Incommensurate magnetic states induced by ordering competition in \( \text{Ba}_{1-x}\text{Na}_x\text{Fe}_2\text{As}_2 \)

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Quantum criticality near a certain magnetic phase transition beneath the superconducting dome of \( \text{Ba}_{1-x}\text{Na}_x\text{Fe}_2\text{As}_2 \) is attentively studied by virtue of a phenomenological theory in conjunction with renormalization group approach. We report that ordering competition between magnetic and superconducting fluctuations is capable of coaxing incommensurate (IC) magnetic states to experience distinct fates depending upon their spin configurations. The \( C_2 \)-symmetry IC magnetic stripe with perpendicular magnetic helix dominates over other \( C_2 \)-symmetry magnetic competitors and hints at a potential candidate for the unknown \( C_2 \)-symmetry magnetic state. Amongst \( C_4 \)-symmetry IC magnetic phases, IC charge spin density wave is substantiated to be superior, shedding light on the significant intertwining of charge and spin degrees of freedom. Meanwhile, ferocious fluctuations render a sharp fall of superfluid density alongside dip of critical temperature as well as intriguing behavior of London penetration depth.

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I. INTRODUCTION

The last dozen years have witnessed considerably intense research devoted to iron pnictides of \( \text{BaFe}_2\text{As}_2 \) family \([1-13]\), whose phase diagrams are ubiquitously borne out of both superconducting (SC) and diverse kinds of magnetic orders mediated by quantum phase transitions (QPTs) \([14]\). Notwithstanding magnetism is an antagonistic state versus superconductivity, they compete and collaborate other than coexist with each other \([12, 13, 15]\). This accordingly poses a substantial challenge as to what the connection is between magnetic and SC states, providing a crucial ingredient to glue Cooper pairing \([12, 13]\). In the light of abundant magnetic states in \( \text{BaFe}_2\text{As}_2 \) \([4, 11]\), one of the most imperative and realistic quests of understanding this very compound, prior to exploring the ultimate SC nature, is how to unambiguously identify concrete configurations of magnetic states around QPTs in that different states are associated with distinguished fluctuations which play a pivotal role in establishing its phase diagram.

Instead of global scenario, the focus of this paper is on finding specific magnetic states that reside close to magnetic QPTs in the phase diagram of \( \text{Ba}_{1-x}\text{Na}_x\text{Fe}_2\text{As}_2 \) \([4, 8, 10]\). This compound provides a versatile platform to investigate ordering-competition impacts on stabilities of magnetic states and relations with SC state. On one hand, it hosts a rather rich phase diagram with typical doping-tuned magnetic QPTs compared to other \( \text{BaFe}_2\text{As}_2 \) systems. It is of unique interest to asseverate there exists an elusive \( C_2 \)-symmetry (\( C_2 \)) magnetic phase in Na-doped system reported recently by Wang et al. \([8]\), which is hitherto enigmatic and remains an open topic. On the other, three commensurate plus kinds of IC magnetic states might all be possible candidates inhabiting in its phase diagram \([3, 6, 16, 19]\). To be specific, the incommensurate magnetic states involve the stripe spin density wave (SDW), charge spin density wave (CSDW), and spin vortex crystal (SVC) \([16, 17, 33, 34]\). In addition, the IC magnetic states cover four different \( C_2 \)-symmetry (\( C_2 \)) IC cases consisting of \( C_2 \) IC stripe (ICS), \( C_2 \) magnetic helix (MH), \( C_2 \) IC magnetic stripe with perpendicular magnetic helix (ICS \( \perp \) MH), and \( C_2 \) double parallel magnetic helix (DPMH), as well as three distinct \( C_4 \)-symmetry (\( C_4 \)) IC situations involving \( C_4 \) IC CSDW, \( C_4 \) IC SVC, and \( C_4 \) IC spin-whirl crystal (SWC) \([10]\). Due to their own peculiarities, these distinct states conventionally bring forward various outcomes. Questions are naturally raised: which one is the prime \( C_4 \)-symmetry (\( C_4 \)) magnetic order in the shadow of some QPT and what is the optimal state characterizing the mystic \( C_2 \) magnetic state? We respond by taking advantage of a phenomenological theory, together with the Wilsonian renormalization group (RG) \([20]\). The answers are of notable help to deeply understand the phase diagram and even offer instructive insights into pairing mechanism. Fig. \( \text{I} \) schematically illustrates our central results driven by ordering competition.

The rest of paper is organized as follows. In Sec. \( \text{II} \) we establish the phenomenological effective theory and provide the coupled RG equations after performing the one-loop momentum-shell RG analysis. Next, within Sec. \( \text{III} \) we endeavor to select the most favorable IC SDW states among all potential candidates under the influence of strong quantum fluctuations induced by the QCP. Sec. \( \text{IV} \) is accompanied by an investigation of unusual physical implications including both superfluid density and London penetration depth caused by the ferocious ordering competition near the QCP. Finally, we briefly summarize the primary conclusions in Sec. \( \text{V} \).
IC magnetic states, ordering vectors are afterwards distributed as \( Q_X = (\pi - \delta, 0) \) and \( Q_Y = (0, \pi - \delta) \) with \( \delta \) being a small correction for generic wavevectors. This indicates that the magnetic order parameters are regarded as a complex quantity \( M_{Q_X,Y} \neq \mathcal{M}_{Q_X,Y} \), which is in striking contrast to the commensurate case with \( \delta = 0 \) and \( M_{Q_X,Y} = M_{Q_X,Y} \).

We begin with the extended Landau-Ginzburg free energy after integrating out the fermionic ingredients [16, 17, 19, 27, 28]

\[
f = \alpha (|M_X|^2 + |M_Y|^2)^2 + \frac{\beta_1}{2}(|M_X|^2 - |M_Y|^2)^2 + \frac{\beta_2}{2}(|M_X|^4 + |M_Y|^4) + \frac{g_1}{2}(|M_X|^2 |M_Y|^2) + \frac{g_2}{2}(|M_X \cdot M_Y|^2) + \frac{g_3}{2}(|M_X \cdot \mathbf{n}_Y|^2 - |M_X \cdot \mathbf{n}_Y|^2).
\]  

with \( \alpha, \beta_1, \beta_2, \) and \( g_1, g_2, g_3 \) being fundamental structure parameters. It deserves to be pointed out that the QCP at \( x_1 \) in Fig. 1 associated with commensurate states was studied previously [16, 18]. In order to determine the unknown \( C_2 \) and \( C_4 \) IC SDWs, we hereafter concentrate on the magnetic QPT denoted by \( x_i \), in Fig. 1.

After designating \( M_X \equiv M_X \cos \theta \mathbf{n}_X \) and \( M_Y \equiv M_Y \sin \theta \mathbf{n}_Y \), where \( \theta \in (0, \pi/2) \) and \( |n_{X,Y}|^2 = 1 \) specify the spin configurations of magnetic states, we go beyond mean-field level and construct a phenomenological effective field theory [16, 22], which captures main information of ordering competition including both \( C_{2,4} \)-symmetric IC magnetic and SC fluctuations [18, 23, 31]. To this end, the phenomenological effective action [16, 22] can be casted as

\[
S = \int d^4L = \int d^4L_{\text{SDW}} + \int d^4L_{\text{SC}} + \int d^4L_{\text{SDW-SC}},
\]  

where \( L_{\text{SDW}}, L_{\text{SC}}, \) and \( L_{\text{SDW-SC}} \) correspond to SDW, SC orders, and their interplay, respectively.

At first, we examine \( L_{\text{SDW}} \). An angle \( \theta \in [0, \pi/2] \) is employed to specify the direction of magnetic order parameter \( \mathbf{M} \) in the spin space. Accordingly, the order parameter can be divided into two components \( M_X \equiv M_X \cos \theta \mathbf{n}_X \) and \( M_Y \equiv M_Y \sin \theta \mathbf{n}_Y \) by projecting \( \mathbf{M} \) onto the spin vectors \( \mathbf{n}_X \) and \( \mathbf{n}_Y \), which characterize the spin configurations of magnetic states with \( |n_{X,Y}|^2 = 1 \) and whose concrete values depending upon the types of candidate states [19]. Inserting them into the free energy density [11] by adding the dynamical terms of magnetic order parameters then gives rise to [16, 18, 22, 27]
We next consider $\mathcal{L}_{\text{SC}}$. In order to obtain SC fluctuations in the ordered state, we bring out the following contribution by employing the condition $\partial_\mu A_\mu = 0$:

$$\mathcal{L}_{\text{SC}} = \partial_\mu \Delta^2 \partial_\mu \Delta + a_s \Delta^2 (k) + \frac{u_s}{2} \Delta^2 (k) + \alpha_A \frac{2}{A^2}$$

$$- \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \lambda_{\Delta A} \Delta^2 A^2. \quad (4)$$

As the system enters the SC ordered state around the SDW QCP, we need to expand the SC order parameter by introducing two new gapless fields

$$\Delta = V_0 + \frac{(\hbar + i \eta)}{\sqrt{2}}, \langle \eta \rangle = 0, V_0 \equiv \langle \Delta \rangle = \sqrt{-\frac{a_s}{u_s}} \lambda_{\Delta A} \Delta^2 A^2 \quad (5)$$

which help us to extract the potential fluctuation of SC order parameter [30], to make the A massive after absorbing the gapless Goldstone particles. Combing Eq. (4) and Eq. (5), after discarding the constant terms and choosing the transformation to make $\eta = 0$ due to the local gauge invariance [30], we obtain

$$\mathcal{L}_{\text{SC}} = \frac{1}{2} (\partial_\mu h)^2 - a_s h^2 + \frac{u_s}{8} h^4 + \frac{\sqrt{2} a_s u_s}{2}$$

$$\left[ - \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \alpha_A \frac{2}{2 A^2} \right]$$

$$+ \lambda_{\Delta A} \sqrt{-\frac{2 a_s}{u_s}} h^2 \Delta^2 A^2, \quad (6)$$

where the “mass” of field $A$ is defined as $\alpha_A \equiv \lambda_{\Delta A} \frac{-2 a_s}{u_s}$.

