Minimizing the mean projections of finite $\rho$-separable packings

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Abstract

A packing of translates of a convex body in the $d$-dimensional Euclidean space $E^d$ is said to be totally separable if any two packing elements can be separated by a hyperplane of $E^d$ disjoint from the interior of every packing element. We call the packing $P$ of translates of a centrally symmetric convex body $C$ in $E^d$ a $\rho$-separable packing for given $\rho \geq 1$ if in every ball concentric to a packing element of $P$ having radius $\rho$ (measured in the norm generated by $C$) the corresponding sub-packing of $P$ is totally separable.

The main result of this paper is the following theorem. Consider the convex hull $Q$ of $n$ non-overlapping translates of an arbitrary centrally symmetric convex body $C$ forming a $\rho$-separable packing in $E^d$ with $n$ being sufficiently large for given $\rho \geq 1$. If $Q$ has minimal mean $i$-dimensional projection for given $i$ with $1 \leq i < d$, then $Q$ is approximately a $d$-dimensional ball. This extends a theorem of K. Böröczky Jr. [Monatsh. Math. 118 (1994), 41–54] from translative packings to $\rho$-separable translative packings for $\rho \geq 1$.

1 Introduction

We denote the $d$-dimensional Euclidean space by $E^d$. Let $B^d$ denote the unit ball centered at the origin $o$ in $E^d$. A $d$-dimensional convex body $C$ is a compact convex subset of $E^d$ with non-empty interior $\text{int} C$. (If $d = 2$, then $C$ is said to be a convex domain.) If $C = -C$, where $-C = \{-x : x \in C\}$, then $C$ is said to be o-symmetric and a translate $c + C$ of $C$ is called centrally symmetric with center $c$.

The starting point as well as the main motivation for writing this paper is the following elegant theorem of Böröczky Jr. \cite{Bo94}: Consider the convex hull $Q$ of $n$ non-overlapping translates of an arbitrary centrally symmetric convex body $C$ in $E^d$ with $n$ being sufficiently large. If $Q$ has minimal mean $i$-dimensional projection for given $i$ with $1 \leq i < d$, then $Q$ is approximately a $d$-dimensional ball. In this paper, our main goal is to prove an extension of this theorem to $\rho$-separable translative packings of convex bodies in $E^d$. Next, we define the concept of $\rho$-separable translative packings and then state our main result.

A packing of translates of a convex domain $C$ in $E^2$ is said to be totally separable if any two packing elements can be separated by a line of $E^2$ disjoint from the interior of every packing element. This notion was introduced by G. Fejes Tóth and L. Fejes Tóth \cite{FT}. We can define a totally separable packing of translates of a $d$-dimensional convex body $C$ in a similar way by requiring any two packing elements to be separated by a hyperplane in $E^d$ disjoint from the interior of every packing element \cite{Bo94,FT}.

Definition 1. Let $C$ be an o-symmetric convex body of $E^d$. Furthermore, let $\|\cdot\|_C$ denote the norm generated by $C$, i.e., let $\|x\|_C := \inf\{\lambda : x \in \lambda C\}$ for any $x \in E^d$. Now, let $\rho \geq 1$. We say that the packing

$P_{\text{sep}} := \{c_i + C \mid i \in I \text{ with } \|c_j - c_k\|_C \geq 2 \text{ for all } j \neq k \in I\}$

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of (finitely or infinitely many) non-overlapping translates of \( C \) with centers \( \{ c_i \mid i \in I \} \) is a \( \rho \)-separable packing in \( \mathbb{E}^d \) if for each \( i \in I \) the finite packing \( \{ c_i + C \mid c_i + C \subseteq c_i + \rho C \} \) is a totally separable packing (in \( c_i + \rho C \)). Finally, let \( \delta_{\text{sep}}(\rho, C) \) denote the largest density of all \( \rho \)-separable translative packings of \( C \) in \( \mathbb{E}^d \), i.e., let
\[
\delta_{\text{sep}}(\rho, C) := \sup_{\mathcal{P}_{\text{sep}}} \left( \limsup_{\lambda \to +\infty} \frac{\sum_{c_i + C \subseteq \mathcal{W}_\lambda^d} \text{vol}_d(c_i + C)}{\text{vol}_d(\mathcal{W}_\lambda^d)} \right),
\]
where \( \mathcal{W}_\lambda^d \) denotes the \( d \)-dimensional cube of edge length \( 2\lambda \) centered at \( o \) in \( \mathbb{E}^d \) having edges parallel to the coordinate axes of \( \mathbb{E}^d \) and \( \text{vol}_d(\cdot) \) refers to the \( d \)-dimensional volume of the corresponding set in \( \mathbb{E}^d \).

