(Non)commutative isotropization in Bianchi I with Barotropic perfect fluid and \(\Lambda\) Cosmological

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We present the classical solutions to the Einstein field equations derived using the WKB-like and Hamilton procedures. The investigation is carried out in the commutative and noncommutative scenario for the Bianchi type I cosmological model coupled to barotropic perfect fluid and \(\Lambda\) Cosmological for two different gauges. Noncommutativity is achieved by modifying the symplectic structure considering that all minisuperspace variables \(q\) does not commute and by a deformation between all the minisuperspace variables. In the gauge \(N=1\), it is possible to obtain that the anisotropic parameter \(\beta_{\pm nc}\) tend to a constant curvature for large period of time considering different values in the noncommutative parameters \(\theta\) and cosmological term. However, this behavior give the idea that is necessary introduce other class of matter in the models, for to have a real isotropization in the model, such as dark energy or dark matter.

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I. INTRODUCTION

Recently, a great interest has been generated in noncommutative spacetimes \[1, 2, 3\], mainly due to the fact that there are of strong motivations in the development of string and M-theories \[4, 5\]. A different approach to noncommutativity is through the introduction of noncommutative fields \[6\], that is, fields of their conjugate momenta are taken as noncommuting. There are several approaches in considering the notion of noncommutativity in cosmology, that could be the best alternative in the absence of a comprehensive and satisfactory theory from string theory. This analysis has been studied in many works \[7, 8\]. Here, taking coordinates as noncommuting, it has been shown that noncommutativity affects the spectrum of Cosmic Microwave Background. For example, in \[7\], noncommutative geometry suggest a non local inflaton field that changes the gaussianity and isotropy property of fluctuations. In cosmological systems, since the scale factors, matter fields and their conjugate momenta play the role of dynamical variables of the system, introduction of noncommutativity by adopting the noncommutativity between all fields, is particularly relevant. The simplest noncommutative classical and quantum cosmology of such models have been studied in different works \[9, 10, 11, 12\].

On the other hand, there is a renewed interest on noncommutative theories to explain the appropriate modification of classical General Relativity, and hence of spacetime symmetries at short-distance scales, that implies modifications at large scales. General quantum mechanics arguments indicate that, it is not possible to measure a classical background spacetime at the Planck scale, due to the effects of gravitational backreaction \[13\]. It is therefore tempting to incorporate the dynamical features of spacetime at deeper kinematical level using the standard techniques of noncommutative classical field theory based in the so called Moyal product in which for all calculations purposes (differentiation, integration, etc.) the space time coordinates are treated as ordinary (commutative) variables and noncommutativity enters into play in the way in which fields are multiplied \[14\]. Using a modified symplectic structure on the space variables in the Hamilton approach, as we are trying with the idea of noncommutative space time, we propose that the minisuperspace variables do not commute, for that purpose we will modified the Poisson structure, this approach does not modify the hamiltonian structure in the noncommutative fields. In the approach used, we choose that the momentas in both spaces, are the same, \(P_{\mu} = P_{\mu}^{nc}\), it is
say, they commute in both spaces.

Another way to extract useful dynamical information is through the WKB semiclassical approximation to the quantum Wheeler-DeWitt equation using the wave function $\Psi = e^{iS(q^\mu)}$. In this sense, we consider the usual approximation in the derivatives and the corresponding relation between the Einstein-Hamilton-Jacobi (EHJ) equation, it was possible to obtain classical solutions at the master equation found by this procedures. The classical field equations were checked for all solutions, using the REDUCE 3.8 algebraic packages.

The main idea in this paper is to obtain both, the commutative ($\Omega, \beta_\pm$) and noncommutative ($\Omega_{nc}, \beta_{\pm nc}$) classical solution of the Einstein field equation in General Relativity for the Bianchi Class A models, without solve these field equations, using two alternative approaches, known as WKB semiclassical approximation and Hamilton approach. Using these solutions in the gauge $N=1$ (the physical gauge), we can infer if the anisotropic parameters $\beta_{\pm nc}$ suffers changes toward isotropic ones (a constant or zero value). This analysis is considered in particular with the Bianchi type I, coupled to barotropic perfect fluid and cosmological term. In this case, we can observe that when the cosmological constant decrease in its value, the isotropization is more notorious for a larger period time.

The paper is then organized as follows. In section II, we obtain the WDW equation including the barotropic matter contribution, and the corresponding commutative classical solutions for the cosmological Bianchi type I, in the gauge $N = 1$, by the WKB semiclassical approximation and Hamilton procedure. Section III is devoted to the noncommutative classical solutions and the analysis of the isotropization is made too, in the physical gauge $N=1$. Final remarks are presented in Section IV. For completeness, we can follow a similar prescription for the gauge $N = 24e^{3\Omega}$, where the noncommutative cosmological model is always anisotropic. So, we present the corresponding solutions for both scenarios, the commutative, appendix A and noncommutative, appendix B.

