Static non-reciprocity in mechanical metamaterials

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Reciprocity is a general, fundamental principle governing various physical systems, which ensures that the transfer function—the transmission of a physical quantity, say light intensity—between any two points in space is identical, regardless of geometrical or material asymmetries. Breaking this transmission symmetry offers enhanced control over signal transport, isolation and source protection2–4. So far, devices that break reciprocity (and therefore show non-reciprocity) have been mostly considered in dynamic systems involving electromagnetic, acoustic and mechanical wave propagation associated with fields varying in space and time. Here we show that it is possible to break reciprocity in static systems, realizing mechanical metamaterials5–16 that exhibit vastly different output displacements under excitation from different sides, as well as one-way displacement amplification. This is achieved by combining large nonlinearities with suitable geometrical asymmetries and/or topological features. In addition to extending non-reciprocity and isolation to statics, our work sheds light on energy propagation in nonlinear materials with asymmetric crystalline structures and topological properties. We anticipate that breaking reciprocity will open avenues for energy absorption, conversion and harvesting, soft robotics, prosthetics and optomechanics.

By pushing an object on side A, we move the other side B by a certain amount. Daily experience tells us that if we now push the opposite side B with the same force, side A moves by the same amount. In other words, if an object transmits motion in one direction, it typically does also in the opposite one. This basic property of static mechanical systems is a direct consequence of the Maxwell–Betti reciprocity theorem, a widely known result at the foundations of mechanical engineering17–19, with important consequences for the analysis and design of a wide range of mechanical systems20,21. The Maxwell–Betti theorem is mathematically formulated as

\[ F_{AB\rightarrow A} = F_B u_{A\rightarrow B} \]  

(1)

where \( F_A \) (\( F_B \)) is the applied force at point A (B) and \( u_{A\rightarrow B} \) (\( u_{B\rightarrow A} \)) the displacement at point B (A) induced by \( F_A \) (\( F_B \)), as shown in Fig. 1a. Similarly to Lorentz’s reciprocity theorem, which governs transmission symmetry for electromagnetic wave phenomena, the Maxwell–Betti theorem stems from time-reversal symmetry and the principle of microscopic reversibility22, and it is thus widely applicable. Breaking reciprocity in statics may substantially extend the functionalities of mechanical systems, in the same way that electromagnetic non-reciprocal devices1,2,4, such as isolators2 and circulators5,6, have become essential components of modern electromagnetic systems. The Maxwell–Betti theorem is derived under a few basic assumptions, including the fact that the system under analysis is linear. We can therefore expect that equation (1) may be violated in suitably designed nonlinear systems. In the following, we explore this path to induce strong static non-reciprocity for moderate applied force intensities using mechanical metamaterials7–16. We further harness their tunable, nonlinear and topological responses to induce giant mechanical isolation for applied static forces of moderate magnitude.

Consider first the ‘fishbone’ mechanical metamaterial shown in Fig. 1b, which consists of horizontal and transverse elastic beams, with the latter clamped to the laboratory frame and tilted by an angle \( \theta \) with respect to the vertical axis. The angle \( \theta \) quantifies the geometrical asymmetry of the structure, with \( \theta = 0 \) corresponding to a symmetric device. Figure 1b shows the experimentally observed deformation of such a structure for an applied force \( F_A = F_B = -1 \text{ N} \) (\( F_B = -F_A = 1 \text{ N} \)) at its left (right) side—see also Supplementary Video 1. We clearly see with the naked eye that the output displacement is much larger for excitation from the left-hand side than from the right-hand side, that is, \( |u_{A\rightarrow B}| > |u_{B\rightarrow A}| \), evidence of a strongly non-reciprocal transmission of the displacement field. Such strong asymmetry results from the instability of the fishbone metamaterial—it’s abrupt switching into a different configuration for a given applied intensity—induced when excited from its left end, and associated with its large nonlinear response. As shown by our measurements and numerical simulations (Fig. 1c), the onset of instability corresponds to the point where the slope of the red curve suddenly becomes very large (\( F_0 \approx -0.3 \text{ N} \)). A comparable instability point also occurs when we push the material from the right end (\( F_0 \approx 0.3 \text{ N} \)). Such a large non-reciprocal response induces mechanical isolation of up to 20 dB (Fig. 1d); here we define mechanical isolation as \( 20 \log_{10} (u_{A\rightarrow B}/u_{B\rightarrow A}) \), where \( u_{A\rightarrow B} \) and \( u_{B\rightarrow A} \) are the ratio of the displacements.

