Complex picture fuzzy $N$-soft sets and their decision-making algorithm

Tahir Mahmood$^1$ · Ubaid ur Rehman$^1$ · Jabbar Ahmmad$^1$

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Abstract

When a decision-maker gives such type of data like yes, no, abstain, or neutral along with 2-dimensional information in grading system, then the existing notions such as fuzzy $N$-soft set (FN-SS), intuitionistic FN-SS are not capable to handle such type of data. To handle such type of data and other problems we introduce the notion of complex picture fuzzy $N$-soft sets (CPFN-SS) which is the fusion of complex picture fuzzy sets (CPFSs) and $N$-soft sets ($N$-SS). CFPN-SS is a vital technique to handle tricky and unreliable information in real-life decision-making problems. We also introduce a novel definition of complex picture fuzzy soft sets which is a combination of complex fuzzy sets and picture fuzzy sets. In CFPN-SS notions of $N$-soft sets, fuzzy $N$-SSs, intuitionistic FN-SSs (IFN-SSs), picture FN-SSs complex FN-SSs, complex IFN-SSs are generalized. Furthermore, we describe some basic operations on CFPN-SSs such as complement, extended union, restricted union, extended intersection, and restricted intersection and illustrate them with the help of examples. After that, we define an algorithm to solve the data in the environment of CFPN-SSs. To show the advantages and usefulness of the proposed CFPN-SSs we represent two real-life applications, i.e., performance assessment of e-waste recycling program and predication about the champions of FIFA world cup 2022 through audience poll of our established model in decision making. In the end, we do a comparison of our established CFPN-SS with some existing methods which shows that CFPN-SS is more effective, better, and generalization of the existing ones.

Keywords

Complex picture fuzzy $N$-soft sets · Complex picture fuzzy sets · $N$-soft sets · Soft sets · Picture fuzzy soft sets · Decision making

1 Introduction

In the real life, there are several difficult issues in management science, engineering, environment, economics, and social science. They are described with dubiousness, fuzziness, and imprecision. We cannot effectively use the old methods to manage these issues because there are different kinds of uncertainties associated with these issues. However, there are several notions like the notion of probability, the notion of FSs (Zadeh 1965), the notion of interval mathematics (Gorzalczyzny 1987), the notion of rough sets (Pawlak et al. 1995; Pawlak 2012; Pawlak and Skowron 2007a, 2007b, 2007c) to be supposed as mathematical instruments to manage uncertainties. In these notions, FSs got a lot of attraction from the researchers and it has been generalized. This notion is valuable apparatus in DM (Riaz and Hashmi 2019). In FSs notion the positive grade belongs to $[0, 1]$. Regardless, because of the uncertainty degree generally, it is not true that the negative grade in FSs is equivalent to 1 minus the positive grade. To clarify this Atanassov (1986) established the notion of intuitionistic FSs (IFSs), which is a reasonable generalization of FSs. Subsequently, Cuong (2014) introduced the notion of picture FSs (PFSs) which is made by the positive grade, neutral grade or abstain grade, negative grade, and refusal grade; accordingly, PFSs can describe the views of decision-makers more precisely than the FS and IFS. Moreover, for solving DM problems Abualigah et al. (2021a) presented the arithmetic optimization algorithm. Abualigah et al. (2021b) introduced aquila optimizer. A comprehensive survey of the grasshopper optimization algorithm was defined by Abualigah and Diabat (2020a). A novel hybrid antlion optimization algorithm for multi-objective task scheduling problems in cloud computing environments was
Another innovative idea complex FS (CFS) is a modification of FSs established by Ramot et al. (2002), where the positive grade \( \rho \) instead of being a real-valued grade belongs to the closed interval [0, 1] is replaced by complex-valued grade \( \phi \). The vital aspect of CFSs is the existence of phase terms and its grade. This provides those CFSs wavelike properties which might result in constructive and destructive intervention relying upon the phase value. Numerous examples are presented in Ramot et al. (2003), which describe the usage of such CFSs. Tamir et al. (2011) defined a new interpretation of CFSs. Akram et al. (2020) initiated the idea of complex FSS (CPFS) by adding the neutral grade and negative grade to the definition of CFS creates CPFS a modification of CFS and complex IFS (CIFS). Mahmood et al. (Mahmood et al. 2020a) presented the notion of complex hesitant FSs (CHFSs). The notion of complex dual hesitant FSs (CDHFSs) was established by Mahmood et al. (2020b).

All the above notions have their drawbacks and restriction to conquer these challenges. Molodtsov (1999) first and foremost established a novel mathematical instrument named SS notion to manage vagueness and uncertainty. This notion is valuable apparatus in numerous applications like game theory, DM, and measurement theory. Maji et al. (2003) developed some algebraic operations in SS. Ali et al. (2009) presented few revised operations in SS.

A great number of authors have grown new models to beat such sort of information and combine the benefits of the established models. These models are created to determine a huge range of DM problems as they have a greater number of benefits than the original ones. The SS model can be mixed with various mathematical models. Maji et al. (2001a) described the notion of fuzzy SS (FSS) by maxing the notion of FS and SS. Feng et al. (2016) presented the FSS approach to the ideal theory of regular AG-groupoids. Additionally, Maji et al. (2009, 2001b) defined the idea of intuitionistic FSS (IFSS) which is the fusion of IFS (Atanassov 1986) and SS. By fusing the interval-valued FS (IVFS) (Gorzalczy and Kerre 2003) and SS, Yang et al. (2009) initiated the idea of the interval-valued FSS (IVFSS). Khan et al. (2019a) presented generalized PFSSs and initiated some improved operations for PFSSs. Khan et al. (2019b) presented the generalized multi-fuzzy bipolar SSs and its application in DM.

N-SSs is the idea of a parameterized representation of objects of a universe that relies upon a finite number of ordered grades. Fatimah et al. (2018) were inspired by real issues. The vast majority of the researchers in SS motivated model performed on the initial binary setting (assessments are either 0 or 1) or otherwise, the situation where assessments are real numbers between 0 and 1 (Alcantud and Laruelle 2014). But we frequently get the information with a non-binary but discrete structure in everyday life, e.g., in the representation of ranking or rating positions or in social judgment system (Ma et al. 2017). From everyday life we can see many examples of rating system where ranking takes shape in the form of the number of stars or dots, etc., which can demonstrate in the shape of natural numbers. Akram et al. (2018) interpreted the notion of FN-SS. The idea of IFN-SS is described by Akram et al. (2019).

There are various generalizations of N-SS to deal with uncertainty and vagueness such as FN-SS, IFN-SS. But when a decision-maker gives the data of the type like yes, no, abstain, or neutral along with 2-dimensional information in the grading system, then these existing notions are not capable to cope with it. To resolve such type of complication we define the notion of CPFN-SS which is a perfect tool to cope with such type of issues. The voting system or audience poll through voting with 2-dimensional information is the best real-life example of such issues, where we divide the voters into 4 parts (vote for, abstain or neutral, vote against, and refusal) and we want 2-dimensional data of every electoral candidate; then, the existing notions are not able to give us any result, but CPFN-SS can solve such type of DM.

Observe that the existing notions such as N-SS, FN-SS, IFN-SS, PFN-SS, CFN-SS, CIFN-SS either deal only with complex positive grade or both complex positive grade and complex negative grade. But from the above situation, we note that complex neutral grade is necessary in many situation like example of voting discuss above, so all above-given notions cannot cope with the complex neutral grade, while the proposed idea is free from above-discussed complexity. The established idea gives full information about existence of ratings and ambiguity under periodic function. It is also handy to obtain optimistic and pessimistic replies by decision-makers. Consequently, for the aim of the modularization of DM issues it gives more pliability when hesitation and intricacy in the parameterizations are take in. We construct the established CPFN-SS significant by providing DM algorithm that appeals to methodologies that have been supported in associated structures. In the notion of CPFN-SS, the complex positive grade, complex neutral grade, and complex negative grade carry 2-dimensional fuzzy information through a magnitude term and phase term. Since the notions PFN-SS, IFN-SS, or FN-SS carry single fuzzy information, CPFN-SS generalize N-SSs, FN-SSs, IFN-SSs, PFN-SSs, CFN-SSs, and CIFN-SSs as follows.

1. If we neglect the complex neutral grade (CN), then CPFN-SS will transform to IFN-SS, and if we neglect the CN and complex negative grade (CNG), then CPFN-SS will transform to CFN-SS.

\[ \text{CPFN-SS} \]
2. If we take the phase term zero in the complex positive grade (CPG), CNG, and CNG, then the CPFN-SS will transform to PFN-SS, and if we neglect the CNG and take the phase term zero in the CPG and CNG, then the CPFN-SS will transform to IFN-SS.

3. If we neglect the CNG and CNG and take the phase term zero in the CPG, then the CPFN-SS will transform to FN-SS, and if we neglect the CPG, CNG, and CNG, then the CPFN-SS will transform to N-SS.

The rest of the article is organized as: In Sect. 2 for better description and understanding we review some fundamental definitions like FSs, IFSs, PFSs, SSs, FSSs, IFSSs, PFSSs, N-SS, and IFN-SS along with some properties. In Sect. 3, we establish the novel notion of CPFS and define its operations. In Sect. 4, we initiated the notion of our model CPFN-SS which is the fusion of CPFS and N-SS. In the same section, we establish some operations of CPFN-SS and explain them by examples. Further, in Sect. 5, we initiate an algorithm to deal with the data in the environment of CPFN-SS and then provide two real-life examples to show the practicability and effectiveness of our established work. The comparison of CPFN-SS with some FN-SS and IFN-SS is presented in Sect. 6. In the end, the conclusion of this article is presented in Sect. 7.

2 Preliminaries

For better understanding, in this section, we recall some underlying definitions like FSs, IFSs, PFSs, SSs, FSSs, IFSSs, PFSSs, N-SS, and IFN-SS. We review the important properties of a few of the above underlying definitions. \( Z \neq \emptyset \) will designate the set of a universe, and \( \emptyset \) will designate a set of attributes in this manuscript.

Definition 1 Zadeh (1965) a FS \( \mathcal{H} \) is elaborated as
\[
\mathcal{H} = \{(z, \rho_{\mathcal{H}}(z)) : z \in Z\},
\]
where \( \rho_{\mathcal{H}} : Z \rightarrow [0, 1] \) is a positive grade. The pair \( \mathcal{H} = (3, \rho_{\mathcal{H}}(3)) \) expresses the fuzzy number (FN).

Definition 2 Zadeh (1965) suppose \( \mathcal{H} = (3, \rho_{\mathcal{H}}(3)) \) and \( \mathcal{J} = (3, \rho_{\mathcal{J}}(3)) \) are two FNs, then
1. \( \mathcal{H} = \mathcal{J} \Leftrightarrow \rho_{\mathcal{H}}(3) = \rho_{\mathcal{J}}(3) \);
2. \( \mathcal{H} \subseteq \mathcal{J} \Leftrightarrow \rho_{\mathcal{H}}(3) \leq \rho_{\mathcal{J}}(3) \);
3. \( \mathcal{H} = (3, 1 - \rho_{\mathcal{H}}(3)) \);
4. \( \mathcal{H} \cup \mathcal{J} = (3, \max(\rho_{\mathcal{H}}(3), \rho_{\mathcal{J}}(3))) \);
5. \( \mathcal{H} \cap \mathcal{J} = (3, \min(\rho_{\mathcal{H}}(3), \rho_{\mathcal{J}}(3))) \).

