On an Algebraic Structure of Dimensionally Reduced Magical Supergravity Theories

Shin Fukuchi∗ and Shun’ya Mizoguchi†

∗†SOKENDAI (The Graduate University for Advanced Studies)
Tsukuba, Ibaraki, 305-0801, Japan and
†Theory Center, Institute of Particle and Nuclear Studies, KEK
Tsukuba, Ibaraki, 305-0801, Japan
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Abstract

We study an algebraic structure of magical supergravities in three dimensions. We show that if the commutation relations among the generators of the quasi-conformal group in the super-Ehlers decomposition are in a particular form, then one can always find a parameterization of the group element in terms of various 3d bosonic fields that reproduces the 3d reduced Lagrangian of the corresponding magical supergravity. This provides a unified treatment of all the magical supergravity theories in finding explicit relations between the 3d dimensionally reduced Lagrangians and particular coset nonlinear sigma models. We also verify that the commutation relations of $E_{6(+2)}$, the quasi-conformal group for $A = \mathbb{C}$, indeed satisfy this property, allowing the algebraic interpretation of the structure constants and scalar field functions as was done in the $F_{4(+4)}$ magical supergravity.

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∗ E-mail:fshin@post.kek.jp
† E-mail:mizoguch@post.kek.jp
I. INTRODUCTION

One of the remarkable discoveries in the history of supergravity theories is that of the existence of five-dimensional exceptional supergravities associated with the Freudenthal-Tits magic square \([1, 2]\). They are a special class of \(D = 5\ N = 2\) Einstein-Maxwell supergravity (having 8 supercharges) that exist in addition to the infinite series of the non-Jordan family of \(D = 5\ N = 2\) supergravity. There are four such theories associated with the four division algebras, whose scalar manifolds arising in the reductions to \(D = 4\) and 3 are coset manifolds of various (real forms of) exceptional and non-exceptional Lie groups, which, surprisingly enough, coincide with every entry of the table of the magic square that shows the symmetries of Jordan algebras (Table I). Since the discovery many works have been done on these mysterious supergravity theories. An incomplete list includes \([3–22]\). (See also \([23]\) for a review.)

More recently, a precise identification was made in \([21]\) between the bosonic fields of the three-dimensional reduced smallest magical supergravity and the parameter functions of the coset space \(F_{4(+4)}/(USp(6) \times SU(2))\), thereby the \(FFA\) couplings \(C_{IJK}\) of the magical supergravity were shown to be identifiable as particular structure constants of the quasi-conformal algebra of the relevant Jordan algebra. It was also clarified there that the scalar fields \(\hat{A}^{I\mathcal{J}}\) and \(\hat{a}_{I\mathcal{J}}\) are nothing but the metric of the reduced dimensions in a particular representation. Since the form of the dimensionally reduced Lagrangian is common to all magical supergravities, with the only differences being the range of the values of the indices of the vector and scalar fields, it was conjectured in \([21]\) that such a Lie algebraic characterization of the coupling constants or the scalar fields also applies to other magical supergravities, not only to the smallest \(J_{3}^{\mathbb{R}}\) magical supergravity.

In this paper, we first show that if the commutation relations among the generators belonging to the respective irreducible components in the super-Ehlers decomposition are in a particular form, then one can always find a parameterization of the group element in terms of various 3d bosonic fields that reproduces the 3d reduced Lagrangian of the corresponding magical supergravity. This provides a unified treatment of all the magical supergravity theories in finding explicit relations between the 3d dimensionally reduced Lagrangians and particular coset nonlinear sigma models. This is done in section 2.

We then verify that the commutation relations of \(E_{6(+2)}\), the quasi-conformal group for \(\mathbb{A} = \mathbb{C}\), allows a decomposition whose generators indeed satisfy this property, which
immediately proves that the 3d reduced $A = \mathbb{C}$ magical supergravity consists of an $E_{6(+2)}/(SU(6) \times SU(2))$ nonlinear sigma model coupled to supergravity. This is done in section 3.

We should mention that $F_4(+4)$, the quasi-conformal group for $A = \mathbb{R}$, has also been shown to have such a property [21]. We conjecture that the remaining quasi-conformal groups in the list ($E_7(-5)$ (for $A = \mathbb{H}$) and $E_8(-24)$ (for $A = \mathbb{O}$) ) also possess such a special algebraic structure.

| TABLE I: The magic square [1] |
|-----------------------------|
| d  | J   | A      |
|----|-----|--------|
|    | $J_3^\mathbb{R}$ | $J_3^\mathbb{C}$ | $J_3^\mathbb{H}$ | $J_3^\mathbb{O}$ |
| 5 (compact) | SO(3) | SU(3) | USp(6) | $F_4$ | $\mathbb{R}$ |
| 5 (non-compact) | SL(3, $\mathbb{R}$) | SL(3, $\mathbb{C}$) | $SU^*(6)$ | $E_6(-26)$ | $\mathbb{C}$ |
| 4 (non-compact) | $Sp(6, \mathbb{R})$ | $SU(3, 3)$ | $SO^*(12)$ | $E_7(-25)$ | $\mathbb{H}$ |
| 3 (non-compact) | $F_4(+4)$ | $E_6(+2)$ | $E_7(-5)$ | $E_8(-24)$ | $\mathbb{O}$ |

