ON A WEIGHTED TRUDINGER-MOSER TYPE INEQUALITY ON THE WHOLE SPACE AND RELATED MAXIMIZING PROBLEM

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Abstract. In this paper, we establish a weighted Trudinger-Moser type inequality with the full Sobolev norm constraint on the whole Euclidean space. Main tool is the singular Trudinger-Moser inequality on the whole space recently established by Adimurthi and Yang, and a transformation of functions. We also discuss the existence and non-existence of maximizers for the associated variational problem.

1. Introduction

Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \) be a domain with finite volume. Then the Sobolev embedding theorem assures that \( W^{1,N}_0(\Omega) \hookrightarrow L^q(\Omega) \) for any \( q \in [1, +\infty) \), however, as the function \( \log (\log(e/|x|)) \in W^{1,N}_0(B) \), \( B \) the unit ball in \( \mathbb{R}^N \), shows, the embedding \( W^{1,N}_0(\Omega) \hookrightarrow L^\infty(\Omega) \) does not hold. Instead, functions in \( W^{1,N}_0(\Omega) \) enjoy the exponential summability:

\[
W^{1,N}_0(\Omega) \hookrightarrow \{ u \in L^N(\Omega) : \int_\Omega \exp \left( \alpha |u|^{N-1} \right) dx < \infty \text{ for any } \alpha > 0 \},
\]

see Yudovich [31], Pohozaev [26], and Trudinger [30]. Moser [22] improved the above embedding as follows, now known as the Trudinger-Moser inequality: Define

\[
TM(N, \Omega, \alpha) = \sup_{u \in W^{1,N}_0(\Omega), \|\nabla u\|_{L^N(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp(\alpha |u|^{N-1}) dx.
\]

Then we have

\[
TM(N, \Omega, \alpha) \begin{cases} < \infty, & \alpha \leq \alpha_N, \\ = \infty, & \alpha > \alpha_N, \end{cases}
\]

here and henceforth \( \alpha_N = N \omega_{N-1}^{1/N} \) and \( \omega_{N-1} \) denotes the area of the unit sphere \( S^{N-1} \) in \( \mathbb{R}^N \). On the attainability of the supremum, Carleson-Chang [6], Flucher [13], and Lin [17] proved that \( TM(N, \Omega, \alpha) \) is attained on any bounded domain for all \( 0 < \alpha \leq \alpha_N \).

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Later, Adimurthi-Sandeep [2] established a weighted (singular) Trudinger-Moser inequality as follows: Let $0 \leq \beta < N$ and put $\alpha_{N,\beta} = \left( \frac{N-\beta}{N} \right) \alpha_N$. Define

$$\widetilde{T}M(N, \Omega, \alpha, \beta) = \sup_{u \in W^{1,N}_0(\Omega), \|\nabla u\|_{L^N(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_{\Omega} \exp(\alpha |u|^\frac{N}{N-1}) \frac{dx}{|x|^\beta}.$$  

Then it is proved that

$$\widetilde{T}M(N, \Omega, \alpha, \beta) \left\{ \begin{array}{ll}
< \infty, & \alpha \leq \alpha_{N,\beta}, \\
= \infty, & \alpha > \alpha_{N,\beta}.
\end{array} \right.$$  

On the attainability of the supremum, recently Csató-Roy [10], [11] proved that $\widetilde{T}M(2, \Omega, \alpha, \beta)$ is attained for $0 < \alpha \leq \alpha_{2,\beta} = 2\pi (2-\beta)$ for any bounded domain $\Omega \subset \mathbb{R}^2$. For other types of weighted Trudinger-Moser inequalities, see for example, [7], [8], [9], [14], [18], [28], [29], [32], to name a few.

On domains with infinite volume, for example on the whole space $\mathbb{R}^N$, the Trudinger-Moser inequality does not hold as it is. However, several variants are known on the whole space. In the following, let

$$\Phi_N(t) = e^t - \sum_{j=0}^{N-2} t^j$$

denote the truncated exponential function.

First, Ogawa [23], Ogawa-Ozawa [24], Cao [5], Ozawa [25], and Adachi-Tanaka [1] proved that the following inequality holds true, which we call Adachi-Tanaka type Trudinger-Moser inequality: Define

$$A(N, \alpha) = \sup_{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}, \|\nabla u\|_{L^N(\mathbb{R}^N)} \leq 1} \frac{1}{\|u\|_{L^N(\mathbb{R}^N)}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^\frac{N}{N-1}) dx.$$  

Then

$$A(N, \alpha) \left\{ \begin{array}{ll}
< \infty, & \alpha < \alpha_N, \\
= \infty, & \alpha \geq \alpha_N.
\end{array} \right.$$  

The functional in (1.1)

$$F(u) = \frac{1}{\|u\|_{L^N(\mathbb{R}^N)}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^\frac{N}{N-1}) dx$$

enjoys the scale invariance under the scaling $u(x) \mapsto u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$, i.e., $F(u_\lambda) = F(u)$ for any $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$. Note that the critical exponent $\alpha = \alpha_N$ is not allowed for the finiteness of the supremum. On the attainability of the supremum, Ishiwata-Nakamura-Wadade [16] proved that $A(N, \alpha)$ is attained for any $\alpha \in (0, \alpha_N)$. In this sense, Adachi-Tanaka type Trudinger-Moser inequality has a subcritical nature of the problem.
On the other hand, Ruf \cite{27} and Li-Ruf \cite{20} proved that the following inequality holds true: Define
\begin{equation}
B(N, \alpha) = \sup_{u \in W^{1,N}(\mathbb{R}^N), \|u\|_{W^{1,N}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^\frac{N}{N-\gamma}) \, dx.
\end{equation}
Then
\begin{equation}
B(N, \alpha) \begin{cases} < \infty, & \alpha < \alpha_N, \\ = \infty, & \alpha > \alpha_N. \end{cases}
\end{equation}
Here \(\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\|\nabla u\|_{L^N(\mathbb{R}^N)}^N + \|u\|_{L^N(\mathbb{R}^N)}^N\right)^{1/N}\) is the full Sobolev norm. Note that the scale invariance \((u \mapsto u_\lambda)\) does not hold for this inequality. Also the critical exponent \(\alpha = \alpha_N\) is permitted to the finiteness of (1.3). Concerning the attainability of \(B(N, \alpha)\), it is known that \(B(N, \alpha)\) is attained for \(0 < \alpha \leq \alpha_N\) if \(N \geq 3\) \cite{27}. On the other hand when \(N = 2\), there exists an explicit constant \(\alpha_0 > 0\) related to the Gagliardo-Nirenberg inequality in \(\mathbb{R}^2\) such that \(B(2, \alpha)\) is attained for \(\alpha_0 < \alpha \leq 2(= 4\pi)\) \cite{27}, \cite{15}. However, if \(\alpha > 0\) is sufficiently small, then \(B(2, \alpha)\) is not attained \cite{15}. The non-attainability of \(B(2, \alpha)\) for \(\alpha\) sufficiently small is attributed to the non-compactness of “vanishing” maximizing sequences, as described in \cite{15}.

In the following, we are interested in the weighted version of the Trudinger-Moser inequalities on the whole space. Let \(N \geq 2\), \(-\infty < \gamma < N\) and define the weighted Sobolev space \(X^{1,N}_\gamma(\mathbb{R}^N)\) as
\[X^{1,N}_\gamma(\mathbb{R}^N) = \dot{W}^{1,N}(\mathbb{R}^N) \cap L^N(\mathbb{R}^N, |x|^{-\gamma} \, dx)\]
\[= \{u \in L^1_{\text{loc}}(\mathbb{R}^N) : \|u\|_{X^{1,N}_\gamma(\mathbb{R}^N)} = \left(\|\nabla u\|_N^N + \|u\|_{L^N(\mathbb{R}^N)}^N\right)^{1/N} < \infty\},\]
where we use the notation \(\|u\|_{N,\gamma}\) for \(\left(\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^{N\gamma}} \, dx\right)^{1/N}\). We also denote by \(X^{1,N}_{\gamma,rad}(\mathbb{R}^N)\) the subspace of \(X^{1,N}_\gamma(\mathbb{R}^N)\) consisting of radial functions. We note that a special form of the Caffarelli-Kohn-Nirenberg inequality in \cite{3}:
\begin{equation}
\|u\|_{N,\beta} \leq C\|u\|_{N,\gamma}^{\frac{N-\beta}{N-\gamma}} \|\nabla u\|_N^{1-\frac{N-\beta}{N-\gamma}}
\end{equation}
implies that \(X^{1,N}_\gamma(\mathbb{R}^N) \subset X^{1,N}_\beta(\mathbb{R}^N)\) when \(\gamma \leq \beta\). From now on, we assume
\begin{equation}
N \geq 2, \quad -\infty < \gamma \leq \beta < N
\end{equation}
and put \(\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right) \alpha_N\).

