Sparse Non-Negative Recovery from Biased
Subgaussian Measurements using NNLS

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Abstract—We investigate non-negative least squares (NNLS) for the recovery of sparse non-negative vectors from noisy linear and biased measurements. We build upon recent results from [1] showing that for matrices whose row-span intersect the positive orthant, the nullspace property (NSP) implies compressed sensing recovery guarantees for NNLS. Such results are as good as for $\ell_1$-regularized estimators but do not require tuning parameters that depend on the noise level. A bias in the sensing matrix improves this auto-regularization feature of NNLS and the NSP then determines the sparse recovery performance only. We show that NSP holds with high probability for biased subgaussian matrices and its quality is independent of the bias.

Index Terms—Compressed sensing, Sparsity, NNLS, bias, subgaussian, nullspace property

I. INTRODUCTION

Compressed sensing (CS) algorithms based on $\ell_1$-regularization, like LASSO or basis pursuit denoising (BPDN) etc., are among the most well-known sparse recovery algorithms today, and as convex programs, the preferred tools in many applications with well-investigated recovery guarantees. The idea of CS itself is based on the fact that the intrinsic dimension of many signals or large data sets is typically far less than their ambient dimensions. In particular, the sparse recovery of images, videos, audio data, network status information, like activity and novel coding techniques for wireless communication.

However, in such practical real world applications the signals and also the sensing matrices itself are also subject to further constraints. Sparsity or, more general, compressibility can be regarded here as first order structure and the signals of interest exhibit additional structure like block-sparse, tree-sparse and, most importantly here, known sign patterns yielding to a non-negativity constraint. In particular such non-negative and sparse structures also arise naturally in certain empirical inference problems, like activity detection [2], [3], network tomography [4], statistical tracking (see e.g. [5]), compressed imaging of intensity patterns [6] and visible light communication based positioning [7]. Interestingly non-negativity itself already provides certain uniqueness guarantees and therefore the underlying mathematical problem has received considerable attention in its own right [8], [9], [10], [11], [12].

Donoho et al. investigated in [13] the noiseless case in the language of convex polytopes. They show that the sparsest non-negative solutions can be found by convex optimization if the sparsity of the sparsest solution is smaller than a fraction of the number of equations. Bruckstein, Elad and Zibulevsky [14] investigated the uniqueness of non-negative and sparse solutions in the noiseless case when also the entries of the sensing matrix are non-negative, or more general, if the matrix has a row-span intersecting the positive orthant (referred to as the $M^+$-criterion defined below).

If both matrices and the entries of the matrix are non-negative, or more general, if the matrix has a row-span intersecting the positive orthant (referred to as the $M^+$-criterion defined below). They found that matrices belonging to the $M^+$-class provide uniqueness of non-negative sparse solutions and reconstruction therefore reduces to just finding a feasible solution. In the noisy case, a meaningful approach consists of replacing the search for an exact non-negative solution of the set of linear equations by a good approximation minimizing the residual in a certain norm (which usually depends on further assumptions like its distribution etc.). In the case of $\ell_2$-norms, it may be therefore sufficient to solve the Non-Negative Least squares (NNLS).

In [15], Slawsky and Hein discussed in the noisy case where the random noise is Gaussian or subgaussian. They show (under a condition similar to the $M^+$-criterion, i.e., the self-regularizing property) that NNLS intrinsically promotes sparsity. They further find for subgaussian noise distributions reconstruction guarantees in the form of upper bounds on the norm of the error vector, including the $\ell_\infty$-case. This bound is important for hard thresholding for sparse recovery. Meinshausen proved in [16] similar results for the $\ell_1$-norm of the error vector under different assumptions on the measurements matrix. It seems also that the idea of non-negativity as a particular conic constraint extends to other cones. For example, Wang et al. [17] established such results also for the cone of positive semi-definite matrices.

Flinth and Keiper investigated in [18] reconstruction of sparse binary signals through box-constrained basis pursuit using biased measurement matrices, and state conditions, under which the solution can be found through box-constrained least squares instead.

In [19], Kueng and Jung established reconstruction guarantees for (adversarial) noise in terms of the Null Space Property (NSP) and the $M^+$-criterion. The conditions in [20] (NSP and the $M^+$ criterion), play a similar role to the conditions in [21] (self regularizing property and the restricted eigenvalues condition). See also [22] for similar steps in the low-rank matrix recovery case. We base most of our analysis on [20]. Note that NSP is a sufficient and necessary condition for the success of $\ell_1$-recovery programs like BPDN given a correct bound on the noise power. In contrast NNLS always succeeds.
without having a-priori knowledge about the noise power and instead the error scales in terms of the instantaneous noise power. To have such a feature for BPDN one usually needs to investigate the quotient property (see for example [18, Ch.11]).

