JACKSON’S INEQUALITY IN THE COMPLEX PLANE AND THE LOJASIEWICZ-SICIAK INEQUALITY OF GREEN’S FUNCTION

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Abstract. We prove a generalization of Dunham Jackson’s famous approximation inequality to the case of compact sets in the complex plane admitting both upper and lower bounds for their Green’s functions, i.e. the well known Hölder Continuity Property (HCP) and the less known but crucial Lojasiewicz-Siciak inequality (LS). Moreover, we show that (LS) is a necessary condition for our Jackson type inequality.

Key words: Jackson inequality, Green function, regularity of sets, approximation by polynomials, holomorphic functions.

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1. Introduction and main result.

Dunham Jackson’s famous inequality which gives some control over the rate of approximation by polynomials of a fixed function, was first proved for the segment [-1,1] in 1911 (see [15] and also [23, sec.5.1; 9, chap.4 sec.6]). There are numerous results in the literature concerning various generalizations of this inequality because of their significant role in approximation theory and in related domains of research. This explains also why sets admitting Jackson type inequalities are especially useful. However, it seems that in the complex case this property was investigated up to now only for particular classes of sets.

The direct reason for our study of Jackson’s inequality was a result by Bos and Milman regarding the equivalence of the local and global Markov inequalities, a Kolmogorov type inequality and an extension property for $C^\infty(K)$ functions (see [7] or L.P. Bos and P.D. Milman, A Geometric Interpretation and the Equality of Exponents in Markov and Gagliardo-Nirenberg (Sobolev) Type Inequalities for Singular Compact Domains, preprint). The proof is hard and proceeds only in the real case making essential use of the Jackson inequality in $\mathbb{R}^N$. We were intrigued to obtain a corresponding result for sets in the complex plane because of the intricate interconnectedness of multiple distinct global and local properties: Markov inequalities, Kolmogorov type inequalities, polynomial approximation, extension operators, geometric properties and, ultimately, the behavior of the Green’s function, i.e. L-regularity, Hölder continuity and the Lojasiewicz-Siciak inequality. However, a simple adaptation to the complex case of the proof given by Bos and Milman is not possible.

In a previous paper [5] we showed that the local Markov property is equivalent to a Kolmogorov type property for any compact set $E \subset \mathbb{C}$. In a subsequent paper (L. Bialas-Ciez and R. Eggink, Equivalence of the global and local Markov inequalities in the complex plane, preprint) we prove that the Global Markov Inequality GMI (i.e. $\|p\|_E \leq M(\deg p)^k \|p\|_E$ with $k, M > 0$ independent of the polynomial $p$) is equivalent to an extension property for functions of the class $s(E)$, which can be rapidly approximated by holomorphic polynomials:

$$s(E) := \{ f \in C(E) : \forall \ell \in \mathbb{N} \lim_{n \to \infty} n^\ell \text{dist}_E(f, P_n) = 0 \},$$

where $\mathbb{N} = \{ 1, 2, \ldots \}$, $\text{dist}_E(f, P_n) := \inf\{\|f - p\|_E : p \in P_n\}$ is the error of approximating the function $f$ on the set $E$ by polynomials of degree $n$ or less and $\| \cdot \|_E$ is the supremum norm on $E$. The latter
extension property requires the existence of an extension, which is bounded together with its derivatives by the following Jackson norms of the extended function:

\[ |f|_\ell := \|f\|_E + \sup_{n \in \mathbb{N}} n^\ell \text{dist}_E(f, \mathcal{P}_n) \quad \text{for} \quad \ell \in \mathbb{N}_0 = \{0, 1, 2 \ldots\}. \]

Sometimes we will use \( | \cdot |_\ell \) also for \( \ell \in \mathbb{R} \), \( \ell \geq 0 \).

In the real case contemplated by Bos and Milman, this extension property implies a Kolmogorov type inequality owing to the fact that the Jackson norms can easily be estimated by quotient norms. This follows from the classical Jackson inequality and therefore we investigated the possibility to generalize this result to the case of compact sets in the complex plane.

Clearly, a lot of work has been done on various "Jackson-properties" for Jordan arcs, domains and other continua, where order of approximation is linked to the regularity of a given function and/or the regularity of the continuum, see for example [20; 24; 17; 11; 2; 16; 1; 12; 10] and many other authors referenced therein. However, our research of the literature leads us to believe that this is not at all the case for compact sets in general, which may even be totally disconnected.

One can envisage different possible generalizations of the Jackson inequality, so we have taken an approach that seems to be best suited to determine a class of sets for which the global and local Markov inequalities are equivalent. This allows us to work only with functions that are holomorphic in open neighborhoods of our compact set and with regular supremum norms in those neighborhoods, while maintaining optimal control over the constants.

For a compact set \( E \subset \mathbb{C} \), let \( \mathcal{H}^\infty(E) := \{ f \in C^\infty(\mathbb{C}) : \frac{\partial f}{\partial z} \equiv 0 \text{ in some open neighborhood of } E \} \) and \( E_\delta := \{ z \in \mathbb{C} : \text{dist}(z, E) \leq \delta \} \). By Taylor’s theorem and Cauchy’s integral formula, we can prove that for a closed disc \( B \subset \mathbb{C} \) and for an arbitrary function \( f \in \mathcal{H}^\infty(B_\delta) \) with some \( \delta \in (0, 1] \), we have \( f|_B \in s(B) \) and

\[ \forall \ell \in \mathbb{N} \quad |f|_{B, \ell} \leq (c\ell)^{\ell+1} \|f\|_{B, \ell+1}, \]

where \( c \) depends only on the diameter of \( B \). Consequently,

\[ \forall \ell \geq 1 \quad |f|_{B, \ell} \leq \left( \frac{c\ell^\ell}{\delta} \right)^{\ell+1} \|f\|_{B, \delta}. \]

The Jackson Property defined below is a generalization of the last inequality.

