A general approach to sample path generation of infinitely divisible processes via shot noise representation

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Abstract

We establish a sample path generation scheme in a unified manner for general multivariate infinitely divisible processes based on shot noise representation of their integrators. The approximation is derived from the decomposition of the infinitely divisible process to three independent components based on jump sizes and timings: the large jumps over a compact time interval, small jumps over the entire time interval and large jumps over an unbounded time interval. The first component is taken as the approximation and is much simpler than simulation of general Gaussian processes, while the latter two components are analyzed as the error. We derive technical conditions for the two error terms to vanish in the limit and for the scaled component on small jumps to converge to a Gaussian process so as to enhance the accuracy of the weak approximation. We provide an extensive collection of examples to highlight the wide practicality of the proposed approach.

Keywords: infinitely divisible laws; Lévy processes; fractional Lévy motions, shot noise representation; infinitely divisible processes.

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1 Introduction

The growing appearance of infinitely divisible processes in applied contexts naturally fuels an increasing demand for simulation methods. Classically, the go-to paradigm for sample path generation of a stochastic process is to partition the time interval into deterministic sample points and recursively simulate the stochastic process from one sample point to the next via increments. This may be appropriate in the case where the driving process is Gaussian, such as the fractional Brownian motion and the Gaussian Ornstein–Uhlenbeck process. However, in the case where the integrator has jump components, simulation by increments may not be viable due to a lack of efficient sampling methods for the corresponding infinitely divisible increments. Applied contexts which demand jump dynamics often also prefer the observation of individual jumps in sample paths. For example, in insurance mathematics, observation of individual jumps is desired for the computation of ruin times. As sample path generation by increments cannot portray individual jumps, the paradigm falls out of favor.

In the recent decades, an increasingly popular simulation method for Lévy processes without Gaussian components is through the truncation of their shot noise series representation [10, 11]. As such representations of the Lévy process are decompositions in terms of its individual jumps [18], this truncation approach fulfills the desire for a jump-based method. The extension of this truncation method for infinitely divisible processes driven by integrators without Gaussian components has made appearances in the literature based on shot noise representations via a decomposition by jump timings and sizes and the corresponding error analysis [13, 14]. In short, the idea is to decompose the infinitely divisible process into three independent components: (i) the simulatable component corresponding to large jumps of the integrator within a bounded time interval, (ii) the component corresponding to small jumps which may be approximated by a Gaussian process, and (iii) the remaining component corresponding to large jumps outside of the bounded time interval. Error analysis is performed by investigating the latter two components.

In this paper, we develop the general framework of simulating infinitely divisible processes driven by Lévy processes without Gaussian components based on their shot noise representations. The proposed framework unifies and generalizes the existing case-by-case approaches, as summarized in [22], and offers a sample path generation scheme that is indeed a lot easier than simulation of general Gaussian processes. We provide technical conditions for error analysis in terms of the integrator and the deterministic kernel and include an extensive collection of infinitely divisible processes of interest in the literature so as to exemplify the effectiveness of the proposed approach. To maintain the flow of the paper, we collect all proofs in the Appendix.

2 Preliminaries

In what follows, we work under the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(\mathbb{N} := \{1, 2, \cdots\}\) and \(\mathbb{N}_0 := \{0, 1, 2, \cdots\}\). Denote \(\mathbb{R}^d := \mathbb{R}^d \setminus \{0\}\) and by \(\text{Leb}(D)\) the Lebesgue measure of a set \(D\). The notation \(\{\Gamma_k\}_{k \in \mathbb{N}}\) will be reserved throughout to denote the arrival times of the standard Poisson process.

We next review some relevant essential notions of Lévy processes, their shot noise representations and infinitely divisible processes. Recall that by the Lévy-Khintchine formula, an infinitely divisible random vector without a Gaussian component has a characteristic
function of the form
\[ \varphi(y) = \exp \left[ i(y, c) + \int_{\mathbb{R}^d} \left( e^{i(y, z)} - 1 - i(y, z) \mathbb{1}_{[0,1]}(\|z\|) \right) v(dz) \right], \quad y \in \mathbb{R}^d, \]  
(2.1)
where \( c \in \mathbb{R}^d \) corresponds to a drift component and the measure \( v \) is referred to as a Lévy measure which satisfies the integrability condition \( \int_{\mathbb{R}^d} (1 + \|z\|^2) v(dz) < +\infty \). A stochastic process is infinitely divisible if its finite dimensional distributions are infinitely divisible. All Lévy processes are thus infinitely divisible, whereas the class of infinitely divisible processes is more general. With a set of time indices \( \mathcal{T} \subseteq \mathbb{R} \), let \( \{X_t : t \in [0, T]\} \) be a stochastic process in \( \mathbb{R}^d \), described by the stochastic integral
\[ X_t = \int_{\mathcal{T}} f(t, s) dL_s, \quad t \in [0, T], \]  
(2.2)
where \( \{L_s : s \in \mathcal{T}\} \) is a Lévy process in \( \mathbb{R}^d \) over time \( \mathcal{T} \), whose unit-time marginal is characterized by (2.1). The focus of this paper is on the simulation of such infinitely divisible processes. Each jump of the underlying Lévy process at time \( s \) is modulated by a mapping \( f \) from \([0, T] \times \mathcal{T}\) to a suitable space, so we have a Lévy-Itô decomposition of the form
\[ X_t = \int_{\mathcal{T}} \int_{\mathbb{R}_0^d} f(t, s) z(\mu - v\mathbb{1}_{[0,1]}(\|z\|))(dz, ds), \quad t \in [0, T], \]  
(2.3)
where \( \mu(dz, ds) \) is the Poisson random measure on \( \mathbb{R}_0^d \times \mathcal{T} \) associated with the Lévy process \( \{L_s : s \in \mathcal{T}\} \) and \( v(dz)ds \) is the corresponding compensator.

From the Lévy-Itô decomposition (2.3) for Lévy processes without Gaussian components, shot noise series representations can be derived by methods of expressing the underlying Poisson random measure as a series of Dirac delta measures scattered at random points in \( \mathbb{R}_0^d \times [0, +\infty) \). We provide the most general formulation [18], as follows.

**Theorem 2.1 (Generalized shot noise method).** Suppose a Lévy measure \( v \) on \( \mathbb{R}_0^d \) can be decomposed as
\[ v(B) = \int_{0^+}^{+\infty} P(H(r, U) \in B) dr, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \]  
(2.4)
where \( U \) is a random vector in some space \( \mathcal{U} \) and \( H : (0, +\infty) \times \mathcal{U} \rightarrow \mathbb{R}^d \) is such that for every \( u \in \mathcal{U} \), \( r \mapsto \|H(r, u)\| \) is nonincreasing. Then, where \( \mathcal{T} \subset \mathbb{R} \) is a bounded time interval, it holds that
\[ \{L_s : s \in \mathcal{T}\} \equiv \left\{ \sum_{k=1}^{+\infty} \left[ H \left( \frac{\Gamma_k}{\operatorname{Leb}(\mathcal{T})}, U_k \right) \mathbb{1}_{\inf \mathcal{T} \neq \{\}}(T_k) - \operatorname{Leb}(\inf \mathcal{T}, s) c_k \right] : s \in \mathcal{T} \right\}, \]  
(2.5)
where \( \{L_s : s \in \mathcal{T}\} \) is the Lévy process characterized by Lévy-Khintchine triple \((0, 0, v)\) over \([0, T]\). \( \{\Gamma_k\}_{k \in \mathbb{N}} \) are the arrival times of the standard Poisson process. \( \{U_k\}_{k \in \mathbb{N}} \) are iid copies of \( U \). \( \{T_k\}_{k \in \mathbb{N}} \) are iid uniform random variables over \( \mathcal{T} \), with mutual independence of the random sequences and \( \{c_k\}_{k \in \mathbb{N}} \) is a sequence of centers in \( \mathbb{R}^d \) with \( c_k := \int_{k-1}^{k} \mathbb{E}[H(s, u) \mathbb{1}_{[0,1]}(\|H(s, u)\|)] ds \) for \( k \in \mathbb{N} \).

