Gauge Invariant Hamiltonian Formalism for Spherically Symmetric Gravitating Shells

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September 1997

Abstract

The dynamics of a spherically symmetric thin shell with arbitrary rest mass and surface tension interacting with a central black hole is studied. A careful investigation of all classical solutions reveals that the value of the radius of the shell and of the radial velocity as an initial datum does not determine the motion of the shell; another configuration space must, therefore, be found. A different problem is that the shell Hamiltonians used in literature are complicated functions of momenta (non-local) and they are gauge dependent. To solve these problems, the existence is proved of a gauge invariant super-Hamiltonian that is quadratic in momenta and that generates the shell equations of motion. The true Hamiltonians are shown to follow from the super-Hamiltonian by a reduction procedure including a choice of gauge and solution of constraint; one important step in the proof is a lemma stating that the true Hamiltonians are uniquely determined (up to a canonical transformation) by the equations of motion of the shell, the value of the total energy of the system, and the choice of time coordinate along the shell. As an example, the Kraus-Wilczek Hamiltonian is rederived from the super-Hamiltonian. The super-Hamiltonian coincides with that of a fictitious particle moving in a fixed two-dimensional Kruskal spacetime under the influence of two effective potentials. The pair consisting of a point of this spacetime and a unit timelike vector at the point, considered as an initial datum, determines a unique motion of the shell.
1 Introduction

One of the most spectacular and at the same time the most mysterious phenomena in nature is the gravitational collapse. It produces the largest fireworks in the sky and it ends up in a singularity, to which no known physical laws seem to be applicable. The enigma is enhanced by the appearance of horizons before the final stages of collapse. The horizon has some regularizing effect; for example, it prevents the divergence of the total energy due to concentration of matter (unlike the electrodynamics). However, the horizon leaks and there is another mystery: does the Hawking effect lead to violation of unitarity or not?

The theory of gravitational collapse is difficult, not least because the phenomenon seems to belong to the high energy regime of quantum gravity and the progress in quantum gravity is slow. Thus, impatient people try to capture the core of the problem by working with simplified models. This is also the line of the present paper.

Let us consider the simplest models of the collapse: spherically symmetric thin shells (in general with a surface tension) falling in a field of a spherical black hole, and Oppenheimer-Snyder stars. Let us restrict the dynamical systems to their minimum: a fixed “amount of matter” without any internal degrees of freedom. For example, the total rest mass of dust shells and of the Oppenheimer-Snyder star is fixed. Let us, finally, suppress all fields different from gravity. The resulting dynamical systems have just one degree of freedom; this can, at least locally, be described by the radius \( r \) of the matter part of the system (of the shell, of the Oppenheimer-Snyder star).

Such or similar simple systems have been often utilized in literature. For instance, in the treatments investigating relativistic dynamics of spherical domain walls (sometimes called bubbles) \( [1] \), models of gravitational collapse \( [2] \), back reaction in the Hawking effect \( [3] \), or models of quantum black holes \( [4], [5], [6] \).

If we study these papers, we find rather surprisingly that the theory of even such simple systems is afflicted with severe problems. Thus, most papers start with guessing the Hamiltonian (or super-Hamiltonian) of the system to be studied from its equations of motion (eg. \( [1], [3], [2] \)). However, it is well-known that a Hamiltonian and, therefore, an action is not uniquely determined by the equations of motion that it is to generate (this is the so-called inverse problem of variational calculus). In some papers, the Hamiltonians are derived from the first principles but the derivation is extremely complicated \( [3] \). The results of both guesses and derivations are, as a rule, very complicated, non-local Hamiltonians. A complicated Hamiltonian is a problem, if we are going to base a quantization on it and to calculate anything of interest. Sometimes, variables are chosen such that the corresponding canonical formalism becomes manageable but this simplicity is often bought by disregarding large portions of the space of solutions (the physical phase space); this seems to be
the case in [6]. Finally, and this is really bad: different choices of time coordinate along the matter boundary lead to very different forms of Hamiltonian for the same system, thus making the corresponding naive quantizations non-equivalent, and so gauge dependent.

The present paper is an attempt to deal with all these problems in the case of the shells. First, we show that a Hamiltonian of a system with one degree of freedom is uniquely determined (up to a canonical transformation) if its value is prescribed to be the total energy of the system. Very simple methods to calculate such Hamiltonians are presented. Second, we present a careful study of all classical solutions of each respective system. This study reveals that the systems possess more states (ie. classical solutions) and admit more symmetry than various quantum approaches usually assume. There are not only bound states, but also scattering ones, and there is a number of states beyond horizon. The symmetries are related to isometries of Kruskal spacetime. This implies that the motion of the system is not (globally) determined by the value of the pair \((r, \dot{r})\), and that the energy of the system is not a (globally) well-defined function of \(r\) and \(\dot{r}\). Another result is a one-to-one correspondence between the set of classical solutions and that of certain trajectories on a certain spacetime. The spacetime is closely associated with the classical solution; for the shell in the field of a black hole, it is the maximal extension of the \(uv\)-surface of the the part of Kruskal spacetime inside the shell and the trajectories coincide with those of the shell in this spacetime. The dynamics of the trajectories can be obtained from a super-Hamiltonian on which three conditions are imposed: it should be gauge invariant, at most quadratic in momenta and it should reproduce all symmetries in the space of solutions. Under these conditions, the super-Hamiltonian is unique (up to an overal factor). In this way, the variables describing the gravitational field—which are pure gauges in the spherically symmetric case—are reduced away without any choice of coordinate conditions. A similar super-Hamiltonian has been obtained and used for the quantization of thin dust shells with the flat spacetime inside [9]. It follows from the uniqueness of the true Hamiltonians that they can be obtained from our super-Hamiltonian by reduction procedures.

The plan of the paper is as follows. In Sec. 2 we briefly describe the relevant properties of Kruskal spacetime, in particular the symmetries. In Sec. 3 we prove the uniqueness of Hamiltonian discussed above, present some simple methods to derive such Hamiltonians and show two examples of complicated and gauge-dependent Hamiltonians. Sec. 4 is devoted to massive thin shells with arbitrary surface tension. By including tension, we can describe shells made from fundamental fields such as

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1S. K. Blau, E. I. Guendelman and A. H. Guth [8], following the suggestion of S. Coleman, discussed the classical solution of the equation of motion of the spherical domain wall as the solution of the equation of motion of a particle moving in one dimension under the influence of a potential.
scalar fields in the bubbles arising in cosmological phase transitions \cite{1}. Sec. 5 collects what we need about null shells. In Sec. 6, we write down a Hamiltonian formalism for the shells. In Sec. 7, starting from our super-Hamiltonian, we derive the Hamiltonians that describe the motion of null shells in the Kruskal spacetime external to the shell (like Ref. \cite{3}).

2 Kruskal spacetime

All generic (not extreme) black hole spacetimes contain a region which has the same topology as the Kruskal spacetime. It includes two identical stationary asymptotically flat submanifolds that are causally separated by two crossing horizons, and two identical dynamical submanifolds that are time inversions of each other. For the astrophysical description of a stellar collapse, only a half of this spacetime is necessary and relevant. What can be the consequence of the existence of the other half? Apart from the discussions of primordial wormholes or white holes, it seems that it may have some importance in the quantum theory: indeed, for example the popular explanation of Hawking effect uses the states beyond the horizon because they have negative energy with respect to infinity. Let us start by briefly describing the Kruskal spacetime, restricting ourselves to the two-dimensional version of it.

Consider $\mathbb{R}^2$ with double null coordinates $u$ and $v$ and the metric

$$ds^2 = \frac{(4GE)^2}{F} e^{-F} (-du \, dv),$$

where $E > 0$, $G$ is the Newton constant and $F = F(-uv)$ is defined by its inverse,

$$F^{-1}(x) := (x - 1)e^x$$

in the interval $x \in (0, \infty)$ and $-uv \in (-1, \infty)$. The metric is well-defined and analytic in the region $-uv \in (-1, \infty)$. This part of $\mathbb{R}^2$ together with the metric \cite{1} is called Kruskal spacetime $(\mathcal{M}, g)$.

The Kruskal spacetime is time and space orientable. We will call future the orientation in which the function $(u + v)$ is increasing, and right in which $(v - u)$ is increasing. There are two independent discrete isometries that invert these orientations: $T$, defined by $u \mapsto -v$, $v \mapsto -u$ and $C$ by $u \mapsto v$, $v \mapsto u$. Their composition $I = CT = TC$ is called the Kruskal inversion; $I$ inverts both orientations. Observe that $C$ is a new transformation, enabled by the topology of the Kruskal manifold. $C$ has nothing to do with parity; indeed, the parity transformation $P$ leaves both $u$ and $v$ unchanged but transforms $\vartheta$ and $\varphi$ by $\vartheta \mapsto \pi/2 - \vartheta$ and $\varphi \mapsto \varphi + \pi$ in the full four-dimensional manifold.
There is also a continuous one-dimensional group of isometries generated by the Killing vector field

$$\xi := \frac{1}{4GE} \left( -u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right).$$

If \(g(s)\) is an arbitrary element of this group, then

$$Tg(s)T = g(-s), \quad Cg(s)C = g(-s), \quad Ig(s)I = g(s).$$

The signature and orientation of \(\xi\) changes over \(M\). Let us define four quadrants in \(M\) as follows:

- \(Q_I := \{(u,v) \mid u < 0, v > 0\}\)
- \(Q_{II} := \{(u,v) \mid u > 0, v < 0\}\)
- \(Q_{III} := \{(u,v) \mid u > 0, v > 0\}\)
- \(Q_{IV} := \{(u,v) \mid u < 0, v < 0\}\)

Then, the orientation and signature of \(\xi\) is given by the table

| Quadrant | Signature | Orientation |
|----------|-----------|-------------|
| \(Q_I\)  | timelike  | future      |
| \(Q_{II}\)| timelike  | past        |
| \(Q_{III}\)| spacelike | right       |
| \(Q_{IV}\)| spacelike | left        |

The group generated by \(T, C,\) and \(\xi\) is called the *Kruskal group*, \(G_K\).

