Apéry-Type Series with Summation Indices of Mixed Parities and Colored Multiple Zeta Values, I

Ce Xu\textsuperscript{a,*} and Jianqiang Zhao\textsuperscript{b,†}

\textsuperscript{a.} School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, PRC
\textsuperscript{b.} Department of Mathematics, The Bishop’s School, La Jolla, CA 92037, USA

Abstract. In this paper, we shall study Apéry-type series in which the central binomial coefficient appears as part of the summand. Let \( b_n = \frac{4^n}{\binom{2n}{n}} \). Let \( s_1, \ldots, s_d \) be positive integers with \( s_1 \geq 2 \). We consider the series

\[
\sum_{n_1 > \cdots > n_d > 0} \frac{b_{n_1}}{n_1^{s_1} \cdots n_d^{s_d}}
\]

and the variants with some or all indices \( n_j \) replaced by \( 2n_j \pm 1 \) and some or all “\( > \)” replaced by “\( \geq \)”, provided the series are defined. We can also replace \( b_{n_1} \) by its square in the above series when \( s_1 \geq 3 \). The main result is that all such series are \( \mathbb{Q} \)-linear combinations of the real and/or the imaginary parts of colored multiple zeta values of level 4, i.e., multiple polylogarithms evaluated at 4th roots of unity.

Keywords: Apéry-type series, colored multiple zeta values, mixed parities, iterated integrals.

AMS Subject Classifications (2020): 11M32, 11B65, 11B37, 44A05, 33B30.

1 Introduction

In his celebrated proof of irrationality of \( \zeta(2) \) and \( \zeta(3) \) in 1979, Apéry used crucially the following two identities

\[
\zeta(2) = 3 \sum_{n \geq 1} \frac{1}{n^2 \binom{2n}{n}} \quad \text{and} \quad \zeta(3) = \frac{5}{2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.
\]  

(1.1)

Motivated by Apéry’s proof, Leshchiner \cite{14} generalized these to higher weight Riemann zeta values and some other analogs. However, no irrationality proof has been found so far for other Riemann zeta values at odd positive integers greater than 4, although a lot of progress has been made (see e.g., \cite{13, 17, 22}. In particular, in 2020, Lai and Yu \cite{13} proved that for any small \( \varepsilon > 0 \), the number of irrationals among the following odd zeta values: \( \zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s) \), is at least \((c_0 - \varepsilon) \sqrt{s / \log(s)}\), provided \( s \) is a sufficiently large odd integer with respect to \( \varepsilon \), with constant \( c_0 = 1.192507 \ldots \).

On the other hand, series generalizing those on the right-hand side of (1.1), including odd-indexed variations (see Remark \cite{4, 2}) have appeared in the calculations of the \( \varepsilon \)-expansions of the Feynman diagrams in recent years (see, e.g., \cite{8, 9, 10}).

\*Email: cexu2020@ahnu.edu.cn, corresponding author, ORCID 0000-0002-0059-7420.
\†Email: zhaoj@ihes.fr, ORCID 0000-0003-1407-4230.
In the meantime, the frequent and sometimes unexpected appearance of colored multiple zeta values (see (1.2)) in quite a few different branches of mathematics and physics has attracted the attention of many mathematicians and physicists alike. These numbers, as vast generalizations of Riemann zeta values, are all conjectured to be not only irrational but also transcendental. One naturally wonders if the multiple sums, which we call Apéry-type series, that generalize those in (1.1) can be related to these numbers. In a series of papers, we will answer some of these questions. As part I of this series, this paper concentrates on Apéry-type series such as those defined by (4.11) and (4.21) in which the central binomial coefficients appear only on the denominators. We will show that a large class of these series can be expressed as \( \mathbb{Q} \)-linear combinations of the real and/or the imaginary parts of the colored multiple zeta values of level 4, i.e., multiple polylogarithms evaluated at 4th roots of unity, see Thm. 4.1. Some related results may be found in [4, 11, 18, 19] and references therein.

1.1 Notation.

Let \( \mathbb{N} \) be the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). A finite sequence \( s := (s_1, \ldots, s_d) \in \mathbb{N}^d \) is called a composition. We define the weight and the depth of \( s \) by

\[
|s| := s_1 + \cdots + s_d, \quad \text{and} \quad \text{dp}(s) := d,
\]

respectively. For any \( N \)th roots of unity \( z_1, \ldots, z_d \) the colored multiple zeta values (CMZVs) of level \( N \) are defined by

\[
\text{Li}_s(z) := \sum_{n_1 > \cdots > n_d > 0} \frac{z_1^{n_1} \cdots z_d^{n_d}}{n_1^{s_1} \cdots n_d^{s_d}}, \tag{1.2}
\]

which converge if \( (s_1, z_1) \neq (1, 1) \) (see [16] and [21, Ch. 15]), in which case we call \( (s; z) \) admissible. The multiple zeta values are CMZVs of level 1, namely, \( \zeta(s) := \text{Li}_{(1_d)} \) where \( 1_d \) is the string of 1’s with \( d \) repetitions. Moreover, CMZVs can be expressed using Chen’s iterated integrals

\[
\text{Li}_s(z) = \int_0^1 a^{s_1-1} x_{\xi_1} \cdots a^{s_d-1} x_{\xi_d}, \tag{1.3}
\]

where \( \xi_j := \prod_{i=1}^j z_i^{-1} \), \( a := dt/t \) and \( x_{\xi} := dt/(\xi - t) \) for any \( N \)th roots of unity \( \xi \), see [21, Sec. 2.1] for a brief summary of this theory. The theory of iterated integrals was developed first by K.T. Chen in the 1960’s. It has played important roles in the study of algebraic topology and algebraic geometry in the past half century. Its simplest form is

\[
\int_0^1 f_1(t) dt f_2(t) dt \cdots f_p(t) dt = \int_0^1 f_1(t) dt \circ f_2(t) dt \circ \cdots \circ f_p(t) dt
\]

\[
:= \int_{1 > t_1 > \cdots > t_p > 0} f_1(t_1) f_2(t_2) \cdots f_p(t_p) dt_1 dt_2 \cdots dt_p.
\]

One can extend these to iterated integrals over any piecewise smooth path on the complex plane via pull-backs. We refer the interested reader to Chen’s original work [6, 7] for more details.

1.2 Akhilesh’s result.

In [2, 3] Akhilesh discovered some very important and surprising connections between MZVs and the following Apéry-type series (which he calls multiple Apéry-like sums and which are
normalized slightly differently here)

\[ \sigma(s; x) := \sum_{n_1 > n_2 > \cdots > n_d > 0} \binom{2n_1}{n_1}^{-1} \frac{(2x)^{2n_1}}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}} \]  

(1.4)

His ingenious idea is to study the \( n \)-tails (and more generally, double tails) of such series. We reformulate one of his key results as follows to make it more transparent. Set

\[ g_s(t) = \begin{cases} \tan t \, dt, & \text{if } s = 1; \\ dt \circ (\cot t \, dt)^{s-2} \circ dt, & \text{if } s \geq 2, \end{cases} \]  

(1.5)

and their non-trigonometric counterpart

\[ G_s(t) = \begin{cases} \omega_2 & \text{if } s = 1; \\ \omega_1 \omega_0^{s-2} \omega_1 & \text{if } s \geq 2, \end{cases} \]  

(1.6)

where \( \omega \)'s are defined by (2.1). Further, we set \( \binom{0}{0} = 1 \),

\[ b_n(x) = 4^n \binom{2n}{n}^{-1} x^{2n} \quad \text{and} \quad b_n = b_n(1) = 4^n \binom{2n}{n}^{-1} \quad \forall n \geq 0. \]

**Theorem 1.1.** ([3 Thm. 4]) For all \( n \in \mathbb{N}_0 \), \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \) we have

\[ \sigma(s; \sin y)_n := \sum_{n_1 > \cdots > n_d > n} \frac{b_{n_1}(\sin y)}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}} = \frac{d}{dy} \int_0^y g_{s_1} \circ \cdots \circ g_{s_d} \circ b_n(\sin t) \, dt, \]

where \( y \in (-\pi/2, \pi/2) \) if \( s_1 = 1 \) and \( y \in [-\pi/2, \pi/2] \) if \( s_1 \geq 2 \). Using non-trigonometric 1-forms, we have for all \( x \in (-1, 1) \)

\[ \sigma(s; x)_n := \sum_{n_1 > \cdots > n_d > n} \frac{b_{n_1}(x)}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}} = \sqrt{1 - x^2} \frac{d}{dx} \int_0^x G_{s_1} \circ \cdots \circ G_{s_d} \circ b_n(t) \omega_1. \]

We will again call the sum \( |s| := s_1 + \cdots + s_d \) the weight of the series \( \sigma(s; \sin y) \) and \( d \) the depth.

### 1.3 Main results.

In Thm 2.3 and Thm 3.1 we will show that the tails of the series

\[ \tau^*(s; x)_n := \sum_{n_1 \geq \cdots \geq n_d \geq n} \binom{2n_1}{n_1}^{-1} \frac{(2x)^{2n_1}}{(2n_1 + 1)^{s_1} \cdots (2n_d + 1)^{s_d}}, \]

\[ \chi(s; x)_n := \sum_{n_1 > \cdots > n_d > n} \binom{2n_1}{n_1}^{-1} \frac{(2x)^{2n_1}}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}} \]

can be written as iterated integrals similar to the one in Thm 1.1. As corollaries, we show that the series \( \sigma(s; 1)_0, \tau^*(s; 1)_0, \chi(s; 1)_0 \) and more general similar series defined by (4.11) and (4.21) with summation indices of any parity pattern can be expressed as \( Q \)-linear combinations of the real and/or the imaginary parts of CMZVs of level 4.
We also consider other Apéry-type series similar to the above by using the square of central binomial coefficients such as

\[
\sigma^{(2)}(s) := \sum_{n_1 > \ldots > n_d > 0} \binom{2n_1}{n_1}^{-2} \frac{16^{n_1}}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}},
\]

and show that they also lie in the \( \mathbb{Q} \)-vector space of CMZVs of level 4.

\section{First variant with odd summation indices}

In this section, we consider a variation of the Apéry-type series studied in [3] by restricting the summation indices to odd numbers only and replacing strict inequalities among them by non-strict ones. Concerning this we have the next well-known result. Define

\[
\omega_0 := \frac{dt}{t}, \quad \omega_1 := \frac{dt}{\sqrt{1-t^2}}, \quad \omega_2 := \frac{t \, dt}{1-t^2},
\]

\[
\omega_3 := \frac{dt}{t \sqrt{1-t^2}}, \quad \omega_5 := \frac{t \, dt}{1-t^2}, \quad \omega_8 := \frac{dt}{1-t^2}.
\]

\begin{lemma}
For all \( d \in \mathbb{N} \) and \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \), we have

\[
ti_s(x)_n := \sum_{n_1 > \ldots > n_d \geq n} \frac{x^{2n_1+1}}{(2n_1+1)^{s_1} \cdots (2n_d+1)^{s_d}} = \int_0^x \omega_0^{s_1-1} \omega_2^{s_2-1} \cdots \omega_8^{s_8-1} \omega_0^{s_d-1} (t^2 \omega_8).
\]
\end{lemma}

\begin{proof}
This follows easily by direct computation. \qed
\end{proof}

For all \( n \in \mathbb{N}_0 \) and \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \) we define

\[
\tau(s; x)_n := \sum_{n_1 > \ldots > n_d \geq n} \binom{2n_1}{n_1}^{-1} \frac{(2x)^{2n_1}}{(2n_1+1)^{s_1} \cdots (2n_d+1)^{s_d}},
\]

\[
\tau^*(s; x)_n := \sum_{n_1 > \ldots > n_d \geq n} \binom{2n_1}{n_1}^{-1} \frac{(2x)^{2n_1}}{(2n_1+1)^{s_1} \cdots (2n_d+1)^{s_d}}.
\]

\begin{theorem}
For all \( d \in \mathbb{N} \) and \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \), the tail

\[
\tau(s; 1/2)_n = \int_0^{1/2} \frac{4t}{\sqrt{1-4t^2}} \omega_0^{s_1-1} \omega_2^{s_2-1} \cdots \omega_8^{s_8-1} \omega_0^{s_d-1} (t^2 \omega_8).
\]
\end{theorem}

\begin{proof}
Note that

\[
\int_0^{1/2} \frac{4t}{\sqrt{1-4t^2}} t^{2n} \, dt = \int_0^{1/4} \frac{2t^n}{\sqrt{1-4t^2}} \, dt = \frac{\Gamma(n+1)^2}{\Gamma(2n+2)} = \binom{2n}{n}^{-1} \frac{1}{2n+1}.
\]

Thus

\[
\sum_{n_1 > \ldots > n_d \geq n} \binom{2n_1}{n_1}^{-1} \frac{1}{(2n_1+1)^{s_1} (2n_2+1)^{s_2} \cdots (2n_d+1)^{s_d}}
\]

\end{proof}
iterated integrals by combining 1-forms and functions as follows. For any $f$

Hence

Observe that

as desired.