**Remark 1.** Let \( \delta(C) \) (resp., \( \delta_{\text{sep}}(C) \)) denote the supremum of the upper densities of all translative packings (resp., totally separable translative packings) of the \( o \)-symmetric convex body \( C \) in \( \mathbb{E}^d \). Clearly, \( \delta_{\text{sep}}(C) \leq \delta_{\text{sep}}(\rho, C) \leq \delta(C) \) for all \( \rho \geq 1 \). Furthermore, if \( 1 \leq \rho < 3 \), then any \( \rho \)-separable translative packing of \( C \) in \( \mathbb{E}^d \) is simply a translative packing of \( C \) and therefore, \( \delta_{\text{sep}}(\rho, C) = \delta(C) \).

Recall that the mean \( i \)-dimensional projection \( M_i(C) \) \((i = 1, 2, \ldots, d - 1)\) of the convex body \( C \) in \( \mathbb{E}^d \), can be expressed (12) with the help of mixed volume via the formula
\[
M_i(C) = \frac{\kappa_i}{\kappa_d} V(C, \ldots, C, B^d, \ldots, B^d),
\]
where \( \kappa_d \) is the volume of \( B^d \) in \( \mathbb{E}^d \). Note that \( M_i(B^d) = \kappa_i \), and the surface area of \( C \) is \( S(C) = \frac{d \kappa_{d-1}}{\kappa_d} M_{d-1}(C) \) and in particular, \( S(B^d) = d \kappa_d \). Set \( M_d(C) := \text{vol}_d(C) \). Finally, let \( R(C) \) (resp., \( r(C) \)) denote the circumradius (resp., inradius) of the convex body \( C \) in \( \mathbb{E}^d \), which is the radius of the smallest (resp., largest) ball that contains (resp., is contained in) \( C \). Our main result is the following.

**Theorem 1.** Let \( d \geq 2 \), \( 1 \leq i \leq d - 1 \), \( \rho \geq 1 \), and let \( Q \) be the convex hull of the \( \rho \)-separable packing of \( n \) translates of the \( o \)-symmetric convex body \( C \) in \( \mathbb{E}^d \) such that \( M_i(Q) \) is minimal and \( n \geq \frac{d \rho^d d^d}{\delta_{\text{sep}}(\rho, C) \pi^{d/2}} \cdot \left( \frac{\rho R(C)}{r(C)} \right)^d \). Then
\[
\frac{r(Q)}{R(Q)} \geq 1 - \frac{\omega}{n^{\frac{d}{d+2}}},
\]
for \( \omega = \lambda(d) \left( \frac{\rho R(C)}{r(C)} \right)^{\frac{d}{d+2}} \), where \( \lambda(d) \) depends only on the dimension \( d \). In addition,
\[
M_i(Q) = \left( 1 + \frac{\sigma}{n^{\frac{d}{d+2}}} \right) M_i(B^d) \left( \frac{\text{vol}_d(C)}{\delta_{\text{sep}}(\rho, C) \kappa_d} \right)^{\frac{d}{2}} \cdot n^{\frac{d}{2}},
\]
where
\[
- \frac{2.25 R(C)^{\rho d}}{\pi^{d/2} \delta_{\text{sep}}(\rho, C)} \leq \sigma \leq \frac{2.1 R(C)^{\rho d}}{\pi^{d/2} \delta_{\text{sep}}(\rho, C)}.
\]

**Remark 2.** It is worth restating Theorem 1 as follows: Consider the convex hull \( Q \) of \( n \) non-overlapping translates of an arbitrary \( o \)-symmetric convex body \( C \) forming a \( \rho \)-separable packing in \( \mathbb{E}^d \) with \( n \) being sufficiently large. If \( Q \) has minimal mean \( i \)-dimensional projection for given \( i \) with \( 1 \leq i < d \), then \( Q \) is approximately a \( d \)-dimensional ball.

