SOME REMARKS ON IDEALS WITH LARGE REGULARITY AND REGULARITY JUMPS

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Abstract. This paper exhibits some new examples of the behavior of the Castelnuovo-Mumford regularity of homogeneous ideals in polynomial rings. More precisely, we present new examples of homogeneous ideals with large regularity compared to the generating degree. Then we consider the regularity jumps of ideals. In particular we provide an infinite family of ideals having regularity jumps at a certain power.

1. Introduction

Castelnuovo-Mumford regularity, or simply regularity, together with the projective dimension are the most important invariants of a homogenous ideal in a polynomial ring $k[x_1,\ldots,x_n]$ (or a closed subscheme of $\mathbb{P}^n$). It measures the extent of cohomological complexity of such an ideal. Explicitly, the regularity is a measure for the Hilbert function of the ideal, or the ideal sheaf, to become polynomial; see [8], §4. Bayer and Mumford [2] point out that the regularity can also be considered as a measure of the complexity of computing the Gröbner bases. More generally, let $S = k[x_1,\ldots,x_n]$ with $k$ a field of characteristic zero and $M$ be a finitely generated graded $S$-module. Consider a minimal graded free resolution of $M$ as follows.

$$
F : \ldots \rightarrow F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} \ldots \rightarrow F_0 \xrightarrow{\delta_0} M
$$

There exists integers $a_{ij}$ such that $F_i = \sum_{j} S(-a_{ij})$. The regularity of $M$, denoted $\text{reg}(M)$, is then defined to be the supremum of the numbers $a_{ij} - i$. For $d \geq \text{reg}(M) + 1$, the Hilbert function $H_M(d)$ agrees with the Hilbert polynomial $P_M(d)$.

Another way of defining the regularity is through graded local cohomology modules $H^i_m(M)$ for each $0 \leq i \leq \dim(M)$, where $m = (x_1,\ldots,x_n)$ denotes the irrelevant maximal ideal of $S$. As such modules are Artinian, one can define $\text{end}(H^i_m(M))$ as the maximum integer $k$ such that $H^i_m(M)_k \neq 0$. Then one can equivalently define

$$
\text{reg}(M) = \max \{\text{end}(H^i_m(M)) + i\}
$$
For equivalent definitions and various algebro-geometric properties of the regularity we refer to [8, 4, 10].

In the case that \( M = I \) is a homogenous ideal in \( S \), we remark that:

**Remark 1.1.** Let \( I \) be a homogenous ideal in the polynomial ring \( S = k[x_1, ..., x_n] \) and \( m \) be the irrelevant maximal ideal of \( S \). If \( I \) is not \( m \)-primary, that is, if \( \sqrt{I} \neq m \), then \( \text{reg}(I) = \min\{\mu | H^i(S/I)_{\mu - i} = 0 \forall i \} \); see [6], Proposition 9.5.

If \( I \) is a homogenous ideal generated in a single degree \( d \), then \( \text{reg}(I) \geq d \). One important problem in studying the Castelnuovo-Mumford regularity of ideals is to find ideals whose regularity is large relative to the generating degree. Mayr and Meyer [12] have given examples of ideals in polynomial rings in \( 10n + 2 \) variables whose regularity is a doubly exponential function of \( n \) and polynomial in the generating degree \( d \); see [12]. Caviglia [5] was probably the first to produce an ideal in a polynomial ring with fixed number of variables and three generators whose regularity is much larger than the generating degree. There have been other attempts to find examples of ideals with large regularity; see for example [3].

Another interesting problem is to consider the regularity of powers of an ideal \( I \). In [7] an interesting notion, namely that of regularity jumps has been defined. An ideal has regularity jump at the \( k \)-th power if \( \text{reg}(I^k) - \text{reg}(I^{k-1}) > d \). In the same article the author mentions many new and known examples of ideals with this property. In [1] the author presented a simple criterion in terms of Rees algebra of a specific ideal to show that high enough powers of certain ideals have linear resolution.

Our aims in this paper are two folds. First we present new results of homogenous ideals with large regularity comparing to their generating degree. Then we focus on the regularity jumps of ideals and provide an infinite family of ideals having regularity jumps at a certain power.