Further insight into the non-reciprocal properties of the structure can be obtained by studying the non-reciprocity parameter \( \Delta u = u_{A\rightarrow B} - u_{B\rightarrow A} \), which is plotted in Fig. 1e for three asymmetry angles, \( \theta = 0, \theta = \pi/16 \) and \( \theta = \pi/8 \). For \( \theta = 0 \), symmetry requires \( \Delta u = 0 \), no matter how large the nonlinearity is. In asymmetric structures, however, \( \Delta u \) has a quadratic dependence on \( F_0 \) for small input forces, which is consistent with the fact that the leading linear terms of the Taylor expansion of \( u_{A\rightarrow B} \) and \( u_{B\rightarrow A} \) with respect to \( F_0 \) should satisfy equation (1). Figure 1e shows that, although the structure with \( \theta = \pi/8 \) is more asymmetric than the one with \( \theta = \pi/16 \), the latter exhibits a much larger \( \Delta u \). These findings suggest that strong static non-reciprocity stems from a delicate balance between asymmetry and nonlinearity, a fact that will become clearer in the following discussion.

The unusual relation between \( \Delta u \) and \( \theta \) in the structure of Fig. 1 can be understood through the simplified model in Fig. 2a, consisting of discrete nodes connected together through linear and torsional springs; these springs model the beams and connections between them, respectively. In the Supplementary Information, we demonstrate that such a structure can be described through the equation

\[ 0 = \frac{d^2u}{dx^2} - \lambda_1 u - \lambda_2 u^2 - \ldots \]  

(2)

where \( u(x) \) is the horizontal displacement of the central nodes located at the coordinate \( x \) and \( \lambda_1 \) are parameters that depend on \( \theta \) and the spring properties. The solution of this equation is presented in Fig. 2a, b for \( \theta = \pi/16 \) without (dashed grey line) and with (black dots) the non-linear terms in equation (2), for excitation from the left and right end, respectively, and input force \( F_0 = -2 \text{ N} \). Despite the geometrical asymmetry, in the linear case the displacement fields \( u(x) \) for excitations...
The fishbone structure.

**Figure 2** | Discrete models for non-reciprocal metamaterials. 

a, b, Top, sketch of the fishbone structure considered in Fig. 1 where \( \theta \) is the asymmetry angle (depicted in green). Bottom, corresponding displacement fields (black dots and coloured arrows) \( u \) versus \( n \) for excitations from the left end (a, red arrows) and the right end (b, blue arrows) for an input force \( F_0 = -2 \) N. The dashed lines correspond to the linearized problem. The shaded area represents \( \Delta u = u_{AB} - u_{BA} \).  

c, Non-reciprocity \( \Delta u \) versus input force \( F_0 \) for different values of \( \theta \) for the fishbone structure.  

d, Non-reciprocal susceptibility \( \kappa \) versus the asymmetry \( \theta \) for the fishbone structure. The metamaterial is topologically trivial.  

e, f, Blueprint of the topological mechanical metamaterial where \( \theta \) is the asymmetry angle (depicted in green). We display both the undeformed (grey) and deformed structures (yellow) and the corresponding displacement fields \( u \) versus \( n \) for excitations (black dots and coloured arrows) from the left end (e, red arrows) and the right end (f, blue arrows) for an input force \( F_0 = -0.15 \) N. The dashed lines correspond to the linearized problem.  

g, Non-reciprocity parameter \( \Delta u \) versus input force \( F_0 \) for different values of the asymmetry \( \theta \).  

h, Non-reciprocal susceptibility \( \kappa \) and topological invariant (called the winding number) \( W \) versus the asymmetry \( \theta \). The metamaterial is topologically non-trivial and its winding number equals 0 (1) for \( \theta > 0 \) (\( \theta < 0 \)).
As a result of such interplay between nonlinearity and asymmetry, $\kappa(\theta)$ exhibits two extrema for which the non-reciprocity is the strongest. Their positions are directly controlled by the properties of the constitutive springs.