Our contribution: First, we review and extend the theory of non-negative sparse recovery using NNLS. We consider then biased subgaussian random \( m \times n \) measurement matrices and illustrate that a bias \( \mu \geq 0 \) improves the self-regularizing property of NNLS. However, proving NSP without taking care of the bias, for example in using the small ball method [19, 20], yields \( \ell_2 \)-recovery guarantees for \( s \)-sparse vectors for \( m \gtrsim s \cdot (\sqrt{\log(n/s)} + \mu)^2 \) observations. To overcome this suboptimal scaling we combine a debiasing step with the small ball method showing that NNLS has indeed the following (non-trivial) \( \ell_q \)-recovery guarantees (informal version):

**Theorem 1.** Let \( A \in \mathbb{R}^{m \times n} \) be a random matrix whose biased rows \( a_i = a_{0,i} + \mu 1^T \) are independent and \( a_{0,i} \) are isotropic \( 1 \)-subgaussian random vectors. Then, for \( q \geq 2 \) and \( s \) solution \( x \) of the NNLS \( (\ref{eq:nnls}) \) for \( y = Ax + n \) satisfies for \( p \in [1, q] \) the following error bound:

\[
\|x - x_0\|_p \leq C \left( \frac{1 + \sigma_s(x)}{s^{1/q}} \right) \|z\|_2,
\]

with constants \( C, D \) and \( \tau \) defined below, and \( \sigma_s(x) \) is given in \( (\ref{eq:sigma}) \).

The complete statement is given in Theorem 6 in Section III-C below.

II. SYSTEM MODEL AND OBJECTIVES

We consider the problem of recovering a non-negative and sparse vector \( x_0 \in \mathbb{R}^n \) from noisy linear observations of the form:

\[
y = Ax_0 + n \in \mathbb{R}^m
\]

where \( A \in \mathbb{R}^{m \times n} \) is the measurement (sensing) matrix and \( n \in \mathbb{R}^m \) denotes additive noise. For example, \( x_0 \) could be \( s \)-sparse meaning that its non-zero entries are supported on a small subset \( S \subset \{1, \ldots, n\} \) of cardinality \( |S| \leq s \). In this work we assume that the sensing matrix \( A \) is a known but random matrix, and we will investigate here distributions of \( A \) which allow a robust and stable reconstruction of \( x_0 \) with overwhelming probability. By “robust and stable” we mean that the error will scale appropriately with respect to the noise power and the algorithm essentially recovers also approximately sparse vectors. Such results are known, for example, for the basis pursuit denoising (BPDN) [18] which is the following convex program:

\[
\hat{x}_{\text{BPDN}} = \arg\min_{x \in \mathbb{R}^n} \|x\|_1 \text{ s.t. } \|y - Ax\|_2 \leq \eta,
\]

Briefly, if \( \eta \) is chosen such that \( \|y - Ax_0\|_2 \leq \eta \) and if the sensing matrix \( A \) has the robust null space property, there are upper bounds on reconstruction error, like \( \|x_0 - \hat{x}_{\text{BPDN}}\|_2 \), as a function of the noise variance and the error of the best \( s \)-term approximation of \( x_0 \), see [18]. However, in a practical setting one often has to estimate the noise level \( \eta \) first and also the optimal tuning of compressed sensing algorithms itself is a difficult task. Even more, there are applications where this assumption is critical since the noise level may depend on the unknown vector to recover. Prototypical examples are sparse recovery under Poisson noise model [21] or covariance matching problems like in [2].

A. Null Space Properties

A well-known tool to characterize the performance of \( \ell_1 \)-regularized programs like the BPDN [1] above is the following formulation of the robust null space property (\( \ell_q \)-NSP) with respect to the \( \ell_2 \)-norm [18].

**Definition 1.** For some \( q \geq 1 \), a matrix \( A \) is said to have the \( \ell_q \)-robust null space property with respect to the \( \ell_2 \)-norm (\( \ell_q \)-NSP) with parameters \( \rho \in (0, 1) \) and \( \tau > 0 \) if \( \forall v \in \mathbb{R}^n \) and \( \forall S \subset \{1, \ldots, n\} \) with \( |S| \leq s \):

\[
\|v_S\|_q \leq \rho \frac{s^{1/q}}{s^{1/q} + \tau} \|v_{S^C}\|_1 + \tau \|Av\|_2
\]

where \( v_S \) is the vector built from \( v \) but all indices not in \( S \) are set to zero, and \( S^C \) is the complement of \( S \).

Due to the inequality \( \|v_S\|_p \leq s^{1/p - 1/q} \|v_S\|_q \) for \( 1 \leq p \leq q \), the \( \ell_q \)-NSP implies the \( \ell_p \)-NSP for \( p \in [1, q] \) w.r.t. the norm \( s^{1/p - 1/q} \| \cdot \|_2 \) in the form

\[
\|v_S\|_p \leq \rho \frac{s^{1/p}}{s^{1/p} + \tau} \|v_{S^C}\|_1 + s^{1/p - 1/q} \|Av\|_2.
\]

Let us recall the definition of the error of the best \( s \)-term approximation of \( x \) in the \( \ell_p \)-norm which is:

\[
\sigma_s(x)_p = \inf_{z: \|z\|_0 \leq s} \|x - z\|_p
\]

Now, [18] Theorem 4.25 states the following:

**Lemma 1.** Given \( 1 \leq p \leq q \), suppose that \( A \in \mathbb{R}^{m \times n} \) has the \( \ell_q \)-NSP with \( 0 \leq \rho < 1 \) and \( \tau > 0 \). Then, for any \( x, z \in \mathbb{R}^n \),

\[
\|z - x\|_p \leq C \left( \frac{1 + \sigma_s(x)}{s^{1/q}} \right) \|z\|_1 + 2\sigma_s(x)\|1\| + Ds^{1/p - 1/q} \|Av\|_2
\]

with \( C = \frac{(1+p)^2}{1-p} \), \( D = \frac{(1+p)^2}{1-p} \).

