COUNTING LINES ON PROJECTIVE SURFACES

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Abstract. We prove a bound on the number of lines on a smooth degree-

d surface in $\mathbb{P}^3(\mathbb{C})$ for $d \geq 3$. This bound improves a bound due to Segre and 
renders some of his arguments rigorous. It is the best known bound for $d \geq 6$.

1. Introduction

The main aim of this note is to prove the following theorem.

Theorem 1.1. Let $X_d \subset \mathbb{P}^3(\mathbb{C})$ be a smooth surface of degree $d \geq 3$ and let $\ell(X_d)$ be the number of lines the surface $X_d$ contains. Then, the following inequality holds

$$\ell(X_d) \leq 11d^2 - 32d + 24.$$  

Let us mention that Theorem 1.1 is the best known bound on the number of lines lying on a degree-$d$ surface for $d \geq 6$.

Recently there has been some interest in configurations of lines on surfaces in $\mathbb{P}^3(\mathbb{C})$ (see e.g. \cite{5}, \cite{9}, \cite{19}, \cite{3}, \cite{12}, \cite{2}). In particular, the picture of the geometry of line configurations on complex projective quartic surfaces is complete (up to \cite{8, Conjecture 4.7}). The claim that the maximal number of lines on a smooth quartic is 64 can be found in \cite{17}, whereas the first correct proof of that fact is given in \cite{14}. The paper \cite{5} contains a complete classification of smooth complex quartic surfaces with many lines. Finally, lines on complex quartics with singular points are considered in \cite{20}, \cite{8}.

By contrast, the maximal number of lines on smooth hypersurfaces in $\mathbb{P}^3(\mathbb{C})$ of a fixed degree $d \geq 5$ remains unknown (see \cite{17}, \cite{3}, \cite{12}). In the case of smooth quintic surfaces the proof of the inequality

$$\ell(X_5) \leq 127$$

can be found in the recent paper \cite{15}, whereas (until now) the best bound for smooth complex surfaces of degree $d \geq 6$ has been the inequality

$$\ell(X_d) \leq (d - 2)(11d - 6)$$

that was stated by Segre in \cite{17, § 4}.

The proof of (2) in \cite{17} is based on various properties of so-called lines of the second kind. In particular, Segre states that every line of the second kind comes up in the flecnodal divisor with multiplicity two (see the next section for details). Unfortunately, several claims made in \cite{17} are false (\cite{13, § 3}) and the proof of the

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claim on the multiplicity of lines of the second kind seems not to fulfill modern standards of rigor (see Remark 2.9.b).

Our main aim while working on this note was to determine whether Segre’s brilliant idea can be proven using modern algebraic geometry (in particular, whether it is correct). Intersection theory [7], combined with Miyaoka’s inequality [11] allows us to obtain a precise proof of the stronger statement (1). Still, the question what is the maximal number of lines on smooth projective surfaces of a fixed degree \( d \geq 5 \) remains open.

Finally, let us recall that the first bound on the number of lines on a smooth degree-\( d \) surface was stated by Clebsch:

\[
\ell(X_d) \leq d(11d - 24)
\]

([11, p. 106]), who used ideas coming from Salmon ([11, p. 95], [16]). A beautiful modern proof of (3) is given in [6] §11.2.1, but we follow a slightly different path to arrive at (1) (see Remark 2.9.b), so the only result from [6] we can use is the fact that general point of the surface \( X_d \) is not flecnodal (Lemma 2.3). Still, for the convenience of the reader, we sketch a proof of the above claim. A bound on the number of lines on complete intersections of codimension two or more can be derived from the orbibundle Miyaoka-Yau-Sakai inequality, but this approach yields no results on the codimension-one case (see [12, Remark 3 on p. 921]).

**Convention:** In this note we work over the base field \( \mathbb{C} \).

### 2. Proof of Theorem 1.1

Let \( X_d \subset \mathbb{P}^3(\mathbb{C}) \) be a smooth degree-\( d \) surface, where \( d \geq 3 \), and let \( f \) be a generator of its ideal \( \mathcal{I}(X_d) \).