Definition 3 Atanassov (1986) an IFS elaborated as
\[
\mathcal{H} = \{(z, (\rho_{\mathcal{H}}(z), v_{\mathcal{H}}(z))) : z \in Z\},
\]
where \( \rho_{\mathcal{H}} : Z \rightarrow [0, 1] \) is a positive grade and \( v_{\mathcal{H}} : Z \rightarrow [0, 1] \) is a negative grade with a condition that \( 0 \leq \rho_{\mathcal{H}}(3) + v_{\mathcal{H}}(3) \leq 1 \) for all \( z \in Z \). The pair \( \mathcal{H} = (3, (\rho_{\mathcal{H}}(3), v_{\mathcal{H}}(3))) \) expresses intuitionistic FN (IFN).

Definition 4 Atanassov (1986) suppose \( \mathcal{H} = (3, (\rho_{\mathcal{H}}(3), v_{\mathcal{H}}(3))) \) and \( \mathcal{J} = (3, (\rho_{\mathcal{J}}(3), v_{\mathcal{J}}(3))) \) are two IFNs, then
1. \( \mathcal{H} = \mathcal{J} \Leftrightarrow \rho_{\mathcal{H}}(3) = \rho_{\mathcal{J}}(3), v_{\mathcal{H}}(3) = v_{\mathcal{J}}(3) \);
2. \( \mathcal{H} \subseteq \mathcal{J} \Leftrightarrow \rho_{\mathcal{H}}(3) \leq \rho_{\mathcal{J}}(3), v_{\mathcal{H}}(3) \geq v_{\mathcal{J}}(3) \);
3. \( \mathcal{H}^c = (3, v_{\mathcal{H}}(3), \rho_{\mathcal{H}}(3)) \);
4. \( \mathcal{H} \cup \mathcal{J} = (3, \max(\rho_{\mathcal{H}}(3), \rho_{\mathcal{J}}(3)), \min(v_{\mathcal{H}}(3), v_{\mathcal{J}}(3))) \);
5. \( \mathcal{H} \cap \mathcal{J} = (3, \min(\rho_{\mathcal{H}}(3), \rho_{\mathcal{J}}(3)), \max(v_{\mathcal{H}}(3), v_{\mathcal{J}}(3))) \).

Definition 5 Cuong and Kreinovich (2014) a PFS elaborated as
\[
\mathcal{H} = \{(3, (\rho_{\mathcal{H}}(3), \eta_{\mathcal{H}}(3), v_{\mathcal{H}}(3))) : z \in Z\},
\]
where \( \rho_{\mathcal{H}} : Z \rightarrow [0, 1] \) is a positive grade, \( \eta_{\mathcal{H}} : Z \rightarrow [0, 1] \) is a neutral grade, and \( v_{\mathcal{H}} : Z \rightarrow [0, 1] \) is a negative grade along with a condition that \( 0 \leq \rho_{\mathcal{H}}(3) + \eta_{\mathcal{H}}(3) + v_{\mathcal{H}}(3) \leq 1 \) for all \( z \in Z \). The triplet \( \mathcal{H} = (3, (\rho_{\mathcal{H}}(3), \eta_{\mathcal{H}}(3), v_{\mathcal{H}}(3))) \) expresses a picture FN (PFN).

Definition 6 Cuong and Kreinovich (2014) suppose \( \mathcal{H} = (3, (\rho_{\mathcal{H}}(3), \eta_{\mathcal{H}}(3), v_{\mathcal{H}}(3))) \) and \( \mathcal{J} = (3, (\rho_{\mathcal{J}}(3), \eta_{\mathcal{J}}(3), v_{\mathcal{J}}(3))) \) are two PFNs, then
1. \( \mathcal{H} = \mathcal{J} \Leftrightarrow \rho_{\mathcal{H}}(3) = \rho_{\mathcal{J}}(3), \eta_{\mathcal{H}}(3) = \eta_{\mathcal{J}}(3), v_{\mathcal{H}}(3) = v_{\mathcal{J}}(3) \);
2. \( \mathcal{H} \subseteq \mathcal{J} \Leftrightarrow \rho_{\mathcal{H}}(3) \leq \rho_{\mathcal{J}}(3), \eta_{\mathcal{H}}(3) \leq \eta_{\mathcal{J}}(3), v_{\mathcal{H}}(3) \geq v_{\mathcal{J}}(3) \);
3. \( \mathcal{H}^c = (3, \eta_{\mathcal{H}}(3), \rho_{\mathcal{H}}(3)) \);
4. \( \mathcal{H} \cup \mathcal{J} = (3, \max(\rho_{\mathcal{H}}(3), \rho_{\mathcal{J}}(3)), \min(\eta_{\mathcal{H}}(3), \eta_{\mathcal{J}}(3)), \min(v_{\mathcal{H}}(3), v_{\mathcal{J}}(3))) \);
5. \( \mathcal{H} \cap \mathcal{J} = (3, \min(\rho_{\mathcal{H}}(3), \rho_{\mathcal{J}}(3)), \max(\eta_{\mathcal{H}}(3), \eta_{\mathcal{J}}(3)), \min(v_{\mathcal{H}}(3), v_{\mathcal{J}}(3))) \).

Definition 7 Molodtsov (1999) A pair \((\Delta, \mathcal{R})\) is called a SS over \( Z \) where \( \Delta : \mathcal{R} \rightarrow P(Z) \), be set-valued function and \( \mathcal{R} \subseteq \beta \).

Definition 8 Ali et al. (2009) Suppose \((\Delta_1, \mathcal{R}_1)\) and \((\Delta_2, \mathcal{R}_2)\) are two SSs with \( \mathcal{R}_1 \cap \mathcal{R}_2 \neq \emptyset \), then their restricted union and intersection are interpreted as:
\( (\Delta_1, \mathcal{R}_1) \cup_r (\Delta_2, \mathcal{R}_2) = (\mathcal{R}, \mathcal{R}_1 \cap \mathcal{R}_2) \) \( \forall r \in \mathcal{R}_1 \cap \mathcal{R}_2 \Rightarrow r' = \Delta_1(r) \cup \Delta_2(r) \);
\( (\Delta_1, \mathcal{R}_1) \cap_r (\Delta_2, \mathcal{R}_2) = (\mathcal{R}, \mathcal{R}_1 \cap \mathcal{R}_2) \) where \( g(r) = \Delta_1(r) \cap \Delta_2(r) \forall r \in \mathcal{R}_1 \cap \mathcal{R}_2 \).

Definition 9 Ali et al. (2009) Suppose \((\Delta_1, \mathcal{R}_1)\) and \((\Delta_2, \mathcal{R}_2)\) are two SSs with \( \mathcal{R}_1 \cap \mathcal{R}_2 \neq \emptyset \), then their extended union and intersection are interpreted as:
Definition 10 Maji et al. (2001a) A pair \((\Delta_1, \mathcal{R}_1) \cup_r (\Delta_2, \mathcal{R}_2) = (f, \mathcal{R}_1 \cup \mathcal{R}_2)\)
where \(\forall r \in \mathcal{R}_1 \cup \mathcal{R}_2 \Rightarrow f(r)\)
\[
= \begin{cases} \\
\Delta_1(r) & \text{if } r \in \mathcal{R}_1 \setminus \mathcal{R}_2 \\
\Delta_2(r) & \text{if } r \in \mathcal{R}_2 \setminus \mathcal{R}_1 \\
\Delta_1(r) \cup \Delta_2(r) & \text{if } r \in \mathcal{R}_1 \cup \mathcal{R}_2 \\
\end{cases}
\]
\[(\Delta_1, \mathcal{R}_1) \cap_r (\Delta_2, \mathcal{R}_2) = (g, \mathcal{R}_1 \cap \mathcal{R}_2)\)
where \(\forall r \in \mathcal{R}_1 \cup \mathcal{R}_2 \Rightarrow g(r)\)
\[
= \begin{cases} \\
\Delta_1(r) & \text{if } r \in \mathcal{R}_1 \setminus \mathcal{R}_2 \\
\Delta_2(r) & \text{if } r \in \mathcal{R}_2 \setminus \mathcal{R}_1 \\
\Delta_1(r) \cap \Delta_2(r) & \text{if } r \in \mathcal{R}_1 \cup \mathcal{R}_2 \\
\end{cases}
\]
\[
\Gamma_1 \cup_r \Gamma_2 = (\Delta_1, \mathcal{R}_1, N_1) \cup_r (\Delta_2, \mathcal{R}_2, N_2) = (\mathcal{A}, \mathcal{R}_1 \cup \mathcal{R}_2, \max(N_1, N_2))
\]
where \(\mathcal{A}(r) = \begin{cases} \\
\Delta_1(r) & \text{if } r \in \mathcal{R}_1 \setminus \mathcal{R}_2 \\
\Delta_2(r) & \text{if } r \in \mathcal{R}_2 \setminus \mathcal{R}_1 \\
(3, x_r) & \text{such that } x_r = \max (x^1_r, x^2_r) \text{, where } (3, x^1_r) \in \Delta_1(r) \text{ and } (3, x^2_r) \in \Delta_2(r)
\end{cases}
\]
\[
\Gamma_1 \cap_r \Gamma_2 = (\Delta_1, \mathcal{R}_1, N_1) \cap_r (\Delta_2, \mathcal{R}_2, N_2) = (\mathcal{L}, \mathcal{R}_1 \cap \mathcal{R}_2, \max(N_1, N_2))
\]
where \(\mathcal{L}(r) = \begin{cases} \\
\Delta_1(r) & \text{if } r \in \mathcal{R}_1 \setminus \mathcal{R}_2 \\
\Delta_2(r) & \text{if } r \in \mathcal{R}_2 \setminus \mathcal{R}_1 \\
(3, x_r) & \text{such that } x_r = \min (x^1_r, x^2_r) \text{, where } (3, x^1_r) \in \Delta_1(r) \text{ and } (3, x^2_r) \in \Delta_2(r)
\end{cases}
\]

Definition 11 Maji et al. (2001b) A pair \((\Delta'', \mathcal{R})\) characterizes an IFSS over \(Z\) if \(\Delta'' : \mathcal{R} \rightarrow \text{IFS}(Z), \mathcal{R} \subseteq \mathcal{B}\), where \(\text{IFS}(Z)\) signifies the set of all IFSSs over \(Z\).

Definition 12 Khan et al. (2019a) Suppose \(\text{PFS}(Z)\) is a set of all PFSs over \(Z\), then a pair \((\Delta''', \mathcal{R})\) is said to be PFS over \(Z\), where \(\Delta''' : \mathcal{R} \rightarrow \text{PFS}(Z), \mathcal{R} \subseteq \mathcal{B}\).

Definition 13 Fatimah et al. (2018) Suppose a universal set \(Z\) and the attributes set \(\mathcal{B}\), a triplet \((\Delta, \mathcal{P}, N)\) is called N-SS over \(Z\), where \(\Delta : \mathcal{R} \rightarrow 2^{Z \times X = P(Z \times X)}, \mathcal{R} \subseteq \mathcal{B}\), with the property that for each \(r \in \mathcal{R}\) and \(\exists\) a unique \((3, x_r) \in P(Z \times X)\) such that \((3, x_r) \in \Delta(r), 3 \in Z, x_r \in X = \{0, 1, 2, \ldots, N - 1\}\) be ordered grades set.