Let us begin by recalling canonical formulation of the ADM formalism to the diagonal Bianchi Class A cosmological models. The metrics have the form

$$ds^2 = -(N^2 - N_iN_j)dt^2 + e^{2\Omega(t)}e^{2\beta_{ij}(t)}\omega^i\omega^j,$$

where $N$ and $N_i$ are the lapse and shift functions, respectively, $\Omega(t)$ is a scalar and $\beta_{ij}(t)$ a 3x3 diagonal matrix, $\beta_{ij} = \text{diag}(\beta_+, \beta_-, \sqrt{3}\beta_-, -2\beta_+)$, $\omega^i$ are one-forms that characterize
each cosmological Bianchi type model, and that obey \( \text{d}\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k \), \( C^i_{jk} \) the structure constants of the corresponding invariance group [15]. The metric for the Bianchi type I, takes the form

\[
\text{ds}_I^2 = -N^2 \text{dt}^2 + e^{2\Omega} e^{2\beta_+} + 2\sqrt{3} \beta_+ \text{dx}^2 + e^{2\Omega} e^{2\beta_+} - 2\sqrt{3} \beta_- \text{dy}^2 + e^{2\Omega} e^{-4\beta_+} \text{dz}^2, \tag{2}
\]

The corresponding lagrangian density is

\[
L_{\text{Total}} = \sqrt{-g} (\text{R} - 2\Lambda) + L_{\text{matter}}, \tag{3}
\]

and using (2), this have the following form

\[
L = 6e^{3\Omega} \left[ -\frac{\dot{\Omega}^2}{N} + \frac{\dot{\beta}_+^2}{N} - \frac{\Lambda}{3} N + \frac{8}{3} \pi GN \rho \right]. \tag{4}
\]

where the overdot denotes time derivatives. The canonical momenta to coordinate fields are defined in the usual way

\[
P_\Omega = \frac{\partial L}{\partial \dot{\Omega}} = -12e^{3\Omega} \frac{\dot{\Omega}}{N}, \quad P_+ = \frac{\partial L}{\partial \dot{\beta}_+} = 12e^{3\Omega} \frac{\dot{\beta}_+}{N}, \quad P_- = \frac{\partial L}{\partial \dot{\beta}_-} = 12e^{3\Omega} \frac{\dot{\beta}_-}{N}, \tag{5}
\]

and the correspondent Hamiltonian function is

\[
H = \frac{Ne^{-3\Omega}}{24} \left[ -P_\Omega^2 + P_+^2 + P_-^2 - 48\Lambda e^6 \Omega + 384\pi GM_\gamma e^{-3(\gamma - 1)\Omega} \right] = 0, \tag{6}
\]

together with barotropic state equation \( p = \gamma \rho \), the Hamilton-Jacobi equation is obtained when we substitute \( P_{q^\mu} \rightarrow \frac{\text{d}s_i}{\text{d}q^\mu} \) into (6). In what follows, we should consider the gauge \( N = 1 \).

II. COMMUTATIVE CLASSICAL SOLUTIONS

A. Commutative Classical Solutions á la WKB

The quantum Wheeler-DeWitt (WDW) equation for these models is obtained by making the canonical quantization \( P_{q^\mu} \) by \( -i\partial_{q^\mu} \) in (6) with \( q^\mu = (\Omega, \beta_+, \beta_-) \) is

\[
e^{-3\Omega} \frac{1}{24} \left[ \frac{\partial^2}{\partial \Omega^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} - \lambda e^6 \Omega + b_\gamma e^{-3(\gamma - 1)\Omega} \right] \Psi = 0. \tag{7}
\]

where \( \lambda = 48\Lambda, \quad b_\gamma = 384\pi GM_\gamma \). We now proceed to apply the WKB semiclassical approximation using the ansatz

\[
\Psi (\Omega, \beta_\pm) = e^{i[S_1(\Omega) + S_2(\beta_+) + S_3(\beta_-)]}, \tag{8}
\]
into (7), and without any loss of generality, one can consider the condition $\frac{d^2S_i}{dq_i^2}$ be small i.e.,

$$\left(\frac{dS_1}{d\Omega}\right)^2 >> \frac{d^2S_1}{d\Omega^2}, \quad \left(\frac{dS_2}{d\beta_+^2}\right)^2 >> \frac{d^2S_2}{d\beta_+^2}, \quad \left(\frac{dS_2}{d\beta_-^2}\right)^2 >> \frac{d^2S_2}{d\beta_-^2},$$

(9)

to get the classical Einstein-Hamilton-Jacobi equation

$$- \left(\frac{dS_1}{d\Omega}\right)^2 + \left(\frac{dS_2}{d\beta_+}\right)^2 + \left(\frac{dS_3}{d\beta_-}\right)^2 - \lambda e^{6\Omega} + b e^{-(\gamma-1)\Omega} = 0,$$

