Stability of transition waves and positive entire solutions of Fisher-KPP equations with time and space dependence

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Abstract
This paper is concerned with the stability of transition waves and strictly positive entire solutions of random and nonlocal dispersal evolution equations of Fisher-KPP type with general time and space dependence, including time and space periodic or almost periodic dependence as special cases. We first show the existence, uniqueness, and stability of strictly positive entire solutions of such equations. Next, we show the stability of uniformly continuous transition waves connecting the unique strictly positive entire solution and the trivial solution zero and satisfying certain decay property at the end close to the trivial solution zero (if it exists). The existence of transition waves has been studied in Liang and Zhao (2010 J. Funct. Anal. 259 857–903), Nadin (2009 J. Math. Pures Appl. 92 232–62), Nolen et al (2005 Dyn. PDE 2 1–24), Nolen and Xin (2005 Discrete Contin. Dyn. Syst. 13 1217–34) and Weinberger (2002 J. Math. Biol. 45 511–48) for random dispersal Fisher-KPP equations with time and space periodic dependence, in Nadin and Rossi (2012 J. Math. Pures Appl. 98 633–53), Nadin and Rossi (2015 Anal. PDE 8 1351–77), Nadin and Rossi (2017 Arch. Ration. Mech. Anal. 223 1239–67), Shen (2010 Trans. Am. Math. Soc. 362 5125–68), Shen (2011 J. Dynam. Differ. Equ. 23 1–44), Shen (2011 J. Appl. Anal. Comput. 1 69–93), Tao et al (2014 Nonlinearity 27 2409–16) and Zlatoš (2012 J. Math. Pures Appl. 98 89–102) for random dispersal Fisher-KPP equations with quite general time and/or space dependence, and in Coville et al (2013 Ann. Inst. Henri Poincare 30 179–223), Rawal et al (2015 Discrete Contin. Dyn. Syst. 35 1609–40) and Shen and Zhang (2012 Comm. Appl. Nonlinear Anal. 19 73–101) for nonlocal dispersal Fisher-KPP equations with time and/or space periodic dependence. The stability result established in this paper implies that the transition waves obtained in many of the above...
mentioned papers are asymptotically stable for well-fitted perturbation. Up to the author’s knowledge, it is the first time that the stability of transition waves of Fisher-KPP equations with general time and space dependence is studied.

Keywords: Fisher-KPP equation, random dispersal, nonlocal dispersal, transition wave, positive entire solution, stability, almost periodic

Mathematics Subject Classification numbers: 35B08, 35C07, 35K57, 45J05, 47J35, 58D25, 92D25

1. Introduction

The current paper is devoted to the study of the stability of transition waves and entire positive solutions of dispersal evolution equations of the form,

$$\frac{\partial u}{\partial t} = Au + uf(t, x, u), \quad x \in \mathbb{R},$$  

(1.1)

where $Au(t, x) = u_{xx}(t, x)$ or $Au(t, x) = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$ for some nonnegative smooth function $\kappa(\cdot)$ with $\kappa(z) > 0$ for $\|z\| < r_0$ and some $r_0 > 0$, $\kappa(z) = 0$ for $\|z\| \geq r_0$, and $\int_{\mathbb{R}} \kappa(y)dy = 1$, and $f(t, x, u)$ is of Fisher-KPP type in $u$. More precisely, we assume that $f(t, x, u)$ satisfies the following standing assumption.

(H0) $f(t, x, u)$ is globally Hölder continuous in $t$ uniformly with respect to $x \in \mathbb{R}$ and $u$ in bounded sets, is globally Lipschitz continuous in $x$ uniformly with respect to $t \in \mathbb{R}$ and $u$ in bounded sets, and is differentiable in $u$ with $f_u(t, x, u)$ being bounded and uniformly continuous in $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $u$ in bounded sets. There are $\beta_0 > 0$ and $P_0 > 0$ such that $f(t, x, u) \leq -\beta_0$ for $t, x \in \mathbb{R}$ and $u \geq P_0$ and $\frac{uf_u(t, x, u)}{f_u(t, x, u)} \leq -\beta_0$ for $t, x \in \mathbb{R}$ and $u \geq 0$. Moreover,

$$-\infty < \inf_{t \in \mathbb{R}, x \in \mathbb{R}, 0 \leq u \leq M} f(t, x, u) \leq \sup_{t \in \mathbb{R}, x \in \mathbb{R}, 0 \leq u \leq M} f(t, x, u) < \infty$$  

(1.2)

for all $M > 0$, and

$$\lim_{t-s \to +\infty} \frac{1}{t-s} \int_s^t \inf_{x \in \mathbb{R}} f(\tau, x, 0)d\tau > 0.$$  

(1.3)

Equations (1.1) appears in the study of population dynamics of species in biology (see [1, 2, 9, 19, 30]), where $u(t, x)$ represents the population density of a species at time $t$ and space location $x$, $Au$ describes the dispersal or movement of the organisms and $f(t, x, u)$ describes the growth rate of the population. When $Au(t, x) = u_{xx}$, it indicates that the movement of the organisms occurs between adjacent locations randomly and the dispersal in this case is referred to as random dispersal. When $Au(t, x) = \int_{\mathbb{R}} \kappa(y - x)u(t, y)dy - u(t, x)$, it indicates that the movement of the organisms occurs between adjacent as well as non-adjacent locations and the dispersal in this case is referred to as nonlocal dispersal. The time and space dependence of the equation reflects the heterogeneity of the underlying environments.

Because of biological reason, only nonnegative solutions of (1.1) will be considered throughout this paper. Also, by a solution of (1.1) in this paper, we always mean a classical solution, i.e. a solution satisfies (1.1) in the classical sense, unless otherwise specified. A function $u(t, x)$ is called an entire solution if it is a bounded and continuous function on $\mathbb{R} \times \mathbb{R}$ and satisfies (1.1) for $(t, x) \in \mathbb{R} \times \mathbb{R}$. An entire solution $u(t, x)$ is called strictly positive if $\inf_{(t, x) \in \mathbb{R} \times \mathbb{R}} u(t, x) > 0$. In the following, we may call a strictly positive entire solution a positive entire solution if no confusion occurs.
Equation (1.1) satisfying (H0) is called in the literature a Fisher-KPP type equation due to the pioneering papers of Fisher [19] and Kolmogorov et al [30] on the following special case of (1.1),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u), \quad x \in \mathbb{R}. \quad (1.4)$$

It is clear that $u(t, x) \equiv 1$ is a unique strictly positive entire solution of (1.4). Moreover, it is globally stable with respect to strictly positive initial data $u_0(x)$.

The existence, uniqueness, and stability of positive entire solutions of (1.1) is one of the central problems about the dynamics of (1.1). Assume (H0). We show in this paper that

- (1.1) has a unique stable strictly positive entire solution $u^+(t, x)$. If, in addition, $f(t, x, u)$ is periodic in $t$ and/or $x$, then so is $u^+(t, x)$, and if $f(t, x, u)$ and $f_u(t, x, u)$ are almost periodic in $t$ and/or $x$, then so if $u^+(t, x)$ (see theorem 2.1).

It should be pointed out that when the dispersal is random, there are many studies on the positive entire solutions of (1.1) for various special cases (see [5, 6, 13, 37, 38, 53], etc). When the dispersal is nonlocal, there are also several studies on the positive entire solutions of (1.1) for some special cases (see [3, 4, 31, 47, 56, 57], etc). It should also be pointed out that, in the random dispersal case, by the regularity and a priori estimates for parabolic equations (see [21]), it is easy to prove the continuity of bounded solutions. In the nonlocal dispersal case, due to the lack of regularity of solutions, the proof of the continuity of bounded solutions is not trivial. Thanks to the existence, uniqueness, and stability of a unique strictly positive entire solution, (1.1) is also said to be of monostable type.

The traveling wave problem is also among the central problems about the dynamics of (1.1). This problem is well understood for the classical Fisher or KPP equation (1.4). For example, Fisher in [19] found traveling wave solutions $u(t, x) = \phi(x - ct)$, $\phi(-\infty) = 1$, $\phi(\infty) = 0$ of all speeds $c \geq 2$ and showed that there are no such traveling wave solutions of slower speed. He conjectured that the take-over occurs at the asymptotic speed 2. This conjecture was proved in [30] for some special initial distribution and was proved in [2] for the general case. More precisely, it is proved in [30] that for the nonnegative solution $u(t, x)$ of (1.4) with $u(0, x) = 1$ for $x < 0$ and $u(0, x) = 0$ for $x > 0$, $\lim_{\tau \to \infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$.

It is proved in [2] that for any nonnegative solution $u(t, x)$ of (1.4), if at time $t = 0$, $u$ is 1 near $-\infty$ and 0 near $\infty$, then $\lim_{\tau \to \infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$. Put $c^* = 2$, $c^*$ is of the following spatially spreading property: for any nonnegative solution $u(t, x)$ of (1.4), if at time $t = 0$, $u(0, x) \geq \sigma$ for some $\sigma > 0$ and $x \ll 1$ and $u(0, x) = 0$ for $x \gg 1$, then

$$\inf_{x \leq c^*} |u(t, x) - 1| \to 0 \quad \forall c^{'} < c^* \quad \text{and} \quad \sup_{x \geq c^*} u(t, x) \to 0 \quad \forall c^{''} > c^* \quad \text{as} \quad t \to \infty.$$  

In literature, $c^*$ is hence called the spreading speed for (1.4). The results on traveling wave solutions of (1.4) have been well extended to general time and space independent monostable equations (see [1, 2, 12, 22, 29, 49, 59], etc).

Due to the inhomogeneity of the underlying media of biological models in nature, the investigation of the traveling wave problem for time and/or space dependent dispersal evolution equations is gaining more and more attention. The notion of transition waves or generalized traveling waves has been introduced for dispersal evolution equations with general time and space dependence (see definition 2.2 and remark 2.2), which naturally extends the notion of traveling wave solutions for time and space independent dispersal evolution equations to the equations with general time and space dependence. A huge amount of research has been carried out toward the transition waves or generalized traveling waves of various time and/or space
dependent monostable equations. See, for example, [7–11, 20, 22, 25, 27, 28, 32–34, 36, 39, 41–46, 51–53, 58, 60–63], and references therein for space and/or time dependent Fisher-KPP type equations with random dispersal, and see, for example, [15–17, 35, 48, 54–57], and references therein for space and/or time dependent Fisher-KPP type equations with nonlocal dispersal.

It should be pointed out that the works [23, 52, 54, 56] considered the stability of transition waves in spatially periodic and time independent or spatially homogeneous and time dependent Fisher-KPP type equations with random and nonlocal dispersal. In particular, the stability of transition waves in spatially periodic and time independent Fisher-KPP type equations with random dispersal (resp. nonlocal dispersal) is studied in [23] (resp. [56]) and the stability of transition waves in spatially homogeneous and time almost periodic Fisher-KPP type equations with random dispersal (resp. nonlocal dispersal) is investigated in [52] (resp. [54]). The paper [23] also considered the stability of traveling waves in spatially and temporally periodic Fisher-KPP equations with random dispersal (see [23, section 1.4]).

However, as long as the equations depend on both time and space variables non-periodically, all the existing works are on the existence of transition waves or generalized traveling waves and there is little study on the stability of transition waves. In the current paper, we consider the stability of transition waves of Fisher-KPP equations with general time and space dependence (see definition 2.2 for the definition of transition waves). We show that

- Any transition wave of (1.1) connecting \( u^+ (t, x) \) and 0 and satisfying certain decaying property near 0 is asymptotically stable for well-fitted perturbation (see theorem 2.2 for detail).

We point out that the existence of transition waves of (1.1) with non-periodic time and/or space dependence has been studied in [35, 41–43, 52–54, 63]. Applying the above stability result for general transition waves of (1.1), we prove

- The non-critical transition waves established in [35, 41–43, 52–54, 63] are asymptotically stable for well-fitted perturbation (see theorem 2.3 and remark 2.3 for detail).

