PHOTON STATISTICS FOR MULTIMODE SQUEEZED SCHRODINGER CAT STATES

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Abstract

Particle distributions in squeezed states, even and odd coherent states are given in terms of multivariable Hermite polynomials. The Q–function and Wigner function for nonclassical field states are discussed.

1 Introduction

The aim of the talk is to give a review of such nonclassical states of light as squeezed states [1], [2], correlated states [3], even and odd coherent states [4] (Schrödinger cat states [5]), squeezed Schrödinger cat states [6]. First we discuss the photon distribution function for the generalized correlated states [7] of multimode light. For finding solutions of Schrödinger equation with time–dependent Hamiltonians which are generic quadratic forms in position and momentum operators the integrals of motion which are linear forms have been constructed in Refs. [8], [9] and [10]. Such integrals of motion have been analyzed and applied to general problems of quantum mechanics and statistics in Refs. [11],[12].

The nonstationary Hamiltonians are appropriate models for the physical conditions in which nonclassical states of fields (photons, phonons, pions, etc.) may be created. Following [13] we will give the result for photon distribution function of generic mixed squeezed and correlated Gaussian state. Initially the state is taken to be standard coherent one (partial case of such state is photon vacuum) and, for example, due to nonstationary Casimir effect it becomes multimode mixed correlated state.
2 Multimode Hermite Polynomials and Mixed Correlated Light

The most general mixed squeezed state of the N–mode light with a Gaussian density operator $\hat{\rho}$ is described by the Wigner function $W(p, q)$ of the generic Gaussian form,

$$W(p, q) = (\det M)^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (Q - <Q>) M^{-1} (Q - <Q>) \right], \quad (1)$$

where 2N–dimensional vector $Q = (p, q)$ consists of N components $p_1, \ldots, p_N$ and N components $q_1, \ldots, q_N$, operators $\hat{p}$ and $\hat{q}$ being the quadrature components of the photon creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators (we use dimensionless variables and assume $\hbar = 1$):

$$\hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{i \sqrt{2}}, \quad \hat{q} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}. \quad (2)$$

2N parameters $<p_i>$ and $<q_i>$, $i = 1, 2, \ldots, N$, combined into vector $<Q>$, are the average values of the quadratures,

$$<p> = \text{Tr} \hat{\rho} \hat{p}, \quad <q> = \text{Tr} \hat{\rho} \hat{q}. \quad (3)$$

A real symmetric dispersion matrix $M$ consists of $2N^2+N$ variances

$$M_{\alpha\beta} = \frac{1}{2} \left< \hat{Q}_\alpha \hat{Q}_\beta + \hat{Q}_\beta \hat{Q}_\alpha \right> - \left< \hat{Q}_\alpha \right> \left< \hat{Q}_\beta \right>, \quad \alpha, \beta = 1, 2, \ldots, 2N. \quad (4)$$

They obey certain constraints, which are nothing but the generalized uncertainty relations [12].

The photon distribution function of this state

$$P_n = \text{Tr} \hat{\rho} |n><n|, \quad n = (n_1, n_2, \ldots, n_N), \quad (5)$$

where the state $|n>$ is photon number state, which was calculated in [13], and it is

$$P_n = P_0 \frac{H_{nn}^{(R)}(y)}{n!}. \quad (6)$$

The function $H_{nn}^{(R)}(y)$ is multidimensional Hermite polynomial. The probability to have no photons is

$$P_0 = \left[ \det \left( M + \frac{1}{2} I_{2N} \right) \right]^{-\frac{1}{2}} \exp \left[ - <Q> (2M + I_{2N})^{-1} <Q> \right]. \quad (7)$$
where we introduced the matrix
\[ R = 2U^\dagger (1 + 2M)^{-1} U^* - \sigma_{Nx}, \] (8)
and the matrix
\[ \sigma_{Nx} = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}. \] (9)
The argument of Hermite polynomial is
\[ y = 2U^\dagger (I_{2N} - 2M)^{-1} < Q >, \] (10)
and the 2N–dimensional unitary matrix
\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} -iI_N & iI_N \\ I_N & I_N \end{pmatrix} \] (11)
is introduced, in which \( I_N \) is the N\times N identity matrix. Also we use the notation
\[ n! = n_1!n_2!...n_N!. \]

