An optimal control problem of forward-backward stochastic Volterra integral equations with state constraints

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Abstract. This paper is devoted to the stochastic optimal control problems for systems governed by forward-backward stochastic Volterra integral equations (FBSVIEs, for short) with state constraints. Using Ekeland’s variational principle, we obtain one kind of variational inequality. Then, by dual method, we derive a stochastic maximum principle which gives the necessary conditions for the optimal controls.

Keyword. Forward-backward stochastic Volterra integral equations (FBSVIEs); M-solution; Terminal perturbation method; State constraints; Ekeland’s variational principle; Stochastic maximum principle.

1 Introduction

As we known, with the exception of the applications in biology, physical, etc, Volterra integral equations often appear in some mathematical economic problems, for example, the relationships between capital and investment which include memory effects (in [24], the present stock of capital depends on the history of investment strategies over a period of time). And the simplest way to describe such memory effects is through Volterra integral operators. Based on the importance of Volterra integral equations, we will study an stochastic optimal control problem about a class of nonlinear stochastic equations—forward-backward stochastic Volterra integral equations (FBSVIEs, for short). First we review the backgrounds of these two kinds of Volterra integral equations: forward stochastic Volterra integral equations (FSVIEs, for short) and backward stochastic Volterra integral equations (BSVIEs, for short).

Let \( B(\cdot) \) be a standard \( d \)-dimensional Brownian motion defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \mathcal{F} = \{ \mathcal{F}_t \}_{t \geq 0} \) is its natural filtration generated by \( B(\cdot) \) and augmented by all the \( \mathbb{P} \)-null sets in \( \mathcal{F} \).

Consider the following FSVIE:

\[
X(t) = \phi(t) + \int_0^t b(t, s, X(s)) \, ds + \int_0^t \sigma(t, s, X(s)) \, dB_s, \quad t \in [0, T].
\]

The readers may refer to [2, 3, 23, 24, 30] and the reference cited therein, for the general results on FSVIEs. When studying the stochastic optimal control problems for FSVIEs, we need one kind of adjoint equation in order to derive a stochastic maximum principle. This new adjoint equation is actually a linear BSVIE.

This motivates the investigation of the theory and applications of BSVIEs.

The following BSVIE was firstly introduced by Yong [30]:

\[
Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(s, t)) \, ds - \int_t^T Z(t, s) \, dB_s, \quad t \in [0, T],
\]

where \( g : \Delta_c \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \Omega \rightarrow \mathbb{R}^m \) and \( \psi : [0, T] \times \Omega \rightarrow \mathbb{R}^m \) are given maps with \( \Delta_c = \{ (t, s) \in [0, T]^2 | t < s \} \). For each \( t \in [0, T] \), \( \psi(t) \) is \( \mathcal{F}_T \)-measurable (Lin [27] studied \( \psi \) when \( \psi(\cdot) \equiv \xi \)). It is obvious that BSVIE is a natural generalization of backward stochastic differential equation (BSDE, for
short). Comparing with BSDEs, BSVIE still has its own features as listed in Yong [36, 38]. One of the advantages is to study time-inconsistent phenomenon. As shown in Laibson [26] and Strotz [33], in the real world, time-inconsistent preference usually exists. At this point, one needs BSVIEs to generalize the so-called stochastic differential utility in [8] and dynamic risk measures (see [1, 6, 31, 34]). Other applications are in the non-exponential discounting problems (see Ekeland, Lazrak [10], Ekeland, Pirvu [11]) and time-inconsistent optimal control problem (see Yong [33, 40]). In [33, 40], Yong solved a time-inconsistent optimal control problem by introducing a family of N-person non-cooperative differential games, and got an equilibrium control which was represented via a forward ordinary differential equation with a backward Riccati-Volterra integral equation.

As stated in Yong [37], \( \psi(t) \) in BSVIE \( (1) \) could represent the total (nominal) wealth of certain portfolio which might be a combination of certain contingent claims (for example, European style, which is mature at time \( T \)), some current cash flows, positions of stocks, mutual funds, and bonds, and so on, at time \( t \). So, in general, the position process \( \psi(\cdot) \) is not necessarily \( \mathcal{F} \)-adapted, but a stochastic process merely \( \mathcal{F}_T \)-measurable. And Yong gave an example to make this point more clear in [37]. Focusing on this kind of position process \( \psi(\cdot) \), a class of convex/coherent dynamic risk measures was introduced by Yong in [37] to measure the risk dynamically. Hence, one kind of control problem appears: how to minimize the risk, or how to maximize the utility. Wang, Shi [34] obtained a maximum principle for FBSVIEs without state constraints. In this paper, we study one kind of optimal control problem in which the state equations are governed by the following FBSVIEs:

\[
\begin{align*}
X(t) &= f(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dB_s, \quad t \in [0, T], \\
Y(t) &= \psi(t) + \int_t^T g(t, s, X(s), Y(s), Z(s, t), u(s))ds - \int_t^T Z(t, s)dB_s.
\end{align*}
\]

By choosing admissible controls \((u, \psi)\), we shall maximize the following objective functional

\[
J(\psi, u) := E\left[\int_0^T \int_t^T l_1(t, s, X(s), u(s))dsdt + \int_0^T l_2(t, s, X(s), Y(s), Z(s, t), u(s))dsdt + \int_0^T q(\psi(t))dt + h(X(T)) + \int_0^T k(Y(s))ds\right].
\]

Our formulation has the following new features:

(i) A strong assumption that \( g(t, \cdot, \cdot, \cdot, \cdot, \cdot) \) in \( (2) \) is \( \mathcal{F}_t \)-measurable is given in [37]. By applying the duality principle introduced in Yong [37], we overcome this restriction and assume a natural condition that \( g(\cdot, s, \cdot, \cdot, \cdot, \cdot) \) is \( \mathcal{F}_s \)-measurable.

(ii) \( \psi \) in \( (2) \) is the terminal state of the BSVIE. In our formulation \( \psi \) is also regarded as a control and our control is a pair \((u, \psi)\). In mathematical finance, such kind of controls often appears as “consumption-investment plan” (see [32]). For the recent progress of studying this kind of control we refer the reader to [12, 16, 21, 23]. We also impose constraints on the state process \( Y(\cdot) \) and \( \psi \).

(iii) We consider the double integral in the cost functional \( (3) \) in theory. Some further studies on the applications are still under consideration.

In order to solve this optimal control problem, we adopt the terminal perturbation method, which was introduced in [5, 12, 15, 22]. Recently, the dual approach is applied to utility optimization problem with volatility ambiguity (see [12, 11]). The basic idea is to perturb the terminal state \( \psi \) and \( u \) directly. By applying Ekeland’s variational principle to tackle the state constraints, we derive a stochastic maximum principle which characterizes the optimal control. It is worth to point out that in place of Itô’s formula, we need two duality principles established by Yong in [37, 38] to obtain the above results.

This paper is organized as follows. First, we recall some elements of the theory of BSVIEs in Section 2. In Section 3, we formulate the stochastic optimization problem and prove a stochastic maximum principle. In Section 4, we give two examples. The first example is associated with the model we studied. The last example is about the ‘terminal’ control \( \psi(\cdot) \).

## 2 Preliminaries

Let \( B(\cdot) \) be a \( d \)-dimensional Brownian motion defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}, P)\), where \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \) is natural filtration generated by \( B(\cdot) \) and augmented by all the \( P \)-null sets in \( \mathcal{F} \), i.e.,

\[
\mathcal{F}_t = \sigma\{B_r, r \leq t\} \vee \mathcal{N}_r, \quad t \in [0, T],
\]

where \( \mathcal{N}_r = \{A \in \mathcal{F}_r : A \cap [0, r] \in \mathcal{P}\} \), \( \mathcal{P} \) is the class of \( P \)-null sets in \( \mathcal{F} \).
where $\mathcal{N}_P$ is the set of all $P$-null sets.

### 2.1 Notations

Here we keep on the definitions and notations for the spaces introduced in Yong [38].