Up to the author’s knowledge, it is the first time that the stability of transition waves of Fisher-KPP type equations with general time and space dependence is studied. Among the technical tools used in the proofs of the main results are spectral theory for linear dispersal evolution equations with time and space dependence, comparison principle, and the very non-trivial application of the so called part metric.

The rest of the paper is organized as follows. In section 2, we will introduce the standing notations, definitions, and state the main results of the paper. We study the existence, uniqueness and stability of positive entire solutions in section 3. Sections 4 and 5 are devoted to the proofs of the main results on transition waves.

2. Notations, definitions, and main results

In this section, we introduce the standing notations, definitions, and state the main results of the paper. Throughout this section, we assume that (H0) holds.

First of all, we recall the definition of almost periodic functions.

**Definition 2.1 (Almost periodic function).**

(1) A continuous function \( g : \mathbb{R} \to \mathbb{R} \) is called almost periodic if for any \( \epsilon > 0 \), the set

\[
T(\epsilon) = \{ \tau \in \mathbb{R} \mid |g(t + \tau) - g(t)| < \epsilon \text{ for all } t \in \mathbb{R} \}
\]

is relatively dense in \( \mathbb{R} \).
(2) Let $g(t, x, u)$ be a continuous function of $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$. $g$ is said to be almost periodic in $t$ uniformly with respect to $x \in \mathbb{R}^m$ and $u$ in bounded sets if $g$ is uniformly continuous in $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, and $u$ in bounded sets and for each $x \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$, $g(t, x, u)$ is almost periodic in $t$.

(3) For a given almost periodic function $g(t, x, u)$, the hull $H(g)$ of $g$ is defined by

$$H(g) = \{ \tilde{g}(\cdot, \cdot) | \exists n \to \infty \text{ such that } g(t + t_n, x, u) \to \tilde{g}(t, x, u) \text{ uniformly in } t \in \mathbb{R} \text{ and } (x, u) \text{ in bounded sets} \}.$$  

Remark 2.1. Let $g(t, x, u)$ be a continuous function of $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$. $g$ is almost periodic in $t$ uniformly with respect to $x \in \mathbb{R}^m$ and $u$ in bounded sets if and only if $g$ is uniformly continuous in $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, and $u$ in bounded sets and for any sequences $\{\alpha_n\}$, $\{\beta_n\} \subset \mathbb{R}$, there are subsequences $\{\alpha_n\} \subset \{\alpha'_n\}$, $\{\beta_n\} \subset \{\beta'_n\}$ such that

$$\lim_{n \to \infty} \lim_{m \to \infty} g(t + \alpha_n + \beta_m, x, u) = \lim_{n \to \infty} g(t + \alpha_n + \beta_n, x, u)$$

for each $(t, x, u) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$ (see [18, theorems 1.17 and 2.10]).

Next, let

$$C^h_{\text{unif}}(\mathbb{R}) = \{ u \in C(\mathbb{R}, \mathbb{R}) | u(x) \text{ is uniformly continuous and bounded on } \mathbb{R} \}$$

endowed with the norm $\|u\|_{\infty} = \sup_{x \in \mathbb{R}} |u(x)|$. By general semigroup theory (see [26]), for any $u_0 \in C^h_{\text{unif}}(\mathbb{R})$, (1.1) has a unique (local) solution $u(t, x, t_0, u_0)$ with $u(t_0, x, t_0, u_0) = u_0(x)$ for $x \in \mathbb{R}$.

We then consider the existence, uniqueness, and stability of strictly positive entire solutions of (1.1). The following is the main result of the paper on the existence, uniqueness, and stability of strictly positive entire solution of (1.1).

Theorem 2.1. There is a unique bounded strictly entire solution $u^+(t, x)$ of (1.1) with $u^+(t, x)$ being uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$. Moreover, for any given $u_0 \in C^h_{\text{unif}}(\mathbb{R})$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0$,

$$\lim_{t \to \infty} \|u(t + t_0, \cdot, t_0, u_0) - u^+(t + t_0, \cdot)\|_{\infty} = 0$$

uniformly in $t_0 \in \mathbb{R}$. If, in addition, $f(t, x, u)$ is periodic in $t$ (resp. periodic in $x$), then so is $u^+(t, x)$. If $f(t, x, u)$ and $f_\alpha(t, x, u)$ are almost periodic in $t$ (resp. almost periodic in $x$), then so is $u^+(t, x)$.

We now consider transition waves of (1.1) connecting $u^+(t, x)$ and 0.

Definition 2.2.

(1) An entire solution $u = U(t, x)$ of (1.1) is a transition wave (connecting 0 and $u^+(\cdot, \cdot)$) if $U(t, x) \in (0, u^+(t, x))$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, and there exists a function $X : \mathbb{R} \to \mathbb{R}$, called interface location function, such that

$$\lim_{t \to -\infty} U(t, x + X(t)) = u^+(t, x + X(t)) \text{ and } \lim_{t \to \infty} U(t, x + X(t)) = 0 \text{ uniformly in } t \in \mathbb{R}.$$  

(2) Assume that $u(t, x) = U(t, x)$ is a transition wave of (1.1) with $X : \mathbb{R} \to \mathbb{R}$ being an interface location function. $c := \lim_{x \to \infty, f \to x} \frac{X(t) - X(s)}{t - s}$ is called the least mean speed of the transition wave. If $\lim_{x \to \infty, f \to x} \frac{X(t) - X(s)}{t - s}$ exists, $c := \lim_{x \to \infty, f \to x} \frac{X(t) - X(s)}{t - s}$ is called the average speed or mean speed of the transition wave.
(3) Assume that \( u(t, x) = U(t, x) \) is a transition wave of (1.1). It is called asymptotically stable if for any \( t_0 \in \mathbb{R} \) and \( u_0 \in C^\infty(\mathbb{R}) \) satisfying that \( u_0(x) > 0 \) for all \( x \in \mathbb{R} \) and

\[
\inf_{x \leq t_0} u_0(x) > 0 \quad \forall \ x_0 \in \mathbb{R}, \quad \lim_{x \to \infty} U(t_0, x) = 1,
\]

there holds

\[
\lim_{t \to \infty} \| U(t + t_0, \cdot) - U(t_0, \cdot) \|_{C^\infty(\mathbb{R})} = 0.
\]

**Remark 2.2.**

(1) The interface location function \( X(t) \) of a transition wave \( u = U(t, x) \) tells the position of the transition front of \( U(t, x) \) as time \( t \) elapses, while the uniform-in-\( t \) limits (the essential property in the definition) shows the bounded interface width, that is,

\[
\forall \ 0 < \epsilon_1 \leq \epsilon_2 < 1, \quad \sup_{t \in \mathbb{R}} \text{diam} \{ x \in \mathbb{R} | \epsilon_1 \leq U(t, x) \leq \epsilon_2 \} < \infty.
\]

Notice, if \( \xi(t) \) is a bounded function, then \( X(t) + \xi(t) \) is also an interface location function. Thus, interface location function is not unique. But, it is easy to check that if \( Y(t) \) is another interface location function, then \( X(t) - Y(t) \) is a bounded function. Hence, interface location functions are unique up to addition by bounded functions and the least mean speed of a transition wave is well defined.

(2) When \( f(t + T, x, u) = f(t, x + p, u) = f(t, x, u) \), an entire solution \( u = U(t, x) \) of (1.1) is called a periodic traveling wave solution with speed \( c \) and connecting \( u^+(t, x) \) and 0 if there is \( \Phi(x, t, y) \) such that

\[
U(t, x) = \Phi(x - ct, t, ct),
\]

\[
\Phi(x, t + T, y) = \Phi(x, t, y + p) = \Phi(x, t, y),
\]

and

\[
\lim_{x \to -\infty} \left( \Phi(x, t, y) - u^+(t, x + y) \right) = 0, \quad \lim_{x \to \infty} \Phi(x, t, y) = 0
\]

uniformly in \( t \in \mathbb{R} \) and \( y \in \mathbb{R} \). It is clear that if \( u = U(t, x) \) is a periodic traveling wave solution, then it is a transition wave.

(3) When \( f(t, x, u) \) is almost periodic in \( t \) and periodic in \( x \) with period \( p \), an entire solution \( u = U(t, x) \) of (1.1) is called an almost periodic traveling wave solution with average speed \( c \) and connecting \( u^+(t, x) \) and 0 if there are \( \xi(t) \) and \( \Phi(x, t, y) \) such that

\[
U(t, x) = \Phi(x - \xi(t), t, \xi(t)),
\]

\[
\Phi(x, t, y) \quad \text{is almost periodic in} \ t \quad \text{and periodic in} \ y,
\]

\[
\lim_{x \to -\infty} \left( \Phi(x, t, y) - u^+(t, x + y) \right) = 0, \quad \lim_{x \to \infty} \Phi(x, t, y) = 0
\]

uniformly in \( t \in \mathbb{R} \) and \( y \in \mathbb{R} \), and

\[
\lim_{t \to t + \infty} \frac{\xi(t) - \xi(s)}{t - s} = c.
\]
It is clear that if \( u = U(t,x) \) is an almost periodic traveling wave solution, then it is a transition wave.

(4) The reader is referred to \([10, 11]\) for the introduction of the notion of transition waves in the general case, and to \([36, 50, 52, 53]\) for the time almost periodic or space almost periodic cases.

(5) In the case that \( Au = u_{tt} \), by the regularity and \textit{a priori} estimates for parabolic equations, any continuous transition wave \( u = U(t,x) \) of \((1.1)\) is uniformly continuous in \((t,x) \in \mathbb{R} \times \mathbb{R} \). In the case that \( Au(t,x) = \int_0^1 \kappa(y-x)u(t,y)dy - u(t,x) \), it is proved in \([55]\) that a transition wave \( u = U(t,x) \) of \((1.1)\) is uniformly continuous under quite general conditions.

We have the following general theorem on the stability of transition waves of \((1.1)\).

**Theorem 2.2.** Assume that \( u = U(t,x) \) is a transition wave of \((1.1)\) with interface location \( X(t) \) satisfying the following properties: \( U(t,x) \) is uniformly continuous in \((t,x) \in \mathbb{R} \times \mathbb{R} \),

\[
\forall \tau > 0, \quad \sup_{t,s \in \mathbb{R}, |t-s| \leq \tau} |X(t) - X(s)| < \infty,
\]

and there are positive continuous functions \( \phi(t,x) \) and \( \phi_1(t,x) \) such that

\[
\liminf_{x \to -\infty} \phi(t,x) = \infty, \quad \liminf_{x \to -\infty} \phi_1(t,x) = \infty, \quad \lim_{x \to -\infty} \phi(t,x) = 0, \quad \lim_{x \to -\infty} \phi_1(t,x) = 0, \tag{2.4}
\]

exponentially, and the second limit in \((2.6)\) is uniformly in \( t \);\n
\[
d^* \phi(t,x) - d_1^* \phi_1(t,x) \leq U(t,x) \leq d^* \phi(t,x) + d_1^* \phi_1(t,x) \tag{2.5}
\]

for some \( d^*, d_1^* > 0 \) and all \( t,x \in \mathbb{R} \); and for any given \( t_0 \in \mathbb{R} \) and \( u_0 \in C^0_{ubd} (\mathbb{R}) \) with \( u_0(x) \geq 0 \), if

\[
u_0(x) \geq d\phi(t_0,x) - d_1 \phi_1(t_0,x) \quad \text{resp.} \quad \nu_0(x) \leq d\phi(t_0,x) + d_1 \phi_1(t_0,x)
\]

for some \( 0 < d < 2d^* \), \( d_1 \gg 1 \), and all \( x \in \mathbb{R} \), then

\[
u(t,x; t_0, u_0) \geq d\phi(t,x) - d_1 \phi_1(t,x) \quad \text{resp.} \quad \nu(t,x; t_0, u_0) \leq d\phi(t,x) + d_1 \phi_1(t,x)
\]

for all \( t \geq t_0 \) and \( x \in \mathbb{R} \). Then the transition wave \( u = U(t,x) \) is asymptotically stable.

