Relationship between Polytropic Index and Temperature Anisotropy in Space Plasmas

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Abstract

The paper develops a theoretical relationship between the polytropic index and the temperature anisotropy that may characterize space plasmas. The derivation is based on the correlation among the kinetic energies of particles with velocities described by anisotropic kappa distributions. The correlation coefficient depends on the effective dimensionality of the velocity distribution, which is determined by the temperature anisotropy caused by the ambient magnetic field; on the other hand, the effective dimensionality is directly dependent on the polytropic index. This analysis leads to the connection between the correlation coefficient, effective dimensionality of the velocity space, and the polytropic index, with the temperature anisotropy. Moreover, a data and statistical analysis is performed to test the developed model in the solar wind proton plasma near 1 au. The derived theoretical relationship is in good agreement with observations, showing that the lowest and classical value of the adiabatic polytropic index occurs in the isotropic case, while higher values of the adiabatic index characterize more anisotropic plasmas. Finally, possible extensions of the theory considering (i) nonadiabatic polytropic behavior and (ii) more general distributions, are further discussed.

Unified Astronomy Thesaurus concepts: Astronomy data analysis (1858); Space plasmas (1544); Plasma astrophysics (1261); Solar wind (1534); Slow solar wind (1873); Heliosphere (711)

1. Introduction

Polytropic behavior is a thermodynamic property of the particle system—the plasma. The value of the polytropic index indicates a certain thermodynamic process of the plasma, i.e., the transition of the plasma from one thermodynamic state to another; the polytropic index determines the heat flux during this transition (e.g., Parker 1963; Chandrasekhar 1967; Verma et al. 1995; Sorriso-Valvo et al. 2007; Vasquez et al. 2007; Marino et al. 2008; Livadiotis 2019a).

The thermal pressure $p$, density $n$, and temperature $T$ of particle systems, driven by a dynamical potential energy with a positional dependence, are characterized by a polytropic relationship, such as

$$p(r) \propto n(r)^{\gamma}, \quad \text{or} \quad n(r) \propto T(r)^{\nu}, \tag{1a}$$

where the polytropic index, determined by the exponent $\gamma$, or sometimes, by the exponent $\nu$, is

$$\gamma = 1 + 1/\nu \quad \text{or} \quad \nu = 1/(\gamma - 1). \tag{1b}$$

Many space plasmas exhibit a positive correlation between their density and temperature (i.e., $\gamma > 1$), with their most frequent polytropic index close to the value of the adiabatic process, i.e., $\gamma = 5/3$. Some examples are solar wind proton and electron plasma (e.g., Totten et al. 1995; Newbury et al. 1997; Nicolau et al. 2014a; Livadiotis & Desai 2016; Livadiotis 2018a), solar flares (e.g., Garcia 2001; Wang et al. 2015, 2016), and planetary bow shocks (Tatraway et al. 1984; Winterhalter et al. 1984).

In several cases, space plasmas are characterized by polytropic indices with sub-adiabatic ($1 < \gamma < 5/3$) or superadiabatic values ($5/3 < \gamma < +\infty$); for example, in coronal mass ejections (CMEs) (e.g., Liu et al. 2006; Mishra & Wang 2016), coronal plasma (e.g., Prasad et al. 2018), Earth’s plasma sheet (e.g., Zhu 1990; Goertz & Baumjohann 1991; Borovsky et al. 1998), planetary magnetospheres (e.g., Dialynas et al. 2018), and even in galaxy clusters and superclusters (e.g., Markevitch et al. 1998; Ettori et al. 2000; Grandi & Molendi 2002; Bautz et al. 2009).

However, there are rarer cases where space plasmas have negative correlations between their density and temperature (i.e., $\gamma < 1$); these were found in the outer heliosphere (Elliott et al. 2019; Livadiotis 2021), inner heliosheath (e.g., Livadiotis et al. 2011; Livadiotis & McComas 2013), and the planetary magnetospheres, namely, the terrestrial low latitude boundary layer (e.g., Sckopke et al. 1981), central plasma sheet (e.g., Pang et al. 2015), and bow shock (e.g., Pang et al. 2020); Jovian ionosphere (Allegrini et al. 2021), magnetosheath and boundary layer (e.g., Nicolau et al. 2014b, 2015); and Saturnian magnetosphere (e.g., Dialynas et al. 2018).

Adiabatic processes characterize plasma flows with nearly zero heat transition; such an example is the solar wind plasma particles that flow throughout the supersonic heliosphere under expansive cooling. However, turbulent heating disturbs this hypothetically ideal adiabatic process. The turbulence in solar wind has two sources: (i) the solar-origin large-scale energy fluctuations and (ii) the excitation of plasma waves by newborn interstellar pickup ions. Other secondary sources may exist, e.g., turbulence can also be generated in situ in the solar wind by velocity shears (e.g., Roberts et al. 1992; Zank et al. 2017). The turbulence affects the polytropic behavior of the solar wind plasma.

On the other hand, nonadiabatic polytropic indices can be connected with space plasmas residing in thermodynamic stationary states, namely, their particle distribution of velocity does not vary significantly with time. Particles of these plasmas are typically described by kappa distributions (Livadiotis & McComas 2010, 2013; Livadiotis 2017, 2018b); in particular, it has been theoretically shown that there is an equivalence between the polytropic behavior and the formalism of kappa distributions (e.g., Livadiotis 2019b).
Both the general cases of (i) adiabatic plasma flows, and (ii) non-adiabatic plasmas consistent with the formalism of kappa distributions, can be connected with temperature anisotropy caused by the presence of interplanetary/magnetospheric magnetic field. In general, space plasmas are described by anisotropic velocity distributions. Namely, the perpendicular and parallel directions, with respect to the magnetic field, have different thermal properties, while the respective temperature-like parameters are noted by $T_\perp$ and $T_\parallel$, and the anisotropy is defined by the ratio $\alpha = T_\perp / T_\parallel$.

As of now there is no known theoretical connection between the polytropic index and the temperature anisotropy. Surely, theoretical models exist separately for the description of polytropic behavior (e.g., Livadiotis 2019b) and temperature anisotropy (e.g., Ao et al. 2003) in plasmas, but without any connection between them. For instance, histograms of the measurements of the polytropic indices and anisotropies for the solar wind proton plasma near 1 au, computed using data sets taken from Wind S/C (e.g., Livadiotis & Desai 2016; Livadiotis 2018a; Livadiotis et al. 2018; Nicolaou & Livadiotis 2019; Nicolaou et al. 2019), are shown in Figure 1. The polytropic indices are distributed with a mode near the adiabatic index, $\gamma \sim 1.67$, while the anisotropies are distributed with a mode near $\alpha \sim 0.8$. Then, we may ask: is there a connection between the adiabatic polytropic indices included in this histogram and the values of temperature anisotropies? Is there a theoretical basis of such a relationship?