Definition 14 Yang et al. (2009) suppose \(\Gamma_1 = (\Delta_1, \mathcal{R}_1, N_1)\) and \(\Gamma_2 = (\Delta_2, \mathcal{R}_2, N_2)\) are two N-SSs with \(\mathcal{R}_1 \cap \mathcal{R}_2 \neq \emptyset\), then their restricted union and intersection are interpreted as:

Definition 15 Fatimah et al. (2018) Suppose \(\Gamma_1 = (\Delta_1, \mathcal{R}_1, N_1)\) and \(\Gamma_2 = (\Delta_2, \mathcal{R}_2, N_2)\) are two N-SSs, then their extended union and intersection are interpreted as:

3 A novel notion of complex picture fuzzy sets
In this part of the article, we establish a novel definition of CPFSs which is the fusion of CFSs and PFSs. We also establish some properties like a complement, union, and intersection.

Definition 17 A CPFS \(\mathcal{H}\) is elaborated by:
\[
\mathcal{H} = \{(3, (\phi_\mathcal{H}(3), \varphi_\mathcal{H}(3), \psi_\mathcal{H}(3))) : 3 \in Z\}
\]
where \(\phi_\mathcal{H}(3) = \rho_\mathcal{H}(3)e^{i2\pi N(3)}\), \(\varphi_\mathcal{H}(3) = \eta_\mathcal{H}(3)e^{i2\pi N(3)}, \psi_\mathcal{H}(3) = \nu_\mathcal{H}(3)e^{i2\pi N(3)}\) expresses the positive grade, neutral or abstain grade, and negative grade in the form of a...
complex number with some rules that are $\rho_H(\bar{z}), \eta_H(\bar{z}), v_H(\bar{z}) \in [0, 1]$ are the amplitude terms and $\alpha_H(\bar{z}), \beta_H(\bar{z}), \gamma_H(\bar{z}) \in [0, 1]$ are the phase terms, respectively, by using the conditions $0 \leq \rho_H(\bar{z}) + \eta_H(\bar{z}) + v_H(\bar{z}) \leq 1$ and $0 \leq \alpha_H(\bar{z}) + \beta_H(\bar{z}) + \gamma_H(\bar{z}) \leq 1$ for all $\bar{z} \in \mathbb{Z}$. The triplet $\mathcal{H} = (\phi_H(\bar{z}), \varphi_H(\bar{z}), \psi_H(\bar{z}))$ expressed the complex PFN (CPFN).

Definition 18 Suppose $\mathcal{H} = (\bar{z}, (\rho_H(\bar{z}), \eta_H(\bar{z}), v_H(\bar{z})))$ and $\mathcal{J} = (\bar{z}, (\rho_J(\bar{z}), \eta_J(\bar{z}), v_J(\bar{z})))$ are two CPFNs, then

1. $\mathcal{H} ^ c = (\bar{z}, (\psi_H(\bar{z}), \varphi_H(\bar{z}), \phi_H(\bar{z}))) = (\bar{z}, (v_H(\bar{z})e^{i2\pi(\gamma_H(\bar{z}))}, \eta_H(\bar{z})e^{i2\pi(\beta_H(\bar{z}))}, \rho_H(\bar{z})e^{i2\pi(\alpha_H(\bar{z}))}));$

2. $\mathcal{H} \cup \mathcal{J} = \left( \bar{z}, \left( \max(\rho_H(\bar{z}), \rho_J(\bar{z})), e^{i2\pi(\max(\beta_H(\bar{z}), \beta_J(\bar{z})))), \min(\eta_H(\bar{z}), \eta_J(\bar{z})), e^{i2\pi(\min(\gamma_H(\bar{z}), \gamma_J(\bar{z})))} \right), \min(v_H(\bar{z}), v_J(\bar{z})), e^{i2\pi(\min(\alpha_H(\bar{z}), \alpha_J(\bar{z})))} \right) \right)$

3. $\mathcal{H} \cap \mathcal{J} = \left( \bar{z}, \left( \min(\rho_H(\bar{z}), \rho_J(\bar{z})), e^{i2\pi(\min(\beta_H(\bar{z}), \beta_J(\bar{z})))), \min(\eta_H(\bar{z}), \eta_J(\bar{z})), e^{i2\pi(\min(\gamma_H(\bar{z}), \gamma_J(\bar{z})))} \right), \max(v_H(\bar{z}), v_J(\bar{z})), e^{i2\pi(\max(\alpha_H(\bar{z}), \alpha_J(\bar{z})))} \right) \right) \right)$

4 The notion of complex picture fuzzy $N$-soft sets

In this segment of the manuscript, we define the idea of CPFN-SS and also define some fundamental properties of CPFN-SS like a complement, intersection, union, etc., along with examples. As we know that our proposed method is the generalization of CIFN-SS, CFN-SS, PFN-SS, IFN-SS, FN-SS, and N-SS. But it cannot generalize spherical FN-SS, T-spherical FN-SS and cannot handle such kind of data. Throughout this article, the universal set be designated by $\mathbb{Z}$, the set of attributes be designated by $\mathbb{B}$, and the set of ordered grades be designated by $\mathbb{X} = \{0, 1, 2, \ldots, N - 1\}$, and $N$-SS over $\mathbb{Z}$ by $\Gamma = (\Delta, \mathbb{R}, \mathbb{N})$, where $\mathbb{R} \subset \mathbb{B}$.

Definition 19 Suppose $\mathbb{Z}$ is a universal set with $\mathbb{R} \subset \mathbb{B}$ is a set of attributes, a pair $(\Psi, \Gamma)$ is called CPFN-SS over $\mathbb{Z}$, where $\Gamma = (\Delta, \mathbb{R}, \mathbb{N})$ be an $N$-SS over $\mathbb{Z}$ with $N \in \{2, 3, \ldots\}$ and $\Psi$ is a mapping from $\mathbb{R}$ to $C PF(\mathbb{Z})$, i.e., $\Psi : \mathbb{R} \rightarrow CPF(\mathbb{Z})$, where $C PF(\mathbb{Z})$ be the set of all CPFNs over $\mathbb{Z}$.

In Definition 19, we assign a CPFS on the image of the parameter under the mapping $\Delta$ by a mapping $\Psi$ to each parameter. Thus, for all $r \in \mathbb{R}$ and $\bar{z} \in \mathbb{Z}$, a unique $(\bar{z}, x_r) \in \mathbb{Z} \times \mathbb{X}$ such that $x_r \in \mathbb{X}$ and $((\bar{z}, x_r), (\Psi, \Gamma)) = (\phi_H(\bar{z}), \varphi_H(\bar{z}), \psi_H(\bar{z})) = (\rho_H(\bar{z})e^{i2\pi(\alpha_H(\bar{z}))}, \eta_H(\bar{z})e^{i2\pi(\beta_H(\bar{z}))}, \gamma_H(\bar{z})e^{i2\pi(\gamma_H(\bar{z}))})) \in \Psi(\bar{z})$, which is the notation that summarizes to $\Psi(\bar{z}) = \Psi(\bar{z}, x_r)$.

Example 1 Suppose $\mathbb{Z} = \{\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4\}$ is a set of attributes and $\mathbb{R} = \{r_1, r_2, r_3\}$ is a set of attributes. An expert allots grades to these alternatives based on attributes $r_1, r_2$ and $r_3$ in the shape of circle stars and circle hole which is represented in Table 1 where four circle stars are for “Excellent,” three circle stars are for “very good,” two circle stars are for “good,” one circle star is for “normal,” circle hole is for “poor.”

We can link the set $\mathbb{X} = \{0, 1, 2, 3, 4\}$ with the circle stars and circle hole displayed in Table 1 by saying that 4 characterizes “$\bigcirc \bigcirc \bigcirc \bigcirc$,” 3 characterizes “$\bigcirc \bigcirc \bigcirc$,” 2 characterizes “$\bigcirc \bigcirc$,” 1 signifies “$\bigcirc$,” and 0 characterizes “$\bigcirc$.” Then the 5-SS ($\Delta, \mathbb{R}, \mathbb{N}$) is defined as $\Delta(1) = \{(\bar{z}_1, 2), (\bar{z}_2, 3), (\bar{z}_3, 1), (\bar{z}_4, 1)\}$

| $\mathbb{Z}$  | $r_1$  | $r_2$  | $r_3$  |
|---|---|---|---|
| $\bar{z}_1$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc \bigcirc \bigcirc \bigcirc$ |
| $\bar{z}_2$ | $\bigcirc \bigcirc$ | $\bigcirc$ | $\bigcirc \bigcirc \bigcirc$ |
| $\bar{z}_3$ | $\bigcirc \bigcirc \bigcirc$ | $\bigcirc \bigcirc$ | $\bigcirc \bigcirc \bigcirc \bigcirc$ |
| $\bar{z}_4$ | $\bigcirc \bigcirc \bigcirc \bigcirc$ | $\bigcirc \bigcirc \bigcirc$ | $\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ |
Table 2 5-SS in tabular form

| (Δ, R, 5) | r1 | r2 | r3 |
|-----------|----|----|----|
| Δ(1)     | 2  | 0  | 4  |
| Δ(2)     | 3  | 1  | 0  |
| Δ(3)     | 1  | 4  | 1  |
| Δ(4)     | 1  | 2  | 3  |

In Table 1 we display the tabular form of the 5-SS (Table 2).

ϕ, φ, and ψ will satisfy the following grading criteria. If the grade is 0, then $0.0 \leq \phi(\frac{1}{2}) \leq 0.2$, (i.e. $0.0 \leq \phi(\frac{1}{2}) \leq 0.2$) and $0 \leq \phi(\frac{1}{2}) + \phi(\frac{1}{2}) + \psi(\frac{1}{2}) \leq 1$, where $0 \leq \phi(\frac{1}{2}) + \psi(\frac{1}{2}) \leq 1$, and $\phi(\frac{1}{2}) + \psi(\frac{1}{2}) \leq 1$ for each $\frac{1}{2} \in \mathbb{Z}$. Similarly, if the grade is 1 then $0.2 \leq \phi(\frac{1}{2}) \leq 0.6$, if the grade is 2 then $0.4 \leq \phi(\frac{1}{2}) \leq 0.6$, if the grade is 3 then $0.6 \leq \phi(\frac{1}{2}) \leq 0.8$, and if the grade is 4 then $0.8 \leq \phi(\frac{1}{2}) \leq 1.0$. The CPF5-SS is established below:

**Definition 20** A CPFN-SS $(\Psi, \Gamma = (\Delta, R, N))$ over $\mathbb{Z}$ is called an efficient if for some $x_0 \in \mathbb{Z}, y_k \in \mathbb{R}$ we have $\Psi(x_k) = \left(\frac{3}{2}, N - 1\right), 1,0.0e^{2\pi(0.0)}, 0.0e^{2\pi(0.0)}, 0.0e^{2\pi(0.0)}$.

