Intervals between numbers that are sums of two squares

Abstract. In this paper, we improve the moment estimates for the gaps between numbers that can be represented as a sum of two squares of integers. We consider certain sum of Bessel functions and prove the upper bound for its weighted mean value. This bound provides estimates for the $\gamma$-th moments of gaps for all $\gamma \leq 2$.

§ 1. Introduction

Let $S = \{s_1 < s_2 < \ldots < s_n < \ldots\} = \{1, 2, 4, 5, 8, \ldots\} \subset \mathbb{N}$ be the set of all natural numbers that are expressible as the sum of two squares of integers. One of the classical areas in the research of the properties of this set is the study of the distribution of the gaps between consecutive elements, i.e. the quantity $s_{n+1} - s_n$ or, equivalently, the value distribution of the distance function of $S$

$$R(x) = \min_{n \in S} |x - n|$$

for $x \to +\infty$.

The following fact is well known (cf.[3],[1]):

**Theorem 1.** The inequality

$$R(x) \ll x^{1/4}$$

holds.

**Proof.** Let us note that if $f(x) = x - \lfloor \sqrt{x} \rfloor^2$, where $[y]$ is an integer part of the number $y$, then $0 \leq f(x) \ll \sqrt{x}$ for $x \gg 1$ and $x - f(x) = \lfloor \sqrt{x} \rfloor^2$ is a square of integer. Therefore,

$$x^{1/4} \gg f(f(x)) = x - (x - f(x) + f(x) - f(f(x))) \geq 0.$$  

But $x - f(x)$ and $f(x) - f(f(x))$ are squares of integers, so for some integers $a$ and $b$ we have

$$|x - a^2 - b^2| \ll x^{1/4},$$

which was to be proved.

The author is partially supported by Laboratory of Mirror Symmetry NRU HSE, RF Government grant, ag. NSh 14.641.31.0001, the Simons Foundation and the Moebius Contest Foundation for Young Scientists

© Alexander Kalmynin, 2017
This estimate was probably known to L. Euler and, unfortunately, was not improved after that (it is still unknown if the identity $R(x) = o(x^{1/4})$ is true). Conjecturally, the correct order of growth of $R(x)$ is much smaller:

**Conjecture 1.** For any $\varepsilon > 0$ the inequality

$$R(x) \ll \varepsilon x^\varepsilon$$

is true.

As for the large values of $R(x)$, by the work [4] of I. Richards, we have the following:

**Theorem 2.** For any $\varepsilon > 0$ there exist infinitely many positive integers $x$ such that

$$R(x) > \left(\frac{1}{4} - \varepsilon\right) \ln x.$$

**Remark 1.** If we assume that $S \cap [1, x]$ is a typical trajectory of a Possion point process with intensity

$$\frac{\#S \cap [1, x]}{x} \asymp \frac{1}{\sqrt{\ln x}},$$

then we obtain $R(x) \ll \ln x$.

In addition to the the upper and lower bounds, one can also consider the mean values of our function. The best result of this type is due to C. Hooley [2]:

**Theorem 3.** For any $0 < \gamma < \frac{5}{3}$ we have

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^\gamma \ll x (\ln x)^{(\gamma - 1)/2}.$$

Therefore, for almost all $n$ the inequality $s_{n+1} - s_n \ll \sqrt{\ln s_n}$ holds. More precisely:

**Corollary 1.** Let $g(x)$ be any function that tends to infinity. Then the number of $s_n \leq x$ with $s_{n+1} - s_n \gg g(x) \sqrt{\ln x}$ is $o\left(\frac{x}{\sqrt{\ln x}}\right)$.

The main goal of the present work is to improve Hooley’s theorem, i.e. to prove analogous (but somewhat weaker) estimate for the wider range of values of $\gamma$.

