A note on Dirichlet spectrum

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Abstract

We prove that the points of Dirichlet spectrum $D_{\|\cdot\|}^{[2]}$ for two-dimensional simultaneous approximation with respect to Euclidean norm can be attained by numbers which are not badly approximable and that the set of Dirichlet constants for badly approximable numbers is dense in $D_{\|\cdot\|}^{[2]}$.

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1 STRUCTURE OF DIRICHLET SPECTRUM

Let $g(y), y \in \mathbb{R}^n$ be an arbitrary norm in $\mathbb{R}^n$. Dirichlet spectrum $D_{g}^{[n]}$ for simultaneous approximation with respect to the norm $g(\cdot)$ is defined as follows. For $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ consider the irrationality measure function

$$\psi_{g, \theta}(t) = \min_{q \in \mathbb{Z}^n : q \leq t} \min_{p = (p_1, \ldots, p_n) \in \mathbb{Z}^n} g(q\theta - p)$$

and define

$$d_{g}^{[n]}(\theta) = \limsup_{t \to \infty} t(\psi_{g, \theta}(t))^n.$$ 

Then

$$D_{g}^{[n]} = \{d \in \mathbb{R} : \exists \theta \in \mathbb{R}^n \setminus \mathbb{Q}^n \text{ such that } d = d_{g}^{[n]}(\theta)\}.$$ 

We should note that if we consider the critical determinant $\mathcal{K}_g$ of the cylinder

$$\mathcal{C}_g = \{z = (x, y) \in \mathbb{R}^{n+1} : g(y) \leq 1, |x| \leq 1\}$$
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(for the definitions see [5]), then

\[ D_g^{[n]} \subset \left[ 0, \frac{1}{\mathfrak{S}_g} \right]. \]

In particular it follows from Minkowski’s convex body theorem that

\[ \mathfrak{S}_g \geq \frac{\Omega(g)}{2^n}, \text{ where } \Omega(g) = \int_{x \in \mathbb{R}^n : g(x) \leq 1} dx. \]

So

\[ D_g^{[n]} \subset \left[ 0, \frac{2^n}{\Omega(g)} \right]. \]

For \( n = 1 \) and the standard norm \( g(y) = |y| \) the spectrum \( D = D_1^{[1]} \) was studied by many authors ([8, 10, 11, 13–15, 24], see also some recent results concerning related metric settings for uniform approximation to one number [4, 12, 16]). In the case \( n \geq 2 \) not much is known, however, a complete structure of the spectrum \( D_2^{[2]} \) for Euclidean norm

\[ |y| = \sqrt{y_1^2 + y_2^2} \]

in \( \mathbb{R}^2 \) was discovered by Akhunzhanov and Shatskov in [1]. It turned out that \( D_1^{[2]} \) is a segment, namely

\[ D_1^{[2]} = \left[ 0, \frac{2}{\sqrt{3}} \right]. \]

Here the value of \( \max D_1^{[2]} = \frac{2}{\sqrt{3}} \) is related to the critical determinant of the cylinder

\[ \{(x, y_1, y_2) \in \mathbb{R}^3 : y_1^2 + y_2^2 \leq 1, |x| \leq 1\} \]

calculated by Mahler [20].

As far as the authors know, \( D_1^{[2]} \) is the only Diophantine spectrum with completely known structure.† The authors believe that for \( n \geq 2 \) and arbitrary norm \( g(\cdot) \) the spectrum \( D_g^{[n]} \) should have the same simple structure, namely it should be the segment of the form \( [0, \frac{1}{\mathfrak{S}_g}] \); however, up to now they are not able to prove this even in the case \( n = 3 \) for Euclidean norm in \( \mathbb{R}^3 \).

We would like to mention that very recently a series of results concerning spectrum \( D_g^{[2]} \) for arbitrary norm \( g \) in \( \mathbb{R}^2 \) was obtained in [19]. In particular it was proven there that the maximal point \( \max D_g^{[2]} \) of the spectrum \( D_g^{[2]} \) is not isolated in \( D_g^{[2]} \).

† After this paper was submitted, J. Schleischitz [22] proved that for any \( n \geq 2 \) and sup-norm \( |x|_{\infty} = \max_{1 \leq j \leq n} |x_j| \), Dirichlet spectrum \( D_{|\cdot|_{\infty}}^{[n]} \) is just the segment \([0,1]\). He showed that for any \( d \in [0,1] \) there exists a Liouville (and hence non badly approximable) vector \( \theta \in \mathbb{R}^n \) with \( d_{|\cdot|_{\infty}}^{[n]}(\theta) = d \).
In this paper we are interested in a more detailed analysis of distribution of the values \( d^{[2]}_g(\theta) = d^{[2]}_g(\theta_1, \theta_2) \) in the two-dimensional case \( \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \).

2 BADLY APPROXIMABLE POINTS, BEST APPROXIMATIONS AND DIRICHLET IMPROVABILITY

Let us use the notation \( |y| \) for the Euclidean norm of \( y \in \mathbb{R}^n \). A point \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \) is called **badly approximable** if there exists a positive \( \gamma \) such that

\[
\min_{p \in \mathbb{Z}^n} |q\theta - p| > \frac{\gamma}{q^{1/n}} \quad \forall q \in \mathbb{Z}^+, \tag{1}
\]

or equivalently for any norm \( g(\cdot) \) in \( \mathbb{R}^n \) one has

\[
\inf_{t \geq 1} t(\psi_{g, \theta}(t))^n > 0.
\]

We should note that \( \psi_{g, \theta}(t) \) is a piecewise constant function and in the case \( \theta \in \mathbb{R}^n \setminus \mathbb{Q}^n \) one has \( \psi_{g, \theta}(t) > 0 \) for every \( t \). So we can define the unique infinite sequence of integers

\[
q_1 = 1 < q_2 < \ldots < q_{\nu-1} < q_\nu < \ldots
\]

such that

\[
\psi_{g, \theta}(t) = \psi_{g, \theta}(q_{\nu-1}) \quad \text{for} \quad q_{\nu-1} \leq t < q_{\nu}.
\]

Moreover for all \( \nu \) large enough (\( \nu \geq \nu_0 [g] \)) there exists a unique \( p_\nu \in \mathbb{Z}^n \) such that

\[
\psi_{g, \theta}(q_\nu) = g(q_\nu \theta - p_\nu).
\]

Of course the sequence (2) of the best approximations depends on the norm \( g \). However, all the norms in \( \mathbb{R}^n \) are equivalent. So Theorem 1 from the paper [2] leads to the following result. Let us fix a norm \( g \) and consider the sequence \( q_\nu \) of denominators of the best approximations with respect to this norm. Then \( \theta \in \mathbb{R}^n \) is badly approximable if and only if the inequality

\[
\sup_{\nu \in \mathbb{Z}_+} \frac{q_\nu}{q_{\nu-1}} < \infty
\]

holds. This result has an obvious quantitative form which we formulate below. Define

\[
m_g(\theta) = \limsup_{\nu \to \infty} \frac{q_\nu}{q_{\nu-1}}.
\]

**Theorem A.** If the inequality (1) holds for a certain positive \( \gamma \) for all \( q \) large enough then for the best approximations with respect to norm \( g \), one has

\[
m_g(\theta) < M \tag{3}
\]
with

$$M = c_1(n; g)\gamma^{-n}.$$  

Conversely, if (3) holds for the best approximations with respect to norm $g$ for some $M$ and for all $n$ large enough then for all $q$ large enough, we have (1) with

$$\gamma = c_2(n; g)M^{-1}.$$  

Here $c_j(n; g), j = 1, 2$ are explicit constant depending on dimension $n$ and norm $g$.

Another interesting phenomenon is related to singularity and Dirichlet improvability. A vector $\theta \in \mathbb{R}^n$ is called singular if $d_\| \cdot \|_\infty (\theta) = 0$. Note that this definition does not depend on the norm $g$ in $\mathbb{R}^n$. For $n = 1$ a real number $\theta$ is singular if and only if it is rational. On the other hand, for sup-norm $|y|_\infty = \max_{1 \leq j \leq n} |y_j|$ in $\mathbb{R}^n$ it is clear that $D_\| \cdot \|_\infty \subset [0, 1]$ and for almost all $\theta \in \mathbb{R}^n$ one has $d_\| \cdot \|_\infty (\theta) = 1$ (see classical paper by Davenport and Schmidt [9] as well as recent papers [17–19] with many metric results and the references therein). A vector $\theta \in \mathbb{R}^n$ is called Dirichlet improvable (with respect to sup-norm) if $d_\| \cdot \|_\infty (\theta) < 1$. In the case $n = 1$ H. Davenport and W. Schmidt showed [8] that a number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is Dirichlet improvable if and only if it is badly approximable. So for $n = 1$ a number $\theta$ is Dirichlet improvable if it is either badly approximable or singular.

The following result was obtained in [3].

**Theorem B.** For $n \geq 2$, the set of Dirichlet improvable vectors $\theta \in \mathbb{R}^n$ has continuum many points which are neither badly approximable nor singular.

In [3] the authors mentioned that the method of the paper [1] uses the theory of best approximations and can be adapted to construct Dirichlet improvable points in $\mathbb{R}^2$ that are not simultaneously singular or badly approximable. In the next section we formulate our main results and in particular explain this phenomenon.

At the end of this section we would like to formulate a quantitative statement which immediately follows from the argument of a familiar paper [25] by Jarník (see also discussion in [21, Section 4.1] as well as Section 2.6 from [7] and [6]).

