TWISTED FOURIER–MUKAI PARTNERS
OF ENRIQUES SURFACES

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Abstract. Bridgeland and Maciocia showed that a complex Enriques surface $X$ has no Fourier–Mukai partners apart from itself: that is, if $D^b(X) \cong D^b(Y)$ then $X \cong Y$. We extend this to twisted Fourier–Mukai partners: if $\alpha$ is the non-trivial element of $\text{Br}(X) = \mathbb{Z}/2$ and $D^b(X, \alpha) \cong D^b(Y, \beta)$, then $X \cong Y$ and $\beta$ is non-trivial. Our main tools are twisted topological K-theory and twisted Mukai lattices.

Introduction

Two smooth projective varieties $X$ and $Y$ are called Fourier–Mukai partners if they have equivalent derived categories of coherent sheaves $D^b(X) \cong D^b(Y)$, and twisted Fourier–Mukai partners if they have equivalent derived categories of twisted sheaves $D^b(X, \alpha) \cong D^b(Y, \beta)$ for some Brauer classes $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$. In the early 2000s, Bridgeland, Maciocia, and Kawamata showed that among complex surfaces, only K3, abelian, and elliptic surfaces have non-trivial Fourier–Mukai partners; see [13, Ch. 12] for a textbook account. Recently several authors have been interested in extending this result to positive characteristic and to twisted Fourier–Mukai partners. Here we carry out one step in this program:

Theorem. Let $X$ be a complex Enriques surface, let $\alpha \in \text{Br}(X) = \mathbb{Z}/2$, and let $Y$ be another smooth complex projective variety and $\beta \in \text{Br}(Y)$. If $D^b(X, \alpha) \cong D^b(Y, \beta)$, then $X \cong Y$, and via that isomorphism $\alpha = \beta$.

If $\alpha$ and $\beta$ are trivial then this was proved by Bridgeland and Maciocia in [6, Prop. 6.1]. Some special cases of the twisted result were obtained by Martinez Navas in [17, Ch. 3].

Dimension, order of the canonical bundle, and Hochschild homology are invariant under twisted derived equivalence, just as they are under untwisted equivalence: the proofs of [13, Prop. 4.1 and Rem. 6.3] go through unchanged, relying on existence and especially uniqueness of kernels for twisted equivalences due to Canonaco and Stellari [8, Thm. 1.1]. Thus $Y$ is an Enriques surface. In §1 we use twisted topological K-theory to show that we cannot have $\alpha$ trivial and $\beta$ non-trivial. In §2 we show that if $\alpha$ and $\beta$ are both non-trivial then $X \cong Y$; our proof follows the outline of Bridgeland and Maciocia’s, but is more delicate.
Acknowledgements. This paper grew out of conversations with K. Honigs and S. Tirabassi, related to [12]. We thank them for a fruitful exchange. We also thank A. Beauville and D. Dugger for helpful advice. Our first attempts at Proposition 2.6 made heavy use of Macaulay2 [11]; we thank B. Young for computer time.

1. One twisted, one untwisted

Given a smooth complex projective variety $X$ and a class $\alpha \in \text{Br}(X)$ with image $\bar{\alpha} \in H^3(X, \mathbb{Z})$, we let $K^i_{\text{top}}(X, \bar{\alpha})$ denote twisted topological K-theory; for the definition and first properties we refer to Atiyah and Segal [1, 2]. It is a 2-periodic sequence of finitely generated Abelian groups, and can be computed using an Atiyah-Hirzebruch spectral sequence.

An untwisted derived equivalence induces an isomorphism on topological K-theory, and recent work of Moulinos [18, Cor. 1.2], together with uniqueness of dg enhancements [9, §6.3], extends this to the twisted case: if $D^b(X, \alpha) \sim D^b(Y, \beta)$ then $K^i_{\text{top}}(X, \bar{\alpha}) \sim K^i_{\text{top}}(Y, \bar{\beta})$.

We will show that if $X$ is an Enriques surface then $K^1_{\text{top}}(X) = \mathbb{Z}/2$, but if $\alpha$ is the non-trivial element of $\text{Br}(X) = \mathbb{Z}/2$ then $K^1_{\text{top}}(X, \bar{\alpha}) = 0$, and thus an untwisted Enriques surface cannot be derived equivalent to a twisted one.