**Theorem 4.** For any $1 < \gamma \leq 2$ the inequality

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^\gamma \ll x (\ln x)^{3/2(\gamma - 1)} \delta(x, \gamma)$$

is true, where

$$\delta(x, \gamma) = \begin{cases} 1, & \text{if } \gamma < 2; \\ \ln x, & \text{if } \gamma = 2. \end{cases}$$
§ 2. Proof of the main theorem

In this section, we will prove Theorem 4 by reducing the original problem to the question about the distribution of values of certain sum of Bessel functions. To do this, we need some lemmas. Let us start with the transformation formula for the theta-function.

**Lemma 1.** Let $M$ be a positive real number. For any real $x$ the equality

$$\vartheta_M(x) := \sum_{n \in \mathbb{Z}} e^{-\pi M(x+n)^2} = \frac{1}{\sqrt{M}} \sum_{m \in \mathbb{Z}} e^{2\pi imx} e^{-\pi m^2/M}$$

is true.

**Proof.**

Consider the function $g(x) = e^{-\pi Mx^2}$ on the real line. It is easy to show that $g$ is a Schwartz function and

$$\vartheta_M(x) = \sum_{n \in \mathbb{Z}} g(x+n).$$

By the Poisson summation formula, for any $x \in \mathbb{R}$ we get

$$\sum_{n \in \mathbb{Z}} g(x+n) = \sum_{m \in \mathbb{Z}} e^{2\pi imx} \hat{g}(m),$$

where $\hat{g}(\xi) = \int_\mathbb{R} g(x)e^{-2\pi i\xi x}dx$ is a Fourier transform of our function. On the other hand, it is well known that

$$\int_\mathbb{R} e^{-\pi Mx^2-2\pi i\xi x}dx = \frac{1}{\sqrt{M}} e^{-\pi \xi^2/M}.$$

Using this relation, we obtain the required result.

With the help of Lemma 1 we will prove the following identity, which will be crucial for the subsequent considerations:

**Lemma 2.** Let $M$ and $N$ be some positive real numbers. Then

$$I(N, M) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \vartheta_M(\sqrt{N}\sin \varphi) \vartheta_M(\sqrt{N}\cos \varphi) d\varphi = \frac{1}{M} \sum_{n \geq 0} r_2(n) J_0(2\pi \sqrt{N}n) e^{-\pi n/M} =: \frac{1}{M} S(N, M),$$

where $J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i x \cos \varphi} d\varphi$ is the Bessel function of the first kind of order zero and $r_2(n)$ is the number of pairs $(a, b)$ of integers such that $a^2 + b^2 = n$.

**Proof.**

Using lemma 1, we find

$$I(N, M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{a \in \mathbb{Z}} e^{2\pi i a \sqrt{N} \sin \varphi} e^{-\pi a^2/M} \right) \left( \sum_{b \in \mathbb{Z}} e^{2\pi i b \sqrt{N} \cos \varphi} e^{-\pi b^2/M} \right) d\varphi.$$
Both series are absolutely and uniformly convergent, so we can replace the product of their sums by the double sum and interchange summation and integration. Consequently,

\[ I(N, M) = \frac{1}{2\pi M} \sum_{(a,b) \in \mathbb{Z}} e^{-(a^2 + b^2)\pi/M} \int_{-\pi}^{\pi} e^{2\pi i \sqrt{N}(a \sin \varphi + b \cos \varphi)} d\varphi. \]

Let us now compute the inner integral for all integers \(a\) and \(b\). If \((a, b) = (0, 0)\), then the integrand is equal to 1 and so the integral equals \(2\pi\). If, in contrast, \(a^2 + b^2 \neq 0\), then there exists some \(\theta \in [-\pi, \pi]\) such that

\[ \sin \theta = \frac{a}{\sqrt{a^2 + b^2}}, \cos \theta = \frac{b}{\sqrt{a^2 + b^2}}. \]

Therefore, \(a \sin \varphi + b \cos \varphi = \sqrt{a^2 + b^2} \cos(\varphi - \theta)\). Applying the change of variables \(\varphi = \varphi_1 + \theta\), we obtain

\[ \int_{-\pi}^{\pi} e^{2\pi i \sqrt{N}(a \sin \varphi + b \cos \varphi)} d\varphi = \int_{-\pi-\theta}^{\pi-\theta} e^{2\pi i \sqrt{N}(a^2 + b^2)} \cos \varphi_1 d\varphi_1. \]