Recently, Ishiwata-Nakamura-Wadade \cite{16} proved that the following weighted Adachi-Tanaka type Trudinger-Moser inequality holds true: Define
\begin{equation}
\tilde{A}_{\text{rad}}(N, \alpha, \beta, \gamma) = \sup_{u \in X^{1,N}_{\gamma,rad}(\mathbb{R}^N) \setminus \{0\}, \|\nabla u\|_{L^N(\mathbb{R}^N)} \leq 1} \frac{1}{\|u\|_{N,\gamma}^{N\left(\frac{N-\beta}{N-\gamma}\right)}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^\frac{N}{N-\gamma}) \, dx / |x|^{\beta}.
\end{equation}
Then for $N, \beta, \gamma$ satisfying \((1.6)\), we have

\[
\tilde{A}_{rad}(N, \alpha, \beta, \gamma) \left\{ \begin{array}{ll}
< \infty, & \alpha < \alpha_{N,\beta}, \\
= \infty, & \alpha \geq \alpha_{N,\beta}.
\end{array} \right.
\]

Later, Dong-Lu \cite{12} extends the result in the non-radial setting. Let

\[
\tilde{A}(N, \alpha, \beta, \gamma) = \sup_{u \in X^{1,N}_{\gamma}(\mathbb{R}^N) \setminus \{0\}, \|\nabla u\|_{L^N(\mathbb{R}^N)} \leq 1} \left\{ \begin{array}{ll}
\|u\|_{L^N(\mathbb{R}^N)}^N & = \infty, \\
\|u\|_{L^N(\mathbb{R}^N)}^N & \leq 1.
\end{array} \right.
\]

Then the corresponding result holds true also for $\tilde{A}_{rad}(N, \alpha, \beta, \gamma)$. Attainability of the best constant \((1.7), (1.9)\) is also considered in \cite{16} and \cite{12}: $\tilde{A}_{rad}(N, \alpha, \beta, \gamma)$ and $\tilde{A}(N, \alpha, \beta, \gamma)$ are attained for any $0 < \alpha < \alpha_{N,\beta}$.

The first purpose of this note is to establish the weighted Li-Ruf type Trudinger-Moser inequality on the weighted Sobolev space $X^{1,N}_{\gamma}(\mathbb{R}^N)$ with $N, \beta, \gamma$ satisfying \((1.6)\). Define

\[
\tilde{B}_{rad}(N, \alpha, \beta, \gamma) = \sup_{u \in X^{1,N}_{\gamma}(\mathbb{R}^N), \|u\|_{X^{1,N}_{\gamma}(\mathbb{R}^N)} \leq 1} \Phi_N(\alpha|u|_{X^{N-1}_{\gamma}(\mathbb{R}^N)}) \frac{dx}{|x|^\beta},
\]

\[
\tilde{B}(N, \alpha, \beta, \gamma) = \sup_{u \in X^{1,N}(\mathbb{R}^N), \|u\|_{X^{1,N}(\mathbb{R}^N)} \leq 1} \Phi_N(\alpha|u|_{X^{N-1}(\mathbb{R}^N)}) \frac{dx}{|x|^\beta}.
\]

**Theorem 1.** (Weighted Li-Ruf type inequality) Assume \((1.6)\) and put $\alpha_{N,\beta} = \left(\frac{N-\beta}{N}\right) \alpha_N$. Then we have

\[
\tilde{B}_{rad}(N, \alpha, \beta, \gamma) \left\{ \begin{array}{ll}
< \infty, & \alpha \leq \alpha_{N,\beta}, \\
= \infty, & \alpha > \alpha_{N,\beta}.
\end{array} \right.
\]

Furthermore if $0 \leq \gamma \leq \beta < N$, we have

\[
\tilde{B}(N, \alpha, \beta, \gamma) \left\{ \begin{array}{ll}
< \infty, & \alpha \leq \alpha_{N,\beta}, \\
= \infty, & \alpha > \alpha_{N,\beta}.
\end{array} \right.
\]

We also study the existence and non-existence of maximizers for the weighted Trudinger-Moser inequalities \((1.12)\) and \((1.13)\).

**Theorem 2.** Assume \((1.6)\). Then the following statements hold.

(i) If $N \geq 3$ then $\tilde{B}_{rad}(N, \alpha, \beta, \gamma)$ is attained for any $0 < \alpha \leq \alpha_{N,\beta}$.

(ii) If $N = 2$ then $\tilde{B}_{rad}(2, \alpha, \beta, \gamma)$ is attained for any $0 < \alpha \leq \alpha_{2,\beta}$ if $\beta > \gamma$, while there exists $\alpha_* > 0$ such that $\tilde{B}_{rad}(2, \alpha, \beta, \gamma)$ is attained for any $\alpha_* < \alpha < \alpha_{2,\beta}$.

(iii) $\tilde{B}_{rad}(2, \alpha, \beta, \gamma)$ is not attained for sufficiently small $\alpha > 0$.

**Theorem 3.** Let $N \geq 2$, $0 \leq \gamma \leq \beta < N$. Then the following statements hold.

(i) If $N \geq 3$ then $\tilde{B}(N, \alpha, \beta, \gamma)$ is attained for any $0 < \alpha \leq \alpha_{N,\beta}$.
(ii) If \( N = 2 \) then \( \tilde{B}(2, \alpha, \beta, \gamma) \) is attained for any \( 0 < \alpha \leq \alpha_{2,\beta} \) if \( \beta > \gamma \), while there exists \( \alpha_* > 0 \) such that \( \tilde{B}(2, \alpha, \beta, \beta) \) is attained for any \( \alpha_* < \alpha < \alpha_{2,\beta} \).

(iii) \( \tilde{B}(2, \alpha, \beta, \beta) \) is not attained for sufficiently small \( \alpha > 0 \).

Next, we study the relation between the suprema of Adachi-Tanaka type and Li-Ruf type weighted Trudinger-Moser inequalities, along the line of Lam-Lu-Zhang \cite{19}. Set \( \tilde{B}_{rad}(N, \beta, \gamma) = \tilde{B}_{rad}(N, \alpha N, \beta, \gamma) \) in \((1.10)\), and \( \tilde{B}(N, \beta, \gamma) = \tilde{B}(N, \alpha N, \beta, \gamma) \) in \((1.11)\), i.e.,

\[
\tilde{B}_{rad}(N, \beta, \gamma) = \sup_{u \in X^{1,N}_{1,\gamma, rad}(\mathbb{R}^N), \|u\|_{X^{1,N}_{1,\gamma}} \leq 1} \int_{\mathbb{R}^N} \Phi_N(\alpha N, \beta, \gamma) \left|u\right|_{N-1}^{\frac{N}{N-1}} \frac{dx}{|x|}^{N-\beta},
\]

\[
\tilde{B}(N, \beta, \gamma) = \sup_{u \in X^{1,N}_{1,\gamma, rad}(\mathbb{R}^N), \|u\|_{X^{1,N}_{1,\gamma}} \leq 1} \int_{\mathbb{R}^N} \Phi_N(\alpha N, \beta, \gamma) \left|u\right|_{N-1}^{\frac{N}{N-1}} \frac{dx}{|x|}^{N-\beta},
\]

for \( N, \beta, \gamma \) satisfying \((1.6)\). Then \( \tilde{B}_{rad}(N, \beta, \gamma) < \infty \), and \( \tilde{B}(N, \beta, \gamma) < \infty \) if \( \gamma \geq 0 \), by Theorem 1.

**Theorem 4.** (Relation) Assume \((1.6)\). Then we have

\[
\tilde{B}_{rad}(N, \beta, \gamma) = \sup_{\alpha \in (0, \alpha_{N,\beta})} \left( 1 - \left( \frac{\alpha}{\alpha_{N,\beta}} \right)^{N-1} \right)^{\frac{N-\beta}{N-\gamma}} A_{rad}(N, \alpha, \beta, \gamma).
\]

Furthermore if \( \gamma \geq 0 \), we have

\[
\tilde{B}(N, \beta, \gamma) = \sup_{\alpha \in (0, \alpha_{N,\beta})} \left( 1 - \left( \frac{\alpha}{\alpha_{N,\beta}} \right)^{N-1} \right)^{\frac{N-\beta}{N-\gamma}} A(N, \alpha, \beta, \gamma).
\]

Note that this implies \( A_{rad}(N, \alpha, \beta, \gamma) < \infty \) for \( N, \beta, \gamma \) satisfying \((1.6)\), and \( A(N, \alpha, \beta, \gamma) < \infty \) if \( 0 \leq \gamma \leq \beta < N \), by Theorem 1.

