su(N) tensor product multiplicities
and
virtual Berenstein-Zelevinsky triangles

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Abstract
Information on su(N) tensor product multiplicities is neatly encoded in Berenstein-Zelevinsky triangles. Here we study a generalisation of these triangles by allowing negative as well as non-negative integer entries. For a fixed triple product of weights, these generalised Berenstein-Zelevinsky triangles span a lattice in which one may move by adding integer linear combinations of so-called virtual triangles. Inequalities satisfied by the coefficients of the virtual triangles describe a polytope. The tensor product multiplicities may be computed as the number of integer points in this convex polytope. As our main result, we present an explicit formula for this discretised volume as a multiple sum. As an application, we also address the problem of determining when a tensor product multiplicity is non-vanishing. The solution is represented by a set of inequalities in the Dynkin labels. We also allude to the question of when a tensor product multiplicity is greater than a given non-negative integer.
1 Introduction

The decomposition of tensor products of modules of simple Lie algebras has been studied for a long time now. Many elegant results have been found for the multiplicities of the decompositions, the so-called tensor product multiplicities. The relatively recent Berenstein-Zelevinsky method of triangles [1] is an example. Although it is a powerful, symmetric method, it is not explicit: triangles are constructed according to certain rules, and their number is the required tensor product multiplicity. Here we show that a generalisation of these Berenstein-Zelevinsky (BZ) triangles allows us to work out a very explicit expression for the multiplicities. A tensor product multiplicity is expressed as a multiple sum, counting the number of integer points in a particular convex polytope, to be defined below. BZ triangles and our results pertain to the $A$-series; $A_r = su(r+1)$. We will sometimes write $su(N)$ with $N = r + 1$.

We are interested in describing decompositions of tensor products of irreducible highest weight modules of simple Lie algebras. They are usually written

$$M_\lambda \otimes M_\mu = \bigoplus_\nu T_{\lambda,\mu}^{\nu} M_\nu , \quad (1)$$

where $M_\lambda$ is the module of highest weight $\lambda$. $T_{\lambda,\mu}^{\nu}$ is the tensor product multiplicity. We shall study the equivalent but more symmetric problem of determining the multiplicity of the singlet in the expansion of the triple product

$$M_\lambda \otimes M_\mu \otimes M_\nu \supset T_{\lambda,\mu,\nu} M_0 . \quad (2)$$

If $\nu^\perp$ denotes the highest weight conjugate to $\nu$, we have $T_{\lambda,\mu,\nu} = T_{\lambda,\mu}^{\nu^\perp}$. We will use the shorthand notation $\lambda \otimes \mu \otimes \nu$ to represent the left hand side of (2).

An $su(3)$ BZ triangle, describing a particular coupling (to the singlet) associated to the triple product $\lambda \otimes \mu \otimes \nu$, is a triangular arrangement of 9 non-negative integers:

$$m_{13} \quad m_{12} \quad m_{23} \quad n_{12} \quad l_{12} \quad n_{23} \quad l_{23} \quad l_{13} \quad n_{13} \quad (3)$$

These integers are related to the Dynkin labels of the three integrable highest weights by

$$m_{13} + n_{12} = \lambda_1 , \quad n_{13} + l_{12} = \mu_1 , \quad l_{13} + m_{12} = \nu_1 ,$$

$$m_{23} + n_{13} = \lambda_2 , \quad n_{23} + l_{13} = \mu_2 , \quad l_{23} + m_{13} = \nu_2 . \quad (4)$$

We call these relations outer constraints. The entries further satisfy the so-called hexagon conditions

$$n_{12} + m_{23} = n_{23} + m_{12} ,$$

$$m_{12} + l_{23} = m_{23} + l_{12} ,$$

$$l_{12} + n_{23} = l_{23} + n_{12} . \quad (5)$$

of which only two are independent. They say that the length of opposite sides of the hexagon must be equal, if the length of a segment is defined to be the sum of the two integers associated to its endpoints. An $su(3)$ BZ triangle is thus composed of one hexagon and three corner points.
For $su(4)$ the BZ triangle is defined in a similar way, in terms of 18 non-negative integers:

\[
\begin{array}{cccccc}
m_{14} & m_{24} & m_{34} & n_{12} & n_{13} & l_{12} \\
n_{24} & l_{23} & n_{23} & m_{13} & m_{23} & n_{14} \\
l_{24} & m_{34} & l_{14} & n_{24} & n_{34} & m_{14}
\end{array}
\]