It turns out that the star version $\tau^*$ behaves better. To study this, we will extend Chen’s iterated integrals by combining 1-forms and functions as follows. For any $r \in \mathbb{N}$, 1-forms $f_1(t) \, dt, \ldots, f_{r+1}(t) \, dt$ and functions $F_1(t), \ldots, F_r(t)$, we define recursively

$$
\begin{align*}
&\int_0^1 (f_1(t) \, dt + F_1(t)) \circ \cdots \circ (f_r(t) \, dt + F_r(t)) \circ f_{r+1}(t) \, dt \\
&:= \int_0^1 (f_1(t) \, dt + F_1(t)) \circ \cdots \circ (f_{r-1}(t) \, dt + F_{r-1}(t)) \circ f_r(t) \, dt \circ f_{r+1}(t) \, dt \\
&+ \int_0^1 (f_1(t) \, dt + F_1(t)) \circ \cdots \circ (f_{r-1}(t) \, dt + F_{r-1}(t)) \circ (F_r(t) \circ f_{r+1}(t)) \, dt.
\end{align*}
$$

We now set the 1-forms

$$h_s(t) = \begin{cases} 2 \csc 2t \, dt, & \text{if } s = 1; \\ \csc t \, dt \circ (\cot t \, dt)^{s-2} \circ \csc t \, dt, & \text{if } s \geq 2, \end{cases} \quad (2.3)$$

and their non-trigonometric counterpart

$$H_s(t) = \begin{cases} \omega_2 := \omega_0 + \omega_2, & \text{if } s = 1; \\ \omega_3 \omega_0^{-2} \omega_3, & \text{if } s \geq 2. \end{cases} \quad (2.4)$$

**Theorem 2.3.** For all $n \in \mathbb{N}_0$, $s = (s_1, \ldots, s_d) \in \mathbb{N}^d$ we have the tail

$$\tau^*(s; \sin y)_n = \sum_{n_1 \geq \cdots \geq n_d \geq n} \frac{b_{n_1}(\sin y)}{(2n_1 + 1)^{s_1} \cdots (2n_d + 1)^{s_d}} = \frac{d}{dy} \int_0^y h_{s_1} \circ \cdots \circ h_{s_d} \circ b_n(\sin t) \, dt. \quad (2.5)$$

Hence

$$\tau^*(s; x)_n = \sqrt{1 - x^2} \frac{d}{dx} \int_0^x H_{s_1} \circ \cdots \circ H_{s_d} \circ b_n(\sin t) \, dt \omega_1.$$

**Proof.** When $d = s_1 = 1$, the right-hand side of (2.5) is equal to

$$\frac{d}{dy} \int_0^y 2 \csc 2t \, dt b_n(\sin t) \, dt = \sec y \csc y \int_0^y b_n(\sin t) \, dt. \quad (2.6)$$

Observe that

$$\left( \frac{2k + 2}{k + 1} \right)^{-1} \frac{1}{k + 1} = \frac{(k + 1)!^2}{(2k + 2)(2k + 1)(2k)!} \frac{1}{k + 1} = \frac{1}{2} \binom{2k}{k}^{-1} \frac{1}{2k + 1}.$$
By Thm. 1.1 we get
\[
\sum_{n_1 \geq n} \left( \frac{2n_1 + 2}{n_1 + 1} \right)^{-1} \frac{(4 \sin^2 y)^{n_1+1}}{n_1 + 1} = 2 \tan y \int_0^y b_n(\sin t) \, dt.
\]

Hence
\[
\sum_{n_1 \geq n} \frac{b_{n_1} \sin^{2n_1+1} y}{2n_1 + 1} = \sec y \int_0^y b_n(\sin t) \, dt,
\] (2.7)

which is exactly (2.6). Thus the case \( d = s_1 = 1 \) of the theorem is proved.

Now, repeatedly multiplying (2.7) by \( \cot y \) and integrating \( s - 1 \) times, we get
\[
\sum_{n_1 \geq n} b_{n_1} \sin^{2n_1+1} y (2n_1 + 1)^s = \int_0^y (\cot t \, dt)^{s-2} (\csc t \, dt) (b_n(\sin t) \, dt),
\] (2.8)

Replacing \( n \) by \( n_2 \), multiplying by \( 1/(2n_2 + 1) \) and taking the sum \( \sum_{n_1 \geq n} \), we get
\[
\sum_{n_1 \geq n_2 \geq n} \frac{b_{n_1} \sin^{2n_1+1} y}{(2n_1 + 1)^s (2n_2 + 1)^s} = \int_0^y (\cot t \, dt)^{s-2} (\csc t \, dt) \sum_{n_2 \geq n} \frac{b_{n_2}(\sin t)}{2n_2 + 1} \, dt
\]
\[
= 2 \int_0^y (\cot t \, dt)^{s-2} (\csc t \, dt) (\csc 2t \, dt) (b_n(\sin t) \, dt).
\]

Multiplying by \( 1/(2n_2 + 1)^{s_2} \) for \( s_2 \geq 2 \) we get
\[
\sum_{n_1 \geq n_2 \geq n} \frac{b_{n_1} \sin^{2n_1+1} y}{(2n_1 + 1)^{s_1} (2n_2 + 1)^{s_2}} = \int_0^y (\cot t \, dt)^{s_1-2} (\csc t \, dt) \sum_{n_2 \geq n} \frac{b_{n_2}(\sin t)}{(2n_2 + 1)^{s_2}} \, dt
\]
\[
= \int_0^y (\cot t \, dt)^{s_1-2} (\csc t \, dt)(\csc t \, dt)(\cot t \, dt)^{s_2-2} (\csc t \, dt) (b_n(\sin t) \, dt).
\]

The theorem follows from doing these repeatedly and can be easily proved by induction. We leave the details to the interested reader. \( \square \)

Corollary 2.4. For all admissible \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \) with \( s_1 \geq 2 \), we have
\[
t^*(s)_n := \sum_{n_1 \geq \cdots \geq n_d \geq n} \frac{1}{(2n_1 + 1)^{s_1} \cdots (2n_d + 1)^{s_d}} = \frac{2}{\pi} \int_0^{\pi/2} h_{s_1} \circ \cdots \circ h_{s_d} \circ b_n(\sin t) \, dt.
\]

In particular,
\[
t^*(s) := \sum_{n_1 \geq \cdots \geq n_d \geq 0} \frac{1}{(2n_1 + 1)^{s_1} \cdots (2n_d + 1)^{s_d}} = \frac{2}{\pi} \int_0^{\pi/2} h_{s_1} \circ \cdots \circ h_{s_d} \, dt.
\]
Proof. Integrating (2.5) over $(0, \pi/2)$ and noticing the fact that
\[
\int_0^{\pi/2} (\sin t)^{2n} dt = \frac{\pi}{2b_n},
\]
we obtain the corollary immediately. \qed

Example 2.5. Let $s = (2^d)$ be the string of 2’s with $d$ repetitions. Then we see that
\[
t^*(2^d) = \frac{2}{\pi} \int_0^{\pi/2} \left( \frac{dt}{\sin t} \right)^{2d} dt = \frac{2}{\pi} i \left( \text{Li}_{2d+1}(-i) - \text{Li}_{2d+1}(i) \right) = \frac{4}{\pi} \beta(2d + 1),
\]
where $\beta$ is the Dirichlet beta function
\[
\beta(s) = \sum_{k \geq 0} \frac{(-1)^k}{(2k + 1)^s}.
\]
Moreover, we have
\[
t^*(2^a, 3, 2^b) = \frac{2}{\pi} \int_0^{\pi/2} (\csc t dt)^{2a + 1} \circ \cot t dt \circ (\csc t dt)^{2b + 1} \circ dt,
\]
where $a, b \in \mathbb{N}_0$.

Remark 2.6. In [15], T. Murakami first proved the analog of Zagier’s 2-3-2 formula of MZVs for multiple $t$-values $t(2^a, 3, 2^b)$. Thereafter, several other proofs have appeared in the literature, see for example [12].

Proposition 2.7. For every positive integer $d$, we have
\[
\tau^*(1^d; \sin y) = 2 \csc 2y \text{ Im } \text{Li}_d(i \tan y),
\]
\[
\tau^*(2^d; \sin y) = 2 \csc y \text{ Im } \text{Li}_{2d}(i \tan(y/2)).
\]

Proof. By Thm. 2.3 we obtain
\[
\tau^*(1^d; \sin y) = 2 \csc 2y \int_0^{\pi/2} \left( \frac{dt}{\sin 2t} \right)^{d-1} dt = \frac{2 \csc 2y}{(d-1)!} \int_0^{\pi/2} \left( \frac{dx}{\sin 2x} \right)^{d-1} dt
\]
\[
= \frac{2 \csc 2y}{(d-1)!} \int_0^{\pi/2} \log^{d-1} \frac{\csc 2t + \cot 2t}{\csc 2y + \cot 2y} dt
\]
\[
= \frac{2 \csc 2y}{(d-1)!} \int_0^{\pi/2} \log^{d-1} | \tan y \cot t | dt. \tag{2.10}
\]
By routine differentiation and induction it can be proved easily that
\[
\frac{1}{(d-1)!} \int_0^{\pi/2} \log^{d-1} | \tan y \cot t | dt = \text{ Im } \text{Li}_d(i \tan y) = \frac{i}{2} \left( \text{Li}_d(-i \tan y) - \text{Li}_d(i \tan y) \right)
\]
for all $d \geq 1$. To see that both sides $\to 0$ as $y \to 0$ we can use (2.10). Similarly,
\[
\tau^*(2^d; \sin y) = \csc y \int_0^{\pi/2} \left( \frac{dt}{\sin t} \right)^{2d-1} dt = \frac{\csc y}{(2d-1)!} \int_0^{\pi/2} \left( \frac{dx}{\sin x} \right)^{2d-1} dt
\]
\[
= \frac{\csc y}{(2d-1)!} \int_0^{\pi/2} \log^{2d-1} | \tan y/2 \cot t/2 | dt = 2 \csc y \text{ Im } \text{Li}_{2d}(i \tan(y/2)).
\]
This completes the proof of the proposition. \qed
Example 2.8. Specializing at \( y = \pi/4 \) and \( \pi/2 \) in the two identities of Prop. 2.7 respectively, we see that

\[
\sum_{n_1 \geq \cdots \geq n_d \geq 0} \left( \frac{2n_1}{n_1} \right)^{-1} \frac{2^{n_1}}{(2n_1 + 1) \cdots (2n_d + 1)} = 2 \text{Im} \operatorname{Li}_d(i) = 2\beta(d),
\]

\[
\sum_{n_1 \geq \cdots \geq n_d \geq 0} \left( \frac{2n_1}{n_1} \right)^{-1} \frac{4^{n_1}}{(2n_1 + 1)^2 \cdots (2n_d + 1)^2} = 2 \text{Im} \operatorname{Li}_{2d}(i) = 2\beta(2d),
\]

where \( \beta \) is the Dirichlet beta function defined by (2.9).