It should be pointed out that, when \( Au = u_{tt} \), by \([24, \text{proposition 4.2,} (2.4)\) holds for any transition wave of \((1.1)\). In the above theorem, the existence of transition waves is assumed. The existence of transition waves of \((1.1)\) with non-periodic time and/or space dependence has been studied in \([35, 41–43, 52–54, 63]\). Applying theorem 2.2 or the arguments in the proof of theorem 2.2, we can establish the asymptotic stability of the transition waves proved in \([35, 41–43, 48, 52–54, 63]\). For convenience, we introduce the following assumptions.

\textbf{(H1)} \( f(t,x+pu,u) = f(t,x,u) \) for some \( p > 0 \), \( f(t,x,1) = 0 \), \( f(t,x,u) \leq f(t,x,0) \) for \( 0 \leq u \leq 1 \), \( \inf_{(t,x) \in \mathbb{R} \times \mathbb{R}} f(t,x,u) > 0 \) for \( u \in (0,1) \), and \( f(t,x,u) \geq f(t,x,0)u - Cu^{1+\nu} \) for some \( C > 0 \), \( \delta, \nu \in (0,1) \), \( u \in (0,\delta) \).
\textbf{(H2)} \( f(t, x, u) = a(x)(1 - u), \inf_{x \in \mathbb{R}} a(x) > 0, \) \( a(x) \) is almost periodic in \( x, \) and there exists an almost periodic positive function \( \phi \in C^2(\mathbb{R}) \) such that \( \phi'' + a(x)\phi(x) = \lambda_0 \phi(x) \) for \( x \in \mathbb{R}, \) where

\[ \lambda_0 = \inf \{ \lambda \in \mathbb{R} \mid \exists \phi \in C^2(\mathbb{R}), \phi > 0, \phi'' + a(x)\phi(x) \leq \lambda \phi(x) \text{ for } x \in \mathbb{R} \}. \]

\textbf{(H3)} \( f(t, x, u) = f(x, u), f(x, 1) = 0, f_u(x, u) < 0, \)

\[ a(x)g(u) \leq af(x, u) \leq a(x)u, \quad u \in [0, 1], \]

\( g \in C^1([0, 1]), \) \( g(0) = g(1) = 0, \) \( g'(0) = 1, \) \( 0 \leq g(u) \leq u \) for \( u \in (0, 1), \)

\[ \int_0^1 \frac{u - g(u)}{u^2} du < \infty, \quad \left( \frac{g(u)}{u} \right)' < 0 \text{ for } u \in (0, 1), \]

and \( 0 < a_- := \inf a(x) \leq \sup a(x) := a_+ < \infty. \)

Observe that, assuming one of (H1), (H2), and (H3), \( u^+(t, x) \equiv 1. \) We have the following theorem on the asymptotic stability of the transition waves established in \([41-43, 48, 52, 53, 63].\)

**Theorem 2.3.**

1. Assume that \( Au = u_{xx} \) and that \( f(t, x, u) \) satisfies (H1). Then there is \( c^* \in \mathbb{R} \) such that for any \( c > c^*, (1.1) \) has an asymptotically stable transition wave with least mean speed \( c. \)

2. Assume that \( Au = u_{xx} \) and that \( f(t, x, u) \) satisfies (H2). Then there is \( c^* \in \mathbb{R} \) such that for any \( c > c^*, (1.1) \) has an asymptotically stable transition wave with mean speed \( c. \)

3. Assume that \( Au = u_{xx} \) and that \( f(t, x, u) \) satisfies (H3). Let \( \lambda_0 = \sup \sigma[\phi'' + a(\cdot)] \) and \( \lambda_1 = 2a_- \). If \( \lambda_0 < \lambda_1, \) then for any \( \lambda \in (\lambda_0, \lambda_1), \) there is an asymptotically stable transition wave solution \( U_\lambda(t, x) \) of (1.1) satisfying that

\[ U_\lambda(t, x) \leq v_\lambda(t, x), \]

where \( v_\lambda(t, x) = \phi_\lambda(x)e^{\lambda t} \) and \( \phi_\lambda(x) \) is the unique solution of

\[ \phi_\lambda''(x) + a(x)\phi_\lambda(x) = \lambda\phi_\lambda(x) \quad \text{for } x \in \mathbb{R}, \quad \phi_\lambda(0) = 1, \quad \lim_{x \to \infty} \phi_\lambda(x) = 0. \]

4. Assume that \( Au = \int_0^\infty \kappa(y - x)u(t, y)dy - u(t, x), \) and that \( f(t, x, u) \) is periodic in both \( t \) and \( x, \) then there is \( c^* > 0 \) such that for any \( c > c^*, \) there is an asymptotically stable periodic traveling wave of (1.1) with speed \( c. \)

**Remark 2.3.**

1. The existence of transition waves in theorem 2.3(1) is proved in \([42].\) In fact, the authors of \([42] \) studied a more general case. With the same techniques for the proof of theorem 2.2 the asymptotic stability of transition waves in this general case can also be established. The following special cases should be mentioned. The existence of transition waves in the case that \( f(t, x, u) \equiv f(t, u) \) is proved in \([41].\) When \( f(t, x, u) \) is almost periodic in \( t \) or recurrent and unique ergodic in \( t, \) the existence of transition waves is proved in \([53].\) In \([52],\) stability, uniqueness, and almost periodicity of transition waves are also proved when \( f(t, x, u) \equiv f(t, u) \) is almost periodic in \( t. \)
(2) The existence of transition waves in theorem 2.3(2) is proved in [43]. Again, the authors of [43] actually studied a more general case, and with the same techniques for the proof of theorem 2.2 the asymptotic stability of transition waves in this general case can also be established.

(3) The existence of transition waves in theorem 2.3(3) is proved in [63]. By the similar arguments as those in theorem 2.3(3), it can be proved that the transition waves established in [35] for the case that $\mathcal{A}u = \int_{\mathbb{R}} \kappa(y-x)u(t,y)dy - u(t,x)$ and $f(t,x,u) \equiv f(x,u)$ are asymptotically stable, and that the transition waves established in [54] for the case that $\mathcal{A}u = \int_{\mathbb{R}} \kappa(y-x)u(t,y)dy - u(t,x)$ and $f(t,x,u) \equiv f(t,u)$ are asymptotically stable (the stability of transition waves in this case has been proved in [54] by the ‘squeezed’ techniques).

(4) The existence of transition waves in theorem 2.3(4) is proved in [48]. In the case that $f(t,x,u) \equiv f(x,u)$ is periodic in $x$, the stability of transition waves is proved in [56] by the ‘squeezed’ techniques.

(5) Transition waves with least mean speed $c^*$ are related to the so called critical traveling waves in literature (see [40, 50]). Both the existence and stability of critical transition waves are much more difficult to study. The reader is referred to [23] and references therein for the study of the stability of critical traveling waves in the space and/or time periodic case with random dispersal. The reader is referred to [39, 41–43] for some results on the existence of critical transition waves of Fisher-KPP equations with random dispersal, and to [17] and reference therein for the existence of critical transition waves in time independent and space periodic or independent Fisher-KPP equations with nonlocal dispersal. The stability of critical transition waves in the general time and space dependent Fisher-KPP equations with random dispersal and in time and/or space periodic Fisher-KPP equations with nonlocal dispersal remains open.

3. Positive entire solutions

In this section, we study the existence, uniqueness, and stability of positive entire solutions and prove theorem 2.1.

For a given continuous and bounded function $u : [t_1, t_2] \times \mathbb{R} \to \mathbb{R}$, it is called a super-solution (sub-solution) of (1.1) on $[t_1, t_2]$ if

$$u(t,x) \geq (\leq) (\mathcal{A}u)(t,x) + u(t,x)f(t,x,u(t,x)) \quad \forall (t,x) \in (t_1, t_2) \times \mathbb{R}. \quad (3.1)$$

**Proposition 3.1 (Comparison principle).**

1. Suppose that $u^1(t,x)$ and $u^2(t,x)$ are sub- and super-solutions of (1.1) on $[t_1, t_2]$ with $u^1(t_1,x) \leq u^2(t_1,x)$ for $x \in \mathbb{R}$. Then $u^1(t,x) \leq u^2(t,x)$ for $t \in (t_1, t_2)$ and $x \in \mathbb{R}$. Moreover, if $u^1(t_1,x) \neq u^2(t_1,x)$ for $x \in \mathbb{R}$, then $u^1(t,x) < u^2(t,x)$ for $t \in (t_1, t_2)$ and $x \in \mathbb{R}$.

2. If $u_{01}, u_{02} \in C^0_{\text{unif}}(\mathbb{R})$ and $u_{01} \leq u_{02}$, then $u(t,:;t_0,u_{01}) \leq u(t,:,;t_0,u_{02})$ for $t > t_0$ at which both $u(t,:,;t_0,u_{01})$ and $u(t,:,;t_0,u_{02})$ exist. Moreover, if $u_{01} \neq u_{02}$, then $u(t,x;t_0,u_{01}) < u(t,x;t_0,u_{02})$ for all $x \in \mathbb{R}$ and $t > t_0$ at which both $u(t,:,;t_0,u_{01})$ and $u(t,:,;t_0,u_{02})$ exist.

3. If $u_{01}, u_{02} \in C^0_{\text{unif}}(\mathbb{R})$ and $u_{01} \leq u_{02}$ (i.e. $\inf_{x \in \mathbb{R}} (u_{02}(x) - u_{01}(x)) > 0$), then $u(t,:,;t_0,u_{01}) < u(t,:,;t_0,u_{02})$ for $t > t_0$ at which both $u(t,:,;t_0,u_{01})$ and $u(t,:,;t_0,u_{02})$ exist.
Proof. When $Au = u_{xx}$, the proposition follows from comparison principle for parabolic equations (see [21]). When $Au(t, x) = \int_{\Omega} \kappa(y - x)u(t, y)dy - u(t, x)$, it follows from comparison principle for nonlocal dispersal evolution equations (see, for example, [57, propositions 2.1 and 2.2]).

Note that, by (H0), for $M \gg 1$, $u(t, x) \equiv M$ is a super-solution of (1.1) on $\mathbb{R}$. The following proposition follows directly from proposition 3.1 and (H0).

Proposition 3.2. For any $t_0 \in \mathbb{R}$ and $u_0 \in C_{unif}^b(\mathbb{R})$ with $u_0(\cdot) \geq 0$, $u(t, x; t_0, u_0)$ exists for all $t \geq t_0$ and $\sup_{t \geq t_0} \|u(t + t_0, \cdot; t_0, u_0)\|_{\infty} \leq \infty$.

For given $u, v \in C_{unif}^b(\mathbb{R})$ with $u, v \geq 0$, if there is $\alpha_0 \geq 1$ such that

$$\frac{1}{\alpha_0} v(x) \leq u(x) \leq \alpha_0 v(x) \quad \forall x \in \mathbb{R},$$

then we can define the so called part metric $\rho(u, v)$ between $u$ and $v$ by

$$\rho(u, v) = \inf \{ \ln \alpha \mid \alpha \geq 1, \frac{1}{\alpha} v(\cdot) \leq u(\cdot) \leq \alpha v(\cdot) \}.$$ 

Note that for any given $u, v \in C_{unif}^b(\mathbb{R})$ with $u, v \geq 0$, the part metric between $u$ and $v$ may not be defined.

Proposition 3.3.