Furthermore, we prove how \( A_{rad}(N, \alpha, \beta, \gamma) \) and \( A(N, \alpha, \beta, \gamma) \) behaves as \( \alpha \) approaches to \( \alpha_{N,\beta} \) from the below:

**Theorem 5.** (Asymptotic behavior of Adachi-Tanaka supremum) Assume \((1.6)\). Then there exist positive constants \( C_1, C_2 \) (depending on \( N, \beta, \) and \( \gamma \)) such that for \( \alpha \) close enough to \( \alpha_{N,\beta} \), the estimate

\[
\left( \frac{C_1}{1 - \left( \frac{\alpha}{\alpha_{N,\beta}} \right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \leq A_{rad}(N, \alpha, \beta, \gamma) \leq \left( \frac{C_2}{1 - \left( \frac{\alpha}{\alpha_{N,\beta}} \right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}}
\]

holds. Corresponding estimates hold true for \( A(N, \alpha, \beta, \gamma) \) if \( \gamma \geq 0 \).
Note that the estimate from the above follows from Theorem 4. On the other hand, we will see that the estimate from the below follows from a computation using the Moser sequence.

The organization of the paper is as follows: In section 2, we prove Theorem 1. Main tools are a transformation which relates a function in $X^{1,N}_{\gamma}(\mathbb{R}^N)$ to a function in $W^{1,N}(\mathbb{R}^N)$, and the singular Trudinger-Moser type inequality recently proved by Adimurthi and Yang [3], see also de Souza and de O [29]. In section 3, we prove the existence part of Theorems 2, 3 (i) (ii). In section 4, we prove the nonexistence part of Theorem 2, 3 (iii). Finally in section 5, we prove Theorem 4 and Theorem 5. The letter $C$ will denote various positive constant which varies from line to line, but is independent of functions under consideration.

2. Proof of Theorem 1

In this section, we prove Theorem 1. We will use the following singular Trudinger-Moser inequality on the whole space $\mathbb{R}^N$: For any $\beta \in [0, N)$, define

\begin{equation}
\tilde{B}(N, \alpha, \beta, 0) = \sup_{\|u\|_{W^{1,N}(\mathbb{R}^N)}} \int_{\mathbb{R}^N} \Phi_N(\alpha |u|^{\frac{N}{N-\gamma}}) \frac{dx}{|x|^\beta}.
\end{equation}

Then it holds

\begin{equation}
\tilde{B}(N, \alpha, \beta, 0) \begin{cases}
< \infty, & \alpha \leq \alpha_{N,\beta}, \\
= \infty, & \alpha > \alpha_{N,\beta}.
\end{cases}
\end{equation}

Here $\|u\|_{W^{1,N}} = (\|\nabla u\|_N^N + \|u\|_N^{N/1})^{1/N}$ denotes the full norm of the Sobolev space $W^{1,N}(\mathbb{R}^N)$. Note that the inequality (2.2) was first established by Ruf [27] for the case $N = 2$ and $\beta = 0$. It then was extended to the case $N \geq 3$ and $\beta = 0$ by Li and Ruf [20]. The case $N \geq 2$ and $\beta \in (0, N)$ was proved by Adimurthi and Yang [3], see also de Souza and de O [29].

Proof of Theorem 1 We define the vector-valued function $F$ by

\[ F(x) = \left( \frac{N - \gamma}{N} \right)^{\frac{N}{N-\gamma}} |x|^{\frac{N}{N-\gamma}} x. \]

Its Jacobian matrix is

\begin{align*}
DF(x) &= \left( \frac{N - \gamma}{N} \right)^{\frac{N}{N-\gamma}} |x|^{\frac{N}{N-\gamma}} \left( Id_N + \frac{\gamma}{N - \gamma} \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \\
&= \frac{N - \gamma}{N} |F(x)|^{\frac{N}{N-\gamma}} \left( Id_N + \frac{\gamma}{N - \gamma} \frac{x}{|x|} \otimes \frac{x}{|x|} \right).
\end{align*}

where $Id_N$ denotes the $N \times N$ identity matrix and $v \otimes v = (v_i v_j)_{1 \leq i, j \leq N}$ denotes the matrix corresponding to the orthogonal projection onto the line generated by the unit vector $v = (v_1, \cdots, v_N) \in \mathbb{R}^N$, i.e., the map $x \mapsto (x \cdot v)v$. Since a matrix of the form $I + \alpha v \otimes v$,
\[ \alpha \in \mathbb{R}, \text{ has eigenvalues } 1 \text{ (with multiplicity } N - 1) \text{ and } 1 + \alpha \text{ (with multiplicity 1), we see that} \]
\[ (2.3) \quad \det(DF(x)) = \left( \frac{N - \gamma}{N} \right)^{N-1} |F(x)|^\gamma. \]

Let \( u \in X_\gamma^1(\mathbb{R}^N) \) be such that \( \|u\|_{X_\gamma^1} \leq 1 \). We introduce a change of functions as follows.
\[ (2.4) \quad v(x) = \left( \frac{N - \gamma}{N} \right)^{\frac{N - 1}{N}} u(F(x)). \]

A simple calculation shows that
\[
\nabla v(x) = \left( \frac{N - \gamma}{N} \right)^{\frac{N - 1}{N}} DF(x)(\nabla u(F(x)))
\]
\[
= \left( \frac{N - \gamma}{N} \right)^{\frac{2N - 1}{N}} |F(x)|^{\frac{2}{N}} \left( \nabla u(F(x)) + \frac{\gamma}{N - \gamma} \left( \nabla u(F(x)) \cdot \frac{x}{|x|} \right) \frac{x}{|x|} \right),
\]
and hence
\[
|\nabla v(x)|^2 = \left( \frac{N - \gamma}{N} \right)^{\frac{2(2N - 1)}{N}} |F(x)|^{\frac{4}{N}} \left( |\nabla u(F(x))|^2 + \frac{\gamma(2N - \gamma)}{(N - \gamma)^2} \left( \nabla u(F(x)) \cdot \frac{x}{|x|} \right)^2 \right).
\]

Since \( \left( \nabla u(F(x)) \cdot \frac{x}{|x|} \right)^2 \leq |\nabla u(F(x))|^2 \), we then have
\[ (2.5) \quad |\nabla v(x)| \leq \left( \frac{N - \gamma}{N} \right)^{\frac{N - 1}{N}} |F(x)|^{\frac{2}{N}} |\nabla u(F(x))| = (\det(DF(x)))^{\frac{1}{N}} |\nabla u(F(x))| \]
if \( \gamma \geq 0 \), with equality if and only if \( \left( \nabla u(F(x)) \cdot \frac{x}{|x|} \right)^2 = |\nabla u(F(x))|^2 \) when \( \gamma > 0 \). If \( \gamma = 0 \) the inequality (2.5) is an equality. Note that the inequality (2.5) does not hold if \( \gamma < 0 \) and \( u \) is not radial function. In fact, a reversed inequality occurs in this case. Moreover, (2.5) becomes an equality if \( u \) is a radial function for any \(-\infty < \gamma < N\). Integrating both sides of (2.5) on \( \mathbb{R}^N \), we obtain
\[ (2.6) \quad \|\nabla v\|_N \leq \|\nabla u\|_N. \]

Moreover, for any function \( G \) on \([0, \infty)\), using the change of variables, we get
\[ (2.7) \quad \int_{\mathbb{R}^N} G \left( |u(x)|^{\frac{N}{N-1}} \right) |x|^{-\delta} \, dx \]
\[ = \left( \frac{N - \gamma}{N} \right)^{N - 1 + \frac{N(\gamma - \delta)}{N - \gamma}} \int_{\mathbb{R}^N} G \left( \frac{N}{N - \gamma} |v(y)|^{\frac{N}{N-1}} \right) |y|^{\frac{N(\gamma - \delta)}{N - \gamma}} \, dy. \]

Consequently, by choosing \( G(t) = t^{N-1} \) and \( \delta = \gamma \), we get \( \|u\|_{N, \gamma} = \|v\|_N \) and hence
\[ (2.8) \quad \|u\|_{X_\gamma^1}^N = \|\nabla u\|_N^N + \int_{\mathbb{R}^N} |u(x)|^N |x|^{-\gamma} \, dx \geq \|\nabla v\|_N^N + \|v\|_N^N = \|v\|_{W^{1,N}}^N. \]
We remark again that (2.6) and (2.8) become equalities if \( u \) is radial function for any \( \gamma < N \). Thus \( \|v\|_{W^{1,N}} \leq 1 \) if \( \|u\|_{X^{1,N}_\gamma} \leq 1 \). By choosing \( G(t) = \Phi_N(\alpha t) \) and \( \delta = \beta \geq \gamma \), we get

\[
(2.9) \quad \int_{\mathbb{R}^N} \Phi_N\left( \frac{\alpha |u(x)|^N}{N} \right) |x|^{-\beta} \, dx
= \left( \frac{N - \gamma}{N} \right)^{N-1+\frac{N(\gamma-\beta)}{N-}\gamma} \int_{\mathbb{R}^N} \Phi_N\left( \frac{N}{N - \gamma} \alpha |v(y)|^N \right) |y|^{-\frac{N(\beta-\gamma)}{N-\gamma}} \, dy.
\]

