T-systems and Y-systems in integrable systems*

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Abstract

T- and Y-systems are ubiquitous structures in classical and quantum integrable systems. They are difference equations having a variety of aspects related to commuting transfer matrices in solvable lattice models, q-characters of Kirillov–Reshetikhin modules of quantum affine algebras, cluster algebras with coefficients, periodicity conjectures of Zamolodchikov and others, dilogarithm identities in conformal field theory, difference analog of L-operators in KP hierarchy, Stokes phenomena in 1D Schrödinger problem, AdS/CFT correspondence, Toda field equations on discrete spacetime, Laplace sequence in discrete geometry, Fermionic character formulas and combinatorial completeness of Bethe ansatz, Q-system and ideal gas with exclusion statistics, analytic and thermodynamic Bethe ansätze, quantum transfer matrix method and so forth. This review is a collection of short reviews on these topics which can be read more or less independently.

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1. Introduction

1.1. T- and Y-systems

The T-system is a difference equation among commuting variables $T_m^{(a)}(u)$, most typically appearing as ($m \in \mathbb{Z}_{\geq 0}$)

$$T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u).$$

Originally it was found as a functional relation in 2D solvable lattice models in statistical mechanics [1]. In this context, $T_m^{(a)}(u)$ is a commuting row transfer matrix in the sense of Baxter [2] labeled with $(a, m)$ and having the spectral parameter $u$.

The Y-system is another difference equation, typically like ($m \in \mathbb{Z}_{\geq 1}$):

$$Y_m^{(a)}(u - 1)Y_m^{(a)}(u + 1) = \frac{(1 + Y_{m}^{(a-1)}(u))(1 + Y_{m}^{(a+1)}(u))}{(1 + Y_{m-1}^{(a)}(u^{-1}))(1 + Y_{m+1}^{(a)}(u^{-1}))}.$$

It was extracted as a universal functional relation in thermodynamic Bethe ansatz (TBA) for solvable lattice models as well as (1+1)D integrable quantum field theory models [3–5]. In this context, $Y_m^{(a)}(u)$ stands for the Boltzmann factor of an excitation mode in the sense of Yang–Yang [6] labeled with $(a, m)$ and having the rapidity $u$.

As such, both systems originate in Yang–Baxter quantum integrable systems but are apparently concerned with the objects that are not related too directly. The first curiosity is nevertheless that the formal substitution

$$Y_m^{(a)}(u) = \frac{T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)}$$

provides a solution to the Y-system in terms of the T-system. Moreover, such a canonical pair of companion systems can be formulated uniformly for all the classical simple Lie algebras $g$ [1]. Now we can give a deferred explanation of the superscript $a$; it runs over the vertices of the Dynkin diagram of $g$. The above formulas are just the examples from type $A$, where the case $g = A_1$ goes back to [7].

In the relevant developments across the centuries, the T- and Y-systems have turned out to be ubiquitous structures with a wealth of applications. For instance, they emerge in $q$-characters for Kirillov–Reshetikhin modules of quantum affine algebras, exchange relations in cluster algebras with coefficients, periodicity conjectures of Zamolodchikov and others, dilogarithm identities in conformal field theory (CFT) and their functional generalizations, dressed vacuum forms in analytic Bethe ansatz, Stokes phenomena in ordinary differential equations, anomalous scaling dimensions of $N = 4$ super-Yang–Mills operators, area of minimal surface in AdS, Laplace sequence of quadrilateral lattice in discrete geometry, tau functions in lattice Toda field equations, Fermionic formulas for branching coefficients and weight multiplicities for Lie algebra characters, combinatorial completeness of string

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4 By $T$ we meant transfer matrices, but it can either be thought as Toda or Tau.

5 Actually to be understood as Yangian $Y(g)$ or untwisted quantum affine algebra $\mathcal{U}_q(g)$. The twisted case is also known. See remark 2.1.

6 The T-system for type $A$ formally coincides with what is known as the Hirota–Miwa equation in soliton theory, which was an unexpected link also to classical integrable systems.
hypothesis in Bethe ansatz, *Q*-system and grand partition function of ideal gas with exclusion statistics, quantum transfer matrix approach to finite-temperature problems and so on.

This review is a collection of brief expositions of these topics where the *T*- and *Y*-systems have played key roles. It consists of sections of moderate length which are not too mutually dependent. A more detailed account of the contents can be found in section 1.2.

As an overview, *T*-systems are fundamental structures reflecting symmetries and algebraic aspects of the problems rather directly. They can also accommodate various gauge/normalization freedom of concrete models. On the other hand, *Y*-systems are more universal being more or less free from such degrees of freedom. They are suitable for practical applications with appropriate analyticity input. In fact, the connection between the *T*- and *Y*-systems mentioned previously has opened a route to establish TBA-type integral equations directly from transfer matrices without recourse to the TBA itself. In this sense, *Y*-systems are the format in which the symmetries encoded in the *T*-systems are most efficiently utilized as a practical implement.

In the light of ever growing perspectives, what sort of equations or structures are to be recognized as *T*- or *Y*-systems is actually a matter of time-dependent option. For instance from an algebraic point of view (leaving analytic aspects), *T*-systems have been generalized broadly to the quantum affinization of quantum Kac–Moody algebras by Hernandez [8] (section 4.6). Cluster algebra with coefficients by Fomin and Zelevinsky [9] offers a comprehensive scheme to generalize and control the *T*- and *Y*-systems simultaneously by quivers (section 5). Nonetheless, this paper is mostly devoted to the description of basic results concerning the aforementioned ‘classic’ *T*- and *Y*-systems associated with *g*. We therefore look forward to the next review to come, hopefully someday by some author, bringing a delightful renewal.

1.2. Contents and brief guide

Here are abstracts of the subsequent sections. They will be followed by another brief guide to the paper.

Section 2. The *T*- and *Y*-systems for untwisted and twisted quantum affine algebras are presented. They have unrestricted and level *ℓ* restricted versions. Those for Yangian are formally the same with the unrestricted ones for the untwisted quantum affine algebra $U_q(\hat{g})$, where *g* denotes a finite-dimensional simple Lie algebra throughout the review. We also include the $U_q(sl(r|s))$ case. This section is meant to be the reference of these systems throughout the review. The first property, *T*-system provides a solution to *Y*-system, is stated. Subsequent sections will mainly be concerned with the untwisted case $U_q(\hat{g})$.7

Section 3. The *T*-system was originally discovered as functional relations among commuting transfer matrices for solvable lattice models in statistical mechanics. We give an elementary exposition of such contexts for the both vertex and restricted solid-on-solid (RSOS) models along with their fusion procedure. The two types of models are related to the unrestricted and restricted *T*-systems, respectively.

Section 4. We describe the background of the *T*-system in the representation theory of quantum affine algebras such as classification of irreducible finite-dimensional representations, Kirillov–Reshetikhin modules and *q*-characters. The fundamental results are that *q*-characters of the Kirillov–Reshetikhin modules satisfy the *T*-system (theorem 4.8) and the description of the Grothendieck ring $\text{Rep} U_q(\hat{g})$ by the *T*-system (theorem 4.9). A broad extension of the *T*-system to the quantum affinization of quantum Kac–Moody algebras is also mentioned. The results of this section are not necessary elsewhere except the basics of *q*-characters which

7 Thus, in most situations we will simply say *T*- and *Y*-systems for *g* instead of $U_q(\hat{g})$. 
will be mentioned in tableau sum formulas (section 7), analytic Bethe ansatz (section 8) and $Q$-system (section 13).

Section 5. The cluster algebra with coefficients is built upon cluster variables and coefficient tuples obeying certain exchange relations controlled by a quiver. We demonstrate how such a setup encodes the T- and Y-systems simultaneously in an essential way. It opens a fruitful link with the cluster category theory, which led to a final proof of the dilogarithm identities in CFT and the periodicity conjecture on the both systems for arbitrary level and g.

Section 6. Jacobi–Trudi-type determinant formulas are listed for T-systems for non-exceptional g. The type $C_r$ and $D_r$ cases involve Pfaffians as well.

Section 7. Tableau sum formulas are presented for T-systems for nonexceptional g along the context of $q$-characters.

Section 8. We argue the relation between $q$-characters and eigenvalue formulas (dressed vacuum forms) of transfer matrices in solvable lattice models by analytic Bethe ansatz. Combined with the results in section 7, it leads to solutions of T-systems in terms of the Baxter $Q$-functions. We mainly concern vertex models and include a brief argument on RSOS models.

Section 9. We introduce a difference analog of $L$-operators in soliton theory to construct solutions to the T-systems for $g = A_r$ and $C_r$ by Casoratians (difference analog of Wronskians). The Baxter $Q$-functions are identified with a special class of Casoratians and generalized to a wider family of functions that admit Bäcklund transformations. Analogous difference $L$-operators are also presented for $B_r$, $D_r$ and $sl(r|s)$.

Section 10. A restricted T-system for $A_1$ emerges in Stokes phenomena of 1D Schrödinger equation with a specific potential. Similar facts also hold for the T-system for $A_r$ and a class of $(r+1)$st order ordinary differential equations (ODE). Wronskians for these equations evaluated at the origin play an analogous role to the Casoratians in section 9 (Wronskian–Casoratian duality). We describe these features that stay within an elementary algebraic part in the so-called ODE/IM (integrable models) correspondence.

Section 11. This section is most hep-th oriented. We briefly digest applications of some specific T- and Y-systems in the two topics from the AdS/CFT correspondence. The first is from the gauge theory about the anomalous scaling dimensions (planar AdS/CFT spectrum) of $\mathcal{N} = 4$ super-Yang–Mills operators. The second is the area of the minimal surface in AdS from the string theory, which is relevant to gluon planar scattering amplitudes. The analysis in the latter topic involves the Stokes phenomena related to a generalized sinh–Gordon equation, which may be viewed as a generalization of the ODE/IM correspondence mentioned in section 10.

Section 12. Continuous limits of the T-system for g yield the difference-differential or 2D differential equations known as the (lattice) Toda field equation. Their Hamiltonian structure is presented for general g. We also discuss an aspect from classical discrete geometry, where the Y-system for $A_\infty$ arises as the Laplace sequence of quadrilateral lattice, the discrete geometry analog of the conjugate net.

Section 13. T-system without a spectral parameter is called the $Q$-system. We systematically construct certain power series solutions to the (generalized) $Q$-system by multivariable Lagrange inversion. As a corollary of this and results from section 4, the so-called Fermionic character formula for the Kirillov–Reshetikhin modules is fully established for all g. Physically, this problem is also connected to the grand partition function of ideal gas with exclusion statistics. These results are reviewed in conjunction with the intimately related

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8 This $Q$ is unrelated with Baxter’s $Q$-functions. See section 13.8 for the origin of the name.
subject known as combinatorial completeness of Bethe ansatz for $U_q(\hat{\mathfrak{g}})$ both at $q = 1$ and $q = 0$, where the case $q = 1$ goes back to Bethe [10], the godfather of the subject, himself.

Section 14. We explain how the $Y$-system for $\mathfrak{g}$ emerges from the TBA equation associated with $U_q(\hat{\mathfrak{g}})$ with $q$ being a root of unity derived in section 15. Various relations among the TBA kernels are summarized. The constant $Y$-system is introduced and related to the $Q$-system. They are essential ingredients in the dilogarithm identity (section 5.1) and the TBA analysis of RSOS models (section 15). As a related issue, we briefly discuss the $Q$-system at the root of unity including conjecture 14.2.

Section 15. The $U_q(\hat{\mathfrak{g}})$ Bethe equation with $q$ being a root of unity is relevant to the critical RSOS models sketched in section 3.3. We outline the TBA analysis to evaluate the high-temperature entropy by the level restricted $Q$-system (sections 14.5 and 14.6) and central charges by the dilogarithm identity (section 5.1). The TBA equation obtained here uniformly for general $\mathfrak{g}$ is the origin of our $Y$-system as shown in sections 14.1 and 14.3.

Section 16. The finite-size or finite-temperature problems in solvable lattice models are analyzed efficiently by the use of $T$- and $Y$-systems without relying on the TBA approach and string hypothesis. We illustrate various such methods along the simplest vertex and RSOS models based on $\mathfrak{g} = A_1$. We also include a simple application of the periodicity of the level 0 restricted $T$-system to the calculation of correlation lengths of vertex models in section 16.1.

Let us conclude the introduction with yet another brief guide of the contents. As we already mentioned, section 2 is the collection of the basic data, and concrete forms of the $T$- and $Y$-systems that will be considered in the review and definitions/notations concerning the root system of $\mathfrak{g}$. With regard to the subsequent sections, it is too demanding to assume the familiarity of the contents in earlier sections. So we have avoided such a style and tried to make each section into a more or less independently readable review on a specific topic around ten pages. Most of them contain bibliographical notes at the end, which hopefully help the readers gain more perspectives into the subjects and activities around.

There are nevertheless several sections that are intimately related or partly dependent of course. Roughly, they may be grouped (nonexclusively) under the following theme.

- Solvable lattice models and their analysis: sections 3, 8, 15 and 16.
- Kirillov–Reshetikhin modules and their $q$-characters: sections 4, 7, 8 and 13.
- Variety of solutions to the $T$-system: sections 6, 7, 8 and 9.
- Stokes phenomena: sections 10 and 11.
- $Q$-system and constant $Y$-system: sections 13 and 14.
- $Y$-system and TBA: sections 11, 14 and 15.

2. $T$- and $Y$-systems for quantum affine algebras and Yangians

We present the $T$- and $Y$-systems associated with untwisted and twisted quantum affine algebras. They have unrestricted and level restricted versions. Those for Yangian are formally the same with the unrestricted ones for the untwisted quantum affine algebras. We also include the case $U_q(sl(r|s))$. This section is devoted to the presentation of these systems with the basic data on root systems. Thus we will only state their first property, the $T$-system provides a solution to the $Y$-system, in theorem 2.5, leaving the exposition of variety of aspects in subsequent sections.

2.1. Untwisted case

Let $\mathfrak{g}$ be a simple Lie algebra associated with a Dynkin diagram of finite type. We set $I = \{1, \ldots, r\}$ with $r = \text{rank} \mathfrak{g}$ and enumerate the vertices of the Dynkin diagrams as
We follow [11], except for $E_6$, for which we choose the one naturally corresponding to the enumeration of the twisted affine diagram $E_6^{(2)}$ in section 2.4. With a slight abuse of notation, we will write for example $g = A_r$ to mean that $g$ is the one associated with the Dynkin diagram of type $A_r$. The cases $A_r, D_r, E_6, E_7$ and $E_8$ are referred to as simply laced.

We set numbers $t$ and $t_a$ ($a \in I$) by

$$
t = \begin{cases} 
1 & g : simply laced, \\
2 & g = B_r, C_r, F_4, \\
3 & g = G_2,
\end{cases} \quad t_a = \begin{cases} 
1 & g : simply laced, \\
1 & g : nonsimply laced, \\
\alpha_a : long root, \\
1 & g : nonsimply laced, \\
\alpha_a : short root.
\end{cases}$$

Let $\alpha_a, \omega_b (a \in I)$ be the simple roots and the fundamental weights of $g$. We fix a bilinear form $(\cdot | \cdot)$ on the dual space of the Cartan subalgebra normalized as

$$
(\alpha_a | \alpha_a) = \frac{2}{t_a}, \quad (\alpha_a | \omega_b) = \frac{b_{ab}}{t_a}.
$$

Let $C = (C_{ab}), C_{ab} = 2(\alpha_a | \alpha_b)/2(\alpha_a | \alpha_a)$, be the Cartan matrix of $g$. We have $C_{ab} = t_a(\alpha_a | \alpha_b)$, $\alpha_a = \sum_{b=1}^{\alpha_a} C_{ab} \omega_b$ and $(C^{-1})_{ab} = t_b(\omega_a | \omega_b)$. We denote by $h$ and $h^\vee$ the Coxeter number and the dual Coxeter number of $g$, respectively. They are listed as follows with the dimension of $g$:

$$
\begin{array}{cccccccc}
g & A_r & B_r & C_r & D_r & E_6 & E_7 & E_8 & F_4 & G_2 \\
\dim g & r(r + 2) & r(2r + 1) & r(2r + 1) & r(2r - 1) & 78 & 133 & 248 & 52 & 14 \\
h & r + 1 & 2r & 2r & 2r - 2 & 12 & 18 & 30 & 12 & 6 \\
h^\vee & r + 1 & 2r - 1 & r + 1 & 2r - 2 & 12 & 18 & 30 & 9 & 4
\end{array}
$$

The relation $\dim g = (1 + h)\text{rank} g$ holds as is well known.

The unrestricted $T$-system for $g$ is the following relations among the commuting variables $\{ T_{m}^{(a)}(u) | a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U \}$, where $T_{m}^{(b)}(u) = T_{0}^{(b)}(u) = 1$ if they occur on the rhs.

For simply laced $g$, we have

$$
T_{m}^{(a)}(u - 1)T_{m}^{(a)}(u + 1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{b \in I \cap \alpha_a = 1} T_{m}^{(b)}(u).
$$

Figure 1. The Dynkin diagrams for $g$ and their enumerations.
For example in type $A_r$, it has the form
\[ T_m^{(a)}(u) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u) \] (2.5)
for $1 \leq a \leq r$ with $T_m^{(r+1)}(u) = 1$. In particular, for $A_1$ it reads
\[ T_m(u)T_m(u+1) = T_{m-1}(u)T_{m+1}(u+1) + 1 \] (2.6)
with the simplified notation $T_m(u) = T_m^{(1)}(u)$.

For $g = B_r$,
\[ T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_m^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u) \] (1 \leq a \leq r - 2),
\[ T_m^{(r-1)}(u - 1)T_m^{(r-1)}(u + 1) = T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u) + T_m^{(r-2)}(u)T_{m+1}^{(r)}(u), \]
\[ T_2^{(r)}(u - \frac{1}{2})T_2^{(r)}(u + \frac{1}{2}) = T_2^{(r)}(u)T_2^{(r+1)}(u) + T_2^{(r-1)}(u - \frac{1}{2})T_2^{(r-1)}(u + \frac{1}{2}) \]
\[ T_{2m+1}^{(r)}(u - \frac{1}{2})T_{2m+1}^{(r)}(u + \frac{1}{2}) = T_{2m+1}^{(r)}(u)T_{2m+2}^{(r)}(u) + T_m^{(r-1)}(u)T_m^{(r+1)}(u). \] (2.7)

For $g = C_r$,
\[ T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_m^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u) \] (1 \leq a \leq r - 2),
\[ T_{2m}^{(r-1)}(u - 1)T_{2m}^{(r-1)}(u + 1) = T_{2m}^{(r-1)}(u)T_{2m+1}^{(r-1)}(u) + T_{2m}^{(r-2)}(u)T_{2m+1}^{(r)}(u), \]
\[ T_{2m+1}^{(r)}(u - 1)T_{2m+1}^{(r)}(u + 1) = T_{2m+1}^{(r)}(u)T_{2m+2}^{(r)}(u) + T_{2m+1}^{(r-2)}(u)T_{2m+2}^{(r)}(u) \]
\[ T_{m}^{(r)}(u - 1)T_m^{(r)}(u + 1) = T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_m^{(r-1)}(u)T_{m+1}^{(r)}(u). \] (2.8)

For $g = F_4$,
\[ T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_m^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u), \]
\[ T_m^{(2)}(u - 1)T_m^{(2)}(u + 1) = T_m^{(2)}(u)T_{m+1}^{(2)}(u) + T_m^{(1)}(u)T_{m+1}^{(3)}(u), \]
\[ T_{2m}^{(3)}(u - 1)T_{2m}^{(3)}(u + 1) = T_{2m}^{(3)}(u)T_{2m+1}^{(3)}(u) + T_{2m}^{(2)}(u)T_{2m}^{(4)}(u), \]
\[ T_{2m+1}^{(3)}(u - 1)T_{2m+1}^{(3)}(u + 1) = T_{2m+1}^{(3)}(u)T_{2m+2}^{(3)}(u) + T_{2m+1}^{(2)}(u)T_{2m+1}^{(4)}(u), \]
\[ T_m^{(4)}(u - 1)T_m^{(4)}(u + 1) = T_{m-1}^{(4)}(u)T_{m+1}^{(4)}(u) + T_m^{(3)}(u). \] (2.9)

For $g = G_2$,
\[ T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_m^{(1)}(u)T_{m+1}^{(1)}(u) + T_{3m}^{(2)}(u), \]
\[ T_m^{(2)}(u - 1)T_m^{(2)}(u + 1) = T_m^{(2)}(u)T_{3m-1}^{(2)}(u) + T_{m}^{(1)}(u - \frac{1}{2})T_{m}^{(1)}(u)T_{m+1}^{(1)}(u + \frac{1}{2}), \]
\[ T_{3m+1}^{(2)}(u - 1)T_{3m+1}^{(2)}(u + 1) = T_{3m+1}^{(2)}(u)T_{3m+2}^{(2)}(u) + T_{m}^{(1)}(u - \frac{1}{2})T_{m}^{(1)}(u + \frac{1}{2})T_{m+1}^{(1)}(u), \]
\[ T_{3m+2}^{(2)}(u - 1)T_{3m+2}^{(2)}(u + 1) = T_{3m+2}^{(2)}(u)T_{3m+3}^{(2)}(u) + T_{m}^{(1)}(u)T_{m+1}^{(1)}(u - \frac{1}{2})T_{m+1}^{(1)}(u + \frac{1}{2}). \] (2.10)

We note that these relations are not bilinear in general under the boundary condition stated before (2.4). The second terms on the rhs can be of order $0, 1, 2$ and $3$ in $T_m^{(1)}(u)$.

The variable $u \in U$ is called the spectral parameter. The set $U$ can be either the complex plane $\mathbb{C}$ or the cylinder $\mathbb{C}_{\xi} := \mathbb{C}/(2\pi\sqrt{-1}/\xi)\mathbb{Z}$ such that $2\pi\sqrt{-1}/\xi \notin \mathbb{Q}$. The choice will not matter seriously, but reflects the underlying algebra.

**Remark 2.1.** In section 4, we will see that the $T$-system for $g$ is actually associated with the untwisted quantum affine algebra $U_q(\hat{g})$ with $q = \xi^8$ when $U = \mathbb{C}_{\xi^8}$. The choice $U = \mathbb{C}$ corresponds to the Yangian $Y(g)$ in a similar sense. In this review we will mostly be concerned with the $U_q(\hat{g})$ case. Thus, we have simply chosen to say $T$-system for $g$ rather than $T$-system for $U_q(\hat{g})$. The latter terminology is more balanced when the twisted case is considered in section 2.4. Note that the choice $U = \mathbb{C}_{\xi}$ effectively imposes an additional periodicity
\(T_m^{(a)}(u) = T_m^{(a)}(u + \frac{2\pi \sqrt{-1}}{\xi})\). By the assumption \(2\pi \sqrt{-1}/\xi \notin \mathbb{Q}\), this does not interfere with the \(T\)-system. Similar remarks apply to the \(Y\)-system in what follows.

The unrestricted \(Y\)-system for \(g\) is the following relations among commuting variables \(\{Y_m^{(a)}(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U\}\), where \(Y_m^{(0)}(u) = Y_0^{(0)}(u)^{-1} = 0\) if they occur on the rhs.

For simply laced \(g\),

\[
Y_m^{(a)}(u - 1)Y_m^{(a)}(u + 1) = \prod_{\ell \in \Omega \omega_{-1,2}} \frac{(1 + Y_m^{(a)(u)}) (1 + Y_m^{(a)}(u))}{(1 + Y_m^{(a-1)}(u^{-1})(1 + Y_m^{(a)}(u^{-1}))}
\]

(2.11)

For \(g = B_r\),

\[
Y_m^{(a)}(u - 1)Y_m^{(a)}(u + 1) = \frac{1 + Y_m^{(a-1)}(u)(1 + Y_m^{(a+1)}(u))}{(1 + Y_m^{(a-1)}(u^{-1})(1 + Y_m^{(a+1)}(u^{-1}))}
\]

(1 \leq a \leq r - 2),

\[
Y_m^{(r-1)}(u - 1)Y_m^{(r-1)}(u + 1) = \frac{1 + Y_m^{(r-2)}(u)(1 + Y_m^{(r)}(u))}{(1 + Y_m^{(r-1)}(u^{-1})(1 + Y_m^{(r+1)}(u^{-1}))}
\]

\[
Y_m^{(r)}(u - 1)Y_m^{(r)}(u + 1) = \frac{1}{(1 + Y_m^{(r)}(u^{-1})(1 + Y_m^{(r)}(u^{-1}))}
\]

(2.12)

For \(g = C_r\),

\[
Y_m^{(a)}(u - 1)Y_m^{(a)}(u + 1) = \frac{1 + Y_m^{(a-1)}(u)(1 + Y_m^{(a+1)}(u))}{(1 + Y_m^{(a-1)}(u^{-1})(1 + Y_m^{(a+1)}(u^{-1}))}
\]

(1 \leq a \leq r - 2),

\[
Y_m^{(r-1)}(u - 1)Y_m^{(r-1)}(u + 1) = \frac{1 + Y_m^{(r-2)}(u)(1 + Y_m^{(r)}(u))}{(1 + Y_m^{(r-1)}(u^{-1})(1 + Y_m^{(r+1)}(u^{-1}))}
\]

\[
Y_m^{(r)}(u - 1)Y_m^{(r)}(u + 1) = \frac{1 + Y_m^{(r)}(u)}{(1 + Y_m^{(r)}(u^{-1})(1 + Y_m^{(r)}(u^{-1}))}
\]

\[
Y_m^{(r-1)}(u - 1)Y_m^{(r-1)}(u + 1) = \frac{(1 + Y_m^{(r-1)}(u^{-1})(1 + Y_m^{(r-1)}(u^{-1}))}{(1 + Y_m^{(r+1)}(u^{-1})(1 + Y_m^{(r+1)}(u^{-1}))}
\]

(2.13)

For \(g = F_4\),

\[
Y_m^{(1)}(u - 1)Y_m^{(1)}(u + 1) = \frac{1 + Y_m^{(2)}(u)}{(1 + Y_m^{(1)}(u^{-1})(1 + Y_m^{(3)}(u^{-1}))}
\]

\[
Y_m^{(2)}(u - 1)Y_m^{(2)}(u + 1) = \frac{(1 + Y_m^{(1)}(u)(1 + Y_m^{(3)}(u)))}{(1 + Y_m^{(1)}(u^{-1})(1 + Y_m^{(3)}(u^{-1}))}
\]

\[
Y_m^{(3)}(u - 1)Y_m^{(3)}(u + 1) = \frac{(1 + Y_m^{(2)}(u)(1 + Y_m^{(4)}(u)))}{(1 + Y_m^{(3)}(u^{-1})(1 + Y_m^{(4)}(u^{-1}))}
\]

(2.14)

\[
Y_m^{(4)}(u - 1)Y_m^{(4)}(u + 1) = \frac{1 + Y_m^{(4)}(u)}{(1 + Y_m^{(3)}(u^{-1})(1 + Y_m^{(3)}(u^{-1}))}
\]
We write down the level 2 restricted \(Y(\cdot)\) with \(n\):

\[
Y_{m}^{(1)}(u - 1)Y_{m}^{(1)}(u + 1) = \frac{1 + Y_{m}^{(3)}(u)}{(1 + Y_{m-1}^{(3)}(u)) (1 + Y_{m+1}^{(3)}(u))},
\]

(2.14)

For \(g = G_{2}\),

\[
Y_{m}^{(1)}(u - 1)Y_{m}^{(1)}(u + 1) = \frac{1 + Y_{m}^{(3)}(u)}{(1 + Y_{m-1}^{(3)}(u)) (1 + Y_{m+1}^{(3)}(u))},
\]

\[
Y_{3m}^{(1)}(u - 1)Y_{3m}^{(1)}(u + 1) = \frac{1 + Y_{m}^{(3)}(u)}{(1 + Y_{3m-1}^{(3)}(u)) (1 + Y_{3m+1}^{(3)}(u))},
\]

\[
Y_{3m+1}^{(1)}(u - 1)Y_{3m+1}^{(1)}(u + 1) = \frac{1}{(1 + Y_{3m}^{(3)}(u)) (1 + Y_{3m+2}^{(3)}(u))},
\]

and

\[
Y_{3m+2}^{(1)}(u - 1)Y_{3m+2}^{(1)}(u + 1) = \frac{1}{(1 + Y_{3m+1}^{(3)}(u)) (1 + Y_{3m+3}^{(3)}(u))}.
\]

(2.15)

We stress that the \(T\)- and \(Y\)-systems for nonsimply laced \(g\) are not just a folding of simply laced cases.

We also remark that both \(T\) and \(Y\)-systems for \(B_{2}\) and \(C_{2}\) are equivalent and transformed to each other by \(T_{m}^{(1)}(u) \leftrightarrow T_{m}^{(2)}(u)\) and \(Y_{m}^{(1)}(u) \leftrightarrow Y_{m}^{(2)}(u)\) reflecting the fact that \(B_{2} \simeq C_{2}\).

### 2.2. Restriction

We fix an integer \(\ell \geq 2\) called the level. Let \(t_{\ell}\) be the number in (2.1). The level \(\ell\) restricted \(T\)-system for \(g\) (with the unit boundary condition) is relations (2.4)–(2.10) naturally restricted to \(T_{m}^{(\ell)}(u)\) with \(1 \leq m \leq t_{\ell} - 1, u \in U\) by imposing \(T_{t_{\ell}}^{(\ell)}(u) = 1\).

The level \(\ell\) restricted \(Y\)-system for \(g\) is relations (2.11)–(2.15) naturally restricted to \(Y_{m}^{(\ell)}(u)\) with \(1 \leq m \leq t_{\ell} - 1, u \in U\) by imposing \(Y_{t_{\ell}}^{(\ell)}(u)^{-1} = 0\).

Note that for \(g\) nonsimply laced, the above restriction makes sense also at \(\ell = 1\). The resulting \(T\) and \(Y\)-systems become equivalent to the level \(t\) restricted \(T\)- and \(Y\)-systems for \(A_{n}\) with \(n = \frac{1}{2} \sum_{i \in I} t_{i} = t\) under the rescaling of the spectral parameter \(u \to u/t\). One can also consider the level 0 case formally. See the text around (16.2).

**Example 2.2.** We write down the level 2 restricted \(T\) and \(Y\)-systems for \(A_{2}\):

\[
T_{1}^{(1)}(u - 1)T_{1}^{(1)}(u + 1) = 1 + T_{1}^{(2)}(u), \quad T_{1}^{(2)}(u - 1)T_{1}^{(2)}(u + 1) = 1 + T_{1}^{(1)}(u),
\]

\[
Y_{1}^{(1)}(u - 1)Y_{1}^{(1)}(u + 1) = 1 + Y_{1}^{(2)}(u), \quad Y_{1}^{(2)}(u - 1)Y_{1}^{(2)}(u + 1) = 1 + Y_{1}^{(1)}(u).
\]

Thus they are identical.

**Example 2.3.** We write down the level 2 restricted \(T\)-system for \(C_{2}\):

\[
T_{1}^{(1)}(u - \frac{1}{2})T_{1}^{(1)}(u + \frac{1}{2}) = T_{1}^{(2)}(u) + T_{1}^{(2)}(u),
\]

\[
T_{2}^{(1)}(u - \frac{1}{2})T_{2}^{(1)}(u + \frac{1}{2}) = T_{2}^{(2)}(u)T_{3}^{(1)}(u) + T_{1}^{(2)}(u)T_{1}^{(2)}(u - \frac{1}{2})T_{1}^{(2)}(u + \frac{1}{2}).
\]

\[
T_{3}^{(1)}(u - \frac{1}{2})T_{3}^{(1)}(u + \frac{1}{2}) = T_{2}^{(2)}(u) + T_{1}^{(2)}(u),
\]

\[
T_{1}^{(2)}(u - 1)T_{1}^{(2)}(u + 1) = 1 + T_{2}^{(1)}(u).
\]
Example 2.4. Level $\ell$ restricted $T$-system for $A_{r-1}$ has the form

$$T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a)}(u)T_{m+1}^{(a)}(u),$$

for $1 \leq a < r - 1$ and $1 \leq m < \ell - 1$. It is invariant under the simultaneous transformation $T_m^{(a)}(u) \mapsto T_m^{(a)}(\pm u + \text{const})$ and $r \leftrightarrow \ell$. The similar property holds also for the level $\ell$ restricted $Y$-system for $A_{r-1}$. This symmetry is called the level-rank duality.

2.3. Relation between $T$- and $Y$-systems

The unrestricted $T$-system for $g$ has the form

$$T_m^{(a)}(u) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{(b,k,v)} T_k^{(b)}(v)^{N(a,m,u;b,k,v)},$$

where the last term is a finite product. Then, it is easy to see that the unrestricted $Y$-system for the same $g$ takes the form

$$Y_m^{(a)}(u) = \left( 1 + Y_{m+1}^{(a)}(u) \right) \left( 1 + \sum_{(b,k,v)} Y_k^{(b)}(v) \right)^{N(a,m,u;b,k,v)}.\tag{2.17}$$

The same relation holds also between the level $\ell$ restricted $T$- and $Y$-systems.

Let us write (2.16) simply as

$$T_m^{(a)}(u) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + M_m^{(a)}(u).\tag{2.18}$$

Theorem 2.5 ([1]). Suppose $T_m^{(a)}(u)$ satisfies the unrestricted $T$-system for $g$. Then

$$Y_m^{(a)}(u) = \frac{M_m^{(a)}(u)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)}$$

is a solution of the unrestricted $Y$-system for $g$. The same claim holds between the level $\ell$ restricted $T$- and $Y$-systems.

Sketch of proof. This can be directly verified by substituting the resulting relations

$$1 + Y_m^{(a)}(u) = \frac{T_m^{(a)}(u - 1)T_m^{(a)}(u + 1)}{T_m^{(a)}(u)T_{m+1}^{(a)}(u)},$$

$$1 + Y_m^{(a)}(u)^{-1} = \frac{T_m^{(a)}(u - 1)T_m^{(a)}(u + 1)}{M_m^{(a)}(u)}$$

into the $Y$-system. Here we demonstrate the calculation for simply laced $g$:

$$Y_m^{(a)}(u) = \frac{\prod_{b,C_m=1} T_{m+1}^{(b)}(u - 1)T_{m+1}^{(b)}(u + 1)}{T_{m-1}^{(a)}(u - 1)T_{m+1}^{(a)}(u - 1)T_{m+1}^{(a)}(u + 1)T_{m}^{(a)}(u)}.$$

This simplifies to

$$Y_m^{(a)}(u) = \left( 1 + Y_{m+1}^{(a)}(u) \right) \left( 1 + \sum_{(b,k,v)} Y_k^{(b)}(v) \right)^{N(a,m,u;b,k,v)}.\tag{2.17}$$
This calculation is valid also at \( m = 1 \) by formally setting \( T^0_{-1}(u) = 0 \). For the level \( \ell \)
restricted case, it is valid similarly by formally setting \( T^0_{\ell+1}(u) = 0 \).

Theorem 2.5 has a natural account from the viewpoint of cluster algebra with coefficients.
See remark 5.5.

**Example 2.6.** We write down relation (2.19) for the level 2 restricted \( T \)-system for \( C_2 \). From
example 2.3, they read
\[
\begin{align*}
Y_1^{(1)}(u) &= \frac{T_1^{(2)}(u)}{T_2^{(1)}(u)}, \\
Y_2^{(1)}(u) &= \frac{T_1^{(2)}(u - \frac{i}{\ell})T_1^{(2)}(u + \frac{i}{\ell})}{T_1^{(1)}(u)T_3^{(1)}(u)}, \\
Y_3^{(1)}(u) &= \frac{T_1^{(2)}(u)}{T_2^{(1)}(u)}, \\
Y_1^{(2)}(u) &= T_2^{(1)}(u).
\end{align*}
\]

Thus, the specific construction (2.19) automatically imposes the condition \( Y_1^{(1)}(u) = Y_3^{(1)}(u) \).
However, the level restricted \( Y \)-system alone does not restrict itself to such a situation in
general.

**Remark 2.7.** Consider a slight modification of the general \( T \)-system relation (2.18) into
\[
T_m^{(a)} \left( u \right) T_m^{(a)} \left( u + \frac{1}{I_a} \right) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + g_m^{(a)}(u)M_m^{(a)}(u),
\]
where \( g_m^{(a)}(u) \) is any function satisfying
\[
g_m^{(a)} \left( u - \frac{1}{I_a} \right) g_m^{(a)} \left( u + \frac{1}{I_a} \right) = g_{m-1}^{(a)}(u)g_{m+1}^{(a)}(u).
\]

Then it is easily checked that the substitution
\[
Y_m^{(a)}(u) = g_m^{(a)}(u)M_m^{(a)}(u) \frac{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)}
\]
is still a solution of the same \( Y \)-system.

**2.4. Twisted case**

Let us proceed to the \( T \)- and \( Y \)-systems associated with the twisted quantum affine algebras
following [12, 13]. In this subsection and the next, \( X_N \) exclusively denotes a Dynkin diagram
of type \( A_N (N \geq 2) \), \( D_N (N \geq 4) \) or \( E_N \). We keep the enumeration of the nodes of \( X_N \) by
the set \( I = \{ 1, \ldots, N \} \) as in figure 1. For a pair \( (X_N, \kappa) = (A_N, 2), (D_N, 2), (E_6, 2) \) or \( (D_4, 3), \)
we define the diagram automorphism \( \sigma : I \to I \) of \( X_N \) of order \( \kappa \) as follows: \( \sigma(a) = a \) except
for the following cases in our enumeration:
\[
\begin{align*}
\sigma(1) &= 3, \quad \sigma(3) = 4, \quad \sigma(4) = 1 \quad (X_N, \kappa) = (D_4, 3),
\sigma(1) &= 6, \quad \sigma(2) = 5, \quad \sigma(5) = 2, \quad \sigma(6) = 1 \quad (X_N, \kappa) = (E_6, 2),
\sigma(1) &= 5, \quad \sigma(2) = 6, \quad \sigma(4) = 1 \quad (X_N, \kappa) = (D_N, 2),
\sigma(N - 1) &= N, \quad \sigma(N) = N - 1 \quad (X_N, \kappa) = (D_N, 2),
\sigma(N) &= N + 1 - a(a \in I) \quad (X_N, \kappa) = (A_N, 2),
\sigma(a) &= N + 1 - a(a \in I) \quad (X_N, \kappa) = (A_N, 2).
\end{align*}
\]

Let \( I/\sigma \) be the set of the \( \sigma \)-orbits of nodes of \( X_N \). We choose, at our discretion, a complete set
of representatives \( I_\sigma \subset I \) of \( I/\sigma \) as
\[
I_\sigma = \begin{cases}
[1, 2, \ldots, r] & (X_N, \kappa) = (A_{2r - 1}, 2), (A_{2r}, 2), (D_{r+1}, 2), \\
[1, 2, 3, 4] & (X_N, \kappa) = (E_6, 2), \\
[1, 2] & (X_N, \kappa) = (D_4, 3).
\end{cases}
\]
Let $X_N^{(c)} = A_{2r-1}^{(2)} (r \geq 2), A_{2r}^{(2)} (r \geq 1), D_{r+1}^{(2)} (r \geq 3), E_6^{(2)}$ or $D_4^{(3)}$ be a Dynkin diagram of twisted affine type [11]. We enumerate the nodes of $X_N^{(c)}$ with $I_\sigma \cup \{0\}$ as in figure 2, where $I_\sigma$ is the one for $(X_N, \kappa)$. By this, we have established the identification of the non-0th nodes of the diagram $X_N^{(c)}$ with the nodes of the diagram $X_N$ belonging to the set $I_\sigma$. For example, for $E_6^{(2)}$, the correspondence is as follows:

The filled nodes 3 and 4 in $E_6^{(2)}$ correspond to the fixed nodes by $\sigma$ in $E_6$. We use this identification throughout. (The 0th node of $X_N^{(c)}$ is irrelevant in our setting here.)

We define $\kappa_a (a \in I_\sigma)$ as

$$\kappa_a = \begin{cases} 1 & \sigma(a) \neq a, \\ \kappa & \sigma(a) = a. \end{cases}$$

(2.27)

Note that $X_N^{(c)} = A_{2r-1}^{(2)}$ is the unique case in which $\kappa_a = 1$ for any $a \in I_\sigma$. By $U_q(X_N^{(c)})$, we mean the quantized universal enveloping algebra [14] of the twisted affine Lie algebra of type $X_N^{(c)}$ [11].

Let us proceed to the unrestricted $T$-systems. Choose $h \in \mathbb{C} \setminus 2\pi \sqrt{-1}\mathbb{Q}$ arbitrarily. The unrestricted $T$-system for $U_q(X_N^{(c)})$ is the following relations for commuting variables $\{T_m^{(a)}(u) \mid a \in I_\sigma, m \in \mathbb{Z}_{\geq 1}, u \in \mathbb{C}_{\mathbb{C}_h}\}$, where $\Omega = 2\pi \sqrt{-1}/xh$, and $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$ if they occur on the rhs in the relations as follows.

For $X_N^{(c)} = A_{2r-1}^{(2)}$,

$$T_m^{(a)}(u)T_m^{(a)}(u + 1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u) \quad (1 \leq a \leq r - 1),$$

$$T_m^{(r)}(u)T_m^{(r)}(u + 1) = T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_{m}^{(r-1)}(u)T_{m}^{(r+1)}(u) + \Omega.$$

(2.28)

For $X_N^{(c)} = A_{2r}^{(2)}$,

$$T_m^{(a)}(u)T_m^{(a)}(u + 1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u) \quad (1 \leq a \leq r - 1),$$

$$T_m^{(r)}(u)T_m^{(r)}(u + 1) = T_{m-1}^{(r)}(u)T_{m+1}^{(r)}(u) + T_{m}^{(r-1)}(u)T_{m}^{(r+1)}(u) + \Omega.$$ 

(2.29)
For $X_N^{(e)} = D_{r+1}^{(2)}$,

$$T_m^{(a)}(u - 1) T_m^{(a)}(u + 1) = T_{m-1}^{(a)}(u) T_m^{(a+1)}(u) + T_m^{(a+1)}(u) T_{m+1}^{(a)}(u) \quad (1 \leq a \leq r - 2),$$

$$T_m^{(r-1)}(u - 1) T_m^{(r-1)}(u + 1) = T_{m-1}^{(r-1)}(u) T_m^{(r)}(u) + T_m^{(r)}(u) T_{m+1}^{(r)}(u + \Omega),$$

$$T_m^{(r)}(u - 1) T_m^{(r)}(u + 1) = T_{m-1}^{(r)}(u) T_m^{(r)}(u) + T_m^{(r-1)}(u).$$  \hfill (2.30)

For $X_N^{(e)} = E_{2}^{(2)}$,

$$T_m^{(1)}(u - 1) T_m^{(1)}(u + 1) = T_{m-1}^{(1)}(u) T_{m+1}^{(1)}(u) + T_m^{(2)}(u),$$

$$T_m^{(2)}(u - 1) T_m^{(2)}(u + 1) = T_{m-1}^{(2)}(u) T_{m+1}^{(2)}(u) + T_m^{(3)}(u),$$

$$T_m^{(3)}(u - 1) T_m^{(3)}(u + 1) = T_{m-1}^{(3)}(u) T_{m+1}^{(3)}(u) + T_m^{(2)}(u) T_m^{(4)}(u) T_m^{(4)}(u),$$

$$T_m^{(4)}(u - 1) T_m^{(4)}(u + 1) = T_{m-1}^{(4)}(u) T_{m+1}^{(4)}(u) + T_m^{(3)}(u).$$  \hfill (2.31)

For $X_N^{(e)} = D_{2}^{(3)}$,

$$T_m^{(1)}(u - 1) T_m^{(1)}(u + 1) = T_{m-1}^{(1)}(u) T_{m+1}^{(1)}(u) + T_m^{(2)}(u),$$

$$T_m^{(2)}(u - 1) T_m^{(2)}(u + 1) = T_{m-1}^{(2)}(u) T_{m+1}^{(2)}(u) + T_m^{(1)}(u - \Omega) T_m^{(1)}(u + \Omega).$$  \hfill (2.32)

The domain $\mathbb{C}_{\kappa,h}$ of the parameter $u$ effectively imposes the following periodicity:

$$T_m^{(a)}(u) = \begin{cases} T_m^{(a)}(u + \kappa \Omega) & \sigma(a) \neq a, \\ T_m^{(a+1)}(u + \Omega) & \sigma(a) = a. \end{cases}$$  \hfill (2.33)

**Remark 2.8.** The $T$-system for $U_q(X_N^{(e)})$ is obtainable from the $T$-system for $g = X_N$ by a folding in the following sense. Denoting the variable in the latter by $T_m^{(a)}(u)$ with $a \in I$, one imposes the condition $T_{m}^{(r-1)}(u) = T_{m}^{(a)}(u + k \Omega)$ and identifies $T_m^{(r)}(u)$ with $a \in I_{r} \subset I$ as the variable $T_m^{(a)}(u)$ in the former. The same remark applies also to the $Y$-system given in what follows.

The unrestricted $Y$-system for $U_q(X_N^{(e)})$ is the following relations for the commuting variables $\{ Y_m^{(a)}(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in \mathbb{C}_{\kappa,h} \}$, where $\Omega = 2\pi \sqrt{-1}/k h$, and $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = 0$ if they occur on the rhs in the relations as follows.

For $X_N^{(e)} = A_{2r-1}^{(2)}$,

$$Y_m^{(a)}(u - 1) Y_m^{(a)}(u + 1) = \frac{1 + Y_{m}^{(a-1)}(u)}{1 + Y_{m}^{(a)}(u)} \frac{1 + Y_{m+1}^{(a+1)}(u)}{1 + Y_{m+1}^{(a)}(u)} \quad (1 \leq a \leq r - 1),$$

$$Y_m^{(r)}(u - 1) Y_m^{(r)}(u + 1) = \frac{1 + Y_{m}^{(r-1)}(u)}{1 + Y_{m}^{(r)}(u + \Omega)} \frac{1 + Y_{m+1}^{(r-1)}(u + \Omega)}{1 + Y_{m+1}^{(r)}(u + \Omega)}. $$  \hfill (2.34)

For $X_N^{(e)} = A_{2r}^{(2)}$,

$$Y_m^{(a)}(u - 1) Y_m^{(a)}(u + 1) = \frac{1 + Y_{m}^{(a-1)}(u)}{1 + Y_{m}^{(a)}(u)} \frac{1 + Y_{m+1}^{(a+1)}(u)}{1 + Y_{m+1}^{(a)}(u)} \quad (1 \leq a \leq r - 1),$$

$$Y_m^{(r)}(u - 1) Y_m^{(r)}(u + 1) = \frac{1 + Y_{m}^{(r-1)}(u)}{1 + Y_{m}^{(r)}(u + \Omega)} \frac{1 + Y_{m+1}^{(r-1)}(u + \Omega)}{1 + Y_{m+1}^{(r)}(u + \Omega)}. $$  \hfill (2.35)
2.6. \( U_q(sl(r|s)) \) case

Among a variety of Lie super-algebras, we present the \( T^-\) and \( Y^-\) systems related to \( U_q(sl(r|s)) \) as a typical example. For brevity we employ the following notation within this subsection:

\[
H_{r,s} = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \setminus (\mathbb{Z}_{\geq r} \times \mathbb{Z}_{\geq s}), \quad \overline{H}_{r,s} = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \setminus (\mathbb{Z}_{\geq r} \times \mathbb{Z}_{\geq s}).
\]
These sets are often called fat hook. The $T$-system for $U_q(sl(r|s))$ is the following relations among the commuting variables \{$T_{m}^{(a)}(u)| (a, m) \in \mathcal{H}_{r,s}, u \in U \}$:

\[ T_{m}^{(a)}(u - 1)T_{m}^{(a)}(u + 1) = T_{m}^{(a-1)}(u)T_{m+1}^{(a+1)}(u) + T_{m}^{(a)}(u)T_{m+1}^{(a)}(u), \] \[ (2.40) \]

Thus, \( T_{r+1}^{(a)}(u) = T_{r}^{(a+1)}(u) \). \[ (2.41) \]

Relation (2.40) is imposed for all \( (a, m) \in \mathcal{H}_{r,s} \setminus \{(0, 0)\} \), where if any \( T_{k}^{(b)}(u) \) with \( (b, k) \not\in \mathcal{H}_{r,s} \) is contained on the rhs, it should be understood as 0:

\[ T_{k}^{(b)}(u) = 0 \quad \text{if} \quad (b, k) \not\in \mathcal{H}_{r,s}. \] \[ (2.42) \]

This leads to the simple recursion relations for the sequences corresponding to the boundary \( \mathcal{H}_{r,s} \setminus \mathcal{H}_{r,s} \):

\[ T_{m}^{(a)}(u - 1)T_{m}^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) \quad (a, m) \in (r, \mathbb{Z}_{>0}) \cup (0, \mathbb{Z}_{>0}), \] \[ (2.43) \]

The extra relation (2.41) leads by induction to

\[ T_{r+1}^{(a)}(u) = T_{r}^{(a+1)}(u) \quad a \geq 0. \] \[ (2.44) \]

In applications, the variables appearing in (2.43) and (2.44) are chosen appropriately reflecting the normalization of the system. Relation (2.40) is the same as the type A case. The essential difference from it lies in (2.42) and (2.44).

Let us proceed to the $Y$-system. We assume $r \geq s \geq 1$ first. The $Y$-system for $U_q(sl(r|s))$ is the following relations among the commuting variables \{$Y_{m}^{(a)}(u)| (a, m) \in \mathcal{H}_{r,s}, u \in U \}$:

\[ Y_{m}^{(a)}(u - 1)Y_{m}^{(a)}(u + 1) = \frac{(1 + Y_{m+1}^{(a)}(u))(1 + Y_{m-1}^{(a)}(u))}{(1 + Y_{m}^{(a+1)}(u))^{2}}(1 + Y_{m+1}^{(a)}(u))^{-1} \quad (a, m) \in \mathcal{H}_{r,s}, \] \[ (2.45) \]

\[ Y_{1}^{(a)}(u - 1)Y_{1}^{(a)}(u + 1) = Y_{2}^{(a)}(u)(1 + Y_{s-1}^{(a)}(u)) \quad a \geq 2, \] \[ (2.46) \]

\[ Y_{1}^{(a)}(u - 1)Y_{1}^{(a)}(u + 1) = Y_{2}^{(a+1)}(u)Y_{1}^{(a-1)}(u)(1 + Y_{s-1}^{(a)}(u)) \quad a \geq 2, \] \[ (2.47) \]

\[ Y_{1}^{(a)}(u) = Y_{2}^{(a)}(u), \quad Y_{1}^{(r)}(u) = Y_{2}^{(r)}(u). \] \[ (2.49) \]

On the rhs of these relations, any factor \((1 + Y_{k}^{(b)}(u))^{\pm 1}\) with \((b, k) \not\in \mathcal{H}_{r,s}\) is to be understood as 1. When $r > s \geq 1$, equations (2.46) and (2.48) are absent. The $Y$-system for $s \geq r \geq 2$ is given by (2.45)–(2.49) by interchanging $r$ and $s$.

There is a simple relation between the $T$- and $Y$-systems analogous to theorem 2.5.

Suppose that \( T_{m}^{(a)}(u) \) is a solution to the $T$-system. Then the combinations

\[ Y_{m}^{(a)}(u) = \frac{T_{m+1}^{(a)}(u)T_{m}^{(a+1)}(u)}{T_{m}^{(a-1)}(u)T_{m+1}^{(a)}(u)} \quad (a, m) \in \mathcal{H}_{r,s}, \] \[ (2.50) \]

\[ Y_{1}^{(a)}(u) = \frac{T_{s}^{(a-1)}(u)}{T_{s}^{(a+1)}(u)}, \quad Y_{2}^{(a)}(u) = \frac{T_{s}^{(a)}(u)}{T_{s}^{(a+1)}(u)}. \] \[ (2.51) \]
satisfy the $Y$-system. In particular, (2.49) holds due to (2.41). When $s \geq r \geq 2$, the parallel fact holds by interchanging $r$ and $s$ and the role of indices $a$ and $m$ in $T_m^{(a)}(u)$ and $Y_m^{(a)}(u)$ everywhere. In view of the symmetry of sets (2.39), we do not introduce the level restriction.

Remark 2.9. The above set of relations seem different from those given in [15] for $gl(2|2)$, where a special relation $T_m^{(a)}(u) \propto T_{m-2}^{(r-2)}$ valid only for this case is utilized. Thanks to this, $\Upsilon_2^{(a)}(u) \neq \Upsilon_2^{(r)}(u)$ are not necessarily needed. The two sets of $Y$-systems nevertheless lead to an identical set of TBA equations. The $Y$-system (2.45)–(2.49) is consistent with the TBA equations in [16] under the identification $N, K \leftrightarrow r, s$ and

\begin{align*}
Y^{(a)}_{s-m} &= e^{-\zeta(a)/T} (1 \leq a, 1 \leq m \leq s - 1), \\
Y^{(a)}_{s-j} &= e^{-\epsilon(a)/T} (1 \leq j, 1 \leq a \leq r - 1).
\end{align*}

2.7. Bibliographical notes

The Hirota relation (2.5) for transfer matrices in the $A_r$ case first appeared in [1], where the $T$-system for $g$ was introduced as functional relations among the commuting transfer matrices $\{T_m^{(a)}(u)\}$. The models relevant to the unrestricted and restricted versions are the vertex and the RSOS-type models, respectively. In such a setting, the $T$-system acquires some scalar coefficients depending on the normalization of $T_m^{(a)}(u)$ as in remark 2.7. The unit boundary condition is also modified accordingly. Actually in [1], the restricted $T$-system was introduced by imposing a slightly weaker condition $T_m^{(a)}(u) = 0$. The $T$-system for the twisted case was introduced in [12] in a similar context. Our presentation here follows [13, 17]. The $T$-system unifies the many functional relations studied earlier individually. See sections 3 and 4 for more details.

The level $\ell$ restricted $Y$-system for $g$ was introduced in [3] for simply laced $g$ with $\ell = 2$ as a universal property of the TBA equation in the context of integrable perturbations of CFT. Then, it was extended to the general case in [4] based on the TBA equation related to RSOS models for $U_q(\hat{g})$ [18]. This procedure is detailed in section 14. The $Y$-system for simply laced $g$ was also given in [5] independently. For more literature in a similar context, see section 14.7. The transformation (2.19) between the $T$- and $Y$-systems first appeared in [7] for the simplest case $g = A_1$ and extended in [1] to general $g$. $T$-systems related to Lie super-algebras and super-symmetric models have been studied in various contexts. See for example [15, 19–24] and references therein.

3. $T$-system among commuting transfer matrices

The aim of this section is to introduce the basic examples of solvable lattice models, both vertex and RSOS type, and demonstrate how the $T$-system is obtained for their transfer matrices in connection to the fusion procedure. Although these issues are nowadays well recognized as being intimately related to the representation theory of quantum groups, we defer such a description to section 4 avoiding too many definitions from the beginning. Our presentation here is based on explicit calculations in trigonometric parameterization along the simplest example from $g = A_1$. The exception is the last subsection 3.7, where we will formally argue the general features of those models associated with general $g$, quoting known facts on Kirillov–Reshetikhin modules and $Q$-system from sections 4, 13.6 and 14.6.

\footnote{There are types in [15] for $gl(2|2)$, in the text around (5.4) and (5.5).}
3.1. Vertex models and fusion

We recall the six-vertex model and its fusion without much recourse to the representation theory\(^\text{10}\). Consider the two-dimensional square lattice, where each edge is assigned with a local variable belonging to \{1, 2\}. Around each vertex, we allow the following six configurations with the respective Boltzmann weights:

\[
\begin{array}{cccccc}
1 & 1 & 2 & 2 & 1 & 2 \\
2 & 2 & 1 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 2 & 1 \\
\end{array}
\]

\[1 - q^2 z \quad 1 - q^2 z \quad q(1 - z) \quad q(1 - z) \quad z(1 - q^2) \quad 1 - q^2.\]  

(3.1)

The other ten configurations are assigned with zero Boltzmann weight. Let \(V = \mathbb{C}v_1 \oplus \mathbb{C}v_2\). Then (3.1) is arranged in the quantum R matrix \(R(z) \in \text{End}(V \otimes V)\) as

\[
R(z) = a(z) \sum_i E_{ii} \otimes E_{ii} + b(z) \sum_{i \neq j} E_{ii} \otimes E_{jj} + c(z) \left( \sum_{i < j} + \sum_{i > j} \right) E_{ji} \otimes E_{ij},
\]

(3.2)

where the \(z\) dependence is exhibited. The Yang–Baxter equation

\[
R_{23}(z')R_{13}(z)R_{12}(z/z') = R_{12}(z/z')R_{13}(z)R_{23}(z')
\]

holds \([2]\), where the indices signify the components in the tensor product as \(V \otimes V \otimes V\) on which both sides act. It is depicted as

\[
\begin{align*}
\begin{array}{c}
\text{z/z'} \\
\text{z} \\
\text{z'/z} \\
\end{array}
\end{align*}
\]

(3.4)

Starting from the six-vertex model \([25, 26]\), one can construct higher spin solvable vertex models by the fusion procedure \([27]\). Let \(V_m\) be the irreducible \(U_q\) module spanned by the \(m\) fold \(q\)-symmetric tensors. Concretely, \(V_1 = V\) and \(V_m\) with \(m \geq 2\) is realized as the quotient \(V^{\otimes m}/A\), where \(A = \sum_j V^{\otimes j} \otimes \text{Im} \hat{R}(q^{-2}) \otimes V^{\otimes m-2-j}\). It is easy to see \(\text{Im} \hat{R}(q^{-2}) = \text{Ker} \hat{R}(q^{-2}) = \mathbb{C}(v_1 \otimes v_2 - q v_2 \otimes v_1)\). We take the base vector of \(V_m\) as

\(^{10}\) Some terminology will be refined after (3.16).

\(^{11}\) The asymmetry between the last two in (3.1) is due to our choice of the coproduct (4.9). It fits the crystal base theory making the limit \(q \to 0\) of (3.7) well defined, although this fact will not be used in this review.
\(v_2^{\otimes x_2} \otimes v_1^{\otimes x_1} \mod A\), where \(x_i \in \mathbb{Z}_{\geq 0}\) and \(x_1 + x_2 = m\). The base will also be denoted by \(x = (x_1, x_2)\) for brevity. Obviously \(\dim V_m = m + 1\).

The Yang–Baxter equation (3.4) with \(z' = zq^2\) shows that \(\text{Im} \tilde{R}(q^{-2}) \subset V \otimes V\) is preserved under the action of \(R_{13}(zq^2)R_{23}(z)\). Therefore, its action on \((V \otimes V) \otimes V\) can be restricted to \(V_2 \otimes V_1 = ((V \otimes V) / \text{Im} \tilde{R}(q^{-2})) \otimes V\). Similarly, by using (3.4) repeatedly, it is shown that the composition
\[
\frac{R_{1,m+1}(zq^{m-1})R_{2,m+1}(zq^{m-3}) \cdots R_{m,m+1}(zq^{-m+1})}{a(zq^{m-3})a(zq^{m-5}) \cdots a(zq^{-m+1})}
\]

(3.5)
can be restricted to \(V_m \otimes V_1\). The resulting operator, the fusion matrix \(R^{(m,1)}(z) \in \text{End}(V_m \otimes V_1)\), is given by
\[
R^{(m,1)}(z)(x \otimes v_j) = \sum_{k=1,2} \left( \begin{array}{c} j \\ k \end{array} \right) y \otimes v_k,
\]

(3.6)
where \(y = (y_1, y_2)\) is specified by the weight conservation (the so-called ice rule) as \(y_i = x_i + \delta_{ij} - \delta_{ik}\). By the definition \(R^{(1,1)}(z) = R(z)\) and (3.7) reduces to (3.1) for \(m = 1\). In the case \((j, k) = (1, 2)\) for example, the matrix element \(1 - q^{2x_2}\) is obtained from the following calculation (\(D = \text{denominator in (3.5)}\):
\[
\frac{1}{D} \sum_{i=1}^{x_2} q^{i-1} \prod_{i=1}^{x_2} \frac{a(zq^{m-1})}{a(zq^{m-1+2n})}.
\]

(3.8)
The red and blue edges are assigned with the local states 1 and 2, respectively. The incoming state (left column) represents \(v_2^{\otimes x_2} \otimes v_1^{\otimes x_1}\). The factor \(q^{i-1}\) accounts for the effect of rearranging the outgoing state into the base form by using the relation \(v_1 \otimes v_2 = qv_2 \otimes v_1 \mod A\) as
\[
v_2^{\otimes x_2} \otimes v_1^{\otimes x_1} = q^{x_2-1} v_2^{\otimes x_2} \otimes v_1^{\otimes x_1} = q^{y_2-x_2-1} v_2^{\otimes x_2} \otimes v_1^{\otimes x_1} \in V_m.
\]

where \(y = (y_1, y_2) = (x_1 + 1, x_2 - 1)\) for \((j, k) = (1, 2)\).

One can fuse \(R^{(m,1)}(z)\) further along the other component of the tensor product in a completely parallel fashion. The composition
\[
R^{(m,1)}_{0,0}(zq^{m-1}) \cdots R^{(m,1)}_{0,2}(zq^{m-n+3}) R^{(m,1)}_{0,1}(zq^{m-n+1}) \in \text{End}(V_m \otimes V_1^{\otimes n})
\]

(3.9)
can be restricted to \(V_m \otimes V_m\). The result yields the quantum R matrix \(R^{(m,n)}(z) \in \text{End}(V_m \otimes V_n)\).

The R matrices so obtained again satisfy the Yang–Baxter equation in \(\text{End}(V \otimes V_m \otimes V_n)\):
\[
R^{(m,n)}_{23}(z) R^{(n,m)}_{13}(z) R^{(m,1)}_{12}(z/z') = R^{(m,1)}_{12}(z/z') R^{(n,m)}_{13}(z) R^{(m,n)}_{23}(z').
\]

(3.10)
It is depicted as (3.4) with the three lines to be interpreted as representing $V_l$, $V_m$ and $V_n$.

The quantum $R$ matrix $R^{(m,n)}(z)$ gives rise to a fusion vertex model on a planar square lattice by the same rule as diagrams (3.3) and (3.6). The local variables on the horizontal and vertical edges are taken from $V_m$ and $V_n$, respectively.

3.2. Transfer matrices

Here we use the additive spectral parameter $u$ as well as the multiplicative one $z$. They are related as $z = q^u$. We introduce the row to row transfer matrix

$$T_m(u) = \text{Tr}_{V_m} \left( R_{0,N}^{(m,s_N)}(z/w_N) \cdots R_{0,1}^{(m,s_1)}(z/w_1) \right) = \sum_{x \in V_m} x \left| \frac{z}{w_1} \ldots \frac{z}{w_N} \right| x.$$

(3.11)

The horizontal line is associated with $V_m$ which is called the auxiliary space. The trace over it corresponds to the periodic boundary condition. There are $N$ vertical lines corresponding to $V_s \otimes \cdots \otimes V_s$ which is called the quantum space. The $T_m(u)$ is a linear operator acting on the quantum space. The data $s_i$, $w_i$ represent the inhomogeneity in the spins and coupling constants.

The first consequence of the Yang–Baxter equation (3.10) is the commutativity of the transfer matrices acting on the common quantum space (common $s_i$ and $w_i$ in the present context)

$$[T_m(u), T_n(v)] = 0.$$

(3.12)

Let us take $s_i = 1$ for all $i$ for simplicity and demonstrate the functional relation

$$T_1(u + 1)T_1(u - 1) = T_0(u)T_2(u) + g_1(u)\text{id},$$

$$T_0(u) = \prod_{i=1}^{N} a(z_i/q), \quad g_1(u) = \prod_{i=1}^{N} a(z_i/q)b(z_i/q),$$

(3.13)

where $z_i = z/w_i$. This corresponds to the $T$-system for $A_1$ (2.6) with $m = 1$ modified by model-dependent factors $T_0(u)$ and $g_1(u)$. Consider the diagram for $T_1(u + 1)T_1(u - 1)$ corresponding to the matrix element for the transition $v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N} \mapsto v_{\beta_1} \otimes \cdots \otimes v_{\beta_N}$:

$$\sum_{k,l=1,2} \begin{array}{c|c|c|c|c}
\alpha_1 & \cdots & \alpha_N \\
\hline
\frac{z_1}{q} & \frac{z_2}{q} & \cdots & \frac{z_N}{q} \\
\hline
\beta_1 & \cdots & \beta_N \\
\end{array}$$

(3.14)

Given $\alpha_i, \beta_i$, the sum over $k, l$ is regarded as the trace of an operator acting on the auxiliary space $V_1 \otimes V_1$ horizontally. The space $V_1 \otimes V_1$ possesses the invariant subspace $\text{Im} \tilde{R}(q^{-2}) = \mathbb{C}(v_1 \otimes v_2 - q v_2 \otimes v_1)$ which propagates to the right owing to the Yang–Baxter
The fusion procedure and the critical case means the trigonometric case of the eight-vertex solid-on-solid model discussed in the previous subsection. For simplicity we concentrate on the (8VSOS) model [34]. It is the fundamental example associated with \( U_q(A_1^{(1)}) \) at \( q \) a root of unity and serves as the prototype of RSOS models. It generalizes to \( U_q(\hat{g}) \) for any \( g \) in principle. We illustrate the fusion procedure [35] and the derivation of the simplest case of the fusion procedure [35] and the derivation of the simplest case of the T-system for the commuting transfer matrices [36, 37]. The contents are parallel with the six-vertex model discussed in the previous subsection. For simplicity we concentrate on the critical case\(^{12}\).