The purpose of this paper is to (i) develop a theoretical model that connects the adiabatic polytropic index and the temperature anisotropy; a similar connection will be developed for the non-adiabatic polytropic indices; and (ii) perform a data and statistical analysis in order to examine the validity of the theoretically developed model in the case of the solar wind proton plasma near 1 au; this will be performed using data sets taken from Wind S/C.

Section 2 presents the formalism of anisotropic kappa distributions and particle correlation, i.e., the basic models of anisotropic kappa distributions of particles’ velocities or kinetic energies, and the derivation of the correlation coefficient among particle’s kinetic energies; the latter will lead to the theoretically developed relationship between the adiabatic polytropic index and the anisotropy. Section 3 presents the application of this developed relationship to solar wind proton plasma near 1 au. We use plasma bulk parameters and interplanetary magnetic field data sets taken from Wind S/C and perform a data and statistical analysis to compare the observations with the developed relationship. In Section 4, we discuss the relationship of non-adiabatic polytropic indices with the anisotropy, which is the case for plasma flows under a dynamical potential; we also discuss related theoretical developments for future analyses, e.g., finding similar results using other models of anisotropic kappa distributions that may be less frequently or rarely used in space plasmas. Finally, conclusions summarize the results.

2. Formalism of Anisotropic Kappa Distributions and Particle Correlation

2.1. Basic Model of Anisotropic Kappa Distributions

The typical 3D anisotropic kappa distribution is given by

$$P(u_\parallel, u_\perp; \theta_\parallel, \theta_\perp, \kappa) = \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{-\frac{3}{2}} \cdot \frac{\Gamma(3\kappa + 1)}{\Gamma(\kappa + \frac{1}{2})} \cdot \theta_\parallel^{-1} \theta_\perp^{-2} \times \left[ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \left( \frac{u_\parallel^2}{\theta_\parallel^2} + \frac{u_\perp^2}{\theta_\perp^2} \right) \right]^{-\kappa - 1},$$

with normalization

$$\int_0^\infty \int_{-\infty}^{+\infty} P(u_\parallel, u_\perp; \theta_\parallel, \theta_\perp, \kappa) du_\parallel 2\pi u_\perp du_\perp = 1. \tag{2b}$$

For applications of this model, see, e.g., Summers & Thorne 1991, Summers et al. 1994, Švěrák et al. 2008, Astfalk et al. 2015, Khokhar et al. 2017, Wilson et al. 2019; Liu & Chen 2019; Khan et al. 2020). The velocity components are defined in correspondence to the direction of the magnetic field, i.e., $u_\parallel$ and $u_\perp$ are set to be parallel and perpendicular to the field, respectively. The bulk velocities are taken into zero; namely, we consider the co-moving reference frame, for simplicity. The thermal speed of a particle of mass $m$, i.e., $\theta = \sqrt{2k_B T / m}$, expresses the temperature $T$ in speed dimensions; $\theta^2 / 2$ provides the second statistical moment of
the velocities (in the co-moving reference frame). Note that the expression of distribution 2(a) has been generalized within the framework of nonextensive statistical mechanics by Livadiotis et al. (2021) in another empirical functional form, which includes the pitch angle and an anisotropy factor relative to the magnetic field has been introduced by Mauk & Fox (2010). This functional form was based on the form of kappa distribution function and on earlier observations from Baker & Van Allen (1976).

In order to calculate the correlation coefficient, we need the formulation of multidimensional kappa distributions, i.e., for describing a distribution of N particles, where each particle is d-dimensional, (Livadiotis & McComas 2011; Livadiotis 2015a).

Let the velocity vector of the i-th particle be \( \mathbf{u}_i^t = u_{i1}^t + u_{i2}^t + u_{i3}^t \). In a d-dimensional velocity space, this is \( u_i^t = u_{i1}^t + u_{i2}^t + \cdots + u_{id}^t \); therefore, the general case of an N-particle N·d-dimensional kappa distribution is

\[
P(\{\mathbf{u}_i\}_{i=1}^N; \theta, \kappa_0) = \left( \pi \kappa_0 \right)^{-N/2} \cdot \frac{\Gamma(\kappa_0 + 1 + \frac{N}{2}d)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-Nd} \times \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{u_i^t + u_j^t + \cdots + u_N^t}{\theta^2} \right]^{-\kappa_0 - 1 - \frac{N}{2}d},
\]

or in the specific case of N 3d-particles,

\[
P(\{\mathbf{u}_i\}_{i=1}^N; \theta, \kappa_0) = \left( \pi \kappa_0 \right)^{-N/2} \cdot \frac{\Gamma(\kappa_0 + 1 + \frac{3}{2}N)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-3Nd} \times \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{u_i^t + u_j^t + \cdots + u_N^t}{\theta^2} \right]^{-\kappa_0 - 1 - \frac{3}{2}Nd},
\]

where we denote \( \{\mathbf{u}_i\}_{i=1}^N = \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_N \). The involved kappa index \( \kappa_0 \) is independent of the dimensionality or the number of correlated particles involved in the kappa distribution. The standard kappa index \( \kappa \) depends on the dimensionality, i.e., \( \kappa(d) = \kappa_0 + \frac{1}{2d} \). The involved constant is noted by \( \kappa_0 \), and indicates an invariant expression of the kappa index; thus, the kappa index that remains invariant under variations of the dimensionality or the number of the correlated degrees of freedom. We typically use the invariant kappa index in order to express the kappa distributions of higher dimensionality.

The anisotropic kappa distribution shown in Equation 2(a) is a one-particle distribution that refers to systems with homogeneous correlation among the particle velocity components. The respective N-particle anisotropic kappa distribution is

\[
P(\{\mathbf{u}_i\}_{i=1}^N; \theta_{ij}, \theta_{\perp}, \kappa_0; N) = \left( \pi \kappa_0 \right)^{-N/2} \cdot \frac{\Gamma(\kappa_0 + 1 + \frac{N}{2}d)}{\Gamma(\kappa_0 + 1)} \cdot \theta_{\perp}^{-2Nd} \times \left[ 1 + \frac{1}{\kappa_0} \cdot \left( \frac{N}{\theta_{\perp}} \sum_{i=1}^N u_i^t \right) + \frac{N}{\theta_{ij}} \sum_{i=1}^N u_i^t \right]^{-\kappa_0 - 1 - \frac{N}{2}d},
\]

with normalization

\[
\int_{-\infty}^{\infty} \cdots (3N \text{ integrals}) \cdots \int_{-\infty}^{\infty} P(\{\mathbf{u}_i\}_{i=1}^N; \theta_{ij}, \theta_{\perp}, \kappa_0; N) d\mathbf{u}_1 \cdots d\mathbf{u}_N = 1,
\]

where \( d\mathbf{u}_i = du_{i1}d\mathbf{u}_i \cdots d\mathbf{u}_{id} \).

