Geometric Random Edge

Friedrich Eisenbrand*  Santosh Vempala†
EPFL  Georgia Tech

September 2, 2014

Abstract

We show that a variant of the random-edge pivoting rule results in a strongly polynomial time simplex algorithm for linear programs max{c^T x: Ax ≤ b}, whose constraint matrix A satisfies a geometric property introduced by Brunsch and Röglins: The sine of the angle of a row of A to a hyperplane spanned by n − 1 other rows of A is at least δ.

This property is a geometric generalization of A being integral and all sub-determinants of A being bounded by Δ in absolute value (since δ ≥ 1/(Δ^2 n)). In particular, linear programs defined by totally unimodular matrices are captured in this framework (δ ≥ 1/n) for which Dyer and Frieze previously described a strongly polynomial-time randomized algorithm.

The number of pivots of the simplex algorithm is polynomial in the dimension and 1/δ and independent of the number of constraints of the linear program. Our main result can be viewed as an algorithmic realization of the proof of small diameter for such polytopes by Bonifas et al., using the ideas of Dyer and Frieze.

*Email: friedrich.eisenbrand@epfl.ch
†Email: vempala@gatech.edu
1 Introduction

Our goal is to solve a linear program

\[
\begin{align*}
\max & \quad c^T x \\
Ax & \leq b
\end{align*}
\]  

where \( A \in \mathbb{R}^{m \times n} \) is of full column rank, \( b \in \mathbb{R}^m \) and \( c \in \mathbb{R}^n \). The rows of \( A \) are denoted by \( a_i, 1 \leq i \leq m \). We assume without loss of generality that \( \|a_i\| = 1 \) holds for \( i = 1, \ldots, m \). They shall have the following \( \delta \)-distance property (we use \( \langle \cdot \rangle \) to denote linear span):

For any \( I \subseteq [m] \), and \( j \in [m] \), if \( a_j \notin \langle a_i : i \in I \rangle \) then \( d(a_j, \langle a_i : i \in I \rangle) \geq \delta \). In other words, if \( a_j \) is not in the span of the \( a_i, i \in I \), then the distance of \( a_j \) to the subspace that is generated by the \( a_i, i \in I \) is at least \( \delta \).

The \( \delta \)-distance property is a geometric generalization of the algebraic property of a matrix being integral to having small sub-determinants. Suppose, for example, that \( A \) is totally unimodular, i.e., all \( k \times k \) sub-determinants are bounded by one in absolute value and let \( a_1, \ldots, a_n \) be \( n \) linearly independent rows of \( A \). The adjoint matrix \( A = (b_1, \ldots, b_n) \) of the matrix with rows \( a_1, \ldots, a_n \) is an integer matrix with all components in \( \{0, \pm 1\} \). The vector \( b_1 \) is orthogonal to \( a_2, \ldots, a_n \) and \( |a_i^T b_1| \geq 1 \), since the vectors are integral. The distance of \( a_1 \) to the sub-space generated by \( a_2, \ldots, a_n \) is thus at least \( 1/\|b_1\| \geq 1/\sqrt{n} \) which means that totally unimodular matrices have the \( \delta \)-distance property for \( \delta = 1/n \). Integral matrices whose sub-determinants are bounded by \( \Delta \) in absolute value have the \( \delta \)-distance property with \( \delta \geq 1/(\Delta^2 n) \).

In this paper, we analyze the simplex algorithm [8] with a variant of the random edge pivoting rule. Our main result is a strongly polynomial running time bound for linear programs satisfying the \( \delta \)-distance property.

Theorem 1. There is a random edge pivot rule that solves a linear program using \( \text{poly}(n, 1/\delta) \) pivots in expectation. The expected running time of this variant of the simplex algorithm is polynomial in \( n, m \) and \( 1/\delta \).

One of the most important consequences of the ellipsoid method [18] is that it results in a polynomial time algorithm for linear programming, if a polynomial-time algorithm for the separation problem is known [14]. This is particularly important for classes of linear programs with an exponential number of constraints. The ellipsoid method does not require an explicit description of \( Ax \leq b \). A similar situation holds for our simplex-variant. Its expected running time is polynomial in \( 1/\delta, n \) and the running time of an algorithm that computes the neighbors of a given vertex of a non-degenerate polyhedron. We discuss this in greater detail in the remarks section.

Our result is an extension of a randomized simplex-type algorithm of Dyer and Frieze [9] that solves linear programs [1] for totally unimodular \( A \in \{0, \pm 1\}^{m \times n} \) and arbitrary \( b \) and \( c \). Also, our algorithm is a strengthening of a recent randomized algorithm of Brunsch and Röglin [3] who compute a path between two given vertices of a polytope that is defined by matrix satisfying the \( \delta \)-distance property. Their algorithm cannot be used to solve a linear program [1] since both vertices have to be given. The expected length of the path is bounded by a polynomial in \( n, m \) and \( 1/\delta \). Bonifas et al. [2] have shown that the diameter of a polytope defined by an integral constraint matrix \( A \) whose sub-determinants are bounded by \( \Delta \) is polynomial in \( n \) and \( \Delta \) and independent of the number of facets. In the setting of the \( \delta \)-distance property, their proof yields a polynomial bound in \( 1/\delta \) and the dimension \( n \) on the diameter that is independent of \( m \). Our result is an extension of this result in the setting of linear programming. We show that there is a variant of the simplex algorithm that uses a number of pivots that is polynomial in \( 1/\delta \) and the dimension \( n \).
The number $\delta$ can be seen as a measure of how far the bases of $A$ are from being singular. More precisely, if $A_B$ is an $n \times n$ minor of $A$ that is non-singular, then each column of $A_B^{-1}$ has Euclidean length $\geq \delta$. In a way, $\delta$ can be seen as a conditioning number of $A$. Conditioning plays an increasing role in the intersection of complexity theory and optimization, see, e.g. [4].

It is also interesting that the notion of $\delta$-distance property is independent of the numbers in $A$. The matrix $A$ can be irrational even. Also, there are examples of combinatorial optimization problems that have exponentially large sub-determinants, but $\delta$ is a polynomial. An edge-node incidence matrix of an undirected graph has two ones in each row. The remaining entries are zero. Although the largest subdeterminant can be exponential (it is $2$ raised to the odd-cycle-packing number of $G$ [24]) one has $\delta = \Omega(1/\sqrt{|V|})$.