For any $0 \leq R < S \leq T$, we denote
\[
\begin{align*}
\Delta[R,S] &= \{(t,s) \in [R,S]^2 | R \leq s \leq t \leq S \}, \\
\Delta^c[R,S] &= \{(t,s) \in [R,S]^2 | R \leq t < s \leq S \} = [R,S]^2 \setminus \Delta[R,S],
\end{align*}
\]

For any $A, B \in \mathbb{R}^{m \times d}$, define the inner product $\langle A, B \rangle := \text{tr}[AB^T]$ and
\[
|A|^2 = \sum_{j=1}^d |a_j|^2 = \sum_{i=1}^m \sum_{j=1}^d a_{ij}^2, \quad \forall A \equiv (a_1, \cdots, a_d) \equiv (a_{ij}) \in \mathbb{R}^{m \times d}.
\]

Let $S \in [0, T]$, define the following spaces:

- $L^p_{\mathcal{F}_S}(0,T) := \{ \varphi : [0,T] \times \Omega \to \mathbb{R}^m | \varphi(\cdot) \text{ is } \mathcal{B}([0,T]) \otimes \mathcal{F}_S\text{-measurable and } E \int_0^T |\varphi(t)|^p dt < \infty \}$;
- $L^p_\mathbb{F}(0,T) := \{ \varphi : [0,T] \times \Omega \to \mathbb{R}^m | \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } E \int_0^T |\varphi(t)|^p dt < \infty \}$;
- $L^p(0,T;L^2_\mathbb{F}(0,T)) := \{ Z : [0,T]^2 \times \Omega \to \mathbb{R}^{m \times d} | \text{for almost all } t \in [0,T], Z(t,\cdot) \in L^2_\mathbb{F}(0,T), \int_0^T E(\int_0^T |Z(t,s)|^2 ds) \, dt < \infty \};$
- $L^\infty_\mathbb{F}(0,T;\mathbb{R}^n) := \{ \varphi : [0,T] \times \Omega \to \mathbb{R}^n | \text{esssup} \sup_{\omega \in \Omega} \sup_{s \in [0,T]} \varphi(s,\omega) < \infty \};$
- $L^\infty([0,T];L^\infty_\mathbb{F}(0,T;\mathbb{R}^{n \times n})) := \{ Z(t,\cdot) \in L^\infty_\mathbb{F}(0,T;\mathbb{R}^{n \times n}) | \text{esssup} \sup_{\omega \in \Omega} \sup_{t \in [0,T]} Z(t,s,\omega) < \infty \};$
- $\mathcal{H}^p[S,T] := L^p_\mathbb{F}(S,T) \times L^p(S,T;L^2_\mathbb{F}(S,T)).$

### 2.2 Backward Stochastic Volterra Integral Equations

For the reader’s convenience, we present some results of BSVIEs which we will use later.

Consider the following integral equation
\[
Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(s,t)) \, ds - \int_t^T Z(t,s) \, dB_s, \quad t \in [0,T],
\]

where $\psi(\cdot) \in L^2_{\mathcal{F}_S}(0,T)$.

We assume:

\[ (H) \] Let $g : \Omega \times \Delta^c[0,T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ be $\mathcal{F}_T \otimes \mathcal{B}(\Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$-measurable such that $s \mapsto g(t,s,y,\zeta)$ is $\mathbb{F}$-progressively measurable for all $(t,y,\zeta) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ and
\[
E \int_0^T \left( \int_t^T |g(t,s,0,0)|^2 ds \right)^2 dt < \infty.
\]
Moreover, $\forall (t,s) \in \Delta^c[0,T]$, $(y,\zeta)$ and $(\tilde{y},\tilde{\zeta}) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}$,
\[
|g(t,s,y,\zeta) - g(t,s,\tilde{y},\tilde{\zeta})| \leq L(t,s)(|y - \tilde{y}| + |\zeta - \tilde{\zeta}|), \quad \text{a.s.,}
\]
where $L : \Delta^c[0,T] \to \mathbb{R}$ is a deterministic function such that
\[
\sup_{t \in [0,T]} \int_t^T L(t,s)^2 + \varepsilon \, ds < \infty, \quad \text{for some } \varepsilon > 0.
\]
The following $M$-solution of BSVIEs was introduced by Yong [38].

**Definition 1.** Let $S \in [0, T]$. A pair $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[S, T]$ is called an adapted $M$-solution of BSVIE [4] on $[S, T]$ if [4] holds in the usual Itô sense for almost all $t \in [S, T]$ and, in addition, the following equation holds:

$$Y(t) = E[Y(t)|\mathcal{F}_S] + \int_S^t Z(t,s)dB_s, \quad a.e., \; t \in [S, T].$$

For the proof of the following wellposedness results, the readers are referred to Yong [38].

**Lemma 2.** Let (H) holds. Then for any $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0, T)$, BSVIE [4] admits a unique adapted $M$-solution $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[0, T]$ on $[0, T]$. Moreover the following estimate holds: $\forall S \in [0, T],$

$$\|Y(\cdot), Z(\cdot, \cdot)\|_{\mathcal{H}^2[S,T]}^2 \equiv E\left\{ \int_S^T |Y(t)|^2 dt + \int_S^T \int_S^T |Z(t,s) - Z(t,s)|^2 dsdt \right\} \leq CE\left\{ \int_S^T |\psi(t)|^2 dt + \int_S^T \left( \int_S^T |g(t,s,Y(s),Z(s,t)) - \tilde{g}(t,s,Y(s),Z(s,t))|ds \right)^2 dt \right\}.$$  \hspace{1cm} (5)

Let $\tilde{g} : \Omega \times [0, T] \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ also satisfies (H), $\psi(\cdot) \in L^2_{\mathcal{F}_T}(0, T)$ and $(\tilde{Y}(\cdot), \tilde{Z}(\cdot, \cdot)) \in \mathcal{H}^2[0, T]$ is the adapted $M$-solution of [4] with $g$ and $\psi(\cdot)$ replaced by $\tilde{g}$ and $\psi(\cdot)$, respectively, then $\forall S \in [0, T]$,

$$E\left\{ \int_S^T |Y(t) - \tilde{Y}(t)|^2 dt + \int_S^T \int_S^T |Z(t,s) - \tilde{Z}(t,s)|^2 dsdt \right\} \leq CE\left\{ \int_S^T |\psi(t) - \tilde{\psi}(t)|^2 dt + \int_S^T \left( \int_S^T |g(t,s,Y(s),Z(s,t)) - \tilde{g}(t,s,Y(s),\tilde{Z}(s,t))|ds \right)^2 dt \right\}.$$  \hspace{1cm} (5)

Yong proved the following two duality principles for linear SVIE and linear BSVIE in [37, 38] respectively. And they play a key role in deriving the maximum principle.

**Lemma 3.** Let $A_i(\cdot, \cdot) \in L^\infty([0, T]; L^\infty_T(0, T; \mathbb{R}^{d \times d}))$ $(i = 0, 1 \cdots d)$, $\varphi(\cdot) \in L^2_T(0, T; \mathbb{R}^d)$, and $\psi(t) \in L^2((0, T) \times \Omega; \mathbb{R}^d)$. Let $\xi(\cdot) \in L^2_T(0, T; \mathbb{R})$ be the solution of the following FSVIE:

$$\xi(t) = \varphi(t) + \int_0^T A_0(t,s)\xi(s)ds + \int_0^T \sum_{i=1}^d A_i(t,s)\xi(s)d\xi_i(s), \quad t \in [0, T].$$

$(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[0, T]$ be the adapted $M$-solution to the following BSVIE:

$$Y(t) = \psi(t) + \int_t^T [A_0(s,t)Y(s) + \sum_{i=1}^d A_i(s,t)Z_i(s,t)]ds - \int_t^T Z(t,s)dB_s, \quad t \in [0, T].$$

Then the following relation holds:

$$E\left\{ \int_0^T \langle \xi(t), \psi(t) \rangle dt \right\} = E\left\{ \int_0^T \langle \varphi(t), Y(t) \rangle dt \right\}.$$  \hspace{1cm} (5)

**Lemma 4.** Let $A_i(\cdot, \cdot) \in L^\infty([0, T]; L^\infty_T(0, T; \mathbb{R}^{d \times d}))$ $(i = 0, 1 \cdots d)$, $\varphi(\cdot) \in L^2_T(0, T; \mathbb{R}^d)$, and $\psi(t) \in L^2((0, T) \times \Omega; \mathbb{R}^d)$. Suppose $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2(0, T)$ is the solution of the following linear BSVIE:

$$Y(t) = \psi(t) + \int_t^T [A_0(t,s)Y(s) + \sum_{i=1}^d A_i(t,s)Z_i(s,t)]ds - \int_t^T Z(t,s)dB_s, \quad t \in [0, T],$$

and $X(\cdot)$ is the solution of the following FSVIE:

$$X(t) = \varphi(t) + \int_0^T A_0(s,t)X(s)ds + \int_0^t X(s) \sum_{i=1}^d E\{A_i(s,t)|\mathcal{F}_s\}dB_i(s), \quad t \in [0, T].$$

