Separable games

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We introduce the notion of separable games, which refines and generalizes that of graphical games. We prove that there exists a minimal splitting with respect to which a game is separable. Moreover we prove that in every strategic equivalence class, there is a game separable with respect to the minimal splitting in the class. This game is also graphical with respect to the smallest graph in the class, which represent a minimal complexity graphical description for the game. We prove a symmetry property of the minimal splitting of potential games and we describe how this property reflects to a decomposition of the potential function. In particular, these last results strengthen the ones recently proved for graphical potential games. Finally, we study the interplay between separability and the classical decomposition of games proposed by [5], characterizing the separability properties of each part of the decomposition.

1 INTRODUCTION

Graphical games (see [9] and [1, Chapter 7]) are games equipped with a network structure among players that specifies the pattern of dependence of their utilities. More precisely, in a graphical game players are identified with the nodes of a graph and the utility of each player depends only on her own action and the action of players that correspond to her neighbors in the graph.

They have recently become prominent as a unifying theory to study the emergence of global phenomena in socio-economic networks like peer effects, technology adoption, and consensus formation [8]. They are also a natural model in engineering and computer science to describe the interaction of multi-agent systems and a powerful tool to design distributed algorithms [6].

While there is already a large amount of literature focusing on specific graphical games (e.g. coordination and anti-coordination games), it is still missing a general theory. How the graphicality of a game reflects on its properties is still largely unexplored. A remarkable exception is constituted by the paper [3] where authors prove that a potential graphical game admits a potential reflecting the graphical structure, precisely, one that can be decomposed as a sum of terms defined on the maximal cliques of the graph.

The contribution of this paper is threefold. First of all, we study graphical games up to the so called strategic equivalence, meaning that we are only concerned with variations of the utility of a player when he modifies its action rather then their absolute values. This is typical in technological contexts where the game rather than an intrinsic model is the result of an explicit design. Classical evolutionary dynamics associated to games like the best-response dynamics or the log-likelihood dynamics are invariant with respect to this equivalence. Second, we consider a refined version of the concept of graphicality, called separability, that takes into consideration the way utility functions decompose as sum of functions only depending on specified subsets (separations) of neighbors’ actions. This generalizes the concept of pairwise graphical games (like coordination games) where the utility of each player \(i\) is representable as the sum of atomic utilities of 2-players games only depending on the action of \(i\) and the action of one of her neighbors. Finally, we focus on a classical decomposition of games in terms of non-strategic, potential and harmonic parts introduced in [5]. An analysis of some strategical aspect of such decomposition has been carried out in [2] where authors point out some drawback of it and propose a generalization to overcome those issues. In our work, instead, we undertake a fundamental analysis on how such concepts interact with the separability of a game.

We present three fundamental results that are Theorem 1, Theorem 2 and Theorem 3. In Theorem 1 we prove that in any strategically equivalent class of games there exists a game that is separable with respect to a minimal splitting and consequently graphical with respect to a minimal graph. This
game has a simple expression and satisfies a certain normalization condition. Theorem 2 concerns potential games. It asserts that the minimal splitting considered in Theorem 1 must satisfy in this case a symmetry property and that the potential admits a minimal decomposition with respect to the corresponding separation. This, in particular, implies, and in fact refines, the result in [3]. Remarkably, our result also implies the Hammersley-Clifford theorem, a classical result for Markov random fields [10, Chapter 3]. Theorem 3 regards the way the decomposition in non-strategic, potential and harmonic components reflects the separability of a game. While the non-strategic part is always separable with respect to the same splitting (and then graphical with respect to the same graph) than the original game, the same is not true in general for the potential and harmonic parts that are instead separable with respect to a larger splitting, and then graphical with respect to a larger graph where out-neighbourhoods have become cliques. Intuitively, this means that there are short range hidden strategic interactions which involve only players that directly influence the utility of some common player in the original game. This “interaction enlargement” not only happens and we show that for the important class of pairwise graphical games (where utility of players is the sum of utilities of 2-players games played with their neighbors) actually the potential and harmonic components maintain the original graphicality. We present explicit examples of where this phenomenon shows up.

In the remaining part of this section, we present some notational assumptions that will be used throughout the paper. Section 2 is devoted to the introduction of the concept of separable game with respect to a given splitting, to the discussion of its relation with graphical games and to the presentation of few explicative examples. In Section 3 we study the question of the minimal splitting with respect to which a game, or a game up to strategic equivalence, is separable. Main result of this section is Theorem 1 that also discusses the minimality of the corresponding graphical representation. Section 4 is devoted to potential graphical games and contains the main result of this paper that is Theorem 2. It consists of a complete analysis of the relation between minimal separability of a game and that of its corresponding potential. A discussion of how this result relates to Hammersley-Clifford theorem is finally presented. In Section 5 we analyse the interplay between the classical decomposition of games and their separability. The main result of this section is Theorem 3, which provides a general characterization of the separability property of each component. Corollaries 4 and 5 instead characterize the components of a $\mathcal{G}$-game and a pairwise separable $\mathcal{G}$-game respectively, which are the two extreme cases of the weakest and the strongest separability possible for a $\mathcal{G}$-game. The paper ends with a conclusive section and an appendix where a technical lemma on separable functions is proven.

1.1 Notation

A (directed) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the pair of a finite node set $\mathcal{V}$ and a link set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, whereby a link $(i, j) \in \mathcal{E}$ is meant as directed from its tail node $i$ to its head node $j$. We will consider graphs containing no selfloops, i.e., that $(i, i) \notin \mathcal{E}$ for every $i \in \mathcal{V}$ and we shall denote by $\mathcal{N}_i = \{ j \in \mathcal{V} : (i, j) \in \mathcal{E} \}$ and $\mathcal{N}_i^* = \mathcal{N}_i \cup \{ i \}$ the open and, respectively, closed out-neighborhoods of a node $i$ in $\mathcal{G}$. The intersection of two graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ with the same node set $\mathcal{V}$ is the graph $\mathcal{G}_1 \cap \mathcal{G}_2 = (\mathcal{V}, \mathcal{E})$ where $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$. We shall say that $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ is a subgraph of $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ (equivalently that $\mathcal{G}_2$ is a supergraph of $\mathcal{G}_1$) if $\mathcal{E}_1 \subseteq \mathcal{E}_2$. Note that we shall consider undirected graphs as a special case where $(i, j) \in \mathcal{E}$ if and only of $(j, i) \in \mathcal{E}$. Finally, we introduce the symbol $\subseteq$ to denote the element-wise inclusion of two sets, i.e., for two sets $U = \{ U_i \}$ and $V = \{ V_j \}$, $U \subseteq V$ means that

$$\forall U_i \in U, \quad \exists V_j \in V : U_i \subseteq V_j.$$
2 GRAPHICAL AND SEPARABLE GAMES

Throughout the paper, we shall consider strategic form games with finite nonempty player set $\mathcal{V}$ and finite nonempty action set $\mathcal{A}_i$ for each player $i \in \mathcal{V}$. We shall denote by $X = \prod_{i \in \mathcal{V}} \mathcal{A}_i$ the space of all players’ strategy profiles and, for every player $i \in \mathcal{V}$, let $X_{-i} = \prod_{j \neq i} \mathcal{A}_j$ be the set of strategy profiles of all players except for player $i$. As customary, for a strategy profile $x$ in $X$, the strategy profile of all players except for $i$ is denoted by $x_{-i}$ in $X_{-i}$. We shall refer to two strategy profiles $x$ and $y$ in $X$ as equivalent if their difference is a non-strategic game, i.e., when $x$ and $y$ coincide except for possibly in their $i$-th entry. Let each player $i \in \mathcal{V}$ be equipped with a utility function $u_i : X \rightarrow \mathbb{R}$. We shall identify a game with player set $\mathcal{V}$ and strategy profile space $X$ with the vector $u$ assembling all the players’ utilities. Notice that, in this way, the set of all games with player set $\mathcal{V}$ and strategy profile space $X$, to be denoted by $\mathcal{U}$, is isomorphic to the vector space $\mathbb{R}^{\mathcal{V} \times X}$.

