EXPLICIT ENUMERATION OF 321,HEXAGON–AVOIDING PERMUTATIONS

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Abstract. The 321,hexagon–avoiding (321–hex) permutations were introduced and studied by Billey and Warrington in [4] as a class of elements of $S_n$ whose Kazhdan–Lusztig and Poincaré polynomials and the singular loci of whose Schubert varieties have certain fairly simple and explicit descriptions. This paper provides a 7–term linear recurrence relation leading to an explicit enumeration of the 321–hex permutations. A complete description of the corresponding generating tree is obtained as a by–product of enumeration techniques used in the paper, including Schensted’s 321–subsequences decomposition, a 5–parameter generating function and the symmetries of the octagonal patterns avoided by the 321–hex permutations.

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1. INTRODUCTION

We start by describing the 321,hexagon–avoiding permutations in two ways; first in the context of pattern–avoidance, which is the viewpoint we will be exploiting in our enumeration, and then in the context of reduced expressions, which explains the introduction of the term 321,hexagon–avoiding (for brevity, 321–hex). Finally, we explain briefly the connection with Kazhdan–Lusztig polynomials in [4] and how this motivates the work on the present paper.

1.1. Pattern–avoidance. From the first viewpoint, we consider permutations in $S_n$ as bijections $w : [n] \rightarrow [n]$, and write them in one–line notation as the image of $w$, $[w_1, w_2, \ldots, w_n]$. For $n < 10$ we suppress the commas without causing confusion.

Key words and phrases. 321–hexagon–avoiding permutations, forbidden subsequences, heaps, linear recurrence, Kazhdan–Lusztig polynomials.
Definition 1. Let $v \in S_k$ and $w \in S_n$ for some $k \leq n$. We say that $w$ contains $v$ if there is a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that the sequences $w' = [w_{i_1}, w_{i_2}, \ldots, w_{i_k}]$ and $[v_1, v_2, \ldots, v_k]$ obey the same pairwise relations, i.e. $w_{i_j} < w_{i_m}$ exactly when $v_j < v_m$. In such a case, we write $w' \sim v$. If $w$ does not contain $v$ then we say that $w$ avoids $v$. We denote by $S_n(v)$ the set of all $v$–avoiding permutations of length $n$.

For example, the permutation $\omega = (52687431)$ avoids (2413) but does not avoid (3142) because of its subsequence (5283). For a classification of forbidden subsequences up to length 7, we direct the reader to [1, 2, 12, 13, 14, 17, 18].

Definition 2. The 321–hex permutations are those permutations which simultaneously avoid each of the following five patterns:

\begin{align*}
[321], & \quad P_1 = [46718235], P_2 = [46781235], P_3 = [56718234], P_4 = [56781234].
\end{align*}

We denote by $\mathcal{P}$ the set of the four length–8 (octagonal) permutations $P_1, P_2, P_3, P_4$. In order to make sense of the above definition, consider the following equivalent, but perhaps more insightful, reformulation in terms of matrices.

Definition 3. Let $w \in S_n$. The permutation matrix $M_w$ is the $n \times n$ matrix having a 1 in position $(i, w_i)$ for $1 \leq i \leq n$, and 0 elsewhere. (To keep the resemblance with the “shape” of $w$, we coordinatize $M_w$ from the bottom left corner.) Given two permutation matrices $M$ and $N$, we say that $M$ avoids $N$ if no submatrix of $M$ is identical to $N$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{octagonal_patterns.png}
\caption{Octagonal Patterns}
\end{figure}

A permutation matrix $M$ of size $n$ is simply a transversal of an $n \times n$ matrix. Clearly, $w \in S_n$ contains $v \in S_k$ if and only if $M_w$ contains $M_v$ as a submatrix. Under this reformulation, Fig. 1 presents the four octagonal patterns \{$P_i$\} which must be avoided by the 321–hex permutations. The fifth pattern $P$ does not come from a permutation because it is not a transversal. Yet, $P$ is a union of the four previous permutation patterns, and it can be easily checked that a permutation matrix $M_w$ avoids all $P_i$’s if...
and only if no $8 \times 8$ permutation submatrix of $M_w$ can be completely “covered” by $P$.
Thus, by abuse of notation, we can say that the 321–hex permutations are defined as
the permutations avoiding both [321] and $P$.

Our enumeration makes use of this last interpretation, exploiting the symmetries in
the set $\mathcal{P}$ of octagonal patterns plus a convenient structural representation of all 321–avoiding permutations. The close relations between the four octagonal permutations
are even more clearly revealed from a group–theoretical viewpoint when examining
their reduced expressions. Below, we briefly review the construction of the heaps of
321–avoiding permutations and their relation to the octagonal patterns $P_i$; for more
details, see [4]. The remainder of the introduction, except for its conclusion, can be
skipped by the reader who is interested only in the pattern–avoidance interpretation.

1.2. Heaps of 321–permutations. The permutation group $S_n$ can be regarded as
generated by the set of the adjacent transpositions $\{s_i\}_{i=1}^{n-1}$, where $s_i = (i, i+1)$ in cyclic
notation. In this presentation, the generators $s_i$ and $s_j$ commute if $|i - j| > 1$; else
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. An expression is any product of generators. A
reduced expression $\langle w \rangle$ for $w \in S_n$ is a shortest–possible expression yielding $w$.
(It is well–known that the number of generators in $\langle w \rangle$ equals the number of inversions in $w$.) For example, the
octagonal pattern $P_1 = [46718235]$ has a reduced expression
$\langle P_1 \rangle = s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_6 s_5 s_4 s_3 s_7 s_6 s_5$.
Reduced expressions for the other avoided octagonal patterns are
$\langle P_2 \rangle = \langle P_1 \rangle \cdot s_4$, $\langle P_3 \rangle = s_4 \cdot \langle P_1 \rangle$ and $\langle P_4 \rangle = s_4 \cdot \langle P_1 \rangle \cdot s_4$. These expressions can be easily verified by
considering the dashed lines in $P$ in Fig. 1. The study of reduced expressions is a major
subject of the representation theory of $S_n$.

After work of Viennot [16], the 321–avoiding permutations $w$ can be represented
by special ranked posets called heaps. The elements of Heap($w$) are identified with
the transpositions $\{s_{i,j}\}$ in a fixed reduced expression $\langle w \rangle$ for $w$. By Billey–Jockusch–
Stanley [3], the 321–avoiding permutations are those in which no reduced expression
contains a substring of the form $s_i s_{i\pm 1} s_i$; and by Tits [15], all reduced expressions for
a 321–avoiding permutation $w$ are equivalent up to moves $s_i s_j \rightarrow s_j s_i$ for $|i - j| > 1$.
Thus, the set of elements in Heap($w$) is independent of the choice of $\langle w \rangle$.

![Figure 2. String diagrams for $s_3 \in S_5$, $s_2 s_1 s_2 \in S_3$ and $s_1 s_2 s_1 \in S_3$](image)

We now describe the rank function of Heap($w$), along with a Hasse diagram for
the poset by embedding its elements in the integer lattice. One way to define and
visualize this embedding is via string diagrams. To form a string diagram of $w \in S_n$,
write the row of numbers $[w_1, w_2, ..., w_n]$ above the row $[1, 2, ..., n]$, thus mimicking
the two–line notation for a permutation. Connect each number $i$ on the bottom line to the
corresponding number \( i \) on the top line, drawing a "string" which may change direction, but which at all times is running either due north, northwest or northeast. Strings may cross and recross, but do not run over top of one another, nor do they stray beyond the rectangular bounds formed by the two rows of numbers. For example, \( s_i \in S_n \) can be realized as the crossing of two strings, as shown in Fig. 2a. In general, two or more adjacent transpositions can be applied simultaneously, provided they commute, i.e. \( s_i \) and \( s_j \) can occur on the same horizontal level of a string diagram unless \( |i - j| = 1 \).