This paper is structured as follows. In the first section we discuss several variants of Caviglia’s example and give further examples of ideals with stronger regularities. In particular, we explain (see Remark 2.7) why we expect that a generalization of our example would produce polynomially large regularities of arbitrary degree. In the second section we consider the problem of ideals with regularity jumps and show that an infinite family of ideal \( I_n \) for \( n \geq 3 \) have regularity jump at \( k = 2 \). The ideals \( I_n \) define Cohen-Macaulay rings of minimal multiplicity indicating that even among such ideals one can find examples whose squares do not have linear resolution. The ideal \( I_3 \) has been shown in [7] to have such a regularity jump by declaring the existence of a non-linear second syzygy. Our contribution here is to show that for all \( n \geq 3 \) the ideal \( I_n^2 \) has regularity strictly greater than 4. We achieve this by local cohomological methods.
2. Ideals with large regularity

In this section we are going to construct homogenous ideals with large Castelnuovo-Mumford regularity. Of course, by large regularity we mean that \( \text{reg}(I) \) could be made arbitrarily large, where \( d = d(I) \) is the degree of generators of \( I \). Note that we only consider *equigenerated* ideals, i.e., homogenous ideals all of whose generators are of the same degree. As it is mentioned in [8], there are only few known examples in small number of variables of such ideals and as the author in [8] mentions it is interesting to construct more such ideals especially with fix number of variables.

Our main tool to produce ideals with large regularity is the notion of weakly stability which was developed in [5]. In fact, we also compute the initial ideal of the ideals we consider. Although this imposes lengthy computations even in relatively simple cases, this approach has the advantage of demonstrating the Bayer-Mumford philosophy in [2] which we mentioned in the introduction that *regularity is a measure of complexity of computing the Gröbner basis* and hence ideals with larger regularity give rise to more complicated initial ideals. Let us also remark that whereas the following ideals with large regularity are not prime ideals, it is expected that for prime ideals much smaller upper bounds should exist. See [9]. We first recall the definition and some properties that we will need later.

**Definition 2.1.** A monomial ideal \((u_1, ..., u_n)\) is called weakly stable if for each generator \(u_i\), there exist \(a_j \in \mathbb{N}\) such that \(x_j^{a_j} \left( x_m^{u_i} \right) \in I\) for every \(j < m(u_i)\). Where \(m(u_i)\) is the maximum of all \(j\) such that \(x_j\) divides \(u_i\) and \(x_m^{u_i}\) is the highest power of \(x_m^{u_i}\) dividing the monomial \(u_i\).

Perhaps the best known and simplest example of ideals with large regularity in a fixed polynomial ring was given by Caviglia in [5]. This is the ideal \(I = (x_1^d, x_2^d, x_3^{d-1} - x_2x_4^{d-1})\) for which \(\text{reg}(I) = d^2 - 1\). Here we first investigate variants of this ideal to obtain more examples of ideals with large regularity. Let us recall two results from [5] that relate the regularity of an ideal to that of its initial ideal:

**Theorem 2.2.** Let \(I \subseteq k[x_1, ..., x_n]\) be a weakly stable ideal generated by the minimal system \(u_1, ..., u_r\). Assume that \(u_1 > u_2 > ... > u_r\) with respect to the reverse lexicographical order (which is different from revlex). Then \(\text{reg}(I) = \max\{\deg(u_i) + C(u_i)\}\), where \(C(u_i)\) is the highest degree of a monomial \(\nu\) in \(k[x_1, ..., x_j]\) such that \(\nu \notin ((u_1, ..., u_{i-1} : u_i)\).
Theorem 2.3. Let $I$ be a homogenous ideal such that the initial ideal $in(I)$ with respect to the rev-lex order is weakly stable, then $\text{reg}(I) = \text{reg}(in(I))$.