A game $u$ is referred to as non-strategic if the utility of each player $i \in \mathcal{V}$ does not depend on her own action, i.e., if

$$u_i(x) = u_i(y), \quad \forall x, y \in X \text{ s.t. } y \sim_i x. \quad (1)$$

The set of non-strategic games will be denoted by $\mathcal{N}$ and it constitutes a linear subspace of $\mathcal{U}$. Two games $u$ and $\tilde{u}$ are referred to as strategically equivalent if their difference is a non-strategic game, i.e., if

$$u_i(x) - \tilde{u}_i(x) = u_i(y) - \tilde{u}_i(y), \quad \forall x, y \in X \text{ s.t. } y \sim_i x. \quad (2)$$

Strategical equivalence is in fact an equivalence relation on games and we shall denote the strategical equivalence class of a game $u$ by $[u]$. A game $u$ is referred to as normalized if

$$\sum_{y \sim_i x} u_i(y) = 0, \quad \forall x \in X, i \in \mathcal{V}. \quad (3)$$

The normalized version of a game $u$ is the game $\overline{u} \in \mathcal{U}$ with utilities

$$\overline{u}_i(x) = u_i(x) - \frac{1}{|\mathcal{A}_i|} \sum_{y \sim_i x} u_i(y), \quad \forall x \in X, i \in \mathcal{V}. \quad (4)$$

It is then easily verified that the game $\overline{u}$ is both normalized and strategically equivalent to $u$. In fact, $\overline{u}$ is the unique normalized game in the strategic equivalence class $[u]$. [5, Lemma 4.6.]

A class of games that play a key role in the theory are potential games [11]. A game $u \in \mathcal{U}$ is referred to as an (exact) potential game if there exists a potential function $\phi : X \rightarrow \mathbb{R}$ such that

$$u_i(x) - u_i(y) = \phi(x) - \phi(y), \quad (5)$$

for every player $i \in \mathcal{V}$ and every pair of $i$-comparable strategy profiles $x \sim_i y$ in $X$. Graphical games [9] are defined imposing suitable restrictions on the way utilities depend on the strategy profiles. Precisely, a game $u$ is said to be graphical on a graph $G = (\mathcal{V}, \mathcal{E})$ (or, briefly, a $G$-game) if the utility of each player $i \in \mathcal{V}$ depends only on her own action and the actions of fellow players in her neighborhood in $G$, i.e., if

$$u_i(x) = u_i(y), \quad \forall x, y \in X \text{ s.t. } x_i = y_i. \quad (6)$$

Notice that if a game $u$ is graphical with respect to two graphs $G_1 = (\mathcal{V}, \mathcal{E}_1)$ and $G_2 = (\mathcal{V}, \mathcal{E}_2)$, it is also graphical with respect to their intersection $G_1 \cap G_2$. Since every game is trivially graphical on the complete graph on $\mathcal{V}$, we can conclude that to each game $u \in \mathcal{U}$ one can always associate the smallest graph on which $u$ is graphical. We shall refer to such graph as the graph of the game $u$ and denote it as $G_u$. 
Example 1. Consider a game with three players $\mathcal{V} = \{1, 2, 3\}$, binary action spaces $\mathcal{A}_i = \{-1, 1\}$ for $i = 1, 2, 3$, and utility functions

$$u_1(x) = x_1x_2, \quad u_2(x) = x_2(x_1 - x_3), \quad u_3(x) = -x_3x_2.$$ \hfill (7)

It is then straightforward to verify that this is a normalized potential game with potential function $\phi(x) = x_1x_2 - x_2x_3$. Notice that for this game $\mathcal{G}_u$ is an undirected graph with link set $\mathcal{E} = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$ (see Figure 1a).

Example 2. Consider a game with player set and actions spaces as in Example 1 and utility functions

$$u_1(x) = x_1x_2 + (1 - x_1)x_2x_3, \quad u_2(x) = x_2(x_1 - 2x_3) + (x_2 + 1)x_1x_3, \quad u_3(x) = x_1x_2.$$ \hfill (8)

In this case $\mathcal{G}_u$ is the complete graph of order 3 displayed in Figure 1b. The normalized version $\overline{u}$ of $u$ has utilities

$$\overline{u}_1(x) = x_1x_2(1 - x_3), \quad \overline{u}_2(x) = x_2(x_1 - 2x_3 + x_1x_3), \quad \overline{u}_3(x) = 0.$$ \hfill (9)

Its graph $\mathcal{G}_{\overline{u}}$ has link set $\mathcal{E} = \{(1, 2), (1, 3), (2, 1), (2, 3)\}$ and is displayed in Figure 1c. Notice that, albeit the normalized game $\overline{u}$ is strategically equivalent to $u$, in this case $\mathcal{G}_{\overline{u}} \subseteq \mathcal{G}_u$ is a strict subgraph.

3 MINIMAL SEPARABILITY OF GAMES

In this section, we first introduce the notion of separable game (Section 3.1) that refines and generalizes that of graphical game. Then, in Section 3.2 we show that every game admits a minimal splitting with respect to which it is separable. Finally, in Section 3.3 we show that a normalized game is separable on the finest possible splitting and graphical on the smallest possible graph in its equivalence class.

3.1 Separable games

In this subsection we introduce a notion of separability for games that refines and generalizes the notion of graphicality.

First, we introduce the notion of separability for function as follows.

DEFINITION 1. Given a family $J = \{\mathcal{J}_k\}_k$ of subsets of $\mathcal{V}$, a function $f : \mathcal{X} \to \mathbb{R}$ is $J$-separable (equivalently, $J$ is a separation for $f$) if there exist functions $g_k : \prod_{j \in \mathcal{J}_k} \mathcal{A}_j \to \mathbb{R}$, for $\mathcal{J}_k \in J$, such that

$$f(x) = \sum_k g_k(x; \mathcal{J}_k), \quad \forall x \in \mathcal{X},$$ \hfill (10)

i.e., if $f(x)$ can be decomposed as the sum of functions $g_k(x; \mathcal{J}_k)$ each depending on a subset $\mathcal{J}_k \subseteq \mathcal{V}$ of the players.
Now, let $S_i = \{S_{i,h}\}_{1 \leq h \leq k_i}$ be a family of subsets of $\mathcal{V}$, for every player $i$ in $\mathcal{V}$. We shall refer to the array of such families $S = \{S_{i,h}\}_{i \in \mathcal{V}, 1 \leq h \leq k_i}$ as a splitting. Then, we have the following definition of separability for a game.

**Definition 2.** Given a splitting $S = \{S_{i,h}\}_{i \in \mathcal{V}, 1 \leq h \leq k_i}$, a game $u$ is $S$-separable if the utility of each player $i$ in $\mathcal{V}$ is $S_i$-separable, i.e., if it can be decomposed as

$$u_i(x) = \sum_{h=1}^{k_i} u_i^h(x_{S_{i,h}}), \quad (11)$$

where, for $1 \leq h \leq k_i$, $u_i^h : \prod_{j \in S_{i,h}} \mathcal{A}_j \rightarrow \mathbb{R}$ is a function that depends on the actions of players in the subset $S_{i,h}$ only.

Observe that separability of a game is both a generalization and a refinement of the notion of graphicality. Indeed, every $S$-separable game is graphical on the graph $\mathcal{G}(S) = (\mathcal{V}, \mathcal{E})$, whereby each node $i$ in $\mathcal{V}$ has neighborhood $\mathcal{N}_i = \bigcup_{h=1}^{k_i} S_{i,h} \setminus \{i\}$. On the other hand, for a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, every $\mathcal{G}$-game is $S$-separable where $S = \{S_i\}_{i \in \mathcal{V}}$ is the splitting with $k_i = 1$ and $S_i = \{\mathcal{N}_i^*\}$ coinciding with the closed neighborhood of node $i$ in $\mathcal{G}$, for every $i$ in $\mathcal{V}$.

In fact, important classes of graphical games have finer separability properties than the aforementioned trivial one. In particular, for a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a pairwise-separable network game (cf. [4, 7]) on $\mathcal{G}$ is such that the utility of player $i$ in $\mathcal{V}$ can be decomposed in the form

$$u_i(x) = \sum_{j \in \mathcal{N}_i} u_{ij}(x_i, x_j) \quad \forall x \in \mathcal{X}, \quad (12)$$

where $u_{ij} : \mathcal{A}_i \times \mathcal{A}_j \rightarrow \mathbb{R}$ for $i, j \in \mathcal{E}$. Notice that the utilities in (12) clearly define a $\mathcal{G}$-game. In fact, such a game can be interpreted as one in which the players are located at the nodes of the undirected graph $\mathcal{G}^{\leftrightarrow}$ whose links are to be interpreted as two-player games between their endpoints (with the convention that $u_{ij}(x_j, x_i) = 0$ for every link $(i, j) \in \mathcal{E}$ whose reverse is not a link, i.e., such that $(j, i) \not\in \mathcal{E}$). Every player $i$ in $\mathcal{V}$ can chose a unique action $x_i \in \mathcal{A}_i$ to be used in all games she simultaneously participates in and gets a utility that is the aggregate of the utilities from all her outgoing links. Clearly, a pairwise-separable network game with utilities as in (12) is $S$-separable with respect to the granular splitting $S = \{S_{i,h}\}_{i \in \mathcal{V}, 1 \leq h \leq k_i}$, where, for $i$ in $\mathcal{V}$, $S_{i,h} = \{i, j_h\}$ for $1 \leq h \leq k_i$ where $k_i$ is the degree of node $i$ and $\mathcal{N}_i = \{j_h\}_{1 \leq h \leq k_i}$ is the neighborhood of node $i$ in $\mathcal{G}$.