We remark that the decomposition (2.4) exists in a similar form to the so-called the inverse Lévy measure method [7], whereas the resulting expression often does not offer a viable numerical method [11]. In fact, the decomposition (2.4) is not unique but can be employed to derive a few distinct shot noise representations of an infinitely divisible random vector via, not only the aforementioned inverse Lévy measure method but also, the rejection, thinning and Bondesson’s methods [22]. Thanks to the non-uniqueness, each of the three typical Lévy measures (Section 4) admits at least one implementable expression for simulation purposes [10, 14].

### 3 Decomposition based on jump sizes and timings

Consider the infinitely divisible process \( \{X_t : t \in [0, T]\} \) in \( \mathbb{R}^d \), described by the stochastic integral (2.2). We look to truncate the Lévy measure based on a shot noise representation of the integrator to obtain simulatable and error components. To this end, we exclusively consider the case when the Lévy measure \( v \) is infinite, as otherwise, exact methods are readily available.

**Assumption 3.1.** \( v(\mathbb{R}_0^d) = +\infty \).

To illustrate the significance of this assumption, consider a subordinator \( \{L_s : s \in [0, 1]\} \) with \( c_0 \equiv 0 \) and \( v(dz) = e^{-z} dz \) on \((0, +\infty)\), violating Assumption 3.1 with \( v(0, +\infty) = 1 \). We then have \( H(r, u) = -\ln(r) \), which has only to be defined on \([0, 1]\), irrespective of the second argument. Hence, with \( \mathcal{T} = [0, 1] \), a shot noise representation in the framework (2.5) is given by
\[ \{L_s : s \in [0, 1]\} \equiv \left\{ \sum_{k \in [0; T]} \ln(\Gamma_k) \mathbb{1}_{[0,1]}(T_k) : s \in [0, 1]\right\}, \]  
(3.1)
which is indeed an exact simulation method without the need for truncation. Assumption 3.1 implies that
\[ \lim_{r \to +\infty} \|H(r, U(\omega))\| = 0, \]  
(3.2)
for almost every \( \omega \in \Omega \), since the origin is the only possible source of infinite mass of the Lévy measure. Note that the converse is not true. For instance, with the aforementioned example of a subordinator with \( v(dz) = e^{-z}dz \) on \( (0, +\infty) \) and \( H(r, u) = -\ln(r) \) on \( [0, 1] \), it still holds that \( |H(r, u)| \downarrow 0 \) as \( r \uparrow 1 \).

As in the case of Lévy processes, shot noise representations for infinitely divisible processes converge almost surely uniformly \([2]\). Moreover, if the probability space is rich enough, then one can choose the random sequences such that the shot noise representation is almost surely equal to the infinitely divisible process \([20]\).

Now, let \( \{ \mathcal{T}_n \}_{n \in \mathbb{N}} \) denote a nondecreasing sequence of connected open intervals over the time indices such that \( \cup_{n \in \mathbb{N}} \mathcal{T}_n = \mathcal{T} \) and \( \text{Leb}(\mathcal{T}_n) < +\infty \), so \( n \) represents a truncation based on jump timings. Suppose a shot noise representation of the driving Lévy process \( \{ L_s : s \in \mathcal{T}_n \} \) is given by (2.5) with the time set \( \mathcal{T}_n \) in lieu of \( \mathcal{T} \). For each \( m \in \mathbb{N} \), denote by \( \nu_m \) the finite Lévy measure defined by

\[
\nu_m(B) := \int_{0^+}^{m} P(H(r, U) \in B) dr, \quad B \in \mathcal{B}(\mathbb{R}_0^d),
\]

as a truncation of the decomposition (2.4), corresponding to the Poisson truncation approximation

\[
\left\{ \sum_{k \in \mathbb{N}} \left( \frac{\Gamma_k}{\text{Leb}(\mathcal{T}_n)} U_k \right) 1_{[\inf \mathcal{T}_n, \cdot]}(T_k) - \text{Leb}(\inf \mathcal{T}_n, s) c_k \right\} : s \in \mathcal{T}_n,
\]

via a similar truncation to the representation (3.1) on the basis of the Poisson arrival times. In view of (3.3), the sequence of finite Lévy measures \( \{ \nu_m \}_{m \in \mathbb{N}} \) increases to the original infinite Lévy measure \( \nu \) as \( m \to +\infty \), in the sense of \( \nu_m(\mathbb{R}_0^d) = \int_{0^+}^{m} P(H(r, U) \in \mathbb{R}_0^d) dr = m \) for all \( m \in \mathbb{N} \). Hence, we hereafter write \( m \) instead of \( \nu_m(\mathbb{R}_0^d) \) in the region of the summation (3.4). It is worth stressing again \([11, \text{Theorem 3.1}]\) that all shot noise representations, including the inverse Lévy measure, rejection, thinning and Bondesson’s methods, are well in the framework of the finite truncation (3.3).

Armed with the truncation parameters \( m \) and \( n \) in (3.4), we decompose the infinitely divisible process as

\[
X_t = X_t(m,n) + Q_t(m) + R_t(m,n), \quad t \in [0,T],
\]

where

\[
X_t(m,n) := \int_{\mathcal{T}_n} \int_{\mathbb{R}_0^d} f(t,s) \mathbf{1}_{[0,1]}(\mu_m - \nu_m \mathbf{1}_{[0,1]}(\|z\|))(dz,ds),
\]

\[
Q_t(m) := \int_{\mathcal{T}} \int_{\mathbb{R}_0^d} f(t,s) \mathbf{1}_{[0,1]}(\mu - \mu_m - (\nu - \nu_m) \mathbf{1}_{[0,1]}(\|z\|))(dz,ds),
\]

\[
R_t(m,n) := \int_{\mathcal{T} \setminus \mathcal{T}_n} \int_{\mathbb{R}_0^d} f(t,s) \mathbf{1}_{[0,1]}(\mu_m - \nu_m \mathbf{1}_{[0,1]}(\|z\|))(dz,ds),
\]

where \( \mu_m(dz,ds) \) denotes a Poisson random measure with intensity measure \( \nu_m(dz,ds) \). The stochastic processes on the right hand side of (3.5) are mutually independent, thanks to the independent scattering of the Poisson random measure \( \mu(dz,ds) \). As \( \text{Leb}(\mathcal{T}_n) \nu_m(\mathbb{R}_0^d) < +\infty \) for every \( (m,n) \in \mathbb{N}^2 \), the first component \( \{ X_t(m,n) : t \in [0,T] \} \) can be treated as the principal component of the approximation by the Poisson truncation of shot noise representation. The remaining two components \( \{ Q_t(m) : t \in [0,T] \} \) and \( \{ R_t(m,n) : t \in [0,T] \} \) are treated as the error processes for further analysis. The stochastic process \( \{ Q_t(m) : t \in [0,T] \} \) comprising of small jumps can sometimes be approximated by a Gaussian process under suitable technical conditions \([1, 5]\). If \( \text{Leb}(\mathcal{T}) < +\infty \), then one may set \( \mathcal{T} \equiv \mathcal{T} \), that is, the component \( \{ R_t(m,n) : t \in [0,T] \} \) vanishes. In fact, we often have \( \mathcal{T} = [0,T] \), in which the temporal truncation via the sequence \( \{ \mathcal{T}_n \}_{n \in \mathbb{N}} \) is irrelevant, or \( \mathcal{T} = \mathcal{T} \). It is worth noting that the limiting process \( \lim_{m \to +\infty} R_t(m,n) : t \in [0,T] \), which is certainly well defined, can be considered as the residual component after time truncation alone and is discussed later in Section 5.9 when the integrator is the stable process.