Thus, the crossings of a string diagram can be identified with the \( s_i \)'s, labelled by the column (from 1 to \( n - 1 \)) in which they occur, and can be seen to form a poset in the obvious way. The linear extensions of this partial order are the expressions for \( w \) defined above. If the string diagram has the smallest possible number of crossings, then it is is minimal and the linear extension is a reduced expression for \( w \).

A short braid is a configuration obtained by applying the non–commuting transpositions \( s_i \) and \( s_{i+1} \) in the following order:
\[
\begin{align*}
s_i & s_{i+1} s_i \\
& \text{or} \\
& s_{i+1} s_i s_{i+1}
\end{align*}
\]
(cf. Fig. 2b–2c). The string diagram for a braid shows crossings at three of the four points of a small diamond, omitting either the eastern or the western point. As mentioned earlier, a permutation is 321–avoiding exactly when its minimal string diagrams avoid such configurations; such permutations are therefore also called short–braid–avoiding in the literature of Coxeter groups. It is now clear how to embed canonically the poset of string crossings in a minimal string diagram of \( w \) into the integer lattice. The resulting Hasse diagram is the heap of \( w \), Heap(\( w \)). It is independent of the choice of \( \langle w \rangle \) as long as \( w \) is 321–avoiding.

The heap of the special reduced expression \( \langle P_1 \rangle \) for the octagonal pattern \( P_1 \) resembles a hexagon: Heap(\( P_1 \)) has horizontal and vertical symmetries, with respectively 2,3,4,3,2 lattice points on its five ranks (see Fig. 3a–3b). The string diagram of \( P_4 \) features one extra point on top and one extra point on bottom, corresponding to the crossings of the strings 1 and 8, and 4 and 5 (see Fig. 3c). The string diagrams for \( P_2 \) and \( P_3 \) have either the top or the bottom extra crossing. In all cases, Heap(\( P_i \)) contains the hexagonal Heap(\( P_1 \)).

**Figure 3.** String diagrams for \( P_1 \) and \( P_4 \), and Heap(\( P_1 \)) where \( \langle P_1 \rangle = \prod_{i=1}^{14} p_i \)

Furthermore, it can be shown that the (321,\( P \))–avoiding permutations are exactly those 321–avoiding permutations whose minimal string diagrams avoid the hexagonal string diagram of \( P_1 \). This justifies the descriptive term 321,hexagon–avoiding and yields the following alternative description (cf. [9]):

**Definition 2′.** A 321–hex permutation is a permutation whose reduced expressions do not contain a substring of the form \( s_j s_{j+1} s_j \) for all \( j \geq 1 \), and do not contain any of the translates \( \prod_{k=1}^{14} s_k + j \) of Heap(\( P_1 \)) for all \( j \geq 0 \).
1.3. Kazhdan–Lusztig polynomials. We now turn our attention to the Kazhdan–Lusztig polynomials. For definitions and a detailed introduction, we direct the reader to [8, 9]. In short, the Hecke algebra \( H \) of a finite Weyl group \( W \) (such as \( S_n \)) is an algebra over \( \mathbb{Q}(\sqrt{q}) \), with basis \( \{ T_w \}_{w \in W} \), relations for all generators \( s \in W \):

\[
\begin{align*}
T_s T_w &= T_{sw} \quad \text{if} \quad l(sw) > l(w), \\
T_s^2 &= (q - 1) T_s + q T_1,
\end{align*}
\]

and well-defined inverses due to the presence of \( q^{-1} \). Moreover, the involution of \( \mathbb{Q}(\sqrt{q}) \) sending \( \sqrt{q} \mapsto 1/\sqrt{q} \) extends to an involution \( \iota \) of \( H \). The Kazhdan–Lusztig polynomials arise in search of a new basis \( \{ C_w' \} \) for \( H \) of \( \iota \)-invariant elements which are linear combinations of “lower–terms” \( T_x \) for \( x \leq w \) under the Bruhat–Chevalley order: \( x \leq w \) if every reduced expression for \( w \) contains a subexpression for \( x \).

**Theorem 1** (Kazhdan–Lusztig). For any \( w \in W \), \( H \) has a unique \( \iota \)-invariant element \( C_w' = q^{-(l(w)/2)} \sum_{x \leq w} P_{x,w} T_x \), where the degrees of the polynomials \( P_{x,w}(q) \in \mathbb{Z}[q] \) are at most \( \frac{1}{2}(l(w) - l(x) - 1) \) if \( x < w \), and \( P_{w,w} = 1 \), \( P_{x,w} = 0 \) if \( x \not\leq w \).

The Kazhdan–Lusztig polynomials \( P_{x,w} \) are of fundamental importance in Lie Theory. It has been proven that their coefficients are non–negative for Weyl groups (see Kazhdan–Lusztig [10], Brylinski–Kashiwara [8]), but this question remains open for arbitrary Coxeter groups. Neither the degrees nor the coefficients of \( P_{x,w} \) are readily computable; indeed, only partial results have been derived in certain cases (see [8] for detailed references.

1.4. 321–hex permutations in Kazhdan–Lusztig polynomials. In [4], Billey and Warrington introduce the 321–hex permutations \( w \in W = S_n \), derive simple combinatorial formulas for their Kazhdan–Lusztig and Poincaré polynomials, and a simple method for determining the singular loci of their Schubert varieties \( X_w \). Concretely, let \( \langle w \rangle = s_{i_1} \cdots s_{i_r} \) be a reduced expression for \( w \), and let \( d(\sigma) \) denote the defect statistic of a mask \( \sigma \) on \( \langle w \rangle \) whose product is a fixed \( x \in S_n \). (For definitions of these and other related concepts, see [8].)

**Theorem 2** (Billey–Warrington). A permutation \( w \in S_n \) is 321–hexagon avoiding if and only if one of the following equivalent conditions is satisfied:

(a) The Kazhdan–Lusztig polynomial \( P_{x,w} = \sum_\sigma q^{d(\sigma)} \) for \( x \leq w \).
(b) The Poincaré polynomial for the full intersection cohomology group of \( X_w \) equals \( (1 + q)^{l(w)} \).
(c) The Kazhdan–Lusztig basis element \( C_w' = C_{s_{i_1}}' \cdots C_{s_{i_r}}' \).
(d) The Bott–Samelson resolution \( Y \) of \( X_w \) is small.
(e) \( IH_* (X_w) \ast H_*(Y) \).

1.5. Enumeration of 321–hex permutations. As Theorem 2 shows, the 321–hex permutations have properties that make certain algebraic and combinatorial computations easier to perform compared to arbitrary permutations. Another interesting aspect of the 321–hex permutations was examined earlier in the introduction: they are one of the few families that are known to be describable both in terms of pattern–avoidance and heap–avoidance. Yet, until now, there was no known recursive, exact or other
closed form for the number of 321–hex permutations. For instance, it would be useful to know to how many permutations Theorem 2 applies; how the number of 321–hex permutations changes asymptotically, and how it compares to sizes of other well–known sets of permutations, such as $S_n(321)$, which is enumerated by the Catalan numbers.

Answers to all these questions are obtained in the present paper, where we find a 7–term linear recursive relation and derive from it an explicit exact formula. With this formula at hand, answering any enumeration questions about the 321–hex permutations becomes a matter of simple observation and calculation.

**Definition 4.** Let $H_n$ denote the set of all 321–hex permutations in $S_n$, and let $\alpha_n = |H_n|$ be the number of such permutations.

**Theorem 3.** The sequence $\alpha_n$ satisfies the following recursive relation for all $n \geq 6$:

$$\alpha_n = 6\alpha_{n-1} - 11\alpha_{n-2} + 9\alpha_{n-3} - 4\alpha_{n-4} - 4\alpha_{n-5} + \alpha_{n-6}.$$  

Of the six roots of the corresponding characteristic polynomial, four are real: $R_i$ for $i = 1, 2, 3, 4$, and two are complex conjugates: $R_5 = R_6$. This implies the same description of the six coefficients below: $c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$, and $c_5 = \overline{c_6} \in \mathbb{C}$.