**Theorem C.** Suppose that the components $\theta_1, \theta_2$ of a vector $\theta \in \mathbb{R}^2$ are linearly independent over $\mathbb{Q}$ together with $1$. Suppose that

$$d_\| \cdot \|_2 (\theta) \leq \epsilon.$$  

Then

$$m_\| \cdot \|_2 (\theta) \geq \frac{1}{36\epsilon^2}.$$  

## 3 | MAIN RESULTS

Here we formulate our main results dealing with non-badly approximable and badly approximable cases.
**Theorem 1.** Let $\varphi(t)$ be an arbitrary function increasing to $+\infty$ as $t \to \infty$. For any $\lambda \in \mathbb{D}_{2}\mid \cdot \mid = [0, \frac{2}{\sqrt{3}}]$ there exists $\theta \in \mathbb{R}^2$ such that

1. $d_{\mid \cdot \mid}^{[2]}(\theta) = \lambda$;
2. for the sequence (2) associated to $\theta$ one has $\frac{q_{n+1}}{q_n} \geq \varphi(n)$ for all $n$, and in particular $m_{\mid \cdot \mid}(\theta) = \infty$.

In particular Theorem 1 together with the result from [2] shows that for any $\lambda \in \mathbb{D}_{2}\mid \cdot \mid$, there exists $\theta \in \mathbb{R}^2$ which is not badly approximable and $d_{\mid \cdot \mid}^{[2]}(\theta) = \lambda$.

**Theorem 2.** Let $\varepsilon \in (0, \frac{1}{2.5 \cdot 10^4})$. Then for any $\lambda$ under the condition $\varepsilon < \lambda \leq \frac{2}{\sqrt{3}}$ there exists $\theta \in \mathbb{R}^2$ such that

1. $\lambda - \varepsilon \leq d_{\mid \cdot \mid}^{[2]}(\theta) \leq \lambda$;
2. $m_{\mid \cdot \mid}(\theta) < 10^6 \cdot \varepsilon^{-2}$.

Of course we do not take care about the optimality of the constant $10^6$ in Theorem 2. However, considering Theorem C we see that the order of the upper bound $O(\varepsilon^{-2})$ from Theorem 2 is optimal.

In particular Theorem 2 shows that $\mathbb{D}_{2}\mid \cdot \mid = [0, \frac{2}{\sqrt{3}}]$ is the closure of the set

$$\{\lambda : \exists \text{ badly approximable } \theta \in \mathbb{R}^2 \text{ such that } d_{\mid \cdot \mid}^{[2]}(\theta) = \lambda\}.$$  

The method of the proofs of Theorems 1 and 2 rely on the construction from [1].

The structure of the paper is as follows. In Section 4 we introduce all necessary parameters. In Section 5 we describe the inductive construction and formulate general Theorem 3. In Section 6 we deduce Theorems 1 and 2 from Theorem 3. The rest of the paper (Sections 7–11) is devoted to a complete proof of Theorem 3.

### 4 | PARAMETERS

Suppose that $\varepsilon_\nu > 0$ form a decreasing sequence

$$\frac{1}{10^4} \geq \varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_\nu \geq \varepsilon_{\nu+1} > \cdots.$$  

(4)

We consider a sequence of intervals

$$\Delta_\nu = (\alpha_\nu, \omega_\nu) \subset \left[0, \frac{2}{\sqrt{3}}\right], \quad \nu = 1, 2, 3, \ldots$$

of lengths $4\varepsilon_\nu = \omega_\nu - \alpha_\nu$ and construct the values of $V_\nu$ satisfying certain properties and such that $\frac{V_\nu}{\pi} \in \Delta_\nu$. Instead of intervals $\Delta_\nu$ it is convenient consider subintervals

$$\Delta^*_\nu = (\alpha^*_\nu, \omega^*_\nu) \subset \Delta_\nu$$
of length $\varepsilon_\nu$ such that either

$$(\alpha^*_\nu, \omega^*_\nu) \subset [\varepsilon_\nu, 1 - \varepsilon_\nu] \quad \text{(case 1)}$$

or

$$(\alpha^*_\nu, \omega^*_\nu) \subset \left[1, \frac{2}{\sqrt{3}} - \varepsilon_\nu\right] \quad \text{(case 2)}.$$ 

The arguments from the proofs below in cases 1 and 2 differ. In the sequel we will not write * for the endpoints $\alpha^*_\nu, \beta^*_\nu$ in order to avoid cumbersome notation. So in each case we denote the corresponding interval $(\alpha^*_\nu, \beta^*_\nu)$ of length $\varepsilon_\nu$ simply as $(\alpha_\nu, \beta_\nu)$ and refer to the condition of the case. Condition

$$\varepsilon_\nu \leq \alpha_\nu < \omega_\nu$$

(5)

gives the inequality

$$\omega_\nu = \alpha_\nu + \varepsilon_\nu \leq 2\alpha_\nu. \quad \text{(6)}$$

Define

$$B^-_\nu := \frac{\alpha^2_\nu}{\omega_{\nu-1}} \omega_\nu, \quad B^+_\nu := \frac{5\omega^2_\nu}{\alpha_{\nu-1}}, \quad \text{(7)}$$

and

$$H^-_\nu := \frac{\alpha_\nu \alpha^2_{\nu-1}}{5\omega_{\nu-1}}, \quad H^+_\nu := \frac{\sqrt{5} \omega_{\nu-1} \omega^2_\nu}{\alpha_{\nu-1} \alpha^2_\nu}, \quad \text{(8)}$$

it follows from (7), (8) and (5), (6) that

$$B^-_\nu < B^+_\nu, \quad \frac{1}{40} \leq H^-_\nu \leq \frac{1}{5} < \sqrt{5} < H^+_\nu \leq 8 \sqrt{5}. \quad \text{(from (4) and the bounds for } H^-_\nu, H^+_\nu \text{ we see that } K^-_\nu < K^+_\nu\text{). Here we should note that}$$

$$B^-_\nu k^2_\nu \geq \frac{30}{\varepsilon_\nu} \quad \text{and} \quad \frac{H^+_\nu}{k_\nu} \leq \frac{\varepsilon_\nu}{3}. \quad \text{(10)}$$

Let us consider, for example, the situation when

$$\alpha_\nu = \alpha, \quad \omega_\nu = \omega, \quad \varepsilon_\nu = \varepsilon_4 = \frac{\varepsilon}{4} = \omega - \alpha, \quad k_\nu = k$$

\(11\)
are constant sequences. We explain the restrictions on our parameters. We use (5) and (6) to obtain

\[ B_v^+ = \frac{\alpha^2}{\omega^2} \geq \frac{1}{4}, \quad B_v^- = \frac{5\omega^2}{\alpha^2} \leq 20, \quad K_v^- = \frac{24\sqrt{5}}{\varepsilon^*}, \quad K_v^+ \geq \frac{1}{160\varepsilon^*}. \]

If we take \( \varepsilon^* \leq 10^{-4} \) and choose

\[ k = \left\lceil \frac{24\sqrt{5}}{\varepsilon^*} \right\rceil + 1, \quad (12) \]

Condition (9) is satisfied.

So in the case (11) when the sequences of parameters are constant sequences and \( k \) is chosen as in (12), we have the inequality

\[ B_v^+ k_v^2 \leq 6 \cdot 10^4 \frac{\varepsilon^2}{\varepsilon^2^*} < 10^6 \varepsilon^2. \]

(13)

5 \quad \text{INDUCTIVE CONSTRUCTION}

For \( Q, R > 0 \) and \( v = (1, v_1, v_2) \in \mathbb{R}^3 \) we define the cylinder

\[ \Pi(v, Q, R) = \left\{ z = (x, y) = (x, y_1, y_2) \in \mathbb{R}^3 : 0 \leq x \leq Q, \ |xv - z| \leq R \right\}. \]

In this section we describe a variant of a standard inductive procedure of constructing a sequence integer points

\[ w_v = (q_v, p_v) = (q_v, p_{1,v}, p_{2,v}) \in \mathbb{Z}^3, \quad p_v = (p_{1,v}, p_{2,v}) \in \mathbb{Z}^2, \quad v = 1, 2, 3, \ldots, \]

corresponding rational points

\[ v_v = (1, v_v) = \left(1, \frac{p_{1,v}}{q_v}, \frac{p_{2,v}}{q_v}\right) \in \mathbb{Q}^3, \quad v_v = \left(\frac{p_{1,v}}{q_v}, \frac{p_{2,v}}{q_v}\right) \in \mathbb{Q}^2, \quad v = 1, 2, 3, \ldots, \]

cylinders

\[ \Pi_v = \Pi(v_v, q_v, R_v), \quad \Pi^-_v = \Pi(v_v, q_{v-1}, R^-_v), \]

(15)

where

\[ R_v = |q_{v-1}v_v - w_{v-1}| = q_{v-1}|v_v - v_{v-1}|, \quad R^-_v = |q_{v-2}v_v - w_{v-2}| = q_{v-2}|v_v - v_{v-2}| \]

and extended cylinders

\[ \overline{\Pi}_v = \Pi(v_v, q_v, R_v(1 + \varepsilon_v)), \quad \overline{\Pi}_v^- = \Pi(v_v, q_{v-1}, R^-_v(1 + \varepsilon^-_v)), \quad \text{where} \ \varepsilon^-_v = \varepsilon^2_{v-1}. \]

(16)
It is clear that
\[ \Pi_\nu \subset \Pi_\nu^-, \quad \Pi^-_\nu \subset \Pi_\nu^- . \]

Our objects for every \( \nu \) should satisfy the following Conditions 1)–6).