II. EXPLICIT CONSTRUCTION OF 3D NONLINEAR SIGMA MODELS USING THE SUPER-EHLERS DECOMPOSITION

The bosonic Lagrangian of a magical supergravity associated with the division algebra $A$ is given by

$$\mathcal{L} = \frac{1}{2} E^{(5)} R^{(5)} - \frac{1}{4} E^{(5)} \hat{a}_{IJ} F_{MN}^{I} F^{JMNP} - \frac{1}{2} E^{(5)} s_{xy}(\partial_{M} \phi^{x})(\partial^{M} \phi^{y}) + \frac{1}{6 \sqrt{6}} C_{IKL} \epsilon^{MNPQR} F_{MN}^{I} F_{PQ}^{J} A_{R}^{K}.$$  (1)

$E^{(5)}$ is the determinant of the fünfbein, and $R^{(5)}$ is the scalar curvature in $D = 5$. $F_{MN}^{I}$ is the $I$th Maxwell field strength $2\partial_{[\mu} A_{\nu]}^{I}$, where $I, J, \cdots = 1, 2, \ldots, n_{A} + 1$ with

$$n_{A} = 3(1 + \dim A) - 1.$$  (2)

$\hat{a}_{IJ}$ and $s_{xy}$ are functions of $n_{A}$ scalar fields $\phi^{x}$ satisfying the relations $\hat{a}_{IJ} = \hat{a}_{JI}$ and $s_{xy} = s_{yx}$.

Following a standard procedure of dimensional reduction and field dualization, one finds
a 3d dimensionally reduced dualized Lagrangian for all magical supergravities as \[\mathcal{L} = \frac{1}{2} E R + \frac{1}{8} E \partial_{\mu} g^{m n} \partial_{\mu} g_{m n} - \frac{1}{2} E e^{-2} \partial_{\mu} e \partial_{\mu} e - \frac{1}{2} E s_{x y} (\partial_{\mu} \phi^{x}) (\partial_{\mu} \phi^{y}) - \frac{1}{2} E \partial_{I} g^{m n} \partial_{\mu} A_{m}^{I} \partial_{\mu} A_{n}^{I} - 2 E e^{-2} g^{m n} \left( \partial_{\mu} \varphi_{I} - \frac{1}{\sqrt{6}} C_{I J K} e^{m n} \partial_{\mu} A_{m}^{J} A_{n}^{K} \right) \left( \partial_{\mu} \varphi_{I'} - \frac{1}{\sqrt{6}} C_{I' J' K'} e^{m' n'} \partial_{\mu} A_{m'}^{J'} A_{n'}^{K'} \right) \right. \\
\left. - E e^{-2} g^{m n} \left( \partial_{\mu} \psi_{m} + \partial_{\mu} A_{m}^{I} \varphi_{I} - A_{m}^{I} \partial_{\mu} \varphi_{I} + \frac{2}{3 \sqrt{6}} C_{I J K} e^{p q} \partial_{\mu} A_{p}^{I} A_{q}^{J} A_{m}^{K} \right) \right) \\
\times \left( \partial_{\mu} \psi_{n} + \partial_{\mu} A_{n}^{I'} \varphi_{I'} - A_{n}^{I'} \partial_{\mu} \varphi_{I'} + \frac{2}{3 \sqrt{6}} C_{I' J' K'} e^{p' q'} \partial_{\mu} A_{p'}^{I'} A_{q'}^{J'} A_{n}^{K'} \right), \tag{3} \right. 
\]

where $\mu, \nu, \ldots$ are the 3d spacetime indices and $m, n, m', n', \ldots$ are the reduced two-dimensional indices. Note that this form is common to all the four magical supergravities; the only difference is the ranges the indices $x, y, \ldots$ and $I, J, \ldots$ run over.

Every magical supergravity contains in its 5d Lagrangian a nonlinear sigma model associated with the coset $\text{Str}_{0}(J_{3}^{A}) \subset \text{Aut}(J_{3}^{A})$. When it is dimensionally reduced to three dimensions, the scalar coset is enlarged to $\text{qConf}(J_{3}^{A}) / \text{Mø}(J_{3}^{A}) \times \text{SU}(2)$, in which all the non-gravity bosonic degrees of freedom are contained. To show explicitly how the various terms arising through the dimensional reduction are gathered to form a single coset, it is convenient to decompose the Lie algebra of the quasi-conformal group $\text{qConf}(J_{3}^{A})$, which is the numerator group of the 3d coset, in terms of representations of the Lie algebra of the subgroup $\text{SL}(3, \mathbb{R}) \times \text{Str}_{0}(J_{3}^{A})$, the latter of which is the numerator group of the 5d coset. The decomposition is always in the same form for all magical supergravities [18]:

\[\text{qConf}(J_{3}^{A}) \supset \text{SL}(3, \mathbb{R}) \times \text{Str}_{0}(J_{3}^{A}),\]

\[\text{adj}(\text{qConf}(J_{3}^{A})) = (8, 1) \oplus (3, \overline{n}_{A} + 1) \oplus (3, n_{A} + 1) \oplus (1, \text{adj}(\text{Str}_{0}(J_{3}^{A}))). \tag{4}\]