The mean photon number for j–th mode is expressed in terms of photon quadrature means and dispersions
\[ < n_j > = \frac{1}{2}(\sigma_{p_jp_j} + \sigma_{q_jq_j} - 1) + \frac{1}{2}(< p_j >^2 + q_j^2 >). \] (12)

We introduce a complex 2N–vector \( B = (\beta_1, \beta_2, \ldots, \beta_N, \beta_1^*, \beta_2^*, \ldots, \beta_N^*) \). Then the Q–function [14] is the diagonal matrix element of the density operator in the coherent state basis \( | \beta_1, \beta_2, \ldots, \beta_N > \). This function is the generating function for the matrix elements of the density operator in the Fock basis \( | n > \) which has been calculated in [15]. In notations corresponding to the Wigner function (1) the Q–function is
\[ Q(B) = \mathcal{P}_0 \exp \left[ -\frac{1}{2} B(R + \sigma_{Nx}) B + BRy \right]. \] (13)
Thus, if the Wigner function (1) is given one has the Q–function. Also, if one has the Q–function (13), i. e. the matrix \( R \) and the vector \( y \), the Wigner function may be obtained due to the relations
\[ M = U^* (R + \sigma_{Nx})^{-1} U^\dagger - \frac{1}{2}, \]
\[ < Q > = U^* [1 - (R + \sigma_{Nx})^{-1} \sigma_{Nx}] y. \] (14)

For pure squeezed and correlated state with the wave function
\[ \Psi = N \exp[-m_k k + c_k], \] (15)
where
\[ N = \left[ \det(m + m^*) \right]^{1/4} \pi^{-N/4} \exp \left\{ \frac{1}{4} (c + c^*)(m + m^*)^{-1} (c + c^*) \right\}, \tag{16} \]
the symmetric $2N \times 2N$–matrix $R$ determining Q–function has the block–diagonal form
\[ R = \begin{pmatrix} r & 0 \\ 0 & r^* \end{pmatrix}. \tag{17} \]
The $N \times N$–matrix $r$ is expressed in terms of symmetric matrix $m$
\[ r^* = 1 - (m + 1/2)^{-1}, \tag{18} \]
and the $2N$–vector $y = (Y, Y^*)$ is given by the relation
\[ Y^* = \frac{1}{\sqrt{2}} (m - 1/2)^{-1} c. \tag{19} \]
The corresponding blocks of the dispersion matrix
\[ M = \begin{pmatrix} \sigma_{pp} & \sigma_{pq} \\ \sigma_{qp} & \sigma_{qq} \end{pmatrix} \tag{20} \]
are
\[ \sigma_{pp} = 2 (m^{-1} + m^*^{-1})^{-1}, \]
\[ \sigma_{qq} = \frac{1}{2} (m + m^*)^{-1}, \]
\[ \sigma_{pq} = \frac{i}{2} (m - m^*) (m + m^*)^{-1}. \tag{21} \]
The probability to have no photons is
\[ P_0 = \frac{[\det(m + m^*)]^{1/2}}{\left| \det(m + 1/2) \right|} \otimes \exp \left\{ \frac{1}{2} (c + c^*) (m + m^*)^{-1} (c + c^*) + \frac{1}{4} [c(m + 1/2)^{-1} c + c^* (m^* + 1/2)^{-1} c^*] \right\}. \tag{22} \]
The multivariable Hermite polynomials describe the photon distribution function for the multimode mixed and pure correlated light [13], [16], [17]. The nonclassical state of the light may be created due to nonstationary Casimir effect [18], and the Husimi oscillator is the model to describe the behaviour of the squeezed and correlated photons.
3 Multimode Even and Odd Coherent States

We define the multimode even and odd coherent states (Schrödinger cat male states and Schrödinger cat female states, respectively) as [19]

\[ | \mathbf{A}_\pm > = N_\pm (| \mathbf{A} > \pm | -\mathbf{A} >), \]

where the multimode coherent state \(| \mathbf{A} >\) is

\[ | \mathbf{A} > = | \alpha_1, \alpha_2, \ldots, \alpha_n > = D(\mathbf{A}) | \mathbf{0} >, \]

and the multimode coherent state is created from multimode vacuum state \(| \mathbf{0} >\) by the multimode displacement operator \(D(\mathbf{A})\). The definition of multimode even and odd coherent states is the obvious generalization of the one–mode even and odd coherent state given in [4], [11]. The normalization constants for multimode even and odd coherent states become

\[ N_+ = \frac{e^{\frac{1}{2}|\mathbf{A}|^2}}{2\sqrt{\cosh |\mathbf{A}|^2}}, \]
\[ N_- = \frac{e^{\frac{1}{2}|\mathbf{A}|^2}}{2\sqrt{\sinh |\mathbf{A}|^2}}, \]

where \(\mathbf{A} = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) is a complex vector and its modulus is

\[ |\mathbf{A}|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 = \sum_{m=1}^{n} |\alpha_m|^2. \]