**Example 2** Suppose a CPF5-SS over $\mathbb{Z}$ in tabular representation which is demonstrated in Table 4. This CPF5-SS is efficient because for $y_k \in \mathbb{Z}$ and $x_0 \in \mathbb{R}$ we have $\Psi(x_k) = (4, 1.0e^{2\pi(1.0)}, 0.0e^{2\pi(0.0)}, 0.0e^{2\pi(0.0)})$.

**Definition 21** An efficient CPFN-SS $(\Psi, \Gamma = (\Delta, R, N))$ is called minimzed efficiently signed by $(\Psi, \Gamma = (\Delta, R, N))$ and illustrated as $\Psi^\prime = \max \Delta(x_k) \left(\frac{3}{2}, N - 1\right), 1,0.0e^{2\pi(1.0)}, 0.0e^{2\pi(0.0)}, 0.0e^{2\pi(0.0)}$ for some $x_0$ and $x_k \in \mathbb{R}, y_k \in \mathbb{Z}$.

**Definition 22** Suppose a CPFN-SS $(\Psi, \Gamma = (\Delta, R, N))$ on $\mathbb{Z}$, then the CPF weak complement of the CPFN-SS is signed by $(\Psi, \Gamma = (\Delta, R, N))$, where $(\Delta^T, R, N)$ is the weak complement of $N-SS$ that is, $\Delta(x_k) \cap \Delta^T(x_k) = \emptyset$ $\forall x_k \in \mathbb{R}$, and $\Psi^\prime(x_k) = \left(\frac{3}{2}, N - 1\right), 1,0.0e^{2\pi(1.0)}, 0.0e^{2\pi(0.0)}, 0.0e^{2\pi(0.0)}$ for some $y_k \in \mathbb{Z}, y_k \in \mathbb{R}$. $\Psi^\prime(x_k) = \left(\frac{3}{2}, N - 1\right), 1,0.0e^{2\pi(1.0)}, 0.0e^{2\pi(0.0)}, 0.0e^{2\pi(0.0)}$ $\forall x_k \in \mathbb{R}$.

**Example 3** Suppose a CPFN-SS $(\Psi, \Gamma = (\Delta, R, 5))$ of

In Table 3 we display the tabular form of CPF5-SS.

Every CPFN-SS can be expressed as CPF(N + 1)-SS. For instance, in Example 1 we have CPF5-SS, but we can also call it CPF6-SS.
Table 3 The tabular representation of CPF5-SS of Example 1

| (Ψ, (Δ, ℝ, 5)) | r₁ | r₂ | r₃ |
|----------------|----|----|----|
| δ₁ | 2, (0.4e²(0.5), 0.2e²(0.01), 0.3e²(0.15)) | 0, (0.05e²(0.18), 0.27e²(0.2)) | 4, (0.84e²(0.8), 0.09e²(0.12)) |
| δ₂ | 3, (0.65e²(0.7), 0.1e²(0.1), 0.1e²(0.05)) | 1, (0.32e²(0.26), 0.14e²(0.09), 0.5e²(0.6)) | 0, (0.15e²(0.17), 0.4e²(0.5), 0.3e²(0.3)) |
| δ₃ | 1, (0.3e²(0.35), 0.4e²(0.2), 0.2e²(0.3)) | 4, (0.9e²(0.85), 0.04e²(0.08), 0.02e²(0.03)) | 1, (0.2e²(0.5), 0.6e²(0.3), 0.19e²(0.16)) |
| δ₄ | 1, (0.35e²(0.45), 0.3e²(0.03), 0.25e²(0.1)) | 2, (0.2e²(0.23), 0.15e²(0.06)) | 3, (0.75e²(0.65), 0.1e²(0.01), 0.1e²(0.02)) |

Table 4 The tabular form of the CPF5-SS is presented in Example 2

| (Ψ, (Δ, ℝ, 5)) | r₁ | r₂ | r₃ |
|----------------|----|----|----|
| δ₁ | 0, (0.05e²(0.18), 0.6e²(0.05), 0.27e²(0.2)) | 2, (0.2e²(0.5), 0.3e²(0.15)) | 4, (0.1e²(1.0), 0.0e²(0.0), 0.0e²(0.0)) |
| δ₂ | 1, (0.32e²(0.26), 0.14e²(0.09), 0.5e²(0.6)) | 3, (0.65e²(0.7), 0.1e²(0.01), 0.1e²(0.05)) | 0, (0.15e²(0.17), 0.4e²(0.5), 0.3e²(0.3)) |
| δ₃ | 1, (0.3e²(0.35), 0.4e²(0.2), 0.2e²(0.3)) | 4, (0.9e²(0.85), 0.04e²(0.08), 0.02e²(0.03)) | 1, (0.2e²(0.5), 0.6e²(0.3), 0.19e²(0.16)) |
| δ₄ | 1, (0.35e²(0.45), 0.3e²(0.03), 0.25e²(0.1)) | 2, (0.2e²(0.23), 0.15e²(0.06)) | 3, (0.75e²(0.65), 0.1e²(0.01), 0.1e²(0.02)) |

Table 5 The tabular form of CPF weak complement of CPF5-SS of Example 1

| (Ψ⁺, (Δ⁺, ℝ, 5)) | r₁ | r₂ | r₃ |
|-------------------|----|----|----|
| δ₁ | 4, (0.3e²(0.15), 0.2e²(0.01), 0.4e²(0.05)) | 3, (0.27e²(0.2), 0.6e²(0.05), 0.05e²(0.18)) | 2, (0.09e²(0.12), 0.06e²(0.07), 0.84e²(0.8)) |
| δ₂ | 4, (0.1e²(0.05), 0.1e²(0.01), 0.65e²(0.07)) | 3, (0.5e²(0.6), 0.14e²(0.09), 0.32e²(0.26)) | 2, (0.3e²(0.3), 0.4e²(0.5), 0.15e²(0.17)) |
| δ₃ | 0, (0.2e²(0.3), 0.4e²(0.2), 0.3e²(0.35)) | 3, (0.02e²(0.03), 0.04e²(0.08), 0.02e²(0.05)) | 2, (0.19e²(0.16), 0.6e²(0.3), 0.2e²(0.5)) |
| δ₄ | 0, (0.25e²(0.1), 0.3e²(0.03), 0.35e²(0.45)) | 3, (0.15e²(0.06), 0.2e²(0.23), 0.55e²(0.65)) | 2, (0.1e²(0.01), 0.1e²(0.01), 0.75e²(0.65)) |

Example 24 Suppose a CPF5-SS (Ψ, Γ = (Δ, ℝ, 5)) of Example 1, then the top CPF weak complement of CPF5-SS is demonstrated in tabular representation in Table 6.

Definition 25 For a CPFN-SS (Ψ, Γ = (Δ, ℝ, N)), the bottom CPF weak complement of a CPFN-SS (Ψ, Γ = (Δ, ℝ, 5)) is (Ψ⁺, Γ⁺ = (Δ⁺, ℝ, 5)) where,
Table 6 The tabular representation of top CPF week complement of CPF5-SS of Example 1

| \((\Psi^*, (A^T, R, 5))\) | \(r_1\) | \(r_2\) | \(r_3\) |
|--------------------------|--------|--------|--------|
| \(\delta_1\) | 4, \((0.3e^{2e(0.15)}, 0.2e^{2e(0.1)}, 0.4e^{2e(0.5)})\) | 4, \((0.27e^{2e(0.2)}, 0.6e^{2e(0.5)}, 0.05e^{2e(0.16)})\) | 0, \((0.09e^{2e(0.12)}, 0.06e^{2e(0.1)}, 0.84e^{2e(0.18)})\) |
| \(\delta_2\) | 4, \((0.1e^{2e(0.05)}, 0.1e^{2e(0.1)}, 0.65e^{2e(0.7)})\) | 4, \((0.5e^{2e(0.6)}, 0.14e^{2e(0.09)}, 0.32e^{2e(0.26)})\) | 0, \((0.3e^{2e(0.3)}, 0.4e^{2e(0.5)}, 0.15e^{2e(0.17)})\) |
| \(\delta_3\) | 4, \((0.2e^{2e(0.3)}, 0.4e^{2e(0.2)}, 0.3e^{2e(0.35)})\) | 0, \((0.02e^{2e(0.03)}, 0.04e^{2e(0.08)}, 0.9e^{2e(0.85)})\) | 4, \((0.19e^{2e(0.16)}, 0.6e^{2e(0.3)}, 0.2e^{2e(0.5)})\) |
| \(\delta_4\) | 4, \((0.25e^{2e(0.1)}, 0.3e^{2e(0.3)}, 0.35e^{2e(0.45)})\) | 4, \((0.15e^{2e(0.06)}, 0.2e^{2e(0.23)}, 0.55e^{2e(0.65)})\) | 0, \((0.1e^{2e(0.2)}, 0.1e^{2e(0.1)}, 0.75e^{2e(0.65)})\) |

Table 7 The tabular representation of bottom CPF week complement of CPF5-SS of Example 1

| \((\Psi^*, (A^B, R, 5))\) | \(r_1\) | \(r_2\) | \(r_3\) |
|--------------------------|--------|--------|--------|
| \(\delta_1\) | 0, \((0.3e^{2e(0.15)}, 0.2e^{2e(0.1)}, 0.4e^{2e(0.5)})\) | 4, \((0.27e^{2e(0.2)}, 0.6e^{2e(0.5)}, 0.05e^{2e(0.16)})\) | 0, \((0.09e^{2e(0.12)}, 0.06e^{2e(0.1)}, 0.84e^{2e(0.18)})\) |
| \(\delta_2\) | 0, \((0.1e^{2e(0.05)}, 0.1e^{2e(0.1)}, 0.65e^{2e(0.7)})\) | 0, \((0.5e^{2e(0.6)}, 0.14e^{2e(0.09)}, 0.32e^{2e(0.26)})\) | 4, \((0.3e^{2e(0.3)}, 0.4e^{2e(0.5)}, 0.15e^{2e(0.17)})\) |
| \(\delta_3\) | 0, \((0.2e^{2e(0.3)}, 0.4e^{2e(0.2)}, 0.3e^{2e(0.35)})\) | 0, \((0.02e^{2e(0.03)}, 0.04e^{2e(0.08)}, 0.9e^{2e(0.85)})\) | 0, \((0.19e^{2e(0.16)}, 0.6e^{2e(0.3)}, 0.2e^{2e(0.5)})\) |
| \(\delta_4\) | 0, \((0.25e^{2e(0.1)}, 0.3e^{2e(0.3)}, 0.35e^{2e(0.45)})\) | 0, \((0.15e^{2e(0.06)}, 0.2e^{2e(0.23)}, 0.55e^{2e(0.65)})\) | 0, \((0.1e^{2e(0.2)}, 0.1e^{2e(0.1)}, 0.75e^{2e(0.65)})\) |

\(\Delta^B(r_k) = \begin{cases} 0 & \text{if } x_k > 0 \\ N - 1 & \text{if } x_k = 0 \end{cases} \quad (5)\)

Example 5 Suppose a CPF5-SS \((\Psi^*, \Gamma = (\Delta, R, 5))\) of Example 1, then the bottom CPF weak complement of CPF5-SS is demonstrated in tabular representation in Table 7.

