PALINDROMES AND ORDERINGS IN ARTIN GROUPS

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Abstract. The braid group $B_n$, endowed with Artin’s presentation, admits two distinguished involutions. One is the anti-automorphism $\text{rev} : B_n \to B_n, v \mapsto \bar{v}$, defined by reading braids in the reverse order (from right to left instead of left to right). Another one is the conjugation $\tau : x \mapsto \Delta^{-1}x\Delta$ by the generalized half-twist (Garside element).

More generally, the involution rev is defined for all Artin groups (equipped with Artin’s presentation) and the involution $\tau$ is defined for all Artin groups of finite type. A palindrome is an element invariant under rev. We classify palindromes and palindromes invariant under $\tau$ in Artin groups of finite type. The tools are elementary rewriting and the construction of explicit left-orderings compatible with rev.

Finally, we discuss generalizations to Artin groups of infinite type and Garside groups.

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1. Introduction

1.1. Palindromes in Artin groups. Let $n \geq 2$. The free group $F_n$ on $n$ generators $s_1, \ldots, s_n$ supports the involution rev: $w \mapsto \overline{w}$ defined by

$$s_1^{\alpha_1} \cdots s_r^{\alpha_r} \mapsto s_r^{\alpha_r} \cdots s_1^{\alpha_1},$$

which consists in reversing the reading of the word $w$ with respect to the prescribed set of generators. Any group $G$ presented by generators and
relations which are rev-invariant admits such an anti-automorphism. The
induced involution will still be denoted by rev. The elements of $G$ which are
rev-invariant are called palindromes. This paper studies palindromes in the
class of Artin groups.

A Coxeter matrix of rank $n$ is a square symmetric matrix $M$ of size $n$
with integer entries $m_{ij} \in \mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $1 \leq i \leq n$ and
$m_{ij} = m_{ji} \geq 2$ for all $1 \leq i \neq j \leq n$. Given two generators $a$ and $b$ of $F_n$
and $k \geq 2$, denote by $w_k(a, b)$ the word of length $k$ defined recursively by

$$w_2(a, b) = ab, \quad w_k(a, b) = \begin{cases} w_{k-1}(a, b)b & \text{if } k \text{ is even} \\ w_{k-1}(a, b)a & \text{if } k \text{ is odd.} \end{cases}$$

Given a Coxeter matrix $M$, the Artin group $A_M$ of type $M$ is the group
defined by the presentation

$$(1.1) \quad A(M) = \langle s_1, \ldots, s_n \mid w_{m_{ij}}(s_i, s_j) = w_{m_{ji}}(s_j, s_i) \text{ for all } i \neq j \text{ and } m_{ij} \neq \infty \rangle.\tag{1.1}$$

A group equipped with the presentation (1.1) will be called an Artin system
of type $M$. The set $S = \{s_1, \ldots, s_n\}$ is the set of positive Artin generators.
Clearly, rev fixes the word $w_k(a, b)$ if $k$ is even and sends it to $w_k(b, a)$ if $k$ is
odd. It follows that all Artin systems carry an involutive anti-automorphism
$\text{rev}: x \mapsto \text{rev}(x) = \bar{x}$ induced from the involution rev on $F_n$. Accordingly,
elements of the Artin group invariant under rev will be called palindromes.

Given a Coxeter matrix $M$, one can similarly define the Coxeter group
$W_M$ of type $M$ as the quotient of the Artin group of type $M$ by the subgroup
normally generated by the relations $s_i^2 = e$, $1 \leq i \leq n$. The kernel of the
natural epimorphism $A_M \to W_M$ sending the generator $s_i \in A_M$ to the
corresponding generator $s_i \in W_M$ is called the pure Artin group.

It is traditional to encode Artin systems in the form of a diagram (see Figure 1.1). Given a Coxeter matrix $M$, a Coxeter diagram is a graph $\Gamma$
whose set of vertices is $\{s_1, \ldots, s_n\}$ such that two vertices $s_i, s_j$ are joined
by an edge if and only if $m_{ij} \geq 3$ and an edge between two vertices $s_i, s_j$
is labelled by $m_{ij}$. (It is customary to omit the label if $m_{ij} = 3$.) We shall
index Artin and Coxeter groups indifferently with the Coxeter diagram $\Gamma$ or
with the Coxeter matrix $M$. Since the Artin and Coxeter groups associated
to disjoint Coxeter diagrams are direct products, we shall assume in this
paper that all Coxeter diagrams are connected. An Artin group $A_\Gamma$ (resp.
a Coxeter diagram $\Gamma$) is said to be of finite type if the associated Coxeter
group $W_\Gamma$ is finite.

As an example, $A_{A_n} = B_{n+1}$, the classical braid group on $n + 1$ strands.
In this case, the associated Coxeter group is the symmetric group $S_{n+1}$.

Let $A_\Gamma$ be an Artin system with set $S$ of Artin generators. The Artin
monoid $A_\Gamma^+$ is the monoid presented by the same generators and relations
as the Artin system $A_\Gamma$. Let $I$ be a subset of $S$. Suppose that all generators
in $I$ have a (right) common multiple. Then there is a uniquely determined
least common multiple, called the fundamental element $\Delta_I \in A_\Gamma^+$ for
the subset $I$. Such a fundamental element is always palindromic. A basic result
in the theory of Artin monoids asserts that the fundamental element $\Delta_S$
exists for the whole set $S = \{s_1, \ldots, s_n\}$ of Artin generators if and only if
$\Gamma$ is of finite-type. It is known that $\Delta_S$ or $\Delta_S^2$ lies in the center of $A_\Gamma$. In
particular, the automorphism

$$\tau : A_\Gamma \to A_\Gamma, \ x \mapsto \Delta_S^{-1} x \Delta_S$$

is always of order at most two. In this paper, we study and give classification results for palindromes in Artin groups and for $\tau$-invariant palindromes in Artin groups of finite type. We also indicate generalizations to Artin groups of infinite type and Garside groups.

