Internal Symmetry Group and Density Matrix of Fields with Spins 0, 1

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Abstract

The internal symmetry group $U(3,1)$ of the neutral vector fields with two spins 0 and 1 is investigated. Massless fields correspond to the generalized Maxwell equations with the gradient term. The symmetry transformations in the coordinate space are integro-differential transformations. Using the method of the Hamiltonian formalism the conservation tensors are found, and the quantized theory is studied. The necessity to introduce an indefinite metric is shown. The internal symmetry group $U(3,1)$ being considered, after the transition to electrodynamics, reduces to the $U(2)$ group. It is shown that the group of dual transformations is the subgroup of the group under consideration. All the linearly independent solutions of the equation for a free particle obtained in terms of the projection matrix-dyads.

1 Canonical formalism

In the general case, without any constraints, the vector field $B_{\mu}(x)$ realizes the $(0,0) \oplus (1/2,1/2)$ representation of the Lorentz group and describes four degrees of freedom which correspond to states with spins $s = 0$ and $s = 1$ (with three spin projections $s_z = 0, \pm 1$). The massive field functions $B_{\mu}(x)$ satisfy the Klein-Gordon-Fock equation

$$\left(\partial_{\mu}^2 - m^2\right) B_{\nu}(x) = 0,$$

where $\partial_{\mu} = (\partial/\partial x_m, \partial/\partial i\partial t) \; , \; B_{\nu}(x) = (B_n(x), iB_0(x))$.

The corresponding Lagrangian for the neutral fields can be rewritten as follows (within unimportant divergent-type terms):

$$\mathcal{L} = -\frac{1}{2} \left[ (\partial_{\mu} B_{\nu})^2 + m^2 B_{\mu}^2 \right].$$

The Lagrangian (2) can be connected also with the Stueckelberg formulation of the vector field [1]. A Lagrangian of the form (2) also was used [2] in a gauge-invariant formulation for a massive neutral vector field.

Eq. (1) can be represented in the form of first-order equations [3]

$$\partial_{\nu}\psi_{\mu\nu} - \partial_{\mu}\psi_0 + m^2\psi_\mu = 0,$$

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\[
\psi_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu, \\
\psi_0 = \partial_\mu \psi_\mu,
\]

(3)

with \( \psi_\mu = B_\mu, \psi_0 = \partial_\mu B_\mu \). In the case \( m = 0 \) equations (3) are the generalized Maxwell equations with the gradient term (see [4-6]). Now we investigate the symmetry group of the four-component neutral vector field \( B_\mu \) which describes two spins 0, 1 (without the Lorentz condition, i.e. \( \partial_\mu B_\mu \neq 0 \) (see Eq.(1)) with the Lagrangian (2) [7]. We call this field “the Stueckelberg field” [8].

Let us consider the transformations \( \Lambda = (\Lambda_{\mu\nu}) \) of the field functions \( B'_\mu(x) = \Lambda_{\mu\nu} B_\nu(x) \), which belong to the group \( SO(3,1) \) but which leave the coordinates \( x_\mu \) unchanged, i.e. \( x'_\mu = x_\mu \). This is a different case from that of the Lorentz group where the coordinates \( x_\mu \) are transformed. It is easy to verify that the Lagrangian (2) is invariant under this group of symmetry transformations as \( \partial_{\mu'} B_{\mu'} = 0 \) and \( B^2_\mu(x) = B^2_\mu(x) \). In accordance with the Noether theorem [9,10] the invariance of the action integral under the group of the transformations under consideration one yields the law of conservation of the following antisymmetric tensor

\[
S_{[\mu[\alpha \beta]} = B_{\beta} \partial_\mu B_{\alpha} - B_{\alpha} \partial_\mu B_{\beta}.
\]

(4)

This tensor coincides with the density of spin momentum corresponding to the conservation law of spin momentum (4): \( \partial_\mu S_{[\mu[\alpha \beta]} = 0 \).

The invariance of the action integral (2) under the Lorentz transformations of the coordinates \( x_\mu' = L_{\mu\nu} x_\nu \) induces transformations of the field functions \( B'_\mu(x') = L_{\mu\nu} B_\nu(x) \) which lead to a conservation law for angular momentum \( M_{\mu[\alpha \beta]} [9,10] \), this being the sum of the orbital, \( T_{\mu[\alpha \beta]} \), and spin, \( S_{[\mu[\alpha \beta]} \), momenta:

\[
T_{\mu\nu} = - (\partial_\mu B_\alpha) (\partial_\nu B_\alpha) - L_\delta_{\mu\nu}, \\
M_{\mu[\alpha \beta]} = T_{\mu[\alpha \beta]} + S_{[\mu[\alpha \beta]}.
\]

(5)

Therefore we also have here the law of conservation of orbital angular momentum: \( \partial_\mu T_{\mu[\alpha \beta]} = 0 \). It should be noted that the Lorentz calibration \( \partial_\mu B_\mu = 0 \) is not invariance under our group of symmetry \( (\Lambda_{\mu\nu}) \). As the Lorentz calibration extracts the pure spin 1 of particles we come to the conclusion that the laws of conservation of the orbital, \( T_{\mu[\alpha \beta]} \), and spin, \( S_{[\mu[\alpha \beta]} \), angular momenta separately: \( \partial_\mu T_{\mu[\alpha \beta]} = 0, \partial_\mu S_{[\mu[\alpha \beta]} = 0 \) are due to the presence of two spins 0 and 1; i.e. multi-spin 0, 1.