Such multimode even and odd coherent states can be decomposed into multimode number states as

\[ | \mathbf{A}_\pm > = N_\pm \sum_n \frac{e^{-\frac{1}{2}|\mathbf{A}|^2} \alpha_1^{n_1} \ldots \alpha_n^{n_n}}{\sqrt{n_1! \ldots n_n!}} (1 \pm (-1)^{n_1+n_2+\ldots+n_n}) | \mathbf{n} >, \]

where \(| \mathbf{n} > = | n_1, n_2, \ldots, n_n >\) is the multimode number state. Also from Eq. (23), we can derive an important relation for the multimode even and odd coherent states, namely,

\[ a_i | \mathbf{A}_+ > = \alpha_i \sqrt{\tanh |\mathbf{A}|^2} | \mathbf{A}_- >, \]
\[ a_i | \mathbf{A}_- > = \alpha_i \sqrt{\coth |\mathbf{A}|^2} | \mathbf{A}_+ >. \]
The probability of finding \( n \) photons in multimode even and odd coherent states can be worked out with the help of Eq. (27)

\[
P_+ (n) = \frac{\vert \alpha_1 \vert^{2n_1} \vert \alpha_2 \vert^{2n_2} \ldots \vert \alpha_n \vert^{2n_n}}{(n_1!) (n_2!) \ldots (n_n!) \cosh \vert A \vert^2}, \quad n_1 + n_2 + \ldots + n_n = 2k, \\
P_- (n) = \frac{\vert \alpha_1 \vert^{2n_1} \vert \alpha_2 \vert^{2n_2} \ldots \vert \alpha_n \vert^{2n_n}}{(n_1!) (n_2!) \ldots (n_n!) \sinh \vert A \vert^2}, \quad n_1 + n_2 + \ldots + n_n = 2k + 1.
\]

(29)

Multimode coherent states are the product of independent coherent states of each mode, and photon distribution function is the product of independent Poissonian distribution functions. But in the present case of multimode even and odd coherent states we cannot factorize their multimode photon distribution functions due to the presence of the nonfactorizable \( \cosh \vert A \vert^2 \) and \( \sinh \vert A \vert^2 \). This fact implies the phenomenon of statistical dependences of different modes of these states on each other.

In order to describe the properties of the distribution functions from Eq. (29) we will calculate the symmetric \( 2N \times 2N \) dispersion matrix for multimode field quadrature components. For even and odd coherent states we have

\[
< A_\pm | a_i a_k | A_\pm > = \alpha_i^* \alpha_k,
\]

(30)

and complex conjugate values of the above equation for \( < A_\pm | a_i^\dagger a_k^\dagger | A_\pm > \). Since the quantity \( < A_\pm | a_i | A_\pm > \) is equal to zero the above equation gives two \( N \times N \) blocks of the dispersion matrix. For other two \( N \times N \) blocks of this matrix we have

\[
\sigma^+_{(a_i^\dagger a_k)} = < A_+ | \frac{1}{2} (a_i^\dagger a_k + a_k a_i^\dagger) | A_+ >= \alpha_i^* \alpha_k \tanh \vert A \vert^2 + \frac{1}{2} \delta_{ik},
\]

(31)

and for multimode odd coherent states

\[
\sigma^-_{(a_i^\dagger a_k)} = < A_- | \frac{1}{2} (a_i^\dagger a_k + a_k a_i^\dagger) | A_- >= \alpha_i^* \alpha_k \coth \vert A \vert^2 + \frac{1}{2} \delta_{ik}.
\]

(32)

For the dispersion matrix, the mean values of the photon numbers \( n_i = a_i^\dagger a_i \) for multimode even and odd coherent states are the following

\[
< A_+ | n_i | A_+ > = \vert \alpha_i \vert^2 \tanh \vert A \vert^2, \\
< A_- | n_i | A_- > = \vert \alpha_i \vert^2 \coth \vert A \vert^2.
\]

