RANKIN-SELBERG CONVOLUTIONS FOR $GL(n) \times GL(n)$ AND $GL(n) \times GL(n-1)$ FOR PRINCIPAL SERIES REPRESENTATIONS

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Abstract. Let $k$ be a local field. Let $I_\nu$ and $I_{\nu'}$ be smooth principal series representations of $GL_n(k)$ and $GL_{n-1}(k)$ respectively. The Rankin-Selberg integrals yield a continuous bilinear map $I_\nu \times I_{\nu'} \to \mathbb{C}$ with a certain invariance property. We study integrals over a certain open orbit that also yield a continuous bilinear map $I_\nu \times I_{\nu'} \to \mathbb{C}$ with the same invariance property, and show that these integrals equal the Rankin-Selberg integrals up to an explicit constant. Similar results are also obtained for Rankin-Selberg integrals for $GL_n(k) \times GL_n(k)$.

1. Introduction and the main results

Although Rankin-Selberg convolution is a well-established theory, explicit calculations of Rankin-Selberg integrals are usually not easy. These explicit calculations are often crucial for the arithmetic study of Rankin-Selberg L-functions.

Let $n$ be a positive integer, and $n' := n$ or $n - 1$ throughout this article. The Rankin-Selberg convolutions for $GL(n) \times GL(n')$ are viewed as the basic cases of the general Rankin-Selberg theory. In these basic cases, at least for principal series representations, we aim to calculate the Rankin-Selberg integrals as explicitly as possible. More precisely, we will express the Rankin-Selberg integral as a more explicit integral over a certain $GL(n')$-torsor.

The archimedean case of the main result (Theorem 1.6) of this article is used as a key ingredient in [LLS21] to prove the period relations for the critical values of Rankin-Selberg L-functions, which is an automorphic analog of Deligne’s conjecture ([D79]). In a paper under preparation, the non-archimedean case of Theorem 1.6 will be used to calculate the modified Euler factor at place $p$ for $p$-adic Rankin-Selberg L-functions (as predicted by J. Coates, see [C-PR89, C89a, C89b]).

1.1. Principal series representations. Fix an arbitrary local field $k$. Write $| \cdot |_k : k \to \mathbb{R}$ for the normalized absolute value. Fix an arbitrary nontrivial unitary character $\psi : k \to \mathbb{C}^\times$. We equip $k$ with the self-dual Haar measure associated to $\psi$.

For every $k \in \mathbb{N} := \{0, 1, 2, \cdots \}$, write $G_k := GL_k(k)$. It contains $\bar{B}_k N_k$ as an open dense subset, where $\bar{B}_k$ is the subgroup of the lower triangular matrices, and

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$N_k$ is the subgroup of the unipotent upper triangular matrices. The group $N_k$ is equipped with the Haar measure

(1) \[ du := \prod_{1 \leq i < j \leq k} du_{i,j}, \quad u = [u_{i,j}]_{1 \leq i,j \leq k} \in N_k, \]

the group $G_k$ is equipped with the Haar measure

\[ dg := |\det g|^{-1} \cdot \prod_{1 \leq i,j \leq k} dg_{i,j}, \quad g = [g_{i,j}]_{1 \leq i,j \leq k} \in G_k. \]

and the group $\bar{B}_k$ is equipped with the left invariant Haar measure

(2) \[ d\bar{b} := \prod_{i=1}^{k} |\det \bar{b}_{i,i}|^{-1} \cdot \prod_{1 \leq j \leq i \leq k} d\bar{b}_{i,j}, \quad \bar{b} = [\bar{b}_{i,j}]_{1 \leq i,j \leq k} \in \bar{B}_k. \]

Here $du_{i,j}, dg_{i,j}$ and $d\bar{b}_{i,j}$ indicate the Haar measure on $k$ associate to $\psi$ as before.

The coset $N_k \backslash G_k$ is equipped with the invariant quotient measure. Unless otherwise mentioned, all measures appearing in integrals in this article are the specified measures as above.

As usual, a continuous homomorphism from a topological group to $\mathbb{C}^\times$ is called a character of the topological group. Write $\widehat{k}^\times$ for the set of all characters of $k^\times$. For every $\mu \in \widehat{k}^\times$, we view it as a character of $\bar{B}_k$ in the obvious way, and write

$\text{Ind}_{\bar{B}_k}^{G_k} \mu := \{ f \in C^\infty(G_k) \mid f(\bar{b}x) = \mu(\bar{b}) \cdot \hat{\rho}_k(\bar{b}) \cdot f(x) \text{ for all } \bar{b} \in \bar{B}_k, \ x \in G_k \}$

for the corresponding smooth principal series representation, on which $G_k$ acts by right translation. Here

$\hat{\rho}_k := (|\cdot|^{\frac{1-k}{2}}, |\cdot|^{\frac{2-k}{2}}, \cdots, |\cdot|^{\frac{k-1}{2}}) \in (\widehat{k}^\times)^k$. 

In the archimedean case, $I_\mu$ is naturally a Fréchet space. In the non-archimedean case, $I_\mu$ is countable-dimensional. We view every countable-dimensional complex vector space as a locally convex topological vector space with the finest locally convex topology. In particular, every linear functional on $I_\mu$ is continuous in the non-archimedean case.

1.2. Rankin-Selberg integrals. Write $S(X)$ for the space of Schwartz functions on $X$ when $X$ is a Nash manifold (see [AG08]), and the space of compactly supported locally constant functions on $X$ when $X$ is a totally disconnected locally compact topological space. All functions in this article are complex-valued.

We review some basic facts concerning the Rankin-Selberg convolutions (see [JPSS83] and [J09] for more details). Define a character

$\psi_k : N_k \rightarrow \mathbb{C}^\times, \quad [u_{i,j}]_{1 \leq i,j \leq k} \mapsto \psi \left( \sum_{1 \leq i \leq k-1} u_{i,i+1} \right)$.
When no confusion is possible, we will not distinguish a character with the corresponding representation on $\mathbb{C}$. The space $\text{Hom}_{N_k}(I_\mu, \psi_k)$ of the $N_k$-equivariant continuous linear functionals is one-dimensional, and there is a unique element of it, to be denoted by $\lambda_\mu$, such that (see [W92, Theorem 15.4.1])

$$\lambda_\mu(f) = \int_{N_k} f(u) \overline{\psi_k(u)} \, du$$

for all $f \in I_\mu$ such that $f|_{N_k} \in \mathcal{S}(N_k)$. Here and henceforth, an overline over a character indicates its complex conjugation. Similarly, denote by $\lambda'_\mu$ the unique element of $\text{Hom}_{N_k}(I_\mu, \psi_k)$ such that

$$\lambda'_\mu(f) = \int_{N_k} f(u) \psi_k(u) \, du$$

for all $f \in I_\mu$ such that $f|_{N_k} \in \mathcal{S}(N_k)$. Then we have the homomorphisms

$$W : I_\mu \to \text{Ind}_{N_k}^G \psi, \quad f \mapsto W_f := (g \mapsto \lambda_\mu(g.f))$$

and

$$\overline{W} : I_\mu \to \text{Ind}_{N_k}^G \overline{\psi}, \quad f \mapsto \overline{W}_f := (g \mapsto \lambda'_\mu(g.f)).$$

Recall that $n' = n$ or $n - 1$. Throughout this article we fix

$$\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in (k^\times)^n \quad \text{and} \quad \nu' = (\nu'_1, \nu'_2, \cdots, \nu'_{n'}) \in (k^\times)^{n'}.$$

Put

$$L(s, \nu \times \nu') := \prod_{1 \leq i \leq n, 1 \leq j \leq n'} L(s, \nu_i \cdot \nu'_j).$$

Here and as usual, for every character $\chi$ of $k^\times$, $L(s, \chi)$ denotes the local $L$-function of $\chi$. For all $k, l \in \mathbb{N}$, denote by $k^{k \times l}$ the space of $k \times l$ matrices with entries in $k$. Let $f \in I_\nu$, $f' \in I_{\nu'}$ and $\phi \in \mathcal{S}(k^{1 \times n})$.

