Sharper bounds on the box-counting dimension of the singularities in the hyperdissipative Navier-Stokes equations

Min Jun Jo

December 6, 2022

Abstract

We provide an improved upper bound on the box-counting dimension of the set of potential singular points in suitable weak solutions to the $\alpha$-fractional Navier-Stokes equations for the hyperdissipation case $1 < \alpha < \frac{5}{4}$.

1 Introduction

We study the hyperdissipative Navier-Stokes equations in $\mathbb{R}^3 \times \mathbb{R}_+$

\[
\begin{cases}
\partial_t u + (-\Delta)^\alpha u + (u \cdot \nabla) u + \nabla p = 0 \\
\text{div} \, u = 0 \\
\quad u(\cdot, 0) = u_0
\end{cases}
\] (1.1)

which describes the motion of incompressible viscous fluids. Here $u$ is the velocity field, $p$ is the associated pressure, and $(-\Delta)^\alpha$ denotes the operator defined by its Fourier symbol $|\xi|^{2\alpha}$ for $\alpha \geq 0$.

The case $\alpha = 1$ corresponds to the classical Navier-Stokes system and it is still unknown whether such a system always admits a unique global smooth solution for any given smooth sufficiently decaying (or compactly supported) initial data $u_0$. To put the regularity problem into perspective, seeing the evolution of the fluid as a competition between dissipation and convection, we may ask

"how much dissipation do we need for global regularity?"

Assuming that $\alpha$ encodes the intensity of the dissipation, the above question is equivalent to asking which $\alpha$ would lead to the global regularity. Partial answers to this question are given in [5, 12]: if $\alpha \geq \frac{5}{3}$, the system (1.1) possesses a unique global smooth solution for smooth $u_0$ which decays sufficiently fast at infinity. Therefore, it is particularly important to investigate the case

$$1 < \alpha < \frac{5}{4}$$

which might bridge the gap between the classical Navier-Stokes equations and the hyperdissipative ones. Throughout this paper, we focus on the above case $\alpha \in (1, \frac{5}{4})$. 

__Key words:__ Partial regularity, box-counting dimension, hyperdissipation, the Navier-Stokes equations

__2020 AMS Mathematics Subject Classification:__ 76D03, 76D05

*Department of Mathematics, The University of British Columbia. E-mail: mijjo@math.ubc.ca
One can expect that the dissipation effect gets stronger as $\alpha$ grows, and such a phenomenon of regularization could be well-measured by the notion of *partial regularity*. That is, to compute the fractal dimension of the set of the potential singularities. It would provide a geometric interpretation of the distribution of such singularities. In the recent work [3], which inspired this article, Colombo, De Lellis, and Massaccesi gave an upper bound

$$L(\alpha) := \frac{15 - 2\alpha - 8\alpha^2}{3} \quad \text{for} \quad 1 < \alpha \leq \frac{5}{4} \tag{1.2}$$

for the box-counting dimension of the potential singular set for *suitable weak solutions* (see Definition 2.3 and Section 1.2) to the system (1.1), which provides a quantitative explanation on how singularities can be excluded as $\alpha$ gets larger. See also [9] where the upper bound (1.2) is shown to be valid for the hypodissipation case $\frac{3}{4} < \alpha < 1$ as well. It is worth pointing out that the hypodissipative case $\frac{3}{4} < \alpha < 1$ was considered earlier in [15], where the $(5 - 4s)$-dimensional Hausdorff measure of the singular set is proved to be zero for any suitable weak solution.

Our main target is to give a more precise description on the potential singularities of suitable weak solutions by sharpening the upper bound $L(\alpha)$ given in [3]. In particular, we prove that the box-counting dimension of the set of potential singular points is bounded by

$$J(\alpha) := \frac{36(3 - \alpha)(3 + 2\alpha)(5 - 4\alpha)}{-64\alpha^3 + 272\alpha^2 - 300\alpha + 369} \tag{1.3}$$

for all $\alpha \in (1, 5/4)$. See the following figure:

![Graph showing the comparison between L(\alpha) and J(\alpha)](image)

This bound $J(\alpha)$ generalizes the result in [17] which was restricted to the case $\alpha = 1$. Indeed, our approach is based on the iteration scheme that was firstly introduced in [6]. To see more results for the case $\alpha = 1$, one may refer to [7, 8, 17, 18].

We point out that in [3] Colombo, De Lellis, and Massaccesi actually gave the full extension of the celebrated Caffarelli-Kohn-Nirenberg partial regularity theory [1]. It would be also worth mentioning that the idea of examining space-time fractal dimension of singular set was initiated by Scheffer [13, 14] while Leray worked on time singular set in his novel work [10].

### 1.1 The box-counting dimension

Instead of directly proving singularity formation, we measure the geometric information about how the potential singularities can be distributed. This provides another window called partial...
regularity, through which we can view singularities as a whole. To state our result, we define the usual parabolic cylinders
\[ Q_r(z) = Q_r(x,t) = B_r(x) \times (t - r^{2\alpha}, t] \]
for any \( z \in \mathbb{R}^3 \times (0,T) \). Throughout the paper, we set
\[ Q_r := Q_r(0,0) \text{ and } B_r := B_r(0). \]
Then we recall the parabolic Hausdorff dimension as follows. Here we let \( E \subset \mathbb{R}^3 \times \mathbb{R} \).

**Definition 1.1.** For fixed \( \delta > 0 \), denote by \( \mathcal{C}(E,\delta) \) the collection of all coverings of \( \{Q_{r_j}(z_j)\} \) that covers \( E \) with \( 0 < r_j \leq \delta \). The **parabolic Hausdorff measure** of \( E \) is defined as
\[ \mathcal{H}^\beta(E) = \lim_{\delta \to 0} \inf_{\mathcal{C}(E,\delta)} \sum_j r_j^\beta \]
and the **parabolic Hausdorff dimension** of \( E \) is defined by
\[ \dim_{\mathcal{H}}(E) = \inf\{\beta : \mathcal{H}^\beta(E) = 0\}. \]
We notice that the above usual Hausdorff dimension sometimes doesn’t capture certain important features of sets. For example, let us consider the analogous 1-dimensional Hausdorff dimension.

Then for \( A = \{ \frac{1}{n} \} \cup \{0\} \subset \mathbb{R} \) which has an accumulation point at 0 we still have that \( A \)'s Hausdorff dimension is zero. We introduce the box-counting dimension which would provide more information on \( E \) compared to the usual Hausdorff dimension.

**Definition 1.2.** The (upper) **parabolic box-counting dimension** of \( E \) is defined as
\[ \dim_{\mathcal{B}}(E) = \limsup_{r \to 0} \frac{\log \mathcal{N}(A,r)}{-\log r} \]
where we denote by \( \mathcal{N}(A,r) \) the minimum number of parabolic cylinders \( Q_r(z) \) required to cover \( A \).
We see that if we consider the analogous 1-dimensional box-counting dimension, as we did in the previous example, then we conclude that the box-counting dimension of \( A \) is 1/2, not zero. In general, one can check that
\[ \dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{B}}(E) \]
and the inequality could be strict. See [11] for more details.

### 1.2 Main result

Throughout the paper, we call a point \( z \in \mathbb{R}^3 \times \mathbb{R} \) **regular** if a given Leray-Hopf weak solution \((u,p)\) is continuous in some cylinder \( Q_r(z) \). Any point which is not regular is called **singular**. We denote by \( \mathcal{S} \) the set of all singular points.

**Theorem 1.3.** Let \( 1 < \alpha < 5/4 \). If \((u,p)\) is a suitable weak solution of \((1.1)\), then the box-counting dimension of \( \mathcal{S} \) is bounded by \( J(\alpha) \).