Then the following relation holds:

$$E\left\{ \int_0^T \langle X(t), \psi(t) \rangle dt \right\} = E\left\{ \int_0^T \langle \varphi(t), Y(t) \rangle dt \right\}.$$  \hspace{1cm} (5)

For the proofs of Lemmas 3 and 4, the readers are referred to Theorem 5.1 in [38] and Theorem 3.1 in [37], respectively.
3 Stochastic optimization problem

3.1 One kind of stochastic optimization problem

Let $K$, $\tilde{K}$ be a nonempty convex subset of $\mathbb{R}^m$, set

$$U[0,T] = \{ u : [0,T] \times \Omega \to \mathbb{R}^m | u(\cdot) \in L^2_{\mathbb{F}}(0,T), u(s) \in K, s \in [0,T], a.e., a.s. \},$$

and

$$\mathcal{U} = \{ (\psi(\cdot), u(\cdot)) | \psi(\cdot) \in L^2_{\mathbb{F}}(0,T), \psi(t) \in \tilde{K}, t \in [0,T], a.e., a.s., u(\cdot) \in U[0,T] \}.$$ 

For any given control pair $(\psi(\cdot), u(\cdot)) \in \mathcal{U}$, we consider the following controlled integral equation:

$$\begin{cases}
  X(t) = f(t) + \int_t^T b(s,x,Y(s),u(s))ds + \int_t^T g(s,x,Y(s),u(s))dB_s, & t \in [0,T], \\
  Y(t) = \psi(t) + \int_t^T \sigma(s,x,Y(s),Z(s,t),u(s))ds - \int_t^T Z(t,s)dB_s,
\end{cases} \quad (6)$$

where $f(\cdot) \in L^2_{\mathbb{F}}(0,T)$ and $b : \Omega \times \Delta[0,T] \times \mathbb{R}^m \times K \to \mathbb{R}^m$, $\sigma : \Omega \times \Delta[0,T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times K \to \mathbb{R}^{m \times d}$, $g : \Omega \times \Delta[0,T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times K \to \mathbb{R}^m$.

For each $(\psi(\cdot), u(\cdot)) \in \mathcal{U}$, define the following objective functional:

$$J(\psi(\cdot), u(\cdot)) := E \left[ \int_0^T l_1(t,s,Y(s),Z(s,t),u(s))dsdt + \int_0^T l_2(t,s,Y(s),Z(s,t),u(s))dsdt + \int_0^T g(\psi(t))dt + h(X(T)) + \int_0^T k(Y(t))dt \right], \quad (7)$$

where $l_1 : \Delta[0,T] \times \mathbb{R}^m \times K \to \mathbb{R}$, $l_2 : \Delta[0,T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times K \to \mathbb{R}$, $q : \mathbb{R}^m \to \mathbb{R}$, $h : \mathbb{R}^m \to \mathbb{R}$, $k : \mathbb{R}^m \to \mathbb{R}$.

We assume:

(A1) $b$, $\sigma$, $g$, $l_1$, $l_2$, $q$, $h$, $k$ are continuous in their argument, and continuously differentiable in the variables $(x, y, \zeta, u)$;

(A2) the derivatives of $b$, $\sigma$, $g$ in $(x, y, \zeta, u)$ are bounded;

(A3) the derivatives of $l_1$, $l_2$ in $(x, y, \zeta, u)$ are bounded by $C(1 + |x| + |y| + |\zeta| + |u|)$, and the derivatives of $l_1$, $l_2$ in $x$ are bounded by $C(1 + |x|)$;

(A4) $g(t,s,x,y,\zeta,u)$ is $\mathcal{F}_T \otimes \mathcal{B}(\Delta[0,T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times K)$-measurable such that $s \to g(t,s,x,y,\zeta,u)$, $s \to g_i(t,s,x,y,\zeta,u)$ are $\mathcal{F}$-progressively measurable for all $(t,x,y,\zeta,u) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times K$, $i = x$, $y$, $\zeta$, $u$, and $E \left[ \int_0^T (\int_0^T g_i(t,s,0,0,0,0)ds)^2 dt \right] < \infty$.

Under the assumptions (A1), (A2) and (A4), for any given $u(\cdot) \in U[0,T]$, the FSVIE in (6) has a unique solution $X^u(\cdot) \in L^2_{\mathbb{F}}(0,T)$. For any given $\psi(\cdot) \in L^2_{\mathbb{F}}(0,T)$, the BSVIE has a unique M-solution $(Y^{\psi,u}(\cdot), Z^{\psi,u}(\cdot)) \in \mathcal{H}^2[0,T]$ associated with $(\psi(\cdot), u(\cdot))$. Hence, there exists a unique triple $(X^u(\cdot), Y^{\psi,u}(\cdot), Z^{\psi,u}(\cdot))$ satisfying (8).

Now we formulate the optimization problem:

$$\begin{align*}
\text{Maximum} & \quad J(\psi(\cdot), u(\cdot)) \\
\text{subject to} & \quad (\psi(\cdot), u(\cdot)) \in \mathcal{U}, \int_0^T EY^{\psi,u}(s)ds = a, \\
& \quad EY^{\psi,u}(t) = \rho(t), a.e.
\end{align*} \quad (8)$$

where $\rho : [0,T] \rightarrow \mathbb{R}^m$ is continuous and satisfies $\int_0^T \rho(t)dt = a$, $\int_0^T |\rho(t)|^2 dt < \infty$.

3.2 Variational equation

For $(\psi^1(\cdot), u^1(\cdot)), (\psi^2(\cdot), u^2(\cdot)) \in \mathcal{U}$, we define a metric in $\mathcal{U}$ by

$$d((\psi^1(\cdot), u^1(\cdot)), (\psi^2(\cdot), u^2(\cdot))) := (E \int_0^T |\psi^1(s) - \psi^2(s)|^2 ds + E \int_0^T |u^1(s) - u^2(s)|^2 ds)^{1/2}.$$
It is obvious that \((\mathcal{U}, d(\cdot, \cdot))\) is a complete metric space.

Let \((\psi^\ast(\cdot), u^\ast(\cdot))\) be an optimal control pair to problem \((\text{S})\) and \((X^\ast(\cdot), Y^\ast(\cdot), Z^\ast(\cdot, \cdot))\) be the corresponding state processes of \((\text{S})\). For any \((\psi(\cdot), u(\cdot)) \in \mathcal{U}, 0 \leq p \leq 1,\) using the convexity of \(\mathcal{U}\), we have
\[
(\psi^p(\cdot), u^p(\cdot)) := ((1 - p)\psi^\ast(\cdot) + p\psi(\cdot), (1 - p)u^\ast(\cdot) + pu(\cdot))
\]
\[
= (u^\ast(\cdot) + p(\psi(\cdot) - \psi^\ast(\cdot)), u^\ast(\cdot) + p(u(\cdot) - u^\ast(\cdot))) \in \mathcal{U}.
\]
We denote \((X^p(\cdot), Y^p(\cdot), Z^p(\cdot, \cdot))\) by the solution of the corresponding FBSVIE \((\text{S})\) with \((\psi(\cdot), u(\cdot)) = (\psi^p(\cdot), u^p(\cdot))\).

Consider the following FBSVIE
\[
\left\{
\begin{array}{l}
\delta X(t) = \int_0^t b_0^p(t, s) \dot{u}(s) ds + \int_0^t \sigma_0^p(t, s) \dot{u}(s) dB_s \\
\quad + \int_0^t b_1^p(t, s) \delta X(s) ds + \int_0^t \sigma_1^p(t, s) \delta X(s) dB_s, \\
\quad t \in [0, T],
\end{array}
\right.
\]
\[
\delta Y(t) = \dot{\psi}(t) + \int_0^t [g_0^p(t, s) \delta X(s) + g_1^p(t, s) \delta Y(s) + g_2^p(t, s) \delta Z(s, t)] ds \\
\quad + g_3^p(t, s) \dot{u}(s) ds - \int_0^T \delta Z(t, s) dB_s,
\]
where \(\dot{\psi}(s) = \psi(s) - \psi^\ast(s), \dot{u}(s) = u(s) - u^\ast(s), \int_0^T b_k(t, s) = f_k(t, s, X^\ast(s), Y^\ast(s), Z^\ast(s, t), u^\ast(s)), k = x, y, z, u, f = b, \sigma, g,\) respectively. This equation is called the variational equation.

