THE NON-LEFSCHETZ LOCUS

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Abstract. We study the weak Lefschetz property of artinian Gorenstein algebras and in particular of artinian complete intersections. In codimension four and higher, it is an open problem whether all complete intersections have the weak Lefschetz property.

For a given artinian Gorenstein algebra $A$ we ask what linear forms are Lefschetz elements for this particular algebra, i.e., which linear forms $\ell$ give maximal rank for all the multiplication maps $\times \ell : [A]_i \to [A]_{i+1}$. This is a Zariski open set and its complement is the non-Lefschetz locus.

For monomial complete intersections, we completely describe the non-Lefschetz locus. For general complete intersections of codimension three and four we prove that the non-Lefschetz locus has the expected codimension, which in particular means that it is empty in a large family of examples. For general Gorenstein algebras of codimension three with a given Hilbert function, we prove that the non-Lefschetz locus has the expected codimension if the first difference of the Hilbert function is of decreasing type. For completeness we also give a full description of the non-Lefschetz locus for artinian algebras of codimension two.

1. Introduction

If $A = R/I$ is an artinian standard graded algebra over the polynomial ring $R = k[x_1, \ldots, x_n]$, where $k$ is a field, then $A$ is said to have the Weak Lefschetz Property (WLP) if the homomorphism induced by multiplication by a general linear form, from every degree to the next, has maximal rank. In this paper we will always assume that $k$ has characteristic zero.

A famous result in commutative algebra says that an artinian monomial complete intersection over a field of characteristic zero has the WLP (and even a stronger condition called the Strong Lefschetz Property). This was proved in [20], [21], and [19]. A consequence of this is that if the generator degrees are specified, a general complete intersection with those generator degrees has the WLP. It is an open question whether every complete intersection has the WLP. Notice that the result above fails to distinguish between a monomial complete intersection and a general one (always with fixed generator degrees).

We give a finer measure of the Lefschetz property that does distinguish between these (conjecturally in all cases, and we give a proof in $\leq 4$ variables).

Suppose that such a standard graded algebra $A$ is given. For any pair of consecutive components $A_i$ and $A_{i+1}$, we can consider the locus $\mathcal{L}_i$ of linear forms that fail to induce a homomorphism of maximal rank on these components. We will observe that for each $i$ the variety $\mathcal{L}_i$ is a determinantal variety, so depending on the absolute value of the difference $\dim[A]_{i+1} - \dim[A]_i$, there is an expected codimension. If the variety achieves this codimension, its degree (as a possibly non-reduced scheme) is also known. One can then ask further questions about $\mathcal{L}_i$, such as what are its irreducible components. If $\mathcal{L}_i$ fails to have the expected codimension, it is still determinantal but its degree is less clear.

We define the non-Lefschetz locus $\mathcal{L}_I$ to be the union of these loci $\mathcal{L}_i$, viewed as subvarieties of the corresponding projective space $(\mathbb{P}^{n-1})^*$, over all possible sets of consecutive components. The algebra $A$ fails to have the WLP if and only if $\mathcal{L}_I = (\mathbb{P}^{n-1})^*$. The
variety $\mathcal{L}_I$ is thus a union of determinantal varieties in general. If $A$ is Gorenstein (e.g. a complete intersection), there is a natural sequence of inclusions of the $\mathcal{L}_i$, so $\mathcal{L}_I$ is in fact itself a determinantal variety. (See Proposition 2.5.)

In this paper we will study the non-Lefschetz locus for specific algebras (monomial algebras) and we will consider it in the case of the general element of an irreducible family (complete intersections of prescribed generator degrees). Much more difficult is the question of whether every element of an irreducible family (specifically complete intersections) has the WLP, i.e. whether the non-Lefschetz locus is always of positive codimension for such algebras.

In Section 3 we completely characterize the non-Lefschetz locus of monomial complete intersections (Proposition 3.1) and we also find all the possible Jordan types of linear forms in such algebras (Proposition 3.7).

In Section 4 we conjecture that the non-Lefschetz locus of a general complete intersection has the expected codimension in the sense that will be made precise in Section 2. We prove this conjecture for complete intersections of codimension three (Theorem 4.10) and codimension four (Theorem 4.13).

In Section 5 we study the non-Lefschetz locus of a general artinian Gorenstein algebra of codimension three with a given Hilbert function. In Theorem 5.1 we prove that the non-Lefschetz locus has the expected codimension if the $g$-vector associated to the Hilbert function is of decreasing type, while it is of codimension one otherwise.

In Section 6 we give a complete description of the situation for algebras in codimension two.

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2. Preliminaries

Let $R = k[x_1, x_2, \ldots, x_n]$ where $k$ is an algebraically closed field of characteristic zero. Let $M$ be a graded $R$-module of finite length. We first briefly recall an idea, originally due to Joe Harris, dealing with an isomorphism invariant of $M$. For further details see [15].

The module structure of $M$ is determined by a collection of homomorphisms $\phi_i : [R]_i \rightarrow \text{Hom}_k(M_i, M_{i+1})$ as $i$ ranges from the initial degree of $M$ to the penultimate degree where $M$ is not zero. Since $\phi_i$ is trivial if either $[M]_i$ or $[M]_{i+1}$ is zero, we assume that this is not the case (we do not assume that $M$ is generated in the first degree, so a zero component could lie between non-zero ones). Let $\ell = a_1x_1 + \cdots + a_nx_n$, and let us refer to the $a_i$ as the dual variables. If we choose bases for $[M]_i$ and for $[M]_{i+1}$, we can view $\phi_i$ as a $(\dim [M]_{i+1}) \times (\dim [M]_i)$ matrix $B_i$ whose entries are linear forms in the dual variables. For any fixed $t$ we can thus consider the ideal of $(t+1) \times (t+1)$ minors of $B_i$, and this is an isomorphism invariant of $M$. However, for our purposes it is enough to consider the ideal of maximal minors of $B_i$. Denoting by $Y_i$ the scheme defined by the ideal of maximal minors of $B_i$, we can view $Y_i$ as lying in the dual projective space $(\mathbb{P}^{n-1})^* = \text{Proj}(k[a_1, \ldots, a_n])$. 


We have an expected codimension for $Y_i$, and if that codimension is achieved then we also have a formula for $\deg Y_i$:

**Lemma 2.1.** Without loss of generality assume that $\dim [M]_i \leq \dim [M]_{i+1}$ (otherwise consider the transpose of $B_i$). For sufficiently general entries of $B_i$, the codimension of $Y_i$ is $\dim [M]_{i+1} - \dim [M]_i + 1$. If this codimension is achieved, then $\deg Y_i = \left(\frac{\dim M_{i+1}}{\dim M_i} - 1\right)$.

**Example 2.2 ([15]).** Harris’s motivation was to apply this machinery to liaison theory. For instance, let $C \subseteq \mathbb{P}^3$ be the union of four general lines. Let

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(t)),$$

the Hartshorne-Rao module of $C$. We have

$$\dim M(C)_t = \begin{cases} 3 & \text{if } t = 0; \\ 4 & \text{if } t = 1; \\ 2 & \text{if } t = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Taking $M = M(C)$, the expected codimension of $Y_0$ is $4 - 3 + 1 = 2$, and the expected degree is $(\binom{4}{2}) = 6$. One can show that in fact $Y_0$ is the curve in $(\mathbb{P}^3)^*$ obtained as the duals of the four components of $C$ together with the duals of the two 4-secant lines of $C$. It then follows from the fact that $Y_0$ is an isomorphism invariant, and some now-classical results of liaison theory (with a small argument), that $C$ is the only union of skew lines in its even liaison class.

Our idea now is to apply this machinery to the study of the Weak Lefschetz property. Traditionally, we say that an artinian algebra $A = R/I$ has the Weak Lefschetz property (WLP) if there is a linear form $\ell \in [A]_1$ such that, for all integers $i$, the multiplication map

$$\times \ell : [A]_i \to [A]_{i+1}$$

has maximal rank, i.e. it is injective or surjective. In this case, the linear form $\ell$ is called a Lefschetz element of $A$. (We will often abuse terminology and say that the corresponding ideal has the WLP.) The Lefschetz elements of $A$ form a Zariski open, possibly empty, subset of $[A]_1$, which as above we will projectivize and view in $(\mathbb{P}^{n-1})^*$. This open set is nothing but $(\mathbb{P}^{n-1})^* \setminus L_I$. This is our primary focus in this paper, but we note that $A$ is said to have the Strong Lefschetz property (SLP) if the analogous statements are true for the multiplication maps

$$\times \ell^d : [A]_i \to [A]_{i+d}$$

for all $i$ and $d$.

If we consider $A$ as an $R$-module, to say that $A$ satisfies the WLP is equivalent to saying that none of the varieties $Y_i$ is all of $(\mathbb{P}^{n-1})^*$. We first relabel the $Y_i$ with a more descriptive notation for our application.

**Definition 2.3.** Given an artinian graded algebra $A = R/I$, we define

$$\mathcal{L}_I := \{[\ell] \in \mathbb{P}([A]_1) \mid \ell \text{ is not a Lefschetz element}\} \subset (\mathbb{P}^{n-1})^*$$

and we call it the non-Lefschetz locus of $I$ (or of $A$). For any integer $i \geq 0$, we define

$$\mathcal{L}_{I,i} := \{\ell \in [A]_1 \mid \times \ell : [A]_i \to [A]_{i+1} \text{ does not have maximal rank}\} \subset (\mathbb{P}^{n-1})^*.$$
In order to study the non-Lefschetz locus from a scheme-theoretic perspective, we view \( \mathcal{L}_{I,i} \) not as a set but rather as the subscheme of \((\mathbb{P}^{n-1})^*\) defined by the maximal minors of a suitable matrix, as explained above, taking \( M = A \). The size of this matrix is determined by the Hilbert function of \( A \). More precisely, we introduce \( S = k[a_1, a_2, \ldots, a_n] \) as the homogeneous coordinate ring of the dual projective space \((\mathbb{P}^{n-1})^*\), where we think of the coordinates \( a_1, a_2, \ldots, a_n \) as the coefficients in \( \ell = a_1x_1 + a_2x_2 + \cdots + a_nx_n \). For each degree \( i \), the multiplication by \( \ell \) on \( S \otimes_k A \) gives the map

\[
\times \ell: S \otimes_k [A]_i \to S \otimes_k [A]_{i+1}
\]

of free \( S \)-modules which is represented by a matrix of linear forms in \( S \) given a choice of bases for \([A]_i\) and \([A]_{i+1}\). The locus \( \mathcal{L}_{I,i} \subseteq (\mathbb{P}^{n-1})^* \) is scheme-theoretically defined by the ideal of maximal minors of this matrix and we denote this ideal by \( I(\mathcal{L}_{I,i}) \). Observe that this ideal is independent of the choice of bases. In this way, we have \( \mathcal{L}_i = \bigcup_{i \geq 0} \mathcal{L}_{I,i} \), and \( \mathcal{L}_i \subseteq (\mathbb{P}^{n-1})^* \) is defined by the homogeneous ideal \( I(\mathcal{L}_i) = \bigcap_{i \geq 0} I(\mathcal{L}_{I,i}) \).

**Definition 2.4.** If \( \text{codim} \mathcal{L}_{I,i} \) takes the value prescribed by Lemma 2.1, where now \( \dim[M]_i \) is the value of the Hilbert function of \( A \) in degree \( i \), (and hence the degree of \( \mathcal{L}_{I,i} \) is also determined by the Hilbert function), then we say that \( \mathcal{L}_i \) has the expected codimension and the expected degree.

