FLUCTUATIONS, GRAVITY, AND THE QUANTUM POTENTIAL

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Abstract. We show how the quantum potential arises in various ways and trace its connection to quantum fluctuations and Fisher information along with its realization in terms of Weyl curvature. It represents a genuine quantization factor for certain classical systems as well as an expression for quantum matter in gravity theories of Weyl-Dirac type. Many of the facts and examples are extracted from the literature (with references cited) and we mainly provide connections and interpretation, with a few new observations. We deliberately avoid ontological and epistemological discussion and resort to a collection of contexts where the quantum potential plays a visibly significant role. In particular we sketch some recent results of F. and A. Shojai on Dirac-Weyl action and Bohmian mechanics which connects quantum mass to the Weyl geometry. Connections à la Santamato of the quantum potential with Weyl curvature arising from a stochastic geometry, are also indicated for the Schrödinger equation (SE) and Klein-Gordon (KG) equation. Quantum fluctuations and quantum geometry are linked with the quantum potential via Fisher information. Derivations of SE and KG from Nottale’s scale relativity are sketched along with a variety of approaches to the KG equation. Finally connections of geometry and mass generation via Weyl-Dirac geometry with many cosmological implications are indicated, following M. Israelit and N. Rosen.

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1. THE SCHRÖDINGER EQUATION

The quantum potential seems to have achieved prominence via the work of L. de Broglie and D. Bohm plus many others on what is often now called Bohmian mechanics. There have been many significant contributions here and we refer to [40, 41, 42, 43, 44, 47, 50, 54] for a reasonably complete list of references. A good picture of the current theory can be obtained from the papers by an American-German-Italian (AGI) group of Allori, Barut, Berndl, Daumer, Dürr, Georgi, Goldstein, Lebowitz, Teufel, Tumulka, and Zanghi (cf. [8, 9, 19, 22, 23, 24, 25, 26, 69, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 189, 190, 194]). We refer also to Holland [103, 104, 105, 106], Nikolić [134, 135, 136, 137, 138], Floyd [83, 84], and Bertoldi, Faraggi, and Matone [27, 81, 82] for other approaches and summaries. Other specific references will arise as we go along but we emphasize with apologies that there are many more interesting papers omitted here which hopefully are covered in [54].

First in a simple minded way one can look at a Schrödinger equation (SE)

\[ -\frac{\hbar^2}{2m} \psi'' + V \psi = i\hbar \psi_t; \quad \psi = Re^{iS/\hbar} \]

leading to

\[ S_t + \frac{S_x^2}{2m} + V - \frac{\hbar^2 R''}{R} = 0; \quad \partial_t(R^2) + \frac{1}{m}(R^2 S')' = 0 \]

Here \( Q \) is the quantum potential and in 3-dimensions for example one expresses this as

\[ Q = -\frac{\hbar^2}{2m} \frac{R''}{R} = S_t + \frac{(S')^2}{2m} + Q + V = 0; \quad P_t + \frac{1}{m}(PS')' = 0 \]

In a hydrodynamic mode one can write (1-dimension for simplicity and with the proviso that \( S \neq \text{const.} \)) \( p = S' = m\dot{q} = mv \) (v a velocity or collective velocity) and \( \rho = mP \) (\( \rho \) an unspecified mass density) to obtain an Euler type hydrodynamic equation (\( \partial \sim \partial_x \))

\[ \partial_t(\rho v) + \partial(\rho v^2) + \frac{\rho}{m} \partial V + \frac{\rho}{m} \partial Q = 0 \]

**REMARK 1.1.** Given a wave function \( \psi \) with \( |\psi|^2 \) representing a probability density as in conventional quantum mechanics (QM) it is not unrealistic to imagine an ensemble picture emerging here (as a “cloud” of particles for example). This will be analogous to diffusion or fluid flow of course but can also be modeled on a Bohmian particle picture and this will be discussed later in more detail. We note also that \( Q \) appears in the Hamilton-Jacobi (HJ) type equation (1.3) but is not present in the SE (1.1). If one were to interpret \( \partial V \) as a hydrodynamical pressure term \(-\frac{1}{\rho}\partial P\) then the SE would be unchanged and the hydrodynamical equation (with no Q term) would be meaningful in the form

\[ \partial_t(\rho v) + \partial(\rho v^2) = \frac{1}{m} \partial P \]

Thinking of \( Q \) as a quantization of (1.5) yielding (1.4) leads then to the SE (1.1).

**REMARK 1.2.** The development of the AGI school involves now

\[ \dot{q} = v = \frac{\hbar}{m} \Im \frac{\psi^* \psi'}{|\psi|^2} \]
and this is derived as the simplest Galilean and time reversal invariant form for velocity transforming correctly under velocity boosts. This is a nice argument and seems to avoid any recourse to Floydian time (cf. \[53\] [54]).

Next we consider relations of diffusion to QM following Nagasawa, Nelson, et al (cf. \[131\] [132] [133] - see also e.g. \[67\] [70] [86] [119] [120] [121]) and sketch some formulas for a simple Euclidean metric where \(\Delta = \sum (\partial/\partial x)^2\). Then \(\psi(t, x) = \exp[R(t, x) + iS(t, x)]\) satisfies a SE \(i\partial_t \psi + (1/2)\Delta \psi + ia(t, x) \cdot \nabla \psi - \nabla V(t, x) \psi = 0\) \((h = m = 1)\) if and only if

\[
(1.7) \quad V = -\frac{\partial S}{\partial t} + \frac{1}{2} \phi R + \frac{1}{2} (\nabla R)^2 - \frac{1}{2} (\nabla S)^2 - a \cdot \nabla S;
0 = \frac{\partial R}{\partial t} + \frac{1}{2} \Delta S + (\nabla S) \cdot (\nabla R) + a \cdot \nabla R
\]

in the region \(D = \{(s, x) : \psi(s, x) \neq 0\}\). Solutions are often referred to as weak or distributional but we do not belabor this point. From \[131\] there results

**THEOREM 1.1.** Let \(\psi(t, x) = \exp[R(t, x) + iS(t, x)]\) be a solution of the SE above; then \(\phi(t, x) = \exp[R(t, x) + S(t, x)]\) and \(\phi(t, x) = \exp[R(t, x) - S(t, x)]\) are solutions of

\[
(1.8) \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} \Delta \phi + a(t, x) \cdot \nabla \phi + c(t, x, \phi) \phi = 0;
- \frac{\partial \phi}{\partial t} + \frac{1}{2} \Delta \phi + a(t, x) \cdot \nabla \phi + c(t, x, \phi) \phi = 0
\]

where the creation and annihilation term \(c(t, x, \phi)\) is given via

\[
(1.9) \quad c(t, x, \phi) = -V(t, x) - \frac{2}{2} \frac{\partial S}{\partial t} (t, x) - (\nabla S)^2 (t, x) - 2a \cdot \nabla S(t, x)
\]

Conversely given \((\phi, \phi)\) as above satisfying (1.8), it follows that \(\psi\) satisfies the SE with \(V\) as in (1.9) (note \(R = (1/2)\log(\phi \phi)\) and \(S = (1/2)\log(\phi / \phi)\) with \(\exp(R) = (\phi \phi)^{1/2}\)).

From this one can conclude that nonrelativistic QM is diffusion theory in terms of Schrödinger processes (described by \((\phi, \phi)\) - more details later). Further it is shown that key postulates in Nelson’s stochastic mechanics or Zambrini’s Euclidean QM (cf. \[202\]) can both be avoided in connecting the SE to diffusion processes (since they are automatically valid). Look now at Theorem 1.1 for one dimension and write \(T = \hbar t\) with \(X = (\hbar / \sqrt{m})x\); then some simple calculation leads to

**COROLLARY 1.1.** Equation (1.8), written in the \((X, T)\) variables becomes

\[
(1.10) \quad \hbar \partial_T + \frac{\hbar^2}{2m} \phi_{XX} + A \Psi_X + \ddot{c} \phi = 0; \quad -\hbar \partial_T + \frac{\hbar^2}{2m} \dot{\phi}_{XX} - A \dot{\phi} X + \ddot{c} \dot{\phi} = 0;
\]

\(\ddot{c} = -V(X, T) - 2\hbar S_T - \frac{\hbar^2}{m} S_X^2 - 2AS_X\)

Thus the diffusion processes pick up factors of \(\hbar\) and \(\hbar / \sqrt{m}\).

Next we sketch a derivation of the SE following scale relativity à la Nottale (cf. \[58\] [139] [140] [141] [142] [143] and \[56\] [64] [65] [66] for some refinements and variations); this material is expanded in \[40\] [54].

**REMARK 1.3.** One considers quantum paths à la Feynman so that \(\lim_{t \to t'} (X(t) - X(t'))^2 / (t - t')\) exists. This implies \(X(t) \in H^{1/2}\) where \(H^a\) means \(\alpha \leq |X(t) - X(t')| \leq C\alpha^a\) and from \[80\] for example this means \(\text{dim}_H X[a, b] = 1/2\). Now one “knows” (see e.g. \[1\])
that quantum and Brownian motion paths (in the plane) have H-dimension 2 and some clarification is needed here. We refer to [126] where there is a paper on Wiener Brownian motion (WBM), random walks, etc. discussing Hausdorff and other dimensions of various sets. Thus given $0 < \lambda < 1/2$ with probability 1 a Brownian sample function $X$ satisfies $|X(t + h) - X(t)| \leq b|h|^\lambda$ for $|h| \leq h_0$ where $b = b(\lambda)$. This leads to the result that with probability 1 the graph of a Brownian sample function has Hausdorff and box dimension 3/2. On the other hand a Brownian trail (or path) in 2 dimensions has Hausdorff and box dimension 2 (note a quantum path can have self intersections, etc.).

Now fractal spacetime here will mean some kind of continuous nonsmooth pathspace so that a bivelocity structure is defined. One defines first

\begin{align}
\frac{d_{+}}{dt}y(t) &= \lim_{\Delta t \to 0^+} \left\langle \frac{y(t + \Delta t) - y(t)}{\Delta t} \right\rangle; \\
\frac{d_{-}}{dt}y(t) &= \lim_{\Delta t \to 0^+} \left\langle \frac{y(t) - y(t - \Delta t)}{\Delta t} \right\rangle
\end{align}

(1.11)

Applied to the position vector $x$ this yields forward and backward mean velocities, namely $(d_{+}/dt)x(t) = b_{+}$ and $(d_{-}/dt)x(t) = b_{-}$. Here these velocities are defined as the average at a point $q$ and time $t$ of the respective velocities of the outgoing and incoming fractal trajectories; in stochastic QM this corresponds to an average on the quantum state. The position vector $x(t)$ is thus “assimilated” to a stochastic process which satisfies respectively after $(dt > 0)$ and before $(dt < 0)$ the instant $t$ a relation $dx(t) = b_{+}[x(t)]dt + d\xi_{+}(t) = b_{-}[x(t)]dt + d\xi_{-}(t)$ where $\xi(t)$ is a Wiener process (cf. [133]). It is in the description of $\xi$ that the $D = 2$ fractal character of trajectories is inserted; indeed that $\xi$ is a Wiener process means that the $d\xi$’s are assumed to be Gaussian with mean 0, mutually independent, and such that

\begin{align}
< d\xi_{+}(t)d\xi_{+}(t) > &= 2D\delta_{ij}dt; \\
< d\xi_{-}(t)d\xi_{-}(t) > &= -2D\delta_{ij}dt
\end{align}

(1.12)

where $\langle \rangle$ denotes averaging ($D$ is now the diffusion coefficient). Nelson’s postulate (cf. [133]) is that $D = h/2m$ and this has considerable justification (cf. [139]). Note also that (1.12) is indeed a consequence of fractal (Hausdorff) dimension 2 of trajectories follows from $< d\xi^2 > /dt^2 = dt^{-1}$, i.e. precisely Feynman’s result $< v^2 > \sim \delta t^{-1/2}$. Note that Brownian motion (used in Nelson’s postulate) is known to be of fractal (Hausdorff) dimension 2. Note also that any value of $D$ may lead to QM and for $D \to 0$ the theory becomes equivalent to the Bohm theory. Now expand any function $f(x,t)$ in a Taylor series up to order 2, take averages, and use properties of the Wiener process $\xi$ to get

\begin{align}
\frac{d_{+}f}{dt} &= (\partial_{t} + b_{+} \cdot \nabla + D\Delta)f; \\
\frac{d_{-}f}{dt} &= (\partial_{t} + b_{-} \cdot \nabla - D\Delta)f
\end{align}

(1.13)

Let $\rho(x,t)$ be the probability density of $x(t)$; it is known that for any Markov (hence Wiener) process one has $\partial_{t}\rho + div(\rho V_{\rho}) = D\Delta\rho$ (forward equation) and $\partial_{t}\rho + div(\rho U) = -D\Delta\rho$ (backward equation). These are called Fokker-Planck equations and one defines two new average velocities $V = (1/2)[b_{+} + b_{-}]$ and $U = (1/2)[b_{+} - b_{-}]$. Consequently adding and subtracting one obtains $\partial_{t}\rho + div(\rho V) = 0$ (continuity equation) and $div(\rho U) - D\Delta\rho = 0$ which is equivalent to $div[(\rho U - D\Delta log(\rho))] = 0$. One can show, using (1.13) that the term in square brackets in the last equation is zero leading to $U = D\nabla log(\rho)$. Now place oneself in the $(U, V)$ plane and write $V = V - iU$. Then write $(dV/dt) = (1/2)(d_{+} + d_{-})/dt$ and $(dU/dt) = (1/2)(d_{+} - d_{-})/dt$. Combining the equations in (1.13) one defines $(dV/dt) = \partial_{t} + V \cdot \nabla$ and
\( (dU/dt) = D\Delta + U \cdot \nabla; \) then define a complex operator \( (d'/dt) = (dV/dt) - i(d\ell/dt) \) which becomes

\[
\frac{d'}{dt} = \left( \frac{\partial}{\partial t} - iD\Delta \right) + \nabla \cdot \nabla
\]  

One now postulates that the passage from classical mechanics to a new nondifferentiable process considered here can be implemented by the unique prescription of replacing the standard \( d/dt \) by \( d'/dt \). Thus consider \( S = \int L(x, V, t)dt \) yielding by least action \( (d'/dt)\partial L/\partial V_i = \partial L/\partial x_i \). Define then \( P_i = \partial L/\partial V_i \) leading to \( P = \nabla S \) (recall the classical action principle with \( dS = pdq - Hdt \)). Now for Newtonian mechanics write \( L(x, v, t) = (1/2)mv^2 - U \) which becomes \( L(x, V, t) = (1/2)mv^2 - U \) leading to \(-\nabla U = m(d'/dt)V \). One separates real and imaginary parts of the complex acceleration \( \gamma = (d'V/dt) \) to get

\[
d'V = (dV - id\ell)(V - iU) = (dV - d\ell U) - i(d\ell V + dV U)
\]

The force \( F = -\nabla U \) is real so the imaginary part of the complex acceleration vanishes; hence

\[
\frac{d\ell}{dt}V + dV U = \frac{\partial U}{\partial t} + U \cdot \nabla V + V \cdot \nabla U + D\Delta V = 0
\]

from which \( \partial U/\partial t \) may be obtained. This is a weak point in the derivation since one has to assume e.g. that \( U(x, t) \) has certain smoothness properties. Now considerable calculation leads to the SE \( i\hbar \psi_t = -(\hbar^2/2m)\Delta \psi + \Delta i \psi \) and this suggests an interpretation of QM as mechanics in a nondifferentiable (fractal) space. In fact (using one space dimension for convenience) we see that if \( \mathcal{U} = 0 \) then the free motion \( m(d'/dt)V = 0 \) yields the SE \( i\hbar \psi_{tt} = -(\hbar^2/2m)\psi_{xx} \) as a geodesic equation in “fractal” space. Further from \( \mathcal{U} = (\h/m)(\partial \sqrt{\rho}/\sqrt{\rho}) \) and \( Q = -(\h^2/2m)(\Delta \sqrt{\rho}/\sqrt{\rho}) \) one arrives at a lovely relation, namely

**PROPOSITION 1.1.** The quantum potential \( Q \) can be written in the form \( Q = -(m/2)U^2 - (\hbar/2)\partial U \). Hence the quantum potential arises directly from the fractal nonsmooth nature of the quantum paths. Since \( Q \) can be thought of as a quantization of a classical motion we see that the quantization corresponds exactly to the existence of nonsmooth paths. Consequently smooth paths imply no quantum mechanics.

**REMARK 1.4.** In (13) one writes again \( \psi = R \exp(iS/\hbar) \) with field equations in the hydrodynamical picture (1-D for convenience)

\[
d_t(m_0\rho v) = \partial_v(m_0\rho v) + \nabla(m_0\rho v) = -\rho \nabla(u + Q); \quad \partial_t \rho + \nabla \cdot (\rho v) = 0
\]

where \( Q = -(\h^2/2m_0)(\Delta \sqrt{\rho}/\sqrt{\rho}) \). The Nottale approach is used as above with \( d_u \sim d_{\sigma} \) and \( d_{\rho} \sim d_t \). One assumes that the velocity field from the hydrodynamical model agrees with the real part \( v \) of the complex velocity \( V = v - iu \) so \( v = (1/m_0)\nabla s \sim 2D\partial s \) and \( u = -(1/m_0)\nabla \sigma \sim D\partial \log(\rho) \) where \( D = \h/m_0 \). In this context the quantum potential \( Q = -(\h^2/2m_0)\Delta \sqrt{\rho}/\sqrt{\rho} \) becomes

\[
Q = -m_0 D \nabla \cdot u - (1/2)m_0 u^2 \sim -\hbar/2 \partial u - (1/2)m_0 u^2
\]

Consequently \( Q \) arises from the fractal derivative and the nondifferentiability of spacetime again, as in Proposition 1.1. Further one can relate \( u \) (and hence \( Q \)) to an internal stress tensor whereas the \( v \) equations correspond to systems of Navier-Stokes type.

**REMARK 1.5.** We note that it is the presence of \( \pm \) derivatives that makes possible the introduction of a complex plane to describe velocities and hence QM; one can think of
this as the motivation for a complex valued wave function and the nature of the SE.

REMARK 1.6. In [56] one extends ideas of Nottale and Ord (cf. [149, 150, 151]) in order to derive an interesting nonlinear Schrödinger equation (NLSE) using a complex diffusion coefficient and a hydrodynamic model.

1.1. THE SCHRODINGER EQUATION IN WEYL SPACE. We go now to Santamato [171] and derive the SE from classical mechanics in Weyl space (i.e. from Weyl geometry - cf. also [18, 32, 33, 55, 108, 172, 199]). The idea is to relate the quantum force (arising from the quantum potential) to geometrical properties of spacetime; the Klein-Gordon (KG) equation is also treated in this spirit in [55, 172]. One wants to show how geometry acts as a guidance field for matter (as in general relativity). Initial positions are assumed random (as in the Madelung approach) and thus the theory is statistical and is really describing the motion of an ensemble. Thus assume that the particle motion is given by some random process \( q^i(t, \omega) \) in a manifold \( M \) (where \( \omega \) is the sample space tag) whose probability density \( \rho(q(t), t) \) exists and is properly normalizable. Assume that the process \( q^i(t, \omega) \) is the solution of differential equations

\[
\dot{q}^i(t, \omega) = (dq^i/dt)(t, \omega) = v^i(q(t, \omega), t)
\]

with random initial conditions \( q^i(0, \omega) = q_0^i(\omega) \). Once the joint distribution of the random variables \( q_0^i(\omega) \) is given the process \( q^i(t, \omega) \) is uniquely determined by (1.19). One knows that in this situation \( \partial_t \rho + \partial_i (\rho v^i) = 0 \) (continuity equation) with initial Cauchy data \( \rho(q, t) = \rho_0(q) \). The natural origin of \( v^i \) arises via a least action principle based on a Lagrangian \( L(q, \dot{q}, t) \) with

\[
L^*(q, \dot{q}, t) = L(q, \dot{q}, t) - \Phi(q, \dot{q}, t); \quad \Phi = \frac{dS}{dt} = \partial_t S + \dot{q}^i \partial_i S
\]

Then \( v^i(q, t) \) arises by minimizing

\[
I(t_0, t_1) = E[\int_{t_0}^{t_1} L^*(q(t, \omega), \dot{q}(t, \omega), t)dt]
\]

where \( t_0, t_1 \) are arbitrary and \( E \) denotes the expectation (cf. [40, 41, 131, 132, 133] for stochastic ideas). The minimum is to be achieved over the class of all random motions \( q^i(t, \omega) \) obeying (1.20) with arbitrarily varied velocity field \( v^i(q, t) \) but having common initial values. One proves first

\[
\partial_t S + H(q, \nabla S, t) = 0; \quad v^i(q, t) = \frac{\partial H}{\partial p_i}(q, \nabla S(q, t), t)
\]

Thus the value of \( I \) in (1.21) along the random curve \( q^i(t, q_0(\omega)) \) is

\[
I(t_1, t_0, \omega) = \int_{t_0}^{t_1} L^*(q(t, q_0(\omega), t)dt
\]

Let \( \mu(q_0) \) denote the joint probability density of the random variables \( q_0^i(\omega) \) and then the expectation value of the random integral is

\[
I(t_1, t_0) = E[I(t_1, t_0, \omega)] = \int_{\mathbb{R}^n} \int_{t_0}^{t_1} \mu(q_0) L^*(q(t, q_0), \dot{q}(t, q_0), t)d^n q_0 dt
\]

Standard variational methods give then

\[
\delta I = \int_{\mathbb{R}^n} d^n q_0 \mu(q_0) \left[ \frac{\partial L^*}{\partial \dot{q}^i}(q(t_1, q_0), \partial_t q(t_1, q_0), t)\delta q^i(t_1, q_0) - \partial_i S(q(t_1, q_0), \partial_t q(t_1, q_0), t)\delta q^i(t_1, q_0) \right]
\]
\[- \int_{t_0}^{t_1} dt \left( \frac{\partial}{\partial q^i} L^*(q(t, q_0), \partial q(t, q_0), t) - \frac{\partial}{\partial \dot{q}^i} L^*(q(t, q_0), \partial \dot{q}(t, q_0), t) \right) \delta q^i(t, q_0) \]

where one uses the fact that \( \mu(q_0) \) is independent of time and \( \delta q^i(t_0, q_0) = 0 \) (recall common initial data is assumed). Therefore

\[
\text{(A)} \quad \left( \frac{\partial L^*}{\partial \dot{q}^i} \right)(q(t, q_0), \partial \dot{q}(t, q_0), t) = 0;
\]

\[
\text{(B)} \quad \frac{\partial}{\partial t} \left( \frac{\partial L^*}{\partial \dot{q}^i} \right)(q(t, q_0), \partial \dot{q}(t, q_0), t) - \frac{\partial L^*}{\partial q^i}(q(t, q_0), \partial q(t, q_0), t) = 0
\]

are the necessary conditions for obtaining a minimum of I. Conditions (B) are the usual Euler-Lagrange (EL) equations whereas (A) is a consequence of the fact that in the most general case one must retain varied motions with \( \delta q^i(t_1, q_0) \) different from zero at the final time \( t_1 \). Note that since \( L^* \) differs from \( L \) by a total time derivative one can safely replace \( L^* \) by \( L \) in (B) and putting (1.20) into (A) one obtains the classical equations

\[
\rho = \left( \frac{\partial L}{\partial \dot{q}^i} \right)(q(t, q_0), \dot{q}(t, q_0), t) = 0; \quad \dot{\rho} = \left( \frac{\partial L}{\partial q^i} \right)(q(t, q_0), \dot{q}(t, q_0), t)
\]

It is known now that if \( \det[(\partial^2 L/\partial q^i \partial q^j)] \neq 0 \) then the second equation in (1.22) is a consequence of the gradient condition (1.27) and of the definition of the Hamiltonian function \( H(q, p, t) = p^i \dot{q}^i - L \). Moreover (B) in (1.20) and (1.21) enslave the HJ equation in (1.22). In order to show that the average action integral (1.22) actually gives a minimum one needs \( \delta^2 I > 0 \) but this is not necessary for Lagrangians whose Hamiltonian \( H \) has the form

\[
H_C(q, p, t) = \frac{1}{2m} g^{ik}(p_i - A_i)(p_k - A_k) + V
\]

with arbitrary fields \( A_i \) and \( V \) (particle of mass \( m \) in an EM field \( A \) which is the form for nonrelativistic applications; given positive definite \( g_{ik} \) such Hamiltonians involve sufficiency conditions \( \det[(\partial^2 L/\partial q^i \partial q^j)] = mg > 0 \). Finally (B) in (1.20) with \( L^* \) replaced by \( L \) shows that along particle trajectories the EL equations are satisfied, i.e., the particle undergoes a classical motion with probability one. Notice here that in (1.22) no explicit mention of generalized momenta is made; one is dealing with a random motion entirely based on position. Moreover the minimum principle (1.21) defines a 1-1 correspondence between solutions \( S(q, t) \) in (1.22) and minimizing random motions \( q^i(t, \omega) \). Provided \( \nu^i \) is given via (1.22) the particle undergoes a classical motion with probability one. Thus once the Lagrangian \( L \) or equivalently the Hamiltonian \( H \) is given, \( \partial_i \rho + \partial_i(\nu^i) = 0 \) and (1.22) uniquely determine the stochastic process \( q^i(t, \omega) \). Now suppose that some geometric structure is given on \( M \) so that the notion of scalar curvature \( R(q, t) \) of \( M \) is meaningful. Then we assume (ad hoc) that the actual Lagrangian is

\[
L(q, \dot{q}, t) = L_C(q, \dot{q}, t) + \gamma(t) \dot{q} H(q, t)
\]

where \( \gamma = (1/6)(n - 2)/(n - 1) \) with \( n = \text{dim}(M) \). Since both \( L_C \) and \( R \) are independent of \( \hbar \) we have \( L \rightarrow L_C \) as \( \hbar \rightarrow 0 \).