**Remark 3.** The nature of the analogue question on minimizing \( M_d(Q) = \text{vol}_d(Q) \) is very different. Namely, recall that Betke and Henk [4] proved L. Fejes Tóth’s sausage conjecture for \( d \geq 42 \) according to which the smallest volume of the convex hull of \( n \) non-overlapping unit balls in \( \mathbb{E}^d \) is obtained when the \( n \) unit balls form a sausage, that is, a linear packing (see also [2] and [3]). As linear packings of unit balls are \( \rho \)-separable therefore the above theorem of Betke and Henk applies to \( \rho \)-separable packings of unit balls in \( \mathbb{E}^d \) for all \( \rho \geq 1 \) and \( d \geq 42 \). On the other hand, the problem of minimizing the volume of the convex hull of \( n \) unit balls forming a \( \rho \)-separable packing in \( \mathbb{E}^d \) remains an interesting open problem for \( \rho \geq 1 \) and \( 2 \leq d < 42 \). Last but not least, the problem of minimizing \( M_d(Q) \) for \( o \)-symmetric convex bodies \( C \) different from a ball in \( \mathbb{E}^d \) seems to be wide open for \( \rho \geq 1 \) and \( d \geq 2 \).
Remark 4. Let \( d \geq 2 \), \( 1 \leq i \leq d - 1 \), \( n > 1 \), and let \( C \) be a given \( o \)-symmetric convex body in \( \mathbb{E}^d \). Furthermore, let \( Q \) be the convex hull of the totally separable packing of \( n \) translates of \( C \) in \( \mathbb{E}^d \) such that \( M_i(Q) \) is minimal. Then it is natural to ask for the limit shape of \( Q \) as \( n \to +\infty \), that is, to ask for an analogue of Theorem 4 within the family of totally separable translative packings of \( C \) in \( \mathbb{E}^d \). This would require some new ideas besides the ones used in the following proof of Theorem 7.

In the rest of the paper by adopting the method of Böröczky Jr. \( 8 \) and making the necessary modifications, we give a proof of Theorem 7.

2 Basic properties of finite \( \rho \)-separable translative packings

The following statement is the \( \rho \)-separable analogue of the Lemma in 5 (see also Theorem 3.1 in 2).

Lemma 1. Let \( \{c_i + C \mid 1 \leq i \leq n\} \) be an arbitrary \( \rho \)-separable packing of \( n \) translates of the \( o \)-symmetric convex body \( C \) in \( \mathbb{E}^d \) with \( \rho \geq 1 \), \( n \geq 1 \), and \( d \geq 2 \). Then

\[
\frac{n\text{vol}_d(C)}{\text{vol}_d(\cup_{i=1}^n c_i + 2\rho C)} \leq \delta_{\text{sep}}(\rho, C) .
\]

Proof. We use the method of the proof of the Lemma in 5 (resp., Theorem 3.1 in 2) with proper modifications. The details are as follows. Assume that the claim is not true. Then there is an \( \epsilon > 0 \) such that

\[
\text{vol}_d(\cup_{i=1}^n c_i + 2\rho C) = \frac{n\text{vol}_d(C)}{\delta_{\text{sep}}(\rho, C)} - \epsilon \quad (2)
\]