Finally, we introduce $\mathcal{L}_{\text{SDW}}$. The interplay between SC and SDW order parameters can be written as [18],

$$\mathcal{L}_{\text{SDW-SC}} = \kappa (|M_X|^2 + |M_Y|^2)^2 A^2 + \kappa (|M_X \cdot M_Y| + |M_X \cdot M_Y^\dagger|) \Delta^2. \quad (7)$$

Based on the information of $\mathcal{L}_{\text{SDW}}$ and $\mathcal{L}_{\text{SC}}$, we are left with our effective theory

$$\mathcal{L}_{\text{eff}} = \left[ \frac{1}{2} (\partial_\mu M_X/C)^2 + \alpha_X M_X^2 + \frac{\beta_X}{2} M_X^4 \right] + \left[ \frac{1}{2} (\partial_\mu M_Y/S)^2 + \alpha_Y M_Y^2 + \frac{\beta_Y}{2} M_Y^4 \right]$$

$$+ \frac{1}{2} (\partial_\mu h)^2 + a_s h^2 + \frac{\beta_h}{2} h^4 + \gamma h^3 \right] + \alpha_X M_X MY + \gamma_{XY h} M_X MY h + \gamma_{X h} M_X M_Y h + \gamma_{Y h} M_Y M_Y h \right]$$

$$+ \lambda_{XY} M_X^2 M_Y^2 + \lambda_{X h} M_X h^2 + \lambda_{Y h} M_Y h^2 + \lambda_{XY h} M_X M_Y h^2 + \lambda_{h A} h^2 A^2, \quad (8)$$

with $C \equiv 1/|n_X \cos \theta|^2$ and $S \equiv 1/|n_Y \sin \theta|^2$. $M_{X,Y}$ point to magnetic fluctuations and $h, A$ are auxiliary fields to absorb SC fluctuations. We here dub factors in [3] such as $\alpha_X$ etc. the effective parameters to prevent them from being confused with fundamental parameters appearing in Eq. (8). Two series of parameters are bridged by virtue of following relationships,

$$\alpha_h = (-a_s), \beta_h = \frac{u_s}{4} \gamma_h = \sqrt{-\frac{2 a_s u_s}{2}}, \alpha_A = \frac{-2 \lambda_{\Delta A} a_s}{u_s}, \gamma_{h A} = \lambda_{\Delta A} \sqrt{-\frac{2 a_s}{u_s}} \lambda_{h A} = \frac{\lambda_{\Delta A}}{2}, \quad (9)$$

$$\alpha_X = \left( a - \frac{\lambda a_s}{u_s} \right) (|n_X|^2 \cos^2 \theta), \beta_X = \beta_2 \left( |n_X|^4 \cos^4 \theta \right), \quad (10)$$

$$\alpha_Y = \left( a - \frac{\lambda a_s}{u_s} \right) (|n_Y|^2 \sin^2 \theta), \beta_Y = \beta_2 \left( |n_Y|^4 \sin^4 \theta \right), \quad (11)$$

$$\alpha_{XY} = \frac{-a_s \kappa}{u_s} \left( |\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^\dagger| \right), \quad (12)$$

$$\gamma_{XY} = \kappa \sqrt{-\frac{2 a_s}{u_s}} (|\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^\dagger|), \quad (13)$$

$$\gamma_{X h} = \lambda \sqrt{-\frac{2 a_s}{u_s}} (|\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^\dagger|), \quad (14)$$

$$\lambda_{XY} = g_1 \cos ^2 \theta \sin ^2 \theta \left( |n_X|^2 |n_Y|^2 \right) + \frac{g_2}{2} \cos ^2 \theta \sin ^2 \theta \left( |n_X \cdot n_Y|^2 + |n_X \cdot n_Y^\dagger|^2 \right), \quad (15)$$

$$\lambda_{X h} = \frac{\lambda}{2} \left( |n_X|^2 \cos^2 \theta \right), \lambda_{Y h} = \frac{\lambda}{2} \left( |n_Y|^2 \sin^2 \theta \right), \quad (16)$$

$$\lambda_{X Y h} = \frac{\kappa}{2} \left( |\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^\dagger| \right), \quad (17)$$
where $\kappa$ and $\lambda_{\Delta A}$ cannot be represented by original parameters $a$, $a_5$, $u_5$, $\lambda$, $\beta_1$, $\beta_2$, $g_1$, and $g_2$ appearing in the free energy $F$, and therefore comes up with two supplementary fundamental parameters. It is necessary to point out these effective parameters are intermediate auxiliary variables but instead the fundamental parameters play a central role in pinning down the specific SDW states.

B. RG analysis

As aforementioned in Sec. II A the concrete spin configuration state would be essentially determined by the fundamental parameters. In order to examine the stabilities of all potential states, we need to construct the energy-dependent coupled RG equations of the fundamental parameters. To this end, we compute one-loop corrections to all effective parameters in Eq. (18) and derive the corresponding RG evolutions within Wilsonian RG framework $^{18}$ $^{20}$ $^{31}$ via integrating out the fast fields in the momentum shell $e^{-\Lambda} < k < \Lambda$ with the running scale $l > 0$. Since the fundamental parameters defined in Eq. (14) dictate the physical properties, it heralds undeviatingly that a pillar of task consists in refining auxiliary variables but instead the fundamental parameters play a central role in pinning down the specific SDW states.

III. STABILITIES OF INCOMMENSURATE MAGNETIC STATES

With the help of energy-dependent flows of fundamental parameters, we are now in a suitable situation to study the stabilities of IC magnetic states triggered by some magnetic QCP. As to BaFe$_2$As$_2$ compounds, many experimental efforts $^{4}$ $^{11}$ corroborate that magnetism occupies major space of phase diagram in terms of various states with distinguished symmetries and spin configurations. In particular, compound Ba$_{1-x}$Na$_x$Fe$_2$As$_2$ $^{4}$ $^{8}$ $^{10}$ harbors a complicated but fascinating phase diagram sketched in Fig. 1 indicating a string of magnetic states for both $C_2$ and $C_4$ symmetries are allowed with proper variations of temperature and doping. Besides three commensurate states, i.e., stripe spin density wave (SSDW), charge spin density wave (CSDW), and spin vortex crystal (SVC) $^{16}$ $^{17}$ $^{33}$ $^{34}$, Christensen et al. $^{15}$ recently advocated that potential IC magnetic states are clustered into nine inequivalent breeds. Moreover, seven of them can be realized with confined parameters of mean-field free energy in the phase diagram $^{19}$, which cover four kinds of $C_2$ IC cases involving $C_2$ IC stripe (ICS), $C_2$ magnetic helix (MH), $C_2$ IC magnetic stripe with perpendicular magnetic helix (ICS $\perp$ MH), and $C_2$ double parallel magnetic helix (DPMH), as well as three distinct $C_4$ IC situations consisting of $C_4$ IC CSDW, $C_4$ IC SVC, and $C_4$ IC spin-whirl crystal (SWC). Their spin configurations as well as their stability constraints and final fates are catalogued point-to-point in Table I. In order to roughly capture the structural information of distinct types of SDW states, Fig. 2 presents the relevant schematic illustrations of related spin configurations for potential IC magnetic states.

A. Setup and Strategy

Despite being an underlying antagonist against SC state, magnetism is assumed to be of intimate relevance to superconductivity as they are closely adjacent to each other or even coexist near the magnetic QPT. To be con-
TABLE I: Collections of low-energy fates for IC magnetic states in Ba$_{1-x}$Na$_x$Fe$_2$As$_2$. The first line enumerates seven distinguished types of IC magnetic states as well as the second and third lines provide their related spin configurations and schematic illustrations, respectively. In addition, the fourth line shows stable constraints as functions of fundamental interaction parameters [10] and the last line presents the corresponding low-energy stabilities. Herein, ✔ and ✘ stand for a stable state (i.e., the prevailing candidate by the side of the magnetic QCP) and an unstable state, respectively.

| Magnetic states | ICS | MHI | ICS ∥ MHI | DPFM | IC CSDW | IC SVC | SWC |
|----------------|-----|-----|-----------|-------|----------|--------|-----|
| Spin configurations | $n_x = (0, 0, 1), n_y = (0, 0, 0)$ | $n_x = (0, 0, 1), n_y = (0, 0, 0)$ | $n_x = (0, 0, 1), n_y = (0, 0, 0)$ | $n_x = (0, 0, 1), n_y = (0, 0, 0)$ | $n_x = (0, 0, 1), n_y = (0, 0, 0)$ | $n_x = (0, 0, 1), n_y = (0, 0, 0)$ | $n_x = (0, 0, 1), n_y = (0, 0, 0)$ |
| Schematic illustrations | Fig. 2(a) | Fig. 2(d) | Fig. 2(f) | Fig. 2(g) | Fig. 2(b) | Fig. 2(c) | Fig. 2(h) |
| Stable constraints | $\beta_1 - \beta_2 < 0$ with $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < -1$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ | $\beta_1 - \beta_2 > 0$ with $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < -1$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ | $\beta_1 - \beta_2 > 0$ with $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < -1$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ | $\beta_1 - \beta_2 > 0$ with $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < -1$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ | $\beta_1 - \beta_2 < 0$ with $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < -1$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ | $\beta_1 - \beta_2 > 0$ with $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < -1$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ | $\beta_1 - \beta_2 > 0$ with $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < -1$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ or $\frac{g_1 - \beta_2}{|\beta_1 - \beta_2|} < 0$ |
| Fates of magnetic states | ✗ | ✗ | ✔ | ✗ | ✔ | ✗ | ✗ | ✗ |

Concrete, we concentrate on a particular point in Fig. 1, namely the QCP at $T = 0$ that separates $C_2$ and $C_4$ IC magnetic states labeled by $x_c$. Generally, the related magnetic fluctuations compete so furiously that they are always responsible for physics in the shadow of QPT including quantum critical regime with higher temperatures [14, 22, 23]. Considering that individualities of diverse states, in spite of hosting common magnetic generalities, have different consequences, we thereby contemplate the magnetic states on both sides of this QPT.

