Low bounds for distribution of sums of independent centered random variables belonging to Grand Lebesgue Spaces.

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Abstract

We deduce in this short report the non-asymptotic lower bounds for exponential tail of distribution for sums of independent centered random variables.

Key words and phrases.

Probability space, centered random variable (r.v.), Lebesgue - Riesz and Grand Lebesgue Spaces (GLS) and norms, natural function, independence, tail of distribution, generating function, rearrangement invariant Banach functional spaces, anti-norm, anti-triangle inequality, upper and lower estimates.

1 Definitions. Notations. Statement of problem.

Let \((\Omega, B, \mathbf{P})\) be certain probability space with expectation \(\mathbb{E}\) and dispersion \(\text{Var}; X, Y\) be independent centered (mean zero) random variables (r.v.) and \(q = \text{const} \in [1, \infty]\). The ordinary Lebesgue - Riesz, or \(L(q)\) norm of the arbitrary r.v. \(Z\) will be denoted by \(\|Z\|_q:\)

\[ |Z|_q := [\mathbb{E}|Z|^q]^{1/q}, \quad 1 \leq q < \infty, \]

and

\[ |Z|_{\infty} := \text{vraisup}_{\omega \in \Omega} |Z(\omega)|. \]
Assaf Naor and Krzysztof Oleszkiewicz in a recent article [15] proved in particular
the following inequality for the r.v.-s. belonging to some Lebesgue - Riesz space
\[ |X + Y|_q \geq \left( \left| X \right|^q + \left| Y \right|^q \right)^{1/q}, \quad q \in [2, \infty], \] (1)
in our notations. More generally, let \{X_i\}, i = 2, 3, \ldots, n be a family of (common)
independent centered r.v.-s; then by induction
\[ \left| \sum_{i=1}^n X_i \right|_q \geq \left( \sum_{i=1}^n |X_i|^q \right)^{1/q}, \quad q \in [2, \infty]. \] (2)

If in addition the r.v. \(X_i\) are identical distributed,
\[ n^{-1/2} \sum_{i=1}^n X_i \mid_q \geq n^{1/q-1/2} |X_1|_q, \quad q \in [2, \infty]. \] (3)
The estimation (3) may be named as power level.
Note that this result is weak if \(q > 2\) as \(n \to \infty\); later we will improve it.

Our purpose in this short article is to extend the last inequality into
the r.v. belonging the so - called Grand Lebesgue Spaces (GLS).

We obtain as a consequence an exact non - uniform lower exponential estimations
for tail of distribution for the sums of independent centered r.v.

BRIEF note about Grand Lebesgue Spaces (GLS).

A classical approach.
Let \(\lambda_0 \in (0, \infty]\) and let \(\phi = \phi(\lambda)\) be an even strong convex function in \((-\lambda_0, \lambda_0)\)
which takes positive values, twice continuously differentiable; briefly \(\phi = \phi(\lambda)\) is a
Young-Orlicz function, such that
\[ \phi(0) = 0, \quad \phi'(0) = 0, \quad \phi''(0) \in (0, \infty). \] (4)

We denote the set of all these Young-Orlicz function as \(\Phi : \Phi = \{\phi(\cdot)\}.\)

Definition 1.1.
Let \(\phi \in \Phi\). We say that the centered random variable \(\xi\) belongs to the space \(B(\phi)\) if there exists a constant \(\tau \geq 0\) such that
\[ \forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau)). \] (5)
The minimal non-negative value \(\tau\) satisfying (5) for any \(\lambda \in (-\lambda_0, \lambda_0)\) is named
\(B(\phi)\)-norm of the variable \(\xi\) and we write
\[ \|\xi\|_{B(\phi)} \overset{def}{=} \inf\{\tau \geq 0 : \forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau))\}. \] (6)
For instance if $\phi(\lambda) = \phi_2(\lambda) := 0.5 \lambda^2$, $\lambda \in \mathbb{R}$, the r.v. $\xi$ is subgaussian and in this case we denote the space $B(\phi_2)$ with Sub. Namely we write $\xi \in \text{Sub}$ and

$$
||\xi||_{\text{Sub}} \overset{\text{def}}{=} ||\xi||_{B(\phi_2)}.
$$

It is known, see [11], [2] that if the r.v. $\xi_i$ are independent and subgaussian, then

$$
||\sum_{i=1}^{n} \xi_i||_{\text{Sub}} \leq \sqrt{\sum_{i=1}^{n} ||\xi_i||_{\text{Sub}}^2}.
$$

