Existence of Coupled Solutions of BVP for $\phi$-Laplacian Impulsive Differential Equations

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Abstract: In this paper, we study the existence of coupled solutions of anti-periodic boundary value problems for impulsive differential equations with $\phi$-Laplacian operator. Based on a pair of coupled lower and upper solutions and appropriate Nagumo condition, we prove the existence of coupled solutions for anti-periodic impulsive differential equations boundary value problems with $\phi$-Laplacian operator.

Keywords: Boundary Value Problems, Coupled Solutions, Impulsive Differential Equations, $\phi$-Laplacian Operator

1. Introduction

In recent years, the study boundary value problems (BVPs for short) with $p$-Laplacian operator has been emerging as an important area and obtained a considerable attention. Since $p$-Laplacian operator appears in the study of flow through porous media ($p = 3/2$), nonlinearity ($p \geq 2$), glaciology ($1 \leq p \leq 4/3$) and so on, there are many works about existence of solutions for differential equations with $p$-Laplacian operator [24, 25]. Usually, $p$-Laplacian operator is replaced by abstract and more general version $\phi$-Laplacian operator, which lead to clearer expositions and a better understanding of the methods which ware employed to derive the existence results [12, 22, 23].

Moreover, impulsive differential equations have become an important aspect in some mathematical models of real processes and phenomena in science. There has a significant development in impulsive differential equations and impulse theory (see [2, 3, 14]). Moreover, $p$-Laplacian operator arises in turbulent filtration in porous media, non-Newtonian fluid flows and in many other application areas [10, 12].

Furthermore, the study of anti-periodic problem for nonlinear evolution equations is closely related to the study of periodic problem which was initiated by Okochi [17]. Anti-periodic problem which is a very important area of research has been extensively studied during the past decades, such as anti-periodic trigonometric polynomials [11] and anti-periodic wavelets [4]. Moreover, anti-periodic boundary conditions also appear in physics in a variety of situations (see [1, 13]) and difference and differential equations (see [6, 8, 19, 20]). The anti-periodic problem is a very important area of research.

In addition, we known that every $T$-anti-periodic solution gives rise to a $2T$-periodic solution if the nonlinearity $f$ satisfy some symmetry condition. Indeed, the periodic and anti-periodic boundary value problems have attracted many researchers great interest (see [6, 8, 9, 15, 16, 19, 20, 21] and references therein). Recently, Guo and Gu [22] study a class of nonlinear impulsive differential equation with anti-periodic boundary condition:

\[
\begin{align*}
(\phi(u'(t)))' &= f(t, u(t), u'(t)) \quad \text{a.e. } t \in [0, T], \\
I_k(u(t_k), u(t'_k)) &= 0, \\
M_k(u(t_k), u(t'_k), u(t_0), u'(t_0), u(0)) &= 0, \\
&= k = 1, 2, \ldots, p, \\
M_k &\in C^0(R^2 \times C^1_p), k = 1, \ldots, p \text{ are impulsive functions. } C^1_p \\
\text{will be given later. In [22], the authors obtained the existence of solution for anti-periodic boundary value problems (1)-(3)}
\end{align*}
\]
for impulsive differential equations with $\phi$-Laplacian operator. In this paper, we will continuous to consider the existence of coupled solutions for boundary value problems (1)-(3).

This paper is organized as follows: In section 2, we will state some preliminaries that will be used throughout the paper. In section 3, we will obtain the existence of coupled solutions for anti-periodic $\phi$-Laplacian impulsive differential equations boundary value problems (1)-(3).

2. Preliminaries

In this section, we will introduce some definitions and preliminaries which are used throughout this paper.

For a given Banach space $E$, let $C(I) = \{ f : E \to R \}$ be the set of all continuous functions $f : E \to R$. Let $C^m(I)$ be the set of functions $u$ which are $m$ times continuously differentiable on $I$ with finite norm

$$
\| u \|_{C^m(I)} = \max_{k=0, \ldots, m} \| u^{(k)} \|_I .
$$

For $1 \leq q \leq \infty$, let $L^q(I)$ be the set of Lebesgue measurable functions $u$ on $I$ such that $\| u \|_q$ is finite.