If $n' = n$, the Rankin-Selberg integral is defined by

$$Z(s, f, f', \phi) := \int_{N_n \setminus G_n} W_f(g) \cdot \overline{W}_{f'}(g) \cdot \phi(e_n g) \cdot |\det g|^s \, dg,$$

where $e_n := [0, 0, \cdots, 0, 1] \in k^{1 \times n}$. The integral (3) is absolutely convergent when the real part $\text{Re}(s)$ of the complex variable $s$ is sufficiently large, and extends to a holomorphic multiple of $L(s, \nu \times \nu')$ (see [J09, Section 8.1]). More precisely, there exists a unique continuous map

$$Z^s : \mathbb{C} \times I_\nu \times I_{\nu'} \times \mathcal{S}(k^{1 \times n}) \to \mathbb{C}$$

with the following properties:

- it is holomorphic in the first variable and linear in the last three variables;
there exists a constant \( c_{\nu,\nu'} \in \mathbb{R} \) such that whenever \( \text{Re}(s) > c_{\nu,\nu'} \), the integral is absolutely convergent and

\[
Z(s, f, f', \phi) = L(s, \nu \times \nu') \cdot Z^\circ(s, f, f', \phi)
\]

for all \( f \in I_\nu^\times, f' \in I_{\nu'}^\times \) and \( \phi \in S(k^{1 \times n}) \).

If \( n' = n - 1 \), the Rankin-Selberg integral is defined by

\[
Z(s, f, f') := \int_{N_{n-1} \setminus G_{n-1}} W_f(g) \cdot \overline{W}_{f'}(g) \cdot |\det g_k|^{s-\frac{1}{2}} dg.
\]

Similarly, the integral is absolutely convergent when \( \text{Re}(s) \) is sufficiently large, and extends to the multiplication of \( L(s, \nu \times \nu') \) with a continuous map

\[
Z^\circ : \mathbb{C} \times I_\nu \times I_{\nu'} \rightarrow \mathbb{C}
\]

that is holomorphic in the first variable and linear in the last two variables.

1.3. The integrals over the open orbits. The right action of \( G_n \) on \( (\overline{B}_n \setminus G_n) \times (\overline{B}_n \setminus G_n) \times k^{1 \times n} \) has a unique open orbit. Likewise, the right action of \( G_{n-1} \) on \( (\overline{B}_{n-1} \setminus G_{n-1}) \times (\overline{B}_{n-1} \setminus G_{n-1}) \) has a unique open orbit. We will introduce integrals over these open orbits and relate them to the Rankin-Selberg convolutions. For this purpose, we introduce some auxiliary matrices as follows. For each \( k \in \mathbb{N} \), write

\[
w_k := \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \cdots & \vdots & \vdots \\
1 & 0 & \cdots & 0
\end{bmatrix} \in \text{GL}_k(k).
\]

Define a family \( \{ z_k \in \text{GL}_k(k) \}_{k \in \mathbb{N}} \) of matrices inductively by

\[
z_0 := \emptyset \quad \text{(the unique element of } \text{GL}_0(k)), \quad z_1 := [1],
\]

and

\[
z_k := \begin{bmatrix}
w_{k-1} & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
z_{k-2}^t & 0 \\
0 & 1_2
\end{bmatrix} \begin{bmatrix}
{1}z_{k-1}w_{k-1}z_{k-1} & {1}e_{k-1} \\
0 & 1
\end{bmatrix}, \quad \text{for all } k \geq 2.
\]

Here and as usual, a left superscript \( t \) over a matrix indicates the transpose, a right superscript \( ^t \) indicates the inverse transpose of an invertible matrix, \( 1_2 \) stands for the \( 2 \times 2 \) identity matrix, and \( e_{k-1} := [0, \cdots, 0, 1] \in k_{0 \times (k-1)} \). In particular,

\[
z_2 := \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}.
\]

**Lemma 1.1.** (a) The right action of \( G_n \) on \( (\overline{B}_n \setminus G_n) \times (\overline{B}_n \setminus G_n) \times k^{1 \times n} \) has a unique open orbit represented by

\[
(\zeta_n, \begin{bmatrix} \zeta_{n-1} & 0 \\ 0 & 1 \end{bmatrix}, e_n).
\]
(b) The right action of $G_{n-1}$ on $(\bar{B}_n \backslash G_n) \times (\bar{B}_{n-1} \backslash G_{n-1})$ has a unique open orbit represented by

\[(z_n, z_{n-1}).\]

Proof. We first prove inductively the claim that the stabilizer (in $G_n$ or $G_{n-1}$) of the element represented by (7) or (8) is trivial. The claim is trivial for $n = 1$, and we assume that $n \geq 2$. Since the stabilizer of $e_n$ in $G_n$ is the mirabolic subgroup

\[
\left\{ \begin{bmatrix} g & v \\ 0 & 1 \end{bmatrix} \mid g \in G_{n-1}, \ v \in k^{(n-1) \times 1} \right\},
\]

it is easy to see that the claim in case (a) follows from that of case (b).

We will show that the claim in case (b) follows from the validity of the claim in case (a) for $n - 1$, which thereby finishes the proof by induction. Consider the diagonal action of $G_{n-1}$ on

\[\left( \bar{B}_{n-1} \backslash G_{n-1} \right) \times \left( \bar{B}_{n-1} \backslash G_{n-1} \right) \times k^{(n-1) \times 1},\]

where the right action of $g \in G_{n-1}$ on $k^{(n-1) \times 1}$ is given by $v \mapsto g^{-1}v$. Direct computation shows that the stabilizer $H$ of $(\bar{B}_n z_n, \bar{B}_{n-1} z_{n-1})$ in $G_{n-1}$ is contained in the stabilizer of

\[\left( \bar{B}_{n-1} w_{n-1} \ \begin{bmatrix} z_{n-2}^- & 0 \\ 0 & 1 \end{bmatrix}^{t} \ z_{n-1}^- \ w_{n-1} \ z_{n-1}^- \ w_{n-1}^{t} \ e_{n-1} \right),\]

under the above action. The latter stabilizer is conjugate to the stabilizer of

\[\left( \bar{B}_{n-1} w_{n-1} \ \begin{bmatrix} z_{n-2}^- & 0 \\ 0 & 1 \end{bmatrix}^{t} \ z_{n-1}^- \ w_{n-1}^{t} \ e_{n-1} \right),\]

which is the matrix transpose of the stabilizer $H'$ of

\[\left( \bar{B}_{n-1} \ \begin{bmatrix} z_{n-2}^- & 0 \\ 0 & 1 \end{bmatrix} \ e_{n-1} \right).
\]

If the claim in case (a) holds for $n - 1$, then the group $H'$ is trivial, hence $H$ is trivial as well, which implies that the claim in case (b) holds for $n$.