(10)

which can be separate in a set of differential equations

$$- \left(\frac{dS_1}{d\Omega}\right)^2 + a_1^2 - \lambda e^{6\Omega} + b e^{-(\gamma-1)\Omega} = 0,$$

(11)

$$\left(\frac{dS_2}{d\beta_+}\right)^2 = n_1^2,$$

(12)

$$\left(\frac{dS_3}{d\beta_-}\right)^2 = p_1^2,$$

(13)

where $a_1^2, n_1^2$ and $p_1^2$ are the separation constants and their relations is $a_1^2 = n_1^2 + p_1^2$. Therefore using the relations between (5), (11), (12) and (13) we have the following equations of motion

$$\pm \sqrt{a_1^2 - \lambda e^{6\Omega} + b e^{-(\gamma-1)\Omega}} \equiv \frac{-12 e^{3\Omega} \dot{\Omega}}{N},$$

(14)

$$\pm n_1 \equiv \frac{12 e^{3\Omega} \dot{\beta}_+}{N},$$

(15)

$$\pm p_1 \equiv \frac{12 e^{3\Omega} \dot{\beta}_-}{N}.$$  

(16)

The main master equation to solved in the gauge $N = 1$, is

$$\frac{dt}{12} = \frac{d\Omega}{\sqrt{a_1^2 e^{-6\Omega} + b e^{-3(\gamma+1)\Omega}}} = \lambda,$$

(17)

the other two equations (15) and (16) are trivially integrable. For particular stadium of the universe evolution, given by the $\gamma$ parameter, we present these classical solutions in table II A.
Commutative solutions

| Case                                                                 | Commutative solutions                                                                 |
|---------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| $\gamma = -1, \Lambda \neq 0, \rho_{-1} = M_{-1}$                   | $\Omega = \frac{1}{3} \ln \left[ e^{\eta t} - 4a_1^2 \right], q^2 = 24\pi \Lambda - 3\Lambda,$ |
|                                                                     | $\beta_+ = \pm \frac{2a_1}{3a_1} \arctanh \left[ \frac{e^{\eta t}}{2a_1} \right], a_1^2 = n_1^2 + p_1^2,$ |
|                                                                     | $\beta_- = \pm \frac{2a_1}{3a_1} \arctanh \left[ \frac{e^{\eta t}}{2a_1} \right].$       |
| $\gamma = 1, \Lambda < 0, \rho_1 = M_1 e^{-6\Omega}$                | $\Omega = \frac{1}{3} \ln \left[ e^{\eta t} - 4a_1^2 \right], q = \sqrt{3|\Lambda|},$ |
|                                                                     | $\beta_+ = \pm \frac{2a_1}{3a_1} \arctanh \left[ \frac{e^{\eta t}}{2a_1} \right], a_1^2 = n_1^2 + p_1^2 + 384\pi \Lambda,$ |
|                                                                     | $\beta_- = \pm \frac{2a_1}{3a_1} \arctanh \left[ \frac{e^{\eta t}}{2a_1} \right].$       |
| $\gamma = 1, \Lambda = 0, \rho_1 = M_1 e^{-6\Omega}$                | $\Omega = \frac{1}{3} \ln \left[ \frac{2}{3} t \right], a_1^2 = n_1^2 + p_1^2 + 384\pi \Lambda,$ |
|                                                                     | $\beta_+ = \pm \ln \left[ \frac{t}{3a_1} \right],$                                       |
|                                                                     | $\beta_- = \pm \ln \left[ \frac{t}{3a_1} \right].$                                       |
| $\gamma = 0, \Lambda = 0, \rho_0 = M_0 e^{-3\Omega}$                | $\Omega = \frac{1}{3} \ln \left[ \frac{b_0 t^2}{a_0^2} + \frac{a_0 t}{4} \right], b_0 = 384\pi \Lambda,$ |
|                                                                     | $\beta_+ = \pm \frac{a_0}{3a_1} \ln \left[ \frac{16a_1 + b_0 t}{t} \right], a_1^2 = n_1^2 + p_1^2,$ |
|                                                                     | $\beta_- = \pm \frac{a_0}{3a_1} \ln \left[ \frac{16a_1 + b_0 t}{t} \right].$            |

Table II. Classical Solutions for $\gamma = -1, 1, 0$, and constraints $q, a_1$ and $b_0$. 

**B. Classical Solutions via Hamiltonian Formalism**

In order to find the commutative equation of motion, we use the classical phase space variables $(\Omega, \beta_\pm)$, where the Poisson algebra for these minisuperspace variables are

$$\{\Omega, \beta_\pm\} = \{\beta_+, \beta_-\} = \{P_\Omega, P_\pm\} = \{P_+, P_-\} = 0, \quad \{q^\mu, p_\nu\} = 1, \quad (18)$$

and recalling the Hamiltonian equation (20), we obtain the classical solutions with the following procedure.

