EXISTENCE AND OBSTRUCTIONS FOR THE CURVATURE ON MANIFOLDS WITH BOUNDARY

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Abstract. We consider the problem of studying the set of curvature functions which a given compact and non-compact manifold with non-empty boundary can possess. First we prove that the sign demanded by the Gauss-Bonnet Theorem is a necessary and sufficient condition for a given function to be the geodesic curvature or the Gaussian curvature of some conformally equivalent metric. Our proof conceptually differs from [17] since our approach allow us to solve problems where the conformal method cannot solve. Also, we prove new existence and nonexistence of metrics with prescribed curvature in the conformal setting which depends on the Euler characteristic. After this, we present a higher order analogue concerning scalar and mean curvatures on compact manifolds with boundary. We also give conditions for Riemannian manifolds not necessarily complete or compact to admit positive scalar curvature and minimal boundary, without any auxiliary assumptions about its “infinity”, which is an extension of those proved by Carlotto-Li [11].

1. INTRODUCTION

Motivated by the Uniformization Theorem, M. S. Berger [5] has studied the problem of finding in a given closed surface $M$ a Riemannian metric with prescribed Gaussian curvature $f \in C^\infty(M)$. He has found a sufficient condition when the Euler characteristic is negative, namely, $f < 0$. In a series of papers [35, 36, 37, 38, 39], J. Kazdan and F. Warner have solved completely this problem and have investigated the problem of prescribing the Gaussian curvature (if $\text{dim } M = 2$) and scalar curvature (if $\text{dim } M \geq 3$) in a closed or non-compact manifold. They also addressed the problem of, given a function $f$ in an $n$-dimensional Riemannian manifold $(M^n, g)$, finding a metric conformally equivalent or pointwise conformal to $g$ which has $f$ as its scalar curvature.

Given an $n$-dimensional manifold $M$, we say that two metrics $g$ and $g_0$ are pointwise conformal if there exists $u \in C^\infty(M)$ such that $g = e^ug_0$, while we say they are conformally equivalent if there exists a diffeomorphism $\varphi$ of $M$ such that $\varphi^*g$ and $g_0$ are pointwise conformal. Note that pointwise

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conformal is a special case if \( \varphi \) is the identity. Due to the Uniformization Theorem any two metrics in a closed surface \( M \) are pointwise conformal. For dimension greater than two this is no longer true. The advantage of conformally equivalent setting is that the diffeomorphism \( \varphi \) gives some flexibility to solve problems where the pointwise conformal case is unsolvable.

After the complete resolution of the Yamabe problem, J. F. Escobar [23, 24, 25] raised the problem of finding a metric of constant scalar curvature and constant mean curvature on the boundary in a compact Riemannian manifold \((M^n, g)\) with nonempty boundary and dimension \( n \geq 3 \), the well-known Yamabe problem with boundary. This naturally leads to ask whether we can realize functions, instead of constants, for this problem, the so-called prescribed curvature problem on manifolds with boundary. In [26] J. F. Escobar has worked in this problem in the case that the scalar curvature vanishes. He has found necessary and sufficient condition for a smooth function to be the mean curvature of a pointwise conformal scalar flat metric.

Although conformal deformation seems restrictive, it is useful from a geometric viewpoint and leads us to solve a deceptively innocent-looking nonlinear elliptic equation, whose existence theory has been given using different techniques, including direct methods of calculus of variations, blow-up analysis and Liouville theorems, see e.g. [1, 3, 6, 13, 19, 20, 22, 32, 44, 50].

Our main task in this work is to investigate, in an \( n \)-dimensional Riemannian manifold \((M^n, g)\) with boundary, conditions to realize functions either as Gaussian and geodesic curvatures (for \( n = 2 \)) or scalar and mean curvatures (for \( n \geq 3 \)), of a pointwise conformal or conformally equivalent metric. To better describe our results, first we present the following definitions.

(a) \( \text{CE}(g) := \{ f \in C^\infty(M) : R_{\overline{g}} = f \text{ and } H_{\overline{g}} = 0 \text{ for some metric } \overline{g} \text{ conformally equivalent to } g \} \).

(b) \( \text{CE}^0(g) := \{ h \in C^\infty(\partial M) : R_{\overline{g}} = 0 \text{ and } H_{\overline{g}} = h \text{ for some metric } \overline{g} \text{ conformally equivalent to } g \} \).

(c) \( \text{PC}(g) := \{ f \in C^\infty(M) : R_{\overline{g}} = f \text{ and } H_{\overline{g}} = 0 \text{ for some } \overline{g} \in [g] \} \).

(d) \( \text{PC}^0(g) := \{ h \in C^\infty(\partial M) : R_{\overline{g}} = 0 \text{ and } H_{\overline{g}} = h \text{ for some } \overline{g} \in [g] \} \).

Here \( R_{\overline{g}} \) and \( H_{\overline{g}} \) are the scalar and the mean curvatures of \( \overline{g} \), respectively, for \( n \geq 3 \), and they are replaced by the Gaussian and geodesic curvatures, respectively, for \( n = 2 \). \([g]\) denotes the set of Riemannian metrics on \( M \) pointwise conformal to \( g \).

We first assume that \((M, g)\) is a compact Riemannian surface with boundary. It is known that the Gauss-Bonnet Theorem gives a necessary sign condition on the Gaussian curvature \( K_g \) and geodesic curvature \( \kappa_g \) in terms of the Euler characteristic \( \chi(M) \). Supposing that \( K_g \equiv 0 \) (or \( \kappa_g \equiv 0 \)) we have the following sign condition.

\begin{align*}
\text{If } \chi(M) > 0, \text{ then } \kappa_g \text{ (resp. } K_g) \text{ is positive somewhere.}
\end{align*}

\begin{align*}
\text{(1.1) If } \chi(M) = 0, \text{ then either } \kappa_g \text{ (resp. } K_g) \text{ changes sign or vanishes.}
\end{align*}

\begin{align*}
\text{If } \chi(M) < 0, \text{ then } \kappa_g \text{ (resp. } K_g) \text{ is negative somewhere.}
\end{align*}
Our first result gives the following converse to this sign condition.

**Theorem 1.1.** Let \((M, g)\) be a compact Riemannian surface with boundary.

(a) If \(K_g \equiv 0\), \(\text{CE}^0(g)\) is precisely the set of functions that satisfies the sign condition (1.1).

(b) If \(\kappa_g \equiv 0\), \(\text{CE}(g)\) is precisely the set of functions that satisfies the sign condition (1.1).

Theorem 1.1 shows that the results of J. Kazdan and F. Warner [36] can be extended to surfaces with boundary. We highlight that a similar problem was investigated in [17], but, in addition to recover it as a particular case, new informations are obtained about the desired metric using a different conformal approach.

Let \((M^n, g)\) be an \(n\)-dimensional compact Riemannian manifold with non-empty boundary. If \(g\) is a pointwise conformal metric to \(g\) either with Gaussian and geodesic curvatures equal to \(K\) and \(\kappa\), respectively, or with scalar and mean curvatures equal to \(R\) and \(H\), respectively, then there exists a smooth function \(u\) such that \(\overline{g} = e^{2u}g\), for \(n = 2\), and \(\overline{g} = u^{\frac{4}{n-2}}g\), where \(u > 0\), for \(n \geq 3\). It is well-known that the relations between its curvatures are given by the followings transformations laws

\[
\begin{align*}
\mathcal{L}_g u &\equiv -\Delta_g u + K_g = Ke^{2u} \quad \text{in } M, \\
\mathcal{B}_g u &\equiv \frac{\partial u}{\partial \nu_g} + \kappa_g = \kappa e^u \quad \text{on } \partial M,
\end{align*}
\]

for \(n = 2\), where \(K_g\) and \(\kappa_g\) are the Gaussian and geodesic curvatures of \(g\), respectively, and

\[
\begin{align*}
\mathcal{L}_g u &\equiv -\frac{4(n-1)}{n-2} \Delta_g u + R_g u = R u^{\frac{n+2}{n-2}} \quad \text{in } M, \\
\mathcal{B}_g u &\equiv \frac{2}{n-2} \frac{\partial u}{\partial \nu_g} + H_g u = h u^{\frac{n}{n-2}} \quad \text{on } \partial M,
\end{align*}
\]

for \(n \geq 3\), where \(R_g\) and \(H_g\) are the scalar and mean curvatures of \(g\), respectively. Here \(\nu_g\) is the outward unit normal on \(\partial M\).

For \(n \geq 3\), we consider the following two lowest eigenvalues of the boundary linear problem \((\mathcal{L}_g, \mathcal{B}_g)\), see Section 4.1,

\[
\begin{align*}
\mathcal{L}_g \varphi &\equiv \lambda_1(\mathcal{L}_g) \varphi \quad \text{in } M \quad \text{and} \quad \mathcal{B}_g \varphi = 0 \quad \text{on } \partial M \\
\mathcal{L}_g \varphi &\equiv 0 \quad \text{in } M \quad \text{and} \quad \mathcal{B}_g \varphi = \sigma_1(\mathcal{B}_g) \varphi \quad \text{on } \partial M.
\end{align*}
\]

The signs of \(\lambda_1(\mathcal{L}_g)\) and \(\sigma_1(\mathcal{B}_g)\) are crucial in our study because besides being conformal invariants (see [24, 25]), we will see that they play the same role as did the sign of the Euler characteristic in our investigation of the geodesic and Gaussian curvatures on compact surfaces with boundary.

For high dimension, we present the following two extensions, to compact manifold with boundary, of the results due to J. Kazdan and F. Warner [39, Theorem 3.3] and [38, Theorem 6.2].
Theorem 1.2. Let \((M^n, g)\) be a compact Riemannian manifold with non-empty boundary and dimension \(n \geq 3\).

(a) If \(\sigma_1(L_g) < 0\), then \(\text{CE}^0(g)\) is the set of smooth functions on \(\partial M\) that are negative somewhere.

(b) If \(\sigma_1(L_g) = 0\), then \(\text{CE}^0(g)\) is the set of smooth functions on \(\partial M\) that either change sign or are identically zero on \(\partial M\).

(c) If \(\sigma_1(L_g) > 0\), then \(\text{CE}^0(g)\) is the set of smooth functions on \(\partial M\) that are positive somewhere.

Similarly, we obtain results for \(\lambda_1(L_g)\).

Theorem 1.3. Let \((M^n, g)\) be a compact Riemannian manifold with non-empty boundary and dimension \(n \geq 3\).

(a) If \(\lambda_1(L_g) < 0\), then \(\text{CE}(g)\) is the set of smooth functions in \(M\) that are negative somewhere.

(b) If \(\lambda_1(L_g) = 0\), then \(\text{CE}(g)\) is the set of smooth functions in \(M\) that either change sign or are identically zero on \(\partial M\).

(c) If \(\lambda_1(L_g) > 0\), then \(\text{CE}(g)\) is the set of smooth functions in \(M\) that are positive somewhere.

We also remark that with minor modifications we can obtain Theorem 1.3 and 1.4 of [17], concerning the existence of metrics with prescribed scalar and mean curvature.

The next step is to present a more subtle information on the pointwise conformal deformation. Again, given a compact Riemannian surface \((M, g)\) with boundary and \(K_g \equiv 0\) (resp. \(\kappa_g \equiv 0\)), we show that

1. \(\chi(M) < 0\): In constrast with Theorem 1.1, there exists \(\kappa \in C^\infty(\partial M)\) (resp. \(K \in C^\infty(M)\)) such that \(\int_{\partial M} \kappa da < 0\) (resp. \(\int_M K dv < 0\)) and \(\kappa \notin \text{PC}^0(g)\) (resp. \(K \notin \text{PC}(g)\)). See Theorem 3.5.

2. \(\chi(M) = 0\): There are necessary and sufficient conditions to a function \(\kappa \in C^\infty(\partial M)\) (resp. \(K \in C^\infty(M)\)) belongs to \(\text{PC}^0(g)\) (resp. \(\text{PC}(g)\)). See Theorem 3.3.

3. \(\chi(M) > 0\): We give conditions to assure that functions belong to \(\text{PC}(g)\) and \(\text{PC}^0(g)\). In this case, there are nontrivial obstructions. See Section 3.2.3.

In [12] K. C. Chang and J. Q. Liu have found condition under a positive function \(\kappa\) defined in the boundary of the disk to assure it belongs to \(\text{PC}^0(g)\). Later in [43] P. Liu and W. Huang study this problem when \(\kappa\) possesses some kind of symmetries. This case also was addressed in [40, 42], where they proved a Moser-Trudinger inequality on the boundary of a compact surface. A blow-up analysis has been performed by Y. X. Guo and J. Q. Liu in [31] in order to study the function space \(\text{PC}^0(g)\).

The case \(\text{PC}(g)\) was treated by S. Y. A. Chang and P. C. Yang [14]. They have obtained sufficient condition on \(K\) defined on a domain \(D\) of the round sphere, satisfying \(\text{Area}(D) < 2\pi\) or \(D = \text{hemisphere}\), to assure that
$K \in PC(g)$. The case $2\pi < \text{Area}(D) < 4\pi$ was addressed by K. Guo and S. Hu in [30].

Using a variational approach and blow-up analysis, the problem of prescribing at the same time the Gaussian and geodesic curvature was obtained in [44], also depending on the Euler characteristic. See also [18, 19, 20, 32].