For both (a) and (b), dimension counting shows that an orbit is open if and only if the stabilizers are finite groups. In particular, the above argument implies that the element represented by (7) or (8) belongs to an open orbit.

We next prove the uniqueness of the open orbit (This is known to experts, and we include a proof for completeness.).

For (a), if a $G_n$-orbit in $(\bar{B}_n \backslash G_n) \times (\bar{B}_n \backslash G_n) \times k^{1 \times n}$ is open, then its image in $(\bar{B}_n \backslash G_n) \times (\bar{B}_n \backslash G_n)$ under the natural projection is also open, which has to be the orbit of $(\bar{B}_n, \bar{B}_n w_n)$, as is well-known. The stabilizer of $(\bar{B}_n, \bar{B}_n w_n)$ in $G_n$ is the diagonal torus, whose action on $k^{1 \times n}$ has a unique open orbit $(k^*)^{1 \times n}$. This shows the uniqueness in case (a).
For (b), note that we have a $G_{n-1}$-equivariant open embedding
\[ \overline{B}_{n-1} \setminus G_{n-1} \times k^{1\times(n-1)} \hookrightarrow \overline{B}_n \setminus G_n, \quad (\overline{B}_{n-1}g, v) \mapsto \overline{B}_n {w_{n-1}g \choose v} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
with dense image. Then the uniqueness of the open orbit in (b) follows from applying the uniqueness assertion in (a) for $n-1$. \hfill \square

We have some remarks for the elements $z_k$. Note that these elements are rational, which is required in the study of period relations in [LLS21]. Below we will introduce certain integrals over the open orbits in Lemma 1.1. The inductive choice of $z_k$, which looks complicated at first glance, will yield nice recurrence relations for these integrals in Section 3.

We are concerned with the following two integrals.

**Definition 1.2.** Let $f \in I_{\nu}, f' \in I'_{\nu}$ and $\phi \in S(k^{1\times n}).$

(a) Suppose that $n' = n$. For every $s \in \mathbb{C}$, define
\[ \Lambda(s, f, f', \phi) := \int_{G_n} f(z_n g) \cdot f' \left( \begin{bmatrix} z_n^{-1} \\ 0 \\ 0 \end{bmatrix} g \right) \cdot \phi(e_n g) \cdot |\det g|_k^{s} \, dg. \]

(b) Suppose that $n' = n - 1$. For every $s \in \mathbb{C}$, define
\[ \Lambda(s, f, f') := \int_{G_{n-1}} f \left( z_n \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \right) \cdot f'(z_n^{-1} h) \cdot |\det h|_k^{s-\frac{1}{2}} \, dh. \]

The following lemma is clear.

**Lemma 1.3.** Let $f \in I_{\nu}, f' \in I'_{\nu}$ and $\phi \in S(k^{1\times n}).$

(a) If $n' = n$, then for $g \in G_n$,
\[ \Lambda(s, g.f, g.f', g.\phi) = |\det g|_k^{-s} \Lambda(s, f, f', \phi) \]
and
\[ Z(s, g.f, g.f', g.\phi) = |\det g|_k^{-s} Z(s, f, f', \phi). \]

(b) If $n' = n - 1$, then for $h \in G_{n-1},$
\[ \Lambda(s, h.f, h.f', \phi) = |\det h|_k^{-s+\frac{1}{2}} \Lambda(s, f, f') \]
and
\[ Z(s, h.f, h.f') = |\det h|_k^{-s+\frac{1}{2}} Z(s, f, f'). \]

More precisely, the left hand side integral of (11) is absolutely convergent if and only if so is the right one, and when this is the case the equality (11) holds. Similar interpretation applies to other equalities for integrals or double integrals in this article (for example, in Proposition 3.1 and 3.2) without further comments.

For every character $\omega$ of $k^\times$, denote by $\text{ex}(\omega)$ the unique real number such that
\[ |\omega(a)| = |a|_k^{\text{ex}(\omega)}, \quad \text{for all } a \in k^\times. \]
Proposition 1.4. Assume that $s$ lies in the vertical strip
\[ \Omega_{\nu,\nu'} := \left\{ s \in \mathbb{C} \mid \begin{array}{c} \text{ex}(\nu_i) + \text{ex}(\nu'_j) + \text{Re}(s) < 1 \text{ whenever } i + j \leq n, \\ \text{ex}(\nu_i) + \text{ex}(\nu'_j) + \text{Re}(s) > 0 \text{ whenever } i + j > n \end{array} \right\}. \]
Then the integrals (9) and (10) are absolutely convergent.

We remark that the set $\Omega_{\nu,\nu'}$ in the above proposition may or may not be empty.

1.4. An equality of two integrals. Define a sign
\[ \text{sgn}(\nu; \nu') := \prod_{j < i, \ i + j \leq n} (\nu_i \cdot \nu'_j)(-1), \]
which by convention equals 1 for $n \leq 2$.

Define a meromorphic function
\[ \gamma_{\psi}(s; \nu; \nu') := \prod_{i + j \leq n} \gamma(s, \nu_i \cdot \nu'_j, \psi), \]
and likewise an entire function
\[ \varepsilon_{\psi}(s; \nu; \nu') := \prod_{i + j \leq n} \varepsilon(s, \nu_i \cdot \nu'_j, \psi), \]
which by convention is equal to 1 if $n = 1$. Here $\gamma(s, \nu_i \cdot \nu'_j, \psi)$ and $\varepsilon(s, \nu_i \cdot \nu'_j, \psi)$ are respectively the local gamma factor and the local epsilon factor defined following the standard references [T79, J79, K03], which will be recalled below.

Given a character $\omega$ of $k^\times$, Tate’s local zeta integral ([T50]) is defined by
\[ Z(s, \omega, \phi) = \int_{k^\times} \phi(x)\omega(x)|x|^s d^\times x, \quad \phi \in \mathcal{S}(k), \]
which converges absolutely when $\text{Re}(s) > -\text{ex}(\omega)$. Here $d^\times x := \frac{dx}{|x|^k}$, which is a Haar measure of $k^\times$. The local epsilon factor $\varepsilon(s, \omega, \psi)$ is an entire function defined by the local functional equation
\[ (12) \quad \frac{Z(1-s, \omega^{-1}, \hat{\phi})}{L(1-s, \omega^{-1})} = \varepsilon(s, \omega, \psi) \cdot \frac{Z(s, \omega, \phi)}{L(s, \omega)}, \quad \phi \in \mathcal{S}(k), \]
where $\hat{\phi} := \mathcal{F}_\psi(\phi) \in \mathcal{S}(k)$ is the Fourier transform of $\phi$ with respect to $\psi$ defined by
\[ \mathcal{F}_\psi(\phi)(x) := \int_k \phi(y)\psi(xy) dy, \quad x \in k. \]
The meromorphic function
\[ \gamma(s, \omega, \psi) := \varepsilon(s, \omega, \psi) \cdot \frac{L(1-s, \omega^{-1})}{L(s, \omega)} \]
is called the local gamma factor attached to $\omega$ and $\psi$. 

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Remark 1.5. A different convention is used in [J09] by setting \( \widehat{\phi} = F_\psi(\phi) \) in (12). This changes the definition of \( \varepsilon(s, \omega, \psi) \) by a factor \( \omega(-1) \), thanks to the relation

\[
\varepsilon(s, \omega, \psi) = \omega(-1) \cdot \varepsilon(s, \omega, \psi).
\]

We will translate the results in [J09] according to the convention of this article.