We will show that if the generators of the Lie algebra of the quasi-conformal group $\text{qConf}(J_{3}^{A})$ take a particular form [5] as assumed below, then one can always reproduce the Lagrangian [3] as a coset nonlinear sigma model coupled to gravity.
Let the generators of the Lie algebra of the quasi-conformal group qConf(J^3_A) satisfy

\[ [\hat{E}_i^j, \hat{E}_k^l] = \delta_j^k \hat{E}_i^l - \delta_i^l \hat{E}_j^k, \]
\[ [\hat{E}_i^j, E_I^k] = \delta_j^k E_I^i, \]
\[ [\hat{E}_i^j, E_I^l] = -\delta_i^l E_I^j, \]
\[ [T_i, T_j] = f_{ij}^k T_k, \]
\[ [T_i, E_I^k] = \tilde{t}_{iI}^j E_J^k, \]
\[ [T_i, E_I^l] = t_{iI}^j E_J^l, \]
\[ [E_I^k, E_J^l] = C^{IJK} \epsilon_{ijk} E_K^l, \]
\[ [E_I^k, E_J^l] = -C^{IJK} \epsilon_{ijk} E_K^l, \]
\[ [E_I^k, E_J^l] = -2\delta_J^i \hat{E}_i^l + \delta_i^l D^j_I \hat{T}_l, \]
otherwise = 0.

(5)

\( \epsilon_{ijk} \) and \( \epsilon_{ijk} \) are completely antisymmetric tensors with \( \epsilon_{123} = \epsilon^{123} = 1. \)

\( \hat{E}_i^j \ (i, j = 1, \ldots, 3) \) with \( \hat{E}_1^1 + \hat{E}_2^2 + \hat{E}_3^3 = 0 \) are the generators of \( SL(3, \mathbb{R}) \), the first irreducible component \( (\mathbf{8}, \mathbf{1}) \) of \( \mathfrak{h} \). They are defined modulo \( \hat{E}_1^1 + \hat{E}_2^2 + \hat{E}_3^3 \), that is, \( \hat{E}_i^j \) is an element of a quotient space of \( GL(3, \mathbb{R}) \) divided by the center generated by the overall \( U(1) \) generator. \( T_i \ (\tilde{i} = 1, \ldots, \dim \text{Str}_0(J^3_A)) \) are the generators of \( \text{Str}_0(J^3_A) \) of the respective magical supergravity, which is the last irreducible component of \( \mathfrak{h} \). Finally, \( E_I^k \) and \( E_I^l \) \( (i, j = 1, \ldots, 3; I, J = 1, \ldots, n_A + 1) \) are the generators of \( (\mathbf{3}, \mathbf{n_A + 1}) \) and \( (\bar{\mathbf{3}}, \mathbf{n_A + 1}) \), respectively.

\( f_{ij}^k, \tilde{t}_{iI}^j, t_{iI}^j, C_{IJK}, C^{IJK} \) and \( D^j_I \) are real structure constants; they are not all inde-
pendent but are constrained by the Jacobi identities. The full set of constraints are

\[
C_{IJK} = C^{IJK} = C^{IKJ} = C^{KIJ},
\]

\[
C_{IJK} = C_{JIK} = C_{JKI} = C_{KIJ},
\]

\[
t_i^I = -t_i^J,
\]

\[
t_i^I K C^{KJI} + t_i^J K C^{KLI} + t_i^L K C^{KIJ} = 0,
\]

\[
t_i^I K C^{KJI} + t_i^J K C^{KLI} + t_i^L K C^{KIJ} = 0,
\]

\[
D^I_i t_j^j M - D^J_j t_i^i M = 2(\delta^I_I \delta^J_J - \delta^I_J \delta^J_I),
\]

\[
D^I_i t_j^j M + D^J_j t_i^i M = 2(\delta^I_I \delta^J_J + \delta^I_J \delta^J_I - C^{IJK} C_{LKM}),
\]

\[
[t_i^I, t_j^J] = -f_{ij}^k t_k^I,
\]

Note that the symmetricity of $C_{IJK}$ or $C^{IJK}$ is required from the Jacobi identities of the above algebra. Of course, this property is one of the virtues of magical supergravity theories.