(6)

related to the Dynkin labels by

\[
\begin{align*}
m_{14} + n_{12} &= \lambda_1, & n_{14} + l_{12} &= \mu_1, & l_{14} + m_{12} &= \nu_1, \\
m_{24} + n_{13} &= \lambda_2, & n_{24} + l_{13} &= \mu_2, & l_{24} + m_{13} &= \nu_2, \\
m_{34} + n_{14} &= \lambda_3, & n_{34} + l_{14} &= \mu_3, & l_{34} + m_{14} &= \nu_3.
\end{align*}
\]

(7)

Furthermore, the $su(4)$ BZ triangle contains three hexagons:

\[
\begin{align*}
n_{12} + m_{24} &= m_{13} + n_{23}, & n_{13} + l_{23} &= l_{12} + n_{24}, & l_{24} + n_{23} &= l_{13} + n_{34}, \\
n_{12} + l_{34} &= l_{23} + n_{23}, & n_{13} + m_{34} &= n_{24} + m_{23}, & n_{23} + m_{23} &= m_{12} + n_{34}, \\
m_{24} + l_{23} &= l_{34} + m_{13}, & m_{34} + l_{12} &= l_{23} + m_{23}, & l_{13} + m_{23} &= l_{24} + m_{12}.
\end{align*}
\]

(8)

It is a general feature for any $N$ that only two out of the three hexagon identities associated to a particular hexagon are independent.

The $su(N)$ generalisation is obvious; the triangle is built out of $(N-1)(N-2)/2$ hexagons and three corner points. Simple examples of lower rank BZ triangles and their applications may be found in Ref. [2].

## 2 Generalised and virtual Berenstein-Zelevinsky triangles

The generalisation of the BZ triangles we shall consider is obtained by weakening the constraint that all entries are non-negative integers to arbitrary integers, negative as well as non-negative. The hexagon identities and the outer constraints are still enforced. A triangle will be called a true BZ triangle if all its entries are non-negative.

We consider a generalised BZ triangle associated to $su(r+1)$. Denoting the number of entries $E_r$ and the number of hexagons $H_r$, we have

\[
E_r = \frac{3}{2} r (r + 1), \quad H_r = \frac{1}{2} r (r - 1).
\]

(9)

For a given triple product $\lambda \otimes \mu \otimes \nu$, the set of associated triangles spans an $H_r$-dimensional lattice. Each hexagon corresponds to two independent constraints on the triangle entries while there are $3r$ outer constraints. This leaves

\[
E_r - (2H_r + 3r) = H_r
\]

(10)

parameters labelling the possible triangles. Among these, only a finite number are true BZ triangles. This number is precisely the tensor product multiplicity of the triple coupling. For example, when the singlet does not occur in the decomposition of the triple product, there are no true BZ triangles in the lattice.
A special class of generalised BZ triangles is associated to the triple product $0 \otimes 0 \otimes 0$. We say they have weight $(\lambda, \mu, \nu) = (0, 0, 0)$. According to the general argument above, $H_r = \frac{1}{2} r(r - 1)$ such triangles are linearly independent. We shall call them virtual triangles, and denote them using $\mathcal{V}$. It is natural to exclude the triangle with all entries equal to zero from the set of virtual triangles. It is the unique true BZ triangle in the lattice associated to the triple product $0 \otimes 0 \otimes 0$. In the cases of $su(3)$ and $su(4)$, virtual triangles have appeared in Ref. [3] and Ref. [4], respectively.

A convenient basis for the virtual triangles is given by associating a simple distribution of plus and minus ones (written 1 and $\bar{1}$, respectively) to a given hexagon. All other entries are zero. The distribution is

$$
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

(11)

Thus, a basis virtual triangle will always have 6 entries equalling $-1$, and between 3 and 6 entries equalling $+1$. The number of $+1$ entries depends on where the associated hexagon is situated in the generalised BZ triangle and on the rank of $su(r + 1)$ ($3$ entries equal $+1$ for $su(3)$ only). That these virtual triangles are indeed linearly independent is obvious.