Since in this article we are studying Gorenstein algebras, especially complete intersections, it will be useful to know that the non-Lefschetz locus is determined by the failure of injectivity of the multiplication by linear forms in a single degree. It is clear on a set-theoretical level that this is true (cf. [16 Proposition 2.1]). We will now look at the question when there is an inclusion of the ideals \( I(\mathcal{L}_{I,i+1}) \subseteq I(\mathcal{L}_{I,i}) \) which will ensure that we only have to consider the middle degree even when we look at the non-Lefschetz locus defined scheme-theoretically and not only set-theoretically.

**Proposition 2.5.** If \( h_A(i) \leq h_A(i+1) \leq h_A(i+2) \) and \([\text{soc} A]_i = 0\), then \( I(\mathcal{L}_{I,i+1}) \subseteq I(\mathcal{L}_{I,i}) \).

**Proof.** The ideal \( I(\mathcal{L}_{I,i+1}) \) is generated by the maximal minors of the matrix representing the map \( \times \ell: S \otimes_k [A]_{i+1} \to S \otimes_k [A]_{i+2} \), where \( \ell = a_1x_1 + a_2x_2 + \cdots + a_nx_n \). Each such minor equals the determinant of the matrix representing the map

\[
\times \ell: S \otimes_k [B]_{i+1} \to S \otimes_k [B]_{i+2}
\]

where \( B = A/J \) and \( J \) is an ideal generated by \( h_A(i+2) - h_A(i+1) \) forms of degree \( i+2 \). Since \([A]_i = [B]_i\) and \([A]_{i+1} = [B]_{i+1}\), we can prove the inclusion \( I(\mathcal{L}_{I,i+1}) \subseteq I(\mathcal{L}_{I,i}) \) for \( A \) by proving the inclusion for all such quotients \( B = A/J \). Therefore, we will now assume that \( h_A(i+1) = h_A(i+2) \).

Suppose that \( \mathcal{L}_{I,i+1} = (\mathbb{P}^{n-1})^* \). Then \( I(\mathcal{L}_{I,i+1}) = \langle 0 \rangle \) and the inclusion of ideals is trivial. If \( \mathcal{L}_{I,i} = (\mathbb{P}^{n-1})^* \) we will also have that \( \mathcal{L}_{I,i+1} = (\mathbb{P}^{n-1})^* \) since \( A \) by assumption does not have socle in degree \( i \) and the inclusion of ideals is again trivial. Thus we only have to consider the case when \( \mathcal{L}_{I,i} \neq (\mathbb{P}^{n-1})^* \) and \( \mathcal{L}_{I,i+1} \neq (\mathbb{P}^{n-1})^* \). In this case, we can change coordinates so that \( x_n: [A]_i \to [A]_{i+1} \) and \( x_n: [A]_{i+1} \to [A]_{i+2} \) both have maximal rank. Consider the diagram
some cases we will have to use two degrees instead of one since we cannot apply duality. (cf. [16, Proposition 2.1] for the set-theoretic statement.)

The injectivity of the two vertical maps shows that we can choose monomial cobases for $[A]_i$, $[A]_{i+1}$ and $[A]_{i+2}$ in such a way that the matrix representing the map $\times \epsilon: S \otimes_k [A]_i \rightarrow S \otimes_k [A]_{i+1}$ is a submatrix of the matrix representing the map $\times \epsilon: S \otimes_k [A]_{i+1} \rightarrow S \otimes_k [A]_{i+2}$. The ideal $I(\mathcal{L}_{i+1})$ is principal, generated by the determinant of the matrix representing the map $\times \epsilon: S \otimes_k [A]_{i+1} \rightarrow S \otimes_k [A]_{i+2}$. Since the two matrices have the same number of rows, the Laplace expansion of the determinant of the larger matrix shows that this determinant is in the ideal generated by the maximal minors of the submatrix, which proves the inclusion $I(\mathcal{L}_{i+1}) \subseteq I(\mathcal{L}_{i})$.

**Corollary 2.6.** If $A = R/I$ is Gorenstein of socle degree $e$ then $\mathcal{L}_i = \mathcal{L}_{i,i}$ scheme-theoretically, where $i = \lfloor \frac{e+1}{2} \rfloor$.

**Proof.** If $A$ does not have the WLP, we have $\mathcal{L}_i = \mathcal{L}_{i,i} = (\mathbb{P}^{n-1})^*$. If $A$ has the WLP the Hilbert function is unimodal and by Proposition 2.5 and the duality of the Gorenstein algebra we get the equality.

**Remark 2.7.** If $A = R/I$ has socle in degree $i$, we need not have the inclusion $I(\mathcal{L}_{i+1}) \subseteq I(\mathcal{L}_{i,i})$ since we then have that $I(\mathcal{L}_{i,i}) = (0)$ while $I(\mathcal{L}_{i+1})$ might be non-trivial.

If $A$ is not Gorenstein, but level, we can get a similar result as Corollary 2.6 but in some cases we will have to use two degrees instead of one since we cannot apply duality. (cf. [16] Proposition 2.1) for the set-theoretic statement.)

### 3. The non-Lefschetz locus of a monomial complete intersection

In this section and the next we will restrict ourselves to the case of complete intersections. In this section we study monomial complete intersections.

Notice that to say that an artinian ideal $I \subset R$ has the WLP is equivalent to saying that codim $\mathcal{L}_I \geq 1$. The aim of this section is to study codim $\mathcal{L}_I$ when $I = \langle F_1, \cdots, F_n \rangle \subset R$ is a monomial complete intersection. We know that for a monomial complete intersection, hence for a general choice of $F_1, \cdots, F_n$, $R/I$ has the WLP, thanks to the main result of [20, 21] and [19]; and the same holds for any choice of $F_i$ if $n \leq 3$ (cf. [10]). Nevertheless, we will see in this section and the next that the non-Lefschetz locus behaves very differently for monomial complete intersections than it does for general complete intersections.

**Proposition 3.1.** Let $I = \langle x_1^{d_1}, \cdots, x_n^{d_n} \rangle \subset R := k[x_1, \cdots, x_n]$ be an artinian monomial complete intersection, with socle degree $e = d_1 + \cdots + d_n - n$. Assume without loss of generality that $d_n \geq \cdots \geq d_1 \geq 2$. Then the following characterization of the Lefschetz elements holds.

1. If $d_n > \lfloor \frac{e+1}{2} \rfloor$ then $\ell = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ is a Lefschetz element if and only if $a_n \neq 0$.

2. If $e$ is even and $d_n \leq \lfloor \frac{e+1}{2} \rfloor$ then $\ell = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ is a Lefschetz element if and only if $a_i = 0$ for at most one index $i$ and $a_j \neq 0$ for all indices $j$ with $d_j > 2$.

3. If $e$ is odd and $d_n \leq \lfloor \frac{e+1}{2} \rfloor$ then $\ell = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ is a Lefschetz element if and only if $a_1 a_2 \cdots a_n \neq 0$. 


Proof. We start by fixing the linear form \( \ell = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \). Let \( A = R/I \), let \( S = \{ i : a_i \neq 0 \} \subseteq \{1, 2, \ldots, n\} \) and define the subrings \( A' \) and \( A'' \) of \( A \) as the subrings generated by \( \{x_i\}_{i \in S} \) and by \( \{x_i\}_{i \notin S} \), respectively. Both \( A' \) and \( A'' \) are monomial complete intersections and \( \ell \) acts trivially on \( A'' \) while it is a Lefschetz element on \( A' \).

In order to determine whether or not \( \ell \) is a Lefschetz element on \( A \), it is sufficient to consider the injectivity of the multiplication map in the middle degree, i.e.,

\[ \times \ell : [A]_{[\frac{e+1}{2}]} \rightarrow [A]_{[\frac{e+1}{2}]} . \]

For any integer \( j \), we have that

\[ [A]_j = ([A']_j \otimes [A'']_0) \oplus ([A']_j \otimes [A'']_1) \oplus \cdots \oplus ([A']_j \otimes [A'']_i) \]

and since \( \ell \) acts trivially on \( A'' \), the injectivity of \( (3.1) \) is equivalent to injectivity in each component

\[ \times \ell : [A']_{[\frac{e+1}{2}]} \otimes [A'']_i \rightarrow [A']_{[\frac{e+1}{2}]} \otimes [A'']_i, \quad \text{for all } i \geq 0. \]

Now, injectivity in the top degree \( \times \ell : [A']_{[\frac{e+1}{2}]} \rightarrow [A']_{[\frac{e+1}{2}]} \) implies injectivity in the lower degrees of \( A' \). Since \( \ell \) is a Lefschetz element on \( A' \), we have injectivity of the latter map if and only if

\[ \dim_k [A']_{[\frac{e+1}{2}]} \leq \dim_k [A']_{[\frac{e+1}{2}]} . \]

Since \( [\frac{e+1}{2}] \) is above the middle degree if \( S \neq \{1, 2, \ldots, n\} \), we must have a flat top in the Hilbert function of \( A' \) between degree \( e' - [\frac{e+1}{2}] \) and degree \( [\frac{e+1}{2}] \) in this situation, where \( e' \) is the socle degree of \( A' \). If this forced flat top has length two, we must have \( [\frac{e+1}{2}] - 1 = e' - [\frac{e+1}{2}] \) which is only possible if \( e \) is even and \( e' = e - 1 \). In this case \( A'' \) is generated by one variable \( x_i \) with \( d_i = 2 \) and \( \ell \) is a Lefschetz element on \( A \).

If there is a flat of length at least three, it follows from \([19] \) Theorem 1\) that one of the generators of the defining ideal of \( A' \) must have a degree which is above the end of the flat. There can be at most one \( d_i \) which is greater than \( [\frac{e+1}{2}] \), so in this case we must have \( d_n > [\frac{e+1}{2}] \). In this case, \( \ell \) is a Lefschetz element of \( A \).

We now relate what we have shown with the statements of our proposition.

In the case \( \{1\} \), we get that \( \ell \) is a Lefschetz element if and only if \( d_n \) is the degree of one of the generators of the defining ideal of \( A' \), which is equivalent to \( a_n \neq 0 \).

If \( d_n \leq [\frac{e+1}{2}] \), the only case when \( \ell \) is a Lefschetz element and \( A' \neq A \) when \( e \) is even and \( A'' = k[x_i]/(x_i^2) \). This shows \( (2) \) and \( (3) \). \( \square \)

Remark 3.2. Case \( \{2\} \) of Proposition 3.1 shows that the non-Lefschetz locus does not need to be unixed. The smallest example is for \( d_1 = d_2 = 2 \) and \( d_3 = 3 \) where we get

\[ I(\mathcal{L}_I) = \langle a_1 a_2 a_3 a_4, a_1 a_3 a_2 a_5, a_1 a_2 a_3 a_5, a_1 a_2 a_5 a_4, a_1 a_2 a_4 a_1, a_1 a_2 a_4 a_1, a_1 a_2 a_4 a_1, a_1 a_2 a_4 a_1, a_1 a_2 a_4 a_1, a_1 a_2 a_4 a_1 \rangle \]

with radical \( \sqrt{I(\mathcal{L}_I)} = \langle a_1 a_2 a_3 a_4, a_2 a_3 a_4 a_2, a_1 a_2 a_3 a_4, a_2 a_3 a_4 a_2, a_1 a_2 a_3 a_4, a_2 a_3 a_4 a_2 \rangle \) with \( \langle a_1, a_2, a_3 \rangle \).