$Str_0(J^h_3)$ and $qConf(J^h_3)$ are symmetric spaces for all $\mathbb{A}$. The symmetric space involution $\tau$ defined on the former coincides with that on the latter if regarded as the action on its subgroup $Str_0(J^h_3)$. Using this fact, one can show that

\[
H = (\oplus_{i,j=1,2,3} \mathbb{R}(\hat{E}^i_j - \hat{E}^j_i)) + (\oplus_{i=1,2,3; I=1,\ldots,n_\mathbb{A}+1} \mathbb{R}(E^I_i - E^{*I}_i)) + H_{Str_0(J^h_3)}
\]

and

\[
K = (\oplus_{i,j=1,2,3} \mathbb{R}(\hat{E}^i_j + \hat{E}^j_i)) + (\oplus_{i=1,2,3; I=1,\ldots,n_\mathbb{A}+1} \mathbb{R}(E^I_i + E^{*I}_i)) + K_{Str_0(J^h_3)}
\]

satisfy

\[
[H, H] \subset H,
\]

\[
[K, K] \subset H,
\]

\[
[H, K] \subset K,
\]

so that one can define the symmetric space involution $\tau$ as

\[
\tau(H) = +H, \quad \tau(K) = -K.
\]
Now the construction of the coset sigma model is straightforward. Defining
\[ \mathcal{V} = \mathcal{V}_- \mathcal{V}_+ , \] (19)
\[ \mathcal{V}_+ = \mathcal{V}_+^{(grav)} \mathcal{V}_+^{(scalar)} , \] (20)
\[ \mathcal{V}_+^{(grav)} = \exp \left( \log e_1^1 \hat{E}_1^1 + \log e_2^2 \hat{E}_2^1 + \log e \hat{E}_3^1 \right) \cdot \exp \left( -e_2^2 \hat{E}_2^1 \right) \exp \left( \psi_1 \hat{E}_3^1 + \psi_2 \hat{E}_2^2 \right), \] (21)
\[ \mathcal{V}_+^{(scalar)} = \exp \left( (\log \tilde{\mathcal{E}}^{-1})^i T_i \right) \quad ( \tilde{i} = 1, \ldots, \text{dimStr}(J^3_3) ) , \] (22)
\[ \mathcal{V}_- = \exp \left( A_m^I E_I^m + \varphi_I E_3^I \right) \quad (m = 1, 2; \ I = 1, \ldots, n_A + 1) , \] (23)
the Maurer-Cartan 1-form is computed as
\[ \partial_\mu \mathcal{V}^{-1} = \partial_\mu \mathcal{V}_+ \mathcal{V}_+^{-1} + \mathcal{V}_+ (\partial_\mu \mathcal{V}_- \mathcal{V}_+^{-1}) \mathcal{V}_+^{-1} , \] (24)
\[ \partial_\mu \mathcal{V}_+ \mathcal{V}_+^{-1} = (e_1^1)^{-1} \partial_\mu e_1^1 \hat{E}_1^1 + (e_2^2)^{-1} \partial_\mu e_2^2 \hat{E}_2^1 + e^{-1} \partial_\mu e \hat{E}_3^1 \]
\[ -e_1^1 (e_2^2)^{-1} \partial_\mu B \hat{E}_2^1 + e^{-1} \left( e_1^1 (\partial_\mu \psi_1 - B \partial_\mu \psi_2) \hat{E}_3^1 + e_2^2 \partial_\mu \psi_2 \hat{E}_2^1 \right) \]
\[ + (\partial_\mu \tilde{\mathcal{E}}^{-1} \cdot \tilde{\mathcal{E}})^i T_i , \] (25)
where
\[ e_\dot{m}^m = \begin{pmatrix} e_1^1 & e_1^2 \\ 0 & e_2^2 \end{pmatrix} , \] (26)
\[ (e^{-1})_a^i = \begin{pmatrix} e_\dot{m}^m & e_\dot{m}^m \psi_m \\ 0 & e \end{pmatrix} , \] (27)
\[ e = \text{det} e_\dot{m}^m = (e_1^1 e_2^2)^{-1} \] (28)
for the \( \mathcal{V}_+ \) term, whereas
\[ \partial_\mu \mathcal{V}_- \mathcal{V}_-^{-1} = \partial_\mu A_m^I E_I^m + \left( \partial_\mu \varphi_I - \frac{1}{2} C_{JKI} \epsilon^{mn} A_m^j A_n^k \right) E_3^K \]
\[ + \left( A_m^I \partial_\mu \varphi_I - \partial_\mu A_m^I \varphi_I - \frac{1}{3} C_{JKI} \epsilon^{np} A_m^I A_n^j A_p^k \right) \hat{E}_3^m \] (29)
for the \( \mathcal{V}_- \) term. Here \( m, n, p \) are the two-dimensional curved indices taking 1 or 2, while \( \dot{m} \)
is the two-dimensional flat index taking \( \dot{1} \) or \( \dot{2} \).
We can compute the $V_+^{(gravg)}$ conjugations as

$$
V_+^{(gravg)} E_i^{sj} V_+^{(gravg)-1} = E_t^a (e^{-1})^j_a,
V_+^{(gravg)} E_j^{I} V_+^{(gravg)-1} = (e)_j^a E_a^I,
V_+^{(scalar)} E_i^{sj} V_+^{(scalar)-1} = f_I^A E_j^a,
V_+^{(scalar)} E_j^{I} V_+^{(scalar)-1} = f_I^A E_j^A,
$$

(30)

where

$$
f_I^A = (\exp((\log e^{-1})^i t_i^A))_I^A,
$$

(31)

$$
f_I^A = (\exp((\log e^{-1})^i t_i^I))_I^A.
$$

(32)

Note that, due to the relation $\bar{t}_i^j = -t_i^J$ [8], $f_I^A$ is the transpose of the inverse of $f_I^A$.

