The Cosmological Constant as a Zero Action Boundary

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Abstract
The cosmological constant \( \Lambda \) is usually interpreted as Dark Energy (DE) or modified gravity (MG). Here we propose instead that \( \Lambda \) corresponds to a boundary term in the action of classical General Relativity. The action is zero for a perfect fluid solution and this fixes \( \Lambda \) to the average density \( \rho \) and pressure \( p \) inside a primordial causal boundary: \( \Lambda = 4\pi G (\rho + 3p) \). This explains both why the observed value of \( \Lambda \) is related to the matter density today and also why other contributions to \( \Lambda \), such as DE or MG, do not produce cosmic expansion. Cosmic acceleration results from the repulsive boundary force that occurs when the expansion reaches the causal horizon. This universe is similar to the \( \Lambda \)CDM universe, except on the largest observable scales, where we expect departures from homogeneity/isotropy, such as CMB anomalies and variations in cosmological parameters indicated by recent observations.

Key words: Cosmology: dark energy, cosmic background radiation, cosmological parameters, early Universe, inflation

1 INTRODUCTION

Causality is a key property for any physical explanation. General Relativity (GR) is built as a covariant theory but this does not warrant a causal structure (see e.g. Howard 2010). Standard solutions to the Friedmann-Lemaître-Robertson-Walker (FLRW) metric lack a proper causal structure: the energy-density and the expansion rate are the same everywhere, in comoving coordinates, no matter how distant. This is not physically possible and has no explanation.

In the FLRW universe, the Hubble Horizon \( r_H \) is defined as \( r_H = c / H(t) \), where \( H(t) = \dot{a} / a \) is the expansion rate and \( a = a(t) \) is the scale factor. The corresponding comoving coordinate is \( \chi_H = r_H/a \). In normal circumstances \( r_H \) has to grow with time because gravity is attractive and slows down the expansion rate \( H(t) \). Scales larger than \( r_H \) can’t evolve because the time a perturbation takes to travel that distance is shorter than the expansion time. This means that \( r > r_H \) scales are “frozen out” (structure can not evolve) and are causally disconnected from the rest. Thus, \( \chi_H \) represents a dynamical causal horizon that is evolving.

This does not mean that we can’t observe scales larger than \( \chi_H \). The comoving particle horizon \( \chi \), defined as the distance traveled by light since the beginning of time, is the integral of the Hubble Horizon: \( \chi = c \int_0^t \! dt/a(t) = \int d\ln a \chi_H \) and is therefore larger than \( \chi_H \). Thus we can see parts of our universe which are causally disconnected from its evolution. This happens with CMB observations, where the largest angular scales are sometimes called “superhorizon”. In fact, if we look back in time \( \chi_H \) becomes smaller and smaller and all observable patches of the universe become causally disconnected! This is a very important inconsistency of the FLRW Universe if we seek a causal explanation.

These contradictions can be addressed with inflation. According to inflation (Starobinskiï 1979; Guth 1981; Linde 1982; Albrecht & Steinhardt 1982) a small primordial (quantum) patch of size \( r_S \), which is causally disconnected from the rest of space-time, starts a De Sitter phase of exponential expansion during which the energy density \( \rho_i \) and Hubble rate \( H_i \) were constant \( (8\pi G \rho_i = 3H_i^2) \). The causal horizon \( \chi_S \) is identified with the particle horizon during inflation: \( \chi_S = c / (a_i H_i) \) or the Hubble horizon when inflation begins. This is the largest causally connected scale at the beginning of inflation \( a_i \). This initial patch grows exponentially in physical units, \( r_S = a x \chi_S \), while the corresponding comoving scale \( \chi_S = c / (a_i H_i) \) and Hubble horizon \( r_H = c / H_i \) remain constant. After 60-70 e-folds \( r_S \) becomes cosmological in size while \( \chi_H \) is microscopically small. This is the beginning of the standard FLRW expansion, where the initial Hubble horizon \( \chi_H \) is negligible as if time had just started. When inflation ends, \( \rho_i \) converts into \( \rho \) and \( p \) (reheating) and \( \chi_H \) starts to grow inside the primordial causal horizon \( \chi_S \).

Fig. 1 illustrates this causal structure. Models of inflation expect that \( \chi_S \) is larger than the particle horizon \( \chi_0 \) today so that \( a_S >> 1 \). The second inflation today is then attributed to DE or some other exotic explanation but not to \( \chi_S \). In our model we will show instead that \( a_S \approx 1 \) and \( \chi_0 = \chi_S \), so the second inflation today is a consequence of the first one. In such scenario, we can observe causally disconnected regions (\( \chi > \chi_S \) in Fig. 1) and this should result in cosmic anisotropy and inhomogeneities. Either way, inflation mostly solves the horizon, homogeneity and flatness problems (see e.g. Dodelson 2003; Liddle 1999) because we encounter a homogeneous universe that was causally frozen before.

How can we distinguish if \( \Lambda \) is related to \( \chi_S \), to DE or to some other change in GR? observationally, the boundary explanation predicts anisotropies and inhomogeneities on scales corresponding to \( \chi_S \). Theoretically, we will show that the value of \( \Lambda \) is determined by \( \chi_S \) using the zero action principle. The least action principle tells us that arbitrary variations around the true field solutions produce no changes to the action. The action is a functional that assigns a number to each possible metric evolution. The action on-shell (AoS), \( S^{on-sh} \), is the value that corresponds to the actual solution of the field equations. We will show that \( S^{on-sh} \) only depends on
the value at the boundary of space-time. We are interested here in asymptotically flat (Minkowski) space-time: for a universe of finite age, we expect no effect at scales larger than the true (primordial) dynamical causal horizon (created by inflation). In such case, we will show that $S^{n-sh} = 0$. We call this new property for the Universe the zero action principle. This fixes $\Lambda$ to the average density $\rho$ and pressure $p$ inside the primordial causal horizon set by inflation: $\Lambda = 4\pi G < \rho + 3p >$.

To explain cosmic acceleration, scientists have thought of alternative ways (such as DE, MG or massive gravity) that can contribute to $\Lambda$ or suppress gravity. But each new alternative can't explain why the other contributions can be neglected. The zero action principle implies that other contributions to $\Lambda$ do not produce cosmic expansion because its value is renormalized by the boundary. So the contribution of vacuum energy $\rho_{\text{vac}}$ to $\Lambda$ could be as large as the Planck mass ($\rho_{\text{vac}} \approx M_{\text{Pl}}^4$) and still produce no observational effects. This provides a true solution to the $\Lambda$ problem.