Now for a differential manifold with \( ds^2 = g_{ik}(q) dq^i dq^k \) it is standard that in a transplan- 

tation \( q^i \rightarrow q^i + \delta q^i \) one has \( \delta A^k = \Gamma_{kl}^i A^l dq^k \) with \( \Gamma_{kl}^i \) general affine connection coefficients on \( M \) (Riemannian structure is not assumed). In (1.74) it is assumed that for \( \ell = (g_{ik} A^k A^k)^{1/2} \) one has \( \delta \ell = t \phi_0 dq^k \) where the \( \phi_0 \) are covariant components of an arbitrary vector (Weyl geometry). Then the actual affine connections \( \Gamma_{kl}^i \) can be found by comparing this with \( \delta \ell^2 = \delta(g_{ik} A^k A^k) \) and using \( \delta A^k = \Gamma_{kl}^i A^l dq^k \). A little linear algebra gives then

\[
\Gamma_{kl}^i = - \left( \begin{array}{c}
{i} \\
{k} \\
{l}
\end{array} \right) + g^{im} (\dot{g}_{mk} \phi_0 + \dot{g}_{ml} \phi_k - \dot{g}_{kl} \phi_m)
\]
Thus we may prescribe the metric tensor $g_{ik}$ and $\phi_i$ and determine via (1.30) the connection coefficients. Note that $\Gamma^i_{kl} = \Gamma^i_{lk}$ and for $\phi_i = 0$ one has Riemannian geometry. Covariant derivatives are defined via

$$(1.31) \quad A^k_{\ell} = \partial_i A^k - \Gamma^k_{\ell i} A^i,$$

for covariant and contravariant vectors respectively (where $\partial_i$ derivatives are defined via $A^k_{\ell} = \partial_i A^k$). Then observing that the term in (1.21) containing the gauge vector is the curvature term. Then observing that $\Gamma^i_{\ell k}$ evaluated now with respect to the class of all Weyl geometries having arbitrarily varied action principle (1.21) giving the motion of the particle. The minimum in (1.21) is to be no longer holds (i.e. $g_{ik,\ell} \neq 0$) so covariant differentiation and operations of raising or lowering indices do not commute. The curvature tensor $R^i_{\ell m k}$ in Weyl geometry is introduced via $A^1_{\ell} = \partial_i A^1 + \Gamma^1_{\ell i} A^i$ from which arises the standard formula of Riemannian geometry

$$(1.32) \quad R^i_{\ell m k} = -\partial_k \Gamma^i_{\ell m} + \partial_m \Gamma^i_{\ell k} + \Gamma^i_{\ell n} \Gamma^m_{nk} - \Gamma^i_{\ell m} \Gamma^n_{nk}$$

where (1.30) is used in place of the Christoffel symbols. The tensor $R^i_{\ell m k}$ obeys the same symmetry relations as the curvature tensor of Riemann geometry as well as the Bianchi identity. The Ricci symmetric tensor $R_{ik}$ and the scalar curvature $R$ are defined by the same formulas also, viz. $R_{ik} = R^j_{\ell jk}$ and $R = g^{ik} R_{ik}$. For completeness one derives here

$$(1.33) \quad R = \hat{R} + (n - 1)(n - 2)\phi_i \phi^i - 2(1/\sqrt{g})\partial_i (\sqrt{g} \phi^i)$$

where $\hat{R}$ is the Riemannian curvature built by the Christoffel symbols. Thus from (1.30) one obtains

$$(1.34) \quad g^{ik} \Gamma^i_{\ell k} = -g^{ik} \left\{ \begin{array}{c} i \\
\ell 
\end{array} \right\} - (n - 2)\phi^i, \quad \Gamma^i_{\ell k} = -\left\{ \begin{array}{c} i \\
\ell 
\end{array} \right\} + n\phi_k$$

Since the form of a scalar is independent of the coordinate system used one may compute $R$ in a geodesic system where the Christoffel symbols and all $\partial_i g_{ik}$ vanish; then (1.30) reduces to $\Gamma^i_{\ell k} = \phi_k \kappa_i^{\ell} + \phi_i \delta^\ell_k - g_{\ell k} \phi^i$ and hence

$$(1.35) \quad R = -g^{km} \partial_m \Gamma^i_{\ell k} + \partial_k (g^{km} \Gamma^i_{\ell m}) + g^{km} \Gamma^i_{\ell m} \Gamma^m_{ni} - g^{m t} \Gamma^i_{m t} \Gamma^m_{nl} \Gamma^l_{nk}$$

Further one has $g^{km} \Gamma^i_{\ell m} \Gamma^m_{ni} = -(n - 2)(\phi_k \phi^k)$ at the point in consideration. Putting all this in (1.30) one arrives at

$$(1.36) \quad R = \hat{R} + (n - 1)(n - 2)(\phi_k \phi^k) - 2(n - 1)\partial_k \phi^k$$

which becomes (1.33) in covariant form. Now the geometry is to be derived from physical principles so the $\phi_i$ cannot be arbitrary but must be obtained by the same averaged least action principle (1.21) giving the motion of the particle. The minimum in (1.21) is to be evaluated now with respect to the class of all Weyl geometries having arbitrarily varied gauge vectors but fixed metric tensor. Note that once (1.20) is inserted in (1.21) the only term in (1.21) containing the gauge vector is the curvature term. Then observing that $\gamma > 0$ when $n \geq 3$ the minimum principle (1.21) may be reduced to the simpler form $E[R(q(t, \omega), t)] = min$ where only the gauge vectors $\phi_i$ are varied. Using (1.33) this is easily done. First a little argument shows that $\hat{\rho}(q, t) = (\rho(q, t)/\sqrt{g})$ transforms as a scalar in a coordinate change and this will be called the scalar probability density of the random motion of the particle (statistical determination of geometry). Starting from $\partial_i \rho + g_{ik} \phi^i \partial_k \rho = 0$ a manifestly covariant equation for $\rho$ is found to be $\partial_i \rho + (1/\sqrt{g}) \partial_i (\sqrt{g} \phi^i \rho) = 0$. Now return to the minimum problem $E[R(q(t, \omega), t)] = min$; from (1.33) and $\hat{\rho} = \rho/\sqrt{g}$ one obtains

$$(1.37) \quad E[R(q(t, \omega), t)] = E[\hat{R}(q(t, \omega), t)] +$$

$$+ (n - 1) \int_M [\rho(q, t) - 2(1/\sqrt{g}) \partial_i (\sqrt{g} \phi^i) \rho(q, t)] \sqrt{g} d^m q$$
Assuming fields go to 0 rapidly enough on \( \partial M \) and integrating by parts one gets then
\[
E[R] = E[\dot{R}] - \frac{n-1}{n-2}E[g^{ik}\partial_i(\log(\hat{\rho}))\partial_k(\log(\hat{\rho})) + \frac{n-1}{n-2}E[g^{ik}((n-2)\phi_i + \partial_i(\log(\hat{\rho}))))((n-2)\phi_k + \partial_k(\log(\hat{\rho})))]
\]
Since the first two terms on the right are independent of the gauge vector and \( g^{ik} \) is positive definite \( E[R] \) will be a minimum when
\[
\phi_i(q,t) = -[1/(n-2)]\partial_i\log(\hat{\rho})(q,t)]
\]
This shows that the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum force \( f_i = \gamma(h^2/m)\partial_iR \) which according to \( (1.33) \) depends on the gauge vector and its derivatives. It is this peculiar feedback between the geometry of space and the motion of the particle which produces quantum effects.

In this spirit one goes now to a geometrical derivation of the SE. Thus inserting \( (1.39) \) into \( (1.33) \) one gets
\[
R = \dot{R} + \frac{1}{2}\gamma \sqrt{\hat{\rho}}[1/\sqrt{g}]\partial_i(\sqrt{gg^{ik}\partial_k}\sqrt{\hat{\rho}}]
\]
where the value \( (n-2)/6(n-1) \) for \( \gamma \) is used. On the other hand the HJ equation \( (1.20) \) can be written as
\[
\partial_tS + H_C(q, \nabla S, t) - \gamma(h^2/m)R = 0
\]
where \( (1.29) \) has been used. When \( (1.40) \) is introduced into \( (1.41) \) the HJ equation and the continuity equation \( \partial_t\hat{\rho} + (1/\sqrt{g})[\sqrt{gg^{ik}\partial_k}\sqrt{\hat{\rho}}] = 0 \), with velocity field given by \( (1.22) \), form a set of two nonlinear PDE which are coupled by the curvature of space. Therefore self consistent random motions of the particle (i.e. random motions compatible with \( (1.33) \) ) are obtained by solving \( (1.41) \) and the continuity equation simultaneously. For every pair of solutions \( S(q,t,\hat{\rho}(q,t)) \) one gets a possible random motion for the particle whose invariant probability density is \( \hat{\rho} \). The present approach is so different from traditional QM that a proof of equivalence is needed and this is only done for Hamiltonians of the form \( (1.28) \) (which is not very restrictive). The HJ equation corresponding to \( (1.28) \) is
\[
\partial_tS + \frac{1}{2m}g^{ik}(\partial_iS - A_i)(\partial_kS - A_k) + V - \gamma(h^2/m)R = 0
\]
with \( R \) given by \( (1.40) \). Moreover using \( (1.22) \) as well as \( (1.33) \) the continuity equation becomes
\[
\partial_t\hat{\rho} + (1/m\sqrt{g})\partial_i[\sqrt{\hat{\rho}}g^{ik}(\partial_kS - A_k)] = 0
\]
Owing to \( (1.40) \), \( (1.42) \) and \( (1.38) \) form a set of two nonlinear PDE which must be solved for the unknown functions \( S \) and \( \hat{\rho} \). Now a straightforward calculations shows that, setting
\[
\psi(q,t) = \sqrt{\hat{\rho}(q,t)}exp[i/h]S(q,t),
\]
the quantity \( \psi \) obeys a linear PDE (corrected from \( (1.71) \))
\[
(1.45) \quad ih\partial_t\psi = \frac{1}{2m}\left\{ \frac{i\hbar}{\sqrt{g}}\left[ A_i + g^{ik}(i\hbar\partial_k + A_k)\right] \right\} \psi + \left[ V - \gamma(h^2/m)R \right] \psi
\]
where only the Riemannian curvature \( \dot{R} \) is present (any explicit reference to the gauge vector \( \phi_i \) having disappeared). \( (1.45) \) is of course the SE in curvilinear coordinates whose invariance under point transformations is well known. Moreover \( (1.44) \) shows that \( |\psi|^2 = \hat{\rho}(q,t) \) is
the invariant probability density of finding the particle in the volume element $d^nq$ at time $t$. Then following Nelson’s arguments that the SE together with the density formula contains QM the present theory is physically equivalent to traditional nonrelativistic QM. One sees also from (1.44) and (1.45) that the time independent SE is obtained via $S = S_0(q) - Et$ with constant $E$ and $\rho(q)$. In this case the scalar curvature of space becomes time independent; since starting data at $t_0$ is meaningless one replaces the continuity equation with a condition $\int_M \rho(q)\sqrt{g} d^nq = 1$.

**REMARK 1.6.** We recall that in the nonrelativistic context the quantum potential has the form $Q = -\langle h^2/2m \rangle (\partial^2/\sqrt{\rho}) (\rho \sim \hat{\rho}$ here) and in more dimensions this corresponds to $Q = -(h^2/2m)(\Delta \sqrt{\rho}/ \sqrt{\rho})$. Here we have a SE involving $\psi = \sqrt{\rho} \exp(i/h)S$ with corresponding HJ equation (1.42) which corresponds to the flat space 1-D $\partial_0$ with continuity equation $\rho S + (s^i)^2/2m + V + Q = 0$ with continuity equation $\partial_t \rho + \partial(\rho S'/m) = 0$ (take $A_k = 0$ here). The continuity equation in (1.44) corresponds to $\partial_t \rho + (1/m\sqrt{g})\partial_i[\rho \sqrt{g} g^{ik}(\partial_k S)] = 0$. For $A_k = 0$ (1.42) becomes

$$\partial_t S + (1/2m)g^{ik} \partial_k S \partial_i S + V - \gamma(h^2/m)R = 0$$

This leads to an identification $Q \sim -\gamma(h^2/m)R$ where $R$ is the Ricci scalar in the Weyl geometry (related to the Riemannian curvature built on standard Christoffel symbols via (1.33)). Here $\gamma = (1/6)(n-2)(n-2)$ as above which for $n = 3$ becomes $\gamma = 1/12$; further the Weyl field $\phi_i = -\partial_i \log(\rho)$. Consequently (see below).

**PROPOSITION 1.2.** For the SE (1.45) in Weyl space the quantum potential is $Q = -(h^2/12m)R$ where $R$ is the Weyl-Ricci scalar curvature. For Riemannian flat space $R = 0$ this becomes via (1.40)

$$R = \frac{1}{2}\gamma \sqrt{\rho} \partial_i g^{ik} \partial_k \sqrt{\rho} \sim \frac{1}{2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{h^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}$$

as is should and the SE (1.45) reduces to the standard SE in the form $i\hbar \partial_t \psi = -(h^2/2m)\Delta \psi + V\psi (A_k = 0)$.

**REMARK 1.7.** In [172] (first paper) one begins with a generic 4-dimensional manifold with torsion free connections and a metric tensor $g_{\mu\nu}$ ($h = c = 1$ for convenience). Then working with an average action principle based on [95] the particle motion and (Weyl) spacetime geometry are derived in a gauge invariant manner (cf. Section 3.2). Thus an integrable Weyl geometry is produced from a stochastic background via an extremization procedure (see Section 3). An effective particle mass is taken as $m^2 - (R/6) \sim m^2(1 + Q) \approx m^2 \exp(Q)$ corresponding to $R/6 = -m^2 Q = -\Box \sqrt{g}/ \sqrt{g}$ (here $h = c = 1$ and one has signature $(-,+,+,+)$ while the term $\exp(Q)$ arises from $\Box 2$). We refer to [12] 43 54 55 172 and Section 2 for details (for various other approaches see 17, 109).

1.2. **FISHER INFORMATION REVISITED.** We recall first that the classical Fisher information associated with translations of a 1-D observable $X$ with probability density $P(x)$ (related to a quantum geometry probability measure $ds^2 = \sum((dp_j)^2/p_j)$) is

$$F_X = \int dx P(x)([\log(P(x))]^2 > 0$$

(cf. 40 43 82 90 97 98 99 100 160 161). One has a well known Cramer-Rao inequality $\text{Var}(X) \geq F_X^{-1}$ where $\text{Var}(X) \sim$ variance of $X$. A Fisher length for $X$ is defined via $\delta X = F_X^{-1/2}$ and this quantifies the length scale over which $p(x)$ (or better $\log(p(x))$) varies appreciably. Then the root mean square deviation $\Delta X$ satisfies $\Delta X \geq \delta X$. Let now $P$
be the momentum observable conjugate to \( X \), and \( p_{cl} \) a classical momentum observable corresponding to the state \( \psi \) given via \( p_{cl}(x) = (\hbar/2i)((\psi')/\psi) - (\bar{\psi}'/\bar{\psi}) \). One has then the identity \( \langle p \rangle = \langle p_{cl} \rangle \langle \psi \rangle \) following via integration by parts. Now define the nonclassical momentum by \( p_{nc} = p - p_{cl} \) and one shows then

\[
\Delta X \Delta p \geq \delta X \Delta p_{nc} = \hbar/2
\]

Then consider a classical ensemble of \( n \)-dimensional particles of mass \( m \) moving under a potential \( V \). The motion can be described via the HJ and continuity equations

\[
\frac{\partial s}{\partial t} + \frac{1}{2m} |\nabla s|^2 + V = 0;
\]

\[
\frac{\partial P}{\partial t} + \nabla \cdot \left[ P \nabla s \right] = 0
\]

for the momentum potential \( s \) and the position probability density \( P \) (note that there is no quantum potential and this will be supplied by the information term). These equations follow from the variational principle

\[
\delta L = 0
\]

with Lagrangian

\[
L = \int dt \, d^n x \, P \left[ (\partial s/\partial t) + (1/2m)|\nabla s|^2 + V \right]
\]

It is now assumed that the classical Lagrangian must be modified due to the existence of random momentum fluctuations. The nature of such fluctuations is immaterial and one can assume that the momentum associated with position \( x \) is given by \( p = \nabla s + N \) where the fluctuation term \( N \) vanishes on average at each point \( x \). Thus \( s \) changes to being an average momentum potential. It follows that the average kinetic energy \( \langle |\nabla s|^2 \rangle / 2m \) appearing in the Lagrangian above should be replaced by \( \langle |\nabla s + N|^2 \rangle / 2m \) giving rise to

\[
L' = L + (2m)^{-1} \int dt \, < N \cdot N > = L + (2m)^{-1} \int dt \, (\Delta N)^2
\]

where \( \Delta N = < N \cdot N >^{1/2} \) is a measure of the strength of the quantum fluctuations. The additional term is specified uniquely, up to a multiplicative constant, by the three assumptions

1. Action principle: \( L' \) is a scalar Lagrangian with respect to the fields \( P \) and \( s \) where the principle \( \delta L' = 0 \) yields causal equations of motion. Thus

\[
(\Delta N)^2 = \int d^n x \, pf(P, \nabla P, \partial P/\partial t, s, \nabla s, \partial s/\partial t, x, t)
\]

for some scalar function \( f \).
2. Additivity: If the system comprises two independent noninteracting subsystems with \( P = P_1 P_2 \) then the Lagrangian decomposes into additive subsystem contributions; thus \( f = f_1 + f_2 \) for \( P = P_1 P_2 \).
3. Exact uncertainty: The strength of the momentum fluctuation at any given time is determined by and scales inversely with the uncertainty in position at that time. Thus \( \Delta N \to k\Delta N \) for \( x \to x/k \). Moreover since position uncertainty is entirely characterized by the probability density \( P \) at any given time the function \( f \) cannot depend on \( s \), nor explicitly on \( t \), nor on \( \partial P/\partial t \).

This leads to the result that (cf. \[40, 54, 96\])

\[
(\Delta N)^2 = c \int d^n x \, P \left| \nabla \log(P) \right|^2
\]

where \( c \) is a positive universal constant. Further for \( \hbar = 2\sqrt{c} \) and \( \psi = \sqrt{P} \exp(is/\hbar) \) the equations of motion for \( p \) and \( s \) arising from \( \delta L' = 0 \) are

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial x} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi.
\]
A second derivation is given in [161, 161]. Thus let \( P(y^i) \) be a probability density and \( P(y^i + \Delta y^i) \) be the density resulting from a small change in the \( y^i \). Calculate the cross entropy via

\[
(1.54) \quad J(P(y^i + \Delta y^i) : P(y^i)) = \int P(y^i + \Delta y^i) \log \frac{P(y^i + \Delta y^i)}{P(y^i)} dy \simeq
\]

\[
\frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^i} \frac{\partial P(y^i)}{\partial y^i} dy \Delta y^i \Delta y^k = I_{jk} \Delta y^i \Delta y^k
\]

The \( I_{jk} \) are the elements of the Fisher information matrix. The most general expression has the form

\[
(1.55) \quad I_{jk}(\theta^i) = \frac{1}{2} \int \frac{1}{P(x^i|\theta^i)} \frac{\partial P(x^i|\theta^i)}{\partial \theta^j} \frac{\partial P(x^i|\theta^i)}{\partial \theta^k} dx
\]

where \( P(x^i|\theta^i) \) is a probability distribution depending on parameters \( \theta^i \) in addition to the \( x^i \). For \( P(x^i|\theta^i) = P(x^i + \theta^i) \) one recovers (1.54). If \( P \) is defined over an \( n \)-dimensional manifold with positive inverse metric \( g^{ik} \) one obtains a natural definition of the information associated with \( P \) via

\[
(1.56) \quad I = g^{ik} I_{ik} = \frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^i} \frac{\partial P(y^i)}{\partial y^k} dy \]

Now in the HJ formulation of classical mechanics the equation of motion takes the form

\[
(1.57) \quad \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + V = 0
\]

where \( g^{\mu\nu} = \text{diag}(1/m, \ldots, 1/m) \). The velocity field \( u^\mu \) is given by \( u^\mu = g^{\mu\nu}(\partial S/\partial x^\nu) \). When the exact coordinates are unknown one can describe the system by means of a probability density \( P(t, x^\mu) \) with \( \int P dx = 1 \) and

\[
(1.58) \quad (\partial P/\partial t) + (\partial/\partial x^\mu)(P g^{\mu\nu}(\partial S/\partial x^\nu) = 0
\]

These equations completely describe the motion and can be derived from the Lagrangian

\[
(1.59) \quad L_{CL} = \int P \left\{ (\partial S/\partial t) + (1/2) g^{\mu\nu}(\partial S/\partial x^\mu) (\partial S/\partial x^\nu) + V \right\} dt dx
\]

using fixed endpoint variation in \( S \) and \( P \). Quantization is obtained by adding a term proportional to the information \( I \) defined in (1.56). This leads to

\[
(1.60) \quad L_{QM} = L_{CL} + \lambda I = \int P \left\{ \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} \right) + \frac{\lambda}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} \right\} dt dx
\]

Fixed endpoint variation in \( S \) leads again to (1.58) while variation in \( P \) leads to

\[
(1.61) \quad \frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} \right] + \lambda \left( \frac{1}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} + \frac{2}{P} \frac{\partial^2 P}{\partial x^\mu \partial x^\nu} \right) = 0
\]

These equations are equivalent to the SE if \( \psi = \sqrt{P} \exp(iS/\hbar) \) with \( \lambda = (2\hbar)^2 \).

