Topological QCD with a Twist

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Non-supersymmetric Yang-Mill gauge theory in 4-dimension is shown to be dual to 4-dimensional non-supersymmetric string theory in a twisted $AdS^2(n) \times T_2$ spacetime background. The partition function of a generic hadron is calculated to illustrate the mathematical structure of the twisted QCD topology. Some physical implications of the twisted QCD topology are discussed.

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I. INTRODUCTION

Originally string theory was invented to explain hadron physics in the 70’s. It was thought from the beginning that there is a correspondence between string theory and gauge theory [1]. Hadronic string theory was eventually replaced by QCD in the 80’s. More recently interest in gauge-string duality is revived when more precise statements about the duality between the $AdS^5 \times S^5$ superstring theory and the $\mathcal{N} = 4$ super-Yang-Mill gauge theory in the large $N_c$ limit are made [2, 3, 4]. Despite the amazing advances in superstring theory, the theoretical foundation of gauge-string duality is still not more than a conjecture. Although supersymmetry is a beautiful symmetry, it is not yet clear that it will make connection with the real world. In low energy applications such as hadron physics, supersymmetry is assumed to be irrelevant. Given these considerations, statements of non-supersymmetric gauge-string duality may find a niche in theoretical physics. This work aims to illustrate (but not prove) 4-dimensional gauge-string duality without supersymmetry. The language of a gauge theory spoken in this work is mainly in QCD even though the arguments can be generalized to other types of Yang-Mill gauge theories. By the way of illustrating the relevance of the present model, a sample calculation that approximates the partition function of a generic hadron is given at the end.

II. GAUGE-STRING DUALITY

The geometrical interpretation of gauge theory is quite old. In lattice gauge theory, a hadron is modelled by a torus with periodic and/or anti-periodic boundary conditions. The minimal coupling term in the total derivative

$$D_\mu = \partial_\mu + ig A_\mu$$

is analogous to the Christoffel symbol $\Gamma^\nu_{bc}$ in general relativity. (In this paper, color indices are suppressed. Lorentz indices are represented by greek letters. Roman indices stand for abstract indices in the context of general relativity.) The correspondance between the gauge field and the Christoffel symbol is easily derived as

$$A_\mu \rightarrow \frac{1}{f 4g} \Gamma^\nu_{\mu\nu}.$$  

In general relativity, Christoffel symbols are contracted with all components of a vector being derived as in $\nabla_a t^b = \partial_a t^b + \Gamma^b_{ac} t^c$. A covariant derivative generally involves the entire volume around a point. Eq. 2 on other hand suggests that a total derivative in Yang-Mill theory depends on only 2 components, $D_\mu t_\nu = \partial_\mu t_\nu + ig A_\mu t_\nu$. If $D_\mu$ is interpreted as a translation operator on a 1-form $t_\mu$, the motion will be generated on a 2-dimensional plane defined by the $\mu, \nu$ indices. It means that any given Yang-Mill gauge configuration has a stratified topology by default. So far the theoretical foundation of a worldsheet description of planar Yang-Mill theory was simply assumed [5]. Although the
present argument is not a proof of the existence of planar Yang-Mill theory, it hopes to offer additional justifications for the worldsheet approach of gauge theory.

A loop on a 2-dimensional spacetime hyper-surface can be generated by the commutation bracket