That $\Lambda$ is related to some cosmic scale, rather than quantum vacuum is hinted by its observed value of $(\Lambda/8\pi G)^{1/2}$ which is close to $\chi_{S} \approx c/H_0$ (this is sometimes called the coincidence problem). The holographic principle (Susskind 1995; Maartens 2004) provides a way to connect Planck and $\Lambda^{-1/2}$ via string theories. But note how in comoving coordinates $\chi_S$ is both scales at the same time! The primordial causal horizon can start as a Planck scale and become cosmological $c/H_0$ through cosmic expansion (Gaztañaga 2020).

It is generally accepted that GR is not a complete theory, both because of its singularities and also because of its apparent incompatibilities with Quantum Field renormalization theory. There are some approaches that relate $\Lambda$ to infrared boundary conditions within Quantum Gravity (see e.g. Stenflo 2020; Banks & Fischer 2018; He & Cao 2015; Huang et al. 2012; Wudka 1987 and references therein), but they do not seem to bear a direct connection to the classical GR approach presented here. A more direct connection exist with the discussion in §10 of Padmanabhan (2010), which relates boundary conditions in the action to singularities in the metric. Here we also find that the boundary condition in the action (i.e. $\Lambda$) relates to a singularity in the deSitter metric (see Appendix C).

Observationally, there is no motivation to modify gravity (MG) on cosmological scales other than to understand $\Lambda$. MG models have been proposed to replace GR using $f(R)$ theories (Nojiri et al. 2017). These models also require screening mechanisms to avoid Solar System and Astrophysical tests of gravity (Brax 2013). On cosmological scales, MG models are equivalent to DE models (Joyce et al. 2016). But neither approach solves the cosmological constant or coincidence problems. The value of $\Lambda$ is added and fixed by hand and there is no explanation as to why its value is so small and so close to the matter density today. As we can also add and fix $\Lambda$ by hand in GR, these MG and DE models seem an unnecessary complication (Occam’s razor). But the same principles presented in this work can be easily applied to these GR extensions.

These ideas are similar to the ones in Gaztañaga (2020), but they are cast here in terms of $S^{n-sh}$. Section 2 presents the zero action principle. Section 3 introduces $\Lambda$ as a boundary term. Section 4 explores implications for cosmological observations. Appendix C shows how we live inside a Black Hole whose horizon corresponds to $r_h$ (Gaztanaga 2021). Appendix D presents a proposal to start our Universe based on $S^{n-sh} = 0$. We use natural units of $c = 1$, but show $c$ explicitly when useful. We use sign conventions as in Padmanabhan (2010) with Greek (Latin) indexes for 4D (3D).

2 ZERO ACTION PRINCIPLE

Consider the Einstein-Hilbert action (Hilbert 1915):

$$ S = \int_{M} dM \left( \frac{R}{16\pi G} + \mathcal{L}_m \right) , \quad (1) $$

where $dM = \sqrt{-\tilde{g}} d^4x$ is the invariant volume element, $M$ is the 4D spacetime manifold, $R = R^\mu_\nu g^\mu\nu R_{\mu\nu}$ the Ricci scalar curvature and $\mathcal{L}_m$ the Lagrangian of the energy-matter content. We can obtain Einstein’s field equations for the metric field $g_{\mu\nu}$ from this action by requiring $\delta S = 0$ under arbitrary variations of the metric $\delta g^{\mu\nu}$. The solution is (Einstein 1916):

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} = -\frac{16\pi G}{\sqrt{-g}} \frac{\delta (\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} . \quad (2) $$

Note that Eq.2 requires that boundary terms vanish (e.g. see §95 in Landau & Lifshitz 1971, Carroll 2004, Padmanabhan 2010). In the notation of Hawking & Horowitz (1996) the boundary term is:

$$ S = \int_{\partial M} \frac{R}{16\pi G} + \mathcal{L}_m \left| \frac{1}{8\pi} \oint_{\partial M} K \right| \quad (3) $$

where $K$ is the trace of the extrinsic curvature at the boundary. We can add terms to the action that do not depend on dynamical fields, as they do not change the field equations. As mentioned in the introduction we are interested in asymptotically flat (Minkowski) space-times. We can then define the action as:

$$ S = \int_{M} \frac{R}{16\pi G} + \mathcal{L}_m \left| \frac{1}{8\pi} \oint_{\partial M} (K - K_0) \right| , \quad (4) $$

where $K_0$ is the trace of the extrinsic curvature of the boundary embedded into asymptotically Minkowski space. This will cancel the boundary term. We will show next that the AoS $S^{n-sh}$ (i.e. the action for the solution of the field equations) for Eq.1 is also a boundary term for a perfect fluid. As we require boundary terms to vanish, we have $S^{n-sh} = 0$. We call this the zero action principle.
2.1 Perfect fluid
For a perfect isotropic fluid with $\bar{\rho} = \rho(t, r)$ isotropic pressure and $\bar{\rho} = \rho(t, r)$ energy-matter density:

$$T_{\mu\nu} = (\bar{\rho} + \bar{p})u_\mu u_\nu + \bar{p} g_{\mu\nu},$$

(5)

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity ($u_\mu u^\mu = -1$). For a generic barotropic fluid with equation of state $\Pi(\rho) = \int dp(r)/\rho = w \ln \rho$ we have $\bar{\rho} = w \rho$ and $\bar{\rho} = \rho[1 + w \ln \rho]$. This fluid is in general made of several components, each with a different equation of state. We describe the metric for an observer comoving with the fluid (so that the spatial velocity is $u^\mu = 0$). We then have:

$$T_0^0 = -\bar{p} \quad ; \quad T_1^1 = T_2^2 = T_3^3 = \bar{\rho},$$

$$T = T_\mu^\mu = 3\bar{\rho} - \bar{p}.$$  

(6)

and using Eq.2:

$$R_0^0 = -4\pi G (\bar{\rho} + 3\bar{p}) \quad ; \quad R_1^1 = R_2^2 = R_3^3 = -4\pi G (\bar{\rho} - \bar{p}),$$

$$R = -8\pi G T = 8\pi G (\bar{\rho} - 3\bar{p}).$$

(7)

The Lagrangian on-shell for a perfect fluid is $L_{on-sh} = -\bar{\rho}$ (Minazzoli & Harko 2012) so that Eq.1 on-shell becomes:

$$S_{on-sh} = -\frac{1}{2} \int_M dM \left[ \bar{\rho} + 3\bar{p} \right] = \int_M dM \frac{R_0^0}{8\pi G} \equiv -m/2$$

(8)

where $m$ is the relativistic mass inside the manifold. Note that this result also holds for anisotropic stress energy with heat flux. In this case we can have different pressure components $T_1^1 \neq T_2^2 \neq T_3^3$ and $\bar{\rho}$ above just corresponds to the mean value.