An important question to address is how to engineer large values of the susceptibility $\kappa$, in order to induce giant non-reciprocal responses for small input forces. A straightforward approach would be to use constitutive materials with smaller elastic moduli, resulting in overall smaller input forces, but in many practical cases we are restricted in terms of available material properties. An alternative strategy consists in realizing metamaterials with a large structural compliance. We can do this by replacing the linear spring elements of the fishbone (Fig. 2a, b) by freely hinging squares and bars (Fig. 2e, f). The transverse bars play the same role as in the fishbone, and their tilt angle $\theta$ quantifies the structure asymmetry. As we discuss in the Supplementary Information, this structure is a mechanism, that is, it supports a single internal hinging motion, where all squares and bars pivot harmoniously. Actuating this mechanism requires elastic energy, which we model using torsional springs located at the central nodes. For a small input force $F_0 = -0.15$ N, we observe a strongly asymmetric response, which amplifies (respectively curbs) the displacement when actuated from the left (respectively right) end (Fig. 2e, f), in contrast with the fishbone structure, in which the displacement fields always decay from the excitation point. While the metamaterial still exhibits the same quadratic scaling with input intensity $\Delta u = \kappa(\theta)F_0^2$ (Fig. 2g), as expected from equation (1), the susceptibility $\kappa$ is several orders of magnitude larger than for the fishbone structure, and its dependence on the asymmetry angle $\theta$ is different: $\kappa(\theta)$ diverges at $\theta \to 0$ and monotonically decreases away from this singular point (Fig. 2h).

Considering that $\theta \to 0$ is the limit of a spatially symmetric structure, the divergence of the non-reciprocal parameter $\kappa$ is particularly interesting. As shown in the Supplementary Information, to linear order the displacement profile takes the form $u(n) = u(0)[g(\theta)]^n$, where $n$ is the (discrete) node index of the lattice and $g(\theta)$ is a characteristic constant of the structure. In the limit of zero torsional stiffness, the structure is isostatic and it supports this displacement distribution at zero energy, a property directly related to its nontrivial topological properties, which are akin to those found in topological mechanical metamaterials at zero frequency$^{23–26}$. It can indeed be shown (see Supplementary Information) that the constant $g(\theta)$ is associated with the topological invariant $W$, as defined in ref. 23 and discussed in Methods. The zero-energy displacement distribution is exponentially localized on the left (right, as in the example shown in Fig. 2e, f) of the structure, associated with $W = 1$ ($W = 0$) for negative (positive) values of $\theta$, with a decay length proportional to $1/|\log g(\theta)|$. In the limit of a symmetric configuration ($\theta = 0$), this topologically induced edge mode is delocalized, with its decay length going to infinity, while the topological invariant exhibits a discontinuity, consistent with the observations in ref. 23. Around this topological transition, the mode delocalization strongly connects the two metamaterial edges, leading to the divergence of the non-reciprocal susceptibility $\kappa$, and implying that the condition $\theta \approx 0$ is ideally suited to enable a strong static non-reciprocal response for small applied forces. In contrast, the fishbone structure can be proven to be topologically trivial in the low-intensity regime—see Supplementary Information and Methods—leading to a larger threshold for observing non-reciprocal responses, enabled by the nonlinearities for larger forces. These results indicate that suitably tailored asymmetry and nonlinearity are the two fundamental requirements to achieve non-reciprocity in statics. Topological order is not necessary to realize this effect, but it provides an efficient framework to enhance it substantially.

Inspired by the architecture in Fig. 2e, f, we designed and built a 2D topological mechanical metamaterial (see Methods) that shows non-reciprocity for small input forces, as well as displacement amplification (Fig. 3a and Supplementary Video 2). Such amplification occurs when the structure is actuated from its left end (red curve in Fig. 3b). In contrast, the response to actuation from its right end induces a decreasing displacement field (blue curve in Fig. 3b). Such difference builds up a strong non-reciprocal response, which can be probed both in simulations and in experiments (Fig. 3c), and that is qualitatively similar to the model discussed above. Since topological properties are inherently robust to continuous perturbations, the overall non-reciprocal...
response of the designed metamaterial is expected to be robust to continuous changes, defects and imperfections. The use of instabilities controlled by confinement\textsuperscript{11} or by the texture of the probe\textsuperscript{16} further provide a promising strategy to change these topological properties in a controlled fashion, achieving tunable topological and non-reciprocal responses.

In contrast to wave-based systems, which are inherently dynamic, a vast number of mechanical systems primarily operate in the static regime. In this Letter, we have shown how non-reciprocal metamaterials with large nonlinearities and suitably tailored asymmetries combined with topologically non-trivial features can support strong non-reciprocity and isolation for static applied forces. These systems offer unique opportunities to design devices with unprecedented static one-way functionalities, which are of relevance to shock and low-frequency vibration damping\textsuperscript{15,27,28}, mechanical energy harvesting\textsuperscript{29}, prosthetics and opto-mechanics\textsuperscript{30}.

Online Content Methods, along with any additional Extended Data display items and Source Data, are available in the online version of the paper; references unique to these sections appear only in the online paper.