$AC(I,q)$ denotes the set of absolutely continuous functions $u$ on $I$ satisfy $u \in L^q(I)$. $W^{m,q}(I)$ denotes the set of functions $u \in C^{m-1}(I)$ and $u^{(m-1)} \in AC(I,q)$ with finite norm

$$
\| u \|_{W^{m,q}(I)} = \max_{k=0, \ldots, m} \| u^{(k)} \|_q .
$$

It is easy to see that $C^m(I)$ and $W^{m,q}(I)$ are Banach spaces and $W^{m,q}(I)$ is a usual Sobolev space.

Let $p \in N$. A finite subset $P$ of the interval $[0,T]$ and defined by

$$
P = \{ t_0, \ldots, t_p : 0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T \} .
$$

Let $J_0 = [0,t_1)$ and $J_k = (t_k, t_{k+1}]$ for all $k = 1, \ldots, p$. For $m \in N \cup \{0\}$ and $1 \leq q \leq \infty$, we denote

$$
C^m = \{ u : [0,T) \to R : \text{for all } k = 0, \ldots, p, u \in C^m(J_k), \text{there exist } u^{(k)}(t_k), k = 1, \ldots, p; p+1 = 0, \cdots, m, \}
$$

$W^{m,q} = \{ u : [0,T) \to R : u_{\xi} \in W^{m,q}(J_k), k = 0, \ldots, p \} .

It is easy to verify that the spaces $C^m_p$ and $W^{m,q}_p$ are Banach spaces with the norms

$$
\| u \|_{C^m_p} = \max_{k=0, \ldots, p} \| u_{\xi} \|_{C^m(I_k)} \quad \text{and} \quad \| u \|_{W^{m,q}_p} = \max_{k=0, \ldots, p} \| u_{\xi} \|_{W^{m,q}(I_k)} .
$$

We say that $f : [0,T] \times S \to R (S \subset R^2)$ satisfies the restricted Carathéodory conditions on $[0,T] \times S$ if

i. for each $x \in S$ the function $f(t, \cdot, x)$ is measurable on $[0,T]$;

ii. the function $f(t, \cdot)$ is continuous on $S$ a.e. $t \in [0,T]$;

iii. for every compact set $K \subset S$, there exists a nonnegative function $\mu(x) \in L^1(0,T)$ such that

$$
| f(t, x) | \leq \mu(x) \quad \text{for a.e. } t \in [0,T] \quad \text{and all } x \in K .
$$

In this paper, we use $Car([0,T] \times S)$ to denote the set of functions satisfying the restricted Carathéodory conditions on $[0,T] \times S$. In what follows, $D^k$ and $D_k$ denote the Dini derivatives.

Definition 1. The functions $\alpha, \beta \in W^{p,1}_p$ such that $\alpha \leq \beta$ are said to be a pair of coupled lower and upper solutions of problem (1)-(3) if $\alpha, \beta$ satisfy the following conditions:

(i) $D_\alpha(t) \leq D_\alpha(t)$ for all $t \in [0,T] . \quad \text{Moreover, if} \quad \tau \in [0,T] . \quad \text{Moreover, if}$

(ii) $D_\beta(t) \geq D_\beta(t)$ for all $t \in [0,T] . \quad \text{Moreover, if} \quad \tau \in [0,T] . \quad \text{Moreover, if}$

(iii) For all $k = 1, \cdots, p$, $I_k(\alpha(t_k), \cdot)$ are injective and there exist $D^\alpha(t_k), D^\alpha(t_k), D^\beta(t_k), D^\beta(t_k) \in R$ such that

$$
\alpha(0) + \beta(T) = 0 \leq D^\alpha(0) + D^\beta(T). \quad \text{and there exist} \quad D^\alpha(0), D, \alpha(T), D, \beta(T) \in R . \quad \text{such that}
$$

Definition 2. Given a function $u \in C^1_p$ is called a solution of the problem (1)-(3) if $\phi \cdot u' \in W^{p,1}_p$ and $u$ satisfies (1) and fulfills conditions (2) and (3).

