A New Expansion for Nucleon-Nucleon Interactions

David B. Kaplan
Institute for Nuclear Theory 351550, University of Washington, Seattle, WA 98195-1550, USA

Martin J. Savage
Department of Physics 351560, University of Washington, Seattle, WA 98195-1560, USA

Mark B. Wise
California Institute of Technology, Pasadena, CA 91125, USA

Abstract

We introduce a new and well defined power counting for the effective field theory describing nucleon-nucleon interactions. Because of the large $NN$ scattering lengths it differs from other applications of chiral perturbation theory and is facilitated by introducing an unusual subtraction scheme and renormalization group analysis. Calculation to subleading order in the expansion can be done analytically, and we present the results for both the $^1S_0$ and $^3S_1 - ^3D_1$ channels.
Effective field theories are standard for dealing with strong interaction physics in the nonperturbative regime. The idea is to construct the most general Lagrangian consistent with the symmetries of quantum chromodynamics out of fields that create and destroy the relevant degrees of freedom. To have predictive power an expansion scheme must be found that limits the number of terms that occur in the effective Lagrangian. In the case of the effective Lagrangian for pion self interactions the expansion is in powers of the typical momentum $p \sim m_\pi$ divided by a typical hadronic scale, of order the rho mass. At leading order, tree diagrams dominate with either two derivatives or one insertion of the light quark mass matrix at the vertex. At next order one includes both four derivative operators at tree level, as well as one loop diagrams with the leading operators at the vertices, and so forth. This perturbative expansion is known as chiral perturbation theory, and has also been applied to nucleon-pion interactions. However, attempts to apply chiral perturbation theory to nuclear physics, where nucleon-nucleon interactions are of primary interest, have encountered difficulties, stemming from the large scattering lengths in the $^1S_0$ and $^3S_1$ nucleon-nucleon scattering amplitudes. The large scattering lengths generally imply finely tuned cancellations between graphs in the effective theory, with the result that a consistent expansion has so far proven elusive.

In this letter we introduce a new method for applying effective field theory to nuclear physics. Our method uses dimensional regularization with a novel subtraction scheme which permits a consistent expansion procedure yielding analytic expressions for nucleon-nucleon scattering amplitudes. We apply it to scattering amplitudes in both the $^1S_0$ and $^3S_1 - ^3D_1$ channels and compare with the results of the Nijmegen partial wave analysis.

We begin by considering nucleon-nucleon scattering in the $^1S_0$ channel in a nonrelativistic effective field theory with only nucleon fields, as is appropriate for momentum much less than the pion mass. The tree level amplitude is

$$iA_{\text{tree}} = -i\left(\frac{\mu}{2}\right)^{4-D} \sum_{n=0}^{\infty} C_{2n}(\mu) p^{2n} = -i\left(\frac{\mu}{2}\right)^{4-D} C(p^2, \mu) ,$$

where $p$ is the magnitude of the nucleon momentum in the center of mass frame; the total center of mass energy is $E = p^2/M + \ldots$, where $M$ is the nucleon mass. In eq. (1) $C_{2n}(\mu)$ is a linear combination of coefficients of four nucleon operators with $2n$ spatial derivatives, $\mu$ is the subtraction point and $D$ is the dimension of spacetime. In order to calculate the full amplitude we must sum a bubble chain of Feynman diagrams with the tree amplitude giving the vertices. The integral required is

$$I_n = -i\left(\frac{\mu}{2}\right)^{4-D} \int \frac{d^D q}{(2\pi)^D} q^{2n} i \frac{E/2 - q^0 - q^2/2M + i\varepsilon}{(E/2 + q^0 - q^2/2M + i\varepsilon)} \cdot \frac{i}{(E/2 + q^0 - q^2/2M + i\varepsilon)}$$