Condition 1) For any \( \nu \) vectors \( w_{\nu-2}, w_{\nu-1}, w_\nu \) form a basis of \( \mathbb{Z}^3 \).

Condition 2) For every \( \nu \) we have \( B^-_\nu k^2_\nu \leq \frac{q_\nu}{q_{\nu-1}} \leq B^+_\nu k^2_\nu \) where \( B^\pm_\nu \) are defined in (7).

Condition 3) \( H^-_\nu R^-_{\nu-1} k^-_\nu \leq R_\nu \leq H^+_\nu R^+_{\nu-1} k^+_\nu \), where \( H^\pm_\nu \) are defined in (8).

Condition 4) \( \Pi_\nu \cap \mathbb{Z}^3 = \Pi^-_\nu \cap \mathbb{Z}^3 = \{0, w_{\nu-1}, w_\nu, w_\nu - w_{\nu-1}\} \).

Condition 5) \( \Pi^-_\nu \cap \mathbb{Z}^3 = \Pi^-_\nu \cap \mathbb{Z}^3 = \{0, w_{\nu-2}, w_{\nu-1}\} \).

Condition 6) The volume \( V_\nu = \text{vol} \Pi_\nu = \pi q_\nu (R_\nu)^2 \) for every \( \nu = 1, 2, 3, \ldots \) satisfies \( V_\nu / \pi \in (\alpha_\nu, \omega_\nu) \).

**Theorem 3.** For a given sequence (4) of \( \varepsilon_\nu \) and for a sequence of parameters \( k_\nu \) which are chosen according to (9), there exists a sequence (14) of integer points \( w_\nu \) such that all Conditions 1)–6) above are valid.

We give a proof of Theorem 3 in Sections 7–11. Namely, in Sections 6–10 we introduce all necessary objects and constructions and prove all the auxiliary statements and in Section 11 we complete the inductive step. In the next section we show that from the existence of a sequence of integer points \( w_\nu, \nu = 1, 2, 3, \ldots \) satisfying Conditions 1)–6) above, Theorems 1 and 2 follow. For this purpose we need to use Conditions 2), 3) and 5), 6), only. The remaining conditions (Conditions 1) and 4)) can be considered as auxiliary conditions for the first ones; however, they have clear geometric interpretation. Moreover, they can clarify the construction.

Here we would like to formulate few more remarks.

First of all it follows from Conditions 2) and 3) and inequalities (10) that
\[
\frac{q_\nu}{q_{\nu-1}} \geq \frac{45}{\varepsilon_\nu} \quad \text{and} \quad \frac{R_\nu}{R_{\nu-1}} \leq \frac{\varepsilon_\nu}{3}.
\]

As cylinder \( \Pi_\nu \) does not have integer points inside, by Mahler’s theorem on the critical determinant [20] we have
\[
V_\nu \leq \frac{2\pi}{\sqrt{3}}.
\]

### 6 PROOF OF THEOREMS 1 AND 2

Here we deduce Theorems 1 and 2 from Theorem 3.

Because of (17) points \( v_\nu \) form a fundamental sequence and the limit point
\[
\Theta = (1, \emptyset) = (1, \emptyset_1, \emptyset_2) = \lim_{n \to \infty} v_\nu
\]
satisfies
\[
|\Theta - v_\nu| = |\Theta - v_{\nu}| \leq \sum_{j=\nu}^{\infty} |v_j - v_{j+1}| \leq \sum_{j=\nu}^{\infty} \frac{R_{j+1}}{q_j} \leq \frac{\varepsilon_\nu \cdot R_\nu}{3} \sum_{j=\nu}^{\infty} \frac{1}{q_j} \leq \frac{\varepsilon_\nu \cdot R_\nu}{2q_\nu}.
\]
Now we define cylinders

\[ \Pi_y(\theta) = \Pi(\Theta, q_y, R_y(\theta)), \text{ where } R_y(\theta) = |q_{y-1}\theta - p_{y-1}| = q_{y-1}|\theta - v_{y-1}|. \]

By the definition, \( w_{y-1} \) belongs to the boundary of the cylinder \( \Pi_y(\theta) \). Now we show that

\[ \Pi_y(\theta) \cap \mathbb{Z}^3 = \{0, w_{y-1}, w_y\}. \]  

To prove this, it is enough to show that

\[ \Pi_{y-1}(\theta) \subset \overline{\Pi}_y \] (21)

and that

\[ w_y \text{ belongs to the facet } \{x = q_y\} \text{ of cylinder } \Pi_y. \] (22)

holds for every \( y \). Indeed (20) follows from (21, 22) by Condition 5).

To obtain (21) by triangle inequality it is enough to prove the inequality

\[ q_{y-1}|\theta - v_y| + R_{y-1}(\theta) \leq R_y(1 + \varepsilon_y^-) \] (23)

(here \( q_{y-1}|\theta - v_y| \) is the distance between the centres of the sections \( \{x = q_{y-1}\} \) of the cylinders under consideration and we want the section of the cylinder \( \Pi_{y-1}(\theta) \) to be inside the section of the cylinder \( \Pi_y \)). We deduce from (19) and (17) the inequality

\[ |R_{y-1}(\theta) - R_y^-| \leq q_{y-2}|\theta - v_y| \leq \frac{q_y}{q_{y-1}}q_{y-1}|\theta - v_y| \leq \frac{\varepsilon_y^- R_y^2}{4050} \leq \frac{\varepsilon_y^2 R_y^-}{6075}. \]

Here in the next estimate we take into account that \( R_y \leq \frac{\varepsilon_y R_y^-}{3} \leq \frac{2\varepsilon_y^-}{3} R_y^- \). On the other hand,

\[ q_{y-1}|\theta - v_y| \leq \frac{\varepsilon_y q_y R_y}{45}|\theta - v_y| \leq \frac{\varepsilon_y^2 R_y}{90} \leq \frac{\varepsilon_y^2 R_y^-}{225}. \]

So

\[ q_{y-1}|\theta - v_y| + R_{y-1}(\theta) \leq R_y^-(1 + \frac{2\varepsilon_y^-}{225}), \]

and this gives (23) and (21). As for (22), it immediately follows form the inequality (19) as for every \( y \) we have

\[ R_{y+1}(\theta) = q_y|\theta - v_y| < \frac{\varepsilon_y R_y(\theta)}{4} < R_y(\theta). \]

Relation (20) means that vectors \( w_y \) form the sequence of all best simultaneous approximations to \( \theta \). So to prove Theorems 1 and 2 it is enough to choose the corresponding values of parameters
and to understand what is
\[
\limsup_{n \to \infty} \frac{\text{Vol} \, \Pi_n(\theta)}{\pi}.
\]

For the volumes \(\text{Vol} \, \Pi_n(\theta) = \pi q_y R_y(\theta)^2\) of the cylinders \(\Pi_n(\theta)\) and in view of (18) we have

\[
|\text{Vol} \, \Pi_n(\theta) - V_y| = \pi q_y |R_y(\theta)^2 - R_y|^2 \leq 3\pi q_y R_y(\theta) - R_y| \leq \frac{\varepsilon^2_v V_y}{30} < \varepsilon_v,
\]

as

\[
|R_y(\theta) - R_y| < q_{v-1} |\theta - v_{v-1}| \leq \frac{\varepsilon^2_v R_y}{900}.
\]

Let us take \(\lambda \in [0, \frac{2}{\sqrt{3}}]\). If we take the sequences \(\alpha_y, \omega_y\) converging to \(\lambda\) and \(k_y\) large enough such that \(B^{-k_y^2} \geq \varphi(v)\) by Conditions 2) and 6), we get a proof of Theorem 1.

To prove Theorem 2 we can take constant parameters \(\alpha_y = \lambda - 3 \varepsilon_y, \omega_y = \lambda - 2 \varepsilon_y\) or \(\alpha_y = \lambda - 2 \varepsilon_y, \omega_y = \lambda - \varepsilon_y\) depending on case 10 or 20 with \(\varepsilon_y = \frac{\varepsilon}{4}\) in (11) and with \(k_y = k\) defined in (12). Then by Condition 6) and (24) we have \(\text{Vol} \, \Pi_n(\theta) \in (\lambda - \varepsilon, \lambda)\). We take into account Condition 2) and (13), and this gives Theorem 2.