**Remarks on Theorem 1.3.**

1. This improves the previous bound \( L(\alpha) = \frac{15-2\alpha-8\alpha^2}{3} \) that was obtained in [3], generalizing the result in [17].
2. The box-counting dimension of the singular set can be effectively measured via the so-called $\varepsilon$-regularity theorem, which is Proposition [1.4] in our case.

For any $f : \mathbb{R}^3 \times (0, T) \to [0, \infty)$, we denote by $\mathcal{M}f$ the maximal function

$$\mathcal{M}f(x, t) = \sup_{r > 0} \frac{1}{r^3} \int_{B_r(x)} f(y, t) \, dy.$$ 

We also define the extended cylinder $Q^*(r)$ by $Q^*(r) = Q(r) \times [0, r)$. Then our key proposition is stated as the following.

**Proposition 1.4.** Let $1 < \alpha < 5/4$ and set $b := 3 - 2\alpha$. There exists a positive constant $\varepsilon$, depending only on $\alpha$, such that the following holds: for each $\gamma < L(\alpha) - J(\alpha)$, there exists $0 < \rho_0 < 1$ such that if $(u, p)$ is a suitable weak solution which satisfies

$$\int_{Q^*(z, 8\rho)} y^b |\Delta_b u|^2 \, dz^* + \int_{Q(z, 8\rho)} \left(\mathcal{M}|u|^2\right)^{\frac{3+2\alpha}{4}} \, dy + \int_{Q(z, 8\rho)} |\nabla u|^2 \, dz + |p - [p]_{2\rho}|^{\frac{3+2\alpha}{4}} + |\nabla p|^{\frac{3+2\alpha}{4}} \, dz \leq (8\rho)^{L(\alpha) - \gamma}$$

for some $0 < \rho < \rho_0$, then $z$ is a regular point.

**Remark 1.5.** The extension $u^*$ of $u$ is defined in Proposition [2.2].

**Remark 1.6.** Non-locality of the fractional Laplacian is encoded by the first two quantities.

### 1.3 Proof of Theorem [1.3]

**Proof of Theorem [1.3]** Our strategy is to employ Proposition [1.4]. Assume, for a contradiction, that

$$\dim_B(S) > J(\alpha).$$

Then we can find a constant $\delta \in (J(\alpha), \dim_B(S))$. Thanks to the definition of the box-counting dimension, there exists a positive sequence $a_i \to 0$ such that

$$\mathcal{N}(S, a_i) > \left(\frac{1}{a_i}\right)^\delta.$$  \hspace{1cm} (1.4)

Denote by $(z_{j})_{j=1}^{N(S, a_i)} \subset S$ an arbitrary collection of $a_i$-separated points in $S$. We observe that Theorem [1.4] implies

$$\int_{Q_{a_i}(z_j)} y^b |\Delta_b u|^2 \, dz^* + \int_{Q_{a_i}(z_j)} |\nabla u|^2 + \left(\mathcal{M}|u|^2\right)^{\frac{3+2\alpha}{4}} + |p - [p]_{2\rho}|^{\frac{3+2\alpha}{4}} + |\nabla p|^{\frac{3+2\alpha}{4}} \, dz > a_i^{L(\alpha) - \gamma}$$

for any $z_j$ and any $\gamma < L(\alpha) - J(\alpha)$. Next, summing it up over $j$ gives

$$\sum_{j=1}^{\mathcal{N}(S, a_i)} \left(\int_{Q_{a_i}(z_j)} y^b |\Delta_b u|^2 \, dz^* + \int_{Q_{a_i}(z_j)} |\nabla u|^2 + \left(\mathcal{M}|u|^2\right)^{\frac{3+2\alpha}{4}} + |p - [p]_{2\rho}|^{\frac{3+2\alpha}{4}} + |\nabla p|^{\frac{3+2\alpha}{4}} \, dz\right)$$

$$\geq \mathcal{N}(S, a_i)a_i^{L(\alpha) - \gamma}.$$

Since $z_j$'s are $a_i$-separated points, we notice that there exists an absolute constant $K > 0$ such that the left-hand side is bounded by $K$: for example, we have $\nabla p \in L^{\frac{3+2\alpha}{4}}(\mathbb{R}^3 \times (0, T))$, see
Proposition A.1 in Appendix. The finiteness of $\|\nabla u\|_{L^2(\mathbb{R}^3 \times (0,T))}$ comes from the simple inequality $\|\nabla u\|_{L^2(\mathbb{R}^3 \times (0,T))}^2 \leq \|u\|_{L^2(\mathbb{R}^3 \times (0,T))}^2 + \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{R}^3 \times (0,T))}^2$ by decomposing the Fourier side into $|\xi| \leq 1$ and $|\xi| > 1$, combined with the definition of Leray-Hopf weak solution.

For the right-hand side, we use (1.4) to get

$$N(S,a_i)\alpha_i L^2(\alpha) - \gamma_i \epsilon > a_i \alpha_i - \gamma_i - \delta_i \epsilon$$

which leads to

$$K > a_i \alpha_i - \gamma_i - \delta_i \epsilon.$$

Pick $\gamma = L(\alpha) - \delta_i J(\alpha)$. Since $\delta$ has been chosen to be bigger than $J(\alpha)$, we immediately see that such $\gamma$ satisfies $\gamma < L(\alpha) - J(\alpha)$. Moreover, we know that $\delta \leq L(\alpha)$ because $L(\alpha)$ is the previously known upper bound for $\dim_S(S)$. It means that $\gamma \geq 0$, which guarantees the validity of our choice of $\gamma$. Therefore, with the chosen $\gamma$, we have

$$K > a_i \frac{J(\alpha)}{2} \epsilon.$$

Letting $i \to \infty$ gives a contradiction against the above inequality. The proof is complete.

\[\square\]

2 Preliminaries

We start by introducing the usual notion of Leray-Hopf weak solutions.

**Definition 2.1.** Let $u_0 \in L^2(\mathbb{R}^3)$ be a given divergence-free initial condition. A pair $(u, p)$ is a Leray-Hopf weak solution of (1.1) on $\mathbb{R}^3 \times (0,T)$ if the followings hold:

1. $u \in L^\infty((0,T), L^2(\mathbb{R}^3)) \cap L^2((0,T), H^\alpha)$;
2. $u$ solves (1.1) in the sense of distributions;
3. $p$ is the potential-theoretic solution of $-\Delta p = \div (\div (u \otimes u))$;
4. $u$ satisfies the global energy inequality

$$\frac{1}{2} \int |u|^2(x,t) \, dx + \int_s^t \int |(-\Delta)^{\alpha/2} u|^2(x,\tau) \, dx \, d\tau \leq \frac{1}{2} \int |u_0|^2(x) \, dx$$

for almost every $s$ and every $t > s$.

2.1 Suitable weak solutions

To define suitable weak solutions of (1.1), we should leverage the following extension theorem. Here we use $\nabla$, $\Delta$ for operators defined on $\mathbb{R}_+^4$ that are analogous to $\nabla$, $\Delta$ respectively.