Consider the two-dimensional square lattice, where each site is assigned with a local state belonging to \( \mathbb{Z} \). On the two local states \( a, b \) on neighboring sites, the condition \( |a - b| = 1 \) is

\[
\begin{pmatrix}
2 & \text{cyclic} & k & \alpha \\
1 & \text{cyclic} & l & \beta
\end{pmatrix} - q \times
\begin{pmatrix}
1 & \text{cyclic} & k & \alpha \\
2 & \text{cyclic} & l & \beta
\end{pmatrix} = \delta_{\alpha \beta} a(zq)b(z/q) \times \begin{cases} 1 & (k, l) = (2, 1), \\ -q & (k, l) = (1, 2), \\ 0 & \text{otherwise}. \end{cases}
\]

Thus, \( \text{Im} \hat{R}(q^{-2}) \) contributes to \( \text{Tr}_{V_1 \otimes V_1} \) (3.14) as \( \prod_{i=1}^{N} \delta_{\alpha_i \beta_i} a(z_iq)b(z_i/q) \), giving the second term on the rhs of (3.13). The other contribution to the trace is from \( (V_1 \otimes V_1)/\text{Im} \hat{R}(q^{-2}) = V_2 \).

This is equal to \( T_0(u)T_1(u) \) by the definition, where the factor \( T_0(u) \) is due to the denominator in (3.5) with \( m = 2 \). In this way one observes that the exact sequence

\[
0 \to \text{Im} \hat{R}(q^{-2}) \to V_1 \otimes V_1 \to V_2 \to 0
\]

plays a key role in deriving (3.13).

In section 4.2, we will introduce the Kirillov–Reshetikhin module \( W_m^{(n)}(u) \) for general quantum affine algebra \( U_q(\hat{g}) \). The case \( g = A_1 \) relevant here, denoted by \( W_m(u) = W_m^{(1)}(u) \), will be described explicitly in section 4.3. In such a formalism, one endows each line in the diagrams like (3.14) and (3.15) with a spectral parameter \( z = q^z \) which corresponds to a Kirillov–Reshetikhin module \( W_m(u) \). The \( R \) matrix \( R^{(m,n)}(z) \) \( \in \text{End}(V_m \otimes V_n) \) is actually to be understood as \( R^{(m,n)}(z_1/z_2) \) \( \in \text{End}(W_m(u_1) \otimes W_n(u_2)) \) with \( z_i = q^{\alpha_i} \). Up to an overall scalar, it is characterized by the intertwining property \( A(g) P R^{(m,n)}(z_1/z_2) = P R^{(m,n)}(z_1/z_2) \Delta(g) \) where \( g \) is any element from \( U_q(A_1^{(1)}) \) and \( \Delta \) is the coproduct (4.9) [14]. Accordingly, we say that the transfer matrix \( T_m(u) \) (3.11) has the auxiliary space \( W_m(u) \) and acts on the quantum space \( W_{\alpha_i}(v_1) \otimes \cdots \otimes W_{\alpha_i}(v_N) \) with \( w_i = q^{\alpha_i} \).

The exact sequence (3.16) will also be refined into the one among the tensor product of Kirillov–Reshetikhin modules. See (4.16). The \( T \)-system relation \( T_m(u + 1)T_m(u - 1) = T_{m+1}(u)T_{m-1}(u) + g_m(u) \) for general \( m \) follows from theorem 4.2 with \( n = j = m \). An additional feature here is that one actually needs to consider the central extension of \( U_q(A_1^{(1)}) \) to properly cope with the factor \( g_m(u) \). We refer to [1, section 2.2] for this point. See also [28].

To summarize, the Kirillov–Reshetikhin module of the quantum affine algebra and their exact sequence form the representation theoretical background for the \( R \) matrix, fusion procedure and the \( T \)-system among commuting family of transfer matrices.

### 3.3. Restricted solid-on-solid models and fusion

Besides vertex models, there is another class of solvable lattice models called interaction round face (IRF or simply face) models [2]. The relation of the two classes of models has been studied from various viewpoints [29–33]. Here we recall the eight-vertex solid-on-solid (8VSOS) model [34]. It is the fundamental example associated with \( U_q(A_1^{(1)}) \) at \( q \) a root of unity and serves as the prototype of RSOS models. It generalizes to \( U_q(\hat{g}) \) for any \( g \) in principle. We illustrate the fusion procedure [35] and the derivation of the simplest case of the \( T \)-system for the commuting transfer matrices [36, 37]. The contents are parallel with the six-vertex model discussed in the previous subsection. For simplicity we concentrate on the critical case\(^{12}\).

\(^{12}\) The RSOS models allow elliptic Boltzmann weights in general. The critical case means the trigonometric case of them. The fusion procedure and the \( T \)-system are equally valid in the elliptic case as well.
imposed. With the allowed configuration round a face, the following Boltzmann weights are assigned [34]:

\[
W\left( \begin{array}{ccc}
  a & a & 1 \\
  a & 1 & a \\
  a & a & 1 \\
\end{array} \right| u \right) = \frac{[2 + u]_{q^{1/2}}}{[2]_{q^{1/2}}}, \quad W\left( \begin{array}{ccc}
  a & a & 1 \\
  a & 1 & a \\
  a & 1 & a \\
\end{array} \right| a \pm 1 & u \right) = \frac{[2a + 2a + u]_{q^{1/2}}}{[2a + 2a]_{q^{1/2}}}. \\
(3.17)
\]

where \( u \) is the spectral parameter, \( q \) and \( \xi \) are generic constants which will be specialized when considering the restriction in section 3.5. The function \([u]_{q^{1/2}}\) is given by replacing \( q \rightarrow q^{1/2}\) in

\[
[u]_q = q^{u - u} - q^{-u}. \\
(3.18)
\]

The Boltzmann weights (3.17) are depicted as

\[
\begin{array}{c}
  b \\
\hline
  a \\
\end{array}
\begin{array}{c}
  u \\
\hline
  c \\
\end{array}
= W\left( \begin{array}{ccc}
  b & c \\
  a & d \\
\end{array} \right| u \right). \\
(3.19)
\]

It satisfies the (generalized) star–triangle relation [2] which plays the role of the Yang–Baxter equation in face models:

\[
\sum_g W\left( \begin{array}{ccc}
  f & g \\
  a & b \\
\end{array} \right| u \right) W\left( \begin{array}{ccc}
  e & d \\
  g & f \\
\end{array} \right| v \right) W\left( \begin{array}{ccc}
  g & d \\
  e & c \\
\end{array} \right| u - v \right) = \sum_g W\left( \begin{array}{ccc}
  e & d \\
  g & c \\
\end{array} \right| u \right) W\left( \begin{array}{ccc}
  g & c \\
  e & a \\
\end{array} \right| v \right) W\left( \begin{array}{ccc}
  f & e \\
  a & g \\
\end{array} \right| u - v \right). \\
(3.20)
\]

The sum over \( g \) consists of at most two terms on each side because of the neighboring condition, e.g. \(|f - g| = |b - g| = |d - g| = 1\) for the lhs. We depict (3.20) as

\[
\begin{array}{c}
  e \\
\hline
  v \\
\end{array}
\begin{array}{c}
  d \\
\hline
  c \\
\end{array}
\begin{array}{c}
  u \\
\hline
  a \\
\end{array}
= \begin{array}{c}
  e \\
\hline
  u \\
\end{array}
\begin{array}{c}
  a \\
\hline
  b \\
\end{array}
\begin{array}{c}
  c \\
\hline
  d \\
\end{array}
(3.21)
\]

where \( \bullet \) stands for the sum over the local state. The faces drawn together are to be understood as the product of the attached Boltzmann weights.

One can apply the fusion procedure to the 8VSOS model [35]. Note the properties

\[
\begin{array}{c}
  \bullet \\
\hline
  -2 \\
\end{array} = 0, \quad a + 1 \\
\bullet \\
\hline
  a - 1 \\
\end{array}
\begin{array}{c}
  -2 \\
\hline
  a \\
\end{array}
\begin{array}{c}
  -2 \\
\hline
  a \\
\end{array}
= \begin{array}{c}
  a \\
\hline
  -2 \\
\end{array}
\begin{array}{c}
  a + 1 \\
\hline
  a - 1 \\
\end{array}
\begin{array}{c}
  a \\
\hline
  -2 \\
\end{array}
= 0 \\
(3.22)
\]
where the second equality from the right is due to the star–triangle relation. This implies that for $m = 2$, the quantity

$$
\begin{array}{c}
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3.4. Relation to vertex models

The trigonometric face models under consideration are related to the six-vertex model and its fusion in section 3.1 [30]. Let us explain it along the simplest cases (3.17) and (3.2). Let \( a \in \mathbb{Z}_{\geq 1} \) and \( V_{a-1} \) be the spin \( \frac{a-1}{2} \) representation of \( U_q(A_1) \) in section 3.1.13. We use coproduct (4.9) and the concrete form (4.10). In the irreducible decomposition \( V_{a-1} \otimes V_1 = \bigoplus_{b=1}^{a-1} V_{b-1} \), the highest weight vector \( v_{a,b} \in V_{b-1} \) is given by \( v_{a,b} = v_{a-1} \otimes v_1 \) and \( v_{a,a-1} = v_{a-1} \otimes v_1 - q^{a-1} v_{a} \otimes v_1 \). Repeating this once more, one obtains the highest weight vectors \( v_{a,b,c} \) in the irreducible component \( V_{a-1} \) in the decomposition of \( V_{a-1} \otimes V_1 \) labeled with \( a, b, c \) such that \( |a - b| = |b - c| = 1 \). Explicitly, they read

\[
v_{a,a+1,a+2} = v_{a-1} \otimes v_1^2 \otimes v_1^2,
\]
\[
v_{a,a-1,a} = [a - 1] q (v_{a-1} \otimes v_1^2 \otimes v_1^2 - q^{a-1} v_{a} \otimes v_1^2 \otimes v_1^2),
\]
\[
v_{a,a+1,a} = [a] q (v_{a-1} \otimes v_1^2 \otimes v_1^2 - q^{a-1} [a - 1] q v_{a} \otimes v_1^2 \otimes v_1^2 - q^{a-1} v_{a} \otimes v_1^2 \otimes v_1^2),
\]
\[
v_{a,a-1,a-2} = v_{a-1} \otimes v_1^2 \otimes v_1^2 - q^{a-1} v_{a} \otimes v_1^2 \otimes v_1^2 - q^{a-1} v_{a-1} \otimes v_1^2 \otimes v_1^2 - q^{a-2} v_{a-1} \otimes v_1^2 \otimes v_1^2 + q^{2a-4} v_{a-2} \otimes v_1 \otimes v_1^2 \otimes v_1^2. \tag{3.28}
\]

Now consider the operator \( 1 \otimes \hat{R}(z) \) acting on \( V_{a-1} \otimes V_1 \otimes V_1 \). Since it commutes with \( U_q(A_1) \), the images of the highest weight vectors are again highest. The face Boltzmann weights can be extracted from the matrix elements between those highest weight vectors as

\[
(1 \otimes \hat{R}(q^x)) v_{a,b,c} = -(q - q^{-1}) q^{(a+1) \xi} \sum_d W\left( \begin{array}{c|c|c}
  b & c & d \\
  a & & \\
 & u & \\
\end{array} \right) v_{a,d,c}. \tag{3.29}
\]

Here \( \xi = 0 \) on the rhs and the sum is over \( d \) such that \( |a - d| = |d - c| = 1 \). A similar relation holds also between the fusion models.

Conversely, one can deduce the \( R \) matrix from the face Boltzmann weights as a limit where the site variables or effectively \( \xi \) tends to infinity. For instance, (3.7) is obtained from (3.24) as

\[
(q - q^{-1}) q^{\frac{(a+1)u}{2}} (a, b)_m (b, c) \frac{(d, c)_m (a, d)}{q^{d-\frac{a}{2}}} \left( \begin{array}{c|c|c}
  b & c & d \\
  a & & \\
 & u & \\
\end{array} \right) = x \frac{j}{k} y, \tag{3.30}
\]

\[
(a, b)_m = q^{\frac{(a-b)^2+m(a+b)}{2}},
\]

\[
x = (x_1, x_2) = \left( \frac{m - a + b}{2}, \frac{m + a - b}{2} \right),
\]

\[
j = \frac{3 + b - c}{2},
\]

\[
k = \frac{3 + a - d}{2}. \tag{3.32}
\]

The factor on the lhs of (3.30) does not spoil the star–triangle relation.

3.5. Restriction

The (fusion) face models constructed thus far possess local states ranging over the infinite set \( \mathbb{Z} \) and are called unrestricted. To obtain a model with finitely many local states, we make restriction. We introduce the integer \( \ell \in \mathbb{Z}_{\geq 2} \) called level, and specialize the parameters as follows:

\[
\xi = 0, \quad q = \exp \left( \frac{\pi \sqrt{-1}}{\ell + 2} \right), \quad [u]_{q^{1/2}} = \frac{\sin \frac{\pi u}{2(\ell + 2)}}{\sin \frac{\pi}{2(\ell + 2)}}. \tag{3.33}
\]

13 Actually, \( V_{a-1} \) can be the Verma module with the highest weight vector \( v_{a-1} \) such that \( k_1 v_{a-1} = q^{a-1} v_{a-1} \) for generic \( a \).
We further set $W_{m,n}(b \begin{array}{c} a \\ c \end{array} d | u) = 0$ unless the pairs $(a, b), (d, c)$ (resp. $(a, d), (b, c)$) are $m$-admissible (resp. $n$-admissible). We say that a pair $(a, b)$ is $m$-admissible if
\begin{equation}
 b-a \in \{-m, -m+2, \ldots, m\},
\end{equation}
\begin{equation}
a+b \in \{m+2, m+4, \ldots, 2\ell+2-m\}.
\end{equation}
Note that the admissibility forces $a, b \in \{1, 2, \ldots, \ell+1\}$. The resulting $W_{m,n}(b \begin{array}{c} a \\ c \end{array} d | u)$ with $a, b, c, d \in \{1, 2, \ldots, \ell+1\}$ is called the restricted Boltzmann weight. One may wonder if $[0]_{q^{1/2}} = [2\ell+4]_{q^{1/2}} = 0$ may cause a divergence somewhere in the construction. However, it has been proved [35] that the restricted Boltzmann weights are well defined and satisfy the star–triangle relation (3.27) among themselves\(^{14}\). In this way one obtains the level $\ell$ RSOS model whose local states belong to $\{1, 2, \ldots, \ell+1\}$ and the fusion degree specified by $m$ and $n$.

Let us comment on the admissibility condition among which the first one (3.34) already appeared in (3.26). When $\ell \to \infty$, the admissibility reduces to the Clebsch–Gordan rule:
\begin{equation}
V_{a-1} \otimes V_m = \bigoplus_{b=1}^{b-1=\ell-1-m, \ldots, \ell+3, a+m-1} V_{b-1}.
\end{equation}
The rhs contains precisely those $b$ such that $(a, b)$ is $m$-admissible at $\ell = \infty$. For $\ell$ finite, the necessity of $a+b \leq 2\ell+2-m$ can be seen for example in the first Boltzmann weight in (3.24). Under the specialization (3.33), it contains the factor $\sin\left(\frac{\pi(a+b+m)}{2\ell+4}\right)$ in the numerator. Thus the ‘next’ $b$ for which $a+b = 2\ell+4-m$ ‘cannot be reached’. Such a truncation is also observed at the level of characters associated with (3.36). Denoting the $q$-dimension of $V_{a-1}$ at the root of unity by $\dim_q V_{a-1} = \sin\left(\frac{\pi a}{2}\right) / \sin\left(\frac{\pi}{12}\right)$, we have
\begin{equation}
(\dim_q V_m)(\dim_q V_{a-1}) = \sum_{b:\{(a,b)\text{ is }m\text{-admissible}\}} \dim_q V_{b-1}.
\end{equation}
This truncated decomposition is also known as the fusion rule in the SU(2) level $\ell$ WZW CFT [38].

Finally we remark that given $\ell$, one cannot fuse too much. In fact, (3.34) and (3.35) fix the admissible pairs to $\{(a, a) \mid 1 \leq a \leq \ell+1\}$ at $m = 0$ and to $\{(1, \ell+1), (\ell+1, 1)\}$ at $m = \ell$. They lead to completely frozen models. Nontrivial situations correspond to the fusion degrees in the range $1 \leq m \leq \ell - 1$. This is an origin of the truncation condition in the restricted $T$-system (section 2.2) for $g = A_1$.

3.6. Transfer matrices

We consider the row to row transfer matrix $T_m(u)$ with periodic boundary condition whose elements $T_m(u)(b_{N+1} \cdots b_N \begin{array}{c} b_1 \\ a_1 \end{array} \begin{array}{c} a_N \\ b_N \end{array} | u)$ are given by
\begin{equation}
W_{m,N}(b_1 \begin{array}{c} a_1 \\ b_2 \end{array} u - v_1) \cdots W_{m,N-1}(b_N \begin{array}{c} a_N \end{array} \begin{array}{c} b_{N-1} \\ u - v_{N-1} \end{array} ) W_{m,N}(b_N \begin{array}{c} a_N \end{array} \begin{array}{c} b_1 \\ u - v_N \end{array} ).
\end{equation}
No sum is involved. It is depicted as
\begin{equation}
T_m(u) = \begin{array}{c c c c}
 b_1 & b_2 & b_{N-1} & b_N \\
 a_1 & a_2 & a_{N-1} & a_N
\end{array}^{u-v_1} \cdots \begin{array}{c c c c}
 u-v_1 & u-v_2 & \cdots & u-v_N
\end{array} =
\begin{array}{c c c c c}
 b_1 & b_2 & b_{N-1} & b_N & b_N \\
 a_1 & a_2 & a_{N-1} & a_N & a_N
\end{array}^{u-v_1} \cdots \begin{array}{c c c c c}
 u-v_1 & u-v_2 & \cdots & u-v_N & u-v_N
\end{array}.
\end{equation}

\(^{14}\) Actually the statement holds for appropriately symmetrized $W_{m,n}$. See [35, section 2.2].

26
Here \((a_i, a_{i+1}), (b_i, b_{i+1})\) are \(s_i\)-admissible \((a_{N+1} = a_1, b_{N+1} = b_1)\) and \((a_i, b_i)\) is \(m\)-admissible for all \(i\). The inhomogeneity \(s_i, v_i\) in fusion degrees and coupling constants are fixed and suppressed in the notation. The \(T_m(u)\) is zero unless the parity condition \(\sum_{i=1}^N s_i \equiv 0 \mod 2\) is satisfied. The star–triangle relation (3.27) implies the commutativity [2]

\[
[T_m(u), T_n(v)] = 0.
\]

Let us take \(s_i = 1\) for all \(i\) for simplicity and demonstrate the functional relation

\[
T_i(u + 1)T_i(u - 1) = T_i(u)T_i(u) + g_i(u)\text{id},
\]

\[
T_0(u) = \prod_{i=1}^N [u_i + 1q_1^{u_i}/2], \quad g_i(u) = \prod_{i=1}^N [u_i + 3q_1^{u_i}/2][u_i - 1q_1^{u_i}/2],
\]

where \(u_i = u - v_i\). We first consider the case \(a_N = b_N = a\).

\[
L_{c,d} = \begin{pmatrix}
    a_N-1 & a_N & \ldots & b_N-1 \\
    a_1 & b_1 & \ldots & a_N-1 \\
    a_2 & b_2 & \ldots & a_1 \\
    a_{N-1} & b_{N-1} & \ldots & a_2 \\
    b_N & a_N-1 & \ldots & b_2 \\
    b_{N+1} & a_1 & \ldots & b_1 \\
\end{pmatrix}
\]

where each face stands for \(W = W_{1,1}\). To the difference \(L_{a+1,d} - L_{a-1,d}\), one can apply the same trick as (3.22). In particular, the repeated use of the star–triangle relation and the property \(W(b, c) = W(a, d)\) shows that it vanishes unless \(a_i = b_i\) for all \(i\). Then the induction on \(N\) leads to the identity

\[
L_{a+1,d} - L_{a-1,d} = \frac{[2a]_{q_1}^{u_i}/2}{[2][a_N]_{q_1}^{u_i}/2} \prod_{i=1}^N \left( \delta_{b_i,b} \frac{[u_i + 3q_1^{u_i}/2][u_i - 1q_1^{u_i}/2]}{[2][a_N]_{q_1}^{u_i}/2} \right) \begin{pmatrix} 1 & d = a_N + 1 \\ -1 & d = a_N - 1 \end{pmatrix}. \tag{3.42}
\]

Now we are ready to evaluate the matrix elements of \(T_i(u + 1)T_i(u - 1)\). When \(a_N = b_N = a\), we have

\[
(T_i(u + 1)T_i(u - 1))_{b_1,\ldots,b_N}^{a_1,\ldots,a_N} = L_{a-1,a+1} + L_{a+1,a+1} = L_{a-1,a+1} + L_{a-1,a+1} + L_{a-1,a+1} = L_{a-1,a+1} + L_{a-1,a+1}.
\]

The first two terms yield \(T_0(u)T_2(u)\) by definition (3.23). The other two terms are equal to \((g_1(u)\text{id})_{b_1,\ldots,b_N}^{a_1,\ldots,a_N}\) due to (3.42) with \(a \equiv a_N\). When \(a_N = b_N = \pm 2\), one can more easily check (3.40) since \((g_1(u)\text{id})\) does not contribute.

### 3.7. Vertex and RSOS models for general \(g\)

We include a formal and partly conjectural description of solvable vertex and RSOS models and their \(T\)-system for general \(g\). We will use the terminology introduced in later sections. (Therefore, this technical section may better be skipped on the first reading.)

Let \(W^m(u)\) be the Kirillov–Reshetikhin module (section 4.2), where \(a \in I\) (set of vertices on the Dynkin diagram of \(g\)) and \(m \in \mathbb{Z}_{\geq 1}\). It is an irreducible finite-dimensional representation of untwisted quantum affine algebra \(U_q\). Up to an overall scalar, there is the unique element, the \(R\) matrix \(R \in \text{End}(W^m(a_1) \otimes W^m(b_2))\) characterized by the intertwining property \(\Delta(U_q)PR = PR\Delta(U_q)\), where \(P\) is the transposition. It can in principle be constructed concretely by solving this linear equation or by the fusion of the simpler cases \(m = n = 1\) (cf theorem 4.3) or by taking the image of the universal \(R\).
denote the resulting $R$ matrix by $R^{(a,m;b,n)}(z_1/z_2)$, where $z_i = q^{t_i}$, $t$ is defined by (2.1) and the dependence through $z_1/z_2$ is due to the general theory:

$$R^{(a,m;b,n)}(z_1/z_2) = W_m(\ell)_{(a_1)(a_2)}$$

(3.43)

As in (3.11), one introduces the row to row transfer matrix with the auxiliary space $W_m^{(a)}(u)$ by $(z = q^{u})$:

$$T_m^{(a)}(u) = Tr_{W_m^{(a)}(u)}(R_{0,N}^{(a,m;r_N,s_N)}(z/w_N) \cdots R_{0,1}^{(a,m;r_1,s_1)}(z/w_1)),$$

(3.44)

which acts on the quantum space $W(t_1)(v_1) \otimes \cdots \otimes W(t_N)(v_N)$ with $w_i = q^{t_i}$. They are all commutative, i.e. $[T_m^{(a)}(u), T_n^{(b)}(v)] = 0$ thanks to the Yang–Baxter relation. It is a corollary of the exact sequence underlying theorem 4.8 and the argument on the central extension (cf [1, section 2.2]) that $T_m^{(a)}(u)$ satisfies the unrestricted $T$-system for $\mathfrak{g}$ (2.22) with some scalars $T_0^{(a)}(u)$ and $g_m^{(a)}(u)$ appropriately chosen depending on the normalization of $T_m^{(a)}(u)$.

Let $\ell \in \mathbb{Z}_{>2}$. From the $R$ matrix one can in principle construct the face Boltzmann weights for level $\ell U_q(\hat{\mathfrak{g}})$ RSOS model at $q = \exp \left( \frac{\pi \sqrt{-1}}{\ell(t+h')^2} \right)$. Let us introduce

$$P_\ell = \mathbb{Z}_{>0} \Omega_1 + \cdots + \mathbb{Z}_{>0} \Omega_n, \quad P_\ell = \{ \lambda \in P_\ell | (\lambda)_{\text{maximal root}} \leq \ell \},$$

(3.45)

where $\Omega_\ell$ is a fundamental weight of $\mathfrak{g}$ (section 2.1). $P_\ell$ is the classical projection of the set of level $\ell$-dominant integral weights of the affine Lie algebra $\hat{\mathfrak{g}}$ at level $\ell$ [11]. For $\lambda \in P_\ell$, let $V_\lambda$ be the irreducible $U_q(\mathfrak{g})$-module with the highest weight $\lambda$. Let res $W_m^{(a)}$ be the (not necessarily irreducible) $U_q(\hat{\mathfrak{g}})$-module obtained by restricting the $U_q(\hat{\mathfrak{g}})$-module $W_m^{(a)}(u)$. It is independent of $u$. See the text around (4.22). When $q$ is not a root of unity, one has the irreducible decomposition

$$V_\lambda \otimes \text{res } W_m^{(a)} \otimes \text{res } W_n^{(b)} = \bigoplus_{\mu \in P_\ell} \Omega(\lambda)_\mu \otimes V_\mu,$$

(3.46)

where $\Omega(\lambda)_\mu$ is the space of highest weight vectors of weight $\mu$. Since $\bar{R}(z) = PR^{(a,m;b,n)}(z)$ commutes with $U_q(\mathfrak{g})$, the space $\Omega(\lambda)_\mu$ is invariant under $id \otimes \bar{R}(z)$. Thus its matrix elements yield the Boltzmann weights of the unrestricted SOS model as in (3.29). The star–triangle relation for them follows from this construction.

To make the restriction, we consider the case $q = \exp \left( \frac{\pi \sqrt{-1}}{\ell(t+h')^2} \right)$, where the decomposition (3.46) no longer holds [39, 40]. However, based on the observation for $\mathfrak{g} = A_1$ [30], we conjecture that if $\lambda$ is taken from $P_\ell$ and $m \leq t_\ell, n \leq t_\ell$, the quotient of the rhs of (3.46) by type I modules [41, 42] reduces the sum $\mu \in P_\ell$ to $\mu \in P_\ell$, and $id \otimes \bar{R}(z)$ remains well defined on it. Then, the RSOS Boltzmann weights are defined as the matrix elements of $id \otimes \bar{R}(z)$ on the quotient space and satisfy the star–triangle relation.

The RSOS model so constructed has the fluctuating variables on edges as well as sites in general (cf [43, figure 1]):

$$\begin{array}{c}
\beta \\
\nu \\
\delta \\
\kappa \\
\gamma \\
\alpha \\
\alpha' \end{array} \begin{array}{c}
\end{array}
\begin{array}{c}
\delta \\
\kappa \\
\gamma \\
\alpha \\
\alpha' \\
\beta' \\
\nu' \\
\mu' \end{array}$$

(3.47)

15 Actually any primitive $2(\ell + h')$th root of unity. $h'$ is the dual Coxeter number of $\mathfrak{g}$ (2.3).
16 Indecomposable modules with $\dim_q = 0$. See (14.49).
The site variables belong to \( P_t \). In fact for \( g = A_1 \), one may regard the set of site variables \( \{ 1, 2, \ldots, l + 1 \} \) as \( P_t = \{ 0, \alpha_1, \ldots, \ell \omega_1 \} \). To describe the edge variables, we consider the decomposition \( V_\ell \otimes \text{res} \, W_m^{(a)} = \bigoplus_{\mu \in P_t} \Omega_{\mu \ell}^{(a,m)} \otimes V_\mu \) at generic \( q \). When \( q = \exp \left( \frac{\pi \sqrt{-1}}{l \omega_1} \right) \), we need to take the quotient of the rhs by type I modules, and this induces the quotient \( \Omega_{\mu \ell}^{(a,m)} \) of \( \Omega_{\mu \ell}^{(a,m)} \). The edge variable associated with \( W_m^{(a)} \) belongs to the space \( \Omega_{\mu \ell}^{(a,m)} \). We set \( \mathcal{A}_{\mu \ell}^{(a,m)} = \dim \Omega_{\mu \ell}^{(a,m)} \) and say that an (ordered) pair of site variables \( (\lambda, \mu) \in P_t \times P_t \) is admissible under \( W_m^{(a)} \) if \( \mathcal{A}_{\mu \ell}^{(a,m)} \geq 1 \).\(^{17}\) The matrix \( \mathcal{A}^{(a,m)} = (\mathcal{A}_{\mu \ell}^{(a,m)})_{\lambda, \mu \in P_t} \) is called the admissibility matrix of \( W_m^{(a)} \).

Let us formulate the row to row transfer matrix \( T_m^{(a)}(u) \) that corresponds to the dual of the one for the vertex model (3.44). It acts on the space of paths

\[
\mathcal{H}(N) = \bigoplus_{\lambda_i \in P_t} \Omega_{\lambda_i \lambda_{i+1}}^{(r,\epsilon_{i+1})} \otimes \cdots \otimes \Omega_{\lambda_N \lambda_{N+1}}^{(r,\epsilon_{N+1})},
\]

\[
\dim \mathcal{H}(N) = \text{Tr}(\mathcal{A}^{(r,\epsilon_{1})} \cdots \mathcal{A}^{(r,\epsilon_{N})}). \tag{3.49}
\]

The matrix elements are depicted as follows (\( u_i = u - u_i, \lambda_i = \lambda_i + \lambda_{i+1} \));

\[
T_m^{(a)}(u)^{\mu_1 \beta_1 \mu_2 \beta_2 \cdots, \mu_N \beta_N}_{\lambda_1 \epsilon_1, \lambda_2 \epsilon_2, \cdots, \lambda_N \epsilon_N} = \sum_{\gamma \in \Omega_{\mu \ell}^{(a,m)}} \gamma \begin{array}{c} \mu_1 \beta_1 \mu_2 \beta_2 \cdots, \mu_N \beta_N \\ \lambda_1 \epsilon_1, \lambda_2 \epsilon_2, \cdots, \lambda_N \epsilon_N \end{array} \begin{array}{c} u_1 \quad u_2 \quad \cdots \quad u_N \\ \lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_N \end{array} \tag{3.50}
\]

Here the symbols \( \alpha_i \) and \( \beta_i \) denote a basis of \( \Omega_{\lambda_i \lambda_{i+1}}^{(r,\epsilon_{i+1})} \) and \( \Omega_{\lambda_i \lambda_{i+1}}^{(r,\epsilon_{i+1})} \), respectively. The pairs \( (\lambda_i, \beta_i) \) and \( (\mu_i, \epsilon_i) \) are both admissible under \( W_m^{(a)} \), whereas \( (\lambda_i, \mu_i) \) is so under \( W_m^{(a)} \).

The rhs stands for the product of the \( N \) Boltzmann weights attached to the elementary squares summed over the states on the vertical edges accommodating \( \Omega_{\lambda_i \lambda_{i+1}}^{(a,m)} \) for \( i = 1, \ldots, N \). As for the weights, \( \lambda_{i+1} = \lambda_i = \mu_{i+1} - \mu_i = \epsilon_i \alpha_i \gamma_i \) mod the root lattice; therefore, the \( T_m^{(a)}(u) \) under consideration is vanishing unless

\[
\sum_{i=1}^{N} s_i (C^{-1})_{\alpha_i \gamma_i} \in \mathbb{Z} \quad \text{for all } a \in I, \tag{3.51}
\]

where \( C \) is the Cartan matrix of \( g \) (section 2.1). Due to the star–triangle relation (including sums over edge variables), the commutativity \( [T_m^{(a)}(u), T_n^{(b)}(v)] = 0 \) holds. We conjecture that \( T_m^{(a)}(u) \) satisfies the level \( \ell \) restricted \( T \)-system for \( g \) of the form (2.22) with some scalars \( T_0^{(a)}(u) \) and \( g^{(a)}(u) \) appropriately chosen depending on the normalization. In particular, this implies that the \( |P_t| \) by \( |P_t| \) matrices \( \mathcal{A}^{(a,m)} \) with \( a \in I, 0 \leq m \leq t_\ell \) are commutative and satisfy the level \( \ell \) restricted \( Q \)-system (cf section 14.5) with the boundary condition

\[
\mathcal{A}^{(a,0)} = 1, \quad \mathcal{A}^{(a,\ell+1)} = 0. \tag{3.52}
\]

Let \( \text{dim}_q V_\lambda \) be the \( q \)-dimension of \( V_\lambda \) at \( q = \exp \left( \frac{\pi \sqrt{-1}}{l \omega_1} \right) \) defined in (14.49). We set \( Q_m^{(a)} = \text{dim}_q \text{res} \, W_m^{(a)} \), which supposedly satisfies the level \( \ell \) restricted \( Q \)-system (14.5) (conjecture 14.2). Now the generalization of (3.37) is given as

\[
Q_m^{(a)} \text{dim}_q V_\lambda = \sum_{\mu \in P_t} A_{\mu \ell}^{(a,m)} \text{dim}_q V_\mu \quad (\lambda \in P_t). \tag{3.53}
\]

17. The type \( A_1 \) is a bit special in that \( A_{\mu \ell}^{(a,m)} \in \{ 0, 1 \} \) holds for any \( (a, m) \) and \( \lambda, \mu \); hence, effectively no edge variable exists. However, the situation \( A_{\mu \ell}^{(a,m)} > 2 \) still occurs for the fusion types more general than those specified by Kirillov–Reshetikhin modules (43).

18. This leads to \( \prod_{a \in I} (1 - q^{A_{\mu \ell}^{(a,m)}})_{\mathbb{C}^*} = 1 \) for any \( a \in I \), which is a weaker constraint than \( A_{\mu \ell}^{(a,\ell+1)} = 1 \) employed in the definition of the level \( \ell \) restricted \( Q \)-system in section 14.5.
Since \( \dim_q V_\lambda > 0 \) for any \( \lambda \in P_\ell \), the Perron–Frobenius theorem states that \( Q^{(p)}_m \) is the largest eigenvalue of the admissibility matrix \( A^{(n,m)} \). Therefore in the homogeneous case where \((r_i, s_i) = (p, s)\) for all \( i \), we find from (3.49) that
\[
\lim_{N \to \infty} \left( \frac{\dim H(N)}{N} \right)^{1/N} = Q^{(p)}_s.
\] (3.54)

This property will be re-derived in the TBA analysis in (15.20).

In general, the Boltzmann weights (3.47) are expressed in terms of the function \( u_q \alpha \propto \sin \left( \frac{n\pi}{\ell+\kappa} \right) \). (\( t \) is defined in (2.1).) This is indeed the case for \( A_1 \) as in (3.17) and in the other known examples. It is also consistent with the Bethe equation (8.25).

Consequently, the transfer matrix with an appropriate normalization possesses the periodicity
\[
T_m^{(a)}(u + 2(\ell + \kappa)) = T_m^{(a)}(u).
\] (3.55)

We will see in theorem 5.7 that the level \( \ell \) restricted \( T \)-system in section 2.2 19 alone compels this property.

### 3.8. Bibliographical notes

The integrability of the six-vertex model (3.1) (first solved in [25, 26]) has been formulated in terms of the Yang–Baxter equation and commuting transfer matrices in [2]. Solutions of the Yang–Baxter equation that have been known by 1980 are surveyed in [44] from the perspective of the quantum inverse scattering method. Subsequent generalizations of trigonometric vertex models to type \( A, B, C, D \) [45–47] and many other \( g \) [48, 49] have been assembled in the reprint volume [50]. The fusion of vertex models is formulated in [27]. See also [51]. The idea of utilizing the functional relations of transfer matrices goes back to Baxter [2, 52]. Some simplest examples of the \( T \)-system have been obtained for the XXZ chain [53], the \( O(n) \)-symmetry models [54] and vertex models associated with some other \( g \) [55].

With regard to the RSOS models, the 8VSOS model is the fundamental example containing the Ising and (generalized) hard hexagon models as the level \( \ell = 2, 3 \) cases, respectively. The one-point function [34] essentially gives rise to the character of the Virasoro minimal series, and this fact inspired intensive studies on the relations with CFT and representation theory of quantum affine algebras. In the terminology in section 3.7, the 8VSOS model corresponds to the level \( \ell \) RSOS model for \( g = A_1 \) with fusion type \( W_1^{(1)} \) (both on the horizontal and vertical edges).

Beyond the \( A_1 \) case, concrete constructions of RSOS models for untwisted affine Lie algebra \( \hat{g} \) have been done for nonexceptional series \( g = A_r, B_r, C_r, D_r \) [56, 57] associated with \( W_1^{(1)} \) (‘vector representation’) and \( g = G_2 \) [58] with \( W_1^{(2)} \). The fusion of RSOS models have been worked out explicitly only for type \( A \) [35, 43]. One of the earliest examples of the \( T \)-system for RSOS models (except the Ising) is [36] for the generalized hard hexagon model. It was systematized to the general level restricted \( T \)-system for \( A_1 \) in [7]. See also [37] where the relation of the form \( 'T_m T_1 = T_{m-1} + T_{m+1}' \) was given. In [59], the Jacobi–Trudi-type functional relations (cf theorems 6.1 and 6.2) were given for the fusion RSOS models of type \( A_n \). The \( T \)-system for \( A_n \) is extracted from them in [1], where the extension to all \( g \) was proposed based on the connection to the \( Y \)-system and the \( Q \)-system. Finally, one can construct the quantum field theory analog of the commuting transfer matrices that act on Virasoro–Fock spaces and satisfy the \( T \)-system. See [60] for the original construction for \( g = A_1 \) and [61] for a recent application.

19 In this case, the normalization is \( T_0^{(0)}(u) = T_0^{(0)}(u) = 1 \).
4. T-system in quantum group theory

4.1. Quantum affine algebra

For simplicity we concentrate on the untwisted quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) until section 4.5. We assume that \( q \in \mathbb{C}^* \) is not a root of unity and set \( q = e^\theta ; \) therefore, the domain \( U \) of the spectral parameter \( u \) should be understood as \( U = \mathbb{C}_u \). See section 2.1. We set \( I = \{0\} \cup I \) and let \( \hat{C} = (\hat{C}_{ij})_{i,j \in I} \) be the Cartan matrix of the affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) [11]. For \( i, j \in I \), one has \( \hat{C}_{ij} = C_{ij} \) where the latter is an element of the Cartan matrix \( C \) of \( \mathfrak{g} \). By definition, the (untwisted) quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) [14, 62] is the associative algebra over \( \mathbb{C} \) with generators \( x_i^\pm, k_i^{\pm 1} (i \in I) \) and the relations

\[
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad k_i x_j^\pm k_i^{-1} = q_i^{\pm \hat{C}_{ij}} x_j^\pm, \quad [x_i^+, x_j^-] = \delta_{ij} k_i - k_i^{-1} q_i - q_i^{-1}, \quad \text{(4.1)}
\]

The algebra \( U_q(\hat{\mathfrak{g}}) \) admits a Hopf algebra structure [14, 62]. The algebra \( U_q(\hat{\mathfrak{g}}) \) is denoted by \( U_q(\hat{\mathfrak{g}}) \) in some literature indicating that the analog of the derivation operator in \( \hat{\mathfrak{g}} \) has not been included. There are \( 2^{r+1} \) algebra automorphisms of \( U_q(\hat{\mathfrak{g}}) \) given on generators by

\[
k_i \mapsto \sigma_i k_i, \quad x_i^+ \mapsto \sigma_i x_i^+, \quad x_i^- \mapsto x_i^-
\]

for any set of signs \( \sigma_0, \ldots, \sigma_r \in \{\pm 1\} \). Obviously, \( U_q(\hat{\mathfrak{g}}) \) contains \( U_q(\mathfrak{g}) \) as a subalgebra. There is another realization of \( U_q(\hat{\mathfrak{g}}) \) called the Drinfeld new realization [63, 64]. Namely, \( U_q(\hat{\mathfrak{g}}) \) is isomorphic to the algebra with generators \( x_i^\pm (i \in I, n \in \mathbb{Z}) \), \( k_i^{\pm 1} (i \in I) \), \( h_{i,n} (i \in I, n \in \mathbb{Z}) \) and central elements \( e^{\pm 1/2} \), with the following relations:

\[
h_{i,n} k_i = k_i h_{i,n}, \quad k_i x_j^\pm k_i^{-1} = q_i^{\pm \hat{C}_{ij}} x_j^\pm, \quad [h_{i,n}, x_{j,m}] = \delta_{i,j} q_i^{-n} h_{i,n} q_i^n - q_i^n h_{i,n} q_i^{-n}, \quad \text{(4.4)}
\]

\[
x_{i,n+1}^\pm x_{j,m}^\pm = \delta_{i,j} q_i^{\pm \hat{C}_{ij}} x_{i,m}^\pm x_{j,n+1}^\pm - x_{i,m}^\pm x_{j,n+1}^\pm - q_i^{\pm \hat{C}_{ij}} x_{i,n+1}^\pm x_{j,m}^\pm, \quad [x_i^+, x_j^-] = \delta_{ij} q_i^{-n} h_{i,n} q_i^n - q_i^n h_{i,n} q_i^{-n}, \quad \text{(4.4)}
\]

\[
\sum_{\pi \in \Sigma} \sum_{k=0}^\infty (-1)^k \left( \frac{S}{k!} \right) x_{i_{k+1}}^\pm \cdots x_{i_{k+1}}^\pm x_{j_{k+1}}^\pm \cdots x_{j_{k+1}}^\pm x_{i_{k+1}}^\pm = 0, \quad i \neq j
\]

for all sequences of integers \( n_1, \ldots, n_s \), where \( \Sigma = 1 - C_{ij} \), \( \Sigma_s \) is the symmetric group on \( s \) letters and \( \phi^+_{i,m}, \phi^-_{i,m} \) are determined by the formal power series

\[
\sum_{n=0}^\infty \phi^\pm_{i,m} x^\pm_n = k_i^{\pm 1} \exp (\pm (q_i - q_i^{-1}) \sum_{m=1}^\infty h_{i,m} x^\pm_m). \quad \text{(4.5)}
\]

In the two realizations (4.1) and (4.4), the symbol \( k_i^{\pm 1} (i \in I) \) stands for the same generator under the isomorphism. \( U_q(\hat{\mathfrak{g}}) \) admits a Hopf algebra structure [14, 62].
4.2. Finite-dimensional representations

A representation $W$ of $U_q(\hat{\mathfrak{g}})$ is called type 1 if the generators $k_0, k_1, \ldots, k_r$ act semi-simply on $W$ with eigenvalues in $q^Z$ and $c^{1/2}$ in (4.4) acts as 1 on $W$. A vector $v \in W$ is called a highest weight vector if

$$x^+_{i,n} \cdot v = 0, \quad \phi^\pm_{i,n} \cdot v = \psi^\pm_{i,n}v, \quad c^{1/2}v = v \quad (4.6)$$

for some complex numbers $\psi^\pm_{i,n}$. A type 1 representation $W$ is called a highest weight representation if $W = U_q(\hat{\mathfrak{g}}) \cdot v$ for some highest weight vector $v$.

**Theorem 4.1** ([65, 66]).

1. Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ can be obtained from a type 1 representation by a twisting with an automorphism (4.3).
2. Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 is a highest weight representation.
3. A type 1 highest weight representation with the highest weight vector $v$ in (4.6) is finite dimensional if and only if there exist polynomials $P_a(\zeta) \in \mathbb{C}[\zeta] (a \in I)$ such that $P_a(0) = 1$ and

$$\sum_{n \geq 0} \psi^\pm_{i,n} \xi^\pm_n = q^{\text{deg} P_a} P_a(\zeta q^{-1}) P_a(\zeta q) \in \mathbb{C}[\zeta^{\pm 1}]. \quad (4.7)$$

The polynomials $P_a(\zeta)$ are called Drinfeld polynomials after the analogous classification theorem by Drinfeld for Yangians [63].

The Kirillov–Reshetikhin module $W^{(a)}(u) (a \in I, m \in \mathbb{Z}_{\geq 1}, u \in \mathbb{C}_{\neq 0})$ is the irreducible finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$ that corresponds to the Drinfeld polynomial

$$P_a(\zeta) = \begin{cases} \prod_{i=1}^m (1 - \zeta q^{m+1-2r}) & \text{if } b = a, \quad (4.8) \\ 1 & \text{otherwise.} \end{cases}$$

This $W^{(a)}(u)$ is equal to $W^{(a)}_{m,q^m,q^{-m}}$, in [67, 68]. In particular, $W^{(1)}(u), \ldots, W^{(r)}(u)$ are called fundamental representations.

4.3. Example

Consider the simplest example $U_q = U_q(A_1^{(1)})$. In realization (4.1), $\hat{I} = \{0, 1\}$ and the Cartan matrix is $\hat{C} = \left( \begin{array}{cc} 2 & -2 \\ -2 & 2 \end{array} \right)$. The coproduct is given by

$$\Delta x_i^+ = x_i^+ \otimes 1 + k_i \otimes x_i^+, \quad \Delta x_i^- = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^+, \quad \Delta k_i^\pm = k_i^\pm \otimes k_i^\pm. \quad (4.9)$$

For $m \in \mathbb{Z}_{\geq 0}$, let $W_m(u) = W_m^{(1)}(u)$ be the Kirillov–Reshetikhin module. Plainly, it is the $(m + 1)$-dimensional (i.e. spin $\frac{m}{2}$) irreducible representation $W_m(u) = \mathbb{C} v_1^m \oplus \cdots \oplus \mathbb{C} v_{m+1}^m$ given by ($z = q^m$)

$$x_1^- v_j^m = [m + 1 - j] v_j^m, \quad x_1^+ v_j^m = [j - 1] v_j^m, \quad k_1^\pm v_j^m = q^{\pm (m+2-j)} v_j^m, \quad (4.10)$$

$$x_0^- v_j^m = z^{-1} (j - 1) v_j^m, \quad x_0^+ v_j^m = z (m + 1 - j) v_j^m, \quad k_0^\pm v_j^m = q^{\pm (m+2-j)} v_j^m. \quad (4.11)$$

32
where $[j] = [j]_q = \frac{q^{j} - q^{-j}}{q - q^{-1}}$ as in (3.18). In the Drinfeld new realization (4.4), the highest weight vector is identified with $v_1^m$ and the eigenvalues in (4.6) read

$$
\psi_{1,\pm n}^\pm = \begin{cases} 
q^\pm n & n = 0, \\
\pm (q^n - q^{-n})(q^m)^\pm n & n \geq 1.
\end{cases}
$$

Relation (4.7) holds with the Drinfeld polynomial

$$
\mathcal{P}_1(\zeta) = (1 - \zeta q^{m+1})(1 - \zeta q^{m+3}) \cdots (1 - \zeta q^{m+1})
$$
in agreement with (4.8).

The exact sequence (3.16) is refined along the definitions here. The vectors $v_i \in V_i$ and $x = (x_1, x_2) \in V_m$ in section 3.1 are to be identified with $v_1$ and $v_m^m$ in (4.10) and (4.11), respectively. We introduce the base of $W_1(u) \otimes W_1(v)$ as

$$
\begin{align*}
\mathbf{u}_1 &= v_1^1 \otimes v_1^1, \\
\mathbf{u}_2 &= \left[\frac{1}{2}\right] \Delta(x_1^+) \mathbf{u}_1 = v_1^1 \otimes v_1^1 + q^{-1} v_1^0 \otimes v_1^1, \\
\mathbf{u}_1' &= v_1^1 \otimes v_1^1 - q v_1^0 \otimes v_1^1, \\
\mathbf{u}_3 &= \Delta(x_1^-) \mathbf{u}_2 = v_1^1 \otimes v_1^1.
\end{align*}
$$

Under the action of $x_1^\pm, k_1^\pm$, the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{u}_1'\}$ behave as the triplet and the singlet representations as usual. On the other hand, with regard to $x_0^0$, they are mixed as follows ($x = q^u, y = q^v$):

The diagram means $\Delta(x_0^0) \mathbf{u}_1 = (x + y) \mathbf{u}_2 + x^{-1} q^{-1} \mathbf{u}_1'$ for instance. From (4.13), we find that $W_1(u) \otimes W_1(v)$ is irreducible if and only if $\frac{x}{y} \neq q^{2u}$, namely $u - v \neq \pm 2$. In the reducible cases, (4.13) appears as

(i) $z := qx = q^{-1} y$  
(ii) $z := q^{-1} x = q y$.

In both cases, $W_1(u) \otimes W_1(v)$ is indecomposable and the subspace $\mathbb{C} \mathbf{u}_1 \oplus \mathbb{C} \mathbf{u}_2 \oplus \mathbb{C} \mathbf{u}_3$ becomes isomorphic to $W_2\left(\frac{q^u}{q^v}\right)$ corresponding to the multiplicative spectral parameter $z$. The difference is that $W_2\left(\frac{q^u}{q^v}\right)$ is the reducible submodule in the case of (i) while it is the irreducible quotient.
for (ii). Denoting the trivial one-dimensional module $\mathbb{C}u_1$ by $W_0$, we thus get the exact sequences of $U_q$-modules:

\[
\begin{align*}
(\text{i}) & \quad 0 \to W_2(u) \to W_1(u) \otimes W_1(u + 1) \to W_0 \to 0, \\
(\text{ii}) & \quad 0 \to W_0 \to W_1(u + 1) \otimes W_1(u - 1) \to W_2(u) \to 0.
\end{align*}
\]

The general case, which was first worked out in the context of Yangian, is summarized in

**Theorem 4.2** ([69]). $W_m(u) \otimes W_n(v)$ is reducible if and only if $|u - v| = m + n - 2j + 2$ for some $1 \leq j \leq \min(m, n)$. In these cases, the following exact sequences are valid:

\[
0 \to W_{m-j}(u + m - j + 1) \otimes W_{m+n-j+1}(v - m + j - 1) \to W_m(u) \otimes W_n(v) \\
\to W_{m-j}(u - j) \otimes W_{n-j}(v + j) \to 0
\]

for $v - u = m + n - 2j + 2$ and

\[
0 \to W_{m-j}(u + j) \otimes W_n(v) \to W_m(u) \otimes W_n(v) \\
\to W_{m-j}(u - m + j - 1) \otimes W_{m+n-j+1}(v - m - j + 1) \to 0
\]

for $u - v = m + n - 2j + 2$.

### 4.4. q-characters

Let $\text{Rep} U_q(\hat{\mathfrak{g}})$ be the Grothendieck ring of the category of the type 1 finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules. Such a module $W$ allows the direct sum decomposition

\[
W = \bigoplus_{\gamma \in \mathbb{C}^\times} W_{\gamma}, \quad W_{\gamma} = \{ v \in W | \exists a \in I, n \geq 0, (\phi_a^{\pm} - \psi_a^{\pm})^n v = 0 \}.
\]

It can be shown [70] that the generating function of the (generalized) eigenvalues is expressed as

\[
\sum_{n>0} y_{a,n}^\pm s^{\pm n} = q_a^{\deg R_a^+ - \deg R_a^-} R_a^+ (\frac{\xi a^{-1}}{q_a}) R_a^- (\frac{\xi a}{q_a}) \in \mathbb{C}[\xi^{\pm 1}]
\]

in terms of some polynomials $R_a^\pm (\xi)$ in $\xi$ with constant term 1.

Let $Z[Y^{\pm 1}_{a,z}]_{a \in I, z \in \mathbb{C}^\times}$ be the ring of integer coefficient Laurent polynomials in infinitely many algebraically independent variables $\{Y_{a,z} \mid a \in I, z \in \mathbb{C}^\times\}$. The Frenkel–Reshetikhin $q$-character $\chi_q$ is the injective ring morphism

\[
\chi_q : \text{Rep} U_q(\hat{\mathfrak{g}}) \to Z[Y^{\pm 1}_{a,z}]_{a \in I, z \in \mathbb{C}^\times}, \quad \chi_q(W) = \sum_{\gamma} \dim(W_{\gamma}) m_{\gamma},
\]

where the monomial $m_{\gamma}$ is specified from $R_a^\pm (\xi)$ (4.18) by

\[
m_{\gamma} = \prod_{a \in I, z \in \mathbb{C}^\times} Y_{a,z}^{\pm 1} - \xi z, \quad R_a^\pm (\xi) = \prod_{z \in \mathbb{C}^\times} (1 - \xi z)^{\pm 1}.
\]

Suppose that $W$ is the irreducible representation with Drinfeld polynomials $P_a(\xi) = \prod_{i=1}^{m_a} (1 - \xi a^{(i)})$. Comparing (4.7) with (4.18) and (4.20), one finds that its $q$-character $\chi_q(W)$ contains the monomial $\prod(Y_{a,z})^{m_a}$ corresponding to the highest weight vector. Such a monomial is called a highest weight monomial. Thus, in particular, the $q$-character

\[34\]

\[20\] The variable $Y_{a,z}$ is unrelated to the $Y$ of $Y$-systems.
of the Kirillov–Reshetikhin module $W_m^{(u)}(u)$ is a Laurent polynomial containing the highest weight monomial as
\[ \chi_q(W_m^{(u)}(u)) = \sum_{s=1}^{m} Y_{2,2q^{s+1}} + \cdots, \]
where we have set $z = q^m$. The case $m = 1$ is called the fundamental $q$-character. For an analogous treatment of the Yangians, see [71].

Example 4.5. For $g = A_1$, the $q$-character of the Kirillov–Reshetikhin module $W_1^{(u)}(u)$ is given by ($z = q^m$, $Y_z = Y_1,z$)
\[ \chi_q(W_1^{(1)}(u)) = Y_{1,z} + Y_{1,2q} + Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_2^{(1)}(u)) = Y_{2q} + Y_{2,2q} + Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_1^{(2)}(u)) = Y_{1,z} + Y_{1,2q}Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_2^{(2)}(u)) = Y_{2,z} + Y_{1,2q}Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_1^{(3)}(u)) = (\chi_q(W_1^{(1)}(u))) \text{ for } B_2 |_{Y_1,z \to Y_2,z} \quad (a = 1, 2), \]
\[ \chi_q(W_2^{(3)}(u)) = (\chi_q(W_2^{(1)}(u))) \text{ for } B_2 |_{Y_1,z \to Y_2,z} \quad (a = 1, 2). \]

Example 4.5. We write down the fundamental $q$-characters $\chi_q(W_1^{(a)}(u))$ for $g$ with rank 2 ($z = q^m$):
\[ A_2 : \quad \chi_q(W_1^{(1)}(u)) = Y_{1,z} + Y_{1,2q}Y_{2,2q} + Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_2^{(1)}(u)) = Y_{1,z} + Y_{1,2q}Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_1^{(2)}(u)) = Y_{2,z} + Y_{1,2q}Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_2^{(2)}(u)) = Y_{1,z} + Y_{1,2q}Y_{2,2q} + Y_{2,2q}, \]
\[ \chi_q(W_1^{(3)}(u)) = (\chi_q(W_1^{(1)}(u))) \text{ for } B_2 |_{Y_1,z \to Y_2,z} \quad (a = 1, 2), \]
\[ \chi_q(W_2^{(3)}(u)) = (\chi_q(W_2^{(1)}(u))) \text{ for } B_2 |_{Y_1,z \to Y_2,z} \quad (a = 1, 2). \]

More examples will be given in sections 7.1–7.4.

Any finite-dimensional $U_q(\hat{g})$-module $W$ defines a representation of the subalgebra $U_q(g)$, which we denote by $\text{res } W$. The (usual) character $\chi$ of the latter lives in $\mathbb{Z}[x_1^{-1}]_{x_1 \in \mathbb{C}}$ with $y_a = e^{iu_a}$ with $u_a$ being a fundamental weight. The $q$-character is a deformation of the character by $z$ in that
\[ \text{res } \chi_q(W) = \chi(\text{res } W). \]
where res on the lhs is to be understood as
\[ \text{res} : \mathbb{Z}[Y_{a,z}^{\pm 1}]_{a \in I, z \in \mathbb{C}^\times} \rightarrow \mathbb{Z}[Y_a^{\pm 1}]_{a \in I} \]
\[ Y_{a,z} \mapsto y_a. \]
Note that res \( W \) is not necessarily an irreducible \( U_q(\mathfrak{g}) \)-module even if \( W \) is so as a \( U_q(\mathfrak{g}) \)-module. Therefore, the irreducible \( q \)-character \( \chi_q(W_m^{(a)}(u)) \) does not restrict to an irreducible character in general. In fact in example 4.5, one observes
\[ \text{res} \chi_q(W_1^{(1)}(u)) = \begin{cases} \chi(V_m) + \chi(V_0) & \text{if } g = G_2 \text{ and } a = 1, \\ \chi(V_m) & \text{otherwise}, \end{cases} \]
where \( V_0 \) denotes the irreducible \( U_q(\mathfrak{g}) \)-module with highest weight \( \lambda \). The algebra \( g = A_r \) is exceptional in that \( \text{res} \chi_q(W_m^{(a)}(u)) = \chi(V_m) \) holds for all \( a \) and \( m \). See (7.7) and (13.63). A systematic treatment of such decompositions is related to the Kirillov–Reshetikhin conjecture which has been fully solved by now. See section 13, especially section 13.7.

For \( a \in I \) and \( z \in \mathbb{C}^\times \), set
\[ A_{a,z} = Y_{a,z,q}^1 Y_{a,z,q} \prod_{b \in C_{\mathfrak{a}=1}} Y_{b,z,q}^{-1} \prod_{b \in C_{\mathfrak{a}=2}} Y_{b,z,q}^{-1} Y_{b,z,q}^{-1} \prod_{b \in C_{\mathfrak{a}=3}} Y_{b,z,q}^{-1} Y_{b,z,q}^{-1}. \]
By the definition, one has \( A_{a,z} = \prod_{b \in C} Y_{b}^{\pm 1} \) with \( \alpha_b \) being a simple root.

Let \( S_a \) (\( a \in I \)) be the screening operator [70]. Namely, \( S_a \) sends \( \mathbb{Z}[Y_{a,z}^{\pm 1}]_{a \in I, z \in \mathbb{C}^\times} \) to the extended ring adjoined with the extra symbols \( S_{a,z} \) with \( a \in I, z \in \mathbb{C}^\times \). The action is given by
\[ S_a \cdot Y_{b,z} = \delta_{ab} Y_{a,z} S_{a,z} \]
and the Leibniz rule \( S_a \cdot (YZ) = (S_a \cdot Y)Z + Y(S_a \cdot Z) \). Thus, for example, \( S_a \cdot Y_{2,z}^{-1} = -\delta_{ab} Y_{a,z}^{-1} S_{a,z} \). The symbol \( S_{a,z} \) is assumed to obey the relation
\[ S_{a,z} = A_{a,z} S_{a,z} \]
in the extended ring.

**Theorem 4.6** ([70, 72]).

1. The \( q \)-character of an irreducible finite-dimensional \( U_q(\mathfrak{g}) \)-module \( W \) has the form
   \[ \chi_q(W) = m_+ + \sum_{\alpha} M_{\alpha}, \]
   where \( m_+ \) is the highest weight monomial and each \( M_{\alpha} \) is a monomial in \( A_{a,z}^+ \) \( a \in I, z \in \mathbb{C}^\times \) (i.e. it does not contain any positive power factors of \( A_{a,z} \)).
2. The image \( \text{Im} \chi_q(\simeq \text{Ch } U_q(\mathfrak{g})) \) of the \( q \)-character morphism (4.19) is equal to \( \bigcap_{i=1}^{12} \text{Ker } S_i \).

Assertion (1) is a natural analog of its undeformed version \( \text{res} \chi(W) \in \text{res}(m_+ + \sum_{\alpha} Z_{\geq 0} e^{-\alpha}) \), where \( \text{res}(m_+) = e^{\text{highest weight}} \) and the \( \alpha \)-sum runs over \( \mathbb{Z}_{\geq 0} \alpha_1 + \cdots + \mathbb{Z}_{\geq 0} \alpha_t \setminus \{0\} \).

Assertion (2) has a background in the characterization of the (deformed) \( W \)-algebra as the intersection of the kernel of screening operators [70].

**Example 4.7.** Let us illustrate theorem 4.6 along \( g = A_2 \). Definition (4.25) reads
\[ A_{1,z} = Y_{1,z}^{-1} Y_{1,z} Y_{2,z}^{-1}, \quad A_{2,z} = Y_{2,z}^{-1} Y_{2,z} Y_{1,z}^{-1}. \]
Take \( \chi_q = \chi_q(W_1^{(1)}(u)) = Y_{1,z} + Y_{1,z}^{-1} + Y_{2,z} + Y_{2,z}^{-1} \) for \( A_2 \) in example 4.5. The highest weight monomial is \( Y_{1,z} \). \( \chi_q \) is expressed as
\[ \chi_q = Y_{1,z} \left( 1 + A_{1,z}^{-1} + A_{1,z}^{-1} A_{2,z}^{-1} \right) \]
in agreement with (1). With regard to (2), let us check that \( \chi_q \) belongs to \( \text{Ker } S_1 \cap \text{Ker } S_2 \):
\[ S_1 \cdot \chi_q = Y_{1,z} S_{1,z} - Y_{1,z}^{-1} S_{2,z} Y_{2,z}^{-1} S_{1,z} = Y_{1,z} S_{1,z} - Y_{1,z}^{-1} Y_{2,z} A_{1,z} S_{1,z} = 0, \]
\[ S_2 \cdot \chi_q = Y_{1,z}^{-1} Y_{2,z} S_{2,z} - Y_{2,z}^{-1} S_{2,z} Y_{2,z}^{-1} S_{2,z} = Y_{1,z}^{-1} Y_{2,z} S_{2,z} - Y_{2,z}^{-1} A_{2,z} S_{2,z} = 0. \]
4.5. T-system and q-characters

We continue to set \( u \in U = C_{gh} \) in this subsection. The following is the fundamental result that relates the Kirillov–Reshetikhin modules with the T-system.

**Theorem 4.8** ([67, 68]). For any \( \mathfrak{g} \), \( T_m^{(a)} (u) = \chi_q (W_m^{(a)} (u)) \) satisfies the unrestricted T-system for \( \mathfrak{g} \).

In fact, the exact sequence corresponding to the \( g \)-version of the \( j = n = m \) case of (4.17) has been obtained. It is an elementary exercise to check that the \( q \)-characters for \( \mathfrak{g} = A_1 \) in example 4.4 satisfy the T-system (2.6).

Theorem 4.8 leads to a description of the ring \( \text{Rep} U_q (\hat{\mathfrak{g}}) \simeq \text{Ch} U_q (\hat{\mathfrak{g}}) \) by the \( q \)-characters of the Kirillov–Reshetikhin modules and the unrestricted T-system, which we shall now explain. Let \( T = \{ T_m^{(a)} (u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U \} \) denote the family of variables. Let \( T(\mathfrak{g}) \) be the ring with generators \( T_m^{(a)} (u)^{\pm 1} \) with the relations given by the T-system for \( \mathfrak{g} \). Define \( T^- \) to be the subring of \( T(\mathfrak{g}) \) generated by \( T \).

**Theorem 4.9** ([117]). The ring \( T^- \) is isomorphic to \( \text{Rep} U_q (\hat{\mathfrak{g}}) \) by the correspondence \( T_m^{(a)} (u) \mapsto W_m^{(a)} (u) \).

4.6. T-system for quantum affinizations of quantum Kac–Moody algebras

The T-systems have been generalized by Hernandez [8] to the quantum affinizations of a wide class of quantum Kac–Moody algebras studied in [63, 73–77]. The most distinct feature compared to the setting so far is that the category \( \text{Rep} U_q (\hat{\mathfrak{g}}) \) and the tensor product \( \otimes \) need to be replaced by \( \text{Mod} \left( U_q (\hat{\mathfrak{g}}) \right) \) consisting of not necessarily finite-dimensional modules and the fusion product \( \ast_f \), respectively. Nevertheless, with an appropriate definition of the Kirillov–Reshetikhin modules and their \( q \)-characters, the latter satisfy the (generalized) T-system [8].

Here we only give the definition of the quantum affine fusion of quantum Kac–Moody algebras and write down the T-system, leaving many details to [8]. Instead, we include the explicit form of the corresponding Y-system [78] on which our presentation is mainly based.

We begin by resetting the definitions and notations such as \( C, t, q_i, \mathfrak{h}, \hat{\mathfrak{g}} \), \( U_q (\mathfrak{g}) \) and \( U_q (\hat{\mathfrak{g}}) \) introduced so far. Let \( I = \{ 1, \ldots, r \} \) and let \( C = (C_{ij})_{i,j \in I} \) be a generalized Cartan matrix in [11]; namely, it satisfies \( C_{ij} \in \mathbb{Z} \), \( C_{ii} = 2 \), \( C_{ij} \leq 0 \) for any \( i \neq j \), and \( C_{ij} = 0 \) if and only if \( C_{ji} = 0 \). We assume that \( C \) is symmetrizable, i.e. there is a diagonal matrix \( D = \text{diag}(d_1, \ldots, d_r) \) with \( d_i \in \mathbb{Z}_{\geq 1} \) such that \( B = DC \) is symmetric. We assume that there is no common divisor for \( d_1, \ldots, d_r \) except for 1.

Let \( (\mathfrak{h}, \Pi, \Pi') \) be a realization of the Cartan matrix \( C \) [11], namely \( \mathfrak{h} \) is a \( (2r - \text{rank} \ C) \)-dimensional \( \mathbb{C} \)-vector space, and \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \subset \mathfrak{h}^* \), \( \Pi' = \{ \alpha'_1, \ldots, \alpha'_r \} \subset \mathfrak{h} \) such that \( \alpha_j (\alpha'_{j'}) = C_{ji} \). Let \( q \in \mathbb{C}^* \) be not a root of unity. We set \( q_i = q^{d_i} (i \in I) \) and use the symbols defined in (4.2). Let \( U_q (\mathfrak{g}) \) be the quantum Kac–Moody algebra [14, 62], which is a \( q \)-analogue of the Kac–Moody algebra \( \mathfrak{g} \) associated with \( C \) [11].