$$[D_\mu, D_\nu] = ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu],$$

where $\mu \neq \nu$. Non-commutative geometry suggests that spacetime has structures. Figure 1 illustrates two kinds of topological structures (curvature and torsion) traced out by displacement vectors. Curvature is a measure of the geometrical features of the worldsheet while torsion measures the rotational motion perpendicular to a tower of worldsheets. The first term on the right hand side of Eq. (3) involves derivatives that rotate. Rotational vector is perpendicular to the plane such that it is taken to represent torsion. The second term involves a series of Christoffel symbols connecting points around a loop so that it is taken to represent the curvature of spacetime. Together the two terms on the right hand side of Eq. (3) represent the parallel (curvature) and perpendicular (torsion) components of a so-called “structure vector” generated by the commutation bracket on the left hand side and is pointing away from the worldsheet. On a carefully chosen surface, two equal and opposite structure vectors generated by a loop and a counter-loop are cancelled as shown in Figure 2(a). The parallel component of the structure vector can in principle be gauged away by twisting the the surface as shown in Figure 2(b). In retrospect, it is understood that any arbitrary manipulation of surfaces can always be performed because the string action will be integrated over all random Riemann surfaces using the path-integral method. The twist degree of freedom is limited at the edge of a folded surface so that a cylinder is introduced underneath the folded surface to provide a means to generate counter-loops to cancel the structure vectors of the outer surface (see Figure 2(b)). The gauge degree of freedom cannot be illustrated in a spacetime diagram. The structure of worldsheets in Figure 2 implicitly defines the gauge field configuration by correspondance based on the principle of gauge-string duality. The parallel component of the structure vector has a geometric interpretation of a force generated by the gauge field. Its removal on the twisted surface implies that the surface is equi-potential. Torsion is a unique feature of the non-Abelian nature of the gauge field. The cancellation of the perpendicular components of the structure vectors means that a non-Abelian theory is projected onto an Abelian theory. Although this conclusion is a surprise, it is consistent with observations in lattice gauge calculations where the full $SU(2)$ string tension is shown to be calculable in a maximally Abelian $U(1)$ gauge [4]. The size of the outer twisted surface is limited by the embedded toroidal surface. The smallest outer surface cannot overlap with the largest inner surface. It means that both the inner and outer surfaces have fuzzy supports and a fuzzy thickness $L^2$. The support of the worldsheets effectively defines the physical locations of the gauge fields. Since the inner and outer gauge surfaces are disjoint, the gauge fields around the two surfaces are also disjoint by default.

Naively Eq. (3) may be taken to be the same as $F_{\mu\nu} = 0$ because both curvature and torsion are gauged away by the twisted topology when $\mu \neq \nu$. For $\mu = \nu$, $F_{\mu\nu} = 0$ is true only in untwisted space when the loop collapses into a pair of lines running in opposite directions in a flat background. In twisted QCD topology, the pair of lines is separated by the twist. A possible parameterization of the stress tensor is

$$F_{\mu\nu} = \frac{1}{2} g^2 [\delta^2(\Sigma_i) + \delta^2(\Sigma_o)] \delta_{\mu\nu},$$

$$F^{\mu\nu} = \frac{1}{2} g^3 \frac{1}{L^2} \delta^{\mu\nu},$$

where $\delta^2(\Sigma)$ is a 2-dimensional delta-function restricting the the separation of the line pair to the surface $\Sigma$. The inner and outer equi-potential gauge surfaces are labelled as $\Sigma_i$ and $\Sigma_o$ respectively. $\delta_{\mu\nu}$ and $\delta^{\mu\nu}$ are Kronecker delta-functions. The choice of the forms of Eqs. (4) and (5) essentially picks a gauge for the induced metric $h_{\mu\nu}$ on the twisted string worldsheet.

The combination of the twist, periodical boundary conditions in space and time, and the number of edges on the closed string worldsheet (taken to be the same as the number of constituent quarks) completes the picture of the global structure of the gauge field as shown in Figure 3. The topology of the gauge field of a hadron consists of a twisted torus with negative curvature and $n$ edges $AdS^2(n)$ and an embedded torus $T_2$. It is assumed that the matter fields of the constituent quarks live inside the torus $T_2$ from symmetry considerations. The gauge-string topology of a meson corresponds to a twisted torus with 2 edges ($n = 2$). The closed mesonic worldsheet is not a Möbius strip because the surface is orientable. The baryonic topology ($n = 3$) has a special name called the “triniton.” The twisted topology is analogous to the Möbius strip in that multiple strips are joined into one around the loop. Since all the edges are one, it is reasonable to assume that there is only one embedded torus $T_2$. This condition restricts $n$ to prime numbers. As shown in Fig. 2, if $n = 6$, the identification of the twisted topology makes possible 2 embedded tori with 3 coils each or 3 embedded tori with 2 coils each. Only prime $n$ will prevent a factorization of the embedded torus $T_2$. This restriction may offer a possible explanation for the experimental observation that only pentaquark is observed, but not tetraquark.
III. TWISTED PARTITION FUNCTION