Palindromes and $\tau$-invariant palindromes have nice geometric interpretations for $A_\Gamma = B_n$ (the classical braid group on $n$ strands). Given a geometric braid $x$, denote by $\widehat{x}$ its closure into a link inside a fixed solid torus $D^2 \times S^1$. The solid torus admits the involution

$$\text{inv} : D^2 \times S^1 \to D^2 \times S^1, \ (re^{it}, \theta) \mapsto (re^{-it}, -\theta),$$

whose set of fixed points consists of two segments ($t \equiv 0 \pmod{\pi}$ and $\theta \equiv 0 \pmod{\pi}$), which is the intersection of the axis of the $180^\circ$ rotation with the solid torus. Restricted to the boundary, this is just the Weierstrass involution of the standard torus. Observe that $\text{rev}(\widehat{x})$ is nothing else than $\text{inv}(\widehat{x})$ with the opposite orientation. In particular, if a braid $x \in B_n$ is
palindromic then $\tilde{x}$ coincides with $\text{inv}(\tilde{x})$ with the opposite orientation, see Figure 1.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.2.png}
\caption{The Weierstrass involution and palindromic braids.}
\end{figure}

The fundamental element coincides with the Garside element $\Delta = \Delta_n \in B_n$, the generalized half-twist on $n$ strands, defined inductively by

$$\Delta_2 = s_1, \quad \text{and} \quad \Delta_n = s_1 s_2 \cdots s_{n-1} \cdot \Delta_{n-1},$$

where $s_1, \ldots, s_{n-1}$ are the Artin generators of $B_n$. It turns out, as is directly verified, that $\tau(s_i) = s_{n-i}$ for $1 \leq i \leq n - 1$. Thus any braid commutes with $\Delta^2$ (a full twist); $\tau$-invariant braids are those which commute with $\Delta$. See Figure 1.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.3.png}
\caption{The involution $\tau$ is induced by a vertical $180^\circ$ rotation of the cylinder.}
\end{figure}
1.2. **Statement of results.** Let $A_\Gamma$ be an Artin group. There is a canonical way of producing palindromes from $A_\Gamma$. It consists in applying the map

$$\text{pal} : A_\Gamma \to A_\Gamma, \quad x \mapsto x \cdot \overline{x}$$

which we call the **palindromization** map.

**Theorem 1.1.** Let $\Gamma$ be of finite type. The palindromization map $\text{pal} : A_\Gamma \to A_\Gamma$ is injective and its image is the subset of pure palindromes.

The injectivity of $\text{pal}$ was first proved in [6] in the case of the classical braid group $B_n$, using the Jacquemard algorithm.

The proof of Theorem 1.1 is given in §2. Here is an outline. The proof is partially based on the existence of a left-invariant ordering of a certain type on Artin groups (Theorem 1.4 below). Recall that a group $G$ is **left-ordered** (or has a left-invariant ordering) if there exists a total ordering $<$ on the set $G$ such that $x < y$ implies $ax < ay$ for all $x, y, a \in G$.

**Lemma 1.2.** Let $G$ be a left-ordered group, equipped with an involutive anti-automorphism $G \to G, x \mapsto \overline{x}$, such that $x > e_G$ if and only if $\overline{x} > e_G$. Then the palindromization map $x \mapsto x \cdot \overline{x}$ is injective.

**Remark 1.3.** Actually, Lemma 1.2 is true for any automorphism or anti-automorphism $x \mapsto \overline{x}$.

We show the existence of an explicit left-invariant ordering on Artin groups of type $A$, $B$ and $D$, “compatible” with the involution $\text{rev}$:

**Theorem 1.4.** Let $\Gamma = A_n, B_n$ or $D_n$. There exists a left-invariant ordering $<$ on $A_\Gamma$ such that $x > e$ if and only if $\overline{x} > e$ for all $x \in A_\Gamma$.

In fact, there exists such an ordering on $B_n$, namely the Dehornoy ordering. (Thurston-type orderings all have that property.) We use it to prove Theorem 1.4. Then we embed other Artin groups of finite type into these in such a way that we can apply Lemma 1.2. We do not know about the left-orderability of the Artin group associated to $E_8$. Hence we cannot conclude at the present time for some Artin groups of finite type by this method. Hence we resort to the Jacquemard algorithm and the combinatorial method developed in [6] to prove that Theorem 1.1 is true for $\Gamma = E_8$. Then we embed all remaining Artin groups of finite type into $A_{E_8}$ and finish the proof.

**Remark 1.5.** We also give in §5 a proof outline valid for all Artin groups of infinite type using the Jacquemard algorithm.

**Remark 1.6.** It is readily seen that not all palindromes are in the image of $\text{pal}$, even for the classical braid group. For instance, $\Delta \notin \text{Im}(\text{pal})$ (in fact, $\Delta$ is not even a pure braid) and $\overline{\Delta} = \Delta$. Fig. 1.4 below displays the equality $\Delta = \overline{\Delta}$ for $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$, the generalized half-twist on four strands. More generally, for all Artin groups of finite type, $\Delta_S$ is square-free (it is not represented by a word which contains a square $s^2$, $s \in S$), so $\Delta_S$ is never in the image of $\text{pal}$.

We note the following consequences of Theorem 1.1.

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**Footnotes:**

[6] Reference to a specific paper or text.

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**Figure 1.4:** Display of the equality $\Delta = \overline{\Delta}$ for a specific word $\Delta$ in $B_4$, the generalized half-twist on four strands.
Corollary 1.7. Let $x \in A_\Gamma$ such that $\text{rev} \circ \tau(x) = x$. Assume that $x$ has the same image in $W_\Gamma$ as $\Delta = \Delta_S$. Then there exists $\delta \in A_\Gamma$ such that $x = \Delta \delta \delta = \tau(\delta) \Delta \delta$.

Proof. Observe that $y = \Delta^{-1} x$ is pure and

$$y = \tau \Delta^{-1} = \tau \Delta = \tau(x) \Delta = \Delta^{-1} x = y.$$  

We can therefore apply Theorem 1.1: there exists $\delta \in A_\Gamma$ such that $x = \Delta^{-1} x = \delta \delta$. □

Corollary 1.8. Let $x \in A_\Gamma$ pure and invariant under $\text{rev}$ and $\tau$. Then there exists a unique $\delta \in A_\Gamma$ such that $x = \delta \delta$ and $\tau(\delta) = \delta$.