For a line \( L \subset \mathbb{P}^3 \) we put \( i(P, L, X_d) \) to denote the order of vanishing of the restriction \( f|_L \) at the point \( P \) if \( L \nsubseteq X_d \). We define \( i(P, L, X_d) := \infty \) when \( P \in L \subset X_d \).

Recall that a line \( L \subset X_d \) is called a line of the second kind iff it meets every plane curve \( \Gamma \in |\mathcal{O}_{X_d}(1) - L| \) only in inflection points of the latter (see [17, p. 87], [13]). Otherwise, the line \( L \) is called a line of the first kind.

In the proof of Thm 1.1 the following proposition plays crucial role.

**Proposition 2.1.** (c.f. [17, p. 90]) There exists an (effective) Weil divisor \( \mathcal{F}(X_d) \in |\mathcal{O}_{X_d}(11d - 24)| \) such that the following equality holds

\[
\text{supp}(\mathcal{F}(X_d)) = \{ P \in X_d : \text{there exists a line } L \text{ such that } i(P, L, X_d) \geq 4 \}
\]

and each line \( L \subset X_d \) of the second kind appears in \( \mathcal{F}(X_d) \) with multiplicity at least two.

In the sequel, we call \( \mathcal{F}(X_d) \) the flecnodal divisor of the surface \( X_d \).

The proof of Prop. 2.1 will be preceded by several lemmata. First, we introduce the necessary notation.

For \( j = 1, 2, 3 \) we define polynomials \( t^{(j)} \in \mathbb{C}[w_0, \ldots, w_3, z_0, \ldots, z_3] \) by the formula

\[
t^{(j)} := \sum_{0 \leq i_1, \ldots, i_j \leq 3} \frac{\partial f^j}{\partial w_{i_1} \ldots \partial w_{i_j}} (w_0, \ldots, w_3) \cdot z_{i_1} \cdots z_{i_j}.
\]
In order to simplify our notation, given a point \( P = (p_0, \ldots, p_3) \in \mathbb{C}^4 \) (resp. \( P = (p_0 : \ldots : p_3) \in \mathbb{P}^3 \)) we put
\[
t^{(j)}_P(z_0, \ldots, z_3) := t^{(j)}_P(p_0, \ldots, p_3, z_0, \ldots, z_3).
\]
Observe that the zero set of the polynomial \( t^{(1)}_P \) (resp. \( t^{(2)}_P \)) is the projective tangent space \( T_PX_d \) (resp. the Hessian quadric \( V_P = V_PX_d \)).

We consider the variety
\[
\mathcal{P} := \mathcal{P}(X_d) \subset X_d \times G(2, 4)
\]
defined as
\[
\mathcal{P} := \{ (P, L) : P \in X_d \text{ and the line } L \text{ satisfies the condition } i(P, L.X_d) \geq 3 \},
\]
where \( G(2, 4) \) is the Grassmanian of lines in \( \mathbb{P}^3(\mathbb{C}) \). The variety \( \mathcal{P} \) is endowed with the projections
\[
\pi_1 : \mathcal{P} \rightarrow X_d \quad \text{and} \quad \pi_2 : \mathcal{P} \rightarrow G(2, 4).
\]

**Lemma 2.2.**

(a) If \( P \in X_d \), then \( \#\pi_1^{-1}(P) \in \{1, 2, \infty\} \).

(b) The set \( \{ P \in X_d : \#\pi_1^{-1}(P) = \infty \} \) is finite.

**Proof.**

(a) We fix a point \( P \in X_d \). Obviously, either \( t^{(2)}_P|_{T_PX_d} \equiv 0 \) or \( t^{(2)}_P|_{T_PX_d} \) vanishes along exactly one or two lines.

Let \( L \subset T_PX_d \) be a line. By direct computation, \( t^{(2)}_P \) vanishes along \( L \) iff \( i(P, L.X_d) \geq 3 \). Thus a finite fiber \( \pi_1^{-1}(P) \) consists of at most two points, which yields (a).