Given a compact Riemannian manifold $(M, g)$ with boundary of dimension $n \geq 3$ and $R_g \equiv 0$ (resp. $H_g = 0$), we show that

1. $\lambda_1(\mathcal{L}_g) < 0$: There is a constant $k < 0$ such that any smooth function $H$ with $k \leq H < 0$ (resp. $k \leq R < 0$) it holds $H \in PC^0(g)$ (resp. $R \in PC(g)$). See Proposition 4.8.
2. $\lambda_1(\mathcal{L}_g) = 0$: There are necessary and sufficient conditions to a function $H \in C^\infty(\partial M)$ (resp. $R \in C^\infty(M)$) belongs to $PC^0(g)$ (resp. $PC(g)$), respectively. See Theorem 4.10.
3. $\lambda_1(\mathcal{L}_g) > 0$: There are nontrivial obstructions to a smooth function to belong to $PC^0(g)$ or $PC(g)$. See Section 4.2.3.

This pointwise conformal problem has been extensively studied in the last few decades. For a general description of the problem see [18, Section 1.3] and the references contained therein.

Another natural problem is to study global constraints that the topology of a manifold with possibly nonempty boundary imposes on its differential geometric structures. The most basic is the Gauss-Bonnet theorem which gives topological obstructions on the curvature depending on the Euler characteristic. Such obstructions are not obvious in dimension $n \geq 3$, and just few results are known. For example, by Theorem 1.4 of [17] (which also can be obtained by the methods of this paper) there are no topological obstructions to scalar curvatures (resp. mean curvature) which may change sign as long as they are negative somewhere.

Using the Bochner technique A. Lichnerowicz [41] found the first topological obstructions to metrics with positive scalar curvature in closed manifolds. An analogous scenario on manifolds with boundary was established by M. Gromov and H.B. Lawson [29] which showed that the problem of characterizing compact manifolds with boundary endowing a metric with positive scalar curvature and strictly mean convex boundary reduces to a double argument (see [11] for a nice exposition of this doubling scheme), see also [4, 21] for more results.

We also address the problem of deforming a metric of a Riemannian manifold not necessarily complete or compact into one with positive scalar curvature, which was proved by A. Carlotto and C. Li [11, Section 2] for compact manifolds with boundary (see also [17, Theorem 1.5]).

**Theorem 1.4.** Let $(M^n, g)$ be a Riemannian manifold (not necessarily complete or compact) with nonempty boundary and dimension $n \geq 3$. Suppose that $g$ is scalar flat and that it has minimal boundary.
(a) If the Ricci tensor is not identically zero, then \( M \) carries a metric \( g_1 \) with positive scalar curvature with minimal boundary and also satisfying \( c_1 g \leq g_1 \leq c_2 g \) for some positive constants \( c_1, c_2 \).

(b) If the second fundamental form of \( \partial M \) is not identically zero, then \( M \) carries a metric \( g_1 \) on \( M \) with non-negative scalar curvature and strictly mean convex boundary. Besides \( g_1 \) satisfies \( c_1 g \leq g_1 \leq c_2 g \) for some positive constants \( c_2 > c_1 > 0 \).

In any case, \( g_1 \) is complete if and only if \( g \) is complete.

In particular we have

**Corollary 1.5.** Let \( M \) be a manifold with nonempty boundary and dimension \( n \geq 3 \) endowed with a complete Riemannian metric of non-negative scalar curvature and minimal boundary. Then either \( M \) carries a complete metric of positive scalar curvature and minimal boundary or \( M \) carries a Ricci flat metric with minimal boundary.

Together with the splitting theorem [16, Theorem 1], we obtain.

**Corollary 1.6.** Let \( X \) be a manifold (compact or non-compact) with dimension \( n \geq 3 \). Suppose that \( g \) is a complete metric on \( M = \mathbb{T}^{n-1} \times [0,1]\#X \) with non-negative scalar curvature and minimal boundary. Then either \( M \) admits a metric of positive scalar curvature and minimal boundary or \( g \) is Ricci flat, in this case \( M \) is compact.

The condition on the Ricci tensor in Theorem 1.4 is not necessary, since the metric \( g = 1/(1 + |x|^2)g_0 \) on \( \mathbb{R}^n \), where \( g_0 \) is standard flat metric, has scalar curvature \( R_g > \text{const} > 0 \) and minimal boundary.

Theorem 1.4 is a counterpart in the setting of complete manifolds with boundary of those of [7, 35]. The idea is to deform the initial metric to a nearby metric with certain good properties, and then to deform this new metric pointwise conformally to one with the desired properties. As an application, we extend the trichotomy result proved in [11, Proposition 2.5] for non-compact manifolds (see Proposition 5.5).

Let us conclude this introduction with a brief description of the structure of this work. In Section 2 we present a technical perturbation result useful to guarantees the invertibility of a certain operator related to the curvature. In Section 3 we prove the Theorem 1.1 and address the problem of prescribing the curvatures of surfaces with boundary in a conformal class depending on the sign of the Euler characteristic. We also study a higher order analogous in Section 4, proving Theorem 1.2 and 1.3, and establishing some existence and nonexistence of conformal metric depending on sign of the lowest eigenvalues of (1.4) and (1.5). Lastly, the problem of deforming a Riemannian metric with zero scalar curvature and zero mean curvature into one with positive scalar curvature is treated in Section 5, with the proof of Theorem 1.4. We also present a trichotomy result as an application (see Proposition 5.5).
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2. A perturbation result

Throughout this paper, we let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold with nonempty boundary \(\partial M\). With respect to this metric, the volume and area element will be denoted by \(dv\) and \(da\), respectively. If \(n = 2\) the Gaussian and the geodesic curvature will be denoted by \(K_g\) and \(\kappa_g\), respectively. While, if \(n \geq 3\), the scalar and the mean curvature will be denoted by \(R_g\) and \(H_g\), respectively. Given a function \(f \in C^\infty(M)\), the Laplacian with respect to \(g\) will be denoted by \(\Delta_g f = \text{div}\nabla f\).

If \(\overline{g}\) is a pointwise conformal metric to \(g\), then written \(\overline{g} = e^{2u} g\), for \(n = 2\) the relations between their Gaussian and mean curvature is given by (1.2). While for \(n \geq 3\), written \(\overline{g} = u^{-2} g\), with \(u > 0\), then the relation between their scalar and mean curvature is given by (1.3). For \(n = 2\) define

\[
T_1(u) := e^{-2u} \mathcal{L}_g u \quad \text{and} \quad Q_1(u) := e^{-u} \mathcal{B}_g u,
\]

and for \(n \geq 3\) and \(u > 0\), define

\[
T_2(u) := u^{-a} \mathcal{L}_g u \quad \text{and} \quad Q_2(u) := u^{-b} \mathcal{B}_g u.
\]

Finally, define

\[
F_i(u) = (T_i(u), Q_i(u)), \quad i = 1, 2.
\]

In order to solve the equation \(F_i(u) = (f, h)\) for functions \(f\) and \(h\), we will consider the linearization of \(F\) with respect to \(u\), which is given by \(F'_i(u)v = (T'_i(u)v, Q'_i(u)v)\), where

\[
\begin{aligned}
T'_1(u)v &= -e^{-2u}[\Delta_g v - 2(\Delta_g u - K_g)v] = -e^{-2u} A_1(u)v, \\
Q'_1(u)v &= -e^{-u} \left( -\frac{\partial v}{\partial v_g} + \left( \frac{\partial u}{\partial v_g} + \kappa_g \right) v \right) = -e^{-u} B_1(u)v,
\end{aligned}
\]

for \(n = 2\), and for \(n \geq 3\) and \(u > 0\) we have

\[
\begin{aligned}
T'_2(u)v &= -\alpha u^{-a} \left( \Delta_g v - \left( \frac{\Delta_g u}{u} + (a - 1) \frac{R_g}{\alpha} \right) v \right) = -\alpha u^{-a} A_2(u)v, \\
Q'_2(u)v &= -\beta u^{-b} \left( -\frac{\partial v}{\partial v_g} + \left( \frac{b}{u} \frac{\partial u}{\partial v_g} + \frac{b - 1}{\beta} H_g \right) v \right) = -\beta u^{-b} B_2(u)v,
\end{aligned}
\]

where \(\beta = 2/(n - 2)\), \(\alpha = 2(n - 1)\beta\), \(a = (n + 2)/(n - 2)\) and \(b = n/(n - 2)\).

It is standard to check that each pair \((A_i(u), B_i(u)), i = 1, 2\), is a self-adjoint linear elliptic operator satisfying \(\ker(A_i(u), B_i(u)) = \ker F'_i(u)\). Before we state our next lemma we observe that if \(v \in \ker(A_i(u), B_i(u))\) is a nontrivial function, then \(v\) is a solution of the problem

\[
\begin{aligned}
A_i(u)v &= 0 \quad \text{in} \ M \\
B_i(u)v &= 0 \quad \text{on} \ \partial M
\end{aligned}
\]
which is the well-known eigenvalue problem under Robin boundary condition. The first eigenvalue has unit multiplicity and the associated eigenfunction can be taken to be everywhere positive on \( M \) (See [11, Appendix A]). This implies that \( \dim \ker (A_1(u), B_1(u)) = 1 \). Now we state the following Perturbation lemma.

**Lemma 2.1 (Perturbation Lemma).** Let \( (M^n, g) \) be a compact Riemannian manifold with nonempty boundary and dimension \( n \geq 2 \). Suppose that

\[
F_t'(u) : W^{2,p}(M) \to L^p(M) \oplus W^{1,p}_0(\partial M)
\]
is not invertible. Then there exists a smooth function \( z \) such that

\[
\dim \ker F'(u + tz) = 0,
\]

for \( |t| > 0 \) sufficiently small.

**Proof.** Suppose that \( \dim \ker T_t'(u) = 1 \). Given a smooth function \( z \), for each \( i = 1 \) and \( 2 \), there exist normalized functions \( v_i(t) \) and numbers \( \lambda_i(t) \), such that \( v_i(0) \) is a nonzero function in \( \ker T_t'(u) \), \( \lambda_i(0) = 0 \) and

\[
A_i(u + tz)v_i(t) = \lambda_i(t)v_i(t) \quad \text{in} \ M
\]
\[
B_i(u + tz)v_i(t) = 0 \quad \text{on} \ \partial M.
\]

By [45, Lemma A.1] (see also [11, Lemma A.2]) the maps \( t \mapsto \lambda_i(t) \) and \( t \mapsto v_i(t) \) are smooth. We will use the following notation

\[
A_i'(\varphi) = \frac{d}{dt} \bigg|_{t=0} A_i(u + tz)\varphi \quad \text{and} \quad B_i'(\varphi) = \frac{d^2}{dt^2} \bigg|_{t=0} B_i(u + tz)\varphi.
\]

where \( \varphi \) is any smooth function. Since \( \lambda_i(0) = 0 \), taking the derivatives of (2.4) at \( t = 0 \), we get

\[
A_i'(v_i) + A_i(v_i') = \lambda_i'v_i \quad \text{and} \quad B_i'(v_i) + B_i(v_i') = 0
\]

Since \( \|v_i\|_{L^2(M)} = 1 \), then \( \langle v_i, v_i' \rangle = 0 \). At \( t = 0 \) we have \( A_i(v_i) = 0 \) and \( B_i(v_i) = 0 \). Since the pair \( (A_i(u), B_i(u)) \) is self-adjoint we get \( \langle v_i, A_i(v_i') \rangle + \langle v_i, B_i(v_i') \rangle = 0 \). Therefore, by (2.5) and (2.5) we obtain

\[
\lambda_i' = \langle A_i'(v_i), v_i \rangle + \langle A_i(v_i'), v_i \rangle = \langle A_i'(v_i), v_i \rangle + \langle B_i'(v_i), v_i \rangle.
\]

Now we have two cases.

**Case 1:** \( i = 1 \). In this case, for any smooth function \( \varphi \) we have \( A_1'(u)\varphi = -2\varphi\Delta_g z \) and \( B_1'(u)\varphi = \frac{\partial z}{\partial v_1} \varphi \). Thus, by (2.6) at \( t = 0 \) we get

\[
\lambda_1' = -2\langle \Delta_g z, v_1^2 \rangle + \left\langle \frac{\partial z}{\partial v_1}, v_1^2 \right\rangle.
\]

Now if we choose \( z \) identically zero in a neighborhood of \( \partial M \), using integration by parts we get \( \lambda_1' = -2\langle \Delta_g z, v_1^2 \rangle = -2\langle z, \Delta_g v_1^2 \rangle \).

Thus, if \( v_1(0) \) is not constant we can choose \( z = \varphi \Delta_g v_1(0)^2 \), where \( \varphi \) is a nonnegative smooth function which vanishes in a neighborhood of \( \partial M \). This implies that \( \lambda_1' = -2\langle \varphi, (\Delta_g v_1(0)^2)^2 \rangle < 0 \).
Suppose $v_1(0)$ is constant, using integration by parts in (2.7) we obtain

$$
\lambda_1' = -v_1(0)^2 \int_{\partial M} \frac{\partial z}{\partial \nu_g}.
$$

It is not difficult to prove the existence of a function $z$ with $\partial z/\partial \nu_g > 0$.