The quantum affinization (without central elements) of the quantum Kac–Moody algebra \( U_q (\mathfrak{g}) \), denoted by \( U_q (\hat{\mathfrak{g}}) \), is the \( \mathbb{C} \)-algebra with generators \( x_{i,n}^\pm (i \in I, n \in \mathbb{Z}) \), \( k_h (h \in \mathfrak{h}) \), \( h_{i,n} (i \in I, n \in \mathbb{Z} \setminus \{ 0 \}) \) and the following relations:

\[
\begin{align*}
k_h k_h &= k_{h+h}, \quad k_0 = 1, \quad k_h \phi^\pm_0 (z) = \phi^\pm_0 (z) k_h, \\
k_h x^\pm_i (z) &= q^{k_{x_i} (h)} x^\pm_i (z) k_h, \\
\phi^+_i (z) x^+_j (w) - q^{ - \alpha_i (\alpha'_j)} w z &= \frac{q^\pm \alpha_i (\alpha'_j) w - z}{w - q^\pm \alpha_i (\alpha'_j) z} \phi^+_j (w) \phi^+_i (z),
\end{align*}
\]

21 This reset is only for the current subsection.
\[ \phi_i^-(z)x_i^+(w) = \frac{q^{\pm R_i}w - z}{w - q^{\pm R_i}z}x_i^+(w)\phi_i^-(z), \]

\[ (w - q^{\pm R_i}z)x_i^+(z)x_j^-(w) = \frac{\delta_{ij}}{q_i - q_j} \left( \delta \left( \frac{w}{z} \right) \phi_i^+(w) - \delta \left( \frac{z}{w} \right) \phi_j^-(z) \right), \]

\( (w - q^{\pm R_i}w - z)x_j^+(w)x_i^+(z), \)

\[ \sum_{\pi \in \Sigma} \sum_{k=1}^{1-C_{ij}} (-1)^k \left[ 1 - C_{ij} \right] \]

\[ \sum_{\pi \in \Sigma} \sum_{k=1}^{1-C_{ij}} (-1)^k \left[ 1 - C_{ij} \right] x_i^+(w_{\pi(1)}) \cdots x_i^+(w_{\pi(k)}) \]

\[ = 0 \quad (i \neq j). \quad (4.29) \]

In (4.29) \( \Sigma \) is the symmetric group for the set \( \{1, \ldots, 1 - C_{ij}\} \). We have also used the following formal series:

\[ x_i^\pm(z) = \sum_{n \in \mathbb{Z}} x_i^{\pm n}, \quad \phi_i^\pm(z) = k_{\pm d_i a_i} \exp \left( \pm (q - q^{-1}) \sum_{n \geq 1} h_{i, \pm n} z^{\pm n} \right) \]

and the formal delta function \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \).

When \( C \) is of finite type, the above \( U_q(\hat{\mathfrak{g}}) \) is called an (untwisted) quantum affine algebra (without central elements) or quantum loop algebra; it is isomorphic to a subquotient of the previously introduced one (4.4) by the ideal generated by \( e_i^{1/2} - 1 \) \[63, 64\]. When \( C \) is of affine type, the quantum Kac–Moody algebra \( U_q(\hat{\mathfrak{g}}) \) is the one in (4.1). Its quantum affinization \( U_q(\hat{\mathfrak{g}}) \) is called a quantum toroidal algebra (without central elements). In general, if \( C \) is not of finite type, \( U_q(\hat{\mathfrak{g}}) \) is no longer isomorphic to a subquotient of any quantum Kac–Moody algebra and has no Hopf algebra structure.

From now on we shall exclusively consider a symmetrizable generalized Cartan matrix \( C \) satisfying the following condition due to Hernandez [8]:

\[ \text{if } C_{ij} < -1, \quad \text{then } d_i = -C_{ji} = 1, \quad (4.30) \]

where \( D = \text{diag}(d_1, \ldots, d_r) \) is the diagonal matrix symmetrizing \( C \). We say that a generalized Cartan matrix \( C \) is tamely laced if it is symmetrizable and satisfies condition (4.30). A generalized Cartan matrix \( C \) is simply laced if \( C_{ij} = 0 \) or \(-1\) for any \( i \neq j \). If \( C \) is simply laced, then it is symmetric, \( d_a = 1 \) for any \( a \in I \), and it is tamely laced.

With a tamely laced generalized Cartan matrix \( C \), we associate a Dynkin diagram in the standard way \[11\]. For any pair \( i \neq j \in I \) with \( C_{ij} < 0 \), the vertices \( i \) and \( j \) are connected by \( \max(|C_{ij}|, |C_{ji}|) \) lines, and the lines are equipped with an arrow from \( j \) to \( i \) if \( C_{ij} < -1 \). Note that condition (4.30) means

(i) the vertices \( i \) and \( j \) are not connected if \( d_i, d_j > 1 \) and \( d_i \neq d_j \),

(ii) the vertices \( i \) and \( j \) are connected by \( d_i \) lines with an arrow from \( i \) to \( j \) or not connected if \( d_i > 1 \) and \( d_j = 1 \) and

(iii) the vertices \( i \) and \( j \) are connected by a single line or not connected if \( d_i = d_j \).

**Example 4.10.**

1. Any Cartan matrix of finite or affine type is tamely laced except for types \( A_n^{(1)} \) and \( A_{2l}^{(2)} \).
2. The following generalized Cartan matrix \( C \) is tamely laced:

\[ C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & -2 & -2 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \]
The corresponding Dynkin diagram is

\[ \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \]

\[ \begin{array}{c}
1 \\
2 \\
3 \\
\end{array} \]

Define the integer \( t \) by

\[ t = \text{lcm}(d_1, \ldots, d_r). \]

For \( a, b \in I \), we write \( a \sim b \) if \( C_{ab} < 0 \), i.e. \( a \) and \( b \) are adjacent in the corresponding Dynkin diagram. Let \( U \) be either \( \frac{1}{2} \mathbb{Z} \), the complex plane \( \mathbb{C} \) or the cylinder \( \mathbb{C}_\xi := \mathbb{C}/(2\pi \sqrt{-1}/\xi)\mathbb{Z} \) for some \( \xi \in \mathbb{C} \setminus 2\pi \sqrt{-1} \mathbb{Q} \), depending on the situation under consideration.

For a tamely laced generalized Cartan matrix \( C \), the unrestricted \( T \)-system associated with \( C \) is the following relations among the commuting variables \( \{ T_m^{(a)}(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U \} \):

\[ T_m^{(a)} \left( u - \frac{d_a}{l} \right) T_m^{(a)} \left( u + \frac{d_a}{l} \right) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + \prod_{b, b \sim a} S(b)_m(u) \quad \text{if} \quad d_a > 1, \quad (4.31) \]

\[ T_m^{(a)} \left( u - \frac{d_a}{l} \right) T_m^{(a)} \left( u + \frac{d_a}{l} \right) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + \prod_{b, b \sim a} S(b)_m(u) \quad \text{if} \quad d_a = 1, \quad (4.32) \]

where \( T_0^{(a)}(u) = 1 \) if they occur on the rhs in the relations. The symbol \( S(b)_m(u) \) is defined by

\[ S(b)_m(u) = \prod_{k=1}^{d_b} T^{(b)}_{m + E \left( \frac{m-k}{d_b} \right)} \left( u + \frac{1}{l} \left( 2k - 1 - m + E \left( \frac{m-k}{d_b} \right) \right) \right), \quad (4.33) \]

and \( E[x] \) (\( x \in \mathbb{Q} \)) denotes the largest integer not exceeding \( x \).

Explicitly, \( S(b)_m(u) \) is written as follows. For \( 0 \leq j < d_b \),

\[ S_{d_b m + j}^{(b)}(u) = \left\{ \prod_{k=1}^{j} T^{(b)}_{m+1} \left( u + \frac{1}{l} (j+1-2k) \right) \right\} \left\{ \prod_{k=1}^{d_b-j} T^{(b)}_{m} \left( u + \frac{1}{l} (d_b-j+1-2k) \right) \right\}. \]

For example, for \( d_b = 1 \),

\[ S_m^{(b)}(u) = T_m^{(b)}(u), \]

for \( d_b = 2 \),

\[ S_{2m}^{(b)}(u) = T_m^{(b)} \left( u - \frac{1}{l} \right) T_m^{(b)} \left( u + \frac{1}{l} \right), \]

\[ S_{2m+1}^{(b)}(u) = T_{m+1}^{(b)}(u) T_m^{(b)}(u), \]

for \( d_b = 3 \),

\[ S_{3m}^{(b)}(u) = T_m^{(b)} \left( u - \frac{2}{l} \right) T_m^{(b)}(u) T_m^{(b)} \left( u + \frac{2}{l} \right), \]

\[ S_{3m+1}^{(b)}(u) = T_{m+1}^{(b)}(u) T_m^{(b)} \left( u - \frac{1}{l} \right) T_m^{(b)} \left( u + \frac{1}{l} \right), \]

\[ S_{3m+2}^{(b)}(u) = T_{m+1}^{(b)} \left( u - \frac{1}{l} \right) T_{m+1}^{(b)} \left( u + \frac{1}{l} \right) T_m^{(b)}(u). \]
and so on. The second terms on the rhs of (4.31) and (4.32) can be written in a unified way as follows [8]:
\[
\prod_{b \neq a} \prod_{k=1}^{C_{ab}} T^{(b)}_{a,b} \left( \frac{u}{\tau} \right) \left( u + \frac{d_{a}}{\tau} \left( \frac{2k + 1}{\tau} C_{ab} - C_{ba} + E \left[ \frac{d_{a} (m - k)}{\tau} \right] - 1 \right) - \frac{d_{a} m}{\tau} \right). 
\]

When \( C \) is of finite type \( \mathfrak{g} \), the above \( T \)-system coincides with the one for \( U_{q}(\mathfrak{g}) \) in section 2.1. For \( C \) of affine type, it was also studied in [79] as a discrete Toda field equation.

Let us proceed to the \( Y \)-system. For a tamely laced generalized Cartan matrix \( C \), the unrestricted \( Y \)-system associated with \( C \) is the following relations among the commuting variables \( \{ Y^{(a)}(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U \} \), where \( Y^{(a)}(u)^{-1} = 0 \) if they occur on the rhs in the relations:
\[
Y_{m}^{(a)} \left( u - \frac{d_{a}}{\tau} \right) Y_{m}^{(a)} \left( u + \frac{d_{a}}{\tau} \right) = \frac{\prod_{b \neq a}^{(b)} Z_{m,u}^{(b)}(u)}{(1 + Y^{(a)}_{m-1}(u)^{-1}) (1 + Y^{(a)}_{m+1}(u)^{-1})} \quad \text{if} \quad d_{a} > 1, \quad (4.34)
\]
\[
Y_{m}^{(a)} \left( u - \frac{d_{a}}{\tau} \right) Y_{m}^{(a)} \left( u + \frac{d_{a}}{\tau} \right) = \frac{\prod_{b \neq a}^{(b)} (1 + Y^{(b)}_{m-1}(u)^{-1}) (1 + Y^{(a)}_{m+1}(u)^{-1})}{(1 + Y^{(a)}_{m-1}(u)^{-1}) (1 + Y^{(a)}_{m+1}(u)^{-1})} \quad \text{if} \quad d_{a} = 1, \quad (4.35)
\]
where for \( p \in \mathbb{Z}_{\geq 1} \)
\[
Z_{p,m}^{(b)}(u) = \prod_{j=1}^{p-1} \left\{ \prod_{k=1}^{p-|j|} \left( 1 + Y^{(b)}_{m+j} \left( u + \frac{1}{\tau} (p - |j| + 1 - 2k) \right) \right) \right\},
\]
and \( Y^{(b)}_{m}(u) = 0 \) in (4.35) if \( \frac{m}{\tau} \notin \mathbb{Z}_{\geq 1} \).

The \( Y \)-systems here are formally in the same form as (2.11)–(2.15) for the quantum affine algebras. However, \( p \) in \( Z_{p,m}^{(b)}(u) \) here may be greater than 3. On the rhs of (4.34), \( \frac{d_{a}}{\tau} \) is either 1 or \( d_{a} \) due to (4.30). The term \( Z_{p,m}^{(b)}(u) \) is written more explicitly as follows: \( p = 1 \)
\[
Z_{1,m}^{(b)}(u) = 1 + Y_{m}^{(b)}(u),
\]
for \( p = 2 \)
\[
Z_{2,m}^{(b)}(u) = \left( 1 + Y^{(b)}_{2m-1}(u) \right) \left( 1 + Y^{(b)}_{2m} \left( u - \frac{1}{\tau} \right) \right) \left( 1 + Y^{(b)}_{2m} \left( u + \frac{1}{\tau} \right) \right) \left( 1 + Y^{(b)}_{2m+1}(u) \right),
\]
for \( p = 3 \)
\[
Z_{3,m}^{(b)}(u) = \left( 1 + Y^{(b)}_{3m-2}(u) \right) \left( 1 + Y^{(b)}_{3m-1} \left( u - \frac{1}{\tau} \right) \right) \left( 1 + Y^{(b)}_{3m-1} \left( u + \frac{1}{\tau} \right) \right) \left( 1 + Y^{(b)}_{3m}(u) \right) \left( 1 + Y^{(b)}_{3m+1} \left( u - \frac{2}{\tau} \right) \right) \left( 1 + Y^{(b)}_{3m+1} \left( u + \frac{2}{\tau} \right) \right) \left( 1 + Y^{(b)}_{3m+2}(u) \right),
\]
and so on. There are \( p^{2} \) factors in \( Z_{p,m}^{(b)}(u) \).

The \( T \)- and \( Y \)-systems in this subsection satisfy formally the same relations as those explained in section 2.3. Their restricted versions have also been formulated in [78].
4.7. Bibliographical notes

The origin of the Kirillov–Reshetikhin modules (they are named so in [80, definition 1.1]) goes back to [81], where the spectral parameter dependence was not considered. The idea of treating them as one family of $Y(g)$ or $U_q(\hat{g})$ modules with a spectral parameter satisfying the $T$-system in the Grothendieck ring was initiated in [1], where the identification by Drinfeld polynomials was also given in the context of Yangian based on the result of [69]. Meanwhile, the representation theory of finite-dimensional $U_q(\hat{g})$ modules was pushed forward in [65, 82], where the Kirillov–Reshetikhin modules were characterized and studied as minimal affinizations of $U_q(\hat{g})$ modules [83–87].

The relation between the Kirillov–Reshetikhin modules and $T$-systems became transparent after the introduction of the $q$-character in [70]. The case of Yangian goes back to [71].

Theorem 4.8 is due to [67] for simply laced $g$ and [68] for general $g$. Under certain circumstances, there are algorithms to compute $q$-characters [72] or its further generalization called $t$-analog of $q$-characters $\chi_{q,t}$ [88, 89] for the ADE case. In particular, $\chi_{q,t}$ of all the fundamental representations has been produced [89], among which the $E_8$ case requires a supercomputer.

The $T$-systems for the quantum affinizations of quantum Kac–Moody algebras in section 4.6 are due to [8]. The corresponding $Y$-system and formulation by cluster algebra are given in [78].

5. Formulation by cluster algebras

5.1. Dilogarithm identities in conformal field theory

Let $L(x)$ be the Rogers dilogarithm function [90, 91]

$$ L(x) = -\frac{1}{2} \int_0^x \left\{ \frac{\ln(1-y)}{y} + \frac{\ln y}{1-y} \right\} dy \quad (0 \leq x \leq 1). \quad (5.1) $$

It is well known that the following properties hold ($0 \leq x, y \leq 1$):

$$ L(0) = 0, \quad L(1) = \frac{\pi^2}{6}, \quad (5.2) $$

$$ L(x) + L(1-x) = \frac{\pi^2}{6}, \quad (5.3) $$

$$ L(x) + L(y) + L(1-xy) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2}. \quad (5.4) $$

In the series of works by Bazhanov, Kirillov and Reshetikhin [37, 53, 59, 81, 92], they reached a remarkable conjecture on identities expressing the central charges of CFT in terms of $L(x)$, and partly established it.

In what follows, $g$ denotes any one of the simple Lie algebras $A_r$, $B_r$, $D_r$, $G_2$ as in the previous sections. In section 2.2, we defined the level $\ell$ restricted $Y$-system for $g$ for $\ell \in \mathbb{Z}_{\geq 1}$. Let us introduce the system of relations for the variable $\{Y_{m}(a)|a \in I, 1 \leq m \leq t_\ell \ell - 1\}$ obtained from the level $\ell$ restricted $Y$-system by setting $Y_{m}(a)(u) = Y_{m}(a)$ dropping the dependence on the spectral parameter $u$. We call it the level $\ell$ restricted constant $Y$-system.

**Theorem 5.1** ([93, 94]). There exists a unique solution of the level $\ell$ restricted constant $Y$-system for $g$ satisfying $Y_{m}(a) \in \mathbb{R}_{>0}$ for all $a \in I, 1 \leq m \leq t_\ell \ell - 1$. 41
Theorem 5.1 was proved in [93] for the simply laced case, and extended to the nonsimply laced case in [94] using the same method. For more information on the constant \( Y \)-system, see sections 14.4 and 14.6.

The following theorem was originally conjectured in [81] and [59] for the simply laced case, and conjectured in [92] and properly corrected in [18] for the nonsimply laced case.

**Theorem 5.2** (Dilogarithm identities [92, 94–96]). Suppose that a family of positive real numbers \( \{ Y_m(a) \mid a \in I, 1 \leq m \leq t_a \ell - 1 \} \) satisfy the level \( \ell \) constant \( Y \)-system for \( g \). Then, the following identities hold:

\[
\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{t_a \ell - 1} L \left( \frac{Y_m(a)}{1 + Y_m(a)} \right) = \frac{\ell \dim g}{\ell + h^\vee} - \text{rank } g,
\]

where \( h^\vee \) is the dual Coxeter number of \( g \) (2.3).

The rational number of the first term on the rhs of (5.5) is the central charge of the Wess–Zumino–Witten CFT associated with \( g \) with level \( \ell \) [97, 98]. The rational number on the rhs of (5.5) itself is also the central charge of the parafermion CFT associated with \( g \) with level \( \ell \) [99, 100]. The identity (5.5) is crucial to establish the connection between CFTs and various types of nonconformal integrable models in various limits (cf section 15.3).

**Example 5.3** ([53]). Consider the case \( g = A_1 \) and any \( \ell \), which is equivalent to the case \( g = A_{\ell - 1} \) and \( \ell = 2 \) by the level-rank duality. Then, one has the solution

\[
Y_m^{(1)} = \frac{\sin^2 \frac{m \pi}{2\ell}}{\sin \frac{m \pi}{\ell}},
\]

and the corresponding identity (5.5) reads

\[
\frac{6}{\pi^2} \sum_{m=1}^{\ell - 1} L \left( \frac{\sin^2 \frac{m \pi}{2\ell}}{\sin^2 \frac{m \pi}{\ell}} \right) = \frac{3\ell}{2 + \ell} - 1.
\]

This identity has been known and studied by various authors from various points of view. See [101, 102] and references therein. In particular, the identity is derived [103, 104] from the following \( q \)-series expression [105] for the parafermion conformal character (‘string function’ in [11]) multiplied with Dedekind’s eta function:

\[
\sum_{n_1, \ldots, n_{\ell-1}=0}^{\infty} q^{\sum_{m=1}^{\ell - 1} mn_m} n_1 n_2 \cdots n_{\ell - 1} (q)^{n_1 - 1} \prod_{m=1}^{\ell - 1} (q)_{n_m}^{-1}, \quad (q)_k := \prod_{j=1}^{k} (1 - q^j),
\]

where the sum is under the constraint \( \sum_{m=1}^{\ell - 1} mn_m \equiv 0 \mod 2\ell \). In fact, a crude estimate by a saddle point method shows that as \( q \to 1 \), this series diverges as \( \text{const} \cdot (\zeta q)^{-c/24} \) where \( c \) is the lhs of (5.7) and \( \overline{\zeta} \to 0 \) is the modular conjugate specified by \( (\ln q)(\ln \overline{q}) = 4\pi^2 \). Comparing this with the known asymptotics of the string function [11] yields (5.7). For general \( g \), see around (14.43).

For \( g = A_r \), Kirillov [92] gave the explicit expression of the solution (cf example 14.4) and proved the corresponding identity (5.5) by the analytic method, but an extension of the proof to the other cases seemed difficult.

In the 1990s, people pursued a proof through lifting the dilogarithm identities to the Rogers–Ramanujan-type identities as example 5.3 (e.g. [106–110]). This created a new subject called the fermionic formula of conformal characters and their variants, which turned out to be a rich subject itself, and it has been intensively studied to this day in its own right. See
5.2. Cluster algebras with coefficients

Here we recall the definition of the cluster algebras with coefficients and some of their basic properties, following the convention in [9] with slight change of notations and terminology. See [9] for more details and information.

Fix an arbitrary semifield \( P \), i.e., an Abelian multiplicative group endowed with a binary operation of addition \( \oplus \) which is commutative, associative and distributive with respect to the multiplication \([112]\). Let \( \mathbb{Q}P \) denote the quotient field of the group ring \( \mathbb{Z}P \) of \( P \). Let \( I \) be a finite set\(^{22}\), and let \( B = (B_{ij})_{i,j \in I} \) be a skew symmetrizable (integer) matrix, namely there is a diagonal positive integer matrix \( D \) such that \( i(DB) = -DB \). Let \( x = (x_i)_{i \in I} \) be an \( I \)-tuple of formal variables, and let \( y = (y_i)_{i \in I} \) be an \( I \)-tuple of elements in \( P \). For the triplet \((B, x, y)\), called the initial seed, the cluster algebra \( \mathcal{A}(B, x, y) \) with coefficients in \( P \) is defined as follows.

Let \((B', x', y')\) be a triplet consisting of skew symmetrizable matrix \( B' \), an \( I \)-tuple \( x' = (x'_i)_{i \in I} \) with \( x'_i \in \mathbb{Q}P(x) \) and an \( I \)-tuple \( y' = (y'_i)_{i \in I} \) with \( y'_i \in P \). For each \( k \in I \), we define another triplet \((B'', x'', y'') = \mu_k(B', x', y')\), called the mutation of \((B', x', y')\) at \( k \), as follows.

(i) **Mutations of the matrix:**

\[
B''_{ij} = \begin{cases} 
-B'_{ij} & \text{if } i = k \text{ or } j = k, \\
B'_{ij} + \frac{1}{2}(B'_{ik}B'_{kj} + B'_{ki}B'_{kj}) & \text{otherwise}. 
\end{cases}
\]  

(ii) **Exchange relation of the coefficient tuple:**

\[
y''_i = \begin{cases} 
y'_i^{-1} & \text{if } i = k, \\
\frac{1}{1 \oplus y'_k^{-1}B'_{ii}} & \text{if } i \neq k, B'_{ii} \geq 0, \\
y'_i(1 \oplus y'_k)^{-B'_{ii}} & \text{if } i \neq k, B'_{ii} \leq 0. 
\end{cases}
\]

(iii) **Exchange relation of the cluster:**

\[
x''_i = \begin{cases} 
x'_i \prod_{j, B'_{ij} > 0} x'_j^{B'_{ij}} + \prod_{j, B'_{ij} < 0} x''_j^{-B'_{ij}}(1 \oplus y'_k)x'_k^{-1} & \text{if } i = k, \\
x'_i & \text{if } i \neq k. 
\end{cases}
\]

It is easy to see that \( \mu_k \) is an involution, namely \( \mu_k(B'', x'', y'') = (B', x', y') \). Now, starting from the initial seed \((B, x, y)\), iterate mutations and collect all the resulted triplets \((B', x', y')\). We call \((B', x', y')\) the seeds, \( y' \) and \( y'_i \) a coefficient tuple and a coefficient, and \( x' \) and \( x'_i \), a cluster and a cluster variable, respectively. The cluster algebra \( \mathcal{A}(B, x, y) \) with coefficients in \( P \) is the \( \mathbb{Z}P \)-subalgebra of the rational function field \( \mathbb{Q}P(x) \) generated by all the cluster variables.

It is standard to identify a skew-symmetric (integer) matrix \( B = (B_{ij})_{i,j \in I} \) with a quiver \( Q \) without loops or 2-cycles. The set of the vertices of \( Q \) is given by \( I \), and we put \( B_{ij} \) arrows

---

\(^{22}\) This \( I \) does not necessarily correspond to the \( I \) in section 2.1 for the index set of Dynkin diagrams.
from $i$ to $j$ if $B_{ij} > 0$. The mutation $Q'' = \mu_k(Q')$ of a quiver $Q'$ is given by the following rule. For each pair of an incoming arrow $i \to k$ and an outgoing arrow $k \to j$ in $Q'$, add a new arrow $i \to j$. Then, remove a maximal set of pairwise disjoint 2-cycles. Finally, reverse all arrows incident with $k$.

Let $P_{\text{univ}}(y)$ be the universal semifield of the $I$-tuple of generators $y = (y_i)_{i \in I}$, namely the semifield consisting of the subtraction-free rational functions of formal variables $y$ with usual multiplication and addition in the rational function field $\mathbb{Q}(y)$. We write $\oplus$ in $P_{\text{univ}}(y)$ as $+$ for simplicity.

From now on, unless otherwise mentioned, we set the semifield $P$ for $A(B, x, y)$ to be $P_{\text{univ}}(y)$, where $y$ is the coefficient tuple in the initial seed $(B, x, y)$.

Let $P_{\text{trop}}(y)$ be the tropical semifield of $y = (y_i)_{i \in I}$, which is the Abelian multiplicative group freely generated by $y$ endowed with the addition $\oplus$:

$$\prod_i y_i^a_i \oplus \prod_i y_i^b_i = \prod_i y_i^{\min(a_i, b_i)}. \quad (5.12)$$

There is a canonical surjective semifield homomorphism $\pi_T$ (the tropical evaluation) from $P_{\text{univ}}(y)$ to $P_{\text{trop}}(y)$ defined by $\pi_T(y) = y$. For any coefficient $y_i'$ of $A(B, x, y)$, let us write $[y_i']_T := \pi_T(y_i')$ for simplicity. We call $[y_i']_T$ the tropical coefficients (the principal coefficients in [9]). They satisfy the exchange relation (5.10) by replacing $y_i'$ with $[y_i']_T$ with $\oplus$ being the addition in (5.12). We also extend this homomorphism to the homomorphism of fields $\pi_T : (\mathbb{Q}P_{\text{univ}}(y))(x) \to (\mathbb{Q}P_{\text{trop}}(y))(x)$.

To each seed $(B', x', y')$ of $A(B, x, y)$ we attach the $F$-polynomials $F_j(y) \in \mathbb{Q}(y) (i \in I)$ by the specialization of $[x_i']_T$ at $x_j = 1 (j \in I)$. It is, in fact, a polynomial in $y$ with integer coefficients due to the Laurent phenomenon [9, proposition 3.6]. For definiteness, let us take $I = \{1, \ldots, n\}$. Then, $x'$ and $y'$ have the following factorized expressions [9, proposition 3.13, corollary 6.3] by the $F$-polynomials:

$$x_i' = \left(\prod_{j=1}^{n} x_j'^{g_{ij}}\right) \frac{F_j(y_1, \ldots, y_n)}{F_j(y_1, \ldots, y_n)}, \quad y_i' = y_i \prod_{j=1}^{n} x_j'^{b_{ij}}. \quad (5.13)$$

The integer vector $g_i' = (g_i'_{1}, \ldots, g_i'_{n}) (i = 1, \ldots, n)$ uniquely determined by (5.13) for each $x_i'$ is called the $g$-vector for $x_i'$.

Let $i = (i_1, \ldots, i_r)$ be an $I$-sequence, namely $i_1, \ldots, i_r \in I$. We define the composite mutation $\mu_i$ by $\mu_i = \mu_{i_1} \cdots \mu_{i_r}$, where the product means the composition.

**Lemma 5.4.** Let $B = (B_{ij})_{i,j \in I}$ be a skew-symmetric matrix and let $i = (i_1, \ldots, i_r)$ be an $I$-sequence. Suppose that $B_{ia} = 0$ for any $1 \leq a, b \leq r$. Then, the following facts hold.

(a) For any permutation $\sigma$ of $\{1, \ldots, r\}$, we have

$$\mu_i(B, x, y) = \mu_{\sigma(i_1 \cdots i_r)}(B, x, y). \quad (5.15)$$

(b) Let $B' = \mu_i(B)$. Then, $B'_{a,b} = 0$ holds for any $1 \leq a, b \leq r$.

(c) Let $(B', x', y') = \mu_i(B, x, y)$. Then, $(B, x, y) = \mu_i(B', x', y')$. 


5.3. T- and Y-systems in cluster algebras

All T- and Y-systems in sections 2.1–2.5 are regarded as relations among a cluster among cluster variables and coefficients in certain cluster algebras \( \mathcal{A}(B, x, y) \).

Let us mention two big advantages of cluster algebra formulation.

(a) The T- and Y-systems are integrated in one algebra \( \mathcal{A}(B, x, y) \) and commonly controlled by \( F \)-polynomials (together with tropical coefficients and \( g \)-vectors) through formulas (5.13) and (5.14). This fact may be hardly realized just by treating the T- and Y-systems only.

(b) The cluster algebra \( \mathcal{A}(B, x, y) \) itself is further controlled by the (generalized) cluster category developed in [113–118].

Here we concentrate on an example of level 4 restricted T- and Y-systems for \( A_4 \) to present a basic idea. Let \( Q \) be the following quiver with the index set \( I = \{1, 2, 3, 4\} \times \{1, 2, 3\} \).

![Quiver diagram](image)

Below we identify \( Q \) with the corresponding skew symmetric matrix \( B \) as described in section 5.2.

Let \( \mathbf{i}_+ \) (resp. \( \mathbf{i}_- \)) be a sequence of all the distinct elements of \( I \) with property + (resp. −) where the order of the sequence is chosen arbitrarily thanks to lemma 5.4. Then, the quiver \( Q \) has the following periodicity under the sequences of mutation \( \mathbf{i}_+ \) and \( \mathbf{i}_- \):

\[
Q \overset{\mu_{\mathbf{i}_+}}{\rightarrow} Q_{\text{opp}} \overset{\mu_{\mathbf{i}_-}}{\leftarrow} Q,
\]

(5.17)

where \( Q_{\text{opp}} \) is the opposite quiver of \( Q \), namely the quiver obtained from \( Q \) by inverting all the arrows.

Now we set \((Q(0), x(0), y(0)) := (Q, x, y)\) (the initial seed of \( \mathcal{A}(Q, x, y) \)) and consider the corresponding infinite sequence of mutations of seeds

\[
\cdots \overset{\mu_{\mathbf{i}_+}}{\rightarrow} (Q(-1), x(-1), y(-1)) \overset{\mu_{\mathbf{i}_-}}{\leftarrow} (Q(0), x(0), y(0)) \overset{\mu_{\mathbf{i}_+}}{\rightarrow} (Q(1), x(1), y(1)) \overset{\mu_{\mathbf{i}_-}}{\leftarrow} (Q(2), x(2), y(2)) \overset{\mu_{\mathbf{i}_+}}{\rightarrow} \cdots,
\]

(5.18)

\[
Q(u) = \begin{cases} Q & \text{if } u \text{ is even}, \\ Q_{\text{opp}} & \text{if } u \text{ is odd}, \end{cases}
\]

(5.19)

thereby introducing a family of clusters \( x(u) (u \in \mathbb{Z}) \) and coefficient tuples \( y(u) (u \in \mathbb{Z}) \).

For \(((i, i'), u) \in \mathcal{I} \times \mathbb{Z}, \text{ we write } ((i, i'), u) : \mathbf{p}_u \text{ if } i + i' + u \text{ is even, or equivalently, if } u \text{ is even and } (i, i') \text{ has the property } +, \text{ or } u \text{ is odd and } (i, i') \text{ has the property } -. \) Plainly speaking, \(((i, i'), u) : \mathbf{p}_u \) is a forward mutation point in (5.18).

For \(((i, i'), u) \in \mathcal{I} \times \mathbb{Z}, \text{ we set } ((i, i'), u) : \tilde{\mathbf{p}}_u \text{ if } ((i, i'), u + 1) : \mathbf{p}_u. \) Consequently, we have

\[
((i, i'), u) : \tilde{\mathbf{p}}_u \iff ((i, i'), u \pm 1) : \mathbf{p}_u,
\]

(5.20)

First, we explain how the Y-system appears in cluster algebra. The sequence of mutations (5.18) gives various relations among coefficients \( y_{i,i'}(u) \) \(((i, i'), u) \in \mathcal{I} \times \mathbb{Z}\) by the exchange...
relation (5.10). Then, all these coefficients are products of the ‘generating’ coefficients \(y_{i,j'}(u)\) and \(1 + y_{j,j'}(u)\) \(((i, i'), u) : \mathbf{p}_s\). Furthermore, these generating coefficients obey some relations, which are the T-system.

Let us write down the relations explicitly. Take \(((i, i'), u) : \mathbf{p}_s\) and consider the mutation at \(((i, i'), u)\), where \(y_{i,j'}(u)\) is exchanged to \(y_{i,j'}(u + 1) = y_{i,j'}(u)\) by (5.10). In the next step going from \(Q(u + 1)\) to \(Q(u + 2)\), the (forward) mutation points are those satisfying \(((j,j'), u + 1) : \mathbf{p}_s\). Therefore, the above \(y_{i,j'}(u + 1)\) gets multiplied by factors \((1 + y_{j,j'}(u + 1))\) if the quiver \(Q(u + 1)\) has an arrow from \((i, i')\) to \((j, j')\), and \((1 + y_{j,j'}(u + 1)^{-1})^{-1}\) if the quiver \(Q(u + 1)\) has an arrow from \((j, j')\) to \((i, i')\). The result coincides with the coefficient \(y_{i,j'}(u + 2)\). In summary, we have the following relations. For \(((i, i'), u) : \mathbf{p}_s\),

\[
y_{i,j'}(u)y_{i,j'}(u + 2) = \frac{(1 + y_{i-1,j}(u + 1))(1 + y_{i+1,j}(u + 1))}{(1 + y_{i,j-1}(u + 1))^{-1}(1 + y_{i,j+1}(u + 1))^{-1}},
\]

where \(y_{0,j'}(u + 1) = y_{0,j'}(u + 1) = 0\) and \(y_{i,0}(u + 1)^{-1} = y_{i,0}(u + 1)^{-1} = 0\) on the rhs. Or, equivalently, for \(((i, i'), u) : \mathbf{q}_s\),

\[
y_{i,j'}(u - 1)y_{i,j'}(u + 1) = \frac{(1 + y_{i-1,j}(u))(1 + y_{i+1,j}(u))}{(1 + y_{i,j-1}(u))^{-1}(1 + y_{i,j+1}(u))^{-1}}.
\]

This certainly agrees with the level 4 restricted T-system for \(A_2\) under the identification of \(y_{i,j'}(u)\) with \(Y_{i,j'}\) (u).

Next, we explain how the T-system appears in cluster algebra. The sequence of mutations (5.18) gives various relations among cluster variables \(x_{i,j'}(u)\) \(((i, i'), u) \in \mathcal{I} \times \mathbb{Z}\) by the exchange relation (5.11). All these coefficients are represented by the ‘generating’ cluster variables \(x_{i,j'}(u)\) \(((i, i'), u) : \mathbf{p}_s\). Furthermore, these generating cluster variables obey some relations, which are the T-system.

Let us write down the relations explicitly. Take \(((i, i'), u) : \mathbf{p}_s\) and consider the mutation at \(((i, i'), u)\). Then, by (5.11) and the fact that \(((i \pm 1, i'), u)\) and \(((i, i' \pm 1), u)\) are not forward mutation points, we have

\[
x_{i,j'}(u)x_{i,j'}(u + 2) = \frac{y_{i,j'}(u)}{1 + y_{i,j'}(u)}x_{i-1,j'}(u + 1)x_{i+1,j'}(u + 1)
\]

\[
+ \frac{1}{1 + y_{i,j'}(u)}x_{i,j'-1}(u + 1)x_{i,j'+1}(u + 1),
\]

where \(x_{0,j'}(u + 1) = x_{0,j'}(u + 1) = x_{i,0}(u + 1) = x_{i,j}(u + 1) = 1\) on the rhs. By introducing the ‘shifted cluster variables’ \(\hat{x}_{i,j'}(u) := x_{i,j}(u + 1)\) \(((i, i'), u) : \mathbf{p}_s\), these relations can be written in a more ‘balanced’ form and become parallel to (5.22) as follows. For \(((i, i'), u) : \mathbf{p}_s\),

\[
\hat{x}_{i,j'}(u - 1)\hat{x}_{i,j'}(u + 1) = \frac{y_{i,j'}(u)}{1 + y_{i,j'}(u)}\hat{x}_{i-1,j'}(u)\hat{x}_{i+1,j'}(u)
\]

\[
+ \frac{1}{1 + y_{i,j'}(u)}\hat{x}_{i,j'-1}(u)\hat{x}_{i,j'+1}(u).
\]

Let \(\mathcal{A}(B, x)\) be the cluster algebra with trivial coefficients with the initial seed \((B, x)\). Namely, we set every coefficient to be 1 in the trivial semifield \(1 = \{1\}\). Let \(\pi_1 : \mathbb{P}_{\text{univ}}(y) \to 1\) be the projection. Let \([x_i(u)]_1\) be the image of \(x_i(u)\) by the algebra homomorphism \(\mathcal{A}(B, x, y) \to \mathcal{A}(B, x)\) induced from \(\pi_1\). By the specialization of (5.24), we have

\[
[x_{i,j'}(u - 1)]_1[x_{i,j'}(u + 1)]_1 = [\hat{x}_{i-1,j'}(u)]_1[\hat{x}_{i+1,j'}(u)]_1 + [\hat{x}_{i,j'-1}(u)]_1[\hat{x}_{i,j'+1}(u)]_1.
\]

This certainly agrees with the level 4 restricted T-system for \(A_2\) under the identification of \([x_{i,j'}(u)]_1\) with \(T_{i,j'}^{(l)}(u)\).
For $\mathfrak{g}$ simply laced, the quiver relevant to the level $\ell$ restricted $T$- and $Y$-systems is drawn similarly to (5.16) on the vertex set $\mathcal{I} = \{\text{nodes of the Dynkin diagram}\} \times \{1, 2, \ldots, \ell - 1\}$. For $\mathfrak{g}$ nonsimply laced, it is slightly more involved [94, 96]. Here we only give examples for $B_3$ with levels 2 (left) and 3 (right):

![Quiver diagrams]

**Remark 5.5.** Once we realize that the $T$- and $Y$-systems are integrated in a single cluster algebra with coefficients as above, the relation between $T$- and $Y$-systems in theorem 2.5 becomes an immediate consequence of a more general relation between cluster variables and coefficients in [9, proposition 3.9], where (2.19) is a special case of [9, equation (3.7)] with the specialization of the base semifield $\mathbb{P}$ therein to the trivial semifield. See also [119, proposition 5.11] for the relation between more general $T$- and $Y$-systems.

### 5.4. Application to periodicity and dilogarithm identities

As remarkable applications of the cluster algebra formulation, one can prove the periodicities of $T$- and $Y$-systems and dilogarithm identities (5.5).

The following periodicity property was originally conjectured for type $A_1$ in [3], for the simply laced case by Ravanini–Tateo–Valleriani [5] and for the nonsimply laced case by Kuniba–Nakanishi–Suzuki [1].

**Theorem 5.6** (Periodicity [17, 94, 96, 115, 120–125]). For any family of variables \(\{Y_m^{(a)}(u) \mid a \in \mathcal{I}, 1 \leq m \leq t_u \ell - 1, u \in \mathbb{Z}\}\) satisfying the level $\ell$ restricted $Y$-system for $\mathfrak{g}$, one has the periodicity

\[
Y_m^{(a)}(u + 2(h^+ + \ell)) = Y_m^{(a)}(u).
\] (5.26)

To prove theorem 5.6 in full generality, the use of the categorification of the cluster algebra by the cluster category in [117, 118] is essential.

Since the $T$-system is integrated in the same cluster algebra, one can simultaneously prove the periodicity of the $T$-system as well, which was overlooked in the literature until recently [17, 126].

**Theorem 5.7** (Periodicity [9, 17, 94, 96, 115, 124, 127]). For any family of variables \(\{T_m^{(a)}(u) \mid a \in \mathcal{I}, 1 \leq m \leq t_u \ell - 1, u \in \mathbb{Z}\}\) satisfying the level $\ell$ restricted $T$-system for $\mathfrak{g}$, one has the periodicity

\[
T_m^{(a)}(u + 2(h^+ + \ell)) = T_m^{(a)}(u).
\] (5.27)

Closely related to the periodicity of $Y$-systems, the following (significant) functional generalization of the dilogarithm identities (5.5) was originally conjectured for the simply laced case by Gliozzi–Tateo [128].
Theorem 5.8 (Functional dilogarithm identities [94–96, 120, 121, 129]). Suppose that a family of positive real numbers \( \{ Y_m(a)(u) \}_{a \in I, 1 \leq m \leq t_0 \ell - 1, u \in \mathbb{Z}} \) satisfy the level \( \ell \) restricted Y-system for \( g \). Then, the following identities hold:

\[
\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{t_0 \ell - 1} \sum_{u=0}^{L} \frac{Y_m(a)(u)}{1 + Y_m^{(a)}(u)} = 2t(\ell h - h^{\vee}) \text{rank} \ g, \tag{5.28}
\]

where \( h \) is the Coxeter number of \( g \) (2.3).

Example 5.9 ([128]).

(i) In the simplest case, type \( A_1 \), identity (5.28) is equivalent to (5.3).

(ii) In the next simplest case, type \( A_2 \), identity (5.28) is equivalent to the five-term relation (5.4).

Theorem 5.8 implies theorem 5.2; namely, take a constant solution \( Y_m(a)(u) \) of the Y-system with respect to the spectral parameter \( u \). Or equivalently, take a solution to the constant Y-system in section 14.4. Then, one obtains (5.5) from (5.28).

See section 5.5 for a more precise account of contributions to theorems 5.6, 5.7 and 5.8.

5.5. Bibliographical notes

The cluster algebraic formulation of Y-systems was given for the simply laced case with level 2 in [122], for the simply laced case with the general level in [115], for the nonsimply laced case in [94, 96] and for the quantum affinizations of the tamely laced quantum Kac–Moody algebras in [78, 130]. The recognition of T-systems in the cluster algebras was made a little later than Y-systems in [17, 131, 132], though the simply laced case with level 2 clearly appeared in [133]. The formulation here is due to [78, 94, 96]. See [119] for a further generalization of T- and Y-systems in view of cluster algebras.

Theorem 5.6 was proved for type \( A_r \) with level 2 in [120, 121], for the simply laced case with level 2 in [133], for type \( A_r \) with the general level in [123] and [124], for the simply laced case with the general level in [115, 125] and for all cases with a unified method in [94, 96].

Theorem 5.7 was proved for the simply laced case with level 2 in [133], for type \( A_r \) with the general level in [127] and [124], for the simply laced case with the general level in [115] and [17], for type \( C_r \) with the general level in [17] and for all cases with a unified method in [94, 96]. Actually in [17, 94, 96], refinements of theorem 5.6 and 5.7 have been obtained concerning the property under the half shift \( u \rightarrow u + h^{\vee} + \ell \).

Theorem 5.8 was proved for type \( A_r \) with level 2 in [120, 121], for the simply laced case with level 2 in [129], for the simply laced case with the general level in [95] and for the nonsimply laced case in [94, 96]. See [119] for a further generalization of dilogarithm identities in view of cluster algebras.

There is a dilogarithm conjecture that generalizes (5.5) involving \(-24 \times \text{(scaling dimensions)}\) in addition to the central charge on the rhs. See [4] and [134, appendix D]. Some of them have been proved in [101, section 1.3, 1.4].

6. Jacobi–Trudi-type formula

6.1. Introduction: type \( A_r \)

In this section we exclusively consider unrestricted T-systems. By theorem 4.3, we know that \( T_m^{(\alpha)}(u) \) is expressible as a polynomial in the fundamental ones \( T_1^{(1)}(v), \ldots, T_1^{(r)}(v) \) with
various $v$. Such formulas can be derived directly. Consider for instance the unrestricted $T$-system for $A_2$:

\[ T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_m^{(2)}(u), \]
\[ T_m^{(2)}(u - 1)T_m^{(2)}(u + 1) = T_{m-1}^{(2)}(u)T_{m+1}^{(2)}(u) + T_m^{(1)}(u). \]

Setting $m = 1, 2$ and noting $T_0^{(1)}(u) = T_0^{(2)}(u) = 1$, one obtains

\[ T_2^{(1)}(u) = T_1^{(1)}(u - 1)T_1^{(1)}(u + 1) - T_1^{(2)}(u), \]
\[ T_2^{(2)}(u) = T_1^{(2)}(u - 1)T_1^{(2)}(u + 1) - T_1^{(1)}(u), \]
\[ T_3^{(1)}(u) = T_1^{(1)}(u - 2)T_1^{(1)}(u)T_1^{(1)}(u + 2) - T_1^{(1)}(u - 2)T_1^{(2)}(u + 1) - T_1^{(1)}(u + 2)T_1^{(2)}(u) + 1. \]

The formulas generated in this manner are systematized in a determinant form:

\[ T_2^{(1)}(u) = \begin{vmatrix} T_1^{(1)}(u - 1) & T_1^{(2)}(u) \\ 1 & T_1^{(1)}(u + 1) \end{vmatrix}, \quad T_2^{(2)}(u) = \begin{vmatrix} T_1^{(2)}(u - 1) & 1 \\ T_1^{(1)}(u) & T_1^{(2)}(u + 1) \end{vmatrix}, \\
T_3^{(1)}(u) = \begin{vmatrix} T_1^{(1)}(u - 2) & T_1^{(2)}(u - 1) \\ 1 & T_1^{(1)}(u) \end{vmatrix} \begin{vmatrix} 1 & T_1^{(2)}(u + 1) \\ 0 & T_1^{(1)}(u + 2) \end{vmatrix}. \]

Proceeding similarly, one obtains

**Theorem 6.1** ([59]). *For the unrestricted $T$-system for $A_r$, the following formula is valid:*

\[ T_m^{(a)}(u) = \det(T_1^{(a - i + j)}(u + i + j - m - 1))_{1 \leq i, j \leq m}, \quad (6.1) \]

where $T_1^{(a)}(u) = 0$ unless $0 \leq a \leq r + 1$, and $T_1^{(0)}(u) = T_1^{(r+1)}(u) = 1$.

The proof reduces to the Jacobi identity among the determinants

\[ D \begin{bmatrix} m + 1 \\ m + 1 \end{bmatrix} D \begin{bmatrix} 1 \\ 1 \end{bmatrix} = D \begin{bmatrix} 1, m + 1 \\ 1, m + 1 \end{bmatrix} D + D \begin{bmatrix} 1 \\ m + 1 \end{bmatrix} D \begin{bmatrix} m + 1 \\ 1 \end{bmatrix}, \quad (6.2) \]

where $D[i, j, \ldots]$ is the minor of $D$ removing $i$’s rows and $j$’s columns.

Alternatively, one can also solve the $T$-system to express everything by $T_j^{(v)}(u)$ with various $v$ and $k$. By the same method as before, one can easily systematize such formulas and establish

**Theorem 6.2** ([59]). *For the unrestricted $T$-system for $A_r$ (2.5) without assuming $T_m^{(r+1)}(u) = 1$, the following formula is valid:*

\[ T_m^{(a)}(u) = \det(T_1^{(1)}(u + i + j - a - 1))_{1 \leq i, j \leq a} \quad (1 \leq a \leq r + 1), \quad (6.3) \]

where $T_0^{(1)}(u) = 1$ and $T_m^{(1)}(u) = 0$ for $m < 0$.

Formulas (6.1) and (6.3) are quantum analogs of the Jacobi–Trudi formula for Schur functions [135].

In the remainder of this section, we present the Jacobi–Trudi-type formulas analogous to (6.1) for the $T$-systems for $B_r$, $C_r$, and $D_r$. The result involves not only determinants but also Pfaffians for $T_m^{(v)}(u)$ in $C_r$ and $T_m^{(v-1)}(u)$ and $T_m^{(v)}(u)$ in $D_r$.
6.2. Type $B_r$

For any $k \in \mathbb{C}$, set

$$x_k^a = \begin{cases} T^{(a)}_1(a + k) & 1 \leq a \leq r, \\ 1 & a = 0. \end{cases} \quad (6.4)$$

We introduce the infinite-dimensional matrices $T = (T_{ij})_{i,j \in \mathbb{Z}}$ and $E = (E_{ij})_{i,j \in \mathbb{Z}}$ as follows:

$$T_{ij} = \begin{cases} \frac{i+1}{2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i+1}{2} \in \{1, 0, \ldots, 2 - r\}, \\ \frac{i+2r-2}{2} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i+2r-2}{2} \in \{1 - r, -r, \ldots, 2 - 2r\}, \\ -x_{r+1} & \text{if } i \in 2\mathbb{Z} \text{ and } j = i + 2r - 3, \\ 0 & \text{otherwise} \end{cases}$$

$$E_{ij} = \begin{cases} \pm 1 & \text{if } i = j - 1 \pm 1 \text{ and } i \in 2\mathbb{Z}, \\ x_{i-1} & \text{if } i = j - 1 \text{ and } i \in 2\mathbb{Z} + 1, \\ 0 & \text{otherwise}. \end{cases} \quad (6.6)$$

For instance for $B_3$, they read

$$T_{ij}(i,j \geq 1) = \begin{pmatrix} x_0^1 & 0 & x_1^1 & 0 & -x_2^1 & 0 & -x_3^1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -x_3^{3/2} & 0 & 0 & 0 & 0 \\ 1 & 0 & x_1^1 & 0 & x_2^1 & 0 & -x_4^1 & 0 & -x_5^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_4^{3/2} & 0 & 0 \\ 0 & 0 & 1 & 0 & x_4^1 & 0 & x_5^1 & 0 & -x_6^2 \\ \vdots & & & & & & & & \ddots \end{pmatrix}$$

$$E_{ij}(i,j \geq 1) = \begin{pmatrix} 0 & x_0^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & x_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & & & & & & \ddots \end{pmatrix}.$$

Let $T_{|u-x^a|}$ be the overall shift of the lower index $x_k^a \rightarrow x_k^{a+u}$ in $T$ in accordance with (6.4). As is evident from this example, the quantity $x_k^a$ is contained in $T_{|u-x^a|}$ at most once as its matrix element for any $1 \leq a \leq r$ and $k$. For example, the shift $s = 1$ is needed to accommodate $x_1^1$ as the (1,1) element of $T_{|u-x^a|}$. In view of this, we employ the notation $T_m(i, j, \pm x_k^a)$ to mean the $m$ by $m$ sub-matrix of $T_{|u-x^a|}$, where $s$ is chosen so that its $(i, j)$ element becomes exactly $\pm x_k^a$. For example in (6.7),

$$T_3(1, 1, x_0^3) = \begin{pmatrix} x_0^3 & 0 & x_1^3 \\ 0 & 0 & 0 \\ 1 & 0 & x_2^3 \end{pmatrix}, \quad T_3(1, 1, x_1^3) = \begin{pmatrix} x_1^3 & 0 & x_2^3 \\ 0 & 0 & 0 \\ 1 & 0 & x_3^3 \end{pmatrix},$$

$$T_3(1, 2, -x_3^{3/2}) = \begin{pmatrix} 0 & -x_3^{3/2} \\ 0 & x_3^{3/2} \end{pmatrix}, \quad T_3(1, 2, -x_3^3) = \begin{pmatrix} 0 & -x_3^3 \\ 0 & x_3^3 \end{pmatrix}.$$  

We also use the similar notation $E_m(i, j, \pm x_k^a)$. Now the result for $B_r$ is stated as
Theorem 6.3 ([136]). For unrestricted T-system for \( B_\alpha \), the following formula is valid:

\[
 T_m^{(a)}(u) = \det \left( T_{2m-1}(1, 1, x_{m+1}^-) + \mathcal{E}_{2m-1}(1, 2, x_{m+r-a+1}^-) \right) \quad (1 < a < r),
\]

\[
 T_m^{(r)}(u) = (-1)^{m(m-1)/2} \det \left( T_m(1, 2, -x_{m+1}^-) + \mathcal{E}_m(1, 1, x_{m+1}^-) \right).
\]

6.3. Type \( C_\alpha \)

Here we introduce the infinite-dimensional matrix \( T \) by

\[
 T_{ij} = \begin{cases}
 x_j^{i-j+1} \frac{x_i^{j-1}}{x_i^{j-1}} & \text{if } i - j \in \{1, 0, \ldots, 1-r\}, \\
 -x_j^{i-j+2r+1} & \text{if } i - j \in \{-1-r, -2-r, \ldots, -1-2r\}, \\
 0 & \text{otherwise.}
\end{cases}
\]

(6.9)

For instance, for \( C_2 \), it reads

\[
 (T_{ij})_{i,j \geq 1} = \begin{pmatrix}
 x_0^1 & x_1^{1/2} & 0 & -x_2^{1/2} & -x_1^1 & -1 & 0 & 0 \\
 x_1^1 & x_2^{1/2} & 0 & -x_3^{1/2} & -x_2^1 & -1 & 0 & 0 \\
 0 & 1 & x_2^{1/2} & 0 & -x_3^{1/2} & -x_2^1 & -1 \\
 0 & 0 & 1 & x_3^{1/2} & 0 & -x_4^{1/2} & -x_3^1 & \cdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}.
\]

We keep the notation (6.4) and \( T_m(i, j, \pm x_k^a) \) (1 \( \leq \alpha \leq r \)) as in section 6.2. Note that \( T_m(1, 2, -x_k^a) \) is an anti-symmetric matrix for any \( m \).

Theorem 6.4 ([136]). For the unrestricted T-system for \( C_\alpha \), the following formula is valid:

\[
 T_m^{(a)}(u) = \det \mathcal{T}_m(1, 1, x_{-\frac{a+1}{2}}) \quad (1 \leq a < r),
\]

(6.10)

\[
 T_m^{(r)}(u) = (-1)^m \text{pf } T_{2m}(1, 2, -x_{m+1}^-).
\]

(6.11)

As an additional result, we have the following relations:

\[
 T_m^{(r)}(u - \frac{1}{2}) T_m^{(r)}(u + \frac{1}{2}) = \det \mathcal{T}_{2m}(1, 1, x_{m+1}^r),
\]

(6.12)

\[
 T_m^{(r)}(u) T_m^{(r)}(u) = \det \mathcal{T}_{2m+1}(1, 1, x_{m}^-).
\]

(6.13)

If one extends the definition of \( x_k^a \) (6.4) by \( x_k^a + x_k^{2-r-a} = 0 \) in accordance with (9.31), then (6.10) is identical with the result (6.1) for \( A_{2r+1} \).

As remarked at the end of section 2.1, the T-systems for \( B_2 \) and \( C_2 \) are equivalent by the interchange \( T_m^{(1)}(u) \leftrightarrow T_m^{(2)}(u) \). Therefore, theorems 6.3 and 6.4 supply these T-systems with two kinds of Jacobi–Trudi-type formulas.
6.4. Type $D_r$

Here we define the infinite-dimensional matrices $T$ and $E$ by

\[
T_{ij} = \begin{cases} 
\frac{x_{m+1}^2}{x_{m+1}^2 - 1} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i - j}{2} \in \{1, 0, \ldots, 3 - r\}, \\
-\frac{x_{m+1}^r}{x_{m+1}^r - 1} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i - j}{2} = \frac{5}{2} - r, \\
-\frac{x_{m+1}^r}{x_{m+1}^r - 1} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i - j}{2} = \frac{3}{2} - r, \\
-\frac{x_{m+1}^r + 2r - 3}{x_{m+1}^2 - 1} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i - j}{2} \in \{1 - r, -r, \ldots, 3 - 2r\}, \\
0 & \text{otherwise}.
\end{cases}
\]  

(6.14)

For the unrestricted $T$-system for $D_r$, the following formula is valid:

\[
E_{ij} = \begin{cases} 
\pm 1 & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z}, \\
x_{m+1}^r & \text{if } i = j - 3 \text{ and } i \in 2\mathbb{Z}, \\
x_{m+1}^r & \text{if } i = j - 1 \text{ and } i \in 2\mathbb{Z}, \\
0 & \text{otherwise}.
\end{cases}
\]  

(6.15)

For instance for $D_1$, they read

\[
(T_{ij})_{i,j \geq 1} = \begin{pmatrix}
x_0^4 & 0 & x_0^2 & -x_1^2 & 0 & -x_2^2 & 0 & -x_3^2 & 0 & -x_4^2 & 0 & -x_5^2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & x_0^4 & 0 & x_0^2 & -x_1^2 & 0 & -x_2^2 & 0 & -x_3^2 & 0 & -x_4^2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},
\]

\[
(E_{ij})_{i,j \geq 1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & x_0^4 & 0 & x_0^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & x_0^4 & 0 & x_0^2 & -1 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}.
\]

We keep notations (6.4), $T_m(i, j, \pm x_k^a)$ (1 $\leq$ $a$ $\leq$ $r - 2$) and $T_m(i, j, -x_k^a)$, $E_m(i, j, x_k^a)$ ($a = r - 1, r$) as in section 6.2.

**Theorem 6.5 ([136]).** For the unrestricted $T$-system for $D_n$ the following formula is valid:

\[
T_m^{(a)}(u) = \det (T_{m+1}(1, 1, x_{m+1}^a) + E_{2m-1}(2, 3, x_{m-r+1}^r)) \quad (1 \leq a \leq r - 2),
\]

(6.16)

\[
T_m^{(r-1)}(u) = \text{pf}(T_{2m}(2, 1, -x_{m+1}^r) + E_{2m}(1, 2, x_{m-r+1}^{r-1})),
\]

(6.17)

\[
T_m^{(r)}(u) = (-1)^m \text{pf}(T_{2m}(1, 2, -x_{m+1}^r) + E_{2m}(2, 1, x_{m-r+1}^r)).
\]

(6.18)

The matrices in (6.17) and (6.18) are indeed anti-symmetric. The following relations also hold:

\[
T_m^{(r-1)}(u)T_m^{(r)}(u) = (-1)^m \det (T_{2m}(1, 1, -x_{m+1}^{r-1}) + E_{2m}(2, 2, x_{m-r+1}^r)),
\]

\[
T_m^{(r-1)}(u + 1)T_m^{(r)}(u - 1) = (-1)^m \det (T_{2m}(1, 1, -x_{m+1}^r) + E_{2m}(2, 2, x_{m-r+1}^{r-1})),
\]

\[
T_{m+1}^{(r-1)}(u)T_m^{(r)}(u - 1) = (-1)^m \det (T_{2m+1}(1, 1, -x_{m+1}^{r-1}) + E_{2m+1}(2, 2, x_{m+1}^r)),
\]

\[
T_{m+1}^{(r-1)}(u + 1)T_m^{(r)}(u) = (-1)^m \det (T_{2m+1}(2, 1, x_{m-r+1}^2) + E_{2m+1}(1, 1, x_{m+1}^r)).
\]

Theorems 6.3–6.5 can only be proved by using (6.2) and the fact $(\text{pf})^2 = \det$. 

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6.5. Another Jacobi–Trudi-type formula for $B_r$

For $B_r$ and $D_r$, a variant of the Jacobi–Trudi-type formula is known which has a quite similar structure to the $A_r$ case. Compared with the rather sparse matrices $T$ and $E$, the relevant matrices are dense and involve some auxiliary variables. Here we present the result for $B_r$. The $D_r$ case is similar although slightly more involved.

Given $T_1^{(1)}(u), \ldots, T_1^{(r)}(u)$, we introduce the auxiliary variable $T^a(u)$ for all $a \in \mathbb{Z}$ by

\[
T^a(u) = \begin{cases} 
0 & a < 0, \\
1 & a = 0, \\
T_1^{(a)}(u) & 1 \leq a \leq r - 1,
\end{cases}
\]

(6.19)

\[
T^a(u) + T^{2r-1-a}(u) = T_1^{(r)}(u + r + a - \frac{1}{2}) T_1^{(r)}(u + r - a - \frac{1}{2}) \quad \text{for all } a \in \mathbb{Z}.
\]

(6.20)

Recall that $t_a = 1$ for $a \neq r$ and $t_r = 2$ for $B_r$ according to (2.1).

**Theorem 6.6** ([137]). For the unrestricted $T$-system for $B_r$, the following formula is valid:

\[
T_{2m+1}^{(r)}(u) = \det(T^{a+b}(u + i + j - m - 1))_{1 \leq i, j \leq m} \quad (1 \leq a \leq r),
\]

(6.21)

where the matrix (6.22) is of size $m + 1$, its $(i + 1, 1)$ element is $T_1^{(r)}(u - m + 2i)$ and the rest has the same pattern as (6.21) for $T_{2m+2}^{(r)}(u - \frac{1}{2})$.

6.6. Bibliographical notes

Formulas (6.1)–(6.3) for $A_r$ in theorem 6.1 first appeared in [59] before the $T$-system was formulated. There, transfer matrices more general than $T_m^{(a)}(u)$ were considered. Theorems 6.3–6.5 supplemented the determinant conjectures in [1] with Pfaffians. A result for $D_4$, analogous to theorem 6.6 is available in [138].

7. Tableau sum formula

7.1. Type $A_r$

Let $[a], \ldots, [r+1]$ be variables depending on $u$. If we set $T_1^{(1)}(u) = \sum_{a=1}^{r+1} [a]$, then

\[
T_1^{(1)}(u - 1) T_1^{(1)}(u + 1) = \sum_{a \leq b} \frac{[a]}{u-1} [b]_{u+1} + \sum_{a > b} \frac{[b]}{a-1} [a]_{u+1},
\]

(7.1)

where both arrays of the boxes stand for the product. Comparing this with the $T$-system relation $T_1^{(1)}(u - 1) T_1^{(1)}(u + 1) = T_2^{(1)}(u) + T_1^{(2)}(u)$, one may identify $T_2^{(1)}(u)$ and $T_1^{(2)}(u)$
individually with the two terms in (7.1) and try to further establish similar formulas for higher $T(a)_m(u)$. Such a procedure leads to a solution of the $T$-system expressed as a sum of tableaux. In fact, if one forgets the spectral parameter $u$ in (7.1), it can be viewed as the identity among Schur functions corresponding to the irreducible decomposition of the $A_r$-modules:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Tableau 1}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Tableau 2}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Sum of Tableaux}
\end{array}
\end{array}
\end{array}
\end{array}
\quad (7.2)
\]

In this sense the result presented in what follows for $A_r$ is a deformation of the classical tableau sum formula for the Schur functions [135].

Consider the Young diagram $(m^a)$ of $a \times m$ rectangular shape. Let $\text{Tab}(m^a)$ be the set of semistandard tableaux on $(m^a)$ with numbers $\{1, 2, \ldots, r+1\}$. The inscribed numbers are strictly increasing to the bottom and nondecreasing to the right. For example when $r = 2$,

\[
\text{Tab}(2) = \{ 11, 12, 13, 23 \},
\text{Tab}(2^2) = \{ 11 12, 11 33, 12 23, 13 23, 22 33 \}.
\]

Note that $\text{Tab}(m^a)$ is empty for $a > r + 1$. We define

\[
T_u(a) = \prod_{i=1}^{a} \prod_{j=1}^{m} t_{ij}^{a_i - m - 2i + 2j}
\]

for $T = (t_{ij}) \in \text{Tab}(m^a)$, where $t_{ij}$ denotes the entry of the box in the $i$th row and the $j$th column from the top left.

**Theorem 7.1.**

\[
T_m(a)(a) = \sum_{T \in \text{Tab}(m^a)} T_u (1 \leq a \leq r + 1)
\]

is a solution of the $T$-system for $A_r$ (2.5).

We note that $T_m(r+1)(a)$ here is not just 1 but nontrivially chosen as (7.4) as opposed to the original definition of the $T$-system. However, $\text{Tab}(m^{r+1})$ consists of a unique tableau; therefore, (7.4) states that $T_m(r+1)(a)$ is a monomial:

\[
T_m(r+1)(a) = \prod_{i=1}^{m} T_1^{(r+1)}(u - m - 1 + 2j),
T_1^{(r+1)}(u) = \prod_{i=1}^{r+1} t_{i+r+2-2i}.
\]

Thus the situation $T_m(r+1)(a) = 1$ can be restored if the variables $1, 2, \ldots, r+1$, are chosen so as to satisfy the simple relation $T_1^{(r+1)}(u) = 1$. Theorem 7.1 yields the $q$-characters by the special choice

\[
\mathbf{a}_m = z_m(a) := Y_{a-1,q^{-m}} Y_{q^{a-m-1}}, \quad Y_{0,q^{-m}} = Y_{r+1, q^{a-m}} = 1,
\]

which indeed satisfies the condition $T_m^{(r+1)}(u) = 1$. The restrictions (4.22) and (4.23) of the resulting $q$-character $T_m(a)(u) = \chi_m(W_m(a))$ are given by

\[
\text{res } T_m^{(a)}(u) = \chi(W_{m,a}),
\]

in the notation of (4.24), since the $a \times m$ rectangle Young diagram corresponds to the highest weight $m_0 a_m$.

In the rest of this section, we present the tableau sum formulas for $g = B_r, C_r, D_r$ along the context of the $q$-characters $T_m(a)(u) = \chi_m(W_m(a))$. The contents cover all the fundamental ones $T_1^{(1)}(u), \ldots, T_1^{(r)}(u)$, which is enough in principle to determine all the higher ones $T_m^{(a)}(u)$ due to theorem 4.3. Some $T_m^{(a)}(u)$ allowing a relatively simple description will also be included.
7.2. Type $B_r$

Let us introduce the index set and a total order on it as

$$J = \{1, 2, \ldots, r, 0, \bar{r}, \ldots, \bar{r}, \bar{r}, \bar{r}\}, \quad 1 < \cdots < r < 0 < \bar{r} < \cdots < \bar{r}. \quad (7.8)$$

We introduce the variables corresponding to single-box tableaux:

$$z_a(u) = Y_{a,q^2+2a}Y^{-1}_{a-1,q^2+2a} \quad (1 \leq a \leq r-1),$$

$$z_r(u) = Y_{r,q^2+2r}Y^{-1}_{r-1,q^2+2r},$$

$$z_0(u) = Y_{r,q^2+2r-1}Y_{r,q^2+2r-2}Y^{-1}_{r-1,q^2+2r-1},$$

$$z_T(u) = Y_{r-1,q^2+2r-1}Y^{-1}_{r,q^2+2r},$$

$$z_T(u) = Y_{a-1,q^2+2a-1}Y_{a,q^2+2a}Y^{-1}_{a+1,q^2+2a} \quad (1 \leq a \leq r-1),$$

where $Y_{0,q^i} = 1$ (see p 1427 of [139] contains a misprint.) Consider the Young diagram $(m')$ of $a \times m$ rectangular shape. Let $\text{Tab}(B_r, m')$ be the set of tableaux on $(m')$ with entries from $J$. The letter $t_{i,j} \in J$ inscribed on the $i$th row and the $j$th column from the top-left corner should satisfy the following conditions for any adjacent pair:

$$t_{i,j} \leq t_{i,j+1} \quad \text{and} \quad (t_{i,j}, t_{i,j+1}) \neq (0, 0),$$

$$t_{i,j} < t_{i+1,j} \quad \text{or} \quad (t_{i,j}, t_{i+1,j}) = (0, 0). \quad (7.10)$$

Given a tableau $T = (t_{i,j}) \in \text{Tab}(B_r, m')$ we set

$$T_a = \prod_{j=1}^m \prod_{i=1}^a z_{t_{i,j}}(u + a - m + 2i + 2j). \quad (7.11)$$

This is an analog of the $A_r$ case (7.3).