Note that the requirement [2] in Definition 1 holds for any subset \( S \) of cardinality at most \( s \) if it holds for the subset \( S_{\text{max}} \) containing the strongest \( s \) components. Let us define \( v_S := v_{S_{\text{max}}} \) and \( v_S = \v_S \). Property [2] is also invariant with respect to re-scaling, i.e., wlog we may assume here \( \|v\|_q = 1 \).

In addition, any vector \( v \) which satisfies \( \|v_S\|_q \leq \frac{s^{1/p}}{s^{1/q} + \tau} \|v_{S^C}\|_1 \) fulfills this condition independently of \( A \), so we restrict our attention to the set:

\[
T^q_{\rho, s} = \left\{ v \in \mathbb{R}^n : \|v_S\|_q > \frac{\rho s^{1/q}}{s^{1/q} + \tau} \|v_{S^C}\|_1, \|v\|_q = 1 \right\}.
\]
Notice, that a matrix \( A \) has the \( \ell_q \)-NSP of order \( s \) with parameters \( \rho \) and \( \tau > 0 \) if the following bound holds:

\[
\inf_{v \in T^{s}_{\rho,\tau}} \| A v \|_2 \geq \frac{1}{\tau}.
\]

(5)

Then we have for \( v \in T^{s}_{\rho,\tau} \):

\[
\| v \|_q \leq \| A v \|_2 \leq \frac{\rho}{s^{1-1/q}} \| v \|_1 + \tau \| A v \|_2.
\]

B. Non-negative Sparse Recovery via NNLS

Theorem 4 in [1] states that non-negative \( s \)-sparse signals can be robustly and stably recovered (precise statement is Theorem 2 below) with the non-negative least squares (NNLS):

\[
\hat{x}_{\text{NNLS}} = \arg \min_{x \in \mathbb{R}^n} \| y - Ax \|_2,
\]

(6)

provided that the sensing matrix \( A \) has both the \( \ell_2 \)-NSP of order \( s \) and it satisfies the following \( M^+ \)-criterion.

**Definition 2.** A matrix \( A \in \mathbb{R}^{m \times n} \) satisfies the \( M^+ \) criterion if

\[
A \in M^+ = \left\{ M \in \mathbb{R}^{m \times n} : \exists t \in \mathbb{R}^n : M^T t > 0 \right\}.
\]

(7)

We like to mention here that this can be formulated for complex matrices in the usual way. The NNLS in (6) has the appealing advantage, that it does not require a-priori knowledge of some \( \eta \) such that \( \| y - Ax_0 \|_2 \leq \eta \). We repeat Theorem 4 from [1] because of its importance to this work, but first we need to define the following condition number of a matrix as:

\[
\kappa(A) = \min \left\{ \| W \| \| W^{-1} \| : \exists t \in \mathbb{R}^n \text{ with } W = \text{diag} \left( A^T t \right) > 0 \right\}
\]

(8)

with \( \| W \| \) being the spectral norm of the diagonal matrix \( W \).

**Theorem 2 (Theorem 4 in [1]).** Let \( A \) be a matrix satisfying both \( \ell_q \)-NSP of order \( s \) with constants \( 0 < \rho < 1 \) and \( \tau > 0 \) and the \( M^+ \) criterion with \( \kappa \) achieved for \( t \). Assume in addition that \( \kappa \rho < 1 \), then for \( 1 \leq p \leq \kappa q \):

\[
\| \hat{x} - x_0 \|_p \leq \frac{C}{s^{1-1/p} \sigma_s(x)} + D \frac{s^{1-1/p}}{s^{1-1/q}} (\| t \|_2 + \tau) \| n \|_2,
\]

where \( C = \frac{2^{(1+q)p/2}}{1-\rho}, \quad D = \frac{q+\rho}{q-\rho} \max \{ \kappa, \| W^{-1} \| \} \) and \( \hat{x} \) is the solution of NNLS in (6) for \( y = Ax_0 + n \).

Note that this theorem has been presented in [1] Theorem 4] only for \( p = q = 2 \). However, its extension to \( 1 \leq p \leq q \) is immediate by using the fact (3) that \( \ell_p \)-robust nullspace property with respect to a norm \( \| \cdot \|_p \) implies \( \ell_p \)-robust nullspace property with respect to the norm \( s^{1/p-1/q} \| \cdot \|_p \) (see the proof of [1] Theorem 4) and combine this with Lemma 1 or [18] Theorem 4.25, respectively). Furthermore, a closer inspection of the proof of [1] Theorem 4 also shows that one could easily replace \( \| t \|_2 \) in Theorem 2 with \( \| t \|_2/s^{1-1/q} \) which may have impact when \( \| t \|_2 \) has some undesired scaling.