Definition 26 Suppose \((\Psi_1, \Gamma_1 = (\Delta_1, R_1, N_1))\) and \((\Psi_2, \Gamma_2 = (\Delta_2, R_2, N_2))\) are two CPFN-SS over \(Z\), then their restricted intersection is signified by \((\Omega, \Gamma_1 \cap \Theta, \Gamma_2)\), where \(\Gamma_1 \cap \Theta, \Gamma_2 = (\delta, R_1 \cap R_2, \min(N_1, N_2))\) \(\forall x_k \in R_1 \cap R_2\), \(\delta \in Z\), \((\vec{3}, \vec{1}^{2e}x_k)\) \(\sigma \in \Omega(k) \leftrightarrow x_k = \min x_k\), \(\sigma = \min(\phi_1(\vec{3}, \vec{1}^{2e}x_k))\), \(\phi_2(\vec{3}, \vec{2}^{2e}x_k)\) = \(\min(\rho_1(\vec{3}, \vec{1}^{2e}x_k))\), \(\phi_2(\vec{3}, \vec{2}^{2e}x_k)\) = \(\min(\rho_1(\vec{3}, \vec{1}^{2e}x_k))\), \(q = \min(\phi_1(\vec{3}, \vec{1}^{2e}x_k), \phi_2(\vec{3}, \vec{2}^{2e}x_k))\)

Example 6 Suppose \(Z = \{3_1, 3_2, 3_3\}\) is a universal set and \(R_1, R_2 \subseteq Z\), where \(R_1 = \{r_1, r_2, r_3\}\), and \(R_2 = \{r_1, r_2, r_4\}\). Suppose \((\Psi_1, \Gamma_1 = (\Delta_1, R_1, 5))\) and \((\Psi_2, \Gamma_2 = (\Delta_2, R_2, 4))\) are CPF5-SS and CPF4-SS over \(Z\), respectively. The tabular representation of CPF5-SS \((\Psi_1, \Gamma_1 = (\Delta_1, R_1, 5))\) is demonstrated in Table 8, and tabular representation of CPF4-SS \((\Psi', \Gamma_2 = (\Delta_2, R_2, 4))\) is demonstrated in Table 9. Then their restricted intersection is demonstrated in Table 10.
Table 8 The CPF5-SS of Example 6

| (Ψ, (A, R, 5)) | r₁ | r₂ | r₃ |
|----------------|----|----|----|
| δ₁            |    | 2  | 0  | 4  |
| δ₂            |    | 3  | 1  | 0  |
| δ₃            |    | 1  | 4  | 1  |

Table 9 The CPF4-SS of Example 6

| (Ψ, (A, R, 4)) | r₁ | r₂ | r₃ |
|----------------|----|----|----|
| δ₁            |    | 1  | 0  | 2  |
| δ₂            |    | 2  | 0  | 1  |
| δ₃            |    | 3  | 0  | 2  |

Table 10 The restricted intersection of CPF5-SS and CPF4-SS of Example 6

(Ω, Γ₁ ∩ R Γ₂, 4) | r₁ | r₂ |
|-----------------|----|----|
| δ₁              | 1  | 0  |
| δ₂              | 2  | 0  |
| δ₃              | 3  | 0  |

Definition 27 Suppose (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) and (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) are two CPF5-SS over Z, then their restricted union is signified by (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) ∪ R (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) and is defined as (Ω, Γ₁ ∪ R Γ₂), where Γ₁ ∪ R Γ₂ = {e, R₁ ∩ R₂, max(N₁, N₂)} ∪ R (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) and is defined as (Ω, Γ₁ ∪ R Γ₂), where Γ₁ ∪ R Γ₂ = {e, R₁ ∩ R₂, max(N₁, N₂)} ∪ R (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) and is defined as (Ω, Γ₁ ∪ R Γ₂), where

Example 7 Suppose Z = {z₁, z₂, z₃} is a universal set and R₁, R₂ ⊆ Z, where R₁ = {v₁, v₂, v₃} and R₂ = {v₁, v₂, v₄}. Suppose (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) and (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) are CPF5-SS and CPF4-SS over Z, respectively. The tabular representation of CPF5-SS (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) is demonstrated in Table 8, and the tabular representation of CPF4-SS (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) is demonstrated in Table 9. Then their restricted union is demonstrated in Table 11.

Definition 28 Suppose (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) and (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) are two CPF5-SS over Z, then their extended intersection is signified by (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) ∩ C (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) and is defined as
(Ω, Γ₁ ∩c Γ₂), where Γ₁ ∩c Γ₂ = Table 9. Then their extended intersection is demonstrated in Table 12.

**Definition 29** Suppose (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) and (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) are two CPFN-SS over Ω, then their extended union is signified by (Ψ₁, Γ₁ = (Δ₁, R₁, N₁)) ∪c (Ψ₂, Γ₂ = (Δ₂, R₂, N₂)) and is defined as (Ω, Γ₁ ∪c Γ₂), where Γ₁ ∪c Γ₂ = (∈, R₁ ∩ R₂, max(N₁, N₂)) and

\[
\Omega(\tau) = \begin{cases} 
\Psi_1(\tau) & \text{if } \tau \in R_1 - R_2 \\
\Psi_2(\tau) & \text{if } \tau \in R_2 - R_1 \\
\left(\delta_j, x_{\tau_k}\right), \sigma, \phi, \tau & \text{such that } x_{\tau_k} = \min(x_{\Omega_k}, x_{\Omega_2}), \sigma = \min\left(\phi_1(3_j, x_{\Omega_k}), \phi_2(3_j, x_{\Omega_2})\right) = \left(\min\left(\rho_1(3_j, x_{\Omega_k}), \rho_2(3_j, x_{\Omega_2})\right) e^{i2\pi \min(\sigma_1(3_j, x_{\Omega_k}), \sigma_2(3_j, x_{\Omega_2}))}\right) \\
\phi = \min\left(\psi_1(3_j, x_{\Omega_k}), \psi_2(3_j, x_{\Omega_2})\right) = \left(\min\left(\eta_1(3_j, x_{\Omega_k}), \eta_2(3_j, x_{\Omega_2})\right) e^{i2\pi \min(\phi_1(3_j, x_{\Omega_k}), \phi_2(3_j, x_{\Omega_2}))}\right) \\
\tau = \max\left(\psi_2(3_j, x_{\Omega_2}), \psi_2(3_j, x_{\Omega_2})\right) = \left(\max\left(\nu_1(3_j, x_{\Omega_k}), \nu_2(3_j, x_{\Omega_2})\right) e^{i2\pi \max(\phi_1(3_j, x_{\Omega_k}), \phi_2(3_j, x_{\Omega_2}))}\right) & \text{if } \left(\delta_j, x_{\Omega_k}\right), \phi_1(3_j, x_{\Omega_k}), \phi_2(3_j, x_{\Omega_2}) \in \Psi_1(\tau_k) \\
\left(\delta_j, x_{\Omega_k}\right), \phi_2(3_j, x_{\Omega_2}), \phi_2(3_j, x_{\Omega_2}) \in \Psi_2(\tau_k) \end{cases}
\]

(6)

**Example 8** Suppose Ω = \{3₁, 3₂, 3₃\} is a universal set and R₁, R₂ ⊆ B, where R₁ = \{r₁, r₂, r₃\}, and R₂ = \{r₁, r₂, r₄\}. Suppose (Ψ₁, Γ₁ = (Δ₁, R₁, 5)) and (Ψ₂, Γ₂ = (Δ₂, R₂, 4)) are CPFN-SS and CPFN-SS over Ω, respectively. The tabular representation of CPFN-SS (Ψ₁, Γ₁ = (Δ₁, R₁, 5)) is demonstrated in Table 8, and tabular representation of CPFN-SS (Ψ₂, Γ₂ = (Δ₂, R₂, 4)) is demonstrated in Table 9, and their extended intersection is demonstrated in Table 12.
Table 11 The restricted union of CPF5-SS and CPF4-SS of Example 7

| (Ω, Γ 1 ∪ Γ 2, 5) | r 1 | r 2 |
|--------------------|-----|-----|
| 3 1                | 0.5 e 2(0.5), 0.1 e 2(0.1), 0.2 e 2(0.05) | 0.15 e 2(0.2), 0.5 e 2(0.4), 0.2 e 2(0.2) |
| 3 2                | 0.69 e 2(0.7), 0.1 e 2(0.1), 0.1 e 2(0.05) | 0.32 e 2(0.26), 0.14 e 2(0.09), 0.4 e 2(0.35) |
| 3 3                | 0.8 e 2(0.8), 0.1 e 2(0.1), 0.05 e 2(0.05) | 0.9 e 2(0.85), 0.04 e 2(0.08), 0.02 e 2(0.03) |

Example 9 Suppose Z = {3 1, 3 2, 3 3} is a universal set and R 1, R 2 ⊆ B, where R 1 = {r 1, r 2, r 3}, and R 2 = {r 1, r 2, r 4}. Suppose (Ψ 1, Γ 1 = (Δ 1, R 1, 5)) and (Ψ 2, Γ 2 = (Δ 2, R 2, 4)) are CPF5-SS and CPF4-SS over Z, respectively. The tabular representation of CPF5-SS (Ψ 1, Γ 1 = (Δ 1, R 1, 5)) is demonstrated in Table 8, and tabular representation of CPF4-SS (Ψ 2, Γ 2 = (Δ 2, R 2, 4)) is demonstrated in Table 9. Then their extended union is demonstrated in Table 13.

In this section, we discussed the basic notion of CPFN-SS which is free from all those contemporary complexities faced by CFN-SS, PFN-SS, IFN-SS, and N-SS. When a decision-maker gives such type of data like yes, no, abstain, or neutral along with 2-dimensional information in grading system, then the existing notions are not capable to handle such type of data. The only tool to handle such type of data is our proposed CPFN-SS. The proposed algorithm of the method is given in the next section.

5 Applications of the CPFN-SS

In this section, we define the algorithm for the real-world issues that are existing in the environment of CPFN-SS. We give two real-life examples (performance assessment of e-waste recycling programs and predication about champions of FIFA world cup 2022 through audience poll) of DM in the environment of CPFN-SS to show the functionality and practicality of our established work (Fig. 1).

5.1 Algorithm for the data in the form CPFN-SS

For the selection of alternatives in the environment of CPFN-SS, we initiated the following algorithm.

1. Utilize Z = {3 1, 3 2, 3 3, · · · , 3 n} as universal set
2. Utilize \( R = \{ r_1, r_2, r_3, \ldots, r_m \} \subseteq \mathbb{B} \) as a set of attributes.
3. Make a CPFN-SS in tabular form.

4. Make the tables of complex positive pole (CPP) \( \phi \), complex neutral pole (CnP) \( \varphi \), and complex negative pole (CNP) \( \psi \).
5. Compute the comparison tables for CPP, CnP, and CNP. Since the CPP, CnP, and CNP are in the form of
CFS so for the comparison of the two CFNs we will use lexicographical order for example, if $\phi_1 = \rho_1 e^{i\pi(x_1)}$, $\phi_2 = \rho_2 e^{i\pi(x_2)}$ are two PP. If $\rho_1 < \rho_2$ then we say that $\phi_1 < \phi_2$. But if $\rho_1 = \rho_2$ then we see the imaginary part, i.e., if $x_1 < x_2$ then $\phi_1 < \phi_2$. If both $\rho_1 = \rho_2$ and $x_1 = x_2$ then we have that $\phi_1 = \phi_2$.

