1. Introduction

Trying to make sense of integrals that are not absolutely convergent is an old endeavour in mathematics and physics, that, despite its apparent meaninglessness, has been surprisingly fruitful and useful in many subjects. Hadamard defined the

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finite part of a divergent integral by first introducing a cutoff, namely by integrating over the complement of a small neighbourhood of the singular set of the integrand, say of size $\epsilon$, and then letting $\epsilon$ tend to zero, after subtraction of divergent terms, see [3], Book III, Chapter I. His motivation was to give a meaning to formal solutions of differential equations, integrals that would be solutions if one were allowed to differentiate under the integral sign. This sort of questions as well as the related questions on the asymptotic behaviour of level set integrals, were also one of the motivations for the theory of generalized functions, see [2], Section II.4.

The same strategy of subtracting infinities from divergent integrals was followed by Feynman to make sense of divergent integrals in perturbative quantum field theory. Renormalised integrals are obtained from divergent integrals by first regularizing them, by introducing a cutoff as above or by a similar procedure, and then removing the cutoff after subtracting divergent terms, see, e.g., Chapter 8 of [4]. In this context the question of dependence on the choice of cutoff arises, and the goal is to show that the final results for physical quantities such as scattering amplitudes, are independent of the choice of regularization. Since the Feynman integrals are integrals over a Euclidean space, it is natural to restrict to regularizations that use this structure, such as cutting off points at distance $< \epsilon$ from the integration region. The question becomes more subtle in perturbative string theory, where Feynman integrals are replaced by integrals over moduli spaces of curves. They can be interpreted as integrals of singular differential forms on the Deligne–Mumford compactification of moduli spaces with singularities on the compactification divisor. Again the integrals are defined by cutting off a small neighbourhood of the divisor and study the asymptotic behaviour as the size of the neighbourhood tends to zero. Since there is no natural way to choose the family of shrinking neighbourhoods, the question of dependence on the regularization is subtle. In fact, in the case of superstrings the limit as the size goes to zero exists without subtracting divergent terms, but it depends on the regularization in a calculable way, showing that changes of regularization can be absorbed into redefinition of the coupling constants, see [5], Section 7.

Inspired by these calculations in superstring theory, in [1] we considered integrals of products of holomorphic and antiholomorphic differential forms on complex manifolds with poles on hypersurfaces. In the case where the antiholomorphic form has a simple pole, we gave formulae for the dependence on the choice of cutoff function, generalizing a calculation of [5]. To treat the general case it is useful to consider a more general setting, which is the approach of this paper.

We consider integrals of differential forms on an oriented $n$-dimensional manifold $X$ that are singular on a submanifold $Y$. The kind of singularities we allow are determined by a conformal class of nonnegative Morse–Bott functions vanishing on $Y$: a nonnegative Morse–Bott function with zero set $Y$ is a nonnegative function $\mu$ on $X$ vanishing exactly on $Y$ with non-degenerate Hessian in the normal direction. Given such a Morse–Bott function $\mu$ we consider the space $A_\mu(X)$ of differential forms $\omega$ on $X \setminus Y$ such that, for some integer $N \geq 0$, $\mu^N \omega$ extends smoothly to $X$. Clearly $A_\mu(X) = A_{f \mu}(X)$ for any positive smooth function $f$ on $X$, so that only the conformal class of $\mu$, consisting of all $f \mu$ with $f$ everywhere positive, plays a role. An important special case is when $Y$ is a hypersurface in a complex manifold $X$, the setting of [1]. In this case we have a canonical conformal class of Morse–Bott functions, consisting of nonnegative Morse–Bott functions locally divisible by $|f|^2$ for any holomorphic function $f$ with a simple zero on $Y$.

Returning to the general case, we wish to give a meaning to the divergent integral $\int_X \omega$ of a top differential form $\omega \in A^{\dim X}_\mu(X)$ whose support has compact closure in $X$. For small $\epsilon > 0$, the inequality $\mu < \epsilon^2$ defines a tubular neighbourhood of $Y$
and the integral over its complement is well-defined. It is then not difficult to see
that, as $\epsilon \to 0$,
\[
\int_{\mu \geq \epsilon^2} \omega = \sum_{k=1}^{2N-m} I_{-k} \epsilon^{-k} + I_0 \log \frac{1}{\epsilon} + I_{\text{finite}} + O(\epsilon),
\]
where $m$ is the codimension of the submanifold $Y \subset X$. The Hadamard finite part
of the divergent integral $\int_X \omega$ is then by definition $I_{\text{finite}} = I_{\text{finite}}(\mu, \omega)$. In general it
depends on the choice of nonnegative Morse–Bott function vanishing on $Y$ and
the question is to describe the dependence.

For this purpose it is useful to introduce the zeta function $\zeta(s; \mu, \omega)$ defined as
the meromorphic continuation of the absolutely convergent integral
\[
\zeta(s; \mu, \omega) = \int_X \mu^{s/2} \omega, \quad \text{Re } s \gg 0.
\]
It turns out that the zeta function, as a function of $s$, has only simple poles and that
$I_0 = \text{res}_{s=0} \zeta(s; \mu, \omega)$ is independent of $\mu$ within its conformal class, see Theorem
2.4. The first result expresses the finite part and describes its dependence on the
Morse–Bott function in its conformal class in terms of the zeta function and its
residue at 0.

**Theorem 1.1.** Let $\omega \in A^m_\mu(X)$, $\psi \in A^{n-1}_\mu(X)$.

(i) $I_{\text{finite}}(\mu, \omega) = \lim_{s \to 0} \big( \zeta(s; \mu, \omega) - \frac{I_0(\mu, \omega)}{s} \big)$.

(ii) For any smooth function $\varphi$, $I_{\text{finite}}(e^{2s} \mu, \omega) = I_{\text{finite}}(\mu, \omega) + I_0(\mu, \omega \varphi)$.

(iii) $I_{\text{finite}}(\mu, d\psi) = \frac{1}{2} I_0(\mu, \psi \wedge d\mu/\mu)$.

Part (i) of this Theorem is proved in Section 2.3, see Theorem 2.4. Part (ii) is
discussed in Section 2.4, see Theorem 2.7. It extends the result of [1], where the
case of complex hypersurfaces was studied. Finally Part (iii) is proved in Section
3.1, Proposition 3.2.

We see that a key role is played by the map $I_0$. It turns out that $I_0$ vanishes
if the codimension $m$ is odd, so in that case the finite part is independent of the
choice of Morse–Bott function within a conformal class. Moreover, because of (iii),
the finite part is a well defined function on the cohomology of the complex $A_\mu(X)$.
Thus this story is mostly interesting if $m = 2r$ is even. In this case we derive a
local formula for $I_0$ in terms of a residue map. For this it is useful to extend the
setting and consider differential forms $\omega \in A_\mu$ not necessarily of top degree. As we
show in Theorem 3.3, the linear form
\[
I_0(\mu, \omega \wedge \varphi) : \omega \mapsto I_0(\mu, \omega \wedge \varphi)
\]
on smooth compactly supported forms of complementary degree, defines a de Rham
current with support in $Y$ and the map $\omega \mapsto I_0(\mu, \omega \wedge \varphi)$ is a morphism of complexes
$A_\mu(X) \to \mathcal{D}'(X)$ to the complex of currents. We then observe that $A_\mu(X)$ is the
algebra of global sections of a sheaf $\mathcal{A}_{X,\mu}$ of differential graded algebras. We show
that $\mathcal{A}_{X,\mu}$ has a quasi-isomorphic subcomplex $A_{X,\mu}^{\text{tame}}$ of differential forms with tame
singularities. By definition, $\omega \in A_{X,\mu}$ has tame singularities if $\mu^r \omega$ and $\mu^{r-1} d\mu \wedge \omega$
extend to smooth forms on $X$.

**Theorem 1.2.** Let $m = 2r$ be even and $i: Y \to X$ denote the inclusion map. Assume also that both $X$ and $Y$ are oriented. There is a morphism of complexes of sheaves $R: A_{X,\mu}^{\text{tame}} \to i_* A_Y [-m]$ such that for any global differential form $\omega$ with tame singularities, and compactly supported smooth form $\varphi$ of complementary
degree,
\[
I_0(\mu, \omega \wedge \varphi) = \int_Y R(\omega) \wedge \varphi.
\]
We prove this result in Section 3, see Theorem 3.7. The orientability of \( Y \), assumed here for simplicity of exposition, is not really needed and is dropped there at the cost of involving the orientation bundle of \( Y \). The “residue map” \( R \) can be given a fairly explicit formula, see Theorem 3.9. We then address the question of comparing \( \mathcal{A}_{X,\mu} \) for \( \mu \) belonging to different conformal classes. We show that the sheaves of differential graded algebras \( \mathcal{A}_{X,\mu} \) are essentially independent of the conformal class of \( \mu \): they come with a quasi-isomorphism—unique up to a contractible space of choices—to a homotopy colimit over the category of simplices of the singular set of the cone of Morse–Bott functions, see Theorem 3.11.

In the last two Sections we focus on the important special cases where \( Y \) has codimension 2 and 1 in \( X \), respectively.

In Section 4 we specialize our results to the case of complex hypersurfaces. The complex structure gives rise to a canonical class of conformal structure and a canonical orientation of both \( X \) and \( Y \). We recover and generalize results of [1] to the case of arbitrary order of poles. We also extend the results to the case where \( Y \) is a divisor with normal crossings: Theorem 1.1 has a natural generalization, see Theorem 5.4, which is however combinatorially slightly more involved.

In Section 6 we treat the case of codimension 1. As mentioned above, the odd codimension case is less involved as the zeta function is regular at \( s = 0 \). However in this case it is natural to extend our setting and consider \( Y \) to be the boundary of an oriented manifold with boundary \( X \). Then we have only one conformal class of Morse–Bott functions (defined as squares of functions vanishing to first order on the boundary) and a canonical orientation of \( Y \). It turns out that the zeta function has a pole at zero and that all our results in the even codimension case have an analogue in the case of manifolds with boundary, see Theorems 6.1, 6.3.

