Dynamical crossover from weak to infinite randomness in the one-dimensional transverse-field Ising model at criticality

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We study the critical dynamics of a many-body quantum system after a quantum quench between two quantum critical points of different universality classes. We achieve this by switching on weak disorder in a one-dimensional transverse-field Ising model initially prepared at its clean quantum critical point. We formulate a nonequilibrium dynamical renormalization group for the time-evolution operator that is capable of analytically capturing the full crossover from weak to infinite randomness. We analytically study signatures of localization both in real space and Fock space. We establish a general necessary criterion for ergodicity in Loschmidt echos.

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In noninteracting one-dimensional quantum systems the effect of arbitrarily weak randomness is substantial: all single-particle states become localized [1]. As a consequence, particles separated over distances larger than the localization length cannot exchange information and are therefore essentially unentangled. The localization dynamics for unentangled initial states in one-dimensional systems show universal behavior [2] that can be attributed to a dynamical renormalization-group fixed point [3] where localization not only happens in real but also in many-body Hilbert space [4, 5]. But how is information propagating within a disordered landscape when the system is highly entangled initially?

We study the localization dynamics out of quantum correlated states in the one-dimensional random transverse-field Ising model

\[ H = -\frac{J}{2} \sum_{i=1}^{N} [\sigma_i^x \sigma_{i+1}^x + h_i \sigma_i^z]. \]  

In equilibrium, the clean system with \( h_i = h \) shows a quantum phase transition at \( h = 1 \) separating a ferromagnetic (\( h < 1 \)) from a paramagnetic (\( h > 1 \)) phase [6]. According to the Harris criterion [9] the quantum critical point is unstable against weak disorder, and it has been shown that the system flows to an infinite-randomness fixed point instead [10].

In this paper, we develop a dynamical theory for this flow from weak to infinite randomness in nonequilibrium real-time evolution where progressing time itself drives this crossover. The quantum correlated state is initialized by preparing the system in the ground state \( |\psi_0\rangle \) of the clean critical model at \( h_i = h = 1 \). The localization dynamics is generated by switching on weak disorder suddenly inducing nonequilibrium real-time evolution that is formally solved by

\[ |\psi_0(t)\rangle = P(t) |\psi_0\rangle, \quad P(t) = e^{-iHt/h}. \]  

The distribution for the random fields \( h_i \) is chosen such that \( \langle \log(h_i) \rangle_{\text{dis}} = 0 \) with \( \langle \ldots \rangle_{\text{dis}} \) the disorder average. Thus the ground state of the system is located right at the infinite-randomness critical point [10] [11]. Contrary to typical condensed-matter systems where disorder is ubiquitous, systems of cold atoms in optical lattices are clean, and disorder has to be imposed, for example, by laser speckle patterns [12] providing an ideal candidate for the implementation of the anticipated nonequilibrium protocol. Moreover, the model in Eq. (1) can also be simulated within circuit QED [13] where disorder is also tunable [14].

For the analytical description of the quantum quench dynamics we present a novel nonequilibrium dynamical renormalization-group (ndRG) technique for the full time-evolution operator that is applicable to weakly perturbed nonequilibrium systems. In particular, it is capable of describing the dynamics also in cases when the initially weak perturbation flows to strong coupling as will be demonstrated below. It is based on the observation that energy resolution during nonequilibrium time evolution is limited. This is a consequence of the energy-time uncertainty relation [15]

\[ \Delta \varepsilon \Delta t \gtrsim \frac{\hbar}{2}, \]  

that expresses a fundamental limit onto the law of conservation of energy within a scattering process monitored over a time span \( \Delta t \) [16]. Thus, it is impossible to distinguish precisely energy-conserving processes from those with energy transfers \( \Delta \varepsilon \lesssim h/(2\Delta t) \) such that we will classify all processes with \( \Delta \varepsilon \lesssim h/(2\Delta t) \) as “resonant”.

In equilibrium, RG approaches constitute one of the central concepts for the theoretical description and understanding of many-body systems. One major challenge in the field of nonequilibrium physics is the development of appropriate out-of-equilibrium generalizations. Recently, a nonequilibrium RG for spin systems with strong
disorder has been formulated [4,5] that is applicable for initial states that carry little entanglement. For particular one-dimensional clean quantum systems RG techniques have been developed [17,18] that transfer the idea of scale separation to out-of-equilibrium dynamics.

**ndRG.**—During coarse-graining the ndRG scheme isolates resonant from offresonant processes. The classification as “resonant” is dictated by the energy-time uncertainty relation in Eq. (1) where \( \Delta \) must be identified with the time \( t \) of nonequilibrium real-time evolution, see Eq. (2). The offresonant processes are eliminated iteratively based on scale separation within a perturbative RG scheme. The resonant processes, nonperturbative in nature, cannot be eliminated in this way. We show, however, that their contribution to the dynamics is still analytically accessible using a general decoupling mechanism for the time-evolution operator even though the system might flow to strong coupling.

