Uniqueness Solution Of Abstract Fractional Order Nonlinear Dynamical Control Problems

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Abstract. The aim of this paper is to investigate the Uniqueness solution of Abstract Cauchy Problem represented for fractional order nonlinear dynamical control system involving certain control input and their approach of investigated depended on commutative composite semigroup and some certain conditions in certain space.

1. Introduction
The semilinear and nonlinear equations appearing in variety of theories and applications in particular in the theory of fractional ordinary and fractional partial differential equations as well as integral equations with different types of derivatives have recently been addressed by several researchers for different problems and provided excellent tool for the description of memory and hereditary properties of various materials and processes.

In [12],[14],[15],[17],[19],[20], the authors had been studied some classes of nonlinear and semiliner equation without ordinary or fractional derivatives with projectively compact and which among others contains completely continuous, quasi compact and monotone operators with general fixed point theorems as well as the nonlinear and semilinear equation studied with closed linear operator in Hilbert space, self adjoint operator also some time with perturbed operator that has densely defined domain in Banach space, moreover studied with monotonicity and compactness of the linear operator on reflexive Banach space, the strongly positive operator and maximal monotonicity linear operator with nonlinear functions presented with existence and uniqueness approach.

In [7],[1],[11],[21],[16],[5],[6],[13],[2], the authors had been studied the solvability of fractional order nonlinear and semilinear control differential equations by using fractional integral formulation with properties of calculus of fractional derivative and integration and the existence and uniqueness obtained by using classical fixed point theorems with initial values as well as boundary values and integral boundary condition also some of them involving nonlocal initial condition.

Our intersect in this paper to study the fractional order nonlinear dynamical feedback control system involve sum of N- unbounded operators with feedback perturbation as a generators of N-semigroup with new definitions depended on no expansive prosperity, maximal accretive, maximal monotone, resolvent set, fractional derivative and fixed point theorem also presented some results for solvability without using fractional calculus and equivalent integral formulation. main interest on nonlinear functional analysis and some new properties defined on special space,

\[ L^\alpha_2(0,T) = \{ x: x \in L_2[0,T], \mathcal{D}^\alpha x \in L_2[0,T] \}, \quad T>0. \]

Also appear the role of feedback control operator as a perturbation for the generators still a challenge for many researchers up to our knowledge.
Our aim establish necessary and sufficient conditions on sum of nonlinearity operator interacts suitably their system:

$$\sum_{i=1}^{n} F_i (t, x, D_{\alpha}^{x} x) = \sum_{i=1}^{n} A_i x + \sum_{i=1}^{n} B_i u_i$$  \hspace{1cm} (1)

$$u_i = K_i (x) \text{ for all } x \in \cap_{i=1}^{n} D(A_i)$$  \hspace{1cm} (2)

Where $A_i : D(A_i) \subseteq L^2 \rightarrow L^2, i = 1, 2, 3, \ldots, n$ are linear unbounded operators generators of $C_0$-semigroups $T_i (t) : L^2 \rightarrow L^2, i = 1, 2, \ldots, n , 0 < \infty \leq 1, B_i : D(B_i) \subseteq L^2 \rightarrow L^2$. $F_i : R_0^+ \times L^2 \times L^2 \rightarrow L^2, \quad i = 1, 2, \ldots, n$, are nonlinear operators. The input control functions $u_i (.) \in L^2 [0, T]$ such that $K_i : L^2 \rightarrow L^2$ is a feedback linear operators, $i = 1, 2, \ldots, n$.

2. Preliminaries

Some necessary mathematical concepts for semigroup theory as well as some non-linear fractional calculus concepts have been presented.

**Definition (2.1), [18]:**
The family of bounded linear operators $T(t), 0 \leq t < \infty$ defined on the Banach space $X$ is a semigroup if $T(0) = I$ is identity operator on $X$, and $T(t) = T(s)T(t)$ for every $t, s \geq 0$.

**Definition (2.2), [18]:**
Let $T(t)$ be a semigroup then $T(t)$ is called strongly continuous and which denoted by $C_0$ on a Banach space $X$ if $\lim_{t \to 0} \|T(t)x - Ix\|_X = 0$.

**Definition (2.3), [18]:**
The domain of the linear operator $A$ is defined as follows:

$$D(A) = \{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \}$$  \hspace{1cm} (3)

$$A x = \frac{d}{dt} \bigg|_{t=0} T(t)x - x$$

**Remarks (2.4), [18]:**
There exists a constant $w \geq 0$, such that $\|T(t)\|_{X} \leq Me^{\omega t}$ for $M \geq 1$.

The family of linear operator $T(t)$ is differentiable which is $\frac{dT(t)}{dt} = AT(t) = T'(0)A$.

**Lemma (2.5), [18]:**
A bounded linear operator $A$ is the generator of a uniformly continuous semigroup.

A strongly continuous semigroup of bounded linear operators on a Banach space $X$ will be called a semigroup of class $C_0$.