(33)

Taking into account the above equation the symmetric \( N \times N \) dispersion matrices for photon number operators can be obtained from the above given distribution functions for multimode even and odd coherent states. By defining

\[
\sigma^\pm_{ik} = < A_\pm | n_i n_k | A_\pm >,
\]

(34)
the corresponding expressions in such states are

\[
\sigma_{ik}^+ = |\alpha_i|^2 |\alpha_k|^2 \text{sech}^2 |A|^2 + |\alpha_i|^2 \tanh |A|^2 \delta_{ik},
\]

\[
\sigma_{ik}^- = - |\alpha_i|^2 |\alpha_k|^2 \text{csch}^2 |A|^2 + |\alpha_i|^2 \coth |A|^2 \delta_{ik}.
\] (35)

As the nondiagonal matrix elements of the dispersion density matrix are not equal to zero so we can predict that different modes of these states are correlated with each other. In other words, as we have mentioned before, there exist some statistical dependences of different modes on each other.

Another interesting property for the multimode even and odd coherent states is the Q–function, and it can be obtained in the following manner. First of all the density matrices for the multimode even and odd coherent states are

\[
\rho_{\pm} = |A\pm><A\pm|,
\] (36)

then the Q–function can be calculated as

\[
Q_+(B, B^*) = <B | \rho_+ | B> = 4N^2 e^{-|A|^2+|B|^2} |\cosh(AB^*)|^2
\]

\[
Q_-(B, B^*) = <B | \rho_- | B> = 4N^2 e^{-|A|^2+|B|^2} |\sinh(AB^*)|^2,
\] (37)

where \( |B| = |\beta_1, \beta_2, \ldots, \beta_n >\) is another multimode coherent state with multimode eigenvalue \( B = (\beta_1, \beta_2, \ldots, \beta_n)\). We call these functions for even and odd coherent states as the Q–functions for the Schrödinger cat states. The Q–function for single–mode odd coherent state shows the crater type behaviour for small values of the quantity \( |\alpha| \) and for its larger values the Q–function begins to split into two peaks in a similar manner as in case of even coherent states [13].

The Wigner function for the multimode coherent states is [12]

\[
W_{A,B} = 2^N \exp[-2ZZ^* + 2AZ^* + 2B^*Z - AB^* - |A|^2 - |B|^2],
\] (38)

where

\[
Z = \frac{q+ip}{\sqrt{2}}.
\] (39)

For even and odd coherent states the Wigner function is

\[
W_{A\pm}(q, p) = |N_\pm|^2 [W_{(A,B=A)}(q, p) \pm W_{(A,B=-A)}(q, p)]
+ W_{(-A,B=A)}(q, p) + W_{(-A,B=-A)}(q, p)],
\] (40)

where the explicit forms of \( N_\pm \) are given in Eq. (25). For multimode case we use the following notations

\[
AZ^* = \alpha_1 Z_1^* + \alpha_2 Z_2^* + \ldots + \alpha_n Z_n^*,
\]

\[
ZZ^* = Z_1 Z_1^* + Z_2 Z_2^* + \ldots + Z_n Z_n^*.
\] (41)
The photon distribution function gives the probability of finding $2k$ photons for two–mode even coherent state, and is defined as

$$P_+(2k) = \frac{(|\alpha_1|^2 + |\alpha_2|^2)^{2k}}{(2k)! \cosh(|\alpha_1|^2 + |\alpha_2|^2)}.$$  \hspace{1cm} (42)

where $2k = n_1 + n_2$, for both $n_1$ and $n_2$ to be even or odd numbers. Similarly for two–mode odd coherent state it gives the probability of finding $2k+1$ photons

$$P_-(2k + 1) = \frac{(|\alpha_1|^2 + |\alpha_2|^2)^{2k+1}}{(2k + 1)! \sinh(|\alpha_1|^2 + |\alpha_2|^2)}.$$  \hspace{1cm} (43)

For this case, $n_1$ is even and $n_2$ is odd number, or vice versa. For single–mode case the photon distribution functions demonstrate super and sub–Poissonian properties for even and odd coherent states, respectively. The same conclusions may be drawn for two–mode (and multimode) even and odd coherent states.

