An Upper Bound for Wiretap Multi-way Channels

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Abstract

A general model for wiretap multi-way channels is introduced that includes several previously studied models in information theoretic security as special cases. A new upper bound is developed that generalizes and unifies previous bounds. We also introduce a multivariate dependence balance bound which is of independent interest.

1 Introduction

A wiretap multi-way channel (WiMWC) with $k$ transceiver terminals and an eavesdropper is defined by $p(y[k], z|x[k])$ where $x_i$ and $y_i$ are the respective channel inputs and outputs of the $i$-th legitimate transmitter, and $z$ is the eavesdropper channel output. Let $[k] = \{1, 2, \ldots, k\}$ and $x[k] = (x_1, \ldots, x_k)$. Extra public or private discussion/feedback links can be included by $L$ parallel channels $q_\ell(y[k], z|x[k]), \ell \in [L]$, that the legitimate terminals can use in addition to the main channel $p(y[k], z|x[k])$. For instance, to model a noiseless public channel one may add a parallel channel $Y_1 = Y_2 = \cdots = Y_k = Z = X[k]$.

A WiMWC code of length $n$ is defined as follows: at time instance $j \in [n]$, the $i$-th legitimate terminal uses local (private) random strings $W_i$ and transmits the symbol

$$X_{ij} = f_{ij}(W_i, Y_{i|j-1}), \quad j \in [n]$$

over the main channel $p(y[k], z|x[k])$ or over one of the parallel channels $q_\ell(y[k], z|x[k])$. Here, $n$ is the number of transmissions and $f_{ij}(\cdot)$ is the encoding function at terminal $i$ for time $j$, and $Y_{ij}$ is the channel output symbols seen by terminal $i$ at time $j$. Random variable $Y_{i|j-1}$ (also sometimes denoted by $Y_i^{j-1}$ in the paper) is the collection of past outputs of terminal $i$ at time $j$. Suppose that the main channel is used $m \leq n$ times during the $n$ transmissions, while the channel $q_\ell(y[k], z|x[k])$ is used $m_\ell$ times for $\ell = 1, 2, \cdots, L$. Thus, $m + \sum_{\ell=1}^L m_\ell = n$.

After transmission, a subset of the terminals – without loss of generality assumed to be the first $r$ terminals – generate the secret keys

$$S_i = g_i(W_i, Y_{i[n]}), \quad i \in [r]$$

where $S_i \in [2^{mR_s}]$ for some $R_s > 0$. In an $(n, \epsilon)$ code, the generated keys satisfy

$$\frac{1}{m}H(S_1) \geq R_s - \epsilon, \quad P[S_1 = S_2 = \cdots = S_r] \geq 1 - \epsilon, \quad \frac{1}{m}I(S_1; Z^n) \leq \epsilon.$$  

The number $R_s$ is called the secret key rate, and

$$\alpha_\ell = \frac{m_\ell}{m}$$  

(2)
is the rate of channel use for \( q_\ell(y[k], z|x[k]) \). Given \( \alpha \geq 0 \) for \( \ell \in [L] \), we are interested in the supremum of rates \( R_\alpha \) that can be achieved for any \( \varepsilon > 0 \) as \( m \) tends to infinity.

Observe that while terminals \( r + 1, r + 2, \ldots, k \) do not generate secret keys, they have channel inputs and can participate as *helper terminals*. If the secret key generated by the first \( r \) terminals must be kept private from a collection of helper terminals, then the outputs of these terminals can be included as part of the eavesdropper’s output variable \( Z \).

The WiMWC includes several models as special cases.

- **Source model**: consider \( k = 2 \) and set \( X_1, X_2 \) to be constant in the main channel \( p(y_1, y_2, z|x_1, x_2) \). Take an extra channel to allow for public discussion, and let \( \alpha_1 \) (as defined in (2)) tend to infinity. Similarly, the multiaccess channel model ( [2, 3]) for which each legitimate terminal is either a receiver or transmitter can be enforced by choosing the alphabets of either \( X_i \) or \( Y_i \) to have only one element. As shown in [2, 3], the secret key capacity is related to MACs with feedback for the special cases considered in these works. Our result provides connections between the key agreement problem and MACs with feedback in a more general setting.