**Corollary 1.** The number of the 321–hexagon avoiding permutations of length $n$ equals

$$c_1R_1^n + c_2R_2^n + c_3R_3^n + c_4R_4^n + c_5R_5^n + \overline{c_5R_5^n},$$

where the roots and coefficients are rounded off below to 5 digits after the decimal point:

$R_1 \approx -0.49890$  
$R_2 \approx 0.21989$  
$R_3 \approx 1.95627$  
$R_4 \approx 3.43526$  
$R_5 \approx 0.44375 - 1.07682i$  
$c_1 \approx 0.00164$  
$c_2 \approx 0.13776$  
$c_3 \approx 0.57156$  
$c_4 \approx 0.24149$  
$c_5 \approx 0.02378 + 0.00080i$

For further discussion of this and other results, we refer the reader to Sections 4–5.

The proof of Theorem 3 follows several steps. First, we describe the nodes in the generating tree of $H_n$ by using Schensted’s algorithm for 321–avoiding permutations and by introducing 5 parameters for the generating function $h_n(x, k, l, m)$. We next observe that this function depends on fewer parameters, yielding therefore relatively few distinct values. We organize these values in five sequences, $\alpha_n, \beta_n, \gamma_n, \delta_n$ and $\epsilon_n$. Using the intrinsic symmetries of the set $\mathcal{P}$ of octagonal patterns, we deduce recursive relations expressing each sequence in terms of $\alpha_n$. The latter turns out to be the number we are looking for, $|H_n|$. Finally, putting together all information about the generating function and the five sequences results in the desired formula.

## 2. The generating tree

### 2.1. What is a generating tree?

We turn now to the development of a recurrence for the 321–hex permutations. A standard tool in the enumeration of restricted permutations is the generating tree $T$ introduced in [3]. Begin with an infinite tree whose nodes on level $n$ are identified with the permutations in $S_n$. The node $w$ is a child of $\hat{w} = [w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n]$ where the omitted value is $w_j = n$. Looking at this
from the point of view of the parent, we can form all the children of \( w \in S_n \) by inserting the element \( n + 1 \) into each of the \( n + 1 \) sites of \( w \).

Then, for a given set \( \Sigma \) of forbidden subsequences, prune the tree by deleting all nodes containing any of the forbidden subsequences. What remains is still connected because if \( w \) does not contain any forbidden subsequence then clearly \( \hat{w} \) does not either.

For any node on level \( n \) of this pruned tree \( T(\Sigma) \), we call a site in the corresponding permutation active if inserting \( n + 1 \) at that site yields a node of the tree; conversely an inactive site is one where the insertion of \( n + 1 \) creates one or more of the forbidden subsequences in \( \Sigma \).

To give a complete description of a generating tree, we need to associate to each node an appropriate label, and then describe a succession rule for deriving the labels attached to the set of children of each node. For instance, we might characterize the original tree \( T \) generating all permutations as having a root labelled \((2)\) and a succession rule \((n) \rightarrow (n + 1)^n\). In this instance, the label can be interpreted as revealing directly how many children each node has in the generating tree.

In the particular instance of the 321–hex permutations, it will turn out that we need a label containing four integers. Although this is more complicated than the single integer of our motivating example, it is nevertheless a major progress to reduce the amount of information recorded at a node from a full permutation to a label of any bounded size. In particular, it is possible to apply the succession–rule recursively to determine the entire downward structure of any node given only its label, regardless of whether that node is on a level corresponding to permutations on four symbols, or four thousand.

2.2. Schensted algorithm for \( S_n(321) \) and active regions. Let \( w \) be any 321–hexagon–avoiding permutation on \( n \) symbols. We divide the elements \( w_1, w_2, \ldots, w_n \) into two categories: the set of right–to–left minima (including \( w_n \), and the rest. Adapting the terminology of Schensted [1], we refer to these as the first basic subsequence and second basic subsequence of \( w \), and denote them by \( \mathcal{B}_1(w) \) and \( \mathcal{B}_2(w) \), respectively. For example, when \( w = P_1 \), \( \mathcal{B}_1(w) = [4, 6, 7, 8] \) and \( \mathcal{B}_2(w) = [1, 2, 3, 5] \). Note that both \( \mathcal{B}_1(w) \) and \( \mathcal{B}_2(w) \) decrease from right to left, the former – by construction, and the latter – because \( w \in S_n(321) \). Now let \( K = w_{i_1}, L = w_{i_2} \) and \( M = w_{i_3} \) be the three largest (i.e. rightmost) elements in \( \mathcal{B}_2(w) \) \((K < L < M)\), with each one set to zero if the corresponding element does not exist.

Since \( M \) is the largest (i.e. rightmost) element in \( \mathcal{B}_2(w) \), it follows that every element to the right of \( M \) belongs to \( \mathcal{B}_1(w) \). There are \( x := n - i_3 \) elements in this region, which we will call the active region. If \( M = 0 \) because \( \mathcal{B}_2(w) \) is empty, then we consider the entire permutation to be the active region. Let \( k \) be the number of elements in the active region which are larger than \( K \); as the elements in the active region decrease from right to left, these \( k \) elements are \( w_{n - k + 1}, \ldots, w_n \). Similarly, \( l \) be the number of elements in the active region which are larger than \( L \), and \( m \) be the number of such elements larger than \( M \) (cf. Fig. [1]).

Now assign the label \((x, k, l, m)\) to \( w \). By construction, \( x \geq k \geq l \geq m \). Let the \( X \)–elements of \( w \) be those that are counted in \( x \), and similarly define the \( K \)–, \( L \)–, and \( M \)–elements. Further, let the \( X \backslash K \)–elements of \( w \) be the set of \( X \)–elements minus the
set of $K$–elements, and similarly define the $K\setminus L$– and $L\setminus M$–elements. For example, the $X$–elements in $P_1$ are 2, 3, 5, and $k = l = m = 0$.

2.3. The succession rule for 321–hex permutations. All children of $w$ must avoid the subsequence $[321]$. This restriction by itself renders inactive all sites to the left of $M$, but none of the sites in the active region. Therefore when considering the children of $w$ and their labels, we need only consider insertions taking place in the active region to the right of $M$. Thus, if we insert $n+1$ into a site in the active region with $j$ elements to its right, it is easy to verify that the resulting permutation will have label:

$$
(1) \begin{cases}
(x + 1, k + 1, l + 1, m + 1) & \text{if } j = 0 \\
(j, j, j, 0) & \text{if } 0 < j \leq m \\
(j, j, m, 0) & \text{if } m < j \leq l \\
e(j, l, m, 0) & \text{if } l < j \leq x
\end{cases}
$$

Furthermore, the number of $w$’s children can be computed as follows. Set

$$
T := \min(k + 2, \max(k + 1, l + 2)).
$$

Then the node $w$ has $S + 1$ children, corresponding to the $S + 1$ rightmost insertion sites, where

$$
(2) \begin{cases}
S = T & \text{if } T \leq x - 2, \\
S = x & \text{if } T > x - 2.
\end{cases}
$$

This allows a more compact and complete succession rule for the labels of all $S + 1$ children of $w$:

$$
(3) (x, k, l, m) \mapsto \begin{cases}
(x + 1, k + 1, l + 1, m + 1), \\
(i, \min(i, l), \min(i, m), 0) & \text{for } i = 1, ..., S.
\end{cases}
$$

We shall not use directly any of the above formulas (1)–(3) in our calculations, so we leave their verification to the reader, who will find this easier after mastering the material in the rest of the paper. The importance of the above discussion is that it completely describes the structure of the generating tree $T(321, \mathcal{P})$, and hence explains in principle why the enumeration in this paper works. Why the resulting final formula for the 321–hex permutations is so simple – a linear recursive relation with constant coefficients – is a completely different matter and can be explained only by the structure of the forbidden set $\mathcal{P}$ of four octagonal patterns, as we shall see later.