From Lemma 5 and \((\text{A}_1), (\text{A}_2), (\text{A}_4)\), it’s easy to check that the variational equation \((\text{S})\) has a unique solution \((\delta X(\cdot), \delta Y(\cdot), \delta Z(\cdot, \cdot)) \in L^2_0(0, T) \times H^2[0, T]\).

Now we define
\[
\bar{X}^p(t) = p^{-1}[X^p(t) - X^\ast(t)] = -\delta X(t),
\]
\[
\bar{Y}^p(t) = p^{-1}[Y^p(t) - Y^\ast(t)] = -\delta Y(t),
\]
\[
\bar{Z}^p(t, s) = p^{-1}[Z^p(t, s) - Z^\ast(t, s)] = -\delta Z(t, s).
\]
To simplify the proof, we use the following notations:
\[
f^p(t, s) = f(t, s, X^p(s), Y^p(s), Z^p(s, t), u^p(s)),
\]
\[
f^\ast(t, s) = f(t, s, X^\ast(s), Y^\ast(s), Z^\ast(s, t), u^\ast(s)),
\]
where \(f = b, \sigma, g,\) respectively. Similar to the arguments in \((\text{I})\), we have the following lemma:

**Lemma 5.** Assume that \((\text{A}_1), (\text{A}_2), (\text{A}_4)\) hold. We have
\[
\lim_{p \to 0} E \int_0^T |\bar{X}^p(t)|^2 dt = 0, \quad \lim_{p \to 0} E \int_0^T |\bar{Y}^p(t)|^2 dt = 0,
\]
\[
\lim_{p \to 0} E \int_0^T |\bar{Z}^p(t)|^2 ds dt = 0.
\]
**Proof.** (1) We prove the first equality. By the FSVIES \((\text{S})\) and \((\text{S})\), we have
\[
\bar{X}^p(t) = \int_0^t \frac{1}{p}[\bar{b}^p(t, s) - b^\ast(t, s) - pb^\ast(t, s) \delta X(s) - pb^\ast(t, s) \dot{u}(s)] ds \\
\quad + \int_0^t \frac{1}{p}[\bar{\sigma}^p(t, s) - \sigma^\ast(t, s) - p\sigma^\ast(t, s) \delta X(s) - p\sigma^\ast(t, s) \dot{u}(s)] dB_s,
\]
where
\[
A^1_p(t, s) := \int_0^s b^p(t, s, L(p, \lambda, s), M(p, \lambda, s)) d\lambda,
\]
\[
A^2_p(t, s) := \int_0^s \sigma^p(t, s, L(p, \lambda, s), M(p, \lambda, s)) d\lambda,
\]
\[
B^1_p(t, s) := \int_0^s b(t, s, L(p, \lambda, s), M(p, \lambda, s)) d\lambda,
\]
\[
B^2_p(t, s) := \int_0^s \sigma(t, s, L(p, \lambda, s), M(p, \lambda, s)) d\lambda,
\]
\[
D^1_p(t, s) := |A^1_p(t, s) - b^\ast(t, s) \delta X(s)| + |B^1_p(t, s) - b^\ast(t, s) \dot{u}(s)| ds,
\]
\[
D^2_p(t, s) := |A^2_p(t, s) - \sigma^\ast(t, s) \delta X(s)| + |B^2_p(t, s) - \sigma^\ast(t, s) \dot{u}(s)| ds,
\]
and
\[
L(p, \lambda, s) := X^\ast(s) + \lambda(X^p(s) - X^\ast(s)),
\]
\[
M(p, \lambda, s) := u^\ast(s) + \lambda(u^p(s) - u^\ast(s)).
\]
Therefore, we have
\[
E \int_0^T e^{-rt} |\bar{X}^p(t)|^2 dt \\
\leq CE \int_0^T e^{-rt} \int_0^T (|A^1_p(t, s)|^2 + |A^2_p(t, s)|^2) |\bar{X}^p(s)|^2 ds dt \\
+ CE \int_0^T e^{-rt} \int_0^T (|D^1_p(t, s)|^2 + |D^2_p(t, s)|^2) ds dt \\
\leq CE \int_0^T e^{-rt} |\bar{X}^p(t)|^2 dt + CE \int_0^T e^{-rt} (|D^1_p(t, s)|^2 + |D^2_p(t, s)|^2) ds dt.
\]
By choosing a proper \( r \) such that \( \frac{C}{r} < 1 \), we have
\[
E \int_0^T |\tilde{X}^P(t)|^2 dt \leq CE \int_0^T \int_0^T e^{(T-t)} (|D_1^P(t,s)|^2 + |D_2^P(t,s)|^2) dsdt.
\]
Applying Lebesgue’s dominated convergence theorem, we have
\[
\lim_{p \to 0} E \int_0^T \int_0^T |D_i^P(t,s)|^2 dsdt = 0, \quad i = 1, 2.
\]
So,
\[
\lim_{p \to 0} E \int_0^T |\tilde{X}^P(t)|^2 dt = 0.
\]
(2). By the BSVIEs in (6) and (9), we have
\[
\tilde{Y}^P(t) = \int_t^T \frac{1}{2} [g^P(t,s) - g^*(t,s) - pg^*_g(t,s)\delta Y(s) - pg^*_g(t,s)\delta Z(s,t)]
- pg^*_u(t,s)\tilde{u}(s))ds - \int_t^T \tilde{Z}^P(t,s)dB_s, \quad t \in [0,T].
\]
Let
\[
N(p,\lambda,s) := Y^*(s) + \lambda(Y^P(s) - Y^*(s)),
\]
\[
P(p,\lambda,t,s) := Z^*(s,t) + \lambda(Z^P(s,t) - Z^*(s,t))
\]
and
\[
C_1^P(t,s) := \int_0^1 g_x(t,s, L(p,\lambda,s), N(p,\lambda,s), P(p,\lambda,t,s), M(p,\lambda,s))d\lambda,
\]
\[
C_2^P(t,s) := \int_0^1 g_y(t,s, L(p,\lambda,s), N(p,\lambda,s), P(p,\lambda,t,s), M(p,\lambda,s))d\lambda,
\]
\[
C_3^P(t,s) := \int_0^1 g_z(t,s, L(p,\lambda,s), N(p,\lambda,s), P(p,\lambda,t,s), M(p,\lambda,s))d\lambda,
\]
\[
D^P(t,s) := [C_1^P(t,s) - g^*_x(t,s)]\delta X(s) + [C_2^P(t,s) - g^*_y(t,s)]\delta Y(s) + [C_3^P(t,s) - g^*_z(t,s)]\delta Z(s,t)
\]
Thus,
\[
\tilde{Y}^P(t) = \int_t^T [C_1^P(t,s)\tilde{X}^P(s) + C_2^P(t,s)\tilde{Y}^P(s) + C_3^P(t,s)\tilde{Z}^P(s,t) + D^P(t,s)]ds
- \int_t^T \tilde{Z}^P(t,s)dB_s, \quad t \in [0,T].
\]
In Lemma 2 we take \( \psi = 0 \), \( g_0(t,s) = C_1^P(t,s)\tilde{X}^P(s) + D^P(t,s) \). Then
\[
\|\tilde{Y}^P(t), \tilde{Z}^P(t,s)\|_{H^2[0,T]} = E[\int_0^T |\tilde{Y}^P(t)|^2 dt + \int_0^T T^T |\tilde{Z}^P(t,s)|^2 dsdt]
\leq CE \int_0^T (\int_0^T |C_1^P(t,s)\tilde{X}^P(s)| ds)^2 + (\int_0^T |D^P(t,s)| ds)^2) dt.
\]
Applying Lebesgue’s dominated convergence theorem, we have
\[
\lim_{p \to 0} E \int_0^T (\int_0^T |D_i^P(t,s)| ds)^2 dt \to 0.
\]
Using the obtained first result, we can get the desired results.

3.3 Variational inequality

In this subsection, using Ekeland’s variational principle (3), we get the variational inequality.