7 INTEGER BASES AND NATURAL COORDINATES

Let

\[
\mathbf{\Omega} = (g_1, g_2, g_3), \quad g_j = (q_j, a_{1,j}, a_{2,j}) \in \mathbb{Z}^3, \quad q_j > 0
\]

be a basis of the integer lattice \(\mathbb{Z}^3\). We consider the vectors \(g_1^*, g_2^*, g_3^*\) defined by

\[
g_1^* = \left(1, \frac{a_{1,1}}{q_1}, \frac{a_{2,1}}{q_1}\right), \quad g_2^* = \left(0, \frac{a_{1,2}}{\sqrt{a_{1,2}^2 + a_{2,2}^2}}, \frac{a_{2,2}}{\sqrt{a_{1,2}^2 + a_{2,2}^2}}\right), \quad |g_2^*| = 1
\]

and

\[
g_3^* = (0, a, b), \quad |g_3^*| = 1, \quad g_3^* \text{ is orthogonal to } g_2^* \text{ and } g_3^* = x g_1 + y g_2 + z g_3 \text{ with } z > 0
\]

which form a basis in \(\mathbb{R}^3\). Coordinates \((x, y, z)\) in \(\mathbb{R}^3\) with respect to the basis \(g_1^*, g_3^*, g_3^*\) are called natural coordinates associated with \(\mathbf{\Omega}\). Consider unit vectors

\[
e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)
\]

and the unique linear mapping \(G\) such that \(G g_j^* = e_j, j = 1, 2, 3\). It is clear that \(G\) preserves volume in \(\mathbb{R}^3\) and moreover in any affine subspace of the form \(\{x = \text{const}\}\), it preserves Euclidean
distances between points. We see that

\[ \mathcal{G}g_1 = (q_1, 0, 0), \quad \mathcal{G}g_2 = (q_2, d, 0), \quad d > 0, \quad \mathcal{G}g_3 = (q_3, f, h), \quad h > 0 \]  \tag{25} 

with some \( d, f, h \). Here in the right-hand side of all the equalities we have the natural coordinates of the vectors \( g_j \) with respect to \( \emptyset \). It is clear from orthogonality that

\[ q_1 d h = 1. \]  \tag{26} 

We associate with \( \emptyset \) an unimodular lattice

\[ \Gamma_{\emptyset} = \mathcal{G}\mathbb{Z}^3. \]

It is clear that the vectors \( \mathcal{G}g_1, \mathcal{G}g_2, \mathcal{G}g_3 \) form a basis of \( \Gamma_{\emptyset} \).

In particular in natural coordinates \( (x, y, z) \) with respect to the basis

\[ g_1 = w_\nu, \quad g_2 = w_{\nu-1}, \quad g_3 = w_{\nu-2}, \]  \tag{27} 

the images \( w'_j = \mathcal{G}w_j \) satisfy

\[ w'_\nu = (q, 0, 0), \quad w'_{\nu-1} = (a_0, d, 0), \quad w'_{\nu-2} = (g, f, h) \] with \( q = q_\nu, \quad a_0 = q_{\nu-1}, \quad g = q_{\nu-2}, \quad d = R_\nu. \]  \tag{28} 

We consider the hyperplane \( \pi_1 \) defined by

\[ \pi_1 = \{(x, y, z) : \quad z = h\}, \]

so \( w_{\nu-2} \in \pi_1 \).

In natural coordinates cylinders \( \Pi'_\nu = \mathcal{G}\Pi_{\nu} \) and \( \Pi''_\nu = \mathcal{G}\Pi''_{\nu} \) from (15) can be defined by

\[ \Pi'_\nu = \{(x, y, z) : \quad 0 \leq x \leq q, \quad y^2 + z^2 \leq d^2\}, \quad \Pi''_\nu = \{(x, y, z) : \quad 0 \leq x \leq a_0, \quad y^2 + z^2 \leq f^2 + h^2\}, \]

respectively.

In the sequel we need to consider the mapping \( F : (x_2, y_2) \mapsto (x_1, y_1) \) introduced in [1] and defined by

\[ x_1 = \frac{h(x_2^2 + q^2)}{y_2q}, \quad y_1 = \frac{hx_2}{q}, \]  \tag{29} 

which depends on \( q \) and \( h \) as parameters. Here we would like to explain the meaning of this mapping. Consider the unique cylinder \( \Pi = \Pi(\nu, Q, R) \) such that the point \( w_\nu = (q, 0, 0) \) belongs to its boundary and the point \( (x_1, y_1, h) \in \pi_1 \) is the centre of the facet \( \{x = Q\} \) of \( \Pi \). So

\[ \nu = \left(1, \frac{y_1}{x_1}, \frac{h}{x_1}\right), \quad Q = x_1 \quad \text{and} \quad R = \frac{q}{x_1} \sqrt{y_1^2 + h^2}. \]

This cylinder \( \Pi \) can be characterised in a rather different way: cylinder \( \Pi \) is the unique cylinder of the form \( \Pi(\nu, Q, R) \) such that the centre of its facet \( \{x = Q\} \) belongs to \( \pi_1 \), the point \( w_\nu = (q, 0, 0) \)
belongs to its boundary and the line

\[ \{(x, y, z) : x = x_2, z = 0\} \]

in the coordinate plane \( \{z = 0\} \) is tangent to the boundary of \( \Pi \) in a certain point \( (x_2, y_2, 0) \) (for the details see [1]). The values \( (x_1, y_1) \) and \( (x_2, y_2) \) just satisfy the relation (29).

8 | ELLIPSES AND HYPERBOLAS

We consider the coordinate plane \( \mathbb{R}^2(x, y) \) and points

\[ W = (q, 0), \; A = (a, d), \; q, d > 0; \; Z = (x, y), \; Z_2 = (x_2, y_2). \]

We consider equations

\[ (xy_2 - yx_2)^2 + (qy)^2 = (qy_2)^2, \]  \hspace{1cm} (30)

\[ (ay_2 - dx_2)^2 + (qd)^2 = (qy_2)^2. \]  \hspace{1cm} (31)

Direct calculations provide the following lemma.

**Lemma 1.** Let us fix the points \( W = (q, 0), Z_2 = (x_2, y_2) \). Let \( \mathcal{E} \) be the 0-symmetric ellipse which passes through the points \( W, Z_2 \), that is \( W, Z_2 \in \mathcal{E} \). Suppose that the tangent line to \( \mathcal{E} \) is parallel to the 0x coordinate axis, so it is of the form

\[ \{(x, y) \in \mathbb{R}^2 : y = y_2\}. \]

Then the points \( W, Z_2 \) define ellipse \( \mathcal{E} = \mathcal{E}(q; x_2, y_2) \) uniquely and

\[ \mathcal{E} = \{(x, y) \in \mathbb{R}^2 : (x, y) \text{ satisfy (30)} \}. \]  \hspace{1cm} (32)

For points \( W = (q, 0), A = (a, d) \) we consider the branch of 0-symmetric hyperbola

\[ \mathcal{H} = \mathcal{H}(q; a, d) = \{(x_2, y_2) \in \mathbb{R}^2 : y_2 \geq 0, \; (x_2, y_2) \text{ satisfy (31)} \}, \]

and the family \( \mathcal{E}(q; a, d) \) of 0-symmetric ellipses which pass through the points \( W = (q, 0) \) and \( A = (a, d) \), that is,

\[ \mathcal{E} = \{\mathcal{E} : A, W \in \mathcal{E}\}. \]  \hspace{1cm} (33)

**Lemma 2.** Let us fix the points \( W = (q, 0), A = (a, d) \). For any ellipse \( \mathcal{E} \in \mathcal{E} \) consider the unique point \( Z_2 = (x_2, y_2) = Z_2(\mathcal{E}), y_2 \geq 0 \) where the tangent line to the ellipse \( \mathcal{E} \) is parallel to 0x axis. Then

\[ \{(x_2, y_2) : \exists \mathcal{E} \in \mathcal{E} \text{ such that } (x_2, y_2) = Z_2(\mathcal{E})\} = \mathcal{H}(q; a, d). \]

**Proof.** We substitute \( (a, d) \) into (30) and obtain (31). \( \square \)
Consider the set
\[ \mathfrak{B}_2(a, d) = \{(x_2, y_2) \in \mathbb{R}^2 : y_2 \geq 0, \ (ay_2 - dx_2)^2 + (qd)^2 \geq (qy_2)^2\} . \]
The boundary of the set \( \mathfrak{B}_2(a, d) \) is the union of the coordinate line \( 0x \) and hyperbola \( \mathfrak{H}(q; a, d) \).

**Lemma 3.** Let us take three points
\[ W = (q, 0), \ A = (a, d), \ A' = (a + q, d) \]
and consider two hyperbolas
\[ \mathfrak{H} = \mathfrak{H}(q; a, d) \text{ and } \mathfrak{H}' = \mathfrak{H}(q; a + q, d). \]

Then
1. the intersection \( \mathfrak{H} \cap \mathfrak{H}' \) consists just of one point \( Z' = (x', y') \) and \( x' = \frac{2}{\sqrt{3}} d \);
2. for any \( \lambda \in \left[ 1, \frac{2}{\sqrt{3}} \right) \) and for curvilinear triangle
   \[ \mathfrak{T} \subset \mathfrak{B}_2(a, d) \cap \mathfrak{B}_2(a + q, d) \]
   with vertices \( A, A', Z' \) and bounded by the curves \( \mathfrak{H}, \mathfrak{H}' \) and \( \ell_1 \), the intersection
   \[ \ell_\lambda \cap \mathfrak{T}, \text{ where } \ell_\lambda = \{(x_2, y_2) \in \mathbb{R}^2 : y_2 = d\lambda\} \]
is a segment \( [R_\lambda, S_\lambda] \subset \ell_\lambda \) of length \( \varepsilon_\lambda \eta \), where
\[ \varepsilon_\lambda = \lambda - 2\sqrt{\lambda^2 - 1}. \] (34)

The endpoints of this segment have coordinates
\[ R_\lambda = (r, d\lambda), \ r = a\lambda + q\sqrt{\lambda^2 - 1}, \ S_\lambda = (s, d\lambda), \ s = (a + q)\lambda - q\sqrt{\lambda^2 - 1}; \] (35)
(3) for any \( a_0 \in \mathbb{R} \) the intersection
\[ \mathfrak{B}_2 = \bigcap_{k \in \mathbb{Z}} \mathfrak{B}_2(a_0 + kq, d) \] (36)
lies in the strip
\[ \Sigma = \left\{(x, y) : 0 \leq x \leq \frac{2}{\sqrt{3}} d\right\} \] (37)
and includes the strip
\[ \Sigma_0 = \{(x, y) : 0 \leq x \leq d\}, \]
so \( \Sigma_0 \subset \mathfrak{B}_2 \subset \Sigma \).
We should note that in the case \( \lambda \in (1, \frac{2}{\sqrt{3}}) \) for \( \varepsilon_\lambda \) defined in (34) we have bounds
\[
3 \cdot \left( \frac{2}{\sqrt{3}} - \lambda \right) < \varepsilon_\lambda \leq 1.
\] (38)

Proof of Lemma 3. Statements 1) and 2) follow from direct calculation. Statement 3) follows from Statement 1) as for every \( k \in \mathbb{Z} \) the intersection \( \mathcal{H}(q; a_0 + kq, d) \cap \mathcal{H}(q; a_0 + (k + 1)q, d) \) consists just of one point \( Z'_k = (x'_k, y'_k) \) and \( x'_k = \frac{2\sqrt{3}}{3}d \).