In the Appendix we calculate the cohomology sheaf of \( \mathcal{A}_\mu \). This calculation is used in Section 3 to show that \( \mathcal{A}_\mu \) is quasi isomorphic to the subcomplex of differential forms with tame singularities.

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## 2. Divergent integrals

### 2.1. Nonnegative Morse–Bott functions

Let \( X \) be an \( n \)-dimensional smooth oriented manifold and \( Y \subset X \) a closed submanifold of dimension \( n - m \). We consider regularizations of divergent integrals \( \int_X \omega \) of smooth differential \( n \)-forms \( \omega \) on \( X \setminus Y \). To do this, following Hadamard, we cut out a small neighbourhood of \( Y \) from the integration and study the behaviour of the integral as the size of the neighbourhood tends to zero. The neighbourhoods we cut out are parametrized by a class of smooth functions that we now introduce.

**Definition 2.1.** A *nonnegative Morse–Bott function* on \( X \) with zero set \( Y \) is a smooth function \( \mu : X \to \mathbb{R}_{\geq 0} \) vanishing exactly on \( Y \) and such that the rank of the Hessian on \( Y \) is equal to the codimension of \( Y \). Two nonnegative Morse–Bott functions are called *conformally equivalent* if they vanish on the same submanifold \( Y \) and their ratio is an everywhere positive function. The equivalence classes for this equivalence relations are called *conformal classes*.

By the Morse–Bott lemma, for each nonnegative Morse–Bott function \( \mu \) vanishing on \( Y \) and point \( x \in Y \) there are coordinate functions \( x_1, \ldots, x_m \) on some neighbourhood of \( x \) in \( X \) such that \( Y \) is given there by the equations \( x_1 = \cdots = x_m = 0 \) and such that \( \mu = x_1^2 + \cdots + x_m^2 \).
Let $\mu$ be a nonnegative Morse–Bott function vanishing on $Y$. We denote by $A_\mu(X)$ the de Rham complex of differential forms $\omega$ on $X \setminus Y$ such that, for some integer $N$, $\mu^N \omega$ extends to a smooth differential form on $X$. Clearly $A_\mu(X)$ only depends on the conformal class of $\mu$.

2.2. Zeta functions and level set integrals. To an $n$-form $\omega \in A_\mu^n(X)$ whose support has compact closure in $X$ we associate the zeta function

$$\zeta(s; \mu, \omega) = \int_X \mu^s \omega,$$

and to an $n-1$-form $\alpha \in A^{n-1}_\mu(X)$ the level set integral

$$I(t; \mu, \alpha) = \int_{\mu = t} \alpha, \quad t > 0.$$  

The zeta function is defined and holomorphic for sufficiently large $\text{Re } s$. It is related to the level set integral by a Mellin transform:

**Lemma 2.2.** Let $\alpha \in A^{n-1}_\mu(X)$ with support in a sufficiently small neighbourhood of $Y$. Set

$$\omega = \frac{d\mu}{2\mu} \wedge \alpha.$$  

Then

$$\zeta(s; \mu, \omega) = \int_0^\infty t^{s-1} I(t; \mu, \alpha) dt.$$  

The regularized integral is

$$\int_{\mu \geq \epsilon^2} \omega = \int_\epsilon^\infty I(t; \mu, \alpha) dt.$$  

**Proof.** The assumption on the support allows us choose spherical coordinates on the support of $\alpha$ such that $r = \sqrt{\mu}$ is a radial coordinate. By a partition of unity argument we may assume that $\alpha$ has support in a coordinate neighbourhood of a point of $Y$ and that $\mu = x_1^2 + \cdots + x_m^2$ in a suitable coordinate system. In spherical coordinates $r > 0, y \in S^{m-1}$ in the normal direction, we can replace $X \setminus Y$ by $\mathbb{R}_{>0} \times S^{m-1} \times \mathbb{R}^{n-m}$ and write

$$\omega = f r dx_1 \wedge \cdots \wedge dx_n$$

$$= f(r, y, x_{m+1}, \cdots, x_n) r^{m-1} dr \wedge d\Omega \wedge dx_{m+1} \wedge \cdots \wedge dx_n.$$  

Here $d\Omega$ is the volume form on the unit sphere. The zeta function can be then evaluated in polar coordinates:

$$\zeta(s, \mu, \omega) = \int_0^\infty r^{s-1} J(r) dr,$$

with

$$J(r) = \int_{S^{m-1} \times \mathbb{R}^{n-m}} f(r, y, x) r^m d\Omega \wedge dx_{m+1} \wedge \cdots \wedge dx_n.$$  

The integrand is the coordinate expression of $\alpha$, so $J(r) = I(r; \mu, \alpha)$. The same calculation with $s = 0$ and integration range $(\epsilon, \infty)$ gives the formula for the regularized integral.

**Lemma 2.3.** Assume that $\omega$ has support in a sufficiently small neighbourhood of $Y$ and that $\mu^N \omega$ extends to a smooth form on $X$. Then $t^{2N-m} I(t; \mu, \alpha)$ extends to a smooth, compactly supported, even function of $t \in \mathbb{R}$. 


Proof. By a partition of unity we may assume that the support of $\omega$ is contained in a small neighbourhood of a point of $Y$. By the Morse–Bott lemma we may also assume that there are local coordinates on that neighbourhood so that $Y$ is given by $x_1 = \cdots = x_m = 0$ and

$$\mu = x_1^2 + \cdots + x_m^2.$$  

Then $\omega = \mu^{-N} f \, dx_1 \ldots dx_m$ for some smooth function $f$. We use spherical coordinates in the normal direction: locally $U$ is $\mathbb{R}_{>0} \times S^{m-1} \times \mathbb{R}^{n-m}$, with radial coordinate $r = \sqrt{\mu}$, and

$$\omega = fr^{-2N+m-1} \, dr \wedge d\Omega \wedge dx_{m+1} \wedge \cdots \wedge dx_n.$$  

Here $d\Omega$ is the volume form on the unit sphere. We can therefore choose $\alpha$ to be

$$\alpha = fr^{-2N+m}d\Omega \wedge dx_{m+1} \wedge \cdots \wedge dx_n,$$

and

$$I(t; \mu, \alpha) = t^{-2N+m} \int_{S^{m-1} \times \mathbb{R}^{n-m}} f(ty, x) d\Omega(y) \wedge dx_{m+1} \wedge \cdots \wedge dx_n.$$  

It is clear that the integral on the right-hand side is defined for all $t \in \mathbb{R}$ and is a smooth compactly supported function of $t$. Since the involution $y \mapsto -y$ of the sphere maps $d\Omega$ to $(-1)^m d\Omega$ and preserves/reverses the orientation if $m$ is even/odd, we get

$$I(-t; \mu, \alpha) = (-1)^m I(t; \mu, \alpha).$$

\[\Box\]

2.3. Hadamard finite part. Here we consider the regularization of the divergent integral of a differential forms in $A_\mu(X)$ for some nonnegative Morse–Bott function $\mu$ on $X$ vanishing on a submanifold $Y$ and define its Hadamard finite part. While $A_\mu(X)$ depends only on the conformal class of $\mu$, the finite part depends on $\mu$ and we describe its dependence within the conformal class.

**Theorem 2.4.** Let $\mu$ be a nonnegative Morse–Bott function vanishing on $Y$. Let $\omega$ be an $n$-form on $X \setminus Y$ such that $\mu^N \omega$ extends to a smooth form on $X$ with compact support. Then

(i) $\int_X \mu^{s/2} \omega$ is holomorphic for $\text{Re} \, s > 2N - m$ and has a meromorphic continuation $\zeta(s; \mu, \omega)$ with at most simple poles on the arithmetic progression $s = 2N - m, 2N - m - 2, \ldots.$

(ii) As $\epsilon \to 0$ we have an expansion

$$\int_{\mu \geq \epsilon^2} \omega = \sum_{k=1}^{2N-m} I_{-k}(\mu, \omega) \epsilon^{-k} + I_0(\mu, \omega) \log \frac{1}{\epsilon} + I_{\text{finite}}(\mu, \omega) + O(\epsilon)$$

and $I_k = 0$ unless $k \equiv m \pmod{2}$.

(iii) For $k = 1, \ldots, 2N - m$, $k \equiv m \pmod{2},$

$$I_{-k}(\mu, \omega) = \frac{1}{k} \text{res}_{s=k} \zeta(s; \mu, \omega),$$

and

$$I_0(\mu, \omega) = \text{res}_{s=0} \zeta(s; \mu, \omega)$$

is independent of $\mu$ within its conformal class; it vanishes if $m$ is odd. The finite part is

$$I_{\text{finite}}(\mu, \omega) = \lim_{s \to 0} \left( \zeta(s; \mu, \omega) - \frac{I_0(\mu, \omega)}{s} \right).$$
Proof. We may assume that \( \omega \) has support in an arbitrary small neighbourhood of \( Y \) since we can achieve this by adding to \( \omega \) a form on \( X \) with support disjoint from \( Y \). Let
\[
I(t; \mu, \alpha) = \sum_{k=-2N+m}^{p} b_k t^k + R_p(t)
\]
be the Laurent expansion of \( I \). Since \( t^{2N-m} I \) extends to a smooth even function,
\[
b_k = 0, \quad \text{if } k \not\equiv m \mod 2.
\]
For any \( M > 0 \), the remainder \( R_p(t) \) is bounded by \( C t^{p+1} \) for \( t \in [0, M] \). For sufficiently large \( M \) and \( \text{Re}(s) \) we have
\[
\zeta(s; \mu, \omega) = \int_0^\infty t^{s-1} I(t; \mu, \alpha) dt
\]
(2.1)
\[
= \int_0^M t^{s-1} I(t; \mu, \alpha) dt + \sum_{k=-2N+m}^{p} b_k M^{k+s} k + \int_0^M t^{s-1} R_p(t) dt
\]
The last term is holomorphic for \( \text{Re}(s) > -p - 1 \). This proves (i) and gives the formula
\[
b_k = \text{res}_{s=-k} \zeta(s; \mu, \omega)
\]
for the residues at the poles. A similar calculation can be done for the integral with cutoff if we expand the integral on level sets up to order \( p = 0 \) giving the proof of (ii):
\[
\left( e^{2\varphi} \mu \right)^{s-1} - \int_X \mu^{s-1} \omega = \int_X \mu^{s-1} \omega(s)
\]
where
\[
\omega(s) = \frac{e^{s\varphi} - 1}{s} \omega
\]
is an entire function of \( s \) and has a convergent expansion at \( s = 0 \) with coefficients in \( A_\mu(X) \). Thus \( \int_X \mu^{s/2} \omega(s) \) has an analytic continuation with at most a simple pole at \( s = 0 \) and the right-hand side of (2.2) is regular there. It follows that \( \zeta(s; e^{2\varphi} \mu, \omega) \) and \( \zeta(s; \mu, \omega) \) have the same residue at \( s = 0 \).

