Characterizing $S$-projective modules and $S$-semisimple rings by uniformity

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Abstract

Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $P$ is called uniformly $S$-projective provided that the induced sequence $0 \rightarrow \text{Hom}_R(P, A) \rightarrow \text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, C) \rightarrow 0$ is $u$-$S$-exact for any $u$-$S$-short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. Some characterizations and properties of $u$-$S$-projective modules are obtained. The notion of $u$-$S$-semisimple modules is also introduced. A ring $R$ is called a $u$-$S$-semisimple ring provided that any free $R$-module is $u$-$S$-semisimple. Several characterizations of $u$-$S$-semisimple rings are provided in terms of $u$-$S$-semisimple modules, $u$-$S$-projective modules, $u$-$S$-injective modules and $u$-$S$-split $u$-$S$-exact sequences.

Key Words: $u$-$S$-projective module, $u$-$S$-injective module, $u$-$S$-split $u$-$S$-exact sequence, $u$-$S$-semisimple ring.

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1. Introduction

Throughout this article, $R$ is always a commutative ring with identity and $S$ is always a multiplicative subset of $R$, that is, 1 $\in$ $S$ and $s_1s_2$ $\in$ $S$ for any $s_1$ $\in$ $S$ and any $s_2$ $\in$ $S$. Let $S$ be multiplicative subset of $R$. In 2002, Anderson and Dumitrescu [1] introduced the notion of an $S$-Noetherian ring $R$, that is, for any ideal $I$ of $R$ there exists a finitely generated sub-ideal $K$ of $I$ and an element $s$ $\in$ $S$ such that $sI \subseteq K$, i.e., $I/K$ is $u$-$S$-torsion by [18]. Since then, $S$-analogues of other well-known classes of rings such as Artinian rings, coherent rings, almost perfect rings, GCD domains and strong Mori domains, were introduced and studied extensively in [2, 3, 4, 10, 11, 9, 15].

Now let’s go back to the definition of $S$-Noetherian rings. Notice that the element $s$ $\in$ $S$ such that $sI \subseteq K$ is decided by the ideal $I$ for $S$-Noetherian rings. This situation makes it difficult to characterize $S$-Noetherian rings from the perspective of module-theoretic viewpoint. In order to overcome this difficulty, Qi and Kim
et al. [13] recently introduced the notion of uniformly $S$-Noetherian rings $R$ for which there exists an element $s \in S$ such that for any ideal $I$ of $R$, $sI \subseteq K$ for some finitely generated sub-ideal $K$ of $I$. They also introduced the notion of $u$-$S$-injective modules and finally showed that a ring $R$ is uniformly $S$-Noetherian if and only if any direct sum of injective modules is $u$-$S$-injective in the case that $S$ is a regular multiplicative set. Another “uniform” case is that of uniformly $S$-von Neumann regular rings introduced by the first author of this paper (see [18]).

The author in [18] first introduced $u$-$S$-flat modules using $u$-$S$-torsion modules, and then gave the notion of uniformly $S$-von Neumann regular rings extending von Neumann regular rings with uniformity on the multiplicative subset $S$. Finally, he characterized uniformly $S$-von Neumann regular rings by using $u$-$S$-flat modules.

The main motivation of this paper is to introduce and study the uniformly $S$-versions of projective modules and semisimple rings. In Section 2 of this article, we first introduce the notion of $u$-$S$-split $u$-$S$-exact sequence (see Definition 2.3). Dualizing the $u$-$S$-injective modules, we introduce the notion of $u$-$S$-projective module and show that an $R$-module $P$ is $u$-$S$-projective if and only if $\text{Ext}_R^1(P, M)$ is $u$-$S$-torsion for any $R$-module $M$, if and only if any $u$-$S$-exact sequence ending at $P$ is $u$-$S$-split (see Theorem 2.9). We also give a new local characterization of projective modules in Proposition 2.10. In Section 3 of this article, we first give the notion of a $u$-$S$-semisimple module $M$, that is, any $u$-$S$-shortly exact sequence with middle term $M$ is $u$-$S$-split. And then we introduced the notion of $u$-$S$-semisimple ring over which every free module is $u$-$S$-semisimple. We prove that a ring $R$ is a $u$-$S$-semisimple ring if and only if every $R$-module is $u$-$S$-semisimple, if and only if every $u$-$S$-short exact sequence is $u$-$S$-split, if and only if every $R$-module is $u$-$S$-projective, if and only if every $R$-module is $u$-$S$-injective (see Theorem 3.5). By Corollary 3.6 a $u$-$S$-semisimple ring is both uniformly $S$-Noetherian and uniformly $S$-von Neumann regular. We also show that if $S$ is a regular multiplicative subset of $R$, then $R$ is a $u$-$S$-semisimple ring if and only if $R$ is semisimple (see Proposition 3.8). A non-trivial example of a $u$-$S$-semisimple ring which is not semisimple is given in Example 3.11. Finally, we give a new characterization of semisimple rings (see Proposition 3.12).