**C. Nullspace Properties through the Small Ball Method**

A well-known tool to prove that a random matrix \( A \) (with independent rows) has the nullspace property, i.e. (5) holds then with high probability, is Mendelson’s small ball method [19] (see here also [20]). This method is essentially the following theorem:

**Theorem 3 ([19], [20]).** Fix a set \( E \subset \mathbb{R}^n \). Let the rows of a matrix \( A \in \mathbb{R}^{m \times n}, \quad a_1, ..., a_m \), be independent copies of a random vector \( a \in \mathbb{R}^n \). Define \( h = \frac{1}{\sqrt{m}} \sum_{k=1}^m \epsilon_k a_k \), where \( \{ \epsilon_k \}_{k=1}^m \) is a Rademacher sequence. Then, for \( t > 0 \) and \( \xi > 0 \), the bound

\[
\inf_{v \in E} \| Av \|_2 \geq \xi \sqrt{m} Q_{2\xi} (E, a) - \xi t - 2W_m (E, a)
\]

with:

\[
Q_{\xi} (E, a) = \inf_{u \in E} \mathbb{P} ( \| a, u \| \geq \xi )
\]

and

\[
W_m (E, a) = E \left[ \sup_{u \in E} \langle h, u \rangle \right],
\]

holds with probability at least \( 1 - e^{-2t^2} \).

This theorem was used by Mendelson in [19] in the context of learning theory, to obtain sharp bounds on the performance of empirical risk minimization. It was later adopted by the compressed sensing community. Tropp used it in [20] to bound the minimum conic singular value of matrices in certain recovery problems. It was used successfully in [22] and [11] to establish the NSP for certain classes of sensing matrices. In [23] a version for the complex setting has been established as well.

1) Impact of the Bias: In certain cases, this method – directly applied without further adaptation – may provide sub-optimal results. For illustration, we bring here a concrete example, sufficiently biased matrices, and look at the number of measurements sufficient for stable recovery. We shall demonstrate for the case \( q = 2 \) that for a random matrix \( A \in \mathbb{R}^{m \times n} \), whose entries are i.i.d. and distributed as \( \mathcal{N} (\mu, 1) \), the bound on the number of measurements obtained from Theorem 3 scales with \( \mu \) in an undesired manner.

To use Theorem 3, we take therefore the rows of \( A \) as independent copies of \( a = g + \mu 1 \) where \( g \sim \mathcal{N} (0, I) \) is a standard iid. Gaussian vector. We will only sketch the steps since this can be found in several works [22, 11, 23].

First, one needs to bound \( Q_{2\xi} (T_{p,\tau}^q, a) \). Using Paley-Zygmund inequality, we show in Appendix V-A that, for any unit vector \( z \in \mathbb{R}^n \) and \( \theta \in [0, 1/2] \), we have:

\[
\mathbb{P} \left( \| a^T z \| \geq \theta \right) \geq \frac{(1 - \theta^2)^2}{3}.
\]

\(^{1}\) \( X \) is a Rademacher variable if \( X = 1 \) or \(-1\) with equal probability.
Now, we will bound $W_m(T_{ρ,s}^q,a)$. For this, first notice that $T_{ρ,s}^q$ contains all normalized $s$-sparse vectors, i.e
\[ T_{ρ,s}^q \supset \{ x \in \mathbb{R}^n : \|x\|_0 \leq s, \|x\|_q = 1 \} . \]
A converse result is also known (Lemma 3.2 in [22]):
\[ T_{ρ,s}^q \subset (2 + ρ^{-1}) \text{conv}(Σ_q^s) , \]
from which we can have the bound
\[ W_m(T_{ρ,s}^q,a) \leq \frac{3}{ρ} W_m(Σ_q^s,a) \leq \frac{3s^{1/2 - 1/q}}{ρ} W_m(Σ_q^s,a), \]
where the last bound is due to the relation $\|x\|_2 \leq s^{1/2 - 1/q} \|x\|_q$ for $s$-sparse vectors (Hoelder inequality). In Appendix V we then show that
\[ W_m(Σ_q^s,a) \leq \sqrt{2s \log \left( \frac{n}{s} \right) + 2s + μ\sqrt{2s}}. \]

One could think that this bound is too pessimistic, but Figure 1 shows indeed the linear dependency of $W_m(T_{ρ,s}^q,a)$ in $μ$, which shifts $W_m(T_{ρ,s}^q,a)$ away from zero. Nevertheless, summarizing the results, for $ξ = θ/2$:
\[ \inf_{v \in T_{ρ,s}^q} \|Av\|_2 \geq ξ \sqrt{m}Q_ξ(T_{ρ,s}^q,a) - ξt - 2W_m(T_{ρ,s}^q,a) \]
\[ \geq \frac{θ(1 - θ^2)}{6} \sqrt{m} - \frac{θ}{2}t \]
\[ - \frac{6}{ρ} \left( \sqrt{2s \log \left( \frac{n}{s} \right) + 2s + μ\sqrt{2s}} \right) . \]

If we choose, for example, $ξ^2 = 1/8$ and $t = \sqrt{m}/24$, we have that the right hand side is positive if:
\[ m \geq \left( \frac{6 \cdot 24 \sqrt{2}}{ρ} \right)^2 2s \left( \sqrt{\log \left( \frac{en}{s} \right) + μ} \right)^2 \]
Thus, from this result one might think that $μ$ directly affects the number of measurements. A main purpose of our work is to show that this is not the case.

In the next section we will therefore present a debiased version of Theorem 3, which indeed brings back the known results on the number of measurements for centered matrices, independent of $μ$ but at the cost of doubling $m$.