Related work

General linear programming problems can be solved in weakly polynomial time [18, 17]. This means that the number of basic arithmetic operations performed by the algorithm is bounded by the binary encoding length of the input. It is a longstanding open problem whether there exists a strongly polynomial time algorithm for linear programming. Such an algorithm would run in time polynomial in the dimension and the number of inequalities on a RAM machine. Tardos [28] gave a strongly polynomial time algorithm for linear programs whose constraint matrix has integer entries that are bounded by a constant. For general linear programming, the simplex method is sub-exponential [21, 16].

Many combinatorial optimization problems can be formulated as linear programming problems with totally unimodular constraint matrices and the problem of finding polynomial time simplex-methods to solve some of these problems received a lot of attention in the literature, see, e.g. [23, 1, 13]. Dyer and Frieze [9] showed that a linear program with totally unimodular constraint matrix can be solved in strongly polynomial time with a randomized algorithm. The matrix $A$ being totally unimodular implies the $1/n$-distance property.

Spielman and Teng [27] have shown that the simplex algorithm with the shadow-edge pivoting rule runs in expected polynomial time if the input is randomly perturbed. This smoothed analysis paradigm was subsequently applied to many other algorithmic problems. Brunsch and Röglin [3] have shown that, given two vertices of a linear program satisfying the $\delta$-distance property, one can compute in expected polynomial time a path joining these two vertices with $O(mn^2/\delta^2)$ edges in expectation. However, the two vertices need to be known in advance. The authors state the problem of finding an optimal vertex w.r.t. a given objective function vector $c$ in polynomial time as an open problem. We solve this problem and obtain a path whose length is independent in the number $m$ of inequalities.

Klee and Minty [19] have shown that the simplex method is exponential if Dantzig’s original pivoting rule is applied. More recently, Friedmann, Hansen and Zwick [11] have shown that the random edge results in a superpolynomial number of pivoting operations. Here, random edge means to choose an improving edge uniformly at random. The authors also show such a lower bound for random facet. Friedmann [10] also recently showed a superpolynomial lower bound for Zadeh’s pivoting rule. Nontrivial, but exponential upper bounds for random edge are given in [12].

We also want to mention a classical result of Clarkson [5] who showed that a linear program defined by $m$ constraints in dimension $n$ can be reduced to $O(n \cdot \ln m)$ linear programs with $O(n^2)$ constraints each in expected polynomial time. Clarkson’s method shows that there exists a strongly polynomial time algorithm for linear programming if and only if there exists one for linear programs defined by $O(n^2)$ constraints. In contrast, our algorithm operates on the full set of constraints and performs a number of pivots which does not depend on the number of
Assumptions
Throughout we assume that \( c \) and the rows of \( A \), denoted by \( a_i, 1 \leq i \leq m \), have Euclidean norm \( \| \cdot \|_2 \) one. We also assume that the linear program is non-degenerate, meaning that for each feasible point \( x^* \), there are at most \( n \) constraints that are satisfied by \( x^* \) with equality. It is well known that this assumption can be made without loss of generality \([25]\).

2 Identifying an element of the optimal basis
Before we describe our variant of the random-edge simplex algorithm, we explain the primary goal, which is to identify one inequality of the optimal basis. Then, we can continue with the search for other basis elements by running the simplex algorithm in one dimension lower.

Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) be the polyhedron of feasible solutions of (1). Without loss of generality, see Section 6, we can assume that \( P \) is a polytope, i.e., that \( P \) is bounded. Let \( v \in P \) be a vertex. The normal cone \( C_v \) of \( v \) is the set of vectors \( w \in \mathbb{R}^n \) such that, if \( c^T x \) is replaced by \( w^T x \) in (1), then \( v \) is an optimal solution of that linear program. Equivalently, let \( B \subseteq \{1, \ldots, m\} \) be the \( n \) row-indices with \( a_i^T v = b_i \), \( i \in B \), then the normal cone of \( v \) is the set \( \{ \sum_{i \in B} \lambda_i a_i : \lambda_i \geq 0, i \in B \} \). The cones \( C_u \) and \( C_v \) of two vertices \( u \neq v \) intersect if and only if \( u \) and \( v \) are neighboring vertices of \( P \). In this case, they intersect in the common facet cone \( \{ a_i : i \in B_u \cap B_v \} \), where \( B_u \) and \( B_v \) are the indices of tight inequalities of \( u \) and \( v \) respectively, see Figure 1.

Figure 1: Two neighboring vertices \( u \) and \( v \) and their normal-cones \( C_u \) and \( C_v \).

Suppose now that we found a point \( c' \) with \( \| c' \| = 1 \) such that:

a) \( \| c - c' \| < \delta/(2 \cdot n) \), and

b) we know a vertex \( v \) of \( P \) with \( c' \in C_v \).

How to compute such a \( c' \) and \( v \) is described in the next section, where we show that this can be done with a randomized simplex algorithm. The following variant of a lemma proved by Cook et al. \([6]\) allows us then to identify at least one index of the optimal basis of (1). We provide a proof in the appendix.
Lemma 2. Let $B \subseteq \{1, \ldots, m\}$ be the optimal basis of the linear program (7) and let $B'$ be an optimal basis of the linear program (7) with $c$ being replaced by $c'$. Consider the conic combination

$$c' = \sum_{j \in B'} \mu_j a_j.$$  

(2)

For $k \in B' \setminus B$, one has

$$\|c - c'\| \geq \delta \cdot \mu_k.$$  

Following the notation of the lemma, let $B'$ be the optimal basis of the linear program with objective function $c'x$. We know this basis because of (11). Also, since (11) holds and since $\|c\| = 1$ we have $\|c'\| > 1 - \delta/(2 \cdot n)$. This means that there exists a $\mu_k$ with $\mu_k > 1/n \cdot (1 - \delta/(2 \cdot n))$. But then $k$ must be in $B$ since $\delta \cdot \mu_k > \delta/n \cdot (1 - \delta/(2 \cdot n)) \geq \delta/(2 \cdot n)$.