Definition 3. Assume that $f \in Car([0,T] \times R^2)$ and
\[\alpha, \beta \in W_0^{1} \quad \text{satisfying } \alpha(t) \leq \beta(t) \quad \text{for all } t \in [0, T].\]

We say that \( f \) satisfies a Nagumo condition with respect to \( \alpha \) and \( \beta \) if, for \( k = 1, \ldots, p \), there exist \( \phi_k \in C([0, \infty)) \) and \( w \in L^r(0, T) \), \( 1 \leq q < \infty \), such that \( \phi_k > 0 \) on \([0, \infty)\),

\[
| f(t, u, v) | \leq w(t)\phi_k(\|v\|) \quad \text{on } J_\times \times (\alpha(t), \beta(t)) \times R.
\]

Moreover, there exists a constant \( K = K(\alpha, \beta) \) with \( K > \max\{\|\phi_k\|_{\infty, L^q} \} \), \( k = 1, \ldots, p \), such that

\[
\int_{\phi_k(\alpha^{-1}(x))}^{\phi_k(\beta^{-1}(x))} \phi_k(x)^{q-1-\alpha} dx > 0 \quad \text{and} \quad \int_{\phi_k(\alpha^{-1}(x))}^{\phi_k(\beta^{-1}(x))} \phi_k(x)^{q-1-\beta} dx > 0
\]

for each \( k = 1, \ldots, p \). Any constant such \( K > \max\{\|\phi_k\|_{\infty, L^q} \} \), \( k = 1, \ldots, p \) will be called a Nagumo constant.

Throughout this paper, we impose the following hypotheses:

- \((H_1)\) The function \( \phi : R \to R \) is a continuous and strictly increasing function.
- \((H_2)\) The BVP (1)-(3) has a pair of coupled lower and upper solutions \( \alpha \) and \( \beta \).
- \((H_3)\) \( f \in \text{Car}(R \times R^2) \) and satisfies a Nagumo condition with respect to \( \alpha \) and \( \beta \).
- \((H_4)\) The functions \( I_k \in C^{0}(R^2) \) are non-decreasing in the first variable for \( k = 1, \ldots, p \), and the functions \( M_k \in C^{0}(R^2 \times C^1_p) \) are non-increasing in the third variable and non-decreasing in the fourth and fifth variables.

### 3. Existence Results of Coupled Solutions

This section is devoted to proving the existence of coupled solutions for anti-periodic impulsive differential equations boundary value problems with \( \phi \)-Laplacian operator. Firstly, we state the following existence and uniqueness result.

**Lemma 2.** For given \( u, u_0 \in C^1_p \) such that \( u_0 \to u \in C^1_p \), then

\[
(i) \quad \frac{d}{dt} \rho(t, u(t)) \text{ exists for a.e. } t \in [0, T], P;
\]

\[
(ii) \quad \frac{d}{dt} \rho(t, u(t)) \to \frac{d}{dt} \rho(t, u(t)) \text{ for a.e. } t \in [0, T], P.
\]

Now, we can define a strictly increasing homeomorphism \( \Phi : R \to R \) by:

\[
x \in R \to \Phi(x) = \left\{\begin{array}{ll}
\frac{\phi(x)}{2K} & |x| \leq K, \\
\frac{\phi(K) - \phi(-K)}{2K} & |x| > K.
\end{array}\right.
\]

In the following, we are in a position to prove the existence theorem for our considering problems.

**Lemma 3.** (Theorem 3.3 of [22]) Assume that \((H_1)-(H_4)\) hold. Then there exists at least one solution \( u \) of the problem (1)-(3) such that

\[
\alpha(t) \leq u(t) \leq \beta(t)
\]

and

\[
|u'(t)| \leq K, \quad t \in [0, T].
\]
where $K = K(\alpha, \beta)$ is the constant introduced in Definition 2.3.

Next, we are devoted to the existence of coupled solutions. We first introduce the following definition.

**Definition 4.** The functions $x, y$ are called coupled solutions of problems (1)-(3) if $x, y \in C_p$ and satisfy (1)-(2) and

\[
x(0) = -y(T), \quad (5)
\]
\[
x'(0) = -y'(T), \quad (6)
\]
\[
y(0) = -x(T), \quad (7)
\]
\[
y'(0) = -x'(T). \quad (8)
\]

**Remark** If the coupled solutions $x$ and $y$ of problem (1)-(3) satisfy $x = y$, the $x = y$ is a solution of problem (1)-(3).

Next, we give the existence of coupled solutions for problems (1)-(3).

**Theorem 5.** Assume hypotheses (H$_1$)-(H$_4$) hold. Then there exists at least a pair of coupled solutions $x, y \in C_p$ of the impulsive differential equations boundary value problem (1)-(3) such that

\[
x, y \in [\alpha, \beta] = \{u : \alpha(t) \leq u(t) \leq \beta(t), t \in [0, T]\}, \quad (9)
\]

and

\[
|x'(t)| \leq K \quad \text{for} \quad t \in [0, T],
\]
\[
|y'(t)| \leq K \quad \text{for} \quad t \in [0, T],
\]

where $K = K(\alpha, \beta)$ is the constant introduced in Definition 2.3.