As it concerns the issue on intricate relationship between magnetism and superconductivity, a hallmark of fathoming overall phase diagram is tantamount to pinpointing the specific construction of each magnetic state. As a corollary, it is appropriate that one investigates how the ordering competition affects the magnetic state at the edge of the QCP by means of RG flows [13] in collaboration with the stable magnetic criteria itemized in the second line of Table I.

To proceed, we briefly address our strategy to judge which magnetic candidate is the most stable/favorable state. In principle, there exist two distinct routines to access the QCP as schematically illustrated in Fig. 3. Although the essential physics should be captured by either routine-A or routine-B, we are supposed to witness the physical behaviors along routine-B to examine the stabilities of all potential states. In order to be relevant with the schematic phase diagram, we adopt $T = T_0 e^{-\beta}$ with $T_0$ the initial temperature to measure the evolution variable [18, 31, 33, 44]. Subsequently, several procedures are followed to investigate whether a certain SDW state is a suitable candidate. At first, one needs to tune the initial values of fundamental interaction parameters to satisfy the corresponding stable constraint and hence make sure that the starting point ($T_0$) is located at such SDW state in the quantum critical regime (QCR) of Fig. 3. While we assume the constraint condition is developed in a regime...
away from the QCP, this SDW state is always stable owing to the absence of quantum fluctuations. In comparison, as approaching the QCP, we have to carefully check whether the stable constraints are still satisfied as the quantum fluctuations become more and more important and play a dominant role in selecting the potential states. To this end, one presents the energy-dependent behaviors of these restrictions after extracting the information from the coupled RG equations of all fundamental interaction parameters [18]. With these in hand, it is suitable to determine the stability of such SDW for accessing the QCP. It would be a preferable state once the constraint is well preserved by lowering the energy scale (approaching the QCP). Otherwise, the state is easily melted by ferocious fluctuations and henceforth not a good candidate.

Accordingly, parallelling the similar steps above, we can examine the stabilities of all candidate states one by one on an equal footing and finally select the most favorable SDW states nearby the QCP, which are schematically summarized in Fig. 4 and analyzed in the forthcoming sections.

B. Fates of magnetic states

By employing the strategy in Sec. IIIA we within this subsection endeavor to inspect the fates of all magnetic candidate states one by one via performing the RG analysis with respect to the coupled RG evolutions that involve the fluctuations and ordering competitions as well as their interplay.

1. Warm-up

Let us consider the $C_2$ IC stripe (ICS) magnetic state for an instance and show how to determine whether it is a good candidate (stable/favorable SDW state) for warm-up. The configurations of spin vectors for such state read $\mathbf{n}_x = (0, 0, 1)$ and $\mathbf{n}_y = (0, 0, 0)$ [19]. As shown in Table I its stable constraints can be either $(\beta_1 - \beta_2) < 0$, $g_2/|\beta_1 - \beta_2| > 0$, $(g_1 - \beta_2)/|\beta_1 - \beta_2| > -1$ or $(\beta_1 - \beta_2) < 0$, $g_2/|\beta_1 - \beta_2| < 0$, $(g_1 - \beta_2 - 0.09g_2)/|\beta_1 - \beta_2| > -1$ [19]. If this state is favorable around the QCP in the real materials, it must be adequately stable against the quantum fluctuations when approaching the QCP. Accordingly, we can initially choose a higher temperature $T_0$ away from the QCP as our starting point, at which the initial interaction parameters are supposed to satisfy the stable constraints of $C_2$ ICS state. Next, we take some representative initial values of fundamental parameters that obey the stable constraint at $T_0$ and then perform numerical RG analysis of related RG equations by approaching the QCP along Routine-B as schematically shown in Fig. 4 (namely, by lowering the temperature).

Due to the differences of spin configurations and fluctuations, we recall that all candidate states in Table I possess their own RG equations for the fundamental parameters collected in Appendix A which dictate the fates of stable constraints of certain states. It is therefore the related RG equations that are in charge of the stability of ICS state when approaching the QCP. After extracting the energy-dependent information from such RG equations, the corresponding numerical results in Fig. 4 display the temperature dependence of flows for the associated fundamental parameters.

In order to compare the robustness with other candidate states, it is helpful to denominate the very temperature as $T_{col}$ at which the candidate state’s stable constraint is collapsed. From Fig. 4 we can infer that the sign change of $\beta_1 - \beta_2$ is occurred explicitly once temperature is slightly lowered at $T = T_{col}$ owing to the effects of ordering competition, hinting at the destruction of the stable constraint. In addition, the basic results are insensitive to the specific values of $\theta$, which are generally rooted in the symmetry of a candidate state [19]. As for the ICS state that does not satisfy the $C_4$ symmetry, the magnetic components are inequivalent in two directions indicating $\theta \neq \pi/4$, and hence, without loss of generality, three representative values $\theta = \pi/12, \pi/6, \pi/3$ are chosen to perform the numerical calculations. In princi-
ple, there exists another critical point that describes the fixed point of parameters can be accessed at $l = l^*$ or a related $T = T^*$ beyond which the parameters are divergent or unphysical as labeled in Fig. 4. As presented in Appendix A different types of candidate SDW states exhibit distinct quantum fluctuations, which give rise to their own sets of RG equations. Henceforth, it is of particular necessity to address that the state with $T < T_{\text{col}}$ is no longer the ICS state but an uncertain state, which possesses unknown but distinct RG equations compared to those of the ICS state. Accordingly, the evolutions of parameters obtained by obeying RG equations of ICS state are unphysical at $T < T_{\text{col}}$ in Fig. 1. This implies that one can neglect the behaviors of parameters at $T < T_{\text{col}}$ in that whether a candidate state is robust can be determined as $T$ approaches $T_{\text{loc}}$ from $T > T_{\text{loc}}$. For convenience, the curves with $T < T_{\text{col}}$ are preserved for comparison with the numerical results of other states.

For completeness, it is worth inspecting whether the fate of an SDW state is robust against the initial fundamental parameters. To this end, we regard the initial condition in Fig. 4 as a reference point and tune an initial parameter of this point but keep all others invariant to form distinct representative groups of initial conditions, all of which are required to meet the stable criteria of $C_2$ ICS state. The numerical results in Fig. 5 share the similar tendency of $\beta_1 - \beta_2$ to its counterpart in Fig. 4 evincing the robustness of stability against the variation of initial condition. As for all other types of candidates states, the basic results are analogous and thus not shown hereby.

To wrap up, in the spirit of strategy addressed in Sec. III A we can infer that $C_2$ ICS is not a stable state against the quantum fluctuations in the low-energy regime and hence not a good candidate for IC magnetic state nearby the QCP.

2. Stable states

After paralleling the procedures for $C_2$ ICS state in Sec. III B 1 we check the stabilities of all candidate states collected in Table 1. This not only bears witness to the crucial role of ordering competition but also sheds light on fates of all types of IC magnetic states.

To be concrete, Fig. 5 exhibits the temperature (energy) dependence of correlated fundamental parameters, which carry the low-energy characteristics for both $C_2$ ICS $\perp$ MH and $C_1$ IC CSDW. At the outset, we find that stable constraints for $C_2$ ICS $\perp$ MH shown in Fig. 5(a) are well protected with a decrease of temperature. They are sabotaged by extremely strong fluctuations only until the magnetic QCP is sufficiently accessed at the collapsed temperature $T_{\text{col}} \sim 10^{-4}T_0$ (taking $T_0 = 100$ K for instance, $T_{\text{col}} \sim 10^{-2}$ K). This evidently signals that $C_2$ ICS $\perp$ MH is of particular robustness withstanding ordering competition. In reminiscence of the unknown $C_2$ magnetic state, which is located at a little deviation from the magnetic QCP portrayed in Fig. 1 we are aware that $C_2$ ICS $\perp$ MH is therefore deemed to be a reasonable candidate for this mysterious $C_2$ state that differs substantially from conventional $C_2$ stripe state. In addition, Fig. 5(b) proposes firmly robust temperature-dependent constraints for $C_4$ IC CSDW. Moreover, the basic results bearing the similarities to the $C_2$ ICS presented in Sec. III B 1 are insusceptible to the variances of starting...
parameters as long as they satisfy the restricted conditions listed in Table I. (the related further discussions will be briefly delivered in the forthcoming subsection IIIC). On the basis of these, we then come to a conclusion that IC CSDW, like its commensurate counterpart [18], behaves dominantly compared to other types of IC C magnetic states. This C4 magnetic state is hence the most applicable choice on the left side of magnetic QCP $x_c$ in Fig. I which compete, coexist, and cooperate with SC state. Furthermore, apart from the two applicable states including $C_2$ IC $\perp$ MH and $C_4$ IC CSDW, ordering competition surrounded by magnetic QCP is not in favour of all other types of IC magnetic states listed in Table I. In terminological language, given these states are prone to easily feel plus efficiently receive the fluctuation corrections even far away from a magnetic QCP, they are fairly sensitive and fragile to ordering competition, resulting in undeviating breakdown themselves as temperature is reduced. This broadly suggests that one is unable to solely fix the configuration of $C_2$ IC SDW above $C_4$ IC CSDW and $C_2$ IC $\perp$ MH as displayed in Fig. I which may either be $C_2$ ICS, $C_2$ DPMH, or $C_2$ MH. Details of verifying stabilities of IC magnetic states are provided in Appendix B. Last but not the least important, we deliver that, as for the region close enough to the QCP with $T < T_{col}$, ordering competition is so ferociously that no magnetic state can exist alone but instead there might be a coexistence of multiple IC magnetic states.