These two statements allow us to compute the regularity of an ideal, if the initial ideal is weakly stable. Of course the easiest variants of Caviglia’s example are ideals of the form $I_{ij} = (x_1^{d_1}x_2^{d_2}, x_1^{d_1-1}x_3^{d_3} - x_2^{d_4}x_4^{d_4})$. It is straightforward to see that $in(I_{ij})$ is weakly stable and that regularity of this ideal is large. For example $\text{reg}(I_{11}) = d^2 - 1$ and if $i = j = 2$, we have:

$$\text{reg}(I_{22}) = \begin{cases} \frac{(d+2)(d-1)}{2} & \text{if } d \text{ is odd} \\ \frac{d^2 - 2}{2} & \text{if } d \text{ even} \end{cases}$$

Note that $I_{22}$ has weaker regularity than $I = I_{11}$ which is Caviglia’s example. The proofs of the above claims can be seen by the same method as in the following.

In [6], the regularity of powers of $I$ is computed using local cohomological arguments. Here we compute the regularity of $I^2$ by computing its initial ideal. In particular, one sees that $I^2$ is also an ideal with large regularity.

Theorem 2.4. Let $I = (x_1^d, x_2^d, x_1x_3^{d-1} - x_2x_4^{d-1})$. Consider the ideal $J = I^2$ given by

$$(x_1^{2d}, x_2^{2d}, x_1x_2^d, x_1^d x_3^{d-1} - x_2x_4^{d-1}, x_2^d (x_1x_3^{d-1} - x_2x_4^{d-1}), (x_1x_3^{d-1} - x_2x_4^{d-1})^2)$$

in $k[x_1, ..., x_4]$. Then it holds that: $\text{reg}(J) = d^2 + d - 1$ for all $d \geq 2$.

Proof. We wield the Buchberger’s algorithm to find a Gröbner basis for the ideal $J$ which yields the following set of generators for $in(J)$:

$$\{x_1^{2d}, x_2^{2d}, x_1^d x_2^d, x_1^{d+1} x_3^{d-1}, x_1 x_2^d x_3^{d-1}, x_2^2 x_3^{2(d-1)}\} \cup$$

$$\{x_1^{d-i} x_2^{d+i} x_4^{i(d-1)} | i = 1, ..., d-1\} \cup$$

$$\{x_1^{d-i} x_2^{d+i} x_3^{d-1} x_4^{i+1(d-1)} | i = 0, ..., d-2\} \cup$$

$$\{x_1^{2d-1} x_2^{d-1} x_4^{i(d-1)} | i = 1, ..., d-1\}$$

Note that unlike the case for $I$, it is not the case here that all S-polynomials are reduced, nor is it true that the S-polynomials are all monomials. For the sake of completeness we present the computation of $in(J)$ in what follows. This also has the
advantage of showing our general method in the later results. Set
\[ g_1 = (x_1 x_3^{d-1} - x_2 x_4^{d-1})^2, \]
\[ g_2 = x_2^d (x_1 x_3^{d-1} - x_2 x_4^{d-1}) , \]
\[ g_3 = x_1^d (x_1 x_3^{d-1} - x_2 x_4^{d-1}), \]
\[ g_4 = x_2^{2d}, \]
\[ g_5 = x_2^{2d} , \]
\[ g_6 = x_2^{2d}, \]
\[ G = \{g_1, g_2, g_3, g_4, g_5, g_6\}. \]

Let us compute the S-polynomials to find a Gröbner basis. One sets \( H_7 := S(g_1, g_3) = -x_1^d x_2 x_3^{d-1} x_4^{d-1} + x_1^{d-1} x_2 x_4^{2(d-1)} \) which gives rise to \( g_7 = x_1^d x_2 x_3^{d-1} x_4^{d-1} \).