Thus we find

$$
V_+((\partial_\mu V_\tau V_{\tau}^{-1}) V_+^{-1} = e_m^m \int d^A \partial_\mu A^I m E_3^m
+ e^{-1} \int d^A \left( \partial_\mu \varphi_I - \frac{1}{2} C_{JKI} e^{m} A^J m \partial_\mu A^K \right) E_3^A
+ e^{-1} \int d^m \left( A^I m \partial_\mu \varphi_I - \partial_\mu A^I m \varphi_I - \frac{1}{3} C_{JKI} e^{m} A^J m A^K \partial_\mu A^K \right) \hat{E}_3^m.
$$

(33)

Defining $M \equiv \tau(V\tau^{-1})V$ as usual, we have

$$
- \frac{1}{4} E^{(3)} \text{Tr} \partial_\mu M^{-1} \partial_\mu M = E^{(3)} \text{Tr} \left( \frac{1}{2} (\partial_\mu VV^{-1} - \tau (\partial_\mu VV^{-1})) \right)^2,
$$

(34)

thereby the $H$ pieces of $\partial_\mu V \tau V^{-1}$ are projected out. This amounts to the replacements of the generators in $\partial_\mu V \tau V^{-1}$ as

$$
\hat{E}_j^i \rightarrow \frac{1}{2}(\hat{E}_j^i + \hat{E}_j^i),
E_I^j \rightarrow \frac{1}{2}(E_I^j + E_I^j),
E_i^{\nu j} \rightarrow \frac{1}{2}(E_i^{\nu j} + E_i^{\nu j}),
T_i \rightarrow \frac{1}{2}(T_i - \tau(T_i)).
$$

(35)
Using the traces with the normalizations

\[
\frac{1}{2\hbar} \text{Tr} \tilde{E}_a^c \tilde{E}_b^d = \delta_b^c \delta_d^a \quad (a, b, c, d = 1, 2, 3),
\]

\[
\frac{1}{2\hbar} \text{Tr} E_A^a E_B^b = 2\delta_A^a \delta_B^b \quad (a, b = 1, 2, 3; \, A, B = 1, \ldots, n_h + 1),
\]

\[
\frac{1}{2\hbar} \text{Tr}(T_i - \tau(T_i))(T_j - \tau(T_j)) = \gamma_{ij} \quad (\tilde{i}, \tilde{j} = 1, \ldots, \dim\text{Str}_0(J^h_3)),
\]

\[
\text{otherwise} = 0,
\]

where \( h \) is the dual Coxeter number of \( q\text{Conf}(J^h_3) \), we obtain the final result

\[
\frac{1}{8\hbar} \text{Tr} \partial_\mu \mathcal{M}^{-1} \partial^\mu \mathcal{M} = \frac{1}{4} \partial_\mu g^{mn} \partial_\mu g_{mn} - e^{-2} \partial_\mu e \partial^\mu e - g^{mn} \tilde{a}_{IJ} \partial_\mu A_n^I \partial^\mu A_n^J
\]

\[
+ \frac{1}{2} \gamma_{ij} (\partial_\mu \tilde{e}^{-1} \cdot \tilde{e})^j (\partial^\mu \tilde{e^{-1}} \cdot \tilde{e})^i
\]

\[
- e^{-2} \partial_I^J \left( \partial_\mu \varphi_I - \frac{1}{2} C_{KLI} \epsilon^{mn} A^K_m \partial_\mu A_n^L \right)
\]

\[
\cdot \left( \partial_\mu \varphi_J - \frac{1}{2} C_{K'L'J} \epsilon^{m'n'} A^K_m \partial_\mu A_{n'}^L \right)
\]

\[
- \frac{1}{2} e^{-2} g^{mn} \left( \partial_\mu \psi_m - \varphi_I \partial_\mu A_n^I + \partial_\mu \varphi A_n^I - \frac{1}{3} C_{KLI} \epsilon^{np} A^K_n \partial_\mu A_p^L A_m^I \right)
\]

\[
\cdot \left( \partial_\mu \psi_n - \varphi_J \partial_\mu A_n^J + \partial_\mu \varphi A_n^J - \frac{1}{3} C_{K'L'J} \epsilon^{n'p'} A^K_{n'} \partial_\mu A_{p'}^L A_n^J \right).
\]

The second line is the \( \text{Str}_0(J^h_3) \) \( \text{Aut}(J^h_3) \) sigma model contained in the original \( D = 5 \) magical supergravity Lagrangian. The result \((37)\) coincides with the sigma-model part of \((3)\) after appropriate rescalings of \( \varphi_I, \psi_m \) and \( C_{IJK} \).

### III. \( E_6^{(+2)} \) Algebra

\( E_6^{(+2)} \) is one of the real forms of the exceptional Lie algebra \( E_6 \). This is different from the more familiar split real form \( E_6^{(+6)} \) encountered as a U-duality group in type II string theory, which is conveniently realized as a Lie subalgebra of \( E_8^{(+8)} \) by using the generators in Freudenthal’s realization \([24, 26]\). Thus we first determine the generators of \( E_6^{(+6)} \) in \( E_8^{(+8)} \), take its complexification \( E_6^{(+6)} \otimes \mathbb{C} \), and then we identify the generators of the real Lie algebra \( E_6^{(+2)} \).