- **Channel model**: consider \( k = 2 \) and set \( Y_1 \) and \( X_2 \) to be constant in the main channel \( p(y_1, y_2, z|x_1, x_2) \). Take an extra channel to allow for public discussion and let \( \alpha_1 \) tend to infinity. Similarly, the multiaccess channel model ( [2, 3]) for which each legitimate terminal is either a receiver or transmitter can be enforced by choosing the alphabets of either \( X_i \) or \( Y_i \) to have only one element. As shown in [2, 3], the secret key capacity is related to MACs with feedback for the special cases considered in these works. Our result provides connections between the key agreement problem and MACs with feedback in a more general setting.

- **Wiretap channels with or without (private or public) feedback**: For instance, for a secure rate-limited feedback link as in [4], we can set \( k = 2 \) and consider a parallel channel where \( Y_2 \) and \( Z \) are constant while \( Y_1 = X_2 \) with the desired feedback rate. The Gaussian wiretap two-way channel [5] is obtained when we do not have any parallel channels and \( k = 2 \). Similarly, one can obtain the models considered in [6–9] as special cases of the WiMWC model. The channel model of [10] also reduces to the model considered here if the parallel channels model a public channel available to all parties.

## 2 New Upper Bound

**Definition 1** (Fractional Partition). Let \( \mathcal{B} \) be the set of all non-empty proper subsets of \([k]\). A fractional partition of \([k]\) is a collection of non-negative weights associated to non-empty proper subsets of \([k]\), i.e., \( \lambda_B \) for every \( B \in \mathcal{B} \) such that

\[
\sum_{B: i \in B} \lambda_B = 1, \quad \forall i \in [k].
\] (3)

**Definition 2** (Multivariate Mutual Information). Let \( (\lambda_B : B \in \mathcal{B}) \) be an arbitrary fractional partition of \([k]\). The \( \lambda \)-mutual information among variables \( X_i, i \in [k] \) conditioned on another random variable \( T \) is

\[
I_\lambda(X_1;X_2;\cdots;X_k|T) = H(X[k]|T) - \sum_B \lambda_B H(X_B|X_B^c, T).
\]

The above definition first appeared in [1] Equation 6]. Observe that when \( k = 2 \) and \( \lambda_{\{1\}} = \lambda_{\{2\}} = 1 \), the \( \lambda \)-mutual information reduces to the ordinary conditional mutual information. Basic properties of \( I_\lambda \) are discussed in Appendix A] In particular, a dependence balance lemma for \( I_\lambda \) is given in Appendix C.

Take some arbitrary alphabet set \( \mathcal{T} \) and an auxiliary random variable \( T \in \mathcal{T} \) defined by a conditional distribution \( q(t|y[k], z, x[k]) \). We call \( T \) an auxiliary receiver.
**Definition 3.** Given a fractional partition $\lambda$, a multiway wiretap channel $q(y_{[k]}; z|x_{[k]})$ and an auxiliary receiver $T$ described by $q(t|y_{[k]}, z, x_{[k]})$, let

$$V_{\lambda}(q) = \max I_{\lambda}(X_1Y_1; X_2Y_2; \cdots; X_kY_k|T) - I_{\lambda}(X_1; X_2; \cdots; X_k) + I(V; T|U) - I(V; Z|U)$$

where the maximum is over all $p(x_{[k]}, u, v, y_{[k]}, z, t)$ of the form:

$$p(x_{[k]}, u, v, y_{[k]}, z, t) = p(x_{[k]})q(y_{[k]}, z, t|x_{[k]})p(u, v|x_{[k]}, y_{[k]})$$

**Theorem 4.** Take an arbitrary fractional partition such that $\lambda_B = 0$ when $[r] \subset B$. Then, for any arbitrary auxiliary receiver $q_T|x_{[k]}, y_{[k]}, z$, the secret key capacity of the multiway wiretap channel $q(y_{[k]}, z|x_{[k]})$ is bounded from above by

$$V(p(y_{[k]}, z, t|x_{[k]}))q(t|x_{[k]}, y_{[k]}) + \sum_{\ell=1}^L \alpha_{\ell}V(q_\ell(y_{[k]}, z, t|x_{[k]}))q(t|x_{[k]}, y_{[k]}),$$

where $\alpha_\ell$ is defined in [2].

**Remark 5.** This converse recovers the best known upper bound for the source and channel models as special cases (for this special case, $V(q(y_{[k]}, z,t|x_{[k]}))q(t|x_{[k]}, y_{[k]}, z)$ vanishes) [11][12], and also recovers the converse for the result in [3]. If we set $T = Z$, we get a bound that can be also deduced from Theorem 3.1 of [13]. The converse utilizes an auxiliary receiver $T$. See [13] for some further applications of auxiliary receivers.