Finally, define the meromorphic function

\[
\Gamma_\psi(s; \nu; \nu') := \text{sgn}(\nu; \nu') \cdot \gamma_\psi(s; \nu; \nu').
\]

Now we state the main result of this paper.

**Theorem 1.6.** Assume that \( s \in \Omega_{\nu, \nu'} \) as in Proposition 1.4. Let \( f \in I_\nu, f' \in I_{\nu'} \) and \( \phi \in \mathcal{S}(k^{1 \times n}) \).

(a) If \( n' = n \), then

\[
\Lambda(s, f, f', \phi) = \Gamma_\psi(s; \nu; \nu') \cdot Z(s, f, f', \phi).
\]

(b) If \( n' = n - 1 \), then

\[
\Lambda(s, f) = \Gamma_\psi(s; \nu; \nu') \cdot Z(s, f, f', \phi).
\]

**Remark 1.7.** In Theorem 1.6 if \( n' = n \) then

\[
\Gamma_\psi(s; \nu; \nu') \cdot Z(s, f, f', \phi)
= \text{sgn}(\nu; \nu') \cdot \gamma_\psi(s; \nu; \nu') \cdot \prod_{i+j \leq n} L(1 - s, \nu_i^{-1} \cdot \nu_j'^{-1}) \cdot \prod_{i+j > n} L(s, \nu_i \cdot \nu_j') \cdot Z(s, f, f', \phi),
\]

which is easily seen to be holomorphic in \( s \in \Omega_{\nu, \nu'} \).

Likewise, if \( n' = n - 1 \) then \( \Gamma_\psi(s; \nu; \nu') \cdot Z(s, f, f') \) is holomorphic in \( s \in \Omega_{\nu, \nu'} \) as well.

When \( k \) is Archimedean, \( n' = n - 1 \), and \( f \) and \( f' \) lie in the minimal \( K \)-types (in the sense of Vogan), the Rankin-Selberg integrals \( Z(s, f, f') \) have been explicitly calculated by Ishii and Miayzaki in [IM22]. They also obtain similar result for \( n' = n \). Still in the Archimedean case, the Rankin-Selberg integrals for minimal \( K \)-type vectors of irreducible generalized principal series representations of \( GL(3) \times GL(2) \) have been explicitly calculated by in Hirano, Ishii and Miyazaki in [HIM22].

This article is organized as follows. In Section 2 we recall the Godement sections and their basic properties. In Section 3 we prove the recurrence relations for our integrals in terms of Godement sections. Proposition 1.4 and Theorem 1.6 will be proved in Sections 4 and 5 respectively, by using induction and the recurrence relations.
2. The Godement sections

We do not claim any originality of the results in this section. See [J09] and [IM22]. Recall that \( n \) is a positive integer, \( n' = n \) or \( n - 1 \), \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in (\hat{k}^\times)^n \) and \( \nu' = (\nu_1', \nu_2', \ldots, \nu_{n'}') \in (\hat{k}^\times)^{n'} \). Let \( \chi \in \hat{k}^\times \).

2.1. A convergence result. When \( X \) is a Nash manifold or a totally disconnected locally compact topological space, we say that a function \( f \) on \( X \) is rapidly decreasing if

\[
|f(x)| \leq \phi(x), \quad \text{for all } x \in X,
\]

for some real valued function \( \phi \in S(X) \).

**Lemma 2.1.** Assume that

\[ \text{ex}(\nu_i') > i - 1 \quad \text{for all } 1 \leq i \leq n'. \]

Then the integral

\[
\int_{B_{n'}} \phi(\bar{b}) \cdot \nu'(\bar{b}) \, d\bar{b}
\]

is absolutely convergent for all continuous functions \( \phi \) on \( k^{n' \times n'} \) that are rapidly decreasing.

**Proof.** For every \( a := (a_1, a_2, \ldots, a_{n'}) \in k^{n'} \), put

\[
\phi_1(a) = \int_{\tilde{n}_{n'}} |\phi(\tilde{u} + \text{diag}(a_1, a_2, \ldots, a_{n'}))| \, d\tilde{u},
\]

where \( \tilde{n}_{n'} \subset \mathfrak{gl}_{n'}(k) \) is the subspace of the lower triangular nilpotent matrices, \( d\tilde{u} \) is the product measure on \( \tilde{n}_{n'} \) similar to (1), and \( \text{diag} \) indicates the diagonal matrix. Then \( \phi_1 \) is a continuous function on \( k^{n'} \) that is rapidly decreasing in the archimedean case and has compact support in the non-archimedean case.

Note that

\[
\int_{B_{n'}} |\phi(\bar{b}) \cdot \nu'(\bar{b})| \, d\bar{b}
\]

\[
= \int_{(k^\times)^{n'}} \phi_1(a) \cdot |\nu'(a)| \cdot \prod_{i=1}^{n'} |a_i| k^{-i+1} \, d^\times a,
\]

where \( d^\times a \) is the product of the Haar measures on \( k^\times \). Then the lemma follows by the usual argument in Tate’s thesis. \( \square \)
2.2. The Godement sections $G_{n-1} \to G_n$. In this subsection, we assume that $n' = n - 1$. For all $f' \in I_{n'}$ and $\phi \in S(k^{n' \times n})$, put

$$g^+(\nu', \chi, f', \phi) := \int_{G_{n'}} f'(h^{-1}) \cdot \phi([h, 0]) \cdot \chi(\det h) \cdot |\det h|_k^{\frac{n'}{2}} \, dh.$$  

\textbf{Proposition 2.2.} Assume that

$$\text{ex}(\chi) > \text{ex}(\nu_i') - 1 \quad \text{for all } 1 \leq i \leq n - 1.$$  

Then the integral \[(15)\] is absolutely convergent.

\textit{Proof.} This is proved in [J09, Proposition 7.1 (i)]. \hfill \Box

Suppose that \[(16)\] holds. Then we have a well-defined map (see [J09, Proposition 7.1 (iv)])

$$g_{\nu', \chi}^+ : I_{n'} \times S(k^{n' \times n}) \to I_{(\nu', \chi)}$$

given by

$$(g_{\nu', \chi}^+(f', \phi))(g) := \chi(\det g) \cdot |\det g|_{k}^{\frac{n'}{2}} \cdot g^+(\nu', \chi, f', g.\phi),$$

where $g \in G_n$ and $g.\phi$ indicates the right translation.

\textbf{Proposition 2.3.} The image of the map \[(17)\] spans the vector space $I_{(\nu', \chi)}$.

\textit{Proof.} This directly follows from [J09, Proposition 7.1 (v)]. \hfill \Box

2.3. The Godement sections $G_n \to G_n$. For all $f \in I_{n}$ and $\phi \in S(k^{n \times n})$, put

$$g^0(\nu, \chi, f, \phi) := \int_{G_n} f(h) \cdot \phi(h) \cdot \chi(\det h) \cdot |\det h|_k^{\frac{n}{2}} \, dh.$$  

\textbf{Proposition 2.4.} Assume that

$$\text{ex}(\chi) > -\text{ex}(\nu_i) \quad \text{for all } i = 1, 2, \cdots, n.$$  

Then the integral \[(18)\] is absolutely convergent.

\textit{Proof.} The proof is similar to that of Proposition 2.2. Fix a maximal compact subgroup $K_n$ of $G_n$, and fix a Haar measure on it such that

$$\int_{G_n} \varphi(\tilde{b}k) = \int_{\tilde{B}_n} \int_{K_n} \varphi(\tilde{b}k) \, dk \, d\tilde{b}$$

for all $\varphi \in S(G_n)$.