(b) Suppose that there exists a curve \( C \subset X_d \) such that
\[
t^{(2)}_P|_{T_PX_d} \equiv 0 \quad \text{for every point } P \in C.
\]
Then the Gauss map is constant on the curve \( C \), because its differential vanishes for all \( P \in \text{reg}(C) \). Thus the curve \( C \) is contained in a fiber of the Gauss map, which is impossible by [21, Prop. 2.1]. \( \square \)

In particular, we have shown that the variety \( \mathcal{P} \) is two-dimensional.

For the sake of completeness we sketch below a proof of [10, Lemma 2.10] (c.f. [10, p. 270], [6, Prop. 11.8]).

**Lemma 2.3.** ([10, Lemma 2.10])

\( X_d \neq \{ P \in X_d : \text{there exists a line } L \text{ such that } i(P, L.X_d) \geq 4 \} \).

**Proof.** Let \( (P, L) \in \text{reg}(\mathcal{P}) \). By direct computation if \( i(P, L.X_d) \geq 4 \), then
\[
\text{rank}(d_{(P, L)}\pi_2) \leq 1.
\]
Suppose that the set
\[
\{ P \in X_d : \text{there exists a line } L \text{ such that } i(P, L.X_d) \geq 4 \}
\]
coincides with \( X_d \). Then \( \text{dim}(\pi_2(\mathcal{P})) \leq 1 \), so \( \pi_2(\mathcal{P}) \) is a curve in \( G(2, 4) \). For infinitely many lines \( L \in \pi_2(\mathcal{P}) \), we have \( L \subset X_d \). Contradiction. \( \square \)
In the sequel, the divisor defined by $t^{(j)}$ in $\mathbb{P}^3 \times \mathbb{P}^3$ is denoted by $\mathcal{T}^{(j)}$ and we put

$$Y_d := (X_d \times \mathbb{P}^3) \cap \bigcap_{j=1}^{j=3} \text{supp}(\mathcal{T}^{(j)}).$$

Moreover, $\psi : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ stands for the projection onto the first factor and we define

$$\Delta_{X_d} := \{(P, P) \in \mathbb{P}^3 \times \mathbb{P}^3 : P \in X_d\}.$$

**Lemma 2.4.** The variety $Y_d$ is two-dimensional.

**Proof.** By Lemma 2.2.b, only finitely many fibers of the projection $\psi|_{Y_d}$ are two-dimensional.

Fix a point $P \in X_d$ such that the fiber $\pi_1^{-1}(P)$ is finite, i.e. $t^{(2)}_P$ vanishes on $T_P X_d$ only along two (not necessarily distinct) lines $L_1, L_2$. By direct computation

$$L_1 \subset V(t^{(3)}_P) \iff i(P, L_1.X_d) \geq 4.$$

Thus, by Lemma 2.3 for generic choice of $P \in X_d$ the fiber of $(\psi|_{Y_d})^{-1}(P)$ is finite. \hfill $\square$

**Lemma 2.5.** The diagonal $\Delta_{X_d}$ comes up in the intersection cycle (6) with multiplicity 6.

**Proof.** We are to show that, for generic choice of the point $P \in X_d$, the intersection multiplicity of the curves $V(t^{(j)}_P|_{T_P X_d})$, where $j = 2, 3$, in $P$ equals 6. By Lemma 2.2(b) we can assume that $V(t^{(3)}_P|_{T_P X_d})$ consists of two lines $L_1, L_2$. Moreover, Lemma 2.3 and (5) allow us to require that $t^{(3)}_P|_{L_k}$ does not vanish identically for $k = 1, 2$.