(1) For given $u_0, v_0 \in C_{unif}^b(\mathbb{R})$ with $u_0, v_0 \geq 0$, if $\rho(u_0, v_0)$ is defined, then for any $t_0 \in \mathbb{R}$, $\rho(u(t + t_0, \cdot; t_0, u_0), u(t + t_0, \cdot; t_0, v_0))$ is also defined for all $t > 0$. Moreover, $\rho(u(t + t_0, \cdot; t_0, u_0), u(t + t_0, \cdot; t_0, v_0))$ is non-increasing in $t$.

(2) For any $\epsilon > 0$, $\sigma > 0$, $M > 0$, and $r > 0$ with $\epsilon < M$ and $\sigma \leq \ln \frac{M}{\epsilon}$, there is $\lambda > 0$ such that for any $u_0, v_0 \in C_{unif}^b(\mathbb{R})$ with $\epsilon \leq u_0(x) \leq M$, $\epsilon \leq v_0(x) \leq M$ for $x \in \mathbb{R}$ and $\rho(u_0, v_0) \geq \sigma$, there holds

$$\rho(u(\tau + t_0, \cdot; t_0, u_0), u(\tau + t_0, \cdot; t_0, v_0)) \leq \rho(u_0, v_0) - \delta \quad \text{for all} \quad t_0 \in \mathbb{R}.$$

Proof. It follows from the similar arguments as those in [31, proposition 3.4].

Proof of theorem 2.1. We divide the proof into two steps.

Step 1. In this step, we prove the existence, uniqueness, and stability of bounded strictly positive entire solutions $u^+(t, x)$ with $u^+(t, x)$ being uniformly continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Note that the existence, uniqueness, and stability of bounded strictly positive entire solutions follows from the similar arguments as those in [14, theorem 1.11]. We outline the proof of existence in the following for the use in the proof of uniform continuity.

Clearly, $u(t, x) \equiv M$ is a super-solution of (1.1) for any $M \gg 1$. For given $\delta > 0$, let $\psi_\delta \equiv \delta$. By the similar arguments of [14, theorem 1.11], there are $\delta_0 > 0$ and $T > 0$ such that for any $0 < \delta \leq \delta_0$,

$$u(t_0 + T, \cdot; t_0, \psi_\delta) \geq \psi_\delta \quad \forall t_0 \in \mathbb{R}. \quad (3.2)$$

Fix $M \gg 1$ and $0 < \delta \ll 1$. Let $u_M \equiv M$ and $u_\delta \equiv \delta$. For given $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, let $u_{m-n, m}(\cdot) = u(MT, \cdot; (m - n)T, u_M)$ and $u_{m-n, m}(\cdot) = u(mT, \cdot; (m - n)T, u_M)$. Then for any $m \in \mathbb{Z}$ and $n \geq 0$,

$$\delta \leq u_{m-1, m}(\cdot) \leq u_{m-2, m}(\cdot) \leq \ldots, \quad M \geq u_{m-1, m}(\cdot) \geq u_{m-2, m}(\cdot) \geq \ldots.$$
Moreover, by proposition 3.3, it is not difficult to prove that
\[
\rho(u^{m-n,m}, u_{m-n,m}) \to 0 \quad \text{as } n \to \infty \quad \text{uniformly in } m \in \mathbb{Z}.
\] (3.3)

Hence there are \( u^{*,m} \in C^0_{\text{uni}}(\mathbb{R}) \) (\( m \in \mathbb{Z} \)) such that
\[
u^{*,m}(x) = \lim_{n \to \infty} u_{m-n,m}(x) = \lim_{n \to \infty} u^{m-n,m}(x) \quad \text{uniformly in } x \in \mathbb{R}.
\]

It is then not difficult to see that \( u(t, \cdot; 0, u^*,0) \) has backward extension for all \( t < 0 \) and hence \( u^+(t,x) := u(t, \cdot; 0, u^*,0) \) is an entire solution. Moreover,
\[
\delta \leq u^+(kT, x) \leq M \quad \forall \ x \in \mathbb{R}, \ k \in \mathbb{Z}
\] (4.4)

and
\[
u^+(kT, x) = \lim_{n \to \infty} u(kT, x; (k-n)T, uM) \quad \text{uniformly in } x \in \mathbb{R}, \ k \in \mathbb{Z}.
\] (3.5)

By (4.4), \( u^+(t,x) \) is a bounded strictly positive entire solution of (1.1).

We now prove the uniform continuity of \( u^+(t,x) \) in \( (t,x) \in \mathbb{R} \times \mathbb{R} \). In the case that \( \Delta u = u_{xx} \), by the regularity and \textit{a priori} estimates for parabolic equations, we have that \( u^+(t,x) \) is uniformly continuous in \( (t,x) \in \mathbb{R} \times \mathbb{R} \). We then only need to prove that \( u^+(t,x) \) is uniformly continuous in \( (t,x) \in \mathbb{R} \times \mathbb{R} \) in the case that \( \mathcal{A} \) is a nonlocal dispersal operator. In this case, by the boundedness of \( u^+(t,x) \), \( u^+(t,x) \) is uniformly bounded. This implies that \( u^+(t,x) \) is uniformly continuous in \( t \) uniformly with respect to \( x \in \mathbb{R} \). We claim that \( u^+(t,x) \) is also uniformly continuous in \( x \) uniformly with respect to \( t \in \mathbb{R} \). Indeed, By (3.5), for any \( \epsilon > 0 \), there is \( K \in \mathbb{N} \) such that
\[
|u^+(t + kT, x) - u(t + kT, x; (k-K)T, uM)| < \epsilon \quad \forall \ x \in \mathbb{R}, \ 0 \leq t \leq T, \ k \in \mathbb{Z}.
\] (3.6)

It then suffices to prove that \( u(t + kT, x; (k-K)T, uM) \) is uniformly continuous in \( x \) uniformly with respect to \( t \in [0,T] \) and \( k \in \mathbb{Z} \). For any given \( h > 0 \) and \( k \in \mathbb{Z} \), let
\[
v(t,x;h,k) = u(t,x + h; (k-K)T, uM) - u(t,x; (k-K)T, uM).
\]

Then \( v(t,x;h,k) \) satisfies
\[
\begin{aligned}
\begin{cases}
\frac{\partial v}{\partial t} = \mathcal{A}v(t,x) + p(t,x)v(t,x; h,k) + q(t,x), & \ x \in \mathbb{R}, \ t > (k-K)T \\
v((k-K)T, x; h,k) = 0, & \ x \in \mathbb{R},
\end{cases}
\end{aligned}
\]

where
\[
p(t,x) = f(t,x + h, u(t,x + h; (k-K)T, uM)) + u(t,x; (k-K)T, uM) - u(t,x; (k-K)T, uM).
\]

and
\[
q(t,x) = u(t,x; (k-K)T, uM) \left[ f(t,x + h, u(t,x + h; (k-K)T, uM)) - f(t,x,u(t,x + h; (k-K)T, uM)) \right]
\]
Then
\[
\psi(t, \cdot; h, k) = \int_{(k-\varepsilon)T}^{t} e^{A(t-\tau)} p(\tau, \cdot) \psi(\tau, \cdot; h, k) d\tau + \int_{(k-\varepsilon)T}^{t} e^{A(t-\tau)} q(\tau, \cdot) d\tau.
\]

This together with the assumption (H0) implies that there is \( C > 0 \) such that
\[
|\psi(t, x; h, k)| \leq C \quad \forall 0 \leq t \leq T, \ x \in \mathbb{R}, \ k \in \mathbb{Z}.
\]

Then by (3.6) and Gronwall’s inequality, \( u^+(t, x) \) is uniformly continuous in \( x \) uniformly with respect to \( x \in \mathbb{R} \) and then \( u^+(t, x) \) is uniformly continuous in \((t, x) \in \mathbb{R} \times \mathbb{R} \).

By the similar arguments as those in [14, theorem 1.11], we have that bounded strictly positive entire solutions of (1.1) are stable and unique. This completes the proof of Step 1.

**Step 2.** In this step, we show that \( u^+(t, x) \) is almost periodic in \( t \) (resp., in \( x \)) if \( f(t, x, u) \) is almost periodic in \( t \) (resp., in \( x \)).

Assume that \( f(t, x, u) \) is almost periodic in \( t \). For any given sequences \( \{\alpha_n^+\}, \{\beta_n^+\} \subset \mathbb{R} \), there are subsequences \( \{\alpha_n\} \subset \{\alpha_n^+\}, \{\beta_n\} \subset \{\beta_n^+\} \) such that
\[
\lim_{m \to \infty} \lim_{n \to \infty} f(t + \alpha_n + \beta_m, x, u) = \lim_{n \to \infty} f(t + \alpha_n + \beta_n, x, u).
\]

By the uniform continuity of \( u^+(t, x) \), without loss of generality, we may assume that \( \lim_{n \to \infty} u^+(t + \alpha_n, x) \) exists locally uniformly in \((t, x) \in \mathbb{R} \times \mathbb{R} \). Let
\[
\tilde{f}(t, x, u) = \lim_{n \to \infty} f(t + \alpha_n, x, u) \quad \text{and} \quad \tilde{u}^+(t, x) = \lim_{n \to \infty} u^+(t + \alpha_n, x).
\]

It is clear that \( \tilde{f}(t, x, u) \) satisfies (H0) and \( \tilde{u}^+(t, x) \) is uniformly continuous in \((t, x) \in \mathbb{R} \times \mathbb{R} \) and \( \inf_{t \in \mathbb{R}, x \in \mathbb{R}} \tilde{u}^+(t, x) > 0 \). Moreover, \( \tilde{u}^+(t, x) \) is a bounded positive entire solution of (1.1) with \( f \) being replaced by \( \tilde{f} \). Similarly, without loss of generality, we may assume that \( \lim_{m \to \infty} \tilde{u}^+(t + \beta_m, x) \) and \( \lim_{n \to \infty} u^+(t + \alpha_n + \beta_n, x) \) exist locally uniformly in \((t, x) \in \mathbb{R} \times \mathbb{R} \). Let
\[
\tilde{f}(t, x, u) = \lim_{m \to \infty} f(t + \beta_m, x, u), \quad \tilde{f}(t, x, u) = \lim_{n \to \infty} f(t + \alpha_n + \beta_n, x, u)
\]
and
\[
\tilde{u}^+(t, x) = \lim_{m \to \infty} \tilde{u}^+(t + \beta_m, x, u), \quad \tilde{u}^+(t, x) = \lim_{n \to \infty} u^+(t + \alpha_n + \beta_n, x, u).
\]

Then \( \tilde{f}(t, x, u) \) and \( \tilde{f}(t, x, u) \) satisfy (H0), \( \tilde{u}^+(t, x) \) and \( \tilde{u}^+(t, x) \) are uniformly continuous in \((t, x) \in \mathbb{R} \times \mathbb{R} \), \( \tilde{u}^+(t, x) \) is a bounded positive entire solution of (1.1) with \( f \) being replaced by \( \tilde{f} \), and \( \tilde{u}^+(t, x) \) is a bounded positive entire solution of (1.1) with \( f \) being replaced by \( \tilde{f} \). Note that \( \tilde{f} = \tilde{f} \). Then by the uniqueness of positive entire solutions of (1.1) with \( f \) being replaced by \( \tilde{f} \), we have
\[
\tilde{u}^+(t, x) = \tilde{u}^+(t, x).
\]

This together with remark 2.1 implies that \( u^+(t, x) \) is almost periodic in \( t \) uniformly with respect to \( x \in \mathbb{R} \). In particular, if \( f \) is periodic in \( t \), so is \( u^+(t, x) \).
Assume that \( f(t,x,u) \) is almost periodic in \( x \). Similarly, we can proved that \( u^+(t,x) \) is almost periodic in \( x \) uniformly with respect to \( t \in \mathbb{R} \). In particular, if \( f \) is periodic in \( x \), so is \( u^+(t,x) \). The theorem is thus proved.

### 4. Stability of transition waves

In this section, we study the stability of transition waves in the general case and prove theorem 2.2.

