$Z_{Kup} = |Z_{Henn}|^2$ FOR SEMISIMPLE HOPF ALGEBRAS

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Abstract. Hennings and Kuperberg defined quantum invariants $Z_{Henn}$ and $Z_{Kup}$ of closed oriented 3-manifolds based on certain Hopf algebras, respectively. We prove that $Z_{Kup} = |Z_{Henn}|^2$ for any closed oriented 3-manifold when both invariants are based on finite dimensional semisimple factorizable Hopf algebras.

1. Introduction

Since Turaev-Viro invariant $Z_{TV}$ and Reshetikhin-Turaev invariant $Z_{RT}$ were introduced in 1990s based on certain semisimple categories, more quantum invariants have been constructed using Hopf algebras instead of categories. A type of such 3-manifold invariants $Z_{Henn}$ was defined by Hennings and reformulated by Kauffman and Radford based on finite dimensional ribbon Hopf algebras. More generally, Kuperberg defined topological invariants $Z_{Kup}$ for closed framed 3-manifolds ([Ku]), which are based on any finite dimensional Hopf algebras. Afterwards, Kerler conjectured that with certain choice of framing, $Z_{Kup} = |Z_{Henn}|^2$ ([Ke]). This can be considered as a non-semisimple generalization of the relation between Turaev-Viro invariants $Z_{TV}$ and Reshetikhin-Turaev invariants $Z_{RT}$. Kerler’s conjecture was verified to be true for lens spaces in [ChaWa].

When the Hopf algebras are semisimple, the equality $Z_{Kup} = |Z_{Henn}|^2$ was proved based on several works. First, [Ro] and [BarWe] showed $Z_{Kup} = Z_{TV}$ and $Z_{Henn} = Z_{RT}$ whenever each is defined for certain semisimple Hopf algebras, respectively. Then $Z_{TV} = |Z_{RT}|^2$ ([TuVi2], [BalKi]) implies Kerler’s conjecture. To define $Z_{TV}$ and $Z_{RT}$, one needs to work on the representation categories of given Hopf algebras. In this paper, we prove this directly by working on Hopf algebras and choosing suitable framing on 3-manifolds.

Theorem. Let $H$ be a finite dimensional semisimple factorizable Hopf algebra and $M$ be an oriented closed 3-manifold. Then $Z_{Kup}(M, f, H) = |Z_{Henn}(M, H)|^2$ for some suitably chosen framing $f$ on $M$.

Another motivation to consider these two types of 3-manifold invariant is that for a given manifold, $Z_{Kup}$ and $Z_{Henn}$ may provide gauge invariants for Hopf algebras. Two Hopf algebras are said to be gauge equivalent if and only if their representation categories are equivalent as tensor categories. For example, the Frobenius-Schur indicators of Hopf algebras are a type of gauge invariants ([KMN],...
which have important application in representation theory of Hopf algebras and coincide with \( Z_{K_{up}} \) for lens space. It is expected that \( Z_{K_{up}} \) provides generalized Frobenius-Schur indicators and more general resource of gauge invariants for any finite dimensional Hopf algebra. This is true for certain ribbon Hopf algebras because of recent result in [CheKe]. That is, \( Z_{H_{enn}} \) turns out to be gauge invariant for any factorizable ribbon Hopf algebra ([CheKe]) and is related to \( Z_{K_{up}} \) by the theorems in this paper and [ChaWa].

The paper is organized as follows. In Section 2, we collect recall the definitions of the Hennings and Kuperberg invariants and set up our notations. After this, we prove our theorem in Section 3.

### 2. Hennings and Kuperberg invariants

#### 2.1. Some facts about Hopf algebras

In this section, we recall necessary notations and structures on finite dimensional semisimple Hopf algebras over \( \mathbb{C} \). Detail can be found in [Ra1], [Ra2] and [KaRa2].