### 3.1 Simulatable component

We first consider the simulatable process \( \{ X_t(m,n) : t \in [0,T] \} \), where

\[
X_t(m,n) = \int_{\mathcal{T}_n} \int_{\mathbb{R}_0^d} f(t,s) \mathbf{1}_{[0,1]}(\mu_m - \nu_m \mathbf{1}_{[0,1]}(\|z\|))(dz,ds), \quad t \in [0,T].
\]

As the Lévy measure \( \nu_m \) is finite with \( \nu_m(\mathbb{R}_0^d) = m \) and the time set \( \mathcal{T}_n \) is bounded, the term \( \{ X_t(m,n) : t \in [0,T] \} \) has almost surely finite number of jumps. Since this component is thus exactly simulatable, we regard \( X_t(m,n) \) as the principal approximation. We provide its shot noise representation as follows, which provides an exact simulation method for this component:

\[
\{ X_t(m,n) : t \in [0,T] \} \overset{\text{d}}{=} \left\{ \sum_{k \in \mathbb{N}} \left( \frac{\Gamma_k}{\text{Leb}(\mathcal{T}_n)} U_k \right) f(t,T_k) H \left( \frac{\Gamma_k}{\text{Leb}(\mathcal{T}_n)} U_k \right) - c_k \int_{\mathcal{T}_n} f(t,s) ds \right\} : t \in [0,T],
\]

where the sequences are the same as in (3.5). For generating sample paths of \( \{ X_t(m,n) : t \in [0,T] \} \) based on the Poisson truncation (3.6) for \( J \) sample points \( 0 = t_0 < t_1 < \cdots < t_{J-1} < t_J = T \) for some \( J \in \mathbb{N} \), we provide the numerical recipe:
Step 1. Generate a standard exponential random variable \( E_1 \). If \( E_1 \leq \text{Leb}(\mathcal{T}_n)m \), then assign \( \Gamma_1 \leftarrow E_1 \). Otherwise, return the degenerate zero process as the approximate sample path and terminate the algorithm.

Step 2. While \( \Gamma_k \leq \text{Leb}(\mathcal{T}_m)n \), generate a standard exponential random variable \( E_{k+1} \) and assign \( \Gamma_{k+1} \leftarrow \Gamma_k + E_{k+1} \). Denote this as \( \{\Gamma_k\}_{k \in \{1, \ldots, N\}} \), where \( N \) satisfies \( \Gamma_N \leq \text{Leb}(\mathcal{T}_m)n < \Gamma_{N+1} \).

Step 3. Generate a sequence \( \{T_k\}_{k \in \{1, \ldots, N\}} \) of iid uniform random variables on \( \mathcal{T}_n \).

Step 4. Generate a sequence \( \{U_k\}_{k \in \{1, \ldots, N\}} \) of suitable iid random variables.

Step 5. For every \( j \in \{1, \ldots, J\} \), assign \( X_j(m,n) \leftarrow \sum_{k=1}^{N_j} (f_n(t_j, T_k) H(\Gamma_k/\text{Leb}(\mathcal{T}_n), U_k) - c_k \int_{\mathcal{T}_m} f(t_j, s) ds) \).

Step 6. Return \( \{0, X_1(m,n), \ldots, X_J(m,n)\} \) as the positions of the approximate sample path at the sample times \( \{0, t_1, \ldots, t_J\} \).

The proposed scheme is built on rather elementary operations based on straightforward iterations without the need for sophisticated coding or a big matrix operation, such as the Cholesky decomposition of a thousand-dimensional covariance matrix for simulating general Gaussian processes by increments. Note that the deterministic integral \( \int_{\mathcal{T}_m} f(t_j, s) ds \) needs to be computed either exactly if possible or by approximation, which we leave to the user’s discretion.

### 3.2 Error components

In what follows, we provide analysis of the error components with all proofs collected in the Appendix so as to maintain the flow. We begin with assumptions to establish the limiting degeneracy of the error components.

**Assumption 3.2.** (a) \( \text{esssup}_{x \in \mathcal{F}} \|f(t, s)\| < +\infty \) for \( t \in [0, T] \);
(b) \( \int_{\mathcal{T}} \|f(t, s)\|^2 ds < +\infty \) for \( t \in [0, T] \);
(c) \( \lim_{m \to +\infty} \int_{\Omega} \|H(r, U(\omega))\|^2 \mathbb{P}(d\omega) dr = 0. \)

**Proposition 3.3.** It holds under Assumption 3.2 that \( \{Q_i(m) : t \in [0, T]\} \) and \( \{R_i(m,n) : t \in [0, T]\} \) converge to degenerate zero processes on \( [0, T] \) in the sense of finite dimensional distributions as \( m \wedge n \to +\infty \).

We have provided Assumption 3.2 (a) as a sufficient condition for Proposition 3.3 as well as Theorem 3.5 later, whereas it is indeed an essential requirement for sample boundedness. That is, sample paths would otherwise be almost surely unbounded on every finite interval of positive length. In such a case, it would be nonsensical to use truncation of shot noise representation to simulate the stochastic integral process, and even misleading if done so without the knowledge of sample path unboundedness. This is because truncation to a finite \( \text{Lévy} \) measure will almost surely produce a bounded sample path, which would otherwise be unbounded in the absence of truncation. Hence, it is highly advisable to check that the stochastic process is almost surely bounded over \([0, T]\) prior to generating its sample paths.

The notion of approximating small jumps by a Gaussian process [1, 5] to improve the quality of the approximation can carry over to the case of some infinitely divisible processes under some technical conditions. Define the covariance matrix

\[
\sigma_m^2 := \int_{d_0} \mathcal{G} \cdot \mathbb{P}(d\omega) dr,
\]

which is finite valued under Assumption 3.2 (c), and further denote by \( \sigma_m \) the lower triangular matrix of the Cholesky decomposition of the matrix \( \sigma_m^2 \), so that \( (\sigma_m)^{\infty \infty} = \sigma_m^2 \). We scale the integrator in the \( \mathcal{Q} \) component by the matrix \( \sigma_m^{-1} \) to obtain

\[
\tilde{Q}_i(m; m) := \int_{d_0} \int_{d_0} f(t, s) \sigma_m^{-1} (\mu - \mu_m - (v - v_m))(dz, ds),
\]

for some \( \mathcal{F} \subseteq \mathcal{T} \), of which convergence to a Gaussian process is our present interest. Clearly, if two time domains, say, \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are disjoint, then the associated \( \text{Lévy} \) measures are disjoint, that is, the two resultant stochastic processes \( \{Q_i(m_1; \mathcal{T}_1) : t \in [0, T]\} \) and \( \{Q_i(m_2; \mathcal{T}_2) : t \in [0, T]\} \) are independent of each other, even irrespective of \( m_1 \) and \( m_2 \). To ease the notation, we below use the notation \( \mathbb{I}_d \) for the identity matrix in \( \mathbb{R}^{d \times d} \) and \( [c] := c \mathbb{I}_d \) for \( c \in \mathbb{R}^d \) and \( B \in \mathcal{A}(0, +\infty) \) and let \( A \preceq B \) indicate that the matrix \( A - B \) is positive semidefinite for two square matrices \( A \) and \( B \) of a common order.

**Assumption 3.4.** (a) The matrix \( \sigma_m^2 \) is positive definite for sufficiently large \( m \);
(b) For every \( k > 0 \), \( \lim_{m \to +\infty} \int_{d_0} \int_{d_0} \|\sigma_m^{-1} H(r, U(\omega))\|^2 \mathbb{P}(d\omega) dr = 0 \);
(c) There exist \( c_1 > 0 \), \( c_2 > 0 \) and \( \mathcal{F} \subseteq \mathcal{T} \) such that \( \int_{\mathcal{F}} \int_{[0, T]} (f(t_2, s) - f(t_1, s))^2 ds \leq c_2 (t_2 - t_1)^4 \mathbb{I}_d \) for \( 0 \leq t_1 \leq t_2 \leq T \).

We remark that the time domain \( \mathcal{F} \subseteq \mathcal{T} \) corresponds to that in the definition (3.7). Indeed, we have specified the time domain \( \mathcal{F} \) in (3.7) so as to address the case where independent processes \( \{Q_i(m_1; \mathcal{F}_1) : t \in [0, T]\} \) on disjoint time domains \( \{\mathcal{F}_k\} \) satisfy Assumption 3.4 (c) with different degrees of continuity, that is, different degrees of \( \text{Hölder} \) continuity in Theorem 3.5 (iii) below. We will later discuss such an example in Section 5.4 of two independent components with \( \mathcal{F} = \mathbb{R} \), \( \mathcal{F}_1 = (-\infty, 0] \) and \( \mathcal{F}_2 = (0, +\infty) \).