Proof. The existence of $\delta \in A_\Gamma$ such that $x = \delta \delta$ follows from Theorem 1.1. Since $\text{rev}$ and $\tau$ commute, we have $\tau(x) = \tau(\delta) \tau(\delta) = \delta \delta = x$. Applying again Theorem 1.1 (injectivity of $\text{pal}$) yields $\tau(\delta) = \delta$. □

We now describe a general decomposition for palindromes in an Artin group $A_\Gamma$ of finite type. Recall that $S$ denotes the set of positive Artin generators.

Theorem 1.9. Let $x \in A_\Gamma$ be a palindrome. Then there exist $y \in A_\Gamma$ and $I \subseteq S$ such that

$$(1.3) \quad x = y \Delta_I \gamma.$$  

Suppose that $A_\Gamma$ is left-ordered with $<$ and the restriction of the order to $A_\Gamma^+$ is a well-ordering. Then the decomposition (1.3) is unique provided that $(\Delta_I, y)$ is minimal with respect to the lexicographic ordering $\prec = (\prec, \prec)$ on $A_\Gamma \times A_\Gamma$.

The decomposition (1.3) with the foregoing requirement on $(\Delta_I, y)$ will be called the canonical decomposition. If we denote by $<_{\text{opp}}$ the oppositive ordering on $A_\Gamma$ ($x <_{\text{opp}} y$ if and only if $y < x$), then there is also a similar decomposition with the requirement that $(\Delta_I, y)$ be minimal with respect to the ordering $(<_{\text{opp}}, \prec)$.

Remark 1.10. The main result of [13] asserts that there exists a left-ordering (the Dehornoy ordering, see [2] for a definition) on $B_n$ whose restriction to $B_n^+$ is a well-ordering. Any left-ordering of Thurston type has also this property [9, Prop. 7.3.1]. This result can be extended to Artin groups $A_\Gamma$ for $\Gamma = B_n$ or $D_n$: there is a left-ordering on $A_\Gamma$ whose restriction to $A_\Gamma^+$ is a well-ordering. I do not know if this is true for other Artin groups.
Remark 1.11. Theorem \textbf{1.9} yields a partial ordering on the set of palindromes. It does not coincide with the initial left-invariant ordering restricted to the subset of palindromes. In the case when $A_F = B_n$, if the ordering on $B_n$ is the Dehornoy ordering, then the ordering of elements $(e, y) < (\Delta_J, y) < (\Delta_J, y)$ does coincides with the Dehornoy ordering restricted to the corresponding palindromes: $y\bar{y} < y\Delta_I \bar{y} < y\Delta_J \bar{y}$ for $\Delta_I < \Delta_J$. (Note that if we identify $I, J \subseteq \{s_1, \ldots, s_{n-1}\}$ to subsets of $\{1, \ldots, n-1\}$, then $\Delta_I < \Delta_J$ is equivalent to $I > J$ in the usual lexicographic ordering of subsets of $\{1, \ldots, n-1\}$.)

Remark 1.12. Suppose that $\alpha \in B_n^+$ is palindromic. To ensure uniqueness of the decomposition (1.3) for $\alpha$, it is not enough to require that the length $\ell(y)$ of $y$ be extremal, as the following example shows:

$$\alpha = (\sigma_3 \sigma_5)\Delta_{\{1,2\}}(\sigma_5 \sigma_3) = (\sigma_5 \sigma_1)\Delta_{\{2,3\}}(\sigma_1 \sigma_5) \in B_6.$$ 

In fact, it is not even enough to fixe $\Delta_J$ in the decomposition to ensure uniqueness, as the following example shows.

Example 1.13. Consider the braid $x = (\sigma_2 \sigma_3 \sigma_1)^2 \in B_4^+$. It is readily verified that $x = \Delta_{\{1,2,3\}}$, hence $x$ is palindromic. However, this is not the canonical decomposition. The following equality shows that the map $y \mapsto y\Delta_I \bar{y}$ is not injective in general:

$$x = \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 \sigma_3 = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3 \sigma_1 = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_2 \sigma_1 = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1.$$ 

Therefore, $x = \sigma_3 \sigma_2 \Delta_{\{1,3\}} \sigma_2 \sigma_3 = \sigma_1 \sigma_2 \Delta_{\{1,3\}} \sigma_2 \sigma_1$. If we endow $B_n$ with the Dehornoy ordering, then $\sigma_3 \sigma_2 < \sigma_1 \sigma_2$, so the second decomposition is not the canonical decomposition. We leave it to the reader to verify that the first decomposition is the canonical decomposition.

2. Pure palindromes

Although it is possible to present a slightly more direct proof of Theorem \textbf{1.1}, the argument we present establishes stronger results about orderings on Artin groups of finite type.

Proof of Lemma \textbf{1.2}. Assume for instance that $x > e_G$. By assumption, $x > e_G$. Then $x \bar{x} > e_G$. We have just proved that $x \bar{x} = e_G$ implies $x = e_G$. Assume now that $x \bar{x} = y\bar{y}$. Write $y = xz$, $z \in A_F$. Then we have $y\bar{y} = x \gamma \bar{z} \bar{x} = x\bar{x}$. Therefore $z\bar{x} = e_G$. By our previous argument, $z = e_G$. Hence $\text{pal}$ is one-to-one. $\blacksquare$

We now turn to the proof of Theorem \textbf{1.4} proper. We focus in this section on the proof of the first statement (injectivity of $\text{pal}$). The proof of the second statement (that the image of $\text{pal}$ coincides with pure palindromes) is postponed after the proof of Theorem \textbf{1.9}. 

\[ \text{Remark 1.13. Consider the braid } x = (\sigma_2 \sigma_3 \sigma_1)^2 \in B_4^+. \text{ It is readily verified that } x = \Delta_{\{1,2,3\}}, \text{ hence } x \text{ is palindromic. However, this is not the canonical decomposition. The following equality shows that the map } y \mapsto y\Delta_I \bar{y} \text{ is not injective in general:} 
\]

\[ x = \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 \sigma_3 = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_3 \sigma_1 = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_2 \sigma_1 = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1. \]

Therefore, $x = \sigma_3 \sigma_2 \Delta_{\{1,3\}} \sigma_2 \sigma_3 = \sigma_1 \sigma_2 \Delta_{\{1,3\}} \sigma_2 \sigma_1$. If we endow $B_n$ with the Dehornoy ordering, then $\sigma_3 \sigma_2 < \sigma_1 \sigma_2$, so the second decomposition is not the canonical decomposition. We leave it to the reader to verify that the first decomposition is the canonical decomposition.