**Theorem 7.2** ([68, 137]). The $q$-character $T^{(a)}_{i,m}(u) = \chi_q(W^{(a)}_{i,m}(u))$ is given by

$$T^{(a)}_{i,m}(u) = \sum_{T \in \text{Tab}(B_r,m')} T_a \quad (1 \leq a \leq r). \quad (7.12)$$

Recall that $t_0$ (2.1) is 1 except $t_0 = 2$ for $B_r$. Formula (7.12) is related to (6.21) in a parallel way with the $A_r$ case explained in the previous subsection. A similar result is available for the remaining case $T^{(r)}_{2m+1}(u)$ based on (6.22) [137]. Theorem 7.2 follows by combining the facts that the rhs and the $T^{(r)}_{2m+1}(u)$ satisfy the $T$-system [137], $q$-characters also satisfy the $T$-system [68] and the $T^{(a)}_{m}(u)$ is uniquely determined by the $T$-system and $T^{(a)}_{m}(u) \quad (a \in I)$. See also [140].

Here we only give the formula for $T^{(r)}_{1}(u)$. It is known that the $U_q(B^{(1)}_r)$-module $W^{(r)}_1(u)$ is isomorphic as a $U_q(B_r)$-module to the spin representation of the latter. Its weights are multiplicity-free and naturally labeled with the arrays $(\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r$. Accordingly we introduce

$$\rho_a = 2(\sigma_1 + \cdots + \sigma_{a-1}) + \frac{\sigma_a - \sigma_{a+1}}{t_a}, \quad \sigma_{r+1} = -\sigma_r. \quad (7.14)$$

Then we have

$$T^{(r)}_1(u) = \sum_{\sigma_1, \ldots, \sigma_r \in \{\pm 1\}} (\sigma_1, \ldots, \sigma_r)_u. \quad (7.15)$$

For $r = 2$, $T^{(2)}_1(u) = \chi_q(W^{(2)}_1(u))$ has been written in example 4.5.
7.3. Type $C_r$

Let us introduce the index set and a total order on it as

$$J = \{1, 2, \ldots, r, \bar{r}, \ldots, \bar{2}, \bar{1}\}, \quad 1 \prec \cdots \prec r \prec \bar{r} \prec \cdots \prec \bar{1}. \quad (7.16)$$

For $1 \leq a \leq r$ we set

$$z_a(u) = Y_{a, q^{a+1}}^{-1} Y_{a-1, q^{a+1}}^{-1},$$

$$z_{\bar{a}}(u) = Y_{a-1, q^{a+1}}^{-1} Y_{a, q^{a+1}}^{-1}, \quad (7.17)$$

where $Y_{0, q^1} = 1$. Here we present the tableau sum formulas for $T_{(1)}(u)$ and $T_{(a)}(u)$. Consider the Young diagram $(m)$ with length $m$ one row shape. Let $\text{Tab}(C_r, (m))$ be the set of tableaux on it with entries from $J$ having the following form:

$$\begin{matrix}
    i_1 & \cdots & i_k \\
    \bar{r} & \cdots & \bar{r} & r_j & \cdots & f_j \\
    1 \leq i_1 \leq \cdots \leq i_k \leq r \leq \bar{r}_j \leq \cdots \leq \bar{r}_1 \leq T.
\end{matrix} \quad (7.18)$$

Here $k$, $l$ and $n$ are any nonnegative integers satisfying $k + 2n + l = m$. Let those tableaux be denoted simply by the array of entries as $(i_1, \ldots, f_j) \in J^m$. Then we have

$$T_{(1)}(u) = \sum_{(i_1, \ldots, i_k) \in \text{Tab}(C_r, (m))} \prod_{k=1}^{m} z_{i_k} \left( u + \frac{2k - m - 1}{2} \right). \quad (7.19)$$

Consider the Young diagram $(1^n)$ with length $a$ one column shape. Let $\text{Tab}(C_r, (1^n))$ be the set of tableaux on it with entries from $J$. Let the $k$th row from the top should satisfy the conditions

$$i_1 < \cdots < i_k, \quad (7.20)$$

$$r + k - l \geq c \text{ for any } k, l, c \text{ such that } i_k = c, i_l = \bar{r}.$$  

Denote such a tableau by the array $(i_1, \ldots, i_k) \in J^n$. Then we have

$$T_{(a)}(u) = \sum_{(i_1, \ldots, i_k) \in \text{Tab}(C_r, (1^n))} \prod_{k=1}^{a} z_{i_k} \left( u + \frac{a + 1 - 2k}{2} \right) \quad (1 \leq a \leq r). \quad (7.21)$$

We note that $T_{(1)}(u)$ and $T_{(a)}(u)$ are the simplest cases in that the tableau rules can actually be described just by arrays without introducing a tableau.

7.4. Type $D_r$

Here we treat $T_{(1)}(u)$ and the fundamental $q$-characters $T_{(a)}(u)$. Let us introduce the index set and a partial order on it as

$$J = \{1, 2, \ldots, r, \bar{r}, \ldots, \bar{2}, \bar{1}\}, \quad 1 \prec \cdots \prec r \prec \bar{r} \prec \cdots \prec \bar{1}, \quad (7.22)$$

where no order is assumed between $r$ and $\bar{r}$. For $i \in J$, define $z_i(u)$ by

$$z_a(u) = Y_{a, q^{a+1}}^{-1} Y_{a-1, q^{a+1}}^{-1} \quad (1 \leq a \leq r - 2),$$

$$z_{r-1}(u) = Y_{r-1, q^{a+1}}^{-1} Y_{r-2, q^{a+1}}^{-1} Y_{r-1, q^{a+1}}^{-1},$$

$$z_r(u) = Y_{r, q^{a+1}}^{-1} Y_{r-1, q^{a+1}}^{-1},$$

$$z_{r-1}(u) = Y_{r-2, q^{a+1}}^{-1} Y_{r-1, q^{a+1}}^{-1} Y_{r, q^{a+1}}^{-1},$$

$$z_{\bar{a}}(u) = Y_{a-1, q^{a+1}}^{-1} Y_{a, q^{a+1}}^{-1} \quad (1 \leq a \leq r - 2). \quad (7.23)$$
where $Y_{0,q^k} = 1$.

Let $\text{Tab}(D_r, (m))$ be the set of one-row tableaux $(i_1, \ldots, i_m) \in J^m$ obeying the condition

$$i_1 \prec \cdots \prec i_m,$$

$r$ and $\mathcal{F}$ do not appear simultaneously. \hfill (7.24)

Then we have

$$T_{m}^{(1)}(u) = \sum_{(i_1, \ldots, i_m) \in \text{Tab}(D_r, (m))} \prod_{k=1}^{m} z_i(u + 2k - m - 1). \hfill (7.25)$$

For $1 \leq a \leq r - 2$, let $\text{Tab}(D_r, (1^a))$ be the set of one-column tableaux $(i_1, \ldots, i_a) \in J^a$ obeying the condition

$$i_k \prec i_{k+1} \text{ or } (i_k, i_{k+1}) = (r, r) \text{ or } (i_k, i_{k+1}) = (\mathcal{F}, r) \quad \text{ for } 1 \leq k \leq a - 1. \hfill (7.26)$$

Then we have

$$T_{a}^{(a)}(u) = \sum_{(i_1, \ldots, i_a) \in \text{Tab}(D_r, (1^a))} \prod_{k=1}^{a} z_i(u + a + 1 - 2k) \quad (1 \leq a \leq r - 2). \hfill (7.27)$$

It is known that the $U_q(D^{(1)})$-modules $W_1^{r-1}(u)$ and $W_1^{r}(u)$ are isomorphic as $U_q(D_r)$-modules to the spin representations of the latter. Their weights are multiplicity-free and naturally labeled with the arrays $(\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r$. Accordingly we introduce

$$(\sigma_1, \ldots, \sigma_r)_u = (Y_{r,q^{a-1}-1}, \rho_a) \prod_{a=1}^{r-1} (Y_{a,q^{a-1}-1}, \rho_a), \hfill (7.28)$$

$$\rho_a = \begin{cases} \sigma_1 + \cdots + \sigma_a - 1 \frac{\sigma_a - \sigma_{a+1}}{2} & 1 \leq a \leq r - 1, \\ \sigma_1 + \cdots + \sigma_r - 2 \frac{\sigma_r + \sigma_{a-1}}{2} & a = r. \end{cases} \hfill (7.29)$$

It follows that

$$(\sigma_1, \ldots, \sigma_{r-1}, -\sigma_r)_u = (\sigma_1, \ldots, \sigma_r)_u |_{Y_{r,q^k} \leftrightarrow Y_{r-1,q^{k-1}}}. \hfill (7.30)$$

We have

$$T_{1}^{(r-1)}(u) = \sum_{(\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r} (\sigma_1, \ldots, \sigma_r)_u, \quad T_{1}^{(r)}(u) = \sum_{(\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r} (\sigma_1, \ldots, \sigma_r)_u. \hfill (7.31)$$

7.5. Bibliographical notes

Tableau sums in theorems 7.1 and 7.2 were respectively given in [59] and [137] in the context of analytic Bethe ansatz for more general skew-shape Young diagrams. A uniform proof of the equality between the Jacobi–Trudi-type determinant and the tableau sum is available in [141].

For type $A_r$, see also [142] for an account from the viewpoint of Macdonald’s ninth variation of Schur functions [143]. The tableau sums in sections 7.3 and 7.4 first appeared in the analytic Bethe ansatz [144]. The sums of the same structure are used in the deformed $W$-algebras [145]. Tableau constructions of higher $T^{(a)}(u)$ for $C_r$ and $D_r$, which are significantly more involved than $A_r$ and $B_r$, have been achieved in [140, 146]. In this section we have only treated the untwisted case $U_q(C_r)$. For tableau sum formulas for $T$-systems in the twisted case, see [12, 13] and references therein.
8. Analytic Bethe ansatz

Let $T_m^{(a)}(u)$ be the commuting transfer matrix of a solvable lattice model in the sense of section 3. There is an empirical method called the analytic Bethe ansatz to produce eigenvalues of $T_m^{(a)}(u)$ in many cases. Those eigenvalue formulas possess a specific 'dressed vacuum form' which necessarily satisfy the $T$-system in remark 2.7 with a nontrivial $g_m^{(a)}(u)$. Here we consider the Bethe equation and dressed vacuum forms for general $g$ and $T_m^{(a)}(u)$ and reformulate the conventional analytic Bethe ansatz via its connection with $q$-characters.

8.1. $A_1$ case

Consider the six-vertex model (3.1). Here we employ the normalization

\[
\begin{array}{ccccccccc}
1 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 + u & \quad & \quad & \quad & \quad & \quad & z^{1/2} & z & z^{-1/2}
\end{array}
\]

which is obtained by dividing (3.1) by $(zq)^{-1/2}(1-q)$ and setting $z = q^u$. For the definition of the symbol $[u]_q$, see (3.18). Let $T_1(u)$ be the transfer matrix (3.11) with $m = 1$ and $w_j = q^{v_j}$. Its eigenvalue (denoted by the same symbol) is given by

\[
T_1(u) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix},
\]

(8.2)

\[
\begin{align*}
\prod_{j=1}^N [u - v_j]_q & = \phi(u+2) \frac{Q(u-1)}{Q(u+1)}, \\
\prod_{j=1}^N [u]_q & = \phi(u) \frac{Q(u+3)}{Q(u+1)}.
\end{align*}
\]

(8.3)

Here $\phi(u) = \prod_{j=1}^N [u - v_j]_q^{1/2}$ and $Q(u) = Q(u) = \prod_{j=1}^N [u - u_j]_q^{1/2}$ with $u_1, \ldots, u_n$ determined from the Bethe equation

\[
\frac{\phi(u_j + 1)}{\phi(u_j - 1)} = \frac{Q(u_j + 2)}{Q(u_j - 2)} (1 \leq j \leq n).
\]

(8.4)

Here, $n$ is the number of down-spins preserved under $T_1(u)$. The factors $\phi(u+2)$ and $\phi(u)$ in (8.3) are called vacuum parts in the sense that they are already present in the vacuum sector $n = 0$ where $Q(u) = 1$. In fact, the vector $11 \ldots 1$ is obviously the unique eigenvector with the vacuum eigenvalue:

\[
\prod_{j=1}^N [u - v_j + 2]_q^{1/2} + \prod_{j=1}^N [u - v_j]_q^{1/2} = \phi(u + 2) + \phi(u).
\]

(8.5)

The factors involving $Q$-functions in (8.3) are called dress parts, and the eigenvalue formula of the form (8.2) and (8.3) is called a dressed vacuum form. The vacuum part is nonuniversal in that it is directly affected by the normalization of the Boltzmann weights (relevant $R$ matrix) and also depends on the quantum space data such as inhomogeneity $\{v_j\}$ entering $\phi(u)$. On the other hand, the dress part encodes the structure of the auxiliary space essentially as we will see below.

The dressed vacuum form has an apparent pole at $u = -1 + u_j$ because of $Q(u_j) = 0$. The Bethe equation (8.4) shows that it is actually spurious provided that $u_0$ is distinct from the other roots. This is compatible with the property that eigenvalues of the transfer matrix are regular functions of $u$ if the local Boltzmann weights are so.
The analytic Bethe ansatz is a hypothesis that one can reverse these arguments to reproduce the eigenvalue formula from its characteristic properties bypassing the construction of eigenvectors. One starts with the ansatz-dressed vacuum form with the prescribed vacuum part

\[ T_1(u) = \phi(u + 2) \frac{Q(u + a)}{Q(u + b)} + \phi(u) \frac{Q(u + c)}{Q(u + d)}. \]  

(8.6)

Then \( a, b, c, d \) are determined by demanding that the pole-freeness is formally guaranteed by the Bethe equation (8.4) which one somehow admits from the outset. In the present example, this certainly fixes \( a, b, c, d \) uniquely as in (8.3). Further supplementary conditions may also be taken into account such as the asymptotic behavior as \( |u| \to \infty \) and the symmetry under complex conjugation, etc. It is not known whether such a procedure indeed leads to the unique and correct eigenvalue formula in general. Instead, we propose in section 8.2 a constructive way of producing the dressed vacuum form for general \( U_q(\hat{g}) \) by utilizing \( q \)-characters.

In the remainder of this subsection, we illustrate the simplest solution of the \( T \)-system for \( A_1 \) in the dressed vacuum form. Although the result is obtainable by specializing the tableau sum formula (7.3), we re-derive it here for later convenience. For simplicity \( T_m(u) \) will be denoted by \( T_m(u) \). Then the product of (8.2) is written as

\[ T_1(u) = \sum_{1 \leq i_1 \leq \cdots \leq i_{m+1}} \frac{1}{i_1 u_{-m+1} - 1} \prod_{j=1}^{m+1} \frac{1}{i_j u_{-m+1} + 1}. \]  

(8.7)

By (8.3), the last term becomes \( \phi(u + 1)\phi(u + 3) \), which is independent of \( Q(u) \). Identifying the other three terms with \( T_2(u) \), one has

\[ T_1(u) = \sum_{1 \leq i_1 \leq \cdots \leq i_{m+1}} \frac{1}{i_1 u_{-m+1} - 1} \prod_{j=1}^{m+1} \frac{1}{i_j u_{-m+1} + 1}. \]  

(8.8)

(8.9)

Explicitly, (8.7) reads as

\[ T_m(u) = \sum_{k=0}^{m} \frac{Q(u - m)Q(u + m + 1 - 2k)}{Q(u + m - 2j)Q(u + m + 2 - 2j)} \phi(u + m + 1 - 2k) \phi(u + m - 2j) \phi(u + 2k + 2). \]  

(8.10)

The summands in (8.7) are naturally labeled with the semistandard tableaux of length \( m \) row shape \((m)\) on numbers \( \{1, 2\} \). Note that

\[ g_m(u - 1)g_m(u + 1) = g_{m-1}(u)g_{m+1}(u) \]  

(8.11)

is satisfied with \( g_0(u) = 1 \). Although the explicit form (8.10) is not particularly more illuminating than (8.7), one can easily check that it is formally pole-free in the same manner.
as before thanks to the Bethe equation (8.4). Another way of seeing this is of course by the Jacobi–Trudi-type formula (6.1) with \( r = 1 \) modified as \( T_j^{(1)}(u) = g_j(u) \), e.g.

\[
T_j(u) = \begin{vmatrix}
T_1(u - 2) & g_1(u - 1) & 0 \\
1 & T_1(u) & g_1(u + 1) \\
0 & 1 & T_1(u + 2)
\end{vmatrix}.
\]

Thus, the pole-freeness of \( T_m(u) \) is an obvious corollary of that for \( T_1(u) \).

### 8.2. Dressed vacuum form and \( q \)-characters

The analytic Bethe ansatz is extended to the general \( U_q(\hat{\mathfrak{g}}) \) and further sharpened by a connection with the theory of \( q \)-characters. First we make a motive observation on the simplest example. Recall the \( q \)-character of \( W^{(1)}(u) \), the 'spin-\( \frac{1}{2} \) representation' of \( U_q(A_1^{(1)}) \), in example 4.4:

\[
\chi_q(W^{(1)}(u)) = Y_z + Y^{-1}_{zq^2} \quad (z = q^u). \tag{8.12}
\]

On the other hand, the dressed vacuum form (8.2) and (8.3) of the six-vertex model transfer matrix reads

\[
T^{(1)}_1(u) = \phi(u + 2) \frac{Q(u - 1)}{Q(u + 1)} + \phi(u) \frac{Q(u + 3)}{Q(u + 1)}. \tag{8.13}
\]

Upon substitution

\[
Y_{q^v} \rightarrow \frac{\eta(u - 1) Q(u - 1)}{\eta(u + 1) Q(u + 1)},
\]

the \( q \)-character (8.12) becomes

\[
\frac{\eta(u - 1) Q(u - 1)}{\eta(u + 1) Q(u + 1)} + \frac{\eta(u + 3) Q(u + 3)}{\eta(u + 1) Q(u + 1)}.
\]

Thus the above substitution with the overall renormalization

\[
\phi(u + 2) \frac{\eta(u + 1)}{\eta(u - 1)} \chi_q(W^{(1)}(u)) = \phi(u + 2) \frac{Q(u - 1)}{Q(u + 1)} + \phi(u + 2) \frac{\eta(u + 3) Q(u + 3)}{\eta(u - 1) Q(u + 1)}
\]

reproduces the dressed vacuum form (8.13) if \( \eta(u) \) is assumed to obey the difference equation

\[
\frac{\phi(u + 1)}{\phi(u - 1)} = \frac{\eta(u - 2)}{\eta(u + 2)}. \tag{8.14}
\]

Note that this equation has the form of the Bethe equation (8.4):

\[
\frac{\phi(u_j + 1)}{\phi(u_j - 1)} = \frac{Q(u_j + 2)}{Q(u_j - 2)}
\]

without the sign factor, and \( Q \) and \( u_j \) being replaced by \( \eta^{-1} \) and \( u \), respectively. The same feature will be adopted in (8.19). The connection of (8.12) and (8.13) originates in the fact that the former is the \( q \)-character of \( W^{(1)}(u) \) which is the auxiliary space of the transfer matrix relevant to the latter.

Now we generalize these observations to \( U_q(\hat{\mathfrak{g}}) \). Consider the trigonometric vertex model associated with \( U_q(\hat{\mathfrak{g}}) \) under the periodic boundary condition. Let \( T_m^{(a)}(u) \) be the transfer matrix (3.44) with the auxiliary space \( W^{(a)}(u) \) and the quantum space \( W_{s_1}^{(r_1)}(v_1) \otimes \cdots \otimes W_{s_N}^{(r_N)}(v_N) \):

\[
T_m^{(a)}(u) = \mathbb{Tr}_{W_m^{(a)}(u)}(R_{0,N}^{(a,m;r_N,s_N)}(z) | W_N) \cdots R_{0,1}^{(a,m;r_1,s_1)}(z) | W_1) \\
\in \text{End}(W_{s_1}^{(r_1)}(v_1) \otimes \cdots \otimes W_{s_N}^{(r_N)}(v_N)). \tag{8.15}
\]
where \( z = q^{u_0}, \) \( w_i = q^{v_i}. \) Due to the Yang–Baxter equation, they are commutative, i.e. \( [T_m^{(a)}(u), T_n^{(b)}(v)] = 0. \) The problem is to find their joint spectrum.

Let us construct a relevant dressed vacuum form \( \Lambda_m^{(a)}(u) \) for \( T_m^{(a)}(u). \) In the following, a simple identity

\[
\Lambda_{a,z}|_{Y_{c,q} \to Y_{c,q} \chi_{q}} = \prod_{b=1}^{r} f_b(u - (\alpha_a|\alpha_b)) \quad (z = q^{u_0}) \quad (8.16)
\]

for any functions \( f_1, \ldots, f_r \) will be useful. See (2.1) and (4.25) for the definitions of \( t_b, t \) and \( A_{a,z}. \) First we introduce an ‘unnormalized’ dressed vacuum form:

\[
\tilde{\Lambda}_m^{(a)}(u) = \chi_q(W_m^{(a)}(u)) \quad \text{with substitution}\quad Y_{c,q} \to \eta_c(v - \frac{1}{z}) Q_c(v - \frac{1}{z}) \quad \eta_c(v + \frac{1}{z}) Q_c(v + \frac{1}{z}).
\]

Let \( \tilde{\Lambda}_c,q^\alpha \) be the result of the same substitution into \( \Lambda_c,q^\alpha. \) By the definition we have

\[
\tilde{\Lambda}_m^{(a)}(u) = \eta_a(u - \frac{m}{z}) Q_a(u - \frac{m}{z}) \eta_a(u + \frac{m}{z}) Q_a(u + \frac{m}{z}) \left( 1 + \sum_{c,v} \text{monomial in } \tilde{\Lambda}_c^{-1},q^\alpha \right).
\]

Here the factor \( \eta_a Q_a / (\eta_a Q_a) \) is the top term specified by (4.21) and (8.17). The appearance of \( \tilde{\Lambda}_c,q^\alpha \) is due to theorem 4.6 (1). As for the functions \( \eta_1, \ldots, \eta_r, \) we postulate, as the generalization of (8.14), the following difference equation:

\[
\prod_{k=1}^{N} \frac{[u - v_k + \frac{1}{z}]}{[u - v_k - \frac{1}{z}]} \eta_a(u - (\alpha_a|\alpha_v)) = \prod_{b=1}^{r} \eta_b(u + (\alpha_a|\alpha_v)) \quad (1 \leq a \leq r),
\]

where \([u]_p \) is defined in (3.18). Then using (8.16) and (8.19) we find

\[
\tilde{\Lambda}_m^{(a)}(u) = \prod_{b=1}^{r} \eta_b(u - (\alpha_a|\alpha_v)) Q_b(u - (\alpha_a|\alpha_v)) \eta_b(u + (\alpha_a|\alpha_v)) Q_b(u + (\alpha_a|\alpha_v))
\]

\[
= \prod_{k=1}^{N} \frac{[u - v_k + \frac{1}{z}]}{[u - v_k - \frac{1}{z}]} \eta_a(u - (\alpha_a|\alpha_v)) Q_a(u - (\alpha_a|\alpha_v)) \eta_a(u + (\alpha_a|\alpha_v)) Q_a(u + (\alpha_a|\alpha_v)) \left( 1 + \sum_{c,v} \text{monomial in } \tilde{\Lambda}_c^{-1},q^\alpha \right).
\]

Next we adjust the overall normalization. Consider the \( R \) matrix on \( W_m^{(a)}(u) \otimes W_n^{(b)}(v) \) and write its unique diagonal matrix element between the tensor product of the highest weight vectors as \( \phi_m^{(a,b)}(u - v) \). Namely,

\[
\phi_m^{(a,b)}(u - v) = \text{Boltzmann weight of the vertex } m_{\omega_d} \left\{ \begin{array}{c} s_{\omega_d} \\ s_{\omega_d} \end{array} \right. m_{\omega_d}.
\]

Now we define the normalized dressed vacuum form by

\[
\Lambda_m^{(a)}(u) = \left( \prod_{k=1}^{N} \phi_m^{(a,c)}(u - v_k) \right) \frac{\eta_a(u + \frac{m}{z})}{\eta_a(u - \frac{m}{z})} \tilde{\Lambda}_m^{(a)}(u)
\]

\[
= \left( \prod_{k=1}^{N} \phi_m^{(a,c)}(u - v_k) \right) \frac{Q_a(u + \frac{m}{z})}{Q_a(u - \frac{m}{z})} \left( 1 + \sum_{c,v} \text{monomial in } \tilde{\Lambda}_c^{-1},q^\alpha \right).
\]

Besides the (in principle) known Boltzmann weights \( \phi_m^{(a,b)} \), this only contains the \( Q \)-functions \( Q_1, \ldots, Q_r \) and the l.h.s of (8.19).
Recall that the transfer matrices preserve the subspaces (sectors) of the quantum space specified by the weight. Let us parameterize the weight by the nonnegative integers \( n_1, \ldots, n_r \), as
\[
\sum_{k=1}^{N} s_k \omega_k - \sum_{a=1}^{r} n_a \alpha_a, 
\]  
(8.23)
where \( \omega_1, \ldots, \omega_r \) denote the fundamental weights of \( g (2.2) \). Given \( n_a \), we set
\[
Q_a(u) = \prod_{j=1}^{n_a} \left[ u - u_j^{(a)} \right]^{q_z/2}, 
\]  
(8.24)
by introducing the unknowns \( \{ u_j^{(a)} | 1 \leq a \leq r, 1 \leq j \leq n_a \} \).

**Conjecture 8.1.** Let \( T_m^{(a)}(u) \) (8.15) be the transfer matrix normalized as (8.21). Then its eigenvalues in sector (8.23) are given by the dressed vacuum form \( \Lambda_m^{(a)}(u) \) (8.22), (8.24) with the numbers \( \{ u_j^{(a)} | 1 \leq a \leq r, 1 \leq j \leq n_a \} \) satisfying the Bethe equation:
\[
\prod_{k=1}^{N} \frac{[u_j^{(a)} - v_k + \frac{\omega}{t_k}]^{q_z/2}}{[u_j^{(a)} - v_k - \frac{\omega}{t_k}]^{q_z/2}} = - \prod_{b=1}^{r} \frac{Q_b(u_j^{(a)} + (\alpha_a|\alpha_b))}{Q_b(u_j^{(a)} - (\alpha_a|\alpha_b))}. 
\]  
(8.25)

Practically, the results in section 7 serve as a large input to prescription (8.17) to produce \( \Lambda_m^{(a)}(u) \). The functions \( Q_a(u) \) are called the (generalized) Baxter \( Q \)-functions. In view of theorem 4.6 (2), we expect that their zeros, if in a generic position, do not cause a pole in \( \Lambda_m^{(a)}(u) \) due to the Bethe equation.

Let \( P_a(\zeta) \) be the product of the \( a \)-th Drinfeld polynomial (4.8) for each component in the quantum space \( W_{s_1/(v_1)} \otimes \cdots \otimes W_{s_N/(v_N)} \):
\[
P_a(\zeta) = \prod_{i=1}^{s_a} \prod_{r_i=d}^{r_a} \left( 1 - \zeta q^{(u_i + u_i + 1 - 2i)/2} \right), \quad \deg P_a = \sum_{k=1}^{N} s_k. 
\]  
(8.26)
We remark that the lhs of (8.19) is expressed as
\[
\prod_{k=1}^{N} \frac{[u - v_k + \frac{\omega}{t_k}]^{q_z/2}}{[u - v_k - \frac{\omega}{t_k}]^{q_z/2}} = q_a \deg P_a \frac{T_m^{(a)}(\zeta q_a)}{P_a(\zeta q_a)}, \quad (\zeta = q^{-u}), 
\]  
(8.27)
which further becomes the lhs of the Bethe equation (8.25) by the specialization \( u = u_j^{(a)} \). This has formally the same form as (4.7). Note however that the quantum space \( W_{s_1/(v_1)} \otimes \cdots \otimes W_{s_N/(v_N)} \) under consideration is not necessarily irreducible in general, and the above \( P_a(\zeta) \) is the \( a \)-th Drinfeld polynomial of its irreducible quotient containing the tensor product of the highest weight vectors.

By construction (8.17) and theorem 4.8, the unnormalized dressed vacuum form \( \tilde{\Lambda}_m^{(a)}(u) \) satisfies the unrestricted \( T \)-system for \( g \). It follows that the normalized one \( T_m^{(a)}(u) = \Lambda_m^{(a)}(u) \) (8.22) satisfies the modified \( T \)-system containing an extra factor \( g_m^{(a)}(u) \) as (2.22):
\[
T_m^{(a)} \left( u - \frac{1}{l_{a}} \right) T_m^{(a)} \left( u + \frac{1}{l_{a}} \right) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + g_m^{(a)}(u) M_m^{(a)}(u), 
\]
where the original \( T \)-system corresponds to \( g_m^{(a)}(u) = 1 \) as in (2.18). The scalar factor \( g_m^{(a)}(u) \) has the following properties.
(i) Apart from \((a, m, u)\), it only depends on the quantum space data \(W_{s_i}^{(r_1)}(v_1) \otimes \cdots \otimes W_{s_N}^{(r_N)}(v_N)\).

(ii) It satisfies relation (2.23):

\[
g^{(a)}_m(u - \frac{1}{t_a}) g^{(a)}_m(u + \frac{1}{t_a}) = g^{(a)}_{m-1}(u) g^{(a)}_{m+1}(u).
\]

In fact this has been encountered for \(g = A_1\) in (8.11). To derive these properties, note that the fusion construction implies that the diagonal element of the trigonometric level \(J. Phys. A: Math. Theor.\) data only, and so does the contribution from \(\text{Remark 8.2.}\) Conjecturally, it is covered by the dressed vacuum form in remark 8.2 specialized to commutativity and the RSOS model. (ii) The integers \((a, m, u)\) of the transfer matrix (8.21) is factorized as \(\phi_{m,s}^{(a,b)}(u) = \prod_{i=1}^m \phi_{1,s}^{(a,b)}(u + (m + 1 - 2i)/t_a)\). Thus, the first relation in (8.22) is written as

\[
\tilde{\Lambda}_m^{(a)}(u) = \Lambda_m^{(a)}(u) \prod_{i=1}^m \gamma_a(u + \frac{m + 1 - 2i}{t_a}), \quad (8.28)
\]

\[
\gamma_a(u) = \frac{n_a(u - \frac{1}{t_a})}{n_a(u + \frac{1}{t_a})} \prod_{k=1}^N \phi_{1,tc}^{(a,tc)}(u - v_k)^{-1}. \quad (8.29)
\]

In view of (8.28), replace \(T_m^{(a)}(u)\) in the original \(T\)-system with \(T_m^{(a)}(u) \prod_{i=1}^m \gamma_a(u + (m + 1 - 2i)/t_a)\). After removing the common factor, the result is indeed reduced to the form (2.22) with

\[
g_m^{(a)}(u) = \prod_{i=1}^m g_1^{(a)}(u + \frac{m + 1 - 2i}{t_a}), \quad (8.30)
\]

\[
g_1^{(a)}(u) = A^{-1}_{m,z} \gamma_{c,z} \gamma_{c,(u)} \quad (z = q^u). \quad (8.31)
\]

The property (ii) directly follows from (8.30) without using the concrete form of \(g_1^{(a)}(u)\). The property (i) is essentially due to the remark after (8.22). In fact, it is attributed to \(g_1^{(a)}(u)\) (8.31). With regard to \(\gamma_c(u)\) therein, \(\phi_{1,tc}^{(a,tc)}(u - v_k)\) in (8.29) depends on the quantum space data only, and so does the contribution from \(n_a\) because of (8.16) and (8.19).

**Remark 8.2.** The transfer matrix (8.15) can be generalized by the ‘magnetic field’ as \(T_m^{(a)}(u) = \text{Tr}_{W_{s_i}^{(r_1)}(v_1) \otimes \cdots \otimes W_{s_N}^{(r_N)}(v_N)}(R_{0,N}^{(a)}(z/w_1) \cdots R_{0,1}^{(a)}(z/w_1))\) without spoiling the commutativity and the \(T\)-system. Here \(\text{Tr}\) is any element in the Cartan subalgebra of \(U_q(g)\) acting on the auxiliary space. The dressed vacuum form for such \(T_m^{(a)}(u)\) is obtained by modifying substitution (8.17) into \(Y_{c,q}^{(a)} \rightarrow e^{\omega_{c,z}}(h)\) \(n_{c,z}(v_{c,z}^{(a)}) \rightarrow e^{\omega_{c,z}}(h)\) \(n_{c,z}(v_{c,z}^{(a)})\). Accordingly \(\tilde{\Lambda}_m^{(a)}(u)\) (8.20) and the lhs of the Bethe equation (8.25) get multiplied by the extra factor \(e^{\omega_{c,z}}(h)\).

**8.3. RSOS models**

We consider the spectrum of the transfer matrix \(T_m^{(a)}(u)\) \((1 \leq m \leq t_a)\) (3.50) for the trigonometric level \(\ell\) RSOS models sketched in section 3.7. \((T_m^{(a)}(u)\) corresponds to a frozen model.) Conjecturally, it is covered by the dressed vacuum form in remark 8.2 specialized along (i)–(iii) in what follows.

(i) The parameter \(q\) entering through \([u]_{q^{1/2}}\) is set as \(q = \exp \left( \frac{\sqrt{-1}}{\pi(s+1)} \right)\), where \(h^\ast\) is the dual Coxeter number of \(\gamma\) (2.3).

(ii) The integers \(n_1, \ldots, n_r\) entering (8.24) are fixed by demanding that (8.23) be 0, which is possible thanks to (3.51).
(iii) The magnetic field is taken so that \( \omega_c(\mathcal{H}) = \frac{2\pi \sqrt{1 - (\omega_c|/\Lambda_1 + \rho)}}{\ell h} \), where \( \rho = \sum_{\alpha \in I} \omega_\alpha \) and \( /\Lambda_1 \) is an element from \( P_\ell (3.45) \).

Introduce the specialized \( q \)-character \( Q_m^{(\alpha)}(\Lambda) := \chi_q(W_{\alpha,\ell}^{(\alpha)}(u))|_{Y_{c,q} \to e^{\omega_c(\mathcal{H})}} \), where \( \Lambda \)-dependence enters through the above \( \mathcal{H} \). Then according to the conjecture in [1, (A.8)–(A.9)], the relation \( \prod_{b \in I} Q_m^{(b)}(\Lambda) = 1 \) holds. The quantity \( Q_m^{(\alpha)} \) in section 14.6 is equal to \( Q_m^{(\alpha)}(0) \) in the notation here. The above relation is a generalization of \( Q_m^{(\alpha)}(0) = 1 \) in section 14.6.

8.4. Bibliographical notes

The analytic Bethe ansatz was proposed in [54] by extracting the idea from Baxter’s solution of the eight-vertex model [52]. It was applied systematically in [55, 137, 144] to a wide class of solvable vertex models. Formulation of the Bethe equation by the root system goes back, for instance, to [55, 147]. A relation between dressed vacuum forms and \( q \)-characters similar to section 8.2 has also been argued in [70, section 6].

9. Wronskian-type (Casoratian) formula

Here we present the solution of the \( T \)-system for \( A_r \) and \( C_r \) in terms of Casoratian (difference analog of Wronskian). It is most naturally done by introducing a difference analog of \( L \)-operators in soliton theory. It also provides a Casoratian interpretation and generalization of the Baxter \( Q \)-functions. Our description is along the context of \( q \)-characters; hence, the identification of the variables

\[
Y_{a,q} = \frac{Q_a(u - \frac{1}{q})}{Q_a(u + \frac{1}{q})}
\]  

(9.1)

is assumed. See (8.17). \( t, t_0 \) are defined in (2.1). Resulting formulas can suitably be modified to fit transfer matrices with specific normalizations according to the argument in section 8.2. We will also give analogous \( L \)-operators for \( B_r, D_r \) and \( sl(r|s) \).

9.1. Difference \( L \) operators

We treat the \( A_r \) case first as an illustration. Let \( D = e^{2\lambda h} \) be the shift operator \( D f(u) = f(u + 2). \) Using \( z_a(u) \) (7.6), we introduce the difference \( L \)-operator:

\[
L(u) = (1 - z_{r+1}(u)D) \cdots (1 - z_1(u)D)(1 - z_1(u)D).
\]  

(9.2)

Expanding the product, one identifies the coefficients with the \( m = 1 \) case of (7.4) to find

\[
L(u) = \sum_{a=0}^{r+1} (-1)^a T_1^{(a)}(u + a - 1) D^a.
\]  

(9.3)

where \( T_1^{(0)} = T_1^{(r+1)} = 1 \). Thus, \( L(u) \) is a generating function of the fundamental \( q \)-characters \( T_1^{(a)}(u) = \chi_q(W_1^{(a)}(u)) \).

Define the action of the screening operator \( S_\alpha \) (4.26) on difference operators by \( S_\alpha \cdot \sum f_i(u) D^i = \sum (S_\alpha \cdot f_i(u)) D^i \). Let us calculate \( S_\alpha \cdot L(u) \) by using the factorized form (9.2). According to the rule (4.26), \( S_\alpha \) acts nontrivially only on the variable \( Y_{a,q} \). From (7.6), it is contained only in \( z_a(u) \) and \( z_{a+1}(u) \). The action on this part is calculated as
$$S_a \cdot (1 - z_{a+1}(u)D)(1 - z_a(u)D)$$
$$= S_a \cdot (1 - Y_{a,q+\epsilon}^{-1} Y_{a+1,q+\epsilon} D - Y_{a,q+\epsilon}^{-1} Y_{a,q+\epsilon} Y_{a+1,q+\epsilon} D)$$
$$= S_{a,q+\epsilon} Y_{a,q+\epsilon}^{-1} Y_{a+1,q+\epsilon} D - S_{a,q+\epsilon} Y_{a,q+\epsilon} Y_{a+1,q+\epsilon} D = 0,$$
where the last equality is due to (4.27) and (4.25):
$$S_{a,q+\epsilon} = A_{a,q+\epsilon} S_{a,q+\epsilon-1} = Y_{a,q+\epsilon}^{-1} Y_{a,q+\epsilon} Y_{a+1,q+\epsilon}^{-1} S_{a,q+\epsilon-1}.$$  
In this way one obtains
$$S_a \cdot L(u) = 0 \quad (1 \leq a \leq r). \quad (9.4)$$
In view of (9.3), this offers a simple way of checking $T_1^{(a)}(u) \in \bigcap_{b=1}^r \text{Ker} S_b$ in agreement with theorem 4.6 (2). When $r = 1$, the change of variables from $[z_a(u)]$ to $[T_1^{(a)}(u)]$ is a difference analog of the Miura transformation $q = q(u) \rightarrow f = f(u) = q^2 - \partial u q$ by
$$(\partial u - q) (\partial u + q) = \partial u^2 - f.$$  
With regard to the inverse
$$L(u)^{-1} = (1 - z_1(u)D)^{-1} (1 - z_2(u)D)^{-1} \cdots (1 - z_{r+1}(u)D)^{-1},$$
the simple expansion formula
$$L(u)^{-1} = \sum_{m \geq 0} T_m^{(1)}(u + m - 1)D^m \quad (9.5)$$
holds due to (7.4), confirming similarly that $T_m^{(1)}(u) \in \bigcap_{b=1}^r \text{Ker} S_b$. The product of (9.3) and (9.5) leads to the two types of TT relations:
$$\sum_{0 \leq a \leq \min(r+1,m)} (-1)^a T_1^{(a)}(u + a) T_{m-a}^{(1)}(u + m + a) = \delta_{m0},$$
$$\sum_{0 \leq a \leq \min(r+1,m)} (-1)^a T_1^{(a)}(u + m - a) T_{m-a}^{(1)}(u - a) = \delta_{m0}$$
for $m \geq 0$.

9.2. Casoratian formula

Consider the linear difference equation on $w(u)$
$$L(u) w(u) = 0. \quad (9.6)$$
This is of the order $r + 1$ with respect to $D$. Letting $\{w_1(u), \ldots, w_{r+1}(u)\}$ be a basis of the solution, we denote the Casoratian by
$$C_\ell[i_1, \ldots, i_k] = \text{det} \begin{bmatrix} w_1(u + i_1) & \cdots & w_1(u + i_k) \\ \vdots & & \vdots \\ w_k(u + i_1) & \cdots & w_k(u + i_k) \end{bmatrix} \quad (9.7)$$
for $1 \leq k \leq r+1$. Thus, for example, $C_{w_2}[i_1, \ldots, i_k] = C_\ell[i_1 + 2, \ldots, i_k + 2]$. By using (9.3), the relations $L(u) w_k(u) = 0$ with $k = 1, \ldots, r + 1$ are expressed in the matrix form:
$$\begin{bmatrix} w_1(u) \\ w_2(u) \\ \vdots \\ w_{r+1}(u) \end{bmatrix} = \begin{bmatrix} w_1(u + 2) & w_1(u + 4) & \cdots & w_1(u + 2r + 2) \\ w_2(u + 2) & w_2(u + 4) & \cdots & w_2(u + 2r + 2) \\ \vdots & \vdots & & \vdots \\ w_{r+1}(u + 2) & w_{r+1}(u + 4) & \cdots & w_{r+1}(u + 2r + 2) \end{bmatrix} \begin{bmatrix} T_1^{(1)}(u) \\ (-1) T_1^{(2)}(u + 1) \\ \vdots \\ (-1)^r T_1^{(r+1)}(u + r) \end{bmatrix}.$$
where \( T^{(r+1)}_1(u) = 1 \) in our normalization here (\( q \)-characters) as noted under (7.6). By Cramer’s formula, we have

\[
T^{(a)}_1(u+a-1) = \frac{C_a[0, \ldots, 2a - 2, 2a + 2, \ldots, 2r + 2]}{C_a[2, \ldots, 2r + 2]} \quad (0 \leq a \leq r + 1),
\]

(9.8)

where \( \ldots \) signifies that the omitted arrays are consecutive with difference 2. The relation \( L(u)w_k(u) = 0 \) means that \( w_k(u + 2r + 2) = (-1)^j w_k(u) + \text{terms involving } w_k(u + 2), \ldots, w_k(u + 2r) \). It follows the periodicity

\[
C_a[0, 2, \ldots, 2r] = C_{a+2}[0, 2, \ldots, 2r].
\]

(9.9)

Its actual value becomes important in physical applications, and the resulting relation on \( C_a[0, 2, \ldots, 2r] \) is called the quantum Wronskian condition. See for example [148, 149].

The solution to the \( T \)-system for \( A_t \), that matches (9.8) is given by

\[
T^{(a)}_m(u+a+m-2) = \frac{C_a[0, \ldots, 2a - 2, 2a + 2m, \ldots, 2r + 2m]}{C_a[0, \ldots, 2r]} \quad (0 \leq a \leq r + 1).
\]

(9.10)

This satisfies the boundary conditions \( T^{(0)}_m(u) = T^{(a)}_0(u) = 1 \) and \( T^{(a)}_r(u) = 0 \). In fact, if (9.10) is substituted into (2.5), the denominator can be removed as an overall factor owing to (9.9). Then (2.5) is identified with the simplest Plücker relation

\[
\xi^{(a)}_m(u)\xi^{(a)}_m(u+2) - \xi^{(a+1)}_m(u)\xi^{(a)}_m(u+2) + \xi^{(a+1)}_m(u+2)\xi^{(a)}_m(u+2) = 0 \quad (9.11)
\]

among the determinant \( \xi^{(a)}(u) = C_a[0, \ldots, 2a - 2, 2a + 2m, \ldots, 2r + 2m] \).

The Casoratian formula (9.10) is a Yang–Baxterization (\( u \)-dependent generalization) of the Weyl character formula. To see this, recall the restriction map \( \text{res} \) (4.23). From (7.6) we have \( \text{res}(z_a(u)) = x_a \), where the latter is defined by \( x_a = y_a/y_{a-1} = e^{a_0-a_{a-1}} \) with \( a_0 = a_{r+1} \). We extend \( \text{res} \) naturally to the difference \( L \) operator and wave functions as

\[
\text{res} L(u) = (1 - x_{r+1}D) \cdots (1 - x_1D), \quad \text{res}(w_i(u)) = x_i^{-u/2}.
\]

(9.12)

The latter is certainly annihilated by the former. By using \( x_1 \cdots x_{r+1} = 1 \), it is straightforward to see that the restriction of (9.10) becomes

\[
\text{res} \left( \frac{C_a[0, \ldots, 2a - 2, 2a + 2m, \ldots, 2r + 2m]}{C_a[0, \ldots, 2r]} \right) = \frac{\det(x_{i,j}^{r+1-j})_{1 \leq i, j \leq r+1}}{\det(x_{i,j}^{r+1-j})_{1 \leq i, j \leq r+1}}.
\]

(9.13)

where \( (\lambda_j) \) corresponds to the \( a \times m \) rectangular Young diagram, namely \( \lambda_j = m \) if \( 1 \leq j \leq a \) and \( \lambda_j = 0 \) otherwise. The rhs is the Weyl character formula of the Schur function for \( (\lambda_j) \) as is well known.

The Casoratian formula here and the tableau sum formula (section 7.1) are connected by the following general fact.

**Proposition 9.1** ([142]). Let \( C_a[i_1, \ldots, i_k] \) be as in (9.7). \( (L(u)w_j(u) = 0 \) is not assumed.)

Given even integers \( 0 = i_0 < i_1 < \cdots < i_{N-1} \), let \( \mu = (\mu_j) \) be the Young diagram with depth less than \( N \) specified by \( \mu_j = \frac{i_{j+1} - i_{j-1}}{2} + j - N \). Take any \( d \geq \mu_1 \). Then

\[
\frac{C_a[0, i_1, i_2, \ldots, i_{N-1}]}{C_{a+2d}[0, 2, \ldots, 2N - 2]} = \sum_T \frac{\bar{x}_{T(\alpha, \beta)}(u + 2\alpha + 2\beta - 4)}{\prod_{(a, b) \in (d^N)/\mu} \bar{x}_{T(\alpha, \beta)}(u + 2\alpha + 2\beta)} - 2C_{a}[1,4,6, \ldots, 2d]\]

where \( \bar{x}_j(u) = \frac{C_a[0, \ldots, j - 2, 2j, \ldots, 2d]}{C_a[0, \ldots, j, 2, \ldots, 2d]} \) and the sum \( \sum_T \) extends over the semistandard tableaux on the skew Young diagram \( (d^N)/\mu \) [135] on letters \( \{1, \ldots, N\} \). \( T(\alpha, \beta) \) denotes the entry of \( T \) at the \( \alpha \)-th row and the \( \beta \)-th column from the bottom-left corner.
According to proposition 9.1, the rhs of (9.10) equals the sum over semistandard tableaux on $a \times m$ Young diagram on letters $\{1, \ldots, r+1\}$. The building block of the tableau variable $\bar{Y}_j(u)$ is the principal minors of the Casoratian (quantum Wronskian) $C_u[0, 2, \ldots, 2r]$. Combined with (9.6), they are identified with the Baxter $Q$-functions as we will see in the next subsection.

9.3. $Q$-functions

From the full $L$ operator (9.2), we extract the partial ones by

$$L_j(u) = (1 - z_j(u)D) \cdots (1 - z_2(u)D)(1 - z_1(u)D) \quad (1 \leq j \leq r + 1). \quad (9.14)$$

The original one corresponds to $L_{r+1}(u)$. By the definition we have

$$\text{Ker } L_1(u) \subset \text{Ker } L_2(u) \subset \cdots \subset \text{Ker } L_{r+1}(u). \quad (9.15)$$

Choose the basis of $\text{Ker } L_j(u)$ according to this flag structure as

$$\{w_1(u)\} \subset \{w_1(u), w_2(u)\} \subset \cdots \subset \{w_1(u), \ldots, w_{r+1}(u)\}. \quad (9.16)$$

As the simplest example, $w_1(u) \in \text{Ker } L_1(u)$ is the condition $0 = (1 - z_1(u)D)w_1(u)$. In view of (7.6) and (9.7), this is the $j = 1$ case of

$$(1 - Y_{j, q^{u+1}}, D)C_u[0, \ldots, 2j - 2] = 0 \quad (1 \leq j \leq r). \quad (9.17)$$

To derive this, note that a direct calculation using (7.6) leads to

$$L_j(u) = 1 + (-1)^j Y_{j, q^{u+1}}D^j + \text{terms involving } D, \ldots, D^{j-1}.$$

Therefore, $L_j(u)w_k(u) = 0$ ($1 \leq k \leq j$) implies

$$Y_{j, q^{u+1}}, w_k(u + 2j) = (-1)^{j-1}w_k(u) + \sum_{l=1}^{j-1} c_{j,l}(u)w_k(u + 2l),$$

where $c_{j,l}(u)$ is independent of $k$. The second term in (9.17) is equal to

$$Y_{j, q^{u+1}}C_u[2, \ldots, 2j - 2, 2j].$$

Applying the above relation to the last column of this, we find that the result is equal to $C_u[0, \ldots, 2j - 2]$, hence (9.17).

If we express the variable $Y_{\alpha, q^u}$ in $q$-characters in terms of $Q$-functions as in (9.1), the solution of the first order difference equation (9.17) is given by

$$C_u[0, \ldots, 2j - 2] = \sigma_j(u)Q_j(u + j - 2) \quad (1 \leq j \leq r), \quad (9.18)$$

where $\sigma_j(u)$ is any variable satisfying $\sigma_j(u + 2) = \sigma_j(u)$. In this way, the $Q$-functions are identified with the principal minors of the Casoratian $C_u[0, \ldots, 2r]$ made of the wavefunctions $[w_1(u)]$ especially chosen along the scheme (9.16). The simplest case $j = 1$ of (9.18) is $w_1(u) = \sigma_1(u)Q_1(u - 1)$. Thus, $L(u)w_1(u) = 0$ is rephrased as

$$\sum_{i=0}^{r+1} (-1)^iT^{(a)}_i(u + a)Q_1(u + 2a) = 0, \quad (9.19)$$

which is an example of TQ-relations.

9.4. Bäcklund transformations

Here we remove the boundary condition $T^{(a)}_0(u) = T^{(0)}_m(u) = 1$ and redefine $T^{(a)}_m(u)$ in (9.10) and $Q_j(u)$ in (9.18) as

$$T^{(a)}_m(u + a + m - 2) = C_u[0, \ldots, 2a - 2, 2a + 2m, \ldots, 2r + 2m], \quad (9.20)$$

...
These functions are special cases of more general ones:
\[ T_m^{(s,a)}(u+a+m-2) = \begin{vmatrix}
  w_1(u) & \cdots & w_1(u+2a-2) & w_1(u+2a+2m) & \cdots & w_1(u+2s+2m) \\
  \vdots & & \vdots & \vdots & & \vdots \\
  w_{s+1}(u) & \cdots & w_{s+1}(u+2a-2) & w_{s+1}(u+2a+2m) & \cdots & w_{s+1}(u+2s+2m)
\end{vmatrix}.
\]

\[ Q_{[i_1, \ldots, i_n]}(u+a-1) = \begin{vmatrix}
  w_{i_1}(u) & \cdots & w_{i_1}(u+2a-2) \\
  \vdots & & \vdots \\
  w_{i_n}(u) & \cdots & w_{i_n}(u+2a-2)
\end{vmatrix},
\]

where \( \cdots \) in determinants signify that \( u \) increases by 2. \( T_m^{(s,a)}(u) \) is defined for \( 0 \leq a \leq s+1, 0 \leq s \leq r \) and \( m \geq 0 \). The set \([i_1, \ldots, i_n]\) is any subset of \([1, \ldots, r+1]\). By the definition, \( T_m^{(0,0)}(u) = T_m(u) \) and \( Q_{[1, \ldots, a]}(u) = Q_a(u) \). These functions obey various relations as the consequence of identities among determinants. Let us mention a few of them that have analogy with soliton theory.

The symmetric group \( S_{a+1} \) acts on the basis \( w_1(u), \ldots, w_{a+1}(u) \) as their permutations keeping \( L(u) \) invariant. This can be viewed as Bäcklund transformations generating the functions \( Q_{[i_1, \ldots, i_n]} \) from \( Q_1, \ldots, Q_{r+1} \). Its generator, the transposition \( s_a \) of \( w_a(u) \) and \( w_{a+1}(u) \), acts trivially as \( s_a(Q_b) = Q_b \) for \( a > b \) and similarly as \( s_a(Q_b) = -Q_b \) for \( a < b \). The nontrivial case \( s_a(Q_a) = Q_{[1, \ldots, a-1, a+1]} \) satisfies the QQ relation:
\[ D(Q_a)s_a(Q_a) + s_a(D(Q_a))Q_{a+1} = 0, \]

where the first term denotes \( Q_{(a+2)}s_a(Q_a)(u) \) for instance. This is derived by applying the Jacobi identity (6.2) to the \( a, a+1 \) rows and \( 1, a+1 \) columns for the determinant of \( Q_{a+1} \).

With regard to \( T_m^{(r,a)}(u) \), it is the \( T \)-function for \( A_r \subset A_s \). Writing \( T_m^{(r,a)}(u) \) and \( T_m^{(r-1,a)}(u) \) simply as \( T_m^{(r)}(u) \) and \( T_m^{(r-1)}(u) \), respectively, one can derive
\[ T_m^{(r)}(u)T_m^{(r-1)}(u-1) = T_m^{(r-1)}(u-1)T_m^{(r)}(u) + T_m^{(r)}(u-1)T_m^{(r-1)}(u), \]
\[ T_{m+1}^{(r)}(u-1)T_m^{(r)}(u) = T_{m+1}^{(r)}(u)T_m^{(r)}(u) + T_{m+1}^{(r)}(u-1)T_m^{(r)}(u) \]
from the Plücker relation. This is a Bäcklund transformation between \( T \)-functions associated with \( A_r \) and \( A_{r-1} \). The \( T \)-system for \( T_m^{(a)}(u) \) arises as a compatibility of the two linear equations on \( T_m^{(r)}(u) \) [150]. For more examples, see [22, 23, 151–153] and references therein. It is an open problem to construct such a Lax representation of the \( T \)-system for general \( g \).

9.5. Type \( C_r \)

Let \( D \) be the difference operator \( Df(u) = f(u+1)D \). We use the variable \( z_a(u) (a \in J) \) (7.17) which are related to the \( Q \)-functions by (9.1). We also introduce the variables \( x_1(u), \ldots, x_{2r+2}(u) \) by
\[ x_a(u) = z_a(u), \quad x_{2r+3-a}(u) = z_{r+a}(u) \quad (1 \leq a \leq r), \]
\[ x_{r+1}(u) = -x_{r+2}(u) = \frac{Q_r(u + \frac{r+1}{2})}{Q_r(u + \frac{r-1}{2})}. \]

Note that \( x_{r+1}(u) \) and \( x_{r+2}(u) \) are not contained in \( \mathbb{Z}[T_{a,z}, \mathbb{Z} \cup \mathbb{Z}^+] \). With the notation
\[ \prod_{1 \leq i \leq k} X_i = X_1X_2 \cdots X_k, \quad \prod_{1 \leq i \leq k} X_i = X_1 \cdots X_kX_1, \]

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the difference $L$-operator is

$$L(u) = \prod_{1 \leq a \leq r} (1 - z_{\pi}(u) D) \cdot (1 - z_{\pi}(u) z_r(u + 1) D^2) \cdot \prod_{1 \leq a \leq r} (1 - z_{\alpha}(u) D). \quad (9.27)$$

One can easily check $S_{a} \cdot L(u) = 0$ as in type $A$. The middle quadratic operator can be factorized as

$$1 - Y_{r,q} Y_{r,q}^{-1} D^2 = 1 - \frac{Q_r(u + \frac{r+1}{2}) Q_r(u + \frac{r-1}{2})}{Q_r(u + \frac{r+1}{2}) Q_r(u + \frac{r-1}{2})} D^2$$

$$= (1 \pm x_{r+2}(u) D)(1 \pm x_{r+1}(u) D).$$

Thus, (9.27) is expressed as

$$L(u) = \prod_{1 \leq i \leq 2r+2} (1 - x_i(u) D), \quad (9.28)$$

which curiously resembles the $A_{2r+1}$ case rather than $A_{2r-1}$. The operator $L(u)$ generates each fundamental $q$-character ‘twice’.

**Theorem 9.2** ([139]).

$$L(u) = \sum_{a=0}^{r} (-1)^{i} T_{i}^{(a)} \left( u + \frac{a - 1}{2} \right) D^a - \sum_{a=r+2}^{2r+2} (-1)^{a} T_{r+2}^{(a)} \left( u + \frac{a - 1}{2} \right) D^a,$$

where $T_{1}^{(0)} = 1$.

From theorem 9.2 and (9.28), we obtain another tableau sum formula for the fundamental $q$-characters:

$$T_{i}^{(a)} \left( u + \frac{a - 1}{2} \right) = \sum_{1 \leq i_1 \leq \ldots \leq i_a \leq 2r+2} \prod_{k=1}^{a} x_i(u + a - k) \quad (1 \leq a \leq r). \quad (9.29)$$

Although this is formally the same form as the $A_{2r+1}$ case (7.4), the variable $x_{r+2}(u)$ (9.25) is ‘negative’ here. It is highly nontrivial that the cancellation due to the sign yields the previous formula (7.21) described by the rule (7.20), which constitutes a substantial part of the proof of theorem 9.2. On the other hand it is easy to see

$$L(u)^{-1} = \sum_{m \geq 0} T_{m}^{(1)} \left( u + \frac{m - 1}{2} \right) D^m \quad (9.30)$$

from (7.19), (7.18) and (9.27).

The rest of this subsection will be brief as the content is more or less parallel with the $A_{2r+1}$ case. We formally extend the fundamental $q$-characters $T_{i}^{(a)}(u)$ to $1 \leq a \leq 2r+2$ by

$$T_{i}^{(a)}(u) + T_{r+2}^{(a-2r-2)}(u) = 0 \quad (0 \leq a \leq 2r+2). \quad (9.31)$$

Then theorem 9.2 is rephrased as

$$L(u) = \sum_{a=0}^{2r+2} (-1)^{a} T_{i}^{(a)} \left( u + \frac{a - 1}{2} \right) D^a. \quad (9.32)$$

We consider the difference equation $L(u) w(u) = 0$ and a basis of the solution \{ $w_1(u), \ldots, w_{2r+2}(u)$ \}. With the same notation $C_{q}[i_1, \ldots, i_k]$ as (9.7), we have the Casoratian formula

$$T_{i}^{(a)} \left( u + \frac{a - 1}{2} \right) = \frac{C_{q}[0, \ldots, a-1, a+1, \ldots, 2r+2]}{C_{q}[1, \ldots, 2r+2]} \quad (0 \leq a \leq 2r+2), \quad (9.33)$$
where . . . signifies that the omitted arrays are consecutive with difference 1. The denominator possesses the periodicity
\[ C_n[0, 1, \ldots, 2r + 1] = -C_{n+1}[0, 1, \ldots, 2r + 1], \]
which is a \( C_r \) analog of the quantum Wronskian condition.

Set
\[ \xi_m^{(a)}(u) = C_m[0, \ldots, a - 1, a + m, \ldots, 2r + 1 + m], \]
\[ \xi(u) = C_m[0, 1, \ldots, 2r + 1]. \]

The solution of the unrestricted \( T \)-system for \( C_r \) that matches (9.33) is given by

**Theorem 9.3** ([139]). The following is a solution of the \( T \)-system for \( C_r \):
\[
\begin{align*}
T_m^{(a)}(u + a + m - \frac{3}{2}) &= (-1)^{m-\frac{1}{2}} \frac{\xi_m^{(a)}(u)}{\xi(u + 1)} (1 \leq a \leq r - 1), \\
T_m^{(r)}(u + r + 2m - \frac{3}{2}) &= \frac{\xi_m^{(r)}(u)}{\xi(u + 1)}, \\
T_m^{(r)}(u + r + 2m - \frac{3}{2}) &= \frac{\xi_m^{(r)}(u)}{\xi(u + 1)}, \\
T_m^{(r)}(u + r + 2m - \frac{3}{2}) &= \frac{\xi_m^{(r)}(u)}{\xi(u + 1)}. \\
\end{align*}
\]

As for the first three, there is an alternative expression derived by using the identity
\[ \xi_m^{(a)}(u) = (-1)^{a + m + r + 1} \xi_m^{(2r + 2 - a)}(u + a - r - 1). \]
See proposition 4.3 in [139] for details.

### 9.6. Types \( B_r \) and \( D_r \)

Here we only give the \( L \)-operators and their expansions. Let \( D \) be the difference operator \( Df(u) = f(u + 2)D \). We use the variables \( z(u) \) for \( B_r \) (7.9) and \( D_r \) (7.23) which are related to the \( Q \)-function by (9.1). The difference \( L \)-operators are
\[
\begin{align*}
B_r : \quad L(u) &= \prod_{1 \leq a \leq r} (1 - z_a(u)D) \cdot (1 + z_0(u)D)^{-1}. \\
D_r : \quad L(u) &= \prod_{1 \leq a \leq r} (1 - z_a(u)D) \cdot (1 - z_0(u)D). 
\end{align*}
\]

One can check \( S_L \cdot L(u) = 0 \) by expanding the middle factor into a power series in \( D \). Introduce the expansion coefficients of \( L(u) \) as
\[ L(u) = \sum_{a \geq 0} (-1)^a T^a(u + a - 1) D^a, \quad L(u)^{-1} = \sum_{m \geq 0} T_m(u + m - 1) D^m. \]

They are related to the previous tableau constructions as follows:
\[
\begin{align*}
T_m(u) &= T_m^{(1)}(u) \quad (7.12) \text{ for } B_r, \\
T_m(u) &= T_m^{(a)}(u) \quad (7.12) \text{ for } B_r, 1 \leq a \leq r \text{ and } (7.27) \text{ for } D_r, 1 \leq a \leq r - 2. \\
\end{align*}
\]

With the convention \( T^a(u) = 0 \) for \( a < 0 \), the coefficient \( T^a(u) \) beyond these upper bound is characterized by the following relations with the \( q \)-characters of spin representations:
\[
\begin{align*}
B_r : \quad T^a(u) + T^{h^r - a}(u) &= T^{(1)}_{(1)} \left( u + \frac{h^r}{2} - a \right) T^{(1)}_{(1)} \left( u - \frac{h^r}{2} + a \right), 
\end{align*}
\]
Here $a \in \mathbb{Z}$ is arbitrary and $\delta = 0$ if $a \equiv r \mod 2$ and $\delta = 1$ otherwise. $h^\vee$ is the dual Coxeter number (2.3), i.e. $h^\vee = 2r - 1$ for $B$, and $h^\vee = 2r - 2$ for $D_r$. In particular, one has $T^{r-1}(u) = T^{(r)}_1(u)T^{(r-1)}_1(u)$ for $D_r$.

9.7. Type $sl(r|s)$

There are two kinds of roots, odd and even, for the graded algebra $sl(r|s)$. The choice of simple roots is not unique. The most standard one is called distinguished, where all roots but $\alpha_r$ are even. Here we follow [19] and set $I = \{1, \ldots, r + s\} = I_1 \cup I_2$, $I_1 = \{1, 2, \ldots, r\}$, $I_2 = \{r + 1, r + 2, \ldots, r + s\}$, and assign the grading $p_a$ by $p_a = 1$ if $a \in I_1 (I_2)$. The Cartan matrix is expressed by the grading as

$$ (\alpha_k, \alpha_j) = (p_k + p_{k+1})\delta_{kj} - p_{k+1}\delta_{k+1,j} - p_k\delta_{k,j+1}. $$

Now the analog of (7.6) is

$$ z_a(u) = Y^{-p_a}_{a-1,q^a}Y_{a,q^a}^{p_a} \quad (a \in I), $$

where $s_a = \sum_{j=1}^n p_j$ and $Y_{0,q^a} = Y_{r+s,q^a} = 1$. Let $D$ be the difference operator $Df(u) = f(u+2)D$. Then the analog of (9.3) and (9.5) are given as

$$ (1 + z_{r+s}(u)D)^{-m}\cdots(1 + z_1(u)D)^{p_1} = \sum_{a=0}^{\infty} T^{(a)}_1(u + a - 1)D^a, $$

$$ (1 - z_1(u)D)^{-m}\cdots(1 - z_{r+s}(u)D)^{p_1} = \sum_{m=0}^{\infty} T^{(m)}_1(u + m - 1)D^m. $$

Example 9.4.

$sl(2|1)$, $p_1 = p_2 = -p_3 = 1$:

$$ T^{(1)}_1(u) = Y_{1,z} + Y_{1,z}^{-1}Y_{2,z} - Y_{2,z}, $$

$$ T^{(2)}_1(u) = Y_{2,z} - Y_{1,z}Y_{1,z}^{-1}Y_{2,z} + Y_{2,z}^{-1}Y_{2,z}, $$

$$ T^{(3)}_1(u) = -Y_{2,z}^{-1}Y_{2,z} + Y_{1,z}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}Y_{2,z}^{-1} - Y_{2,z}^{-1}Y_{2,z}Y_{2,z}^{-1}. $$

$sl(2|1)$, $p_1 = -p_2 = p_3 = 1$:

$$ T^{(1)}_1(u) = Y_{1,z} - Y_{1,z}Y_{2,z}^{-1} + Y_{2,z}^{-1}, $$

$$ T^{(2)}_1(u) = -Y_{1,z}Y_{1,z}^{-1}Y_{2,z} + Y_{1,z}Y_{2,z}^{-1} + Y_{1,z}Y_{1,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1} - Y_{1,z}Y_{2,z}^{-1}Y_{2,z} - Y_{1,z}Y_{1,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}, $$

$$ T^{(3)}_1(u) = Y_{1,z}Y_{1,z}^{-1}Y_{2,z}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}Y_{2,z}^{-1} - Y_{1,z}Y_{1,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}Y_{2,z}^{-1}, $$

For the formulas for the general case, see [152, 153].
9.8. Bibliographical notes

The Casoratian solution (9.10) for $A_r$ has been known in various contexts. For the $T$-system of transfer matrices, a slightly more general solution than (9.20) was given in equation (2.25) in [150] containing $2r + 2$ arbitrary functions. It does not satisfy the natural boundary condition $T^{(r)}_{ij}(u) = 0$ for fusion transfer matrices in general. As usual, such a ‘Dirichlet’ condition halves the arbitrary functions to $w_1(u), \ldots, w_{r+1}(u)$, which brings one back to (9.20). Casoratian solutions are also known for the restricted $T$-systems for $A_r$ [124] and $C_r$ [17].