For \( i = 1, \ldots, d - 2 \), set recursively
\[ H_{7+i} = S(g_1, H_{6+i}) = -x_1^{d-i} x_2 x_3^{d-1} x_4^{(i+1)(d-1)} + x_1^{(d-i-1)} x_2 x_4^{(i+2)(d-1)} \]
which yields the generator \( g_{7+i} = x_1^{d-i} x_2 x_3^{d-1} x_4^{(i+1)(d-1)} \) for \( i = 0, \ldots, d - 2 \). Note that for \( i = d - 1 \), \( H_{7+(d-1)} = H_{6+d} \rightarrow 0 \), i.e., \( H_{6+d} \) reduces to zero with respect to the set \( G \) and so this sequence stops. We compute further that \( g''_d := S(g_2, g_4) = x_1^{d-1} x_2 x_3^{d-1} x_4^{i(d-1)} \) and for \( i = 2, \ldots, d - 1 \), we set: \( g''_{6+i} := S(g_2, g''_{5+i}) = x_1^{d-i} x_2^{i+2} x_4^{i(d-1)} \).

Note that for \( i = d \), \( g''_{6+d} = S(g_2, g''_{5+d}) \rightarrow 0 \). Finally note that: \( g''_{d} = S(g_3, g_6) = x_1^{2d-1} x_2 x_4^{d-1} \) and for \( i = 2, \ldots, d - 1 \), one sets \( g''_{6+i} = S(g_3, g''_{5+i}) = x_1^{2d-1} x_2^{i} x_4^{(d-1)} \). We remark that for \( i = d \), \( g''_{6+d} = S(g_3, g_{5+d}) \rightarrow 0 \). By adding these new generators to \( G \) one sees that all other S-polynomials can be reduced to zero with respect to the new set and hence we have found a Gröbner Basis for \( J \) and all of the generators of \( \text{int}(J) \) are as above. We order the generators with respect to the reverse lexicographical order as follows:
\[ x_1^{2d} < \ldots < x_1^{d-1} x_2^{d-1} x_3^{d-1} x_4^{(d-1)^2} \]

Note that \( \text{int}(J) \) is a weakly stable ideal and therefore its regularity is equal to the maximum of the numbers \( \text{deg}(u_i) + C(u_i) \). This maximum is obtained at the last generator and \( C(u_i) \) is given by \( x_3^{d-2} \) and therefore \( \text{reg}(J) = (d - 1)^2 + 2d + (d - 2) = d^2 + d - 1 \).

**Theorem 2.5.** In \( S = k[x_1, x_2, x_3, x_4, x_5] \), let \( I \) be the ideal:
\[ I = (x_1^d, x_2^d, x_3^d, x_1 x_2^{d-1}, x_1 x_3^{d-1}, x_2 x_4^{d-1} - x_3 x_5^{d-1}) \]

Then \( \text{reg}(I) = d^2 - 1 \) for \( d \geq 2 \).
Proof. By the same method as above, one can show that

\[ \text{in}(I) = \{x_1^d, x_2^d, x_3^d, x_1 x_2^d, x_1 x_3^d, x_1 x_3^d x_2 x_4^d, x_1 x_3^d x_2 x_4^d x_5^d, x_4 x_5^d x_6^d, x_4 x_5^d x_6^d x_7^d, x_4 x_5^d x_6^d x_7^d x_8^d, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13}, \ldots \} \]

This ideal is weakly stable. We order the generators as follows

\[ x_1^d < \ldots < x_2(x_3 x_5^{d-1})^{d-1} \]

The maximum of the degrees is \( 1 + d(d - 1) \) and \( C(u_i) \) can be given by \( x_4^{d-2} \). It follows that: \( \text{reg}(I) = 1 + d(d - 1) + d - 2 = d^2 - 1. \)

In what follows, we give an example of an ideal in the polynomial ring \( S = k[x_1, x_2, x_3, x_4, x_5, x_6] \) with six generators whose regularity is a polynomial of degree 3 in the generating degree \( d \) of the ideal.