Let parameters \( \alpha, \omega \) and \( d, v \) be fixed. We consider the segment
\[
[U^{[1]}, V^{[1]}] \text{ with } U^{[1]} = (v, d\alpha), \ V^{[1]} = (v, d\omega).
\] (39)
Here we should note that the endpoints of the segment \([U^{[1]}, V^{[1]}]\) do not depend on parameter \( \lambda \). In case 1\(^{0}\) when \( \alpha, \omega \in (0, 1) \) this segment belongs to the strip \( \Sigma_0 \) and so to the set \( \mathfrak{B}_2 \). The situation in case 2\(^{0}\) when parameter \( \alpha, \omega \in (1, \frac{2}{\sqrt{3}}) \) is a little bit more complicated. In the next lemma for any \( \lambda \) we define a segment \([U^{[2]}(\lambda), V^{[2]}(\lambda)] \subset \mathfrak{B}_2 \) which belongs to the segment \([U^{[1]}, V^{[1]}]\) with endpoints depending on \( \lambda \) and under certain conditions (see formula (44) from the next section).

Lemma 4. Suppose that \( a \geq 0 \). Let \( \lambda \in [1, \frac{2}{\sqrt{3}}) \) and
\[
u = \frac{dv\lambda}{s}, \ v = \frac{r + s}{2} = \left( a + \frac{q}{2}\right)\lambda,
\] (40)
where \( r \) and \( s \) are defined in (35). Consider the points
\[
V^{[2]}(\lambda) = (v, d) \in [R_\lambda, S_\lambda], \ U^{[2]}(\lambda) = (v, u).
\]
Then
\[
[U^{[2]}(\lambda), V^{[2]}(\lambda)] \subset \mathfrak{B}_2(a, d) \cap \mathfrak{B}_2(a + q, d)
\]
and the length of the segment \([U^{[2]}(\lambda), V^{[2]}(\lambda)]\) is equal to \( d\omega(1 - \frac{v}{s}) \).

Proof. Both sets \( \mathfrak{B}_2(a, d) \) and \( \mathfrak{B}_2(a + q, d) \) are star bodies with respect to the origin 0. So the intersection \( \mathfrak{B}_2(a, d) \cap \mathfrak{B}_2(a + q, d) \) is also a star body and
\[
\text{conv} \{0, R_\lambda, S_\lambda\} \subset \mathfrak{B}_2(a, d) \cap \mathfrak{B}_2(a + q, d).
\]
The point \( U^{[2]}(\lambda) \) is just the intersection of the segment \([0, S_\lambda]\) and the line \( \{x, y: x = v\} \). As \( a \geq 0 \) we see that \([U^{[2]}(\lambda), V^{[2]}(\lambda)] \subset \text{conv} \{0, R_\lambda, S_\lambda\} \subset \mathfrak{B}_2(a, d) \cap \mathfrak{B}_2(a + q, d) \). So the length of the segment \([U^{[2]}(\lambda), V^{[2]}(\lambda)]\) is equal to \( d\lambda(1 - \frac{v}{s}) \) and everything is clear.

Now we specify a little bit the choice of parameters.
Let
\[ 0 \leq a_0 < q, \quad a_k = a_0 + kq, \quad k \in \mathbb{Z}, \quad k \geq 2 \] (41)

and let
\[ V^{[2]}_k(\lambda) = (v_k, d\lambda), \quad U^{[2]}_k(\lambda) = (v_k, u_k) \]

be a point with coordinates \( v_k = v, u_k = u \) depending on \( \lambda \) and defined by (35, 40) by taking \( a = a_k \), so in particular
\[ u_k = \frac{dv_k\lambda}{s_k}, \quad v_k = \left( a_0 + \left( k + \frac{1}{2} \right)q \right)\lambda, \quad s_k = (a_0 + (k + 1)q)\lambda - q\sqrt{\lambda^2 - 1} \]

and
\[ \frac{v_k}{s_k} \geq a_0 + \left( k + \frac{1}{2} \right)q \geq a_0 + (k + 1)q \geq 1 - \frac{1}{2k}. \] (42)

Here we should note that the length of the segment \([U^{[2]}_k(\lambda), V^{[2]}_k(\lambda)]\) from Lemma 4 by (42) has an upper bound
\[ \text{length}[U^{[2]}_k(\lambda), V^{[2]}_k(\lambda)] = d\lambda \left( 1 - \frac{v_k}{s_k} \right) \leq \frac{d\lambda}{2k}. \] (43)

We consider the segment \([U^{[1]}, V^{[1]}]\) where \( v = v_k \) depends on \( \lambda \) and is chosen afterwards. Then
\[ [U^{[2]}_k(\lambda), V^{[2]}_k(\lambda)] \subset [U^{[1]}, V^{[1]}] \quad \text{provided} \quad \frac{1}{4k} < \omega - \alpha. \] (44)

9 | ELLIPSES \( \mathfrak{E} \) AND A TWO-DIMENSIONAL LATTICE

We consider lattice
\[ \Lambda_{q,a_0,d} = \{ g_1 m + g_2 n : m, n \in \mathbb{Z} \}, \quad \text{where} \quad g_1 = (q, 0), \quad g_2 = (d, a_0). \] (45)

By \( \widehat{\mathfrak{E}} \) we denote the ellipse \( \mathfrak{E} \) together with its interior, so \( \mathfrak{E} \) is the boundary of \( \widehat{\mathfrak{E}} \). For our further consideration we need two ellipses \( \mathfrak{E}_1 \) and \( \mathfrak{E}_2 \) which depend on the parameters \( q, a, d\lambda \).

We are interested in dependence on parameter \( \lambda \in (0, \frac{2}{\sqrt{3}}\sqrt{\lambda^2 - 1}) \). However, we should distinguish two cases.

In case 1\(^0\) we consider a segment \([\alpha, \omega] \subset (0, 1)\) and we simply deal with \( \lambda \in [\alpha, \omega] \). In this case we define
\[ t_1 = \frac{1}{\omega} = 1 + \delta_1 > 1, \quad \delta_1 = \frac{1}{\omega} - 1 > 0. \] (46)
We take arbitrary $a \in \mathbb{R}$ and for $\lambda \in [\alpha, \omega]$ we consider the ellipse

$$\mathcal{E}_1 = \mathcal{E}(q; a, d\lambda).$$

The second ellipse is defined as a dilated ellipse of the form

$$\mathcal{E}_2 = t_1 \cdot \mathcal{E}(q; a, d\lambda),$$

where the parameter of dilatation $t_1$ is defined by (46).

**Case 2** is a little bit more complicated. We take $[\alpha, \omega] \subset (1, \frac{2}{\sqrt{3}})$ and deal with $\lambda \in [\alpha, \omega]$.

In case 2 we define ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ for these values of $\lambda$. The first ellipse is defined by

$$\mathcal{E}_1 = \mathcal{E}(q; v_k, d\lambda).$$

The second ellipse is defined as a dilated ellipse of the form

$$\mathcal{E}_2 = t_2 \cdot \mathcal{E}(q; v_k, d\lambda),$$

where parameter of dilatation $t_2$ does not depend on $\lambda$ and is defined by the formula

$$t_2 = 1 + \delta_2, \quad \delta_2 = \frac{2}{\sqrt{3}} - \omega.$$  

(50)

We should note that the values of the coefficients of dilatations $t_1$ and $t_2$ do not depend on $\lambda$. They depend on the endpoints of segment $[\alpha, \omega]$.

**Lemma 5.** In both cases 1 and 2 for any integer $k \geq 2$ and for both ellipses $\hat{\mathcal{E}}_j, j = 1, 2$ defined above, one has

$$\hat{\mathcal{E}}_j \cap \Lambda_{q, a_0, d} = \{0, \pm g_1\},$$

where $g_1$ is defined in (45).

**Proof.** As $\hat{\mathcal{E}}_1 \subset \hat{\mathcal{E}}_2$ it is enough to proof Lemma 5 for ellipse $\hat{\mathcal{E}}_2$ only.

In the case 1 we see from (46, 47) that ellipse $\hat{\mathcal{E}}_2$ lies in the strip $\{(x, y) : -d < x < d\}$, and lemma follows immediately.