The $L$-operator for type $A$ has been studied from the viewpoint of difference analog of Drinfeld–Sokolov reduction [154]. The concrete forms for type $BCD$ and their application to $q$-characters were given in [139]. Analogous difference $L$-operators for all the twisted cases except $E(6)$ have been constructed in [155]. Results (9.39) and (9.40) are taken from theorem 2.3 in [137] and proposition 2.3 in [138], respectively.

10. $T$-system in ODE

$T$-system appears also in the connection problem of 1D Schrödinger equation, which is a typical example of the ODE/IM correspondence. As a comprehensible review on the ODE/IM correspondence is already available in [149], we only discuss the issue briefly in view of the $T$-system. Wronskians appear naturally in the context of ODE. They will be shown to coincide with the analogous object, the Casoratian (9.7) in the difference equation in section 9.

10.1. Generalized Stokes multipliers—the second order case

As the simplest example, we consider the 1D Schrödinger equation on the real axis with a potential term:

$$\left(-\frac{d^2}{dx^2} + x^{2M}\right)\psi(x) = E\psi(x), \quad (10.1)$$

where $M \in \mathbb{Z}_{>0}$. The boundary condition $\psi(\pm\infty) = 0$ is imposed. We find it convenient to extend $x$ into the complex plane.\(^23\)

Since the Schrödinger equation has the irregular singularity at $\infty$, we expect a sudden change of $\psi(x)$ when crossing the border line of sectors defined below. This is called the Stokes phenomenon. The change is characterized by the Stokes multiplier $\tau_1$. Below we will introduce a set of generalized Stokes multipliers $\{\tau_j\}_{j=M}^{2M}$ and show that they satisfy the level $2M$ restricted $T$-system for $A_1$.

First, let $S_j$ be a sector in the complex plane defined by

$$S_j = \left\{ x \mid \arg x - \frac{j\pi}{M+1} < \frac{\pi}{2M+2} \right\}. \quad (10.1)$$

The sector $S_0$ thus includes the positive real axis. We then introduce a solution $\phi(x, E)$ to (10.1) which decays exponentially as $x$ tends to $\infty$ inside $S_0$ as

$$\phi(x, E) \sim \frac{x^{-M/2}}{\sqrt{2i}} \exp \left(-\frac{x^{M+1}}{M+1}\right), \quad x \in S_0. \quad (10.2)$$

This is referred to as the subdominant solution. There should be another solution to (10.1) which diverges exponentially in $S_0$ as $x$ tends to $\infty$. We call it dominant. It is also represented

\(^23\) For a general reference to ODE in the complex domain, we recommend [156].
by \( \phi \). To see this, note the invariance of (10.1) under the simultaneous transformations \( x \to q^{-1}x \) and \( E \to Eq^2 \), where \( q = \exp \left( \frac{\pi \text{i}u}{M+1} \right) \). We call this ‘discrete rotational symmetry’. We thus introduce \( y_j = q^{-j/2} \phi(q^{-j}x, q^2 E) \) so that \( y_0 = \phi \). The above observation states that any \( y_j \) is a solution to (10.1). Moreover, we can show that the pair \( (y_j, y_{j+1}) \) forms the fundamental system of solutions (FSS) in \( S_j \). This is easily seen by introducing the Wronskian matrix \( \Phi_j \) and the Wronskian \( W[y_j, y_{j+1}] \):

\[
\Phi_j = \begin{pmatrix} y_j & y_{j+1} \\ \partial y_j / \partial y_{j+1} & \partial y_j / \partial y_{j+1} \end{pmatrix}, \quad W[y_j, y_{j+1}] = \det \begin{pmatrix} y_j & y_{j+1} \\ \partial y_j / \partial y_{j+1} & \partial y_j / \partial y_{j+1} \end{pmatrix}.
\]

By using the asymptotic form (10.2), one can check \( W[y_j, y_{j+1}] = 1 \); hence, the pair \( (y_j, y_{j+1}) \) is independent. Thus, \( y_0 \) (equals to \( \phi \)) is the subdominant solution in \( S_0 \), while \( y_1 \) is a dominant one. We are interested in the relation among FSS in different sectors. Let us start from \( S_0 \) and \( S_1 \). Obviously \( y_j \) must be represented by the linear combination of \( y_0 \) and \( y_2 = a_0 y_0 + a_1 y_1 \). As \( W[y_j, y_{j+1}] = 1 \) for any \( j \), we find \( a_0 = -1 \). The coefficient \( a_1 \) can be regarded as a function of \( E \) and we write it as \( \tau_1(E) = a_1 \), which is referred to as the Stokes multiplier. The result can be neatly represented in the matrix form

\[
\Phi_0 = \Phi_1 \mathcal{M}_{1,0}, \quad \mathcal{M}_{1,0} = \begin{pmatrix} \tau_1(E) & 1 \\ -1 & 0 \end{pmatrix}.
\]

The general adjacent FSS \( \Phi_j \) and \( \Phi_{j+1} \) are connected by \( \Phi_j = \Phi_{j+1} \mathcal{M}_{j+1, j} \), and the ‘discrete rotational symmetry’ leads to \( \mathcal{M}_{j+1, j} = \mathcal{M}_{1, 0}|_{E \to Eq^2} \). We introduce the matrix connecting well-separated sectors

\[
\Phi_0 = \Phi_j \mathcal{M}_{j, j},
\]

(10.3)

By the definition, the recursion relation

\[
\mathcal{M}_{j, 0} = \mathcal{M}_{j+1, 1} \mathcal{M}_{1, 0}
\]

holds. The solution to this takes the form

\[
\mathcal{M}_{j, 0} = \begin{pmatrix} \tau_j(E) & \tau_{j-1}(Eq^2) \\ -\tau_{j-1}(E) & -\tau_{j-2}(Eq^2) \end{pmatrix}.
\]

(10.5)

Here \( \tau_j \) is the function uniquely determined from \( \tau_1 \) and the recursion relation

\[
\tau_j(Eq^2 E) \tau_1(E) = \tau_{j+1}(E) + \tau_{j-1}(q^2 E)
\]

with \( \tau_0(E) = 1 \). We set \( \tau_{-1}(E) = E \) so that this holds also at \( j = 0 \). In addition we have \( \tau_{2M}(E) = 1 \), \( \tau_{2M+1}(E) = 0 \) as after \( 360^\circ \) rotation, FSS must coincide with the original ones \((-1)\) (cf [156, (21.31)]). We call \( \tau_j (j \geq 2) \) the generalized Stokes multipliers. The generalized Stokes multipliers satisfy the relation

\[
\tau_j(E) \tau_j(Eq^2) = \tau_{j-1}(Eq^2) \tau_{j+1}(E) + 1.
\]

(10.7)

This is equivalent to det \( \mathcal{M}_{j, 0} = 1 \). It is shown either by (10.3) or by induction on \( j \) using (10.6). See also the discussion in section 10.3. Setting

\[
T_j(u) = \tau_j(Eq^{-j-1}) \quad \text{where} \quad E = \exp \left( \frac{\pi \text{i}u}{M+1} \right),
\]

we therefore have

**Proposition 10.1.** \( \{T_j(u)\} \) satisfies the level \( 2M \) restricted \( T \)-system for \( A_1 \)

\[
T_j(u + 1)T_j(u - 1) = T_{j-1}(u)T_{j+1}(u) + 1 \quad (j = 1, \ldots, 2M),
\]

(10.8)

where \( T_0(u) = 1 \) and \( T_{2M+1}(u) = 0 \).
Example 10.2. By (10.3), (10.5) and det $M_{j,0} = 1$, one has
\[ \tau_j(E) = W[y_0, y_{j+1}], \]
where the rhs is independent of $x$. The consistency of $\tau_{2M} = 1$ and $\tau_{2M+1} = 0$ with $y_{2M+1} = -y_1$ and $y_{2M+2} = -y_0$ is reconfirmed. Relation (10.7) is also re-derived from the simple identity among Wronskians $[y_0, y_\beta][y_\alpha, y_\lambda] = [y_\alpha, y_\beta][y_0, y_\lambda] + [y_0, y_\lambda][y_\alpha, y_\beta]$ by the specialization $\alpha = 0, \beta = j + 1, \gamma = 1, \delta = j + 2$. Note $W[y_k, y_{k+1}] = 1$ for any $k$. \(\Box\)

10.2. Higher order ODE

One can extend the observation on the second order ODE to the higher order case corresponding to $g = A, [157–160]$. Consider a natural generalization of (10.1):
\[ (-1)^r \frac{d^{r+1} y}{dx^{r+1}} + x^\ell y = E y = \lambda^{r+1} y. \tag{10.9} \]
Let $q = e^{i\theta}$ with $\theta = \frac{2\pi}{r+2}$. The sector $S_k$ is now defined by $|\arg x - k\theta| \leq \frac{\theta}{2}$. We pay attention to the solution $\phi(x, \lambda)$ in $S_0$ which decays most rapidly as $x \to \infty$ as
\[ \phi(x, \lambda) \sim C x^{-r\ell/(2r+2)} \exp(-x\nu), \quad \nu = \frac{\ell + r + 1}{r + 1}. \]
The normalization factor $C$ will be determined later. As in the second order ODE case, (10.9) is invariant under $x \to xq^{-1}, E \to Eq^{r+1}$. Thus in terms of $\lambda, y_k = q^{r/2}\phi(xq^{-k}, \lambda q^k)$ is also a solution to (10.9) for any $k \in \mathbb{Z}$.

The FSS in $S_k$ consists of $(y_k, \ldots, y_{k+r})$. It is convenient to introduce a Wronskian matrix
\[ \Phi_k = \begin{pmatrix} y_k & y_{k+1} & \cdots & y_{k+r} \\ \vdots & \vdots & & \vdots \\ \partial^r y_k & \partial^r y_{k+1} & \cdots & \partial^r y_{k+r} \end{pmatrix}. \]
We write the determinant of a slightly more general matrix (for $m \leq r$) as
\[ W[y_0, y_1, \ldots, y_m] = \det \begin{pmatrix} y_0 & y_1 & \cdots & y_m \\ \vdots & \vdots & & \vdots \\ \partial^m y_0 & \partial^m y_1 & \cdots & \partial^m y_m \end{pmatrix}. \tag{10.10} \]
Due to (10.9), the Wronskians ($m = r$ cases) are independent of $x$. In particular, the normalization constant $C$ can be fixed so that $\det \Phi_k = W[y_k, \ldots, y_{k+r}] = 1$ for any $k$. We introduce the connection matrix $M_{k+1,k}$ by
\[ \Phi_k = \Phi_{k+1} M_{k+1,k}. \tag{10.11} \]
It has the form
\[ M_{k+1,k} = \begin{pmatrix} \tau_1^{(1)}(\lambda q^k) & 1 & 0 & 0 & \cdots & 0 \\ \tau_1^{(2)}(\lambda q^k) & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \tau_1^{(r)}(\lambda q^k) & 0 & 0 & 0 & \cdots & 1 \\ \tau_1^{(r+1)}(\lambda q^k) & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \]
By using Cramer’s formula, $\tau_1^{(a)}(\lambda q^k)$ is expressed as the Wronskian
\[ \tau_1^{(a)}(\lambda q^k) = W[y_k, \ldots, y_{k+a-1}, y_k, y_{k+a+1}, \ldots, y_{k+r+1}]. \]
Especially, one finds \( \tau_{r+1}^{(r+1)}(\lambda q^k) = (-1)^r \). We further introduce the generalized Stokes multipliers \( \tau_m^{(a)}(\lambda) \) for \( m \geq 2 \) by

\[
\tau_m^{(a)}(\lambda) = W[y_1, y_2, \ldots, y_{a-1}, y_0, y_{a+m}, y_{a+m+1}, \ldots, y_{m+1}].
\] (10.12)

Note that \( m \) does not extend to infinity. Due to \( y_{r+1} = (-1)^r y_0 \), one has \( \tau_{r+1}^{(a)}(\lambda) = 0 \). This causes a truncation analogous to the level restriction in the quantum group at root of unity. It is elementary to prove

**Proposition 10.3.** The generalized Stokes multipliers \( \tau_m^{(a)}(\lambda) \) satisfy the level \( t \) restricted T-system for \( \Lambda_a \):

\[
\tau_m^{(a)}(\lambda) = \tau_{m+1}^{(a)}(\lambda) + \tau_{m-1}^{(a)}(\lambda) + \tau_m^{(a+1)}(\lambda) + \tau_m^{(a-1)}(\lambda) \quad (1 \leq a \leq r),
\]

where the boundary conditions are modified as \( \tau_0^{(a)}(\lambda) = 1, \tau_0^{(r+1)}(\lambda) = (-1)^r \) and \( \tau_0^{(a)}(\lambda) = (-1)^{a-1} \).

**Remark 10.4.** One might expect that \( \tau_m^{(a)}(\lambda) \) may appear in the generalized connection matrix \( \mathcal{M}_{k+m,k} \) connecting \( \Phi_k \) and \( \Phi_{k+m} \) (\( m \geq 2 \)). This is not the case. As the Schur functions, one can define generalized Stokes multipliers associated with (skew) Young tableaux of a general shape. Entries of \( \mathcal{M}_{k+m,k} \) are generally identified with such objects. Especially the \((a,1)\) component of \( \mathcal{M}_{k+m,k} \) corresponds to the Young tableau of the hook shape of width \( m \) and height \( a \).

### 10.3. Wronskian–Casoratian duality

The \((i + 1, 1)\) element from the matrix relation (10.11) with \( k = 0 \) reads \( \partial^i y_0 = \tau_1^{(1)}(\lambda) \partial^i y_1 + \cdots + \tau_i^{(r+1)}(\lambda) \partial^i y_{r+1} \). Remember that \( y_0 = q^{\ell i/2} \phi(xq^{-k}, \lambda q^{k}) \) involves \( x \) but \( \tau_1^{(1)}(\lambda) \) does not. Thus, one obtains an \( x \)-independent relation by setting \( x = 0 \) as

\[
\partial^i y_0|_{x=0} = \tau_1^{(1)}(\lambda) \partial^i y_1|_{x=0} + \cdots + \tau_i^{(r+1)}(\lambda) \partial^i y_{r+1}|_{x=0} \quad (0 \leq i \leq r). \] (10.13)

In view of \( y_0 = q^{\ell i/2} \phi(xq^{-k}, \lambda q^{k}) \), this has the same form as the difference equation (TQ relation) (9.6) with (9.3):

\[
w(u) - T_1^{(1)}(u) w(u + 2) + \cdots + (-1)^{r+1} T_1^{(r+1)}(u + r) w(u + 2r + 2) = 0. \] (10.14)

In fact, under the formal (ODE/IM) correspondence between the Stokes multipliers and the transfer matrix eigenvalues

\[
\tau_1^{(a)}(\lambda) = (-1)^{a-1} T_1^{(a)}(u + a - 1) \quad (1 \leq a \leq r + 1), \] (10.15)

the identification \( w(u + 2j) = \partial^i y_j|_{x=0} \) provides a solution to (10.14) for any \( 0 \leq i \leq r \). The variables \( u \) and \( \lambda \) are related so that the shift \( u \rightarrow u + 2 \) corresponds to \( \lambda \rightarrow \lambda q \). Now we are entitled to substitute

\[
w_i(u + 2j) = \partial^{i-1} y_j|_{x=0} \quad (1 \leq i \leq r + 1) \] (10.16)

into the Casoratian \( \mathcal{C}_u \) (9.7). The result is the equality

\[
W[y_1, \ldots, y_{r+1}]|_{x=0} = \mathcal{C}_u[2i_1, \ldots, 2i_k]. \] (10.17)

which we call the Wronskian–Casoratian duality. One can remove ‘|\text{x=0}’ when \( k = r + 1 \). Remember that in sections 9.1–9.3, a variety of generalizations of \( T_{1}^{(a)} \) are expressed in terms of Casoratians \( \mathcal{C}_u \). Relations (10.15) and (10.17) enable us to import those results to establish a number of Wronskian formulas for the generalized Stokes multipliers. For example, formula (9.10) leads to (10.12).
The Wronskian–Casoratian duality further provides the Stokes multipliers with dressed vacuum forms like the ones for \( A_r \) in section 8. Recall that proposition 9.1 expresses the Casoratians as the sums over semistandard tableaux like (skew) Schur functions. The variables attached to tableau letters are ratios of the principal minors of \( C_n[0, 2, \ldots, 2r] \), namely \( Q_n[u + a - 1] = C_n[0, \ldots, 2a - 2] \) (9.21), which are called Baxter’s \( Q \)-functions. Vi a the Wronskian–Casoratian duality, this is translated to a dressed vacuum form for Stokes multipliers. The tableau variables are ratios of \( W[y_{k+1}, y_{k+2}, \ldots, y_{k+a}]|_{x=0} \), which are to be identified with Baxter’s \( Q \)-functions in the present context.

As explained in section 9.4 for Casoratians, the solutions \( w_1, \ldots, w_{r+1} \) to (10.14) may be renumbered arbitrarily, and this freedom generates Bäcklund transformations among \( Q \)-functions. Even more generally, one may consider arbitrary linear combinations of (10.13) instead of (10.16) as

\[
    w_i(u + 2j) = \sum_{a=0}^{r} A_{ia} \partial^a y_j|_{x=0} \quad (1 \leq i \leq r + 1),
\]

where \((A_{ia})_{1 \leq i \leq r+1, 0 \leq a \leq r}\) is any invertible matrix. In the Wronskian language, this corresponds to identifying \( Q_n(\lambda_{q+1}) \) with

\[
    \sum_{0 \leq n_1 < \cdots < n_a \leq r} \det(A_{i,n_j})_{1 \leq i,j \leq a} \det \begin{pmatrix}
    \partial^{n_1}y_{k+1} & \partial^{n_1}y_{k+2} & \cdots & \partial^{n_1}y_{k+a} \\
    \vdots & \vdots & \ddots & \vdots \\
    \partial^{n_a}y_{k+1} & \partial^{n_a}y_{k+2} & \cdots & \partial^{n_a}y_{k+a}
\end{pmatrix}
\]

evaluated at \( x = 0 \). In this way the same Stokes multiplier acquires a variety of representations.

We note that in the simple cases like \( \tau_1(1)(\lambda) \), the recursion relation (see for example [157, 158])

\[
    \begin{bmatrix} y_0, y_2, \ldots, y_m \\ y_1, \ldots, y_m \end{bmatrix} = \begin{bmatrix} y_0, y_2, \ldots, y_{m-1} \\ y_1, \ldots, y_{m-1} \end{bmatrix} \begin{bmatrix} y_0, y_1, \ldots, y_{m-1} \\ y_0, y_1, \ldots, y_{m-1} \end{bmatrix}
\]

is handy to derive the dressed vacuum forms without recourse to proposition 9.1 and the Wronskian–Casoratian duality (10.17).

10.4. Bibliographical notes

The functional relations have appeared in ODE in the context of asymptotic analysis [156] or of the complex WKB method [161]. The connection to integrable models has been realized in [162] and the machineries of the latter have been applied since then [163–165]. The connection not only provides the information on Stokes multipliers but also solves the spectral problem of ODE. With an assumption on analyticity, one can transform (10.8) to the thermodynamic Bethe ansatz equation that describes a CFT in the ground state. It provides a quantitative tool to obtain the eigenvalues of (10.1). A more direct relation can be established between the spectral determinant associated with ODE and the vacuum expectation value of the Baxter’s \( Q \) operator in CFT [164, 165].

It is tempting to consider Schrödinger operators with more general polynomial potentials. Although we can argue the algebraic part in an almost same manner, the problem with the analyticity defies most attempts up to now. The case with \( V(x) = ax^{M-1} + x^{2M} \) is exceptionally treated nicely [166]. The underlying model seems to possess \( g(2|1) \) symmetry. The fundamental reason why this symmetry appears remains to be clarified. This case seems interesting in its relation to \( PT \) symmetric quantum systems [167] and spontaneous breakdown of the symmetry [168]. The integro-differential systems corresponding to nonexceptional classical Lie algebras in the similar sense are proposed in [169].

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The role played by the excited states of CFT is studied in [170]. The corresponding Schrödinger operators with potentials possessing singularities are identified. A further argument from the viewpoint of the Langlands correspondence is given in [171].

In general, CFTs are realized as scaling limits of lattice models. Then one may wonder if there exists an ODE which corresponds to a lattice model on a finite system. This is investigated in [172, 173] for particular cases. As for generalizations related to massive deformations of CFT, see [174, 175].

11. Applications in gauge/string theories

The AdS/CFT correspondence is a huge subject in theoretical and mathematical physics. Here we pick just two topics rather briefly, planar AdS/CFT spectrum (sections 11.1–11.4) and area of minimal surface in AdS (sections 11.5–11.8), from the gauge and the string theory sides, respectively. These subjects have been growing rapidly during the last couple of years where some specific $T$- and $Y$-systems have found notable applications.

11.1. Planar AdS/CFT spectrum

Recall the AdS$_5$/CFT$_4$ correspondence between the type IIB superstring on the curved spacetime AdS$_5 \times S^5$ and the large $N$ conformal $\mathcal{N} = 4$ super-Yang–Mills (SYM) gauge theory in four dimensions on the boundary of AdS$_5$ [176–178]. The correspondence implies that the energies of specific string states should coincide with anomalous scaling dimensions of local gauge invariant operators in the SYM. We call the sought common spectrum the planar AdS/CFT spectrum.

To be concrete, let us consider the simplest examples from the SYM side, linear combinations of single trace scalar operators without derivatives

$$\sum_{i_1,\ldots,i_L} e^{i_1\ldots i_L} \text{Tr} \Phi_{i_1} \cdots \Phi_{i_L}, \quad (11.1)$$

where $\Phi_i$ ($i = 1, \ldots, 6$) denote the six scalar fields of $\mathcal{N} = 4$ SYM in the adjoint representation of SU($N$). They contain important examples like chiral primary and BMN operators [179] as special cases and form an interesting sector that are mixed only among themselves at one-loop renormalization. In fact, the last property reduces the one-loop calculation of scaling dimensions of (11.1) to the diagonalization of the Wilson matrix $\left( \frac{\partial \ln \Lambda}{\partial \ln Z_{ij}} \right)$ consisting of the wavefunction renormalization factors $Z = (Z_{ij})$, where $\Lambda$ is the UV cutoff. This problem turns out rather remarkably identical with a periodic spin chain of length $L$ associated with the rational $R$ matrix for $SO(6)$. Thus, in the large $L$ limit, one can evaluate, for example, the largest possible scaling dimension of (11.1) by the Bethe ansatz as [180]

$$L + \frac{\lambda L}{8\pi^2} \left( \frac{\pi}{2} + \ln 2 \right) + O(\lambda^2),$$

where $\lambda = g_{YM}^2 N$ is the ’t Hooft coupling. One sees how the bare dimension $L$ (first term) acquires the anomalous correction.

Although this is a one-loop perturbative approximation to the planar AdS/CFT spectrum in a very limited sector, the connection to the Bethe ansatz is a signal of the integrability of the full problem. In fact, this theme has been explored both from the gauge and string theory perspectives extensively by enormous amount of works. We do not intend to cover them here, but refer to the literature that will be cited in the next subsection and [179–192] for example (and references therein). See also [193, 194] for earlier observations before AdS/CFT.
11.2. T- and Y-system for AdS/CFT

The planar AdS/CFT spectrum is accessible from the gauge theory side via an integrable long-range quantum spin chain with $PSU(2, 2|4)$ symmetry [195]. This is actually so, at least asymptotically, if the relevant quantum numbers like the bare scaling dimension are large enough. In the language of spin chains, such situations correspond to the thermodynamic limit where ‘impurities’ (Bethe roots) are kept dilute.

Complementally, the exact spectrum including ‘finite-size effects’ may be encoded in some $T$- and $Y$-systems together with an appropriate, albeit highly elaborate, analyticity input. A candidate for such a $Y$-system has been proposed in [196–198] based on the ground state TBA equation associated with the asymptotic Bethe ansatz (ABA) equation [195, 199, 200] in the mirror form [201].

The underlying symmetry of the ABA equation is $PSU(2, 2|4)$ [195]. Reflecting this fact, the $Y$-system in question contains two copies of the $Y$-systems for the subgroup $SU(2|2)$ denoted by $SU(2|2)_L$ and $SU(2|2)_R$. Apparently it takes the same form as the type A case:

$$\frac{Y_{a,s}(u)}{Y_{a+1,s}(u)} = \frac{(1 + Y_{a+1}(u))(1 + Y_{a+1}(u))}{(1 + Y_{a+1}(u))(1 + Y_{a-1}(u))}.$$ (11.2)

A peculiarity here is that $Y_{a,s}(u)$ is defined for those $(a, s)$ that correspond to the black nodes in the following T-shaped fat hook:

![Diagram](attachment:image.png)

(11.3)

The relevant $T$-system [197] is also formally of type A:

$$T_{a,s}(u) = T_{a,s} \left( u - \frac{i}{2} \right) T_{a,s} \left( u + \frac{i}{2} \right) = T_{a,s}(u) + T_{a-1,s}(u) + T_{a+1,s}(u),$$ (11.4)

where this time $(a, s)$ ranges over black as well as red nodes in (11.3). The relation to the $Y$-system $Y_{a,s}(u) = T_{a,s}(u) + T_{a-1,s}(u)$ is as usual. The diagram (11.3) is meant to capture the structure of equations (11.2) and (11.4).

Recall that the $Y$-system for $U_q(sl(2|2))$ in section 2.6 involves the variables $Y_m(a)$ with $(a, m)$ ranging over $H_{2,2}$ (2.39) which is an L-shaped ‘thin’ hook. This and its copy are embedded into (11.3) as $Y_{a,m}$ and $Y_{a,-m}$. The extra variables $Y_{a,0}(u)$ on the middle vertical array $(a, 0)_{a \geq 1}$ are the carriers of the ‘momentum’ (cf (11.5)). The two wings $s < 0$ and $s > 0$ correspond to $SU(2|2)_L$ and $SU(2|2)_R$ mentioned earlier. The range $m \in \mathbb{Z}$ for the ‘fusion degree’ or ‘string length’ for $T_{a,m}$ and $Y_{a,m}$ is a natural convention in those systems equipped with doubled symmetry, e.g. the $O(4)$ nonlinear sigma model ($SU(2)$ principal chiral field) having the global $SU(2)_L \times SU(2)_R$ symmetry [202].

24 Such features are illustrated along the elementary example of the XXZ chain in section 16.
25 It is essentially the $Y$-system for $U_q(sl(2|2))$ in section 2.6.
26 Another, yet more intrinsic way of encoding the $Y$-system together with the $T$-system is by the quiver in the cluster algebra formulation in section 5.3.
11.3. Formula for the planar AdS/CFT spectrum

Now the planar AdS/CFT spectrum (with $R$-charge subtracted) is given in terms of the solutions to the $Y$-system in the previous subsection by the formula

$$\sum_{j=1}^{K_0} \epsilon_1(u_{0,j}) + \sum_{a \geq 1} \int_{-\infty}^{\infty} \frac{da}{2\pi i} \frac{\partial \epsilon_a(u)}{\partial u} \ln(1 + Y^*_a(u)).$$  \hfill (11.5)\)

Here $K_0$ is specified from the sector in question (see (11.9) and (11.10)) and $\epsilon_a(u)$ is defined by $\epsilon_a(u) = a + \frac{2g}{\lambda^2} = \frac{2g}{\lambda^2}$ in terms of $x(u)$ satisfying $\frac{2}{g} = x(u) + x(u^{-1})$ and $|x(u \pm \frac{2}{g})| > 1$.

The parameter $g$ is related to the 't Hooft coupling $\lambda$ by $\lambda = (4\pi g)^2$. The above choice of the branch is called physical kinematics. On the other hand, $Y_a^*(u)$ with $a > 1$ is defined by the same formula but with another branch called mirror kinematics (cf [192, 197, 198]). The function $Y^*_{a,0}(u)$ is defined by the mirror kinematics. Finally, the rapidities $u_{0,j}$ are determined by the Bethe equation

$$Y_{1,0}(u_{0,j}) = -1 \quad (j = 1, \ldots, K_0).$$  \hfill (11.6)\)

This description of the planar AdS/CFT spectrum has been claimed exact for any 't Hooft coupling (i.e. to all loop orders) and operators of any finite $L$ [197, 203].

11.4. Asymptotic Bethe ansatz

To be consistent with the ABA equation [195], the $Y$-system (11.2) should split into the left and right wings in the limit $L \to \infty$. Compatibly with this, the middle series should behave as

$$Y_{a \geq 1,0}(u) \simeq \frac{x(u - \frac{i}{2})}{x(u + \frac{i}{2})}^L \frac{\phi(u - \frac{i}{2})}{\phi(u + \frac{i}{2})} T^L_{a,-1}(u) T^R_{a,1}(u),$$  \hfill (11.7)\)

where $\phi$ is a function obeying relation (11.15). The last two factors represent the $T$-functions for the decoupled $SU(2)_L$ and $SU(2)_R$. They are constructed from the $a = 1$ case [19, 22] in a way analogous to (9.2), (9.3) and (9.5). Explicitly, the $a = 1$ case is given as the dressed vacuum form

$$T^L_{1,-1}(u) = \frac{K_0^{(+)}(u - \frac{i}{2})}{K_0^{(-)}(u - \frac{i}{2})} \left( \frac{Q_{\pm 2}(u - i)Q_{\pm 3}(u + \frac{i}{2})}{Q_{\pm 2}(u)Q_{\pm 3}(u - \frac{i}{2})} + \frac{Q_{\pm 2}(u + i)Q_{\pm 1}(u - \frac{i}{2})}{Q_{\pm 2}(u)Q_{\pm 1}(u + \frac{i}{2})} \right)$$

$$- \frac{K_0^{(-)}(u - \frac{i}{2})Q_{\pm 3}(u + \frac{i}{2})}{K_0^{(+)}(u - \frac{i}{2})Q_{\pm 3}(u - \frac{i}{2})} - \frac{B_0^{(+)}(u + \frac{i}{2})Q_{\pm 1}(u + \frac{i}{2})}{B_0^{(-)}(u + \frac{i}{2})Q_{\pm 1}(u + \frac{i}{2})}. \hfill (11.8)\)

where $Q_1(u) = \prod_{j=1}^{K_1}(u - u_{1,j})$. In addition we introduce

$$\begin{align*}
R_1(u) &= \prod_{j=1}^{K_1} \frac{x(u) - x(u_{1,j})}{\sqrt{x(u_{1,j})}}, \\
R^{(+)}_1(u) &= \prod_{j=1}^{K_1} \frac{x(u) - x(u_{1,j} + \frac{i}{2})}{\sqrt{x(u_{1,j} + \frac{i}{2})}}, \\
R^{(-)}_1(u) &= \prod_{j=1}^{K_1} \frac{x(u) - x(u_{1,j} - \frac{i}{2})}{\sqrt{x(u_{1,j} - \frac{i}{2})}}, \\
R^{(+)}_1(u) &= \prod_{j=1}^{K_1} \frac{x(u) - x(u_{1,j} + \frac{i}{2})}{\sqrt{x(u_{1,j} + \frac{i}{2})}}.
\end{align*} \hfill (11.9)\)

$$\begin{align*}
B_1(u) &= \prod_{j=1}^{K_1} \frac{x(u)^{-1} - x(u_{1,j})}{\sqrt{x(u_{1,j})}}, \\
B^{(+)}_1(u) &= \prod_{j=1}^{K_1} \frac{x(u)^{-1} - x(u_{1,j} + \frac{i}{2})}{\sqrt{x(u_{1,j} + \frac{i}{2})}}, \\
B^{(-)}_1(u) &= \prod_{j=1}^{K_1} \frac{x(u)^{-1} - x(u_{1,j} - \frac{i}{2})}{\sqrt{x(u_{1,j} - \frac{i}{2})}}.
\end{align*} \hfill (11.10)\)

27 The Bethe roots $u_{1,j}$, $u_{2,j}$, $u_3$, $u_{0,j}$, $u_{-3,j}$, $u_{-2,j}$, $u_{-1,j}$ and the $T$-functions $T^L_{1,-1}$, $T^R_{1,1}$ here denote $u_{12}, j, u_{22}, j, u_3, u_{0,j}, u_{-3,j}, u_{-2,j}, u_{-1,j}$ and $T^L_{1,-1}$, $T^R_{1,1}$ in [197], respectively. The notation for the $Q$-functions is also slightly modified accordingly. These Bethe roots further correspond to $u_{1,j}, u_{2,j}, u_3, u_{0,j}, u_{-3,j}, u_{-2,j}, u_{-1,j}$ in [195].
for \(-3 \leq l \leq 3\). They are factorized pieces of \(Q_l(u)\) in that

\[ R_l(u)B_l(u) = (-g)^{-K_l}Q_l(u), \quad R_l^{(-)}(u)B_l^{(-)}(u) = (-g)^{-K_l}Q_l(u \pm \frac{i}{2}). \] (11.11)

The numbers \(K_l\) specify the relevant sectors. As usual in the analytic Bethe ansatz (cf section 8), analyticity of \(T_{l,R}^2(u)\) leads to the equations

\[ 1 = \frac{Q_{l+2}(u_{\pm,1,k} + \frac{i}{2})B_{l+2}^{(-)}(u_{\pm,1,k})}{Q_{l+2}(u_{\pm,1,k} - \frac{i}{2})B_{l+2}^{(+)}(u_{\pm,1,k})}, \quad 1 = \frac{Q_{l+2}(u_{\pm,3,k} + \frac{i}{2})R_{l+2}^{(-)}(u_{\pm,3,k})}{Q_{l+2}(u_{\pm,3,k} - \frac{i}{2})R_{l+2}^{(+)}(u_{\pm,3,k})}. \] (11.12)

\[ -1 = \frac{Q_{l+1}(u_{\pm,2,k} - \frac{i}{2})Q_{l+2}(u_{\pm,2,k} + i)Q_{l+3}(u_{\pm,2,k} - \frac{i}{2})}{Q_{l+1}(u_{\pm,2,k} + \frac{i}{2})Q_{l+2}(u_{\pm,2,k} - i)Q_{l+3}(u_{\pm,2,k} + \frac{i}{2})}. \] (11.13)

In addition, the cyclicity of the single trace operator in SYM is to be reflected as the ‘zero momentum’ condition \(\prod_{j=1}^{K_2} \sigma(x(u), x_0,j) = 1\). Upon a convention adjustment, these relations coincide with the ABA equation in [195, section 5.1] except the most complicated one

\[ -1 = \left( \frac{x(u_{0,k} - \frac{i}{2})}{x(u_{0,k} + \frac{i}{2})} \right)^L \left( \frac{B_{l+2}^{(-)}B_{l+3}R_{l+3}R_{l+3}^{(-)}(u_{0,k} + \frac{i}{2})}{B_{l+2}^{(+)}B_{l+3}R_{l+3}R_{l+3}^{(-)}(u_{0,k} - \frac{i}{2})} \right)S(u_{0,k})^2. \] (11.14)

which involves the dressing factor \(\sigma\) [200] via \(S(u) = \prod_{j=1}^{K_2} \sigma(x(u), x_0,j)\). The ABA equation (11.14) is to be reproduced in the present scheme as the large \(L\) limit of equation (11.6). In view of \(T_{l,R}^2(u_0,j) = \frac{Q_{l+1}(u_0,j + \frac{i}{2})}{Q_{l+1}(u_0,j - \frac{i}{2})}\) and (11.7), this amounts to postulating that \(\phi\) therein should satisfy the difference equation

\[ \frac{\phi(u - \frac{i}{2})}{\phi(u + \frac{i}{2})} = \frac{B_{l+2}^{(+)}B_{l+3}R_{l+3}R_{l+3}^{(-)}(u + \frac{i}{2})B_{l+2}^{(-)}B_{l+3}R_{l+3}R_{l+3}^{(-)}(u - \frac{i}{2})}{B_{l+2}^{(+)}B_{l+3}R_{l+3}R_{l+3}^{(-)}(u + \frac{i}{2})B_{l+2}^{(-)}B_{l+3}R_{l+3}R_{l+3}^{(-)}(u - \frac{i}{2})} S(u)^2. \] (11.15)

The asymptotics (11.7) with (11.15) specifies the large \(L\) solution of the \(Y\)-system.

With regard to the finite \(L\) effects, the above formulation reproduces wrapping corrections at weak coupling for twist 2 operators obtained by other methods such as the Lüscher formula. For instance in the case of the Konishi operator \(\text{Tr}(D^2 Z^2 - DZDZ)\), one gets the scaling dimension from ABA as \(E_{\text{ABA}} = 4 + 12g^2 - 48g^4 + 336g^6 - (2820 + 288\zeta(3))g^8\). The above \(Y\)-system approach yields the result \(E_{\text{ABA}} + E_{\text{wrapping}}\) with the correction \(E_{\text{wrapping}} = (324 + 864\zeta(3) - 1440\zeta(5))g^8\) starting at four-loop in agreement with [192].

11.5. Area of minimal surface in \(\text{AdS}\)

Now we turn to the second topic of this section. The \(T\)- and \(Y\)-systems play an essential role in calculating the action of classical open string solutions, i.e. the area of minimal surface, in \(\text{AdS}\) space. Via the \(\text{AdS}/\text{CFT}\) correspondence, this yields the planar amplitudes of gluon scattering in \(\mathcal{N} = 4\) SYM at strong coupling. The gluon momenta are incorporated in null polygonal configurations at the \(\text{AdS}\) boundary. The first important step in this problem is to linearize the equation of motion of the \(\text{AdS}\) sigma model (section 11.5). Once this is achieved, the \(T\)- and \(Y\)-systems come into the game naturally through the Stokes phenomena of the auxiliary linear problem around the irregular singularity at the boundary of the worldsheet (section 11.6). This part is close in spirit to section 10.1. Extra complication can occur when passing to the TBA-type nonlinear integral equations most typically due to the complex nature of the driving terms (‘complex mass’ appearing in asymptotics of \(Y\)-functions). They are determined by period integrals of the Riemann surface reflecting the null polygonal boundary and the cross ratios of gluon momenta. The regularized area is formally expressed in the same form as the
free energy in the conventional TBA analysis (section 11.8). Sections 11.5–11.8 are quick
digest of the recent progress [204–207] along a simple version of AdS3.

The AdS3 is given in terms of the global coordinate $\tilde{Y} = (Y_{-1}, Y_{0}, Y_{1}, Y_{2}) \in \mathbb{R}^{2,2}$ as

$$\tilde{Y} \cdot \tilde{Y} := -Y_{-1}^{2} - Y_{0}^{2} + Y_{1}^{2} + Y_{2}^{2} = -1.$$  

(11.16)

The general product $\tilde{A} \cdot \tilde{B}$ in $\mathbb{R}^{2,2}$ is defined similarly with the signature $-1, -1, 1, 1$. The
equation of motion and the Virasoro constraint read

$$\partial \tilde{A} \cdot \tilde{Y} - (\partial Y \cdot \tilde{A}) \tilde{Y} = 0, \quad \partial \tilde{Y} \cdot \tilde{Y} = \tilde{d} \tilde{Y} \cdot \tilde{A} \tilde{Y} = 0,$$

(11.17)

where $\partial = \frac{\partial}{\partial z}$, $\tilde{\partial} = \frac{\partial}{\partial \bar{z}}$ and $z$ is a complex coordinate parameterizing the worldsheet. This
classical motion of strings in AdS3 is integrable. In fact, it is transformed into a $\mathbb{Z}_{2}$-projected
$SU(2)$ Hitchin system through a Pohlmeyer-type reduction [208, 209]. To see this, introduce the
new variables $\alpha$ and $p$ by

$$e^{2\alpha(z, \bar{z})} = \frac{1}{2} \tilde{A} \cdot \tilde{Y}, \quad N_{a} = \frac{1}{2} \epsilon_{abcd} Y^{b} \partial Y^{c} \partial Y^{d},$$  

(11.18)

$$p = \frac{1}{2} \tilde{N} \cdot \partial^{2} \tilde{Y}, \quad \bar{p} = -\frac{1}{2} \tilde{N} \cdot \tilde{\partial}^{2} \tilde{Y}.$$  

(11.19)

Note that $\tilde{N} \cdot \tilde{Y} = \tilde{N} \cdot \partial \tilde{Y} = \tilde{N} \cdot \tilde{\partial} \tilde{Y} = 0$ and $\tilde{N} \cdot \tilde{N} = 1$. The variable $\alpha = \alpha(z, \bar{z})$ is real
and $\tilde{N}$ is pure imaginary. Moreover it can be shown from (11.16)–(11.19) that $p = p(z)$
is holomorphic. The area is given by $\oint dz e^{2\alpha}$. The $\alpha$ satisfies the sinh-Gordon equation
modified with $p$ as $\partial \alpha - e^{2\alpha} + |p(z)|^{2} e^{-2\alpha} = 0$. As this fact indicates, equations (11.17) are
expressible as the flatness condition of the connections:

$$\partial B_{-}^{L} - \tilde{\partial} B_{-}^{L} + [B_{-}^{L}, B_{-}^{L}] = 0, \quad \partial B_{-}^{R} - \tilde{\partial} B_{-}^{R} + [B_{-}^{R}, B_{-}^{R}] = 0,$$

(11.20)

where the connections are given by

$$B_{-}^{L} = B_{-}(1), \quad B_{-}^{L} = B_{1}(1), \quad B_{-}^{R} = U B_{-}(i) U^{-1}, \quad B_{-}^{R} = U B_{1}(i) U^{-1},$$

(11.21)

$$B_{-}(\zeta) = \left(\begin{array}{cc}
\frac{1}{\xi} \partial \alpha & -\zeta^{-1} e^{\alpha} \\
-\zeta^{-1} e^{-\alpha} p(z) & -\frac{1}{2} \partial \alpha
\end{array}\right), \quad B_{1}(\zeta) = \left(\begin{array}{cc}
\frac{1}{\xi} \partial \alpha & -\zeta e^{\alpha} p(z) \\
-\zeta^{-1} e^{\alpha} & \frac{1}{2} \partial \alpha
\end{array}\right),$$

(11.22)

with $U = \begin{pmatrix} 0 & e^{\pi i/4} \\ e^{-\pi i/4} & 0 \end{pmatrix}$. Here $\zeta$ is the spectral parameter. Actually the relation
$\partial B_{-}(\zeta) - \tilde{\partial} B_{-}(\zeta) + [B_{-}(\zeta), B_{-}(\zeta)] = 0$ including $\zeta$ is satisfied. Splitting the connection into
the $\zeta$-dependent part and the rest as $B_{-}(\zeta) = A_{-} + \zeta^{-1} \Phi_{-}$ and $B_{1}(\zeta) = A_{1} + \zeta \Phi_{1}$, one finds that the
flatness conditions form the Hitchin system with the gauge field $A$ and Higgs field $\Phi$. The
gauge group is $SU(2)$ but the system is $\mathbb{Z}_{2}$-projected in the sense that the above form (11.22)
belongs to the invariant subspace under the involution $A_{-} \rightarrow \sigma^{3} A_{-} \sigma^{3}, \Phi_{-} \rightarrow -\sigma^{3} \Phi_{-} \sigma^{3}$ and
similarly for $A_{1}$ and $\Phi_{1}$. ($\sigma^{3}$ is a Pauli matrix.)

With each zero curvature condition in (11.20), there is associated a pair of auxiliary linear
problems whose compatibility yields it. Thanks to relations (11.21), one can combine and
promote them into the $\zeta$-dependent versions $(\partial + B_{-}(\zeta))\psi = 0$ and $(\tilde{\partial} + B_{-}(\zeta))\psi = 0$
or equivalently

$$\left(d + \frac{\Phi \cdot d z}{\zeta} + A + \zeta \Phi \cdot d \bar{z}\right) \psi = 0$$

(11.23)

with $A = A_{-} d z + A_{1} d \bar{z}$ for $\psi = \psi(z, \bar{z}; \zeta)$. A useful property is that if $\psi(\zeta)$ is a flat section
with the spectral parameter $\zeta$, then so is $\sigma^{3} \psi(e^{\pi i} \zeta)$ by the $\mathbb{Z}_{2}$-symmetry.

Given two solutions $\psi, \psi'$ to (11.23), define their $SL(2)$-invariant pairing as $\langle \psi, \psi' \rangle = e^{i \theta} \psi_{a}^{\dag} \psi'_{a}$, where $\psi = (\psi_{1}, \psi_{2})^{T}$, etc. This is a constant function on the worldsheet playing the
role analogous to Wronskians in section 10. Let $\psi_{a}^{L} = (\psi_{1a}, \psi_{2a})^{T} (a = 1, 2)$ be
the two solutions \( \psi(z, \bar{z}, \zeta = 1) \) normalized as \( \langle \psi^L, \psi^L \rangle = \langle \psi^L, \psi^R \rangle = \langle \psi^R, \psi^R \rangle = \epsilon_{ab} \). Fix also the solutions \( \psi^R_a = \langle \psi^R, \psi^R \rangle a \rangle (a = 1, 2) \) which are similarly normalized at \( \zeta = i \). Then, the original \( \text{AdS}_3 \) coordinate \( \tilde{Y} = (Y_1, 0, Y_1, Y_2) \) is reproduced from the auxiliary linear problem by

\[
\begin{pmatrix}
Y_{-1} + Y_2 \\
Y_{1} - Y_0 \\
Y_1 + Y_0 \\
Y_{-1} - Y_2
\end{pmatrix}
= \psi^L_{1, a} \psi^L_{1, a} + \psi^L_{2, a} \psi^L_{2, a}.
\]

(11.24)

This substantially achieves the linearization of the problem.

### 11.6. Stokes phenomena, T- and Y-system

Scattering amplitudes for \( 2n \) gluons correspond to open string solutions having polygonal shapes with \( 2n \) cusps at the \( \text{AdS}_3 \) boundary. This translates into the following boundary condition:

\[
\alpha \to -\frac{1}{4} \ln |p(z)|^2 (z \to \infty), \quad p(z) = z^{n-2} + \cdots \quad \text{(polynomial of degree \( n-2 \)).}
\]

(11.25)

We assume that \( n \) is odd for simplicity. From (11.22), solutions of the auxiliary linear problem (11.23) as \( |z| \to \infty \) behave as

\[
\psi \sim \left( \frac{p/p}{p/p} \right)^{\pm} \exp \left( \pm \frac{1}{\zeta} \int \sqrt{p} \, dz \pm \zeta \int \sqrt{\bar{p}} \, d\bar{z} \right).
\]

(11.26)

Since \( \exp \left( \pm \int / \int \right) \) holds asymptotically, there are \( n \) Stokes sectors which are separated by \( n \) rays in the \( z \) plane. We label them consecutively anticlockwise.

Let \( s_k(\zeta) \) be the small (subdominant in the terminology of section 10) solution in the \( k \)th Stokes sector. Then we have the properties like \( \alpha^3 s_k(e^{i\eta} \zeta) \propto s_{k+1}(\zeta) \) and \( s_j(s_k) = s_{k+1}(\zeta) \). Fixing the small solution \( s_k(\zeta) \) in the first Stokes sector, we define the others by

\[
s_{k+1}(\zeta) = (\alpha^3 s_k(e^{i\eta} \zeta)).
\]

Set \( T_k(\zeta) = (s_0(s_k))(e^{-\pi ik+1/2}\zeta) \) in the normalization \( \langle s_j, s_{k+1} \rangle(\zeta) = 1 \). Then from the simplest Plücker relation or Schouten identity \( \langle s_i, s_j \rangle(\zeta) = \langle s_j, s_i \rangle(\zeta) + \langle s_i, s_k \rangle(\zeta) \), one finds

\[
T_k(e^{\pi \zeta})T_k(e^{-\pi \zeta}) = T_{k-1}(\zeta)T_{k+1}(\zeta) + 1.
\]

(11.27)

This is a version of the level \( n-2 \) restricted \( T \)-system for \( A_1 \) where the conditions \( T_0(\zeta) = 1 \) and \( T_{n-1}(\zeta) = 0 \) are imposed. Setting further \( Y_k(\zeta) = T_{k-1}(\zeta)T_{k+1}(\zeta) \) as usual, one gets the level \( n-2 \) restricted \( Y \)-system (for \( \text{Y}^{-1} \)-variables in (2.11))

\[
Y_k(e^{\pi \zeta})Y_k(e^{-\pi \zeta}) = (1 + Y_{k-1}(\zeta))(1 + Y_{k+1}(\zeta))
\]

(11.28)

with the boundary condition \( Y_0(\zeta) = Y_{n-2}(\zeta) = 0 \) in the \( k \) direction.

### 11.7. Asymptotics, WKB and TBA

As is well known, relation (11.28) determines the \( Y \)-functions effectively only with the information on their analyticity. By the definition, \( Y_k(\zeta) \)'s are analytic away from \( \zeta^{\pm 1} = 0 \) where they possess essential singularities. One can deduce the asymptotic behavior around them using the WKB approximation regarding \( \zeta^{\pm 1} \) as the Planck constant. For example when \( \zeta \to 0 \), the solutions of (11.23), after a simple similarity transformation making \( \Phi \) into \( \sqrt{\bar{p}} \text{diag}(1, -1) \), behave as \( \exp \left( \pm \frac{1}{2} \int \sqrt{\bar{p}} \, dz \right) \) times constant vectors. Thus they are well approximated by performing the integral along the Stokes (steepest descent) lines defined by \( \text{Im}(\sqrt{\bar{p}}(z) dz/\zeta) = 0 \). At a generic point in the \( z \) plane, there is one Stokes line passing

---

28 The latter is a slightly weaker condition than \( T_{n-2}(\zeta) = 1 \) in the definition of section 2.2.
through it. Exceptions are zeros of \( p(z) \) (turning points). From a single zero, there emanate three Stokes lines. They go toward infinity along certain directions corresponding to Stokes sectors or flow into another turning point. The family of these infinitely many noncrossing lines constitute the WKB foliations. See figure 3.

First consider the case in which the zeros of \( p(z) \) are aligned on the real axis. Then one obtains the estimate like

\[
\langle s_1, s_2 \rangle \sim \exp \left( -\frac{1}{\zeta} \int_{C_1} \sqrt{p} \, dz \right).
\]

Therefore, the \( Y \)-variables (without the normalization constraint on \( s_i \))

\[
Y_{2k}(\zeta) = \langle s_{k-1}, s_k \rangle \langle s_{k-1}, s_{k+1} \rangle \langle s_{k+1}, s_{k+2} \rangle \cdots \langle s_{n-1}, s_n \rangle \exp \left( \frac{2\pi i}{\zeta} \right),
\]

have the asymptotics

\[
\ln Y_{2k}(\zeta) \sim \frac{Z_{2k}}{\zeta} + \cdots, \quad \ln Y_{2k+1}(\zeta) \sim \frac{Z_{2k+1}}{i\zeta} + \cdots (\zeta \to 0),
\]

where \( Z_k = -\int_{y_k} \sqrt{p} \, dz \) is the period integral along the cycle \( y_k \) going around the \( k \)th and \((k+1)\)st largest zeros of \( p(z) \) (cf figure 5 in [206]). The asymptotics as \( \zeta \to \infty \) is similarly investigated. Together with the \( \zeta \to 0 \) case, the result is summarized as

\[
\ln Y_k(e^{i\theta}) = -m_k \cosh \theta + \cdots (\theta \to \pm \infty),
\]

where \( m_{2k} = -2Z_{2k} \) and \( m_{2k+1} = 2iZ_{2k+1} \) are both positive. Now that the combination \( \ln Y_k(e^{i\theta})/e^{-m_k \cosh \theta} \) is analytic in the strip \( |\Im \theta| \leq \frac{\pi}{2} \) and decays as \( |\theta| \to \infty \) within it, the standard argument leads to the integral equation:

\[
\ln Y_k(e^{i\theta}) = -m_k \cosh \theta + \int_{-\infty}^{\infty} \frac{\ln(1 + Y_{k-1}(e^{i\theta}))(1 + Y_{k+1}(e^{i\theta}))}{2\pi \cosh(\theta - \theta')} \, d\theta'
\]

for \( 1 \leq k \leq n-3 \) \( (Y_0(\zeta) = Y_{n-2}(\zeta) = 0) \). Up to the driving (mass) term, this has the same form with the integral equation in TBA or QTM analyses associated with the level \( n-2 \) restricted \( Y \)-system for \( A_1 \). See for example (15.14) and (16.28).

So far, we have considered the case where the zeros of \( p(z) \) are on the real axis. When they deviate from it, the \( T \)- and \( Y \)-systems remain unchanged. On the other hand, the asymptotics is
modified as \( \ln Y_k(\zeta) \sim -\frac{m_k}{\Phi_1} (\zeta \to 0) \) and \( \ln Y_k(\zeta) \sim -\frac{\eta_k}{\pi} \zeta (\zeta \to \infty) \), where \( m_k = |m_k|e^{\im \eta_k} \) is complex in general. Consequently, the integral equation (11.31) is replaced with

\[
\ln \tilde{Y}_k(\e^\theta) = -|m_k| \cosh \theta + \sum_{j=k \pm 1}^{\infty} \frac{\ln(1 + \tilde{Y}_j(\e^\theta))}{2\pi \cosh(\theta - \theta' + \im \eta_k - \im \eta_j)},
\]

(11.32)

where \( \tilde{Y}_j(\e^\theta) = Y_j(\e^{\theta+i\eta_j}) \). This holds for \( |\eta_k - \eta_{k\pm 1}| < \frac{\pi}{2} \). If the phases go beyond this range (so-called wall crossing), the integral equation acquires extra terms corresponding to the contributions of the poles from the convolution kernel. A simple illustration of such a situation has been given in [206, appendix B].

### 11.8. Area and free energy

The interesting part A of the area is given by29

\[
A = 2 \int d^2z \Tr(\Phi_1 \Phi_2) = i \int \sqrt{p} dz \wedge \Phi_1^{j1} dz = -i \sum_{j,k=1}^{n-3} w_{jk} \oint_{Y_j} \sqrt{p} dz \oint_{Y_k} \Phi_1^{j1} dz,
\]

(11.33)

where the gauge \( \Phi_2 = \sqrt{p} \diag(1, -1) \) is taken and \( \Tr \Phi_2 = 0 \) is used. In the last equality we have dropped the contribution from infinity. The matrix \( (w_{jk}) \) is the inverse of the intersection forms30 \( \langle \gamma_j, \gamma_k \rangle \) specified by \( \langle \gamma_2, \gamma_{2k} \pm 1 \rangle = 1 \). Set \( \tilde{Y}_2k(\zeta) = Y_{2k}(\zeta) \) and \( \tilde{Y}_{2k+1}(\zeta) = Y_{2k+1}(\e^{-\frac{1}{2} \zeta}) \) somehow reconciling the shift in (11.29). The factor \( \int_{Y_j} \Phi_1^{j1} dz \) in (11.33) also appears as the coefficient of \(-\zeta\) in the small \( \zeta \) expansion of \( \ln \tilde{Y}_k(\zeta) \) based on the perturbative solution of (11.23). On the other hand, the small \( \zeta = \e^\theta \) expansion of (11.32) gives

\[
\ln \tilde{Y}_k(\zeta) = \frac{Z_k}{\zeta} + \zeta \left[ \frac{Z_k}{\pi} + \sum_j \frac{\langle \gamma_j, \gamma_j \rangle}{\pi i} \int \frac{d\zeta'}{\zeta'} \ln(1 + \tilde{Y}_j(\zeta')) \right] + \cdots,
\]

(11.34)

where the appearance of \( \langle \gamma_j, \gamma_j \rangle \) is the effect of using \( \tilde{Y}_k(\zeta) \) rather than \( Y_k(\zeta) \). Thus one can substitute \( \int_{Y_j} \Phi_1^{j1} dz \) in (11.33) by \([ \ldots ] \) here times \((-1)\). As a result the area is expressed as

\[
A = A_{\text{periods}} + A_{\text{free}}\text{ with}
\]

\[
A_{\text{periods}} = -i \sum_{j,k} w_{jk} Z_k \tilde{Z}_j, \quad A_{\text{free}} = -\frac{1}{\pi} \sum_k Z_k b \int \frac{d\zeta}{\zeta^2} \ln(1 + \tilde{Y}_k(\zeta)).
\]

(11.35)

Actually one should replace \( A_{\text{free}} \) by the average \( A_{\text{free}} \) taking the contribution from large \( \zeta \) into account. Thus the final result reads \( A = A_{\text{periods}} + A_{\text{free}} \) with

\[
A_{\text{free}} = \sum_k |m_k| \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta \ln(1 + \tilde{Y}_k(\e^\theta))
\]

(11.36)

in terms of \( \tilde{Y}_k(\e^\theta) \) defined after (11.32). This has the same form as the free energy in the conventional TBA. See for example (15.15).

To summarize, the symmetry aspects of the problem (AdS, Virasoro constraints, null-cusp boundary) are incorporated into the restricted \( T \)- and \( Y \)-systems. Then, all the dynamical information (gluon momenta, Riemann surface, cycles) are remarkably integrated in the ’complex mass’ parameters \( m_1, \ldots, m_{n-3} \).

29 Our \( \Phi_1 \) here is \( \hat{\Phi}_1 \). In [206].

30 The inverse exists under our assumption of \( n \) being odd. The intersection form \( \langle \cdot, \cdot \rangle \) here should not be confused with the \( SL(2) \)-invariant pairing of spinors.
11.9. Bibliographical notes

The subjects in this section are currently in the course of rapid development. For various aspects of the planar AdS/CFT spectrum, see the literature given at the end of section 11.1 and references therein. We have only dealt with the limited issues related to $T$- and $Y$-systems. The contents in sections 11.2–11.4 are mainly based on [197]. For numerical studies, it is important to formulate the analyticity precisely and to derive the TBA (or other type of) integral equations including excited states. We refer to [196–198, 203, 210] for this problem. Similar analyses have been made in [211–213] for the AdS$_d$/CFT$_3$ duality proposed recently [214].

Calculation of gluon scattering amplitudes at strong coupling using gauge/string duality was initiated in [204] and developed in a series of works [205–207, 215–217]. For classical integrability of AdS sigma models and their connection to Hitchin system, see also [218]. An auxiliary linear problem in section 11.7 is a special case of that for the general SU(2) Hitchin system [219], where a number of aspects in the Riemann–Hilbert problem have been discussed including WKB triangulations, the Fock– Goncharov coordinates, the Kontsevich–Soibelman wall-crossing formula, TBA and so forth. The contents of sections 11.5–11.8 are mainly taken from [206]. We have treated the $n$ (number of gluons) odd case. For the case $n$ even, see [216, 217]. In [217], further effect of operator insertion is studied, and the (slightly deformed) level 2 restricted $Y$-system for $D_n$ has been obtained. For a similar appearance of the $D$ type $Y$-system in $A_1$-related lattice models, see remark 16.8. The generalized sinh-Gordon equation has also been studied in the context of generalized ODE/IM correspondence in [175].

12. Aspects as the classical integrable system

Besides the quantum integrable systems, $T$- and $Y$-systems also have interesting aspects as classical nonlinear difference equations. For instance, the $T$-system relation (2.5) is presented in the form

$$\tau_{123} - \tau_{231} + \tau_{312} = 0$$

with a suitable redefinition up to the boundary condition. Here the indices signify a shift of the independent vector variable in the respective directions ($\tau_{ij} = \tau_{ji}$). This is a version of Hirota–Miwa equation on tau functions in the theory of discrete KP equations [220–223]. A simplest account for its integrability is the Lax representation, namely the compatibility of the linear system:

$$\psi_i - \psi_j = \frac{\tau_{ij}}{\tau_{ij}} \psi_j (i < j).$$

The Hirota–Miwa equation serves as a master equation generating a variety of soliton equations under suitable specializations and boundary conditions. See for instance [222, 224, 225]. Apart from this, there are numerous aspects in type $A$ $T$-system, sometimes called octahedron recurrence, related to discrete geometry [226–228], Littlewood–Richardson rule [229], perfect matchings and partition functions on a network [230, 231] and so forth. For types other than $A$ however, such results are relatively few.

Our presentation in this section is necessarily selective. In section 12.1, we explain that the $T$-system for $g$ is a discretized Toda field equation that has decent continuous limits with a known Hamiltonian structure. In section 12.2, a connection of the $Y$-system for $A_{\infty}$ with discrete geometry is reviewed.
12.1. Continuum limit

We present a simple continuous limit of the $T$-system for general $g$ known as the lattice Toda field equation \cite{232}. It is a difference-differential system containing continuous time and discrete space variables. Further continuous limit on the latter yields the Toda field equation on $(1+1)$-dimensional continuous spacetime \cite{233}.

We begin by making a slight change of variables in the $T$-system as

$$T_m^{(a)}(u) = \tau_a \left( \frac{u + m}{t_a}, s + \varepsilon \frac{m}{t_a} \right) \quad (1 \leq a \leq r, u \in \mathbb{Z}/t, m \in \mathbb{Z}). \quad (12.2)$$

Here $\varepsilon$ is a small parameter and $s$ is going to be the continuous time variable soon. For the symbols $t, t_a$ and root system data, see around (2.1). We substitute (12.2) into the $T$-system (2.22) $T_m^{(a)}(u - \varepsilon) T_m^{(a)}(u + \varepsilon) - T_m^{(a)}(u) T_m^{(a)}(u) = \delta_m^{(a)}(u) M_m^{(a)}(u)$ with $m \in t_a \mathbb{Z}$.

For each $g$ of rank $r$ there are $r$ such equations. (The case $m \notin t_a \mathbb{Z}$ leads to the same continuum limit as the one considered in the following.) For example, the $B_2$ case reads

$$\tau_1(n - 1, s) \tau_1(n + 1, s) = \tau_1(n - 1, s - \varepsilon) \tau_1(n + 1, s + \varepsilon) = \tau_2(n, s),$$

$$\tau_1(n - 1, s) \tau_1(n + 1, s) = \tau_1(n - 1, s - \varepsilon) \tau_1(n + 1, s + \varepsilon) = \tau_2(n, s).$$

where we have chosen $g_a = \delta_m^{(a)}(u)$ to be a constant. We take the continuum limit in the time variable $s$ keeping $n \in \mathbb{Z}/t$ as the coordinate of a one-dimensional lattice without the boundary. Namely, we replace $g_a \rightarrow \varepsilon g_a / t_a$ and set $\varepsilon \rightarrow 0$. The result reads

$$D_s \tau_1(n - 1) \cdot \tau_1(n + 1) = g_1 \tau_2(n),$$

$$D_s \tau_2(n - \frac{1}{2}) \cdot \tau_2(n - \frac{1}{2}) = g_2 \tau_2(n - \frac{1}{2}) \tau_2(n + \frac{1}{2}).$$

Here we suppressed the time dependence as $\tau_a(n) = \tau_a(n, s)$, which we shall also do in the remainder of this subsection. $D_s$ denotes the Hirota derivative:

$$D_s f \cdot g = \frac{\partial f}{\partial s} g - f \frac{\partial g}{\partial s}.$$

Similarly, the general $g$ case is given by

$$D_s \tau_a \left( n - \frac{1}{t_a} \right) \cdot \tau_a \left( n + \frac{1}{t_a} \right) = g_a \mathcal{M}_a(n),$$

$$\mathcal{M}_a(n) := \prod_{b \in C_{a} = -1} \tau_b(n) \prod_{b \in C_{a} = -2} \tau_b \left( n - \frac{1}{2} \right) \tau_b \left( n + \frac{1}{2} \right) \times \prod_{b \in C_{a} = -3} \tau_b \left( n - \frac{2}{3} \right) \tau_b(n) \tau_b \left( n + \frac{2}{3} \right). \quad (12.3)$$

where $n \in \mathbb{Z}/t$. We call this the lattice Toda field equation for $g$. In some case, it actually splits into disjoint sectors. For instance in types ADE, one has $t_a = t = 1$ for any $a \in I$; hence, (12.3) closes among $\{\tau_a(n)\}_{a \in I_{-1}^\cup I}$ or $\{\tau_a(n)\}_{a \in I_{-1}^\cup \text{other}}$, where $I_{-1}$ is the bipartite decomposition of the Dynkin diagram nodes $I = \{1, \ldots, r\} = I_{-1} \cup I_{-}$. One can rewrite (12.3) in a form that looks more like the Toda equation and explore its Hamiltonian structure. As an illustration, we first treat the $A_1$ case. Let us introduce the dynamical variables $x(n)$ and $\beta(n)$ by

$$x(n) = \frac{\partial}{\partial s} \ln \frac{\tau_1(n - 1)}{\tau_1(n + 1)}, \quad \beta(n) = \frac{x(n - 1)}{x(n + 1)} \quad (n \in \mathbb{Z}). \quad (12.4)$$

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Equation (12.3) for \( A_1 \) reads
\[
\frac{\partial \tau_1(n-1)}{\partial s} \tau_1(n+1) - \tau_1(n-1) \frac{\partial \tau_1(n+1)}{\partial s} = g_1. \tag{12.5}
\]
This allows us to rewrite (12.4) as
\[
x(n) = \frac{g_1}{\tau_1(n-1) \tau_1(n+1)}, \quad \beta(n) = \frac{\tau_1(n+2)}{\tau_1(n-2)} \tag{12.6}
\]
From the expression of \( x(n) \) in (12.4) and \( \beta(n) \) in (12.6), one obtains another form of the lattice Toda field equation for \( A_1 \):
\[
\frac{\partial \ln \beta(n)}{\partial s} = -x(n-1) - x(n+1), \tag{12.7}
\]
which is a discrete analog of the Liouville equation. It is derived as the equation of motion
\[
\frac{\partial \beta(n)}{\partial s} = \{ \mathcal{H}, \beta(n) \}, \tag{12.8}
\]
with the following Hamiltonian and Poisson bracket:
\[
\mathcal{H} = \sum_{m \in \mathbb{Z}} x(m), \quad \{ x(m), x(n) \} = x(m) x(n) \text{sgn}_{2}(n-m). \tag{12.9}
\]
See (12.13) for the definition of \( \text{sgn}_{2}(n) \). We remark that (12.5), (12.7) and their relation explained in the above are difference-differential analog of the \( T \)-system, \( Y \)-system and their transformation stated in theorem 2.5 for \( A_1 \), respectively.