**Example 3 (Network coordination game).** Relevant examples of pairwise-separable network games can be constructed by combining the 2-player coordination games, which are described by the following utility functions.

$$u_{12}(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ -1 & \text{if } x_1 \neq x_2 \end{cases}, \quad u_{21}(x_1, x_2) = u_{12}(x_1, x_2). \quad (13)$$

The coordination game is an example of symmetric 2-player games with binary actions. Notice that it is normalized and potential. Given an undirected graph $\mathcal{G}$ we construct a pairwise-separable graphical game where each pair of adjacent players is involved in a coordination game. In this way we obtain the well known majority game, which is used to model the behaviour of a population of conformist agents whose social interactions are described by $\mathcal{G}$. 

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Example 4 (Best-shot public good game). Consider a graph $G$ and the game where every player $i$ in $V$ has binary action set $A_i = \{0, 1\}$ and utility:

$$u_i(x) = \begin{cases} 
1 - c & \text{if } x_i = 1 \\
1 & \text{if } x_i = 0 \text{ and } x_j = 1 \text{ for some } j \in N_i \\
0 & \text{if } x_i = 0 \text{ and } x_j = 0 \text{ for every } j \in N_i.
\end{cases}$$

The game $u$ constructed in this way is the so called “public good game”. It models a more complex behaviour for the population $V$: players benefit form acquiring some good, represented by taking action 1 and which is public in the sense that it can be lent from one player to another. Taking action 1 has a cost $c$, so players would prefer that one of their neighbors takes that action, but taking the action and paying the cost is still the best choice if no one of their neighbors does. The public good game is a graphical game on $G$ and it is separable with respect to the splitting $S$ where for every node $i$, $k_i = 1$ and $S_i = \{N_i^\ast\}$.

3.2 Minimal separability of a game

Recall that in Section 2 we defined the graph $G_u$ of the game $u$ as the minimal graph $G$ such that $u$ is a $G$-game. It is less trivial to see that the concept of minimal splitting of a game is well defined, as we will show in this subsection.

First, notice the following general fact about separation of functions. Clearly, if a function $f$ is $A$-separable, then it is also $\bar{A}$-separable, where $\bar{A}$ is the family of maximal sets of $A$ (with respect to inclusion). Moreover, for two families $A$ and $B$ of subsets of $V$, let $A \cap B$ be the family

$$A \cap B := \{ C = A \cap B | A \in \bar{A}, B \in \bar{B} \}.$$

Then, the following fundamental result holds true, as proven in Appendix A.

Lemma 1. If a function $f$ is both $A$-separable and $B$-separable, then it is also $C$-separable for $C = A \cap B$.

Lemma 1 allows us to prove the following result guaranteeing the existence of a minimal splitting of a game and showing that the minimal graph of a game is the one associated to its minimal splitting.

Proposition 1. Let $u$ be a game. Then, there exists a minimal splitting $S_u = \{S_{i,h}\}_{i,h}$ such that $u$ is $S_u$-separable. Moreover

$$G_u = G(S_u)$$

where $G(S_u)$ is the graph with node set $V$ and where for every $i \in V$ the neighborhood of player $i$ is

$$N_i(S_u) = \bigcup_h S_{i,h} \setminus \{i\}.$$

Proof. Lemma 1 ensures that the family of splittings of a game $u$ is closed under the intersection $\cap$. In particular, consider a game $u$ which is separable with respect to two splittings $S_1 = \{B_i\}_{i \in V}$ and $S_2 = \{D_i\}_{i \in V}$, where $B_i$ and $D_i$ are separations of $u_i$. Then Lemma 1 directly implies that $u$ is separable with respect to the splitting $S = \{S_i\}_{i \in V}$ where $S_i = A_i \cap B_i$. Together with the fact that every game is trivially separable with respect to the splitting $S = \{S_i\}$ with $S_i = \{V\}$, this implies the first claim.

To prove the inclusion $G_u \subset G(S_u)$ it is sufficient to show that $u$ is graphical on $G(S_u)$. This follows from the fact that $u$ is $S_u$ separable. Indeed we have that for every $i \in V$

$$u_i(x) = \sum_h u_{i,h}(x_{S_{i,h}})$$
so that $u_i$ only depends on $N(S_u)$. To prove the remaining inclusion $G(S_u) \subset G_u$, observe that since $u$ is a $G_u$-game, then it is separable with respect to the splitting $\hat{S} = \{\hat{S}_i\}_{i \in V}$ where $\hat{S}_i = \{N_i^*\}$ and $N_i$ denotes the out-neighborhood of $i$ in $G_u$. Since $S_u$ is minimal, $S_u \leq \hat{S}$, that is

$$\forall i \in V, \forall h, S_{i,h} \subset N_i^*.$$ 

This implies that $N_i(S_u) \subset N_i$, that is $G(S_u) \subset G_u$. \hfill $\square$

### 3.3 Minimal separability of a strategic equivalence class

As illustrated by Example 2 in Section 3.1, two strategically equivalent games $u$ and $\tilde{u}$ might have quite different graphs $G_u$ and $G_{\tilde{u}}$. Indeed, it is easy to see that every game $u$ admits a strategically equivalent game $\tilde{u}$ such that $G_{\tilde{u}}$ is the complete graph. Analogously, strategically equivalent game can have quite different separability properties.

It is not obvious whether a strategic equivalence class $[u]$ always contains a game whose graph is minimal and whose splitting is the finest in the class. This property turns out to hold true, as a consequence of the following result.

**Theorem 1.** Let $u$ be an $S$-separable game. Then its normalization $\bar{u}$ is separable with respect to the splitting $\hat{S} \leq S$ such that

$$\hat{S}_i = \{S_{i,h} : i \in S_{i,h}\}.$$ 

Moreover, for the minimal splitting $S_u$ of $u$, $\hat{S}_u$ is minimal for $\bar{u}$, i.e. $S_u = \hat{S}_u \leq S_u$. Finally, $G_{\hat{S}_u} \subset G_u$.

**Proof.** Let the game $u$ have utilities satisfying (11). Then, for $i \in V$ and $1 \leq h \leq k_i$, observe that the quantity

$$n_{i,h}(x_{S_{i,h} \setminus \{i\}}) = \frac{1}{|A_i|} \sum_{y \in X} u_{i,h}(y_{S_{i,h}})$$

depends only on the actions of players in $S_{i,h} \setminus \{i\}$. Then, define a new game $u^*$ with utilities

$$u^*_i(x) = \sum_{h=1}^{k_i} u^*_i(x_{S_{i,h}}), \quad u^*_i(x_{S_{i,h}}) = u^*_i(x_{S_{i,h}}) - n_{i,h}(x_{S_{i,h} \setminus \{i\}}),$$

for every player $i \in V$ and strategy profile $x$ in $X$.

Now, notice that, since the terms $n_{i,h}(x_{S_{i,h} \setminus \{i\}})$ do not depend on the action $x_i$ of player $i$, the game $u^*$ defined by (15) is strategically equivalent to $u$. We show that each $u^*_i$ satisfies

$$\sum_{y \in X : y_{i} \neq x_{i}} u^*_i(y_{S_{i,h}}) = \sum_{y \in X : y_{i} \neq x_{i}} \left( u^*_i(x_{S_{i,h}}) - n_{i,h}(x_{S_{i,h} \setminus \{i\}}) \right) = \sum_{y \in X : y_{i} \neq x_{i}} u^*_i(x_{S_{i,h}}) - |A_i|n_{i,h}(x_{S_{i,h} \setminus \{i\}}) = 0.$$ 

Then, by linearity, $u^*$ is normalized. But since there exists just one normalized game $\bar{u}$ in the strategic equivalence class $[u]$, this shows that $u^* = \bar{u}$. Moreover, notice that if $i \notin S_{i,h}$, then $n_{i,h}(x_{S_{i,h}}) = u^*_i(x_{S_{i,h}})$. Then from (15) we have that $u^*_i \neq 0$ only if $i \in S_{i,h}$, that is separable with respect to $\hat{S}$. Finally, assume that $\hat{S}$ is minimal for $u$, i.e. $S = S_u$. To show that $\hat{S}$ is minimal for $\bar{u}$, we call $\hat{S}$ the minimal splitting of $\bar{u}$ and we show that $\hat{S} \leq \hat{S}$. Recall that since $u$ is strategically
equivalent to \( \tilde{u} \) we have
\[
    u_i(x) = v_i(x_{-i}) + \tilde{u}_i(x) = v_i(x_{-i}) + \sum_k \tilde{u}_i^h(x_{S_{i,k}}).
\]
for some function \( v_i \) depending only on \( x_{-i} \). The last shows that a separation for \( u_i \) is given by
\[
    \{ V \setminus \{ i \}, \tilde{S}_{i,k} \}_k.
\]
Since by assumption \( S_i \) is the minimal separation for \( u_i \), we have that \( \forall S_{i,h} \in S_i \), either \( i \not\in S_{i,h} \) or \( S_{i,h} \subseteq \tilde{S}_{i,k} \) for some \( k \). Equivalently, if \( i \in S_{i,h} \) (which by definition means that \( S_{i,h} \in \tilde{S} \)) then \( S_{i,h} \subseteq \tilde{S}_{i,k} \). This shows that \( \tilde{S} \) is minimal for \( \tilde{u} \).

