mid \sigma(G)] = \frac{1}{|G|} \sum_{\phi \in G} f(T_\phi(X)) \quad \text{almost surely},
\]

so \( \xi(f) \) is indeed given explicitly. The groups arising in the context of sampling are typically countably infinite. In this case, the average on the right is no longer defined. It is then natural to ask whether \( \xi(f) \) can be approximated by finite averages, i.e. whether there are finite sets \( A_1, A_2, \ldots \subset G \) such that

\[
\frac{1}{|A_k|} \sum_{\phi \in A_k} f(T_\phi(X)) \xrightarrow{n \to \infty} \mathbb{E}[f(X) \mid \sigma(G)] \quad \text{almost surely}.
\]

Since \( \mathbb{E}[f(X) \mid \sigma(G)] \) is invariant under each \( \phi \in G \), each average on the left must be invariant at least approximately: A necessary condition for convergence is certainly that, for any \( \phi \in G \), the relative size of the displacement \( \phi A_k \Delta A_k \) can be made arbitrarily small by choosing \( k \) large. That is formalized in the next condition, (17)(i).
A countable group is **amenable** if there is a sequence $A_1, A_2, \ldots$ of finite subsets of $G$ with the property: For some $c > 0$ and all $\phi \in G$, 
\[
(17) \quad (i) \quad |\phi A_k \cap A_k| \xrightarrow{n \to \infty} |A_k| \quad \text{and} \quad (ii) \quad |\bigcup_{j<k} A_j^{-1} A_k| \leq c|A_k| \quad \text{for all } k \in \mathbb{N}.
\]

A sequence $(A_k)$ satisfying (i) is called **almost invariant**. This first condition turns out to be the crucial one: If a sequence satisfying (i) exists, it is always possible to find a sequence satisfying (i) and (ii), by passing to a suitable subsequence if necessary [42, Proposition 1.4]. Thus, $G$ is amenable if it contains a sequence satisfying (i). Amenable groups arise first and foremost in ergodic theory [e.g. 24], but also, for example, in hypothesis testing, as the natural class of groups satisfying the Hunt-Stein theorem [13]. If (17) holds, and $T$ is a measurable action of $G$ on $X$, we call the measurable mapping 
\[
(x, k) \mapsto F^x_k(\cdot) := \frac{1}{|A_k|} \sum_{\phi \in A_k} \delta_{T^k \phi}(x)(\cdot)
\]

an **empirical measure** for the action $T$.

**Theorem 5.** Let $X$ be a random element of a Polish space $X$, and $f, f_1, f_2, \ldots$ functions $X \to \mathbb{R}$ in $L_1(X)$, such that $f_k \to f$ almost surely under the law of $X$. Let $T$ be a measurable action of a countable group satisfying (17), and $F$ the empirical measure defined by $(A_k)$. If $X$ is invariant under $T$, then 
\[
F^X_k(f_k) \xrightarrow{n \to \infty} \xi(f) \quad \text{almost surely},
\]

where $\xi$ is the random ergodic measure in (12). If moreover there is a function $g \in L_1(X)$ such that $|f_k| \leq g$ for all $k$, convergence in (19) also holds in $L_1(X)$.

The finitary symmetric group $S_F$ satisfies (17) for $A_k := S_k$. The law of large numbers (19) hence holds generically for any “exchangeable random structure”, i.e. for any measurable action of $S_F$. Special cases include the law of large numbers for de Finetti’s theorem, the continuity of Kingman’s correspondence [46, Theorem 2.3], and Kallenberg’s law of large numbers for exchangeable arrays [34]. They can be summarized as follows:

**Corollary 6.** If a random element $X$ of a Polish space is invariant under a measurable action $T$ of $S_F$, the empirical measure $\frac{1}{k!} \sum_{\phi \in S_k} \delta_{T^k \phi}(X)$ converges weakly to $\xi$ as $k \to \infty$, almost surely under the law of $\xi$.

For sequences, the empirical measure can be broken down further into a sum over sequence entries, and redundancy of permutations then shrinks the sum from $k!$ to $k$ terms. Now suppose that $X$ is specifically the output $S_\infty$ of a sampling algorithm:

**Corollary 7.** Let $S : \mathcal{Y} \to \mathcal{X}$ be a sampling algorithm whose prefix densities exist for all $y \in \mathcal{Y}$. Suppose a countable amenable group $G$ is a symmetry group of $S$ under a measurable action $T$. If $S$ samples from a random input graph $Y$, then 
\[
\mathbb{P}_{S_\infty}^Y(f_k) \xrightarrow{k \to \infty} P_Y(f) \quad \mathcal{L}(Y)\text{-a.s.}
\]

holds for any functions $f, f_1, f_2, \ldots$ satisfying $f \in L_1(P_y)$ and $(f_k) \to f$ $P_y$-a.s. for $\mathcal{L}(Y)$-almost all $y$. 

For example, one can fix a finite structure $x_j$ of size $j$, and choose $f$ as the indicator $f(x) := \mathbb{I}\{x|_j = x_j\}$. Corollary 7 then implies

$$\frac{1}{A_k} \sum_{\phi \in A_k} \mathbb{I}\{T_\phi(S_\infty(y))|_j = x_j\} \xrightarrow{k \to \infty} t_{x_j}(y),$$

which makes $\mathbb{P}^{S_\infty(y)}(\{\cdot = x_j\})$ a (strongly) consistent estimator of the prefix density $t_{x_j}$ from output generated by the sampler. Here, $S_\infty(y)$ is still an infinite structure. If the action $T$ is such that the elements of each set $A_k$ affect only the initial substructure of size $k$, we can instead define $f_k(x_k) := \mathbb{I}\{x_k|_j = x_j\}$ for graphs $x_k$ of size $k \geq j$. Thus, $f_k : X_k \to \{0, 1\}$, and $f_k(x|_k) = f(x)$. If a sample $S_1(y) \leq S_2(y) \leq \ldots$ is generated from $y$ using $S$,

$$\frac{1}{A_k} \sum_{\phi \in A_k} \mathbb{I}\{T_\phi(S_k(y))|_j = x_j\} \xrightarrow{k \to \infty} t_{x_j}(y)$$

consistently estimates $t_{x_k}(y)$ from a finite sample of increasing size. The sampling algorithms discussed in the next section admit such estimators.

7. Sampling by random transformation. We now consider group actions where each element $\phi$ of the group $G$ changes only a finite substructure: $T_\phi(x)$ replaces a prefix $x|_k$ of $x$ by some other structure of size $n$. We can hence subdivide the group into subsets $G_n$, for each $n$, consisting of elements which only affect the prefix of size $n$. Thus, $G_n \subset G_{n+1}$. If $\phi$ only affects a prefix of size $\leq n$, then typically so does its inverse, and each subset $G_n$ is itself a group. If each subgroup $G_n$ is finite, the group $G$ is hence of the form

$$G = \bigcup_{n \in \mathbb{N}} G_n \quad \text{for some finite groups } G_1 \subset G_2 \subset \ldots .$$

A group satisfying (20) is called direct limit or direct union of finite groups. Since it is countable, any measurable action satisfies Theorem 3. Plainly, $G$ also satisfies (17), with $A_n = G_n$. Thus, for any measurable action $T$,

$$(x, n) \mapsto \mathbb{P}_n^{x}(\cdot) = \sum_{\phi \in G_n} \delta_{T_\phi(x)}(\cdot)$$

is an empirical measure, and satisfies the law of large numbers (19).

If each $\phi$ affects only a finite substructure, the action must commute with restriction, in the sense that

$$T_n(\phi, x|_n) = T(\phi, x)|_n \text{ for an action } T_n : G_n \times X_n \to X_n \text{ and all } \phi \in G_n, x \in X.$$

We call any action of a direct limit group that satisfies (21) a prefix action. In most cases, one can think of a prefix action $T_\phi$ as a map that removes the subgraph $x|_n$ from $x$ by some form of “surgery”, and then pastes in another graph $T_n(\phi, x|_n) \in X_n$ of the same size. The action $T_n$ is hence a subset of the group $S_{X_n}$ of all permutations of $X_n$. If $X_n$ is finite, so is $S_{X_n}$, which is hence a valid choice for $G_n$. Prefix actions include, for example, the case where $T_n$ is the action of $S_n$ on the first $n$ vertices, but it is worth noting that $G_n$ can be much larger: $S_{X_n}$ is typically of size exponential in $S_n$. We observe:

**Proposition 8.** Prefix actions on almost discrete spaces are continuous.
7.1. Random transformations. Transformation invariance can be built into a sampling algorithm by constructing the algorithm from a random transformation. For a random element $\Phi_n$ of $G_n$, we define

\begin{equation}
S_{n\rightarrow k}(y) := T(\Phi_n, y)_{|k} \quad \text{for each } y \in \mathcal{Y} \subset \mathcal{X}.
\end{equation}

If $T$ is prefix action, one can equivalently substitute $y|_n$ for $y$ on the right-hand side. Algorithm 1 can for instance be represented in this manner, by choosing $\Phi_n$ as a uniform random permutation of the first $m$ vertices. The next results assume the following conditions:

(i) $T$ is a prefix action of a direct limit $G$ on an almost discrete space $\mathcal{X}$.

(ii) The sampling algorithm $S$ is defined by (22), where each $\Phi_n$ is uniformly distributed on the finite group $G_n$.

(iii) Its prefix densities $t$ exist for all $y$ in a $T(G)$-invariant subset $\mathcal{Y} \subset \mathcal{X}$.

The uniform random elements $\Phi_n$ used in the construction are only defined on finite subgroups, but whenever prefix densities exist, one can once again take the limit in input size and obtain a limiting sampler $S_\infty$. These samplers are particularly well-behaved:

**Theorem 9.** Let $S$ be a sampling algorithm satisfying (23). Then for all $\phi \in G$,

(i) $t \circ T_\phi = t$ for $\phi \in G$

(ii) $t(S_\infty(y)) \overset{=}{} t(y)$

(iii) $t(y) = t(y')$ iff $y \equiv_S y'$.

Each output distribution $P_y$ is $G$-invariant, and the law of a sample $S_k(y)$ of size $k$ is $G_k$-invariant. The algorithm is idempotent and resolvent, and any two output distributions $P_y$ and $P_{y'}$ are either identical, or mutually singular.