Let us consider the case 2. Ellipse $\hat{\mathcal{E}}_1$ can be defined by Equation (30) in coordinates $(x, y)$ with parameters

$$x_2 = v_k = \left(a_0 + \left(k + \frac{1}{2}\right)q\right)\lambda, \quad y_2 = d\lambda,$$

that is,

$$\left(xd\lambda - y\left(a_0 + \left(k + \frac{1}{2}\right)q\right)\lambda\right)^2 + (qy)^2 = (qd\lambda)^2.$$
Applying linear transformation
\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - \tau y \\ y \end{pmatrix}, \quad \tau = \frac{a_0 + \left( k + \frac{1}{2} \right) q \lambda}{d \lambda}.
\]
we see that it is enough to consider ellipse \( \mathcal{E}_2 \) defined by the equation
\[ (xd\lambda)^2 + (qy)^2 = (qd\lambda)^2 \]
and lattice \( \Lambda_{q,\frac{q}{2},d} \). The only ‘dangerous’ lattice points for the dilated ellipse \( \hat{\mathcal{E}}_2 = t_2 \cdot \mathcal{E}_1 \) are the lattice points \((\pm \frac{q}{2}, \pm d)\in \Lambda_{q,\frac{q}{2},d}\). These points do not belong to \( \hat{\mathcal{E}}_2 \) if and only if
\[
\left( \left( \frac{a_0}{q} + k \right) \lambda - \left( \frac{a_0}{q} + \left( k + \frac{1}{2} \right) \right) \lambda \right)^2 + 1 > (t_2\lambda)^2,
\]
or \( t_2 < \frac{1}{4} + \frac{1}{\lambda^2} \leq \frac{1}{4} + \frac{1}{\lambda^2} \). The choice of \( t_2 \) by \((50)\) satisfies this condition.

\section*{10 | MAPPING F}

We consider mapping \( F : (x_2, y_2) \mapsto (x_1, y_1) \) defined by \((29)\) in the end of Section 5 and introduced in \([1]\).

\textbf{Lemma 6.} Let \( 0 < \alpha < \omega < 1 \) and
\[
\nu^2 > q^2 \cdot \left( \frac{\alpha \omega}{\omega - \alpha} \cdot \frac{d}{h} - 1 \right).
\]
Consider the segment \([U^{(1)}, V^{(1)}]\) defined in \((39)\). Then the image \( F([U^{(1)}, V^{(1)}]) \) is the segment \([F(U^{(1)}), F(V^{(1)})]\) parallel to 0y axis, its endpoints have coordinates
\[
F(U^{(1)}) = \left( \frac{h \nu^2 + q^2}{dq\alpha}, \frac{h \nu}{q} \right), \quad F(V^{(1)}) = \left( \frac{h \nu^2 + q^2}{dq\omega}, \frac{h \nu}{q} \right)
\]
and its length is bounded below by
\[
\text{length } [F(U^{(1)}), F(V^{(1)})] > q.
\]
\textbf{Proof.} We use formulas \((29)\) to see that
\[
\text{length } [F(U^{(1)}), F(V^{(1)})] = \frac{h \nu^2 + q^2}{dq\alpha} - \frac{h \nu^2 + q^2}{dq\omega} \geq \frac{h \nu^2 + q^2}{dq} \cdot \frac{\omega - \alpha}{\alpha \omega} > q.
\]
Everything is proven. \(\square\)
Lemma 7. \( \lambda \in [\alpha, \omega] \subset (1, \frac{2}{\sqrt{3}}) \). Consider segment \([U^{[2]}(\lambda), V^{[2]}(\lambda)]\) defined in Lemma 4. Then this image \( F([U^{[2]}(\lambda), V^{[2]}(\lambda)]) \) is the segment \([F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))]\) parallel to \(0y\) axis and its endpoints have coordinates

\[
F(U^{[2]}(\lambda)) = \left( \frac{h(v^2 + q^2)}{dq\lambda} \cdot \frac{s}{v}, \frac{hu}{q} \right), \quad F(V^{[2]}(\lambda)) = \left( \frac{h(v^2 + q^2)}{dq\lambda}, \frac{hu}{q} \right).
\] (53)

Moreover, suppose that

\[
a > \frac{3\sqrt{3}}{2\varepsilon_\omega} \cdot q
\] (54)

and

\[
\frac{h}{d} \geq \frac{\sqrt{3}}{2}
\] (55)

Then for any \( \lambda \) under the consideration the length of the segment \([U^{[2]}(\lambda), V^{[2]}(\lambda)]\) is bounded below by

\[
\text{length} \ [F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))] > q.
\] (56)

Proof. Equations (53) follow from (40) and (29). Then

\[
\text{length} \ [F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))] = \frac{h(v^2 + q^2)}{dq\lambda} \cdot \frac{s}{v} - \frac{h(v^2 + q^2)}{dq\lambda} \geq \frac{h(v^2 + q^2)}{dq\lambda} \cdot \frac{s - r}{s + r}.
\]

But \( s - r = \varepsilon_1 q \) and \( s + r = (2a + q)\lambda \). Now to get the lower bound for the length of \([F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))]\) we apply inequalities \( v \geq r \geq a \omega, \) (55) and condition (54). So we obtain

\[
\text{length} \ [F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))] \geq \frac{3\sqrt{3}a}{2\varepsilon_\lambda} \geq \frac{3\sqrt{3}a}{2\varepsilon_\omega} \geq q.
\]

This is the desired lower bound (56).

\[ \square \]

11 | INDUCTIVE STEP

We suppose that the points \( w_j, 1 \leq j \leq \nu \) satisfying Conditions 1)–6) from Section 5 are constructed. We must explain how to construct point \( w_{\nu+1} \) such that the objects from Section 5 satisfy the required conditions for \((\nu + 1)\)th step.

First of all we will take vector \( w_{\nu+1} \) of the form

\[
w_{\nu+1} = w_{\nu-2} + mw_{\nu-1} + kw, \quad m, k \in \mathbb{Z},
\]

so \( w_{\nu+1} \) belongs to the affine hyperplane

\[
\pi_1 = w_{\nu-2} + \langle w_{\nu-1}, w \rangle_R
\]
and lies in the lattice \( \Lambda_1 = w_{y-2} + \Lambda_{q,a_0,d} \) which is congruent to the lattice \( \Lambda_{q,a_0,d} \). We see that Condition 1) is satisfied.

Then we will use natural coordinates \((x, y, z)\) with respect to the basis \((27)\) described in Section 6. As it was mentioned in Section 6 in these coordinates for the basis vectors \( w'_{y-2}, w'_{y-1}, w'_y \), we have \((28)\) and for the values \( q, h, d \) we have formula \((26)\). Let \( \Gamma = \Gamma_{q, h, d} \). We define the two-dimensional lattice

\[
\Lambda = \mathbb{Z}^3 \cap \text{span} (w'_{y-1}, w'_y)
\]

with basis \( w'_{y-1}, w'_y \) and determinant

\[
\det_2 \Lambda = \frac{1}{h} = qd.
\]

The lattice \( \Lambda \) lies in the coordinate plane \( \{z = 0\} \). We can identify it with the lattice \( \Lambda_{q,a_0,d} \) defined in \((45)\) in Section 9. Now by \((26)\) we get

\[
\lambda_- := qd^2 = \frac{d}{h} = \frac{\text{Vol} \Pi_{\nu}}{\pi} \in (\alpha_\nu, \omega_\nu)
\]

by inductive assumption. In coordinates \((x, y, z)\) plane \( \pi_1 \) has equation \( \{z = h\} \). The lattice \( \Lambda_1 = \pi \cap \Gamma \subset \pi_1 \) consists of all the points with coordinates

\[
x = f + lq, \quad y = g + md, \quad z = h, \quad l, m \in \mathbb{Z}^3.
\]

Lattice \( \Lambda_1 \) splits into one-dimensional lattices parallel to \( w'_y = (q, 0, 0) \), namely

\[
\Lambda_1 = \bigcup_{m \in \mathbb{Z}} \{w'_{y-2} + mw'_{y-1} + lw'_y, l \in \mathbb{Z}\}.
\]

Each of the one-dimensional lattices

\[
\{w'_{y-2} + mw'_{y-1} + lw'_y, l \in \mathbb{Z}\}
\]

belongs to the line

\[
\ell_m = \{w'_{y-2} + mw'_{y-1} + \zeta w'_y, \zeta \in \mathbb{R}\}.
\]

Euclidean distance between neighbouring lines \( \ell_m \) and \( \ell_{m+1} \) is equal to \( d \). So these lines can be enumerated as

\[
\ell_t = \{(x, y, z) : y = td + \eta, z = h\}, \quad t \in \mathbb{Z}
\]

with certain \( \eta \in \mathbb{R} \).

Our aim is to find the point \( w_{y+1} \) as a pre-image of a non-zero point from \( \Lambda_1 \). We will define this lattice point by means of its image under the mapping \( \emptyset \), that is, we explain how to choose \( w' \in \Gamma \) such that \( w_{y+1} = \emptyset^{-1} w' \).