(7)

At the same inequality holds true in the more general case in the $B(\phi)$ norm, when the function $\lambda \rightarrow \phi(\sqrt{\lambda})$ is convex, see [11].

As a slight corollary: in this case and if in addition the r.v. - s $\{\xi_i\}$ are i., i.d., then

$$
\sup_{n=1,2,...} ||n^{-1/2} \sum_{i=1}^{n} \xi_i||_{B(\phi)} = ||\xi_1||_{B(\phi)}.
$$

(8)

It is proved in particular that $B(\phi)$, $\phi \in \Phi$, equipped with the norm (6) and under the ordinary algebraic operations, are Banach rearrangement invariant functional spaces, which are equivalent the so-called Grand Lebesgue spaces as well as to Orlicz exponential spaces. These spaces are very convenient for the investigation of the r.v. having an exponential decreasing tail of distribution; for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous and weak compactness of random fields, study of Central Limit Theorem in the Banach space, etc.

Let $g : R \rightarrow R$ be numerical valued measurable function, which can perhaps take the infinite value. Denote by $\text{Dom}[g]$ the domain of its finiteness:

$$
\text{Dom}[g] := \{y, \ g(y) \in (-\infty, +\infty) \}.
$$

(9)

Recall the definition $g^*(u)$ of the Young-Fenchel or Legendre transform for the function $g : R \rightarrow R$:

$$
g^*(u) \overset{\text{def}}{=} \sup_{y \in \text{Dom}[g]} (yu - g(y)),
$$

(10)

but we will use further the value $u$ to be only non-negative.

In particular, we denote by $\nu(\cdot)$ the Young-Fenchel or Legendre transform for the function $\phi \in \Phi$:

$$
\nu(x) = \nu(\phi)(x) \overset{\text{def}}{=} \sup_{\lambda : |\lambda| \leq \lambda_0} (\lambda x - \phi(\lambda)) = \phi^*(x).
$$

(11)

It is important to note that if the non-zero r.v. $\xi$ belongs to the space $B(\phi)$ then
\[ P(\xi > x) \leq \exp \left( -\nu(x/||\xi||_{B(\phi)}) \right). \] (12)

The inverse conclusion is also true up to a multiplicative constant under suitable conditions.

Furthermore, assume that the centered r.v. \( \xi \) has in some non-trivial neighborhood of the origin finite moment generating function and define

\[ \phi_\xi(\lambda) \overset{\text{def}}{=} \max_{a=\pm 1} \ln E \exp(a \lambda \xi) < \infty, \lambda \in (-\lambda_0, \lambda_0) \] (13)

for some \( \lambda_0 = \text{const} \in (0, \infty] \). Obviously, the last condition (12) is quite equivalent to the well known Cramer’s one.

We agree that \( \phi_\xi(\lambda) := \infty \) for all the values \( \lambda \) for which

\[ E \exp(|\lambda| \xi) = \infty. \] (14)

The function \( \phi_\xi(\lambda) \) introduced in (13) is named natural function for the r.v. \( \xi \); herewith \( \xi \in B(\phi_\xi) \) and moreover we assume

\[ ||\xi||_{B(\phi_\xi)} = 1. \]

We recall here for reader convenience some known definitions and facts about Grand Lebesgue Spaces (GLS) using in this article.