C. Relevant comments and explanations

Before going further, we stop to address three relevant issues with comments and explanations.

To begin with, we highlight the major concerns between Ref. [19] and this work are different and then explain the reason for adjusting the initial values not very largely. Concisely, the authors of Ref. [19] focus on how many possible SDW states can be generated and where do they reside in the parameter space via tuning a series of energy-independent parameters. The potential states are separated by several boundaries that are developed by the related parameters and not directly associated with the QCP. In comparison, the phase diagram in Fig. I with a QCP is constructed by the temperature and doping, which indicates that the boundaries of these two situations are not the same thing. Additionally, our target is to examine and determine which are the most favorable states among all candidates neighboring the QCP. Following the strategy in Sec. IIIA, we confine the initial parameters to satisfy the related stable constraints of a candidate state and judge whether such a state is suitable to exist nearby QCP with the help of the corresponding RG equations. In order to make sure the starting point is 100% of the candidate state and avoid the possible influence of other states as different states are associated with different RG equations, it is more suitable to choose the initial parameters a little away from the very boundary of Ref. [19] but near the QCP in Fig. I. This may be ascribed to a shortage of our strategy in that the RG equations of parameters are based upon the quantum fluctuations around the QCP and we can only deal with the candidate states one by one but cannot tackle two or more mixed states simultaneously.

Afterward, we move on to deliver several comments on the underlying fixed points (FPs) of parameters in...
the lowest-energy limit. For convenience, let us suppose that the FPs can be accessed at $T = T^*$, beyond which the parameters are divergent or unphysical as labeled in Fig. 4 for an example. From Fig. 4 or Fig. 9 (whose $T^*$ can be designated analogously to Fig. 4s and have not been shown for brevity), we can infer that the FPs can be accessed either at a much or a little lower energy scale for an unstable ($T^* \ll T_{\text{col}}$) or a stable ($T^* < T_{\text{col}}$) candidate state, implying the parameters do not satisfy the restricted conditions within $T^* < T < T_{\text{col}}$. As a consequence, one can already judge whether some candidate state survives and which are the most favorable states among potential candidates around the QCP before the FPs are exactly approached. Indeed, the FPs may be instructive to other interesting behaviors which are out scope of our main target and worth systematically studying in future.

Furthermore, it is necessary to present some words on the IC parameter $\delta$ in Sec. II A which does not directly appear in the effective action but is indirectly reflected by imposing the order parameters $M_{Q_{X,Y}} \neq M_{Q_{X,Y}}$ described in Sec. II A. There exists a little distinction from Puga et al.’s pioneering work on the sine-Gordon model [45], in which the parameter $\delta$ is explicit in their effective theory. Henceforth, one can regard such parameter as an interaction parameter and examine the transition between a commensurate and an IC state via tracking the evolution of parameter $\delta$. However, all potential states hereby are restricted to IC states and the focus is put on the stability of certain IC state without involving the transition in Ref. [45].

IV. SUPERFLUID DENSITY AND LONDON PENETRATION DEPTH

Generally, the quantum critical region accompanied by a certain QCP is a fertile ground for generating unusual physical behaviors caused by the strong fluctuations, which are of qualitative distinction from the scopes out of control by the QCP. According to Sec. III B, the most favorable SDW states for the left and right sides of the QCP correspond to the $C_4$ IC CSDW and $C_2$ ICs $\perp$ MH states, respectively. These different sorts of magnetic states would be responsible for distinct fates of physical implications around the QCP. In order to make the logic self-consistent, we follow this clue and endeavor to evaluate the temperature dependence of superfluid density and London penetration depth around both sides of the QCP by taking into account the quantum fluctuations of these two preferable states.

As magnetic states steadily compete and coexist with a SC order, it is of great temptation to examine how the superfluid density ($\rho_s$) and London penetration depth ($\lambda_L$) are influenced in the presence of ordering competition, which has two particularly important implications. In principle, $\rho_s(T)$ can be evaluated as $\rho_s(T) = \rho_s^c(T) - \rho_n(T)$, where $\rho_s^c(T) \propto \alpha_A(T)$ stems from the mass of vector field $A$ that obey RG equations due to Anderson-Higgs mechanism [29] and $\rho_n(T)$ grasps the density of thermally excited normal (non-SC) fermionic quasiparticles (QPs), respectively. Approaching the QCP, ordering competition is dominant and thus the normal QPs effects can be neglected implying $\rho_s(T) \sim \rho_s^c(T)$.

Fig. 7 clearly shows that $\rho_s(T)$ is notably suppressed by the ordering competition [18, 44, 47]. Because critical temperature $T_c$ is nominated by $\rho_s(T_c) = \rho_s^c(T_c) - \rho_n(T_c) = 0$, one can infer that it would be intensively reduced in the absence of $\rho_n(T)$. As explicitly delineated in the inset of Fig. 7 it is worth declaring that the drop of $T_c$ caused by the $C_4$ IC CSDW is a little more than its $C_2$ ICs $\perp$ MH’s counterpart, which is also apparently exposed in Fig. 1. Albeit a slight splitting, principal tendencies are qualitatively compatible with recent experiments [8, 48, 49]. As for the unconventional derivation of temperature-dependent $\rho_s$ from usual $s$-wave gap symmetry’s, there are two underlying reasons. A major concern is the possible alteration of the pairing gap symmetry driven by so ferocious fluctuations around the QCP. In addition, the fundamental interaction parameters can also display anomalous energy-dependence behaviors as accessing the QCP, which enter into the analytical expression of $\rho_s$ and hence can indirectly influence the tendency of superfluidity.
For qualitative discussions, we single out the s-wave gap symmetry as a toy and tentative substitute. In this respect, the London penetration depth is expressed as \( \lambda_L(0)/\lambda_L(T) = \sqrt{\rho_c(T)} \) [18]. As a consequence, \( \lambda_L(0)/\lambda_L(T) \) shares an analogous temperature-dependent trajectory with \( \rho_c \) under the impact of ordering competition as depicted in Fig. 1. Although BaFe\(_2\)As\(_2\) system possesses a more intricate gap structure [19], this primitive result might uncover parts of central ingredients that are in charge of \( \lambda_L \)'s property. For completeness, we adopt the method in Sec. III B 2 and check that the basic conclusions concerning the superfluid density in Fig. 1 are robust under the variation of couplings between SDW and SC, which are embodied by the initial fundamental parameters.

As a consequence, the behaviors of physical observables indirectly corroborate \( C_1 \) IC CSDW and \( C_2 \) IC states are favorable SDW states compared to the other candidates in Table I. This implies that the phenomenological theory can qualitatively capture the key information around the QCP. In addition, the primary conclusions concerning the preferable SDW states neighboring the putative QCP are relatively stable and self-consistent.

V. SUMMARY

To recapitulate, we study and discern the probable IC magnetic states induced by subtle ordering competition in the vicinity of certain QPT below the SC dome of Ba\(_{1-x}\)Na\(_x\)Fe\(_2\)As\(_2\). Specifically, we find that \( C_2 \) IC survives to be a good candidate for the obscure \( C_2 \) magnetic state and IC CSDW points to the reasonable IC state in the vicinity of the magnetic QPT. In addition, we address that superfluid density in tandem with critical temperature and London penetration depth manifest critical behaviors attesting to ordering competition around the QCP.

Fig. 1 schematically presents our primary conclusions, whose overall structure is borrowed from the experimental results in Ref. [8]. However, it is worth pointing out that the spin configurations of magnetic states for both sides of the QCP are unclear and hence only labeled by SDW state in Wang et al.’s work [8]. In sharp contrast, we explicitly determine that the most favorable candidates for the left and right sides correspond to the \( C_2 \) IC CSDW and \( C_2 \) IC states by virtue of one-loop RG analysis. In addition, we theoretically address that the \( C_2 \) IC CSDW state is more harmful to the superconductivity. The conclusions are qualitatively concomitant with recent experiments [6, 8]. In this sense, we offer a relatively operable strategy to select out the most favorable states around the QCP, with which one can in principle examine whether some magnetic state is a preferable state against the influence of quantum fluctuations. We expect our results are profitable to further understand the phase diagram of Ba\(_{1-x}\)Na\(_x\)Fe\(_2\)As\(_2\) and explore the correspondence between SC and magnetic states in the iron-based superconductors.

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Appendix A: Coupled RG equations of fundamental interaction parameters

After performing one-loop analysis of effective theory [18, 20, 31] via integrating out the fields in the momentum shell \( e^{-l}A \) with \( l > 0 \) the running scale, we can derive flows of effective parameters in Eq. (5). Combining these equations and connections [9, 17, 18, 51], the coupled RG equations for fundamental parameters can be derived.

Before going further, it is necessary to highlight that the fundamental parameters \( g_{1,2} \) only appear in Eq. (15). This implies that they do not evolve independently. In this sense, it is hereafter convenient to introduce a parameter

\[
\tilde{g} = g_1 \cos^2 \theta \sin^2 \theta (|n_X|^2 |n_Y|^2) + \frac{g_2}{2} \cos^2 \theta \sin^2 \theta (|n_X| \cdot n_Y|^2) + |n_X| \cdot n_Y|^2 \cdot (A1)
\]


to describe the information of \( g_{1,2} \).

