Novel Position-Space Renormalization Group for Bond Directed Percolation in Two Dimensions

Hüseyin Kaya\textsuperscript{a} and Ayşe Erzan\textsuperscript{a,b}
\textsuperscript{a} TUBITAK, Feza Gursey Institute, P.O. Box 6
Çengelköy 81220, Istanbul - TURKEY
\textsuperscript{b} İstanbul Technical University, Faculty of Science and Letters
Department of Physics, Maslak 80626, İstanbul-TURKEY

A new position-space renormalization group approach is investigated for bond directed percolation in two dimensions. The threshold value for the bond occupation probabilities is found to be $p_c = 0.6443$. Correlation length exponents on time (parallel) and space (transverse) directions are found to be $\nu_\parallel = 1.719$ and $\nu_\perp = 1.076$, respectively, which are in very good agreement with the best known series expansion results.

\textbf{I. INTRODUCTION}

Directed percolation has been studied extensively [1-6], since it plays very important role in a large variety of non-equilibrium systems with a single absorbing state [7]. A wide array of dynamic processes as fluid flow through a porous medium in a external field [8], forest fires [9,10] or epidemic growth models [11], reaction-diffusion systems [12,13], damage spreading [14], self-organized criticality [15], models of growing surfaces with roughening transition [16,17], etc. fall into the same universality class [18], models of growing surfaces with roughening transition [17,19], etc. fall into the same universality class as directed percolation. There is no exact result for the critical exponents characterising the dynamical phase transition separating the absorbing steady state from the active phase. The correlation length exponents in the longitudinal and transverse directions, defined via $\xi_\parallel \sim |p-p_c|^{-\nu_\parallel}$ and $\xi_\perp \sim |p-p_c|^{-\nu_\perp}$ and the percolation threshold $p_c$ have been computed from the steady state distribution found from the fixed point of the FST. These scale factors are determined independently, without having to make any additional assumptions.

The fixed-scale transformation (FST) [20], introduced by Pietronero, Erzan and Evertsz [21,22], has been used to investigate the self-similar properties of irreversible growth models, namely, diffusion limited aggregation [23] and dielectric breakdown [24]. The FST has also been applied to systems such as percolation [25], Ising or Potts models [26], invasion percolation [27,28] and to directed percolation [29] at the critical point. Renormalization group ideas have been used in conjunction with the FST approach [30], in order to identify the scale invariant growth rules in terms of which the finite cell fixed scale transformation matrix should be computed, with good results. Forest fires [31] (which fall into the same universality class as directed percolation) and the sandpile model [32,33,34], have been studied by means of the “dynamically driven renormalization group” [33,34], which combines real space renormalization group ideas with a dynamical steady state condition reminiscent of the fixed scale transformation approach.

In this paper, we would like to introduce a novel position space renormalization group (PSRG) treatment of the directed percolation problem. This approach modifies conventional PSRG methods in two ways: i) the weights of different initial states in the RG cell are computed from the steady state distribution found from the fixed point of the FST; ii) a “dynamical” coarse graining procedure is defined which allows for the appearance of two different scale factors in the longitudinal (time-like) and transverse (space-like) directions, thus taking into account the self-affine nature of the problem. These scale factors are determined independently, without having to make any additional assumptions.

The paper is organized as follows. In the next section, we construct the renormalization group transformation, subject to the dynamical fixed point condition, and compute the critical parameter as well as the transverse and longitudinal correlation length exponents. In the last section we provide a short discussion.

\textbf{II. RG TRANSFORMATION WITH DYNAMICAL COARSE GRAINING}

We consider 1-time and 1-space dimensional bond directed percolation as a growth process. Let us recall the growth rules: if a site at time $t$ is active, its two neighbours at time $t+1$ may be activated, each independently, with a probability $p$. If $p$ is larger than the threshold value $p_c$, there is a finite probability that the growth process is continued indefinitely.

We now outline a renormalization group procedure which takes into account fluctuations in regions larger...
than the renormalization group cell by using a steady-state distribution of initial conditions, obtained from the FST. It also introduces self-consistently determined rescaling lengths, rather than pre-set scale factors between the original and coarse grained lattices.

