On Special Differential Subordinations Using Fractional Integral of Sălăgean and Ruscheweyh Operators

Alina Alb Lupaș and Georgia Irina Oros

Department of Mathematics and Computer Science, University of Oradea, 1 Universitatii Street, 410087 Oradea, Romania; alblupas@gmail.com
* Correspondence: georgia_oros_ro@yahoo.co.uk
† These authors contributed equally to this work.

Abstract: In the present paper, a new operator denoted by $D_z^{-1}L_n^α$ is defined by using the fractional integral of Sălăgean and Ruscheweyh operators. By means of the newly obtained operator, the subclass $S_n(α, λ)$ of analytic functions in the unit disc is introduced, and various properties and characteristics of this class are derived by applying techniques specific to the differential subordination concept. By studying the operator $D_z^{-1}L_n^α$, some interesting differential subordinations are also given.

Keywords: differential subordination; convex function; best dominant; differential operator; fractional integral

1. Introduction

The concept of differential subordination was introduced by P.T. Mocanu and S.S. Miller in two articles in 1978 and 1981. Since then, an entire theory has developed around this concept, and many approaches using it have emerged. One important research direction was established by involving different differential and integral operators in the studies. Obtaining subordination properties using operators is a vast topic of research which has its roots at the beginning of the study using this theory and still presents interest for researchers. One of the most usual outcomes of the study involving operators is defining new classes of functions and studying properties related to them. This paper follows this line of research.

The common notations are used for the basic notions involved in the present study.

Denote by $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and by $\mathcal{H}(U)$ to understand the class of holomorphic functions in $U$. Consider the subclass $\mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U \}$, and write $A_1$ as $A$. Denote by $\mathcal{K}(α) = \{ f \in A : \Re \left( \frac{zf''(z)}{f'(z)} \right) > α \}$, the class of convex functions of order $α$ when $0 < α < 1$. The class of convex functions denoted by $\mathcal{K}$ is obtained for $α = 0$. For $a \in \mathbb{C}$, $n \in \mathbb{N}^*$; let $\mathcal{H}[a,n] = \{ f \in \mathcal{H}(U) : f(z) = a + az^n + a_{n+1}z^{n+1} + \ldots, z \in U \}$ and use $\mathcal{H}_0 = \mathcal{H}[0,1]$.

We used some definitions related to the theory of differential subordination synthesized in [1].

Definition 1 ([1], p. 4). If $f$ and $g$ are analytic functions in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$, if there is an analytic function $w$ in $U$, with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Definition 2 ([1], p. 7). Let $ψ : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and $h$ be univalent functions in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination, then the following is the case:

$$ψ(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U,$$

(1)
then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all $p$ satisfying (1).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of $U$.

Two lemmas which are useful for proving the original results of the theorems in the next section are next given.

**Lemma 1.** (Hallenbeck and Ruscheweyh ([1], Th. 3.1.6, p. 71)) Let $h$ be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}\backslash\{0\}$ be a complex number with $\text{Re}\ \gamma \geq 0$. If $p \in H[a,n]$ and the following is the case:

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad z \in U,$$

where $g(z) = \frac{\gamma}{n^{2+\gamma}} \int_0^z h(t)t^{\gamma/n-1}dt, \ z \in U$.

**Lemma 2** (Miller and Mocanu [1]). Let $g$ be a convex function in $U$ and let $h(z) = g(z) + n\alpha z g'(z), z \in U$, where $\alpha > 0$ and $n$ is a positive integer.

If $p(z) = g(0) + pnz^n + pn+1z^{n+1} + \ldots, z \in U$, is holomorphic in $U$ and the following is the case:

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z), \quad z \in U,$$

and this result is sharp.

The well-known definitions for Sâlâgean and Ruscheweyh operators are also reminded.

