Guess your neighbour’s input: no quantum advantage but an advantage for quantum theory

Antonio Acín, Mafalda L. Almeida, Remigiusz Augusiak, and Nicolas Brunner

Abstract Quantum mechanics dramatically differs from classical physics, allowing for a wide range of genuinely quantum phenomena. The goal of quantum information is to understand information processing from a quantum perspective. In this mindset, it is thus natural to focus on tasks where quantum resources provide an advantage over classical ones, and to overlook tasks where quantum mechanics provides no advantage. But are the latter tasks really useless from a more general perspective? Here we discuss a simple information-theoretic game called ‘guess your neighbour’s input’, for which classical and quantum players perform equally well. We will see that this seemingly innocuous game turns out to be useful in various contexts. From a fundamental point of view, the game provides a sharp separation between quantum mechanics and other more general physical theories, hence bringing a deeper understanding of the foundations of quantum mechanics. The game also finds unexpected applications in quantum foundations and quantum information theory, related to Gleason’s theorem, and to bound entanglement and unextendible product bases.
1 Introduction

Quantum theory is arguably the most accurate scientific theory designed so far. However, despite this success, we still lack a deep understanding of the foundations of the theory. An important goal in the foundations of quantum mechanics is therefore to recover quantum theory from alternative sets of axioms, motivated by physical principles rather than mathematical ones [1].

In particular, a case that attracted considerable attention recently is that of quantum nonlocal correlations. Quantum nonlocality [2], a valuable resource for information processing [3, 4, 5, 6], is the strongest manifestation of quantum correlations; distant observers performing local measurements on a shared entangled state, may observe correlations between their measurement outcomes which could provably not have been obtained in any local theory. The strength of quantum correlations appears however to be limited, in a way that cannot be yet explained by any physical principle. Consider for instance the principle stating that information cannot be transmitted instantaneously, the so-called no-signaling principle. Although this principle is satisfied by all quantum correlations, preventing from a direct conflict with relativity, it does not single out quantum correlations. Indeed, there exist no-signaling correlations which are stronger than those allowed in quantum mechanics [7], usually referred to as super-quantum correlations. Why such correlations would be unlikely to exist in nature and whether there exist a physical principle singling out quantum correlations are important issues in the foundations of quantum mechanics [8, 9, 10].

Several approaches have been investigated to discuss this problem. The first consists in investigating the capabilities for information processing of super-quantum correlations, and to compare them with that of quantum correlations. Interestingly, it was shown that the availability of certain super-quantum correlations, instead of quantum correlations, would tremendously increase the communication power of classical communication. In particular, it was shown that some of them would collapse communication complexity [11, 12, 13] (hence dramatically reducing the amount of classical communication required to solve a large class of problems [4]) or violate the principles of information causality [14, 15] and macroscopic locality [16]. A second approach, perhaps less demanding, starts from assuming 'local quantum mechanics'. In other words the statistics of local measurements are assumed to follow Born’s rule. What other principle should then be imposed in order for the global statistics to be quantum? In the bipartite case, it is proven that the no-signaling principle is enough to single out quantum correlations [17, 18]. Importantly, while both of these approaches have proven to be (at least partially) successful in the case of two parties, none of them can tackle the general multipartite scenario.

Here we present 'Guess your neighbour’s input' (GYNI) [19], a simple multipartite game, the rules of which can be understood intuitively from its name. Despite its innocuous appearance, the game captures crucial features of multipartite quantum correlations. The main aspect of the game is the following. Whereas players sharing quantum resources do not have any advantage over players sharing clas-
Guess your neighbour’s input ...

... classical resources, it turns out that players sharing super-quantum correlations have an advantage over players sharing either classical or quantum resources. In other words, the limitation of quantum resources is here not a mere consequence of the no-signaling principle. Hence, the game of GYNI provides a natural separation between quantum and super-quantum correlations. More generally these results point towards a strengthening of the no-signaling principle, in the general multipartite case, obeyed by quantum mechanics. Therefore, whereas the game of GYNI may seem a priori useless from a quantum perspective, it does in fact bring a novel and fresh perspective on the foundations of quantum theory [20].

Although it is not clear yet what fundamental principle lies behind the quantum limitations for GYNI, several important features of such a principle can already be identified. In particular, this principle must be genuinely multipartite, which can be shown directly from the GYNI game. This is because there exist multipartite super-quantum correlations, that will nevertheless satisfy any bipartite principle [21] (see also [22]). Hence quantum correlations can provably not be recovered from any principle that is inherently bipartite (such as no trivial communication complexity or information causality).