Two important functions, \(t\) and \(r\), on \(M\) are defined as follows:

- \(r := 2GE F(-uv)\) everywhere,

$$t := 4GE \arctanh \frac{u + v}{-u + v}$$

in \(Q_I \cup Q_{II}\), and

$$t := 4GE \arctanh \frac{-u + v}{u + v}$$

in \(Q_{III} \cup Q_{IV}\). The pair \((t, r)\) of functions can be considered as coordinates within each of the four quadrants; these are usual Schwarzschild coordinates. We have the following transformations:

$$dr = -2GE \frac{e^{-F}}{F} (vdu + udv)$$

on \(M\), and

$$dt = -2GE \frac{vdu - udv}{uv}$$
on $Q_1 \cup Q_{II}$, and
\[ dt = 2GE \frac{vdu - udv}{uv} \]
on $Q_{III} \cup Q_{IV}$, where we can also set
\[ F(-uv) = \frac{r}{2GE}, \quad -uv = \frac{r - 2GE}{2GE} e^{2GE}. \]

Within the quadrants and in the Schwarzschild coordinates, we have $\xi = \partial/\partial t$.

The behaviour of the functions $t$ and $r$ with respect to the Kruskal group is as follows. Let $p \in M$ and $g(s)$ be an element of the component of identity of $G_K$. Then
\[
(t(g(s)p), r(g(s)p)) = (t(p) + s, r(p)), \\
(t(Tp), r(Tp)) = (-t(p), r(p)), \\
(t(Cp), r(Cp)) = (-t(p), r(p)), \\
(t(Ip), r(Ip)) = (t(p), r(p)).
\]

Thus, $r$ is an invariant function of $G_K$, and any invariant function of $G_K$ must be a function of $r$. Moreover, on the part of $M$ covered by the four quadrants, the fixed value $(t, r)$ of the pair of functions $t$ and $r$ defines exactly two points, $p$ and $I(p)$.

## 3 Hamiltonians of simple systems

In this section, we consider spherically symmetric systems that a) are sufficiently simple so that they have just one degree of freedom and b) the matter has a well defined external boundary so that the geometry outside this boundary coincides with a portion of the Kruskal spacetime. We study the ways, in which a Hamiltonian generating the motion of such systems can be obtained from first principles (that is, from the Einstein-Hilbert action to which some matter action is coupled). Our aim is to show that such Hamiltonians are a) unique in certain sense if the time coordinate is chosen, b) very complicated, and c) gauge dependent.

In order to obtain a true Hamiltonian, we have to reduce the system. This, roughly, is the following procedure. First, we have to adapt the coordinates to the spherical symmetry. Second, we have to choose a foliation given by levels of a function $T$ on the phase space and a radial coordinate $X$ along the folii; then, we have to solve the constraints for the momenta $p_T$ and $p_X$ conjugate to $T$ and $X$. The result can be written as
\[ p_T + H(q, p, T) = 0, \]
where \( q \) and \( p \) are the coordinates on the physical phase space of the system; the true Hamiltonian is \( H(q, p, T) \) (see, eg. [10]).

Since the spacetime outside the matter has the Kruskal geometry, we can choose the foliation such that it asymptotically coincides with the Schwarzschild foliation. It is well-known ([10], [11]) that the value of the true Hamiltonian then coincides with the ADM energy \( E \) (\( E = GM \), where \( M \) is the Schwarzschild mass parameter of the external Kruskal spacetime). Observe that the foliation can be different from the foliation by the Schwarzschild time coordinate at the boundary of the matter. Thus, the foliation will determine a time parametrization of the boundary. Let us denote the time parameter by \( s \); the value \( s(p) \) of the parameter at a given point \( p \) of the boundary coincides with the value \( T(p) \) of the function \( T \) at \( p \). In fact, \( s \) can be an arbitrary time parameter along the boundary. In this way, each motion of the system determines a function \( r(s) \), where \( r \) is the Schwarzschild radius of the boundary at the value \( s \) of the time parameter. As the ADM energy is always conserved, \( H \) cannot depend explicitly on \( T \).

Since the system has just one degree of freedom, the function \( r(s) \) can contain all information about the motion (at least locally). In this case, we can choose \( r \) and its conjugate momentum \( p \) as the canonical coordinates on an open set in the physical phase space of the system. The reduction procedure will then result in the true Hamiltonian of the form \( H(r, p) \).

On the other hand, one can also study the dynamical equations of the system. They follow from Einstein’s equations and from the stress-energy tensor conservation for the matter. Solving the gravity part of these equations by the Schwarzschild metric, a non-trivial radial equation remains:

\[
\dot{r} = \rho(r, E),
\]

where the dot denotes the derivative with respect to the parameter \( s \). The system can possess different sectors, so \( \rho \) can also depend on a few discrete parameters. We stress that the function \( \rho \) is completely determined by the dynamical equations; to derive its form, one does not need to employ the canonical formalism and its reductions.

The crucial question now is: How many Hamiltonians of the form \( H(r, p) \) will lead to the equation of motion (3)? It is well-known that there are many such Hamiltonians ([12]). However, if we impose the condition on \( H \) that its value coincides with the total energy \( E \), then the possible Hamiltonians are strongly limited. Indeed, suppose that \( H(r, p) \) is such a Hamiltonian. Then, we must have

\[
\dot{r} = \frac{\partial H}{\partial p}(r, p).
\]
This equation represents the transformation between the variables \((r, \dot{r})\) and \((r, p)\). Thus, the Eq. (3) implies that

\[
\frac{\partial H}{\partial p} = \rho(r, H).
\]

This can be considered as a differential equation for \(H\); it is even a separable first-order ordinary differential equation as \(r\) plays the role of a parameter:

\[
dp = \frac{dH}{\rho(r, H)}.
\]

Hence, the general solution has the form:

\[
p = p_0(r) + \int \frac{dH}{\rho(r, H)},
\]

where \(p_0(r)\) is an arbitrary function of \(r\). Solving Eq. (4) for \(H\), we obtain a family of Hamiltonians of the form:

\[
H = H(r, p - p_0(r)).
\]

The result is clearly non-unique, but this non-uniqueness, represented by the function \(p_0(r)\), can be regarded as the symmetry of the formalism under the transformation

\[
r \mapsto r, \quad p \mapsto p - p_0(r).
\]

It is clear that we always have the freedom to perform such a canonical transformation in the classical Hamiltonian formalism (to simplify the equations, for example). More important, however, is the fact that the transformation (3) belongs to the small class of canonical transformations that can be implemented by a unitary transformation in the quantum theory. Indeed, suppose that we represent the Heisenberg commutation relations for \(r\) and \(p\) as usual:

\[
(\hat{r}_1 \psi_1)(r) = r \psi_1(r), \quad (\hat{p}_1 \psi_1)(r) = -i \psi_1'(r),
\]

and that the wave functions \(\psi_1(r), \phi_1(r)\), etc. span the Hilbert space with the scalar product

\[
(\psi_1, \phi_1) = \int_0^\infty dr \, \psi_1^*(r) \phi_1(r).
\]

Next, we consider another copy of the same Hilbert space, but denote its elements by \(\psi_2(r), \phi_2(r)\), etc. and define a unitary transformation between the spaces by

\[
\psi_2(r) = e^{-iY(r)} \psi_1(r),
\]
where $Y(r)$ is a fixed differentiable function. The form of the operators $\hat{r}_2$ and $\hat{p}_2$ is given by

$$
\hat{r}_2 = e^{iY(r)}\hat{r}_1 e^{-iY(r)} = \hat{r}_1,
$$

$$
\hat{p}_2 = e^{iY(r)}\hat{p}_1 e^{-iY(r)} = \hat{p}_1 - Y'.
$$

Hence, the choice $Y(r) = \int dr \rho_0(r)$ shows our claim. In this sense, the Hamiltonian $H$ can be considered as uniquely determined by 1) the equations of motion, 2) the value of energy and 3) the gauge.

The description of dynamics based on a true Hamiltonian has, however, the two disadvantages discussed in the Introduction: complicated form and gauge dependence.

Let us consider a null dust shell (see Sec. 5); it always moves along a null geodesic in both Kruskal spacetimes, the internal as well as the external one with respect to the shell. In the Schwarzschild coordinates, Eq. (10) holds, and so the functions $t_+(s)$ and $r(s)$ that describe the trajectory of the shell in the external Kruskal spacetime must satisfy the equation

$$
-f_+(r)\dot{t}_+^2 + f_+^{-1}(r)\dot{r}^2 = 0.
$$

Hence,

$$
\left( \frac{dr}{dt_+} \right)^2 = \left( 1 - \frac{2GE}{r} \right)^2,
$$

and

$$
\rho(r, E) = \epsilon \left| 1 - \frac{2GE}{r} \right|,
$$

where $\epsilon = \pm 1$ distinguishes the ingoing from the outgoing shells. In this case, Eq. (5) becomes

$$
p = p_0 - \epsilon \frac{r}{2G} \ln \left| 1 - \frac{2GE}{r} \right|,
$$

and its solution with respect to $E$ yields the form of the Hamiltonian:

$$
H = \frac{r}{2G} \left\{ 1 - \exp \left[ -\epsilon \frac{2G}{r} (p - p_0(r)) \right] \right\}.
$$

This is a very complicated, non-local Hamiltonian.