Let \( H(m, \Delta, S, 1, \varepsilon) \) be a finite dimensional Hopf algebra over \( \mathbb{C} \) with multiplication \( m \), comultiplication \( \Delta \), antipode \( S \), unit 1, and counit \( \varepsilon \). In the following, we will apply the Sweedler notion and omit the summation symbol, i.e., we write \( \Delta(x_{(n-1)}x_{(1)} \otimes \cdot \cdot \cdot \otimes x_{(n)}) \).

For finite dimensional semisimple Hopf algebras, there exists \( \lambda \in H^* \), called two sided integral (integral, for short), satisfying
\[
(id \otimes \lambda)\Delta(x) = (\lambda \otimes id)\Delta(x) = \lambda(x) \cdot 1
\]

Dually, there exist \( \Lambda \in H \) called two sided cointegral (cointegral, for short), satisfying
\[
x\Lambda = \Lambda x = \varepsilon(x)\Lambda.
\]

Such elements are unique up to a scalar multiple. Further more, a integral plays the role of invariant trace. Namely, for all \( x, y \in H \),
\[
\lambda(xy) = \lambda(yx), \lambda(S(x)) = \lambda(x)
\]

We may choose a normalization so that \( \lambda(\Lambda) = \lambda(S(\Lambda)) = 1 \).

Recall that a Hopf algebra \( H \) is quasitriangular if there exists an \( R \)-matrix \( R \in H \otimes H \) such that
\[
R\Delta(x) = R^{op}\Delta(x), (id \otimes \Delta)R = R_{13}R_{12}, (\Delta \otimes id)R = R_{13}R_{23},
\]
where \( R_{ij} \in H \otimes H \otimes H \) be obtained from \( R = \sum s_k \otimes t_k \) by inserting the unit 1 into the tensor factor labeled by the index in \( \{1, 2, 3\}\setminus\{i, j\} \). The Drinfeld element \( u = \sum S(t_k)s_k \) satisfies \( S^2(x) = uxu^{-1} \) for \( x \in H \). Note that if \( H \) is semisimple, \( u \) is a central element.

A quasitriangular Hopf algebra is said to be ribbon if there exists a central element \( \theta \) such that
\[
\Delta(\theta) = (R^* R)^{-1}(\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad S(\theta) = \theta.
\]
Here $R^\tau = \sum_k t_k \otimes s_k$. It can be shown that the balancing element $G = u\theta^{-1}$ induces $S^2$. That is $S^2(x) = GxG^{-1}$ for $x \in H$. If $H$ is semisimple, it has a canonical ribbon element $\theta = u$ and $G = 1$.

With $R$-matrix, one can define an algebra anti-homomorphism $f_{R^\tau R} : H^* \rightarrow H$, called Drinfeld map. For any $p \in H^*$,

$$f_{R^\tau R}(p) = \sum_{i,j} p(t'_j s_i) s'_j t_i$$

where $R = \sum_i s_i \otimes t_i = \sum_j s'_j \otimes t'_j$. If $f_{R^\tau R}$ is a linear isomorphism, then $H$ is called factorizable. For a semisimple factorizable Hopf algebra $H$, the Drinfeld map sends a cointegral to a cointegral. That is $f_{R^\tau R}(\lambda) = \Lambda$ (see [CoWe2]).

The following are examples of finite dimensional semisimple factorizable ribbon Hopf algebras.

1. The group algebra $\mathbb{C}(\mathbb{Z}_n)$ becomes quasitriangular equipped with

$$R = \frac{1}{n} \sum_{a,b=0}^n e^{-2\pi i \frac{n}{a} b} g^a \otimes g^b.$$ 

Then $\mathbb{C}(\mathbb{Z}_n)$ is ribbon since it is semisimple. Note that $\mathbb{C}(\mathbb{Z}_n)$ is factorizable if and only if $n$ is odd.