2.4. The generating function for $T(321, \mathcal{P})$. Let $h_n(x, k, l, m)$ and $\mathcal{H}_n(x, k, l, m)$ be the number, resp. the set, of 321–hex permutations in $S_n$ labelled $(x, k, l, m)$. According to this definition, $h_n(0, 0, 0, 0)$ is not defined since no permutation is labelled $(0, 0, 0, 0)$. For convenience, denote by $h_n(0, 0, 0, 0)$ and $\mathcal{H}_n(0, 0, 0, 0)$ the number and the set of all 321–hex permutations in $S_n$ which end in their largest element: $w_n = n$. It is worth noting that $\mathcal{H}_n(x, x, x, m)$ for $x > m$ corresponds to permutations $w$ in which either $L$ is smaller than the final “tail” of $w$, or $L$ does not exist and $\mathcal{B}_2(w) = \{M\}$.

Naively, there are $O(n^5)$ enumerations to be performed for $h_n(x, k, l, m)$. However, many have the same answers; indeed, only $5n$ different values of $h_{n+1}(x, k, l, m)$ appear:
Lemma 1. The operation of deleting all $K$-elements in a 321-hex permutation provides a bijection $d_K : \mathcal{H}_n(x, k, l, m) \rightarrow \mathcal{H}_{n-k}(x-k, 0, 0, 0)$. Hence, for all $n, x, k, l, m$ we have $h_n(x, k, l, m) = h_{n-k}(x-k, 0, 0, 0)$.

Proof: Let $w \in S_n(321)$. The $K$-elements of $w$ lie in $B_1(w)$ and are part of the final increasing “tail” of $w$ (since they are to the right of $M$). Thus, if a $K$-element $w_i$ were part of an octagonal pattern $P_j$ in $w$, then $w_i$ would lie in $B_1(P_j)$. But then $P_j$, and hence $w$, would contain at least 3 elements larger than and to the left of $w_i$: this contradicts the definition of a $K$-element, for which only $L$ and $M$ are larger than and to the left of it.

The above discussion shows that $K$-elements cannot participate in octagonal patterns in $w$, and hence they can be deleted without losing any relevant 321-hex information about $w$. (Of course, we have to rescale down appropriately $M$ and $L$ of $w$ to arrive at a permutation of smaller size.) The resulting map

$$d_K : \mathcal{H}_n(x, k, l, m) \rightarrow \mathcal{H}_{n-k}(x-k, 0, 0, 0)$$

is bijective: to obtain $w \in \mathcal{H}_n(x, k, l, m)$ from its image $\tilde{w} = d_K(w) \in \mathcal{H}_{n-k}(x-k, 0, 0, 0)$, insert the necessary number of $M, L, K$-elements into $\tilde{w}$ and increase appropriately $L$ and $M$ to fit their definitions in $\tilde{w}$. This procedure works because we can identify $M$ as the largest element in $\tilde{w}$ (after the deletion of all $K$-elements in $w$), and then identify $L$ as the second largest element of $\tilde{w}$, which will be necessarily to the left of $M$. Then insertion of the appropriate $M, L, K$-elements at the end of $\tilde{w}$ requires rescaling $M$ and $L$ only (not $K$), and hence transforms the 321-Schensted decomposition of $\tilde{w}$ into that of $w$: whether $L$ was in $B_1(\tilde{w})$ or in $B_2(\tilde{w})$ does not prevent $L$ from becoming an element of $B_1(w)$ after applying $d_K^{-1}$. \[\square\]
2.5. **Octagonal conditions.** Consider a family $\mathcal{F}$ of permutations in $\mathcal{H}_n$ which are described by certain configuration conditions imposed on their basic subsequence decomposition $\mathcal{B}_1 \sqcup \mathcal{B}_2$. For example, $\mathcal{H}_n(0,0,0,0)$ is such a family defined by the condition $w_n = n \in \mathcal{B}_1(w)$. Let $a$ be an element of the permutations in $\mathcal{F}$ which is identified uniquely in each $w \in \mathcal{F}$ by the configuration description of $\mathcal{F}$.

**Definition 5.** We say that $a$ is **pattern-free** if its deletion in each $w \in \mathcal{F}$ (and appropriate rescaling of $w$) results in a numerically equivalent family $\tilde{\mathcal{F}}$ of permutations in $\mathcal{H}_n$; i.e. $d_a : \mathcal{F} \to \tilde{\mathcal{F}}$ with $|\mathcal{F}| = |\tilde{\mathcal{F}}|$. For $w \in \mathcal{F}$, denote by $\tilde{w}$ the image $d_a(w) \in \tilde{\mathcal{F}}$.

For example, in $\mathcal{H}_n(0,0,0,0)$, the largest element $a = n$ is identified by being in the last position in each $w$, and clearly it is pattern-free (no octagonal pattern $P_i$ has 8 in its last position):

$$d_n : \mathcal{H}_n(0,0,0,0) \sim \to \mathcal{H}_{n-1}.$$ 

Establishing pattern-free elements and identifying the image set $\tilde{\mathcal{F}}$ is the basis of the enumeration of $\mathcal{H}_n$. The following technical lemma summarizes the pattern-free situations which will be used later in the proof of Theorem 3.

**Lemma 2.** Let $\mathcal{F}$ be a family in $\mathcal{H}_n$, $a_1, \ldots, a_{x-k}$ be the $X \backslash K$–elements of $w \in \mathcal{F}$, and $H$ be the fourth largest element in $\mathcal{B}_2(w)$. The set $\mathcal{F}$ imposes the following octagonal conditions on $\mathcal{F}$:

- (P1) All $K$–elements are pattern-free.
- (P2) If $w_n = n$ in $\mathcal{F}$, or $w_n = n - 1$ in $\mathcal{F}$, then $w_n$ is pattern-free.
- (P3) If $M = n$ and $x \leq 2$ in $\mathcal{F}$, then $M$ is pattern-free.

Assume now that $x \geq 3$ and $k = l = m = 0$ in $\mathcal{F}$.

- (P4) If $H < a_2$ in $\mathcal{F}$ ($H$ may not exist in $w$,) then $M, a_2, \ldots, a_x$ are pattern-free.
- (P5) If $a_2 < H < a_3$ in $\mathcal{F}$, then $L, M, a_3, \ldots, a_x$ are pattern-free.
- (P6) If $H > a_3$ (forcing $x = 3$) in $\mathcal{F}$, then $M$ is pattern-free.

**Proof:** (P1) follows from the proof of Lemma 1. (P2) says that if all $w \in \mathcal{F}$ end in their largest element $n$ (or in their second largest element $n-1$), then this last element is pattern-free (cf. Fig. 3); (P3) says that if $M$ happens to be the largest element in all $w \in \mathcal{F}$ and is followed by at most two $X$–elements, then $M$ is pattern-free (cf. Fig. 3b) Both (P2) and (P3) follow from the facts:

- Each octagonal pattern $P_i$ has an empty $3 \times 3$ upper–right corner (see Fig. 3a.)
For (P2), reinserting \( n \) or \( n - 1 \) in the last position in \( \tilde{w} \in \mathcal{H}_{n-1} \) does not create 321–patterns. Similarly, for (P3), reinserting \( M = n \) into \( \tilde{w} \in \mathcal{H}_{n-1} \) as the first (rightmost) element in \( \mathcal{B}_2(w) \) does not create 321–patterns (since the tail of \( w \) after \( M \) is increasing as part of \( \mathcal{B}_1(\tilde{w}) \)).

For (P4), cf. Case 1 in Fig. [10]. Start by deleting the crosses of the elements in the bottom row and in the rightmost column of each \( P_i \); this leaves the permutation matrix \( M(345612) \), which decomposes into a \( 4 \times 4 \) block \( I_4 \) in the upper left corner and the fixed \( 2 \times 2 \) block \( I_2 \) in lower left corner (corresponding to the original \( p_6 = 2 \) and \( p_7 = 3 \), see Fig. [3]).