There are no virtual triangles in the case of $su(2)$. In the case of $su(3)$ there is one basis virtual triangle

$$
\mathcal{V} = \begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

(12)

while in the case of $su(4)$ the three basis virtual triangles $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are

$$
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

$$
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

$$
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

$$
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

(13)

The generalisation to higher rank $su(r + 1)$ is straightforward. In Section 3 we shall use another choice of indices on $\mathcal{V}$.

We are now in a position to generate all generalised BZ triangles associated to a given triple coupling. Once a single generalised triangle has been found, the lattice of triangles associated
to the triple coupling is spanned by adding integer linear combinations of the virtual triangles. The choice of initial triangle is not important. We shall denote the integer coefficients linear coefficients. We emphasise that the euclidean spaces spanned by the lattices are all of dimension $H_r$, i.e., the dimension is independent of the triple coupling and depends only on the rank of $su(r + 1)$.

Negative entries in (generalised) BZ triangles appear also in Ref. [5]. That work is mainly devoted to the construction of tensor-product generating functions. A new method is proposed based on elementary solutions to certain sets of linear equations related to the BZ triangles, and for $su(3)$ one of these solutions corresponds to a triangle with negative entries. The appearance of negative entries is expected to be a general feature for higher $su(N)$ as well. The elementary solutions are closely related to the so-called elementary couplings.

3 Polytopes, multiple sums and tensor product multiplicities

We shall now focus on the $H_r$-dimensional linear coefficient space and seek an algebraic description of the tensor product multiplicities. The latter are computed by counting true BZ triangles. Demanding that the entries of a true BZ triangle be non-negative, we obtain $E_r$ inequalities the linear coefficients must satisfy. The inequalities depend on the choice of initial triangle, and they correspond to a polyhedral combinatorial expression for the multiplicities in linear coefficient space. The structure of the basis virtual triangles (cf. (11), (12) and (13)) ensures that all linear coefficients have upper as well as lower bounds. The polyhedron is therefore bounded and such a polyhedron is called a polytope. It is easily seen to be convex.

In order to specify the polytope, we must find an initial triangle of weight $(\lambda, \mu, \nu)$. It is convenient to break the symmetry among the 3 weights, and first look at the unique true triangle of weight $(\lambda, \mu, \lambda^+ + \mu^+)$. For $su(3)$, this triangle is

$$\begin{align*}
\lambda_1 & \quad 0 \\
\lambda_2 & \quad \mu_1 \\
0 & \quad \mu_2
\end{align*}$$

and the generalisation to $su(r + 1)$ is clear. Every highest weight $\nu$ in a coupling $\lambda \otimes \mu \otimes \nu$ satisfies

$$\nu = \lambda^+ + \mu^+ - \sum_{i=1}^{r} n_i \alpha_i,$$

with $n_i \in \mathbb{Z}_{\geq 0}$, where $\alpha_i$ is the $i$-th simple root. The coefficients $n_i$ are conveniently expressed using dual Dynkin labels. A weight $\lambda$ can be written

$$\lambda = \sum_{i=1}^{r} \lambda_i \Lambda^i = \sum_{i=1}^{r} \lambda_i^\dagger \alpha_i^\vee,$$

where $\{\Lambda^i\}$ and $\{\alpha_i^\vee\}$ are the sets of fundamental weights and simple co-roots, respectively. The $\lambda_i^\dagger$ are the dual Dynkin labels, while the ordinary Dynkin labels are the $\lambda_i$. For simply-laced
algebras, like $su(N)$, $\alpha_i$ is identical to the co-root $\alpha_i^\vee$ (with standard normalisation $\alpha^2 = 2$, for $\alpha$ a long root). Taking the scalar product of (13) with $\Lambda^i$ therefore gives

$$n_i = (\lambda^+)^i + (\mu^+)^i - \nu^i .$$

(17)

Generalised triangles of weight $(0, 0, \alpha_i)$ are also easily constructed. An $su(3)$ example is

$$
\begin{array}{cccc}
1 & 1 & & \\
\bar{1} & 0 & & \\
0 & 0 & 0 & 0
\end{array}
$$