**Theorem 2.6.** In \( S = k[x_1, x_2, x_3, x_4, x_5, x_6] \) consider the ideal

\[ I = (x_1^d, x_2^d, x_3^d, x_4^d, x_1 x_2^d, x_1 x_3^d, x_1 x_3^d x_2 x_4^d, x_1 x_3^d x_2 x_4^d x_5^d, x_4 x_5^d x_6^d, x_4 x_5^d x_6^d x_7^d, x_4 x_5^d x_6^d x_7^d x_8^d, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13}, \ldots) \]

then \( \text{reg}(I) = d(d - 1)(d - 2) + 3d - 3 = d^3 - 3d^2 + 5d - 3. \)

**Proof.** One could show that in \( (I) \) is generated by the following set:

\[ \{x_1^d, x_2^d, x_3^d, x_4^d, x_5^d, x_1 x_2^d, x_1 x_3^d, x_1 x_3^d x_2 x_4^d, x_1 x_3^d x_2 x_4^d x_5^d, x_4 x_5^d x_6^d, x_4 x_5^d x_6^d x_7^d, x_4 x_5^d x_6^d x_7^d x_8^d, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12}, x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13}, \ldots \} \)

This ideal is weakly stable. Moreover, one sees that the maximum of the numbers in Theorem 2.2 is obtained at the generator \( x_1^d x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} \) and an example of \( C(u_j) \) could be: \( x_5^{d-2} \). This means that: \( \text{reg}(I) = d(d - 2)(d - 1) + d - 2 + d - 1 + d - 2 = d(d - 1)(d - 2) + 3d - 3 = d^3 - 3d^2 + 5d - 3. \)

**Remark 2.7.** Presumably by a similar argument as above one can show that in the ring \( S = k[x_1, \ldots, x_{2n}] \) the ideal given by \( I = (x_1^d, \ldots, x_{2n-2}^d) + (x_{2n+1} x_{2n+2} x_{2n+3} \ldots x_{2n+2} x_{2n+3} \ldots x_{2n+2}) | 0 \leq i \leq n - 2 \) is a polynomial of degree \( n \). We expect that in \( (I) \) is weakly stable. Note that for this ideal, in \( (I) \) is very plausible to be weakly stable as
the ideal already contains pure powers of $x_1, ..., x_{2n-2}$. That is, pure powers of all of the variables except two. For example for $n = 4$, then

$$I = (x_1^d, x_2^d, x_3^d, x_4^d, x_5^d, x_6^d, x_1x_3^{d-1} - x_2x_4^{d-1}, x_3x_5^{d-1} - x_4x_6^{d-1}, x_5x_7^{d-1} - x_6x_8^{d-1})$$

and one could show that $\text{in}(I)$ is weakly stable and $\text{reg}(I) = (d - 2)(d(d - 1)^2 + 3) + 3$.

In order to do some computational experiments, let us consider this example in the following simple code in CoCoA to compute the $\text{BettiDiagram}$ and hence the regularity of $I$ when $d = 3, \ldots, 10$ for example.

Use R ::= Q[x[1..8]]; For D:=3 To 10 Do

$I := \text{Ideal}(x[1]^D, x[2]^D, x[3]^D, x[4]^D, x[5]^D, x[6]^D, x[1]x[3]^{(D-1)} - x[2]x[4]^{(D-1)}, x[3]x[5]^{(D-1)} - x[4]x[6]^{(D-1)}, x[5]x[7]^{(D-1)} - x[6]x[8]^{(D-1)})$;

$\text{BettiDiagram}(I)$;
EndFor;

and here are the regularities:

- If $d = 3$ then $\text{reg}(I) = 18$
- If $d = 4$ then $\text{reg}(I) = 81$
- If $d = 5$ then $\text{reg}(I) = 252$
- If $d = 6$ then $\text{reg}(I) = 615$
- If $d = 7$ then $\text{reg}(I) = 1278$
- If $d = 8$ then $\text{reg}(I) = 2373$
- If $d = 9$ then $\text{reg}(I) = 4056$
- If $d = 10$ then $\text{reg}(I) = 6507$

and all of them satisfy the presented formula.

**Remark 2.8.** It can be seen that the large regularity of the ideal in Theorem 2.5 is revealed in the first syzygy of the ideal. In fact if one defines $t_i(I) = \max\{j|\beta_{i,j}(S/I) \neq 0\}$, where the $\beta_{i,j}$ are the Betti numbers of $S/I$. Then it follows that $t_1(S/I) = d$ and $t_2(S/I) = \text{reg}(I)$; see [11].
3. Ideals with Regularity Jumps

In this section we give several examples of the jump phenomenon introduced in [7]. The notion of regularity jump has been defined in [7] as follows:

**Definition 3.1.** An equigenerated ideal $I$ in degree $d$ is said to have regularity jump at $k$ (or that the regularity of powers of $I$ jumps at place $k$) if $\text{reg}(I^k) - \text{reg}(I^{k-1}) > d$.