**Case 2:** $i = 2$. In this case $u > 0$ and for any smooth function $\varphi$ we have $A'_2(u)\varphi = -\frac{\Delta u}{u} W(z)$ and $B'_2(u)\varphi = -\frac{b}{2} V(z)$, where $W(z) = \Delta_g z - \Delta g u$ and $V(z) = \frac{z}{u} \frac{\partial u}{\partial \nu} - \frac{\partial \nu}{\partial \nu}$. By (2.6) at $t = 0$ we have

$$
\lambda_2' = -a \left\langle \frac{v_2^2}{u}, W(z) \right\rangle - b \left\langle \frac{v_2^2}{u}, V(z) \right\rangle.
$$

(2.8)

Similar to the case 1, if $W(v_2^2/u)$ is not identically zero, we consider $z = \varphi W(v_2^2/u)$, where $\varphi$ is a nonnegative smooth function which vanishes in a neighborhood of $\partial M$. Using integration by parts we get $\lambda_2' = -a(\varphi, W(v_2^2/u)^2) < 0$. If $W(v_2^2/u) \equiv 0$, then by (2.8) we obtain

$$
\lambda_2' = - \int_{\partial M} \left( (a - b) \frac{v_2^2}{u} \frac{\partial z}{\partial \nu_g} + z \left( \frac{v_2^2}{u^2} \frac{\partial u}{\partial \nu_g} - a \frac{\partial}{\partial \nu_g} \left( \frac{v_2^2}{u} \right) \right) \right),
$$

with $a - b \neq 0$. It is not difficult to find a smooth function $z$ such that $z = 0$ on $\partial M$ and $\frac{\partial z}{\partial \nu} < 0$ (for instance, a smooth function that coincides in a neighborhood of the boundary with the distance function of a point to the boundary $\partial M$).

**Theorem 2.2** (Perturbation Theorem). Let $(M^n, g)$ be a Riemannian manifold with nonempty boundary of dimension $n \geq 2$. Then, the operator

$$
F_1'(u) : W^{2,p}(M) \to L^p(M) \oplus W^{1,p}(\partial M)
$$

is invertible on an open dense set of functions $u \in C^2(M)$, where we assume that $u$ is positive if $n \geq 3$.

**Proof.** Since $F_1'(u)$ depends continuously on $u$ (See [11, Appendix A]), then the openness assertion follows. By Theorem 2.20 in [27], the operator $F'(u)$ is invertible if and only if $\ker F_1'(u) = \ker (A_i(u), B_i(u)) = 0$. The density condition follows by Lemma 2.1.

Consider a surface with boundary endowed with a flat metric and geodesic boundary (e.g. the half torus). Notice that $F_1'(0) \cdot v = (-\Delta_g v, \partial v/\partial \nu_g)$ is an example of non-invertible operator, since its kernel is composed by constants. However, Perturbation Theorem 2.2 is useful to guarantees that for any $\varepsilon > 0$ there is a smooth function $u_0$ so close to zero that $F_1'(u_0)$ is invertible and $\|F(u_0)\|_{\infty} \leq \varepsilon$. In that regard, we highlight that in some situations this Perturbation Theorem is not needed, as for example, for a surface $M$ with negative Gaussian curvature and geodesic boundary, since $F_1'(u_0)$ is already invertible.
3. Curvature functions for surfaces with boundary

The main goal of this section is to study the problem of prescribe the Gaussian and geodesic curvature in surfaces with boundary from a conformal viewpoint. The focus of our results concerns the case in which at least one of the curvatures is zero. First we will prove the Theorem 1.1, and then we will address the problem of prescribe the curvatures in a conformal class.

In order to prove the Theorem 1.1, some preliminary results are needed. The first describes the $L^p$ and the $W^{1/2,p}$ closure of the orbit of functions under the group of diffeomorphisms of $M$.

**Lemma 3.1** (Approximation Lemma [17]). Let $(M^n, g)$ be a Riemannian manifold with nonempty boundary of dimension $n \geq 2$.

(a) Let $f \in C^\infty(\partial M) \cap W^{1/2,p}(\partial M)$. Given $h \in W^{1/2,p}(\partial M)$, if $\min f \leq h(x) \leq \max f$ on $\partial M$, then for all $\varepsilon > 0$ there exists a diffeomorphism $\varphi$ of $M$ such that, for $p > 2n$, we have that

$$\|f \circ \varphi - h\|_{W^{1/2,p}(\partial M)} < \varepsilon.$$

(b) Let $f \in C^\infty(M) \cap L^p(M)$. Given $h \in L^p(M)$, if $\min f \leq h(x) \leq \max f$ on $M$, then for all $\varepsilon > 0$ there is a diffeomorphism $\varphi$ of $M$ such that, for $p > n$, we have that

$$\|f \circ \varphi - h\|_{L^p(M)} < \varepsilon.$$

Another fundamental tool is the Inverse Function Theorem for Banach Spaces (see for instances [28]). Since $F_i'(u) = (T_i'(u), Q_i'(u))$ is a linear elliptic operator and

$$F_i'(u) : W^{2,p}(M) \to L^p(M) \oplus W^{1/2,p}(\partial M), \quad i = 1, 2,$$

is a $C^1$ map if $n > 2p$ and $F_i'(u)$ is invertible, then there is an $\delta > 0$ such that given $(f, h) \in C^\infty(M) \times C^\infty(\partial M)$ with

$$\|(f, g) - F_i(u)\|_{L^p \oplus W^{1/2,p}} < \delta,$$

then there is a $v \in C^\infty(M)$ such that $F(v) = (f, h)$.

3.1. **Proof of Theorem 1.1.**

*Proof of Theorem 1.1.* We prove the sufficiency condition of item (a) since the necessity of (1.1) is clear. Assume that $\kappa$ satisfies (1.1). Because the celebrated Osgood, Phillips, and Sarnak uniformization theorem for surfaces with boundary [47] (see also Brendle [8, 9]) we can assume that $g$ is a Gauss flat metric with constant geodesic curvature equal to $c$. In this case we have $F_1(0) = (T_1(0), Q_1(0)) = (0, c)$, where $F_1$ is defined in (2.3). This implies that the result is trivial if $\kappa$ is constant.

So, let us assume that $\kappa$ is not constant. First let us suppose that

$$\min \kappa < c < \max \kappa.$$
The Perturbation Theorem 2.2 shows that given any $\varepsilon > 0$, there exists a smooth function $u_1$ sufficiently close to $u_0 = 0$ such that
\[
\|F_1(0) - F_1(u_1)\|_{L^p@W^{1,p}} = \|(0, c) - F_1(u_1)\|_{L^p@W^{1,p}} < \varepsilon,
\]
and $F_1'(u_1)$ is invertible. By (3.1), for $\varepsilon > 0$ small enough we have
\[
\min \kappa < Q_1(u_1) < \max \kappa.
\]
If $p > 4$, the Inverse Function Theorem [28] implies that there exists $\delta > 0$ such that if $(f, h) \in C^\infty(M) \times C^\infty(\partial M)$ and $\|(f, h) - F_1(u_1)\|_{L^p@W^{1,p}} < \delta$ then there is $u \in C^\infty(M)$ such that $F_1(u) = (f, h)$. We can take $u$ smooth because the operator is elliptic.

By Lemma 3.1 there is a diffeomorphism $\varphi$ of $M$ such that
\[
\|\kappa \circ \varphi - Q_1(u_1)\|_{W^{1/2,p}} < \delta/2.
\]
Thus, for $\varepsilon < \delta/2$ we have $\|(0, \kappa \circ \varphi) - F_1(u_1)\|_{L^p@W^{1,p}} < \delta$. Therefore, we can find a smooth function $u$ such that $F_1(u) = (0, \kappa \circ \varphi)$.

Finally, suppose that (3.1) does not hold. In this case we use (1.1) to conclude that $\kappa$ and $c$ have the same sign at some point of $M$. Thus, there is a constant $m > 0$ such that the function $m\kappa$ satisfies (3.1). By the previous case, there exists a smooth function $u$ and a diffeomorphism $\varphi$ such that $F_1(u) = (0, m\kappa \circ \varphi)$.

The item (b) is similar since Osgood, Phillips, and Sarnak uniformization theorem also provides a conformal metric of constant Gaussian curvature and geodesic boundary.

3.2. Pointwise conformal metrics in surfaces with boundary. Let $(M, g)$ be a compact Riemannian surface with nonempty boundary. In contrary to the conformally equivalent case, let us divide the pointwise conformal analysis for surfaces with boundary in three parts depending on the sign of the Euler characteristic: $\chi(M) = 0$, $\chi(M) < 0$ and $\chi(M) > 0$. Here we extend to the manifold with boundary context results for closed manifold contained in [36].

3.2.1. Case 1: $\chi(M) = 0$. In this case we will prove a sufficient and necessary condition to functions belong to PC($g$) or PC°($g$), see Definition in introduction. We start the analysis by proving the following result.

**Proposition 3.2.** Let $(M, g)$ be a compact Riemannian surface with non-empty boundary.

(a) Given $\kappa \in C^\infty(\partial M)$, with $\kappa \neq 0$, there exists a solution $u \in C^\infty(M)$ to the problem $\Delta_g u = 0$ in $M$ with $\partial u/\partial \nu_g = \kappa e^u$ on $\partial M$ if and only if, $\kappa$ changes sign and $\int_{\partial M} \kappa da < 0$.

(b) Given $K \in C^\infty(M)$, with $K \neq 0$, there exists a solution $u \in C^\infty(M)$ to the problem $-\Delta_g u = Ke^{2u}$ in $M$ with $\partial u/\partial \nu_g = 0$ on $\partial M$ if and only if, $K$ changes sign and $\int_{\partial M} K dv < 0$. 
Proof. Let us prove item (a). Suppose $u \in C^\infty(M)$ is such that $\Delta_g u = 0$ in $M$ and $\partial u / \partial \nu_g = \kappa e^u$ on $\partial M$. Using integration by parts we get

$$0 = \int_M \Delta_g u dv = \int_{\partial M} \frac{\partial u}{\partial \nu_g} da = \int_{\partial M} \kappa e^u da,$$

which implies that $\kappa$ changes sign. Since $u \not\equiv 0$, then

$$\int_{\partial M} \kappa da = \int_{\partial M} e^{-u} \frac{\partial u}{\partial \nu_g} da = \int_M e^{-u} \Delta_g u dv - \int_M e^{-u} |\nabla_g u|^2 dv < 0. \quad (3.2)$$

Now, suppose that $\kappa$ changes sign and $\int_{\partial M} \kappa da < 0$. Define

$$D := \left\{ w \in W^{1,2}(M) : \int_{\partial M} \kappa e^w da = 0 \quad \text{and} \quad \int_M w dv = 0 \right\}.$$

Using the hypothesis, it is not difficulty to show that $D \neq \emptyset$ and $w \neq 0$ on $\partial M$ for all $w \in D$. Now define the functional $F : D \to \mathbb{R}$ given by

$$F(w) := \int_M |\nabla_g w|^2 dv \geq 0.$$

Let $a := \inf_{w \in D} F(w)$ and let $\{w_k\}$ be a minimizing sequence in $D$. Note that

$$\|w_k\|_{W^{1,2}}^2 = F(w_k) + \int_M w_k^2 dv.$$

By Poincaré-Sobolev inequality, there exists a constant $c > 0$ such that for any $w \in W^{1,2}(M)$ with $\int_M w dv = 0$, it holds

$$\int_M w^2 dv \leq c \int_M |\nabla_g w|^2 dv, \quad (3.3)$$

Thus

$$\|w_k\|_{W^{1,2}}^2 \leq c F(w_k),$$

which implies that $\{w_k\}$ is bounded in $W^{1,2}(M)$. Since a ball in any Hilbert space is weakly compact, there exists a function $w \in W^{1,2}(M)$ and a sub-sequence of $\{w_k\}$, still denoted by $\{w_k\}$, converging weakly to $w$. Thus $\int_M w dv = 0$.

By [18, Proposition 3.16] we have that $e^{w_k} \to e^w$ in $L^2(\partial M)$. This implies that $\int_{\partial M} \kappa e^w = 0$. Therefore $w \in D$. Since $w_k \to w$ in $W^{1,2}(M)$, it is well known that $\|w\|_{W^{1,2}} \leq \liminf \|w_k\|_{W^{1,2}}$. Since $W^{1,2}(M)$ is compactly embedding in $L^2(M)$, then $w_k \to w$ in $L^2(M)$. This implies that $F(w) \leq \liminf F(w_k)$. Therefore, $w$ minimizes $F$ in $D$.

By Lagrange multiplier method we have the existence of constants $\lambda_1$ and $\lambda_2$ such that for all $\varphi \in W^{1,2}(M)$ it holds

$$\int_M (2\langle \nabla_g w, \nabla_g \varphi \rangle + \lambda_1 \varphi) dv + \lambda_2 \int_{\partial M} \kappa \varphi e^w da = 0, \quad (3.4)$$

which implies that $w$ is a weak solution of $\Delta_g w = \lambda_1$ in $M$ and $\partial w / \partial \nu_g = \lambda_2 \kappa e^w$ on $\partial M$. By regularity one finds that $w$ is a smooth function. Choosing $\varphi \equiv 1$ in (3.4) we obtain $\lambda_1 = 0$, and choosing $\varphi = e^{-w}$ we get $\lambda_2 < 0$, since
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The item (b) follows in an analogous way. Use the same functional but now defined in the space
\[ D := \left\{ w \in W^{1,2}(M) : \int_M Ke^w da = 0 \text{ and } \int_M wdv = 0 \right\}. \]

□

The main result in this case reads as follows.