Let \( \psi = \psi(p), \ p \in [1, b] \) where \( b = \text{const}, 1 \leq b \leq \infty \) be positive measurable numerical valued function, not necessary to be finite in every point, such that \( \inf_{p \in [1, b]} \psi(p) > 0 \). For instance

\[ \psi_m(p) := p^{1/m}, \ m = \text{const} > 0, \ p \in [1, \infty) \]

or

\[ \psi^{(b, \beta)}(p) := (b - p)^{-\beta}, \ p \in [1, b), \ b = \text{const}, \ 1 \leq b < \infty; \ \beta = \text{const} \geq 0. \]

**Definition 1.2.**

By definition, the (Banach) Grand Lebesgue Space (GLS) \( G\psi = G\psi(b) \), consists on all the real (or complex) numerical valued random variable (measurable functions) \( f : \Omega \to R \) defined on whole our space \( \Omega \) and having a finite norm

\[ || f || = ||f||_{G\psi} \overset{\text{def}}{=} \sup_{p \in [1, b]} \left[ \frac{|f|_p}{\psi(p)} \right]. \] (15)

The function \( \psi = \psi(p) \) is named as the generating function for this space. If for instance
\[ \psi(p) = \psi^{(r)}(p) = 1, \; p = r; \; \psi^{(r)}(p) = +\infty, \; p \neq r, \]

where \( r = \text{const} \in [1, \infty), \; C/\infty := 0, \; C \in \mathbb{R}, \) (an extremal case), then the correspondent \( G_{\psi^{(r)}}(p) \) space coincides with the classical Lebesgue - Riesz space \( L_r = L_r(\Omega, \mathbb{P}). \)

These spaces are investigated in many works, e.g. in [4], [6], [7], [9], [10], [11], [14], [16] - [20] etc. They are applied for example in the theory of Partial Differential Equations [6], [7], in the theory of Probability [8], [18] - [20], in Statistics [16], chapter 5, theory of random fields [11], [19], in the Functional Analysis [16], [17], [19] and so one.

These spaces are rearrangement invariant (r.i.) Banach functional spaces; its fundamental function is considered in [19]. They not coincides in general case with the classical spaces: Orlicz, Lorentz, Marcinkiewicz etc., see [14] [17].

The belonging of some r.v. \( f : \Omega \to \mathbb{R} \) to some \( G_{\psi} \) space is closely related with its tail behavior

\[ T_f(t) = \text{meas}\{x; \; x \in \mathbb{R}^d, \; |f(x)| > t\} \]

as \( t \to \infty, \) see [11], [12].

Let a family of the functions \( \{f_w\} = \{f_w(\omega)\}, \; x \in \mathbb{R}^d, \; w \in W, \) where \( W = \{w\} \) is arbitrary set, be such that

\[ \exists b \in [1, \infty] \Rightarrow \psi[W](p) := \sup_{p \in (a,b)} |f|_p < \infty. \]  

(16)

The function \( \psi[W](p) \) is named as a natural function for the family \( \{f_w\}, \; w \in W. \) It may be considered as a generating function for certain Grand Lebesgue Space \( G_{\psi[W]} \). Obviously,

\[ \sup_{w \in W} ||f_w||G_{\psi[W]} = 1. \]

Notice that the family \( \{f_w\} \) may consists on the single function \( f_w = f, \) if course it satisfied the condition (16); we will write then

\[ \psi[f](p) := |f|_p, \; 1 \leq p < b, \]

one can take \( 1 \leq p \leq b, \) if \( B < \infty \) and \( |f|_b < \infty. \)

\[ 2 \quad \text{Main result: lower estimate. Anti - norms.} \]

The theory of GLS allows in particular to deduce the upper bound for distribution of sums of random variables, independent, centered or not. The norm in these spaces is defined by means of the operation \( \sup, \) see (15).
It is reasonable to assume that for an obtaining of the lower bounds for these sums we must apply for definition of some functionals of a type "norm" use the operator $\inf$. In detail:

Definition 2.1. Let $\psi = \psi(p), \ p \in [1,b)$ be certain generating function: $\psi(\cdot) \in \Psi_b$. The following functional is named as the $AG\psi$- anti - norm $V(X) = V(X)AG\psi = V(X)_\psi$ of the r.v. $X$:

$$V(X) = V(X)AG\psi \overset{def}{=} \inf_{p \in [1,b)} \left[ \frac{|X|_p}{\psi(p)} \right],$$

(17)
in contradiction with the classical definition of the GLS norms, see (15).