**Proposition 2.2 ([19]).** Let $u \in H^\alpha(\mathbb{R}^3)$ with $\alpha \in (1,2)$ and set $b := 3 - 2\alpha$. Define the differential operator $\Delta_b$ by

$$\Delta_b u^* := \Delta u^* + \frac{b}{y} \partial_y u^* = \frac{1}{y^b} \div (y^b \nabla u^*).$$

Then there exists a unique extension $u^*$ of $u$ in the weighted space $L^2_{\text{loc}}(\mathbb{R}^4, y^b)$ which satisfies $\Delta_b u^* \in L^2(\mathbb{R}^4, y^b)$ and

$$\Delta_b^2 u^*(x,y) = 0$$

(2.1)
and the boundary conditions

\[ u^*(x, 0) = u(x) \]
\[ \lim_{y \to 0} y^{1-\alpha} \partial_y u^*(x, y) = 0. \]

Moreover, there is a constant \( c_\alpha \) depending only on \( \alpha \), such that

1. the fractional Laplacian \((-\Delta)^\alpha u\) is given by the formula

\[ (-\Delta)^\alpha u(x) = c_\alpha \lim_{y \to 0} y^b \partial_y \Delta_b u^*(x, y); \]

2. the following energy identity holds

\[ \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2} u^2 \, dx = \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 \, d\xi = c_\alpha \int_{\mathbb{R}^4} y^b |\Delta_b u^*|^2 \, dy \] (2.2)

3. for every extension \( v \in L^2_{loc}(\mathbb{R}^4, y^b) \) of \( u \) with \( \Delta_b v \in L^2(\mathbb{R}^4, y^b) \)

\[ \int_{\mathbb{R}^4} y^b |\Delta_b u^*|^2 \, dy \leq \int_{\mathbb{R}^4} y^b |\Delta_b v|^2 \, dy. \] (2.3)

The above extension allows us to define suitable weak solutions. In the below definition, we use Einstein’s convention for the sum on repeated indices in the following definition.

**Definition 2.3.** Let \( 1 < \alpha < 2 \). A Leray-Hopf weak solution \((u, p)\) on \( \mathbb{R}^3 \times [0, T] \) is a suitable weak solution if the following inequality holds a.e. \( t \in [0, T] \) and all nonnegative test functions \( \varphi \in C^\infty_c(\mathbb{R}^4_+ \times (0, T)) \) with \( \partial_y \varphi(\cdot, 0, \cdot) = 0 \) in \( \mathbb{R}^3 \times (0, T) \):

\[
\int_{\mathbb{R}^3} \varphi(x, 0, t) |u|^2 \, dx + 2c_\alpha \int_0^t \int_{\mathbb{R}^4_+} y^b |\Delta_b u^*|^2 \varphi \, dy \, ds \\
\quad \leq \int_0^t \int_{\mathbb{R}^3} \left[ |u|^2 \partial_t \varphi |_{y=0} + (|u|^2 + p) u \cdot \nabla \varphi |_{y=0} \right] \, dx \, ds \\
\quad - 2c_\alpha \int_0^t \int_{\mathbb{R}^4_+} y^b \Delta_b u^*_i (2 \nabla \varphi \cdot \nabla u^*_i + u^*_i \Delta_b \varphi) \, dy \, ds,
\]

where the constant \( c_\alpha \) depends only on \( \alpha \) and comes from the previous extension proposition.

### 2.2 \( \varepsilon \)-regularity theorem

We end this section by stating the important \( \varepsilon \)-regularity theorem that will serve as our key ingredient for proving Theorem 1.4.

**Proposition 2.4** ([3]). There exists a positive constant \( \varepsilon_0 \), depending only on \( \alpha \), such that the following holds: if \((u, p)\) is a suitable weak solution to (1.1) and satisfies

\[
\frac{1}{r^{6-4\alpha}} \int_{Q_r} |u|^3 + |p - [p]_r|^{3/2} \, dx \, dt + T(r) < \varepsilon_0,
\]

then \( u \in C^{0,\kappa}(Q_{r/2}; \mathbb{R}^3) \) for some \( \kappa > 0 \).
3 Basic lemmas

Here we gather some known estimates. One is the local-in-time supremum of $L^2$ energy of $u$ measured on a ball, and the other is the local-in-space fractional Sobolev embedding in the fashion of Yang's extension introduced in the previous section.

Lemma 3.1. If $(u, p)$ is a suitable weak solution of (1.1) in $\mathbb{R}^3 \times [-\frac{4}{3} r^{2\alpha}, 0]$, then we have

$$\sup_{-r^{2\alpha} \leq t \leq 0} \frac{1}{r^{5-4\alpha}} \int_{B_r} |u|^2 \, dx + \frac{1}{r^{5-4\alpha}} \int_{Q_r^*} y^b |\Delta u|^2 \, dy \, dt$$

$$\leq \frac{C}{r^{5-2\alpha}} \int_{Q_{\frac{r}{3^\alpha}}} |u|^2 \, dx \, dt + \frac{C}{r^{6-4\alpha}} \int_{Q_{\frac{r}{3^\alpha}}} |u| \left( |u|^2 - g(t) + |p| \right) \, dx \, dt$$

$$+ C r^{5\alpha - 2} \int_{-\frac{4}{3} r^{2\alpha}}^0 \sup_{R \geq r} R^{-3\alpha} \int_{B_R} |u|^2 \, dx \, dt$$

for any $g \in L^1([0, (4r/3)^{2\alpha}])$.

Proof. It is enough to show that

$$\sup_{-\frac{3}{4} r^{2\alpha} \leq t \leq 0} \int_{B_{3/4}} |u|^2 + \int_{Q_{3/4}^*} y^b |\Delta u|^2 \, dx \, dt$$

$$\leq C \int_{Q_1} |u| + C \int_{Q_1} |u| \left( |u|^2 - f(t) + |p| \right) + C \int_{-1}^0 \sup_{R \geq 1} R^{-\alpha} \int_{B_R} |u|^2 \, dx$$

(3.2)

because we can recover (3.1) by applying (3.2) to the scaled solution

$$u_{\frac{3}{4} r}(x, t) := (4r/3)^{2\alpha - 1} u \left( (4r/3) x, (4r/3)^{2\alpha} t \right)$$

supplemented with

$$f(t) := (4r/3)^{4\alpha - 2} g((4r/3)^{2\alpha} t).$$

Indeed, (3.2) is a direct consequence of Lemma 3.2 of [3] and Lemma 3.4 in the same reference. Notice that $\left( \int_{B_R} |u| \, dx \right)^2 \leq \int_{B_R} |u|^2 \, dx$ by Hölder’s inequality. The proof is complete.

To exploit the fractional dissipation in a natural way, we need certain fractional Sobolev embeddings. The issue is that the spatially-local Sobolev embeddings for the scale-invariant quantities cannot be directly applied due to the non-locality of $(-\Delta)^\alpha$. This can be overcome by the extension-wise embedding lemma. See Lemma 4.4 in [3].

Lemma 3.2 ([3]). Let $(u, p)$ be a suitable weak solution. For $0 < r \leq \rho$, there holds

$$\|u\|_{L^{\frac{6}{\alpha-2\alpha}}(B_r)}^2 \lesssim \int_{B_r^*} y^b |\Delta u|^2 \, dy + \rho^{\alpha+3} \sup_{R \geq \rho} R^{-3\alpha} \int_{B_R} |u|^2 \, dx \quad \forall \alpha \in (1, \frac{5}{4}).$$
4 Iteration lemmas

We introduce the following scaling-invariant quantities:
\[
\mathcal{A}(r) := \sup_{-r^{2\alpha} \leq t \leq 0} \frac{1}{r^{5-4\alpha}} \int_{B(r)} |u|^2 \, dx \\
\mathcal{C}(r) := \frac{1}{r^{6-4\alpha}} \int_{Q_r} |u|^3 \, dx \, dt \\
\mathcal{D}(r) := \frac{1}{r^{6-4\alpha}} \int_{Q_r} |p - [p]_r|^{3/2} \, dx \, dt \\
\mathcal{E}(r) := \frac{1}{r^{6-4\alpha}} \int_{Q^*_r} y^b |\nabla u|^2 \, dy \, dt \\
\mathcal{T}(r) := r^{5\alpha - 2} \int_0^1 \frac{1}{R^{3\alpha}} \int_{B_R} |u|^2 \, dt.
\]

The system (1.1) enjoys the scaling property: if \((u, p)\) is a solution pair to (1.1), then \(u_\lambda(x, t) := \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t)\) and \(p_\lambda(x, t) := \lambda^{4\alpha-2} p(\lambda x, \lambda^{2\alpha} t)\) solve (1.1) as well for any \(\lambda > 0\). Such a scaling property gives rise to all the quantities in (4.1), except for \(\mathcal{E}\) which involves the extension \(u^*\) of \(u\). Setting up \(u^*_\lambda := \lambda^{2\alpha-1} u^*(\lambda x, \lambda y, \lambda^{2\alpha} t)\), we see that \(u^*_\lambda\) satisfies not only the boundary condition \(u^*_\lambda|_{y=0} = u_\lambda\), but also the weighted Laplacian (2.1) required for extensions. Due to the uniqueness of the extension \((u_\lambda)^*\), we obtain that \((u_\lambda)^* = u^*_\lambda\). From this spatially 4-dimensional scaling for \(u^*\) and the scaling for the weight \(y^b\) via \(y^b = (\lambda y)^b\lambda^{-b}\), the scaling-invariant quantity \(\mathcal{E}(r)\) can be derived as in (4.1).

