FINITE-SAMPLE GUARANTEES FOR HIGH-DIMENSIONAL DML

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June 16, 2022

ABSTRACT  Debiasing machine learning (DML) offers an attractive way to estimate treatment effects in observational settings, where identification of causal parameters requires a conditional independence or unconfoundedness assumption, since it allows to control flexibly for a potentially very large number of covariates. This paper gives novel finite-sample guarantees for joint inference on high-dimensional DML, bounding how far the finite-sample distribution of the estimator is from its asymptotic Gaussian approximation. These guarantees are useful to applied researchers, as they are informative about how far off the coverage of joint confidence bands can be from the nominal level. There are many settings where high-dimensional causal parameters may be of interest, such as the ATE of many treatment profiles, or the ATE of a treatment on many outcomes. We also cover infinite-dimensional parameters, such as impacts on the entire marginal distribution of potential outcomes. The finite-sample guarantees in this paper complement the existing results on consistency and asymptotic normality of DML estimators, which are either asymptotic or treat only the one-dimensional case.

1 Introduction

A recent strand of literature in econometrics has considered estimation of treatment effects and causal or structural parameters using machine learning (ML) methods (Chernozhukov et al., 2018b; Belloni et al., 2017; Athey and Imbens, 2019; Farrell et al., 2021). In many observational settings, the treatment or policy whose impact we wish to quantify was not randomly assigned, so a simple comparison of treatment and control groups is confounded by factors that are correlated both with the outcome and with the treatment. We may still be able to identify the causal effect of the treatment, however, if we are willing to make an unconfoundedness assumption, i.e., that the treatment is exogenous conditional on an appropriate set of controls. There are several ways in which ML can be useful in such quasi-experimental research designs. On the one hand, it allows to control flexibly for a large number of covariates, as are typically available in modern datasets. On the other hand, it opens a wide range of possibilities in terms of which types of data can be used as controls, e.g., textual or image data.\footnote{This is increasingly relevant in applied economics. For example, see Dube et al. (2020), who use data from job ads in Amazon MTurk to study whether online employers have monopsonistic labor market power, controlling for job characteristics in the form of features learned by a random forest trained on the ad...}
The first section of this paper considers inference on a set of parameters that can be expressed as averages of a functional,

\[ \theta_{0j} = E_P [m_j(W, \gamma_{0j})], \quad j = 1, \ldots, p, \]

where \( \gamma_{0j} \) is an infinite-dimensional nuisance parameter (for example, a conditional expectation or regression function), and \( p \) is potentially large. This setting encompasses, for example, joint inference on the effects of many different treatments or on the impact of a treatment on many different outcomes.

Modern ML methods perform very well in predictive settings, typically by trading off variance and bias through some form of explicit or implicit regularization (and so they offer an attractive methodology to estimate \( \gamma_{0j} \) when it is some form of “best predictor”). At the same time, that trade-off means that their convergence rates are slower than the parametric \( \sqrt{n} \)-rate, causing a first-order bias in the estimation of the target parameters \( \theta_{0j} \) that does not, in general, vanish asymptotically.

Chernozhukov et al. (2018a) and subsequent work show how to construct estimators of \( \theta_{0j} \) that are correctly centered asymptotically by “debiasing” the moment conditions above, making them first-order robust to the ML estimation error. These are known as double or debiased machine learning (DML) estimators.

An applied research may be interested in conducting joint inference about a high-dimensional parameter \( \{\theta_{0j}\}_{j=1,\ldots,p} \). That can be done by constructing simultaneous confidence bands that cover the true parameter with a pre-specified probability (approximately) \( 1 - \alpha \). Based on DML estimators \( \{\hat{\theta}_j\}_{j=1,\ldots,p} \) of \( \{\theta_{0j}\}_{j=1,\ldots,p} \), and an estimator \( \hat{\sigma}_j^2 = \text{Var}(\sqrt{n}(\hat{\theta}_j - \theta_{0j})) \) for each \( j = 1, \ldots, p \), we will consider a joint confidence band of the form \( \times_{j=1}^p [\hat{\theta}_j \pm \frac{n^{-1/2}\hat{\sigma}_j c_\alpha}{n}] \), where the critical value \( c_\alpha \) is chosen so that the sup-\( t \)-statistic satisfies:

\[ \mathbb{P}_P \left( \max_{1 \leq j \leq p} \sqrt{n} \frac{|\hat{\theta}_j - \theta_{0j}|}{\hat{\sigma}_j} \leq c_\alpha \right) \approx 1 - \alpha. \]

One way to choose \( c_\alpha \) is to rely on the joint asymptotic normality of the \( t \)-statistics, which is a well-established fact in the literature (Chernozhukov et al., 2018a,b, 2021c), even in the high-dimensional case and for continua of parameters (Belloni et al., 2017, 2018).

The goal of this paper is to provide finite-sample guarantees for the normal approximation above, in the form of a bound on the Kolmogorov distance between the finite-sample distribution of the sup-\( t \)-statistic and \( \max_{1 \leq j \leq p} Z_j \) for a suitable jointly Gaussian distribution \( (Z_1, \ldots, Z_p) \sim \mathcal{N}(0, \Sigma) \), i.e.:

\[ \sup_{t \in \mathbb{R}} \left| \mathbb{P}_P \left( \max_{1 \leq j \leq p} \sqrt{n} \frac{|\hat{\theta}_j - \theta_{0j}|}{\hat{\sigma}_j} \leq t \right) - \mathbb{P} \left( \max_{1 \leq j \leq p} Z_j \leq t \right) \right|. \]

The dependence of the bound on \( n \) and \( p \) will be made explicit. These guarantees are useful to applied researchers, as they are informative about how far off the coverage of joint con-

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2 More precisely, Neyman-orthogonal, as defined below.
Confidence bands can be from the nominal level for a given sample size and complexity of the problem. The closest existing result to ours is Chernozhukov et al. (2021b), who provide finite-sample guarantees in the single-parameter case, $p = 1$, when using pure sample splitting. We extend their results in two ways, allowing for a potentially large $p$ and for no sample splitting. To that end, we leverage results from the literature on high-dimensional estimation, including a maximal inequality for empirical processes (Chernozhukov et al., 2014b) and new normal approximation results for high-dimensional vectors (Chernozhukov et al., 2021a).

In the second section of this paper, we consider a more general moment problem,

$$E_P [\psi_u(W, \theta_{0u}, \eta_{0u})] = 0, \quad u \in U,$$

where $U$ is potentially an uncountable set. In the continuum of target parameters case, the $\{\theta_{0u}\}_{u \in U}$ could represent, for example, the marginal distribution of an outcome under a treatment, which allows to derive many other interesting statistics (e.g., quantile treatment effects, Gini coefficients, Oaxaca-Blinder decompositions of distributional shifts, etc.). Again, we wish to construct a simultaneous confidence band for $\{\theta_{0u}\}_{u \in U}$ using DML, based on a normal approximation to the sup-$t$-statistic. We provide finite-sample guarantees for that normal approximation also in this setting. To the best of our knowledge, ours is the first paper to provide non-asymptotic guarantees for the DML estimator of a continuum of parameters, which extend and complement the asymptotic results of Belloni et al. (2017, 2018).

**Notation** Throughout the paper, we use the following notation. For a random variable $W \in \mathcal{W}$, distributed according given probability measure $P$ on $W$, we denote by $E_P [\cdot]$ the expectation with respect to $P$, i.e., $E_P [f(W)] = \int f(w) dP(w)$ for a suitably measurable and integrable $f$. We denote by $E_n [\cdot]$ the average of a sample $\{W_i\}_{i=1}^n$ of size $n$, i.e., $E_n [f(W)] = n^{-1} \sum_{i=1}^n f(W_i)$. We use $G_n [\cdot] = G_{n,p} [\cdot]$ for an empirical process $\sqrt{n}(E_n [\cdot] - E_P [\cdot])$ over a class $F$ of measurable and integrable functions $f : \mathcal{W} \to \mathbb{R}$, i.e.,

$$G_n [f] = G_n [f(W)] = \frac{1}{\sqrt{n}} \sum_{i=1}^n (f(W_i) - E_P [f(W)]).$$

We denote by $L^q = L^q(P)$ the space of functions with finite $q$-th absolute moments with respect to $P$, i.e., $f \in L^q (P)$, we denote by $\|f\|_P = (E_P [\|f(W)\|^q])^{1/q}$, the $L^q$ norm. For a bounded function $f : \mathcal{W} \to \mathbb{R}$, we denote by $\|f\|_\infty$ the sup norm, i.e., $\|f\|_\infty = \sup_{w \in W} |f(w)|$.

For a function $f : \mathbb{R} \to \mathbb{R}$, we denote by $\partial_r f(r) = f'(r)$ the derivative with respect to $r$. For a functional $F : \mathcal{F} \to \mathbb{R}$ over a class of functions $\mathcal{F}$, we define the Gateaux derivative of $F$ at $f$ in the direction $u \in \mathcal{F}$ as $\partial_u F(f + ru)|_{r=0}$. We say that $F$ is Gateaux differential at $F$ if that derivative exists for all $u \in \mathcal{F}$.

For a function class $\mathcal{F}$ endowed with a norm $\|\cdot\|_\mathcal{F}$, and for any $\varepsilon > 0$, we define the covering number $N(\varepsilon, \mathcal{F}, \|\cdot\|_\mathcal{F})$ as the smallest number of closed balls with radius $\varepsilon$.  

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3 A precise definition of what we mean by sample splitting will be provided in Remark 2.6.
that could cover $F$. Denote by $F$ a measurable envelope for $\mathcal{F}$, i.e., a function such that $F \geq \sup_{f \in \mathcal{F}} |f|$. The uniform entropy number is, for any $\varepsilon > 0$, $\log \sup_Q N(\varepsilon, \mathcal{F}, \|\cdot\|_{Q,2})$, where the supremum is taken over any finitely discrete probability measure $Q$ such that $\|F\|_{Q,2} > 0$.

2 Averages of Many Linear Functionals

In this section we extend the results of Chernozhukov et al. (2021b) to the high-dimensional case. Suppose we have access to an i.i.d. sample \{(Y_i, W_i)\}_{i=1}^n from a probability distribution $P$, where $Y \in \mathcal{Y}$ denotes outcomes of interest and $W \in \mathcal{W}$ are other observed data. Our goal is to construct a simultaneous confidence band for the set of (scalar) parameters $\{\theta_{0j}\}_{j=1,\ldots,p}$, satisfying the following moment condition:

$$\theta_{0j} = \mathbb{E}_P [m_j(W, \gamma_{0j})], \quad j = 1, \ldots, p,$$

where the moment functional $m_j(\cdot, \cdot)$ may depend on the unknown true value of an infinite-dimensional nuisance parameter $\gamma_{0j} \in \Gamma$, e.g., a conditional expectation. The set $\Gamma \subset L^2$ is assumed to be a linear function space, and can be used to encode restrictions on $\gamma_{0j}$, such as smoothness (Chernozhukov et al., 2021b). We assume that an ML estimator $\hat{\gamma}_j$ of $\gamma_{0j}$ can be obtained from the same data.

Below we give some examples where this setting may apply.

Example 2.1 (ATE of many treatments). Researchers may be interested in estimating the average treatment effect (ATE) of many different treatments, combinations of treatments or dosages. In observational settings, where treatments are not randomly assigned, ATEs may still be identified under an unconfoundedness assumption (Rosenbaum and Rubin, 1983).