The (linear) space of all the random variables having non trivial $AG\psi$- norms forms by definition the Anti - Grand Lebesgue space $AG\psi$.

The following properties of introduced anti - norm are evident: $V(X) \geq 0$; and if in addition the generating function $\psi(\cdot)$ is bounded: $\sup_p \psi(p) < \infty$, then

$$V(X) \geq 0, \ V(X) = 0 \Leftrightarrow X = 0;$$

$$\forall C \in R \Rightarrow V(CX) = |C| \ V(X);$$

$$V(X + Y) \geq V(X) + V(Y) -$$

anti - triangle inequality.

Remark 2.1. If the generating function $\psi(\cdot)$ coincides with the natural function of some r.v. $X$, $\psi(p) = \psi[X](p) = |X|_p, \ 1 \leq p < b$, then obviously the ordinary and anti- GLS norms of r.v. $X$ coincides:

$$||X||G\psi = ||X||AG\psi = V(X) = 1.$$

Let us now investigate the strengthening of the anti - triangle inequality for independent centering r.v. $X$ and $Y$. Define for this purpose the following functions

$$\theta(p,q) := \inf_{a,b>0} \left[ \frac{(a^q + b^q)^{1/q}}{(a^p + b^p)^{1/p}} \right] = \inf_{z>0} \left[ \frac{(z^q + 1)^{1/q}}{(z^p + 1)^{1/p}} \right], \ p, q \geq 1;$$

$$\kappa(p) = \kappa_b(p) := \min_{q \in [1,b)} \theta(p,q);$$

then

$$\forall p \geq 1 \Rightarrow (a^q + b^q)^{1/q} \geq \theta(p,q) (a^p + b^p)^{1/p}, \quad (18)$$

$$\theta(p,q) = \min \left( 1, 2^{1/q - 1/p} \right). \quad (19)$$
\[ \kappa_b(p) = \min \left( 1, 2^{1/b - 1/p} \right), \quad (20) \]

so that
\[ \kappa_b(p) = 2^{1/b - 1/p}, \quad 1 \leq p \leq b, \quad (21) \]

and
\[ \kappa_b(p) = 1, \quad p > b. \quad (22) \]

Let now the centered independent random variables \( X, Y \) belong to some Anti - Grand Lebesgue space \( AG\psi, \exists \psi \in \Psi(b) : \)
\[ |X|_q \geq V(X)\psi(q), \quad |Y|_q \geq V(Y)\psi(q), \quad 1 \leq q < b. \]

We apply the Naor and Oleszkiewicz inequality (1) for the values \( q \in [1, b) \)
\[ |X + Y|_q \geq \left[ |X|^q_q + |Y|^q_q \right]^{1/q} \geq \psi(q) \left[ |V(X)|^q + |V(Y)|^q \right]^{1/q}; \quad (23) \]
\[ \frac{|X + Y|_q}{\psi(q)} \geq \left[ |V(X)|^q + |V(Y)|^q \right]^{1/q}. \]

Let now \( p = \text{const} \geq 1; \) we obtain using (18) and (20)
\[ V(X + Y) \geq \kappa_b(p) \left( V^p(X) + V^p(Y) \right)^{1/p}, \quad p \geq 1. \quad (24) \]

Highlight a particularly very important case \( p = 2 : \)
\[ V(X + Y) \geq \kappa_b(2) \left( V^2(X) + V^2(Y) \right)^{1/2}. \quad (25) \]