3  Second variant with odd summation indices

In this section, we consider another variation of the Apéry-type series (1.4) by restricting the summation indices to odd numbers only and keeping the strict inequalities among them. These series do not behave as well as the first variant studied in the last section but are still of interest.

Define the 1-forms

\[
\kappa_s(t) = \begin{cases} 
\sin t \, dt \csc t \, dt + \tan t \, dt, & \text{if } s = 1; \\
\sin t \, dt(\cot t \, dt + 1)(\cot t \, dt)^{s-2} \csc t \, dt, & \text{if } s \geq 2,
\end{cases}
\]

and their non-trigonometric counterpart

\[
K_s(t) = \begin{cases} 
\omega_5\omega_3 + \omega_2, & \text{if } s = 1; \\
\omega_5(\omega_0 + 1)\omega_0^{s-2}\omega_3, & \text{if } s \geq 2.
\end{cases}
\]

Theorem 3.1. For all \( n \in \mathbb{N}_0 \) and \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \) the tail

\[
\chi(s; \sin y) := \sum_{n_1 > \cdots > n_d > n} \frac{b_{n_1}(\sin y)}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}} = \frac{d}{dy} \int_0^y \kappa_{s_1} \circ \cdots \circ \kappa_{s_d} \circ b_n(\sin t) \, dt.
\]

In the above \( y \in [-\pi/2, \pi/2] \) if \( s_1 > 1 \) and \( y \in (-\pi/2, \pi/2) \) if \( s_1 = 1 \). Using non-trigonometric 1-forms, for \( x \in (-1, 1) \) we have

\[
\chi(s; x)_n := \sum_{n_1 > \cdots > n_d > n} \frac{b_{n_1}(x)}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}} = \sqrt{1 - x^2} \frac{d}{dx} \int_0^x K_{s_1} \circ \cdots \circ K_{s_d} \circ b_n(t) \omega_1.
\]

Proof. With \( s = 2 \) the identity (2.8) yields that

\[
\sum_{n_1 > n} \left( \frac{2n_1 - 2}{n_1 - 1} \right)^{-1} \frac{4^{n_1-1}\sin^{2n_1-1} y}{(2n_1 - 1)^2} = \int_0^y (\csc t \, dt) b_n(\sin t) \, dt.
\]

Noting that

\[
\left( \frac{2n_1 - 2}{n_1 - 1} \right)^{-1} = \left( \frac{2n_1}{n_1} \right)^{-1} \frac{2(2n_1 - 1)}{n_1},
\]

we have

\[
\sum_{n_1 > n} \frac{b_{n_1}(\sin y)}{n_1(2n_1 - 1)} = 2 \sin y \int_0^y (\csc t \, dt) b_n(\sin t) \, dt.
\]
Differentiating, we obtain
\[ \sum_{n_1 > n} b_{n_1} \frac{\sin^{2n_1 - 1} y \cos y}{2n_1 - 1} = \cos y \int_{0}^{y} (\csc t \, dt) b_{n}(\sin t) \, dt + \int_{0}^{y} b_{n}(\sin t) \, dt. \] (3.3)

Multiplying (3.3) by \( \tan y \) we get
\[ \sum_{n_1 > n} b_{n_1} \frac{\sin y}{2n_1 - 1} = \sin y \int_{0}^{y} (\csc t \, dt + \sec y) b_{n}(\sin t) \, dt. \] (3.4)

Dividing (3.3) by \( \sin y \) and integrating
\[ \sum_{n_1 > n} b_{n_1} \frac{\sin^{2n_1 - 1} y}{(2n_1 - 1)^2} = \int_{0}^{y} (\cot t \, dt)(\csc t \, dt)b_{n}(\sin t) \, dt + \int_{0}^{y} (\csc t \, dt)b_{n}(\sin t) \, dt. \]

Repeatedly multiplying by \( \cot y \) and integrating, we see that for all \( s \geq 2 \)
\[ \sum_{n_1 > n} b_{n_1} \frac{\sin^{2n_1 - 1} y}{(2n_1 - 1)^s} = \int_{0}^{y} (\cot t \, dt)^{s-1}(\csc t \, dt)b_{n}(\sin t) \, dt + \int_{0}^{y} (\cot t \, dt)^{s-2}(\csc t \, dt)b_{n}(\sin t) \, dt. \]

Hence if \( s \geq 2 \) then
\[ \sum_{n_1 > n} b_{n_1} \frac{\sin y}{(2n_1 - 1)^s} = \sin y \int_{0}^{y} (\cot t \, dt + 1)(\csc t \, dt)^{s-2}(\csc t \, dt)b_{n}(\sin t) \, dt. \] (3.5)

The theorem now follows from repeatedly applying (3.4) or (3.5) at each depth. This concludes the proof of the theorem. \( \square \)

4 Variant with summation indices of mixed parities

By combining Thm. 1.1, Thm. 2.3, and Thm. 3.1 we obtain the following result easily.

**Theorem 4.1.** Suppose \( d \in \mathbb{N} \) and \( s = (s_1, \ldots, s_d) \in \mathbb{N}^d \). Let \( y \in (-\pi/2, \pi/2) \) if \( s_1 = 1 \) and \( y \in [-\pi/2, \pi/2] \) if \( s_1 \geq 2 \). Set \( \lambda_{2n,s}(t) = g_s(t), \lambda_{2n+1,s}(t) = h_s(t) \) and \( \lambda_{2n-1,s}(t) = \kappa_s(t) \) which are defined by (1.5), (2.3) and (3.1), respectively. Then for any \( l_1(n), \ldots, l_d(n) = 2n, 2n \pm 1 \) we have the tails
\[ \sum_{n_1 \geq 1 \atop \ldots \atop n_d \geq d} \frac{b_{n_1}(\sin y)}{l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}} = \frac{d}{dy} \int_{0}^{y} \lambda_{l_1,s_1} \circ \cdots \circ \lambda_{l_d,s_d} \circ b_{n}(\sin t) \, dt, \] (4.1)

where \( \succ \) is \( \succeq \) if \( l_j(n) = 2n + 1 \) and is \( \succ \) otherwise. Using non-trigonometric 1-forms, we get for all \( x \in (-1, 1) \)
\[ \sum_{n_1 \geq 1 \atop \ldots \atop n_d \geq d} \frac{b_{n_1}(x)}{l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}} = \sqrt{1 - x^2} \frac{d}{dx} \int_{0}^{x} \Lambda_{l_1,s_1} \circ \cdots \circ \Lambda_{l_d,s_d} \circ b_{n}(t) \omega_1, \] (4.2)

where \( \Lambda_{2n,s}(t) = G_s(t), \Lambda_{2n+1,s}(t) = H_s(t) \) and \( \Lambda_{2n-1,s}(t) = K_s(t) \) which are defined by (1.6), (2.4) and (3.2), respectively.
Proof. Define

\[ f_1(t) := 1, \quad f_2(t) := \frac{t}{\sqrt{1-t^2}}, \quad f_3(t) = \frac{1}{t}, \quad f_20(t) := \frac{1}{t\sqrt{1-t^2}}, \quad f_5(t) = t. \]

By Thm. 1.1, Thm. 2.3, and Thm. 3.1

\[ \sum_{n_1 > n} \frac{b_{n_1}(x)}{2n_1} = f_2(x) \int_0^x b_n(t) \omega_1, \quad (4.3) \]
\[ \sum_{n_1 > n} \frac{b_{n_1}(x)}{(2n_1)^s} = f_1(x) \int_0^x \omega_0^{s-2} \omega_1 b_n(t) \omega_1 \forall s \geq 2, \quad (4.4) \]
\[ \sum_{n_1 \geq n} \frac{b_{n_1}(x)}{2n_1 + 1} = f_20(x) \int_0^x b_n(t) \omega_1, \quad (4.5) \]
\[ \sum_{n_1 \geq n} \frac{b_{n_1}(x)}{(2n_1 + 1)^s} = f_3(x) \int_0^x \omega_0^{s-2} \omega_3 b_n(t) \omega_1 \forall s \geq 2, \quad (4.6) \]
\[ \sum_{n_1 > n} \frac{b_{n_1}(x)}{(2n_1 - 1)} = f_5(x) \int_0^x \omega_3 b_n(t) \omega_1 + f_2(x) \int_0^x b_n(t) \omega_1, \quad (4.7) \]
\[ \sum_{n_1 > n} \frac{b_{n_1}(x)}{(2n_1 - 1)^s} = f_5(x) \int_0^x (\omega_0 + 1) \omega_0^{s-2} \omega_3 b_n(t) \omega_1 \forall s \geq 2. \quad (4.8) \]

For convenience, we call the right-hand side of (4.3) and (4.4) (resp. (4.5) and (4.6), resp. (4.7) and (4.8)) a \( \sigma \)-block (resp. \( \tau^* \)-block, resp. \( \chi \)-block). In (4.2), each \( s_j \) corresponds to (a variation of) such a block. Observing that \( f_j(t) \omega_1 = \omega_j \) we find that after starting with a block in (4.3)-(4.8), all the middle blocks should be modified as follows: (i) change \( f_j(x) \) to \( \omega_j \), (ii) remove the integral sign, and (iii) drop the \( 1 \)-form \( b_n(t) \omega_1 \). Repeating this until the end block, for which only operations (i) and (ii) are required.

This concludes the constructive proof of the theorem. \( \square \)

Remark 4.2. We note that Ap\'ery-type series with indices of mixed parity already appeared implicitly in \cite{9} (1.1)). Indeed, using their notation one need to consider, for e.g., the following series:

\[ \sum_{j=1}^{\infty} \frac{1}{(2j)^c} S_a(2j - 1) = 2c \sum_{j=1}^{\infty} \frac{1}{(2j)^c} \sum_{k=0}^{j-1} \frac{1}{(2k + 1)^a} \sum_{k=1}^{j-1} \frac{1}{(2k)^a} \]
\[ = 2c \sum_{j > k \geq 0} \frac{1}{(2j)^c(2k + 1)^a} + 2c \sum_{j > k \geq 0} \frac{1}{(2j)^c(2k)^a}. \]

Example 4.3. By composing (4.4) and (4.5) we see that

\[ \sum_{n_1 \geq n_2 > 0} \frac{b_{n_1}(x)}{(2n_1 + 1)(2n_2)^2} = \frac{1}{x\sqrt{1-x^2}} \int_0^x \omega_1^3. \]

Taking \( x = 1/2, \sqrt{3}/2 \) we see immediately that

\[ \sum_{n_1 \geq n_2 > 0} \binom{2n_1}{n_1}^{-1} \frac{1}{(2n_1 + 1)(2n_2)^2} = \frac{4}{\sqrt{3}} \frac{(\sin^{-1}(1/2))^3}{3!} = \frac{\pi^3}{4 \cdot 81\sqrt{3}}. \]
\[
\sum_{n_1 \geq n_2 > 0} \binom{2n_1}{n_1}^{-1} \frac{3^n}{(2n_1 + 1)(2n_2)^2} = \frac{4}{\sqrt{3}} \frac{(\sin^{-1}(\sqrt{3}/2))^3}{3!} = \frac{2\pi^3}{81\sqrt{3}}.
\]

These are consistent with the first two identities at the beginning of [11]. Many other evaluations in the loc. cit. can be verified using similar ideas by repeatedly applying (4.3)–(4.8). Taking \(x = \sqrt{2}/2\) we also get
\[
\sum_{n_1 \geq n_2 > 0} \binom{2n_1}{n_1}^{-1} \frac{2^n}{(2n_1 + 1)(2n_2)^2} = \frac{4}{\sqrt{3}} \frac{(\sin^{-1}(\sqrt{2}/2))^3}{3!} = \frac{\pi^3}{96\sqrt{3}}.
\]

By specializing at \(y = \pi/2\) (or taking limit as \(x \to 1^-\)) we obtain the following theorem, which helps answer two questions at the end of [20] affirmatively in Cor. 5.1.

