ASYMPTOTIC LEVEL STATE DENSITY FOR PARABOSONIC STRINGS

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Abstract

Making use of some results concerning the theory of partitions, relevant in number theory, the complete asymptotic behavior, for large $n$, of the level density of states for a parabosonic string is derived. It is also pointed out the similarity between parabosonic strings and membranes.

PACS number(s): 03.70 Theory of quantized fields
11.17 Theory of strings and other extended objects

1 Introduction

It is quite standard nowadays to describe the quantum field theory in terms of operators obeying canonical commutation relations. However, there exists the alternative logical possibility of para-quantum field theory [1, 2], where parafields satisfy tri-linear commutation relations. Later, the Green’s proposal was investigated in ref. [3]. We also would like to remind that parastatistics is one of the possibilities found by Haag and coworkers [4, 5] in a general study of particle statistics within the algebraic approach to quantum field theory. Despite the efforts to apply parastatistics for the description of internal symmetries (for example, in paraquark models [2]) or even in solid state physics for the description of quasiparticles, no experimental evidence in favour of the existence of parafields has been found so far.

Nevertheless, parasymmetry still can of be of some interest from the mathematical point of view. For example, it can be considered as formal extension of the supersymmetry algebra. Furthermore, some connections with W-symmetry can also be found. Moreover, parasymmetry may find some physical application in string theory, where parastrings [6] have been constructed. It has been showed there that these parastrings possess some interesting properties, like the existence of critical dimensions different from the standard ones, i.e. $D = 10$ and 26.

The present work is devoted to the evaluation of the asymptotic behaviour of the level state density for parabosonic strings. A complete and rigorous mathematical result is obtained, on the basis of Meinardus’s theorem. Connections with membranes and some applications are briefly discussed.

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2 Partition Function for Parabosonic Strings

Here, we will briefly review the paraquantization for parabose harmonic oscillators relevant to the parabosonic string, in the limit $p = \infty$, where $p$ is the order of the paraquantization. The Hamiltonian and the zero point energy for the free parabose system has the form

\[
H = \sum_n \frac{\omega_n}{2} (a_n^\dagger a_n + a_n a_n^\dagger) - E_0
\]

\[
E_0 = \frac{p}{2} \sum_n \omega_n.
\]

The operators $a_n$ and $a_n^\dagger$ obey the following tri-commutation relations [1, 2]

\[
\begin{align*}
[a_n, \{a_m^\dagger, a_l\}] &= 2 \delta_{nm} a_l, \\
[a_n \{a_m, a_l\}] &= 0.
\end{align*}
\]

The vacuum will be chosen to satisfy the relations

\[
a_n |0 > = 0, \quad \{a_n^\dagger, a_m\} |0 > = p \delta_{nm} |0 > ,
\]

so that $H|0 >= 0$. The paracreation operators $a_n^\dagger$'s do not commute and therefore the Fock space is complicated [4]. For the D-dimensional harmonic oscillators $a_n^i$ of parabosonic string with frequencies $\omega_n^i = n$, Eqs. (4) leads to the Hamiltonian

\[
H = \sum_{i=1}^{D} \sum_{n=1}^{\infty} \frac{n}{2} \{a_n^i, a_n^i\} - E_0.
\]

A closed form for the partition function $Z(t) = \text{Tr} e^{-tH}$, the trace being computed over the entire Fock space, in the limit $p \to \infty$, reads (see, for example, [7])

\[
Z(t) = \text{Tr} e^{-tH} = \left\{ \prod_{n=1}^{\infty} \frac{1}{1 - e^{-tn}} \right\}^D \left\{ \prod_{n,m=1}^{\infty} \frac{1}{1 - e^{-t(n+m)}} \right\} \left\{ \prod_{n=1}^{\infty} (1 - e^{-t2n}) \right\}^D.
\]

3 Asymptotic Behavior of the Partition Function

Our aim is to evaluate, asymptotically, the degeneracy or state level density corresponding to a parabosonic string, in the limit of infinite paraquantization order parameter. As a preliminary result we need the asymptotic expansion of the partition function for $t \to 0$. To this aim, it may be convenient to work with the quantity

\[
F(t) = \log Z(t) = -DF_1(t) + \frac{D}{2} F_1(2t) - \frac{D^2}{2} F_2(t),
\]

where we have introduced the definitions

\[
F_1(t) = \sum_{n=1}^{\infty} \log(1 - e^{-tn})
\]

\[
F_2(t) = \sum_{n,m=1}^{\infty} \log(1 - e^{-t(n+m)}).
\]
With regards to the two first contributions, one may use the following result, known in the theory of elliptic modular function (Hardy-Ramanujan)

\[
F_1(t) = \frac{\pi^2}{6t} - \frac{1}{2} \log \left(\frac{t}{2\pi}\right) + \frac{t}{24} + F_1(4\pi^2).
\]

(10)

A simple proof of the above identity is presented in Appendix.