Now we consider a wider group of symmetry using the method of the Hamiltonian formalism.

The generalized coordinates in this scheme are

\[
q_\mu(x) = B_\mu(x).
\]

(6)

The density of the momenta found from Eq.(2) is given by

\[
\pi_\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{q}_\mu(x)} = \dot{q}_\mu(x),
\]

(7)
where \( \dot{q}_\mu(x) = \partial q_\mu(x)/\partial t \). The density of the Hamiltonian is defined by the relationship

\[
\mathcal{H}(x) = \pi_\mu(x) \dot{q}_\mu(x) - \mathcal{L}(x).
\]

Inserting the density of the Lagrangian (2) and the densities of the momenta (7) into Eq.(8) we arrive at

\[
\mathcal{H}(x) = \frac{1}{2} \left[ (\partial_m B_\mu(x))^2 + B_\mu^2(x) + m^2 B_\mu^2(x) \right].
\]

In momentum space the real fields \( B_\mu(x) \) are given by

\[
B_\mu(x) = L^{-3/2} \sum_k \left[ B_\mu(k, t)e^{ikx} + B_\mu^+(k, t)e^{-ikx} \right],
\]

where \( L \) is the normalizing length so that the energy of a quantum is \( k_0 = 2\pi/L \) and the normalizing volume is \( V = L^3 \); \( B_\mu^+(k, t) \) is the Hermitian conjugated quantity. The time dependence of fields in the momentum space is

\[
B_\mu(k, t) \sim e^{-ik_0 t}, \quad B_\mu^+(k, t) \sim e^{ik_0 t},
\]

where \( k_0^2 = k^2 + m^2 \). Taking into account Eqs.(9), (10) the Hamiltonian \( H = \int \mathcal{H}(x)d^3x \) takes the form

\[
H = \sum_k 2k_0^2 B_\mu(k, t)B_\mu^+(k, t).
\]

Introducing the canonical variables in the momentum space

\[
q_\mu(k, t) = B_\mu(k, t) + B_\mu^+(k, t), \quad \pi_\mu(k, t) = \dot{q}_\mu(k, t) = -ik_0 \left[ B_\mu(k, t) - B_\mu^+(k, t) \right],
\]

the Hamiltonian (12) is rewritten as

\[
H = \frac{1}{2} \sum_k \left[ (\pi_\mu(k, t))^2 + k_0^2 (q_\mu(k, t))^2 \right] = \sum_k 2k_0^2 \varphi^+ \varphi,
\]

where

\[
\varphi = \begin{pmatrix} B_1(k, t) \\ B_2(k, t) \\ B_3(k, t) \\ iB_0(k, t) \end{pmatrix}, \quad \varphi^+ = \begin{pmatrix} B_1^+(k, t) & B_2^+(k, t) & B_3^+(k, t) & iB_0^+(k, t) \end{pmatrix}.
\]

It is obvious that the Hamiltonian (14) is invariant (see [11, 12]) under the transformations of the pseudounitary group \( U(3, 1) \) (for real fields \( B_m(k, t), B_0(k, t) \)):

\[
\varphi' = U \varphi, \quad \varphi^+' = \varphi^+U^+,
\]

where

\[
\varphi = \begin{pmatrix} B_1(k, t) \\ B_2(k, t) \\ B_3(k, t) \\ iB_0(k, t) \end{pmatrix}.\]
where the complex $4 \times 4$-matrix $U$ obeys the equation $U^+ U = 1$. The matrix of the infinitesimal transformations (16) can be represented as

$$U = (1 - i \omega_0) I_4 + \omega_{[\mu\nu]} I_{[\mu\nu]} - \omega_{(\mu\nu)} I_{(\mu\nu)},$$

where the antisymmetric $I_{[\mu\nu]}$ and symmetric $I_{(\mu\nu)}$ generators of the group are given by

$$I_{[\mu\nu]} = \varepsilon^{\mu\nu} - \varepsilon^{\nu\mu}, \quad I_{(\mu\nu)} = i \left( \varepsilon^{\mu\nu} + \varepsilon^{\nu\mu} - \frac{1}{2} \delta_{\mu\nu} \right).$$

Here $I_4$ is the unit $4 \times 4$-matrix and the elements of the entire algebra $\varepsilon^{\mu\nu}$ satisfy the relations [9]

$$(\varepsilon^{\mu\nu})_{\alpha\beta} = \delta_{\mu\alpha} \delta_{\nu\beta}, \quad \varepsilon^{\mu\nu} \varepsilon^{\alpha\beta} = \delta_{\nu\alpha} \varepsilon^{\mu\beta},$$

where indexes $\mu, \nu, \alpha, \beta = 1, 2, 3, 4$. The parameters of the $U(3.1)$ group obey the equations $\omega^*_0 = \omega_0, \omega^*_{[ab]} = \omega_{[ab]}, \omega^*_{[a4]} = -\omega_{[a4]}, \omega^*_{(ab)} = \omega_{(ab)}, \omega^*_{(a4)} = -\omega_{(a4)}$ ($a, b = 1, 2, 3$). The generators of the $U(3.1)$ group: $i I_4, I_{[\mu\nu]}, I_{(\mu\nu)}$ have the correct commutation relations [13] as may be verified using Eqs.(19). The $I_{[\mu\nu]}$ are the generators of the subalgebra corresponding to the $SO(3.1)$ group and $i I_4$ corresponds to the subgroup $U(1)$. Transformations (16) act in the space of the functions (15) and the momentum $k$ is not transformed.