Our renormalization group cell is shown in Fig. 1(a). The boxes $AA'B'B'$ and $CC'A'A''$ will coarse grain, under a dynamic coarse graining procedure we describe below, to the bonds $ab$ and $ac$, as shown in Fig. 1(b). This process conserves the transverse and longitudinal directions, but does not conserve the lattice angles, since these two directions are scaled differently. Rescaling by the appropriate longitudinal and transverse scale factors will restore the original lattice (Fig. 1(c)). The lattice spacing is taken to be $\sqrt{2}$ for convenience, yielding $\ell_\perp = 1$ and $\ell_\parallel = 1$ for the transverse and perpendicular distances between the nearest neighbors, such as $AA'$ and $BB'$. Clearly, we only have to consider one of the boxes, e.g., $AA'B'B'$ for our renormalization procedure, by symmetry.

To obtain the renormalization transformation for the bond occupation probability $p$, we compute the total probability $P(p)$ that a spanning path exists across the box $AA'B'B'$, by considering all the different initial configurations (states of $A$ and $A'$) and the spanning configurations that can be obtained from them. A spanning configuration is defined as one where a path starts from either $A$, or $A'$ or both, and ends on $B$ or $B'$ or both.

A. Steady state distribution of initial conditions

Since there are two possible origins which can be active both together or by themselves, four different initial configurations have to be considered as illustrated in Fig.2. Any spanning configuration may have more than one path which connects the origins to the end points. For instance, two different paths are possible for connecting the sites $A$ and $B'$ in the spanning configuration shown in Fig.3.

In order to compute the probabilities of finding each initial configuration, say $W_i$, $i = 1, ..., 4$, we make use of the time invariant probabilities for the relative frequency of doubly or singly occupied boxes in a box covering of the equal-time transverse subsets on the cluster of active sites, in terms of the probabilities $p$.

![FIG. 1. The dynamical coarse-graining procedure. (a) The RG cells $AA'B'B'$ and $CC'A'A''$ (the bold lines) in the original lattice are coarse-grained (b) to the bonds $ab$ and $ac$ respectively, which are rescaled in the next step (c) to preserve the lattice angles.](image)

![FIG. 2. Four different initial configurations are represented. In computing their respective weights, (see text) the bold lines in the each configuration are replaced with the probability $p$, and the dashed lines with $1 - p$.](image)

![FIG. 3. Two different paths belonging to the same spanning configuration.](image)
the critical point, as \( t \to \infty \), the infinite transverse subsets of active sites at any given \( t \) are statistically similar under translation in \( t \). In particular, they can be modeled by Generalized Cantor Sets generated by a random sequence of fragmentations obeying a one-parameter, scale invariant distribution as shown in Fig.4. Given a hierarchical partitioning of the transverse subsets into cells of size \( 2^{-k} \), the relative probability of encountering singly occupied or doubly occupied cell among nonempty cells at a fixed arbitrary scale is equal to \( C \) or \( 1 - C \). This probability can be computed from the fixed point of the fixed-scale transformation as,

\[
C = \frac{2 - 3p + 5p^2 - \sqrt{36 - 108p + 109p^2 - 30p^3 + 9p^4}}{2p^2 + 4p - 4},
\]

up to first order in the FST approach.

![Figure 4](image_url)

**FIG. 4.** The generator for the random Cantor set corresponding to a transverse subset of the incipient infinite cluster. Cells of type-1 and type-2 participate in the fine graining process with corresponding probabilities \( C_1 \) and \( C_2 \). We represent the activated sites with black circle and inactive sites with white circle.

We now make use of this FST results to compute the normalized weights \( W_i \) of the initial configurations \( i \) in terms of \( C \) and \( p \),

\[
W_1 = pC/2
\]

(2)

\[
W_2 = (1-p)C/2 + (1-p)^2(1-C)
\]

(3)

\[
W_3 = p(1-C)/2
\]

(4)

\[
W_4 = p(1-p)(1-C)
\]

(5)

In the following subsection we proceed to obtain the renormalization transformation in terms of which the critical value of \( p \) can be determined.

