A two-weight Sobolev inequality for Carnot-Carathéodory spaces

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Received: 3 August 2020 / Revised: 10 October 2020 / Accepted: 15 October 2020 / Published online: 4 November 2020

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Abstract
Let \( X = \{X_1, X_2, \ldots, X_m\} \) be a system of smooth vector fields in \( \mathbb{R}^n \) satisfying the Hörmander’s finite rank condition. We prove the following Sobolev inequality with reciprocal weights in Carnot-Carathéodory space \( G \) associated to system \( X \)

\[
\left( \frac{1}{\int_{B_R} K(x) \, dx} \int_{B_R} \left| u \right|^t K(x) \, dx \right)^{1/t} \leq C R \left( \frac{1}{\int_{B_R} \frac{1}{K(x)} \, dx} \int_{B_R} \frac{|Xu|^2}{K(x)} \, dx \right)^{1/2},
\]

where \( Xu \) denotes the horizontal gradient of \( u \) with respect to \( X \). We assume that the weight \( K \) belongs to Muckenhoupt’s class \( A_2 \) and Gehring’s class \( G_\tau \), where \( \tau \) is a suitable exponent related to the homogeneous dimension.

Keywords Carnot-Carathéodory spaces · Weighted Sobolev inequalities · Muckenhoupt and Gehring weights

Mathematics Subject Classification 35R03 · 39B62

1 Introduction

This paper is devoted to study some basic functional and geometric properties of general families of vector fields that include the Hörmander’s type as a special case.

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Similar to their Euclidean counterparts, such properties play an important role in the analysis of the relevant differential operators (both linear and nonlinear).

We are concerned with a two-weight Sobolev type inequality on $\mathbb{G}$, where $\mathbb{G}$ denotes the Carnot-Carathéodory space $(\Omega, d)$ (suitably defined - see Sect. 2.1) associated to a system of smooth vector fields $X = \{X_1, X_2, \ldots, X_m\}$ on $\mathbb{R}^n$ satisfying the Hörmander’s finite rank condition. This fact introduces a kind of degeneracy different from that Euclidean one. Here, $\Omega$ is an open (Euclidean) bounded and connected set of $\mathbb{R}^n$, $n \geq 2$, and $d$ is the metric generated by $X$.

Let $u \in \text{Lip}(\mathbb{G})$. We denote by $X u = (X_1 u, \ldots, X_m u)$ the horizontal gradient of $u$ with respect to the system $X$, where $X_j$ plays the role of the first order differential operator acting on $u$ given by

$$X_j u(x) = \langle X_j(x), \nabla u(x) \rangle \quad \text{for } j = 1, \ldots, m.$$ 

Set

$$|X u| = \left( \sum_{j=1}^{m} (X_j u)^2 \right)^{1/2},$$

the length of the horizontal gradient of $u$. We refer to [5,12] for more details.

In our paper we prove a two-weight Sobolev type inequality where the weights $K$ and $K^{-1}$ form a $2$-admissible pair $(K^{-1}, K)$, namely

1) $K$ is locally doubling in $\Omega$ and $K^{-1}$ belongs to $A_2(\mathbb{G})$.
2) Given a compact set $V \subset \Omega$ there exist $t > 2$ and $\overline{C} \geq 1$ such that, for every ball $B$ with center in $V$ and $0 < r < 1$, it holds

$$r \left( \frac{\int_B K(x) \, dx}{\int_B K^{-1}(x) \, dx} \right)^{1/t} \leq \overline{C} \left( \frac{\int_B K^{-1}(x) \, dx}{\int_B K^{-1}(x) \, dx} \right)^{1/2}. \quad (1.1)$$

Note that inequality (1.1) is the Chanillo-Wheeden condition (see [8]), with exponents $t$ and 2, adapted to the Carnot-Carathéodory geometry (see [18]).

Our main result reads as follows.

**Theorem 1.1** Let $K$ be in $A_2(\mathbb{G}) \cap G_t(\mathbb{G})$ with $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$. Let $t > 2$. Then, for every $u \in \mathcal{C}_{0}^{1}(B_R)$, there exists a constant $C \geq 1$ such that

$$\left( \frac{1}{\int_{B_R} K(x) \, dx} \int_{B_R} |u|^t K(x) \, dx \right)^{1/t} \leq CR \left( \frac{1}{\int_{B_R} \frac{1}{K(x)} \, dx} \int_{B_R} \frac{|X u|^2}{K(x)} \, dx \right)^{1/2} \quad (1.2)$$

with

$$C = c(Q, n, t, q) \overline{C} \left[ K^{-1} \right]_{A_2}^{\frac{1}{2}} \left[ K \right]_{A_2}^{\frac{1}{2} - \frac{1}{q}},$$

where $\overline{C}$ is the constant in (1.1), $2 < q < t$, and $B_R$ denotes the ball centered at the origin with radius $R > 0$. Here, $[K^{-1}]_{A_2}$ and $[K]_{A_2}$ stand for $A_2$ constants of $K^{-1}$ and $K$, respectively.
By properties of Muckenhoupt’s class $A_p(G)$, we have that since $K \in A_2(G)$, then $K^{-1} \in A_2(G)$. Moreover, by [12, Theorem 4.8], the assumption that $K$ belongs to $A_2(G) \cap G_\tau(G)$, with $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$, guarantees that the pair $(K^{-1}, K)$ satisfies condition (1.1). Thus, one deduces that $(K^{-1}, K)$ is a 2-admissible pair in $\Omega$. We emphasize that the 2-admissible property of $(K^{-1}, K)$ will be used in the proof of Theorem 1.1.