**Proof of theorem 2.2.** Suppose that \( u = U(t,x) \) is a transition wave of (1.1) satisfying (2.4) and (2.7).

Note that, for given \( u_0(\cdot) \) satisfying (2.1) and given \( t_0 \in \mathbb{R} \), the part metric \( \rho(u_0,U(t_0)) \) is well defined and then \( \rho(u(t,\cdot;0,0),U(t+t_0,\cdot)) \) is well defined for all \( t \geq 0 \). To prove (2.2), it suffices to prove that for any \( \epsilon > 0 \), there is \( T > 0 \) such that

\[
\rho(u(t+t_0;\cdot;0,0),U(t+t_0,\cdot)) < \epsilon \quad \text{for all} \quad t \geq T.
\] (4.1)

In fact, by proposition 3.3(1), we only need to prove that for any \( \epsilon > 0 \),

\[
\rho(u(t+t_0;\cdot;0,0),U(t+t_0,\cdot)) < \epsilon \quad \text{for some} \quad t > 0.
\] (4.2)

Assume by contradiction that there is \( \epsilon_0 > 0 \) such that

\[
\rho(u(t+t_0;\cdot;0,0),U(t+t_0,\cdot)) \geq \epsilon_0
\] (4.3)

for all \( t \geq 0 \). Fix a \( \tau > 0 \). We claim that if (4.3) holds, then there is \( \delta > 0 \) such that

\[
\rho(u(t+s+t_0;\cdot;0,0),U(t+s+t_0,\cdot)) \leq \rho(u(s+t_0;\cdot;0,0),U(s+t_0,\cdot)) - \delta
\] (4.4)

for all \( s \geq 0 \).

Before proving (4.4), we prove that (4.4) gives rise to a contradiction. In fact, assume (4.4). Then we have

\[
\rho(u(n\tau+t_0;\cdot;0,0),U(n\tau+t_0,\cdot)) \leq \rho(u(t_0;\cdot;0,0),U(t_0,\cdot)) - n\delta
\]

for all \( n \geq 0 \). Letting \( n \to \infty \), we have \( \rho(u(n\tau+t_0;\cdot;0,0),U(n\tau+t_0,\cdot)) \to -\infty \), which is a contradiction. Therefore, (4.3) does not hold and then for any \( \epsilon > 0 \),

\[
\rho(u(t+t_0;\cdot;0,0),U(t+t_0,\cdot)) < \epsilon \quad \text{for some} \quad t > 0.
\]

This together with proposition 3.3 implies that

\[
\lim_{t \to \infty} \rho(u(t+t_0;\cdot;0,0),U(t+t_0,\cdot)) = 0.
\]

The theorem then follows.

We now prove that if (4.3) holds, then there is \( \delta > 0 \) such that (4.4) holds.

First, for any \( 0 < \epsilon < \frac{\mu}{2d_1\phi_1} \), by (2.1), (2.5), and (2.6), there is \( d_1 \gg d_1' \) such that

\[
d'(1-\epsilon)\phi(t_0,x) - d_1\phi_1(t_0,x) \leq u_0(x) \leq d'(1+\epsilon)\phi(t_0,x) + d_1\phi_1(t_0,x).
\]
By (2.8), there holds
\[ d^*(1 - \epsilon) \phi(t, x) - d_1 \phi_1(t, x) \leq u(t, x; t_0, u_0) \leq d^*(1 + \epsilon) \phi(t, x) + d_1 \phi_1(t_0, x) \quad \forall \ t \geq t_0. \]

Then by (2.6) and (2.7), for \( x - X(t) \gg 1 \), we have that \( 0 < \frac{\phi_1(t, x)}{\phi(t, x)} \ll 1 \),
\[ u(t, x; t_0, u_0) \leq d^*(1 + \epsilon) \phi(t, x) \left(1 + \frac{d_1}{d^*(1 + \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)} \right) \leq (1 + \epsilon) U(t, x) \left(1 + \frac{d_1}{d^*(1 + \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)} \right) \left(1 - \frac{d_1}{d^*} \frac{\phi_1(t, x)}{\phi(t, x)} \right)^{-1}, \]

and
\[ u(t, x; t_0, u_0) \geq d^*(1 - \epsilon) \phi(t, x) \left(1 - \frac{d_1}{d^*(1 - \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)} \right) \geq (1 - \epsilon) U(t, x) \left(1 - \frac{d_1}{d^*(1 - \epsilon)} \frac{\phi_1(t, x)}{\phi(t, x)} \right) \left(1 + \frac{d_1}{d^*} \frac{\phi_1(t, x)}{\phi(t, x)} \right)^{-1}. \]

This together with (2.4) and (2.6) implies that, for any \( s \geq 0 \), there is \( x_s(\geq X(t_0 + s)) \) such that
\[ \sup_{s \geq 0, x \in [b + s_0 + s + \tau]} |x_s - X(t)| < \infty \quad (4.5) \]

and
\[ \frac{1}{1 + \epsilon_0/2} U(t, x) \leq u(t, x; t_0, u_0) \leq (1 + \epsilon_0/2) U(t, x) \quad \forall \ t \in [t_0 + s, t_0 + s + \tau], x \geq x_s. \quad (4.6) \]

Next, we claim that
\[ \inf_{s \geq 0, x \in [b + s_0 + s + \tau], x \leq x_s} U(t, x) > 0. \quad (4.7) \]

In fact, if this is not true, then there are \( s_n \geq 0, \ t_n \in [s_n + t_0, \tau + s_n + t_0] \), and \( x_n \leq x_n \) such that
\[ \lim_{n \to \infty} U(t_n, x_n) = 0. \]

By remark 2.2(1) and (4.5), there are \( \beta > 0 \) and \( \tilde{x}_n \leq x_n \) such that
\[ U(s_n + t_0, \tilde{x}_n) \geq \beta \quad \text{and} \quad \sup_{n \geq 1} |x_n - \tilde{x}_n| < \infty. \]

Let
\[ U_n(t, x) = U(t + s_n + t_0, x + \tilde{x}_n). \]

By the uniform continuity of \( U(t, x) \) in \( (t, x) \in \mathbb{R} \times \mathbb{R}, \) without loss of generality, we may assume that
\[
\lim_{n \to \infty} U_n(t,x) = U^*(t,x)
\]
locally uniformly in \((t,x) \in \mathbb{R} \times \mathbb{R}\). Without loss of generality, we may also assume that
\[
\lim_{n \to \infty} (t_n - s_n - t_0) = t^*, \quad \lim_{n \to \infty} (x_n - \tilde{x}_n) = x^*,
\]
and
\[
\lim_{n \to \infty} f(t + s_n + t_0, x + \tilde{x}_n, u) = f^*(t,x,u)
\]
locally uniformly in \((t,x,u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}\). Then
\[
U^*(t,x) \geq 0, \quad U^*(t^*,0) \geq \beta, \quad U^*(t^*,x^*) = 0,
\]
and \(U^*(t,x)\) is a solution of (1.1) with \(f(t,x,u)\) being replaced by \(f^*(t,x,u)\). Then by comparison principle, we have either \(U^*(t,x) \equiv 0\) or \(U^*(t,x) > 0\) for all \((t,x) \in \mathbb{R} \times \mathbb{R}\), which contradicts to (4.8). Therefore, (4.7) holds.

Now for given \(s \geq 0\), let
\[
\rho(s + t_0) = \rho(u(s + t_0, \cdot; t_0, u_0), U(s + t_0, \cdot)).
\]
By (4.3),
\[
\rho(t_0) \geq \rho(s + t_0) \geq \epsilon_0
\]
and
\[
\frac{1}{\rho(s + t_0)} U(s + t_0, \cdot) \leq u(s + t_0, \cdot; t_0, u_0) \leq \rho(s + t_0) U(s + t_0, \cdot).
\]
It follows from (4.10) and proposition 3.1 that
\[
u(t+s+t_0, \cdot; t_0, u_0) \leq u(t+s+t_0, \cdot; s+t_0, \rho(s+t_0)U(s+t_0, \cdot)) \quad \text{for} \quad t \geq 0.
\]
Let
\[
\tilde{u}(t,x) = u(t+s+t_0, \cdot; s+t_0, \rho(s+t_0)U(s+t_0, \cdot)),
\]
\[
\tilde{u}(t,x) = \rho(s+t_0)u(t+s+t_0, \cdot; s+t_0, U(s+t_0, \cdot)) = \rho(s+t_0)U(t+s+t_0,x),
\]
and
\[
\bar{u}(t,x) = \tilde{u}(t,x) - \tilde{u}(t,x).
\]
Then
\[
\dot{u}(t,x) = A\tilde{u}(t,x) + \tilde{u}(t,x)f(t+s+t_0,x, U(t+s+t_0,x)) - \bar{u}(t,x)f(t+s+t_0,x, \bar{u}(t,x))
\]
\[
= A\tilde{u}(t,x) + p(t,x)\bar{u}(t,x) + b(t,x),
\]
(4.11)
where
\[ p(t, x) = f(t + s + t_0, x, \tilde{u}(t, x)) + \tilde{u}(t, x) \int_0^1 f_s(t + s + t_0, x, r\tilde{u}(t, s) + (1 - r)\tilde{u}(t, x)) \, dr, \]

and
\[ b(t, x) = \tilde{u}(t, x) \left[ f(t + s + t_0, x, U(t + s + t_0, x)) - f(t + s + t_0, x, \tilde{u}(t, x)) \right] \]
\[ = \tilde{u}(t, x) \left[ f(t + s + t_0, x, U(t + s + t_0, x)) - f(t + s + t_0, x, \rho(s + t_0)U(t + s + t_0, x)) \right]. \]

By (H0) and (4.7), there is \( b_0 > 0 \) such that for any \( s \geq 0, \)
\[ \inf_{t \in [\tau + s + t_0, \tau + s + t_0 + h], x \leq x_s} b(t, x) \geq b_0 > 0. \] (4.12)

In the case that \( \mathcal{A}u = u_{tt}, \) by (4.11) and comparison principle for parabolic equations, we have
\[ \tilde{u}(\tau, \cdot) \geq \int_{\tau + s + t_0}^{\tau + s + t_0 + \tau} e^{(\tau + s + t_0 - r)} p_{\text{inf}} T(\tau + s + t_0 - r) b(r, \cdot) \, dr, \]
where \( p_{\text{inf}} = \inf_{t \in [\tau + s + t_0, \tau + s + t_0 + \tau], x \in \mathbb{R}} p(t, x), \]
\[ T(t)u(x) = \int_{\mathbb{R}^n} G(x - y, t)u(y) \, dy \]
and
\[ G(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}. \] (4.13)

This together with (4.12) implies that there is \( \delta_0 > 0 \) such that for any \( s \geq 0, \)
\[ \tilde{u}(\tau, x) \geq \delta_0 \quad \forall \ x \leq x_s \]
and then
\[ u(\tau + s + t_0, x; t_0, u_0) \leq \rho(s + t_0) U(\tau + s + t_0, x) - \delta_0 \quad \forall \ x \leq x_s. \] (4.14)

By (4.6) and (4.14), then there is \( \delta_1 > 0 \) such that
\[ u(\tau + s + t_0, \cdot; t_0, u_0) \leq (\rho(s + t_0) - \delta_1) U(\tau + s + t_0, \cdot) \quad \text{for all} \ s \geq 0. \]

Similarly, we can prove that there is \( \delta_2 > 0 \) such that
\[ \frac{1}{\rho(s + t_0) - \delta_2} U(\tau + s + t_0, \cdot) \leq u(\tau + s + t_0, \cdot; t_0, u_0) \quad \text{for all} \ s \geq 0. \]

The claim (4.4) then holds for \( \delta = \min \{\delta_1, \delta_2\}. \)

In the case that \( \mathcal{A}u(t, x) = \int_{\mathbb{R}} \kappa(y - x)u(t, y) \, dy - u(t, x), \) we have that \( \tilde{u}(t, x) \) satisfies
\[ \bar{u}_t = \int_{\mathbb{R}} \kappa(y-x) \bar{u}(t,y) dy - \bar{u}(t,x) + p(t,x) \bar{u}(t,x) + b(t,x) \quad \forall \, x \in \mathbb{R}. \]

Note that
\[ \int_{\mathbb{R}} \kappa(y-x) \bar{u}(t+s+t_0, y) dy \geq 0. \]

Hence for \( x \leq x_s \),
\[ \bar{u}_t(t,x) \geq (-1 + p(t,x)) \bar{u}(t,x) + b_0. \]

This implies that
\[ \bar{u}(\tau, x) \geq \int_{\tau+s+t_0}^{\tau+s+\bar{t}_0} e^{(-1+p_{\inf})(\tau+s+\bar{t}_0-r)} b_0 dr \quad \forall \, x \leq x_s. \]

Then by the similar arguments as in the above, the claim \((4.4)\) then holds for some \( \delta > 0 \). \( \square \)

5. Existence and stability of transition waves

In this section, we consider the stability of transition waves of \((1.1)\) established in the literature and prove theorem 2.3.