Example 3.3. Proposition 3.1 only gives us that \( \mathcal{L}_I \) is defined set-theoretically by the equation \( a_1 \cdots a_n = 0 \) in the cases given by \( (3) \). Scheme-theoretically, \( \mathcal{L}_I \) is defined by an ideal generated by maximal minors of certain matrices as seen in Section 2. For instance, if \( n = 3 \) and \( d_1 = d_2 = d_3 = 4 \), the Hilbert function is \( (1, 3, 6, 10, 12, 12, 10, 6, 3, 1) \) and the defining polynomial of \( \mathcal{L}_I \) is \( a_1^4 a_2^2 a_3^4 \). More generally, if \( n = 3 \) and \( d_1 = d_2 = d_3 = d \) where \( d \) is even, then the Hilbert function of \( R/I \) is \( (1, h_1, \ldots, h_e) \) with \( e = 3d - 3 \) and \( h_{\frac{d+1}{2}} = h_{\frac{d+3}{2}} = 3 \left( \frac{d}{2} \right)^2 \) and the defining polynomial of \( \mathcal{L}_I \) is \( (a_1 a_2 a_3)^e \left( \frac{d}{2} \right)^2 \).
This example leads to the following two immediate corollaries.

**Corollary 3.4.** If \( \langle F_1, \ldots, F_n \rangle \) is any complete intersection in \( k[x_1, \ldots, x_n] \) with \( \deg F_1 = \cdots = \deg F_n = d \), if \( n(d-1) \) is even, then the value of the Hilbert function in degrees \( \frac{n(d-1)-1}{2} \) and \( \frac{n(d-1)+1}{2} \) is divisible by \( n \).

**Corollary 3.5.** Let \( I = \langle x_1^{d_1}, \ldots, x_n^{d_n} \rangle \). If \( d_1 = \cdots = d_n = 2 \) and \( n \) is even then the non-Lefschetz locus \( L_I \) has codimension 2. In all other cases, it has codimension 1. Furthermore, if \( d_1 = \cdots = d_n = d \) where \( n \) is odd and \( d \) is even, then \( I(L_I) = (a_1^\alpha \cdots a_n^\alpha) \) where \( \alpha = \frac{1}{n} h_n(2d-1) \) and \( h_n(2d-1) = h_n(2d+1) \). When \( d = 2 \), this is equal to \( \binom{n}{2} \).

**Proof.** The ideas are contained in the proof of Proposition 4.1. In particular, under the hypothesis \( d_1 = \cdots = d_n = d \) where \( n \) is odd and \( d \) is even, the expected codimension of the non-Lefschetz locus is achieved, namely codimension 1. In this case the degree of the non-Lefschetz locus is equal to \( h_n(2d-1) \), and the generating polynomial has to be symmetric with respect to all \( n \) variables. The fact that \( \alpha \) is an integer is guaranteed by Corollary 3.4.

**Remark 3.6.** One can also study the non-Lefschetz locus with respect to the Strong Lefschetz Property. Junzo Watanabe has communicated to us that he has extended Corollary 3.5 for the question of the Strong Lefschetz Property, showing that \( a_1 x_1 + \cdots + a_n x_n \) is a Strong Lefschetz element for \( R/\langle x_1^2, \ldots, x_n^2 \rangle \) if and only if \( a_1 a_2 \cdots a_n \neq 0 \). Thus the non-Lefschetz locus for \( R/\langle x_1^2, \ldots, x_n^2 \rangle \) for the Strong Lefschetz Property has codimension 1, not 2 as it was for the non-Lefschetz locus for the Weak Lefschetz Property.

Notice that if a linear form \( \ell \) is a non-Weak-Lefschetz element for \( R/I \) then of course it is a non-Strong-Lefschetz element, so Watanabe’s case is the only one left open by Corollary 3.5.

3.1. **Jordan types.** Multiplication by a linear form \( \ell \) corresponds to a nilpotent linear operator on the artinian algebra \( A \). The Jordan type of this nilpotent operator is an integer partition \( P_L \) of \( \dim_k A \).

The study of Jordan types refines the study of Lefschetz properties as we have the following:

- \( \ell \) is a weak Lefschetz element if and only if the number of parts of \( P_L \) equals the maximal value of the Hilbert function of \( A \).
- \( \ell \) is a strong Lefschetz element if and only if \( P_L \) equals the dual partition the partition given by the Hilbert function of \( A \).

Here we investigate the possible Jordan types of linear forms for the case when \( A \) is a monomial complete intersection.

For a degree sequence \( d_1, d_2, \ldots, d_n \) let \( P_{d_1,d_2,\ldots,d_n} \) denote the dual partition to the partition given by the Hilbert function of an artinian complete intersection of type \( (d_1, d_2, \ldots, d_n) \). For a partition \( P \) we denote by \( P^k \) the partition given by repeating all parts of \( P \) \( k \) times.

**Proposition 3.7.** Let \( A = k[x_1, x_2, \ldots, x_n]/\langle x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n} \rangle \) be a monomial complete intersection in characteristic zero. The possible Jordan types for linear forms \( \ell \) are \( P_{d_1,d_2,\ldots,d_n}^m \), where \( m = \prod_{j=1}^k d_j/\prod_{j=1}^k d_{i_j} \), for all non-empty subsequences \( d_{i_1}, d_{i_2}, \ldots, d_{i_k} \) of \( d_1, d_2, \ldots, d_n \).

**Proof.** From the action of the torus \( (k^*)^n \) we see that the Jordan type of a linear form \( \ell = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \) depends only on which coefficients are non-zero. Let \( \{i_1, i_2, \ldots, i_k\} \)
be the indices for which the coefficients are non-zero and let \( \{j_1, j_2, \ldots, j_{n-k}\} \) be the remaining indices.

Let \( A' \) be the artinian monomial complete intersection of type \((d_1, d_2, \ldots, d_k)\) and let \( A'' \) be the artinian monomial complete intersection of type \((d_j, d_{j+1}, \ldots, d_{n-k})\). We now have \( A \cong A' \otimes A'' \) and \( \ell = \sum_{j=1}^{k} a_i x_{ij} \) is a strong Lefschetz element acting on the first factor while it acts trivially on the second factor. Thus the Jordan type of \( \ell \) is \( P_{d_{i_1}, d_{i_2}, \ldots, d_{i_k}}^m \), where \( m = \dim_k A'' = \prod_{j=1}^{n} d_j / \prod_{j=1}^{k} d_{i_j} \).

Example 3.8. The situation is easiest to summarize when all degrees are equal. Consider for example the case \( n = 4 \) and \( d_1 = d_2 = d_3 = d_4 = 2 \). There are combinatorially just four possible subseqences and the four possible Jordan types are
\[
\begin{align*}
[53^21^2], & \quad [42^21^2], & \quad [3^41^4] & \quad \text{and} & \quad [2]^8,
\end{align*}
\]
corresponding to the linear forms \( x_1 + x_2 + x_3 + x_4, x_1 + x_2 + x_3, x_1 + x_2 \) and \( x_1 \), respectively.

4. THE NON-LEFSCHETZ LOCUS OF A GENERAL COMPLETE INTERSECTION

In the previous section we considered the non-Lefschetz locus of a monomial complete intersection, and saw that it has codimension 1. We also know that a general complete intersection is a general complete intersection, and has the Jordan type \( [\cdots] \) \( [\cdots] \) \( [\cdots] \) \( [\cdots] \) of degree \( 1 \). We will now prove that for \( n = 3 \) and \( n = 4 \), the non-Lefschetz locus has the expected codimension.

Notation 4.1. We begin in the setting of \( R = k[x_1, \ldots, x_n] \), and then turn to the case \( n = 3, 4 \). Throughout this section we will fix integers \( 2 \leq d_1 \leq \cdots \leq d_n \), and \( I \) will be a complete intersection ideal, \( I = \langle F_1, \ldots, F_n \rangle \), where \( \deg F_i = d_i \) and \( F_i \) is a general form of degree \( d_i \). We will denote by \( e \) the socle degree of \( R/I \), namely \( e = (\sum_{i=1}^{n} d_i) - n \). We will denote by \( (1, h_1, \ldots, h_{e-1}, h_e) \) the \( h \)-vector (i.e. Hilbert function) of \( R/I \).

We will describe the expected codimension of the non-Lefschetz locus in Conjecture 4.3. One of our goals is to prove that for \( n = 3 \) or \( 4 \), and for a general choice of \( F_i, 1 \leq i \leq n \), the non-Lefschetz locus \( \mathcal{L}_I \) of \( I = \langle F_1, \ldots, F_n \rangle \) has the expected codimension.

Remark 4.2. When the \( F_i \) are general, we know that \( R/I \) has the WLP, so \( \mathcal{L}_I \neq (\mathbb{P}^{n-1})^* \). In the case where the socle degree \( e \) is odd, the Hilbert function of \( R/I \) has at least two values in the middle that are equal. Thanks to Corollary 2.6, this means that \( \mathcal{L}_I \) is defined by the vanishing of the determinant of a square matrix of size \( h_{e-1} \times h_{e+1} \), hence (since \( \mathcal{L}_I \neq (\mathbb{P}^{n-1})^* \)) \( \mathcal{L}_I \) is a hypersurface of degree \( \delta_I = h_{e-1}^{-1} \). So the case of odd socle degree is completely understood, and from now on we will assume without loss of generality that \( e \) is even.

Based on computer experiments \[8\] and our results in four or fewer variables, we make the following conjecture.

Conjecture 4.3. Let \( I = \langle F_1, \ldots, F_n \rangle \subset R \) be a complete intersection ideal of general forms as in Notation 4.1 and assume that \( e \) is even (see Remark 4.2). Then
\[
\text{codim } \mathcal{L}_I = \min\{h_{\frac{e}{2}} - h_{\frac{e}{2} - 1} + 1, n\}
\]
where we consider the empty set to have codimension \( n \) in \( \mathbb{P}^{n-1} \). In particular, \( \mathcal{L}_I \subset (\mathbb{P}^{n-1})^* \) is non-empty if and only if \( h_{\frac{e}{2}} - h_{\frac{e}{2} - 1} \leq n - 2 \) and in that case \( \delta_I := \deg(\mathcal{L}_I) = (h_{\frac{e}{2}} - h_{\frac{e}{2} - 1} + 1) \).
Remark 4.4. Notice that in Conjecture \ref{e:1} the hypothesis that the complete intersection artinian ideal \( I \subset R \) is generated by general forms cannot be dropped. In fact, a complete intersection \( I \subset k[x_1, x_2, x_3] \) of type \((3, 3, 3)\) has \( h \)-vector \((1, 3, 6, 7, 6, 3, 1)\), so the expected codimension of the non-Lefschetz locus \( \mathcal{L}_I \) is 2; and we will see later that indeed it is true for a general choice of 3 cubics \( F_1, F_2, F_3 \in k[x_1, x_2, x_3] \) (cf. Theorem \ref{e:4}). But unfortunately it is not true for every choice. For instance, we saw in the last section that if we take \( I = \langle F_1, F_2, F_3 \rangle = \langle x_1^3, x_2^3, x_3^3 \rangle \) we get that \( \text{codim} \mathcal{L}_I = 1 \) since a line \( a_1x_1 + a_2x_2 + a_3x_3 \in k[x_1, x_2, x_3] \) fails to be a Lefschetz element of \( k[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3) \) if and only if \( a_1a_2a_3 = 0 \). Therefore, if we fix coordinates \( a_1, a_2 \) and \( a_3 \) in \((\mathbb{P}^2)^*\), the support of \( \mathcal{L}_I \) is the union of the lines \( \ell_1 : a_1 = 0, \ell_2 : a_2 = 0 \) and \( \ell_3 : a_3 = 0 \).

Remark 4.5. We will see shortly that to measure the non-Lefschetz locus in \((\mathbb{P}^{n-1})^*\), it will be enough to measure how many such algebras fail the WLP in a suitable irreducible parameter space. As noted in Section \ref{e:2} if \( R/I \) is a complete intersection and the WLP fails, it must fail ”in the middle”, and possibly also in other degrees. By semicontinuity and under the hypothesis that \( e \) is even, to measure the dimension of the set of algebras failing the WLP in an irreducible parameter space we can assume that WLP fails from degree \( h_{2^3-1} \) to \( h_{2^3} \) (and, by duality, from \( h_{2^3} \) to \( h_{2^3+1} \)), and that the failure is just by one.