**Theorem 4.4.** Suppose \(d \in \mathbb{N}, s = (s_1, \ldots, s_d) \in \mathbb{N}^d\) and \(s_1 \geq 2\). Set \(\delta(l) = 0\) if \(l(n) = 2n\) and \(\delta(l) = 1\) if \(l(n) = 2n \pm 1\).

(a) Suppose \(l_1(n), \ldots, l_d(n) = 2n, 2n + 1\). Then we have
\[
\sum_{n_1 \geq \cdots \geq n_d > 0} b_{n_1} \frac{l_1(n_1)^{s_1} \cdots l_d(n_d)^{s_d}}{l_1(n_1) \cdots l_d(n_d)^{s_d}} \in \delta(l_1) \text{CMZV}_{[s]}^4,
\]
where “\(\succ\)” is “\(\geq\)” if \(l_j(n) = 2n + 1\) and is “\(>\)” otherwise.

(b) Suppose \(l_1(n), \ldots, l_d(n) = 2n, 2n + 1\). If for all \(l_j(n) = 2n - 1\) (\(j \geq 2\)) we have \(l_{j-1}(n) \neq 2n\), then we have
\[
\sum_{n_1 \geq \cdots \geq n_d > 0} b_{n_1} \frac{l_1(n_1)^{s_1} \cdots l_d(n_d)^{s_d}}{l_1(n_1) \cdots l_d(n_d)^{s_d}} \in \delta(l_1) \left(\text{CMZV}_{[s]}^4 + \nu(l_1) \text{CMZV}_{[s+1]}^4\right), \tag{4.9}
\]
where \(\nu(l) = 1\) if \(l(n) = 2n - 1\) and \(\nu(l) = 0\) otherwise. In particular, if \(l_j(n) \neq 2n\) for all \(j\) then (4.9) holds.

(c) More generally, \(l_1(n), \ldots, l_d(n) = 2n, 2n + 1\) then we have
\[
\sum_{n_1 \geq \cdots \geq n_d > 0} b_{n_1} \frac{l_1(n_1)^{s_1} \cdots l_d(n_d)^{s_d}}{l_1(n_1) \cdots l_d(n_d)^{s_d}} \in \text{CMZV}_{[s]}^4 \otimes \mathbb{Q}[i]. \tag{4.10}
\]

(d) Moreover, the claim in (4.10) still holds if one changes any of the strict inequalities \(n_j > n_{j+1}\) to \(n_j \geq n_{j+1}\) and vice versa, provided the series is defined. Here we set \(n_{d+1} = 0\). In particular, if \(l_1(n), \ldots, l_d(n) = 2n, 2n + 1\) then
\[
\sum_{n_1 \geq n_2 \geq \cdots \geq n_d > 0} b_{n_1} \frac{l_1(n_1)^{s_1} \cdots l_d(n_d)^{s_d}}{l_1(n_1) \cdots l_d(n_d)^{s_d}} \in \text{CMZV}_{[s]}^4 \otimes \mathbb{Q}[i], \tag{4.11}
\]
where “\(\succ\)” can be either “\(\geq\)” or “\(>\)” provided the series is defined.

**Remark 4.5.** Let \(q = \max\{j : l_j(n) \neq 2n + 1\}\). The series is defined if and only if “\(\succ\)” is “\(>\)”.
Proof. Put \( x_\xi = dt/(\xi - t) \) for any \( \xi \in \mathbb{C} \) and \( d_{\xi,\xi'} = x_\xi - x_{\xi'} \). First, we observe that under the change of variables
\[
t \rightarrow \sin^{-1} t \quad \text{then} \quad t \rightarrow \frac{1 - t^2}{1 + t^2},
\] we have
\[
\begin{align*}
cot t \, dt \rightarrow \omega_0 &= a := \frac{dt}{t} \rightarrow y, \quad \csc t \, dt \rightarrow \omega_3 := \frac{dt}{t\sqrt{1 - t^2}} \rightarrow d_{-1,1}, \\
dt \rightarrow \omega_1 &= \frac{dt}{\sqrt{1 - t^2}} \rightarrow id_{-i,i}, \quad \sec t \csc t \, dt \rightarrow \omega_20 := \frac{dt}{t(1 - t^2)} \rightarrow y + z, \\
\tan t \, dt \rightarrow \omega_2 &= \frac{dt}{1 - t^2} \rightarrow z, \quad \sec t \, dt \rightarrow \omega_8 := \frac{dt}{1 - t^2} \rightarrow -a,
\end{align*}
\] where \( y = x_{-i} + x_i - x_{-1} - x_1 \) and \( z = -a - x_{-i} - x_i \). Furthermore, we notice
\[
\sin t \, dt \rightarrow \omega_5 = \frac{t \, dt}{\sqrt{1 - t^2}} \rightarrow \frac{dt}{(i - t)^2} + \frac{dt}{(i + t)^2}.
\] Let
\[
O := \mathbb{Q} \langle \omega_j : 0 \leq j \leq 3 \textrm{ or } j = 5 \rangle.
\] By repeatedly using the six cases (4.3)–(4.8) we see that every sum in (4.10) can be expressed as \( \mathbb{Q} \)-linear combinations of the following form
\[
\int_0^1 \alpha_1 \ldots \alpha_m \quad (4.17)
\] with \( m \leq |s| \) and \( \alpha_j \in O \).

\textbf{(a)} In this case \( \omega_5 \) never appears and the weight in (4.3)–(4.6) is always the same as the number of 1-forms appearing on the right-hand side so that there is no weight drop. Thus we only need to consider the number \( \omega_1 \)'s appearing in (4.2). From (4.13)–(4.15) we see that only \( \omega_1 \) produces \( i \) after the change of variables \( t \rightarrow (1 - t^2)/(1 + t^2) \).

Among the four cases (4.3)–(4.6) only iteration (4.4) affects the number of \( \omega_1 \)'s in (4.10), by adding two \( \omega_1 \)'s. Then ending block \( b_0(t)\omega_1 = \omega_1 \) when \( n = 0 \). So we need to consider the starting 1-form inside the iterated integral of (4.2), which is chopped off after taking the derivative \( d/dx \) and then multiplied by \( \sqrt{1 - x^2} \). This 1-form is \( \omega_1 \) if and only if it is a \( \sigma \)-block with \( s_1 \geq 2 \). Hence, after taking \( d/dx \) we find that on the right-hand side of (4.2) the total number of \( \omega_1 \)'s is even for a starting \( \sigma \)-block (i.e., \( l_1(n) = 2n \)) and the number is odd for a starting \( \tau^* \)-block (i.e., \( l_1(n) = 2n + 1 \)). The claim of \textbf{(a)} is thus proved.

\textbf{(b)} We first claim that we may reduce this case to the case where \( l_j(n) = 2n - 1 \) appears only when \( j = 1 \). We will prove this by induction on the depth. We have nothing to do when the depth is 1. In general, we change the index \( n_j \rightarrow n_j + 1 \) for all \( j \geq 2 \) such that \( l_j(n) = 2n - 1 \). Then we need to consider the following three possible blocks in front of \( j \)-th block. Setting \( k = n_{j-1}, m = n_j, r = s_{j-1}, s = s_j \), we see that
\[
\sum_{k > m} \frac{1}{(2k - 1)^r(2m - 1)^s} \Rightarrow \sum_{k > m} \frac{1}{(2k + 1)^r(2m + 1)^s} = \sum_{k \geq m} \frac{1}{(2k + 1)^r(2m + 1)^s} - \frac{1}{(2k + 1)^{r+s}},
\]
\[
\sum_{k \geq m} \frac{1}{(2k + 1)^s(2m - 1)^s} \implies \sum_{k \geq m+1} \frac{1}{(2k + 1)^s(2m + 1)^s} = \sum_{k \geq m} \frac{1}{(2k + 1)^s(2m + 1)^r} - \frac{1}{(2k + 1)^r+s}.
\]

For the possible \( \tau^s \)-block after the \( j \)-th block, setting \( k = n_{j+1}, m = n_j, r = s_{j+1}, s = s_j \) we have
\[
\sum_{k > m} \frac{1}{(2m - 1)^s(2k + 1)^r} \implies \sum_{k+1 > m} \frac{1}{(2m + 1)^s(2k + 1)^r} = \sum_{k \geq m} \frac{1}{(2m + 1)^s(2k + 1)^r},
\]
which looks in good shape. We also need to consider the possible \( \sigma \)-blocks after the \( j \)-th block: (setting \( k = n_{j+1}, m = n_j, r = s_{j+1}, s = s_j \))
\[
\sum_{k > m} \frac{1}{(2k - 1)^s(2m)^r} \implies \sum_{k+1 > m} \frac{1}{(2k + 1)^s(2m)^r} = \sum_{k \geq m} \frac{1}{(2k + 1)^s(2m)^r},
\]
which looks in good shape, too.

To summarize, the above shows that if no \( \sigma-\chi \)-block chain appears then we see that no weight drops can occur in the decomposed sums.

Furthermore, from the above, we can assume the \( \chi \)-block appears only as the first block, if it ever does. By the explicit iterated integral expressions of the \( \sigma \)- and \( \tau^s \)-blocks \( (4.3) \)–(4.6), we see that if \( \chi \)-block does not appear then all the CMZVs involved are of the same weight. If a \( \chi \)-block appears at the beginning then \( (4.8) \) shows that the weight can increase by one for some CMZVs and the counting of \( \omega_1 \) is the same as the case with a starting \( \tau^s \)-block. This completes the proof of \((b)\).

\((c)\) As the proof of \((b)\), we first claim that we may reduce the general case to the case where \( l_j(n) = 2n - 1 \) appears, if it ever does, then \( j = 1 \). We will prove this by induction on the depth. We have nothing to do when the depth is 1. In general, we change the index \( n_j \rightarrow n_j + 1 \) for all \( j \geq 2 \) such that \( l_j(n) = 2n - 1 \). Then we need to consider the following possible block \( \sigma \) in front of \( j \)-th block since the other two possibilities have been already handled by case \((b)\) (setting \( k = n_{j-1}, m = n_j, r = s_{j-1}, s = s_j \))
\[
\sum_{k > m} \frac{1}{(2k)^s(2m - 1)^s} \implies \sum_{k > m+1} \frac{1}{(2k)^s(2m + 1)^s} = \sum_{k > m} \frac{1}{(2k)^r(2m + 1)^s} - \frac{1}{(2k)^r(2k - 1)^s}.
\]
Thus by partial fractions we may decompose the second term above as pure powers of either \( 2k \) or \( 2k - 1 \), thus reducing the depth by 1. We point out that this is also the reason why the weight may drop due to the partial fractions when \( \sigma-\chi \)-block chain appears.