4 Some Relations for Wigner and Q–functions

The Wigner function of a system $W(p, q) = W(Q)$ is expressed in terms of density matrix in coordinate representation as (see, for example, [12])

$$W(p, q) = \int \rho(q + \frac{u}{2}, q - \frac{u}{2}) \exp(-ipu) \, du.$$  \hspace{1cm} (44)

The inverse transform is

$$\rho(x, x') = \frac{1}{(2\pi)^N} \int W(\frac{x + x'}{2}, p) \exp[ip(x - x')] \, dp.$$  \hspace{1cm} (45)

The Q–function is expressed in terms of the Wigner function through the 3N–dimensional integral transform

$$Q(B) = \frac{1}{(2\pi)^N} \int \Phi_B(x, x', p) W(\frac{x + x'}{2}, p) \, dx \, dx' \, dp$$  \hspace{1cm} (46)

with the kernel

$$\Phi_B(x, x', p) = \pi^{-N/2} \exp[ip(x - x')] - \frac{1}{2}B(\sigma_{Nz} + I_{2N})B - \frac{X^2}{2} + \sqrt{2}B\sigma_{Nx}X,$$  \hspace{1cm} (47)

where the 2N–vector $X = (x, x')$ is introduced. If one has the Q–function the Wigner function is given by the integral transform

$$W(p, q) = \frac{1}{\pi^{2N}} \int \{ \prod_{k=1}^N d^2\beta_k \, d^2\gamma_k \, du_k \tilde{\Phi}_k(u_k, \tilde{B}) \} Q(\tilde{B}),$$  \hspace{1cm} (48)
where the argument of the Q–function $\mathbf{B}$ is replaced by the $2N$–vector with complex components
\[ \tilde{\mathbf{B}} = (\beta_1, \beta_2, \ldots, \beta_N, \gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*), \]
and the kernel has the form
\[ \tilde{\Phi}_k(u_k, \tilde{\mathbf{B}}) = \pi^{-1/2} \exp[-|\beta_k|^2 - |\gamma_k|^2 + \sqrt{2}(q_k + \frac{u_k}{2})\beta_k + \sqrt{2}(q_k - \frac{u_k}{2})\gamma_k^*] \]
\[ - \frac{1}{2}(q_k + \frac{u_k}{2})^2 - \frac{1}{2}(q_k - \frac{u_k}{2})^2 - \frac{\beta_k^2}{2} - \frac{\gamma_k^*}{2} - ip_ku_k + \gamma_k\beta_k^*. \] (49)
The density matrix in coordinate representation is related to the Q–function
\[ \rho(x, x', t) = \pi^{-2N} \int \prod_{k=1}^N d^2\beta_k d^2\gamma_k \phi_k(\tilde{\mathbf{B}}) \exp[-\frac{1}{2}(x_k^2 + x_k'^2)]Q(\tilde{\mathbf{B}}), \] (50)
where the kernel of the transform is
\[ \phi_k(\tilde{\mathbf{B}}) = \pi^{-1/2} \exp[-|\beta_k|^2 - |\gamma_k|^2 + \sqrt{2}x_k\beta_k + \sqrt{2}x_k^*\gamma_k^* - \frac{\beta_k^2}{2} - \frac{\gamma_k^*}{2} + \gamma_k\beta_k^*]. \] (51)
The evolution of the Wigner function and Q–function for systems with quadratic Hamiltonians for any state is given by the following prescription. Given the Wigner function $W(p, q, t = 0)$ for the initial time $t = 0$. Then the Wigner function for the time $t$ is obtained by the replacement
\[ W(p, q, t) = W(p(t), q(t), t = 0), \]
where the time–dependent arguments are the linear integrals of motion of the quadratic system found in [20], [12]. The same ansatz is used for the Q–function. Namely, given the Q–function of the quadratic system $Q(\mathbf{B}, (t = 0))$ for the initial time $t = 0$. Then the Q–function for the time $t$ is given by the replacement
\[ Q(\mathbf{B}, t) = Q(\mathbf{B}(t), t = 0), \]
where the $2N$–vector $\mathbf{B}(t)$ is the integral of motion linear in annihilation and creation operators found in [20], [12]. This ansatz follows from the statement that the density operator of the Hamiltonian system is the integral of motion, and its matrix elements in any basis must depend on the appropriate integrals of motion. In particular, the Wigner function and Q–function depend just on the linear invariants found in [20], [12].