Let us now consider the quantity \( F_2(t) \). A Mellin representation gives

\[
\log(1 - e^{-at}) = -\frac{1}{2\pi i} \int_{\text{Re} z = c > 2} dz \Gamma(z) \zeta(1 + z) a^{-z} t^{-z}.
\]

(11)

As a result,

\[
F_2(t) \equiv \sum_{n,m} \log(1 - e^{-t(n+m)}) = -\frac{1}{2\pi i} \int_{\text{Re} z = c > 2} dz \Gamma(z) \zeta(1 + z) \zeta_2(z) t^{-z},
\]

(12)

where

\[
\zeta_2(z) \equiv \sum_{n,m} \frac{(n + m)^{-z}}{n},
\]

(13)

\( \zeta(z) \) being the Riemann zeta function. Now it is easy to show that (see for example [8])

\[
\zeta_2(z) = \zeta(z - 1) - \zeta(z).
\]

(14)

So we have

\[
F_2(t) = G_2(t) - F_1(t),
\]

(15)

where

\[
G_2(t) = -\frac{1}{2\pi i} \int_{\text{Re} z = c > 2} dz \Gamma(z) \zeta(1 + z) \zeta(z - 1) t^{-z}.
\]

(16)

Thus, we can deal only with \( G_2(t) \). Concerning this quantity, we note that one can present it in the form

\[
G_2(t) = \sum_{n=1}^{\infty} \log \left(1 - e^{-tn}\right)^n.
\]

(17)

The related generating function is

\[
g_2(t) = \prod_{n=1}^{\infty} (1 - e^{-tn})^n.
\]

(18)

For the estimation of the small \( t \) behavior, we can use the following result on the asymptotic of the partition functions which admit infinite product as associated generating function. To be specific, we shall employ results due to Meinardus [3, 10].

Let us introduce the generating function

\[
f(z) = \prod_{n=1}^{\infty} [1 - e^{-zn}]^{-a_n},
\]

(19)
where $\text{Re } z > 0$ and $a_n$ are non-negative real numbers. Let us consider the associated Dirichlet series
\[ D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s = \sigma + ir, \] (20)
which converges for $0 < \alpha < \sigma$. We assume that $D(s)$ can be analytically continued in the region $\sigma \geq -C_o$ ($0 < C_o < 1$) and here $D(s)$ is analytic except for a pole of order one at $s = \alpha$ with residue $A$. Besides we assume that
\[ D(s) = O(|t|^C_1), \] (21)
uniformly in $\sigma \geq -C_o$ as $|t| \to \infty$, where $C_1$ is a fixed positive real number. The following lemma \[9, 10\] is useful with regard to the asymptotic properties of $f(z)$, for $z \to 0$, $z = t + 2\pi ix$.

**Lemma**

If $f(z)$ and $D(s)$ satisfy the above assumptions, then
\[ f(z) = \exp \{ A \Gamma(\zeta(1 + \alpha)z^{-\alpha} - D(0) \log z + D'(0) + O(t^{C_o}) \} \] (22)
uniformly in $x$ as $t \to 0$, provided $|\arg z| \leq \pi/4$ and $|x| \leq 1/2$; there exists a positive $\varepsilon$ such that
\[ f(t + 2\pi ix) = O(\exp \{ A \Gamma(\alpha)\zeta(1 + \alpha)t^{-\alpha} - Ct^{-\varepsilon} \}), \] (23)
uniformly in $x$ with $y^\rho \leq |x| \leq 1/2$, as $y \to 0$, where
\[ \rho = 1 + p/2 - p\nu/4, \quad 0 < \nu < 2/3, \] (24)
and $C$ a fixed real number.

We sketch the proof of the Lemma. One has to use the Mellin-Barnes representation of the function $\log f(z)$, namely
\[ \log f(z) = \frac{1}{2\pi i} \int_{1+\alpha-i\infty}^{1+\alpha+i\infty} z^{-s} \zeta(1+s) \Gamma(s) D(s) ds. \] (25)
The integrand in the above equation has a first order pole at $s = \alpha$ and a second order pole at $s = 0$. Therefore shifting the vertical contour from $\text{Re } z = 1 + \alpha$ to $\text{Re } z = -C_o$ (due to the conditions of the Lemma the shift of the line of integration is permissible) and making use of the theorem of residues, one obtains
\[ \log f(z) = A \Gamma(\alpha) \zeta(1 + \alpha)z^{-\alpha} - D(0) \log z + D'(0) + \frac{1}{2\pi i} \int_{-C_o-i\infty}^{-C_o+i\infty} z^{-s} \zeta(s+1) \Gamma(s) D(s) ds. \] (26)
The first part of the Lemma follows from Eq. (26), since the absolute value of the integral in the above equation can be estimated to behave as $O(y^{C_o})$. In a similar way, one can prove the second part of the Lemma but we do not dwell on this derivation, and we refer to ref. \[11\] for details.