**Definition 3** (Sâlâgean [2]). For $f \in A, n \in \mathbb{N}$, the operator $S^n$ is defined by $S^n : A \to A$.

$$S^0f(z) = f(z)$$

$$S^1f(z) = zf'(z)$$

$$S^{n+1}f(z) = z(S^n f(z))', \ z \in U.$$

**Remark 1.** For $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in A$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, z \in U$.

Sâlâgean differential operator inspired many researchers to generalize it, as it can be observed, for example, in [3,4]. Quantum calculus has also been added to the studies for obtaining extensions of different types of operators. A quantum symmetric conformable differential operator is introduced in [5] as the generalization of known differential operators among which the Sâlâgean differential operator is included. Recently, in a new study [6], the authors have formulated a symmetric differential operator and its integral which has the Sâlâgean differential operator as the special case. The new type of operator is introduced by making use of the concept of symmetric derivative of complex variables. A modified symmetric Sâlâgean $g$-differential operator is obtained by combining the quantum calculus and the symmetric Sâlâgean differential operator. This new operator is introduced and studied in [7]. The results which follow in the next section of this paper could be adapted by using this new symmetric Sâlâgean differential operator combined with the Ruscheweyh differential operator, as observed in the same cited paper [6].
** Remark 2. For \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A} \), then \( R^n f(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(n+j)}{\Gamma(n+1) \Gamma(j)} a_j z^j, \quad z \in U \), where \( \Gamma \) is the gamma function and \( \Gamma(n) = (n-1)! \) for \( n \in \mathbb{N}^* \).

Using operators derived as a combination of those two operators, interesting results can be obtained, as it can be observed in [9–12]. We can also refer to [13,14] for applications of differential operators in the analysis of phenomena from mathematical biology and physics.

The operator introduced in [9], which inspired the study shown in this paper, is the following.

** Definition 5 ([9]). ** Let \( \alpha \geq 0, \ n \in \mathbb{N} \). Denote by \( L^n_{\alpha} \) the operator given by \( L^n_{\alpha} : \mathcal{A} \to \mathcal{A} \).

\[
L^n_{\alpha} f(z) = (1-\alpha) R^n f(z) + \alpha S^n f(z), \quad z \in U.
\]

** Remark 3. ** \( L^n_{\alpha} \) is a linear operator and if \( f \in \mathcal{A}, f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( L^n_{\alpha} f(z) = z + \sum_{j=2}^{\infty} \left( a_j + (1-\alpha) \frac{\Gamma(n+j)}{\Gamma(n+1) \Gamma(j)} \right) a_j z^j, \quad z \in U \).

** Definition 6 ([15]). ** The fractional integral of order \( \lambda (\lambda > 0) \) is defined for a function \( f \) by the following:

\[
D^{-\lambda}_z f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{2}
\]

where \( f \) is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of \( (z-t)^{\lambda-1} \) is removed by requiring \( \log(z-t) \) to be real, when \( (z-t) > 0 \).

Fractional integral was used intensely for obtaining new operators which have generated interesting subclasses of functions providing useful and inspiring outcome related to them [16–21]. Similar methods are used in the present investigation for obtaining the original results shown in the next section.

Using Definitions 5 and 6, the fractional integral associated with the linear differential operator \( L^n_{\alpha} f \) is introduced. Using this operator, a new subclass of analytic functions is introduced and investigated by applying means of the theory of differential subordinations.

2. Main Results

We introduce the fractional integral of the operator \( L^n_{\alpha} f \).