6. Make the score tables for CPP, CnP, and CNP.

7. Adding the scores of CnP and CNP.

8. Evaluate the final score by subtracting the sum of scores of CnP and CNP from the score of PP.

9. Consider the highest score with the greatest grades, if it is in $k$–th row, then we will select option $z_k, 1 \leq k \leq n$.

5.2 Performance assessment of e-waste recycling programs

E-waste is short for electronic waste, i.e., waste generated from surplus, broken, and obsolete electronic gadgets. People also called it e-scraps. Ordinarily, these electronics frequently have harmful chemicals and risky materials. And if we do not dispose of these electronics accurately, it can cause the discharge of harmful materials into our natural environment. E-waste recycling then suggests the reprocessing of these e-wastes. It is a procedure to recover material from e-waste. Thusly, we can utilize them in new electronic items. Today e-waste recycling is one of the most debated problems in the world because of its capabilities to decrease environmental perils and contamination. In the following example, we will use the CPFN-SS to evaluate the performance of e-waste recycling programs (e-WRPs) (Fig. 2).

Example 10 Suppose an electronics company needs the most appropriate e-WRP which decreases the harmful environmental influence of e-waste, and simultaneously, developing its business. For selection and assessment of the e-WRP the company hires a decision-maker. The decision-maker found various potential e-WRPs and assessment criteria through a complete analysis. Four alternative e-WRPs include.

- $z_1$ = Retrieval of plastic and metals from e-waste
- $z_2$ = Retrieval of glass and other recycling substances from e-waste
- $z_3$ = Retrieval of mercury and valuable plastic
- $z_4$ = Retrieval of circuit boards and other valuable substances

The three most significant criteria are found for assessing the performance of appropriate e-WRP including.

- $r_1$ = Social criteria
- $r_2$ = Technical criteria
- $r_3$ = Environmental criteria
The e-WRP performance assessment producer starts with directing the decision-maker to give grades in the shape of circle stars and circle holes to the alternatives based on criteria which is displayed in Table 14.

We can link the set $X = \{0, 1, 2, 3, 4\}$ with the circle stars and circle hole displayed in Table 14 by saying that $4$ characterizes "$\otimes \otimes \otimes \otimes \otimes\$", $3$ characterizes "$\otimes \otimes \otimes \otimes\$", $2$ characterizes "$\otimes \otimes \otimes\$", $1$ signifies "$\otimes\$", and $0$ characterizes "$\otimes\$". Then the 5-SS ($\Delta, \mathbb{R}, 5$) is defined as

\[ \Delta(r_1) = \{(3_1, 2), (3_2, 3), (3_3, 1), (3_4, 1)\} \]
\[ \Delta(r_2) = \{(3_1, 2), (3_2, 1), (3_3, 4), (3_4, 0)\} \]
\[ \Delta(r_3) = \{(3_1, 4), (3_2, 1), (3_3, 0), (3_4, 3)\} \]

In Table 15 we display the tabular form of 5-SS.

\[ \phi, \varphi, \text{ and } \psi \text{ will satisfy the following grading criteria. If the grade is 0, then } 0.0 \leq \phi(3) < 0.2, \text{ (i.e., } 0.0 \leq \frac{\rho(3)+\eta(3)}{\rho(3)+\eta(3)+v(3)} \leq 0.2) \text{ and } 0 \leq \psi(3) + \varphi(3) + \psi(3) \leq 1, \text{ i.e., } 0 \leq \rho(3) + \eta(3) + v(3) \leq 1 \text{ and } 0 \leq \alpha(3) + \beta(3) + \gamma(3) \leq 1, \]

where $0 \leq \rho(3), \eta(3), v(3), \alpha(3), \beta(3), \gamma(3) \leq 1$ for each $3 \in Z$. Similarly, if the grade is 1 then $0.2 \leq \phi(3) < 0.4$, if the grade is 2 then $0.4 \leq \phi(3) < 0.6$, if the grade is 3 then $0.6 \leq \phi(3) < 0.8$, and if the grade is 4 then $0.8 \leq \phi(3) \leq 1.0$. Then the CPF5-SS is established below:

\[ \Psi(r_1) = \left\{ \left(3_1, 2 \right), \left(0.4e^{2\pi(0.5)}, 0.2e^{2\pi(0.1)}\right), \left(3_2, 3 \right), \left(0.65e^{2\pi(0.7)}, 0.1e^{2\pi(0.1)}\right), \left(3_3, 1 \right), \left(0.3e^{2\pi(0.35)}, 0.4e^{2\pi(0.2)}\right), \left(3_4, 1 \right), \left(0.25e^{2\pi(0.1)}\right) \right\} \]

\[ \Psi(r_2) = \left\{ \left(3_1, 2 \right), \left(0.55e^{2\pi(0.65)}, 0.2e^{2\pi(0.23)}\right), \left(3_2, 1 \right), \left(0.32e^{2\pi(0.26)}, 0.14e^{2\pi(0.09)}\right), \left(3_3, 4 \right), \left(0.9e^{2\pi(0.85)}, 0.04e^{2\pi(0.08)}\right), \left(3_4, 0 \right), \left(0.05e^{2\pi(0.18)}, 0.6e^{2\pi(0.5)}\right), \left(0.27e^{2\pi(0.2)}\right) \right\} \]

\[ \Psi(r_3) = \left\{ \left(3_1, 4 \right), \left(0.84e^{2\pi(0.8)}, 0.06e^{2\pi(0.07)}\right), \left(3_2, 1 \right), \left(0.2e^{2\pi(0.5)}, 0.6e^{2\pi(0.3)}\right), \left(3_3, 0 \right), \left(0.15e^{2\pi(0.17)}, 0.4e^{2\pi(0.5)}\right), \left(3_4, 3 \right), \left(0.1e^{2\pi(0.1)}, 0.75e^{2\pi(0.65)}\right), \left(0.1e^{2\pi(0.2)}\right) \right\} \]

In Table 16 we display the tabular form of CPF5-SS.

The tabular representation of CPP is demonstrated in Table 17. We make the comparison table for CPP which is demonstrated in Table 18. Afterward, we determine the CPP score for each e-WRP with the sum of grades $\sum_{i=1}^{3} x_i \Delta(r_i)$ by subtracting the column sum from the row sum of Table 18 and demonstrated in Table 19.

Now similarly the tabular representation of CnP is demonstrated in Table 20. We make the comparison table for CnP which is demonstrated in Table 21. Afterward, we determine the CnP score for each e-WRP with the sum of grades $\sum_{i=1}^{3} x_i \Delta(r_i)$ by subtracting the column sum from the row sum of Table 21 and demonstrated in Table 22.
Table 16 The tabular representation of CPF5-SS of Example 10

| \(\Psi, (A, R; 5)\) | \(r_1\) | \(r_2\) | \(r_3\) |
|---------------------|---------|---------|---------|
| \(\delta_1\)       | 0.4\(e^{2x(0.5)}\) | 0.55\(e^{2x(0.65)}\) | 0.84\(e^{2x(0.8)}\) |
| \(\delta_2\)       | 0.65\(e^{2x(0.7)}\) | 0.32\(e^{2x(0.26)}\) | 0.2\(e^{2x(0.5)}\) |
| \(\delta_3\)       | 0.3\(e^{2x(0.35)}\) | 0.1\(e^{2x(0.05)}\) | 0.26\(e^{2x(0.16)}\) |
| \(\delta_4\)       | 0.35\(e^{2x(0.45)}\) | 0.05\(e^{2x(0.18)}\) | 0.75\(e^{2x(0.65)}\) |

Table 17 The tabular representation of CPP of Example 10

| \(\phi\) | \(r_1\) | \(r_2\) | \(r_3\) |
|---------|---------|---------|---------|
| \(\delta_1\)       | 0.4\(e^{2x(0.5)}\) | 0.55\(e^{2x(0.65)}\) | 0.84\(e^{2x(0.8)}\) |
| \(\delta_2\)       | 0.65\(e^{2x(0.7)}\) | 0.32\(e^{2x(0.26)}\) | 0.2\(e^{2x(0.5)}\) |
| \(\delta_3\)       | 0.3\(e^{2x(0.35)}\) | 0.1\(e^{2x(0.05)}\) | 0.26\(e^{2x(0.16)}\) |
| \(\delta_4\)       | 0.35\(e^{2x(0.45)}\) | 0.05\(e^{2x(0.18)}\) | 0.75\(e^{2x(0.65)}\) |

Table 18 The comparison table of CPP of Example 10

| \(\delta_1\) | \(\delta_2\) | \(\delta_3\) | \(\delta_4\) |
|-------------|-------------|-------------|-------------|
| \(\delta_1\) | 3           | 2           | 2           | 3           |
| \(\delta_2\) | 1           | 3           | 2           | 2           |
| \(\delta_3\) | 1           | 3           | 1           | 3           |
| \(\delta_4\) | 0           | 1           | 2           | 3           |

Table 19 The CPP score of e-WRP

| Grade sum \(\sum_{i=1}^{3} x_i\) | Row sum \((\mathcal{R}S_1)\) | Column sum \((\mathcal{C}S_1)\) | \(\nabla_1 = \mathcal{R}S_1 - \mathcal{C}S_1\) |
|-------------------------------|-----------------------------|-----------------------------|---------------------------------|
| \(\delta_1\) | 8    | 10  | 5   | 5   |
| \(\delta_2\) | 6    | 8   | 7   | 1   |
| \(\delta_3\) | 5    | 6   | 9   | -3  |
| \(\delta_4\) | 4    | 6   | 9   | -3  |

Next, the tabular representation of CNP is demonstrated in Table 23. We make the comparison table for CNP which is demonstrated in Table 24. Afterward, we determine the CNP score for each e-WRP with the sum of grades \(\sum_{i=1}^{3} x_i\) by subtracting the column sum from the row sum of Table 24 and demonstrated in Table 25.

Table 20 The tabular representation of CnP of Example 10

| \(\phi\) | \(r_1\) | \(r_2\) | \(r_3\) |
|---------|---------|---------|---------|
| \(\delta_1\)       | 0.2\(e^{2x(0.1)}\) | 0.2\(e^{2x(0.23)}\) | 0.06\(e^{2x(0.07)}\) |
| \(\delta_2\)       | 0.1\(e^{2x(0.1)}\) | 0.14\(e^{2x(0.09)}\) | 0.6\(e^{2x(0.5)}\) |
| \(\delta_3\)       | 0.4\(e^{2x(0.2)}\) | 0.04\(e^{2x(0.08)}\) | 0.4\(e^{2x(0.5)}\) |
| \(\delta_4\)       | 0.3\(e^{2x(0.3)}\) | 0.6\(e^{2x(0.5)}\) | 0.1\(e^{2x(0.1)}\) |

Table 21 The comparison table of CnP of Example 10

| \(\delta_1\) | \(\delta_2\) | \(\delta_3\) | \(\delta_4\) |
|-------------|-------------|-------------|-------------|
| \(\delta_1\) | 3           | 2           | 1           | 0           |
| \(\delta_2\) | 1           | 3           | 2           | 1           |
| \(\delta_3\) | 2           | 1           | 3           | 2           |
| \(\delta_4\) | 3           | 2           | 1           | 3           |

In the end, add the scores of CnP and CNP which are displayed in Table 26 and then subtract it by the score of PP as demonstrated in Table 27. After the whole process of the assessment of the e-WRPs, the decision-maker concluded that the e-WRP \(\delta_1\) has good performance as compared to other e-WRPs as it has maximum grades, i.e., 8 with the highest score of 9 as
demonstrated in Table 27. This leads the electronics company in a present situation to select the most suitable e-WRP.

