Relations among $B_c \to J/\psi, \eta_c$ form factors

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Abstract
We analyze the form factors parametrizing the $B_c \to J/\psi, \eta_c$ matrix elements of the operators in a generalized low-energy $b \to c$ semileptonic Hamiltonian. We consider an expansion in nonrelativistic QCD, classifying the heavy quark spin symmetry breaking terms and expressing the form factors in terms of universal functions in a selected kinematical range. Using as an input the lattice QCD results for the $B_c \to J/\psi$ matrix element of the SM operator, we obtain information on other form factors. The extrapolation to the full kinematical range is also presented.

1 Introduction

The hadronic uncertainty affecting the description of several weak processes represents an important limitation for the Standard Model (SM) predictions. Such a theoretical error needs to be reduced or (at least) controlled in view of the ongoing and planned high precision measurements. Moreover, the error assessment is a preliminary step before interpreting the deviations of experimental results from the SM expectations, an important issue considering the recently observed tensions, the so-called flavour anomalies. In the time in which the search for phenomena beyond the Standard Model (BSM) through the direct production of new particles at the colliders has not produced compelling evidence, the search for BSM signals relies on the precision analysis of virtual effects distorting the SM predictions, as well as on the improved measurements of the fundamental parameters.

In the heavy flavour sector, anomalies have been detected in charged current $b \to c\ell\nu$ induced transition of $B_{d,u}$ mesons, comparing the $\ell = e, \mu$ with the $\ell = \tau$ modes \cite{1,2}. If such deviations are due to genuine new physics (NP) phenomena producing violation of lepton flavour universality (LFU), related deviations should be found in $B_s$, $B_c$ and $b$-baryon decay modes, both exclusive and inclusive \cite{3,4}. However, the various processes are affected by hadronic uncertainties in different ways, hence it is necessary to analyze each mode separately.

The present study is devoted to the exclusive semileptonic $b \to c\ell\nu$ decays of the $B_c$ meson, in particular $B_c \to J/\psi\ell\bar{\nu}_\ell$ and $B_c \to \eta_c\ell\bar{\nu}_\ell$ which are under experimental scrutiny \cite{5}. For such processes possible BSM effects can be analyzed in terms of a low-energy Hamiltonian generalizing the Standard Model one with the inclusion of the full set of dimension-6 operators,
as done for other modes \cite{3,6–12}. The matrix elements of the various operators in the effective Hamiltonian, parametrized in terms of form factors, introduce the hadronic uncertainty. The peculiarity of the processes is that they involve mesons each one comprising two heavy quarks, strengthening the interest in $B_c$, a meson with quarkonium structure and only weak decays.

The form factors for the $B_c \rightarrow J/\psi$ matrix elements of SM operators have been computed in the full kinematical range of dilepton invariant mass $q^2$ by lattice nonrelativistic QCD, by the HPQCD Collaboration \cite{13}. Other QCD-based computations employ QCD sum rules in the low $q^2$ range \cite{14,15}. In the same kinematical range, using standard NRQCD methods the $\langle J/\psi|\bar{c}\Gamma_i b|B_c \rangle$ matrix elements are expressed in the form $\langle J/\psi|\bar{c}\Gamma_i b|B_c \rangle \simeq \psi_{B_c}(0)\psi_{J/\psi}(0)T_i$, in terms of meson wave functions at the origin $\psi_{B_c}(0), \psi_{J/\psi}(0)$, and of perturbatively calculable hard-scattering kernels $T_i$ describing spectator interaction corrections and vertex corrections to factorization \cite{16–20}. Perturbative QCD calculations use an analogous principle \cite{21–24}. Light-Cone sum rules have been applied for all $q^2$ \cite{25}. These approaches are affected by their own theoretical uncertainties. As for the determinations based on quark models, beyond the variation of the input parameters the uncertainty attached to the model can hardly be quantified \cite{26,27}.

In Ref. \cite{28} it has been observed that the semileptonic $B_c$ form factors can be expressed in terms of universal functions in selected kinematical regions, on the basis of the heavy quark spin symmetry for large heavy quark masses. This remark has prompted a number of phenomenological analyses of $B_c$ decays \cite{29–31}. The relations to the universal functions have been provided at the leading order in the heavy quark mass expansion. Here we extend the analysis at the next-to-leading order in the expansion, to establish relations among the form factors and a set of universal functions based on the heavy quark spin symmetry and the power counting rules of nonrelativistic QCD (NRQCD). The relations obtained in such a systematic expansion have two applications. They can be used to test the form factors obtained by different methods, for a quantitative assessment of their theoretical uncertainty. Moreover, information can be gained on form factors that have not been computed yet, which are needed for analyses based on the generalized low-energy Hamiltonian. For this purpose, the available lattice QCD results can be employed as input information.

We shall use the power counting of NRQCD, the effective QCD theory relevant for mesons comprising two heavy quarks \cite{32,33}. A classification of the NRQCD operators is in \cite{34}. Using this effective theory and the potential-NRQCD effective theory (pNRQCD) formulated at a lower scale, fundamental QCD parameters have been precisely computed, namely the beauty and charm quark mass and the coupling constant $\alpha_s$ \cite{35}. Precise determinations of the $B_c$ mass \cite{36,37} and lifetime have also been obtained \cite{38,39}. Here we focus on semileptonic form factors based on a systematic expansion.

The plan of the paper is as follows. In Sec.2 we introduce the $b \rightarrow c\ell\nu$ generalized low-energy Hamiltonian in terms of $D = 6$ operators. In Sec.3 we describe the formalism of the NRQCD expansion and in Sec.4 we define the universal functions. In Sec.5 we present a set of numerical results. In App. A we define the parametrization used for the hadronic matrix elements. In App. B we provide the form factors in terms of universal functions. In App. C we give the results of the fit of some form factors obtained through the universal functions.
2 Generalized $b \to c$ semileptonic effective Hamiltonian and $B_c \to J/\psi, \eta_c$ form factors

The low-energy Hamiltonian comprising the full set of $D = 6$ semileptonic $b \to c$ operators with left-handed neutrinos can be written in the form:

\[
H_{eff}^{b\to c\ell \nu} = \frac{G_F}{\sqrt{2}} V_{cb} \left[ (1 + \epsilon_1^c) (\bar{c} \gamma_\mu (1 - \gamma_5) b) (\bar{\ell} \gamma_\mu (1 - \gamma_5) \nu_\ell) + \epsilon_2^c (\bar{c} \gamma_\mu (1 + \gamma_5) b) (\bar{\ell} \gamma_\mu (1 - \gamma_5) \nu_\ell) \\
+ \epsilon_3^c (\bar{c} \bar{b}) (\bar{\ell} (1 - \gamma_5) \nu_\ell) + \epsilon_4^c (\bar{c} \bar{b}) (\bar{\ell} (1 - \gamma_5) \nu_\ell) + \epsilon_5^c (\bar{c} \bar{b}) (\bar{\ell} (1 - \gamma_5) \nu_\ell) \right].
\]  

(1)

$G_F$ is the Fermi constant and $V_{cb}$ an element of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix. $O_{SM} = 4(\bar{c} L \gamma^\mu b_L) (\bar{\ell} L \gamma^\mu \nu_L)$ is the SM operator. The Hamiltonian Eq. (1) also comprises the operator $O_R = 4(\bar{c} R \gamma^\mu b_R) (\bar{\ell} L \gamma^\mu \nu_L)$, the scalar $O_S = (\bar{c} \bar{b}) (\bar{\ell} (1 - \gamma_5) \nu_\ell)$, pseudoscalar $O_P = (\bar{c} \bar{b}) (\bar{\ell} (1 - \gamma_5) \nu_\ell)$ and tensor $O_T = (\bar{c} \bar{b}) (\bar{\ell} \sigma^{\mu\nu} (1 - \gamma_5) \nu_\ell)$ operators. The Wilson coefficients $\epsilon_{V,R,S,P,T}^c$ are complex and lepton-flavour dependent, in general. Eq. (1) reduces to the SM for $\epsilon_i^c = 0$. The Hamiltonian Eq. (1) has been considered in connection with the anomalies in $B \to D^{(*)} \tau \nu_\tau$ vs $B \to D^{(*)} \ell \nu_\ell$ decays, obtaining information on the various operators and bounding the parameter space of the Wilson coefficients [3, 6–12]. Exclusive $B$ semileptonic modes induced by the $b \to u$ transition and inclusive $b$–baryon modes have been studied analogously [4, 13].

The $B_c$ and $J/\psi, \eta_c$ matrix elements of the operators in Eq. (1) require hadronic form factors, for which different parametrizations can be used. The $B_c \to \eta_c$ matrix elements of the vector $Q' \gamma_\mu Q$, scalar $Q' Q$, and tensor $Q' \sigma_{\mu\nu} Q$ and $Q' \sigma_{\mu\nu} \gamma_5 Q$ currents can be written in terms of form factors $f_i$ as

\[
\langle P(p') | \bar{Q}' \gamma_\mu Q | B_c(p) \rangle = f_{B_c \to P}^{B_c \to P}(q^2) \left( p_\mu + p'_\mu - \frac{m_{B_c}^2 - m_P^2}{q^2} q_\mu \right) + f_{B_c \to P}^{B_c \to P}(q^2) \frac{m_{B_c}^2 - m_P^2}{q^2} q_\mu ,
\]

\[
\langle P(p') | \bar{Q}' Q | B_c(p) \rangle = f_{S}^{B_c \to P}(q^2) ,
\]

\[
\langle P(p') | \bar{Q}' \sigma_{\mu\nu} Q | B_c(p) \rangle = -i \frac{2 f_{T}^{B_c \to P}(q^2)}{m_{B_c} + m_P} (p_\mu p'_\nu - p_\nu p'_\mu) ,
\]

\[
\langle P(p') | \bar{Q}' \sigma_{\mu\nu} \gamma_5 Q | B_c(p) \rangle = - \frac{2 f_{T}^{B_c \to P}(q^2)}{m_{B_c} + m_P} \epsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta ,
\]

where $P = \eta_c$, $q = p - p'$ is the momentum transfer to the lepton pair, and the condition $f_{B_c \to P}^{B_c \to P}(0) = f_{B_c \to P}^{B_c \to P}(0)$ holds. We use $\epsilon^{0123} = +1$ and the relation $\epsilon_{\mu\nu} \gamma_5 = \frac{i}{2} \epsilon_{\mu\rho\nu\sigma} a^{\rho\sigma}$. $f_{S}^{B_c \to P}$ is related to $f_{B_c \to P}^{B_c \to P}$: $f_{S}^{B_c \to P}(q^2) = \frac{m_{B_c}^2 - m_P^2}{m_Q - m_{Q'}} f_{B_c \to P}^{B_c \to P}(q^2)$ with quark masses $m_Q$ and $m_{Q'}$. The

\[1\] The operator $O_R$ is included in the set of dimension 6 operators with left-handed neutrinos. In the Standard Model Effective Field Theory the only dimension-6 operator with right-handed quark current is nonlinear in the Higgs field [40, 42].
$B_c \to J/\psi$ matrix elements can be parametrized as

\[
\langle V(p', \epsilon) | \bar{Q} \gamma \mu Q | B_c(p) \rangle = -\frac{2 V^{B_c \to V}(q^2)}{m_{B_c} + m_V} i \epsilon_{\mu \nu \alpha \beta} \epsilon^* \nu p^\alpha p^\beta ,
\]