$^1$Dimensional regularization is the most convenient regularization scheme, as it preserves chiral symmetry and gauge symmetry; it also preserves Galilean invariance, which makes the Feynman integrals relatively simple to evaluate. However, physical results do not depend upon the choice of regulator.
This amplitude has no pole at $D = 4$, but it does have a pole at $D = 3$ resulting from what would be a linear ultraviolet divergence in the integration if a momentum cutoff were used. The residue of the pole at $D = 3$ equals $M(\mathcal{M}E)^n \mu/4\pi$. In the minimal subtraction scheme, $MS$, counter terms are added which subtract the poles at $D = 4$. Here we use a new subtraction procedure which we call power divergence subtraction, $PDS$, where the poles at $D = 4$ and $D = 3$ are both subtracted by counter terms. In the case of $I_n$ this subtraction amounts to adding the counter term contribution

$$\delta I_n = -\frac{M(\mathcal{M}E)^n \mu}{4\pi(D - 3)},$$

and the subtracted integral is

$$I_n^{PDS} = I_n + \delta I_n = -(\mathcal{M}E)^n \left(\frac{M}{4\pi}\right)(\mu + ip).$$

The choice $\mu = 0$ corresponds to minimal subtraction. Using the $PDS$ subtraction scheme the full amplitude, including loop effects is

$$iA = \frac{-iC(p^2, \mu)}{1 + MC(p^2, \mu)(\mu + ip)/4\pi}.$$

The scattering amplitude $A$ is related to the $S$-matrix by

$$S - 1 = e^{2i\delta} - 1 = i\left(\frac{pM}{2\pi}\right)A,$$

where $\delta$ is the phase shift. It is convenient to consider the quantity

$$p \cot \delta = ip + \frac{4\pi}{MA} = -\frac{4\pi}{MC(p^2, \mu)} - \mu,$$

which has an expansion in $p^2$ of the form

$$p \cot \delta = \frac{1}{a} + \frac{1}{2}r_0p^2 + \ldots,$$

where $a$ is called the scattering length and $r_0$ is the effective range. If there is a state that is very near zero binding energy then $|a|$ will be very large, which is the case in the $^1S_0$ channel.

---

2Some recent papers have criticized the applicability of dimensional regularization to $NN$ scattering [7,11,14]. Since $C$ is an arbitrary polynomial in $p^2$, over a range in $p$, eq. (6) gives the most general form for $S$ consistent with unitarity. Clearly dimensional regularization can be used for $NN$ scattering. Relativistic corrections have been neglected in eq. (5). They can be included perturbatively by inserting higher derivative one-body operators and corrections to the energy-momentum relation for external legs.
channel where \( a = -23.714 \pm 0.013 \text{ fm} \) and \( r_0 = 2.73 \pm 0.03 \text{ fm} \). The effective range \( r_0 \) and the higher coefficients in the momentum expansion are bounded by the range of the nuclear potential.

The coefficients \( C_{2n}(\mu) \) are determined by \( a, r_0, \ldots \) etc. Their \( \mu \) dependence cancels the explicit \( \mu \) dependence in eq. (\ref{eq:5}) so that the physical scattering amplitude \( A \) is \( \mu \) independent. The \( \mu \) independence of \( A \) implies a set of renormalization group equations for the \( C_{2n} \)'s.

\[
\mu \frac{d}{d\mu} C_{2n} = \beta_{2n},
\]

where, for example,

\[
\beta_0 = \frac{M}{4\pi} C_0^2 \mu \quad \text{and} \quad \beta_2 = \frac{M}{2\pi} C_0 C_2 \mu,
\]

are the exact beta-functions for \( C_0 \) and \( C_2 \). Solving these equations with boundary conditions supplied by eqs. (\ref{eq:7},\ref{eq:8}) yields

\[
C_0(\mu) = \frac{4\pi}{M} \left( \frac{1}{-\mu + 1/a} \right) \quad \text{and} \quad C_2(\mu) = \frac{2\pi}{M} \left( \frac{1}{-\mu + 1/a} \right)^2 r_0.
\]

In a system with a short range interaction characterized by the momentum scale \( \Lambda \), the natural size for the parameters \( a, r_0, \ldots \) is set by the scale \( \Lambda \). When \( a \) is of natural size (i.e. in magnitude of order \( 1/\Lambda \)), a simple power counting like that used in chiral perturbation theory for pion self interactions is possible. It is then convenient to take \( \mu = 0 \) which corresponds to the MS scheme. The amplitude \( A \) can be expanded in powers of \( p/\Lambda \) with each loop contributing a factor of \( (Mp/4\pi) \) and each vertex a factor of, \( C_{2n} p^{2n} \sim (4\pi p^{2n})/(M\Lambda^{2n+1}) \).