The tools used to obtain inequality (1.2) are the classical ones of the Euclidean case. Nevertheless, here we deal with a degeneracy into the geometry due to the presence of a differential operator $Xu$ different from the classical gradient $\nabla u$. In particular, this fact causes a change of metric on $\mathbb{R}^n$ and consequently some of the results valid for Euclidean metric have been enlarged to Carnot-Carathéodory metric.

Let us emphasize that more general weighted inequalities for Euclidean case have been extensively investigated, and are the subject of a rich literature (see e.g. [1–4,6,8–11,14,15,26]).

In the Euclidean setting, Theorem 1.1 generalizes similar result contained in [2], where the authors prove a weighted Sobolev inequality of the same type as (1.2), with the weight $K(x)$ related to the function $|u|^t$ and the weight $K^{-1}(x)$ to the gradient $|\nabla u|^2$.

Problems of this kind, involving weighted Sobolev inequalities for Carnot-Carathéodory space $G$, have been systematically studied in the literature (see e.g. [7,12,16,17,19]).

The result of Theorem 1.1 is a particular case of that contained in [12, Corollary 3.4] with $v(x)$ replaced by $K(x)$ and $w(x)$ replaced by $K^{-1}(x)$. In [12] the authors show the following more general weighted Sobolev inequality

\[
\left( \int_{B_R} \frac{1}{v(x)} \frac{1}{w(x)} \frac{1}{w(x)} dx \int_{B_R} |u|^t v(x) dx \right)^{1/t} \leq C R \left( \int_{B_R} \frac{1}{w(x)} \frac{1}{w(x)} \frac{1}{w(x)} dx \int_{B_R} |Xu|^p w(x) dx \right)^{1/p},
\]

where $1 < p < t < \infty$, $C > 0$ is a constant, and $(w, v)$ is a $p$-admissible pair in $\Omega$. Herein, we prove inequality (1.2) by using different techniques which rely upon a combination of an estimate for fractional integral of first order with other some properties of $A_2(G)$ and $G_\tau(G)$ classes. Moreover, in contrast with the result in [12, Corollary 3.4], we give the explicit value of constant $C$ in our inequality (1.2).

Our paper is organized as follows. In Sect. 2 we give some preliminary results. Actually, in Sect. 2.1 we recall definition and basic properties of Hörmander vector fields, including Carnot-Carathéodory spaces; in Sect. 2.2 we discuss the theory of Muckenhoupt’s and Gehring’s weights. In Sect. 3 we present the machinery we need to work with the inequality we are interested in. Finally, we prove our main theorem.
2 Preliminary results

2.1 Carnot Carathéodory spaces

Let $\Omega$ be an open (Euclidean) bounded and connected subset in $\mathbb{R}^n$, with $n \geq 2$. Let $X = \{X_1, \ldots, X_m\}$ be a system of $C^\infty$ vector fields on $\mathbb{R}^n$.

We denote by $\text{Lie} \{X_1, \ldots, X_m\}$ the Lie algebra generated by $X_1, \ldots, X_m$ and by their commutators of any order. We say that a field $Z$ belongs to $\text{Lie} \{X_1, \ldots, X_m\}$ if and only if $Z$ is a finite linear combination of terms of this type

$$[X_{i_1}[X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]]]$$

for $k \in \mathbb{N}$, $1 \leq i_h \leq m$, $1 \leq h \leq k$.

We define, for any fixed $x \in \mathbb{R}^n$, the Lie rank as

$$\text{rank Lie} \{X_1, \ldots, X_m\} = \dim V(x),$$

where $V(x) = \{Z(x) : Z \in \text{Lie} \{X_1, \ldots, X_m\}\}$ is a subspace of $\mathbb{R}^n$. Henceforth, we assume that $X$ satisfies the following Hörmander’s finite rank condition in $\Omega$

$$\text{rank Lie} \{X_1, \ldots, X_m\} = n, \quad (2.1)$$

namely there exist a neighborhood $\Omega_0$ of $\overline{\Omega}$ and $m \in \mathbb{N}$ such that the family of commutators of the vector fields in $X$ up to length $m$ span $\mathbb{R}^n$ at every point of $\Omega_0$.