Once we have identified an index $k$ of the optimal basis $B$ of (11) we set this inequality to equality and let the simplex algorithm search for the next element of the basis on the induced face of $P$. This is, in fact, a $n - 1$-dimensional linear program with the $\delta$-distance property. We explain the details.

Suppose that the element from the optimal basis is $a_1$. Let $U \in \mathbb{R}^{n \times n}$ be a non-singular matrix that rotates $a_1$ into the first unit vector, i.e.

$$a_1^T \cdot U = e_1^T.$$  

The linear program $\max \{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ is equivalent to the linear program $\max \{c^T U \cdot x : x \in \mathbb{R}^n, A \cdot U \cdot x \leq b\}$. Notice that this transformation preserves the $\delta$-distance property. Therefore we can assume that $a_1$ is the first unit vector.

Now we can set the constraint $x_1 \leq b_1$ to equality and subtract this equation from the other constraints such that they do not involve the first variable anymore. The $a_i$ are in this way projected into the orthogonal complement of $e_1$. We scale the vectors and right-hand-sides with a scalar $\geq 1$ such that they lie on the unit sphere and are now left with a linear program with $n - 1$ variables that satisfies the $\delta$-distance property as we show now.

Lemma 3. Suppose that the vectors $a_1, \ldots, a_m$ satisfy the $\delta$-distance property, then $a_2^*, \ldots, a_m^*$ satisfy the $\delta$-distance property as well, where $a_i^*$ is the projection of $a_i$ onto the orthogonal complement of $a_1$.

Proof. Let $I \subseteq \{2, \ldots, m\}$ and $j \in \{2, \ldots, m\}$ such that $a_j^*$ is not in the span of the $a_i^*$, $i \in I$. Let $d(a_j^*, \langle a_i^* : i \in I \rangle) = \gamma$. Clearly, $d(a_j^*, \langle a_i : i \in I \cup \{1\} \rangle) \leq \gamma$ and since $a_j^*$ stems from $a_j$ by subtracting a suitable scalar multiple of $a_1$, we have $d(a_j, \langle a_i : i \in I \cup \{1\} \rangle) \leq \gamma$ and consequently $\gamma \geq \delta$. \qed

Now it is clear what we want to achieve with the simplex algorithm: Find a point $c'$ in the unit sphere that is close to $c$, together with a vertex $v$ of $P$ with $c' \in C_v$. Which pivoting rule makes this happen? This question is answered in the next section.

1A similar fact holds for totally unimodular constraint matrices, see, e.g.,[22] Proposition 2.1, page 540] meaning that after one has identified an element of the optimal basis, one is left of a linear program in dimension $n - 1$ with a totally unimodular constraint matrix. A similar fact fails to hold for integral matrices with sub-determinants bounded by 2.
3 The random edge pivoting algorithm

We now describe how the simplex algorithm can be used to find a point $c'$ satisfying (a) and (b) from above. The classical random edge simplex pivot rule can be stated as follows: At a vertex $v$ of the polytope $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$, identify the adjacent vertices of higher objective value, pick one at random and move to it. A more general process is the following random walk from vertex to vertex of $P$ along its edges.

| At vertex $v$: |
| Go to a neighboring vertex $u$ with probability $p_v(u)$. |

When the $p_v(u)$ are set to be equal and positive only for neighbors $u$ with better objective value than $v$, then this is the random edge simplex pivot rule. If the $p_v(u)$ are uniform over all neighbors, it is a simple random walk on the 1-skeleton of $P$. The algorithm we analyze is a random edge pivot rule, with pivot probabilities $p_v(u)$ such that their distribution can be sampled locally at the vertex $v$.

A cone-walk is triggering the pivots

The pivots of the simplex algorithm are triggered by a random walk on a sub-division of the cones. Here is a sketch of the approach. Consider the function $g : \mathbb{R}^n \to \mathbb{R}_+$ defined as $g(x) = e^{-\|x-(c/8)\|^2/2t_0}$ proportional to the Gaussian density with mean $c/8$ and variance $t_0$.

We partition each cone into countably infinitely many parallelepipeds whose axes are parallel to the unit vectors defining the cone. The lengths of each edge these parallelepipeds is $1/N$ with $N = 3n/t_0$, see Figure 2. More precisely, a given cone $C_v = \{ \sum_{i \in B} \lambda_i a_i : \lambda_i \geq 0, i \in B \}$ is partitioned into translates of the parallelepiped

$$1/N \cdot \left\{ \sum_{i \in B} \lambda_i \cdot a_i : 0 \leq \lambda_i \leq 1, i \in B \right\}.$$  

The volume of such a parallelepiped $P$ is $\text{vol}(P) = 1/N^n \cdot |\det(A_B)|$. For a parallelepiped $P$, we define

$$f(P) = g(z(P)) \text{vol}(P)$$

where $z(P)$ is the center of $P$.

Figure 2: The partitioning of $\mathbb{R}^2$ into parallelepipeds for the polytope in Figure 1 together with the illustration of a parallelepiped (blue) and its neighbors (green).
The state space of the random walk will be all parallelopipeds used to partition all cones, a countably infinite collection. At a parallelopiped \( P \), we pick a neighboring parallelopiped \( P' \) uniformly at random, i.e. with probability \( \frac{1}{2^n} \), and go to it with probability \( \frac{1}{2 \min\{1, f(P')/f(P)\}} \). This will imply (see Section 4.1) that the steady state distribution of the Markov chain is proportional to \( f(P) \). We call this distribution \( Q \).

### 3.1 Sampling close to \( c \)

In the next section we show that a suitable random walk that is coupled with the simplex algorithm can be used to sample a parallelopiped \( P \in \mathbb{R}^n \) with density proportional to \( f(P) = \text{vol}(P)g(z(P)) \) where \( x^* = z(P) \) is the center of \( P \), along with the optimum basis \( B' \) of the linear program (1) with \( c \) replaced by \( x^* \).