The classical equations of motion for the phase variables $\Omega, \beta_\pm, P_\pm, \text{and} P_\Omega$ are

$$\dot{\Omega} = \{\Omega, H\} = -\frac{1}{12} e^{-3\Omega} P_\Omega, \quad (19)$$

$$\dot{\beta}_- = \{\beta_-, H\} = \frac{1}{12} e^{-3\Omega} P_-, \quad (20)$$

$$\dot{\beta}_+ = \{\beta_+, H\} = \frac{1}{12} e^{-3\Omega} P_+, \quad (21)$$

$$\dot{P}_\Omega = \{P_\Omega, H\} = \frac{1}{8} e^{-3\Omega} \left[ -P_\Omega^2 + P_-^2 + P_+^2 + \lambda e^{6\Omega} + \gamma b_4 e^{-3(\gamma-1)\Omega} \right], \quad (22)$$

$$\dot{P}_- = \{P_-, H\} = 0, \quad \rightarrow \quad P_- = \pm p_1 = \text{const.} \quad (23)$$

$$\dot{P}_+ = \{P_+, H\} = 0, \quad \rightarrow \quad P_+ = \pm n_1 = \text{const.} \quad (24)$$
Introducing (6) into (22), we have
\[ 8e^{-3\Omega}P' = 2\lambda + (\gamma - 1)b\gamma e^{-3(\gamma+1)\Omega}, \] (25)
which can be integrate to obtain the relation for \( P_\Omega \)
\[ P_\Omega = \pm \sqrt{a_1^2 - \lambda e^{6\Omega} + b\gamma e^{-3(\gamma-1)\Omega}}, \] (26)
where \( a_1^2 = n_1^2 + p_1^2 \).

The set of equations (19), (20) and (21) are equivalents to the set of equations (14), (15) and (16), equations used to obtain the classical solutions.

In summary, the anisotropic parameters are a crescent function of the time (see table II A), and the solutions obtained with the Hamiltonian formalism and the WKB-like procedure are equivalents in Classical General Relativity.

**III. NONCOMMUTATIVE SOLUTIONS**

Let us begin introducing the noncommutative deformation of the minisuperspace \[9\] in the WDW equation, this time, between all the variables of the minisuperspace, assuming that \( \Omega_{nc} \) and \( \beta_{\pm nc} \) obey the commutation relation
\[ [\Omega_{nc}, \beta_{-nc}] = i\theta_1, \quad [\Omega_{nc}, \beta_{+nc}] = i\theta_2, \quad [\beta_{-nc}, \beta_{+nc}] = i\theta_3. \] (27)
Instead of working directly with the physical variables \( \Omega \) and \( \beta_{\pm} \) we may achieve all the above solutions by making use of the auxiliary canonical variables \( \Omega_{nc} \) and \( \beta_{\pm nc} \) defined as
\[ \Omega_{nc} \equiv \Omega - \frac{\theta_1}{2}P_\Omega - \frac{\theta_2}{2}P_+, \] (28)
\[ \beta_{-nc} \equiv \beta_+ - \frac{\theta_1}{2}P_\Omega + \frac{\theta_3}{2}P_+, \] (29)
\[ \beta_{+nc} \equiv \beta_+ + \frac{\theta_2}{2}P_\Omega + \frac{\theta_3}{2}P_. \] (30)
maintaining the usual commutation relations between the fields, i.e., \([q^\mu, q^\nu] = 0\). A shift generalization for the commutative symplectic structure can be made it through the change
\[ q^\mu \equiv q^\mu_{nc} + \frac{1}{2}\theta^{\mu\nu}P_\nu, \] (31)
where $\theta^{\mu\nu}$ is an antisymmetric matrix, and the identifications $P_{\Omega} = P_{\Omega\text{nc}}$ and $P_{\pm} = P_{\pm\text{nc}}$. With this shift and the usual canonical quantization $P_{q^\mu} \rightarrow -i\partial_{q^\mu}$, we arrive to the noncommutative WDW equation

$$
\left[ \frac{\partial^2}{\partial \Omega^2_{\text{nc}}} - \frac{\partial^2}{\partial \beta^2_{+\text{nc}}} - \frac{\partial^2}{\partial \beta^2_{-\text{nc}}} - \lambda e^{6\Omega_{\text{nc}}} + b_\gamma e^{-3(\gamma-1)\Omega_{\text{nc}}} \right] \Psi(\Omega, \beta_{\pm}) = 0,
$$

(32)

where $\lambda = 48\Lambda$, $b_\gamma = 384\pi G M_\gamma$. At this point we have a noncommutative WDW equation and noncommutative Hamiltonian. In what follows, we shall consider a wave function and apply the WKB procedure to obtain classical solutions.