**Definition 1.1.** A compact set \( E \subset \mathbb{C} \) admits the **Jackson Property** \( \text{JP}(s) \), where \( s \geq 1 \), if \( \mathcal{H}^\infty(E)|_E \subset s(E) \) and there exist constants \( c, v \geq 1 \) such that

\[ \forall \ell \in \mathbb{N} \quad \forall \delta \in (0, 1] \quad \forall f \in \mathcal{H}^\infty(E_\delta) \quad : \quad |f|_{E, \ell} \leq \left( \frac{c\ell^v}{\delta} \right)^{\ell+c} \|f\|_{E_\delta}. \]

Note that every closed disc admits JP(1). Note also that if \( \mathcal{H}^\infty(E)|_E \subset s(E) \) then the set \( E \) must obviously be polynomially convex, i.e. \( E = \hat{E} \) where \( \hat{E} := \{ z \in \mathbb{C} : \forall n \in \mathbb{N} \forall p \in \mathcal{P}_n \ | p(z) | \leq ||p||_E \} \) is the polynomial hull of \( E \).

The interesting thing is that the Jackson Property defined above turns out to be intimately connected with the rate of growth of the Green’s function \( g_E \) (with logarithmic pole at infinity) of the unbounded complement of the compact set \( E \).

**Definition 1.2.** The set \( E \) admits the **Lojasiewicz-Siciak inequality** \( \text{LS}(s) \), where \( s \geq 1 \), if

\[ \exists M > 0 \quad \forall z \in E_1 \quad : \quad g_E(z) \geq M \text{dist}(z, E)^s. \]

We will write that the set \( E \) admits LS if it admits LS(s) for some \( s \geq 1 \).

As far as we know, the term Lojasiewicz-Siciak inequality was first coined by Gendre, who used it to obtain advanced approximation results [14] (see also [22]). The interested reader is referred to [6] for basic information.

We set out (without proofs) the following examples:

- if \( E \) is a compact set in \( \mathbb{R} \) then \( E \) admits LS(1),
- the set \( E := \{ z \in \mathbb{C} : |z - 1| \leq 1 \text{ or } |z + 1| \leq 1 \} \) does not admit LS(s) for any \( s \),
• if E is the starlike set $E = E(n) := \{z = r \exp^{\frac{2\pi ij}{n}} \in \mathbb{C} : 0 \leq r \leq 1, \ j = 1, \ldots, n\}$ then E admits $\text{LS}(\frac{2}{n})$ whenever $n \in \mathbb{N} \setminus \{1\}$,
  
• a simply connected compact set $E \subset \mathbb{C}$ with nonempty interior, admits $\text{LS}(s)$ with some $s \geq 1$ if and only if its complement to the Riemann sphere is a Hölder domain, i.e. a conformal map $\varphi : \{z \in \mathbb{C} : |z| < 1\} \to \hat{\mathbb{C}} \setminus E$ such that $\varphi(0) = \infty$ is Hölder continuous in $\{z \in \mathbb{C} : \frac{1}{2} \leq |z| \leq 1\}$ with exponent $1/s$.

The Lojasiewicz-Siciak inequality is the opposite of the well known Hölder Continuity Property (HCP), which gives an upper bound of the Green’s function (see e.g. [8; 3; 19]).

**Definition 1.3.** A compact set $E \subset \mathbb{C}$ admits the Hölder Continuity Property $\text{HCP}(k)$, where $k \geq 1$, if

$$\exists M \geq 1 \ \forall z \in E : g_E(z) \leq M \text{dist}(z, E)^{1/k}.$$ 

We will write that the set $E$ admits HCP if it admits HCP($k$) for some $k \geq 1$.

The connection between the Jackson property and the rate of growth of the Green’s function is evidenced by our main result:

**Theorem 1.4.** Let $s' > s \geq 1$. Any polynomially convex compact set $E \subset \mathbb{C}$ admitting $\text{LS}(s)$ and HCP, admits $\text{JP}(s)$. Moreover, any compact set $E \subset \mathbb{C}$ admitting $\text{JP}(s)$, admits $\text{LS}(s')$.

This finding allowed us to construct an example of a compact set in the complex plane which admits the Global Markov Inequality, while it does not admit any Local Markov Property, nor the Lojasiewicz-Siciak inequality (L. Bialas-Ciez and R. Eggink, *Equivalence of the global and local Markov inequalities in the complex plane*, preprint).

This paper is organized as follows. In Section 2 we introduce the notations used throughout the paper, some of which are standard while others are more specific to our work. Section 3 contains the proof of the main result. In Section 4 we give some remarks and additional results concerning the Jackson Property. We wrap up with some open problems.

### 2. Preliminaries and notations.

In our further deliberations we make active use of Siciak’s extremal function for a compact set $E \subset \mathbb{C}$ (see [21])

$$\Phi_E(z) := \limsup_{n \to \infty} \sqrt[n]{\Phi_n(z)} \text{ for } z \in \mathbb{C},$$

where for $n \in \mathbb{N}$

$$\Phi_n(z) = \Phi_n(E, z) := \sup\{|p(z)| : p \in \mathcal{P}_n, \|p\|_E \leq 1\}$$

denotes the $n$-th extremal function. It is well known that $\Phi_E = e^{g_E}$, where $g_E$ stands for the Green’s function of $\mathbb{C} \setminus E$ with logarithmic pole at infinity. For convenience we extend $g_E$ to the entire complex plane by putting $g_E(z) := 0$ for all $z \in \hat{E}$.

The set $E$ is called L-regular if its extremal function $\Phi_E$ is continuous on the entire complex plane. Similarly we speak of regularity in a boundary point $z_0 \in E$ when $\Phi_E$ is continuous at this point. Note that whenever the cardinality of the set $E$ is bigger than $n$, then the $n$-th extremal function $\Phi_n$ of $E$ is necessarily continuous on the entire complex plane. Since the extremal function $\Phi_E$ is always continuous on $\mathbb{C} \setminus \hat{E}$, L-regularity is really determined by the behavior of $\Phi_E$ at the outer boundary of the set $E$.