2.2 Gaussian flux and Zero Action
Consider the geodesic acceleration $g^\mu = (g^\mu_i, \bar{g})$ defined from the geodesic deviation equation (Padmanabhan 2010):

$$g^\mu = \frac{D^2 \xi^\mu}{D\tau^2} = R^\mu_{\alpha\beta\gamma} u^\alpha u^\beta v^\gamma,$$

(9)

where $v^\mu$ is the separation vector between neighbouring geodesics and $u^\alpha$ is the tangent vector to the geodesic. For an observer following the trajectory of the geodesic $u^\mu = (1, 0)$ and $g^\mu = (0, \bar{g})$:

$$g^I = R_{000}^I \bar{v}^I.$$ 

(10)

and we can choose the separation vector $v^\mu$ to be the spatial coordinate. The spatial divergence of $\bar{g}$ is then:

$$\bar{\nabla} \bar{g} = R_{00}^0 = -4\pi G (\bar{\rho} + 3\bar{p}).$$

(11)

where for the second equality we have used Eq.7. This equation does not require any approximation and is always valid for a comoving observer (see Eq.6.105 in Padmanabhan 2010). Newtonian gravity is reproduced for the case of non-relativistic matter ($\bar{\rho} = 0$). The covariant version of Eq.11 is the relativistic version of Poisson’s Equation (see also Gaztañaga 2020):

$$\nabla_\mu g^\mu = R_0^0 = -4\pi G (\bar{\rho} + 3\bar{p}).$$

(12)

The solution to these equations is given by an integral over the usual propagators or retarded Green functions which account for causality and indicates that the AoS has to be estimated on the light-cone. From Eq.12 and Eq.8 we can calculate the value of the AoS using Stoke’s theorem to convert the 4D $M$ integral into a 3D $\partial M$ boundary integral:

$$S^{on-sh} = \int_M dM \frac{R_0^0}{8\pi G} = \int_M d\bar{M} \frac{\nabla_\mu g^\mu}{8\pi G} = \int_{\partial M} d\nu M g^{\mu\nu},$$

(13)

where $V_\mu$ is a 4D vector whose components are 3D volume elements normal to $\partial M$ (e.g. $dV_0 = dx dy dz$). So the AoS is just the relativistic version of the Gauss flux, which results in a boundary term or a total charge. Traditionally, we take such boundary terms to be zero at infinity (see Eq.4.7.8 in Weinberg 1972) because there is no causal connection at infinity. In other words: we expect space-time to be asymptotically Minkowski because particles should be free at spatial infinity for a finite time of evolution. This means that boundary terms should be zero and therefore $S^{on-sh} = 0$.

To see this explicitly, consider the boundary $\partial M$ as two spacelike surfaces at $t = t_1$ and $t = t_2$ and one timelike surface at spatial infinity. The usual assumption is that the fields do not contribute at spatial infinity. This is the case for asymptotically flat space-times. So Eq.13 reduces to two volume $dV_0$ integrals at constant $t$ over $g^\mu$.

But $g^0 = 0$ for a comoving observer and we then have $S^{on-sh} = 0$.

We also need $S^{on-sh} = 0$ because boundary terms must vanish to reproduce Eq.2 from Eq.1. In Appendix A we present a different derivation for $S^{on-sh} = 0$ as a boundary condition. In Appendix B we estimate $S^{on-sh}$ explicitly for the FLRW metric and in Appendix C we show how it is possible to combine the FLRW with asymptotically Minkowski boundary conditions. But note that Eq.13 does not assumed a FLRW metric or homogeneity.

3 THE Λ TERM AS A BOUNDARY
As first pointed out by Einstein, we can add a fundamental constant, that here we call $\Lambda_F$, to the action in Eq.1 (vacuum energy or Dark Energy are included in the energy content via $L_m$ or $T_{\mu\nu}$):

$$S = \int_M dM \left[ \frac{R - 2\Lambda_F}{16\pi G} + L_m \right].$$

(14)

We can fulfill the zero action condition $S^{on-sh} = 0$ (see above) by subtracting a constant term $B$ from the action:

$$S = \int_M dM \left[ \frac{R - 2\Lambda_F}{16\pi G} + L_m \right] - B.$$

(15)

This is sometimes called “background subtraction” (another way to write Eq.4). The boundary density $B/M \equiv \Lambda_{B}/8\pi G$ plays the role of a new cosmological constant $\Lambda_B$, so we can write Eq.14 as:

$$S = \int_M dM \left[ \frac{R - 2\Lambda}{16\pi G} + L_m \right].$$

(16)

where $\Lambda \equiv \Lambda_F + \Lambda_B$. So the value of $\Lambda_F$ is renormalized by the boundary term $\Lambda_B$, so that its actual value is irrelevant. The new field equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$

(17)

so that Eq.7 transforms into:

$$R_0^0 = \Lambda - 4\pi G (\bar{\rho} + 3\bar{p}) \quad ; \quad R_1^1 = R_2^2 = R_3^3 = \Lambda - 4\pi G (\bar{\rho} - \bar{p})$$

$$R = 4\Lambda - 8\pi G T = 4\Lambda + 8\pi G (\bar{\rho} - 3\bar{p}).$$

(18)

The geodesic equation Eq.12 changes to:

$$\nabla_\mu g^\mu = R_0^0 = \Lambda - 4\pi G (\bar{\rho} - 3\bar{p}).$$

(19)

We can see here how $\Lambda > 0$ corresponds to a repulsive force which opposes gravity, given by the $G$ term. This is the analog of the
classical Casimir effect, which induces geometrical forces as a result of a boundary condition (e.g. Elizalde & Romeo 1991; Griffiths & Ho 2001; Boyer 2003). Replacing Eq.18 back in the action of Eq.16, the condition that \( S^{\text{on-\theta}} = 0 \) fixes \( \Lambda \) to:

\[
\Lambda = 4\pi G < \bar{\rho} + 3\bar{p} > \equiv \frac{4\pi G}{M} \int_M dM(\bar{\rho} + 3\bar{p}).
\]  

(20)

Thus \( \Lambda = \langle R_0^0 \rangle \) is the action density on-shell (for \( \Lambda = 0 \)). As the Aos does not evolve (see Appendix A) and the primordial causal volume \( M_8 \) is fixed (see Fig.1 and §3.2), \( \Lambda \) is also constant.