(18)

of weight $(0, 0, \alpha_2)$. So, one can find a generalised triangle of weight $(\lambda, \mu, \nu)$ by subtracting non-negative integer multiples of triangles of weight $(0, 0, \alpha_i)$, such as (18), from a triangle like (14), of weight $(\lambda, \mu, \lambda^+ + \mu^+)$. The result for $su(r + 1)$ is the following generalised BZ triangle associated to the triple product $\lambda \otimes \mu \otimes \nu$:

$$
\begin{array}{cccccccc}
N_r' & N_r & & & & & & \\
n_r & N_r & & & & & & \\
\lambda_2 & 0 & \mu_1 & n_{r-1} & N_{r-1} & & & \\
\lambda_3 & 0 & \mu_1 & n_{r-1} & & N_{r-2} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\lambda_{r-2} & 0 & \mu_1 & \lambda_{r-1} & & & & N_3' \\
\lambda_{r-1} & 0 & \mu_1 & \lambda_{r-1} & & & & \\
\lambda_r & \lambda_r & \lambda_r & \lambda_r & \lambda_r & \lambda_r & \lambda_r & \\
0 & \mu_1 & \mu_2 & \mu_3 & \mu_1 & \mu_2 & \mu_3 & \\
\end{array}
$$

(19)

The entries $n_i$, $N_i$ and $N'_i$ are defined by

$$
n_i = \lambda^{r-i+1} + \mu^{r-i+1} - \nu^i ,$$
$$N_i = (1 - \delta_{i1})n_{i-1} - n_i + \mu_{r-i+1}$$
$$= -\lambda^{r-i+1} + (1 - \delta_{i1})\lambda^{r-i+2} - (1 - \delta_{ir})\mu^{r-i} + \mu^{r-i+1} - (1 - \delta_{i1})\nu^{i-1} + \nu^i ,$$
$$N'_i = \nu_i - N_i$$
$$= \lambda^{r-i+1} - (1 - \delta_{i1})\lambda^{r-i+2} + (1 - \delta_{ir})\mu^{r-i} - \mu^{r-i+1} + \nu^i - (1 - \delta_{ir})\nu^{i+1} .$$

(20)

In order to be able to describe the polytope explicitly, we need to label the virtual triangles. Our choice is to write them as $V_{i,j}$ or $V_l$ depending on where the associated hexagons are
situated. The corresponding linear coefficients are denoted $d_{i,j}$ and $\eta_l$:

\[
\begin{array}{cccccccccccc}
& & & & & & & & \ast & & & \\
& & & & & & & \ast & & & & \\
& & & & & & \ast & & \ast & & & \\
& & & & \ast & & & & & \ast & & \\
& & & \ast & & & & & & \ast & \\
& & \ast & & & & & & & & \ast & \\
& \ast & & & & & & & & & \ast & \\
\end{array}
\]

(21)

Here a $\ast$ indicates an unspecified entry while $d_{i,j}$ and $\eta_l$ are the linear coefficients of the virtual triangles. They are depicted at the centres of the hexagons associated to the corresponding virtual triangles. We have chosen two different notations for the virtual triangles (and their associated linear coefficients) to reflect the positions of the corresponding hexagons in the asymmetric initial triangle (19).

Now, denoting the initial triangle $T_0$, any triangle in the lattice of general triangles of weight $(\lambda, \mu, \nu)$ may be written as

\[
T = T_0 + \sum_{i=1}^{r-1} \eta_i V_i + \sum_{i,j \geq 1}^{i+j=r-1} d_{i,j} V_{i,j}.
\]

(22)

The associated polytope of interest is in the $H_r$-dimensional space spanned by $d_{i,j}$ and $\eta_l$. It is bounded by the inequalities requiring that all entries in $T$ are non-negative (whereby $T$ is ensured to be a pure BZ triangle). Hence, the position of the polytope depends on the initial triangle $T_0$. Nevertheless, the volume of the polytope, the number of integer points bounded by the polytope, is independent of $T_0$. By construction, this number is the tensor product multiplicity $T_{\lambda,\mu,\nu}$ of the triple coupling $\lambda \otimes \mu \otimes \nu$ to the singlet.