Finally, the finite part is
\[
I_{\text{finite}}(\mu, \omega) = \sum_{k=-2N+m}^{-1} b_k \frac{M^k}{k} + b_0 \log M + \int_0^M R_0(t) \frac{dt}{t},
\]
for any sufficiently large $M$. It coincides with the value at $s = 0$ of the expression above for $\zeta(s; \mu, \omega)$ after subtraction of the pole $I_0/s = b_0/s$ (the logarithmic term comes from $\lim_{s \to 0}(M^s/s - 1/s) = \log M$).

\begin{proof}
Let, as in the proof of Theorem 2.4, subcomplex in terms of a residue map. Theorem 2.4 (iii). It is also clear from Theorem 2.4 (ii): Replacing $\mu$ by $\mu + \epsilon$ in $\int_{\mu \geq e^\epsilon} \omega$ is homoge-

uous of degree $k/2$ as a function of $\mu$.

\begin{proof}
This follows from the obvious identity $\zeta(s; t\mu, \omega) = t^{s/2}\zeta(s; \mu, \omega)$, $t > 0$, and Theorem 2.4 (iii). It is also clear from Theorem 2.4 (ii): Replacing $\mu$ by $t\mu$ is the same as replacing $\epsilon$ by $t^{-1/2}\epsilon$.
\end{proof}

\begin{definition}
The Hadamard finite part of the divergent integral $\int_X \omega$ with Morse–Bott function $\mu$ is $I_{\text{finite}}(\mu, \omega)$, as defined in Theorem 2.4 (ii) or (iii).
\end{definition}

\section{Dependence on the Morse–Bott function.}
The following result says how the finite part depends on the Morse–Bott function $\mu$ in its conformal class.

\begin{theorem}
For any function $\varphi \in C^\infty(X)$,
\[ I_{\text{finite}}(\mu e^{2\varphi}, \omega) = I_{\text{finite}}(\mu, \omega) + I_0(\mu, \varphi \omega), \]
where $I_0(\mu, \varphi \omega) = \text{res}_{s=0} \zeta(s; \mu, \varphi \omega)$ is independent of $\mu$.
\end{theorem}

\begin{proof}
Let, as in the proof of Theorem 2.4
\[ \omega(s) = \frac{e^{s\varphi} - 1}{s} \omega. \]
Since $\omega(0) = \varphi \omega$, we have by (2.2)
\[ I_{\text{finite}}(\mu e^{2\varphi}, \omega) - I_{\text{finite}}(\mu, \omega) = \lim_{s \to 0} \left( \zeta(s; e^{2s} \mu, \omega) - \zeta(s; \mu, \omega) \right) = \text{res}_{s=0} \zeta(s; \mu, \varphi \omega). \]

By Theorem 2.4 (iii), the right-hand side is independent of $\mu$.
\end{proof}

\begin{corollary}
Suppose that $\mu^N \omega$ extend to a smooth form on $X$ and that $\varphi \leq \text{const} \mu^{-1/2}$. Then
\[ I_{\text{finite}}(\mu e^{2\varphi}, \omega) = I_{\text{finite}}(\mu, \omega) \]
\end{corollary}

\begin{proof}
In this case $\varphi \omega = f \text{vol}$ for some smooth volume form $\text{vol}$ and a function $f$ such that $|f| \leq \text{const} \mu^{-1/2}$ which is integrable. Thus $\zeta(s; \mu, \varphi \omega)$ is smooth and given by its absolutely convergent integral representation at $s = 0$.
\end{proof}

\begin{remark}
If the codimension $m$ of $Y$ is odd, then $I_0 = 0$ and the finite part is independent of the choice of $\mu$ in its conformal class.
\end{remark}

\section{The de Rham complex of differential forms with singularities}

Let $\mu$ be a nonnegative Morse–Bott function on an $n$-dimensional manifold $X$ vanishing on a submanifold $Y$ of codimension $m$. Let $\omega$ be a top degree differential on $X \setminus Y$ such that $\mu^N \omega$ extends to a form on $X$ with compact support for some $N$. The residue at zero of the zeta function
\[ I_0(\mu, \omega) = \text{res}_{s=0} \zeta(s; \mu, \omega) \]
is independent of the choice of the Morse–Bott function $\mu$ within a fixed conformal class. It vanishes if the codimension of $Y$ is odd, so in this section we assume that $m$ is even. We define a sheaf of differential forms whose global sections are the forms $A_\mu(X)$ on which $I_0$ is defined and give a local formula for $I_0$ on a quasi-isomorphic subcomplex in terms of a residue map.
3.1. The de Rham complex. Let $Y \subset X$ as above, $U = X \setminus Y$ and denote $j : U \to X$ the inclusion map. We write $\mathcal{A}_Z = \oplus_j \mathcal{A}_Z^j$ for the complex of sheaves of differential forms on a manifold $Z$ with de Rham differential. Let $\mu$ be a nonnegative Morse–Bott function vanishing on $Y$. We identify $\mathcal{A}_X$ as the subcomplex of $j_* \mathcal{A}_U$ consisting of forms on $U$ that extends to $X$. We set

$$
\mathcal{A}_{X, \mu} = \bigcup_{N \geq 0} \mathcal{A}_{X, \mu, N},
$$

$$
\mathcal{A}_{X, \mu, N} = \{ \omega \in j_* \mathcal{A}_{X \setminus Y} : \mu^N \omega \in \mathcal{A}_X \}.
$$

Lemma 3.1.

(i) The de Rham differential maps $\mathcal{A}_{X, \mu, N}$ to $\mathcal{A}_{X, \mu, N+1}$. In particular, $\mathcal{A}_{X, \mu}$ is a subcomplex of $j_* \mathcal{A}_{X \setminus Y}$.

(ii) $\mathcal{A}_{X, \mu, N}$ depends only on the conformal class of $\mu$.

Proof. (i) Suppose $\mu^N \omega$ extends smoothly to $X$ then $d(\mu^{N+1} \omega)$ is also smooth and therefore also

$$
\mu^{N+1} d\omega = d(\mu^{N+1} \omega) - (N + 1) d\mu \wedge \mu^N \omega.
$$

The support condition is preserved by the differential. (ii) It is clear that $\mathcal{A}_{X, \mu} = \mathcal{A}_{X, f_* \mu}$ for any everywhere positive function $f$. \hfill \Box

In the notation of the preceding sections, $\mathcal{A}_\mu(X) = \Gamma(X, \mathcal{A}_{X, \mu})$ is the differential graded algebra of global sections.

We may then view $I_0$ as a map on compactly supported sections of $\mathcal{A}_{X, \mu}^\infty$:

$$
I_0: \Gamma_c(X, \mathcal{A}_{X, \mu}^\infty) \to \mathbb{C}.
$$

Proposition 3.2. Let $\psi \in \Gamma_c(X, \mathcal{A}_{X, \mu}^n)$. Then

(i) $I_0(\mu, d\psi) = 0$.

(ii) $I_{\text{finite}}(\mu, d\psi) = I_0(\mu, \psi \wedge \theta)$, where $\theta = \frac{d\mu}{2\mu}$.

Proof. By Stokes’s theorem we have

$$
\int_X \mu^s d\psi = -\frac{s}{2} \int_X \mu^{s-1} d\mu \wedge \psi,
$$

provided Re $s$ is sufficiently large. Since $d\mu/\mu \wedge \psi$ belongs to $\mathcal{A}_{X, \mu}$, we get the identity of meromorphic functions

$$
(3.1) \quad \zeta(s; \mu, d\psi) = -s \zeta(s; \mu, \frac{d\mu}{2\mu} \wedge \psi).
$$

Since the zeta function on the right has only simple poles, the left-hand side is regular at zero. Since $I_0(\mu, d\psi) = 0$, the finite part is just the value of the zeta function at $s = 0$. By (3.1),

$$
\zeta(0; \mu, d\psi) = -\lim_{s \to 0} s \zeta(s; \mu, \theta \wedge \psi) = -\text{res}_{s=0} \zeta(s; \mu, \theta \wedge \psi),
$$

which is $-I_0(\mu, \theta \wedge \psi)$ by definition. \hfill \Box

3.2. The homomorphism to de Rham currents. Let $\mathcal{D}^p(X) = \Gamma_c(X, \mathcal{A}_{X, \mu}^p)$ be the space of compactly supported differential $p$-forms with the usual Fréchet topology. Recall that the space $\mathcal{D}'(X)^p$ of de Rham currents of degree $p$ is the space of continuous linear forms on $\mathcal{D}^{n-p}(X)$. The complex of currents is the direct sum $\mathcal{D}'(X) = \bigoplus_{p=0}^n \mathcal{D}'(X)^p$ with differential $d: \mathcal{D}'(X)^p \to \mathcal{D}'(X)^{p+1}$ defined by

$$
d\kappa(\varphi) = (-1)^{p+1} \kappa(d\varphi), \quad \kappa \in \mathcal{D}'(X)^p, \quad \varphi \in \mathcal{D}^{n-p}(X).
$$

For any smooth differential $p$-form $\omega$, the map $\varphi \mapsto \int_X \omega \wedge \varphi$ defines a current of degree $p$ and this defines an injective morphism of complexes $\mathcal{D}(X) \hookrightarrow \mathcal{D}'(X)$. A current is said to be supported on a closed subset $Y$ if it vanishes on all forms with support in its complement.
Theorem 3.3. Let $\omega \in \Gamma(X, A^p_{X, \mu})$. Then

$$I_0(\mu, \omega \wedge \cdot): \varphi \mapsto I_0(\mu, \omega \wedge \varphi), \quad \varphi \in D^{n-p}(X),$$

is a de Rham current supported on $Y$. The map $\omega \mapsto I_0(\mu, \omega \wedge \cdot)$ is a morphism of complexes $\Gamma(X, A^p_{X, \mu}) \to D'(X)$.