2. $u$-$S$-SPLIT $u$-$S$-EXACT SEQUENCES AND $u$-$S$-PROJECTIVE MODULES

Let $R$ be a ring and $S$ a multiplicative subset of $R$. Recall from [17, Definition 1.6.10] that an $R$-module $T$ is called a $u$-$S$-torsion module provided that there exists an element $s \in S$ such that $sT = 0$. Suppose $M$, $N$ and $L$ are $R$-modules.
(1) An $R$-homomorphism $f : M \to N$ is called a $u$-$S$-monomorphism (resp., $u$-$S$-epimorphism) provided that $\text{Ker}(f)$ (resp., $\text{Coker}(f)$) is a $u$-$S$-torsion module.

(2) An $R$-homomorphism $f : M \to N$ is called a $u$-$S$-isomorphism provided that $f$ is both a $u$-$S$-monomorphism and a $u$-$S$-epimorphism.

(3) An $R$-sequence $M \xrightarrow{f} N \xrightarrow{g} L$ is called $u$-$S$-exact provided that there is an element $s \in S$ such that $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$.

(4) A $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$ is called a short $u$-$S$-exact sequence.

It is easy to verify that $f : M \to N$ is a $u$-$S$-monomorphism (resp., $u$-$S$-epimorphism) if and only if $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$) is $u$-$S$-exact.

Suppose $M$ and $N$ are $R$-modules. We say $M$ is $u$-$S$-isomorphic to $N$ if there exists a $u$-$S$-isomorphism $f : M \to N$. A family $\mathcal{C}$ of $R$-modules is said to be closed under $u$-$S$-isomorphisms if $M$ is $u$-$S$-isomorphic to $N$ and $M$ is in $\mathcal{C}$, then $N$ is also in $\mathcal{C}$. One can deduce from the following Lemma 2.1 that the $u$-$S$-isomorphism is actually an equivalence relation.

**Lemma 2.1.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Let $M$ and $N$ be $R$-modules. Suppose there is a $u$-$S$-isomorphism $f : M \to N$. Then there is a $u$-$S$-isomorphism $g : N \to M$ and $t \in S$ such that $f \circ g = t\text{Id}_N$ and $g \circ f = t\text{Id}_M$.

**Proof.** Let $f : M \to N$ be a $u$-$S$-isomorphism. Then there is $s \in S$ such that $sN \subseteq \text{Im}(f)$ and $s\text{Ker}(f) = 0$. For $y \in N$ pick $x \in M$ with $f(x) = sy$ and define $g(y) = sx$. Suppose $y \in N$ and pick $x_1, x_2 \in M$ such that $f(x_1) = sy = f(x_2)$. Then $x_1 - x_2 \in \text{Ker}(f)$. So $sx_1 = sx_2$. Thus $g$ is well-defined. One can check that $g$ is also linear. Trivially, $g$ is a $u$-$S$-isomorphism with $f \circ g = s^2\text{Id}_N$ and $g \circ f = s^2\text{Id}_M$. □

**Remark 2.2.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ and $N$ $R$-modules. Then the condition “there is an $R$-homomorphism $f : M \to N$ such that $f_S : M_S \to N_S$ is an isomorphism” does not mean “there is an $R$-homomorphism $g : N \to M$ such that $g_S : N_S \to M_S$ is an isomorphism”.

Indeed, let $R = \mathbb{Z}$ be the ring of integers, $S = R - \{0\}$ and $\mathbb{Q}$ the quotient field of integers. Then the embedding map $f : \mathbb{Z} \to \mathbb{Q}$ satisfies $f_S : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism. However, since $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, there does not exist any $R$-homomorphism $g : \mathbb{Q} \to \mathbb{Z}$ such that $g_S : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism.

Recall that an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split provided that there is an $R$-homomorphism $f' : B \to A$ such that $f' \circ f = \text{Id}_A$. 

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Definition 2.3. Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-short exact sequence. Then $\xi$ is said to be $u$-$S$-split (with respect to $s$) provided that there is $s \in S$ and $R$-homomorphism $f' : B \to A$ such that $f'(f(a)) = sa$ for any $a \in A$, that is, $f' \circ f = s \text{Id}_A$.

Obviously, any split exact sequence is $u$-$S$-split. Certainly, an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits if and only if there exists an $R$-homomorphism $g' : C \to B$ such that $g \circ g' = \text{Id}_C$. Now, we give the uniformly $S$-version of this result.

Lemma 2.4. Let $R$ be a ring and $S$ a multiplicative subset of $R$. A $u$-$S$-short exact sequence $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is $u$-$S$-split if and only if there is $s \in S$ and $R$-homomorphism $g' : C \to B$ such that $g(g'(c)) = sc$ for any $c \in C$, that is, $g \circ g' = s \text{Id}_C$ for some $s \in S$.

Proof. Choose the $R$-homomorphism $f' : B \to A$ and $s' \in S$ such that $f' \circ f = s' \text{Id}_A$, $s'C \subseteq \text{Im}(g)$, $s'\text{Ker}(g) \subseteq \text{Im}(f)$ and $s'\text{Im}(f) \subseteq \text{Ker}(g)$. Define the map $g' : C \to B$ as follows. For $z \in C$, pick $y \in B$ with $g(y) = s'z$ and define $g'(z) = s'^2 y - s'f(f'(y)))$. When $g(y) = g(y') = s'z$, we pick $x \in A$ with $f(x) = s'(y - y')$, so $(s'^2 y - s'f(f'(y))) - (s'^2 y' - s'f(f'(y'])) = s'^2 (y - y') - s'f(f'(y - y')) = s'f(x) - f(f'(f(x))) = s'f(x) - s'f(x) = 0$, thus $g'$ is well-defined. It also can be checked that $g'$ is linear. Finally, if $g(y) = s'z$, we have $g(g'(z)) = g(s'^2 y - s'f(f'(y))) = s'^3 z$ because $s'g \circ f = 0$. Setting $s = s'^3$, we have $g \circ g' = s \text{Id}_C$. The sufficiency can be proved similarly.