Next, we choose the Kraus-Wilczek coordinates $\tau$ and $r$ on the external Kruskal spacetime which are defined by

$$
\tau = t_+ + 2\sqrt{2GE} r + 2GE \ln \frac{\sqrt{r} - \sqrt{2GE}}{\sqrt{r} + \sqrt{2GE}},
$$

so that the metric takes the spatially flat form

$$
\text{ds}^2 = -f_+ d\tau^2 + 2\sqrt{\frac{2GE}{r}} dr d\tau + dr^2 + r^2 d\Omega^2.
$$
The equation of a radial null geodesic in this metric reads

\[
\left( \frac{dr}{d\tau} \right)^2 + 2 \sqrt{\frac{2GE}{r}} \frac{dr}{d\tau} - 1 + \frac{2GE}{r} = 0
\]

and its solution is

\[
\frac{dr}{d\tau} = \epsilon - \sqrt{\frac{2GE}{r}},
\]

where \(\epsilon\) has the same meaning as above. The Eq. (5) now reads

\[
G_p = G_{p0}(r) - \sqrt{2GE} \epsilon - \epsilon r \ln \left| \frac{\sqrt{2GE} - \epsilon \sqrt{r}}{\sqrt{r}} \right|. \tag{9}
\]

This will coincide with the Kraus-Wilczek momentum \(p_c\) if one matches the units and the notation and if \(p_0(r)\) is chosen properly:

\[
G_{p0}(r) = \sqrt{2GE} - \epsilon r \ln \left| \frac{\sqrt{2GE} - \epsilon \sqrt{r}}{\sqrt{r}} \right|,
\]

where \(E_-\) is the Schwarzschild energy of the Kruskal spacetime inside the shell. We observe again that the Hamiltonian is very complicated (there is no explicit formula this time that would express the solution of the Eq. (5) with respect to \(E\), and it is also very different from Eq. (7).

In order to solve this problem, we shall try to find a super-Hamiltonian, which will be gauge independent, at most quadratic in momenta, and such that its reduction based on a suitable gauge will give each of the complicatedHamiltonians. Such a super-Hamiltonian can either be itself directly used for quantization, or define the factor ordering of the complicated Hamiltonians.

\section{Massive shells}

In this section, a careful study of the states of massive spherically symmetric dust shells (in general with arbitrary surface tension) will be given. In the following section, the null dust shells will emerge as a (degenerate) simple limit case.

\subsection{The trajectory of the shell}

Consider two Kruskal manifolds \(\mathcal{M}_1\) and \(\mathcal{M}_2\) with energies \(E_-\) and \(E_+\) containing isometric timelike hyper-surfaces \(\Sigma_1\) and \(\Sigma_2\) which divide each \(\mathcal{M}_i\) in two parts, left \(\mathcal{M}_{i-}\), and right \(\mathcal{M}_{i+}\). Let us solder \(\mathcal{M}_{1-}\) to \(\mathcal{M}_{2+}\) along \(\Sigma_1\) and \(\Sigma_2\). Most points in the resulting surface \(\Sigma := \Sigma_1 \cap \Sigma_2\) in the resulting spacetime will have a
neighbourhood in which we can introduce Schwarzschild coordinates on both sides. The metric left (−) and the metric right (+) is given by \((\epsilon = \pm 1)\)

\[
ds^2_\epsilon = -f_\epsilon(r)dt_\epsilon^2 + f_\epsilon^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2),
\]

where

\[
f_\epsilon := 1 - \frac{2GE_\epsilon}{r};
\]

we assume that \(r, \theta\) and \(\phi\) are continuous functions through \(\Sigma\), however, in general, \(t_+ \neq t_-\) at \(\Sigma\). The surface \(\Sigma\) is given by

\[
t_\epsilon = t_\epsilon(s), \quad r = r(s), \quad \theta = \vartheta, \quad \phi = \varphi,
\]

where \(s, \vartheta\) and \(\varphi\) are coordinates on \(\Sigma\) and \(s\) is the proper time along the radial generators of \(\Sigma\). Let us denote by

\[
T_\epsilon^\mu := (i_\epsilon, \dot{r}, 0, 0)
\]

the tangent vector along these generators. Since \(s\) is the proper time, we obtain

\[
f_\epsilon i_\epsilon = \tau_\epsilon \sqrt{f_\epsilon + \dot{r}^2},
\]

where \(\tau_\epsilon := \text{sign}(f_\epsilon i_\epsilon)\). The meaning of the left hand side is \(f_\epsilon i_\epsilon = -g^\epsilon_{\mu\nu} T_\epsilon^\mu \xi^\nu\), where \(\xi^\mu\) is the Killing vector \(\xi^\epsilon\). This would be a total energy with respect to the right infinity for a particle moving in the respective Kruskal spacetime, and it would be a conserved quantity for a geodesic motion. Let \(m_\epsilon^\mu\) be the normal vector to \(\Sigma\), \(m_+\) being external to \(\mathcal{M}_{2+}\), and \(m_-\) to \(\mathcal{M}_{1-}\); we assume \(m_\epsilon^\mu\) to be normalized, or

\[
f_\epsilon i_\epsilon m_\epsilon^0 - f_\epsilon^{-1}\dot{r}m_\epsilon^1 = 0,
\]

\[
f_\epsilon (m_\epsilon^0)^2 - f_\epsilon^{-1}(m_\epsilon^1)^2 = -1.
\]

It follows that

\[
m_\epsilon^0 = \sigma_\epsilon \frac{\dot{r}}{f_\epsilon}, \quad m_\epsilon^1 = \sigma_\epsilon \sqrt{f_\epsilon + \dot{r}^2},
\]

where \(\sigma_\epsilon := \text{sign}(m_\epsilon^\mu \partial_\mu r)\).

There is an important relation between \(\sigma\) and \(\tau\) that we are going to derive (we omit all epsilons in the indices). The Killing vector \(\xi^\mu\) is tangential to \(r = \text{const}\) curves; hence, \(\xi^\mu\) is orthogonal to the vectors \(\rho^\mu\) defined by \(\rho^\mu := g^{\mu\nu} \partial_\nu r\). Thus, there are positive numbers \(\xi\) and \(\rho\), almost everywhere along \(\Sigma\), such that

\[
\left(\frac{\xi^\mu}{\xi}, \rho^\mu \right) \quad \text{or} \quad \left(\frac{\rho^\mu}{\rho}, \xi^\mu\right)
\]

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(the timelike vector first) is an orthonormal dyad. Similarly, \((T^\mu, m^\mu)\) is an orthonormal dyad at the same point so that
\[
T^\mu = a \frac{\xi^\mu}{\xi} + b \frac{\rho^\mu}{\rho},
\]
\[
\kappa m^\mu = b \frac{\xi^\mu}{\xi} + a \frac{\rho^\mu}{\rho},
\]
where \(a\) and \(b\) are some reals and \(\kappa = \pm 1\); if \(\kappa = +1 (−1)\), the two dyads have the same (opposite) orientation. Then,
\[
-g_{\mu\nu} T^\mu \xi^\nu = -a (\text{sign} \xi^\mu) \xi,
\]
\[
\sigma = \text{sign}(g_{\mu\nu} m^\mu \rho^\nu) = \text{sign}(\kappa a (\text{sign} \rho^\mu)),
\]
where \(\text{sign} v^\mu = +1 (−1)\), if \(v^\mu\) is spacelike (timelike). Thus,
\[
\sigma = \kappa (\text{sign} \rho^\mu)(\text{sign} \xi^\mu)(−\tau) = \kappa \tau,
\]
because \((\text{sign} \rho^\mu)(\text{sign} \xi^\mu) = −1. \kappa\) can easily be determined. We distinguish two cases: \(T^\mu\) is future or past oriented, and introduce the corresponding sign \(\eta = \pm 1.\)

Consider first the future oriented case. In \(Q_I\), \(\xi^\mu\) and \(T^\mu\) are both timelike and future oriented, so \(a > 0\). \(m^\mu\) and \(\rho^\mu\) are both spacelike, \(\rho^\mu\) is oriented to the right and \(em^\mu\) to the left; hence, \(\kappa^\epsilon = −\epsilon\.\) In \(Q_{II}\), \(\rho^\mu\) and \(T^\mu\) are both timelike and both future oriented, so \(b > 0\). \(m^\mu\) and \(\xi^\mu\) are spacelike, \(\xi^\mu\) oriented right and \(em^\mu\) left; again, \(\kappa^\epsilon = −\epsilon\.\) A similar analysis for the remaining quadrants confirms the relation \(\sigma^\epsilon = −\epsilon \tau^\epsilon\) for future oriented shells. Suppose now that \(T^\mu\) is past oriented. Then \(\tau\) just changes sign, but everything else (in particular, the \(\sigma^\epsilon\)'s) remains as before; hence, \(\sigma^\epsilon = \epsilon \tau^\epsilon\) for past oriented shells. Thus, the results can be summarized by the equation
\[
\sigma^\epsilon = −\eta \epsilon \tau^\epsilon. \quad (14)
\]