Proof. Suppose $\varphi$ has support in a sufficiently small neighbourhood of a point of $Y$. Then a local calculation with Morse–Bott coordinates shows that $I_0(\mu, \omega \wedge \varphi)$ is a finite sum of terms of the form $\int_Y \alpha \wedge \partial_1^{n-p} \varphi$, for some differential operators $\partial_1$. By a partition of unity argument, the same holds for general $\varphi$. This is certainly a well-defined de Rham current. The fact that the map is a morphism of complexes follows from Prop. 3.2. Indeed let $\omega \in \Gamma(X, A^p_{X, \mu})$, $\kappa_\omega = I_0(\mu, \omega \wedge \cdot)$ and $\varphi \in D^{n-p-1}(X)$.

Then

$$d\kappa_\omega(\varphi) = (-1)^p + I_0(\mu, \omega \wedge d\varphi) = -I_0(\mu, d(\omega \wedge \varphi)) + I_0(\mu, d\omega \wedge \varphi) = 0 + \kappa_d\omega(\varphi).$$

3.3. The subcomplex of differential forms with tame singularities. Let $m = 2r$ be the even codimension of $Y$. We introduce a subcomplex of the de Rham complex $A^p_{X, \mu}$ which is quasi-isomorphic to it. It is analogous to the complex of logarithmic forms.

Definition 3.4. A differential form $\omega \in A^p_{X, \mu}$ has tame singularities if

(i) $\mu^r \omega \in A_X$,
(ii) $\mu^{r-1} d\mu \wedge \omega \in A_X$,

for the half-codimension $r$. We denote by $A^{tame}_{X, \mu}$ the sheaf of differential forms with tame singularities.

In fact $A^{tame}_{X, \mu}$ is a subcomplex, as we now show. More generally we prove that it is part of a filtration of the complex $A^p_{X, \mu}$:

$$\cdots \subset F_p A^p_{X, \mu} \subset F_{p+1} A^p_{X, \mu} \subset \cdots \subset A^p_{X, \mu} = \cup_{p \in \mathbb{Z}} F_p A^p_{X, \mu}$$

by subspaces

$$F_p A^p_{X, \mu} = \{ \omega \in A^p_{X, \mu} : \mu^p \omega, \mu^{p-1} d\mu \wedge \omega \in A \}.$$

Lemma 3.5. Each $F_p A^p_{X, \mu}$, in particular $F_r A^p_{X, \mu} = A^{tame}_{X, \mu}$, is a subcomplex.

Proof. Suppose $\omega \in F_p A^p_{X, \mu}$. Then

$$\mu^p d\omega = d(\mu^p \omega) - p \mu^{p-1} d\mu \wedge \omega,$$

$$\mu^{p-1} d\mu \wedge d\omega = -d(\mu^{p-1} d\mu \wedge \omega).$$

The right-hand sides are regular on $X$ by assumption. Thus $F_p A_{X, \mu}$ is a subcomplex.

Proposition 3.6. Let $Y \subset X$ have codimension $m = 2r$. The inclusion $A^{tame}_{X, \mu} \hookrightarrow A^p_{X, \mu}$ is a quasi-isomorphism.

This is a local statement, so it is sufficient to prove it on a small ball in $\mathbb{R}^n$ with $\mu = \sum_{i=1}^m x_i^2$. In this case it follows from the calculation of the cohomology done in the Appendix, see Corollary A.11.
3.4. The residue map. Let \( i: Y \hookrightarrow X \) be the inclusion map and denote by \( \text{ory}\) the orientation bundle of \( Y \). We define a residue map \( R: \mathcal{A}_{X,\mu}^{\text{tame}} \to i_* (\text{ory} \otimes \mathcal{A}_Y [-m]) \) such that for any \( \omega \in \Gamma_c(X, \mathcal{A}_{X,\mu}^{\text{tame}}) \) with tame singularities and relatively compact support. We denote by \( C_X^\infty \) the sheaf of smooth functions on \( X \).

**Theorem 3.7.** Let \( m = 2\tau \) be the codimension of \( Y \). There is a unique morphism of graded \( C_X^\infty \)-modules

\[
R: \mathcal{A}_{X,\mu}^{\text{tame}} \to i_* (\text{ory} \otimes \mathcal{A}_Y [-m])
\]

such that for any \( p \)-form \( \omega \in \Gamma(X, \mathcal{A}_{X,\mu}^{\text{tame}}) \) with tame singularities and smooth compactly supported \( (n-p) \)-form \( \varphi \in \Gamma(X, \mathcal{A}_X) \),

\[
I_0(\mu, \omega \wedge \varphi) = \int_Y R(\omega) \wedge \varphi.
\]

Moreover \( R \) is a morphism of complexes of sheaves.

The proof of Theorem 3.7 occupies the rest of this section.

We first discuss uniqueness. First of all for any global section \( \omega \in \Gamma_c(X, \mathcal{A}_{X,\mu}^{\text{tame}}) \), \( R(\omega) \) is uniquely determined by (3.2) since a de Rham current is represented by at most one smooth form. It remains to show that \( R \) is uniquely determined by its action on global sections. This follows from the fact that it is linear over the algebra of functions: let \( \omega \) be a section on an open set \( U \subset X \). Then for any \( f \in C^\infty(U) \) with compact support, \( f \omega \) extends (by zero) to \( X \) and since \( R \) is a map of sheaves we obtain that \( I_0(\mu, f \omega \wedge \varphi) = \int_Y R(f \omega) \wedge \varphi \) for all \( \varphi \) with support in \( U \). We may now choose \( f \) to be 1 on the support of \( \varphi \), so that \( f \varphi = \varphi \). By the \( C_X^\infty \)-linearity of \( R \), it follows that (3.2) holds for sections \( \omega \) on any open subset \( U \) and \( \varphi \) with compact support in \( U \). Therefore \( R \) is uniquely defined as a map of sheaves. The uniqueness also implies that \( R \) is a morphism of complexes: by Prop. 3.2,

\[
I_0(\mu, dw \wedge \varphi) = \int_Y R(\omega) \wedge d\varphi
\]

and therefore \( R(d\omega) = (-1)^m dR(\omega) \), where \( (-1)^m d \) is the differential of \( i_* \mathcal{A}_Y [-m] \).

To prove existence, we claim that we may assume that \( X \) is a small ball in \( \mathbb{R}^n \) with the standard orientation and that \( \mu = x_1^2 + \cdots + x_m^2 \). To reduce the general case to this local statement, notice that we may an open cover \( (U_i) \), such that on each \( U_i \) we have Morse–Bott coordinates. Assuming the local statement, we obtain, for each global section \( \omega \), forms \( R_i(\omega) \) defined on \( U_i \) such that

\[
I_0(\mu, \omega \wedge \varphi) = \int_{Y_i} R_i(\omega) \wedge \varphi
\]

for all forms \( \varphi \) with support on \( U_i \). By uniqueness, these forms \( R_i(\omega) \) must agree on intersections and are thus restrictions of a unique form \( R(\omega) \) on \( Y \).

From now on, we thus assume that \( X \) is small ball around the origin of \( \mathbb{R}^n \) and that \( \mu = \sum_{i=1}^m x_i^2 \), so that \( Y \) is given by the equations \( x_1 = \cdots = x_m = 0 \).

**Lemma 3.8.** Let \( \omega \) be a differential form with tame singularities. Then

\[
\omega = \frac{1}{(x_1^2 + \cdots + x_m^2)^{\tau}} (dx_1 \wedge \cdots \wedge dx_m \wedge \alpha + \sum_{i=1}^m x_i \alpha_i)
\]

for some smooth forms \( \alpha, \alpha_i \).
Thus $\psi$ compatible orientation $dx$ implying that the second condition may be written as

$$\sum_{i=1}^{m} x_i dx_i \wedge \psi \equiv 0 \mod x_1^2 + \cdots + x_m^2,$$

implying that $\sum_{i=1}^{m} x_i dx_i \wedge \psi|_{Y=0} = 0$. This condition has the form

$$\sum_{i \in I} \pm x_i \psi_{1\sim(i)}|_{Y} = 0.$$

Thus $\psi_J|_{Y}$ vanishes for all $J$ of the form $I \sim \{i\}$, namely such that $|J| < m$. \hfill $\square$

We can now compute the residue of the zeta function in spherical coordinates. Let $x = (x', x''')$ with $x'$ the first $m$ coordinates. Write $x' = ry$ with $y \in S^{m-1}$ on the unit sphere with volume form $d\Omega(y)$. Then, in the notation of Lemma 3.8

$$\zeta(s; \mu, \omega \wedge \varphi) = \int_{R \times S^{m-1} \times R^{n-1}} r^{s-1} dr \wedge d\Omega(y) \wedge (\alpha \wedge \varphi + O(r)).$$

This integral is a holomorphic function of $s$ in the right half-plane and has a simple pole at $s = 0$. Its residue can be computed as in the proof of Theorem 2.4 by first integrating over $r$. To do this calculation we need to choose an orientation of $Y$ (a trivialization of $\text{or}_Y$), which we take to be defined by $dx_{m+1} \wedge \cdots \wedge dx_n$ and the compatible orientation $dx_1 \wedge \cdots \wedge dx_m$ of the fibres.

$$I_0(\mu, \omega \wedge \varphi) = \int_{S^{m-1}} d\Omega \int_{R^{n-1}} \alpha \wedge \varphi = \frac{2\pi^\frac{m}{2}}{(m/2 - 1)!} \int_Y \alpha \wedge \varphi.$$ 

Thus the claim of the Theorem holds with $R(\omega) = \frac{2\pi^\frac{m}{2}}{(m/2 - 1)!} \alpha|_Y$. If we change the trivialization of $\text{or}_Y$, the orientation of the fibres, and thus $R$, change sign, and we get a well defined form on $Y$ twisted by the orientation bundle. It is clear from the definition in Lemma 3.8 that $R$ is $C^\infty$-linear. The proof of Theorem 3.7 is complete.