**Lemma 4.1.** Let \(\alpha \in (1, 5/4)\). Then, for \(0 < r \leq \frac{1}{2}\rho\), there holds
\[
\mathcal{C}(r) \lesssim \left(\frac{\rho}{r}\right)^{15-6\alpha} \mathcal{A}^{1/2}(\rho) \left(\mathcal{E}(\rho) + \mathcal{T}(\rho)\right) + \left(\frac{r}{\rho}\right)^{6\alpha - 3} \mathcal{A}^{3/2}(\rho).
\]

**Proof.** Let \(0 < r \leq \frac{1}{2}\rho\). By the triangular inequality we observe that
\[
\int_{B_r} |u|^3 \, dx \lesssim \int_{B_r} |u - [u]_\rho|^3 \, dx + \int_{B_r} |[u]_\rho|^3 \, dx.
\]
The first summand on the right-hand side can be estimated by Hölder’s inequality and the Poincaré-Sobolev inequality
\[
\int_{B_r} |u - [u]_\rho|^3 \, dx \lesssim \left(\int_{B_r} |u - [u]_\rho|^2 \, dx\right)^{1/2} \left(\int_{B_r} |u - [u]_\rho|^4 \, dx\right)^{1/4}
\lesssim \|u\|_{L^2(B_r)}^{2\alpha - \frac{3}{2}} \left(\int_{B_r} |u - [u]_\rho|^\frac{6}{3-2\alpha} \, dx\right)^{\frac{3-2\alpha}{4}}
\lesssim \|u\|_{L^2(B_r)}^{2\alpha - \frac{3}{2}} \left(\int_{B^*_r} \int_{B_9} y^b |\nabla u|^2 \, dy \, dx + \rho^{\alpha + 3} \sup_{R \geq \frac{r}{2}} \int_{B_R} |u|^2 \, dx\right).
\]
For the second summand, we notice that \(|[u]_\rho|^3 \lesssim \rho^{-9/2}(\int_{B_r} |u|^2)^{3/2}\) by Hölder’s inequality and so
\[
\int_{B_r} |[u]_\rho|^3 \, dx \lesssim \frac{r^3}{\rho^{9/2}} (\int_{B_r} |u|^2 \, dx)^{3/2}.
\]
Integrating (4.3) in time over \((-r^{2\alpha}, 0)\) and multiplying it by \(\frac{1}{r^{6-4\alpha}}\), the above estimates finish the proof. □
Lemma 4.2. For $0 < r \leq \frac{1}{2} \rho$, there holds
\[
\mathcal{D}(r) \lesssim \left( \frac{r}{\rho} \right)^{4\alpha - \frac{3}{2}} \mathcal{D}(\rho) + \left( \frac{\rho}{r} \right)^{6-4\alpha} C(\rho).
\]

Proof. Let $\phi$ be a non-negative function supported in $B_\rho$, which is identically 1 on $B_{\rho/2}$. We decompose $p$ into the sum $p = p_1 + p_2$ where $p_1$ is defined by
\[
p_1(x,t) := \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \left\{ \partial_i \partial_j \left[ \left( u_i - (u_i)_\rho \right) \left( u_j - (u_j)_\rho \right) \phi \right] \right\}(y,t) dy.
\]
By Calderon-Zygmund theory, $p_1$ can be estimated as
\[
\int_{B_\rho} |p_1|^{3/2} dx \leq c \int_{B_\rho} |u|^3 dx.
\]
It leads to
\[
\int_{B_r} |p_1 - [p_1]_r|^{3/2} dx \lesssim \int_{B_r} |p_1|^{3/2} dx \lesssim \int_{B_\rho} |u|^3 dx.
\]
Therefore, we obtain
\[
\frac{1}{r^{6-4\alpha}} \int_{Q_r} |p_1 - [p_1]_r|^{3/2} dx dt \lesssim \frac{1}{r^{6-4\alpha}} \int_{-r^{2\alpha}}^0 \int_{B_r} |u|^3 dx dt \leq \left( \frac{\rho}{r} \right)^{6-4\alpha} \int_{Q_\rho} |u|^3 dx dt \quad (4.4)
\]
On the other hand, we have $\Delta p = \Delta p_1$ in $B_{\rho/2}$ and therefore
\[
\Delta p_2 = 0 \quad \text{in } B_{\rho/2}.
\]
Being harmonic, $p_2$ satisfies
\[
\sup_{B_r} |p_2 - [p_2]_r| \leq cr \sup_{B_{\rho/2}} |\nabla p_2(y,t)|
\]
\[
\leq cr \frac{1}{\rho^4} \int_{B_\rho} |p_2 - [p_2]_\rho| dy
\]
\[
\leq cr \frac{1}{\rho^4} \left( \int_{B_\rho} |p_2 - [p_2]_\rho|^{3/2} dy \right)^{2/3}.
\]
It follows that
\[
\frac{1}{r^{6-4\alpha}} \int_{Q_r} |p_2 - [p_2]_r|^{3/2} dx dt \leq \frac{1}{r^{6-4\alpha}} \int_{-r^{2\alpha}}^0 \int_{B_r} \left( \sup_{y \in B_r} |p_2 - [p_2]_r| \right)^{3/2} dx dt
\]
\[
\lesssim \frac{1}{r^{6-4\alpha}} \cdot r^{3/2} \cdot \frac{1}{\rho^9/2} \int_{-r^{2\alpha}}^0 \int_{B_\rho} |p_2 - [p_2]_\rho|^{3/2} dy dt
\]
\[
= \frac{r^{4\alpha - \frac{3}{2}}}{\rho^9/2} \int_{-r^{2\alpha}}^0 \int_{B_\rho} |p_2 - [p_2]_\rho|^{3/2} dy dt \quad (4.5)
\]
By the simple identity $p - [p]_r = p_1 - [p_1]_r + p_2 - [p_2]_r$, combining it with (4.4) and (4.5), we conclude that
\[
D(r) \lesssim \left( \frac{r}{\rho} \right)^{4\alpha - \frac{3}{2}} D(\rho) + \left( \frac{\rho}{r} \right)^{6-4\alpha} C(\rho)
\]
as desired.
5 Proof of Proposition 1.4

Proof of Proposition 1.4. Without loss of generality, we assume that $z = (0, 0)$. We further assume that for some fixed $0 < \rho < \rho_0 < 1$

$$
\int_{Q_{8\rho}} y^b |\Sigma_b u^*|^2 \; dz^* + \int_{Q_{8\rho}} |\nabla u|^2 \; dz^* + \int_{Q_{8\rho}} \left( M|u|^2 \right)^{\frac{3+2\alpha}{4}} + |p - [p]_{2\rho}|_{\frac{3+2\alpha}{4}} + |\nabla p|_{\frac{3+2\alpha}{4}} \; dz 
\leq (8\rho)^{L(\alpha) - \gamma \varepsilon},
$$

(5.1)

where $L(\alpha) = \frac{15 - 2\alpha - 8\alpha^2}{3}$. It suffices to find maximal $\gamma = \gamma(\alpha)$ such that $z$ would be a regular point and then check that $J(\alpha) = L(\alpha) - \gamma(\alpha)$. We notice that the suitable $\rho$ will be selected later as well.