Using the explicit choice of initial triangle (19) and the basis of virtual triangles discussed above, it is simple to write down the inequalities defining the polytope. To illustrate, we list the three inequalities given by the three entries located furthest to the right on the bottom line of $T$:

\[
\mu_{r-1} + d_{1,r-2} - \eta_1 \geq 0, \quad n_1 - \eta_1 \geq 0, \quad N_1 + \eta_1 \geq 0.
\]

(23)

A similar polyhedral combinatorial expression is discussed in Ref. [6]. Convex polytopes constructed there lie in the space of Gelfand-Tsetlin patterns (see e.g. [8, 9]), while ours lie in spaces associated to BZ triangles. Hence, for $su(N)$ their polytopes are embedded in the euclidean vector space $\mathbb{R}^{N(N+1)/2}$, while ours may be embedded in the smaller space $\mathbb{R}^{(N-2)(N-1)/2}$. A more universal method of constructing polyhedral combinatorial expressions for tensor product multiplicities, generalising that of Ref. [6] and making sense for any simple Lie algebra, may be found in Refs. [10].
3.1 Explicit multiple sum formula

As already stated, our polyhedral expression differs from the one discussed in Refs. \[6, 7\]. Its structure allows us to extract an explicit multiple sum formula counting the integer points bounded by the polytope. The multiple sum is over the linear coefficients, so different orders of summation give a total of \(H_r!\) possible representations of the polytope volume. For practical purposes, however, there are considerably fewer appropriate summation orders. Let us illustrate our procedure for choosing an appropriate order of summation by considering the following simple planar example.

Let a planar polytope be defined by the set of inequalities

\[
\begin{align*}
1 &\leq x \leq 4, & 8 \leq x + y \leq 14, \\
6 &\leq y, & 4 \leq y - x \leq 8 .
\end{align*}
\]

(24)

The volume or area \(A\) of the polytope (the number of integer points bounded by the inequalities) can be written in two ways:

\[
A = 16 = \sum_{x=1}^{4} \left( \sum_{y=\max\{6,x+4,8-x\}}^{\min\{x+8,14-x\}} 1 \right) = \sum_{y=\max\{1,y-8,8-y\}}^{11} \left( \sum_{x=\max\{6,x+4,8-x\}}^{\min\{x+8,14-x\}} 1 \right) .
\]

(25)

The second expression is slightly more difficult to write, since the upper limit 11 on \(y\) must be calculated from the intersection of faces (lines). Here the bounding lines \(x + y = 14\) and \(y - x = 8\) intersect at the point \((x, y) = (3, 11)\). This is a complication to avoid when writing our formula, since it will involve many sums. In the first expression the explicitly written lower limit \(y = 6\) is redundant since the remaining two intersect at the point \((2, 6)\). However, including redundant limits does not change the result, so we may choose to keep the limit \(y = 6\).

It is clear that an important difference between the two orders of summation is that we have \(1 \leq x \leq 4\), but only \(6 \leq y\). For example, if (24) is supplemented with \(y \leq 15\), then the upper limit 11 can be replaced simply by 15, and the formula is still valid. While it is true that the new formula contains single sums with lower limits greater than upper limits, this is a relatively small inconvenience. Those sums simply contribute 0.

So, when choosing an appropriate order of summation (over the summation variables \(\eta_r\) and \(d_{i,j}\)), it is crucial that for any summation variable, all subsequent summation variables have upper as well as lower bounds parallel to (or independent of) the one under consideration.

This is a non-trivial consideration: the procedure does not apply to all polytopes. A trivial example is provided by (24) with the inequality \(x \leq 4\) removed.

Fortunately, the simplifying procedure applies to our polytope. A simple inspection reveals that not all summation orders are appropriate, however. Nevertheless, in the general case and in accordance with our procedure, we may express the volume of the polytope as

\[
T_{\lambda,\mu,\nu} = \left( \sum_{d_{1,1}} \right) \left( \sum_{d_{2,1}} \sum_{d_{1,2}} \right) \ldots \left( \sum_{d_{r-2,1}} \sum_{d_{r-2,2}} \sum_{d_{1,r-2}} \right) \left( \sum_{\eta_{r-1}} \sum_{\eta_{r-2}} \ldots \sum_{\eta_{1}} \right) 1 .
\]

(26)

where the summation variables are bounded according to

\[
\max\{-N_1, d_{1,r-2}, -N'_2 + \eta_2, -\mu_{r-2} + d_{1,r-2} - d_{2,r-3} + \eta_2\}
\]
The summation (29) is non-vanishing if and only if the upper limit is greater than or equal to symmetric in the weights. This is simply because we have chosen an asymmetric initial triangle. The weights are subject to the integer constraint (28). Note that the summation limits are not algebras, and hope to report more general results later.