**Remark 1.8.** Following ideas in [55, 56, 139] we note in (1.60) for \( \phi_\mu \sim A_\mu = \partial_\mu \log(P) \) (cf. (1.39)) and \( P_\mu = \partial A_\mu \), a complex velocity can be envisioned leading to

\[
(1.62) \quad |P_\mu + i\sqrt{2} A_\mu|^2 = P_\mu^2 + \lambda A_\mu^2 \sim g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, + \frac{\lambda}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} \right)
\]

Further I in (1.56) is exactly known from \( \phi_\mu \) so one has a direct connection between Fisher information and the Weyl field \( \phi_\mu \), along with motivation for a complex velocity. ☐
REM 1.9. Comparing now with (1.63) and quantum geometry in the form \( ds^2 = \sum (dy_j^2/p_j) \) on a space of probability distributions we can define (1.66) as a Fisher information metric in the present context. This should be positive definite in view of its relation to \((\Delta N)^2\) in (1.53) for example. Now for \( \psi = R\exp(i\nu/h) \) one has (\( \rho \sim \rho \))

\[
\frac{-h^2}{8m} \frac{\partial^2 \phi}{\partial \rho^2} = \frac{-h^2}{8m} \frac{2\rho''}{\rho} - \left( \frac{\rho'}{\rho} \right)^2
\]

in 1-D while in more dimensions we have a form (\( \rho \sim \rho \))

\[
Q \sim -2h^2g^{\alpha\beta} \left[ \frac{1}{P^2} \frac{\partial P}{\partial x^\alpha} \frac{\partial P}{\partial x^\beta} - \frac{2}{P} \frac{\partial^2 P}{\partial x^\alpha \partial x^\beta} \right]
\]

as in (1.63) (arising from the Fisher metric I of (1.60) upon variation in \( P \) in the Lagrangian). It can also be related to an osmotic velocity field \( u = D\nabla \log(\rho) \) via \( Q = (1/2)u^2 + D\nabla \cdot u \) connected to Brownian motion where \( D \) is a diffusion coefficient (cf. \( [56, 67, 86, 139] \)). For \( \phi_{\mu} = -\partial_\mu \log(P) \) we have then \( u = -D\phi \) with \( Q = D^2(1/2)(|u|^2 - \nabla \cdot \phi) \), expressing \( Q \) directly in terms of the Weyl vector. This enforces the idea that QM is built into Weyl geometry!

2. BOHMIAN MECHANICS AND WEYL GEOMETRY

From Chapters 1 and 2 we know something about Bohmian mechanics and the quantum potential and we go now to the papers \([178, 179, 180, 181, 182, 183, 184, 185, 186]\) by A. and F. Shojai to begin the present discussion (cf. also \([2, 28, 29, 68, 129, 130, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 183, 184, 185, 186]\). For linking of dBB theory with Weyl geometry). In nonrelativistic de Broglie-Bohm theory the quantum potential is \( Q \sim -\hbar^2/(2m)(\nabla^2 \Psi/|\Psi|) \). The particles trajectory can be derived from Newton’s law of motion in which the quantum force \( -\nabla Q \) is present in addition to the classical force \( -\nabla V \). The enigmatic quantum behavior is attributed here to the quantum force or quantum potential (with \( \Psi \) determining a “pilot wave” which guides the particle motion). Setting \( \Psi = \sqrt{\rho} \exp(i\nu/h) \) one has

\[
\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V + Q = 0; \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \frac{\nabla S}{m} \right) = 0
\]

The first equation in (2.1) is a Hamilton-Jacobi (HJ) equation which is identical to Newton’s law and represents an energy condition \( E = (|p|^2/2m) + V + Q \) (recall from HJ theory \(-\partial S/\partial t) = E (= H) \) and \( \nabla S = p \). The second equation represents a continuity equation for a hypothetical ensemble related to the particle in question. For the relativistic extension one could simply try to generalize the relativistic energy equation \( \eta_{\mu\nu}P^\mu P^\nu = m^2c^2 \) to the form

\[
\eta_{\mu\nu}P^\mu P^\nu = m^2c^2 + 1 + Q = \mathcal{M}^2c^2; \quad Q = (h^2/m^2c^2)(\Box/|\Psi|/|\Psi|)
\]

\[
\mathcal{M}^2 = m^2 \left( 1 + \alpha \frac{\Box|\Psi|}{|\Psi|} \right); \quad \alpha = \frac{\hbar^2}{m^2c^2}
\]

This could be derived e.g. by setting \( \Psi = \sqrt{\rho} \exp(i\nu/h) \) in the Klein-Gordon (KG) equation and separating the real and imaginary parts, leading to the relativistic HJ equation \( \eta_{\mu\nu} \partial^\mu S \partial^\nu S = \mathcal{M}^2c^2 \) (as in (2.1) - note \( P^\mu = -\partial^\mu S \)) and the continuity equation \( \partial_\mu (\rho \partial^\mu S) = 0 \). The problem of \( \mathcal{M}^2 \) not being positive definite here (i.e. tachyons) is serious however and in fact (2.2) is not the correct equation (see e.g. \([180, 182, 183]\)). One
must use the covariant derivatives $\nabla_{\mu}$ in place of $\partial_{\mu}$ and for spin zero in a curved background there results

\begin{equation}
\nabla_{\mu}(\rho \nabla^{\mu} S) = 0; \quad g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} S = M^{2} c^{2};
\end{equation}

To see this one must require that a correct relativistic equation of motion should not only be Poincaré invariant but also it should have the correct nonrelativistic limit. Thus for a relativistic particle of mass $M$ (which is a Lorentz invariant quantity)

\begin{equation}
A = \int d\lambda [1/2] M(\delta(1/2) r_{\mu}/d\lambda)(dr^{\nu}/d\lambda)
\end{equation}

is the action functional where $\lambda$ is any scalar parameter parametrizing the path $r_{\mu}(\lambda)$ (it could e.g. be the proper time $\tau$). Varying the path via $r_{\mu} \rightarrow r_{\mu}' = r_{\mu} + \epsilon_{\mu}$ one gets

\begin{equation}
A \rightarrow A' = A + \delta A = A + \int d\lambda \left[ \frac{dr_{\mu}}{d\lambda} \frac{d\epsilon_{\mu}}{d\lambda} + \frac{1}{2} \frac{d^{2} r_{\mu}}{d\lambda^{2}} \epsilon_{\nu} \partial^{\nu} M \right]
\end{equation}

By least action the correct path satisfies $\delta A = 0$ with fixed boundaries so the equation of motion is

\begin{equation}
(d/d\lambda)(M u_{\mu}) = (1/2) u_{\nu} u^{\nu} \partial_{\nu} M;
\end{equation}

\begin{equation}
M(du_{\mu}/d\lambda) = ((1/2) \eta_{\mu\nu} u_{\alpha} - u_{\mu} u_{\nu}) \partial^{\nu} M
\end{equation}

where $u_{\mu} = dr_{\mu}/d\lambda$. Now look at the symmetries of the action functional via $\lambda \rightarrow \lambda + \delta$. The conserved current is then the Hamiltonian $\delta = -L + u_{\mu}(\partial L/\partial u_{\mu}) = (1/2) M u_{\mu} u^{\mu} = E$. This can be seen by setting $\delta A = 0$ where

\begin{equation}
0 = \delta A = A' - A = \int d\lambda \left[ \frac{1}{2} u_{\nu} u^{\nu} \partial_{\nu} M + M u_{\mu} \frac{du_{\mu}}{d\lambda} \right] \delta
\end{equation}

which means that the integrand is zero, i.e. $(d/d\lambda)((1/2) M u_{\mu} u^{\mu}) = 0$. Since the proper time is defined as $c^{2} d\tau^{2} = dr_{\mu} dr^{\mu}$ this leads to $(d\tau/d\lambda) = \sqrt{(2E/Mc^{2})}$ and the equation of motion becomes

\begin{equation}
M(du_{\mu}/d\tau) = (1/2)(c^{2} \eta_{\mu\nu} - v_{\mu} v_{\nu}) \partial^{\nu} M
\end{equation}

where $v_{\mu} = dr_{\mu}/d\tau$. The nonrelativistic limit can be derived by letting the particles velocity be ignorable with respect to light velocity. In this limit the proper time is identical to the time coordinate $\tau = t$ and the result is that the $\mu = 0$ component is satisfied identically via $(r \sim \bar{r})$

\begin{equation}
M \frac{d^{2} r}{d\tau^{2}} = -\frac{1}{2} c^{2} \nabla M \Rightarrow m \left( \frac{d^{2} r}{d\tau^{2}} \right) = -\nabla \left[ \frac{m c^{2}}{2} \log \left( \frac{M}{\mu} \right) \right]
\end{equation}

where $\mu$ is an arbitrary mass scale. In order to have the correct limit the term in parenthesis on the right side should be equal to the quantum potential so

\begin{equation}
M = \mu \exp[-(\hbar^{2}/m^{2}c^{2})(\nabla^{2}|\Psi|/|\Psi|)]
\end{equation}

The relativistic quantum mass field (manifestly invariant) is $M = \mu \exp[(\hbar^{2}/2m)(\square|\Psi|/|\Psi|)]$ and setting $\mu = m$ we get

\begin{equation}
M = m \exp[(\hbar^{2}/m^{2}c^{2})(\square|\Psi|/|\Psi|)]
\end{equation}

If one starts with the standard relativistic theory and goes to the nonrelativistic limit one does not get the correct nonrelativistic equations; this is a result of an improper decomposition of the wave function into its phase and norm in the KG equation (cf. also [27] for related procedures). One notes here also that (2.12) leads to a positive definite mass
squared. Also from [180] this can be extended to a many particle version and to a curved spacetime. In summary, for a particle in a curved background one has (cf. [182] which we continue to follow)

\[(2.13) \quad \nabla_\mu (\rho \nabla^\mu S) = 0; \quad g^{\mu\nu} \nabla_\mu S \nabla_\nu S = \mathcal{M}^2 c^2; \quad \mathcal{M}^2 = m^2 c^2; \quad \Omega = \frac{\hbar^2}{m^2 c^2} \frac{\Box_{\mu}|\Psi|}{|\Psi|}\]

Since, following deBroglie, the quantum HJ equation (QHJE) in (2.13) can be written in the form \((m^2/\mathcal{M}^2) g^{\mu\nu} \tilde{\nabla}_\mu S \tilde{\nabla}_\nu S = m^2 c^2\) the quantum effects are identical to a change of spacetime metric

\[(2.14) \quad g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\mathcal{M}^2/m^2) g_{\mu\nu}\]

which is a conformal transformation. The QHJE becomes then \(\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu S \tilde{\nabla}_\nu S = m^2 c^2\) where \(\tilde{\nabla}_\mu\) represents covariant differentiation with respect to the metric \(\tilde{g}_{\mu\nu}\) and the continuity equation is then \(\tilde{g}_{\mu\nu} \tilde{\nabla}_\mu (\rho \tilde{\nabla}_\nu S) = 0\). The important conclusion here is that the presence of the quantum potential is equivalent to a curved spacetime with its metric given by (2.14). This is a geometrization of the quantum aspects of matter and it seems that there is a dual aspect to the role of geometry in physics. The spacetime geometry sometimes looks like “gravity” and sometimes reveals quantum behavior. The curvature due to the quantum potential may have a large influence on the classical contribution to the curvature of spacetime. The particle trajectory can now be derived from the guidance relation via differentiation of (2.13) leading to the Newton equations of motion

\[(2.15) \quad \mathcal{M} \frac{d^2 x^\mu}{d\tau^2} + \mathcal{M} \Gamma^\mu_{\nu\kappa} u^\nu u^\kappa = (c^2 g^{\mu\nu} - u^\mu u^\nu) \nabla_\nu \mathcal{M}\]

Using the conformal transformation above (2.15) reduces to the standard geodesic equation.

Now a general “canonical” relativistic system consisting of gravity and classical matter (no quantum effects) is determined by the action

\[(2.16) \quad A = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} R + \int d^4 x \sqrt{-g} \frac{\hbar^2}{2m} \left( \frac{\rho}{\hbar \kappa} \mathcal{D}_\mu S \mathcal{D}^\mu S - \frac{m^2}{\hbar \kappa} \rho\right)\]

where \(\kappa = 8\pi G\) and \(c = 1\) for convenience. It was seen above that via deBroglie the introduction of a quantum potential is equivalent to introducing a conformal factor \(\Omega^2 = \mathcal{M}^2/m^2\) in the metric. Hence in order to introduce quantum effects of matter into the action (2.16) one uses this conformal transformation to get \((1 + Q \sim exp(Q))\)

\[(2.17) \quad A = \frac{1}{2\kappa} \int d^4 x \sqrt{-\bar{g}} \left[ (\mathcal{R} \Omega^2 - 6 \nabla_\mu \Omega \nabla^\mu \Omega) + \right.\]

\[+ \left. \int d^4 x \sqrt{-\bar{g}} \left( \frac{\rho}{m} \Omega^2 \nabla_\mu S \nabla^\mu S - m \rho \Omega^2 \right) \right] + \int d^4 x \sqrt{-\bar{g}} \lambda \left[ \Omega^2 - \left( 1 + \frac{\hbar^2}{m^2} \Box_{\rho} \frac{\sqrt{\rho}}{\sqrt{\rho}} \right) \right]\]

where a bar over any quantity means that it corresponds to the nonquantum regime. Here only the first two terms of the expansion of \(\mathcal{M}^2 = m^2 \exp(\Omega)\) in (2.13) have been used, namely \(\mathcal{M}^2 \sim m^2 (1 + \Omega)\). No physical change is involved in considering all the terms. \(\lambda\) is a Lagrange multiplier introduced to identify the conformal factor with its Bohmian value. One uses here \(g_{\mu\nu}\) to raise of lower indices and to evaluate the covariant derivatives; the physical metric (containing the quantum effects of matter) is \(\tilde{g}_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}\). By variation of the action with respect to \(\tilde{g}_{\mu\nu}, \Omega, \rho, S, \) and \(\lambda\) one arrives at the following quantum equations of motion:
(1) The equation of motion for $\Omega$
\begin{equation}
\tag{2.18}
\mathcal{R}\Omega + 6\mathcal{G}\Omega + \frac{2\kappa}{m}\rho\Omega(\nabla_\mu S\nabla^\mu S - 2m^2\Omega^2) + 2\kappa\lambda\Omega = 0
\end{equation}

(2) The continuity equation for particles $\nabla_\mu (\rho\Omega^2\nabla_\nu S) = 0$

(3) The equations of motion for particles (here $\alpha' \equiv \bar{a}$)
\begin{equation}
\tag{2.19}
(\nabla_\mu S\nabla^\mu S - m^2\Omega^2)\sqrt{\rho} + \frac{\hbar^2}{2m}\left[\nabla'\left(\frac{\lambda}{\sqrt{\rho}}\right) - \frac{\nabla'\sqrt{\rho}}{\rho}\right] = 0
\end{equation}

(4) The modified Einstein equations for $\bar{g}_{\mu\nu}$
\begin{equation}
\tag{2.20}
\Omega^2 \left[ \mathcal{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\mathcal{R} \right] - [\bar{g}_{\mu\nu}\nabla' - \nabla_\mu \nabla_\nu]\Omega^2 - 6\nabla_\mu \Omega\nabla_\nu \Omega + 3\bar{g}_{\mu\nu}\nabla_\alpha \Omega \nabla^\alpha \Omega +
\end{equation}
\begin{equation}
+ \frac{2\kappa}{m}\rho\Omega^2 \nabla_\mu S\nabla_\nu S - \frac{\kappa}{m}\rho\Omega^2 \bar{g}_{\mu\nu}\nabla_\alpha S\nabla^\alpha S + \kappa m\rho\Omega^4 g_{\mu\nu} + 
\end{equation}
\begin{equation}
\frac{\kappa\hbar^2}{m^2} \left[ \nabla_\mu \sqrt{\rho} \nabla_\nu \left( \frac{\lambda}{\sqrt{\rho}} \right) + \nabla_\nu \sqrt{\rho} \nabla_\mu \left( \frac{\lambda}{\sqrt{\rho}} \right) \right] - \frac{\kappa\hbar^2}{m^2} g_{\mu\nu} \nabla_\alpha \left[ \frac{\nabla^\alpha \sqrt{\rho}}{\sqrt{\rho}} \right] = 0
\end{equation}

(5) The constraint equation $\Omega^2 = 1 + (\hbar^2/m^2)(\Box/\sqrt{\rho})$

Thus the back reaction effects of the quantum factor on the background metric are contained in these highly coupled equations. A simpler form of (2.20) can be obtained by taking the trace of (2.20) and using (2.18), which produces $\lambda = (\hbar^2/m^2)\nabla_\mu [\lambda(\nabla_\mu \sqrt{\rho})/\sqrt{\rho}]$. A solution of this via perturbation methods using the small parameter $\alpha = \hbar^2/m^2$ yields the trivial solution $\lambda = 0$ so the above equations reduce to
\begin{equation}
\tag{2.21}
\nabla_\mu (\rho\Omega^2\nabla_\nu S) = 0; \; \nabla_\mu S\nabla_\nu S = m^2\Omega^2; \; \mathcal{G}_{\mu\nu} = -\kappa\mathcal{T}^{(m)}_{\mu\nu} - \kappa\mathcal{T}^{(\Omega)}_{\mu\nu}
\end{equation}

where $\mathcal{T}^{(m)}_{\mu\nu}$ is the matter energy-momentum (EM) tensor and
\begin{equation}
\tag{2.22}
\kappa\mathcal{T}^{(\Omega)}_{\mu\nu} = \frac{[g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu]\Omega^2}{\Omega^2} + 6\frac{\nabla_\mu \Omega \nabla_\nu \Omega}{\omega^2} - 2\bar{g}_{\mu\nu} \frac{\nabla_\alpha \Omega \nabla^\alpha \Omega}{\Omega^2}
\end{equation}

with $\Omega^2 = 1 + \alpha(\Box/\sqrt{\rho})$. Note that the second relation in (2.21) is the Bohmian equation of motion and written in terms of $g_{\mu\nu}$ it becomes $\nabla_\mu S\nabla_\nu S = m^2c^2$.

**REMARK 2.1.** In the preceding one has tacitly assumed that there is an ensemble of quantum particles so what about a single particle? One translates now the quantum potential into purely geometrical terms without reference to matter parameters so that the original form of the quantum potential can only be deduced after using the field equations. Thus the theory will work for a single particle or an ensemble. One notes that the use of $\psi^\dagger \psi$ automatically suggests or involves an ensemble if (or its square root) it is to be interpreted as a probability density. Thus the idea that a particle has only a probability of being at or near $x$ seems to mean that some paths take it there but others don’t and this is consistent with Feynman’s use of path integrals for example. This seems also to say that there is no such thing as a particle, only a collection of versions or cloud connected to the particle idea. Bohmian theory on the other hand for a fixed energy gives a one parameter family of trajectories associated to $\psi$ (see here [17, 50, 54] for details). This is because the trajectory arises from a third order differential while fixing the solution $\psi$ of the second order stationary Schrödinger equation involves only two “boundary” conditions. As was shown in [50] this automatically generates a Heisenberg inequality $\Delta x \Delta p \geq \hbar c$; i.e. the uncertainty is built in when using the wave function $\psi$ and amazingly can be expressed by the operator theoretical framework of quantum mechanics. Thus a one parameter family of paths can
be associated with the use of $\psi\psi^*$ and this generates the cloud or ensemble automatically associated with the use of $\psi$. In fact (cf. Remark 3.2) one might conjecture that upon using a wave function description of quantum particle motion, one opens the door to a cloud of particles, all of whose motions are incompletely governed by the SE, since one determining condition for particle motion is ignored. Thus automatically the quantum potential will give rise to a force acting on any such particular trajectory and the “ensemble” idea naturally applies to a cloud of identical particles.

**REMARK 2.2.** Now first ignore gravity and look at the geometrical properties of the conformal factor given via

\[(2.23) \quad g_{\mu\nu} = e^{4\Sigma} \eta_{\mu\nu}, \quad e^{4\Sigma} = \frac{9m^2}{\alpha \nabla^2} = \exp \left( \frac{\alpha \nabla^2 \rho}{\sqrt{\rho}} \right) = \exp \left( \alpha \frac{\nabla^2 \sqrt{|\Sigma|}}{\sqrt{|\Sigma|}} \right) \]

where $\Sigma$ is the trace of the EM tensor and is substituted for $\rho$ (true for dust). The Einstein tensor for this metric is

\[(2.24) \quad \Theta_{\mu\nu} = 4g_{\mu\nu} \nabla_\eta \exp(-\Sigma) + 2\exp(-2\Sigma)\nabla_\mu \eta \exp(2\Sigma); \quad \Sigma = \alpha \frac{\nabla^2 \rho}{\sqrt{\rho}} \]

Hence as an Ansatz one can suppose that in the presence of gravitational effects the field equation would have a form

\[(2.25) \quad R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa \Sigma_{\mu\nu} + 4g_{\mu\nu} \nabla^2 e^{-\Sigma} + 2\exp(-2\Sigma)\nabla_\mu \nabla_\nu e^{2\Sigma} \]

This is written in a manner such that in the limit $\Sigma_{\mu\nu} \to 0$ one will obtain $\square \rho = 0$. Taking the trace of the last equation one gets $-R = \kappa \Sigma - 12\Sigma + 24(\nabla \Sigma)^2$ which has the iterative solution $\Sigma = \kappa \Sigma - R + 12\alpha \nabla^2 \sqrt{|\Sigma|}/\sqrt{|\Sigma|}$ leading to

\[(2.26) \quad \Sigma \propto \alpha \left( \frac{\nabla^2 \sqrt{|\Sigma|}}{\sqrt{|\Sigma|}} \right) \approx \alpha \left( \frac{\nabla^2 \sqrt{|R|}}{\sqrt{|R|}} \right) \]

to first order in $\alpha$. One goes now to the field equations for this toy model. First from the above one sees that $\Sigma$ can be replaced by $R$ in the expression for the quantum potential or for the conformal factor of the metric. This is important since the explicit reference to ensemble density is removed and the theory works for a single particle or an ensemble. So from (??) for a toy quantum gravity theory one assumes the following field equations

\[(2.27) \quad \Theta_{\mu\nu} - \kappa \Sigma_{\mu\nu} - 3\epsilon_{\alpha\beta\gamma} \exp \left( \frac{\alpha}{2} \Phi \right) \nabla^\alpha \nabla^\beta \exp \left( -\frac{\alpha}{2} \Phi \right) = 0 \]

where $\epsilon_{\alpha\beta\gamma} = 2\left[ g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} \right]$ and $\Phi = \sqrt{|\nabla^2 \Sigma|}/\sqrt{|\Sigma|}$. The number 2 and the minus sign of the second term are chosen so that the energy relation derived later will be correct. Note that the trace of (2.27) is

\[(2.28) \quad R + \kappa \Sigma + 6\exp(\alpha \Phi/2) \nabla \exp(-\alpha \Phi/2) = 0 \]

and this represents the connection of the Ricci scalar curvature of space time and the trace of the matter EM tensor. If a perturbative solution is admitted one can expand in powers of $\alpha$ to find $R^{(0)} = -\kappa \Sigma$ and $R^{(1)} = -\kappa \Sigma - 6\exp(\alpha \Phi/2) \nabla \exp(-\alpha \Phi/2)$ where $\Phi^{(0)} = \sqrt{|\Sigma|}/\sqrt{|\Sigma|}$. The energy relation can be obtained by taking the four divergence of the field equations and since the divergence of the Einstein tensor is zero one obtains

\[(2.29) \quad \kappa \nabla^\nu \Sigma^\mu = \alpha R_{\mu\nu} \nabla^\nu \Phi - \frac{\alpha^2}{4} \nabla_\mu (\nabla^\nu \Phi)^2 + \frac{\alpha^2}{2} \nabla_\mu \Phi \nabla \Phi \]

For a dust with $\Sigma_{\mu\nu} = \rho u_\mu u_\nu$ and $u_\mu$ the velocity field, the conservation of mass law is $\nabla^\nu (\rho \nabla u_\nu) = 0$ so one gets to first order in $\alpha$ $\nabla_\mu \Sigma^\mu /\Sigma = -(\alpha/2) \nabla_\mu \Phi$ or $\Sigma^\mu = m^2 \exp(-\alpha \Phi)$
where $m$ is an integration constant. This is the correct relation of mass and quantum potential.