One can ask whether it is even possible to recover properties of the input graph: If $\mathcal{Y} \subset \mathcal{X}$ and $f : \mathcal{X} \rightarrow \mathbb{R}$ is a statistic, can $f(y)$ be estimated based on $S_\infty(y)$? Since the sampling algorithm does not resolve differences between to equivalent input graphs $y \equiv_S y'$, a minimal requirement is that $f$ be constant on equivalence classes,

\begin{equation}
f(y) = f(y') \quad \text{whenever } y \equiv_S y'.
\end{equation}

For algorithms defined by random transformations, the law of large numbers strengthens to:

**Corollary 10.** Suppose a sampling algorithm $S$ satisfies (23), and $f : \mathcal{X} \rightarrow \mathbb{R}$ is a Borel function satisfying (24). Require $S_\infty(y)$ is $T(G)$-ergodic. Let $(f_m)$ be a sequence of functions on $\mathcal{X}$. Then for every $y$ with (i) $f \in L_1(P_y)$ and (ii) $f_m \rightarrow f \text{ } P_y$-a.s.,

\[
\frac{1}{|G_k|} \sum_{\phi \in G_k} f_k(S_\infty(y)) \overset{k \rightarrow \infty}{\longrightarrow} f(y) \quad P_y\text{-a.s.}
\]

If $y$ is replaced by a $\mathcal{Y}$-valued random variable $Y$, and (i) and (ii) hold $\mathcal{L}(Y)$-a.s., convergence holds $\mathcal{L}(Y)$-a.s.
7.2. The topology induced by a sampling algorithm. Any sampling algorithm \( S \) whose prefix densities exist on a set \( \mathcal{Y} \) induces a topology on this set, the weak topology of \( t \) (i.e. the smallest topology on \( \mathcal{Y} \) that makes each prefix density \( t_{xk} : \mathcal{Y} \rightarrow [0,1] \) continuous). Informally speaking, if the equivalence classes of \( \equiv_S \) coincide with the fibers of \( t \) (as is the case in Theorem 9), this is the smallest topology that distinguishes input points whenever they are distinguishable by the sampler. If \( S \) is defined by Algorithm 1, the prefix densities are precisely the “homomorphism densities” of graph limit theory—depending on the definition, possibly up to normalization [22]. The weak topology of \( t \) is hence the cut norm topology [14]. The cut norm topology is defined on the set of undirected, simple graphs with vertex set \( \mathbb{N} \), and coarsens the almost discrete topology on this set. One may hence ask how this property depends on the sampler: Under what conditions on the subsampling algorithm does the topology induced by the sampler coarsen the topology of the input space? If the algorithm is defined by random transformation as above, that is always the case:

**Proposition 11.** Let \( S \) be a sampling algorithm defined as in (22) by a prefix action \( T \) on an almost discrete space \( \mathbb{X} \). Let \( \mathcal{Y} \) be any topological subspace of \( \mathbb{X} \) such that the prefix densities exist for each \( y \in \mathcal{Y} \). Then \( t \) is continuous on \( \mathcal{Y} \).

8. Selecting vertices independently. Throughout this section, we choose both input space \( \mathcal{Y} \) and the output space \( \mathbb{X} \) as the set of simple, undirected graphs with vertex set \( \mathbb{N} \), and \( \cdot \mid_k \) extracts the induced subgraph on the first \( k \) vertices.

8.1. Exchangeability and graphons. Algorithm 1 selects a subgraph uniformly from the set of all subgraphs of size \( n \) of the input graph \( y \mid_n \). Such uniform random subgraphs are integral to the definition of graphons [14, 15], and the prefix densities are in this case precisely the homomorphism densities of graph limit theory (up to normalization). It is thus a well-known fact that Algorithm 1 induces the class of graphon models, whose relationship to exchangeable random graphs has in turn be clarified by Diaconis and Janson [22] and Austin [6].

Applied to this case, our results take the following form: Algorithm 1 can equivalently be represented as a random transformation (22). Define \( T \) as the action of \( S_F \) that permutes the vertex labels of a graph, and rewrite Algorithm 1 as:

**Algorithm 1’**.

1. Draw \( \Phi_n \sim \text{Uniform}(S_n) \).
2. Generate the permuted graph \( X_n := \Phi_n(y \mid_n) \).
3. Report the subgraph \( S_n \rightarrow_k (y) := X_n \mid_k \).

Clearly, Algorithm 1’ and 1 are equivalent. It is possible to construct pathological input graphs \( y \) for which prefix densities do not exist; we omit details, and simply define \( \mathcal{Y} := \{ y \in \mathcal{Y} \mid \text{Algorithm 1’ has prefix densities} \} \). Then \( \mathcal{Y} \) is invariant under \( S_F \), and we obtain from Theorems 5 and 9:

**Corollary 12.** Algorithm 1 is idempotent, and the limiting random graph \( S_\infty(y) \) is exchangeable. Let \( f \in L_1(\mathbb{P}_y) \) be a function constant on each equivalence class of \( \equiv_S \). Then
Fig 1. (i) Input graph. (ii) A subsample $X_k$ generated by an algorithm that extracts the $n$-neighborhood of a random vertex, here $n = 2$. The sample $X_k$ has 16 vertices. (iii) A sample of the same size generated by Algorithm 1. (iv) Reconstruction of the input graph (i), i.e. a sample of with the same number of vertices as (i), from a graphon model fit to the sample (ii).

if functions $f_k : X_k \rightarrow \mathbb{R}$ satisfy $f_k(x|_k) \rightarrow f(x)$ for all $x$ outside a $P_g$-null set,

$$\frac{1}{k!} \sum_{\pi \in S_k} f_k(T_\pi(S_k(y))) \xrightarrow{k \to \infty} f(y) \quad \text{almost surely.}$$

The equivalence classes of $\equiv_S$ are the fibers of $t$.

Let $w$ be a graphon, i.e. a measurable function $w : [0, 1]^2 \rightarrow [0, 1]$ symmetric in its arguments [14]. Let $X_w$ be a random graph with the canonical distribution defined by $w$: The (symmetric) adjacency matrix of $X_w$ is given by

$$\left( \mathbb{I}(U_{ij} < w(U_i, U_j)) \right)_{i,j \in \mathbb{N}} \quad \text{where } (U_i)_{i \in \mathbb{N}} \text{ and } (U_{ij})_{i,j \in \mathbb{N}} \text{ are i.i.d. Uniform}[0, 1].$$

Let $t_w$ denote the (suitably normalized) vector of homomorphism densities of $w$ [14, 22]. Comparing the definitions of homomorphism and prefix densities, we have $t(X_w) = t_w$ almost surely. Since the fibers of $t$ are the equivalence classes of $\equiv_S$, we can choose a fixed graph $y \in t^{-1}(t_w)$, and obtain

$$X_w \equiv_S y \quad \text{a.s. and } \quad X_w \overset{d}{=} S_\infty(X_w) \overset{d}{=} S_\infty(y).$$

For every graphon $w$, there is hence a graph $y$ such that $X_w \overset{d}{=} S_\infty(y)$. It is well-known that the law of $X_w$ remains unchanged if a Lebesgue-measure preserving transformation $\psi$ of $[0, 1]$ is applied to $w$: If $w' = w \circ (\psi \otimes \psi)$, then $X_{w'} \overset{d}{=} X_w$. Equivalence classes of graphons hence correspond to equivalence classes of graphs under $\equiv_S$,

$$X_{w'} \equiv_S X_w \equiv_S S_\infty(X_w) \quad \text{almost surely.}$$

8.2. Misspecification of graphon models as a sample selection bias. A model $P$ is misspecified for data generated by a random variable $X$ if $L(X) \not\in P$. If $X$ is with positive probability a sparse graph, that implies, without any further knowledge of $X$, that a graphon model is misspecified for $X$. This fact is well known [e.g. 31, 19, 45].
A **sample selection bias** is an erroneous assumption on what properties of the underlying population a sample is representative of. One way in which such biases occur is if the sampling protocol is known, but either its behavior or the underlying population are insufficiently understood—for example, in opinion polling problems if the protocol has a systematic tendency to exclude individuals with certain properties. Another is if false assumptions are made about which sampling protocol has been used. A sampling algorithm $S_\infty$ with input set $\mathcal{Y}$ induces a model $\mathcal{P} = \{P_y|y \in \mathcal{Y}\}$. Analyzing data generated by another algorithm $S'_\infty$ using $\mathcal{P}$ thus constitutes a selection bias. Such a selection bias results in misspecification if $S'_\infty(y) \notin \mathcal{P}$.

Explicitly considering sampling makes the nature of graphon misspecification more precise: As noted above, Algorithm 1 induces the model $\mathcal{P} = \{\mathcal{L}(X_w)|w \text{ graphon}\}$. Explaining data by a graphon model hence implicitly assumes the data is an outcome of Algorithm 1. One might argue, for example, that the distinction between sparse and dense is of limited relevance for small graphs, and that a small sample can hence be fit without concern using a graphon model. Fig. 1 illustrates the implicit sampling assumption can have drastic consequences, even for small samples.

Consider a sample $X_k := S_k(y)$ generated by the limit of Algorithm 1 from an infinite input graph $y$. If $X_k$ contains a subgraph $x_j$ of size $j \leq k$, $y$ must contain infinitely many copies of $x_j$. There are $\binom{k}{j}$ possible subgraphs of size $k$ in $X_k$. If $x_j$ occurs $m$ times in $X_k$, a graphon model assumes a proportion $m/\binom{k}{j}$ of all subgraphs of size $j$ in $y$ match $x_j$. A consequence of the rapid growth of $\binom{k}{j}$ is:

- Small observed patterns are assigned much higher probability than larger ones.

For example, if a sample $X_{20}$ contains a subgraph $x_k$ exactly once, a graphon reconstruction of $X_{20}$ contains $x_k$ with probability $1/1140$ if $j = 3$, compared to $1/38760$ if $j = 6$.

A single edge in $X_k$ is a subgraph of size two. An isolated edge, however, is a subgraph of size $k$: One edge is present, all other edges between its terminal vertices and the remaining graph are absent. A further consequence of the above is hence:

- Graphon models tend not to reproduce (partially) isolated subgraphs.

That is illustrated by Fig. 2, which compares a protein-protein interaction network to its reconstruction from a graphon model. The semi-isolated chains in the input graph, for example, are not visible in the reconstruction; they are present, but not isolated.

### 8.3. Sparsified graphon models.

To address denseness, “sparsified” graphon models have been proposed as a remedy, and recent work in mathematical statistics frequently invokes these random graphs as network models. They are equivalent to random graphs originally introduced in [12], and are defined as follows: Fix a graphon $w$ and a monotonically decreasing function $\rho: \mathbb{N} \rightarrow [0, 1]$. A graph of size $k$ is then generated as

$$\left(\mathbb{I}\{U_{ij} < \rho(k)w(U_i, U_j)\}\right)_{i<j\leq k} \quad \text{where} \ (U_i)_{i\leq k} \text{ and } (U_{ij})_{i,j\leq k} \text{ are i.i.d. Uniform}[0, 1].$$

A graph of this form can equivalently be generated by generating $k$ vertices of the graph $X_w$ defined by the graphon, followed i.i.d. bond percolation, where each present edge is deleted independently with probability $1 - \rho(k)$. This model is hence generated by the following sampling algorithm:
Algorithm 2.

i.) Select $k$ vertices of $y|_k$ independently and uniformly without replacement.

ii.) Extract the induced subgraph $S_{n\rightarrow k}(y|_n)$ of $y|_n$ on these vertices.

iii.) Label the vertices of $S_{n\rightarrow k}(y|_n)$ by $1, \ldots, k$ in order of appearance.

iv.) Delete each edge in $S_{n\rightarrow k}(y|_n)$ independently with probability $p(k)$.

Although the random graphs generated by such models can be made sparse by suitable choice of $\rho$, this modification clearly does not alleviate the misspecification problems discussed above. To emphasize:

Sparsified graphon models still assume implicitly that the input graph is dense, and explain sparsity by assuming the data has been culled after it was sampled.

Note the application of Algorithm 1 and the bond percolation step (iv) in Algorithm 2 cannot be exchanged: One cannot equivalently assume the input graph has been sparsified first, and then sampled.