Finally, we calculate the acceleration \(a^\epsilon\) of the shell in the spacetime \(M^\epsilon\) directed out of \(M^\epsilon\). It is defined by
\[
a^\epsilon = m^\epsilon \rho \nabla^\epsilon T^\rho^\epsilon,
\]
where
\[
\nabla^\epsilon T^\rho^\epsilon := \dot{T}^\rho^\epsilon + \Gamma^\rho^\epsilon_{\mu\nu} T^\mu^\epsilon T^\nu^\epsilon,
\]
and \(\Gamma^\rho^\epsilon_{\mu\nu}\) are the Christoffel symbols of the Kruskal metric \(g^\epsilon_{\mu\nu}\). Substituting for \(\dot{t}^\epsilon\) from Eq. (12) and using Eq. (13), we obtain after some rearrangements
\[
a^\epsilon = \sigma^\epsilon \left(\frac{\sqrt{f^\epsilon + r^2}}{\dot{r}}\right). \quad (15)
\]

These relations will be important for the exact counting of possible solutions of Israel’s equation, which determines the motion of the shell.
4.2 The matter of the shell

We assume the shell to be made of ideal fluid. A spherically symmetric shell matter is “isotropic”, and so it is always “ideal fluid”, though the fluid could have some internal degrees of freedom (cf. [13]), which we are suppressing. Thus, we set

\[ T^{kl} = (\rho + p)T^kT^l + p\gamma^{kl}, \]

where \( \gamma_{kl} := g_{\mu\nu}X_{\epsilon k}X^\nu_{\epsilon l} \) is the induced metric on \( \Sigma \),

\[ x^\mu_{\epsilon}(y^0, y^1, y^2) \]

are the embedding functions of \( \Sigma \) in \( M_{1,2} \). If we choose functions (11) as embedding functions we obtain

\[ \gamma_{ss} = -1, \quad \gamma_{\vartheta\vartheta} = r^2(s), \quad \gamma_{\varphi\varphi} = r^2(s) \sin^2 \vartheta; \]

\( \rho \) is the surface mass density and \( p \) is the surface pressure (negative of surface tension). We assume that the state of the shell matter depends only on its total surface area \( A := 4\pi r^2 \) and that the kind of the material is defined by the barotropic equation of state \( p = p(\rho) \). We will also assume that the stress-energy in the shell is conserved (which follows, if there is vacuum outside the shell, from Israel’s equation—see later). Thus,

\[ T^{kl}_{|l} = 0, \]

where the vertical bar denotes the covariant derivative associated with the metric \( \gamma_{kl} \). If one substitutes for the stress-energy tensor, one obtains the energy equation

\[ (\rho T^k)_{|k} + p T^k_{|k} = 0. \]

This is equivalent to

\[ A \frac{d\rho}{dA} + \rho + p(\rho) = 0. \] (16)

The solution to this equation will be of the form \( \rho(A) \) and it will depend on one constant. We define our system to be just one particular solution to Eq. (16). We assume that the solution defines a positive density \( \rho \) for all \( r \in (0, \infty) \). Thus, there is a well-behaved unique rest mass density at any radius. In this way, the system is as elementary as possible: it has no internal degree of freedom and its total energy depends only on its radius and velocity. Finally, we define the so-called mass function \( M(r) \) by

\[ M(r) := A(r)\rho(A(r)); \]

this definition is justified by the fact that \( M(r) \) will appear in many equations at a prominent place (in some cases, it represents the proper mass of the shell).
As an example, consider the equation of state

\[ p = k\rho, \]

where \( k \in (-1, +1) \). The corresponding solution to Eq. (16) reads

\[ \rho = m_1 A^{-k-1}, \]

where \( m_1 \) is a constant. If \( k > 0 \) (positive pressure), then \( M(0) = \infty \) and \( M(\infty) = 0 \); if \( k < 0 \) (positive surface tension), then \( M(0) = 0 \) and \( M(\infty) = \infty \); for \( k = 0 \) (dust), \( M = \text{const.} \)

We summarize: the non-zero components of the stress-energy tensor have the form

\[ T_{ss} = \rho, \quad T_{\vartheta\vartheta} = \frac{p(\rho)}{r^2}, \quad T_{\varphi\varphi} = \frac{p(\rho)}{r^2 \sin^2 \vartheta}. \]  

\[ (17) \]

### 4.3 Israel’s equation

The dynamics of a (massive) shell is governed by equations found by Dautcourt [14] and brought to a nice geometrical form by Israel [15]. Israel’s equation can be written in the form

\[ Q_{+}^{kl} + Q_{-}^{kl} = 8\pi GT^{kl}, \]  

where \( Q_{kl}^{\pm} := L_{k}^{\pm} \gamma_{kl} - L_{k}^{kl} \) and \( L_{k}^{kl} := -m_{\ell}^\mu_{\nu}X_{\ell k}^\mu X_{\ell l}^\nu \) is the second fundamental form of \( \Sigma \) in \( M_{1-} \) or \( M_{2+} \).

The way (18) of writing the Israel equation makes the independence of the equation on the orientation of the surface \( \Sigma \) manifest. It is also invariant with respect to any coordinate change of \( y \)'s, in particular with respect to the time orientation change \( y^0 \mapsto -y^0 \). If we use the embedding formulas (14), we obtain

\[ L_{ss}^\epsilon = a_{\epsilon}, \quad L_{\vartheta\vartheta}^\epsilon = -r \sigma_{\epsilon} \sqrt{f_{\epsilon} + \dot{r}^2}, \quad L_{\varphi\varphi}^\epsilon = -r \sigma_{\epsilon} \sqrt{f_{\epsilon} + \dot{r}^2 \sin^2 \vartheta} \]

where \( a_{\epsilon} \) is given by Eq. (15). Then the trace reads

\[ L_{\epsilon} = -a_{\epsilon} - 2\sigma_{\epsilon} \frac{\sqrt{f_{\epsilon} + \dot{r}^2}}{r}, \]

and

\[ Q_{ss}^{\epsilon} = 2\sigma_{\epsilon} \frac{\sqrt{f_{\epsilon} + \dot{r}^2}}{r}, \quad Q_{\vartheta\vartheta}^{\epsilon} = -\frac{1}{r^2} \left( a_{\epsilon} + \sigma_{\epsilon} \frac{\sqrt{f_{\epsilon} + \dot{r}^2}}{r} \right), \]

\[ Q_{\varphi\varphi}^{\epsilon} = -\frac{1}{r^2 \sin^2 \vartheta} \left( a_{\epsilon} + \sigma_{\epsilon} \frac{\sqrt{f_{\epsilon} + \dot{r}^2}}{r} \right). \]
If we substitute these formulas into the L.H.S. and Eqs. (17) into the R.H.S. of Eq. (18), we obtain Israel’s equation for our model:

\[ \sigma_+ + \sqrt{f_+ + \dot{r}^2} + \sigma_- \sqrt{f_- + \dot{r}^2} = 4\pi G r \rho, \]

(19)

and

\[ a_+ + a_- + \sigma_+ \frac{\sqrt{f_+ + \dot{r}^2}}{r} + \sigma_- \frac{\sqrt{f_- + \dot{r}^2}}{r} = -8\pi G p. \]

(20)

If we substitute for \( a \) from Eq. (15) into (20), then the resulting equation together with the time derivative of Eq. (19) implies the energy equation (16), or the time derivative of Eq. (19) together with Eq. (16) implies the resulting equation. We can summarize: the full dynamics of the shell is contained in the following equations:

\[ f_\epsilon \dot{t}_\epsilon = -\eta \epsilon \sigma_\epsilon \sqrt{f_\epsilon + \dot{r}^2}, \]

(21)

\[ \frac{\sigma_+ \sqrt{f_+ + \dot{r}^2} + \sigma_- \sqrt{f_- + \dot{r}^2}}{r} = \frac{GM(r)}{r}, \]

(22)

where the meaning of the paremeters \( \eta, \epsilon, \sigma \) (all equal to \( \pm 1 \)) was explained above, \( f_\epsilon = 1 - 2GE_\epsilon/r \), and \( M(r) \) is the mass function defined in Sec. 4.2.

4.4 The radial equation

Double squaring Eq. (22), we obtain the radial equation

\[ \dot{r}^2 + V(r) = 0, \]

(23)

where

\[ V(r) = -\frac{G^2 M^2(r)}{4 r^2} - \frac{G(E_+ + E_-)}{r} - \frac{(E_+ - E_-)^2}{M^2(r)} + 1. \]

(24)

From the point of view of the existence and properties of solutions, Eq. (23) is easier to study than Eq. (22). Let us do this study first and postpone the question of how the solutions of these two equations are exactly related.

Eq. (23) is invariant with respect to the transformations

\[ s \leftrightarrow -s, \text{ or } s \mapsto s + s_0, \]

and

\[ E_+ \leftrightarrow E_-, \text{ or } M \leftrightarrow -M. \]

Suppose that the pair \( (E_-, E_+) \) is given. Eq. (23) together with the initial condition \( r(0) = r_0 \) has either no solution, if \( V(r_0) > 0 \), or two solutions, \( r(s) \) and \( r(-s) \), if \( V(r_0) < 0 \). Thus, for each maximal interval \( (r_1, r_2) \) in which \( V(r) < 0 \), we obtain just one or just two maximal solutions, depending on what happens at the
points \( r_1 \) and \( r_2 \), where \( V(r_1) = V(r_2) = 0 \). If \( V'(r_2) \neq 0 \), then we can extend the solution \( r(s) \) by \( r(-s) \) at this point; similarly, if \( r_1 \neq 0 \) and \( V'(r_1) \neq 0 \). In each of these cases, there is only one solution in the interval. If none of these cases takes place, then we have two maximal solutions, \( r(s) \) and \( r(-s) \), in the interval; they can be chosen in some standard way for each interval. Any other solution within the interval can be obtained by the shift \( s \mapsto s + s_0 \) so that it satisfies the initial condition \( r(s_0) = r_0 \) for any \( r_0 \in (r_1, r_2) \). The solutions that reach infinity can only exist if \((E_1, E_2)\) is such that \( V(\infty) \leq 0 \).