2. The Drinfeld double $D(H)$ of a finite dimensional semisimple Hopf algebra $H$ is factorizable and semisimple.

In the following, we review Hennings and Kuperberg invariants in the setting of finite dimensional semisimple factorizable Hopf algebras. They are ribbon Hopf algebras with the canonical ribbon elements.

### 2.2. Hennings invariant.

In 1990s, Hennings constructed invariant for any closed oriented 3-manifold using ribbon Hopf algebras with certain non-degenerated condition [He]. Then Kauffman and Radford improved Hennings’ construction via unoriented surgery diagrams [KaRa]. Given a semisimple ribbon Hopf algebra $H$, one can associate a regular isotopy invariant $TR(L,R)$ to a framed link $L$ as follows: given any link diagram of $L$, decorate each crossing with the tensor factors from the $R$-matrix $R = \sum_i s_i \otimes t_i$ as below.

$$\leftrightarrow \sum_i s_i \otimes t_i \leftrightarrow \sum_i S(s_i) \otimes t_i$$

Once all the crossings of have been decorated, the resulting is a labeled diagram immersed in the plane, where all crossings became 4-valent vertices. The Hopf algebra elements may slide across maxima or minima on the same component at
the expense of the action of the antipode or its inverse as below.

\[
x = S(x)
\]

Afterwards, slide all the Hopf algebra elements on the same component into one vertical portion of the same component. Along a vertical line, all the Hopf algebra elements on the same component are multiplied together from bottom to top so that a product \( w_i \in H \) is on each component.

\[
y = xy
\]

Define

\[
TR(L, H) = \lambda(w_1) \cdots \lambda(w_{c(L)}),
\]

where \( c(L) \) denotes the number of components of \( L \). The full definition of \( TR(L, H) \) evolves Whitney degree of each component and powers of balancing element \( G \). (See [KaRa], [ChaWa]). Here \( G = 1 \) in our setting of canonical ribbon structure.

If \( \lambda(\theta)\lambda(\theta^{-1}) \neq 0 \), which is always true when \( H \) is factorizable [CoWe2], then

\[
Z_{\text{Henn}}(M(L), H) = [\lambda(\theta)\lambda(\theta^{-1})]^{-\frac{c(L)}{2}} [\lambda(\theta)/\lambda(\theta^{-1})]^{-\frac{\sigma(L)}{2}} TR(L, H)
\]

is an invariant of the closed oriented 3-manifold \( M(L) \) obtained from surgery on the framed link \( L \), where \( \sigma(L) \) denotes the signature of the framing matrix of \( L \).

Let \( \overline{M} \) be the manifold with the opposite orientation as \( M \), then we have the following from [He]:

\[
(1) \quad Z_{\text{Henn}}(M_1 \# M_2, H) = Z_{\text{Henn}}(M_1, H)Z_{\text{Henn}}(M_2, H),
\]

\[
(2) \quad Z_{\text{Henn}}(\overline{M}, H) = Z_{\text{Henn}}(M, H)
\]

### 2.3. Kuperberg invariant.

From a finite dimensional Hopf algebra \( H \), Kuperberg constructed invariant \( Z_{Kup}(M, f, H) \) for any closed oriented 3-manifold \( M \) with framing \( f \) ([Ku]). To define the invariant, he presented the framing \( f \) on a Heegaard diagram of \( M \). In the following, we recall \( Z_{Kup}(M, f, H) \) in the setting of semisimple Hopf algebra. For full definition, see [Ku] and [ChaWa].

Given a closed oriented 3-manifold \( M \), it is obtained by gluing two handlebodies of genus \( g \) along their boundaries. A Heegaard diagram consists of two families of \( g \) simple closed curves on a genus \( g \) closed oriented surface \( F \) which tell us how to glue the two handlebodies. One family of simple closed curve are referred as lower circles and the other family as upper circles. Note that this choice is arbitrary. The orientation of \( M \) induces an orientation on its Heegaard surface \( F \), by the convention that a positive tangent basis at a point on \( F \) extends to a positive basis
for $M$ by appending a normal vector that points from the lower side to the upper side. Heegaard circles are also oriented.