Thus we will assume the \( \chi \)-block appears only as the first block. Then this case is proved again by the explicit formula \( (4.8) \).

\((d)\) Observe that for any fixed \( n \),
\[
\sum_{m \geq n} \frac{1}{m^s(2n + 1)^t} = \frac{1}{n^s(2n + 1)^t} + \sum_{m > n} \frac{1}{m^s(2n + 1)^t},
\]
\[
\sum_{m > n} \frac{1}{(2m + 1)^t n^s} = -\frac{1}{n^s(2n + 1)^t} + \sum_{m \geq n} \frac{1}{(2m + 1)^t n^s}.
\]
such that $0 < x \leq 1$ the value

$$\Xi(s, l; x) := \sum_{n_1 > n_2 > \ldots > n_d > 0} \frac{b_{n_1}(x)}{l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}},$$

if it exists, can be expressed as a $\mathbb{Q}[i, x, \sqrt{1-x^2}]$-linear combination of the multiple polylogarithms evaluated at algebraic points.

Proof. By the proof of Thm. 4.4 up to factors of $x$ and $\sqrt{1-x^2}$ in front (which can be seen more easily from (4.3)–(4.8)), $\Xi(s, l; x)$ can be expressed as a $\mathbb{Q}$-linear combinations of

$$\int_0^x \left[ \omega_j : j = 0, 1, 2, 3 \right]$$

where $[\omega_j : j = 0, 1, 2, 3]_{|s|}$ is an iteration of 1-forms of length $\ell \leq |s| + 1$. Here the 1-form $\omega_5$ is not needed since the proof of Thm. 4.4 shows that if $l_j(n) = 2n - 1$ then we may assume $j = 1$ (i.e., $\chi$-block only appears at the beginning). Therefore, after applying the change of variables $t \to \frac{1-t^2}{1+t^2}$, by (4.13)–(4.15) we see that (4.18) is transformed to a $\mathbb{Q}[i]$-linear combination of iterated integrals of the form

$$\int_{\lambda(x)}^1 \alpha_1 \ldots \alpha_\ell,$$

where $\lambda(x) = \sqrt{\frac{1-x}{1+x}}$, $\alpha_j \in \{x_0, x_\mu : \mu^8 = 1\}$. Note $\lambda(x) \to x$ under the change of variables $t \to \frac{1-t^2}{1+t^2}$. If $x \neq 1$, to convert this to multiple polylogs we generally need to use the regularization process. Thus, for an arbitrarily small $\varepsilon > 0$ we write

$$\int_{\lambda(x)}^1 \alpha_1 \ldots \alpha_\ell = \sum_{j=0}^\ell \int_{\lambda(x)}^\varepsilon \alpha_1 \alpha_2 \ldots \alpha_j \int_{\varepsilon}^1 \alpha_{j+1} \ldots \alpha_\ell.$$
\[ = \sum_{j=0}^{\ell} (-1)^{j} \int_{\varepsilon}^{1} \alpha_{j} \ldots \alpha_{2} \alpha_{1} \int_{\varepsilon}^{1} \alpha_{j+1} \ldots \alpha_{\ell} \]

\[ = \sum_{j=0}^{\ell} (-1)^{j} \int_{\varepsilon}^{1} \alpha'_{j} \ldots \alpha'_{2} \alpha'_{1} \int_{\varepsilon}^{1} \alpha_{j+1} \ldots \alpha_{\ell} \]

where \( \alpha'_{k} = x_{k/\lambda(x)} \) if \( \alpha_{k} = x_{k} \) where \( \xi = 0 \) or \( \xi^8 = 1 \). Note that \( \xi/\lambda(x) \) is still algebraic. By the usual regularization procedure we see that the last expression can be written as a polynomial \( P(\log(\varepsilon)) \) plus \( O(\varepsilon \log^{\ell}(\varepsilon)) \), such that all the coefficients of \( P \) are \( \mathbb{Q} \)-linear combination of the multiple polylogarithms evaluated at algebraic points. Here we have used the fact that \( \log(\lambda(x)) = \frac{B}{2}(\text{Li}_{1}(-x) - \text{Li}_{1}(x)) \). Finally, taking \( \varepsilon \to 0 \) yields the corollary at once.

**Theorem 4.7.** Keep notation as in Thm. 4.4. Assume \( s_{1} \geq 3 \).

(a) Let \( l_{1}(n), \ldots, l_{d}(n) = 2n, 2n \pm 1 \). If \( l_{1}(n) \neq 2n - 1 \) and for all \( l_{j}(n) = 2n - 1 \) \( (j \geq 2) \) we have \( l_{j-1}(n) \neq 2n \), then

\[ \sum_{n_{1} \geq 1 \ldots \geq n_{d} \geq 0} \frac{b_{n_{1}}}{l_{1}(n_{1})^{s_{1}} \ldots l_{d}(n_{d})^{s_{d}}} \in \text{CMZV}_{|s|}^{4}. \quad (4.19) \]

In particular, if \( l_{j}(n) \neq 2n \) for all \( j \) then \((4.19)\) holds. If \( l_{1}(n) = 2n - 1 \) and for all \( l_{j}(n) = 2n - 1 \) \( (j \geq 2) \) we have \( l_{j-1}(n) \neq 2n \), then

\[ \sum_{n_{1} \geq 1 \ldots \geq n_{d} \geq 0} \frac{b_{n_{1}}^{2}}{l_{1}(n_{1})^{s_{1}} \ldots l_{d}(n_{d})^{s_{d}}} \in \text{CMZV}_{|s|}^{4} + \text{CMZV}_{|s|+1}^{4} + \text{CMZV}_{|s|+2}^{4}. \]

(b) More generally, if \( l_{1}(n), \ldots, l_{d}(n) = 2n, 2n \pm 1 \) then we have

\[ \sum_{n_{1} \geq 1 \ldots \geq n_{d} \geq 0} \frac{b_{n_{1}}^{2}}{l_{1}(n_{1})^{s_{1}} \ldots l_{d}(n_{d})^{s_{d}}} \in \text{CMZV}_{|s|+2\nu(t_{1})}^{4} \otimes \mathbb{Q}[i]. \quad (4.20) \]

(c) Moreover, the claim in \((b)\) still holds if one changes any of the strict inequalities \( n_{j} > n_{j+1} \) to \( n_{j} \geq n_{j+1} \) in \((4.20)\) and vice versa, provided the series is defined. In particular,

\[ \sum_{n_{1} \geq 1 \ldots \geq n_{d} \geq 0} \frac{b_{n_{1}}^{2}}{l_{1}(n_{1})^{s_{1}} \ldots l_{d}(n_{d})^{s_{d}}} \in \text{CMZV}_{|s|+2\nu(t_{1})}^{4} \otimes \mathbb{Q}[i]. \quad (4.21) \]

where \( \succ \) can be either \( \succ \) or \( \succ \), provided the series is defined.

**Proof.** The key observation is that

\[ \int_{0}^{1} \frac{x^{2n+1}}{\sqrt{1-x^{2}}} \, dx = \int_{0}^{\pi/2} \sin^{2n+1} t \, dt = B\left(n + 1, \frac{1}{2}\right) = \frac{b_{n}}{2n+1}. \]

(a) When \( l_{1}(n) = 2n + 1 \) by \((4.2)\) we see that the sum

\[ \sum_{n_{1} \geq 1 \ldots \geq n_{d} \geq 0} \frac{b_{n_{1}}(x)}{l_{1}(n_{1})^{s_{1}} \ldots l_{d}(n_{d})^{s_{d}}} = \frac{1}{x^{\sqrt{1-x^{2}}}} \int_{0}^{x} \omega_{0}^{s_{1}} \ldots \omega_{d}^{s_{d}} \circ \Lambda_{l_{1},s_{2}}(t) \circ \ldots \Lambda_{l_{d},s_{d}}(t) \circ \omega_{1} \]

15
when \( s_1 \geq 2 \). Thus multiplying by \( x \) on both sides and integrating over \((0, 1)\) we get

\[
\sum_{n_1 \geq 1 \ldots \geq n_d \geq 0} \frac{b_{n_1}^2}{l_1(n_1)^{s_1+1} \ldots l_d(n_d)^{s_d}} = \int_0^1 \omega_1 \omega_0^{s_1-2} \omega_3 \circ \Lambda_{l_2,s_2}(t) \circ \ldots \circ \Lambda_{l_d,s_d}(t) \circ \omega_1.
\]

The claim follows immediately since there are even number of \( \omega_1 \)'s in this case.

(b) If \( l_1(n) = 2n \) then we see that

\[
\sum_{n_1 \geq 1 \ldots \geq n_d \geq 0} \frac{b_{n_1}(x)}{l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}} = \int_0^1 \omega_0^{s_1-2} \omega_1 \circ \Lambda_{l_2,s_2}(t) \circ \ldots \circ \Lambda_{l_d,s_d}(t) \circ \omega_1
\]

when \( s_1 \geq 2 \). Then we can divide by \( x\sqrt{1-x^2} \) and integrate over \((0, 1)\) to get

\[
\sum_{n_1 \geq 1 \ldots \geq n_d \geq 0} \frac{b_{n_1}^2}{l_1(n_1)^{s_1+1} \ldots l_d(n_d)^{s_d}} = \sum_{n_1 \geq 1 \ldots \geq n_d \geq 0} \frac{b_{n_1}b_{n_1-1}}{(2n_1-1)l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}}
\]

\[
= \int_0^1 \omega_3 \omega_0^{s_1-2} \omega_1 \circ \Lambda_{l_2,s_2}(t) \circ \ldots \circ \Lambda_{l_d,s_d}(t) \circ \omega_1
\]

since

\[
\frac{b_{n_1-1}}{2n_1-1} = \frac{b_{n_1}}{2n_1}.
\]

The theorem holds as well in this case as the number of \( \omega_1 \)'s is still even.

(c) If \( l_1(n) = 2n - 1 \) then since \( s_1 \geq 3 \) by \( \text{(4.2)} \) we have

\[
\sum_{n_1 \geq 1 \ldots \geq n_d \geq 0} \frac{b_{n_1}(x)}{l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}} = \int_0^x (\omega_0 + 1)\omega_0^{s_1-2} \omega_3 \circ \Lambda_{l_2,s_2}(t) \circ \ldots \circ \Lambda_{l_d,s_d}(t) \circ b_n(t)\omega_1.
\]

We first differentiate this to get

\[
\sum_{n_1 \geq 1 \ldots \geq n_d \geq 0} \frac{2nb_{n_1}x^{2n_1-1}}{l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}} = \int_0^x (\omega_0 + 1)^2\omega_0^{s_1-3} \omega_3 \circ \Lambda_{l_2,s_2}(t) \circ \ldots \circ \Lambda_{l_d,s_d}(t) \circ b_n(t)\omega_1.
\]

As in the \( l_1(n) = 2n \) case, we can divide by \( \sqrt{1-x^2} \) and integrate over \((0, 1)\) to get

\[
\sum_{n_1 \geq 1 \ldots \geq n_d \geq 0} \frac{b_{n_1}^2}{l_1(n_1)^{s_1} \ldots l_d(n_d)^{s_d}} = \int_0^1 \omega_1 (\omega_0 + 1)^2\omega_0^{s_1-3} \omega_3 \circ \Lambda_{l_2,s_2}(t) \circ \ldots \circ \Lambda_{l_d,s_d}(t) \circ b_n(t)\omega_1
\]

by using \( \text{(4.22)} \) again. This completes the proof of the theorem.

\[\square\]