5 Multivariable Hermite Polynomials

For parametric forced oscillator the transition amplitude between its energy levels has been calculated as overlap integral of two generic Hermite polynomials with a Gaussian function (Frank–Condon factor) and expressed in terms of Hermite polynomials
of two variables \[3\]. For N–mode parametric oscillator the analogous amplitude has
been expressed in terms of Hermite polynomials of 2N variables, i. e. the overlap
integral of two generic Hermite polynomials of N variables with a Gaussian function
(Frank–Condon factor for a polyatomic molecule) has been evaluated in \[20\]. The
corresponding result uses the formula

\[
\int H_n^{(R)}(x)H_m^{(r)}(\Lambda x + d) \exp(-xm + cx) \, dx = \frac{\pi^{N/2}}{\sqrt{\det m}} \exp\left(\frac{1}{4}cm^{-1}c\right) H_{\rho}^{(\rho)}(y),
\]

where the symmetric 2N×2N–matrix

\[
\rho = \begin{pmatrix} R_1 & R_{12} \\ \bar{R}_{12} & R_2 \end{pmatrix}
\]

with N×N–blocks \( R_1, R_2, \bar{R}_{12} \) is expressed in terms of symmetric N×N–matrices \( R, r, m \) and N×N–matrix \( \Lambda \) in the form

\[
R_1 = R - \frac{1}{2}Rm^{-1}R,
\]

\[
R_2 = r - \frac{1}{2}r\Lambda m^{-1}\bar{\Lambda}r,
\]

\[
\bar{R}_{12} = -\frac{1}{2}r\Lambda m^{-1}R.
\]

Here the matrix \( \bar{\Lambda} \) is transposed matrix \( \Lambda \) and \( \bar{R}_{12} \) is transposed matrix \( R_{12} \). The
2N–vector \( y \) is expressed in terms of N–vectors \( c \) and \( d \) in the form

\[
y = \rho^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]

where the N–vectors \( y_1 \) and \( y_2 \) are

\[
y_1 = \frac{1}{4}(Rm^{-1} + m^{-1}R)c
\]

\[
y_2 = \frac{1}{4}(r\Lambda m^{-1} + m^{-1}\bar{\Lambda}r)c + rd.
\]

For matrices \( R = 2, r = 2 \) the above formula (52) yields

\[
\int \left\{ \prod_{i=1}^{N} H_{n_i}(x_i)H_{m_i}\left(\sum_{k=1}^{N} \Lambda_{ik}x_k + d_i\right) \right\} \exp(-xm + cx) \, dx
\]

\[
= \frac{\pi^{N/2}}{\sqrt{\det m}} \exp\left(\frac{1}{4}cm^{-1}c\right) H_{\rho}^{(\rho)}(y),
\]

(57)
with $N \times N$–blocks $R_1$, $R_2$, $R_{12}$ expressed in terms of $N \times N$–matrices $m$ and $\Lambda$ in the form

\[
R_1 = 2(1 - m^{-1}), \quad R_2 = 2(1 - \Lambda m^{-1} \tilde{\Lambda}), \quad \tilde{R}_{12} = -2 \Lambda m^{-1}.
\]

The $2N$–vector $y$ is expressed in terms of $N$–vectors $c$ and $d$ in the form (55) with

\[
y_1 = m^{-1}c, \quad y_2 = \frac{1}{2}(\Lambda m^{-1} + m^{-1} \tilde{\Lambda})c + 2d.
\]

If the symmetric matrix $\rho$ has the block–diagonal structure

\[
\rho = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}
\]

with the symmetric $S \times S$–matrix $R_1$ and the symmetric $(2N-S) \times (2N-S)$–matrix $R_2$ the multivariable Hermite polynomial is represented as the product of two Hermite polynomials depending on $S$ and $2N-S$ variables, respectively,

\[
H_{k}^{(\rho)}(y) = H_{n_S}^{(R_1)}(y_1) H_{n_{2N-S}}^{(R_2)}(y_2),
\]

where the $2N$–vector $y$ has the vector–components

\[
y = (y_1, y_2),
\]

and $2N$–vector $k$ has the components

\[
k = (n_S, n_{2N-S}) = (n_1, \ldots, n_S, n_{S+1}, \ldots, n_{2N}).
\]

The partial case of this relation is the relation for the Hermite polynomials with the matrix $R$ with complex conjugate blocks $R_1 = r$, $R_2 = r^*$, and complex conjugate vector–components $y_1 = y_2^*$

\[
H_{k}^{(\rho)}(y) = |H_{n_S}^{(R_1)}(y_1)|^2, \quad S = N.
\]

The calculated integrals are important to evaluate the Green function or density matrix for the systems with quadratic Hamiltonians. The partial cases of multivariable Hermite polynomials determine some other special functions $[16]$, $[17]$. 