To get the integral of motion (corresponding to the group of symmetry (16) in the framework of this formalism one needs to consider the canonical transformations which leave the Hamiltonian (14) invariant. Taking into account Eqs.(17), (18), infinitesimal transformations (16) can be cast into the form

$$\delta B_{\mu}(k, t) = -i \omega_0 B_{\mu}(k, t) + 2 \omega_{[\mu\nu]} B_{\nu}(k, t) -$$

$$- 2i \left[ \omega_{(\mu\nu)} - \frac{1}{4} \omega_{(\alpha\alpha)} \delta_{\mu\nu} \right] B_{\nu}(k, t),$$

$$\delta B^+_{\mu}(k, t) = i \omega_0 B^+_{\mu}(k, t) + 2 \omega_{[\mu\nu]} B^+_{\nu}(k, t) +$$

$$+ 2i \left[ \omega_{(\mu\nu)} - \frac{1}{4} \omega_{(\alpha\alpha)} \delta_{\mu\nu} \right] B^+_{\nu}(k, t).$$

With the help of Eq.(13), transformations (20), (21) can be rewritten in the canonical form

$$\delta q_{\mu}(k, t) = \frac{1}{k_0} \pi_{\mu}(k, t) + 2 \omega_{[\mu\nu]} q_{\nu}(k, t) +$$

$$+ \frac{2}{k_0} \left[ \omega_{(\mu\nu)} - \frac{1}{4} \omega_{(\alpha\alpha)} \delta_{\mu\nu} \right] \pi_{\nu}(k, t),$$

$$\delta \pi_{\mu}(k, t) = -k_0 \omega_0 q_{\mu}(k, t) + 2 \omega_{(\mu\nu)} \pi_{\nu}(k, t) -$$

$$- 2k_0 \left[ \omega_{(\mu\nu)} - \frac{1}{4} \omega_{(\alpha\alpha)} \delta_{\mu\nu} \right] q_{\nu}(k, t).$$
To get integrals of motion we use the method of generating functions [14]. It is verified that the generating function corresponding to transformations (22), (23) is

\[ F = \sum_{k} \left\{ q_{\mu}(k, t) \pi^{\nu}_{\mu}(k, t) + \frac{\omega_{\mu}}{2} \left[ \left( \frac{\pi^{\nu}_{\mu}(k, t)}{k_{0}} \right)^{2} + k_{0} (q_{\mu}(k, t))^{2} \right] + \right. \]

\[ + \omega_{[\mu\nu]} \left[ \pi^{\nu}_{\mu}(k, t) q_{\nu}(k, t) - \pi^{\nu}_{\nu}(k, t) q_{\mu}(k, t) \right] + \left. \left( \omega_{(\mu)} - \frac{1}{4} \omega^{(\alpha\alpha)} \delta_{\mu\nu} \right) \left[ \frac{\pi^{\nu}_{\mu}(k, t) \pi^{\nu}_{\nu}(k, t)}{k_{0}} + k_{0} q_{\mu}(k, t) q_{\nu}(k, t) \right] \right\}, \tag{24} \]

so that \( q^{\nu}_{\mu}(k, t) = \frac{\partial F}{\partial \pi^{\nu}_{\mu}(k, t)} \), \( \pi_{\mu}(k, t) = \frac{\partial F}{\partial q_{\mu}(k, t)} \). From Eq.(24) we find the conservation law of the following tensors

\[ J_{[\mu\nu]} = \sum_{k} \left[ \pi_{\mu}(k, t) q_{\nu}(k, t) - \pi_{\nu}(k, t) q_{\mu}(k, t) \right] = \]

\[ = i \sum_{k} \left[ b^{+}_{\mu}(k) b_{\nu}(k) - b^{+}_{\nu}(k) b_{\mu}(k) \right], \tag{25} \]

\[ J_{(\mu\nu)} = \sum_{k} \left\{ \frac{\pi_{\mu}(k, t) \pi_{\nu}(k, t)}{k_{0}} + k_{0} q_{\mu}(k, t) q_{\nu}(k, t) - \frac{1}{4} \delta_{\mu\nu} \left[ \left( \pi_{\alpha}(k, t) \right)^{2} + k_{0} \left( q_{\alpha}(k, t) \right)^{2} \right] \right\} = \]

\[ = \sum_{k} \left[ b^{+}_{\mu}(k) b_{\nu}(k) + b^{+}_{\nu}(k) b_{\mu}(k) - \frac{1}{2} \delta_{\mu\nu} \left( b^{+}_{\alpha}(k) b_{\alpha}(k) \right) \right], \tag{26} \]

\[ J = \frac{1}{2} \sum_{k} \left[ \left( \pi_{\alpha}(k, t) \right)^{2} + k_{0} \left( q_{\alpha}(k, t) \right)^{2} \right] = \sum_{k} b^{+}_{\mu}(k) b_{\nu}(k), \tag{27} \]