### A. Noncommutative Classical Solutions á la WKB

In order to find noncommutative classical solutions through the WKB approximation, we use the fact that $e^{i\theta_{\mu\nu} x^\nu} \equiv e^{i\theta_{\mu} x^\mu}$, and the ansatz for the wavefunction $\Psi(\Omega_{\text{nc}}, \beta_{\pm\text{nc}}) = e^{i[S_1(\Omega_{\text{nc}}) \pm n_1 + \beta_{+\text{nc}} \pm p_1 - \beta_{-\text{nc}}]}$, where we use explicitly $S_2(\beta_{+\text{nc}}) = \pm n_1 \beta_{+\text{nc}}$ and $S_3(\beta_{-\text{nc}}) = \pm p_1 \beta_{-\text{nc}}$ to get the classical noncommutative Einstein-Hamilton-Jacobi (EHJ) equation

$$
-(\frac{dS_1}{d\Omega_{\text{nc}}} + \frac{dS_2}{d\beta_{+\text{nc}}} + \frac{dS_3}{d\beta_{-\text{nc}}})^2 - \lambda e^{6\Omega_{\text{nc}}} + b e^{-3(\gamma-1)\Omega_{\text{nc}}} = 0,
$$

(33)

which can be separate in a set of differential equations with $n_1^2 = n_1^2 + p_1^2$. We have the following noncommutative equations of motion

$$
\pm \sqrt{-a_1^2 - \lambda e^{6\Omega_{\text{nc}}} + b e^{-3(\gamma-1)\Omega_{\text{nc}}}} \equiv -12e^{3\Omega_{\text{nc}}} \frac{\dot{\Omega}_{\text{nc}}}{N},
$$

(34)

$$
\pm n_1 \equiv 12e^{3\Omega_{\text{nc}}} \frac{\dot{\beta}_{+\text{nc}}}{N},
$$

(35)

$$
\pm p_1 \equiv 12e^{3\Omega_{\text{nc}}} \frac{\dot{\beta}_{-\text{nc}}}{N}.
$$

(36)

One just need to be careful in (34), (35) and (36), and apply the chain rule to the variables (28), (29) and (30), in order to get the right solution, $
\dot{\beta}_{-\text{nc}} = \frac{\partial \beta_{-\text{nc}}}{\partial \Omega_{\text{nc}}} \frac{\partial P_{\Omega}}{\partial \Omega_{\text{nc}}} + \frac{\partial \beta_{-\text{nc}}}{\partial P_{+}} \frac{\partial P_{+}}{\partial \Omega_{\text{nc}}} + \frac{\partial \beta_{-\text{nc}}}{\partial P_{-}} \frac{\partial P_{-}}{\partial \Omega_{\text{nc}}} = \dot{\beta}_{-} + \frac{\theta_{\Omega}}{2} \dot{P}_{\Omega}$. In this sense, all solutions to find in the commutative case, remain for the noncommutative case with the corresponding shift, as we show in the table III A.
In the following plots we present as example, the second line in the table [III A] using different small values to the cosmological constant, we can see that the anisotropic parameters $\beta_{\pm nc}$ for some particular value in the $\theta$ noncommutative parameter, tends to a constant curvature in some range of the cosmological time. When this occur, the anisotropic scale factors tend to anisotropic one, but next go to anisotropic again. We should have a process where this anisotropic behavior does not appear again, for example, introducing in the model other class of matter or energy, such as dark matter and dark energy. In the noncommutative space, it is possible to find one range in the $\theta$ parameter that produce the dynamical isotropization to the model. But this isotropization does not occur in the other gauge $N = 24e^{3\Omega}$, when the $\beta_{\pm nc}$, always is a crescent function of the time (see table [B I]), independent to the values in the $\theta$ parameters.

### B. Noncommutative Classical Solutions á la Hamilton

Now the natural extension is to consider the noncommutative version of our model, with the idea of noncommutative between the three variables ($\Omega_{nc}, \beta_{\pm nc}$), for that purpose we have two approaches the first one is, to modify the Poisson structure, in this approach the

| Case | Noncommutative Solutions |
|------|--------------------------|
| $\gamma = -1, \Lambda \neq 0, \rho_1 = M_1e^{-6\Omega}$ | $\beta_{+ nc} = \pm \frac{2 \pi a_1}{3 a_1} \arctanh \left[ \frac{q t}{2 a_1} \right] + \frac{\theta_2}{8} \left( \frac{q t}{4} + a_1^2 e^{-qt} \right) - \frac{\theta_2}{2} p_1$, $\beta_{- nc} = \pm \frac{2 \pi a_1}{3 a_1} \arctanh \left[ \frac{q t}{2 a_1} \right] + \frac{\theta_2}{8} \left( \frac{q t}{4} + a_1^2 e^{-qt} \right) + \frac{\theta_2}{2} n_1$, $\Omega_{nc} = \frac{1}{3} \ln \left[ \frac{b t^{n_1}}{64} + \frac{a t}{4} \right] - \frac{\theta_1}{2} p_1 - \frac{\theta_2}{2} n_1$, where $|q| = |\Lambda|$ and $a_1 = 384$, $b_0 = 384\pi GM_0$, $a_1^2 = n_1^2 + p_1^2$, $q^2 = 24\pi GM_1 - 3\Lambda$, $A_1 = 6\Omega$, $\beta_{\pm nc} = \pm \frac{1}{4} \sqrt{\frac{16a_1 + b_0}{t}} + \frac{\theta_2}{2} \sqrt{a_1^2 + \frac{b_0 t^2}{64} + a_1^2} - \frac{\theta_2}{2} p_1$, $\beta_{- nc} = \pm \frac{1}{4} \sqrt{\frac{16a_1 + b_0}{t}} + \frac{\theta_2}{2} \sqrt{a_1^2 + \frac{b_0 t^2}{64} + a_1^2} + \frac{\theta_2}{2} n_1$, $\beta_{+ nc} = \pm \frac{1}{4} \sqrt{\frac{16a_1 + b_0}{t}} + \frac{\theta_2}{2} \sqrt{a_1^2 + \frac{b_0 t^2}{64} + a_1^2} + \frac{\theta_2}{2} n_1$, |