Lemma 6 (Ekeland’s variational principle). Let \( (V,d(\cdot,\cdot)) \) be a complete metric space and \( F(\cdot) : V \rightarrow R \) be a proper lower semi-continuous function bounded from below. Suppose that for some \( \varepsilon > 0 \), there exists
\[ u \in V \] satisfying \( F(u) \leq \inf_{v \in V} F(v) + \varepsilon \). Then there exists \( u_\varepsilon \in V \) such that

(i) \( F(u_\varepsilon) \leq F(u) \),

(ii) \( d(u,u_\varepsilon) \leq \varepsilon \),

(iii) \( F(v) + \sqrt{\varepsilon}d(v,u_\varepsilon) \geq F(u_\varepsilon), \quad \forall v \in V \).
Given the optimal control pair \((\psi^*(\cdot), u^*(\cdot)) \in \mathcal{U}\), introduce a mapping \(F_{\varepsilon} : \mathcal{U} \to \mathbb{R}\) by

\[
F_{\varepsilon}(\psi, u) := \left\{ \int_0^T EY(t)dt - \alpha^2 + \int_0^T EY(t) - \rho(t)^2 dt + \max(0, \int_0^T EK(Y^*(s)) ds - \int_0^T EK(Y(s)) ds + \varepsilon) \right\}^2 \\
+ \left\{ \max(0, \int_0^T EK(X^*(T)) - \rho(t) dt - \int_0^T EK(Y(t)) dt + \varepsilon) \right\}^2 \\
+ \left\{ \max(0, \int_0^T EK(X^*(T)) - \rho(t) dt - \int_0^T EK(Y(t)) dt + \varepsilon) \right\}^2 \\
+ \left\{ \max(0, \int_0^T EK(X^*(T)) - \rho(t) dt - \int_0^T EK(Y(t)) dt + \varepsilon) \right\}^2 \\
+ \left\{ \max(0, \int_0^T EK(X^*(T)) - \rho(t) dt - \int_0^T EK(Y(t)) dt + \varepsilon) \right\}^2 \\
\right\}^{1/2},
\]

where \(l_i^*(t, s) = l_i(t, s, X^*(s), Y^*(s), Z^*(s, t), u^*(s)), l_i(t, s) = l_i(t, s, X(s), Y(s), Z(s, t), u(s)), i = 1, 2, \varepsilon \) is an arbitrary positive constant, \(l_i, q, h, k\) satisfy \((A_1), (A_2), (A_3)\).

**Remark 7.** Under \((A_1) - (A_4)\), from the well-posedness of BSIEs (Lemma 6) as well as the proof of Lemma 6 we know that \(F_{\varepsilon} : \mathcal{U} \to \mathbb{R}\) is a continuous function on \(\mathcal{U}\).

**Theorem 8.** Let \((\psi^*(\cdot), u^*(\cdot)) \in \mathcal{U}\) be the optimal control pair. Under the assumptions \((A_1) - (A_4)\), there exist a deterministic function \(h_0(\cdot) \in \mathbb{R}^m, h_0 \in \mathbb{R}^m, h_1, h_1, h_2, h_3, h_4 \in \mathbb{R}, h_1, h_2, h_3, h_4 \leq 0, |h_0| + |h_0(\cdot)| + |h_1| + |h_2| + |h_3| + |h_4| \neq 0\) such that the following variational inequality holds

\[
\int_0^T E(h_0(t, \tilde{t}) + h_0 \cdot \delta Y(t)) dt + \int_0^T E(q_\varepsilon(\psi(t), \tilde{\psi}(t))) dt + h_1 \cdot h_1(\psi(t), \tilde{\psi}(t)) dt + \int_0^T E(h_2(\psi(t), \tilde{\psi}(t)), \delta X(t)) dt + h_3 \cdot h_3(\psi(t), \tilde{\psi}(t), \delta Y(t)) dt + h_4 \cdot h_4(\psi(t), \tilde{\psi}(t), \delta Z(t)) dt \\
+ h_4 \cdot h_4(\psi(t), \tilde{\psi}(t), \delta Z(t)) dt + h_4 \cdot h_4(\psi(t), \tilde{\psi}(t), \delta Z(t)) dt \\
+ h_4 \cdot h_4(\psi(t), \tilde{\psi}(t), \delta Z(t)) dt + h_4 \cdot h_4(\psi(t), \tilde{\psi}(t), \delta Z(t)) dt \\
+ h_4 \cdot h_4(\psi(t), \tilde{\psi}(t), \delta Z(t)) dt + h_4 \cdot h_4(\psi(t), \tilde{\psi}(t), \delta Z(t)) dt \\
\right\}^{1/2},
\]

where \(l_i^*(t, s) = l_i(t, s, X^*(s), Y^*(s), Z^*(s, t), u^*(s)), l_i(t, s) = l_i(t, s, X(s), Y(s), Z(s, t), u(s)), i = 1, 2, \varepsilon \) is the derivative of \(l_i^*(t, s)\) with respect to \(k\), respectively.

**Proof.** It is easy to check that the following properties hold

(i) \(F_{\varepsilon}(\psi^*(\cdot), u^*(\cdot)) = \sqrt{2}\varepsilon\),
(ii) \(F_{\varepsilon}(\psi, u) > 0, \forall (\psi, u) \in \mathcal{U}\),
(iii) \(F_{\varepsilon}(\psi^*(\cdot), u^*(\cdot)) \leq \inf_{(\psi, u) \in \mathcal{U}} F_{\varepsilon}(\psi, u) + \sqrt{2}\varepsilon\).

Then from Lemma 6 (Ekeland’s variational principle), we can find a \((\psi^*(\cdot), u^*(\cdot)) \in \mathcal{U}\), such that

(i) \(F_{\varepsilon}(\psi^*(\cdot), u^*(\cdot)) \leq F_{\varepsilon}(\psi^*(\cdot), u^*(\cdot))\),
(ii) \(d(\psi^*(\cdot), u^*(\cdot)), (\psi^*(\cdot), u^*(\cdot)) \leq \sqrt{2}\varepsilon\),
(iii) \(F_{\varepsilon}(\psi, u) + \sqrt{2}\varepsilon d((\psi, u), (\psi^*(\cdot), u^*(\cdot))) \geq F_{\varepsilon}(\psi^*(\cdot), u^*(\cdot)), \forall (\psi, u) \in \mathcal{U}\).

For each \((\psi(\cdot), u(\cdot)) \in \mathcal{U}\), we define

\[
(\tilde{\psi}(\cdot), \tilde{u}(\cdot)) := (\psi(\cdot) - \psi^*(\cdot), u(\cdot) - u^*(\cdot)), (\tilde{\psi}(\cdot), \tilde{u}(\cdot)) := (\psi^*(\cdot) - \psi^*(\cdot), u^*(\cdot) - u^*(\cdot)),
\]

then \((\psi^*_p, u^*_p) := (\psi^*(\cdot) + p\tilde{\psi}(\cdot), u^*(\cdot) + p\tilde{u}(\cdot)) \in U\). Indeed, \((\psi^*(\cdot), u^*(\cdot)) \in U\), \((\tilde{\psi}(\cdot) + \psi^*(\cdot), \tilde{u}(\cdot) + u^*(\cdot)) \in U\), then

\[
(\psi^*_p, u^*_p) := (\psi^*(\cdot) + p\tilde{\psi}(\cdot), u^*(\cdot) + p\tilde{u}(\cdot)) = (1 - p)^{(\psi^*(\cdot) + \psi^*(\cdot)} + (1 - p)^{u^*(\cdot) + u^*(\cdot)}) \in U.
\]

Let \((X^*_p, Y^*_p, Z^*_p, \cdot)\) (resp. \((X^*_p, Y^*_p, Z^*_p, \cdot)\) be the solution of BSIE \(\psi^*(\cdot), u(\cdot) = (\psi^*_p, u^*_p)\) (resp. \(\psi(\cdot), u(\cdot) = (\psi^*(\cdot), u^*(\cdot))\)). From Ekeland’s variational principle, it follows that

\[
F_{\varepsilon}(\psi^*_p, u^*_p) + \sqrt{2}\varepsilon d((\psi^*_p, u^*_p), (\psi^*(\cdot), u^*(\cdot))) - F_{\varepsilon}(\psi^*(\cdot), u^*(\cdot)) \geq 0.
\]

We consider the following variational equation:

\[
\left\{
\begin{array}{l}
\delta X(t) = \int_0^t b_0^*(t, s) \delta X(s) ds + \int_0^t \sigma_0^*(t, s) \delta X(s) dB_s \\
+ \int_0^t \sigma_0^*(t, s) \delta X(s) ds + \int_0^t \sigma_0^*(t, s) \delta X(s) dB_s, \\
\delta Y(t) = \tilde{\psi}(t) + \int_0^t g_0^*(t, s) \delta Y(s) ds + \int_0^t g_0^*(t, s) \delta Y(s) ds + \int_0^t g_0^*(t, s) \delta Z(s, t) + g_0^*(t, s) \delta Z(s, t) + g_0^*(t, s) \delta Z(s, t) dB_s,
\end{array}
\right.
\]
where \( f_k(t, s) = f_k(t, s, X^\varepsilon(s), Y^\varepsilon(s), Z^\varepsilon(s, t), u^\varepsilon(s)), k = x, y, \zeta, u, f = b, \sigma, g \), respectively.