Remark 4.6. We have \( d_1 \leq \cdots \leq d_n \). For large values of \( d_n \) the question of the non-Lefschetz locus for a complete intersection generator with generator degrees \( d_1, \ldots, d_n \) is clear.

\begin{enumerate}
\item If \( d_n \geq d_1 + \cdots + d_{n-1} - (n-1) + 2 = d_1 + \cdots + d_{n-1} - n + 3 \) then \( h_{2^3-1} = h_{2^3} \) (remembering that we are assuming \( e \) even), and the conjecture is clear (with the non-Lefschetz locus consisting of the linear forms through individual points).

\item If \( d_n = d_1 + \cdots + d_{n-1} - (n-1) + 1 = d_1 + \cdots + d_{n-1} - n + 2 \) then \( R/(F_1, \ldots, F_{n-1}) \) is the coordinate ring of the reduced complete intersection set of points, \( Z \), in \( \mathbb{P}^{n-1} \) defined by \((F_1, \ldots, F_{n-1})\), which reaches the multiplicity in degree \( d_1 + \cdots + d_{n-1} - (n-1) \). If \( \{h_i\} \) is the Hilbert function of \( R/(F_1, \ldots, F_n) \), then clearly

\begin{itemize}
\item \( d_1 + \cdots + d_{n-1} - (n-1) = \frac{e}{2} \);
\item \( h_{2^3} - h_{2^3-1} = 1 \);
\item \( h_{2^3} = d_1d_2 \cdots d_{n-1} \).
\end{itemize}

The Hilbert function of \( R/I \) agrees with that of \( R/I_Z \) in degrees \( \leq \frac{e}{2} \).
\end{enumerate}

Notice that \( Z \) has the Uniform Position Property, since the \( F_i \) are general. We claim that

\( a \) linear form \( \ell \) fails to have maximal rank from degree \( \frac{e}{2} - 1 \) to degree \( \frac{e}{2} \) if and only if \( \ell \) vanishes on (any) two points of \( Z \).

Indeed, if \( P_1, P_2 \in Z \), notice first that the Hilbert function of \( Z\setminus\{P_1\} \) agrees with that of \( Z \) up to and including degree \( \frac{e}{2} - 1 \), and is one less than that of \( Z \) from then on. The Hilbert function of \( Z\setminus\{P_1, P_2\} \) agrees with that of \( Z \) up to and including degree \( \frac{e}{2} - 2 \), is one less than that of \( Z \) in degree \( \frac{e}{2} - 1 \), and is two less than that of \( Z \) from degree \( \frac{e}{2} \) on. In particular, there is a form of degree \( \frac{e}{2} - 1 \) vanishing on all of \( Z \) except \( P_1 \cup P_2 \), but the same is not true for all of \( Z \) except only \( P_1 \).

Since \( R/I_Z \) has depth 1, a linear form \( \ell \) not vanishing on any point of \( Z \) is a non-zerodivisor, so the resulting multiplication from degree \( \frac{e}{2} - 1 \) to degree \( \frac{e}{2} \) is injective. If \( \ell \) vanishes at just one point, \( P_1 \), of \( Z \), then for a form \( F \) of degree
Let $\ell = 1$, $\ell \cdot F = 0$ in $R/I$ means that $F$ vanishes at all points of $Z$ except $P_1$. But we know that any form of degree $\frac{n}{2} - 1$ vanishing at all but one point must in fact vanish on all of $Z$, so $F = 0$ in $R/I$. On the other hand, any linear form vanishing on the line spanned by $P_1$ and $P_2$ lies in the non-Lefschetz locus, which then has codimension $2$ and degree $(\frac{h}{2})$ as claimed in Conjecture 4.3.

(3) If $d_n = d_1 + \cdots + d_{n-1} - n + 1$ then $R/I$ has odd socle degree, so the non-Lefschetz locus has codimension $1$ and degree $d_1 \cdots d_{n-1} - 1$.

(4) Finally, assume that $d_n = d_1 + \cdots + d_{n-1} - n$. In this case $\langle F_1, \ldots, F_{n-1} \rangle$ defines a complete intersection set of $d_1 \cdots d_{n-1}$ points, $Z$, and its Hilbert function reaches its multiplicity in degree $d_1 + \cdots + d_{n-1} - n + 1 = \deg F_n + 1$. More precisely, letting $s = d_1 + \cdots + d_{n-1}$ and $d = d_1 \cdots d_{n-1}$, its Hilbert function is

\[
\begin{array}{cccccccc}
\text{degree} & 0 & 1 & 2 & \ldots & (s-n-1) & (s-n) & (s-n+1) & (s-n+2) & \ldots \\
1 & n & h_2 & \ldots & d-n & d-1 & d & d & \ldots
\end{array}
\]

and the Hilbert function of $R/I$ is

\[
\begin{array}{cccccccc}
\text{degree} & 0 & 1 & 2 & \ldots & (s-n-1) & (s-n) & (s-n+1) & \ldots & e-1 & e \\
1 & n & h_2 & \ldots & d-n & d-2 & d-n & \ldots & n & 1
\end{array}
\]

For a linear form $\ell \in R$, the failure of $\times \ell : [R/I]_{s-n-1} \to [R/I]_{s-n}$ to be injective is equivalent to the condition that the restriction $F_n$ of $F_n$ to $R/(\ell)$ is in the restricted ideal $\langle F_1, \ldots, F_{n-1} \rangle$. Since $I$ is artinian, it follows then that $\langle F_1, \ldots, F_{n-1} \rangle$ is a complete intersection. In particular, $\ell$ is a non-zero divisor on $R/(F_1, \ldots, F_{n-1})$. We also note that in this situation, the conjectured codimension of $\mathcal{L}_\ell$ is $(d-2) - (d-n) + 1 = n - 1$ in $(\mathbb{P}^{n-1})^*$, i.e. there should only be a finite number of linear forms failing to induce an injective homomorphism from degree $s-n-1$ to degree $s-n$.

Thus from now on we may assume that $d_n \leq d_1 + \cdots + d_{n-1} - n$, and if equality holds we have an equivalent condition for failure to have maximal rank.

Our goal in this section is to prove Conjecture 4.3 in the cases $n = 3$ and $n = 4$. We begin with a description of the approach that we will take except for Theorem 4.10. Fix degrees $d_1, \ldots, d_n$ for the complete intersections in $R = k[x_1, \ldots, x_n]$, with $d_1 \leq d_2 \leq \cdots \leq d_n$. Let $CI(d_1, \ldots, d_n)$ be the irreducible space parametrizing all such complete intersections. Let $(\mathbb{P}^{n-1})^*$ be the projective space parametrizing the linear forms of $R$ (up to scalar multiple). For each complete intersection $I$ and linear form $\ell$, we consider the pair $(\ell, I) \in (\mathbb{P}^{n-1})^* \times CI(d_1, \ldots, d_n)$. Let $X$ be the set of such pairs such that $\ell$ is not a Lefschetz element for $A = R/I$.

Since the $d_i$ are given, there is a precise degree where this latter condition must be checked: $(\ell, I) \in X$ if and only if $\times \ell : [R/I]_{\frac{s}{2}-1} \to [R/I]_{\frac{s}{2}}$ fails to be injective (recall that the socle degree $e$ is assumed to be even, thanks to Remark 4.2). Since the general element of $CI(d_1, \ldots, d_n)$ has the WLP, there are expected values for the Hilbert function of $R/(I, \ell)$ in degrees $\frac{s}{2}$ and $\frac{s}{2} + 1$ (the latter being $0$), and $(\ell, I) \in X$ if and only if these values are not achieved.

Consider the projections $\phi_1$ and $\phi_2$:

\[
(\ell, I) \in (\mathbb{P}^{n-1})^* \times CI(d_1, \ldots, d_n) \supset X
\]

\[
\phi_1 \quad \phi_2
\]

\[
(\mathbb{P}^{n-1})^* \quad CI(d_1, \ldots, d_n)
\]
We need to show that there is a non-empty open set $U \subset CI(d_1, \ldots, d_n)$ such that if $I \in U$ then the closure of $\phi_1(\phi_2^{-1}(I) \cap X)$ has the expected codimension as described in Conjecture 4.3. Thus we want to show that the intersection of $X$ with the generic fibre of $\phi_2$ has the expected dimension (computed from Conjecture 4.3). More precisely, let $m = (n-1) - \min\{h_{\frac{d}{2}} - h_{\frac{d}{2}-1} + 1, n\}$, the expected dimension of $L_I$, and let $I$ be a general element of $CI(d_1, \ldots, d_n)$. Then Conjecture 4.3 says that

$$\dim(\phi_2^{-1}(I) \cap X) = m.$$  

We will reformulate this. Let $p = \dim CI(d_1, \ldots, d_n)$. We want to show that there is an open subset $U \subset CI(d_1, \ldots, d_n)$ such that

$$\dim(\phi_2^{-1}(U) \cap X) = m + p.$$  

Now, $\phi_1$ is surjective, and the fibres all have the same dimension (since we can always do a change of variables). Thus we want to show that for any linear form $\ell$ (viewed as an element of $(\mathbb{P}^{n-1})^*$),

$$\dim(\phi_2^{-1}(U) \cap X \cap \phi_1^{-1}(\ell)) = m + p - (n-1).$$

So from now on we fix a linear form $\ell$. We denote by $ACI_\ell(d_1, \ldots, d_n)$ the irreducible space of ideals in $S = R/(\ell)$ with generators in degrees $d_1, \ldots, d_n$. We note that an ideal in $ACI_\ell(d_1, \ldots, d_n)$ may have only $n-1$ minimal generators. This would happen for instance if $d_n > d_1 + \cdots + d_{n-1} - (n-1)$ and $\ell$ is a non-zerodivisor on $R/(F_1, \ldots, F_{n-1})$, but we have assumed this not to be the case in Remark 4.6. But even avoiding this situation, it may happen that an ideal in $CI(d_1, \ldots, d_n)$ restricts to an ideal in $ACI_\ell(d_1, \ldots, d_n)$ with only $n-1$ minimal generators. Let $V \subset ACI_\ell(d_1, \ldots, d_n)$ be the open subset consisting of restricted ideals $(\bar{F}_1, \ldots, \bar{F}_n)$ such that all the $\bar{F}_i$ are minimal generators.

Consider the morphism

$$CI(d_1, \ldots, d_n) \xrightarrow{\phi} ACI_\ell(d_1, \ldots, d_n).$$

We want to study a certain subvariety, $Y \subset ACI_\ell(d_1, \ldots, d_n)$. The precise definition of $Y$ will depend on the value of $d_n$, breaking into two cases, but the treatment of $Y$ will be the same in both cases.

**Case 1:** $d_n = d_1 + \cdots + d_{n-1} - n$. We have seen in Remark 4.6 (4) that in this case $m = 0$, and that failure of maximal rank is equivalent to $\bar{F}_n \in (\bar{F}_1, \ldots, \bar{F}_{n-1})$, which then is a complete intersection. By Remark 4.5 or by direct observation in this case, we can assume that the Hilbert function of the restricted ideal differs by one, in degrees $d_1 + \cdots + d_{n-1} - n$ and $d_1 + \cdots + d_{n-1} - (n-1)$, from the expected one. Let $Y \subset ACI_\ell(d_1, \ldots, d_n)$ be the subset in the complement of $V$ consisting of those ideals such that the first $n-1$ generators form a regular sequence, and the last generator is not minimal.

**Case 2:** $d_n < d_1 + \cdots + d_{n-1} - n$. In this case we let $Y \subset V$ be the set of ideals $I$ such that $h_{S/I}(\frac{d}{2} + 1) > 0$. (The distinction between the cases is that the ideals of $Y$ are complete intersections in Case 1, and are not complete intersections in Case 2.)