**Theorem (2.6), [3]:**
A linear (unbounded) operator $A$ is the generator of a strong semigroup of contraction family $\{T(t)\}_{t \geq 0}$ if and only if:

(i) $A$ is closed and densely defined, and

(ii) The resolvent set $\rho(A)$ of $A$ contains $R^+$ and $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ for every $\lambda > 0$.

**Remark (2.7), [18]:**
1. If $B$ is a bounded linear operator on $X$, then $A+B$ with $D(A+B) = D(A)$ is the generator of $C_0$-semigroup $S(t)$ on $X$, satisfying $\|S(t)\| \leq Me^{(w+M||B||)t}$ for $t \geq 0$.

2. For $x \in X, \lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} T(s)x \, ds = T(t)x, h \in (0, t)$. 

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Definitions (2.8), [22]:
1. Let $X$ be a real Banach space and let $A:X\to X^*$ be an operator. Then $A$ is called monotone if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x,y \in X$.
2. Assume operator $A:D(A)\subseteq H \to H$ defined on real Hilbert space $H$.
   a. $A$ is called maximal monotone if $A$ is monotone and $\langle b - Ay, x - y \rangle \geq 0$ for $y \in D(A)$. Implies $Ax=b$ which is $A$ has no proper monotone extension.
   b. $A$ is accretive if $(I+\mu A):D(A)\to H$ is injective also $(I+\mu A)^{-1}$is nonexpansive for $\mu > 0$.
   c. $A$ is maximal accretive if $A$ is accretive also $(I+\mu A)^{-1}$exists on $H$ for $\mu > 0$.

Definition (2.9), [8]:
The For a function $g:[0,\infty)\to \mathbb{R}$, the Caputo derivative of fractional order $\alpha$ is defined as
$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds, \quad n-1 < \alpha < n,$$
where $\Gamma$ denotes the gamma function.

Definition (2.10), [10]:
The Riemann-Liouville fractional integral of order $\alpha$ for a function $g$ is defined as
$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds, \quad \alpha > 0,$$
provided the right hand side is pointwise defined on $(0,\infty)$.

Lemma(2.11),[22]:
Let an operator $A:D(A)\subseteq H \to H$ on the real Hilbert space $H$. the statements are equivalent:
A is monotone and $R(I-A) = H \iff A$ is maximal monotone.

Lemma(2.12),[22]:
Let a linear operator $A:D(A)\subseteq H \to H$ on real Hilbert space $H$
1. $A$ is the generator of a linear nonexpansive semigroup.
2. $-A$ maximal accretive and $\overline{D}(\overline{A}) = X$.

Lemma(2.13),[4]:
Let $A$ be the generator of $C_0$- semigroup of contraction (nonexpansive semigroup) on a Banach space $X$. A bounded linear operator $B$ is a perturbation of $A$ such that $D(A) \subseteq D(B)$ and
i. Let $F$ denoted the duality on $Y$ Banach space to $Y^*$ defined as
$$F(g) = \{ y \in Y, \langle y, g \rangle = ||g||^2 = ||y||^2 \}$$
So for every $x \in D(\lambda I - (A+B))$ there is $g \in F\left( (\lambda I - (A+B))x \right)$, for every $y \in Y$, thus $\langle (\lambda I - (A+B))x, y \rangle \geq \langle -B||I||, g \rangle$.

ii. $c ||(\lambda I - (A+B))^{-1}|| + a ||(\lambda I - (A+B))^{-1}|| + b < 1$

iii. $\lambda > \||B||$.Then $A+B$ is the generator of $C_0$-semigroup of contraction (nonexpansive semigroup) in $X$.

Lemma(2.14),[22]:
Let the mapping $A,B:X \to X^*$ be maximal monotone on the real reflexive Banach space $X$ (where $X^*$ is the dual space of $X$) and let $D(A) \cap \text{int}D(B) \neq \emptyset$. Then the sum $A+B:X \to X^*$ is also maximal monotone.

Lemma(2.15),[9]:
Let $f$ be a contraction on complete metric space $X$. Then $f$ has a unique fixed point $\overline{x} \in X$.
Our problem investigated on the following space that which denoted by $L^2_{\alpha}$,
$$L^2_{\alpha}([0,T]) = \{ x: x \in L^2_{\alpha}[0,T], \quad \mathcal{D}^\alpha x \in L^2_{\alpha}[0,T], \quad 0 < \alpha \leq 1 \}.$$
2. Main Results:

Lemma (3.1):

Let \( A_i + B_i K_i: D(A_i) \subseteq L^p_2 \to L^q_2, i = 1, 2, 3, \ldots, n \) are linear unbounded operators generators of \( C_0 \)-semigroups \( S_i(t): L^p_2 \to L^q_2 \), respectively. Then from Remark (2.8), we have that

\[
\int_0^\infty (\lambda - \sum_{i=1}^n \|B_i K_i\|)^{-1} I - \sum_{i=1}^n (A_i + B_i K_i)^{-1} \leq \lambda \sum_{i=1}^n \|B_i K_i\| , \text{forall } \lambda \geq \sum_{i=1}^n \|B_i K_i\| \tag{4}
\]

where \( \sum_{i=1}^n \|B_i K_i\| = \frac{1}{\sum_{i=1}^n \|B_i K_i\|} \).