Similarly to Lemma 3 we have

\[
\lim_{p \to 0} E \int_0^T \frac{|X^\varepsilon_p(t) - X^\varepsilon(t)|^2}{p} - \delta X^\varepsilon(t) dt = 0, \quad \lim_{p \to 0} E \int_0^T \frac{|Y^\varepsilon_p(t) - Y^\varepsilon(t)|^2}{p} - \delta Y^\varepsilon(t) dt = 0,
\]

which lead to the following expansions:

\[
EX^\varepsilon(t) - EX^\varepsilon(t) = pE\delta X^\varepsilon(t) + o(p), \\
EY^\varepsilon(t) - EY^\varepsilon(t) = pE\delta Y^\varepsilon(t) + o(p), \\
\int_0^T |EY^\varepsilon_p(t) - \rho(t)|^2 dt - \int_0^T |EY^\varepsilon(t) - \rho(t)|^2 dt = \int_0^T 2p(EY^\varepsilon(t) - \rho(t), E\delta Y^\varepsilon(t)) dt + o(p).
\]

From (A1), we have

\[
\int_0^T E\{E(\psi^\varepsilon(t)) dt\} - \int_0^T E\{E(\psi^\varepsilon(t)) dt\} = p \int_0^T E\{g_2(\psi^\varepsilon(t)), \hat{\psi}^\varepsilon(t) dt\} + o(p), \\
Eh(X^\varepsilon(T)) - E(X^\varepsilon(T)) = pE(h_x(X^\varepsilon(T)), \delta X^\varepsilon(T)) + o(p), \\
\int_0^T E\{Ek(Y^\varepsilon_p(t)) dt\} - \int_0^T E\{Ek(Y^\varepsilon(t)) dt\} = p \int_0^T E\{k_2(Y^\varepsilon(s)), \delta Y^\varepsilon(s) ds + o(p), \\
\int_0^T E\{El^\varepsilon_1(t, s)|dsdt - \int_0^T E\{El^\varepsilon_1(t, s)|dsdt = p \int_0^T E\{El^\varepsilon_1(t, s) + El^\varepsilon_2(t, s)|dsdt + o(p), \\
\int_0^T E\{El^\varepsilon_2(t, s)|dsdt - \int_0^T E\{El^\varepsilon_2(t, s)|dsdt = p \int_0^T E\{El^\varepsilon_2(t, s) + El^\varepsilon_3(t, s)|dsdt + o(p), \\
\int_0^T E\{El^\varepsilon_3(t, s)|dsdt - \int_0^T E\{El^\varepsilon_3(t, s)|dsdt = p \int_0^T E\{El^\varepsilon_3(t, s) + El^\varepsilon_4(t, s)|dsdt + o(p).
\]

Furthermore, the following expansions hold:

\[
\int_0^T E\{E(\psi^\varepsilon(t)) dt\} - \int_0^T E\{E(\psi^\varepsilon(t)) dt\} = p \int_0^T E\{g_2(\psi^\varepsilon(t)), \hat{\psi}^\varepsilon(t) dt\} + o(p), \\
2(\int_0^T E\{E(Y^\varepsilon(t)) dt\} - \int_0^T E\{E(Y^\varepsilon(t)) dt\}) + o(p), \\
\int_0^T E\{E(\psi^\varepsilon(t)) dt\} - \int_0^T E\{E(\psi^\varepsilon(t)) dt\} + \int_0^T E\{E(\psi^\varepsilon(t)) dt\} + o(p), \\
2(\int_0^T E\{E(Y^\varepsilon(t)) dt\} - \int_0^T E\{E(Y^\varepsilon(t)) dt\}) + o(p), \\
\int_0^T E\{E(\psi^\varepsilon(t)) dt\} - \int_0^T E\{E(\psi^\varepsilon(t)) dt\} + \int_0^T E\{E(\psi^\varepsilon(t)) dt\} + o(p), \\
2(\int_0^T E\{E(Y^\varepsilon(t)) dt\} - \int_0^T E\{E(Y^\varepsilon(t)) dt\}) + o(p), \\
\int_0^T E\{E(\psi^\varepsilon(t)) dt\} - \int_0^T E\{E(\psi^\varepsilon(t)) dt\} + \int_0^T E\{E(\psi^\varepsilon(t)) dt\} + o(p). \\
\]

For the given \( \varepsilon \), we consider the following cases:

Case 1. There exists \( r > 0 \) such that, for any \( p \in (0, r) \),

\[
\int_0^T E\{E(\psi^\varepsilon(t)) dt\} - \int_0^T E\{E(\psi^\varepsilon(t)) dt\} + \varepsilon > 0, \\
Eh(X^\varepsilon(T)) - E(X^\varepsilon(T)) + \varepsilon > 0, \\
\int_0^T E\{E(Y^\varepsilon(t)) dt\} - \int_0^T E\{E(Y^\varepsilon(t)) dt\} + \varepsilon > 0, \\
\int_0^T E\{El^\varepsilon_1(t, s)|dsdt - \int_0^T E\{El^\varepsilon_1(t, s)|dsdt + \varepsilon > 0, \\
\int_0^T E\{El^\varepsilon_2(t, s)|dsdt - \int_0^T E\{El^\varepsilon_2(t, s)|dsdt + \varepsilon > 0, \\
\int_0^T E\{El^\varepsilon_3(t, s)|dsdt - \int_0^T E\{El^\varepsilon_3(t, s)|dsdt + \varepsilon > 0.
\]
Then
\[
\begin{align*}
\lim_{p \to 0} \frac{E_{\nu}(\psi^p_0(-), u^p_0(-)) - F_{\nu}(\psi^p_0(-), u^p_0(-))}{p} & = \lim_{p \to 0} \frac{F^2(\psi^p_0(-), u^p_0(-)) - F^2(\psi^p_0(-), u^p_0(-))}{p} \\
& = \frac{1}{F_{\nu}(\psi^p_0(-), u^p_0(-))} \left\{ \int_0^T EY^* \langle t \rangle dt - a \int_0^T E\delta Y^* \langle t \rangle dt + \int_0^T (EY^* - \rho(t), E\delta Y^* \langle t \rangle dt \\
& - \left[ \int_0^T E\rho(t) \langle t \rangle dt - \int_0^T E\rho(t) \langle t \rangle dt \right] + \int_0^T E\rho(t) \langle t \rangle dt + \int_0^T E\rho(t) \langle t \rangle dt + \int_0^T E\rho(t) \langle t \rangle dt \right\}.
\end{align*}
\]

Set
\[
\begin{align*}
\tilde{h}_0^0 &= \frac{\int_0^T EY^* \langle t \rangle dt - a \int_0^T E\delta Y^* \langle t \rangle dt}{F_{\nu}(\psi^p_0(-), u^p_0(-))}, \\
\tilde{h}_0^1 &= -\frac{1}{F_{\nu}(\psi^p_0(-), u^p_0(-))} \left[ \int_0^T E\rho(t) \langle t \rangle dt - \int_0^T E\rho(t) \langle t \rangle dt \right],
\end{align*}
\]

Then it follows from (10),
\[
\begin{align*}
& \int_0^T E\rho(t) \langle t \rangle dt - \int_0^T E\rho(t) \langle t \rangle dt + \int_0^T E\rho(t) \langle t \rangle dt - \int_0^T \int_0^t E\rho(t) \langle t \rangle dt + \int_0^T \int_0^t E\rho(t) \langle t \rangle dt \leq 0,
\end{align*}
\]

Case 2. There exists a positive sequence \( \{p_n\} \), which satisfies \( p_n \to 0 \) such that
\[
\begin{align*}
& \int_0^T E\rho(t) \langle t \rangle dt - \int_0^T E\rho(t) \langle t \rangle dt + \int_0^T E\rho(t) \langle t \rangle dt - \int_0^T \int_0^t E\rho(t) \langle t \rangle dt + \int_0^T \int_0^t E\rho(t) \langle t \rangle dt \leq 0.
\end{align*}
\]

From the definition of \( F_{\nu} \), for enough large \( n \),
\[
F_{\nu}(\psi^p_{p_n}(-), u^p_{p_n}(-)) = \| Y^p_{p_n}(-) \| - \rho(0))^2 + \int_0^T | EY^p_{p_n}(t) - \rho(t) |^2 dt \}
\]