- The fact that \( I_4 \) is to the left of and higher than \( I_2 \) proves that \( a_2, \ldots, a_x \) are pattern–free: for them, only \( M, L, K \) can fit in such an \( I_4 \). In detail, if one of \( a_2, \ldots, a_x \) participates in an octagonal pattern \( P_i \), then without loss of generality we may assume that \( a_x \) is the rightmost element of \( P_i \). This forces \( M, L, K \) to participate in \( P_i \) too as the only elements larger than \( a_x \). Now \( M \), being the largest element of \( P_i \), forces at least three elements after it to participate in \( P_i \); one is \( a_x \), one could be \( a_1 \), hence the third one must be among \( a_2, \ldots, a_{x-1} \). But none of these elements can fit into an \( I_2 \) as in Fig. [3] because only \( L, M, N \) are larger and to the left of them. Further, reinsertion of \( a_2, \ldots, a_x \) into \( \tilde{w} \) as the largest elements of \( \mathcal{B}_1(w) \) cannot create 321–patterns. Thus, \( a_2, a_3, \ldots, a_x \) are indeed pattern–free.

- After deleting \( a_2, \ldots, a_x \), the largest element \( M \) lands in second to last position in \( \tilde{w} \), and by (P3), it is pattern–free.

For (P5), cf. Case 2 in Fig. [10]. The reasoning here is similar to the case for (P4).

- First note that there can be no element \( a_0 \) of \( w \) between \( L \) and \( M \), or else \([H, L, a_0, M, a_1, a_2, a_3] \sim P_1 \).

- If one of \( a_3, \ldots, a_x \) participates in a pattern \( P_i \), this forces \( M, L, K \) to participate too, and in order not to run into contradiction with the \( I_4 \times I_2 \) argument above, we must assume that \( a_3 \) and \( a_2 \) are also in \( P_i \), and only one among \( a_3, \ldots, a_x \) is in \( P_i \). Then the ”1” in \( P_i \) will have to be between the two largest elements \( L \) and \( M \), which was ruled earlier. Thus, \( a_3, \ldots, a_x \) are pattern–free.

- Deletion of \( a_3, \ldots, a_x \) leaves \( M \in \mathcal{B}_2(\tilde{w}) \) in third to last position, so by (P3), \( M \) is pattern–free. But \( L \) is the largest element in \( \tilde{w} = d_M(w) \), with only \( a_1 \) and \( a_2 \) after it, so by (P3) again, \( L \) is pattern–free.

For (P6), cf. Case 3 in Fig. [11]. Again note that there can be no element \( a_0 \) of \( w \) between \( L \) and \( M \), or else \([H, L, a_0, M, a_1, a_2, a_3] \sim P_3 \). If \( M \) participates in a pattern \( P_i \), then it forces \( a_1, a_2, a_3 \) also to participate. Since the ”1” in \( P_i \) cannot come from between \( L \) and \( M \), we conclude that \( L \) does not participate in \( P_i \). But then we can replace \( M \) by \( L \) and argue that there is a pattern \( P_i \) in \( \tilde{w} = d_M(w) \), a contradiction. Hence \( M \) is pattern–free.

3. Relations among \( \alpha, \beta, \gamma, \delta \) and \( \epsilon \)

Lemma [1] shows that \( h_n(x, k, l, m) \) does not depend on \( l \) or \( m \), but rather on the differences \( n - k \) and \( x - k \). We shall see further that there are only 5 ranges for \( x - k \), that completely determine the values of \( h \): each of \( x - k = 0, 1, 2, 3 \) and \( x - k \geq 4 \).
corresponds to exactly one of the 5 sequences listed earlier, and defined as follows:

\[
\begin{align*}
\{ h_n(x, x-0, l, m) &= h_{n-x+0}(0,0,0,0) =: \alpha_{n-x-1} \\
h_n(x, x-1, l, m) &= h_{n-x+1}(1,0,0,0) =: \beta_{n-x-1} \\
h_n(x, x-2, l, m) &= h_{n-x+2}(2,0,0,0) =: \gamma_{n-x+1} \\
h_n(x, x-3, l, m) &= h_{n-x+3}(3,0,0,0) =: \delta_{n-3} \\
h_n(x, x-\bar{x}, l, m) &= h_{n-x+\bar{x}}(\bar{x},0,0,0) =: \epsilon_{n-x+3}
\end{align*}
\]

where \( \bar{x} \geq 4 \). All of these definitions are justified by Lemma 4, except for the definition of \( \epsilon_n \), which depends only on \( n \) but not on \( \bar{x} \) and whose explanation will be given later.

In our search for relations between these sequences, it will be useful to define each sequence via an alternative description in the terminology of the families \( \mathcal{F} \) discussed in Section 2.

3.1. The sequence \( \alpha \). By definition, \( \alpha_n = h_{n+1}(0,0,0,0) \). \( M = n + 1 \) is the largest and the rightmost element of any \( w \in \mathcal{H}_{n+1}(0,0,0,0) \), and hence it is pattern–free by (P2) (see Fig. 3). Deleting \( M \) without any further restrictions, i.e. we have a bijection

\[
d_M : \mathcal{H}_{n+1}(0,0,0,0) \sim \mathcal{H}_n = \bigcup_{x,k,l,m} \mathcal{H}_n(x,k,l,m).
\]

This justifies the alternative description of \( \alpha_n \), stated in the Introduction:

\[
\alpha_n = \# \{ 321–hex permutations in \mathcal{S}_n \} = |\mathcal{H}_n|.
\]

![Figure 6. Alternative Description of \( \alpha_n \)](image)

3.2. The sequence \( \beta \). By definition, \( \beta_n = h_{n+1}(1,0,0,0) \). Here \( M = n + 1 \) is second from right to left in \( w \in \mathcal{H}_{n+1}(1,0,0,0) \), and by (P3) it is pattern–free (see Fig. 4.) Thus, deleting \( M \) imposes only one extra condition: in the new 321–hex permutation \( \bar{w} \in \mathcal{H}_n \), the rightmost element in \( \mathcal{B}_1(\bar{w}) \) is a \( K\backslash L \)-element or lower, i.e. the largest two elements (the original \( L = n \) and \( K = n - 1 \)) belong to \( \mathcal{B}_2(\bar{w}) \). This justifies the alternative description:

\[
\beta_n = \# \{ w \in \mathcal{H}_n \mid \{ n, n-1 \} \subset \mathcal{B}_2(w) \} \text{ for } n \geq 3.
\]

In order to find a recursive description of \( \beta_n \), note that each \( w \in \mathcal{H}_n \) either ends in \( n \) or \( n-1 \), or both \( n \) and \( n-1 \) belong to \( \mathcal{B}_2(w) \):

\[
\alpha_n = \underbrace{\# \{ w \in \mathcal{H}_n \mid \text{n last} \}}_{\alpha_{n-1}} + \underbrace{\# \{ w \in \mathcal{H}_n \mid \text{n-1 last} \}}_{\alpha_{n-1}} + \beta_n.
\]
By (P2), if \( n \) or \( n-1 \) is the last element in \( w \in \mathcal{H}_n \), then it is pattern-free, so deleting it results in a bijection:

\[
d_n : \{ w \in \mathcal{H}_n \mid n \text{ last} \} \sim \mathcal{H}_{n-1}
\]

\[
d_{n-1} : \{ w \in \mathcal{H}_n \mid n-1 \text{ last} \} \sim \mathcal{H}_{n-1}.
\]