\[
\langle V(p', \epsilon) | \bar{Q} \gamma \gamma_5 Q | B_c(p) \rangle = (m_{B_c} + m_V) \left( \epsilon^* - \frac{(\epsilon^* \cdot q)}{q^2} q_\mu \right) A_{1}^{B_c \to V}(q^2) - \frac{(\epsilon^* \cdot q)}{m_{B_c} + m_V} \left( p + p' \right)_\mu - \frac{m_{B_c}^2 - m_V^2}{q^2} q_\mu \right) A_{2}^{B_c \to V}(q^2) + (\epsilon^* \cdot q) \frac{2m_V}{q^2} q_\mu A_0^{B_c \to V}(q^2),
\]

\[
\langle V(p', \epsilon) | \bar{Q} \sigma_{\mu \nu} Q | B_c(p) \rangle = T_0^{B_c \to V}(q^2) \frac{\epsilon^* \cdot q}{(m_{B_c} + m_V)^2} \epsilon_{\mu \nu \alpha \beta} p^\alpha p^\beta \\
+ T_1^{B_c \to V}(q^2) \epsilon_{\mu \nu \alpha \beta} p^\alpha \epsilon^* \beta + T_2^{B_c \to V}(q^2) \epsilon_{\mu \nu \alpha \beta} p^\alpha \epsilon^* \beta,
\]

\[
\langle V(p', \epsilon) | Q \sigma_{\mu \nu} \gamma_5 Q | B_c(p) \rangle = i T_0^{B_c \to V}(q^2) \frac{\epsilon^* \cdot q}{(m_{B_c} + m_V)^2} (p_\mu p'_\nu - p_\nu p'_\mu) \\
+ i T_1^{B_c \to V}(q^2) (p_\mu \epsilon^*_\nu - \epsilon_\mu p'_\nu) + i T_2^{B_c \to V}(q^2) (p'_\mu \epsilon^*_\nu - \epsilon_\mu p'_\nu),
\]

with $V = J/\psi$ and $\epsilon$ the $J/\psi$ polarization vector. The condition holds:

\[
A_0^{B_c \to V}(0) = \frac{m_{B_c} + m_V}{2m_V} A_1^{B_c \to V}(0) - \frac{m_{B_c} - m_V}{2m_V} A_2^{B_c \to V}(0).
\]

The results for $V$ and $A_{1,2,0}$ computed in [13] will be exploited in our numerical analysis.

In our study it is convenient to use a different basis of form factors defined in Appendix A For $B_c \to \eta_c$ the relations between the two basis are:

\[
\frac{m_{B_c}^2 - m_P^2}{q^2} (f_0(q^2) - f_+(q^2)) = \frac{1}{2 \sqrt{m_{B_c} m_P}} \left( (m_{B_c} + m_P) h_-(w) - (m_{B_c} - m_P) h_+(w) \right)
\]

\[
f_+(q^2) = \frac{1}{2 \sqrt{m_{B_c} m_P}} \left( (m_{B_c} + m_P) h_+(w) - (m_{B_c} - m_P) h_-(w) \right)
\]

\[
f_T(q^2) = -\frac{m_{B_c} + m_P}{\sqrt{m_{B_c} m_P}} h_T(w) ,
\]

where $v = \frac{p}{m_{B_c}}$, $v' = \frac{p'}{m_P}$ and $w = v \cdot v'$, hence $q^2 = m_{B_c}^2 + m_P^2 - 2m_{B_c}m_Pw$. For $B_c \to J/\psi$
The relations are:

\[ V(q^2) = \frac{m_{Bc} + m_V}{2\sqrt{m_Bm_V}}h_V(w) \]

\[ A_1(q^2) = \sqrt{m_Bm_V} \frac{w + 1}{m_{Bc} + m_V}h_{A_1}(w) \]

\[ A_2(q^2) = \frac{m_{Bc} + m_V}{2\sqrt{m_Bm_V}} \left( h_{A_3}(w) + \frac{m_V}{m_B}h_{A_2}(w) \right) \]

\[ A_0(q^2) = \frac{1}{2\sqrt{m_Bm_V}} \left( m_{Bc}(w + 1)h_{A_3}(w) - (m_{Bc} - m_Vw)h_{A_2}(w) - (m_{Bc}w - m_V)h_{A_3}(w) \right) \]

\[ T_0(q^2) = -\frac{(m_{Bc} + m_V)^2}{m_B\sqrt{m_Bm_V}}h_{T_3}(w) \]  \hspace{1cm} (6)

\[ T_1(q^2) = \frac{m_V}{\sqrt{m_Bm_V}} \left( h_{T_1}(w) + h_{T_2}(w) \right) \]

\[ T_2(q^2) = \frac{m_{Bc}}{\sqrt{m_Bm_V}} \left( h_{T_1}(w) - h_{T_2}(w) \right) \]

with \( q^2 = m_{Bc}^2 + m_V^2 - 2m_{Bc}m_Vw \). The form factors \( h_i \) can be related to a set of universal functions in a kinematical range close to \( w = 1 \). For hadrons comprising a single heavy quark, this has been done in [44–46]. The modifications for the heavy quarkonium are discussed in the next sections.

3 Expansion of the heavy quark field and the QCD Lagrangian

To construct the heavy quark expansion, the heavy quark QCD field \( Q(x) \) with mass \( m_Q \) is written factorizing a fast oscillation mass term:

\[ Q(x) = e^{-im_Qv^\mu x} \psi(x) = e^{-im_Qv^\mu x} \left( \psi_+(x) + \psi_-(x) \right) \]  \hspace{1cm} (7)

with \( \psi_\pm = P_\pm \psi(x) \) and \( P_\pm = \frac{1 \pm \gamma}{2} \). \( \psi_\pm \) is the positive energy component of the field (we use the notation adopted in [38]). \( v \) is identified with the heavy meson (quarkonium) 4-velocity with \( v^2 = 1 \). The equation of motion allows us to relate \( \psi_- \) to \( \psi_+ \),

\[ \psi_-(x) = \frac{1}{2m_Q + iv \cdot D} i\partial_\perp \psi_+(x) \]  \hspace{1cm} (8)

where \( D_\perp \mu = D_\mu - (v \cdot D)v_\mu \). In the rest frame \( v = (1, 0, 0, 0) \) we have \( v \cdot D = D_t \) and \( D_\perp \mu = (0, D_\perp) \).

Using (7) and (8) \( Q(x) \) can be expressed in terms of \( \psi_+(x) \),

\[ Q(x) = e^{-im_Qv^\mu x} \left( 1 + \frac{1}{2m_Q + iv \cdot D} i\partial_\perp \right) \psi_+(x), \]  \hspace{1cm} (9)
a nonlocal expression which can be expanded:

\[
Q(x) = e^{-imQv \cdot x} \left(1 + \frac{iD_\perp}{2m_Q} + \frac{(-iv \cdot D)}{2m_Q} \frac{iD_\perp}{2m_Q} + \ldots \right) \psi_+(x).
\] (10)

The power counting of the various operators is set within NRQCD: \(D_i \sim \tilde{v}^2\), \(D_\perp \sim \tilde{v}\) and \(\psi_+ \sim \tilde{v}^{3/2}\), where \(\tilde{v} = |\vec{v}| \ll 1\) is the relative heavy quark 3-velocity in the hadron rest frame \[33\]. Therefore, the second term in Eq. (10) is \(O(\tilde{v} \times \tilde{v}^{3/2})\), the third one is \(O(\tilde{v}^3 \times \tilde{v}^{3/2})\).

From now on, the power \(\tilde{v}^{3/2}\) for each quark field will be omitted in the power counting of the operators.

The QCD Lagrangian expressed in terms of \(\psi_+\)

\[
L_{QCD} = \overline{\psi}_+(x) \left(\frac{i v \cdot D + i D_\perp}{2m_Q} + i D_\perp \frac{v \cdot D}{2m_Q} \right) \psi_+(x)
\] (11)

can be expanded:

\[
L_{QCD} = \overline{\psi}_+(x) \left(\frac{i v \cdot D + (iD_\perp)^2}{2m_Q} + \frac{g}{4m_Q} \sigma \cdot G_\perp + \frac{iD_\perp}{2m_Q} \left(-\frac{i v \cdot D}{2m_Q} \right) (iD_\perp) + \ldots \right) \psi_+(x)
\] = \(L_0 + L_1 + \ldots\). (12)

In this expression \(G_{\perp \mu \nu}\) is \(G_{\perp \mu \nu} = (g_{\mu \alpha} - v_\mu v_\alpha)(g_{\nu \beta} - v_\nu v_\beta)G^{\alpha \beta}\). In the rest frame \(G_{\perp \mu \nu} = G_{ij}\) for \(\mu = i = 1, 2, 3\) and \(\nu = j = 1, 2, 3\), while the other components vanish. The power counting of the chromoelectric field components \(E_i = G_{0i}\) and of the chromomagnetic ones \(B_i = \frac{1}{2} \epsilon_{ijk}G_{jk}\) is \(\tilde{v}^3\) and \(\tilde{v}^4\), respectively \[32\].

The first and second term in Eq. (12) are \(O(\tilde{v}^2)\) and provide the leading order Lagrangian

\[
L_0 = \overline{\psi}_+(x) \left(\frac{i v \cdot D + (iD_\perp)^2}{2m_Q} \right) \psi_+(x)
\] (13)

giving the equation of motion for \(\psi_+(x)\)

\[
\left(\frac{i v \cdot D + (iD_\perp)^2}{2m_Q} \right) \psi_+(x) = 0.
\] (14)

The third and fourth term in Eq. (12) are \(O(\tilde{v}^4)\) and give the NLO Lagrangian

\[
L_1 = L_{1,1} + L_{1,2}
\]

where

\[
L_{1,1} = \overline{\psi}_+(x) \frac{g\sigma \cdot G_\perp}{4m_Q} \psi_+(x)
\]

\[
L_{1,2} = \overline{\psi}_+(x) \frac{iD_\perp}{2m_Q} \left(-\frac{i v \cdot D}{2m_Q} \right) (iD_\perp) \psi_+(x).
\] (15)
Using the equation of motion together with \([iD_\perp, iv \cdot D] = ig\gamma^\mu v^\nu G_{\mu\nu}\) and \(D_\perp D_\perp = D_\perp^2 - \frac{1}{2}g\sigma \cdot G_\perp\), \(\mathcal{L}_{1,2}\) can be expressed in the form

\[
\mathcal{L}_{1,2} = -\frac{1}{4m_Q^2} \left( \bar{\psi}_+(x) \left( -\left(\frac{iD_\perp)^4}{2m_Q} \right) \psi_+(x) + \bar{\psi}_+(x) \frac{g}{2} \sigma \cdot G_\perp \left( -\frac{(iD_\perp)^2}{2m_Q} \right) \psi(x) \right) + igv^\alpha \bar{\psi}_+(x)iD_\perp^\alpha G_{\alpha\sigma} \psi_+(x) + gv^\alpha \bar{\psi}_+(x)iD_\perp\sigma^\sigma G_{\alpha\sigma} \psi_+(x) \right)
\]

\[= \mathcal{L}_{1,2}^{(1)} + \mathcal{L}_{1,2}^{(2)} + \mathcal{L}_{1,2}^{(3)} + \mathcal{L}_{1,2}^{(4)}
\]

\(\mathcal{L}_{1,2}^{(2)}\) is of higher order in the \(\tilde{v}\) expansion. \(\mathcal{L}_1\) can be arranged in the form

\[
\mathcal{L}_1 = \mathcal{L}_1^A + \mathcal{L}_1^B
\]

with

\[
\mathcal{L}_1^A = \mathcal{L}_{1,1} + \mathcal{L}_{1,2}^{(4)} = \frac{1}{4m_Q^2} \bar{\psi}_+(x) A_{\tau\sigma} \sigma^\tau \sigma \psi_+(x)
\]

\[
\mathcal{L}_1^B = \mathcal{L}_{1,1}^{(1)} + \mathcal{L}_{1,2}^{(3)} = \frac{1}{4m_Q^2} \bar{\psi}_+(x) B \psi_+(x),
\]

where in \((18)\) we have factorized the leading \(1/m_Q\) power. To deal with the antiquark, the QCD field \(Q(x)\) is written as

\[
Q(x) = e^{im_\perp v \cdot x} X(x) = e^{im_\perp v \cdot x} \left( X_+(x) + X_-(x) \right) = e^{im_\perp v \cdot x} \left( 1 + \frac{1}{2m_Q - iv \cdot D} i\bar{D}_\perp \right) X_-(x),
\]

with \(X_-\) containing the negative energy component. The QCD Lagrangian written in terms of \(X_-\)

\[
\mathcal{L}_{QCD} = \bar{X}_-(x) \left( -iv \cdot D + \frac{1}{2m_Q - iv \cdot D} i\bar{D}_\perp \right) X_-(x)
\]

is expanded as

\[
\mathcal{L}_{QCD} = \bar{X}_-(x) \left( -iv \cdot D + \frac{(iD_\perp)^2}{2m_Q} + \ldots \right) X_-(x).
\]

The above expressions define the effective theory in which to work out the meson form factors.