Let $D \in D$ denote the treatment variable, where $D = \{d_0, \ldots, d_p\}$ is the set of different treatment profiles, with $d_0$ denoting no treatment. Let $Y(D)$ for $d \in D$ denote potential outcomes, so that the observed outcome is $Y = Y(D)$. Suppose that the researcher has access to a set of control variables $X$ such that $D$ is independent of $Y(d)$ given $X$ for all $d \in D$. In that case, $\mathbb{E}_P [Y(d)] = \mathbb{E}_P [\gamma_0(d_j, X)]$, where $\gamma_0(d, x) = \mathbb{E}_P [Y \mid D = d, X = x]$. (Notice that, with this formulation, $\gamma_0$ does not depend on $j$.) The ATE of treatment profile $d_j$ with respect to no treatment is:

$$\theta_{0j} = \mathbb{E}_P [m_j(W, \gamma_0)], \quad m_j(w, \gamma_0) = \gamma_0(d_j, x) - \gamma_0(d_0, x), \quad j = 1, \ldots, p.$$

In modern datasets, the number of available controls $X$ may be large, so that ML methods may be especially well suited to estimate the nuisance parameter $\gamma_{0j}$.

Example 2.2 (ATE on many outcomes). Our framework also provides guarantees for uniform inference on the ATE of a binary treatment $D \in \{0, 1\}$ on a large set of outcomes $Y = (Y_1, \ldots, Y_p)$ using the sup-$t$-statistic. In this case, under the appropriate unconfoundedness assumption, we have:

$$\theta_{0j} = \mathbb{E}_P [m_j(W, \gamma_{0j})], \quad m_j(w, \gamma_{0j}) = \gamma_{0j}(1, x) - \gamma_{0j}(0, x), \quad j = 1, \ldots, p.$$
where now $\gamma_0(j,d,x) = E_P [Y_j \mid D = d, X = x]$.

**Example 2.3 (Policy optimization).** Consider again a binary treatment $D \in \{0,1\}$ and an outcome of interest $Y$. Policymakers may wish to select the best treatment assignment policy based on a set of characteristics $X$, where a treatment assignment policy is a mapping $\pi : X \to \{0,1\}$. Suppose we only have access to observational data, collected under an unknown treatment policy. We want to evaluate and compare the effects of an alternative set of candidate policies $\{\pi_1, \ldots, \pi_p\}$, where the average effect of policy $\pi_j$ is:

$$\theta_0 j = E_P [m_j(W, \gamma_0)], \quad m_j(w, \gamma_0) = \gamma_0(0,x) + \pi_j(X)(\gamma_0(1,x) - \gamma_0(0,x)), \quad j = 1, \ldots, p.$$ 

Here, $\gamma_0(d, x) = E_P [Y \mid D = d, X = x]$ again does not depend on $j$. Other recent literature has also considered doubly-robust approaches to policy optimization based on observational data (e.g., Athey and Wager, 2021).

Given an estimator $\hat{\gamma}_j$ of $\gamma_0$, it may seem natural to estimate $\theta_0 j$ by the empirical analog of (1),

$$\hat{\theta}_j = E_n [m_j(W, \hat{\gamma}_j)], \quad j = 1, \ldots, p.$$ 

However, when $\hat{\gamma}_j$ is obtained using modern ML methods, $\hat{\gamma}_j$ typically converges to $\gamma_0$ more slowly than the parametric $\sqrt{n}$-rate. In this formulation, Chernozhukov et al. (2018a) show that the bias from using $\hat{\gamma}_j$ instead of $\gamma_0$ is of first-order magnitude, so that $\hat{\theta}_j$ is not asymptotically centered around the true value $\theta_0 j$. Inference based on $\hat{\theta}_j$ will thus be incorrect if one fails to account for that.

Following Chernozhukov et al. (2018a) and subsequent work, we proceed by adjusting the moment condition (1) to make it immune, to a first order, against the estimation error in $\hat{\gamma}_j$. This method is known as double or debiased machine learning (DML). Below we collect some existing results (Propositions 2.1 and 2.2) that show how to “debias” the moment condition (1) in the particular case of linear, mean-square continuous functionals of $\gamma$.

**Assumption 2.1 (Linearity and mean-square continuity).** For all $j = 1, \ldots, p$, the moment functional $\gamma \mapsto m_j(w, \gamma)$ is linear and mean-square continuous, i.e., there exists $Q < \infty$ such that

$$E_P [m_j(W, \gamma)^2] \leq Q^2 E_P [\gamma(W)^2] \quad \text{for all } \gamma \in \Gamma, j = 1, \ldots, p.$$ 

**Proposition 2.1 (Riesz representation).** Suppose Assumption 2.1 holds. Then, there exists a unique $\alpha_0 j \in \text{cl} (\text{span}(\Gamma))$ such that

$$E_P [m_j(W, \gamma)] = E_P [\alpha_0 j(W)\gamma] \quad \text{for all } \gamma \in \Gamma, j = 1, \ldots, p.$$ 

**Proof.** This is a consequence of the Riesz representation theorem. For a proof under more general conditions in the context of classic semiparametric theory, see, e.g., Newey (1994), Ichimura and Newey (2022). See also, e.g., Chernozhukov et al. (2018b) for a more detailed discussion of the role of the Riesz representer in DML. \qed

**Remark 2.1.** It is easy to see that the functionals in examples 2.1 to 2.3 are linear. One can also show mean-square continuity under appropriate regularity conditions (e.g., an overlap condition, $0 < \rho \leq P_P (D = d \mid X) \leq P < 1$ a.s. for all possible treatments or treatment
profiles \(d\), see e.g., Chernozhukov et al. (2018b). As a consequence, the Riesz representer exists in examples 2.1 to 2.3.

In general, the true Riesz representer \(\alpha_{0j}\) will be unknown. We assume that an estimator \(\hat{\alpha}_{j}\) can be obtained from the same data. In some cases, the explicit form of \(\alpha_{0j}\) can be derived. For instance, it can be shown that

\[
\alpha_{0j}^*(D, X) = \frac{1 \{D = d_j\}}{PP(D = d_j \mid X)} - \frac{1 \{D = d_0\}}{PP(D = d_0 \mid X)}
\]

is a Riesz representer for the functional Example 2.1.\(^4\) An estimator \(\hat{\alpha}_{j}\) can be then constructed by plugging in a non-parametric estimate of the propensity scores \(PP(D = d \mid X)\).

A more recent strand of literature considers automatic estimation of \(\alpha_{0j}\), where knowledge of the explicit form of \(\alpha_{0j}\) is not required (Chernozhukov et al., 2018b, 2020, 2021c).

We consider a point estimator for \(\{\theta_{0j}\}_{j=1,\ldots,p}\) based on the following augmented moment condition:

\[
\theta_{0j} = EP[m_j(W, \gamma_{0j}) + \alpha_{0j}(W)(Y - \gamma_{0j}(W))], \quad j = 1, \ldots, p, \tag{2}
\]

For the ATE examples, this is the augmented inverse propensity-weighted estimator (AIPW) of Robins et al. (1994). The addition of the term \(\alpha_{0j}(W)(Y - \gamma_{0j}(W))\) can be seen as “debiasing” the moment condition (1), since it makes it robust to estimation errors in \(\hat{\gamma}_{0j}\) and \(\hat{\alpha}_{0j}\) in a sense made explicit by the proposition below.

**Proposition 2.2** (Neyman orthogonality and double robustness). Let \(Z = (Y, W)\), and \(\psi_j(Z, \theta, \gamma, \alpha)\) denote the augmented score:

\[
\psi_j(Z, \theta, \gamma, \alpha) = m_j(W, \gamma) + \alpha(W)(Y - \gamma(W)) - \theta.
\]

We have:

(i) (Neyman orthogonality) The Gateaux derivative maps of \(EP[\psi_j(Z, \theta, \gamma, \alpha)]\) with respect to \(\gamma\) and \(\alpha\) are 0 at \((\theta_{0j}, \gamma_{0j}, \alpha_{0j})\):

\[
\frac{\partial}{\partial r}EP[\psi_j(Z, \theta_{0j}, \gamma_{0j} + r(\gamma - \gamma_{0j}), \alpha_{0j})] \bigg|_{r=0} = 0 \quad \text{for all } \gamma \in \Gamma,
\]

\[
\frac{\partial}{\partial r}EP[\psi_j(Z, \theta_{0j}, \gamma_{0j}, \alpha_{0j} + r(\alpha - \alpha_{0j}))] \bigg|_{r=0} = 0 \quad \text{for all } \alpha \in \Gamma.
\]

(ii) (Double robustness) Moreover,

\[
EP[\psi_j(Z, \theta_{0j}, \gamma, \alpha)] = -EP[(\alpha(W) - \alpha_{0j}(W))(\gamma(W) - \gamma_{0j}(W))]
\]

so that the augmented score is mean zero for all \(\alpha \in \Gamma\) whenever \(\gamma = \gamma_{0j}\), or for all \(\gamma \in \Gamma\) whenever \(\alpha = \alpha_{0j}\).

**Proof.** See, e.g., Chernozhukov et al. (2018b). \(\square\)

\(^4\)With a restricted semiparametric model \(\Gamma\), it is possible that \(\alpha_{0j}^* \notin cl(span(\Gamma))\), in which case the *unique* or minimal Riesz representer of Proposition 2.1 would be \(\alpha_{0j} = \text{Proj}(\alpha_{0j}^* \mid cl(span(\Gamma)))\).
Remark 2.2. Proposition 2.2 (ii) hints at a trade-off between the quality of the estimates for $\gamma_{0j}$ and $\alpha_{0j}$. In situations where $\gamma_{0j}$ can be estimated very well, it may be possible to achieve $\sqrt{n}$-convergence and asymptotic normality even when the rate of convergence of $\hat{\alpha}_j$ is slow, and vice versa.

Consider a simultaneous confidence band for $\{\theta_{0j}\}_{j=1,\ldots,p}$ constructed as $\times_{j=1}^p [\hat{\theta}_j \pm n^{-1/2} \hat{\sigma}_j c_\alpha]$, where the point estimates $\{\hat{\theta}_j\}_{j=1,\ldots,p}$ are based on an empirical analog of (2),

$$
\hat{\theta}_{0j} = E_n \left[ m_j(W, \hat{\gamma}_j) + \hat{\alpha}_j(W)(Y - \hat{\gamma}_j(W)) \right], \quad j = 1, \ldots, p,
$$

and $\hat{\sigma}_j^2$ is an estimator of $\sigma_j^2 = \text{Var}(\sqrt{n}(\hat{\theta}_j - \theta_{0j}))$. The critical value $c_\alpha$ will be chosen such that

$$
P_P \left( \max_{1 \leq j \leq p} \sqrt{n} \frac{\hat{\theta}_j - \theta_{0j}}{\hat{\sigma}_j} \leq c_\alpha \right) \approx 1 - \alpha,
$$

using a normal approximation for the sup-$t$-statistic. The goal of this section is to provide finite-sample guarantees for this normal approximation in the form of a bound on the Kolmogorov distance between the finite-sample distribution of the sup-$t$-statistic and $\max_{1 \leq j \leq p} Z_j$ for a suitable jointly Gaussian distribution $(Z_1, \ldots, Z_p) \sim \mathcal{N}(0, \Sigma)$:

$$
\sup_{t \in \mathbb{R}} \left| P_P \left( \max_{1 \leq j \leq p} \sqrt{n} \frac{\hat{\theta}_j - \theta_{0j}}{\hat{\sigma}_j} \leq t \right) - P \left( \max_{1 \leq j \leq p} Z_j \leq t \right) \right|.
$$

Towards that goal, we list a set of sufficient regularity conditions in the following assumptions.