The partition function of the string theory in an $AdS^2(n) \times T_2$ background is calculated in this section as an example to illustrate the mathematical structure of the twisted QCD topology. It will soon be obvious that there are still many unsolved mathematical problems so that reasonable approximations are made along the way. In order to simplify the discussion, the Yang-Mill gauge action $S_g$ contains only the free gauge term,

$$S_g = \frac{1}{4} \int d^4x \, \text{tr} F_{\mu\nu} F^{\mu\nu}. \quad (6)$$

The partition function in gauge theory is defined as

$$Z_g = N \int D A \, e^{-S_g} \rightarrow \int D A_i \, D A_o \, e^{-S_g} \quad (7)$$

where $N$ is a normalization constant. Numerical constants will be absorbed into $N$ at various stages of the calculation. For the sake of simplicity, the presence of the normalization constant is understood implicitly so that $N$ will be dropped explicitly from now on. $A_o$ and $A_i$ are the gauge fields around the $AdS^2(n)$ and $T_2$ surfaces respectively. The factorization $D A \rightarrow D A_i \, D A_o$ is chosen because the gauge fields on the inner and outer surfaces are considered disjoint in an effective sense. This choice also makes it possible to factorize the partition function to yield simple results at a later point. By choosing the lightcone gauge ($A^+ = 0$) in the Faddeev-Popov procedure, the ghost fields are eliminated

$$Z_g = \int D A_i \, D A_o \, \delta(A^+) \, e^{-S_g} \quad (8)$$

so that the string worldsheet topology of the gauge field remains simple in the discussion to follow. Substitution of Eqs. (4) and (5) into Eq. (8) shows that the partition function can be factorized as

$$Z_g = Z_i Z_o, \quad (9)$$

where

$$Z_{(i,o)} = \int D A_{(i,o)} \, \exp \left( -\frac{g^2 f_c}{4L^2} \int d\Sigma_{(i,o)} \right). \quad (10)$$

$f_c$ is the color factor calculable from the structure constants of the Yang-Mill algebra. Let $h$ be the induced metric on the worldsheet, $x$ the displacement vector and $R$ the radius of the hadron. The Jacobian in $D A \rightarrow J \, D h \, D x$ is approximated through dimensional analysis as

$$J \sim \frac{1}{gR^2}. \quad (11)$$

The coefficient of the action in Eq. (10) is interpreted as the string tension. It can be absorbed into $x$ by a change of variable. At last Eq. (10) is transformed as

$$Z_{(i,o)} = \frac{L}{g^3 R^2 \sqrt{J_c}} \int D h_{(i,o)} \, D x \, \exp \left( -\frac{1}{2} \int d\Sigma_{(i,o)} \right), \quad (12)$$

Eq. (12) has the form of a string partition function. A path-integral over Yang-Mill gauge configurations is now transformed to a path-integral over string worldsheets. This procedure is similar to a Penrose transform.

In topological QCD, both $AdS^2(n)$ and $T_2$ are genus 1 surfaces. The former differs from the latter by a twist and a negative curvature. Curvature is a local property in twisted QCD topology. A path-integral sums over all the topological equivalent surfaces globally. Genus 1 surfaces of both signs of local curvature are topologically equivalent and are therefore indistinguishable to the path-integral. The induced metric on an untwisted space $h_{\mu\nu}$ is related to that on a twisted space $\tilde{h}_{\mu\nu}$ by a twist function $Z_n$. It is also reasonable to assume that the area of the string worldsheet is unaffected by the twist so that the action is invariant in the twist degree of freedom. This way the partition function of the twisted outer surface $Z_o$ is related to that of the untwisted inner surface by the twist function $Z_n$ as in

$$Z_o = Z_n Z_i. \quad (13)$$
With Eq. (13), Eq. (9) can be rewritten as

\[ Z_g = Z_n Z_i^2. \]  

Typically the Polyakov version of the string action

\[ S_p = \frac{1}{2} \int d\sigma d\tau \sqrt{-g} g^{ab} \partial_a x^\mu \partial_b x_\mu \]

is used in the worldsheet path-integral

\[ Z_i = \int Dg_{ab} Dx^\mu e^{-S_p}. \]

The evaluation of Eq. (16) for bosonic string in the critical dimension \((D = 26)\) is well-known [7] and is given as

\[ Z_c = \frac{N}{2} \int_{D(\Gamma)} \frac{d^2\tau}{2\pi\tau_2^2} e^{4\pi\tau_1(2\pi\tau_2)^{-12}} \prod_{n=1}^{\infty} \left| 1 - e^{2i\pi n\tau} \right|^{-48}, \]

with \(\tau = \tau_1 + i\tau_2\). Let \(U\) be the upper half of the complex plane, then

\[ D(\Gamma) = \left\{ \tau \in U \mid \text{Re} \tau \leq \frac{1}{2}, |\tau| \geq 1 \right\}. \]