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Therefore, $x = \sigma_3 \sigma_2 \Delta_{\{1,3\}} \sigma_2 \sigma_3 = \sigma_1 \sigma_2 \Delta_{\{1,3\}} \sigma_2 \sigma_1$. If we endow $B_n$ with the Dehornoy ordering, then $\sigma_3 \sigma_2 < \sigma_1 \sigma_2$, so the second decomposition is not the canonical decomposition. We leave it to the reader to verify that the first decomposition is the canonical decomposition.
Step 1 (Proof of Theorem 1.1 for the braid group \( B_n \)). There is a total left-ordering \(<\) on \( B_n \), called the Dehornoy order. A word of the form

\[ x_0 s_1 x_1 s_i \cdots x_k s_i x_k \]

where \( x_1, \ldots, x_k \) are words in the letters \( s_{i+1}^{\pm 1}, \ldots, s_{n-1}^{\pm 1} \), is called a \( s_i\)-positive word. A braid \( x \) is \( s_i\)-positive if \( x \) can be represented by a \( s_i\)-positive word. A braid is called \( s_i\)-negative if its inverse is \( s_i\)-positive. Call a braid \( s\)-positive or \( s\)-negative if it is \( s_i\)-positive or \( s_i\)-negative for some \( i \).

**Theorem 2.1** (Dehornoy, [7]). There is a set partition of \( B_n \) into three classes: \( s\)-positive braids, \( s\)-negative braids, and the trivial braid.

The (left) Dehornoy order is defined by setting \( x < y \) if and only if \( x^{-1} y \) is \( s\)-positive. It is obvious that \( B_n \) endowed with the Dehornoy ordering satisfies the condition of Lemma 1.2. Therefore, Lemma 1.2 applies.

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**Step 2** (Construction of left-orderings and proof of Theorem 1.1 for Artin groups of type \( B \) and \( D \)). Let \((G, <)\) be a left-invariant ordered group and let \( \varphi \) be an automorphism or an anti-automorphism of \( G \). A pair \((\varphi, <)\) will be said to have property (PPC)* (resp. (SPPC)†) when \( \varphi(x) > e \) if (resp. and only if) \( x > e \) for all \( x \in G \). Our first goal is to construct a pair satisfying (SPPC), so as to apply again Lemma 1.2.

**Lemma 2.2.** Let \( G \) be a group equipped with an automorphism or anti-automorphism \( \varphi \), sitting in a short exact sequence

\[ 1 \longrightarrow H \overset{i}{\longrightarrow} G \overset{p}{\longrightarrow} K \longrightarrow 1 \]

of groups such that \((H, <)\) and \((K, <)\) are both left-invariant ordered groups and \( \varphi(i(H)) \subseteq i(H) \). Suppose that the automorphism or anti-automorphism \( \varphi_H = i^{-1} \circ \varphi \circ i \) has the property (PPC) (resp. (SPPC)) and that \( p \circ \varphi(x) > e \) if and only if \( p(x) > e \) for all \( x \in G \). Then there is a left-invariant order \(<\) on \( G \) such that \((\varphi, <)\) is (PPC) (resp. (SPPC)) pair.

**Proof.** Let \( x, y \in G \). We declare \( x < y \) if \( p(x) < p(y) \) or else \( p(x) = p(y) \) and \( e < i^{-1}(x^{-1}y) \). This defines a left-invariant order on \( G \). The claimed properties are left to the reader to verify.

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**Remark 2.3.** If \( \varphi \) is periodic, that is \( \varphi^k = \text{id} \) for some \( k \geq 1 \), then (PPC) is equivalent to (SPPC). This is the case in particular for \( \varphi = \text{rev} \).

Let \( G \) be an Artin system of type \( B \) or \( D \). There is a natural projection \( \pi : G \to B_n \), easily described in terms of the Artin generators. For convenience, denote by \( \beta_1, \ldots, \beta_n \) (resp. \( \delta_1, \ldots, \delta_n \)) the Artin generators of the Artin system \( A_{B_n} \) (resp. \( A_{D_n} \)). We keep the notation \( s_1, \ldots, s_{n-1} \) for the generators of the braid group \( B_n \).

If \( G = A_{B_n} \), then

\[ \pi(\beta_j) = \begin{cases} \sigma_j & \text{if } 1 \leq j \leq n-1 \\ e & \text{if } j = n \end{cases} \]

*preserves the positive cone.
†strongly preserves the positive cone.
If $G = \mathbb{A}_n^D$ then
\[
\pi(\delta_j) = \begin{cases} 
\sigma_j & \text{if } 1 \leq j \leq n - 2 \\
\sigma_{n-1} & \text{if } j = n - 1, n
\end{cases}
\]

Therefore there is a short exact sequence
\[
1 \rightarrow \text{Ker}(\pi) \rightarrow G \xrightarrow{\pi} B_n \rightarrow 1.
\]

It is shown in [5] that $\text{Ker}(\pi)$ is a free group of rank $n$ or $n - 1$ (according to whether $G = \mathbb{A}_n^B$ or $G = \mathbb{A}_n^D$).

**Lemma 2.4.** Let $F_n$ be a free group of order $n$. There is a left-invariant order $<$ on $F_n$ such that $(\text{rev}, <)$ has (SPPC).

**Proof.** We use the Magnus ordering defined as follows. Let $F_n$ be freely generated by $x_1, \ldots, x_n$ and $\Lambda = \mathbb{Z}\langle\langle X_1, \ldots, X_n \rangle\rangle$ be the ring of formal power series in the non-commuting indeterminates $X_1, \ldots, X_n$. The Magnus map $\mu : F_n \rightarrow \Lambda$ defined by
\[
\mu(x_j) = 1 + X_j, \quad \mu(x_j^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots
\]
is an injective mapping of $F_n$ into the multiplicative subgroup of series whose first coefficient (degree 0 coefficient) is 1. Now order $\Lambda$ as follows. We first list formal power series according to the total degree of monomials. Now monomials $X_{j_1}^{k_1} \cdots X_{j_r}^{k_r}$ of a given degree $d = \sum 1 \leq i \leq r, k_i$ are ordered lexicographically according to the $r$-uple of subscripts $(j_1, \ldots, j_r)$. Then two series are compared by looking at the first term at which their coefficient differ and order them according to that coefficient. This defines a total order on $\Lambda$ whose restriction to the image of $\mu$ is left-invariant (in fact even bi-invariant). There is a known sufficient condition for the Magnus ordering to have (SPPC).