Moreover, it can be shown, using GYNI, that there exist multipartite super-quantum correlations obeying the Born rule locally [18]. Therefore, in the multipartite case, the no-signaling principle is not enough to recover quantum correlations from local quantum mechanics. This result also has fundamental consequences on extensions of Gleason’s theorem [23] to composite systems.

Finally, we shall see that GYNI has also applications beyond quantum foundations. In particular, the game turns out to be strongly related [24, 25] to topics of quantum information theory, namely bound entanglement [26] and unextendible product bases [27]. This is surprising since these subjects seem to be completely unconnected at first sight. This connection deepens our understanding of Bell inequalities with no quantum advantage. In particular it allows us to derive such inequalities from unextendible product bases.

This chapter is structured as follows. In Section 2, after giving a brief background introduction to nonlocal correlations, we present the GYNI game and derive the winning probabilities for various types of correlations (local, quantum, and no-signaling). Applications of GYNI are presented in Sections 3 and 4. First, in Section 3 we discuss results on the extension of Gleason’s theorem for composite systems. Then, in Section 4 we shall see that any information-theoretic principle capturing quantum correlations must be genuinely multipartite. In Section 5, after presenting in detail the connection between GYNI and unextendible product bases, we will make use of this connection to go beyond GYNI, and to better understand the structure of Bell inequalities with no quantum advantage. Finally, we will conclude in Section 6.
2 Guess your neighbour’s input

2.1 Background: classical, quantum and no-signalling correlations

The definition of (non)locality was introduced by Bell, as a rigorous physical and mathematical framework to test the Einstein-Podolsky-Rosen paradox. Consider two distant observers, Alice and Bob, sharing a physical system, and performing local measurements on their subsystems. Alice and Bob’s choice of observables are labeled by $x_1$ and $x_2$ respectively, and take outcomes $a_1$ and $a_2$ (hereafter the subscripts will be omitted). The joint probability distribution of outcomes, conditioned on the choice of observables, is represented by $P(a_1,a_2|x_1,x_2)$. This set of data is described as local (or classical) if and only if $P(a_1,a_2|x_1,x_2)$ can be reproduced by a local hidden-variable model\(^1\), that is, iff it can be written in the form

$$P_L(a_1,a_2|x_1,x_2) = \sum_\lambda P(\lambda)P(a_1|x_1,\lambda)P(a_2|x_2,\lambda).$$

(1)

Here individual outcomes are completely specified by the choice of local observable and the shared (hidden) variable $\lambda$. Indeed, Alice and Bob’s outcomes may be correlated via the hidden-variable $\lambda$, which is distributed with probability density $P(\lambda)$.

The probability distribution $P(a_1,a_2|x_1,x_2)$ is said realizable in quantum mechanics (or in short, to be quantum) if and only if it can be written in the following form:

$$P_Q(a_1,a_2|x_1,x_2) = \text{tr}(\rho_{AB}M_{a_1}^{x_1} \otimes M_{a_2}^{x_2}),$$

(2)

where the state of system $\rho_{AB}$ is defined by a density operator on the joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and $M_{a_1}^{x_1}, M_{a_2}^{x_2}$ are the local generalized measurements (positive semidefinite operators on the local Hilbert space such that $\sum_j M_{a_j}^{x_j} = 1$ ($j = 1,2$) with $1$ denoting an identity matrix of the dimension following from the context). Indeed quantum correlations are stronger than classical ones, hence there exist quantum distributions $P_Q(a_1,a_2|x_1,x_2)$ which cannot be written in the form (1).

A crucial feature of both classical and quantum correlations is that they satisfy the no-signalling principle: instantaneous information transmission is impossible. More formally the principle says that Alice’s measurement outcome is uncorrelated to Bob’s choice of measurement, that is

$$\forall x_1, x_2, \sum_{a_2} P_{NS}(a_1,a_2|x_1,x_2) = \sum_{a_2} P_{NS}(a_1,a_2|x_1,x_2') \equiv P_a(a_1|x_1).$$

(3)

Similar equations must be obeyed for Bob’s marginal distribution. Correlations satisfying this principle, as well as normalization and positivity, are referred to as

\(^1\) By simplicity, we consider $\lambda$ to be discrete, but all the formulation can be extended to the continuous case.
nonsignaling correlations [28]. Interestingly, there exist nonsignaling correlations that are not quantum [7], i.e. cannot be written in the form (5).