To illustrate the method, we discuss the inequalities for three weights, determining when the associated tensor product multiplicity is non-vanishing.

4 An application

It is of interest to know whether or not a coupling of a certain weight \( (\lambda, \mu, \nu) \) exists, without having to work out the tensor product multiplicity. Based on our multiple sum formula (28) and (29) one may derive a set of inequalities in the dual and ordinary Dynkin labels of the three weights, determining when the associated tensor product multiplicity is non-vanishing. To illustrate the method, we discuss the inequalities for \( su(3) \) and \( su(4) \) and outline their derivation. In principle, it is possible to repeat the procedure for higher rank, but even for \( su(4) \) the derivation is very cumbersome. We believe that similar results exist for all simple Lie algebras, and hope to report more general results later.

The dimension of the linear coefficient space for \( su(3) \) is one, so the tensor product multiplicity may be represented by a single sum:

\[
T_{\lambda, \mu, \nu} = \min \{ \mu_1, \lambda_2, \lambda^2 + \mu^2 - \nu^2, \lambda^1 + \mu^1 - \nu^2, -\lambda^1 + \lambda^2 + \mu^2 - \nu^2, -\lambda^1 + \lambda^2 + \mu^1 + \nu^2, \lambda^2 + \mu^2 - \nu^1 + \nu^2 \} 
\sum_{\eta = \max \{ 0, \lambda^2 + \mu^1 - \nu^2 - \lambda^1 + \lambda^2 + \mu^1 - \nu^2 \}} 1.
\]

The weights are subject to the integer constraint (28). Note that the summation limits are not symmetric in the weights. This is simply because we have chosen an asymmetric initial triangle. The summation (29) is non-vanishing if and only if the upper limit is greater than or equal to the lower limit. This condition yields \( 6 \cdot 3 = 18 \) inequalities:

\[
0 \leq \lambda_i, \mu_i, \nu_i, \quad \text{for} \quad i = 1, 2,
\]
\[
\max\{\lambda^1 - \lambda^2 + \mu^1 - \mu^2, -\lambda^1 + \mu^1, \lambda^2 - \mu^1\} \leq \nu^1 \leq \lambda^2 + \mu^2, \\
\max\{-\lambda^1 + \lambda^2 - \mu^1 + \mu^2, \lambda^1 - \mu^2, -\lambda^2 + \mu^1\} \leq \nu^2 \leq \lambda^1 + \mu^1, \\
\max\{-\lambda^2 - \mu^1 + \mu^2, -\lambda^1 + \lambda^2 - \mu^2\} \leq \nu^1 - \nu^2 \\
\leq \min\{\lambda^1 - \mu^1 + \mu^2, -\lambda^1 + \lambda^2 + \mu^1\}. \quad (30)
\]

Expressing the inequalities in terms of ordinary Dynkin labels, one should bear in mind that the summation variable increases in steps of one while the quadratic-form matrix involves a factor of $1/3$. A similar factor $1/N$ is present for higher rank $su(N)$.

In the case of $su(4)$ the tensor product multiplicity may be written as a triple sum:

\[
T_{\lambda,\mu,\nu} = \sum_{d=0}^{N} \min\{\lambda_1,\lambda_3\} \min\{\lambda_2+d,\lambda^1+\nu^3,\lambda^1+\lambda^2+\mu^1-\nu^2+\nu_3\} \\
\times \min\{\nu^2,\lambda^2+\nu^3,\lambda^2+\mu^1-\nu^2+\nu_3+\nu_2,\min\lambda^2+\lambda^3+\mu^1-\nu^2+\nu_3+\nu_2\} \\
\times \min\{\lambda^1,\lambda^3,\mu_2\}.
\]