where variables \( b_{\mu}(k) = \sqrt{2k_{0}} B_{\mu}(k) \) and \( b^{+}_{\mu}(k) = \sqrt{2k_{0}} B^{+}_{\mu}(k) \) in the quantized theory are annihilation and creation operators, respectively. The conserved variables (25)-(27) satisfy the equations \( \{ J_{[\mu\nu]}, H \} = \{ J_{(\mu\nu)}, H \} = \{ J, H \} = 0 \), where \( H \) is given by (14) and the \( \{ \ldots \} \) are the classical Poisson brackets. In the quantum case, operators \( b_{\mu}(k), b^{+}_{\nu}(k) \) obey the commutation relation \( [b_{\mu}(k), b^{+}_{\nu}(k')] = \delta_{\mu\nu} \delta(k - k') \) and the \( J_{[\mu\nu]}, J_{(\mu\nu)}, J \) are the generators of the group of internal symmetry \( U(3, 1) \) so that \( [J_{[\mu\nu]}, H] = [J_{(\mu\nu)}, H] = [J, H] = 0 \). Using the expansion

\[ B_{\mu}(x) = L^{-3/2} \sum_{k} \left[ B_{\mu}(k) e^{i(kx - k_{0}t)} + B^{+}_{\mu}(k) e^{-i(kx - k_{0}t)} \right], \]
it is easy to verify that the integral of motion, $J_{\mu\nu}$ (Eq. (25)), coincides with the conserved spin momentum tensor:

$$J_{\mu\nu} = i \int S_{\mu\nu} d^3x.$$  \hspace{1cm} (28)

So the group $SO(3,1)$ under consideration ($\Lambda$) with the generators $J_{\mu\nu}$ is the subgroup of the general group $U(3,1)$. The conserved quantity $J$ is the number of quanta of the field. The generator $J$ defines the subgroup $U(1)$ of the phase transformations and the generators $J_{\mu\nu}$, $J_{\mu}$ correspond to the group $SU(3,1)$. In the general case, transformations (16) in the coordinate space are integro-differential transformations and therefore the Lagrangian formalism is not convenient for studying this symmetry. The use of the canonical formalism and the method of generating functions allow us to investigate in a simple manner the group of “addition” symmetry of the vector field which possesses two spin values $0, 1$ (massive and massless fields). The analogous group of symmetry $SU(n)$ of the $n$-dimensional oscillator has been investigated [11, 12]. The analysis performed is readily generalized to arbitrary vector fields.

So the integral of motion (25)-(27) corresponds to the transformations of internal-symmetry (16) which are not induced by the space-time transformations.

The canonical variables $q_{\mu}(x), \pi_{\mu}(x)$ satisfy the commutation relation

$$\{q_{\mu}(x), \pi_{\nu}(x')\}_{t=t'} = \delta_{\mu\nu} \delta (x-x').$$  \hspace{1cm} (29)

Using Eq.(7) we arrive at the relationships

$$\left\{ q_{\mu}(x), \dot{q}_{\nu}(x') \right\}_{t=t'} = i \delta_{\mu\nu} \delta (x-x').$$  \hspace{1cm} (30)

To quantize the fields we should transfer to quantum commutators in accordance with the procedure $\{.,.\}_{t=t'} \rightarrow i[.,.]$, where $[q, \pi] = q\pi - \pi q$. Taking into account Eqs.(6), (30) we find the commutation relations

$$\left[ B_{\mu}(x), \dot{B}_{\nu}(x') \right]_{t=t'} = i \delta_{\mu\nu} \delta (x-x'),$$  \hspace{1cm} (31)

It is obvious that commutators (31) correspond to Bose-Einstein statistics.

2 Quantized fields and indefinite metric

Now we will consider the quantized theory of the fields with multi-spin 0,1. In accordance with the general rule that the translation generator $P_{\mu}$ and the generator of four-dimensional rotations $M_{\rho\sigma}$ are given by [9, 10]

$$P_{\mu} = i \int T_{\mu\nu} d^3x,$$
\[ M_{\mu\sigma} = -i \int M_{4[\mu\sigma]} d^3 x. \] (32)

Using Eqs.(5) and commutation relations (31), it is not difficult to check that generators (32) satisfy the commutation relations of the Poincaré group.

Let us discuss the eigenvalues of the operator of the energy \( P_0 = -i P_4 \) of the fields considered. In the momentum representation, the energy operator has the form:

\[ P_0 = \int k_0 \left( b_a^+(k)b_a(k) - b_0^+(k)b_0(k) \right) d^3 k, \] (33)

where operators

\[ b_\mu^+(k) = \sqrt{2k_0}B_\mu^+(k), \quad b_\mu(k) = \sqrt{2k_0}B_\mu(k) \quad (B_\mu(k,t) = B_\mu(k) \exp(-ik_0t)), \]

as follows from Eqs.(31), must obey the following commutation relations:

\[ \left[ b_\mu(k), b_\nu^+(k') \right] = \delta_{\mu\nu}\delta(k-k'). \] (34)

Commutation relation (34) is invariant under the group of symmetry (16) because the operator of finite transformations \( U \) is unitary. It is seen from Eq.(33) that the field energy, in classical theory, is not positive-definite. In commutation relations (34) there is a minus sign (as \( b_4(k) = ib_0(k), b_4^+(k) = ib_0^+(k) \)), and the operators \( b_0(k), b_0^+(k) \) obey the “incorrect” commutation relation

\[ \left[ b_0(k), b_0^+(k') \right] = -\delta(k-k'). \]