** Definition 7. ** Let \( \alpha \geq 0, \lambda > 0 \) and \( n \in \mathbb{N} \). The fractional integral of order \( \lambda \) is defined by the following.

\[
D^{-\lambda}_z L^n_{\alpha} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{L^n_{\alpha} f(t)}{(z-t)^{1-\lambda}} dt = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt + \sum_{j=2}^{\infty} \left( \alpha a_j + (1-\alpha) \frac{\Gamma(n+j)}{\Gamma(n+1) \Gamma(j)} \right) \int_0^z \frac{t^j}{(z-t)^{1-\lambda}} dt.
\]
The following form can be obtained easily:

\[
D_z^{-\lambda} L^\eta_a f(z) = \frac{1}{\Gamma(\lambda + 2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \left[ \frac{a^j \Gamma(j + 1)}{\Gamma(j + \lambda + 1)} \frac{\Gamma(m + j)}{\Gamma(m + 1)\Gamma(j + \lambda + 1)} \right] \eta_j z^j, \]

considering \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in A \). We note that \( D_z^{-\lambda} L^\eta_a f(z) \in \mathcal{H}[0, \lambda + 1] \).

Using this operator, the subclass of analytic functions studied in this paper is defined as follows.

**Definition 8.** The subclass \( S_n(\delta, \alpha, \lambda) \) consists of functions \( f \in A \) which satisfy the inequality:

\[
\Re \left( D_z^{-\lambda} L^\eta_a f(z) \right)' > \delta, \quad z \in U,
\]

when \( \delta \in [0, 1) \), \( \alpha \geq 0 \), \( \lambda > 0 \), and \( n \) is a natural number.

We begin the study of the class by proving that the set of functions contained in it is convex.

**Theorem 1.** The set \( S_n(\delta, \alpha, \lambda) \) is convex.

**Proof.** Consider the following functions:

\[
f_j(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad k = 1, 2, \quad z \in U,
\]

from the class \( S_n(\delta, \alpha, \lambda) \). We have to show that the function as follows

\[
h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)
\]

belongs to the class \( S_n(\delta, \alpha, \lambda) \) with \( \eta_1, \eta_2 \geq 0 \), \( \eta_1 + \eta_2 = 1 \).

We have \( h(z) = z + \sum_{j=2}^{\infty} (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j \), \( z \in U \), so the following is the case:

\[
D_z^{-\lambda} L^\eta_a h(z) = \frac{1}{\Gamma(\lambda + 2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \left[ \frac{a^j \Gamma(j + 1)}{\Gamma(j + \lambda + 1)} \frac{\Gamma(m + j)}{\Gamma(m + 1)\Gamma(j + \lambda + 1)} \right] (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j, \quad z \in U,
\]

and by differentiating this relation, we obtain

\[
(D_z^{-\lambda} L^\eta_a h(z))' = \frac{1}{\Gamma(\lambda + 2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \left[ \frac{a^j \Gamma(j + 1)}{\Gamma(j + \lambda + 1)} \frac{\Gamma(m + j)}{\Gamma(m + 1)\Gamma(j + \lambda + 1)} \right] (\eta_1 a_{j1} + \eta_2 a_{j2}) (j + \lambda) z^{j+\lambda-1}, \quad z \in U \text{ and the following,}
\]

\[
\Re \left( D_z^{-\lambda} L^\eta_a h(z) \right)' = \Re \left( \frac{1}{\Gamma(\lambda + 1)} z^\lambda \right) + \Re \left( \eta_1 \sum_{j=2}^{\infty} (j + \lambda) \left[ \frac{a^j \Gamma(j + 1)}{\Gamma(j + \lambda + 1)} \frac{\Gamma(m + j)}{\Gamma(m + 1)\Gamma(j + \lambda + 1)} \right] a_{j1} z^{j+\lambda-1} \right) \]

\[
+ \Re \left( \eta_2 \sum_{j=2}^{\infty} (j + \lambda) \left[ \frac{a^j \Gamma(j + 1)}{\Gamma(j + \lambda + 1)} \frac{\Gamma(m + j)}{\Gamma(m + 1)\Gamma(j + \lambda + 1)} \right] a_{j2} z^{j+\lambda-1} \right).
\]