Let $C_X$ be the family of absolutely continuous curves $\gamma : [a, b] \rightarrow \mathbb{R}^n$ such that there exist measurable functions $c_j : [a, b] \rightarrow \mathbb{R}$, with $j = 1, \ldots, m$, fulfilling

$$\sum_{j=1}^m c_j(t)^2 \leq 1 \quad \text{and} \quad \gamma'(t) = \sum_{j=1}^m c_j(t)X_j(\gamma(t)) \quad \text{for a.e. } t \in [a, b].$$

We define Carnot-Carathéodory distance $d$ as

$$d(x, y) = \inf \{T > 0 : \exists \gamma \in C_X, \gamma(0) = x, \gamma(T) = y\} \quad \text{for } x, y \in \Omega. \quad (2.2)$$

Note that, owing to Hörmander’s finite rank condition (2.1), $d$ is a metric. This fact is not true in general. The Carnot-Carathéodory space $G$ is the pair $(\Omega, d)$ associated to a system of $C^\infty$ vector fields $X = \{X_1, \ldots, X_m\}$ on $\mathbb{R}^n$ fulfilling (2.1).

For $x \in \mathbb{R}^n$ and $R > 0$, set $B(x, R) = \{y \in \mathbb{R}^n : d(x, y) < R\}$. The basic properties of these balls have been obtained by Nagel, Stein and Wainger in [25]. In particular, in the following proposition, the authors prove that the metric $d$ is locally Hölder continuous with respect to the Euclidean metric.

**Proposition 2.1** ([25, Proposition 1.1]) Let $X_1, \ldots, X_m$ be as above. Then, for any compact set $E \subseteq \Omega$, there are positive constants $c_1, c_2$ and $\lambda \in (0, 1]$ such that

$$c_1|x - y| \leq d(x, y) \leq c_2|x - y|^{\lambda}$$
for every $x, y \in E$.

Thanks to Proposition 2.1, the topology of Carnot-Carathéodory induced by $d$ on $\Omega$ coincides with the Euclidean ones. In the sequel, all the distances will be understood in the sense of the Carnot-Carathéodory metric $d$. In particular, all the balls will be defined with respect to $d$.

We denote by $\cdot$ the Lebesgue measure in $(\mathbb{R}^n, d)$ and, by $\frac{1}{|B|} \int_B f(x) \, dx$, the average of a function $f$ on the ball $B$, i.e.

$$\int_B f(x) \, dx = \frac{1}{|B|} \int_B f(x) \, dx.$$ 

Note that the Lebesgue measure locally satisfies the following \textit{doubling condition} (see e.g. [25]).

**Proposition 2.2** For any compact set $E \subset \subset \Omega$, if $x_0 \in E$, there exist a constant $C_d \geq 1$, called doubling constant, and $R_0 > 0$ such that

$$|B(x_0, 2R)| \leq C_d |B(x_0, R)|$$

for $0 < R < R_0$.

Let $Y_1, \ldots, Y_l$ be the collection of the $X_j$’s and of those commutators which are needed to generate $\mathbb{R}^n$. To each $Y_i$ it is associated a formal “degree” $\deg(Y_i) \geq 1$, namely the corresponding order of the commutator. Set $I = (i_1, \ldots, i_n)$, with $1 \leq i_j \leq l$, an $n$-tuple of integers. We define (see also [25]) the degree of $I$ as

$$\tilde{d}(I) = \sum_{j=1}^n \deg(Y_{i_j}).$$

For a given compact set $E \subset \mathbb{R}^n$, we define $Q$ by

$$Q = \sup \{ \tilde{d}(I) : |a_I(x)| \neq 0, x \in E \},$$

the local homogeneous dimension of $E$ with respect to system $X$, where $a_I(x) = \det(Y_{i_1}, \ldots, Y_{i_n})$.

We define by

$$Q(x) = \inf \{ \tilde{d}(I) : |a_I(x)| \neq 0 \}$$

the homogeneous dimension at $x \in \mathbb{R}^n$ with respect to $X$. It is obvious that $3 \leq n \leq Q(x) \leq Q$.

Just to give an idea, we consider in $\mathbb{R}^3$ the system (see [13])

$$X = \{X_1, X_2, X_3\} = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, x_1 \frac{\partial}{\partial x_3} \right\}. $$
It is easy to see that \( l = 4 \) and
\[
\{Y_1, Y_2, Y_3, Y_4\} = \{X_1, X_2, X_3, [X_1, X_3]\}.
\]

Moreover, \( Q(x) = 3 \) for all \( x \neq 0 \), whereas for any compact set \( E \) containing the origin, \( Q(0) = Q = 4 \).

Let \( Y \) be a metric space and \( \mu \) a Borel measure in \( Y \). Assume \( \mu \) finite on bounded sets and satisfying the doubling condition on every open, bounded subset \( \Omega \) in \( Y \). We say that \( Q \) is a homogeneous dimension relative to \( \Omega \), if there exists a positive constant \( C \) such that
\[
\frac{\mu(B)}{\mu(B_0)} \geq C \left( \frac{R}{R_0} \right)^Q
\]
for any ball \( B_0 \) having center in \( \Omega \) and radius \( R_0 < \text{diam} \), and any ball \( B \) centered in \( x_0 \in B_0 \) and having radius \( R \leq R_0 \).

It is well known that the doubling condition implies the existence of the homogeneous dimension \( Q \). However, \( Q \) is not unique and it may change with \( \Omega \). Obviously, any \( Q' \geq Q \) it is also a homogeneous dimension.