The first example of such an ideal was given by Terai.

**Example 3.2.** (Terai) This ideal is

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6)$$

for which $\text{reg}(I) = 3$ and $\text{reg}(I^2) = 7$. The non-linear syzygy of $I^2$ appears at the end of the resolution.

The example that we consider is a generalization of example 2.10 in [7]. In the aforementioned paper [7] it is guessed that this family of ideals has regularity jumps at $k = 2$ by declaring that there are some experimental evidences that $I_n^2$ has non-linear syzygy. Here we take a different approach and prove that specific graded pieces of the graded local cohomology modules do not vanish, leading to the fact that $\text{reg}(I_n^2) > 4$.

Note that ideals $I_n$ define Cohen-Macaulay rings of minimal multiplicity and our result shows that even among such ideals one can find infinitely many examples whose squares do not have linear resolution. The example is as follows.

**Example 3.3.** Let

$$I_n = (x_1^2, \ldots, x_{n+1}^2, x_1x_2, \ldots, x_{n+1}x_n + 1, x_2x_3 - x_1x_{n+1} + 2, x_2x_4 - x_1x_{n+3}, \ldots, x_2x_{n+1} - x_1x_{2n}, \ldots, x_3x_4 - x_1x_{2n+1}, \ldots, x_3x_{n+1} - x_1x_{3n-2}, \ldots, x_nx_{n+1} - x_1x_{n+1})$$

where $s = \frac{n(n+1)}{2} + 1$.

In this description, a typical generator apart from $x_1^2, \ldots, x_{n+1}^2, x_1x_2, \ldots, x_{n+1}x_n$ is of the form $x_i x_j - x_1 x_{(n, i, j)}$, where $2 \leq i < j \leq n + 1$ and $t(n, i, j) = (i - 1)n - \frac{(i-1)(i-2)}{2} + 1 + (j - i)$.

Another description of this ideal as given in [7] is as follows:

$$I_n = (x^2, y_1^2, y_2^2, \ldots, y_n^2, xy_1, \ldots, xy_n, y_iy_j - xz_{i,j})$$

for $1 \leq i < j \leq n$.

Then it holds that:

**Theorem 3.4.** For $I_n$ as above, $\text{reg}(I_n) = 2$ and $\text{reg}(I_n^2) > 4$. Therefore we get an infinite family of ideals with regularity jumps at $k = 2$. 

Proof. Although the first description seems more complicated than the second one, we prefer, in order to avoid complicated indices, to work with the first description. It is straightforward to check that 
\[ in(I_n) = (x_1^2, ..., x_{n+1}^2, x_1x_2, ..., x_1x_{n+1}, x_2x_3, x_2x_4, ..., x_nx_{n+1}) \] 
In fact the $S$-polynomials are all reduced. It follows that \( in(I_n) \) is weakly stable and therefore regularity of \( I_n \) and \( in(I_n) \) are both equal to the maximum degree of generators of \( in(I_n) \), i.e., equal to 2. Note however that \( in(I_2^3) \) is not weakly stable.

In order to show that \( \text{reg}(I_n^3) > 4 \), setting \( J_n = I_n^2 \), we show that there always exists an integer \( l \) such that \( H^1_n(S/J_{n-1}) \neq 0 \). By Remark 1.1 it follows that \( \text{reg}(J_n) > 4 \). To this end, we use the fact that local cohomology can be computed via Čech complex. That is, the following complex:

\[
0 \rightarrow S/J_n \xrightarrow{d^0} \oplus(S/J_n)_{x_i} \xrightarrow{d^1} \oplus(S/J_n)_{x_i^2} \rightarrow \cdots \xrightarrow{d^{n-1}} (S/J_n)_{x_1x_2} \rightarrow 0
\]