**Theorem 3.3.** Let \((M, g)\) be a compact Riemannian surface with boundary and \(\chi(M) = 0\). Let \(\kappa \in C^\infty(\partial M)\) and \(K \in C^\infty(M)\).

(a) Suppose \(K_g = 0\). \(\kappa \in PC^0(g)\) if and only if either \(\kappa \equiv 0\) or \(\kappa\) changes sign and \(\int_{\partial M} \kappa e^{-v} da < 0\), where \(v\) is such that \(\Delta_g v = 0\) in \(M\) and \(\partial v/\partial \nu_g = \kappa_g\) on \(\partial M\).

(b) Suppose \(\kappa_g = 0\). \(K \in PC(g)\) if and only if either \(K \equiv 0\) or \(K\) changes sign and \(\int_{\partial M} Ke^{2v} dv < 0\), where \(v\) is such that \(\Delta_g v = K_g\) in \(M\) and \(\partial v/\partial \nu_g = 0\) on \(\partial M\).

**Proof.** Let us prove item (a). Using Gauss-Bonnet Theorem we find that \(\int_{\partial M} \kappa da = 0\). Let \(v \in C^\infty(M)\) the solution of the equation \(\Delta_g v = 0\) in \(M\) and \(\partial v/\partial \nu_g = \kappa_g\) on \(\partial M\), which is given by Lemma [18, Lemma 3.1].

Since \(\kappa \in PC^0(g)\), there exists \(u \in C^\infty(M)\) such that

\[ \Delta_g u = 0 \text{ in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_g} + \kappa_g = \kappa e^u \text{ on } \partial M. \] (3.5)

Define \(w := v + u\). Thus

\[ \Delta_g w = 0 \text{ in } M \quad \text{and} \quad \frac{\partial w}{\partial \nu_g} = \kappa e^{-v} e^w \text{ on } \partial M. \] (3.6)

The inequality \(\int_{\partial M} \kappa e^{-v} da < 0\), follows by a similar way as (3.2).

Conversely, if \(\kappa \equiv 0\), then the metric \(e^{-2v}g\) belongs to \(PC^0(g)\). If \(\kappa\) changes sign and \(\int_{\partial M} \kappa e^{-v} da < 0\) holds, then by Proposition 3.2 the problem (3.6) has a solution \(w\). Therefore, the function \(u := w - v\) satisfies (3.5).

The item (b) follows by a similar manner. □

3.2.2. **Case 2:** \(\chi(M) < 0\). To begin with the study of this case, given constants \(K_0\) and \(\kappa_0\) consider the two following equations

\[ \Delta_g u = 0 \text{ in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_g} + \kappa_0 = \kappa e^u \text{ on } \partial M \] (3.7)

and

\[ -\Delta_g u + K_0 = Ke^{2u} \text{ in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_g} = 0 \text{ on } \partial M. \] (3.8)

**Proposition 3.4.** Let \((M^2, g)\) be a compact Riemannian surface with non-empty boundary.
(a) Suppose $\kappa_0 < 0$. If (3.7) has a positive solution $u$ for some function $\kappa \in C^\infty(\partial M)$, then the unique solution of
\begin{equation}
\Delta_g \varphi = 0 \text{ in } M \quad \text{with} \quad \frac{\partial \varphi}{\partial \nu_g} - \kappa_0 \varphi = -\kappa \text{ on } \partial M
\end{equation}
is positive and $\int_{\partial M} \kappa da < 0$. Moreover, there exists $\kappa \in C^\infty(\partial M)$ with $\int_{\partial M} \kappa da < 0$ such that the problem (3.7) has no solution.

(b) Suppose $K_0 < 0$. If (3.8) has a positive solution for some function $K \in C^\infty(M)$, then the unique solution of
\begin{equation}
\Delta_g \varphi + K_0 \varphi = K \text{ in } M \quad \text{with} \quad \frac{\partial \varphi}{\partial \nu_g} = 0 \text{ on } \partial M
\end{equation}
is positive and $\int_M K dv < 0$. Moreover, there exists $K \in C^\infty(M)$ with $\int_M K dv < 0$ such that the problem (3.8) has no solution.

Proof. We prove the item (a). Set $v = e^{-u}$. Thus (3.7) becomes
\[ \Delta_g v = \frac{|\nabla_g v|^2}{v} \text{ in } M \quad \text{and} \quad \frac{\partial v}{\partial \nu_g} - \kappa_0 v = -\kappa \text{ on } \partial M. \]

Let $\varphi$ be the unique solution of (3.9). Setting $\Psi = \varphi - v$, we obtain $\Delta_g \Psi < 0$ in $M$ and $\partial \Psi/\partial \nu_g = \kappa_0 \Psi$ on $\partial M$, which implies that the minimum of $\Psi$ is achieved at the boundary. Moreover, at the minimum point it holds $\partial \Psi/\partial \nu_g \leq 0$. Thus $\Psi \geq 0$ at this point, and so $\varphi \geq v > 0$. Observe also that $\int_{\partial M} \kappa da < 0$ follows directly by integrating (3.9).

For the second part, note that by the first part it is enough to find $\kappa$ such that the solution of (3.9) is negative somewhere. Let $\psi \in C^\infty(\partial M)$ be any nontrivial function with $\int_{\partial M} \psi da = 0$. Consider its harmonic extension to $M$, which we still denote it by $\psi$. Let $\alpha > 0$ be a constant such that $\varphi := \psi + \alpha$ is negative somewhere. Defining
\[ \kappa := -\frac{\partial \psi}{\partial \nu_g} + \kappa_0 (\psi + \alpha), \]
we find that $\Delta_g \varphi = 0$ in $M$, $\partial \varphi/\partial \nu_g - \kappa_0 \varphi = -\kappa$ on $\partial M$ and $\int_{\partial M} \kappa da < 0$.

Item (b) follows similarly. Setting $v = e^{-2u}$, we obtain that (3.8) becomes
\[ \Delta_g v = \frac{|\nabla_g v|^2}{v^2} - 2K_0 v + 2K \text{ in } M \quad \text{and} \quad \frac{\partial v}{\partial \nu_g} = 0 \text{ on } \partial M. \]

Setting $\Psi = 2\varphi - v$, we obtain $\Delta_g \Psi + 2K_0 \Psi = -|\nabla_g v|^2/v < 0$ in $M$ and $\partial \Psi/\partial \nu_g = 0$ on $\partial M$. Thus, we can argue as before. If the minimum of $\Psi$ is interior, then $\Delta_g \Psi \geq 0$ at this point, and so $-2K_0 \Psi > \Delta_g \Psi \geq 0$. Therefore, since $K_0 < 0$, we get $2\varphi > v > 0$. Suppose the minimum point $p$ of $\Psi$ is in the boundary. If $\Psi(p) \geq 0$, then we get the result. If $\Psi(p) < 0$, using the Hopf’s Lemma we obtain $\partial \Psi/\partial \nu_g < 0$ at $p$, which is a contradiction.

The rest of the proof follows as before. □

The main result of this case reads as the following two theorems.
**Theorem 3.5.** Let \((M, g)\) be a compact Riemannian surface with nonempty boundary, \(\chi(M) < 0\) and vanishing Gaussian curvature.

(a) If \(\kappa \in PC^0(g)\), then

\[
\int_{\partial M} \kappa e^{-v} da < 0,
\]

where \(v\) is a solution of \(\Delta_g v = 0\) in \(M\) and \(\partial v / \partial \nu_g = \kappa_g - \overline{\kappa}_g\) on \(\partial M\). Moreover, the unique solution \(\varphi\) of

\[
\Delta_g \varphi = 0\quad \text{in } M \quad \text{with} \quad \frac{\partial \varphi}{\partial \nu_g} - \overline{\kappa}_g \varphi = -\kappa e^{-v} \quad \text{on } \partial M
\]

must be positive. Here \(\overline{\kappa}_g := \text{Area}(\partial M)^{-1} \int_{\partial M} \kappa_g da\).

(b) There exists \(\kappa \in C^\infty(\partial M)\) satisfying (3.11) such that \(\kappa \notin PC^0(g)\).

(c) \(\kappa \in PC^0(g)\) if and only if there exists a solution of

\[
\Delta_g u \leq 0\quad \text{in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_g} + \kappa_g \geq \kappa e^u \quad \text{on } \partial M.
\]

**Proof.** If \(\kappa \in PC^0(g)\), then there exists a function \(u\) satisfying (3.5). By [18, Lemma 3.1] there exists \(v\) such that \(\Delta_g v = 0\) in \(M\) and \(\partial v / \partial \nu_g = \kappa_g - \overline{\kappa}_g\) on \(\partial M\). Thus, if we define \(w := v + u\) then we get

\[
\Delta_g w = 0\quad \text{in } M \quad \text{and} \quad \frac{\partial w}{\partial \nu_g} + \overline{\kappa}_g = (\kappa e^{-v})e^w \quad \text{on } \partial M.
\]

By the Gauss Bonnet Theorem we have \(\overline{\kappa}_g < 0\). Thus item (a) and (b) follow from item (a) of the Proposition 3.4. For item (c), note that \(\kappa \in PC^0(g)\) if and only if there exists a solution of (3.13). Also, observe that if \(u\) is a small constant, then

\[
\frac{\partial u}{\partial \nu_g} + \overline{\kappa}_g - \kappa e^u < 0,
\]

that is, a small constant is a lower solution for the problem (3.13). Otherwise, a solution to (3.12) give rise a upper solution for (3.13). Thus, the result follows by [48, Theorem 2.3.1].

We also have

**Theorem 3.6.** Let \((M, g)\) be a compact Riemannian surface with nonempty boundary, \(\chi(M) < 0\) and geodesic boundary.

(a) If \(K \in PC(g)\), then

\[
\int_M K e^{2v} dv < 0,
\]

where \(v\) is a solution of \(\Delta_g v = K_g - \overline{K}_g\) in \(M\) and \(\partial v / \partial \nu_g = 0\) on \(\partial M\). Moreover, the unique solution \(\varphi\) of

\[
\Delta_g \varphi + \overline{K}_g = K e^{2v}\quad \text{in } M \quad \text{and} \quad \frac{\partial \varphi}{\partial \nu_g} = 0 \quad \text{on } \partial M
\]

must be positive. Here \(\overline{K}_g := \text{Vol}(M)^{-1} \int_M K_g dv\).
(b) There exists \( K \in C^\infty(M) \) satisfying (3.14) such that \( K \notin PC(g) \).
(c) \( K \in PC(g) \) if and only if there exists a solution of
\[
\Delta_g u - K \leq -Ke^{2u} \text{ in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_g} \geq 0 \text{ on } \partial M.
\]

**Remark 3.7.** It is important to mention here that the Gaussian curvature of the metric \( g \) in Theorem 3.6 is allowed to be positive somewhere, on the contrary of [44, Theorem 1.1] where it is necessary that \( K_g < 0 \).

### 3.2.3. Case 3: \( \chi(M) > 0 \)

Let us recall the following Theorem. The idea of proofs comes from [36].

**Theorem 3.8 (P. Liu [42]).** Let \((M,g)\) be a compact Riemannian surface with boundary, then there exists a constant \( C \), which depends only on the geometry of \( M \), such that for all \( u \in W^{1,2}(M) \)
\[
\text{log} \int_{\partial M} e^u \, da \leq \frac{1}{4 \pi} \int_M |\nabla u|^2 \, dv + \int_{\partial M} u \, da + C.
\]

The value \( 1/4\pi \) is sharp.

**Theorem 3.9.** Let \((M,g)\) be a compact Riemannian surface with nonempty boundary, \( \chi(M) = 1 \) and vanishing Gaussian curvature. Let \( \kappa \in C^\infty(\partial M) \).

(a) If \( \kappa \in PC^0(g) \), then \( \kappa \) is positive somewhere.
(b) Suppose \( \kappa_g = c > 0 \) is constant and \( \kappa \) is positive somewhere. Let \( \gamma \in (0,1) \). Then there exists a smooth solution \( u \) of the problem
\[
\Delta_g u = 0 \text{ in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_g} + c = \kappa e^{\gamma u} \text{ on } \partial M
\]
such that \( \kappa e^{(\gamma-1)u} \in PC^0(g) \).