For a bounded open set \( \Omega \) containing a family of vector fields satisfying the Hörmander’s finite rank condition, the homogeneous dimension of the Carnot-Carathéodory space \( \mathbb{G} \), defined with the Lebesgue measure, is given by \( Q = \log_2 C_d \), where \( C_d \) is the doubling constant.

### 2.2 Some properties of \( A_p \) and \( G_q \) classes

In this section, we recall a few properties of Muckenhoupt’s and Gehring’s classes (see [22,24,27,28]).

We recall that a weight is a positive function in \( L^1_{loc}(\mathbb{R}^n) \). We say that a weight \( w \) is doubling in \( \Omega \) if
\[
\int_{2B} w(x) \, dx \leq C \int_B w(x) \, dx,
\]
where the constant \( C \) is independent by the ball \( B \subset \Omega \).

We say that \( w \) is locally doubling in \( \Omega \) if for each compact set \( V \subset \Omega \) and \( \tilde{R} > 0 \) there exists \( C_{V, \tilde{R}} \) such that
\[
\int_{2B} w(x) \, dx \leq C_{V, \tilde{R}} \int_B w(x) \, dx,
\]
where the ball \( B \) has center in \( V \) and radius \( R < \tilde{R} \) and \( 2B \) is the ball concentric with \( B \) and having radius 2-times that of \( B \).

We say that a weight \( w \) belongs to the class \( A_p(\mathbb{G}) \) (briefly, \( w \in A_p(\mathbb{G}) \)) for some \( p \in (1, +\infty) \) if
\[
[w]_{A_p} = \sup_B \left( \int_B w(x) \, dx \right) \left( \int_B w(x)^{1-p'} \, dx \right)^{p-1}
\]
is finite, where the supremum is taken over all balls $B \subset \Omega$. Here, $p'$ denotes the Hölder conjugate of $p$. The quantity $[w]_{A_p}$ is called the $A_p$ constant of $w$.

When $p = 1$, we say that $w \in A_1(\mathbb{G})$ if there exists a constant $c \geq 1$ such that, for every ball $B \subset \Omega$,

$$
\int_B w(x) \, dx \leq c \inf_B \text{ess inf } w.
$$

If a weight belongs to a class $A_p$, it is called a Muckenhoupt weight.

A weight $w$ is said to belong to the class $G_q(\mathbb{G})$ (briefly, $w \in G_q(\mathbb{G})$) for some $q \in (1, +\infty)$ if

$$
[w]_{G_q} = \sup_B \left( \frac{\left( \int_B w(x)^q \, dx \right)^{\frac{1}{q}}}{\int_B w(x) \, dx} \right)
$$

is finite. The quantity $[w]_{G_q}$ is called the $G_q$ constant of $w$.

If a weight belongs to a class $G_q$, it is called a Gehring weight.

Here, we recall some properties of $A_p$ classes with respect to dyadic cubes which we will be used to prove Theorem 1.1.

We use a grid $\mathcal{D}_h$ of dyadic cubes $Q$, which are “almost balls”, where $h$ is a large negative integer which indexes the edgelengths $l(Q)$ of the smallest cubes $Q \in \mathcal{D}_h$.

In other words, the smallest edgelengths are $\lambda^h$ for an appropriate geometric constant $\lambda > 1$ and each cube in the grid has edgelength $\lambda^k$ for some $k \geq h$.

In particular, we will make use of a grid of dyadic cubes in the ball $B_R$ in the same spirit of [29], where it is proved that there exists a constant $\lambda > 1$ such that, for every $h \in \mathbb{Z}$, there are points $x_j^k \in B_R$ and a family of cubes $\mathcal{D}_h = \{Q_j^k\}$ for $j \in \mathbb{N}$ and $k = h, h + 1, \ldots$ such that

i) $B(x_j^k, \lambda^k) \subset Q_j^k \subset B(x_j^k, \lambda^{k+1})$.

ii) For each $k = h, h + 1, \ldots$, the family $\{Q_j^k\}$ is pairwise disjoint in $j$ and $B_R = \bigcup_j Q_j^k$.

iii) If $h \leq k < l$, then either $Q_j^k \cap Q_j^l = \emptyset$ or $Q_j^k \subset Q_j^l$.

We call the family $\mathcal{D} = \bigcup_{h \in \mathbb{Z}} \mathcal{D}_h$ a dyadic cube decomposition of $B_R$ and we refer to its sets as dyadic cubes which will be denoted by $Q$. We observe explicitly that being $\mathcal{D}$ a decomposition of $B_R$, then any dyadic cube $Q \in \mathcal{D}$ is contained in the ball $B_R$.

By [29], making use of (2.3), one can deduce the following lemma.