Assumption 2.2 (Moment conditions). Suppose the following moment conditions hold:

(i) (Bounded heteroskedasticity of the outcome) $E_P [(Y - \gamma_{0j}(W))^2 \mid W] \leq \bar{\sigma}$ for all $j = 1, \ldots, p$.

(ii) (Variance bounded away from 0) Let $\bar{\psi}_{0j}(Z) = m_j(W, \gamma_{0j}) + \alpha_{0j}(Y - \gamma_{0j}(W)) - \theta_{0j}$ denote the oracle score (that is, the score evaluated at the true value of the parameters). We assume $\sigma_j^2 = E_P \left[ \bar{\psi}_{0j}(Z)^2 \right] \geq \sigma_{\min}^2 > 0$ for all $j = 1, \ldots, p$. Moreover, let $\Sigma$ denote the correlation matrix of the $\psi_{0j}(Z)$, with $(j, k)$-th entry given by

$$
\Sigma_{jk} = E_P \left[ \frac{\bar{\psi}_{0j}(Z) \bar{\psi}_{0k}(Z)}{\sigma_j \sigma_k} \right], \quad j, k = 1, \ldots, p.
$$

We assume that the smallest eigenvalue of $\Sigma$ is bounded below by some $\lambda_{\min} \geq 0$.

(iii) (Higher-order moments) For some $q \geq 4$, there exists $b_n < \infty$ such that

$$
\| \max_{1 \leq j \leq p} \bar{\psi}_{0j}(Z) / \sigma_j \|_{p,q} \leq b_n,
$$

and, for all $j = 1, \ldots, p$,

$$
E_P \left[ \left( \frac{\bar{\psi}_{0j}(Z)}{\sigma_j} \right)^4 \right] \leq b_n^2.
$$
Remark 2.3 (On the eigenvalue condition). The assumption that the minimum eigenvalue of $\Sigma$ is bounded below allows us to obtain nearly-optimal rates with respect to the sample size in the normal approximation we use (Chernozhukov et al., 2021a). In practice, it implies that the identifying moments do not become perfectly correlated as the number of parameters grows. This restriction precludes certain applications, e.g., using a grid of $p$ points to approximate the CDF of an outcome, with the grid becoming dense asymptotically. The case where there can possibly be a continuum of parameters will be covered in the next section.

Assumption 2.3 (Nuisance parameters). Suppose the following:

(i) (RMSE convergence rates) We have $\|\hat{\gamma}_j - \gamma_{0j}\|_{P,2} \leq R_n(\hat{\gamma})$ and $\|\hat{\alpha}_j - \alpha_{0j}\|_{P,2} \leq R_n(\hat{\gamma})$ for all $j = 1, \ldots, p$.

(ii) (Boundedness of Riesz Representer) We have $\|\alpha_{0j}\|_{\infty} \leq \bar{\alpha}$, $\|\hat{\alpha}_j\|_{\infty} \leq \bar{\alpha}$ for all $j = 1, \ldots, p$.

(iii) (Envelope and entropy conditions) The class of functions $\mathcal{F} = \{(Y, W) \mapsto (m_j(W, \gamma) + \alpha(W)(Y - \gamma(W)) - m_j(W, \gamma_{0j}) - \alpha_{0j}(W)(Y - \gamma_{0j}(W)) : \|\gamma - \gamma_{0j}\|_{P,2} \leq R_n(\hat{\gamma}), \|\alpha - \alpha_{0j}\|_{P,2} \leq R_n(\hat{\alpha}), j = 1, \ldots, p\}$ is suitably measurable, with a measurable envelope $F \geq \sup_{f \in \mathcal{F}} |f|$ that satisfies $\|F\|_{P,2+\delta} \leq M_n$ for some $\delta \geq 0$ and some sequence of constants $M_n$. There exist sequences $v_n \geq 1$, $a_n \geq n \vee M_n$, such that the uniform entropy numbers of $\mathcal{F}$ obey

$$\log \sup_{Q} N(\varepsilon \|F\|_{Q,2}, \mathcal{F}, \| \cdot \|_{Q,2}) \leq v_n \log(a_n/\varepsilon), \quad \text{for all } 0 < \varepsilon \leq 1.$$ 

Remark 2.4 (On entropy conditions). The goal of entropy conditions is to control the complexity of the class of functions used to estimate nuisance parameters. On the one hand, this class needs to be rich enough for it to be possible to obtain good MSE convergence rates. On the other hand, if the class is too complex, it may lead to an overfitting bias when using the same data to estimate the nuisance parameters $\gamma_{0j}$, $\alpha_{0j}$ and the target parameter $\theta_{0j}$. An alternative to restricting the entropy of the class of functions considered is using some form of sample splitting, as discussed below in Remark 2.6.

The following is one of the main theoretical results of this paper. It provides a bound on the Kolmogorov distance between the finite-sample distribution of the sup-$t$-statistic of the DML estimators and $\max_{1 \leq j \leq p} Z_j$ for the Gaussian limit distribution $(Z_1, \ldots, Z_p) \sim \mathcal{N}(0, \Sigma)$, where $\Sigma$ is as defined in Assumption 2.2.

Theorem 2.3. Suppose Assumptions 2.1, 2.2 and 2.3 hold. Then,

$$\sup_{t \in \mathbb{R}} \left| P_P \left( \max_{1 \leq j \leq p} \sqrt{n} \frac{\hat{\theta}_j - \theta_{0j}}{\sigma_j} \leq t \right) - P \left( \max_{1 \leq j \leq p} Z_j \leq t \right) \right| \leq \varrho(n, p),$$

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where
\[
\varrho(n, p) = C(q) \left\{ \frac{b_n (\log p)^{3/2} \log n}{\sqrt{n} \lambda_{\min}} + \frac{b_n^2 (\log p)^2 \log n}{n^{1-2/q} \lambda_{\min}} + \left[ \frac{b_n^q (\log d)^{3q/2-4} \log n \log (pm)}{n^{q/2-1}(\lambda_{\min})^{q/2}} \right]^{1/2} \right\}
\]
\[+ \frac{6\sqrt{\log p}}{\sigma_{\min}} \left\{ \Delta_{1n} + \Delta_{2n} \right\}
\]
\[+ \frac{c}{\log n} \]
\[\Delta_{1n} = K \left( 2 + \delta, \frac{c}{3} \right) \left( \left[ (2 + \sqrt{2})\bar{\alpha} + \sqrt{2}Q \right] \mathcal{R}_n(\hat{\gamma}) + \sigma \mathcal{R}_n(\hat{\alpha}) \right) \sqrt{3v_n \log(3a_n)}
\]
\[+ 3v_n^{1/2} - \frac{\log(3a_n)}{25} M_n \log(3a_n) \right),
\]
\[\Delta_{2n} = \sqrt{n} \mathcal{R}_n(\hat{\gamma}) \mathcal{R}_n(\hat{\alpha}),
\]
for any \( c > 0 \), some constant \( C(q) > 0 \) depending on \( q \), and some constant \( K \left( 2 + \delta, \frac{c}{3} \right) > 0 \) that may depend on \( \delta \) and \( c \).

**Proof.** The full proof is in Appendix A.1. Here we discuss the heuristics, which may help understand each of the terms (A), (B) and (C).

The first step of the proof is a decomposition of \( \hat{\theta}_j \) into an oracle estimator,

\[\hat{\theta}_j = \mathbb{E}_n [m(W, \gamma_{0j}) + \alpha_{0j}(W)(Y - \gamma_{0j}(W))]
\]

(i.e., the sample average of the augmented moment condition (2) if \( \gamma_{0j} \) and \( \alpha_{0j} \) were known), and a deviation from that oracle estimator, \( \mathbb{E}_n [m(W, \hat{\gamma}_j) + \hat{\alpha}_j(W)(Y - \hat{\gamma}_j(W)) - m(W, \gamma_{0j}) - \alpha_{0j}(W)(Y - \gamma_{0j}(W))] \).

On the one hand, the oracle estimator satisfies a high-dimensional version of a Berry-Esseen type of inequality, which allows us to quantify the Kolmogorov distance between its finite-sample distribution and the distribution of the corresponding multivariate normal distribution. In particular, we use the nearly-optimal rates in Chernozhukov et al. (2021a), which yield term (A). We can obtain refinements on this term by assuming stronger conditions on the higher-order moments of \( \psi_{0j}(Z) \), as discussed in Remark 2.5.

On the other hand, the deviation from the oracle estimator can be bounded using empirical process techniques. In the Appendix, we show that, with probability no more than \( c/\log n \) (C), the deviation is upper bounded by \( \Delta_{1n} + \Delta_{2n} \) (B). Improvements on (B) can be obtained by using some form of sample splitting, as discussed in Remark 2.6.

**Remark 2.5 (Stronger tail conditions).** We could obtain a simpler bound for term (A) by assuming stronger tail conditions than the ones in Assumption 2.2 (iii), as made clear in Lemma B.3, which collects results from Chernozhukov et al. (2021a). In particular,

(i) Assuming that \( \psi_{0j}(Z) \) is sub-Gaussian with Orlicz norm upper-bounded by \( b_n \), (A)
could be replaced by:
\[
C \left\{ \frac{b_n (\log p)^{3/2} \log n}{\sqrt{n} \lambda_{\min}} + \frac{b_n^2 (\log p)^2}{\sqrt{n} \lambda_{\min}} \right\}
\]
for an absolute constant \( C > 0 \).

(ii) Assuming that \( \overline{\psi}_{0j}(Z) \) is almost-surely bounded by \( b_n \), (A) could be replaced by:
\[
C b_n (\log p)^{3/2} \log n
\]
for an absolute constant \( C > 0 \).

These stronger assumptions may be satisfied in certain economic applications, for example if outcomes are binary or naturally bounded (e.g., hours worked in a labor supply example).

Remark 2.6 (Removing entropy conditions by sample splitting). The role of the entropy conditions in Assumption 2.3 is to prevent overfitting bias, due to the same data being used in the nuisance parameter estimators \( \hat{\gamma}_j, \hat{\alpha}_j \) and the estimator of the target parameter \( \hat{\theta}_j \). Another way to overcome this overfitting problem is, as discussed in Chernozhukov et al. (2018a) or Newey and Robins (2018), to use sample splitting.

1. With pure sample splitting, observations 1 to \( n \) are divided randomly into \( L \) data folds of roughly equal size, \( I_\ell, \ell = 1, \ldots, L \). For a given \( \ell \), estimators \( \hat{\gamma}_{j\ell}, \hat{\alpha}_{j\ell} \) of \( \gamma_{0j} \) and \( \alpha_{0j} \) are constructed using the data \textit{not} in \( \ell \), \( I_\ell^c \). An estimator for \( \theta_{0j} \) is then constructed as:
\[
\hat{\theta}_j = \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_\ell^c} \left[ m_j(W_i, \hat{\gamma}_{j\ell}) + \hat{\alpha}_{j\ell}(W_i)(Y_i - \hat{\gamma}_{j\ell}) \right].
\]
In that case, within the \( \ell \)-th fold and after conditioning on \( I_\ell^c \), the summands are i.i.d., and so we can set \( v_n = 1, a_n = e \) in Assumption 2.3 (iii).