III. MAIN RESULTS

We have shown above that usual small ball method for establishing the nullspace property suffers from a bias $μ$ in the measurement matrices whereby a bias will be essential for self-regularizing property (the $M^+$-criterion in (7) of NNLS). Our main result, however, shows that this is not the case and NSP is essentially independent of bias. To establish this result we present next a debiased version of the small ball result.

A. Debiased Mendelson’s small balls method

The following results parallels Theorem 3 but handles better a bias $e \in \mathbb{R}^n$ (constant offset) in the random measurement vectors (rows of the measurement matrix).

**Theorem 4.** Fix a set $E \subset \mathbb{R}^n$. Let the rows of a matrix $A \in \mathbb{R}^{m\times n}$, $a_1,...,a_m$, be independent copies of a random vector $a$. Define the matrix $B \in \mathbb{R}^{[m/2] \times n}$ whose rows are $b_i = (a_{2i-1} - a_{2i})/\sqrt{2}$ and $h = \frac{1}{\sqrt{[m/2]}} \sum_{k=1}^{[m/2]} \epsilon_k b_k$, where $\{ \epsilon_k \}$ is a Rademacher sequence. Then, for $t > 0$ and $ξ > 0$, The bound
\[ \inf_{v \in E} \|Av\|_2 \geq ξ \sqrt{\frac{m - 1}{2}} Q_ξ(E, b) - ξt - 2W_{[m/2]}(E, b) \]
with $Q_ξ$ and $W_m$ as defined in Theorem 3 holds with probability at least $1 - e^{-2t^2}$.

A first version, proving the results in [24] and only valid for distributions which are symmetric around $e$, has been presented by the authors already in the first arXiv version [25] of this work.

**Proof.** The following technique is motivated by a similar debiasing step in [26, Sec. IV(b)].
\[ \|Av\|_2^2 = \sum_{i=1}^{2[m/2]} |\langle a_i, v \rangle|^2 \geq \sum_{i=1}^{2[m/2]} |\langle a_{2i-1}, v \rangle|^2 + |\langle a_{2i}, v \rangle|^2 \]
\[ \geq \sum_{i=1}^{[m/2]} |\langle a_{2i-1}, v \rangle - a_{2i}, v \rangle|^2 \]
\[ = \sum_{i=1}^{[m/2]} \|b_i, v\|^2 = \|Bv\|_2^2 , \]
where in (a) we used Cauchy-Schwarz inequality and (b) the reverse triangle inequality. The last missing step is just an application of Theorem 2 on the matrix $B$, and the conclusion follows.

This theorem will allow to prove the NSP of biased matrices even if $\mu \propto \sqrt{m}$, and even for any fixed bias, it improves the previous bounds (up to constants).

**Corollary 1.** Let $A \in \mathbb{R}^{m \times n}$ be a matrix whose rows can be written in the form $a_i = a_{0,i} + \mu_1$ where $\{a_{0,i}, 1 \leq i \leq m\}$ are independent copies of a random vector $a_0$ with the following properties:

- is sub-isotropic, i.e., $\mathbb{E} \left[ (a_{0,i}, v)^2 \right] \geq \|v\|^2_2$ for all $v \in \mathbb{R}^n$,
- 1-subgaussian, i.e., $\mathbb{E} \left[ \exp (t (a_{0,i}, v)) \right] \leq \exp (t^2)$ for all $v \in \mathbb{R}^n$ with $\|v\|^2_2 \leq 1$ and $t \in \mathbb{R}$,

then, $A$ has the $\ell_q$-NSP for all $q \geq 2$ with probability at least $1 - e^{-\omega(m^{-1/64})}$ if

$$m \geq \frac{2 \cdot 384^2}{\rho^2} s^{-2/2q} \left( 2 + \sqrt{\log \left( \frac{en}{s} \right)} \right)^2 + 1. \quad (10)$$

**Proof.** We will show that Equation (5) holds by using Theorem 4 and bounding $\inf_{v \in T_{\rho,s}^q} \|Bv\|_q$. By the definition of $B$, the fact that the rows of $A$ are sub-Gaussian and Theorem 7.27 in [18], we know that the rows of $B$ are also sub-Gaussian with the same sub-Gaussian norm. In addition:

$$\mathbb{E} \left[ (a_{0,i}, v)^2 \right] = \mathbb{E} \left[ \frac{(a_{0,2i-1} - a_{2i})}{\sqrt{2}}, v \right]^2 \leq \left( \frac{(a_{0,2i-1} - a_{2i})}{\sqrt{2}} \right)^2 + \frac{1}{2} \mathbb{E} \left[ (a_{0,2i-1}, v)^2 \right] - \mathbb{E} \left[ (a_{0,2i}, v) (a_{0,2i-1}, v) \right] = \frac{1}{2} \mathbb{E} \left[ (a_{0,2i-1}, v)^2 \right] + \frac{1}{2} \mathbb{E} \left[ (a_{0,2i}, v)^2 \right] \geq \|v\|^2_2,$$

so the rows of $B$ are sub-isotropic as well. Thus, we can use results from the proof of Corollary 5.2 from [22]. There, they first use the sub-isotropic and sub-Gaussian properties to bound $Q_\xi (T_{\rho,s}^q, b)$. Notice that for $q \geq 2$ and some $u \in T_{\rho,s}^q$ we have $1 = \|u\|_q \leq \|u\|_2$. Therefore:

$$\mathbb{P} \left( |(b, u)| > \xi \right) = \mathbb{P} \left( \frac{(b, u)}{\|u\|_2} > \frac{\xi}{\|u\|_2} \right) \geq \mathbb{P} \left( \frac{(b, u)}{\|u\|_2} > \xi \right).$$