8.4. Graphon models on \(\sigma\)-finite spaces. Generating a draw from a graphon model involves a generic sequence of i.i.d. random variables. The joint law of these variables is a probability measure. Caron and Fox [19] have proposed a random graph model that, loosely speaking, samples from a \(\sigma\)-finite measure by substituting a Poisson process, which makes it possible to generate sparse graphs. Veitch and Roy [48], Borgs, Chayes, Cohn, and Holden [16] and Janson [31, 32] have extended this idea to a generalization of graph limits that can represent certain sparse graphs. It can be shown that this approach substitutes Algorithm 1 by a form of site percolation:

Algorithm 3.

i.) Select each vertex in $y|_n$ independently, with a fixed probability $p \in [0, 1]$.

ii.) Extract the induced subgraph $X_m$ of $y|_n$ on the selected vertices.

iii.) Delete all isolated vertices from $X_m$, and report the resulting graph.

Note the size of the output graph is now random. Algorithm 3 is derived by Veitch and Roy [49] (who refer to the site percolation step as \(p\)-sampling) as the sampling scheme that describes the relation between \(\sigma\)-finite graphon models at different sizes. Borgs, Chayes, Cohn, and Veitch [17] show it defines the model class, and characterize the associated topology.
8.5. **Biasing by degree.** Algorithm 1 and 2 fail to resolve sparse input graphs since they select vertices independently of the edge set. Algorithm 3 circumvents the problem by oversampling—it selects a much larger number of vertices, and then weeds out insufficiently salient bits. We compare this to an example that still selects vertices independently, but includes some information about the edge set, in the form of the vertex degrees:

**Algorithm 4.**

i.) Select \( n \) vertices of \( y|_{n} \) independently w/o replacement from the degree-biased distribution.

ii.) Extract the induced subgraph \( S_{n}(y|_{n}) \) of \( y|_{n} \) on these vertices.

iii.) Label the vertices of \( S_{n}(y|_{n}) \) by 1, \ldots, \( n \) in order of appearance.

We observe immediately this algorithm is not idempotent: Suppose an input graph \( y_{4} \) for \( S_{1\rightarrow 3} \) is chosen as follows:

If \( S_{1\rightarrow 3} \) is defined by Algorithm 1, it generates \( x_{3} \) and \( x_{4} \) with equal probability. Under Algorithm 4, the probabilities differ. See also Fig. 3. This effect becomes more pronounced in the input size limit, and permits the limiting algorithm \( S_{\infty}(y) \) to distinguish sparse from empty input graphs. Consider an input graph \( y \) with vertex set \( N \), defined as follows:

In \( y|_{n} \), vertex 1 has degree \( m - 1 \); all other vertices have degree 1. Under Algorithm 1, the probability of vertex 1 being selected converges to 0 as \( n \to \infty \). Consequently, \( S_{\infty}(y) \) is empty almost surely. If Algorithm 4 is used instead, each step \( S_{m}(y) \) selects vertex 1 with probability \( \geq 1/2 \) until it is selected, and \( S_{\infty}(y) \) is almost surely connected. Thus, Algorithm 4 resolves sparse graphs.

On the other hand, analysis of the algorithm becomes considerably more complicated than for Algorithm 1. It is not clear, for example, which input graphs have prefix density, although the example of the graph in (33) shows graphs with prefix densities exist.

8.6. **Reporting a shortest path.** In a sparse input graph, uniformly selected vertices are not connected by an edge with (limiting) probability 1, but they may be connected by a path of finite length. A possible way sample vertices independently of the edge and to resolve sparse graphs is hence to report path length, rather than presence or absence of edges. Choose \( \mathcal{Y} \subset \mathcal{Y} \) as the set of undirected, simple graphs that are connected and have finite diameter. In other words, these are graphs on \( N \) in which any two vertices are connected by a path of finite length.
Algorithm 5.

i.) Select \( n \) vertices of \( y|_n \) independently and uniformly w/o replacement.

ii.) Choose \( S_{n\rightarrow k}(y) \) as the complete graph on \( n \) vertices, and mark each edge \((i,j)\) by the length of the shortest path between \( i \) and \( j \) in \( y|_n \).

iii.) Label the vertices of \( S_{n\rightarrow k}(y) \) by 1,\( \ldots, n \) in order of appearance.

The example graph in (33) again illustrates this algorithm can resolve sparse graphs.

In this case, \( Y \neq X \), since the output graphs have weighted edges. The algorithm does generate exchangeable output: If \( S_{n\rightarrow k}(y) \) is regarded as a symmetric, \( n \times n \) adjacency matrix with values in \( \mathbb{N} \), its distribution is invariant under joint permutations of rows and columns. Thus, the output is a jointly exchangeable array. Since \( Y \neq X \), the algorithm cannot be represented as a random permutation, and Theorem 9 is not directly applicable. One can, however, define a map \( h: Y \rightarrow X \) that takes an infinite graph \( y \) to a graph \( h(y) \) on the same vertex set, with each edge marked by the corresponding shortest path. Then Algorithm 5 is equivalent to application of Algorithm 1 to \( h(y) \) (where Algorithm 1 additionally reports edge marks).

9. Subsampling sequences and partitions. If a sampler selects edges rather than vertices of a graph, it produces a sequence of edges. If this sequence is exchangeable, the output graph properties are closely related to those of exchangeable sequences and partitions. It is hence helpful to first consider sequences and partitions from a subsampling perspective, even though the results so obtained are known [see e.g. 46, 9].

9.1. Exchangeable sequences vs exchangeable partitions. A partition \( \pi \) subdivides \( \mathbb{N} \) into subsets, called blocks, such that each element of \( \mathbb{N} \) is contained in one and only one block. The blocks of \( \pi \) can be ordered uniquely by their smallest elements, and then numbered consecutively; denote the set of ordered partitions of \( \mathbb{N} \) so defined by \( \mathcal{P} \). A partition can be identified with the sequence of block labels of its elements, i.e. the sequence \( x(\pi) \) with

\[
x_n(\pi) = \text{number of the block containing } n \text{ in } \pi.
\]
Any sequence in \( \mathbb{N} \) thus defines a partition, though not necessarily with ordered blocks—the blocks are ordered iff

\[
x_n \leq |\{x_1, \ldots, x_n\}| \quad \text{for all } n.
\]

Two partitions \( \pi \) and \( \pi' \) are isomorphic, \( \pi \cong \pi' \), if they differ only in the enumeration of their blocks. A sequence representing a possibly unordered partition can be turned into an isomorphic, ordered one by means of the relabeling map

\[
\rho(x) := x^* \quad \text{where } x^*_n := \begin{cases} x_j^* & \text{if } x_n = x_j \text{ for some } j < n \\ |\{x_1^*, \ldots, x_{n-1}^*\}| + 1 & \text{otherwise} \end{cases}
\]

Then \( \mathcal{P} = \rho(\mathbb{N}^\infty) = \mathbb{N}^\infty/\cong \). The permutation group \( \mathbb{S}_\infty \) naturally acts on \( \mathbb{N}^\infty \) as

\[
T(\phi, x) = (x_{\phi(1)}, x_{\phi(2)}, \ldots),
\]

and on \( \mathcal{P} \) by acting on the underlying set \( \mathbb{N} \), which defines an action \( T' \) as

\[
i \text{ in block } j \text{ of } T'(\phi, \pi) \iff \phi(i) \text{ in block } j \text{ of } \pi.
\]

For simplicity, we write \( \phi(\pi) = T'(\phi, \pi) \) and \( \phi(x) = T(\phi, x) \). The two actions correspond as

\[
x(\phi(\pi)) = \rho(\phi(x(\pi))).
\]

In other words, if a sequence \( x \) is an ordered representation of \( \pi \in \mathcal{P} \), then \( \rho(\phi(x)) \) is an ordered representation of \( \phi(\pi) \), and the maps \( x \mapsto \rho(\phi(x)) \) define an action of \( \mathbb{S}_\infty \) on the subset of sequences in \( \mathbb{N}^\infty \) that satisfy (26).

A random sequence \( X \) in \( \mathbb{N}^\infty \) and a random partition \( \Pi \) in \( \mathcal{P} \) are respectively called exchangeable if \( \phi(X) \overset{d}{=} X \) and \( \phi(\Pi) \overset{d}{=} \Pi \), for all \( \phi \in \mathbb{S}_\infty \). Kingman’s theory of exchangeable partitions \([38]\) exposes a subtle difference between the two cases: Suppose \( X \) is a random sequence such that \( \pi(X) \in \mathcal{P} \) almost surely. Then

\[
X \text{ exchangeable sequence } \implies \pi(X) \text{ exchangeable partition},
\]

but the converse is not true: Since \( X \) is exchangeable, de Finetti’s theorem implies there is a random probability measure \( \mu \) on \( \mathbb{N} \) such that \( X_1, X_2, \ldots \mid \mu \sim_{\text{iid}} \mu \). Hence, every value that occurs in \( X \) also reoccurs infinitely often with probability 1, and every block of the partition \( \pi(X) \) is almost surely of infinite size. Kingman’s representation theorem shows that an exchangeable partition can contain two types of blocks, infinite blocks and singletons. Aldous’ proof of Kingman’s theorem \([\text{e.g. } 9]\) shows that every exchangeable partition \( \Pi \) can be encoded as an exchangeable sequence \( X' \) in \([0, 1]\): The numbers \( i \) and \( j \) are in the same block of \( \Pi \) iff \( X'_i = X'_j \). Again by de Finetti’s theorem, there is a random probability measure \( \nu \) on \([0, 1]\) such that \( X'_1, X'_1, \ldots \mid \nu \sim_{\text{iid}} \nu \). Atoms of \( \nu \) account for infinite blocks of \( \Pi = \pi(X') \), whereas the continuous component of \( \nu \) generates singleton blocks. Thus, de Finetti’s theorem on \([0, 1]\) subsumes Kingman’s theorem; de Finetti’s theorem on \( \mathbb{N} \) does not. The latter fails precisely for those partitions that contain singleton blocks.
9.2. Subsampling sequences. Now consider both cases from a subsampling perspective. To generate exchangeable sequences, we define an algorithm with input $y \in \mathbb{N}^\infty$ as:

**Algorithm 6.**

i.) Select $k$ indices $J_1, \ldots, J_k$ in $[n]$ uniformly without replacement.

ii.) Extract the subsequence $S_{n \rightarrow k}(y) = (y_{J_1}, \ldots, y_{J_k})$.

The algorithm can be represented as a random transformation, using the action (27),

$$S_{n \rightarrow k}(y) = \Phi_n(y)\big|_k$$ for $\Phi_n$ uniform on $\mathbb{S}_m$.