The condition in Eq.20 looks similar to that in Sobral-Blanco & Lombriser (2020); Lombriser (2019), who found \( \Lambda/4\pi G = cT \gg < 3\bar{p} - \bar{\rho} > \) (see also Kaloper et al. 2016; Carroll & Remmen 2017). Eq.20 was first obtained by Gaztañaga (2020) starting from the relativistic version of Poisson equation Eq.19. Our new derivation here shows how \( \Lambda \) emerges as a boundary condition to the action and not as a fundamental change to gravity (i.e. \( \Lambda_F \)) above. A further interpretation as a BH Universe can be seen in Gaztanaga (2021) and Appendix C.

3.1 Vacuum energy and DE do not gravitate

It is well known that the effects of \( \Lambda \) in Eq.17 are equivalent to that of vacuum energy, which has constant energy density and equation of state: \( \omega = -1 \). Replacing \( \rho_{\text{vac}} \) and \( p_{\text{vac}} \) in \( T_{\mu\nu} \) by

\[
\rho_{\Lambda} \equiv \rho_{\text{vac}} + \frac{\Lambda}{8\pi G}; \quad p_{\Lambda} = \rho_{\text{vac}} - \frac{\Lambda}{8\pi G} = -\rho_{\Lambda},
\]  

(21)

we can make Eq.17 look the same as Eq.2. Equivalently we can define a new effective energy-momentum tensor \( T_{\mu\nu}' \equiv T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu} \) (Peebles & Ratra 2003) so that Eq.17 looks the same as Eq.2. So it looks as if we could not separate the effects of \( \Lambda \) from \( \rho_{\text{vac}} \). But both quantities are related via Eq.20, which results in a cancellation of terms in \( \rho_{\Lambda} \). If we combine \( \rho_{\text{vac}} \) with matter \( \rho_m \) (with \( p_m = 0 \)) and radiation \( \rho_g \) (with \( p_R = \rho_R / 3 \)), Eq.20 gives:

\[
\Lambda = 4\pi G (< \rho_m + 2\rho_R > -2\rho_{\text{vac}}). \tag{22}
\]

so that Eq.21 results in:

\[
\rho_{\Lambda} = \frac{\Lambda}{8\pi G} + \rho_{\text{vac}} < \rho_m / 2 + \rho_R >. \tag{23}
\]

This shows that vacuum energy cancels out and can not affect the observed value of \( \rho_{\Lambda} \) or cosmic expansion. As we will show, Eq.23 also explains the coincidence problem (Peebles & Ratra 2003; Carroll 2004) connecting the measured values for \( \rho_m \) and \( \rho_{\Lambda} \approx 2.3\rho_m \).

Another way to see this is to combine Eq.18 and Eq.20:

\[
\rho_0^0 = -4\pi G [(\rho_\omega - \rho_p + 3(\rho_\omega - \rho_p))] \tag{24}
\]

which explicitly shows that constant \( \rho \) or \( p \) components to the energy-momentum do not produce cosmic acceleration, giving \( \rho_0^0 = 0 \) (recall that \( R_0^0 = \frac{\nabla^2 \bar{\rho}}{3\dot{\bar{a}}/a} \)). The contribution of DE with equation of state \( \rho_{DE}(a) = \omega \rho_{DE}(a) \), is:

\[
\rho_{0}^0 = -4\pi G (1 + 3\omega) \rho_{DE}(a) - \omega^{-3}(1 + \omega) > - < \omega^{-3}(1 + \omega) > \tag{25}
\]

where \( \rho_{DE} \) is the value today. So, for \( \omega \) close to \( -1 \), DE produces negligible cosmic acceleration or expansion. This removes the original motivation to have DE, as it represents an unnecessary complication of the model (Occam’s razor). Vacuum energy or DE violate the strong \( \omega > -1/3 \) and weak \( \omega > 0 \) energy conditions in GR (e.g. see Visser 1997 and references therein). As shown above, in our interpretation where \( \Lambda \) is a boundary term given by Eq.20, vacuum energy does not gravitate and DE is not needed, so these conditions are not necessarily violated.

In Unimodular Gravity, a modification of classical GR, \( \Lambda \) appears as a Lagrange multiplier or, equivalently, the determinant of the metric is fixed. In this situation \( \Lambda \) does not gravitate (see Anderson 1971; Smolin 2009; García-Bellido et al. 2011; Percacci 2018 and references therein). This is different from what we find here, in classical GR, where \( \Lambda \) gravitates but it cancels \( \rho_{\text{vac}} \), DE or \( \Lambda_F \).

3.2 Causal Boundary and inflation

Without inflation the different parts of the observable Universe are causally disconnected. According to inflation, our Universe is inside a large cosmic horizon \( \chi_8 \), which could be much larger than our observable Universe today (so that \( \chi_8 \gg \chi_0 \)). In such case we expect the flux to be zero at \( \chi_8 \), i.e. \( \text{Son-\theta} = 0 \) in Eq.13, because sources produce no flux at distances that are out of causal reach. During inflation, the dynamics of the Universe are frozen out. After inflation our new Hubble horizon \( \chi_H \) is very small \( \chi_H \ll \chi_8 \) so we can assume that the Universe inside \( \chi_8 \) follows the (flat) FLRW metric (see Fig.1). For the FLRW Universe, we have \( \rho_{\Lambda} = \rho_m / (2 < a^3 >) \) (see Appendix B). The average \( < a^3 > > 0 \) is over the comoving \( \chi \) in the light-cone. If \( \chi \) is large compared to \( \chi_0 \) (\( \chi > 1 \)) then \( \rho_{\Lambda} \to 0 \) as \( a \to \infty \). This is contrary to current observations which find: \( \rho_{\Lambda} \approx 2.3\rho_m \), which corresponds to \( < a^3 > > 0.22 \) and therefore \( \chi_8 \approx \chi_0 \). So current measurements of \( \rho_{\Lambda} > 0 \) are the smoking gun for primordial inflation with \( \chi_8 \gg \chi_0 \).