Then we have that

$$\int_{G_{n'}} \left| f(h) \cdot \phi(h) \cdot \chi(\det h) \cdot |\det h|_k^{\frac{n'-1}{2}} \right| \, dh$$

$$= \int_{\tilde{B}_n} \phi_1(\tilde{b}) \cdot |\nu(\tilde{b})| \cdot |\chi(\det \tilde{b})| \cdot |\det \tilde{b}|_k^{\frac{n-1}{2}} \cdot \tilde{\rho}_n(\tilde{b}) \, d\tilde{b},$$

where

$$\phi_1(\tilde{b}) = \int_{K_n} |f(k) \cdot \phi(\tilde{b}k)| \, dk.$$
The Proposition then follows from Lemma 2.1.

Assume that (19) holds. Define a map
\[(21) \quad g_{\nu, \chi} : I_{\nu} \times S(k^{n \times n}) \rightarrow I_{\nu}\]
by
\[(g_{\nu, \chi}(f, \phi))(g) := \chi(\det g^{-1}) \cdot |\det g|^{\frac{1-n}{2}} \cdot g_{\nu, \chi}(\nu, \chi, f, L_g \phi),\]
where \(g \in G_n\), and \(L_g\) stands for the left translation so that \((L_g \phi)(x) = \phi(g^{-1}x)\) for all \(x \in k^{n \times n}\). It is easy to see that this map is well-defined and bilinear (cf. [LM22, Proposition 3.2]).

**Proposition 2.5.** Assume the condition (19). Then the image of the map (21) spans \(I_{\nu}\).

**Proof.** By change of variable, we have that
\[(g_{\nu, \chi}(f, \phi))(g) = \int_{G_n} \phi(h) \cdot f(gh) \cdot \chi(\det h) \cdot |\det h|^{\frac{n-1}{2}} \, dh.\]
Note that \(S(G_n) \subset S(k^{n \times n})\), and then the proposition easily follows by Dixmier-Malliavin Theorem [DM78].

3. **Recurrence relations**

We continue with the notation of the last section. Recall that \(\chi \in \widehat{k}^\times\). For \(s \in \mathbb{C}\), write \(\chi_s := \chi \cdot |\cdot|^s \in \widehat{k}^\times\). Suppose that \(n' = n - 1\) in this section, so that \(\nu \in (\widehat{k}^\times)^n\) and \(\nu' \in (\widehat{k}^\times)^{n-1}\).

3.1. **The first recurrence relation.**

**Proposition 3.1.** Let \(\phi_1 \in S(k^{(n-1) \times n})\) and \(\phi_2 \in S(k^{1 \times n})\), and write \(\phi_0 := \phi_1 \otimes \phi_2 \in S(k^{n \times n})\). Then for all \(f \in I_{\nu}\) and \(f' \in I_{\nu'}\),
\[\Lambda(s, f, g_{\nu, \chi}^+(f', \phi_1), \phi_2) = \Lambda(s, g_{\nu, \chi}^0(f, \phi_0), f').\]

As explained right below Lemma 1.3, the equation in Proposition 3.1 should be understood that both sides have the same range of absolute convergence. This will be clear from the proof below.

**Proof.** We have that
\[(22) \quad \Lambda(s, g_{\nu, \chi}^0(f, \phi_0), f') \]
\[= \int_{G_{n-1}} g_{\nu, \chi}^0(f, \phi_0) \left( z_{n-1} \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \right) f'(z_{n-1}h) \cdot |\det h|^{\frac{n-1}{2}} \, dh \]
\[= \int_{G_{n-1}} \int_{G_n} \phi_0(g) \cdot f \left( z_n \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \right) g \cdot \chi(\det g) \cdot |\det g|^{\frac{n-1}{2}} \, dg \cdot f'(z_{n-1}h) \cdot |\det h|^{\frac{n-1}{2}} \, dh.\]
By the change of variables \( g \mapsto \begin{bmatrix} h^{-1} & 0 \\ 0 & 1 \end{bmatrix} g \), the above inner integral equals
\[
\chi(\det h^{-1}) \cdot \det h^{-\frac{n+1}{2}} \cdot \int_{G_n} \phi_0 \left( \begin{bmatrix} h^{-1} & 0 \\ 0 & 1 \end{bmatrix} g \right) \cdot f \left( z_ng \right) \cdot \chi(\det g) \cdot \det g^{s+\frac{n-1}{2}} dg.
\]
Hence
\[
(22) = \int_{G_{n-1}} \int_{G_n} \phi_0 \left( \begin{bmatrix} h^{-1} & 0 \\ 0 & 1 \end{bmatrix} g \right) \cdot f' \left( z_{n-1}h \right) \cdot \chi(\det h^{-1}) \cdot \det h^{-\frac{n}{2}} dh \\
\quad \quad \quad \cdot f \left( z_n g \right) \cdot \chi(\det g) \cdot \det g^{s+\frac{n-1}{2}} dg
\]
\[
= \int_{G_n} \int_{G_{n-1}} \phi_1 \left( \begin{bmatrix} h^{-1} & 0 \\ 0 & 1 \end{bmatrix} g \right) \cdot f' \left( z_{n-1}h \right) \cdot \chi(\det h^{-1}) \cdot \det h^{-\frac{n}{2}} dh \\
\quad \quad \quad \cdot f \left( z_n g \right) \cdot \phi_2(e_ng) \cdot \chi(\det g) \cdot \det g^{s+\frac{n-1}{2}} dg.
\]
By the change of variables \( h \mapsto z_{n-1}^{-1}h \), the above inner integral equals
\[
\int_{G_{n-1}} \phi_1 \left( \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \right) \cdot f' \left( h^{-1} \right) \cdot \chi(\det h) \cdot \det h^{-\frac{n}{2}} dh \\
\quad \quad = \chi(\det g^{-1}) \cdot \det g^{\frac{n-1}{2}} \cdot (g_\phi^+)(f', \phi_1) \cdot \left( \begin{bmatrix} z_{n-1} & 0 \\ 0 & 1 \end{bmatrix} g \right).
\]
Therefore
\[
(22) = \int_{G_n} f \left( z_n g \right) \cdot (g_\phi^+)(f', \phi_1) \cdot \left( \begin{bmatrix} z_{n-1} & 0 \\ 0 & 1 \end{bmatrix} g \right) \cdot \phi_2(e_ng) \cdot \det g^{s} dg \\
\quad \quad = \Lambda(s, f, g_\phi^+)(f', \phi_1, \phi_2).
\]
This finishes the proof of the proposition. \( \square \)

3.2. The second recurrence relation. For every \( \phi \in S(k^{l\times l}) \) \( (k, l \in \mathbb{N}) \), write \( t\phi \in S(k^{l\times k}) \) for the function
\[
t\phi(x) = \phi(tx).
\]
For every \( f \in C^\infty(G_k) \), write \( \hat{f} \in C^\infty(G_k) \) for the function
\[
\hat{f}(g) = f(w_kg^w_k)
\]
Proposition 3.2. Let $\hat{\phi}^\ast g$ Thus we have that

As in the proof of Proposition 3.1, we are free to change the order of the integrals.

Then $\hat{f} \in I_{\hat{\alpha}}$ whenever $f \in I_{\alpha}$.