3.5. An explicit formula. The proof of Lemma 3.8 gives an explicit formula for $R(\omega)$ in terms of Morse–Bott coordinates. We may formulate it more invariantly as follows. The Hessian of the Morse–Bott function $\mu$ defines a euclidean metric on the normal bundle. Each local trivialization of $\text{or}_Y$ defines an orientation of the normal bundle. These two data define a volume form $V(\mu)$ on the normal bundle, which we may view as a section of $\text{or}_Y \otimes \wedge^m T^*X|_Y$ vanishing on vectors tangent to $Y$.

**Theorem 3.9.** Let $m = 2r$ be the even codimension of $Y$. Let $\omega \in \Gamma(X, A^\text{tame}_{X, \mu})$. Then

$$R(\omega) = \frac{2\pi^r (\mu' \omega)|_Y}{(r-1)! V(\mu)}.$$ 

More properly, the restriction of $\mu' \omega$ to $Y$ is of the form $V(\mu) \wedge \alpha$ and $R(\omega) = \frac{2\pi^r}{(r-1)!} \alpha$. 

3.6. Dependence on the conformal class of the Morse–Bott function. To compare the complexes $A_{X,\mu}$ for different $\mu$ we notice that for any two such $\mu_0, \mu_1$, the function $\mu: (x,t) \mapsto t\mu_1(x) + (1-t)\mu_0(x)$ is a nonnegative Morse–Bott function on $X \times I$ vanishing on $Y \times I \subset X \times I$, where $I = [0,1]$ and restricting to $\mu_j$ at the endpoints. Let $p: X \times I \to X$ be the projection to the first factor. We then have maps

$$(3.3) \quad A_{X,\mu_0} \leftarrow p_* A_{X \times I, \mu} \to A_{X,\mu_1}.$$ 

Proposition 3.10. These maps are quasi-isomorphisms of complexes of sheaves.

It follows that we have a canonical isomorphism between the cohomology sheaves for $\mu_0$ and $\mu_1$.

To obtain a more precise information, in particular to show that the composition of isomorphisms is again an isomorphism of this form, we prove a slightly stronger version of this proposition: we denote by $\Delta_p = \{ t \in \mathbb{R}^p_{\geq 0} : t_0 + \cdots + t_p = 1 \}$ the geometric $p$-simplex.

Let $MB$ be the convex cone of nonnegative Morse–Bott functions $Y S(MB)$ the category of simplices of the affine singular set of $MB$. Its objects are affine $p$-simplices in $MB$, i.e., affine maps from the geometric $p$-simplex $\Delta_p = \{ t \in \mathbb{R}^p_{\geq 0} : \sum t_i = 0 \}$ to $MB$, and the morphisms are compositions of face and degeneracy maps.

Theorem 3.11. There is a functor

$$F: S(MB) \to \text{ShDGA}(X)$$

to the category of sheaves of differential graded algebras, such that on vertices

$$F(\mu) = A_{X,\mu},$$

and sending all morphisms to quasi-isomorphisms.

Since $MB$ is contractible, it follows that all $A_{X,\mu}$ are quasi-isomorphic to the homotopy colimit $\text{hocolim} F$, and that the quasi-isomorphism is unique up to a contractible space of choices.

To prove this theorem we begin by defining the functor. Let $\mu_0, \ldots, \mu_p$ be nonnegative Morse–Bott functions vanishing on $Y$. They are vertices of a $p$-simplex $\Delta(\mu_0, \ldots, \mu_p)$, an object of $S(MB)$. We set

$$F(\Delta(\mu_0, \ldots, \mu_p)) = p_* A_{X \times \Delta_p, \sum t_i\mu_i}.$$  

Here $p: X \times \Delta_p \to X$ is the projection onto the first factor and we view the convex linear combination $\sum_{i=0}^p t_i\mu_i$ as a nonnegative Morse–Bott function on $X \times \Delta_p$ vanishing on $Y \times \Delta_p$. The face and degeneracy maps are mapped to the pull-backs of corresponding face and degeneracy maps on the $\Delta_p$. The first example is (3.3).

We turn to the proof of Theorem 3.11. Since the claim is a local statement we may assume that $X$ is a small ball centered at the origin in $\mathbb{R}^n$. The non-trivial case is when the origin is in $Y$. To prove the proposition in this case we need a slight generalization of the Morse–Bott lemma.

Lemma 3.12. Let $\mu_0, \ldots, \mu_p$ be nonnegative Morse–Bott functions on an open ball $B \subset \mathbb{R}^n$ centered at the origin and vanishing on the same smooth submanifold $0 \in Y \subset B$ of codimension $m$. For $t \in \Delta_p$ set

$$\mu_t = \sum_{i=0}^p t_i\mu_i.$$  

1 This approach was suggested to us by Tomer Schlank
Then there are smooth functions $z_1, \ldots, z_m$ on $B' \times \Delta_p$ for some possibly smaller ball $B' \subset B$, such that

$$\mu_t = z_1^2 + \cdots + z_m^2 \text{ on } B' \times \Delta_p.$$ 

**Proof.** By the Morse–Bott lemma there exist functions $x_1^{(i)}, \ldots, x_m^{(i)}$ vanishing on $Y$ and defined on a possibly smaller ball $B' \subset B$ and with linear independent differentials on $Y$, such that $\mu_i = (x_1^{(i)})^2 + \cdots + (x_m^{(i)})^2$, for $i = 0, \ldots, p$. Let $x_1, \ldots, x_m$ be generate the ideal of functions vanishing on $Y$, for instance $x_i = x_i^{(0)}$. Then we can write $x_j^{(i)} = \sum_{k=1}^m x_k a_{kj}^{(i)}$, for some smooth functions $a_{kj}^{(i)}$ forming, for each $i$, a non-degenerate matrix. After possibly rotating $x_j^{(i)}$ by a linear orthogonal transformation, we may assume that $x_j^{(i)} = x_j g_j^{(i)}$ for some smooth functions $g_j^{(i)}$ with $g_j^{(i)}|_Y = 1$. Then

$$\mu_t = \sum_{j=1}^m x_j^2 \sum_{i=0}^p t_i (g_j^{(i)})^2.$$ 

Since $\sum_{i=0}^p t_i (g_j^{(i)})^2$ is close to 1 in the vicinity of $Y$, it is positive and we can define, again after making $B'$ smaller, new functions

$$z_j = x_j \sqrt{\sum_{i=0}^p t_i (g_j^{(i)})^2},$$

such that $\mu_t = \sum_{j=1}^m z_j^2$. \hfill \square

**Proof of Theorem 3.11.** As we saw, it is sufficient to assume that $X$ is a small ball centered at the origin in $\mathbb{R}^n$. To prove that morphisms are mapped to quasi-isomorphisms it is sufficient to prove that face maps and degeneracy maps are mapped to quasi-isomorphisms. These maps involve simplices with a fixed set of vertices, say $\mu_0, \ldots, \mu_p$. By Lemma 3.12 we may assume that $\mu_i = \bar{\mu} = x_1^2 + \cdots + x_m^2$ for all $\mu$. Thus

$$\sum_{i=0}^p t_i \mu_i = \bar{\mu},$$

is a constant function of $t \in \Delta_p$. Thus the algebras $F(\Delta(\mu_0, \ldots, \mu_p))$ are all equal to $p_*A_{X,\Delta_k, p}$. The maps $A_{X, \Delta_k, p} \to p_*A_{X, \Delta_k, p}$ sending a form to its pull back by $p$ are quasi-isomorphisms commuting with face and degeneracy maps. The maps induced by face and degeneracy maps in cohomology are thus the identity maps in $H(A_{X, \mu})$. \hfill \square

We conclude this section by stating an elementary consequence.

**Corollary 3.13.** For any two nonnegative Morse–Bott functions $\mu_0, \mu_1$ we have a canonical isomorphism of the cohomology sheaves

$$I(\mu_0, \mu_1): \mathcal{H}(A_{X, \mu_0}) \to \mathcal{H}(A_{X, \mu_1}).$$

For any three $\mu_0, \mu_1, \mu_2$ we have

$$I(\mu_1, \mu_2) \circ I(\mu_0, \mu_1) = I(\mu_0, \mu_2).$$

The first statement follows from Proposition 3.10. The second statement follows from the case $p = 2$ of Theorem 3.11. The sheaves $A_i = A_{X, \mu_i}$, $i = 0, 1, 2$ are
related by a commutative diagram of quasi-isomorphisms:

\[ \begin{array}{c}
A_0 \\
\downarrow \\
A_1 \\
\downarrow \\
A_2 \\
\end{array} \]

where \( A_{i_0, \ldots, i_k} = p_* A_{X, \sum s_i \mu_{i_s}} \).

4. Complex Hypersurfaces

4.1. The canonical conformal class of Morse–Bott functions. Suppose \( D \subset X \) is a smooth divisor (complex hypersurface) in a \( d \)-dimensional complex manifold. Thus we have \( n = 2d \) and \( m = 2 \). The complex structure defines a canonical conformal class of nonnegative Morse–Bott functions \( \mu \) vanishing on \( D \): locally on an open set \( U \subset X \), \( D \) is defined by \( f = 0 \) for some holomorphic function \( f \) on \( U \) such that \( df|_{D\cap U} \neq 0 \). We call such a function a local equation for \( D \).