To conclude the proof, notice that by Proposition 1 we have \( G_u = G(S) \) and \( G_{\tilde{u}} = G(\tilde{S}) \). Since by construction \( \tilde{S} \subseteq S \), it holds \( G(\tilde{S}) \subseteq G(S) \) so \( G_{\tilde{u}} \subseteq G_u \).

**COROLLARY 1.** Let \( u \) be a game and \( \tilde{u} \) be its normalized version. Then,
\[
    S_{\tilde{u}} \subseteq S_u \quad \text{and} \quad G_{\tilde{u}} \subseteq G_u \tag{16}
\]
for every strategically equivalent game \( \tilde{u} \in [u] \).

**Proof.** It follows immediately from Theorem 1 and the fact that the normalized version of any strategically equivalent game \( \tilde{u} \in [u] \) coincides with the normalized version \( \tilde{u} \) of \( u \).

Corollary 1 implies the existence of a game, the only normalized game \( \tilde{u} \) in \([u]\), that is simultaneously separable with respect to the intersection
\[
    S_{[u]} = \bigcap_{\tilde{u} \in [u]} S_{\tilde{u}}
\]
of the minimal splittings of all the strategically equivalent games in the class \([u]\) and graphical on the intersection
\[
    G_{[u]} = \bigcap_{\tilde{u} \in [u]} G_{\tilde{u}}
\]
of the minimal graphs of all the strategically equivalent games in the class \([u]\). In fact, the graph
\[
    G_{[u]} = G_{\tilde{u}}
\]
may be interpreted as the minimal topological complexity needed to represent a game in the class \([u]\), namely a game up to non-strategic equivalence.

## 4 ON THE STRUCTURE OF POTENTIAL GAMES

In this section we focus on potential games and derive our main result Theorem 2 and Corollaries 2–3, which improve the existing results about the structure of potential functions and games.

As anticipated by Examples 1 and 2, the minimal graph associated to a game can be either directed or undirected. Corollary 2 shows that this is not the case when we restrict to normalized potential games, which only exhibit undirected minimal graphs and thus are characterized by symmetry of interactions between players. Theorem 2 is an extension of this result, showing that the symmetry of interactions in normalized potential games concerns not only binary but also higher order interactions.

To make this formal, we first introduce the following definition.

**DEFINITION 3.** A splitting \( S \) is said to be symmetric if for all \( i \in V \) and \( S_{i,h} \in S \) we have that
\[
    j \in S_{i,h} \implies \exists h' : S_{j,h'} = S_{i,h} \tag{17}
\]
Then, point 2 of Theorem 2 tells that the minimal splitting of normalized potential game is symmetric, meaning that if the utility of a player \( i \) depends jointly on the actions of players \( j \) and \( k \) in the minimal splitting, than \( j \) and \( k \) are jointly dependent on each other and on \( i \). This first structural result allows us to characterize the separability properties of the potential function and to describe the relation between the minimal splitting and the minimal graph. In particular, point 2 shows how the minimal splitting of a normalized potential game is connected to the minimal separation of its potential function. Then, in point 2 we deduce the relation between minimal splitting and minimal graph and thanks to point 2 we are able to translate this property to obtain a corresponding result on the potential function.

**Theorem 2.** Let \( u \) be a normalized potential game with potential \( \phi \). Denote by \( S \) the minimal splitting of \( u \) and by \( T \) the minimal separation of \( \phi \). Then

1. \( S \) is symmetric,
2. \( T = \bigcup_{i \in V} S_i \) and \( \forall i \in V : S_i = \{ T \in T : i \in T \} \),
3. \( S \preceq C(G_u) \) and \( T \preceq C(G_u) \).

**Remark 1.** Point 2 of Theorem 2 implies the result of [3] on the structure of the potential and of the utility functions in graphical games. Indeed, the characterization of potential function of graphical potential games proposed in [3] directly follows from point 2, as we characterize the minimal separation of \( \phi \). Similarly, the characterization of the utility functions proposed in [3] can be directly obtained from point 2 by exploiting that \( u \) is strategically equivalent to its normalization \( \bar{u} \) and by the fact that \( G_{\bar{u}} \subseteq G_u \). The result we propose is actually stronger than the one of [3], since we don’t just show that one possible splitting of \( u \) can be constructed with the cliques of \( G_u \), but instead we show that every set in the minimal splitting of a normalized potential game is a clique of \( G_u \). Moreover, our proof is essentially self-contained and, in contrast to [3], it does not rely on the Hammersleyâ€“Clifford theorem.

**Remark 2.** In fact, Theorem 2 implies the Hammersleyâ€“Clifford theorem. To see this, for any positive probability distribution \( P \) on \( X \) that satisfies the Markov property on an undirected graph \( G \), let \( \phi(x) = \log P(x) \). Then, upon interpreting \( \phi \) as the potential of a normalized potential \( G \)-game, Theorem 2 implies that \( P(x) = \exp(\phi(x)) \) factorizes on the cliques of \( G \).

**Proof of Theorem 2.** Proof of point 2. As a first step, we prove one side of the inclusion in (17):

\[
j \in S_{i,h} \implies \exists h' : S_{i,h} \subseteq S_{j,h'}.
\]

By hypothesis \( u \) is \( S \)-separable, i.e.,

\[
u_j(x) = \sum_{h'} u_j^{h'}(x_{S_{j,h'}}).
\]

\( u \) is strategically equivalent to the game \( u_\phi \), where every player’s utility coincides with the potential function. As a consequence, we can express the utility of players \( i \) and \( j \) in the form:

\[
u_i(x) = v_i(x_{-i}) + \phi(x)
\]

\[
u_j(x) = v_j(x_{-j}) + \phi(x) = \sum_{h'} u_j^{h'}(x_{S_{j,h'}}).
\]

By combining the two equations we obtain

\[
u_i(x) = v_i(x_{-i}) - v_j(x_{-j}) + \sum_{h'} u_j^{h'}(x_{S_{j,h'}}).
\]
This shows that $u_i$ admits the following separation:

$$\tilde{S}_i = \{V \setminus \{i\}, V \setminus \{j\}, S_{j,h'}\}.$$ 

Since $S$ is minimal, $S_i$ must be a finer separation that $\tilde{S}_i$. In particular, $S_{i,h}$ must be contained in some element of $\tilde{S}_i$. Then one of the following inclusions must be satisfied

$$S_{i,h} \subseteq V \setminus \{i\}$$
$$S_{i,h} \subseteq V \setminus \{j\}$$
$$S_{i,h} \subseteq S_{j,h'}$$

for some $h'$. The first inclusion is false, since it follows from Theorem 1 that the minimal splitting $S$ of a normalized game satisfies $i \in S_{i,h}$ for all $i$ and $h$. The second inclusion is also false since by assumption $j$ is contained in $S_{i,h}$. Then the last is true, which proves (18). To conclude the proof, observe that

$$j \in S_{i,h} \implies \exists h': S_{i,h} \subseteq S_{j,h'}$$
$$\implies \exists h': i \in S_{j,h'}$$
$$\implies \exists h'': S_{j,h'} \subseteq S_{i,h''}$$
$$\implies S_{i,h} \subseteq S_{i,h''}.$$ 

Since $S$ contains only maximal sets, we deduce that $h'' = h$ and $S_{i,h} = S_{j,h'}$.