We are now ready to give the results on the Gaussian approximation. On the one hand, since the regularity of the kernel (Assumption 3.4 (c)) does not affect the convergence of finite dimensional distributions in any way, we can retain the entire time domain \( \mathcal{F} \) in Theorem 3.5 (i). In Theorem 3.5 (ii) and (iii), on the other hand, we address the weak convergence of sample paths, thus the time domain needs to be restricted to where Assumption 3.4 (c) holds. It is also worth mentioning that Assumption 3.4 (a) and (b) are not only sufficient conditions but also necessary conditions for Theorem 3.5 to hold true [5]. Note that \( \mathcal{T}([0, T]; \mathbb{R}^d) \) denotes the space of càdlàg functions from \([0, T]\) to \( \mathbb{R}^d \) endowed with the Skorohod topology.
Theorem 3.5. Let Assumption 3.2 hold.
(i) It holds under Assumption 3.4 (a)-(b) that \( \tilde{Q}_t(m; \mathcal{D}) : t \in (0, T] \) converges to \( \{ \int_\mathcal{F} f(t, s)dB_s : t \in [0, T] \} \) in the sense of finite dimensional distributions, as \( m \to +\infty \).
(ii) It holds that under Assumption 3.4 (a)-(c) that \( \{ \tilde{Q}_t(m; \mathcal{D}) : t \in [0, T] \} \) converges to \( \{ \int_\mathcal{F} f(t, s)dB_s : t \in [0, T] \} \) in \( \mathcal{D}([0, T]; \mathbb{R}^d) \), as \( m \to +\infty \).
(iii) Assume there exists a continuous version of \( \tilde{Q}_t(m; \mathcal{D}) : t \in [0, T] \) for \( m \in \mathbb{N} \) and let Assumption 3.4 (a)-(c) hold with the exponent \( c_1 \) in (c) strictly greater than \( d \). Then, the weak convergence in (ii) can be replaced with the weak convergence in \( \mathcal{C}([0, T]; \mathbb{R}^d) \) and the limiting process is almost surely locally Hölder continuous with exponent in \( (0,(c_1 - d)/2) \).

Note that Assumption 3.4 (c) is the only regularity condition on the kernel throughout. Despite sample paths of an infinitely divisible process without Gaussian components cannot be smoother than its kernel \cite{17}, the smoothness of the kernel is rather naturally irrelevant for the Gaussian approximation in the sense of finite dimensional distributions (Theorem 3.5 (i)).

Looking closely at Assumptions 3.2 and 3.4, the analysis of the error terms depends on the integrator and the kernel, while interestingly it suffices to investigate those two factors separately, which we do in Sections 4 and 5, respectively.

4 Lévy measures

In this section, we illustrate three typical integrators against Assumption 3.2 (c) and Assumption 3.4 (a) and (b). To avoid overloading the paper, we omit nonessential details in some instances by referring to relevant work in the literature.

4.1 Gamma law

We start with the one-dimensional Lévy measure \( \nu(dz) = ae^{-\beta z}/zdz \) on \( (0, +\infty) \) with \( a > 0 \) and \( \beta > 0 \), corresponding to the gamma law. The preferred shot noise representation for numerical purposes is the one by Bondesson’s method, based on the decomposition (2.4) with \( H(r,u) = \beta^{-1}e^{-r/a}u \) where \( U \) is the standard exponential random variable. Then, we have

\[
\sigma^2_m = \int_m^{+\infty} \int_\Omega (H(r,U(\omega)))^2 \mathbb{P}(d\omega) \, dr = \int_m^{+\infty} \int_\Omega \left( \frac{1}{\beta} e^{-r/a} U(\omega) \right)^2 \mathbb{P}(d\omega) \, dr = \frac{a}{\beta} e^{-2m/a}, \quad m \in \mathbb{N},
\]

which justifies Assumption 3.2 (c) and Assumption 3.4 (a). Assumption 3.4 (b) is however violated, since for every \( \kappa > 0 \),

\[
\int_m^{+\infty} \int_\Omega \left[ \sigma_m^{-1}H(r,U(\omega)) \right]^2 \mathbb{P}(d\omega) \, dr = \int_0^\infty \int_\Omega \left[ \frac{1}{\beta} e^{-2r/a} U(\omega) \right]^2 \mathbb{P}(d\omega) \, dr > 0,
\]

which no longer depends on the index \( m \). The Gaussian approximation (Theorem 3.5) fails, which supports, as mentioned earlier, that Assumption 3.4 (b) is not only sufficient but also necessary. We refer the reader to \cite[Section 5.4]{14} for many other shot noise representations of the gamma law as well as error analysis.

4.2 Stable law

Consider the Lévy measure of a stable law

\[
\nu(B) = \int_{S^{d-1}} \int_0^{+\infty} \mathbb{P}_B(r\xi) \frac{\alpha}{\rho + 1} q(r, \xi) \, dr \lambda(d\xi), \quad B \in \mathcal{D}(\mathbb{R}_0^d),
\]

with \( \alpha \in (0, 2) \), a finite measure \( \lambda \) on the unit sphere \( S^{d-1} \) and \( q \equiv 1 \). We have \( H(r, u) = (r/\|\lambda\|^a u, \|\lambda\|^a r) \), where \( \|\lambda\| := \lambda(S^{d-1}) \) and \( U \) is a random vector in \( S^{d-1} \) with the distribution \( \lambda/\|\lambda\| \). Interestingly, the inverse Lévy measure, rejection and Bondesson’s methods yield this shot noise representation. While the thinning method may deduce a different shot noise representation, it is significantly elapsd by the representation above in terms of elegance and practicality.

Now, Assumption 3.2 (c) holds true, due to

\[
\sigma^2_m = \int_m^{+\infty} \int_\Omega (H(r,U(\omega)))^2 \mathbb{P}(d\omega) \, dr = m^{1-2/\alpha} \lambda(\|\lambda\|^{2-\alpha})\lambda, \quad m \to +\infty,
\]

as \( m \to +\infty \), with \( \Lambda := \int_{S^{d-1}} \xi \otimes \lambda(d\xi) \). If the measure \( \lambda \) is not concentrated on a proper linear subspace of \( \mathbb{R}^d \), Assumption 3.4 (a) is satisfied, since then the matrix \( \Lambda \) is positive definite. In addition, Assumption 3.4 (b) holds true, since for every \( \kappa > 0 \),

\[
\int_m^{+\infty} \int_\Omega \left[ \|\sigma_m^{-1} H(r,U(\omega))\|^2 \right] \mathbb{P}(d\omega) \, dr = \int_1^\infty \int_\Omega \left[ \frac{2/\alpha - 1}{m} r^{-2/\alpha} \|\lambda\| \langle \xi, \lambda^{-1} \xi \rangle \right] \lambda(d\xi) \|\lambda\| \, dr \to 0,
\]

as \( m \to +\infty \), where the term \( [\cdot]_{(\kappa, +\infty)} \) eventually vanishes for almost every \( (r, \lambda) \in (1, +\infty) \times S^{d-1} \).

In the literature, there exist a few variants of the stable law and their error analysis that can proceed in a similar manner. We refer the reader to, for instance, \cite{9, 16} for layered stable and multistable Lévy processes.
4.3 Tempered stable law

Consider again the Lévy measure (4.1), where \(q(\cdot, \xi)\) here is completely monotone with \(q(0^+, \xi) = 1\) and \(\lim_{r \to +\infty} q(r, \xi) = 0\) for \(\xi \in S^{d-1}\). Then, it is the Lévy measure of a (proper) tempered stable law [19]. The most well known representation is the one developed in [19], which can be described as

\[
H(r, u) = \left[ \frac{r}{\|\lambda\|} \right]^{-1/\alpha} \land \frac{u_1}{\|u_2\|^{1/\alpha}} \frac{u_3}{\|u_3\|}
\]

where \(u = (u_1, u_2, u_3)\) corresponds to a random vector taking values in \((0, +\infty) \times [0, 1] \times \mathbb{R}^d\) with the law \(e^{-u_1 du_1} \otimes du_2 \otimes (Q/\|\lambda\|, \nu)\), such that \(Q(B) := \int_{Q_0(B)} \int_{u_1 > 0} \|\xi\|^2 h_m(d\xi)\lambda(d\xi)\) and \(q(r, \xi) = \int_{Q_0} e^{-rQ} Q(d\xi)\).