From this equation it is seen that the term which appears in the energy operator with a (−) sign satisfies the “incorrect” commutation relation. The “incorrect” commutation relations are not compatible, however, with the assumption that the fields are real. As in the case of the electromagnetic field, this difficulty is surmounted by introducing an indefinite metric (see e.g., [15]). There are two possibilities. According to the first one we can consider operators \( b_0(k), b_0^+(k') \), as the creation operators, and \( b_a(k), b_a^+(k') \), as the annihilation operators of particles. The vacuum state \( | 0 \rangle_1 \) is defined by the requirement:

\[ b_a(k) \mid 0 \rangle_1 = b_0^+(k) \mid 0 \rangle_1 = 0, \]

\[ \langle 0 \mid 0 \rangle_1 = 1. \] (35)

The basis in Hilbert space is given by

\[ \mid m, n \rangle_1 = \frac{1}{\sqrt{m!n!}} \left( b_a^+(k) \right)^m (b_0(k))_n \mid 0 \rangle_1 \] (36)

with the normalization condition

\[ \langle m, n \mid m', n' \rangle_1 = \delta_{mm'}\delta_{nn'}. \] (37)
In this case there is Hilbert space but the eigenstates of the energy operator (33) are not positive defined and the interpretation of the states with negative energy is problematical. The second possibility [15] is more favorable and connected with introducing an indefinite metric. In this case the algebra \((34)\) can be represented as the operator algebra in the states with the indefinite metric and here operators \(b_\mu^+(k)\) are the creation and \(b_\mu(k)\) the annihilation operators. As the basis of the irreducible representation in Hilbert space we choose the following vectors:

\[
|m,n\rangle = \frac{1}{\sqrt{mn!}} \left( b_\mu^+(k) \right)^m \left( b_0^+(k) \right)^n |0\rangle,
\]

\[
b_\mu(k) |0\rangle = 0, \quad \langle 0 | 0 \rangle = 1. \quad (38)
\]

With the help of Eqs.(34) we arrive at

\[
\langle m, n | m', n' \rangle = (-1)^n \delta_{mm'} \delta_{nn'} \quad (39)
\]

In this way, the state space of vector fields has an indefinite metric, i.e. the square vector norm can be negative. Using Eqs.(34), it is easy to verify that the eigenvalues of the operator \(P_0\) are positive; however, the metric is still indefinite.

As usual, in the theory with an indefinite metric, it is necessary to divide the space into “physical” and “nonphysical” subspaces (see [15, 16]). The “physical” subspace corresponds to a positive square norm, and the “nonphysical” to a negative square norm. The state vectors \(|m,0\rangle\) create the “physical” subspace \(H_p\) and \(|m,n\rangle\) (at \(n \neq 0\)) create the “nonphysical” subspace \(H_n\). The physical states permit the usual probability interpretation. It is apparent from Eq.(39) that the “physical” and “nonphysical” subspaces are orthogonal \(\langle m, 0 | m', n' \rangle = 0\), and therefore the state space is the direct sum of the two subspaces \(H_p\) and \(H_n\). We can then represent any vector in the form

\[
|\rangle = |\rangle_p + |\rangle_n \quad (40)
\]

where \(|\rangle_p \in H_p, |\rangle_n \in H_n\). In scattering processes, states \(|in\rangle, |out\rangle\) should belong to the “physical” subspace \(H_p\). It is possible also to consider transitions between states which belong to the “nonphysical” subspace \(H_n\). But the consideration of transitions between the vectors \(|\rangle_p, |\rangle_n\) presents difficulties [15] because such transitions violate the unitarity of the \(S\)-matrix and the usual probability interpretation.

3 Internal symmetry of the electromagnetic field

It will be shown that the symmetry-group of an electromagnetic field is \(U(2)\). When we impose the constraints \(\partial_\mu B_\mu = 0\) (The Lorentz condition), \(m = 0\), we arrive at the transition to the formulation of electrodynamics, and the Hamiltonian (33) takes the form

\[
H = \sum_k k_0 \left( b_1^+(k)b_1(k) + b_2^+(k)b_2(k) \right). \quad (41)
\]
In this case the Hamiltonian (41) is invariant under the transformations of the $U(2)$ group:

$$\begin{pmatrix} b_1'(k) \\ b_2'(k) \end{pmatrix} = \exp \left( i \frac{\alpha}{2} + i n_\tau \frac{\theta}{2} \right) \begin{pmatrix} b_1(k) \\ b_2(k) \end{pmatrix},$$  \hspace{1cm} (42)

where $n^2 = 1$, $\alpha$, $\theta$ are real group parameters, $\tau$ are the Pauli matrices. So, the internal symmetry group $U(3,1)$ being considered, after the transition to electrodynamics, reduces to the $U(2)$ group. Using the procedure described above, after finding the generalizing function, we come to the following conserved quantities:

$$J_1 = \frac{1}{2} \sum_k \left[ b_1^+(k) b_2(k) + b_2^+(k) b_1(k) \right],$$

$$J_2 = \frac{i}{2} \sum_k \left[ b_2^+(k) b_1(k) - b_1^+(k) b_2(k) \right],$$

$$J_3 = \frac{1}{2} \sum_k \left[ b_1^+(k) b_1(k) - b_2^+(k) b_2(k) \right],$$

$$J_0 = \frac{1}{2} \sum_k \left[ b_1^+(k) b_1(k) + b_2^+(k) b_2(k) \right].$$  \hspace{1cm} (43)

The generators $J_i$ satisfy the commutation relation of the $SU(2)$ group, and commute with $J_0$: $[J_i, J_0] = 0$. There is a representation of the rotation generators in the Hilbert space in the form of Eqs.(43) in [17, 18]. The average values of the operators (43) are identified in [19] with the Stokes parameters characterizing the polarization of the electromagnetic wave.