Proof:

From Lemma (2.12), we have that \(- (A_i + B_i K_i)\) are a maximal monotone for \( i = 1, \ldots, n \). Since \( D(\sum_{i=1}^n (A_i + B_i K_i) \cap \text{Int } D(A_n + \sum_{i=1}^n B_i K_n) = D(\sum_{i=1}^n (A_i + B_i K_i)) \subseteq H \), a maximal monotone, then by Lemma (2.13) and Definition (2.9), we get

\[
\left| \frac{1}{\lambda - \sum_{i=1}^n \|B_i K_i\|} I - \sum_{i=1}^n (A_i + B_i K_i)^{-1} \right| \leq 1 \quad \text{for } \lambda > \sum_{i=1}^n \|B_i K_i\| \tag{5}
\]

Hence,

\[
\left| (\lambda - \sum_{i=1}^n \|B_i K_i\|)^{-1} I - \sum_{i=1}^n (A_i + B_i K_i)^{-1} \right| \leq \lambda - \sum_{i=1}^n \|B_i K_i\| , \text{for } \lambda > \sum_{i=1}^n \|B_i K_i\| \tag{6}
\]

Lemma (3.2):

Let \( A_i + B_i K_i: D(A_i) \subseteq L^p_2 \to L^q_2, i = 1, 2, 3, \ldots, n \) be linear unbounded operators generators of \( C_0 \)-semigroups \( S_i(t): L^p_2 \to L^q_2, i = 1, 2, 3, \ldots, n \), respectively. Then by Lemma (2.15), we have that

\[
\left| \frac{1}{\lambda - \sum_{i=1}^n \|B_i K_i\|} I - \sum_{i=1}^n (A_i + B_i K_i)^{-1} \right| \leq 1 \quad \text{for } \lambda > \sum_{i=1}^n \|B_i K_i\| \tag{7}
\]

and for \( \lambda > \|B_i K_i\| \), we get \( \frac{\lambda}{\lambda - \sum_{i=1}^n \|B_i K_i\|} \to \sum_{i=1}^n \frac{\|B_i K_i\|}{n} = \sum_{i=1}^n \|B_i K_i\| \) then by Lemma (2.15) we have that
\[-\sum_{i=1}^{n}(A_i + B_i K_i) : D(\sum_{i=1}^{n}(A_i + B_i K_i)) \subseteq H \rightarrow H\]
is also a maximal monotone, then by lemma (3.1), we have that
\[
\left\| \left( I - \left( \lambda - \sum_{i=1}^{n} \| B_i K_i \| \right)^{-1} \right) \sum_{i=1}^{n}(A_i + B_i K_i) \right\|^{-1} \leq 1 \text{ for } \lambda > \sum_{i=1}^{n} \| B_i K_i \| .
\]
Thus,
\[
\left\| \left( \lambda - \sum_{i=1}^{n} \| B_i K_i \| \right) I - \sum_{i=1}^{n}(A_i + B_i K_i) \right\|^{-1} \leq 1 \text{ for } \lambda > \sum_{i=1}^{n} \| B_i K_i \| .
\]
Hence,
\[
\left\| \left( \lambda - \sum_{i=1}^{n} \| B_i K_i \| \right) I - \sum_{i=1}^{n}(A_i + B_i K_i) \right\| \leq \left( \lambda - \sum_{i=1}^{n} \| B_i K_i \| \right)^{-1} , \text{ for } \lambda > \sum_{i=1}^{n} \| B_i K_i \| .
\]

**Lemma (3.3):**

Let \( F_i : R^2 \times L_2 \times L_2 \rightarrow L_2^a \), \( i = 1, 2, ..., n \), are nonlinear operators satisfy the following

1. \( \langle F_i(t,x,D_a^x - x) - F_i(t,y,D_a^x y), y - x \rangle \geq m_i \| x - y \| + \| D_a^x - D_a^y \| \| x - y \| \)
2. \( \langle D_a^x F_i(t,x,D_a^x x) - D_a^y F_i(t,y,D_a^y y), y - x \rangle \geq m_i \| x - y \| + \| D_a^x x - D_a^y y \| \)
   for all \( x, y \in H \) and some \( m_i > 0; m^* = \min \{ m_i, i = 1, ..., n \} \),
3. \( \| F_i(t,x,D_a^x x) - F_i(t,x,D_a^x x) \|_{L_2^a} \leq K_i \| x - y \| + \| D_a^x x - D_a^y y \| \leq K_i \| x - y \| \) for all \( x, y \in H \), and some \( K_i > 0 \).
4. \( K = \min \{ K_i, i = 1, ..., n \} \)

Then there exists an interval of \( \lambda \) such that \( \sum_{i=1}^{n} \| B_i K_i \| < \lambda < \min \{ \frac{2m}{nk^2} + \sum_{i=1}^{n} \| B_i K_i \|, \frac{2m^*}{nk^2} + \sum_{i=1}^{n} \| B_i K_i \| \} \) for some \( m^*, k^* > 0 \), \( n \in N \) such that \( S_i : L_2^a \rightarrow L_2^a \)
\[
S_i(x) = x - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x x))
\]
\[
D_a^x S_i(x) = x - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} D_a^x F_i(t_1,x,D_a^x x))
\]
\( S_i(x) \) is a contraction operator in \( L_2^a \) space.