Step 1. Setting up the iteration: To invoke the $\varepsilon$-regularity criterion in Proposition 2.4, we want to find sufficiently small $r_N > 0$ such that

$$
\mathcal{C}(r_N) + D(r_N) + T(r_N) < \varepsilon_0.
$$

(5.2)

It proves that $z$ is a regular point. To this end, we design an iteration procedure as follows. For $\eta \geq 1$ and $\zeta > 0$ that would be determined later, define the sequence $r_k = \rho^\eta + k\zeta$ for $k = 0, 1, \ldots, N$.

The common ratio of this geometric sequence is denoted by $\theta = \rho^\zeta$. We notice that this sequence is a strictly decreasing sequence, and so the proof reduces from finding small $r_N$ to finding large $N > 0$ such that (5.2) is fulfilled. By applying Lemma 4.2 repeatedly, with the aid of the simple fact that $\mathcal{C}(r) \lesssim \theta^{4\alpha - 6} \mathcal{C}((\theta^{-1})r)$, we have

$$
\mathcal{C}(r_N) + D(r_N) \lesssim \theta^{4\alpha - 6} \mathcal{C}(r_{N-1}) + D(r_N) \\
\lesssim \theta^{4\alpha - 6} \mathcal{C}(r_{N-1}) + \theta^{4\alpha - \frac{3}{2}} D(r_{N-1}) \\
\vdots \\
\lesssim \sum_{i=1}^{N} \theta^{(4\alpha - \frac{3}{2})(i-1) + 4\alpha - 6} C(r_{N-i}) + \theta^{(4\alpha - \frac{3}{2})N} D(r_0) =: I + II.
$$

(5.3)

Therefore, it suffices to estimate $I$, $II$, and $T(r_N)$.

Step 2. Estimate on the tail term $T(r_N)$. We can leverage the maximal function to control the non-locality of $T(r_N)$. We first observe that

$$
\sup_{R \geq \frac{1}{4}} \int_{B_R} |u|^2 \; dy \leq C \int_{B_{\frac{1}{2}(x)}} |u|^2 \; dy \; dx \leq C \int_{B_{\frac{1}{2}}} M|u|^2 \; dx \quad \text{for any } r > 0.
$$

(5.4)
Then we obtain

\[ T(r_N) = r_N^{5\alpha-2} \int_{(r_N)^{2\alpha}}^{0} \sup_{R \geq \frac{r_N}{2}} R^{-3\alpha} \int_{B_R} |u|^2 \, dt \]

\[ \leq C r_N^{5\alpha-2} \int_{(r_N)^{2\alpha}}^{0} \sup_{R \geq \frac{r_N}{3}} R^{-3\alpha} \int_{B_{\frac{r_N}{2}}} \mathcal{M}|u|^2 \, dt \]

\[ \leq C r_N^{2\alpha-2} \int_{(r_N)^{2\alpha}}^{0} \left( \int_{B_{\frac{r_N}{2}}} \left( \mathcal{M}|u|^2 \right)^{\frac{3+2\alpha}{3+2\alpha}} \right)^{\frac{3}{3+2\alpha}} \frac{6\alpha}{r_N^{3+2\alpha}} \, dt \]

\[ \leq C \rho^{(\eta+N\zeta)(4\alpha-2)+(L-\gamma)\frac{3}{3+2\alpha}} \]

by Hölder’s inequality and the assumption (5.1). Thus we have

\[ T(r_N) \leq C \rho^{J_1 \varepsilon \frac{3}{3+2\alpha}} \] (5.6)

where we set

\[ J_1 := (\eta + N\zeta)(4\alpha - 2) + (L-\gamma)\frac{3}{3+2\alpha}. \]

**Step 3. Energy estimates.** Our goal is to use Lemma 4.2 to control the pressure term $II$. To this end, we need the estimates on $A(\rho)$ and $E(\rho)$ in advance. Both terms can be estimated by the local energy inequality which implicitly encodes certain non-locality of $(-\Delta)^\alpha$ with the extension $u^*$. For the term $A(\rho)$, we employ (3.1) to get

\[ A(\rho) = \sup_{t \in [-\rho^{2\alpha}, 0]} \frac{1}{\rho^{5-4\alpha}} \int_{B_\rho} |u|^2 \]

\[ \leq \frac{C}{\rho^{5-2\alpha}} \int_{Q_{2\rho}} |u|^2 + \frac{C}{\rho^{6-4\alpha}} \int_{Q_{2\rho}} |u| \left( |u|^2 - |[u]^2| \right)_{2\rho} + \frac{C}{\rho^{6-4\alpha}} \int_{Q_{2\rho}} |u| p - |p|_{2\rho} \]

\[ + C \rho^{5\alpha-2} \int_{-(2\rho)^{2\alpha}}^{0} \sup_{R \geq \rho} R^{-3\alpha} \int_{B_R} |u|^2 \, dx \, dt. \] (5.7)

The first term on the right-hand side is estimated as

\[ \frac{1}{\rho^{5-2\alpha}} \int_{Q_{2\rho}} |u|^2 \leq \frac{1}{\rho^{5-2\alpha}} \left( \int_{Q_{2\rho}} |u|^\frac{6+4\alpha}{3} \right)^{\frac{3}{3+2\alpha}} \left( \int_{Q_{2\rho}} |u|^\frac{2\alpha}{3+2\alpha} \right)^{\frac{3}{3+2\alpha}} \]

\[ \leq C \rho^{4\alpha-5+(L-\gamma)\frac{3}{3+2\alpha} \varepsilon \frac{3}{3+2\alpha}} \] (5.8)

by Hölder’s inequality and the assumption (5.1). For the second term, we use Hölder’s inequality and the Poincaré-Sobolev inequality combined with (5.1) to yield

\[ \frac{1}{\rho^{6-4\alpha}} \int_{Q_{2\rho}} |u| \left( |u|^2 - |[u]^2| \right)_{2\rho} \leq C \rho^{6\alpha-\frac{15}{2}} \left( \int_{Q_{2\rho}} |u \nabla u|^{\frac{3+2\alpha}{4+\alpha}} \right)^{\frac{6+2\alpha}{4+\alpha}(3+2\alpha)} \left( \int_{Q_{2\rho}} |u|^{\frac{6+4\alpha}{3+\alpha}} \right)^{\frac{9-3\alpha}{\alpha(3+\alpha)}} \]

\[ \leq \rho^{6\alpha-\frac{15}{2}+L-\gamma \varepsilon}. \] (5.9)
The term involving the pressure is computed similarly:

\[
\frac{1}{\rho^{6-4\alpha}} \int_{Q_{2r}} |u||p - [p]_{2r}| \\
\leq C \rho^{4\alpha - 6 + \frac{4\alpha - 3}{2}} \left( \int_{Q_{2r}} |u|^{\frac{4+4\alpha}{3}} \right)^{\frac{3}{3+4\alpha}} \left( \int_{Q_{2r}} |p - [p]_{2r}|^{\frac{3+2\alpha}{3}} \right)^{\frac{15-12\alpha}{8+4\alpha}} \left( \int_{Q_{2r}} |\nabla p|^{\frac{3+2\alpha}{4}} \right)^{\frac{16\alpha - 12}{6+4\alpha}} \\
\leq C \rho^{6\alpha - \frac{15}{2} + L - \gamma \varepsilon}.
\]

(5.10)

Here we have used Hölder’s inequality, the Poincaré-Sobolev inequality, and the assumption (5.1).