More detail:
\[ V(X + Y) \geq \min \left( 1, 2^{1/b - 1/2} \right) \left( V^p(X) + V^p(Y) \right)^{1/p}. \quad (26) \]

To summarize:

**Theorem 2.1.** Let \( \{X_i\}, \ i = 1, 2, \ldots, n \) be a sequence of centered independent random variables belonging to some Anti - Grand Lebesgue space \( AG\psi, \psi \in \Psi(b), \ 1 < b \leq \infty. \) Our proposition:
\[ V \left( \sum_{i=1}^{n} X_i \right) \geq \min \left( 1, 2^{1/b - 1/p} \right) \left[ \sum_{i=1}^{n} V^p(X_i) \right]^{1/p}, \quad p \in [1, \infty]. \quad (27) \]

In particular:
\[ V \left( \sum_{i=1}^{n} X_i \right) \geq \min \left( 1, 2^{1/b - 1/2} \right) \left[ \sum_{i=1}^{n} V^2(X_i) \right]^{1/2}. \quad (28) \]
If in addition \( b = \infty \), then

\[
V \left( \sum_{i=1}^{n} X_i \right) \geq 2^{-1/2} \left[ \sum_{i=1}^{n} V^2(X_i) \right]^{1/2}.
\]  

(29)

3 Examples.

A. Let us consider the symmetrical distributed subgaussian r.v \( X \) defined on some sufficiently rich probability space having the density

\[
f_x(x) = 0.5 \left| x \right| e^{-x^2/2}, \ x \in (-\infty, \infty).
\]  

(30)

We have for non-negative values \( p \)

\[
E|X|^p = 2^{p/2} \Gamma(p/2 + 1),
\]

therefore the natural function for this r.v. is following

\[
\psi[X](p) = |X|_p = 2^{1/2} [\Gamma(p/2 + 1)]^{1/p}.
\]  

(31)

Note that as \( p \in [1, \infty) \)

\[
\psi_X(p) \asymp (p/e)^{1/2}.
\]

B. Let the centered r.v. \( X \) be bilateral subgaussian:

\[
C_1 p^{1/2} \leq \psi[X](p) \leq C_2 p^{1/2}, \ \exists C_1, C_2 \in (0, \infty), \ C_1 \leq C_2, \ 1 \leq p < \infty.
\]

Let also \( X_i, \ i = 1, 2, \ldots \) be independent copies of \( X \). Define the classical normed sum

\[
S_n := n^{-1/2} \sum_{i=1}^{n} X_i.
\]

We deduce by virtue of theorem 2.1 that for \( u \geq 1 \)

\[
\exists C_3, C_4 = \text{const} \in (0, \infty), \ 0 < C_4 \leq C_3 \Rightarrow \exp(-C_5 u^2) \leq P(S_n > u) \leq \exp(-C_4 u^2)
\]  

(32)

and the same estimate there holds for left-hand side tail \( P(S_n < -u) \).

C. Let us consider a more general case of the sequence of centered independent r.v. \( \{X_1, X_2, \ldots, X_n\} \) such that
\[ \exists m > 0, \exists C_5, C_6 \in (0, \infty), C_6 \leq C_5, \forall u \geq 1 \Rightarrow \]
\[ \exp(-C_5 u^m) \leq P(|X_i| > u) \leq \exp(-C_6 u^m), \]

or equally
\[ C_7 p^{1/m} \leq \inf_i \psi[X_i](p) \leq \sup_i \psi[X_i](p) \leq C_8 p^{1/m}, p \in [1, \infty). \]

We propose
\[ \exists C_9, C_{10} \in (0, \infty), C_{10} \leq C_9 \Rightarrow \exp\left(-C_9 u^{\min(m,2)} \right) \leq \]
\[ P\left( n^{-1/2} \left| \sum_{i=1}^{n} X_i \right| > u \right) \leq \exp\left(-C_{10} u^{\min(m,2)} \right), u \geq 1. \quad (33) \]

Note that the upper estimate in (33) is known, see [11], [16], chapter 2, section 2.1.

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