After long but straightforward calculations [18, 31], we eventually obtain the coupled RG equations of all fundamental interaction parameters around the magnetic QCP, which include \( \alpha, \beta_{1,2}, \tilde{g} \) and \( \kappa \) specifying the characters of spin configurations as well as \( \alpha_s, u_s, \lambda_{\Delta A} \) stemming from SC fluctuations. These coupled RG evolutions are closely dependent upon the spin configurations of magnetic fluctuations, namely the relationships between \( |n_X|^2, |n_Y|^4, |n_X^2|^2, |n_Y|^4 \), which are divided into two main sorts of situations.

For type-I case, at which \( |n_X|^2 \neq |n_Y|^4 \) and \( |n_X|^2 = |n_Y|^4 \) or \( |n_X|^2 = |n_Y|^4 \) and \( |n_X^2|^2 = |n_Y|^4 \), both \( \beta_1 \) and \( \beta_2 \) flow independently and thus the coupled evolutions are written as
\[
\frac{da_s}{dl} = 2a_s - \frac{1}{4\pi^2} \left\{ \frac{9a_s u_s (1 + 4a_s)}{2} + \frac{2SE_1^2 a_s \lambda^2}{u_s} [1 - 4SE_1 (a - \frac{\lambda a_s}{u_s})] + \frac{2CD_1^2 a_s \lambda^2}{u_s} [1 - 4CD_1 (a - \frac{\lambda a_s}{u_s})] \\
+ \frac{32a_s \lambda^3 \Delta (1 + 4\lambda \Delta a_s)}{3u_s} + (SE_1 + CD_1) \lambda + \frac{3u_s (1 + 2a_s)}{4} - (a - \frac{\lambda a_s}{u_s}) (E_1^2 S^2 + D_1^2 (C^2)) \lambda \\
+ \lambda \Delta \Lambda (1 + 2\lambda \Delta \Lambda a_s) + \frac{CSF^2 \alpha^2 \kappa^2}{4u_s} [1 - 2(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})] \right\}, \\
(A2)
\]

\[
\frac{da}{dl} = \frac{2(a - \frac{\lambda a_s}{u_s})}{\frac{1}{4\pi^2} \left\{ \frac{\lambda}{2} + \frac{SCEF_1 \Delta^2 D_1 + 3C[\beta_2 D_1 + (\beta_1 - \beta_2) D_2]}{D_1} - \frac{2SE_1 \Delta \lambda^2}{a_s} [1 - 2(CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \\
- \frac{SCEF^2 \kappa^2}{8D_1} [1 - 2(SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \right\} + \left( \frac{\lambda}{u_s} \frac{da_s}{dl} \right) \frac{a_s d\lambda}{dl} - \frac{a_s \lambda a_s}{6u_s^2} \right), \\
(A3)
\]

\[
\frac{du_s}{dl} = u_s + \frac{1}{2\pi^2} \left\{ \frac{-18a_s u_s^2 (1 + 6a_s)}{2} + \frac{-16C^2 D_1 a_s \lambda^2}{3u_s} \right\} \left[ 1 - 6C D_1 (a - \frac{\lambda a_s}{u_s}) D_1 \right] - \frac{16SE_1^2 (\beta_2 D_1 + (\beta_1 - \beta_2) D_2)}{3u_s} \left[ 1 - 6S (a - \frac{\lambda a_s}{u_s}) \right] E_1 \\
+ \frac{8\lambda^3 (S^3 E_1^2 + C^2 D_1^2) (a - \frac{\lambda a_s}{u_s})}{3u_s} \left[ 1 - 6C D_1 (a - \frac{\lambda a_s}{u_s}) \right] E_1 \\
+ \frac{11072a_s \lambda^3 \Delta (1 + 6\lambda \Delta a_s)}{105u_s} - \frac{CSF^2 \kappa^2}{2(1 - (CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})}] \\
- \frac{C^2 S^2 F^2 \alpha^2 \kappa^2 (F^2 \kappa^2 + 2D_1 \lambda^2 \lambda^2)}{6u_s^2} [1 - 4(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})] \\
- \frac{2C^2 S^2 F^2 \alpha_2 \lambda^2 \lambda^2}{u_s} [1 - 2(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})], \\
(A4)
\]

\[
\frac{d\lambda}{dl} = \lambda + \frac{1}{2\pi^2} \left\{ \frac{8\lambda^3 S^2 \lambda^2 \lambda^2}{3D_1 u_s} \right\} \left[ 1 - 6SE_1 (a - \frac{\lambda a_s}{u_s}) - \frac{8C^2 D_1 a_s \lambda^2 [\beta_2 D_1^2 + (\beta_1 - \beta_2) D_2]}{u_s} \right] \\
\times \left[ 1 - 6C D_1 (a - \frac{\lambda a_s}{u_s}) - 4C D_1 a_s \lambda^2 [1 - 2CD_1 (a - \frac{\lambda a_s}{u_s})] - \frac{8C^2 D_1 a_s \lambda^2}{2} [1 - 2CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s] \\
+ 3a_s u_s \lambda (1 + 6a_s) - \frac{SCEF^2 \kappa^2}{D_1} [1 - 2(SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \right\} + \frac{S^2 E_1 \lambda^2 \hat{g}}{D_1} [4SE_1 (a - \frac{\lambda a_s}{u_s})] - 1 \right] \\
- \frac{3u_s (1 + 4\lambda \Delta a_s)}{4} - \frac{3\lambda^3 \lambda^2 [\beta_2 D_1^2 + (\beta_1 - \beta_2) D_2]}{3u_s} [4C D_1 (a - \frac{\lambda a_s}{u_s}) - 1] \\
+ \frac{2S^2 F^2 \lambda^2 \lambda^2 \lambda^2}{3D_1 u_s} \left[ 1 - 2(2SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s) \right] - \frac{2CSF^2 a_s \lambda^2 \kappa^2}{3u_s} [1 - 2(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s}) + 2a_s] \\
- \frac{C^2 S^2 E_1 \lambda^2 \lambda^2 \lambda^2 \lambda^2 [\beta_2 D_1^2 + (\beta_1 - \beta_2) D_2]}{2D_1 u_s^2} [1 - 4(CD_1 (a - \frac{\lambda a_s}{u_s}) + SE_1 (a - \frac{\lambda a_s}{u_s})] \\
- \frac{C^2 S^2 \lambda^2 \lambda^2 \lambda^2 \lambda^2 \lambda^2}{6u_s^2} [1 - 4(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})], \\
(A5)
\]

\[
\frac{d\beta_1}{dl} = \frac{\left\{ (D_1^2 - D_2 E_2 - (E_1^2 - E_2) D_2) \beta_1 + \frac{2(D_1^2 - D_2)}{2\pi^2(D_1^2 E_2 - E_1^2 D_2)} \right\} \left( \frac{C^2 \hat{g}^2}{4CD_1 (a - \frac{\lambda a_s}{u_s}) - 1} \right) \\
- \frac{E_1^2 \lambda^2 (1 + 4a_s)}{4} - \frac{9S^2 [\beta_2 E_1^2 + (\beta_1 - \beta_2) E_2^2 (1 - 4SE_1 (a - \frac{\lambda a_s}{u_s}) - \frac{4S^2 E_1^2 a_s \lambda^2 [\beta_2 E_1^2 + (\beta_1 - \beta_2) E_2^2]}{u_s}] \\
\times [1 - 2(2SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \right\} - \frac{4SE_1^2 a_s \lambda^2}{3u_s} [1 - 2(SE_1 (a - \frac{\lambda a_s}{u_s}) - 2a_s)] - \frac{2C^2 F^2 a_s \lambda^2 \hat{g}}{3u_s} \\
\times [1 - 2(2CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \right\} - \frac{2(E_1^2 - E_2)}{2\pi^2(D_1^2 E_2 - E_1^2 D_2)} \left\{ -4C^2 D_1 a_s \lambda^2 [\beta_2 D_1^2 + (\beta_1 - \beta_2) D_2] \right\}_{\frac{u_s}{u_s}} \\
\times [1 - 2(2CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \right\} - \frac{9C^2 [\beta_2 D_1^2 + (\beta_1 - \beta_2) D_2]}{4[4CD_1 (a - \frac{\lambda a_s}{u_s}) - 1] \\
+ \frac{S^2 \hat{g}^2 [4SE_1 (a - \frac{\lambda a_s}{u_s}) - 1] - \frac{D_2 \lambda^2 (1 + 4a_s)}{4}}{3u_s} \right\} \left[ 1 - 2(CD_1 (a - \frac{\lambda a_s}{u_s}) - 2a_s)] \\
- \frac{2S^2 \lambda^2 a_s \lambda^2 \hat{g}}{3u_s} [1 - 2(2SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \right\}, \\
(A6)
\]