Let \( C_n = \{c_i \mid i = 1, \ldots, n\} \) and let \( \Lambda \) be a packing lattice of \( C_n + 2\rho C = \cup_{i=1}^n c_i + 2\rho C \) such that \( C_n + 2\rho C \) is contained in a fundamental parallelootope of \( \Lambda \) say, in \( P \), which is symmetric about the origin. Recall that for each \( \lambda > 0 \), \( W^d_\lambda \) denotes the \( d \)-dimensional cube of edge length \( 2\lambda \) centered at the origin \( o \) in \( \mathbb{E}^d \) having edges parallel to the coordinate axes of \( \mathbb{E}^d \). Clearly, there is a constant \( \mu > 0 \) depending on \( P \) only, such that for each \( \lambda > 0 \) there is a subset \( L_\lambda \) of \( \Lambda \) with

\[
W^d_\lambda \subseteq L_\lambda + P \quad \text{and} \quad L_\lambda + 2P \subseteq W^d_{\lambda+\mu} .
\]

The definition of \( \delta_{\text{sep}}(\rho, C) \) implies that for each \( \lambda > 0 \) there exists a \( \rho \)-separable packing of \( m(\lambda) \) translates of \( C \) in \( \mathbb{E}^d \) with centers at the points of \( C(\lambda) \) such that

\[
C(\lambda) + C \subset W^d_\lambda
\]

and

\[
\lim_{\lambda \to +\infty} \frac{m(\lambda)\text{vol}_d(C)}{\text{vol}_d(W^d_{\lambda})} = \delta_{\text{sep}}(\rho, C) .
\]

As \( \lim_{\lambda \to +\infty} \frac{\text{vol}_d(W^d_{\lambda+\mu})}{\text{vol}_d(W^d_{\lambda})} = 1 \) therefore there exist \( \xi > 0 \) and a \( \rho \)-separable packing of \( m(\xi) \) translates of \( C \) in \( \mathbb{E}^d \) with centers at the points of \( C(\xi) \) and with \( C(\xi) + C \subset W^d_\xi \) such that

\[
\frac{\text{vol}_d(P)\delta_{\text{sep}}(\rho, C)}{\text{vol}_d(P) + \epsilon} < \frac{m(\xi)\text{vol}_d(C)}{\text{vol}_d(W^d_{\xi+\mu})} \quad \text{and} \quad \frac{n\text{vol}_d(C)\text{card}(L_{\xi})}{\text{vol}_d(W^d_{\xi+\mu})} ,
\]

where \( \text{card}(\cdot) \) refers to the cardinality of the given set. Now, for each \( x \in P \) we define a \( \rho \)-separable packing of \( m(x) \) translates of \( C \) in \( \mathbb{E}^d \) with centers at the points of

\[
\mathcal{C}(x) := \{x + L_\xi + C_n\} \cup \{y \in C(\xi) \mid y \notin x + L_\xi + C_n + \text{int}(2\rho C)\} .
\]

Clearly, \( d \) implies that \( \mathcal{C}(x) + C \subset W^d_{\xi+\mu} \). Now, in order to evaluate \( \int_{x \in P} m(x) dx \), we introduce the function \( \chi_y \) for each \( y \in C(\xi) \) defined as follows: \( \chi_y(x) = 1 \) if \( y \notin x + L_\xi + C_n + \text{int}(2\rho C) \) and \( \chi_y(x) = 0 \).
for any other \( x \in P \). Based on the origin symmetric \( P \) it is easy to see that for any \( y \in C(\xi) \) one has
\[
\int_{x \in P} \chi_y(x)dx = \text{vol}_d(P) - \text{vol}_d(C_n + 2\rho C).
\]
Thus, it follows in a straightforward way that
\[
\int_{x \in P} \overline{m}(x)dx = \int_{x \in P} \left( n\text{card}(L_\xi) + \sum_{y \in C(\xi)} \chi_y(x) \right)dx = n\text{vol}_d(P)\text{card}(L_\xi) + m(\xi)\left( \text{vol}_d(P) - \text{vol}_d(C_n + 2\rho C) \right).
\]
Hence, there is a point \( p \in P \) with
\[
\overline{m}(p) \geq m(\xi) \left( 1 - \frac{\text{vol}_d(C_n + 2\rho C)}{\text{vol}_d(P)} \right) + n\text{card}(L_\xi)
\]
and so
\[
\frac{\overline{m}(p)\text{vol}_d(C)}{\text{vol}_d(W^d_{\xi+\mu})} \geq \frac{m(\xi)\text{vol}_d(C)}{\text{vol}_d(W^d_{\xi+\mu})} \left( 1 - \frac{\text{vol}_d(C_n + 2\rho C)}{\text{vol}_d(P)} \right) + \frac{n\text{vol}_d(C)\text{card}(L_\xi)}{\text{vol}_d(W^d_{\xi+\mu})}.
\] (5)
Now, (2) implies in a straightforward way that
\[
\frac{\text{vol}_d(P)\delta_{\text{sep}}(\rho, C)}{\text{vol}_d(P)} + \epsilon \left( 1 - \frac{\text{vol}_d(C_n + 2\rho C)}{\text{vol}_d(P)} \right) + \frac{n\text{vol}_d(C)}{\text{vol}_d(P)} + \epsilon = \delta_{\text{sep}}(\rho, C).
\] (6)
Thus, (4), (5), and (6) yield that
\[
\frac{\overline{m}(p)\text{vol}_d(C)}{\text{vol}_d(W^d_{\xi+\mu})} > \delta_{\text{sep}}(\rho, C).
\]
As \( \overline{C}(p) + C \subset W^d_{\xi+\mu} \) this contradicts the definition of \( \delta_{\text{sep}}(\rho, C) \), finishing the proof of Lemma 1.