\(^1\) There was a typo in the formula of rescalings \((64)\) in \([21]\).
The $E_{8(+8)}$ generators are (The numbers in the parentheses are the total numbers of the respective generators.)

\[
E^I_J \quad (I, J = 1, \ldots, 9; \ I \neq J) \quad (72), \\
E^{IJK} \quad (I, J, K = 1, \ldots, 9) \quad (84), \\
E^*_{IJK} \quad (I, J, K = 1, \ldots, 9) \quad (84), \\
h_I \quad (I = 1, \ldots, 8) \quad (= E^I_I - E^J_J) \quad (8),
\]

which are assumed to satisfy the commutation relations

\[
\begin{align*}
[E^I_J, E^K_L] &= \delta^K_J E^I_L - \delta^K_L E^I_J, \\
[E^I_J, E^{KLM}] &= 3\delta^{[M}_{I} E^{KL]J}, \\
[E^I_J, E^*_{KLM}] &= -3\delta^{[M}_{I} E^{*KL]J}, \\
[E^{IJK}, E^{LMN}] &= -\frac{1}{3} \sum_{P,Q,R=1}^{9} \epsilon^{IJKLMNPQR} E^{*PQR}, \\
[E^*_{IJK}, E^*_{LMN}] &= +\frac{1}{3} \sum_{P,Q,R=1}^{9} \epsilon^{IJKLMNPQR} E^{PQR}, \\
[E^{IJK}, E^*_{LMN}] &= 6\delta^{[M}_{I} \delta^K_N E^J_L \quad \text{(if } I \neq L, M, N), \\
[E^{IJK}, E^*_{IJK}] &= h_{IJK},
\end{align*}
\]

where

\[
h_{IJK} \equiv E^I_I + E^J_J + E^K_K - \frac{1}{3} \sum_{L=1}^{9} E^L_L.
\]

Among them, $E_{6(+6)}$ is generated by the following 78 generators:

\[
\begin{align*}
E^i_j (1 \leq \hat{i} \neq \hat{j} \leq 6), \\
E^1_1 - E^2_2, E^2_2 - E^3_3, E^3_3 - E^4_4, E^4_4 - E^5_5, E^5_5 - E^6_6 \quad (35), \\
E^{ijk} (1 \leq \hat{i} < \hat{j} < \hat{k} \leq 6) \quad (20), \\
E^*_{ijk} (1 \leq \hat{i} < \hat{j} < \hat{k} \leq 6) \quad (20), \\
E^{789}, E^*_{789}, h_{789} \quad (3).
\end{align*}
\]

The first line is the generators of the $SL(6, \mathbb{R})$ subalgebra, whereas the bottom line is the ones of the $SL(2, \mathbb{R})$ subalgebra. They form a real Lie algebra with all real structure constants, whose complexification is the complex Lie algebra $E_6$. Among the complex generators thus obtained, we can find another set of generators forming a basis of a different real form of $E_6$ as follows:

\[\text{Remark: the sign factor of } [E^*_{IJK}, E^*_{LMN}] \text{ is different from that in ref.} \quad (25) \text{ because the metric used there was the "mostly negative" one.}\]
1. The generators $E^{123}$, $E^{456}$, $E^{789}$, $E_{123}^*$, $E_{456}^*$, $E_{789}^*$, $h_{123}$ and $h_{456}$ form an $SL(3, \mathbb{R})$ algebra. We identify them as the $\hat{E}_{ij}$ generators in the previous section as

$$
\begin{align*}
\hat{E}_1^1 &= E_{123}^*, \\
\hat{E}_2^3 &= E_{456}^*, \\
\hat{E}_3^7 &= E_{789}, \\
\hat{E}_1^2 &= E_{123}, \\
\hat{E}_2^3 &= E_{456}, \\
\hat{E}_3^1 &= E_{789}, \\
\hat{E}_1^1 &= -\frac{1}{3}(2h_{123} + h_{456}), \\
\hat{E}_2^2 &= \frac{1}{3}(h_{123} - h_{456}).
\end{align*}
$$

(Note that we have defined the $\hat{E}_{ij}$ generators modulo $\hat{E}_1 + \hat{E}_2 + \hat{E}_3$, so for instance $\hat{E}_1$ is equal to $\frac{1}{3}(2\hat{E}_1 - \hat{E}_2 - \hat{E}_3)$.)