**Lemma 2.3** Let $w \in A_2(\mathbb{G})$ and let $Q$ and $Q_0$ dyadic cubes in $\mathbb{R}^n$ such that $Q \subset Q_0$. If $\beta > 1$, then

$$
\sum_{Q \subset Q_0} \left( \int_Q w \, dx \right)^{\beta} \leq (c(Q, n)[w]_{A_2})^{\beta - 1} \left( \int_{Q_0} w \, dx \right)^{\beta}. \quad (2.4)
$$

Another important property of $A_p(\mathbb{G})$ classes is given by the following proposition (see [20], [30, Chapter 5, p. 195]).
Proposition 2.4 If \( w \in A_p(\mathbb{G}) \), then, for any nonnegative \( f \),
\[
\left( \frac{1}{|Q|} \int_Q f(x) \, dx \right)^p \leq [w]_{A_p} \frac{1}{|Q|} \int_Q w(x) \, dx \int_Q |f(x)|^p w(x) \, dx \quad \forall Q \subset \mathbb{G}.
\]

(2.5)

2.3 Some preliminary estimates

In order to prove our main theorem, let us prove some preliminary results.

The first lemma yields an estimate of the fractional integral of order 1 (see e.g. [5]). In general, the fractional integral of order \( \alpha \in (0, Q) \) of a locally integrable function \( g \) in \( \mathbb{R}^n \) is defined as
\[
I_\alpha g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{d(x, y)^{Q-\alpha}} \, dy \quad \text{for} \ x \in \mathbb{R}^n.
\]

(2.6)

Lemma 2.5 Let \( g \in L^1_{loc}(\mathbb{G}) \) and assume that \( g \geq 0 \). Then
\[
I_1 g(x) \leq c_0 \sum_{Q \in D} \left( \frac{1}{|Q|} \int_{3Q} g(y) \, dy \right) \chi_Q(x) \quad \forall x \in \Omega,
\]

(2.7)

where \( c_0 \) is an absolute constant.

Proof Thanks to a dyadic cube decomposition, we discretize the operator \( I_1 \)
\[
I_1 g(x) = \sum_{k \in \mathbb{Z}} \left( \int_{2^{k-1} < d(x, y) \leq 2^k} \frac{g(y)}{d(x, y)^{Q-1}} \, dy \right)
\leq c_0 \sum_{k \in \mathbb{Z}} \sum_{Q \in D} \left[ \left( \frac{1}{l(Q)^Q} \int_{d(x, y) \leq l(Q)} g(y) \, dy \right) \chi_Q(x) \right]
\leq c_0 \sum_{Q \in D} \left[ \left( \frac{1}{Q} \int_{3Q} g(y) \, dy \right) \chi_Q(x) \right],
\]

where the last inequality follows by \( |Q| = l(Q)^Q \) and, moreover, by \( B(x, l(Q)) \subset 3Q \) if \( x \in Q \). Hence, inequality (2.7) is proved.

Let us consider a Dirichlet problem in this form
\[
\begin{cases}
\Delta_G \varphi = f(x) \quad &\text{in } B_R \\
\varphi = 0 \quad &\text{on } \partial B_R,
\end{cases}
\]

(2.8)

where \( \Delta_G \) denotes the canonical sub-Laplacian operator defined as \( \Delta_G = \sum_{j=1}^m X_j^2 \), with \( \{X_1, \ldots, X_m\} \) the family of smooth vector fields on \( \mathbb{R}^n \) satisfying the Hörmander’s finite rank condition.
Let \( \mathcal{F}_\alpha(B_R) \) be the anisotropic Hölder space, with \( \alpha \in (0, 1) \), defined by

\[
\mathcal{F}_\alpha(B_R) = \left\{ f : B_R \to \mathbb{R} : \sup_{x, y \in B_R \atop x \neq y} \frac{f(x) - f(y)}{d(x, y)^\alpha} < \infty \right\},
\]  
\[(2.9)\]

where \( d \) is the Carnot-Carathéodory distance given by (2.2).

In [21, Theorem 3.2], the authors proved that, if \( f \in \mathcal{F}_\alpha(B_R) \), then there exists a unique solution \( \varphi \in C^2(B_R) \cap C^1(\overline{B_R}) \) to problem (2.8), represented by the formula

\[
\varphi(x) = \int_{B_R} \Delta G \varphi \Gamma_x(y) \, dy.
\]  
\[(2.10)\]

Here, \( \Gamma_x(y) \) is the fundamental solution of the sub-Laplacian. Thanks to [21, Theorem 2.2], there exists a positive constant \( c \) such that

\[
\Gamma_x(y) = c \, d(x, y)^{2-Q}.
\]  
\[(2.11)\]

Consequently, combining (2.10) and (2.11) yields

\[
\varphi(x) = c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-2}} \, dy.
\]  
\[(2.12)\]

The next lemma gives an estimate of the gradient of the solution to problem (2.8) through the fractional integral of order 1.

**Lemma 2.6** Let \( f \in \mathcal{F}_\alpha(B_R) \) and let \( \varphi \) be the solution to problem (2.8). Then, there exists a positive constant \( c \) such that

\[
|X\varphi(x)| \leq c \, I_1 f(x),
\]  
\[(2.13)\]

where \( I_1(f) \) denotes the fractional integral of order 1 of \( f \).