Table [III A]. Noncommutative solutions for, $\gamma = -1, 1, 0$, and constraints $q$, $a_1$ and $b_0$. |
FIG. 1: Plots of $\beta_{\pm nc}$ that appear in the second line in the table III A, using the values in the parameters $n_1 = 1, p_1 = 1, b_0 = 10$ and $\theta = 0, 0.05, 0.2$, from bottom to top in the figure. The possible isotropization is saw in function of the curvature, but it appears again in this fields.

Hamiltonian is not modify; in the second approach we modify the hamiltonian via shift in the variables but the symplectic structure stay intact. For the first case, as we said we have the usual hamiltonian (6), but the symplectic structure is modify as follow

$$\{P_\Omega, P_+\}_s = \{P_+, P_-\}_s = 0, \quad \{q^\mu, P_{q^\nu}\}_s = 1,$$

$$\{\Omega, \beta_-\}_s = \theta_1, \quad \{\Omega, \beta_+\}_s = \theta_2, \quad \{\beta_-, \beta_+\}_s = \theta_3.$$
where the $\ast$ is the Moyal product $[14]$. In the second case, the Hamiltonian is modified by the shift (28), (29) and (30) resulting
\[
H_{nc} = \frac{Ne^{-3\Omega_{nc}}}{24} \left[ -P^2_\Omega + P^2_+ + P^2_- - \lambda e^{6\Omega_{nc}} + b\gamma e^{-3(\gamma - 1)\Omega_{nc}} \right] = 0,
\]
but the symplectic structure is the one that we know, the commutative one (18).

The noncommutative equations of motion, for the first formalism that we exposed have the original variables, but with the modified symplectic structure,
\[
\begin{align*}
\dot{q}^\mu_{nc} &= \{q^\mu, H\}_\ast, \\
\dot{P}^\mu_{nc} &= \{P^\mu, H\}_\ast,
\end{align*}
\]
and for the second formalism we use the shifted variables but with the original (commutative) symplectic structure
\[
\begin{align*}
\dot{q}^\mu_{nc} &= \{q^\mu_{nc}, H_{nc}\}, \\
\dot{P}^\mu_{nc} &= \{P^\mu_{nc}, H_{nc}\},
\end{align*}
\]
in both approaches we have the same result. Therefore the equations of motion take the form
\[
\begin{align*}
\dot{\Omega}_{nc} &= \{\Omega, H\}_\ast = \{\Omega_{nc}, H_{nc}\} = -\frac{e^{-3\Omega_{nc}}}{12} P_\Omega, \\
\dot{\beta}^-_{nc} &= \{\beta^-, H\}_\ast = \{\beta^-_{nc}, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{12} P^- + \frac{\theta_1}{2} \dot{P}_\Omega, \\
\dot{\beta}^+_{nc} &= \{\beta^+, H\}_\ast = \{\beta^+_{nc}, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{12} P^+ + \frac{\theta_2}{2} \dot{P}_\Omega, \\
\dot{P}_\Omega &= \{P_\Omega, H\}_\ast = \{P_{\Omega}, H_{nc}\} = \frac{e^{-3\Omega_{nc}}}{8} \left[ 6\lambda e^{6\Omega_{nc}} + 3(\gamma - 1)b\gamma e^{-3(\gamma - 1)\Omega_{nc}} \right], \\
\dot{P}^- &= \{P^-, H\}_\ast = \{P^-, H_{nc}\} = 0, \quad \rightarrow \quad P^- = p_1, \\
\dot{P}^+ &= \{P^+, H\}_\ast = \{P^+, H_{nc}\} = 0, \quad \rightarrow \quad P^+ = n_1.
\end{align*}
\]
if we proceed as in the commutative case we get the solutions showed in the table IVA.