The above definitions are naturally generalized to the multipartite case: local correlations between \(N\) parties are described by

\[
P_L(a_1, \ldots, a_N|x_1, \ldots, x_N) = \sum_{\lambda} P(\lambda)P(a_1|x_1, \lambda)P(a_2|x_2, \lambda) \cdots P(a_N|x_N, \lambda),
\]

(4)

Quantum correlations are given by

\[
P_Q(a_1, \ldots, a_N|x_1, \ldots, x_N) = \text{tr}(\rho M_{a_1}^{x_1} \otimes \cdots \otimes M_{a_N}^{x_N}),
\]

(5)

where \(\rho\) denotes the quantum state shared between the parties. Finally nonsignalling correlations are defined such that no party is allowed to signal to others through his choice of measurement, that is

\[
\forall j, \ j' \sum_{a_j} P_{NS}(a_1, \ldots, a_N|x_1, \ldots, x_N) = \sum_{a'_j} P_{NS}(a_1, \ldots, a_N|x_1, \ldots, x_N).
\]

(6)

and similar relations for any two sets of parties.

In order to distinguish between these three kinds of correlations (local, quantum, and nonsignaling) one devises a Bell test, involving a certain number (usually finite) of parties, observables and outcomes. It is convenient to represent a probability distribution \(P(a_1, \ldots, a_N|x_1, \ldots, x_N)\) as a vector of probabilities \(P\), with entries

\[
P(a|\mathbf{x}) = P(a_1, \ldots, a_N|x_1, \ldots, x_N).
\]

In this vector space, Bell inequalities are given by linear expressions

\[
S = \sum_{j} \alpha_j P_j \leq \omega_c.
\]

(7)

The coefficients \(\alpha_j\) are real. The bound of the inequality, i.e. \(\omega_c\), is the largest value of the Bell polynomial \(S\) for any local probability distribution, i.e. of the form (4).

The set of local correlations defines a convex polytope. Hence it can be described by a finite set of linear inequalities, that are called tight Bell inequalities.

The local set is a strict subset of the set of quantum correlations. The latter is still a convex set, although no longer a polytope. It can, however, be described by an infinite set of quantum Bell inequalities, similar to (7) but replacing the classical bounds by quantum ones, \(\omega_q\), which may in general exceed the classical one, i.e. \(\omega_q \geq \omega_c\).

Finally, the set of no-signalling correlations is also a convex polytope, which is strictly larger than the quantum set. Its facets are given by positivity inequalities, stating that joint probabilities are positive. The largest value of a Bell polynomial \(S\) for any no-signaling probability distribution is denoted \(\omega_{ns}\); indeed, in general \(\omega_{ns} \geq \omega_q\).

The scene being set, let us bring in the protagonists.
2.2 The GYNI game

Consider $N$ players disposed on a ring. The game starts with each player receiving a (private) input bit $x_i$ (say from a referee), distributed according to the probability density $q(x)$. Now, the name of the game says it all: the goal is that each player makes a correct guess $a_i$ of his (say) right-hand side neighbour’s input bit (see Fig. 1), that is

$$\forall i \quad a_i = x_{i+1},$$

where $x_{N+1} \equiv x_1$. Importantly, the players are successful if and only if all the parties make a correct guess.

![Fig. 1 The GYNI game. The goal is that each party outputs its right-neighbour’s input: $a_i = x_{i+1}$.](image)

The winning probability is defined as

$$\omega = \sum_x q(x) P(a_i = x_{i+1} | x_i)$$

with $P(a_i = x_{i+1} | x_i) = P(a_1 = x_2, \ldots, a_N = x_1 | x_1, \ldots, x_N)$. Note that no communication between the players is allowed during the game. However, during the preparation stage of the game, the players are informed of the distribution $q(x)$ of the inputs. They are allowed to establish a common strategy, which will consist in utilizing in a judicious way physical resources they are allowed to share. Here our aim will be to find out how good the parties can perform at the game when sharing respectively classical, quantum, and no-signaling correlations. Formally, the game represents a multipartite Bell test, and eq. (9) has the structure of a multipartite Bell inequality (see (7)). Hence our goal will be to determine the bounds $\omega_c$, $\omega_q$ and $\omega_{ns}$, corresponding to the classical, quantum and no-signalling bounds of the GYNI Bell inequality.