In [132], there is then some discussion about making the conformal factor dynamical via a general scalar tensor action (cf. also [176]) and subsequently one makes both the conformal factor and the quantum potential into dynamical fields and creates a scalar tensor theory with two scalar fields. Thus first start with a general action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ \phi R - \omega \nabla_\mu \phi \nabla^\mu \phi - \nabla_\mu Q \nabla^\mu Q + 2\Lambda \phi + \mathcal{L}_m \right]$$

The cosmological constant generally has an interaction term with the scalar field and here one uses an ad hoc matter Lagrangian

$$\mathcal{L}_m = \frac{\rho}{m} \phi^a \nabla_\mu S \nabla^\mu S - m \rho \phi^b = \Lambda (1 + Q) + \alpha \rho (e^{\ell Q} - 1)$$

(only the first two terms $1 + Q$ from $exp(Q)$ are used for simplicity in the third term). Here $a, b, c$ are constants to be fixed later and the last term is chosen (heuristically) in such a manner as to have an interaction between the quantum potential field and the ensemble density (via the equations of motion); further the interaction is chosen so that it vanishes in the classical limit but this is ad hoc. Variation of the above action yields

(1) The scalar fields equation of motion

$$\nabla^\nu \nabla_\nu \phi - \frac{\alpha}{\phi} \nabla^\mu \phi \nabla_\mu \phi + 2\Lambda +$$

$$\nabla^\nu Q \nabla_\nu Q + \frac{a}{m} \rho^a \nabla^\mu S \nabla_\mu S - mb \rho \phi^{b-1} = 0$$

(2) The quantum potential equations of motion

$$(\Box Q/\phi) - (\nabla_\mu Q \nabla^\mu \phi/\phi^2) - \Lambda c (1 + Q)^{c-1} + \alpha \rho \phi e^{\ell Q} = 0$$

(3) The generalized Einstein equations

$$\mathcal{G}^\mu \nu - \Lambda g^\mu \nu = -\frac{1}{\phi} \Box g^\mu \nu - \frac{1}{\phi} \nabla^\mu \nabla^\nu - g^\mu \nu \Box \phi + \frac{\omega}{\phi^2} \nabla^\mu \phi \nabla^\nu \phi -$$

$$\frac{\omega}{\phi^2} g^{\mu \nu} \nabla^\alpha \phi \nabla_\alpha \phi + \frac{1}{\phi^2} \nabla^\mu Q \nabla_\nu Q - \frac{1}{2\phi^2} g^{\mu \nu} \nabla^\alpha Q \nabla_\alpha Q$$

(4) The continuity equation $\nabla_\mu (\rho \phi^a \nabla^\mu S) = 0$

(5) The quantum Hamilton Jacobi equation

$$\nabla_\nu S \nabla^\nu \phi = m^2 \rho \phi^{b-2} - \alpha m \rho \phi^{-a} (e^{\ell Q} - 1)$$

In [2.32], the scalar curvature and the term $\nabla^\mu S \nabla_\mu S$ can be eliminated using [2.31] and [2.35]; further on using the matter Lagrangian and the definition of the EM tensor one has

$$\Box (\phi) = (\alpha + 1) \rho \phi (e^{\ell Q} - 1) - 2\Lambda (1 + Q)^c + 2\Lambda \phi - \frac{2}{\phi} \nabla_\mu Q \nabla^\mu Q$$

where $b = a + 1$. Solving [2.31] and [2.35] with a perturbation expansion in $\alpha$ one finds

$$Q = Q_0 + \alpha Q_1 + \cdots; \phi = 1 + \alpha Q_1 + \cdots; \sqrt{\rho} = \sqrt{\rho_0} + \alpha \sqrt{\rho_1} + \cdots$$

where the conformal factor is chosen to be unity at zeroth order so that as $\alpha \to 0$, [2.34] goes to the classical HJ equation. Further since by [2.35] the quantum mass is $m^2 \phi + \cdots$ the first order term in $\phi$ is chosen to be $Q_1$ (cf. [2.13]). Also we will see that $Q_1 \sim \Box \sqrt{\rho}/\sqrt{\rho}$ plus corrections which is in accord with $Q$ as a quantum potential field. In any case after some
computation one obtains $a = 2 \omega k$, $b = a + 1$, and $\ell = (1/4)(2\omega k + 1) = (1/4)(a + 1) = b/4$ with $Q_0 = [1/e(2c - 3)][(1/2\omega k + 1)/2\Lambda k^2(2 - c^2 + 1)]$ while $\rho_0$ can be determined (cf. [182] for details). Thus heuristically the quantum potential can be regarded as a dynamical field and perturbatively one gets the correct dependence of quantum potential upon density, modulo some corrective terms.

**REMARK 2.3.** The gravitational effects determine the causal structure of spacetime as long as quantum effects give its conformal structure. This does not mean that quantum effects have nothing to do with the causal structure; they can act on the causal structure through back reaction terms appearing in the metric field equations. The conformal factor of the metric is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. One has shown that the presence of the quantum potential is equivalent to a conformal mapping of the metric. Thus in different conformally related frames one feels different quantum masses and different curvatures. In particular there are two frames with one containing the quantum mass field and the classical metric while the other contains the classical mass and the quantum metric. In general frames both the spacetime metric and the mass field have quantum properties so one can state that different conformal frames are identical pictures of the gravitational and quantum phenomena. We feel different quantum forces in different conformal frames. The question then arises of whether the geometrization of quantum effects implies conformal invariance just as gravitational effects imply general coordinate invariance. One sees here that Weyl geometry provides additional degrees of freedom which can be identified with quantum effects and seems to create a unified geometric framework for understanding both gravitational and quantum forces. Some features here are: (i) Quantum effects appear independent of any preferred length scale. (ii) The quantum mass of a particle is a field. (iii) The gravitational constant is also a field depending on the matter distribution via the quantum potential (cf. [176, 183]). (iv) A local variation of matter field distribution changes the quantum potential acting on the geometry and alters it globally; the nonlocal character is forced by the quantum potential (cf. [177]).

### 2.1. DIRAC-WEYL ACTION

Next (still following [182]) one goes to Weyl geometry based on the Weyl-Dirac action

$$\mathcal{A} = \int d^4x \sqrt{-g} \left( F_{\mu \nu} F^{\mu \nu} - \beta^2 W R + (\sigma + 6) \beta_{\mu} \beta^{\mu} + \Sigma_{\text{matter}} \right)$$

Here $F_{\mu \nu}$ is the curl of the Weyl 4-vector $\phi_\mu$, $\sigma$ is an arbitrary constant and $\beta$ is a scalar field of weight $-1$. The symbol "\" represents a covariant derivative under general coordinate and conformal transformations (Weyl covariant derivative) defined as $X_{\mu} = W_{\lambda}^{\cdots} X^\lambda - N X^\mu$ where $N$ is the Weyl weight of $X$. The equations of motion are then

$$\phi^{\mu \nu} = -\frac{8\pi}{\beta^2} (\Sigma^{\mu \nu} + M^{\mu \nu}) + \frac{2}{\beta} (g^{\mu \nu \rho} \nabla_\rho \beta_\alpha \nabla_\alpha - W \nabla_\mu W \nabla_\nu \beta) +$$

$$+ \frac{1}{\beta^2} (4 \nabla_\mu \beta \nabla_\nu \beta - g^{\mu \nu} \nabla_\alpha \beta_\alpha) + \sigma \frac{\beta^2}{\beta^2} (\beta^\mu \beta^\nu - \frac{1}{2} g^{\mu \nu} \beta^\alpha \beta_\alpha);$$

$$W \nabla_\mu F^{\mu \nu} = \frac{1}{2} \sigma (\beta^2 \phi^\mu + \beta \nabla^\mu \beta) + 4 \pi J^\mu;$$

$$\mathcal{R} = \frac{\psi}{\beta}$$
where
\[ M^{\mu\nu} = (1/4\pi)\left((1/4)g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu}_{\alpha}F^{\nu\alpha}\right) \]
and
\[ 8\pi \Sigma^{\mu\nu} = \frac{1}{\sqrt{-g}} \left( \frac{\delta \sqrt{-g} \Sigma_{\text{matter}}}{\delta g^{\mu\nu}} \right); \quad 16\pi J^\mu = \frac{\delta \Sigma_{\text{matter}}}{\delta \phi_\mu}; \quad \psi = \frac{\delta \Sigma_{\text{matter}}}{\delta \beta} \]

For the equations of motion of matter and the trace of the EM tensor one uses invariance of the action under coordinate and gauge transformations, leading to
\[ W_{\nu} \Sigma^{\mu\nu} - \frac{\delta}{\beta} = J_\alpha \phi'^\alpha - \left( \phi'^\mu + \frac{\nabla^{\mu}}{\beta} \right) W_{\alpha} J^\alpha; \quad 16\pi G - 16\pi W_{\mu} J^\mu - \beta \psi = 0 \]

The first relation is a geometrical identity (Bianchi identity) and the second shows the mutual dependence of the field equations. Note that in the Weyl-Dirac theory the Weyl vector does not couple to spinors so \( \phi_\mu \) cannot be interpreted as the EM potential; the Weyl vector is used as part of the spacetime geometry and the auxiliary field (gauge field) \( \beta \) represents the quantum mass field. The gravity fields \( g_{\mu\nu} \) and \( \phi_\mu \) and the quantum mass field determine the spacetime geometry. Now one constructs a Bohmian quantum gravity which is conformally invariant in the framework of Weyl geometry. If the model has mass this must be a field (since mass has non-zero Weyl weight). The Weyl-Dirac action is a general Weyl invariant action as above and for simplicity now assume the matter Lagrangian does not depend on the Weyl vector so that \( J_\mu = 0 \). The equations of motion are then
\[ \phi'^\mu = -\frac{8\pi}{\beta^2} (\Sigma^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta} (g^{\mu\nu} W^{\alpha} \nabla_\alpha \beta - W^{\mu} W^{\nu} \nabla_\nu \beta) + \frac{1}{\beta^2} (4\nabla_{\mu} \beta \nabla_{\nu} \beta - g^{\mu\nu} \nabla_{\alpha} \beta \nabla_{\beta} \beta) + \frac{\sigma}{\beta^2} \left( \beta \phi'^{\mu} - \frac{2}{\beta} g^{\mu\nu} \beta \alpha \beta \alpha \right); \]
\[ W_{\nu} F^{\mu\nu} = \frac{1}{\beta^2} \left[ \beta \phi'^{\mu} + \beta \nabla^{\nu} \beta \right]; \quad \mathcal{R} = -(\sigma + 6) \frac{W_{\beta} \beta}{\beta} + \frac{\sigma \beta \phi'^{\alpha} + \sigma W_{\nu} \phi_{\alpha} + \psi}{\beta^2} \]

The symmetry conditions are
\[ W_{\nu} \Sigma^{\mu\nu} - \frac{\delta}{\beta} = 0; \quad 16\pi \Sigma - \beta \psi = 0 \]
(recall \( \Sigma = \Sigma_{\mu\nu}^{\nu} \)). One notes that from (2.43) results \( W_{\mu} (\beta^2 \phi'^{\mu} + \beta \nabla^{\mu} \beta) = 0 \) so \( \phi_\mu \) is not independent of \( \beta \). To see how this is related to the Bohmian quantum theory one introduces a quantum mass field and shows it is proportional to the Dirac field. Thus using (2.43) and (2.44) one has
\[ \Box \beta + \frac{1}{6} \beta \mathcal{R} = \frac{4\pi}{\beta^2} \Sigma - \sigma \beta \phi' + 2(\sigma - 6) \phi' \nabla \gamma \beta + \frac{\sigma}{\beta^2} \nabla_{\nu} \beta \nabla_{\mu} \beta \]
This can be solved iteratively via
\[ \beta^2 = (8\pi \Sigma/\mathcal{R}) - \left\{ 1/\left( \mathcal{R}/6 \right) - \sigma \phi' \right\} \beta \Box \beta + \cdots \]
Now assuming \( \Sigma^{\mu\nu} = \rho u^{\mu} u^{\nu} \) (dust with \( \Sigma = \rho \)) we multiply (2.44) by \( u_\mu \) and sum to get
\[ W_{\nu} (\rho u^{\mu}) - \rho (u_\mu \nabla^{\mu} \beta/\beta) = 0 \]
Then put (2.44) into (2.47) which yields
\[ u^{\mu} W_{\mu} u^{\nu} = (1/\beta)(g^{\mu\nu} - u^{\mu} u^{\nu}) \nabla_{\nu} \beta \]
To see this write (assuming $g^\mu\nu\nabla_\nu\beta = \nabla^\mu\beta$)

\[
W \nabla_\nu(\rho u^\mu u^\nu) = u^\mu W \nabla_\nu u^\mu + \rho u^\nu W \nabla_\nu u^\mu \Rightarrow \]

\[
\Rightarrow u^\mu \left(\frac{u^\nu W \nabla_\nu\beta}{\beta}\right) + u^\nu W \nabla_\nu u^\mu - \frac{\nabla^\mu\beta}{\beta} = 0 \Rightarrow u^\nu W \nabla_\nu u^\mu = (1 - u^\mu u_\mu) \frac{\nabla^\mu\beta}{\beta} = \]

\[
(g^{\mu\nu} - u^\mu u^\nu g^{\mu\nu}) \nabla_\nu\beta = (g^{\mu\nu} - u^\mu u^\nu) \nabla_\nu\beta \]

which is (2.47). Then from (2.46)

\[
(2.49)
\]

\[
\beta^{2(1)} = \frac{8\pi \Sigma}{R}; \quad \beta^{2(2)} = \frac{8\pi \Sigma}{R} \left(1 - \frac{1}{(R/6) - \sigma_\alpha \phi^\alpha} \sqrt{\Sigma}\right); \quad \cdots
\]

Comparing with (2.48) and (2.3) shows that we have the correct equations for the Bohmian theory provided one identifies

\[
(2.50)
\]

\[
\beta \sim \delta R; \quad \frac{8\pi \Sigma}{R} \sim m^2; \quad \frac{1}{\sigma_\alpha \phi^\alpha - (R/6)} \sim \alpha
\]

Thus $\beta$ is the Bohmian quantum mass field and the coupling constant $\alpha$ (which depends on $\hbar$) is also a field, related to geometrical properties of spacetime. One notes that the quantum effects and the length scale of the spacetime are related. To see this suppose one is in a gauge in which the Dirac field is constant; apply a gauge transformation to change this to a general spacetime dependent function, i.e.

\[
(2.51)
\]

\[
\beta = \beta_0 \rightarrow \beta(x) = \beta_0 \exp(-\Xi(x)); \quad \phi_\mu \rightarrow \phi_\mu + \partial_\mu \Xi
\]

Thus the gauge in which the quantum mass is constant (and the quantum force is zero) and the gauge in which the quantum mass is spacetime dependent are related to one another via a scale change. In particular $\phi_\mu$ in the two gauges differ by $-\nabla_\mu(\beta/\beta_0)$ and since $\phi_\mu$ is a part of Weyl geometry and the Dirac field represents the quantum mass one concludes that the quantum effects are geometrized (cf. also (2.43) which shows that $\phi_\mu$ is not independent of $\beta$ so the Weyl vector is determined by the quantum mass and thus the geometrical aspect of the manifold is related to quantum effects).

### 3. MORE ON KLEIN GORDON EQUATIONS

We give several approaches here, from various points of view.

#### 3.1. BERTOLDI-FARAGGI-MATONE THEORY

The equivalence principle (EP) of Faraggi-Matone (cf. [27, 44, 45, 51, 82]) is based on the idea that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy (i.e. all potentials are equivalent under coordinate transformations). This automatically leads to the quantum stationary Hamilton-Jacobi equation (QSHJE) which is a third order nonlinear differential equation providing a trajectory representation of quantum mechanics (QM). The theory transcends in several respects the Bohm theory and in particular utilizes a Floydian time (cf. [36, 38]) leading to $\dot{q} = p/m_Q \neq p/m$ where $m_Q = m(1 - \partial_E Q)$ is the “quantum mass” and $Q$ the “quantum potential”. Thus the EP is reminiscent of the Einstein equivalence of relativity theory. This latter served as a midwife to the birth of relativity but was somewhat inaccurate in its original form. It is better put as saying that all laws of physics should be invariant under general coordinate transformations (cf. [146]). This demands that not only the form but also the content of the equations be unchanged. More precisely the equations should be covariant and all absolute constants in the equations are to be left unchanged (e.g. $c, \hbar, e, m$ and $\eta_{\mu\nu} = \text{Minkowski tensor}$).
Now for the EP, the classical picture with $S^c(q,Q^0,t)$ the Hamilton principal function ($p = \partial S^c/\partial q$) and $P^0$, $Q^0$ playing the role of initial conditions involves the classical HJ equation (CHJE) $H(q,p) = (\partial S^c/\partial q), t) + (\partial S^c/\partial t) = 0$. For time independent $V$ one writes $S^c = S^c_0(q,Q^0) - Et$ and arrives at the classical stationary HJ equation (CSHJE) 

$$\left(1/2m\right) \left(\dot{W}'\right)^2 + V = E - \frac{\hbar^2 R''}{2mR} = 0; \ (R^2 \dot{W}')' = 0$$

where $\dot{Q} = -\hbar^2 R'/2mR$ was called the quantum potential; this can be written in the Schwartzian form $\dot{Q} = (h^2/4m)(\dot{W}; q)$ (via $R^2 \dot{W}' = c$). Here $\{f; q\} = (f''/f') - (3/2)(f''/f')^2$.

Writing $\mathfrak{W} = V(q) - E$ as in above we have the quantum stationary HJ equation (QSHJE)

$$\left(1/2m\right) (\partial \mathfrak{W}'/\partial q)^2 + \mathfrak{W}(q) + \dot{Q}(q) = 0 \equiv \mathfrak{W} = -(\hbar^2/4m)\left\{\exp(2iS_0/\hbar); q\right\}$$

This was worked out in the Bohm school (without the Schwarzian connections) but $\psi = Rexp(i\dot{W}/\hbar)$ is not appropriate for all situations and care must be taken ($\dot{W}$ = constant must be excluded for example - cf. [82, 83, 84]). The technique of Faraggi-Matone (FM) is completely general and with only the EP as guide one exploits the relations between Schwarzsians, Legendre duality, and the geometry of a second order differential operator $D^2_x + V(x)$ (Möbius transformations play an important role here) to arrive at the QSHJE in the form

$$\left(1/2m\right) \left(\partial S_0^c/\partial q''\right)^2 + \mathfrak{W}(q'') + \mathfrak{Q}''(q'') = 0$$

where $v : q \rightarrow q''$ represents an arbitrary locally invertible coordinate transformation. Note in this direction for example that the Schwarzian derivative of the the ratio of two linearly independent elements in $\ker(D_x^2 + V(x))$ is twice $V(x)$. In particular given an arbitrary system with coordinate $q$ and reduced action $S_0(q)$ the system with coordinate $q^0$ corresponding to $V - E = 0$ involves $\mathfrak{W}(q) = (q' q; q)$ where $(q', q)$ is a cocycle term which has the form $(q''; q'^0) = -(\hbar^2/4m)\{q''; q'^0\}$. In fact it can be said that the essence of the EP is the cocycle condition

$$\{q''; q'^0\} = (\partial_{q''} q'^0)^2[(q''; q'^0) - (q'; q'^0)]$$

In addition FM developed a theory of $(x, \psi)$ duality (cf. [81]) which related the space coordinate and the wave function via a prepotential (free energy) in the form $\mathfrak{F} = (1/2)\psi \psi + iX/\epsilon$ for example. A number of interesting philosophical points arise (e.g. the emergence of space from the wave function) and we connected this to various features of dispersionless KdV in [44, 51] in a sort of extended WKB spirit. One should note here that although a form $\psi = Rexp(i\dot{W}/\hbar)$ is not generally appropriate it is correct when one is dealing with two independent solutions of the Schrödinger equation $\psi$ and $\bar{\psi}$ which are not proportional. In this context we utilized some interplay between various geometric properties of KdV which involve the Lax operator $L^2 = D_x^2 + V(x)$ and of course this is all related to Schwartzians, Virasoro algebras, and vector fields on $S^1$ (see e.g. [44, 55, 51, 52, 53]). Thus the simple presence of the Schrödinger equation (SE) in QM automatically incorporates a host of geometrical properties of $D_x = d/dx$ and the circle $S^1$. In fact since the FM theory exhibits the fundamental nature of the SE via its geometrical properties connected to the QSHJE one could speculate about trivializing QM (for 1-D) to a study of $S^1$ and $\partial_x$. 


We import here some comments based on [27] concerning the Klein-Gordon (KG) equation and the equivalence principle (EP) (details are in [27] and cf. also [72]). One starts with the relativistic classical Hamilton-Jacobi equation (RCHJE) with a potential $V(q,t)$ given as

$$\frac{1}{2m} \sum_{i=1}^{D} (\partial_i S_{\text{cl}}(q,t))^2 + \mathcal{M}_{\text{rel}}(q,t) = 0;$$

$$\mathcal{M}_{\text{rel}}(q,t) = \frac{1}{2mc^2} [m^2 c^4 - (V(q,t) + \partial_i S_{\text{cl}}(q,t))^2]$$

In the time-independent case one has $S_{\text{cl}}(q,t) = S_{0\text{cl}}(q) - Et$ and \( \mathcal{M}_{\text{rel}} \) becomes

$$\frac{1}{2m} \sum_{i=1}^{D} (\partial_i S_{0\text{cl}})^2 + \mathcal{M}_{\text{rel}} = 0; \quad \mathcal{M}_{\text{rel}}(q) = \frac{1}{2mc^2} [m^2 c^4 - (V(q) - E)^2]$$

In the latter case one can go through the same steps as in the nonrelativistic case and the relativistic quantum HJ equation (RQHJE) becomes

$$(1/2m)(\nabla S_{0\text{cl}})^2 + \mathcal{M}_{\text{rel}} - (\hbar^2/2m)(\Delta R/R) = 0; \quad \nabla \cdot (R^2\nabla S_{0\text{cl}}) = 0$$

these equations imply the stationary KG equation

$$-\hbar^2 c^2 \Delta \psi + (m^2 c^4 - V^2 + 2EV - E^2)\psi = 0$$

where $\psi = \text{Rexp}(iS_{0\text{cl}}/\hbar)$. Now in the time dependent case the $(D+1)$-dimensional RCHJE is $(\eta^{\mu\nu} = \text{diag}(-1,1,\cdots,1))$

$$\frac{(1/2m)\eta^{\mu\nu}\partial_\mu S_{\text{cl}} \partial_\nu S_{\text{cl}} + \mathcal{M}'_{\text{rel}}}{(\hbar^2/2m)} = 0;$$

$$\mathcal{M}'_{\text{rel}} = (1/2mc^2)[m^2 c^4 - V^2(q) - 2cV(q)\partial_0 S_{\text{cl}}(q)]$$

with $q = (ct, q_1, \cdots, q_D)$. Thus (3.3) has the same structure as (3.6) with Euclidean metric replaced by the Minkowskian one. We know how to implement the EP by adding $Q$ via $(1/2m)(\partial S)^2 + \mathcal{M}_{\text{rel}} + Q = 0$ (cf. [82] and remarks above). Note now that $\mathcal{M}_{\text{rel}}$ depends on $S_{\text{cl}}$ requires an identification

$$\mathcal{M}_{\text{rel}} = (1/2mc^2)[m^2 c^4 - V^2(q) - 2cV(q)\partial_0 S(q)]$$

(S replacing $S_{\text{cl}}$ and implementation of the EP requires that for an arbitrary $\mathfrak{m}_a$ state ($q \sim q^a$) one must have

$$\mathcal{M}_{\text{rel}}^b(q^b) = (p_0^b|p^a)\mathcal{M}_{\text{rel}}(p^a) + (q^a; q^b); \quad Q^b(q^b) = (p_0^b|p^a)Q(p^a) - (q^a; q^b)$$

where

$$(p_0^b|p) = [\eta^{\mu\nu}\rho_\mu^b\rho_\nu^p/\eta^{\mu\nu}\rho_\mu p_\nu] = p^T J_0 J^T p/p^T \eta p; \quad J_\mu = \partial q^\mu/\partial q^b$$

(J is a Jacobian and these formulas are the natural multidimensional generalization - see [27] for details). Furthermore there is a cocycle condition $(q^a; q^b) = (p_0^b|p^a)((q^a; q^b) - (q^a; q^b))$.