Recall from [13] Definition 4.1 that an $R$-module $E$ is called $u$-$S$-injective provided that the induced sequence

$$0 \to \text{Hom}_R(C, E) \to \text{Hom}_R(B, E) \to \text{Hom}_R(A, E) \to 0$$

is $u$-$S$-exact for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$. By [13] Theorem 4.3, an $R$-module $E$ is $u$-$S$-injective, if and only if for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence $0 \to \text{Hom}_R(C, E) \xrightarrow{g_*} \text{Hom}_R(B, E) \xrightarrow{f_*} \text{Hom}_R(A, E) \to 0$ is $u$-$S$-exact, if and only if $\text{Ext}^1_R(M, E)$ is $u$-$S$-torsion for any $R$-module $M$, if and only if $\text{Ext}^n_R(M, E)$ is $u$-$S$-torsion for any $R$-module $M$ and any $n \geq 1$. We can characterize $u$-$S$-injective modules using $u$-$S$-exact sequences.

Proposition 2.5. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $E$ an $R$-module. Then the following statements are equivalent:

(1) $E$ is $u$-$S$-injective;
(2) for any $u$-$S$-monomorphism $A \xrightarrow{f} B$ there exists $s \in S$ such that for any $R$-homomorphism $h : A \to E$, there exists an $R$-homomorphism $g : B \to E$ satisfying $sh = g \circ f$; 

Proof. $(1) \Rightarrow (2)$: Set $C = \text{Coker}(f)$. As $0 \to A \xrightarrow{f} B \to C \to 0$ is $u$-$S$-exact, we get the $u$-$S$-epimorphism $f^* : \text{Hom}(B, E) \to \text{Hom}(A, E)$. Pick $s \in S$ with $s\text{Hom}(A, E) \subseteq \text{Im}(f^*)$. Then $sh = g \circ f$ for some linear map $g : B \to E$.

$(2) \Rightarrow (1)$: Let $M$ be an $R$-module and $0 \to N \xrightarrow{i} P \xrightarrow{m} M \to 0$ a short exact sequence of $R$-modules with $P$ projective. Then we have a long exact sequence $0 \to \text{Hom}_R(M, E) \to \text{Hom}_R(P, E) \xrightarrow{i^*} \text{Hom}_R(N, E) \xrightarrow{\text{Ext}_R^1(M, E)} 0$. By $(2)$, $i^*$ is a $u$-$S$-epimorphism. Thus $\text{Ext}_R^1(M, E)$ is $u$-$S$-torsion. So $E$ is $u$-$S$-injective. □

Lemma 2.6. The following two statements are equivalent:

1. any $u$-$S$-short exact sequence $0 \to E \to B \to C \to 0$ beginning at $E$ is $u$-$S$-split;
2. any short exact sequence $0 \to E \to B \to C \to 0$ beginning at $E$ is $u$-$S$-split.

Proof. $(1) \Rightarrow (2)$: Obvious.

$(2) \Rightarrow (1)$: Let $0 \to E \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-short exact sequence. Then there is a short exact sequence $0 \to \text{Im}(f) \to B \to \text{Coker}(f) \to 0$. If we denote by $f' : E \to \text{Im}(f)$ to be the natural epimorphism, then there is an $u$-$S$-isomorphism $h : \text{Im}(f) \to E$ such that $h \circ f' = s_1\text{Id}_E$ for some $s_1 \in S$ by Lemma 2.1. Consider the following push-out:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & E & \xrightarrow{v} & X & \xrightarrow{x} & \text{Coker}(f) & \longrightarrow & 0 \\
& & \downarrow{h} & & \downarrow{x} & & \uparrow{\cong} & & \\
0 & \longrightarrow & \text{Im}(f) & \xrightarrow{i} & B & \longrightarrow & \text{Coker}(f) & \longrightarrow & 0,
\end{array}
\]

By $(2)$, there exists $s_2 \in S$ and an $R$-homomorphism $w : X \to E$ such that $w \circ v = s_2\text{Id}_E$. So $w \circ x \circ f = w \circ x \circ i \circ f = w \circ v \circ h \circ f = s_1s_2\text{Id}_E$. Hence $0 \to E \to B \to C \to 0$ is $u$-$S$-split with respect to $s_1s_2$. □

Corollary 2.7. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $E$ an $R$-module. Then the following two statements hold:

1. If $E$ is $u$-$S$-injective, then any $u$-$S$-short exact sequence $0 \to E \to B \to C \to 0$ beginning at $E$ is $u$-$S$-split;
2. If there exists $s \in S$ such that any short exact sequence $0 \to E \to B \to C \to 0$ beginning at $E$ is $u$-$S$-split with respect to $s$, then $E$ is $u$-$S$-injective.
Proof. (1) Let $0 \to E \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-short exact sequence of $R$-modules. Set $h = \text{Id}_E : E \to E$ be the identity map of $E$. Then there exists $s \in S$ and $g : B \to E$ such that $s\text{Id}_E = g \circ f$. Hence $0 \to E \to B \to C \to 0$ is $u$-$S$-split.