Since \( f_1, f_2 \in S_n(\delta, \alpha, \lambda) \), we have the following:

\[
\Re \left( \eta_k \sum_{j=2}^{\infty} \left[ a^j + (1 - \alpha)c^j_{n+j-1} \right] a_{j1} z^{j-1} \right) > \eta_k \left( \delta - \Re \left( \frac{1}{\Gamma(\lambda + 1)} z^\lambda \right) \right), \quad k = 1, 2,
\]
which implies the following.

\[
Re \left( D_z^{-\lambda} L^n_u h(z) \right)' > Re \left( \frac{1}{\Gamma(\lambda + 1)} z^\lambda \right) + \eta_1 \left( \delta - Re \left( \frac{1}{\Gamma(\lambda + 1)} z^\lambda \right) \right) + \eta_2 \left( \delta - Re \left( \frac{1}{\Gamma(\lambda + 1)} z^\lambda \right) \right) = \delta, \quad z \in U,
\]

Equivalently, \( S_n(\delta, \alpha, \lambda) \) is convex. \( \square \)

Next, several interesting differential subordinations are proved involving the operator \( D_z^{-\lambda} L^u_n f \) following the idea used in [16].

**Theorem 2.** Taking \( g \) as a convex function in \( U \) and defining \( h(z) = g(z) + \frac{1}{c+2} z g'(z) \), with \( c > 0, z \in U \), if \( f \in S_n(\delta, \alpha, \lambda) \) and \( F(z) = L_c(f)(z) = \frac{c+2}{z+c+1} \int_0^z t^\lambda f(t)dt, z \in U \), the following differential subordination

\[
\left( D_z^{-\lambda} L^u_n f(z) \right)' < h(z), \quad z \in U,
\]

implies

\[
\left( D_z^{-\lambda} L^u_n F(z) \right)' < g(z), \quad z \in U,
\]

and this result is sharp.

**Proof.** From the definition of \( F \), we have \( z^{c+1} F(z) = (c+2) \int_0^z t^\lambda f(t)dt \) and by differentiating it we obtain \( (c+1) F(z) + z F'(z) = (c+2) f(z) \). Applying the operator \( D_z^{-\lambda} \), it is equivalent to the following.

\[
(c+1) D_z^{-\lambda} L^u_n f(z) + z \left( D_z^{-\lambda} L^u_n F(z) \right)' = (c+2) D_z^{-\lambda} L^u_n f(z), \quad z \in U.
\]

Differentiating the last relation, the following equality is obtained:

\[
\left( D_z^{-\lambda} L^u_n F(z) \right)' + \frac{1}{c+2} z \left( D_z^{-\lambda} L^u_n F(z) \right)'' = \left( D_z^{-\lambda} L^u_n f(z) \right)', \quad z \in U,
\]

which transforms relation (4) into the following.

\[
\left( D_z^{-\lambda} L^u_n F(z) \right)' + \frac{1}{c+2} z \left( D_z^{-\lambda} L^u_n F(z) \right)'' < g(z) + \frac{1}{c+2} z g'(z).
\]

By using the following notation

\[
p(z) = \left( D_z^{-\lambda} L^u_n F(z) \right)', \quad z \in U,
\]

we obtain \( p \in \mathcal{H}(0, \lambda] \) and the previous differential subordination becomes the following.

\[
p(z) + \frac{1}{c+2} z p'(z) < g(z) + \frac{1}{c+2} z g'(z), \quad z \in U.
\]

Applying Lemma 2, we obtain

\[
p(z) < g(z), \quad z \in U, \quad i.e., \quad \left( D_z^{-\lambda} L^u_n F(z) \right)' < g(z), \quad z \in U,
\]

and \( g \) is the best dominant. \( \square \)