A sequence $y$ has prefix densities under this algorithm if, for each $k \in \mathbb{N}$ and every finite sequence $(x_1, \ldots, x_k)$ in $\mathbb{N}$, the scalar

$$p(x_1, \ldots, x_k) := \lim_{n \to \infty} \frac{|\{\phi \in \mathbb{S}_n : \phi(y)\big|_k = (x_1, \ldots, x_k)\}|}{|\mathbb{S}_n|}$$

exists. Let $\mathcal{Y} \subset \mathbb{N}^\infty$ be the set of all sequences $y$ satisfying this condition. Then Theorem 9 holds on $\mathcal{Y}$, which shows that $S_{\infty}(y)$ is an exchangeable random sequence in $\mathbb{N}$. Moreover, the algorithm is idempotent, which implies

$$p(x_1, \ldots, x_k) = p(x_1) \cdots p(x_k).$$

By Fatou’s lemma,

$$\bar{p} := \sum_{m \in \mathbb{N} | p(m) \text{ exists}} p(m) \text{ satisfies } \bar{p} \leq 1.$$

If $X_{n,1}$ denotes the first entry of the sequence output by the sampler on input of length $n$, then $p(m) = \lim_n P\{X_{n,1} = m\}$, and as these events are mutually exclusive for different $m$,

$$y \in \mathbb{N}^\infty \text{ has prefix densities } \iff \bar{p} = 1 \iff \sum_{m \in \mathbb{N}} p(m) = 1.$$

We note every ergodic exchangeable sequence can be obtained in this way: For $y \in \mathcal{Y}$ fixed, $S_{\infty}(y)$ is ergodic, with de Finetti measure $\sum_{m \in \mathbb{N}} p(m)\delta_m$. It is not hard to see that for every choice of scalars $p(m)$ with $\sum_{m} p(m) = 1$, one can construct a sequence $y \in \mathcal{Y}$ that yields these scalars as prefix limits. By Theorem 3, all exchangeable sequences can be obtained by randomization, as $S_{\infty}(Y)$, for some (not necessarily exchangeable) random sequence $Y$. The factorization (29) can be read as a combinatorial explanation of de Finetti’s theorem on $\mathbb{N}$.

9.3. Subsampling partitions. For an input partition $\pi \in \mathfrak{P}$, define subsampling as:

**Algorithm 7.**

i.) Select $k$ indices $J_1, \ldots, J_k$ in $[n]$ uniformly without replacement.

ii.) Extract the block labels $(x_{J_1(\pi)}, \ldots, x_{J_k(\pi)})$.

iii.) Report the ordered sequence $S_{n \rightarrow k}(\pi) = r(x_{J_1(\pi)}, \ldots, x_{J_k(\pi)})$. 

...
The algorithm can again be represented as a random transformation,

\[ S_{n \to k}(\pi) := \Phi_n(\pi)|_k = r(\Phi_n(x(\pi)))|_k \quad \text{for} \quad \Phi_n \sim \text{Uniform}(S_n). \]

The second identity shows that it can be reduced to Algorithm 6: Transform \( \pi \) to the sequence \( x(\pi) \), apply Algorithm 6, and reorder the output sequence.

If the input partition \( \pi \) is regarded as a sequence \( y = x(\pi) \), its prefix densities under Algorithm 7 clearly exist if they do under Algorithm 6, and hence if (30) holds. That condition is sufficient, not necessary: Suppose for every element \( m \in \mathbb{N} \), the limit \( p(m) \) as above exists for \( x(\pi) \), and let \( N_0(\pi) \subset \mathbb{N} \) be the set of all \( m \) with \( p(m) = 0 \). Require that

\[ p_0 := \lim_{n \to \infty} \frac{\left| \{ j \leq n : x_j(\pi) \in N_0(\pi) \} \right|}{n} \quad \text{exists, hence} \quad p_0 + \bar{p} = p_0 + \sum_{m} p(m) = 1. \]

If \( p_0 > 0 \), the sequence \( x(\pi) \) does not have prefix densities under Algorithm 6, since \( \bar{p} < 1 \). In contrast, Algorithm 7 has prefix densities even for \( p_0 > 0 \), since it does not resolve differences between elements of \( N_0(\pi) \).

To make this more precise from the algorithmic perspective, assume Algorithm 6 is applied to the sequence representation \( x(\pi) \) of \( \pi \), and similarly regard the partition output by Algorithm 7 as a sequence. Fix an index \( i \leq k \), and suppose the \( i \)-th entry of the sequence generated in step (ii) of either algorithm takes a value in \( N_0(\pi) \). We can distinguish two cases: A specific value \( m \in N_0(\pi) \) is reported, or we only report whether or not the value is in \( N_0(\pi) \). These correspond to the events

\[ A_{im} := \{ x_i(\pi) = m \} \quad \text{for some} \quad m \in N_0(\pi) \quad \text{and} \quad A_{i0} := \{ x_i(\pi) \in N_0(\pi) \}. \]

Under Algorithm 6, the events \( A_{im} \) are observable, and hence measurable in \( \sigma(S_{n \to k}(y)) \) for every \( i \leq k \leq n \). Since \( p_0 \) is the probability of \( A_{i0} \) under the limiting output distribution \( P_y \), where \( y = x(\pi) \), we have

\[ p_0 = P_x(\pi)(A_{i0}) = \sum_{m \in N_0(\pi)} P_x(\pi)(A_{im}) = \sum_{m \in N_0(\pi)} p(m). \]

Since \( p(m) = 0 \) by assumption and \( N_0(\pi) \) is countable, that excludes the case \( p_0 > 0 \) for Algorithm 6. Under Algorithm 7, the events \( A_{im} \) are not measurable, since they are masked by step (iii), and so \( p_0 > 0 \) does not lead to contradictions. In summary, the set of sequences which have prefix densities under Algorithm 6 is

\[ \mathcal{V} = \{ y \in \mathbb{N}^\infty \mid p(m) \text{ exists for all } m \in \mathbb{N} \text{ and } \sum_{m} p(m) = 1 \}. \]

Algorithm 7 has prefix densities on the strictly larger set

\[ \mathcal{V}' = \{ y \in \mathbb{N}^\infty \mid p(m) \text{ exists for all } m \in \mathbb{N} \text{ and } \sum_{m} p(m) \leq 1 \}, \]

and the input sequences in \( \mathcal{V}' \setminus \mathcal{V} \) are precisely those that generate ergodic exchangeable partitions with singleton blocks. Theorem 9 holds on \( \mathcal{V}' \) for Algorithm 7, which in particular implies idempotence of the algorithm, and this property is indeed used by Kingman: Note identity (1.3) in [38] is precisely idempotence, if not by the same name.
10. Selecting edges independently. We next consider undirected input graphs \( y \), with vertex set \( \mathbb{N} \) and infinitely many edges, represented as sequences \( y = ((i_k, j_k)_{k \in \mathbb{N}}) \) of edges as described in Section 2. The sequence representation makes this case similar to the sequences and partitions in the previous section. In analogy to the set \( \mathcal{P} \) of partitions represented by sequences with ordered labels, we define

\[
\mathcal{G} := \{ y \in (\mathbb{N}^2)^\infty \mid i_k < j_k \text{ for all } k \in \mathbb{N} \text{ and } (i_1, j_1, i_2, j_2, \ldots) \text{ satisfies (26)} \}.
\]

These are those undirected graphs on \( \mathbb{N} \) that have infinitely many edges, and whose vertices are labeled in order of appearance in the sequence. The set contains both simple graphs (if each edge occurs only once), and multigraphs. We select edges uniformly by applying a version of Algorithm 7 to the edge sequence \( y \). That requires a suitable adaptation of the relabeling map \( r \): For any finite sequence \( x = ((i_1, j_1), \ldots, (i_n, j_n)) \) of edges, define

\[
r'(x) := \text{swap}(r(x)) \quad \text{where} \quad \text{swap}((i_k, j_k)_{k \leq n}) = (\min\{i_k, j_k\}, \max\{i_k, j_k\})_{k \leq n}.
\]

In words, apply the relabeling map \( r \) for sequences to the sequence of edges (read as a sequence of individual vertices, rather than of pairs), and then swap each pair of vertices if necessary to ensure \( i_k \leq j_k \). The sampling algorithm is defined as follows:

**Algorithm 8.**

i.) Select \( k \) indices \( J_1, \ldots, J_k \) in \([n]\) uniformly without replacement.

ii.) Extract the sequence \( ((i_{J_1}, j_{J_1}), \ldots, (i_{J_k}, j_{J_k})) \).

iii.) Report the relabeled graph \( S_{n \to k}(y) := r'((i_{J_1}, j_{J_1}), \ldots, (i_{J_k}, j_{J_k})) \).

As in the partition case, the algorithm can be represented by a permutation followed by an application of \( r' \): Define

\[
T(\phi, ((i_k, j_k)_k)) := r'((i_{\phi(k)}, j_{\phi(k)})_k) \quad \text{for } \phi \in S_F.
\]

Then \( T \) is a prefix action of \( S_F \) on the set \( \mathcal{G} \), and leaves the set invariant, \( T(\mathcal{G}) = \mathcal{G} \). Algorithm 8 satisfies \( S_{n \to k}(y) \overset{d}{=} T(\Phi_n, y)|_{n} \) if \( \Phi_n \) is uniformly distributed on \( S_n \). For a sequence \( x_k \) of \( k \) edges, labeled in order, the prefix density is

\[
t_{x_k}(y) = \lim_{n \to \infty} \frac{\# \text{ of } k\text{-edge subgraphs of } y|_{n} \text{ isomorphic to } x_k}{\# \text{ of } k\text{-edge subgraphs of } y|_{n}},
\]

provided the limit exists.

10.1. Sampling edges of simple graphs. Suppose first the input graphs are simple: We choose the input set \( \mathcal{Y} \) as

\[
\mathcal{Y} := \{ y \in \mathcal{G} \mid (i_k, j_k) \neq (i_l, j_l) \text{ whenever } k \neq l \}.
\]

Since \( T \) does not change the multiplicities of edges, \( \mathcal{Y} \) is a measurable, \( T \)-invariant subset of \( \mathcal{G} \), and \( T \) restricted to \( \mathcal{Y} \) is again a prefix action of \( S_F \). Define the limiting relative degree of vertex \( i \) as

\[
\overline{d}(i, y) := \lim_{n \to \infty} \frac{1}{2n} \deg(i, y|_{n}).
\]
If $d(i, y) = 0$ for all vertices $i$ in $y$—for example, if all degrees are finite—the limiting probability of any vertex reoccurring in $S_\infty(y)$ is zero. If so, sequences of isolated edges occur almost surely, and $t_{x_k}(y) = 1$ if $x_k = ((1, 2), (3, 4), \ldots, (2k - 1, 2k))$, or 0 otherwise.

In analogy to Section 9, define

$$\bar{p}(y) = \sum_{i \in \mathbb{N}} d(i, y).$$

If $\bar{p}(y) > 0$, there is at least one vertex $i$ with $d(i, y) > 0$, and as for partitions, $\bar{p}(y) \leq 1$. An infinite simple graph $y$ with $\bar{p}(y) > 0$ satisfies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} \mathbb{1}\{\bar{d}(i_k, y)\bar{d}(j_k, y) > 0\} = 0,$$

i.e. those edges contained in the induced subgraph on all vertices with positive relative degree make up only an asymptotically vanishing fraction of all edges in the graph. Roughly speaking, many more vertices than those with positive relative degree are needed to account for such large degrees on some vertices.