Proof of point 2. Denote by $\text{split}(T)$ the splitting obtained from $T$ by setting

$$\text{split}(T)_i = \{T \in T : i \in T\}$$

and denote by $\text{join}(S)$ the separation obtained by the splitting $S$ as

$$\text{join}(S) = \bigcup_{i \in V} S_i.$$

First, notice that $\phi$ is separable with respect to $\text{join}(S)$. Indeed, for every player $i$, we can write

$$u_i(x) = \sum_h u_i^h(x_{S_{i,h}}) = v_i(x_{-i}) + \phi(x)$$

for some function $v_i$, and this shows that for every $i \in V$, $\phi$ is separable with respect to

$$\{S_{i,h}, V \setminus \{i\}\}_h.$$ 

(19)

Since $T$ is the minimal separation for $\phi$, $T$ is finer than the separation in (19) for each $i$. This means that $\forall i \in V$ and $\forall T \in T$, either $T \subset S_{i,h}$ for some $h$ or $T$ does not contain $i$. Since $T$ is non-empty, we can consider $j \in T$ and we have $T \subset S_{j,k}$ for some $k$. This shows that for all $T \in T$, $T \in \bigcup_{j \in V} S_i$, i.e., $\phi$ is separable with respect to $\text{join}(S)$. Moreover, observe that $u$ is separable with respect to $\text{split}(T)$. This simply follows from the fact that, since $u$ is normalized, we have

$$u = \tilde{u}_\phi,$$

i.e., $u$ coincides with the normalization of the game where each player has utility equal to the potential function $\phi$. So each utility function $u_{\phi,i}$ is trivially separable with respect to $T$ and by Theorem 1 its normalization $u_i$ is separable with respect to $\text{split}(T)_i$. Comparing the minimal splitting $S$ with $\text{split}(T)$, we clearly have that

$$S \leq \text{split}(T).$$
where \( \preceq \) stands for pointwise inclusion of sets. Moreover, both \( S \) and \( \text{split}(T) \) are symmetric splittings (the former by construction, the latter by point 2). This implies that

\[
\text{join}(S) \preceq \text{join}(\text{split}(T)) = T
\]

but since \( T \) is by assumption the minimal splitting for \( \phi \), this implies

\[
\text{join}(S) = T, \tag{20}
\]

which is the first part of the thesis. Moreover, from (20) we have

\[
S = \text{split(} \text{join}(S) \text{)} = \text{split}(T),
\]

which concludes the proof. The result is represented in Figure 2.

Proof of point 2. By applying Proposition 1, we have that the minimal graph of \( u \) coincides with the graph induced by the minimal splitting, that is \( \mathcal{G}(S) = \mathcal{G}_u \). Since \( u \) is normalized, \( i \in S_{i,h} \) for every set \( S_{i,h} \in S \). Moreover, by point 2 \( S_u \) is symmetric so we have that \( \forall i \in \mathcal{V}, \forall h \) and \( \forall j, k \in S_{i,h}, \mathcal{G}_u \) contains the undirected link \( \{j, k\} \) and \( S_{i,h} \) is a clique in \( \mathcal{G}_u \), even if not necessarily a maximal one. This shows that \( S \preceq C(\mathcal{G}_u) \). Since every set \( S_{i,h} \) is a clique in \( \mathcal{G}_u \), point 2 implies that also \( T = \bigcup_{i \in \mathcal{V}} S_i \) only contains cliques of \( \mathcal{G}_u \), so that \( T \preceq C(\mathcal{G}_u) \). \( \square \)

Another simple but relevant consequence of Theorem 2 is the following result, showing that the minimal graph of a normalized potential game is undirected. As previously mentioned, this fact can be interpreted as a first order symmetry of the interactions among players. While this pairwise symmetry is weaker than the higher order symmetry property of Theorem 2, it is significant since it can be directly read on the minimal graph of the game.

**Corollary 2.** Let \( u \) be a normalized potential game. Then \( \mathcal{G}_u \) is undirected.

**Proof.** Denote by \( S \) the minimal splitting of \( u \). As a consequence of Proposition 1 \( \mathcal{G}_u = \mathcal{G}(S) \), where \( \mathcal{G}(S) \) is the graph induced by \( S \). Since \( u \) is normalized, by Theorem 2 \( S \) is symmetric. This implies that \( \forall i \in \mathcal{V}, \text{ if } j \text{ is a neighbor of } i \text{ in } \mathcal{G}_u, \text{ i.e. } j \in \mathcal{N}_i, \text{ then } j \text{ is in } S_{i,h} \text{ for some } h \text{ and by symmetry there exists } h' \text{ such that } i \in S_{j,h'}. \text{ Then } i \in \mathcal{N}_j, \text{ that is, } i \text{ is a neighbor of } j \text{ in } \mathcal{G}_u. \square \]

**Corollary 3.** Let \( u \) be a pairwise-separable network potential game. Then \( u \) admits a pairwise separable potential function with minimal separation

\[
\phi(x) = \sum_{\{i,j\} \in \overline{E}} \phi_{i,j}(x_i, x_j)
\]

where \( \overline{E} \) is the set of the undirected links of the graph \( \mathcal{G}_u \).
Proof. Without loss of generality we can assume that \( u \) is normalized. Since \( u \) is pairwise-separable, every set \( S_{i,h} \) in the minimal splitting \( S \) of \( u \) contains exactly two elements, one of which is \( i \). By Proposition 1, \( G_u = \mathcal{G}(S) \), which means that every set \( S_{i,h} \in S \) corresponds to a link of \( G_u \), which is undirected per Corollary 2. By applying point 2 of Theorem 2 we obtain that \( \phi \) has minimal separation \( T = \bigcup_{i \in V} S_i = \mathcal{E} \). If \( u \) is not normalized, we can apply the above procedure to \( \bar{u} \), which is still a pairwise-separable game with the same potential function \( \phi \). □

5 DECOMPOSING GRAPHICAL GAMES

5.1 Classical game decomposition

Another class of games that play a key role in the theory are harmonic games [5]. A game \( u \in \mathcal{U} \) is referred to as harmonic if

\[
\sum_{i \in V} \sum_{y \sim i} [u_i(x) - u_i(y)] = 0, \tag{21}
\]

for every strategy profile \( x \in \mathcal{X} \). Notice that a normalized game is harmonic if and only if

\[
\sum_{i \in V} |\mathcal{A}_i| u_i(x) = 0, \quad \forall x \in \mathcal{X}. \tag{22}
\]

Hence, in particular, if the action sets of all players have the same cardinality \( |\mathcal{A}_i| = a \), then a normalized game \( u \) is harmonic if and only if it is a 0-sum game, i.e.,

\[
\sum_{i \in V} u_i(x) = 0, \quad \forall x \in \mathcal{X}. \tag{23}
\]

Example 5 (Anticoordination and discoordination games.). We already introduced the 2-player coordination game in Example 3. Similarly, we now introduce the 2-player binary anticoordination game, with utility functions

\[
u_{12}(x_1, x_2) = \begin{cases} 
-1 & \text{if } x_1 = x_2 \\
1 & \text{if } x_1 \neq x_2
\end{cases}, \quad u_{21}(x_1, x_2) = u_{12}(x_1, x_2) \quad \tag{24}\]

and discoordination games, which is described by the following utility functions

\[
u_{12}(x_1, x_2) = \begin{cases} 
1 & \text{if } x_1 = x_2 \\
-1 & \text{if } x_1 \neq x_2
\end{cases}, \quad u_{21}(x_1, x_2) = -u_{12}(x_1, x_2) \quad \tag{25}\]

where \( x_1, x_2 \in \{0, 1\} \).

Anticoordination games is a symmetric potential game. Discoordination game, instead, does not admit any potential, as it does not have Nash equilibria. It can be verified that it is an harmonic game. Notice that both games are normalized.

It has been proven in [5, Theorem 4.1.] that the space of games can be decomposed as a direct sum

\[
P = \mathcal{N} \oplus \mathcal{P} \oplus \mathcal{H}, \quad \tag{26}\]

where \( \mathcal{P} \) is the space of normalized potential games, \( \mathcal{N} \) is the space of non-strategic games, \( \mathcal{H} \) is the space of normalized harmonic games. In other terms, (26) means that every finite game can be decomposed in a unique way as the sum of a nonstrategic game, a normalized potential game, and a normalized harmonic game.

Example 6. Consider a game with player set and actions spaces as in Example 1 and utility functions

\[
u_1(x) = -x_1 x_2 x_3, \quad \nu_2(x) = x_2 x_3 (x_1 - 1), \quad \nu_3(x) = x_3 x_2. \quad \tag{27}\]
It is then easily verified that this game is a zero-sum normalized game, hence a harmonic game since all
action sets have the same cardinality. The graph $G_u$ of this game has link set $E = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1)\}$,
hence it is not undirected, and is displayed in Figure 3.

Notice that the game of Example 2 game can be decomposed as $u = u_N + u_P + u_H$, where $u_N$ is a
is non-strategic game with utilities

$$u_{N,1}(x) = x_2 x_3, \quad u_{N,2}(x) = x_1 x_3, \quad u_{N,3}(x) = x_1 x_2,$$

$u_P$ is the normalized potential game with utilities as in (7), and $u_H$ is the normalized harmonic game
with utilities as in (27). In particular, the normalized version $\bar{u}$ of $u$ has utilities as in 9 and graph $G_{\bar{u}}$
as in Figure 1c. Notice that, in spite of the fact that $\bar{u}$ can be decomposed as $\bar{u} = u_P + u_H$, neither the
graph $G_{u_P}$ of the normalized potential component (Figure 1a) nor the graph $G_{u_H}$ of the normalized
harmonic component (Figure 3) are subgraphs of $G_{\bar{u}}$.