**Proof.** Let us define $\rho, A_k, B_{k+1}$ for each $k = 1, \cdots, p$ in the same way as above, and construct a modified problem ($P'$) similar to the proof of Lemma 3, that is

\[
\begin{align*}
& (\Phi(x'(t)))' = f_j(t), \quad a.e. t \in [0, T], P, \\
& (\Phi(y'(t)))' = f_j(t), \quad a.e. t \in [0, T], P, \\
& x(t_k) = B_{k-1}(x), \quad y(t_k) = B_{k-1}(y), \quad k = 1, 2, \cdots, p, \\
& x(t'_k) = A_k(x), \quad y(t'_k) = A_k(y), \quad k = 1, 2, \cdots, p, \\
& x(0) = A_0(x), \quad y(0) = A_0(y), \\
& x(T) = B_p(x), \quad y(T) = B_p(y),
\end{align*}
\]

where

\[
A_k(x) = \rho(0, -y(T)), \quad B_p(x) = \rho(T, x(T) - y'(0) - x'(T)), \\
A_k(y) = \rho(0, -x(T)), \quad B_p(y) = \rho(T, y(T) - x'(0) - y'(T)).
\]

From the proof of the Lemma 3, there exists a couple of solutions $x, y \in C_p$ such that

\[
\alpha \leq x \leq \beta, \\
\alpha \leq y \leq \beta,
\]

and

\[
|x'(t)| \leq K, \quad |y'(t)| \leq K \quad \text{for} \quad t \in [0, T].
\]

Furthermore, $x, y$ satisfy the condition (2). Now, to prove that (5)-(8) is verified, it suffices to prove that

\[
\begin{align*}
\alpha(0) &\leq -y(0) \leq \beta(0), \quad (10) \\
\alpha(0) &\leq -x(0) \leq \beta(0), \quad (11) \\
\alpha(T) &\leq x(T) - y'(0) - x'(T) \leq \beta(T), \quad (12) \\
\alpha(T) &\leq y(T) - x'(0) - y'(T) \leq \beta(T). \quad (13)
\end{align*}
\]

Firstly, we will prove (10), by contradiction, if $\alpha(0) > -y(0)$, then by $\alpha \leq x \leq \beta$, we have

\[
\alpha(0) > -y(0) \geq -\beta(T),
\]

which contradict to $\alpha(0) + \beta(T) = 0$. Moreover, $-y(0) \leq \beta(0)$ can be proved similarly.

As the same way, we can obtain that the inequality (10) is holds. Thus we have

\[
\begin{align*}
x(0) = -y(T), \quad y(0) = -x(T). \quad (14)
\end{align*}
\]

Assume that the first inequality if (11) isn’t holds, as a consequence, we have

\[
x(T) = \alpha(T)
\]

and

\[
y'(0) + x'(T) > 0.
\]

From (14) and $\alpha(T) + \beta(0) = 0$, we have

\[
y(0) = -x(T) = -\alpha(T) = \beta(0).
\]

From these facts and the relation $\alpha \leq x, y \leq \beta$, we have

\[
x'(T) \leq D_x \alpha(T), \quad y'(0) \leq D_y \beta(0),
\]

thus

\[
0 \leq y'(0) + x'(T) \leq D_x \beta(0) + D_y \alpha(T) \leq 0.
\]

It is a contradiction. Moreover, the inequality in (13) be obtain in a similar way. Hence inequalities (11)-(12) are hold, that is to say $x, y$ satisfy (5)-(8).

Therefore, the functions $x, y$ is a coupled solutions of the problem (1)-(3), which completes the proof.
4. Conclusion

In this paper, we mainly discuss the existence of coupled solutions of anti-periodic boundary value problems for impulsive differential equations with \( \phi \)-Laplacian operator. To give the existence results of coupled solutions for the problem (1)-(3), we first introduce a pair of coupled lower and upper solutions (see Definition 1), then, we provide and prove the existence results of coupled solutions for anti-periodic \( \phi \)-Laplacian impulsive differential equations boundary value problems based on a pair of coupled lower and upper solutions and appropriate Nagumo condition (Theorem 5).

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