Notice that in both cases,

$$\dim \phi^{-1}(Y) = \dim(\phi_2^{-1}(U) \cap X \cap \phi_1^{-1}(\ell)).$$
Notice also that the fibres of \( \phi \) over \( V \cup Y \) all have the same dimension, namely
\[
p - \dim ACI_\ell(d_1, \ldots, d_n).
\]
So we want to show that
\[
\dim Y + p - \dim ACI_\ell(d_1, \ldots, d_n) = m + p - (n - 1),
\]
i.e. that
\[
\dim Y = m - (n - 1) + \dim ACI_\ell(d_1, \ldots, d_n).
\]
Equivalently, we want to show that
\[
(4.4) \quad \text{The codimension of } Y \text{ in } ACI_\ell(d_1, \ldots, d_n) \text{ is } \min\{h_{\frac{m}{2}} - h_{\frac{m}{2} - 1} + 1, n\}.
\]
This is what we will prove in the results below.

We have noted above that without loss of generality we can assume that \( d_n \leq d_1 + \cdots + d_{n-1} - n \), and that the case of equality is handled slightly differently from the case of strict inequality. We now consider equality.

**Proposition 4.7.** Let \( I = \langle F_1, \ldots, F_n \rangle \subset R = k[x_1, \ldots, x_n] \) be a complete intersection generated by general forms of degrees \( 2 \leq d_1 \leq d_2 \leq \cdots \leq d_n \). Assume that \( d_n = d_1 + \cdots + d_{n-1} - n \). Then Conjecture 4.3 is true.

**Proof.** We have defined the quasi-projective variety \( Y \) in Case 1 above. From what we said in Remark 4.6 (4) and in Case 1 of the discussion above, we want to show that the codimension of \( Y \) in \( ACI_\ell(d_1, \ldots, d_n) \) is \( n - 1 \). We recall that a complete intersection of type \( (d_1, \ldots, d_{n-1}) \) in \( R/\langle \ell \rangle \) with \( d_1 \geq 2 \) has Hilbert function with value \( n - 1 \) in degree \( d_n = d_1 + \cdots + d_{n-1} - n \).

Now, let \( M_\ell(d_1, \ldots, d_{n-1}) \) be the variety parametrizing the ideals with generator degrees \( d_1, \ldots, d_{n-1} \), and let \( U' \subset M_\ell \) be the dense open subset consisting of complete intersections of type \( (d_1, \ldots, d_{n-1}) \). Consider
\[
\begin{array}{c}
Y \subseteq ACI_\ell(d_1, \ldots, d_n) \\
\downarrow \phi \\
U' \subseteq M_\ell(d_1, \ldots, d_{n-1}).
\end{array}
\]
We have that \( Y \) is contained in \( \phi^{-1}(U') \), and \( \phi^{-1}(U') \) is a dense open subset of \( ACI_\ell(d_1, \ldots, d_n) \). For any \( J \in U' \), the codimension of \( \phi^{-1}(J) \cap Y \) in \( \phi^{-1}(J) \) is \( n - 1 \), thanks to the Hilbert function observation above. The desired conclusion \((4.4)\) follows from this. \(\square\)

Thus from now on we can assume that \( d_n < d_1 + \cdots + d_{n-1} - n \), and that we are in Case 2 above. To fix the ideas for most of the rest of the paper in a simple first case, we first state the case \( n = 3 \) and \( \deg(F_i) = d \) for \( 1 \leq i \leq 3 \) and prove the analogous case \( n = 4 \) and \( \deg(F_i) = d \) for \( 1 \leq i \leq 4 \).

**Proposition 4.8.** Let \( I = \langle F_1, F_2, F_3 \rangle \subset R = k[x_1, x_2, x_3] \) be a complete intersection generated by general forms of degree \( (d, d, d), d \geq 2 \). Then we have
\begin{enumerate}
  
  \item If \( e \) is odd then \( \text{codim } L_\ell = 1 \) and \( L_\ell \subset (\mathbb{P}^2)^* \) is a curve of degree \( 3(\frac{d}{2})^2 \).
  
  \item If \( e \) is even then \( \text{codim } L_\ell = h_{\frac{d}{2}} - h_{\frac{d}{2} - 1} + 1 = 2 \) and \( \deg L_\ell = (\frac{h_{\frac{d}{2}}}{2}) = (\frac{d^2 + 1}{2}) \).
\end{enumerate}

**Proposition 4.9.** Let \( I = \langle F_1, F_2, F_3, F_4 \rangle \subset R = k[x_1, x_2, x_3, x_4] \) be a complete intersection generated by general forms of degree \( (d, d, d, d), d \geq 2 \).
(1) If $d = 2$ then $\text{codim } \mathcal{L}_I = 3$, and in particular $\mathcal{L}_I \subset (\mathbb{P}^3)^*$ is a set of 20 different points.
(2) If $d \geq 3$ then $\mathcal{L}_I = \emptyset$.

Proof. (1) The $h$-vector of $R/I$ is $(1, 4, 6, 4, 1)$ and $\mathcal{L}_I$ is a scheme defined by the maximal minors of a $4 \times 6$ matrix with linear entries. We will prove that $\mathcal{L}_I \subset (\mathbb{P}^3)^*$ has codimension 3 and consists of 20 different points which shows that $\mathcal{L}_I$ is a standard determinantal scheme. If $\ell$ fails to give an injection from degree 1 to degree 2, then there is a linear form $M$ such that $\ell M \in I$. So we first want to know how many reducible quadrics lie in the projectivization of the 4-dimensional vector space generated by $F_1, F_2, F_3, F_4$ inside $\mathbb{P}[R]_2 = \mathbb{P}^9$. The dimension of the space of such reducible quadrics is 6, and its degree is 10 (\cite{1}, top of page 300). Thus its intersection with a general 3-dimensional linear space $\ell$ is a set of 10 points in $\mathbb{P}^9$. Such a linear space avoids the locus of double planes, and each of the 10 points is of the form $\ell_1 \ell_2$ where either $\ell_1$ or $\ell_2$ could play the role of $\ell$ for us. Thus there are 20 such linear forms, or 20 points in $(\mathbb{P}^3)^*$.

(2) Let $(1, h_1, h_2, \cdots, h_{e-1}, h_e)$ be the $h$-vector of $R/I$. Therefore, $e = 4d - 4$.

Claim: $h_{2} - h_{2-1} = d$.

We will prove a more general result in Lemma 4.41, but here we give a completely different proof to illustrate a different approach.

Proof of the Claim: We consider the rank 3 vector bundle $\mathcal{E}$ on $\mathbb{P}^3$

$$\mathcal{E} := \ker(\mathcal{O}_{\mathbb{P}^3}(-d))^4 (F_1, F_2, F_3) \rightarrow \mathcal{O}_{\mathbb{P}^3}.$$

Using the exact sequences

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^3}(-d)^4 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0,$$

we get

$$H^0(\mathbb{P}^3, \mathcal{E}(t)) = 0 \quad \text{for all} \quad t < 2d$$
$$H^2(\mathbb{P}^3, \mathcal{E}(t)) = 0 \quad \text{for all} \quad t \in \mathbb{Z}$$
$$H^3(\mathbb{P}^3, \mathcal{E}(t)) = 0 \quad \text{for all} \quad t \geq d - 3.$$

Therefore, we have

$$h_2 - h_{2-1} = h^1(\mathbb{P}^3, \mathcal{E}(2d - 2)) - h^1(\mathbb{P}^3, \mathcal{E}(2d - 3))$$
$$= -\chi(\mathcal{E}(2d - 2)) + \chi(\mathcal{E}(2d - 3)) = d$$

where the last equality follows applying the Riemann-Roch Theorem, and the Claim is proved.

Since $h_2 - h_{2-1} = d$, $\mathcal{L}_I$ is expected to be empty and this is what we will prove. To this end, we set $S = k[x_1, x_2, x_3, x_4]/(\ell) \cong k[x_1, x_2, x_3]$ where $\ell = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 \in [R]_1$ is a linear form. Call $\mathcal{A}_{d,d,d,d}$ the set of almost complete intersection ideals $J \subset S$ of type $(d, d, d, d)$. It holds that

$$\dim \mathcal{A}_{d,d,d,d} = \dim \text{Gr} \left( 4, \binom{d + 2}{2} \right) = 4 \binom{d + 2}{2} - 16 = 2d^2 + 6d - 12.$$
ideal \( J \) in \( B_{d,d,d,d} \) can be linked by means of a complete intersection \( J' \) of type \((d, d, d)\) to a Gorenstein ideal \( J_1 \) with socle degree \(2d - 3\) and h-vector
\[
(1, 3, 6, \cdots, \binom{d-1}{2}, \binom{d}{2} - 1, \binom{d}{2} - 1, \binom{d-1}{2}, \cdots, 6, 3, 1).
\]
Observe that

(i) the dimension of the Gorenstein ideals \( J_1 \) with h-vector
\[
(1, 3, 6, \cdots, \binom{d-1}{2}, \binom{d}{2} - 1, \binom{d}{2} - 1, \binom{d-1}{2}, \cdots, 6, 3, 1)
\]
is \(2d - 2 = 2d^2 - 4d - 1\) (see Example 5.2),

(ii) the dimension of complete intersections \( J' \) of type \((d, d, d)\) contained in \( J_1 \) is
\[
\dim Gr(3, 3d) = 3(3d - 3) \quad \text{(note that } \dim[J_1]_d = \binom{d+2}{2} - \binom{d-1}{2} = 3d)\), and

(iii) the dimension of complete intersections of type \((d, d, d)\) contained in \( J \) is \( \dim Gr(3, 4) = 3 \).

To compute \( \dim B_{d,d,d,d} \) we use liaison. The computation is
\[
\dim B_{d,d,d,d} = \left( \binom{2d-1}{2} - d - 2 \right) + (9d - 9) - 3 = 2d^2 + 5d - 13.
\]
We have only to justify subtracting the value from (iii) in this computation. Indeed, this is to remove over-counting, since the same ideal \( J \) can be reached from many different ideals \( J_1 \) using different complete intersections in \( J \). Now subtracting, we see that the difference of the dimensions is
\[
(2d^2 + 6d - 12) - (2d^2 + 5d - 13) = d + 1 = h_2^e - h_{e-1}^e + 1.
\]
Since this is \( > n - 1 \) for \( d \geq 3 \), the locus is empty according to [4.4]. \qedhere

**Theorem 4.10.** Let \( I = \langle F_1, F_2, F_3 \rangle \subset R = k[x_1, x_2, x_3] \) be a complete intersection artinian ideal generated by general forms of degree \((d_1, d_2, d_3)\). Assume that \( d_1 \leq d_2 \leq d_3 \). Let \( e \) be the socle degree of \( R/I \) and let \((1, h_1, \cdots, h_{e-1}, h_e)\) be the h-vector of \( R/I \). Then

\(1\) If \( e \) is odd then \( \dim \mathcal{L}_I = 1 \) and \( \mathcal{L}_I \subset (\mathbb{P}^2)^* \) is a plane curve of degree
\[
\begin{cases}
  d_1d_2 & \text{if } d_3 \geq d_1 + d_2 \\
  d_1d_2 - \frac{(d_1 + d_2 - d_3)^2}{4} & \text{if } d_3 < d_1 + d_2.
\end{cases}
\]

\(2\) If \( e \) is even then
\[
\dim \mathcal{L}_I = h_2^e - h_{e-1}^e + 1 = \begin{cases}
  1 & \text{if } d_3 \geq d_1 + d_2 + 1, \\
  2 & \text{if } d_3 \leq d_1 + d_2 - 1.
\end{cases}
\]
Moreover, if \( d_3 \geq d_1 + d_2 + 1 \) then \( \mathcal{L}_I \subset (\mathbb{P}^2)^* \) is a plane curve of degree \( d_1d_2 \); and if \( d_3 \leq d_1 + d_2 - 1 \) then \( \mathcal{L}_I \subset (\mathbb{P}^2)^* \) is a finite set of \( \binom{n_i}{2} \) points, where
\[
n_I = \frac{2d_1d_2 + 2d_1d_3 + 2d_2d_3 + 1 - d_1^2 - d_2^2 - d_3^2}{4}
\]
Proof. It is well known that $I$ has WLP and hence codim $\mathcal{L}_I \geq 1$.