Consequently the leading contribution of order \( p^0 \) comes from \( C_0 \) at tree level. At order \( p^1 \) a one loop diagram with two \( C_0 \)'s contributes. At order \( p^2 \) one must include a two loop diagram with \( C_0 \) at each vertex and a tree level diagram with a vertex proportional to \( C_2 \), etc.

In contrast, for the special case where \( |a| \) is very large and \( \mu = 0 \), the coefficients are very large, \( C_{2n} \sim (4\pi a^{n+1})/(M\Lambda^n) \), and the momentum expansion used for \( |a| \sim 1/\Lambda \) is valid only over a small momentum range. This difficulty is cured by taking \( \mu \) nonzero in which case the coefficients \( C_{2n} \) need not be large. As \( |a\mu| \to \infty \) one finds, \( C_{2n} \sim 4\pi/(M\Lambda^n \mu^{n+1}) \). If one were to set \( \mu = \Lambda \), this would be similar to the result of \cite{11} where a momentum cut off \( \Lambda \) is used.

The \( \mu \) dependence induced by the additional finite subtraction in the PDS scheme makes it possible to establish a power counting that allows a systematic expansion of the scattering amplitude over a much larger range of momentum. Choosing \( \mu \sim p \), all Feynman diagrams with \( C_0 \)'s at each vertex give a contribution of the same order. This occurs because \( C_0 \) is of order \( p^{-1} \) while each loop integration is of order \( p \). With \( \mu \sim p \) the coefficients \( C_{2n} \) are of order \( p^{-(n+1)} \) and \( C_{2n} p^{2n} \sim p^{n-1} \). Consequently only \( C_0 \) must be treated to all orders. The other coefficients can be treated perturbatively. The scattering amplitude thus has an expansion of the form

\[
A = A_{-1} + A_0 + A_1 + \ldots
\]
\[ A_{-1} = -C_0 \left[ 1 + \frac{C_0 M}{4\pi}(\mu + ip) \right] \] and \[ A_0 = -C_2 p^2 \left[ 1 + \frac{C_0 M}{4\pi}(\mu + ip) \right]^2, \]

which arise from the graphs shown in Fig. 1.

As we are interested in momenta \( p \sim m_\pi \) explicit pion fields must be included in the effective Lagrangian. The pion couplings are determined from the usual chiral Lagrangian for pion-nucleon interactions. Since we are considering nucleon-nucleon interactions in a theory where the nucleons are treated nonrelativistically, it is convenient to consider separately the pion interaction which gives rise to a static \( NN \) potential and the radiation pion field responsible for the remaining effects. This is similar to the treatment of the photon field in nonrelativistic QED \[ \text{[10],[18]} \]. The resulting effective field theory amounts to expanding the dependence of nucleon propagators on the 3-momentum of the pion fields when the energy loop integrations get a contribution from the pole in the pion propagator. In the \( ^1S_0 \) channel the momentum space potential from one pion exchange is

\[ V_\pi(p, p') = -\frac{g_A^2}{2f^2} \left( \frac{m_\pi^2}{q^2 + m_\pi^2} - 1 \right), \]

where \( q = p' - p, g_A = 1.25 \) and \( f = 132 \text{ MeV} \) is the pion decay constant. It gives an order \( p^0 \), spatially extended, nucleon-nucleon interaction since we are treating \( p \sim m_\pi \). Hence including pions does not change the order \( A_{-1} \) term, and the graphs summed at leading order are still those in Fig. 1. For the order \( p^0 \) contribution there are additional terms from pion exchange: aside from the contribution to \( A_0 \) shown in Fig. 1, one must also include the
exchange of one potential pion dressed by $C_0$ to all orders (radiation pions do not contribute at this order).