Consider a system whose Hamiltonian \( H_\Lambda = H_0^\Lambda + V_\Lambda \) at a given UV-cutoff \( \Lambda \) can be decomposed into an exactly solvable part \( H_0^\Lambda \) and a weak perturbation \( V_\Lambda \) with an associated time-evolution operator \( P_\Lambda = \exp[-iH_\Lambda t] \). Given the UV cutoff takes a value \( \Lambda + \Delta \) we now aim at reducing it to \( \Lambda \) by eliminating offresonant contributions \( V_{\Lambda+\Delta} \) of \( V_{\Lambda+\Delta} = V_{\Lambda+\Delta}^L + V_{\Lambda+\Delta}^R \) involving particles with momenta \( q \) in the shell \( |q| \in [\Lambda, \Lambda + \Delta] \).

The ndRG achieves this by constructing explicitly an infinitesimal unitary transformation yielding \( P_{\Lambda+\Delta} = e^{-S_\Lambda e^{S_\Lambda}} \). Notice that \( \Lambda \) could alternatively be chosen as an energy cutoff as opposed to momentum used here.

First, we pass over to an interaction picture \( P_{\Lambda+\Delta} = \exp[-iH^\Lambda t/\hbar] W_\Lambda(t) \) with respect to the renormalized free part \( H^\Lambda_0 \) after this RG step. We eliminate \( V_{\Lambda+\Delta} \) using an exact disentangling theorem for time-ordered exponential operators [19] \( W_\Lambda(t) = W_\Lambda^>(t) W_\Lambda^<(t) \) where \( W_\Lambda^>(t) = T \exp[-i \int_0^t dt' K_\Lambda(t')] \) with \( T \) the time-ordering prescription yielding \( W_\Lambda^>(t) = T \exp[-i \int_0^t dt'[W_\Lambda^>(t')][H_\Lambda^>(t') - K_\Lambda(t')][W_\Lambda^<(t')] \) for any \( K_\Lambda \). Up to this point all steps have been exact. The main approximation consists in making \( W_\Lambda^>(t) \) local in time through a Magnus expansion controlled by the strength of perturbation. Choosing \( K_\Lambda = V_{\Lambda+\Delta}^\Lambda + [V_{\Lambda+\Delta}^\Lambda, S_\Lambda]/2 \) one obtains second order accuracy in the perturbation \( W_\Lambda(t) = e^{-S_\Lambda e^{S_\Lambda}} \) with

\[
S_\Lambda(t) - S_\Lambda = i \int_0^t dt' V_{\Lambda+\Delta}^>(t').
\]

Switching back to the Heisenberg picture one obtains \( P_{\Lambda+\Delta} = e^{-S_\Lambda} P_\Lambda e^{S_\Lambda} \) with the renormalized Hamiltonian \( H_\Lambda = H_0^\Lambda + V_{\Lambda+\Delta}^\Lambda + \frac{1}{2}[S_\Lambda, V_{\Lambda+\Delta}^\Lambda] \) taking into account all contributions up to second order in the perturbation. Repeated iteration of this RG step leads to a time-evolution operator \( P(t) \) formally given by

\[
P(t) = U^*_\Lambda e^{-iH_\Lambda t/\hbar} U_\Lambda, \quad H_\Lambda = H_0^\Lambda + V_\Lambda^R.
\]

The unitary transformation \( U_\Lambda = T_\Lambda \exp[i\Lambda^\Lambda d\Lambda' S_{\Lambda'}] \) is a \( \Lambda \)-ordered exponential with \( \Lambda' \) the initial value of the UV-cutoff. Its final value \( \Lambda_\text{f} \) at the end of the RG transformation depends on the details of the resulting scaling equations as we will discuss below. By construction we have not eliminated all processes of the perturbation \( V \), but kept the resonant contributions \( V^R_\Lambda \) that still have to be accounted for. This seemingly complicated problem, however, can be simplified substantially because the processes in \( V^R_\Lambda \) are resonant as the renormalized time-evolution operator approximately factorizes:

\[
e^{-iH_\Lambda t/\hbar} e^{-iH^0_\Lambda t/\hbar} e^{-iV^R_\Lambda t/\hbar}.
\]

In the interaction picture the renormalized time-evolution operator obeys \( \exp[-iH_\Lambda t/\hbar] = \exp[-iH^0_\Lambda t/\hbar] T \exp[-i \int_0^t dt' V^R_\Lambda(t')/\hbar] \). As \( V^R_\Lambda \) contains the resonant processes with \( \Delta \epsilon < \hbar / (2t) \) we have that \( V^R_\Lambda(t) = \exp[iH^0_\Lambda t/\hbar] V^R_\Lambda \exp[-iH^0_\Lambda t/\hbar] \approx V^R_\Lambda \) is approximately independent of time leading directly to the factorization in Eq. (6).

In typical problems the complexity of \( H \) originates from the noncommutativity of \( H^0 \) and \( V \) while their individual properties are much easier to determine. The major advantage of Eq. (6) is the separation of the resonant processes of the perturbation from the dynamics of the (renormalized) unperturbed system whose individual time evolution can be determined much easier as will be demonstrated for the Ising model with disorder below.