5 A corollary and some examples

In this last section, we will apply our main theorems to compute a few typical Apéry type series to illustrate the power of our method. We can also see how the regularization process is needed in some of the examples.
First, we can answer affirmatively a few questions we posted at the end of [20]. For $k \in \mathbb{N}^d$ and $l \in \mathbb{N}^e$ we define

$$\zeta_n(k) := \sum_{n \geq m_1 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}},$$

$$t_n(l) := \sum_{n \geq r_1 > \cdots > r_e > 0} \frac{1}{(2r_1 - 1)^{l_1} \cdots (2r_e - 1)^{l_e}}.$$

**Corollary 5.1.** For all $m \in \mathbb{N}$, $p \in \mathbb{N}_{\geq 2}$, $q \in \mathbb{N}_{\geq 3}$, and all compositions of positive integers $k$ and $l$ (including the cases $k = \emptyset$ or $l = \emptyset$), we have

$$(a) \sum_{n=1}^{\infty} b_n \frac{\zeta_n(k) t_n(l)}{n^p} \in \text{CMZV}_{|k|+|l|+p}, \quad (b) \sum_{n=1}^{\infty} b_n^2 \frac{\zeta_n(k) t_n(l)}{n^q} \in \text{CMZV}_{|k|+|l|+q},$$

$$(c) \sum_{n=0}^{\infty} b_n \frac{\zeta_n(k) t_n(l)}{(2n+1)^p} \in i\text{CMZV}_{|k|+|l|+p}, \quad (d) \sum_{n=0}^{\infty} b_n^2 \frac{\zeta_n(k) t_n(l)}{(2n+1)^q} \in \text{CMZV}_{|k|+|l|+q}.$$

**Proof.** Write

$$\zeta_n(k) = \sum_{n \geq m_1 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}, \quad t_n(l) = \sum_{n \geq r_1 > \cdots > r_e > 0} \frac{1}{(2r_1 - 1)^{l_1} \cdots (2r_e - 1)^{l_e}}.$$

We only need to note the following facts: (i) for any summation index $m$ for $\zeta_n(k)$ and summation index $r$ for $t_n(l)$ there are only two possibilities: $m > r$ or $r \geq m$; (ii) we can re-write

$$\sum_{n > r_1} \frac{1}{(2n+1)^q(2r_1 + 1)^{l_1}} = \sum_{n \geq r_1} \frac{1}{(2n+1)^q(2r_1 + 1)^{l_1}} - \frac{1}{(2n+1)^q+l_1}$$

and obtain similar identities when $n$ and $r_1$ are replaced by $r_j$ and $r_{j+1}$. Therefore, we see that (a) and (c) are special cases of Thm. 4.4(a) (b) and (d) are special cases of Thm. 4.7(a). \[\square\]

In the following we will compute a series of examples using our main theorems.

**Example 5.2.** When depth $d = 1$, by (4.6) we see that for all $x \in [-1, 1]$

$$\sum_{n \geq 0} \frac{b_n(x)}{(2n+1)^m+2} = \frac{1}{x} \int_0^x \omega_0^m \omega_3 \omega_1$$

(5.1)

for all $m \geq 0$. Applying $t \to \frac{1 - t^2}{1 + t^2}$ to (5.1) we have

$$\sum_{n \geq 0} \frac{b_n(x)}{(2n+1)^m+2} = \frac{i(-1)^m}{x} \int_{\lambda(x)} (x_i - x_{-i})(x_1 - x_{-1})y^m,$$

where $\lambda(x) = \frac{1-x}{1+x}$ and $y = x_{-i} + x_i - x_{-1} - x_1$ as defined in (4.13). Taking $x = 1$ and applying Au's Mathematica package [4] we get

$$\sum_{n \geq 0} \frac{b_n}{(2n+1)^2} = 2 \text{Im}(\text{Li}_{1,1}(i, -i) + \text{Li}_{1,1}(-i, -i)) = 2G \approx 1.83193119,$$

(5.2)
\[
\sum_{n \geq 0} \frac{b_n}{(2n + 1)^3} = 2 \text{ Im} \left( \text{Li}_{13}(-i, i, i) + \text{Li}_{13}(-i, i, -i) - \text{Li}_{13}(-i, i, -1) - \text{Li}_{13}(-i, i, 1) \right) \\
- \text{Li}_{13}(-i, -i, -i) - \text{Li}_{13}(-i, -i, i) + \text{Li}_{13}(-i, -i, -1) + \text{Li}_{13}(-i, -i, 1) \right) \\
= -\frac{\pi^3}{32} - \frac{1}{8} \pi \log^2 2 + 4 \text{Im} \text{Li}_3 \left( \frac{1 + i}{2} \right) \approx 1.122690025,
\]

where \( G = \beta(2) \) is Catalan’s constant. This sum appears in [5, Example 2.12], too.

**Example 5.3.** As an application of Cor. 4.6, we now compute

\[
\sum_{n \geq 0} \frac{1}{(2n + 1)^2} = \sum_{n \geq 0} b_n(1/2).
\]

From the previous example we see that

\[
\sum_{n \geq 0} \frac{1}{(2n + 1)^2} = = \int_1^1 (x_i - x_{-i})(x_1 - x_{-1}) \\
= 2i \left( \int_0^1 (x_i - x_{-i})(x_1 - x_{-1}) + \int_0^1 (x_i - x_{-i}) \int_{1/\sqrt{3}}^1 (x_1 - x_{-1}) + \int_0^1 (x_i - x_{-i})(x_1 - x_{-1}) \right) \\
= 2i \left( \int_0^{1/\sqrt{3}} (x_1 - x_{-1})(x_i - x_{-i}) - \int_0^1 (x_i - x_{-i}) \int_0^{1/\sqrt{3}} (x_1 - x_{-1}) \right) + 4G \text{ (by (5.2))} \\
= 4 \text{ Im} \left( \text{Li}_{1,1} \left( \left( \frac{1}{\sqrt{3}} \right) \right) - \text{Li}_{1,1} \left( \left( \frac{1}{\sqrt{3}} \right) - i \right) \right) - \pi \log(2 + \sqrt{3}) + 4G \approx 1.063459833.
\]

**Example 5.4.** As an easy example of Thm. 4.4(a) by (4.4) and (4.5) we have

\[
\sum_{n_1 > n_2 \geq 0} \frac{b_{n_1}}{n_1^2 (2n_2 + 1)} = 4 \int_0^{\pi/2} dt \circ (\text{csc} t \sec t \text{dt}) \circ dt \\
= 4 \int_0^1 \omega_1 \circ \omega_20 \circ \omega_1 \quad \text{(by } t \rightarrow \sin^{-1} t) \\
= -4 \int_0^1 (x_{-i} - x_i) \circ (x_0 + x_{-1} + x_1) \circ (x_{-i} - x_i) = 7\zeta(3)
\]

by the change of variables \( t \rightarrow (1 - t^2)/(1 + t^2) \) then using Au’s package [4].

**Example 5.5.** For a pure \( \chi \)-sum, by (4.8) we have

\[
\sum_{n \geq 0} \frac{b_n}{(2n - 1)^2} = \int_0^1 (\omega_0 + 1) \omega_3 \omega_1 = i \int_0^1 d_{-i,1,1}(1 - y) \\
= 2G - \frac{1}{32} \pi^3 + 4 \text{Im} \text{Li}_3 \left( \frac{1 + i}{2} \right) - \frac{1}{8} \pi \log^2 2 \approx 2.954621213,
\]

where we see the weight can increase by one as predicted by Thm. 4.4(b). Similarly,

\[
\sum_{n > 0} \frac{b_n}{(2n - 1)^3} = \int_0^1 (\omega_0 + 1) \omega_0 \omega_3 \omega_1 = i \int_0^1 d_{-i,1,1,1}y(y - 1)
\]

18
\[-4\beta(4) + \frac{1}{96} \left( 2 \text{Im} \text{Li}_4 \left( \frac{1+i}{2} \right) + 4\pi \log^3 2 + 3\pi^3 \log 2 \right)
- 12\pi \log^2 2 - 3\pi^3 - \text{Im} \text{Li}_3 \left( \frac{1+i}{2} \right) \approx 2.1543060048.
\]

**Example 5.6.** For a sum of mixed parities as examples of Thm. 4.4(b) by (4.5) and (4.8) we have

\[
\sum_{n_1 > n_2 \geq 0} \frac{b_{n_1}}{(2n_1 - 1)^2(2n_2 + 1)} = \int_0^1 (\omega_0 + 1)\omega_3 \omega_2 \omega_1 = i\int_0^1 d_{-i,i}(a - x - x_1)d_{-1,1}(y - 1)
= 14\beta(4) - 16 \text{Im} \text{Li}_4 \left( \frac{1+i}{2} \right) - \frac{1}{12} \pi \log^3 2 - \frac{3}{16} \pi^3 \log 2
+ \frac{1}{8} \pi \log^2 2 + \frac{5}{32} \pi^3 - 4 \text{Im} \text{Li}_3 \left( \frac{1+i}{2} \right) \approx 3.937040753.
\]

So we see the weight can increase by one with a starting $\chi$-block.

**Example 5.7.** For a sum of mixed parities without $\sigma$-block but with a starting $\tau^*$-block, by (4.6) and (4.7) we have

\[
\sum_{n_1 \geq n_2 > 0} \frac{b_{n_1}}{(2n_1 + 1)^2(2n_2 - 1)} = \int_0^1 \omega_3 (\omega_5 \omega_3 + \omega_2) \omega_1
= \int_0^1 \omega_3 \omega_2 \omega_1 - \int_0^1 \omega_0 \omega_3 \omega_1 \quad \text{(since $\omega_5 = -d\sqrt{1 - t^2}, \sqrt{1 - t^2} \omega_3 = \omega_0$)}
= -i\int_0^1 d_{-i,i}(y + z)d_{-1,1} + i\int_0^1 d_{-i,i}d_{-1,1}y
= \frac{3}{16} \pi^3 - 8 \text{Im} \text{Li}_3 \left( \frac{1+i}{2} \right) + \frac{1}{4} \pi \log^2 2 \approx 1.630404535576,
\]

where $y + z = -a - x - x_1$. Thus the weight is unchanged as predicted by Thm. 4.4(b).