**Proof:**

We have
\[
\| S_i(x) - S_i(y) \|^2 = \langle x - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,x,D_a^x x) - y - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y), x - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,x,D_a^x x) - y - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) \rangle
\]
\[
= \langle x - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x x) - y - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y), x - y - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x x) - y - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) \rangle
\]
\[
+ \langle x - y, (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x x) + (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y), x - y \rangle
\]
\[
+ (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) \rangle
\]
\[
+ (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) \rangle
\]
\[
(x - y) = \langle x - y, (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x x) + (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y), x - y \rangle
\]
\[
+ (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t_1,x,D_a^x y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) - (\lambda - \sum_{i=1}^{n} \| B_i K_i \|) \sum_{i=1}^{n} F_i(t,y,D_a^y y) \rangle
\]
Let \( \|B_i, K_i\| \subseteq \sum_{i=1}^{n} F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} F_i(t, x, D^\infty_alpha x) \)

Thus,

\[
\|S_j(x) - S_j(y)\|^2 = \|x - y\|^2 - 2(\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} D^\infty_alpha F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} D^\infty_alpha F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} D^\infty_alpha F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} F_i(t, x, D^\infty_alpha x)
\]

Then by (11), we have that

\[
\|D^\infty_alpha S_j(x) - D^\infty_alpha S_j(y)\|^2 = \|D^\infty_alpha(x - y)\|^2 - 2(\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} D^\infty_alpha F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} D^\infty_alpha F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} D^\infty_alpha F_i(t, x, D^\infty_alpha x) - (\lambda - \sum_{i=1}^{n} \|B_i, K_i\|) \sum_{i=1}^{n} D^\infty_alpha F_i(t, x, D^\infty_alpha x)
\]

From conditions (1-4), we obtain

\[
\|S_j(x) - S_j(y)\| \leq \left( 1 - 2(\lambda - \sum_{i=1}^{n} \|B_i, K_i\|)\right)^{\frac{1}{2}} + \left( 1 - 2(\lambda - \sum_{i=1}^{n} \|B_i, K_i\|)\right)^{\frac{1}{2}} < 1
\]

Thus,\[
\|D^\infty_alpha S_j(x) - D^\infty_alpha S_j(y)\| < \|x - y\| \Rightarrow S_j(x) \text{ is a contraction operator in } L^2_alpha space.
\]

Consider the following semilinear sum of N-perturbed unbounded operators equations discussed in the following equations.

**Theorem (3.4):** Let \( A_i + B_i K_i : D(A_i) \subseteq L^2 \rightarrow L^2, i = 1, 2, ..., n \), are linear unbounded operators generators of \( c \)-semigroups \( S_i(t) : L^2 \rightarrow L^2, i = 1, 2, ..., n \), respectively and \( B_i D(A_i) \subseteq L^2 \rightarrow L^2 \) satisfies the following condition for every \( x \in D(\lambda I - (A_1 + B_1 K_1)) \) there is \( g \in F(\lambda I - (A_1 + B_1 K_1)x) \) such that

\[
\langle (\lambda I - (A_1 + B_1 K_1)x), g \rangle \geq c\|x\|^2 - A\|D^\infty alpha x - D^\infty alpha y\|<0
\]

Hence, \( S_j(x) \) is contraction operator in \( L^2_alpha space. \)
such that

\[ \lambda > \| B_i K_i \| \]

for all \( x \in H \) and some \( m_i > 0 \); \( m^* = \min \{ m_i, i = 1, \ldots, n \} \),

3. \( \| F_i(t, x, D^{n}_a x) - F_i(t, x, D^{n}_a x) \|_{L^2} \leq K_i(\| x - y \| + \| D^{n}_a x - D^{n}_a y \|) \)

\[ \leq K(\| x - y \| + \| D^{n}_a x - D^{n}_a y \|) \leq K^* x - y \|_{L^2} \]

for all \( x, y \in L^2 \) \( x, y \in H \), and some \( K_i > 0 \), \( K = \min \{ K_i, i = 1, \ldots, n \} \) for all \( x, y \in H \).