To complete inductive step we need the following easy lemma.
Lemma 8. Suppose that $\xi, d > 0, \eta \in \mathbb{R}$ and $\omega > \alpha$. Let

$$\xi(\omega - \alpha) > d. \quad (59)$$

Then there exist $\lambda \in (\alpha, \omega)$ and an integer $t$ such that

$$\xi \lambda = dt + \eta. \quad (60)$$

Proof. When $\lambda$ changes in an interval $J$ of length $\delta = \omega - \alpha > 0$ the value of $\xi \lambda$ changes in the dilated interval $\xi J$ of length $> d$. $\square$

Now we consider the next interval $(\alpha, \omega) = (\alpha \nu+1, \omega \nu+1)$ from Condition 8) from the $(\nu+1)$th step of inductive process. We deal with $k_{\nu+1}$ which satisfies (9) and

$$\xi = \left(\frac{a_0}{q} + \left(k_{\nu+1} + \frac{1}{2}\right)\right). \quad (61)$$

By (8) and (9) we have

$$\xi = \left(\frac{a_0}{q} + \left(k_{\nu+1} + \frac{1}{2}\right)\right) \lambda \geq k_{\nu+1} \geq \frac{3H_{\nu+1}}{\epsilon_{\nu+1}} \geq \frac{3\sqrt{5}}{\epsilon_{\nu+1}} = \frac{3\sqrt{5}}{\omega_{\nu+1} - \alpha_{\nu+1}}.$$

So we can use Lemma 8 to define $\lambda$ for the specified value of $\xi$ and $\eta$ from (58) to have (60). Now we can consider the value

$$v = \left(a_0 + \left(k_{\nu+1} + \frac{1}{2}\right) q\right) \lambda = \left(a + \frac{q}{2}\right) \lambda, \quad k_{\nu+1} q \lambda \leq v \leq (k_{\nu+1} + 2) q \lambda \leq 2 k_{\nu+1} q \lambda, \quad (61)$$

which depends on $k_{\nu+1}$ and on the chosen value of $\lambda$. For some $t \in \mathbb{Z}$ we have $td + \eta = \frac{hv}{q}$. Then we deal with the segment $[F(U^{[1]}), F(V^{[1]})]$ and the segment $[F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))]$ from Section 9. It is clear that

$$[F(U^{[1]}), F(V^{[1]})] \times \{z = h\} \subset \ell_t \quad \text{(case 1)}, \quad [F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))] \times \{z = h\} \subset \ell_t \quad \text{(case 2)}.$$

Recall that the one-dimensional line $\ell_t$ contains the one-dimensional affine lattice

$$\{(x, y, z) : y = \frac{hv}{q}, z = h\} \cap \Lambda_1 = \{(x, y, z) : x = x_0 + l q, l \in \mathbb{Z}, y = \frac{hv}{q}, z = h\}.$$ 

This means that each segment of the line $\ell_t$ of length $q$ contains a point of the lattice $\Lambda \subset \Gamma$. From (8) and (9) we see that

$$\left(\frac{v}{q}\right)^2 \geq k_{\nu+1}^2 q^2 \lambda^2 \geq \frac{9 (H_{\nu+1}^+)^2 \alpha_{\nu+1}^2}{\epsilon_{\nu+1}^2} \geq \frac{45 \alpha_{\nu+1}}{\epsilon_{\nu+1}} \geq \frac{\alpha_{\nu+1} \omega_{\nu+1} \lambda}{\omega_{\nu+1} - \alpha_{\nu+1}} - 1,$$

and condition (51) of Lemma 6 is satisfied. By the same argument

$$\frac{a}{q} \geq k_{\nu+1} \geq \frac{1}{\epsilon_{\nu+1}} \geq \frac{1}{2 \sqrt{3} - \omega_{\nu+1}} \geq \frac{3\sqrt{3}}{2 \epsilon_{\nu+1}}.$$
because of $\frac{2}{\sqrt{3}} - \omega_{y+1} \geq \varepsilon_{y+1}$. So condition (54) of Lemma 7 is satisfied. As for condition (55), it follows from (57). So all the conditions of Lemmas 6 and 7 are satisfied. By Lemmas 6 and 7 we see that the segments $[F(U^{[1]}), F(V^{[1]}))] \times \{z = h\} \subset \ell'$ and $[[F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))) \times \{z = h\} \subset \ell'$ have length $> q$. We see that in case 1° segment $[F(U^{[1]}), F(V^{[1]}))] \times \{z = h\}$ and in case 2° segment $[[F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))) \times \{z = h\}$ has an integer point $\mathbf{w'} \in \Lambda_1 \subset \Gamma$. This integer point is just what we need and it defines the next point $\mathbf{w}_{y+1} = \mathcal{O}^{-1} \mathbf{w'}$ of our inductive process.

Here we should note that $y$-coordinate of the point $\mathbf{w'}$ is positive and equal to $\frac{hv}{q}$. This will help us to establish Condition 6).

Up to now we have verified only Condition 1) of $(\nu + 1)$th step. Now we verify all other conditions.

To check Condition 2) we take into account that

$$\mathbf{w'} = \mathbf{w'}_{y+1} = \left(q_{y+1}, \frac{hv}{q}, h\right) \in [F(U^{[j]}), F(V^{[j]}))] \times \{z = h\}, \quad \text{where } j = 1 \text{ or } 2 \quad (62)$$

with $v$ depending on the chosen $\lambda$ and satisfying (61). However, in the case 2° by (44) we know that

$$[F(U^{[2]}(\lambda)), F(V^{[2]}(\lambda))] \subset [F(U^{[1]})), F(V^{[1]}))]$$

(the last segment here is defined for the special value of $v$ depending on $\lambda$). So in all the cases

$$\mathbf{w'} = \mathbf{w'}_{y+1} \in [F(U^{[1]}), F(V^{[1]}))] \times \{z = h\}, \quad (63)$$

and we evaluate $q_{y+1}$ by means of endpoints of the segments $[F(U^{[1]}), F(V^{[1]}))]$ by taking into account the inequality from (61).

Recall that $q = q_y$. We use the formulas for the endpoints of the segment $[F(U^{[1]}), F(V^{[1]}))]$ and obtain the bounds

$$B_{y+1}^{2} q_{y+1} = \frac{\alpha_{y+1}^{2} k_{y+1}^{2} q_{y}}{\omega_{y+1}^{2}} \leq \frac{k_{y+1}^{2} q_{y}^{2}}{q \lambda_{\omega_{y+1}}} \leq \frac{\alpha_{y+1}^{2} k_{y+1}^{2} q_{y}}{\omega_{y+1}^{2}} \leq \frac{v^{2}}{d q \omega_{y+1}} \leq q_{y+1} \leq \frac{h(v^{2} + q^{2})}{d q \alpha_{n+1}} \leq \frac{4 k_{y+1}^{2} q_{y}^{2} \lambda_{\omega_{y+1}}^{2}}{q \alpha_{y+1}^{2} \lambda_{y+1}} \leq \frac{5 k_{y+1}^{2} \omega_{y+1} q_{y}}{q} = B_{y+1}^{2} k_{y+1}^{2} q_{y}$$

(we take into account definitions of $B_{y}, B_{\nu}$ from (7), bounds for $v$ from (61), bounds for $\lambda_{\omega_{y+1}}$ from (57) and the inequality $\alpha_{y+1} \leq \lambda \leq \omega_{y+1}$; in the upper bound here we also use the inequality $k_{y+1}^{2} \omega_{y+1} \geq 1$).

We have checked the inequalities from Condition 2) in all the cases.

Now we verify Condition 3). Let us consider points $\mathbf{w'} = \mathcal{O} \mathbf{w}_{y+1} = (q_{n+1}, y_{0}, z_{0}), y_{0} = \frac{hv}{q}, z_{0} = h$ and $\mathcal{O} \mathbf{w}_{y}$ and the corresponding points $\mathbf{b'}_{y+1} = \left(\frac{v_{0}}{q_{y+1}}, \frac{z_{0}}{q_{y+1}}\right)$ and $\mathbf{b'}_{n} = (0, 0)$ in $\mathbb{R}^{2}$. As $|\mathbf{b}_{y+1} - \mathbf{b'}_{1}| = |\mathbf{v}_{y+1} - \mathbf{v}_{1}|$, we omit $'$ in the notation of the points $\mathbf{b'}_{y+1}, \mathbf{b'}_{y+1}$. For

$$|\mathbf{v}_{y+1} - \mathbf{v}_{1}| = \frac{1}{q_{y+1}} \sqrt{y_{0}^{2} + z_{0}^{2}} = \frac{1}{q_{y+1}} \sqrt{\left(\frac{hv}{q}\right)^{2} + h^{2}} = \frac{d}{q_{y+1} \lambda_{y+1}} \sqrt{\left(\frac{v}{q}\right)^{2} + 1} = \frac{R_{y}}{q_{y+1} \lambda_{y+1}} \sqrt{\left(\frac{v}{q}\right)^{2} + 1}$$
we have upper and lower bounds
\[
\frac{\alpha_{\nu+1} R_{\nu} k_{\nu+1}}{\omega_{\nu} q_{\nu+1}} \leq |v_{\nu+1} - v_{\nu}| \leq \frac{\sqrt{5} \omega_{\nu+1} R_{\nu} k_{\nu+1}}{\alpha_{\nu} q_{\nu+1}}
\]
as \(k_{\nu+1} \omega_{\nu+1} \geq 1\). We take into account Condition 2) to get
\[
\frac{H^-_{\nu+1} R_{\nu}}{k_{\nu+1}} = \frac{\alpha_{\nu+1} R_{\nu}}{\omega_{\nu} B^+_{\nu+1} k_{\nu+1}} \leq R_{\nu+1} = q_{\nu} |v_{\nu+1} - v_{\nu}| \leq \frac{\sqrt{5} \omega_{\nu+1} R_{\nu}}{\alpha_{\nu} B^-_{\nu+1} k_{\nu+1}} = \frac{H^+_{\nu+1} R_{\nu}}{k_{\nu+1}}.
\]
This gives us Condition 3).