Proof of theorem 2.3 (1). As it is mentioned in remark 2.3(1), the existence of transition waves is established in [42]. In the following, we outline the construction of transition waves from [42] and show that they satisfy the conditions in theorem 2.2 and hence are asymptotically stable.

Assume (H1) and let \( a(t,x) = f(t,x,0) \). For any \( \mu > 0 \), by [42, lemma 3.1], the equation
\[ u_t = u_{xx} + a(t,x)u \]
has a positive solution of the form
\[ u_\mu(t,x) = e^{-\mu x} \eta_\mu(t,x), \quad \text{where} \quad \eta_\lambda(t,x+p) = \eta_\lambda(t,x). \]

By lemma [42, lemma 3.2], there are \( \beta > 0 \) and a uniformly Lipschitz continuous function \( S_\mu(t) \) such that
\[ |S_\mu(t) - \frac{1}{\mu} \ln \| \eta_\mu(\cdot,t) \|_{L^\infty} | \leq \beta \quad \forall \, t \in \mathbb{R}. \]

Let
\[ c_\mu = \lim_{t \to +\infty} \frac{S_\mu(t) - S_\mu(s)}{t-s}. \]

By [42, lemma 3.4], there is \( \mu^* > 0 \) such that for \( 0 < \mu < \mu^* \), \( c_\mu \) is decreasing for \( \mu \in (0,\mu^*) \) and this does not hold for \( \mu \in (0,\bar{\mu}^*) \) for any \( \bar{\mu}^* > \mu^* \). Let \( c^* = c_\mu^* \). Note that when \( a(x) \equiv a \), \( c^* = 2\sqrt{a} \). We show that theorem 2.3(1) holds with this \( c^* \).
To this end, for fixed \( \mu > 0 \), let
\[
\tilde{\phi}_\mu(t,x) = e^{-\mu S_\mu(t)} \eta_\mu(t,x).
\]
By equation (38) in [42], there is \( C_\mu > 0 \) such that
\[
C_\mu \leq \tilde{\phi}_\mu(t,x) \leq e^{\mu \beta} \quad \forall \ x \in \mathbb{R}, \ t \in \mathbb{R}.
\]
Note that for any \( c > c^* \), there is \( 0 < \mu < \mu^* \) such that \( c_\mu = c \). Let \( \mu' \) be such that \( \mu < \mu' < (1 + \nu)\mu \). By [41, lemma 3.2], there is \( \sigma(\cdot) \in W^{1,\infty}(\mathbb{R}) \) such that
\[
\inf \{ \sigma'(t) + \mu'(c_\mu(t) - c_{\mu'}(t)) \} > 0,
\]
where \( c_{\mu}(t) = S'_\mu(t) \) and \( c_\mu(t) = S'_\mu(t) \). Let
\[
\phi(t,x) = e^{-\mu(x-S_\mu(t))} \tilde{\phi}_\mu(t,x) \quad \text{and} \quad \phi_1(t,x) = e^{\sigma(t) - \mu'(x-S_\mu(t))} \tilde{\phi}_{\mu'}(t,x).
\]
By the arguments of [42, theorem 2.1], for any \( u_0 \in C^{\beta}_{\text{unif}}(\mathbb{R}) \) \( (u_0(x) \geq 0) \) and \( t_0 \in \mathbb{R} \) with \( u_0(x) \leq \delta \phi(t_0,x) + d_1 \phi_1(t_0,x) \) \( (\text{resp.} \ u_0(x) \geq \delta \phi(t_0,x) - d_1 \phi_1(t_0,x) \) \( \forall \ x \in \mathbb{R} \)
for some \( d > 0 \) and \( d_1 > 0 \) \( (\text{resp.} \ for \ some \ d > 0 \ and \ d_1 \gg 1) \), there holds
\[
\phi(t,x) - d_\mu \phi_1(t,x) \leq U_\mu(t,x) \leq \phi(t,x) \quad \forall \ x \in \mathbb{R}
\]
for all \( t \geq t_0 \). Moreover, there is a transition wave solution \( u = U_\mu(t,x) \) of (1.1) satisfying that
\[
\phi(t,x) - d_\mu \phi_1(t,x) \leq U_\mu(t,x) \leq \phi(t,x) \quad \forall \ t,x \in \mathbb{R}
\]
for some \( d_\mu \gg 1 \). Clearly, \( X(t) = S_\mu(t) \) is an interface location of \( u = U_\mu(t,x) \) satisfying (2.4), and \( \phi(t,x) \) and \( \phi_1(t,x) \) satisfy (2.5) and (2.6). It then follows from theorem 2.2 that \( u = U_\lambda(t,x) \) is asymptotically stable. Clearly, \( u = U_\mu(t,x) \) has least mean speed \( c \). Theorem 2.3(1) thus follows. \( \square \)

**Proof of theorem 2.3 (2).** As it is mentioned in remark 2.3(2), the existence of transition waves is established in [43]. Similarly, we outline the construction of transition waves from [43] and show that they satisfy the conditions in theorem 2.2 and hence are asymptotically stable.

Assume (H2) and let \( a(x) = f(x,0) \). By [43, proposition 1.3], for any \( \lambda > \lambda_0 \), there is a unique positive \( \phi_\lambda \in C^2(\mathbb{R}) \) such that
\[
\phi_\lambda'' + a(x)\phi_\lambda(x) = \lambda \phi_\lambda(x) \quad \text{in} \ \mathbb{R}, \ \phi_\lambda(0) = 1, \ \lim_{x \to \pm \infty} \phi_\lambda(x) = 0,
\]
and there exists the limit
\[
\mu(\lambda) := - \lim_{x \to \pm \infty} \frac{1}{x} \ln \phi_\lambda(x) > 0. \quad (5.1)
\]
By [43, lemma 2.3], \( \phi_\lambda(x) \) is unbounded, and by [43, lemma 2.4], \( \phi'_\lambda(x)/\phi_\lambda(x) \) is almost periodic in \( x \). Let
\[c^* = \inf_{\lambda > \lambda_0} \frac{\lambda}{\mu(\lambda)}.
\]

In the following we show that theorem 2.3(2) holds with this \(c^*\).

By [43, lemma 3.2], for any \(c > c^*\), there is \(\lambda > \lambda_0\) such that
\[c = \frac{\lambda}{\mu(\lambda)} \quad \text{and} \quad c > \frac{\lambda'}{\mu(\lambda')} \quad \text{for} \quad \lambda' - \lambda > 0 \quad \text{small enough.}
\]

Let \(\sigma_\lambda(x) = -\frac{\phi_\lambda'(x)}{\phi_\lambda(x)}\). By [43, proposition 3.3], there exist \(\delta > 0, \epsilon \in (0, 1)\), and a function \(\theta \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\) such that
\[\inf_{x \in \mathbb{R}} \theta(x) > 0, \quad -\theta'' + 2\sigma_\lambda \phi' - (\sigma_\lambda^2 - \sigma_\lambda' + a) \theta \geq (\delta + (1 + \epsilon)\lambda) \theta \quad \text{in} \quad \mathbb{R}.
\]

By the arguments of [43, proposition 3.4], for any \(t_0 \in \mathbb{R}\) and \(u_0 \in C^b_{\text{unif}}(\mathbb{R}) (u_0 \geq 0)\) with
\[u_0(x) \leq d\phi_\lambda(x)e^{\lambda t_0} + d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t_0}
\]
\[(\text{resp.,} \ u_0(x) \geq d\phi_\lambda(x)e^{\lambda t_0} - d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t_0})
\]
for some \(d > 0\) and \(d_1 > 0\) (resp., for \(d > 0\) and \(d_1 \gg 1\)), there holds
\[u(t, x; t_0, u_0) \leq d\phi_\lambda(x)e^{\lambda t} + d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t}
\]
\[(\text{resp.,} \ u(t, x; t_0, u_0) \geq d\phi_\lambda(x)e^{\lambda t} - d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t})
\]
for all \(t \geq t_0\). Moreover, there is a transition wave \(u(t, x) = \Upsilon_\lambda(t, x)\) of (1.1) with mean speed \(c\) and satisfying that
\[\phi_\lambda(x)e^{\lambda t} - d_1\theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t} \leq \Upsilon_\lambda(t, x) \leq \phi_\lambda(x)e^{\lambda t}
\]
for some \(d_1 \gg 1\).

Let \(X(t)\) be such that \(\phi_\lambda(X(t)) = e^{-\lambda t}\), that is, \(\int_{X(t)}^{X(t) + \phi_\lambda'(y)} dy = -\lambda t\). By [43, lemma 3.5], \(X(t)\) is an interface location of \(u = \Upsilon_\lambda(t, x)\). By [24, proposition 4.2], \(X(t)\) satisfies (2.4). Let
\[\phi(t, x) = \phi_\lambda(x)e^{\lambda t}, \quad \phi_1(t, x) = \theta(x)\phi_\lambda^{1+\epsilon}e^{(1+\epsilon)\lambda t}.
\]

It is clear that \(\phi(t, x)\) and \(\phi_1(t, x)\) satisfy (2.5). Note that
\[\frac{\phi_1(t, x + X(t))}{\phi(t, x + X(t))} = \left(\frac{e^{\epsilon X(t)} \phi_\lambda'(y)}{\frac{\phi_\lambda'(y)}{\phi_\lambda(y)} e^{\lambda y}}\right)^{1+\epsilon} = e^{\epsilon \int_{X(t)}^{X(t) + \phi_\lambda'(y)} dy}. \tag{5.2}
\]

By the almost periodicity of \(\frac{\phi_\lambda'(y)}{\phi_\lambda(y)}\), (5.1), and (5.2),
\[\lim_{|x| \to \infty} \frac{1}{x} \int_{X(t)}^{x+X(t)} \frac{\phi_\lambda'(y)}{\phi_\lambda(y)} dy = \mu(\lambda) < 0
\]
uniformly in $t \in \mathbb{R}$. This implies that $\phi_1(t,x)$ and $\phi_2(t,x)$ satisfy (2.6). It then follows from theorem 2.2 that $u = U_\lambda(t,x)$ is an asymptotically stable transition wave of (1.1) with average speed $c$. \hfill \Box

**Proof of theorem 2.3(3).** The existence of transition waves is established in [63]. In the following, we outline the construction of transition waves from [63] and show that they are asymptotically stable by using the arguments in the proof of theorem 2.2.

Recall that $\lambda_0 = \sup \sigma[\phi''_\lambda + a(\cdot)]$, $\lambda_1 = 2a_-$.