To estimate the fourth term, we observe the simple fact that

\[
\sup_{R \geq \frac{r}{4}} \int_{B_R} |u|^2 \leq C \int_{B_{\frac{r}{4}}} \int_{B_{\frac{r}{4}}(x)} |u|^2 \, dy \, dx \leq C \int_{B_{\frac{r}{4}}} M|u|^2 \, dx \quad \text{for any } r > 0
\]

which leads to

\[
\rho^{5\alpha - 2} \sup_{R \geq \frac{r}{2}} \int_{B_{\frac{r}{2}}} |u|^2 \, dx \, dt \leq C \rho^{2\alpha - 2} \int_{Q_{2r}} |u|^2 \, dx \\
\leq C \rho^{4\alpha - 2} \left( \int_{Q_{2r}} \left( M|u|^2 \right)^{\frac{3+2\alpha}{3}} \right)^{\frac{3}{3+2\alpha}} \\
\leq \rho^{4\alpha - 2 + (L - \gamma) \frac{3}{3+2\alpha} \varepsilon} \frac{3}{3+2\alpha}
\]

by Hölder’s inequality and the assumption (5.1). If we further assume that

\[
\gamma \leq \frac{4\alpha - 3}{4\alpha} L,
\]

(5.12)

which allows the exponent of \( \rho \) in (5.8) to be smaller than that of (5.9), from the previous estimates (5.8), (5.9), (5.10), and (5.11) all together, we conclude that

\[
A(\rho) \leq C \rho^{4\alpha - 5 + (L - \gamma) \frac{3}{3+2\alpha} \varepsilon} \frac{3}{3+2\alpha}
\]

(5.13)

For the term \( E(\rho) \), our assumption (5.1) immediately implies

\[
E(\rho) \leq C \rho^{4\alpha - 5 + (L - \gamma) \varepsilon}
\]

(5.14)

**Step 4. Estimate on the pressure term II.** We first claim that

\[
D(r) \leq C r^{6\alpha - \frac{15}{2}} \left( \int_{Q_r} |\nabla p|^{\frac{3+2\alpha}{3}} \right)^{\frac{8\alpha - 6}{3+2\alpha}} \left( \int_{Q_r} |p - [p]_r|^{\frac{3+2\alpha}{3}} \right)^{\frac{9-6\alpha}{3+2\alpha}}
\]

(5.15)

for any \( r > 0 \). By Hölder’s inequality and the Sobolev-Poincaré inequality, we observe that

\[
D(r) = \frac{1}{r^{6-4\alpha}} \int_{Q_r} |p - [p]_r|^2 \, dx \, dt \\
\leq \frac{1}{r^{6-4\alpha}} \int_{r^{2\alpha}}^{0} \left( \int_{B_r} |p - [p]_r|^{\frac{3+2\alpha}{3}} \, dx \right)^{\frac{3}{3+2\alpha}} \left( \int_{B_r} |p - [p]_r|^{3+2\alpha} \, dx \right)^{\frac{3}{3+2\alpha}} \, dt \\
\leq C \frac{1}{r^{6-4\alpha}} \int_{r^{2\alpha}}^{0} \left( \int_{B_r} |\nabla p|^{\frac{3+2\alpha}{3}} \, dx \right)^{\frac{3}{3+2\alpha}} \left( \int_{B_r} |p - [p]_r|^{3+2\alpha} \, dx \right)^{\frac{3}{3+2\alpha}} \, dt \\
\leq C r^{6\alpha - \frac{15}{2}} \left( \int_{Q_r} |\nabla p|^{\frac{3+2\alpha}{3}} \right)^{\frac{8\alpha - 6}{3+2\alpha}} \left( \int_{Q_r} |p - [p]_r|^{\frac{3+2\alpha}{3}} \right)^{\frac{9-6\alpha}{3+2\alpha}}
\]
as claimed. A direct consequence of the above claim \eqref{5.15} is the following estimate on $II$

$$II = \theta^{4\alpha - \frac{3}{2}} D(r_0) \leq C \theta^{(4\alpha - \frac{3}{2}) N} \frac{r_0^{6\alpha - \frac{15}{2}}}{r_0} \left( \int_{Q_{r_0}} |\nabla p|^{\frac{3+2\alpha}{2}} \right)^{\frac{8\alpha - 6}{3+2\alpha}} \left( \int_{Q_{r_0}} |p - [p]_r|^{\frac{3+2\alpha}{2}} \right)^{\frac{9-6\alpha}{3+2\alpha}}$$

\begin{equation}
\leq C \rho^{(4\alpha - \frac{3}{2}) N \zeta + (6\alpha - \frac{15}{2}) \eta + (L - \gamma) \varepsilon} \tag{5.16}
\end{equation}

thanks to the assumption \eqref{5.1}. By our future choice of $\eta$ in \eqref{5.21}, we will further obtain

$$II \leq C \rho J_2 \varepsilon \tag{5.17}$$

with

$$J_2 := \left( 4\alpha - \frac{3}{2} \right) N \zeta + \left( 6\alpha - \frac{15}{2} \right) \eta + L - \gamma.$$  

5. Control over $I$. Now we have the estimates for $A(\rho)$ and $E(\rho)$, so we are in the position to complete the estimate on $C(r_k)$. By Lemma \eqref{4.1}, we start by

\begin{equation}
C(r_k) \leq C \left( \frac{\rho}{r_0} \right)^{\frac{15}{2} - 6\alpha} A^{1/2}(\rho) E(\rho) + C \left( \frac{r_k}{\rho} \right)^{6\alpha - 3} A^{3/2}(\rho). \tag{5.18}
\end{equation}

Noting that $r_k = \rho^{\eta + k \zeta}$, thanks to the estimates for $A(\rho)$ and $E(\rho)$ we get

$$C(r_k) \leq C \varepsilon^{\frac{2}{6+4\alpha}} \left( \rho^{(\eta + k \zeta)(6\alpha - \frac{15}{2}) + (L - \gamma) \frac{2\alpha}{3+2\alpha} + \rho^{(\eta + k \zeta)(6\alpha - 3) + (L - \gamma) \frac{2}{3+2\alpha}} \right).$$

Then some elementary comparison between exponents of $\rho$ leads to

\begin{equation}
I = \sum_{i=1}^{N} (4\alpha - \frac{3}{2})(i-1) + 4\alpha - 6 C(r_{N-i})
\leq C \varepsilon^{\frac{2}{6+4\alpha}} \rho^{-\frac{3}{2} + \frac{\eta + N \zeta}{2} + (L - \gamma) \frac{2\alpha}{3+2\alpha}} \sum_{i=1}^{N} \left( \rho^{(6-2\alpha)i + (L - \gamma) \frac{2\alpha}{3+2\alpha} + \rho^{(\eta + N \zeta) - \frac{2}{2}}} \right)^{\frac{2}{6\alpha + 4\alpha}} \tag{5.19}
\end{equation}

\begin{equation}
\leq C \varepsilon^{\frac{2}{6+4\alpha}} \rho^{-\frac{3}{2} + \frac{\eta + N \zeta}{2} + (L - \gamma) \frac{2\alpha}{3+2\alpha}} \left( \rho^{(6-2\alpha)\zeta + (L - \gamma) \frac{2\alpha}{3+2\alpha} + \rho^{N \zeta} \left( \frac{3}{2} - 2\alpha + \frac{9}{2} (\eta + N \zeta) - \frac{9}{2} \right) \right).
\end{equation}