**Proof.** Item (a) follows by the Gauss-Bonnet Theorem. To prove item (b) consider the functional \( J : D \to \mathbb{R} \) given by
\[
J(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dv + c \int_{\partial M} u \, da,
\]
where \( D := \{ u \in W^{1,2}(M) : \int_{\partial M} \kappa e^{\gamma u} = 2\pi \} \). By Gauss Bonnet-Theorem it holds \( c \text{Area}(\partial M) = 2\pi \). Since \( \kappa \) is positive somewhere, it is not difficult to show that \( D \) is nonempty. If \( w := u - \overline{u} \), where \( \overline{u} := \text{Area}(\partial M)^{-1} \int_{\partial M} u \, da, \) we can solve for \( \overline{u} \) in \( \int_{\partial M} \kappa e^{\gamma u} = 2\pi \) and one sees that \( J \) can be express as
\[
J(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dv - \frac{2\pi}{\gamma} \log \int_{\partial M} \kappa e^{\gamma u} \, da + \frac{2\pi}{\gamma} \log(2\pi).
\]

By (3.15) we find that
\[
J(u) \geq \frac{1 - \gamma}{2} \int_M |\nabla u|^2 + \text{const}.
\]
This implies that \( J \) is bounded from below. Using that the trace embedding \( W^{1,2}(M) \to L^q(\partial M) \) is continuous (in fact, compact) for all \( q \geq 1 \), as in the proof of Proposition 3.2 we can find a minimizer \( u_0 \) for \( J \), since \( 1 - \gamma > 0 \).
By Lagrange multiplier method, there exists a constant $\lambda$ such that for all $\varphi \in W^{1,2}(M)$ it holds
\[
\int_M \langle \nabla u, \nabla \varphi \rangle dv + c \int_{\partial M} \varphi da + \lambda \int_{\partial M} \varphi Ke^{\gamma u} da = 0.
\]
By choosing $\varphi \equiv 1$, we get that $\lambda = -1$. Thus $u$ is a solution to (3.16). □

In [12] the authors have found sufficient condition to assure that $k \in PC^0(g)$. They have constructed a new functional which satisfies the Palais Smale condition in a suitable space and possesses the same critical set as $J$.

**Theorem 3.10.** Let $(M, g)$ be a compact Riemannian surface with non-empty boundary, $\chi(M) = 1$ and geodesic boundary. Let $K \in C^\infty(M)$

(a) If $K \in PC(g)$, then $K$ is positive somewhere.

(b) Suppose $K_g = k > 0$ is constant and $K$ is positive somewhere. Let $\gamma \in (0, 1)$. Then there exists a smooth solution $u$ of the problem
\[
-\Delta_g u + k = Ke^{2\gamma u} \text{ in } M \quad \text{and} \quad \frac{\partial u}{\nu_g} = 0
\]
such that $Ke^{2(\gamma-1)u} \in PC(g)$.

The proof is similar to the previous one. Use the functional
\[
G(u) = \int_M \left( \frac{1}{2} |\nabla u|^2 + ku \right) dv
\]
defined in $\mathcal{H} := \{u \in W^{1,2}(M) : \int_M Ke^{2\gamma u} = 2\pi \text{ and } \frac{\partial u}{\nu_g} = 0 \}$. For all $u \in W^{1,2}(M)$ it holds
\[
\log \int_M e^u dv \leq \frac{1}{8\pi} \int_M |\nabla u|^2 dv + \text{Area}(M)^{-1} \int_M udv + C,
\]
for some constant $C$ which depends only on $M$, see [18, Corollary 3.5]. Using this and the fact that $\gamma \in (0, 1)$ we can find a minimizer for $G$ as before.

**Remark 3.11.** Theorems 3.9 and 3.10 is not easy to deal with the case $\gamma = 1$. If the surface is closed, this is the well known Nirenberg problem, which is still a open problem and it has attracted a lot of attention in the last decades. In the previous case, the functionals $J$ and $G$ are still bounded from below when $\gamma = 1$, but in general it is not possible to find a minimizer. This follows by [26, Proposition 4.5], see Section 4.2.3, which give us a nontrivial obstruction for functions to belong to $PC(g)$ and $PC^0(g)$.

4. Prescribed curvature problems in higher dimension

4.1. Existence of conformally equivalent metric. Let $(M^n, g)$ be a compact connected manifold with nonempty boundary and dimension $n \geq 3$. The content of this section extend some results of [38] to this context.

Remember the definition of the eigenvalues $\lambda_1(L_g)$ and $\sigma_1(B_g)$ in (1.4) and (1.5), respectively. From the variational characterization of these eigenvalues
it follows that $\lambda_1(\mathcal{L}_g)$ is positive (negative, zero) if and only if $\sigma_1(\mathcal{B}_g)$ is positive (negative, zero). Also the first eigenfunction for problems (1.4) and (1.5) are strictly positive (or negative), see [25, Proposition 1.3]. Moreover, its importance follows from the fact that the sign of $\lambda_1(\mathcal{L}_g)$ is uniquely determined by the conformal structure (see [24, Proposition 1.3] and [25, Proposition 1.2]). More precisely we have the following proposition which is the content of [24, Proposition 1.3 and Lemma 1.1] and [25, Proposition 1.2 and Proposition 1.4].

**Proposition 4.1** (Escobar [24, 25]). Let $(M^n, g)$ be a compact Riemannian manifold with nonempty boundary and dimension $n \geq 3$.

(a) There exists a metric pointwise conformal to $g$ whose scalar curvature is zero and the mean curvature of the boundary does not change sign. The sign is the same of $\lambda_1(\mathcal{L}_g)$ which is uniquely determined by the conformal structure.

(b) There exists a metric pointwise conformal to $g$ whose scalar curvature does not change sign and the boundary is minimal. The sign is the same of $\sigma_1(\mathcal{B}_g)$ which is uniquely determined by the conformal structure.

Before we prove the main results of this section, we will need to prove the following lemma.

**Lemma 4.2.** Let $(M^n, g)$ be a compact Riemannian manifold with boundary and dimension $n \geq 3$.

(a) Suppose $R_g \equiv 0$ and $H_g \equiv H_0$ is constant. If there exists a constant $k > 0$ satisfying $\min H \leq kH_0 \leq \max H$, then $H$ is the mean curvature of a scalar flat metric conformally equivalent to $g$.

(b) Suppose $R_g \equiv R_0$ is constant and $H_g \equiv 0$. If there exists a constant $k > 0$ satisfying $\min R \leq kR_0 \leq \max R$, then $R$ is scalar curvature of a metric with minimal boundary conformally equivalent to $g$.

**Proof.** We prove only item (a) since item (b) is entirely analogous.

It is enough to prove the result for $k = 1$, since we can multiply the metric by an appropriate constant, see (1.3). By the Perturbation Theorem 2.2 given $\varepsilon > 0$, there exists $u_1 \in C^\infty(M)$ sufficiently close to $u_0 = 1$ such that

$$\|F_2(1) - F_2(u_1)\|_{L^p \otimes W^{1,p}} < \varepsilon,$$

and $F'_2(u_1)$ is invertible, where $F_2(1) = (0, H_0)$ and $F_2$ is defined in (2.3). Arguing as proof of Theorem 1.1 (Section 3.1), using the approximation Lemma 3.1 and the Inverse Function Theorem for Banach spaces we conclude our result. □

For $\lambda_1(\mathcal{L}_g) < 0$ we have the following result.

**Theorem 4.3.** Let $(M^n, g)$ be a compact Riemannian manifold with boundary, dimension $n \geq 3$ and $\lambda_1(\mathcal{L}_g) < 0$. Let $h \in C^\infty(\partial M)$ and $f \in C^\infty(M)$.

(a) $h \in CE^0(g)$ if and only if $h$ is negative somewhere.
(b) $f \in CE(g)$ if and only if $f$ is negative somewhere.

**Proof.** We prove only item (a) since item (b) is entirely analogous. Suppose $h \in CE^0(g)$. Then there exists a diffeomorphism $\varphi$ and a smooth positive function $u$ such that $\overline{g} := \frac{1}{u^{4-n/2}} \varphi^*g$ is scalar flat and has mean curvature equal to $h$. Thus $L_{\varphi^*g}u = 0$ and $B_{\varphi^*g}u = hu^{\frac{4}{n-2}}$. Let $\psi$ be the first eigenfunction of (1.5). This implies that

$$\sigma_1(B_{\varphi^*g}) \langle \psi, u \rangle_{L^2(\partial M)} = \langle B_{\varphi^*g} \psi, u \rangle_{L^2(\partial M)} = \langle \psi, B_{\varphi^*g}u \rangle_{L^2(\partial M)} = \langle \psi, hu^{\frac{4}{n-2}} \rangle_{L^2(\partial M)}.$$

By Proposition 4.1, $\sigma_1(B_{\varphi^*g}) < 0$. Since $\psi$ and $u$ are positive function, then $h$ must be negative somewhere.

Now, suppose that $h \in C^\infty(M)$ is negative somewhere. By [17, Remark 6.4], there exists a pointwise conformal scalar flat metric $g_0$ with mean curvature $H_{g_0} \equiv -1$. The result follows by Lemma 4.2.

For $\lambda_1(L_g) = 0$ we have the following result.

**Theorem 4.4.** Let $(M^n, g)$ be a compact Riemannian manifold with boundary, dimension $n \geq 3$ and $\lambda_1(L_g) = 0$. Let $h \in C^\infty(\partial M)$ and $f \in C^\infty(M)$.

(a) $h \in CE^0(g)$ if and only if $h$ changes sign or is identically zero.
(b) $h \in CE(g)$ if and only if $f$ changes sign or is identically zero.

**Proof.** We prove only item (a) since item (b) is entirely analogous. Let $\varphi > 0$ be the eigenfunction of the problem (1.5). By Proposition 4.1, it holds $\sigma_1(B_g) = 0$. Thus, there exists a metric conformally equivalent to $g$ which is scalar flat and has minimal boundary. The rest of the prove is similar to Theorem 4.3. $\square$

Finally, for $\lambda_1(L_g) > 0$ we have the following.

**Theorem 4.5.** Let $(M^n, g)$ be a compact Riemannian manifold with boundary, dimension $n \geq 3$ and $\lambda_1(L_g) > 0$. Let $h \in C^\infty(\partial M)$ and $f \in C^\infty(M)$.

(a) $h \in CE^0(g)$ if and only if $h$ is positive somewhere.
(b) $f \in CE(g)$ if and only if $f$ is positive somewhere.

**Proof.** The proof is entirely analogous to Theorem 4.3, using the solution of the Yamabe Problem with boundary (see [2, 10, 24, 46, 15]). $\square$

The Theorems 4.3, 4.4 and 4.5 prove the Theorems 1.2 and 1.3.

### 4.2. Existence and nonexistence of pointwise conformal metric.

In this section we extend some results of [39] to the context of manifold with boundary. Let $(M^n, g)$ be a compact manifold with nonempty boundary and dimension $n \geq 3$. The existence of a scalar flat metric in the conformal class $[g]$ with mean curvature $H \in C^\infty(\partial M)$ is equivalent to the existence of a positive solution to the problem

$$L_g(u) = 0 \text{ in } M \quad \text{and} \quad B_g(u) = Hu^{\frac{4}{n-2}} \text{ on } \partial M.$$
Similarly, the existence of a metric in the conformal class $[g]$ with scalar curvature $R \in C^\infty(M)$ and minimal boundary is equivalent to the existence of a positive solution to the problem

\[(4.2) \quad \mathcal{L}_g(u) = R u^{n+2 \over n-2} \text{ in } M \quad \text{and} \quad \mathcal{B}_g(u) = 0 \text{ on } \partial M.\]

Here $\mathcal{L}_g$ and $\mathcal{B}_g$ are defined in (1.3). Remember from the introduction, if such solution exists then $H \in PC^0(g)$ and $R \in PC(g)$, respectively.

Again, our analysis will be divided in three cases.

4.2.1. **Negative curvature case:** $\lambda_1(\mathcal{L}_g) < 0$. As it was pointed out in Section 4.1, $\sigma_1(\mathcal{B}_g) < 0$ if only if $\lambda_1(\mathcal{L}_g) < 0$ (see [24, 25]). This implies that there exists a pointwise conformal scalar flat metric with constant negative mean curvature, and also a pointwise conformal metric with constant negative scalar curvature with minimal boundary (see for instance [17, Remark 6.4]). We start with the following proposition.

**Proposition 4.6.** Let $(M^n, g)$ be a compact manifold with nonempty boundary and dimension $n \geq 3$.

(a) Suppose that $g$ is scalar flat metric with constant mean curvature $H_0 < 0$. If (4.1) has a positive solution for some $H \in C^\infty(\partial M)$, then the unique solution of the problem

\[(4.3) \quad \Delta_g \varphi = 0 \text{ in } M \quad \text{and} \quad \frac{\partial \varphi}{\partial \nu_g} - H_0 \varphi = -H \text{ on } \partial M\]

must be positive and $\int_{\partial M} H da < 0$. Moreover, there exists a function $H \in C^\infty(\partial M)$ with $\int_{\partial M} H da < 0$ such that $H \notin PC^0(g)$.

(b) Suppose that $g$ has constant scalar curvature $R_0 < 0$ and minimal boundary. If (4.2) has a positive solution for some $R \in C^\infty(M)$, then the unique solution of the problem

\[(4.4) \quad \Delta_g \varphi + \frac{R_0}{n-1} \varphi = \frac{R}{n-1} \text{ in } M \quad \text{and} \quad \frac{\partial \varphi}{\partial \nu_g} = 0 \text{ on } \partial M\]

must be positive and $\int_M R dv < 0$. Moreover, there exists a function $R \in C^\infty(M)$ with $\int_M R dv < 0$ such that $R \notin PC(g)$.