4. \( \| D^{n}_a F_i(t, x, D^{n}_a x) - D^{n}_a F_i(t, x, D^{n}_a x) \|_{L^2} \leq K_i(\| x - y \| + \| D^{n}_a x - D^{n}_a y \|) \leq K^* x - y \|_{L^2} \)

for all \( x, y \in H \) and some \( m_i > 0 \); \( m^* = \min \{ m_i, i = 1, \ldots, n \} \),

Proof:

The Equation (12) can be written as

\[
(1 - (\lambda - \sum_{i=1}^{n} [B_i K_i]) x - (\lambda - \sum_{i=1}^{n} [B_i K_i]) \sum_{i=1}^{n} A_i x = 0 \]

for \( \lambda > \sum_{i=1}^{n} [B_i K_i] \) and \( x \in H \). Or

\[
-(\lambda - \sum_{i=1}^{n} [B_i K_i]) x = S_k(x), \text{ for } \lambda > \sum_{i=1}^{n} [B_i K_i] \text{ and } x \in H. \quad (13)
\]

Where \( S_k(x) = x - (\lambda - \sum_{i=1}^{n} [B_i K_i]) \sum_{i=1}^{n} F_i(t, x, D^{n}_a x) \).

for \( \lambda > \sum_{i=1}^{n} [B_i K_i] \) and \( x \in H. \)

\[
(I(\lambda - \sum_{i=1}^{n} [B_i K_i])\bigg) \bigg(\lambda - \sum_{i=1}^{n} (A_i + B_i K_i)\bigg) x = (\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} S_k(x) \quad (\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} > 0
\]

From lemma (2.15), we have

\( x = (\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} (\lambda - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} S_k(x) \quad (14) \)

To show that, \( (\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} (\lambda - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} S_k(x) \) is a contraction operator

\( (\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} (\lambda - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} S_k(x) - \)

\[
(\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} (\lambda - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} S_k(y) \leq \lambda - \sum_{i=1}^{n} [B_i K_i]^{-1} \bigg((\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} (\lambda - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} \bigg)(S_k(x) - S_k(y))
\]

By lemmas (3.1) and equation (3.2), we get

\( \lambda - \sum_{i=1}^{n} [B_i K_i]^{-1} (\lambda - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} S_k(x) - \)

\[
((\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1} (\lambda - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} S_k(y)) \leq \bigg((\lambda - \sum_{i=1}^{n} [B_i K_i])^{-1}\bigg)(\lambda - \sum_{i=1}^{n} [B_i K_i] b \| x - y \|
\]

Then the following equation

\[
\sum_{i=1}^{n} F_i(t, x, D^{n}_a x) = \sum_{i=1}^{n} (A_i + B_i K_i) x + \sum_{i=1}^{n} B_i u_i
\]

\( u_i = K_i x, \text{ for all } x \in \bigcap_{i=1}^{n} D(A_i) \) (12)

has an unique solution.
Hence, 
\[
(\lambda - \sum_{i=1}^{n} ||B_{i}K_{i}||)^{-1} \left( (\lambda - \sum_{i=1}^{n} ||B_{i}K_{i}||)^{-1} \mathbf{1} - \sum_{i=1}^{n} (A_{i} + B_{i}K_{i}) \right)^{-1} \mathbf{S}_{\lambda}(x) - (\lambda - \sum_{i=1}^{n} ||B_{i}K_{i}||)^{-1} \sum_{i=1}^{n} (A_{i} + B_{i}K_{i})^{-1} \mathbf{S}_{\lambda}(y) \leq b ||x - y||
\] (15)

From lemma (3.2), we have
\[
b = (1 - 2(\lambda - \sum_{i=1}^{n} ||B_{i}K_{i}||)(nm') + (\lambda - \sum_{i=1}^{n} ||B_{i}K_{i}||)^{2}(nK^{*})^{2})^{-1} < 1
\]
for \(\sum_{i=1}^{n} ||B_{i}K_{i}|| < \lambda < \frac{2nm'}{nK^{2}} + \sum_{i=1}^{n} ||B_{i}K_{i}||\) by theorem (1.7.8), we have that

\[
(\lambda - \sum_{i=1}^{n} ||B_{i}K_{i}||)^{-1} - \sum_{i=1}^{n} (A_{i} + B_{i}K_{i})^{-1} \mathbf{S}_{\lambda}(x)
\]
has an unique fixed point, thus (14) and consequently (12) has an unique solution.

Definition (3.5):
Let \(X\) be a real separable Banach space a one-parameter family \(\{S_{a}(t)S_{n-1}(t)\}_{t} \subseteq L(X), t \in [0, \infty)\) of a perturbed \(C_{a}\)-semigroups of bounded linear operators \(\{S_{a}(t)\}_{t} \subseteq L(X)\) are commutative and generated by \((A_{i} + B_{i}K_{i})\) for \(i = 1, ..., n\) respectively and \(t \in [0, \infty)\) is called commutative perturbed semigroup if

1. \(S_{a}(0)S_{n-1}(0)S_{0}(0) = I\) (\(I\) is the identity operator on \(X\)).
2. \(S_{a}(t + s)S_{n-1}(t + s) = \left(S_{a}(t)S_{n-1}(t)\right)\left(S_{a}(s)S_{n-1}(s)\right)\) for every \(t, s \geq 0\).