Let us consider an example: the dust matter, which has \( p = 0 \) and thus \( M = \text{const}. \) For the potential (24) we obtain

\[
V(r) = -\frac{G^2M^2}{4} \left( \frac{1}{r} \right)^2 - G(E_+ + E_-) \left( \frac{1}{r} \right) - \frac{(E_+ - E_-)^2}{M^2} + 1.
\]

Then

\[
\frac{dV}{dr} = \frac{G^2M^2}{2} \left( \frac{1}{r} \right)^3 + G(E_+ + E_-) \left( \frac{1}{r} \right)^2,
\]

so that

\[
\frac{dV}{dr} > 0
\]
in the whole range \( r \in (0, \infty) \). Further,

\[
V(0) = -\infty, \quad V(\infty) = 1 - \frac{(E_+ - E_-)^2}{M^2}.
\]

For given \( E_+ \) and \( E_- \), there is a unique \( r_m \) such that \( V(r_m) = 0 \) (allowing for the value \( r_m = \infty \)), and the range of possible initial values \( r_0 \) for which a solution exists is then \([0, r_m]\). Hence, we always have a unique solution satisfying \( r(0) = 0 \), if \( r_m < \infty \). There are two solutions, \( r(s) \) and \( r(-s) \), satisfying \( r(0) = 0 \) if \( r_m = \infty \).

We shall have hyperbolic solutions if \( V(\infty) < 0 \), a parabolic solution if \( V(\infty) = 0 \), and elliptic solutions if \( V(\infty) > 0 \). For the elliptic (bound) solutions, \( r_m < \infty \). The unique non-negative solution of the equation \( V(r) = 0 \) is

\[
\frac{G}{r_m} = \frac{2}{M^2} \left( \sqrt{4E_+E_- + M^2} - E_+ - E_- \right).
\]

The parabolic solutions satisfy \( E_+ - E_- = \pm M \). If the maximal radius reaches a horizon, \( r_m = 2GE_\epsilon \), the condition

\[
E_- - E_\epsilon = -\frac{M^2}{4E_\epsilon}
\]

must be satisfied.

We can easily understand the spectrum of \( E_+ \) for a fixed \( E_- \). Three cases can be recognized according to the value of \( E_- \):
Case $E_- < M/2$

If $E_+ \in (E_- + M, \infty)$, we have the hyperbolic solutions, if $E_+ = E_- + M$ parabolic, and for $E_+ \in (0, E_- + M)$ elliptic. At $E_+ = (E_- + \sqrt{E_-^2 + M^2})/2$, $r_m = 2GE_+$; if $E_+$ decreases beyond this value, then $r_m$ also decreases but it does not reach $2GE_-$; the minimal value of $r_m$ is $(GM^2)/(2(M - E_-))$.

Case $E_- \in [M/2, M)$

This is the same as the previous case but if $E_+$ decreases after $r_m$ reaches the value $2GE_+$, $r_m$ can also reach the value $2GE_-$—at $E_+ = E_- - M^2/(4E_-)$; then, as $E_+$ decreases further, $r_m$ increases again reaching the value $(GM^2)/(2(M - E_-)) > 2GE_-$ at $E_+ = 0$.

Case $E_- \in [M, \infty)$

The hyperbolic solutions fill two intervals: for $E_+ \in (E_- + M, \infty)$ they reach the right infinity, and for $E_+ \in (0, E_- - M)$ they reach the left one. We have two parabolic solutions, $E_+ = E_- \pm M$, and in the interval $E_+ \in (E_- - M, E_- + M)$ the solutions are elliptic.

4.5 Relation between solutions of the radial equation and the shell spacetime

The radial equation (23) is a consequence of Israel’s equation (22). Hence, every solution of Eq. (22) must solve Eq. (23). However, as (23) is obtained by squaring (22) twice, some solutions to (23) need not solve (22): one could lose the information about the signs $\sigma$.

For the understanding of this problem, the following observation is crucial. If one calculates the expression $f_\epsilon + \dot{r}^2$ from the radial equation (23), one obtains:

$$f_\epsilon + \dot{r}^2 = J_\epsilon^2,$$

(25)

$$J_\epsilon := \frac{GM(r)}{2r} - \epsilon \frac{E_+ - E_-}{M(r)}.$$  

(26)

Thus, the expression on the left hand side of Israel’s equation,

$$\sigma_+ \sqrt{f_+ + \dot{r}^2} + \sigma_- \sqrt{f_- + \dot{r}^2}$$

(27)

with some signs $\sigma_\pm$ equals to

$$\sigma_+ |J_+| + \sigma_- |J_-|.$$
It follows immediately that the only choice of the signs that satisfies Israel’s equation (22) is
\[ \sigma_\epsilon = \text{sign } J_\epsilon. \] (28)
To obtain a valid shell spacetime, we have to satisfy also Eqs. (12) and (14); this leads to
\[ f_\epsilon(r(s))\dot{t}_\epsilon = -\eta_\epsilon J_\epsilon. \] (29)
Thus, quite surprisingly, each solution of the radial equation (23) generates a solution of Eqs. (21) and (22) which describe the full dynamics of the shell.

Let us construct a shell spacetime with a future oriented shell motion from a solution of the radial equation:

1. Choose an oriented pair \((E_1, E_2)\) of positive numbers; this defines two Kruskal spacetimes, and we will construct a shell spacetime by having \(\mathcal{M}_1\) with \(E_1 = E_-\) to the left and \(\mathcal{M}_2\) with \(E_2 = E_+\) to the right.

2. Choose a solution \(r(s)\) to the radial equation. Then the expressions \(\sigma_\epsilon\) given by Eq. (28) are well-defined for each \(s\) and \(\epsilon\). The function \(r(s)\) solves then Eq. (22) with the signs \(\sigma_\epsilon\).

3. Eq. (29) determines the functions \(t_\epsilon(s)\) up to an additive constant, \(t_\epsilon(s) \mapsto t_\epsilon(s) + c_\epsilon\).

4. The pair of functions \(r(s), t_\epsilon(s)\) given on some common interval of \(s\) determines two different curves in each Kruskal spacetime \(\mathcal{M}_1\) and \(\mathcal{M}_2\). They are related by the inversion map \(I\) (cf. Sec. 2), so only one of them is future oriented. Thus, in each of \(\mathcal{M}_1\) and \(\mathcal{M}_2\), there is a unique surface \(\Sigma_1\) and \(\Sigma_2\), respectively. We cut the spacetimes along it and solder the parts together. The geometry of the resulting spacetime does not depend on \(c_\epsilon\): the constants can be made zero by continuous isometries in each half.

The construction also revealed some ambiguities. If we did not fix the orientation of the shell movement and the order of the pair \((E_- , E_+)\), we would have obtained four non isometrical spacetimes in general. This corresponds to the number of elements in the discrete subgroup of the Kruskal group \(G_K\). Reversing the order of \((E_- , E_+)\) corresponds to the map \(C\) (cf. Sec. 3) performed in each spacetime \((\mathcal{M}_1, E_1)\) and \((\mathcal{M}_2, E_2)\)—the dependence of \(r\) on \(s\) along any curve cannot be changed by such a map. Similarly, if we allowed also for past oriented shells, then the two functions \(t_\epsilon(s)\) and \(r(s)\) would determine two curves in each spacetime that are related by the map \(I\); by inverting the parameter \(s \mapsto -s\), we obtain a future-oriented shell motion. Let us denote the parameter inversion by \(T_s\). It is clear from this discussion that a description of the dynamics that is based solely on the radial
trajectory \( r(s) \) will not distinguish physically different states. More concretely: the radial equation (23) is quadratic in the total energy \( E_+ \); if considered as an equation for \( E_+ \), it has two roots in general. Hence, \( E_+ \) is not a well-defined function on the space of allowed pairs \((r, \dot{r})\).

In fact, there are even more states to be distinguished: sometimes, two spacetimes that are otherwise completely isometric to each other can describe different physical states. The crucial point is whether the isometry moves also the observers or not. Let us assume that the observers in our model are assembled at the right spacelike infinity. If we move the shell by the continuous subgroup of \( G_K \), \( t_+ \mapsto t_+ + c_+ \), without moving the observers, then the physical state will change: the observer will be able to detect \( c_+ \) as a change of the shell arrival time at a fixed radius, for example. Thus, one and the same radial trajectory \( r(s) \) represents \( 4 \times \infty \) different physical states (or \( 2 \times \infty \), if one does not like the past-oriented shells). Observe that the time shift \( t_- \mapsto t_- + c_- \) does not do anything to change the physical state with respect to the right family of observers. Thus, we have to couple the internal and the external times, if we wish to describe the shell dynamics by means of variables that refer to the left (internal) spacetime.