(2) Let $f : A \to B$ be a $u$-$S$-monomorphism. Then there is $s_1 \in S$ such that $s_1\text{Ker}(f) = 0$. Let $g : A \to E$ be an $R$-homomorphism. Consider the following push-out:

$$
\begin{array}{ccccccccc}
E & \xrightarrow{h} & X & \xrightarrow{\text{Coker}(h)} & 0 \\
\downarrow{g} & & \downarrow{f} & & \downarrow{\approx} \\
A & \xrightarrow{\text{Coker}(f)} & 0,
\end{array}
$$

we have Ker$(h)$ can be seen as a quotient of Ker$(f)$. So $s_1\text{Ker}(f) = 0$. Hence $0 \to E \to X \to \text{Coker}(h) \to 0$ is $u$-$S$-exact, and thus $u$-$S$-split with respect to $s$. So there is an $R$-homomorphism $h' : X \to E$ such that $h' \circ h = s\text{Id}_E$. Hence $h' \circ l \circ f = h' \circ h \circ g = sg$. Note that $s$ is independent with $g$. So $E$ is $u$-$S$-injective. □

Recall that an $R$-module $P$ is said to be projective provided that the induced sequence $0 \to \text{Hom}_R(P, A) \to \text{Hom}_R(P, B) \to \text{Hom}_R(P, C) \to 0$ is exact for any exact sequence $0 \to A \to B \to C \to 0$. Now we give a uniformly $S$-analogue of projective modules.

**Definition 2.8.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $P$ is called $u$-$S$-projective provided that the induced sequence

$$
0 \to \text{Hom}_R(P, A) \to \text{Hom}_R(P, B) \to \text{Hom}_R(P, C) \to 0
$$

is $u$-$S$-exact for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$.

In common with the classical cases, we have the following characterizations of $u$-$S$-projective modules. Since the proof is very similar with that of characterizations of $u$-$S$-injective modules (see Proposition 2.5 and [13, Theorem 4.3]), we omit the proof.

**Theorem 2.9.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $P$ an $R$-module. Then the following statements are equivalent:

1. $P$ is $u$-$S$-projective;
2. for any $u$-$S$-epimorphism $B \xrightarrow{g} C$ there exists $s \in S$ such that for any $R$-homomorphism $h : P \to C$, there exists an $R$-homomorphism $\alpha : P \to B$ satisfying $sh = g \circ \alpha$;
3. for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence $0 \to \text{Hom}_R(P, A) \xrightarrow{f^*} \text{Hom}_R(P, B) \xrightarrow{g^*} \text{Hom}_R(P, C) \to 0$ is $u$-$S$-exact;
(4) \( \text{Ext}_R^1(P, M) \) is \( u\)-\( S\)-torsion for any \( R \)-module \( M \);
(5) \( \text{Ext}_R^n(P, M) \) is \( u\)-\( S\)-torsion for any \( R \)-module \( M \) and \( n \geq 1 \).

Similar to the proof of Corollary 2.7 we have the following result.

**Corollary 2.10.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \) and \( P \) an \( R \)-module. Then the following statements hold:

1. If \( P \) is \( u\)-\( S\)-projective, then any \( u\)-\( S\)-short exact sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0 \) is \( u\)-\( S\)-split;
2. If there is \( s \in S \) such that any short exact sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0 \) is \( u\)-\( S\)-split with respect to \( s \), then \( P \) is \( u\)-\( S\)-projective.

By Theorem 2.9 projective modules are \( u\)-\( S\)-projective. Moreover, \( u\)-\( S\)-torsion modules are \( u\)-\( S\)-projective by [13, Lemma 4.2].

**Corollary 2.11.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Let \( P \) be a \( u\)-\( S\)-torsion \( R \)-module or a projective \( R \)-module. Then \( P \) is \( u\)-\( S\)-projective.

**Proposition 2.12.** Let \( R = R_1 \times R_2 \) be direct product of rings \( R_1 \) and \( R_2 \), \( S = S_1 \times S_2 := \{(s_1, s_2) | s_1 \in S_1, s_2 \in S_2 \} \) a direct product of multiplicative subsets of \( R_1 \) and \( R_2 \). Set \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). Then \( P \) is a \( u\)-\( S\)-projective \( R \)-module if and only if \( Pe_i \) is a \( u\)-\( S_i\)-projective \( R_i \)-module for each \( i = 1, 2 \).

**Proof.** Suppose \( P \) is \( u\)-\( S\)-projective. Then \( P \cong Pe_1 \times Pe_2 \). Let \( N \) be an \( R_1 \)-module. Then, as \( R_1 \)-modules, we have \( \text{Ext}^1_R(M, N \times 0) \cong \text{Ext}^1_{R_1}(Pe_1, N) \) which is \( u\)-\( S_1\)-torsion. Consequently, \( Pe_1 \) is a \( u\)-\( S_1\)-projective \( R_1 \)-module. Similarly, \( Pe_2 \) is a \( u\)-\( S_2\)-projective \( R_2 \)-module.