2. An alternative is to use a “dirty” version of sample splitting, in which data in the \( \ell \)-th fold is used to \textit{select} amongst a finite set of estimators \( \{(\hat{\gamma}_{j\ell}^{(1)}, \hat{\alpha}_{j\ell}^{(1)}), \ldots, (\hat{\gamma}_{j\ell}^{(r)}, \hat{\alpha}_{j\ell}^{(r)})\} \) trained on data \textit{not} in \( \ell \). In that case, because a covering number for a finite class of functions is at most its cardinality, we can set \( v_n = 1, a_n = e \vee r \). \( \diamond \)

Finally, the asymptotic validity of the simultaneous confidence band follows as a corollary of Theorem 2.3 under two additional assumptions. This is not a new result, as it could be seen as a particular case of Belloni et al., 2018, but we present it here for completeness.

Assumption 2.4 (Consistent variance estimation). Suppose that we have an estimator \( \hat{\sigma}_j \) of \( \sigma_j \) for all \( j = 1, \ldots, p \) such that
\[
\max_{1 \leq j \leq p} \left( \frac{\hat{\sigma}_j}{\sigma_j} \right) \xrightarrow{p} 1.
\]

Assumption 2.5 (Growth conditions). Suppose the following growth conditions as \( n, p \to \infty \):
(i) (Nuisance parameters converge fast enough) $\sqrt{\log(p)nR_n(\hat{\gamma})R_n(\hat{\alpha})} \to 0$.

(ii) (Complexity characteristics do not grow too fast)

$$\sqrt{\log(p)v_n \log(a_n)[R_n(\hat{\gamma}) \vee R_n(\hat{\alpha})]} \to 0 \quad \text{and} \quad \sqrt{\log(p)v_n n^{2/3-1/2} M_n \log(a_n)} \to 0.$$

(iii) (Moment bounds do not grow too fast)

$$\frac{b_n(\log p)^{3/2} \log n}{\sqrt{n}\lambda_{\min}} + \frac{b_2^2(\log p)^2 \log n}{n^{1-2/3}\lambda_{\min}} + \left[\frac{b_2^2(\log d)^{3q/2-4} \log n \log(pn)}{n^{q/2-1}(\lambda_{\min})^{q/2}}\right]^{1/q-2} \to 0.$$

Remark 2.7. As we pointed out in Remark 2.2, there is a tradeoff between the RMSE convergence rate of $\hat{\gamma}$ and of $\hat{\alpha}$, which is made explicit in (i). In the case of a single parameter, $p = 1$, a sufficient condition for (i) is that both $\hat{\gamma}$ and $\hat{\alpha}$ converge faster than $n^{-1/4}$, a rate that is typically attainable by non-parametric estimators (Chernozhukov et al., 2018a). Note that the dependence of $p$ in the bound is only logarithmic, allowing for very high dimensional cases (potentially $p \gg n$).

Corollary 2.4 (Validity of the simultaneous confidence band). Under Assumptions 2.1 to 2.5, we have

$$P_P \left( \hat{\theta}_j - n^{-1/2} \hat{\sigma}_j c_\alpha \leq \theta_{0j} \leq \hat{\theta}_j + n^{-1/2} \hat{\sigma}_j c_\alpha, \forall 1 \leq j \leq p \right) \to 1 - \alpha.$$

3 Continua of Parameters

In this section, we consider the same setting as Belloni et al. (2017). Again, suppose we have access to an i.i.d. sample $\{W_i\}_{i=1}^n$ from a probability distribution $P$ on $W$. Now, we are interested in constructing a simultaneous confidence band for the set of (scalar) parameters $\{\theta_{0u}\}_{u \in U}$, satisfying the following moment condition:

$$E_P [\tilde{\psi}_u(W, \theta_{0u}, \eta_{0u})] = 0, \quad u \in U$$

for a possibly uncountable set $U$, where $\eta_{0u} \in T_u$ is the unknown true value of an infinite-dimensional nuisance parameter. Here we also assume that an ML estimator $\tilde{\eta}_u$ of $\eta_{0u}$ can be obtained from the same data.

Below we give a leading example where this framework may be appropriate.

Example 3.1 (Distributional treatment effects). Consider a setting where we want to evaluate the effect of a binary treatment $D \in \{0, 1\}$ on an outcome $Y$. We work with observational data and, as in Examples 2.1 and 2.2, we suppose that we have access to a rich enough set of controls $X$ such that an unconfoundedness assumption holds.

In some cases, features of the marginal distributions of potential outcomes beyond the mean may be of interest. For example, policymakers may care about how the treatment impacts inequality. In other cases, economic theory will make predictions about how different regions of the outcome distribution should be affected by the treatment, so looking at features other than the mean can be used to probe or validate the theory.
Under the unconfoundedness assumption, the marginal distribution of $Y(d)$ can be identified as

$$\theta_{0u} = F_Y(d)(u) = E_P [\gamma_{0u}(d, X)],$$

where $\gamma_{0u}(d, x) = F_Y(u \mid D = d, X = x) = P_F(Y \leq u \mid D, X)$ is the conditional distribution of $Y$ given $D$ and $X$ at point $u$. A non-parametric estimator of $\gamma_{0u}$ may be constructed using different techniques, for example, distribution regression (Chernozhukov et al., 2013).

Having access to estimates of $\{\theta_{0u}\}_{u \in U}$ for a suitable range $U$ allows to construct many interesting statistics, such as quantile treatment effects, Gini indices, and Oaxaca-Blinder type of decompositions.

Again, our goal in this section is to give finite-sample guarantees for a simultaneous confidence band for $\{\theta_{0u}\}_{u \in U}$ using a set of DML point estimates $\{\hat{\theta}_u\}_{u \in U}$ based on an empirical analog of (3). We assume that each $\theta_{0u} \in \Theta_u$ for some $\Theta_u \subset \mathbb{R}$, and that we can find, for each $u \in U$, an approximate solution to the empirical analog of (3), i.e., a $\hat{\theta}_u$ such that

$$\mathbb{E}_n \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) \right] \leq n^{-1/2} \epsilon_n, \quad (4)$$

for some sequence of $\epsilon_n > 0$ such that $n^{-1/2} \epsilon_n \to 0$ as $n \to \infty$. Again, we want to choose a critical value $c_\alpha$ such that

$$P_P \left( \sup_{u \in U} \sqrt{n} \frac{|\hat{\theta}_u - \theta_{0u}|}{\sigma_u} \leq c_\alpha \right) \approx 1 - \alpha,$$

using a normal approximation for the sup-$t$-statistic. As in the previous section, our objective is to provide finite-sample guarantees for this normal approximation in the form of a bound on the Kolmogorov distance between the finite-sample distribution of the sup-$t$-statistic and the supremum of a suitable Gaussian process. We begin by giving some sufficient regularity conditions towards this result.

**Assumption 3.1 (Moment problem).** For all $u \in U$, the following conditions hold.

(i) The true parameter satisfies $\theta_{0u} \in \text{int} \Theta_u$.

(ii) The map $(\theta, \eta) \mapsto E_P [\psi_u(W, \theta, \eta)]$ is twice continuously Gateaux-differentiable on $\Theta_u \times \mathcal{T}_u$.

(iii) (Neyman orthogonality) For $\bar{r} \in [0, 1)$, let $D_{\theta u}[\eta - \eta_{0u}]$ denote the Gateaux derivative map of $E_P [\psi_u(W, \theta, \eta)]$ with respect to $\eta$ at $(\theta_{0u}, \eta_{0u})$ in the direction $\eta - \eta_{0u}$,

$$D_{\theta u}[\eta - \eta_{0u}] = \partial_\eta E_P [\psi_u(W, \theta_{0u}, \eta_{0u} + r(\eta - \eta_{0u}))] |_{r = \bar{r}}.$$

Then, $D_{\theta u}[\eta - \eta_{0u}] = 0$ for all $\eta \in \mathcal{T}_u$.

(iv) (Bounded derivatives with respect to $\theta$) Let $J_{\theta u} = \partial_\theta E_P [\psi_u(W, \theta, \eta_{0u})] |_{\theta = \theta_{0u}}$. Then $c_0 \leq |J_{\theta u}| \leq C_0$. Moreover, $|\partial_\theta E_P [\psi_u(W, \theta, \eta_{0u})]| > c_1$ for all $\theta \in \Theta_u$.

(v) (Lipschitz-continuity at the true parameters) For all $\theta \in \Theta_u$ and $\eta \in \mathcal{T}_u$,

$$E_P \left[ (\psi_u(W, \theta, \eta) - \psi_u(W, \theta_{0u}, \eta_{0u}))^2 \right] \leq C_0(|\theta - \theta_{0u}| + \|\eta - \eta_{0u}\|_{P, 2})^\omega.$$
(vi) (Bounded derivatives with respect to $\eta$) For all $r \in [0, 1)$, $\theta \in \Theta_u$ and $\eta \in \mathcal{T}_u$,

$$|\partial_r E_P [\psi_u(W, \theta_0u + r(\eta - \eta_0u))]| \leq B_{1n} \|\eta - \eta_0u\|_{P,2}.$$  

(vii) (Bounded second derivatives) For all $r \in [0, 1)$, $\theta \in \Theta_u$ and $\eta \in \mathcal{T}_u$,

$$|\partial^2_{rr} E_P [\psi_u(W, \theta_0u + r(\theta - \theta_0u), \eta_0u + r(\eta - \eta_0u))]| \leq B_{2n} (|\theta - \theta_0u| \lor \|\eta - \eta_0u\|_{P,2})^2.$$  

Remark 3.1 (On the Neyman orthogonality condition). As opposed to the previous section, here we take Neyman orthonality (iii) as a primitive condition, and hence we assume that $\eta_{0ij}$ contains all nuisance parameters needed to make the moment functional Neyman-orthogonal. In the case of a linear, mean-square continuous functional of a regression, the same construction as in Section 2 is valid, and so $\eta_{0u} = (\gamma_{0u}, \alpha_{0u})$ for $\alpha_{0u}$ the Riesz representer. This is true, for instance, in Example 3.1. More generally, Chernozhukov et al. (2018b) discuss how to orthogonalize non-linear functionals, and Chernozhukov et al. (2018a), Belloni et al. (2018) cover many other important models, such as conditional moment restrictions.

Assumption 3.2 (Nuisance parameters). Suppose the following:

(i) (RMSE convergence rates) For all $u \in \mathcal{U}$ we have $\|\hat{\eta}_u - \eta_{0u}\|_{P,2} \leq R_n(\hat{n})$.

(ii) (Envelope and entropy conditions) The class of functions

$$\mathcal{F} = \{ W \mapsto \psi_u(W, \theta, \eta) : \theta \in \Theta_u, \eta \in \mathcal{T}_u, u \in \mathcal{U} \}$$

is suitably measurable, with a measurable envelope $F \geq \sup_{f \in \mathcal{F}} |f|$ that satisfies $\|F\|_{P,2+\delta} \leq M_n$ for some $\delta \geq 0$ and some sequence of constants $M_n$. There exist sequences $v_n \geq 1$, $a_n \geq n \lor M_n$, such that the uniform entropy numbers of $\mathcal{F}$ obey

$$\log \sup_Q N(\varepsilon\|F\|_{Q,2}, \mathcal{F}, \|\cdot\|_{Q,2}) \leq a_n \log(a_n/\varepsilon), \quad \text{for all } 0 < \varepsilon \leq 1.$$  

Finally, for all $f \in \mathcal{F}$, we have $c_0 \leq \|f\|_{P,2} \leq C_0$.