For the calculation of the covariance between two-particle energies, we use the two-particle 3D distributions (e.g., Swaczyna et al. 2019), i.e., the isotropic distribution

\[
P(\mathbf{u}_1, \mathbf{u}_2; \theta, \kappa_0) = \left( \pi \kappa_0 \right)^{-3} \cdot \frac{\Gamma(\kappa_0 + 4)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-6} \times \left( 1 + \frac{1}{\kappa_0} \cdot \frac{u_1^t + u_2^t}{\theta^2} \right)^{-\kappa_0 - 4},
\]

while the respective anisotropic distribution is expressed as

\[
P(\mathbf{u}_1, \mathbf{u}_2; \theta_{ij}, \theta_{\perp}, \kappa_0) = \left( \pi \kappa_0 \right)^{-3} \cdot \frac{\Gamma(\kappa_0 + 4)}{\Gamma(\kappa_0 + 1)} \cdot \theta_{ij}^{-2} \theta_{\perp}^{-4} \times \left[ 1 + \frac{1}{\kappa_0} \cdot \left( \frac{u_1^t + u_2^t}{\theta_{ij}} + \frac{u_1^t + u_2^t}{\theta_{\perp}} \right) \right]^{-\kappa_0 - 4}.
\]

In terms of the standard 3D kappa index \( \kappa \), the two-particle isotropic and anisotropic 3D distributions are respectively given by

\[
P(\mathbf{u}_1, \mathbf{u}_2; \theta, \kappa) = \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{-3} \cdot \frac{\Gamma(\kappa + \frac{5}{2})}{\Gamma(\kappa - \frac{1}{2})} \cdot \theta^{-6} \times \left( 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{u_1^t + u_2^t}{\theta^2} \right)^{-\kappa - \frac{4}{2}},
\]

and

\[
P(\mathbf{u}_1, \mathbf{u}_2; \theta_{ij}, \theta_{\perp}, \kappa) = \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{-3} \cdot \frac{\Gamma(\kappa + \frac{5}{2})}{\Gamma(\kappa - \frac{1}{2})} \cdot \theta_{ij}^{-2} \theta_{\perp}^{-4} \times \left[ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \left( \frac{u_1^t + u_2^t}{\theta_{ij}} + \frac{u_1^t + u_2^t}{\theta_{\perp}} \right) \right]^{-\kappa - \frac{4}{2}}.
\]

### 2.2. Correlation Coefficient

Here we present the derivation of the correlation coefficient that characterizes the particle energies in a population described by anisotropic kappa distributions. The Pearson’s correlation coefficient (Abe 1999; Livadiotis & McComas 2011; Livadiotis 2015a, 2017, Ch.5.4; Livadiotis et al. 2021, in press) is given by

\[
\rho = \frac{\sigma_{\varepsilon_1 \varepsilon_2}}{\sigma_{\varepsilon_1} \sigma_{\varepsilon_2}},
\]

where

\[
\sigma_{\varepsilon_1 \varepsilon_2}^2 = \langle \varepsilon_1 \varepsilon_2 \rangle - \langle \varepsilon_1 \rangle \langle \varepsilon_2 \rangle = \frac{1}{2} \text{m} (\langle u_1^2 u_2^2 \rangle - \langle u_1^2 \rangle \langle u_2^2 \rangle) = \frac{1}{2} \text{m} \sigma_{u_1^2 u_2^2}^2,
\]

and

\[
\sigma_{\varepsilon_1}^2 = \langle \varepsilon_1^2 \rangle - \langle \varepsilon_1 \rangle^2 = \frac{1}{2} \text{m} (\langle u_1^4 \rangle - \langle u_1^2 \rangle^2) = \frac{1}{2} \text{m} \sigma_{u_1^4}^2.
\]

Using the isotropic kappa distributions (Livadiotis & McComas 2011), the correlation coefficient was found:

\[
\rho = \frac{\frac{3}{\kappa_0 + \frac{1}{2}}}{\frac{3}{\kappa_0 + \frac{1}{2}}} \text{ or } \rho = \frac{\frac{3}{\kappa_0 + \frac{1}{2}}}{\frac{3}{\kappa_0 + \frac{1}{2}}},
\]

or, in the case of 3D distributions,

\[
\rho = \frac{3}{\kappa_0 + \frac{1}{2}} \text{ or } \rho = \frac{3}{\kappa_0 + \frac{1}{2}}.
\]

It is important to note that while the mean particle kinetic energy provides the kinetic definitions of thermal energy, \( k_B T \),
the (normalized) covariance (i.e., the correlation coefficient) of the particle kinetic energy provides the kinetic definition of the (inverse) kappa index, $1/\kappa$.

In the case of the anisotropic kappa distributions, we find the variance

$$
\sigma_{u^2}^2 = \langle u^4 \rangle - \langle u^2 \rangle^2 = \left( \frac{1}{4} \theta_{||}^4 + \theta_{\perp}^4 \right) + \left( \frac{3}{4} \theta_{||}^4 + 2 \theta_{||}^2 \theta_{\perp}^2 \right) \frac{1}{\kappa - 1},
$$

and covariance

$$
\sigma_{u^2u^2} = \langle u_1^2 u_2^2 \rangle - \langle u^2 \rangle^2 = \left( \frac{1}{4} \theta_{||}^4 + \theta_{||}^2 \theta_{\perp}^2 \right) \cdot \frac{1}{\kappa - 1}.
$$

Substituting Equations (9)(a), (b) in Equation (7)(a), we obtain

$$
\rho = \frac{\sigma_{u^2u^2}}{\sigma_{u^2u^2}} = \frac{\frac{1}{\kappa - 1} \frac{1}{\kappa - 1} + \frac{1}{\kappa - 1} \frac{1}{\kappa - 1}}{\frac{1}{\kappa - 1} + \frac{1}{\kappa - 1}}.
$$

Introducing the temperature anisotropy by

$$\alpha \equiv \frac{T_1}{T_0}$$

and given that the temperature can be written as $T = \frac{3}{4}(T_1 + 2T_0)$, then, the temperature-like components are expressed in terms of the actual temperature and anisotropy as

$$T_0 = \frac{3}{1 + 2\alpha} \cdot T, \quad T_1 = \frac{3\alpha}{1 + 2\alpha} \cdot T.$$

In terms of thermal speeds, Equation (12)(a) becomes

$$\theta_{||}^2 = \frac{3}{1 + 2\alpha} \cdot \theta^2, \quad \theta_{\perp}^2 = \frac{3\alpha}{1 + 2\alpha} \cdot \theta^2.$$ 