Note that $\lambda_0 \geq a_-$. By the arguments of [63, theorem 1.1], for any $\lambda > \lambda_0$, there is a unique $\phi_\lambda(x)$ such that

$$\phi''_\lambda(x) + a(x)\phi_\lambda(x) = \lambda \phi_\lambda(x) \quad \text{for} \quad x \in \mathbb{R}$$

and

$$\phi_\lambda(0) = 1, \quad \lim_{x \to \infty} \phi_\lambda(x) = 0.$$

Fix $\lambda \in (\lambda_0, \lambda_1)$ and $1 - \frac{\lambda - \lambda_0}{\lambda_1 - \lambda_0} < \alpha < 1$. Let $U_{g,\sqrt{\alpha}}(x)$ be the traveling front profile for the PDE

$$u_t = u_{xx} + g(u)$$

with propagating speed $c_1, \sqrt{\alpha} = \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} > 2$ and

$$\lim_{x \to \infty} U_{g,\sqrt{\alpha}}(x)e^{\sqrt{\alpha}x} = 1.$$

Let

$$h_{g,\alpha}(v) = U_{g,\sqrt{\alpha}}(-\alpha^{-\frac{1}{2}} \ln v)$$

for $v > 0$ and $h_{g,\sqrt{\alpha}}(0) = 0$. Then $h_{g,\sqrt{\alpha}}'(0) = 1$, and by [63, 2.5],

$$h_{g,\sqrt{\alpha}}(v) \leq v \quad \forall v \in [0, \infty).$$

For any $\lambda \in (\lambda_0, \lambda_1)$, let

$$u^+(t,x) = \phi_\lambda(x)e^{\lambda t} \quad \text{and} \quad u^-(t,x) = h_{g,\sqrt{\alpha}}(\phi_\lambda(x)e^{\lambda t}).$$

By [63, lemma 2.1, theorem 1.1], $u^+(t,x)$ is a super-solution of (1.1) and $u^-(t,x)$ is a subsolution of (1.1), and there is a transition wave $u(t,x) = U_\lambda(t,x)$ of (1.1) satisfying

$$u^-(t,x) \leq U_\lambda(t,x) \leq u^+(t,x). \quad (5.3)$$

We show that this transition wave is asymptotically stable by applying the arguments in the proof of theorem 2.2.

To this end, first, let

$$w^+(t,x) = u^+(t,x) - u^-(t,x) (\geq 0).$$
We have
\[ w_1^+(t,x) - w_{xx}^+ - a(x) w_1^+(t,x) = -(u_t^+ - u_x^+ - a(x) u_1^+(t,x)) \]
\[ \geq -u_1^-(t,x) f(x,u_1^-(t,x)) + a(x) u_1^-(t,x) \]
\[ = u_1^+(t,x) (a(x) - f(x,u_1^-(t,x))) \]
\[ \geq 0. \]

This implies that
\[ d_1 w_1^+(t,x) \leq d_1 w_{xx}^+ + d_1 a(x) w_1^+(t,x) \]
for any \( d_1 > 0 \). Let
\[ \phi(t,x) = u_1^+(t,x), \quad \phi_1(t,x) = w_1^+(t,x). \]

We have that \( u(t,x) = d \phi(t,x) + d_1 \phi_1(t,x) \) is a super-solution of (1.1) for any \( d, d_1 > 0 \). Note that
\[ \lim_{x \to \pm \infty} \phi(t,x) = \lim_{x \to \pm \infty} \phi_1(t,x) = 0 \]
locally uniformly in \( t \).

Next, for any \( M > 0 \), let
\[ g_M(u) = g(u) - Mu^2 \]
and \( U_{g_\sqrt{\alpha\pi}} \) be the traveling front profile of
\[ u_t = u_{xx} + g_M(u) \]
with propagating speed \( \sqrt{\alpha + \frac{1}{\sqrt{\alpha}}} \) and \( \lim_{x \to \pm \infty} U_{g_\sqrt{\alpha\pi}}(x)e^{\sqrt{\alpha}t} = 1 \). Let
\[ h_{g_\sqrt{\alpha\pi}}(v) = U_{g_\sqrt{\alpha\pi}}(-\alpha^{\frac{1}{2}} \ln v) \]
for \( v > 0 \) and \( h_{g_\sqrt{\alpha\pi}}(0) = 0 \). Then
\[ h'_{g_\sqrt{\alpha\pi}}(0) = 1 \quad \text{and} \quad h_{g_\sqrt{\alpha\pi}}(v) \leq h_{g_\sqrt{\alpha\pi}}(v) \leq v. \]

Similarly, by [63, lemma 2.1, theorem 1.1], we have that \( \psi(t,x) := h_{g_\sqrt{\alpha\pi}}(\phi_\lambda(x)e^{\lambda t}) \) is a sub-solution of (1.1). This also implies that \( u = d \psi(t,x) \) is a sub-solution of (1.1) for any \( 0 < d < 1 \).

Now, note that for any \( M > 0 \) and \( d_1 > 0 \),
\[ \psi_M(t,x) \leq U_{\lambda}(t,x) \leq \phi(t,x) + d_1 \phi_1(t,x) \quad \forall \ t, x \in \mathbb{R} \]
and
\[ \lim_{M \to \infty} \psi_M(t,x) = 0 \]
uniformly in \((t,x) \in \mathbb{R} \times \mathbb{R}\). Hence for any given \( u_0 \) satisfying (2.1) and (2.2), for any \( \epsilon > 0 \), there are \( M > 0 \) and \( d_1 > 0 \) such that
\[(1 - \epsilon)\psi_M(t_0, x) \leq u_0(x) \leq (1 + \epsilon)\phi(t_0, x) + d_1\phi_1(t_0, x)\]

and then
\[(1 - \epsilon)\psi_M(t, x) \leq u(t, x; t_0, u_0) \leq (1 + \epsilon)\phi(t, x) + d_1\phi_1(t, x) \quad \forall \ t \geq t_0, \ x \in \mathbb{R}. \quad (5.5)\]

By (5.4) and (5.5), we have that
\[u(t, x; t_0, u_0) \leq (1 + \epsilon)U_\lambda(t, x)\frac{\phi(t, x)}{\psi_M(t, x)} \left(1 + d_1\frac{\phi_1(t, x)}{\phi(t, x)}\right) \quad \forall \ t \geq t_0, \ x \in \mathbb{R} \quad (5.6)\]

and
\[u(t, x; t_0, u_0) \geq (1 - \epsilon)U_\lambda(t, x)\frac{\psi_M(t, x)}{\phi(t, x)} \left(1 + d_1\frac{\phi_1(t, x)}{\phi(t, x)}\right)^{-1} \quad \forall \ t \geq t_0, \ x \in \mathbb{R}. \quad (5.7)\]

Let \(X(t)\) be an interface location of \(U_\lambda(t, x)\). By [24, proposition 4.2], \(X(t)\) satisfies (2.4). Note that
\[\lim_{x \to \infty} U_\lambda(t, x + X(t)) = 0\]
uniformly in \(t \in \mathbb{R}\). By (5.3) and (5.4),
\[\lim_{x \to \infty} h_{\epsilon_0, \sqrt{\lambda}}(\phi_\lambda(x + X(t))e^{\lambda t}) = \lim_{x \to \infty} h_{\epsilon_0, \alpha}(\phi_\lambda(x + X(t))e^{\lambda t}) = 0\]
uniformly in \(t \in \mathbb{R}\). This implies that
\[\lim_{x \to \infty} \phi_\lambda(x + X(t))e^{\lambda t} = 0\]
uniformly in \(t \in \mathbb{R}\). Hence
\[\lim_{x \to \infty} \frac{\psi_M(t, x + X(t))}{\phi(t, x + X(t))} = \lim_{x \to \infty} \frac{h_{\epsilon_0, \sqrt{\lambda}}(\phi_\lambda(x + X(t))e^{\lambda t})}{\phi_\lambda(x + X(t))e^{\lambda t}} = 1\]
and
\[\lim_{x \to \infty} \frac{\phi_1(t, x + X(t))}{\phi(t, x + X(t))} = \lim_{x \to \infty} \left(1 - \frac{h_{\epsilon_0, \sqrt{\lambda}}(\phi_\lambda(x + X(t))e^{\lambda t})}{\phi_\lambda(x + X(t))e^{\lambda t}}\right) = 0\]
uniformly in \(t \in \mathbb{R}\). This together with (5.6) and (5.7) implies that, for any given \(\epsilon_0 > 0\) with \(\frac{\epsilon_0}{1 + \epsilon_0/2} > \epsilon\), for any \(s > 0\), there is \(x_s(\geq X(t_0 + s))\) such that
\[\sup_{t \geq 0, x \in [t_0 + s, t_0 + s + \tau]} |x_s - X(t)| < \infty \quad (5.8)\]
and
\[\frac{1}{1 + \epsilon_0/2} U_\lambda(t, x) \leq u(t, x; t_0, u_0) \leq (1 + \epsilon_0/2)U_\lambda(t, x) \quad \forall \ t \in [t_0 + s, t_0 + s + \tau], \ x \geq x_s. \quad (5.9)\]
It then follows from the arguments after (4.6) in the proof of theorem 2.2 that, for any \( \epsilon > 0 \),
\[
\rho(u(t + t_0; \cdot; t_0, u_0), U_\lambda(t + t_0; \cdot)) < \epsilon \quad \text{for some } t > 0.
\]

Then by proposition 3.3(1),
\[
\rho(u(t + t_0; \cdot; t_0, u_0), U_\lambda(t + t_0; \cdot)) < \epsilon \quad \text{for some } t \gg 1.
\]

Therefore, \( u = U_\lambda(t, x) \) is asymptotically stable. \( \square \)

**Proof of theorem 2.3(4).** Note that the existence of transition waves is established in [48].

In the following, we outline the construction of transition waves from [48] and show that they satisfy the conditions in theorem 2.2 and hence are asymptotically stable.

First of all, consider
\[
v_t = \int_{\mathbb{R}} e^{-\mu(y-x)} \kappa(y-x) v(t,y) dy - v(t,x) + a(t,x) v(t,x), \quad x \in \mathbb{R}, \quad (5.10)
\]
where \( \mu \in \mathbb{R}, \ a(t,x) = f(t,x,0) \). Assume that \( a(t,x + p) = a(t + T, x) = a(t,x) \). By [48, propositions 3.2, 3.4, 3.5], we have

(i) For given \( \mu > 0 \), there are \( \lambda(\mu) \in \mathbb{R} \) and a continuous function \( v(t,x; \mu) \) such that
\[
v(t + T, x; \mu) = v(t,x + p; \mu) = v(t,x; \mu), \quad \inf_{(t,x) \in \mathbb{R} \times \mathbb{R}} v(t,x; \mu) > 0,
\]
\[
\|v(\cdot, \cdot; \mu)\|_\infty = 1,
\]

and
\[
v(t,x) := e^{\lambda(\mu)t} v(t, \cdot; \mu)
\]
is a solution of (5.10).

(ii) There is \( \mu^* > 0 \) such that
\[
\frac{\lambda(\mu^*)}{\mu^*} = \inf_{\mu > 0} \frac{\lambda(\mu)}{\mu}
\]
and
\[
\frac{\lambda(\mu)}{\mu} > \frac{\lambda(\mu^*)}{\mu^*} \quad \text{for } 0 < \mu < \mu^*.
\]

Let \( e^* = \frac{\lambda(\mu^*)}{\mu^*} \). We show that theorem 2.3 (4) holds with this \( e^* \).