This justifies the use of \( \alpha_{n-1} = |\mathcal{H}_{n-1}| \) in (6). Consequently, we have the recursive relation

\[
\beta_n = \alpha_{n-1} - \alpha_{n-2}
\]

for \( n \geq 3 
\]

\[3.3. \textbf{The sequence } \gamma. \text{ By definition, } \gamma_n = h_{n+1}(2,0,0,0). \text{ Here } M = n+1 \text{ is third from right to left in } w \in \mathcal{H}_{n+1}(2,0,0,0), \text{ and by (P3) it is pattern-free (see Fig. 8.) Deleting } M \text{ imposes the following extra conditions: in the image } \tilde{w} \in \mathcal{H}_n, \text{ the first two elements in } B_1(\tilde{w}) \text{ are } K \setminus L\text{-elements or lower, i.e. the largest two elements (the original } L = n \text{ and } K = n-1) \text{ belong to } B_2(w) \text{ and there are at least 2 numbers after } L = n \text{. This justifies the alternative description for } n \geq 4:
\]

\[
\gamma_n = \# \{ w \in \mathcal{H}_n \mid \{ w_s = n, \ w_t = n-1 \} \subset B_2(w), \ t < s \leq n-2 \}.
\]

\[\beta_n = \# \{ w \in \mathcal{H}_n \mid \{ w_{n-1} = n, n-1 \} \subset B_2(w) \} + \gamma_n.
\]

The underbraced set in (8) is depicted in the LHS of Fig. 9. By (P3), deletion of \( L = n \) results in the numerically equivalent set \( S \) in the RHS of Fig. 9. The permutations in \( S \)
can be described as having their largest element $K = n - 1 \in \mathcal{B}_2(\tilde{w})$. On the other hand, $\mathcal{H}_{n-1}$ breaks into two disjoint groups: group $A$ consists of the permutations having the largest element $n - 1$ in last position, and group $B$ is the set $S: \mathcal{H}_{n-1} = A \sqcup S$. Finally, note that $d_{n-1} : A \to \mathcal{H}_{n-2}$, so that $|S| = \alpha_{n-1} - \alpha_{n-2}$. This justifies the underbrace notation in (8), and implies the formulas: $\gamma_n = \beta_n - (\alpha_{n-1} - \alpha_{n-2}) = \alpha_n - 3\alpha_{n-1} + \alpha_{n-2}$ for $n \geq 4$.

\[ \begin{array}{c}
L = \bullet \\
M = \bullet \\
K = \bullet \\
\end{array} \]

**Figure 9.** Calculation of $\gamma_n$

### 3.4. The sequence $\epsilon$

By definition, $\epsilon_n = h_{n+x-3}(x, 0, 0, 0)$ for $x \geq 4$. Let $a_1, \ldots, a_x$ be the $X \setminus K$–elements of $w \in \mathcal{H}_{n+x-3}(x, 0, 0, 0)$, where $x \geq 4$. There are two cases to consider (see Fig. 10).

**Case 1.** Except for $M, L, K$, all other elements of $\mathcal{B}_2(w)$ are smaller than $a_2$. Our drawing shows the next element $H \in \mathcal{B}_2(w)$ s.t. $H < a_2$ (H may not exist in w.) By (P4), $a_2, a_3, a_4, \ldots, a_x$ and $M$ are pattern–free. After deletion, the remaining configuration in $\mathcal{H}_{n-3}$ is identical to the alternative description of $\beta_{n-3}$.

\[ \begin{array}{c}
M = \bullet \\
L = \bullet \\
K = \bullet \\
\end{array} \]

**Figure 10.** Calculation of $\epsilon_n$
**Case 2.** After \(M, L, K\), the fourth element \(H\) of \(B_2(w)\) is between \(a_2\) and \(a_3\): \(a_2 < H < a_3\). Recall from (P5) that there can be no element \(a_0\) of \(w\) between the vertical lines of \(L\) and \(M\); or else, \(a_0 < a_1\), and hence \([H, K, L, a_0, M, a_1, a_2, a_3] \sim P_1\). Further, \(a_3, a_4, ..., a_x, M, L\), are pattern–free. After deletion, the remaining configuration in \(\mathcal{H}_{n-3}\) is identical to the alternative description of \(\gamma_{n-3}\).

Note that the case \(H > a_3\) is not allowable, or else \([H, K, L, M, a_1, a_2, a_3, a_4] \sim P_2\) or \(\sim P_4\) (when \(a_3 < H < a_4\) or \(H > a_4\), respectively.)

Incidentally, the above discussion shows that \(h_{n+3}(x, 0, 0, 0)\) does not depend on \(x\) as long as \(x \geq 4\), and hence justifies the definition of \(\epsilon_n\). We conclude that \(\epsilon_n = \beta_{n-3} + \gamma_{n-3} = 2\alpha_{n-3} - 5\alpha_{n-4} + \alpha_{n-5}\) for \(n \geq 7\). \(\square\)

### 3.5. The sequence \(\delta\)

By definition, \(\delta_n = h_{n+1}(3, 0, 0, 0)\). As above, denote by \(a_1, a_2, a_3\) the \(X\backslash K\)–elements of \(w \in \mathcal{H}_{n+1}(3, 0, 0, 0)\). There are three cases to consider (see Fig. [11].)

![Diagram](image-url)

**Figure 11.** Calculation of \(\delta_n\)
Case 1. Except for $M, L, K$, all other elements of $B_2(w)$ are smaller than $a_2$. Our drawing shows the next element $H \in B_2(w)$ s.t. $H < a_2$ ($H$ may not exist in $w$.) By (P4), $a_2, a_3$ and $M$ are pattern–free. The remaining configuration in $\mathcal{H}_{n-2}$ is identical to the alternative description of $\beta_{n-2}$.

Case 2. After $M, L, K$, the fourth element $H$ of $B_2(w)$ is between $a_2$ and $a_3$: $a_2 < H < a_3$. By (P5), there can be no element $a_0$ of $w$ between the vertical lines of $L$ and $M$; or else, $a_0 < a_1$, $a_0 \in B_2(w)$, and hence $[H, K, a_0, M, a_1, a_2, a_3] \sim P_1$. Further, $a_3, M$ and $L$ are pattern–free. After its deletion, the remaining configuration in $\mathcal{H}_{n-2}$ is identical to the alternative description of $\gamma_{n-2}$.

Case 3. In contrast to the discussion of $\epsilon$, in the case of $\delta$ it is possible to have $H > a_3$ as long as there is no element $a_0$ of $w$ between the vertical lines of $L$ and $M$; otherwise, $[H, K, a_0, M, a_1, a_2, a_3] \sim P_3$. By (P6), the largest element $M = n + 1$ is pattern–free. After its deletion, the remaining configuration in $\mathcal{H}_{n-1}$ is identical to the original description of $\delta_{n-1}$.

We conclude that $\delta_n = \beta_{n-2} + \gamma_{n-2} + \delta_{n-1} = \epsilon_{n+1} + \delta_{n-1}$ for $n \geq 6$.

We summarize the results in this Section in

**Lemma 3.** The sequences $\alpha, \beta, \gamma, \delta$ and $\epsilon$ satisfy the following recursive relations:

\[
\begin{align*}
\beta_n &= \alpha_n - 2\alpha_{n-1} \quad \text{for } n \geq 3; \\
\gamma_n &= \alpha_n - 3\alpha_{n-1} + \alpha_{n-2} \quad \text{for } n \geq 4; \\
\delta_n &= \epsilon_{n+1} + \delta_{n-1} \quad \text{for } n \geq 5; \\
\epsilon_n &= 2\alpha_{n-3} - 5\alpha_{n-4} + \alpha_{n-5} \quad \text{for } n \geq 6.
\end{align*}
\]

4. **Enumeration of 321–hex permutations**

We are now in a position to combine the recurrence formulas for $\alpha, \beta, \gamma, \delta$ and $\epsilon$ into a single recurrence for $\alpha$. We first use the interpretation of $\alpha_n$ as $|\mathcal{H}_n|$ and expand this in terms of the individual values of $h_n(x, k, l, m)$. Thus, for a fixed $n$

\[
\alpha_n = \sum_{x, k, l, m} h_n(x, k, l, m),
\]

where the sum is taken over $n \geq x \geq k \geq l \geq m \geq 0$, $x \neq 0$. Next, we break the sum into five separate sums depending on the value of $x - k$; each such sum corresponds to the definitions of $\alpha, \beta, \gamma, \delta$ and $\epsilon$, respectively. Note that the sum for $\alpha$ (where $x = k$) requires two extra special cases for $x = k = n - 1$ and $x = k = n$; in both cases $h_n(x, k, l, m) = 1$.