**Ising Model.**—The transverse-field Ising model in Eq. (1) can be diagonalized exactly by a mapping to a free fermionic theory [8]. For the numerical implementation we parametrize \( h_1 = e^{x_1} \) with \( x_1 \in [-\delta, \delta] \) drawn from uncorrelated uniform distributions, i.e., \( (x_1|x_2) \text{dis} = \delta_{x_1} \delta_{x_2} / 3 \), yielding \( \langle \log(h_1) \rangle \text{dis} = 0 \). In the analytical treatment we use the parametrization \( g = \langle h_1 \rangle \text{dis} \) and \( g_1 = h_1 - \langle h_1 \rangle \text{dis} \). The strength of the disorder we characterize via the variance \( \sigma^2 = \langle g_1^2 \rangle \text{dis} \) in the weak-disorder limit this becomes \( \sigma^2 \rightarrow \delta^2 / 3 \). In the diagonal basis of the clean system the full model reads

\[
H = H^0 + V, \quad H^0 = \sum_k \varepsilon_k \gamma^+_k \gamma_k, \quad V = \sum_{kk'} \omega_{kk'} \gamma^+_k \gamma_{k'}, \quad \left[ m_{kk'} \gamma^+_k \gamma_k, \right] + \text{h.c.},
\]

with \( \gamma_k \) fermionic annihilation operators. The matrix elements in \( V \) are given by \( \omega_{kk'} = J g_{k-k'} \cos[\theta_k + \theta_{k'}] \) and \( m_{kk'} = -i J g_{k-k'} \sin[\theta_k - \theta_{k'}] / 2 \) where \( g_k = N^{-1} \sum_l e^{-ik_l g_l} \). The Bogoliubov angles \( \theta_0 \in [0, \pi/2] \) are determined by the equation \( \tan(2\theta_0) = \sin(\theta_k) / (g - \cos(k)) \) yielding the spectrum \( \varepsilon_k = J[(g - \cos k)^2 + \sin^2 k]^{1/2} \). In equilibrium this model has been solved in the weak-disorder limit analytically [20].

The influence of randomness onto the nonequilibrium dynamics of quantum many-body systems is drastic es-
especially for one-dimensional systems where local observables can show strong fluctuations [24]. Recent interest has also been on systems with a potential many-body localization transition [24, 25] manifesting, for example, in an unbounded logarithmic growth of entanglement [24–26] contrasting the integrable Ising case with a double-logarithmic entanglement growth [23].

In applying the ndRG scheme to the Ising model in Eq. (7), the disorder amplitude $\sigma$ takes the role of the perturbation strength. One obtains for the generator $S_\lambda$ of the unitary transformation, see Eq. (4):

$$S_\lambda = \sum_{kq} \left[ \frac{\omega_{kq}}{\varepsilon_k - E_q} \gamma_k \gamma_q + \frac{2m_{kq}}{\varepsilon_k + E_q} \gamma_k \gamma_q - \text{h.c.} \right]$$  \hspace{1cm} (8)

with the restricted sum defined as

$$\sum_{kq} = \sum_{|q| \in \Lambda, \Lambda + \Delta \Lambda} \sum_{|k| < \Lambda} \sum_{E_k - E_q > \Omega} \text{,}$$  \hspace{1cm} (9)

where we use the following notation: Energies $E_q$ denote the final renormalized ones whereas the longer wavelength energies $\varepsilon_k = \varepsilon_k(\Lambda)$ depend on the UV-cutoff. Additionally, we have introduced the scale $\Omega$ that is supposed to distinguish between resonant $|E_k - E_q| < \Omega$ and off-resonant processes $|E_k - E_q| > \Omega$, i.e., $\Omega \sim \hbar/t$. When physical quantities have been calculated we replace $\Omega$ by $\Omega = C_\hbar/t$ in the end with $C$ a nonuniversal constant. By keeping $\Omega$ instead of $\hbar/t$ we are able to identify whether some properties depend on the nonuniversal details of the RG cutoff and have therefore be interpreted with care.

Within the ndRG we identify three regimes. For times $t < \hbar/J \sigma$ the dynamics are accessible using time-dependent perturbation theory. On intermediate times $\hbar/J \sigma < t < \hbar/J \sigma^2$ the quasi-particles of the critical Ising system are renormalized, and the system acquires only weak corrections at low energies [15]. We stop the coarse-graining at a UV-cutoff $\Lambda_\epsilon$ with $\varepsilon_{\Lambda_\epsilon} = \Omega$ because the slow modes with $\varepsilon_k(\Lambda_\epsilon) < \Omega$ are essentially inert. The infinite-randomness character of the equilibrium critical point manifests on long times $t > \hbar/J \sigma^2$, revealing signatures of localization both in real and Fock space. The RG transformation stops when $\varepsilon_\Lambda \to J \sigma^2/2$ because the remaining modes are all resonant [15].

For the full time evolution of the system it is also necessary to determine the influence of the resonant processes in $V^\alpha$, see Eq. (4), establishing a criterion for the relevance of a perturbation which may be different from the equilibrium classification. We have that $V^\alpha = \sum_{k} \bar{g}_{kk'} c_k^\dag c_{k'}$, with $c_k = \cos(\theta_k)\gamma_k - i \sin(\theta_k)\gamma_k^\dag$, $\bar{g}_{kk'} = g_{k-k'}\Theta(\Omega - |\varepsilon_k - \varepsilon_{k'}|)$, and $\Theta(x)$ is the Heaviside step function. The diagonalization of the full $V^\alpha$ is difficult. Concentrating onto the low-energy degrees of freedom that are supposed to contribute dominantly at large times, analytical insight can be obtained by approximating $V^\alpha$ via $\bar{g}_{kk'} \approx g_{k-k'}\Theta(\Omega - J|k-k'|)$. Then, $V^\alpha$ can be diagonalized analytically. The probability distribution $P(g, \Omega)$ of its random diagonal elements $g$ is

$$P(g, \Omega) = \sqrt{\frac{\pi J}{2\sigma \Omega^2}} e^{-\sigma^2 J/(4\sigma^2 \Omega)}.$$  \hspace{1cm} (10)