A non-vanishing tangent vector field on $M$ is referred as a combing on $M$. Any combing on $M$ can be presented completely by a combing on the Heegaard diagram, which is a vector field on $F$ with $2g$ singularities of index $-1$, one on each circle, and one singularity of index $+2$ disjoint from all circles. The singularity of index $-1$ on a given circle, which is called the base point of the circle, should not be a crossing and the two outward-pointing vectors should be tangent to the circle. Kuperberg showed that any combing on a Heegaard diagram of $M$ can be extended to a combing on $M$; conversely, any combing on $M$ is homotopic to an extension of some combing on the Heegaard diagram.

A framing on $M$ consists of three orthogonal non-vanishing vector fields on $M$. It suffices to described a framing by two orthogonal non-vanishing vector field $b_1$ and $b_2$ (the third one is determined by the orientation of $M$). Suppose $b_1$ has been represented on the Heegaard diagram. Then $b_2$ be described using twist front that encodes how $b_2$ rotates relative to $b_1$. For a factorizable Hopf algebra, $b_2$ contributes in the 3-manifold invariant powers of antipode square $S^2$. See [Ku] and [ChaWa] for detail. When the Hopf algebra is semisimple, $S^2 = id$ and so Kuperberg invariant only depends on a combing rather than a framing.

To define Kuperberg invariant, orient all Heegaard circles according to the orientation of $M$. Let $b_1$ be a combing on the Heegaard diagram. For each point $p$ on a circle $c$ with base point $o_c$, $\psi(p)$ is defined to be the counterclockwise rotation of the tangent to $c$ relative to $b_1$ from $o_c$ to $p$ in units of $1 = 360^\circ$. If $p$ is a crossing of a lower circle and a upper circle, then two rotation angles $\psi_l(p)$ and $\psi_u(p)$ are defined, respectively. Then

$$a_p = 2(\psi_l(p) - \psi_u(p)) - \frac{1}{2}$$

is called the exponent of the crossing $p$.

The algorithm to write down Kuperberg invariant is as follows: assign a coinTEGRAL to each lower circle; do comultiplication for each cointegral and label each crossing with tensor product factors in the direction of lower circles starting from their based points; apply $S^{a_p}$ to the element labeled at crossing $p$, where $a_p$ is the exponent of $p$; multiply all labels in the direction of upper circles starting from their based points; evaluate the resulting products using integrals. In short, the invariant is a big summation:

$$Z_{Kup}(M, f, H) = \sum_{(\Lambda)} \prod_{\text{upper circles}} \lambda(\cdots S^{a_i}(\Lambda_{(i)}) \cdots)$$

In next section, there is an example of Kuperberg invariant for the framed Heegaard diagram in Fig. [7]
In this section, we prove the theorem. To compute the Kuperberg invariant for a closed 3-manifold $M$, we construct a framing $f$ on its Heegaard diagram. On the other hand, we calculate the Hennings invariant for $M\#\overline{M}$ using chain-mail link that is a surgery diagram of $M\#\overline{M}$ [Ro].

3.1. $Z_{Kup}(M, f, H)$. In the following, a 2-sphere $S^2$ is regarded as the plane together with the point at infinity. A genus $g$ Heegaard surface is obtained by attaching $g$ 1-handles (not drawn) to $2g$ discs in the plane. To do Heegaard decomposition, we can glue one genus $g$ handle body from below and another one from above. By isotope, the attaching circles of the lower handle body (called lower circles) can be always chosen to be the meridians of the handles. In our figures, they are drawn as horizontal lines from the left disk to the right disk. Parts of them go through the handles above the plane and so are not drawn. For instance, Fig. 1 and Fig. 7 are the Heegaard diagrams of Lens space $L(5, 2)$ and Poincare homology 3-sphere, respectively.