**Lemma 2.5.** Let $\varphi : F_n \rightarrow F_n$ be an automorphism or an anti-automorphism. If the induced map $\varphi_{ab} : F/[F_n, F_n] \rightarrow F/[F_n, F_n]$ on the abelianization of $F_n$ is the identity, then the Magnus ordering has (SPPC).

A proof can be found in [9], Proposition 9.2.5. The proof there is given for an automorphism $\varphi$ but works as well if $\varphi$ is an anti-automorphism. Since $\text{rev}$ clearly satisfies the hypotheses of Lemma 2.5, application of Lemma 2.5 concludes the proof of Lemma 2.4.

We shall apply Lemma 2.2 to $(G, \text{rev})$. Clearly, the hypotheses pertaining to $H = F_n$ (or $F_{n-1}$), equipped with the Magnus ordering, and to $K = B_n$, equipped with the Dehornoy ordering, are satisfied. Applying Lemma 2.2 we obtain a left-ordering on $G$ with respect to which $\text{rev}$ has (SPPC). Therefore, we can apply Lemma 1.2 to derive the desired conclusion.

**Step 3** (Embeddings of Artin groups). One general method to construct a left-ordering on a group $G$ is to embed it into a left-orderable group. For our purpose, it is sufficient to construct an embedding with a special property.

**Lemma 2.6.** Let $i : (H, \varphi_H) \hookrightarrow (G, \varphi)$ be an embedding of groups equipped with anti-automorphisms such that $i \circ \varphi_H = \varphi \circ i$. If $G$ is left-ordered and has (SPPC), then $H$ has the same properties.
It is known that the Artin groups of type \( H_3 \) and \( I_2 \) inject into Artin groups of type \( D \) and \( A \) respectively. Furthermore, the embeddings can be realized \([1]\) so as to ensure that the images of the Artin generators are palindromes (invariant under \( \text{rev} \)). Therefore, since Artin groups of type \( D \) and \( A \) equipped with \( \text{rev} \) have (SPPC), Lemma \([2.6]\) applies for Artin groups associated to \( H_3 \) and \( I_2 \) with \( \varphi = \text{rev} \). Then Lemma \([1.2]\) applies.

\[\text{Step 4 (Other Artin groups of finite type).} \]\ If we knew that the Artin group associated to \( \Gamma \) is left-orderable with property (SPPC), then we could apply the previous argument (step 3) to all remaining Artin groups of finite type. Unfortunately, we do not know whether this is true. So we use a different method for the remaining cases. It is based on the following observation.

**Lemma 2.7.** Let \( i : A_{\Gamma'} \hookrightarrow A_{\Gamma} \) be an embedding of Artin groups such that \( i \circ \text{rev} = \text{rev} \circ i \). If the palindromization map \( \text{pal} : A_{\Gamma} \to A_{\Gamma} \) is one-to-one, then the palindromization map \( \text{pal} : A_{\Gamma'} \to A_{\Gamma'} \) is also one-to-one.

The remaining Artin groups (for which the previous steps do not apply) are associated to Coxeter diagrams \( E_6, E_7, E_8, F_4 \) and \( H_4 \). All these groups admit an embedding into \( A_{E_8} \) such that the images of Artin generators are palindromes. By Lemma \([2.7]\) it is therefore sufficient to prove that \( \text{pal} : A_{E_8} \to A_{E_8} \) is one-to-one. Then by the usual argument (briefly recalled below), since the natural map \( A^+_{E_8} \to A_{E_8} \) is an embedding, it is sufficient to prove that the restriction of \( \text{pal} \) to \( A^+_{E_8} \) is one-to-one.

We recall an algorithm due to A. Jacquemard \([12]\). It is originally expressed for the classical braid monoid \( B_n^+ \), but all his arguments go mutatis mutandis for \( A^+_{E_8} \) (and other Artin monoids). The input is a pair \((w, s)\) where \( w \) is a word representing an element in \( A^+_{E_8} \) written in letters representing positive Artin generators and \( s \in S \) is one letter representing a positive Artin generator. The algorithm decides whether there is a new word \( w' = sww' \) that represents the same element in \( A^+_{E_8} \) and that starts with the letter \( s \). The corresponding Artin generator \( s \) is said to be left-extractible from \( w \in A^+_{E_8} \). If this is the case, it returns the new word \( w' \). If not, it returns "false". The algorithm is recursive and based on two steps: 1) Swap \( s \) with the immediate left neighbor \( s' \) as long as \( s \) and \( s' \) commute when regarded in \( A^+_{E_8} \). If \( s \) becomes the first letter of the word, we are done. This step stops if and only if \( w = w_0s'sw_1 \) where \( w_0 \) is a non-empty word and \( s'ss' = ss's \) in \( A^+_{E_8} \). 2) This step occurs only after a call to the first step has been performed. The algorithm calls itself recursively with the pair \((w_1, s')\). If the call is successful, the new word is of the form \( w = w_0s'ssw_2 \), so that we can apply the Coxeter relation to turn it into \( w = w_0ss'sw_2 \) and return to the first step. If the call fails to rewrite \( w_1 = s'w' \), then the algorithm stops and returns false.

Define the blocking left-index \( I_s(w) \) of \( s \) in the word \( w \) to be the number of letters \( s' \) in the word \( w \) to the left of \( s \) such that \( s'ss' = ss's \) in \( A^+_{E_8} \). (This notion does depend on the word \( w \), not on the element it represents.) Since the blocking left-index of \( s \) strictly decreases [step 1] or the length of the word \( w \) (to which the algorithm applies) strictly decreases [step 2], the algorithm terminates.
The crucial point that requires to be verified is the following claim. Let 
\( w = w_0s'sw_1 \) be a word such that \( w_0 \) is a non-empty word that does not contain the letter \( s \) and where (when viewed in \( A_{E_6}^+ \)) \( s'ss' = ss's \). The claim is: if \( s' \) is not left-extractible in \( w_1 \), then \( s \) is not left-extractible in \( w \). Following Jacquemard’s original argument, we see that though the blocking left-index of \( s \) may decrease, it cannot be zero because after each word transformation, there will always be a letter \( s' \) occurring on the left before the leftmost letter \( s \).