In non-critical dimensions, conformal anomaly contributes an extra factor \(Z_L\) so that \(Z_i = Z_L Z_c\) and

\[ Z_L = \int D\phi \exp \left( -\frac{26-D}{48\pi} S_L \right). \]

The Liouville action \(S_L\) has very complicated mathematical structures [8] and is still an unsolved problem today. Nevertheless approximate solutions of the Liouville field theory have been constructed based on reasonable guesses [9]. The approximate contribution to \(Z_i\) from the Liouville action is given as

\[ Z_L \approx \left[ \left( \frac{26-D}{48} \right) \mu \frac{b^{-2i\beta}}{\Gamma(1+b^2)} \frac{\Gamma(1+b^2)}{\Gamma(1-b^2)} \right]^{Q/b} \frac{Y_0}{Y(-Q)}. \]

(See Appendix A.) In the standard model, \(D = 4\). The twist function \(Z_n\) of a spin-1 field is approximated from results in reference [10] as

\[ Z_n = -e^{-i\pi\epsilon^2i\sin\pi\epsilon} \prod_{n=1}^{\infty} \left( 1 - e^{2i\pi\epsilon k^n} \right) \left( 1 - e^{-2i\pi\epsilon k^n} \right), \]

where \(k = \exp(2i\pi\tau)\) and \(\tau\) is the period matrix in the \(\Theta\)-function. The parameter \(\epsilon = m/n\) where \(m \in \{Z \mid 0 < m < n\}\) measures the twist. At last the combination of Eqs. (14), (20) and (21) give the final expression of the partition function

\[ Z_g = Z_n Z_L^2 Z_c^2 \sim \frac{L^2}{g^b R^4 f_c} Z_n. \]

The last part of Eq. (22) highlights the dependencies of \(Z_g\).

**IV. DISCUSSIONS AND CONCLUSION**

Non-supersymmetric Yang-Mill gauge theory in 4-dimension is shown to be dual to non-supersymmetric 4-dimensional string theory in an \(AdS^2(n) \times T^2\) background. The topology of the gauge field represented by spacetime worldsheet has a twisted structure. On any given time slice, the topology shows multiple gauge surfaces with negative curvature. These multiple surfaces trace out a single surface globally in a way analogous to a Möbius strip. The \(T^2\) torus embedded underneath the twisted outer surface along the edge is thought to contain the matter field of the constituent quark. In the example of a baryon, the triunity structure of the gauge field suggests the possibility that there is only one global quark field that manifests itself as 3 separate matter fields locally. In this case, the
quark field couples to itself via back-reaction. If mulitple constituent quarks are simply mirror images of a single quark, confinement is automatically obtained because a quark cannot deconfine from itself. Since the purpose of this work is to illustrate the concept of twisted QCD topology in the simplest manner, matter fields are excluded from this analysis in order to focus solely on the dynamics of the gauge field. However, in order to have meaningful discussions on hadron phenomenology, matter fields and interaction terms must be included into a realistic action. Nevertheless it is encouraging to see that confinement emerges naturally in the present model without any need to tweak the action. The coupling of the loops and counter-loops via the structure vectors may provide a mechanism for color superconductivity. The subjects of matter fields, the interaction terms, color superconductivity, confinement and asymptotic freedom will be discussed in future works. In the present work, the partition function of a generic hadron is approximated in Eq. (22) as a sample calculation to illustrate the mathematical structure of twisted QCD topology which is still work in progress. It is expected that the final phenomenologically meaningful results will take on slightly different forms as the theory becomes more mature. The solutions of many difficult mathematical problems are still wanting and a more mathematically rigorous proof of gauge-string duality is still in need, possibly in the language of twistor algebra and Penrose transform. Future works will test the validity of twisted QCD topology by calculating physical observables such as Regge trajectories, generalized parton distributions and electromagnetic form factors $G_E/G_M$ to be compared with experimental data. The present work predicts the non-existence of tetraquark and hexaquark as opposed to the prediction of their existence by standard model calculations [11]. Future experiments of exotic bound states with either positive or negative discoveries will provide further insights into the structure of hadrons.