2. Besides, there are two commuting $SL(3, \mathbb{R})$’s in the $SL(6, \mathbb{R})$ subalgebra of $E_{6(+6)}$, and by complexification one can construct two commuting sets of Gell-Mann matrices generating $SU(3) \oplus SU(3)$. Let these generators be $\lambda_r$ and $\tilde{\lambda}_r$ ($r = 1, \ldots, 8$), then they can be taken to be

$$
\begin{align*}
\lambda_1 &= E_2^1 + E_2^2, & \lambda_2 &= i(-E_{2}^1 + E_{2}^2), & \lambda_3 &= E_{1}^1 - E_{2}^2, & \lambda_4 &= E_{3}^1 + E_{2}^2, \\
\lambda_5 &= i(-E_{3}^1 + E_{2}^2), & \lambda_6 &= E_{1}^1 + E_{1}^2, & \lambda_7 &= i(-E_{1}^1 + E_{1}^2), & \lambda_8 &= E_{2}^3 - E_{3}^3, \\
\tilde{\lambda}_1 &= E_{4}^1 + E_{5}^4, & \tilde{\lambda}_2 &= i(-E_{4}^1 + E_{5}^4), & \tilde{\lambda}_3 &= E_{4}^1 - E_{5}^4, & \tilde{\lambda}_4 &= E_{6}^5 + E_{5}^6, \\
\tilde{\lambda}_5 &= i(-E_{6}^5 + E_{5}^6), & \tilde{\lambda}_6 &= E_{6}^4 + E_{4}^6, & \tilde{\lambda}_7 &= i(-E_{6}^4 + E_{4}^6), & \tilde{\lambda}_8 &= E_{5}^5 - E_{6}^6,
\end{align*}
$$

where $\lambda_8$ and $\tilde{\lambda}_8$ are not what should correspond to the original eighth Gell-Mann matrix, but are generators corresponding to

$$
\begin{pmatrix}
0 \\
1 \\
-1
\end{pmatrix}
$$

for respective $SU(3)$ algebras.

Defining

$$
\mu_r = i(\lambda_r + \tilde{\lambda}_r), \quad \nu_r = -\lambda_r + \tilde{\lambda}_r \quad (r = 1, \ldots, 8),
$$

then we see that these 16 generators form a basis of a real Lie algebra $SL(3, \mathbb{C})$.

As the $T_i$ generators for $\text{Str}_0(J^C_3) = SL(3, \mathbb{C})$, it is convenient to consider another set of 16 generators defined by taking the following real linear combinations of $\mu_r$ and $\nu_r$:
3. Finally, the generators transforming as \((\mathbf{3}, \mathbf{n}_A + \mathbf{1}) \oplus (\overline{\mathbf{3}}, \mathbf{n}_A + \mathbf{1})\) with \(n_A = 8\) under \(\text{Str}_0(J^c_3) = SL(3, \mathbb{C})\) (44) are identified to be:

\[
E_I^{*i}:
\]

\[
\begin{align*}
E_1^{*1} &= \sqrt{2}E_{234}^*, & E_1^{*2} &= \sqrt{2}E_{14}^*, & E_1^{*3} &= -\sqrt{2}E_{156}^*, \\
E_2^{*1} &= \sqrt{2}E_{315}^*, & E_2^{*2} &= \sqrt{2}E_{25}^*, & E_2^{*3} &= -\sqrt{2}E_{264}^*, \\
E_3^{*1} &= \sqrt{2}E_{126}^*, & E_3^{*2} &= \sqrt{2}E_{36}^*, & E_3^{*3} &= -\sqrt{2}E_{345}^*, \\
E_4^{*1} &= E_{316}^* + E_{125}^*, & E_4^{*2} &= E_{26}^* + E_{35}^*, & E_4^{*3} &= -(E_{364}^* + E_{245}^*), \\
E_5^{*1} &= E_{124}^* + E_{236}^*, & E_5^{*2} &= E_{34}^* + E_{16}^*, & E_5^{*3} &= -(E_{145}^* + E_{356}^*), \\
E_6^{*1} &= E_{235}^* + E_{314}^*, & E_6^{*2} &= E_{15}^* + E_{24}^*, & E_6^{*3} &= -(E_{256}^* + E_{164}^*), \\
E_7^{*1} &= i(E_{316}^* - E_{125}^*), & E_7^{*2} &= i(E_{26}^* - E_{35}^*), & E_7^{*3} &= i(E_{364}^* - E_{245}^*), \\
E_8^{*1} &= i(E_{124}^* - E_{236}^*), & E_8^{*2} &= i(E_{34}^* - E_{16}^*), & E_8^{*3} &= i(E_{145}^* - E_{356}^*), \\
E_9^{*1} &= i(E_{235}^* - E_{314}^*), & E_9^{*2} &= i(E_{15}^* - E_{24}^*), & E_9^{*3} &= i(E_{256}^* - E_{164}^*).
\]