All these features are generalized to \( g \) straightforwardly. The relevant dynamical variables are
\[
x_a(n), \quad \beta_a(n) = \frac{x_a(n - \frac{1}{2})}{x_a(n + \frac{1}{2})} \quad (a \in I, \ n \in \mathbb{Z}/t), \tag{12.10}
\]
which are functions of the continuous time \( s \). We keep the notation \( I, t, t_a, C, (\alpha_a | \alpha_b) \) around (2.1) and set
\[
B_{ab} = B_{ba} = \frac{t_b}{\max(t_a, t_b)} C_{ab} = \begin{cases} 2 & C_{ab} = 2, \\ -1 & C_{ab} < 0, \\ 0 & C_{ab} = 0, \end{cases} \tag{12.11}
\]
where \( (B_{ab}) \) is the Cartan matrix for the simply laced Dynkin diagram obtained by forgetting the multiplicity of oriented edges in that for \( g \). We specify the Poisson bracket of \( x_a(n) \) as
\[
\{ x_a(m), x_b(n) \} = \frac{1}{2} B_{ab} x_a(m) x_b(n) \text{sgn}_{B_{ab}}(\max(t_a, t_b)(n-m)), \tag{12.12}
\]
where \( \text{sgn}_{k}(v) \) with \( k \in \{2, -1\} \) is the odd function of \( v \in \mathbb{R} \) defined by
\[
\text{sgn}_{k}(v) = \begin{cases} 1 & \text{if } v > 0 \text{ and } v \in 2\mathbb{Z} + k, \\ -1 & \text{if } v < 0 \text{ and } v \in 2\mathbb{Z} + k, \\ 0 & \text{otherwise.} \end{cases} \tag{12.13}
\]
Consequently, the Poisson bracket concerning \( \beta_a(n) \) becomes local in that it is nonvanishing only with finitely many opponents:
\[
\{ x_a(m), \beta_b(n) \} = \begin{cases} -x_a(m) \beta_b(n) \left( \frac{\delta_{m,n+1} + \delta_{m,n-1}}{t_a} \right) & C_{ab} = 2, \\ x_a(m) \beta_b(n) \sum_{j=C_{ab}+1}^{-C_{ab}-1} \delta_{m, n+j} & C_{ab} < 0, \\ 0 & C_{ab} = 0, \end{cases} \tag{12.14}
\]
\( \text{sgn}_0(v) \) is not necessary since the rhs of (12.12) contains the factor \( B_{ab} \).
\[ \{\beta_a(m), \beta_b(n)\} = \beta_a(m)\beta_b(n)(\delta_{m+(\alpha_a|\alpha_b),n} - \delta_{m-(\alpha_a|\alpha_b),n}). \tag{12.15} \]

In (12.14), the \(j\)-sum is taken with the condition \(j \equiv C_{ab} + 1 \mod 2\). The equation of motion with the Hamiltonian

\[ \frac{\partial \beta_a(n)}{\partial s} = \{\mathcal{H}, \beta_a(n)\}, \quad \mathcal{H} = \sum_{a \in I, n \in \mathbb{Z}/t} x_a(n) \tag{12.16} \]

leads to the differential-difference system

\[ \frac{\partial \ln \beta_a(n)}{\partial s} = -x_a \left( n - \frac{1}{t_a} \right) - x_a \left( n + \frac{1}{t_a} \right) \]

\[ + \sum_{b : C_{ab} = -3} x_b(n) + \sum_{b : C_{ab} = -2} \left( x_n \left( n - \frac{1}{2} \right) + x_n \left( n + \frac{1}{2} \right) \right) \]

\[ + \sum_{b : C_{ab} = -1} \left( x_n \left( n - \frac{2}{3} \right) + x_n \left( n + \frac{2}{3} \right) \right). \tag{12.17} \]

For \( g = A_1 \) this reduces to (12.7). Equation (12.17) with \( x_a(n) \) and \( \beta_a(n) \) related as (12.10) is another form of the lattice Toda field equation (12.3). In fact, the transformation between (12.3) and (12.17) is parallel with the \( A_1 \) case (12.4)–(12.7). Generalizing (12.4) we relate \( x_a(n) \) and \( \tau_a(n) \) by

\[ x_a(n) = \frac{g}{\partial s} \ln x_a(n - \frac{1}{t_a}) = \frac{g_{a,b} M_a(n)}{\tau_a(n - \frac{1}{t_a}) \tau_a(n + \frac{1}{t_a})}, \tag{12.18} \]

where the latter equality is due to the lattice Toda field equation (12.3). Substituting the latter form into (12.10), we find

\[ \beta_a(n) = \prod_{b \in I} \frac{\tau_b(n + (\alpha_a|\alpha_b))}{\tau_b(n - (\alpha_a|\alpha_b))}, \tag{12.19} \]

This can also be derived from (8.16) by noting the same structure in \( A_{1,2}^{-1}z = q^n \) (4.25) and \( M_a(n) / (x_a(n - \frac{1}{t_a})x_a(n + \frac{1}{t_a})) \) given by (12.3). Anyway, \( \frac{\partial \ln \beta_a(n)}{\partial s} \) is expressed as a linear combination of \( x_a(n) \) by using the first formula in (12.18). The result reproduces (12.17).

A further continuous limit on \( n \) can be taken by letting

\[ x_a(n) \rightarrow 2\varepsilon \exp(\phi_a(z + \varepsilon n)), \quad \ln \beta_a(n) \rightarrow -\frac{2\varepsilon}{t_a} \phi_a', \tag{12.20} \]

where \( \varepsilon = \frac{a}{\partial z} \). Then the limit \( \varepsilon \rightarrow 0 \) of (12.17) leads to a version of the Toda field equation for \( \phi_a = \phi_a(z, s) \):

\[ \frac{\partial^2 \phi_a}{\partial z \partial s} = \sum_{b \in I} t_b \alpha_b \alpha_b e^{\phi_b}. \tag{12.21} \]

The case \( g = A_1 \) is the Liouville equation. Switching to \( \psi_a \) by \( \phi_a = \sum_{b \in I} C_{a b} \psi_b - \ln t_a \), one may rewrite it in the form

\[ \frac{\partial^2 \psi_a}{\partial z \partial s} = \exp \left( \sum_{b \in I} C_{a b} \psi_b \right) \]

studied in [233]. An explicit construction of the general solution is known containing \( 2r \) arbitrary functions [233]. We see that (12.16) and (12.14) are lattice analog of the Hamiltonian formulation of the Toda field equation:

\[ \frac{\partial \phi_a}{\partial s} = \{\mathcal{H}, \phi_a\}, \quad \mathcal{H} = \sum_{a \in I} \int dz e^{\phi_a(z)}, \quad \{\phi_a(z), \phi_b(z')\} = t_a t_b (\alpha_a|\alpha_b) \delta(z - z'). \]
The Poisson structures (12.12)–(12.15) have an origin in the lattice analog of the \( W \)-algebras going back to [234]. In particular, they may be deduced from the Poisson relations among appropriate constituent fields in the \( q \)-deformed \( W \)-algebra. See for example [154, 232, 235–237] and references therein. Here we only mention, as an example, that (12.15) is a lattice analog of the Poisson relation
\[
\{ A_\alpha(z), A_\beta(w) \} = \left( \delta \left( q^{\alpha_\lambda | \alpha_\beta} \frac{w}{z} \right) - \delta \left( q^{-\alpha_\lambda | \alpha_\beta} \frac{z}{w} \right) \right) A_\alpha(z) A_\beta(w)
\]
among the fields \( A_\alpha(z) \) corresponding to the exponential simple root \( e^{\alpha_\lambda} \) whose counterpart in the theory of \( q \)-character has appeared in (4.25). See equation (3.1) in [237] and also equation (8.8) in [70] for the logarithmic form.

12.2. Discrete geometry

As we have seen in the previous subsection, continuous limits of the \( T \)-system lead to Toda-type differential equations. On the other hand, geometric origins of many differential equations of such kind have been known from the days of Darboux. Like the continuous case, it is natural to seek discrete geometry responsible for the integrability of discrete integrable equations. In fact, if we let such geometric objects speak of themselves, they would say ‘We exist, therefore it is integrable’ [32]. There are many results in this direction. See for example [226–228, 238, 239] and references therein. In a sense they provide a most natural framework to set up Lax formalisms of the integrable difference equations from geometric points of view. Here we only include a simple exposition of the basic example [240, 241] connecting the \( Y \)-system for \( A_\infty \) to a discrete analog of the Laplace sequence of conjugate nets.

We begin by recalling the appearance of the Toda field equation in projective differential geometry. Consider a surface in the real projective space \( \mathbb{P}^3 \) which has the homogeneous coordinate vector \( z = z(x, y) \in \mathbb{P}^3 \). A local coordinate \((x, y)\) of the surface is called a conjugate net if
\[
z_{xy} + a(x, y)z_x + b(x, y)z_y + c(x, y)z = 0 \tag{12.22}
\]
is valid for some functions \( a, b, c \), where the indices mean the derivatives. Although \( z \) and \( w \) specify the same surface if they are related by \( z = \lambda w \), the above equation is not invariant but changed into
\[
w_{xy} + \tilde{a}(x, y)w_x + \tilde{b}(x, y)w_y + \tilde{c}(x, y)w = 0 \tag{12.23}
\]
with \( \tilde{a} = a + (\ln \lambda)_x, \tilde{b} = b + (\ln \lambda)_y, \tilde{c} = c + a(\ln \lambda)_y + b(\ln \lambda)_x + \lambda_{xy}/\lambda \). A characteristic of a surface independent of the gauge \( \lambda \) is the Laplace invariant
\[
h = a_x + ab - c, \quad k = b_x + ab - c, \tag{12.24}
\]
satisfying \( \tilde{h} = h \) and \( \tilde{k} = k \). In what follows we consider the generic situation that they are nonzero.

For the homogeneous coordinate vector \( z \) satisfying (12.22), the Laplace transformation \( \mathcal{L}_\pm \) is defined by
\[
\mathcal{L}_+(z) = z_y + az, \quad \mathcal{L}_-(z) = z_x + bz. \tag{12.25}
\]
This is compatible with the defining property (12.22) of the conjugate net in that \( \mathcal{L}_+(\lambda w) = \lambda (w_y + \tilde{a}w) \) and \( \mathcal{L}_-(\lambda w) = \lambda (w_x + \tilde{b}w) \) hold with \( \tilde{a} \) and \( \tilde{b} \) given in the above equation. Any component \( z \) of \( z \) transforms as \( \mathcal{L}_- \circ \mathcal{L}_+(z) = h z \) and \( \mathcal{L}_+ \circ \mathcal{L}_-(z) = k z \), meaning that \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) are inverse to each other as transformations in \( \mathbb{P}^3 \). The family of surfaces in \( \mathbb{P}^3 \) generated

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32 V V Bazhanov, talk at Newton Institute, Cambridge, UK, March 2009.
from \( \mathbf{z}^{(0)} = \mathbf{z} \) as \( \mathbf{z}^{(k+n)} = (\mathcal{L}_k)^n(\mathbf{z}) \) \((n \geq 1)\) is called a Laplace sequence. Denote by \( h_n, k_n \) the Laplace invariant associated with \( \mathbf{z}^{(0)} \). It is easy to see that \( \mathbf{z}^{(k+n)} \) satisfies (12.22) with \( a, b, c \) replaced by \( a^{(\pm)}, b^{(\pm)}, c^{(\pm)} \) given by

\[
\begin{align*}
  a^{(1)} &= a + \frac{h}{h}, & b^{(1)} &= b, & c^{(1)} &= ab - h + \frac{h}{h}, \\
  a^{(-1)} &= a, & b^{(-1)} &= b, & c^{(-1)} &= ab - k + \frac{a}{k}.
\end{align*}
\] (12.26)

Substituting this into (12.24), one can express \( h_{\pm 1} \) and \( k_{\pm 1} \) in terms of \( h_0 = h \) and \( k_0 = k \). The result shows that the sequence of Laplace invariants satisfy a Toda field equation for \( A_{\infty} \):

\[
\frac{\partial^2 \ln h_n}{\partial x \partial y} = -h_{n-1} + 2h_n - h_{n+1}, \quad h_n = k_{n+1}.
\] (12.27)

Now we move onto the discrete analog of these constructions. The first step is to observe that (12.22) implies that the four infinitesimally neighboring points are coplanar. This motivates us to introduce a map \( \mathbf{x} : \mathbb{Z}^2 \to \mathbb{R}^3 \) such that the four points \( \mathbf{x}(n, m), \mathbf{x}(n+1, m), \mathbf{x}(n, m+1), \mathbf{x}(n+1, m+1) \) are coplanar for any \((n, m) \in \mathbb{Z}^2 \). Such a map is called two-dimensional quadrilateral lattice, which serves as a discrete analog of the conjugate net. In the inhomogeneous coordinate of the projective space, a two-dimensional quadrilateral lattice is represented by a map \( \mathbf{x} : \mathbb{Z}^2 \to \mathbb{R}^3 \) satisfying the discrete analog of (12.22) as follows:

\[
\Delta_1 \Delta_2 \mathbf{x} = (T_1 A) \Delta_1 \mathbf{x} + (T_2 B) \Delta_2 \mathbf{x}.
\] (12.28)

Here \( \Delta_i = T_i - 1 \) and \( T_i \) changes \( n_i \) in any function \( f(n_1, n_2) \) to \( n_i + 1 \). The functions \( A, B \) on \( \mathbb{Z}^2 \) are ‘gauge potentials’ analogous to \( A, B \) in the continuum case. The Laplace transformation, denoted by the same symbol as before, reads

\[
\mathcal{L}_+ = x - \frac{\Delta_1 x}{B}, \quad \mathcal{L}_- = x - \frac{\Delta_2 x}{A}.
\] (12.29)

To see the geometric meaning of this, note that the four points \( x, T_1 x, T_2 x, T_1 T_2 x \) form a quadrilateral on a plane due to (12.28). The points \( T_1 \mathcal{L}_+(x) \) and \( T_2 \mathcal{L}_-(x) \) are intersections of the two lines extending the opposite sides of the quadrilateral.

\[ T_1 \mathcal{L}_-(x) \]

\[ T_2 \mathcal{L}_+(x) \]

As in (12.26), the postulate \( \Delta_1 \Delta_2 \mathcal{L}_\pm(z) = T_1 \mathcal{L}_\pm(A) \Delta_1 \mathcal{L}_\pm(z) + T_2 \mathcal{L}_\pm(B) \Delta_2 \mathcal{L}_\pm(z) \) fixes the Laplace transformation of the gauge potentials as

\[
\begin{align*}
\mathcal{L}_+(A) &= \frac{B}{T_2 B} (1 + T_1 A) - 1, \quad \mathcal{L}_+(B) = T_2^{-1} \left( \frac{T_1 \mathcal{L}_+(A)}{\mathcal{L}_+(A)} (1 + B) \right) - 1, \\
\mathcal{L}_-(A) &= T_1^{-1} \left( \frac{T_2 \mathcal{L}_-(B)}{\mathcal{L}_-(B)} (1 + A) \right) - 1, \quad \mathcal{L}_-(B) = \frac{A}{T_1 A} (1 + T_2 B) - 1.
\end{align*}
\] (12.30)
It follows that the Laplace transformation is invertible, i.e. $\mathcal{L}_+\circ\mathcal{L}_- = \mathcal{L}_-\circ\mathcal{L}_+ = \text{id}$. Introduce the Laplace sequence as the continuous case by $x^{(0)} = x$ and $x^{(±n)} = (\mathcal{L}_±)^n(x)$ ($n \geq 1$).

Now we are going to assign a cross ratio to each member of the Laplace sequence. For the four colinear points $q_1, q_2, q_3, q_4$ in $\mathbb{R}^3$, we define the cross ratio as

$$\text{cr}(q_1, q_2, q_3, q_4) = \text{cr}(q_2, q_1, q_3, q_4) = \frac{(q_3 - q_1)(q_4 - q_2)}{(q_3 - q_2)(q_4 - q_1)},$$

which is invariant under projective transformations. Define the sequence of the cross ratio by

$$Y^{(n)} = -\text{cr}(x^{(n)}, \mathcal{L}_+(x^{(n)}), T_1 x^{(n)}, T_2 \mathcal{L}_+(x^{(n)})) \quad (n \in \mathbb{Z}),$$

or equivalently, by setting $Y^{(0)} = Y$ and $Y^{(±n)} = (L_±)^n(Y)(n \geq 1)$ with $Y^{(0)} = Y = -\text{cr}(x, \mathcal{L}_+(x), T_1 x, T_2 \mathcal{L}_+(x))$. The four points in $\text{cr}$ are colinear. Using (12.28)–(12.30) one can derive various formulas, e.g.

$$Y = \frac{T_2 B - (1 + T_1 A)B}{(1 + B)(1 + T_1 A)} = -\frac{\mathcal{L}_+(A)}{1 + \mathcal{L}_+(A)} \frac{B}{1 + B},$$

$$Y^{(-1)} = -\text{cr}(x, \mathcal{L}_-(x), T_2 x, T_1 \mathcal{L}_-(x)).$$

The sequence $Y^{(n)}$ satisfies the functional relation [240, 241]

$$(T_1 T_2 Y^{(n)}) Y^{(n)} = T_1 \left(1 + \frac{Y^{(n-1)}}{1 + Y^{(n)}}\right) T_2 \left(\frac{1 + Y^{(n+1)}}{1 + Y^{(n)}}\right).$$

With a suitable identification, this coincides with the $Y$-system for $A_\infty$ (2.11):

$$Y^{(a)}(u - 1) Y^{(a)}(u + 1) = \frac{(1 + Y^{(a-1)}(u))(1 + Y^{(a+1)}(u))}{(1 + Y^{(a-1)}(u)) (1 + Y^{(a+1)}(u))},$$

with no boundary conditions on $a$ and $m$.

12.3. Bibliographical notes

The contents of sections 12.1 and 12.2 are mainly taken from [232, 237] and [240, 241], respectively.

13. $Q$-system and Fermionic formula

13.1. Introduction

Consider the $T$-system for $g$. If one formally forgets the spectral parameter $u$ in $T^{(a)}_m(u)$, the resulting variable is conventionally denoted by $Q^{(a)}_m$ and the $T$-system reduces to the relation among them called the $Q$-system. In the context of $q$-characters, $T^{(a)}_m(u)$ is the $q$-character $\chi_q(\mathcal{W}_m^{(a)}(u))$ of the Kirillov–Reshetikhin module $\mathcal{W}_m^{(a)}(u)$ (theorem 4.8). Therefore,

$$Q^{(a)}_m = \text{res}_m T^{(a)}_m(u) \quad (13.1)$$

is the usual character of $g$ obtained by the restriction defined in (4.23). Consider an arbitrary product of $Q^{(a)}_m$’s and the two kinds of decompositions (we assume $v^{(a)}_m \in \mathbb{Z}_{>0}$ for the time being)

$$\prod_{a,m} (Q^{(a)}_m)^{v^{(a)}_m} = \sum_{\lambda} b_\lambda \chi(V_\lambda) = \sum_{\lambda} c_\lambda e^\lambda.$$  

(13.2)

Here $\chi(V_\lambda)$ denotes the (usual) character of the irreducible $g$-module $V_\lambda$ with the highest weight $\lambda$. The multiplicities $b_\lambda$ of the irreducible representation $V_\lambda$ (branching coefficients)
and the multiplicities $c_{\lambda}$ of weights $\lambda$ (dimensions of weight spaces) are two basic quantities characterizing the decompositions. It turns out that analyses of the $Q$-system provide them with fermionic formulas $b_{\lambda} = M_{\lambda}$ and $c_{\lambda} = N_{\lambda}$. They possess fascinating forms that symbolize the formal completeness of the string hypothesis in the Bethe ansatz at $q = 1$ and $q = 0$, respectively.

In sections 13.2 and 13.3 we explain how $M_{\lambda}$ and $N_{\lambda}$ emerge from the Bethe ansatz along the simplest setting in $g = A_1$. Precise statements for $A_1$ are formulated in section 13.4 and the proof by a unified perspective of the multivariable Lagrange inversion method is outlined in section 13.5. All the essential ingredients are given by this point. In section 13.6, we introduce the $Q$-system for $g$ and write down the associated Fermionic formulas $M_{\lambda}$ and $N_{\lambda}$. The main theorem 13.11 in the general case is stated. In section 13.7, the expansion of $Q_m^{(a)}$ into classical characters is given for nonexceptional algebras $A_r$, $B_r$, $C_r$ and $D_r$. There are a lot of further aspects which are beyond the scope of this review. They will be mentioned briefly in section 13.8. For simplicity we restrict ourselves to untwisted affine Lie algebras in this section. Analogous results are also available in the twisted cases.

### 13.2. Simplest example of $\mathcal{M}_k$

Recall the Bethe equation (8.4) for the six-vertex model. In the rational limit $q \to 1$, it takes the form

$$\left( u_j + \sqrt{-1} \right)^L = \prod_{k=1}^{n} \frac{u_j - u_k + 2\sqrt{-1}}{u_j - u_k - 2\sqrt{-1}},$$

(13.3)

where we have set all the inhomogeneity $w_j = 0$ and replaced $u_j$ by $\sqrt{-1} u_j$. The string hypothesis [10] is that the roots $u_1, \ldots, u_n$ are arranged as (called originally ‘WellenKomplex’ in [10])

$$\bigcup \bigcup \bigcup \{ u_{ma} + \sqrt{-1}(m+1-2i) + \epsilon_{mai} \mid 1 \leq i \leq m \}$$

(13.4)

for each partition $n = \sum_{m \geq 1} m N_m$ $(N_m \in \mathbb{Z}_{\geq 0})$. Here $\epsilon_{mai}$ stands for a small deviation. The $m$-tuple configuration (with negligible $\epsilon_{mai}$) is called a length $m$ string with the string center $u_{ma}$. The $N_m$ is the number of length $m$ strings. The string hypothesis is not literally true as exemplified for instance when $n = 2$ and $L > 21$ (cf. [242]). Nevertheless, a formal count of the number of solutions to (13.3) is done as follows [10, 243]. First one rewrites the Bethe equation into the one for the string centers. This is done by replacing $u_j$ by a member of a string $u_{ma} + \sqrt{-1}(m+1-2i) + \epsilon_{mai}$ and taking the product over $1 \leq i \leq m$. The resulting equation in the logarithmic form $\ln(\text{LHS/RHS}) \in 2\pi \sqrt{-1} \mathbb{Z}$ is cast, if $\epsilon_{mai}$ is negligible, into the form $f_m(u_{ma}) \in \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ $(1 \leq \alpha \leq N_m)$ which depends on $m$ and the partition $\{N_m\}$. Explicitly, $f_m(u)$ is given by

$$f_m(u) = L \theta_{m,1}(u) - \sum_{k \geq 1} \sum_{\beta=1}^{N_k} (\theta_{m,k-1} + \theta_{m,k+1})(u - u_{1\beta}),$$

(13.5)

$$\theta_{m,k}(u) = \frac{1}{\pi} \sum_{\alpha=1}^{\min(m,k)} \tan^{-1} \left( \frac{u}{|m-k| + 2\alpha - 1} \right).$$

(13.6)
Let us employ the principal branch \(-\frac{\pi}{2} \leq \tan^{-1}(u) \leq \frac{\pi}{2}\). Then from \(\theta_{m,k}(\pm \infty) = \pm \min(m, k)/2\) and \((\theta_{m,k-1} + \theta_{m,k+1})(\pm \infty) = \pm (\min(m, k) - \delta_{m,k}/2)\), we get \(f_m(\pm \infty) = \pm (P_m + N_m)/2\). Here \(P_m\), called the vacancy number, is given by

\[
P_m = L - 2 \sum_{k \geq 1} \min(m, k)N_k
\]

(13.7)

and will play a significant role in what follows. The bold argument is then that if \(P_m \geq 0\), the solutions \(\{u_{m}\}\) (up to permutations of \(u_{m1}, \ldots, u_{mN_m}\) for each \(m\)) are in one to one correspondence with the sequences \((I_1, \ldots, I_{N_m}) \in (\mathbb{Z} + \frac{P_m + N_m + 1}{2})^{N_m}\) such that

\[
-f_m(\infty) + \frac{1}{2} \leq I_1 < \cdots < I_{N_m} \leq f_m(\infty) - \frac{1}{2}.
\]

There are \((P_m + N_m)\) such sequences for each \(m\). Accordingly if one admits the argument, the number of solutions is

\[
\mathcal{M}_n = \sum_{\{N_m\} m \geq 1} \left( \begin{array}{c} P_m + N_m \\ N_m \end{array} \right).
\]

(13.8)

where the sum extends over all the partitions of \(n\), namely those \(N_m \geq 0\) satisfying \(n = \sum_{m \geq 1} mN_m\). (We understand \(M_0 = 1\).)

What number should we expect for \(\mathcal{M}_n\)? The quantum space for the rational six-vertex model is \((V_{\omega_0})^\otimes L\), where \(V_{\omega_0} \cong \mathbb{C}^3\) is the spin-\(\frac{1}{2}\) representation whose highest weight is the fundamental weight \(\omega_0\). As a result of the global \(A_1 = sl_2\) symmetry, the Bethe vectors become by construction the highest weight vectors in the quantum space \([244]\). The sector labeled by \(n\) carries the weight \((L - 2n)\omega_0\). Thus, for Bethe’s string hypothesis to be complete, one should have \(\mathcal{M}_n = b_n\) for \(0 \leq n \leq L/2\), where \(b_n\) is the branching coefficient in the irreducible decomposition \((V_{\omega_0})^\otimes L = \bigoplus_{0 \leq n \leq L/2} b_nV(L - 2n)\omega_0\).\(^{33}\) Explicitly, \(b_n = \left(\binom{\frac{L}{2}}{n}\right) - \left(\binom{\frac{L}{2}}{n-1}\right)\). Note that the condition \(0 \leq n \leq L/2\) and (13.7) imply that \(P_1 \geq P_2 \geq \cdots \geq P_{\infty} = L - 2n \geq 0\), which automatically guarantees the condition \(P_m \geq 0\).

Example 13.1. For \(L = 6\), one has \((V_{\omega_0})^\otimes 6 = V_{\omega_0} \oplus 5V_{4\omega_0} \oplus 9V_{2\omega_0} \oplus 5V_0\). Accordingly one can check \((\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) = (1, 5, 9, 5)\). In fact, the nontrivial cases are checked as

\[
\mathcal{M}_1 = \begin{pmatrix} 4 + 1 \\ N_{l=1} \end{pmatrix} = 5, \quad \mathcal{M}_2 = \begin{pmatrix} 2 + 1 \\ N_{l=1} \end{pmatrix} + \begin{pmatrix} 2 + 2 \\ N_{l=2} \end{pmatrix} = 9,
\]

\[
\mathcal{M}_3 = \begin{pmatrix} 0 + 1 \\ N_{l=1} \end{pmatrix} + \begin{pmatrix} 2 + 1 \\ N_{l=1} = N_{l=2} \end{pmatrix} + \begin{pmatrix} 0 + 3 \\ N_{l=3} \end{pmatrix} = 5.
\]

We postpone what can be proved mathematically in a more general setting to section 13.4.

13.3. Simplest example of \(N_k\)

Here we return to the trigonometric Bethe equation (8.4). After setting the inhomogeneity \(w_j = 0, q = e^{-2\pi h}\) and replacing \(u_j\) by \(u_j/(\sqrt{-1h})\), it reads

\[
\left( \frac{\sin \pi (u_j + \sqrt{-1h})}{\sin \pi (u_j - \sqrt{-1h})} \right)^L = -\prod_{k=1}^n \frac{\sin \pi (u_j - u_k + 2\sqrt{-1h})}{\sin \pi (u_j - u_k - 2\sqrt{-1h})}, \quad (13.9)
\]

In this convention, the analog of the string configuration (13.4) is

\[
\bigcup_{m \geq 1} \bigcup_{1 \leq u \leq N_m} \bigcup_{a_{m} \in \mathbb{R}} [u_{m0} + \sqrt{-1}(m + 1 - 2i)h + \epsilon_{m0} \mid 1 \leq i \leq m], \quad (13.10)
\]

\(^{33}\) This argument lacks the consideration on the associated Bethe vectors.
where $N_m$ is again the number of length $m$ strings. Apart from $q = 1$ treated in the previous subsection, there is a point $q = 0$, i.e. the limit $h \to \infty$ where one can make another formal but systematic counting of the string solutions [245]. Leaving the precise definitions and statements to [245], we just state here casually that at $q = 0$ the Bethe equation (13.9) becomes the following linear congruence equation on the string centers:

$$\sum_{k \geq 1} \sum_{\beta=1}^{N_k} A_{m,k\beta} u_{k\beta} \equiv \frac{P_m + N_m + 1}{2} \mod \mathbb{Z}. \quad (13.11)$$

Here the coefficient $A_{m,k\beta}$ is given by

$$A_{m,k\beta} = \delta_{mk} \delta_{\alpha\beta} (P_m + N_m) + 2 \min(m, k) - \delta_{mk} \quad (13.12)$$

with the same $P_m$ as in (13.7). Equation (13.11) is called the string center equation. The concrete form of its rhs will not matter in the counting problem considered in what follows. Given a string pattern $(N_m)$, one should actually regard the solutions to (13.11) as belonging to

$$(u_{k1}, u_{k2}, \ldots, u_{kN_k}) \in (\mathbb{R}/\mathbb{Z})^{N_k} / \mathcal{S}_N$$

for each $k$, where $\mathcal{S}_N$ denotes the degree $N$ symmetric group. This is because the Bethe vector is a symmetric function of $e^{\pm \sqrt{-1} \theta_{\alpha k}}$, $\ldots$, $e^{\pm \sqrt{-1} \theta_{\beta k}}$ for each $k$. We say that a solution $(u_{k\beta})$ to (13.11) is off-diagonal if $u_{k1}, u_{k2}, \ldots, u_{kN_k} \in \mathbb{R}/\mathbb{Z}$ are all distinct for each $k$. This definition is motivated by the fact that the Bethe vectors vanish unless the associated Bethe roots are all distinct [246].

For $0 \leq n \leq L/2$ we define

$$N_n = \sum_{\{N_m\}} [\text{off-diagonal solutions to the string center equation (13.11)}], \quad (13.13)$$

where the sum is taken over $N_m \in \mathbb{Z}_{\geq 0}$ satisfying $n = \sum_{m \geq 1} m N_m$ as in (13.8). (We understand $N_0 = 1$.)

**Example 13.2.** We derive $N_0 = \binom{L}{2}$ for $n = 1, 2$ as an illustration. When $n = 1$, the only possible string pattern $(N_m)$ is $N_m = \delta_{m1}$. Equation (13.11) is just $Lu_{11} \equiv \text{const mod } \mathbb{Z}$; hence, there are $N^1_1 = L$ off-diagonal solutions.

For $n = 2$ (hence $L \geq 4$), there are two possible string patterns (i) $N_m = \delta_{m2}$ and (ii) $N_m = 2\delta_{m1}$. In (i), equation (13.11) is $Lu_{11} \equiv \text{const mod } \mathbb{Z}$, which again yields $L$ off-diagonal solutions. In (ii), equation (13.11) reads in the matrix notation as

$$\left( \begin{array}{cc} L-1 & 1 \\ 1 & L-1 \end{array} \right) \equiv \bar{c} \mod \mathbb{Z}^2$$

for some $\bar{c}$. The number of solutions equals the determinant $L(L-2)$ of the coefficient matrix, which is positive by the assumption $L \geq 4$. However, they contain the collision $(u_{11} = u_{12}) L$ times which should be excluded from the off-diagonal solutions. Thus, there are $(L(L-2) - L)/2 \equiv \text{const mod } \mathbb{Z}$, which again yields $L$ off-diagonal solutions for (ii), where the division by 2 is due to the identification by $\mathcal{S}_2$. Collecting the contributions from (i) and (ii), one obtains $N_2 = L + (L(L-2) - L)/2 = L(L-1)/2$ as desired.

It is possible to generalize the calculations in example 13.2 by a systematic application of the inclusion–exclusion principle. The final result reads [245]

$$N_n = \sum_{\{N_m\}} \det (F_{m,k}) \left( \prod_{m \in J} \frac{1}{N_m} \left( \frac{P_m + N_m - 1}{N_m - 1} \right) \right),$$

$$F_{m,k} = \delta_{mk} P_m + 2 \min(m, k) N_k, \quad (13.14)$$
where \( J = \{ j \in \mathbb{Z}_{\geq 1} \mid N_j \geq 1 \} \) and \( P_m \) is defined by (13.7). Again the sum in (13.14) is taken in the same way as (13.13). As noted before example 13.1, the assumption \( 0 \leq n \leq L/2 \) implies \( P_m \geq 0 \) \((m \geq 1)\). By using this property it can be shown that \( \det_{m,k \in J} (F_{m,k}) > 0 \) and the rhs of the first equality in (13.14) is a positive integer.

What number should we expect for \( N_n \)? Unlike the rational case in the previous subsection, the six-vertex model with \( q \neq 1 \) under the periodic boundary condition does not possess the global \( sl_2 \)-symmetry. Thus for the string solutions (13.10) to be complete, one should have \( N_n = c_n \), where \( c_n \) is the weight multiplicity of the quantum space \( (V_{\omega_1})^{\otimes L} \) with weight \((L - 2n)\omega_1\). Explicitly, \( c_n = \binom{L}{n} \). This has been confirmed for \( n = 1, 2 \) in example 13.2.

The next case is checked as
\[
N_3 = L + \binom{L - 2}{N_1} + \binom{L - 2}{N_1} + \binom{L - 2}{N_1} + \binom{L - 2}{N_1} = \frac{L(L - 1)(L - 2)}{6}.
\]

One may wonder what happens for \( n > L/2 \) where \( c_n \) still makes sense. The answer will be given in the next subsection in a more general setting together with the analogous result for \( b_n \). The only preliminary we mention here is that such considerations necessarily involve the situation \( P_m < 0 \), hence the binomial coefficients \( \binom{X}{N} \) with \( X < N \).

13.4. Theorems for type A

We have hitherto argued about three kinds of quantities:

(i) number of string solutions in the Bethe ansatz,
(ii) fermionic forms \( M_n \) and \( N_n \),
(iii) representation theoretical data \( b_n \) and \( c_n \),

especially without a much distinction between (i) and (ii). Here we redefine (ii) without recourse to (i) and formulate the theorems on the relations between (ii) and (iii). We treat the general spin case \( \bigotimes_{m \geq 1} (V_{m\omega_1})^{\otimes n_m} \) and present the Fermionic character formulas. As power series formulas, they are actually valid for arbitrary \( n_m \in \mathbb{C} \). The proof of the theorem, which will be outlined in the next subsection, does not lean on the string hypotheses but is solely derived from the \( Q \)-system. As such, it does not prove nor disprove the completeness of the string hypothesis.

Let \( Q_m \) \((Q_m)\) be the character (normalized character) of the irreducible \((m+1)\)-dimensional representation \( V_{m\omega_1} \). Namely,
\[
Q_m = \chi(V_{m\omega_1}) = y^m + y^{m-2} + \cdots + y^{-m} = \frac{y^{m+1} - y^{-m-1}}{y - y^{-1}} \quad (y = e^{\omega_1}),
\]
\[
Q_m = y^{-m} Q_m. \tag{13.15}
\]

The \( Q_m \) is a simplified notation for the variable \( Q_m^{(1)} \) (13.1) in the \( Q \)-system for \( A_1 \):
\[
Q_m^2 = Q_{m-1} Q_{m+1} + 1. \tag{13.16}
\]

See (13.41). The \( Q_m \) expressed as a function of \( Q_1 \) is the Chebyshev polynomial of the second kind. In section 13.5, we will utilize the one adapted to the normalized character \( Q_m \):
\[
\frac{Q_{m-1} Q_{m+1} + y^{-2m} Q_m^2}{Q_m^2} = 1. \tag{13.18}
\]

\(^{34}\) The same remark as the previous footnote applies here.
Let \( v_m \in \mathbb{C} \) (\( m \in \mathbb{Z}_{\geq 1} \)) be arbitrary except that \( v_m = 0 \) for all but finitely many \( m \). We define the branching coefficient \( b_n \) and the weight multiplicity \( c_n \) for all \( n \in \mathbb{Z}_{\geq 0} \) by

\[
\prod_{m \geq 1} (Q_m)^{v_m} = \frac{\sum_{n \geq 0} b_n y^{-2n}}{1 - y^{-2}} = \sum_{n \geq 0} c_n y^{-2n}.
\]  

(13.19)

By this definition, the normalized character \( Q_m \) is a polynomial in \( y^{-2} \) with a unit constant term. \( (Q_m)^{v_m} \) denotes its \( v_m \)th power with the unit constant term \( 1 + v_m (Q_m - 1) + \frac{v_m(v_m - 1)}{2} (Q_m - 1)^2 + \cdots \), which is a polynomial or a power series in \( y^{-2} \) according as \( v_m \in \mathbb{Z}_{\geq 0} \) or not. When \( v_m \in \mathbb{Z}_{\geq 0} \) for any \( m \geq 1 \), this definition of \( b_n \) agrees with the one for the branching coefficient of \( V_{\sum_{m \geq 1} (V_m)^{v_m}} \) in \( \mathbb{Z}_{m \geq 1} (V_m)^{v_m} \) for \( 0 < n \leq \sum_m m v_m / 2 \). The above \( b_n \) is an extension of this by \( b_n = -b_{-n+1+\sum_m m v_m} \), which is the skew symmetry under the Weyl group.

As for the fermionic forms, we redefine \( \mathcal{M}_n \) (13.8) and \( \mathcal{N}_n \) (13.14) by replacing \( P_m \) (13.7) and the binomial coefficient therein with the generalized ones:\footnote{In sections 13.2 and 13.3, the symbol \( \binom{z}{n} \) was used only for \( 0 \leq N \leq X \).}

\[
(P_m = \sum_{k \geq 1} \min(m, k) (v_k - 2N_k),
\]

(13.20)

\[
\binom{X}{N} = \prod_{i=1}^N \frac{(X - i + 1)}{N!} \quad (X \in \mathbb{C}, N \in \mathbb{Z}_{\geq 0}).
\]

(13.21)

The sum over \( \{N_m | m \in \mathbb{Z}_{\geq 1} \} \) is taken in the same way as (13.8) and (13.14). Namely, it is the finite sum over those \( N_m \in \mathbb{Z}_{\geq 0} \) satisfying \( \sum_{m \geq 1} m N_m = n \). There is no condition like \( P_m \geq 0 \) which does not make sense in the general setting \( v_m \in \mathbb{C} \) under consideration. The generalized binomial (13.21) is nonzero except the \( N \) points \( X = 0, 1, \ldots, N - 1 \), and appears in the expansion

\[
(1 - x)^{-\beta - 1} = \sum_{N=0}^{\infty} \binom{\beta + N}{N} x^N,
\]

(13.22)

for any \( \beta \in \mathbb{C} \). With these definitions we have

**Theorem 13.3** [243, 245]. The equalities (1) \( \mathcal{M}_n = b_n \) and (2) \( \mathcal{N}_n = c_n \) hold for all \( n \in \mathbb{Z}_{\geq 0} \). Namely, the following power series formulas hold:

\[
\prod_{m \geq 1} (Q_m)^{v_m} = \frac{\sum_{n \geq 0} \mathcal{M}_n y^{-2n}}{1 - y^{-2}} = \sum_{n \geq 0} \mathcal{N}_n y^{-2n}.
\]

(13.23)

Formulas (1) and (2) are due to [243] and [245], respectively. The theorem reproduces the observations in sections 13.2 and 13.3 in the special case \( v_m = L \delta_{m1} \) and \( 0 \leq n \leq L / 2 \), where \( P_m \geq 0 \) for any \( m \geq 1 \) automatically holds. However, even for this simple choice \( v_m = L \delta_{m1} \), it further claims infinitely many nontrivial identities including \( \mathcal{M}_n = 0 \) for \( n \geq L + 2 \) and \( \mathcal{N}_n = 0 \) and \( n \geq L + 1 \).

**Example 13.4.** Assume that \( v_m = 0 \) for \( m \geq 4 \). Then, thelhs of (13.23) is

\[
(1 + y^{-2})^{\nu1} (1 + y^{-2} + y^{-4})^{\nu2} (1 + y^{-2} + y^{-4} + y^{-6})^{\nu3}. \]

Setting \( \gamma_m = \sum_{k=1}^3 \min(m, k) v_k \), we write down \( \mathcal{M}_n \) (13.8) and \( \mathcal{N}_n \) (13.14) for \( n = 1, 2, 3 \):

\[
\mathcal{M}_1 = \gamma_1 - 1, \quad \mathcal{M}_2 = \gamma_2 - 3 + \frac{1}{2} (\gamma_1 - 2) (\gamma_1 - 3),
\]

\[
\mathcal{M}_3 = (\gamma_1 - 5) (\gamma_2 - 5) + \frac{1}{6} (\gamma_1 - 3) (\gamma_1 - 4) (\gamma_1 - 5),
\]

\[35\text{In sections 13.2 and 13.3, the symbol } \binom{z}{n} \text{ was used only for } 0 \leq N \leq X.\]
\( N_1 = \gamma_1, \quad N_2 = \gamma_2 + \frac{1}{2} \gamma_1 (\gamma_1 - 3), \quad N_3 = \gamma_3 + \left| \begin{array}{cc} \gamma_1 - 2 & 2 \\ 2 & \gamma_2 - 2 \end{array} \right| + \frac{1}{6} \gamma_1 (\gamma_1 - 4)(\gamma_1 - 5). \)

One can directly check these coefficients in the power series expansions (13.23). For instance in the simplest case \( v_m = 0 \), hence \( y_m = 0 \) for all \( m \geq 1 \), all these coefficients vanish except \( M_1 = -1 \) as they should.

In the case \( v_m \in \mathbb{Z}_{\geq 0} (m \geq 1) \), \( P_m \) in (13.20) can be a nonnegative integer for some \( \{N_m\} \).

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In the case \( v_m \in \mathbb{Z}_{\geq 0} (m \geq 1) \), \( P_m \) in (13.20) can be a nonnegative integer for some \( \{N_m\} \).

Then it makes sense to introduce the following variant of \( M_n \):

\[
\overline{M}_n = \sum_{\{N_m\}} \frac{P_m + N_m}{N_m},
\]

(13.24)

where \( P_m \) and \( \binom{N_m}{N} \) are again specified by (13.20) and (13.21) as for \( M_n \). The only difference from it is that the sum \( \sum_{\{N_m\}} \) extends over those \( N_m \in \mathbb{Z}_{\geq 0} \) satisfying \( n = \sum_{m \geq 1} mN_m \) with the extra condition \( P_m \geq 0 \) if \( N_m \geq 1 \).

Given \( \{v_m\} \), \( n \) and \( \{N_m\} \) satisfying \( \sum_{m \geq 1} mN_m = n \), let \( m_0 \) be the maximal \( m \) such that \( N_m \geq 1 \). Then we have \( P_m = \sum_{k \geq 1} \min(m_0, k)N_{m-k} - 2n \leq \sum_{k \geq 1} kN_{m-k} - 2n \). Thus, we see that \( \overline{M}_n = 0 \) if \( n > \frac{1}{2} \sum_{m \geq 1} mN_m \).

**Theorem 13.5** ([247, 248]). For any \( v_m \in \mathbb{Z}_{\geq 0} \), the equality \( \overline{M}_n = b_n \) holds for \( 0 \leq n \leq \frac{1}{2} \sum_{m \geq 1} mN_m \).

As remarked after theorem 13.3, theorem 13.5 is equivalent to theorem 13.3 (1) in the special case \( v_m = L\delta_{m1} \) and \( 0 \leq n \leq L/2 \). In general, they imply that the contributions to \( N_n \) involving \( P_m < 0 \) cancel out.

**Example 13.6.** Take \( v_m = 2\delta_{m3} \) in example 13.4. Then, \( (\gamma_1, \gamma_2, \gamma_3) = (2, 4, 6) \). The three terms in \( M_3 \) correspond to choosing nonzero \( N_m \) as \( N_3 = 1, N_1 = N_2 = 1 \) and \( N_1 = 3 \). The relevant \( P_m \)'s are \( P_3 = 0, P_1 = P_2 = -2 \) and \( P_1 = -4 \), respectively. Thus, \( \overline{M}_3 \) is given by the first term only \( \gamma_3 - 5 = 1 \). This coincides with \( M_3 \) since the other two terms cancel.

### 13.5. Multivariable Lagrange inversion

Here we outline the proof of theorem 13.3. We describe an essential step of deriving (13.23) from (13.18) in a generalized setting applicable to the case [249].

Let \( H \) denote a finite-index set. Let \( w = (w_i)_{i \in H} \) and \( v = (v_i)_{i \in H} \) be complex multivariables, and let \( G = (G_{ij})_{i,j \in H} \) be a complex square matrix of size \( |H| \). We consider a holomorphic map \( D \to \mathbb{C}^H, v \mapsto w(v) \) with

\[
w_i(v) = v_i \prod_{j \in H} (1 - v_j)^{-G_{ij}},
\]

(13.25)

where \( D \) is some neighborhood of \( v = 0 \) in \( \mathbb{C}^H \). The Jacobian \( (\partial w/\partial v)(v) \) is 1 at \( v = 0 \), so that the map \( w(v) \) is bijective around \( v = w = 0 \). Let \( v(w) \) be the inverse map around \( v = w = 0 \). Inverting (13.25), we obtain the following functional equation for \( v_i(w) \)’s:

\[
v_i(w) = w_i \prod_{j \in H} (1 - v_j(w))^{G_{ij}}.
\]

(13.26)

By introducing new functions

\[
Q_r(w) = 1 - v_i(w),
\]

(13.27)
equation (13.26) is written as
\[ Q_i(w) + w_i \prod_{j \in H} Q_j(w)^{G_{ij}} = 1. \] (13.28)

From now on, we regard (13.28) as equations for a family \((Q_i(w))_{i \in H}\) of power series of \(w = (w_i)_{i \in H}\) with the unit constant terms. The procedure from (13.25)–(13.28) can be reversed; therefore, the power series expansion of \(Q_i(w)\) in (13.27) gives the unique family \((Q_i(w))_{i \in H}\) of power series of \(w\) with the unit constant terms which satisfies (13.28).

We define the (finite) \(Q\)-system to be the following equations for a family \((Q_i(w))_{i \in H}\) of power series of \(w\) with the unit constant terms:
\[ \prod_{j \in H} Q_j(w)^{D_{ij}} + w_i \prod_{j \in H} Q_j(w)^{G_{ij}} = 1 \quad (i \in H), \] (13.29)
where \(D = (D_{ij})_{i,j \in H}\) and \(G = (G_{ij})_{i,j \in H}\) are arbitrary complex matrices with \(\det D \neq 0\).

Equation (13.28), which is the special case of (13.29) with \(D = I\) (the identity matrix) is called a standard \(Q\)-system. By setting \(Q'_i(w) = \prod_{j \in H} Q_j(w)D_{ij}\), (13.29) is always transformed to the standard one (13.28) with \(G\) replaced by \(G' = GD^{-1}\) and vice versa. Therefore, the \(Q\)-system (13.29) also has the unique solution.

Given the \(Q\)-system (13.29) and \(\nu = (\nu_i)_{i \in H} \in \mathbb{C}^H\), we define two power series of \(w\)
\[ M^\nu(w) = \sum_N M(\nu, N) w^N, \quad N^\nu(w) = \sum_N N(\nu, N) w^N, \] (13.30)
where \(w^N = \prod_{i \in H} w_i^{N_i}\) and the sums run over \(N = (N_i)_{i \in H} \in (\mathbb{Z}_{\geq 0})^H\). The coefficients are given by
\[ M(\nu, N) = \prod_{i \in H(N)} \left( P_i + N_i \right), \] (13.31)
\[ N(\nu, N) = \left( \det_{H(N)} F_{ij} \right) \prod_{i \in H(N)} \frac{1}{N_i} \left( P_i + N_i - 1 \right), \] (13.32)
where the binomial is defined by (13.21) and we have set \(H(N) = \{ i \in H \mid N_i \neq 0 \}\),
\[ P_i = P_i(\nu, N) := -\sum_{j \in H} \nu_j (D^{-1})_{ji} - \sum_{j \in H} N_j (GD^{-1})_{ji}, \] (13.33)
\[ F_{ij} = F_{ij}(\nu, N) := \delta_{ij} P_i + (GD^{-1})_{ij} N_j, \] (13.34)
\(\det_{H(N)}\) is a shorthand notation for \(\det_{i,j \in H(N)}\). In (13.31) and (13.32), \(\det_{\emptyset}\) and \(\prod_{\emptyset}\) mean 1; therefore, \(M^\nu(w)\) and \(N^\nu(w)\) are power series with the unit constant terms. See [249, section 2] for the convergence radius. Note a similarity to (13.8) and (13.14).

**Theorem 13.7** ([249]). Let \((Q_i(w))_{i \in H}\) be the unique solution of (13.29). For \(\nu = (\nu_i)_{i \in H} \in \mathbb{C}^H\), the following formulas are valid:
\[ \prod_{i \in H} Q_i(w)^{\nu_i} = \frac{M^\nu(w)}{M^0(w)} = N^\nu(w). \] (13.35)
\(Q_i(w)\) itself is obtained by setting \(\nu_j = \delta_{ij}\).

**Example 13.8.** Let \(|H| = 1\). Then, (13.29) is an equation for a single power series \(Q(w)\):
\[ Q(w)^D + wQ(w)^G = 1, \]
where \( D \neq 0 \) and \( G \) are complex numbers and theorem 13.7 shows that
\[
Q(w)^\nu = N^\nu(w) = \frac{v}{D} \sum_{N=0}^{\infty} \frac{\Gamma((\nu + N G)/D)(-w)^N}{\Gamma((\nu + N G)/D - N + 1)N!}.
\]

This power series formula is well known and has a very long history since Lambert (e.g. [250, pp 306–7]).

As noted before, the \( Q \)-system (13.29) is bijectively transformed to the standard one (13.28). Under the corresponding changes \( D \rightarrow I \), \( \nu_i \rightarrow \sum_{j \in H} \nu_j(D^{-1})_{ji} \) and \( G \rightarrow G D^{-1} \), quantities (13.33) and (13.34) remain invariant, hence so are \( M(\nu, N) \) and \( M(\nu, N) \). Thus we have only to prove theorem 13.7 for the standard case \( D = I \), where \( Q_i(w) \) is described by (13.25)–(13.27). Therefore, theorem 13.7 follows from

**Proposition 13.9** ([249] proposition 2.8). *Let \( v = v(w) \) be the inverse map of (13.25). Let \( M^+(\nu) \) and \( M^-(\nu) \) be those for \( D = I \) in (13.33) and (13.34). Then, the power series expansions
\[
\det H \left( \frac{w_j}{v_i} \frac{\partial v_i}{\partial w_j}(w) \right) \prod_{i \in H} (1 - v_i(w))^{\nu_i - 1} = M^+(\nu),
\]
(13.36)
\[
\prod_{i \in H} (1 - v_i(w))^{\nu_i} = M^-(\nu)
\]
(13.37)
hold around \( w = 0 \).

This is a particularly nice example of the multivariable Lagrange inversion formula (e.g. [251]), where all the calculations can be carried through by a multivariable residue analysis.

**Proof.** The first formula (13.36). We evaluate the coefficient for \( w^N \) on the lhs of (13.36) as follows:
\[
\text{Res}_{w=0} \frac{\partial v}{\partial w}(w) \prod_{i \in H} (1 - v_i(w))^{\nu_i - 1} (v_i(w))^{-1} (w_i)^{-1} N_i^{-1} \text{ d}w
\]
\[
= \text{Res}_{v=0} \prod_{i \in H} \left\{ (1 - v_i)^{\nu_i - 1} (v_i)^{-1} \left( v_i \prod_{j \in H} (1 - v_j)^{-G_{ij}} \right)^{-N_i} \right\} \text{ d}v
\]
\[
= \text{Res}_{v=0} \prod_{i \in H} \left\{ (1 - v_i)^{-P(v, N)^{-1}} (v_i)^{-N_i^{-1}} \right\} \text{ d}v
\]
\[
= \prod_{i \in H} \left( \frac{P_i(v, N) + N_i}{N_i} \right) = M(v, N),
\]
where we used (13.22) to get the last line. Thus, (13.36) is proved.

The second formula (13.37). By a simple calculation, we have
\[
\det H \left( \frac{v_i}{w_j} \frac{\partial v_i}{\partial v_j}(v) \right) \prod_{i \in H} (1 - v_i) = \det(\delta_{ij} + (-\delta_{ij} + G_{ij})v_i) = \sum_{J \subset I} d_J \prod_{i \in J} v_i,
\]
(13.38)
where \( d_J := \det(-\delta_{ij} + G_{ij}) \), and the sum is taken over all the subsets \( J \) of \( I \). Therefore, the lhs of (13.37) is written as
\[
\det H \left( \frac{v_i}{w_j} \frac{\partial v_i}{\partial v_j}(v) \right) \sum_{J \subset I} d_J \prod_{i \in H} \left\{ (1 - v_i(w))^{\nu_i - 1} v_i(w)^{\theta(i \in J)} \right\}.
\]
(13.39)
By a similar residue calculation as above, the coefficient for \( w^N \) of (13.39) is evaluated as 
\( \theta(\text{true}) = 1 \) and \( \theta(\text{false}) = 0 \):
\[
\sum_{J \subset H} d_J \text{Res}_{v \to 0} \prod_{i \in H} \left( (1 - v_j) - P_i(w, N) - (v_j) - N_i + \theta(i \in J) - 1 \right) \text{d}v
\]
\[
= \sum_{J \subset H(N)} d_J \prod_{i \in H(N)} \left( P_i(w, N) + N_i - \theta(i \in J) \right)
\]
\[
= \left( \sum_{J \subset H(N)} d_J \prod_{i \in J} N_i \prod_{i \in H(N) \setminus J} (P_i + N_i) \right) \prod_{i \in H(N)} \frac{1}{N_i} \left( P_i + N_i - 1 \right)
\]
\[
= \det_{H(N)} (\delta_{ij} (P_j + N_j) + (-\delta_{ij} + G_{ij}) N_j) \prod_{i \in H(N)} \frac{1}{N_i} \left( P_i + N_i - 1 \right)
\]
\[
= N(v, N).
\]
This completes the proof of theorem 13.7. What is left to prove theorem 13.3 from it.

Comparing the \( Q \)-systems (13.29) and (13.18) and also \( P_m \) in (13.33) and (13.20), we see that theorem 13.3 formally corresponds to taking
\[
H = \mathbb{Z}_{\geq 1}, \quad w_i = y^{-2i},
\]
\( (D^{-1})_{ij} = -\min(i, j) \), \( D_{ij} = \delta_{i,j+1} + \delta_{i,j-1} - 2\delta_{ij} \), \( G_{ij} = -2\delta_{ij} \) (13.40)
in theorem 13.7, and claiming \( \mathcal{M}(w) = 1 - y^{-2} \) thereunder. Since we started with the assumption that \( H \) is a finite set, it is nontrivial how to make sense of these choices and claims. We refer to [249] for a proper treatment of such an infinite (\( |H| = \infty \)) \( Q \)-system as a projective limit of the finite \( Q \)-systems. According a result therein, theorem 13.3 is shown, among other things, from the convergence property: the limit \( \lim_{m \to \infty} \mathcal{Q}_m(w_i) = y^{-2i} \) exists in \( \mathbb{C}[[y^{-2}]] \).

### 13.6. \( Q \)-system and theorems for \( g \)

Here we present the \( Q \)-system and analog of theorems 13.3 and 13.5 for general \( g \). We use the notations in section 2.1 such as \( I, \imath, t_\imath, C = (C_{ab}), a_\imath \) and \( \omega_\imath \). The unrestricted \( Q \)-system for \( g \) is the following relations among the variables \( \{ Q_m^{(a)} \}_{a \in I, m \geq 1} \), where \( Q_m^{(0)} = 0 \) if they occur on the rhs.

For simply laced \( g \),
\[
(Q_m^{(a)})^2 = Q_m^{(a-1)} Q_m^{(a+1)} + \prod_{b \in I \setminus C_{a,m-1}} Q_b^{(b)}, \quad (13.41)
\]

For \( g = B_r \),
\[
(Q_m^{(a)})^2 = Q_m^{(a-1)} Q_m^{(a+1)} + Q_m^{(a-1)} Q_m^{(a+1)} \quad (1 \leq a \leq r - 2), \quad (13.42)
\]

For \( g = C_r \),
\[
(Q_m^{(a)})^2 = Q_m^{(a-1)} Q_m^{(a+1)} + Q_m^{(a-1)} Q_m^{(a+1)} \quad (1 \leq a \leq r - 2), \quad (13.43)
\]
For $g = F_4$, 
\[(Q_{m}^{(1)})^2 = Q_{m-1}^{(1)}Q_{m+1}^{(1)} + Q_{m}^{(2)},\]
\[(Q_{m}^{(2)})^2 = Q_{m-1}^{(2)}Q_{m+1}^{(2)} + Q_{m}^{(1)}Q_{2m+1},\]
\[(Q_{2m}^{(3)})^2 = Q_{2m-1}^{(3)}Q_{2m+1} + (Q_{m}^{(2)})^2 Q_{2m},\]
\[(Q_{2m+1}^{(3)})^2 = Q_{2m}^{(3)}Q_{2m+2} + Q_{m}^{(2)}Q_{2m+1}^{(4)},\]
\[(Q_{m}^{(4)})^2 = Q_{m-1}^{(4)}Q_{m+1}^{(4)} + Q_{m}^{(3)}.\]

For $g = G_2$, 
\[(Q_{m}^{(1)})^2 = Q_{m-1}^{(1)}Q_{m+1}^{(1)} + Q_{3m}^{(2)},\]
\[(Q_{3m}^{(2)})^2 = Q_{3m-1}^{(2)}Q_{3m+1}^{(2)} + (Q_{m}^{(1)})^2,\]
\[(Q_{3m+1}^{(2)})^2 = Q_{3m}^{(2)}Q_{3m+2} + (Q_{m}^{(1)})^2 Q_{m+1},\]
\[(Q_{3m+2}^{(2)})^2 = Q_{3m+1}^{(2)}Q_{3m+3} + Q_{m}^{(1)}(Q_{m+1})^2.\]

These relations are uniformly written as 
\[(Q_{m}^{(a)})^2 = Q_{m-1}^{(a)}Q_{m+1}^{(a)} + (Q_{m}^{(a)} \prod_{(b,k) \in H} (Q_{k}^{(b)})^{G_{am,bk}})^{1},\]

by using notations (13.48) and (13.51). We shall introduce the restricted $Q$-system in section 14.5.

As mentioned around (13.1), these relations follow from the $T$-systems by forgetting the spectral parameter $u$. Recall that $\text{res}_q(W_{m}(u))$ denotes the classical character of the Kirillov–Reshetikhin module $W_{m}(u)$. See (4.23) for the definition of res. Since res removes the dependence on $u$, we will simply write as $\text{res}_q(W_{m}(a))$ in what follows. The following is a corollary of theorem 4.8.

**Proposition 13.10.** The substitution $Q_{m}^{(a)} = \text{res}_q(W_{m}(a))$ satisfies the unrestricted $Q$-system.

From now on, we understand the symbol $Q_{m}^{(a)}$ as representing $\text{res}_q(W_{m}(a))$. By theorem 4.6 (1), the normalized character $Q_{m}^{(a)} = e^{-mau} Q_{m}^{(a)}$ is a polynomial in $e^{-a_1}$, $\ldots$, $e^{-a_\alpha}$ with a unit constant term and coefficients from $\mathbb{Z}_{\geq 0}$. In terms of $Q_{m}^{(a)}$, the $Q$-system is expressed as 
\[
\prod_{(b,k) \in H} (Q_{k}^{(b)})^{D_{am,bk}} + e^{-mau} \prod_{(b,k) \in H} (Q_{k}^{(b)})^{G_{am,bk}} = 1
\]

for $(a, m) \in H$. Here $H, D_{am,bk}$ and $G_{am,bk}$ are defined by 

\[H = \{(a, m) | a \in I, m \in \mathbb{Z}_{\geq 1}\},\]

\[D_{am,bk} = -\delta_{ab}(2\delta_{mk} - \delta_{m,k+1} - \delta_{m,k-1}),\]

\[(D^{-1})_{am,bk} = -\delta_{ab} \min(m, k).\]

\[G_{am,bk} = \begin{cases} 
-\delta_{ab} \delta_{m,k-1} + 2\delta_{m,2k} + \delta_{m,2k+1} & t_a/t_b = 2, \\
-\delta_{ab} \delta_{m,3k-2} + 2\delta_{m,3k-1} + 3\delta_{m,3k} & t_a/t_b = 3, \\
+2\delta_{m,3k+1} + \delta_{m,3k+2} & \text{otherwise}.
\end{cases}\]
For $g = A_1$, the data $H$, $D$, $G$ here reduce to (13.40), hence (13.47) to (13.18). By an analysis parallel with the $A_1$ case, one can establish the power series formulas involving Fermionic forms. They are read off (13.30)–(13.34) by formally replacing the single indices by double ones as $i \to (a, m)$, $j \to (b, k)$, etc. To be concrete, let $v = (v_m^{(a)})_{(a, m) \in H} \in \mathbb{C}^H$, where $v_m^{(a)} = 0$ for all but finitely many $(a, m)$. For $N = (N_m^{(a)})_{(a, m) \in H} \in (\mathbb{Z}_{\geq 0})^H$, we define

$$\mathcal{M}(v, N) = \prod_{(a, m) \in H(N)} \left( \frac{P_m^{(a)} + N_m^{(a)}}{N_m^{(a)}} \right).$$  \hspace{1cm} (13.52)

$$\mathcal{N}(v, N) = \left( \det_{H(N)} F_{am,bk} \right) \prod_{(a, m) \in H(N)} \frac{1}{N_m^{(a)}} \left( \frac{P_m^{(a)} + N_m^{(a)}}{N_m^{(a)}} - 1 \right).$$  \hspace{1cm} (13.53)

where the binomial is the generalized one (13.21). We have set $H(N) = \{ (a, m) \in H : N_m^{(a)} \neq 0 \}$ and $\det_{H(N)}$ denotes $\det_{(a, m), (b, k) \in H(N)}$. Define further

$$P_m^{(a)} = \sum_{(b, k) \in H} \min(m, k)v_k^{(b)} - \sum_{(b, k) \in H} (\alpha_a|\alpha_b) \min(t_b m, t_k k) N_k^{(b)}.$$  \hspace{1cm} (13.54)

$$F_{am,bk} = \delta_{ab} \delta_{mk} P_m^{(a)} + (\alpha_a|\alpha_b) \min(t_b m, t_k k) N_k^{(b)}.$$  \hspace{1cm} (13.55)

With these definitions we have

**Theorem 13.11** ([68, 80, 81, 249, 252]). The following power series formulas are valid:

$$\prod_{(a, m) \in H} (Q_m^{(a)})_{v_m^{(a)}} = \sum_{\lambda} \mathcal{M}(v, N) e^{-\sum_{a=1}^r \omega_a N_m^{(a)}} = \sum_{\mathcal{N}(v, N)} e^{-\sum_{a=1}^r \omega_a N_m^{(a)}}$$  \hspace{1cm} (13.56)

where the sums run over $N = (N_m^{(a)})_{(a, m) \in H} \in (\mathbb{Z}_{\geq 0})^H$ without any constraints. The symbol $\Delta_+$ denotes the set of positive roots of $g$.

See section 13.8 for how this theorem was established by integrating many works.

Let us turn to the special case $v_m^{(a)} \in \mathbb{Z}_{\geq 0}$ for any $(a, m) \in H$. Then the power series (13.56) actually truncates to a polynomial, and theorem 13.11 implies the Fermionic formulas for the branching coefficient $b_\lambda$ and the weight multiplicity $c_\lambda$ in (13.2). To write them down, we introduce

$$\mathcal{M}_\lambda = \sum_{N} \mathcal{M}(v, N), \quad \mathcal{N}_\lambda = \sum_{N} \mathcal{N}(v, N) \quad (\lambda \in \sum_{a=1}^r \mathbb{Z} \omega_a).$$  \hspace{1cm} (13.57)

where the sums run over $N = (N_m^{(a)})_{(a, m) \in H} \in (\mathbb{Z}_{\geq 0})^H$ satisfying the weight condition

$$\lambda = \sum_{(a, m) \in H} m v_m^{(a)} \omega_a - \sum_{(a, m) \in H} m N_m^{(a)} \alpha_a.$$  \hspace{1cm} (13.58)

Then the following is a corollary of theorem 13.11:

$$\prod_{a,m} (Q_m^{(a)})_{v_m^{(a)}} = \sum_{\lambda} b_\lambda \chi(V_\lambda), \quad b_\lambda = \mathcal{M}_\lambda \quad \text{for} \quad \lambda \in \sum_{a=1}^r \mathbb{Z} \omega_a,$$  \hspace{1cm} (13.59)

$$\prod_{a,m} (Q_m^{(a)})_{v_m^{(a)}} = \sum_{\lambda} c_\lambda e^\lambda, \quad c_\lambda = \mathcal{N}_\lambda \quad \text{for} \quad \lambda \in \sum_{a=1}^r \mathbb{Z} \omega_a.$$  \hspace{1cm} (13.60)
As the generalization of (13.24), we further introduce
\[
\mathcal{M}_\lambda = \sum_N \mathcal{M}(\nu, N),
\]
where the sum \(\sum_N\) extends over \(N = \left(N_m^{(\nu)}\right)_{\nu,m \in \mathbb{H}} \in (\mathbb{Z}_{\geq 0})^H\) satisfying (13.58) and the extra condition that \(P_m \geq 0\) whenever \(N_m \geq 1\). Then the following is the \(g\) version of theorem 13.5.