Then, by direct computation, the restriction $t^{(3)}_P|_{L_k}$ has a triple root in $P$ for $k = 1, 2$ and the proof is complete. \hfill $\square$

Consequently, all components appear in the cycle

$$\mathfrak{W} := \mathcal{T}^{(1)} \cdot \mathcal{T}^{(2)} \cdot \mathcal{T}^{(3)} \cdot (X_d \times \mathbb{P}^3) - 6\Delta_{X_d}.$$  

with non-negative coefficients. Moreover, by definition, the set $\psi(\text{supp}(\mathfrak{W}))$ consists of the points $P$ such that $t^{(1)}_P, \ldots, t^{(3)}_P$ vanish simultaneously along a line. From (5), we obtain the equality

$$\psi(\text{supp}(\mathfrak{W})) = \{P \in X_d : \exists \text{ a line } L \text{ such that } i(P, L.X_d) \geq 4\}.$$

Let $L \subset X_d$ be a line. Recall that the linear system $|O_{X_d}(1) - L|$ endows the surface in question with a fibration

$$\pi : X_d \to \mathbb{P}^1.$$
Let us follow [17] and put $\Gamma_P$ to denote its fiber that is contained in the tangent space $T_P X_d$ for a point $P \in L$. One can easily check that if the Hessian quadric $V_P$ does not contain the tangent space in question, then the line residual to $L$ in the (scheme-theoretic) intersection $T_P X_d \cap V_P$ is tangent to the curve $\Gamma_P$ in the point $P$.

For the proof of Prop. 2.1 we will need the following observation.

Lemma 2.6. Let $L \subset X_d$ be a line. Then the set

$$\{ P \in L : V(t_P^{(1)}) \cap V(t_P^{(2)}) \text{ does not consist of two distinct lines} \}$$

is finite.

Proof. Let $P \in L$ be a point. By Lemma 2.2, we can assume that the Hessian quadric $V_P$ does not contain the tangent space $T_P X_d$. If the Hessian quadric and the tangent space meet along the line $L$ with multiplicity two, then $L$ is tangent to the curve $\Gamma_P$ in the point $P$. Thus $P$ is the ramification point of the degree-$(d-1)$ map $\pi|_L : L \to \mathbb{P}^1$ (i.e. the restriction of the fibration $\pi$ to the line $L$). There are only finitely many such points, so the claim follows. □

One has the following property of lines of the second kind.

Lemma 2.7. Let $L \subset X_d$ be a line and let $P \in L$ be a point such that the Hessian quadric $V_P$ does not contain the tangent space $T_P X_d$. If the line $L$ is of the second kind, then the form $t_P^{(3)}$ vanishes along the (set-theoretic) intersection $T_P X_d \cap V_P$.

Proof. We can assume that the plane and the quadric meet along two distinct lines, one of which is not contained in $X_d$ (otherwise the claim is obvious, because $t_P^{(3)}$ vanishes along $L \subset X_d$).

Let $L'$ be the line residual to $L$ in $T_P X_d \cap V_P$. As we already explained, the line $L'$ is tangent to the fiber $\Gamma_P$ of $\pi$ in the point $P$. Since $L$ is of the second kind, $P$ is an inflection point of the curve $\Gamma_P$, so $L'$ meets $\Gamma_P$ with multiplicity at least 3 in the point $P$. But $L \subset X_d$ also meets $L'$ in the point $P$, so we have

$$i(P, L', X_d) \geq 4$$

The claim follows directly from [13]. □

Now we are in position to give a proof of Prop. 2.1. In the proof below we maintain the notation of this section. In particular the cycle $\mathfrak{W}$ is given by (7), and $\psi$ denotes the projection $\mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$ onto the first factor.

Proof of Prop. 2.1. Let $H \subset \mathbb{P}^3$ be a generic hyperplane. We claim that the effective divisor $\mathcal{F}(X_d) := \psi_* (\mathfrak{W} \cdot (\mathbb{P}^3 \times H))$ has the required properties.

Indeed, one can easily see that

$$\psi(\text{supp}(\mathfrak{W})) = \psi(\text{supp}(\mathfrak{W}) \cap (\mathbb{P}^3 \times H)),$$

so [13] implies that the support of $\mathcal{F}(X_d)$ satisfies the claim of the proposition.