We then require \( \mu/|f|^2 \) to extend to a positive smooth function. This condition is independent of \( f \) as the ratio of any two \( f \)'s is a nowhere vanishing function. Any two nonnegative Morse–Bott functions with this property differ by multiplication by a positive function and thus define a conformal class. The differential forms we consider can then be defined as those locally of the form \( \omega/|f|^{2N} \) with \( \omega \) smooth and \( f \) a local equation for \( D \).

The de Rham complex \( A_{X, \mu} \) is quasi-isomorphic to the subcomplex of differential forms with tame singularities. Here is a description of these forms, which in this context could be called bilogarithmic.

**Proposition 4.1.** Let \( \omega \in \Gamma(X, A_{X, \mu}) \). Then \( \omega \) has tame singularities if and only for any local equation \( f \) of \( D \),

\[ \omega = \frac{df}{f} \wedge \frac{df}{f} \wedge \omega_{1,1} + \frac{df}{f} \wedge \omega_{1,0} + \frac{df}{f} \wedge \omega_{0,1} + \omega_{0,0}, \]

for some smooth forms \( \omega_{i,j} \). The residue map is

\[ R(\omega) = -4\pi i \omega_{1,1}. \]

**Proof.** We may choose local complex coordinates \( z_1, \ldots, z_d \) so that \( f = z_1 \) and \( \mu = |z_1|^2 \). Let us write

\[ |z_1|^2 \omega = dz_1 \wedge d\bar{z}_1 \wedge \alpha_{1,1} + dz_1 \wedge \alpha_{1,0} + d\bar{z}_1 \wedge \alpha_{0,1} + \alpha_{0,0}, \]

for some forms \( \alpha_{i,j} \) not involving \( dz_1 \) or \( d\bar{z}_1 \). The first condition for tameness implies that \( \alpha_{i,j} \) are smooth. The second condition in real codimension 2 states that \( d\mu \wedge \omega \) is smooth. Since \( d\mu = z_1 d\bar{z}_1 + \bar{z}_1 dz_1 \) this translates to the smoothness of

\[ \omega_{1,0} = \frac{1}{z_1} \alpha_{1,0}, \quad \omega_{0,1} = \frac{1}{\bar{z}_1} \alpha_{0,1}, \quad \omega_{0,0} = \frac{1}{z_1 \bar{z}_1} \alpha_{0,0}. \]

Comparing with the explicit formula of \[ \omega \], we see that \( R(\omega) \) is proportional to \( \alpha \).

The Morse–Bott coordinates \( x_1, x_2 \) are given by \( z_1 = x_1 + ix_2 \) and \( d\bar{z}_1/dz_1/|z_1|^2 = -2i dx_1 \wedge dx_2/(x_1^2 + x_2^2) \). Thus

\[ R(\omega) = 2\pi(-2i)\omega_{1,1} = -4\pi i \omega_{1,1}. \]

\( \square \)
In \[\mathbb{I}\], inspired by calculations in perturbative superstring theory, we considered the case where \(\omega = \alpha \wedge \bar{\beta}\) where \(\beta\) is a holomorphic \(d\)-form with simple pole on \(Y\) and \(\alpha\) is a smooth \((d,0)\)-form on \(X \setminus D\) so that locally \(f^N \alpha\) extends to a compactly supported smooth form on \(X\) for some \(N\). There we defined a Dolbeault residue \(\text{Res}_\partial\) defined on this class of \((d,0)\)-forms \(\alpha\) and taking values in \(\partial\)-cohomology classes of forms of type \((d-1,0)\) on \(D\). The Dolbeault residue vanishes on \(\partial\)-exact forms and coincides with the Poincaré residue \(\text{Res}\) for forms with first order pole. Comparing with Prop. \[\mathbb{I}\] we obtain
\[
R(\alpha \wedge \bar{\beta}) = 4\pi i (-1)^d \text{Res}_\partial \alpha \wedge \overline{\text{Res}_\partial \beta}.
\]
The dependence on the Morse–Bott function of the Hadamard finite part is thus
\[
I_{\text{finite}}(\mu e^{2\varphi}, \alpha \wedge \bar{\beta}) = I_{\text{finite}}(\mu, \alpha \wedge \bar{\beta}) + (-1)^d 4\pi i \int \text{Res}_\partial \alpha \wedge \overline{\text{Res}_\partial \beta},
\]
in agreement with \([\mathbb{I}]\).

5. Normal crossing divisor

It is desirable to extend our results to the case where the singularities of the differential forms are not smooth submanifolds. We consider here the special case of a divisor \(D\) with normal crossings in a complex manifold. We first focus on the case of two components \(D = D_1 \cup D_2\). Away from the intersection the theory of Section \([\mathbb{II}]\) applies, so it is sufficient to consider a neighbourhood of the intersection, which is locally given by \(z_1 = 0\), \(z_2 = 0\), for some local coordinate functions \(z_1\), \(z_2\). Let \(\omega\) be a top degree form on \(X \setminus D\) and assume that \([z_1 z_2]^{2N} \omega\) extends to a smooth form on \(X\) with compact support. The zeta function is
\[
\zeta(s_1, s_2; \mu_1, \mu_2, \omega) = \int_X \frac{\mu_1^{s_1} \mu_2^{s_2}}{s_1 s_2} \omega.
\]
It depends on nonnegative Morse–Bott functions \(\mu_1, \mu_2\) and vanishing on \(D_1\) and \(D_2\) respectively. As in the case of a smooth divisor we take \(\mu_1\) and \(\mu_2\) in the canonical conformal class defined by the complex structure. As a function of \(s_1, s_2\) the zeta function is holomorphic for \(\text{Re}(s_i)\) large enough and extends to a meromorphic function on \(\mathbb{C}^2\) with at most simple poles on the lines \(s_i = 2k, \ k \in \mathbb{Z}\).

We define the finite part of \(\int_X \omega\) as the constant term of the Laurent expansion of \(\zeta\) at 0.

**Definition 5.1.** The finite part of the divergent integral \(\int_X \omega\) is
\[
I_{\text{finite}}(\mu_1, \mu_2, \omega) = \text{res}_{s_1 = 0} \text{res}_{s_2 = 0} \frac{1}{s_1 s_2} \zeta(s_1, s_2; \mu_1, \mu_2, \omega).
\]

**Remark 5.2.** If \(\omega\) is regular on one of the components, say \(D_2\), then \(\zeta\) is regular at \(s_2 = 0\) and our definition of the finite part reduces to the one for smooth divisors.

To describe the dependence on the Morse–Bott functions it is useful to introduce coefficients of divergent terms:
\[
I_{j,k}(\mu_1, \mu_2, \omega) = \text{res}_{s_1 = 0} \text{res}_{s_2 = 0} \frac{1}{s_1 s_2} \zeta(s_1, s_2; \mu_1, \mu_2, \omega).
\]
We only care about \(j, k = 0\) or 1. The Laurent expansion looks like
\[
\zeta(s_1, s_2; \mu_1, \mu_2, \omega) = \frac{I_{0,0}}{s_1 s_2} + \frac{I_{0,1}}{s_1} + \frac{I_{1,0}}{s_2} + I_{\text{finite}} + \cdots
\]
in the dots there are other divergent terms such as \(s_1/s_2\). Note that \(I_{\text{finite}} = I_{1,1}\).

**Proposition 5.3.**
(1) \( I_{0,0} \) is independent of \( \mu_1, \mu_2 \), \( I_{0,1} \) is independent of \( \mu_1 \) and \( I_{1,0} \) is independent of \( \mu_2 \).

(2) Let \( \varphi \in C^\infty(X) \). Then

\[
\begin{align*}
I_{\text{finite}}(e^{2\varphi} \mu_1, \mu_2, \omega) &= I_{\text{finite}}(\mu_1, \mu_2, \omega) + I_{0,1}(\mu_2, \varphi \omega), \\
I_{\text{finite}}(\mu_1, e^{2\varphi} \mu_2, \omega) &= I_{\text{finite}}(\mu_1, \mu_2, \omega) + I_{1,0}(\mu_1, \varphi \omega), \\
I_{1,0}(e^{2\varphi} \mu_1, \omega) &= I_{1,0}(\mu_1, \omega) + I_{0,0}(\varphi \omega), \\
I_{0,1}(e^{2\varphi} \mu_2, \omega) &= I_{0,1}(\mu_2, \omega) + I_{0,0}(\varphi \omega).
\end{align*}
\]

Proof. For \( s_1, s_2 \) with large real part

\[
\zeta(s_1, s_2; e^{2\varphi} \mu_1, \mu_2, \omega) = \zeta(s_1, s_2; \mu_1, \mu_2, \omega) - s_1 \int_X \frac{\mu_1^s \mu_2^s e^{s\varphi} - 1}{s_1 \omega}.
\]

The analytic continuation is regular on the line \( s_1 = 0 \) and \( (\ldots) \) vanishes there. For \( j \in \{0, 1\} \) we get

\[
\begin{align*}
I_{0,j}(e^{2\varphi} \mu_1, \mu_2, \omega) - I_{0,j}(\mu_1, \mu_2, \omega) &= 0, \\
I_{1,j}(e^{2\varphi} \mu_1, \mu_2, \omega) - I_{1,j}(\mu_1, \mu_2, \omega) &= I_{0,j}(\mu_1, \mu_2, \varphi \omega),
\end{align*}
\]

and similarly for \( \mu_2 \).