Assumption 3.3 (Entropy and moments of the score at the truth). Suppose $\sigma_u^2 = J_{0u}^{-1}E_P [\psi_u(W, \theta_0u, \eta_{0u})]^2 \geq C_0^{-2}c_0^2$, and let $\bar{\psi}_{0u}(W) = -(\sigma_u J_{0u})^{-1} \psi_u(W, \theta_0u, \eta_{0u})$ denote the re-scaled score evaluated at the true values $(\theta_0u, \eta_{0u})$. Then:

(i) (Envelope and entropy conditions) The class of functions

$$\mathcal{F}_0 = \{ W \mapsto \bar{\psi}_{0u}(W) : u \in \mathcal{U} \}$$

is suitably measurable, with a measurable envelope $F_0 \geq \sup_{f \in \mathcal{F}_0} |f|$ that satisfies $\|F_0\|_{P,q} \leq b_n$ for some $q \geq 4$ and some sequence of constants $b_n$. There exist sequences $V_n \geq 1$, $A_n \geq n$, such that the uniform entropy numbers of $\mathcal{F}_0$ obey

$$\log \sup_Q N(\varepsilon\|F_0\|_{Q,2}, \mathcal{F}_0, \|\cdot\|_{Q,2}) \leq V_n \log(A_n/\varepsilon), \quad \text{for all } 0 < \varepsilon \leq 1.$$
(ii) (Moments) For all \( f \in \mathcal{F}_0 \) and \( k = 3, 4 \), \( \mathbb{E}_P \left[ |f(W)|^k \right] \leq C_0 b_n^{k-2} \).

The following theorem is the second main result of this paper. Again, it provides a bound on the Kolmogorov distance between the finite-sample distribution of the sup-t-statistic of the DML estimators and its limiting distribution. In the statement of the theorem, \( G_P \) denotes a tight mean-zero Gaussian process indexed by the class of functions in Assumption 3.3, with covariance function \( \mathbb{E} \left[ G_P[\psi_{0u}]G_P[\psi_{0u'}] \right] = \mathbb{E}_P \left[ \psi_{0u}(W)\psi_{0u'}(W) \right] \) for all \( u, u' \in U \).

**Theorem 3.1.** Suppose Assumptions 3.1, 3.2 and 3.3 hold. Then,

\[
\sup_{t \in \mathbb{R}} \left| P_P \left( \sup_{u \in U} \sqrt{n} \frac{\left( \hat{\theta}_u - \theta_{0u} \right)}{\sigma_u} \leq t \right) - P \left( \sup_{u \in U} G_P[\psi_{0u}] \leq t \right) \right| \leq \kappa r_{1n} \left( \chi \sqrt{V_n \log(A_n b_n)} + \sqrt{1 \log(1/r_{1n})} \right) + r_{2n},
\]

where \( \kappa, \chi > 0 \) are universal constants,

\[ r_{1n} = c_0^{-1} \epsilon_n + \Delta_{1n} + \Delta_{2n} + \Delta_{3n} \]

for:

\[
\Delta_{1n} = C_0^{-1} K \left( 2 + \delta, \frac{C}{2} \right) \left( \sqrt{C_0} R_n^{\gamma}(\hat{\eta}) \right)^{\frac{1}{2\gamma}} \sqrt{2v_n \log(2a_n) + 2v_n \frac{1}{2\gamma + \frac{1}{2}} M_n \log(2a_n)}.
\]

\[
\Delta_{2n} = C_0^{-1} \frac{1}{2} \sqrt{n} B_{2n} [R_n^{\gamma}(\hat{\eta})]^2.
\]

\[
\Delta_{3n} = \frac{b_n L_n^{\gamma/2} n^{1/2 - 1/q}}{\gamma^{1/2} n^{1/4}} + \frac{(b_n)^{1/2} L_{n}^{3/4}}{\gamma^{1/2} n^{1/4}} + \frac{\gamma^{1/3} n^{1/6}}{\gamma^{1/3} n^{1/6}}.
\]

\[
R_n^{\gamma}(\hat{\eta}) = \left\{ c_1^{-1} n^{-1/2} \epsilon_n + c_1^{-1} n^{-1/2} K \left( 2 + \delta, \frac{C}{2} \right) \left( C_0 \sqrt{v_n \log(a_n)} + v_n \frac{1}{2\gamma + \frac{1}{2}} M_n \log(a_n) \right) \right\} \wedge \mathcal{R}_n(\hat{\eta}),
\]

\[
L_n = d(q) V_n (\log n \vee \log(A_n b_n)),
\]

and

\[
r_{2n} = D(q) (\gamma + \log n/n) + c / \log n,
\]

where \( d(q), D(q) \) are constants depending only on \( q \), for any \( c > 0 \) and \( \gamma \in (0, 1) \).

**Proof.** The full proof is in Appendix A.2. As above, we discuss the heuristics here. First, \( R_n^{\gamma}(\hat{\eta}) \) is the maximum of two objects: the rate \( \mathcal{R}_n(\hat{\eta}) \) for \( \hat{\eta} \) and a preliminary rate for \( \hat{\theta} \) (an upper bound for how far \( \hat{\theta} \) can be from \( \theta_0 \) based only on the smoothness conditions). Notice that this step becomes unnecessary whenever the moment function \( \psi_u \) is linear in \( \theta \), which will be the case in many applications, including Example 3.1.

Second, the Kolmogorov distance-based statement of the theorem is related by Lemma...
B.6 to another kind of approximation, of the form:

\[ P\left( \sup_{u \in \mathcal{U}} \sqrt{n} \left( \frac{\hat{\theta}_u - \theta_{0u}}{\sigma_u} - G_P[\tilde{\psi}_{0u}] \right) > r_{1n} \right) \leq r_{2n}. \]

As an intermediate step, we first approximate \( \sqrt{n}(\hat{\theta}_u - \theta_{0u})/\sigma_u \) by the empirical process on the re-scaled oracle score, \( G_n[\bar{\psi}_{0u}] \). With high probability, the distance between these two objects is bounded by \( \Delta_{1n} + \Delta_{2n} \). The first term quantifies the size of the deviation between \( \psi_u(W, \hat{\theta}, \hat{\eta}) \) and \( \psi_u(W, \theta_{0u}, \eta_{0u}) \) when \( \psi_u(W, \hat{\theta}, \hat{\eta}) \) is in the class of functions \( \mathcal{F} \). The second term bounds the error that we incur by linearizing the score. In particular, if the score is linear in both \( \theta \) and \( \eta \), this term can be ignored.

Finally, the term \( \Delta_{3n} \) arises when approximating the supremum of the empirical process on the oracle score, \( G_n[\bar{\psi}_{0u}] \), by the supremum of the corresponding Gaussian process, \( G_P[\bar{\psi}_{0u}] \), and it is a consequence of Lemma B.5.

As in the previous section, we give two additional conditions for the asymptotic validity of the simultaneous confidence band. This is also not a new result, but was shown in Belloni et al., 2017, 2018. We present it below for completeness.

**Assumption 3.4** (Consistent variance estimation). Suppose that we have an estimator \( \hat{\sigma}_u \) of \( \sigma_u \) for all \( u \in \mathcal{U} \) such that \( \sup_{u \in \mathcal{U}} (\hat{\sigma}_u/\sigma_u) \xrightarrow{P} 1 \).

**Assumption 3.5** (Growth conditions). Suppose the following growth conditions as \( n \to \infty \):

(i) (Nuisance parameters converge fast enough) \( \sqrt{n}[R_n(\hat{\eta})]^2 \to 0 \).

(ii) (Complexity characteristics do not grow too fast)

\[
\sqrt{v_n \log(a_n)[R_n(\hat{\eta})]^2} = 0, \quad v_n n^{1/3} = 0, \quad v_n M_n \log(a_n) \to 0 \quad \text{and} \quad r_{1n} \sqrt{V_n \log(a_nb_n)} \to 0.
\]

(iii) (Moment bounds do not grow too fast)

\[
\frac{b_n L_n}{(n^{1/2} n^{1/2-1/q})} + \frac{(b_n)^{3/4} L_n^{3/4}}{(n^{1/2} n^{1/4})} + \frac{(b_n L_n^{3/4})^{3/4}}{(n^{1/3} n^{1/6})} \to 0.
\]

**Corollary 3.2** (Validity of the simultaneous confidence band). Under Assumptions 3.1 to 3.5, we have

\[ P_P \left( \hat{\theta}_u - n^{-1/2} \hat{\sigma}_u c_\alpha \leq \theta_{0u} \leq \hat{\theta}_u + n^{-1/2} \hat{\sigma}_u c_\alpha, \forall u \in \mathcal{U} \right) \to 1 - \alpha. \]

**4 Conclusions**

In many applications, researchers are interested in the causal impact of a treatment or policy that was not randomly assigned. Inference in such non-experimental settings is still possible by controlling for a set of covariates, conditional on which the treatment becomes plausibly exogenous. In modern settings, DML offers an alternative way to leverage a large
number of potential controls with regularization and model selection, which does not bias the estimates of the target parameters thanks to a Neyman orthogonality condition. Often, we want to make joint inference on a high-dimensional set of parameters, such as the ATE of many treatments or combinations thereof, the ATE of a treatment on many outcomes, or effects on the entire marginal distribution of the potential outcomes. In this paper we have complemented existing asymptotic results for high-dimensional DML (Belloni et al., 2017, 2018) with finite-sample guarantees. These finite sample guarantees can be useful to applied researchers, as they are informative of how far off the coverage of simultaneous confidence bands can be from the nominal levels.

There is one natural extension of the paper that would be an interesting avenue for future research. Our guarantees are based on the true standard error, $\sigma_j$ or $\sigma_u$, respectively, which in general will not be known and will have to be estimated. For our asymptotic corollaries, we have simply assumed that estimators $\hat{\sigma}_j$ or $\hat{\sigma}_u$ exist. We leave it to future work to provide finite-sample guarantees for variance estimation in the high dimensional case, although we note that such guarantees are available for one-dimensional DML with sample splitting (Chernozhukov et al., 2021b).
References

Athey, Susan and Guido W. Imbens, “Machine Learning Methods That Economists Should Know About,” *Annual Review of Economics*, 2019, 11 (1), 685–725.

_ and Stefan Wager, “Policy Learning with Observational Data,” *Econometrica*, 2021, 89 (1), 133–161.

Belloni, Alexandre, Victor Chernozhukov, Denis Chetverikov, and Ying Wei, “Uniformly valid post-regularization confidence regions for many functional parameters in Z-estimation framework,” *Annals of statistics*, 2018, 46 (6B), 3643.

_ , _ , Iván Fernández-Val, and Christian Hansen, “Program Evaluation and Causal Inference with High-dimensional Data,” *Econometrica*, 2017, 85 (1), 233–298.

Chernozhukov, Victor, Denis Chetverikov, and Kengo Kato, “Comparison and anti-concentration bounds for maxima of Gaussian random vectors,” *Probability theory and related fields*, 2014, 162 (1-2), 47–70.

_ , _ , and _ , “Gaussian approximation of suprema of empirical processes,” *The Annals of Statistics*, 2014, 42 (4), 1564–1597.

_ , _ , and Yuta Koike, “Nearly optimal central limit theorem and bootstrap approximations in high dimensions,” 2021.

_ , _ , Mert Demirer, Esther Duflo, Christian Hansen, Whitney K Newey, and James Robins, “Double/debiased Machine Learning for Treatment and Structural Parameters,” *The Econometrics Journal*, 2018, 21 (1), C1–C68.

_ , Iván Fernández-Val, and Blaise Melly, “Inference on Counterfactual Distributions,” *Econometrica*, 2013, 81 (6), 2205–2268.

_ , Whitney K Newey, and Rahul Singh, “Automatic debiased machine learning of causal and structural effects,” *arXiv preprint arXiv:1809.05224*, 2018.