(1) If $d_3 \geq d_1 + d_2$ arguing as in Remark 4.6 we see that $J = \langle F_1, F_2 \rangle$ is the ideal of a set of $d_1d_2$ different points in $\mathbb{P}^2$, $h_{\ell+1} = h_{\ell+1} = d_1d_2$ and
\[ \times \ell: (R/I)_{\ell+1} \rightarrow (R/I)_{\ell+1} \]
with $\ell = ax + by + cz$ fails to be injective if and only if $\ell$ passes through one of the $d_1d_2$ points defined by $J$. Therefore, codim($\mathcal{L}_I$) = 1 and deg($\mathcal{L}_I$) = $d_1d_2$.

Assume $d_3 < d_1 + d_2$. In this case we consider the syzygy bundle associated to $I$, i.e. the rank 2 vector bundle $E$ on $\mathbb{P}^2$ defined by
\[ E := \ker(\oplus_{i=1}^{3}O_{\mathbb{P}^2}(-d_i) \xrightarrow{(F_1,F_2,F_3)} O_{\mathbb{P}^2}). \]

By [1, Corollary 2.7], $E$ is $\mu$-stable. By [3, Theorem 2.2.3], the linear form $\ell = ax + by + cz$ fails to be a Lefschetz element of $I$ if and only if $\ell = 0$ is a jumping line of $E$ and only if $E|_{\ell} \cong O_{\ell}(a_1^1) \oplus O_{\ell}(a_2^1)$ with $|a_1^1 - a_2^1| \geq 2$. Since the first Chern class $c_1(E(d_1 + d_2 + d_3)) = 0$, we can apply [17, Theorem 2.2.3], and we get that the set $J_E$ of jumping lines of $E$ is a curve of degree $c_2(E(d_1 + d_2 + d_3))$ in $(\mathbb{P}^2)^*$. Therefore, the non-Lefschetz locus $\mathcal{L}_I$ of $I$ is a plane curve of degree
\[ c_2(E(d_1 + d_2 + d_3)) = \frac{(d_1 + d_2 - d_3)(d_1 - d_2 + d_3)}{4} + \frac{(d_1 + d_2 - d_3)(-d_1 + d_2 + d_3)}{4} + \frac{(d_1 - d_2 + d_3)(-d_1 + d_2 + d_3)}{4} = 2d_1d_2 + 2d_1d_3 + 2d_2d_3 - d_1^2 - d_2^2 - d_3^2. \]

(2) If $d_3 \geq d_1 + d_2 + 1$ the result follows from Remark 4.6. So, let us assume that $d_3 \leq d_1 + d_2 - 1$. Let $(1, h_1, h_2, \ldots, h_{e-1}, h_e)$ be the $h$-vector of $R/I$.

Claim: $h_{\frac{e}{2}} - h_{\frac{e-1}{2}} = 1$. To prove the claim, we consider the rank 2 vector bundle $E$ on $\mathbb{P}^2$
\[ E := \ker\left(\oplus_{i=1}^{3}O_{\mathbb{P}^2}(-d_i) \xrightarrow{(F_1,F_2,F_3)} O_{\mathbb{P}^2}\right). \]

By [1, Corollary 2.7], $E$ is $\mu$-stable. Using the fact that $E$ is a $\mu$-stable rank 2 vector bundle on $\mathbb{P}^2$, $c_1(E) = -d_1 - d_2 - d_3$ and $E_{\text{norm}} = E(d_1 + d_2 + d_3)$, we get
\[ H^0(\mathbb{P}^2, E(h_{\frac{e}{2}})) = H^2(\mathbb{P}^2, E(h_{\frac{e}{2}})) = H^0(\mathbb{P}^2, E(h_{\frac{e-1}{2}})) = H^2(\mathbb{P}^2, E(h_{\frac{e-1}{2}})) = 0. \]

Therefore, we have
\[ h_{\frac{e}{2}} - h_{\frac{e-1}{2}} = h^1(\mathbb{P}^2, E(h_{\frac{e}{2}})) - h^1(\mathbb{P}^2, E(h_{\frac{e-1}{2}})) = -\chi(E(h_{\frac{e}{2}})) + \chi(E(h_{\frac{e-1}{2}})) = 1 \]
where the last equality follows applying the Riemann-Roch Theorem, and the claim is proved.

Thanks to the claim, the expected codimension of $\mathcal{L}_I \subset (\mathbb{P}^2)^*$ is two and, in fact, we are going to prove that $\mathcal{L}_I \subset (\mathbb{P}^2)^*$ is a set of $\binom{n_I}{2}$, $n_I := \frac{2d_1d_2 + 2d_1d_3 + 2d_2d_3 + 1 - d_1^2 - d_2^2 - d_3^2}{2}$, different points. To this end, we consider the rank 2 vector bundle $E$ on $\mathbb{P}^2$.
\[ E := \ker\left(\oplus_{i=1}^{3}O_{\mathbb{P}^2}(-d_i) \xrightarrow{(F_1,F_2,F_3)} O_{\mathbb{P}^2}\right). \]

By [1, Corollary 2.7], $E$ is $\mu$-stable. By [3, Theorem 2.2], the linear form $\ell = ax + by + cz$ fails to be a Lefschetz element of $I$ if and only if $\ell = 0$ is a jumping line of $E$ and only if $E|_{\ell} \cong O_{\ell}(a_1^1) \oplus O_{\ell}(a_2^1)$ with $|a_1^1 - a_2^1| \geq 2$. Since the first Chern class $c_1(E(d_1 + d_2 + d_3)) =
−1, we can apply [12, Corollary 10.7.1], and we get that \( \mathcal{E} \) has exactly \( c_2(\mathcal{E}(\frac{d_1 + d_2 + d_3 - 1}{2})) \) jumping lines. Let us compute \( c_2(\mathcal{E}(\frac{d_1 + d_2 + d_3 - 1}{2})) \). From the exact sequence

\[
0 \to \mathcal{E} \to \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^2}(-d_i) \to \mathcal{O}_{\mathbb{P}^2} \to 0
\]

we get that \( c_1(\mathcal{E}) = -d_1 - d_2 - d_3 \) and \( c_2(\mathcal{E}) = d_1 d_2 + d_1 d_3 + d_2 d_3 \). Since

\[
c_2 \left( \mathcal{E} \left( \frac{d_1 + d_2 + d_3 - 1}{2} \right) \right) = c_2(\mathcal{E}) + c_1(\mathcal{E}) \left( \frac{d_1 + d_2 + d_3 - 1}{2} \right) + \left( \frac{d_1 + d_2 + d_3 - 1}{2} \right)^2,
\]

we have

\[
c_2 \left( \mathcal{E} \left( \frac{d_1 + d_2 + d_3 - 1}{2} \right) \right) = \frac{2d_1 d_2 + 2d_1 d_3 + 2d_2 d_3 + 1 - d_1^2 - d_2^2 - d_3^2}{4},
\]

and we conclude that the set \( J_\mathcal{E} \) of jumping lines of \( \mathcal{E} \) is a set of \( \binom{n_I}{2} \) points in \( \mathbb{P}^2 \), where

\[
n_I := \frac{2d_1 d_2 + 2d_1 d_3 + 2d_2 d_3 + 1 - d_1^2 - d_2^2 - d_3^2}{4},
\]

which proves what we want.

Our next goal is to prove Conjecture \([4.3]\) for \( n = 4 \). To this end the following lemmas will be very useful.

**Lemma 4.11.** Let \( I = \langle F_1, F_2, F_3, F_4 \rangle \subset R = k[x_1, x_2, x_3, x_4] \) be a complete intersection artinian ideal generated by general forms of degree \((d_1, d_2, d_3, d_4)\). Assume that \( d_1 \leq d_2 \leq d_3 \leq d_4 \). Let \( e = d_1 + d_2 + d_3 + d_4 - 4 \) be the socle degree of \( R/I \) and let \((1, h_1, \cdots, h_{e-1}, h_e)\) be the h-vector of \( R/I \). Then

1. If \( e \) is odd then

\[
h_{\frac{e-1}{2}} = \frac{d_1 d_2 d_3}{3} - \frac{1}{4} \left( d_1 + d_2 + d_3 - d_4 + 1 \right)
\]

2. If \( e \) is even then

\[
h_{\frac{e}{2}} - h_{\frac{e-2}{2}} = \begin{cases} 
0 & \text{if } d_4 \geq d_1 + d_2 + d_3 \\
\frac{d_1 d_2 + d_3 - d_4}{2} & \text{if } -d_1 + d_2 + d_3 \leq d_4 \leq d_1 + d_2 + d_3 \\
\frac{d_1}{2} & \text{if } d_4 \leq -d_1 + d_2 + d_3.
\end{cases}
\]

**Proof.** Let \( h'_i \) be the h-vector of \( R/\langle f_1, f_2, f_3 \rangle \). Then for \( e \) odd we get

\[
h_{\frac{e-1}{2}} = \sum_{i=0}^{\frac{e-1}{2}-1} h'_{\frac{e-1}{2}-i} = \sum_{j=\frac{e+1}{2}-d_4}^{\frac{e-1}{2}} h'_j = d_1 d_2 d_3 - \sum_{j=0}^{\frac{e-1}{2}-d_4} h'_j - \sum_{j=\frac{e+1}{2}}^{\frac{e-1}{2}} h'_j
\]

\[
= d_1 d_2 d_3 - \sum_{j=0}^{\frac{e+1}{2}-d_4} h'_j - \sum_{j=\frac{e+1}{2}}^{\frac{e-1}{2}} h'_j
\]

\[
= d_1 d_2 d_3 - \sum_{j=0}^{\frac{e+1}{2}-d_4} h'_j - \sum_{j=\frac{e+1}{2}}^{\frac{e-1}{2}} h'_j
\]

\[
= d_1 d_2 d_3 - \sum_{j=0}^{\frac{e+1}{2}-d_4} h'_j - \sum_{j=\frac{e+1}{2}}^{\frac{e-1}{2}} h'_j
\]

\[
= d_1 d_2 d_3 - \sum_{j=0}^{\frac{e+1}{2}-d_4} h'_j - \sum_{j=\frac{e+1}{2}}^{\frac{e-1}{2}} h'_j
\]
In the range $0 \leq j \leq \frac{d_1 + d_2 + d_3 - d_4 - 5}{2}$ we have that $h_j' = \left(\frac{j+2}{2}\right) - \left(\frac{j-d_2+2}{2}\right)$ since $d_2 > \frac{d_1 + d_2 + d_3 - d_4 - 5}{2}$. For any $m$ we have that

$$
\sum_{j=0}^{m} \left(\frac{j+2}{2}\right) \sum_{j=0}^{m-1} \left(\frac{j+2}{2}\right) = \left(m + 3\right) \left(m + 2\right) = \frac{1}{4} \left(2m + 4\right).
$$

Thus we conclude that the maximum value of the $h$-vector is

$$h_1 = h_2 = d_1d_2d_3 - \frac{1}{4}\left(d_1 + d_2 + d_3 - d_4 - 1\right) + \frac{1}{4}\left(-d_1 + d_2 + d_3 - d_4 - 1\right).$$