Definition (3.6):
The generator \(\sum_{i=1}^{n}(A_{i} + B_{i}K_{i})\) of a semigroup of commutative composite perturbed semigroups \(\{S_{a}(t)S_{n-1}(t)\}_{t} \subseteq L(X)\), defined as the Limit

\[
\sum_{i=1}^{n}(A_{i} + B_{i}K_{i}) = \lim_{t \to 0} \frac{S_{a}(t)S_{n-1}(t)-I}{t}, x \in D(\sum_{i=1}^{n}(A_{i} + B_{i}K_{i})) = D(A_{1} + B_{1}K_{1}) \cap D(A_{2} + B_{2}K_{2}) \cap \cdots \cap D(A_{n} + B_{n}K_{n}) = D(A_{1}) \cap D(A_{2}) \cdots \cap D(A_{n})
\]
where \(D(\sum_{i=1}^{n}(A_{i} + B_{i}K_{i})) \subseteq X\) is a domain of \(\sum_{i=1}^{n}(A_{i} + B_{i}K_{i})\) has a countable subset which is dense in \(X\) and defined as follows

\[
D(\sum_{i=1}^{n}(A_{i} + B_{i}K_{i})) = \{x \in X: \lim_{t \to 0} \frac{S_{a}(t)S_{n-1}(t)-I}{t} \text{ exist in } X\}
\]

Lemma (3.7):
Let \(H\) be a real separable Hilbert space, and
\[
\sum_{i=1}^{n}(A_{i} + B_{i}K_{i}) : \cap_{i=1}^{n} D(A_{i}) \subseteq H \to H
\]
be a generator of a semigroup of a commutative composite perturbed semigroups. Then

\[
\left\| (\lambda I - \sum_{i=1}^{n}(A_{i} + B_{i}K_{i}))^{-1} x \right\| \leq \frac{1}{\lambda - \sum_{i=1}^{n} ||B_{i}K_{i}||} \left\| x \right\| \quad \text{and} \quad x \in \cap_{i=1}^{n} D(A_{i}) \quad \text{for} \quad i = 1, ..., n.
\] (16)

Proof:
\[
F_{\lambda}x = L(S_{a}(t)S_{n-1}(t) \cdots S_{1}(t)x) = \int_{0}^{\infty} e^{-\lambda t} S_{a}(t)S_{n-1}(t) \cdots S_{1}(t)x \, dt, \text{ for } \lambda > \sum_{i=1}^{n} ||B_{i}K_{i}||
\]
and \(x \in X\).

Since \(t \to S_{i}(t)x\) are continuous for \(i = 1, 2, ..., n\) the integral exists and defines a bounded linear operator \(F_{\lambda}\) satisfying

\[
\left\| F_{\lambda}x \right\| \leq \int_{0}^{\infty} e^{-\lambda t} \left\| S_{a}(t)S_{n-1}(t) \cdots S_{1}(t)x \right\| \, dt
\]
\[
\left\| F_{\lambda}x \right\| \leq \int_{0}^{\infty} e^{-\lambda t} \left\| S_{a}(t) \right\| \left\| S_{n-1}(t) \right\| \cdots \left\| S_{1}(t) \right\| \left\| x \right\| \, dt
\]

But

\[
\left\| S_{i}(t) \right\| \leq e^{||B_{i}K_{i}||t}, \text{ for } i = 1, 2, ..., n,
\]
then

\[
\left\| F_{\lambda}x \right\| \leq \int_{0}^{\infty} e^{-\lambda t} e^{||B_{n}K_{n}||t} \cdots e^{||B_{1}K_{1}||t} \left\| x \right\| \, dt
\]
\[
\left\| F_p(\lambda) x \right\| \leq \int_0^\infty e^{-(\lambda - \sum_{i=1}^n \|B_i K_i\|)t} \left\| x \right\| dt \\
\left\| F_p(\lambda) x \right\| \leq \frac{1}{\lambda - \sum_{i=1}^n \|B_i K_i\|} \left\| x \right\|
\]
(18)

Furthermore, for \( h > 0 \)
\[
S_n(h)S_{n-1}(h)...S_1(h) - I \rightarrow F(\lambda)x = S_n(h)S_{n-1}(h)...S_1(h) - I \int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t)...S_1(t) x dt
\]
\[
= \frac{1}{h} \int_0^\infty e^{-\lambda h t} \left( (S_n(h)S_{n-1}(h)...S_1(h))(S_n(t)S_{n-1}(t)...S_1(t))x - S_n(t)S_{n-1}(t)...S_1(t)x \right) dt
\]

Since \( S_j(t) \) are commutative then
\[
= \frac{1}{h} \int_0^\infty e^{-\lambda h t} \left( S_n(t+h)S_{n-1}(t+h)...S_1(t+h)x - S_n(t)S_{n-1}(t)...S_1(t)x \right) dt
\]
\[
= \frac{1}{h} \int_0^\infty e^{-\lambda h t} \left( S_n(t+h)S_{n-1}(t+h)...S_1(t+h)x - \frac{1}{h} \int_0^\infty e^{-\lambda h t} S_n(t)S_{n-1}(t)...S_1(t) x dt \right) dt
\]