An interesting inclusion result is proved for the class \( S_n(\delta, \alpha, \lambda) \) in the next theorem:
Theorem 3. Let \( h(z) = \frac{1+(2\delta-1)z}{1+z^\delta} \), \( \delta \in [0,1) \) and \( c > 0 \). If \( \alpha \geq 0 \), \( \lambda > 0 \), \( n \in \mathbb{N} \) and \( I_c(f)(z) = \mathcal{C}_{\lambda} \int_0^z t^\lambda f(t)dt , z \in U \), then we have the following:

\[
I_c[S_n(\delta, \alpha, \lambda)] \subset S_n(\delta^*, \alpha, \lambda),
\]

where \( \delta^* = 2\delta - 1 + (c + 2)(2 - 2\delta) \int_0^1 \frac{t^{\alpha+1}}{t+1}dt . \)

Proof. By using the same reasoning as in the proof of Theorem 2, since \( h \) is convex, we obtain the following:

\[
p(z) + \frac{1}{c+2}zp'(z) < h(z),
\]

with \( p(z) = (D^{-\lambda}L^\alpha_f(z))^\prime \).

By applying Lemma 1, we obtain the following:

\[ p(z) < g(z) < h(z), \text{ i.e., } (D^{-\lambda}L^\alpha_f(z))^\prime < g(z) < h(z), \]

where

\[
g(z) = \frac{c + 2}{z^{\alpha+2}} \int_0^z \frac{1+(2\delta-1)t}{1+t}dt = 2\delta - 1 + \frac{(c + 2)(2 - 2\delta)}{z^{\alpha+2}} \int_0^z \frac{t^{\alpha+1}}{t+1}dt.
\]

From convexity of \( g \) and using the fact that \( g(U) \) is symmetric with respect to the real axis, we obtain the following,

\[
Re\left(D^{-\lambda}L^\alpha_f(z)^\prime\right) \geq \min_{|z|=1} Re\ g(z) = Re\ g(1) = \delta^* = 2\delta - 1 + (c + 2)(2 - 2\delta) \int_0^1 \frac{t^{\alpha+1}}{t+1}dt.
\]

\[ \square \]

Theorem 4. Let \( h(z) = g(z) + zg'(z) , z \in U \), when \( g \) is a convex function in \( U \) with \( g(0) = 0 \). If a function \( f \in A \) satisfies the following:

\[
(D^{-\lambda}L^\alpha_f(z))^\prime < h(z), \quad z \in U,
\]

then the following results

\[
\frac{D^{-\lambda}L^\alpha_f(z)}{z} < g(z), \quad z \in U,
\]

for \( \alpha \geq 0, \lambda > 0, n \in \mathbb{N} \), and this result is sharp.

Proof. Denote by \( p(z) = \frac{D^{-\lambda}L^\alpha_f(z)}{z} \in \mathcal{H}[0, \lambda] \) and we obtain \(zp(z) = D^{-\lambda}L^\alpha_f(z) , z \in U \). After differentiating it, we obtain \( p(z) + zg'(z) = (D^{-\lambda}L^\alpha_f(z))^\prime , z \in U \), and relation (6) becomes the following.

\[ p(z) + zg'(z) < h(z) = g(z) + zg'(z) , \quad z \in U. \]

By applying Lemma 2, we obtain the following.

\[ p(z) < g(z), \quad z \in U, \text{ i.e., } \frac{D^{-\lambda}L^\alpha_f(z)}{z} < g(z), \quad z \in U. \]

\[ \square \]

Theorem 5. Consider \( h \) the convex function of order \(-\frac{1}{2}\) with \( h(0) = 0 \). If a function \( f \in A \) satisfies

\[
(D^{-\lambda}L^\alpha_f(z))^\prime < h(z), \quad z \in U,
\]
Let $h$.

**Example 1.**

where $q$.