\section*{IV. CONCLUSIONS}

In this work we present the equivalence in General Relativity between the WKB-approximation and Hamilton formalism, in the commutative and noncommutative scenarios,
this was achieved by means of a comparative study of the exact solutions for the Bianchi type I cosmological model coupled to barotropic perfect fluid and cosmological term. As we can see the solution $\Omega_{nc}$ is the commutative solution plus a function on $\theta_i$, independent of time. However, we have that in the physical gauge $N = 1$, the $\beta_{\pm nc}$ noncommutative solutions suffers drastic changes with respect to the $\beta_{\pm}$ commutative evolution. These changes give the possibility that in some ranges on the parameter $\theta_i$ and cosmological constant, occurs a dynamical isotropization, i.e., $\beta_{nc} \rightarrow$ a constant curvature. In other hand, in the gauge $N = 24e^{3\Omega}$ (see appendix A and B), in all cases considered the influence of the noncommutativity is encoded as an addition smooth function on time, to the classical solutions and the change is qualitatively very remarkable only for certain ranges on the $\theta_i$ parameters, but in general, the anisotropization is not modified, in sense commutativity is recovered dynamically. Besides we show that the definitions of the noncommutative commutators can be applied to all the variables of the minisuperspace. This approach can be used for other Bianchi cosmological model, which will be reported elsewhere.

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APPENDIX A: COMMUTATIVE CLASSICAL SOLUTIONS IN THE GAUGE $N = 24e^{3\Omega}$

1. Commutative Classical Solutions á la WKB

The master equation becomes

\[ 2dt = \frac{d\Omega}{\sqrt{\alpha_i^2 - \lambda e^{6\Omega} + b_{\gamma}e^{-3(\gamma - 1)\Omega}}} \tag{A1} \]

and the other two equations are immediately integrable. Again for particular cases in the $\gamma$ parameter, we present the classical solutions, table A1.
Table A1. Classical Solutions for $\gamma = -1, 1, 0, \frac{1}{3}$, and constraints $a_1, b_0$ and $b_1$.

| Case | Commutative solutions |
|------|------------------------|
| $\gamma = -1, \Lambda \neq 0, \rho = M_{-1}$ | $\Omega = \frac{1}{6} \ln \left[ -\frac{a_1^2}{384\pi GM_{-1} - 48\Lambda} \sech^2 (6a_1 t) \right], \quad a_1^2 = n_1^2 + p_1^2,$ $\beta_+ = \pm 2n_1 t,$ $\beta_- = \pm 2p_1 t.$ |
| $\gamma = 1, \Lambda \neq 0, \rho = M_{1}e^{-6\Omega}$ | $\Omega = \frac{1}{6} \ln \left[ \frac{a_1^2}{48\Lambda} \sech^2 (6a_1 t) \right], \quad a_1^2 = n_1^2 + p_1^2 + 384\pi GM_{1},$ $\beta_+ = \pm 2n_1 t,$ $\beta_- = \pm 2p_1 t.$ |
| $\gamma = 1, \Lambda = 0, \rho = M_{1}e^{-6\Omega}$ | $\Omega = 2\sqrt{a_1^2 + b_1 t}, \quad a_1^2 = n_1^2 + p_1^2 + 384\pi GM_{1},$ $\beta_+ = -2n_1 t,$ $\beta_- = -2p_1 t.$ |
| $\gamma = 0, \Lambda = 0, \rho = M_{0}e^{-3\Omega}$ | $\Omega = \frac{1}{3} \ln \left[ -\frac{a_1^2}{b_0} \sech^2 (3a_1 t) \right], \quad b_0 = 384\pi GM_{0},$ $\beta_+ = \pm 2n_1 t,$ $a_1^2 = n_1^2 + p_1^2,$ $\beta_- = \pm 2p_1 t.$ |
| $\gamma = \frac{1}{3}, \Lambda = 0, \rho = M_{\frac{1}{3}}e^{-4\Omega}$ | $\Omega = \frac{1}{2} \ln \left[ -\frac{a_1^2}{b_1^3} \sech^2 (2a_1 t) \right], \quad b_1^3 = 384\pi GM_{\frac{1}{3}},$ $\beta_+ = \pm 2n_1 t,$ $b_1^3 = 384\pi GM_{\frac{1}{3}}.$ |

2. Classical Solutions via Hamiltonian formalism

With the gauge fixed to $N = 24e^{3\Omega}$ we can see that the hamiltonian takes the form

$$H = -P_\Omega^2 + P_+^2 + P_-^2 - \lambda e^{6\Omega} + b_0 e^{-3(\gamma - 1)\Omega} = 0. \quad (A2)$$