It is obvious, that the transformations (42), and the integrals of motion (43), (44), can be written in coordinate space but in nonlocal form. So, in the coordinate representation the expression (44) takes the nonlocal form [20]

$$J_0 = \frac{1}{2(2\pi)^3} \int \int \frac{E(x)E(y) + H(x)H(y)}{|x - y|^2} d^3x d^3y. \hspace{1cm} (45)$$

The transformations (42) at $\theta = 0$, which lead to the integral of motion (45) are local only in the momentum representation, and in the coordinate representation are integro-differential transformations.

It should be noted that the transformations (42) with $n_1 = n_3 = 0$, $\alpha = 0$, $n_2 = 1$, lead to the dual transformations of the strengths of electric $E$, and magnetic $H$ fields:

$$E' = E \cos \frac{\theta}{2} + H \sin \frac{\theta}{2},$$

$$H' = H \cos \frac{\theta}{2} - E \sin \frac{\theta}{2}. \hspace{1cm} (46)$$

Hence, the well known group of dual transformations is the subgroup of the $SU(2)$ group under consideration.
4 First-order equations and density matrix

Now we consider the matrix formulation of the first-order of the Stueckelberg fields (3) which is convenient for constructing the density matrix and for some electrodynamics calculations. All the linearly independent solutions of the equation for a free particle will be obtained in terms of the projection matrix-dyads.

Let us introduce the 11-dimensional function

\[
\Psi(x) = \{\psi_A(x)\} = \frac{1}{m} \begin{pmatrix}
-\psi_0 \\
m\psi_\mu \\
\psi^{[\mu\nu]}
\end{pmatrix} \quad (A = 0, \mu, [\mu\nu]), \tag{47}
\]

where \(\mu, \nu = 1, 2, 3, 4\). Using the elements of the entire algebra Eq. (19), equations (3) can be written in the form of one equation

\[
\partial_\nu \left( \varepsilon^{[\mu[\nu]+\varepsilon^{[\mu],[\nu]}+\delta^{0,0}} \right)_{AB} \Psi_B(x) + \\
+ m \left[ \varepsilon^{\mu,\mu} + \varepsilon^{0,0} + \frac{1}{2} \varepsilon^{[\mu],[\nu]} \right]_{AB} \Psi_B(x) = 0. \tag{48}
\]

After introducing 11-dimensional matrices

\[
\alpha_\nu = \varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu],[\nu]} + \varepsilon^{0,0},
\]

\[
I_{11} = \varepsilon^{\mu,\mu} + \varepsilon^{0,0} + \frac{1}{2} \varepsilon^{[\mu],[\nu]}, \tag{49}
\]

Eq. (48) takes the form of the relativistic wave equation of the first order:

\[
(\alpha_\mu \partial_\mu + m) \Psi(x) = 0. \tag{50}
\]

We took into account that \(I_{11}\) in Eq. (49) is the unit matrix in 11-dimensional space.

Eq. (50) represents the Stueckelberg equation for massive fields in the matrix form. When fields \(\Psi_A(x)\) are complex values, Eq. (50) describes charged particles with multi-spin 0, 1.

It should be noted that the matrices \(\alpha_\mu\) can be represented as

\[
\alpha_\mu = \beta^{(1)}_\mu + \beta^{(0)}_\mu,
\]

\[
\beta^{(1)}_\nu = \varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu],[\nu]},
\]

\[
\beta^{(0)}_\nu = \varepsilon^{0,0} + \varepsilon^{0,\nu}, \tag{51}
\]

where the 10-dimensional \(\beta^{(1)}_\mu\) and 5-dimensional \(\beta^{(0)}_\mu\) matrices obey the Petiau-Duffin-Kemmer [21-23] algebra:

\[
\beta_\mu \beta_\nu \beta_\alpha + \beta_\alpha \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\alpha + \delta_{\alpha\nu} \beta_\mu, \tag{52}
\]
so that the equations for spin-1 and spin-0 particles are

\[
\begin{align*}
\left(\beta^{(1)}_\mu \partial_\mu + m\right) \Psi^{(1)}(x) &= 0, & \Psi^{(1)}(x) &= \frac{1}{m} \begin{pmatrix} m\psi_\mu \\ \psi_{[\mu\nu]} \end{pmatrix}, \\
\left(\beta^{(0)}_\mu \partial_\mu + m\right) \Psi^{(0)}(x) &= 0, & \Psi^{(0)}(x) &= \frac{1}{m} \begin{pmatrix} -\psi_0 \\ m\psi_\mu \end{pmatrix}.
\end{align*}
\]