Note that both properties HCP and LS can be defined equivalently in terms of Siciak’s extremal function instead of Green’s function, because for arbitrary $t > 0$ we have

$$1 + g_E(z) \leq e^{g_E(z)} = \Phi_E(z) \leq 1 + \frac{e^t - 1}{t} g_E(z)$$

for all $z \in \mathbb{C} \setminus \hat{E}$ such that $0 \leq g_E(z) \leq t$.

For a compact set $E \subset \mathbb{C}$ and $\rho \geq 1$ we denote the level set of the extremal function

$$C(E, \rho) := \{z \in \mathbb{C} : \Phi_E(z) = \rho\} \cup \{z \in \mathbb{C} : g_E(z) = \log \rho\}.$$ 

In order to control the behavior of the extremal function $\Phi_E$ near the boundary of $E$ we introduce

$$\phi_n(t) := \inf_{z \in \partial E_t} \Phi_n(z),$$
for $n \in \mathbb{N}_0$ and $t \in [0, \infty)$. Here and further we denote by $dE_t$ the set $\{z \in \mathbb{C} : \text{dist}(z, E) = t\}$, which may be a slightly bigger set than just the boundary $\partial E_t$ of $E_t$. Note that for $t > 0$ the function $\phi_n$ is continuous or equal to $+\infty$. Furthermore, $\phi_n(t)$ is an increasing function with respect to $n$ and moreover, the maximum principle for subharmonic functions, applied to the function $\log \Phi_n$, implies that $\phi_n(t)$ is increasing also with respect to $t > 0$.

For $\delta > 0$ we denote by $K(E, \delta)$ a compact neighborhood constructed as follows. First we cut up the entire complex plane into closed squares of size $\delta \times \delta$, starting at the origin of the plane. Next we select all squares having a non-empty intersection with the set $E$ and by $K(E, \delta)$ we denote the sum of those squares. Clearly we have $E \subset K(E, \delta) \subset E_{\delta/\sqrt{2}}$. Also it is easy to see that the set $K(E, \delta)$ consists of at most $(\frac{\text{diam} E}{\delta} + 2)^2$ squares and therefore the length of its border $\partial K(E, \delta)$ is definitely less than

$$4\delta \left(\frac{\text{diam} E}{\delta} + 2\right)^2 = 4(\text{diam} E)^2.$$ 

For a compact set $E \subset \mathbb{C}$ we denote the family of smooth functions that are $\overline{\partial}$-flat on $E$:

$$\mathcal{A}^\infty(E) := \left\{ f \in \mathcal{C}^\infty(\mathbb{C}) : \text{the function } \frac{\partial f}{\partial \overline{z}} \text{ is flat on } E \right\},$$

where a function $g \in \mathcal{C}^\infty(\mathbb{C})$ is said to be flat in the point $z_0$ if $D^\alpha g(z_0) = 0$ for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, $D^\alpha = \frac{\partial^{\alpha_1}}{\partial \overline{z}_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \overline{z}_2^{\alpha_2}}$ and $|\alpha| = \alpha_1 + \alpha_2$. This definition is slightly different than in [22], where $\mathcal{A}^\infty(E)$ stood for functions defined on $E$ only, which will be denoted here as $\mathcal{A}^\infty(E)|_E := \{f|_E : f \in \mathcal{A}^\infty(E)\}$.

3. Proof of the main result.

Our goal in this section is to establish the main result of the paper, which is a general version of Jackson's inequality in the complex plane. For a fixed compact set $E \subset \mathbb{C}$ and $\zeta \notin E$ we put $f_\zeta(z) := \frac{1}{\zeta - z}$ for $z$ in some open neighborhood of $E$ and extend it to a function of class $\mathcal{C}^\infty(\mathbb{C})$ so that $f_\zeta \in \mathcal{H}^\infty(E)$.

Lemma 3.1. For all $\zeta \notin E \subset \mathbb{C}$ and $n \in \mathbb{N}_0$ we have

$$\frac{1}{(\text{dist}(\zeta, E) + \text{diam} E)} \Phi_{n+1}(\zeta) \leq \text{dist}_E(f_\zeta, \mathcal{P}_n) \leq \frac{1}{\text{dist}(\zeta, E)} \Phi_{n+1}(\zeta).$$

Proof. Fix $n \in \mathbb{N}$ and take an arbitrary polynomial $q \in \mathcal{P}_{n+1}$ such that $\|q\|_E = 1$ and $q(\zeta) \neq 0$. Define $p(z) := \frac{q(\zeta) - q(z)}{q(\zeta)}$ so that $p \in \mathcal{P}_n$. We obtain

$$\text{dist}_E(f_\zeta, \mathcal{P}_n) \leq \|f_\zeta - p\|_E \sup_{z \in E} \left| \frac{q(z)}{(\zeta - z)q(\zeta)} \right| \leq \frac{1}{\|q\|_E} \frac{1}{\text{dist}(\zeta, E) |q(\zeta)|}.$$ 

We take the infimum over all $q \in \mathcal{P}_{n+1}$ to arrive at $\text{dist}_E(f_\zeta, \mathcal{P}_n) \leq \frac{1}{\text{dist}(\zeta, E) \Phi_{n+1}(\zeta)}$.

On the other hand for fixed $n \in \mathbb{N}$ find $p \in \mathcal{P}_n$ such that $\text{dist}_E(f_\zeta, \mathcal{P}_n) = \|f_\zeta - p\|_E$. Define $q(z) := 1 - (\zeta - z) p(z)$ so that $q \in \mathcal{P}_{n+1}$. We see that

$$\|q\|_E = \sup_{z \in E} |1 - (\zeta - z) p(z)| \leq \sup_{z \in E} |\zeta - z| \cdot \sup_{z \in E} |f_\zeta(z) - p(z)| \leq (\text{dist}(\zeta, E) + \text{diam} E) \text{dist}_E(f_\zeta, \mathcal{P}_n)$$

and hence

$$\Phi_{n+1}(\zeta) \geq \frac{|q(\zeta)|}{\|q\|_E} \geq \frac{1}{(\text{dist}(\zeta, E) + \text{diam} E) \text{dist}_E(f_\zeta, \mathcal{P}_n)}. \quad \square$$

The next results were inspired by the proof of Runge’s theorem (see e.g. [13, chap.III §3, chap.III §1]).