The expression for $A_0$ is conveniently expressed as the sum of the five graphs shown in Fig. 2; $A_0^{(I)}$ is the contribution to $A_0$ in eq. (13), while

$$A_0^{(II)} = \left( \frac{g_A^2}{2f^2} \right) \left( -1 + \frac{m^2_\pi}{4p^2} \ln \left( 1 + \frac{4p^2}{m^2_\pi} \right) \right),$$

$$A_0^{(III)} = \frac{g_A^2}{f^2} \left( \frac{m_\pi M A_{-1}}{4\pi} \right) \left( - \frac{1}{m_\pi} + \frac{m_\pi}{2p} \left[ \tan^{-1} \left( \frac{2p}{m_\pi} \right) + i \frac{1}{2} \ln \left( 1 + \frac{4p^2}{m^2_\pi} \right) \right] \right),$$

$$A_0^{(IV)} = \frac{g_A^2}{2f^2} \left( \frac{m_\pi M A_{-1}}{4\pi} \right)^2 \left( - \left( \frac{1}{m_\pi} \right)^2 + \left[ i \tan^{-1} \left( \frac{2p}{m_\pi} \right) - \frac{1}{2} \ln \left( \frac{m^2_\pi + 4p^2}{\mu^2} \right) + 1 \right] \right),$$

$$A_0^{(V)} = -\frac{D_2 m^2_\pi}{\left[ 1 + \frac{C_0 M}{4\pi} (\mu + ip) \right]^2}. \quad (15)$$

Note that $A_0^{(IV)}$ has a logarithmic dependence on $\mu$. Since $m^2_\pi$ is proportional to the light quark masses we are required to include the vertex in $A_0^{(V)}$ proportional to $D_2 m^2_\pi$. We have absorbed into $A_0^{(V)}$ some of the finite, $\mu$-independent, part of $A_0^{(IV)}$ that arises in $PDS$ (e.g. the part involving Euler’s constant and a logarithm of $4\pi$). In our power counting $p, \mu$ and $m_\pi$ are considered to be of the same order. Comparing $A_{-1}$ with $A_0$ indicates that the expansion parameter is $p/\Lambda_{NN}$ where

$$\Lambda_{NN} = (8\pi f^2/g_A^2 M) \sim 300 \text{ MeV}. \quad (16)$$

The presence of such a divergence indicates that the power counting suggested by Weinberg is not consistent. The influence of this operator is not subdominant to one pion exchange when $C_0$ is treated to all orders. Its presence is needed to get a subtraction point independent amplitude.
At leading order in our expansion, for large scattering length, \( C_0(m_\pi) = -4\pi/(Mm_\pi) = -3.7 \text{ fm}^2 \). Our power counting leads us to expect that \(|C_2(m_\pi)|\) and \(|D_2(m_\pi)|\) are of order \( 4\pi/(M\Delta NN m_\pi^2) \approx 3.5 \text{ fm}^4 \).

The inclusion of the pions changes the renormalization group equations for \( C_0 \) and \( C_2 \) from what is given in eq. (11). To the order we are working the new beta-functions are

\[
\beta_0 = \frac{M}{4\pi} \left( C_0^2 + \frac{C_0 g_A^2}{f^2} \right) \mu \quad \text{and} \quad \beta_2 = \frac{M}{2\pi} C_0 C_2 \mu .
\]

(17)

In addition the constant \( D_2(\mu) \) satisfies the renormalization group equation

\[
\mu \frac{d}{d\mu} D_2 = \frac{M}{2\pi} D_2 C_0 \mu + \frac{M^2 g_A^2}{32\pi^2 f^2} C_0^2 .
\]

(18)

The solutions to these renormalization group equations are the running coupling constants \( C_0(\mu) \), \( C_2(\mu) \) and \( D_2(\mu) \). When expressed in terms of the running couplings, the full amplitude \( \mathcal{A} \) is independent of \( \mu \); however, as in perturbative QCD, our expression for \( \mathcal{A}_{-1} + \mathcal{A}_0 \) will be \( \mu \) independent only up to the order we are working, \( p^0 \), with residual \( \mu \) dependence canceled by higher order terms in the expansion of \( \mathcal{A} \). The free parameters can be taken to be the value of the running coupling constants at a particular scale, which we choose to be \( \mu = m_\pi \).