To this end, for given \( 0 < \mu < \mu^* \), choose \( \mu_1 \) such that \( \mu < \mu_1 < \min\{\mu^*, 2\mu\} \). Let
\[
e_\mu = \frac{\lambda(\mu)}{\mu}.
\]

Let
\[
\phi(t,x) = e^{-\mu_1 \kappa(x-d_1 t)} v(t,x; \mu), \quad \phi_1(t,x) = e^{-\mu_1 \kappa(x-d_1 t)} v(t,x; \mu_1),
\]
where \( v(t,x; \mu) \) and \( v(t,x; \mu_1) \) are as in (i). By the arguments of [56, propositions 3.2 and 3.5] and [48, propositions 5.1 and 5.2], there is \( d_0 > 0 \) such that for any \( 0 < d < 1, d_1 > d_0d, \) and any \( t_0 \in \mathbb{R}, u_0 \in C_0^b (\mathbb{R}) \) \((u_0(x) > 0)\) satisfying
there holds
\[
\phi(t, x) - d_1 \phi_1(t, x) \leq u(t, x; t_0, u_0) \leq \phi(t, x) + d_1 \phi_1(t, x) \quad \forall \ t > t_0. \tag{5.11}
\]

Fix \(0 < d^* < 1\) and \(d_1^* > d_0 d^*\). By the arguments of \([56, \text{ theorem 2.4}]\) and \([48, \text{ theorem 5.1}]\), there is a uniformly continuous periodic transition wave solution \(u = U(t, x)\) satisfying
\[
d^* \phi(t, x) - d_1^* \phi_1(t, x) \leq U(t, x) \leq d^* \phi(t, x) + d_1^* \phi_1(t, x). \tag{5.12}
\]

Clearly, \(X(t) = \frac{c_{\mu} t}{\mu} \) is an interface location of \(U(t, x)\) and satisfies (2.4), and \(\phi(t, x), \phi_1(t, x)\) satisfy (2.5) and (2.6). By (5.11), (5.12), and theorem 2.2, we have that \(u = U(t, x)\) is a periodic wave solution with speed \(c_{\mu} = \frac{\lambda(\mu)}{\mu}\) and is asymptotically stable. □

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References

[1] Aronson D G and Weinberger H F 1975 Nonlinear diffusion in population genetics, combustion and nerve pulse propagation Partial Differential Equations and Related Topics ed J Goldstein (Lecture Notes in Mathematics vol 466) (New York: Springer) pp 5–49
[2] Aronson D G and Weinberger H F 1978 Multidimensional nonlinear diffusions arising in population genetics Adv. Math. 30 33–76
[3] Bates P W and Zhao G 2007 Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal J. Math. Anal. Appl. 332 428–40
[4] Berestycki H, Coville J and Vo H H 2016 Persistence criteria for populations with non-local dispersion J. Math. Biol. 72 1693–745
[5] Berestycki H, Hamel F and Roques L 2005 Analysis of the periodically fragmented environment model. I. Species persistence J. Math. Biol. 51 75–113
[6] Berestycki H, Hamel F and Rossi L 2007 Liouville-type results for semilinear elliptic equations in unbounded domains Ann. Mat. Pura Appl. 186 469–507
[7] Berestycki H, Hamel F and Nadirashvili N 2005 The speed of propagation for KPP type problems, I—periodic framework J. Eur. Math. Soc. 7 172–213
[8] Berestycki H, Hamel F and Nadirashvili N 2010 The speed of propagation for KPP type problems, II—General domains J. Am. Math. Soc. 23 1–34
[9] Berestycki H, Hamel F and Roques L 2005 Analysis of periodically fragmented environment model: II—biological invasions and pulsating traveling fronts J. Math. Pures Appl. 84 1101–46
[10] Berestycki H and Hamel F 2007 Generalized travelling waves for reaction-diffusion equations Perspectives in Nonlinear Partial Differential Equations (Contemporary Mathematics vol 446) (Providence, RI: American Mathematical Society) pp 101–23
[11] Berestycki H and Hamel F 2012 Generalized transition waves and their properties Comm. Pure Appl. Math. 65 592–648
[12] Bramson M 1983 Convergence of Solutions of the Kolmogorov Equations to Traveling Waves (Memoirs of the American Mathematical Society vol 44) (Providence, RI: American Mathematical Society) ([https://doi.org/10.1090/memo/0285](https://doi.org/10.1090/memo/0285))
[13] Cantrell R S and Cosner C 1998 On the effects of spatial heterogeneity on the persistence of interacting species J. Math. Biol. 37 103–45
[14] Cao F and Shen W 2017 Spreading speeds and transition fronts of lattice KPP equations in time heterogeneous media Discrete Cont. Dyn. Syst. 37 4697–727
[15] Carr J and Chmaj A 2004 Uniqueness of travelling waves for nonlocal monostable equations Proc. Am. Math. Soc. 132 2433–9
[16] Coville J and Dupaigne L 2005 Propagation speed of travelling fronts in non local reaction-diffusion equations Nonlinear Anal. 60 797–819
[17] Coville J, Dávila J and Martínez S 2013 Pulsating fronts for nonlocal dispersion and KPP nonlinearity Ann. Inst. Henri Poincare 30 179–223
[18] Fink A M 1974 Almost Periodic Differential Equations (Lecture Notes in Mathematics vol 377) (Berlin: Springer)
[19] Fisher R 1937 The wave of advance of advantageous genes Ann. Eugenics 7 335–69
[20] Fink A M 1974 Almost Periodic Differential Equations (Lecture Notes in Mathematics vol 377) (Berlin: Springer)
[21] Friedman A 1964 Partial Differential Equations of Parabolic Type (Englewood Cliffs, NJ: Prentice-Hall)
[22] Hamel F 2008 Qualitative properties of monostable pulsating fronts: exponential decay and monotonicity J. Math. Pures Appl. 89 355–99
[23] Hamel F and Roques L 2011 Uniqueness and stability properties of monostable pulsating fronts J. Eur. Math. Soc. 13 345–90
[24] Hamel F and Rossi L 2016 Transition fronts for the Fisher-KPP equation Trans. Am. Math. Soc. 368 8975–713
[25] Heinze S, Papanicolaou G and Stevens A 2001 A variational principle for propagation speeds in space-time periodic media Trans. Am. Math. Soc. 354 8675–713
[26] Henry D 1981 Geometric Theory of Semilinear Parabolic Equations (Lecture Notes in Mathematics vol 840) (Berlin: Springer)
[27] Huang J H and Shen W 2009 Speeds of spread and propagation for KPP models in time almost and space periodic media SIAM J. Appl. Dyn. Syst. 8 790–821
[28] Hudson W and Zinner B 1995 Existence of traveling waves for reaction diffusion equations of Fisher type in periodic media Boundary Value Problems for Functional-Differential Equations (River Edge, NJ: World Scientific) pp 187–99
[29] Kametaka Y 1976 On the nonlinear diffusion equation of Kolmogorov–Petrovskii–Piskunov type Osaka J. Math. 13 11–66
[30] Kolmogorov A, Petrovsky I and Piscunov N 1937 A study of the equation of diffusion with increase in the quantity of matter and its application to a biological problem Bjul. Mosk. Gos. Univ. 1 1–26
[31] Kong L and Shen W 2014 Liouville type property and spreading speeds of KPP equations in periodic media with localized spatial inhomogeneity J. Dyn. Differ. Equ. 26 181–215
[32] Liang X, Yi Y and Zhao X-Q 2006 Spreading speeds and traveling waves for periodic evolution systems J. Differ. Equ. 231 57–77
[33] Liang X and Zhao X-Q 2007 Asymptotic speeds of spread and traveling waves for monotone semiflows with applications Commun. Pure Appl. Math. 60 1–40
[34] Liang X and Zhao X-Q 2010 Spreading speeds and traveling waves for abstract monostable evolution systems J. Funct. Anal. 259 857–903
[35] Lim T and Zlatoš A 2016 Transition fronts for inhomogeneous Fisher-KPP reactions and non-local diffusion Trans. Am. Math. Soc. 368 8615–31
[36] Matano H 2003 Traveling waves in spatially random media RIMS Kokyuroku 1337 1–9
[37] Mierczynski J and Shen W 2008 Spectral Theory for Random and Nonautonomous Parabolic Equations and Applications (Chapman and Hall/CRC Monographs and Surveys in Pure and Applied Mathematics) (Boca Raton, FL: Chapman and Hall)
[38] Mierczynski J and Shen W 2006 Lyapunov exponents and asymptotic dynamics in random Kolmogorov models J. Evol. Equ. 4 377–90
[39] Nadin G 2009 Traveling fronts in space-time periodic media J. Math. Pures Appl. 92 232–62
[40] Nadin G 2015 Critical travelling waves for general heterogeneous one-dimensional reaction-diffusion equations Ann. Inst. Henri Poincare 32 841–73
[41] Nadin G and Rossi L 2012 Propagation phenomena for time heterogeneous KPP reaction-diffusion equations J. Math. Pures Appl. 98 633–53
[42] Nadin G and Rossi L 2015 Transition waves for Fisher-KPP equations with general time-heterogeneous and space-periodic coefficients Anal. PDE 8 1351–77
[43] Nadin G and Rossi L 2017 Generalized transition fronts for one-dimensional almost periodic Fisher-KPP equations Arch. Ration. Mech. Anal. 223 1239–67

3490
[44] Nolen J, Roquejoffre J-M, Ryzhik L and Zlatoš A 2012 Existence and non-existence of Fisher-KPP transition fronts Arch. Ration. Mech. Anal. 203 217–46
[45] Nolen J, Rudd M and Xin J 2005 Existence of KPP fronts in spatially-temporally periodic advective and variational principle for propagation speeds Dyn. PDE 2 1–24
[46] Nolen J and Xin J 2005 Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle Discrete Contin. Dyn. Syst. 13 1217–34
[47] Rawal N and Shen W 2012 Criteria for the existence and lower bounds of principal eigenvalues of time periodic nonlocal dispersal operators and applications J. Dynam. Differ. Equ. 24 927–54
[48] Rawal N, Shen W and Zhang A 2015 Spreading speeds and traveling waves of nonlocal monostable equations in time and space periodic habitats Discrete Contin. Dyn. Syst. 35 1609–40
[49] Sattinger D H 1976 On the stability of waves of nonlinear parabolic systems Adv. Math. 22 312–55
[50] Shen W 2004 Traveling waves in diffusive random media J. Dynam. Differ. Equ. 16 1011–60
[51] Shen W 2010 Variational principle for spatial spreading speeds and generalized wave solutions in time almost and space periodic KPP models Trans. Am. Math. Soc. 362 5125–68
[52] Shen W 2011 Existence, uniqueness and stability of generalized traveling solutions in time dependent monostable equations J. Dynam. Differ. Equ. 23 1–44
[53] Shen W 2011 Existence of generalized traveling waves in time recurrent and space periodic monostable equations J. Appl. Anal. Comput. 1 69–93
[54] Shen W and Shen Z 2016 Transition fronts in nonlocal Fisher-KPP equations in time heterogeneous media Commun. Pure Appl. Anal. 15 1193–213
[55] Shen W and Shen Z 2016 Regularity of transition fronts in nonlocal dispersal evolution equations J. Dyn. Diff. Equat. (accepted) (https://doi.org/10.1007/s10884-016-9528-4)
[56] Shen W and Zhang A 2012 Traveling wave solutions of spatially periodic nonlocal monostable equations Comm. Appl. Nonlinear Anal. 19 73–101
[57] Shen W and Zhang A 2010 Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats J. Differ. Equ. 249 747–95
[58] Tao T, Zhu B and Zlatoš A 2014 Transition fronts for inhomogeneous monostable reaction-diffusion equations via linearization at zero Nonlinearity 27 2409–16
[59] Uchiyama K 1978 The behavior of solutions of some nonlinear diffusion equations for large time J. Math. Kyoto Univ. 18–3 453–508
[60] Weinberger H F 1982 Long-time behavior of a class of biology models SIAM J. Math. Anal. 13 353–96
[61] Weinberger H F 2002 On spreading speeds and traveling waves for growth and migration models in a periodic habitat J. Math. Biol. 45 511–48
[62] Xin J 2000 Front propagation in heterogeneous media SIAM Rev. 42 161–230
[63] Zlatoš A 2012 Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations J. Math. Pures Appl. 98 89–102