**Proof.** Let $p(z) = \frac{D^{-\lambda}L_{\alpha}f(z)}{z} \in \mathcal{H}[0, \lambda]$, $z \in U$. By differentiating it, we obtain \((D^{-\lambda}L_{\alpha}f(z))' = p(z) + zp'(z), z \in U\) and the relation (7) can be written as follows.

\[ p(z) + zp'(z) < h(z), \quad z \in U. \]

By applying Lemma 1, we obtain the following:

\[ p(z) < q(z), \quad z \in U, \]

i.e.,

\[ \frac{D^{-\lambda}L_{\alpha}f(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t)dt, \quad z \in U, \]

and $q$ is the best dominant. \[
\]

**Corollary 1.** Let $h(z) = \frac{2(\beta-1)z}{1+z}$ a convex function in $U$, where $0 \leq \beta < 1$. If $f \in \mathcal{A}$ and satisfies the differential subordination

\[ \left(D^{-\lambda}L_{\alpha}f(z)\right)' < h(z), \quad z \in U, \] (8)

then

\[ \frac{D^{-\lambda}L_{\alpha}f(z)}{z} < q(z), \quad z \in U, \]

where $q(z) = 2(\beta-1) + 2(1-\beta) \frac{\ln(1+z)}{z}$, $z \in U$. The function $q$ is convex, and it is the best dominant.

**Proof.** Using the same reasoning as in the proof of Theorem 5 for $p(z) = \frac{D^{-\lambda}L_{\alpha}f(z)}{z}$, the differential subordination (8) becomes the following.

\[ p(z) + zp'(z) < h(z) = \frac{2(\beta-1)z}{1+z}, \quad z \in U. \]

By applying Lemma 1, we obtain $p(z) < q(z)$. In other words, we have the following.

\[ \frac{D^{-\lambda}L_{\alpha}f(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t)dt = \frac{1}{z} \int_0^z \frac{2(\beta-1)t}{1+t}dt = 2(\beta-1) + 2(1-\beta) \frac{1}{z} \ln(z+1), \quad z \in U. \]

**Example 1.** Let $h(z) = \frac{2z}{1+z}$ with $h(0) = 0$, $h'(z) = -\frac{2}{(1+z)^2}$ and $h''(z) = -\frac{4}{(1+z)^3}$. Since $Re\left(\frac{z\pi'(z)}{\pi(z)} + 1\right) = Re\left(1 + \frac{1}{1+z}\right) = Re\left(1 - \frac{1}{1+z}\right) = \frac{1}{1+z} > 0 > -\frac{1}{4}$, the function $h$ is convex in $U$.

Let $f(z) = z + z^2, z \in U$. For $\alpha = 2, n = 1$, we have $L_{1/2}f(z) = zf'(z) = z + 2z^2$ and $D^{-\lambda}L_{1/2}f(z) = \frac{1}{\Gamma(\lambda+1)} \int_0^z \frac{L_{1/2}f(t)}{(1-t)^{\lambda+1}}dt = \frac{1}{\Gamma(\lambda+1)} \int_0^z \frac{L_{1/2}f(t)}{(1-t)^{\lambda+1}}dt$. After simple computation, we obtain $D^{-\lambda}L_{1/2}f(z) = \frac{1}{\Gamma(\lambda+1)} z^{\lambda+1} + \frac{4}{\Gamma(\lambda+2)} z^{\lambda+2}$ and $(D^{-\lambda}L_{1/2}f(z))' = \frac{1}{\Gamma(\lambda+1)} z^{\lambda} + \frac{4}{\Gamma(\lambda+2)} z^{\lambda+1}$.

Moreover, we obtain $q(z) = \frac{1}{z} \int_0^z \frac{2z}{1+z}dt = -2 + 2 \frac{\ln(1+z)}{z}$. Using Theorem 5, we obtain the following.

\[ \frac{1}{\Gamma(\lambda+1)} z^{\lambda} + \frac{4}{\Gamma(\lambda+2)} z^{\lambda+1} < \frac{2z}{1+z}, \quad z \in U, \]
The following result is induced.

\[
\frac{1}{\Gamma(\lambda + 2)}z^{\lambda + 1} + \frac{4}{\Gamma(\lambda + 3)}z^{\lambda + 2} < -2 + \frac{2\ln(1 + z)}{z}, \quad z \in U.
\]