6 Squeezing in Parametric Oscillator

For the parametric oscillator with the Hamiltonian
\[ H = -\frac{\partial^2}{\partial x^2} + \frac{\omega^2(t)x^2}{2}, \]  
where we take \( \hbar = m = \omega(0) = 1 \), there exists the time-dependent integral of motion
\[ A = \frac{i}{\sqrt{2}}[\epsilon(t)p - \dot{\epsilon}(t)x], \]  
where
\[ \ddot{\epsilon}(t) + \omega^2(t)\epsilon(t) = 0, \quad \epsilon(0) = 1, \quad \dot{\epsilon}(0) = i, \]  
satisfying the commutation relation
\[ [A, A\dagger] = 1. \]

It is easy to show that the packet solutions of the Schrödinger equation may be introduced and interpreted as coherent states [8], since they are eigenstates of the operator \( A (61) \), of the form
\[ \Psi_\alpha(x, t) = \Psi_0(x, t) \exp\left\{ -\frac{\alpha^2}{2} - \frac{\alpha^2 \dot{\epsilon}(t)}{2 \epsilon(t)} + \frac{\sqrt{2} \alpha x}{\epsilon(t)} \right\}, \]  
where
\[ \Psi_0(x, t) = \pi^{-1/4} \epsilon(t)^{-1/2} \exp\left( \frac{i \dot{\epsilon}(t)x^2}{2 \epsilon(t)} \right) \]  
is analog of the ground state of the oscillator and \( \alpha \) is a complex number. The variances of the position and momentum of the parametric oscillator in the state (65) are
\[ \sigma_x = \frac{|\epsilon(t)|^2}{2}, \quad \sigma_p = \frac{|\dot{\epsilon}(t)|^2}{2}, \]  
and the correlation coefficient of the position and momentum has the value corresponding to minimization of the Schrödinger uncertainty relation [21]
\[ \sigma_x \sigma_p = \frac{1}{4} \frac{1}{1 - r^2}. \]

Another normalized solution to the Schrödinger equation
\[ \Psi_{\alpha m}(x, t) = 2N_m \Psi_0(x, t) \exp\left\{ -\frac{|\alpha|^2}{2} - \frac{\epsilon^*(t)\alpha^2}{2\epsilon(t)} \right\} \cosh \frac{\sqrt{2} \alpha x}{\epsilon(t)}, \]  
where
\[ N_m = \frac{\exp(|\alpha|^2/2)}{2\sqrt{\cosh |\alpha|^2}} \]
is the even coherent state \([4]\) (the Schrödinger cat male state). The odd coherent state of the parametric oscillator (Schrödinger cat female state)

\[
\Psi_{\alpha f}(x, t) = 2N_f\Psi_0(x, t) \exp\left\{-\frac{|\alpha|^2}{2} - \frac{\varepsilon^*(t)\alpha^2}{2\varepsilon(t)}\right\} \sinh\frac{\sqrt{2}\alpha x}{\varepsilon(t)},
\]

(70)

where

\[
N_f = \frac{\exp(|\alpha|^2/2)}{2\sqrt{\sinh |\alpha|^2}}
\]

(71)

satisfies the Schrödinger equation and is the eigenstate of the integral of motion \(A^2\) (as well as the even coherent state) with the eigenvalue \(\alpha^2\). These states are one-mode examples of squeezed and correlated Schrödinger cat states constructed in \([6]\).

7 Conclusion

The discussed nonclassical states of the fields (photons, phonons, pions) may be created in nonlinear interactions. The particle statistics with squeezing and correlations of quadrature components may give an experimental evidence of producing the new types of the field states. The Schrödinger–like equations are used also in other branches of physics like fiber optics. In \([22]\) the Schrödinger–like equations has been introduced to describe the charged particle beam in accelerator. The approach described in the talk may be also applied in the classical physics using the quantum–mechanical methods.

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