**Proposition 3.2.** Let $\mu \in (k^\times)^{n-1}$. Let $\phi_1 \in S(k^{(n-1) \times (n-1)})$ and $\phi_2 \in S(k^{(n-1) \times n})$, and write $\phi_0 := \phi_1 \otimes \phi_2 \in S(k^{(n-1) \times n})$. Then for all $f_\mu \in I_\mu$ and $f' \in I_{\mu'}$,

$$\Lambda(s, g^\circ_{\mu, \chi}(f_\mu, \phi_0), f') = \Lambda(1 - s, g^\circ_{\mu', \chi}(f', \phi_1), \hat{f}_\mu, w_{n-1} \cdot \phi_2).$$

**Proof.** The proposition is trivial when $n = 1$. Thus we assume that $n \geq 2$. We have that

$$\Lambda(1 - s, g^\circ_{\mu, \chi}(f', \phi_1), \hat{f}_\mu, w_{n-1} \cdot \phi_2) = \int_{G_{n-1}} |\det g|^{1-s} \cdot (g^\circ_{\mu, \chi}(f', \phi_1))(w_{n-1} z_{n-1} g^t w_{n-1})$$

$$\cdot f_\mu \left( w_{n-1} \begin{bmatrix} z_{n-2}^t & 0 & 0 \\ 0 & 1 \end{bmatrix} g^t w_{n-1} \right) \cdot \phi_2(\epsilon_{n-1} g w_{n-1}) \, dg$$

$$= \int_{G_{n-1}} |\det g|^{1-s} \cdot \phi_2(w_{n-1} g^t \epsilon_{n-1}) \cdot f_\mu \left( w_{n-1} \begin{bmatrix} z_{n-2}^t & 0 & 0 \\ 0 & 1 \end{bmatrix} g^t w_{n-1} \right)$$

$$\cdot \int_{G_{n-1}} \phi_1(h) \cdot f'(w_{n-1} z_{n-1} g^t w_{n-1} h) \cdot \chi(\det h) \cdot |\det h|^{s+\frac{n-2}{2}} dh \, dg.$$

By the change of variables $h \mapsto w_{n-1} g^t z_{n-1} w_{n-1} z_{n-1} h$, the above inner integral equals

$$\chi(\det g) \cdot |\det g|^{s+\frac{n-2}{2}} \int_{G_{n-1}} \phi_1(w_{n-1} g^t z_{n-1} w_{n-1} z_{n-1} h) \cdot f'(z_{n-1} h) \cdot \chi(\det h) \cdot |\det h|^{s+\frac{n-2}{2}} dh.$$

As in the proof of Proposition 3.1, we are free to change the order of the integrals. Thus we have that

$$\chi(\det g) \cdot |\det g|^{s+\frac{n-2}{2}} \int_{G_{n-1}} \phi_1(w_{n-1} g^t z_{n-1} w_{n-1} z_{n-1} h) \cdot f'(z_{n-1} h) \cdot \chi(\det h) \cdot |\det h|^{s+\frac{n-2}{2}} dh \, dh,$$

where

$$\chi(\det g) \cdot |\det g|^{s+\frac{n-2}{2}} \int_{G_{n-1}} \phi_1(w_{n-1} g^t z_{n-1} w_{n-1} z_{n-1} h) \cdot f'(z_{n-1} h) \cdot \chi(\det h) \cdot |\det h|^{s+\frac{n-2}{2}} dh \, dh,$$

$$\xi(h) := \int_{G_{n-1}} \chi(\det g) \cdot |\det g|^{s+\frac{n-2}{2}} \cdot \phi_1(w_{n-1} g^t z_{n-1} w_{n-1} z_{n-1} h) \cdot \phi_2(w_{n-1} g^t \epsilon_{n-1})$$

$$\cdot f_\mu \left( w_{n-1} \begin{bmatrix} z_{n-2}^t & 0 & 0 \\ 0 & 1 \end{bmatrix} g^t w_{n-1} \right) \, dg.$$
By the change of variable $g \mapsto \begin{bmatrix} z_{n-2}^{-1} & 0 \\ 0 & 1 \end{bmatrix} w_{n-1}^t gw_{n-1}$, we have that

$$
\xi(h) = \int_{G_{n-1}} \chi(\det g) \cdot |\det g|_k^{\frac{n}{2}} \cdot \phi_1 \left( gw_{n-1} \begin{bmatrix} z_{n-2}^{-1} & 0 \\ 0 & 1 \end{bmatrix} t z_{n-1} w_{n-1} z_{n-1} h \right) \cdot \phi_2 \left( gw_{n-1} \begin{bmatrix} z_{n-2}^{-1} & 0 \\ 0 & 1 \end{bmatrix} t e_{n-1} \right) \cdot f_\mu(g^{-1}) \, dg.
$$

Recall that

$$
z_n := \begin{bmatrix} w_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_{n-2}^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t z_{n-1} w_{n-1} z_{n} \end{bmatrix} e_{n-1},
$$

This implies that

$$
[g, 0] z_n \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} = gw_{n-1} \begin{bmatrix} z_{n-2}^t & 0 \\ 0 & 1 \end{bmatrix} t z_{n-1} w_{n-1} z_{n-1} h, gw_{n-1} \begin{bmatrix} z_{n-2}^t & 0 \\ 0 & 1 \end{bmatrix} t e_{n-1},
$$

which further implies that

$$
\phi_1 \left( gw_{n-1} \begin{bmatrix} z_{n-2}^t & 0 \\ 0 & 1 \end{bmatrix} t z_{n-1} w_{n-1} z_{n-1} h \right) \cdot \phi_2 \left( gw_{n-1} \begin{bmatrix} z_{n-2}^t & 0 \\ 0 & 1 \end{bmatrix} t e_{n-1} \right) = \phi_0 \left( [g, 0] z_n \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \right).
$$

Hence

$$
\xi(h) = \int_{G_{n-1}} \chi(\det g) \cdot |\det g|_k^{\frac{n}{2}} \cdot \phi_0 \left( [g, 0] z_n \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \right) \cdot f_\mu(g^{-1}) \, dg,
$$

and

$$
\chi(\det h) \cdot |\det h|_k^{\frac{n-1}{2}} \cdot \xi(h) = (g^+_{\mu, \chi}(f_\mu, \phi_0)) \left( z_n \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \right).
$$

Finally,

$$
\int_{G_{n-1}} f'(z_{n-1} h) \cdot |\det h|_k^{\frac{n-1}{2}} \cdot (g^+_{\mu, \chi}(f_\mu, \phi_0)) \left( z_n \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \right) \, dh = \Lambda(s, g^+_{\mu, \chi}(f_\mu, \phi_0)), f').
$$

This finishes the proof of the proposition. \hfill \Box

4. **Absolute convergence**

In this section we prove Proposition 1.4, which is restated below. Let $f \in I_\nu$, $f' \in I_{\nu'}$ and $\phi \in S(k^{1 \times n})$, as in Theorem 1.6.
Proposition 4.1. Assume that $s$ lies in
\[
\Omega_{\nu,\nu'} := \left\{ s \in \mathbb{C} \left| \begin{array}{l}
ex(\nu_i) + \ex(\nu'_j) + \Re(s) < 1 \text{ whenever } i + j \leq n, \\
ex(\nu_i) + \ex(\nu'_j) + \Re(s) > 0 \text{ whenever } i + j > n \end{array} \right. \right\}.
\]

(a) Assume that $n' = n$. Let $\mu' := (\nu'_1, \nu'_2, \cdots, \nu'_{n-1})$, $f_{\mu'} \in I_{\mu'}$, and $\phi_1 \in \mathcal{S}(k^{(n-1)\times n})$. Then the double integral
\[
\Lambda(s, g_{\nu,\nu'}^0 (f, \phi_0), f_{\mu'}),
\]
where $\phi_0 := \phi_1 \otimes \phi \in \mathcal{S}(k^{n \times n})$, is absolutely convergent.