Now, it is rather straightforward to verify Assumption 3.2 (c) and Assumption 3.4 (a) with the aid of (4.2) for the tempered stable law, whereas, in more general terms, it seems very difficult to go through all the conditions, particularly in higher dimensions. We refer the reader to [5, Theorems 2.4 and 2.5] for sufficient conditions to ensure Assumption 3.2 (c) and Assumption 3.4 (a) and (b) for the Lévy measure in polar coordinates, just like (4.1), by which the tempered stable law is proved to satisfy the conditions for the Gaussian approximation, as long as the matrix \(\lambda\) is positive definite. In our notation, if the Lévy measure \(\nu_m\) of the discarded jumps can be decomposed in the polar form \((\nu - \nu_m)(d\xi) = h_m(dr; \xi) \lambda(d\xi)\) for \((r, \xi) \in (0^+, +\infty) \times S^{d-1}\), then those sufficient conditions read as follows: There exists a sequence \(\{b_m\}_{m \in \mathbb{N}}\) in \((0^+, +\infty)\) such that \(\liminf_{m \to +\infty} b_m^{-1} \int_{u_1 > 0} r^2 h_m(d\xi) > 0\) for almost every \(\xi \in S^{d-1}\), as well as such that for every \(\kappa > 0\), \(\lim_{m \to +\infty} b_m^{-2} \int_{|u_1| > \kappa b_m} \|\xi\|^2 (\nu - \nu_m)(d\xi) = 0\) (in fact, without requiring the polar form for the latter).

For many other shot noise representations and error analysis of the tempered stable law, we refer the reader to [10].

5 Kernels

In what follows, we illustrate a variety of kernels. The first three examples have \(\mathcal{T} = [0, T]\), so the term \(\{R_i(m, n) : t \in [0, T]\}\) is simply irrelevant. In the next four examples, since the support of their time integrands are only bounded from above by \(T\), we decompose the stochastic integral processes with, for instance, \(\mathcal{T}_n = (-n, T)\). In the last one, since the support is unbounded \(\mathcal{T} = \mathbb{R}\), the time truncation needs to be performed from both sides, for instance, \(\mathcal{T}_n = (-n, +n)\). Moreover, in Section 5.4, we illustrate the relevance of further splitting the \(Q\) term into multiple independent components (with disjoint time sets in the sense of (3.7)), resulting in different degrees of the limiting Hölder continuity (Theorem 3.5 (iii)).

5.1 Lévy processes

We first consider the simple case of a Lévy process \(\{X_t : t \in [0, T]\}\), which is an infinitely divisible process with the trivial stochastic integral representation \(X_t = \int_0^t \mathbb{1}_{[0,t]}(s) dL_s\) for \(t \in [0, T]\). As the stochastic integral only integrates over, at most, \([0, T]\), the error term \(\{R_i(m, n)\}\) is irrelevant here, that is, the only error component of the approximation is \(\{Q_i(m) : t \in [0, T]\}\) of the discarded jumps. Clearly, the kernel \(\mathbb{1}_{[0,t]}(s)\) is uniformly bounded and square-integrable on \([0, T]\) (Assumptions 3.2 (a) and (b)). Moreover, Theorem 3.5 (ii) holds as Assumption 3.4 (c) is satisfied with \(c_1 = 1\). In fact, the Gaussian approximation for the discarded jumps of Lévy processes is well known in the literature [5]. We remark that in the case of Lévy processes, the weak convergence in \(\mathcal{D}([0, T]; \mathbb{R}^d)\) is equivalent to the weak convergence of a marginal law [12, Exercise 16.10].

5.2 Lévy-driven Ornstein—Uhlenbeck processes

The Lévy-driven Ornstein—Uhlenbeck process \(\{X_t : t \in [0, T]\}\) is described by the stochastic differential equation \(dX_t = \lambda (\mu - X_t) dt + dL_t\), with \(\lambda > 0\) and \(\mu \in \mathbb{R}\), where its explicit solution is available as follows:

\[
X_t = e^{-\lambda t} X_0 + \mu \left(1 - e^{-\lambda t}\right) + \int_0^t e^{-\lambda(t-s)} dL_s. \tag{5.1}
\]

In light of the representation (5.1), it is an infinitely divisible process with the kernel \(f(t, s) = e^{-\lambda(t-s)} \mathbb{1}_{[0,t]}(s)\), that is, it only integrates over a compact time interval. Hence, we have \(\mathcal{T}_n \equiv \mathcal{T}\) and the sole error component is \(\{Q_i(m) : t \in [0, T]\}\), corresponding to the discarded jumps. The kernel is clearly square-integrable (Assumption 3.2 (b)) and is uniformly bounded (Assumption 3.2 (a)). Moreover, it satisfies Assumption 3.4 (c) with \(c_1 = 1\) for Theorem 3.5 (ii), as

\[
\int_0^T \left( f(t_2, s) - f(t_1, s) \right) ds = \frac{1}{2\lambda} \left[ 2 \left(1 - e^{-\lambda(t_2-t_1)}\right) + e^{-2\lambda t_1} \left(1 - e^{-\lambda(t_2-t_1)}\right)^2 \right] < \frac{3}{2} (t_2 - t_1), \quad 0 \leq t_1 \leq t_2 \leq T,
\]

where the inequality holds as \((1 - e^{-x})^2 \leq 1 - e^{-x} \leq x\) for all \(x \geq 0\).
5.3 A fractional Lévy motion

Consider the following kernel defined on a bounded time interval:
\[ K_{H,\alpha}(t,s) := c_{H,\alpha} \left[ \left( \frac{1}{s} \right)^{H-1/\alpha} (t-s)^{H-1/\alpha} - \left( H - \frac{1}{\alpha} \right) s^{1/\alpha-H} \int_s^t u^{H-1/\alpha-1} (u-s)^{H-1/\alpha} \, du \right] \mathbb{1}_{[0,1]}(s), \]
where \( \alpha \in (0,2) \), \( H \in (1/\alpha - 1/2, 1/\alpha + 1/2) \) and \( c_{H,\alpha} \) is a suitable constant. We highlight that when \( H \in (1/\alpha - 1/2, 1/\alpha) \), it holds that \( \lim_{s \to 0^-} |K_{H,\alpha}(t,s)| = +\infty \), thus leading to failure of Assumption 3.2 (a) and thus sample unboundedness. Hence, it suffices to focus on \( H \in (1/\alpha, 1/\alpha + 1/2) \). Then, there exists a continuous modification of the infinitely divisible process which is almost surely locally Hölder continuous with exponent \( \gamma < H - 1/\alpha \). Interesting features of this kernel are its self-similarity \( K_{H,\alpha}(ht,s) = h^{H-1/\alpha} K_{H,\alpha}(t,s/h) \) for all \( h > 0 \), as well as the second order one [8, Lemmas 2.1 and 2.3]:
\[ \int_0^T (K_{H,\alpha}(t_2,s) - K_{H,\alpha}(t_1,s))^2 \, ds = c_{H,\alpha}(t_2-t_1)^{2H-2/\alpha+1}, \quad 0 \leq t_1 \leq t_2 \leq T, \]
which verifies Assumption 3.2 (b) and Assumption 3.4 (c) with \( c_1 = 2H - 2/\alpha + 1 \). Thus, by Theorem 3.5 (iii), the limiting Gaussian process of the scaled term \( \{ \tilde{Q}(m): t \in [0,T] \} \) is almost surely locally Hölder continuous with exponent in \( (0,H-1/\alpha) \). We refer the reader to [8] for typical sample paths with various parameter sets generated by shot noise representations.

5.4 Linear fractional Lévy motions

Let \( n \in \mathbb{N} \) and \( \alpha \in (0,2) \). The \( n \)-th order moving average kernel with Hurst parameter \( H \in (n-1,n) \setminus \{1/\alpha\} \), such that \( H - 1/\alpha \) is not an integer, is given by
\[ f_n(t,s;H,\alpha) := \frac{1}{(H-1/\alpha+1)} \left[ (t-s)^{-H-1/\alpha} - \frac{n-1}{(H-1/\alpha+1)} \sum_{k=0}^{n-1} \left( \frac{H-1/\alpha}{k} \right) \right] (-s)^{-H-1/\alpha-k}, \quad s \in \mathbb{R}. \]
If \( H = 1/\alpha \), then the kernel reduces to \( \mathbb{1}_{[0,1]}(s) \) by the zero-power convention, corresponding to Lévy processes (Section 5.1). If \( n = 1 \), then it is the kernel for typical linear fractional Lévy motions, whereas it is a higher order one [13] with \( n \in \{2,\cdots\} \). Much like the kernel of Section 5.3, this moving average kernel captures self-similarity, with strong self-similarity with index \( H \) if the integrator is self-similar with index \( \alpha \), and second-order self-similarity in the case where the integrator is not necessarily self-similar but has finite second-order moments (see, for instance, [4]).