We thus obtain the formula

\[
I_{\text{finite}}(e^{2\varphi} \mu_1, e^{2\varphi} \mu_2, \omega) = I_{\text{finite}}(\mu_1, \mu_2, \omega) + I_{1,0}(\mu_1, \varphi \omega) + I_{1,0}(\mu_2, \varphi \omega).
\]

It is easy to generalize this result to the case of an arbitrary number of components. Let \( D = D_1 \cup \cdots \cup D_m \) be a divisor in \( X \) with normal crossings and irreducible components \( D_i \). Take a nonnegative Morse–Bott function \( \mu_i \) for \( D_i \) in the canonical conformal class for each \( i \). Let \( \omega \) be a top form on \( X \setminus D \) so that, at a generic point of each component \( D_i \), \( \mu_i \omega \) is smooth for sufficiently large \( N \). We then have a zeta function

\[
\zeta(s; (\mu_i)_{i=1}^m, \omega) = \int_X \frac{\mu_1^s \cdots \mu_m^s}{s_1 \cdots s_m} \omega
\]

which has an analytic continuation to a meromorphic function of \( s \in \mathbb{C}^m \) with at most simple poles on the hyperplanes \( s_i = 2k, i = 1, \ldots, m, k \in \mathbb{Z} \). We let \( [m] = \{1, \ldots, m\} \) and for any \( M \subset [m] \),

\[
I_M((\mu_i)_{i=1}^m, \omega) = \sum_{\omega_{s_i=0} \cdots \omega_{s_m=0}} \left( \prod_{i \in M} \frac{1}{s_i} \right) \zeta(s; (\mu_i)_{i=1}^m, \omega)
\]

and define the finite part as

\[
I_{\text{finite}}((\mu_i)_{i=1}^m, \omega) = I_{[m]}((\mu_i)_{i=1}^m, \omega).
\]

**Theorem 5.4.**

1. \( I_M \) is independent of \( \mu_j, j \notin M \).
2. Let \( i \in M \) and \( \varphi \in C^\infty(X) \). Then

\[
I_M(\ldots, e^{2\varphi} \mu_i, \ldots, \omega) = I_M(\ldots, \mu_i, \ldots, \omega) + I_M(\{i\}) (\mu_1, \ldots, \mu_m, \varphi \omega)
\]

We write \( I_M = I_M((\mu_i)_{i \in M}, \omega) \) accordingly.
Corollary 5.5.
\[ I_{\text{finite}}((e^{2\varphi_i} \mu_i)_{i=1}^m, \omega) = \sum_{M \subset [m]} I_M((\mu_i)_{i \in M}, \prod_{i \in M} \varphi_i \omega) \]
\[ = I_{\text{finite}}((\mu_i)_{i=1}^m, \omega) + \sum_{M \subset [m]} I_M((\mu_i)_{i \in M}, \prod_{i \in M} \varphi_i \omega) \]
Theorem 6.1. Let $M \in \mathbb{Z}_{\geq 0}$ and $\lambda$ be a nonnegative smooth function vanishing on $Y$ to first order. Let $\omega \in \Gamma_c(X, A_{X,Y})$ have polar singularity of order $M$ at $Y \subset \partial X$. Then

(i) $\int_X \lambda^s \omega$ is holomorphic for $\text{Re } s > M - 1$ and has a meromorphic continuation $\tilde{\zeta}(s; \lambda, \omega)$ with at most simple poles on the arithmetic progression $s = M - 1, M - 2, \ldots$.

(ii) As $\epsilon \to 0$ we have an expansion

\[
\int_{\lambda \geq \epsilon} \omega = \sum_{k=1}^{M-1} I_{-k}(\lambda, \omega) \epsilon^{-k} + I_0(\omega) \log \frac{1}{\epsilon} + I_{\text{finite}}(\lambda, \omega) + O(\epsilon).
\]

(iii) For $k = 1, \ldots, M - 1$,

\[
I_{-k}(\lambda, \omega) = \frac{1}{k} \text{res}_{s=k} \tilde{\zeta}(s; \lambda, \omega),
\]

and

\[
I_0(\omega) = \text{res}_{s=0} \tilde{\zeta}(s; \lambda, \omega)
\]

is independent of $\lambda$. The finite part is

\[
I_{\text{finite}}(\lambda, \omega) = \lim_{s \to 0} \left( \tilde{\zeta}(s; \lambda, \omega) - \frac{I_0(\omega)}{s} \right).
\]

(iv) For any function $\varphi \in C^\infty(X)$,

\[
I_{\text{finite}}(\lambda e^\varphi, \omega) = I_{\text{finite}}(\lambda, \omega) + I_0(\varphi \omega).
\]

The analogue of the tame differential forms are (the real version) of logarithmic forms. By definition, a logarithmic form in $A_{X,Y}$ is a form $\omega$ such that $\lambda \omega$ and $d\lambda/\lambda \wedge \omega$ extend to smooth forms on $X$ for one (and thus any) choice of a nonnegative function $\lambda$ vanishing to first order on $Y$. As in the complex case, and in the case of forms with tame singularities, logarithmic forms form a subcomplex of sheaves $A_{X,Y}^{\log}$ which is quasi-isomorphic to $A_{X,Y}$. Given a choice of the function $\lambda$ vanishing to first order on $Y \subset \partial X$, any logarithmic form can locally be written as

\[
\omega = \frac{d\lambda}{\lambda} \wedge \sigma + \tau,
\]

for some smooth forms $\sigma, \tau$. Moreover it is standard to check that $\sigma|_Y$ is independent of the choice of the decomposition and of the choice of $\lambda$. Thus the map $\omega \mapsto \sigma|_Y$ is well-defined and is the real analogue of the Poincaré residue map.

Definition 6.2. The residue is the map $R: A_{X,Y} \to i_* A_Y[-1]$ such that

\[
R \left( \frac{d\lambda}{\lambda} \wedge \sigma + \tau \right) = \sigma|_Y.
\]

Theorem 6.3.

(i) The residue map $R$ is a morphism of complexes of sheaves.

(ii) For any logarithmic $p$-form $\omega \in \Gamma(X, A_{X,Y}^{\log})$ smooth compactly supported $(n-p)$-form $\varphi \in \Gamma(X, A_X)$,

\[
I_0(\omega \wedge \varphi) = \int_Y R(\omega) \wedge \varphi.
\]

The real analogue of a normal crossing divisor is the boundary of a manifold with corners. We leave it to the reader to extend the results of Section 5 to this case.
Appendix A. Cohomology: local calculation

Let $X$ be an open ball in $\mathbb{R}^n$ centered at the origin and $Y \subset X$ its intersection with the subspace $x_1 = \cdots = x_m = 0$. Let $\mu = x_1^2 + \cdots + x_m^2$. We compute the cohomology of the complex $A_\mu(X)$ of differential forms $\alpha$ on $X \setminus Y$ such that $\mu^N \alpha$ is smooth for some $N$.

A.1. Cohomology of $A_\mu(D^n)$. We denote by $\Lambda(t_1, \ldots, t_k)$ the exterior algebra with generators $t_1, \ldots, t_k$.

**Proposition A.1.** Let $X = D^n$ be an open ball in $\mathbb{R}^n$ centered at the origin, $\mu = x_1^2 + \cdots + x_m^2$, and $Y = \mu^{-1}(0) \cap X \subset X$.

(i) If $m$ is odd,

$$H(A_\mu(X)) \cong \Lambda(\alpha), \quad \deg(\alpha) = 1.$$ 

(ii) If $m$ is even,

$$H(A_\mu(X)) \cong \Lambda(\alpha, \beta), \quad \deg(\alpha) = 1, \quad \deg(\beta) = m - 1.$$ 

Here $\alpha$ is the class of $\alpha = d\mu/\mu$ and $\beta$ (in the even case) is the class of

$$\beta = \sum_{i=1}^{m} (-1)^{i-1} dx_1 \wedge \cdots \wedge (dx_i \wedge \cdots \wedge dx_m)/(x_1^2 + \cdots + x_m^2)^{m/2}.$$ 

It is the basic representative of a rotation invariant volume form on the $m - 1$-dimensional real projective space.

To prove this result we first notice that $A_\mu(X)$ has a subcomplex $B(X)$ of differential forms vanishing to infinite order at $Y$. By Borel’s lemma, the quotient $A_\mu(X)/B(X)$ is isomorphic to the complex of differential forms that are formal power series in the normal direction:

$$C_\mu(X) = A_\mu(X)/B(X) = A(Y)[[x_1, \ldots, x_m]](\frac{1}{\mu}) \otimes \Lambda(dx_1, \ldots, dx_m).$$

The Euler vector field $e = \sum_{i=1}^{m} x_i \partial_{x_i}$ acts on $C_\mu$ via the Lie derivative $L_e = d \circ e + e \circ d$.

**Lemma A.2.** The inclusion map $\text{Ker}(L_e) \hookrightarrow C_\mu(X)$ induces an isomorphism in cohomology.

**Proof.** The complex $C_\mu$ splits into a direct sum $C_\mu(X) = \text{Ker}(L_e) \oplus \text{Im}(L_e)$ of subcomplexes invariant under $L_e$, such that $L_e$ is invertible on $\text{Im}(L_e)$. \hfill $\square$

**Lemma A.3.** Any form $\omega \in C_\mu(X)$ can uniquely be written as

$$\omega = \frac{d\mu}{\mu} \wedge \sigma + \tau,$$

where $\iota_e \sigma = 0 = \iota_e \tau$.

**Proof.** We have

$$\iota_e d\mu = \sum x_i \iota_{e x_i} 2 \sum x_i dx_i = 2\mu.$$ 

Thus, given $\omega$ we set $\sigma = \frac{1}{2} \iota_e \omega$ and $\tau = \omega - \frac{d\mu}{\mu} \wedge \sigma$. Then $\iota_e \sigma = 0$ (since $\iota_e^2 = 0$) and

$$\iota_e \tau = \iota_e \omega - \frac{1}{2} \iota_e \frac{d\mu}{\mu} \wedge \sigma = 0.$$ 

This proves existence. To check uniqueness, suppose $0 = \frac{d\mu}{\mu} \wedge \sigma + \tau$ with $\sigma, \tau \in \text{Ker}(\iota_e)$. Then applying $\iota_E$ we get $0 = \iota_E \frac{d\mu}{\mu} \wedge \sigma$ and thus $\sigma = 0$, and therefore also $\tau = 0$. \hfill $\square$
Let $C_{\mu, \text{basic}}(X) = \text{Ker}(L_e) \cap \text{Ker}(\iota_e)$. It is the subcomplex of basic differential forms for the action of the group of dilations in the normal direction. Let us denote by $C[-1]$ the $(-1)$-shift of a cochain complex $C$: $C[-1] = C^{\cdot -1}$ with differential $d_{C[-1]} = -d_C$.