The weights are subject to the integer constraint (28). For a multiple sum like this, the inequalities are obtained by first considering the interior summation over $\eta_1$, leading to $6 \cdot 4 = 24$ inequalities which may depend on the remaining two summation variables: 4 of them depend only on the weights, 6 depend on $d$ but not $\eta_2$, while 14 depend on $\eta_2$. Treating the latter in the same way as the upper and lower bounds on the $\eta_2$-summation, we obtain a total of $13 \cdot 6 = 78$ inequalities from the $\eta_2$-consideration. Repeating the procedure for $d$ leads to a total of $54 \cdot 10 = 540$ inequalities in addition to the ones already derived from the $\eta_1$- and $\eta_2$-considerations. This huge set of inequalities may be reduced considerably and we find the following constraints on the Dynkin labels (expressed in terms of dual as well as ordinary ones)

\[
0 \leq \lambda_i, \mu_i, \nu_i, \quad \text{for} \quad i = 1, 2, 3, \\
\max\{\lambda^3 - \mu^1, \lambda^3 - \mu_3 - \mu_1 + \mu_3, -\lambda^3 + \mu_1 + \mu_1, -\lambda^1 + \lambda^1 + \mu^1 + \mu^3 - \mu_3\} \leq \nu^1 \leq \lambda^3 + \mu^3, \\
\max\{\nu^2, |\lambda^2 - \lambda^2 - \mu^2 + \mu_2|, |\lambda^1 - \lambda^3 + \mu^1 - \mu^3|\} \leq \nu^2 \leq \lambda^2 + \mu^2, \\
\max\{\lambda^1 - \mu^3, \lambda^1 - \lambda^1 + \mu_3 + \mu_3, -\lambda^3 + \mu^1, -\lambda^3 + \lambda^3 + \mu^1 - \mu_1\} \leq \nu^3 \leq \lambda^1 + \mu^1, \\
\max\{\lambda^2 - \lambda^3 - \mu^1, -\lambda^1 + \mu^2 - \mu^3, \lambda^1 + \lambda^2 + \mu^1 - \mu^2\} \leq \nu^1 - \nu_1 \\
\leq \min\{\lambda^2 - \lambda^3 + \mu^3, \lambda^3 + \mu^2 - \mu^3\}, \\
\max\{-\lambda^1 + \lambda^3 + \mu_2 - \mu^2, \lambda_2 - \lambda^2 - \mu^1 + \mu^3, \lambda^1 - \lambda^3 + \mu_2 - \mu^2, \\
\lambda_2 - \lambda^2 + \mu^1 - \mu^3, -\lambda^2 + \mu^2 - \mu_2, \lambda^2 - \lambda_2 - \mu^2\} \leq \nu^2 - \nu_2 \\
\leq \min\{\lambda^2 - \lambda_2 + \mu^2, \lambda^2 + \mu^2 - \mu_2\}, \\
\max\{\lambda^2 - \lambda^1 - \mu^3, -\lambda^3 + \mu^2 - \mu_1, \lambda^3 - \lambda^2 + \mu^3 - \mu^2\} \leq \nu^3 - \nu_3 \leq \min\{\lambda^2 - \lambda^1 + \mu^1, \lambda^1 + \mu^2 - \mu^1\}, \\
\max\{-\lambda^1 + \lambda^3 - \mu^2, -\lambda^2 - \mu^1 + \mu^3, \lambda_2 - \lambda^2 + \mu_2 - \mu^2\} \leq \nu^1 - \nu_3 \\
\leq \min\{-\lambda^1 + \lambda^3 + \mu_2, \lambda^2 - \mu^1 + \mu_3, \lambda^2 - \lambda_2 + \mu^2 - \mu_2\}. \quad (32)
\]

A recent discussion [8] includes a brief review of the problem of determining when a tensor product multiplicity is non-vanishing. The focus there is on $gl(N)$ (and therefore also on $su(N)$).
and results are provided in the form of polyhedral combinatorial expressions. Our prescription described above is in general more explicit than results previously obtained. For lower rank $su(N)$, though, the explicit inequalities may be obtained using various alternative approaches.

4.1 A refinement

Here we shall indicate how one may derive sets of inequalities determining when a tensor product multiplicity is greater than a given non-negative integer $K$

$$T_{\lambda,\mu,\nu} > K. \quad (33)$$

The case $K = 0$ was discussed above.