Another important point is that the validity of our equations is not limited to spacetimes with just one shell; as far as there are no crossings, there can be other shells and/or sources. Interesting examples of two-shell spacetimes can be easily constructed using the symmetries. Let us take three Kruskal spacetimes, \((M_1, E_1)\), \((M_2, E_2)\), and \((M_3, E_3)\) such that \( E_1 = E_3 \). The shell \( \Sigma_{12} \) between \( M_1 \) and \( M_2 \) can be taken as the C- or \( \text{IoT}_+ \)-image of the shell \( \Sigma_{23} \) between \( M_2 \) and \( M_3 \). Consider a source lying far left in \((M_1, E_1)\) so that both shells are to the right from it. Such a source can be constructed explicitly by taking for example a suitable Oppenheimer-Snyder star. The motion of both shells is not changed by this source. We can interpret this as follows. The difference \( e_{12} := E_2 - E_1 \) is the contribution of the energy of the shell \( \Sigma_{12} \) to the total energy of the system, and similarly \( e_{23} := E_3 - E_2 \) for \( \Sigma_{23} \); these contributions exactly cancel each other:

\[
e_{12} = -e_{23}.
\]

One can consider \( e \) as the gravitational charge of the shell. Then \( \Sigma_{12} \) and \( \Sigma_{23} \) have opposite gravitational charges. It is interesting to observe that the opposite charges give rise to opposite accelerations. Let us calculate the acceleration \( a_{1-} \) of \( \Sigma_{12} \) in \( M_1 \) directed out of \( M_1 \) and the acceleration \( -a_{3+} \) of \( \Sigma_{23} \) in \( M_3 \) directed into \( M_3 \) (both pointing right). Using the formulas (15), (25), (26) and (28), we obtain an important formula for \( a_\epsilon \):

\[
a_\epsilon = \frac{d}{dr} J_\epsilon.
\]
Substituting $e_{12}$ or $e_{23}$, respectively, for $E_+ - E_-$ in Eq. (26) we obtain:

$$
a_{1-} = \frac{d}{dr} \left( \frac{GM(r)}{2r} + \frac{e_{12}}{M(r)} \right),
$$

$$
-a_{3+} = -\frac{d}{dr} \left( \frac{GM(r)}{2r} - \frac{e_{23}}{M(r)} \right).
$$

However, $e_{12} = -e_{23}$, so $a_{1-} = -(-a_{3+})$ as claimed.

An interesting example is the following. Let $\Sigma_{23}$ be a hyperbolic shell outgoing to the right of the event horizons of both spacetimes $(M_2, E_2)$ and $(M_3, E_3)$; it will reach the right asymptotic region sometimes in future. The other shell, $\Sigma_{12}$, can be constructed by the map $I \circ T_s$ of the first one in $(M_2, E_2)$ and $(M_3, E_3)$, where the map $I$ is interpreted as a map $I : M_3 \rightarrow M_1$. Thus, $\Sigma_{12}$ is a hyperbolic outgoing (with respect to the right infinity) state lying left from the event horizons of both $(M_1, E_1)$ and $(M_2, E_2)$. If we could construct a quantum theory in such a way that these will be a pair of shells whose classical interpretation will coincide with our construction, and if we can get creation of such pairs, spontaneous or induced by some agent, then this may give a model of Hawking effect with some sort of back reaction (cf. [3]). There seem to be other possibilities, too: $\Sigma_{12}$ can be constructed from $\Sigma_{23}$ by $C$. One should try to implement these speculations by a suitable quantization of the system.

## 5 Null shells

The spherically symmetric shell from null matter moving in vacuum has a very simple theory [16], [17]. Its motion with respect to each of the embedding spacetimes $M_1$ and $M_2$ is that of radial null geodesics. Thus, if $x^a_\epsilon$, $a = 0, 1$, are two coordinates in the Kruskal $uv$-plane, and $g_{\epsilon ab}(x)$ the corresponding metric, the dynamical equation of the shell takes the form

$$g_{\epsilon ab} \ddot{x}^a_\epsilon \dot{x}^b_\epsilon = 0,$$

where the dot is the derivative with respect to an arbitrary parameter (in a two-dimensional spacetime, every smooth null curve is a null geodesic). In local Schwarzschild coordinates [10] we obtain

$$-f_\epsilon \dot{r}_\epsilon^2 + f_\epsilon^{-1} \dot{r}^2 = 0. \quad (30)$$

There are only six different types of null geodesics in any two-dimensional Kruskal spacetime (we consider only the future oriented shells):

1. generic geodesics, that reach the right (left) asymptotic region; we distinguish them by the sign parameter $\zeta = +1$ ($\zeta = -1$).
2. outgoing (\(\zeta r > 0\)) and ingoing (\(\zeta r < 0\)) for each \(\zeta\)-type (out- and ingoing is defined with respect to the right infinity).

3. outgoing and ingoing horizons.

Along any generic null geodesics, the coordinate \(r\) is an affine parameter and it behaves strictly monotonically, acquiring all values from the interval \((0, \infty)\). There are only two exceptions: the horizons.

Given two generic null geodesics of the same type, one in \(\mathcal{M}_1\) and the other in \(\mathcal{M}_2\), they define the surfaces \(\Sigma_1\) and \(\Sigma_2\), and there is only one soldering of \(\mathcal{M}_{1-}\) with \(\mathcal{M}_{2+}\) along \(\Sigma_1\) and \(\Sigma_2\): the function \(r\) must be continuous. This is possible for any values of \(E_-\) and \(E_+\). In particular, the outgoing (ingoing) horizon in \(\mathcal{M}_1\) can be soldered only to the outgoing (ingoing) horizon in \(\mathcal{M}_2\), and this only if \(E_1 = E_2\). However, for a null dust shell (which is pressureless), the soldering must be affine (the shell is “affinely conciliable” — cf. [17]), and this is not unique. For the outgoing horizon, using the double null Kruskal coordinates \(u\) and \(v\), a general soldering is given by \(u_\pm = 0, v_+ = v_- + \alpha, \) where \(\alpha\) is an arbitrary constant, and similarly for the ingoing case: \(v_\pm = v_+ = 0, u_+ = u_- + \beta, \) \(\beta\) arbitrary.

It follows that we can define the quantity \(e_{\lambda}\) for all affine parameters \(\lambda\) along any null shell. We define first \(e_{\lambda\epsilon}\) by

\[
e_{\lambda\epsilon} := \frac{dx_\epsilon^a}{d\lambda} \xi_a,
\]

where \(\xi^a\) is the Killing vector (2) of the respective Kruskal spacetime. As \(x_\epsilon^a(\lambda)\) is a null geodesic in each of the Kruskal spacetimes, \(e_{\lambda\epsilon}\) is conserved along the curve on both sides. If the shell reaches an asymptotic region, both geodesics reach the same (right or left) asymptotic region in the respective spacetime, and so both \(e_{\lambda+}\) and \(e_{\lambda-}\) have the same sign. From Eq. (30), we obtain

\[
\left( f, \frac{dt}{d\lambda} \right)^2 = \left( \frac{dr}{d\lambda} \right)^2;
\]

but, in the Schwarzschild coordinates,

\[
e_{\lambda\epsilon} = f \frac{dt}{d\lambda};
\]

Hence, \(e_{\lambda+} = e_{\lambda-}\). Moreover, if \(\lambda = r\), then \(e_{\lambda r} = \zeta\). For null shells, we thus have \(e_{\lambda-} = e_{\lambda+}\), and we can define

\[
e_{\lambda} := e_{\lambda-} = e_{\lambda+}.
\]

The structure of matter of the given null shell constructed according to the recipe above is given by the Barrabès-Israel equations. In particular, for a generic shell, Eqs. (52) and (53) of [17] imply
1) the pressure $p$ inside the shell vanishes (we have null dust if the shell is surrounded by vacuum),

2) the shell stress-energy tensor $T^{ab}$ is given by

$$T^{ab} = \zeta \frac{E_+ - E_-}{4\pi r^2} l^a l^b,$$

where $l^a$ is the tangent vector to the radial null geodesics affinely parametrized by $r$; in the shell coordinates $y^k$,

$$l^k = \frac{\partial y^k}{\partial r}.$$

For a horizon shell (see [16]), the only non-vanishing components of the stress-energy tensor with respect to the coordinates $(v, \vartheta, \varphi)$ (or $(u, \vartheta, \varphi)$) are

$$T^{vv} = \frac{e}{(32GE)^2 \pi^2} \alpha, \quad (T^{uu} = \frac{e}{(32GE)^2 \pi^2} \beta),$$

where $e$ is the basis of natural logarithms and $E = E_+ = E_-$. The energy density of the shell is a measurable quantity (see [17]). In order that this be positive, the condition

$$E_+ - E_- = \zeta |E_+ - E_-|$$

must hold for generic shells, and some conditions on the sign of $\alpha$ and $\beta$ for the horizon shells.

From the formulas (30), (33) and (34) it follows that the matter content of the null shell is independent of their trajectories; they seem to have an internal degree of freedom. However, it is possible, at least formally, to create a relation between $\dot{r}$ and $E_+ - E_-$ by hand, using the arbitrariness of an affine parameter $\lambda$. This will turn out to be necessary in the Hamiltonian formalism where the final justification for this step lies.

Let us define: the affine parameter $\lambda$ along shells is a physical parameter, if

$$f_e \frac{dt_e}{d\lambda} = \eta (E_+ - E_-)$$

where, as in the case of massive shells, $\eta = +1(-1)$ if $l^a$ is future (past) oriented. (This also keeps the mass density positive for generic shells.) Then, Eqs. (31) and (32) imply

$$\left( \frac{dr}{d\lambda} \right)^2 = (E_+ - E_-)^2.$$

This is the “radial equation for the null shells” corresponding to the Eqs. (23) and (24) for massive shells. Together with the time equation

$$f_e \frac{dt_e}{d\lambda} = \eta \zeta \left| \frac{dr}{d\lambda} \right|,$$
it determines the trajectory (the left hand side does not depend on $\epsilon$).