With Paley-Zygmund inequality we then have [22] proof of Corollary 5.2:

$$\mathbb{P} \left( \frac{(b, u)}{\|u\|_2} > \xi \right) \geq (1 - \xi^2)^2$$

for $0 \leq \xi \leq 1$. Recall equation (9), and combine it with the following bound:

$$W_{[m/2]} (\Sigma_{\frac{n}{s}}, b) \leq 4 \sqrt{2} \left( 2 \sqrt{\xi} + \sqrt{s \log \left( \frac{en}{s} \right)} \right).$$

The last bound is due to Dudley’s inequality, taken from [18] Theorem 8.23, and, again, from the proof of Corollary 5.2 from [22]. Now, by choosing for example $t = \sqrt{m - 1} / (8 \sqrt{2})$ and $\xi^2 = 1/2$, we have with probability at least $1 - e^{-\omega(m^{-1/64})}$ (Theorem 4).

$$\inf_{v \in T_{\rho,s}^q} \|Av\|_q \geq \xi \sqrt{\frac{m - 1}{2}} Q_{2\xi} (T_{\rho,s}^q, b) - \xi t - 2W_{[m/2]} (T_{\rho,s}^q, b) \geq \xi \sqrt{\frac{m - 1}{2}} (1 - \xi^2)^2 - \xi t \geq \frac{3\xi - 1 - \xi}{\rho} = \frac{3\xi - 1 - q}{\rho} \sqrt{2} \left( 2 + \sqrt{s \log \left( \frac{en}{s} \right)} \right) = \frac{\sqrt{m - 1}}{16} = \frac{3\xi - 1 - q}{\rho} 8 \sqrt{2} \left( 2 + \sqrt{s \log \left( \frac{en}{s} \right)} \right).$$

If the bound is positive, we can set it to $1/\rho$ and we have the NSP. This bound is positive if

$$m \geq \frac{2 \cdot 384^2}{\rho^2} s^{-2/2q} \left( 2 + \sqrt{\log \left( \frac{en}{s} \right)} \right)^2 + 1.$$

Notice that this bound is independent of the bias now.

**B. Establishing the $\mathcal{M}^+$ criterion**

**Theorem 5.** Let $A \in \mathbb{R}^{m \times n}$ be a random matrix with independent rows. Assume that the columns of $A$ can be written as $a_{i}^{\text{col}} = a_{i}^{\text{col}} + \mu_1$ for $1 \leq i \leq n$, where $a_{i}^{\text{col}}$ are zero-mean 1-subgaussian random vectors (with independent components), i.e., satisfying $\mathbb{E} \left[ \exp (t (a_{i}^{\text{col}}, v)) \right] \leq \exp (t^2)$ for all $v \in \mathbb{R}^m$ with $\|v\|^2_2 \leq 1$ and $t \in \mathbb{R}$ for all $1 \leq i \leq n$.

Then $A \in \mathcal{M}^+$ with probability at least $1 - 2n \exp \left( -\frac{\omega m}{16} \right)$.

**Proof.** We choose the vector $t$ to be $t = \frac{1}{\mu_{1}}$ and have $w = A^T t$. Compute now

$$|w_i - 1| = \left| \mathbb{E} (A^T t)_i - 1 \right| = \left| \frac{1}{n\mu_1} \sum_{k=1}^{m} A_{k,i} - 1 \right| = \left| \frac{1}{n\mu_1} \sum_{k=1}^{m} (A_{k,i} - \mu_1) \right| = \left| \frac{1}{n\mu_1} \sum_{k=1}^{m} (A_{k,i} - \mathbb{E} \left[ A_{k,i} \right]) \right|.$$

We can now use Hoeffding’s inequality (Theorem 7.27 from [18]) to bound this term by:

$$\mathbb{P} \left( \left| \frac{1}{n\mu_1} \sum_{k=1}^{m} (A_{k,i} - \mathbb{E} \left[ A_{k,i} \right]) \right| \geq \frac{1}{2} \right) \leq 2 \exp \left( -\frac{\mu^2 m}{16} \right).$$
This guarantees that \( w_i \) is positive with probability better than the one written above, and by applying the union bound we have that all the components of \( w \) are positive with probability at least \( 1 - 2n \exp \left( -\frac{\epsilon^2 m}{16} \right) \).

Thus, for \( \mu \to \infty \) the probability \( \mathbb{P} (A \in M^+) \) converges to 1. This also proves that \( \kappa (A) \leq 3 \) with probability as in Theorem 5.

C. Main theorem

Now we combine here the results for NSP and \( M^+ \) criterion.