**Proof.** Take some arbitrary $q(t|x_{[k]}, y_{[k]}, z)$. Take some arbitrary code and let $T^n$ be defined by passing $X_{[k]j}, Y_{[k]j}, Z_j$ through the memoryless channel $q(t|x_{[k]}, y_{[k]}, z)$ for $j \in [n]$.

Let $S_i$ for $i \in [r]$ be the key produced by the first $r$ terminals. Set $S_i$ for $r < i \leq k$ to be constants. We have

$$H(S_1) - nk_1(\epsilon) \leq H(S_1) - I(S_1; Z^n)$$

$$= H(S_1|T^n) + I(S_1; T^n) - I(S_1; Z^n) - nk_2(\epsilon)$$

where $k_1(\epsilon)$ and $k_2(\epsilon)$ are functions that tend to zero as $\epsilon$ tends to zero. Observe that

$$I(S_1; T^n) - I(S_1; Z^n) = \sum_i I(S_1; T_j|Z^n_{j+1}, T^{j-1}) - I(S_1; Z_j|Z^n_{j+1}, T^{j-1})$$

$$= \sum_j I(V_j; T_j|U_j T_j) - I(V_j; Z_j|U_j T_j)$$

where $V_j = S_1$, $U_j = Z^n_{j+1}$ and $T_j = T^{j-1}$. Observe that

$$U_j V_j T_j \rightarrow X_{[k]j} Y_{[k]j} \rightarrow T_j Z_j$$

forms a Markov chain. Next, we have

$$H(S_1|T^n) - nk_3(\epsilon) \leq I_{\lambda}(S_1; S_2; \cdots; S_k|T^n)$$

$$\leq I_{\lambda}(W_1 Y_1^n; W_2 Y_2^n; \cdots; W_k Y_k^n|T^n)$$

$$= I_{\lambda}(W_1 Y_1^n; W_2 Y_2^n; \cdots; W_k Y_k^n|T^n) - I_{\lambda}(W_1; W_2; \cdots; W_k$$
\[
\leq \sum_{j=1}^{n} I_{\lambda}(X_{1j}Y_{1j}; X_{2j}Y_{2j}; \ldots; X_{kj}Y_{kj}|T_j, T_{j-1}) - \sum_{i=1}^{n} I_{\lambda}(X_{1j}; X_{2j}; \ldots; X_{kj}|T_{j-1}) \tag{11}
\]

where (8) and (9) follow from the Fano and data processing inequalities for \( I_{\lambda} \), as shown in Proposition 6 in Appendix A, and (11) follows from Lemma 9 in Appendix C, and \( k_3(\epsilon) \) is a function that tends to zero as \( \epsilon \) tends to zero.

Collecting the above results, we obtain
\[
H(S_1) - nk_1(\epsilon) - nk_2(\epsilon) 
\leq \sum_{j=1}^{n} \left( I_{\lambda}(X_{1j}Y_{1j}; X_{2j}Y_{2j}; \ldots; X_{kj}Y_{kj}|T_j, T_{j}) - I_{\lambda}(X_{1j}; X_{2j}; \ldots; X_{kj}|T_{j}) 
+ I(V_j; T_j|U_jT_j) - I(V_j; Z_j|U_jT_j) \right). \tag{12}
\]

Finally, observe that if at time \( j \), the main channel \( p(y_{[k]}, z, t|x_{[k]}) \) is used, the expression inside parenthesis in (12) is bounded from above by
\[
V\left(p(y_{[k]}, z, t|x_{[k]})q(t|x_{[k]}, y_{[k]}, k)\right)
\]
while if the \( q_\ell(y_{[k]}, z, t|x_{[k]}) \) is used, the expression inside parenthesis in (12) is bounded from above by
\[
V\left(q_\ell(y_{[k]}, z, t|x_{[k]})q(t|x_{[k]}, y_{[k]}, k)\right).
\]

This completes the proof. \( \square \)

**Appendices**

**A Properties of \( \lambda-k \)-mutual information**

The properties in the following proposition essentially follow from the arguments in [1]. We include their proofs for completeness.