Suppose now that \( c' \) with \( \|c'\| = 1 \) is obtained from \( x^* \) by scaling with a positive number. If \( \|c - c'\| < \delta/(2 \cdot n) \), then \( c' \) satisfies \([\text{a}] \) and \([\text{b}] \) and we have identified an element of the optimal basis. We are now answering the following question:

How small do we have to choose \( t_0 \) such that \( \|c - c'\| < \delta/(2 \cdot n) \) holds with high probability?

If our algorithm were instead to sample points according to the Gaussian density \( \mathcal{N}(c, t_0) \), we could use the following standard bound on the concentration of a Gaussian density.

**Lemma 4.** Let \( X \sim \mathcal{N}(\mu, \sigma^2 I) \). Then for any \( \alpha \geq 2 \),

\[
P(\|X - \mu\| > \alpha \sigma \sqrt{n}) \leq 2e^{-\alpha^2 n/4}.
\]

We derive the following concentration inequality for the distribution \( Q \) on parallelopipeds, where the probability of a parallelopiped is proportional to \( f(P) \).

**Lemma 5.** For any \( \alpha \geq 3 \) and \( \sqrt{nt_0} > n/N \),

\[
P_Q(\|x - (c/8)\| \geq \alpha \sqrt{nt_0}) \leq 2(3\alpha)^n e^{-\alpha^2 n/4}.
\]

**Proof.** Let \( P_{\text{out}} \) be the set of parallelopipeds with centers outside a ball of radius \( d \) centered at \( (c/8) \). Then, the quantity to estimate is

\[
P_Q(\|x - (c/8)\| \geq d) = \frac{\sum_{P \in P_{\text{out}}} f(P)}{\sum_P f(P)}.
\]

We will bound the numerator from above and the denominator from below.

For the numerator, we partition \( P_{\text{out}} \) according to the distance of centers from \( (c/8) \), into buckets of width \( d \). So the first bucket \( B_1 \) is all \( P \) with \( d \leq \|z(P) - (c/8)\| < 2d \), then \( 2d \) to \( 3d \) and so on; the \( k \)’th bucket \( B_k \) has all \( P \) with centers at distance between \( kd \) and \( (k + 1)d \) from \( c/8 \). For any \( P \) in \( B_k \), the value of \( g(z(P)) \) is at most \( e^{-(kd)^2/(2t_0)} \). Moreover,

\[
B_k \subset \left( (k + 1)d + \frac{n}{N} \right) ((c/8) + \mathcal{B}_n)
\]

i.e., \( B_k \) is contained in a ball of radius slightly larger than \( (k + 1)d \) centered at \( c/8 \). The quantity
\[ \frac{n}{N} \] is a bound on the diameter of any parallelopiped. Therefore,

\[ \sum_{P \in \mathcal{P}} f(P) \leq \sum_{k=1}^{\infty} \text{vol} \left( \left( (k+1)d + \frac{n}{N} \right) \mathcal{B}_n \right) e^{-(kd^2)/(2t_0)} \]

\[ = \text{vol}(\mathcal{B}_n) \sum_{k=1}^{\infty} \left( (k+1)d + \frac{n}{N} \right)^n e^{-(kd^2)/(2t_0)} \]

\[ \leq \text{vol}(\mathcal{B}_n) \sum_{k=1}^{\infty} ((k+2)d)^n e^{-(kd^2)/(2t_0)} \]

\[ \leq \text{vol}(\mathcal{B}_n) d^n \sum_{k=1}^{\infty} (k+2)^n e^{-(kd^2)/(2t_0)} \]

Here we have used the bound that \( d \geq \frac{n}{N} \). The ratio of consecutive terms in the summation above is

\[ \left( 1 + \frac{1}{k+2} \right) e^{-(k+1)^2d^2+k^2d^2)/(2t_0)} \leq e^{(n/k)-(kd^2)/t_0} \leq e^{-n} \]

since \( d^2 > 2nt_0 \). Therefore, the summation is at most twice the first term, giving

\[ \sum_{P \in \mathcal{P}} f(P) \leq 2\text{vol}(\mathcal{B}_n)(3d)^n e^{-d^2/(2t_0)}. \]

Next, to lower bound the sum of \( f(P) \) over all parallelopipeds, consider a ball of radius \( r \) around \( c/8 \) and all parallelopipeds with centers inside this ball. Then for any such \( P \), we have \( g(z(P)) \geq e^{-r^2/(2t_0)}. \) Moreover, the ball of radius \( r - (n/N) \) is completely covered by such parallelopipeds, i.e., no parallelopiped outside the ball of radius \( r \) can intersect the ball of radius \( r - (n/N) \). Therefore, we have the following lower bound:

\[ \sum_{P \in \mathcal{P}} f(P) \geq \text{vol} \left( \left( r - \frac{n}{N} \right) \mathcal{B}_n \right) e^{-r^2/(2t_0)} \]

\[ = \text{vol}(\mathcal{B}_n) \left( r - \frac{n}{N} \right)^n e^{-r^2/(2t_0)}. \]

Taking \( r^2 = 4nt_0 \) and noting that \( n/N < \sqrt{nt_0} \), we have

\[ \sum_{P \in \mathcal{P}} f(P) \geq \text{vol}(\mathcal{B}_n)(nt_0)^{n/2} e^{-2n}. \]

Combining the upper bound and lower bound,

\[ \mathbb{P}_{\mathcal{Q}}(\mathcal{P}_{out}) \leq 2 \left( \frac{9d^2}{nt_0} \right)^{n/2} e^{2n-d^2/(2t_0)} \leq 2(3\alpha)^n e^{-\alpha^2n/4}, \]

where we used \( 2 \leq \frac{9d}{t_0} \) which follows from \( \alpha \geq 3. \)