5.2 Decomposition of separable games

In this section, we shall state our result concerning the decomposition of graphical games into their
non-strategic, potential, and harmonic components.

Certain supergraphs of a graph $G = (V, E)$ will play a relevant role in our analysis: they are all
undirected and obtained by keeping the same node set $V$ and augmenting the link set $E$ as follows.

First, let $G^{\leftrightarrow} = (V, E^{\leftrightarrow})$ be the minimal undirected supergraph of $G$. Clearly, $G^{\leftrightarrow}$ is obtained
from $G$ by making all its links undirected, i.e., it has link set $E^{\leftrightarrow} = E \cup \{(i, j) : (i, j) \in E\}$.

Moreover, let $G^{\triangle} = (V, E^{\triangle})$ be the graph obtained from $G^{\leftrightarrow}$ by adding links between every pair
of outneighbors of every node, i.e., $G^{\triangle}$ has link set

$$E^{\triangle} = E^{\leftrightarrow} \cup \bigcup_{i \in V} \{(j, l) : j \neq l \in N_i\}.$$
We will also consider suitable enlargements of splittings according to the following definition.

**Definition 4.** For a given splitting $S = \{S_{i,h}\}_{i \in \mathcal{V}, h}$ we define the symmetrized version of $S$ as the splitting $S^*$ such that $\forall i \in \mathcal{V}$

$$S^*_i = S_i \bigcup \{S_{j,h} | i \in S_{j,h}\}. \quad (28)$$

Observe that $S^*$ is by definition a symmetric splitting. Actually, it is the smallest symmetric splitting which contains $S$, i.e., such that $S \leq S^*$. In particular, if $S$ is symmetric, then $S^* = S$.

We start with the following result that is a simple consequence of [5, Theorem 4.1] that will nevertheless prove very useful in proving our main result of this section.

**Lemma 2.** For $i \in \mathcal{V}$ and $\mathcal{W} \subseteq \mathcal{V} \setminus \{i\}$, let $\hat{\mathcal{V}} = \{i\} \cup \mathcal{W}$ and let $\mathcal{G} = (\mathcal{V}, E)$, where $E = \{(i, j) : j \in \mathcal{W}\}$ be a directed star graph with center $i$ and leave set $\mathcal{W}$. Then, every normalized $\mathcal{G}$-game $u \in \mathcal{U}$ can be decomposed in a unique way as

$$u = u_\mathcal{P} + u_\mathcal{H},$$

where $u_\mathcal{P}$ and $u_\mathcal{H}$ are, respectively, normalized potential and normalized harmonic games that are graphical on the $\hat{\mathcal{V}}$-clique $\hat{\mathcal{G}}^\triangleright$. Moreover, $u_\mathcal{P}$ admits potential function $\phi : \mathcal{X} \to \mathbb{R}$ such that $\phi(x)$ depends on $x_{\hat{\mathcal{V}}}$ only.

**Proof.** Let $\hat{\mathcal{U}}$ be the space of games with player set $\hat{\mathcal{V}}$ and strategy profile space $\hat{\mathcal{X}} = \prod_{i \in \hat{\mathcal{V}}} \mathcal{A}_i$. Then, define the game $\hat{u} \in \hat{\mathcal{U}}$ as the restriction of $u$ to the player set $\hat{\mathcal{V}}$. Notice that $\hat{u}$ is normalized, so that by [5, Theorem 4.1] it can be decomposed uniquely as $\hat{u} = \hat{u}_\mathcal{P} + \hat{u}_\mathcal{H}$ where $\hat{u}_\mathcal{P} \in \hat{\mathcal{U}}$ is normalized potential with potential function $\hat{\phi} : \hat{\mathcal{X}} \to \mathbb{R}$ and $\hat{u}_\mathcal{H} \in \hat{\mathcal{U}}$ is normalized harmonic. Let then $u_\mathcal{P} \in \mathcal{U}$ and $u_\mathcal{H} \in \mathcal{U}$ be the extensions of $\hat{u}_\mathcal{P}$ and $\hat{u}_\mathcal{H}$, respectively, obtained by defining, $(u_\mathcal{P})_i(x) = (\hat{u}_\mathcal{P})_i(x_{\hat{\mathcal{V}}})$ and $(u_\mathcal{H})_k(x) = 0$ for $\forall \mathcal{F} \in \{\mathcal{H}, \mathcal{P}\}$, $i \in \hat{\mathcal{V}}$, and $k \in \mathcal{V} \setminus \hat{\mathcal{V}}$. Then, notice that both $u_\mathcal{P}$ and $u_\mathcal{H}$ are normalized $\mathcal{G}^\triangleright$-games. Moreover, $u_\mathcal{P}$ is a potential game with potential function $\phi(x) = \hat{\phi}(x_{\hat{\mathcal{V}}})$, while $u_\mathcal{H}$ is a harmonic game and $u = u_\mathcal{P} + u_\mathcal{H}$. \hfill $\square$

We are now ready to state and prove the following, that is our main result providing a refinement of the one in [5].

**Theorem 3.** Let $u \in \mathcal{U}$ be a finite game with minimal splitting $S$. Then $u$ can be decomposed as

$$u = u_N + u_\mathcal{P} + u_\mathcal{H}, \quad (29)$$

where

- $u_N$ is a non-strategic $S$-separable game;
- $u_\mathcal{P}$ is a normalized potential $S^*$-game admitting a potential function $\phi : \mathcal{X} \to \mathbb{R}$ that can be decomposed as

$$\phi(x) = \sum_{i \in \mathcal{V}} \sum_{h} \phi_{i,h}(x_{S_{i,h}}); \quad (30)$$

- $u_\mathcal{H}$ is a normalized harmonic $S^*$-game.

Moreover, $\mathcal{G}(\hat{S}^*) = (\mathcal{V}, E^\hat{S})$ where

$$E^\hat{S} = E^\hat{S} \cup \bigcup_{i \in \mathcal{V}} \bigcup_{h} \{(j,l) : j \neq l \in \overline{S}_{i,h} \setminus \{i\}\} \quad (31)$$

where $E$ is the link set of $\mathcal{G}_\pi = \mathcal{G}(\hat{S}) = (\mathcal{V}, E)$. 

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(a) Construction of $G_u\leftrightarrow$.  
(b) Construction of $G_u\triangle$.  
(c) Neighborhood of node $i$ in $G(\bar{S}^*)$. Colours represent two groups $\bar{S}_i,h \in S$. The construction must be repeated for every node.

Fig. 5. Supergraphs of the graph $G_u$ represented in Figure 4.

Observe that the graph $G(\bar{S}^*)$ is obtained from $G_u$ by making all existing links undirected and by adding an undirected link between every couple of players belonging to a common group in $\bar{S}$. It is then clear that $G(\bar{S}^*)$ defines a more general notion of extension of $G_u$ than the ones previously introduced.

Indeed, notice that $G_u\leftrightarrow$ and $G_u\triangle$ are special cases of $G(\bar{S}^*)$. In particular $G(\bar{S}^*) = G_u\leftrightarrow$ when the game $\bar{u}$ has minimal splitting $\bar{S}$ such that for every node $i$, $k_i = |N_i|$ and $\bar{S}_{i,h} = \{i,j_h\}$ for $1 \leq h \leq k_i$, where $N_i = \{j_h|1 \leq h \leq k_i\}$ is the neighborhood of $i$ in $G_u$. Similarly $G_u\triangle = G(\bar{S}^*)$ when $k_i = 1$ and $\bar{S}_{i,1} = N_i^*$. This fact is illustrated in Figure 5.