In order to show that $\mathcal{F}(X_d) \in |\mathcal{O}_{X_d}(11d - 24)|$ we compute the class

$$[\psi_* (\mathfrak{W} \cdot (\mathbb{P}^3 \times H))].$$
in the Chow ring $A^*(\mathbb{P}^3)$. We put $h_1 := [H \times \mathbb{P}^3]$ and $h_2 := [\mathbb{P}^3 \times H]$. At first we compute in $A^*(\mathbb{P}^3 \times \mathbb{P}^3)$:

$$[\mathcal{T}^{(1)}] \cdot [\mathcal{T}^{(2)}] \cdot [\mathcal{T}^{(3)}] = ((d-1)h_1 + h_2) \cdot ((d-2)h_1 + 2h_2) \cdot ((d-3)h_1 + 3h_2)$$

$$= 6h_2^3 + (11d - 18)h_2^3 \cdot h_1 + (\text{terms of degree at most one w.r.t } h_2)$$

Since $[\Delta_{\mathbb{P}^3}, \psi^*([X_d])] = [\Delta_{X_d}]$ we have

$$(9) \quad \psi_*([\mathcal{W}] \cdot h_2) = \psi_*(([\mathcal{T}^{(1)}] \cdot [\mathcal{T}^{(2)}] \cdot [\mathcal{T}^{(3)}] - 6[\Delta_{\mathbb{P}^3}] \cdot \psi^*[X_d] \cdot h_2).$$

Recall that, by [7, Ex. 8.4.2], the class $[\Delta_{\mathbb{P}^3}]$ of the diagonal in $\mathbb{P}^3 \times \mathbb{P}^3$ can be expressed as

$$[\Delta_{\mathbb{P}^3}] = h_1^3 + h_2^3 + h_1h_2^2 + h_2^3,$$

whereas $h_1 = \psi^*[H]$. Therefore, from (9), we obtain that

$$\psi_*([\mathcal{W}] \cdot h_2) = (\psi_*([\mathcal{T}^{(1)}] \cdot [\mathcal{T}^{(2)}] \cdot [\mathcal{T}^{(3)}]) \cdot h_2) - 6 \psi_*([\Delta_{\mathbb{P}^3}] \cdot h_2) \cdot [X_d]$$

$$= (11d - 24) \mathcal{O}_{\mathbb{P}^3}(1) \cdot [X_d]$$

and the proof of that part of the proposition is complete.

Finally, let $L \subset X_d$ be a line of the second kind and let $P \in L$ be a point such that $T_PX_d$ and the Hessian quadric $V_P$ meet along two distinct lines. As in Lemma 2.7 we put $L'$ to denote the line residual to $L$ in $T_PX_d \cap V_P$. We can assume that the hyperplane $H$ meets $L$ (resp. $L'$) in the point $Q \neq P$ (resp. $Q' \neq P$). Obviously we have $(P, Q) \in \text{supp}(\mathcal{W}) \cap (\mathbb{P}^3 \times H)$. Moreover, by Lemma 2.7 the point $(P, Q')$ also belongs to the set $\text{supp}(\mathcal{W}) \cap (\mathbb{P}^3 \times H)$. Since $Q \neq Q'$, Lemma 2.6 implies that the restriction of the projection $\psi$

$$\psi^{-1}(L) \cap \text{supp}(\mathcal{W}) \cap (\mathbb{P}^3 \times H) \to L$$

is of degree at least two, so the claim on the multiplicity follows from the definition of the map $\psi_*$. \hfill \Box

**Example 2.8.** An elementary computation shows that the Schur quartic

$$x_0^4 - x_0x_3^3 = x_3^4 - x_3x_4^3$$

contains exactly 64 lines: 48 lines of the first kind and 16 lines of the second kind. Since the flecnodal divisor of a quartic surface has degree 80, each line of the second kind must come in the flecnodal divisor precisely with multiplicity two. Thus, the lower bound of Prop. 2.1 is sharp.