We can now answer the question above. If we set \( t_0 \) so that \( \|x - (c/8)\| < \delta/(16n) \) whp, then \( \|8 \cdot x - c\| < \delta/(2n) \) holds whp. We still need to scale \( 8 \cdot x \) with a positive number between 1/2 and 2 such that the result has length one. Thus, if we set \( t_0 \) such that \( \|x - (c/8)\| < \delta/(32n) \) w.h.p. then \( \|c - c^\prime\| < \delta/(2n) \) holds with high probability. Applying Lemma 5 with \( \alpha = 4 \), we get

\[ \mathbb{P}(\|x - (c/8)\| > 3\sqrt{nt_0}) \leq 2 \cdot (12)^n e^{-2n} \leq e^{-n}, \]

for \( n > 4 \). Thus, it suffices to have \( 4\sqrt{nt_0} \leq \delta/32n \), i.e., \( t_0 = \delta^2/(10^4 n^3) \) suffices.
The description of the algorithm

The algorithm is described in Figure 3. We assume that we are given some vertex of the polytope, $x^0$ and its associated basis $B_0 \subseteq \{1, \ldots, m\}$ to start with. We explain in the appendix how this assumption can be removed. Next, we choose a particular parallelepiped in $C_{x^0}$ to start the random walk. For this, consider $c_0 = \sum_{i \in B} a_i / \| \sum_{i \in B} a_i \|$. The point $x^0$ is the unique optimal solution of the linear program $\max \{ c_0^T x : x \in \mathbb{R}^n, Ax \leq b \}$. The random walk starts with a parallelopiped $P_0$ in the cone $C_{x^0}$ that contains $c_0/8$.

Parallelepipeds that have a facet within a facet in $C_{x^0}$ have neighboring parallelepipeds in the corresponding neighboring cone of $C_{x^0}$. The first time that the random walk moves into such a neighboring parallelepiped, we perform the first random-edge pivot of the simplex algorithm.

**Input:** An LP specified by $A, b, c$; a basic feasible solution $x^0$ of $Ax \leq b$ and an associated basis $B_0$.

1. Let $f(P) = e^{-\|z(P) - (c/8)\|^2/2t_0} \text{vol}(P)$. Start with the parallelopiped $P_0$ containing the point $c_0/8$ in the cone $C_{x^0}$.

2. Repeat for $\ell$ iterations:
   - If $c$ is in the current cone, the algorithm stops. It has found the optimal basis.
   - Otherwise pick a neighboring parallelopiped, say $P'$, uniformly at random.
   - Go to $P'$ with probability $\frac{1}{2} \min \left\{ 1, \frac{f(P')}{f(P)} \right\}$.
     (this is a pivot whenever $P'$ and $P$ are in different cones.)

3. In the cone obtained, let $u$ be the unit vector closest to $c$.

4. Find $\hat{y}$ such that $A^T \hat{y} = u, \hat{y} \geq 0$.

5. Let $l = \arg \max \hat{y}_i$. Project all other constraints orthogonal to $a_l$ and recurse in one lower dimension.

Figure 3: Geometric Random Edge

This algorithm implements the randomized simplex pivot rule. The analysis has two parts. We will show that:

i) the random walk converges quickly to its target distribution, and

ii) w.h.p. there exists a point in the cone obtained that is close to $c$.

4 Convergence of the random walk

4.1 Some preliminaries on Markov chains

A Markov chain is said to be lazy if at each step, with probability $\geq 1/2$ it does nothing. The rejection sampling step where we step from $x$ to $x'$ with probability $\min \{1, f(x')/f(x)\}$ is called a Metropolis filter.
Proposition 6. The stationary distribution of a lazy Markov chain with state space $V$ and a Metropolis filter applied to a nonnegative function $f$ is

$$Q(x) = \frac{f(x)}{\sum_{x' \in V} f(x')}.$$ 

This folklore proposition is proved by verifying that $Q$ is stationary. Any lazy, connected chain with the above property has $Q$ as its unique stationary distribution. Note that our Markov chain is time reversible, i.e., for two states $x$ and $y$ one has $Q(x) \cdot p(x, y) = Q(y) \cdot p(y, x)$.

The ergodic flow for a subset of states $S$ is defined as

$$\Phi(S, \bar{S}) = \sum_{x \in S, y \in \bar{S}} f(x) p(x, y).$$

Flow-conservation implies that $\Phi(S, \bar{S}) = \Phi(\bar{S}, S)$ holds. To prove convergence of the walk, we bound the conductance of the underlying Markov chain [26]. For a discrete random walk with support $V$, stationary distribution $Q$ and transition matrix $p$, the conductance is defined as

$$\phi = \min_{S \subseteq V: 0 < Q(S) \leq \frac{1}{2}} \sum_{x \in S, y \in V \setminus S} Q(x) p(x, y) / Q(S).$$

Jerrum and Sinclair [26] related conductance to the convergence of a finite Markov chain to its stationary distribution. Lovász and Simonovits [20] extended this to general Markov chains and to the notion of $s$-conductance which allows us to ignore small subsets when analyzing the convergence. The $s$-conductance is the following:

$$\phi_s = \min_{S: s \leq Q(S) \leq 1} \sum_{x \in S, y \in V \setminus S} Q(x) p(x, y) / Q(S) - s.$$ 

In their theorem, $Q^0$ is an initial distribution on the states of the Markov chain. In our case, this initial distribution is 1 for the starting parallelepiped $P_0$ and 0 for the other parallelepipeds. The number $H_s$ is defined as

$$H_s = \sup \left\{ \left| \sum_{x \in V} Q^0(x) \cdot w_x - s \right|: 0 \leq w \leq 1, \sum_{x \in V} Q(x) \cdot w_x = s \right\}.$$ 

Theorem 7 ([20]). The distribution $Q^\ell$ obtained after $\ell$ steps of the Markov chain satisfies

$$|Q^\ell(T) - Q(T)| \leq H_s + \frac{H_s}{s} \left( 1 - \frac{\phi_s^2}{2} \right)^\ell.$$ 

for any subset $T \subseteq V$.

The rate of convergence is thus $O(1/\phi_s^2)$. In our case $H_s$ can be bounded from above by $s/Q(P_0)$ such that the factor $H_s/s$ then becomes $1/Q(P_0)$ which we bound from above in the next section.
Figure 4: An illustration of Theorem 8 in the setting in which it is applied later. The set $A$ is the union of some parallelepipeds and the convex body $K$ is the unit ball $B_n$. The boundary of $A$ relative to $B_n$, $\partial_{B_n}A$ is drawn in blue.