**Proposition 6.** \( \lambda-k \)-mutual information satisfies the following properties:

- (Nonnegativity): \( I_{\lambda}(X_1; X_2; \ldots; X_k) \geq 0 \)
- (Fano): Let \( A = \{i_1, i_2, \ldots, i_r\} \) be an arbitrary subset of \([k]\). Assume that \( X_j \) is constant when \( j \notin A \), and
  \[
  \mathbb{P}[X_{i_1} = X_{i_2} = \cdots = X_{i_r}] \geq 1 - \epsilon.
  \]
  Take a fractional partition such that \( \lambda_B = 0 \) when \( A \subset B \). Then
  \[
  I_{\lambda}(X_1; X_2; \cdots; X_k) \geq H(X_A) - \left( \sum_B \lambda_B \right) \left( H_2(\epsilon) + \epsilon \sum_{i \in A} \log(|X_i|) \right),
  \]
  where \( H_2(\cdot) \) is the binary entropy function.
• (Data processing): if we locally produce $Y_i$ from $X_i$, then

$$I_\lambda(X_1; X_2; \cdots; X_k) \geq I_\lambda(Y_1; Y_2; \cdots; Y_k).$$

**Proof.** For non-negativity, we have

$$H(X_1 X_2 \cdots X_k) = \sum_i H(X_i|X^{i-1})$$

$$= \sum_i (\sum_{B: i \in B} \lambda_B) H(X_i|X^{i-1})$$

$$= \sum_B \sum_{i \in B} \lambda_B H(X_i|X^{i-1})$$

$$\geq \sum_B \sum_{i \in B} \lambda_B H(X_i|X_{[i-1]|B}X_{B^c})$$

$$= \sum_B \lambda_B H(X_B|X_{B^c}).$$

To show the Fano’s property, observe that $\lambda_B \neq 0$ implies $B^c \cap A \neq \emptyset$. Using Fano’s inequality we have

$$H(X_B|X_{B^c}) \leq H(X_B|X_{B^c} \cap A) \leq H_2(\epsilon) + \epsilon \sum_{i \in [k]} \log(|X_i|).$$

Finally, we show the data processing property: since adding private noise to variables does not change the $\lambda$-k-information, it suffices to show this claim when $Y_i$ is a function of $X_i$. In this case, we have

$$I_\lambda(X_1; X_2; \cdots; X_k) = H(X_{[k]}) - \sum_B \lambda_B H(X_B|X_{B^c})$$

$$= H(Y_{[k]}) - \sum_B \lambda_B H(Y_B|Y_{B^c}X_{B^c})$$

$$+ H(X_{[k]}|Y_{[k]}) - \sum_B \lambda_B H(X_B|Y_BX_{B^c})$$

$$= H(Y_{[k]}) - \sum_B \lambda_B H(Y_B|Y_{B^c}) + \sum_B \lambda_B I(Y_B; X_{B^c}|Y_{B^c})$$

$$+ H(X_{[k]}|Y_{[k]}) - \sum_B \lambda_B H(X_B|Y_{[k]}X_{B^c})$$

$$= I_\lambda(Y_1; Y_2; \cdots; Y_k) + I_\lambda(X_1; X_2; \cdots; X_k|Y_{[k]}) + \sum_B \lambda_B I(Y_B; X_{B^c}|Y_{B^c})$$

$$\geq I_\lambda(Y_1; Y_2; \cdots; Y_k).$$

where

$$I_\lambda(X_1; X_2; \cdots; X_k|Y_{[k]}) = \sum_{y_{[k]}} p(y_{[k]}|y_{[k]}) I_\lambda(X_1; X_2; \cdots; X_k|Y_{[k]} = y_{[k]})$$

is defined just like the ordinary mutual information.

It is further shown in [1, Lemma A.1] that $I_\lambda(X_1; X_2; \cdots; X_k)$ is concave in $p(x_1)$ for a fixed $p(x_2, x_3, \cdots, x_k|x_1)$. An interactive communication property of $I_\lambda$ can be also deduced from Lemma 6 of [3].
B  Relation to another definition of \( k \)-mutual information

Define the \( k \)-mutual information for the random variables \( X_1, X_2, \ldots, X_k \) as

\[
J(X_1; X_2; \cdots; X_k) = -H(X_1 \cdots X_k) + \sum_i H(X_i).
\]