### 4 Meson form factors in the effective theory

To obtain the meson form factors in the effective theory, we expand the weak current involving two heavy quarks \(\bar{Q}'\Gamma Q\), with \(\Gamma\) a generic Dirac matrix:

\[
\bar{Q}'(x)\Gamma Q(x) = \bar{\psi}_+(x) \left( 1 - \frac{i\bar{D}_\perp}{2m_Q} - \frac{1}{4m_Q^2} (i\bar{D}_\perp)(iv' \cdot \bar{D}') + \ldots \right) \psi_+(x)
\]

\[
\Gamma \left( 1 + \frac{i\bar{D}_\perp}{2m_Q} + \frac{1}{4m_Q^2} (-iv \cdot \bar{D}) i\bar{D}_\perp + \ldots \right) \psi_+(x)
\]

\[\text{(23)}\]
where \( \mathcal{D}_{\perp \mu} = D_\mu - (v' \cdot D)v'_\mu \). Keeping terms up to \( \mathcal{O}(1/m_Q^2) \), the current can be written as

\[
\bar{Q}'(x) \Gamma Q(x) = J_0 + \left( \frac{J_{1,0}}{2m_Q} \right) + \left( - \frac{J_{2,0}}{4m_Q^2} \right) + \left( \frac{J_{1,1}}{4m_Q m_{Q'}} \right),
\]

with \( J_i \) terms

\[
J_0 = \bar{\psi}_+ \Gamma \psi_+
\]

\[
J_{1,0} = \bar{\psi}_+ \Gamma i \slashed{D}_\perp \psi_+
\]

\[
J_{0,1} = \bar{\psi}_+ \left( -i \slashed{D}_\perp \right) \Gamma \psi_+
\]

\[
J_{2,0} = \bar{\psi}_+ \Gamma \left( iv \cdot \slashed{D} \right) i \slashed{D}_\perp \psi_+
\]

\[
J_{0,2} = \bar{\psi}_+ i \slashed{D}_\perp \left( iv' \cdot \slashed{D} \right) \Gamma \psi_+
\]

\[
J_{1,1} = \bar{\psi}_+ \left( -i \slashed{D}_\perp \right) \Gamma \left( i \slashed{D}_\perp \right) \psi_+.
\]

Considering the power counting in \( \tilde{v} \), Eq. \([24]\) comprises terms up to \( \mathcal{O}(\tilde{v}^3) \). The \( \mathcal{O}(1/m_Q^2) \) terms involving three derivatives have not been included in \([24]\), even though they can be of the same order in \( \tilde{v} \) of some terms appearing in the (nonlocal) corrections discussed in the following: we assume that they provide numerically suppressed effects.

We have neglected the perturbative \( \alpha_s \) corrections. Considering such corrections the short distance expansion of the current in \([24]\) would contain more operators and a set of matching coefficients would appear:

\[
Q'(x) \Gamma Q(x) = \sum_i C_i(\mu, w)(J_0)_i + \sum_j \left[ B_j(\mu, w) (J_{1,0})_j + B'_j(\mu, w) (J_{0,1})_j \right] + \ldots. \]

The various structures in Eq. \([24]\) can be identified with the first terms in each of the sums in \([26]\). The coefficients \( C_i \) and \( B'_j \) are perturbatively expanded in \( \alpha_s \). Only the coefficients of the operators in \([24]\) contribute at leading order in \( \alpha_s \), the others start at \( \mathcal{O}(\alpha_s) \). At leading order in the inverse HQ mass expansion and at \( \mathcal{O}(\alpha_s) \) one finds for the various currents:

\[
\bar{Q}' Q = \bar{\psi}_+ \left( 1 + \frac{\alpha_s}{\pi} C_S \right) \psi_+ + \ldots
\]

\[
\bar{Q}' \gamma_5 Q = \bar{\psi}_+ \left( 1 + \frac{\alpha_s}{\pi} C_P \right) \gamma_5 \psi_+ + \ldots
\]

\[
\bar{Q}' \gamma_\mu Q = \bar{\psi}_+ \left[ \left( 1 + \frac{\alpha_s}{\pi} C_{V_1} \right) \gamma_\mu + \frac{\alpha_s}{\pi} C_{V_2} v_\mu + \frac{\alpha_s}{\pi} C_{V_3} v'_\mu \right] \psi_+ + \ldots
\]

\[
\bar{Q}' \gamma_\mu \gamma_5 Q = \bar{\psi}_+ \left[ \left( 1 + \frac{\alpha_s}{\pi} C_{A_1} \right) \gamma_\mu + \frac{\alpha_s}{\pi} C_{A_2} v_\mu + \frac{\alpha_s}{\pi} C_{A_3} v'_\mu \right] \gamma_5 \psi_+ + \ldots
\]

\[
\bar{Q}' \sigma^{\mu\nu} Q = \bar{\psi}_+ \left[ \left( 1 + \frac{\alpha_s}{\pi} C_{T_1} \right) \sigma^{\mu\nu} + \frac{\alpha_s}{\pi} C_{T_2} i(v^{\mu} \gamma^{\nu} - v^{\nu} \gamma^{\mu}) + \frac{\alpha_s}{\pi} C_{T_3} i(v^{\mu} \gamma^{\nu} - v^{\nu} \gamma^{\mu}) \right] \psi_+ + \ldots
\]
The coefficients $C_i$ have been computed in [47][49], the results for $B_j^{(l)}$ for the vector and axial vector currents are in [50]. In Sec. 5 we comment on the accuracy of using (24) instead of (26).

The $B_c$ and $J/\psi, \eta_c$ matrix elements of the various terms in the expansion (24) can be expressed using the trace formalism [51]. In this formalism, the lowest-lying S-wave $\bar{b}c$ and $\bar{c}c$ bound states are described by $4 \times 4$ matrices [28]

$$H^{\bar{b}c}(v) = 1 + \frac{\gamma}{2} \left[ B_{\bar{b}c}^{* \mu} \gamma_{\mu} - B_{\bar{b}c} \gamma_{5} \right] \frac{1 - \gamma}{2}$$

$$H^{\bar{c}c}(v') = 1 + \frac{\gamma}{2} \left[ \Psi^{* \mu} \gamma_{\mu} - \eta_c \gamma_{5} \right] \frac{1 - \gamma}{2}$$

satisfying the relations $\not{H}(v) = H(v) = -H(v')$ and $H(v')\gamma^\mu = H(v) = -\gamma^\mu H(v')$. $B_{\bar{b}c}^{* \mu}, B_{\bar{b}c}$ and $\Psi^{* \mu}, \eta_c$ annihilate vector and pseudoscalar $\bar{b}c$ and $\bar{c}c$ mesons of velocity $v$ and $v'$, respectively. They are normalized to $\sqrt{m_M}$ with $M$ one of the mesons in the spin doublet. The trace formalism has been used to write the effective Lagrangians governing strong and radiative heavy quarkonium transitions in the soft-exchange approximation [52][54].

The $x$-dependence of the matrix element $M_0(x) = \langle M'(v')|\bar{\psi}^\alpha_+(x)\Gamma \psi^\alpha_+(x)|M(v)\rangle$ can be obtained exploiting the dependence in the effective theory [55][56]:

$$M(x) = e^{-i\tilde{\Lambda}v \cdot x} M(0)$$

where, for a $Q\bar{Q'}$ meson,

$$\tilde{\Lambda} = m_H - m_Q - m_{Q'}.$$  

(31)

Eq. (31) means that for the heavy quarkonium under scrutiny the heavy quark binding is generated by nonperturbative effects. $B_c$ and $J/\psi, \eta_c$ are not considered as purely Coulombic states. The small binding energy scale $\tilde{\Lambda}$ can display a residual heavy quark mass dependence. From (30) and (31) we have

$$M_0(x) = e^{-i\phi \cdot x} M_0(0)$$

(32)

with $\phi = \tilde{\Lambda}v - \tilde{\Lambda}'v'$. For $B_c \rightarrow J/\psi(\eta_c)$, the binding energies $\tilde{\Lambda}^{(l)}$ are given by $\tilde{\Lambda} = m_{B_c} - m_b - m_c$ and $\tilde{\Lambda}' = m_{J/\psi(\eta_c)} - 2m_c$.