**Proof** Owing to (2.12), it follows that

\[
X_j \varphi(x) = c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} X_j(d(x, y)) \, dy.
\]  
\[(2.14)\]

Thus,

\[
|X\varphi(x)| = \left( \sum_{j=1}^{n} |X_j \varphi(x)|^2 \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_{j=1}^{n} \left| c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} X_j(d(x, y)) \, dy \right|^2 \right)^{\frac{1}{2}}.
\]  
\[(2.15)\]
Since $|X_j(d(x, y))| = 1$ (see [23]), by (2.15) and (2.6) one can deduce that

$$\left| X \varphi(x) \right| \leq \left( \sum_{j=1}^{n} \left| c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} dy \right|^2 \right)^{\frac{1}{2}}$$

$$\leq nc \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} dy = c I_1 f(x), \quad (2.16)$$

where the second inequality is due to the fact that $a^2 + b^2 \leq (a + b)^2$. \hfill \Box

## 3 Proof of main result

The following preliminary lemma will be use in the proof of Theorem 1.1.

**Lemma 3.1** If $K \in A_2(\mathbb{G})$ and $u \in C_0^1(B_R)$, then

$$S_1 = \left[ \sum_{Q \in D} \left( \int_Q K(x) \ dx \right)^{\frac{q}{t'}} \left( \frac{1}{\int_Q K(x) \ dx} \int_{3Q} |u|^{t-1} K(x) \ dx \right)^{\frac{1}{q'}} \right]^{\frac{1}{q}} \leq C \left( \int_{B_R} |u|^{t} K(x) \ dx \right)^{\frac{1}{t}}, \quad (3.1)$$

where $2 < q < t$ and $C = c(Q, n, t, q)[K]_{A_2}^{\frac{1}{t} - \frac{1}{q'}}$.

**Proof** For each $h \in \mathbb{Z}$, we set

$$C^h = \left\{ Q \ dyadic \ cube: 2^h < \frac{1}{\int_Q K(x) \ dx} \int_Q |u|^{t-1} K(x) \ dx \leq 2^{h+1} \right\}. \quad (3.2)$$

Note that, if $Q$ is any dyadic cube such that $|u|^{t-1} K(x)$ is not identically zero on $Q$, then $Q$ belongs to only one collection $C^h$.

For each $h \in \mathbb{Z}$, let us build the collection $\{Q^h_j\}$ of pairwise disjoint maximal dyadic cubes (maximal with respect to inclusion) in $C^h$. If $Q \in C^h$, then there exists $j \in \mathbb{N}$ such that $Q \subset Q^h_j$. Note also that for each fixed $h$, the cubes $Q^h_j$ are disjoint with respect to $j$. Nevertheless, they may not be disjoint for different values of $h$. \hfill \Box
By (3.2),

\[
S_1 \leq \left( \sum_{h \in \mathbb{Z}} 2^{(h+1)q'} \sum_{Q \in \mathcal{C}^h} \left( \int_Q K(x) \, dx \right)^{q' \over q} \right)^{1 \over q}. \tag{3.3}
\]

By Lemma 2.3, since \( q' \over t' > 1 \), we have

\[
\sum_{Q \subset Q_j^h} \left( \int_Q K(x) \, dx \right)^{q' \over q'} \leq \left( c(Q, n)[K]_{A_2} \right)^{q' \over t'} \left( \int_{Q_j^h} K(x) \, dx \right)^{q' \over q} \tag{3.4}
\]

By (3.3) and (3.4), we deduce

\[
S_1 \leq \left( \sum_{h \in \mathbb{Z}} 2^{(h+1)q'} \sum_{j \in \mathbb{N}} \sum_{Q \subset Q_j^h} \left( \int_Q K(x) \, dx \right)^{q' \over q} \right)^{1 \over q}. \tag{3.5}
\]

Since \( Q_j^h \in \mathcal{C}^h \), by (3.2)

\[
\frac{1}{\int_{Q_j^h} K(x) \, dx} \int_{Q_j^h} |u|^{t-1} K(x) \, dx > 2^h.
\]

Thus,

\[
\frac{1}{\int_{Q_j^h} K(x) \, dx} \int_{Q_j^h \cap \{x \in B_R : |u| > 2^{h-10}\}} |u|^{t-1} K(x) \, dx \geq C_1 2^h, \tag{3.6}
\]

where \( C_1 = C_1(Q, n) \) is a constant. Consequently,

\[
\int_{Q_j^h} K(x) \, dx \leq C_1 2^{-h} \int_{Q_j^h \cap \{x \in B_R : |u| > 2^{h-10}\}} |u|^{t-1} K(x) \, dx. \tag{3.7}
\]

Owing to (3.5) and (3.7),

\[
S_1 \leq C_2 \left( \sum_{h \in \mathbb{Z}} 2^{(h+1)q'} \sum_{j \in \mathbb{N}} 2^{-h} \int_{Q_j^h \cap \{x \in B_R : |u| > 2^{h-10}\}} |u|^{t-1} K(x) \, dx \right)^{q' \over q} \tag{3.8}
\]
where $C_2 = C_1(Q, n) (c(Q, n)[K]_{A_2})^{\frac{1}{q'} - \frac{1}{q'}}$.