We now review the key lemma (Lemma 3.2) of [6]. The Lemma is proved there in the setting of the braid monoid, but the proof continues mutatis mutandis to be valid in the case of \( A_{E_8}^+ \). Suppose that \( x\bar{x} = syys \) in \( A_{E_8}^+ \) for some positive Artin generator \( s \). Let \( w_0 = x\bar{x}, w_1, \ldots, w_k = sy' \) be a finite sequence of positive words representing \( x\bar{x} \) such that each word \( w_i \) is obtained from \( w_{i-1} \) by a relation in \( A_{E_8}^+ \) according to Jacquemard’s algorithm. Then each relation is performed only within the first half of the word \( w_i, i = 0, 1, \ldots, k \), which implies that all relations involve only letters from \( x \).

Using this lemma, an induction on the length of the positive word \( w \) shows that \( v\bar{v} = w\bar{w} \) (for a positive word \( v \)) in \( A_{E_8}^+ \) if and only if \( v = w \) in \( A_{E_8} \). (Here we use the fact that \( A_{E_8}^+ \) embeds in \( A_{E_8} \).) The general case follows since given any \( x \in A_{E_8} \), there exists a central element \( \Delta^N \in A_{E_8}^+ \) such that \( \Delta^N x \in A_{E_8}^+ \). Therefore, Theorem [1.1] is true for \( \Gamma = E_8 \) and finally, for the remaining \( \Gamma = E_6, E_7, F_4 \) and \( H_4 \). This achieves the proof.

3. General palindromes

This section is devoted to the proof of Theorem [1.9] and the fact that the image of \( \text{pat} \) consists of the subset of pure palindromes.

3.1. Proof of Theorem [1.9]

3.1.1. Existence. The existence is the consequence of the following two lemmas. Given \( x \in A_\Gamma^+ \), denote by \( S(x) = \{ s \in S \mid x = sy \text{ for some } y \in A_\Gamma^+ \} \) (starting set) and \( F(x) = S(\bar{x}) \) (finishing set). The first lemma follows from the divisibility theory for Artin monoids.

**Lemma 3.1.** Let \( x \in A_\Gamma^+ \). If \( J \subseteq S(x) \), then there exists \( y \in A_\Gamma^+ \) such that \( x = \Delta_J y \).

**Lemma 3.2.** Let \( x \in A_\Gamma^+ \) be a palindrome. Then there exists \( J \subseteq S \) such that \( x = \Delta_J \) or there exists \( s \in S \) such that \( x = sas \) for some \( a \in A_\Gamma^+ \).

**Proof of lemma [3.2]** Apply Lemma [3.1] to \( x \) and \( J = S(x) \), thus \( x = \Delta_{s(x)} y \) for some \( y \in A_\Gamma^+ \). If \( y = e \), we are done. Otherwise \( F(y) \neq \emptyset \), so there exists \( s \in F(y) \). Since \( \bar{x} = x \), we have \( F(y) \subseteq S(x) \). Hence \( s \in S(x) \). Thus there are \( \delta, y' \in A_\Gamma^+ \) such that \( s\delta = \Delta_{S(x)} \) and \( y's = y \). Then \( x = sas \) for some \( a = \delta y' \).

Let now \( x \in A_\Gamma^+ \) such that \( \mathfrak{q} = x \). We apply Lemma [3.2] to \( x \). If \( x = \Delta_J \) for some \( J \subseteq S \), we are done. Otherwise \( x = sa_1s \) for some \( a_1 \in A_\Gamma^+ \). Since
We deduce that $\overline{\Delta} = a_1$. So Lemma \ref{lemma:palindrome} applies to $a_1$. Denote by $\ell(x)$ the length (in positive Artin generators) of an element $x \in A_1^\Gamma$. An immediate induction using repeatedly Lemma \ref{lemma:palindrome} shows that either $x = y\Delta_1\overline{\gamma}$ for some $y \in A_1^\Gamma$ with $\ell(y) < \ell(x)/2$ or $x = ya\overline{\gamma}$ where $y, a \in A_1^\Gamma$ and $\ell(y) = \ell(x)/2$. In the latter case, since $\ell(x) = \ell(ya\overline{\gamma}) = \ell(\gamma) + \ell(a) + \ell(\overline{\gamma}) = 2\ell(y) + \ell(a) = \ell(x) + \ell(a)$, we deduce that $\ell(a) = 0$. Therefore $a = e$ and we are done.

In the general case, let $x$ be a palindrome. There is a central element $\Delta^N \in A_1^\Gamma$ such that $\Delta^{2N}x \in A_1^\Gamma$. Since $\Delta^{2N}x$ is still a palindrome, the previous argument applies. There is $y \in A_\Gamma$ and $I \subseteq S$ such that $\Delta^{2N}x = y\Delta_I\overline{y}$. Thus $x = \Delta^{-N}y\Delta_I\overline{y}\Delta^{-N}$ is a desired decomposition. \hfill $\blacksquare$

### 3.1.2. Uniqueness

Let $x$ be a positive palindrome. The set

$$M(x) = \left\{ J \subseteq S \mid x = y\Delta_1\overline{\gamma} \text{ for some } y \in A_1^\Gamma \right\}$$

is non-empty and finite. Since the ordering is total, there exists a unique smallest element $\Delta_I$ such that $\Delta_I < \Delta_J$ for all $J \in M(x)$, $J \neq I$. The subset

$$N(x; I) = \left\{ y \in A_1^+ \mid x = y\Delta_I\overline{\gamma} \right\}$$

is nonempty. Since $<$ restricted to $A_1^+$ is a well-ordering, $N(x; I)$ contains a unique smallest element with respect to $<$. Consider now a general palindrome $x \in A_\Gamma$. Then there exists a central element $\Delta^N \in A_1^\Gamma$ such that $\Delta^{2N}x \in A_1^\Gamma$. Clearly, $\Delta^{2N}x = \overline{x}\Delta^{2N} = x\Delta^{2N} = \Delta^{2N}x$. Applying the previous argument, we obtain a canonical decomposition $\Delta^{2N}x = y\Delta_I\overline{y}$ with $(\Delta_I, y)$ minimal among all other such decompositions. Therefore,

$$x = \Delta^{-N}y\Delta_I\overline{y}\Delta^{-N}.$$  \hfill (3.1)