\[
\Rightarrow \alpha_n = 1 + (n - 1) + \sum_{n-2 \geq x > 0 \atop x \geq l \geq m \geq 0} h_n(x, x, l, m) + \sum_{n \geq x \atop x - 2 \geq l \geq m \geq 0} h_n(x, x - 1, l, m) + \sum_{n \geq x \atop x - 2 \geq l \geq m \geq 0} h_n(x, x - 2, l, m) + \sum_{n \geq x \atop x - 3 \geq l \geq m \geq 0} h_n(x, x - 3, l, m) + \sum_{n \geq x \atop x - 4 \geq l \geq m \geq 0} h_n(x, k, l, m).
\]


In the next step, we replace the $h_n$’s by the appropriate values of $\alpha, \beta, \gamma, \delta$ and $\epsilon$. The case of $\alpha$ requires some care as we must observe the condition that not all the numerical parameters $x, k, l, m$ can be simultaneously equal; indeed, this only happens in the special case $x = k = l = m = n$, which was broken out from the main sum earlier.

$$\Rightarrow \alpha_n = n + \sum_{n \geq x, x > m} \alpha_{n-x-1} + \sum_{n \geq x} \beta_{n-x} + \sum_{n \geq x} \gamma_{n-x+1} + \sum_{n \geq x} \delta_{n-x+2} + \sum_{n \geq x} \epsilon_{n-x+3}.$$ 

We replace the indices $k, l$ and $m$ by binomial coefficients:

$$\alpha_n = n + \sum_{i=1}^{n-1} \left( \binom{n+1-i}{2} - 1 \right) \alpha_i + \sum_{i=3}^{n-1} \binom{n+1-i}{2} \beta_i + \sum_{i=4}^{n-1} \binom{n+1-i}{2} \gamma_i + \sum_{i=5}^{n-1} \binom{n+1-i}{2} \delta_i + \sum_{i=6}^{n-1} \binom{n+2-i}{3} \epsilon_i.$$ 

Finally, we use the recurrences from Sect. 3 to obtain a summation in terms of $\alpha$ alone:

$$\alpha_n = n + \sum_{i=1}^{n-1} \binom{n+1-i}{2} \alpha_i + \sum_{i=3}^{n-1} \binom{n+1-i}{2} (\alpha_i - 2\alpha_{i-1}) + \sum_{i=4}^{n-1} \binom{n+1-i}{2} (\alpha_i - 3\alpha_{i-1} + \alpha_{i-2})$$
$$+ \sum_{i=5}^{n-1} \binom{n+1-i}{2} (2\alpha_{i-2} - 3\alpha_{i-3} - 2 \sum_{j=3}^{i-4} \alpha_j - 4\alpha_2 + \alpha_1) + \binom{n+2-i}{3} \delta_5$$
$$+ \sum_{i=6}^{n-1} \binom{n+2-i}{3} (2\alpha_{i-3} - 5\alpha_{i-4} + \alpha_{i-5}).$$

We simplify this large expression into the following full–history, linear recurrence relation with cubic polynomial coefficients, valid for $n \geq 6$:

$$\alpha_n = 2\alpha_{n-1} + 3\alpha_{n-2} + \sum_{k=3}^{n-4} Q(k)\alpha_{n-k} + R(n),$$

where $Q(k) = -\frac{2}{3} k^3 + \frac{11}{2} k^2 - \frac{71}{6} k + 9$ and $R(n) = -\frac{17}{3} n^3 + 85n^2 - \frac{1207}{3} n + 704$. The polynomial $R(n)$ arises from three special cases for the polynomial coefficients of $\alpha_3, \alpha_2$ and $\alpha_1$. Since $\deg Q(n) = \deg R(n) = 3$, we need 4 successive history eliminations of the form $\alpha_n - \alpha_{n-1}$; combined with the 3 initial terms $\alpha_n, \alpha_{n-1}$ and $\alpha_{n-2}$, this produces the desired order–six constant–coefficient linear recurrence for all $n \geq 6$:

$$\alpha_n = 6\alpha_{n-1} - 11\alpha_{n-2} + 9\alpha_{n-3} - 4\alpha_{n-4} - 4\alpha_{n-5} + \alpha_{n-6}.$$

Consequently, the number of the 321–hexagon avoiding permutations of length $n$ is given by the formula:

$$c_1 R_1^n + c_2 R_2^n + c_3 R_3^n + c_4 R_4^n + c_5 R_5^n + c_6 R_6^n,$$

where the roots and coefficients are listed in the Introduction. This completes the proof of Theorem 3. \qed
The sixth-degree characteristic polynomial of our recurrence relation (9) is irreducible over \( \mathbb{Q} \) and has Galois group \( S_6 \), as calculated by Maple. This means that there are no further algebraic relations among the roots \( R_i \) in (11), and thus we cannot hope for any better closed-form results.

However, our numerical approximations of the roots and coefficients can yield exact values for the number of 321-hex permutations for any fixed length \( n \), exploiting the fact that the value being approximated is known to be an integer. Furthermore, since the two roots of modulus less than 1 make such small contributions, they can be dropped and the following formula is exact for all \( n \):

\[
\alpha_n = [c_3 R_3^n + c_4 R_4^n + c_5 R_5^n + c_6 R_6^n],
\]

where the braces denote rounding to the nearest integer.

5. Extensions and Further Discussion

The study of the octagonal patterns \( P_i \) in the present paper was motivated by their appearance in the representation theory of \( S_n \) via heap-avoidance and thus Kazhdan-Lusztig polynomials and Schubert varieties. We refer to the enumeration of \( S_n(321, P) \) as the “8 \times 8 case”. From a purely combinatorial viewpoint, it is natural to ask what happens in the analogous smaller 6 \times 6 and 4 \times 4 cases whose generalized patterns are depicted in Fig. 12.

**Figure 12.** 6 \times 6 and 4 \times 4 Patterns

To obtain these cases, for each \( i = 1, 2, 3, 4 \) shorten by one the lengths of both \( \mathcal{B}_1(P_i) \) and \( \mathcal{B}_2(P_i) \) by removing a chosen fixed point appearing in all \( P_i \)'s. Thus, define

\[
\mathcal{P}^6 = \{[351624], [356124], [451623], [456123]\} \quad \text{and} \quad \mathcal{P}^4 = \{[2143], [3142], [2413], [3412]\}
\]

to be the families of avoided octagonal patterns in the 6 \times 6 and the 4 \times 4 cases, respectively. Both of these cases lead again to linear recursive relations with constant coefficients. The proofs below follow closely the method described in the 8 \times 8 case, so we leave the details for verification to the reader.

5.1. The 6 \times 6-case.

**Theorem 4.** Let \( \mathcal{H}_n^6 = S_n(321, \mathcal{P}^6) \), and \( \alpha_n = |\mathcal{H}_n^6| \). Then \( \{\alpha_n\} \) satisfies a 6-term linear recursive relation with constant coefficients:

\[
\alpha_{n+1} = 4\alpha_n - 4\alpha_{n-1} + 3\alpha_{n-2} + \alpha_{n-3} - \alpha_{n-4} \quad \text{for all } n \geq 1.
\]

Consequently, for all \( n \geq 1 \),

\[
\alpha_n = c_1 R_1^n + c_2 R_2^n + c_3 R_3^n + c_4 R_4^n + c_5 R_5^n + c_6 R_6^n
\]
where the roots and coefficients are rounded off below to 5 digits after the decimal point:

\[
\begin{align*}
R_1 &\approx -0.49569 & c_1 &\approx 0.63205 \\
R_2 &\approx 0.51154 & c_2 &\approx 0.53110 \\
R_3 &\approx 3.03090 & c_3 &\approx 0.50154 \\
R_4 &\approx 0.47662 - 1.03635i & c_4 &\approx -0.19482 + 0.11092i
\end{align*}
\]

The degree–5 characteristic polynomial of the recurrence is irreducible over \( \mathbb{Q} \) and has the largest possible Galois group, \( S_5 \), as calculated by Maple. We can again drop the small roots \( R_1 \) and \( R_2 \), rounding off the remainder to the nearest integer:

\[(14) \quad \alpha_n = \lfloor c_3 R_3^n + c_4 R_4^n \rfloor \text{ for all } n \geq 1.
\]

The first values of \( \alpha_n \) are: 1, 2, 5, 14, 42, 128, 389, 1179, 3572, 10825, 32810, 99446.