By (3.8), we have

$$S_1 \leq C_2 \left( \sum_{h \in \mathbb{Z}} 2^{(h+1) - \frac{h}{q'}} \sum_{j \in \mathbb{N}} \int_{Q_j^h \cap \{x \in B_R : |x| > 2^h - 10\}} |u|^{t-1} K(x) \, dx \right)^{\frac{1}{q'}}$$

$$= 2^{\frac{1}{q'}} C_2 \left( \sum_{h \in \mathbb{Z}} 2^h \sum_{j \in \mathbb{N}} \int_{Q_j^h \cap \{x \in B_R : |x| > 2^h - 10\}} |u|^{t-1} K(x) \, dx \right)^{\frac{1}{q'}}$$

$$\leq 2^{\frac{1}{q'}} C_2 \left( \sum_{h \in \mathbb{Z}} 2^h \int_{\{x \in B_R : |x| > 2^h - 10\}} |u|^{t-1} K(x) \, dx \right)^{\frac{1}{q'}}$$

$$= 2^{\frac{1}{q'}} C_2 \left( \int_{B_R} |u|^{t-1} K(x) \sum_{\{h \in \mathbb{Z} : 2^h < 2^{10}|u|\}} 2^h \, dx \right)^{\frac{1}{q'}} ,$$

(3.9)

where the first inequality is a consequence of the fact that $\sum_h \frac{q'_j}{q'} a_j \leq \left[ \sum_h a_h \right]^{\frac{q'}{q'}}$, the third one holds because, fixed $h \in \mathbb{Z}$, $Q_j^h$ are disjoint in $j$, the fourth one is due to Fubini’s type Theorem. To conclude the proof, we have to evaluate the quantity

$$\sum_{\{h \in \mathbb{Z} : 2^h < 2^{10}|u|\}} 2^h .$$

(3.10)

Set $H = \log_2 (2^{10}|u|)$. Thus, (3.10) yields

$$\sum_{h=-\infty}^{H} 2^h = \sum_{h=-H}^{+\infty} \left( \frac{1}{2} \right)^h = \sum_{h=-H}^{+\infty} \left( \frac{1}{2} \right)^{h+H} = \left( \frac{1}{2} \right)^{-H} \sum_{m=0}^{+\infty} \left( \frac{1}{2} \right)^m = \left( \frac{1}{2} \right)^{-H} 2 = 2^{\log_2 (2^{10}|u|) - 2} = 2^{11|u|} .$$

(3.11)

Then, by (3.9) and (3.11), we obtain

$$S_1 \leq C_3 \left( \int_{B_R} |u|^{t} K(x) \, dx \right)^{\frac{1}{q'}} ,$$

with $C_3 = 2^{\frac{1}{q'} + 11} C_1(Q, n) (c(Q, n)[K]_{A_2})^{\frac{1}{q'} - \frac{1}{q'}}$ and inequality (3.1) is proved. □

Now we are in position to prove our main result.
Proof of Theorem 1.1. By Theorem 3.2 of [21], there exists a solution $\varphi$ to the following Dirichlet problem for sub-Laplacian

$$\begin{cases}
\Delta_G \varphi = |u|^{t-1} K(x) & \text{in } B_R \\
\varphi = 0 & \text{on } \partial B_R,
\end{cases}$$

with $u \in C_0^1(B_R)$. By Lemma 2.6, we get

$$|X\varphi(x)| \leq c I_1(|u|^{t-1} K(x)) \quad \forall x \in B_R,$$

(3.12)

where $c$ is a positive constant.

Thanks to Lemma 2.5, it follows that

$$I_1(|u|^{t-1} K)(x) \leq c_0 \sum_{Q \in \mathcal{D}} \left( |Q|^{-1} \int_{3Q} |u(y)|^{t-1} K(y) \, dy \right) \chi_Q(x) \quad \forall x \in B_R,$$

(3.13)

where $c_0$ is an absolute constant.

Combining (3.12) and (3.13) yields

$$\int_{B_R} |u(x)|^{t} K(x) \, dx$$

$$= \int_{B_R} |u(x)||u(x)|^{t-1} K(x) \, dx = \int_{B_R} |u(x)|\Delta_G \varphi \, dx$$

$$\leq \int_{B_R} |Xu||X\varphi| \, dx \leq c \int_{B_R} |Xu|I_1(|u|^{t-1} K)(x) \, dx$$

$$\leq C_6 \int_{B_R} |Xu(x)| \sum_{Q \in \mathcal{D}} \left( |Q|^{-1} \int_{3Q} |u(y)|^{t-1} K(y) \, dy \right) \chi_Q(x) \, dx$$

$$= C_6 \int_{B_R} \sum_{Q \in \mathcal{D}} |Q|^{-1} |Xu(x)| \chi_Q(x) \left( \int_{3Q} |u(y)|^{t-1} K(y) \, dy \right) \, dx$$

$$= C_6 \int_{B_R} |Xu(x)| \sum_{Q \in \mathcal{D}} |Q|^{-1} \left( \frac{1}{|Q|} \int_Q |Xu| \, dx \right) \left( \int_{3Q} |u(y)|^{t-1} K(y) \, dy \right) \, dx$$

(3.14)

where $C_6 = c c_0$. Note that the last inequality is the consequence of the fact that $B_R \cap Q = Q$.