(47)
$E_i$:

\begin{align*}
E_1 &= \sqrt{2}E^{234}, & E_2 &= \sqrt{2}E^4, & E_3 &= -\sqrt{2}E^*_1, \\
E_2 &= \sqrt{2}E^{315}, & E_2 &= \sqrt{2}E^5, & E_3 &= -\sqrt{2}E^*_2, \\
E_3 &= \sqrt{2}E^{126}, & E_2 &= \sqrt{2}E^6, & E_3 &= -\sqrt{2}E^*_3, \\
E_4 &= E^{125} + E^{316}, & E_4 &= E^5_3 + E^6_2, & E_3 &= -(E^*_3 + E^*_{245}), \\
E_5 &= E^{236} + E^{124}, & E_5 &= E^6_1 + E^4_3, & E_3 &= -(E^*_1 + E^*_{356}), \\
E_6 &= E^{314} + E^{235}, & E_6 &= E^4_2 + E^5_1, & E_3 &= -(E^*_2 + E^*_{164}), \\
E_7 &= i(E^{125} - E^{316}), & E_7 &= i(E^5_3 - E^6_2), & E_3 &= -i(E^*_3 - E^*_{245}), \\
E_8 &= i(E^{236} - E^{124}), & E_8 &= i(E^6_1 - E^4_3), & E_3 &= -i(E^*_1 - E^*_{356}), \\
E_9 &= i(E^{314} - E^{235}), & E_9 &= i(E^4_2 - E^5_1), & E_3 &= -i(E^*_2 - E^*_{164}).
\end{align*}

(48)

One can verify that the 78 generators (42), (46), (47) and (48) form a closed algebra, which is a real form of $E_6$ by construction. By examining the Killing form, it can be easily seen that the $SL(3,\mathbb{C})$ generators (46) as well as the $E^*_i$ and $E^I_i$ generators (47) (48) consist of the same number of positive and negative generators. This leaves the $SL(3,\mathbb{R})$ (42), which is $A_{2(+2)}$. Thus we see that the whole algebra is $E_{6(+2)}$.

One can also show that it satisfies the commutation relations of the form (5), whose actual values of the structure constants are

\begin{align*}
C^{123} &= +\sqrt{2}, \\
C^{144} &= C^{177} = C^{255} = C^{288} = C^{366} = C^{399} = -\sqrt{2}, \\
C^{456} &= +1, \\
C^{489} &= C^{579} = C^{678} = -1.
\end{align*}

(49)
and

\[
D_{5}^{1,3} = D_{6}^{1,2} = D_{4}^{2,6} = D_{6}^{2,4} = D_{4}^{3,8} = D_{5}^{3,7} = D_{6}^{4,8} = D_{3}^{4,6} = D_{5}^{5,7} = D_{1}^{5,3} = D_{4}^{6,4} = D_{2}^{6,2} = + \sqrt{2},
\]

\[
D_{4}^{4,5} = D_{3}^{4,7} = D_{5}^{5,2} = D_{6}^{5,8} = D_{4}^{6,3} = D_{6}^{6,6} = D_{6}^{6,1} = D_{6}^{6,5} = D_{9}^{9,1} = D_{9}^{9,5} = + 1,
\]

\[
D_{4}^{4,1} = D_{5}^{5,5} = D_{7}^{7,1} = D_{7}^{7,4} = D_{9}^{7,9} = D_{8}^{8,8} = D_{8}^{8,5} = D_{9}^{9,8} = D_{9}^{9,3} = D_{8}^{8,9} = D_{7}^{9,3} = D_{9}^{9,6} = - 1,
\]

\[
D_{1}^{1,1} = D_{2}^{2,5} = + 2,
\]

\[
D_{3}^{3,1} = D_{3}^{3,5} = - 2,
\]

\[
D_{9}^{1,10} = D_{9}^{2,14} = D_{8}^{3,15} = D_{3}^{7,14} = D_{1}^{8,15} = D_{2}^{9,10} = + \sqrt{2},
\]

\[
D_{8}^{1,11} = D_{9}^{2,12} = D_{7}^{3,16} = D_{2}^{7,16} = D_{3}^{8,11} = D_{1}^{9,12} = - \sqrt{2},
\]

\[
D_{8}^{4,12} = D_{8}^{5,13} = D_{9}^{5,16} = D_{7}^{6,11} = D_{9}^{6,13} = D_{4}^{7,9} = D_{5}^{7,12} = D_{6}^{8,16} = D_{4}^{9,11} = D_{6}^{9,9} = + 1,
\]

\[
D_{7}^{4,9} = D_{4}^{9,15} = D_{7}^{5,10} = D_{8}^{6,14} = D_{9}^{6,9} = D_{6}^{7,15} = D_{4}^{8,10} = D_{5}^{8,13} = D_{5}^{9,14} = D_{6}^{9,13} = - 1,
\]

\[
D_{8}^{5,9} = D_{4}^{7,13} = + 2,
\]

\[
D_{7}^{4,13} = D_{5}^{8,9} = - 2,
\]

otherwise 0. Thus we have confirmed that \( E_{6(\pm 2)} = q\text{Conf}(J_{3}^{C}) \) and \( F_{4(\pm 4)} = q\text{Conf}(J_{3}^{R}) \) share the common algebraic structure in terms of the \( SL(3, \mathbb{R}) \times \text{Str}_{0}(J_{3}^{A}) \) decomposition, whose coset necessarily leads, as we proved in section 2, to the general form of the three-dimensional sigma model of a dimensionally reduced magical supergravity. This means that the second, \( J_{3}^{C} \) magical supergravity also allows the algebraic interpretation of the structure constants and scalar field functions as was done in the first, \( J_{3}^{R} \) magical supergravity. We naturally expect that the remaining magical supergravities also possess such a structure. We hope to report on this problem elsewhere.
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