The limiting probability that step (ii) of the algorithm selects an edge connecting two vertices with positive relative degree is hence zero. Every edge $(i_k, j_k)$ thus has at least one terminal vertex with $d = 0$, and by definition of the relabeling map $r'$, this implies $j_k$ always corresponds to a vertex with $d = 0$. It then follows by comparison to Algorithm 7 that the sequence $(i_1, i_2, \ldots)$ represents an exchangeable random partition, and all arguments in Section 9 apply, with $p(m) := \bar{d}(m, y)$ and $p_0 := 1 - \bar{p}(y)$. By Theorem 5, the normalized number of edges connected to a vertex converges to $d$. We have shown:

**Corollary 13.** If the input graph $y \in \mathcal{G}$ of Algorithm 8 is simple, each connected component is of $S_\infty(y)$ is either a star or an isolated edge. Let $D_k(i)$ be the number of edges in the $i$th-largest star $S_k(y)$, and $D_k(0)$ the number of isolated edges. Then

$$(32) \quad D_k(0) \to 1 - \bar{p}(y) \quad \text{and} \quad D_k(i) \xrightarrow{k \to \infty} \Delta_i \quad \text{almost surely,}$$

where $\Delta_i$ is $i$th-largest limiting relative degree of $y$.

Since the algorithm essentially extracts an exchangeable partition from $y$ whose block sizes correspond to the relative limiting degrees, but no other information, a statistic of $y$ can be estimated using Algorithm 8 if and only if it is a measurable function of the limiting degree sequence of $y$. In conclusion: **Uniform sampling of edges from large simple graphs is useful if and only if the objective is to estimate a property of the degree sequence.**

10.2. Sampling edges of multigraphs. Now suppose the input $y$ is a multigraph, i.e. edges may reoccur in the sequence $y = ((i_k, j_k)_k)$. The **limiting relative multiplicity** of a multiedge is

$$\overline{m}(i, j, y) := \lim_{n \to \infty} \frac{1}{n} \sum_{k \leq m} \mathbb{1}\{(i, j) = (i_k, j_k)\}.$$
We assume there exists at least one edge with \( m(i, j, y) > 0 \). In analogy to \( \bar{p}(y) \) above, we define a similar quantity in terms of multiplicities as
\[
\bar{\mu}(y) := \sum_{i < j} m(i, j, y) .
\]
Since we are effectively sampling from a sequence, the discussion in Section 9 shows we have to distinguish input graphs with \( \bar{p}(y) = 1 \) and \( \bar{p}(y) < 1 \). The next few results apply only to the former, i.e. we consider the input set
\[
\mathcal{Y} := \{ y \in \mathcal{G} | \bar{p}(y) = 1 \} .
\]

**Corollary 14.** If Algorithm 8 applied to input graph with \( \bar{\mu}(y) = 1 \), then \( S_\infty(y) \) is “edge-exchangeable” in the sense of [21] and [18], and all edge-exchangeable graphs can be obtained in this manner, with \( y \) randomized if the output graph is non-ergodic.

That follows immediately from the definition of edge-exchangeable graphs as exchangeable sequences of pairs \( i < j \) in \( \mathbb{N} \) in [21], and our analysis of Algorithm 6 above. We also conclude from Section 9 that Theorem 9 and Corollary 10 are applicable, hence:

**Corollary 15.** On \( \mathcal{Y} \), the action (31) is a symmetry of Algorithm 8, and the algorithm is idempotent and resolvent on \( \mathcal{Y} \). Moreover:

(i) If two input graphs \( y \) and \( y' \) have identical prefix densities, then \( P_y = P_{y'} \); otherwise, there exists a \( T \)-invariant Borel set \( A \subset X \) such that \( P_y(A) = 1 \) and \( P_{y'}(A) = 0 \).

(ii) Let \( f \) be a function in \( L_1(P_y) \), and \( (f_k) \) a sequence of Borel functions \( f_k : X_k \to \mathbb{R} \) with \( f_k(x_k) \to f(x) \) as \( k \to \infty \), for all \( x \) outside a \( P_y \)-null set. Then almost surely
\[
\frac{1}{\pi} \sum_{\phi \in \mathcal{S}_k} f(T_\phi(S_k(y))) \to f(y) \quad \text{as} \quad k \to \infty .
\]

Statement (ii) implies in particular that multiplicities converge: If \( \nu_k(y) \) denotes the \( k \)-th largest limiting relative multiplicity in a graph \( y \), then
\[
\nu_k(S_\infty(y)) = \nu_k(y) \quad \text{almost surely}.
\]

The behavior of Algorithm 8 changes significantly compared to simple input graphs: \( S_\infty(y) \) discovers an edge location \( (i, j) \) in \( y \) if and only if \( m(i, j, y) > 0 \), and a vertex \( i \) in \( y \) if and only if it is the terminal vertex of such an edge. Thus, the output distribution is now governed entirely by edge multiplicities, rather than by vertex degrees as in the simple case.

**Remark 16.** If one allows more generally input graphs \( y \) with \( \bar{\mu}(y) \leq 1 \), the situation becomes more complicated: Singleton blocks in partitions now correspond to edges with \( m = 0 \). In partitions, singleton blocks can exist only because they are not individually identifiable (see Section 9), but edges with \( m = 0 \) become partly identifiable if they share a terminal vertex with an edge with positive relative multiplicity. The following example shows, however, that there are input graphs with \( \bar{\mu}(y) < 1 \) whose prefix densities exist:
Suppose the input graph \( y \) of Algorithm 8 is chosen such that \( y|_n \), for \( n \) even, is

\[
(33) \quad y|_n = \begin{array}{c}
\begin{array}{c}
\bigcirc \quad \bigcirc \quad \bigcirc \\
| \quad | \\
1 \quad 1 \\
\end{array}
\end{array}
\] \( \frac{n}{2} \) times

For finite \( n \), each draw in step (ii) of the algorithm selects the single “large” multiedge with multiplicity \( n/2 \) with probability \( 1/2 \), and each of the other edges with probability \( \frac{1}{2n} \). Clearly, the output graph is of the same form—a single edge whose relative multiplicity converges to \( 1/2 \), and \( \sim n/2 \) edges with multiplicity 1.

11. Selecting neighborhoods and random rooting. Instead of extracting individual edges or induced subgraphs, one can draw neighborhoods around given vertices. One such problem, where one extracts the immediate neighborhoods of multiple vertices, is of great practical interest, but we do not know its invariance properties. Another, where one extracts a large neighborhood of a single random vertex, reduces to well-known results, but turns out to be of limited use for “statistical” problems.

11.1. Multiple neighborhoods. Let \( B_k(v) \) denote \( k \)-neighborhood of vertex \( v \) in the graph, i.e. the ball of radius \( k \) around \( v \).

**Algorithm 9.**

i.) Select \( k \) vertices \( V_1, \ldots, V_k \) in \( y|_n \) uniformly at random.

ii.) Report the 1-neighborhoods of these vertices, \( S_{n\to k}(y) := \{B_1(V_1), \ldots, B_1(V_n)\} \).

The networks literature often refers to the 1-neighborhood as the **ego network** of \( v \). Data consisting of a collection of such ego networks is of considerable practical relevance, and it is hence an interesting question what the invariance properties of Algorithm 10 are; at present, we have no answer. This question is related to a number of open problems; for example, how many vertices need to be sampled to obtain a good reconstruction of an input graph, known as the **shotgun assembly problem** [44].

11.2. The Benjamini-Schramm algorithm. A different strategy is to report not small neighborhoods of many vertices, but a large neighborhood of a single vertex; in this case, a number of facts are known. A **rooted graph** is a pair \( x_* = (x, v) \), where \( x \) is a graph and \( v \) a distinguished vertex in \( x \), the **root**. Let \( X \) be the space of undirected, connected, rooted graphs with finite vertex degrees on the vertex set \( \mathbb{N} \). For \( x_* \in X \), define \( x_*|_k \) as the induced subgraph on those vertices with distance \( \leq n \) to the root of \( x \); that is, the ball \( B_n(v, x) \) of radius \( n \) in \( x \) centered at the root \( v \). Then \( X_n = X|_k \) is the set of finite, rooted, undirected, connected graphs with diameter \( 2n + 1 \) (rooted at a vertex “in the middle”). As \( X_n \) is countably infinite, \( X \) is procountable but not compact. It is separable, and hence almost discrete, by Lemma 18.
Benjamini and Schramm [8] introduced this algorithm to define a notion of convergence of graphs: A graph sequence \( (y^{(k)}) \) in \( \mathcal{Y} \) is locally weak convergent to \( y \) if \( S_k(y^{(k)}) \to S_k(y) \) in distribution as \( k \to \infty \), for every \( n \in \mathbb{N} \).

Aldous and Lyons [1] have studied invariance properties of such limits, and their work provides—from our perspective—a symmetry analysis of Algorithm 10. An automorphism \( \phi \) of a graph \( x \), with vertex set \( V \) and edge set \( E \), is a pair of bijections \( \phi_V : V \to V \) and \( \phi_E : E \to E \) that cohere in the sense that, for each edge \( e \in E \), \( \phi_V \) maps terminal vertices of \( e \) to the terminal vertices of \( \phi_E(e) \). Let \( \text{Aut}(x) \) denote the set of automorphisms of \( x \); clearly, it is a group. Let \( \mathcal{T} \) be the family of all maps from \( V \times \{i, j \in \mathbb{N} | i < j \} \) to itself. Then \( \text{Aut}(x) \subseteq \mathcal{T} \). For each \( \phi \in \mathcal{T} \), define

\[
t_\phi : X \to X \quad \text{as} \quad t_\phi(x) := \begin{cases} 
\phi(x) & \text{if } \phi \in \text{Aut}(x) \\
x & \text{otherwise}
\end{cases}
\]

Then the set \( \mathcal{G} := \{t_\phi | \phi \in \mathcal{T}\} \) is a group of measurable bijections of \( X \). Aldous and Lyons [1] call a probability measure on \( X \) involution invariant if it is \( \mathcal{G} \)-invariant. Involution invariance can be understood as a form of stationarity: Draw a rooted graph \( X \) with root \( V \) at random. Perform a single step of simple random walk: By definition, the root has at least one and at most finitely many neighbors. Choose one of these neighbors uniformly at random, and shift the root to that neighbor, which results in a random rooted graph \( X' \). Then \( X \) is involution invariant if \( X \overset{d}{=} X' \). In other words, in an involution invariant graph, the distribution of neighborhoods of arbitrary diameter around the current root remains invariant under simple random walk on the graph. Involution invariance is closely related to the concept of unimodularity, which can be formulated on unrooted graphs as a “mass-transport principle”; we omit details and refer to [8, 1].

**Fact** (Aldous and Lyons [1]). Let \( S_\infty \) be defined by Algorithm 10, and \( Y \) a random element of \( \mathcal{Y} \). Then \( S_\infty(Y) \) is \( \mathcal{G} \)-invariant if and only if the random (rootless) graph \( Y \) is unimodular. For every \( y \in \mathcal{Y} \), the output distribution \( P_y = \mathcal{L}(S_\infty(y)) \) is \( \mathcal{G} \)-ergodic.

Whether all \( \mathcal{G} \)-invariant measures can be obtained as \( P_y \) for some fixed \( y \in \mathcal{Y} \), or more generally as weak limits \( P = \lim_k P_{y^{(k)}} \) for some sequence \( (y^{(k)}) \) in \( \mathcal{Y} \), is an open problem [1].