_ , _ , and _ , “A Simple and General Debiased Machine Learning Theorem with Finite Sample Guarantees,” 2021.

_ , _ , and Vasilis Syrgkanis, “Adversarial Estimation of Riesz Representer,” *arXiv preprint arXiv:2101.00009*, 2020.

_ , _ , Victor Quintas-Martinez, and Vasilis Syrgkanis, “Automatic Debiased Machine Learning via Neural Nets for Generalized Linear Regression,” 2021.

Dube, Arindrajit, Jeff Jacobs, Suresh Naidu, and Siddharth Suri, “Monopsony in Online Labor Markets,” *American Economic Review: Insights*, 2020, 2 (1), 33–46.

Farrell, Max H, Tengyuan Liang, and Sanjog Misra, “Deep Neural Networks for Estimation and Inference,” *Econometrica*, 2021, 89 (1), 181–213.
Ichimura, Hidehiko and Whitney K Newey, “The influence function of semiparametric estimators,” Quantitative Economics, 2022, 13 (1), 29–61.

Newey, Whitney K, “The asymptotic variance of semiparametric estimators,” Econometrica, 1994, pp. 1349–1382.

— and James R Robins, “Cross-fitting and Fast Remainder Rates for Semiparametric Estimation,” arXiv preprint arXiv:1801.09138, 2018.

Olken, Benjamin, Edward Glaeser, Rema Hanna, Nikhil Naik, and Scott Kominers, “Evidence to inform the JKN health insurance programme: Analysis and collection of GIS data,” 2017. https://www.theigc.org/project/evidence-to-inform-the-jkn-health-insurance-programme-analysis-and-collection-of-gis-data/ [Accessed: December 29, 2021].

Robins, James M, Andrea Rotnitzky, and Lue Ping Zhao, “Estimation of regression coefficients when some regressors are not always observed,” Journal of the American Statistical Association, 1994, 89 (427), 846–866.

Rosenbaum, Paul R and Donald B Rubin, “The Central Role of the Propensity Score in Observational Studies for Causal Effects,” Biometrika, 1983, 70 (1), 41–55.

van der Vaart, Aad and Jon A Wellner, Weak convergence and empirical processes, Springer, 1996.
Appendix A  Proofs

A.1 Proof of Theorem 2.3

The DML estimator of $\theta_{0j}$ is, for all $j = 1, \ldots, p$,

$$\hat{\theta}_j = E_n \left[ m(W, \hat{\gamma}_j) + \hat{\alpha}_j(W)(Y - \hat{\gamma}_j(W)) \right].$$

Decompose it as in Chernozhukov et al. (2021b):

$$\sqrt{n}(\hat{\theta}_j - \theta_{0j}) = \sqrt{n}(\theta_j - \theta_{0j}) + A_{jn} + B_{jn} + C_{jn} + D_{jn},$$  \hspace{1cm} (5)

where:

(i) $\hat{\theta}_j = E_n \left[ m(W, \gamma_{0j}) + \alpha_{0j}(W)(Y - \gamma_{0j}(W)) \right]$ is the oracle estimator.

(ii) $A_{jn} = G_n[m(W, \hat{\gamma}_j - \gamma_{0j}) + \alpha_{0j}(W)(\hat{\gamma}_j - \gamma_{0j}(W))]$,

(iii) $B_{jn} = G_n[(\hat{\alpha}_j(W) - \alpha_{0j}(W))(Y - \gamma_{0j}(W))]$,

(iv) $C_{jn} = G_n[-(\hat{\alpha}_j(W) - \alpha_{0j}(W))(\hat{\gamma}_j - \gamma_{0j}(W))]$,

(v) $D_{jn} = -\sqrt{n} E_P[(\hat{\alpha}_j(W) - \alpha_{0j}(W))(\hat{\gamma}_j - \gamma_{0j}(W))]$.

The first term in the right-hand side of (5) is an average of random variables. We can quantify, using a multivariate Berry-Esseen inequality, how far its finite-sample distribution is from that of a Gaussian random variable. The terms $A_{jn}$, $B_{jn}$ and $C_{jn}$ are more challenging to bound, since they are empirical process of functions that themselves depend on the data through the estimated nuisance parameters $\hat{\gamma}_j$ and $\hat{\alpha}_j$. Chernozhukov et al. (2021b) proceed by using cross-fitting — i.e., splitting the sample in folds and, for each fold, use an estimate of $\gamma_{0j}$ and $\alpha_{0j}$ based on the remaining data. In this paper, we employ empirical process inequalities instead. Finally, the last term, $D_{jn}$, can be bounded using simple moment inequalities.

Bounds on $A_{jn}$, $B_{jn}$ and $C_{jn}$ We need to obtain high-probability bounds for $A_{jn}$, $B_{jn}$, $C_{jn}$ that hold uniformly over $j = 1, \ldots, p$. To do so, we will apply Lemma B.1 to the following empirical processes:

$$\max_{1 \leq j \leq p} |A_{jn}| \leq \sup_{f \in \mathcal{F}_A} G_n[f], \quad \max_{1 \leq j \leq p} |B_{jn}| \leq \sup_{f \in \mathcal{F}_B} G_n[f], \quad \max_{1 \leq j \leq p} |C_{jn}| \leq \sup_{f \in \mathcal{F}_C} G_n[f],$$

with the corresponding classes of functions:

$$\mathcal{F}_A = \{ m_j(W, \gamma - \gamma_{0j}) - \alpha_{0j}(W)(\gamma(W) - \gamma_{0j}(W)) : \|\gamma - \gamma_{0j}\|_{P,2} \leq R_n(\hat{\gamma}), j = 1, \ldots, p \},$$

$$\mathcal{F}_B = \{ (\alpha(W) - \alpha_{0j}(W))(Y - \gamma_{0j}(W)) : \|\alpha - \alpha_{0j}\|_{P,2} \leq R_n(\hat{\alpha}), j = 1, \ldots, p \},$$

$$\mathcal{F}_C = \{ -(\alpha(W) - \alpha_{0j}(W))(\gamma(W) - \gamma_{0j}(W)) : \|\gamma - \gamma_{0j}\|_{P,2} \leq R_n(\hat{\gamma}), \quad \|\alpha - \alpha_{0j}\|_{P,2} \leq R_n(\hat{\alpha}), j = 1, \ldots, p \}.$$
Recall the class of functions defined in Assumption 2.3,
\[ F = \{ (m_j(W, \gamma) + \alpha(Y - \gamma) - m_j(W, \gamma_0) - \alpha_0(Y - \gamma_0) : \| \gamma - \gamma_0 \|_{p, 2} \leq \mathcal{R}_n(\hat{\gamma}), \| \alpha - \alpha_0 \|_{p_2} \leq \mathcal{R}_n(\hat{\alpha}), j = 1, \ldots, p \}. \]

We have that \( F_A \subset F, F_B \subset F \) and \( F_C \subset F - F_A - F_B \). By assumption, there exists an envelope function \( F \) for \( F \), such that \( F \geq \sup_{f \in F} |f| \) with \( \| F \|_{p, 2 + \delta} \leq M_n \) for some \( \delta \geq 0 \). Hence, the corresponding envelopes for \( F_A, F_B \) and \( F_C \) are \( F, F \) and \( 3F \). By Assumptions 2.1, 2.2 and 2.3, the classes \( F_A, F_B \) and \( F_C \) satisfy \( \sup_{f \in F_k} \| f \|_{p, 2} \leq \sigma_k^2 \) for \( k = A, B, C \), where:
\[ \sigma_A^2 = 2(\hat{Q}^2 + \alpha^2)\mathcal{R}_n(\hat{\gamma})^2, \quad \sigma_B^2 = \hat{\sigma}^2 \mathcal{R}_n(\hat{\alpha})^2, \quad \sigma_C^2 = 4\hat{\alpha}^2 \mathcal{R}_n(\hat{\gamma})^2. \]

Finally, the uniform covering entropy of \( F_A \) and \( F_B \) is upper bounded by that of \( F \). For \( F_C \), we can obtain an upper bound using Lemma B.2:
\[ \log \sup_Q N(\varepsilon \| 3F \|_{Q, 2}, \mathcal{F}_C, \| \cdot \|_{Q, 2}) \leq 3v_n \log(3a_n/\varepsilon) \quad \text{for all } 0 < \varepsilon \leq 1. \]

By Lemma B.1, we conclude that, with probability at least \( 1 - c/\log n \),
\[ \max_{1 \leq j \leq p} |A_{jn}| + \max_{1 \leq j \leq p} |B_{jn}| + \max_{1 \leq j \leq p} |C_{jn}| \leq \Delta_n = K \left( 2 + \delta, \frac{c}{3} \right) \left( \left[ (2 + \sqrt{2})\hat{\alpha} + \sqrt{2}\hat{Q} \right] \mathcal{R}_n(\hat{\gamma}) + \hat{\sigma} \mathcal{R}_n(\hat{\alpha}) \right) \sqrt{3v_n \log(3a_n)} + 3v_n \frac{1}{2} \log M_n \log(3a_n). \]

**Bound on** \( D_{jn} \) By the Cauchy-Schwarz inequality, \( |D_{jn}| \leq \Delta_2 = \sqrt{n} \mathcal{R}_n(\hat{\gamma}) \mathcal{R}_n(\hat{\alpha}) \) for each \( j = 1, \ldots, p \). Hence, \( \max_{1 \leq j \leq p} |D_{jn}| \leq \Delta_{2n} \).

**Normal Approximation** Combining the two bounds, we have:
\[ \left| \mathbb{P} \left( \max_{1 \leq j \leq p} \sqrt{n} \left( \frac{\hat{\theta}_j - \theta_{0j}}{\sigma_j} \right) \leq z \right) - \mathbb{P} \left( \max_{1 \leq j \leq p} Z_j \leq z \right) \right| \leq \mathbb{P} \left( \max_{1 \leq j \leq p} \sqrt{n} \left( \frac{\hat{\theta}_j - \theta_{0j}}{\sigma_j} \right) \leq z + \frac{\Delta_n}{\sigma_{\min}} \right) - \mathbb{P} \left( \max_{1 \leq j \leq p} Z_j \leq z + \frac{\Delta_n}{\sigma_{\min}} \right) \]
\[ + \mathbb{P} \left( \max_{1 \leq j \leq p} Z_j \leq z + \frac{\Delta_n}{\sigma_{\min}} \right) - \mathbb{P} \left( \max_{1 \leq j \leq p} Z_j \leq z \right) + \frac{c}{\log n}, \]
where \( \Delta_n = \Delta_1 + \Delta_2 \) and \( Z = (Z_1, \ldots, Z_p) \sim \mathcal{N}(0, \Sigma) \), for \( \Sigma \) and \( \sigma_{\min} \) defined in Assumption 2.2.

By the results of Chernozhukov et al. (2021a), which we collect in Lemma B.3, we have that the first term on the right-hand side of the inequality is bounded above by:

20
for a constant $C(q) > 0$ depending only on $q$, where the quantities $b_n$, $q$ and $\lambda_{\text{min}}$ are defined in Assumption 2.2.