4 LATE TIME COSMIC ACCELERATION

If we assume that the Universe is isotropic and homogeneous inside \( \chi_8 \) we have the FLRW metric in comoving coordinates \((t, \chi)\):

\[
d s^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \left[ d\chi^2 + \frac{\sin^2(\chi)}{k^2} d\chi^2 \right] \tag{26}
\]

where \( d\chi^2 = \cos^2(\theta) d\theta^2 + d\theta^2 \) is the solid angle element. For a closed (or open) Universe of radius \( \chi_k \) we have \( \chi_k = k \sin[\chi / \chi_k] \) (or \( k \sin[\chi / \chi_k] \)). For the flat case \( \chi_k \to \infty \) so that \( \chi_k = \chi \). The FLRW Universe in Eq.26, filled with matter \( \rho_m \approx a^{-3} (p_m = 0) \), radiation \( \rho_R \approx a^{-4} \) (\( p_R = \rho_R / 3 \)) and \( \rho_{\Lambda} \) has an expansion \( H \) and acceleration rate \( q \) (e.g. Peebles & Ratra 2003; Weinberg 2008):

\[
3H^2 = 8\pi G (\rho_m + \rho_R + \rho_{\Lambda}) - \frac{3}{\dot{\chi_0}^2} \chi_0^2 \tag{27}
\]

\[
3qH^2 = 3\dot{a} / a = R_0^0 = \Lambda - 4\pi G (\rho + 3p) \tag{28}
\]

where \( \Omega_k = \frac{\rho_m}{\rho_{\Lambda}} \) and \( \rho_{\Lambda} = \frac{3H^2}{8\pi G} \). The observed positive value for the cosmic acceleration \( q \approx 0.55 \) indicates that \( \rho_{\Lambda} > 0 \) and corresponds to \( \Omega_k \approx 2.3\Omega_m \approx 0.7 \) today for the flat case \( \Omega_k \approx 0 \). These values produce \( \chi_H \) as shown numerically in Fig.1. The Hubble radius corresponding to the causal horizon is \( \chi_k = \Omega_k^{1/2} \approx 1.2 \), in units of c/H0, which corresponds to a Black Hole with \( m \approx 5.8 \times 10^{22} M_\odot \) (see Eq.25). In terms of the FLRW frame \((t, \chi)\) what is relevant is the corresponding comoving particle horizon coordinate \( \chi_8 \) (see Fig.1). We can estimate \( \chi_8 \) from the average in Eq.23 with the information we have from observations after inflation (see Fig.1).

Inflation creates a large causal primordial horizon \( \chi_8 \) inside which the Universe is homogeneous and nearly flat. The particle horizon after inflation is \( \chi \) in Eq.26, \( t = a(t) \chi \), so we can write:

\[
\chi(a) = \int_{\chi_0}^a \frac{dt}{a(t)} = \int_{\chi_0}^a \frac{dlna}{aH(a)} = \chi_0 - \dot{\chi}(a), \tag{29}
\]
where $t_c$ and $a_c$ represent the time and scale factor when inflation ends. The particle horizon today is $\chi_0 \equiv \chi(a = 1) = 3H_0$ and $\chi(\bar{a}) = \int_0^\bar{a} da/(a^2 H)$ is the radial lookback time coordinate so that $d_A(\bar{a}) = S_\bar{a}(\bar{a})$ is the comoving angular diameter distance. We can now estimate the average in Eq. 23 in the light-cone to $\chi_S$:

$$\rho_\Lambda = \frac{\int_0^{\chi_0} d\chi S_\chi^2(\chi) (1 - \cos \frac{\chi}{\chi_0}) a^3 (\rho_m a^{-3} + 2\rho_R a^{-4})}{\int_0^{\chi_0} d\chi S_\chi^2(\chi) (1 - \cos \frac{\chi}{\chi_0}) a^3},$$  \hspace{1cm} (30)$$

where $a = a(\chi)$ is the light-cone crossing corresponding to the inverse of $\chi(a)$ in Eq. 29, which also depends on $\rho_\Lambda$ via the Hubble rate $H$. The term $(1 - \cos \frac{\chi}{\chi_0})$ comes from the limits of $\chi_S$ on the solid angle $d\Omega$ and was not included in Gaztañaga (2020), which explains the small differences in the final results. To do this integral, it is easier to change variables to $a$:

$$\rho_\Lambda = \frac{\int_0^{a_S} da S_\chi^2(\chi) (1 - \cos \frac{\chi}{\chi_0}) (\Omega_m a^{-3} + 2\Omega_R a^{-4})}{\int_0^{a_S} da S_\chi^2(\chi) (1 - \cos \frac{\chi}{\chi_0})},$$  \hspace{1cm} (31)$$

where $\chi = \chi(a)$ in Eq. 29 and $a_S \equiv a(\chi_S)$. We have divided by $\rho_c$ to express the result in terms of adimensional ratios $\Omega$. For each value of $\chi_S$ (and $\Omega_k$, with $\Omega_k \approx 0.25E-5$) we can find numerically the corresponding $\Omega_\Lambda$. The result is shown in Fig. 2. For $\Omega_\Lambda \approx 0.7 \pm 0.1$ and $\Omega_k \approx 0$ we find:

$$\chi_S = (3.34 \pm 0.18) \frac{c}{H_0}$$  \hspace{1cm} (32)$$

compared to: $\chi_0 = 3.26c/H_0$. Because $\chi_S < \pi \chi_0$ we can actually measure $\chi_S$ on the sky. At the CMB, $\chi_S$ corresponds to an angle: $\theta_S(\chi_S) \equiv \chi_S/d_A \approx 60$ deg. This boundary could therefore result in CMB anomalies or tensions in measurements. Because $\chi_S$ in Eq. 32 is closed to $\chi_0$, we could wonder if inflation lasted enough time to make $\Omega_\Lambda \approx 0$ today. In fact, $\Omega_k < 0$ is not ruled out by observations (Di Valentino et al. 2020). For $\Omega_k = 0.06$ and $\Omega_\Lambda \approx 0.7 \pm 0.1$ we find $\chi_S = (3.39 \pm 0.27) \frac{c}{H_0}$ compared to $\chi_0 = 3.31$ and $\theta_S(\chi_S) \approx 68$ deg. Fig. 3 shows the angle on the sky $\theta_S(z) = \chi_S/d_A(z)$ corresponding to a transverse $\chi_S$.