**Theorem 13.12** ([247, 248, 253, 254]). For \(\lambda \in \sum_{\nu,m} \mathbb{Z}_{\geq 0} \omega_a\), the equality \(b_\lambda = \mathcal{M}_\lambda\) is valid.

### 13.7. \(Q^{(a)}_m\) as a classical character

Here we present the expansion of \(Q^{(a)}_m\) into classical characters. Such an example has already been given in (4.24) for the rank 2 algebras \(g = A_2, B_2, C_2\) and \(G_2\). Here are a few examples from \(E_8\):

\[
Q^{(a)}_1(\lambda) = \chi(V_\omega(a)) + \chi(V_0),
\]
\[
Q^{(a)}_2(\lambda) = \chi(V_{2\omega(a)}) + \chi(V_0),
\]
\[
Q^{(a)}_3(\lambda) = \chi(V_{2\omega(a)}) + 3\chi(V_{2\omega(a)}) + 4\chi(V_0) + \chi(V_{\omega(a)+\omega(a)}).
\]

which satisfy a \(Q\)-system relation \((Q^{(1)}_1)^2 = Q^{(2)}_1 + Q^{(2)}_2\) for instance. In general from (13.59) and (13.58), the expansion takes the form

\[
Q^{(a)}_m = \chi(V_{m\omega(a)}) + \sum_{\lambda < m\omega(a)} \text{called } \left[\begin{array}{c}
\text{"children"} \\
\lambda \end{array}\right],
\]

where \(b_\lambda\) is obtained by specializing \(v^{(a)}_m\) in (13.59) or theorem 13.12. As we see in the above example, the description of the children is complicated in general for \(g\) of exceptional types. However, for nonexceptional \(g\), they can be described by simple combinatorial rules given below. For simplicity we write \(\chi(V_\lambda)\) as \(\chi(\lambda)\).

For \(g = A_r\), there are no children:

\[
Q^{(a)}_m = \chi(m\omega_a).
\]

To check the relation \((Q^{(a)}_m)^2 = Q^{(a)}_{m-1}Q^{(a)}_m + Q^{(a-1)}_mQ^{(a+1)}_m\) is an easy exercise on Schur functions. It is customary to depict the weights \(m_1\omega_1 + \cdots + m_r\omega_r\) \((m_i \in \mathbb{Z}_{\geq 0})\) as a Young diagram. The rule is to regard each \(\omega_a\) as a depth \(a\) column. Thus, (13.63) is represented as the \(a \times m\) rectangle Young diagram. As we will see, in the other nonexceptional algebras, the children for most \(Q^{(a)}_m\) are described by removals of dominos from the \(a \times m\) rectangle.

For \(g = C_r\), we have

\[
Q^{(a)}_m = \left\{\begin{array}{ll}
\chi(k_1\omega_1 + \cdots + k_a\omega_a) & 1 \leq a \leq r - 1,
\chi(m\omega_r) & a = r,
\end{array}\right.
\]

where the sum is taken over nonnegative integers \(k_1, \ldots, k_a\) that satisfy \(k_1 + \cdots + k_a \leq m\), \(k_j \equiv m\delta_{ja} \mod 2\) for all \(1 \leq j \leq a\). The summands correspond to the removals of horizontal dominos (shape \(1 \times 2\) Young diagram).
For $g = B_r$ and $D_r$, we have

$$Q_m^{(a)} = \sum \chi(k_{a_0}a_0 + \cdots + k_{a-2}a_{-2} + \cdots + k_0a_0) \quad 1 \leq a \leq r',$$

$$r' = r \quad \text{for} \quad B_r, \quad r' = r - 2 \quad \text{for} \quad D_r, \quad a_0 \equiv a \mod 2, \quad a_0 = 0 \text{ or } 1,$$

(13.65)

Here $\omega_0 = 0$. The sum extends over nonnegative integers $k_{a_0}, k_{a_0+2}, \ldots, k_0$ obeying the constraint $t_0(k_{a_0} + k_{a_0+2} + \cdots + k_{a-2}) + k_a = m$. The summands correspond to the removals of vertical dominos (shape $2 \times 1$ Young diagram).

### 13.8. Bibliographical notes and further aspects

The $Q$-system$^{36}$ for $g$ first appeared in [81, 92]. In [81], it was claimed that (in a nowadays terminology) $Q_m^{(a)}$ satisfies the $Q$-system, and the generalization of Bethe’s fermionic formula $b_{\lambda} = \mathcal{M}_{\lambda}$ (theorem 13.12) holds. These assertions became known as the Kirillov–Reshetikhin conjecture. Together with the closely related formulas $b_{\lambda} = \mathcal{M}_{\lambda}$, $c_{\lambda} = \mathcal{N}_{\lambda}$ and theorem 13.11, they have now been established by the integration of numerous works since then. Here we shall only mention the literature that is most relevant to our presentation in this section. More detailed accounts are available in [249, section 5.7] and [13, section 1].

The method of the multivariable residue analysis was initiated in [243, 255] for $A_1$, $A_r$ and extended to $g$ in [252]. The main conclusion from this approach is that the Fermionic formula $b_{\lambda} = \mathcal{M}_{\lambda}$ follows from the $Q$-system and a convergence property of $Q_m^{(a)}$ as $m \to \infty$. It was found in [80, 245] that these properties also lead to another version of the Fermionic formula $c_{\lambda} = \mathcal{N}_{\lambda}$. The two stories $b_{\lambda} = \mathcal{M}_{\lambda}$ (‘XXX type’) and $c_{\lambda} = \mathcal{N}_{\lambda}$ (‘XXZ type’) were put in a unified perspective by a version of multivariable Lagrange inversion [249] with a proper passage from the finite to infinite $Q$-systems. Last but a crucial input that $\chi_q(W_m^{(a)})$ actually satisfies the $Q$-system for any $g$ was proved as a corollary of theorem 4.8 [67, 68] together with the convergence property [68, theorem 3.3(2)]. Thus, theorem 13.3 (1) and (2) for $A_1$ are due to [243] and [245], respectively. Its $g$ version, theorem 13.11, is an outcome of [68, 80, 81, 249, 252].

The identity $b_{\lambda} = \mathcal{M}_{\lambda}$ (theorem 13.12) has been proved by combinatorial methods in [247, 248, 253] for $A_r$. Thanks to $b_{\lambda} = \mathcal{M}_{\lambda}$, it suffice to show $\mathcal{M}_{\lambda} = \mathcal{M}_{\lambda}$ for dominant $\lambda$. A uniform proof of the latter for all $g$ is given in [254] by a generating function method.

The expansion of $Q_m^{(a)}$ into classical characters as in section 13.7 also has a long history going back to [147]. By many works, e.g. [65, 82, 86, 252], such formulas have been established for all $Q_m^{(a)}$’s for $A_r, B_r, C_r, D_r$ and many ones from $E_{6,7,8}, F_4$ and $G_2$.

We conclude with a few remarks on further aspects which have not been discussed in this section.

(i) The series $\mathcal{M}(w)$ (13.30) has an interpretation of the grand partition function of the ideal gas with the Haldane exclusion statistics [256]. The finite $Q$-system (13.29) appeared in [256] as the thermal equilibrium condition for the distribution functions of the same system.$^{37}$ See also [257] for another interpretation. The one-variable case (example 13.8) also appeared in [258] as the thermal equilibrium condition for the distribution function of the Calogero–Sutherland model. As an application of our second formula in theorem

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$^{36}$ They are named so in [1] after the notation $Q_m^{(a)}$ due to [81, 92], which was adopted to mean ‘quantum character’ [304].

$^{37}$ For the translation, substitute $w_i = Q_i/(1 - Q_i)$ in equation (10) in [256].
13.7, we can quickly reproduce the ‘cluster expansion formula’ in [259, equation (129)]. Setting $D = I$ in (13.30)–(13.34), we have

$$\ln Q_i(w) = \left[ \frac{\partial}{\partial \nu_i} N^\nu_i(w) \right]_{\nu_i = 0} = \sum_{N} \det_{H(N)} F_{jk}(0, N) \prod_{j \in H(N)} \left( P_j(0, N) + N_j - 1 \right) N_j, \quad (13.66)$$

where $\{Q_i(w)\}_{i \in H}$ is the solution of (13.28). The Sutherland–Wu equation also plays an important role for the CFT spectra. See [260] and references therein.

(ii) There are decent $q$-analogs of $b_\lambda$ and $c_\lambda$ by using the crystal base of $U_q(\hat{g})$ [261]. A typical one for $A_r$ is the Kostka–Foulkes polynomial [135]. Correspondingly, there is a $q$-analog of the Fermionic formula $b_\lambda = M_\lambda$ known as the ‘$X = M$ conjecture’ [252, 262], which has been solved for $A_r$ [247, 248] and some other cases. There is also a conjectural $q$-analog of $\mathcal{M}_\lambda = M_\lambda$ [252, equation (4.21)]. These formulas have the level restricted versions and are related to RSOS models and CFT characters. For a historical survey, see [262, section 1] and [263].

(iii) The $Q$-system, theorem 13.11 and the expansion formula as in section 13.7 have been generalized to twisted quantum affine Lie algebras $U_q(X_\kappa N)$ [13, 249, 262, 264].

14. $Y$-system and thermodynamic Bethe ansatz

In this section we explain how the level $\ell$ restricted $Y$-system for $\mathfrak{g}$ (2.11)–(2.15) emerges from the TBA equation associated with $U_q(\hat{g})$ at $q = \exp \left( \frac{\pi \sqrt{-1}}{t(\ell + h^\vee)} \right)$. (See (2.1) and (2.3) for $t$ and $h^\vee$.) The TBA equation is relevant to level $\ell$ RSOS models and quoted from section 15. We also introduce the constant $Y$-system and explain its relation to the $Q$-system in the both unrestricted and level restricted versions. Conjecturally, the level restricted $Q$-system allows a solution via a specialization of characters to the $q$-dimension with $q$ being the root of unity. They play important roles in the dilogarithm identity related to conformal field theory and the TBA analysis of RSOS models. We use the notation

$$\ell_a = t_a \ell, \quad L = \ell + h^\vee, \quad (14.1)$$

$$H_\ell = \{(a, m) \mid a \in I, 1 \leq m \leq \ell_a - 1, m \in \mathbb{Z}\}, \quad (14.2)$$

where $t_a$ is defined in (2.1) and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (2.3). The set $H_\ell$ is a level truncation of $H$ (13.48). We will further use

$$t_{ab} = \max(t_a, t_b), \quad (14.3)$$

$$N_{ab} = 2\delta_{ab} - B_{ab}, \quad B_{ab} = B_{ba} = \frac{t_b}{t_{ab}} C_{ab} = \begin{cases} 2 & C_{ab} = 2, \\ -1 & C_{ab} < 0, \\ 0 & C_{ab} = 0. \end{cases} \quad (14.4)$$

This $B_{ab}$ is the same as (12.11).

14.1. $Y$-system for ADE and deformed Cartan matrices

For simplicity we first deal with the simply laced algebras $\mathfrak{g} = A_r, D_r$ and $E_{6,7,8}$. In section 15.1, we obtain the TBA equation for the level $\ell (\ell \in \mathbb{Z}_{\geq 2}$ critical RSOS model in
It is the following nonlinear integral equation on the functions \( \{ e_m^{(a)}(u) \} (a, m) \in H_\ell, u \in \mathbb{R} \):

\[
\frac{\epsilon \beta \gamma \delta_m \delta_m^{(i)}}{4 \cosh(\pi u/2)} = \beta e_m^{(a)}(u) + \int_{-\infty}^{\infty} \ln \left[ \frac{\prod_{b \in I} (1 + \exp(-\beta E_m^{(b)}(u)))^{N_{ab}}}{(1 + \exp(\beta E_m^{(a)}(u)))^{N_{ab}}} \right] dv.
\]

Here \( \beta, \gamma > 0, \epsilon = \pm 1 \) and \( (p, s) \in H_\ell \) are model parameters specifying the temperature, normalization of energy, two critical regimes and representation \( W_\beta^{(p)} \) (fusion type) with which the model is associated, respectively. The physical meaning of \( e_m^{(a)}(u) \) is the pseudo-energy defined by \( \exp(-\beta e_m^{(a)}(u)) = \rho_m^{(a)}(u)/\sigma_m^{(a)}(u) \) in terms of the color \( a \), length \( m \), string density \( \rho_m^{(a)}(u) \) and hole density \( \sigma_m^{(a)}(u) \). More details can be found in section 15.1, but we do not need that background here.

We assume that (14.5) can be analytically continued off the real axis of \( u \) until \( \Im u \leq 1 \). Setting \( u \rightarrow u + \mp i \sigma_0 \), take the sum of the resulting two equations. The lhs vanishes and the rhs is evaluated by means of

\[
\frac{1}{4 \cosh \frac{\pi u}{2}(u - v + i 0)} + \frac{1}{4 \cosh \frac{\pi u}{2}(u - v - i 0)} = \delta(u - v)
\]

as the convolution kernel. By introducing the variable \( Y_m^{(a)}(u) = \exp(-\beta e_m^{(a)}(u)) \), the Boltzmann factor of the pseudoenergy, the result is the logarithm of

\[
Y_m^{(a)}(u - i) Y_m^{(a)}(u + i) = \prod_{b \in I} \left( 1 + Y_m^{(b)}(u) \right)^{N_{ab}}.
\]

This is the \( Y \)-system for \( g = A_\ell, D_\ell \) and \( E_{6,7,8} \),\(^{2.11}\) in the convention that \( Y_m^{(a)}(u + k) \) there becomes \( Y_m^{(a)}(u + ik) \). It is level \( \ell \) restricted since only \( Y_m^{(a)}(u) \) with \( (a, m) \in H_\ell \) are present.

Let us observe another aspect of the \( Y \)-system (14.7). It is written as

\[
\frac{(1 + Y_m^{(a)}(u - i))^{-1}(1 + Y_m^{(a)}(u + i))^{-1}}{(1 + Y_{m-1}^{(a)}(u))^{-1}(1 + Y_{m+1}^{(a)}(u))^{-1}} = \prod_{b \in I} \left( 1 + Y_m^{(b)}(u) \right)^{N_{ab}}.
\]

The lhs and rhs of (14.7) possess parallel structures related to \( A_{\ell-1} \) and \( g \), respectively. In the Fourier space they are encoded in the deformed Cartan matrices with indices corresponding to the length \( m \) and the color \( a \), respectively. To see it, define the Fourier transformation

\[
\hat{f} = \hat{f}(x) \text{ of } f = f(u) \text{ by}
\]

\[
f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x) e^{iux} dx, \quad \hat{f}(x) = \int_{-\infty}^{\infty} f(u) e^{-iux} du.
\]

If we formally interpret the multiplication with \( e^{ix} \) \( (c \in \mathbb{R}) \) in the Fourier \( (x) \) space as the difference operator \( u \rightarrow u - ic \) in the ‘real’ \( (u) \) space, the logarithm of the rhs of (14.8) is assigned with the Fourier component \( \sum_{b \in I} N_{ab} \hat{f}(x) \ln(1 + Y_m^{(b)}) \), where

\[
\hat{N}_{ab}(x) = 2\delta_{ab} \cosh x - N_{ab} \quad \text{(for ADE)}
\]

is the deformed Cartan matrix of \( g \). Actually the Fourier transformation of the TBA equation (14.5) contains \( \sum_{a,b \in I} \hat{N}_{ab}(x) \ln(1 + Y_m^{(b)}) \) so that identity (14.6) works in the real space. Parallel remarks apply to the lhs of (14.8).
We call the functions like $\hat{\mathcal{N}}_{a b}(x)$ TBA kernels as they emerge in the TBA calculation (section 15.1) and play important roles as building blocks of integral kernels in the TBA equation.

### 14.2. TBA kernels

Here we summarize the definitions and useful properties of the TBA kernels for general $g$. In place of (14.10), we redefine $\mathcal{N}_{a b}(x)$ and introduce $\mathcal{K}_{a b}(x)$ as

$$\hat{\mathcal{N}}_{a b}(x) = 2 \delta_{a b} \cosh \left( \frac{x}{l_a} \right) - N_{a b} = B_{a b} + 2 \delta_{a b} \left( \cosh \left( \frac{x}{l_a} \right) - 1 \right),$$

(14.11)

$$\hat{\mathcal{K}}_{a b}(x) = \delta_{a b} - \frac{\delta_{a, m - 1} + \delta_{a, n + 1}}{2 \cosh \left( \frac{x}{l_a} \right)}.$$  

(14.12)

For $(a, m), (b, k) \in H$, we further introduce

$$\hat{\mathcal{A}}_{a b}^{m k}(x) = \frac{\sinh \left( \min \left( \frac{m}{2}, \frac{k}{2} \right) \right) \sinh \left( (\ell - \min \left( \frac{m}{2}, \frac{k}{2} \right)) x \right)}{\sinh \left( \frac{x}{l_a} \right) \sinh(\ell x)},$$

(14.13)

$$\hat{\mathcal{K}}_{a b}^{m k}(x) = \hat{\mathcal{A}}_{a b}^{m k}(x) \hat{\mathcal{N}}_{a b}(x),$$

(14.14)

$$\hat{\mathcal{J}}_{a b}^{m k}(x) = \sum_{n=1}^{\ell_x - 1} \mathcal{K}_{a b}^{m n}(x) \hat{\mathcal{N}}_{a b}^{k n}(x) = \frac{\hat{\mathcal{N}}_{a b}(x) \hat{\mathcal{P}}_{a b}^{m k}(x)}{2 \cosh \left( \frac{x}{l_a} \right)},$$

(14.15)

$$\hat{\mathcal{P}}_{a b}^{m k}(x) = 2 \cosh \left( \frac{x}{l_a} \right) \sum_{n=1}^{\ell_x - 1} \mathcal{A}_{a b}^{m n}(x) \hat{\mathcal{N}}_{a b}^{k n}(x) = \delta_{m k}.$$  

(14.16)

The sum $\sum_{j=1}^{\ell_x - 1}$ in (14.16) is to be understood as zero if $l_a \geq L$. Since the latter expressions in (14.15) and (14.16) do not contain $\ell$, we can and do extend the definition of $\mathcal{J}_{a b}^{m k}(x)$ and $\hat{\mathcal{P}}_{a b}^{m k}(x)$ to all the nonnegative integers $m, k \geq 0$. The inverse Fourier transform $\mathcal{J}_{a b}^{m k}(u)$ is an even function of $u$ but $\mathcal{J}_{a b}^{m k}(u) \neq \mathcal{J}_{a b}^{m k}(-u)$ in general as opposed to $\hat{\mathcal{A}}_{a b}^{m k}(x) = \hat{\mathcal{A}}_{a b}^{m k}(x)$ and $\hat{\mathcal{K}}_{a b}^{m k}(x) = \hat{\mathcal{K}}_{a b}^{m k}(x)$. The $\mathcal{K}_{a b}^{m n}(x)$ in (14.12) should be distinguished from $\hat{\mathcal{K}}_{a b}^{m n}(x)$ in (14.14).

The following relations are easily checked:

$$2 \cosh \left( \frac{x}{l_a} \right) \sum_{n=1}^{\ell_x - 1} \mathcal{A}_{a b}^{m n}(x) \hat{\mathcal{N}}_{a b}^{k n}(x) = \delta_{m k},$$

(14.17)

$$2 \cosh \left( \frac{x}{l_a} \right) \sum_{n=1}^{\ell_x - 1} \mathcal{A}_{a b}^{m n}(x) \hat{\mathcal{K}}_{a b}^{k n}(x) = \delta_{m k},$$

(14.18)

$$\mathcal{J}_{a b}^{m k}(x) = \delta_{a b} \delta_{m k} - \frac{N_{a b} \hat{\mathcal{P}}_{a b}^{m k}(x)}{2 \cosh \left( \frac{x}{l_a} \right)}.$$  

(14.19)

All the TBA kernels (14.11)–(14.16) are deduced from $\hat{\mathcal{A}}_{a b}^{m n}(x)$ and $\hat{\mathcal{N}}_{a b}(x)$ by using these relations. The basic ones $\hat{\mathcal{A}}_{a b}^{m n}(x)$ and $\hat{\mathcal{N}}_{a b}(x)$ are obtained as

$$\int_{-\infty}^{\infty} du \ e^{-iux} \frac{\partial}{\partial u} \left( \hat{\mathcal{A}}_{a b}^{m n}(u) \right) = \hat{\mathcal{A}}_{a b}^{m n}(x) |_{\ell \rightarrow L},$$

(14.20)
\[
\int_{-\infty}^{\infty} du e^{-iu} \frac{\partial}{\partial u} \Theta_m^{ab}(u, (\alpha_a|\alpha_b)) = -\delta_{ab}\delta_{mk} + \hat{N}_{ab}(x)\hat{A}_{ab}^m(x)|_{x \rightarrow L}, \tag{14.21}
\]

where \(\Theta_m^{ab}(u, (\alpha_a|\alpha_b))\) (15.3) and \(\Theta_{ab}^{mk}(u, (\alpha_a|\alpha_b))\) (15.4) are the logarithm of the lhs and the rhs of the Bethe equation under the string hypothesis, respectively. See (15.1)–(15.4).

When \(g\) is simply laced, the TBA kernels simplify as

\[
\hat{A}_{ab}^m(x) \equiv \frac{\sinh(\min(m, k)x) \sinh(\ell - \max(m, k)x)}{\sinh x \sinh(\ell x)}, \tag{14.22}
\]

\[
\hat{N}_{ab}(x) \delta_{mk} = \frac{\hat{N}_{ab}(x)\delta_{mk}}{2 \cosh x} = \left( \delta_{ab} - \frac{N_{ab}}{2 \cosh x} \right) \delta_{mk}, \tag{14.23}
\]

\[
\hat{P}_{ab}^m(x) = \delta_{mk}. \tag{14.24}
\]

### 14.3. \(Y\)-system for \(g\) from the TBA equation

Let us derive the level \(\ell\) restricted \(Y\)-system for general \(g\) from the TBA equation. We quote the latter obtained in (15.13) with the notation \(Y_m^{(a)}(u) = \exp (-\beta\epsilon^{(a)}_m(u))\):

\[
\frac{e\beta \gamma_m \delta_{sm}}{4t_p^{-1}\cosh(t_p\pi u/2)} = -\ln Y_m^{(a)}(u) - \int_{-\infty}^{\infty} dv \frac{\ln \left[ (1 + Y_m^{(a)}(v)^{-1})(1 + Y_m^{(a)}(v)^{-1}) \right]}{4t_v^{-1}\cosh(t_v\pi (u - v)/2)} + \sum_{(b,k) \in H_l} N_{ab} \int_{-\infty}^{\infty} dv \frac{P_{ab}^{mk} \ast \ln \left[ (1 + Y_k^{(b)}(v)^{-1}) \right]}{4t_v^{-1}\cosh(t_v\pi (u - v)/2)}. \tag{14.25}
\]

\(P_{ab}^{mk}\) is defined via its Fourier component (14.16) and \(\ast\) denotes the convolution

\[
(f_1 \ast f_2)(u) = \int_{-\infty}^{\infty} dv f_1(u - v) f_2(v). \tag{14.26}
\]

As the simply laced case, we assume that (14.25) can be analytically continued off the real axis of \(u\) until \(|\Im u| \leq t_a^{-1}\). Then the sum after the shifts \(u \rightarrow u \pm t_a^{-1}i \mp 0i\) eliminates the lhs, giving

\[
\ln \left[ Y_m^{(a)}(u - \frac{i}{t_a}) \right] Y_m^{(a)}(u + \frac{i}{t_a}) = -\ln \left[ (1 + Y_m^{(a)}(u)^{-1})(1 + Y_m^{(a)}(u)^{-1}) \right] + \sum_{(b,k) \in H_l} N_{ab} \ln \left[ (1 + Y_k^{(b)}(u)) \right]. \tag{14.27}
\]

For simply laced algebras, \(P_{ab}^{mk}(u) = \delta_{mk} \delta(u)\) by (14.24), and we are done. To illustrate the general case, take \(g = G_2\) with \((a, b) = (1, 2)\) as an example. Then \((t_a, t_b) = (1, 3)\) and (14.16) reads

\[
\hat{P}_{ab}^{mk}(x) = \hat{P}_{12}^{mk}(x) = (e^{\frac{i}{3}} + 1 + e^{-\frac{i}{3}})\delta_{3m,k} + \delta_{3m-2,k} + \delta_{3m+2,k} + (e^{\frac{i}{3}} + e^{-\frac{i}{3}})(\delta_{3m-1,k} + \delta_{3m+1,k}). \tag{14.28}
\]

\[
\hat{P}_{12}^{mk}(u) = \delta \left( u - \frac{2i}{3} \right) \delta(u) + \delta \left( u + \frac{2i}{3} \right) \delta_{3m,k} + \delta(u)(\delta_{3m-2,k} + \delta_{3m+2,k}) + \delta \left( u - \frac{i}{3} \right) + \delta \left( u + \frac{i}{3} \right) (\delta_{3m-1,k} + \delta_{3m+1,k}). \tag{14.29}
\]
If \( \ln \left( 1 + Y_k^{(2)}(v) \right) \) is analytic in the strip \( |\text{Im} v| \leq \frac{2}{5} \), and decays rapidly as \( |\text{Re} v| \to \infty \), one can shift the convolution integral \( \int dv \mathcal{P}_{12}^{mk}(a - v) \ln(1 + Y_k^{(2)}(v)) \) off the real axis of \( v \) to pick the support of delta functions. In this way the last term in (14.27) gives the logarithm of

\[
\left( 1 + Y_{3m}^{(2)} \left( u - \frac{2i}{3} \right) \right) \left( 1 + Y_{3m}^{(2)}(u) \right) \left( 1 + Y_{3m-2}^{(2)}(u) \right) \left( 1 + Y_{3m+2}^{(2)}(u) \right)
\]

\[
\times \left( 1 + Y_{3m-1}^{(2)} \left( u - \frac{i}{3} \right) \right) \left( 1 + Y_{3m-1}^{(2)}(u + \frac{i}{3}) \right)
\]

\[
\times \left( 1 + Y_{3m+1}^{(2)} \left( u - \frac{i}{3} \right) \right) \left( 1 + Y_{3m+1}^{(2)}(u + \frac{i}{3}) \right).
\]

This is the numerator of the rhs in the first relation of the \( Y \)-system for \( G_2 \) (2.15) with the shift unit multiplied by \( i \).

The general case is similar and (14.27) gives rise to the logarithmic form of the (restricted) \( Y \)-system for \( g \). On account of (14.16), in general it suffices to assume that \( \ln \left( 1 + Y_m^{(a)}(u) \right) \) is analytic in the strip \( |\text{Im} u| \leq \frac{2}{5} \), and decays rapidly as \( |\text{Re} u| \to \infty \).

If the analyticity argument can be left out, the \( Y \)-system is deduced more quickly from the TBA kernels in the Fourier space. In fact, one can start with the TBA equation (15.12) without the lhs\(^{39} \):

\[
\sum_{n=1}^{\ell-1} \mathcal{K}_m^{mn}(x) \ln(1 + (Y_n^{(a)}))^{-1} = \sum_{(b,k) \in H_t} \mathcal{J}_m^{bk}(x) \hat{Y}_m^{(b)}(x).
\]  

Multiply with \( 2 \cosh \left( \frac{x}{2} \right) \) and use (14.12) and (14.19) to rearrange it slightly as

\[
2 \cosh \left( \frac{x}{l_a} \right) \ln Y_m^{(a)} = \sum_{(b,k) \in H_t} N_{ab} \hat{Y}_m^{bk}(x) \ln(1 + Y_k^{(b)})
\]

\[
- \ln \left[ 1 + (Y_n^{(a)})^{-1} \right] (1 + (Y_{n+1}^{(a)})^{-1}).
\]  

This is the \( Y \)-system if \( \cosh \left( \frac{x}{l_a} \right) \) and \( \hat{Y}_m^{bk}(x) \) (14.16) are regarded as the difference operators as mentioned after (14.9).

We have demonstrated that the \( Y \)-system is a difference equation whose structure is governed by the TBA kernels. On the other hand, recall that theorem 2.5 offers another route to obtain the \( Y \)-system by invoking its connection to the \( T \)-system. It is yet to be understood why the two ‘characterizations’ of the \( Y \)-system coincide.

### 14.4. Constant \( Y \)-system

In either unrestricted or level \( \ell \) restricted \( Y \)-system, one can discard the dependence of \( Y_m^{(a)}(u) \) on \( u \). The resulting algebraic equation on \( Y_m^{(a)} = Y_m^{(a)}(u) \) is called the unrestricted or level \( \ell \) restricted constant \( Y \)-system\(^{40} \).

The unrestricted constant \( Y \)-system for \( g \) is the set of algebraic equations on \( \{ Y_m^{(a)} \mid (a, m) \in H \} \). (\( H \) is defined in (13.48).)

For simply laced \( g \), it has the form

\[
(Y_m^{(a)})^2 = \frac{\prod_{b \in T \cap C_{a(a)}} \left( 1 + Y_m^{(b)} \right)}{(1 + (Y_m^{(a)})^{-1}) (1 + (Y_m^{(a)})^{-1})},
\]

\[
(14.30)
\]

\(^{38}\) Actually \( |\text{Im} v| \leq \frac{2}{5} \) for \( \ln \left( 1 + Y_m^{(a)}(v) \right) \) and \( |\text{Im} v| \leq \frac{1}{5} \) for \( \ln \left( 1 + Y_{m+1}^{(2)}(v) \right) \) suffice.

\(^{39}\) According to our previous argument, it is actually more proper to suppress the lhs after multiplying \( 2 \cosh \left( \frac{x}{l_a} \right) \).

\(^{40}\) The level \( \ell \) restricted constant \( Y \)-system here is the same as the one introduced in section 5.1.
where $\left(Y_{0}^{(a)}\right)^{-1} = 0$. See (2.11). The nonsimply laced case is similarly written down from (2.12)–(2.15).

For $g = B_r$,

\[
\left(Y_{m}^{(a)}\right)^2 = \frac{1 + Y_{m}^{(a-1)}}{(1 + Y_{m-1}^{(a)}) (1 + Y_{m+1}^{(a)})} (1 \leq a \leq r - 2),
\]

\[
\left(Y_{m}^{(r-1)}\right)^2 = \frac{1 + Y_{m}^{(r-2)}}{(1 + Y_{m-1}^{(r-2)}) (1 + Y_{m+1}^{(r-2)})} \left(1 + Y_{2m}^{(r)}\right)^2 (1 + Y_{2m+1}^{(r)}),
\]

\[
\left(Y_{2m}^{(r)}\right)^2 = \frac{1 + Y_{2m}^{(r-1)}}{(1 + Y_{2m-1}^{(r-1)}) (1 + Y_{2m+1}^{(r-1)})} (1 + Y_{2m+1}^{(r)}),
\]

\[
\left(Y_{2m+1}^{(r)}\right)^2 = \frac{1 + Y_{2m+1}^{(r-2)}}{(1 + Y_{2m+1}^{(r-2)}) (1 + Y_{2m+1}^{(r-2)})} (1 + Y_{2m+1}^{(r-1)}),
\]

(14.31)

For $g = C_r$,

\[
\left(Y_{m}^{(a)}\right)^2 = \frac{1 + Y_{m}^{(a-1)}}{(1 + Y_{m-1}^{(a)}) (1 + Y_{m+1}^{(a)})} (1 \leq a \leq r - 2),
\]

\[
\left(Y_{2m}^{(r-1)}\right)^2 = \frac{1 + Y_{2m}^{(r-2)}}{(1 + Y_{2m-1}^{(r-2)}) (1 + Y_{2m+1}^{(r-2)})} (1 + Y_{2m+1}^{(r-1)}),
\]

(14.32)

For $g = F_4$,

\[
\left(Y_{m}^{(1)}\right)^2 = \frac{1 + Y_{m}^{(2)}}{(1 + Y_{2m}^{(1)}) (1 + Y_{m+1}^{(2)})},
\]

\[
\left(Y_{m}^{(2)}\right)^2 = \frac{1 + Y_{m}^{(3)}}{(1 + Y_{2m}^{(3)}) (1 + Y_{m+1}^{(2)})} (1 + Y_{2m+1}^{(3)}),
\]

\[
\left(Y_{m}^{(3)}\right)^2 = \frac{1 + Y_{m}^{(4)}}{(1 + Y_{2m}^{(4)}) (1 + Y_{m+1}^{(2)})} (1 + Y_{2m+1}^{(4)}),
\]

\[
\left(Y_{2m+1}^{(1)}\right)^2 = \frac{1 + Y_{2m+1}^{(4)}}{(1 + Y_{2m}^{(4)}) (1 + Y_{2m+1}^{(4)})},
\]

\[
\left(Y_{2m+1}^{(2)}\right)^2 = \frac{1 + Y_{2m+1}^{(4)}}{(1 + Y_{2m}^{(4)}) (1 + Y_{2m+1}^{(4)})},
\]

\[
\left(Y_{2m+1}^{(3)}\right)^2 = \frac{1 + Y_{2m+1}^{(4)}}{(1 + Y_{2m}^{(4)}) (1 + Y_{2m+1}^{(4)})},
\]

(14.33)

For $g = G_2$,

\[
\left(Y_{m}^{(1)}\right)^2 = \frac{1 + Y_{m}^{(2)}}{(1 + Y_{m}^{(1)}) (1 + Y_{m+1}^{(2)})} (1 + Y_{m}^{(1)}) (1 + Y_{m+1}^{(2)}) (1 + Y_{m+2}^{(2)}),
\]

\[
\left(Y_{m}^{(2)}\right)^2 = \frac{1 + Y_{m}^{(1)}}{(1 + Y_{m}^{(1)}) (1 + Y_{m+1}^{(2)})} (1 + Y_{m}^{(1)}) (1 + Y_{m+1}^{(2)}) (1 + Y_{m+2}^{(2)}),
\]

\[
\left(Y_{3m}^{(1)}\right)^2 = \frac{1 + Y_{3m}^{(2)}}{(1 + Y_{3m}^{(1)}) (1 + Y_{3m+1}^{(2)})},
\]

\[
\left(Y_{3m}^{(2)}\right)^2 = \frac{1 + Y_{3m}^{(1)}}{(1 + Y_{3m}^{(1)}) (1 + Y_{3m+1}^{(2)})},
\]

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\[
\begin{align*}
(y^{(2)}_{3m+1})^2 &= \frac{1}{\left(1 + (y^{(2)}_{3m})^{-1}\right)\left(1 + (y^{(2)}_{3m+2})^{-1}\right)}, \\
(y^{(2)}_{3m+2})^2 &= \frac{1}{\left(1 + (y^{(2)}_{3m+1})^{-1}\right)\left(1 + (y^{(2)}_{3m+2})^{-1}\right)}.
\end{align*}
\]

The level \( \ell \) restricted constant \( Y \)-system for \( g \) is obtained from (14.30)-(14.34) by setting \( (Y^{(a)}_{\ell,\ell})^{-1}_m = 0 \) and naturally restricting the variables \( \{Y^{(a)}_m \mid (a, m) \in H\} \) to \( \{Y^{(a)}_m \mid (a, m) \in H_\ell\} \). \( H_\ell \) is defined in (14.2).)

For the TBA analysis, it is useful to recognize that the level \( \ell \) restricted constant \( Y \)-system is expressed in terms of the 0th Fourier component \( (x = 0) \) of the TBA kernels. We prepare the notations for them:

\[
\begin{align*}
\tilde{C}_m &= 2\tilde{K}_m(0), \\
K_{ab} &= \tilde{K}_{ab}(0) = \left(\min(t_m, t_n) - \frac{mk}{\ell}\right)(\alpha_a|\alpha_b), \\
P_{ab} &= \tilde{P}_{ab}(0) = \frac{t_a}{t_m} \delta_{m+n,k} + \sum_{j=1}^{t_m-t_a} j.(\delta_{a,m}^j + \delta_{b,(m-1)+j,k}), \\
J_{ba} &= \tilde{J}_{ba}(0) = \frac{1}{2} \sum_{n=1}^{t_a-1} \tilde{C}_n K_{mn} - \delta_{ab} \delta_{mk} - \frac{1}{2} N_{ab} P_{km} = -\frac{1}{2} G_{am,bk},
\end{align*}
\]

where (14.14)-(14.19) are used. \( G_{am,bk} \) is defined in (13.51). The sum \( \sum_{j=1}^{t_m-t_a} \) in (14.37) is to be understood as zero if \( t_a \geq t_m \) as in (14.16). Note that \( K_{ab} = K_{ba} \) but \( P_{ab} \neq P_{ba} \) and \( J_{ab} \neq J_{ba} \) in general. We have \( P_{ab} \in \mathbb{Z} \). From (14.35), the specialization \( x = 0 \) of (14.17) gives

\[
\sum_{n=1}^{t_a-1} \tilde{A}_{na}(0) \tilde{C}_n = \delta_{mk}.
\]

Using \( N_{ab} \) and \( P_{ab} \) in the above, the level \( \ell \) restricted constant \( Y \)-system is expressed uniformly for all \( g \) as

\[
(y^{(a)}_m)^2 = \frac{\prod_{(b,k) \in H_\ell} (1 + y^{(b)}_k)^N_{ab} P_{ab}}{(1 + (y^{(a)}_m)^{-1}_k)(1 + (y^{(a)}_m)^{-1}_m)} \quad ((a, m) \in H_\ell),
\]

where \( (Y^{(a)}_m)^{-1}_m = 0 \). This is easily seen from (14.29). The unrestricted version is similarly presented by replacing \( H_\ell \) here with \( H \).

The level \( \ell \) restricted constant \( Y \)-system is expressed in several guises:

\[
\sum_{n=1}^{t_a-1} \tilde{A}_{na}(0) \ln (1 + (y^{(a)}_m)^{-1}) = \sum_{(b,k) \in H_\ell} \tilde{J}_{ab}(0) \ln (1 + y^{(b)}_k),
\]

\[
f^{(a)}_m = \prod_{(b,k) \in H_\ell} (1 - f^{(b)}_k)^{K_{ab}}, \quad \text{where } f^{(a)}_m = \frac{Y^{(a)}_m}{1 + Y^{(a)}_m}.
\]

The form (14.41) directly follows from (14.28) and shows up naturally as the TBA equation in a certain asymptotic limit. See (15.18). On the other hand, (14.42) is deduced from (14.35)
and (14.38). It is related to the conjectural $q$-series formula [106] for the string function $c_{kn}^{(a)}(q)$ [11] of the level $\ell$ vacuum module of $\hat{g}$ up to a power of $q$:

$$\prod_{j=1}^{\infty} (1 - q^j)^{\text{rank} \, g} \sum (N_a^{(a)})_{(a,m) \in H_\ell} q^{|\text{rank} \, g|} \ell^{(a)}_{(a,m) \in H_\ell} \sum (N_a^{(a)})_{(a,m) \in H_\ell} q^{|\text{rank} \, g|} \ell^{(a)}_{(a,m) \in H_\ell}$$

(14.43)

The outer sum is over $N_a^{(a)} \in \mathbb{Z}_{\geq 0}$ such that $\sum_{(a,m) \in H_\ell} m N_a^{(a)} \alpha_\ell \equiv \lambda \mod \ell \sum_{a \in \ell} \mathbb{Z} \alpha_a$. In fact, the crude approximation of the extremum condition on the summand is

$$q \sum_{a,b,h} K_{ab}^{(a)} N_a^{(a)} = 1 - q^{|\text{rank} \, g|} \ell^{(a)}_{(a,m) \in H_\ell},$$

which is cast into (14.42) upon setting $q^{|\text{rank} \, g|} \ell^{(a)}_{(a,m) \in H_\ell} = 1 - f^{(a)}_{ba}$.

The level $\ell$ restricted constant $Y$-system is the set of $|H_\ell|$ algebraic equations on the same number of unknowns $\{Y_m^{(a)}(\alpha_m) \in H_\ell\}$. With regard to its solution, the uniqueness of the positive real one (theorem 5.1) is fundamental. The concrete construction of the solution is a subject of the subsequent sections 14.5 and 14.6.

### 14.5. Relation with the $Q$-system

Recall that the unrestricted $Q$-system for $\mathfrak{g}$ (13.45) is

$$(Q_m^{(a)})^2 = Q_{m-1}^{(a)} Q_m^{(a)} + (Q_m^{(a)})^2 \prod_{(b,k) \in H_\ell} (Q_{k}^{(b)})^{-2 j_{ba}^m},$$

(14.44)

where we have replaced the notation of the power $G_{am,bk}$ by (14.38). Given $\ell \in \mathbb{Z}_{\geq 1}$, we define the level $\ell$ restricted $Q$-system for $\mathfrak{g}$ to be the relations obtained from (14.44) by restricting the variables $Q_m^{(a)}$ to those with $(a, m) \in H_\ell$ by imposing $Q_{m}^{(a)} = 1$. Thus it reads

$$(Q_m^{(a)})^2 = Q_{m-1}^{(a)} Q_m^{(a)} + (Q_m^{(a)})^2 \prod_{(b,k) \in H_\ell} (Q_{k}^{(b)})^{-2 j_{ba}^m} \quad \text{for} \quad (a, m) \in H_\ell.$$

(14.45)

**Proposition 14.1.** Suppose $Q_m^{(a)}$ satisfies the level $\ell$ restricted $Q$-system for $\mathfrak{g}$. Then

$$Y_m^{(a)} = \frac{(Q_m^{(a)})^2 \prod_{(b,k) \in H_\ell} (Q_{k}^{(b)})^{-2 j_{ba}^m}}{Q_{m-1}^{(a)} Q_m^{(a+1)}}$$

(14.46)

is a solution of the level $\ell$ restricted constant $Y$-system for $\mathfrak{g}$. The same holds between the unrestricted $Q$-system and the unrestricted constant $Y$-system if the product $\prod_{(b) \in H_\ell}$ in (14.46) is replaced by $\prod_{b \in \ell, k \geq 1}$.

This is a corollary (constant version) of theorem 2.5. For instance in the restricted case, it can also be verified directly by noting

$$1 + (Y_m^{(a)})^{-1} = \prod_{(b,a) \in H_\ell} (Q_{k}^{(b)})^{-2 j_{ba}^m}, \quad 1 + Y_m^{(b)} = \prod_{n=1}^{\ell_n} (Q_n^{(b)})^{\tilde{C}_{kn}^{(a)}},$$

(14.47)

where $\tilde{C}_{kn}^{(a)}$ is defined by (14.35). By virtue of (14.42), the assertion is reduced to $2 j_{ba}^m = \sum_{n=1}^{\ell_n} \tilde{C}_{kn}^{(a)} K_{ab}^{(a)}$, which indeed holds by (14.38). For $\mathfrak{g}$ simply laced, (14.46) reads

$$Y_m^{(a)} = \frac{\prod_{b \in \ell, \ell_n = 1} Q_{n}^{(b)}}{Q_{m-1}^{(a)} Q_m^{(a+1)}}.$$

(14.48)
14.6. $Q_m(a)$ at root of unity

We fix the level $\ell \in \mathbb{Z}_{>1}$. Let $\chi(V_\omega)$ be the character of the irreducible finite-dimensional representation $V_\omega$ of $\mathfrak{g}$ with the highest weight $\omega = \sum_{a \in I} \mathbb{Z}_{>0} a\omega_a$. We introduce the following specialization of $\chi(V_\omega)$:

$$
dim_q V_\omega = \prod_{a \in \Delta_+} \sin \frac{\pi (\omega_{a} + \rho)}{\ell h'} / \sin \frac{\pi \omega_{a}}{\ell h'},
$$

(14.49)

where $h'$ is the dual Coxeter number (2.3), $\Delta_+$ is the set of positive roots of $\mathfrak{g}$ and $\rho = \frac{1}{2} \sum_{a \in \Delta_+} a = \sum_{a \in \Delta_+} a\omega_a$. The quantity $\prod_{a \in \Delta_+} \frac{\omega_{a} + \rho}{\omega_{a}}$ is a $q$-analogue of the dimension of $V_\omega$. Thus, (14.49) is the $q$-dimension at the root of unity $q = \exp(q \sqrt{-1})$.

By proposition 13.10, we know that the classical character of the Kirillov–Reshetikhin module $Q_m(a) = \dim_q \text{res} W_m(a)$ satisfies the unrestricted $Q$-system. As shown in (13.62) and (13.59), $\chi_0(W_m(a))$ is a linear combination of various $\chi(V_\omega)$’s. The specialization of $\chi_0(W_m(a))$ to the $q$-dimension will be denoted by $\dim_q \text{res} W_m(a)$. By definition, $Q_m(a) = \dim_q \text{res} W_m(a)$ still satisfies the unrestricted $Q$-system. Furthermore, it seems to match the level truncation as follows.

**Conjecture 14.2.** $Q_m(a) = \dim_q \text{res} W_m(a)$ satisfies the level $\ell$ restricted $Q$-system. More strongly, the following properties hold for any $a \in I$:

$$
Q_m^{(a)} = \begin{cases} 
Q_{\ell_a}^{(a)} & \text{for } 0 \leq m \leq \ell_a, \\
Q_{m+1}^{(a)} & \text{for } 0 \leq m < \lceil \ell_a/2 \rceil, \\
0 & \text{for } 1 \leq j \leq \ell_a h' - 1,
\end{cases}
$$

(14.50)

where $\lceil \ell_a/2 \rceil$ is the largest integer not exceeding $\ell_a/2$ (not $q$-integer).

**Remark 14.3.** Conjecture 14.2 implies $Q_m^{(a)} > 0$ for all $(a, m) \in H_\ell$. Thus $Y_m^{(a)}$ constructed by (14.46) with the substitution $Q_m^{(a)} = \dim_q \text{res} W_m(a)$ is real positive for all $(a, m) \in H_\ell$. Therefore, it must coincide with the unique solution characterized in theorem 5.1.

We note that (14.50) implies $Q_{\ell_a}^{(a)} = Q_{0}^{(a)} = 1$; therefore, $j = 1$ case of (14.52) as well because of the $Q$-system relation $Q_{\ell_a}^{(a)} = Q_{\ell_a-1}^{(a)}Q_{\ell_a+1}^{(a)} + \prod_{b \neq a} (Q_{\ell_b}^{(b)})^{-c_{ab}}$ and the fact that $Q_{\ell_a-1}^{(a)} \neq 0$ by (14.51).

**Example 14.4.** For $\mathfrak{g} = A_r$, one has $Q_m^{(a)} = \dim_q \text{res} W_m^{(a)} = \dim_q V_\omega$ from (16.63). Thus

$$
Q_m^{(a)} = \prod_{i=1}^{r+1-a} \prod_{j=1}^{\ell_m} \sin \frac{\pi (a(j-1) - 1)}{\ell_m} / \sin \frac{\pi j}{\ell_m+1}.
$$

(14.53)

The property (14.50) and $Q_m^{(a)} > 0$ for $(a, m) \in H_\ell$ are easily checked. Substitution of this into (14.48) gives the real positive solution of the level $\ell$ restricted constant $Y$-system:

$$
Y_m^{(a)} = \frac{\sin \frac{\pi a}{\ell_m} \sin \frac{\pi (r+1-a)}{\ell_m+1}}{\sin \frac{\pi a}{\ell_m+1} \sin \frac{\pi (r+1-a)}{\ell_m+1}}, \quad 1 + Y_m^{(a)} = \frac{\sin \frac{\pi (a+m)}{\ell_m+1} \sin \frac{\pi (r+1-m)}{\ell_m+1}}{\sin \frac{\pi (a-m)}{\ell_m+1} \sin \frac{\pi (r+1+m)}{\ell_m+1}}.
$$

(14.54)

Obviously $(Y_m^{(a)})^{-1} = (Y_{\ell}^{(a)})^{-1} = 0$ and $Y_m^{(a)} > 0$ hold for $(a, m) \in H_\ell$. When $r = 1$, this reduces to $Y_m^{(1)}$ in example 5.3.

One of the most remarkable features of the level $\ell$ restricted constant $Y$-system and $Q$-system is their connection with the dilogarithm identity (5.5) in theorem 5.2. The lhs emerges from the TBA analysis (section 15). The $Y_m^{(a)}$ in the dilogarithm is characterized by the $Y$-system as in theorem 5.1 or constructed by the $Q$-system as in remark 14.3.
14.7. Bibliographical notes

The idea of converting TBA equations into difference equations (Y-system) as described in this section was put into practice in [3] for factorized scattering theories describing integrable perturbations of CFTs. The TBA equation treated there corresponds to the simply laced $\mathfrak{g}$ with level $\ell = 2$ in the terminology here up to the driving term. There are numerous Y-systems or related nonlinear integral equations in the similar TBA approaches to various integrable field theories, e.g. [5, 265–270]. The Y-systems considered here appear as typical building blocks in these theories in many cases.

There are also exotic variants and applications of Y-systems related to Takahashi–Suzuki’s continued fraction TBA [271] in the context of polymers [272], the sine-Gordon model [273] and the $T$-system for the XXZ model [274]. Intricate examples of $T$- and $Y$-systems are also worked out for the dilute AL models [275].

With regard to the $Q$-system, there are conjectures concerning more general specialization than $\dim q$ and related dilogarithm sum rules. See [1, appendix A], [134, appendix D], [4] and [101, section 1.4].

15. TBA analysis of RSOS models

We digest the TBA analysis of the $U_q(\widehat{\mathfrak{g}})$ Bethe equation, which is a natural candidate for the level $\ell$ critical RSOS model associated with the representation $W_{t(p)}^\ell$ of $U_q(\widehat{\mathfrak{g}})$ ($\ell \in \mathbb{Z}_{\geq 1}$, $(p, s) \in H_\ell$ (14.2)). The basic features of the model have been sketched in section 3.3. The derivation of high-temperature entropy and central charges in two critical regimes is outlined. The level $\ell$ restricted $Q$-system, the constant $Y$-system and the dilogarithm identity described in sections 5.1 and 14.4–14.6 play a fundamental role.

We make a uniform treatment for general $\mathfrak{g}$ elucidating the origin of the $Y$-system. The results cover rational vertex models formally as the limit $\ell \to \infty$. The TBA equation (15.13) also applies to a number of situations in other contexts, most notably, integrable perturbations of conformal field theories (cf section 14.7) with a suitable modification of the lhs.

Apart from the relatively well-known results in the ADE case, a curious aspect in nonsimply laced $\mathfrak{g}$ is that the central charges in one of the regimes correspond to the Goddard–Kent–Olive construction of Virasoro modules [276] involving the embeddings

$$B^{(1)}_r \hookrightarrow D^{(1)}_{r+1}, \quad C^{(1)}_r \hookrightarrow A^{(1)}_{2r-1}, \quad F^{(1)}_4 \hookrightarrow E^{(1)}_6, \quad G^{(1)}_2 \hookrightarrow B^{(1)}_3.$$ See (15.28)–(15.34). These results have stimulated notable developments in the crystal basis theory of quantum groups [262]. The content of this section is based on [59] for the ADE case and [18] for general $\mathfrak{g}$.

15.1. TBA equation

We keep the notations $t, t_a, \alpha_a, C$ in (2.1) and (2.2) and $L, \ell, H_\ell$ in (14.1) and (14.2). The Bethe equation is the following for the unknowns $\{ u_j^{(a)} | a \in I, 1 \leq j \leq n_a \}$:

$$\left( \frac{\sinh \frac{\pi}{L} (u_j^{(a)} - \sqrt{-1} \frac{d}{\ell} s_{\langle p, s \rangle})}{\sinh \frac{\pi}{L} (u_j^{(a)} + \sqrt{-1} \frac{d}{\ell} s_{\langle p, s \rangle})} \right)^N = \Omega_a \prod_{b=1}^r \prod_{k=1}^{n_b} \frac{\sinh \frac{\pi}{L} (u_j^{(a)} - u_k^{(b)} - \sqrt{-1} (\alpha_a | \alpha_b))}{\sinh \frac{\pi}{L} (u_j^{(a)} - u_k^{(b)} + \sqrt{-1} (\alpha_a | \alpha_b))}. \quad (15.1)$$

Here $n_a = \dim C^{-1}_{\langle p, s \rangle}$ as in (3.51) with $(r_i, s_i) = (p, s)$ for all $i$, and $\Omega_a$ is a root of unity without which (15.1) is essentially the same as the Bethe equation for the vertex model (8.25).
at \( q = \exp \left( \frac{2\pi x}{L} \right) \).\(^{41}\) The Bethe equation (15.1) is indeed valid \([59]\) for the \( U_q (A_1^{(1)}) \) RSOS model \([43]\).

It is a well-known mystery that the TBA analysis yields supposedly correct results in the end despite that it involves arguments that can hardly be justified mathematically\(^{42}\). Our arguments in what follows are no exception.

We employ a string hypothesis. Suppose that \( \{ u_{j}^{(a)} \mid a \in I, 1 \leq j \leq n_a \} \) is approximately grouped as the union of \( \{ u_{m,j}^{(a)} + \sqrt{-1} t_a^{-1} (m + 1 - 2n) \mid 1 \leq n \leq m, 1 \leq i \leq N_{m,i}^{(a)}, u_{m,i}^{(a)} \in \mathbb{R} \} \) and the rest. Here \( u_{m,i}^{(a)} \) is the center of a color \( a \) length \( m \) string and \( N_{m,i}^{(a)} \) is the number of such strings. Then the hypothesis is that \( \lim_{N \to \infty} \sum_{m=1}^{l_u} m N_{m,i}^{(a)}/n_a = 1 \) for all \( a \in I \). It means that for color \( a \), only those strings with length \( \leq \ell_a \) contribute to the thermodynamic quantities. This is a peculiar feature in the RSOS model and one of the most significant effects of the phase factor \( \Omega_a \). Substituting the string forms into (15.1) and taking the product over the internal coordinate of strings, one obtains

\[
N_{m,i}^{(a)} \Theta_{a}^{m} \left( \frac{u_{m,i}^{(a)}}{t_a} \right) = I_{m,i}^{(a)} + \sum_{b,k,j} N_{b,j}^{(b)} \left( \rho_{b}^{(a)} - \rho_{b}^{(b)} \right), \tag{15.2}
\]

Here \( I_{m,i}^{(a)} \in \mathbb{Z} + \text{constant} \), and \( \Theta_{a}^{m} \), \( \Theta_{ab}^{m} \) are defined by

\[
\Theta_{a}^{m} (u, \Delta) = \frac{1}{2\pi \sqrt{-1}} \sum_{n=1}^{m} \ln \frac{\sinh \pi (u + \sqrt{-1} t_a^{-1} (m + 1 - 2n) - \sqrt{-1} \Delta)}{\sinh \pi (u + \sqrt{-1} t_a^{-1} (m + 1 - 2n) + \sqrt{-1} \Delta)}. \tag{15.3}
\]

\[
\Theta_{ab}^{m} (u, \Delta) = \frac{\Theta_{ba}^{m} (u, \Delta)}{\Theta_{b}^{m} (u, \Delta)} = \sum_{j=1}^{k} \Theta_{a}^{m} (u + \sqrt{-1} t_b^{-1} (k + 1 - 2j), \Delta). \tag{15.4}
\]

One assumes that each solution satisfying \( u_{m,1}^{(a)} < u_{m,2}^{(a)} < \cdots < u_{m,N_{m,i}^{(a)}}^{(a)} \), which corresponds to an array such that \( I_{m,1}^{(a)} < I_{m,2}^{(a)} < \cdots < I_{m,N_{m,i}^{(a)}}^{(a)} \) and introduces the string density \( \rho_{m}^{(a)} (u) \) and the hole density \( \sigma_{m}^{(a)} (u) \) for \( u \sim u_{m,i}^{(a)} \) with large enough \( N \) by

\[
\rho_{m}^{(a)} (u) = \frac{1}{N \left( u_{m,i}^{(a)} - u_{m,i-1}^{(a)} \right)}, \quad \sigma_{m}^{(a)} (u) = \frac{I_{m,i}^{(a)} - I_{m,i-1}^{(a)} - 1}{N \left( u_{m,i}^{(a)} - u_{m,i-1}^{(a)} \right)}. \tag{15.5}
\]

Then (15.2) is converted into an integral equation. A little inspection of it shows a characteristic property \( \sigma_{m}^{(a)} (u) = 0 \), which enables one to eliminate the density of the ‘longest strings’ \( \rho_{m}^{(a)} (u) \). For such calculations, it is convenient to work in the Fourier components. We attach ‘\(^{\sim}\)’ to them. See (14.9). We shall flexibly present formulas either in the Fourier or original variables. By means of the basic formulas (14.20) and (14.21), the resulting integral equation is expressed in the Fourier space as\(^{43}\)

\[
\delta_{pa} \hat{A}_{pa}^{mn} (x) = \hat{\sigma}_{m}^{(a)} (x) + \sum_{(b,k) \in H_c} \hat{k}_{ab}^{mk} (x) \hat{\rho}_{k}^{(b)} (x) \quad \text{for} \quad (a, m) \in H_c. \tag{15.6}
\]

The ‘TBA kernels’ \( \hat{A}_{ab}^{mk} (x), \hat{k}_{ab}^{mk} (x) \), etc and their useful properties are summarized in section 14.2. By (14.17) and (14.15), (15.6) is also written as

\[
\frac{\delta_{pa} \delta_{im}}{2 \cosh \left( \frac{\Delta}{L} \right)} = \sum_{n=1}^{\ell_a-1} \hat{k}_{a}^{mm} (x) \hat{\sigma}_{n}^{(a)} (x) + \sum_{(b,k) \in H_c} \hat{J}_{ab}^{mk} (x) \hat{\rho}_{k}^{(b)} (x). \tag{15.7}
\]

\(^{41}\) \( \Omega_a = e^{-\alpha_a (T)} \) in the notation of (iii) in section 8.3.

\(^{42}\) A more reliable derivation based on the T-system is given in section 16.3.

\(^{43}\) The replacement \( \ell \to L \) in (14.20) and (14.21) has become unnecessary here due to the elimination of \( \rho_{m}^{(a)} (u) \).
for \((a, m) \in H_l\). Equation (15.6) or equivalently (15.7) is the Bethe equation for the string and hole densities.

We will actually consider the thermodynamics of the 'quantum spin' chain associated with the row to row transfer matrix \(T_i^{(p)}(u)\) of the RSOS model. We chose its Hamiltonian density \(\mathcal{H}\) as

\[
\mathcal{H} = -\frac{\gamma}{N} \left. \frac{\partial}{\partial u} \ln T_t^{(p)}(u) \right|_{u=0}, \quad (\epsilon = \pm 1),
\]

where \(\gamma > 0\) is a normalization constant and \(\epsilon = \pm 1\) specifies the two critical regimes in the RSOS model. The point \(u_0\) is such that \(T_t^{(p)}(u_0)\) becomes a cyclic shift (generator of momentum) up to an overall multiple, i.e. (3.50) becomes (scalar) \(\prod_{s=1}^{N} \delta_{u_s, u_{s+1}} \delta_{\alpha_s, \beta_{s+1}}\). In view of section 8.3, it is natural to assume that the spectrum \(\mathcal{E}\) of \(\mathcal{H}\) is obtained from the derivative of the top term \(Q_p(u - \frac{1}{t_p})/Q_p(u + \frac{1}{t_p})\) therein up to an overall factor independent of the Bethe roots. Thus up to an additive constant we obtain

\[
\mathcal{E} = \frac{\gamma}{N} \sum_{m=1}^{N_p} \frac{N_p}{m} \left. \frac{\partial}{\partial u} \ln \left( u, \frac{s}{t_p} \right) \right|_{u=0}, \quad \epsilon \in \{0, \pm 1\}.
\]

We will actually consider the thermodynamics of the 'quantum spin' chain associated

The Yang–Yang-type entropy density \(\mathcal{S}\) responsible for the arrangement of strings and holes is

\[
\mathcal{S} = \sum_{(a, m) \in H_l} \int_{-\infty}^{\infty} du \left( \mu_m^{(a)}(u) + \sigma_m^{(a)}(u) \right) \ln \left( \rho_m^{(a)}(u) + \sigma_m^{(a)}(u) \right) - \rho_m^{(a)}(u) \ln \rho_m^{(a)}(u) - \sigma_m^{(a)}(u) \ln \sigma_m^{(a)}(u). \]

The thermal equilibrium condition at temperature \(T = \beta^{-1}\) is obtained by demanding that the free energy density \(\mathcal{F} = \mathcal{E} - T \mathcal{S}\) be the extremum with respect to \(\rho_m^{(a)}(u)\), namely

\[
\delta \mathcal{F}/\delta \rho_m^{(a)}(u) = 0,
\]

under constraint (15.6). Setting \(\sigma_m^{(a)}(u)/\rho_m^{(a)}(u) = \exp(-\beta \epsilon_m^{(a)}(u))\), the result reads

\[
\beta \gamma \delta_{\alpha \beta} \delta_{\mu \nu} = \frac{4t_p^{-1}}{t_p \cosh(t_p \pi u/2)} \sum_{n=1}^{\ell_p-1} \int_{-\infty}^{\infty} dv K_{\alpha \beta}^{(a)}(u - v) \ln \left(1 + \exp(-\beta \epsilon_m^{(a)}(v))\right) - \sum_{(b, k) \in H_l} \int_{-\infty}^{\infty} dv J_{ak}^{(b)}(u - v) \ln \left(1 + \exp(-\beta \epsilon_k^{(b)}(v))\right).
\]

The nonlinear integral equation (15.12) is an example of the TBA equation, which serves as the basis in studying thermodynamic quantities. Using (14.12) and (14.19) it can be slightly

\[44\] The sign \((- 1)\) in (15.8) is absent here since \(T_t^{(a)}(u)\) is related to \(\frac{1}{\pi} \Theta_p \left( v, s/t_p \right) \mid_{v = -s/t_p} \cdot\]
rearranged as
\[
\frac{\epsilon \beta \gamma \delta p_{\gamma} \delta_{m}}{4 t_p^{-1} \cosh(t_p \pi u/2)} = \beta \epsilon_{m}^{(a)}(u) - \int_{-\infty}^{\infty} dv \frac{\ln \left[ (1 + \exp(\beta \epsilon_{m-1}^{(a)}(v))) (1 + \exp(\beta \epsilon_{m+1}^{(a)}(v))) \right]}{4 t_u^{-1} \cosh(t_u \pi (u - v)/2)}
\]
\[+ \sum_{(b,k) \in H} N_{ab} \int_{-\infty}^{\infty} dv \frac{\ln \left[ \prod_{m} (1 + \exp(\beta \epsilon_{m}^{(b)}(v))) \right]}{4 \cosh(t_u \pi (u - v)/2)}. \quad (15.13)
\]

When \( g \) is simply laced, one has \( \pi_{ab}^{mk}(u) = \delta_{mk} \delta(u) \) from (14.24) and (14.9). Therefore, (15.13) simplifies considerably to
\[
\frac{\epsilon \beta \gamma \delta p_{\gamma} \delta_{m}}{4 \cosh(\pi u/2)} = \beta \epsilon_{m}^{(a)}(u) - \int_{-\infty}^{\infty} dv \frac{\ln \left[ \prod_{m} (1 + \exp(-\beta \epsilon_{m}^{(b)}(v))) \right]}{4 \cosh(\pi (u - v)/2)}. \quad (15.14)
\]

15.2. High-temperature entropy

The free energy density is expressed as
\[
\mathcal{F} = \epsilon \epsilon_0 - T \sum_{m=1}^{\epsilon_{\pi}^{-1}} du \hat{A}_{pp}^{mm}(u) \ln \left( 1 + \exp(-\beta \epsilon_{m}^{(p)}(u)) \right) \quad (15.15)
\]
by means of (15.12), (14.17) and (14.18). Let us evaluate the high-temperature limit of the entropy density
\[
S_{\text{high}} = - \lim_{T \to \infty} \frac{\mathcal{F}}{T}. \quad (15.16)
\]

When \( T \to \infty \), the leading part of the asymptotic of \( \epsilon_{m}^{(a)}(u) \) is expected to become independent of \( u \). Thus we set \( Y_{m}^{(a)} = \exp(-\beta \epsilon_{m}^{(a)}(u)) \) to be a constant and obtain from (15.15) that
\[
S_{\text{high}}^{(a)} = \sum_{m=1}^{\epsilon_{\pi}^{-1}} \hat{A}_{pp}^{mm}(0) \ln \left( 1 + Y_{m}^{(p)} \right). \quad (15.17)
\]

Here \( \hat{A}_{pp}^{mm}(0) \) is the 0th Fourier component of \( \hat{A}_{pp}^{mm}(u) \) given by (14.13). Similarly the TBA equation (15.12) tends to
\[
\sum_{n=1}^{\epsilon_{\pi}^{-1}} \hat{K}_{n}^{mm}(0) \ln \left( 1 + Y_{n}^{(a)} \right) = \sum_{(b,k) \in H} \hat{J}_{ab}^{mk}(0) \ln \left( 1 + Y_{k}^{(b)} \right). \quad (15.18)
\]

This is the logarithmic form of the level \( \ell \) restricted constant \( Y \)-system (14.41). Thus we employ the solution \( Q_{a}^{(a)} = \dim_q \text{Res} W_{a}^{(a)} \) explained in remark 14.3 constructed from the \( q \)-dimension at a root of unity (14.49). Substituting the latter formula in (14.47) into (15.17) and applying (14.39), we find
\[
S_{\text{high}}^{(a)} = \ln \hat{Q}_{a}^{(p)}. \quad (15.19)
\]

This is consistent with the dimension of the space of states \( \mathcal{H}(N) \) of the RSOS spin chain (3.49). Namely, (15.19) implies
\[
\lim_{N \to \infty} \frac{\dim \mathcal{H}(N)}{N} = \dim_q \text{Res} W_{s}^{(p)}, \quad (15.20)
\]
which agrees with (3.54).
15.3. Central charges

The central charge $c$ of the underlying CFT is extracted from the low-temperature asymptotics of the entropy as $S_{\text{low}} \simeq \frac{c}{3\pi v_F^2}$ [277, 278], where $v_F$ is the Fermi velocity of the low-lying massless excitations. In each regime $\epsilon = \pm 1$, the result is expressed as

$$c = \frac{6}{\pi^2} \sum_{(a,m) \in H_\ell} \left( L \left( f_m^{(a)}(\infty) \right) - L \left( f_m^{(a)}(-\infty) \right) \right),$$  \hspace{1cm} (15.21)$$

where $L(x)$ is the Rogers dilogarithm (5.1). The number $f_m^{(a)}(\infty)$ is the positive real solution of $\ln f_m^{(a)}(\infty) = \sum_{(b,k) \in H_\ell} K_{mk}(a) \ln \left( 1 - f_b^{(k)}(\infty) \right)$ in both regimes $\epsilon = \pm 1$, where $K_{mk}(a)$ is the 0th Fourier component of $K_{mk}$ (14.36). By theorem 5.1, $f_m^{(a)}(\infty)$ equals $f_m(a)$ in (14.42) constructed from the unique real positive solution of the level $\ell$ restricted constant $Y$-system for $\mathfrak{g}$.