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APPENDIX A: LIOUVILLE ACTION

The Liouville action $S_L$ is defined as

$$S_L = \frac{1}{4\pi} \int d^2x \left[ (\partial_a \phi)^2 + 4\pi \mu e^{2b\phi} \right], \quad (A1)$$

where $\mu$ is called the cosmological constant and $b$ is a dimensionless coupling constant. Approximate solutions of the 3-point function of the Liouville field theory have been constructed [8]. An example is

$$C(\alpha_1, \alpha_2, \alpha_3) = \int \mathcal{D}\phi e^{-S_L} V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3)$$

$$= \left[ \pi \mu b^{-2b^2} \frac{\Gamma(1 + b^2)}{\Gamma(1 - b^2)} \right]^{(Q - \sum \alpha_i)/b}$$

$$\times \frac{\Gamma(2\alpha_1)\Gamma(2\alpha_2)\Gamma(2\alpha_3)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 - Q)\Gamma(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_1)\Gamma(\alpha_2 + \alpha_3 - \alpha_1)\Gamma(\alpha_3 + \alpha_1 - \alpha_2)}, \quad (A3)$$

The special function $\Upsilon(x)$ is defined as

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left( \frac{Q}{2} - x \right) t}{\sinh \frac{Q}{2} \sinh \frac{Q}{2t}} \right], \quad (A4)$$

with

$$Q = b + \frac{1}{b}. \quad (A5)$$

Given that

$$\Upsilon_0 = \frac{d\Upsilon(x)}{dx} \bigg|_{x=0}, \quad (A6)$$
and

\[ V_\alpha(x) = e^{2\alpha \phi(x)}, \]  

(A7)

it can be seen that \( Z_L \) can almost be obtained from Eq. (A3) by setting \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) if it were not the factor of \((D - 26)/48\pi\) in Eq. (19). It turns out that the factor can be absorbed into the cosmological constant \( \mu \) and the field variable \( \phi \) of the Liouville action \( S_L \) in Eq. (A1) by a change of variables. At last an approximate solution of \( Z_L \) is estimated from the 3-point function as given by Eq. (20). The variable \( b \) in Eq. (20) is changed to

\[ b \rightarrow \sqrt{\frac{26 - D}{48\pi}} b. \]  

(A8)

It is noted that \( Z_n \neq 1 \) as \( D \to 26 \). The reason is that Eq. (20) is an approximation based on an educated guess and is valid for non-critical dimensions only.

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FIGURES
FIG. 1: The non-commutative properties of the displacement operator, $D_\mu = \partial_\mu + igA_\mu$ can be interpreted as (a) curvature and (b) torsion. The arrows represent the spacetime translation generated by $D_\mu$. Curvature causes the loops to translate laterally as shown in (a). Torsion is signified by the presence of a spiral translation of the loops as in (b).
FIG. 2: The spacetime displacement generated by the loop \([D_\mu, D_\nu]\) on a 2-dimensional surface creates a residual vector called the “structure vector” representing curvature and torsion pointing away from the surface. Sets of loops and counter loops on 2 surfaces cancel the equal and opposite structure vectors as shown in (a). The components of the structure vectors parallel to the generating loops can be eliminated effectively by twisting the surfaces as shown in (b). The cylinder in (b) provides counter loops to cancel the structure vectors of the embedding outer surface at the edge.
The topology of the gauge field of a hadron consists of a twisted torus with negative curvature and \( n \) edges (2 edges for a meson and 3 for a baryon and so on) called \( AdS^2(n) \) and an embedded torus \( T_2 \). There are two possible twists in the case of the baryon—one corresponding to a twist of 120° and another to 240°. The identification in part (b) corresponds to a twist of 120°. 

FIG. 3: Sketches of the topologies of (a) a meson and (b) a baryon. The numbers show the identification of the boundaries.
FIG. 4: The factorizations of a 6-quark bound state \( n = 6 \). There are two possible factorizations depending on the different identifications of the boundaries—(a) a twist of 120° resulting in 2 loops of 3 coils each and (b) a twist of 240° giving 3 loops of 2 coils each. Each bar represents an edge of the twisted torus and corresponds to one coil.