**Theorem 6.** Let \( h(z) = g(z) + zg'(z), z \in U, \) when \( g \) is a convex function in \( U \) with \( g(0) = 0. \) If a function \( f \in A \) satisfies

\[
\left( \frac{zD^{-\lambda}L^n f(z)}{D^{-\lambda}L^n f(z)} \right)' < h(z), \quad z \in U,
\]

then

\[
\frac{D^{-\lambda}L^nf(z)}{D^{-\lambda}L^nf(z)} < g(z), \quad z \in U,
\]

for \( \alpha \geq 0, \lambda > 0, n \in \mathbb{N}, \) and this result is sharp.

**Proof.** Let \( p(z) = \frac{D^{-\lambda}L^nf(z)}{D^{-\lambda}L^nf(z)}, \) and we obtain \( p(z) + zp'(z) = \left( \frac{z^n f(z)}{L^n f(z)} \right)' \). With this notation relation, (9) can be written as follows:

\[
p(z) + zp'(z) < h(z) = g(z) + zg'(z), \quad z \in U,
\]

and by applying Lemma 2, we obtain the following.

\[
p(z) < g(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{D^{-\lambda}L^nf(z)}{D^{-\lambda}L^nf(z)} < g(z), \quad z \in U.
\]

\( \square \)

**Theorem 7.** Consider \( g \) a convex function with \( g(0) = 0 \) and \( h(z) = g(z) + \lambda zg'(z), z \in U, \) \( \delta, \lambda > 0, \alpha \geq 0. \) If \( f \in A \) and satisfies the differential subordination

\[
\left( \frac{D^{-\lambda}L^nf(z)}{z} \right)^{\delta-1} \left( D^{-\lambda}L^nf(z) \right)' < h(z), \quad z \in U,
\]

then

\[
\left( \frac{D^{-\lambda}L^nf(z)}{z} \right)^{\delta} < g(z), \quad z \in U,
\]

and, this result is sharp.

**Proof.** Let \( p(z) = \left( \frac{D^{-\lambda}L^nf(z)}{z} \right)^{\delta} \in \mathcal{H}[0, \lambda \delta], \) \( z \in U. \) By differentiating it, we have the following.

\[
zp'(z) = \delta \left( \frac{D^{-\lambda}L^nf(z)}{z} \right)^{\delta-1} \left( D^{-\lambda}L^nf(z) \right)' - \delta \left( \frac{D^{-\lambda}L^nf(z)}{z} \right)^{\delta}.
\]

\[
= \delta \left( \frac{D^{-\lambda}L^nf(z)}{z} \right)^{\delta-1} \left( D^{-\lambda}L^nf(z) \right)' - \delta p(z),
\]

Therefore, the following is the case: \( p(z) + \frac{1}{\delta}zp'(z) = \left( \frac{D^{-\lambda}L^nf(z)}{z} \right)^{\delta-1} \left( D^{-\lambda}L^nf(z) \right)' \), \( z \in U. \)

Relation (10) can be written as follows.

\[
p(z) + \frac{1}{\delta}zp'(z) < h(z) = g(z) + \lambda zg'(z), \quad z \in U.
\]
Let $h$. Theorem 9. Consider $h$ the convex function of order $-\frac{1}{2}$ with $h(0) = 0$. If a function $f \in \mathcal{A}$ satisfies
\[
\left( \frac{D_z^{-\lambda} L_n^nf(z)}{z} \right)^{\delta-1} \left( D_z^{-\lambda} L_n^nf(z) \right)' \prec h(z), \quad z \in U,
\] (11)
then
\[
\left( \frac{D_z^{-\lambda} L_n^nf(z)}{z} \right)^{\delta} \prec q(z), \quad z \in U,
\]
for $\alpha \geq 0, \delta, \lambda > 0, n \in \mathbb{N}$, and $q(z) = \frac{1}{2} \int_0^z h(t)dt$ is convex and is the best dominant.