5.3 Predication about champions of FIFA world cup 2022 through audience poll

The FIFA world cup is an international football competition between men’s national teams of different countries who are associated with the Federation Internationale de Football Association (FIFA). The championship has been given every four years since the opening tournament in 1930. In this subsection, we will show through the following example that how we utilize CPFN-SS to conduct an audience poll for the next champions of FIFA world cup 2022.

Example 11 Suppose we want to conduct an audience poll to get the people’s point of view that which team will win the world cup 2022. We provided top three ranked football teams, i.e., \( z_1 = \) Belgium, \( z_2 = \) France, and \( z_3 = \) Brazil, and asked people to vote for these teams in the form of Table 22 The CnP score of e-WRP

| \( z_1 \) | \( \sum_{i=1} \xi_i \) | \( \text{Row sum} \) \( \mathcal{R}_2 \) | \( \text{Column sum} \) \( \mathcal{C}_2 \) | \( \nabla_2 = \mathcal{R}_2 - \mathcal{C}_2 \) |
|---|---|---|---|---|
| \( z_1 \) | 8 | 6 | 9 | -3 |
| \( z_2 \) | 6 | 7 | 8 | -1 |
| \( z_3 \) | 5 | 8 | 7 | 1 |
| \( z_4 \) | 4 | 9 | 6 | 3 |

Table 23 The tabular representation of CNP of Example 10

| \( \psi \) | \( r_1 \) | \( r_2 \) | \( r_3 \) |
|---|---|---|---|
| \( z_1 \) | 0.3e^{2x(0.15)} | 0.15e^{2x(0.06)} | 0.09e^{2x(0.12)} |
| \( z_2 \) | 0.1e^{2x(0.05)} | 0.5e^{2x(0.16)} | 0.19e^{2x(0.12)} |
| \( z_3 \) | 0.2e^{2x(0.13)} | 0.02e^{2x(0.03)} | 0.3e^{2x(0.3)} |
| \( z_4 \) | 0.25e^{2x(0.1)} | 0.27e^{2x(0.2)} | 0.1e^{2x(0.2)} |

Table 24 The comparison table of CNP of Example 10

| \( \psi \) | \( r_1 \) | \( r_2 \) | \( r_3 \) |
|---|---|---|---|
| \( z_1 \) | 3 | 1 | 2 | 1 |
| \( z_2 \) | 2 | 3 | 1 | 2 |
| \( z_3 \) | 1 | 1 | 3 | 1 |
| \( z_4 \) | 2 | 1 | 2 | 3 |

Table 25 The CNP score of each e-WRP

| \( \psi \) | \( \xi_i \) | \( \text{Row sum} \) \( \mathcal{R}_1 \) | \( \text{Column sum} \) \( \mathcal{C}_1 \) | \( \nabla_3 = \mathcal{R}_3 - \mathcal{C}_3 \) |
|---|---|---|---|---|
| \( z_1 \) | 8 | 7 | 8 | -1 |
| \( z_2 \) | 6 | 8 | 6 | 2 |
| \( z_3 \) | 5 | 6 | 8 | -2 |
| \( z_4 \) | 4 | 8 | 7 | 1 |

Table 26 Addition of CnP and CNP scores of Example 10

| \( \psi \) | \( \xi_i \) | \( \nabla_2 \) | \( \nabla_3 \) | \( \nabla_4 = \nabla_2 + \nabla_3 \) |
|---|---|---|---|---|
| \( z_1 \) | 8 | -3 | -1 | -4 |
| \( z_2 \) | 6 | -1 | 2 | 1 |
| \( z_3 \) | 5 | 1 | -2 | -1 |
| \( z_4 \) | 4 | 3 | 1 | 4 |

demonstrated in Table 27. This leads the electronics company in a present situation to select the most suitable e-WRP.

Table 27 Table of final score with grades of Example 10

| \( \psi \) | \( \xi_i \) | \( \nabla_1 \) | \( \nabla_4 \) | \( \text{Final score} = \nabla_1 - \nabla_4 \) |
|---|---|---|---|---|
| \( z_1 \) | 8 | 5 | -4 | 9 |
| \( z_2 \) | 6 | 1 | 1 | 0 |
| \( z_3 \) | 5 | -3 | -1 | -2 |
| \( z_4 \) | 4 | -3 | 4 | -7 |

Table 28 The information given by the audience

| \( \psi \) | \( r_1 \) | \( r_2 \) | \( r_3 \) |
|---|---|---|---|
| \( z_1 \) | \( \bigstar \bigstar \) | \( \bigstar \bigstar \bigstar \bigstar \) | \( \bigstar \bigstar \) |
| \( z_2 \) | \( \bigstar \) | \( \bigstar \bigstar \bigstar \) | \( \bigstar \bigstar \bigstar \bigstar \) |
| \( z_3 \) | \( \bigstar \bigstar \bigstar \bigstar \) | \( \bigstar \bigstar \bigstar \) | \( \bigstar \bigstar \bigstar \bigstar \) |

The FIFA world cup is an international football competition between men’s national teams of different countries who are associated with the Federation Internationale de Football Association (FIFA). The championship has been given every four years since the opening tournament in 1930. In this subsection, we will show through the following example that how we utilize CPFN-SS to conduct an audience poll for the next champions of FIFA world cup 2022.

Example 11 Suppose we want to conduct an audience poll to get the people’s point of view that which team will win the world cup 2022. We provided top three ranked football teams, i.e., \( z_1 = \) Belgium, \( z_2 = \) France, and \( z_3 = \) Brazil, and asked people to vote for these teams in the form of
circle stars and circle hole based on the following attributes.

- \( r_1 \) = Game plan
- \( r_2 \) = Skills, speed, and a shot accuracy of the players of each team
- \( r_3 \) = Performances of each team in previous world cups

which is demonstrated in Table 28.

We can relate the set \( X = \{0, 1, 2, 3\} \) with the circle stars and circle hole displayed in Table 28 by saying that 3 characterizes “\( \bigcirc \) \( \bigcirc \) \( \bigcirc \),” 2 characterizes “\( \bigcirc \) \( \bigcirc \),” 1 signifies “\( \bigcirc \),” and 0 characterizes “\( \bigcirc \).” Then the 4-SS (\( \Delta, \mathcal{R}, 4 \)) is demonstrated as

\[
\begin{align*}
\Delta(r_1) &= \{(31, 2), (32, 3), (33, 3)\} \\
\Delta(r_2) &= \{(31, 1), (32, 0), (33, 3)\} \\
\Delta(r_3) &= \{(31, 3), (32, 2), (33, 3)\}
\end{align*}
\]

In Table 29 we display the tabular form of the 4-SS.

\( \phi, \varphi, \text{and} \psi \) will satisfy the following grading criteria. If the grade is 0, then \( 0.0 \leq \phi(\delta) < 0.25 \), (i.e., \( 0.0 \leq \rho(\delta) + \varphi(\delta) + \psi(\delta) \leq 0.25 \) and \( 0 \leq \phi(\delta) + \varphi(\delta) + \psi(\delta) \leq 1 \), i.e., \( 0 \leq \rho(\delta) + \eta(\delta) \leq 1 \) and \( 0 \leq \alpha(\delta) + \beta(\delta) + \gamma(\delta) \leq 1 \), where \( 0 \leq \rho(\delta), \eta(\delta), \nu(\delta), \alpha(\delta), \beta(\delta), \gamma(\delta) \leq 1 \) for each \( \delta \in \mathcal{Z} \). Similarly, if the grade is 1 then \( 0.25 \leq \phi(\delta) < 0.75 \), if grade is 2 then \( 0.5 \leq \phi(\delta) < 0.75 \), if grade is 3 then \( 0.75 \leq \phi(\delta) \leq 1.0 \) Then the CPF4-SS is established below:

In Table 30 we display the tabular form of CPF5-SS.

The tabular representation of CPP is demonstrated in Table 31. We make the comparison table for CPP which is demonstrated in Table 32. Afterward, we determine the CPP score for each football team with the sum of grades \( \left( \sum_{i=1}^{3} x_i \right) \) by subtracting the column sum from the row sum of Table 32 and demonstrated in Table 33.

Now similarly the tabular representation of CNP is demonstrated in Table 34. We make the comparison table for CNP which is demonstrated in Table 35. Afterward, we determine the CNP score for each football team with the sum of grades \( \left( \sum_{i=1}^{3} x_i \right) \) by subtracting the column sum from the row sum of Table 35 and demonstrated in Table 36.

Next, the tabular representation of CNP is demonstrated in Table 37. We make the comparison table for CNP which is demonstrated in Table 38. Afterward, we determine the CNP score for each football team with the sum of grades

\[
\begin{array}{c|c|c|c}
\hline
& r_1 & r_2 & r_3 \\
\hline
\delta_1 & 2 & 3 & 3 \\
\delta_2 & 1 & 0 & 3 \\
\delta_3 & 3 & 2 & 3 \\
\hline
\end{array}
\]
Brazil will win the world cup 2022.

In this part of the article, we will do a comparison of our established work with existing work such as IFN-SS. Using the following example we will show that our established work CPFN-SS is more general and effective than the IFN-SS.

**Example 12** Reconsider Example 11 in Sect. 4 where we conducted an audience poll to get the point of view of people about the winner of the world cup 2022. We used the data in the form of CPFN-SS and got the result that the team Brazil will win the world cup of 2022. In Example 11 some people cast their votes in favor of teams (which is the positive grade of these teams), some remain neutral, i.e., may be in the favor or against these teams (which is neutral grade), and some cast their votes against the teams (which is negative grade for these teams). If we neglect the neutral grade (the people who remain neutral) in the data given in Example 11, then the data will transform in the form of complex intuitionistic fuzzy N-SS. Now if we take phase term zero in both positive and negative grades and neglect the neutral grade, then the data will convert in the environment of intuitionistic fuzzy N-SS (IFN-SS) which is demonstrated in Table 42.

Now we will utilize the algorithm defined by Akram et al. (Akram et al. 2019) for solving data in the environment of IFN-SS.

The tabular representation of the positive pole (PP) is demonstrated in Table 43. We make the comparison table for PP which is demonstrated in Table 44. Afterward, we determine the PP score for each football team with the sum of grades \( \sum_{i=1}^{3} x_{r_i} \) by subtracting the column sum from the row sum of Table 44 and demonstrated in Table 45.