Although $P(g, \Omega)$ becomes increasingly narrow on large times as $\Omega \sim \hbar/t$ this is not true for the product $gt$. Consequently, for times $t > \hbar/J \sigma^2$ the resonant processes become relevant. Note that we have neglected the additional disorder matrix elements at low energies as we have stopped the scaling equations at $\varepsilon_k = J \sigma^2$. Their corrections to $P(g, \Omega)$ are of the order $O(\sigma^4)$ and thus $O(\sigma^2)$ smaller than those contributing to Eq. (10).

Fock space and ergodicity.- Unlike in classical systems a general understanding of ergodicity for quantum many-body systems has not yet been achieved [24]. Recently, however, a framework addressing this fundamental problem has been proposed, many-body localization [6, 7], where the transition from ergodic to nonergodic is associated with an Anderson localization transition in Fock space. In the following, we propose a necessary and general criterion for Fock-space delocalization and thus quantum ergodicity which involves the long-time behavior of the Loschmidt echo

$$\mathcal{L}(t) = \left| \langle \psi_0 | e^{-iHt} | \psi_0 \rangle \right|^2 .$$  \hspace{1cm} (11)

Its long-time average $\tilde{\mathcal{L}} = \lim_{T \to \infty} T^{-1} \int_0^T dt \mathcal{L}(t)$ can be bounded by the inverse participation ratio (IPR) via $\tilde{\mathcal{L}} \geq \sum_n |\langle \psi_0 | \phi_n \rangle|^4 = \text{IPR}^{-1}$ with $\langle \psi_0 | \phi_n \rangle$ a complete basis set of the final Hamiltonian $H$ [25]. As $\mathcal{L}(t)$ approaches a constant for $t \to \infty$ (provided that $N \to \infty$ is taken before or that the disorder average has been performed), see Fig. 1 we have that $\tilde{\mathcal{L}} = 1$. Moreover, in the imaginary time axis $\tilde{\mathcal{L}}(t)$ is connected to the fidelity $F = |\langle \psi_0 | \phi_n \rangle|^4$ via $\tilde{\mathcal{L}}(\tau = i\tau) \to F^\tau$ for $\tau \to \infty$. If $\tilde{\mathcal{L}}(t)$ supports analytic continuation one thus obtains that $\mathcal{L}(t \to \infty) = F^\tau$ implying that $|\langle \psi_0 | \phi_n \rangle|^4 \geq \sum_{n \neq 0} |\langle \psi_0 | \phi_n \rangle|^4$ (such that $\text{IPR}^{-1} = |\langle \psi_0 | \phi_n \rangle|^4$ in the large-deviation theory sense) and the system is not ergodic. Thus, ergodicity can emerge only in case when $\mathcal{L}(t \to \infty) \neq F^\tau$, i.e., $\mathcal{L}(t)$ is nonanalytic in the complex time plane.

For global quenches that affect an extensive part of the system the Loschmidt echo shows a large-deviation scaling $\mathcal{L}(t) = \exp[-N \lambda(t)]$ [26]. The rate function $\lambda(t)$ can be obtained on the basis of a cumulant expansion [27] yielding for the typical values $\lambda(t) = \langle \lambda(t) \rangle_{\text{dis}}$ at times $t \gg \hbar/J \sigma$: [15]

$$\lambda(t) = -\frac{\sigma^2}{\pi^2 \lambda_\epsilon^2} \int_0^\pi \int_0^{2\pi} \frac{dk dq \cos(|\varepsilon_k + \varepsilon_q| t/\hbar J \sigma^2) - 1}{(\varepsilon_q + \varepsilon_k)^2}.$$  \hspace{1cm} (12)

with $T_{kq} = 1 - \cos(2\theta_k) \cos(2\theta_q)$ and $\lambda_\epsilon$ the final value of the UV-cutoff. The influence of the resonant contributions is contained in the function $a(t)$ that has the
property $a(t) \to 1$ for $h/J\sigma \ll t \ll h/J\sigma^2$ and $a(t) \to 0$ for $t \gg h/J\sigma^2$. Using Eq. (10) the functional form of $a(t)$ can be estimated as $a(t) = \exp[-\sigma^2 J^2 t^2/h^2]$ decaying exponentially as a function of time. As the dynamics during the crossover at a time scale $t \sim h/J\sigma^2$ depends on the details of the RG cutoff function via $\Omega$ we expect that this crossover cannot be described quantitatively. Fig. 1 shows a comparison of the result in Eq. (12) with the exact numerics obtained by the methods outlined in Ref. 28 showing very good agreement. At intermediate times $h/J\sigma \ll t \ll h/J\sigma^2$ the rate function $\lambda(t)$ satisfies to a constant value $\lambda_\infty = \lambda(t \to \infty) \sim (\sigma/\pi)^2 \log(\sigma^2 D)/\pi^2$ that fulfills $\lambda_\infty = 4\tilde{F}$, see Fig. 1(a) with $f$ given by $F = \exp[-Nf]$. Thus, the system is not ergodic and shows localization in Fock space.