By [4, Lem. VIII.15.1], an Enriques surface has $\pi_1 = \mathbb{Z}/2$ and Hodge diamond

\[
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 10 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & & & \\
\end{array}
\]

By the universal coefficient theorem and Poincaré duality, it follows that

\[H^i(X, \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & i = 0 \\
0 & i = 1 \\
\mathbb{Z}^{10} \oplus \mathbb{Z}/2 & i = 2 \\
\mathbb{Z}/2 & i = 3 \\
\mathbb{Z} & i = 4.
\end{cases}\]

Now the claims about $K^1_{\text{top}}$ above follow from:

**Proposition 1.1.** If $X$ is any compact complex surface, then

\[K^1_{\text{top}}(X) \cong H^1(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z}).\]

If $\alpha \in \text{Br}(X)$ has image $\bar{\alpha} \in H^3(X, \mathbb{Z})$, then

\[K^1_{\text{top}}(X, \bar{\alpha}) \cong H^1(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z})/\bar{\alpha}.

---

1This reference is very $\infty$-categorical; a more down-to-earth reader might want to say that for any kernel $P \in D^b(X \times Y, \alpha^{-1} \boxtimes \beta)$, the class $[P] \in K^0_{\text{top}}(X \times Y, \alpha^{-1} \boxtimes \beta)$ induces a map $K^i_{\text{top}}(X, \bar{\alpha}) \rightarrow K^i_{\text{top}}(Y, \bar{\beta})$ in a way that’s functorial with respect to composition of kernels. But this would require a compatibility between pushforward on algebraic and topological twisted K-theory, comparable to [3]. This seems to be missing from the literature, and to prove it here would take us too far afield.
Proof. We abbreviate $H^i(X, \mathbb{Z})$ as $H^i$. The $E_3$ page of the Atiyah–Hirzebruch spectral sequence is

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 \\
H^0 & H^1 & H^2 & H^3 & H^4 & \\
0 & 0 & 0 & 0 & 0 & 0 \\
H^0 & H^1 & H^2 & H^3 & H^4 & \\
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

For untwisted K-theory, the map $d_3$ is given by

\[ Sq_3^Z = \beta \circ Sq^2 \circ r, \]

where $r$ is reduction mod 2, $Sq^2$ is the usual Steenrod square, and $\beta$ is the Bockstein homomorphism associated to the short exact sequence of coefficient groups

\[ 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}/2 \rightarrow 0. \]

This vanishes on $H^0$ and $H^1$ for degree reasons, so the spectral sequence degenerates. The filtration of $K^1_{\text{top}}(X)$ splits because $H^1$ is free.

For twisted K-theory, the $E_3$ page has the same terms, but now $d_3(x) = Sq_3^Z(x) - \bar{\alpha} \cup x$

by [2, Prop. 4.6]. This maps $1 \in H^0$ to $-\bar{\alpha} \in H^3$, and vanishes on $H^1$ because $\bar{\alpha}$ is torsion and $H^4$ is free. Thus the $E_4$ page is

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 \\
k \cdot H^0 & H^1 & H^2 & H^3/\bar{\alpha} & H^4 & \\
0 & 0 & 0 & 0 & 0 & 0 \\
k \cdot H^0 & H^1 & H^2 & H^3/\bar{\alpha} & H^4 & \\
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

where $k$ is the order of $\bar{\alpha}$. At this point the spectral sequence degenerates, and again the filtration of $K^1_{\text{top}}(X, \alpha)$ splits because $H^1$ is free. \hfill \square
2. Both twisted

We begin by recalling the outline of Bridgeland and Maciocia’s proof that if $X$ and $Y$ are complex Enriques surfaces and $D^b(X) \cong D^b(Y)$, then $X \cong Y$; see [6, Prop. 6.1] for the original or [13, Prop. 12.20] for another account. Take universal covers $p: \tilde{X} \to X$ and $q: \tilde{Y} \to Y$, so $\tilde{X}$ and $\tilde{Y}$ are K3 surfaces, and let $\tau$ denote the covering involution of either $\tilde{X}$ or $\tilde{Y}$. An equivalence $D^b(X) \to D^b(Y)$ lifts to an equivalence $D^b(\tilde{X}) \to D^b(\tilde{Y})$ that commutes with $\tau^*$. The induced Hodge isometry $H^*(\tilde{X}, \mathbb{Z}) \to H^*(\tilde{Y}, \mathbb{Z})$ commutes with $\tau^*$, and hence restricts to a Hodge isometry between the $\tau^*$-anti-invariant parts $H^2_-(\tilde{X}, \mathbb{Z}) \to H^2_-(\tilde{Y}, \mathbb{Z})$. Using Nikulin’s lattice theory, this extends to a Hodge isometry on all of $H^2$, still commuting with $\tau^*$. Thus $X \cong Y$ by the Torelli theorem for Enriques surfaces.