Then, the correlation coefficient in Equation (10) is written as

$$\rho = \frac{\frac{1}{\kappa - 1} \frac{1}{\kappa - 1} + \frac{1}{\kappa - 1} \frac{1}{\kappa - 1}}{\frac{1}{\kappa - 1} + \frac{1}{\kappa - 1}}.$$

Introducing the effective dimensionality, i.e.,

$$\rho = \frac{\frac{1}{\kappa - 1} \frac{1}{\kappa - 1} + \frac{1}{\kappa - 1} \frac{1}{\kappa - 1}}{\frac{1}{\kappa - 1} + \frac{1}{\kappa - 1}}.$$

The effective dimensionality recovers the dimensionality of the embedded 3D velocity space in the isotropic case $\alpha = 1$. In general, the impact of anisotropy on dimensionality is nontrivial: The dimensionality of the embedded space is $d = 3$—for any anisotropy; however, the limiting cases where the parallel or perpendicular directions are neglected should characterize a degeneration of the velocity distributions so that its dimensionality to be reduced to 2D or 1D, respectively. Then, the effective dimensionality defined with respect to anisotropy, $d_{\text{eff}}$, is expected to satisfy

$$\alpha = 0 \leftrightarrow d_{\text{eff}} = 1, \quad \alpha = 1 \leftrightarrow d_{\text{eff}} = 3, \quad \alpha = \infty \leftrightarrow d_{\text{eff}} = 2.$$ 

The relationship between the effective dimensionality and anisotropy is derived from comparing the correlation coefficient in Equations (13) and (14), i.e.,

$$d_{\text{eff}}(\alpha) = \frac{(2\alpha + 1)^2}{2\alpha^2 + 1},$$

where we observe that it satisfies the extreme conditions of Equation (15).

Next, we connect the expression in Equation (16) with the polytropic index $\gamma$ (blue) as a function of the anisotropy $\alpha$, emphasizing the cases of $\alpha = 0, 1, \infty$, corresponding to $d_{\text{eff}} = 1, 3, 2$, and $\gamma = 3, 5/3, 2$, where the velocity distribution takes the form of a cigar, sphere, pie. (Taken from Livadiotis et al. 2021).

Figure 2 plots the effective dimensionality $d_{\text{eff}}$ (red) and the respective adiabatic polytropic index $\gamma$ (blue) as a function of the anisotropy $\alpha$, emphasizing the cases of $\alpha = 0, 1, \infty$, corresponding to $d_{\text{eff}} = 1, 3, 2$, and $\gamma = 3, 5/3, 2$, where the velocity distribution takes the form of a cigar, sphere, pie.
3. Application to Solar Wind Proton Plasma Near 1 au

3.1. Data

We show the dependence of the adiabatic polytropic index on the anisotropy of the velocity distribution for the solar wind proton plasma near 1 au. We use publicly available data of solar wind proton bulk parameters and interplanetary magnetic field, taken from the Wind mission.

In particular, we use ∼92 s solar wind plasma moments (speed $V_{sw}$, anisotropy $\alpha$, density $n$, and temperature $T$) (Ogilvie et al. 1995) and simultaneous measurements of the interplanetary magnetic field (Lepping et al. 1995), measured, respectively, from Solar Wind Experiment (SWE) and Magnetic Field Investigation (MFI) instruments on board Wind S/C, during the first 73 days of 1995 (see Figure 3). This time period occurred during the declining phase of solar activity cycle 23, and was characterized by corotating interaction regions that are apparent in increases in the solar wind density and magnetic field magnitude that precede the arrival of the high speed streams at 1 au (e.g., Jian et al. 2006a, 2006b, 2009).

3.2. Method

We derive the polytropic index using the proton plasma moments and interplanetary magnetic field time series shown in Figure 3, according to the following steps:

1. Selection of time intervals: Analyze the time intervals for which we fit $T$ and $n$. The intervals must be short enough to minimize the possibility of mixing measurements of different streamlines, but large enough to improve statistics. Following previous studies (e.g., Newbury et al. 1997; Kartalev et al. 2006; Nicolaou et al. 2014a), we select intervals covering eight consecutive measurements of $T$ and $n$.

2. Filtering Bernoulli’s integral: For each interval, we examine the variability of Bernoulli’s energy integral (Kartalev et al. 2006).
The standard deviation of the eight values of the Bernoulli's integral must be <10% of their mean. The threshold was chosen to be low enough to minimize the possibility of having different streamlines in an interval, but also high enough to retain a statistically significant number of data points.

3. Fitting: For each interval, we fit with a linear statistical model the logarithms of temperature and density (for the log-normal distributions of plasma moments see Burlaga & Lazarus 2000; Kasper et al. 2006), i.e., log \( T \) versus log \( n \), where the slope equals \( \gamma - 1 \). (See the example in Figure 4.)

3.3. Statistical Analysis

Having the polytropic indices and their uncertainties derived from the linear fits for all the time intervals of the examined data sets (Figure 3), we analyze the constructed data set of polytropic indices against the anisotropies. The 2D histogram of the polytropic indices against the anisotropies is normalized by dividing it with the 1D histogram of anisotropies (Figure 5); in particular, the number of data points at each \( \Delta\alpha \times \Delta\gamma \) 2D bin is divided by the number of data points in the whole column of the 1D \( \Delta\alpha \) bin. (This type of standardized 2D histogram is called a conditional 2D histogram, corresponding to the ratio of a joint 2D probability \( P(x, y) \) over \( P(x) \); for more details, see Livadiotis & Desai 2016; Park 2018, p.99.)

The constructed normalized \( \alpha \times \gamma \) 2D histogram is plotted in the background of Figure 5. The co-plotted black-dashed curve shows the modeled relationship between adiabatic polytropic indices and anisotropies as described by Equation (18).