2.3 No quantum advantage

A central features of the GYNI game is that the maximum winning probability in the quantum world is exactly the same as in a classical one. In other words, the GYNI
inequalities (9) have the same classical and quantum bound, i.e. \( \omega_c = \omega_q \), for any distribution of inputs \( q(x) \).

**Classical bound.** Let us start by analyzing the best classical performance. Any probabilistic classical strategy (which includes the use of shared randomness), can be decomposed into a convex sum of deterministic strategies. This means that players can achieve the best winning probability \( \omega_c \) by making a definite guess \( a_i \) for each input bit \( x_i \). Hence it is enough to analyze such cases. Imagine that their deterministic strategy allows them to succeed when receiving some input string \( y \), i.e. \( a_i(y_i) = y_i + 1, \forall i \). The input strings have an interesting orthogonality property: for any other input \( x \neq y, \bar{y} \), there is some \( i \) such that \( x_i = y_i \) and \( x_{i+1} \neq y_{i+1} \). Then, for any input-strings \( x \), there is always some player \( i \) which will make a wrong guess. He will receive the bit \( x_i = y_i \), and output \( a_i(y_i) = y_i + 1 \) according to the strategy, while the correct would be \( a_i(\bar{y}_i) = \bar{y}_i + 1 \). It is still possible to score when receiving \( \bar{y} \) by setting the strategy to \( a_i(\bar{y}_i) = \bar{y}_i + 1, \forall i \). The best classical winning probability is then

\[
\omega_c = \max_x [q(x) + q(\bar{x})],
\]

achieved by using \( y \) such that \( q(y) + q(\bar{y}) = \max_x [q(x) + q(\bar{x})] \).

**Quantum bound.** If players have access to quantum systems, the most general protocol involves a quantum state \( \rho \) of arbitrary Hilbert space dimension and general quantum measurements \( M_{ax} \) corresponding to a probability distribution

\[
P(a_1, \ldots, a_N|x_1, \ldots, x_N) = \text{tr}(\rho M_{ax_1}^a \otimes \cdots \otimes M_{ax_N}^a).
\]

The best quantum winning probability is then the maximum expected value of the Bell operator

\[
\omega_q = \max_{\psi, \text{meas}} \sum_x q(x) \langle M_x \rangle.
\]

where \( M_x \equiv M_{ax_1}^a \otimes \cdots \otimes M_{ax_N}^a \). Notice that it is enough to optimize over pure states \( |\psi\rangle \) and projective measurements \( M_{ax}^a \), since there are no restrictions on the size of local Hilbert spaces. Following a similar reasoning to the classical case, take projectors \( M_y \) and \( M_x \), where \( x \neq y, \bar{y} \). Then there is some local projector \( i \), defined on the same basis \( x_i = y_i \), but projecting on orthogonal subspaces \( x_{i+1} \neq y_{i+1} \). Consequently, the measurement projectors also obey an orthogonality condition,

\[
M_x M_y = 0 \quad \text{if} \quad x \neq y, \bar{y}.
\]

This property is sufficient to show that

\[
\sum_x q(x) \langle M_x \rangle \leq \max_x [q(x) + q(\bar{x})],
\]

which proves that the best quantum winning probability is the same as the classical one

\[
\omega_q = \omega_c.
\]
Indeed, the derivation of the best winning probabilities, in both the classical and quantum case, relies on a rather natural orthogonality property (either of deterministic local strategies, or of orthogonal measurement projectors). Interestingly such a property is not a consequence of the no-signaling, and does in general not hold for no-signaling correlations, as we shall see in the next section.

2.4 No-signalling advantage

The game of GYNI is in some sense clearly related to the notion of signaling. Indeed, if all players can guess correctly their input’s neighbour with a high probability, this will lead to signaling. Hence it may come to no surprise that quantum resources, which are indeed no-signaling, give no advantage for GYNI. Surprisingly this intuition is not correct, as we shall see here, since certain super-quantum no-signaling correlations can in fact provide an advantage compared to classical correlations.