Now let $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$ be non-trivial. First we will check that an equivalence $D^b(X, \alpha) \to D^b(Y, \beta)$ lifts to an equivalence $D^b(\tilde{X}, p^*\alpha) \to D^b(\tilde{Y}, q^*\beta)$ that commutes with $\tau^*$. Next we will make a careful choice of $B$-fields in $H^2(\tilde{X}, \mathbb{Q})$ and $H^2(\tilde{Y}, \mathbb{Q})$ that lift $p^*\alpha$ and $q^*\beta$ and satisfy $\tau^*B = -B$. Then we have an induced isometry $\varphi: H^2(\tilde{X}, \mathbb{Z}) \to H^2(\tilde{Y}, \mathbb{Z})$, such that $e^{-B} \circ \varphi \circ e^B$ preserves $H^2_0$ and commutes with $\tau^*$. This yields a Hodge isometry $H^2_-(\tilde{X}, \mathbb{Q}) \to H^2_-(\tilde{Y}, \mathbb{Q})$, and the delicate step is to show that it takes $H^2_-(\tilde{X}, \mathbb{Z})$ into $H^2_-(\tilde{Y}, \mathbb{Z})$. Then we can conclude as in the untwisted case.

2.1. Lifting the kernel.

**Proposition 2.1.** With the notation introduced above, if $D^b(X, \alpha) \cong D^b(Y, \beta)$ then there is a kernel $\tilde{P} \in D^b(\tilde{X} \times \tilde{Y}, p^*\alpha^{-1} \boxtimes q^*\beta)$ that induces an equivalence $D^b(\tilde{X}, p^*\alpha) \to D^b(\tilde{Y}, q^*\beta)$ and satisfies $(\tau \times \tau^*)^*\tilde{P} \cong \tilde{P}$.

**Remark 2.2.** The expert reader might worry that $(\tau \times \tau^*)^*\tilde{P}$ lies a priori in $D^b(\tilde{X} \times \tilde{Y}, \tau^*p^*\alpha^{-1} \boxtimes \tau^*q^*\beta)$, and that in order to identify this with $D^b(\tilde{X} \times \tilde{Y}, p^*\alpha^{-1} \boxtimes q^*\beta)$ we might have to make some non-canonical choice; cf. [10, Rmk. 1.2.9]. But once we fix a cocycle $\{U_i, \alpha_{ijk}\}$ representing $\alpha$, the cocycle $\{p^{-1}(U_i), \alpha_{ijk} \circ p\}$ representing $p^*\alpha$ is actually the same as the cocycle $\{\tau^{-1}(p^{-1}(U_i)), \alpha_{ijk} \circ p \circ \tau\}$ representing $\tau^*p^*\alpha$, not just cohomologous, because $p \circ \tau = p$. The same is true of $\beta$. So the identification is canonical.

**Proof of Proposition 2.1.** By [8, Thm. 1.1], the equivalence is induced by a kernel $P \in D^b(X \times Y, \alpha^{-1} \boxtimes \beta)$. To lift it to a kernel $\tilde{P}$ as in the statement of the proposition, we can follow Bridgeland and Maciocia [7, Thm. 4.5], or Huybrechts’ book [13, Prop. 7.18], or Lombardi and Popa [16, Thm. 10] with no changes. The key point is an equivalence between:

1. $(p^*\alpha^{-1} \boxtimes q^*\beta)$-twisted sheaves on $\tilde{X} \times \tilde{Y}$,
(2) \((p^* \alpha - 1 \boxtimes \beta)\)-twisted sheaves on \(\tilde{X} \times Y\) that are modules over 
\[(1 \times q)_* \mathcal{O}_{\tilde{X} \times Y} = \mathcal{O}_{\tilde{X} \times Y} \oplus \omega_{\tilde{X} \times Y},\]
and

(3) \((p^* \alpha - 1 \boxtimes \beta)\)-twisted sheaves \(F\) on \(\tilde{X} \times Y\) with 
\(F \otimes \omega_{\tilde{X} \times Y} \sim F\).