4.2 Conductance and mixing of the cone walk

Here we analyze the mixing of the random walk. We will use the isoperimetry of a Gaussian restricted to a convex body, see e.g., [7] and Figure 4.

**Theorem 8.** For a convex body $K \subseteq \mathbb{R}^n$, let $g(x) = e^{-\|x-\mu\|^2/2\sigma^2}$ for $x \in K$ and zero outside. For any set $A \subset K$,

$$
\int_{\partial_K A} g(x) \, dx \int_K g(x) \, dx \geq \frac{\ln 2}{\sigma} \int_A g(x) \, dx \int_{K \setminus A} g(x) \, dx.
$$

Here $\partial_K A$ denotes the boundary of $A$ relative to $K$.

In the following, we denote the set of all parallelepipeds by $\mathcal{P}$. Let $\mathcal{P}_+ \mathcal{P}_{\text{out}}$ be the set of parallelepipeds that do not lie completely inside $\mathcal{B}_n$ and $\mathcal{P}_{\text{out}}$ be the set of parallelepipeds that do not intersect $\mathcal{B}_n$. The next lemma lists some observations that will then enable us to prove a lower bound on the conductance.

**Lemma 9.** Let $Q$ be the stationary distribution of our Markov chain with state-space being the parallelepipeds.

1. For two points $x, y \in \mathbb{R}^n$ in the same parallelopiped $P$ that intersects the unit ball, we have

$$
\frac{g(x)}{g(y)} \leq e^{3n/(2Nt_0)} \leq 2
$$

if $N \geq 3n/t_0$.

2. For the starting parallelopiped $P_0$, we have $Q(P_0)$ is at least $p = e^{-1/(16t_0)}$.

3. For two neighboring parallelepipeds $P, P'$ that intersect the unit ball $\mathcal{B}_n$ we have $\delta \leq \frac{f(P)}{f(P')} \leq \frac{2}{3}$.

**Proof.** The first part is a direct application of the definition of $g$.

$$
\frac{g(x)}{g(y)} \leq e^{-1/(2t_0)(\|x\|^2-\|y\|^2-(c/4)(x-y))} \leq e^{3\|x-y\|/(2t_0)} \leq e^{3n/(2Nt_0)}.
$$
For note that
\[ f(P_0) \geq e^{-(1/4)^2/2t_0(\delta/N)^n} \geq e^{-1/(32t_0) - n\ln(N/\delta)}. \]

Meanwhile, \( \sum_P f(P) \leq \sum_{P: z(P) \in (c/8)+B} f(P) + f(P_{out}) \). The first term is at most twice the total measure of the Gaussian \( g \), i.e. bounded by \( 2 \cdot (2\pi t_0)^n/2 \). The second term is \( f(P_{out}) \) with \( \alpha \sqrt{n t_0} = 1 \) in Lemma 5 which is tiny. Thus, \( Q(P_0) \geq e^{-1/(32t_0) - n\ln(N/\delta)} \geq e^{-1/(16t_0)}. \)

The \( \delta \)-distance property implies that the volume ratio of two neighboring parallelepipeds \( P \) and \( P' \) is bounded by \( 1/\delta \). The centers of \( P \) and \( P' \) are within distance \( n/N \) from each other. A similar argument as the one for \( S \) shows that the ratio of their values w.r.t. \( g \) is also bounded by 2 which implies \( 3 \).

**Lemma 10.** Let \( s = Q(P_{out}^+) \). The \( 2s \)-conductance of the random walk on the parallelepipeds is \( \Omega(\delta^3/n^{3.5}) \).

**Proof.** Let \( S \subset \mathcal{P} \) with \( 2 \cdot s \leq Q(S) \leq 1/2 \). Then we need to bound the following expression from below
\[ \phi_{2s}(S) = \frac{\sum_{P \in S, P' \in \mathcal{P}\setminus S} Q(P)p(P, P')}{Q(S) - 2s}. \]

Let \( T \) be the set of parallelepipeds in the neighborhood of \( S \) that intersect the unit ball, see Figure 5. Let \( A \) be the set of ordered tuples
\[ A = \{(P, P') : P \in S, P' \in N(P) \cap T\}, \]
where \( N(P) \) denotes the parallelepipeds in the neighborhood of \( P \). We observe now that the neighboring facets of each such ordered pair \( (P, P') \in A \) partition a subset of the boundary \( \partial S \) of \( S \), where \( S = \cup_{P \in S} P \) and this subset covers the relative boundary \( \partial}_{\mathcal{P}} S \).

By restricting to ordered tuples in \( A \) and by ignoring \(-2s\) in the denominator, we obtain
\[ \phi_{2s}(S) \geq \frac{\sum_{(P, P') \in A} Q(P)p(P, P')} {Q(S)} = \frac{\sum_{(P', P) \in A} Q(P')p(P', P)} {Q(S)} \]
where we used time-reversibility in the equation.

By part 3) of Lemma 9, each nonzero \( p(P', P) \) can be bounded by \( \frac{1}{4n} \cdot \delta / 2 \). Consequently, we have

\[
\phi_2(s) \geq \frac{\delta}{8n} \sum_{(P,P') \in A} \frac{Q(P')}{Q(S)} \geq \frac{\delta}{8n} \sum_{(P,P') \in A} \frac{f(P')}{f(S)}.
\]

We denote the facet separating \( P \) and \( P' \) by \( F(P, P') \). Since \( P' \) intersects the unit ball, we have with part 1) of Lemma 9

\[
f(P') = \text{vol}(P) \cdot g(z(P)) \geq \frac{1}{2} \int_{x \in F(P, P')} g(x) \cdot h,
\]

where \( h \) denotes the height of \( P' \) relative to the facet \( F(P, P') \). This height is, by the \( \delta \)-distance property, at least \( \delta / N \). Consequently, we have

\[
\phi_2(s) \geq \frac{\delta^2}{8nN} \int_{x \in \partial S \cap \mathbb{B}_n} g(x) \, dx.
\]

Next we split \( S \) into \( S = S_1 \cup S_2 \), where \( S_1 \) consists of those parallelepipeds of \( S \) that lie entirely in the unit ball. Since \( f(S_2) \leq s \) and \( f(S) \geq 2 \cdot s \), we have \( f(S) \leq 2 \cdot f(S_1) \). Using again part 1) of Lemma 9 we see that

\[
f(S) \leq 4 \cdot \int_{x \in S \cap \mathbb{B}_n} g(x) \, dx
\]

holds. Now we can apply Theorem 8 and obtain

\[
\phi_2(s) \geq \frac{\delta^2}{32nN} \int_{x \in S \cap \mathbb{B}_n} g(x) \, dx
\]

\[
\geq \frac{\delta^2}{32nN} \ln \frac{2}{\sqrt{t_0}}
\]

\[
\geq \frac{\delta^3}{10^3 n^{3.5}}.
\]

A bound on the convergence of the walk to its stationary distribution follows from the conductance bound above, the bound on \( Q(P_0) \) and Theorem 7.