Denote

\[
\bar{\beta} = \frac{N(\beta - \gamma)}{N - \gamma} \in [0, N).
\]

By using (2.8) and (2.9) and applying the singular Trudinger-Moser inequality (2.2), we get

\[
\sup_{u \in X^{1,N}_\gamma(\mathbb{R}^N), \|u\|_{X^{1,N}_\gamma} \leq 1} \int_{\mathbb{R}^N} \Phi_N\left( \frac{\alpha |u(x)|^N}{N} \right) |x|^{-\beta} \, dx
\leq \left( \frac{N - \gamma}{N} \right)^{N-1+\frac{N(\gamma-\beta)}{N-}\gamma} \sup_{v \in W^{1,N}(\mathbb{R}^N), \|v\|_{W^{1,N}} \leq 1} \int_{\mathbb{R}^N} \Phi_N\left( \frac{N}{N - \gamma} \alpha |v(y)|^N \right) |y|^{-\bar{\beta}} \, dy
= \left( \frac{N - \gamma}{N} \right)^{N-1+\frac{N(\gamma-\beta)}{N-}\gamma} \bar{B} \left( N, \frac{N}{N - \gamma} \alpha, \bar{\beta}, 0 \right)
< \infty,
\]

since \( \frac{N}{N - \gamma} \alpha \leq \frac{N - \gamma}{N - \gamma} \alpha_{N,\beta} = \frac{N - \beta}{N - \gamma} \alpha_N = \left( \frac{N - \beta}{N} \right) \alpha_N = \alpha_{N,\bar{\beta}} \).

If \( u \) is radial then so is \( v \). In this case, (2.5), (2.6) become equalities, and hence so does (2.8). Then the conclusion follows again from the singular Trudinger-Moser inequality (2.2).

We finish the proof of Theorem 1 by showing that \( \bar{B}(N, \alpha, \beta, \gamma) = \infty \) and \( \bar{B}_{\text{rad}}(N, \alpha, \beta, \gamma) = \infty \) when \( \alpha > \alpha_{N,\beta} \). Since \( \bar{B}_{\text{rad}}(N, \alpha, \beta, \gamma) \leq \bar{B}(N, \alpha, \beta, \gamma) \), it is enough to prove that \( \bar{B}_{\text{rad}}(N, \alpha, \beta, \gamma) = \infty \). Suppose the contrary that \( \bar{B}_{\text{rad}}(N, \alpha, \beta, \gamma) < \infty \) for some \( \alpha > \alpha_{N,\beta} \). Using again the transformation of functions (2.4) for radial functions \( u \in X^{1,N}_\gamma \), we then have equalities in (2.5), (2.6), and hence in (2.8). Evidently, the transformation of functions (2.4) is a bijection between \( X^{1,N}_{\gamma,\text{rad}} \) and \( W^{1,N}_{\text{rad}} \) and preserves the equality in (2.8). Consequently, we have

\[
\bar{B}_{\text{rad}}(N, \alpha, \beta, \gamma) = \left( \frac{N - \gamma}{N} \right)^{N-1+\frac{N(\gamma-\beta)}{N-}\gamma} \bar{B}_{\text{rad}} \left( N, \frac{N}{N - \gamma} \alpha, \bar{\beta}, 0 \right),
\]
with \( \tilde{\beta} = \frac{N(\beta - \gamma)}{N - \gamma} \in [0, N) \). Hence \( \tilde{B}_{rad} \left( N, \frac{N}{N - \gamma} \alpha, \tilde{\beta}, 0 \right) < \infty \). By rearrangement argument, we have

\[
\tilde{B} \left( N, \frac{N}{N - \gamma} \alpha, \tilde{\beta}, 0 \right) = \tilde{B}_{rad} \left( N, \frac{N}{N - \gamma} \alpha, \tilde{\beta}, 0 \right) < \infty
\]

which violates the result of Adimurthi and Yang since \( \frac{N}{N - \gamma} \alpha > \alpha_{N, \tilde{\beta}} \).

For the later purpose, we also prove here directly \( \tilde{B}_{rad} \left( N, \alpha, \beta, \gamma \right) = \infty \) when \( \alpha > \alpha_{N, \beta} \) by using the weighted Moser sequence as in [16], [19]: Let \( -\infty < \gamma \leq \beta < N \) and for \( n \in \mathbb{N} \) set

\[
A_n = \left( \frac{1}{\omega_{N-1}} \right)^{1/N} \left( \frac{n}{N - \beta} \right)^{1/N}, \quad b_n = \frac{n}{N - \beta},
\]

so that \( (A_n b_n)^{\frac{N}{N-1}} = n/\alpha_{N, \beta} \). Put

\[
(2.10) \quad u_n = \begin{cases} 
A_n b_n, & \text{if } |x| < e^{-b_n}, \\
A_n \log(1/|x|), & \text{if } e^{-b_n} < |x| < 1, \\
0, & \text{if } 1 \leq |x|.
\end{cases}
\]

Then direct calculation shows that

\[
(2.11) \quad \| \nabla u_n \|_{L^N(\mathbb{R}^N)} = 1,
\]

\[
(2.12) \quad \| u_n \|_{N, \gamma, rad}^N = \frac{N - \beta}{(N - \gamma)^{N+1}} \Gamma(N+1)(1/n) + o(1/n)
\]
as \( n \to \infty \). Thus \( u_n \in X_{\gamma, rad}^{1,N}(\mathbb{R}^N) \). In fact for (2.12), we compute

\[
\| u_n \|_{N, \gamma, rad}^N = \omega_{N-1} \int_0^{e^{-b_n}} (A_n b_n)^N r^{N-1-\gamma} dr + \omega_{N-1} \int_{e^{-b_n}}^1 A_n^N (\log(1/r))^{N-1-\gamma} r^{N-1-\gamma} dr
\]

We see

\[
I = \omega_{N-1} (A_n b_n)^N \left[ \frac{r^{N-\gamma}}{N - \gamma} \right]_{r=e^{-b_n}}^{r=1} = \omega_{N-1} \left( \frac{n}{\alpha_{N, \beta}} \right)^{N-1} \frac{e^{-\left( \frac{N-\gamma}{N-\beta} \right)^n}}{N - \gamma} = o(1/n)
\]
as \( n \to \infty \). Also

\[
II = \left( \frac{N - \beta}{n} \right) \int_{e^{-b_n}}^1 (\log(1/r))^{N-1-\gamma} r^{N-1-\gamma} dr
\]

\[
= \left( \frac{N - \beta}{n} \right) \int_0^{b_n} \rho^{N-\gamma} e^{-(N-\gamma)\rho} d\rho = \frac{N - \beta}{(N - \gamma)^{N+1}} \Gamma(N+1) (1/n) \int_0^{(N-\gamma)b_n} \rho^{N-\gamma} e^{-\rho} d\rho
\]

\[
= \frac{N - \beta}{(N - \gamma)^{N+1}} (1/n) \Gamma(N+1) + o(1/n).
\]

Thus we obtain (2.12).
Now, put $v_n(x) = \lambda_n u_n(x)$ where $u_n$ is the weighted Moser sequence in (2.10) and $\lambda_n > 0$ is chosen so that $\lambda_n^N + \lambda_n^N \|u_n\|_{N, \gamma} = 1$. Thus we have $\|\nabla v_n\|_{L^N} + \|v_n\|_{N, \gamma} = 1$ for any $n \in \mathbb{N}$.