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APPENDIX A: TECHNICAL ADDENDA

A.1. Almost discrete spaces (Section 2). To check whether a procountable space is almost discrete, one has to verify separability. The combinatorial character of these spaces makes it possible to formulate conditions in terms of the elements of $X_n$: One is by verifying each finite structure $x_n \in X_n$ can be extended to an element of $X$ in some canonical way, which is the case if there is a map

$$\tau : \cup_n X_n \to X \quad \text{with} \quad \tau(x_n)|_n = x_n \quad \text{for all } n \in \mathbb{N} \text{ and } x_n \in X_n.$$

For example, $\tau$ may pad a finite graph $x_n$ with isolated vertices (if the graphs in $X$ are not required to be connected).

Lemma 17. (i) A procountable space is compact if and only if it is profinite. If so, it is almost discrete. (ii) Any procountable space admitting a map $\tau$ as in (34) is almost discrete.

Proof. (i) Inverse limits of topological spaces are compact iff each factor is compact. Compact metrizable spaces are Polish. (ii) By definition of the metric $d$, the preimage of $x|_n$ under the map $x|_n$ is the $d$-ball $B_n(x)$ of radius $2^{-n}$ at $x$, so $\tau(x|_n) \in B_n(x)$. Since the metric balls form a base of the topology, the countable set $\tau(\cup_n X_n)$ is dense in $X$.

Discrete spaces and continua can be distinguished by their abundance of sets that are clopen (simultaneously closed and open): In a discrete space, each subset is clopen; in Euclidean space, no proper subset is clopen. An almost discrete spaces can contain sets that are not clopen, but clopen sets determine all topological properties: There is a clopen base of the topology (the metric balls), and the space is hence zero-dimensional in the parlance of descriptive set theory. Informally, an almost discrete space is as close to being discrete as an uncountable space can be if it is Polish. The following lemma adapts standard results on weak convergence and inverse limits to this setting:

Lemma 18. Let $X$ be almost discrete. (i) A sequence $(P_i)$ of probability measures on $X$ converges weakly to a probability measure $P$ if and only if $P_i(B) \to P(B)$ for all metric balls $B$ of finite radius. (ii) Let $X'$ be a topological subspace of $X$, and $P_n$ a probability measure on $X_n$ for each $n \in \mathbb{N}$. Then there exists a probability measure $P$ on $X'$ that satisfies $P|_n = P_n$ for all $n$ if an only if, for all $n$,

$$P_{n+1}|_n = P_n \quad \text{and} \quad \text{if } P_n(x_n) > 0 \text{ then } x_n = x_{n} \quad \text{for some } x \in X'.$$

If $P$ exists, it is tight on $X'$.

A.2. Separability of transformation families. Whether a given family $\mathcal{T}$ is separable (cf. Section 4) can be established by a number of criteria. A general sufficient condition is: If $\mathcal{T}_0 \subset \mathcal{T}$ is dense in the topology of point-wise convergence on $X$, it is separating [25]. If $\mathcal{T}$ is a group action, a sufficient condition can be formulated in terms of the group: If $\mathcal{T}$ is an action of a group $G$, and if there exists a second-countable, locally compact topology on $G$ that makes the action measurable, then $\mathcal{T}$ is separable [25]. Since a second-countable, locally compact group is Polish, each orbit of the action is measurable. Countable groups have countable orbits, and are trivially separable, but that does not imply that a separable
group action has countable orbits—the infinite symmetric group (the set of all bijections of \(\mathbb{N}\)) is a counterexample. There is, however, the following remarkable converse, the Feldman-Moore theorem [26]: If \(\Pi\) is a partition of a standard Borel space \(\mathbb{X}\) into countable sets, there is a measurable action of a countable group on \(\mathbb{X}\) whose orbits are the constituent sets of \(\Pi\). In particular, if each orbit of \(\mathcal{T}\) is countable, then \(\mathcal{T}\) is separable.

A.3. Left-inversion of prefix densities. The proof of Corollary 10 establishes that if \(f(S_\infty(y)) = f(y)\) almost surely, by constructing a map \(f' : [0,1]^\infty \to \mathbb{R}\) that factorizes \(f\) as \(f = f' \circ t\) outside a null set. This map \(f'\) depends on the law \(P_y\), or more generally on \(\mathcal{L}(Y)\) if \(y\) is randomized. A measure-dependent result suffices for Corollary 10, but from an analytic perspective, it is interesting to ask whether the statement can be strengthened to be measure-free. That is possible if \(t\) preserves measurability of open sets:

**Proposition 19.** Let \(S\) be a sampling algorithm defined as in (22) by a prefix action \(T\) on an almost discrete space, and require all prefix densities exist. If \(t(A)\) is Borel whenever \(A\) is open, there exists a Borel map \(\sigma : [0,1]^\infty \to \mathbb{X}\) such that \(\sigma \circ t(y) \equiv_s y\) for all \(y\).

**Proof.** Since \(\equiv_s\) is a partition of a standard Borel space, it follows from the theory of measurable partitions that the map \(\sigma\) exists if (1) each equivalence class is a closed set and (2) for every open set \(A\), the \(\equiv_s\)-saturation \(A^* := \{y \in \mathbb{X} | y \equiv_s y'\text{ for some } y' \in A\}\) is Borel [e.g., 37, Theorem 12.16]. By Theorem 9, the equivalence classes of \(\equiv_s\) are the fibers of \(t\). Since \(t\) is continuous by Proposition 11, the equivalence classes are closed. Observe that the saturation of any set \(A\) is \(A^* = t^{-1}(\sigma(A))\). Thus, if \(t(A)\) is measurable, so is \(A^*\).

APPENDIX B: PROOFS

B.1. Existence of a limiting sampler. We denote be \(\text{PM}(\mathbb{Y})\) the space of probability measures on \(\mathbb{Y}\) (topologized by weak convergence). The proof uses a continuity result of Blackwell and Dubins [11]: If \(\mathbb{Y}\) is Polish, \(\text{PM}(\mathbb{Y})\) the space of probability measures on \(\mathbb{X}\), and \((\mathbb{U}, \lambda)\) a standard probability space, there exists a mapping \(\rho : \text{PM}(\mathbb{Y}) \times \mathbb{U} \to \mathbb{Y}\) and a \(\lambda\)-null set \(N_\rho \subset \mathbb{U}\) such that

\[
\rho(\cdot, u) \text{ is continuous for all } u \not\in N_\rho \quad \text{and} \quad \mathcal{L}(\rho(P, U)) = P ,
\]

where the latter holds for any \(P \in \text{PM}(\mathbb{Y})\) and any random variable \(U\) with law \(\lambda\).

**Proof of Theorem 1.** Abbreviate \(U := (U_1, U_2, \ldots)\) and \(P(n, k, y) := \mathcal{L}(S_{n\to k}(y, U))\). Since \(\mathbb{X}\) is almost discrete, it is Polish, and so is each space \(\mathbb{X}_n\). The spaces of \(\text{PM}(\mathbb{X})\) and \(\text{PM}(\mathbb{X}_n)\) are hence again Polish in their weak topologies.

- First fix \(k\). Since the prefix densities exist, the sequence \(\lim_n P(n, k, y)\) converges weakly to a limit \(P(k, y)\). Since \(\mathbb{Y}\) is measurable and \(\mathbb{X}_k\) Polish, measurability of each \(y \mapsto P(n, k, y)\) implies measurability of the limit as a function \(y \mapsto P(k, y)\).

Now write the restriction explicitly as a map \(\text{pr}_k(x) := x|_{\cdot k}\), and recall it is continuous. It induces a map \(\text{PM}(\mathbb{X}_{k+1}) \to \text{PM}(\mathbb{X}_k)\) on probability measures, as \(P_{k+1} \mapsto P_{k+1} \circ \text{pr}_k^{-1}\), which is again continuous.
• The distributional limits above preserve projectivity of laws: By (8), we have
  \[ \text{pr}_k P(n, k + 1, y) = P(n, k, y) \quad \text{and hence} \quad \text{pr}_k P(k + 1, y) = P(k, y) \]
  by continuity of \text{pr}_k.

• For \( y \) fixed, the measures \((P(k, y))_k\) hence form a projective family, and by Lemma 18,
  there is a probability measure \( P(y) \) on \( X \) satisfying \( \text{pr}_k P(y) = P(k, y) \) for each \( k \).

This already suffices to guarantee that any random variable \( S_\infty \) with law \( P(y) \) satisfies
(1), pointwise in \( y \). What remains to be shown is only that this variable can be chosen as
a measurable function of \((y, u)\). To this end, let \( \rho : \text{PM}(X) \times [0, 1] \to X \) be the rep-
resentation map guaranteed by (35), and \( N \) the associated null set. Additionally, define
\( f : Y \times [0, 1] \to \text{PM}(X) \times [0, 1] \) as \( f(y, u) := (P(y), u) \). Then set
\[ S_\infty(y, u) := \rho(f(y, u)) \quad \text{which implies} \quad S_\infty(y, U) \sim P(y) \quad \text{for each} \quad y . \]

Suppose a function of two arguments, say \( g : \mathbb{Z}_1 \times \mathbb{Z}_2 \to \mathbb{Z}_3 \), is continuous in its first argu-
mament and measurable in the second. If \( \mathbb{Z}_1 \) is separable metrizable, \( \mathbb{Z}_2 \) measurable, and \( \mathbb{Z}_3 \)
metrizable, that suffices to make \( g \) jointly measurable [3, Lemma 4.51].

• The restriction of \( \rho \) to \( \text{PM}(X) \times ([0, 1] \setminus N) \) is continuous in its first argument and
  measurable in the second, hence jointly measurable. Since its range \( \mathbb{R} \) is Polish, it can be
  extended to a measurable function on all of \( \text{PM}(X) \times [0, 1] \) [23, Corollary 4.27].

• Since \([0, 1]\) is Polish, \([0, 1] \setminus N\) is separable metrizable. By the same device as above,
  \( f \) is jointly measurable, which makes \( \rho \circ f \) measurable as a function \( Y \times [0, 1] \to X \).

Thus, \( S_\infty \) is indeed jointly measurable.  

\[ \Box \]

### B.2. Law of large numbers.

The main ingredient of the proof of Theorem 5 is the pointwise ergodic theorem of Lindenstrauss [42]. For countable discrete groups, it states:

**Theorem 20 (E. Lindenstrauss).** Let \( G \) be a countable group, \( (A_n) \) a sequence of finite
subsets of \( G \) satisfying (17), and \( T \) a measurable action of \( G \) on a standard Borel space \( X \).
If \( X \) is a random element invariant under \( T \), and \( f \in L_1(X) \), there is \( T \)-invariant function
\( \tilde{f} \in L_1(X) \) such that \( \|A_k\|^{-1} \sum_{\phi \in A_k} f(T_\phi(X)) \to \tilde{f} \) almost surely as \( k \to \infty \).