\[
\frac{d\beta_2}{dl} = \frac{(E_2 D_1^2 - D_2 E_2^2)}{(D_1^2 E_2 - E_1^2 D_2)} \beta_2 + \frac{2E_2}{2\pi^2(D_1^2 E_2 - E_1^2 D_2)} \left\{ \frac{S^2 \hat{g}^2 [4SE_1 (a - \frac{\lambda a_s}{u_s}) - 1] - \frac{D_2 \lambda^2 (1 + 4a_s)}{4}}{4} \right\} \\
\]
\[ +9C^2[\beta_2 D_1^2 + (\beta_1 - \beta_2) D_2]^2[4CD_1(a - \frac{\lambda a_s}{u_s}) - 1] - \frac{4C^2 D_1^2 a_s \lambda^2 [\beta_2 D_1^2 + (\beta_1 - \beta_2) D_2]}{u_s} \]
\[ \times \left[ 1 - 2 \left( 2CD_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - \frac{4CD_1^3 a_s \lambda^3}{3u_s} \left[ 1 - 2 \left( CD_1(a - \frac{\lambda a_s}{u_s}) - 2a_s \right) \right] - \frac{2S^2 F^2 a_s \kappa^2 \tilde{g}}{3u_s} \]
\[ \times \left[ 1 - 2 \left( 2SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - \frac{2D_2}{2\pi^2 (D_1^2 \tilde{E}^2 - E_1^2 D_2)} \left[ -4S^2 E_1^2 a_s \lambda^2 [\beta_2 E_1^2 + (\beta_1 - \beta_2) E_2] \right] \]
\[ \times \left[ 1 - 2 \left( 2SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - 9S^2 [\beta_2 E_1^2 + (\beta_1 - \beta_2) E_2]^2 \left[ 1 - 4SE_1(a - \frac{\lambda a_s}{u_s}) - 2a_s \right] + C^2 \tilde{g}^2 \]
\[ \times \left[ 4CD_1(a - \frac{\lambda a_s}{u_s}) - 1 \right] - \frac{E_1^2 \lambda^2 (1 + 4a_s)}{4} - \frac{4SE_1^2 a_s \lambda^3}{3u_s} \left[ 1 - 2 \left( SE_1(a - \frac{\lambda a_s}{u_s}) - 2a_s \right) \right] + \frac{-2C^2 F^2 a_s \kappa^2 \tilde{g}}{3u_s} \left[ 1 - 2 \left( 2CD_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] \]
\[ (A7) \]
\[ \frac{d\lambda_{\Delta A}}{dt} = \lambda_{\Delta A} + \frac{2}{2\pi^2} \left\{ \frac{-64a_s a_{\Delta A}^3}{9u_s} \left[ 1 - 2 \left( CD_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - 3a_s u_s \lambda_{\Delta A} (1 + 6a_s) \right\} - \frac{3a_s \lambda_{\Delta A} (4a_s + 1)}{8} \]
\[ -4a_s^2 \Delta A \left[ 1 + 2 \left( CD_1(a - \frac{\lambda a_s}{u_s}) + SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - \frac{3a_s u_s \kappa (1 + 6a_s)}{2} \]
\[ \frac{dk}{dt} = \kappa + \frac{2}{2\pi^2} \left\{ \frac{-2CD_1 E_1 a_s \lambda^2 \kappa}{3u_s} \left[ 1 - 2 \left( CD_1(a - \frac{\lambda a_s}{u_s}) + SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] \right\} - \frac{C S \tilde{g} \kappa \left[ 1 - 2 \left( 2CD_1(a - \frac{\lambda a_s}{u_s}) + SE_1(a - \frac{\lambda a_s}{u_s}) \right) \right]}{8} \]
\[ - \frac{32C^2 D_1^2 \tilde{g} a_s \lambda^2}{3u_s} \left[ 1 - 2 \left( 2CD_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] + S^2 \left( a - \frac{\lambda a_s}{u_s} \right)[\beta_2 E_1^2 + (\beta_1 - \beta_2) E_2] \]
\[ - \frac{32S^2 E_1^2 \tilde{g} a_s \lambda^2}{3u_s} \left[ 1 - 2 \left( SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - \frac{F^2 (1 + 4a_s) \kappa^2}{4} - \frac{D_1 \left[ 4a_s + 1 \right] \lambda^2}{4} \]
\[ + 12 \tilde{g} \left[ C^3 D_1 \left( a - \frac{\lambda a_s}{u_s} \right) \right] \left[ \beta_2 D_1^2 + (\beta_1 - \beta_2) D_2 \right] + 8CS \tilde{g}^2 \left[ 2CD_1(a - \frac{\lambda a_s}{u_s}) \right] \]
\[ + SE_1(a - \frac{\lambda a_s}{u_s}) - 1 \right\] - \frac{3(\tilde{g} - \beta_2 D_1 E_1) \left[ C^2 \beta_2 D_1^2 + (\beta_1 - \beta_2) D_2 \right] + S^2 \left[ \beta_2 E_1^2 + (\beta_1 - \beta_2) E_2 \right]}{2\pi^2} \]
\[ - \frac{2C^2 F^2 \left[ \beta_2 D_1^2 + (\beta_1 - \beta_2) D_2 \right] a_s \kappa^2}{1 - 2 \left( 2CD_1(a - \frac{\lambda a_s}{u_s}) - a_s \right)} - \frac{2S^2 F^2 \left[ \beta_2 E_1^2 + (\beta_1 - \beta_2) E_2 \right] a_s \kappa^2}{1 - 2 \left( SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right)} \]
\[ \times \left[ 1 - 2 \left( 2SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - \frac{4C^2 S^2 F^2 \tilde{g} a_s \lambda^2 \kappa^2 \left[ 1 - 4 \left( CD_1(a - \frac{\lambda a_s}{u_s}) + SE_1(a - \frac{\lambda a_s}{u_s}) \right) \right]}{3u_s} \]
\[ - \frac{32C^2 S^2 E_1^2 \tilde{g} a_s \lambda^2 \kappa^2 \left[ 1 - 4 \left( CD_1(a - \frac{\lambda a_s}{u_s}) + SE_1(a - \frac{\lambda a_s}{u_s}) \right) \right]}{3u_s} \]
\[ - C^2 S^2 \tilde{g}^2 \left[ 1 - 4 \left( CD_1(a - \frac{\lambda a_s}{u_s}) + SE_1(a - \frac{\lambda a_s}{u_s}) \right) \right] \]
\[ (A8) \]
\[ (A9) \]
\[ (A10) \]

where the variable functions are designated as

\[ D_1 \equiv |n_X|^2 \cos^2 \theta, \quad D_2 \equiv |n_X|^2 \cos^4 \theta, \quad E_1 \equiv |n_Y|^2 \sin^2 \theta, \quad E_2 \equiv |n_Y|^2 \sin^4 \theta, \]
\[ \mathcal{F} \equiv \cos \sin \theta n_X \cdot n_Y + | \cos \sin \theta n_X \cdot n_Y |, \quad C \equiv 1 / |n_X \cos \theta|^2, \quad S \equiv 1 / |n_Y \sin \theta|^2. \]

(A11)

(A12)

Here, we will use the notation that \( \theta \in [0, \pi/2] \), and \( \theta = 0, \pi/2 \) serve as single magnetic order parameter with \( Q_X \) or \( Q_Y \), respectively.

For type-II case, at which \( |n_i|^2 \neq |n_i|^4 \) with \( i = X, Y \), only one of \( \beta_1 \) and \( \beta_2 \) flows independently. In this circumstance, the flows of \( a_s, a_s, \lambda, \lambda_{\Delta A} \) and \( \kappa \) share the same evolutions with their type-I counterparts. Nevertheless, the parameter \( \tilde{g} \) evolves under the following way

\[ \frac{d\tilde{g}}{dt} = \tilde{g} + \frac{1}{2\pi^2} \left\{ \frac{-4CD_1^2 E_1 a_s \lambda^3}{3u_s} \left[ 1 - 2 \left( CD_1(a - \frac{\lambda a_s}{u_s}) - 2a_s \right) \right] - \frac{4SD_1 E_1^2 a_s \lambda^3}{3u_s} \left[ 1 - 2 \left( SE_1(a - \frac{\lambda a_s}{u_s}) - 2a_s \right) \right] \right\} \]
\[ \frac{32C^2 D_1^2 \tilde{g} a_s \lambda^2}{3u_s} \left[ 1 - 2 \left( 2CD_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] - \frac{32S^2 E_1^2 \tilde{g} a_s \lambda^2}{3u_s} \left[ 1 - 2 \left( 2SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] \]
FIG. 8: (Color online) (a) Temperature-dependent stable constraints of the $C_2$-symmetry DPMH state under the representative starting values of interaction parameters chosen as $g_1 = -0.015$, $g_2 = -0.01$, $u_s = 0.05$, $\lambda = 0.01$, $\beta_1 = 0.005$, $\beta_2 = 0.01$ (the qualitative results are insensitive to the initial values). Hereby, the angle $\theta$ is designated in Sec. [III A] to specify the direction of magnetic order in the spin space. Insets: the sign-change region of $\beta_1 - \beta_2$ (left panel) and enlarged-region for $\frac{\varphi_1}{\varphi_2} = \frac{\varphi_1}{\varphi_2} = 1$ (right panel). (b) Sign-change regions of $\beta_1 - \beta_2$ under different values of $\theta$. 

\begin{align}
+12g^2C^4D_1(a - \lambda a_s/a_s)[\beta_2 D_2 + (\beta_1 - \beta_2)D_2] + S^3E_1(a - \lambda a_s/a_s)[\beta_2 E_2 + (\beta_1 - \beta_2)E_2] - \frac{D_1E_1(4a_s + 1)\lambda^2}{4}
+8CS\tilde{g}^2[2(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s) - 1) - \frac{F_2(1 + 4a_s)\lambda^2}{4} - 2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2] + (\beta_1 - \beta_2)E_2] + S^2[\beta_2 E_1 + (\beta_1 - \beta_2)E_2] + \frac{2C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{3a_s^2} [1 - 2(2SE_1(a - \lambda a_s/a_s) - a_s)] - \frac{4C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{2a_s^2} \times [1 - 4(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s))] - \frac{2C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{3a_s^2} [1 - 4(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s))] \right\}. 
\end{align}

Furthermore, the RG equations of parameters $\beta_1$ and $\beta_2$ can be broken down into six distinct sorts depending on the concrete conditions. 