The Poisson brackets structure yields to equations of motion

$$\dot{\Omega} = \{\Omega, H\} = -2P_\Omega, \quad (A3)$$
$$\dot{\beta}_- = \{\beta_-, H\} = 2P_-, \quad \rightarrow \quad \beta_- = \pm 2p_1 t, \quad (A4)$$
$$\dot{\beta}_+ = \{\beta_+, H\} = 2P_+, \quad \rightarrow \quad \beta_+ = \pm 2n_1 t, \quad (A5)$$
$$\dot{P}_\Omega = \{P_\Omega, H\} = [+6\lambda e^{6\Omega} + 3(\gamma - 1)b_0 e^{-3(\gamma - 1)\Omega}], \quad (A6)$$
$$\dot{P}_- = \{P_-, H\} = 0, \quad \rightarrow \quad P_- = \pm p_1 = \text{const.} \quad (A7)$$
$$\dot{P}_+ = \{P_+, H\} = 0, \quad \rightarrow \quad P_+ = \pm n_1 = \text{const.} \quad (A8)$$
Using (A2), introducing (A7) and (A8), we obtain the expression for $P_\Omega$

$$P_\Omega = \sqrt{m_1^2 - \lambda e^{6\Omega} + b_\gamma e^{-3(\gamma-1)\Omega}}, \quad (A9)$$

being self-consistent with equation (A6), where $a_1^2 = n_1^2 + p_1^2$. Introducing this equation into (A3) we get the master equation found to solve the Einstein field equation in this gauge, where the classical solutions are presented in Table IIB.

**APPENDIX B: NONCOMMUTATIVE CLASSICAL SOLUTIONS**

1. **Noncommutative Classical Solutions in the Gauge $N = 24e^{3\Omega}$ à la WKB and via Hamiltonian formalism**

The noncommutative solutions in the space $q^\mu$ become

| Case | Noncommutative Solutions |
|------|--------------------------|
| $\gamma = -1, \, \Lambda \neq 0, \, \rho_{-1} = M_{-1}$ | $\Omega_{nc} = \frac{1}{6}\ln \left[ \frac{a_1^2}{38\pi GM_{-1} - 48\Lambda} \right] \text{Sech}^2 \left( 6a_1 t \right) - \theta_1 \frac{1}{2} p_1 - \theta_2 \frac{1}{2} n_1$, |
| $a_1^2 = n_1^2 + p_1^2$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_1 a_1}{2} \tanh(6a_1 t) - \theta_2 \frac{1}{2} p_1$, $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \tanh(6a_1 t) + \theta_2 \frac{1}{2} n_1$, |
| $\gamma = 1, \, \Lambda \neq 0, \, \rho_1 = M_1 e^{-6\Omega}$ | $\Omega_{nc} = \frac{1}{6}\ln \left[ \frac{a_1^2}{48\Lambda} \text{Sech}^2 \left( 6a_1 t \right) \right] - \theta_1 \frac{1}{2} p_1 - \theta_2 \frac{1}{2} n_1$, |
| $a_1^2 = n_1^2 + p_1^2 + 384\pi GM_1$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_1 a_1}{2} \tanh(6a_1 t) - \theta_2 \frac{1}{2} p_1$, $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \tanh(6a_1 t) + \theta_2 \frac{1}{2} n_1$, |
| $\gamma = 1, \, \Lambda = 0, \, \rho_1 = M_1 e^{-6\Omega}$ | $\Omega_{nc} = 2a_1 t - \theta_1 \frac{1}{2} p_1 - \theta_2 \frac{1}{2} n_1$, |
| $a_1^2 = n_1^2 + p_1^2 + 384\pi GM_1$, $\beta_{+nc} = -2n_1 t + \theta_1 a_1 - \theta_2 \frac{1}{2} p_1$, $\beta_{-nc} = -2p_1 t + \theta_1 a_1 + \theta_2 \frac{1}{2} n_1$, |
| $\gamma = 0, \, \Lambda = 0, \, \rho_0 = M_0 e^{-3\Omega}$ | $\Omega_{nc} = \frac{1}{3}\ln \left[ \frac{a_1^2}{b_0 \text{sech}^2(3a_1 t)} \right] - \theta_1 \frac{1}{2} p_1 - \theta_2 \frac{1}{2} n_1$, |
| $b_0 = 384\pi GM_0$, $a_1^2 = n_1^2 + p_1^2$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_1 a_1}{2} \tanh(3a_1 t) - \theta_2 \frac{1}{2} p_1$, $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \tanh(3a_1 t) + \theta_2 \frac{1}{2} n_1$, |
| $\gamma = \frac{1}{3}, \, \Lambda = 0, \, \rho_0 = M_1 e^{-\frac{4\Omega}{3}}$ | $\Omega_{nc} = \frac{1}{3}\ln \left[ \frac{a_1^2}{b_0 \text{sech}^2(2a_1 t)} \right] - \theta_1 \frac{1}{2} p_1 - \theta_2 \frac{1}{2} n_1$, |
| $a_1^2 = n_1^2 + p_1^2$, $\beta_{+nc} = \pm 2n_1 t + \frac{\theta_1 a_1}{2} \tanh(2a_1 t) - \theta_2 \frac{1}{2} p_1$, $\beta_{-nc} = \pm 2p_1 t + \frac{\theta_1 a_1}{2} \tanh(2a_1 t) + \theta_2 \frac{1}{2} n_1$. |

*Table [B7] Noncommutative solutions for $\gamma = -1, \frac{1}{3}, 1, 0$, and constraints $a_1, b_0$ and $b_1$.***
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