Proposition 3.2. For any compact set $E \subset \mathbb{C}$, $0 < \delta \leq 1$ and $f \in \mathcal{H}^\infty(E_\delta)$ we have

$$\forall \frac{1}{2} \leq b < 1 \quad \forall n \in \mathbb{N} : \quad \text{dist}_E(f, \mathcal{P}_n) \leq \frac{c \|f\|_{E_\delta}}{(1 - b)\delta^2 \Phi_{n+1}(b\delta)},$$

where the constant $c := \frac{28}{\pi} (2 + \text{diam} E)^2$ depends only on the set $E$. 
Proof. Fix $\frac{1}{2} \leq b < 1$ and $n \in \mathbb{N}$. If $\phi_{n+1}(b\delta) = +\infty$ then the set $E$ consists of $n + 1$ or less points and $\text{dist}_E(f, \mathcal{P}_n) = 0$, which finishes the proof. Otherwise, find a positive $\delta$ such that $\delta \leq \frac{(1-b)\delta}{4\pi(1-b)\delta}$ and $(1-b)\delta$ is an integer. Let $\Gamma$ be the boundary $\partial K(E_{b\delta}, 1+b\delta)$, with proper orientation, and cut it up into equal intervals $\Gamma_j$, each of length $\delta$, so that $\Gamma = \bigcup_j \Gamma_j$, with $j$ running over a finite index set. As $K(E_{b\delta}, 1+b\delta) \subset E_{b\delta + \frac{1}{4}\delta^2} \subset E_{\frac{1+b\delta}{2}}$, we see that $\Gamma \subset E_{\frac{1+b\delta}{2}} \setminus \text{int} E_{b\delta}$, while for the length of $\Gamma$, denoted $m(\Gamma)$, we have
\[
\sum_j \delta = m(\Gamma) \leq \frac{4(\text{diam } E_{b\delta})^2}{\delta} \leq \frac{4\pi c}{7(1-b)\delta}.
\]

For a fixed $z \in E$ and $f \in \mathcal{H}^\infty(E_{b\delta})$ put $g_z(\zeta) := \frac{f(\zeta)}{\zeta - z}$, which is a holomorphic function in an open neighborhood of the set $E_{b\delta} \setminus \{z\}$. Let $\zeta_0, \zeta_1 \in \Gamma_j$ for some $j$. Then the entire interval $I := [\zeta_0, \zeta_1]$ lies in $\Gamma_j$ and of course $\text{dist}(z, I) \geq b\delta$. By Cauchy’s integral formula, for $\zeta \in I$ we have
\[
|g_z'(|\zeta|) - f(\zeta)| \leq \frac{\|f\|_{E_{\frac{1+b\delta}{2}}} + \|f\|_{E_{\frac{1+b\delta}{2}}}}{b\delta}). (1-\delta)^2 \leq \frac{6\|f\|_{E_{b\delta}}}{(1-b)\delta^2} \leq \frac{6\|f\|_{E_{b\delta}}}{4\delta \phi_{n+1}(b\delta)}.
\]
This leads us to
\[
\int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta \leq \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta \leq \int_{\Gamma_j} \frac{6\|f\|_{E_{b\delta}}}{4\delta \phi_{n+1}(b\delta)} \, |d\zeta| \leq \frac{3\delta}{2\delta \phi_{n+1}(b\delta)} |f|_{E_{b\delta}}.
\]
By summing over $j$ we obtain
\[
|f(z) - R(z)| = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta - R(z) \leq \sum_j \frac{3\delta}{4\pi \phi_{n+1}(b\delta)} = \frac{3m(\Gamma)}{4\pi \phi_{n+1}(b\delta)}.
\]
where we denote
\[
R(z) := \frac{1}{2\pi i} \sum_j \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \sum_j \frac{c_j}{\zeta - z} = \sum_j c_j f(\zeta_j),
\]
where $c_j := \frac{1}{2\pi i} f(\zeta_j) \int_{\Gamma_j} d\zeta$. By the above, we can see that the rational function $R$ approximates uniformly $f$ on the set $E$ and
\[
\|f - R\|_{E} \leq \frac{3m(\Gamma)}{4\pi \phi_{n+1}(b\delta)}.
\]
Simultaneously, by virtue of Lemma 3.1 and by the minimum principle, we have
\[
\text{dist}_E(R, \mathcal{P}_n) \leq \sum_j |c_j| \text{dist}_E(f, \mathcal{P}_n) \leq \sum_j \frac{\delta}{2\pi \text{dist}(\zeta_j, E)} \Phi_{n+1}(\zeta_j) \leq \sum_j \frac{\delta}{2\pi \delta \phi_{n+1}(b\delta)} = \frac{m(\Gamma)}{2\pi \delta \phi_{n+1}(b\delta)},
\]
because $\text{dist}(\zeta_j, E) \geq b\delta$. Consequently, from (2) and (3), since $\frac{1}{2} \leq 2$, we conclude that
\[
\text{dist}_E(f, \mathcal{P}_n) \leq \|f - R\|_E + \text{dist}_E(R, \mathcal{P}_n) \leq \frac{7m(\Gamma)}{4\pi \phi_{n+1}(b\delta)} \leq \frac{c \|f\|_{E_{b\delta}}}{(1-b)\delta^2 \phi_{n+1}(b\delta)}. \quad \square
\]

Lemma 3.3. For any $L$-regular compact set $E \subset \mathbb{C}$, $\zeta \in E_1 \setminus E$, $1 < \rho \leq \Phi_E(\zeta)$ and $n \in \mathbb{N}_0$ we have
\[
\text{dist}_E(f, \mathcal{P}_n) \leq \frac{c(n+1)}{\text{dist}(C(E, \rho), E)} \text{dist}(\zeta, E) \left( \frac{\rho}{\Phi_E(\zeta)} \right)^{n+1},
\]
where $c \geq 1$ depends only on the set $E$. 