**Example 7.** Let \( \lambda_B = 0 \) if \( |B| \neq k - 1 \), and \( \lambda_B = \frac{1}{k-1} \) otherwise. We have

\[
I_\lambda(X_1; X_2; \cdots; X_k) = H(X_1 X_2 \cdots X_k) - \frac{1}{k-1} \sum_i H(X_i|X_{[k]-i})
\]

\[
= \frac{1}{k-1} \left( -H(X_1 \cdots X_k) + \sum_i H(X_i) \right)
\]

\[
= \frac{1}{k-1} J(X_1; X_2; \cdots; X_k).
\]

Next, let \( \Pi = (\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_r) \) be a partition of \( [k] \) into \( r \geq 2 \) sets. Let \( \lambda_B = \frac{1}{r-1} \) if \( B = [k] - \mathcal{P}_i \) for some \( i \in [r] \), and \( \lambda_B = 0 \) otherwise. We have

\[
I_\lambda(X_1; X_2; \cdots; X_k) = \frac{1}{r-1} J(X_{\mathcal{P}_1}; X_{\mathcal{P}_2}; \cdots; X_{\mathcal{P}_r}).
\]

The following theorem complements Example [7]

**Theorem 8.** [15, Theorem 4.1] For any fractional partition \( \lambda_B \) and any \( X_1, X_2, \ldots, X_k \), we have

\[
I_\lambda(X_1; X_2; \cdots; X_k) \geq \min_{\Pi} \frac{1}{r-1} J(X_{\mathcal{P}_1}; X_{\mathcal{P}_2}; \cdots; X_{\mathcal{P}_r})
\]

where the minimum is over all \( r \geq 2 \) and over all partitions \( \Pi = (\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_r) \) of \( [k] \) into \( r \) sets.

C  A Dependence Balance Bound for \( I_\lambda \)

**Lemma 9.** Given random variables \( W_i, X_{ij}, Y_{ij} \) and \( Z_j \) for \( i \in [k], j \in [n] \) satisfying

\[
X_{ij} = f_{ij}(W_i, Y_{i[j-1]}), \quad i \in [k], \ j \in [n]
\]

and the Markov chain

\[
W_{[k]} Y_{[k]}^{j-1} Z_{j-1} \rightarrow X_{[k]j} \rightarrow Z_j Y_{[k]j}, \quad j \in [n]
\]

we have

\[
I_\lambda(W_1 Y_1^n; W_2 Y_2^n; \cdots; W_k Y_k^n|Z^n) - I_\lambda(W_1; W_2; \cdots; W_k) \\
\leq \sum_{j=1}^n I_\lambda(X_{1j} Y_{1j}; X_{2j} Y_{2j}; \cdots; X_{kj} Y_{kj}|Z_j, Z_{j-1}^{j-1}) - \sum_{j=1}^n I_\lambda(X_{1j}; X_{2j}; \cdots; X_{kj}|Z_{j-1}^{j-1}).
\]
Proof. We have

\[
I_\lambda(W_1 Y_1^n; W_2 Y_2^n; \cdots ; W_k Y_k^n | Z^n) - I_\lambda(W_1; \cdots ; W_k) \\
= \sum_{j=1}^{n} \left[ I_\lambda(W_1 Y_1^j; W_2 Y_2^j; \cdots ; W_k Y_k^j | Z^j) - I_\lambda(W_1 Y_1^{j-1}; W_2 Y_2^{j-1}; \cdots ; W_k Y_k^{j-1} | Z^{j-1}) \right] \\
= \sum_{j=1}^{n} \left[ (1 - \sum_B \lambda_B) \left( H(W[k] Y_k^j | Z^j) - H(W[k] Y_k^{j-1} | Z^{j-1}) \right) \right. \\
\left. + \sum_B \lambda_B \left( H(W_{B^c} Y_{B^c}^j | Z^j) - H(W_{B^c} Y_{B^c}^{j-1} | Z^{j-1}) \right) \right] \\
\leq \sum_{j=1}^{n} \left[ (1 - \sum_B \lambda_B) \left( H(Y[k,j] Y_{k,j} Z^j) - I(Z_j; X_{k,j} Z^{j-1}) \right) \\
+ \sum_B \lambda_B \left( H(Y_{B^c,j} Y_{B^c,j} Z^j) - I(Z_j; X_{B^c,j} Z^{j-1}) \right) \right] \\
= \sum_{j=1}^{n} \left[ (1 - \sum_B \lambda_B) \left( H(X[k,j] Y_{k,j} Z^j) - H(X_{k,j} Z^{j-1}) \right) \\
+ \sum_B \lambda_B \left( H(X_{B^c,j} Y_{B^c,j} Z^j) - H(X_{B^c,j} Z^{j-1}) \right) \right] \\
= \sum_{j=1}^{n} \left[ I_\lambda(X_1 Y_1; X_2 Y_2; \cdots ; X_k Y_k | Z_j, Z^{j-1}) - I_\lambda(X_1; X_2; \cdots ; X_k | Z^{j-1}) \right].
\]

\[ \square \]