Next, the tabular representation of the negative pole (NP) is demonstrated in Table 46. We make the comparison table for NP which is demonstrated in Table 47. Afterward, we determine the NP score for each football team with the sum of grades \( \sum_{i=1}^{3} x_{r_i} \) by subtracting the column sum from the row sum of Table 47 and demonstrated in Table 48.
Table 33 The CPP score of each team

| Team | Grade sum $\sum_{i=1}^{3} x_i$ | Row sum ($\mathcal{RS}_1$) | Column sum ($\mathcal{CS}_1$) | $\mathcal{V}_1 = \mathcal{RS}_1 - \mathcal{CS}_1$ |
|------|-------------------------------|-----------------------------|-----------------------------|-----------------------------------------------|
| $\delta_1$ | 8                            | 7                           | 5                           | 2                                             |
| $\delta_2$ | 4                            | 5                           | 7                           | $-2$                                          |
| $\delta_3$ | 8                            | 6                           | 6                           | 0                                             |

Table 34 The tabular representation of CNP of Example 11

| $\psi$ | $r_1$ | $r_2$ | $r_3$ | $\mathcal{V}_1 = \mathcal{RS}_1 - \mathcal{CS}_1$ |
|--------|-------|-------|-------|-----------------------------------------------|
| $\delta_1$ | $0.09e^{2x(0.2)}$ | $0.09e^{2x(0.05)}$ | $0.1e^{2x(0.1)}$ |
| $\delta_2$ | $0.07e^{2x(0.1)}$ | $0.06e^{2x(0.08)}$ | $0.09e^{2x(0.1)}$ |
| $\delta_3$ | $0.04e^{2x(0.08)}$ | $0.05e^{2x(0.05)}$ | $0.03e^{2x(0.1)}$ |

Table 35 The comparison table of CNP of Example 11

| Team | $\delta_1$ | $\delta_2$ | $\delta_3$ |
|------|-------------|-------------|-------------|
| $\delta_1$ | 3           | 3           | 3           |
| $\delta_2$ | 0           | 3           | 3           |
| $\delta_3$ | 0           | 0           | 3           |

Table 36 The CNP score of each team

| Team | Grade sum $\sum_{i=1}^{3} x_i$ | Row sum ($\mathcal{RS}_1$) | Column sum ($\mathcal{CS}_1$) | $\mathcal{V}_2 = \mathcal{RS}_2 - \mathcal{CS}_2$ |
|------|-------------------------------|-----------------------------|-----------------------------|-----------------------------------------------|
| $\delta_1$ | 8                            | 9                           | 3                           | 6                                             |
| $\delta_2$ | 4                            | 6                           | 6                           | 0                                             |
| $\delta_3$ | 8                            | 3                           | 9                           | $-6$                                          |

Table 37 The tabular representation of CNP of Example 11

| $\psi$ | $r_1$ | $r_2$ | $r_3$ | $\mathcal{V}_2 = \mathcal{RS}_2 - \mathcal{CS}_2$ |
|--------|-------|-------|-------|-----------------------------------------------|
| $\delta_1$ | $0.15e^{2x(0.05)}$ | $0.1e^{2x(0.09)}$ | $0.03e^{2x(0.09)}$ |
| $\delta_2$ | $0.25e^{2x(0.15)}$ | $0.35e^{2x(0.3)}$ | $0.03e^{2x(0.09)}$ |
| $\delta_3$ | $0.05e^{2x(0.1)}$ | $0.2e^{2x(0.09)}$ | $0.1e^{2x(0.05)}$ |

Table 38 The comparison table of CNP of Example 11

| Team | $\delta_1$ | $\delta_2$ | $\delta_3$ |
|------|-------------|-------------|-------------|
| $\delta_1$ | 3           | 1           | 1           |
| $\delta_2$ | 3           | 3           | 2           |
| $\delta_3$ | 2           | 1           | 3           |

In the end, subtract the score of NP from a score of PP which is demonstrated in Table 49.

By using the algorithm defined by Akram et al. (Akram et al. 2019) observed from Table 49 that the football team $\delta_1$ and $\delta_3$ has maximum grades but $\delta_1$ has the highest score than the team $\delta_3$ so from the audience poll, we got that the team Belgium will win the world cup 2022.

We know that the voters may be divided into 4 groups, i.e., vote for a team, abstain or neutral, vote against the team, and refusal of the voting. As we noted in Examples 11 and 12 that both are giving different results, in Example 11 we consider the information in the environment of CPFN-SS so we get the result that the team Brazil will win the world cup 2022 and in Example 12 we neglect the complex neutral grade and also take one-dimensional information by taking the phase term zero so we get the result that the team Belgium will win the world cup 2022. From Example 12 we can observe that IFN-SS cannot provide us an accurate answer and cannot carry 2-dimensional information, but our established model PFN-SS can carry 2-dimensional information and complex neutral grade along with grades and provide the accurate answer. When a decision-maker gives such type of data like yes, no, abstain, or neutral along with 2-dimensional information in grading system, then the existing notions such as FN-SS, IFN-SS, are not capable to cope with it. To resolve such types of complications the idea of CPFN-SS is a perfect tool. So our established model CPFN-SS is the generalization of these models and is more beneficial.

Our algorithm established in Sect. 4 is more general than the algorithm established by Akram et al. (Akram et al. 2019). Through our established algorithm one can resolve the data in the form of FN-SS, IFN-SS, complex FN-SS, and complex IFN-SS.

7 Conclusion

In this article, we established a novel notion of CPFN-SSs which is the combination of CPFSs along with N-SSs and explained it with the help of example. We defined a new definition of CPFS along with some of its fundamental operations like complement, union, and
intersection. Additionally, we established a complement of CPFN-SS in the article along with the example. After that, we initiated two types of union, and intersection and explained them with the help of examples. Moreover, we defined an algorithm to solve the data in the environment of CPFN-SSs. To show the advantages and usefulness of the proposed CPFN-SSs we represented two real-life applications, i.e., performance assessment of e-waste recycling program and predication about the champions of FIFA world cup 2022 through audience poll of our established model in decision-making (DM). In the performance assessment of e-WRPs, we noted that \( z_1 \) is the best e-WRP for a company and in predication about the champions of FIFA world cup 2022 through audience poll we noted that the team \( z_3 \) will win the world cup 2022. Finally, we did a comparison of our established CPFN-SS with some existing methods which showed that CPFN-SS is more effective, better and generalization of the existing ones.

As we know that our proposed method is the generalization of CIFN-SS, CFN-SS, PFN-SS, IFN-SS, FN-SS, and N-SS. So our established method solves all kind of data which are given in the environment of CIFN-SS, CFN-SS, PFN-SS, IFN-SS, FN-SS, and N-SS. But this method also has some limitations, i.e., our proposed method cannot apply to the data in the form of complex spherical fuzzy N-soft set, and complex \( T \)-spherical fuzzy N-soft set. Our established method is unable to solve such kind of data. In future we will work on these two notions.

In the future, we want to talk about N-SSs to modify the ideas of bipolar fuzzy sets (Mahmood 2020), complex hesitant fuzzy sets (Chinram et al. 2021), etc. (Mahmood

| Table 39 | The CNP score of each team |
|---|---|---|---|
| Grade sum \( \sum_{i=1}^{3} x_i \) | Row sum \( R_{S_3} \) | Column sum \( C_{S_3} \) | \( \nabla_3 = R_{S_3} - C_{S_3} \) |
| \( z_1 \) | 8 | 5 | 8 | - 3 |
| \( z_2 \) | 4 | 8 | 5 | 3 |
| \( z_3 \) | 8 | 6 | 6 | 0 |

| Table 40 | Addition of CnP and CNP scores of Example 11 |
|---|---|---|
| Grade sum \( \sum_{i=1}^{3} x_i \) | \( \nabla_2 \) | \( \nabla_5 \) | \( \nabla_4 = \nabla_2 + \nabla_3 \) |
| \( z_1 \) | 8 | 6 | - 3 | 3 |
| \( z_2 \) | 4 | 0 | 3 | 3 |
| \( z_3 \) | 8 | - 6 | 0 | - 6 |

| Table 41 | Table of final score with grades of Example 11 |
|---|---|---|---|
| Grade sum \( \sum_{i=1}^{3} x_i \) | \( \nabla_1 \) | \( \nabla_4 \) | Final score = \( \nabla_1 - \nabla_4 \) |
| \( z_1 \) | 8 | 2 | 3 | - 1 |
| \( z_2 \) | 4 | - 2 | 3 | - 5 |
| \( z_3 \) | 8 | 0 | - 6 | 6 |

| Table 42 | The tabular form of IF4-SS |
|---|---|---|---|
| \( \Omega, \mathbb{R}, \mathbb{A} \) | \( r_1 \) | \( r_2 \) | \( r_3 \) |
| \( z_1 \) | 2, (0.75, 0.15) | 3, (0.8, 0.1) | 3, (0.85, 0.03) |
| \( z_2 \) | 1, (0.25, 0.25) | 0, (0.15, 0.35) | 3, (0.86, 0.03) |
| \( z_3 \) | 3, (0.9, 0.05) | 2, (0.6, 0.2) | 3, (0.8, 0.1) |

| Table 43 | The tabular representation of PP of Example 12 |
|---|---|---|---|
| \( \phi \) | \( r_1 \) | \( r_2 \) | \( r_3 \) |
| \( z_1 \) | 0.75 | 0.8 | 0.85 |
| \( z_2 \) | 0.25 | 0.15 | 0.86 |
| \( z_3 \) | 0.9 | 0.6 | 0.8 |

| Table 44 | The comparison table of PP of Example 12 |
|---|---|---|
| \( z_1 \) | 3 | 2 | 2 |
| \( z_2 \) | 1 | 3 | 1 |
| \( z_3 \) | 1 | 2 | 3 |
Table 45 The PP score of each team

| Team | Grade sum (∑xᵢ) | Row sum (RS₁) | Column sum (CS₁) | ∇₁ = RS₁ - CS₁ |
|------|-----------------|---------------|-----------------|-----------------|
| ₁    | 8               | 7             | 5               | 2               |
| ₂    | 4               | 5             | 7               | -2              |
| ₃    | 8               | 6             | 6               | 0               |

Data availability The data utilized in this manuscript are hypothetical and artificial, and one can use these data before prior permission by just citing this manuscript.

Table 46 The tabular representation of NP of Example 12

| ψ  | r₁  | r₂  | r₃  |
|----|-----|-----|-----|
| ₁  | 0.15| 0.1 | 0.03|
| ₂  | 0.25| 0.35| 0.03|
| ₃  | 0.05| 0.2 | 0.1 |

Table 47 The comparison table of NP of Example 12

| Team | ₁ | ₂ | ₃ |
|------|---|---|---|
| ₁    | 3 | 1 | 1 |
| ₂    | 3 | 3 | 2 |
| ₃    | 3 | 1 | 3 |

Table 48 The NP score of each team

| Team | Grade sum (∑xᵢ) | Row sum (RS₁) | Column sum (CS₁) | ∇₃ = RS₃ - CS₃ |
|------|-----------------|---------------|-----------------|----------------|
| ₁    | 8               | 6             | 8               | -3             |
| ₂    | 4               | 8             | 5               | 3              |
| ₃    | 8               | 6             | 6               | 0              |

Table 49 Final score with grades of Example 12

| Team | Grade sum (∑xᵢ) | ∇₁ | ∇₃ | Final score = ∇₁ - ∇₃ |
|------|-----------------|----|----|-----------------------|
| ₁    | 8               | 2  | -3 | 5                     |
| ₂    | 4               | -2 | 3  | -5                    |
| ₃    | 8               | 0  | 0  | 0                     |

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