**Lemma A.4.** The map

$$C_{\mu, \text{basic}}(X)[-1] \oplus C_{\mu, \text{basic}}(X) \to \text{Ker}(L_e)$$

sending $\sigma \oplus \tau$ to $d\mu/\mu \wedge (\sigma + \tau)$ is an isomorphism of complexes.

**Proof.** If $\omega = d\mu/\mu \wedge (\sigma + \tau) + d\tau$, thus the map is compatible with differentials. By the uniqueness part of Lemma A.3 it is injective. To prove surjectivity, suppose $\omega \in \text{Ker}(L_e)$. Then by Lemma A.3 $\omega = d\mu/\mu \wedge (\sigma + \tau)$ and $\sigma, \tau \in \text{Ker}(\iota_e)$ and applying $L_e$ we see that

$$0 = \frac{d\mu}{\mu} \wedge L_e\sigma + L_e\tau,$$

and, since $\iota_e$ commutes with $L_e$, also $L_e\sigma, L_e\tau \in \text{Ker}(\iota_e)$. Again by the uniqueness part of Lemma A.3 it follows that $L_e\sigma, L_e\tau$ both vanish. \[\Box\]

The rotation group $\text{SO}(m)$ acts on $C_{\mu, \text{basic}}(X)$ and by averaging we can replace this complex by the quasi-isomorphic subcomplex of invariants. Combining Lemma A.4 with Lemma A.3 we get:

**Corollary A.5.** The map

$$C_{\mu, \text{basic}}(X)^{\text{SO}(m)}[-1] \oplus C_{\mu, \text{basic}}(X)^{\text{SO}(m)} \to C_{\mu}(X)$$

sending $\sigma \oplus \tau$ to $d\mu/\mu \wedge (\sigma + \tau)$ is a quasi-isomorphism.

**Lemma A.6.** Let $\beta$ be the closed differential form defined in Prop. A.1. As a module over $\mathcal{A}(Y)$,

$$C_{\mu, \text{basic}}(X)^{\text{SO}(m)} = \begin{cases} \mathcal{A}(Y)^1, & \text{if } m \text{ is odd}, \\ \mathcal{A}(Y)^1 \oplus \mathcal{A}(Y)\beta, & \text{if } m \text{ is even}. \end{cases}$$

**Proof.** The complex $C_{\mu, \text{basic}}(X)$ consists of basic homogeneous differential forms in $C_\mu(X)$. They are thus homogeneous rational differential forms in the normal variables $x_1, \ldots, x_m$ with coefficients in $\mathcal{A}(Y)$ and with powers of $\mu$ as denominators. They can be viewed as differential forms on $(\mathbb{R}^m \setminus \{0\}) \times Y$ that are basic for the action of the group $\mathbb{R}_{>0}$ of dilations. They thus define differential forms on the quotient $S^{m-1} \times Y$. The only $\text{SO}(m)$-invariant differential forms on the sphere are the constants and the multiples of a volume form.

If $m$ is even, the form $\beta$ restricts to a volume form on $S^{m-1}$ and belongs to $C_{\mu}(X)$. If $m$ is odd there is still a unique rotation invariant volume form on $S^{m-1}$ up to normalization, its extension to a basic invariant form is given by the same formula as $\beta$, which is however not in $C_{\mu}(X)$ due to the presence of the square root of $\mu$. \[\Box\]

**Lemma A.7.** $H(\mathcal{B}(X)) = 0$.

**Proof.** We imitate a standard proof of the Poincaré lemma. The dilation flow $\varphi_t(x) = tx$ for $t \in [0,1]$ maps balls centered at the origin to themselves. Let $h: \mathcal{B}(X) \to \mathcal{B}^{*-1}(X)$ be the linear map

$$h \omega = \int_0^1 \varphi_t^* \omega \frac{dt}{t},$$
(it is well-defined since \( \varphi_t^* \alpha \) vanishes for \( t = 0 \) if \( \alpha \) vanishes at the origin). It is clear that \( h \) maps forms vanishing to infinite order to forms vanishing to infinite order. Moreover, as in the proof of the Poincaré lemma, we see that \( d \circ h + h \circ d = id - \varphi_0^* \).

But on forms vanishing at 0, \( \varphi_0^* \) vanishes. Thus the identity is homotopic to the zero map and the cohomology vanishes in all degrees. \( \square \)

The long exact sequence associated with

\[
0 \to \mathcal{B}(X) \to \mathcal{A}_\mu(X) \to \mathcal{A}_\mu(X)/\mathcal{B}(X) \to 0
\]

implies:

**Lemma A.8.** \( \mathcal{H}(\mathcal{A}_\mu(X)) \cong \mathcal{H}(\mathcal{C}_\mu(X)) \).

**A.2. Filtration and the tame subcomplex.** Let \( X \) be as above. The complex \( \mathcal{A}_\mu(X) \) has a filtration

\[
\cdots \subset F_p \mathcal{A}_\mu(X) \subset F_{p+1} \mathcal{A}_\mu(X) \subset \cdots \subset \mathcal{A}_\mu(X) = \bigcup_{p \in \mathbb{Z}} F_p \mathcal{A}_\mu(X)
\]

by subspaces

\[
F_p \mathcal{A}_\mu(X) = \{\omega \in \mathcal{A}_\mu(X) : \mu^p \omega, \mu^{p-1} d\mu \land \omega \in \mathcal{A}(X)\}.
\]

Since \( \mathcal{B}(X) \subset \bigcap_{p \in \mathbb{Z}} F_p \mathcal{A}_\mu(X) \), the filtration induces a filtration \( F_p \mathcal{C}_\mu(X) \) of \( \mathcal{C}_\mu(X) = \mathcal{A}_\mu(X)/\mathcal{B}(X) \).

**Lemma A.9.** Each \( F_p \mathcal{A}_\mu(X) \) is a subcomplex preserved by \( \iota_\varepsilon \) and invariant under \( \text{SO}(m) \). The same holds for the quotient complexes \( F_p \mathcal{C}_\mu(X) \).

**Proof.** The fact that \( F_p \mathcal{A}_\mu(X) \) is a subcomplex is a special case of Lemma [A.4]. As for the Euler vector field, we have

\[
\mu^p \iota_\varepsilon \omega = \iota_\varepsilon (\mu^p \omega),
\]

\[
\mu^{p-1} d\mu \land \iota_\varepsilon \omega = -\iota_\varepsilon (\mu^{p-1} d\mu \land \omega) + 2 \mu^p \omega.
\]

The right-hand sides are regular. Thus \( F_p \mathcal{A}_\mu(X) \) is preserved by \( \iota_\varepsilon \). Since \( \mu \) is rotation invariant, the action of \( \text{SO}(m) \) preserves the subcomplexes. Clearly \( \mathcal{B}(X) \) is an \( \text{SO}(m) \)-invariant subcomplex preserved by \( \iota_\varepsilon \), so the same holds for the quotient. \( \square \)

Thus \( F \) induces a filtration on \( \text{Ker}(L_\varepsilon) \) and on \( \mathcal{C}_{\mu, \text{basic}}(X) = \text{Ker}(L_\varepsilon) \cap \text{Ker}(\iota_\varepsilon) \).

**Lemma A.10.** The isomorphism of Lemma [A.4] restricts to an isomorphism

\[
F_p \mathcal{C}_{\mu, \text{basic}}(X)[-1] \oplus F_p \mathcal{C}_{\mu, \text{basic}}(X) \to F_p \text{Ker}(L_\varepsilon)
\]

for all \( p \in \mathbb{Z} \).

**Proof.** It is easy to check that the filtration is preserved. By Lemma [A.4] the map is injective. It remains to prove the surjectivity. Suppose \( \omega \in F_p \text{Ker}(L_\varepsilon) \). Write \( \omega = d\eta/\mu \land \sigma + \tau \) with \( \sigma, \tau \in \mathcal{C}_{\mu, \text{basic}}(X) \). Since \( \mu^{p-1} d\mu \land \omega \) is regular, we deduce that \( \mu^{p-1} d\mu \land \tau \) is regular on \( X \). Applying \( \iota_\varepsilon \) and using that \( \iota_\varepsilon d\mu/\mu = 2 \) we see that \( \mu^p \tau \) is also regular. Thus \( \tau \in F_p \mathcal{C}_{\mu, \text{basic}}(X) \). It follows that \( \mu^{p-1} d\mu \land \sigma = \mu^p (\omega - \tau) \) is regular. Again applying \( \iota_\varepsilon \) we see that \( \mu^p \sigma \) is regular, so that also \( \sigma \) belongs to \( F_p \mathcal{C}_{\mu, \text{basic}}(X) \). \( \square \)

**Corollary A.11.** Let the codimension \( m \) of \( Y \) in \( X \) be even. Then the inclusion

\[
F_{\varphi_{\mu}} \mathcal{A}_\mu(X) \hookrightarrow \mathcal{A}_\mu(X)
\]

is a quasi-isomorphism.

**Proof.** The differential forms \( d\mu/\mu, \beta, d\mu/\mu \land \beta \) are all in \( F_{\varphi_{\mu}} \mathcal{A}_\mu(X) \), which is preserved by multiplication by (pull-backs of forms in) \( \mathcal{A}(Y) \). Thus in Lemma [A.6] and Corollary [A.5] we can replace the complexes by their \( F_{\varphi_{\mu}} \) subcomplexes. \( \square \)
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