By (2.12) with $\beta = \gamma$, we see that $\lambda_n^N = 1 - O(1/n)$ as $n \to \infty$. For $\alpha > \alpha_{N, \beta}$, we calculate

$$
\int_{\mathbb{R}^N} \Phi_N(\alpha|v_n|^{\frac{N}{N-1}}) \frac{dx}{|x|^{\beta}} \geq \int_{\{0 \leq |x| \leq e^{-b_n}\}} \Phi_N(\alpha|v_n|^{\frac{N}{N-1}}) \frac{dx}{|x|^{\beta}}
$$

$$
= \int_{\{0 \leq |x| \leq e^{-b_n}\}} \left( e^{\alpha|v_n|^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{\alpha^j}{j!} |v_n|^{\frac{Nj}{N-1}} \right) \frac{dx}{|x|^{\beta}}
$$

$$
\geq \left\{ \exp \left( \frac{\alpha n^N}{\alpha_{N, \beta} \lambda_n^N} \right) - O(n^{N-1}) \right\} \int_{\{0 \leq |x| \leq e^{-b_n}\}} \frac{dx}{|x|^{\beta}}
$$

$$
\geq \left\{ \exp \left( \frac{\alpha n^N}{\alpha_{N, \beta}} \left( 1 - O \left( \frac{1}{n^{N-1}} \right) \right) \right) - O(n^{N-1}) \right\} \left( \frac{\omega_{N-1}}{N-\beta} \right) e^{-n} \to +\infty
$$

as $n \to \infty$. Here we have used that for $0 \leq |x| \leq e^{-b_n}$,

$$
\alpha|v_n|^{\frac{N}{N-1}} = \alpha \lambda_n^{\frac{N}{N-1}} (A_n b_n)^{\frac{N}{N-1}} = \frac{n\alpha}{\alpha_{N, \beta}} \lambda_n^{\frac{N}{N-1}}
$$

by definition of $A_n$ and $b_n$. Also we used that for $0 \leq |x| \leq e^{-b_n}$,

$$
|v_n|^{\frac{Nj}{N-1}} = \lambda_n^{\frac{Nj}{N-1}} (A_n b_n)^{\frac{Nj}{N-1}} \leq Cn^j \leq Cn^{N-1}
$$

for $0 \leq j \leq N - 2$ and $n$ is large. This proves Theorem 1 completely. \hfill \Box

3. Existence of maximizers for the weighted Trudinger-Moser inequality

As explained in the Introduction, the existence and non-existence of maximizers for (2.1) is well known. Now, let us recall it here.

Proposition 1. The following statements hold,

(i) If $N \geq 3$ then $\tilde{B}(N, \alpha, 0, 0)$ is attained for any $0 < \alpha \leq \alpha_N$ (see [15, 20]).

(ii) If $N = 2$, there exists $0 < \alpha_* < \alpha_2 = 4\pi$ such that $\tilde{B}(2, \alpha, 0, 0)$ is attained for any $\alpha_* < \alpha \leq \alpha_2$ (see [15, 27]).

(iii) If $\beta \in (0, N)$ and $N \geq 2$ then $\tilde{B}(N, \alpha, \beta, 0)$ is attained for any $0 < \alpha \leq \alpha_{N, \beta}$ (see [21]).

(iv) $\tilde{B}(2, \alpha, 0, 0)$ is not attained for any sufficiently small $\alpha > 0$ (see [15]).

The existence part (iii) of Proposition 1 is recently proved by X. Li, and Y. Yang [21] by a blow-up analysis.

Remark 1. By a rearrangement argument, the maximizers for (2.11), if exist, must be a decreasing spherical symmetric function if $\beta \in (0, N)$ and up to a translation if $\beta = 0$.

The proofs of the existence part (i) (ii) of Theorem 2 and 3 are completely similar by using the formula of change of functions [24] and the results on the existence of maximizers.
for (2.1). So we prove Theorem 3 only here. As we have seen from the proof of Theorem 1 that
\[ \tilde{B}(N, \alpha, \beta, \gamma) \leq \left( \frac{N - \gamma}{N} \right)^{N - 1 + \frac{N(\gamma - \beta)}{N - \gamma}} \tilde{B} \left( N, \frac{N}{N - \gamma}, \alpha, \tilde{\beta}, 0 \right) \]
if \( 0 \leq \gamma \leq \beta < N \), where \( \tilde{\beta} = \frac{N(\beta - \gamma)}{N - \gamma} \in [0, N) \). If \( N, \alpha, \beta, \gamma \) satisfy the condition (i) and (ii) of Theorem 3, then \( N, N/\gamma \) and \( \tilde{\beta} \) satisfy the condition (i)–(iii) of Proposition 1, hence there exists a maximizer \( v \in W^{1,N}(\mathbb{R}^N) \) for \( \tilde{B} \left( N, \frac{N}{N - \gamma}, \alpha, \tilde{\beta}, 0 \right) \) with \( \|v\|_N + \|
abla v\|_N^N = 1 \) and
\[ \int_{\mathbb{R}^N} \Phi_N \left( \frac{N}{N - \gamma} |v(y)|^{\frac{N}{N - 1}} \right) |y|^{-\tilde{\beta}} dy = \tilde{B} \left( N, \frac{N}{N - \gamma}, \alpha, \tilde{\beta}, 0 \right). \]
As mentioned in Remark 1, we can assume that \( v \) is a radial function. Let \( u \in X^{1,2}_\gamma(\mathbb{R}^2) \) be a function defined by (2.4). Note that \( u \) is also a radial function, hence (2.5) becomes an equality. So do (2.6) and (2.8). Hence, we get
\[ \|u\|_{X^{1,2}_\gamma} = \|
abla v\|_N^N + \|v\|_N^{N/2} = 1, \]
and by (2.9)
\[ \int_{\mathbb{R}^N} \Phi_N \left( \frac{N}{N - \gamma} |u(x)|^{\frac{N}{N - 1}} \right) |x|^{-\beta} dx = \left( \frac{N - \gamma}{N} \right)^{N - 1 + \frac{N(\gamma - \beta)}{N - \gamma}} \tilde{B} \left( N, \frac{N}{N - \gamma}, \alpha, \tilde{\beta}, 0 \right). \]
This shows that \( u \) is a maximizer for \( \tilde{B}(N, \alpha, \beta, \gamma) \). □

4. Non-existence of maximizers for the weighted Trudinger-Moser inequality

In this section, we prove the non-existence part (iii) of Theorem 3. The proof of (iii) of Theorem 2 is completely similar. We follow Ishiwata’s argument in [15].

Assume \( 0 \leq \beta < 2, 0 < \alpha \leq \alpha_{2,\beta} = 2\pi(2 - \beta) \) and recall
\[ \tilde{B}(2, \alpha, \beta, \beta) = \sup_{u \in X^{1,2}_\beta(\mathbb{R}^2) \atop \|u\|_{X^{1,2}_\beta(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} (e^{au^2} - 1) \frac{dx}{|x|^\beta}. \]
We will show that \( \tilde{B}(2, \alpha, \beta, \beta) \) is not attained if \( \alpha > 0 \) sufficiently small. Set
\[ M = \left\{ u \in X^{1,2}_\beta(\mathbb{R}^2) : \|u\|_{X^{1,2}_\beta} = (\|\nabla u\|_2^2 + \|u\|_{2,\beta}^2)^{1/2} = 1 \right\} \]
be the unit sphere in the Hilbert space \( X^{1,2}_\beta(\mathbb{R}^2) \) and
\[ J_\alpha : M \to \mathbb{R}, \quad J_\alpha(u) = \int_{\mathbb{R}^2} (e^{au^2} - 1) \frac{dx}{|x|^\beta} \]
be the corresponding functional defined on \( M \). Actually, we will prove the stronger claim that \( J_\alpha \) has no critical point on \( M \) when \( \alpha > 0 \) is sufficiently small.
Assume the contrary that there existed \( v \in M \) such that \( v \) is a critical point of \( J_\alpha \) on \( M \). Define an orbit on \( M \) through \( v \) as

\[
v_\tau(x) = \sqrt{\tau}v(\sqrt{\tau}x) \quad \tau \in (0, \infty), \quad w_\tau = \frac{v_\tau}{\|v_\tau\|_{X_\alpha}^{1/2}} \in M.
\]

Since \( w_\tau|_{\tau=1} = v \), we must have

\[
(4.1) \quad \frac{d}{d\tau} \bigg|_{\tau=1} J_\alpha(w_\tau) = 0.
\]

Note that

\[
\|\nabla v_\tau\|^2_{L^2(\mathbb{R}^2)} = \tau \|\nabla v\|^2_{L^2(\mathbb{R}^2)}, \quad \|v_\tau\|_{p, \beta}^p = \tau^{\frac{p+\beta-2}{2}} \|v\|_{p, \beta}^p
\]

for \( p > 1 \). Thus,

\[
J_\alpha(w_\tau) = \int_{\mathbb{R}^2} \left( e^{\alpha \omega^2} - 1 \right) \frac{dx}{|x|^{\beta}} = \int_{\mathbb{R}^2} \frac{\sum_{j=1}^{\infty} \alpha^j v_\tau^{2j}(x)}{\|v_\tau\|^2_{X_\alpha}^{1/2} |x|^{\beta}} dx
\]

\[
= \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \frac{\|v_\tau\|^{2j}_{2, \beta}}{(\|\nabla v\|^2_2 + \|v_\tau\|^2_{2, \beta})^j} = \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \left( \frac{\tau^{j-1+\frac{\beta}{2}}}{\tau a + \tau^{\frac{\beta}{2}} b} \right)^j.
\]