On the other hand, suppose \( Pe_i \) is a \( u\)-\( S_i\)-projective \( R_i \)-module for each \( i = 1, 2 \). Let \( N \) be an \( R \)-module. Then \( N \cong Ne_1 \times Ne_2 \). So \( \text{Ext}^1_R(P, N) \cong \text{Ext}^1_{R_1}(Pe_1, Ne_1) \times \text{Ext}^1_{R_2}(Pe_2, Ne_2) \) which is \( u\)-\( S\)-torsion. Consequently, \( P \) is a \( u\)-\( S\)-projective \( R \)-module.

Recall from [18] that an \( R \)-module \( F \) is \( u\)-\( S\)-flat if and only if \( \text{Tor}^1_R(M, F) \) is \( u\)-\( S\)-torsion for any \( R \)-module \( M \).

**Proposition 2.13.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). If \( P \) is a \( u\)-\( S\)-projective \( R \)-module, then \( P \) is \( u\)-\( S\)-flat.

**Proof.** Let \( P \) be a \( u\)-\( S\)-projective \( R \)-module, \( M \) an \( R \)-module and \( E \) an injective cogenerator. Then there is an element \( s \in S \) such that \( s\text{Ext}^1_R(P, \text{Hom}_R(M, E)) = 0 \) by Theorem 2.9. By [7, Lemma 2.16(b)], we have \( s\text{Hom}_R(\text{Tor}^1_R(P, M), E) = 0 \). Let \( f : \text{Tor}^1_R(P, M) \to E \) be an \( R \)-homomorphism. Since \( E \) is injective, there is an \( R \)-homomorphism \( g : \text{Tor}^1_R(P, M) \to E \) such that \( f = g \circ i \) where \( i : s\text{Tor}^1_R(P, M) \to \text{Tor}^1_R(P, M) \).
$\text{Tor}^R_1(P, M)$ is the embedding map. Since $s\text{Hom}_R(\text{Tor}^R_1(P, M), E)) = 0$, we have $f(sx) = g(sx) = sg(x) = 0$ for any $x \in \text{Tor}^R_1(P, M)$. Thus $\text{Hom}_R(s\text{Tor}^R_1(P, M), E) = 0$. So $s\text{Tor}^R_1(P, M) = 0$ since $E$ is an injective cogenerator. Consequently, $P$ is $u$-$S$-flat.

**Proposition 2.14.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then the following statements hold.

1. Any finite direct sum of $u$-$S$-projective modules is $u$-$S$-projective.
2. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-exact sequence. If $C$ is $u$-$S$-projective, then $A$ is $u$-$S$-projective if and only if so is $B$.
3. Let $A \to B$ be a $u$-$S$-isomorphism. Then $A$ is $u$-$S$-projective if and only if $B$ is $u$-$S$-projective.
4. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-split $u$-$S$-exact sequence. If $B$ is $u$-$S$-projective, then $A$ and $C$ are $u$-$S$-projective.

**Proof.** We only prove (4) since the proof of (1)-(3) is dual to that of [13, Proposition 4.7].

(4): Let $X$ be a module. We will prove that $\text{Ext}^1_R(C, X)$ is annihilated by some element of $S$. Let $h : C \to B$ such that $g \circ h = s\text{Id}_C$ with $s \in S$. Then $g, h$ induce the maps $g' : \text{Ext}^1_R(C, X) \to \text{Ext}^1_R(B, X)$ and $h' : \text{Ext}^1_R(B, X) \to \text{Ext}^1_R(C, X)$ with $h' \circ g' = s\text{Id}_{\text{Ext}^1_R(C, X)}$. As $t\text{Ext}^1_R(B, X) = 0$ for some $t \in S$, we get $s\text{Ext}^1_R(C, X) = 0$. The “$A$-part” of the proof goes similarly.

It is well-known that any direct sum of projective modules is projective. However, the following example shows that a direct sum of $u$-$S$-projective modules is not necessarily $u$-$S$-projective.

**Example 2.15.** Let $R = \mathbb{Z}$ be the ring of integers, $p$ a prime in $\mathbb{Z}$ and $S = \{p^n | n \in \mathbb{N}\}$. Let $M_n = \mathbb{Z}/\langle p^n \rangle$ for each $n \geq 1$. Then $M_n$ is $u$-$S$-torsion and thus $u$-$S$-projective. Set $N = \bigoplus_{n=1}^\infty M_n$. Note that $\text{Ext}^1_\mathbb{Z}(\mathbb{Z}/\langle p^n \rangle, \mathbb{Z}/\langle p^m \rangle) \cong \mathbb{Z}/\langle p^{\min(m,n)} \rangle$.

We have $\text{Ext}^1_\mathbb{Z}(N, N) \cong \prod_{n \in \mathbb{N}} \left( \bigoplus_{m \in \mathbb{N}} \mathbb{Z}/\langle p^{\min(m,n)} \rangle \right) \cong \prod_{n \in \mathbb{N}} \left( \bigoplus_{m \in \mathbb{N}} \mathbb{Z}/\langle p^{\min(m,n)} \rangle \right)$.