(b) Assume that $n' = n-1$. Let $\mu := (\nu_1, \nu_2, \cdots, \nu_{n-1})$, $f_\mu \in I_\mu$, $\phi_2 \in \mathcal{S}(k^{(n-1)\times(n-1)})$, and $\phi_3 \in \mathcal{S}(k^{(n-1)\times 1})$. Then the double integral
\[
\Lambda(1-s, g_{\nu,\nu'}^0 (f', \phi_2), f_\mu, \overline{w_{n-1}} \cdot \phi_3)
\]
is absolutely convergent.

(c) The integral $\Lambda(s, f, f', \phi)$ is absolutely convergent if $n' = n$, and the integral $\Lambda(s, f, f')$ is absolutely convergent if $n' = n-1$. 

Proof. We prove (c) by induction on $n + n'$. The statement (c) is trivial when $n + n' = 1$. So we assume that $n + n' \geq 2$, and that (c) holds when $n + n'$ is smaller. Without loss of generality, we assume that all the $\nu_i$’s and $\nu'_j$’s are positive characters.

We first assume that $n' = n$. The assumptions of the proposition imply that
\[
ex(\nu'_n) > \ex(\nu'_i) - 1, \quad \text{for all } i = 1, 2, \cdots, n-1.
\]
In view of Proposition 2.3, we assume without loss of generality that
\[
f' = g_{\mu',\nu'}^0 (f_{\mu'}, \phi_1),
\]
where $\mu'$, $f_{\mu'}$ and $\phi_1$ are as in (a). By Proposition 3.1, we have that
\[
\Lambda(s, f, f', \phi) = \Lambda(s, g_{\mu',\nu'}^0 (f, \phi_0), f_{\mu'}),
\]
where $\phi_0 := \phi_1 \otimes \phi \in \mathcal{S}(k^{n \times n})$.

Recall that
\[
(g_{\nu,\nu'}^0 (f, \phi_0))(g) = \int_{G_n} \phi_0(h) \cdot f(gh) \cdot \nu'_n(\det h) \cdot |\det h|^{\frac{s+n-1}{2}} dh,
\]
where $g \in G_n$. This is absolutely convergent by Proposition 2.2. Note that the function
\[
g \mapsto \int_{G_n} \left| \phi_0(h) \cdot f(gh) \cdot \nu'_n(\det h) \cdot |\det h|^{\frac{s+n-1}{2}} \right| dh
\]
is bounded by a positive function in $I_\nu$. Thus the integral (24) is absolutely convergent by the induction hypothesis.

Now we assume that $n' = n-1$. The assumptions of the proposition imply that
\[
ex(\nu_n) > \ex(\nu_i) - 1, \quad \text{for all } i = 1, 2, \cdots, n-1.
\]
Proposition 2.3 implies that $|f|$ is bounded by a finite sum of functions of the form
$$|g^+_{\mu,\nu_n}(f_\mu, \phi_2 \otimes \phi_3)|,$$
where $\mu$, $f_\mu$, $\phi_2$ and $\phi_3$ are as in (b), so that $\phi_2 \otimes \phi_3 \in \mathcal{S}(k^{(n-1) \times n})$. Thus we may assume without loss of generality that
$$f = g^+_{\mu,\nu_n}(f_\mu, \phi_2 \otimes \phi_3).$$
Then by Proposition 3.2, we have that
$$\Lambda(s, f, f') = \Lambda(1-s, g^\circ_{\nu',\nu_n}, \nu_n(f', \phi_2), \nu_n(w_{n-1}, \nu_n f_\mu, w_{n-1})).$$

Recall that
$$\Lambda(s, f, f') = \Lambda(1-s, g^\circ_{\nu',\nu_n}, \nu_n(f', \phi_2), \nu_n(w_{n-1}, \nu_n f_\mu, w_{n-1})).$$

Proposition 5.1 implies that Theorem 1.6 holds when $n + n' \leq 2$. In the rest of this subsection we assume that $n + n' \geq 3$. 

5. Proof of Theorem 1.6

In this section we prove Theorem 1.6 by induction on $n + n'$. As before, let $f \in I_\nu$, $f' \in I_{\nu'}$ and $\phi \in \mathcal{S}(k^{1 \times n})$.

Proposition 5.1. If $n = 1$, then
$$\Lambda(s, f, f', \phi) = Z(s, f, f', \phi).$$

Proof. This is straightforward from the definitions of the two integrals. 

Proposition 5.1 implies that Theorem 1.6 holds when $n + n' \leq 2$. In the rest of this subsection we assume that $n + n' \geq 3$. 

Proposition 5.2. Assume that Theorem 1.6 holds for \( G_n \times G_{n-1} \). Then it holds for \( G_n \times G_n \).

Proof. Suppose that \( n' = n \). Write \( \mu' = (\nu_1', \ldots, \nu_{n-1}') \) so that \( \nu' = (\mu', \nu_n') \). Note that

\[
\Gamma_\psi(s; \nu; \nu') = \Gamma_\psi(s; \nu; \mu').
\]

By Proposition 3.1 we have that

\[
\Lambda(s, f, g^\mu_{\nu', \nu_n'}(f_{\mu'}), \phi) = \Lambda(s, g^\mu_{\nu', \nu_n'}(f, \phi_0), f_{\mu'}),
\]

where \( f_{\mu'} \in I_{\mu'} \), \( \phi_1 \in \mathcal{S}(k^{(n-1)\times n}) \), and \( \phi_0 := \phi_1 \otimes \phi \in \mathcal{S}(k^{n\times n}) \). Since \( s \in \Omega_{\nu, \nu'} \), Proposition 4.1 (a) implies that the double integrals in both sides of (26) are absolutely convergent.

On the other hand, by [IM22, Proposition 3.4],

\[
Z(s, f, g^\mu_{\nu', \nu_n'}(f_{\mu'}, \phi_1), \phi) = Z(s, g^\mu_{\nu', \nu_n'}(f, \phi_0), f_{\mu'}).
\]

In view of Proposition 2.3 the proposition follows from the above two equalities.

Proposition 5.3. Assume that Theorem 1.6 holds for \( G_{n-1} \times G_{n-1} \). Then it holds for \( G_n \times G_{n-1} \).

Proof. Suppose that \( n' = n - 1 \). Write \( \mu = (\nu_1, \ldots, \nu_{n-1}) \) so that \( \nu = (\mu, \nu_n) \). By [IM22, Proposition 3.5],

\[
Z(s, g^\mu_{\nu', \nu_n}(f_{\mu}, \phi_0), f') = Z(s, f_{\mu}, g^\mu_{\nu', \nu_n}(f', \phi_1), F_{\psi}(\phi_2)),
\]

where \( f_{\mu} \in I_{\mu} \) and \( \phi_0 = \phi_1 \otimes \phi_2 \in \mathcal{S}(k^{(n-1)\times n}) \) are as in Proposition 3.2 and \( F_{\psi}(\phi_2) \in \mathcal{S}(k^{1\times(n-1)}) \) is the Fourier transform of \( \phi_2 \in \mathcal{S}(k^{(n-1)\times 1}) \) with respect to \( \psi \) defined by

\[
F_{\psi}(\phi_2)(x) := \int_{k^{(n-1)\times 1}} \phi_2(y)\overline{\psi}(xy) \, dy, \quad x \in k^{1\times(n-1)}.
\]

We now apply the functional equation of Rankin-Selberg integrals as in [J09], where the notations are slightly different from ours. Put

\[
\tilde{\phi}_2(x) := (w_{n-1}, t \phi_2)(-x), \quad x \in k^{1\times(n-1)}.
\]

Then \( \tilde{\phi}_2 \in \mathcal{S}(k^{1\times(n-1)}) \). Write

\[
\gamma(s, I_{\mu} \times I_{\nu'}, \psi) := \prod_{1 \leq i, j \leq n-1} \gamma(s, \nu_i \times \nu'_j, \psi),
\]

which is a product of local gamma factors.