All that remains is to fit our free parameters \( C_0(m_\pi) \), \( C_2(m_\pi) \) and \( D_2(m_\pi) \) to data. We compute the phase shift \( \delta \) from the exact expression

\[
\delta = \frac{1}{2i} \ln \left( 1 + i \frac{M_p}{2\pi} \mathcal{A} \right) ,
\]

(19)

expanding both sides to a given order with \( \delta = (\delta_{(0)} + \delta_{(1)} + \ldots) \). For example, to subleading order,

\[
\delta_{(0)} = \frac{1}{2i} \ln \left( 1 + i \frac{M_p}{2\pi} \mathcal{A}_{-1} \right) , \quad \delta_{(1)} = \frac{M_p}{4\pi} \left( \frac{\mathcal{A}_0}{1 + i \frac{M_p}{2\pi} \mathcal{A}_{-1}} \right) .
\]

(20)

We could determine the free parameters analytically by requiring that the theory exactly reproduce effective range theory at low momentum. If we do this, our result reproduces the data very well up to center of mass momentum \( p \sim 150 \text{ MeV} \). However, better agreement with the data at large momenta results from a fit to the phase shift from the Nijmegen partial wave analysis [17] optimized over the momentum range \( p \leq 200 \text{ MeV} \). The result of this procedure are the values

\[
C_0(m_\pi) = -3.34 \text{ fm}^2 , \quad D_2(m_\pi) = -0.42 \text{ fm}^4 , \quad C_2(m_\pi) = 3.24 \text{ fm}^4 .
\]

(21)

and the phase shift plotted in Fig. 3. (The individual values of \( C_0 \) and \( D_2 \) are not particularly meaningful since up to terms higher order in our expansion the amplitude can be written in terms of the linear combination \( C_0 + 2D_2m_\pi^2 \).) As is apparent from Fig. 3, the agreement of the phase shift with data is excellent at quite large values of \( p \). Furthermore, the coupling \( C_0(m_\pi) \) is close to its leading order value (in the limit of large scattering length), \(-3.7 \text{ fm}^2 \) and \( C_2(m_\pi) \) is at the expected size, suggesting that our expansion is valid in this channel.
Fig 3. The phase shifts $\delta$ in the $^1S_0$ channel, and $\delta_0$, $\varepsilon_1$, and $\delta_2$ in the $^3S_1 - ^3D_1$ coupled channels, plotted in degrees versus center of mass momentum $p$. The dot-dashed line represents the leading $p^{-1}$ order calculation for $\delta$ and $\delta_0$; at this order one finds $\varepsilon_1 = \delta_2 = 0$. The dashed lines are the results from the order $p^0$ calculation. Solid lines are results from the Nijmegen partial wave analysis of scattering data [17]. The fits involve three parameters in the $^1S_0$ channel and three parameters in the $^3S_1 - ^3D_1$ coupled channels.

However, for $p > 100$ MeV the magnitude of the ratio $A_0/A_{-1}$ is greater than $\sim 0.5$ and it is difficult to justify the approximations we have made, e.g. neglecting terms suppressed by $(A_0/A_{-1})^2$.

We have performed a similar analysis in the coupled $^3S_1 - ^3D_1$ NN scattering channels. In this channel the amplitude $A$ is a $2 \times 2$ matrix with elements $A_{L,L'}$. The S-matrix in this channel is usually expressed in terms of two phase shifts, $\delta_0$, $\delta_2$ and a mixing angle $\varepsilon_1$,

$$S - 1 = i \frac{pM}{2\pi} A = \begin{pmatrix} e^{2i\delta_0} \cos 2\varepsilon_1 - 1 & i e^{i(\delta_0 + \delta_2)} \sin 2\varepsilon_1 \\ i e^{i(\delta_0 + \delta_2)} \sin 2\varepsilon_1 & e^{2i\delta_2} \cos 2\varepsilon_1 - 1 \end{pmatrix}$$

(22)

The scattering length is large in the $^3S_1$ channel ($a = 5.423 \pm 0.005$ fm) and the power counting is analogous to that of the $^1S_0$ channel. At leading order, $p^{-1}$, $A$ is expressed in terms of a single parameter that is a linear combination of coefficients of four-nucleon operators with no derivatives. Thus $A_{02} = A_{20} = A_{22} = 0$, which implies that $\varepsilon_1 = \delta_2 = 0$. At next order, $p^0$, there are two new parameters analogous to $C_2$ and $D_2$ that occur in the
expression for $A_{00}$. In a future publication [19] analytic formulae for the elements $A_{LL'}$ will be given to order $p^0$. Results of fitting the three parameters that occur in $A_{00}$ to the Nijmegen partial wave analysis [17] are presented in Fig. 3. Note that we have worked to subleading order in calculating $\delta_0$, but only leading order in $\varepsilon_1$ and $\delta_2$, which explains the relative quality of the fits. After fitting $\delta_0$ there are no free parameters in the predictions for $\varepsilon_1$ and $\delta_2$ at this order.