**Proof.** Let us prove item (a). If $v = u^{-n+2 \over 2}$, then (4.1) is equivalent to

\[2\Delta_g v - n \frac{\left| \nabla_g v \right|^2}{v} = 0 \text{ in } M \quad \text{and} \quad \frac{\partial v}{\partial \nu_g} - H_0 v = -H \text{ on } \partial M.\]

Let $\varphi$ be the unique solution of (4.3). Setting $\Psi := \varphi - v$, we obtain that $\Delta_g \Psi < 0$ in $M$ and $\partial \Psi / \partial \nu_g = H_0 \Psi$ on $\partial M$. Similarly to the Proposition 3.4 we obtain that $\varphi \geq v > 0$. Finally, observe that the weaker conditions $\int_{\partial M} H da < 0$ is obtained by integration of (4.3).
In an analogous way, the item (b) is obtained considering
\[ v = u \frac{R_0}{n-1} \]
Then (4.2) is equivalent to
\[ \Delta_g v + \frac{R_0}{n-1} v = \frac{1}{n-1} R + \frac{n + 2 |\nabla g v|^2}{4v} \text{ in } M \quad \text{and} \quad \frac{\partial v}{\partial \nu_g} = 0 \text{ on } \partial M. \]
Thus, if we define \( \Psi := \phi - v \) we get \( \Delta_g \Psi + \frac{R_0}{n-1} \Psi < 0 \) in \( M \) and \( \partial \Psi / \partial \nu_g = 0 \) on \( \partial M \). Arguing as in the Proposition 3.4 we obtain the result. \( \square \)

**Proposition 4.7.** Let \( (M^n, g) \) be a compact manifold with nonempty boundary and dimension \( n \geq 3 \).

(a) Suppose that \( g \) is a scalar-flat metric with constant mean curvature \( H_0 < 0 \). Let \( H, H_1 \in C^\infty(\partial M) \). If \( H \in PC^0(g) \) and either \( H_1 \leq H \) or \( H_1 = k_1 H \), for some constant \( k_1 > 0 \), then \( H_1 \in PC^0(g) \).

(b) Suppose that \( g \) is a metric of constant scalar curvature \( R_0 < 0 \) and minimal boundary. Let \( R, R_1 \in C^\infty(M) \). If \( R \in PC(g) \) and either \( R_1 \leq R \) or \( R_1 = k_2 R \), for some constant \( k_2 > 0 \), then \( R_1 \in PC(g) \).

**Proof.** As before, we prove only item (a). First we prove the following claim.

**Claim:** If \( \sigma_1(B_g) < 0 \), then one can find a positive lower solution of (4.1).

Let \( u_- = \gamma \varphi \), where \( \varphi \) is the first normalized eigenfunction of the problem (1.5), which is positive, and \( \gamma \) is a positive constant. If \( H_1 \in C^\infty(M) \), then \( L_g(u_-) = 0 \) and \( B_g(u_-) - H_1 u_-^{\frac{n}{n-2}} = \gamma \varphi (\sigma_1(B_g) - H_1 (\gamma \varphi)^{\frac{2}{n-2}}) \). Since \( H_1 \) and \( \varphi \) are bounded and \( \gamma \) can be chosen sufficiently small, \( u_- \) is a lower solution of (4.1), with \( H \) replaced by \( H_1 \).

If \( H \in PC^0(g) \), then there exists a positive smooth function \( u^+ \) such that \( L_g(u^+) = 0 \) and \( B_g(u^+) = H(u^+)^{\frac{n}{n-2}} \). Assuming \( H_1 \leq H \), we obtain that \( B_g(u^+) \geq H_1 (u^+)^{\frac{n}{n-2}} \), which implies that \( u^+ \) is a upper solution of (4.1), with \( H \) replaced by \( H_1 \). For \( \gamma > 0 \) small enough we have \( u_- \leq u^+ \). By Theorem 2.3.1 of [48] we conclude that \( H_1 \in PC^0(g) \).

Besides, if \( H_1 = k_1 H \), for some constant \( k_1 > 0 \), since \( H \in PC(g) \), it is enough to choose a proper multiple of the metric \( g_0 \) so that \( R_{g_0} = 0 \) and \( H_{g_0} = H \). \( \square \)

In fact, we can say more.

**Proposition 4.8.** Let \( (M^n, g) \) be a compact manifold with nonempty boundary and dimension \( n \geq 3 \).

(a) Suppose that \( g \) is scalar-flat with mean curvature \( H_0 \in C^\infty(\partial M) \)
satisfying \( \int_M H_0 dv < 0 \). Then there exists a constant \( k(H_0) < 0 \) such that \( H \in PC^0(g) \) for any \( H \in C^\infty(\partial M) \) with \( k(H_0) \leq H < 0 \).

(b) Suppose that \( g \) has minimal boundary and scalar curvature \( R_0 < 0 \) satisfying \( \int_M R_0 dv < 0 \). Then there exists a constant \( k(R_0) < 0 \) such that \( R \in PC(g) \) for any \( R_0 \in C^\infty(M) \) with \( k(R_0) < 0 \).
Proof. We prove only item (a). To prove that \( H \in \mathcal{PC}_0(g) \) we need to find a positive solution \( u \) to (4.1), that is,

\[
\Delta_g u = 0 \quad \text{in} \; M \quad \text{and} \quad \frac{2}{n-2} \frac{\partial u}{\partial \nu_g} + H_0 u = H u^{\frac{n}{n-2}} \quad \text{on} \; \partial M.
\]

As in the proof of Proposition 4.7 there exists a positive lower solution \( u_- \), which can be made small enough. Thus, we need only to find an upper solution \( u^+ \). The result will follow by [48, Theorem 2.3.1].

Similar to the proof of Proposition 4.6, the function \( v = (u^+)^{-\frac{2}{n-2}} \) is a positive lower solution of

\[
2 \Delta_g v - n \frac{|\nabla_g v|^2}{v} \geq 0 \quad \text{in} \; M \quad \text{and} \quad \frac{\partial v}{\partial \nu_g} - H_0 v \leq -H \quad \text{on} \; \partial M.
\]

Let \( \Psi \in C^\infty(M) \) be a solution of the problem \( \Delta_g \Psi = -\frac{1}{\text{Vol}(M)} \int_{\partial M} H da \) in \( M \) and \( \partial \Psi/\partial \nu_g = 2\overline{H} - H \) on \( \partial M \) (see [18, Lemma 3.1]), where \( \overline{H} = \text{Area}(\partial M)^{-1} \int_{\partial M} H da \). Set \( v := \Psi + \gamma \), where the constant \( \gamma \) is taken so large that \( v > 0 \) and

\[
2 \Delta_g v - n \frac{|\nabla_g \Psi|^2}{\Psi + \gamma} > 0.
\]

By choosing \( H_0 < 0 \) so small that \( 2\overline{H} \leq H_0 v \), we obtain that \( v \) is a lower solution to (4.6).

The item (b) is similar. If \( u^+ \) is a upper solution of (4.2), then \( v = (u^+)^{-\frac{2}{n-2}} \) is a lower solution of

\[
\Delta_g v + \frac{R_0}{n-1} v - \frac{R}{n-1} \geq \frac{n + 2}{4} \frac{|\nabla_g v|^2}{v} \quad \text{in} \; M \quad \text{and} \quad \frac{\partial v}{\partial \nu_g} \leq 0.
\]

To find this lower solution, consider \( \Psi \) a solution of \( \Delta_g \Psi = (R - \overline{R})/(n-1) \) in \( M \) and \( \partial \Psi/\partial \nu_g = 0 \) on \( \partial M \), and \( v := \Psi + \gamma \), where \( \overline{R} = \text{Vol}(M)^{-1} \int_M R dv \) and the constant \( \gamma \) is so large that \( v > 0 \) and

\[
-\frac{\overline{R}}{2(n-1)} \geq \frac{n + 2}{4} \frac{|\nabla_g \Psi|^2}{\Psi + \gamma}.
\]

Also, choose \( R_0 < 0 \) small enough such that \( \overline{R} \leq R_0(\Psi + \gamma) \). With this, we have that \( v \) is a lower solution to (4.7). Similarly to the item (a) we find a lower solution to (4.7). \( \square \)

Remark 4.9. It was showed in [19] that if manifold has a metric of negative scalar curvature and minimal boundary, then for any smooth function \( R < 0 \) and \( H \in C^\infty(\partial M) \) satisfying

\[
\frac{H}{\sqrt{|R|}} < \frac{1}{\sqrt{n(n-1)}} \quad \text{on} \; \partial M,
\]

there is a pointwise conformal metric whose scalar curvature is \( R \) and mean curvature is \( H \).
4.2.2. **Zero curvature case:** $\lambda_1(\mathcal{L}_g) = 0$. By Proposition 4.1, the constant function $R \equiv 0$ (resp. $H \equiv 0$) belongs to $PC(g)$ (resp. $PC^0(g)$) if and only if $\lambda_1(\mathcal{L}) = 0$. Our aim in this section is to prove the following result.

**Theorem 4.10.** Let $(M^n, g)$ be a compact Riemannian manifold with boundary of dimension $n \geq 3$ and with $\lambda_1(\mathcal{L}_g) = 0$. Let $H \in C^\infty(\partial M)$ and $R \in C^\infty(M)$.

(a) $H \in PC^0(g)$ if and only if either $H \equiv 0$ or $H$ changes sign and $\int_{\partial M} Hda < 0$.

(b) $R \in PC(g)$ if and only if either $R \equiv 0$ or $R$ changes sign and $\int_M Rdv < 0$.

**Proof.** The item (a) was proved by J. F. Escobar, see [26, Theorem 6.8]. Thus we need only to prove item (b). Since the problem is conformally invariant, by Proposition 4.1 we can suppose that $g$ is scalar flat and has minimal boundary. Let $u > 0$ be a smooth function in $M$ and let $\overline{g} = u^{\frac{4}{n-2}}g$.

If we require the metric $\overline{g}$ to be scalar flat and its mean curvature to be equal to $R$, then transformations laws $(1.3)$ imply that $u$ has to satisfy

$$ \Delta_{\overline{g}} u = -c(n)Ru^{\frac{n+2}{n-2}} \quad \text{in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_{\overline{g}}} = 0 \quad \text{on } \partial M, $$

where $c(n) = \frac{n-2}{2(n-1)}$. Using integration by parts, we obtain

$$ \int_M Rdv = -c(n)^{-1} \int_M u^{\frac{n+2}{n-2}} \Delta_{\overline{g}} u dv = -b(n) \int_M u^{\frac{4}{n-2}} |\nabla u|^2 dv \leq 0 $$

and

$$ \int_M Ru^{\frac{n+2}{n-2}}dv = 0, $$

where $b(n) > 0$. This implies the “only if” part. Let us prove the “if” part using the method of upper and lower solution, see [48, Theorem 2.3.1].

**Claim 1:** There exists a positive upper ersolution $u^+$ to $(4.9)$.

Let $\Psi$ be the smooth solution to the problem (see [18, Lemma 3.1])

$$ \Delta_{\overline{g}} \Psi = \overline{R} - R \quad \text{in } M \quad \text{and} \quad \frac{\partial \Psi}{\partial \nu_{\overline{g}}} = 0 \quad \text{on } \partial M, $$

where $\overline{R} = \text{Vol}(M)^{-1} \int_M Rdv < 0$. Let $\varepsilon > 0$ be a constant and define $u^+ := \varepsilon^{\frac{n+2}{n-2}} \Psi + \varepsilon$. Then $\partial u^+/\partial \nu_g = 0$ and for $\varepsilon > 0$ small enough we have

$$ \Delta_{\overline{g}} u^+ + c(n)R(u^+)^{\frac{n+2}{n-2}} = \varepsilon(\overline{R} - R(1 - c(n)(1 + \varepsilon^{\frac{4}{n-2}} \Psi^{\frac{n+2}{n-2}}))) \leq 0. $$

**Claim 2:** For some constant $m > 0$ the first eigenvalue $\lambda_1$ to the problem

$$ \Delta_g v + mRv + \lambda_1 v = 0 \quad \text{in } M \quad \text{and} \quad \frac{\partial v}{\partial \nu_g} = 0 \quad \text{on } \partial M $$

is negative. The first eigenfunction $\varphi_1$ can be taken to be strictly positive.
The first eigenvalue is characterized variationally by

\[ \lambda_1 = \inf_{u \in W^{1,2}(M) \setminus \{0\}} \frac{\int_M (|\nabla u|^2 - m Ru^2) dv}{\int_M u^2 dv}. \]

Since \( R \) changes sign, there exists some smooth function \( u \) which is positive on some open set in \( \{ x \in M : R(x) > 0 \} \) and \( u \equiv 0 \) otherwise. Then, for \( m > 0 \) sufficiently large, \( \int_M (|\nabla u|^2 - m Ru^2) dv < 0 \). Using standard argument we can prove that there exists a minimizer \( \varphi_1 \) for the variational problem (4.11). Using the inequality \( \int_M |\nabla|u||^2 \leq \int_M |\nabla u|^2 \), the boundary condition and Hopf’s Lemma we can assume \( \varphi_1 > 0 \) in \( M \).