By [J09, Theorem 2.1] and Remark 1.5 and by noting that

\[
((F_{\psi} \circ F_{\psi})(\phi_2))(x) = \phi_2(-x), \quad x \in k^{(n-1)\times 1},
\]

we have

\[
\Lambda(s, f, g^\mu_{\nu', \nu_n}(f_{\mu'}, \phi_1), \phi) = \Lambda(s, g^\mu_{\nu', \nu_n}(f, \phi_0), f_{\mu'}),
\]

where \( f_{\mu'} \in I_{\mu'} \), \( \phi_1 \in \mathcal{S}(k^{(n-1)\times n}) \), and \( \phi_0 := \phi_1 \otimes \phi \in \mathcal{S}(k^{n\times n}) \). Since \( s \in \Omega_{\nu, \nu'} \), Proposition 4.1 (a) implies that the double integrals in both sides of (26) are absolutely convergent.

On the other hand, by [IM22, Proposition 3.4],

\[
Z(s, f, g^\mu_{\nu', \nu_n}(f_{\mu'}, \phi_1), \phi) = Z(s, g^\mu_{\nu', \nu_n}(f, \phi_0), f_{\mu'}).
\]

In view of Proposition 2.3 the proposition follows from the above two equalities.
we obtain that
\[
\omega_\mu(-1) \cdot \omega_{\nu'}(-1)^{n-1} \cdot \gamma(s, I_\mu \times I_{\nu'}, \psi) \cdot Z(s, f_\mu, g_{\nu', \nu, \alpha, \beta}^0(f', \phi_1), \mathcal{F}_{\nu'}(\phi_2))
\]
(28)
\[= Z(1 - s, g_{\nu', \nu, \alpha, \beta}^0(f', \phi_1), \tilde{f}_\mu, \tilde{\phi}_2),
\]
where \(\omega_\mu\) and \(\omega_{\nu'}\) denote the central characters of \(I_\mu\) and \(I_{\nu'}\) respectively, so that
\[
\omega_\mu(-1) = \prod_{1 \leq i \leq n-1} \nu_i(-1), \quad \omega_{\nu'}(-1) = \prod_{1 \leq i \leq n-1} \nu'_i(-1).
\]

Combining (27) and (28), we obtain that
\[
\omega_\mu(-1) \cdot \omega_{\nu'}(-1)^{n-1} \cdot \gamma(s, I_\mu \times I_{\nu'}, \psi) \cdot Z(s, g_{\nu', \nu, \alpha, \beta}^+ (f_\mu, \phi_0), f')
\]
(29)
\[= Z(1 - s, g_{\nu', \nu, \alpha, \beta}^0(f', \phi_1), \tilde{f}_\mu, \tilde{\phi}_2).
\]

By Proposition 3.2 we have that
\[
\Lambda(s, g_{\nu', \nu, \alpha, \beta}^+(f_\mu, \phi_0), f')
\]
(30)
\[= \Lambda(1 - s, g_{\nu', \nu, \alpha, \beta}^0(f', \phi_1), \tilde{f}_\mu, \tilde{\phi}_2)
\]
\[= ((\omega_{\nu'} \cdot \omega_\mu)(-1)) \cdot \Lambda(1 - s, g_{\nu', \nu, \alpha, \beta}^0(f', \phi_1), \tilde{f}_\mu, \tilde{\phi}_2).
\]

Since \(s \in \Omega_{\nu', \nu}\), Proposition 4.1 (b) implies that the three double integrals in (30) are all absolutely convergent.

In view of Proposition 2.3 the proposition follows from (29), (30) and the following lemma.

**Lemma 5.4.** Assume that \(n' = n - 1\). Then it holds that
\[
\Gamma_{\psi}(s; \nu; \nu') = \Gamma_{\psi}(1 - s; \widehat{\nu'}; \widehat{\mu}) \cdot \prod_{1 \leq j \leq n-1} \nu'_j(-1)^n \cdot \prod_{1 \leq i, j \leq n-1} \gamma(s, \nu_i \cdot \nu'_j, \psi),
\]
where \(\mu := (\nu_1, \ldots, \nu_{n-1})\).

**Proof.** We prove the lemma by induction on \(n\). The lemma is easily checked when \(n = 2\). Assume that \(n \geq 3\) and the lemma holds for \(n - 1\). Then by the induction hypothesis, we have that
\[
\Gamma_{\psi}(1 - s; \widehat{\nu'}; \widehat{\mu}) = \Gamma_{\psi}(1 - s; (\nu'_{n-1}, \ldots, \nu'_{2-1}); (\nu_{n-1}, \ldots, \nu_2))
\]
(32)
\[= \Gamma_{\psi}(s; (\nu_2, \ldots, \nu_{n-1}); (\nu_{n-1} - 1, \ldots, \nu_2))
\]
\[\cdot \prod_{2 \leq i \leq n-1} \nu_i(-1)^{n-1} \cdot \prod_{2 \leq i, j \leq n-1} \gamma(1 - s, \nu_i^{-1} \cdot \nu_j^{-1}, \psi).
\]

For \(\omega \in \hat{k}^\times\), it holds that
\[
\gamma(s, \omega, \psi) \cdot \gamma(1 - s, \omega^{-1}, \psi) = \varepsilon(s, \omega, \psi) \cdot \varepsilon(1 - s, \omega^{-1}, \psi) = \omega(-1).
\]


By \((32)\) and \((33)\), the right hand side of \((31)\) equals
\[
\Gamma_\psi(s; (\nu_2, \cdots, \nu_{n-1}); (\nu'_2, \cdots, \nu'_{n-2})) \cdot \prod_{2 \leq i \leq n-1} \nu_i (1)^{n-1} \cdot \prod_{1 \leq j \leq n-1} \nu'_j (1)^n
\]
\[
\cdot \prod_{2 \leq i, j \leq n-1} (\nu_i \nu'_j) (1) \cdot \prod_{1 \leq i, j \leq n-1, \min(i, j) = 1} \gamma(s, \nu_i \cdot \nu'_j, \psi),
\]
\[
= \Gamma_\psi(s; (\nu_2, \cdots, \nu_{n-1}); (\nu'_2, \cdots, \nu'_{n-2})) \cdot \nu' (1)^n \cdot \prod_{2 \leq i \leq n-1} \nu_i (1)
\]
\[
\cdot \prod_{1 \leq i, j \leq n-1, \min(i, j) = 1} \gamma(s, \nu_i \cdot \nu'_j, \psi),
\]
which is easily seen to be equal to \(\Gamma_\psi(s; \nu; \nu')\). This finishes the proof of the lemma. \(\Box\)

Finally, Theorem 1.6 follows from Propositions 5.1–5.3.

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