Since the integrand is periodic with the period \(2\pi\), the last integral is equal to the integral of the same function over the interval \([-\pi, \pi]\). Thus, we finally find

\[ \int_{-\pi}^{\pi} e^{2\pi i \sqrt{N}(a \sin \varphi + b \cos \varphi)} d\varphi = \int_{-\pi}^{\pi} e^{2\pi i \sqrt{N}(a^2 + b^2)} \cos \varphi_1 d\varphi_1 = 2\pi J_0(2\pi \sqrt{N}(a^2 + b^2)). \]

Substituting the obtained result into the formula for \(I(N, M)\), we deduce the identity

\[ I(N, M) = \frac{1}{M} \sum_{(a,b) \in \mathbb{Z}} J_0(2\pi \sqrt{N}(a^2 + b^2)) e^{-\pi(a^2 + b^2)/M} = \frac{1}{M} S(N, M), \]

which was to be proved.

It turns out that if \(R(N)\) is large, then the quantity \(I(N, M)\) is rather small.

**Lemma 3.** If \(N\) and \(M\) are positive real numbers and \(N \geq 36\), then

\[ I(N, M) \leq e^{-\pi MR(N)^2 / 5N} + O(e^{-\pi M/4}) \]

**Proof.** Let us observe that if \(N \geq 36\) and for some \(\varphi \in [-\pi, \pi]\) we have

\[ ||\sqrt{N} \sin \varphi||^2 + ||\sqrt{N} \cos \varphi||^2 = c, \]

where \(||x||\) is the distance from \(x\) to the nearest integer, then \(c \geq \frac{R(N)^2}{5N}\). Indeed, there exist some integers \(a\) and \(b\) such that

\[ (\sqrt{N} \sin \varphi - a)^2 + (\sqrt{N} \cos \varphi - b)^2 = c \]

and \(|\sqrt{N} \sin \varphi - a| < 1, |\sqrt{N} \cos \varphi - b| < 1\). Consequently,
\[ |N - a^2 - b^2| = |(\sqrt{N} \sin \varphi - a)(\sqrt{N} \sin \varphi + a) + (\sqrt{N} \cos \varphi - b)(\sqrt{N} \cos \varphi + b)|. \]

On the other hand,
\[ (\sqrt{N} \sin \varphi - a)^2 + (\sqrt{N} \cos \varphi - b)^2 = c \]
and
\[ (\sqrt{N} \sin \varphi + a)^2 + (\sqrt{N} \cos \varphi + b)^2 \leq (2\sqrt{N} \sin \varphi + 1)^2 + (2\sqrt{N} \cos \varphi + 1)^2 = 4N + 4\sqrt{N}(\sin \varphi + \cos \varphi) + 2 \leq 4N + 4\sqrt{2N} + 2 \leq 5N. \]

Next, using the Cauchy-Bunyakovsky-Schwartz inequality, we find
\[ R(N) \leq |N - a^2 - b^2| \leq \sqrt{5cN}, \]
as needed.

Furthermore, it is easy to see that for any \( x \in \mathbb{R} \) the identity
\[ \vartheta_M(x) = e^{-\pi M ||x||^2} + O(e^{-\pi M/4}) \]
holds. Therefore,
\[ I(N, M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\pi M(||\sqrt{N} \sin \varphi||^2 + ||\sqrt{N} \cos \varphi||^2)} d\varphi + O(e^{-\pi M/4}). \]

But, as we showed before, for any \( \varphi \) we have \( ||\sqrt{N} \sin \varphi||^2 + ||\sqrt{N} \cos \varphi||^2 \geq \frac{R(N)^2}{5N} \).
Thus,
\[ I(N, M) \leq e^{-\pi MR(N)^2/5N} + O(e^{-\pi M/4}), \]
which was to be proved.