Now let us apply the lemma to the generating function $g_2(t)$. Obviously we have $D(s) = -\zeta(s-1)$, $\alpha = 2$, $A = -1$. According to Meinardus’ lemma, for small $t$, we arrive at the following asymptotic expansion
\[ G_2(t)(t) \simeq -\zeta(3) t^{-2} - \frac{1}{12} \log t - \zeta'(-1) + O(t). \] (27)
Collecting all the results, we have proved
**Proposition** The quantity $F(t) = \log Z(t)$ admits the following asymptotic expansion, for $t \to 0$

\[
F(t) = \left(\frac{D^2}{2} - D\right) F_1(t) + \frac{D}{2} F_1(2t) - \frac{D^2}{2} G_2(t)
\]

\[
\simeq \frac{D^2}{2} \zeta(3) t^{-2} + \log(\frac{B D - 5 D^2}{24}) + \frac{D^2}{2} \zeta'(-1) + O(t^{-1}).
\]

(28)

As a consequence, the asymptotic behaviour for small $t$ of the quantity $Z(t)$ reads

\[
Z(t) \simeq A t^B \exp(C t^{-2}),
\]

(29)

where

\[
A = \pi \frac{D^2}{2} \frac{D^2 - 2 D}{4} e^{\frac{D^2}{2} \zeta'(-1)}
\]

\[
B = \frac{6 D - 5 D^2}{24}
\]

\[
C = \frac{D^2}{2} \zeta(3).
\]

(30)

Note that, in ordinary string theory, the asymptotic behaviour of $Z_1(t)$ is of the kind $\exp(\alpha t^{-1})$. We shall see in the next section the consequence of this different asymptotic behaviour.

### 4 Asymptotic Level State Density for Parabosonic String

The degeneracy or density of levels can easily be calculated starting from the above asymptotic behaviour. In fact the density of level for parabosonic string (for a general discussion on parabronics, see [6]) may be defined by

\[
\text{Tr} e^{-z H} = 1 + \sum_{n=1}^{\infty} \hat{v}(n) e^{-z n},
\]

(31)

The Cauchy integral theorem gives

\[
\hat{v}_n = \frac{1}{2\pi i} \oint e^{z n} Z(z) \, dz,
\]

(32)

where the contour integral is a small circle about the origin. For $n$ very large, the leading contribution comes from the asymptotic behavior for $z \to 0$ of $Z(z)$. Thus, making use of the Eq. (29), we may write

\[
\hat{v}_n \simeq \frac{A}{2\pi i} \oint z^B e^{z n + C z^{-2}} \, dz.
\]

(33)

A standard saddle point evaluation, or Meinardus’s main theorem, gives as $n \to \infty$

\[
\hat{v}_n \simeq \hat{C}_1 n^{-\frac{B+2}{2}} \exp\left(\hat{b}_1 n^{\frac{2}{3}}\right)
\]

(34)

with

\[
\hat{C}_1 = A \left(\frac{2C}{6}\right)^{\frac{2B+1}{6}} \sqrt{\frac{6\pi}{\sqrt{6\pi}}}
\]

(35)

and

\[
\hat{b}_1 = \frac{3}{2} (2C)^{\frac{1}{3}} = \frac{3}{2} (D^2 \zeta(3))^{\frac{1}{3}}.
\]

(36)
5 Concluding remarks

Eqs. (29), (30) and (34)-(36) are the main result of this paper. The factor $\hat{b}_1$ is in agreement with ref. [7], where, however, the prefactor $\hat{C}_1$ was missing. Here, with the help of Meinardus’s techniques, we have been able to compute it.

The asymptotic behaviour given by Eq. (34) should be compared with the one of the ordinary bosonic string, whose corresponding partition function reads

$$Z_1(t) = \prod_{n=1}^{\infty} (1 - e^{-tn})^{-a}$$

(37)

with non-negative $a$. For example, for the open bosonic string, $a = D - 2$. Now $D(s) = a\zeta(s)$, $\alpha = 1$ and Meinardus’s theorems lead to the well known level density asymptotic behaviour for large $n$ (a derivation, on this line, can be found in ref. [12])

$$v(n) \simeq C_1 n^{-(a+3)/4} \exp (b_1 \sqrt{n})$$

(38)

where the constants $C_1$ and $b_1$ are given by

$$C_1 = (2\pi)^{-\frac{a-1}{2}} \left(\frac{a}{48}\right)^{1/2}$$

(39)

$$b_1 = \pi \sqrt{\frac{2a}{3}}$$

(40)

In this case, since $n \simeq m^2$, the density of states as a function of mass is (see for example [13])

$$\rho(m) \simeq m^{\frac{a+1}{2}} \exp b_1 m$$

(41)

It is well known that this behavior leads to the existence of the critical Hagedorn temperature, related to the coefficient $b_1$. For example, the inverse of the Hagedorn temperature for the critical bosonic string ($D = 26$) is $\beta_c = b_1 = 4\pi$.