Assume that there is another, distinct decomposition $x = z\Delta_J\overline{z}$. Then we have $\Delta^{2N}x = \Delta^Nz\Delta_J\overline{z}\Delta^N = y\Delta_I\overline{y}$. Since $\Delta^{2N}x = y\Delta_I\overline{y}$ is the canonical decomposition, we have $(\Delta_I, y) < (\Delta_J, \Delta^Nz)$. By left-invariance of $<$, this is equivalent to $(\Delta_I, \Delta^{-N}y) < (\Delta_J, z)$. Hence, the decomposition (3.1) is unique. This is the desired result. \hfill $\blacksquare$

**Remark 3.3.** If $<$ does not restrict to a well-ordering on $A_1^\Gamma$, we can still obtain uniqueness of the decomposition by requiring the length (in Artin generators) of $\gamma$ to be minimal.

### 3.2. Image of the palindromization map

Using Theorem \ref{thm:palindromization}, we must see that a palindrome $x = \gamma\Delta_1\overline{\gamma}$ is pure if and only if $\Delta_I = e$. The anti-automorphisms of $W_\Gamma$, $\textrm{rev} : x \mapsto \overline{x}$ and $x \mapsto x^{-1}$ coincide on images of Artin generators in $W_\Gamma$. Hence they coincide on $W_\Gamma$. Therefore, projecting $x$ to $W_\Gamma$, we have $x = \gamma\Delta_J\gamma^{-1}$ (we abusively keep the same notation for elements in $W_\Gamma$). Now $x \in A_\Gamma$ is pure if and only if $\gamma\Delta_J\gamma^{-1}$ is trivial in $W_\Gamma$. This occurs if and only if $\Delta_I$ is trivial in $W_\Gamma$, hence trivial in $A_\Gamma$ (by Tits’ solution to the word problem). Alternatively, one can verify inductively that
for any $\varnothing \neq J \subseteq S$, $\Delta_J$ is not pure.

\[\text{\textit{Remark 3.4.}}\] The argument above yields a description of the images of palindromes in $W_\Gamma$.

4. Applications

\textbf{Corollary 4.1.} Every element of order at most 2 in $W_\Gamma$ lifts to a palindrome in $A_\Gamma$.

\textbf{Proof.} This is essentially a reformulation of the previous observation (§3.2) coupled with the fact that any element of order 2 is the image of a conjugacy class of $\Delta_I$ for some subset $I$.

The following consequence of Theorem 1.9 yields restrictions on the possible fundamental elements occurring in the canonical decomposition of a palindrome.

\textbf{Proposition 4.2.} Assume that $A_\Gamma$ is equipped with a left-invariant ordering extending the subword order of $A^+_{\Gamma}$. Let $x$ be a palindrome in $A^+_{\Gamma}$ and let $x = y \Delta_I y$ be its canonical decomposition. Then $I = \{s_1\} \cup \{s_2\} \cup \ldots \cup \{s_r\} \subseteq S$ and for all $1 \leq i < j \leq r$, there is no edge in $\Gamma$ between $s_i$ and $s_j$. In particular, $\Delta_I = \prod_{j} s_j$.

\textbf{Proof.} Suppose that $I$ contains two non-commuting Artin generators $s$ and $s'$. Denote by $m_{s,s'}$ the label of the edge between $s$ and $s'$ in the Coxeter diagram. Then $\Delta_{s,s'} = w_{m_{s,s'}}(s,s')$ divides $\Delta_I$. It follows that $\Delta_I = sas$ for some $s \in S$ and $a \in A^+_{\Gamma}$. Since $\overline{\Delta_I} = \Delta_I$, we have $\overline{a} = a$. Applying Theorem 1.9 to $a$, we obtain $a = b \Delta_J b$ for some $b \in A_\Gamma$ and $J$ strictly contained in $I$. Hence $x = a' \Delta_J a'$ with $a' = sb \in A^+_{\Gamma}$ and $\Delta_J$ divides $\Delta_I$. Since the left-ordering of $A_\Gamma$ extends the subword order of $A^+_{\Gamma}$, we deduce that $\Delta_J < \Delta_I$. This contradicts the minimality of the canonical decomposition for $x$.

\textbf{Corollary 4.3.} Let $x = \gamma \Delta_I \gamma$ be the canonical decomposition of $x \in A_\Gamma$. Then $|I|$ is bounded by the number of the maximal subset of $S$ of commuting positive Artin generators.

\textbf{Example 4.4.} If $\Gamma = A_n$, $B_n$ or $D_n$, then $|I| \leq \lfloor \frac{n+1}{2} \rfloor$.

Recall that $\tau(x) = \Delta^{-1} x \Delta$, where $\Delta$ is the fundamental element of $A_\Gamma$.

\textbf{Corollary 4.5.} Let $x \in A_\Gamma$ be a palindrome. There is a decomposition $x = y \Delta_I y$ for some $I \subseteq S$, $y \in A_\Gamma$ such that $\tau(\Delta_I) = \Delta_I$.

\textbf{Proof.} It follows from [2] §7 that if $\tau$ is non trivial then each edge of the Coxeter diagram is labelled by an odd integer. Start with the canonical decomposition $x = y \Delta_I y$. By Proposition 1.2 $\Delta_I = \prod_{j} s_j$ where $\{s_j\}_j$ is a family of commuting positive Artin generators. By definition of $\Delta$ as a left and right least common multiple of $S$, the map $\tau : S \to S$, $s \mapsto \tau(s)$ is a permutation of $S$ of order at most 2. Furthermore, since $\tau$ is a homomorphism, $\tau$ must preserve the Coxeter diagram. Declare a subset $J \subseteq S$ admissible if there exists $z \in A_\Gamma$ such that $x = z \Delta_J z$. Let $J$ be an admissible subset such that there is $s \in J$ commuting with all elements in $J$. Consider the following operations on $J$. 
(A) Adding a positive Artin generator \( s' \) to \( J \) such that \( s' \) commutes with all elements of \( J - \{s\} \) and \( s \) and \( s' \) are joined by a single edge in the Coxeter diagram.