Real space.- Importantly, Fock-space localization also manifests in localization in real space. While in clean systems local correlations decay in time this is not the case in localized systems 29. Signatures for retaining local memory are contained in the long-time behavior of the autocorrelation function 22

$$\chi(t) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i^x(t) \sigma_i^x \rangle_c \right\rangle_{\text{dis}}$$

(13)

where $\langle \sigma_i^x(t) \sigma_i^x \rangle_c = \langle \sigma_i^x(t) \sigma_i^x \rangle - \langle \sigma_i^x(t) \rangle \langle \sigma_i^x \rangle$ denotes the cumulant and $\langle \ldots \rangle = \langle \psi_0 | \ldots | \psi_0 \rangle$.

For times $h/J\sigma < t < h/J\sigma^2$ the autocorrelator $\chi(t)$ shows an oscillatory decay that can be estimated on the basis of the ndRG scheme as:

$$\chi(t) = e^{-2i\pi/4} \left[ \frac{4\sigma^2}{(4\pi J^2 t/h)^{3/2}} + \frac{\sigma^2}{\pi} \left( \frac{e^{i\pi/4}}{\sqrt{4\pi J^2 t/h}} - \frac{1}{4} \right) \right]$$

(14)

A comparison of the numerical and analytical result is shown in Fig. 2. Notice the nondecaying oscillating contribution which persists up to times $t \sim h/J\sigma^2$ beyond which $\chi(t)$ approaches a nonzero value. Thus, initially localized information is trapped within a finite region of space reflecting the localized nature of the underlying infinite-randomness fixed point.

Conclusion and Outlook.- We have introduced a RG method designed for the analytical description of the nonequilibrium dynamics in weakly perturbed integrable systems. We applied this RG scheme to the transverse-field Ising model to study the localization dynamics in consequence of a quantum quench from the clean to the infinite-randomness critical point. We identified a time scale $h/J\sigma^2$ beyond which disorder becomes relevant and the system shows features of localization both in real as well as Fock space. Initial local information stays trapped within a finite spatial region, and propagation in Fock space is highly constrained preventing the system from being ergodic. This manifests in the violation of a general necessary criterion for ergodicity that we identified in the long-time behavior of Loschmidt echoes.

Due to this criterion, ergodicity can only emerge when the Loschmidt echo exhibits a nonanalytic behavior in the complex time plane hinting towards a potential connection between ergodicity and the recently discovered dynamical phase transitions 26.

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Supplemental Material to

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A: ENERGY-TIME UNCERTAINTY RELATION

The energy-time uncertainty relation
\[ \Delta \varepsilon \Delta t \gtrsim \frac{\hbar}{2} \]  
(1)
differs from the uncertainty relations of conjugate operators such as coordinate and momentum
\[ \Delta x \Delta p \geq \frac{\hbar}{2} \]  
(2)
in fundamental ways as already indicated by the use of \( \gtrsim \) instead of \( \geq \). This has led and still leads to intense discussions in the literature [30]. In this appendix we briefly outline the conceptual differences and in the end clarify the precise meaning of the energy-time uncertainty relation used in the main text.

The special role of the energy-time uncertainty relation becomes apparent immediately when recognizing that time itself is a parameter of the dynamics but not an operator. As a consequence, the derivation of this uncertainty relation cannot foot on the same basis as found in textbooks for Eq. (2). Although it is possible to construct time measuring clock operators, a general and generic prescription of how to construct the appropriate energy-time uncertainty relation is still an open question [31]. It is, however, possible to derive an uncertainty relation of the type of Eq. (1) with the same mathematical rigour as for Eq. (2) when identifying \( \Delta t \) not with some time duration but rather with a characteristic time scale \( \Delta t_A \) of some arbitrary observable \( A \) [32]:
\[ \Delta t_A = \left| \frac{d(A(t))}{dt} \right| \]  
(3)
yielding
\[ \Delta \varepsilon \Delta t_A \geq \frac{\hbar}{2} \]  
(4)
In this way, the uncertainty relation gives a bound on the decay rate of an observable \( A \) in terms of its standard deviation and the spread in energies.

From a different perspective an energy-time uncertainty relation emerges when studying the influence of perturbations on the system of interest. This concerns, for example, the fundamental question of whether it is, in principle, possible to verify the law of conservation of energy for a quantum system in an experiment [16]. It turns out that this is not possible as a consequence of an energy-time uncertainty relation of the type in Eq. (1). For an illustration it is suitable to discuss first-order time-dependent perturbation theory according to which the probability for a transition from an initial state labeled by \( i \) to a final state \( f \) is given by [16]
\[ p_{i \rightarrow f} = 4|V_{if}|^2 \frac{\sin^2 \left[ (\varepsilon_i - \varepsilon_f)t/(2\hbar) \right]}{(\varepsilon_i - \varepsilon_f)^2} \]  
(5)
when the perturbation with matrix element \( V_{if} \) acts over a time \( t \). For any nonzero \( t \), the most probable energy measurement outcomes will be spread over a width \( \Delta \varepsilon \sim \hbar/t \). This, of course, does not mean that it is not possible to measure the energy with arbitrary precision. Instead, when the experiment is repeated many times the probability distribution for the final energy will show a width of the order \( \hbar/t \) such that
\[ \Delta \varepsilon \Delta t \gtrsim \frac{\hbar}{2} \]  
(6)
which precisely is Eq. (1) when identifying \( t \) with \( \Delta t \). The law of conservation of energy will be recovered only in the limit \( t \to \infty \) where \( p_{i \rightarrow f} \) reduces to Fermi’s golden rule.