Because $d_A(\chi_{\text{CMB}}) \approx \chi_S$, we expect the cosmological parameters in our local universe to be slightly different from the ones that we see in the CMB, as they come from different primordial causal patches. This could explain tensions in measurements of $H_0$ and other cosmological parameters. The full CMB sky covers transverse scales up to $\pi d_A \approx \pi \chi_S$, so that we are able to measure a few separate causal horizons on the CMB sky (Fosalba & Gaztañaga 2020).

### 5 DISCUSSION AND CONCLUSIONS

We have argued that our universe not only has "zero energy" (i.e. critical density) but also zero AoS. This generates a boundary term for the action which fixes the cosmological constant to $\Lambda = 4\pi G < \rho + 3p >$, i.e. Eq. 20. As a consequence vacuum energy, DE or a fundamental $\Lambda_F$ do not gravitate (see §3.1). This provides a true solution to the cosmological problem. It predicts that the observed value of $\rho_\Lambda$ is related to the mean matter and radiation content inside the boundary $\chi < \chi_S$, see Eq. 23 and 30. In our interpretation, $\Lambda$ is not a dark energy component or a fundamental constant $\Lambda_F$ (Weinberg 1989; Huterer & Turner 1999; Carroll et al. 1992), but just a boundary term (see §3). Measurements of $\rho_\Lambda$ allow us to estimate $\chi_S$ (see Eq. 32 and Fig. 2). For $\chi_S > \chi_0$ we have $\Omega_\Lambda \approx 0$, which is contrary to observations. While $\Omega_\Lambda \approx 1$ indicates that $\chi_S \approx 0$. This can be used to constrain models of inflation (Gaztañaga 2020). In contrast to the Hubble horizon, $\chi_H = c/(aH)$, which increases with $a$, the primordial boundary $\chi_S$ remains constant throughout cosmic evolution (see Fig. 1). The $\Lambda$ boundary corresponds to a singularity in the metric of our Universe. We are dynamically trapped inside $\chi_S$, like observers inside a Black Hole (BH). In Appendix C we show how $\chi_S$ corresponds to the BH horizon, which explains how we can have an asymptotically Minkowski boundary and a FLRW metric at the same time (Gaztanaga 2021). Cosmic acceleration occurs when our particle horizon approaches this primordial boundary. Thus late time cosmic acceleration is the smoking gun of primordial inflation.

Each causally disconnected patch (or horizon) that emerges from inflation evolves like a separate Universe (see Fig. 4 and Appendix
The anomalous lack of correlations (in respect to LCDM model) in the sky on angular scales (see also Appendix D) of why is the reason for Fig.1) with the highest chances to host observers like us. Ultimately, our observable Universe today, the temperature can be similar across nearby disconnected patches in the sky. Continuity across contiguous regions then forces any uncorrelated differences to be small. This could explain why the mean CMB temperature can be similar across nearby primordial patches in the sky.

That the primordial causal boundary scale, $\chi_\delta$, is similar in size to our observable Universe today, $\chi_0$, solves the coincidence problem (Peebles & Ratra 2003; Carroll 2004) of why is $\rho_\Lambda \approx 2.3\rho_m$ or why $(\Lambda/8\pi G)^{-1/2}$ is close to $c/H_0$. But it rises new questions: why $\chi_\delta \approx \chi_0$? Fig.2 shows how very different values of $\Omega_\Lambda$ have similar $\chi_\delta/\chi_0$ values. So may be this is not such a big coincidence. In terms of anthropic reasoning, at earlier times the Universe is dominated by radiation and there are no stars or galaxies to host observers. Moreover, $\chi_0 \approx \chi_\delta$, has the largest possible Hubble radius (see Fig.1) with the highest chances to host observers like us. Ultimately, the reason for $\chi_\delta \approx \chi_0$ resides in the details of the initial conditions (see also Appendix D).

We should be able to find observational evidence of such a primordial boundary $\chi_\delta$ in our sky (see Fig.3). Because separate horizons are not causally connected, we expect to see no statistical correlations in the sky on angular scales $\Theta > \Theta_\delta$. This seems in agreement with the anomalous lack of correlations (respect to LCDM model) in the CMB temperature correlations $w(\theta)$ for $\theta \geq 60$ deg. Such anomaly has been used to predict $\Omega_\Lambda \approx 0.7$ for a flat Universe with independence of any other measurements (Gaztañaga 2020). But note that the value of $\Omega_\Lambda$ that we measure locally, could be different from the one affecting the faraway distance where CMB photons were emitted. Tensions between measurements of cosmological parameters (or fundamental constants) from very different redshifts or different regions of the sky (at high $z > 2$) could be related to such in-homogeneities.

In a recent analysis Fosalba & Gaztañaga (2020) found strong evidence for such horizons in the coherent variation of cosmological parameter in the CMB sky. We should look closer for such differences to further validate or falsify these ideas.

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**DATA AVAILABILITY STATEMENT**

The data and codes used in this article will be shared on request.
and comparing with Eq. A3 we find:

\[ \delta S[g^*, M] = 0. \] (A2)

The AoS is defined as \( S^{on-sh} \equiv S[g^*, M] \).

**Theorem:** \( S^{on-sh} = S[g^*, \partial M] \) where \( \partial M \) is a boundary of \( M \).

**Proof:** Consider a partition of \( M = M_1 + M_2 \):

\[ S[g^*, M] = S[g^*, M_1] + S[g^*, M_2], \] (A3)

it follows from Eq. A2:

\[ S[g^*, M] = S[g^* + \delta g, M_1] + S[g^* + \delta g, M_2]. \] (A4)

Because \( \delta g \) can be arbitrary we can choose:

\[ \delta g = \begin{cases} \delta g_1 & \text{for } M_1 \\ 0 & \text{for } M_2 \end{cases} \] (A5)

so that:

\[ S[g^*, M_1] = S[g^* + \delta g_1, M_1], \] (A6)

and comparing with Eq. A3 we find:

\[ S[g^*, M_1] = S[g^* + \delta g_1, M_1], \] (A7)

so that the action is also stationary as in Eq. A2 for any part of \( M \). Consider a volume element \( dM \) centered around a coordinate \((t, x)\) of the manifold. We can choose \( M_1 = dM(t, x) \) so that each step in the integration in Eq. A1 is stationary:

\[ \delta S[g^*, dM(t, x)] = 0. \] (A8)

The solution \( g^*(t, x) \) evolves when we move one step \( dM \) away from \((t, x)\). We denote this \( \delta g^*(t, x) \) to distinguish it from some arbitrary variation of the field \( \delta g \). Because the action for each volume element \( dM(t, x) \) is stationary around an arbitrary variation of \( \delta g \), it should also be stationary if we choose \( \delta g = \delta g^*(t, x) \), so that:

\[ S[g^*, dM(t, x)] = S[g^* + \delta g^*(t, x), dM(t, x)]. \] (A9)

which shows that the evolution of the field in step \( dM \) does not change the action. As this is true for each step in the action integral, we can conclude that the action integral value on-shell is the same as the one in its boundary: \( S[g^*, M] = S[g^*, \partial M] \). Q.E.D.