It is easy to show that Assumption 3.2 (a) is satisfied only when \( H - 1/\alpha > n - 1 \), while Assumption 3.2 (b) is satisfied only when \( H - 1/\alpha \in (n-3/2,n-1/2) \). Hence, it suffices to focus on \( H - 1/\alpha \in (n-1,n-1/2) \) for our simulation purpose. If \( n = 1 \), on the one hand, then it holds that for \( 0 \leq t_1 \leq t_2 \leq T \),
\[ \int_{\mathbb{R}} (f_1(t_2,s;H,\alpha) - f_1(t_1,s;H,\alpha))^2 \, ds = \frac{(t_2-t_1)^{2H-2/\alpha+1}}{(H-1/\alpha+1)^2} \int_{\mathbb{R}} ((1-s)^{-H-1/\alpha} - (-s)^{-H-1/\alpha})^2 \, ds, \quad (5.2) \]
which is finite valued since \( H - 1/\alpha \in (0,+1/2) \) when \( n = 1 \).

On the other hand, if \( n \in \{2,\cdots\} \), then the kernel is no longer as uniform as (5.2) over the entire domain \( \mathcal{F} \). We split \( \mathcal{F} = \mathbb{R} \) into the two disjoint and exhaustive sets \( \mathcal{F}_1 = (-\infty,0] \) and \( \mathcal{F}_2 = (0,\infty) \) so that
\[ \tilde{Q}(m;\mathcal{F}_k) = \left( \int_{-\infty}^0 + \int_0^{+\infty} \right) \int_{\mathbb{R}} f_n(t,s;H,\alpha) \sigma^{-1} \mathcal{N}((\mu - \mu_m) - (\nu - \nu_m) \mathbb{1}_{[0,1]}(\|z\|))(dz,ds), \]
where the two integrals \( \int_0^{+\infty} \) and \( \int_{-\infty}^0 \) correspond to \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively. Clearly, the resulting two stochastic processes are independent of each other and thus can be treated separately. First, on the negative time line \( (-\infty,0] \subset \mathcal{F}_1 \), we employ the mean value theorem with the aid of the recurrence formula \( \delta_tf_n(t,s;H,\alpha) = f_{n-1}(t,s;H-1,\alpha) \) for \( t > s \), so as to yield that for \( 0 \leq t_1 \leq t_2 \leq T \),
\[ \int_{-\infty}^0 (f_n(t_2,s;H,\alpha) - f_n(t_1,s;H,\alpha))^2 \, ds = (t_2-t_1)^2 \int_{\mathbb{R}} (f_{n-1}(\Theta(t_2,t_1),s;H-1,\alpha))^2 \, ds \leq (t_2-t_1)^2 \int_{\mathbb{R}} (f_{n-1}(t_1,s;H-1,\alpha))^2 \, ds, \]
where \( \Theta(t_1,t_2) \) is a real number in the interval \( (t_1,t_2) \) depending on \( s \). The inequality holds since \( f_{n-1}(.;H-1,\alpha) \) is positive and increasing on \( [0,T] \) as well as the integral is finite valued, both due to \( H - 1/\alpha \in (n-1,n-1/2) \) and \( n \in \{2,\cdots\} \). Therefore, the first component (corresponding to \( \mathcal{F}_1 \subset \mathcal{F} \)) satisfies Assumption 3.4 (c) with \( c_1 = 2 \), that is, Theorem 3.5 (ii) with Hölder exponent in \( (0,1/2) \). In turn, since the terms \( (-s)^{-H-1/\alpha-k} \) all vanish on the positive time line \( (0,\infty) \subset \mathcal{F}_2 \), it holds that for \( 0 \leq t_1 \leq t_2 \leq T \),
\[ \int_0^{+\infty} (f_n(t_2,s;H,\alpha) - f_n(t_1,s;H,\alpha))^2 \, ds \leq \int_{\mathbb{R}} \frac{((t_2-s)^{-H-1/\alpha} - (t_1-s)^{-H-1/\alpha})^2}{(H-1/\alpha+1)^2} \, ds \leq \frac{(t_2-t_1)^{2H-2/\alpha+1}}{(H-1/\alpha+1)^2} \int_{\mathbb{R}} ((1-s)^{-H-1/\alpha} - (-s)^{-H-1/\alpha})^2 \, ds. \]
The Lévy-driven reverse Ornstein-Uhlenbeck process is defined as the stochastic integral (2.2) with the kernel

\[ \int_{\mathcal{H}} \mathcal{C}^{p} dL_s \]

Under suitable technical conditions, the Lévy-driven CARMA process \[ \{Y_t : t \in \mathbb{R}\} \] can be expressed as a linear combination of dependent Ornstein-Uhlenbeck-like processes as follows:

\[ Y_t = \sum_{k=1}^{p} \frac{b(\lambda_k)}{a_d(\lambda_k)} \int_{-\infty}^{t} e^{\lambda_k(t-s)} dL_s, \quad t \in [0, T]. \]

Hence, the kernel here is given by

\[ f(t, s) = \sum_{k=1}^{p} \frac{b(\lambda_k)}{a_d(\lambda_k)} e^{\lambda_k(t-s)} \mathbf{1}_{[-\infty, t]}(s), \quad s \in \mathbb{R}, \]

that is, a linear combination of bounded exponential functions. Clearly, the kernel satisfies boundedness and square-integrability (Assumption 3.2 (a) and (b)), as well as the Lipschitz continuity (Assumption 3.4 (c)), since

\[ \int_{-\infty}^{T} \left( e^{\lambda(t-s)} - e^{\lambda(t_1-s)} \right)^2 ds = \frac{1 - e^{\lambda(t_2-t_1)}}{-\lambda} \leq t_2 - t_1, \quad 0 \leq t_1 \leq t_2 \leq T, \]

for \( \lambda < 0 \). Hence, the kernel satisfies all the relevant conditions for the limiting degeneracy (Proposition 3.2) and the Gaussian approximation (Theorem 3.5 (ii) with \( \mathcal{S} = \mathcal{F} \)). We refer the reader to [14] for the presentation of typical sample paths based on the proposed simulation method and error analysis when the integrator is stable and gamma.

The Lévy-driven reverse Ornstein-Uhlenbeck process is defined as the stochastic integral (2.2) with the kernel \( f(t, s) = e^{-\lambda|t-s|} \mathbf{1}_{[t, +\infty)}(s) \), with \( \lambda > 0 \). Therefore, the kernel satisfies boundedness and square-integrability (Assumption 3.2 (a) and (b)), as well as the Lipschitz continuity (Assumption 3.4 (c)):

\[ \int_{\mathbb{R}} (f(t_2, s) - f(t_1, s))^2 ds = \frac{1}{\lambda} \left( 1 - e^{-\lambda|t_2-t_1|} \right) \leq t_2 - t_1, \quad 0 \leq t_1 \leq t_2 \leq T, \]

satisfying the conditions for Proposition 3.2 and Theorem 3.5 (ii) with \( \mathcal{S} = \mathcal{F} \).