### B. The renormalization transformation for \( p \)

Depending on the initial configurations \( i = 1, \ldots, 4 \), the total number of possible spanning configurations will be different. One sees that in Fig.2 (1) only one; (2) seven; and (3) - (4) eighteen spanning configurations each are possible. It should also be noticed that some spanning configurations can be observed in more than one different initial configuration. For example, the spanning configuration which contains only one path which starts from \( A' \) and ends at \( B' \) can be observed in all the initial configurations except the second one (see Fig.2). The total probability \( f_i \) of the spanning cluster for the \( i \)th initial configuration is given by

\[
f_1(p) = p^2
\]

(6)

\[
f_2(p) = p^2 + p^3 - p^4
\]

(7)

\[
f_3(p) = 2p^2 + p^3 - 3p^4 + p^5
\]

(8)

\[
f_4(p) = f_3(p)
\]

(9)

The renormalization group transformation for \( p \) is then found to be

\[
P(p) = W_1f_1(p) + W_2f_2(p) + W_3f_3(p) + W_4f_4(p) = p'.
\]

(10)

The fixed point of this transformation gives the threshold value, \( p_c = 0.6443 \) which is in agreement up to the third digit with the series expansion result, namely 0.6447.

### C. The Affine transformation in the longitudinal and transverse directions

The system has two independent correlation lengths \( \xi_∥ \) and \( \xi_⊥ \), parallel and perpendicular to the time direction respectively, which diverge with different exponents as \( \xi_∥ \sim |p - p_c|^{-\nu_∥} \) and \( \xi_⊥ \sim |p - p_c|^{-\nu_⊥} \). To compute these exponents, we use the eigenvalue equations,

\[
\frac{dp}{dp} \bigg|_{p=p_c} = b_∥^{1/\nu_∥}, \quad \frac{dp}{dp} \bigg|_{p=p_c} = b_⊥^{1/\nu_⊥}.
\]

(11)

We determine the appropriate rescaling lengths \( b_∥ \) and \( b_⊥ \) from,

\[
b_∥ = \frac{< L_∥ >}{L_∥}, \quad b_⊥ = \frac{< L_⊥ >}{L_⊥},
\]

(12)

in terms of the average projected lengths, \( < L_∥ > \) and \( < L_⊥ > \), of the spanning paths onto the time and transverse directions as shown in Fig.5. These quantities are the amounts by which the coarse grained lattice (see Fig.1(b)) has been dilated in the “dynamical coarse-graining” step. Since we have taken the lattice constant to be \( \sqrt{2} \), the projections of a single bond on the original lattice onto the time and space directions are \( \ell_⊥ = \ell_∥ = 1 \).
FIG. 5. All possible paths and their extremal projections contributing to the dynamical rescaling factors $< L_\parallel >$ and $< L_\perp >$. See Fig.1.

In the time direction, one has to take into account the fact that under coarse graining, different time steps collapse onto each other; that is why we consider each path originating from $A$ or $A'$ and terminating on $B$ or $B'$ as contributing separately to $L_\parallel$. While enumerating the possible paths over which the average is taken, it makes a difference if there is a bond between $A$ and $A'$, since the paths change in case $A'$ is activated by $A$. The same thing holds for the end sites $B$ and $B'$. Note that the result of taking the extremal projections first and then averaging is different from finding the “average path” and then taking its projections. This point will be further discussed below.

For $X \equiv \{\perp, \parallel\}$, we have

$$< L_X > = \sum_i W_i \sum_\beta q_{i,\beta} \sum_\alpha L_X^{i,\beta,\alpha} \tag{13}$$

where $q_{i,\beta}$ is the relative probability of any spanning path $\beta$ belonging to an initial state $i$, and is to be found from $f_i$ by giving equal weights to the distinct paths in any spanning configuration equally. Substituting the value of $p_c$ found from Eq.(10) into Eqs.(2,3), we find,

$$< L_\perp > = 1.7996 \tag{14}$$

and

$$< L_\parallel > = 2.5561 \tag{15}$$

These values yield, together with Eq.(11), the correlation length exponents to be $\nu_\parallel = 1.719$ and $\nu_\perp = 1.076$ which are comparable with the best known results $[5,6]$ $\nu_\parallel = 1.733$ and $\nu_\perp = 1.097$.