By using an elementary computation

\[
f(\tau) = \frac{\tau^{j-1+\frac{\beta}{2}} c}{(\tau a + \tau^{\frac{\beta}{2}} b)^j}, \quad a = \|\nabla v\|^2_2, \quad b = \|v\|^2_{2, \beta}, \quad c = \|v_\tau\|^2_{2, \beta},
\]

\[
f'(\tau) = (1 - \frac{\beta}{2}) \frac{\tau^{j-2+\frac{\beta}{2}} c}{(\tau a + \tau^{\frac{\beta}{2}} b)^{j+1}} \left\{ -\tau a + (j - 1)b \right\},
\]

we estimate \( \frac{d}{d\tau} \bigg|_{\tau=1} J_\alpha(w_\tau) \):

\[
= \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} \left( 1 - \frac{\beta}{2} \right) \frac{\tau^{j-2+\beta/2} \|v\|^{2j}_{2, \beta}}{(\tau a + \tau^{\beta/2} b)^{j+1}} \left\{ -\tau \|\nabla v\|^2_2 + (j - 1)\|v\|^2_{2, \beta} \right\}
\]

\[
\leq \alpha \left( 1 - \frac{\beta}{2} \right) \|\nabla v\|^2_2 \|v\|^2_{2, \beta} \left\{ -\frac{1}{1} + \sum_{j=2}^{\infty} \frac{\alpha^{j-1} \|v\|^{2j}_{2, \beta}}{(j - 1)! \|\nabla v\|^2_2 \|v\|^2_{2, \beta}} \right\},
\]

since \(-\|\nabla v\|^2_2 + (j - 1)\|v\|^2_{2, \beta} \leq j\).

Now, we state a lemma. Unweighted version of the next lemma is proved in [15]:Lemma 3.1, and the proof of the next is a simple modification of the one given there using the
weighted Adachi-Tanaka type Trudinger-Moser inequality:
\[
\tilde{A}(2, \alpha, \beta, \beta) = \sup_{u \in X^\frac{1}{2}(\mathbb{R}^2) \setminus \{0\}} \frac{1}{\|u\|^{2j}_{2, \beta}} \int_{\mathbb{R}^2} (e^{ou^2} - 1) \frac{dx}{|x|^\beta} < \infty
\]
for \( \alpha \in (0, \alpha_2, \beta) \) if \( \beta \geq 0 \), and the expansion of the exponential function.

**Lemma 1.** For any \( \alpha \in (0, \alpha_2, \beta) \), there exists \( C_\alpha > 0 \) such that
\[
\|u\|^{2j}_{2j, \beta} \leq C_\alpha \frac{j!}{\alpha^j} \|\nabla u\|^{2j-2}_{2, \beta} \frac{\|u\|^2_{2, \beta}}
\]
holds for any \( u \in X^\frac{1}{2}(\mathbb{R}^2) \) and \( j \in \mathbb{N}, j \geq 2 \).

By this lemma, if we take \( \alpha < \tilde{\alpha} < \alpha_{2, \beta} \) and put \( C = C_{\tilde{\alpha}} \), we see
\[
\|\nabla v\|^{2j}_{2j, \beta} \leq C \frac{j!}{\alpha^j} \|\nabla v\|^{2j}_{2j, \beta} \leq C \frac{j!}{\alpha^j}
\]
for \( j \geq 2 \) since \( v \in M \). Thus we have
\[
\frac{1}{(j-1)!} \|\nabla v\|^{2j}_{2j, \beta} \leq \sum_{j=2}^{\infty} C \frac{1}{(j-1)!} \frac{j!}{\alpha^j} \left( \frac{\alpha}{\alpha} \right)^{-2} j \leq C'
\]
for some \( C' > 0 \). Inserting this into the former estimate (4.2), we obtain
\[
\frac{d}{dx} \bigg|_{x=1} J_\alpha(w_\gamma) \leq (1 - \frac{\beta}{2}) \alpha \|\nabla v\|^2_{2j, \beta} (-1 + C' \alpha) < 0
\]
when \( \alpha > 0 \) is sufficiently small. This contradicts to (4.1). \( \square \)

5. **Proof of Theorem 4 and 5.**

In this section, we prove Theorem 4 and Theorem 5. As stated in the Introduction, we follow the argument by Lam-Lu-Zhang [19]. First, we prepare several lemmata.

**Lemma 2.** Assume (1.6) and set
\[
\hat{A}(N, \alpha, \beta, \gamma) = \sup_{u \in X^\frac{1}{2}(\mathbb{R}^N) \setminus \{0\}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u|^N) \frac{dx}{|x|^\beta}.
\]

Let \( \hat{A}(N, \alpha, \beta, \gamma) \) be defined as in (1.9). Then \( \hat{A}(N, \alpha, \beta, \gamma) = \hat{A}(N, \alpha, \beta, \gamma) \) for any \( \alpha > 0 \).
Similarly, \( \hat{A}_{rad}(N, \alpha, \beta, \gamma) = \hat{A}_{rad}(N, \alpha, \beta, \gamma) \) for any \( \alpha > 0 \), where \( \hat{A}_{rad}(N, \alpha, \beta, \gamma) \) is defined similar to (5.1) and \( \hat{A}_{rad}(N, \alpha, \beta, \gamma) \) is defined in (1.7).

**Proof.** For any \( u \in X^\frac{1}{2}(\mathbb{R}^N) \setminus \{0\} \) and \( \lambda > 0 \), we put \( u_\lambda(x) = u(\lambda x) \) for \( x \in \mathbb{R}^N \). Then it is easy to see that
\[
\begin{cases}
\|\nabla u_\lambda\|^N_{L^\infty(\mathbb{R}^N)} = \|\nabla u\|^N_{L^\infty(\mathbb{R}^N)}, \\
\|u_\lambda\|^N_{N, \gamma} = \lambda^{-(N-\gamma)} \|u\|^N_{N, \gamma}.
\end{cases}
\]
Lemma 3. Assume (1.6) and set

\[ \widehat{B}(\beta, \gamma) = \frac{\| \nabla u \|_{L^\infty}^\gamma}{\| u \|_{L^\infty}^\gamma} \]

Thus for any \( u \in X_{1,N}^\beta(\mathbb{R}^N) \setminus \{0\} \) with \( \| \nabla u \|_{L^\infty} \leq 1 \), if we choose \( \lambda = \| u \|_{L^\infty}^{\frac{N}{N-\gamma}} \), then \( u_\lambda \in X_{1,N}^\beta(\mathbb{R}^N) \) satisfies

\[ \| \nabla u_\lambda \|_{L^\infty} \leq 1 \quad \text{and} \quad \| u_\lambda \|_{L^\infty} = 1. \]

Thus

\[ \widehat{A}(\alpha, \beta, \gamma) \leq \left( \frac{\alpha}{\alpha_{N,\beta}} \right)^{\frac{N-1}{N-\gamma}} \left( \frac{\alpha_{N,\beta}}{\alpha} \right)^{N-1} \widehat{B}(\beta, \gamma) \]

which implies \( \widehat{A}(\alpha, \beta, \gamma) \geq \widehat{A}(\alpha, \beta, \gamma) \). The opposite inequality is trivial. \( \square \)

Lemma 3. Assume (1.6) and set \( \tilde{B}(\beta, \gamma) \) as in (1.13). Then we have

\[ \tilde{A}(\alpha, \beta, \gamma) \leq \left( \frac{\alpha}{\alpha_{N,\beta}} \right)^{\frac{N-1}{N-\gamma}} \tilde{B}(\beta, \gamma) \]

for any \( 0 < \alpha < \alpha_{N,\beta} \). The same relation holds for \( \tilde{A}_{\text{rad}}(\alpha, \beta, \gamma) \) in (1.7) and \( \tilde{B}_{\text{rad}}(\beta, \gamma) \) in (1.14).