**Corollary 11.** After \( \ell = O(n^{10}/\delta^8) \) steps of the random walk starting at \( P_0 \), the distribution \( Q^\ell \) satisfies for any subset \( T \):

\[
|Q^\ell(T) - Q(T)| \leq e^{-n}.
\]

**Proof.** We set \( s \) to be the \( Q \)-measure of all parallelepipeds that do not lie completely inside \( \mathcal{B}_n \). This set, \( \mathcal{P}^{\text{out}} \), lies completely outside \( (1 - (n/N)) \mathcal{B}_n \). Hence, applying Lemma 8 with \( \alpha \sqrt{n t_0} = 1 - (n/N) \geq 1/2 \), this is at most

\[
2(3\alpha)^n e^{-(1-(n/N))^2/(2t_0)} \leq 2(3/\sqrt{n t_0})^n e^{1/(4t_0)} \leq e^{-1/(8t_0)}.
\]
Next, using Lemma 9, we have $1/Q(P_0) \leq e^{1/(16t_0)}$. Therefore, $H_{2s}$ as at most $2s/Q(P_0) \leq 2e^{-(1/t_0)(1 + \frac{1}{16})} \leq 2e^{-1/(16t_0)}$.

Thus, from Theorem 7 applied with $2s$ in place of $s$,

$$|Q^\ell(T) - Q(T)| \leq 2e^{-1/(16t_0)} + e^{1/(16t_0)} \left(1 - \frac{\phi^2}{2}\right)^\ell$$

The claim follows for $\ell > \frac{C}{\phi^2 t_0}$ steps, i.e., $\ell = O(n^{10}/\delta^8)$ suffices to make the distance between $Q^\ell$ and $Q$ exponentially small.

This implies our main result, Theorem 1. Note that the random walk in our algorithm stops with some cone. We can then go to the basis vector in the cone that is closest to $c$. This point will satisfy conditions $\square$ and $\square$ from Section 2 with high probability.

5 Remarks

Convergence of the random walk and $\delta$-distance property

The $\delta$-distance property is a global condition on the constraint matrix $A \in \mathbb{R}^{m \times n}$ of the linear program. A closer look at the convergence proof of the random walk reveals that the same bounds can be deduced if $A_B$, where $B$ is a feasible basis, satisfies the $\delta$-distance property. For example, this holds if each cone $C_v$ intersected with the unit ball $B_n$ contains a ball of radius $\delta$.

In this case, our algorithm also computes an objective function vector $c'$ of norm one together with a vertex $x'$ that is maximal w.r.t. $c'^T x$ that is close to the original objective function vector $c$, i.e., satisfies the properties $\square$ and $\square$. However, we do not know how to identify an element of the optimal basis under this weaker condition.

Exponentially many constraints

One of the most important theoretical consequences of Khachiyan’s \cite{18} ellipsoid method is the polynomial time equivalence of separation and optimization \cite{14}. It shows that a linear program can be solved in (weakly) polynomial time in the dimension and the largest binary encoding length of a coefficient of $A, b$ and $c$, as long as the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is equipped with a polynomial time separation algorithm. This is an algorithm that decides for a given $x^* \in \mathbb{Q}^n$ whether it lies in $P$ and if not outputs an inequality of $Ax \leq b$ that is violated by $x^*$.

Our result shows that the simplex algorithm also runs in expected polynomial time in the dimension and $1/\delta$ if the non-degenerate polyhedron $P$ is equipped with a polynomial-time algorithm that computes the neighbors of a given vertex and outputs the corresponding bases of these neighbors. This is all that we need to compute the neighbors of a parallelepiped $P$ and to make a transition of the random walk.

Acknowledgment

The authors are grateful to Daniel Dadush and Nicolai Hähnle, who pointed out an error in the sub-division scheme in a previous version of this paper.
References

[1] R. K. Ahuja and J. B. Orlin. The scaling network simplex algorithm. *Operations Research*, 40(1-supplement-1):S5–S13, 1992.

[2] N. Bonifas, M. Di Summa, F. Eisenbrand, N. Hähnle, and M. Niemeier. On sub-determinants and the diameter of polyhedra. In *Proceedings of the 28th annual ACM symposium on Computational geometry*, SoCG ’12, pages 357–362, 2012.

[3] T. Brunsch and H. Rögl. Finding short paths on polytopes by the shadow vertex algorithm. In *Automata, Languages, and Programming*, pages 279–290. Springer, 2013.

[4] P. Bürgisser and F. Cucker. *Condition*, volume 349 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2013. The geometry of numerical algorithms.

[5] K. L. Clarkson. Las Vegas algorithms for linear and integer programming when the dimension is small. *Journal of the Association for Computing Machinery*, 42:488–499, 1995.

[6] W. Cook, A. M. H. Gerards, A. Schrijver, and E. Tardos. Sensitivity theorems in integer linear programming. *Mathematical Programming*, 34:251 – 264, 1986.

[7] B. Cousins and S. Vempala. A cubic algorithm for computing gaussian volume. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, (SODA 2014), pages 1215–1228. SIAM, 2014.

[8] Dantzig, G. B. Maximization of a linear function of variables subject to linear inequalities. In Koopmans, T. C., editor, *Activity Analysis of Production and Allocation*, pages 339–347. John Wiley & Sons, New York, 1951.