In fact there is a subtlety in identifying (2) and (3), which the references above elide, but which Krug and Sosna treat carefully in \([15, \text{Lem. 3.6(ii)}]\). To turn a sheaf as in (3) into a \((\mathcal{O} \oplus \omega)\)-module as in (2), one needs the chain of isomorphisms 
\(F \otimes \omega^{2} \sim F \otimes \omega \sim F\) to agree with the global identification 
\(\omega^{2} \sim \mathcal{O}\). But in our case, the complex \((p \times 1)^* P\) that we wish to lift is simple, 
so any discrepancy can be scaled away before we start lifting cohomology sheaves.

To see that \((p \times 1)^* P\) is simple, first observe that it is the composition of 
the kernels \(P \in D_{b}(X \times Y)\) and \(O_{\Gamma_{p}} \in D_{b}(\tilde{X} \times X)\), where \(\Gamma_{p}\) is the graph of 
\(p\), by \([13, \text{Ex. 5.12 and 5.4(ii)}]\). Moreover, because \(P\) induces an equivalence, 
composition with \(P\) is an equivalence 
\(D_{b}(\tilde{X} \times X) \to D_{b}(\tilde{X} \times Y)\), so 
\(\text{Hom}_{\tilde{X} \times Y}((p \times 1)^* P, (p \times 1)^* P) = \text{Hom}_{\tilde{X} \times X}(O_{\Gamma_{p}}, O_{\Gamma_{p}}) = H^{0}(O_{\Gamma_{p}})\).

This is 1-dimensional because \(\Gamma_{p} \cong \tilde{X}\). \(\square\)

2.2. Choice of B-field. To get induced maps on cohomology from our 
kernel \(\tilde{P}\), we must choose B-field lifts of our Brauer classes, that is, a class 
\(B \in H^{2}(\tilde{X}, \mathbb{Q})\) with \(\exp(B^{0,2}) = p^* \alpha\), and similarly with \(q^* \beta\).

By [4, Lem. VIII.19.1], we can choose an isometry 
\[H^{2}(\tilde{X}, \mathbb{Z}) \cong -E_{8} \oplus -E_{8} \oplus U \oplus U \oplus U\] \hspace{1cm} (1)
under which the involution \(\tau^*\) acts as 
\[(x, y, z_{1}, z_{2}, z_{3}) \mapsto (y, x, z_{2}, z_{1}, -z_{3})\]. \hspace{1cm} (2)

Here \(-E_{8}\) is the unique negative definite even unimodular lattice of rank 8, and \(U\) is the standard hyperbolic lattice, with basis \(e\) and \(f\) satisfying 
\(e^{2} = f^{2} = 0\) and \(e.f = 1\).

**Proposition 2.3** (Beauville [5]). Under the isometry (1), the class 
\[B := (0, 0, 0, 0, \frac{1}{2} e + \frac{1}{2} f) \in H^{2}(\tilde{X}, \mathbb{Q})\]
satisfies \(\exp(B^{0,2}) = p^* \alpha \in \text{Br}(\tilde{X}) \subset H^{2}(\mathcal{O}_{\tilde{X}}^*)\).

**Remark 2.4.** Note that \(p^* \alpha\) may be trivial: it may be that \(B\) has the same 
(0, 2) part as some integral class in \(H^{2}(\tilde{X}, \mathbb{Z})\). Indeed, the point of 
Beauville’s beautiful paper is that set of Enriques surfaces for which this 
happens form a countable union of divisors in the moduli space.
Proof of Proposition 2.3. Beauville’s set-up is a bit different from ours, so we explain how to deduce the proposition from his paper.

Consider the diagram of sheaves

\[
\begin{array}{c}
0 \\ r \\
\end{array}
\begin{array}{c}
\mathbb{Z} \\
\mathbb{Z}/2
\end{array}
\begin{array}{c}
\exp(\frac{1}{2} -) \\
_{z \rightarrow z^2}
\end{array}
\begin{array}{c}
\mathcal{O}^* \\
\mathcal{O}^*
\end{array}
\begin{array}{c}
2\pi i \\
0
\end{array}
\begin{array}{c}
\mathbb{O} \\
\mathbb{O}
\end{array}
\begin{array}{c}
r \\
\exp(\frac{1}{2} -)
\end{array}
\begin{array}{c}
\mathcal{O}^* \\
\mathcal{O}^*
\end{array}
\begin{array}{c}
0 \\
0
\end{array}
\begin{array}{c}
\end{array}
\end{array}
\]

on either $X$ or $\tilde{X}$. On the Enriques surface $X$, we have $H^0.1 = H^0.2 = 0$, so $Br(X) = H^2(\mathcal{O}_X^*) = H^3(X, \mathbb{Z}) = \mathbb{Z}/2$, and taking cohomology we get

\[
\begin{array}{c}
0 \\
Pic(X)
\end{array}
\begin{array}{c}
\xrightarrow{c_1} \\
\xrightarrow{r}
\end{array}
\begin{array}{c}
H^2(X, \mathbb{Z}) \\
H^2(X, \mathbb{Z}/2)
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
Br(X) \\
Br(X)
\end{array}
\]