Note that the abelian group $\prod_{n \in \mathbb{N}} \left( \bigoplus_{m \in \mathbb{N}} \mathbb{Z}/\langle p^{\min(m,n)} \rangle \right)$ contains a subgroup $\prod_{n \in \mathbb{N}} \mathbb{Z}/\langle p^n \rangle$.

Since $\prod_{n \in \mathbb{N}} \mathbb{Z}/\langle p^n \rangle$ is not $u$-$S$-torsion, we have $\text{Ext}^1_\mathbb{Z}(N, N)$ is also not $u$-$S$-torsion. Consequently $N$ is not $u$-$S$-projective.

Let $p$ be a prime ideal of $R$. We say an $R$-module $P$ is (simply) $u$-$p$-projective provided that $P$ is $u-(R \setminus p)$-projective.
Proposition 2.16. Let $R$ be a ring and $P$ an $R$-module. Then the following statements are equivalent:

1. $P$ is projective;
2. $P$ is $u$-$p$-projective for any $p \in \text{Spec}(R)$;
3. $P$ is $u$-$m$-projective for any $m \in \text{Max}(R)$.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ : Trivial.

$(3) \Rightarrow (1)$ : Let $M$ be an $R$-module. Then $\text{Ext}^1_R(P, M)$ is $(R \setminus m)$-torsion. Thus for any $m \in \text{Max}(R)$, there exists $s_m \in R \setminus m$ such that $s_m \text{Ext}^1_R(P, M) = 0$. Since the ideal generated by $\{s_m \mid m \in \text{Max}(R)\}$ is $R$, we have $\text{Ext}^1_R(P, M) = 0$. So $P$ is projective. □

3. $u$-$S$-semisimple modules and $u$-$S$-semisimple rings

Let $R$ be a ring. Recall from [14] that an $R$-module $M$ is semisimple provided that it is a direct sum of simple modules. By [14, Proposition 4.1] an $R$-module $M$ is semisimple if and only if every submodule is a direct summand of $M$. So $M$ is semisimple if and only if any short exact sequence $0 \to A \to M \to C \to 0$ is split.

Utilizing this characterization, we introduce the notion of $u$-$S$-semisimple module.

Definition 3.1. Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $M$ is called $u$-$S$-semisimple provided that any $u$-$S$-short exact sequence $0 \to A \to M \to C \to 0$ is $u$-$S$-split.

Obviously, $u$-$S$-torsion modules are $u$-$S$-semisimple. Certainly, the class of $u$-$S$-semisimple modules is closed under $u$-$S$-isomorphisms. We can deduce that semisimple modules are also $u$-$S$-semisimple from the following lemma.

Lemma 3.2. An $R$-module $M$ is $u$-$S$-semisimple if and only if any short exact sequence $0 \to L \to M \to N \to 0$ is $u$-$S$-split.

Proof. The “only if part” is clear. To prove the converse, let $0 \to A \xrightarrow{f} M \xrightarrow{g} C \to 0$ be a $u$-$S$-short exact sequence. Consider the natural exact sequence $0 \to \text{Ker}(g) \to M \xrightarrow{g} \text{Im}(g) \to 0$. Then there is an $R$-homomorphism $g'_1 : \text{Im}(g) \to M$ and $s \in S$ such that $g_1 \circ g'_1 = s \text{Id}_{\text{Im}(g)}$. Let $i : \text{Im}(g) \to C$ be the embedding map. Then by Lemma 2.1, there exists a $u$-$S$-isomorphism $j : C \to \text{Im}(g)$ such that $i \circ j = s' \text{Id}_C$ for some $s' \in S$. Setting $g' = g'_1 \circ j$, we have $g \circ g' = ss' \text{Id}_C$. So the $u$-$S$-short exact sequence $0 \to A \xrightarrow{f} M \xrightarrow{g} C \to 0$ is $u$-$S$-split by Lemma 2.4. □

Proposition 3.3. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a $u$-$S$-short exact sequence. If $B$ is $u$-$S$-semisimple, then $A$ and $C$ are $u$-$S$-semisimple.
Proof. Let $a : X \to A$ be a $u$-$S$-monomorphism. Since $f$ and $a$ are $u$-$S$-monomorphisms, so is their composition $f \circ a : X \to B$. Indeed, if $t \text{Ker}(f) = 0$ and $t' \text{Ker}(a) = 0$ with $t, t' \in S$, then $tt' \text{Ker}(f \circ a) = 0$. As $B$ is $u$-$S$-semisimple, $w \circ f \circ a = s \text{Id}_X$ for some $R$-homomorphism $w : B \to X$ and some $s \in S$. So $0 \to X \to A \to Y \to 0$ $u$-S-splits, where $Y = \text{Coker}(a)$. Similarly, let $j : C \to K$ be a $u$-$S$-epimorphism. As $g$ and $j$ are $u$-$S$-epimorphisms, so is their composition $j \circ g : B \to K$. Indeed, if $t K \subseteq \text{Im}(j)$ and $t'C \subseteq \text{Im}(g)$ with $t, t' \in S$, then $tt'K \subseteq \text{Im}(j \circ g)$. As $B$ is $u$-$S$-semisimple, $j \circ g \circ w = s \text{Id}_K$ for some linear map $w : K \to B$ and some $s \in S$. Let $M = \text{Ker}(j)$, then $0 \to M \to C \to K \to 0$ $u$-S-splits by Lemma 2.4.