The log-fractional Lévy motion is defined as the stochastic integral (2.2) with the kernel \( f(t, s) = \ln|t-s| - \ln|s| \) for \( s \in \mathbb{R} \). Despite that the kernel is square-integrable (Assumption 3.2 (b)), the log-fractional motion is known to be unbounded on every interval of positive length [21, Example 10.2.6]. Indeed, Assumption 3.2 (a) fails.
5.8 Real harmonizable fractional motions

The real harmonizable fractional motion [3, 21] is a real-valued random field with locally Hölder continuous sample paths and can generalize fractional Gaussian fields. It is within our scope if the time index is one dimensional with \( \mathcal{T} = \mathbb{R} \) and, for instance, \( \mathcal{T}_n = (-n, n) \). On a compact time interval \([0, T]\), the real harmonizable fractional Lévy motion is formulated as

\[
X_t = \int_{\mathbb{R}} \frac{e^{-itx} - 1}{|x|^{1+H}} M(ds), \quad t \in [0, T],
\]

where \( H \in (0, 1) \), \( \alpha \in (0, 2] \) and \( M \) is a suitable random Lévy measure on \( \mathbb{R} \). On the one hand, the kernel is square-integrable (Assumption 3.2 (b)) and satisfies the regularity condition (Assumption 3.4 (c) with \( \mathcal{T} = \mathcal{T}_n \) and \( c_1 = 2H + 2/\alpha - 1 \)), due to

\[
\int_{\mathbb{R}} \left| \frac{e^{-itx} - 1}{|x|^{1+H}} - \frac{e^{-isx} - 1}{|s|^{1+H}} \right|^2 ds = (t_2 - t_1)^{2H+2/\alpha-1} \int_{\mathbb{R}} \frac{2(1 - \cos(s))}{|s|^{2/\alpha+2H}} ds, \quad 0 \leq t_1 \leq t_2 \leq T,
\]

if and only if \( H + 1/\alpha \in (1/2, 3/2) \). On the other hand, Assumption 3.2 (a) fails when \( H + 1/\alpha > 1 \), since then \( \sup_{t \in \mathbb{R}} |f(t,s)| = +\infty \) for all \( t \in [0, T] \). Hence, for our purposes, it suffices to focus on \( H + 1/\alpha \in (1, 3/2) \). Since \( c_1 = 2H + 2/\alpha - 1 \in (1, 2) \), Theorem 3.5 (ii) holds, provided that the random Lévy measure satisfies all the relevant conditions in Assumptions 3.2 and 3.5. We refer the reader to [6, 15, 21] for other simulation methods, error analysis and typical sample paths.

5.9 Pareto tails of an error term with the stable integrator

We close this study by illustrating a possible issue when the integrator is a stable process. We have already seen in Example 4.2 that the stable Lévy measure (4.1) (with \( \theta = \lambda \sum^n \phi_0, N \rightarrow 0 \)), that is, the Gaussian approximation (Assumption 3.4 (a) and (b)), is critical as quiet as a Gaussian process, while the \( R \) term exhibits non-negligible Pareto tails (for instance, [21, Theorem 10.5.1]) in the sense of

\[
\lim_{\theta \rightarrow +\infty} \theta^\alpha \mathbb{P} \left( \sup_{t \in [0,T]} \sup_{m \rightarrow +\infty} R_t(m,n) > \theta \right) = c \int_{\mathcal{T} \setminus \mathcal{T}_n} \sup_{t \in [0,T]} |f(t,s)|^\alpha ds, \quad n \in \mathbb{N},
\]

where \( c \) is a suitable positive constant, independent of \( n \). This is the case whenever the integrand of the right-hand side is finite valued, including the stable CARMA process [14, Theorem 4.1] and the higher order fractional stable motion [13, Theorem 6.4]. It is then desirable to take the time truncation \( \mathcal{T}_n \) sufficiently large to suppress the tails, taking into account a typical tradeoff with additional computing cost for more summands in light of (3.6).

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Define $\varphi_a(x; y) := e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle$ and $\varphi_b(x; y) := e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle$. Note that

$$
\varphi_a(x; y) = \varphi_b(x; y) + i\langle x, y \rangle \mathbb{I}_{[0, 1]}(||x||) \leq \frac{1}{2}||y||^2 ||x||^2 \mathbb{I}_{[0, 1]}(||x||) + 2 \times \mathbb{I}_{[1, +\infty]}(||x||), \quad ||\varphi_b(x; y)|| \leq \frac{1}{2}||x||^2,
$$

(A.1)

for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Recall the notation $[c]_{\mathbb{B}} := c \mathbb{B}([||x||])$ for $c \in \mathbb{R}^q$ and $\mathbb{B} \in \mathcal{B}(0, +\infty)$ for general dimension $q \in \mathbb{N}$.

Proof of Proposition 3.3. Fix $l \in \mathbb{N}$, let $\{t_k\}_{k \in \{1, \ldots, l\}}$ be a sequence of constants in $[0, T]$, and let $\{\theta_k\}_{k \in \{1, \ldots, l\}}$ be a sequence of constants in $\mathbb{R}$. To ease the notation, we denote $\psi(s) := \sum_{k=1}^l \theta_k f(t_k, s)$. For the convergence of $\{Q_t(m) : t \in [0, T]\}$ in $m$, it suffices to show the pointwise convergence: for every $y \in \mathbb{R}^d$,

$$
\mathbb{E}\left[\exp\left(\iint_{\mathcal{F}} \varphi_a(\psi(s)z; y) (v - v_m)(dz) \, ds\right)\right] = \exp\left(\int_{\mathcal{F}} \int_{\mathbb{R}^d} \varphi_a(\psi(s)z; y) (v - v_m)(dz) \, ds\right) \to 1,
$$
as $m \to +\infty$. Fix $y \in \mathbb{R}^d$. It holds by (A.1) that

$$
\left|\int_{\mathcal{F}} \int_{\mathbb{R}^d} \varphi_a(\psi(s)z; y) (v - v_m)(dz) \, ds\right| \leq \int_{\mathcal{F}} \int_{\mathbb{R}^d} \varphi_b(\psi(s)H(r, U(\omega)); y) \mathbb{P}(d\omega) \, dr \, ds \leq \frac{1}{2} \int_{\mathcal{F}} \int_{\mathbb{R}^d} \mathbb{E}[(\psi(s)H(r, U(\omega)))^2 \mathbb{P}(d\omega) \, dr \, ds,
$$

where the first term tends to zero, since

$$
\int_{\mathcal{F}} \int_{m}^{+\infty} \int_{\Omega} \varphi_b(\psi(s)H(r, U(\omega)); y) \mathbb{P}(d\omega) \, dr \, ds \leq \frac{1}{2} \int_{\mathcal{F}} \int_{m}^{+\infty} \int_{\Omega} (\psi(s)H(r, U(\omega)))^2 \mathbb{P}(d\omega) \, dr \, ds
$$

$$
= \frac{1}{2} \int_{\mathcal{F}} \int_{m}^{+\infty} \mathbb{E}\left[\psi(s)\sigma_m^2(\psi(s)) \mathbb{P}(d\omega)\right] \, dr \, ds \to 0,
$$

where the passage to the limit can be justified by the dominated convergence theorem due to Assumption 3.2 (a) and (b). For the second term, the integrand $m[\psi(s)H(r, U(\omega))]\mathbb{P}(d\omega)$ eventually vanishes for $(\mathbb{F} \otimes \mathbb{L} \otimes \mathbb{Leb})$-a.e. $(\omega, r, x) \in \mathbb{X} \times (1, +\infty) \times \mathcal{F}$, due to Assumption 3.2 (c).

Next, noting that the limiting process $\{\lim_{m \to +\infty} R_t(m, n) : t \in [0, T]\}$ exists, it suffices, in a similar manner to (i), to observe that

$$
\mathbb{E}\left[\exp\left(\iint_{\mathcal{F}} \varphi_a(\psi(s)z; y) (v - v_m)(dz) \, ds\right)\right] = \exp\left(\int_{\mathcal{F}} \int_{\mathbb{R}^d} \varphi_a(\psi(s)z; y) v(dz) \, ds\right) \to 1,
$$

for $y \in \mathbb{R}^d$, due to $\mathcal{F}_n \subset \mathcal{F}$.

Proof of Theorem 3.5. Throughout, we let $m$ be large enough to satisfy Assumption 3.4 (a).