III. DISCUSSION

There have been some earlier studies of directed (therefore self-affine) systems via position space renormalization group (PSRG) techniques. Introduction of two different scaling factors $b_\parallel$ and $b_\perp$ due to existence of two independent scaling directions was firstly suggested by Dhar and Phani [36], where they employed a decimation transformation. However, their results are far from the accepted values of the critical exponents and dependent upon their choice of the RG cell, which determines $b_\parallel/b_\perp$. Very large-cell PSRG calculations by Zhang and Yang [37] are able to accurately reproduce the self-affine behaviour of directed self-avoiding walks, but are more appropriately in a class with Monte Carlo RG. A bond-moving and decimation transformation for anisotropic directed bond percolation in arbitrary dimension [38] and its generalization for other directed systems [39] by da Silva and Droz give the critical fugacity very accurately in all dimensions for directed self-avoiding walks. For two dimensional directed percolation, it yields good results for the threshold value $p_c$ and $\nu_\parallel$, but is not as good for $\nu_\perp$. The approach in these papers is in fact very close to ours; however, the way the projections of paths are taken to compute the scaling factors in the two different directions are different.

One may consider two different ways of projecting a path. The approach of da Silva and Droz [38,39] is to draw a vector between the origin and the end point, and take its two components to be the projections of the path. Our is to take the projection to be the transverse or longitudinal distance between the extremal points of the path, as shown in Fig.5.

![FIG. 6. Macroscopic paths with identical end-to-end vectors, but strongly differing extremal projections.](image)

We believe that our method yields a result which corresponds more closely to what is meant by the size of the transverse and longitudinal fluctuations, in that it measures more accurately the actual size of the region over
which a coherent flow takes place. This becomes more evident if we consider the incipient infinite cluster of our system for $p \approx p_c$. In this cluster, there can be many paths which connect some point at time $t_0$ to an end point at $t_0 + t$. Even though the total number of bonds in these different paths are the same, the spatial size of the spanned regions can be radically different (see Fig. 6). Since the transverse (longitudinal) correlation length corresponds to the spatial (time-like) size of the fluctuations, it is more appropriate to take the average over the extremal transverse (longitudinal) extent of each path.

In conclusion, by incorporating the FST fixed point condition in the determination of our distribution of initial configurations, and by a new, dynamical coarse graining procedure which makes use of averages over extremal projections of spanning paths, we have succeeded in computing the percolation threshold and the correlation function exponents much more accurately than before.

Note added in proof: D. Sornette has kindly brought to our attention Refs. [40,41] where the equivalence of directed percolation to a parallel version of the Bak-Sneppen model [42] has been rigorously demonstrated. Also see [43] for a comparison of directed percolation with sandpile models.

[1] W. Kinzel, in Percolating Structures and Processes (Ann. Israel Phys. Soc., 5), edited by G. Deutcher, R. Zallen and J. Adler (Adam Hilger, Bristol 1983, p.425.
[2] D. Ben-Avraham, R. Bidaux and L.S. Schulman, Phys. Rev. A 43(1991) 7093.
[3] K. De'Bell and J.W. Essam, J. Phys. A 16 (1983) 3145.
[4] E. Domany and W. Kinzel, Phys. Rev. Lett. 47 (1981) 5.
[5] R.J. Baxter and A.J. Guttmann, J. Phys. A 21 (1988) 3193.
[6] J.W. Essam, A.J. Guttmann and K. De&Bell, J. Phys A 21 (1988) 3815.
[7] P. Grassberger, Z. Phys. B47 (1982) 365.
[8] S.R. Broadbent and J.M. Hammersley, Proc. Camb. Phil. Soc. 53 (1957) 629.
[9] P. Bak, K. Chen and C. Tang, Phys. Rev. Lett. A147 (1990) 297.
[10] B. Drossel and F. Schwabl, Phys. Rev. Lett. A69 (1992) 1629.
[11] J.L. Cardy, J. Phys. A 16 (1983) L709.
[12] R.M. Ziff, E. Gulari and Y. Barshad, Phys. Rev. Lett. 56 (1986) 2553.
[13] F. Schlogl, Z. Phys. 252 (1972) 147.