Concluding, in presence of a perturbation that acts over a time-interval \( t \) and that induces transitions in the unperturbed eigenstates there is an energy-time uncertainty relation characterizing the spreading in energy of the unperturbed system. This is precisely the notion of energy-time uncertainty that we refer to in the main text.
B: NDRG SCALING EQUATIONS

In this appendix we outline the derivation of the ndRG scaling equations for the one-dimensional random transverse-field Ising model and determine their analytical solution. Using the expression for the generator $S_{\Lambda}$:

$$S_{\Lambda} = \sum_{kq} \omega_{kq}^{\alpha} \frac{\varepsilon_{k} - E_{q}}{\gamma_{k}^{*} \gamma_{q} + \frac{2m_{kq}}{E_{q}} \gamma_{k}^{*} \gamma_{q}^{\dagger} - \text{h.c.}}$$

and that the renormalized Hamiltonian is $H_{\Lambda} = H_{\Lambda+\Delta\Lambda}^{\alpha} + V_{\Lambda+\Delta\Lambda}^{\alpha} + \frac{1}{2} [S_{\Lambda}, V_{\Lambda+\Delta\Lambda}^{\alpha}]$ one obtains for the spectrum in second order of the disorder strength $\sigma$:

$$E_{k} = \varepsilon_{k} + \sum_{q} \omega_{kq}^{\alpha} \left[ \frac{4|m_{kq}|^{2}}{E_{k} + E_{q}} - \frac{\omega_{kq}^{2}}{E_{q} - E_{k}} \right], \quad E_{q} = \varepsilon_{q} + \sum_{k} \omega_{kq}^{\alpha} \left[ \frac{4|m_{kq}|^{2}}{E_{k} + E_{q}} + \frac{\omega_{kq}^{2}}{E_{q} - E_{k}} \right] .$$

where capital energies $E_{k}$ refer to the renormalized ones at scale $\Lambda$ and energies $\varepsilon_{k}$ to the initial ones at $\Lambda + \Delta \Lambda$ before the RG step has been performed. As in the main text we define the restricted sums

$$\sum_{k}^{\Lambda,\Omega} = \sum_{|k|<\Lambda;|E_{k}-E_{\Omega}|<\Omega} \omega_{kq}^{\alpha}, \quad \sum_{q}^{\Lambda,\Omega} = \sum_{|q|<\Omega;|\Omega+\Delta\Omega|} \omega_{kq}^{\alpha},$$

and for what follows we will refer to $k$ as the slow and to $q$ as the fast modes, respectively. Note that Eq. (8) would result in a Schrieffer-Wolff like transformation when replacing capital energies on the right hand side by lowercase ones. The presence of the renormalized energies in the denominators is one of the main advantages of the presented RG procedure because this will prevent the appearance of divergencies due to resonances.

We assume for the moment that initially at sufficiently large UV-cutoffs $\Lambda$ the disorder strength does not renormalize substantially. We will justify this below. As $|E_{k} - \varepsilon_{k}| \sim \Delta \Lambda$ (but not $|E_{q} - \varepsilon_{q}|$) one can replace $E_{k}$ by $\varepsilon_{k}$ on the right-hand side of the above equations. Using the definitions of the amplitudes $\omega_{kq}$ and $m_{kq}$ one obtains the following scaling equation for the slow modes:

$$\frac{d\varepsilon_{k}}{d\Lambda} = \frac{J^{2}\sigma^{2}}{\pi} \frac{1 + \cos(2\theta_{k}) \cos(2\theta_{\Lambda})}{E_{\Lambda} - \varepsilon_{k}} - \frac{J^{2}\sigma^{2}}{\pi} \frac{1 - \cos(2\theta_{k}) \cos(2\theta_{\Lambda})}{E_{\Lambda} + \varepsilon_{k}} .$$

Additionally, we have to determine the renormalization of the eliminated fast modes whose energies $E_{q}$ have to be chosen according to the self-consistency equation:

$$E_{q} = \varepsilon_{q} + \frac{J^{2}\sigma^{2}}{N} \sum_{k>0}^{\Lambda,\Omega} \left[ \frac{1 - \cos(2\theta_{k}) \cos(2\theta_{\Lambda})}{E_{q} + \varepsilon_{k}} + \frac{1 + \cos(2\theta_{k}) \cos(2\theta_{\Lambda})}{E_{q} - \varepsilon_{k}} \right] .$$

In the derivation of these equations we have replaced the sums over the random variables $w_{kk'}$ and $m_{kk'}$ by sums over their means, $\sum |m_{kq}|^{2} = \sum \langle |m_{kq}|^{2} \rangle_{\text{dis}} + \text{corrections}$, for example. As is the case in Wilson RG schemes the width of the momentum shell $\Delta \Lambda$ is small but still large enough to host an extensive number of states. Summing over a large number of random variables is equivalent to performing an effective averaging. The corrections to the mean are random variables themselves with zero mean and typical magnitude $\sim J\sigma^{2}/\sqrt{N}$. Comparing this to the amplitudes $w_{kk'}$ and $m_{kk'}$ with a typical magnitude of $\sim J\sigma/\sqrt{N}$ the corrections will give contributions beyond second order in the interaction strength which can be neglected within the current second order accuracy of the calculation.