One the other hand, the numbers $f_m^{(a)}(-\infty)$ are to satisfy formally the same equation $\ln f_m^{(a)}(-\infty) = \sum_{(b,k) \in H_\ell} K_{mk}(a) \ln \left( 1 - f_b^{(k)}(\infty) \right)$ but with extra condition $f_m^{(a)}(-\infty) = \left( 1 - \epsilon \right)/2$ for $(a, m) \in H_\ell^\epsilon$ in the regime $\epsilon = \pm 1$. Here the subset $H_\ell^\epsilon$ of $H_\ell$ is specified as

$$H_\ell^\epsilon = \{(p, m) \mid 1 \leq m \leq \ell_p - 1\},$$  \hspace{1cm} (15.22)$$

Consequently, the equations governing the remaining $f_m^{(a)}(-\infty)$’s are split into subsets corresponding to the complement $H_\ell \setminus H_\ell^\epsilon$. Their solutions are obtained by the restricted constant $Y$-system associated with various subalgebras of $\mathfrak{g}$ and levels. The details can be found in [18, section 3]. In any case, the dilogarithm identity (5.5) suffices to evaluate the sum (15.21). Below we list the results using the rhs of (5.5)

$$L(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{\ell + \ell^\ell} - \text{rank } \mathfrak{g}$$  \hspace{1cm} (15.24)$$
as the building block.

Regime $\epsilon = +1$.

$\mathfrak{g} = \mathfrak{A}_r$,

$$c = L(\mathfrak{A}_r, \ell) - L(\mathfrak{A}_{p-1}, \ell) - L(\mathfrak{A}_{r-p}, \ell) \hspace{1cm} 1 \leq p \leq r.$$  

$\mathfrak{g} = \mathfrak{B}_r$,

$$c = L(\mathfrak{B}_r, \ell) - L(\mathfrak{B}_{p-1}, \ell) - L(\mathfrak{B}_{r-p}, \ell) \hspace{1cm} 1 \leq p \leq r - 2,$$

$$= L(\mathfrak{B}_r, \ell) - L(\mathfrak{A}_{p-1}, \ell) - L(\mathfrak{A}_{r-p}, 2\ell) \hspace{1cm} p = r - 1, r.$$  

$\mathfrak{g} = \mathfrak{C}_r$,

$$c = L(\mathfrak{C}_r, \ell) - L(\mathfrak{A}_{p-1}, 2\ell) - L(\mathfrak{C}_{r-p}, \ell) \hspace{1cm} 1 \leq p \leq r.$$  

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\( g = D_r, \)
\[
c = \mathcal{L}(D_r, \ell) - \mathcal{L}(A_p-1, \ell) - \mathcal{L}(D_r-p, \ell) \quad 1 \leq p \leq r - 2, \\
= \mathcal{L}(D_r, \ell) - \mathcal{L}(A_{r-1}, \ell) \quad p = r - 1, r.
\]

\( g = E_6, \)
\[
c = \mathcal{L}(E_6, \ell) - \mathcal{L}(D_5, \ell) \quad p = 1, 6, \\
= \mathcal{L}(E_6, \ell) - \mathcal{L}(A_1, \ell) - \mathcal{L}(A_4, \ell) \quad p = 2, 5, \\
= \mathcal{L}(E_6, \ell) - 2\mathcal{L}(A_2, \ell) - \mathcal{L}(A_1, \ell) \quad p = 3, \\
= \mathcal{L}(E_6, \ell) - \mathcal{L}(A_4, \ell) \quad p = 4.
\]

\( g = E_7, \)
\[
c = \mathcal{L}(E_7, \ell) - \mathcal{L}(D_6, \ell) \quad p = 1, \\
= \mathcal{L}(E_7, \ell) - \mathcal{L}(A_1, \ell) - \mathcal{L}(A_5, \ell) \quad p = 2, \\
= \mathcal{L}(E_7, \ell) - \mathcal{L}(A_1, \ell) - \mathcal{L}(A_2, \ell) - \mathcal{L}(A_3, \ell) \quad p = 3, \\
= \mathcal{L}(E_7, \ell) - \mathcal{L}(A_4, \ell) - \mathcal{L}(A_2, \ell) \quad p = 4, \\
= \mathcal{L}(E_7, \ell) - \mathcal{L}(A_1, \ell) - \mathcal{L}(D_5, \ell) \quad p = 5, \\
= \mathcal{L}(E_7, \ell) - \mathcal{L}(E_6, \ell) \quad p = 6, \\
= \mathcal{L}(E_7, \ell) - \mathcal{L}(A_6, \ell) \quad p = 7.
\]

\( g = E_8, \)
\[
c = \mathcal{L}(E_8, \ell) - \mathcal{L}(E_7, \ell) \quad p = 1, \\
= \mathcal{L}(E_8, \ell) - \mathcal{L}(A_1, \ell) - \mathcal{L}(E_6, \ell) \quad p = 2, \\
= \mathcal{L}(E_8, \ell) - \mathcal{L}(A_2, \ell) - \mathcal{L}(D_5, \ell) \quad p = 3, \\
= \mathcal{L}(E_8, \ell) - \mathcal{L}(A_3, \ell) - \mathcal{L}(A_4, \ell) \quad p = 4, \\
= \mathcal{L}(E_8, \ell) - \mathcal{L}(A_4, \ell) - \mathcal{L}(A_2, \ell) - \mathcal{L}(A_1, \ell) \quad p = 5, \\
= \mathcal{L}(E_8, \ell) - \mathcal{L}(A_6, \ell) - \mathcal{L}(A_1, \ell) \quad p = 6, \\
= \mathcal{L}(E_8, \ell) - \mathcal{L}(D_7, \ell) \quad p = 7, \\
= \mathcal{L}(E_8, \ell) - \mathcal{L}(A_7, \ell) \quad p = 8.
\]

\( g = F_4, \)
\[
c = \mathcal{L}(F_4, \ell) - \mathcal{L}(C_3, \ell) \quad p = 1, \\
= \mathcal{L}(F_4, \ell) - \mathcal{L}(A_{p-1}, \ell) - \mathcal{L}(A_{4-p}, 2\ell) \quad p = 2, 3, \\
= \mathcal{L}(F_4, \ell) - \mathcal{L}(B_3, \ell) \quad p = 4.
\]

\( g = G_2, \)
\[
c = \mathcal{L}(G_2, \ell) - \mathcal{L}(A_1, 3\ell) \quad p = 1, \\
= \mathcal{L}(G_2, \ell) - \mathcal{L}(A_1, \ell) \quad p = 2.
\]

*Regime \( \epsilon = -1. \) If \( \frac{s}{t_p} \in \mathbb{Z}, \) the central charge is given by
\[
c = \mathcal{L}\left( \mathfrak{g}, \frac{s}{t_p} \right) + \mathcal{L}\left( \mathfrak{g}, \ell - \frac{s}{t_p} \right) - \mathcal{L}(\mathfrak{g}, \ell) + \text{rank} \mathfrak{g}.
\]

(15.25)

This is the value corresponding to the coset pair
\[
\mathfrak{g} \oplus \hat{\mathfrak{g}} \supset \hat{\mathfrak{g}}
\]
level \( \ell = \frac{s}{t_p} \quad \frac{s}{t_p} \ell. \)

(15.26)
The situation $\frac{r}{p} \not\in \mathbb{Z}$ can take place in nonsimply laced algebras. The central charges for such cases are given as follows:
\begin{equation}
\begin{aligned}
g = B_r \ (p = r, \ 1 \leq s \leq 2\ell - 1, \ s \in 2\mathbb{Z} + 1), \\
c = \mathcal{L} \left( B_r, \frac{s-1}{2} \right) + \mathcal{L} \left( B_r, \frac{s+1}{2} \right) - \mathcal{L} (B_r, \ell) + 2r + 1.
\end{aligned}
\end{equation}

This value corresponds to the following coset pair via the embedding $B_r^{(i)} \hookrightarrow D_{r+1}^{(i)}$:
\begin{equation}
\begin{aligned}
B_r^{(i)} \oplus B_r^{(i)} \oplus B_r^{(i)} & \supset B_r^{(i)} \\
\text{level} \ \ell & = \frac{s+1}{2} \quad \frac{s-1}{2} \quad \ell.
\end{aligned}
\end{equation}

This value corresponds to the following coset pair via the embedding $C_r^{(i)} \hookrightarrow A_{2r-1}^{(i)}$:
\begin{equation}
\begin{aligned}
C_r^{(i)} \oplus C_r^{(i)} \oplus A_{2r-1}^{(i)} & \supset C_r^{(i)} \\
\text{level} \ \ell & = \frac{s+1}{2} \quad \frac{s-1}{2} \quad \ell.
\end{aligned}
\end{equation}

This value corresponds to the following coset pair via the embedding $F_r^{(i)} \hookrightarrow F_6^{(i)}$:
\begin{equation}
\begin{aligned}
F_r^{(i)} \oplus F_r^{(i)} \oplus F_6^{(i)} & \supset F_6^{(i)} \\
\text{level} \ \ell & = \frac{s+1}{2} \quad \frac{s-1}{2} \quad \ell;
\end{aligned}
\end{equation}

This value corresponds to the following coset pair via the embedding $G_r^{(i)} \hookrightarrow B_3^{(i)}$:
\begin{equation}
\begin{aligned}
G_r^{(i)} \oplus G_r^{(i)} \oplus B_3^{(i)} & \supset G_r^{(i)} \\
\text{level} \ \ell & = \frac{s-s_0}{3} \quad \frac{s-s_0}{3} \quad \ell.
\end{aligned}
\end{equation}

In (15.27), (15.29), (15.31) and (15.33), the contributions $2r + 1, 3r - 1, 10$ and $5$ other than the dilogarithm $\mathcal{L}$ are equal to $|H(p, s)|$ in (15.23).

These values of the central charges and coset pairs are consistent with the analyses of RSOS models [35, 56, 279] by Baxter’s corner transfer matrix method [2]. For $A_r$ level $\ell$, the central charges in regime $\epsilon = +1$ and $\epsilon = -1$ are transformed to each other via the interchange $(r - 1, \ell, p, s) \leftrightarrow (\ell, r - 1, s, p)$, which is a manifestation of the level-rank duality [56, 59, 280].

So far we have considered the $N$ site RSOS chain with the homogeneous quantum space, namely the one corresponding to $(W_{k(p)}^\otimes)^N$ in the dual picture of vertex models. One can extend the whole analysis to the inhomogeneous case corresponding to $(W_{k(p)}^\otimes \otimes \cdots \otimes W_{k(p)}^\otimes)^\otimes N$. Then, the lhs of (15.12) becomes nonvanishing for $(a, m) = (p_1, s_1), \ldots, (p_k, s_k)$,
and $H_e^\ell$ in (15.22) and (15.23) gets replaced by $\cup_{i=1}^k \{H_e^\ell \text{ for } (p_i, s_i)\}$. As a result, a broad list of central charges is realized, e.g. the coset pair $(\hat{g})^{[k+1]} \supset \hat{g}$ for the ADE case in the regime $\epsilon = -1$. For more details see [18, section 4.2]. Such a generalization has also been consistently incorporated into the crystal basis theory of one-dimensional configuration sums [262, section 3.2].

16. $T$-system in use

Here we present various applications of the $T$- and $Y$-systems to solvable lattice models.

16.1. Correlation lengths of vertex models

The correlation length $\xi$ is the simplest quantity to characterize ordered states. It is evaluated from the energy gap, which needs a lengthy calculation in the Bethe ansatz approach. As an application of the $T$-system for transfer matrices, we will demonstrate a quick derivation of $\xi$ based on the ‘periodicity at level 0’.

We consider the vertex models associated with quantum affine algebra $U_q(\hat{g})$. The row transfer matrix $T_m^{(0)}(u)$ is given by (3.44). We employ the parameterization $q = e^{-\lambda/t}$ with $t = 1, 2, 3$ defined in (2.1). To simplify the argument, we consider the homogeneous case $(r_i, s_i, w_i) = (p, s, 1)$ for all $i$; thus, $T_m^{(0)}(u)$ acts on the quantum space $W^{(p)}(0)^{\otimes N}$. We assume that $t_p = 1$ and the system size $N$ is even. Possible vertex configurations and the Boltzmann weights are explicitly given in (3.1) for $U_q(A^{(1)}_1)$ for instance. The vertex weights associated with $U_q(\hat{g})$ with $g$ other than $A_1$ have also been written down explicitly in some cases [48, 49]. Based on the concrete example from the $U_q(A^{(1)}_1)$ case, we assume that there is a range of the spectral parameter $u$ in which the model is in anti-ferroelectric order in the sense that those features explained below are realized. For a more detailed account, see [134, section 2.1].

In the ordered regime, the ground state and the first excited state are almost degenerate. The relevant energy gap is thus given by the energy difference between the ground state and the second excited state(s). Let $T_{\text{ground}}$ and $T_{\text{2nd}}$ be the corresponding eigenvalues of the transfer matrix. Consequently, $1/\xi = \ln(T_{\text{ground}}/T_{\text{2nd}})$. We will show that $\xi$ is given as

$$\xi = -\frac{1}{\ln k}$$ (16.1)

where $k$ is determined by the data $U_q(g)$ as

$$\frac{K'(k)}{K(k)} = \frac{\lambda h^\vee}{\pi},$$

where $h^\vee$ is the dual Coxeter number of $g$ (2.3) as before. $K(k)$ ($K'(k)$) stands for the complete elliptic integral of the first (second) kind with modulus $k$.

Recall that the unrestricted $T$-system for $g$ (2.22) has the form

$$T_m^{(a)} \left( u - \frac{1}{I_m} \right) T_m^{(a)} \left( u + \frac{1}{I_m} \right) = T_{m-1}^{(a)}(u) T_{m+1}^{(a)}(u) + g_m^{(a)}(u) M_m^{(a)}(u),$$

where the scalar function $g_m^{(a)}(u)$ depends on the normalization of vertex weights. The factor $M_m^{(a)}(u)$ is a product of $T_k^{(b)}$s. We assume $m \in I_u \mathbb{Z}_{>0}$ and denote the eigenvalues of $T_m^{(a)}(u)$ also by the same symbol. For the ground state in the anti-ferroelectric regime, the second term

$^{45}$ In the parameterization (3.1) for the $U_q(A^{(1)}_1)$ case, the range is $-1 < u < 0$. We assume the same range for general $U_q(\hat{g})$ leaving the precise Boltzmann weights corresponding to it unspecified.
on the rhs is exponentially larger than the first. So it is a good approximation to drop the first term on the rhs. The same is true for the second excited state(s). Let \( L_m^{(a)}(u) \) be the ratio of the eigenvalues

\[
L_m^{(a)}(u) = \frac{(T_m^{(a)}(u))_{2nd}}{(T_m^{(a)}(u))_{ground}}.
\]

Then the above argument implies that it satisfies

\[
L_m^{(a)}(u - \frac{1}{t_a}) L_m^{(a)}(u + \frac{1}{t_a}) = M_m^{(a)}(u)|_{\forall T_b^{(v)}(v) \to L_m^{(b)}(v)}.
\] (16.2)

This is regarded as the level zero restricted \( T \)-system. From (2.4) to (2.10), one can check that it closes among those \( L_m^{(a)}(u) \)'s with \( m \in t_a \mathbb{Z}_{>0} \). Moreover it enforces the following periodicity. (See also (3.55).)

**Proposition 16.1** ([17], theorem 8.8). Suppose that \( L_m^{(a)}(u) \) satisfies (16.2). Then the relation

\[
L_m^{(a)}(u)L_m^{(\omega(a))}(u + h^\vee) = 1
\]

is valid for \( m \in t_a \mathbb{Z}_{>0} \). Here \( \omega \) is the involution on the index set \( I \) such that \( \omega(a) = a \) except for the following cases (see figure 1):46

| \( g \)   | \( \omega(a) \) | Remarks               |
|--------|-----------------|-----------------------|
| \( g = A_r \) | \( \omega(a) = r + 1 - a \) |                        |
| \( g = D_r (r : \text{odd}) \) | \( \omega(r - 1) = r, \omega(r) = r - 1 \) |                      |
| \( g = E_6 \) | \( \omega(1) = 6, \omega(2) = 5, \omega(5) = 2, \omega(6) = 1 \) |                        |

In particular, \( L_m^{(a)}(u) = L_m^{(a)}(u + 2h^\vee) \) holds.

See also [134, Appendix A] for some manipulation leading to the above result. Below we only consider \( a \) such that \( \omega(a) = a \). Obviously \( L_m^{(a)}(u) \) has another periodicity in the imaginary direction

\[
L_m^{(a)}(u) = L_m^{(a)}(u + \frac{2\pi i}{\lambda})
\]

because the vertex weights are rational functions of \( z = q^u = e^{-\lambda u} \). We thus conclude that \( L_m^{(a)}(u) \) is doubly periodic. Introduce two further functions \( h_1, h_2 \) by

\[
\begin{align*}
     h_1(u, u_0) &= \sqrt{k} \text{sn}\left(\frac{i\lambda K(k)}{\pi} (u - u_0)\right), \\
     h_2(u, u_0) &= \sqrt{k} \text{sn}\left(\frac{i\lambda K(k)}{\pi} (u - u_0 + h^\vee)\right).
\end{align*}
\]

These are meromorphic, \( 2h^\vee \)-periodic and \( \frac{2\pi i}{\lambda} \)-anti-periodic functions of \( u \) and satisfy

\[
h_j(u, u_0)h_j(u + h^\vee, u_0) = 1 \quad (j = 1, 2).
\]

We note also that \( h_1(u, u_0)h_2(u, u_0) \) has one simple zero (pole) and no poles (zeros) in the rectangle \( \Omega := [0, h^\vee] \times [0, 2\pi i/\lambda) \) for \( u - u_0 \in \Omega \). We denote by \( \{u_z\} \) the set of zeros47 and poles of \( L_m^{(a)}(u) \) in \( \Omega \), respectively. The ratio defined below is analytic and nonzero for \( 0 \leq \Re u < h^\vee \):

\[
h(u) = \frac{L_m^{(a)}(u)}{\prod_{u_z} h_1(u, u_z) \prod_{u_p} h_2(u, u_p)}.
\]

46 For \( g = D_r (r : \text{even}) \), we set \( \omega(a) = a \) for any \( a \in I \).
47 In the Bethe ansatz, these zeros show up as ‘holes’.
Furthermore we have
\[ h(u)h(u + h^\prime) = 1. \quad (16.3) \]
The Liouville theorem and (16.3) claim that \( h(u) = \pm 1 \). We thus obtain the representation
\[ L_m(a)(u) = \pm \prod_u \sqrt{k} \frac{1}{\pi} \left( \frac{i\lambda K(k)}{\pi} (u - u_1) \right) \frac{1}{\pi} \left( \frac{i\lambda K(k)}{\pi} (u - u_2) \right). \]
The lower excited states are described by only two zeros. The above expression is then simplified to
\[ L_m(a)(u) = L_m(a)(u_1, u_2) := \pm k \frac{1}{\pi} \left( \frac{i\lambda K(k)}{\pi} (u - u_1) \right) \frac{1}{\pi} \left( \frac{i\lambda K(k)}{\pi} (u - u_2) \right). \quad (16.4) \]
The locations of these zeros label the excitations. The energy levels are almost degenerate with slight change in the locations of zeros. Thus, we observe the band structure of second excited states.

16.2. Finite-size corrections
Evaluation of finite-size corrections to the energy spectra of the Hamiltonian or the free energy provides information on the critical behavior such as central charges and scaling dimensions [277, 278, 283]. Numerical approaches often suffer from the smallness of system size and other technical problems such as logarithmic corrections. The evaluation of finite-size corrections is a nontrivial problem even for integrable models. The Bethe equation is highly transcendental and it simplifies only in the thermodynamics limit to an integral equation. For an arbitrary given system size, it is not possible in general to find the exact locations of the Bethe roots. Nevertheless, there are successful results in deriving finite-size corrections based on clever manipulations of Bethe equations [284–287]. Here we demonstrate yet another method utilizing the \( T \)-system in place of the Bethe equation following [7, 288].

As a concrete example we treat a level \( \ell \) critical RSOS model associated with \( A_1^{(1)} \) in sections 3.3–3.6 (\( \ell \in \mathbb{Z}_{\geq 2} \)). Local states on lattice sites range over \( \{1, 2, \ldots, \ell + 1\} \). We consider the fusion model in which any neighboring pair of local states is \( s \)-admissible (1 \( \leq s \leq \ell - 1 \)). See (3.34) and (3.35) for the definition of the admissibility. The transfer matrix \( T_s(u) \) is defined by (3.38) with \( m, s_i \) and \( v_j \) replaced by \( s, s \) and 0, respectively. We assume the system size \( N \) is even and treat the range \( -2 \leq u \leq 0 \) (referred to as the regime III/IV critical line [34]) for simplicity. We set
\[ q = e^{i\theta}, \quad \lambda = \frac{\pi}{\ell + 2}, \]
in the RSOS Boltzmann weights according to (3.33).

Although we are concerned with such an isotropic model, the key in our approach is to embed it in a family of models in which the admissibility (fusion degree) conditions in the
horizontal and vertical directions can be different. We consider the level $\ell$ fusion RSOS model \cite{35} in which neighboring states in the horizontal direction are $s$-admissible while those in the vertical direction are $m$-admissible. The corresponding transfer matrix is denoted by $T_m(u)$ and depicted in (3.38) with $s_i = s$ and $v_j = 0$. The evaluation of the finite-size correction to the largest eigenvalue of $T_s(u)$ utilizing the restricted $T$-system among $\{T_j(u)\}$ will be the main issue in what follows.

First we need to fix the normalizations. Let $W_{1,s}$ be the RSOS Boltzmann weights obtained by the $s$-fold fusion in the horizontal direction (cf (3.24)). Our normalization is such that

$$W_{1,s}(a + s - 1 \quad a - 1 \quad a | u) = \frac{[u + s + 1]_{q^{1/2}}}{[2]_{q^{1/2}}}.$$ 

See (3.33) for the symbol $[u]_{q^{1/2}}$. From now on we use $x = (u + 1)i$ as the spectral parameter, and $T_m(u)$ will also be written as $T_m(x)$. We furthermore define the normalized transfer matrices by $\tilde{T}_0(x) = 1$ and

$$\tilde{T}_m(x) = \begin{cases} T_m(x) & 1 \leq m \leq s, \\ T_m(x) \prod_{j=1}^{m-s} \phi(x + (m - s + 1 - 2j)i) & s + 1 \leq m \leq \ell, \end{cases}$$

where we have introduced

$$\phi(x) = \left( \frac{\sinh \lambda x}{\sin \lambda} \right)^N.$$ 

Thanks to these normalizations $\tilde{T}_j(x)$ is of degree $N \min(j, s)$ in $[ix + \cdots]_{q^{1/2}}$ for $1 \leq j \leq \ell$. One then obtains the level $\ell$ restricted $T$-system for $g = A_1$:

$$\tilde{T}_j(x - i)\tilde{T}_j(x + i) = f_j(x)\tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x) + g_j(x) \quad (1 \leq j \leq \ell - 1). \quad (16.5)$$

Here the scalar factors are given by $f_j(x) = \phi(x)^{q^{1/2}}$ and

$$g_j(x) = \prod_{k=0}^{\min(j,s)-1} \phi(x + (s + j - 2k)i)\phi(x - (s + j - 2k)i).$$

Numerical calculations for small system sizes suggest the following analyticity of $\tilde{T}_j(x)$.

**Assumption 16.2.** $\tilde{T}_j(x) (1 \leq j \leq \ell)$ is analytic and nonzero in the strip $|\Im x| \leq 1$.

We then construct $Y_j(x) (1 \leq j \leq \ell - 1)$ by

$$Y_j(x) = \frac{f_j(x)\tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x)}{g_j(x)}. \quad (16.6)$$

This leads to the $Y$-system

$$Y_j(x - i)Y_j(x + i) = (1 + Y_{j-1}(x))(1 + Y_{j+1}(x)) \quad (1 \leq j \leq \ell - 1), \quad (16.7)$$

where $Y_0(x) = Y_\ell(x) = 0$. The assumption on $T_j(x)$ is inherited to the analyticity of $Y_j(x)$ except for $Y_s(x)$: $Y_s(x)$ has order $N$ zero at the origin due to $f_s(x)$. We thus define the modified $Y$ by

$$\tilde{Y}_j(x) = \frac{Y_j(x)}{(\tanh \frac{x}{2})^N}. \quad (16.8)$$

Then the above assumption is rephrased as follows.

\footnote{We employ the inverse of (2.24) to make the resulting integral equation suitable for numerical investigations.}

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\textbf{Assumption 16.3.} $\tilde{Y}_j(x)$ $(1 \leq j \leq \ell - 1)$ is analytic and nonzero in the strip $|\text{Im} \, x| \leq 1$. Also $1 + Y_j(x)$ is analytic and nonzero in the strip $|\text{Im} \, x| \leq \epsilon$ for small positive $\epsilon$.

$Y$ and $\tilde{Y}$ satisfy

$$\tilde{Y}_j(x - i) \tilde{Y}_j(x + i) = (1 + Y_{j-1}(x))(1 + Y_{j+1}(x)), \quad (16.9)$$

where a simple identity $\tanh \frac{x}{2}(x - i) \tanh \frac{x}{2}(x + i) = -1$ is used. With the above analyticity assumption, one can apply the Fourier transformation to the logarithmic derivative of the $Y$-system\footnote{The derivative here is not essential. It is done only in order to ensure the convergence.}. After solving it with respect to the logarithmic derivative of $\ln Y_j$, the inverse Fourier transformation followed by an integration converts the $Y$-system into the coupled integral equation $(1 \leq j \leq \ell - 1)$:

$$\ln Y_j(x) = \delta_{j,0} \ln \tanh \pi x + \int_{-\infty}^{\infty} K(x - x') \ln[(1 + Y_{j-1}(x'))(1 + Y_{j+1}(x'))] \frac{dx'}{2\pi}, \quad (16.10)$$

$$K(x) = \frac{\pi}{2 \cosh \frac{x}{2}}. \quad (16.11)$$

The integration constant turns out to be zero due to the asymptotic values

$$Y_j(\infty) = \frac{\sin(j \pi)}{\sin^2 \theta}; \quad (16.12)$$

with $\theta = \frac{\pi}{\ell+1}$. Up to the driving term, (16.10) coincides with the TBA equation (15.14) for $g = A_1$ although they originate from completely different contexts. The asymptotic value (16.12) is an example of solutions to the constant $Y$-system. See examples 5.3 and 14.4.

Once $Y_j(x)$ is obtained from (16.10), the quantity $T_j(x)$ in question can be evaluated by using the relation

$$T_j(x - i) T_j(x + i) = g_x(x) (1 + Y_j(x)). \quad (16.13)$$

Note $\tilde{T}_j(x) = T_j(x)$. As numerical data shows $|Y_j(x)| \ll 1$, the bulk contribution $T_j^{\text{bulk}}(x)$ is determined by $T_j^{\text{bulk}}(x - i) T_j^{\text{bulk}}(x + i) = g_x(x)$. To separate the bulk part and finite-size correction, let $\tilde{T}_j(x) = T_j^{\text{bulk}}(x) T_j^{\text{finite}}(x)$. Then, (16.13) yields

$$\ln T_j^{\text{finite}}(x) = \int_{-\infty}^{\infty} K(x - x') \ln(1 + Y_j(x')) \frac{dx'}{2\pi}.$$

So far, all the relations are valid for arbitrary even $N$. We now proceed to the evaluation of $\ln T_j^{\text{finite}}(x)$ in the large $N$ limit for $x \sim O(1)$. The main contribution to the integrals in (16.10) comes from $x' \sim \pm \frac{2}{\pi} \ln 2N$. Thus it is convenient to introduce

$$y_j^\pm(\theta) := \lim_{N \to \infty} Y_j \left( \pm \frac{2}{\pi} (\theta + \ln 2N) \right).$$

The evenness of the original $Y_j(x)$ as a function of $x$ implies $y_j^+(\theta) = y_j^-(\theta)$. We then arrive at simpler expressions for $N$ sufficiently large:

$$\ln y_j^+(\theta) = -\delta_{j,0} e^{-\theta} + \int_{-\infty}^{\infty} K_0(\theta - \theta') \ln \left[ (1 + y_{j-1}^+(\theta'))(1 + y_{j+1}^+(\theta')) \right] \frac{d\theta'}{2\pi},$$

$$\ln T_j^{\text{finite}} \left( \frac{2\theta}{\pi} \right) = \frac{2 \cosh \theta}{N} \int_{-\infty}^{\infty} e^{-\theta} \ln \left( 1 + y_x^+(\theta') \right) \frac{d\theta'}{2\pi}. \quad \text{for small positive } \epsilon.$$
where \( K_\theta(\theta) := \frac{1}{\cosh\frac{\theta}{2}} \). The first equation exactly coincides with the TBA equation in the low-temperature limit. Thus, the dilogarithm trick (cf [7, section 3.3], [134, section 3.2]) is naturally applied to evaluate \( \ln T_f^{\text{finite}}(x) \). The final result of the finite-size correction to the largest eigenvalue of \( T_f(x) \) is given by

\[
\ln T_f^{\text{finite}} \left( \frac{2\theta}{\pi} \right) \simeq \cosh \frac{\theta}{2} \frac{1}{\pi N} \sum_{j=1}^{\ell-1} \int_{y_j^{\text{(-\infty)}}}^{y_j^{\text{(\infty)}}} \left( \frac{\ln(1 + y)}{y} - \ln \frac{1}{y + 1} \right) dy
\]

\[
= \frac{\cosh \theta}{\pi N} \sum_{j=1}^{\ell-1} \left( L_+(y_j^+(\infty)) - L_+(y_j^-(\infty)) \right)
\]

\[
= \frac{\pi \cosh \theta}{6N} \left( \frac{3s}{s + 2} - \frac{6s}{(\ell + 2)(\ell + 2 - s)} \right) =: \frac{\pi \cosh \theta}{6N} c.
\]  

(16.14)

Here \( L_+ (y) \) is related to the Rogers dilogarithm \( L(y) \) in (5.1) by

\[
L_+ (y) = L \left( \frac{y}{1 + y} \right) = L(1) - L \left( \frac{1}{1 + y} \right).
\]

We have also used \( y_j^+(\infty) = Y_j(\infty) = i(j, \frac{\pi}{\ell}) \) as in (16.12) while

\[
y_j^-(\infty) = \begin{cases} 
  \left( j, \frac{\pi}{s + 2} \right) & 1 \leq j \leq s - 1, \\
  \left( j - s, \frac{\pi}{\ell + 2 - s} \right) & s \leq j \leq \ell - 1.
\end{cases}
\]

Then, the dilogarithm identity (5.7) is applied. The quantity \( c \) in the last expression in (16.14) is regarded as the central charge [277]. This value agrees with the TBA result (15.25) obtained from the low-temperature specific heat with \( \theta \equiv A_1 \) and \( p = 1, t_p = 1 \).

The above argument can be generalized to calculate the finite-size correction in excited states with suitable modifications. The major difference from the ground-state case is that assumption 16.3 does not hold any longer. Instead, we assume the following for low-lying excited states.

**Assumption 16.4.** There are finitely many zeros \( \{ \zeta_{j}^{(\ell)} \} \) of \( \tilde{T}_f(x) \) in the strip \( |\Im x| \leq 1 \).

Letting the zeros of \( \tilde{T}_f(x) \) in the strip be \( \{ \zeta_{j}^{(\ell)} \} \), we modify (16.8) as

\[
Y_j(x) = \tilde{Y}_j(x) \left( \tanh \frac{\pi}{4} \right)^{N_{\delta j}} \prod_{\alpha} \tanh \frac{\pi}{4} (x - z_{\alpha}^{(j-1)}) \prod_{\alpha} \tanh \frac{\pi}{4} (x - z_{\alpha}^{(j+1)}),
\]

which still satisfies (16.9). Then it is straightforward to derive the following equation valid for arbitrary \( N \):

\[
\ln Y_j(x) = D_j + \delta_{j1} \ln \tanh \frac{\pi}{4} x
\]

\[
+ \sum_{\alpha} \ln \tanh \frac{\pi}{4} (x - z_{\alpha}^{(j-1)}) + \sum_{\alpha} \ln \tanh \frac{\pi}{4} (x - z_{\alpha}^{(j+1)})
\]

\[
+ \int_{-\infty}^{\infty} K(x - x') \ln [(1 + Y_{j-1}(x'))(1 + Y_{j+1}(x'))] \frac{dx'}{2\pi}.
\]  

(16.15)
The integration constant $D_j$ takes account of the branch of $\ln \tanh$ and it must be fixed case by case. For low-lying excitations in the thermodynamic limit, it is reasonable to assume $|z_a^{(j)}| \gg 1$. Thus, we employ the parameterization

$$z_a^{(j)} = \begin{cases} \frac{2}{\pi} (\theta_{a,+}^{(j)} + \ln 2N) & \text{for } z_a^{(j)} \gg 1 \quad (1 \leq \alpha \leq n_a^{(j)}), \\ -\frac{2}{\pi} (\theta_{a,-}^{(j)} + \ln 2N) & \text{for } z_a^{(j)} \ll -1 \quad (1 \leq \alpha \leq n_a^{(j)}), \end{cases}$$

where $n_a^{(j)}$ denotes the number of $z_a^{(j)}$ near $\pm \frac{\pi}{2} \ln 2N$. Then (16.15) is reduced in the limit $N \to \infty$ to

$$\ln y_j^{(j)}(\theta) = D_j - \delta_{j,1} e^{-\theta} + \sum_a \ln \tanh \frac{1}{2} (\theta - \theta_{a,-}^{(j)}) + \sum_{a'} \ln \tanh \frac{1}{2} (\theta - \theta_{a,+}^{(j+1)})$$

$$+ \int_{-\infty}^{\infty} K_0(\theta - \theta') \ln \left[ (1 + y_{j-1}^{(j)}(\theta'))(1 + y_{j+1}^{(j)}(\theta')) \right] \frac{d\theta'}{2\pi}. \quad (16.16)$$

The constants $D_j^{(\pm)}$ can be in general different and depend on $n_{\pm}^{(j)}$, etc.

The subsidiary conditions $T_j(z_a^{(j)}) = 0$ must also be satisfied. This is rephrased as $Y_{j}(z_a^{(j)} + i) = -1$ or equivalently

$$\ln y_j^{(j)} \left( \theta_{a,\tilde{c}}^{(j)} + \frac{\pi}{2} i \right) = (2I_{a,\tilde{c}}^{(j)} + 1)\pi i$$

in terms of the branch cut integers $\{ I_{a,\tilde{c}}^{(j)} \}$. Thanks to (16.16), this is rewritten as

$$-\int_{-\infty}^{\infty} \frac{1}{\sinh (\theta_{a,\tilde{c}}^{(j)} - \theta - i\epsilon')} \ln \left[ (1 + y_{j-1}^{(j)}(\theta))(1 + y_{j+1}^{(j)}(\theta)) \right] \frac{d\theta}{2\pi}$$

$$= (2I_{a,\tilde{c}}^{(j)} + 1)\pi i + D_j^{(j)} - \delta_{j,1} e^{-\theta_{a,\tilde{c}}^{(j)}} \sum_a \ln \tanh \frac{\theta_{a,\tilde{c}}^{(j)} - \theta_{a,\tilde{c}}^{(j+1)}}{2} + \frac{\pi}{4} i$$

$$+ \sum_{a'} \ln \tanh \left( \frac{\theta_{a',\tilde{c}}^{(j)} - \theta_{a',\tilde{c}}^{(j+1)}}{2} \right) + \frac{\pi}{4} i, \quad (16.17)$$

where $\epsilon' > 0$ is infinitesimally small. The finite part of the eigenvalue is now given by

$$\ln T_{\tilde{s}}^{\text{finite}} \left( \frac{2\theta}{\pi} \right) = \sum_{\tilde{c}=\pm} e^{\epsilon \tilde{c}} N \left[ - \sum_a e^{-\theta_{a,\tilde{c}}^{(j)}} + \int_{-\infty}^{\infty} e^{-\theta} \ln (1 + y_{\tilde{s}}^{(j)}(\theta)) \frac{d\theta}{2\pi} \right].$$

Although the expressions are more involved than the ground-state case, one can still apply the dilogarithm trick to evaluate the above. In particular, (16.17) and the elementary relations ($\ln \frac{\pi}{2} = 1/\sinh x$ and $\ln \tanh(x + \frac{\pi}{2}) + \ln \tanh(-x + \frac{\pi}{2}) = \pi i$ are useful. The final result reads

$$\ln T_{\tilde{s}}^{\text{finite}} \left( \frac{2\theta}{\pi} \right) = \sum_{\tilde{c}=\pm} e^{\epsilon \tilde{c}} N \sum_{j=1}^{\ell-1} \left( L_{a}(y_{\tilde{s}}^{(j)}(\infty)) - L_{a}(y_{\tilde{s}}^{(j)}(-\infty)) \right) + \frac{1}{2} D_j^{(j)} \ln \left( \frac{1 + y_{\tilde{s}}^{(j)}(\infty)}{1 + y_{\tilde{s}}^{(j)}(-\infty)} \right)$$

$$- 2\pi \pi n_a^{(j)} D_j^{(j)} - \pi^2 \sum_{a=1}^{J(j)} (2I_{a,\tilde{c}}^{(j)} + 1). \quad (16.18)$$

The above derivation is based on the first principle. However, it lacks a general prescription to determine the integration constants and to choose the branch cut integers. With regard to
this, an interesting observation has been made in [7, 289]. It is possible to absorb the additional driving terms in (16.15) to integrals by adopting deformed contours \( L_j \) as

\[
\ln Y_j(x) = D_j + \delta j \ln \tanh \frac{N \pi}{4} x + \int_{L_j}^L K(x - x') \ln(1 + Y_{j-1}(x')) \frac{dx'}{2\pi}.
\]

Then the evaluation of the finite-size correction goes parallel to the case of the largest eigenstate. The differences lie in the asymptotic values of \( y^\epsilon_j(x) \) and the nontrivial homotopy in the integration contours of \( L_j \). The authors of [7, 289] have found empirical rules for the choice of homotopy and integration constants to reproduce known scaling dimensions from conformal field theories.

We have seen that the \( T \)-system provides an efficient tool in the analysis of finite-size corrections. It enables one to analytically calculate the central charge (16.14) in the ground state. The scaling dimensions of relevant operators can also be obtained by use of the result in excited states (16.18). The above calculation of the finite-size correction of the largest eigenvalue has been generalized to RSOS models associated with \( g \) in [134, section 3] up to the analyticity argument on auxiliary functions.

**16.3. Quantum transfer matrix approach**

According to Matsubara, finite-size corrections and low-temperature asymptotics are dual pictures of the same physical characteristics of a two-dimensional system on an infinite cylinder of circumference \( N = \beta \). Here \( N \) is the system size in the former picture and \( \beta \) is the inverse temperature in the latter. Our analyses of the \( U_q(A^{(1)}_1) \) RSOS model in sections 15 and 16.2 have been done along these two points of view. What is remarkable there is that beyond the formal coincidence of the two pictures, the two entirely different approaches end up with essentially the same integral equation of TBA type. One then expects a framework to treat the finite-temperature problem in the same manner as the finite-size corrections without recourse to string hypothesis. As we will see in what follows, the quantum transfer matrix (QTM) approach [290] offers such a scheme. For a further detail, see the recent reviews [291, 292].

QTM utilizes the equivalence between \((d + 1)\)-dimensional classical models and \( d \)-dimensional quantum system [293]. To be concrete, we argue along the 1D spin-1/2 XXZ model as a prototypical integrable lattice system:

\[
\mathcal{H} = \frac{J}{4} \sum_{j=1}^{N} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta (\sigma_j^z \sigma_{j+1}^z + 1) \right) \equiv \sum_{j=1}^{N} \hat{h}_{j,j+1},
\]

where \( \sigma^a (a = x, y, z) \) are the Pauli matrices. The periodic boundary condition implies \( \sigma_{N+1}^a = \sigma_1^a \). The anisotropy is parameterized as \( \Delta = \cos \lambda \). The Hamiltonian acts on the physical space \( \mathcal{V}_{\text{phys}} := \bigotimes_{j=1}^{N} V_j \) where \( V_j \) denotes the \( j \)th copy of \( \mathbb{C}^2 = \mathbb{C}e_+ \oplus \mathbb{C}e_- \). The main subject here is to calculate the partition function exactly

\[
Z_{\text{1d}}(\beta, N) = \text{Tr}_{\mathcal{V}_{\text{phys}}} \text{e}^{-\beta \mathcal{H}}.
\]

It would be nice if this task can be done for any finite \( N \), although we do not have satisfactory progress at present. We thus concentrate on the evaluation of the free energy per site in the thermodynamic limit

\[
f = - \lim_{N \to \infty} \frac{1}{\beta N} \ln Z_{\text{1d}}(\beta, N).
\]
We introduce the six-vertex model on the 2D square lattice. Let \( R(u, v) \) be the \( U_q(A_1^{(1)}) \) \( R \) matrix (in a convention different from (3.1)):

\[
R(u, v) = \begin{pmatrix}
a(u, v) & b(u, v) & c(u, v) \\
b(u, v) & c^{-1}(u, v) & b(u, v) \\
c(u, v) & b(u, v) & a(u, v)
\end{pmatrix}
\]

\[
a(u, v) = \frac{[2 + u - v]_{q^{1/2}}}{[2]_{q^{1/2}}}, \quad b(u, v) = \frac{[u - v]_{q^{1/2}}}{[2]_{q^{1/2}}},
\]

\[
c(u, v) = q^{-\frac{1}{2}}, \quad q = e^{i\lambda}.
\]

Define the matrix element \( R^{\alpha\gamma}_{\beta\delta} \) by

\[
R(u, v) = \sum_{\alpha, \beta, \gamma, \delta=1,2} R^{\alpha\gamma}_{\beta\delta} (u, v) E_{\alpha,\beta} \otimes E_{\gamma,\delta}.
\]

The index 1 (2) refers to \( e_+ (e_-) \) in figure 4. The arrows are assigned in order to distinguish this \( R \) matrix from other \( R \) matrices that will appear below. By \( R_{j,j+1}(u,v) \) we mean the \( R \) matrix acting nontrivially only on the tensor product \( V_j(u) \otimes V_{j+1}(v) \). We introduce the row to row (RTR) transfer matrix \( T_{\text{RTR}}(u) \in \text{End}(V_{\text{phys}}) \) by

\[
T_{\text{RTR}}(u) = \text{Tr}_a (R_{a,N}(u,0) R_{a,N-1}(u,0) \cdots R_{a,1}(u,0)),
\]

where the subscript ‘a’ stands for the auxiliary space. With the lattice translation \( e^{iP} \) shifting the sites by one, the Baxter–Lüscher formula [52]

\[
T_{\text{RTR}}(u) = e^{iP} \left( 1 + \frac{\lambda u}{J \sin \lambda} \mathcal{H} + O(u^2) \right)
\]

holds. With a rotated \( R \) matrix \( \tilde{R}^{\alpha\gamma}_{\beta\delta}(u, v) \) (figure 5), we introduce a rotated transfer matrix \( \tilde{T}_{\text{RTR}}(u) \in \text{End}(V_{\text{phys}}) \) by

\[
\tilde{T}_{\text{RTR}}(u) = \text{Tr}_a (\tilde{R}_{a,N}(-u,0) \tilde{R}_{a,N-1}(-u,0) \cdots \tilde{R}_{a,1}(-u,0)).
\]
Figure 5. A graphic representation for $\tilde{R}_{\beta\delta}^{\alpha\gamma}(u,v)$. The spectral parameter $u(v)$ is associated with horizontal (vertical) lines.

Figure 6. Fictitious two-dimensional system.

The expansion analogous to (16.21) holds as $\tilde{T}_{RT}(u) = e^{-i\beta \left(1 + \frac{uM}{J \sin \lambda} \mathcal{H} + O(u^2)\right)}$. We thus obtain an important identity

$$Z_{1d}(\beta, N) = \text{Tr}_{\rho_{\text{phys}}} e^{-\beta \mathcal{H}} = \lim_{M \to \infty} \text{Tr}_{\rho_{\text{phys}}}(T_{\text{double}}(u = u_M) \tilde{T}_{RT}).$$

(16.22)

where $T_{\text{double}}(u) := T_{RT}(u) \tilde{T}_{RT}(u)$ and

$$u_M = -\frac{\beta J \sin \lambda}{M \lambda}. \quad (16.23)$$

The rhs of (16.22) can be interpreted as a partition function of a 2D classical system defined on $M \times N$ sites (figure 6)

$$Z_{1d}(\beta, N) = \lim_{M \to \infty} Z_{2d \text{ classical}}(M, N; u_M).$$

This embodies the equivalence between $(d + 1)$-dimensional classical models and $d$-dimensional quantum system for $d = 1$. Since the spectra of $T_{\text{double}}(u)$ is gapless, we still need a trick to evaluate $Z_{2d \text{ classical}}(M, N; u_M)$.

We follow the observation in [290] and consider the transfer matrix propagating in the horizontal direction, that is, $T_{\text{QTM}}(u = u_M)$ which acts on a virtual space of size $M$. It was shown that this transfer matrix possesses a gap between the largest ($\lambda_0$) and the other eigenvalues $\lambda_j$ ($j \geq 1$). This is a crucial benefit, as one only has to consider the largest
eigenvalue to evaluate the free energy in the thermodynamic limit
\[
\lim_{N \to \infty} Z_{\text{2d classical}}(M, N, u_M) = \lim_{N \to \infty} \left( \text{Tr} T_{\text{QTM}}^N(u = u_M) \right)^\frac{1}{N} = \lim_{N \to \infty} \left( \Lambda_0^N + \Lambda_1^N + \cdots \right)^\frac{1}{N} = \lim_{N \to \infty} \Lambda_0 \left( 1 + \left( \frac{\Lambda_1}{\Lambda_0} \right)^N + \cdots \right) \approx \lim_{N \to \infty} \Lambda_0.
\]

Although we have made use of the integrability for simplicity in the above argument, the same conclusion can be proved in a more general setting.

Theorem 16.5 ([290]). Let \( \Lambda_0 \) be the largest eigenvalue of \( T_{\text{QTM}} \). Then the free energy per site is given by
\[
f = -\frac{1}{\beta} \lim_{M \to \infty} \ln \Lambda_0.
\]

Two problems are still to be overcome. First we must evaluate the largest eigenvalue of \( T_{\text{QTM}}(u_M) \) in which interaction depends on the fictitious system size \( M \). Second we must take the ‘Trotter limit’ \( M \to \infty \). Both of these are highly nontrivial. Nevertheless we stress that the above formulation makes it clear why the finite-size correction and the finite-temperature problem can be treated in the same way. To disentangle the difficulties, we introduce a slight generalization, a commuting QTM \( T_{\text{QTM}}(x, u) \), by assigning the parameter \( ix \) in the ‘horizontal’ direction [294]. We let the transposed matrix \( R^{i \gamma}_{j \beta}(u, v) \) [295] be \((R^i)^{\gamma}_{\beta}(u, v) = R^{i \gamma}_{\beta \gamma}(v, u)\). See figure 7. Then \( T_{\text{QTM}}(x, u) \) is defined by
\[
T_{\text{QTM}}(x, u) = T_{\gamma_0}(R_{a, \gamma}(ix, -u)R_{a, M-1}(ix, u) \cdots R_{a, 2}(ix, -u)R_{a, 1}(ix, u)) .
\]

The parameter \( u \) will always be set to \( u_M \) (16.23); thus, we drop its dependence hereafter. It is the new parameter \( x \) that will play the role of a spectral parameter instead. By this we mean that two QTMs with different values of \( x \) are intertwined by the same \( R \) matrix
\[
R_{a, \gamma}(ix, iy)T_{\gamma_0}(x) \otimes T_{\gamma_0}(x') = T_{\gamma_0}(x') \otimes T_{\gamma_0}(x)R_{a, \gamma}(ix, iy).
\]

Here \( T_{\gamma_0}(x) \) denotes the monodromy matrix associated with \( T_{\text{QTM}}(x, u_M) \). The proof is elementary. Now we are able to introduce the fusion hierarchy of commuting transfer matrices \( T_{I_f}(x) \) which contains \( T_{\text{QTM}}(x, u_M) \) as the first member. (The \( u_M \)-dependence will be suppressed.) By the construction, they satisfy the \( T \)-system
\[
T_{I_f}(x - i + j + 1)i) = T_{I_f}(x - (j + 1)i) \text{ with}
\]

where \( g_j(x) = T_{0}(x + (j + 1)i)T_{0}(x - (j + 1)i) \)

As in section 16.2, we need assumptions on the analyticity of \( T_{I_f}(x) \). For simplicity we consider the case \( \lambda \to 0 \) for a moment. Then the numerical analysis suggests

Conjecture 16.6. The zeros of \( T_{I_f}(x) \) are distributed almost on the line \[ 3m \cdot x \] = \( j + 1 \).

We set \( Y_j(x) = T_{I_f - 1}(x)T_{I_f + 1}(x)/g_j(x) \) and introduce its modification
\[
\tilde{Y}_j(x) = \frac{Y_j(x)}{\left( \tanh \frac{\pi}{\lambda} \right) \left( \tanh \frac{\pi}{\lambda} \right)^j} .
\]

Note that \( u_M \) is a small negative quantity. Then the conjecture is translated into
Conjecture 16.7. \( \tilde{Y}_j(x) \) is analytic and nonzero in the strip \(|\Im x| \leq 1\) and \(1 + Y_{j+1}(x)\) is analytic and nonzero in the strip \(|\Im x| \leq \epsilon\) for small \(\epsilon\).

This immediately leads to the integral equation

\[
\ln Y_j(x) = \delta_{j1} \frac{1}{2} \ln \left[ \tanh^M \frac{\pi}{4} (x - (1 + u) \bar{M}) \tanh^M \frac{\pi}{4} (x + (1 + u) \bar{M}) \right] \\
+ \int_{-\infty}^{\infty} K(x - x') \ln[(1 + Y_{j-1}(x'))(1 + Y_{j+1}(x'))] \frac{dx'}{2\pi}, \tag{16.28}
\]

where \(K(x)\) is defined in (16.11). \(M\) enters only in the first line in (16.28). Therefore, the Trotter limit \(M \to \infty\) can be taken analytically, giving

\[
\ln Y_j(x) = \delta_{j1} D(x) + \int_{-\infty}^{\infty} K(x - x') \ln[(1 + Y_{j-1}(x'))(1 + Y_{j+1}(x'))] \frac{dx'}{2\pi} \quad (j \geq 1),
\tag{16.29}
\]

where \(D(x)\) in the driving term is given by

\[
D(x) = -\frac{\beta \pi J \sin \lambda}{2 \lambda \cosh \frac{\pi}{2} x}. \tag{16.30}
\]

These are nothing but the Gaudin–Takahashi equations for the anti-ferromagnetic Heisenberg model. Also they coincide with (16.10) up to the driving term. The free energy per site is obtained from the solution to the above equations as

\[
f = \frac{1}{\beta} \int_{-\infty}^{\infty} K(x') \ln(1 + Y_1(x')) \frac{dx'}{2\pi}.
\]

Summarizing, we have seen that the \(T\)-system plays the central role for the quantitative studies on both the finite-size system and the finite-temperature system. A wider range of the parameter \(0 < \lambda < \frac{\pi}{2}\) is treated in [274] under the restriction that the continued fractional expansion of \(\pi/\lambda\) terminates at a finite stage. A suitably chosen subset of the fusion QTMs are shown to satisfy a closed set of functional relations and it successfully recovers the well-known Takahashi–Suzuki continued fraction TBA equation [271] without using the string hypothesis. See [274] for details.
16.4. Simplified TBA equations

We continue our discussion on the XXZ spin chain at finite temperatures. We retain the definitions of the symbols such as \( \phi(x), T_j(x), u_M \), etc in the previous subsection. The TBA equation is a coupled set of integral equations with (finitely or infinitely) many unknown functions \( Y_j(x) \). It is known that equations change their forms drastically according to a small change in the coupling constant \( \lambda \) [271]. On the other hand, we expect only small changes in physical quantities. Thus one may hope alternative formulations that are more stable against the change in \( \lambda \). Here we present one such approach which also originates from the \( T \)-system. It is sometimes referred to as a simplified TBA equation [296].

The idea is complementary to the QTM method where one pays attention to the zeros of \( T_j(x) \). In the simplified TBA, one is concerned with singularities of a renormalized \( \tilde{T}_j(x) \). The latter is defined by

\[
\tilde{T}_j(x) = \frac{T_j(x)}{\phi(x + (j + 1 + u_M)i)\phi(x - (j + 1 + u_M)i)},
\]

where \( \phi(x) \) is defined in (16.26). Note that \( \tilde{T}_j(x) \) possesses poles of order \( M/2 \) at \( x \sim \pm(j + 1)i \). Accordingly, the first equation of the \( T \)-system reads

\[
\tilde{T}_1(x + i)\tilde{T}_1(x - i) = \tilde{T}_2(x) + b_1^{(M)}(x), \tag{16.32}
\]

\[
b_1^{(M)}(x) = \frac{\phi(x + (1 - u_M)i)\phi(x - (1 - u_M)i)}{\phi(x + (1 + u_M)i)\phi(x - (1 + u_M)i)}. \tag{16.33}
\]

Let \( \tau_j(x) \) be \( \tilde{T}_j(x) \) after the Trotter limit

\[
\tau_j(x) = \lim_{M \to \infty} \tilde{T}_j(x).
\]

Then \( \tau_1(x) \) develops singularity at \( x = \pm 2i \). By construction, it is periodic under \( x \to x + 2p_0i \), where \( p_0 = \pi/\lambda \). We thus assume the expansion

\[
\tau_1(x) = 2 + \sum_{n \in \mathbb{Z}} \sum_{j=1}^{\infty} \frac{c_j}{(x - 2i - 2p_0ni)^j} + \sum_{n \in \mathbb{Z}} \sum_{j=1}^{\infty} \frac{\bar{c}_j}{(x + 2i - 2p_0ni)^j}. \tag{16.34}
\]

We utilize the \( T \)-system and information on the locations of singularities to fix \( c_j \) and \( \bar{c}_j \). Rewrite the Trotter limit of (16.32) as

\[
\tau_1(x + i) = \frac{b_1(x)}{\tau_1(x - i)} + \frac{\tau_2(x)}{\tau_1(x - i)}, \tag{16.35}
\]

\[
b_1(x) = \lim_{M \to \infty} b_1^{(M)}(x) = \exp \left( \frac{\beta J \sin^2 \lambda}{\cosh \lambda x - \cos \lambda} \right). \tag{16.36}
\]

The lhs possesses the singularities at \( x = i, -3i \), while only the first term on the rhs possesses singularity at \( x = i \). Consequently we have

\[
c_j = \oint_{y=0} \frac{b_1(y)}{\tau_1(y - i)} (y - i)^{j-1} \frac{dy}{2\pi i} = \oint_{y=0} \frac{b_1(y + i)}{\tau_1(y)} y^{j-1} \frac{dy}{2\pi i}.
\]

The contour for the first integral is a small circle centered at \( y = i \) and the same circle centered at \( y = 0 \) for the second. Similarly, by rewriting (16.32) in the form \( \tau_1(x - i) = \frac{b_1(x)}{\tau_1(x + i)} + \frac{\tau_1(x)}{\tau_1(x + i)} \), one finds

\[
\bar{c}_j = \oint_{y=0} \frac{b_1(y - i)}{\tau_1(y)} y^{j-1} \frac{dy}{2\pi i}.
\]
By substituting the expressions for $c_j, \bar{c}_j$ into (16.34) and performing the summation over $j$ and $n$, we arrive at the closed integral equation involving $\tau_1(x)$ only:

$$
\tau_1(x) = 2 + \frac{\lambda}{4\pi i} \left( \oint_{y=0} b_1(y+i) \coth \frac{\lambda}{2}(x - y - 2i) \frac{dy}{\tau_1(y)} \right) + \frac{\lambda}{4\pi i} \left( \oint_{y=0} b_1(y-i) \coth \frac{\lambda}{2}(x - y + 2i) \frac{dy}{\tau_1(y)} \right).
$$

Once the above equation is solved, the free energy is given by $f = -\frac{1}{\beta} \ln \tau_1(0)$.

It turns out that the new equation works efficiently to produce the high-temperature expansion. One assumes $\tau_1(x)$ in the form

$$
\tau_1(x) = \exp \left( \sum_{n=0}^{\infty} a_n(x)(\beta J)^n \right).
$$

Then the coefficients $a_n(x)$ can be iteratively determined.

The simplified TBA equations are applied in many different contexts and they successfully provide high-temperature data of the models [297, 298]. The derivation of the simplified TBA equations requires less information on the analyticity. Therefore, it is quite efficient when the analytic property is difficult to investigate. The noncompact case is such an example. See [191] for the applications to certain sectors of $\mathcal{N} = 4$ super-Yang–Mills theory and [299] for applications to thermodynamics of ladder compounds.

There is however a price to pay. Any eigenvalue of $T_j(x)$ satisfies the same equation after renormalization. Therefore, the equation itself cannot select the right answer. Rather, one has to know a priori the right goal to be achieved and start from a sufficiently near point to the goal in numerical approaches. The convergence becomes also problematic in the low-temperature regime and one needs to apply, e.g., the Padé approximation to improve the accuracy.

### 16.5. Hybrid equations

There is yet further approach to the finite-size and the finite-temperature problems [295, 300, 301]. It also makes use of a finite set of unknown functions and different types of integral equations from those derived in the previous sections. Following [302], we refer to it as NLIE (nonlinear integral equation)\(^{50}\) just in order to distinguish it from the other nonlinear integral equations discussed hitherto. It turns out that a hybridization of TBA and NLIE is possible [303]. The hybrid approach is especially efficient in dealing with thermodynamics of higher spin XXZ models as explained below.

We treat the integrable spin-$s/2$ XXZ model whose Hamiltonian $\mathcal{H}$ is obtained from the fusion $R$ matrix in section 3.1 as

$$
\mathcal{H} = \sum_{i=1}^{N} h_{i,i+1}, \quad h_{i,i+1} \propto \frac{d}{dt} PR^{R(k,k)}(q^u)|_{u=0},
$$

where $P$ is the transposition. A simple generalization of the argument in section 16.3 shows that the free energy per site is obtained from the largest value of QTM $T_s(x = 0)$ consisting of the $R$ matrix acting on $V_s \otimes V_s$. As before we set

$$
q = e^{\lambda}, \quad \lambda = \frac{\pi}{P_0}
$$

\(^{50}\) The equation first appeared in the context of the finite-size problem in the XXZ model [287]. The simplest case is sometimes referred to as the DDV equation in the context of integrable field theories.
and assume $s \leq p_0 - 1$. As in section 16.3, we introduce the auxiliary QTM $T_j(x)$. This time, we prepare only a finitely many ones $\{T_j(x)\}_{j=1}^{\ell}$, where the integer $\ell$ is arbitrary as far as it is in the range

$$s \leq \ell \leq 2p_0 - s - 2. \quad (16.37)$$

With a suitable normalization, we have the $T$-system

$$T_j(x+i)T_j(x-i) = f_j(x)T_{j-1}(x)T_{j+1}(x) + g_j(x) \quad (1 \leq j \leq s - 1),$$

and define its slight modification generalizing (16.27) as

$$\tilde{Y}_j(x) = \frac{Y_j(x)}{(\tanh \frac{\pi}{2}(x + (1 + u)i)\tanh \frac{\pi}{2}(x - (1 + u)i))^{\frac{M}{2}}}.$$

Then, the modified $Y$-system (16.9) holds for $1 \leq j \leq \ell - 2$.

In addition we introduce the auxiliary functions $b(x)$ and $\tilde{b}(x)$. They are defined by the combination of the terms appearing in the dressed vacuum form of $T_{\ell}(x)$. For general $n$, the dressed vacuum form reads $T_m(x) = \sum_{m=1}^{n+1} \lambda_m(x)$, where

$$\lambda_m(x) = \Phi_m(x) \frac{Q(x + (n + 1)i)Q(x - (n + 1)i)}{Q(x + (2m - n - 1)i)Q(x + (2m - n - 3)i)},$$

and define its slight modification generalizing (16.27) as

$$\tilde{b}(x) = \frac{\lambda_1(x) + \cdots + \lambda_\ell(x)}{\lambda_{\ell+1}(x)} \quad (-1 \leq \Im x < 0),$$

$$\tilde{b}(x) = \frac{\lambda_1(x - i) + \cdots + \lambda_\ell(x - i)}{\lambda_{\ell+1}(x - i)} \quad (0 < \Im x \leq 1),$$

which are assumed to be analytic and nonzero in the strips indicated in the parentheses for the largest eigenvalue of the QTM $T_{\ell}(x)$. We also introduce

$$\mathcal{B}(x) = 1 + \bar{b}(x), \quad \mathcal{B}(x) = 1 + \tilde{b}(x)$$

in each analytic strip. There are nice relations among them, e.g.

$$Y_{\ell-1}(x-i)Y_{\ell-1}(x+i) = (1 + Y_{\ell-2}(x))\mathcal{B}(x)\mathcal{B}(x),$$

$$b(x) = \prod_{r=1}^{\ell} \Phi(x + (\ell - s + 2r)i) \frac{Q(x + (\ell + 2)i)}{Q(x - \ell i)} T_{\ell-1}(x),$$

$$\tilde{b}(x) = \prod_{r=1}^{\ell} \Phi(x - (\ell - s + 2r)i) \frac{Q(x - (\ell + 2)i)}{Q(x + \ell i)} T_{\ell-1}(x),$$

which can be easily checked by using the definitions.
By use of the analyticity assumptions, it is straightforward to derive the following equations after the limit $M \to \infty$:

$$\ln Y_j(x) = \delta_{j,1} D(x) + \int_{-\infty}^{\infty} K(x - x') \ln[(1 + Y_{j+1}(x'))(1 + Y_{j-1}(x'))] \frac{dx'}{2\pi},$$

$$1 \leq j \leq \ell - 2,$$  \hfill (16.39)

$$\ln Y_{\ell-1}(x) = \delta_{\ell-1,1} D(x) + \int_{-\infty}^{\infty} K(x - x') \ln(1 + Y_{\ell-2}(x')) \frac{dx'}{2\pi}$$

$$+ \int_{C_-} K(x - x') \ln \mathcal{B}(x) \frac{dx'}{2\pi} + \int_{C_+} K(x - x') \ln \mathcal{B}(x) \frac{dx'}{2\pi},$$  \hfill (16.40)

$$\ln b(x) = \delta_{\ell,1} D(x) + \int_{-\infty}^{\infty} K(x - x') \ln(1 + Y_{\ell-1}(x')) \frac{dx'}{2\pi}$$

$$+ \int_{C_-} F(x - x') \ln \mathcal{B}(x) \frac{dx'}{2\pi} - \int_{C_+} F(x - x' + 2i) \ln \mathcal{B}(x) \frac{dx'}{2\pi} \quad x \in C_-,$$  \hfill (16.41)

$$\ln \bar{b}(x) = \delta_{\ell,1} D(x) + \int_{-\infty}^{\infty} K(x - x') \ln(1 + Y_{\ell-1}(x')) \frac{dx'}{2\pi}$$

$$+ \int_{C_-} F(x - x') \ln \mathcal{B}(x) \frac{dx'}{2\pi} - \int_{C_+} F(x - x' - 2i) \ln \mathcal{B}(x) \frac{dx'}{2\pi} \quad x \in C_+,$$  \hfill (16.42)

where $C_+(C_-)$ is a contour just above (below) the real axis. The kernel $K(x)$ is given in (16.11) and $F$ is related to the spinon $S$ matrix

$$F(x) = \int_{-\infty}^{\infty} \frac{\sinh(p_0 - \ell - 1)x}{2 \cosh k \sinh(k(p_0 - \ell))} e^{-ikx} \, dk.$$ 

The integration constants are found to be zero by comparing asymptotic values of the both sides and $D(x)$ is defined in (16.30).

Obviously (16.39) is a reminiscence of the TBA-type equation (16.29), while (16.41) and (16.42) resemble NLIE were it not for the $\ln(1 + Y_{\ell-1})$ term. In this sense we call the above equations hybrid. They fix the values of $Y_j(x)$. The functional relations similar to (16.13) and the trick mentioned around (16.13) then yield the evaluation of the free energy per site.

**Remark 16.8.** The number $\ell$ is arbitrary under condition (16.37). This is quite different from ‘genuine’ TBA equations at special $\lambda$ [271, 274], where the number of equations is completely determined by $\lambda$. When $\lambda \to 0$, we can formally put $\ell = \infty$, which recovers the usual TBA equation in the rational limit as argued in section 16.3 for $s = 1$. For $s = 1$, one can make $F(x)$ null by choosing $p_0 = \ell + 1$. The resulting system reproduces the known TBA equation corresponding to the level 2 restricted $Y$-system for $D_{s+1}$ for the XXZ chain. See [274, equations (4.10)–(4.12)] for example. For arbitrary $s \in \mathbb{Z}_{\geq 1}$, the choice $\ell = s$ recovers the result in [303].

The above equations are numerically stable and yield a quick convergence to the unique solution. They are efficient in the analysis of the low-temperature regime. It is also known that with a suitable modification, one can derive the equations for excited states. We again have to pay the price. The systematic algorithm to construct the auxiliary functions is still lacking except for $g = A_1$ discussed here. This remains as an interesting future problem.
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