Next one shows that $\mathcal{M}_{\text{rel}} = (\hbar^2/2m)[\Box(\text{Rexp}(iS_{0\text{cl}}/\hbar))]/\text{Rexp}(iS_{0\text{cl}}/\hbar)])$ and hence the corresponding quantum potential is $Q_{\text{rel}} = -\hbar^2/2m)\Box R/R$. Then the RQHJE becomes

$$(1/2m)(\partial S)^2 + \mathcal{M}_{\text{rel}} + Q = 0 \text{ with } \partial \cdot (R^2 \partial S) = 0 \text{ (here } \Box R = \partial \partial R) \text{ and this reduces to the standard SE in the classical limit } c \rightarrow \infty \text{ (note } \partial \sim (\partial_0, \partial_1, \cdots, \partial_D) \text{ with } q_0 = ct, \text{ etc. - cf. [83])}. \text{ To see how the EP is simply implemented one considers the so called minimal prescription for an interaction with an electromagnetic four vector $A_\mu$. Thus set } P_\mu^A = p_\mu^A + eA_\mu \text{ where } p_\mu^A \text{ is a particle momentum and } P_\mu^A = \partial_\mu S_{\text{cl}}^A \text{ is the generalized momentum. Then the RCHJE reads as } (1/2m)(\partial S_{\text{cl}}^A - eA)^2 + (1/2)mc^2 = 0 \text{ where}$$
to get (1/2)m^2 and the critical case \( \mathfrak{M} = 0 \) corresponds to the limit situation where \( m = 0 \). One adds the standard Q correction for implementation of the EP to get \((1/2m)(\partial S - eA)^2 + (1/2)m^2 + Q = 0\) and there are transformation properties (here \((\partial S - eA)^2 \sim \sum (\partial_\mu S - eA_\mu)^2\))

\[
(3.13) \quad \mathfrak{M}(q^b) = (p^b[p^2] \mathfrak{M}^a(q^a) + (q^a; q^b); Q^b(p^b) = (p^b[p^2]Q^a(q^a) - (q^a; q^b))
\]

Here J is a Jacobian \( J^\mu = \partial q^\mu/\partial q^{b\nu} \) and this all implies the cocycle condition again. One finds now that (recall \( \partial \cdot (R^2(\partial S - eA)) = 0 \) - continuity equation)

\[
(3.14) \quad (\partial S - eA)^2 = \hbar^2 \left( \frac{\Box R}{R} - \frac{D^2(Re^{iS/h})}{Re^{iS/h}} \right); \quad D_\mu = \partial_\mu - \frac{i}{\hbar} eA_\mu
\]

and it follows that

\[
(3.15) \quad \mathfrak{M} = \frac{\hbar^2}{2m} \frac{D^2(Re^{iS/h})}{Re^{iS/h}}; \quad Q = \frac{\hbar^2}{2m} \frac{\Box R}{R}; \quad D^2 = \Box - \frac{2ieA\partial}{\hbar} - \frac{e^2A^2}{\hbar^2} - \frac{i\epsilon A}{\hbar}
\]

\[
(3.16) \quad (\partial S - eA)^2 + m^2c^2 - \hbar^2 \frac{\Box R}{R} = 0; \quad \partial \cdot (R^2(\partial S - eA)) = 0
\]

Note also that \( \mathfrak{M}_\text{rel} \) agrees with \((1/2m)(\partial S^\text{rel} - eA)^2 + (1/2)m^2 = 0\) after setting \( \mathfrak{M}_\text{rel} = mc^2/2 \) and replacing \( \partial_\mu S^\text{rel} \) by \( \partial_\mu S^\text{rel} - eA_\mu \). One can check that \( \mathfrak{M}_\text{rel} \) implies the KG equation \((ih\partial + eA)^2 \psi + m^2c^2 \psi = 0\) with \( \psi = Re^{iS/h} \).

**REMARK 3.1.** We extract now a remark about mass generation and the EP from \[19\]. Thus a special property of the EP is that it cannot be implemented in classical mechanics (CM) because of the fixed point corresponding to \( \mathfrak{M} = 0 \). One is forced to introduce a uniquely determined piece to the classical HJ equation (namely a quantum potential Q).

In the case of the RCHJE the fixed point \( \mathfrak{M}(q^b) = 0 \) corresponds to \( m = 0 \) and the EP then implies that all the other masses can be generated by a coordinate transformation. Consequently one concludes that masses correspond to the inhomogeneous term in the transformation properties of the \( \mathfrak{M}^b \) state, i.e. \((1/2)m^2c^2 = (q^b; q)\). Furthermore by \( \mathfrak{M}_\text{rel} \) masses are expressed in terms of the quantum potential \((1/2)m^2c^2 = (p^b[p^2]Q^0(q^0) - Q(q))\).

In particular in \[2\] the role of the quantum potential was seen as a sort of intrinsic self energy which is reminiscent of the relativistic self energy and this provides a more explicit evidence of such an interpretation.

**REMARK 3.2.** In a previous paper \[17\] (working with stationary states and \( \psi \) satisfying the Schrödinger equation \( (SE) \) \( -(\hbar^2/2m)\psi'' + V\psi = E\psi \)) we suggested that the notion of uncertainty in quantum mechanics (QM) could be phrased as incomplete information. The background theory here is taken to be the trajectory theory of Bertoldi-Faraggi-Matone-Floyd as above and the idea in \[17\] goes as follows. First recall that Floydian microstates satisfy a third order quantum stationary Hamilton-Jacobi equation (QSHJE)

\[
(3.17) \quad \frac{1}{2m}(S_0')^2 + \mathfrak{M}(q) + Q(q) = 0; \quad Q(q) = \frac{\hbar^2}{4m} \{S_0; q\}; \quad \mathfrak{M}(q) = -\frac{\hbar^2}{4m} \{exp(2iS_0/h); q\} \sim V(q) - E
\]

where \( \{f; q\} = (f''/f') - (3/2)(f''/f')^2 \) is the Schwarzian and \( S_0 \) is the Hamilton principle function. Also one recalls that the EP of Faraggi-Matone can only be implemented when
$S_0 \neq \text{const}$; thus consider $\psi = R e^{i S_0 / \hbar}$ with $Q = -\hbar^2 R'' / 2m R$ and $(R^2 S'_0)' = 0$ where $S'_0 = p$ and $m Q \dot{q} = p$ with $m_Q = m(1 - \partial_E Q)$ and $t \sim \partial_E S_0$. Thus microstates require three initial or boundary conditions in general to determine $S_0$ whereas the SE involves only two such conditions. Hence in dealing with the SE in the standard QM Hilbert space formulation one is not using complete information about the “particles” described by the wave function. One can then write e.g. $S_0$ it is shown in [82] that one has a general formula

$$e^{2i S_0(\delta)/\hbar} = e^{i \alpha w + i \ell \bar{w}}$$

(3.18)

where $w \sim \psi^D / \psi$ above and $\Omega = \psi' \psi^D - \psi(\psi^D)'$. Note $\psi$ and $\psi^D$ are linearly independent solutions of the SE and one can arrange that $\psi^D / \psi \in \mathbb{R}$ in describing any situation. Here $p$ is determined by the two constants in $\ell$ and has a form

$$p = \pm \hbar \Omega \ell$$

(3.19)

(where $w \sim \psi^D / \psi$ and $\Omega = \psi' \psi^D - \psi(\psi^D)'$). Now let $p$ be determined exactly with $p = p(q, E)$ via the Schrödinger equation and $S'_0$. Then $\dot{q} = (\partial_E p)^{-1}$ is also exact so $\Delta q = (\partial_E p)^{-1}(\tau) \Delta t$ for some $\tau$ with $0 \leq \tau \leq t$ is exact (up to knowledge of $\tau$). Thus given the wave function $\psi$ satisfying the stationary SE with two boundary conditions at $q = 0$ say to fix uniqueness, one can create a probability density $|\psi|^2(q, E)$ and the function $S'_0$. This determines $p$ uniquely and hence $\dot{q}$. The additional constant needed for $S_0$ appears in (3.18) and we can write $S_0 = S_0(\alpha, q, E)$ since from (3.18) one has

$$S_0 - (\hbar / 2) \alpha = -(i \hbar / 2) \log(\beta)$$

(3.20)

and $\beta = (w + i \ell) / (w - i \ell)$ with $w = \psi^D / \psi$ is to be considered as known via a determination of suitable $\psi, \psi^D$. Hence $\partial_\alpha S_0 = -\hbar / 2$ and consequently $\Delta S_0 = \partial_\alpha S_0 \delta \alpha = -(\hbar / 2) \Delta \alpha$ measures the indeterminacy in $S_0$.

Let us expand upon this as follows. Note first that the determination of constants necessary to fix $S_0$ from the QSHJE is not usually the same as that involved in fixing $\ell, \ell$ in (3.18). In particular differentiating in $q$ one gets

$$S'_0 = -\frac{i \hbar \beta'}{\beta}; \quad \beta' = -\frac{2i \ell \Omega w'}{(w - i \ell)^2}$$

(3.21)

Since $w' = -\Omega / \psi^2$ where $\Omega = \psi' \psi^D - \psi(\psi^D)'$ we get $\beta' = -2i i \ell \Omega / (\psi^D - i \ell \psi)^2$ and consequently

$$S'_0 = -\frac{\hbar \ell \Omega}{|\psi^D - i \ell \psi|^2}$$

(3.22)

which agrees with $p$ in (3.19) ($\pm \hbar$ simply indicates direction). We see that e.g. $S_0(x_0) = i \hbar \ell \Omega / |\psi^D(x_0) - i \ell \psi(x_0)|^2 = f(\ell_1, \ell_2, x_0)$ and $S''_0 = g(\ell_1, \ell_2, x_0)$ determine the relation between $(p(x_0), p'(x_0))$ and $(\ell_1, \ell_2)$ but they are generally different numbers. In any case, taking $\alpha$ to be the arbitrary unknown constant in the determination of $S_0$, we have $S_0 = S_0(q, E, \alpha)$ with $q = q(S_0, E, \alpha)$ and $t = t(S_0, E, \alpha) = \partial_E S_0$ (emergence of time from the wave function). One can then write e.g.

$$\Delta q = (\partial q / \partial S_0)(S_0, E, \alpha) \Delta S_0 = (1/p)(\dot{q}, E)\Delta S_0 = -(1/p)(\dot{q}, E)(\hbar / 2)\Delta \alpha$$

(3.23)
(for intermediate values $(\hat{S}_0, \hat{q})$) leading to

**THEOREM 3.1.** With $p$ determined uniquely by two “initial” conditions so that $\Delta p$ is determined and $q$ given via (3.15) we have from (3.18) the inequality $\Delta p \Delta q = O(\hbar)$ which resembles the Heisenberg uncertainty relation.

**COROLLARY 3.1.** Similarly $\Delta t = (\partial t/\partial \hat{S}_0)(\hat{S}_0, E, \alpha)\Delta \hat{S}_0$ for some intermediate value $\hat{S}_0$ and hence as before $\Delta E \Delta t = O(\hbar)$ ($\Delta E$ being precise).

Note that there is no physical argument here; one is simply looking at the number of conditions necessary to fix solutions of a differential equation. In fact (based on some correspondence with E. Floyd) it seems somewhat difficult to produce a physical argument. We refer also to Remark 3.1.2 for additional discussion.

**REMARK 3.3.** In order to get at the time dependent SE from the BFM (Bertoldi-Faraggi-Matone) theory we proceed as follows. From the previous discussion on the KG equation one sees that (dropping the A terms) in the time independent case one has $S^d(q, t) = S^d_0(q) - Et$

$$
(3.24) \quad (1/2m) \sum_1^D (\partial_k S^{cl}_k)^2 + \mathfrak{W}_{rel} = 0; \quad \mathfrak{W}_{rel}(q) = (1/2mc^2)[m^2c^4 - (V(q) - E)^2]
$$

leading to a stationary RQHJE

$$
(3.25) \quad (1/2m)(\nabla S_0)^2 + \mathfrak{W}_{rel} - (\hbar^2/2m)(\Delta R/R) = 0; \quad \nabla \cdot (R^2 \nabla S_0) = 0
$$

This implies also the stationary KG equation

$$
(3.26) \quad -\hbar^2 c^2 \Delta \psi + (m^2 c^4 - V^2 + 2VE - E^2)\psi = 0
$$

Now in the time dependent case one can write $(1/2m)\eta^{\mu\nu} \partial_\mu S^{cl} \partial_\nu S^{cl} + \mathfrak{W}_{rel} = 0$ where $\eta \sim diag(-1, 1, \cdots, 1)$ and

$$
(3.27) \quad \mathfrak{W}_{rel}(q) = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S^{cl}(q)]
$$

with $q \equiv (ct, q_1, \cdots, q_D)$. Thus we have the same structure as (3.1) with Euclidean metric replaced by a Minkowskian one. To implement the EP we have to modify the classical equation by adding a function to be determined, namely $(1/2m)(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0$ ($(\partial S)^2 \sim \sum (\partial_\mu S)^2$ etc.). Observe that since $\mathfrak{W}_{rel}$ depends on $S^{cl}$ we have to make the identification $\mathfrak{W}_{rel} = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S(q)]$ which differs from $\mathfrak{W}_{rel}$ since $S$ now appears instead of $S^{cl}$. Implementation of the EP requires that for an arbitrary $\mathfrak{W}^a$ state

$$
(3.28) \quad \mathfrak{W}_{rel}^b(q^b) = (p^b|p^a)\mathfrak{W}_{rel}^a(q^a) + (q^a; q^b); \quad Q^b(q^b) = (p^b|p^a)Q^{ab}(q^a) - (q^a; q^b)
$$

where now $(p^b|p) = \eta^{\mu\nu} p^b_\mu p^\nu_\eta / \eta^{\mu\nu} p_\mu p_\nu = p^T J_{\eta} J^T p/p^T \eta p$ and $J^{\mu}_\eta = \partial q^\mu / \partial (q^a)^\eta$. This leads to the cocycle condition $(q^a; q^b) = (p^b|q^b)[(q^a; q^b) - (q^a; q^b)]$ as before. Now consider the identity

$$
(3.29) \quad \alpha^2 (\partial S)^2 = \Box(Re\exp(\alpha S))/Re\exp(\alpha S) - (\Box R/R) - (\alpha \partial \cdot (R^2 \partial S)/R^2)
$$

and if $R$ satisfies the continuity equation $\partial \cdot (R^2 \partial S) = 0$ one sets $\alpha = i/\hbar$ to obtain

$$
(3.30) \quad \frac{1}{2m} (\partial S)^2 = -\frac{\hbar^2}{2m} (\Box Re^{iS/\hbar}) + \frac{\hbar^2}{2m} \Box R
$$
Then it is shown that $\mathcal{M}_{rel} = (h^2/2m)(\Box (\text{Re}xp(iS/h))/\text{Re}xp(iS/h))$ so there results $Q_{rel} = -(h^2/2m)(\Box R/R)$. Thus the RQHJE has the form (cf. (3.14) - (3.16))

\[
(3.31) \quad \frac{1}{2m}(\partial S)^2 + \mathcal{M}_{rel} - \frac{\hbar^2}{2m} \frac{\Box R}{R} = 0; \quad \partial \cdot (R^2 \partial S) = 0
\]

Now for the time dependent SE one takes the nonrelativistic limit of the RQHJE. For the classical limit one makes the usual substitution $S = S' = mc^2 t$ so as $c \to \infty \mathcal{M}_{rel} \to (1/2)mc^2 + V$ and $-(1/2m)(\partial S)^2 \to \partial_t S' - (1/2)mc^2$ with $\partial (R^2 \partial S) = 0 \to m\partial_t (R')^2 + \nabla \cdot (\partial (R')^2) = 0$. Therefore (removing the primes) (3.31) becomes $(1/2m)(\nabla S)^2 + V + \partial_t S - (\hbar^2/2m)(\Delta R/R) = 0$ with the time dependent nonrelativistic continuity equation being $m\partial_t R^2 + \nabla \cdot (R^2 \nabla S) = 0$. This leads then (for $\psi \sim \text{Re}xp(iS/h)$) to the SE

\[
(3.32) \quad i \hbar \partial_t \psi = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \psi
\]

One sees from all this that the BFM theory is profoundly governed by the equivalence principle and produces a usable framework for computation. It is surprising that it has not attracted more adherents.

### 3.2. **KLEIN GORDON À LA SANTAMATO.**

The derivation of the SE in [172] (treated in Section 1.1) was modified in [174] to a derivation of the Klein-Gordon (KG) equation via a somewhat different average action principle. Recall that the spacetime geometry in [171] was obtained from the average action principle to obtain Weyl connections with a gauge field $\phi_\mu$ (thus the geometry had a statistical origin). The Riemann scalar curvature $\bar{R}$ was then related to the Weyl scalar curvature $R$ via an equation

\[
(3.33) \quad R = \bar{R} - 3[(1/2)g^{\mu\nu} \phi_\mu \phi_\nu + (1/\sqrt{-g})\partial_\mu (\sqrt{-g}g^{\mu\nu} \phi_\nu)]
\]

Explicit reference to the underlying Weyl structure disappears in the resulting SE and we refer to Remark 1.7 for a few comments (cf. also [54] for an incisive review). We recall here from [156, 157, 158, 159] (cf. [42, 13, 54]) that in the conformal geometry the particles do not follow geodesics of the conformal metric alone; further the work in [156, 157, 158, 159] is absolutely fundamental in exhibiting a correct framework for general relativity via the conformal (Weyl) version. Summarizing from [171] and the second paper in [172] one can say that traditional QM is equivalent (in some sense) to classical statistical mechanics in Weyl spaces. The moral seems to be (loosely) that quantum mechanics in Riemannian spacetime is the same as classical statistical mechanics in a Weyl space. In particular one wants to establish that traditional QM, based on wave equations and ad hoc probability calculus is merely a convenient mathematical construction to overcome the complications arising from a nontrivial spacetime geometric structure. Here one works from first principles and includes gauge invariance (i.e. invariance with respect to an arbitrary choice of the spacetime calibration). The spacetime is supposed to be a generic 4-dimensional differential manifold with torsion free connections $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and a metric tensor $g_{\mu\nu}$ with signature $(+, -, -, -)$ (one takes $\hbar = c = 1$). Here the (restrictive) hypothesis of assuming a Weyl geometry from the beginning is released, both the particle motion and the spacetime geometric structure are derived from a single average action principle. A result of this approach is that the spacetime connections are forced to be integrable Weyl connections by the maximization principle.

The particle is supposed to undergo a motion in spacetime with deterministic trajectories and random initial conditions taken on an arbitrary spacelike 3-dimensional hypersurface; thus the theory describes a relativistic Gibbs ensemble of particles (cf. [152, 172] for all this
in detail and see also [54]). Both the particle motion and the spacetime connections can be obtained from the average stationary action principle

\[
(3.34) \quad \delta \left[ E \left( \int_{\tau_1}^{\tau_2} L(x(\tau), \dot{x}(\tau)) d\tau \right) \right] = 0
\]

This action integral must be parameter invariant, coordinate invariant, and gauge invariant. All of these requirements are met if \( L \) is positively homogeneous of the first degree in \( \dot{x}^\mu = dx^\mu/d\tau \) and transforms as a scalar of Weyl type \( w(L) = 0 \). The underlying probability measure must also be gauge invariant. A suitable Lagrangian is then

\[
(3.35) \quad L(x, dx) = (m^2 - (R/6))^{1/2} ds + A_\mu dx^\mu
\]

where \( ds = (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2} d\tau \) is the arc length and \( R \) is the space time scalar curvature; \( m \) is a parameterlike scalar field of Weyl type (or weight) \( w(m) = -(1/2) \). The factor 6 is essentially arbitrary and has been chosen for future convenience. The vector field \( A_\mu \) can be interpreted as a 4-potential due to an externally applied EM field and the curvature dependent factor in front of \( ds \) is an effective particle mass. This seems a bit ad hoc but some feeling for the nature of the Lagrangian can be obtained from Section 1.1 (cf. also [18]). The Lagrangian will be gauge invariant provided the \( A_\mu \) have Weyl type \( w(A_\mu) = 0 \).

Now one can split \( A_\mu \) into its gradient and divergence free parts \( A_\mu = \tilde{A}_\mu - \partial_\mu S \), with both \( S \) and \( \tilde{A}_\mu \) having Weyl type zero, and with \( \tilde{A}_\mu \) interpreted as and EM 4-potential in the Lorentz gauge. Due to the nature of the action principle regarding fixed endpoints in variation one notes that the average action principle is not invariant under EM gauge transformations \( A_\mu \rightarrow A_\mu + \partial_\mu S \); but one knows that QM is also not invariant under EM gauge transformations (cf. [7]) so there is no incompatibility with QM here.

Now the set of all spacetime trajectories accessible to the particle (the particle path space) may be obtained from (3.34) by performing the variation with respect to the particle trajectory with fixed metric tensor, connections, and an underlying probability measure. Thus (cf. [54, 95, 172]) the solution is given by the so-called Carathéodory complete figure associated with the Lagrangian

\[
(3.36) \quad \bar{L}(x, dx) = (m^2 - (R/6))^{1/2} ds + \tilde{A}_\mu dx^\mu
\]

(note this leads to the same equations as (3.35) since the Lagrangians differ by a total differential \( dS \)). The resulting complete figure is a geometric entity formed by a one parameter family of hypersurfaces \( S(x) = \text{const.} \) where \( S \) satisfies the HJ equation

\[
(3.37) \quad g^{\mu\nu}(\partial_\mu S - \tilde{A}_\mu)(\partial_\nu S - \tilde{A}_\nu) = m^2 - \frac{R}{6}
\]

and by a congruence of curves intersecting this family given by

\[
(3.38) \quad \frac{dx^\mu}{ds} = \frac{g^{\mu\nu}(\partial_\nu S - \tilde{A}_\nu)}{[g^{\rho\sigma}(\partial_\rho S - \tilde{A}_\rho)(\partial_\sigma S - \tilde{A}_\sigma)]^{1/2}}
\]

The congruence yields the actual particle path space and the underlying probability measure on the path space may be defined on an arbitrary 3-dimensional hypersurface intersecting all of the members of the congruence without tangencies (cf. [54]). The measure will be completely identified by its probability current density \( j^\mu \) (see [54, 172]). Moreover, since the measure is independent of the arbitrary choice of the hypersurface, \( j^\mu \) must be conserved, i.e. \( \partial_\mu j^\mu = 0 \). Since the trajectories are deterministically defined by (3.38), \( j^\mu \) must be parallel to the particle 4-velocity (3.38), and hence

\[
(3.39) \quad j^\mu = \rho \sqrt{-g} g^{\mu\nu}(\partial_\nu S - \tilde{A}_\nu)
\]
with some \( \rho > 0 \). Now gauge invariance of the underlying measure as well as of the complete figure requires that \( j^\mu \) transforms as a vector density of Weyl type \( w(j^\mu) = 0 \) and \( S \) as a scalar of Weyl type \( w(S) = 0 \). From (3.39) one sees then that \( \rho \) transforms as a scalar of Weyl type \( w(\rho) = -1 \) and \( \rho \) is called the scalar probability density of the particle random motion.