2.4.1 Correlated inputs

Consider a particular version of the GYNI game in which the inputs are correlated in the following way: \(q(x)\) is uniform on the set of inputs that satisfy the parity condition:

\[
q(x) = \begin{cases} 
1/2^{N-1} & \text{if } x_1 \oplus \cdots \oplus x_N = 0 \\
0 & \text{otherwise,}
\end{cases}
\]  

where \(\hat{N} = N\) if \(N\) is odd and \(\hat{N} = N - 1\) if \(N\) is even. Using Eq. (10), it is easy to check that in classical or quantum theory, the success probability is limited by \(\omega_c = 1/2^{N-1}\). We will see that, allowing for super-quantum correlations, this limit can be beaten: the best winning probability \(\omega_{ns}\) is upper-bounded by \(\omega_{ns} \leq 1/3\).

Unlike the previous example, here, although each party still has absolute uncertainty about his neighbour’s input, no-signalling correlations are able to exploit a global correlation (the parity of the input-string) to increase the chance of correct guess.

3-player game. Let us first consider the simplest game, featuring three players.\(^2\) The GYNI inequality is then simply given by

\[
\omega = \frac{1}{4} |P(000|000) + P(110|011) + P(011|100) + P(101|110)| \leq \frac{1}{4}, \tag{17}
\]

where the bound holds for any local or quantum strategy.

Let us first derive an upper bound on the no-signaling winning probability. Consider the first three terms in (17). The no-signalling principle implies that

\(^2\) Note that for 2 players, no-signaling correlation provide no advantage.
Then, for any \(N\) combinations of three probability terms of Eq. (17), such that we get

\[
P(000|000) \leq \sum_{a_3} P(00a_3|000) = \sum_{a_3} p(00a_3|001),
\]

\[
P(110|011) \leq \sum_{a_2} P(1a_20|011) = \sum_{a_2} p(1a_20|001),
\]

\[
P(011|101) \leq \sum_{a_1} P(a_11|101) = \sum_{a_1} p(a_11|001).
\]

From normalization, we know that the sum of these terms satisfies \(P(000|000) + P(110|011) + P(011|101) \leq 1\). We apply a similar reasoning to the remaining combinations of three probability terms of Eq. (17), such that we get

\[
3|P_{NS}(000|000) + P_{NS}(110|011) + P_{NS}(011|101) + P_{NS}(101|110)| \leq 4.
\]  

Hence we obtain an upper limit on the no-signalling winning probability: \(\omega_{ns} \leq 1/3\). From this derivation, we also conclude that it is only possible to reach this limit if every probability term in the GYNI inequality (17) has the value 1.

Now, it turns out that this upper bound can be reached by an actual no-signaling probability distribution. The latter is rather complicated (see [19]), but it would be interesting to better understand its structure. To be complete, let us mention that there exist two (among 45) inequivalent classes of extremal tripartite no-signaling boxes [29], that reach the best winning no signaling probability \(\omega_{ns} = 1/3\).

Finally note an interesting feature of inequality (17). It is a tight Bell inequality, that is, it defines a facet of the polytope of local correlations [30]. Hence it identifies a portion of the quantum boundary which is of maximal dimension [19].

**N-player game.** Next let us consider the general case of \(N\) players, using the condition (16) on the inputs. For any \(N\), no-signaling correlations provide an advantage. To show this, we prove that resources that provide a winning probability \(\omega/\omega_k\), in the game with \(N\) players, can provide at least the same ratio \(\omega/\omega_k\) for \(N+1\) players. The strategy is very simple: players 1 to \(N\) play exactly as in the \(N\)-player game, while player \(N+1\) outputs his input, \(a_{N+1} = x_{N+1}\). This guess is correct when \(x_{N+1} = x_1\), which happens with probability \(1/2\). Since \(\omega, (N+1) = (1/2)\omega, (N)\), the ratio remains the same:

\[
\frac{\omega}{\omega_k}\ (N) = \frac{\omega}{\omega_k}\ (N+1).
\]

Then, for any \(N \geq 3\), the best no-signalling success probability is at least as good as \((4/3)\omega_k\). This lower bound is achieved if the first 3 players use the optimal no-signalling strategy for the 3-player game, while the remaining output their inputs. They can however do better: using linear programming, we obtained that \(\omega_{ns}/\omega_k = 4/3\), for \(N = 4\); \(\omega_{ns}/\omega_k = 16/11\), for \(N = 5, 6\); and \(\omega_{ns}/\omega_k = 64/42\), for \(N = 7, 8\). Basing on these three values the rough estimation would suggest that the ratio \(\omega_{ns}/\omega_k\) scales with \(N\) as \(4^k/\lfloor(1/3)[(23/3)4^{k-1} + k + 1/3]\rfloor\), where \(k = [(N-1)/2]\). This in the limit of \(N \to \infty\) gives \(\omega_{ns}/\omega_k \to 36/23\).