**Definition 2.** Let \( d \geq 2 \), \( \rho \geq 1 \), and let \( K \) (resp., \( C \)) be a convex body (resp., an \( o \)-symmetric convex body) in \( \mathbb{E}^d \). Then let \( \nu_C(\rho, K) \) denote the largest \( n \) with the property that there exists a \( \rho \)-separable packing \( \{ c_i + C \mid 1 \leq i \leq n \} \) such that \( \{ c_i \mid 1 \leq i \leq n \} \subset K \).

**Lemma 2.** Let \( d \geq 2 \), \( \rho \geq 1 \), and let \( K \) (resp., \( C \)) be a convex body (resp., an \( o \)-symmetric convex body) in \( \mathbb{E}^d \). Then
\[
\left(1 + \frac{2\rho R(C)}{r(K)}\right)^{-d} \frac{\text{vol}_d(C)\nu_C(\rho, K)}{\delta_{\text{sep}}(\rho, C)} \leq \text{vol}_d(K) \leq \frac{\text{vol}_d(C)\nu_C(\rho, K)}{\delta_{\text{sep}}(\rho, C)}.
\]

**Proof.** Observe that Lemma 1 and the containments \( K + 2\rho C \subset \left( 1 + \frac{2\rho R(C)}{r(K)} \right) K \) yield the lower bound immediately.

We prove the upper bound. Let \( 0 < \varepsilon < \delta_{\text{sep}}(\rho, C) \). By the definition of \( \delta_{\text{sep}}(\rho, C) \), if \( \lambda \) is sufficiently large, then there is a \( \rho \)-separable packing \( \{ c_i + C \mid 1 \leq i \leq n \} \) such that \( C_n := \{ c_i \mid 1 \leq i \leq n \} \subset W^d_{\lambda} \) and
\[
\frac{n\text{vol}_d(C)}{\text{vol}_d(W^d_{\lambda})} \geq \delta_{\text{sep}}(\rho, C) - \varepsilon.
\] (7)

**Sublemma 1.** If \( X \) and \( Y \) are convex bodies in \( \mathbb{E}^d \) and \( C \) is an \( o \)-symmetric convex body in \( \mathbb{E}^d \), then
\[
\nu_C(\rho, Y) \geq \frac{\text{vol}_d(Y)\nu_C(\rho, X)}{\text{vol}_d(X - Y)}.
\] (8)

**Proof.** Indeed, consider any finite point set \( X := \{ x_1, \ldots, x_N \} \subset X \). Observe that the following are equivalent for a positive integer \( k \):

- \( k \) is the maximum number a point of \( X - Y \) is covered by the sets \( x_i - Y, x_i \in X \),
- \( k \) is the maximum number such that \( \text{card}((z+Y) \cap X) = k \) for some point \( z \in X - Y \).