When $e$ is even we want to compute the difference

$$h_2 - h_2 - 1 = \sum_{i=0}^{d_1-1} h_{2-i} - h_{2-2-i} = h_{2} - h_{2-2-d_4+1} = h_{2} - h_{2-d_4}$$

$$= h_{2} - h_{2-d_4}.$$
Theorem 4.13. Let $I = \langle F_1, F_2, F_3, F_4 \rangle \subset R = k[x_1, x_2, x_3, x_4]$ be a complete intersection artinian ideal generated by general forms of degree $(d_1, d_2, d_3, d_4)$. Assume that $d_1 \leq d_2 \leq d_3 \leq d_4$. Let $e$ be the socle degree of $R/I$ and let $(1, h_1, \cdots, h_{e-1}, h_{e})$ be the $h$-vector of $R/I$. Then

(1) If $e$ is odd then the non-Lefschetz locus $L_I \subset (\mathbb{P}^3)^*$ is a surface of degree

$$h_{\frac{e-1}{2}} = d_1d_2d_3 - \frac{1}{4}(d_1 + d_2 + d_3 - d_4 - 1) + \frac{1}{4}(-d_1 + d_2 + d_3 - d_4 - 1).$$

(2) If $e$ is even then

$$\text{codim } L_I = \min\{h_{\frac{e}{2}} - h_{\frac{e}{2}-1} + 1, 4\}.$$  

In particular, $L_I \subset (\mathbb{P}^{e-1})^*$ is non-empty if and only if $h_{\frac{e}{2}} - h_{\frac{e}{2}-1} \leq 2$ if and only if $d_4 \geq d_1 + d_2 + d_3$ or $-d_1 + d_2 + d_3 \leq d_4 \leq d_1 + d_2 + d_3$ and $d_1 + d_2 + d_3 - d_4 \leq 4$ or $d_4 \leq -d_1 + d_2 + d_3$ and $d_1 \leq 2$. In these cases $\delta_I := \deg(L_I) = \left(h_{\frac{e}{2}} - h_{\frac{e}{2}-1} + 1\right)^2$.

Proof. Part (1) follows from Lemma 4.11 taking into account that if $R/I$ has the WLP and the socle degree is odd then the non-Lefschetz locus is a surface of degree $h_{\frac{e-1}{2}}$.

We now consider (2). By Remark 4.6 (1) , if $d_4 \geq d_3 + d_2 + d_1$ (remembering that $e$ is even, so $d_4 \neq d_3 + d_2 + d_1 - 1$), then $h_{\frac{e}{2}} = h_{\frac{e}{2}-1} = d_1d_2d_3$ and $L_I \subset (\mathbb{P}^{e-1})^*$ is a surface of degree $d_1d_2d_3$. By Remark 4.6 (2), if $d_4 = d_3 + d_2 + d_1 - 2$, then $h_{\frac{e}{2}} - h_{\frac{e}{2}-1} = 1$, $h_{\frac{e}{2}} = d_1d_2d_3$ and $L_I \subset (\mathbb{P}^{e-1})^*$ is an arithmetically Cohen-Macaulay curve of degree $(d_1d_2, d_3)$. By Proposition 4.7 if $d_4 = d_1 + d_2 + d_3 - 4$ then $\text{codim } L_I = \min\{h_{\frac{e}{2}} - h_{\frac{e}{2}-1} + 1, 4\} = 3$ (where the last equality follows from Lemma 4.11 (2)) and has degree $(h_{\frac{e}{2}})$.

From now on we assume $d_4 \leq d_1 + d_2 + d_3 - 6$. We fix a linear form $\ell$ and we set $S = \text{R}/(\ell)$. We denote by $A_{d_1, d_2, d_3, d_4}$ the set of almost complete intersection ideals $J \subset S$ of type $(d_1, d_2, d_3, d_4)$ and by $B_{d_1, d_2, d_3, d_4}$ the set of almost complete intersection ideals $J \subset S$ of type $(d_1, d_2, d_3, d_4)$ and $h$-vector

$$(1, h_1 - 1, h_2 - h_1, \cdots, h_{\frac{e}{2}-1} - h_{\frac{e}{2}-2}, h_{\frac{e}{2}} - h_{\frac{e}{2}-1} + 1).$$

A general ideal $J$ in $A_{d_1, d_2, d_3, d_4}$ can be linked by means of a complete intersection $K$ of type $(d_1, d_2, d_3)$ to a Gorenstein ideal $G$ with socle degree $s := d_1 + d_2 + d_3 - d_4 - 3$ and $h$-vector $H_G = (1, f_1, \cdots, f_s)$. A general ideal $J'$ in $B_{d_1, d_2, d_3, d_4}$ can be linked by means of a complete intersection $K$ of type $(d_1, d_2, d_3)$ to a Gorenstein ideal $G'$ with socle degree $s$ and $h$-vector $H_{G'} = (1, f'_1, \cdots, f'_s)$. Moreover, we have:

$$f'_i = f_i \text{ for } i \neq \frac{s-1}{2}, \frac{s+1}{2},$$

$$f'_i = f_i - 1 \text{ for } i = \frac{s-1}{2}, \frac{s+1}{2}.$$

According to (4.3), to finish the proof it is enough to demonstrate that

$$\dim \text{Gor}(H_G) - \dim \text{Gor}(H_{G'}) = h_{\frac{e}{2}} - h_{\frac{e}{2}-1} + 1$$

(see also the end of the proof of Proposition 4.9 for the equivalence). Let us prove it. To this end, we denote by $(\tilde{h}_1, \cdots, \tilde{h}_w)$ the $h$-vector of the complete intersection ideal $K$ in $S$ of type $(d_1, d_2, d_3)$. So, $w = d_1 + d_2 + d_3 - 3$. Applying Lemma 4.12 we obtain

$$\dim \text{Gor}(H_G) - \dim \text{Gor}(H_{G'}) = f_{\frac{s+3}{2}+1} - 2f_{\frac{s+3}{2}+3} + f_{\frac{s+5}{2}+4} + 1$$

$$= (f_{\frac{s+3}{2}+1} + f_{\frac{s+5}{2}+2}) + (f_{\frac{s+3}{2}+2} - 2f_{\frac{s+3}{2}+3} + f_{\frac{s+3}{2}+4}) + 1$$

$$= -\Delta f_{\frac{s+3}{2}+2} - \Delta^2 f_{\frac{s+3}{2}+4} + 1.$$
Since \( G \) (resp. \( G' \)) is linked to \( J \) (resp. \( J' \)) by a complete intersection \( K \) of type \( d_1, d_2, d_3 \) we have \( f_i = \tilde{h}_{i+d_4} \) for \( i = \frac{d_1-1}{2}, \frac{d_2-1}{2}, \frac{d_3-1}{2} + 2 \). So, we get:

\[
-\Delta f_{\frac{d_1-1}{2}+2} - \Delta^2 f_{\frac{d_2-1}{2}+4} + 1 = -\Delta \tilde{h}_{\frac{d_1-2+d_2+d_3}{2}+d_4} + \Delta^2 \tilde{h}_{\frac{d_1-2+d_2+d_3}{2}+d_4+2} + 1
\]

\[
= \begin{cases} 
1 & \text{if } d_1 \geq d_1 + d_2 + d_3 \\
\frac{d_1+d_2+d_3-4}{2} + 1 & \text{if } -d_1 + d_2 + d_3 \leq d_4 \leq d_1 + d_2 + d_3 \\
d_1 + 1 & \text{if } d_4 \leq -d_1 + d_2 + d_3.
\end{cases}
\]

Therefore, applying Lemma 3.11 we conclude that

\[
\dim \text{Gor}(H_G) - \dim \text{Gor}(H_{G'}) = h_{\frac{4}{2}} - h_{\frac{4}{2}+1} + 1.
\]

\[
\square
\]

5. The non-Lefschetz locus of a general height three Gorenstein algebra

When the Hilbert function is fixed, the height three Gorenstein algebras with that Hilbert function lie in a flat family [6], so it makes sense to talk about the general Gorenstein algebra in this family. From now on, we will abuse terminology and refer to a general Gorenstein algebra, and assume that it is understood that we have fixed the Hilbert function; we will also assume that it is understood that in this section we refer only to the height three situation, except for a small remark at the end of the section. In this section we will describe the codimension of the non-Lefschetz locus of a general Gorenstein algebra, and in particular describe exactly when it is of the expected codimension (given the Hilbert function) in the sense of the earlier sections. One might expect that just as with complete intersections, the general Gorenstein algebra has non-Lefschetz locus of the expected codimension, but this is not always the case. We give a classification of those Hilbert functions for which the general Gorenstein algebras fail to have non-Lefschetz locus of the expected codimension.

The Hilbert functions of height three Gorenstein algebras are well-understood. They are the so-called Stanley-Iarrobino (SI) sequences of height three. They are characterized as follows. A sequence \( h = (1, 3, h_2, \ldots, h_{e-1}, h_e) \) is an SI-sequence if and only if

(i) \( h \) is symmetric.

(ii) Setting \( g_i = h_i - h_{i-1} \) for \( 1 \leq i \leq \lceil \frac{e}{2} \rceil \), the sequence \( g = (1, 2, g_2, \ldots, g_{\lceil \frac{e}{2} \rceil}) \) satisfies Macaulay’s growth condition.

Condition (ii) says that the sequence \( (1, 3, h_2, \ldots, h_{\lceil \frac{e}{2} \rceil}) \) is the beginning of the Hilbert function of some zero-dimensional scheme in \( \mathbb{P}^2 \) of degree \( h_{\lceil \frac{e}{2} \rceil} \). It is important to note that it does not mean that for every Gorenstein algebra \( R/I \) with this Hilbert function, the components of \( R/I \) up to degree \( \lceil \frac{e}{2} \rceil \) actually coincide with the corresponding components of a zero-dimensional scheme. If such a condition does hold, and if the zero-dimensional scheme is reduced, we will say that \( R/I \) “comes from points.” For any SI-sequence, by taking a suitable Gorenstein quotient of the coordinate ring of a suitable reduced set of points, there is always a subfamily (of the Gorenstein family corresponding to the SI-sequence) that does come from points.

We say that a sequence \( (1, 2, g_2, g_3, \ldots, g_k) \) is of decreasing type if begins with \( (1, 2, 3, \ldots) \) (growing with the polynomial ring \( k[x, y] \)), then is possibly flat, then is strictly decreasing.

**Theorem 5.1.** Fix an SI-sequence \( h = (1, 3, h_2, \ldots, h_{e-2}, 3, 1) \) of socle degree \( e \).
(i) If there are two or more consecutive values of \( h_i \) that are equal then the general Gorenstein algebra with Hilbert function \( h \) has non-Lefschetz locus of the expected codimension, namely one. This holds, in particular, when \( e \) is odd.

(ii) Assume \( e \) is even and

\[
(5.1) \quad h = (1, 3, h_2, \ldots, h_{\frac{e}{2} - 1}, h_{\frac{e}{2}}, h_{\frac{e}{2} + 1}, \ldots, h_{e-2}, 3, 1)
\]

where \( h_{\frac{e}{2} - 1} < h_{\frac{e}{2}} > h_{\frac{e}{2} + 1} \). Let \( g \) be the sequence of positive first differences, as above. Then the general Gorenstein algebra with this Hilbert function has non-Lefschetz locus of the expected codimension if and only if \( g \) is of decreasing type. If \( g \) is not of decreasing type then the non-Lefschetz locus has codimension one.

Proof. Suppose \( h_{|Z|} = h_{|Z|+1} \), for instance if the socle degree \( e \) is odd. Then the expected codimension of the non-Lefschetz locus is one. Since it is known that the general height three artinian Gorenstein algebra with any given Hilbert function has the WLP [9], the non-Lefschetz locus of the general Gorenstein algebra with odd socle degree has the expected codimension. So from now on assume that \( e \) is even, and that \( h_{\frac{e}{2} - 1} < h_{\frac{e}{2}} > h_{\frac{e}{2} + 1} \).