Even though we use Lemma 9 in the context of key agreement rate, it is of independent interest. For example, one can apply Lemma 9 to a k-user MAC with generalized feedback

\[
p(y_{F_1}, y_{F_2}, \cdots , y_{F_k}, y|x_1, x_2, \ldots , x_k).
\]

Here the \(X_i\)'s are channel inputs and the \(Y_{F_i}\)'s are the noisy feedback that they receive. The receiver sees \(Y\). Suppose first that \(Y_{F_i} = Y\) for all \(i\) which gives an ordinary MAC with feedback. In this case, Lemma 9 recovers the “refined dependence balance equations” of [16] as we vary \(\lambda\). Lemma 9 also implies the following outer bound:

**Theorem 10.** Consider a MAC with two-users with generalized feedback of the form \(Y_{F_1} = (Y, \tilde{Y}_{F_1})\) and \(Y_{F_2} = (Y, \tilde{Y}_{F_2})\). Then, for any achievable rate pair \((R_1, R_2)\) satisfies

\[
\{ (R_1, R_2) : R_1 \leq I(X_1; Y, \tilde{Y}_{F_2}| X_2, T_1, T_2) \}
\]

\[
R_2 \leq I(X_2; Y, \tilde{Y}_{F_1}| X_1, T_1, T_2)
\]

\[
R_1 + R_2 \leq I(X_1, X_2; Y, \tilde{Y}_{F_1}, \tilde{Y}_{F_2}| T_1, T_2)
\]

\[
R_1 + R_2 \leq I(X_1, X_2; Y| T_1)
\]
for some \( p(t_1, t_2, x_1, x_2) \) satisfying

\[
I(X_1; X_2|T_1, T_2) \leq I(X_1; X_2|Y_{F_1}, Y_{F_2}, T_1, T_2) \\
I(X_1; X_2|T_1) \leq I(X_1, Y_{F_1}; X_2, Y_{F_2}|T_1, Y)
\] (22)

Moreover, one can assume that \(|T_1| \leq 5\) and \(|T_2| \leq |X_1||X_2| + 3\).

This bound reduces to the bound given in Theorem 1 of [17] if we drop the constraint (23). The bound is proved by choosing \( T_1 = (Q, Y^{Q-1}) \) and \( T_2 = (Y_{F_1}^{Q-1}, Y_{F_2}^{Q-1}) \) where \( Q \) is the time-sharing variable. Equations (18)-(20) and (22) follow from Theorem 1 of [17]. Equations (21) and (23) are new. We show these two equations for an arbitrary \( k \)-user MAC. Let \( M_i \) be the message of transmitter \( i \). Then, Lemma 9 implies

\[
0 \leq I(X_1Y_{F_1}; X_2Y_{F_2}; \cdots; X_kY_{F_k}|Y^n) - I(M_1; M_2; \cdots; M_k) \\
\leq \sum_{j=1}^n I(X_{1j}Y_{F_{1j}}; X_{2j}Y_{F_{2j}}; \cdots; X_{kj}Y_{F_{kj}}|Y_j, Y^{j-1}) - \sum_{j=1}^n I(X_{1j}; X_{2j}; \cdots; X_{kj}|Y^{j-1})
\]

Next, we also have

\[
n \sum_{i=1}^k R_i = H(M_{[k]}) \\
= I(M_{[k]}; Y^n) + nk(\epsilon) \\
= \sum_{j=1}^n I(M_{[k]}; Y_j|Y^{j-1}) \\
\leq \sum_{j=1}^n I(M_{[k]}Y_{F_{[k]}}^{j-1}; Y_j|Y^{j-1}) \\
= \sum_{j=1}^n I(M_{[k]}Y_{F_{[k]}}^{j-1}; X_{[k]}; Y_j|Y^{j-1}) \\
= \sum_{j=1}^n I(X_{[k]}; Y_j|Y^{j-1})
\] (24)

Letting \( T_1 = (Q, Y^{Q-1}) \) for a time sharing variable \( Q \) gives the desired bound.

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