Let \( \delta = t + h \rightarrow d\delta = dt \) if \( 0 \leq t \leq \infty \) then \( h \leq \delta \leq \infty \), we get
\[
= \frac{1}{h} \int_0^\infty e^{-\lambda \delta t} S_n(\delta)S_{n-1}(\delta)...S_1(\delta) x d\delta - \frac{1}{h} \int_0^\infty e^{-\lambda h t} S_n(t)S_{n-1}(t)...S_1(t) x dt
\]
\[
= \frac{e^{-\lambda h t}}{h} \int_0^\infty e^{-\lambda \delta t} S_n(\delta)S_{n-1}(\delta)...S_1(\delta) x d\delta - \frac{e^{-\lambda h t}}{h} e^{-\lambda h t} S_n(t)S_{n-1}(t)...S_1(t) x dt
\]
\[
- \frac{1}{h} \int_0^\infty e^{-\lambda \delta t} S_n(\delta)S_{n-1}(\delta)...S_1(\delta) x d\delta
\]
We get
\[
= \frac{e^{-\lambda h t}}{h} \int_0^\infty e^{-\lambda \delta t} S_n(\delta)S_{n-1}(\delta)...S_1(\delta) x d\delta - \frac{e^{-\lambda h t}}{h} e^{-\lambda h t} S_n(t)S_{n-1}(t)...S_1(t) x dt
\]
(19)

As \( h \downarrow 0 \), from (17) and remarks (2.4) the right-hand side of (19) converges to \( \lambda F_p(\lambda)x - x \).

This implies that for every \( x \in H \) and \( \lambda > 0, F_p(\lambda)x \in D(\sum_{i=1}^n (A_i + B_i K_i)) \) and \( \sum_{i=1}^n (A_i + B_i K_i) F_p(\lambda) = \lambda F_p(\lambda) - I \)
or
\[
(\lambda I - \sum_{i=1}^n (A_i + B_i K_i)) F_p(\lambda) = I
\]
(20)

For \( x \in D(\sum_{i=1}^n (A_i + B_i K_i)) \) we have
\[
F_p(\lambda) \sum_{i=1}^n (A_i + B_i K_i) x = \int_0^\infty e^{-\lambda t} S_n(t)S_{n-1}(t)...S_1(t) x dt
\]
(21)

From remarks (2.4), the Equation (21) become
\[
F_p(\lambda) \sum_{i=1}^n (A_i + B_i K_i) x = \int_0^\infty e^{-\lambda t} \sum_{i=1}^n (A_i + B_i K_i) S_n(t)S_{n-1}(t)...S_1(t) x dt
\]
(22)

From (20) and (22) it follows that
\[
F_p(\lambda) (\lambda I - \sum_{i=1}^n (A_i + B_i K_i)) x = x \text{ for } x \in D(\sum_{i=1}^n (A_i + B_i K_i))
\]
Thus, \( F_p(\lambda) \) is the inverse of \( \lambda I - \sum_{i=1}^n (A_i + B_i K_i) \), it exists for all \( \lambda > \sum_{i=1}^n \|B_i K_i\| \).

**Theorem (3.8):**

Let \( H \) be a real separable Hilbert space, \( \sum_{i=1}^n (A_i + B_i K_i) \) be a generator of commutative composite perturbed semigroup \( \{S_n(t), S_{n-1}(t)..., S_1(t), S_{n-1}(t)..., S_1(t), S_{n-1}(t)..., S_1(t)\}_{t=0}^{\infty} \) and \( F_i: H \rightarrow H, i = 1, 2, ..., n \) are nonlinear operators and there exist \( m_i, k_i > 0, i = 1, 2, ..., n \) such that

1. \( \langle F_i(t_1, x, D_{\alpha}^2 x) - F_i(t_2, y, D_{\alpha}^2 y), x - y \rangle \geq m_i \| x - y \| \) for all \( x, y \in H \) and some \( m_i > 0 \);
2. \( \| F_i(t_1, x, D_{\alpha}^2 x) - F_i(t_2, x, D_{\alpha}^2 x) \| \leq K_i(\| t_1 - t_2 \| + \| x - y \| + \| D_{\alpha}^2 x - D_{\alpha}^2 y \|) \)
\[
\sum_{i=1}^{n} F_i (t, x, D_a^x x) = \sum_{i=1}^{n} A_i x + \sum_{i=1}^{n} B_i u_i, x = K_i x, \text{ for all } x \in \bigcap_{i=1}^{n} \mathcal{D}(A_i)
\]

has an unique solution.

**Proof:**

The Equation (23) can be equivalently written as

\[
(\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i)) x = \sum_{i=1}^{n} F_i (t, x, D_a^x x), x = 0,
\]

or

\[
\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i) x = T_\lambda (x)
\]