Since \( F_{\nu}(-) \) is continuous, we know \( F_{\nu}(\psi^p(-), u^p(-)) = \| \int_0^T EY^p(t) dt - a \|^2 + \int_0^T | EY^p(t) - \rho(t) |^2 dt \}
\]

Now
\[
\begin{align*}
& \lim_{n \to \infty} \frac{F_{\nu}(\psi^p_{p_n}(-), u^p_{p_n}(-)) - F_{\nu}(\psi^p(-), u^p(-))}{p_n} \\
& = \lim_{n \to \infty} \frac{1}{F_{\nu}(\psi^p_{p_n}(-), u^p_{p_n}(-))} F_{\nu}(\psi^p_{p_n}(-), u^p_{p_n}(-)) - F_{\nu}(\psi^p(-), u^p(-)) \\
& = \frac{1}{F_{\nu}(\psi^p(-), u^p(-))} \left\{ \int_0^T EY^p(t) dt - a \int_0^T E\delta Y^p(t) dt + \int_0^T \{ EY^p(t) - \rho(t), E\delta Y^p(t) \} dt \right\}.
\end{align*}
\]

Similar to Case 1, it follows from (10),
\[
\begin{align*}
\int_0^T E\tilde{h}_0^0(t) + h_0^0(t), \delta Y^* \langle t \rangle dt \geq -\sqrt{2E} | E\tilde{h}_0^0(t) \|^2 dt + E \int_0^T | \tilde{u}^* \langle t \rangle dt |^2 dt \}
\end{align*}
\]
where \( \tilde{h}^0 = \int_0^T EY^\varepsilon(t)dt - a \), \( \tilde{h}_0(t) = EY^\varepsilon(t) - g(t) \), \( \tilde{h}^1 = h^1 = h^2 = h^3 = h^4 = 0 \).

Similarly, we can prove (12) still holds for the other thirty cases.

In summary for given \( \varepsilon \), we have

(i) (12) holds,
(ii) \( h_1^2 \leq 0, h_1 \leq 0, h_2 \leq 0 \),
(iii) \( |h_1|^2 + \int_0^T |h_1(t)|^2 dt + |h_2|^2 + |h_3|^2 + |h_4|^2 = 1 \).

Hence there is a subsequence \((\tilde{h}^0, h^0, h^1, h^2, h^3, h^4)\) of \((\tilde{h}^0, h^0, h^1, h^2, h^3, h^4)\), such that \( \tilde{h}^0 \to h_0, h^0 \to h_0, h^1 \to h_1, h^2 \to h_2, h^3 \to h_3, h^4 \to h_4 \). Since \( h_1^2, h_2, h_3, h_4 \leq 0 \), we have \( h_1, h_2, h_3, h_4 \leq 0 \).

Because of \( d((\psi(\cdot), u^\varepsilon(\cdot)), (\psi(\cdot), u^\varepsilon(\cdot))) \leq \sqrt{2} \varepsilon \), we have \( (\psi(\cdot), u^\varepsilon(\cdot)) \to (\psi^*(\cdot), u^*(\cdot)) \) in \( U \).

Therefore, from the wellposedness of FBSVIEs, it is easy to check \( \delta X^\varepsilon(\cdot) \to \delta X(\cdot), \delta Y^\varepsilon(\cdot) \to \delta Y(\cdot) \), as \( \varepsilon \to 0 \). Furthermore, as \( \varepsilon \to 0 \)

\[
|E(h_2(X^\varepsilon(T)), \delta X^\varepsilon(T)) - E(h_2(X^\varepsilon(T)), \delta X(T))| = |E(h_2(X^\varepsilon(T)), \delta X^\varepsilon(T)) + E(h_2(X^\varepsilon(T)) - h_2(X^\varepsilon(T)), \delta X(T))| \to 0.
\]

Indeed, together with the Schwarz inequality, using the boundedness of \( h_2 \), we can get the limit of the first part goes to 0; from the continuity \( h_2 \), we get the second part also goes to 0. Similarly, as \( \varepsilon \to 0 \), we have

\[
\int_0^T E(|\int_0^t (\langle l_x^1, s \rangle, \delta X(s))dsdt|) dt \to \int_0^T E(|\int_0^t (\langle l_x^2, s \rangle, \delta X(s))dsdt|) dt, \int_0^T E(|\int_0^t (\langle l_y^1, s \rangle, \delta X(s))dsdt|) dt \to \int_0^T E(|\int_0^t (\langle l_y^2, s \rangle, \delta X(s))dsdt|) dt, \int_0^T E(|\int_0^t (\langle l^3, s \rangle, \delta X(s))dsdt|) dt \to \int_0^T E(|\int_0^t (\langle l^4, s \rangle, \delta X(s))dsdt|) dt.
\]

Let \( \varepsilon \to 0 \) in (12), the result holds. The proof is completed.

### 3.4 Maximal principle

We introduce the adjoint equation:

\[
\begin{cases}
    m(t) = A(t) + \int_0^T [b_x^0(s, t)^T m(s) + \sigma_x^0(s, t)^T n(s, t)] ds - \int_0^T n(t, s)dB_s, \\
    p(t) = B(t) + \int_0^T g_x^0(s, t)^T p(s) ds + \int_0^T E[g_x^0(s, t)^T |F_s]p(s) ds, \\
    A(t) = h_1 b_x^0(T, t)^T h_x(X^\varepsilon(T)) + h_2 \sigma_x^0(T, t)^T \pi(t) + \int_0^T g_x^0(s, t)^T p(s) ds, \\
    B(t) = h_0(t) + h_1 h_2 k_y(Y^\varepsilon(t)),
\end{cases}
\]

and \( h_x(X^\varepsilon(T)) = Eh_x(X^\varepsilon(T)) + \int_0^T \pi(s) dsB_s \).

By the duality principles, we get the following theorem:

**Theorem 9.** Assume that (A1) - (A4) hold and \( l_1, l_2 = 0 \). Let \((\psi^*(\cdot), u^*(\cdot))\) be the optimal control pair; \((X^*(\cdot), Y^*(\cdot), Z^*(\cdot))\) be the corresponding optimal trajectory. Then there exist a deterministic function \( h_0(\cdot) \in \mathbb{R}^m, h_0 \in \mathbb{R}^m, h_1, h_2 \leq 0 \) such that \( \forall (\psi(\cdot), u(\cdot)) \in U \),

\[
\begin{align*}
    \langle p(t) + h_1 g_x(\psi(t)), \psi(t) - \psi^*(t) \rangle + & \int_0^T \langle g_x^0(t, s)^T (p(t), u(s) - u^*(s))ds + h_1 b_x^0(T, t)^T h_x(X^\varepsilon(T)) + \sigma_x^0(T, t)^T \pi(t), u(t) - u^*(t) \rangle \\
    + & \int_0^T h_x^1(s, t)^T m(s) + \sigma_x^0(s, t)^T n(s, t), u(t) - u^*(t)ds \geq 0, \text{ a.e., a.s.}
\end{align*}
\]