The attaching circles of the upper handle body are called upper circles. There is a pairing $\sigma$ between the set of upper circles and the set of handles. Namely, a circle $c$ can be matched with exactly one handle $\sigma(c)$ such that an arc of $c$ passes between the two attaching discs of the handle $\sigma(c)$ and there are no other upper circles between this arc and the right attaching disc of $\sigma(c)$. This pairing can be obtained by isotopy. Fig. 2 provides an example of such pairing and the isotopy to obtain it. In the figure, a part of the blue arc is moved to the left attaching disc through the handle above the page.

Let us fix a choice of the pairing $\sigma$ and set up the Kuperberg framing on the Heegaard surface, which is described in Fig. 3. First, all lower circles are represented by the horizontal lines. The index $-1$ singularity (base point) on any lower circle $c^L_k$ is placed close to the left attaching disc so that no upper circles pass between them. For a upper circle $c^U_k$, by the pairing $\sigma$ there are no other upper circles.
circle passing between \( c_k^U \) and \( \sigma(c_k^U) \). Then the index \(-1\) singularity (base point) on \( c_k^U \) is placed at the point nearest the right attaching disc of \( \sigma(c_k^U) \). To avoid the right singularity, let \( c_k^L \) turn around slightly when it is close to the singularity of \( c_k^U \). Fig. 3 provides a local picture around one handle. The dashed lines indicate the flow of the vector field. The vector field flows parallel through the one handle. This local vector field can be translated up and down so that a global vector field is formed. An genus two example is drawn in Fig. 7. Finally, the index \(+2\) singularity is located at the infinity.

The orientation of the Heegaard surface is chosen by setting its normal vector upwards through the paper. The lower circles are oriented from left to right while the upper circles are going upwards from the base point. To write down the Kuperberg invariant from the diagram, one needs to split cointegrals into coproduct factors and label them at the intersections of the lower and upper circles and then multiply the coproduct factors following the direction of upper circles. Before the multiplication, \( S^{a_p} \) acts on the coproduct factors according to the angle relative to the combing. Now we calculate the change of the power of \( S \), which is the exponent \( a_p \) of crossing \( p \), when traveling along the circles.

**Lemma 1.** (1) The power of \( S \) remains unchanged when traveling along a vertical arc.

(2) The power of \( S \) increases by 1 when passing through an extremum in a counterclockwise direction;

(3) The power of \( S \) decreases by 1 when passing through an extremum in a clockwise direction;
Proof. (1) is obvious because the angle relative to the combing does not change when traveling transversely along a vertical arc. (2) and (3) can be examined case by case.

Suppose $S^a_j(\Lambda_j)$ and $S^a_{j+1}(\Lambda'_{j+1})$ are successive terms in the product along a upper circle. Here $\Lambda$ and $\Lambda'$ are two copies of cointegral. Then

$$a_{j+1} - a_j = 2(\psi_L(\Lambda'_{j+1}) - \psi_L(\Lambda_j)) - 2(\psi_U(\Lambda'_{j+1}) - \psi_U(\Lambda_j))$$

In the case shown in Fig. 4, we move from the coproduct factor $\Lambda_j$ to $\Lambda_{j+1}$ through an extremum counterclockwise, then

$$a_{j+1} - a_j = 2 \cdot 0 - 2(-\frac{1}{2}) = 1$$

The clockwise case can be verified similarly. If the arc passes through two handles, for example, as Fig. 5 then the calculation can be reduced to the case within one handle as Fig. 4 because the arc goes transversely from one handle to another.
Here is an example of the Kuperberg invariant of Poincare homology 3-sphere whose framed Heegaard diagram is drawn in Fig. 7. Let $\Lambda$ and $\Lambda'$ be two copies of cointegrals. Their coproduct factors are labeled and multiplied going up from the base points of each upper circle respectively. Then

$$Z_{Kup} = \sum_{(\Lambda)} \lambda \left( S(\Lambda'_{(3)})\Lambda_{(1)}\Lambda'_{(5)}S^{-1}(\Lambda_{(3)})\Lambda'_{(1)}S^{-1}(\Lambda_{(5)}) \right)$$

$$\cdot \lambda \left( \Lambda_{(4)}S(\Lambda'_{(4)})S(\Lambda'_{(2)})\Lambda_{(6)}\Lambda'_{(7)}\Lambda_{(8)} \right)$$