**Proof.** Let $u \in U$ be a game and $S = \{S_{i,h}\}_{i \in V, 1 \leq h \leq k_i}$ be its minimal splitting. First, assume that the game $u$ is normalized so that by Theorem 1 for every $i \in V$ we have $i \in S_{i,h}, \forall h$. We can express the game $u$ as a sum

$$u = \sum_{i \in V} \sum_{h=1}^{k_i} u^{(i,h)},$$

where, for every $i \in V$ and $h$, $u^{(i,h)} \in U$ is a normalized game with utilities

$$u^{(i,h)}_i(x) = u^h_i(x_{S^*_i}), \quad u^{(i,h)}_j(x) = 0, \quad j \neq i,$$

i.e, where all players apart from $i$ have zero utility, while player $i$ has utility $u^h_i$, equal to the utility he gets from playing with the group $S_{i,h}$. Clearly, each game $u^{(i,h)}$ satisfies the assumptions of
Lemma 2 with \( \mathcal{W} = S_{i,h} \setminus \{i\} \), so that it can be decomposed as
\[
    u^{(i,h)} = u_p^{(i,h)} + u_{\mathcal{H}}^{(i,h)},
\]
where \( u_p^{(i,h)} \) and \( u_{\mathcal{H}}^{(i,h)} \) are normalized potential and, respectively, harmonic \( G_{(i,h)}^+ \) games where \( G_{(i,h)}^+ \) is the clique graph on \( S_{i,h} \). Also, by Lemma 2, \( u_p^{(i,h)} \) admits a potential function \( \phi^{(i,h)} \) depending only on \( S_{i,h} \). Both games \( u_p^{(i,h)} \) and \( u_{\mathcal{H}}^{(i,h)} \) only involve players in \( S_{i,h} \), whose utility depends only on \( S_{i,h} \), i.e.
\[
    u_{p,j}^{(i,h)} = u_{\mathcal{H},j}^{(i,h)} = 0 \quad \text{if} \ j \notin S_{i,h}.
\]
However, while in \( u^{(i,h)} \) only player \( i \) has non-zero utility, in general for \( u_p^{(i,h)} \) and \( u_{\mathcal{H}}^{(i,h)} \) all players in \( S_{i,h} \) have non-vanishing utility, i.e., for each \( k \in S_{i,h} \), \( u_p^{(i,h)} \) and \( u_{\mathcal{H}}^{(i,h)} \) are not necessarily vanishing. By combining (32) and (34) we then get that
\[
    u = u_p + u_{\mathcal{H}},
\]
where
\[
    u_p = \sum_{i \in \mathcal{V}} \sum_{h=1}^{k_i} u_p^{(i,h)}, \quad u_{\mathcal{H}} = \sum_{i \in \mathcal{V}} \sum_{h=1}^{k_i} u_{\mathcal{H}}^{(i,h)}.
\]
By linearity, \( u_p \in \mathcal{P} \) is a normalized potential game with potential function \( \phi \) satisfying
\[
    \phi(x) = \sum_i \sum_h \phi^{(i,h)}(x_{S_{i,h}})
\]
while \( u_{\mathcal{H}} \in \mathcal{H} \) is a normalized harmonic game. Observe that not only the games \( u_p^{i,h} \) contribute to the utility of a player \( j \) in \( u_p \), but also all games \( u_p^{i,h} \) with \( i \) and \( h \) such that \( j \in S_{i,h} \), i.e., we can write
\[
    u_{p,j}(x) = \sum_{i,h \in S_{i,h}} u_p^{(i,h)}(x_{S_{i,h}})
\]
This shows that \( u_p \) is \( S^* \)-separable, where \( S^* \) is the symmetrization of \( S \). The same reasoning shows that also \( u_{\mathcal{H}} \) is \( S^* \)-separable.

Moreover, both \( u_p \) and \( u_{\mathcal{H}} \) are graphical on the graph \( G^* := \bigcup_{i \in \mathcal{V}} \bigcup_{1 \leq h \leq k_i} G_{(i,h)}^+ \) with nodes \( \mathcal{V} \) and link set
\[
    \bigcup_{i \in \mathcal{V}} \bigcup_{h} \{ (j, l) : j \neq l \in S_{i,h} \} = \mathcal{E}^{S^*} \cup \bigcup_{i \in \mathcal{V}} \bigcup_{h} \{ (j, l) : j \neq l \in S_{i,h} \setminus \{i\} \} = \mathcal{E}^S,
\]
with \( \mathcal{E}^S \) as defined in 31.

To conclude, we show that \( G^* = (\mathcal{V}, \mathcal{E}^S) \) coincides with the graph induced by \( S^* \), \( G(S^*) \). To see this, denote by \( \mathcal{N}_i \) the neighborhood of player \( i \in \mathcal{V} \) in \( G_u = G(S) = (\mathcal{V}, \mathcal{E}) \),
\[
    \mathcal{N}_i = \bigcup_{h} S_{i,h} \setminus \{i\}
\]
and by \( \mathcal{N}_i(S^*) \) the neighborhood of \( i \) in \( G(S^*) \)
\[
    \mathcal{N}_i(S^*) = \left( \bigcup_{h} S_{i,h} \bigcup \{ S_{j,k} \in S | i \in S_{j,k} \} \right) \setminus \{i\}.
\]
First we show that \( G^* \subset G(S^*) \). Indeed, consider a link \( (i, j) \) of \( G^* \) such that \( (i, j) \in \mathcal{E}^{S^*} \). Then either \( (i, j) \in \mathcal{E} \) or \( (j, i) \in \mathcal{E} \). If \( (i, j) \in \mathcal{E} \), \( j \in \mathcal{N}_i \) so that \( j \in S_{i,h} \) for some \( h \). This implies that there exist
We can proceed in the same way if \((j, k)\) are not direct neighbors in \(G\). To see this, let \(i \in S^*_j\) and \(j \in S^*_i\), which again implies that \((j, i)\) and \((i, j)\) are links of \(G(S^*)\). We can proceed in the same way if \((j, i)\) is a link of \(G(S^*)\). Moreover, \((j, k)\) is a link of \(G(S^*)\) if \(j, k \in S_{l, h}\) for some \(l \neq j, k, h\). Then we have that \(S_{l, h} \in S^*_j\) so that \(k \in \mathcal{N}(S^*)\) and \((j, k)\) is a link of \(G(S^*)\).

We prove now the opposite inclusion, that is \(G(S^*) \subseteq G^\ast\). Consider a link \((i, j)\) of \(G(S^*)\). We have that \(j \in \mathcal{N}(S^*)\), which implies that either \(j \in S_{l, h}\) for some \(l \neq j, k, h\) or \(j \in S_{l, k}\) for some \(l \neq j, k, h\) with \(i \in S_{l, k}\). Now, if \(j \in S_{l, h}\) then \(j \in \mathcal{N}_i\) which implies that \((i, j) \in E \subseteq E^S\). If instead \(j \in S_{l, k}\) with \(i \in S_{l, k}\), then \(i, j \in S_{l, k}\) with \(i \neq j\) and then \((i, j) \in E^S\).

This concludes the proof for the case when \(u\) is normalized. For the general case when \(u\) may not be normalized, just observe that one can write \(u = u_N + \bar{u}\) where \(\bar{u}\) is normalized and by Theorem 1 has minimal splitting \(\bar{S} = S_u\), while \(u_N \in \mathcal{N}\) is a non-strategic \(S\)-separable game, thus graphical on \(G(S) = G_u\). \(\square\)

Theorem 3 has a few important direct consequences that we discuss in the following. First, observe that, for every splitting \(S\) of \(u\), we have by Theorem 3 that \(u_F\) and \(u_H\) are graphical on \(G(S^*) \subseteq G^\ast\), so that Theorem 3 directly implies the following.

**Corollary 4.** Every finite game \(u \in \mathcal{U}\) can be decomposed as

\[
u = u_N + u_F + u_H,
\]

where

- \(u_N\) is a non-strategic \(G_u\)-game;
- \(u_F\) is a normalized potential \(G^\ast\)-game;
- \(u_H\) is a normalized harmonic \(G^\ast\)-game.

Corollary 4 states that the minimal graphs \(G_{u_F}\) and \(G_{u_H}\) of the normalized potential and, respectively, normalized harmonic components of a finite game \(u\) are subgraphs of \(G^\ast\). As shown in Figure 5b, in general \(G^\ast\) is a strict supergraph of \(G\), which suggests that the nonstrategic-harmonic-potential decomposition of finite graphical games may not preserve graphicality. In fact, Corollary 4 allows for the possibility that the minimal graphs of the normalized potential and normalized harmonic components of a finite game \(u\) include links between two players \(j\) and \(k\) that are not direct neighbors in \(G\) but share a common in-neighbor \(i\). While not influencing directly their respective normalized utilities, such players \(j\) and \(k\) both directly influence the utility of player \(i\) and this may result in the appearance of a link between them in the minimal graphs of the normalized potential and normalized harmonic components of the game, as shown by the following example.

**Example 7 (Decomposition of the public good game).** Consider the public good game defined in Example 4 over a cycle graph with 6 nodes. Figure 6 shows the minimal interaction graph \(G_u = G\) and the minimal graphs associated to the potential and harmonic components. According to Corollary 4, they are both subgraphs of \(G^\ast\). In this particular case they coincide with \(G^\ast\), showing that the a sharper result cannot be obtained.

On the other hand, such possible links between common outneighbors of a single player in \(G\) may not show up in the minimal graphs \(G_{u_F}\) and \(G_{u_H}\) of the normalized potential and normalized harmonic components of the game. In fact, in the special case of pairwise-separable network games, Theorem 3 reduces to the following.