**Corollary-I:** If the Universe has a asymptotically Minkowski boundary, the AoS must be zero, \( S^{on-sh} = 0 \), for its whole evolution.

**Proof:** Because the Minkowski metric has \( S^{on-sh} = 0 \), we can use the AoS Theorem to conclude that \( S^{on-sh} = 0 \) also for the full evolution. Q.E.D.

An example of Corollary-I is the zero Gaussian flux condition presented in section 2.2. Another example is the Schwarzschild metric, which asymptotically goes to Minkowski and produces \( S^{on-sh} = 0 \) (if we avoid the singularity at \( r = 0 \)). Other examples where \( S^{on-sh} = 0 \) are the Dirac action or the harmonic oscillator action. More generally, the kinetic term of a field theory \( L_m = \psi_i L_j \psi_j \) (where \( L_{ij} \) is a linear operator) is zero on-shell because the equation of motion is \( L_{ij} \psi_j = 0 \). Potential terms are usually assumed to be zero at the boundaries because of causality.

**Corollary-II:** If the Universe started out of nothing, the AoS must be zero, \( S^{on-sh} = 0 \), for its whole evolution.

**Proof:** The solution for the field equation for a Universe without content \( T_{\mu\nu} = 0 \) is the Minkowski metric. Using Corollary-I we conclude that \( S^{on-sh} = 0 \) also for the full evolution. Q.E.D.

**APPENDIX A: AOS THEOREM**

We next enunciate and prove what we call the Action on-shell (AoS) Theorem, which basically says that the AoS is fixed by its boundary. This is reminiscent of the holographic principle (Susskind 1995; Maartens 2004). Indeed the Einstein–Hilbert action in Eq. 1 possesses such holographic relation between the surface and bulk terms (see Eq. 15.45 in Padmanabhan 2010).

This theorem applies to any stationary action principle. But we note that the proof presented here lacks the mathematical rigor of the Stokes’ Theorem which was used to derive the zero action principle in section 2.2. Our arguments below assume continuous and smooth functions and do not account for situations with singularities.

**Definitions:** given an action functional:

\[ S[g, M] = \int_M dM L_m[g], \] (A1)

where \( M \) is a spacetime manifold and \( dM = \sqrt{-g} d^4x \) is the invariant 4D volume and \( g = g(x) \) stands for any dynamical field in our problem, including the metric \( g^{\mu\nu} \). Assume that \( S \) obeys an stationary action principle, i.e. the equations of motion for \( g \) are such that arbitrary small variation \( \delta g \) around the solution \( g^* \) produce no changes in \( S \), so that:

\[ \delta S[g^*, M] = \delta S[g^* + \delta g, M] = 0. \] (A2)

The AoS is defined as \( S^{on-sh} = S[g^*, M] \).

**Theorem:** \( S^{on-sh} = S[g^*, \partial M] \) where \( \partial M \) is a boundary of \( M \).

**Proof:** Consider a partition of \( M = M_1 + M_2 \):

\[ S[g^*, M] = S[g^*, M_1] + S[g^*, M_2], \] (A3)

it follows from Eq. A2:

\[ S[g^*, M_1] = S[g^* + \delta g, M_1] + S[g^* + \delta g, M_2]. \] (A4)

Because \( \delta g \) can be arbitrary we can choose:

\[ \delta g = \begin{cases} \delta g_1 & \text{for } M_1 \\ 0 & \text{for } M_2 \end{cases} \] (A5)

so that:

\[ S[g^*, M_1] = S[g^* + \delta g_1, M_1], \] (A6)

and comparing with Eq. A3 we find:

\[ S[g^*, M_1] = S[g^* + \delta g_1, M_1], \] (A7)

APPENDIX B: THE ACTION FOR THE FLRW METRIC

To estimate Eq. 8 for a flat matter dominated FLRW metric, we choose our location as the origin of the radial coordinate and the end of inflation as the initial condition. For simplicity consider the flat matter dominated Universe in the FLRW metric in Eq. 26 and Eq. 27. The 4D volume element is \( dM = dt \sqrt{-g} d^3x = dt a^4 \pi^2 d^3x \) where we have used: \( \sqrt{-g} = a^3 \). As we integrate in the radial direction we need to update time to the light cone \( dt = d\chi / a \) because the solution of the field equations obey causality (on-shell). The integral them becomes integral over \( \chi = \chi(a) \) (see Eq. 29):

\[ S^{on-sh}_{FLRW} = - \frac{1}{2} \int_0^a d\chi 4\pi^2 a^3 (\rho_m a^{-3}) = -V \rho_m \frac{a^4}{2} \] (B1)

where \( V = 4/3 \pi^3 \) is the 3D comoving volume and \( \rho_m \) the matter density today \( (a = 1) \). This integral is unbounded and diverges as we increase the comoving volume \( V \) (or the time involved). This illustrates the issue with the FLRW metric that we mentioned in the introduction. It produces boundary terms because it has a non compact geometry. This is an issue because we can not reproduce Eq. 2 from Eq. 1, as boundary terms don’t cancel out. In the case of a
FLRW model with $\Lambda$, we can use Eq.16 to find:

$$S_{\text{FLRW}}^n = V \left( \frac{\Lambda - a^3 > \frac{\rho_m}{2}}{8\pi G} \right) \quad (B2)$$

$$< a^3 > = \frac{1}{V} \int_0^\chi d\chi \ 4\pi \chi^2 a^3 \quad (B3)$$

We can see here how we can fix $\Lambda = 4\pi G \rho_m / < a^3 >$ to cancel this boundary term. Note that in this case $V$ is finite because there is a maximum value for $\chi$ (see Fig.1). If we also add $\rho_{vac}$ we find: $\rho_{\Lambda} = \rho_m / (2 < a^3 >)$ which cancels $\rho_{vac}$ (see §3.1).