**Theorem 6.** Set \( \rho \in (0, 1/3) \), \( q \geq 2 \) and \( p \in [1, q) \). Let \( A \in \mathbb{R}^{m \times n} \) be a random matrix whose rows can be written as \( \mathbf{a}_i = \mathbf{a}_{0,i} + \mu \mathbf{1} \), where \( \{\mathbf{a}_{0,i}, 1 \leq i \leq m\} \) are independent copies of a random vector \( \mathbf{a}_0 \) which is:

- **sub-isotropic, i.e.,** \( \mathbb{E} [\langle \mathbf{a}_0, \mathbf{v} \rangle^2] \geq \|\mathbf{v}\|^2 \) for all \( \mathbf{v} \in \mathbb{R}^n \),
- **1-subgaussian, i.e.,** \( \mathbb{E} [\exp (t \langle \mathbf{a}_0, \mathbf{v} \rangle)] \leq \exp (t^2) \) for all \( \mathbf{v} \in \mathbb{R}^n \) with \( \|\mathbf{v}\|^2 \leq 1 \) and \( t \in \mathbb{R} \),

and whose columns can be written for all \( i \) as \( \mathbf{a}_i^{col} = \mathbf{a}_{0,i}^{col} + \mu \mathbf{1} \) with \( \mathbf{a}_{0,i}^{col} \) also being 1-subgaussian. If

\[
m \geq \frac{2 \cdot 384^2}{p^2} s^2 - 2/q \left( 2 + \sqrt{\log \left( \frac{16 n}{p} \right)} \right)^2 + 1,
\]

the following holds with probability at least \( 1 - e^{-(m-1)/64 - 2n \exp \left( -\frac{\epsilon^2 m}{16} \right)} \):

For all \( \mathbf{x}_0 \) and \( \mathbf{n} \), solution \( \hat{\mathbf{x}} \) of the NNLS \((6)\) for \( \mathbf{y} = A \mathbf{x}_0 + \mathbf{n} \) satisfies the following error bound:

\[
\|\hat{\mathbf{x}} - \mathbf{x}_0\|_p \leq \frac{C}{s^{1-1/p} \sigma_s (\mathbf{x})} + \frac{D}{s^{1-\sigma}} \left( \frac{1}{\sqrt{m} \mu} + \tau \right) \|\mathbf{n}\|_2,
\]

with constants \( C, D \) and \( \tau \) defined as in Theorem 2.

**Proof.** This is just an application of the union bound to bound the probability of the intersection of the events that Corollary 1 and Theorem 5 hold together, since the conditions of both are assumed to be satisfied. Since both the \( M^+ \) criterion and the \( \ell_2 \)-NSP hold, NNLS reconstructs the original vector with a reconstruction guarantee according to Theorem 2. Notice that we used our choice of \( t \) from the proof of Theorem 5 to compute \( \|t\|_2 \), and the condition \( \kappa_p < 1 \) is satisfied with high probability because \( \kappa < 3 \) (proof of Theorem 5).

IV. NUMERICAL EXPERIMENTS

In this section we provide numerical experiments to support the results of the previous sections. We measured the recovery performance in terms of the normalized square root of the MSE, given by:

\[
\text{MSE} = \frac{\|\hat{\mathbf{x}} - \mathbf{x}_0\|_2}{\|\mathbf{x}_0\|_2}
\]

for four algorithms: NNLS \((6)\) with biased and with centered sensing matrix, BPDN \((1)\) with biased and with centered sensing matrix. For NNLS we have used either the internal MATLAB routine “lsqnonneg” which is based on the active-set algorithm \((27)\) or a speed-optimized version “bwhiten” \((2)\) and BPDN has been solved using the CVX-toolbox.

For fixed \( n = 100 \) and \( s = 5 \), we take \( m \in \{20, 25, ..., 55, 60, 70, 80\} \), and randomly generate Gaussian matrices with i.i.d. entries \( N(\mu, 1) \) for \( \mu \in \{0, 20\} \). The non-negative signals were either binary vectors \( \mathbf{x}_0 \in \{0, 1\}^n \) or absolute value of a standard normal random variables. We add Gaussian white noise \( \mathbf{n} \) with zero mean and variance \( \sigma_n^2 = -20 \text{dB} \) to the measurements, and reconstruct the signal, either by NNLS \((6)\) or by BPDN \((1)\) using the instantaneous norm \( \|\mathbf{n}\|_2 \). Note that this already reflects some instantaneous extra knowledge for BPDN.

The results are given in Figures 2 and 3. First, we see that the bias is critical for reconstruction with NNLS, since non-biased matrices are likely not to belong to \( M^+ \). Therefore, NNLS with centered matrices performed worse than the other three algorithms. It is known that in general NNLS makes sense for this case only when \( m \geq n/2 \) (see for example comments in \((15)\)). For \( m \leq n/2 \) there is with high probability

\(^2\) B. Whiten, “nnls - Alternative to lsqnonneg”, https://de.mathworks.com/matlabcentral/fileexchange/38003-nnls-non-negative-least-squares.
no unique solution in the noiseless case, i.e., the NNLS performance is determined by the algorithm implementation. In contrast, NNLS reconstruction of a non-negative signal from biased measurements, where the sensing matrix both satisfies the NSP and belong to $\mathcal{M}^+$, achieved the best performance among the four.

Another thing worth noticing, is the equivalence between biased and centered matrices when using BPDN, when $m$ is above some threshold. This can be expected from Corollary \[\text{[1]}\] since, by this result, the bias plays no role for the NSP.