Recall that a ring $R$ is semisimple provided that $R$ is semisimple as an $R$-module. Note that a ring $R$ is semisimple if and only if any free $R$-module is semisimple by [14, Proposition 4.5]. To give a “uniform” version of semisimple rings, we define $u$-$S$-semisimple rings by considering all free $R$-modules.

**Definition 3.4.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. $R$ is called a $u$-$S$-semisimple ring provided that any free $R$-module is $u$-$S$-semisimple.

Obviously, all semisimple rings are $u$-$S$-semisimple for any multiplicative subset $S$ of $R$. The next result gives various characterizations of $u$-$S$-semisimple rings.

**Theorem 3.5.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then the following statements are equivalent:

1. $R$ is a $u$-$S$-semisimple ring;
2. any $R$-module is $u$-$S$-semisimple;
3. any $u$-$S$-short exact sequence is $u$-split;
4. any short exact sequence is $u$-S-split;
5. $\text{Ext}^1_R(M, N) = 0$ is $u$-$S$-torsion for any $R$-modules $M$ and $N$;
6. any $R$-module is $u$-$S$-projective;
7. any $R$-module is $u$-$S$-injective.

**Proof.** (1) $\Rightarrow$ (2): Let $M$ be an $R$-module. There exists an exact sequence $0 \to K \to F \to M \to 0$ with $F$ free $R$-module. By Proposition 3.3, $M$ is $u$-$S$-semisimple.

(2) $\Rightarrow$ (3): Let $\xi : 0 \to A \to B \to C \to 0$ be a $u$-$S$-short exact sequence. Since $B$ is $u$-$S$-semisimple, the $u$-$S$-short exact sequence $\xi$ is $u$-S-split.

(3) $\Rightarrow$ (2): Let $M$ be an $R$-module and $0 \to A \to M \to B \to 0$ a $u$-$S$-short exact sequence. By (3), $0 \to A \to M \to B \to 0$ is $u$-S-split. So $M$ is $u$-$S$-semisimple.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4): Trivial.
(4) $\Rightarrow$ (6): Let $M$ be an $R$-module and $0 \to K \to P \to M \to 0$ be a short exact sequence with $P$ projective. Then $M$ is $u$-$S$-projective by Proposition 2.14.

(5) $\Leftrightarrow$ (6): This equivalence follows from Theorem 2.7.

(5) $\Leftrightarrow$ (7): This equivalence follows from [13, Theorem 4.3].

(6) $\Rightarrow$ (3): Let $0 \to N \to K \to M \to 0$ be a $u$-$S$-short exact sequence. Since $M$ is a $u$-$S$-projective module, then $0 \to N \to K \to M \to 0$ is $u$-$S$-split by Corollary 2.10.

Corollary 3.6. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Suppose $R$ is a $u$-$S$-semisimple ring. Then $R$ is both $u$-$S$-Noetherian and $u$-$S$-von Neumann regular. Consequently, there exists an element $s \in S$ such that for any ideal $I$ of $R$ there is an $R$-homomorphism $f_I : R \to I$ satisfying $f_I(i) = si$ for any $i \in I$.

Proof. Let $\Gamma := \{I\}_{I \subseteq R}$ be the set of all ideals of $R$. Considering the natural short exact sequence $0 \to \bigoplus_{I \in \Gamma} I \xrightarrow{i} \bigoplus_{I \in \Gamma} R \xrightarrow{\pi} \bigoplus_{I \in \Gamma} R/I \to 0$, we have an $R$-homomorphism $i' : \bigoplus_{I \in \Gamma} R \to \bigoplus_{I \in \Gamma} I$ such that $i' \circ i = s\text{Id}_{\bigoplus_{I \in \Gamma} I}$ for some $s \in S$. So the natural embedding map $\text{Im}(i') \hookrightarrow \bigoplus_{I \in \Gamma} I$ is a $u$-$S$-isomorphism. Thus the set $\Gamma := \{I\}_{I \subseteq R}$ is uniformly $S$-finite (see [13] for example) since the $I$-th component of $\text{Im}(i')$ is finitely generated for any ideal $I$ of $R$. So $R$ is a uniformly $S$-Noetherian ring. Since any $R$-module is $u$-$S$-projective by (4) $\Rightarrow$ (6) of Theorem 3.5 we have $R$ is uniformly $S$-von Neumann regular by Proposition 2.13.

Let $\Gamma := \{I\}_{I \subseteq R}$ be the set of all ideals of $R$. Since $R$ is uniformly $S$-Noetherian, there exists an element $s \in S$ such that for any ideal $I \in \Gamma$ there is a finitely generated sub-ideal $K$ of $I$ satisfying $sI \subseteq K$. Since $R$ is uniformly $S$-von Neumann regular, there is an element $s' \in S$ such that for any finitely generated ideal $K$ of $R$ there is an idempotent $e \in K$ such that $s'(K/\langle e \rangle) = 0$ by [13, Theorem 3.13]. Let $f : R \to I$ be the $R$-homomorphism given by $f(1) = ss'e$. Then we have $f(i) = ss'\ i$ for any $i \in I$.