For type-II case-A with $|n_X|^2 = |n_X|^4$, $|n_Y|^2 = |n_Y|^4$, $|n_Y|^2 = 0$ and $|n_Y|^2 \neq 0$, $\beta_1$ evolves but $\beta_2$ is an invariant constant,

\begin{align}
\frac{d\beta_1}{dt} &= \beta_1 + \frac{8C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{3a_s^2} [1 - 4(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s))] - \frac{2C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{3a_s^2} [1 - 4(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s))] \right\}, 
\end{align}

\begin{align}
\frac{d\beta_2}{dt} &= 0. 
\end{align}

For type-II case-B with $|n_X|^2 = |n_X|^4$, $|n_Y|^2 = |n_Y|^4$, $|n_Y|^2 = 0$ and $|n_Y|^2 \neq 0$, $\beta_1$ evolves whereas $\beta_2$ is an invariant constant,

\begin{align}
\frac{d\beta_1}{dt} &= \beta_1 + \frac{12g^2C^4D_1(a - \lambda a_s/a_s)[\beta_2 D_2 + (\beta_1 - \beta_2)D_2] + S^3E_1(a - \lambda a_s/a_s)[\beta_2 E_2 + (\beta_1 - \beta_2)E_2] - \frac{D_1E_1(4a_s + 1)\lambda^2}{4}}{2a_s^2} 
+ S^2[\beta_2 E_1 + (\beta_1 - \beta_2)E_2] + \frac{2C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{3a_s^2} [1 - 4(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s))] - \frac{4C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{2a_s^2} \times [1 - 4(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s))] - \frac{2C^2\tilde{g}^2[\beta_2 D_2 + (\beta_1 - \beta_2)D_2][a_s]\kappa^2}{3a_s^2} [1 - 4(CD_1(a - \lambda a_s/a_s) + SE_1(a - \lambda a_s/a_s))] \right\}, 
\end{align}

\begin{align}
\frac{d\beta_2}{dt} &= 0. 
\end{align}
FIG. 9: (Color online) Temperature-dependent stable constraints \( \langle \beta_1 - \beta_2 \rangle \) of the \( C_2 \)-symmetry MH state under the representative starting values of interaction parameters chosen as \( g_1 = 0.01, g_2 = -0.01, u_s = 0.05, \lambda = 0.01, \beta_1 = 0.01, \beta_2 = 0.005 \) with three representative \( \theta = \pi/12, \pi/6, \pi/3 \) to satisfy the MH’s stable constraint (the qualitative results are insensitive to the initial values). Hereby, the angle \( \theta \) is designated in Sec. II A to specify the direction of magnetic order in the spin space. Insets: (a) the sign-change region of \( \beta_1 - \beta_2 \) and (b) behaviors around \( l_c \).

FIG. 10: (Color online) Temperature-dependent stable constraints of the \( C_2 \)-symmetry ICS \( \perp \) MH state under the representative starting values of interaction parameters chosen as \( g_1 = 0.03, g_2 = 0.05, u_s = 0.05, \lambda = 0.01, \beta_1 = 0.01, \beta_2 = 0.005 \) with three representative \( \theta = \pi/12, \pi/6, \pi/3 \) to satisfy the ICS \( \perp \) MH’s stable constraint (the qualitative results are insensitive to the initial values). Hereby, the angle \( \theta \) is designated in Sec. II A to specify the direction of magnetic order in the spin space: (a) \( \theta = \pi/12 \), (b) \( \theta = \pi/6 \), and (c) \( \theta = \pi/3 \). Insets: the enlarged regions for \( \beta_1 - \beta_2 \) (left panel) and \( g_2/|\beta_1 - \beta_2| \) (right panel).

\[
\frac{d\beta_2}{dt} = 0.05, \quad \beta_2 - \beta_2(0) = 0.05, \beta_2(0) = 0.12, \beta_2(0) = 0.20,
\]

\[
\frac{d\beta_1}{dt} = 0.05, \quad \beta_1 - \beta_1(0) = 0.05, \beta_1(0) = 0.12, \beta_1(0) = 0.20.
\]

\[
\frac{d\beta_3}{dt} = 0.05, \quad \beta_3 - \beta_3(0) = 0.05, \beta_3(0) = 0.12, \beta_3(0) = 0.20.
\]

For type-II case-C with \( |\mathbf{n}_X|^2 = |\mathbf{n}_X|^4, |\mathbf{n}_y|^2 \neq |\mathbf{n}_y|^4, |\mathbf{n}_z|^2 = 0 \), and \( |\mathbf{n}_y|^2 = 0 \), \( \beta_2 \) evolves but \( \beta_1 \) is an invariant constant,

\[
\frac{d\beta_1}{dt} = 0,
\]

\[
\frac{d\beta_2}{dt} = \beta_2 + \frac{2}{2\pi^2} \left\{ -4S^2 a_0 \lambda^2 \beta_2 \beta_2^2 + \beta_1 \beta_2 \beta_2^2 \right\} \frac{1}{u_s} - \frac{4SE_1 \lambda^2 a_0^2}{3u_s} \left[ 1 - 2(2SE_1 - \lambda a_0^2 - 2a_s) \right] - \frac{4SE_1 \lambda^2 a_0^2}{3u_s} \left[ 1 - 2(2SE_1 - \lambda a_0^2 - 2a_s) \right] - \frac{\lambda^2(1 + 4a_s)}{4}. \quad (A18)
\]

\[
\frac{d\beta_3}{dt} = \frac{2C^2 \mathcal{F} a_0 \kappa^2 \tilde{q}}{3E_1^2 u_s} \left[ 1 - 2(2CD_1 - \lambda a_0^2 - 2a_s) \right]. \quad (A19)
\]
FIG. 11: (Color online) (a) Temperature-dependent stable constraints of $C_4$-symmetry SVC state under the representative starting values of interaction parameters chosen as $g_1 = -0.01$, $g_2 = 0.01$, $u_s = 0.05$, $\lambda = 0.01$, $\beta_1 = 0.005$, $\beta_2 = 0.01$ with three representative $\theta = \pi/12$, $\pi/6$, $\pi/3$ to satisfy the SVC’s stable constraint (the qualitative results are insensitive to initial values of parameters). Hereby, the angle $\theta$ is designated in Sec. II A to specify the direction of magnetic order in the spin space. Inset: the enlarge region for $g_2/|\beta_1 - \beta_2|$ and $(g_1 + |\beta_1 - \beta_2|)/|\beta_1 - \beta_2|$. (b) Sign-change regions at different values of $\theta$.

FIG. 12: (Color online) (a) Temperature-dependent stable constraints of symmetric double-$Q$ noncoplanar SVC state under the representative starting values of interaction parameters chosen as $g_1 = -0.005$, $g_2 = 0.005$, $u_s = 0.05$, $\lambda = 0.01$, $\beta_1 = 0.01$, $\beta_2 = 0.005$ with three representative $\theta = \pi/12$, $\pi/6$, $\pi/3$ to satisfy the symmetric SVC’s stable constraint (the qualitative results are insensitive to initial values of parameters). Hereby, the angle $\theta$ is designated in Sec. II A to specify the direction of magnetic order in the spin space. In addition, $\phi$ is introduced by $n_X \cdot n_Y = \sin^2 \phi$ and $\phi = \pi/4$ corresponds to the symmetric noncoplanar SVC [13]. Inset: the enlarge region for $\beta_1 - \beta_2$. (b) Sign-change regions at different values of $\theta$.

For type-II case-D with $|n_Y|^2 = |n_Y|^4$, $|n_X|^2 \neq |n_X|^4$, $|n_X|^2 = 0$, and $|n_X|^2 = 0$, $\beta_2$ evolves but $\beta_1$ is an invariant constant,

$$\frac{d\beta_1}{dl} = 0,$$

$$\frac{d\beta_2}{dl} = \beta_2 + \frac{2}{2\pi^2} \left\{ -\frac{4C^2_2a_s\lambda^2[\beta_2D_1^2 + (\beta_1 - \beta_2)D_2]}{u_s} \right\} \left[ 1 - 2\left( 2CD_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] + 9C^2_2 \beta_2^2 D_1^2 \left[ 4CD_1(a - \frac{\lambda a_s}{u_s}) - 1 \right] + \frac{S^2}{D_1^2} \left[ 4SE_1(a - \frac{\lambda a_s}{u_s}) - 1 \right] - \frac{\lambda^2(1 + 4a_s)}{4} - \frac{4CD_1 a_s \lambda^3}{3u_s} \left[ 1 - 2\left( 2CD_1(a - \frac{\lambda a_s}{u_s}) - 2a_s \right) \right] - \frac{2S^2 \sqrt{2} a_s \lambda^2 \beta_2}{3D_1 u_s} \left[ 1 - 2\left( 2SE_1(a - \frac{\lambda a_s}{u_s}) - a_s \right) \right] \right\}. \tag{A20}$$

$$\frac{d\phi}{dl} = 0. \tag{A21}$$
For type-II case-E with $|n_x^2|^2 \neq |n_y|^4$, $|n_y^2|^2 \neq |n_x|^4$, $|n_x^2|^2 = 0$, and $|n_y^2|^2 = 0$, $\beta_2$ evolves but $\beta_1$ is an invariant constant,

$$\frac{d\beta_1}{dl} = 0,$$

$$\frac{d\beta_2}{dl} = \beta_2 + \frac{2\pi}{\Delta l^2} \left\{ -4C^2a_s\lambda^2[\beta_2 D_1^2 + (\beta_1 - \beta_2)D_2] \right\} \left[ 1 - 2(2CD_1(a - \lambda a_s)) - a_s \right] + 9C^2\beta_2 D_2(4CD_1(a - \lambda a_s) - 1)$$

$$+ \frac{S^2\Delta l^2}{3D_2^3}[4SE_1(a - \lambda a_s) - 1] - \frac{\lambda^2(1 + 4a_s)}{4} - 4CD_1a_s\lambda^3 \cdot \frac{3a_s}{1 - 2(2CD_1(a - \lambda a_s) - 2a_s)}$$

$$- \frac{2S^2\Delta l^2}{3D_2^3}[1 - 2(2SE_1(a - \lambda a_s)) - a_s] \right\}. \quad (A23)$$

For type-II case-F with $E_2D_2^2 - D_2E_2^2 = 0$, both $\beta_1$ and $\beta_2$ are energy-independent constants,

$$\frac{d\beta_1}{dl} = 0, \quad \frac{d\beta_2}{dl} = 0. \quad (A24)$$

**Appendix B: Stabilities of incommensurate magnetic states**

As aforementioned in Sec. III of main text, there are seven different types of IC magnetic states other than three commensurate ones including stripe SDW, CSDW, and SVC [16, 17, 33, 34]. To be concrete, these IC magnetic states cover four different $C_2$ IC cases consisting of $C_2$ IC ICS, $C_2$ MH, $C_2$ ICS $\perp$ MH, and $C_2$ DPMH, as well as three distinct $C_4$ IC situations involving $C_4$ IC CSDW, $C_4$ IC SVC, and $C_4$ IC SWC [19]. In order to examine whether these IC magnetic states are stable against the decrease of energy scales, we within this section lean upon the coupled RG equations (A22)-(A24), which are completely encoded with the information of ordering competition, in conjunction with their stable constraints catalogued in Table III of the main text.