In this form, the null-shell dynamics is just a limit case of the massive-shell dynamics (with the exception of the horizon shells). Indeed, let us define the physical parameter $\lambda$ for massive shells by

$$
\frac{ds}{d\lambda} = M(r(s)).
$$

Then, the radial equation (23) and (24) becomes

$$
\left(\frac{dr}{d\lambda}\right)^2 + \tilde{V}(r) = 0,
$$

(39)

where

$$
\tilde{V}(r) = -\frac{G^2 M^4(r)}{4r^2} - \frac{GM^2(r)(E_+ + E_-)}{r} - (E_+ - E_-)^2 + M^2(r).
$$

(40)

Clearly, $\lim_{M(r)\to 0} \tilde{V}(r) = -(E_+ - E_-)^2$, and Eq. (39) goes over into Eq. (37).

6 Shells as particles on two-dimensional Kruskal spacetimes

In this section, we will try to rewrite the dynamics of the shells as a dynamics of a fictitious particle on a fixed two-dimensional Kruskal background.

Let us study the motion of a point particle on a two-dimensional Kruskal spacetime with the energy $E$ that results from the action

$$
S = \int ds(p_0 \dot{x}^0 + p_1 \dot{x}^1 - \mathcal{N}\mathcal{H}),
$$

(41)

where $x^a$ are some coordinates on the spacetime, which serve as canonical coordinates of the particle, $p_a$ are the conjugate momenta, $\mathcal{N}$ is a Lagrange multiplier,

$$
\mathcal{H} = \frac{1}{2} \left[g^{ab}(p_a - U_{\xi_a})(p_b - U_{\xi_b}) + W^2\right]
$$

(42)

is the super-Hamiltonian, $g^{ab}(x)$ is the Kruskal metric in the coordinates $x^a$, $U(r(x))$ and $W(r(x))$ are some functions. Eq. (42) provides the most general (up to an overall factor that is independent of the momenta) super-Hamiltonian $\mathcal{H}$ with the properties

1. $\mathcal{H}$ is a polynomial of second order in momenta,

2. the momentum dependence of $\mathcal{H}$ has the correct light-cone structure,

3. $\mathcal{H}$ as a function of $x^\mu$ and $p_\mu$ is invariant with respect to $G_K$. 

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The action is clearly invariant with respect to all transformations of coordinates \( x^a \).

Within each quadrant of the Kruskal spacetime, we can transform the canonical coordinates to Schwarzschild coordinates \( t \) and \( r \). The resulting action is

\[
S = \int ds(p_t \dot{t} + p_r \dot{r} - \mathcal{N} \mathcal{H}), \tag{43}
\]

where

\[
\mathcal{H} = \frac{1}{2} \left[ -f^{-1}(p_t + Uf)^2 + f p_r^2 + W^2 \right], \tag{44}
\]

and \( f(r) := 1 - (2GE)/r \).

The action (43) is meaningful only inside the four quadrants but can be considered as a result of a canonical transformation from the variables \( x^a \), which can be global Kruskal coordinates \( u \) and \( v \). The solutions can be matched through the horizons and the matching is well-defined by the transformation to the Kruskal or Eddington-Finkelstein coordinates.

Variation with respect to \( p_t \) and \( p_r \) yields:

\[
\begin{align*}
 f \dot{t} &= -\mathcal{N}(p_t + Uf), \\
 \dot{r} &= \mathcal{N} f p_r.
\end{align*}
\]

We suppose first that \( W \neq 0 \) and choose the gauge \( \mathcal{N} = \eta/W \), where the sign \( \eta \) corresponds, as before, to different time orientations of the shell; then

\[
\begin{align*}
 p_t + Uf &= -\eta W f \dot{t}, \tag{45} \\
 p_r &= \eta W f^{-1} \dot{r}, \tag{46}
\end{align*}
\]

and the constraint (following from the variation of the action with respect to \( \mathcal{N} \))

\[
\mathcal{H} = 0 \tag{47}
\]

implies that the parameter \( s \) is a proper time. Eqs. (43)–(47), and the equation that results from the variation of the action with respect to \( t \),

\[
p_t = \text{const}, \tag{48}
\]

form a complete system of equations of motion. The equation that is obtained by varying the action with respect to the variable \( r \) is a consequence of the system (43)–(48). Inserting Eqs. (46) and (48) into Eq. (47), we obtain

\[
f + \dot{r}^2 = \left( \frac{p_t + Uf}{W} \right)^2. \tag{49}
\]
It is easy to see that Eqs. (49) and (45) are equivalent to the radial equations (25), (26) and the time equation (29), provided that the constants $E$ and $p_t$ are chosen properly. Let $E = E_\epsilon$. Then Eq. (25) with $\epsilon = -1$ will be satisfied if

$$p_t = -E_+,$$

$$Uf_- = E_- - \frac{GM^2(r)}{2r},$$

$$W = M(r).$$

The apparent singularity in the super-Hamiltonian $\mathcal{H}$ can be removed if $\mathcal{H}$ is multiplied by $f(r)$ because $g^{\mu\nu}\xi_{\mu}\xi_{\nu}$ provides another factor $f(r)$ at $U^2$.

There are two important elements in the choices (50)–(52):

1. The choice $E = E_-$: this makes the shell to move on the left spacetime and thus decouples the background along which the shell moves from the gravitational field the shell produces. A great simplification is a consequence.

2. Eq. (50): we require that the total energy of the full system, which is $E_+$, coincides with this conserved quantity.

The above considerations enable us now to formulate the following theorem.

**Theorem 1** There is one-to-one correspondence between the states of the motion of the shell with mass function $M(r)$ and the dynamical trajectories of a point particle with $p_t < 0$ which follow from the variational principle with the action

$$S = \int ds(p_a \dot{x}^a - N \mathcal{H}),$$

where

$$\mathcal{H} = \frac{1}{2} \left[ g^{ab}(p_a - U_-(r)\xi_a)(p_b - U_-(r)\xi_b) + M^2(r) \right],$$

$g^{ab}(x)$ is the Kruskal metric with $E_-$ as energy parameter, and

$$U_-(r) = \frac{1}{f_-} \left( E_- - \frac{GM^2(r)}{2r} \right).$$

**Proof** The construction of the shell spacetime from a given solution of the variational principle (53) goes as follows. The super-Hamiltonian is chosen such that the solution is identical with the shell trajectory in the spacetime $(\mathcal{M}, E) = (\mathcal{M}_1, E_-)$. Then, setting $E_+ = -p_t$, the spacetime $(\mathcal{M}_2, E_+)$ is well-defined and we have to find the shell trajectory there. The function $r(s)$ is the same as in $\mathcal{M}_1$, but $t_+(s)$ must satisfy

$$f_+ \dot{t}_+ = -\eta \left( \frac{GM(r)}{2r} - \frac{E_+ - E_-}{M(r)} \right).$$
By this the trajectory in \( M_2 \) is fixed up to the map given by the discrete isometry \( C \) and a map given by any element of the component of identity of the Kruskal group \( G_K \) (see Sec. 2). The map by \( C \) does not change the resulting spacetime because always only one and the same half \( M_{2+} \) of \( M_2 \) can be soldered to \( M_1^- \). The time shift can be fixed by some coupling of the internal time \( t_- \) to the external time \( t_+ \).

For example, we can require that the point of the trajectory in \( M_1^- \) with \( t_- = 0 \) coincides with the point of the trajectory in \( M_2 \) with \( t_+ = 0 \). (Another example is given in the next section.) We thus obtain a unique state of the shell. On the other hand, given a shell spacetime, the trajectory of the shell in \((M_1, E^-)\) is well-defined and it is a solution of the variational principle (53). There is indeed a one-to-one correspondence, Q.E.D.

Let us show that the variational principle (53) yields also the dynamics of the null shells. If we set \( M(r) \equiv 0 \) in Eqs. (54) and (55), we obtain for the super-Hamiltonian in Schwarzschild coordinates

\[
\mathcal{H} = \frac{1}{2}[-f^{-1}_-(p_t + E_-)^2 + f_- p_r^2].
\]

Let us choose the gauge \( \mathcal{N} = \eta \). Then variations with respect to \( p_t \) and \( p_r \) yield

\[
\begin{align*}
    f_- \dot{t}_- &= -\eta(p_t + E_-), \\
    \dot{r} &= \eta f_- p_r.
\end{align*}
\]

Inserting this into the constraint \( \mathcal{H} = 0 \), we obtain

\[
-f_- \dot{t}_-^2 + f_- \dot{r}^2 = 0.
\]

Thus, we have a null geodesic. Moreover, the comparison of Eqs. (57) with Eq. (34) shows that \( s \) is the physical parameter now. The trajectory of the shell in \((M_2, E_+ = -p_t)\) is determined similarly as for the massive case; the soldering, however, is unique only for the generic shells. For horizon shells, an affine shift stays undetermined; and only this shift determines the spacetime and the energy density of the shell (cf. Eq. (34)).

What is the use of the super-Hamiltonian (54)? The discussion of Sec. 3 allows only the following accurate statement. Any true Hamiltonian \( H \) that results from \( \mathcal{H} \) by some reduction procedure is a meaningful one, if its value coincides with the total Energy of the system (that is with \( E_+ \)). Those reduction procedures of the constraint system (53) that result in a different value of \( H \) are not allowed.