Next, we average all the polytropic indices corresponding in each column of the 1D \( \Delta\alpha \) bin. The averaging is performed in two different ways. First, we estimate the weighted means and errors, each corresponding to each \( \Delta\alpha \) bin, leading to the data points plotted in Figure 5(a). In another way, we use the normalized 2D histogram that provides the probability of having a polytropic index at each \( \Delta\alpha \times \Delta\gamma \) 2D bin; in particular, we estimate the average by summing the occurrence given by the normalized 2D histogram multiplied by the value of the polytropic index at the center of each bin; we count only the bins with a number of data points \( N > 10 \), leading to the averaged data points plotted in Figure 5(b). Note that in both ways of averaging, we include only indices in the interval \( 0 \leq \gamma \leq 3 \), corresponding to a reasonable set of values of dimensionality, i.e., \( 1 \leq d_{eff} \leq \infty \); (Figure 5 demonstrates only the subintervals \( 1 \leq d_{eff} \leq 3 \) and \( 1 \leq \gamma \leq 3 \)). The necessary formulae for the two ways are:

\[ \frac{1}{N} \sum_{i=1}^{N} (\gamma_i - \bar{\gamma}) \]
combined data set \( \{ \gamma_i^{(C)} \pm \delta \gamma_i^{(C)} \}_{i=1}^{N} \) plotted in Figure 6. Moreover, Figure 5(d) plots a double-combined data set, noted by \( \{ \gamma_i^{(CC)} \pm \delta \gamma_i^{(CC)} \}_{i=1}^{N} \), derived from both the weighted averages of \( \{ \gamma_i^{(C)} \pm \delta \gamma_i^{(C)} \}_{i=1}^{N} \) and \( \{ \gamma_i^{(C)} \pm \delta \gamma_i^{(C)} \}_{i=1}^{N} \).

Next, we perform a specific statistical analysis that examines the fit stability and homogenization, which is appropriate for determining whether the fit of the modeled adiabatic polytropic indices to their averaged values suffers from any bias of false statistical weighting caused by confounding variables or other systematic errors. In particular, for each of the given set of \( N_0 \) data points of \( \{ \alpha_i \pm \delta \alpha_i, \gamma_i \pm \delta \gamma_i \}_{i=1}^{N_0} \), we (a) reproduce \( N=1000 \) bi-normally distributed data points, \( \sim\mathcal{N}(\mu_\alpha=\alpha, \sigma_\alpha=\delta \alpha_i) \times \mathcal{N}(\mu_\gamma=\gamma, \sigma_\gamma=\delta \gamma_i) \); (b) homogenize the enriched data set of \( N \times N_0 \) data points by averaging it over the original 1D \( \Delta \alpha \) binning, leading to the new data set \( \{ \gamma_i^{(E)} \pm \delta \gamma_i^{(E)} \}_{i=1}^{N} \); (c) examine the fitting of the derived averaged indices \( \{ \gamma_i^{(E)} \pm \delta \gamma_i^{(E)} \}_{i=1}^{N} \), with the modeled relationship between the adiabatic polytropic indices and the anisotropies; this should lead to the same statistical results as the original data set \( \{ \alpha_i \pm \delta \alpha_i, \gamma_i \pm \delta \gamma_i \}_{i=1}^{N_0} \) otherwise, the presence of potentially confounding variables would have led to a statistically significant difference.

We have performed an overall averaging between the two combined data sets, namely, those combined by (i) \( \{ \gamma_i^{(C)} \pm \delta \gamma_i^{(C)} \}_{i=1}^{N_0} \), i.e., the direct averaging as plotted in Figure 5(c), and (ii) \( \{ \gamma_i^{(CC)} \pm \delta \gamma_i^{(CC)} \}_{i=1}^{N_0} \), i.e., the homogenization of the enriched data set as plotted in Figure 6; the resulting data set, \( \{ \gamma_i^{(CC)} \pm \delta \gamma_i^{(CC)} \}_{i=1}^{N_0} \), is shown in Figure 5(d). (It is noted that the method of homogenization of data is required when deriving polytropic indices; e.g., misleadings measurements of polytropic indices due to inhomogeneities in the solar wind originating at coronal source regions have been detected and mitigated by Newbury et al. 1997.)

3.4. Results

Figure 7 shows a 2D histogram of the enriched \( N \times N_0 \) data set of \( \gamma \) versus \( \alpha \) values, the 1D histograms of \( \alpha \) and \( \gamma \), and the homogenized set of polytropic indices \( \{ \gamma_i^{(CC)} \pm \delta \gamma_i^{(CC)} \}_{i=1}^{N_0} \), together with the modeled relationship in Equation (18). The \( p \)-value of the extremes is a measure of the goodness of fits. The fitting of the model to the optimized data set \( \{ \gamma_i^{(CC)} \pm \delta \gamma_i^{(CC)} \}_{i=1}^{N_0} \) is characterized by \( p \sim 0.082 \) (0.05), corresponding to a statistically confident agreement of the model with the observed data set. All the results of the statistical analysis are shown in Table 1.

The derived theoretical model \( \gamma(\alpha) \) has zero flexible parameters to be fitted; yet, the high \( p \)-value indicates the goodness of the fit and its high statistical confidence.

3.5. Comments

The previous data and statistical analysis was performed to test the application of the theoretically developed model in the solar wind proton plasma near 1 au. The analysis showed that the derived theoretical relationship is in good agreement with observations. It also showed that the lowest and classical value of the adiabatic polytropic index, \( \gamma = 5/3 \), occurs near the isotropic case, while anisotropic plasmas are characterized by higher polytropic indices.

Nicolaou et al. (2020) examined the large-scale variations of the proton plasma density and temperature within the inner...
heliosphere explored by the Parker Solar Probe, and found polytropic indices higher than the adiabatic value, \( \gamma > 5/3 \), with the most frequent value of \( \gamma \sim 2.7 \). In particular, these authors discussed possible combinations of the effective kinetic degrees of freedom and the energy transfer terms in order to interpret the short timescale fluctuations of the solar wind protons in the inner heliosphere, which seem to follow a polytropic law with \( \gamma > 5/3 \). Therefore, while the solar wind plasma near Earth is mostly near adiabatic at \( \gamma = 5/3 \) and thus isotropic (e.g., Livadiotis & Desai 2016), in smaller heliospheric distances the solar wind plasma is super-adiabatic (and thus far from being isotropic).

4. Discussion: What’s next?

4.1. Polytropic Index versus Anisotropy for Plasma Flows Under a Dynamical Potential

In the previous sections (Sections 2.2 and 3), we connected the adiabatic polytropic index with temperature anisotropy. Here we extend the theoretical analysis to derive the relationship between the anisotropy and any polytropic index—not just the adiabatic one for space plasmas residing in stationary states; this is achieved by employing the already known connection between kappa and polytropic indices, for particles driven by potential energy (Livadiotis 2019b).