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Thus, $N \text{vol}_d(Y) \leq \text{card}((z+Y) \cap X) \text{vol}_d(X-Y)$ for some $z \in X-Y$. Hence, if \{x_i + C \mid 1 \leq i \leq N\} is an arbitrary $\rho$-separable packing with $X = \{x_1, \ldots, x_N\} \subset X$, then

$$\nu_C(\rho, Y) \geq \text{card}((z+Y) \cap X) \geq \frac{\text{vol}_d(Y)N}{\text{vol}_d(X-Y)},$$

which implies (8).

Applying (8) to $X = W_\lambda$ and $Y = -K$ and using (7), we obtain

$$\nu_C(\rho, K) \geq \frac{n \text{vol}_d(K)}{\text{vol}_d(W_\lambda + K)} \geq \frac{\text{vol}_d(K)}{\text{vol}_d(W_\lambda + \kappa d)} \cdot \frac{\text{vol}_d(W_\lambda)}{\text{vol}_d(C)},$$

which finishes the proof of Lemma 2.

**Definition 3.** Let $d \geq 2$, $n \geq 1$, $\rho \geq 1$, and let $C$ be an $o$-symmetric convex body in $\mathbb{E}_d$. Then let $R_C(\rho, n)$ be the smallest radius $R > 0$ with the property that $\nu_C(\rho, R B^d) \geq n$.

Clearly, for any $\varepsilon > 0$ we have $\nu_C(\rho, (R_C(\rho, n) - \varepsilon) B^d) < n$, and thus, by Lemma 2 (for $K = R_C(\rho, n) B^d$), we obtain

**Corollary 1.** Let $d \geq 2$, $n \geq 1$, $\rho \geq 1$, and let $C$ be an $o$-symmetric convex body in $\mathbb{E}_d$. Then

$$R_C(\rho, n)^d \leq \frac{\text{vol}_d(C)n}{\delta_{\text{sep}}(\rho, C)\kappa_d} \leq (R_C(\rho, n) + 2\rho R(C))^d. \tag{9}$$

**Lemma 3.** Let $n \geq 4^d \delta_{\text{sep}}(\rho, C)^d (\rho R(C))^d$ and $i = 1, 2, \ldots, d-1$. Then for $R = R_C(\rho, n)$,

$$M_i((R + \rho R(C)) B^d) \leq M_i(B^d) \left(\frac{\text{vol}_d(C)n}{\delta_{\text{sep}}(\rho, C)\kappa_d}\right)^{\frac{i}{n}} \left(1 + \frac{2\delta_{\text{sep}}(\rho, C)^{\frac{1}{2}} \rho R(C)}{r(C)} \cdot \frac{1}{n^\frac{1}{2}}\right)^i.$$

**Proof.** Set $t = R + 2\rho R(C)$. Then the first inequality in (9) yields that

$$R + \rho R(C) \leq \frac{t - \rho R(C)}{t - 2\rho R(C)} \left(\frac{\text{vol}_d(C)n}{\delta_{\text{sep}}(\rho, C)\kappa_d}\right)^{\frac{1}{n}}.$$

Thus, by the second inequality in (9) and the condition that $n \geq 4^d \delta_{\text{sep}}(\rho, C)^d (\rho R(C))^d$, we obtain that

$$\frac{t - \rho R(C)}{t - 2\rho R(C)} = 1 + \left(\frac{t}{\rho R(C)} - 2\right)^{-1} \leq 1 + \frac{2\delta_{\text{sep}}(\rho, C)^{\frac{1}{2}} \rho R(C)\kappa_d}{\text{vol}_d(C)} \cdot \frac{1}{n^\frac{1}{2}} \leq 1 + \frac{2\delta_{\text{sep}}(\rho, C)^{\frac{1}{2}} \rho R(C)}{r(C)} \cdot \frac{1}{n^\frac{1}{2}}.$$

\[\square\]

### 3 Proof of Theorem [1]

In the proof that follows we are going to use the following special case of the Alexandrov-Fenchel inequality ([13]): if $K$ is a convex body in $\mathbb{E}_d$ satisfying $M_i(K) \leq M_i(r B^d)$ for given $1 \leq i < d$ and $r > 0$, then

$$M_j(K) \leq M_j(r B^d) \tag{10}$$

holds for all $j$ with $i < j \leq d$. In particular, this statement for $j = d$ can be restated as follows: if $K$ is a convex body in $\mathbb{E}_d$ satisfying $M_d(K) = M_d(r B^d)$ for given $d \geq 2$ and $r > 0$, then $M_i(K) \geq M_i(r B^d)$ holds for all $i$ with $1 \leq i < d$. 