(B) Replacing \( s \in J \) by another positive Artin generator \( s' \in S \) such that \( s' \) commutes with all elements of \( J - \{s\} \) and \( s \) and \( s' \) are joined by a single edge in the Coxeter diagram.

We claim that these operations do not affect admissibility. Denote by \( m_{s, s'} \in Z \) the odd label of the edge between \( s \) and \( s' \). Set \( k = \frac{m_{s, s'} - 1}{2} \). We have

\[
\Delta_{I \cup \{s'\}} = \Delta_{(I - \{s\}) \cup \{s, s'\}} = \Delta_{I - \{s\}} \Delta_{s, s'} = \Delta_{I - \{s\}} w_{m_{s, s'}}(s, s') = \Delta_{I - \{s\}} w_k(s, s') \Delta_{s, s'} = w_k(s, s') \Delta_{I - \{s\}} w_k(s, s').
\]

(We used the fact that \( m_{s, s'} \) is odd in the fourth equality.) Thus the operation (A) does not affect admissibility. The verification is similar for operation (B). There is a sequence \((J_m)_{0 \leq m \leq k}\) of subsets of \( S \) such that \( J_0 = J \), each subset \( J_{k+1} \) is obtained from \( J_k \) by means of one operation (A) or (B) and the final subset \( I = J_m \) satisfies \( \tau(\Delta_I) = \Delta_I \).

**Theorem 4.6.** Let \( x \in A^+_\Gamma \) such that \( \text{rev}(x) = x \) and \( \tau(x) = x \). Then there exists \( I \subseteq S, y \in A^+_\Gamma \) such that

\[
(4.1) \quad x = y \Delta_I y, \quad \tau(y) = y, \quad \tau(\Delta_I) = \Delta_I.
\]

**Proof.** Suppose first that \( x \in A^+_\Gamma \). Lemma 3.4 yields \( x = \Delta S(x) y \) for some \( y \in A^+_\Gamma \). If \( y = e \), we are done. Otherwise, \( F(y) \) is not empty and there is \( s \in F(y) \). Since \( \bar{x} = x, F(y) \subseteq S(x) \). Hence \( s \in S(x) \). Since \( \tau(x) = x \), \( \tau(S(x)) = S(\tau(x)) = S(x) \). Hence \( \tau(\Delta S(x)) = \Delta S(x) \). It follows that \( \tau(y) = y \). We deduce that \( s, \tau(s) \in F(y) \). Apply Lemma 3.4 to \( y \); we have \( x = \Delta_{s, \tau(s)} a \Delta_{s, \tau(s)} \) for some \( a \in A^+_\Gamma \). Since \( \Delta_{s, \tau(s)} \) is both \( \text{rev-} \) and \( \tau-\)invariant, we have \( \tau(a) = a = \bar{a} \). Furthermore, \( \ell(a) < \ell(x) \). So we can apply the argument again to \( a \). This defines a recursive procedure that stops if and only if the middle element \( a \) either trivial or is \( \Delta_I \) for some subset \( I \subseteq S \).

This is the desired result. For the general case, there is a central element \( \Delta^N \in A^+_\Gamma \) such that \( z = \Delta^N x \in A^+_\Gamma \). Clearly \( z \) is \( \text{rev-} \) and \( \tau-\)invariant. The previous argument applies: there is \( y \in A^+_\Gamma \) such that \( z = y \Delta_I y \) for some \( I \subseteq S \). Hence \( x = \Delta^{-N} y \Delta_I y \Delta^{-N} = y' \Delta_I y' \) with \( y' = \Delta^{-N} y \) is a decomposition with the required properties.

**Remark 4.7.** The two cases when the decomposition \( (4.1) \) is unique are \( I = \emptyset \) and \( I = S \), corresponding to Corollary 1.8.

**Remark 4.8.** The subgroup \( A^+_\Gamma \subseteq A^+_\Gamma \) of \( \tau \)-invariant palindromes is again an Artin group of finite type by a result due to J. Michel and P. Dehornoy – L. Paris [S].

5. Further remarks

5.1. **Left-orderings and palindromization.** Let \( A^+_\Gamma \) be an Artin system equipped with a left-ordering \(<\) that has the property (SPPC). It is tempting
to ask whether the one-to-one palindromization map \( \text{pal} : A_\Gamma \to A_\Gamma \), \( x \mapsto x\bar{x} \) is monotonic. It cannot be decreasing since \( \text{pal}(e) = e < x\bar{x} = \text{pal}(x) \) for any \( x > e \).

Below we show that for the classical braid group equipped with Dehornoy ordering, the map \( \text{pal} \) is not increasing. Set \( x = s_1s_2 \) and \( y = s_1^2 \). It follows from the definition that \( x < y \). However, \( \text{pal}(x) > \text{pal}(y) \). Indeed, we have \( \text{pal}(x) = s_1s_2^2s_1 \) and \( \text{pal}(y) = s_1^3 \). We rewrite \( \text{pal}(y)^{-1}\text{pal}(x) \) so as to find a representative which is \( s \)-positive:

\[
\text{pal}(y)^{-1}\text{pal}(x) = s_1^{-4}s_1s_2^2s_1 = s_1^{-3}s_2s_1s_2s_1^{-1} = s_1^{-3}s_2s_1s_2s_1^{-1} = s_1^{-3}s_1s_2s_1s_2s_1^{-1} = s_1^{-2}s_2s_1s_2^{-1} = s_1^{-2}s_2s_1s_2^{-1} \]

is \( s_1 \)-positive. Thus \( \text{pal}(x) > \text{pal}(y) \).

5.2. Artin groups of infinite type. How much from the previous results remain true for Artin groups of infinite type? It is an open problem to determine which ones are left-orderable (a fortiori to determine whether there is a left-ordering that has (SPPC)). On the other hand, Jacquier’s algorithm is valid for all Artin groups. The key lemma of [4] can be extended to all Artin monoids. Since all Artin monoids embed naturally into their groups [15], the palindromization map is one-to-one for all Artin groups.

5.3. Garside groups. Garside groups are a generalization of Artin groups of finite type [5]. By means of the techniques used in this paper (elementary divisibility theory and rewriting), the decomposition for palindromes admits a generalization to Garside groups (no uniqueness in general). However, the palindromization map is not one-to-one in general. The simplest example is provided by the Garside group

\[ G = \langle x, y \mid x^2 = y^2 \rangle. \]

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