On the K3 surface $\tilde{X}$, we get

\[
\begin{array}{c}
0 \\
Pic(\tilde{X})
\end{array}
\begin{array}{c}
\xrightarrow{c_1} \\
\xrightarrow{r}
\end{array}
\begin{array}{c}
H^2(\tilde{X}, \mathbb{Z}) \\
H^2(\tilde{X}, \mathbb{Z}/2)
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
H^2(\mathcal{O}_X^*) \\
H^2(\mathcal{O}_X^*)
\end{array}
\]

Moreover the pullback $p^*$ maps the first diagram to the second.

In the second diagram, consider $2B \in H^2(\tilde{X}, \mathbb{Z})$. By [5, Prop. 5.3], we can choose $x \in H^2(X, \mathbb{Z}/2)$ with $p^*x = r(2B)$ (= $\varepsilon$ in Beauville’s notation). To prove the proposition, it is enough to show that $x$ maps to $\alpha \in Br(X)$; or equivalently that $x$ is not the reduction of an integral class $y \in H^2(X, \mathbb{Z})$. If $x = r(y)$ then $x^2 = r(y^2) = 0$, because the intersection pairing on $H^2(X, \mathbb{Z})$ is even [4, Lem. VIII.15.1(iii)]. But $(2B)^2 = 2$, so $x^2 = 1$ by [5, Lem. 5.4]. □

2.3. Induced map on cohomology. From here on we fix bases for $H^*(\tilde{X}, \mathbb{Z})$ and $H^*(\tilde{Y}, \mathbb{Z})$ as in (1). We continue to let $\tau^*$ denote the involution on both sides, which in our basis acts by (2). From Proposition 2.3 we get B-fields on both $\tilde{X}$ and $\tilde{Y}$, both denoted $B$.

Following Huybrechts and Stellari [14, §4], the twisted Mukai vector

\[v^{-B\otimes B}(\tilde{P}) \in H^*(\tilde{X} \times \tilde{Y}, \mathbb{Z})\]

induces an isometry

\[\varphi: H^*(\tilde{X}, \mathbb{Z}) \rightarrow H^*(\tilde{Y}, \mathbb{Z}),\]

whose complexification takes $e^B H^0.2(\tilde{X})$ into $e^B H^0.2(\tilde{Y})$. 

Proposition 2.5. \( e^{-B} \circ \varphi \circ e^B \) commutes with \( \tau^* \).

Proof. We have
\[
(\tau \times \tau)^* v^{-B \oplus B}(\tilde{P}) = v^{\tau^*(-B) \oplus \tau^* B}((\tau \times \tau)^* \tilde{P}) = v^{B \oplus (-B)}(\tilde{P}) = \text{ch}^{2B \oplus (-2B)}(\mathcal{O}_{\tilde{X} \times \tilde{Y}}) \cdot v^{-B \oplus B}(\tilde{P}) = e^{2B \oplus (-2B)} \cdot v^{-B \oplus B}(\tilde{P}),
\]
where in the third line we have used [14, Prop. 1.2(iii)], and in the fourth we have used [ibid., Prop. 1.2(ii)].

This implies that \( \tau^* \circ \varphi \circ \tau^* = e^{-2B} \circ \varphi \circ e^{2B} \), which can be manipulated to give the desired result. \( \square \)

2.4. Integrality. If we denote the \( \tau^* \)-invariant and -anti-invariant parts of \( H^* \) by \( H^*_+ \) and \( H^*_-=H^2 \), then we have constructed a Hodge isometry
\[
e^{-B} \circ \varphi \circ e^B: H^2_-(\tilde{X}, \mathbb{Q}) \to H^2_-(\tilde{Y}, \mathbb{Q}).
\]

It remains to show that it maps integral classes to integral classes.