Let \( d \geq 2, 1 \leq i \leq d-1, \rho \geq 1, \) and let \( Q \) be the convex hull of the \( \rho \)-separable packing of \( n \) translates of the \( o \)-symmetric convex body \( C \) in \( \mathbb{E}^d \) such that \( M_i(Q) \) is minimal and

\[
n \geq \frac{4^d d^{4d}}{\delta_{sep}(\rho, C)^{d-1}} \left( \frac{\rho R(C)}{r(C)} \right)^d .
\]

(11)

By the minimality of \( M_i(Q) \) we have that

\[
M_i(Q) \leq M_i(RB^d + C) \leq M_i((R + \rho R(C))B^d)
\]

(12)

with \( R = R_C(\rho, n) \). Note that \([12]\) and Lemma 3 imply that

\[
M_i(Q) \leq \left( 1 + \frac{2\delta_{sep}(\rho, C)^\frac{\rho}{2} \rho R(C)}{r(C)} \cdot \frac{1}{n^{\frac{d-1}{2}}} \right)^i \left( \frac{\text{vol}_d(C)}{\delta_{sep}(\rho, C) \kappa_d} \right)^\frac{\rho}{2} \cdot n^\frac{\rho}{2} .
\]

(13)

We examine the function \( x \mapsto (1+x)^i \), where, by \([11]\), we have \( x \leq x_0 = \frac{1}{2\pi^2} \). The convexity of this function implies that \((1+x)^i \leq 1 + i(1+x_0)^{-i} x \). Thus, from the inequality \((1 + \frac{1}{2\pi^2})^{d-1} \leq \frac{33}{32} < 1.05\), where \( d \geq 2 \), the upper bound for \( M_i(Q) \) in Theorem 1 follows.

On the other hand, in order to prove the lower bound for \( M_i(Q) \) in Theorem 1, we start with the observation that \([10]\) (based on \([12]\), \([11]\), and Lemma 3) yield that

\[
S(Q) \leq S((R + \rho R(C))B^d) \leq d \kappa_d \left( n \frac{\text{vol}_d(C)}{\delta_{sep}(\rho, C) \kappa_d} \right)^{\frac{d}{d-1}} \left( 1 + \frac{2\delta_{sep}(\rho, C)^\frac{\rho}{2} \rho R(C)}{r(C)} \cdot \frac{1}{n^{\frac{d-1}{2}}} \right)^{\frac{d-1}{d}} .
\]

(13)

Thus, \([13]\) together with the inequalities \( S(Q)r(Q) \geq \text{vol}_d(Q) \) (cf. \([11]\)) and \( \text{vol}_d(Q) \geq n \text{vol}_d(C) \) yield

\[
r(Q) \geq \left( 1 + \frac{2\delta_{sep}(\rho, C)^\frac{\rho}{2} \rho R(C)}{r(C)} \cdot \frac{1}{n^{\frac{d-1}{2}}} \right)^{-\frac{(d-1)}{d} \frac{\text{vol}_d(C)}{\delta_{sep}(\rho, C) \kappa_d}} \cdot n^\frac{\rho}{2} .
\]

(14)

Applying the assumption \((11)\) and \( \text{vol}_d(C) \geq \kappa_d \rho(C)^d \) to \((14)\), we get that

\[
r(Q) \geq \left( 1 + \frac{1}{2d^d} \right)^{-\frac{(d-1)}{d} \frac{\text{vol}_d(C)}{\delta_{sep}(\rho, C) \kappa_d}} n^\frac{\rho}{2} \geq \frac{4d^3}{(1 + \frac{1}{2\pi^2})^{d-1}} R(C) \geq 31 R(C) .
\]