By (2.5),

$$\frac{1}{|Q|} \int_Q |Xu| \, dx \leq \left[ K^{-1} \right]^{1/2} \left( \frac{1}{\int_{K(x)}^1 dx} \int_Q \frac{|Xu|^2}{K(x)} \, dx \right)^{1/2}.$$
Coupling inequalities (3.14) and (3.15) tells us that

\[
\int_{B_R} |u|^t K(x) \, dx N \\
\leq C_6 \left[ K^{-1} \right]_{\mathcal{A}_2}^{1/2} \sum_{Q \in \mathcal{D}} |Q|^\frac{1}{2} \left( \frac{1}{\int_{Q} K(x) \, dx} \int_{Q} |Xu|^2 K(x) \, dx \right)^{1/2} \left( \int_{3Q} |u|^{t-1} K(y) \, dy \right).
\]  

(3.16)

By (3.16), the following chain of inequality holds

\[
\int_{B_R} |u|^t K(x) \, dx \\
\leq C_7 |B_R|^{1/n} \left( \frac{\int_{B_R} K(x) \, dx}{\int_{B_R} \frac{1}{K(x)} \, dx} \right)^{1/2} \sum_{Q \in \mathcal{D}} \left( \frac{1}{\int_{Q} K(x) \, dx} \right)^{1/2} \\
\times \left( \frac{1}{\int_{Q} K(x) \, dx} \int_{Q} |Xu|^2 K(x) \, dx \right)^{1/2} \left( \int_{3Q} |u|^{t-1} K(x) \, dx \right) \\
= C_7 |B_R|^{1/n} \left( \frac{\int_{B_R} K(x) \, dx}{\int_{B_R} \frac{1}{K(x)} \, dx} \right)^{1/2} \sum_{Q \in \mathcal{D}} \left( \frac{1}{\int_{Q} K(x) \, dx} \right)^{1/2} \\
\times \left( \frac{1}{\int_{Q} K(x) \, dx} \int_{Q} |Xu|^2 K(x) \, dx \right)^{1/2} \left( \int_{3Q} |u|^{t-1} K(x) \, dx \right)^{1-t} \\
\leq C_7 |B_R|^{1/n} \left( \frac{\int_{B_R} K(x) \, dx}{\int_{B_R} \frac{1}{K(x)} \, dx} \right)^{1/2} \left[ \sum_{Q \in \mathcal{D}} \left( \frac{1}{\int_{Q} K(x) \, dx} \right)^{q/2} \right]^{1/q} \\
\times \left[ \sum_{Q \in \mathcal{D}} \left( \int_{Q} K(x) \, dx \right)^{q'/t'} \left( \frac{1}{\int_{Q} K(x) \, dx} \int_{3Q} |u|^{t-1} K(x) \, dx \right)^{q'} \right]^{1/q'} \\
= C_7 |B_R|^{1/n} \left( \frac{\int_{B_R} K(x) \, dx}{\int_{B_R} \frac{1}{K(x)} \, dx} \right)^{1/2} \left[ \sum_{Q \in \mathcal{D}} \left( \frac{1}{\int_{Q} K(x) \, dx} \int_{3Q} |u|^{t-1} K(x) \, dx \right)^{q'} \right]^{1/q'} \\
\left[ \sum_{Q \in \mathcal{D}} \left( \int_{Q} K(x) \, dx \right)^{q'/t'} \left( \frac{1}{\int_{Q} K(x) \, dx} \int_{3Q} |u|^{t-1} K(x) \, dx \right)^{q'} \right]^{1/q'}.
\]
where the first inequality follows by Chanillo-Wheeden condition (1.1), the second one holds since $1/t = 1 - 1/t'$, the third one is due to Hölder’s inequality, for $2 < q < t$, and the fifty one comes from the fact that $D$ is a decomposition of $B_R$. Here, constant $C_7 = C_6 \overline{C} \left[ K^{-\frac{1}{2}} \right]_{A_2}^\frac{1}{2}$. The quantity $S_1$ is introduced in Lemma 3.1 above. Combining (3.17) and (3.1) shows that

$$
\left( \int_{B_R} |u|^t K(x) \, dx \right)^{1/t} \leq C_8 |B_R|^{1/n} \left( \int_{B_R} \frac{K(x) \, dx}{1} \right)^{1/t} \left( \int_{B_R} \frac{|Xu|^2}{K(x)} \, dx \right)^{1/2} S_1 \tag{3.18}
$$

where $C_8 = c(Q, n, t, q) \overline{C} \left[ K^{-\frac{1}{2}} \right]_{A_2}^\frac{1}{2} \left[ K \right]_{A_2}^{\frac{1}{2} - \frac{1}{q'}}$. Then, inequality (1.2) follows. □

**Acknowledgements** This research was partly supported by GNAMPA of the Italian INdAM (National Institute of High Mathematics).

**Funding** Open access funding provided by Università degli Studi di Salerno within the CRUI-CARE Agreement.

**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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