**Example 5.8.** For a sum of mixed parities with a starting $\sigma$-block, followed by a $\tau$-block then a $\chi$-block, by (4.4), (4.5) and (4.8) we get

\[
\sum_{n_1 > n_2 \geq n_3 > 0} \frac{b_{n_1}}{(2n_1)^2(2n_2 + 1)(2n_3 - 1)} = \int_0^1 \omega_1 \omega_2 \omega_5 (\omega_1 + 1)\omega_3 \omega_1
= \int_0^1 \omega_1 (\omega_2 \omega_3 - \omega_3 \omega_0) \omega_3 \omega_1 \quad \text{(since $\omega_5 = -d\sqrt{1 - t^2}, \sqrt{1 - t^2} \omega_3 = \omega_0$)}
= \int_0^1 d_{-i,i}d_{-1,1}d_{-1,1}(y + z)d_{-i,i} - \int_0^1 d_{-i,i}d_{-1,1}yd_{-1,1}d_{-i,i}
= \frac{G}{4} \left( \frac{\pi^3}{4} - 32 \text{Im} \text{Li}_3 \left( \frac{1+i}{2} \right) + \pi \log^2 2 \right)
- \frac{15}{2} \left( \text{Li}_5 \left( \frac{1}{2} \right) + \log 2 \text{Li}_4 \left( \frac{1}{2} \right) \right)
- \frac{1}{4} \log^5 2
+ 6\beta(4) + 24 \text{Re} \text{Li}_3,1,1(1, 1, I) + \frac{1}{384} \left( 80\pi^2 \log^3 2 - 15\pi^4 \log 2 - 87\pi^2 \zeta(3) - 2250\zeta(5) \right)
\approx 0.98658158829.
\]

So weight is unchanged in every step as predicted by Thm. 4.4(b).
Example 5.9. For a sum of mixed parities with $\sigma$-block followed by a $\chi$-block, by (4.4) and (4.7) we obtain

$$
\sum_{n_1 > n_2 > 0} \frac{b_{n_1}}{(2n_1)^2(2n_2 - 1)} = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \left( \sin \phi \csc \phi d\phi + \tan \phi d\phi \right) d\phi
$$

$$
= \int_0^{\pi/2} d\phi \int_0^{\pi/2} \left( d(-\cos \phi) \csc \phi d\phi + \tan \phi d\phi \right) d\phi
$$

$$
= \int_0^{\pi/2} d\phi \cos \phi \csc \phi \cot \phi d\phi + \int_0^{\pi/2} d\phi \csc \phi \sec \phi d\phi
$$

$$
= \int_0^{\pi/2} d\phi \left( \sin \phi - 1 \right) \csc \phi d\phi + \int_0^{\pi/2} d\phi \csc \phi \sec \phi d\phi
$$

Note that not only the weight is a mix of 2 and 3, but this is a mix of both real and imaginary parts of some CMZVs of level 4. The main complication is brought in by the 1-form $\sin \phi d\phi$ (corresponding to $\omega_5$) appearing in the $\chi$-block, which moves to the front after integration by parts if the block in front is a $\sigma$-block but disappears if the block in front is a $\tau^*$-block.

Example 5.10. We apply the idea of proof of Thm. 4.1 to the sum

$$
S := \sum_{n_1 > n_2 > n_3 \geq 1} \frac{b_{n_1}}{n_1^2(2n_2 + 1)(2n_3 - 1)}.
$$

First, we break the sum in two sub-sums $S = S_1 + S_2$ where

$$
S_1 = \sum_{n_1 > n_2 \geq n_3 \geq 1} \frac{b_{n_1}}{n_1^2(2n_2 + 1)(2n_3 - 1)};
$$

$$
S_2 = \sum_{n_1 \geq n_2 > n_3 \geq 1} \frac{b_{n_1}}{n_1^2(2n_1 + 1)(2n_3 - 1)}.
$$

Then by (4.4), (4.5) and (4.7) we see that

$$
S_1 = 4 \int_0^1 \omega_1 \sum_{n_2 \geq n_3 > 0} \frac{b_{n_2}(t)}{(2n_2 + 1)(2n_3 - 1)} \omega_1
$$

$$
= 4 \int_0^1 \omega_1 \omega_20 \sum_{n_3 > 0} \frac{b_{n_1}(t)}{2n_3^2 - 1} \omega_1
$$

Note that not only the weight is a mix of 2 and 3, but this is a mix of both real and imaginary parts of some CMZVs of level 4. The main complication is brought in by the 1-form $\sin \phi d\phi$ (corresponding to $\omega_5$) appearing in the $\chi$-block, which moves to the front after integration by parts if the block in front is a $\sigma$-block but disappears if the block in front is a $\tau^*$-block.

Example 5.10. We apply the idea of proof of Thm. 4.1 to the sum

$$
S := \sum_{n_1 \geq n_2 \geq n_3 \geq 1} \frac{b_{n_1}}{n_1^2(2n_2 + 1)(2n_3 - 1)}.
$$

First, we break the sum in two sub-sums $S = S_1 + S_2$ where

$$
S_1 = \sum_{n_1 > n_2 \geq n_3 \geq 1} \frac{b_{n_1}}{n_1^2(2n_2 + 1)(2n_3 - 1)};
$$

$$
S_2 = \sum_{n_1 \geq n_2 > n_3 \geq 1} \frac{b_{n_1}}{n_1^2(2n_1 + 1)(2n_3 - 1)}.
$$

Then by (4.4), (4.5) and (4.7) we see that

$$
S_1 = 4 \int_0^1 \omega_1 \sum_{n_2 \geq n_3 > 0} \frac{b_{n_2}(t)}{(2n_2 + 1)(2n_3 - 1)} \omega_1
$$

$$
= 4 \int_0^1 \omega_1 \omega_20 \sum_{n_3 > 0} \frac{b_{n_1}(t)}{2n_3^2 - 1} \omega_1
$$

Note that not only the weight is a mix of 2 and 3, but this is a mix of both real and imaginary parts of some CMZVs of level 4. The main complication is brought in by the 1-form $\sin \phi d\phi$ (corresponding to $\omega_5$) appearing in the $\chi$-block, which moves to the front after integration by parts if the block in front is a $\sigma$-block but disappears if the block in front is a $\tau^*$-block.
By the change of variables $t \to \frac{1 - t^2}{1 + t^2}$ we obtain
\[ S_1 = -4A - 4\bar{A} - 4 \int_0^1 d_{-i,i}(a + x_{-i} + x_i)(a + x_{-1} + x_1)d_{-i,i}, \]
where $\bar{A}$ is the complex conjugation of
\[ A = \int_0^1 d_{-i,i}d_{-1,1} \frac{dt}{(i - t)^2}(a + x_{-1} + x_1)d_{-i,i}. \]
Integration by parts yields
\[ A = \int_0^1 d_{-i,i} \frac{x_{-1} - x_i}{i - t}(a + x_{-1} + x_1)d_{-i,i} - \int_0^1 d_{-i,i}d_{-1,1} \frac{a + x_{-1} + x_1}{i - t}d_{-i,i}. \]
Explicitly, for all fourth roots of unity $\xi \neq i$
\[ \frac{a}{i - t} = i(x_i - a), \quad \frac{x_\xi}{\xi - i} = \frac{d_{i,\xi}}{\xi - i}. \] (5.3)
We obtain
\[ A = \int_0^1 d_{-i,i} \left( \frac{1}{1-i}d_{i,-1} - \frac{1}{1+i}d_{i,1} \right)(a + x_{-1} + x_1)d_{-i,i} \]
\[ - \int_0^1 d_{-i,i}d_{-1,1} \left( -i(x_i + a) - \frac{1}{1+i}d_{i,-1} + \frac{1}{1-i}d_{i,1} \right)d_{-i,i} \]
\[ = \int_0^1 d_{-i,i} \left( -x_i + \frac{1-i}{2}x_{-1} + \frac{1+i}{2}x_1 \right)(a + x_{-1} + x_1)d_{-i,i} \]
\[ - \int_0^1 d_{-i,i}d_{-1,1} \left( -ia + \frac{1-i}{2}x_{-1} - \frac{1+i}{2}x_1 \right)d_{-i,i}. \]
Therefore using Au’s package [4] we find that
\[ S_1 = -4 \int_0^1 \left( d_{-i,i}(a + x_{-1} + x_1)^2d_{-i,i} - d_{-i,i}d_{-1,1}d_{-i,i} \right) = 8G^2 \approx 6.71194375752575. \]
Now we turn to $S_2$. Set
\[ S_2(x) = \sum_{n_1 \geq n_3 \geq 1} \frac{b_{n_1}(x)}{n_1^2(2n_1 + 1)(2n_3 - 1)}. \]
By partial fraction
\[ S_2(x) = \sum_{m \geq n_3 > 0} \left( \frac{4b_m(x)}{(2m + 1)(2n_3 - 1)} - \frac{2b_m(x)}{m(2n_3 - 1)} + \frac{b_m(x)}{m^2(2n_3 - 1)} \right) \]
\[ = \sum_{m \geq n_3 > 0} \frac{4b_m(x)}{(2m + 1)(2n_3 - 1)} - \sum_{n > 0} \left( \frac{2b_m(x)}{m(2n - 1)} - \frac{b_m(x)}{m^2(2n - 1)} \right) - \sum_{n > 0} \left( \frac{2b_n(x)}{n(2n - 1)} - \frac{b_n(x)}{n^2(2n - 1)} \right) \]
\[
= 4 \sum_{n>0} \left( f_{20}(x) \int_0^x \frac{b_n(t)}{2n-1} \omega_1 - f_2(x) \int_0^x \frac{b_n(t)}{2n-1} \omega_1 + \int_0^x \omega_1 \frac{b_n(t)}{2n-1} \omega_1 - \frac{b_n(x)}{4n^2} \right)
= 4 (f_{20}(x) - f_2(x)) \int_0^x (\omega_5 \omega_3 \omega_1 + \omega_2 \omega_1) + 4 \int_0^x \omega_1 (\omega_5 \omega_3 \omega_1 + \omega_2 \omega_1) - 4 \int_0^x \omega_1 \omega_1
\]
by (4.3)–(4.7). Note that
\[
\lim_{x \to 1^-} \int_0^1 \omega_5 \omega_3 \omega_1 = -i \int_0^1 d_{-1,i} d_{-1,1} \left( \frac{dt}{(i-t)^2} + \frac{dt}{(i+t)^2} \right) = 2 \Re \left( -i \int_0^1 d_{-1,i} \frac{d_{-1,1}}{i-t} \right)
= 2 \Im \int_0^1 d_{-1,i} \left( \frac{1}{1-i} d_{i,-1} - \frac{1}{1+i} d_{i,1} \right) = \Im \int_0^1 d_{-1,i} (-2x_i + (1-i)x_{-1} + (1+i)x_1),
\]
which is a finite value in iCMZV. Further, setting \( \lambda(x) = \sqrt{\frac{1-x}{1+x}} \) we get
\[
\int_0^x \omega_2 \omega_1 = -i \int_{\lambda(x)}^1 d_{-1,i} (a + x_{-i} + x_i).
\]
We only need to take care of
\[
\int_{\lambda(x)}^1 x_{-i} a = \int_{\lambda(x)}^1 x_{-i} \int_{\lambda(x)}^1 a - \int_{\lambda(x)}^1 a x_{-i} = -\log \left( \frac{i + \lambda(x)}{i + 1} \right) \log \lambda(x) - \int_{\lambda(x)}^1 a x_{-i}
\]
of which the last term \( \to \text{Li}_2(i) \) as \( x \to 1^- \). Hence
\[
\lim_{x \to 1^-} (f_{20}(x) - f_2(x)) \int_{\lambda(x)}^1 x_{-i} a = -\log \left( \frac{i}{i + 1} \right) \lim_{x \to 1^-} \sqrt{1-x^2} \log \sqrt{\frac{1-x}{1+x}} = -\log \left( \frac{i}{i + 1} \right) \frac{\sqrt{2}}{2} \lim_{\epsilon \to 0^+} \sqrt{\epsilon} \log(\epsilon/2) = 0.
\]
Thus
\[
S_2 = \lim_{x \to 1^-} S_2(x) = 4 \int_0^1 \omega_1 (\omega_5 \omega_3 \omega_1 + \omega_2 \omega_1) - 4 \int_0^1 \omega_1 \omega_1
\]
\[
= 4 \int_0^1 d_{-1,i} \left( z - d_{-1,1} \left( \frac{dt}{(i-t)^2} + \frac{dt}{(i+t)^2} \right) \right) d_{-1,i} - 2 \left( \int_0^1 \omega_1 \right)^2 = -4 \int_0^1 d_{-1,i} (a + x_{-i} + x_i) d_{-1,i} - 2B - 2B - \frac{\pi^2}{2},
\]
where
\[
B = 2 \int_0^1 d_{-1,i} d_{-1,1} d_{-1,i} - 2 \int_0^1 d_{-1,i} d_{-1,1} \frac{d_{-1,i}}{i-t} + 2 \int_0^1 \frac{d_{-1,i}}{i-t} \left( z - d_{-1,1} \left( \frac{dt}{(i-t)^2} + \frac{dt}{(i+t)^2} \right) \right) d_{-1,i} - 2 \left( \int_0^1 \omega_1 \right)^2
\]
\[
= \int_0^1 d_{-1,i} \left( x_{-1} + x_1 - i d_{-1,1} - 2x_i \right) d_{-1,i} - i \int_0^1 d_{-1,i} d_{-1,1} d_{-1,i} + 2 \int_0^1 d_{-1,i} \frac{d_{-1,i}}{i-t} + 2i \int_0^1 d_{-1,i} d_{-1,1},
\]
\[
= \int_0^1 d_{-1,i} \left( x_{-1} + x_1 - i d_{-1,1} - 2x_i \right) d_{-1,i} + i \int_0^1 d_{-1,i} d_{-1,1} d_{-1,i} + 2i \int_0^1 d_{-1,i} d_{-1,1}.
\]
Hence
\[ B + \bar{B} = 2 \left( \int_0^1 d_{-i,i}(x_{-1} + x_1 - x_{-1} - x_i) d_{-i,i} + \int_0^1 d_{-i,i} d_{-i,i} + i \int_0^1 d_{-i,i} d_{-1,1} \right). \]

We can see that \( S_2 = 7\zeta(3) - 8G \approx 1.0866735685 \) by Au’s package \[4\] and therefore
\[ \sum_{n_1 \geq n_2 \geq n_3 \geq 1} \frac{b(n_1)}{n_1^2(2n_2 + 1)(2n_3 - 1)} = 7\zeta(3) + 8G^2 - 8G \approx 7.79861732643. \]

In the next three examples, we consider some Apéry-type series which involve the square of the central binomial coefficients.