We need a lemma relating convergence of a sequence of functions \( f_k \) and of averages \( \mu_k(f) \)
to convergence of the diagonal sequence \( \mu(f_k) \). It involves random measures defined on a
probability space \( X \) that take values in the set of measures on the same space—that is,
measurable mappings \( \mu : X \to \text{PM}(X) \). We denote these \( x \mapsto \mu^x. \)

**Lemma 21.** Let \( (X, \mathcal{B}(X), P) \) be a standard Borel probability space, and \( f, f_1, f_2, \ldots \)
measurable functions \( X \to \mathbb{R} \) such that \( f_i \to f \) uniformly on some set \( A \in \mathcal{B}(X) \).
Let \( \mu_1, \mu_2, \ldots \) be measurable mappings \( X \to \text{PM}(X) \), such that the limits
\[ \tau(x) := \lim_k \mu_k^x(f_i) \quad \text{and} \quad \tau(x) := \lim_k \mu_k^x(f) \quad \text{exist for} \ P \text{-almost all} \ x \in X . \]

Then there is a conull set \( A' \) such that \( \mu_k^x(f_i) \to \tau(x) \) for all \( x \in A \cap A' \), and \( \tau \) is measurable
as a function on \( X \).
For the proof, consider a real-valued net \((\tau_{ik})_{i,k \in \mathbb{N}}\), with the product order on \(\mathbb{N} \times \mathbb{N}\). Recall that such a net is said to converge to a limit \(\tau\) if, for every \(\epsilon > 0\), \(|\tau_{ik} - \tau| < \epsilon\) for all \(i\) exceeding some \(i_\varepsilon\) and \(k\) exceeding some \(k_\varepsilon\). Recall further that a sufficient condition for convergence is that (i) the limits \(\lim_i \tau_{ik}\) exist for all \(i \in \mathbb{N}\), and (ii) convergence to the limits \(\lim_i \tau_{ik}\) holds and is uniform over \(k\). If so, the limits commute, and \(\lim_k (\lim_i \tau_{ik}) = \lim_i (\lim_k \tau_{ik}) = \tau\).

**Proof.** Let \(N\) be the union of all null sets of exceptions in (36), and \(A' := X \setminus N\). On \(A \cap A'\), define \(\tau_{ik}(x) := x_i(f_i)\). Fix some \(x \in A \cap A'\). The net \((\tau_{ik}(x))\) converges uniformly over \(n\) as \(n \to \infty\): Let \(\epsilon > 0\). Uniform convergence of \((f_i)\) implies \(|f_i(z) - f(z)| < \epsilon\) for all \(z \in A \cup A'\) and all \(i\) exceeding some \(i_0\), and therefore
\[
|\mu_k^x(f_i) - \mu_k^x(f)| \leq \mu_k^x(|f_i - f|) \leq \epsilon \quad \text{for all } k \in \mathbb{N}, i \geq i_0.
\]

If \(i\) is fixed instead, \(\lim_k \tau_{ik}(x)\) exists, by (36). The entire net hence converges for \(x \in A \cap A'\), to some limit \(\tau(x)\), and extracting the diagonal sequence yields \(\mu_k^x(f_k) \to \tau(x)\). Since \(\tau\) is a limit of measurable functions into a metrizable space, it is measurable.

Recall that \((f_i)\) converges almost uniformly if, for every \(\delta > 0\), convergence is uniform on a set of probability at least \((1 - \delta)\). To move between almost sure and almost uniform convergence, we use Egorov’s theorem [30, 47]: For a sequence of \(L_1\) functions, almost sure implies almost uniform convergence [30, Theorem 11.32], and vice versa [47, Lemma 1.9.2(ii) and Theorem 1.9.6].

**Proof of Theorem 5.** By Lindenstrauss’ pointwise ergodic theorem above, there exists a \(T\)-invariant function \(\bar{f} \in L_1(P)\) such that \(\mathbb{F}_k^X(f) \xrightarrow{k \to \infty} \bar{f}\) almost surely. For any invariant set \(B \in \sigma_s\), we hence have
\[
\int_B \bar{f}dP = \lim_k \frac{1}{|A_k|} \sum_{\phi \in A_k} \int_B f(\phi(x))P(dx) = \int_B f dP,
\]
since \(P\) is \(T\)-invariant. Since \(G\) is countable, Theorem 3 implies there is a random ergodic measure \(\xi\) such that \(P[\cdot | \sigma_s] = P[\cdot | \xi] = \xi(\cdot)\) almost surely. Thus,
\[
\bar{f} = \mathbb{E}[f|\xi] = \xi(f) \quad \text{and hence } \mathbb{F}_k^X(f) \xrightarrow{a.s.} \xi(f).
\]

**Almost sure convergence.** Consider the sequence \((f_i)\). Since \(f_i \to f\) almost everywhere, Egorov’s theorem implies that for every \(\delta > 0\), there is a set \(B_{\delta/2}\) of measure at least \(1 - \delta/2\), such that \(f_i \to f\) uniformly on \(B_{\delta/2}\). Applying Lemma 21 to the functions \(f_i\) and random measures \(\mu_k = \mathbb{F}_k\) shows there is a conull set \(A\) such that
\[
\tau(x) := \lim_k \mathbb{F}_k^X(f_k) \quad \text{exists for all } x \in B_{\delta/2} \cap A.
\]
Again by Egorov’s theorem, one can find a further set \(B'_{\delta/2}\) with \(P(B'_{\delta/2}) \geq 1 - \delta/2\), such that convergence is even uniform on \(B_{\delta/2} \cap B'_{\delta/2}\). Thus, for every \(\delta > 0\), there is a set \(B\) of probability \(P(B) \geq 1 - \delta\) such that \(\mathbb{F}_k^X(f_k) \to \tau(x)\) holds uniformly on \(B\), and hence almost uniformly on \(X\). As noted above, that implies \(\mathbb{F}_k^X(f_k) \to \tau(X)\) almost surely, and by (39), \(\tau(X) = \xi(f)\) a.s.
Convergence in the mean. Define \( h_k(x) := \mathbb{F}_k^x(g) \). By (39), \( h_k(X) \) converges in \( L_1 \), which makes \((h_k)\) uniformly integrable. There is hence, for any \( \varepsilon > 0 \), an integrable function \( h_{\varepsilon} \geq 0 \) such that \( \int_{\{h_k > h_{\varepsilon}\}} h_k(x)P(dx) < \varepsilon \) holds for all \( k \). The same \( h_{\varepsilon} \) then also satisfies

\[
\int_{\{\|F_k^n\| > h_{\varepsilon}\}} \mathbb{F}_k^n(f_k^n)P(dx) < \varepsilon \quad \text{since} \quad |\mathbb{F}_k^n(f_k^n)| \leq \mathbb{F}_k^n(|f_k^n|) \leq \mathbb{F}_k^n(g) \leq |\mathbb{F}_k^n(g)|
\]

for all \( k \). That likewise makes \((\mathbb{F}_k^n(f_k^n))_{k \in \mathbb{N}}\) uniformly integrable.

\[\square\]

B.3. Sampling by random transformation. Prefix actions on almost discrete spaces are continuous:

\[\textbf{Proof of Proposition 8.}\] Fix \( \phi \in \mathcal{G}_k \). Then \( T_k(\phi, x|_k) = T_n(\phi, x|_n)|_k \) for \( n \geq k \), by hypothesis (21). That makes \( T(\phi, \bullet) \) the inverse limit of the mappings \( T_n(\phi, \bullet) \), for \( n \geq k \). Since each space \( X_n \) is discrete, each such map is continuous; as an inverse limit of continuous maps, \( T(\phi, \bullet) \) is again continuous. As this is true for each \( \phi \in \mathcal{G} \) by (20), and \( \mathcal{G} \) is discrete, the assembled map \( T(\bullet, \bullet) \) is continuous.

\[\square\]

For the proof of Theorem 9, we note an immediate consequence of the definition (21) of prefix actions: If \( \Phi_n \) is a uniform random element of \( \mathcal{G}_n \), for each \( n \in \mathbb{N} \), then

\[(41) \quad T(\Phi_n, T(\Phi_m, y)) \overset{d}{=} T(\Phi_{\max\{m, n\}}, y) \quad \text{and} \quad T(\Phi_k, T(\phi, y)) \overset{d}{=} T(\Phi_k, y)\]

for every \( \phi \in \mathcal{G} \) and all sufficiently large \( k \).

\[\textbf{Proof of Theorem 9.}\] Since \( T \) is (jointly) measurable, \( S_{n \rightarrow k} \) is jointly measurable, and satisfies (8) by construction, so \( S_\infty \) exists by Theorem 1.

Invariance of \( P_y \). Let \( \phi \in \mathcal{G} \); we have to show \( \phi(S_\infty(y)) \overset{d}{=} S_\infty(y) \). Choose \( k(\phi) \) such that \( \phi \in \mathcal{G}_{k(\phi)} \). Then for any \( k \geq k(\phi) \),

\[T(\phi, S_\infty(y)|_k) \overset{(21)}{=} T_k(\phi, \lim_{n \geq k} T(\Phi_n, y)|_k) \overset{\text{continuity}}{=} \lim_{n} T_k(\phi, T_n(\Phi_n, y)|_k) \quad \text{in distribution,}\]

where the second identity holds by continuity of \( T_n \) and \( \bullet|_k \). The limit on the right satisfies

\[\lim_{n} T_k(\phi, T_n(\Phi_n, y)|_k) \overset{(41)}{=} \lim_{n} T_n(\Phi_n, y)|_k = \lim_{n} T(\Phi_n, y)|_k = S_\infty(y)|_k \quad \text{in distribution.}\]

Whenever \( \phi \in \mathcal{G}_{k(\phi)} \), we hence have \( T_\phi P_y|_k = P_y|^k \) for all \( k \geq k(\phi) \), and since the finite-dimensional distributions \( P_y|_k \) completely determine \( P_y \), that implies \( T_\phi P_y = P_y \).

Idempotence. Fix \( k \leq m \leq n \). By definition of the sampler in (22),

\[S_{n \rightarrow k}(S_{n \rightarrow m}(y)) \overset{d}{=} T_m(\Phi_m, T_n(\Phi_n, y)|_k) \overset{d}{=} T(\Phi_n, y)|_k \overset{d}{=} S_{n \rightarrow k}(y) \] ,

where the second identity holds by (41). That implies, using Theorem 1, that also

\[S_{n \rightarrow k}(S_k(y)) \overset{d}{=} S_k(y) ,\]

so idempotence holds.
**Relation between the measures** $P_t$ and hence $k$ so indeed $t \phi$. Fix $\phi \in G$. Since $\mathcal{Y}$ is invariant, $t(T_\phi(y))$ is well-defined. The vector has one entry for each $k \in \mathbb{N}$ and $x_k \in \mathcal{X}_k$, and by definition of prefix densities,

$$t_{x_k}(T_\phi(y)) = \lim_n \mathbb{P}_{x_k}\{T(\Phi_n, T(\phi, y)) = x_k\} \overset{(41)}{=} t_{x_k}(y),$$

so indeed $t = t \circ T_\phi$. For a random draw, idempotence implies

$$t_{x_k}(S_{\infty}(y)) = \lim_n \mathbb{P}_{x_k}\{S_{n} = x_k\} = \lim_n \mathbb{P}_{x_k}\{S_{n} = y_k = x_k\} = t_{x_k}(y) \text{ a.s.,}$$

and hence $t(S_{\infty}(y)) = t(y)$ almost surely.