Where \(T_\lambda (x) = \lambda x - \sum_{i=1}^{n} F_i (t, x, D_a^x x)\), we have

\[
\|T_\lambda (x) - T_\lambda (y)\|^2 = (\lambda x - \sum_{i=1}^{n} F_i (t, x, D_a^x x) - (\lambda y - \sum_{i=1}^{n} F_i (t, y, D_a^x y)), \lambda x - \sum_{i=1}^{n} F_i (t, x, D_a^x x) - (\lambda y - \sum_{i=1}^{n} F_i (t, y, D_a^x y))
\]

\[
= (\lambda x - \lambda y, \lambda x - \lambda y) - (\lambda x - \lambda y, \sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y)) - (\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \lambda x - \lambda y) + (\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y))
\]

\[
\|T_\lambda (x) - T_\lambda (y)\|^2 = 2\|\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \lambda x - \lambda y) + (\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y))
\]

\[
\|T_\lambda (x) - T_\lambda (y)\|^2 = 2\|\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \lambda x - \lambda y) + (\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y))
\]

\[
\|T_\lambda (x) - T_\lambda (y)\|^2 = 2\|\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \lambda x - \lambda y) + (\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y))
\]

\[
\|T_\lambda (x) - T_\lambda (y)\|^2 = 2\|\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \lambda x - \lambda y) + (\sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y), \sum_{i=1}^{n} F_i (t, x, D_a^x x) - \sum_{i=1}^{n} F_i (t, y, D_a^x y))
\]

From conditions(1)(2), we obtain

\[
\|T_\lambda (x) - T_\lambda (y)\|_{L^2} \leq \left( \lambda^2 - 2\lambda (nm) + (nK)^2 \right)^{1/2} + \left( \lambda^2 - 2\lambda (nm) + (nK)^2 \right)^{1/2} \left\| x - y \right\|_{L^2}
\]

(25)

From lemma (3.4.36) the operator \(\sum_{i=1}^{n} (A_i + B_i K_i)\) is generator of a family of linear commutative composite perturbed semigroup.

Then the operator \(\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i)\) is invertible and

\[
\left\| \left( \lambda I - \sum_{i=1}^{n} (A_i + B_i K_i) \right)^{-1} \right\| \leq \left( \lambda - \sum_{i=1}^{n} \| B_i K_i \| \right)^{-1}, \text{for } \lambda > \sum_{i=1}^{n} \| B_i K_i \|
\]

(26)

Now, Equation (24) is equivalent with

\[
x = (\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} T_\lambda (x)
\]

(27)

To show that \(x = (\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} T_\lambda (x)\) is a contraction operator

\[
\left\| \left( \lambda I - \sum_{i=1}^{n} (A_i + B_i K_i) \right)^{-1} T_\lambda (x) - (\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} T_\lambda (y) \right\|
\]

\[
= \left\| (\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} (T_\lambda (x) - T_\lambda (y)) \right\|
\]

\[
\leq \left\| (\lambda I - \sum_{i=1}^{n} (A_i + B_i K_i))^{-1} \right\| \left\| T_\lambda (x) - T_\lambda (y) \right\|_{L^2}
\]

By (24) and (25), we get

\[
\leq (\lambda - \sum_{i=1}^{n} \| B_i K_i \| )^{-1} (\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} (\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} \left\| x - y \right\|_{L^2}
\]

for all \(x, y \in H\).

Now we find when the following inequality is hold

\[
(\lambda - \sum_{i=1}^{n} \| B_i K_i \| )^{-1} (\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} (\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} < 1
\]

\[
(\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} + (\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} < \lambda - \sum_{i=1}^{n} \| B_i K_i \| )^{-1}
\]
(\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} < (\lambda - \sum_{i=1}^{n} ||B_iK_i||)

(\lambda^2 - 2\lambda (nm^*) + (nK^*)^2)^{1/2} < (\lambda - \sum_{i=1}^{n} ||B_iK_i||)

Hence

\lambda^2 - 2\lambda (nm) + (nK)^2 < (\lambda - \sum_{i=1}^{n} ||B_iK_i||)^2

(\lambda^2 - 2\lambda (nm^*) + (nK^*)^2)^{1/2} < (\lambda - \sum_{i=1}^{n} ||B_iK_i||)^2

\lambda > (nK^2 - (\sum_{i=1}^{n} ||B_iK_i||^2)(2nm - 2 \sum_{i=1}^{n} ||B_iK_i||)^{-1}

Also

\lambda > (nK^2 - (\sum_{i=1}^{n} ||B_iK_i||^2)(2nm^* - 2 \sum_{i=1}^{n} ||B_iK_i||)^{-1}

Let us choose

\lambda > \max\{\sum_{i=1}^{n} ||B_iK_i||, (nK^2 - (\sum_{i=1}^{n} ||B_iK_i||^2)^2)(2nm - 2 \sum_{i=1}^{n} ||B_iK_i||)^{-1}, (nK^*)^2 - (\sum_{i=1}^{n} ||B_iK_i||^2)(2nm^* - 2 \sum_{i=1}^{n} ||B_iK_i||)^{-1}\}

it result that

(\lambda - \sum_{i=1}^{n} ||B_iK_i||)^{-1}\left((\lambda^2 - 2\lambda (nm) + (nK)^2)^{1/2} + (\lambda^2 - 2\lambda (nm^*) + (nK^*)^2)^{1/2}\right) < 1

Therefore, \((\lambda I - \sum_{i=1}^{n}(A_i + B_iK_i)^{-1}T_i(x))\) is a contraction in \(L_2^g\). Then by theorem (2.16) the Equation (27) and consequently (23) has a unique solution.

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