The things have been arranged in such a way that the allowed reduction procedures are for example those which use a stationary foliation. That is, given any spacelike surface \( \mathcal{S} \) of the auxiliary Kruskal spacetime, the stationary foliation is defined as the family \( \{g(s)\mathcal{S} \mid s \in \mathbb{R}\} \), where \( g(s) \) is an element of the component of identity of the Kruskal group \( G_K \). By these foliations, however, not all allowed reduction procedures are exhausted, as an example in the next section will show.
7 Out-Side Story

In this section, we shall transform the dynamics of the shell to the form of particle dynamics on the external spacetime \((M_2, E_+)\). Such schemes are in use but they are complicated. Our aim is to show that they can be obtained from the variational principle (53) by a reduction procedure including a choice of gauge. We shall restrict ourselves to the null shells. As an example, we shall rederive the Kraus-Wilczek Hamiltonian \([3]\) from our super-Hamiltonian.

The physical phase space (whose points are maximal classical solutions) of the generic null shell consists of four (topologically separated) sectors.\(^2\) The subsequent analysis applies to the sector of those shells that reach the right infinity and are outgoing. The other sectors can be dealt with in a completely analogous manner.

To describe the dynamics of the outgoing shells, we choose the Eddington-Finkelstein retarded coordinates \(u\) and \(r\). The metric in the internal spacetime reads

\[
ds^2 = -f_- du^2 - 2dudr + r^2 d\Omega^2.
\]

In these coordinates the super-Hamiltonian (54) with \(M(r) \equiv 0\) is given by

\[
\mathcal{H} = -(p_u + E_-)(p_r + f_{-1}E_-) + \frac{1}{2}f_-(p_r + f_{-1}E_-)^2,
\]

but we allow only the solutions with \(p_r = -f_{-1}E_-\) as the others, with \(p_u = (f_{-}p_r - E_-)/2\), belong to the ingoing sectors. Thus, we can simplify the super-Hamiltonian to \(p_r + f_{-1}E_-\). The equations of motion then read

\[
\dot{u} = 0, \quad \dot{r} = \mathcal{N}, \quad \dot{p}_u = 0, \quad p_r = -f_{-1}E_-.
\]

The phase functions \(u\) and \(p_u\) are thus perennials (constants of motion) spanning the physical phase space \(\Gamma\) in this sector (cf. \([18]\)); on \(\Gamma\), the perennials have the ranges

\[
u \in (-\infty, \infty), \quad p_u \in (-\infty, 0).
\]

The constraint surface \(\Gamma\) is the submanifold \(p_r = -f_{-1}E_-\) of the extended phase space \(\tilde{\Gamma}\) with the coordinates \(r, p_r, u\) and \(p_u\). Thus, we can cover the constraint surface \(\Gamma\) by the coordinates \(r, u\) and \(p_u\) \((r \in (0, \infty))\).

The reparametrization invariant system defined above is to be reduced so that a “true” Hamiltonian results. For that purpose, first a family of transversal surfaces \(\{\Gamma_t\}\)—the so-called time levels—is needed (see \([18]\) for more detail): \(\Gamma_t \subset \Gamma\) for each \(t\), and \(\Gamma_t\) have exactly one (transversal) intersection with each c-orbit (trajectory)

\(^2\)The reader is referred to Ref. \([18]\) and Ref. \([10]\) for some reviews of the constrained dynamics of reparametrization invariant systems in which the concepts we use in this section are described in detail.
given by \( u = \text{const}, \ p_u = \text{const}, \ r = \lambda \). (Here \( t \) is any suitable time parameter—not necessarily the Schwarzschild time). \( \Gamma_t \) can be defined by an equation of the form

\[
t = T(r, u, p_u),
\]

where \( T(x, y, z) \) is a suitable function of the three variables. \( \Gamma_t \) is a symplectic space for each \( t \); as a canonical chart on \( \Gamma_t \), \((u|_{r_i}, p_u|_{r_i})\) can be chosen. Second, for a construction of a Hamiltonian dynamics, a family of time shifts \( \theta_{tt'} : \Gamma_t \rightarrow \Gamma_t \) is necessary such that each \( \theta_{tt'} \) is a symplectic diffeomorphism \([18]\). Clearly, in our case a trivial family of \( \theta_{tt'} \) is obtained by mapping the points \((u, p_u) \in \Gamma_t \) to \((u, p_u) \in \Gamma_t \); this would give a trivial (frozen) dynamics. A more suitable possibility is to employ the momentum \( p_u \) as a generator of time shifts. Since

\[
\{r, p_u\} = 0, \quad \{u, p_u\} = 1,
\]

and \((r, u)\) can be considered as Eddington-Finkelstein coordinates, \( p_u \) generates just the Schwarzschild time shift. The requirement that \( \Gamma_{t+\delta} \) is obtained from \( \Gamma_t \) by \( p_u\delta \) leads to the following condition on \( T \):

\[
\{T, p_u\delta\} = \delta \quad \forall \delta \in \mathbb{R},
\]

or

\[
\frac{\partial T}{\partial u} = 1.
\]

The general solution has the form

\[
T(r, u, p_u) = u + \tilde{T}(r, p_u).
\]

The next requirement we impose is that the transversal surfaces can be interpreted as time-surfaces in the external (right) spacetime. To find such surfaces, we have to couple the external time to the internal one in such a way that the time in \((\mathcal{M}_{2+}, E_+)\) can be expressed as a function of our phase space variables. The most straightforward and useful way is to require that the retarded time be continuous across the shell (this can easily be achieved since it is determined up to a constant),

\[
u = u_- = u_+.
\]

Then, using \( E_+ = -p_u \), we can specify the function \( T \). For example, if we want to employ the surfaces of constant Schwarzschild time \( t_+ \), then we use the formula expressing \( t_+ \) in terms of \( u_+ \) and \( r \),

\[
t_+ = u_+ + r + A \ln \left| \frac{r - A}{A} \right|,
\]

(58)
where $A := -2Gp_u$. Eq. (59) then yields the following form of $T$ (satisfying the condition (58)):

$$T(r, u, p_u) = u + r + A \ln \left| \frac{r - A}{A} \right|.$$  

Finally, instead of the perennials $u$ and $p_u$, we would like to write our dynamics by means of the radial variable $r$ and a corresponding conjugate momentum $p$. Let us study this transformation for general $\tilde{T}$. Of course, $\tilde{T}$ must be such that the function $r$ is a non-trivial coordinate along $\Gamma_t$, that is, there must exist a solution $r = R(t - u, p_u)$ of the equation $t = u + \tilde{T}(r, p_u)$. Thus, we have the identity

$$t - u = \tilde{T}(R(t - u, p_u), p_u).$$  

To find the momentum $p(t, u, p_u)$ canonically conjugate to the coordinate $r = R(t - u, p_u)$, we look for a canonical transformation (for each $t$)

$$(u, p_u) \mapsto (r, p).$$

The transformation will be canonical, if

$$\frac{\partial R}{\partial u} \frac{\partial p}{\partial p_u} - \frac{\partial R}{\partial p_u} \frac{\partial p}{\partial u} = 1. \quad (62)$$

Let us search for $p$ in the form

$$p = P(R(t - u, p_u), p_u).$$

Then Eq. (62) is equivalent to

$$\left( \frac{\partial P}{\partial p_u} \right)_r = \left( \frac{\partial R}{\partial u} \right)_{u=t-\tilde{T}(r,p_u)}^{-1},$$

From the identity (61), we obtain the final result for the momentum conjugate to $r$ in the form

$$P(r, p_u) = P_0(r) - \int dp_u \frac{\partial \tilde{T}(r, p_u)}{\partial r}, \quad (63)$$

where $P_0(r)$ is an arbitrary function of $r$.

The Hamiltonian of Kraus and Wilczek. Let us use the formula (63) to rederive the Kraus-Wilczek Hamiltonian obtained in [3]. Kraus and Wilczek employ the system of coordinates $\tau$ and $r$ in the external Kruskal spacetime ($M_2, E_+$) that has been defined in Sec. 3 (formula (8)). The relations (60) and (8) imply that the retarded time $u_+$ can be expressed by means of $\tau$ as follows:

$$u_+ = \tau - r - 2\sqrt{Ar} - 2A \ln \left| \frac{r}{A} - 1 \right|. $$

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Thus, the transversal surfaces are defined by the function $T$ of the form:

$$T(r, u, p_u) = u + r + 2\sqrt{Ar} + 2A \ln \left| \sqrt{\frac{T}{A}} - 1 \right|.$$ 

The function $T$ satisfies the condition (58) because the foliation of the external spacetime $(\mathcal{M}_2, E_+)$ by the Kraus-Wilczek time $\tau$ is stationary. Comparing the last relation with Eq. (58), we find $\tilde{T}$, from which we obtain

$$\frac{\partial \tilde{T}}{\partial r} = \frac{1}{1 - \sqrt{\frac{A}{r}}}.$$ 

Hence, the formula (63) becomes identical with Eq. (64) with $\epsilon = +1$ and $-p_u = H$.

We can conclude: the non-locality of the Hamiltonian of Kraus and Wilczek is not due to any physical property of the system, but to the choice of gauge and to the reduction procedure.

Acknowledgements

Helpful discussions with D. Giulini, G. Lavrelashvili, K. V. Kuchař and J. Louko are gratefully acknowledged. P.H. is thankful to National Science Foundation grant PHY9507719, to The Tomalla Foundation, Zurich, to The Swiss Nationalfonds and to the Max-Planck-Institut für Gravitationsphysik, Potsdam for support; J. B. is grateful to the Institute for Theoretical Physics, University of Berne, for the kind hospitality, and to the grants Nos. GACR-202/96/0206 and GAUK-230/1996 of the Czech Republic and the Charles University for a partial support.

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