Using the trace formalism, we define

$$\langle M'(v')|\bar{\psi}^\alpha_+ \Gamma (i v \cdot \vec{D}) \psi^\alpha_+ |M(v)\rangle = -\phi_K(w) \text{Tr} \left[ H'(v') \Gamma H(v) \right]$$

(33)

$$\langle M'(v')|\bar{\psi}^\alpha_+ (-i v' \cdot \vec{D}) \Gamma \psi^\alpha_+ |M(v)\rangle = -\phi'_K(w) \text{Tr} \left[ H'(v') \Gamma H(v) \right]$$

(34)

with $w = v \cdot v'$. The same formalism allows us to parametrize the matrix elements of the various terms in (24). The matrix element of $J_0$ is written as

$$\langle M'(v')|J_0|M(v)\rangle = -\Delta(w) \text{Tr} \left[ H'(v') \Gamma H(v) \right]$$

(35)

and involves the form factor $\Delta(w)$ obtained in [28]. The $1/m_{Q,Q'}$ terms involve the functions $\Delta_\alpha$ and $\tilde{\Delta}_\alpha$,

$$\langle M'(v')|\bar{\psi}^\alpha_+ \Gamma^\alpha i D_\alpha \psi^\alpha_+ |M(v)\rangle = -\text{Tr} \left[ \Delta_\alpha (v, v') \bar{H}'(v') \Gamma^\alpha H(v) \right]$$

(36)

$$\langle M'(v')|\bar{\psi}^\alpha_+ (-i \vec{D}_\alpha) \Gamma^\alpha \psi^\alpha_+ |M(v)\rangle = -\text{Tr} \left[ \tilde{\Delta}_\alpha (v, v') \bar{H}'(v') \Gamma^\alpha H(v) \right]$$

(37)
which are expressed in general as

$$\Delta_\alpha(v, v') = \Delta_+(w)(v + v')_\alpha + \Delta_-(w)(v - v')_\alpha - \Delta_3(w)\gamma_\alpha$$  \hspace{1cm} (38)$$

$$\bar{\Delta}_\alpha(v, v') = \bar{\Delta}_+(w)(v + v')_\alpha + \bar{\Delta}_-(w)(v' - v)_\alpha - \bar{\Delta}_3(w)\gamma_\alpha.$$  \hspace{1cm} (39)$$

Exploiting the relation

$$i\partial_\alpha(\bar{\psi}_+^\Gamma \psi_+ - \bar{\psi}_+^\Gamma \psi_+) = \bar{\psi}_+^\Gamma (i\vec{D}_\alpha) \Gamma \psi_+ + \bar{\psi}_+^\Gamma \Gamma (i\vec{D}_\alpha) \psi_+$$  \hspace{1cm} (40)$$

and using (32), the functions in (38)-(39) can be connected to $\Delta$ in (35):

$$\bar{\Lambda} - \bar{\Lambda}' w) \Delta = (\Delta_+ - \bar{\Delta}_+)(1 + w) + (\Delta_- + \bar{\Delta}_-)(1 - w) + (\Delta_3 - \bar{\Delta}_3)$$  \hspace{1cm} (41)$$

$$\bar{\Lambda} w - \bar{\Lambda}' \Delta = (\Delta_+ - \bar{\Delta}_+)(1 + w) + (\Delta_- + \bar{\Delta}_-)(w - 1) + (\Delta_3 - \bar{\Delta}_3).$$  \hspace{1cm} (42)$$

$\Delta_i$ satisfy the equations

$$\Delta_+(1 + w) + \Delta_- (1 - w) + \Delta_3 = \phi_K$$  \hspace{1cm} (43)$$

$$\bar{\Delta}_+(1 + w) + \bar{\Delta}_-(1 - w) + \bar{\Delta}_3 = \phi_K'$$  \hspace{1cm} (44)$$

$$\Delta_3 = \bar{\Delta}_3$$  \hspace{1cm} (45)$$

obtained using Eqs. (14) and (32), with solutions

$$\Delta_+(w) = -\frac{\Delta_3(w)}{(1 + w)} + \frac{\Delta(w)}{2(1 + w)} \left(\bar{\Lambda} w - \bar{\Lambda}'\right) + \frac{\phi_K(w) + \phi_K'(w)}{2(1 + w)}$$  \hspace{1cm} (46)$$

$$\bar{\Delta}_+(w) = -\frac{\Delta_3(w)}{(1 + w)} + \frac{\Delta(w)}{2(1 + w)} \left(\bar{\Lambda}' w - \bar{\Lambda}\right) + \frac{\phi_K(w) + \phi_K'(w)}{2(1 + w)}$$  \hspace{1cm} (47)$$

$$\Delta_-(w) = \frac{\Delta(w)}{2(w - 1)} \left(\bar{\Lambda} w - \bar{\Lambda}'\right) - \frac{\phi_K(w) - \phi_K'(w)}{2(w - 1)}$$  \hspace{1cm} (48)$$

$$\bar{\Delta}_-(w) = \frac{\Delta(w)}{2(w - 1)} \left(\bar{\Lambda}' w - \bar{\Lambda}\right) + \frac{\phi_K(w) - \phi_K'(w)}{2(w - 1)}.$$  \hspace{1cm} (49)$$

The functions $\Delta_-$ and $\bar{\Delta}_-$ are finite at $w = 1$ if $\phi_K(w)$ and $\phi_K'(w)$ are related by the condition

$$\phi_K(w) - \phi_K'(w) = \left(\bar{\Lambda} - \bar{\Lambda}'\right) \Delta(w).$$  \hspace{1cm} (50)$$

Solving Eqs. (41)-(44) after an expansion of the universal functions close to $w = 1$ we find that Eq. (50) holds for $w = 1$. There is an interesting analogy to the case of heavy-light mesons, where $\phi_K(w) = \phi_K'(w)$ and $\phi_K(1)$ is proportional to the heavy quark kinetic energy [45]. If the relation holds for all values of $w$, the heavy-light case is recovered for $\bar{\Lambda} = \bar{\Lambda}'$. The condition
allows us to obtain:

\[
\begin{align*}
\Delta_+(w) &= -\frac{\Delta_3(w)}{(w + 1)} + \Delta(w)\frac{\tilde{\Lambda}(w - 1)}{2(1 + w)} + \frac{\phi_K(w)}{w + 1} \\
\tilde{\Delta}_+(w) &= -\frac{\Delta_3(w)}{(w + 1)} + \Delta(w)\frac{\tilde{\Lambda}'(w - 1)}{2(1 + w)} + \frac{\phi_K'(w)}{w + 1} \\
\Delta_-(w) &= \frac{\Delta(w)}{2}\tilde{\Lambda} \\
\tilde{\Delta}_-(w) &= \frac{\Delta(w)}{2}\tilde{\Lambda}'.
\end{align*}
\]

The matrix elements of the operators with \(1/m_{Q,Q'}\) in (54) are expressed in terms of \(\Delta_i\):

\[
\begin{align*}
\frac{1}{2m_Q} \langle M'(v')|J_{1,0}|M(v)\rangle &= -\frac{1}{2m_Q} \left\{ -\text{Tr} \left[ \tilde{H}'(v')\Gamma H(v) \right] \left[ \frac{\Delta_3(w)}{1 + w} + \frac{w}{1 + w}(\phi_K(w) - \Delta(w)\tilde{\Lambda}) \right] \\
&\quad + \text{Tr} \left[ \tilde{H}'(v')\Gamma\gamma\Phi H(v) \right] \left[ -\frac{\Delta_3(w)}{1 + w} + \frac{1}{1 + w}(\phi_K(w) - \Delta(w)\tilde{\Lambda}) \right] \\
&\quad - \left[ \gamma^\beta \tilde{H}'(v')\gamma\beta H(v) \right] \Delta_3(w) \right\}.
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2m_{Q'}} \langle M'(v')|J_{0,1}|M(v)\rangle &= -\frac{1}{2m_{Q'}} \left\{ -\text{Tr} \left[ \tilde{H}'(v')\Gamma H(v) \right] \left[ \frac{\Delta_3(w)}{1 + w} + \frac{w}{1 + w}(\phi_K(w) - \Delta(w)\tilde{\Lambda}) \right] \\
&\quad + \text{Tr} \left[ \tilde{H}'(v')\gamma\Phi H(v) \right] \left[ -\frac{\Delta_3(w)}{1 + w} + \frac{1}{1 + w}(\phi_K(w) - \Delta(w)\tilde{\Lambda}) \right] \\
&\quad - \left[ \gamma^\beta \tilde{H}'(v')\gamma\beta H(v) \right] \Delta_3(w) \right\}.
\end{align*}
\]

To consider the \(1/m_Q^2\) terms we define

\[
\langle M'(v')|\tilde{\psi}_+(-i\tilde{D}_\alpha)\Gamma^{\alpha\beta}(i\tilde{D}_\beta)\psi_+|M(v)\rangle = -\text{Tr} \left[ \psi_{\alpha\beta}(v,v')\tilde{H}'(v')\Gamma^{\alpha\beta}H(v) \right].
\]

The function \(\psi_{\alpha\beta}(v,v')\) is written in terms of its symmetric \(\psi^S\) and antisymmetric \(\psi^A\) parts

\[
\psi_{\alpha\beta} = \frac{1}{2}[\psi^{S}_{\alpha\beta} + \psi^{A}_{\alpha\beta}]
\]

which can be parametrized as

\[
\begin{align*}
\psi^{S}_{\alpha\beta} &= \psi^S_1(w)g_{\alpha\beta} + \psi^S_2(w)(v + v')_\alpha(v + v')_\beta + \psi^S_3(w)(v - v')_\alpha(v - v')_\beta \\
&\quad + \psi^S_4(w)[(v + v')_\alpha\gamma_\beta + (v + v')_\beta\gamma_\alpha] + \psi^S_5(w)[(v - v')_\alpha\gamma_\beta + (v - v')_\beta\gamma_\alpha] \\
&\quad + \psi^S_6(w)[(v + v')_\alpha(v - v')_\beta + (v + v')_\beta(v - v')_\alpha] \\
\psi^{A}_{\alpha\beta} &= \psi^A_1(w)v_\alpha v'_\beta - v_\beta v'_\alpha + \psi^A_2(w)[(v - v')_\alpha\gamma_\beta - (v - v')_\beta\gamma_\alpha] \\
&\quad + \psi^A_3(w)i\sigma_{\alpha\beta} + \psi^A_4(w)[(v + v')_\alpha\gamma_\beta - (v + v')_\beta\gamma_\alpha].
\end{align*}
\]
The hadronic matrix element of $J_{1,1}$ in (24) can be expressed in terms of $\psi_{\alpha\beta}$. For $J_{2,0}$ and $J_{0,2}$ integration by parts is also needed. We obtain:

$$
\langle M'(v')|J_{2,0}|M(v)\rangle = -(\bar{\Lambda} - w\bar{\Lambda}') \left\{ \Delta_+ \left[ \text{Tr} \left( \bar{H}'(v')\Gamma H(v) \right) + \text{Tr} \left( \bar{H}'(v')\gamma^\beta H(v) \right) \right] \right.
+ \Delta_- \left( \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] - \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] \right)
- \Delta_3 \text{Tr} \left( \bar{H}'(v')\gamma^\beta H(v)\gamma_\beta \right) - \phi_K \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] \left. \right\}
- \frac{1}{2} \left( (1 + w)\psi_2^S - (1 - w)\psi_3^S - 2w\psi_6^S + \psi_1^A - (\psi_4^S - \psi_5^S + \psi_2^A - \psi_4^A) \right)
\times \left( \text{Tr} \left[ \bar{H}'(v')\Gamma H(v) \right] + \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] \right)
\right)
(61)

$$

$$
\langle M'(v')|J_{0,2}|M(v)\rangle = (\bar{\Lambda} - \bar{\Lambda}) \left\{ \Delta_+ \left( \text{Tr} \left[ \bar{H}'(v')\Gamma H(v) \right] + \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] \right) \right.
\right.
+ \Delta_- \left( \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] - \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] \right)
- \Delta_3 \text{Tr} \left[ \gamma_\beta \bar{H}'(v')\gamma^\beta \Gamma H(v) \right] - \phi'_K \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] \left. \right\}
- \frac{1}{2} \left( (1 + w)\psi_2^S - (1 - w)\psi_3^S + 2w\psi_6^S + \psi_1^A - \psi_4^S - \psi_5^S - \psi_2^A - \psi_4^A \right)
\times \left( \text{Tr} \left[ \bar{H}'(v')\Gamma H(v) \right] + \text{Tr} \left[ \bar{H}'(v')\gamma^\beta H(v) \right] \right)
\right)
(62)