Notice that although the spectrum does not contain any randomness, this does not mean that the randomness is fully gone. In fact, it is hidden in the unitary transformation connecting the extended states of the clean with the localized wave functions of the disordered system.

At criticality the Bogoliubov angles $\theta_{k}$ acquire a simple form: $\theta_{k} = (\pi - k)/4$ for $k > 0$ and $\theta_{-k} = -\theta_{k}$. The above scaling equations can be solved analytically for the low-energy states where one can use $\varepsilon_{k} \ll E_{\Lambda}$. Assuming that the low-energy states only acquire nonuniversal perturbative corrections (that we will neglect) from the upper and lower band edges at $|k| \approx \pi$ we can effectively start the RG transformation at a UV-cutoff in the regime where the clean spectrum becomes linear $\varepsilon_{k} \approx Jk$. This yields in leading order by approximating $E_{\Lambda} \approx J\Lambda$:

$$\frac{d\varepsilon_{k}}{d\Lambda} = \frac{2\sigma^{2}}{\pi} \frac{\varepsilon_{k}}{\Lambda^{2}} ,$$

implying a renormalization of the sound velocity $\varepsilon_{k}(\Lambda) = v(\Lambda)k$ with $v(\Lambda) = J \exp[-2\sigma^{2}/(\pi\Lambda)]$ which can neglected as long as $\Lambda > 2\sigma^{2}/\pi$. 
Much more important than the scaling of the spectrum is the behavior of the disorder strength under the RG. For that purpose it will be suitable to analyze the superpositions

\[ f_{kk'} = w_{kk'} - 2m_{kk'}, \quad h_{kk'} = w_{kk'} + 2m_{kk'} \]

instead of the amplitudes \( w_{kk'} \) and \( m_{kk'} \) themselves. Initially, before we start the scaling transformation their variances are directly related to the disorder strength \( \langle |f_{kk'}|^2 \rangle_{\text{dis}} = \langle |h_{kk'}|^2 \rangle_{\text{dis}} = J^2 \sigma^2 / N \). At low energies with \( \varepsilon_k, \varepsilon_{k'} \ll E_\lambda \) the scaled variances \( F_{kk'} = N \langle |f_{kk'}|^2 \rangle_{\text{dis}} \) and \( H_{kk'} = N \langle |h_{kk'}|^2 \rangle_{\text{dis}} \) obey the identical scaling equations

\[
df_{kk'}/d\Lambda = -F_{kk'}/\pi E_\lambda^2 \quad \text{and} \quad dH_{kk'}/d\Lambda = -H_{kk'}/\pi E_\lambda^2.
\]

Note that because neither the initial condition nor the scaling equations themselves have an explicit dependence on the momenta \( k \) and \( k' \) both \( H \) and \( F \) do not depend on \( k \) and \( k' \). Due to the initial condition \( F = H = J^2 \sigma^2 \) one obtains the following scaling equation for the disorder strength in the limit \( E_\lambda \approx J \Lambda \):

\[
\frac{d\sigma^2}{d\Lambda} = -\frac{\sigma^4}{\pi \Lambda^2}
\]

which is solved by

\[
\sigma^2(\Lambda) = \frac{\sigma^2}{1 - \frac{\sigma^2}{\pi} \left[ 1 - \frac{1}{\Lambda_0} \right]},
\]

with \( \Lambda_0 \) the initial UV-cutoff. Under the RG the disorder strength thus increases consistent with the equilibrium infinite-randomness critical point and eventually diverges at \( \Lambda \) of the order of \( \sigma^2 \) such that we leave the region of applicability of our RG. We circumvent this divergence by stopping the RG transformation at a scale of the order \( \sigma^2 \) such that the spectrum as well as the disorder only acquire weak corrections. The other modes we keep in the Hamiltonian and do not try to integrate them out, see main text. Note that when \( \Lambda \to \sigma^2 \) the approximation \( E_\lambda \approx \Lambda \) in the derivation of the scaling equations for both the spectrum and the disorder strength is not applicable any more. If one would go beyond this point, this would change the scaling equations for \( \Lambda \sim \sigma^2 \) by shifting the divergence of the disorder strength from \( \Lambda \sim \sigma^2 \) to \( \Lambda \to 0 \). For the description of the dynamics studied in the main text this, however, is not important.

Notice that the above scaling equation for the disorder strength is consistent with Ref. [33] where it has been shown that the bare disorder strength to first order does not show any renormalization for a noninteracting system.

Although the RG method for the time evolution operator \( P(t) \) can be directly transferred to the equilibrium case, the actual calculation of quantities requires knowledge of the state \( |\psi_0\rangle \) onto which \( P(t) \) acts. While for the typical quantum quench protocols \( |\psi_0\rangle \) is known and simple, in equilibrium it is the ground state of the problem and its determination a difficult problem by itself.