Proof. We put
\[ d := \max_{z \in C(E, \Phi E, \|E\|)} \text{dist}(z, E) \geq 1 \]
and \( c := 2d + \text{diam } E \). Fix \( \zeta \in E_1 \setminus \hat{E} \), \( 1 < \rho \leq \Phi E(\zeta) , n \in \mathbb{N}_0 \) and consider any \( \eta \in E_d \setminus \hat{E} \). For the Lagrange interpolation polynomial \( L_n f_\eta \) with knots in \( n + 1 \) Fekete extremal points \( \{ z^{(n)}_j \} \) \( j = 0, \ldots , n \subset E \) and \( \omega_n(z) := \prod_{j=0}^{n} (z - z^{(n)}_j) \), we have
\[ L_n f_\eta (z) = \frac{\omega_n(\eta) - \omega_n(z)}{\omega_n(\eta)(\eta - z)} \]

Consequently, applying the properties of the Fekete extremal points, we see that for all \( z \in E \) we have
\[ \left| \frac{\omega_n(z)}{\omega_n(\eta)} \right| = \left| 1 - (\eta - z) L_n f_\eta (z) \right| \leq 1 + (d + \text{diam } E)(n + 1)\|f_\eta\|_E \leq 1 + \frac{(d + \text{diam } E)(n + 1)}{\text{dist}(\eta, E)} \leq \frac{(n + 1) c}{\text{dist}(\eta, E)} \]

Now put \( h_n := \log |\omega_n| - (n + 1) g_E \), which is a harmonic function on \( \mathbb{C} \setminus \hat{E} \), bounded in \( \mathcal{C} \). If \( \eta \in C(E, \rho) \subset E_d \setminus \hat{E} \) and \( z \in E \), we have
\[ h_n(\eta) \geq \log \frac{\text{dist}(\eta, E) |\omega_n(z)|}{(n + 1) c} - (n + 1) \log \rho \geq \log \left( \frac{\text{dist}(C(E, \rho), E) |\omega_n(z)|}{(n + 1) c} \right) - (n + 1) \log \rho. \]
The L-regularity of the set \( E \) leads us to the fact that the level set \( C(E, \rho) \) is the boundary of the open domain \( \Omega := \{ z \in \mathbb{C} : \Phi_E(z) > \rho \} \). Therefore, the minimum principle for harmonic functions implies that the last inequality holds for all \( \eta \in \Omega \), in particular, for \( \eta = \zeta \). By the definition of \( h_n \) and since \( g_E = \log \Phi_E \), we can easily obtain
\[ \left| \frac{\omega_n(z)}{\omega_n(\zeta)} \right| \leq \frac{(n + 1) c}{\text{dist}(C(E, \rho), E)} \left( \frac{\rho}{\Phi_E(\zeta)} \right)^{n+1}. \]

Returning to the Lagrange interpolation polynomial we have
\[ \text{dist}_E(f_\zeta, P_n) \leq \|f_\zeta - L_n f_\zeta\|_E = \max_{z \in E} \left| \frac{\omega_n(z)}{\omega_n(\zeta)(\zeta - z)} \right| \leq \frac{(n + 1) c}{\text{dist}(C(E, \rho), E) \text{dist}(\zeta, E)} \left( \frac{\rho}{\Phi_E(\zeta)} \right)^{n+1}. \]

Lemma 3.4. Assume that a polynomially convex compact set \( E \subset \mathbb{C} \) admits LS(s) and HCP(k) for some \( s, k \geq 1 \), i.e. there exist \( a_1, a_2 \geq 1 \) such that for all \( y \in E_1 \)
\[ \frac{1}{a_1} \text{dist}(z, E)^s \leq g_E(z) \leq a_2 \text{dist}(z, E)^{1/k}. \]

Then there exist \( c_0, c_1 \geq 1 \) dependent only on \( E \) such that
\[ \forall \ell \geq 1 \quad \forall 0 < t \leq 1 \quad : \sup_{n \in \mathbb{N}} \frac{n^\ell}{\Phi_n(t)} \leq \left( \frac{c_1 t^{\ell+c_0}}{t^s} \right) \]

Proof. Fix \( \ell \geq 1 \) and \( 0 < t \leq 1 \). By Lemma 3.3 for arbitrary \( \zeta \in dE_\rho, \rho := \sqrt{\Phi_E(\zeta)} > 1 \) and \( n \in \mathbb{N} \) we have
\[ n^\ell \text{dist}_E(f_\zeta, P_{n-1}) \leq \frac{c n^{\ell+1} \rho^{-n}}{\text{dist}(C(E, \rho), E)t} \leq \frac{c}{\text{dist}(C(E, \rho), E)t} \left( \frac{\ell + 1}{\ell c \log \rho} \right)^{\ell+1} \leq \frac{c}{\text{dist}(C(E, \rho), E)t} \left( \frac{2t}{g_E(\zeta)} \right)^{\ell+1}, \]

because for \( a, b > 0 \) we have \( \sup_{n \geq 0} a^e b^{-n} = (\frac{a}{e})^a \). We combine this with Lemma 3.1 to obtain
\[ \frac{n^\ell}{\Phi_n(\zeta)} \leq (\text{dist}(\zeta, E) + \text{diam } E)n^\ell \text{dist}_E(f_\zeta, P_{n-1}) \leq \frac{\bar{c}}{\text{dist}(C(E, \rho), E)t} \left( \frac{2t}{g_E(\zeta)} \right)^{\ell+1}, \]

where \( \bar{c} := (1 + \text{diam } E) c \). By the above and from assumption (4),
\[ \frac{n^\ell}{\Phi_n(\zeta)} \leq \frac{\bar{c}}{t} \left( \frac{a_2}{\log \rho} \right)^k \left( \frac{2t}{g_E(\zeta)} \right)^{\ell+1} \leq \frac{\bar{c}}{t} \left( \frac{2a_1}{\log \rho} \right)^k \left( \frac{2t}{g_E(\zeta)} \right)^{\ell+1} \leq \left( \frac{c_1 t^{\ell+c_0}}{t^s} \right), \]