**APPENDIX C: THE BLACK HOLE UNIVERSE (BHU)**

The idea of the causal boundary $\chi_3$ results in a non-homogeneous universe. This makes sense for a universe of finite age because there are causally disconnected regions with different densities (see Fig.4). For $\chi < \chi_3$ the FLRW metric should be a good approximation, but we want an asymptotically flat metric on larger scale. How can we combine these two different metrics?

Consider the flat FLRW metric of Eq.26 in comoving coordinates $(t, \chi)$. When $\rho_\Lambda$ dominates the expansion rate: $H^2 = (a/a)^2 = 8\pi G \rho_\Lambda / 3$, which results in exponential inflation: $a(t) \propto e^{Ht}$. We can change variables from comoving $\chi$ to proper $r = a(t) \chi$ distances, and introduce a new time variable $T = T(t, \chi)$ such that $\partial_T T = (1 - Hr^2)^{-1}$ and $\partial_T T = aHr \partial_r T$. In these new variables $(T, r)$, the FLRW metric becomes DeSitter:

$$ds^2 = -(1 - H^2 r^2) dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 dA^2 \quad (C1)$$

where $H^2 = 8\pi G \rho_\Lambda / 3$ is constant. So in $(T, r)$ coordinates the FLRW expansion is static (see Mitra 2012). DeSitter metric above is very similar to the Schwarzschild metric:

$$ds^2 = -(1 - 2Gm/r) dt^2 + \frac{dr^2}{1 - 2Gm/r} + r^2 dA^2 \quad (C2)$$

with an Event Horizon (EH) at $r_g = 2Gm$. Both metrics are static. DeSitter metric also has an EH at $r_g = 1/H$. But DeSitter solution corresponds to the interior $r < r_{EH}$, while the Schwarzschild metric represents the exterior $r > r_{EH}$. We can match both solutions into a Black Hole Universe (BHU) where we consider DeSitter metric as the interior of the Schwarzschild metric:

$$ds^2 = -[1 - (r/r_g)^n] dt^2 + \frac{dr^2}{1 - (r/r_g)^n} + r^2 dA^2, \quad (C3)$$

where $n = \begin{cases} 2 & \text{for } r < r_g \\ -1 & \text{for } r > r_g \end{cases}$

which is continuous and smoothed everywhere except at the EH (see Fig.C1). This results in $r_g = 1/H = 2mG$, which gives:

$$\rho_\Lambda = \frac{3m}{(4\pi r_g^3)}, \quad (C4)$$

$$m = \frac{4\pi}{3} r_g^3 < \rho_m / 2 + \rho_R >. \quad (C5)$$

where in the second equation we have use Eq.23. The expansion of our universe is not always dominated by $\rho_\Lambda$. But Eq.C1 is still the exact transformation of the FLRW metric in the Schwarzschild frame $(T, r)$, by just replacing $H$ by $H(t)$ (Gaztanaga 2021). Thus, the FLRW metric also has an EH at $r_H \equiv 1/H(t) < r_g$. But this is an expanding horizon that grows inside $r_g$. As $r_H$ approaches $r_g$ a repulsive force (from the $\Lambda$ term or the EH) forces $r_H = r_g$ (see Fig.1). So Eq.C3-C4 is valid throughout cosmic evolution (for more details on the BHU see Gaztanaga 2021).

In summary, we can combine the FLRW metric and the Schwarzschild metric into a BHU to obtain the causal structure required by the zero action principle: $S^{\text{on-sh}} = 0$.

**APPENDIX D: THE APOLLONIAL MULTIVERSE**

The zero action principle could also be used to build a multiverse of BHU. We can picture the start of our universe as an empty Minkowski space-time with quantum fluctuations of some field $\phi_1(a)$ with equation of state $\omega$ or just false vacuum BH (Gaztanaga 2021). Because $S^{\text{on-sh}} = 0$, this fluctuation will generate a boundary $\Lambda$ term (also call a BH). We can consider these fluctuations homogeneous inside some (quantum) causal scale $a_\chi$. The Hubble rate is then:

$$H^2 = \frac{8\pi G}{3} \left[ \rho_\Lambda + \rho_1 (a/\bar{a})^{-3(1+\omega)} \right], \quad (D1)$$

where $\bar{a}_i$ is some reference primordial time and from Eq.30:

$$\rho_\Lambda = \rho_1 - 1 + 3\omega \left[ \int_0^{\chi_3} d\chi S^2_k(\chi) a^3 (a/\bar{a})^{-3(1+\omega)} \right] \quad (D2)$$

where $\chi_3$ $(and \ a_\bar{a} \equiv a(\chi_3))$ corresponds to the causal scale. For $a_\bar{a} < a_i$ and $\omega > -1$ (or $a_\bar{a} > a_i$ and $\omega < -1$) the $\rho_\Lambda$ term dominates over $\rho_1 (a/\bar{a})^{-3(1+\omega)}$. This will generate an inflationary expansion. The quantum fluctuation will be enlarged and become macroscopic. This inflation will also make the Hubble horizon smaller than $\chi_3$ (see Fig.1) so our observable Universe will eventually become quite large but trapped inside $\chi_3$. This will happen in different locations of the initial Minkowski manifold so that we can end up with multiple disconnected and inflating Universes. Each Universe will have different $\chi_3$ and $\rho_\Lambda$ and they will inflate until they touch each other creating an Apollonial fractal, as illustrated by Fig.4.

Another way to picture this, for an external observer, is that we will have a network of interacting BHU (see Appendix SC) which could also be in equilibrium with some external (Hawking) radiation or matter fields. It seems reasonable to predict that their interactions could end inflation inside some BHU (e.g. because of the dynamical collisions) or just because false vacuum decay or false vacuum rolling.
The dynamical energy of the expansion could transform into thermal energy, in a similar way to the reheating mechanism in traditional inflationary models. Inflation will happen under quite general considerations according to the zero action principle. The principle will still apply to each separate Universe (with some influence of its surroundings) because the interaction with nearby horizons will be limited in time and space. Later quantum fluctuations within one universe represent negligible contribution and do not change $\chi^2$ or $H$.

Our universe is now entering a second inflationary phase and it is possible that we could cycle to a situation similar to our previous inflation. So repeated phases of inflation could also play a role to understand our Universe and the value of the cosmological parameters. Could this result in an eternal return? or in natural selection (Smolin 1992)? Details of this model need to be worked out.