So, if \( R(N) \) is sufficiently large, then the quantity \( I(N, M) \) is close to 0. Therefore, the same is true for the sum \( S(N, M) \). For arbitrary \( x \), we split the sum \( S(x, M) \) into several parts:
\[ S(x, M) = 1 + \sum_{0 \leq c \leq \ln N} S_c(x, M) + S_{\infty}(x, M; N), \]
where for \( c \geq 0 \) we have
\[ S_c(x, M) = \sum_{cM < n \leq (c+1)M} r_2(n)J_0(2\pi \sqrt{xn})e^{-\pi n/M} \]
and
\[ S_{\infty}(x, M; N) = \sum_{n > M \ln N} r_2(n)J_0(2\pi \sqrt{xn})e^{-\pi n/M}. \]

The last sum is easily estimated in terms of \( N \) and \( M \):
Lemma 4. If \( N, M \geq 2 \), then for all \( x \geq 0 \) the inequality

\[
|S_{\infty}(x, M; N)| \ll \frac{M}{N^3}
\]
is true.

Proof. Due to the definition of Bessel function, for any real \( y \) we have \(|J_0(y)| \leq 1\). Consequently,

\[
|S_{\infty}(x, M; N)| \leq \sum_{n > M \ln N} r_2(n)e^{-\frac{\pi n}{M}}.
\]

It is clear that the last sum equals

\[
\sum_{c=0}^{\infty} S^*_c(N, M),
\]

where

\[
S^*_c(N, M) = \sum_{M \ln N + cM < n \leq M \ln N + (c+1)M} r_2(n)e^{-\frac{\pi n}{M}}.
\]

For any \( y > 0 \) we have

\[
\sum_{0 < n \leq y} r_2(n) \ll y,
\]

thus, for the sum \( S^*_c(N, M) \) the estimate

\[
S^*_c(N, M) \ll e^{-\frac{\pi (M \ln N + cM)}{M}} \sum_{M \ln N + cM < n \leq M \ln N + (c+1)M} r_2(n) \ll e^{-cN - \frac{\pi}{M}((c + 1)M + M \ln N)}
\]

holds. Summing these inequalities over all values of \( c \), we obtain the required inequality.

Consider next the integral

\[
J(N, M) = \int_0^{+\infty} (S(x, M) - 1)^2 e^{-\frac{\pi x}{N}} dx.
\]

Later we will prove the following estimate for \( J(N, M) \) in the case when \( M \leq \frac{N}{2 \ln^2 N} \):

Lemma 5. If \( M, N \geq 2 \) and \( \frac{N}{2 \ln^2 N} \geq M \), then the inequality

\[
J(N, M) \ll \sqrt{NM} \ln N
\]

holds.

To prove this lemma, we need three more propositions. First of them is very well known:
**Lemma 6.** For any $x \geq 1$ we have

$$\sum_{0 < n < x} r_2^2(n) \ll x \ln x$$

We also need Weber’s second exponential integral to find an explicit expression for the quantity $J(N, M)$:

**Lemma 7.** For arbitrary $\alpha, \beta, \gamma > 0$ the formula

$$\int_{0}^{+\infty} e^{-\alpha x} J_0(2\beta \sqrt{x}) J_0(2\gamma \sqrt{x}) dx = \frac{1}{\alpha} I_0 \left( \frac{2\beta \gamma}{\alpha} \right) \exp \left( -\frac{\beta^2 + \gamma^2}{\alpha} \right)$$

is true, where $I_0(x) = J_0(ix)$ is the modified Bessel function.

**Proof.**
See [5].

Furthermore, we will use the asymptotic formula for the modified Bessel function.