where \( c_0 := k + 2 \) and \( c_1 := 2a_1 a_2 \bar{c} \) depend only on the set \( E \). Finally we conclude that
\[ \sup_{n \in \mathbb{N}} \frac{n^\ell}{\Phi_n(\zeta)} = \sup_{n \in \mathbb{N}} \sup_{\zeta \in dE} \frac{n^\ell}{\Phi_n(\zeta)} \leq \left( \frac{c_1 t^{\ell+c_0}}{t^s} \right). \]
Proposition 3.5. For any compact set $E \subset \mathbb{C}$ and $s \geq 1$ the Jackson Property $\text{JP}(s)$ is equivalent to the following condition:

\begin{equation}
\exists \zeta, v \geq 1 \quad \forall \ell \geq 1 \quad \forall 0 < t \leq 1 \quad \forall n \in \mathbb{N} : \quad \frac{n^\ell}{\phi_{n+1}(t)} \leq \left( \frac{c\ell v}{t^s} \right)^{\ell+\zeta}.
\end{equation}

Proof. First, observe that we can write equivalently $\ell \geq 1$ in condition (1) instead of $\ell \in \mathbb{N}$. Assume that the set $E$ admits $\text{JP}(s)$ and we shall prove (5). Fix $0 < t \leq 1$ and arbitrary $\zeta \in dE_t$. Obviously, $f \in \mathcal{H}^\infty(E_\delta)$ for $\delta := \frac{1}{2}$. $\text{JP}(s)$ implies that

$$n^\ell \text{dist}_E(f, \mathcal{P}_n) \leq |f|_{\ell} \leq \left( \frac{c\ell v}{\delta^s} \right)^{\ell+\zeta} ||f||_{E_\delta} = \left( \frac{2^e c\ell v}{\delta^s} \right)^{\ell+\zeta}.$$

for all $\ell \geq 1$ and $n \in \mathbb{N}$. By Lemma 3.1, we obtain

$$\frac{n^\ell}{\phi_{n+1}(\zeta)} \leq \left( \text{dist}(\zeta, E) + \text{diam}E \right) n^\ell \text{dist}_E(f, \mathcal{P}_n) \leq \left( \frac{c\ell v}{\delta^s} \right)^{\ell+\zeta},$$

where $\bar{c} := \max\{1 + \text{diam}E, 2^e c, c + 1\}$. Therefore, since $\zeta \in dE_t$ was arbitrary, we conclude that

$$\frac{n^\ell}{\phi_{n+1}(t)} = \sup_{\zeta \in dE_t} \frac{n^\ell}{\Phi_{n+1}(\zeta)} \leq \left( \frac{c\ell v}{\delta^s} \right)^{\ell+\zeta}$$

and (5) is proved.

In order to show $\text{JP}(s)$ assuming (5), fix $\ell \geq 1$, $0 < \delta \leq 1$ and $f \in \mathcal{H}^\infty(E_\delta)$. We apply Prop. 3.2 with $b := \frac{1}{2}$ and the assumption with $t := \delta/2$ to obtain for any $n \in \mathbb{N}$

$$n^\ell \text{dist}_E(f, \mathcal{P}_n) \leq \frac{2c\ell^2||f||_{E_\delta}}{\delta^s} \leq \frac{2c\ell v}{\delta^s} \left( \frac{2^e c\ell v}{\delta^s} \right)^{\ell+\zeta} ||f||_{E_\delta} \leq \left( \frac{2^e c\ell v}{\delta^s} \right)^{\ell+\zeta+2} ||f||_{E_\delta}.$$From this it follows that for $c_0 := \max\{1 + 2^e \bar{c}, \bar{c} + 2\}$ we have

$$|f|_{\ell} = ||f||_{E_\delta} + \sup_{n \in \mathbb{N}} n^\ell \text{dist}_E(f, \mathcal{P}_n) \leq \left( \frac{c_0\ell v}{\delta^s} \right)^{\ell+c_0} ||f||_{E_\delta}. \quad \square$$

Theorem 3.6. Let $s \geq 1$. Any polynomially convex compact set $E \subset \mathbb{C}$ admitting $\text{LS}(s)$ and $\text{HCP}$, admits $\text{JP}(s)$ with $v = 1$.

Proof. This is an immediate consequence of Lemma 3.4 and Prop. 3.5. \quad \square

The closest we could find in the literature was an estimate equivalent to $\text{JP}(1)$ with $v = 1$ and $c \geq 2$, proved for all simply connected bounded regions with boundaries that are Jordan curves of class $C^{1+\Delta}$ [17, lemma 4].

Note that as a simple corollary of Theorem 3.6, we can obtain $\text{JP}(1)$ for a disk $E = B(0, r)$, because in this case we have $\Phi_E(z) = |z|/r$.

Proposition 3.7. For any compact set $E \subset \mathbb{C}$ and $s' > s \geq 1$ we have

$$\text{JP}(s) \implies \text{LS}(s').$$

Proof. By Prop. 3.5, for arbitrary $t \in (0,1]$, $\zeta \in dE_t$, $n \in \mathbb{N}$ and $\ell \geq 1$ we get

$$g_E(\zeta) = \log \Phi_E(\zeta) \geq \log n^{\ell+1} \phi_{n+1}(\zeta) \geq \log n^{\ell+1} \phi_{n+1}(t) \geq \frac{1}{n+1} \log \left( n^\ell \left( \frac{t^s}{c\ell v} \right)^{\ell+\zeta} \right).$$

Specifically, by taking $n \in \mathbb{N}$ and $\left[ e^{2\ell v}/(c\ell v)^{1+\zeta/\ell} \right]$ we obtain

$$g_E(\zeta) \geq \frac{\log n^\ell}{2 + e^{2\ell v}/(c\ell v)^{1+\zeta/\ell}} \geq \frac{\ell}{2 + e^{2\ell v}/(c\ell v)^{1+\zeta/\ell}} \left( \text{dist}(\zeta, E) e^{(1+\zeta/\ell)} \right)^{s(1+\zeta/\ell)} = \frac{\ell}{2 + e^{2\ell v}/(c\ell v)^{1+\zeta/\ell}} \text{dist}(\zeta, E)^{s(1+\zeta/\ell)}.$$

If we take $\ell$ sufficiently large then we obtain $\text{LS}(s')$ for any $s' > s$. \quad \square

Note that if we have $\text{JP}(s)$ with $v = 1$ in the assumption of the last proposition, then we can conclude $\text{LS}(s)$ rather than $\text{LS}(s')$ for any $s' > s$, by simply taking the limit for $\ell \to +\infty$ in the last inequality of the proof.
4. Remarks and additional results.