The connection of the polytropic index, adiabatic or non-adiabatic, is achieved through the effective dimensionality. In the case of the adiabatic polytropic index, we used the kinetic effective dimensionality, here denoted by \( d_u \), developed via the concept of correlation coefficient (as shown in Section 2.2 and Equation (17)). In the case of a non-adiabatic polytropic index, the connection is with the positional effective dimensionality, which is denoted by \( d_r \). It was shown that the polytropic behavior of the plasma flow, driven by the dynamics of particle potential energy is one-to-one equivalent with the description of kappa distributions of particle velocities/energies (e.g., Livadiotis 2019b); the connection between polytropic and kappa indices includes the positional dimensionality, \( d_r \). Namely, we set

\[
d_{\text{eff}} \rightarrow \begin{cases} d_u \rightarrow \gamma = 1 + 2 d_u^{-1} \\ d_r \rightarrow \kappa + (\gamma - 1)^{-1} = \frac{1}{2} - \frac{1}{b} d_r, \end{cases}
\]

where the effective dimensionality \( d_{\text{eff}} \) is derived from the correlation coefficient in Equation (16),

\[
d_{\text{eff}} = \frac{(2\alpha + 1)^2}{2\alpha^2 + 1},
\]

and interprets either the kinetic degrees of freedom, \( d_{\text{eff}} = d_u \), leading to the adiabatic polytropic index according to Equations (17), and (18), or the positional degrees of freedom, \( d_{\text{eff}} = d_r \), leading to the nonadiabatic polytropic index according to the relationship between the kappa index and the positional degrees of freedom (Livadiotis 2019b):

\[
\kappa + (\gamma - 1)^{-1} = \frac{1}{2} - \frac{1}{b} d_r.
\]
where \( b \) is the exponent of the position vector in the potential energy. Hence, we find
\[
\kappa = 1 + \frac{1}{2} - \frac{1}{b} \left(\frac{2\alpha + 1}{2}e^{2} + 1\right)^{-\left(\gamma - 1\right)^{-1}},
\] (23)

thus, we have the nonadiabatic polytropic index expressed in terms of the anisotropy
\[
\gamma = \frac{(2\alpha + 1)^{2}}{2\alpha^{2} + 1} + b\left(\kappa - \frac{3}{2}\right),
\] (24a)
or vice versa,
\[
\alpha = \frac{-1 + \sqrt{\frac{\pi}{\kappa} + \left(\gamma - 1\right)^{-1} + \frac{3}{2}\left(\gamma - 1\right)^{-1}}}{\frac{3}{2}\left(\gamma - 1\right)^{-1} + 2}.
\] (24b)

Figure 7 plots the polytropic indices, \( \nu \) and \( \gamma \), as a function of the anisotropy \( \alpha \), for two cases of attractive potentials, i.e., a power law with exponent \( b = -1/2 \) (e.g., interplanetary electric field potential, e.g., Cuperman & Harten 1971; Lacombe et al. 2002; Livadiotis 2018c; Nicolaou & Livadiotis 2019) (upper panels), and \( b = 2 \) (e.g., centrifugal potential Meyer-Vernet et al. 1995; Livadiotis 2015a, 2015c) (lower panels).

4.2. Correlation Coefficients for Anisotropic Distributions with Heterogeneous Correlations

There is a number of different types or models of the anisotropic kappa distributions that have been frequently or rarely used in the past. In this paper, we have dealt with the most frequent one, expressed by Equation 2(a). The correlation coefficient, the effective dimensionality, and the adiabatic polytropic index were derived from considering only this certain type of kappa distribution. However, there are another two associated models that can be examined in a future analysis.

The model in Equation 2(a) characterizes correlated particles, for which the particle velocity components are homogeneously correlated, namely, the correlations between velocity components of each particle, are equal to the correlations between the same components of different particles. On the other hand, another—less frequently used—model considers the case where the heterogeneous correlations are equal to zero; namely, there are no correlations between velocity components of each particle and finite correlations.
between the same components of different particles. These two types of models were generalized by considering arbitrary homogeneous/heterogeneous correlations among the particles’ velocity components. In particular, the generalized model mediates the mentioned two types of anisotropic kappa distributions, where the first considers equal correlations among particles’ velocity components, while the second considers zero correlation among different velocity components. (For more details, see Section 4 in Livadiotis et al. 2021).

Below we present the three associated models of anisotropic kappa distributions. The correlation coefficient between two velocity components is symbolized by the related kappa index; we recall that the correlation coefficient is inversely proportional to the kappa index (Equation (14)). The standard kappa index \( k \) corresponds to the correlation between the same velocity components of different particles, while the kappa index denoted by \( k_{\text{int}} \) corresponds to the correlation between the different velocity components of each particle. Namely, we have:

1. Homogeneous correlations for all velocity components, \( k_{\text{int}} = k \),

\[
P(u; \theta, \kappa) = \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{-\frac{1}{2}} \cdot \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \frac{3}{2})} \cdot \theta^{-2} \cdot \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \left( \frac{u_{\parallel}^2}{\theta_{\parallel}^2} + \frac{u_{\perp}^2}{\theta_{\perp}^2} \right) \right]^{-\kappa - 1},
\]

or, in terms of anisotropy \( \alpha \),

\[
P(u; \alpha, \kappa) = \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{-\frac{1}{2}} \cdot \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \frac{3}{2})} \cdot \alpha^{-1} \left[ \left( 1 + 2\alpha \right) \theta_{\parallel} \right]^3 \theta_{\perp}^{-3} \cdot \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \left( 1 + 2\alpha \cdot \frac{1}{3\alpha \theta_{\parallel}^2 + u_{\parallel}^2} \right) \left( \frac{u_{\parallel}^2}{\theta_{\parallel}^2} + \frac{u_{\perp}^2}{\theta_{\perp}^2} \right) \right]^{-\kappa - 1}.
\]

2. Heterogeneous correlation (between parallel and perpendicular components) equal to zero, \( k_{\text{int}} \rightarrow \infty \),

\[
P(u; \theta, \kappa) = \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{-\frac{1}{2}} \cdot \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \frac{3}{2})} \cdot \theta_{\perp}^{-2} \cdot \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \left( \frac{u_{\parallel}^2}{\theta_{\parallel}^2} + \frac{u_{\perp}^2}{\theta_{\perp}^2} \right) \right]^{-\kappa - 1}.
\]

3. Heterogeneous correlations with arbitrary value, \( k_{\text{int}} < \infty \),

\[
P(u; \theta, \kappa) = C(\kappa_{\text{int}}, \kappa) \cdot \theta_{\perp}^{-2} \theta_{\parallel}^{-1} \cdot \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \left( \frac{u_{\parallel}^2}{\theta_{\parallel}^2} + \frac{u_{\perp}^2}{\theta_{\perp}^2} \right) \right]^{-\kappa - 1}.
\]

where the normalization constant is given by

\[
C(\kappa_{\text{int}}, \kappa) = \int_0^\infty \int_{-\infty}^{\infty} \left[ -1 + \left( 1 + \frac{x^2}{\kappa - \frac{3}{2}} \right) \right]^{-\kappa_{\text{int}} - 1} \frac{dx_{\parallel}}{2\pi \kappa_{\text{int}}} \cdot dx_{\perp}.
\]