In conclusion, we have shown how to describe $NN$ scattering in a dimensionally regulated theory with a consistent expansion, and have demonstrated the failure of Weinberg’s power counting for these systems. The subtraction scheme we have introduced is useful for treating finely tuned theories with power-law divergences such as the standard model with a light Higgs, and the Nambu–Jona-Lasinio model, as well as any nonrelativistic theory with a large scattering length arising from a short-range interaction. In a future publication we will present other applications of the methods developed here, including a discussion of other partial waves, inelastic processes, properties of the deuteron, and the $N$-body problem [19].

We would like to thank P. Bedaque, A. Bulgac and U. van Kolck for useful discussions. This work supported in part by the U.S. Dept. of Energy under Grants No. DOE-ER-40561, DE-FG03-97ER4014, and DE-FG03-92-ER40701.
REFERENCES

[1] S. Weinberg, Phys. Lett. B251 (1990) 288; Nucl. Phys. B363 (1991) 3.
[2] C. Ordonez, U. van Kolck, Phys. Lett. B291 (1992) 459; C. Ordonez, L. Ray, U. van Kolck, Phys. Rev. Lett. 72 (1994) 1982; Phys. Rev. C53 (1996) 2086.; U. van Kolck, Phys. Rev. C49 (1994) 2932.
[3] T.S. Park, D.P. Min, M. Rho, Phys. Rev. Lett. 74 (1995) 4153; Nucl. Phys. A596 (1996) 515.
[4] D.B. Kaplan, M.J. Savage and M.B. Wise, Nucl. Phys. B478 (1996) 629.
[5] T. Cohen, J.L. Friar, G.A. Miller and U. van Kolck, Phys. Rev. C53 (1996), 2661.
[6] D. B. Kaplan, Nucl. Phys. B 494 (1997) 471.
[7] T.D. Cohen, Phys. Rev. C55 (1997) 67. D.R. Phillips, T.D. Cohen, Phys. Lett. B390 (1997) 7. K.A. Scaldeferri, D.R. Phillips, C.W. Kao, T.D. Cohen, Phys. Rev. C56 (1997) 679. S.R. Beane, T.D. Cohen, D.R. Phillips, nucl-th/9709062.
[8] J.L. Friar, Few Body Syst. 99 (1996) 1, nucl-th/9607020.
[9] M.J. Savage, Phys. Rev. C55 (1997) 2185.
[10] M. Luke and A.V. Manohar, Phys. Rev. D55 (1997) 4129.
[11] G.P. Lepage, Nucl-th/970029, Lectures given at 9th Jorge Andre Swieca Summer School: Particles and Fields, Sao Paulo, Brazil, 16-28 Feb 1997.
[12] M.B. Wise, hep-ph/9707522, Talk given at International Conference (7th Blois Workshop) on Elastic and Diffractive Scattering - Recent Advances in Hadron Physics, Seoul, Korea, 10-14 Jun 1997.
[13] S.K. Adhikari and A. Ghosh, J. Phys. A30 (1997) 6553.
[14] K.G. Richardson, M.C. Birse, J.A. McGovern, hep-ph/9708435.
[15] P.F. Bedaque and U. van Kolck, nucl-th/9710073.
[16] T.S. Park, K.Kubodera, D.P. Min, M. Rho, hep-ph/9711463.
[17] M.M. Nagels, T.A. Rijken, J.J. de Swart, Phys. Rev. D17 (1978) 768; V.G.J. Stoks, R.A.M. Klomp, C.P.F. Terheggen, J.J. de Swart, Phys. Rev. C49, (1994) 2950.
[18] P. Labelle, [hep-ph/9608491]; B. Grinstein and I. Rothstein, Phys.Rev. D57 (1998) 78. M. Luke and M.J. Savage, Phys. Rev. D57 (1998) 413. M. Beneke and V.A. Smirnov, hep-ph/9711391. H.W. Griesshammer, hep-ph/9712467.
[19] D.B. Kaplan, M.J. Savage and M.B. Wise, in preparation.