**C: OBSERVABLES**

In the following we will shortly highlight the methodology to compute the dynamics of observables using the ndRG. First, we will focus on the Loschmidt echo. Using

\[
P(t) = U^+_* e^{-i H_* t / \hbar} U_* + V_*^R, \quad e^{-i H_* t / \hbar} = e^{-i H_0^R t / \hbar} e^{-i V_* t / \hbar},
\]

see main text, one obtains for the Loschmidt amplitude:

\[
G(t) = \langle \psi_0 | U^+_*(t) U_* | \psi_0 \rangle
\]

with

\[
U^+_*(t) = e^{i V_* t} e^{i H_0^R t} U^+_* e^{-i H_0^R t} e^{-i V_* t}.
\]

As \( U_* = T_\Lambda \exp[\int d\Lambda S_\Lambda] \) and \( S_\Lambda \) by construction only contains perturbative processes one can perform a cumulant expansion up to second order yielding:

\[
G(t) = \exp \left[ \int d\Lambda d\Lambda' \langle (S_\Lambda - S_\Lambda(t)) S_{\Lambda'} \rangle \right]
\]
with $\langle \ldots \rangle = \langle \psi_0 | \ldots | \psi_0 \rangle$. For the final evaluation of this expression it is necessary to determine the dynamics under the resonant contributions $V^\text{R}$. In terms of the Jordan-Wigner fermions $c_k$, that are connected to the critical quasiparticles $\gamma_k$ via $c_k = \cos(\theta_k)\gamma_k - i\sin(\theta_k)\gamma_k^\dagger$, we have that $V^\text{R} = \sum_{kk'} g_{kk'} \Theta(\Omega - |\varepsilon_k - \varepsilon_{k'}|)$, see main text. This can be diagonalized analytically by Fourier transformation yielding $V^\text{R} = \sum_i g_i c_i^\dagger c_i$ where the random diagonal elements $g_i$ are random variables with the probability distribution $P(g) = \sqrt{\frac{\pi J}{\sigma^2 \Omega}} e^{-g^2 J/(4\Omega^2 \sigma^2)}$, see main text. Thus, we have that $e^{itV^\text{R}} c_k e^{-itV^\text{R}} = \sum_{kk'} \alpha_{kk'}(t) c_{k'}$ with $\alpha_{kk'}(t) = N^{-1}\sum_i e^{-i(k-k')^2 t} e^{-i\varepsilon_{k'} t}$. For the diagonal element $\alpha_{kk}(t)$ one obtains then, for example, after disorder averaging $\langle \alpha_{kk}(t) \rangle_{\text{dis}} = e^{-\sigma^2 J \Omega^2 t^2 / \hbar^2}$. Using Eq. (7) for $S_\lambda$ one then obtains the desired expression for the Loschmidt echo by performing the disorder average.

The autocorrelator $\chi(t)$ can be evaluated in the following way. In terms of the Jordan-Wigner fermions $c_k$ one obtains for $\chi(t)$:

$$\chi(t) = \left\langle \frac{1}{N^2} \sum_{kk'q} \langle c_{k+q}(t)c_k(t) \rangle_{\text{dis}} \right\rangle - \left\langle \frac{1}{N^2} \sum_{kk'} \langle c_k(t)c_k(t) \rangle_{\text{dis}} \right\rangle.$$

(20)

In order to obtain its dynamics we have to determine the action of the unitary transformation $U_*$ onto the quasiparticles $\gamma_k$ which to lowest order is given by:

$$U_\ast^\dagger \gamma_k U_* = \gamma_k + \sum_{|q|=|k|,|E_q - E_k| > \Omega} \frac{|w_{kq}|}{\varepsilon_k - \varepsilon_q} \gamma_q + \frac{2m_{kq}}{\varepsilon_k + \varepsilon_q} \gamma_q^\dagger.$$

(21)

In the end one obtains for the autocorrelator for times $t < \hbar / J \sigma^2$

$$\chi(t) = \chi_c(t) + \chi_d(t)$$

(22)

with

$$\chi_c(t) = \frac{1}{N} \sum_k \cos^2(\theta_k) e^{-iz_k t} \frac{1}{N} \sum_{k'} \sin^2(\theta_{k'}) e^{-iz_{k'} t}$$

(23)

the equilibrium autocorrelation function for the clean system. The additional disorder contribution yields:

$$\chi_d(t) = \frac{1}{N^2} \sum_{kqq'} \sin(\theta_q + \theta_{q'}) \cos(\theta_{k-q'}) C^*_{k-q',q'} \left[ i \cos(\theta_{k-q}) B_{k-q,-q} + \sin(\theta_{k-q}) C^*_{k+q,-q} \right] -$$

$$- \frac{1}{N^2} \sum_{kqq'} \sin(\theta_q + \theta_{q'}) \sin(\theta_{k-q'}) B^*_{k+q,-q} \left[ i \sin(\theta_{k-q}) C^*_{k-q',q'} + \cos(\theta_{k-q}) B^*_{k+q',-q'} \right]$$

(24)

with

$$B_{k,q} = \frac{w_{kq}}{\varepsilon_k - \varepsilon_q} \left[ e^{-iz_q t} - e^{-iz_{k} t} \right], \quad C_{k,q} = \frac{2m_{kq}}{\varepsilon_k + \varepsilon_q} \left[ e^{iz_q t} - e^{-iz_{k} t} \right].$$

(25)