The 10-dimensional Petiau-Duffin-Kemmer equation (53) is equivalent to the Proca equations [24] for spin-1 particles and the 5-dimensional Eq. (54) is equivalent to the Klein-Gordon-Fock equation for scalar particles. The 11-dimensional Eq. (50) describes fields with two spins 0, 1. It is not difficult to verify (using Eqs. (19)) that the 11-dimensional matrices \( \alpha_\mu \) (49) satisfy the algebra (see also [25]):

\[
\begin{align*}
\alpha_\mu \alpha_\nu \alpha_\alpha + \alpha_\alpha \alpha_\nu \alpha_\mu + \alpha_\mu \alpha_\alpha \alpha_\nu + \alpha_\nu \alpha_\alpha \alpha_\mu + \alpha_\nu \alpha_\mu \alpha_\alpha + \alpha_\alpha \alpha_\mu \alpha_\nu = \\
= 2 (\delta_{\mu\nu} \alpha_\alpha + \delta_{\alpha\nu} \alpha_\mu + \delta_{\mu\alpha} \alpha_\nu).
\end{align*}
\]

This algebra is more complicated than the Petiau-Duffin-Kemmer algebra (52). Different representations of the Petiau-Duffin-Kemmer algebra (52) were considered in [26-28].

Now we find the solutions to Eq. (50) corresponding to definite values of the energy and momentum of a quantum of the massive fields. In the momentum space Eq. (50) becomes

\[
- \dot{p} \Psi_p = \varepsilon m \Psi_p,
\]

where \( \dot{p} = \alpha_\mu p_\mu, \ p_\mu = (p, ip_0), \ p^2 = p^2 - p^2_0 = -m^2; \) the value of \( \varepsilon = 1 \) corresponds to positive energy and \( \varepsilon = -1 \) to negative energy. Here \( p \) means the momentum of a field-quantum. It may be verified using (55) that the equality

\[
\dot{p}^3 = p^2 \dot{p}
\]

is valid. Following the general method of projection operators [29, 30], we find solutions to Eq. (56) in the form of the projection matrix

\[
M_\varepsilon = \frac{i\dot{p} (i\dot{p} - \varepsilon m)}{2m^2},
\]

so that

\[
M^2_\varepsilon = M_\varepsilon,
\]

and \( \varepsilon = \pm 1 \). Every column of the matrix \( M_\varepsilon \) can be considered as an eigenvector \( \Psi_p \) of equation (56) with eigenvalue \( \varepsilon m \). Eq. (59) for projection operators tells that matrix \( M_\varepsilon \) can be transformed into diagonal form, with the diagonal containing only ones and zeroes. So the \( M_\varepsilon \) acting on the wave function \( \Psi \) will retain components which correspond to the
eigenvalue \(\varepsilon m\). The generators of the Lorentz group in the 11—dimensional space being considered are given by

\[
J_{\mu\nu} = \beta^{(1)}_\mu \beta^{(1)}_\nu - \beta^{(1)}_\nu \beta^{(1)}_\mu.
\] (60)

It should be noted that matrices (60) act in the 10—dimensional subspace \((m\psi_\mu, \psi_{[\mu\nu]}\)) because the scalar \(\psi_0\) is an invariant of the Lorentz transformations. So matrices (60) are also generators of the Lorentz group for the Petiau-Duffin-Kemmer fields of Eq. (53). Using properties (19), we get the commutation relations

\[
[J_{\rho\sigma}, J_{\mu\nu}] = \delta_{\sigma\mu} J_{\rho\nu} + \delta_{\rho\nu} J_{\sigma\mu} - \delta_{\rho\mu} J_{\sigma\nu} - \delta_{\sigma\nu} J_{\rho\mu},
\] (61)

\[
[\alpha_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu} \alpha_\nu - \delta_{\lambda\nu} \alpha_\mu.
\] (62)

Relationship (61) is a well known commutation relation for generators of the Lorentz group \(SO(3,1)\). Equation (50) is form-invariant under the Lorentz transformations since relation (62) is valid. To guarantee the existence of a relativistically invariant bilinear form

\[
\overline{\Psi} \Psi = \Psi^+ \eta \Psi,
\] (63)

where \(\Psi^+\) is the Hermitian-conjugate wave function, we should construct a Hermitianizing matrix \(\eta\) with the properties \([9, 28, 30]\):

\[
\eta \alpha_i = -\alpha_i \eta, \quad \eta \alpha_4 = \alpha_4 \eta \quad (i = 1, 2, 3).
\] (64)

Such a matrix exists and is given by

\[
\eta = -\varepsilon^{0,0} + 2\beta^{(1)}_4 - I_{10},
\]

\[
I_{10} = \varepsilon^{\mu,\mu} + \frac{1}{2} \varepsilon_{[\mu\nu],\nu},
\] (65)

where the matrix \(\eta^{(1)} = 2\beta^{(1)}_4 - I_{10}\) plays the role of a Hermitianizing matrix for the Petiau-Duffin-Kemmer equation (53) \([9]\). The operator of the squared spin (squared Pauli-Lubanski vector) is given by

\[
\sigma^2 = \left(\frac{1}{2m}\varepsilon_{\mu\alpha\beta} p_\nu J_{\alpha\beta}\right)^2 = \frac{1}{m^2} \left(J_{\mu\nu} p^2 - J_{\mu\sigma} J_{\nu\sigma} p_\mu p_\nu\right).
\] (66)