**Proposition 4.1.** Every compact set $E \subset \mathbb{R}$ admits JP(1).

To the extent that the set $E$ admits HCP, this proposition is a simple corollary of the main theorem and the fact that every compact set in $\mathbb{R}$ admits LS(1). We leave the proof of the general case to the reader. Hint: apply the classical Jackson inequality and appropriate cut-off functions to estimate the quotient norms. Our best estimate gives $v = 6$.

One may ask whether the Jackson Property is maintained after combining two sets into one, or separating one set into two distinct subsets. Before answering this question we need to do some preparations.

**Lemma 4.2.** Assume that the compact set $E \subset \mathbb{C}$ is the sum of two polynomially convex, disjoint compact subsets, i.e. $E = A \cup B$, $A = \tilde{A}$, $B = \tilde{B}$, $A \cap B = \emptyset$. Assume also that the subset $A$ is non-polar, i.e. $\text{cap} A > 0$. Then for any function $f \in C(E)$ such that $f|_A \in s(A)$ and $f|_B \equiv 0$, we have $f \in s(E)$ and furthermore we can estimate its Jackson norms on the set $E$ by its Jackson norms on the subset $A$ as follows:

$$\forall \ell \geq 1 : |f|_\ell \leq (c\ell)^t |f|_{A\ell},$$

where the constant $c \geq 1$ depends only on the subsets $A$ and $B$. Note that these are two different Jackson norms and only the domain of the function indicates which norm is meant.

**Proof.** Like in the proof of [18, Th. 1] we consider

$$\chi_B(z) = \begin{cases} 0 & \text{for } z \in A, \\ 1 & \text{for } z \in B, \end{cases}$$

and we note that this characteristic function can be extended holomorphically so that $\chi_B \in C(\mathbb{C}, 2\rho)$ for some $\rho > 1$. By [13, chap. II §3A Th.1], there exists a constant $M \geq 1$ such that

$$\forall n \in \mathbb{N} : \text{dist}_E(\chi_B, P_n) \leq M \rho^n.$$ 

Put $x := \|\Phi_A\|_B$ and note that $1 < x < +\infty$, because the subset $A$ is non-polar and both subsets $A$ and $B$ are compact. Therefore, we can find an integer $k \in \mathbb{N}$ such that $t := \frac{x}{2^n} > 1$.

Now fix an arbitrary function $f \in C(E)$, such that $f|_A \in s(A)$ and $f|_B \equiv 0$, and also fix a number $\ell \geq 1$. Find two sequences of polynomials of best approximation for the functions $f|_A$ and $\chi_B$ on the sets $A$ and $E$ respectively, i.e. $p_n, q_n \in P_n$, $\|f - p_n\|_A = \text{dist}_A(f, P_n)$ and $\|\chi_B - q_n\|_E = \text{dist}_E(\chi_B, P_n)$ for each $n \in \mathbb{N}_0$. Clearly, $\|p_n\|_A \leq \|f\|_A$, $\|f - p_n\|_A \leq 2\|f\|_A$. By the definition of Siciak’s extremal function, we see that

$$\|p_n\|_B \leq \|\Phi_A\|_B \|p_n\|_A = x^n\|p_n\|_A \leq 2x^n\|f\|_A.$$ 

For each $n \in \mathbb{N}_0$ we put

$$r_n(z) := p_n(z) \left(1 - q_n(z)\right)$$

so that $r_n \in P_{(k+1)n}$. This way we obtain

$$\|f - r_n\|_A \leq \text{dist}_A(f, P_n) + \|p_n\|_A \text{dist}_A(\chi_B, P_{kn}) \leq \frac{|f_A|_\ell}{n^\ell} + 2\|f\|_A \frac{M}{\rho^kn^\ell} < \frac{|f_A|_\ell}{n^\ell} + \frac{2M}{\rho^n\ell^n} \|f\|_A,$$

$$\|f - r_n\|_B = \|r_n\|_B \leq \|p_n\|_B \|\chi_B - q_n\|_B \leq \|p_n\|_B \text{dist}_E(\chi_B, P_{kn}) \leq 2x^n\|f\|_A \frac{M}{\rho^kn^\ell} = \frac{2M}{\rho^n\ell^n} \|f\|_A.$$ 

This then leads us to

$$n^\ell \text{dist}_E(f, P_{(k+1)n}) \leq n^\ell \|f - r_n\|_E \leq |f_A|_\ell + 2M \frac{\ell^n}{\rho^n\ell^n} \|f\|_A \leq |f_A|_\ell + 2M \left(\frac{\ell}{c\log t}\right)^{\ell} \|f\|_A \leq (\tilde{c}\ell)^t |f|_{A\ell},$$

where the constant $\tilde{c} := 1 + \frac{2M}{c\log t}$ depends on the subsets $A$ and $B$ but not on the choice of the function $f$ and the number $\ell$.