\right)\)
Using Eqs. (14), (33) and (34) two relations follow:

\[ \langle M'(v') \rangle |_{J_{1,1}} M(v) \rangle = -\frac{1}{2}(\psi^S_1 - \psi^A_3) \left( w \text{Tr} \left[ H'(v') \Gamma H(v) \right] - \text{Tr} \left[ H'(v') \gamma^\beta \Gamma \gamma_\beta H(v) \right] \right) \]

\[ - \text{Tr} \left[ H'(v') \Gamma \gamma^\beta H(v) \right] + \text{Tr} \left[ H'(v') \gamma^\beta \Gamma \gamma_\beta H(v) \right] \bigg) \]

\[ - \frac{1}{2}(\psi^S_2 - \psi^S_3 + \psi^A_1) \left( w^2 \text{Tr} \left[ H'(v') \Gamma H(v) \right] \right) \]

\[ - w \left( \text{Tr} \left[ H'(v') \gamma^\beta H(v) \right] + \text{Tr} \left[ H'(v') \gamma^\beta \Gamma H(v) \right] \right) + \text{Tr} \left[ H'(v') \gamma^\beta H(v) \right] \bigg) \]

\[ - \frac{1}{2}(\psi^S_4 + \psi^A_2 + \psi^S_4 + \psi^A_4) \left( \text{Tr} \left[ H'(v') (\gamma - w) \Gamma H(v) \right] \right) + \text{Tr} \left[ H'(v') (\gamma - w) \Gamma H(v) \right] \bigg) \]

\[ - \frac{1}{2} \psi^A_5 \left( \text{Tr} \left[ H'(v') \Gamma H(v) \right] + \text{Tr} \left[ \gamma^\beta H'(v') \gamma^\alpha \Gamma \gamma_\alpha H(v) \gamma_\alpha \right] \right) \]

\[ + \text{Tr} \left[ \gamma^\beta H'(v') \gamma^\beta \Gamma H(v) \gamma_\beta \right] \bigg) . \]

Using Eqs. (14), (33) and (34) two relations follow:

\[ 2m_Q \phi_K = \tilde{\Lambda}'(\tilde{\Lambda} \Delta - \phi_K)(w - 1) \]

\[ - \frac{1}{2} \left( 3\psi^S_1 + (1 - w^2)(\psi^S_2 + \psi^S_3 - 2\psi^S_6) + 2(w - 1)(\psi^S_4 - \psi^S_5) \right) \]  

\[ 2m_Q \phi'_K = \tilde{\Lambda}'(\tilde{\Lambda} \Delta - \phi'_K)(w - 1) \]

\[ - \frac{1}{2} \left( 3\psi^S_1 + (1 - w^2)(\psi^S_2 + \psi^S_3 + 2\psi^S_6) + 2(w - 1)(\psi^S_4 + \psi^S_5) \right) . \]

In addition to the corrections obtained by the expansion of the weak currents, we must consider the corrections to the states. Using Eqs. (18), (19), they can be written as

\[ \langle M'(v') | i \int d^4x T[J_0(0)\mathcal{L}_1(x)] | M(v) \rangle = -\frac{1}{2m_Q^2} \chi_1(w) \text{Tr} \left[ H'(v') \Gamma H(v) \right] \]

\[ - \frac{1}{4m_Q} \text{Tr} \left[ \chi_2 \Gamma H'(v') \Gamma P_+ \left( -\frac{i}{2} \right) \sigma^{\mu\nu} H(v) \right] \]

\[ = \mathcal{M}_3 + \mathcal{M}_4 \]

13
and, considering the term analogous to $\mathcal{L}_1$ for the antiquark in Eq. (22),

$$
\langle M'(v')|i \int d^4x T[J_0(0)\mathcal{L}'_1(x)]|M(v)\rangle = -\frac{1}{2m'^2} \bar{\chi}_1(w) \text{Tr} \left[ \bar{H}'(v') \Gamma H(v) \right] \\
- \frac{1}{4m'Q'} \text{Tr} \left[ \bar{\chi}_{2\mu\nu}(w) \bar{H}'(v') \left( -\frac{i}{2} \right) \sigma^{\mu\nu} P_+ \Gamma H(v) \right]
$$

(67)

$$
\bar{\mathcal{M}}_3 + \bar{\mathcal{M}}_4.
$$

We deal with such corrections parametrizing the functions $\chi_i, \bar{\chi}_i$ as

$$
\chi_{2\mu\nu} = \chi^A_{2i} \sigma_{\mu\nu} + \chi^B_{2}(v_{\mu} \gamma_{\nu} - v_{\nu} \gamma_{\mu}) + \chi^C_{2}(v'_{\mu} \gamma_{\nu} - v'_{\nu} \gamma_{\mu})
$$

(68)

$$
\bar{\chi}_{2\mu\nu} = \bar{\chi}^A_{2i} \sigma_{\mu\nu} + \bar{\chi}^B_{2}(v_{\mu} \gamma_{\nu} - v_{\nu} \gamma_{\mu}) + \bar{\chi}^C_{2}(v'_{\mu} \gamma_{\nu} - v'_{\nu} \gamma_{\mu}).
$$

(69)

Using $\not{v} H(v) = H(v)$ and $\not{v}' H'(v') = \not{v}' H(v')$ one sees that $\chi^B_{2}$ and $\bar{\chi}^C_{2}$ do not contribute to the matrix elements. Hence, $\mathcal{M}_4$ and $\bar{\mathcal{M}}_4$ are given by:

$$
\mathcal{M}_4 = -\frac{1}{4m'Q} \chi^A_{2}(w) d_M \text{Tr} \left[ \bar{H}'(v') \Gamma H(v) \right] \\
- \frac{1}{4m'Q} \chi^C_{2}(w) \left[ \text{Tr} \left[ \bar{H}'(v') \Gamma H(v) \right] + \text{Tr} \left[ \gamma' \bar{H}'(v') \Gamma P_+ \not{v}' H(v) \right] \right]
$$

(70)

$$
\bar{\mathcal{M}}_4 = -\frac{1}{4m'Q'} \bar{\chi}^A_{2}(w) d'_M \text{Tr} \left[ \bar{H}'(v') \Gamma H(v) \right] \\
+ \frac{1}{4m'Q'} \bar{\chi}^B_{2}(w) \left[ \text{Tr} \left[ \bar{H}'(v') \Gamma H(v) \right] + \text{Tr} \left[ \gamma' \bar{H}'(v') \gamma_{\mu} \not{v}' P_+ \Gamma H(v) \right] \right],
$$

(71)

with $M = P, V$ and $d_P^{\mu} = 3, d_V^{\mu} = -1$.

The above relations can be used to write the various form factors in terms of universal functions. The resulting expressions are collected in Appendix B.

## 5 Numerical results

We present a numerical analysis based on the relations in App. B. Since the number of universal functions increases including the various terms, we shall limit the study neglecting $\mathcal{O}(1/m^2)$ terms (with $m$ generically $m_b, m_c$). This already produces interesting relations. For example, the combinations

$$
F_1(w) = \frac{\phi_K(w) - \Delta(w) \hat{A}}{2} + \Delta_3(w)
$$

(72)

$$
F_2(w) = \frac{\phi_K(w) - \Delta(w) \hat{A}}{2} - \Delta_3(w)
$$

(73)

$$
K(w) = 3\chi^A_{2}(w) + 2(w - 1)\chi^C_{2}(w)
$$

(74)

$$
\bar{K}(w) = 3\bar{\chi}^A_{2}(w) - 2(w - 1)\bar{\chi}^B_{2}(w)
$$

(75)
allow us to simplify the expressions for the form factors and to establish relations among them. For the \( B_c \rightarrow \eta_c \) form factors we find:

\[
\begin{align*}
 h_+(w) &= \Delta(w) + \frac{1}{4m_b} K(w) + \frac{1}{4m_c} \tilde{K}(w) \\
 h_-(w) &= \left( - \frac{1}{m_b} + \frac{1}{m_c} \right) F_1(w) \\
 h_T(w) &= \Delta(w) - \left( \frac{1}{m_b} + \frac{1}{m_c} \right) F_1(w) + \frac{1}{4m_b} K(w) + \frac{1}{4m_c} \tilde{K}(w) \\
 h_S(w) &= \Delta(w) - \left( \frac{1}{m_b} + \frac{1}{m_c} \right) \frac{w-1}{w+1} F_1(w) + \frac{1}{4m_b} K(w) + \frac{1}{4m_c} \tilde{K}(w) .
\end{align*}
\]

(76)

For \( B_c \rightarrow J/\psi \) form factors we obtain:

\[
\begin{align*}
 h_V(w) &= \Delta(w) + \frac{1}{4m_b} \left[ -4F_1(w) + K(w) \right] - \frac{1}{4m_c} \left[ \bar{\chi}_2^A(w) + 2(F_1(w) + F_2(w)) \right] \\
 h_{A1}(w) &= \Delta(w) - \frac{1}{m_b} \frac{w-1}{w+1} F_1(w) - \frac{1}{2m_c} \frac{w-1}{w+1} (F_1(w) + F_2(w)) + \frac{1}{4m_b} K(w) - \frac{1}{4m_c} \bar{\chi}_2^A(w) \\
 h_{A2}(w) &= \frac{1}{m_c} \left[ \frac{1}{2(1+w)} (F_1(w) + 3F_2(w)) + \frac{1}{2} \bar{\chi}_2^B(w) \right] \\
 h_{A3}(w) &= \Delta(w) - \frac{1}{m_b} F_1(w) + \frac{1}{4m_b} K(w) - \frac{1}{2m_c(1+w)} \left[ wF_1(w) + (w-2)F_2(w) \right] \\
 &\quad - \frac{1}{4m_c} \left[ \bar{\chi}_2^A(w) + 2\bar{\chi}_2^B(w) \right] \\
 h_{T1}(w) &= \Delta(w) + \frac{1}{4m_b} K(w) - \frac{1}{4m_c} \bar{\chi}_2^A(w) \\
 h_{T2}(w) &= - \frac{1}{m_b} F_1(w) + \frac{1}{2m_c} (F_1(w) + F_2(w)) \\
 h_{T3}(w) &= - \frac{1}{2m_c} \left[ \bar{\chi}_2^B(w) - \frac{1}{1+w} (F_1(w) + 3F_2(w)) \right] \\
 h_P(w) &= \Delta(w) + \frac{1}{4m_b} \left[ -4F_1(w) + K(w) \right] - \frac{1}{4m_c} \left[ 4F_2(w) + \bar{K}(w) - 2\bar{\chi}_2^A(w) \right] .
\end{align*}
\]

(77)

The lattice results for \( V(q^2) \) and \( A_{1,2,0}(q^2) \) [13] are translated into \( h_V(w) \) and \( h_{A_{1,2,3}}(w) \) in Fig. [1]. Keeping the leading term in the heavy quark mass expansion leads to \( h_V = h_{A_1} = h_{A_3} = \Delta \) and \( h_{A_2} = 0 \), relations badly violated by the results obtained by lattice QCD, as shown in Fig. [1]. Therefore, subleading terms must be considered. In this respect, the same lattice QCD results can be used to predict other form factors, exploiting Eqs. (76) and (77). A number of universal functions can be determined. In particular, we find:
Figure 1: $B_c \to J/\psi$ form factors $h_V, h_{A_1}, h_{A_2}$ and $h_{A_3}$ obtained using the lattice QCD results for $V, A_1, A_2, A_0$.