For (i), fix $l \in \mathbb{N}$, let $\{t_k\}_{k \in \{1, \ldots, l\}}$ be a sequence of constants in $[0, T]$, and let $\{\theta_k\}_{k \in \{1, \ldots, l\}}$ be a sequence of constants in $\mathbb{R}$. It suffices to show the pointwise convergence: for every $y \in \mathbb{R}^d$,

$$
\ln \mathbb{E}\left[\exp\left(\iint_{\mathcal{F}} \varphi_a(\psi(s)z; y) (v - v_m)(dz) \, ds\right)\right] = \int_{\mathcal{F}} \int_{\mathbb{R}^d} \varphi_b(\psi(s)\sigma_m^{-1}z; y) (v - v_m)(dz) \, ds
$$

$$
+ i \int_{\mathcal{F}} \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\psi(s)\sigma_m^{-1}z\right)_{(1, +\infty)}(v - v_m)(dz) \, ds\right]
$$

$$
\to -\frac{1}{2} \int_{\mathcal{F}} \int_{\mathbb{R}^d} \mathbb{E}\left[\psi(s)^{2^2} \, dz\right],
$$
where \( \psi(s) := \sum_{k=1}^{\ell} \theta_k f(t_k, s) \), as in the proof of Theorem 3.3. Hereafter, we fix \( y \in \mathbb{R}^d \) throughout. First, we show that

\[
\int_{\mathcal{J}} \int_{\mathbb{R}^d} \phi_0 \left( \psi(s) \sigma_m^{-1} z : y \right) (v - v_m)(dz) ds = \int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} \phi_0 \left( \psi(s) \sigma_m^{-1} z : y \right) P(d\omega) dr ds \\
\sim -\frac{1}{2} \int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} \left\langle \psi(s) \sigma_m^{-1} H(mr, U(\omega)) \right\rangle \left( v - v_m \right)(dz) ds \\
\leq -\frac{1}{2} \left\langle y, \int_{\mathcal{J}} \left( \psi(s) \right)^{\otimes 2} ds \right\rangle y,
\]

where the asymptotic equivalence remains to be justified. Assumption 3.4 (a) reads that for every \( \kappa > 0 \),

\[
\int_{|\sigma_m^0 z|^2 > \kappa} \left\| \sigma_m^{-1} z \right\|^2 (v - v_m)(dz) ds = \int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} \left\| \sigma_m^{-1} H(mr, U(\omega)) \right\|^2 \left\langle \psi(s) \right\rangle P(d\omega) dr ds \\
= \frac{1}{2} \int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} \left\langle y, \psi(s) \sigma_m^{-1} H(mr, U(\omega)) \right\rangle^2 P(d\omega) dr ds < \epsilon,
\]
as \( m \to +\infty \). This, along with Assumption 3.2 (a), ensures that for each \( \epsilon > 0 \), there exists \( m_\epsilon \in \mathbb{N} \) such that for every \( m \geq m_\epsilon \),

\[
\left\| \phi_0 \left( \psi(s) \sigma_m^{-1} H(mr, U(\omega)) ; y \right) \left( v - v_m \right)(dz) ds \right\| < \frac{1}{4} \left\langle y, \int_{\mathcal{J}} \left( \psi(s) \right)^{\otimes 2} ds \right\rangle y,
\]

that is, for every \( m \geq m_\epsilon \),

\[
\int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} m \phi_0 \left( \psi(s) \sigma_m^{-1} H(mr, U(\omega)) ; y \right) \left\| \sigma_m^{-1} H(mr, U(\omega)) \right\|^2 \left\langle \psi(s) \right\rangle P(d\omega) dr ds \\
\leq \frac{1}{2} \epsilon \int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} \left\langle y, \psi(s) \sigma_m^{-1} H(mr, U(\omega)) \right\rangle^2 P(d\omega) dr ds = \frac{1}{2} \epsilon \left\langle y, \int_{\mathcal{J}} \left( \psi(s) \right)^{\otimes 2} ds \right\rangle y,
\]

again due to (A.1), which justifies the desired asymptotic equivalence since \( \epsilon \) can be chosen arbitrarily small. It remains to observe that

\[
\left\| \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( \psi(s) \sigma_m^{-1} z \right) (v - v_m)(dz) ds \right\| \leq \int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} \left\| \psi(s) \sigma_m^{-1} H(r, U(\omega)) \right\|^2 \left\langle \psi(s) \right\rangle P(d\omega) dr ds \\
\leq \int_{\mathcal{J}} \int_{m}^{+\infty} \int_{\Omega} \left\| \psi(s) \sigma_m^{-1} H(mr, U(\omega)) \right\|^2 \left\langle \psi(s) \right\rangle P(d\omega) dr ds,
\]

which tends to zero as \( m \to +\infty \), due to Assumption 3.2 (a) and Assumption 3.4 (b).

For (ii) and (iii), recall that the time domain here is \( \mathcal{J} \), which is a subset of \( \mathcal{T} \) of (i). First, consider a càdlàg version of \( \tilde{Q}(m, \mathcal{J}) : t \in [0, T] \). Let \( \{ \tau_m \}_{m \in \mathbb{N}} \) be a sequence of stopping times (with respect to the filtration generated by \( \{ \tilde{Q}(m, \cdot) : t \in [0, T] \} \)) taking values in \([0, T]\) and let \( \{ h_m \}_{m \in \mathbb{N}} \) be a sequence of positive constants decreasing to zero. Then, the desired result follows from [12, Theorem 16.11], if

\[
\left\| \tilde{Q}_{\tau_m + h_m}(m, \mathcal{J}) - \tilde{Q}_{\tau_m}(m, \mathcal{J}) \right\| = \left\| \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( f(\tau_m + h_m, s) - f(\tau_m, s) \right) \sigma_m^{-1} z (\mu - \mu_m) - (v - v_m) \mathbb{I}_{(0,1)}(\|z\|)(dz, ds) \right\| \xrightarrow{P} 0,
\]
as \( m \to +\infty \). To show that the last convergence holds true, it suffices to observe that

\[
\mathbb{E}\left\| \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( f(\tau_m + h_m, s) - f(\tau_m, s) \right) \sigma_m^{-1} z (\mu - \mu_m) - (v - v_m)(dz, ds) \right\| = \mathbb{E}\left[ \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( f(\tau_m + h_m, s) - f(\tau_m, s) \right) \sigma_m^{-1} z (\mu - \mu_m) - (v - v_m)(dz, ds) \right] \\
= \mathbb{E}\left[ \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( f(\tau_m + h_m, s) - f(\tau_m, s) \right) \sigma_m^{-1} z (\mu - \mu_m) - (v - v_m)(dz, ds) \right] \\
= \mathbb{E}\left[ \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( f(\tau_m + h_m, s) - f(\tau_m, s) \right) \sigma_m^{-1} z (\mu - \mu_m) - (v - v_m)(dz, ds) \right] \leq c_2 h_m d \to 0,
\]

where we have applied the Wiener-Itô isometry, Assumption 3.4 (c) and the identity

\[
\int_{\mathbb{R}^d} \left\langle z, (\sigma_m^0 z)^{-1} z \right\rangle (v - v_m)(dz) = \int_{\mathbb{R}^d} \|z\|^2 \rho_m(dz) = \text{tr} \left( \int_{\mathbb{R}^d} z^{\otimes 2} \rho_m(dz) \right) = \text{tr}(\mathbb{I}_d) = d,
\]

where \( \rho_m = (v - v_m) \circ \sigma_m \) denotes the push forward of the Lévy measure \( (v - v_m) \) by the map \( z \to \sigma_m^{-1} z \).

Finally, for a continuous version of \( \{ \tilde{Q}(m, \mathcal{J}) : t \in [0, T] \} \), it holds, as before, that for \( 0 \leq t_1 \leq t_2 \leq T \),

\[
\mathbb{E}\left\| \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( f(t_2, s) - f(t_1, s) \right) \sigma_m^{-1} z (\mu - \mu_m) - (v - v_m)(dz, ds) \right\| \leq c_2 (t_2 - t_1)^c d,
\]

where the last term is independent of \( m \). Hence, it remains to show that the deterministic residual term tends to zero uniformly, as follows:

\[
\sup_{t \in [0, T]} \left\| \int_{\mathcal{J}} \int_{\mathbb{R}^d} \left( f(t, s) \right) \sigma_m^{-1} z (v - v_m) \mathbb{I}_{(1, +\infty)}(\|z\|)(dz, ds) \right\| \leq \int_{\mathcal{J}} \int_{m}^{+\infty} \sup_{t \in [0, T]} \left\| f(t, s) m \sigma_m^{-1} H(mr, U(\omega)) \right\| \mathbb{I}_{(1, +\infty)} P(d\omega) dr ds,
\]

which tends to zero as \( m \to +\infty \), since the term \( H(mr, U(\omega)) \mathbb{I}_{(1, +\infty)} \) eventually vanishes for almost every \( (\omega, r) \), as well as due to Assumption 3.2 (a). Hence, the claim holds by [12, Corollary 16.9].