In particular, if we further determine $\eta \geq 1$ via the relation

\begin{equation}
(6 - 2\alpha)\zeta + (L - \gamma) \frac{2\alpha}{3+2\alpha} = N \zeta \left( \frac{3}{2} - 2\alpha \right) + \frac{9}{2} (\eta + N \zeta) - \frac{9}{2}, \tag{5.20}
\end{equation}

or equivalently,

\begin{equation}
\eta = \frac{2}{9} \left( (6 - 2\alpha)\zeta + (2\alpha - 6)N \zeta + \frac{9}{2} (L - \gamma) \frac{2\alpha}{3+2\alpha} \right), \tag{5.21}
\end{equation}

which lets both exponents of $\rho$ in \eqref{5.19} coincide with each other, then we further obtain

\begin{equation}
I \leq C \varepsilon^{\frac{2}{6+4\alpha}} \rho^{J_3} \tag{5.22}
\end{equation}

where $J_3$ is defined by

\begin{equation}
J_3 := \frac{-16\alpha^2 + 56\alpha - 51}{6} \zeta + \frac{(4\alpha - 5)(4\alpha - 3)}{6} N \zeta + \frac{16\alpha^2 - 8\alpha + 27}{6(3 + 2\alpha)} (L - \gamma) + \frac{3(4\alpha - 5)}{2}.
\end{equation}

Step 6. Choosing parameters. In short, we have obtained

$$C(r_N) + D(r_N) + T(r_N) \leq C_1 r_N^{1/2} + C_2 r_N + C_3 r_N^{1/2}$$  \hspace{1cm} (5.23)

If we can choose \((\eta, \zeta, N, \gamma)\) such that \(J_i \geq 0\) for all \(i \in \{1, 2, 3\}\), then we obtain

$$C(r_N) + D(r_N) + T(r_N) \leq C_\varepsilon r_N^{1/2} \leq \varepsilon_0$$  \hspace{1cm} (5.24)

by taking sufficiently small \(\varepsilon > 0\), because \(\rho\) is strictly smaller than 1. By Proposition 2.4, we conclude that \(z\) is a regular point. Therefore it suffices to find such parameters \((\eta, \zeta, N, \gamma)\).

Since we have also used the assumptions that \(\eta \geq 1\) and \((5.12)\) for the estimates in the previous steps, actually the quadruple \((\eta, \zeta, N, \gamma)\) should satisfy the below five conditions corresponding to \(J_1 \geq 0, J_2 \geq 0, J_3 \geq 0, \eta \geq 1\), and \((5.12)\):

\[
\begin{align*}
\gamma &\leq L + \frac{9(4\alpha - 5)(3 + 2\alpha)}{16\alpha^2 - 8\alpha + 27} + \frac{(3 + 2\alpha)(4\alpha - 5)(4\alpha - 3)}{16\alpha^2 - 8\alpha + 27} \frac{N\zeta}{\alpha} + \frac{2(3 + 2\alpha)(4\alpha - 2)(6 - 2\alpha)}{16\alpha^2 - 8\alpha + 27} \zeta, \\
\gamma &\leq L + \frac{9(4\alpha - 5)(3 + 2\alpha)}{16\alpha^2 - 8\alpha + 18} + \frac{(3 + 2\alpha)(4\alpha - 5)(4\alpha - 3)}{16\alpha^2 - 8\alpha + 18} N\zeta + \frac{2(3 + 2\alpha)(4\alpha - 5)(6 - 2\alpha)}{3(16\alpha^2 - 8\alpha + 18)} \zeta, \\
\gamma &\leq L + \frac{9(4\alpha - 5)(3 + 2\alpha)}{16\alpha^2 - 8\alpha + 27} + \frac{(3 + 2\alpha)(4\alpha - 5)(4\alpha - 3)}{16\alpha^2 - 8\alpha + 27} N\zeta + \frac{9(3 + 2\alpha)(4\alpha - 5)}{16\alpha^2 - 8\alpha + 27} \zeta, \\
\gamma &\leq L + \frac{(3 + 2\alpha)(\alpha - 3)}{\alpha} N\zeta + \frac{(3 + 2\alpha)(3 - \alpha)}{\alpha} \zeta, \\
\gamma &\leq \frac{4\alpha - 3}{4\alpha} L.
\end{align*}
\]  \hspace{1cm} (5.25)

Note that we used the choice of \(\eta\) in \((5.21)\) and also that the conditions are stated in terms of the upper bound of \(\gamma\) because we hope to maximize \(\gamma\) to improve the given bound \(L = L(\alpha)\). Then one can check that it is appropriate to set

$$N\zeta = \frac{27(4\alpha - 5)}{64\alpha^3 - 272\alpha^2 + 300\alpha - 369}. \hspace{1cm} (5.26)$$

The above choice of \(N\zeta\) can be derived from the following heuristic argument. The other three upper bounds for \(\gamma\) in \((5.25)\) are expected to be relatively big compared with the two conditions related to \(J_2 \geq 0\) and \(J_3 \geq 0\). Thus it suffices to consider those two because they are the only plausible candidates for the most restrictive one. We notice that \(\zeta\) vanishes as \(N\) grows, so we temporarily assume \(\zeta \sim 0\) while \(N\zeta\) is still significant. Equating the second upper bound and the third one for \(\gamma\), we get \((5.26)\) as claimed.

Thanks to the particular choice of \(N\zeta\) we can finish the selection process rigorously as follows. For any fixed \(\gamma \geq 0\) which satisfies

\[
\begin{align*}
\gamma &< L(\alpha) + \frac{9(4\alpha - 5)(3 + 2\alpha)}{16\alpha^2 - 8\alpha + 27} + \frac{(3 + 2\alpha)(4\alpha - 5)(4\alpha - 3)}{16\alpha^2 - 8\alpha + 27} N\zeta \\
&= L(\alpha) - \left( \frac{27(3 + 2\alpha)(4\alpha - 5)^2(3 - 4\alpha)}{(16\alpha^2 - 8\alpha + 27)(64\alpha^3 - 272\alpha^2 + 300\alpha - 369)} + \frac{9(3 + 2\alpha)(5 - 4\alpha)}{16\alpha^2 - 8\alpha + 27} \right) \\
&= L(\alpha) - J(\alpha),
\end{align*}
\]  \hspace{1cm} (5.27)

we choose sufficiently large \(N > 0\) such that

$$\zeta = \frac{1}{N} \cdot \frac{27(4\alpha - 5)}{64\alpha^3 - 272\alpha^2 + 300\alpha - 369} \leq \frac{16\alpha^2 - 8\alpha + 27}{9(3 + 2\alpha)(5 - 4\alpha)} (L(\alpha) - J(\alpha) - \gamma).$$

14
This leads to (5.25) as targeted. In other words, such a specific choice of \( N \) allows \( \zeta \) to become as negligible as we need in the previous argument for \( N\zeta \). This rigorously justifies that our fully determined quadruple \((\eta, \zeta, N, \gamma)\) is admissible. Finally, we select sufficiently small \( \rho < 1 \) that the common ratio \( \theta = \rho^k \) is smaller than \( 1/2 \), which is crucial for our auxiliary lemmas (Lemma 4.1 and Lemma 4.2) to be well-applied. We eventually conclude that

\[
C(r_N)+D(r_N)+T(r_N) \leq C\varepsilon^{3/\tau} \leq \varepsilon_0
\]

with sufficiently small \( \varepsilon > 0 \), as desired. Since \( \gamma < L(\alpha)-J(\alpha) \) is arbitrary, the proof is complete.

\[
\square
\]

A Appendix

**Proposition A.1.** Suppose that \((u,p)\) is a suitable weak solution to (1.1) in \(\mathbb{R}^3 \times (0,T)\). Then we have

\[
\int_{\mathbb{R}^3 \times (0,T)} |\nabla p|^{\frac{14+2\alpha}{5+2\alpha}} dx \, dt < \infty.
\]

(A.1)