Remarkably, it turns out that the \(N\)-partite GYNI Bell inequalities (with promise [16]), hereafter referred to as GYNI\(_N\), are tight for an arbitrary odd \(N\) [25] and for \(N = 4, 6\) [19]. It is conjectured that they are tight for any \(N\).
2.4.2 Upper bounds on $\omega_{ns}$

From the winning probability in the classical case (Eq. (10)), we know that $q(x) \leq \omega_c$ for any $x$, from which we get the bound $\omega \leq \omega_c \sum_x P(a_i = x_{i+1} | x_i)$. Something more meaningful is obtained if we now assume the distributions to be no-signalling. Take the summation $\sum_x P(a_i = x_{i+1} | x)$. Repeatedly applying the no-signalling condition (3), (first to party $N$, then to $N-1$ and so on), we get

$$\sum_{x_1, \ldots, x_N} P_{NS}(x_2, \ldots, x_N | x_1, \ldots, x_{N-1})$$

$$\leq \sum_{x_1, \ldots, x_N} P_{NS}(x_2, \ldots, x_N | x_1, \ldots, x_{N-1})$$

$$= \sum_{x_1, \ldots, x_{N-1}} P_{NS}(x_2, \ldots, x_N | x_1, \ldots, x_{N-2}) = \cdots = 2. \tag{21}$$

We conclude that the success probability within no-signalling theories is bounded by

$$\omega_{ns} \leq 2\omega_c, \tag{22}$$

which means that, in general, no-signalling correlations do not allow deterministic success. As we could predict, for some input distributions, perfect guessing is only possible if players communicate. In those cases, it is reasonable to expect that classical, quantum and no-signalling resources provide exactly the same best performance.

2.4.3 Completely uniform distributions of inputs

The counter-example for the previous intuition is the following: the completely uniform distribution over the inputs, i.e. $q(x) = 1/2^N$. We obtain a tight upper bound on $\omega_{ns}$ by noticing that $2q(x) = \omega_c$, which leads to

$$\omega_{ns} = \frac{\omega_c}{2} \sum_x P_{NS}(a_i = x_{i+1} | x_i) \leq \omega_c. \tag{23}$$

Classical and no-signalling resources provide exactly the same best winning probability, in a situation where each player has, a priori, no information about the input of its neighbour.

Once the GYNI Bell inequality has been introduced, we discuss in the next sections the application of this inequality in two different contexts, related to the characterization of quantum correlations.
3 Application 1: Gleason’s theorem for multipartite systems

Gleason’s Theorem [23] is a celebrated theorem in the foundations of quantum mechanics that allows recovering the Born rule for quantum probabilities from the structure of quantum measurements. Recall that a quantum measurement acting on a Hilbert space of dimension $d$ corresponds to a set of $k$ positive operators, $M_i \geq 0$ with $i = 1, \ldots, k$ such that $\sum_i M_i = 1$. Gleason’s Theorem aims at characterizing maps from quantum measurements to probability distributions. The maps $\Lambda$ have to satisfy the following properties:

1. For any positive operator $0 \leq M \leq 1$ one has $\Lambda(M) \geq 0$.
2. Given a quantum measurement, that is, given a set of $k$ positive operators summing up to the identity, one has

$$\sum_{i=1}^k \Lambda(M_i) = 1. \quad (24)$$

Note that the considered maps are non-contextual, as the measurement operators are mapped into probabilities independently of the structure of the measurement they belong to.

Gleason’s Theorem implies that all maps satisfying the two requirements 1 and 2 can be written as $\Lambda(M) = \text{tr}(\rho M)$ for a given quantum state $\rho$, that is, $\rho$ is a positive operator of trace one. We sketch here the idea of the proof, while its detailed version may be found e.g. in Ref. [31]. Notice, however, that the author of [31] imposes an additional condition on $\Lambda$ which, as we show below, can be simply inferred from 1 and 2. Indeed, consider two measurement operators $M_1, M_2$ such that $M_3 = 1 - (M_1 + M_2) \geq 0$. Consider now the two different measurements $\{M_1, M_2, M_3\}$ and $\{M_1 + M_2, M_3\}$. The second measurement is simply a coarse-grained version of the first in which the two first outcomes are grouped together. A direct application of property 2 above implies that $\Lambda(M_1) + \Lambda(M_2) = \Lambda(M_1 + M_2)$. This together with properties 1 and 2 imply that the map $\Lambda$, initially defined for positive operators, can be uniquely extended to a linear map acting on all operators. It immediately follows that it can be written as $\text{tr}(XM)$ for an operator $X$. But then, the condition 1 implies the positivity of the operator $X$ and its normalization follows from condition 2. On the other hand, one checks by hand that any of these maps satisfies conditions 1 and 2.