**Lemma 8.** For any positive real $x$ we have the following asymptotic formula:

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

**Proof.**
Cf. [5].

**Proof of Lemma 5.**
Due to the fact that $S(x, M) - 1$ is decomposed into the sum of $S_c(x, M)$ over all $0 \leq c \leq \ln N$ and $S_\infty(x, M; N)$, it suffices to estimate the integrals

$$J_c(N, M) = \int_{0}^{+\infty} S_c(x, M)^2 e^{-\pi x/N} dx$$

and

$$J_\infty(N, M) = \int_{0}^{+\infty} S_\infty(x, M; N)^2 e^{-\pi x/N} dx.$$ 

The last quantity is easily estimated with the help of Lemma 4. Indeed, for all real $x$ we have

$$|S_\infty(x, M; N)| \ll \frac{M}{N^3}.$$ 

Therefore,

$$J_\infty(N, M) \ll \int_{0}^{+\infty} \frac{M^2}{N^6} e^{-\pi x/N} dx \ll \frac{M^2}{N^5} \ll \sqrt{MN} \ln N.$$ 

Now, let us prove for any $\ln N \geq c \geq 0$ the inequality

$$J_c(N, M) \ll \sqrt{(c + 1)} e^{-2\pi c \sqrt{NM}} \ln N + N^{-4}.$$ 

Using Lemma 7, we find the explicit expression for the $J_c(N, M)$:
\[ J_c(N, M) = \int_0^{+\infty} \sum_{cM < n, m \leq (c+1)M} r_2(n) r_2(m) e^{-\pi(n+m)/M - \pi x/N} J_0(2\pi\sqrt{nx}) J_0(2\pi\sqrt{mx}) \, dx = \]

\[ = \sum_{cM < n, m \leq (c+1)M} r_2(n) r_2(m) e^{-\pi(n+m)/M} \frac{N}{\pi} I_0(2\pi\sqrt{nmN}) \exp(-\pi(n+m)N) \]

Consider the sums over \( n = m \) and \( n \neq m \) separately. We get

\[ J_c(N, M) = \frac{N}{\pi} (S_1 + S_2), \]

where

\[ S_1 = \sum_{cM < n \leq (c+1)M} r_2(n)^2 e^{-2\pi n/M} I_0(2\pi nN) \exp(-2\pi nN) \]

and

\[ S_2 = \sum_{cM < n \neq m \leq (c+1)M} r_2(n) r_2(m) e^{-\pi(n+m)/M} I_0(2\pi\sqrt{nmN}) \exp(-\pi(n+m)N). \]

Let us estimate the sum \( S_1 \) first.

By the Lemma 8, the inequality

\[ I_0(2\pi nN) \exp(-2\pi nN) \ll \frac{1}{\sqrt{nN}} \]

holds. Therefore,

\[ S_1 \ll \sum_{cM < n \leq (c+1)M} \frac{r_2(n)^2}{\sqrt{nN}} e^{-2\pi n/M} \leq e^{-2\pi c} \sum_{0 < n \leq (c+1)M} \frac{r_2(n)^2}{\sqrt{nN}} \ll \]

\[ \ll \sqrt{(c+1)e^{-2\pi c}} \frac{\sqrt{M}}{\sqrt{N}} \ln N. \]

Now, consider arbitrary summand in the sum \( S_2 \).

Due to the Lemma 8, we find

\[ I_0(2\pi\sqrt{nmN}) \exp(-\pi(n+m)N) \ll \frac{\exp(-\pi N(\sqrt{n} - \sqrt{m})^2)}{n^{1/4}m^{1/4}\sqrt{N}}. \]

From the relations \( n \neq m \) and \( n, m < (c+1)M \) we deduce

\[ (\sqrt{n} - \sqrt{m})^2 = \frac{(n - m)^2}{(\sqrt{n} + \sqrt{m})^2} \geq \frac{1}{4(c+1)M}. \]

Consequently, if \( n \neq m \) and \( n, m < (c+1)M \), then

\[ I_0(2\pi\sqrt{nmN}) \exp(-\pi(n+m)N) \ll \frac{1}{n^{1/4}m^{1/4}\sqrt{N}} \exp\left(-\frac{\pi N}{4(c+1)M}\right) \ll \]
\[
\ll \frac{1}{n^{1/4}m^{1/4}\sqrt{N}} \exp \left( -\frac{\pi \ln^2 N}{2(c + 1)} \right).
\]

So, we have the inequality
\[
S_2 \ll \sum_{cM < n, m < (c+1)M} \frac{r_2(n)r_2(m)}{n^{1/4}m^{1/4}\sqrt{N}} e^{-\pi(n+m)/M} \exp \left( -\frac{\pi \ln^2 N}{2(c + 1)} \right).
\]