To bound the second term, we invoke an anti-concentration inequality (Lemma B.4). Since the $Z_j$ are centered and unit-variance, we get:

$$
\sup_{t \in \mathbb{R}} \left| P\left( \max_{1 \leq j \leq p} \frac{\hat{\theta}_j - \theta_{0j}}{\sigma_j} \leq t \right) - P\left( \max_{1 \leq j \leq p} |Z_j| \leq t \right) \right| \leq C(q) \left\{ \frac{b_n (\log p)^{3/2} \log n}{\sqrt{n} \lambda_{\text{min}}} + \frac{b_n^2 (\log p)^2 \log n}{n^{1-2/q} \lambda_{\text{min}}} + \left[ \frac{b_n^3 (\log d)^{3q/2 - 4} \log n \log \left( \max_{1 \leq j \leq p} |\hat{\theta}_j| \right)}{n^{q/2 - 1} \lambda_{\text{min}}^{q/2}} \right]^{1/2} \right\},
$$

for a constant $C(q) > 0$ depending only on $q$, where the quantities $b_n$, $q$ and $\lambda_{\text{min}}$ are defined in Assumption 2.2.

A.2 Proof of Theorem 3.1

As in the proof of Theorem 2.3, we begin by approximating $\hat{\theta}_u$ by an “oracle” estimator. To do that, fix $u \in \mathcal{U}$ and consider a second-order Taylor expansion of the function

$$
f(r) = E_P \left[ \psi_u(W, \theta_{0u} + r(\hat{\theta}_u - \theta_{0u}), \eta_{0u} + r(\hat{\eta}_u - \eta_{0u})) \right]
$$

around $r = 0$:

$$
f(1) - f(0) = E_P \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) \right] - E_P \left[ \psi_u(W, \theta_{0u}, \eta_{0u}) \right] = J_{0u} \left[ \hat{\theta}_u - \theta_{0u} \right] + D_{0u} \left[ \hat{\eta}_u - \eta_{0u} \right] + \frac{1}{2} f''(\bar{r})
$$

for some $r \in [0, 1]$. Because $E_P \left[ \psi_u(W, \theta_{0u}, \eta_{0u}) \right] = 0$ and $D_{0u} \left[ \hat{\eta}_u - \eta_{0u} \right] = 0$ because of the Neyman-orthogonality condition, we have

$$
\sqrt{n} \left( \hat{\theta}_u - \theta_{0u} \right) = \sqrt{n} J_{0u}^{-1} E_P \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) \right] - \sqrt{n} J_{0u}^{-1/2} f''(\bar{r}).
$$

We can further expand the right hand side to obtain:

$$
\sqrt{n} (\hat{\theta}_u - \theta_{0u}) = -J_{0u}^{-1} G_{0u} [\psi_u(W, \theta_{0u}, \eta_{0u})] + A_{un} + B_{un} + C_{un},
$$

where:

(i) The first term on the right-hand side is the oracle score,

(ii) $A_{un} = \sqrt{n} J_{0u}^{-1} E_n \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) \right]$, which is related to the approximation error in $\hat{\theta}_u$,

(iii) $B_{un} = -J_{0u}^{-1} G_{0u} [\psi_u(W, \hat{\theta}_u, \hat{\eta}_u) - \psi_u(W, \theta_{0u}, \eta_{0u})]$, the empirical process controlling the deviation of $\psi_u(W, \hat{\theta}_u, \hat{\eta}_u)$ from $\psi_u(W, \theta_{0u}, \eta_{0u})$,

(iv) $C_{un} = -\sqrt{n} J_{0u}^{-1} f''(\bar{r})$ is a linearization error.
Bound on $A_{un}$ By Assumption 3.1 $|J_{0u}| \geq c_0$ for all $u \in \mathcal{U}$. Moreover, because $\hat{\theta}_u$ is an $n^{-1/2} \epsilon_n$-approximate solution to $\mathbb{E}_n[\psi(W, \theta, \hat{\eta}_u)] = 0$ for all $u \in \mathcal{U}$, we have $\sup_{u \in \mathcal{U}} |A_{un}| \leq c_0^{-1} \epsilon_n$.

Bound on $B_{un}$ We use a similar argument as before. Let

$$\mathcal{F}_B = \{ \psi_u(W, \theta, \eta) - \psi_u(W, \theta_{0u}, \eta_{0u}) : \theta - \theta_{0u} \leq R_n(\theta), \theta \in \Theta_u, \|\eta - \eta_{0u}\|_{p,2} \leq R_n(\hat{\eta}), u \in \mathcal{U} \},$$

where $R_n(\hat{\theta}) = \sup_{u \in \mathcal{U}} |\hat{\theta}_u - \theta_{0u}|$ is a preliminary rate of convergence for $\{\hat{\theta}_u\}_{u \in \mathcal{U}}$. A bound on this quantity will be obtained below.

We have that $\mathcal{F}_B \subset \mathcal{F} - \mathcal{F}$ for the class $\mathcal{F}$ defined in Assumption 3.2. By assumption, there exists an envelope function $F$ for $\mathcal{F}$, such that $F \geq \sup_{f \in \mathcal{F}} |f|$ with $\|F\|_{p,2+\delta} \leq M_n$ for some $\delta \geq 0$. Hence, the corresponding envelope $\mathcal{F}_B$ is $2F$. Moreover, the uniform covering entropy of $\mathcal{F}_B$ can be upper-bounded by Lemma B.2:

$$\log \sup_Q N(\varepsilon \|2F\|_{Q,2}, \mathcal{F}_B, \|\cdot\|_{Q,2}) \leq 2v_n \log(2a_n/\varepsilon) \quad \text{for all } 0 < \varepsilon \leq 1.$$

Moreover, by the Lipschitz-continuity condition in Assumption 3.1, we can bound

$$\sup_{f \in \mathcal{F}_B} \|f\|_{p,2} = \sup_{|\theta - \theta_{0u}| \leq R_n(\hat{\theta}), \theta \in \Theta_u, \|\eta - \eta_{0u}\|_{p,2} \leq R_n(\hat{\eta}), u \in \mathcal{U}} E_\mathcal{P} \left[ \left( \psi_u(W, \theta, \eta) - \psi_u(W, \theta_{0u}, \eta_{0u}) \right)^2 \right] \leq C_0 \sup_{|\theta - \theta_{0u}| \leq R_n(\hat{\theta}), \theta \in \Theta_u, \|\eta - \eta_{0u}\|_{p,2} \leq R_n(\hat{\eta}), u \in \mathcal{U}} (|\theta - \theta_{0u}| \lor \|\eta - \eta_{0u}\|_{p,2}) \omega$$

$$= C_0 [R_n(\hat{\theta}) \lor R_n(\hat{\eta})] \omega.$$

By Lemma B.1, we conclude that, with probability at least $1 - c/(2 \log n)$,

$$\sup_{u \in \mathcal{U}} |B_{un}| \leq c_0^{-1} K \left( 2 + \delta, \frac{c}{2} \right) \left( \sqrt{C_0 [R_n(\hat{\theta}) \lor R_n(\hat{\eta})] \omega^2} \sqrt{2v_n \log(2a_n)} \right)$$

$$+ 2v_n n^{r+\frac{1}{2}} + 2M_n \log(2a_n) \right).$$

Bound on $C_{un}$ By the smoothness condition in Assumption 3.1, we have that $\sup_{u \in \mathcal{U}} |f''(\bar{r})| \leq B_{2n} \sup_{u \in \mathcal{U}} (|\hat{\theta}_u - \theta_{0u}| \lor \|\eta - \eta_{0u}\|_{p,2})^2 \leq B_{2n} [R_n(\hat{\theta}) \lor R_n(\hat{\eta})]^2$. Hence,

$$\sup_{u \in \mathcal{U}} |C_{un}| \leq c_0^{-1} \frac{1}{2} \sqrt{n} B_{2n} [R_n(\hat{\theta}) \lor R_n(\hat{\eta})]^2.$$

Preliminary rate Here we obtain an upper bound for $R_n(\hat{\theta})$ in terms of $R_n(\hat{\eta})$ and other primitive quantities. To that end, consider now a first-order Taylor expansion of (6) around $r = 0$:

$$f(1) - f(0) = E_\mathcal{P} \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) \right] - E_\mathcal{P} \left[ \psi_u(W, \theta_{0u}, \eta_{0u}) \right]$$

$$= E_\mathcal{P} \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) - \psi_u(W, \theta_{0u}, \eta_{0u}) \right].$$
\[ = J_{fu}(\hat{\theta}_u - \theta_{0u}) + D_{fu}[\hat{\eta}_u - \eta_{0u}] \]

for some \( \bar{r} \in [0,1] \). By the smoothness conditions, \(|D_{fu}[\hat{\eta}_u - \eta_{0u}]| \leq B_{1u}\|\eta - \eta_{0u}\|_{p,2} \leq B_{1u}\mathcal{R}_n(\bar{\eta}) \) for all \( u \in U \). Moreover, \(|J_{fu}| \geq c_1 \) for any \( \bar{r} \in [0,1] \) and \( u \in U \). Re-arranging, and adding and subtracting \( \mathbb{E}_n \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) \right] \) we obtain

\[
\begin{align*}
\sup_{u \in U} |\hat{\theta}_u - \theta_{0u}| &\leq c_1^{-1} |\mathbb{E}_n \left[ \psi_u(W, \hat{\theta}_u, \hat{\eta}_u) \right]| + c_1^{-1} n^{-1/2} \sup_{u \in U} |\mathbb{E}_n \psi(W, \hat{\theta}_u, \hat{\eta}_u)| + c_1^{-1} B_{1u}\mathcal{R}_n(\bar{\eta}) \\
&\leq c_1^{-1} n^{-1/2}\epsilon_n + c_1^{-1} n^{-1/2} \sup_{\gamma \in \mathcal{F}} |\mathbb{E}_n f| + c_1^{-1} B_{1u}\mathcal{R}_n(\bar{\eta})
\end{align*}
\]

for the class \( \mathcal{F} \) defined in Assumption 3.2. By Lemma B.1, with probability at least \( 1 - \frac{c}{2(2 \log n)} \)

\[
\sup_{\gamma \in \mathcal{F}} |\mathbb{E}_n f| \leq K \left( 2 + \delta, \frac{C}{2} \right) \left( C_0 \sqrt{v_n \log(a_n)} + v_n n^{\frac{1}{2} + \frac{1}{2} M_n \log(a_n)} \right).
\]

Below, we denote:

\[
\begin{align*}
\mathcal{R}_n^{\gamma}(\bar{\eta}) &= \left\{ c_1^{-1} n^{-1/2}\epsilon_n + c_1^{-1} n^{-1/2}K \left( 2 + \delta, \frac{C}{2} \right) \left( C_0 \sqrt{v_n \log(a_n)} + v_n n^{\frac{1}{2} + \frac{1}{2} M_n \log(a_n)} \right) + c_1^{-1} B_{1u}\mathcal{R}_n(\bar{\eta}) \right\} \vee \mathcal{R}_n(\bar{\eta}). \\
\Delta_{1n} &= C_0^{-1} K \left( 2 + \delta, \frac{C}{2} \right) \left( \sqrt{C_0[\mathcal{R}^{\gamma}(\bar{\eta})]^{\omega/2}} \sqrt{2v_n \log(2a_n)} + 2v_n n^{\frac{1}{2} + \frac{1}{2} M_n \log(2a_n)} \right) \\
\Delta_{2n} &= C_0^{-1} \frac{1}{2} \sqrt{n}B_{2u}[\mathcal{R}^{\gamma}(\bar{\eta})]^{2}.
\end{align*}
\]

**Normal Approximation** To obtain bounds on the Kolmogorov distance between the finite-sample distribution of \( \{\hat{\theta}_u\}_{u \in U} \) and that of the corresponding Gaussian process \( \{Z_u\} \), we will use the results of Chernozhukov et al. (2014b), combining Lemma B.5 and Lemma B.6.