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Supplementary Information is available in the online version of the paper.

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Author Information Reprints and permissions information is available at www.nature.com/reprints. The authors declare no competing financial interests. Readers are welcome to comment on the online version of the paper. Correspondence and requests for materials should be addressed to C.C. (coulais@amolf.nl).

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METHODS

Sample geometries. The first chain is 20 mm wide, 8 mm high and 120 mm long, and is made of 10 repeatedly stacked unit cells (see Extended Data Fig. 1a). The thickness of the horizontal beam (respectively transverse beams) is 2 mm (respectively 1 mm). We control the deformation by imposing the angle $\theta$ between the struts and the vertical axis, and investigate three structures where $\theta = 0, \pi/16$ and $\pi/8$.

The second metamaterial is 75 mm wide, 8 mm high and 120 mm long, and is made of 4 repeatedly stacked unit cells (see Extended Data Fig. 1b). The unit cell is made of squares of diagonal length 10 mm and diamond shaped quadrilaterals of diagonal lengths 16 mm and 8 mm. The thickness of the connections between the vertices of these shapes is 1 mm.

Sample fabrication. We create our structures by casting a two component silicon rubber with well-calibrated elastic properties (PVS, Zhermarck, Elite Double 32, Young's modulus 1.2 MPa) into 3D printed moulds. We let the silicon rubber cure for a few hours, after which we extract the sample by breaking the mould apart. We wait for one week after which the elastic properties of the polymer have settled. Finally, we glue the samples edges onto an aluminium frame, which allows us to confine the sample laterally.

Mechanical testing and data acquisition. In order to minimize alignment bias, we carefully align our structures within a 0.1 mm accuracy in the frame. We position such frame in a uniaxial testing device (Instron 3366), which we equip with a 10 N load cell and which allows us to impose the input position better than 0.01 mm and to measure the input force $F_i$ better than 0.5 mN. The samples are probed mechanically by custom made mechanical tweezers. We subsequently apply point forces at points A and B (as in Fig. 1a) simply by changing the orientation of the frame with respect to the tensile tester. In parallel, we measure the output displacement by using a high-resolution camera (Basler 3,840 pixels $\times$ 2,748 pixels) synchronized with the tensile tester, and use a custom made sub-pixel detection algorithm that allows us to track the output displacement within a 0.002 mm accuracy. In order to optimize the contrast of the image acquisition between the structures and the background, black opaque fabric is glued onto the supporting aluminium frames.

Numerical protocol. For our static finite elements simulations, we use commercial software Abaqus/Standard 6.13 (Dassault Systèmes) and a ‘neo-Hookean’ energy density as a material model; we use shear modulus $G = 0.40$ MPa and bulk modulus $K = 20.0$ GPa in plane strain conditions with hybrid quadratic triangular simulation elements (Abaqus type CPE6H). We perform a mesh refinement study in order to ensure that the thinnest parts of the samples where most of the stress and strain localized are meshed with at least four elements. As a result, the two metamaterials have about $2 \times 10^4$ triangular elements. We apply boundary conditions that correspond closely to the experiments.

Topological properties. The fishbone metamaterial has a topologically trivial static linear response, which is symmetric despite its structural asymmetry. This is confirmed by the fact that its lowest-order modes have a finite non-zero energy, and that the phonon spectrum always has a gap; see Supplementary Information for more details. In contrast, our second (topological) metamaterial, similarly to earlier examples in the literature, is isostatic—its number of degrees of freedom equals its geometric constraints—and it supports a single edge mode, a localized soft mode of deformation at the edge. The response of such a structure is characterized by a topological invariant, the winding number $W$, which takes different values as $\theta$ changes sign, as $W = 0$ for $\theta > 0$ and $W = 1$ for $\theta < 0$, and which determines whether the edge mode is localized on the right (for $W = 0$) or on the left (for $W = 1$). The existence of a topological invariant in such an isostatic mechanical system is intimately related to the absence of inversion-symmetry in the linear response. In addition, such a property indicates that the system is topologically protected: that is, the nature of its response—here the side where the edge mode is localized—is robust to continuous perturbations of the structure, provided that it does not substantially perturb the asymmetry $\theta$, switching the integer topological invariant $W$.

Sample size. No statistical methods were used to predetermine sample size.

Data availability. Source Data for Figs 1–3 are provided in the online version of this Letter.

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Extended Data Figure 1 | Pictures of the mechanical metamaterials in their confining frames. a, b, Fishbone (a) and topological (b) mechanical metamaterials, both with an asymmetry angle $\theta = \pi/16$. 