The actual spacetime affine connections are obtained from (3.34) by performing the variation with respect to the fields \( \Gamma^\lambda_{\mu\nu} \) for a fixed metric tensor, particle trajectory, and probability measure. It is expedient to transform the average action principle to the form of a 4-volume integral

\[
\delta \left[ \int_\Omega d^4x [(m^2 - (R/6))(g_{\mu\nu}j^\mu j^\nu)^{1/2} + A_\mu j^\mu] \right] = 0
\]

where \( \Omega \) is the spacetime region occupied by the congruence (3.38) and \( j^\mu \) is given by (3.39) (cf. [54, 172] for proofs). Since the connection fields \( \Gamma^\lambda_{\mu\nu} \) are contained only in the curvature term \( R \) the variational problem (3.40) can be further reduced to

\[
\delta \left[ \int_\Omega \rho R \sqrt{-g} d^4x \right] = 0
\]

(here the HJ equation (3.37) has been used). This states that the average spacetime curvature must be stationary under a variation of the fields \( \Gamma^\lambda_{\mu\nu} \) (principle of stationary average curvature). The extremal connections \( \Gamma^\lambda_{\mu\nu} \) arising from (3.41) are derived in [172] using standard field theory techniques and the result is

\[
\Gamma^\lambda_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu \\ \nu \end{array} \right\} \left[ \frac{1}{2}(\phi_\mu \delta^\lambda_\nu + \phi_\nu \delta^\lambda_\mu - g_{\mu\nu} g^{\lambda\rho} \phi_\rho) \right]; \quad \phi_\mu = \partial_\mu \log(\rho)
\]

This shows that the resulting connections are integrable Weyl connections with a gauge field \( \phi_\mu \) (cf. [171] and Sections 1.1-1.2). The HJ equation (3.37) and the continuity equation \( \partial_\mu j^\mu = 0 \) can be consolidated in a single complex equation for \( S \), namely

\[
e^{iS} g^{\mu\nu} (iD_\mu - \bar{A}_\mu)(iD_\nu - \bar{A}_\nu)e^{-iS} - (m^2 - (R/6)) = 0; \quad D_\mu \rho = 0
\]

Here \( D_\mu \) is (doubly covariant - i.e. gauge and coordinate invariant) Weyl derivative given by (cf. [18])

\[
D_\mu T_\beta^\alpha = \partial_\mu T_\beta^\alpha + \Gamma^\alpha_{\mu\epsilon} T_\epsilon^\beta - \Gamma^\epsilon_{\mu\beta} T_\epsilon^\alpha + w(T)\phi_\mu T_\beta^\alpha
\]

It is to be noted that the probability density (but not the rest mass) remains constant relative to \( D_\mu \). When written out (3.43) for a set of two coupled partial differential equations for \( \rho \) and \( S \). To any solution corresponds a particular random motion of the particle.

Next one notes that (3.43) can be cast in the familiar KG form, i.e.

\[
[(i/\sqrt{-g})\partial_\mu \sqrt{-g} - \bar{A}_\mu] g^{\mu\nu} (i\partial_\nu - \bar{A}_\nu)\psi - (m^2 - (\dot{R}/6))\psi = 0
\]

where \( \psi = \sqrt{\rho} \exp(-iS) \) and \( \dot{R} \) is the Riemannian scalar curvature built out of \( g_{\mu\nu} \) only. We have the (by now) familiar formula

\[
R = \dot{R} - 3[(1/2) g^{\mu\nu} \phi_\mu \phi_\nu + (1/\sqrt{-g}) \partial_\mu (\sqrt{-g} g^{\mu\nu} \phi_\nu)]
\]

According to point of view (A) above in the KG equation (3.45) any explicit reference to the underlying spacetime Weyl structure has disappeared; thus the Weyl structure is hidden in the KG theory. However we note that no physical meaning is attributed to \( \psi \) or to the KG equation. Rather the dynamical and statistical behavior of the particle, regarded as a
classical particle, is determined by (3.43), which, although completely equivalent to the KG
equation, is expressed in terms of quantities having a more direct physical interpretation.

**REMARK 3.4.** The formula (3.46) goes back to Weyl [198] and the connection of
matter to geometry arises from (3.42). The time variable is treated in a special manner here
related to a Gibbs ensemble and \( \rho > 0 \) is built into the theory.

3.3. **KLEIN GORDON VIA SCALE RELATIVITY.** In [40, 54] and Section 1.1 we
sketched a few developments in the theory of scale relativity. This is by no means the whole
story and we want to give a taste of some further main ideas while deriving the KG
equation in this context (cf. [3, 58, 64, 65, 66, 139, 140, 141, 142, 143, 144, 145]). A main idea here is
that the Schrödinger, Klein-Gordon, and Dirac equations are all geodesic equations in the
fractal framework. They have the form \( D^2/\sigma^2 = 0 \) where \( D/\sigma \) represents the appropriate
covariant derivative. The complex nature of the SE and KG equaton arises from a discrete
time symmetry breaking based on nondifferentiability. For the Dirac equation further dis-
crete symmetry breakings are needed on the spacetime variables in a biquaternionic context
(cf. here [58]). First we go back to [139, 140, 144] and sketch some of the fundamentals
of scale relativity. This is a very rich and beautiful theory extending in both spirit and
generality the relativity theory of Einstein (cf. also [57] for variations involving Clifford
theory). The basic idea here is that (following Einstein) the laws of nature apply whatever
the state of the system and hence the relevant variables can only be defined relative to other
states. Standard scale laws of power-law type correspond to Galilean scale laws and from
them one actually recovers quantum mechanics (QM) in a nondifferentiable space. The
quantum behavior is a manifestation of the fractal geometry of spacetime. In particular the
quantum potential is a manifestation of fractality in the same way as the Newton potential
is a manifestation of spacetime curvature. In this spirit one can also conjecture (cf. [144])
that this quantum potential may explain various dynamical effects presently attributed to
dark matter (cf. also [3]). Now for the KG equation via scale relativity the derivation in the
first paper of [58] seems the most concise and we follow that at first (cf. also [140]). All
of the elements of the approach for the SE remain valid in the motion relativistic case with
the time replaced by the proper time \( s \), as the curvilinear parameter along the geodesics.
Consider a small increment \( dX^\mu \) of a nondifferentiable four coordinate along one of the
geodesics of the fractal spacetime. One can decompose this in terms of a large scale part
\( L_S < dX^\mu > = dx^\mu = v_\mu ds \) and a fluctuation \( d\xi^\mu \) such that \( L_S < d\xi^\mu > = 0 \). One is led to
write the displacement along a geodesic of fractal dimension \( D = 2 \) via

\[
(3.47) \quad dX^\mu_\pm = d_\pm x^\mu + d\xi^\mu_\pm = v^\mu_\pm ds + u^\mu_\pm \sqrt{2D} ds^{1/2}
\]

Here \( u^\mu_\pm \) is a dimensionless fluctuation and the length scale \( 2D \) is introduced for dimensional
purposes. The large scale forward and backward derivatives \( d/ds_+ \) and \( d/ds_- \) are defined via

\[
(3.48) \quad \frac{d}{ds_\pm} f(s) = \lim_{s \to 0_\pm} \frac{L_S}{1} \left( \frac{f(s + \delta s) - f(s)}{\delta s} \right)
\]

Applied to \( x^\mu \) one obtains the forward and backward large scale four velocities of the form

\[
(3.49) \quad (d/dx_+) x^\mu(s) = v^\mu_+; \quad (d/dx_-) x^\mu = v^\mu_-
\]

Combining yields

\[
(3.50) \quad \frac{d'}{ds} = \frac{1}{2} \left( \frac{d}{ds_+} + \frac{d}{ds_-} \right) - \frac{i}{2} \left( \frac{d}{ds_+} - \frac{d}{ds_-} \right)
\]
\[ \mathcal{V}^\mu = \frac{d^\nu x^\mu}{ds} = V^\mu - i U^\mu = \frac{v_+^\mu + v_-^\mu}{2} - i \frac{v_+^\mu - v_-^\mu}{2} \]

For the fluctuations one has
\[ \overline{LS} < d\xi_\mu^a d\xi_\mu^b >= \mp 2 D \eta^\mu\nu ds \]

One chooses here (+,−,−,−) for the Minkowski signature for \( \eta^\mu\nu \) and there is a mild problem because the diffusion (Wiener) process makes sense only for positive definite metrics. Various solutions have been given and they are all basically equivalent, amounting to the transformation a Laplacian into a D'Alembertian. Thus the two forward and backward differentials of \( f(x,s) \) should be written as
\[ (df/ ds_+) = (\partial_+ + v_+^\mu \partial_\mu + D\partial^\mu \partial_\mu) f \]

One considers now only stationary functions \( f \), not depending explicitly on the proper time \( s \), so that the complex covariant derivative operator reduces to
\[ (df/ ds) = (V^\mu + i D\partial^\mu) \partial_\mu \]

Now assume that the large scale part of any mechanical system can be characterized by a complex action \( \mathcal{G} \) leading one to write
\[ \delta \mathcal{G} = -mc \delta \int_a^b ds = 0; \quad ds = \overline{LS} < dX^\mu dX_\mu > \]

This leads to \( \delta \mathcal{G} = -mc \int_a^b \mathcal{V}_\nu (\delta x^\nu) \) with \( \delta x^\nu = \overline{LS} < dX^\nu >. \) Integrating by parts yields
\[ \delta \mathcal{G} = -mc \delta x^\nu \bigg|_a^b + mc \int_a^b \delta x^\nu (d\mathcal{V}_\mu / ds) ds \]

To get the equations of motion one has to determine \( \delta \mathcal{G} = 0 \) between the same two points, i.e. at the limits \( (\delta x^\nu)_a = (\delta x^\nu)_b = 0 \). From \( \ref{eq:5.52} \) one obtains then a differential geodesic equation \( d\mathcal{V}/ ds = 0 \). One can also write the elementary variation of the action as a functional of the coordinates. So consider the point \( a \) as fixed so \( (\delta x^\nu)_a = 0 \) and consider \( b \) as variable. The only admissible solutions are those satisfying the equations of motion so the integral in \( \ref{eq:5.52} \) vanishes and writing \( (\delta x^\nu)_b \) as \( \delta x^\nu \) gives \( \delta \mathcal{G} = -mc \mathcal{V}_\nu \delta x^\nu \) (the minus sign comes from the choice of signature). The complex momentum is now
\[ \mathcal{P}_\nu = mc \mathcal{V}_\nu = -\partial_\nu \mathcal{G} \]

and the complex action completely characterizes the dynamical state of the particle. Hence introduce a wave function \( \psi = \exp(i\mathcal{G}/\mathcal{G}_0) \) and via \( \ref{eq:5.57} \) one gets
\[ \mathcal{V}_\nu = (i\mathcal{G}_0/mc) \partial_\nu \log(\psi) \]

Now for the scale relativistic prescription replace the derivative in \( d/ ds \) by its covariant expression \( d'/ ds \). Using \( \ref{eq:5.58} \) one transforms \( d\mathcal{V}/ ds = 0 \) into
\[ -\frac{\mathcal{G}_0}{m^2 c^2} \partial^\mu \log(\psi) \partial_\mu \partial_\nu \log(\psi) - \frac{\mathcal{G}_0 D}{mc} \partial^\mu \partial_\mu \log(\psi) = 0 \]

The choice \( \mathcal{G}_0 = h = 2mcD \) allows a simplification of \( \ref{eq:5.58} \) when one uses the identity
\[ \frac{1}{2} \left( \partial_\mu \partial^\mu \psi \right) = \left( \partial_\mu \log(\psi) + \frac{1}{2} \partial_\mu \right) \partial^\mu \log(\psi) \]

Dividing by \( D^2 \) one obtains the equation of motion for the free particle \( \partial^\nu [\partial^\mu \partial_\mu \psi / \psi] = 0 \). Therefore the KG equation (no electromagnetic field) is
\[ \partial^\nu \partial_\mu \psi + (m^2 c^2 / h^2) \psi = 0 \]
and this becomes an integral of motion of the free particle provided the integration constant is chosen in terms of a squared mass term $m^2 c^2 / \hbar^2$. Thus the quantum behavior described by this equation and the probabilistic interpretation given to $\psi$ is reduced here to the description of a free fall in a fractal spacetime, in analogy with Einstein’s general relativity. Moreover these equations are covariant since the relativistic quantum equation written in terms of $d'/ds$ has the same form as the equation of a relativistic macroscopic and free particle using $d/\ell$. One notes that the metric form of relativity, namely $V^\mu V_\mu = 1$ is not conserved in QM and it is shown in [155] that the free particle KG equation expressed in terms of $V$ leads to a new equality

$$V^\mu V_\mu + 2iD^\mu V_\mu = 1$$

In the scale relativistic framework this expression defines the metric that is induced by the internal scale structures of the fractal spacetime. In the absence of an electromagnetic field $V^\mu$ and $S$ are related by (3.50) which can be written as $V_\mu = -(1/mc)\partial_\mu S$ so (3.61) becomes

$$\partial^\mu S \partial_\mu S - 2imcD\partial^\mu \partial_\mu S = m^2 c^2$$

which is the new form taken by the Hamilton-Jacobi equation.

**REMARK 3.5.** We go back to [140, 155] now and repeat some of their steps in a perhaps more primitive but revealing form. Thus one omits the $\mathcal{L}_s$ notation and uses $\lambda \sim 2D$; equations (3.47) - (3.53) and (3.54) are the same and one writes now $\mathcal{S}/ds$ for $d'/ds$. Then $\mathcal{S}/ds = V^\mu \partial_\mu + (i\lambda/2)\partial^\mu \partial_\mu$ plays the role of a scale covariant derivative and one simply takes the equation of motion of a free relativistic quantum particle to be given as $(\mathcal{S}/ds)V^\nu = 0$, which can be interpreted as the equations of free motion in a fractal spacetime or as geodesic equations. In fact now $(\mathcal{S}/ds)V^\nu = 0$ leads directly to the KG equation upon writing $\psi = \exp(iS/mc\lambda)$ and $\Psi^\mu = -\partial^\mu S = mcV^\mu$ so that $i\mathcal{S} = mc\lambda \log(\psi)$ and $V^\mu = i\lambda \partial^\mu \log(\psi)$. Then

$$\mathcal{S}/ds \left( V^\mu \partial_\mu + \frac{i\lambda}{2} \partial^\mu \partial_\mu \right) \partial^\nu \log(\psi) = 0 = i\lambda \left( \frac{\partial^\mu \psi}{\psi} \partial_\mu + \frac{1}{2} \partial^\mu \partial_\mu \right) \partial^\nu \log(\psi)$$

Now some identities are given in [155] for aid in calculation here, namely

$$\frac{\partial^\mu \psi}{\psi} \frac{\partial^\nu \psi}{\psi} \partial_\mu \partial_\nu = \frac{\partial^\mu \psi}{\psi} \partial^\nu \left( \frac{\partial_\mu \psi}{\psi} \right) =$$

$$\frac{1}{2} \partial^\nu \left( \frac{\partial^\mu \psi}{\psi} \frac{\partial_\mu \psi}{\psi} \right); \partial_\mu \left( \frac{\partial^\mu \psi}{\psi} \frac{\partial_\mu \psi}{\psi} \right) + \frac{\partial^\mu \psi}{\psi} \frac{\partial_\mu \psi}{\psi} = \frac{\partial^\mu \partial_\mu \psi}{\psi^2}$$

The first term in the last equation of (3.63) is then $(1/2)[(\partial^\mu \psi/\psi)(\partial_\mu \psi/\psi)]$ and the second is

$$(1/2)\partial^\nu (\partial^\mu \partial_\mu \psi/\psi) = (1/2)\partial^\nu \partial_\mu \partial_\mu \log(\psi) =$$

$$(1/2)\partial^\nu (\partial^\mu \partial_\mu \psi/\psi) = (1/2)\partial^\nu \left( \frac{\partial^\mu \partial_\mu \psi}{\psi} - \frac{\partial^\mu \psi \partial_\mu \psi}{\psi^2} \right)$$

Combining we get $(1/2)\partial^\nu (\partial^\mu \partial_\mu \psi/\psi) = 0$ which integrates then to a KG equation

$$-(\hbar^2 / m^2 c^2) \partial^\mu \partial_\mu \psi = \psi$$

for suitable choice of integration constant (note $\hbar/mc$ is the Compton wave length).

Now in this context or above we refer back to Section 3.1 for example and write $Q = -\langle 1/2m \rangle (\Box R/R)$ (where $\hbar = c = 1$ for convenience here). Then recall $\psi = \exp(i\mathcal{S}/m\lambda)$ and
\[ \mathcal{P}_\mu = m\mathcal{Q}_\mu = -\partial_\mu \mathcal{S} \text{ with } i\mathcal{S} = m\lambda \log(\psi). \] Also \( \mathcal{Q}_\mu = -(1/m)\partial_\mu \mathcal{S} = i\lambda \partial_\mu \log(\psi) \) with \( \psi = R\exp(iS/m\lambda) \) so \( \log(\psi) = i\mathcal{S}/m\lambda = \log(R) + iS/m\lambda \), leading to

\[ (3.67) \quad \mathcal{Q}_\mu = i\lambda \partial_\mu \log(R) + (i/m\lambda)\partial_\mu S = -\frac{1}{m} \partial_\mu S + i\lambda \partial_\mu \log(R) = \mathcal{V}_\mu + i\mathcal{U}_\mu \]

Then \( \mathcal{D} = \partial^\mu \partial_\mu \) and \( \mathcal{U}_\mu = \lambda \partial_\mu \log(R) \) leads to

\[ (3.68) \quad \partial^\mu \mathcal{U}_\mu = \lambda \partial^\mu \partial_\mu \log(R) = \lambda \mathcal{D} \log(R) \]

Further \( \partial^\mu \partial_\mu \log(R) = (\partial^\mu \partial_\rho R/R) - (R_\rho R_\mu /R^2) \) so

\[ (3.69) \quad \mathcal{D} \log(R) = \partial^\mu \partial_\mu \log(R) = (\mathcal{D} R/R) - \left( \sum \frac{R_\mu R_\mu}{R^2} \right) = \]

\[ = (\mathcal{D} R/R) - \sum (\partial_\mu R/R)^2 = (\mathcal{D} R/R) - |U|^2 \]

for \( |U|^2 = \sum U_\mu^2 \). Hence via \( \lambda = 1/2m \) for example one has

\[ (3.70) \quad Q = -(1/2m)(\mathcal{D} R/R) = -\frac{1}{2m} \left[ |U|^2 + \frac{1}{\lambda} \mathcal{D} \log(R) \right] = \]

\[ = -\frac{1}{2m} \left[ |U|^2 + \frac{1}{\lambda} \partial^\mu \mathcal{U}_\mu \right] = -\frac{1}{2m} |U|^2 - \frac{1}{2} \text{div}(\mathcal{U}) \]

(cf. Proposition 1.1).

### 3.4. FIELD THEORETIC METHODS.

In trying to imagine particle trajectories of a fractal nature or in a fractal medium we are tempted to abandon (or rather relax) the particle idea and switch to quantum fields (QF). Let the fields sense the bumps and fractality; if one can think of fields as operator valued distributions for example then fractal supports for example are quite reasonable. There are other reasons of course since the notion of particle in quantum field theory (QFT) has a rather fuzzy nature anyway. Then of course there are problems with QFT itself (cf. [137]) as well as arguments that there is no first quantization (except perhaps in the Bohm theory - cf. [134]). Some aspects of particles arising from QF and QFT methods, especially in a Bohmian spirit are reviewed in [11, 53] and here we only briefly indicate one approach due to Nikolić for bosonic fields (cf. [134, 135, 136, 137, 138] (cf. also [37, 103, 104, 105, 106, 107] for other field aspects of KG). We refer also to [100, 107] for interesting philosophical discussion about particles and localized objects in a QFT. Many details are omitted and standard QFT techniques are assumed to be known (see e.g. [101]) and we will concentrate here on derivations of KG type equations. First note that the papers [136] are impressive in producing a local operator describing the particle density current for scalar and spinor fields in an arbitrary gravitational and electromagnetic background. This enables one to describe particles in a local, general covariant, and gauge invariant manner and this is reviewed in [54]. We follow here [135] concerning Bohmian particle trajectories in relativistic bosonic and fermionic QFT. First we recall that there is no objection to a Bohmian type theory for QFT and no contradiction to Bell’s theorems etc. (see e.g. [30, 75]). Without discussing philosophical aspects of such a theory we simply construct one following Nikolic. Thus consider first particle trajectories in relativistic QM and posit a real scalar field \( \phi(x) \) satisfying the Klein-Gordon equation in a Minkowski metric \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) written as \( (\partial^2 - \nabla^2 + m^2)\phi = 0. \) Let \( \psi = \phi^+ \) with \( \psi^* = \phi^- \) correspond to positive and negative frequency parts of \( \phi = \phi^+ + \phi^- \). The particle current is \( j_\mu = i\psi^* \partial_\mu \psi \) and \( N = \int d^4x j_0 \) is the positive definite number of particles (not the charge). This is most easily seen from the plane wave expansion \( \phi^+(x) = \int d^3k a(\kappa) \exp(-ikx)/\sqrt{(2\pi)^32\kappa} \) since then \( N = \int d^3k a^+(\kappa)a(\kappa) \) (see above and [7, 136] where it is shown that the particle current and
the decomposition $\phi = \phi^+ + \phi^-$ make sense even when a background gravitational field or some other potential is present). One can write also $j_\mu = i(\phi^+ \pi^+ - \phi^+ \pi^-)$ where $\pi = \pi^+ + \pi^-$ is the canonical momentum (cf. [103]). Alternatively $\phi$ may be interpreted not as a field containing an arbitrary number of particles but rather as a one particle wave function. Here we note that contrary to a field a wave function is not an observable and so doing we normalize $\phi$ here so that $N = 1$. The current $j_\mu$ is conserved via $\partial_\mu j^\mu = 0$ which implies that $N = \int d^3x j_0$ is also conserved, i.e. $dN/dt = 0$. In the causal interpretation one postulates that the particle has the trajectory determined by $dx^\mu/d\tau = j^\mu/2m\psi^*\psi$. The affine parameter $\tau$ can be eliminated by writing the trajectory equation as $dx/\!\!/dt = j(t, x)/j_0(t, x)$ where $t = x^0$, $x = (x^1, x^2, x^3)$ and $j = (j^1, j^2, j^3)$. By writing $\psi = R \exp(iS)$ where $R$, $S$ are real one arrives at a Hamilton-Jacobi (HJ) form $dx^\mu/d\tau = -(1/m)\partial^\mu S$ and the KG equation is equivalent to

$$\tag{3.71} \partial^\mu (R^2 \partial_\mu S) = 0; \quad \frac{(\partial^\mu S)(\partial_\mu S)}{2m} - \frac{m^2}{2} + Q = 0$$

Here $Q = -(1/2m)(\partial^\mu \partial_\mu R/R)$ is the quantum potential $(c = \hbar = 1)$. From the HJ form and (3.71) plus the identity $d/\!\!/dt = (dx^\mu/\!\!/dt)\partial_\mu$ one arrives at the equations of motion $m(d^2x^\mu/d\tau^2) = \partial^\mu Q$. A typical trajectory arising from $dx/\!\!/dt = j/j_0$ could be imagined as an S shaped curve in the $t-x$ plane (with $t$ horizontal) and cut with a vertical line through the middle of the S. The velocity may be superluminal and may move backwards in time (at points where $j_0 < 0$). There is no paradox with backwards in time motion since it is physically indistinguishable from a motion forwards with negative energy. One introduces a physical number of particles via $N_{phys} = \int d^3x|j_0|$. Contrary to $N = \int d^3x j_0$ the physical number of particles is not conserved. A pair of particles one with positive and the other with negative energy may be created or annihilated; this resembles the behavior of virtual particles in conventional QFT.