To that end, suppose that \( x \in H^2(\tilde{X}, \mathbb{Z}) \) satisfies \( \tau^* x = -x \), and write
\[
\varphi(e^B x) = (r, c, s) \in H^*(\tilde{Y}, \mathbb{Q}).
\]
Then
\[
e^{-B}(r, c, s) = (r, c - rB, s - cB + \frac{1}{2}rB^2)
\]
is \( \tau^* \)-anti-invariant, so \( r = 0 \), and \( s - cB = 0 \): that is,
\[
e^{-B} \varphi(e^B x) = (0, c, 0).
\]

So we wish to show that the degree-2 part of \( \varphi(e^B x) \) is integral. We have
\[
e^B x = (0, x, y) \in H^*(\tilde{X}, \mathbb{Q}),
\]
where \( y = x.B \in \frac{1}{2}\mathbb{Z} \). Since \( x \) is integral and \( y \) is half-integral, we will have proved our main theorem once we prove:

Proposition 2.6. For any isometry \( \varphi: H^*(\tilde{X}, \mathbb{Z}) \to H^*(\tilde{Y}, \mathbb{Z}) \) that commutes with \( T := e^B \circ \tau^* \circ e^{-B} \), the degree-2 part of \( \varphi(0, 0, 1) \) is divisible by 2.

Proof. Observe that \( T \) is integral: \( T = e^{2B} \circ \tau^* \).

By Poincaré duality, the statement of the proposition is equivalent to
\[
\langle \varphi(0, 0, 1), \ell \rangle \equiv 0 \pmod{2}
\]
for all \( \ell \in H^2(\tilde{X}, \mathbb{Z}) \). Because \( T \) is an isometry and \( (0, 0, 1) \) is \( T \)-invariant,
\[
\langle \varphi(0, 0, 1), \ell \rangle = \langle \varphi(0, 0, 1), \frac{1}{2}(\ell + T\ell) \rangle,
\]
so it is enough to show that
\[
\langle \varphi(0, 0, 1), \ell + T\ell \rangle \equiv 0 \pmod{4}.
\]
Now our proof will consist of two calculations:
Claim 1. For any $\ell \in H^2(\tilde{X}, \mathbb{Z})$,
\[(\ell + T\ell)^2 \equiv 0 \pmod{4}.
\]

Claim 2. For any $T$-invariant class $v \in H^*(\tilde{X}, \mathbb{Z})$,
\[
\langle (0, 0, 1), v \rangle \equiv v^2 \pmod{4}.
\]

Observe that this property is preserved by $T$-equivariant isometries, so $\varphi(0, 0, 1)$ has the same property.

To prove the first claim, write $(\ell + T\ell)^2 = \ell^2 + 2(\ell, T\ell) + (T\ell)^2 = 2\ell^2 + 2\ell.\tau^*\ell$.

Since $\ell^2$ is even, it is enough to show that $\ell.\tau^*\ell$ is even. Using the basis (1), write
\[
\ell = (x, y, z_1, z_2, z_3).
\]

Then
\[
\ell.\tau^*\ell = 2xy + 2z_1z_2 - z_3^2,
\]

which is even because $z_3^2$ is even. Thus the first claim is proved.

To prove the second claim, write
\[
v = (r, x, y, z_1, z_2, ae + bf, s) \in H^0 \oplus H^2 \oplus H^4,
\]

where again we use the basis (1) for $H^2$. Then
\[
Tv = e^{2B}\tau^*v
= e^{2B}(r, y, x, z_2, z_1, -ae - bf, s)
= (r, y, x, z_2, z_1, (r - a)e + (r - b)f, s - a - b + r).
\]

From $Tv = v$ we find that $x = y$, $z_1 = z_2$, $r = 2a$, and $a = b$. Thus
\[
v = (2a, x, x, z_1, z_1, ae + af, s),
\]

so
\[
v^2 = 2x^2 + 2z_1^2 + 2a^2 - 4as.
\]

Since $x^2$ and $z_1^2$ are even,
\[
v^2 \equiv 2a^2 \equiv 2a \pmod{4},
\]

and moreover
\[
\langle (0, 0, 1), v \rangle = -2a,
\]

so the second claim is proved.

\[\square\]

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2What’s going on is that the pairing on the $T$-invariant sublattice of $H^*(\tilde{X}, \mathbb{Z})$ is two times an odd unimodular pairing, and $(0, 0, 1)$ is what’s sometimes called a “characteristic” or “parity” vector.
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