Proof. Choose any \( u \in X_{1,N}^\beta(\mathbb{R}^N) \) with \( \| \nabla u \|_{L^\infty} \leq 1 \) and \( \| u \|_{L^\infty} = 1 \). Put \( v(x) = C u(\lambda x) \) where \( C \in (0, 1) \) and \( \lambda > 0 \) are defined as

\[ C = \left( \frac{\alpha}{\alpha_{N,\beta}} \right)^{\frac{N-1}{N-\gamma}} \quad \text{and} \quad \lambda = \left( \frac{C^N}{1 - C^N} \right)^{1/(N-\gamma)}. \]

Then by scaling rules (5.2), we see

\[ \| u \|_{X_{1,N}^\beta} = \| v \|_{X_{1,N}^\beta} = C^N \| \nabla u \|_{L^\infty}^{\frac{N}{N-\gamma}} + \lambda^{-\frac{N}{N-\gamma}} C^N \| u \|_{L^\infty}^{\frac{N}{N-\gamma}} \leq C^N + \lambda^{-\frac{N}{N-\gamma}} C^N = 1. \]

Also we have

\[ \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta} | v |_{L^\infty}^{\frac{N}{N-\gamma}}) \frac{dx}{|x|^\beta} = \lambda^{-\frac{N}{N-\beta}} \int_{\mathbb{R}^N} \Phi_N(\alpha_{N,\beta} C^N | v |_{L^\infty}^{\frac{N}{N-\gamma}}) \frac{dx}{|x|^\beta} \]

\[ = \lambda^{-\frac{N}{N-\beta}} \int_{\mathbb{R}^N} \Phi_N(\alpha | v |_{L^\infty}^{\frac{N}{N-\gamma}}) \frac{dx}{|x|^\beta}. \]

Thus testing \( \tilde{B}(\beta, \gamma) \) by \( v \), we see

\[ \tilde{B}(\beta, \gamma) \geq \left( \frac{1 - C^N}{C^N} \right)^{\frac{N-1}{N-\gamma}} \int_{\mathbb{R}^N} \Phi_N(\alpha | v |_{L^\infty}^{\frac{N}{N-\gamma}}) \frac{dx}{|x|^\beta}. \]

By taking the supremum for \( u \in X_{1,N}^\beta(\mathbb{R}^N) \) with \( \| \nabla u \|_{L^\infty} \leq 1 \) and \( \| u \|_{L^\infty} = 1 \), we have

\[ \tilde{B}(\beta, \gamma) \geq \left( \frac{1 - \alpha_{N,\beta}}{\alpha_{N,\beta}} \right)^{\frac{N-1}{N-\gamma}} \widehat{A}(\alpha, \beta, \gamma). \]
Finally, Lemma 2 implies the result. The proof of

$$\tilde{B}_{\text{rad}}(N, \beta, \gamma) \geq \left(1 - \frac{\alpha}{\alpha_{N, \beta}}\right)^{N-1} \tilde{A}_{\text{rad}}(N, \alpha, \beta, \gamma)$$

is similar. \(\square\)

**Proof of Theorem 4** We prove the relation between \(\tilde{B}(N, \beta, \gamma)\) and \(\tilde{A}(N, \alpha, \beta, \gamma)\) only. The assertion that

$$\tilde{B}(N, \beta, \gamma) \geq \sup_{\alpha \in (0, \alpha_{N, \beta})} \left(1 - \frac{\alpha}{\alpha_{N, \beta}}\right)^{N-1} \tilde{A}(N, \alpha, \beta, \gamma)$$

follows from Lemma 3. Note that \(\tilde{B}(N, \beta, \gamma) < \infty\) when \(0 \leq \gamma \leq \beta < N\) by Theorem 1.

Let us prove the opposite inequality. Let \(\{u_n\} \subset X^{1,N}_{\gamma}(\mathbb{R}^N), u_n \neq 0, \|\nabla u_n\|_{L^N} + \|u_n\|_{N,\gamma} \leq 1\), be a maximizing sequence of \(\tilde{B}(N, \beta, \gamma)\):

$$\int_{\mathbb{R}^N} \Phi_N(\alpha_{N, \beta} |u_n|^{N-1}) \frac{dx}{|x|^{\beta}} = \tilde{B}(N, \beta, \gamma) + o(1)$$

as \(n \to \infty\). We may assume \(\|\nabla u_n\|_{L^N(\mathbb{R}^N)} < 1\) for any \(n \in \mathbb{N}\). Define

$$\begin{cases} v_n(x) = \frac{u_n(\lambda_n x)}{\|\nabla u_n\|_{L^N}}, & (x \in \mathbb{R}^N) \\ \lambda_n = \left(\frac{1 - \|\nabla u_n\|_{L^N}}{\|\nabla u_n\|_{L^N}}\right)^{1/(N-\gamma)} > 0. \end{cases}$$

Thus by (5.2), we see

$$\|\nabla u_n\|_{L^N(\mathbb{R}^N)} = 1, \quad \|\nabla u_n\|_{L^{N/(N-\beta)}(\mathbb{R}^N)} = \left(\frac{\lambda_n^{-1}}{\|\nabla u_n\|_{L^N}^{N/(N-\gamma)}}\right)^{\frac{N-\beta}{N-\gamma}} \leq \left(\frac{\lambda_n^{-1}}{1 - \|\nabla u_n\|_{L^N}^{N/(N-\gamma)}}\right)^{\frac{N-\beta}{N-\gamma}} \leq 1,$$

since \(\|\nabla u_n\|_{L^N}^{N} + \|u_n\|_{N,\gamma}^{N} \leq 1\). Thus, setting

$$\alpha_n = \alpha_{N, \beta} \|\nabla u_n\|_{L^N}^{N/(N-\gamma)} < \alpha_{N, \beta}$$
for any $n \in \mathbb{N}$, we may test $\tilde{A}(N, \alpha_n, \beta, \gamma)$ by $\{v_n\}$, which results in

\[
\tilde{B}(N, \beta, \gamma) + o(1) = \int_{\mathbb{R}^N} \Phi_N(\alpha_N, \beta | u_n(y) | \frac{N}{N-\beta} ) \frac{dy}{|y|^{N-\beta}}
= \lambda_n^{N-\beta} \int_{\mathbb{R}^N} \Phi_N(\alpha_N, \beta \|\nabla u_n\|_N^{\frac{N}{N-\beta}} | v_n(x) | \frac{N}{N-\beta}) \frac{dx}{|x|^{N-\beta}}
\leq \lambda_n^{N-\beta} \left( \frac{1}{\|v_n\|_{N, \beta}} \right)^{\frac{N-\beta}{N-\gamma}} \int_{\mathbb{R}^N} \Phi_N(\alpha_n | v_n(x) | \frac{N}{N-1}) \frac{dx}{|x|^{N-\beta}}
\leq \lambda_n^{N-\beta} \tilde{A}(N, \alpha_n, \beta, \gamma) = \left( \frac{1 - \|\nabla u_n\|_N^{\frac{N}{N-\beta}}}{\|\nabla u_n\|_N^{\frac{N}{N-\gamma}}} \right)^{\frac{N-\beta}{N-\gamma}} \tilde{A}(N, \alpha_n, \beta, \gamma)
\leq \sup_{\alpha \in (0, \alpha_N, \beta]} \left( \frac{1 - \left( \frac{\alpha}{\alpha_N, \beta} \right)^{N-1}}{\left( \frac{\alpha}{\alpha_N, \beta} \right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \tilde{A}(N, \alpha, \beta, \gamma).
\]

Here we have used a change of variables $y = \lambda_n x$ for the second equality, and $\|v_n\|_{N, \gamma}^{\frac{N(N-\beta)}{N-\gamma}} \leq 1$ for the first inequality. Letting $n \to \infty$, we have the desired result. \hfill \Box

**Proof of Theorem** Again, we prove theorem for $\tilde{A}(N, \alpha, \beta, \gamma)$ only. The assertion that

\[
\tilde{A}(N, \alpha, \beta, \gamma) \leq \left( \frac{C_2}{1 - \left( \frac{\alpha}{\alpha_N, \beta} \right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}}
\]

follows form Theorem and the fact that $\tilde{B}(N, \beta, \gamma) < \infty$ when $0 \leq \gamma \leq \beta < N$.

For the rest, we need to prove that there exists $C > 0$ such that for any $\alpha < \alpha_N, \beta$ sufficiently close to $\alpha_N, \beta$, it holds that

\[
(5.3) \quad \left( \frac{C}{1 - \left( \frac{\alpha}{\alpha_N, \beta} \right)^{N-1}} \right)^{\frac{N-\beta}{N-\gamma}} \leq \tilde{A}(N, \alpha, \beta, \gamma).
\]
For that purpose, we use the weighted Moser sequence (2.10) again. By (2.12), we have $N_1 \in \mathbb{N}$ such that if $n \in \mathbb{N}$ satisfies $n \geq N_1$, then it holds

$$(5.4) \quad \|u_n\|_{N, \gamma}^N \leq \frac{2(N - \gamma)\Gamma(N + 1)}{(N - \beta)^{N+1}} \left( \frac{1}{n} \right).$$

On the other hand,

$$\int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} \geq \omega_{N-1} \int_0^{e^{-bn}} \Phi_N\left((\alpha/\alpha_{N,\beta})n\right) \left[r^{N-\beta}\right]_{r=0}^{r=e^{-bn}} = \frac{\omega_{N-1}}{N - \beta} \Phi_N\left((\alpha/\alpha_{N,\beta})n\right) e^{-n}.$$ 

Note that there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $\Phi_N\left((\alpha/\alpha_{N,\beta})n\right) \geq \frac{1}{2} e^{(\alpha/\alpha_{N,\beta})n}$. Thus we have

$$(5.5) \quad \int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} \geq \frac{1}{2} \left( \frac{\omega_{N-1}}{N - \beta} \right) e^{(1 - \frac{\alpha}{\alpha_{N,\beta}})n}.$$ 

Combining (5.4) and (5.5), we have $C_1(N, \beta, \gamma) > 0$ such that

$$(5.6) \quad \frac{1}{\|u_n\|_{N, \gamma}^{N/(N-1)}} \int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} \geq C_1(N, \beta, \gamma)n^{\frac{N-\beta}{N-\gamma}} e^{(1 - \frac{\alpha}{\alpha_{N,\beta}})n}$$

holds when $n \geq \max\{N_1, N_2\}$.