Now go to relativistic QFT where in the Heisenberg picture the Hermitian field operator $\hat{\phi}(x)$ satisfies

$$\tag{3.72} (\partial_0^2 - \nabla^2 + m^2)\hat{\phi} = J(\hat{\phi})$$

where $J$ is a nonlinear function describing the interaction. In the Schrödinger picture the time evolution is determined via the Schrödinger equation (SE) $H[\phi, -i\delta/\delta\phi][\Psi[\phi, t]] = i\partial_0 \Psi[\phi, t]$ where $\Psi$ is a functional with respect to $\phi(x)$ and a function of $t$. A normalized solution of this can be expanded as $\Psi[\phi, t] = \sum_\infty \tilde{\Psi}_n[\phi, t]$ where the $\tilde{\Psi}_n$ are unnormalized n-particle wave functionals and the analysis proceeds from there (cf. [103]). In the deBroglie-Bohm (dBB) interpretation the field $\phi(x)$ has a causal evolution determined by

$$\tag{3.73} (\partial_0^2 - \nabla^2 + m^2)\phi(x) = J(\phi(x)) - \left(\frac{\delta Q[\phi, t]}{\delta \phi(x)}\right)_{\phi(x) = \phi(x)}$$

where $Q$ is the quantum potential again. However the $n$ particles attributed to the wave function $\Psi_n$ also have causal trajectories determined by a generalization of $dx/\!\!/dt = j/j_0$ as

$$\tag{3.74} \frac{dx_{n, j}}{dt} = \left(\frac{\psi_n^*(x^{(n)}) \nabla \psi_n(x^{(n)})}{\psi_n^*(x^{(n)}) \nabla \psi_n(x^{(n)})}\right)$$
where the n-particle wave function is
\[ \psi_n(x^{(n)}, t) = \langle 0 | \hat{\phi}(t, \mathbf{x}_1) \ldots \hat{\phi}(t, \mathbf{x}_n) | \Psi \rangle \]

These n-particles have well defined trajectories even when the probability (in the conventional interpretation of QFT) of the experimental detection is equal to zero. In the dBB interpretation of QFT we can introduce a new causally evolving parameter \( e_n[\phi, t] \) defined as
\[ e_n[\phi, t] = |\tilde{\Psi}_n[\phi, t]|^2 \sum_{n'} |\tilde{\Psi}_{n'}[\phi, t]|^2 \]

The evolution of this parameter is determined by the evolution of \( \phi \) given via (3.73) and by the solution \( \Psi = \sum \tilde{\Psi} \) of the SE. This parameter might be interpreted as a probability that there are n particles in the system at time t if the field is equal (but not measured!) to be \( \phi(\mathbf{x}) \) at that time. However in the dBB theory one does not want a stochastic interpretation. Hence assume that \( e_n \) is an actual property of the particles guided by the wave function \( \psi_n \) and call it the effectivity of these n particles. This is a nonlocal hidden variable attributed to the particles and it is introduced to provide a deterministic description of the creation and destruction of particles (see [41, 54, 135] for more on this).

**Remark 3.6.** In [134] an analogous fermionic theory is developed but it is even more technical and we refer to [54] for a sketch.

**Remark 3.7.** In [138] one addresses the question of statistical transparency. Thus the probabilistic interpretation of the nonrelativistic SE does not work for the relativistic KG equation \( (\partial^\mu \partial_\mu + m^2)\psi = 0 \) (where \( x = (\mathbf{x}, t) \) and \( \hbar = c = 1 \)) since \( |\psi|^2 \) does not correspond to a probability density. There is a conserved current \( j^\mu = i\psi^* \leftarrow \partial^\mu \psi \) (where \( a \leftarrow \partial^\mu b = a\partial^\mu b - b\partial^\mu a \)) but the time component \( j^0 \) is not positive definite. In [134] the equations that determine the Bohmian trajectories of relativistic quantum particles described by many particle wave functions were written in a form requiring a preferred time coordinate. However a preferred Lorentz frame is not necessary (cf. [25]) and this is developed in [138] following [25] [135]. First note that as in [25] [135] it appears that particles may be superluminal and the principle of Lorentz covariance does not forbid superluminal velocities and conversely superluminal velocities do not lead to causal paradoxes (cf. [25] [138]). As noted in [25] the Lorentz-covariant Bohmian interpretation of the many particle KG equation is not statistically transparent. This means that the statistical distribution of particle positions cannot be calculated in a simple way from the wave function alone without the knowledge of particle trajectories. One knows that classical QM is statistically transparent of course and this perhaps helps to explain why Bohmian mechanics has not attracted more attention. However statistical transparency (ST) may not be a fundamental property of nature as suggested by looking at standard theories (cf. [138]) The upshot is that since statistical probabilities can be calculated via Bohmian trajectories that theory is more powerful than other interpretations of general QM and we refer to [138] for discussion on this, on the KG equation, and on Lorentz covariance.

**3.5. DeDONDER-WEYL AND KG.** We go here to a paper [137] which gives a manifestly covariant canonical method of field quantization based on the classical De Donder-Weyl formulation of field theory. The Bohmian formulation is not postulated for interpretational purposes here but derived from purely technical requirements, namely covariance and consistency with standard QM. It arises automatically as a part of the formalism...
S the local vector (3.78) are equivalent to the standard Euler-Lagrange (EL) equation and by introducing (3.79) (this does not play an important role in classical physics but is important here). That in deriving (3.81) it was necessary to use the space part of the equations of motion (3.79) is given by the Legendre transform where the scalar DeDonder-Weyl (DDW) Hamiltonian (not related to the energy density) is given by 

$$S \equiv \int d^4x \mathfrak{A}; \quad \mathfrak{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - V(\phi)$$

As usual one has

$$\pi^\mu = \frac{\partial \mathfrak{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi; \quad \partial_\mu \phi = \frac{\partial \delta}{\partial \pi^\mu}; \quad \partial_\mu \pi^\mu = -\frac{\partial \delta}{\partial \phi}$$

where the scalar DeDonder-Weyl (DDW) Hamiltonian (not related to the energy density) is given by the Legendre transform \(S(\pi^\mu, \phi) = \pi^\mu \partial_\mu \phi - \mathfrak{L} = (1/2)\pi^\mu \pi_\mu + V\). The equations (3.78) are equivalent to the standard Euler-Lagrange (EL) equations and by introducing the local vector \(S^\mu(\phi(x), x)\) the dynamics can also be described by the covariant DDW HJ equation and equations of motion

$$S(\partial S^\alpha, \phi) + \partial_\alpha S^\mu = 0; \quad \partial^\mu \phi = \pi^\mu = \frac{\partial S^\mu}{\partial \phi}$$

Note here \(\partial_\mu\) is the partial derivative acting only on the second argument of \(S^\mu(\phi(x), x)\); the corresponding total derivative is \(d_\mu = \partial_\mu + (\partial_\mu \phi)(\partial/\partial \phi)\). Further the first equation in (3.79) is a single equation for four quantities \(S^\mu\) so there is a lot of freedom in finding solutions. Nevertheless the theory is equivalent to other formulations of classical field theory. Now following [112] one considers the relation between the covariant HJ equation and the conventional HJ equation; the latter can be derived from the former as follows. Using (3.78), (3.79) takes the form \((1/2)\partial_\alpha S^\mu \partial_\alpha S^\mu + V + \partial_\mu S^\mu = 0\). Then using the equation of motion in (3.78) write the first term as

$$\frac{1}{2} \frac{\partial S_\mu}{\partial \phi} \frac{\partial S^\mu}{\partial \phi} = \frac{1}{2} \frac{\partial S_\mu}{\partial \phi} \frac{\partial S^\mu}{\partial \phi} + \frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi)$$

Similarly using (3.78) the last term is \(\partial_\mu S^\mu = \partial_\mu \phi^0 + d_i S^i - (\partial_i \phi)(\partial^i \phi)\). Now introduce the quantity \(S = \int d^3x S^0\) so that \([\partial S^0(\phi(x), x)/\partial \phi(x)] = [\delta S/\delta \phi(x, t)]/\delta \phi(x, t)\) where \(\delta/\partial \phi(x, t) \equiv [\delta/\partial \phi(x)]_{\phi(x) = \phi(x, t)}\) is the space functional derivative. Putting this together gives

$$\int d^3x \left[ \frac{1}{2} \left( \frac{\delta S}{\delta \phi(x, t)} \right)^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi) \right] + \partial_\mu S = 0$$

which corresponds to the standard noncovariant HJ equation. The time evolution of \(\phi(x, t)\) is given by \(\partial_t \phi(x, t) = \delta S/\delta \phi(x, t)\) which arises from the time component of (3.79). Note that in deriving (3.81) it was necessary to use the space part of the equations of motion (3.79) (this does not play an important role in classical physics but is important here).

Now for the Bohmian formulation look at the SE \(\hat{H} \Psi = i\hbar \partial_t \Psi\) where we write

$$\hat{H} = \int d^3x \left[ -\frac{\hbar^2}{2} \left( \frac{\delta}{\delta \phi(x)} \right)^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi) \right];$$
\[ \Psi(\phi(x), t) = \mathfrak{R}(\phi(x), t) \exp[i \mathfrak{S}(\phi(x), t)] / \hbar \]

Then the complex SE equation is equivalent to two real equations

\begin{equation}
(3.83) \quad \int d^3x \left[ \frac{1}{2} \left( \frac{\delta \mathfrak{S}}{\delta \phi(x)} \right)^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi) + Q \right] + \partial_t \mathfrak{S} = 0;
\end{equation}

\begin{equation}
(3.84) \quad \int d^3x \left[ \frac{\delta \mathfrak{R}}{\delta \phi(x)} \delta \phi(x) + J \right] + \partial_t \mathfrak{R} = 0; \quad Q = -\frac{\hbar^2}{2\mathfrak{R}} \frac{\delta^2 \mathfrak{R}}{\delta \phi^2(x)}; \quad J = \frac{\mathfrak{R}}{2} \frac{\delta^2 \mathfrak{S}}{\delta \phi^2(x)}
\end{equation}

The second equation is also equivalent to

\[ \partial_t \mathfrak{R}^2 \int d^3x \frac{\delta}{\delta \phi(x)} \left( \mathfrak{R}^2 \frac{\delta \mathfrak{S}}{\delta \phi(x)} \right) = 0 \]

and this exhibits the unitarity of the theory because it provides that the norm \( \int [d\phi(x)]^2 \Psi^* \Psi = \int [d\phi(x)]^2 \) does not depend on time. The quantity \( \mathfrak{R}^2(\phi(x), t) \) represents the probability density for fields to have the configuration \( \phi(x) \) at time \( t \). One can take \( (3.83) \) as the starting point for quantization of fields (note \( \exp(i\mathfrak{S}/\hbar) \) should be single valued). Equations \( (3.83) \) and \( (3.84) \) suggest a Bohmian interpretation with deterministic time evolution given via \( \partial_t \phi \). Remarkably the statistical predictions of this deterministic interpretation are equivalent to those of the conventional interpretation. All quantum uncertainties are a consequence of the ignorance of the actual initial field configuration \( \phi(x, t_0) \). The main reason for the consistency of this interpretation is the fact that \( (3.84) \) with \( \partial_t \phi \) as above represents the continuity equation which provides that the statistical distribution \( \rho(\phi(x), t) \) of field configurations \( \phi(x) \) is given by the quantum distribution \( \rho = \mathfrak{R}^2 \) at any time \( t \), provided that \( \rho \) is given by \( \mathfrak{R}^2 \) at some initial time. The initial distribution is arbitrary in principle but a quantum H theorem explains why the quantum distribution is the most probable (cf. [195]). Comparing \( (3.83) \) with \( (3.81) \), we see that the quantum field satisfies an equation similar to the classical one, with the addition of a term resulting from the nonlocal quantum potential \( Q \). The quantum equation of motion then turns out to be

\begin{equation}
(3.85) \quad \partial^\mu \partial_\mu \phi + \frac{\partial V(\phi)}{\partial \phi} + \frac{\delta \Omega}{\delta \phi(x; t)} = 0
\end{equation}

where \( \Omega = \int d^3x Q \). A priori perhaps the main unattractive feature of the Bohmian formulation appears to be the lack of covariance, i.e. a preferred Lorentz frame is needed and this can be remedied with the DDW presentation to follow.

Thus one wants a quantum substitute for the classical covariant DDW HJ equation

\[ (1/2) \partial_\mu S_\mu \partial_\phi S^\mu + V + \partial_\mu S^\mu = 0 \]

Define then the derivative

\begin{equation}
(3.86) \quad \frac{dA(\phi, x)}{d\phi(x)} = \int d^4x \frac{\delta A(\phi, x)}{\delta \phi(x)}
\end{equation}

where \( \delta / \delta \phi(x) \) is the spacetime functional derivative (not the space functional derivative used before in \( (3.81) \)). In particular if \( A(\phi, x) \) is a local functional, i.e. if \( A(\phi, x) = A(\phi(x), x) \) then

\begin{equation}
(3.87) \quad \frac{dA(\phi(x), x)}{d\phi(x)} = \int d^4x' \frac{\delta A(\phi(x'), x')}{\delta \phi(x)} = \frac{\partial A(\phi(x), x)}{\partial \phi(x)}
\end{equation}

Thus \( d/d\phi \) is a generalization of \( \partial / \partial \phi \) such that its action on nonlocal functionals is also well defined. An example of interest is a functional nonlocal in space but local in time so
that

\[
\frac{\delta A([\phi], x')}{\delta \phi(x)} = \frac{\delta A([\phi], x')}{\delta \phi(x, x')} \delta((x')^0 - x^0) \Rightarrow
\]

\[
\Rightarrow \frac{dA([\phi], x)}{d\phi(x)} = \frac{\delta}{\delta \phi(x, x')} \int d^3 x' A([\phi], x', x^0)
\]

Now the first equation in (3.79) and the equations of motion become

\[
(3.89) \quad \frac{1}{2} \frac{dS_\mu}{d\phi} \frac{dS^\mu}{d\phi} + V + \partial_\mu S^\mu = 0; \quad \partial^\mu \phi = \frac{dS^\mu}{d\phi}
\]

which is appropriate for the quantum modification. Next one proposes a method of quantization that combines the classical covariant canonical DDW formalism with the standard spacetime asymmetric canonical quantization of fields. The starting point is the relation between the noncovariant classical HJ equation (3.81) and its quantum analogue (3.88).

Suppressing the time dependence of the field in (3.79) we see that they differ only in the existence of the \(Q\) term in the quantum case. This suggests the following quantum analogue of the classical covariant equation (3.89)

\[
(3.90) \quad \frac{1}{2} \frac{dS_\mu}{d\phi} \frac{dS^\mu}{d\phi} + V + Q + \partial_\mu S^\mu = 0
\]

Here \(S^\mu = S^\mu([\phi], x)\) is a functional of \(\phi(x)\) so \(S^\mu\) at \(x\) may depend on the field \(\phi(x')\) at all points \(x'\). One can also allow for time nonlocalities (cf. [138]). Thus (3.91) is manifestly covariant provided that \(Q\) given by (3.88) can be written in a covariant form. The quantum equation (3.90) must be consistent with the conventional quantum equation (3.88); indeed by using a similar procedure to that used in showing that (3.79) implies (3.81) one can show that (3.90) implies (3.88) provided that some additional conditions are fulfilled. First \(S^0\) must be local in time so that (3.88) can be used. Second \(S^i\) must be completely local so that \(dS^i/d\phi = \partial S^i/d\phi\), which implies

\[
(3.91) \quad d_i S^i = \partial_i S^i + (\partial_i \phi) \frac{dS_i}{d\phi}
\]

However just as in the classical case in this procedure it is necessary to use the space part of the equations of motion (3.79). Therefore these classical equations of motion must be valid even in the quantum case. Since we want a covariant theory in which space and time play equal roles the validity of the space part of the (3.79) implies that its time part should also be valid. Consequently in the covariant quantum theory based on the DDW formalism one must require the validity of the second equation in (3.89). This requirement is nothing but a covariant version of the Bohmian equation of motion written for an arbitrarily nonlocal \(S^\mu\) (this clarifies and generalizes results in [118]). The next step is to find a covariant substitute for the second equation in (3.88). One introduces a vector \(R^\mu([\phi], x)\) which will generate a preferred foliation of spacetime such that the vector \(R^\mu\) is normal to the leaves of the foliation. Then define

\[
(3.92) \quad R([\phi], \Sigma) = \int_\Sigma d\Sigma_\mu R^\mu; \quad S([\phi], x) = \int_\Sigma d\Sigma_\mu S^\mu
\]

where \(\Sigma\) is a leaf (a 3-dimensional hypersurface) generated by \(R^\mu\). Hence the covariant version of \(\Psi = R exp(iS)\) is \(\Psi([\phi], \Sigma) = R([\phi], \Sigma) exp(iS([\phi], \Sigma)/\hbar)\). For \(R^\mu\) one postulates the equation

\[
(3.93) \quad \frac{dR^\mu}{d\phi} \frac{dS^\mu}{d\phi} + J + \partial_\mu R^\mu = 0
\]
In this way a preferred foliation emerges dynamically as a foliation generated by the solution \( R^i \) of the equations (3.93) and (3.90). Note that \( R^\mu \) does not play any role in classical physics so the existence of a preferred foliation is a purely quantum effect. Now the relation between (3.93) and (3.83) is obtained by assuming that nature has chosen a solution of the form \( R^\mu = (R^0, 0, 0, 0) \) where \( R^0 \) is local in time. Then integrating (3.93) over \( d^3x \) and assuming again that \( S^0 \) is local in time one obtains (3.83). Thus (3.93) is a covariant substitute for the second equation in (3.83). It remains to write covariant versions for \( Q \) and \( J \) and these are

\[
Q = -\frac{\hbar^2}{2\mathcal{R}} \frac{\delta^2 \mathcal{R}}{\delta \Sigma \phi^2(x)}; \quad J = \frac{\mathcal{R}}{2} \frac{\delta \mathcal{E}}{\delta \Sigma \phi^2(x)}
\]

where \( \delta / \delta \Sigma \phi(x) \) is a version of the space functional derivative in which \( \Sigma \) is generated by \( R^\mu \). Thus (3.93) and (3.90) with (3.94) represent a covariant substitute for the functional SE equivalent to (3.84). The covariant Bohmian equations (3.90) imply a covariant version of (3.85), namely

\[
\partial^\mu \partial_\mu \phi + \frac{\partial V}{\partial \phi} + \frac{dQ}{d\phi} = 0
\]

Since the last term can also be written as \( \delta (\int d^4xQ) / \delta \phi(x) \) the equation of motion (3.96) can be obtained by varying the quantum action

\[
A_Q = \int d^4x \mathcal{L}_Q = \int d^4x (\mathcal{L} - Q)
\]

Thus in summary the covariant canonical quantization of fields is given by equations (3.89), (3.90), (3.93), and (3.94). The conventional functional SE corresponds to a special class of solutions for which \( R^i = 0 \), \( S^i \) are local, while \( R^0 \) and \( S^0 \) are local in time. In [137] a multifield generalization is also spelled out, a toy model is considered, and applications to quantum gravity are treated. The main result is that a manifestly covariant method of field quantization based on the DDW formalism is developed which treats space and time on an equal footing. Unlike the conventional canonical quantization it is not formulated in terms of a single complex SE but in terms of two coupled real equations. The need for a Bohmian formulation emerges from the requirement that the covariant method should be consistent with the conventional noncovariant method. This suggests that Bohmian mechanics (BM) might be a part of the formalism without which the covariant quantum theory cannot be formulated consistently.

4. DIRAC WEYL GEOMETRY

A sketch of Dirac Weyl geometry following [71] was given in [42] in connection with deBroglie-Bohm theory in the spirit of the Tehran school (cf. [28, 29, 129, 130, 163, 164, 165, 166, 167, 168, 169, 170, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188]). We go now to [110, 111, 112, 113, 114, 115, 116, 117, 118] for a very brief discussion of versions of the Dirac Weyl theory involved in discussing magnetic monopoles, dark matter, quintessence, matter creation, etc. (see [54] for more in this direction). Thus go to [111] where in particular an integrable Weyl-Dirac theory is developed (the book [110] is a lovely exposition but the work in [111] is somewhat newer). Note, as remarked in [129] (where twistors are used), the integrable Weyl-Dirac geometry is desirable in order that the natural frequency of an atom at a point should not depend on the whole world line of the atom. The first paper in [111] is designed to investigate the integrable Weyl-Dirac (Int-W-D) geometry and its ability to create massive matter. For example in this theory a
spherically symmetric static geometric formation can be spatially confined and an exterior observer will recognize it as a massive entity. This may be either a fundamental particle or a cosmic black hole both confined by a Schwarzschild surface. Here we only summarize some basic features in order to establish notation, etc. and sketch the preliminary theory (referring to [54], and the work of Israelit and Rosen for many examples). Thus in the Weyl geometry one has a metric $g_{\mu\nu} = g_{\nu\mu}$ and a length connection vector $w_\mu$ along with an idea of Weyl gauge transformation (WGT)

\begin{equation}
(4.1) \quad g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{2\lambda} g_{\mu\nu}; \quad g^{\mu\nu} \rightarrow \tilde{g}^{\mu\nu} = e^{-2\lambda} g^{\mu\nu}
\end{equation}

where $\lambda(x^\mu)$ is an arbitrary differentiable function. One is interested in covariant quantities satisfying $\psi \rightarrow \tilde{\psi} = \exp(n\lambda)\psi$ where the Weyl power $n$ is described via $\pi(\psi) = n$, $\pi(g_{\mu\nu}) = 2$, and $\pi(g^{\mu\nu}) = -2$. If $n = 0$ the quantity $\psi$ is said to be gauge invariant (in-invariant).

Under parallel displacement one has length changes and for a vector

\begin{equation}
(4.2) \quad (i) \quad dB^\mu = -B^\sigma \Gamma^\mu_{\sigma\nu} dx^\nu; \quad (ii) \quad B = (B^\mu B^\nu g_{\mu\nu})^{1/2}; \quad (iii) \quad dB = Bw_\nu dx^\nu
\end{equation}

(note $\pi(B) = 1$). In order to have agreement between (i) and (iii) one requires

\begin{equation}
(4.3) \quad \Gamma^\lambda_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu \\ \nu \end{array} \right\} + g_{\mu\nu} w^\lambda - \delta^\lambda_\nu w_\mu - \delta^\lambda_\mu w_\nu
\end{equation}

where $\left\{ \begin{array}{c} \lambda \\ \mu \\ \nu \end{array} \right\}$ is the Christoffel symbol based on $g_{\mu\nu}$. In order for (iii) to hold in any gauge one must have the WGT $w_\mu \rightarrow \tilde{w}_\mu = w_\mu + \partial_\mu \lambda$ and if the vector $B^\mu$ is transported by parallel displacement around an infinitesimal closed parallelogram one finds

\begin{equation}
(4.4) \quad \Delta B^\lambda = B^\sigma K^\lambda_{\sigma\mu} dx^\mu \delta x^\nu; \quad \Delta B = BW_{\mu\nu} dx^\mu \delta x^\nu;
\end{equation}

\begin{equation}
K^\lambda_{\sigma\mu} = -\Gamma^\lambda_{\sigma\mu\nu} + \Gamma^\lambda_{\sigma\nu\mu} - \Gamma^\sigma_{\mu\nu} \Gamma^\lambda_{\sigma\nu} + \Gamma^\sigma_{\nu\mu} \Gamma^\lambda_{\sigma\nu}
\end{equation}

is the curvature tensor formed from (4.3) and $W_{\mu\nu} = w_{\mu,\nu} - w_{\nu,\mu}$. Equations for the WGT $w_{\mu\nu} \rightarrow \tilde{w}_{\mu\nu}$ and the definition of $W_{\mu\nu}$ led Weyl to identify $w_\mu$ with the potential vector and $W_{\mu\nu}$ with the EM field strength; he used a variational principle $\delta I = 0$ with $I = \int L\sqrt{-gd^4x}$ with $L$ built up from $K^\lambda_{\sigma\mu\nu}$ and $W_{\mu\nu}$. In order to have an action invariant under both coordinate transformations and WGT he was forced to use $R^2$ (R the Riemannian curvature scalar) and this led to the gravitational field.