where \((m(\cdot), n(\cdot, \cdot), p(\cdot))\) is the solution of the adjoint equation (13).
Proof. From the duality principles (Lemma 3, Lemma 4), we have the following relations
\[ E \int_0^T (A(t), \delta X(t))dt \]
and
\[ E \int_0^T \langle B(t), \delta Y(t) \rangle dt = E \int_0^T (p(t), \hat{\psi}(t)) + \int_0^T [g^*_x(t, s)\delta X(s) + g^*_u(t, s)\hat{u}(s)]ds dt. \]
Combined with the variational inequality (Theorem 8), we get
\[ 0 \leq E \int_0^T E(\tilde{h}_0 + h_0(t), \delta Y(t))dt + h_1E \int_0^T E(q_x(\psi^*(t)), \psi(t))dt + h_1E \int_0^T (h_x(X^*(T)), \delta X(T)) \]
\[ + h_2E \int_0^T E(k_p(Y^*(s)), \delta Y(s))ds \]
\[ = E \int_0^T E(\tilde{h}_0 + h_0(t), \delta Y(t))dt + \int_0^T E \langle g_x(\psi^*(t)), \psi(t) \rangle dt + h_1E \int_0^T (h_x(X^*(T)), \delta X(T)) \]
\[ + h_2E \int_0^T E(k_p(Y^*(s)), \delta Y(s))ds + E \int_0^T \int_t^T \sum_{i=1}^d n_i(s, t)\sigma^*_i(s, t)\hat{u}(t)ds dt \]
\[ + E \int_0^T \int_t^T m(s)\sigma^*_i(s, t)\hat{u}(t)ds dt + E \int_0^T \int_t^T \sum_{i=1}^d n_i(s, t)\sigma^*_i(s, t)\hat{u}(t)ds dt \]
\[ - E \int_0^T \langle A(t), \delta X(t) \rangle dt + E \int_0^T \langle p(t), \hat{\psi}(t) \rangle + \int_0^T [g^*_x(t, s)\delta X(s) + g^*_u(t, s)\hat{u}(s)]ds dt \]
\[ - E \int_0^T \langle B(t), \delta Y(t) \rangle dt \]
\[ = E \int_0^T (p(t) + h_1g_x(\psi^*(t)), \hat{\psi}(t))dt + E \int_0^T \int_t^T (g^*_x(t, s)\delta X(s) + g^*_u(t, s)\hat{u}(s))ds dt \]
\[ + h_1E \int_0^T \langle g^*_u(t, T)T\pi_x(t), \hat{u}(s) \rangle ds dt \]
\[ + E \int_0^T \int_t^T (b^*_x(s, t)Tm(s) + \sigma^*_u(s, t)Tn(s, t), \hat{u}(s))ds dt. \]
Since the above holds for all \((\psi(\cdot), u(\cdot)) \in \mathcal{U}\), we obtain
\[ \langle p(t) + h_1g_x(\psi^*(t)), \psi(t) - \psi^*(t) \rangle + \int_0^T [g^*_x(t, s)T\pi(t), u(s) - u^*(s)]ds \]
\[ + h_1[h^*_u(T, T)TX^*(T) + \sigma^*_u(T, T)\pi(t), u(t) - u^*(t)] \]
\[ + \int_0^T [b^*_x(s, t)Tm(s) + \sigma^*_u(s, t)Tn(s, t), u(t) - u^*(t)]ds \geq 0, \text{ a.e., a.s.} \]
\[ \Box \]

When \(l_1, l_2 \neq 0\), the associated adjoint equation is:
\[ \begin{cases}
    m(t) = A(t) + \int_0^T [b^*_x(s, t)Tm(s) + \sigma^*_u(s, t)Tn(s, t)]ds - \int_0^T n(t, s)dB_s, \\
    p(t) = B(t) + \int_0^T g^*_x(s, t)T\pi(t), u(s) - u^*(s)]ds
\end{cases} \quad (14) \]
where
\[ \begin{cases}
    A(t) = h_1\sigma_x^*(T, t)T\pi(t) + h_1b^*_u(T, t)TX^*(T) + \int_0^T g^*_x(s, t)T\pi(t), u(t) - u^*(t)] \\
    B(t) = h_0(t) + \tilde{h}_0 + h_2k_p(Y^*(t)) + \int_0^T h^*_x(T, t)Y(t) + \int_0^T h^*_u(T, t)U(t), ds \\
    + \int_0^T h^*_x(T, t)\pi(t), u(t) - u^*(t)]ds + \int_0^T h^*_u(T, t)\pi(t), u(t) - u^*(t)]ds \geq 0, \text{ a.e., a.s.}
\end{cases} \]

Similarly, we have the following maximum principle:

**Theorem 10.** Assume \((A_1) - (A_4)\) hold. Let \((\psi^*(\cdot), u^*(\cdot))\) be the optimal control pair; \((X^*(\cdot), Y^*(\cdot), Z^*(\cdot, \cdot))\) be the corresponding optimal trajectory. Then there exist a deterministic function \(h_0(\cdot) \in \mathbb{R}^m\),

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\[\tilde{h}_0 \in \mathbb{R}^m, \tilde{h}_1, h_2, h_3, h_4 \leq 0 \text{ such that } \forall (\psi(\cdot), u(\cdot)) \in U,
\]
\[\langle p(t) + \tilde{h}_1 q_x(\psi^*(t)), \psi(t) - \psi^*(t) \rangle + \int_t^T (g^*_w(s, t) p(s) + u(s) - u^*(s)) ds + \int_t^T (\tilde{h}_2 u(s) + \tilde{h}_3 u(s) - u^*(s)) ds \geq 0, \ a.e., a.s.\]

where \((m(\cdot), n(\cdot), p(\cdot))\) is the solution of the adjoint equation \([14]\).

**Remark 11.** When the terminal condition \(\psi(\cdot)\) is replaced by \(\psi(\cdot) + \varphi(X(T))\) in [3], the above methods can still go through.

### 4 Examples

First we will give an example associated with the model studied above.

**Example 12.** Consider the following controlled system \((m = d = 1)\):

\[
\begin{align*}
X(t) &= \int_0^T t u(s) dB_s, \\
Y(t) &= \psi(t) + \int_0^T (t - 1) u(s) ds - \int_t^1 Z(t, s) dB_s, \ t \in [0, 1],
\end{align*}
\]

with the control domain

\[U = \{(\psi(\cdot), u(\cdot))|\psi(\cdot) \in L^2_{\mathcal{F}_T}[0, 1), \ u(\cdot) \in L^2[0, 1), \ \psi(t) \in [0, 1], \ u(t) \in [\frac{1}{2}, 1], \ a.e., a.s.\}\]

and the objective function

\[J(\psi(\cdot), u(\cdot)) = E \{X(1)^2 + Y(0)\}. \quad (16)\]

We will minimize the objective function under the constraints \((\psi(\cdot), u(\cdot)) \in U\). After substituting \(X(1), Y(0)\) into the objective function, we get

\[J(\psi(\cdot), u(\cdot)) = E[\int_0^1 u(s)^2 ds + \psi(0) - \int_0^1 u(s) ds]. \quad (17)\]

From \((17)\), we obtain the optimal control:

\[\psi^*(s) = \begin{cases} 0, & s = 0, \\
\text{values in } [0, 1], & s \in (0, 1], \\
\frac{1}{2}, & s \in [0, 1].\end{cases}\]

So, \[\min_{(\psi(\cdot), u(\cdot)) \in U} J(\psi(\cdot), u(\cdot)) = -\frac{1}{4}.\]

At last we give an example to show the form of the optimal terminal \(\psi(\cdot)\).

**Example 13.** For convenience, we suppose \(m = d = 1\), and consider a simple BSVIE as follows:

\[Y(t) = \psi(t) + \int_t^1 [AY(s) + BZ(s, t)] ds - \int_t^1 Z(t, s) dB_s, \ t \in [0, 1],\]

\[A, B \in \mathbb{R}. \]

We will maximize the objective function \(J(\psi(\cdot)) = \frac{1}{2} E \left[\int_0^1 \psi(s)^2 ds\right]\), subject to \(\psi(\cdot) \in L^2_{\mathcal{F}_T}(0, 1), \ \psi(t) \in [0, 1], \ EY^\psi(t) = \rho(t), \ t \in [0, 1], \ a.e., a.s.\)

From Subsection 3.4, we know the adjoint process \(p(\cdot)\) satisfies

\[p(t) = h_0(t) + (A + B)\tilde{h}_1 + \int_t^T A p(s) ds + \int_t^1 B p(s) dB_s, \ t \in [0, 1].\]

Applying Theorem \([9]\) we have, if \(\psi^*(\cdot)\) is optimal to \(J(\psi(\cdot))\), then there exists a deterministic function \(h_0(\cdot), \tilde{h}_1 \leq 0, |h_0(\cdot)| + |\tilde{h}_1| \neq 0\) such that, for any \(\psi(\cdot)\),

\[(p(t) + \tilde{h}_1 \psi^*(t)) (\psi(t) - \psi^*(t)) \geq 0, \ t \in [0, 1], \ a.s.\]

Similar to the example in Ji, Zhou \([21]\), let \(\Omega_1 := \{\omega, t \in \Omega \times [0, 1] | \psi^*(t, \omega) = 0\}, \ \Omega_2 := \{\omega, t \in \Omega \times [0, 1] | \psi^*(t, \omega) = 1\}, \) we obtain \(\psi^*(\cdot)\) satisfies

\[p(t) + \tilde{h}_1 \psi^*(t) \geq 0, \ \omega, t \in \Omega_1, \ a.s.\]

\[p(t) + \tilde{h}_1 \psi^*(t) \leq 0, \ \omega, t \in \Omega_2, \ a.s.\]

\[p(t) + \tilde{h}_1 \psi^*(t) = 0, \ \omega, t \in \Omega - \Omega_1 - \Omega_2, \ a.s.\]
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