\[
\frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} = \frac{m_c}{2(m_b + 3m_c)}(1 + w) \left( m_b h_{A_1}(w) + m_c (h_{A_2}(w) + h_{A_3}(w)) - (m_b + m_c)h_V(w) \right) \tag{78}
\]

\[
\Delta_3(w) = -\frac{m_c}{2(m_b + 3m_c)}(1 + w) \left( -2m_b h_{A_1}(w) + (m_b + m_c) (h_{A_2}(w) + h_{A_3}(w)) + (m_b - m_c)h_V(w) \right) \tag{79}
\]

\[
\bar{\chi}_B^2(w) = m_c (h_{A_2}(w) - h_{A_3}(w) + h_V(w)). \tag{80}
\]

Such functions are displayed in Fig. 2. In the figure the relations are applied to the full kinematical range, a useful extrapolation for comparing with other calculations. This allows
us to derive other form factors near the zero recoil point. For $B_c \rightarrow J/\psi$ the relations hold:

$$h_{T_3}(w) = \frac{1}{2} \left( (1 + w)h_{A_1}(w) - (w - 1)h_V(w) \right)$$  \hspace{1cm} (81)

$$h_{T_2}(w) = \frac{1 + w}{2(m_b + 3m_c)} \left( (m_b - 3m_c)h_{A_1}(w) + 2m_c(h_{A_2}(w) + h_{A_3}(w)) 
- (m_b - m_c)h_V(w) \right)$$  \hspace{1cm} (82)

$$h_{T_3}(w) = h_{A_3}(w) - h_V(w)$$  \hspace{1cm} (83)

$$h_P(w) = \frac{1}{m_b + 3m_c} \left( (1 + w) \left( m_b h_{A_1}(w) + 2m_c h_V(w) \right) 
+ (-m_b + (w - 2)m_c) h_{A_2}(w) - (w m_b + (2w - 1)m_c) h_{A_3}(w) \right).$$  \hspace{1cm} (84)

These functions are depicted in Fig. 3 extrapolated to the full kinematical range. For $B_c \rightarrow \eta_c$, $h_\omega(w)$ and two form factor differences can be derived:

$$h_\omega(w) = \frac{m_b - m_c}{2(m_b + 3m_c)} (1 + w) \left( 3h_{A_1}(w) - h_{A_2}(w) - h_{A_3}(w) - 2h_V(w) \right)$$  \hspace{1cm} (85)

$$h_T(w) - h_+(w) = -\frac{m_b + m_c}{2(m_b + 3m_c)} (1 + w) \left( 3h_{A_1}(w) - h_{A_2}(w) - h_{A_3}(w) - 2h_V(w) \right)$$  \hspace{1cm} (86)

$$h_T(w) - h_S(w) = -\frac{m_b + m_c}{(m_b + 3m_c)} \left( 3h_{A_1}(w) - h_{A_2}(w) - h_{A_3}(w) - 2h_V(w) \right).$$  \hspace{1cm} (87)
Figure 3: Tensor $B_c \to J/\psi$ form factors obtained applying Eqs. (81), (82), (83) and (84) in the full kinematical range and using lattice QCD results for $V$ and $A_{1,2,0}$.

Their extrapolations are displayed in Fig. 4. The value of the universal functions at $w = 1$ is not predicted, however from Eqs. (81) and (76) the relations $h_{T_1}(w = 1) = h_{A_1}(w = 1)$ and $h_{S}(w = 1) = h_{+}(w = 1)$ are obtained, respectively. It is interesting to observe that if we consider the limit $m_b \to \infty$ keeping the $1/m_c$ terms, some results remain unaffected. This is the case of Eqs. (81) and (83), and of the relations $h_{T_1}(w = 1) = h_{A_1}(w = 1)$ and $h_{S}(w = 1) = h_{+}(w = 1)$. In the $\eta_c$ case, the relations (85)-(87) are rescaled replacing

$$
\frac{m_b \pm m_c}{m_b + 3m_c} \to 1.
$$

The relations among the various form factors are modified using the expansion (27). In
Figure 4: $B_c \to \eta_c$ form factors in Eqs. (85), (86) and (87) extended to the full kinematical range, obtained using lattice QCD results for $V$ and $A_{1,2,0}$.

In particular, the right-hand sides of Eqs. (81)-(87) acquire the extra terms:

\[
\delta h_T^1(w) = -\frac{1}{2} \frac{\alpha_s}{\pi} \left[ 2C_T^1 + (w-1)(C_T^2 - C_T^3 + C_{V^1}) - (w+1)C_{A_1} \right] \Delta(w)
\]

\[
\delta h_T^2(w) = -\frac{1}{2} + \frac{w}{2} \frac{\alpha_s}{\pi} \left[ (C_T^2 + C_T^3) + \frac{m_b - m_c}{m_b + 3m_c} (C_{V^1} - C_{A_1}) - \frac{2m_c}{m_b + 3m_c} (C_{A_2} + C_{A_3}) \right] \Delta(w)
\]

\[
\delta h_T^3(w) = -\frac{\alpha_s}{\pi} \left[ (C_T^2 + C_{V^1} - C_{A_1} - C_{A_3}) \Delta(w) \right]
\]

\[
\delta h_P(w) = -\frac{\alpha_s}{\pi} \left[ C_P - \frac{2m_c}{m_b + 3m_c} (1 + w)C_{V^1} - \frac{m_b - (2w - 1)m_c}{m_b + 3m_c} C_{A_1} \right.
\]

\[
+ \frac{m_b - (w - 2)m_c}{m_b + 3m_c} C_{A_2} + \frac{w m_b + (2w - 1)m_c}{m_b + 3m_c} C_{A_3} \right] \Delta(w)
\]

\[
\delta h_-(w) = -\frac{1}{2} + \frac{w}{2} \frac{\alpha_s}{\pi} \left[ \frac{m_b - m_c}{m_b + 3m_c} (2C_{V^1} - 2C_{A_1} + C_{A_2} + C_{A_3}) + C_{V^2} - C_{V^3} \right] \Delta(w)
\]

\[
\delta h_{T^+}(w) = -\frac{1}{2} \frac{\alpha_s}{\pi} \left[ 2C_{T^1} - 2C_{T^2} + 2C_{T^3} - \frac{2(m_b + 2m_c + (w + 4))}{m_b + 3m_c} \right] C_{V^1}
\]

\[
- (1 + w)(C_{V^2} + C_{V^3}) + \frac{m_b + m_c}{m_b + 3m_c} (1 + w)(2C_{A_1} - C_{A_2} - C_{A_3}) \right] \Delta(w)
\]

\[
\delta h_{T_S}(w) = -\frac{\alpha_s}{\pi} \left[ C_{T^1} - C_{T^2} + C_{T^3} - C_S + \frac{m_b + m_c}{m_b + 3m_c} (2C_{V^1} + 2C_{A_1} - C_{A_2} - C_{A_3}) \right] \Delta(w)
\]
To assess the size of the new contributions we choose \( w = 1 \) and \( \Delta(1) \simeq 1 \) \cite{20}. Setting \( \mu = \sqrt{m_c m_b} \) and \( \alpha_s(\mu) = 0.27 \), we find \( \delta h_T(1) \simeq -0.065 \), \( \delta h_{T_2}(1) \simeq -0.02 \), \( \delta h_{T_3}(1) \sim \mathcal{O}(10^{-4}) \), \( \delta h_P(1) \simeq 0.025 \), \( \delta h_{\perp}(1) \simeq -0.04 \), \( \delta h_{T+}(1) \simeq -0.01 \), \( \delta h_{TS}(1) \simeq -0.09 \). Such results compared to the values of the form factors at \( w = 1 \) in Figs. 3, 4 show that the impact of the matching coefficients is small.

The difference of our approach with other calculations based on NRQCD must be noticed. We have used the heavy quark expansion and the NRQCD power counting to relate the various form factors close to the zero recoil point and exploited lattice QCD to determine, e.g., the tensor form factors. A systematic control of the error can be achieved by the expansion. Then the results are extrapolated to small \( q^2 \). Different analyses are carried out at \( q^2 \simeq 0 \), where a perturbative approach to the form factor calculation can be attempted, and extrapolated to higher values of \( q^2 \) after a normalization to the lattice QCD results \cite{20}. Numerically, the results match for the form factors \( T_1 \) and \( T_2 \) in the basis (3), while \( T_0 \) is affected by a larger uncertainty.

A final remark is in order about the extrapolation from the kinematical range close to the zero-recoil point \( w \sim 1 \) to the full kinematical range. There are methods to constrain the extrapolation, starting from the form factors evaluated in few points and using unitarity constraints and the dispersion matrix \cite{57,64}. The application of such methods to the form factors discussed here is deferred to a dedicated analysis. Their availability strengthens the significance of our study, allowing us to foresee the control of the hadronic uncertainties not only near the zero-recoil point but in the full kinematical range.

6 Conclusions

Using the heavy quark expansion, the heavy quark spin symmetry and NRQCD power counting we have expressed the form factors parametrizing the matrix elements \( \langle J/\psi(\eta_c)|c\Gamma_i b|B_c \rangle \) in terms of universal functions near the zero-recoil point. This can be done for the various operators in the generalized low-energy Hamiltonian Eq. (1), establishing relations among form factors in a kinematical range around the zero-recoil point. Lattice QCD results for the matrix element of the Standard Model operator between \( B_c \) and \( J/\psi \) allow us to predict the pseudoscalar and tensor form factors. \( B_c \to \eta_c \) form factors are also related to the previous ones, obtaining \( h_- \) and the differences between the remaining form factors. We have also presented the results of the extrapolation to the full kinematical range. The relations worked out in our study can be checked if further information from lattice QCD is available. The effort is to efficiently control the hadronic uncertainties affecting the predictions for semileptonic \( B_c \to J/\psi, \eta_c \) decays in the Standard Model and beyond.

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A Form factors basis $h_i$ for $B_c \rightarrow \eta_c, J/\psi$

The basis of form factors $h_i$ is defined below.

$B_c \rightarrow \eta_c$:

\[
\langle P'(v')|\bar{Q}'\gamma_\mu Q|B_c(v)\rangle = \sqrt{m_P m_{B_c}} \left[ h_+(w) (v + v')_\mu + h_-(w) (v - v')_\mu \right] \\
\langle P'(v')|\bar{Q}'Q|B_c(v)\rangle = \sqrt{m_P m_{B_c}} h_S(w) (1 + w) \\
\langle P'(v')|\bar{Q}'\sigma_{\mu\nu} Q|B_c(v)\rangle = -i \sqrt{m_P m_{B_c}} h_T(w) (v_\mu v'_\nu - v_\nu v'_\mu)
\]

with $P = \eta_c$, $v = \frac{p}{m_{B_c}}$, $v' = \frac{p'}{m_P}$ and $w = v \cdot v'$.

$B_c \rightarrow J/\psi$:

\[
\langle V(v', \epsilon)|\bar{Q}'\gamma_\mu Q|B_c(v)\rangle = i \sqrt{m_V m_{B_c}} h_V(w) \epsilon_{\mu\nu\alpha\beta} \epsilon^{*\nu} v'^\alpha v^\beta \\
\langle V(v', \epsilon)|\bar{Q}'\gamma_5 \gamma_\mu Q|B_c(v)\rangle = \sqrt{m_V m_{B_c}} \left[ h_{A_1}(w) (1 + w) \epsilon^*_{\mu} - h_{A_2}(w) (\epsilon^* \cdot v)_\mu - h_{A_3}(w) (\epsilon^* \cdot v)'_\mu \right] \\
\langle V(v', \epsilon)|\bar{Q}'\gamma_5 Q|B_c(v)\rangle = -i \sqrt{m_V m_{B_c}} h_P(w) (\epsilon^* \cdot v) \\
\langle V(v', \epsilon)|\bar{Q}'\sigma_{\mu\nu} Q|B_c(v)\rangle = -i \sqrt{m_V m_{B_c}} \epsilon_{\mu\nu\alpha\beta} \left[ h_{T_1}(w) \epsilon^*_{\alpha} (v + v')_\beta + h_{T_2}(w) \epsilon^*_{\alpha} (v - v')_\beta \right] \\
+ h_{T_3}(w) (\epsilon^* \cdot v) v_\alpha v'^\beta
\]

with $V = J/\psi$, $\epsilon$ the $J/\psi$ polarization vector and $v' = \frac{p'}{m_V}$.