[9] M. Dyer and A. Frieze. Random walks, totally unimodular matrices, and a randomised dual simplex algorithm. *Mathematical Programming*, 64(1, Ser. A):1–16, 1994.

[10] O. Friedmann. A subexponential lower bound for Zadeh’s pivoting rule for solving linear programs and games. In *Integer Programming and Combinatorial Optimization*, pages 192–206. Springer, 2011.

[11] O. Friedmann, T. D. Hansen, and U. Zwick. Subexponential lower bounds for randomized pivoting rules for the simplex algorithm. In *STOC’11—Proceedings of the 43rd ACM Symposium on Theory of Computing*, pages 283–292. ACM, New York, 2011.

[12] B. Gärtner and V. Kaibel. Two new bounds for the random-edge simplex-algorithm. *SIAM Journal on Discrete Mathematics*, 21(1):178–190, 2007.

[13] D. Goldfarb, J. Hao, and S.-R. Kai. Efficient shortest path simplex algorithms. *Operations Research*, 38(4):624–628, 1990.

[14] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[15] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics*. Springer, 1988.

[16] G. Kalai. A subexponential randomized simplex algorithm (extended abstract). In *Proceedings of the 24th Annual ACM Symposium on Theory of Computing (STOC92)*, pages 475–482, 1992.

[17] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984.

[18] L. Khachiyan. A polynomial algorithm in linear programming. *Doklady Akademii Nauk SSSR*, 244:1093–1097, 1979.

[19] V. Klee and G. J. Minty. How good is the simplex algorithm? In *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)*, pages 159–175. Academic Press, New York, 1972.
6 Appendix

Proof of Lemma 2. We denote the normal cones of $B$ and $B'$ by

$$C = \{ \sum_{i \in B} \lambda_i a_i : \lambda_i \geq 0 \} \quad \text{and} \quad C' = \{ \sum_{j \in B'} \mu_j a_j : \mu_j \geq 0 \}.$$ 

By a gift-wrapping technique, we construct a hyperplane $(h^T x = 0)$, $h \in \mathbb{R}^n \setminus \{0\}$ such that the following conditions hold.

i) The hyperplane separates the interiors of $C$ and $C'$.

ii) The row $a_k$ does not lie on the hyperplane.

iii) The hyperplane is spanned by $n - 1$ rows of $A$.

We start with a hyperplane $(h^T x = 0)$ strictly separating the interiors of $C$ and $C'$. The conditions [i][ii] are satisfied. Suppose that [iii] is not satisfied and let $\ell < n - 1$ be the maximum number of linearly independent rows of $A$ that are contained in $(h^T x = 0)$.

We tilt the hyperplane by moving its normal vector $h$ along a chosen equator of the ball of radius $\|h\|$ to augment this number. Since $\ell < n - 1$ there exists an equator leaving the rows of $A$ that are contained in $(h^T x = 0)$ invariant under each rotation.

However, as soon as the hyperplane contains a new row of $A$ we stop. If this new row of $A$ is not $a_k$ then still, conditions [i][ii] hold and the hyperplane now contains $\ell + 1$ linearly independent rows of $A$.

If this new row is $a_k$, then we redo the tilting operation but this time by moving $h$ in the opposite direction on the chosen equator. Since there are $n$ linearly independent rows of $A$ without the row $a_k$ this tilting will stop at a new row of $A$ which is not $a_k$ and we end the first tilting operation.
This tilting operation has to be repeated at most 
\[ n - 1 - |B \cap B'| \] times to obtain the desired hyperplane.

The distance of \( \mu_k \cdot a_k \) to the hyperplane \( (h^T x = 0) \) is at least \( \mu_k \cdot \delta \). Since \( c' \) is the sum of \( \mu_k \cdot a_k \) and a vector that is on the same side of the hyperplane as \( a_k \) it follows that the distance of \( c' \) to this hyperplane is also at least \( \mu_k \cdot \delta \). Since \( c \) lies on the opposite side of the hyperplane, the distance of \( c \) and \( c' \) is at least \( \mu_k \cdot \delta \).

\[ \square \]

**Phase 1**

We now describe an approach to determine an initial basic feasible solution or to assert that the linear program (1) is infeasible. Furthermore, we justify the assumption that the set of feasible solutions is a bounded polytope. This phase 1 is different from the usual textbook method since the linear programs that we need to solve have to comply with the \( \delta \)-distance property.

To find an initial basic feasible solution, we start by identifying \( n \) linearly independent linear inequalities \( \tilde{a}_1^T x \leq \tilde{b}_1, \ldots, \tilde{a}_n^T x \leq \tilde{b}_n \) of \( Ax \leq b \). Then we determine a ball that contains all feasible solutions. This standard technique is for example described in [15]. Using the normal-vectors \( \tilde{a}_1, \ldots, \tilde{a}_n \) we next determine values \( \beta_i, \gamma_i \in \mathbb{R}, i = 1, \ldots, n \) such that this ball is contained in the parallelepiped \( Z = \{ x \in \mathbb{R}^n : \beta_i \leq \tilde{a}_i^T x \leq \gamma_i, i = 1, \ldots, n \} \). We start with a basic feasible solution \( x_0^* \) of this parallelepiped and then proceed in \( m \) iterations. In iteration \( i \), we determine a basic feasible solution \( x_i^* \) of the polytope

\[ P_i = Z \cap \{ x \in \mathbb{R}^n : a_j^T x \leq b_j, 1 \leq j \leq i \} \]

using the basic feasible solution \( x_{i-1}^* \) from the previous iteration by solving the linear program

\[ \min \{ a_i^T x : x \in P_{i-1} \} \]

If the optimum value of this linear program is larger than \( b_i \), we assert that the linear program (1) is infeasible. Otherwise \( x_i^* \) is the basic feasible solution from this iteration.

Finally, we justify the assumption that \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) is bounded as follows. Instead of solving the linear program (1), we solve the linear program \( \max \{ c^T x : x \in P \cap Z \} \) with the initial basic feasible solution \( x_m^* \). If the optimum solution is not feasible for (1) then we assert that (1) is unbounded.