Note that $\lim_{x \to 1} \left(\frac{1-x^{N-1}}{1-x}\right) = N - 1$, thus

$$\frac{1 - (\alpha/\alpha_{N,\beta})^{N-1}}{1 - (\alpha/\alpha_{N,\beta})} \geq \frac{N - 1}{2}$$

if $\alpha/\alpha_{N,\beta} < 1$ is very close to 1. Now, for any $\alpha > 0$ sufficiently close to $\alpha_{N,\beta}$ so that

$$(5.7) \quad \begin{cases} 
\max\{N_1, N_2\} < \frac{2}{1-\alpha/\alpha_{N,\beta}}, \\
\frac{1-(\alpha/\alpha_{N,\beta})^{N-1}}{1-(\alpha/\alpha_{N,\beta})} \geq \frac{N-1}{2},
\end{cases}$$

we can find $n \in \mathbb{N}$ such that

$$(5.8) \quad \begin{cases} 
\max\{N_1, N_2\} \leq n \leq \frac{2}{1-\alpha/\alpha_{N,\beta}}, \\
\frac{1}{1-\alpha/\alpha_{N,\beta}} \leq n.
\end{cases}$$
We fix \( n \in \mathbb{N} \) satisfying (5.8). Then by (5.6) and (5.7), we have

\[
\frac{1}{\|u_n\|_{N,\beta}^N} \int_{\mathbb{R}^N} \Phi_N(\alpha|u_n|^{N/(N-1)}) \frac{dx}{|x|^\beta} \geq C_1(N, \beta, \gamma) n \frac{N-\beta}{N-\gamma} e^{-2} \\
\geq C_2(N, \beta, \gamma) \left( \frac{1}{1 - (\alpha/\alpha_{N,\beta})} \right)^{N-\beta} = \frac{N-1}{2} C_2(N, \beta, \gamma) \frac{1}{1 - (\alpha/\alpha_{N,\beta})^{N-1}}^{N-\beta} \\
= C_3(N, \beta, \gamma) \frac{1}{1 - (\alpha/\alpha_{N,\beta})^{N-1}}^{N-\beta} ,
\]

where \( C_2(N, \beta, \gamma) = e^{-2} C_1(N, \beta, \gamma) \) and \( C_3(N, \beta, \gamma) = \frac{N-1}{2} C_2(N, \beta, \gamma) \). Thus we have (5.3) for some \( C > 0 \) independent of \( \alpha \) which is sufficiently close to \( \alpha_{N,\beta} \). \( \square \)

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References

[1] S. Adachi, and K. Tanaka: A scale-invariant form of Trudinger-Moser inequality and its best exponent, Proc. Am. Math. Soc. 1102, (1999) 148-153.
[2] Adimurthi, and K. Sandeep: A singular Moser-Trudinger embedding and its applications, NoDEA Nonlinear Differential Equations Appl. 13 (2007), no. 5-6, 585–603.
[3] Adimurthi, and Y. Yang, An interpolation of Hardy inequality and Trudinger–Moser inequality in \( \mathbb{R}^N \) and its applications, Int. Math. Res. Not., IMRN 13 (2010) 2394–2426.
[4] L. Caffarelli, R. Kohn, and L. Nirenberg: First order interpolation inequalities with weights, Compositio Math. 53 (1984), no. 3, 259–275.
[5] D. M. Cao: Nontrivial solution of semilinear elliptic equation with critical exponent in \( \mathbb{R}^2 \), Commun. Partial Differ. Equ. 17, (1992) 407-435.
[6] L. Carleson, and S.-Y.A. Chang: On the existence of an extremal function for an inequality of J. Moser, Bull. Sci. Math. 2(110), (1986) 113-127.
[7] M. Calanchi: Some weighted inequalities of Trudinger-Moser type, Analysis and topology in nonlinear differential equations, Progr. Nonlinear Differential Equations Appl. 85, (2014) 163–174.
[8] M. Calanchi, and B. Ruf: On Trudinger-Moser type inequalities with logarithmic weights, J. Differential Equations, 258, no.6, (2015), 1967–1989.
[9] M. Calanchi, and B. Ruf: Trudinger-Moser type inequalities with logarithmic weights in dimension \( N \), Nonlinear Anal., 121, (2015), 403–411.
[10] G. Csató, and P. Roy: Extremal functions for the singular Moser-Trudinger inequality in 2 dimensions, Calc. Var. Partial Differential Equations, 54 (2015), no. 2, 2341–2366.
[11] G. Csató, and P. Roy: Singular Moser-Trudinger inequality on simply connected domains, Comm. Partial Differential Equations, 41 (2016), no. 5, 838–847.
[12] M. Dong, and G. Lu: Best constants and existence of maximizers for weighted Trudinger-Moser inequalities, Calc. Var. Partial Differential Equations 55 (2016), no. 4, Art. 88, 26 pp.
[13] M. Flucher: Extremal functions for the Trudinger-Moser inequality in 2 dimensions, Comment. Math. Helv. 67, (1992) 471–497.
[14] M. F. Furtado, E. S. Medeiros, and U. B. Severo: A Trudinger-Moser inequality in a weighted Sobolev space and applications, Math. Nachr. 287, (2014) 1255–1273.
[15] M. Ishiwata: *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in \( \mathbb{R}^N \),* Math. Ann. **351**, (2011) 781-804.

[16] M. Ishiwata, M. Nakamura and H. Wadade: *On the sharp constant for the weighted Trudinger-Moser type inequality of the scaling invariant form,* Ann. Inst. H. Poincare Anal. Non Lineaire. **31** (2014), no. 2, 297-314.

[17] K. C. Lin: *Extremal functions for Moser’s inequality,* Trans. Am. Math. Soc. **348**, (1996) 2663-2671.

[18] N. Lam, and G. Lu: *Sharp singular Trudinger-Moser-Adams type inequalities with exact growth,* Geometric Methods in PDE’s, (G. Citti et al. (eds.)), Springer INdAM Series **13**, (2015) 43–80.

[19] N. Lam, G. Lu, and L. Zhang: *Equivalence of critical and subcritical sharp Trudinger-Moser-Adams inequalities,* arXiv:1504.04858v1 (2015)

[20] Y. Li, and B. Ruf: *A sharp Trudinger-Moser type inequality for unbounded domains in \( \mathbb{R}^n \),* Indiana Univ. Math. J. **57**, (2008) 451-480.

[21] X. Li, and Y. Yang, *Extremal functions for singular Trudinger–Moser inequalities in the entire Euclidean space,* preprint, arXiv:1612.08241v1.

[22] J. Moser: *A sharp form of an inequality by N. Trudinger,* Indiana Univ. Math. J. **20**, (1970) 1077-1092.

[23] T. Ogawa: *A proof of Trudinger’s inequality and its application to nonlinear Schrodinger equation,* Nonlinear Anal. **14**, (1990) 765-769.

[24] T. Ogawa, and T. Ozawa: *Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem,* J. Math. Anal. Appl. **155**, (1991) 531-540.

[25] T. Ozawa: *On critical cases of Sobolev’s inequalities,* J. Funct. Anal. **127**, (1995) 259–269.

[26] S. Pohozaev: *The Sobolev embedding in the case \( pl = n \),* Proceedings of the Technical Scientific Conference on Advances of Scientific Research (1964/1965). Mathematics Section, Moskov. Energetics Institute, Moscow, (1965) 158–170.

[27] B. Ruf: *A sharp Trudinger-Moser type inequality for unbounded domains in \( \mathbb{R}^2 \),* J. Funct. Anal. 219, (2005) 340-367.

[28] M. de Souza: *On a class of singular Trudinger-Moser type inequalities for unbounded domains in \( \mathbb{R}^N \),* Appl. Math. Lett. **25** (2012), no. 12, 2100–2104.

[29] M. de Souza, and M. do Ó: *On singular Trudinger-Moser type inequalities for unbounded domains and their best exponents,* Potential Anal. **38** (2013), no. 4, 1091–1101.

[30] N. S. Trudinger: *On imbeddings into Orlicz spaces and some applications,* J. Math. Mech. **17**, (1967) 473-483.

[31] V. I. Yudovich: *Some estimates connected with integral operators and with solutions of elliptic equations,* Dok. Akad. Nauk SSSR **138**, (1961) 804-808.

[32] Y. Yuan: *A weighted form of Moser-Trudinger inequality on Riemannian surface,* Nonlinera Anal. **65** no. 3, (2006) 647–659.

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