B Form factors in terms of universal functions

The expressions of the form factors $h_i$ in terms of universal functions are given in the following.

B.1 $B_c \rightarrow \eta_c$

\[
h_+(w) = \Delta(w) + \frac{1}{4m_b} \left( 3\chi_2^A(w) + 2(w - 1)\chi_2^C(w) \right) + \frac{1}{4m_c} \left( 3\bar{\chi}_2^A(w) - 2(w - 1)\bar{\chi}_2^B(w) \right) \\
+ \frac{1}{2m_b^2} \chi_1^B(w) + \frac{1}{2m_c^2} \bar{\chi}_1^B(w) \\
+ \frac{1}{8m_b m_c} \left( - (2 + w)\psi_1^S(w) - (w^2 - 1) \left( \psi_2^S(w) - \psi_3^S(w) + \psi_4^A(w) \right) \\
+ 6(w - 1) \left( \psi_4^S(w) + \psi_2^A(w) \right) + (w - 7)\psi_3^A(w) \right)
\]
\[ h_-(w) = \left( -\frac{1}{m_b} + \frac{1}{m_c} \right) \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) \]
\[ + \frac{1}{2m_b^2} \left[ (\tilde{\Lambda} - w\tilde{\Lambda}')\left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) + \frac{1}{4}(1 + w)^2\psi_2^S(w) + \frac{1}{4}(w^2 - 1)\psi_3^S(w) \right. \]
\[ + (1 + w) \left( -\psi_4^S(w) - \frac{w}{2}\psi_6^S(w) + \frac{1}{4}\psi_1^A(w) - \frac{1}{2}\psi_4^A(w) \right) \]
\[ + \frac{2w - 1}{2}\psi_5^S(w) + \frac{w - 2}{2}\psi_2^A(w) - \frac{3}{4}\psi_3^A(w) \left] \right. \]
\[ + \frac{1}{2m_c^2} \left[ (\tilde{\Lambda}w - \tilde{\Lambda}')\left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) \right. \]
\[ - \frac{1}{4}(1 + w)^2\psi_2^S(w) - \frac{1}{4}(w^2 - 1)\psi_3^S(w) \]
\[ + (1 + w) \left( \psi_4^S(w) - \frac{w}{2}\psi_6^S(w) - \frac{1}{4}\psi_1^A(w) - \frac{1}{2}\psi_4^A(w) \right) \]
\[ + \frac{2w - 1}{2}\psi_5^S(w) - \frac{w - 2}{2}\psi_2^A(w) + \frac{3}{4}\psi_3^A(w) \]
\[ h_T(w) = \Delta(w) - \left( \frac{1}{m_b} + \frac{1}{m_c} \right) \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) \\
+ \frac{1}{4m_b} \left( 3\chi_2^A(w) + 2(w - 1)\chi_2^S(w) \right) + \frac{1}{4m_c} \left( 3\chi_2^A(w) - 2(w - 1)\tilde{\chi}_2^B(w) \right) \\
+ \frac{1}{2m_b^2} \left[ (\tilde{\Lambda} - w\tilde{\Lambda}') \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) + \chi_1(w) \right] \\
+ \frac{1}{4} (1 + w)^2 \psi_3^S(w) + \frac{1}{4} (w^2 - 1) \psi_3^S(w) \\
+ (1 + w) \left( -\psi_4^S(w) - \frac{w}{2} \psi_6^S(w) + \frac{1}{4} \psi_1^A(w) - \frac{1}{2} \psi_4^A(w) \right) \\
+ \frac{2w - 1}{2} \psi_5^S(w) + \frac{w - 2}{2} \psi_2^A(w) - \frac{3}{4} \psi_3^A(w) \right] \\
- \frac{1}{2m_c^2} \left[ (\tilde{\Lambda}w - \tilde{\Lambda}') \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) - \bar{\chi}_1(w) \right] \quad \text{(B.3)} \\
- \frac{1}{4} (1 + w)^2 \psi_3^S(w) - \frac{1}{4} (w^2 - 1) \psi_3^S(w) \\
+ (1 + w) \left( \psi_4^S(w) - \frac{w}{2} \psi_6^S(w) - \frac{1}{4} \psi_1^A(w) - \frac{1}{2} \psi_4^A(w) \right) \\
+ \frac{2w - 1}{2} \psi_5^S(w) - \frac{w - 2}{2} \psi_2^A(w) + \frac{3}{4} \psi_3^A(w) \right] \\
+ \frac{1}{2m_b m_c} \left[ \frac{w + 2}{4} \psi_1^S(w) + \frac{w^2 - 1}{4} (\psi_2^S(w) - \psi_3^S(w) + \psi_1^A(w)) \right. \\
- \frac{3}{2} (w - 1) (\psi_4^S(w) + \psi_2^A(w)) - \frac{w - 7}{4} \psi_3^A(w) \right]}
\[ h_s(w) = \Delta(w) - \left( \frac{1}{m_b} + \frac{1}{m_c} \right) \frac{w - 1}{w + 1} \left( \frac{\phi_K(w) - \Delta(w)\bar{\Lambda}}{2} + \Delta_3(w) \right) \\
+ \frac{1}{4m_b} \left( 3\chi_2^A(w) + 2(w - 1)\chi_2^C(w) \right) + \frac{1}{4m_c} \left( 3\bar{\chi}_2^A(w) - 2(w - 1)\bar{\chi}_2^B(w) \right) \\
+ \frac{1}{2m_b^2} \left[ (\bar{\Lambda} - w\bar{\Lambda}) \frac{w - 1}{w + 1} \left( \frac{\phi_K(w) - \Delta(w)\bar{\Lambda}}{2} + \Delta_3(w) \right) + \chi_1(w) \right] \\
- \frac{1}{2m_c^2} \left[ (\bar{\Lambda}w - \bar{\Lambda}) \frac{w - 1}{w + 1} \left( \frac{\phi_K(w) - \Delta(w)\bar{\Lambda}}{2} + \Delta_3(w) \right) - \bar{\chi}_1(w) \right] \\
+ \frac{1}{8m_b^2} (w - 1) \left[ (w + 1)\psi_2^S(w) + (w - 1)\psi_3^S(w) - 4\psi_4^S(w) + \frac{2w - 1}{w + 1} \psi_5^S(w) \\
- 2w\psi_6^S(w) + \psi_1^A(w) + \frac{2w}{w + 1} \psi_2^A(w) - \frac{3}{w + 1} \psi_3^A(w) - 2\psi_4^A(w) \right] \\
+ \frac{1}{8m_c^2} (w - 1) \left[ (w + 1)\psi_2^S(w) + (w - 1)\psi_3^S(w) - 4\psi_4^S(w) - \frac{2w - 1}{w + 1} \psi_5^S(w) \\
+ 2w\psi_6^S(w) + \psi_1^A(w) + \frac{2w}{w + 1} \psi_2^A(w) - \frac{3}{w + 1} \psi_3^A(w) + 2\psi_4^A(w) \right] \\
+ \frac{1}{8m_b m_c} \left[ (2 + w)\psi_1^S(w) - (w - 7)\psi_3^A(w) - 6(w - 1)\left[ \psi_4^S(w) + \psi_2^A(w) \right] \\
+ (w^2 - 1)\left[ \psi_2^S(w) - \psi_3^S(w) + \psi_1^A(w) \right] \right] \]
\[ h_V(w) = \Delta(w) + \frac{1}{4m_b} \left[ 3\chi_2^A(w) + 2(w-1)\chi_2^C(w) - 4 \left( \Delta_3(w) + \frac{\phi_K(w) - \Delta(w)\bar{\Lambda}}{2} \right) \right] \\
- \frac{1}{4m_c} \left[ \bar{\chi}_2^A(w) + 2 \left( \phi_K(w) - \Delta(w)\bar{\Lambda} \right) \right] \\
+ \frac{1}{2m_b^2} \left[ \left( \bar{\Lambda} - w\bar{\Lambda}' \right) \left( \frac{\phi_K(w) - \Delta(w)\bar{\Lambda}}{2} + \Delta_3(w) \right) + \chi_1(w) - \frac{3}{4}\psi_3^A(w) \right] \\
+ (1 + w) \left( \frac{1}{4}\psi_1^A(w) - \psi_4^S(w) - \frac{1}{2}\psi_4^A(w) - \frac{1}{2}w\psi_6^S(w) \right) + \frac{w - 2}{2}\psi_2^A(w) \\
+ \frac{2w - 1}{2}\psi_3^S(w) + \frac{w^2 - 1}{4}\psi_3^S(w) + \frac{(w + 1)^2}{4}\psi_2^S(w) \right] \\
+ \frac{1}{2m_c^2} \left[ \left( \bar{\Lambda}' - w\bar{\Lambda} \right) \frac{\phi_K(w) - \Delta(w)\bar{\Lambda}}{2} + \chi_1(w) - \frac{1}{4}\psi_3^A(w) - \frac{1}{2}\psi_2^A(w) \right] \\
+ (1 + w) \left( \frac{1}{4}\psi_1^A(w) - \frac{1}{2}\psi_4^S(w) + \frac{1}{2}w\psi_6^S(w) \right) - \frac{1}{2}w\psi_5^S(w) + \frac{w^2 - 1}{4}\psi_3^S(w) + \frac{(1 + w)^2}{4}\psi_2^S(w) \right] \\
+ \frac{1}{8m_bm_c} \left[ w\psi_1^S(w) + (w^2 - 1) \left( \psi_2^S(w) - \psi_3^S(w) + \psi_1^A(w) \right) \right] \\
+ 2(1 - w) \left( 2\psi_3^S(w) + 2\psi_2^A(w) + \psi_4^A + \psi_5^S \right) + (3 - w)\psi_3^A(w) \right] \]
\[ h_F(w) = \Delta(w) + \frac{1}{4m_b} \left[ 3\chi_2^A(w) + 2(w - 1)\chi_2^C(w) - 4 \left( \Delta_3(w) + \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} \right) \right] \]

\[ - \frac{1}{4m_c} \left[ \tilde{x}_2^A(w) + 4 \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} - \Delta_3(w) \right) - 2(w - 1)\tilde{x}_2^B \right] \]

\[ + \frac{1}{2m_b^2} \left[ (\tilde{\Lambda} - w\tilde{\Lambda}) \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) + \chi_1(w) - \frac{3}{4} \psi_3^A(w) \right] \]

\[ + (1 + w) \left( \frac{1}{4} \psi_1^A(w) - \psi_4^S(w) - \frac{1}{2} \psi_4^A(w) - \frac{1}{2} w\psi_6^S(w) \right) + \frac{w - 2}{2} \psi_2^A(w) \]

\[ + \frac{2w - 1}{2} \psi_5^S(w) + \frac{w^2 - 1}{4} \psi_3^S(w) + \frac{(w + 1)^2}{4} \psi_2^S(w) \]