We first assume that \( g \) is of decreasing type. Our strategy will be to construct an explicit Gorenstein algebra having such a Hilbert function and non-Lefschetz locus of expected codimension; then by semicontinuity the general Gorenstein algebra with this Hilbert function has non-Lefschetz locus of the expected codimension.

So consider the SI-sequence (5.1), and assume that its first difference is of decreasing type. Let \( Z \) be a reduced set of \( h_{\frac{e}{2}} \) points in \( \mathbb{P}^2 \) with Hilbert function given by

\[
(1, 3, h_2, \ldots, h_{\frac{e}{2} - 1}, h_{\frac{e}{2}}, h_{\frac{e}{2} + 1}, \ldots).
\]

The \( h \)-vector of \( Z \) is given by the first difference sequence \( g \). Let \( I \) be an artinian Gorenstein ideal obtained as a suitable quotient of \( R/I_Z \), so that the Hilbert function of \( R/I \) is precisely \( h \). (See [2].) This means that \([I]_i = [I_Z]_i\) for \( i \leq \frac{e}{2} \). We want to show:

(i) \( \mathcal{L}_i = \emptyset \) if \( g_{\frac{e}{2}} \geq 2 \);

(ii) \( \text{codim } \mathcal{L}_i = 2 \) if \( g_{\frac{e}{2}} = 1 \);

We have already seen that \( \text{codim } \mathcal{L}_i = 1 \) if \( g_{\frac{e}{2}} = 0 \), so we have assumed \( h_{\frac{e}{2} - 1} < h_{\frac{e}{2}} \).

Because \( g \) is of decreasing type, we must assume that \( Z \) has the Uniform Position Property (UPP) by a result by Maggioni and Ragusa [14]. In particular it has the 2-Cayley-Bacharach Property: the Hilbert functions of \( Z \) minus a point are all the same, and the Hilbert functions of \( Z \) minus two points are all the same. We consider the multiplication on \( R/I_Z \) from degree \( \frac{e}{2} - 1 \) to degree \( \frac{e}{2} \) by a linear form \( \ell \). Notice that by UPP, \( \ell \) vanishes on at most two points since \( h_1 = 3 \) (so not all points lie on a line).

**Case 1:** \( \ell \) does not vanish on any point of \( Z \).

Then \( \ell \) is a non-zerodivisor, so the multiplication is injective and \( \ell \) is a Lefschetz element.

**Case 2:** \( \ell \) vanishes at exactly one point, \( P \), of \( Z \).

Let \( Y = Z \setminus P \), defined by \( I_Y = I_Z : \ell \). Notice that since \( h_{\frac{e}{2} - 1} < h_{\frac{e}{2}} \), and \( Z \) has the UPP, we have \([I_Y]_{\frac{e}{2} - 1} = [I_Z]_{\frac{e}{2} - 1}\). From the diagram

\[
\begin{array}{ccc}
[R/I]_{\frac{e}{2} - 1} & \xrightarrow{\times \ell} & [R/I]_{\frac{e}{2}} \\
\| & & \|
\end{array} \quad \begin{array}{ccc}
[I_Y/I_Z]_{\frac{e}{2} - 1} & \xrightarrow{\times \ell} & [R/I_Z]_{\frac{e}{2} - 1} \\
\| & & \|
\end{array}
\]

\[
0 \to [I_Y/I_Z]_{\frac{e}{2} - 1} \to [R/I_Z]_{\frac{e}{2} - 1} \xrightarrow{\times \ell} [R/I_Z]_{\frac{e}{2}}
\]
we see that multiplication by $\ell$ is again injective, i.e. $\ell$ is a Lefschetz element.

Case 3: $\ell$ vanishes at exactly two points, $P$ and $Q$, of $Z$.

We obtain the same diagram as in Case 2. In this case, though, we have $[I_Y/I_Z]_{i-1} = 0$ if and only $g_2 \geq 2$. If $g_2 = 1$, then $[I_Y/I_Z]_{i-1} \neq 0$. Thus $\mathcal{L}_I = \emptyset$ if $g_2 \geq 2$, and codim $\mathcal{L}_I = 2$ if $g_2 = 1$, both of which correspond to the expected codimension. In the latter case, the degree formula gives $\deg \mathcal{L}_I = \binom{h_2}{2}$, which can be seen directly as all choices of two points of $Z$.

Thus we have constructed an explicit Gorenstein algebra with Hilbert function $h$ and non-Lefschetz locus of the expected codimension, so as noted above, by semicontinuity the general Gorenstein algebra has non-Lefschetz locus of the expected codimension.

It remains to consider the case where $g$ is not of decreasing type. In this case the same approach will not work, since it is a priori possible that the Gorenstein algebras coming from points in this case fail to have non-Lefschetz locus of the expected codimension, but nevertheless the general one does. We will show by a different method that this is not the case.

So assume that $g_{i-1} = g_i$ for some $i \leq \frac{d}{2}$, and that $i - 1$ is the least degree for which $g_{i-2} > g_{i-1} = g_i$. Assume also that $R/I$ is general in the flat family of Gorenstein algebras with this Hilbert function. By a result of Ragusa and Zappala [18] the generators of $I$ of degree $\leq i - 1$ all have a common factor, $F$, of degree $g_i$. Furthermore, the generators of the ideal $I : F$ of degree $\leq i - 1$ span the ideal of a reduced set of points, $Z$, in $\mathbb{P}^2$.

In order to prove our statement on the non-Lefschetz locus, let $\ell$ be a linear form and let $Y$ be defined by $I_Z : \ell$. This time we consider the multiplication from degree $i - 1$ to degree $i$. We have a diagram

$$
\begin{array}{c}
[R/I]_{i-1} \\
\downarrow \\
0
\end{array} \xrightarrow{\times\ell} \begin{array}{c}
[R/I]_i \\
\downarrow \\
0
\end{array}
$$

But $\dim_k [R/I_Z]_j$ reaches its multiplicity in degree $i - 2 - g_i$ [5], so whenever $\ell$ vanishes at a point of $Z$, $[I_Y \cdot F/I_Z \cdot F]_{i-1}$ is not zero and the multiplication fails to have maximal rank. Thus the non-Lefschetz locus in degree $i - 1$ has codimension one. By Proposition 2.5 we are done.

Remark 5.2. Using the ideas from the proof of Theorem 5.1, it is easy to construct an artinian Gorenstein algebra whose non-Lefschetz locus is non-reduced, for almost any SI-sequence $h$. We simply relax the generality condition on $Z$ and allow three points to lie on a line. The only obstacle is when the $h$-vector of $Z$ does not allow this, i.e. when it is $(1, 1, 1)$

Remark 5.3. If $g$ is not of decreasing type and $A$ has odd socle degree, then even though $\mathcal{L}_I$ is of the expected codimension, it is still true that its behavior is not the expected one because in earlier degrees it is a hypersurface when we expect it not to be.

Remark 5.4. Since complete intersections of codimension three have $g$-vectors of decreasing type, Theorem 5.1 is a generalization of Theorem 4.10. However, since the method of proof is completely different, we prefer to give the proof of Theorem 4.10 in Section 4.

6. The non-Lefschetz locus in codimension two

In this short section we describe the situation in codimension two. Let $R = k[x, y]$ and let $I$ be an artinian ideal in $R$. Now the Hilbert function of $R/I$ has the form
(1, 2, h_2, \ldots, h_e), where h_i = i+1 until the initial degree of h_{R/I}, and then is non-decreasing from then on. Furthermore, if h_i = h_{i+1} for some i, this represents maximal growth of the Hilbert function, so Macaulay’s theorem \cite{13} together with Gotzmann’s theorem \cite{7} gives that the greatest common divisor of all the elements in I of degree i and degree i + 1 has degree h_i.

For fixed Hilbert function, the algebras having that Hilbert function form an irreducible family. For a general such algebra, if h_{i+1} < h_i then the elements of I in degree i + 1 do not have a common divisor.

**Lemma 6.1.** Let I be any artinian graded ideal in $R = k[x, y]$. Let $\{h_i\}$ be the h-vector of $R/I$. Fix any degree i. There exists a linear form $\ell$ such that $\times \ell : [R/I]_{i-1} \rightarrow [R/I]_i$ fails to have maximal rank if and only if I has a common factor, say F, between all forms of degree i. We have $\text{deg } F \leq h_i$.

**Proof.** The fact that the degree of a GCD in degree i must be $\leq h_i$ is well known. Assume that the forms of degree i in I have a GCD, say F, of positive degree. If $F \in I$ then $[I]_i$ is the degree i part of a principal ideal (F), we have $h_{i-1} = h_i = \text{deg } F$. But in R, F factors into linear factors. Thus clearly the non-Lefschetz locus consists precisely of the factors of F (counted with multiplicity). That is, the locus in $(\mathbb{P}^1)^*$ of linear forms $\ell$ for which $\times \ell : [R/I]_{i-1} \rightarrow [R/I]_i$ fails to have maximal rank is zero-dimensional of degree equal to $\text{deg } F = h_i$. (This is not quite the same as the non-Lefschetz locus since we are looking only in degrees i − 1 and i.)

Suppose instead that the GCD, F, is not in I and has degree $d \leq h_i$. We have $\dim \langle I \rangle_i = i + 1 - h_i = m$, say. Choose a basis for $\langle I \rangle_i$ of the form $\{FA_1, \ldots, FA_m\}$. Say $F$ factors as $F = \ell_1 \cdots \ell_d$. For each factor of F, for instance $\ell_1$, we have m independent elements of $[R]_{i-1}$ such that multiplication by $\ell_1$ is zero in $R/I$. Now,

$$h_{i-1} - h_i = (i - \dim \langle I \rangle_{i-1}) - (i + 1 - \dim \langle I \rangle_i) = m - 1 - \dim \langle I \rangle_{i-1} < m$$

so multiplication by $\ell_1$ has a larger kernel than expected (surjectivity implies a kernel of dimension $h_{i-1} - h_i$) and so fails to have maximal rank.

Conversely, assume that I does not have a GCD in degree i. We want to show that multiplication by any linear form $\ell$ gives a homomorphism of maximal rank from degree $i - 1$ to degree i. In degrees smaller than the initial degree of I, $R/I$ agrees with the polynomial ring, so the result is clear. If $h_{i-1} = h_i$ then by the result of Davis \cite{5} I has a GCD in degree i. Thus we may assume that $h_{i-1} > h_i$. Suppose that there exists a linear form $\ell$ for which the corresponding multiplication from degree $i - 1$ to degree i is not surjective. Consider the exact sequence

$$0 \rightarrow [R/(I : \ell)(-1)]_i \xrightarrow{\times \ell} [R/I]_i \rightarrow [R/(I, \ell)]_i \rightarrow 0.$$

By assumption, $[R/(I, \ell)]_i \neq 0$. But $R/(\ell) \cong k[x]$. This means that the restriction of $\langle I \rangle_i$ modulo $\ell$ is zero. This can only happen if $\ell$ is a GCD for $\langle I \rangle_i$, contradicting our assumption. The result follows. \qed

**Proposition 6.2.** Let $R = k[x, y]$. Fix a Hilbert function $\{h_i\}$ that exists for artinian graded quotients of R. Let $R/I$ be a general algebra with this Hilbert function. For any i, there exists a linear form $\ell$ such that $\times \ell : [R/I]_{i-1} \rightarrow [R/I]_i$ fails to have maximal rank if and only if $h_{i-1} = h_i$. In particular, if $R/I$ is a general complete intersection of type $(d_1, d_2)$, with $d_1 \leq d_2$, then the non-Lefschetz locus is empty if and only if $d_1 = d_2$. Otherwise, the degree of the non-Lefschetz locus is $d_1$. 22
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