Therefore,
\[
S_2 \ll \frac{1}{\sqrt{N}} \exp \left( -\frac{\pi \ln^2 N}{2(c + 1)} \right) \left( \sum_{cM < n < (c+1)M} \frac{r_2(n)}{n^{1/4}} e^{-\pi n/M} \right)^2.
\]

Estimating the last sum, we obtain
\[
S_2 \ll \frac{(c + 1)^{3/2}M^{3/2}}{\sqrt{N}} \exp \left( -\frac{\pi \ln^2 N}{2(c + 1)} - 2\pi c \right).
\]

Furthermore, the inequality \( c \leq \ln N \) implies \((c + 1)^{3/2}M^{3/2} \ll N^{3/2}\). Also, by the inequality of arithmetic and geometric means,
\[
-\frac{\pi \ln^2 N}{2(c + 1)} - 2\pi c = 2\pi - \frac{\pi \ln^2 N}{2(c + 1)} - 2\pi(c + 1) \leq 2\pi - 2\pi \ln N.
\]

Thus,
\[
S_2 \ll N \exp(-2\pi \ln N) = N^{1-2\pi}.
\]

From this we finally deduce the bound
\[
J_c(N, M) \ll N(S_1 + S_2) \ll \sqrt{(c + 1)e^{-2\pi c}} \sqrt{NM \ln N + N^{2-2\pi}} \ll \sqrt{(c + 1)e^{-2\pi c}} \sqrt{NM \ln N + N^{-4}}.
\]

Now, as the function \( e^{-\pi x/N} \) is positive, the inequality
\[
\sqrt{J(N, M)} \leq \sum_{0 \leq c \leq \ln N} \sqrt{J_c(N, M)} + \sqrt{J_\infty(N, M)}
\]
holds. Therefore,
\[
\sqrt{J(N, M)} \ll \sum_{0 \leq c \leq \ln N} ((c + 1)^{1/4}e^{-\pi c}(NM)^{1/4}\sqrt{\ln N + N^{-2}} + (NM)^{1/4}\sqrt{\ln N}
\]
and so,
\[
J(N, M) \ll \sqrt{NM \ln N}.
\]

This concludes the proof.

The estimate for the quantity \( J(N, M) \) implies the upper bound for the measure of \( N \in [0, x] \) such that \( R(N) \) is large.
Lemma 9. Let $x, H > 3$ and denote by $M(H, x)$ the set of all real $y \leq x$ with $R(y) \geq H$. Then the inequality

$$\mu(M(H, x)) \ll \frac{x(\ln x)^{3/2}}{H}$$

holds, where $\mu$ is the Lebesgue measure.

Proof.

As this bound is trivial for $H \ll (\ln x)^{3/2}$, we can assume that $H \gg (\ln x)^{3/2}$. Now, suppose that $x^{1/2} \ll M = \frac{2x \ln x}{H^2} \ll \frac{x}{\ln^2 x}$. If $y \in M(H, x)$, then due to the Lemma 3 the inequality

$$I(y, M) \ll \exp\left(-\frac{\pi MR(y)^2}{5y}\right) + O(e^{-\pi M/4}) \ll \exp\left(-\frac{2\pi}{5} \ln x\right) \ll x^{-5/4}$$

is true. Therefore,

$$S(y, M) = MI(y, M) \ll \frac{M}{x^{5/4}} \ll \frac{1}{x^{1/4}}$$

and so, for any $y \in M(x, H)$ we have

$$|S(y, M) - 1| \gg 1.$$ 

Thus,

$$J(x, M) \gg \int_{M(x, H)} (S(y, M) - 1)^2 e^{-\pi y/N} dy \gg \mu(M(x, H)).$$

Consequently, by the Lemma 5,

$$\mu(M(x, H)) \ll \sqrt{xM \ln x} = \frac{\sqrt{2x(\ln x)^{3/2}}}{H},$$

which was to be proved.