\[ - \frac{1}{2m_c^2} \left[ (\tilde{\Lambda}' - w\tilde{\Lambda}) \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) \right] \]

\[ - \chi_1(w) - \frac{1}{4} \psi_3^A(w) + \frac{1}{2} \psi_5^S(w) + \frac{1}{2} w\psi_2^A(w) \]

\[ - (1 + w) \left( \frac{1}{4} \psi_1^A(w) - \frac{1}{2} \psi_4^A(w) + \frac{1}{2} w\psi_6^S(w) \right) + \frac{1 - w^2}{4} \psi_3^S(w) - \frac{(1 + w)^2}{4} \psi_2^S(w) \]

\[ + \frac{1}{8m_b m_c} \left[ (w - 2)\psi_1^S(w) + (w^2 - 1) \left( \psi_2^S(w) - \psi_3^S(w) + \psi_1^A(w) \right) \right] \]

\[ + 2(1 - w) \left( 2\psi_4^A(w) + \psi_2^A(w) + \psi_4^S(w) + 2\psi_6^S(w) \right) - (1 + w)\psi_3^A(w) \]
\[ h_{A_1} = \Delta(w) - \frac{1}{m_b w + 1} \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) - \frac{1}{m_c (w - 1)} \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2(w + 1)} \]

\[ + \frac{1}{4m_b} \left( 3\chi^2_2(w) + 2(w - 1)\chi^C_2(w) \right) - \frac{1}{4m_c} \chi^A_2(w) \]

\[ + \frac{1}{2m_b^2} \left[ \left( \tilde{\Lambda} - w\tilde{\Lambda}' \right) \frac{w - 1}{(w + 1)} \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) + \chi_1(w) \right] \]

\[ + \frac{1}{8m_b^2} \left[ (w^2 - 1)\psi^S_2(w) + (w - 1)^2\psi^S_3(w) \right] \]

\[ + (w - 1) \left[ - 4\psi^A_4(w) + 2w^2 - 3w + 2 \psi^A_2(w) \right] \]

\[ + \frac{1}{2m_c^2} \left[ - \left( \tilde{\Lambda}w - \tilde{\Lambda}' \right) \frac{w - 1}{2(w + 1)} \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) + \chi_1(w) \right] \]

\[ (B.7) \]

\[ h_{A_2} = \frac{1}{m_c} \left( \frac{1}{1 + w} \left[ \phi_K(w) - \Delta(w)\tilde{\Lambda} - \Delta_3(w) \right] + \frac{1}{2} \chi^B_2(w) \right) \]

\[ + \frac{1}{m_c^2} \left( \frac{1}{2(w + 1)} \left( \tilde{\Lambda}w - \tilde{\Lambda}' \right) \left[ \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda} - \Delta_3(w)}{2} \right] \right) \]

\[ + \frac{1}{4m_c^2} \left[ - (1 + w)\psi^S_2(w) - (w - 1)\psi^S_3(w) \right] \]

\[ + \psi^S_4(w) + \psi^S_5(w) - 2w\psi^S_6(w) - \psi^A_1(w) + \psi^A_2(w) + \psi^A_4(w) \]

\[ (B.8) \]

\[ + \frac{1}{4m_c} \left[ - \psi^S_1(w) - (1 + w)\psi^S_2(w) + (w + 1)\psi^S_3(w) \right] \]

\[ + 3\psi^S_2(w) + 3\psi^S_5(w) - (1 + w)\psi^A_1(w) + 3\psi^A_2(w) + \psi^A_3(w) + 3\psi^A_4(w) \]
\[ h_{A_3} = \Delta(w) - \frac{1}{m_b} \left( \frac{\phi_K(w) - \Delta(w) \tilde{\Lambda}}{2} + \Delta_3(w) \right) + \frac{1}{4m_b} \left( 3\chi_2^A(w) + 2(w - 1)\chi_2^C(w) \right) \\
- \frac{1}{m_c(1 + w)}[(w - 1)\phi_K(w) - \Delta(w) \tilde{\Lambda}] + \Delta_3(w) - \frac{1}{4m_c} [\bar{\chi}_2^A(w) + 2\bar{\chi}_2^B(w)] \\
+ \frac{1}{2m_b^2} \left[ (\tilde{\Lambda} - w\tilde{\Lambda}) \left( \frac{\phi_K(w) - \Delta(w) \tilde{\Lambda}}{2} + \Delta_3(w) \right) + \chi_1(w) \right] \\
- \frac{1}{2m_c^2} \left[ \frac{1}{1 + w} (\tilde{\Lambda} w - \tilde{\Lambda}') [(w - 1)\phi_K(w) - \Delta(w) \tilde{\Lambda}] + \Delta_3(w) - \tilde{\chi}_1(w) \right] \\
+ \frac{1}{8m_b^3} \left[ (1 + w)^2\psi_2^S(w) + (w^2 - 1)\psi_3^S(w) + 2(2w - 1)\psi_5^S(w) + 2(w - 2)\psi_2^A(w) \\
- 3\psi_3^A(w) + (1 + w)(-4\psi_4^S(w) - 2w\psi_6^S(w) + \psi_4^A(w) - 2\psi_4^A(w)) \right] \\
+ \frac{1}{8m_c^3} \left[ (w^2 - 1)\psi_2^S(w) + (w - 1)^2\psi_3^S(w) - 2w\psi_4^S(w) - 2(w - 1)\psi_5^S(w) \\
- \psi_3^A(w) + 2\psi_4^A(w) + (w - 1)(2w\psi_6^S(w) + \psi_1^A(w)) \right] \\
+ \frac{1}{8m_b m_c} \left[ (2 + w)(\psi_1^S(w) - 2\psi_5^S(w) - 2\psi_4^A(w)) - 2(2w + 1)(\psi_4^S(w) + \psi_2^A(w)) \\
+ (1 + w)^2(\psi_2^S(w) - \psi_3^S(w) + \psi_4^A(w)) - (w - 1)\psi_3^A(w) \right] \] 

\[ h_{T_1} = \Delta(w) + \frac{1}{4m_b} \left( 3\chi_2^A(w) + 2(w - 1)\chi_2^C(w) \right) - \frac{1}{4m_c} \bar{\chi}_2^A(w) \\
+ \frac{1}{2m_b^2} \chi_1(w) + \frac{1}{2m_c^2} \bar{\chi}_1(w) \\
+ \frac{1}{8m_b m_c} \left[ -w\psi_1^S(w) - (w^2 - 1)[\psi_3^S(w) - \psi_3^S(w) + \psi_1^A(w)] \right] \]  

\[ + (w - 1)[4\psi_4^S(w) + 2\psi_5^S(w) + 4\psi_2^A(w) + 2\psi_4^A(w)] + (w - 3)\psi_3^A(w) \]
\[h_{T_2} = -\frac{1}{m_b} \left( \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right) + \frac{1}{2m_c} (\phi_K(w) - \Delta(w)\tilde{\Lambda}) + \frac{1}{2m_b^2} (\tilde{\Lambda} - w\tilde{\Lambda}') \left[ \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \Delta_3(w) \right] + \frac{1}{2m_b^2} (\tilde{\Lambda} w - \tilde{\Lambda}') \frac{\phi_K(w) - \Delta(w)\tilde{\Lambda}}{2} + \frac{1}{8m_b^2} \left[ (w + 1)^2 \psi_2^S(w) + (w^2 - 1)\psi_3^S(w) + (1 + w)[-4\psi_4^S(w) - 2w\psi_6^S(w) + \psi_1^A(w) - 2\psi_4^A(w)] + 2(2w - 1)\psi_5^S(w) + 2(w - 2)\psi_2^A(w) - 3\psi_3^A(w) \right] + 2w\psi_5^S(w) + 2\psi_2^A(w) + \psi_3^A(w) \right] \]

\[h_{T_3}(w) = -\frac{1}{2m_c} \left[ \tilde{\chi}_2^B(w) + \frac{2}{1 + w} \left( \Delta_3(w) - \left( \phi_K(w) - \Delta(w)\tilde{\Lambda} \right) \right) \right] - \frac{1}{4m_c^2} \left[ \frac{2}{w + 1} \left( \tilde{\Lambda} - w\tilde{\Lambda} \right) \left( \left( \phi_K(w) - \Delta(w)\tilde{\Lambda} \right) - \Delta_3(w) \right) \right] + \psi_1^A(w) - \psi_2^A(w) - \psi_4^S(w) - \psi_4^A(w) + (w - 1)\psi_3^S(w) - \psi_5^S(w) + (1 + w)\psi_2^S(w) + 2w\psi_6^S(w) \right] \] (B.12)

\[\left[ \psi_1^S(w) - \psi_3^A(w) - 3 \left( \psi_2^A(w) + \psi_4^A(w) + \psi_4^S(w) + \psi_5^S(w) \right) \right] \right] \}

\[\right] + (1 + w) \left( \psi_1^A(w) + \psi_2^S(w) - \psi_3^S(w) \right) \].

C  Fit of the form factors \(h_{T_i}\) and \(h_P\)

The form factors \(h_{T_i}\) and \(h_P\) depicted in Fig. 3 are fitted using the parametrization adopted by the HPQCD Collaboration \[65\]. The variable \(z(q^2, t_0)\) is defined:

\[z(q^2, t_0) = \frac{\sqrt{t_+ - q^2 - \sqrt{t_+ - t_0}}}{\sqrt{t_+ - q^2 + \sqrt{t_+ - t_0}}} \] (C.1)

with \(t_+ = (m_B + m_{D^*})^2\) and \(t_0 = t_- = (m_{B_c} - m_{J/\psi})^2\). The form factors \(h_i(q^2)\) are fitted using a truncated expansion in powers of \(z\):

\[h_i(q^2) = \frac{1}{P(q^2)} \sum_{n=0}^N a_n z^n \] (C.2)
Table 1: Mass $M_{P_i}$ (in GeV) of the $\bar{b}c$ resonances $P_i$ included in the Blaschke factors (C.3).

| $J^P/i$ | 1$^-$ | 1$^+$ | 0$^-$ |
|---------|-------|-------|-------|
| 1       | 6.336 | 6.745 | 6.275 |
| 2       | 6.926 | 6.75  | 6.872 |
| 3       | 7.02  | 7.15  | 7.25  |
| 4       | 7.28  | 7.15  |       |

where the Blaschke factors $P(t)$ account for the resonances in the $t$-channel:

$$P(q^2) = \prod_{P_i} z(q^2, M_{P_i}^2).$$  \hspace{1cm} (C.3)

$h_{T_3}$ involves poles with $J^P = 1^+$, $h_{T_{1,2}}$ with $J^P = 1^-$, $h_P$ with $J^P = 0^-$, with masses in Table 1. The values of $a_n$ resulting from the fit are collected in Table 2.

| $h_{T_3}$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|-----------|-------|-------|-------|-------|
| 0.058 ± 0.002 | -0.476 ± 0.001 | 1.228 ± 0.131 | 3.419 ± 3.977 |
| -0.021 ± 0.003 | 0.096 ± 0.014 | 0.030 ± 0.665 | 0.473 ± 0.670 |
| -0.047 ± 0.014 | 0.675 ± 0.165 | -0.701 ± 1.693 | 0.894 ± 7.468 |
| 0.142 ± 0.008 | -0.944 ± 0.053 | 0.157 ± 1.629 | -1.495 ± 1.131 |

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