3.2. $Z_{Henn}(M \# \overline{M}, H)$. Now, we use the chain-mail link to evaluate the Hennings invariant for $M \# \overline{M}$. A Heegaard diagram of $M$ can be turned into a surgery diagram of $M \# \overline{M}$ by pushing the upper circles into the lower handle body slightly.
\[ S^{a_j-1}(\Lambda_{j+1}') \]

\[ S^{a_j}(\Lambda_j) \]

Figure 5.
Then the upper circles and the lower circles form a link $L_M$. All these curves are framed by thickening them into thin bands parallel to the Heegaard surface. The resulting link $L_M$ is a surgery presentation for $M \# \overline{M}$ and called a chain-mail link of $M$ ([Ro]). Fig. 6 is the chain-mail link of the Lens space $L(5, 2)$.

Note that, the signature $\sigma(L_M)$ of the framing matrix of the chain-mail link is zero and $\lambda(\theta)\lambda(\theta^{-1}) = 1$ for a factorizable ribbon Hopf algebra (see [CoWe2]), so the normalization factor

$$[\lambda(\theta)\lambda(\theta^{-1})]^{-\frac{\sigma(L_M)}{2}} [\lambda(\theta)/\lambda(\theta^{-1})]^{-\frac{\sigma(L_M)}{2}} = 1.$$ 

It is sufficient to find the link invariant $TR(L_M, H)$. Lemma 2 shows that the contribution of the lower circles is equivalent to decorating the upper circles with coproduct factors of cointegrals. Thus we can work on the cointegral decorated diagram to evaluate the Hennings invariant. Lemma 3 states that the self crossings of the upper circles can be resolved and absorbed by cointegrals. The proof of these lemmas are the same as that for Lens spaces (see [ChaWa]).

**Lemma 2.**

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{lemma2.png}
\end{array}
\]

**Lemma 3.**

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{lemma3.png}
\end{array}
\]
To write down the Hennigs invariant, we push all labeled coproduct factors along the upper circles to the base points. Then we multiply them and evaluate the resulting product by integrals. For Poincare homology 3-sphere $M$, we have cointegral decorated surgery diagram in Fig. 7 (ignore the vector field).

\[
Z_{\text{Henn}} = \sum_{(\Lambda)} \lambda \left( S(\Lambda'_3(3)\Lambda_1(1)\Lambda'_5(5))S^{-1}(\Lambda'_3(3)\Lambda_1(1)S^{-1}(\Lambda_5(5))) \right) \\
\cdot \lambda \left( \Lambda_4(4)S(\Lambda'_4(4)S(\Lambda'_2(2)\Lambda_2(2)\Lambda'_6(6)\Lambda'_7(7)\Lambda'_8(8)) \right)
\]

3.3. **Proof of Theorem.** By Lemma 1, 2 and 3 we see that both $Z_{\text{Kup}}(M, f, H)$ and $Z_{\text{Henn}}(M \# \overline{M}, H)$ are written down by the same algorithm. That is one first creates coproduct factors of cointegrals along the lower circles and labels them at the intersections of the lower and upper circles, then multiplies the factors along upper circles following the same rule of antipode action and evaluates the product by integrals in the end. Therefore,

\[
Z_{\text{Kup}}(M, f, H) = Z_{\text{Henn}}(M \# \overline{M}, H) = |Z_{\text{Henn}}(M, H)|^2.
\]

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Figure 7.

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