On the contrary, if we consider q-branes, higher-dimensional generalizations of strings, compactified on a manifold with the topology $(S^1)^q \times R^{D-q}$, within the semiclassical quantization, one can show that, asymptotically, the level density behaves

$$q(n) \simeq C_q(n) \exp B_q n^{\frac{q}{q+1}}$$

(42)

where the the prefactor $C_q(n)$ and the coefficient $B_q$ can be found in ref. [12]. As a consequence, the parabosonic string, in the limit of infinite paraquantization parameter, behaves as an ordinary membrane ($q=2$)! In this sense, the parabosonic string seems to have similarity with black holes, since the asymptotic behavior of the level density as a function of mass for black holes is similar to the one of q-branes. There is some indication that canonical partition function for q-branes does not exist. Thus, with regard to Hagedorn temperature, the situation for parabosonic strings may be similar to membranes. Hence, the concept of Hagedorn temperature might be meaningless for parastrings. To answer this question, one needs to do careful considerations within the formalism for extended objects at non-zero $T$, which has not developed yet.
Acknowledgments

SDO thanks the Generalitat de Catalunya for financial support

6 Appendix

The proof of the Hardy-Ramanujan formula can be done by making use of the Mellin representation, namely

\[ F_1(t) \equiv \sum_{n=1}^{\infty} \log(1 - e^{-tn}) = -\frac{1}{2\pi i} \int_{\text{Re} z = c} dz \Gamma(z) \xi(1 + z) \zeta(z) t^{-z}, \]  

with \( c > 1 \), and \( \zeta(s) \) is the Riemann zeta-function. Shifting the line of integration from \( \text{Re} z = c > 1 \) to \( \text{Re} z = c' \), \(-1 < c' < 0\), noting that the integrand has a first-order pole at \( z = 1 \) and a second-order pole at \( z = 0 \) one arrives at

\[ F_1(t) = -\frac{\pi^2}{6t} - \frac{1}{2} \log(t\pi) - \frac{1}{2\pi i} \int_{\text{Re} z = c'} dz \Gamma(z) \xi(1 + z) \zeta(z) t^{-z}. \]  

Making the change of variable in the complex integral \( z = -s \) and using the functional equations for the functions \( \Gamma(s) \) and \( \xi(s) \) we get

\[ F_1(t) = -\frac{\pi^2}{6t} - \frac{1}{2} \log(t\pi) - \frac{1}{2\pi i} \int_{\text{Re} z = c''} dz \Gamma(z) \xi(1 + z) \zeta(z) (4\pi^2 t)^{-s}. \]  

where \( 0 < c'' < 1 \). The identity (4) is obtained shifting the line of integration to \( \text{Re} z = c'' > 1 \), and taking into account of the first-order pole at \( z = 1 \).

References

[1] H. S. Green. Phys. Rev., 90, 70, (1953).
[2] Y. Ohnuki and Kamefuchi. Quantum Field Theory and Parastatistics. University of Tokyo Press, (1982).
[3] O.W. Greenberg and A.M.L. Messiah. Phys. Rev., 138, 1155, (1965).
[4] S. Doplicher, R. Haag and J. Roberts. Commun. Math. Phys., 23, 199, (1971).
[5] S. Doplicher, R. Haag and J. Roberts. Commun. Math. Phys., 35, 49, (1974).
[6] F. Ardalan and F. Mansouri. Phys. Rev., 9, 3341, (1987).
[7] M. Hama, M. Sawamura and H. Suzuki,. Progr. Theor. Phys., 88, 149, (1992).
[8] A. A. Actor. J. Phys. A: Math. Gen., 20, 927, (1987).
[9] G. Meinardus. Math. Z., 59, 338, (1954).
[10] G. Meinardus. Math. Z., 61, 289, (1954).
[11] G.E. Andrews. The Theory of Partitions. Addison-Wesley, (1976). Encl. of Math. and applications.
[12] A. Bytsenko, K. Kirsten and S. Zerbini. Phys. Lett., 304, 235, (1993).
[13] M.B. Green, J. Schwarz and E. Witten. Superstring Theory. Cambridge University Press, (1987).