Where the maps are alternating sums of localization maps. See [4], §5.1. Note that since \( x_1^4 = \cdots = x_{n+1}^4 = 0 \) in \( S/J_n \), the only localized summands that contribute to the above complex are localizations at \( x_j \) with \( j = n+2, ..., n \). In other words, the cohomology can be computed by the complex

\[
0 \rightarrow S/J_n \xrightarrow{d^0} \oplus_{n+2 \leq i \leq s}(S/J_n)_{x_i} \xrightarrow{d^1} \oplus_{n+2 \leq i < j \leq s}(S/J_n)_{x_i} \rightarrow \cdots \xrightarrow{d^{s-n-1}} (S/J_n)_{x_{n+2} \cdots x_s} \rightarrow 0
\]

We first describe our method for the simplest case of \( n = 3 \) and then write down the natural generalization.

Let \( n = 3 \). Then in \( S/J_3 \), we have the following equalities: (note that we abuse the notation and show the image of an element \( \beta \in S \) in \( S/J \) again by \( \beta \))

\[
\begin{align*}
x_1x_4(x_2x_3 - x_1x_5) &= 0 \Rightarrow x_1x_2x_3x_4 = x_1^2x_4x_5 \\
x_1x_2(x_3x_4 - x_1x_7) &= 0 \Rightarrow x_1x_2x_3x_4 = x_2^2x_2x_7 \\
x_1x_3(x_2x_4 - x_1x_6) &= 0 \Rightarrow x_1x_2x_3x_4 = x_3^2x_3x_6
\end{align*}
\]

and also:

\[
x_1x_2(x_2x_3 - x_1x_5) = 0 \Rightarrow x_1^2x_2x_5 = 0, \text{ because } x_1x_2x_3 = (x_1x_3)(x_2^2) = 0.
\]

Similarly, \( x_1^2x_3x_5 = x_1^2x_2x_6 = x_1^2x_4x_6 = x_1^2x_3x_7 = x_1^2x_4x_7 = 0 \)
It follows, combining the above sets of equalities, that the element \( \alpha := x_1 x_2 x_3 x_4 \) is annihilated by \( x_5, x_6 \) and \( x_7 \). This shows that \( \alpha \in \ker(d^0)_{4)} = H^{0}_{m}(S/J_{3})_{4} \) and hence \( H^{0}_{m}(S/J_{3})_{4} \neq 0 \).

The above ideas can be generalized to arbitrary \( n \). In fact, in the general case, we have the following in \( S/J_{n} \):

\[
x_1 x_2 x_3 x_4 = x_1^2 x_4 x_{n+2} = x_1^2 x_3 x_{n+3} = x_1^2 x_2 x_{2n+1}
\]

If \( x_i x_j - x_1 x_r \) is a generator of \( I_{n} \) such that \( \{i, j\} \cap \{2, 3, 4\} \neq \emptyset \), then by the same argument as in the \( n = 3 \) case, it follows that

\[
x^2 t x_r = 0 \text{ for all } t \in \{i, j\} \cap \{2, 3, 4\}.
\]

Now let \( x_{i_1} x_{j_1} - x_1 x_{r_1}, \ldots, x_{i_l} x_{j_l} - x_1 x_{r_l} \) be the set of all generators of \( I_{n} \) of the form \( x_i x_j - x_1 x_r \) such that \( \{i, j\} \cap \{2, 3, 4\} = \emptyset \). Set \( \alpha := x_1 x_2 x_3 x_4 \) as before. Then the above equalities show that for

\[
j \in \{n + 2, \ldots, s\} \setminus \{r_1, \ldots, r_l\}, \alpha x_j = 0.
\]

This implies that the element \( \kappa := (0, \ldots, \underbrace{0}_{x_1 \ldots x_{r_1}}, \ldots, 0) \in C^l(S/J_{n}) \) lies in \( \ker(d^4_{4-l}) \). It follows that \( \kappa \in H^{l}_{m}(S/J_{n})_{4-l} \) which is non-zero in this cohomology module and hence \( H^{l}_{m}(S/J_{n})_{4-l} \neq 0 \).

\[\square\]

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