**Claim 3:** There exists a positive lower solution \( u_- \) to (4.9).

The idea to prove this claim comes from [33] (see also [34]). Let \( a > 0 \) be a constant such that \( ma^\frac{n}{n-2} = c(n) \), where \( m \) is given by Claim 2. Thus, \( u_- > 0 \) is a lower solution of (4.9) if and only if \( v_- = \log(au_-) \) satisfies

\[ \Delta_g v_- + |\nabla v_-|^2 + mRe^\frac{a^\frac{n}{n-2}}{2}v_- \geq 0 \text{ in } M \quad \text{and} \quad \frac{\partial v_-}{\partial \nu_g} \leq 0 \text{ on } \partial M. \]

Now choose an eigenfunction \( \varphi_1 > 1 \) of (4.10) and define

\[ v_- : = \gamma(\varphi_1^2 - r^b)^{1-(\log r)/r} + \frac{n-2}{4} \log(\gamma(r^2 - r \log r)(1-r)^{-(\log r)/r}), \]

where \( \gamma > 0 \) and \( b > 0 \) are small enough constants. It is a long calculation to show that \( v_- \) satisfies (4.12), see proof of Theorem 7 in [33]. Also, it is easy to see that for \( \gamma > 0 \) small enough we have \( 0 < u_- = a^{-1} e^{v_-} < u^+ \). □

### 4.2.3 Positive curvature case: \( \lambda_1(L_g) > 0 \)

Let \( B^n \subset \mathbb{R}^n \) be the unit ball. We have the following nontrivial Kazdan-Warner type obstruction discovered by Escobar [26, Proposition 4.6]. If a function \( H \in C^\infty(\partial M) \) is the mean curvature of a pointwise conformal metric, then

\[ \int_{\partial B^n} \langle X, \nabla_y H \rangle \, da = 0, \]

where \( X \) is a conformal Killing vector field on \( \partial B^n \). This condition gives that problem (4.1) has no solutions for \( H = Ax + B, A \neq 0 \). The arguments in [26] also gives a similar obstruction for the Euclidean hemisphere \( \mathbb{S}^n_+ \subset \mathbb{R}^{n+1} \).

### 5. Obstruction result: Non-compact setting

Let \((M, g)\) be a non-compact Riemannian manifold with nonempty boundary \( \partial M \). If \( M \) has a non-compact boundary, we will consider an exhaustion \( \Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \ldots \) of \( M \) by bounded open sets \( \Omega_j \) with piecewise smooth boundary, where part of each of them are composed by pieces in \( \partial M \). Given a bounded set \( \Omega \subset M \) we will distinguish the boundary part of \( \Omega \) on \( \partial M \) and inside \( M \) by writing \( \partial_M \Omega := \partial \Omega \setminus \partial M \) and \( \partial_0 \Omega := \partial \Omega \setminus \partial_M \Omega \subset \partial M \).
Given \( h \in C^\infty(\partial_0 \Omega) \), we define the following functional \( Q^0_g(u) \) for functions \( u \) on \( M \), which is given by

\[
Q^0_g(u) = \frac{\int_{\Omega} |\nabla g u|^2 \, dv + \int_{\partial_0 \Omega} hu^2 \, da}{\int_{\partial_0 \Omega} u^2 \, da}.
\]

Also define

\((5.1)\quad \sigma_1(\Omega) = \inf \{ Q^0_g(u) : u \in H^1(\Omega) \setminus \{0\}, \partial u/\partial \eta_g = 0 \text{ on } \partial_M \Omega \},\)

where \( \eta_g \) is the outward unit normal to \( \partial_M \Omega \).

By variational characterization, it is well-known that \( \sigma_1(\Omega) \) is the lowest eigenvalue of the problem \( \Delta_g u = 0 \) in \( \Omega \) and \( \partial u/\partial \eta_g + hu = \lambda u \) on \( \partial_0 \Omega \).

Now, if \( M \) has a compact boundary, we will consider an exhaustion by open domains \( \Omega_0 \subsetneq \Omega_1 \subsetneq \Omega_2 \subsetneq \ldots \) such that each boundary is smooth and \( \partial M \subset \Omega_0 \). In this case, given a bounded domain \( \Omega \subset M \) and \( f \in C^\infty(\Omega) \), we define the functional \( Q_g(u) \) as

\[
Q_g(u) = \frac{\int_{\Omega} |\nabla g u|^2 \, dv + \int_{\Omega} fu^2 \, dv}{\int_{\Omega} u^2 \, da}
\]

and

\((5.2)\quad \mu_1(\Omega) = \inf \{ Q_g(u) : u \in H^1(\Omega) \setminus \{0\}, \partial u/\partial \eta_g = 0 \text{ on } \partial M \Omega \}
\]

Again, by variational characterization, it is well-known that \( \mu_1(\Omega) \) is the lowest eigenvalue of the problem \( -\Delta_g u + fu = \lambda u \) in \( \Omega \) with Neumann boundary condition \( \partial u/\partial \eta_g = 0 \) on \( \partial M \Omega \). Now if \( M \) has a non-compact boundary, then we can define similarly the functional \( Q_g \) and the eigenvalue

\[
\mu_1(\Omega) = \inf \{ Q_g(u) : u \in H^1(\Omega) \setminus \{0\}, \partial u/\partial \eta_g = 0 \text{ on } \partial M \Omega \}
\]

and \( \partial u/\partial \nu_g = 0 \) on \( \partial_0 \Omega \).

For sake of simplicity we do not differentiate the eigenvalue for the manifolds with compact and non-compact boundary.

**Lemma 5.1.** Let \((M, g)\) be a non-compact Riemannian manifold with non-empty boundary. Let \( h \in C^\infty(\partial M) \) and \( f \in C^\infty(M) \).

(a) If \( \sigma_1(\Omega_0) > 0 \) for some \( \Omega_0 \), and \( h \geq 0 \) in \( \partial M \setminus \partial_0 \Omega_0 \), then there exists a positive function \( q \in C^\infty(\partial M) \), such that for any \( j = 0, 1, 2, \ldots \), and all \( v \in C^\infty(\Omega_j) \) it holds

\[
\int_{\partial_0 \Omega_j} qv^2 \, da \leq \int_{\Omega_j} |\nabla g v|^2 \, dv + \int_{\partial_0 \Omega_j} hv^2 \, da.
\]

(b) If \( \mu_1(\Omega_0) > 0 \) for some \( \Omega_0 \), and \( f \geq 0 \) in \( M \setminus \Omega_0 \), then there exists a positive function \( p \in C^\infty(M) \), such that for any \( j = 0, 1, 2, \ldots \), and all \( v \in C^\infty(\Omega_j) \) it holds

\[
\int_{\Omega_j} pv^2 \, dv \leq \int_{\Omega_j} (|\nabla g v|^2 + fv^2) \, dv.
\]
Proof. We prove item (a). The proof of (b) is similar, except that we consider the variational characterization (5.2). We claim that \( \sigma_1(\Omega_j) > 0 \), for all \( j > 0 \). In fact, if \( \sigma_1(\Omega_j) \leq 0 \), for some \( j > 0 \), then by the variational characterization (5.1) and using that \( \Omega_0 \subset \Omega_j \), an eigenfunction \( \psi \) satisfies

\[
0 \geq \int_{\Omega_j} |\nabla_g \psi|^2 dv + \int_{\partial_0 \Omega_j} h \psi^2 da \geq \int_{\Omega_0} |\nabla_g \psi|^2 dv + \int_{\partial_0 \Omega_0} h \psi^2 da
\]

which is a contradiction to the fact that \( \sigma_1(\Omega_0) > 0 \). Then for all smooth function \( v \) in \( \Omega_j \), we have

\[
(5.4) \quad \int_{\partial_0 \Omega_j} v^2 da \leq \frac{1}{\sigma_1(\Omega_j)} \left( \int_{\Omega_j} |\nabla_g v|^2 dv + \int_{\partial_0 \Omega_j} h v^2 da \right).
\]

Define a function \( q \) such that

\[
0 < q(x) \leq \sum_{i=0}^{\infty} C_i \chi_{E_i}(x), \text{ for all } x \in \partial M,
\]

where \( E_0 = \partial_0 \Omega_0, E_i = \partial_0 \Omega_i \setminus \partial_0 \Omega_{i-1} \) for \( i = 1, 2, \ldots \), \( \chi_{E_i} \) is its characteristic function and \( C_i \) are positive constants to be chosen later (\( q \) can be defined on \( \partial_0 \Omega \) and after we consider its harmonic extension).

Using (5.4) and the fact that \( h \geq 0 \) we obtain

\[
\int_{\partial_0 \Omega_j} q v^2 da \leq \sum_{i=0}^{j} C_i \int_{E_i} v^2 da \leq \sum_{i=0}^{j} C_i \int_{\partial_0 \Omega_i} v^2 da
\]

\[
\leq \sum_{i=0}^{j} \frac{C_i}{\sigma_1(\Omega_i)} \left( \int_{\Omega_i} |\nabla_g v|^2 dv + \int_{\partial_0 \Omega_i} h v^2 da \right)
\]

\[
\leq \left( \sum_{i=0}^{j} \frac{C_i}{\sigma_1(\Omega_i)} \right) \left( \int_{\Omega_j} |\nabla_g v|^2 dv + \int_{\partial_0 \Omega_j} h v^2 da \right).
\]

The conclusion now follows from picking \( C_i = \sigma_1(\Omega_i) / 2^{i+1} \).

Lemma 5.2. Let \((M, g)\) be a non-compact Riemannian manifold with non-empty boundary. Let \( f \in C^\infty(M) \) and \( h \in C^\infty(\partial M) \).

(a) There exists a solution \( u > 0 \) of \( \Delta_g u = 0 \) in \( M \) and \( \partial u / \partial \nu_g + hu > 0 \) on \( \partial M \) if and only if there is a function \( q > 0 \) on \( \partial M \) such that for all \( v \in C_0^\infty(M) \) it holds

\[
(5.5) \quad \int_{\partial M} q v^2 da \leq \int_{M} |\nabla_g v|^2 dv + \int_{\partial M} h v^2 da.
\]

(b) There exists a solution \( u > 0 \) of \( -\Delta_g u + fu > 0 \) in \( M \) and \( \partial u / \partial \nu_g = 0 \) on \( \partial M \) if and only if there is a function \( p > 0 \) in \( M \) such that for
all \( v \in C^\infty_0(M) \) it holds
\[
\int_M pv^2 dv \leq \int_M (|\nabla_g v|^2 + fv^2) \, dv.
\]

Proof. We prove item (a). Assume there exists a harmonic function \( u > 0 \) such that \( \partial u/\partial \nu_g + hu > 0 \) on \( \partial M \). Let \( q = (\partial u/\partial \nu_g + hu)/u \). Define \( B := \partial/\partial \nu_g + h - q \), and note that \( Bu = 0 \). Given \( v \in C^\infty_0(M) \), consider \( \Omega \subset M \) a smoothly bounded domain containing the support of \( v \). Let \( \lambda_1(\Omega) \) be the lowest eigenvalue of \( B \) with Dirichlet condition on \( \partial M \). Let \( \phi > 0 \) be the corresponding eigenfunction. This implies that \( \Delta_g \phi = 0 \) in \( \Omega \), \( B\phi = \lambda_1(\Omega) \phi \) on \( \partial M \). Since \( \phi > 0 \), by the maximum principle we get \( \partial \phi/\partial \eta_g < 0 \) on \( \partial \Omega \), where \( \eta_g \) is the outward unit normal to \( \partial \Omega \). Using integration by parts we get
\[
\lambda_1(\Omega) \int_{\partial \Omega} \phi u = \int_{\partial \Omega} u B \phi = \int_{\partial M} u B \phi \geq 0.
\]
Thus \( \lambda_1(\Omega) \geq 0 \). From the variational characterization we obtain
\[
\lambda_1(\Omega) = \inf_{v \in H^1_\Omega \setminus \{0\}, v = 0 \text{ on } \partial M} \left\{ \frac{\int_\Omega |\nabla_g v|^2 + \int_{\partial M} (h - q)v^2}{\int_\Omega v^2} \right\}.
\]
This implies (5.5).

For the converse, by making \( q > 0 \) smaller if necessary, we may assume that \( q \in L^1(\partial M) \cap C^\infty(M) \). Define a functional as
\[
J(v) = \int_M |\nabla_g v|^2 + \int_{\partial M} (hv^2 - 2qv).
\]
For \( v \in H^1_{loc}(M) \), define
\[
\|v\| := \left( \int_M |\nabla_g v|^2 + \int_{\partial M} hv^2 \right)^{1/2}.
\]
By (5.5) and the fact that \( q > 0 \) we get that \( \| \cdot \| \) is a norm. Consider the Hilbert subspace \( H \subset H^1_{loc}(M) \) given by \( H := \{ v \in H^1_{loc}(M) : \|v\| < \infty \} \).

Using the Schwarz inequality and (5.5), for any \( v \in C^\infty_0(M) \) we obtain
\[
(\int_{\partial M} qv)^2 \leq \|q\|_{L^1(\partial M)} \left( \int_{\partial M} qv^2 \right) \leq \|q\|_{L^1(\partial M)} \|v\|^2.
\]
From this we get
\[
J(v) \geq \|q\|_{L^1(\partial M)}^{-1} \left( \int_{\partial M} qv \right)^2 - 2 \int_{\partial M} qv \geq C,
\]
where \( C \in \mathbb{R} \) is a constant independently of \( v \in C^\infty_0(M) \). This implies that \( J : H \to \mathbb{R} \) is bounded from below. Note that a critical point \( u \) of the functional \( J \) is a weak solution of \( \Delta_g u = 0 \) in \( M \) and \( \partial u/\partial \nu_g + hu = q > 0 \) on \( \partial M \). By classical regularity theory we conclude that \( u \) is a smooth solution of this problem. Thus, let us prove that \( J \) has a minimal point.
Let \( v_j \in H \) such that \( J(v_j) \to a := \inf_{v \in H} J(v) \), which implies that \( |J(v_j)| \leq c \), for some constant \( c > 0 \). Thus, using (5.5) and (5.6) we have

\[
\|v_j\|^2 = J(v_j) + 2\int_{\partial M} qv_j \leq c + 2\|q\|^{1/2}_{L^1(\partial M)}\|v_j\|
\]

which implies that the sequence \( v_j \) is bounded.