From this last lemma we deduce the Theorem 4.

Proof of Theorem 4.

Observe that

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^\gamma \ll \int_0^x R(t)^{\gamma - 1} dt.$$ 

Indeed, for any positive integer $n$ we have

$$\int_{s_n}^{s_{n+1}} R(t)^{\gamma - 1} dt = \int_{s_n}^{s_{n+1}} \min(t - s_n, s_{n+1} - t)^{\gamma - 1} dt = \frac{(s_{n+1} - s_n)^\gamma}{2^{\gamma - 1} \gamma}.$$ 

Summation over all $n$ with condition $s_{n+1} \leq x$ gives the desired result.

Let $k$ be some positive integer. Consider the set $B_k \subset [0, x]$ of all real $y$ with $2^k \leq R(y) \leq 2^{k+1}$.

Due to the Lemma 9, $\mu(B_k) \ll x(\ln x)^{3/2} 2^{-k}$. Therefore,

$$\int_{B_k} R(t)^{\gamma - 1} dt \leq 2^{(k+1)(\gamma - 1)} \mu(B_k) \ll x(\ln x)^{3/2} 2^{(\gamma - 2)k}.$$
Let $B$ be the set of $y \in [0, x]$ such that $R(y) \leq c(\ln x)^{3/2}$ with some sufficiently large constant $c$. Trivially, we have $B \subset [0, x]$ and so $\mu(B) \leq x$. Consequently,

$$\int_B R(t)^{\gamma-1} dt \ll \mu(B)(\ln x)^{3/2(\gamma-1)} \leq x(\ln x)^{3/2(\gamma-1)}.$$ 

Due to the fact that on the interval $[0, x]$ the inequality $R(x) \leq x$ holds, we have

$$[0, x] = B \cup \bigcup_{x \geq 2^k \geq c(\ln x)^{3/2}} B_k.$$ 

Therefore,

$$\int_0^x R(t)^{\gamma-1} dt = \int_B R(t)^{\gamma-1} dt + \sum_{x \geq 2^k \geq c(\ln x)^{3/2}} \int_{B_k} R(t)^{\gamma-1} dt.$$ 

Furthermore, we have

$$\int_0^x R(t)^{\gamma-1} dt \ll x(\ln x)^{3/2(\gamma-1)} + \sum_{x \geq 2^k \geq c(\ln x)^{3/2}} x(\ln x)^{3/2(\gamma-2)k} \ll x(\ln x)^{3/2(\gamma-1)} \delta(x, \gamma).$$

This concludes the proof of Theorem 4.

§ 3. Conclusion

In this work, we constructed certain sum of Bessel functions that is unusually small in the points that are far from numbers that are sums of two squares. The estimate for some mean value of this sum allowed us to prove the upper bound for the measure of the set of points with this property. However, we were not able to prove sharp enough bound for the sum $S(N, M) - 1$. Nontrivial estimates for this quantity would allow us to improve the exponent in the inequality $R(x) \ll x^{1/4}$. One can show that our construction works not only for the sums of two squares, but also for the set of values of arbitrary positive definite quadratic form with integer coefficients. It would be also interesting to generalize this construction to the case of indefinite forms.

References

[1] R. P. Bambah, Chowla, S., «On numbers which can be expressed as a sum of two squares», Proc. Nat. Acad. Sci. India, 13, 1947, 101-103.
[2] C. Hooley, «On the intervals between numbers that are sums of two squares I», Acta Math., 127, 1971, 279-297
[3] A. A. Karatsuba, «Euler and number theory», Proc. Steklov Inst. Math., 274, suppl. 1 (2011), 169-179
[4] I. Richards, «On the gaps between numbers which are sums of two squares», Advances in Math, 46, 1982, 1-2
[5] G. N. Watson, «A Treatise on the Theory of Bessel Functions» (2nd.ed.), Cambridge University Press, 1966
Alexander Kalmynin
National Research University Higher School of Economics, Russian Federation, Math Department, International Laboratory of Mirror Symmetry and Automorphic Forms
E-mail: alkalb1995cd@mail.ru