(15)

Let \( P \) denote the convex hull of the centers of the translates of \( C \) in \( Q \). Then, \((15)\) implies

\[
r(P) \geq r(Q) - R(C) \geq \frac{30}{31} r(Q) \geq \frac{8d \rho R(C)}{9d} \cdot \frac{d}{d-1} R(C) \cdot n^\frac{\rho}{2} .
\]

(16)

Hence, by \((16)\) and Lemma 2

\[
\text{vol}_d(Q) \geq \text{vol}_d(P) \geq \left( 1 + \frac{9d \rho R(C)}{4 \delta_{sep}(\rho, C) r(C)} \cdot \frac{1}{n^{\frac{d-1}{2}}} \right)^{-d} \frac{n \text{vol}_d(C)}{\delta_{sep}(\rho, C) \kappa_d} .
\]

(17)

which implies in a straightforward way that

\[
\text{vol}_d(Q) \geq \left( 1 + \frac{9d \rho R(C)}{4 \delta_{sep}(\rho, C) r(C)} \cdot \frac{1}{n^{\frac{d-1}{2}}} \right)^{-d} \frac{n \text{vol}_d(C)}{\delta_{sep}(\rho, C) \kappa_d} .
\]

(18)

Note that \((10)\) (see the restated version for \( j = d \)) implies that \( M_i(Q) \geq \left( \frac{\text{vol}_d(Q)}{\kappa_d} \right)^\frac{\rho}{2} \kappa_i \). Then, replacing \( \text{vol}_d(Q) \) by the right-hand side of \((18)\), and using the convexity of the function \( x \mapsto (1+x)^{-i} \) for \( x > -1 \) yields the lower bound for \( M_i(Q) \) in Theorem 1.
Finally, we prove the statement about the spherical shape of $Q$, that is, the inequality (1). As in [8], let
\[
\theta(d) = \frac{1}{2 \frac{d+1}{d+3}} \min \left\{ \frac{3}{\pi^2 d(d+2) 2^d}, \frac{16}{(d \pi)^{\frac{d+1}{d}}} \right\}.
\]
Using the inequality $\frac{\kappa_{d+1}}{\kappa_d} \geq \sqrt{\frac{d}{2 \pi}}$ (cf. [1]) and (6) of [10], we obtain
\[
\left( \frac{S(Q)}{S(B^d)} \right)^d \left( \frac{\text{vol}_d(B^d)}{\text{vol}_d(Q)} \right)^{d-1} - 1 \geq \theta(d) \cdot \left( 1 - \frac{r(Q)}{R(Q)} \right)^{\frac{d+3}{2}}
\]
(see also (5) of [8]). Substituting (13) and (17) into this inequality, we obtain
\[
\left( 1 + \frac{2 \delta_{sep}(\rho, C)^{\frac{d}{2}} \rho R(C)}{r(C)} \cdot \frac{1}{n^2} \right)^{d(d-1)} \left( 1 + \frac{9 \rho R(C)}{4 \delta_{sep}(\rho, C)^{\frac{d+1}{2}} r(C)} \cdot \frac{1}{n^2} \right)^{d(d-1)} \geq \left( \frac{S(Q)}{S(B^d)} \right)^d \left( \frac{\text{vol}_d(B^d)}{\text{vol}_d(Q)} \right)^{d-1}
\]
By the assumptions $d \geq 2$ and (11), it follows that
\[
4 d^2 (d - 1) \frac{\rho R(C)}{\delta_{sep}(\rho, C) r(C)} \cdot \frac{1}{n^2} \geq \theta(d) \left( 1 - \frac{r(Q)}{R(Q)} \right)^{\frac{d+3}{2}}. \tag{19}
\]
Note that by [12],
\[
\frac{1}{\delta_{sep}(\rho, C)} \leq \frac{2^{\frac{d}{2}}}{(d+1) \frac{d+1}{2} \pi \Gamma \left( \frac{d+1}{2} \right)}.
\]
This and (19) implies (1), finishing the proof of Theorem 1.

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