Next, we deal with the case of anisotropic kappa distributions with zero heterogeneous correlations (between parallel and perpendicular components), \( k_{\text{int}} \rightarrow \infty \), given by Equation (26). Following steps similar to those in Section 2.2, we find that the correlation coefficient is again written as in Equation (14), where the effective dimensionality is

\[
d_{\text{eff}}(\alpha) = \frac{1 + 4\alpha^2}{1 + 2\alpha^2}.
\]

The effective dimensionality ranges between 1 and d–1 as expected. Namely, when \( \alpha = 0 \), the perpendicular projection is suppressed and the distribution becomes one dimensional, appearing like a cigar; when \( \alpha \rightarrow \infty \), the perpendicular projection is prevailed the parallel and the distribution becomes \( (d–1) \) dimensional, appearing thus, like a \( (d–1) \)-D pie. However, the effective dimensionality of the distribution is never equal to the actual dimensionality of the \( d-D \) velocity space, even in the isotropic case, \( \alpha = 1 \). The reason is the existence of unequal heterogeneous correlations among...
different velocity components; i.e., homogeneous correlation between the perpendicular components, corresponding to $\kappa^\text{int} = \kappa$, and zero correlation between perpendicular and parallel components, $\kappa^\text{int} \to \infty$.

The adiabatic polytropic index, corresponding to the effective dimensionality of unequal heterogeneous correlations, is derived by following the steps in Equations (17) and (18), leading to

$$\gamma(\alpha; d) = 1 + 2 \cdot \frac{1 + (d-1)\alpha^2}{1 + (d-1)\gamma^2},$$  \hspace{1cm} (29b)

where it ranges from $\gamma = 3$ for $\alpha = 0$ to $\gamma = (d+1)/(d-1)$ for $\alpha \to \infty$, while it is $\gamma = (d^2+2)/(1+(d-1)^2)$ for $\alpha = 1$. In the 3D isotropic case ($\alpha = 1$), we find $\gamma = 11/5 = 2.2$, i.e., not a minimum, but slightly above the value of $\gamma = 2$ for $\alpha \to \infty$.

Figure 8 plots the effective dimensionality and the adiabatic polytropic index with respect to the anisotropy for both the distributions given in Equations 25(b) and 26(b).

The data and statistical analysis performed in Section 3, showed that the theoretically developed model is in good agreement with the observations in the solar wind plasma. However, as shown in Figure 6, a possible deviation of the observations from the modeled relationship appears for large anisotropies; indeed, for $\alpha > 2$ the observed adiabatic polytropic index appears larger than its modeled value. This high-anisotropy deviation can be explained by the existence of heterogeneous correlations. The basic model of anisotropic kappa distributions characterizes correlated particles, for which the particle velocity components are homogeneously correlated; namely, the correlations between different velocity components of each particle, are equal to the correlations between the same components of different particles. On the other hand, other, less frequently observed models, consider heterogeneous correlations among particle velocity components. (For more details, see Livadiotis et al. 2021).

Therefore, the results have shown that the adiabatic polytropic index can be larger than the standard value of $\gamma = 5/3$ in the case where the solar wind plasma is characterized by highly anisotropic distributions and/or when heterogeneous correlations exist among particle velocity components.

Nicolaou et al. (2020) examined the large-scale variations of the proton plasma density and temperature within the inner heliosphere explored by the Parker Solar Probe, and found polytropic indices higher than the adiabatic value, $\gamma > 5/3$, with the most frequent value being $\gamma \sim 2.7$. In particular, these authors discussed possible combinations of the effective kinetic degrees of freedom and the energy transfer terms, in order to interpret the short timescale fluctuations of the solar wind protons in the inner heliosphere, which seem to follow a polytropic law with $\gamma > 5/3$. Therefore, while the solar wind plasma near Earth is mostly adiabatic (e.g., Livadiotis & Desai 2016), and thus isotropic (Figure 6), in smaller heliospheric distances the solar wind plasma is superadiabatic, and thus, far from being isotropic.

In summary, the results contained in this paper are outlined as follows:

1. Derived the theoretical relationship between the adiabatic polytropic index and the anisotropy.
2. Performed data and statistical analysis, showing that the theoretically developed model is in good agreement with the observations in the solar wind proton plasma.
3. Shown that the lowest and classical value of the adiabatic polytropic index, $\gamma = 5/3$, occurs in the isotropic case, while anisotropic plasmas (mostly with $\alpha > 2$) are characterized by higher polytropic indices.

5. Conclusions

This paper has shown a relationship between the polytropic index $\gamma$ and the temperature anisotropy $\alpha$ that may characterize space plasmas. In particular, this paper (i) developed a theoretical model that connects the adiabatic polytropic index and the temperature anisotropy (a similar connection is also developed for the nonadiabatic polytropic indices) and (ii) performed a data and statistical analysis in order to examine the validity of the theoretically developed model, in the case of the solar wind proton plasma near 1 au; this was accomplished using data sets taken from Wind S/C.

The derived theoretical relationship is found to be in good agreement with observations. According to the results of the analysis, the classical value of the adiabatic polytropic index, $\gamma = 5/3$, occurs in the case of isotropic distributions, while higher indices characterize anisotropic plasmas. A possible deviation of the observations from the modeled relationship appears for large anisotropies (Figure 6); indeed, for $\alpha > 2$ the observed average adiabatic polytropic index is larger than its modeled value. This high-anisotropy deviation can be explained by the existence of heterogeneous correlations. The basic model of anisotropic kappa distributions characterizes correlated particles, for which the particle velocity components are homogeneously correlated; namely, the correlations between different velocity components of each particle, are equal to the correlations between the same components of different particles. On the other hand, other, less frequently observed models, consider heterogeneous correlations among particle velocity components. (For more details, see Livadiotis et al. 2021).
4. Showed that the adiabatic polytropic index is even larger for highly anisotropic distributions when heterogeneous correlations exist among particle velocity components (mostly with $\kappa < \kappa^{\text{int}} < \infty$).

5. Development of possible extensions of the theory considering (i) nonadiabatic polytropic behavior and (ii) more general distributions.