Finally, for arbitrary $n \in \mathbb{N}_0$ we can find $N \in \mathbb{N}_0$ such that $(k+1)N \leq n < (k+1)(N+1)$ to conclude that

$$n^\ell \text{dist}_E(f, P_n) \leq ((k+1)(N+1))^{\ell} \text{dist}_E(f, P_{(k+1)N}) \leq (4k)\tilde{c}^{N}\text{dist}_E(f, P_{(k+1)N}) \leq (4k\tilde{c})^{\ell}\|f_A|_\ell,$$

$$|f|_\ell = \|f\|_E + \sup_{n \in \mathbb{N}} n^\ell \text{dist}_E(f, P_n) \leq \|f\|_A + (4k\tilde{c})^{\ell}\|f_A|_\ell \leq (\tilde{c}\ell)^t |f|_{A\ell} < +\infty,$$

where the constant $c := 1 + 4k\tilde{c}$ also depends only on the subsets $A$ and $B$. \(\square\)
Corollary 4.3. Assume that the compact set \( E \subset \mathbb{C} \) is the sum of two polynomially convex, disjoint, non-polar compact subsets, i.e. \( E = A \cup B \), \( A = \tilde{A} \), \( B = \tilde{B} \), \( A \cap B = \emptyset \), \( \text{cap} \ A > 0 \) and \( \text{cap} \ B > 0 \). Then for any function \( f \in \mathcal{C}(E) \) such that \( f|_A \in s(A) \) and \( f|_B \in s(B) \), we have \( f \in s(E) \) and furthermore we can estimate its Jackson norms on the set \( E \) by its Jackson norms on the subsets \( A \) and \( B \) as follows:

\[
\forall \ell \geq 1 : |f|_\ell \leq (c\ell)^\ell (|f|_A|_\ell + |f|_B|_\ell),
\]
where the constant \( c \geq 1 \) depends only on the subsets \( A \) and \( B \). Note that these are three different Jackson norms and only the domain of the function indicates which norm is meant.

Proof. We put \( \tilde{f}^A := \chi_A \cdot f \) and \( \tilde{f}^B := \chi_B \cdot f \) so that \( \tilde{f}^A, \tilde{f}^B \in \mathcal{C}(E) \). We apply Lemma 4.2 to obtain

\[
|f|_\ell \leq |\tilde{f}^A|_\ell + |\tilde{f}^B|_\ell \leq (cA\ell)^\ell |f|_A|_\ell + (cB\ell)^\ell |f|_B|_\ell \leq (c\ell)^\ell (|f|_A|_\ell + |f|_B|_\ell) < +\infty
\]
for any \( \ell \geq 1 \) with the constant \( c := \max\{c_A, c_B\} \) depending only on the sets \( A \) and \( B \). \( \square \)

Proposition 4.4. Assume that the compact set \( E \subset \mathbb{C} \) is the sum of two polynomially convex, disjoint compact subsets, i.e. \( E = A \cup B \), \( A = \tilde{A} \), \( B = \tilde{B} \) and \( A \cap B = \emptyset \). If the set \( E \) admits JP(s) with some \( s \geq 1 \), then both subsets \( A \) and \( B \) admit JP(s).

Conversely, if both subsets \( A \) and \( B \) are additionally non-polar and they both admit JP(s) with some \( s \geq 1 \), then the set \( E \) admits JP(s).

Proof. In order to prove the first assertion, we note that the Jackson Property is invariant to an affine change of variable and therefore if necessary we can blow these sets up so that \( \text{dist}(A,B) > 2 \). This way the intersection of the neighborhoods \( A_1 \) and \( B_1 \) of the sets \( A \) and \( B \), respectively, is empty. Next we apply Prop. 3.5 to get condition (5) for the set \( E \). The extremal functions \( \Phi_k \), of the subsets \( A \) and \( B \) are bounded below by the respective extremal functions of the set \( E \) and this way we obtain the condition (5) for the sets \( A \) and \( B \). Finally, we apply Prop. 3.5 again to conclude that they too admit JP with the same coefficients.

The second assertion follows straight from Corollary 4.3 and the definition of the Jackson Property. Indeed, for arbitrary \( \ell \geq 1, 0 < \delta \leq 1 \) and \( f \in \mathcal{H}^\infty(E_\delta) \) we have \( f \in \mathcal{H}^\infty(A_\delta) \), \( f \in \mathcal{H}^\infty(B_\delta) \) and

\[
|f|_\ell \leq (c\ell)^\ell (|f|_A|_\ell + |f|_B|_\ell) \leq (\overline{c}\ell)^\ell \left( \left( \frac{cA\ell^\nu}{\delta^s} \right)^{\ell+cA} \|f\|_{A_\delta} + \left( \frac{cB\ell^\nu}{\delta^s} \right)^{\ell+cB} \|f\|_{B_\delta} \right) \leq (\overline{c}\ell^{\nu+1}) \ell^{\frac{\nu+1}{\delta^s}} \|f\|_{E_\delta},
\]
where \( \overline{c} := 2c \max\{c_A, c_B\} \). \( \square \)

Remark 4.5. We close this paper by offering three open problems for further research:

- The proof of Lemma 3.4 applies the assumption of HCP only in order to make sure that the level sets of the extremal function do not come too close to the compact set \( E \). The coefficient in HCP(k) has no meaningful impact on the coefficients of the Jackson Property, suggesting that we may have used a sledge-hammer to crack a nut. Specifically, due to the intended application of the Jackson Property, it would be interesting to know whether it is sufficient to assume GMI instead of HCP (which implies GMI)? It should be noted though that Lemma 3.3 assumes L-regularity, which is guaranteed by HCP, but it is still not known whether all compact subsets of the complex plane admitting GMI are L-regular. In the real case this follows from the combination of [7] and [4].

- The characterization of compact sets \( E \subset \mathbb{C} \), for which \( \mathcal{A}^\infty(E)|_E = s(E) \), also remains an open problem, especially for totally disconnected sets. Siciak proved this property for simply connected Hölder domains, i.e. admitting LS [22, Th.1.10]. More recently, Gendre proved the same for every compact set \( E \subset \mathbb{C}^N \) that is Whitney 1-regular and admits HCP as well as LS [14, Cor. 7].

- Finally we had a good look at the Wiener type characterization given by Carleson and Totik for pointwise Hölder continuity of Green’s functions. Their Wiener type criterion (i.e. lower bounds for capacities) introduced in [8] implies HCP, but in order to assert the converse they needed an additional assumption, i.e. either a (geometric) cone condition or a quantitative (capacity) condition (upper bounds for capacities). The examples given in the Introduction above suggest that both those conditions could be special cases of LS. It is worth investigating whether HCP in conjunction with LS is sufficient to assert the Wiener type criterion proposed by Carleson and Totik.
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