**Relation between the measures** $P_y$. The above implies $P_y(t^{-1}t(y)) = 1$. Thus, if $y$ and $y'$ are such that $t(y) = t(y')$, then $P_y$ and $P_{y'}$ concentrate on the same set. Otherwise,

$$t(y) \neq t(y') \Rightarrow t^{-1}(y) \cap t^{-1}(y') = \emptyset \Rightarrow P_y(t^{-1}(y')) = 0.$$  

Hence, $t(y) = t(y')$ if and only if $y \equiv s y'$. Moreover, since $P_{y}\{x_k\} = t_{x_k}(y)$, that implies $P_y = P_{y'}$ if and only if $y \equiv s y'$, so the algorithm is resolvent. Since $t = t \circ T_\phi$, the set $t^{-1}(y)$ is invariant, so $P_y$ and $P_{y'}$ are mutually singular on $\sigma(G)$ unless $y \equiv s y'$.  

**Proof of Proposition 11.** Recall that $t$ is a map $X \rightarrow [0, 1]^{\cup k \mathcal{X}_k}$. Denote the subvector of densities of prefixes in $\mathcal{X}_k$ by $t^{(k)} : y \mapsto (t_{x_k}(y))_{x_k \in \mathcal{X}_k}$. The latter is precisely the law $\mathcal{L}(S_k(y))$, represented as a vector of probabilities on the countable set $\mathcal{X}_k$,

$$t^{(k)}(y) = \mathcal{L}(S_k(y)) = \lim_n T(\mathcal{L}(\Phi_n), y)_k.$$  

Clearly, $t$ is continuous if and only if $t^{(k)}$ is continuous for each $k$. Fix $k$, and any sequence $(y_i)$ in $X$ with limit $y$, and define the net

$$\alpha_{\infty} := T(\mathcal{L}(\Phi_n), y_i)_k.$$  

Consider row- and column-wise convergence of the net:

(i) Hold $i$ fixed: Since the prefix densities exist, $\alpha_{\infty}$ converges as $n \rightarrow \infty$.

(ii) Hold $n$ fixed: Since $X$ is almost discrete, $y_i \rightarrow y$ implies, for every $n \in \mathbb{N}$, that $y_i|_n = y|_n$ for all sufficiently large $i$. Since $T$ is a prefix action, that in turn means

$$T(\mathcal{L}(\Phi_n), y_i)_k = T(\mathcal{L}(\Phi_n), y)_k \text{ hence } \alpha_{\infty} \overset{i \rightarrow \infty}{\rightarrow} T(\mathcal{L}(\Phi_n), y)_k \text{ uniformly.}$$

Since $(\alpha_{\infty})$ converges separately in $i$ and $n$, and convergence in $i$ is even uniform, $(\alpha_{\infty})$ converges as a net to a limit $\alpha$, and $\lim_n \lim_i \alpha_{\infty} = \lim_i \lim_n \alpha_{\infty} = \alpha$. Thus,

$$\lim_i t^{(k)}(y_i) = \lim_n T(\mathcal{L}(\Phi_n), y_i)_k = \lim_n T(\mathcal{L}(\Phi_n), y)_k = t^{(k)}(y) \text{ whenever } y_k \rightarrow y,$$

and $t^{(k)}$ is indeed continuous for every $k \in \mathbb{N}$.  

The next lemma is adapted from a standard result on Borel sections [35, A1.3], using the fact that measurable functions between suitable spaces have a measurable graph [3, 4.45]:
Lemma 22. Let $f : X \to Y$ be a Borel map from a standard Borel into a second-countable Hausdorff space, the latter equipped with its Borel $\sigma$-algebra. Then for every probability measure $P$ on $Y$, there exists a Borel map $\sigma_P : Y \to X$ such that

$$f(\sigma_P(y)) = y \quad \text{for } P\text{-almost all } y \in Y.$$ 

The image $f(X)$ is measurable in the joint completion of the Borel sets on $Y$ under all probability measures on $Y$.

Proof of Corollary 10. It suffices to show the randomized case. Abbreviate $Q := \mathcal{L}(Y)$. The vector $t$ is a map $\mathcal{Y} \subset X \to [0, 1]^{\infty}$. The law $Q$ defines an image measure $Q' := t(P_Y)$ on $[0, 1]^{\infty}$. Since $\mathcal{Y} \subset X$ is invariant, it is measurable, so its relative topology makes it a standard Borel space [37, 13.4]. By Lemma 22, there exists a map $\sigma_Q : [0, 1]^{\infty} \to X$ satisfying $t(\sigma_Q(s)) = s$ for $Q'$-a.a. $s \in [0, 1]^{\infty}$ and hence $t(\sigma_Q(t(y))) = t(y)$ for $Q$-a.a. $y \in \mathcal{Y}$.

By (iii) in Theorem 9, the equivalence classes of $\equiv_s$ are the fibers of $t$. Since $f$ is hence constant on each fiber, the map $f' := f \circ \sigma_Q$ satisfies $f = f' \circ t$ almost surely under $Q$. Hence, by (ii), $f(S_\infty(y)) = f(y)$ almost surely. \hfill \Box

REFERENCES

[1] D. Aldous and R. Lyons. Processes on unimodular random networks. Electron. J. Probab., 12:no. 54, 1454–1508, 2007.
[2] D. J. Aldous. Exchangeability and continuum limits of discrete random structures. In Proceedings of the International Congress of Mathematicians, 2010.
[3] C. D. Aliprantis and K. C. Border. Infinite Dimensional Analysis. Springer, 3rd edition, 2006.
[4] N. Alon and J. H. Spencer. The Probabilistic Method. J. Wiley & Sons, 2008.
[5] C. Ambroise and C. Matias. New consistent and asymptotically normal parameter estimates for random-graph mixture models. J. R. Stat. Soc. Ser. B. Stat. Methodol., 74(1):3–35, 2012. ISSN 1369-7412.
[6] T. Austin. On exchangeable random variables and the statistics of large graphs and hypergraphs. Probab. Surv., 5:80–145, 2008.
[7] H. Becker and A. S. Kechris. The descriptive set theory of Polish group actions. Cambridge University Press, 1996.
[8] I. Benjamini and O. Schramm. Recurrence of distributional limits of finite planar graphs. Electron. J. Probab., 6:no. 23, 13 pp., 2001.
[9] J. Bertoin. Random Fragmentation and Coagulation Processes. Cambridge University Press, 2006.
[10] P. J. Bickel, A. Chen, and E. Levina. The method of moments and degree distributions for network models. Ann. Statist., 39(5):2280–2301, 2011.
[11] D. Blackwell and L. E. Dubins. An extension of Skorohod’s almost sure representation theorem. Proc. Amer. Math. Soc., 89(4):691–692, 1983.
[12] B. Bollobás, S. Janson, and O. Riordan. The phase transition in inhomogeneous random graphs. Random Structures Algorithms, 31(1):3–122, 2007.
[13] J. V. Bondar and P. Mihóes. Amenability: a survey for statistical applications of Hunt-Stein and related conditions on groups. Z. Wahrsch. Verw. Gebiete, 57(1):103–128, 1981.
[14] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. Adv. Math., 219(6):1801–1851, 2008. ISSN 0001-8708.
[15] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs II. Multiway cuts and statistical physics. Ann. of Math. (2), 176(1):151–219, 2012. ISSN 0003-486X.
[16] C. Borgs, J. T. Chayes, H. Cohn, and N. Holden. Sparse exchangeable graphs and their limits via graphon processes. 2016. arxiv 1601.07134.
[17] C. Borgs, J. T. Chayes, H. Cohn, and V. Veitch. 2017. arxiv 1708.03237.
36

[18] D. Cai, T. Campbell, and T. Broderick. Edge-exchangeable graphs and sparsity. In Advances in Neural Information Processing Systems 29, pages 4249–4257. 2016.

[19] F. Caron and E. B. Fox. Sparse graphs using exchangeable random measures. J. Roy. Statist. Soc. Ser. B, 2017. To appear.

[20] H. Crane and W. Dempsey. A framework for statistical network modeling. 2015. arxiv 1509.08185.

[21] H. Crane and W. Dempsey. Edge exchangeable models for network data. 2016. Preprint. arxiv 1603.04571.

[22] P. Diaconis and S. Janson. Graph limits and exchangeable random graphs. Rendiconti di Matematica, Serie VII, 28:33–61, 2008.

[23] R. M. Dudley. Real analysis and probability, volume 74 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002. ISBN 0-521-00754-2. . Revised reprint of the 1989 original.

[24] M. Einsiedler and T. Ward. Ergodic theory. Springer, 2011. ISBN 978-0-85729-020-5.

[25] R. H. Farrell. Representation of invariant measures. Illinois J. Math., 6:447–467, 1962. ISSN 0019-2082.

[26] J. Feldman and C. C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. Trans. Amer. Math. Soc., 234(2):289–324, 1977. ISSN 0002-9947.

[27] C. Gao, Y. Lu, and H. H. Zhou. Rate-optimal graphon estimation. Ann. Statist., 43:2624–2652, 2015.

[28] S. Givant and P. Halmos. Introduction to Boolean algebras. Springer, 2009.

[29] G. Greschonig and K. Schmidt. Ergodic decomposition of quasi-invariant probability measures. Colloq. Math., 84(2):405–514, 2000.

[30] E. Hewitt and K. Stromberg. Real and Abstract Analysis. Springer, 1975.

[31] S. Janson. Graphons and cut metric on sigma-finite measure spaces. 2016. arxiv 1608.01833.

[32] S. Janson. On edge exchangeable random graphs. 2017. arxiv 1702.06389.

[33] S. Janson. On convergence for graphexes. 2017. arxiv 1702.06396.

[34] O. Kallenberg. Multivariate sampling and the estimation problem for exchangeable arrays. J. Theoret. Probab., 12:859–883, 1999. ISSN 0894-9840.

[35] O. Kallenberg. Foundations of Modern Probability. Springer, 2nd edition, 2001.

[36] O. Kallenberg. Probabilistic Symmetries and Invariance Principles. Springer, 2005.

[37] A. S. Kechris. Classical Descriptive Set Theory. Springer, 1995.

[38] J. F. C. Kingman. The representation of partition structures. J. London Math. Soc., 2(18):374–380, 1978.

[39] O. Klopp, A. B. Tsybakov, and N. Verzelen. Oracle inequalities for network models and sparse graphon estimation. 2015. arxiv 1507.04118.

[40] E. D. Kolaczyk. Statistical Analysis of Network Data. Springer, 2009.

[41] X. Li and K. Rohe. Central limit theorems for network driven sampling. 2015. arxiv 1509.04704.

[42] E. Mossel and N. Ross. Shotgun assembly of labeled graphs. 2015. arxiv 1504.07682.

[43] P. Orbanz and D. M. Roy. Bayesian models of graphs, arrays and other exchangeable random structures. IEEE Trans. on Pattern Analysis and Machine Intelligence, 37:437–461, 2015.

[44] J. Pitman. Combinatorial stochastic processes. Lecture Notes in Mathematics. Springer, 2006.

[45] A. W. van der Vaart and J. A. Wellner. Weak convergence and empirical processes. Springer-Verlag, New York, 1996.

[46] V. Veitch and D. M. Roy. The class of random graphs arising from exchangeable random measures. 2015. arxiv 1512.03099.

[47] V. Veitch and D. M. Roy. Sampling and estimation for (sparse) exchangeable graphs. 2016. arxiv 1611.00843.

[48] A. M. Vershik. Kolmogorov’s example (an overview of actions of infinite-dimensional groups with an invariant probability measure). Theory Probab. Appl., 48(2):370–378, 2004.