References

Abe, S. 1999, PhyA, 269, 403
Allegri, F., Kurth, W. S., Elliott, S., et al. 2021, JGRA, submitted
Ao, X., Shen, J., & Tu, C. 2003, ScChG, 46, 78
Astfalk, P., Görler, T., & Jenko, F. 2015, JGRA, 120, 7107
Baker, D. N., & Van Allen, J. A. 1976, JGR, 81, 617
Bautz, M. W., Miller, E. D., Saunders, I. S., et al. 2009, PASI, 61, 1117
Borovsky, J. E., Thomsen, M. F., Elphic, R. C., Cayton, T. E., & Bautz, M. W., Miller, E. D., Sanders, J. S., et al. 2009, PASJ, 61, 1117
Burlaga, L. F., & Lazarus, A. J. 2000, JGR, 105, 2357
Livadiotis, G. 2007, PhysA, 375, 518
Livadiotis, G. 2015a, APJ, 809, 111
Livadiotis, G. 2015c, JGRA, 120, 1607
Livadiotis, G. 2016, ApJS, 223, 13
Livadiotis, G. 2017, Kappa Distribution: Theory Applications in Plasmas (1st ed.; Netherlands: Elsevier)
Livadiotis, G. 2018a, Entpr, 20, 799
Livadiotis, G. 2018b, EPL, 122, 50001
Livadiotis, G. 2018c, JGRA, 123, 1050
Livadiotis, G. 2019a, Entpr, 21, 1041
Livadiotis, G. 2019b, ApJ, 874, 10
Livadiotis, G. 2021, RNAAS, 5, 4
Livadiotis, G., & Desai, M. I. 2016, ApJ, 829, 88
Livadiotis, G., Desai, M. L., & Wilson, L. B., III 2018, ApJ, 853, 142
Livadiotis, G., & McComas, D. J. 2010, PhyS, 82, 035003
Livadiotis, G., & McComas, D. J. 2011, ApJ, 741, 88
Livadiotis, G., & McComas, D. J. 2013, SSRv, 175, 183
Livadiotis, G., McComas, D. J., Dayeh, M. A., Funsten, H. O., & Schwadron, N. A. 2011, ApJ, 734, 1
Livadiotis, G., Nicolaou, G., & Allegri, F. 2021, ApJS, 253, 26
Liu, X., & Chen, L. 2019, PhPl, 26, 042902
Liu, Y., Richardson, J. D., Belcher, J. W., Kasper, J. C., & Elliott, H. A. 2006, JGRA, 111, A03105
Liu, Y., Richardson, J. D., Belcher, J. W., Kasper, J. C., & Elliott, H. A. 2006, JGRA, 111, A03102
Livadiotis, G. 2007, PhysA, 375, 518
Livadiotis, G. 2015a, Entpr, 17, 2062
Livadiotis, G. 2015b, ApJ, 809, 111
Livadiotis, G. 2015c, JGRA, 120, 1607
Livadiotis, G. 2016, ApJS, 223, 13
Livadiotis, G. 2017, Kappa Distribution: Theory Applications in Plasmas (1st ed.; Netherlands: Elsevier)
Livadiotis, G. 2018a, Entpr, 20, 799
Livadiotis, G. 2018b, EPL, 122, 50001
Livadiotis, G. 2018c, JGRA, 123, 1050
Livadiotis, G. 2019a, Entpr, 21, 1041
Livadiotis, G. 2019b, ApJ, 874, 10
Livadiotis, G. 2021, RNAAS, 5, 4
Livadiotis, G., & Desai, M. I. 2016, ApJ, 829, 88
Livadiotis, G., Desai, M. L., & Wilson, L. B., III 2018, ApJ, 853, 142
Livadiotis, G., & McComas, D. J. 2010, PhyS, 82, 035003
Livadiotis, G., & McComas, D. J. 2011, ApJ, 741, 88
Livadiotis, G., & McComas, D. J. 2013, SSRv, 175, 183
Livadiotis, G., McComas, D. J., Dayeh, M. A., Funsten, H. O., & Schwadron, N. A. 2011, ApJ, 734, 1
Livadiotis, G., Nicolaou, G., & Allegri, F. 2021, ApJS, 253, 26
Markevitch, M., Forman, W. R., Sarazin, C. L., & Vikhlinin, A. 1998, ApJ, 503, 77
Mauk, B. H., & Fox, N. J. 2010, JGRA, 115, A12220
Meyer-Vernet, N., Moncuquet, M., & Hoang, S. 1995, Icar, 116, 202
Mishra, W., & Wang, Y. 2018, ApJ, 865, 50
Newbury, J. A., Russell, C. T., & Lindsay, G. M. 1997, GeoRL, 24, 1431
Newbury, J. A., Russell, C. T., & Lindsay, G. M. 1997, GeoRL, 24, 1431
Nicolaou, G., & Livadiotis, G. 2019, ApJ, 884, 52
Nicolaou, G., Livadiotis, G., & Moussas, X. 2014a, SoPh, 289, 1371
Nicolaou, G., & Livadiotis, G. 2019, ApJ, 884, 52
Nicolaou, G., Livadiotis, G., Wicks, R. T., Verscharen, D., & Maruca, B. A. 2020, ApJ, 901, 26
Nicolaou, G., McComas, D. J., Bagenal, F., & Elliott, H. A. 2014b, JGRA, 119, 3463
Nicolaou, G., McComas, D. J., Bagenal, F., Elliott, H. A., & Wilson, R. J. 2015, P&SS, 119, 222
Ogilvie, K. W., Chornay, D. J., Fritztenreiter, R. J., et al. 1995, SSRv, 71, 55
Park, K. 2018, Fundamentals of Probability and Stochastic Processes with Applications to Communications (Berlin: Springer)
Park, E. N., 1963, Interplanetary Dynamical Processes (New York: Wiley-Interscience)
Prasad, S. K., Raes, J. O., Van Doorselaere, T., Magyar, N., & Jess, D. B. 2018, ApJ, 868, 149
Roberts, D. A., Goldstein, M. L., Matthaeus, W. H., & Ghosh, S. 1992, JGR, 97, 17115
Scopke, N., Paschmann, G., Haerendel, G., et al. 1981, JGR, 86, 2099
Sorriso-Valvo, L., Marino, R., Carbone, V., et al. 2007, PhRvL, 99, 115001
Summers, D., & Thorne, R. M. 1991, PhFIB, 3, 1835
Summers, D., Xue, S., & Thorne, R. M. 1994, PhP, 1, 2012
Swaczyna, P., McComas, D. J., & Schwadron, N. A. 2019, ApJ, 871, 254
Tatraallyay, M., Russell, C. T., Luhmann, J. G., Barnes, A., & Mihalov, J. G. 1998, Icar, 116, 202
Totten, T. L., Freeman, J. W., & Arya, S. 1995, JGR, 100, 13
Vasquez, B. J., Smith, C. W., Hamilton, K., MacBride, B. T., & Leamon, R. J. 2018, ApJ, 868, 149
Vasquez, B. J., Smith, C. W., Hamilton, K., MacBride, B. T., & Leamon, R. J. 2018, ApJ, 868, 149
Winterhalter, D., Kivelson, M. G., Walker, R. J., & Russell, C. T. 2018, AdSpR, 4, 287
Zank, G. P., Adhikari, L., Hunana, P., et al. 2017, ApJ, 835, 147
Zhu, X. M. 1990, GrRL, 17, 2321

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