It may be verified that this operator obeys the minimal equation

\[
\sigma^2 (\sigma^2 - 2) = 0,
\] (67)

so that eigenvalues of the squared spin operator \(\sigma^2\) are \(s(s+1) = 0\) and \(s(s+1) = 2\). This confirms that the considered fields describe the superposition of two spins \(s = 0\) and \(s = 1\). To separate these states we use the projection operators

\[
S^2_{(0)} = 1 - \frac{\sigma^2}{2}, \quad S^2_{(1)} = \frac{\sigma^2}{2}
\] (68)
having the properties \( S_{(0)}^2 S_{(1)}^2 = 0 \), \( (S_{(0)}^2)^2 = S_{(0)}^2 \), \( (S_{(1)}^2)^2 = S_{(1)}^2 \), \( S_{(0)}^2 + S_{(1)}^2 = 1 \), where \( 1 \equiv I_{11} \) is the unit matrix in \( 11 \)-dimensional space. In accordance with the general properties of the projection operators, the matrices \( S_{(0)}^2, S_{(1)}^2 \) acting on the wave function extract pure states with spin 0 and 1, respectively. Now we introduce the operator of the spin projection on the direction of the momentum \( \mathbf{p} \):

\[
\sigma_p = -\frac{i}{2|\mathbf{p}|} \epsilon_{abc} p_a J_{bc} = -\frac{i}{|\mathbf{p}|} \epsilon_{abc} p_a \beta_b^{(1)} \beta_c^{(1)},
\]

where \(|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}\). The minimal matrix equation for the spin projection operator is

\[
\sigma_p (\sigma_p - 1) (\sigma_p + 1) = 0
\]

and the corresponding projection operators are given by

\[
\hat{S}_{(\pm 1)} = \frac{1}{2} \sigma_p (\sigma_p \pm 1) \quad \hat{S}_{(0)} = 1 - \sigma_p^2.
\]

Operators \( S_{(\pm 1)} \) correspond to the spin projections \( s_p = \pm 1 \) and \( S_{(0)} \) to \( s_p = 0 \). It is easy to verify that the required commutation relations hold:

\[
\begin{align*}
[S_{(0)}^2, \hat{p}] &= [S_{(1)}^2, \hat{p}] = [\hat{S}_{(\pm 1)}, \hat{p}] = [\hat{S}_{(0)}, \hat{p}] = 0, \\
[S_{(0)}^2, \hat{S}_{(\pm 1)}] &= [S_{(1)}^2, \hat{S}_{(\pm 1)}] = [S_{(0)}^2, \hat{S}_{(0)}] = 0.
\end{align*}
\]

Thus the projection matrices extracting pure states with definite spin, spin projection and energy take the form

\[
\Delta_{\varepsilon, \pm 1} = M_{\varepsilon} S_{(\pm 1)} \hat{S}_{(\pm 1)} = \frac{i \hat{p} (i \hat{p} - \varepsilon m)}{2m^2} \frac{1}{2} \sigma_p (\sigma_p \pm 1),
\]

\[
\Delta_{\varepsilon}^{(1)} = M_{\varepsilon} S_{(1)}^2 \hat{S}_{(0)} = \frac{i \hat{p} (i \hat{p} - \varepsilon m)}{2m^2} \frac{\sigma^2}{2} (1 - \sigma_p^2),
\]

\[
\Delta_{\varepsilon}^{(0)} = M_{\varepsilon} S_{(0)}^2 \hat{S}_{(0)} = \frac{i \hat{p} (i \hat{p} - \varepsilon m)}{2m^2} \left(1 - \frac{\sigma^2}{2}\right) \left(1 - \sigma_p^2\right),
\]

where we took into account that \((\sigma^2/2) \sigma_p = \sigma_p\). Projection operators \( \Delta_{\varepsilon, \pm 1}, \Delta_{\varepsilon}^{(1)} \) extract states with spin 1 and spin projections \( \pm 1, 0 \), and \( \Delta_{\varepsilon}^{(0)} \) corresponds to spin 0. The \( \Delta_{\varepsilon, \pm 1}, \Delta_{\varepsilon}^{(1)}, \Delta_{\varepsilon}^{(0)} \) are the density matrices for pure spin states. It is easy to consider impure states by summation of Eqs. (73) over spin projections and spins. Projection operators for pure states can be represented as matrix-dyads [29, 30]:

\[
\Delta_{\varepsilon, \pm 1} = \Psi_{\varepsilon, \pm 1} \cdot \overline{\Psi}_{\varepsilon, \pm 1}, \quad \Delta_{\varepsilon}^{(1)} = \Psi_{\varepsilon} \cdot \overline{\Psi}_{\varepsilon}, \quad \Delta_{\varepsilon}^{(0)} = \Psi_{\varepsilon}^{(0)} \cdot \overline{\Psi}_{\varepsilon}^{(0)},
\]

where the wave functions \( \Psi_{\varepsilon, \pm 1}, \Psi_{\varepsilon} \) are the solution of the field equations for spin 1 and spin projections \( \pm 1 \) and 0, respectively, and \( \Psi_{\varepsilon}^{(0)} \) corresponds to the solution with spin 0. Expressions (73), (74) are convenient for calculating different electrodynamics processes involving polarized vector charged particles.
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