**Remark 2.9.** (a) The idea of studying lines on a surface via points of fourfold contact goes back to work of Salmon and Clebsch on cubic surfaces (see [9] and the bibliography therein). In particular, an equation of the flecnodal divisor is obtained in [4] via projection of the intersection of the varieties $\mathcal{T}^{(j)}$. A beautiful exposition of a modern treatment of this approach can be found in [6, § 11.2.1]. Still, for the proof of Prop. 2.1 we find it more convenient to avoid the use of bundles of relative principal parts. In this way we can control the behaviour of flecnodal divisor along a line of the second kind.

(b) As we already explained, the claim on multiplicities of lines of the second kind in the flecnodal divisor was stated in [17, p. 90]. Segre (see [17, (7) on p. 88]) justified it by giving an explicit formula for an analytic function (defined on an open neighbourhood of a point $P$ on a line $L \subset X_d$ that vanishes along the set $\text{supp}(\mathcal{F}(X_d))$ and showing that the function in question has multiplicity at least
two along the line $L$ provided the latter is of the second kind. Unfortunately, this argument does not explain why the function in question is a local equation of the flecnodal divisor (although it explains why its set of zeroes contains the support $\text{supp}(\mathcal{F}(X_d))$), i.e. why its order of vanishing along the line $L$ yields any information on the multiplicity with which $L$ comes up in the divisor $\mathcal{F}(X_d)$.

To render our exposition self-contained let us recall two facts we need for the proof of Thm 1.1.

Claim 2.10. ([11, § 2.2]) A smooth degree $d$ surface $X_d$ contains at most $2d(d-2)$ pairwise disjoint lines.

Claim 2.11. ([17, p. 88]) A line $L \subset X_d$ of the first kind is met by at most $(8d-14)$ other lines lying on the surface $X_d$.

Let us remark that the proof of Claim 2.11 in [17] is an elementary computation. It can be also found as the proof of [14, Lemma 5.2] (for $d = 4$).

After those preparations we can finally give a proof of Thm 1.1.

Proof of Thm 1.1: Suppose that $X_d$ contains no pair of coplanar lines of the first kind. By Claim 2.10 and Prop. 2.1 the surface $X_d$ contains at most $2d(d-2)$ lines that appear with multiplicity one in the flecnodal divisor $\mathcal{F}(X_d)$. Since $\deg(\mathcal{F}(X_d)) = (11d-24)d$, we have

$$\ell(X_d) \leq 2d(d-2) + \frac{(11d-24)d - 2d(d-2)}{2}$$

and (1) follows.

Thus we can assume that $X_d$ contains two lines of the first kind, say $L_1$, $L_2$ that meet in one point. Let $k_1$, (resp. $k_2$) be the multiplicity of the line $L_1$ (resp. $L_2$) in the divisor $\mathcal{F}(X_d)$ (i.e. the divisor $(\mathcal{F}(X_d) - k_1L_1 - k_2L_2)$ is effective and its support meets both lines $L_1$, $L_2$ properly).

By Claim 2.11 each of the lines $L_1$, $L_2$ meets at most $(8d-14)$ other lines on $X_d$ Recall that $L^2 = -(d-2)$ for each line $L \subset X_d$. Thus we have

$$(\mathcal{F}(X_d) - k_1L_1 - k_2L_2).(L_1 + L_2) = 2(11d-24) + (k_1 + k_2)(d-3) \leq 24d - 54.$$

Therefore, we obtain the following lower bound

$$\deg(\mathcal{F}(X_d)) - \sum_{L_j \subset X_d} L_j \geq (24d - 54) - 2(8d - 15) = 8(d-3).$$

Finally we arrive at the inequality:

$$\ell(X_d) \leq d(11d-24) - 8(d-3),$$

which completes the proof. □

Example 2.12. The Fermat surface

$$x_0^d + x_1^d + x_2^d + x_3^d = 0$$

contains $3d^2$ lines. For $d \neq 4, 6, 8, 12, 20$ this is the best known example of a surface with many lines (see e.g. [3]).
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