PROOF (Theorem 4): We modify the discussion in the proof for the \( 8 \times 8 \) case. Define the generating function \( h_n(x, l, m) \) with one fewer parameter, thus taking into account only the elements \( M \) and \( L \) of \( \mathfrak{A}_n^6 \). As in Lemma 1, the \( L \)-elements are pattern–free, and therefore \( h_n(x, l, m) = h_{n-1}(x-l, 0, 0) \).

Next, define the sequences \( \alpha_n = h_{n+1}(0,0,0), \beta_n+1 = h_n(1,0,0), \gamma_n = h_{n+1}(2,0,0) \) and \( \delta_n = h_{n+x-2}(x,0,0) \) for \( x \geq 3 \). Note that \( \alpha_n = h_{n+1}(0,0,0) = [\mathfrak{A}_n^6] \) since \( \alpha_n \) is the number of \( w \in \mathfrak{A}_n^6 \) in which the largest element \( n+1 \) is at the end and hence it is pattern–free. Further, the relations among the sequences are:

\[
\begin{align*}
\beta_n &= \alpha_n - \alpha_{n-1} \quad \text{for } n \geq 2; \\
\gamma_n &= \alpha_{n-2} + \beta_{n-2} + \gamma_{n-1} \quad \text{for } n \geq 3; \\
\delta_n &= \alpha_{n-3} + \beta_{n-3} \quad \text{for } n \geq 4.
\end{align*}
\]

Now we are ready to express everything in terms of \( \alpha_n = \sum_{x,l,m} h_n(x, l, m) \):

\[
\alpha_n = 1 + \sum h_n(x, x, m) + \sum h_n(x, x-1, m) + \sum h_n(x, x-2, m) + \sum h_n(x, l, m).
\]

Here the “1” counts the identity permutation: no \( M \)-element in \( w \); in the first sum \( L \) is smaller than all \( X \)-elements (or \( L \) does not exist); in the second sum \( L \) is larger than exactly one \( X \)-element; in the third sum \( L \) is larger than exactly two \( X \)-elements; and in the forth sum \( L \) is larger than three or more \( X \)-elements and hence \( x-3 \geq 1 \).

\[
\begin{align*}
&\Rightarrow \quad \alpha_n = 1 + \sum_{l=1}^{n-1} l \alpha_{n-l-1} + \sum_{l=0}^{n-3} (l+1) \beta_{n-l-1} + \sum_{l=0}^{n-4} (l+1) \gamma_{n-l-1} + \sum_{l=0}^{n-5} (l+2) \delta_{n-l-1} \\
&\Rightarrow \quad \alpha_{n+1} - \alpha_n = \sum_{l=1}^{n} \alpha_{n-l} + \sum_{l=0}^{n-2} \beta_{n-l} + \sum_{l=0}^{n-3} \gamma_{n-l} + \sum_{l=0}^{n-4} (l+1) \delta_{n-l} \\
&\Rightarrow \quad \alpha_{n+2} - 2 \alpha_{n+1} + \alpha_n = \alpha_n + \beta_{n+1} + \gamma_n + \sum_{l=0}^{n-3} \delta_{n-l+1} \\
&\Rightarrow \quad \alpha_{n+3} - 3 \alpha_{n+2} + 2 \alpha_{n+1} = (\beta_{n+2} - \beta_{n+1}) + (\gamma_{n+2} - \gamma_{n+1}) + \delta_{n+2}.
\end{align*}
\]

Conveniently, each summand on the RHS can be expressed in terms of \( \alpha \), including \( \gamma_n - \gamma_{n+1} = \alpha_n + \beta_n = 2 \alpha_n - \alpha_{n-1} \):

\[
\Rightarrow \quad \alpha_{n+3} = 4 \alpha_{n+2} - 4 \alpha_{n+1} + 3 \alpha_n + \alpha_{n-1} - \alpha_{n-2}.
\]
5.2. **The 4 × 4 case.**

**Theorem 5.** Let $\mathcal{H}_n^4 = S_n(321, P^4)$, and $\alpha_n = |\mathcal{H}_n^4|$. Then

\begin{equation}
\alpha_n = (n - 1)^2 + 1 \quad \text{for all } n \geq 1.
\end{equation}

In particular, $\{\alpha_n\}$ satisfies the 4–term linear recursive relation

\begin{equation}
\alpha_{n+1} = 3 \alpha_n - 3 \alpha_{n-1} + \alpha_{n-2} \quad \text{for all } n \geq 1.
\end{equation}

Note that in contrast to the $6 \times 6$ and $8 \times 8$ cases, the characteristic polynomial here factors (completely) over $\mathbb{Q}$: $(x - 1)^3$.

**Proof:** As expected, we define the generating function $h_n(x, m)$ with only two parameters, keeping track only of $M$ in $w \in \mathcal{H}_n^4$. Obviously, the $M$–elements are pattern–free, so $h_n(x, m) = h_{n-m}(x - m, 0)$. Further, letting $\beta_n = h_{n+1}(1, 0)$ and $\gamma_n = h_{n+1}(x, 0)$ for $x \geq 2$, one quickly discovers that $\beta_n = \beta_{n-1} + 1$, so $\beta_n = n + 1$ for $n \geq 3$, and $\gamma_n = n - 1$ for $n \geq 1$. Express $\alpha_n = \sum_{x,m} h_n(x, m)$ as the sum

$$
\alpha_n = 1 + \sum h_n(x, x - 1) + \sum h_n(x, m).
$$

As before, here the “1” counts the identity permutation: no $M$–element in $w$; in the first sum $M$ is larger than exactly one $X$–element; and in the third sum $M$ is larger than two or more $X$–elements and hence $x - 2 \geq m$.

\begin{align*}
\Rightarrow & \quad \alpha_n = 1 + \sum_{m=0}^{n-2} \beta_{n-m-1} + \sum_{m=0}^{n-3} \gamma_{n-m-1} \\
\Rightarrow & \quad \alpha_{n+1} - \alpha_n = \beta_n + \gamma_n \Rightarrow \alpha_{n+2} - 2\alpha_{n+1} + \alpha_n = 2 \\
\Rightarrow & \quad \alpha_{n+3} - 3\alpha_{n+2} + 3\alpha_{n+1} + \alpha_n = 0 \quad \text{for all } n \geq 1. \quad \square
\end{align*}

5.3. **Further Discussion.** Until now, there were relatively few known examples of sets of permutations whose avoidance led to linear, polynomial, or exponential formulas (see [12, 18].) After the successful enumeration of the $4 \times 4$, $6 \times 6$ and the $321$–hex $8 \times 8$ cases, it is tempting to generalize the recursive sequence method of this paper to the corresponding larger sets of $2k \times 2k$ patterns. At this point, it is not surprising to conjecture that all these families yield linear recursive relations with constant coefficients. In fact, when the result of the present paper was publicized, Herbert Wilf requested that many more examples of such “linear” families be found. These and other related questions will be answered positively in a forthcoming paper by Stankova–Frenkel.

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Explicit Enumeration of 321–Hexagon–Avoiding Permutations

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