**Proof.** Consider $p(z) = \left( \frac{D_z^{-\lambda} L_n^nf(z)}{z} \right)^{\delta} \in \mathcal{H}[0,\delta\lambda], \quad z \in U$. After short computation, we obtain the following
\[
p(z) + \frac{1}{\delta} z p'(z) = \left( \frac{D_z^{-\lambda} L_n^nf(z)}{z} \right)^{\delta-1} \left( \left( D_z^{-\lambda} L_n^nf(z) \right)' \right), \quad z \in U,
\]
and relation (11) becomes the following,
\[
p(z) + \frac{1}{\delta} z p'(z) \prec h(z), \quad z \in U.
\]
By applying Lemma 1, we obtain the following
\[
p(z) \prec q(z), \quad z \in U, \quad \text{i.e.,} \quad \left( \frac{D_z^{-\lambda} L_n^nf(z)}{z} \right)^{\delta} \prec q(z) = \frac{\delta}{\delta^2} \int_0^z h(t)t^{\delta-1}dt, \quad z \in U,
\]
and $q$ is the best dominant. \[\square\]

**Theorem 9.** Let $h(z) = g(z) + zg'(z), \quad z \in U$, when $g$ is a convex function in $U$ with $g(0) = \frac{1}{x+1}$. If a function $f \in \mathcal{A}$ satisfies
\[
1 - \frac{D_z^{-\lambda} L_n^nf(z)}{z} \left( D_z^{-\lambda} L_n^nf(z) \right)'' \prec h(z), \quad z \in U,
\] (12)
then
\[
\frac{D_z^{-\lambda} L_n^nf(z)}{z} \prec g(z), \quad z \in U,
\]
for $\alpha \geq 0, \lambda > 0, n \in \mathbb{N}$, and this result is sharp.

**Proof.** Consider $p(z) = \frac{D_z^{-\lambda} L_n^nf(z)}{z} \left( D_z^{-\lambda} L_n^nf(z) \right)''$ and, by differentiating it, we obtain $p(z) + z p'(z) = 1 - \frac{D_z^{-\lambda} L_n^nf(z)}{z} \left( D_z^{-\lambda} L_n^nf(z) \right)''$, $z \in U$.

Using this notation the differential subordination becomes
\[
p(z) + z p'(z) \prec h(z) = g(z) + zg'(z),
\]
and, by applying Lemma 2, we obtain the following:

\[ p(z) \prec g(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{D^{-\lambda}_z L^u_n f(z)}{z(D^{-\lambda}_z L^u_n f(z))} \prec g(z), \quad z \in U, \]

and this result is sharp. \( \square \)

3. Conclusions

A new operator \( D^{-\lambda}_z L^u_n f \) is defined by using the fractional integral of the operator \( L^u_n f \) defined in a previously published paper using Sălăgean and Ruscheweyh operators. A new subclass of analytic functions \( S_n(\delta, \alpha, \lambda) \) is introduced and studied using the operator \( D^{-\lambda}_z L^u_n f \) and means related to differential subordination studies. The class \( S_n(\delta, \alpha, \lambda) \) has interesting properties given in proved theorems. New differential subordinations are obtained for \( D^{-\lambda}_z L^u_n f \). An example is also included for showing applications of the results stated and proved. Both the operator \( D^{-\lambda}_z L^u_n f \) and the subclass \( S_n(\delta, \alpha, \lambda) \) can be used for further studies. The operator can be applied for the introduction of other subclasses of analytic functions and further investigations related to coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity or close-to-convexity of functions belonging to the class \( S_n(\delta, \alpha, \lambda) \) can be performed. Symmetry properties for this newly introduced operator can be investigated in the future. Moreover, a similar operator can be introduced by using the fractional integral of an operator defined as a linear combination of symmetric Sălăgean differential operator and Ruscheweyh operator.

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