In principle, the energy variable of RG evolution is expressed by $\Lambda = \Lambda_0 e^{-l}$ with $l > 0$ denoting the running scale. As our study is concerned with the structure of schematic phase diagram, it is herein of remarkable convenience to associate $l$ with temperature via designating $T = T_0/e^{-l}$ with $T_0$ being the initial temperature to measure the evolution of energy scale [18, 31, 32, 34]. On the basis of this transformation and RG equations in conjunction with the strategy addressed in Sec. IIIA we are now in a proper position to judge whether these IC magnets are good candidates residing in the phase diagram of $\text{Ba}_1-x\text{Na}_x\text{Fe}_2\text{As}_2$ one by one.

We start out by considering the $C_2$ IC magnetic states. On one hand, the configurations of spin vectors for $C_2$ ICS magnetic state read $\mathbf{n}_{x} = (0, 0, 1)$ and $\mathbf{n}_{y} = (0, 0, 0)$ [19], which satisfy the restricted conditions of type-II case-A. This indicates the interaction parame-
FIG. 14: (Color online) (a) Temperature-dependent stable constraints $C_4$-symmetry IC CSDW state for case-1 under the representative starting values of interaction parameters chosen as $g_1 = -0.015$, $g_2 = -0.01$, $u_c = 0.05$, $\lambda = 0.01$, $\beta_1 = 0.005$, $\beta_2 = 0.01$ with three representative $\theta = \pi/12$, $\pi/6$, $\pi/3$ to satisfy the IC CSDW’s stable constraint (the qualitative results are insensitive to initial values of parameters). Hereby, the angle $\theta$ is designated in Sec. II A to specify the direction of magnetic order in the spin space. Inset: the sign-change region of $(g_2 + |\beta_1 - \beta_2|)/|\beta_1 - \beta_2|$. (b) Sign-change regions at different values of $\theta$.

Iters obey the RG evolutions of type-II case-A delineated in Eqs. (22, 23, 28, 29, and 33-35). As for $C_2$ ICs, its stable constraints can be either $(\beta_1 - \beta_2) < 0$, $g_2/|\beta_1 - \beta_2| < 0$, $(g_1 - \beta_2)/|\beta_1 - \beta_2| > -1$ or $(\beta_1 - \beta_2) < 0$, $g_2/|\beta_1 - \beta_2| < 0$, $(g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_2| > -1$.

Based on these, we perform numerical RG analysis by taking some initial representative values of parameters and obtain the results shown in Fig. 14. On the other, concerning $C_2$ MH and $C_2$ DPMH, the configurations of spin vectors are characterized by $n_x = \frac{1}{\sqrt{2}}(i, 0, 1)$, $n_y = (0, 0, 0)$, and $n_x = \frac{1}{\sqrt{2}}(i, 0, 1)$, respectively.

Accordingly, this indicates that the interaction parameters are dictated by the evolutions for type-II case-D provided in Eqs. (22, 23, 28, 29, and 33-35). To proceed, we parallel the analogous RG numerical analysis taking advantage of the corresponding constraints.

In a sharp contrast, with respect to $C_2$ ICS $\perp$ MH, whose the configurations of spin vectors are related to $n_x = (0, 0, 1)$ and $n_y = \frac{1}{\sqrt{2}}(i, 0, 0)$, their interaction parameters are therefore subject to type-I coupled RG equations (22, 23, 28, 29, and 33-35). Carrying out the similar numerical analysis gives rise to temperature-dependent evolutions depicted in Fig. 14. It manifestly heralds that stable constraints of $C_2$ ICs $\perp$ MH, i.e., $(\beta_1 - \beta_2) > 0$, $(g_1 - \beta_2)/|\beta_1 - \beta_2| < 0$, and $g_2/|\beta_1 - \beta_2| > f(g_1 - \beta_2) \approx 2$.

In a finite value of $g_1 - \beta_2$ and limit$(g_1 - \beta_2) \to 0$, $f(g_1 - \beta_2) \to 0$.

It is very necessary to point out that $C_2$ ICS $\perp$ MH can be destroyed as long as the magnetic QCP is closely accessed, at which the ordering competition becomes so ferocious that any state cannot present solely.

Next, we go to judge $C_4$ IC magnetic states, which include $C_4$ IC SVC, $C_4$ SCW, and $C_4$ ICS CSDW. In analogy to $C_2$ IC magnetic states, we inspect low-energy fates of these states by combining their RG equations and stable constraints. For $C_4$ SVC with the configurations of spin vectors being $n_x = (0, 0, 1)$ and $n_y = (0, 1, 0)$, the interaction parameters are governed by the type-II case-A RG equations (22, 23, 28, 29, and 33-35) and the stable constraints correspond to $(\beta_1 - \beta_2) < 0$, $g_2/|\beta_1 - \beta_2| > 0$, and $(g_1 - \beta_2)/|\beta_1 - \beta_2| < -1$.

The numerical results presented in Fig. 15 reflect that $C_4$ SVC cannot be a well stable state in the phase diagram caused by the influence of ordering competition.

To proceed, we turn to $C_4$ IC SCW, which is well protected by constraints $(\beta_1 - \beta_2) > 0$, $(g_1 - \beta_2)/|\beta_1 - \beta_2| < 0$, and $0 < g_2/|\beta_1 - \beta_2| < 2$. In addition, the...
FIG. 15: (Color online) Temperature-dependent stable constraints $C_4$-symmetry IC CDW state for case-2 under the representative starting values of interaction parameters chosen as $g_1 = -0.015$, $g_2 = -0.01$, $u_e = 0.05$, $\lambda = 0.01$, $\beta_1 = 0.01$, $\beta_2 = 0.005$ with a representative $\theta = \pi/12$ to satisfy the IC CDW's stable constraint (the qualitative results are insensitive to initial values of parameters). Hereby, the angle $\theta$ is designated in Sec. II A to specify the direction of magnetic order in the spin space.

configurations of spin vectors are equivalent to $\mathbf{n}_X = (i \cos \phi, 0, \sin \phi)$ and $\mathbf{n}_Y = (0, \cos \phi, \sin \phi)$. Before going further, it is of particular interest to address that they can be clustered into two sub-situations distinguished by the parameter $\phi$ which is introduced by $\mathbf{n}_X \cdot \mathbf{n}_Y = \sin^2 \phi$ and characterize the symmetric double-$Q$ noncoplanar SWC with $\phi = \pi/4$ and asymmetric double-$Q$ noncoplanar with $\phi \neq \pi/4$, respectively [19]. As a result, the former interaction parameters are dictated by type-II case-A RG equations (A2)–(A5), (A8), (A9), and (A13)–(A15) but instead the latter ones evolve under type-II case-F RG equations exhibited in Eqs. (A2)–(A5), (A8), (A9), (A13), and (A24). Carrying out analogous RG steps yields to Fig. 12 and Fig. 13 which explicitly signals $C_4$ SWC is not suitable to be present in the phase diagram.

Further, we move to $C_4$ IC CDW state, at which the configurations of spin vectors are of the form $\mathbf{n}_X = (0, 0, 1)$ and $\mathbf{n}_Y = (0, 0, 1)$ [19], and thus type-II case-A RG equations (A2)–(A5), (A8), (A9), and (A13)–(A15) are in charge of the low-energy fates of interaction parameters. Hereby, it is necessary to highlight that $C_4$-symmetry IC CDW [19] can be stabilized by either $|\beta_1 - \beta_2| < 0, g_2/|\beta_1 - \beta_2| < 0, (g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_2| < -1$ (case-1) or $(\beta_1 - \beta_2) > 0, g_2/|\beta_1 - \beta_2| < -1, (g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_2| < -1$ (case-2). Fig. 14 and Fig. 15 collect the central results stemming from RG analysis, which manifestly exhibit the temperature (energy) dependence of associated parameters for $C_4$ IC CDW. In the light of these figures, we are informed that stable constraints for both case-1 and case-2 are considerably robust with the decrease of temperature, which of course can be sabotaged due to sufficiently strong fluctuations so long as the magnetic QCP is closely approached. Consequently, $C_4$ IC CDW, like its $C_2$ ICS $\perp$ MH counterpart, is of fair robustness against ordering competition and an appropriate candidate for $C_4$ magnetic state in phase diagram of Ba$_{1-x}$Na$_x$Fe$_2$As$_2$. 

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