We show that Condition 4) is satisfied by the construction.

We use natural coordinates and consider cylinders \(\Pi'_{\nu+1} = \emptyset \Pi_{\nu+1}, \overline{\Pi'}_{\nu+1} = \emptyset \overline{\Pi}_{\nu+1}\). By Condition 3) according to (17) and by (57) we have
\[
R_{\nu+1} \leq \frac{R_{\nu}}{5} = \frac{d}{5} \leq \frac{2h}{5 \sqrt{3}} < h.
\]
So the intersections \(\Pi'_{\nu+1} \cap \Gamma\) and \(\overline{\Pi'}_{\nu+1} \cap \Gamma\) are covered by the planes \(\{z = 0\}\) and \(\{z = h\}\). By construction the centre of the facet \(x = q_{\nu+1}\) of cylinders \(\Pi'_{\nu+1}\) and \(\overline{\Pi'}_{\nu+1}\) is the point \(w' = (q_{\nu+1}, y_0, z_0)\) and the point \(F^{-1}(q_{\nu+1}, y_0) = (v_k, d\lambda)\) is just the point of the section \(\Pi'_{\nu+1} \cap \{z = 0\}\) where the tangent line is parallel to 0x axis (see the explanation of the geometrical meaning of mapping \(F\) in the end of Section 7). Then
\[
\Pi'_{\nu+1} \cap \{z = 0\} \subset \hat{\mathcal{E}}_1, \quad \overline{\Pi'}_{\nu+1} \cap \{z = 0\} \subset \hat{\mathcal{E}}_2,
\]
where ellipses are defined in Section 8. We explain these inclusions. Ellipse \(\mathcal{E}_1\) belongs to the boundary of \(\Pi'_{\nu+1}\) and this explains the first inclusion in (64). From the definition \(\mathcal{E}_2 = \{j \in \mathcal{E}_1, j = 1, 2\}\), where the coefficients of dilatations \(t_1, t_2\) are different in cases 10 and 20. These coefficients are defined in Section 9 in formulas (46) and (50). The coefficient of dilatation for cylinder \(\Pi'_{\nu+1}\) is defined in (16) and is equal to \(1 + \varepsilon_{\nu+1}\). In both cases we see that
\[
1 + \varepsilon^-_{\nu+1} \leq t_j, \quad j = 1, 2,
\]
because
\[
t_1 = \frac{1}{\omega_{\nu}} \geq 1 + \varepsilon_{\nu} > 1 + \varepsilon^-_{\nu+1}, \quad t_2 = 1 + \frac{2}{\sqrt{3}} \geq \omega_{\nu} \geq 1 + \varepsilon_{\nu} > 1 + \varepsilon^-_{\nu+1}, \quad \varepsilon^-_{\nu+1} = \varepsilon^2_{\nu} < \varepsilon_{\nu}
\]
by the conditions defining case 10 and case 20 in the beginning of Section 4. So we explained the second inclusion in (64). Then we apply Lemma 5 to see that in the plane \(\{z = 0\}\) in the intersection \(\Pi'_{\nu+1} \cap \Gamma\) there are just two integer points — \(0\) and \(w'\). Now we consider ellipses \(\hat{\mathcal{E}}_j, j = 1, 2\) shifted by the vector \(w'_{\nu+1}\). These shifted ellipses lie in the plane \(\{z = h\}\). So in the plane \(\{z = h\}\) in the intersection \(\overline{\Pi'}_{\nu+1} \cap \{z = h\}\) we have two integer points \(w'_{\nu+1}\) and \(w'_{\nu+1} - w'\) only. This gives Condition 4) for the cylinders \(\Pi_{\nu+1} = \emptyset^{-1}\Pi'_{\nu+1}, \overline{\Pi}_{\nu+1} = \emptyset^{-1}\overline{\Pi'}_{\nu+1}\).
Next we explain Condition 5).
First we will show that
\[ \overline{\Pi}^{\nu+1} \subset \Pi_\nu. \]  
(65)

Let us explain how to prove (65). Note that \( R_{\nu+1} \) is the distance between the centres of sections \( \{x = q_\nu\} \) of the cylinders \( \Pi^{\nu+1} \) and \( \Pi_\nu \). We use the triangle inequality to see that
\[ |R_{\nu+1}^- - R_\nu| \leq q_{\nu-1} |v_{\nu+1} - v_\nu| - |v_\nu - v_{\nu-1}| | \leq q_{\nu-1} |v_{\nu+1} - v_\nu|. \]
So
\[ |R_{\nu+1}^- - R_\nu| \leq q_{\nu-1} |v_{\nu+1} - v_\nu| = R_{\nu+1} \leq \frac{R_\nu \epsilon_\nu}{3}. \]  
(66)

We see that
\[ R_{\nu+1} + R_{\nu+1}^- (1 + \epsilon_{\nu+1}^-) \leq R_{\nu+1} + R_\nu (1 + \epsilon_{\nu+1}) + |R_{\nu+1}^- - R_\nu|(1 + \epsilon_{\nu+1}). \]  
(67)

Now substituting (17) and (66) into (67) we get
\[ R_{\nu+1} + R_{\nu+1}^- (1 + \epsilon_{\nu+1}^-) \leq R_\nu \left( 1 + \epsilon_{\nu+1}^- + \frac{2 \epsilon_\nu}{3} (1 + \epsilon_{\nu+1}^-) \right) < R_\nu (1 + \epsilon_\nu). \]

So the section \( \{x = q_\nu\} \) of the cylinder \( \overline{\Pi}^{\nu+1} \) is inside the section \( \{x = q_\nu\} \) of the cylinder \( \Pi_\nu \), and this immediately gives (65).

We have Condition 4) for the \((\nu+1)\)th step of the process yet established. So from (65) and Condition 4) we see that
\[ \overline{\Pi}^{\nu+1} \cap \mathbb{Z}^3 \subset \{0, w_{\nu-1}, w_\nu, w_\nu - w_{\nu-1}\}. \]

To complete the proof of Condition 5) for the \((\nu+1)\)th inductive step, we should show that
\[ w_\nu - w_{\nu-1} \notin \overline{\Pi}^{\nu+1}. \]  
(68)

We use natural coordinates and instead of (68) we prove
\[ w'_\nu - w'_{\nu-1} \notin \Pi^{\nu+1} = \emptyset \Pi^{\nu+1}. \]  
(69)

We deduce a lower bound for the distance \( \rho \) between
\[ w'_\nu - w'_{\nu-1} = (q_\nu - q_{\nu-1}, -d, 0) \]
and the centre
\[ \frac{q_\nu - q_{\nu-1}}{q_{\nu+1}} w'_\nu = \left( q_\nu - q_{\nu-1}, \frac{hv(q_\nu - q_{\nu-1})}{q_\nu q_{\nu+1}}, \frac{h(q_\nu - q_{\nu-1})}{q_{\nu+1}} \right) \]
of the section \( \{x = q_\nu - q_{\nu-1}\} \) of cylinder \( \Pi^{\nu+1} \). Here \( v \) satisfies (61).
By Condition 3) we see that
\[
\rho = \left| \frac{q_y - q_{y-1}}{q_{y+1}} \mathbf{w}_{y+1}' - (\mathbf{w}_y' - \mathbf{w}_{y-1}') \right| \geq R_y \left( 1 + \frac{R_y + 1}{2R_y} \right) \geq R_y \left( 1 + \frac{H^-_{y+1}}{2k_{y+1}} \right) \geq R_y \left( 1 + 2\varepsilon^2_{y+1} \right).
\]  \tag{70}

On the other hand,
\[
R^-_{y+1} \leq R_y + \frac{q_{y-1}}{q_y} R_{y+1} \leq R_y \left( 1 + \frac{\varepsilon^2_y}{135} \right)
\]
and so
\[
R^-_{y+1} (1 + \varepsilon^-_{y+1}) \leq R_y \left( 1 + \frac{\varepsilon^2_y}{135} \right) (1 + \varepsilon^2_y) \leq R_y \left( 1 + 2\varepsilon^2_y \right).
\]  \tag{71}

Now from (70) and (71) and the upper bound from (9) we deduce
\[
R^-_{y+1} (1 + \varepsilon^-_y) \leq \rho,
\]
and this gives (69).

So we proved (69) and hence (68).

We have already mentioned that y-coordinate of the point \( w' \) is positive and equal to \( \frac{hv}{q} \).

Let us check Condition 6). By construction we have (63). This means that for a certain \( \xi = (v, u) \in [U^{[1]}, V^{[1]}] \) we have \( \mathbf{w}' = F(\xi) \). Then for the second coordinate \( u \) we have
\[
d\alpha_{y+1} \leq u \leq d\omega_{y+1}.
\]  \tag{72}

So for the volume \( \text{Vol} \Pi_{y+1} \) we get
\[
\text{Vol} \Pi_{y+1} = \text{Vol} \Pi'_{y+1} = \pi h \text{ Area } \mathbf{G}(q; a, u) = \pi q hu.
\]

Here ellipse \( \mathbf{G}(q; a, u) \) is defined in Lemma 1 in the beginning of Section 8, \( q = q_y \) and the value of \( a \) is not important for the calculation of the area. As the values \( q, h, d \) satisfy (26), by (72) we establish Condition 6).

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