Our approach is straightforward since the problem translates into studying when a convex polytope has a (discretised) volume bigger than $K$. To illustrate, let us consider $su(3)$. In this case the volume is expressed as a single sum (29), so (33) is equivalent to requiring

$$\min\{\mu_1, \lambda_2, \lambda^2 + \mu_1 + \nu_1, -\lambda^1 + \lambda^2 + \mu_1 - \nu_1 + \nu^2, \lambda^2 + \mu_1 - \mu^2 + \nu^1 - \nu^2\}
\geq K. \quad (34)$$

This leaves us the following 18 inequalities refining (30)

$$K \leq \lambda_i, \mu_i, \nu_i, \quad \text{for} \quad i = 1, 2,$$

$$\max\{-\lambda^1 + \lambda^2 + \mu_1 - \mu^2, -\lambda^1 + \mu^1, \lambda^1 - \mu^2, -\lambda^2 + \mu^1\} + K \leq \nu^1 \leq \lambda^2 + \mu^2 - K,$$

$$\max\{-\lambda^2 - \mu^1 + \mu^2, -\lambda^1 + \lambda^2 - \mu^2\} + K \leq \nu^2 \leq \lambda^1 + \mu^1 - K,$$

$$\max\{-\lambda^1 - \mu^1 + \mu^2, -\lambda^2 + \mu^1 - \mu^2\} \leq \min\{\lambda^1 - \mu^1 + \mu^2, -\lambda^1 + \lambda^2 + \mu^1\} - K. \quad (35)$$

To the best of our knowledge, this is a new result.

In the case $su(4)$ the situation is already much more complicated. That is because the polytope is three-dimensional, and we cannot immediately use the triple-sum formula (31). We recall that our simplifying method for obtaining an appropriate order of summation, may include redundant summations contributing zero to the final expression. We might therefore lose crucial information if we only considered the multiple sum formula. The remedy is to consider the original polytope, and require the defining faces to embrace a volume of at least the desired value. We would then be led to consider three-dimensional partitions of $K + 1$, which is beyond the scope of the present work. For lower values of $K$ the problem is straightforward, though.

5 Conclusion

By virtue of virtual BZ triangles we have obtained a polyhedral combinatorial expression for the $su(N)$ tensor product multiplicities, different from the ones discussed in Refs. [3, 4]. The main merit of our expression is that it admits a simple measurement of the convex polytope volume
in terms of a multiple sum formula. The latter is then a new and explicit way of expressing the tensor product multiplicities of $su(N)$.

As an application, one may derive explicit bounds on a triple of weights determining when the associated coupling to the singlet exists. To illustrate, the bounds were written for $su(3)$ and $su(4)$. Also included was a brief discussion on how to generalise this to bounds describing $T_{\lambda,\mu,\nu} > K$.

We believe that our multiple sum representation of the tensor product multiplicities provides a significant computational improvement over previous (combinatorial) results. In particular, it is expected to lead to considerable simplifications when implemented in computer programs.

It is our hope that our results may find applications to the computation of fusion rules in conformal field theory with affine Lie group symmetry, the so-called WZW theories. Since tensor product multiplicities correspond to the infinite-level limit of fusion multiplicities, it is helpful to have simple descriptions of the former in order to understand the latter.

Acknowledgements

We are grateful to G. Flynn for discussions and to C. Cummins and P. Mathieu for commenting on the manuscript.

References

[1] A.D. Berenstein and A.V. Zelevinsky, J. Alg. Comb. 1 (1992) 7.

[2] P. Di Francesco, P. Mathieu and D. Sénéchal, Conformal Field Theory (Springer Verlag, 1997).

[3] L. Bégin, P. Mathieu and M. Walton, Mod. Phys. Lett. A7 (1992) 731.

[4] L. Bégin, A.N. Kirillov, P. Mathieu and M. Walton, Lett. Math. Phys. 28 (1993) 257.

[5] L. Bégin, C. Cummins and P. Mathieu, Generating-function method for tensor products, math-ph/0005003.

[6] I.M. Gelfand and A. Zelevinsky, in Group theoretical methods in physics, Proceedings of the Third Seminar, Yurmala (North-Holland, 1985).

[7] A. Berenstein and A. Zelevinsky, J. Geom. Phys. 5 (1988) 453; Tensor product multiplicities, canonical bases and totally positive varieties, math.RT/9912012.

[8] A. Zelevinsky, Littlewood-Richardson semigroups, math.CO/9704228.