Certainly, if $R$ is a $u$-$S$-semisimple ring, then $R$ is $u$-$S$-semisimple as an $R$-module. However, the following example shows that the converse does not hold in general.

Example 3.7. Let $R = \mathbb{Z}$ be the ring of all integers and the multiplicative subset $S = \mathbb{Z}\setminus\{0\}$. Let $\langle n \rangle$ be the ideal generated by $n \in \mathbb{Z}$, and consider the exact sequence $0 \to \langle n \rangle \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/\langle n \rangle \to 0$. Set $i' : \mathbb{Z} \to \langle n \rangle$ to be the $\mathbb{Z}$-homomorphism satisfying $i'(1) = n$. Then $i'(i(m)) = nm$ for any $m \in \langle n \rangle$. Thus $\mathbb{Z}$ is a $u$-$S$-semisimple $\mathbb{Z}$-module. Since $R$ is not uniformly $S$-von Neumann regular by [13, Example 3.15], $R$ is not a $u$-$S$-semisimple ring by Corollary 3.6.
Moreover, the following result shows that any $u$-$S$-semisimple ring is in fact a semisimple ring in the case that $S$ is regular multiplicative subset of $R$, i.e., the multiplicative set $S$ is composed of non-zero-divisors.

**Proposition 3.8.** Let $R$ be a ring and $S$ a regular multiplicative subset of $R$. Then $R$ is a $u$-$S$-semisimple ring if and only if $R$ is a semisimple ring.

**Proof.** The “if part” is clear. To prove the converse, let $R$ be a $u$-$S$-semisimple ring, where each element in $S$ is a non-zero-divisor. There exists an element $s \in S$ such that for any ideal $I$ of $R$ there is an $R$-homomorphism $f : R \to I$ satisfying $f(i) = si$ for any $i \in I$ by Corollary 3.6. Set $I = \langle s^2 \rangle$. Then $s^2 f(1) = f(s^2) = s^3$. Let $f(1) = s^2 r \in I$ for some $r \in R$. Then $s^4 r = s^3$. Since $s$ is a non-zero-divisor, we have $sr = 1$ and thus $s$ is a unit. Consequently, $R$ is a semisimple ring. \[\Box\]

Suppose $R$ is $u$-$S$-semisimple as an $R$-module for a regular multiplicative subset $S$ of $R$. We must note that $R$ is not necessarily a semisimple ring.

**Example 3.9.** Let $R$ be a non-field domain and $S = R \setminus \{0\}$ the set of all nonzero elements in $R$. Then $R$ is obviously not a semisimple ring. However, $R$ is $u$-$S$-semisimple as an $R$-module. Indeed, let $I$ be an nonzero ideal of $R$ and $0 \neq s \in I$. Let $f : R \to I$ be an $R$-homomorphism satisfying $f(1) = s$. Then we have $f(i) = si$ for any $i \in I$. Hence $R$ is $u$-$S$-semisimple as an $R$-module by Lemma 3.2.

We also have the following direct product property of $u$-$S$-semisimple rings.

**Proposition 3.10.** Let $R = R_1 \times R_2$ be direct product of rings $R_1$ and $R_2$ and $S = S_1 \times S_2$ a direct product of multiplicative subsets of $R_1$ and $R_2$. Then $R$ is a $u$-$S$-semisimple ring if and only if $R_i$ is a $u$-$S_i$-semisimple ring for each $i = 1, 2$.

**Proof.** Follows by Proposition 2.12 and Theorem 3.5. \[\Box\]

The following non-trivial example shows that the condition that “$S$ is a regular multiplicative subset of $R$” in Proposition 3.8 cannot be removed.

**Example 3.11.** Let $R_1$ be a semi-simple ring and $R_2$ a non-semi-simple ring. Denote by $R = R_1 \times R_2$. Then $R$ is not a semi-simple ring. Set $S = \{(1,1),(1,0)\}$ which is a multiplicative subset of $R$. Then $R$ is trivially a $u$-$S$-semi-simple ring by Proposition 3.10.

Let $\mathfrak{p}$ be a prime ideal of $R$. We say a ring $R$ is (simply) a $u$-$\mathfrak{p}$-semisimple ring provided $R$ is a $u$-$(R \setminus \mathfrak{p})$-semisimple ring. The final result gives a new characterization of semisimple rings.

**Proposition 3.12.** Let $R$ be a ring. Then the following statements are equivalent:
(1) $R$ is a semisimple ring;
(2) $R$ is a $u$-$p$-semisimple ring for any $p \in \text{Spec}(R)$;
(3) $R$ is a $u$-$m$-semisimple ring for any $m \in \text{Max}(R)$.

Proof. (1) $\Rightarrow$ (2): Let $P$ be an $R$-module and $p \in \text{Spec}(R)$. Then $P$ is projective, and thus is $u$-$p$-projective. So $R$ is a $u$-$p$-semisimple ring by Theorem 3.5.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $M$ be an $R$-module. Then $M$ is $u$-$m$-projective for any $m \in \text{Max}(R)$. Thus $M$ is projective by Proposition 2.16. So $R$ is a semisimple ring. $\square$

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