Let \( Z = \sup_{u \in U} \sqrt{n}\mathbb{E}_n^{-1}(\hat{\theta}_u - \theta_{0u}), \bar{Z} = \sup_{u \in U} \mathbb{E}_n[\bar{\psi}_{0u}], \) where \( \bar{\psi}_{0u}(W) = -(\sigma_u\mathbb{E}_n^{-1})^{-1}\psi_u(W, \theta_{0u}, \eta_{0u}) \) is the re-scaled score at the true values \( \{\theta_{0u}, \eta_{0u}\} \), and \( \bar{Z} = \sup_{u \in U} \mathbb{E}_n[\psi_{0u}] \), for a tight mean-zero Gaussian process \( G_P \) with covariance function \( \mathbb{E}_n[\psi_{0u}]G_P[\psi_{0u}] = G_P[\bar{\psi}_{0u}] \). Notice that, by construction, \( \mathbb{E}_P[\bar{\psi}_{0u}(W)^2] = 1 \) for all \( u \in U \).

By Lemma B.5 and Assumption 3.3, we have, for any \( \gamma \in (0,1) \), \( \mathbb{P}_P\left( |Z - \bar{Z}| > \Delta_{3n} \right) \leq D(q) \left( \gamma + \log n/n \right) \)

\[
\begin{align*}
\Delta_{3n} &= \frac{b_n L_n}{\gamma^{1/2}n^{1/2 - 1/q}} + \frac{(b_n)^{1/2}L_{n}^{3/4}}{\gamma^{1/2}n^{1/4}} + \frac{(b_n L_{n}^{1/2})^{1/3}}{\gamma^{1/3}n^{1/6}},
\end{align*}
\]

\( L_n = d(q)\mathbb{E}_n[\log n + \log(\mathbb{E}_n[b_n])] \) and \( d(q), D(q) \) are constants that depend only on \( q \).

Combining that with the previous bounds, and using the triangle inequality, we have

\[
\mathbb{P}_P\left( |Z - \bar{Z}| > c_0^{-1}\epsilon_n + \Delta_{1n} + \Delta_{2n} + \Delta_{3n} \right) \leq D(q) \left( \gamma + \log n/n \right) + c/\log n.
\]

Finally, Dudley’s Theorem (see, e.g., Corollary 2.2.8 in Vaart and Wellner (1996)) im-
plies that, for our class of functions,
\[ E[\tilde{Z}] \leq \sqrt{V_n \log(A_n b_n)} \]
for an absolute constant \( \chi > 0 \).

By Lemma B.6,
\[
\sup_{t} \left| P_{\tilde{Z}} (Z \leq t) - P_{\tilde{Z}} (\tilde{Z} \leq t) \right| \leq \kappa r_{1n} \left( \chi \sqrt{V_n \log(A_n b_n)} + \sqrt{1 \log(1/r_{1n})} \right) + r_{2n},
\]
for an absolute constant \( \kappa > 0 \), where

\[
r_{1n} = c_0^{-1} \epsilon_n + \Delta_1 + \Delta_2 + \Delta_3 \quad \text{and} \quad r_{2n} = D(q) (\gamma + (\log n)/n) + c/\log n.
\]

### Appendix B  Auxiliary Results

**Lemma B.1** (Maximal inequality, Lemma M.2 of Belloni et al., 2018). Let \( F \) be a set of suitably measurable functions \( f : W \to \mathbb{R} \), equipped with a measurable envelope \( F : W \to \mathbb{R} \), \( F \geq \sup_{f \in F} |f| \), such that \( \|F\|_{P,q} \leq M < \infty \) for some \( q \geq 2 \). Let \( \sigma^2 \) be any positive constant such that \( \sup_{f \in F} \|f\|_{P,2} \leq \sigma^2 \leq \|F\|_{P,2} \). Suppose that there exist constants \( a \geq e \) and \( v \geq 1 \) such that:

\[
\log \sup_{Q} N(\epsilon \|F\|_{Q,2}, F, \| \cdot \|_{Q,2}) \leq v \log(a/\epsilon), \quad \text{for all } 0 < \epsilon \leq 1.
\]

Then, with probability at least \( 1 - c/\log n \),
\[
\sup_{f \in F} \left| G_{n} f \right| \leq K(q,c) \left( \sigma \sqrt{v \log(a)} + v n^{1/2} M \log(a) \right),
\]
where \( K(q,c) > 0 \) is a constant depending only on \( q \) and \( c \).

**Lemma B.2** (Algebra for covering entropies, Lemma L.1 of Belloni et al., 2018). For any measurable classes of functions \( F \) and \( F' \) mapping \( W \) to \( \mathbb{R} \),

\[
\log N(\epsilon \|F + F'\|_{Q,2}, F + F', \| \cdot \|_{Q,2}) \leq \log N(\frac{\epsilon}{2} \|F\|_{Q,2}, F, \| \cdot \|_{Q,2}) + \log N(\frac{\epsilon}{2} \|F'\|_{Q,2}, F', \| \cdot \|_{Q,2}).
\]

**Lemma B.3** (High-dimensional Gaussian approximation, Corollary 2.1 of Chernozhukov et al., 2021a). Let \( X_1, \ldots, X_n \) be a sequence of centered independent random vectors in \( \mathbb{R}^p \), and \( W = n^{-1/2} \sum_{i=1}^{n} X_i \), where \( E[WW'] = \Sigma \) with unit diagonal entries. Let \( \lambda_{\min} > 0 \) denote the smallest eigenvalue of \( \Sigma \). Consider the class \( A \) of hyperrectangles in \( \mathbb{R}^p \), i.e., sets of the form \( \times_{j=1}^{p} (a_j, b_j) \) for some \( -\infty \leq a_j \leq b_j \leq \infty \), \( j = 1, \ldots, p \). Under the conditions below, we provide a bound on:

\[
g = \sup_{A \in A} |P(W \in A) - P(Z \in A)|, \quad Z \sim \mathcal{N}(0, \Sigma).
\]

Define three alternative conditions (from most to least restrictive):
\((E.1)\) \(|X_{ij}| \leq b_n\) for all \(i = 1, \ldots, n\) and \(j = 1, \ldots, p\) almost surely.

\((E.2)\) \(\|X_{ij}\|_{\psi_2} \leq b_n\) for all \(i = 1, \ldots, n\) and \(j = 1, \ldots, p\), where \(\|X\|_{\psi_2} = \inf \{C > 0 : \mathbb{E} [\exp(X^2/C^2)] \leq 2\}\).

\((E.3)\) \(\max_{1 \leq j \leq p} |X_{ij}| \|_{P,q} \leq b_n\) for all \(i = 1, \ldots, n\) and some \(q \geq 4\).

Define also:

\((M)\) \(n^{-1} \sum_{i=1}^{n} \mathbb{E} [X_{ij}^4] \leq b_n^2\) for all \(j = 1, \ldots, p\).

Then:

(i) Under \((E.1)\), we have:

\[ \varrho \leq \frac{C b_n (\log p)^{3/2} \log n}{\sqrt{n \lambda_{\min}}} \]

for an absolute constant \(C > 0\).

(ii) Under \((E.2)\) and \((M)\), we have:

\[ \varrho \leq C \left\{ \frac{b_n (\log p)^{3/2} \log n}{\sqrt{n \lambda_{\min}}} + \frac{b_n^2 (\log p)^2}{\sqrt{n \lambda_{\min}}} \right\} \]

for an absolute constant \(C > 0\).

(iii) Under \((E.3)\) and \((M)\), we have:

\[ \varrho \leq C \left\{ \frac{b_n (\log p)^{3/2} \log n}{\sqrt{n \lambda_{\min}}} + \frac{b_n^2 (\log p)^2 \log n \log (pn)}{n^{1-2/q} \lambda_{\min}^{q/2}} \right\} \]

for a constant \(C(q) > 0\) depending only on \(q\).

**Lemma B.4** (Anti-concentration inequality, Theorem 3 and Corollary 1 of Chernozhukov et al., 2014a). Let \((X_1, \ldots, X_p)\) be a centered Gaussian random vector, with \(\sigma = \mathbb{E} [X_j^2]\) for all \(j = 1, \ldots, p\). Then, for all \(\epsilon > 0\),

\[ \sup_{x \in \mathbb{R}} \mathbb{P} (\max_{1 \leq j \leq p} X_j \in [x - \epsilon, x + \epsilon]) \leq 12 \epsilon \sqrt{\log p / \sigma}. \]

**Lemma B.5** (Gaussian approximation to suprema of empirical processes, Corollary 2.2 of Chernozhukov et al., 2014b). Let \(\mathcal{F}\) be a set of suitably measurable functions \(f : W \to \mathbb{R}\), equipped with a measurable envelope \(F : W \to \mathbb{R}\), \(F \geq \sup_{f \in \mathcal{F}} |f|\). Let \(\sigma^2\) be any positive constant such that \(\sup_{f \in \mathcal{F}} \|f\|_{L^2} \leq \sigma^2 \leq \|F\|_{L^2}^2\). Assume that, for some \(b \geq \sigma\) and \(q \geq 4\), we have \(\sup_{f \in \mathcal{F}} \mathbb{E}_{\mathcal{P}}[|f|^k] \leq \sigma^2 b^{k-2}\) for \(k = 2, 3, 4\), and \(\|F\|_{p,q} \leq b\). Suppose that there exist constants \(a \geq e\) and \(v \geq 1\) such that:

\[ \log \sup_{Q} N(\varepsilon \|F\|_{Q,2}, \mathcal{F}, \| \cdot \|_{Q,2}) \leq v \log (a/\varepsilon), \quad \text{for all } 0 < \varepsilon \leq 1. \]
Let $Z = \sup_{f \in \mathcal{F}} G_n f$. Then, for every $\gamma \in (0, 1)$ there exists a random variable $\tilde{Z} \overset{d}{=} \sup_{f \in \mathcal{F}} G_P f$ such that

$$P_P\left( |Z - \tilde{Z}| > \frac{b L_n}{\gamma^{1/2} n^{1/2 - 1/4}} + \frac{(b \sigma)^{1/2} L_n^{3/4}}{\gamma^{1/2} n^{1/4}} + \frac{(b \sigma^2 L_n^2)^{1/3}}{\gamma^{1/3} n^{1/6}} \right) \leq C \left( \gamma + \frac{\log n}{n} \right),$$

where $L_n = B v (\log n \vee \log(ab/\sigma))$ and $B, C$ are constants that depend only on $q$. Here, $G_P$ denotes a tight mean-zero Gaussian process with covariance function $E [G_P(f)G_P(g)] = E[f(W)g(W)]$ for all $f, g \in \mathcal{F}$.

Lemma B.6 (Gaussian approximation to suprema of empirical processes, Kolmogorov distance version, Lemma 2.3 of Chernozhukov et al., 2014b). Under the same conditions as the previous lemma, suppose that there exist constants such that $c_0 \leq \|f\|_{P,2} \leq C_0$ for all $f \in \mathcal{F}$. For a random variable $Z$, suppose that there exist constants $r_1, r_2 > 0$ and $\tilde{Z} \overset{d}{=} \sup_{f \in \mathcal{F}} G_P f$ such that $P_P\left( |Z - \tilde{Z}| > r_1 \right) \leq r_2$. Then,

$$\sup_t \left| P_P(Z \leq t) - P(\tilde{Z} \leq t) \right| \leq \kappa r_1 \left( E[\tilde{Z}] + \sqrt{1 \vee \log(c_0/r_1)} \right) + r_2,$$

where $\kappa$ is a constant depending only on $c_0$ and $C_0$.