**Example 5.11.** Since \( \chi \)-block does not appear in this example we see that the weight of the CMZVs is the same as the weight of the series, as predicted by Thm. 4.7(a)
\[ \sum_{n_1 \geq n_2 > 0} \frac{b_{n_1}^2}{(2n_1 + 1)^4(2n_2)} = \int_0^1 \omega_1 \omega_3 \omega_2^2 \omega_1 = \int_0^1 d_{-i,i} y d_{-1,1} y d_{-i,i} = W_5 \approx 0.04433915814, \]
where
\[ W_5 = 5 \log 2 \text{Li}_4 \left( \frac{1}{2} \right) + 21 \text{Li}_5 \left( \frac{1}{2} \right) + \pi \left( 16 \text{Im Li}_4 \left( \frac{1+i}{2} \right) - 17\beta(4) + 8 \text{Im Li}_3 \left( \frac{1+i}{2} \right) \log 2 \right) + \frac{379}{2880} \pi^4 \log 2 + \frac{1}{30} \log^5 2 - 16 \text{Re Li}_{3,1,1}(1,1,i) - \frac{1}{192} \pi^2 \left( 16 \log^3 2 - 29\zeta(3) \right) - \frac{27}{4} \zeta(5). \]

**Example 5.12.** The sum next has weight 4 but due to the \( \sigma-\chi \)-block chain we need to use CMZVs of weight of both 3 and 4 to express it:
\[ \sum_{n_1 > n_2 > 0} \frac{b_{n_1}^2}{(2n_1)^3(2n_2 - 1)} = \int_0^1 \omega_3 \omega_1 (\omega_2 \omega_3 + \omega_2) \omega_1 = \int_0^1 \omega_3 (\omega_1 \omega_0 - d \omega_3) \omega_1 + \omega_3 \omega_1 \omega_2 \omega_1 \]
(since \( \omega_5 = -d \sqrt{1 - t^2} \), \( \sqrt{1 - t^2} \omega_3 = \omega_0 \), \( \sqrt{1 - t^2} \omega_1 = dt \))
\[ = \int_0^1 (\omega_3 \omega_1 - \omega_1 \omega_3) \omega_1 + \omega_3 \omega_1 \omega_0 \omega_1 + \omega_3 \omega_1 \omega_2 \omega_1 \quad \text{(since } t \omega_3 = \omega_1) \]
\[ = \int_0^1 d_{-i,i} (d_{-i,i} d_{-1,1} - d_{-i,i} d_{-i,i}) - d_{-i,i} (y + z) d_{-i,i} d_{-1,1} \]
\[ = 2G^2 - G \pi \log 2 + \frac{1}{64} \pi \left( 3\pi^3 - 128 \text{Im Li}_3 \left( \frac{1+i}{2} \right) + 4\pi \log^2 2 \right) + 2G \pi - \frac{21}{4} \zeta(3) \]
\[ \approx 0.40829155182. \]

**Example 5.13.** The series in this example does not have a \( \sigma-\chi \)-block chain so that there is no weight drop. But the first block is a \( \chi \)-block so the weight can increase by two as predicted by Thm. 4.7(b)
\[ \sum_{n_1 > n_2 > 0} \frac{b_{n_1}^2}{(2n_1 - 1)^3(2n_2)} = \int_0^1 \omega_1 (\omega_0 + 1)^2 \omega_3 \omega_2 \omega_1 \]

23
\[ \int_{0}^{1} d_{-i,i} z d_{-1,1}(y - 1)^2 d_{-i,i} = W_6 + 2W_5 + W_4 \approx 0.38530528471, \]

where \( W_5 \) is defined in Example 5.11 and

\[
W_4 = -2G^2 - \frac{49}{720} \pi^4 + 2 \pi \text{Im} \text{Li}_3 \left( \frac{1+i}{2} \right) - \frac{11}{48} \pi^2 \log^2 2 + \frac{1}{6} \log^4 2 + G \pi \log^2 2 + 4 \text{Li}_4 \left( \frac{1}{2} \right),
\]

\[
W_6 = 68 \text{Li}_6 \left( \frac{1}{2} \right) - \frac{7655}{27648} \pi^6 + \frac{61}{2} \pi \text{Im} \text{Li}_{4,1}(i,1) - \frac{41}{2} \pi \text{Im} \text{Li}_{4,1}(i, -1) + 96 \pi \text{Im} \text{Li}_5 \left( \frac{1+i}{2} \right)
\]

- \( 19 \pi \beta(4) \log 2 + 32 \pi \text{Im} \text{Li}_4 \left( \frac{1+i}{2} \right) \log 2 - \frac{181}{2880} \pi^4 \log^2 2 - \frac{1}{96} \pi^2 \log^4 2 + \frac{1}{90} \log^6 2 \)

- \( \frac{169}{4} \zeta(5,1) - \frac{5}{12} \pi^2 \text{Li}_4 \left( \frac{1}{2} \right) + 10 \log 2 \text{Li}_5 \left( \frac{1}{2} \right) - 24G \beta(4) - 6 \pi^2 \log 2 \zeta(3) \)

- \( 24 \text{Re} \text{Li}_{4,2}(-1,i) + 64 \text{Re} \text{Li}_{3,1,1,1}(1,1,1,i) + \frac{8195}{128} \zeta(3)^2 + \frac{2821}{32} \log 2 \zeta(5). \)

**Acknowledgement.** Ce Xu is supported by the National Natural Science Foundation of China [Grant No. 12101008], the Natural Science Foundation of Anhui Province [Grant No. 2108085Q0A01] and the University Natural Science Research Project of Anhui Province [Grant No. KJ2020A0057]. Jianqiang Zhao is supported by the Jacobs Prize from The Bishop’s School.

**References**

[1] J. Ablinger, Discovering and proving infinite binomial sums identities, *Experimental Math.* 26 (2017), pp. 62–71. [arXiv:1507.01703](https://arxiv.org/abs/1507.01703)

[2] P. Akhilesh, Double tails of multiple zeta values, *J. Number Thy.* 170(2017), pp. 228–249.

[3] P. Akhilesh, Multiple zeta values and multiple Apéry-like sums, *J. Number Thy.* 226(2021), pp. 72–138.

[4] K.C. Au, Evaluation of one-dimensional polylogarithmic integral, with applications to infinite series, [arXiv:2007.03957](https://arxiv.org/abs/2007.03957). A companion Mathematica package available at researchgate.net/publication/342344452

[5] J. M. Campbell, M. Cantararini and J. D’Aurizio, Symbolic computations via Fourier–Legendre expansions and fractional operators, *Integral Transforms and Special Func.* 33(2)(2022), pp. 1–19.

[6] K.-T. Chen, Algebras of iterated path integrals and fundamental groups, *Trans. Amer. Math. Soc.* 156(1971), pp. 359–379.

[7] K.-T. Chen, Iterated path integrals, *Bull. Amer. Math. Soc.* 83(1977), pp. 831–879.

[8] A.I. Davydychev and M. Yu. Kalmykov, New results for the epsilon-expansion of certain one-, two- and three-loop Feynman diagrams, *Nucl. Phys. B* 605 (2001), pp. 266–318. [arXiv:hep-th/0012189](https://arxiv.org/abs/hep-th/0012189)

[9] A.I. Davydychev and M. Yu. Kalmykov, Massive Feynman diagrams and inverse binomial sums, *Nuclear Phys. B* 699 (2004), pp. 3–64. [arXiv:hep-th/0303162v4](https://arxiv.org/abs/hep-th/0303162v4).
[10] F. Jegerlehner, M.Yu. Kalmykov and O. Veretin, \( \overline{\text{MS}} \) versus pole masses of gauge bosons II: two-loop electroweak Fermion corrections, \textit{Nucl. Phys.} \textbf{B658} (2003), pp. 49–112.

[11] M.Yu. Kalmykov, B.F.L. Ward, S.A. Yost, Multiple (inverse) binomial sums of arbitrary weight and depth and the all-order \( \varepsilon \)-expansion of generalized hypergeometric functions with one half-integer value of parameter, \textit{J. High Energy Phys.} \textbf{2007} (10)(2007) 048, 26 pp.

[12] L. Lai, C. Lupu and D. Orr, Elementary proofs of Zagier’s formula for multiple zeta values and its odd variant, \texttt{arXiv:2201.09262}.

[13] L. Lai and P. Yu, A note on the number of irrational odd zeta values, \textit{Compos. Math.} \textbf{156}(2020), no. 8, pp. 1699–1717.

[14] D. Leshchiner, Some new identities for \( \zeta(k) \), \textit{J. Number Theory}, \textbf{13}(1981), pp. 355–362. MR0634205 (83k:10072).

[15] T. Murakami, On Hoffman’s \( t \)-values of maximal height and generators of multiple zeta values, \textit{Math. Ann.}, \textbf{382}(2022), pp. 421–458.

[16] G. Racinet, Doubles mélanges des polylogarithmes multiples aux racines de l’unité (in French), \textit{Publ. Math. IHES} \textbf{95} (2002), pp. 185–231.

[17] T. Rivoal, La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs (in French), \textit{C. R. Acad. Sci. Ser. A. Math.} \textbf{331}(2000), pp. 267–270.

[18] Z.-W. Sun, New series for some special values of \( L \)-functions, \textit{Nanjing Univ. J. Math. Biquarterly} \textbf{32}(2015), no.2, pp. 189–218.

[19] C. Xu, Explicit relations between multiple zeta values and related variants, \textit{Adv. Appl. Math.} \textbf{130}(2021), 102245.

[20] C. Xu and J. Zhao, Apéry-type series and colored multiple zeta values, \texttt{arXiv:2111.10998}.

[21] J. Zhao, \textit{Multiple zeta functions, multiple polylogarithms and their special values}, Series on Number Theory and its Applications, Vol. 12, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

[22] W. Zudilin, One of the numbers \( \zeta(5), \zeta(7), \zeta(9), \zeta(11) \) is irrational, \textit{Russian Math. Surveys} \textbf{56}(4)(2001), pp. 774–776.