V. CONCLUSIONS

We obtained recovery guarantees for NNLS in the case of biased subgaussian matrices for non-negative sparse vectors. For that purpose, we needed to show that these matrices satisfy the NSP. For this, we first used Mendelson’s small ball method, and saw that the bias affects the bound in a negative way. We showed that the NSP of the biased matrix is implied by the NSP of a related centered matrix. This allows to ignore the bias and to find better bounds on the class of biased matrices, even when the bias is much bigger than the variance.

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APPENDIX

A. Bounds on $Q_{2r}(T_{\rho,2,s}^2, a)$ and $W_m(\Sigma_s^2, a)$

We can use Paley-Zygmund inequality to bound $Q_{2r}(T_{\rho,2,s}^2, a)$ from below. Let $z$ be a unit vector, and $\theta \in [0, 1/2]$, and recall that $a = g + \mu 1$ where $g \sim \mathcal{N}(0, I)$. Define $S = (a^T z)^2$. Then for any unit vector $z$:

$$S = (a^T z)^2 = (\mu 1^T z + g^T z)^2 = \mu^2 z^T 11^T z + z^T g g^T z + 2 \mu 1^T z z^T g.$$  

Now,  
$$E[S] = \mu^2 z^T 11^T z + \|z\|_2^2 \geq 1,$$

and therefore

$$\mathbb{P}(|a^T u| \geq \theta) \geq \mathbb{P}(S \geq \theta^2 E[S]) \geq (1 - \theta^2)^2 \frac{E[S]^2}{E[S^2]}.$$  

In addition,

$$S^2 = \mu^4 (z^T 11^T z)^2 + (z^T g)^4 + 4 \mu^2 (1^T z)^2 z^T g g^T z + 2 \mu^2 z^T 11^T z g g^T z + 4 \mu^3 (1^T z)^3 z^T g + 4 \mu 1^T z (z^T g)^3,$$

and since $z^T g \sim \mathcal{N}(0, 1)$:

$$E[S^2] = \mu^4 (z^T 11^T z)^2 + 3 + 4 \mu^2 (1^T z)^2 \|z\|_2^2 + 2 \mu^2 z^T 11^T z \|z\|_2^2 \geq$$  

$$\geq \mu^4 (z^T 11^T z)^2 + 3 + 6 \mu^2 (1^T z)^2 \|z\|_2^2.$$  

Finally:

$$\frac{E^2[S]}{E[S^2]} = \frac{\mu^4 (1^T z)^4 + 2 \mu^2 (1^T z)^2 + 1}{\mu^4 (1^T z)^4 + 6 \mu^2 (1^T z)^2 + 3} = 1 - \frac{4 \mu^2 (1^T z)^2 + 2}{\mu^4 (1^T z)^4 + 6 \mu^2 (1^T z)^2 + 3},$$

and since

$$\mu^4 (1^T z)^4 + 6 \mu^2 (1^T z)^2 + 3 \geq \mu^2 (1^T z)^2 + 3$$

then

$$\frac{4 \mu^2 (1^T z)^2 + 2}{\mu^4 (1^T z)^4 + 6 \mu^2 (1^T z)^2 + 3} \leq \frac{2}{3}$$

and then

$$\frac{E^2[S]}{E[S^2]} = 1 - \frac{4 \mu^2 (1^T z)^2 + 2}{\mu^4 (1^T z)^4 + 6 \mu^2 (1^T z)^2 + 3} \geq \frac{1}{3}$$

and

$$\mathbb{P}(\|a^T u\| \geq \theta) \geq (1 - \theta^2)^2 \frac{E[S]^2}{E[S^2]} \geq \frac{(1 - \theta^2)^2}{3}.$$  

We turn now to look at $W_m(\Sigma_s^2, a)$. Denote:

$$h_0 = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_k g_k$$

with $g_k \sim \mathcal{N}(0, I)$, and from symmetry of $\epsilon_k$ and the normal distribution, $h_0 \sim \mathcal{N}(0, I)$.

$$\mathbb{E} \left[ \sup_{u \in S_s^2} \langle h, u \rangle \right] = \mathbb{E} \left[ \sup_{u \in S_s^2} \langle h_0, u \rangle + \mu \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_k \langle 1, u \rangle \right] \leq \mathbb{E} \left[ \sup_{u \in S_s^2} \langle h_0, u \rangle \right] + \mathbb{E} \left[ \sup_{u \in S_s^2} \frac{\mu}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_k \langle 1, u \rangle \right].$$  

The first term is proportional to the Gaussian width of $\Sigma_s^2$ so we can use known results (Proposition 3.10 in [28]). For the second term

$$\mathbb{E} \left[ \sup_{u \in S_s^2} \frac{\mu}{\sqrt{m}} \sum_{k=1}^{m} \epsilon_k \langle 1, u \rangle \right] \leq \mu \frac{1}{\sqrt{m}} \sqrt{\mathbb{E} \left[ \sum_{k=1}^{m} \epsilon_k \right]} \leq \sqrt{s} \mu,$$

where the last inequality follows from a Khintchine argument (see [18], Corollary 8.7), and from bounds on the Gaussian width of the set $\Sigma_s^2$ (Proposition 3.10 in [28]) we get

$$W_m(\Sigma_s^2, a) = \mathbb{E} \left[ \sup_{u \in S_s^2} \langle h, u \rangle \right] \leq \left( \frac{1}{2s} \log \left( \frac{1}{\delta} \right) + 2s + \mu \sqrt{2s} \right).$$
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