Since \( H \) is a Hilbert space, there is a subsequence, still denoted by \( v_j \), which converges weakly to some function \( u \in H \). The inequality (5.6) implies that the functional \( v \mapsto \int_{\partial M} qv \) is continuous, in particular it is weakly continuous. Hence, \( \int_{\partial M} qv_j \to \int_{\partial M} qv \). From this and \( \|u\| = \lim \inf \|v_j\| \), as in the previous sections, we get \( J(u) \leq \lim \inf J(v_j) \), which implies that \( u \) minimizes \( J \). It easy to see that \( q > 0 \) implies \( J(|u|) \leq J(u) \), so \( u \geq 0 \). Since \( \Delta_g u = 0 \) in \( M \), then the minimal point is in the boundary. If \( x_0 \in \partial M \) is such that \( u(x_0) = 0 \), then \( 0 < q(x_0) = \partial u / \partial \nu_g(x_0) < 0 \), since \( \nu_g \) is the unit outer normal to \( \partial M \). Therefore \( u > 0 \) in \( M \).

The proof of item (b) is similar. First suppose that such a solution exists. Then \( u \) is a positive solution of \( Qu := -\Delta_g u + (f - p)u = 0 \), where \( p = (-\Delta_g u + fu) / u \). Arguing as before, using the eigenvalue of \( Q \) with Dirichlet condition on \( \partial \Omega \), we obtain the result.

For the converse, we can assume that \( p \in L^1(M) \cap C^\infty(M) \). Define the functional \( J(v) = \int_M (|\nabla v|^2 + fv^2 - 2pv) \), as before we can prove that \( J \) has a positive minimum.

**Theorem 5.3.** Let \((M, g)\) be a non-compact Riemannian manifold with nonempty boundary. Let \( \Omega_0 \subset M \) be a bounded open set in \( M \) such that \( \sigma_1(\Omega_0) > 0 \) and \( h \geq 0 \) in \( \partial M \setminus \partial_0 \Omega_0 \). Then there is a solution \( u \) on \( M \) with

\[
\Delta_g u \leq 0 \text{ in } M \quad \text{and} \quad \frac{\partial u}{\partial \nu_g} + hu > 0 \text{ on } \partial M,
\]

where \( 0 < c_2 < u < c_1 \) for some constants \( c_1 \) and \( c_2 \).

**Proof.** Let \( \{\Omega_j\} \) be an exhaustion of \( M \) as in the beginning of this section. By Lemma 5.1, there exists a positive function \( q \in C^\infty(M) \) satisfying (5.3), in particular for all \( u \in C_0^\infty(M) \) we have

\[
\int_{\partial M} qu^2 da \leq \int_M |\nabla_g u|^2 dv + \int_{\partial M} hu^2 da.
\]

By Lemma 5.2 there exists a positive solution \( u \) of \( \Delta_g u = 0 \) in \( M \) and \( \partial u / \partial \nu_g + hu > 0 \) on \( \partial M \).

**Claim:** We can modify \( u \) to a solution \( w \) satisfying \( 0 < c_2 < w < c_1 \), where \( c_1 \) and \( c_2 \) are constants.

Consider \( N = \{ x \in \partial M : h(x) < 0 \} \), which is bounded because is a subset of \( \partial_0 \Omega_0 \). Since the closure of \( N \) is compact, pick a small constant \( \alpha > 0 \) so that \( \partial u / \partial \nu_g + hu + \alpha h > 0 \) on \( N \), and hence on all \( \partial M \). Then the function \( v := u + \alpha > \alpha > 0 \) satisfies \( \partial v / \partial \nu_g + hv > 0 \).
Setting \( w = 1 - e^{-cw} \), where \( c > 0 \) is a small constant to be chosen later. Thus, we have \( 0 < 1 - e^{-cw} < w < 1 \), \( \Delta_g w \leq 0 \) in \( M \) and

\[
\frac{\partial w}{\partial v_g} + hw \geq e^{-cw} \left[ \frac{\partial v}{\partial v_g} + cv(1 - cv) \right] \quad \text{on} \quad \partial M.
\]

Thus on \( \partial M \backslash \overline{N} \) we have that \( \frac{\partial w}{\partial v_g} + hw > 0 \) since \( e^t \geq 1 + t \) for all \( t \in \mathbb{R} \). Now, letting \( v_\infty = \max v(x) \), using the Taylor’s Theorem we have

\[
0 < e^{cv} - 1 - cv \leq \frac{1}{2} c^2 v_\infty^2 e^{cv} \quad \text{in} \quad N.
\]

This implies that

\[
(5.7) \quad \frac{\partial w}{\partial v_g} + hw \geq ce^{-cv} \left[ \frac{\partial v}{\partial v_g} + hv - \frac{1}{2} c^2 v_\infty^2 e^{cv} |h| \right].
\]

Since \( \overline{N} \) is compact, we can choose \( c > 0 \) in order to the right hand side of (5.7) to be positive in \( N \).

A similar argument gives the following Theorem,

**Theorem 5.4.** Let \( (M, g) \) be a non-compact Riemannian manifold with nonempty boundary. Let \( \Omega_0 \subset M \) be a bounded open set in \( M \) such that \( \mu_1(\Omega_0) > 0 \) and \( f \geq 0 \) in \( M \backslash \Omega_0 \). Then there is a solution \( u \) on \( M \) with

\[
-\Delta_g u + fu > 0 \quad \text{in} \quad M \quad \text{and} \quad \frac{\partial u}{\partial v_g} = 0 \quad \text{on} \quad \partial M,
\]

where \( 0 < c_2 < u < c_1 \) for some constants \( c_1 \) and \( c_2 \).

**Proof.** The proof is similar to the previous result, except that we consider the eigenvalue \( \mu_1(\Omega_0) \) and the following computation instead

\[
-\Delta_g w + fw \geq e^{-cw} \left[ c(-\Delta_g v + fv) + f(1 - cv) \right] \quad \text{in} \quad M
\]

and \( \partial w/\partial v_g = 0 \) on \( \partial M \).

It should be remarked that follows from (1.3) that a positive function \( u > 0 \) satisfying \( L_g u > 0 \) (resp. \( L_g u = 0 \)) in \( M \) and \( B_g u = 0 \) (resp. \( B_g u > 0 \)) on \( \partial M \), then the metric \( g_1 = u^{\frac{4}{n-2}} g \) has positive scalar curvature and minimal boundary (resp. scalar-flat metric with strictly mean convex boundary). Moreover, if \( g \) is complete with \( u > \text{const} > 0 \), then \( g_1 \) is complete.

**Proof of Theorem 1.4.** We prove Item (b) since the other item is entirely analogous. We focus only on the case that \( M \) is non-compact, because the compact case was proved in Theorem 1.5 of [17] (For Item (a), see [11, Section 2]). Let \( g \) be a scalar flat metric with zero mean curvature on the boundary. Let \( \Omega_0 \) be a domain in \( M \) such that the second fundamental form \( \Pi \) is not identically zero in \( \partial_0 \Omega_0 \) which is nonempty. Let \( h \) be a symmetric 2-tensor in \( M \) supported in \( \Omega_0 \). In the sequel, we follow the steps in the proof of Theorem 1.5 of [17] (see also [11, Proposition 2.5]). Let \( g(t) \) be a smooth family of metrics with initial condition \( g(0) = g \) and \( \frac{\partial}{\partial t} g(t)|_{t=0} = -f\Pi_g \) on
\[ \partial_0 \Omega_0, \text{ where } f \text{ is a nonnegative smooth function supported on } \Omega_0. \] Observe also that \( g(t) = g \) in \( M \setminus \Omega_0 \). Let \( \sigma_1(\Omega_0) \) defined as in (5.1).

We have (see Appendix of [17])

\[ \frac{d}{dt} \sigma_1(\Omega_0, g(t)) \bigg|_{t=0} = \int_{\partial_0 \Omega_0} f |\Pi_g|^2 d\sigma > 0. \]

Since \( \sigma_1(\Omega_0, g) = 0 \), then \( \sigma_1(\Omega_0, g(t)) > 0 \) for all \( t > 0 \) sufficiently small.

Then by Theorem 5.3, there exists a function \( 0 < c_1 < u < c_2 \) such that \( \Delta g u = 0 \) in \( M \) and \( B_g u > 0 \) on \( \partial M \). Hence the metric \( g_1 = u^{4/(n-2)} g \) is scalar flat, it has strictly mean convex boundary and satisfies \( c_1 g \leq g_1 \leq c_2 g \). □

Now, we extend the trichotomy result in [11, Proposition 2.5].

**Proposition 5.5.** Let \( (M^3, g) \) be a scalar flat orientable complete Riemannian manifold with nonempty minimal boundary.

(a) If \( \text{Ric}_g \neq 0 \), then there exists a small isotopic smooth perturbation \( g_1 \) of \( g \), such that \( R_{g_1} > 0 \) and \( H_{g_1} \equiv 0 \).

(b) If \( \text{Ric}_g \equiv 0 \) and \( \partial M \) is not totally geodesic, then there exists a small isotopic smooth perturbation \( g_1 \) of \( g \), such that \( R_{g_1} \equiv 0 \) and every connected component of \( \partial M \) that is not totally geodesic becomes strictly mean-convex with respect to the outward unit normal vector field in \( g_1 \) (while the other ones are kept totally geodesic);

(c) If \( \text{Ric}_g \equiv 0 \) and all components of \( \partial M \) are totally geodesic, then \( (M, g) \) is isometric to a Riemannian product \( \partial M \times I \) equipped with a flat metric, where \( I \subset \mathbb{R} \) is an interval and \( \partial M \) is a complete flat surface.

**Proof.** We will focus on the case that the boundary \( \partial M \) is non-compact, provided the compact case can be treated as in [11, Proposition 2.5].

Item (a) follows from Theorem 1.4.

For item (b), assume that \( M \) is Ricci-flat and that some boundary component, say \( \partial M_1 \), is not totally geodesic. First, observe that \( \partial M_1 \) is unstable, otherwise arguing as in [49], we could find a positive constant \( C \) such that

\[ \int_{\partial M_1} |A|^2 f^2 \leq \frac{C}{\log R}, \]

where \( A \) is the second fundamental form of \( \partial M_1 \) and \( f(q) = \varphi(r) \) is a radial logarithmic cut-off function given by

\[
\varphi(r) = \begin{cases} 
1 & \text{if } 0 \leq r \leq 1, \\
1 - \log r / \log R & \text{if } 1 \leq r \leq R, \\
0 & \text{if } R \leq r,
\end{cases}
\]

where \( r(q) = \text{dist}(x_0, q) \) denotes the intrinsic distance from \( q \) to \( x_0 \). The right hand side of (5.9) goes to 0 as \( R \) tends to infinity, which shows that \( \partial M_1 \) would be totally geodesic, a contradiction. Then, we can choose \( \dot{\phi} \in C^\infty(\partial M_1) \) an eigenfunction for the lowest eigenvalue \( \lambda < 0 \) of the Jacobi operator \( L = \Delta_{\partial M_1} + |A|^2 \). Consider a vector field \( X \) in \( M \) such that
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Let \( X = \phi \nu_g \) on \( \partial M_1 \), where \( \nu_g \) is the unit normal vector to \( \partial M_1 \). Let \((F_t)_{t \in \mathbb{R}} \) denote by the flow generated by \( X \). Since \( \phi \) can be taken to be strictly positive and
\[
\frac{\partial}{\partial t} \left< \hat{H}(F_t(\partial M_1)), \nu_t \right> \bigg|_{t=0} = L\phi = -\lambda \phi > 0,
\]

where \( \nu_t \) denotes the unit normal vector to \( F_t(\partial M_1) \). Thus for sufficiently small \( t \) there exists a strictly mean convex subdomain \( M_t \) of \( M \). Hence \((M, F_t^*(g))\) is scalar-flat with a strictly mean-convex boundary component.

Finally, assume that \((M^3, g)\) is Ricci flat with totally geodesic boundary. We point out that since \( M \) has dimension three, \( g \) is a flat metric and \( \partial M \) is intrinsically flat by the Gauss equation. Let \( \hat{M} \) be the doubling of \( M \) across its boundary. Observe that the doubled metric is smooth and flat, which implies that the universal cover of \( \hat{M} \) is the Euclidean space \( \mathbb{R}^3 \). Following the same idea in item (c) of Proposition 2.5 of [11] (for \( \partial M \) connected or disconnected) we can conclude that \( M \) is isometric to a product \( \partial M \times I \), where \( I \subset \mathbb{R} \) is a interval non necessarily compact. For completeness, let us show what happens if at least one boundary component of \( \partial M \) is compact. If \( \partial M \) is connected, it follows from [16, Theorem 2] that \( M \) is isometric to \( \partial M \times [0, \infty) \) with the product metric. Now, if \( \partial M \) is disconnected, we can proceed again as above to conclude that \( M \) is isometric to a flat product of the form \( \partial M \times I \), where \( I \) is compact. \( \Box \)

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