**Corollary 5.** Let \(u \in \mathcal{U}\) be a pairwise-separable network game on a graph \(G = (V, E)\), with utilities as in (12). Then, \(u = u_N + u_F + u_H\) where...
Fig. 6. Representation of $G_u$ and $G_{u_H}$ for a public good game on $G$.

- $u_N$ is a non-strategic $G$-game;
- $u_P$ is a pairwise-separable normalized potential $G_u$-game;
- $u_H$ is a pairwise-separable normalized harmonic $G_u$-game.

Observe that when $u$ is pairwise-separable its potential and harmonic components are also pairwise separable. Indeed, Theorem 3 implies that the potential function $\phi$ of $u_P$ is separable on the undirected links of $G_u$ and therefore $u_P$ is pairwise separable as a consequence of Theorem 2 point 2. Then, by linearity also $u_H = \bar{u} - u_P$ is pairwise-separable (since both $\bar{u}$ and $u_P$ are).

Corollary 5 shows that for finite pairwise-separable network games the decomposition (26) preserves the original graphical structure in that no link between players that were not directly interacting in the original game shows up in either $G_u$ or $G_{u_H}$. In fact, when $G$ is undirected, we have that $G_u$ is a subgraph of $G$ so that we can simplify the result to deduce that $u_P$ and $u_H$ are graphical with respect to $G$.

Example 8 (Decomposition of pairwise games). Consider an undirected interaction graph $G$ and construct a pairwise graphical game where each pair of adjacent players is involved either in a coordination, an anticoordination or a discoordination game, as in Examples 3 and 5. Denote by $C$, $A$, $D$ the sets of pairs of players involved in a coordination, anticoordination or discoordination game, respectively.

If we perform the decomposition according to Corollary 5, $u_N = 0$ and we obtain that

- $G_u$ contains the link $\{i, j\} \in E$ if and only if $\{i, j\} \in C$ or $\{i, j\} \in A$, i.e. if $i$ and $j$ are involved in either a coordination or an anticoordination game;
- $G_{u_H}$ contains the link $\{i, j\} \in E$ if and only if $\{i, j\} \in D$, i.e. if $i$ and $j$ are involved in a discoordination game.

So in this case the graphs $G_u$ and $G_{u_H}$ can be determined by inspection and their link sets are disjoint. In Figure 1 we report some example of decomposition. In particular we consider two topologies for $G$. Red (blue) nodes indicate players who anticoordinate (coordinate) with their neighbors.

In fact, Theorem 3 provides a characterization of the minimal graphs of the normalized potential and normalized harmonic components of a finite game $u$. Such minimal graphs $G_u$ and $G_{u_H}$ are contained in $G(S^\Delta)$, which is in general a subgraph of $G_u$ and a supergraph of $G_{u_H}$.

6 CONCLUSION

In this paper we have discussed separable graphical games. These are games where utility functions decompose with respect to a certain splitting of the action variables. We have studied minimal splitting with respect to strategic equivalence and we have shown how, for potential games, the potential function also decomposes with respect to the corresponding minimal separation. Also,
we have studied the classical decomposition of a game into a potential and a harmonic part in the category of separable games. We believe that these fundamental results can be exploited in a variety of directions. Among the issues, left for future research, we indicate

• the study of how such minimal splitting can be used to lower the implementation complexity of distributed learning algorithms;
• the study of the implication of the decomposition of separable games in the field of evolutionary dynamics. We expect that this, as for pure potential games, should have interesting consequences.

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A PROOF OF LEMMA 1

Proof of Lemma 1. In the following we use the notation $[n]$ to denote the set of the first $n$ natural numbers, i.e., $[n] = \{1, 2, \ldots, n\}$. Consider the two separations $A = \{A_i\}_{i \in [l]}$ and $B = \{B_j\}_{j \in [m]}$. 

\begin{table}
\centering
\begin{tabular}{ccc}
(a) Case I & (b) Case II \\
\includegraphics[width=0.4\textwidth]{G}\hspace{1cm} & \hspace{1cm} \includegraphics[width=0.4\textwidth]{G_uP} \\
\includegraphics[width=0.4\textwidth]{G_uH} & \hspace{1cm} \includegraphics[width=0.4\textwidth]{G_uP_H} \\
\end{tabular}
\caption{Representation of $G_{uP}$ and $G_{uH}$ for pairwise-separable graphical game on two different graphs $G$.}
\end{table}
Define the binary set operator ⪯ whose action on $A$ and $B$ gives the set $A \triangleright B$, which contains all the intersections between elements of $A$ and all the intersections of elements of $A$ with elements of $B$:

$$A \triangleright B = \{ A_i \cap A_j \}_{i, j \in [l]} \bigcup \{ A_i \cap B_j \}_{i \in [l], j \in [m]} .$$

(35)

The proof is divided in two steps: in the first part, we show that if $f$ is separable with respect to $A$ and $B$, then it is separable with respect to $A \triangleright B$; in the second part we show that by iteratively joining $A$ with $B$ via $\blacktriangleleft$ we obtain the separability of $f$ on $C$.

**Step one.**

Fix any index $s \in [l]$. For all $x$ we have

$$g_s(x_{A_s}) = - \sum_{i=1}^{l} g_i(x_{A_i}) + \sum_{j=1}^{m} h_j(x_{B_j}) = - \sum_{i=1}^{l} g_i(x_{A_i \cap A_s}, x^*_{A_i \setminus A_s}) + \sum_{j=1}^{m} h_j(x_{B_j \cap A_s}, x^*_{B_j \setminus A_s})$$

(36)

for any $x^* = (x_1^*, \ldots, x_N^*)$.

Indeed, since $g_s(x_{A_s})$ only depends on $x_{A_s}$, also the right side in (36) is independent of the values of $x$ outside $A_s$ which can be arbitrarily fixed in a configuration $x^*$. Then for every $i \in [l], i \neq s$ and every $j \in [m]$ we define

$$g^{(s, i)}(x_{A_i \cap A_s}) = -g_i(x_{A_i \cap A_s}, x^*_{A_i \setminus A_s})$$

(37)

$$h^{(s, j)}(x_{B_j \cap A_s}) = h_j(x_{B_j \cap A_s}, x^*_{B_j \setminus A_s})$$

(38)

so that we can write

$$f(x) = \sum_{s=1}^{l} g_s(x_{A_s})$$

$$= \sum_{s=1}^{l} \sum_{i=1}^{l} g^{(s, i)}(x_{A_i \cap A_s}) + \sum_{s=1}^{l} \sum_{j=1}^{m} h^{(s, j)}(x_{B_j \cap A_s}) .$$

(39)

This shows that $f$ is separable with respect to $A \triangleright B$. If $A$ is a partition, we have that $A_i \cap A_j = \emptyset$ for every $i \neq j$ and (39) directly implies that $f(x) = \sum_k f_k(x_{C_k})$ where for $k = (s, j)$, $f_k = h^{(s, j)}$ and $C_k = B_j \cap A_s$. In general, however, the intersections $A_i \cap A_j$ are non-empty.

**Step two.** Whenever we perform the $\blacktriangleleft$ operation, we implicitly simplify the resulting set by only keeping its maximal elements. It follows from step one that $f$ is separable with respect to $A \blacktriangleleft B \blacktriangleleft \cdots \blacktriangleleft B$, where the $\blacktriangleleft$ operation is performed an arbitrary number of times ($\blacktriangleleft$ associate to the left, so we omit the parenthesis). To conclude the proof it is sufficient to show that, starting from $A$ and iteratively applying $\blacktriangleleft B$, we reduce to a set which is element-wise included in $C$. In particular, we will show that

$$A \blacktriangleleft B \blacktriangleleft \cdots \blacktriangleleft B \leq C .$$

(40)

To see this, observe that, by induction on $n$, we prove that for any $n \in \mathbb{N}$

$$A \blacktriangleleft B \blacktriangleleft \cdots \blacktriangleleft B \leq \{ \cap_{k=1}^{n+1} A_{I_k} \}_{I \subset [l]} \bigcup \cap_{|I| = n+1} C$$

(41)
where \( \left\{ \bigcap_{k=1}^{n+1} \mathcal{A}_{I_k} \right\}_{|I|=n+1} \) is the collection of all intersections of \( n + 1 \) distinct elements of \( A \). So, if \( A \) contains \( l \) elements, the set \( A \kappa B \kappa \cdots \kappa B \) does not contain any member which is an intersection of elements of \( A \) only. Indeed, as per (41),

\[
A \kappa B \kappa \cdots \kappa B \leq \left\{ \bigcap_{k=1}^{l+1} \mathcal{A}_{I_k} \right\}_{|I|=l+1} \bigcup C
\]

\[
= \emptyset \cup C
\]

\[
= C.
\]

In conclusion, this proves that \( f \) is separable with respect to \( C \). \qed