Dirac revised this with a scalar field $\beta(x^\nu)$ which under WGT changes via $\beta \rightarrow \tilde{\beta} = e^{-\lambda}\beta$ (i.e. $\pi(\beta) = -1$). His in-invariant action integral is then $(f_{,\nu} \equiv \partial_{\nu} f)$

\begin{equation}
(4.5) \quad I = \int [W^{\lambda\alpha} W_{\lambda\sigma} - \beta^2 R + \beta^2 (k - 6)w^\sigma w_\sigma + 2(k - 6)\beta w^\sigma \beta_{,\sigma} + \sqrt{-g} d^4x]
\end{equation}

Here $k$ is a parameter, $\Lambda$ is the cosmological constant, $L_M$ is the Lagrangian density of matter, and an underlined index is to be raised with $g^{\mu\nu}$. Now according to (4.1) this is a nonintegrable geometry but there may be situations when geometric vector fields are ruled out by physical constraints (e.g. the FRW universe). In this case one can preserve the WD character of the spacetime by assuming that $w_\nu$ is the gradient of a scalar function $w$ so that $w_\nu = w_{,\nu} = \partial_\nu w$. One has then $W_{\mu\nu} = 0$ and from (4.4) results $\Delta B = 0$ yielding an integrable spacetime (Int-W-D spacetime). To develop this begin with (4.3) but with $w_\nu$ given by $w_\nu = \partial_\nu w$ so the first term in (4.4) vanishes. The parameter $k$ is not fixed.
and the dynamical variables are $g_{\mu\nu}$, $w$, and $\beta$. Further it is assumed that $L_M$ depends on $(g_{\mu\nu}, w, \beta)$. For convenience write
\begin{equation}
(4.6) \quad b_\mu = (\log(\beta))_{,\mu} = \beta_{,\mu}/\beta
\end{equation}
and use a modified Weyl connection vector $W_\mu = w_\mu + b_\mu$ which is a gauge invariant gradient vector. Write also $k - 6 = 16\pi\kappa$ and varying $w$ in (4.5) one gets a field equation
\begin{equation}
(4.7) \quad 2(\kappa\beta^2 W^\nu)_{,\mu} = S
\end{equation}
where the semicolon denotes covariant differentiation with the Christoffel symbols and $S$ is the Weylian scalar charge given by
\begin{equation}
(4.8) \quad G_\nu = -8\pi T^\nu_{\beta^2} + 16\pi\kappa \left( W^\nu W_\mu - \frac{1}{2} \delta^\nu_\mu W^\sigma W_\sigma \right) + 
\end{equation}
\begin{equation}
+ 2(\delta^\nu_\beta\sigma - b^\nu_\beta) + 2b^\nu b_\mu + \sigma^\nu_\mu - \delta^\nu_\beta \beta^2 A
\end{equation}
where $G_\nu$ represents the Einstein tensor and the EM density tensor of ordinary matter is
\begin{equation}
(4.9) \quad 8\pi\sqrt{-g}T^{\mu\nu} = \delta(\sqrt{-g}L_M)/\delta g_{\mu\nu}
\end{equation}
Finally the variation with respect to $\beta$ gives an equation for the $\beta$ field
\begin{equation}
(4.10) \quad R + k(b^\nu_\beta + b^\nu b_\beta) = 16\pi\kappa(w^\sigma w_\sigma - w^\sigma_\sigma) + 4\beta^2 A + 8\pi\beta^{-1} B
\end{equation}
Note in (4.10) $R$ is the Riemannian curvature scalar and the Dirac charge $B$ is a conjugate of the Dirac gauge function $\beta$, namely $16\pi B = \delta L_M/\delta \beta$.

By a simple procedure (cf. [71]) one can derive conservation laws; consider e.g. $I_M = \int L_M \sqrt{-g}d^4x$. This is an in-invariant so its variation due to coordinate transformation or WGT vanishes. Making use of $16\pi S = \delta L_M/\delta w$, (4.9), and $16\pi B = \delta L_M/\delta \beta$ one can write
\begin{equation}
(4.11) \quad \delta I_M = 8\pi \int (T^{\mu\nu} \delta g_{\mu\nu} + 2S\delta w + 2B\delta \beta) \sqrt{-g}d^4x
\end{equation}
Via $x^\mu \to \tilde{x}^\mu = x^\mu + \eta^\mu$ for an arbitrary infinitesimal vector $\eta^\mu$ one can write
\begin{equation}
(4.12) \quad \delta g_{\mu\nu} = g_{\lambda\nu} \eta^\lambda_{,\mu} + g_{\mu\lambda} \eta^\lambda_{,\nu}; \ \delta w = w_{,\nu}\eta^\nu; \ \delta \beta = \beta_{,\nu}\eta^\nu
\end{equation}
Taking into account $x^\mu \to \tilde{x}^\mu$ we have $\delta I_M = 0$ and making use of (4.12) one gets from (4.11) the energy momentum relations
\begin{equation}
(4.13) \quad T^\lambda_{\mu;\nu} - Sw_{\mu} - \beta Bb_\mu = 0
\end{equation}
Further considering a WGT with infinitesimal $\lambda(x^\mu)$ one has from (4.11) the equation $S + T - \beta B = 0$ with $T = T^\nu_\nu$. One can contract (4.13) and make use of (4.7) and $S + T = \beta B$ giving again (4.10), so that (4.10) is a corollary rather than an independent equation and one is free to choose the gauge function $\beta$ in accordance with the gauge covariant nature of the theory. Going back to the energy-momentum relations one inserts $S + T = \beta B$ into (4.13) to get $T^\lambda_{\mu;\nu} - Tb_\mu = SW_{\mu}$. Now go back to the field equation (4.8) and introduce the EM density tensor of the $W_\mu$ field
\begin{equation}
(4.14) \quad 8\pi\Theta^{\mu\nu} = 16\pi\kappa\beta^2[(1/2)g^{\mu\nu}W^\lambda W_\lambda - W^\mu W^\nu]
\end{equation}
Making use of (4.7) one can prove $\Theta^\lambda_{\mu;\nu} - \Theta_{,\mu} = -SW_{\mu}$ and using $T^\lambda_{\mu;\nu} - TB_\mu = SW_{\mu}$ one has an equation for the joint energy momentum density
\begin{equation}
(4.15) \quad (T^\lambda_{\mu;\nu} + \Theta^\lambda_{,\mu})_{,\nu} - (T + \Theta)b_\mu = 0
\end{equation}
One can derive now the equation of motion of a test particle (following [162]). Consider matter consisting of identical particles with rest mass m and Weyl scalar charge $q$, being in the stage of a pressureless gas so that the EM density tensor can be written $T^\mu\nu = \rho U^\mu U^\nu$ where $U^\mu$ is the 4-velocity and the scalar mass density $\rho$ is given by $\rho = m \rho_n$ with $\rho_n$ the particle density. Taking into account the conservation of particle number one obtains from $T^\mu_{\lambda\mu} - T\mu = SW_\mu$ the equation of motion

$$\frac{dU^\mu}{ds} + \left\{ \begin{array}{c} \mu \\ \lambda \\ \sigma \end{array} \right\} U^\lambda U^\sigma = \left( h_\lambda + \frac{q}{m} W_\lambda \right) \left( g^{\mu\lambda} - U^\mu U^\lambda \right)$$

In the Einstein gauge ($\beta = 1$) we are then left with

$$\frac{dU^\mu}{ds} + \left\{ \begin{array}{c} \mu \\ \lambda \\ \sigma \end{array} \right\} U^\lambda U^\sigma = \frac{q_s}{m} w_\lambda (g^{\mu\lambda} - U^\mu U^\lambda)$$

This gives a sketch of a powerful framework capable of treating many problems involving “matter” and geometry. Connections to Section 2 are obvious and we have supplied earlier additional relations to fluctuations via the quantum potential.

5. REMARKS ON QUANTUM GEOMETRY

We gave a “hands on” sketch of quantum geometry in [43] and refer to [10, 11, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 59, 60, 61, 62, 63, 88, 96, 97, 102, 109, 123, 127, 128, 133, 151, 154, 191, 192, 193, 201] for background and extensive theory. Here we follow [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 59, 60, 61, 62, 63, 88, 96, 97, 102, 109, 123, 127, 128, 133, 151, 154, 191, 192, 193, 201] and briefly extract from [43]. Roughly the idea is that for $H$ the Hilbert space of a quantum system there is a natural quantum geometry on the projective space $P(H)$ with inner product $\langle \phi, \psi \rangle = (1/2\hbar) g(\phi, \psi) + (i/2\hbar) \omega(\phi, \psi)$ where $g(\phi, \psi) = 2\hbar R(\phi|\psi)$ is the natural Fubini-Study (FS) metric and $g(\phi, \psi) = \omega(\phi, J\psi)$ ($J^2 = -1$). On the other hand the FS metric is proportional to the Fisher information metric of the form $Cos^{-1} \langle \phi, \psi \rangle$. Moreover (in 1-D for simplicity) $\tilde{\mathfrak{g}} \propto \int \rho Q dx$ is a functional form of Fisher information where $Q$ is the quantum potential and $\rho = |\psi|^2$. Finally one recalls that in a Riemannian flat spacetime (with quantum matter and Weyl geometry) the Weyl-Ricci scalar curvature is proportional to $Q$. Thus assume $H$ is separable with a complete orthonormal system $\{u_n\}$ and for any $\psi \in H$ denote by $[\psi]$ the ray generated by $\psi$ while $\eta_n = (u_n|\psi)$. Define for $k \in \mathbb{N}$

$$U_k = \{[\psi] \in P(H); \eta_k \neq 0\}; \phi_k : U_k \to \ell^2(C); \phi_k([\psi]) = \left( \frac{\eta_1}{\eta_k}, \ldots, \frac{\eta_{k-1}}{\eta_k}, \frac{\eta_{k+1}}{\eta_k}, \ldots \right)$$

where $\ell^2(C)$ denotes square summable functions. Evidently $P(H) = \bigcup_k U_k$ and $\phi_k \circ \phi_j^{-1}$ is biholomorphic. It is easily shown that the structure is independent of the choice of complete orthonormal system. The coordinates for $[\psi]$ relative to the chart $(U_k, \phi_k)$ are $\{z^k_n\}$ given via $z^k_n = (\eta_n/\eta_k)$ for $n < k$ and $z^k_n = (\eta_{n+1}/\eta_k)$ for $n \geq k$. To convert this to a real manifold one can use $z^k_n = (1/\sqrt{2}) (x^k_n + iy^k_n)$ with

$$\frac{\partial}{\partial z^k_n} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^k_n} + i \frac{\partial}{\partial y^k_n} \right); \frac{\partial}{\partial \bar{z}^k_n} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^k_n} - i \frac{\partial}{\partial y^k_n} \right)$$
etc. Instead of nondegeneracy as a criterion for a symplectic form inducing a bundle isomorphism between \( TM \) and \( T^*M \) one assumes here that a symplectic form on \( M \) is a closed 2-form which induces at each point \( p \in M \) a toplinear isomorphism between the tangent and cotangent spaces at \( p \). For \( P(H) \) one can do more than simply exhibit such a natural symplectic form; in fact one shows that \( P(H) \) is a Kähler manifold (meaning that the fundamental 2-form is closed). Thus one can choose a Hermitian metric \( \Phi = \sum g^k_{mn} dz_m^k \otimes d\bar{z}_n^k \) with

\[
(5.3) \quad g^k_{mn} = (1 + \sum_i z_i^k \bar{z}_i^k)^{-1} \delta_{mn} - (1 + \sum_i z_i^k \bar{z}_i^k)^{-2} \bar{z}_m^k \bar{z}_n^k 
\]

relative to the chart \( U_k, \phi_k \). The fundamental 2-form of the metric \( \Phi \) is \( \omega = i \sum_{m,n} g^k_{mn} dz_m^k \wedge d\bar{z}_n^k \) and to show that this is closed note that \( \omega = \partial \bar{\partial} f \) where locally \( f = \log (1 + \sum z_i^k \bar{z}_i^k) \) (the local Kähler function). Note here that \( \partial + \bar{\partial} = d \) and \( d^2 = 0 \) implies \( \partial^2 = \bar{\partial}^2 = 0 \) so \( d\omega = 0 \) and thus \( P(H) \) is a K manifold (cf. [128] for K geometry).

Now \( P(H) \) is the set of one dimensional subspaces or rays of \( H \); for every \( x \in H/\{0\}, [x] \) is the ray through \( x \). If \( H \) is the Hilbert space of a Schrödinger quantum system then \( H \) represents the pure states of the system and \( P(H) \) can be regarded as the state manifold (when provided with the differentiable structure below). One defines the K structure as follows. On \( P(H) \) one has an atlas \( \{(V_h, b_h, C_h)\} \) where \( h \in H \) with \( \|h\| = 1 \). Here \( (V_h, b_h, C_h) \) is the chart with domain \( V_h \) and local model the complex Hilbert space \( C_h \) where

\[
(5.4) \quad V_h = \{x \in P(H); (h|x) \neq 0\}; \quad C_h = [h]^{-1}; \quad b_h : V_h \to C_h; \quad [x] \to b_h([x]) = \frac{x}{(h|x) - h} \]

This produces a analytic manifold structure on \( P(H) \). As a real manifold one uses an atlas \( \{(V_h, R \circ b_h, RC_h)\} \) where e.g. \( R \circ b_h \) is the realification of \( C_h \) (the real Hilbert space with \( R \) instead of \( C \) as scalar field) and \( R : C_h \to RC_h ; \nu \to R \nu \) is the canonical bijection (note \( Rv \neq \bar{R}v \)). Now consider the form of the K metric relative to a chart \((V_h, R \circ b_h, RC_h)\) where the metric \( g \) is a smooth section of \( L_2(TP(H), R_c) \) with local expression \( g^h : RC_h \to L_2(RC_h, R) ; Rz \to g^h_{Rz} \) where

\[
(5.5) \quad g^h_{Rz}(Rv, Rw) = 2
\]

The fundamental form \( \omega \) is a section of \( L_2(TP(H), R_c) \), i.e. \( \omega^h : RC_h \to L_2(RC_h, R) ; Rz \to \omega^h_{Rz} \), given via

\[
(5.6) \quad \omega^h_{Rz}(Rv, Rw) = 2
\]

Then using e.g. [128] for the FS metric in \( P(H) \) consider a Schrödinger Hilbert space with dynamics determined via \( R \times P(H) \to P(H) : (t, [x]) \mapsto [\exp(-i/\hbar t)H]x \) where \( H \) is a (typically unbounded) self adjoint operator in \( H \). One thinks then of Kähler isomorphisms of \( P(H) \) (i.e. smooth diffeomorphisms \( \Phi : P(H) \to P(H) \) with the properties \( \Phi^*J = J \) and \( \Phi^*g = g \)). If \( U \) is any unitary operator on \( H \) the map \([x] \mapsto [Ux] \) is a K isomorphism of \( P(H) \). Conversely (cf. [42]) any K isomorphism of \( P(H) \) is induced by a unitary operator \( U \) (unique up to phase factor). Further for every self adjoint operator \( A \) in \( H \) (possibly unbounded) the family of maps \( (\Phi_t)_{t \in R} \) given via \( \Phi_t : [x] \mapsto [\exp(-itA)x] \) is a continuous one parameter group of K isomorphisms of \( P(H) \) and vice versa (every K isomorphism of \( P(H) \) is induced by a self adjoint operator where boundedness of \( A \) corresponds to smoothness of the \( \Phi_t \)). Thus in the present framework the dynamics of QM is described by a continuous one parameter group of K isomorphisms, which automatically are symplectic isomorphisms.
pure states via (note \( \psi \) | (5.10)

the natural metric on the manifold of Hilbert space \( \mathbb{R} \). Normalization requires \( ds \)

the one dimensional projectors | metric (5.7). The maximum (for optimal dis astounding) is given by the Hilbert space angle \( \cos^{-1}(| \psi \langle \psi > |) \) and the corresponding line element (PS \( \sim \) pure state)

(5.9) \[
\frac{1}{4} ds^2_{PS} = [\cos^{-1}(| \tilde{\psi} \langle \tilde{\psi} > |)]^2 \sim 1 - | \tilde{\psi} \langle \tilde{\psi} > |^2 = < \psi \langle \psi > | \psi \langle \psi > | \psi \langle \psi > |
\]

(called the Fubini-Study (FS) metric) is the natural metric on the manifold of Hilbert space rays. Here

(5.10) \[
| \psi \langle \psi > | = | \psi \langle \psi > | = \theta \text{ then } \cos(\theta) = | \psi \langle \psi > | \text{ and } \cos^2(\theta) = | \psi \langle \psi > |^2 = 1 - \sin^2(\theta) \sim 1 - \theta^2 \text{ for small } \theta. \text{ Hence } \theta^2 \sim 1 - \cos^2(\theta) = 1 - | \psi \langle \psi > |^2. \text{ The term in square brackets (the variance of phase changes) is nonnegative and an appropriate choice of basis makes it zero. In [33] one then goes on to discuss distance formulas in terms of density operators and Fisher information but we omit this here. Generally as in [201] one observes that the angle in Hilbert space is the only Riemannian metric on the set of rays which is invariant under unitary transformations. In any event } ds^2 = \sum (dp_i^2/p_i), \sum p_i = 1 \text{ is referred to as the Fisher metric (cf. [128]). Note in terms of } dp_i = \tilde{p}_i - p_i \text{ one can write } d\sqrt{p} = (1/2)dp/\sqrt{p} \text{ with } (d\sqrt{p})^2 = (1/4)(dp^2/p) \text{ and think of } \sum d\sqrt{p}^2 \text{ as a metric. Alternatively from } \cos^{-1}(| \psi \langle \psi > |) \text{ one obtains } ds_1^2 = 4\cos^{-1}(| \psi_1 \langle \psi_2 > |)^2 \text{ for two nearby distributions (involving N samples with probabilities } p_j, \tilde{p}_j). \text{ This can be generalized to quantum mechanical pure states via (note } \psi \sim \sqrt{\text{exp}(i\phi)} \text{ in a generic manner)

(5.8) \[
| \psi > = \sum \sqrt{p_j}e^{i\phi_j}|j >; | \tilde{\psi} > = | \psi > + |d\psi > = \sum \sqrt{p_j + dp_je^{i(\phi_j + d\phi_j)}}|j >
\]

Normalization requires \( R(< \psi|d\psi >) = -1/2 < d\psi|d\psi > \) and measurements described by the one dimensional projectors | [j > < j] can distinguish | \psi > and | \tilde{\psi} > according to the metric (5.7). The maximum (for optimal distinguishability) is given by the Hilbert space angle \( \cos^{-1}(| \psi \langle \psi > |) \)

and think of \( \sum (\sqrt{p_i}d\phi_i)^2 \). Here one thinks of the central limit theorem and a distance between probability distributions distinguished via a Gaussian \( \text{exp}[-(N/2)(\tilde{p}_j - p_j)^2/p_j] \) for two nearby distributions (involving N samples with probabilities \( p_j, \tilde{p}_j). \text{ This can be generalized to quantum mechanical pure states via (note } \psi \sim \sqrt{\text{exp}(i\phi)} \text{ in a generic manner)

One defines a (Riemann) metric (statistical distance) on the space of probability distributions \( \mathcal{P} \) of the form

(5.7) \[
ds^2_{PD} = \sum (dp_j^2/p_j) = \sum p_j(d\log(p_j))^2
\]

In any event \( \sim 1 \) for the structure defined by the fundamental form) and one has a Hamiltonian system.

We omit this here. Generally as in [201] one observes that the angle in Hilbert space is the only Riemannian metric on the set of rays which is invariant under unitary transformations. In any event \( ds^2 = \sum (dp_i^2/p_i), \sum p_i = 1 \) is referred to as the Fisher metric (cf. [128]). Note in terms of \( dp_i = \tilde{p}_i - p_i \) one can write \( d\sqrt{p} = (1/2)dp/\sqrt{p} \) with \( (d\sqrt{p})^2 = (1/4)(dp^2/p) \) and think of \( \sum d\sqrt{p}^2 \) as a metric. Alternatively from \( \cos^{-1}(| \psi \langle \psi > |) \) one obtains \( ds_1^2 = 4\cos^{-1}(| \psi_1 \langle \psi_2 > |)^2 \equiv 4(< \psi_1|d\psi > - < d\psi|\psi > < \psi|d\psi >) \) begins to look like a FS metric before passing to projective coordinates. In this direction we observe from [128] that the FS metric can be expressed also via

(5.9) \[
d\tilde{\psi} = |d\psi > - |\psi > < \psi|d\psi >
\]

is the projection of \( |d\psi > \) orthogonal to \( |\psi > \). Note that if \( \cos^{-1}(| \psi \langle \psi > |) = \theta \text{ then } \cos(\theta) = | \psi \langle \psi > | \text{ and } \cos^2(\theta) = | \psi \langle \psi > |^2 = 1 - \sin^2(\theta) \sim 1 - \theta^2 \text{ for small } \theta. \text{ Hence } \theta^2 \sim 1 - \cos^2(\theta) = 1 - | \psi \langle \psi > |^2. \text{ The term in square brackets (the variance of phase changes) is nonnegative and an appropriate choice of basis makes it zero. In [33] one then goes on to discuss distance formulas in terms of density operators and Fisher information but we omit this here. Generally as in [201] one observes that the angle in Hilbert space is the only Riemannian metric on the set of rays which is invariant under unitary transformations. In any event \( ds^2 = \sum (dp_i^2/p_i), \sum p_i = 1 \) is referred to as the Fisher metric (cf. [128]). Note in terms of \( dp_i = \tilde{p}_i - p_i \) one can write \( d\sqrt{p} = (1/2)dp/\sqrt{p} \) with \( (d\sqrt{p})^2 = (1/4)(dp^2/p) \) and think of \( \sum d\sqrt{p}^2 \) as a metric. Alternatively from \( \cos^{-1}(| \psi \langle \psi > |) \) one obtains \( ds_1^2 = 4\cos^{-1}(| \psi_1 \langle \psi_2 > |)^2 \equiv 4(< \psi_1|d\psi > - < d\psi|\psi > < \psi|d\psi >) \) begins to look like a FS metric before passing to projective coordinates. In this direction we observe from [128] that the FS metric can be expressed also via

(5.10) \[
d\tilde{\psi} = |d\psi > - |\psi > < \psi|d\psi >
\]

so for \( v \sim \sum v_i \partial_i + \bar{v}_i \partial_i \) and \( w \sim \sum w_i \partial_i + \bar{w}_i \partial_i \) and \( |z|^2 = 1 \) one has \( \phi(v, w) = (v|w) - (v|z)(z|w). \)

Now recall the material on Fisher information in Section 1.2 and the results on the SE in Weyl space in Section 1.1 to confirm the connection of quantum geometry as above to
Fisher information, Weyl curvature, and the quantum potential. Several features arise which deserve emphasis (cf. also [55])

• Philosophically the wave function seems to be inevitably associated to a cloud or ensemble (cf. Remarks 2.1 and 3.2). This provides meaning for \( \psi = R \exp(iS/\hbar) \) with \( R = \sqrt{\rho} \) and \( \rho = \psi^* \psi \) representing a probability density. Connections to hydrodynamics, diffusion, and kinetic theory are then natural and meaningful.

• From the ensemble point of view or by statistical derivations as in Section 1.1 one sees that spacetime geometry should also be conceived of in statistical terms at the quantum level. This is also connected with the relativistic theory and the quantum potential (in various forms) is exhibited as a fundamental ingredient of both QM and spacetime geometry.

• Bohmian type mechanics plays a fundamental role in providing unification of all these ideas. Similarly fractal considerations as in Nottale’s scale relativity lead to important formulas consistent with the pictures obtained via Bohmian mechanics and the quantum potential.

• Quantum geometry in a projective Hilbert space is connected to all these matters as indicated in this section.
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