**Proof.** Denote by \( R_k \) the \( k \)-th Riesz transform with symbol \( i\xi_k/|\xi| \). Since \( \partial_t \) and \( \sum R_i R_j \) commute (consider their Fourier symbols), we write

\[
\partial_k p = \sum R_i R_j (\partial_t u_i u_j + u_i \partial_k u_j)
\]

using the divergence-free condition of \( u \). Since \( \sum R_i R_j \) is \( L^p \)-bounded for any \( 1 < p < \infty \), we obtain

\[
\|\nabla p\|_{L^p(\mathbb{R}^3)} \leq C_p \|u\|\|\nabla u\|_{L^p(\mathbb{R}^3)}.
\]

(A.2)

Observing \( \|\nabla u\|_{L^{\frac{6}{5-2\alpha}}(\mathbb{R}^3)} \leq C\|(-\Delta)^{\frac{\theta}{2}} u\|_{L^2(\mathbb{R}^3)} \) and \( \|u\|_{L^{\frac{6}{5-2\alpha}}(\mathbb{R}^3)} \leq C\|(-\Delta)^{\frac{\theta}{2}} u\|_{L^2(\mathbb{R}^3)} \) by fractional Sobolev’s embeddings, we use (A.2), Hölder’s inequality, and Lebesgue interpolation to compute

\[
\int_0^T \int_{\mathbb{R}^3} |\nabla p|^r \, dx \, dt \leq C \int_0^T \int_{\mathbb{R}^3} |(u \cdot \nabla) u|^r \, dx \, dt
\]

\[
\leq C \int_0^T \left( \int_{\mathbb{R}^3} |u|^\frac{r}{5-r(5-2\alpha)/6} \right)^{\frac{6-r(5-2\alpha)/6}{r}} \left( \int_{\mathbb{R}^3} |\nabla u|^\frac{6}{5-2\alpha} \right)^{\frac{r(5-2\alpha)}{6}} dt
\]

\[
\leq C \int_0^T \left( \|u\|_{L^2}^{\theta} \|u\|_{L^{\frac{6}{5-2\alpha}}}^{1-\theta} \right)^r \left( \int |(-\Delta)^{\frac{\theta}{2}} u|^2 \right)^{\frac{r}{2}} dt
\]

for \( r > 0 \) and \( 0 < \theta < 1 \) such that \( 2 < \frac{6r}{6-r(5-2\alpha)/6} < \frac{6}{5-2\alpha} \). By the uniform in time \( L^2 \)-boundedness of \( u \), fractional Sobolev’s embedding, and Hölder’s inequality again, we get

\[
\int_0^T \int_{\mathbb{R}^3} |\nabla|^r \, dx \, dt \leq C \left( \int_0^T \|u\|_{L^{\frac{6}{5-2\alpha}}}^2 \, dt \right)^{\frac{r(1-\theta)}{2}} \left( \int_0^T \left( \int |(-\Delta)^{\frac{\theta}{2}} u|^2 \, dx \, dt \right)^{\frac{r}{2}} \right)^{\frac{r}{2}} \left( \int_0^T \left( \int |(-\Delta)^{\frac{\theta}{2}} u|^2 \, dx \, dt \right)^{\frac{2-r(1-\theta)}{2}} \right)
\]

\[
\leq C \left( \int_{\mathbb{R}^3 \times (0,T)} |(-\Delta)^{\frac{\theta}{2}} u|^2 \, dx \, dt \right)^{\frac{r(1-\theta)}{2}} \left( \int_0^T \left( \int |(-\Delta)^{\frac{\theta}{2}} u|^2 \, dx \, dt \right)^{\frac{2-r(1-\theta)}{2}} \right)^{\frac{2-r(1-\theta)}{2}}
\]

\[
\leq C \left( \int_0^T \left( \int |(-\Delta)^{\frac{\theta}{2}} u|^2 \, dx \, dt \right)^{\frac{2-r(1-\theta)}{2}} \right)^{\frac{2-r(1-\theta)}{2}}
\]

15
for $r > 0$ and $0 < \theta < 1$ which further satisfies $\frac{2}{r(1-\theta)} > 1$ to apply H"older’s inequality for the first inequality in the above calculation. Now we set

$$r := \frac{3 + 2\alpha}{4}, \quad \theta := \frac{4\alpha - 2}{3 + 2\alpha}.$$  

For such $r$ and $\theta$, not only all the previous computations are valid but also we have $\frac{r}{2-r(1-\theta)} = 1$ so that we conclude that

$$\int_{\mathbb{R}^3 \times (0,T)} |\nabla p|^{\frac{3+2\alpha}{4}} \, dx \, dt \leq C \left( \int_{\mathbb{R}^3 \times (0,T)} \left| (-\Delta)^{\alpha/2} u \right|^2 \, dx \, dt \right)^{\frac{3+2\alpha}{8}} < \infty,$$

as desired. The proof is complete. \qed

References

[1] L. Caffarelli, R. Kohn and L. Nirenberg: Partial regularity of suitable weak solutions of Navier-Stokes equation. Comm. Pure. Appl. Math., 35, 771–831 (1982).

[2] L. Caffarelli and L. Silvestre: An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations, 32(7-9), 1245–1260 (2007).

[3] M. Colombo, C. De Lellis, and A. Massaccesi: The generalized Caffarelli-Kohn-Nirenberg theorem for the hyperdissipative Navier-Stokes system. Comm. Pure Appl. Math., 73(3), 609–663 (2019).

[4] K. Falconer, Fractal geometry, 3rd Edition, John Wiley & Sons, Ltd., Chichester, mathematical foundations and applications (2014).

[5] N. Katz, N. Pavlovic, A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation. Geom. Funct. Anal. 12, no. 2, 355–379 (2002).

[6] Y. Koh and M. Yang: The Minkowski dimension of interior singular points in the incompressible Navier-Stokes equations. J. Differential Equations., 261, 3137–3148 (2016).

[7] I. Kukavica: The fractal dimension of the singular set for solutions of the Navier-Stokes system. Nonlinearity, 22, 2889–2900 (2009).

[8] I. Kukavica and Y. Pei: An estimate on the parabolic fractal dimension of the singular set for solutions of the Navier-Stokes system. Nonlinearity, 25, 2775–2783 (2012).

[9] H. Kwon, W. S. O’zański, Local regularity of weak solutions of the hypodissipative Navier-Stokes equations. Journal of Functional Analysis, 282(7), 109370 (2022).

[10] J. Leray: Sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math., 63(1), 193–248 (1934).

[11] F. Lin: A new proof of the Caffarelli-Kohn-Nirenberg Theorem. Comm. Pure Appl. Math., 51, 241–257 (1998).

[12] J.-L. Lions: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod; Gauthier-Villars, Paris (1969).
[13] V. Scheffer: *Partial regularity of solutions to the Navier-Stokes equations*. Pacific J. Math., 66, 535–552 (1976).

[14] V. Scheffer: *Hausdorff measure and the Navier-Stokes equations*. Comm. Math. Phys., 55(2), 97–112 (1977).

[15] L. Tang and Y. Yu: *Partial regularity of suitable weak solutions to the fractional Navier-Stokes equations*. Comm. Math. Phys., 334(3), 1455–1482 (2015).

[16] T. Tao: *Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation*. Anal. PDE, 2(3), 361–366 (2009).

[17] Y. Wang and G. Wu: *On the box-counting dimension of potential singular set for suitable weak solutions to the 3D Navier-Stokes equations*. Nonlinearity, 30, 1762–1772 (2017).

[18] Y. Wang and M. Yang: *Improved bounds for box dimensions of potential singular points to the Navier-Stokes equations*. Nonlinearity, 32, 4817–4833 (2019).

[19] R. Yang: *On higher order extensions for the fractional Laplacian*. arXiv:1302.4413, (2013).