This theorem is a seminal result in the Foundations of Quantum Physics. In particular, it implies that Born’s rule for the computation of measurement probabilities can be derived from the Hilbert space structure of quantum measurements and the two natural conditions provided above.
3.1 Gleason correlations

Gleason’s Theorem was initially established for single systems. It was later extended to composite systems in Refs. [32, 33]. The scenario consists of \( N \) independent observers. To each observer \( j \), with \( j = 1, \ldots, N \), one associates a Hilbert space of dimension \( d_j \) and a structure of quantum measurements given by sets of positive operators summing up to the identity. For the sake of simplicity, we take in what follows all the local dimensions equal, \( d_i = d, \forall i \). We denote by \( \{M^{(j)}_{i_j}\}, i_j = 1, \ldots, k_j \) the sets of positive operators defining a measurement for each observer, that is, \( \sum_{i_j} M^{(j)}_{i_j} = 1 \). The extension of the theorem then aims at characterizing those maps from measurements by each observer to probability distributions. In what follows, for the ease of notation, we restrict the analysis to the simplest bipartite case, although it can be easily generalized to an arbitrary number of parties. The map is requested to satisfy the following conditions:

1. For pairs of positive operators, \( M^{(1)}_{i_1}, M^{(2)}_{i_2} \), where \( 0 \leq M^{(1)}_{i_1}, M^{(2)}_{i_2} \leq 1 \) one has \( \Lambda(M^{(1)}_{i_1}, M^{(2)}_{i_2}) \geq 0 \).
2. For pairs of measurements, \( \{M^{(1)}_{i_1}\}, \{M^{(2)}_{i_2}\} \), where \( 0 \leq M^{(1)}_{i_1}, M^{(2)}_{i_2} \leq 1 \) one has
   \[
   \sum_{i_1, i_2 = 1}^{k_1, k_2} \Lambda(M^{(1)}_{i_1}, M^{(2)}_{i_2}) = 1. \quad (25)
   \]
3. Given two complete quantum measurements by one of the observers, say the second, \( \{M^{(2)}_{i_2}\} \) and \( \{N^{(2)}_{i_2}\} \), the map has to be such that
   \[
   \sum_{i_2 = 1}^{k_2} \Lambda(M^{(1)}_{i_1}, M^{(2)}_{i_2}) = \sum_{i_2 = 1}^{k_2} \Lambda(M^{(1)}_{i_1}, N^{(2)}_{i_2}). \quad (26)
   \]

The new condition, i.e., the third one, can be understood as the natural formalization of the no-signalling principle in the considered framework: the marginal probability distribution seen by one of the observers cannot depend on the measurement performed by the other observer. The generalization to an arbitrary number of parties of these requirements is straightforward. Now \( \Lambda \) maps tuple of positive operators \( M^{(1)}_{i_1}, \ldots, M^{(N)}_{i_N} \) into non-signalling probability distributions.

The generalization of the theorem to this scenario, that we call multipartite Gleason’s Theorem, states that all such maps can be written as

\[
\Lambda(M^{(1)}_{i_1}, \ldots, M^{(N)}_{i_N}) = \text{tr} \left( W M^{(1)}_{i_1} \otimes \ldots \otimes M^{(N)}_{i_N} \right), \quad (27)
\]

where \( W \) is an operator which is positive on product states \( |\psi_1\rangle \ldots |\psi_N\rangle \). These operators are also known as entanglement witnesses [34]. As above, it is clear that maps of the form (27) satisfy the previous three requirements and the non-trivial part of the result is proving the opposite direction.
As the set of entanglement witnesses is larger than the set of quantum states (or, in other words, there exist operators $W$ that are non-positive, but positive on product states) the set of distributions \((27)\), called in what follows Gleason correlations, is in principle larger than the quantum set. However, it was shown in \([17, 18]\) that the two sets actually coincide for two parties. Thus, as it happens for single-party systems, imposing the structure of quantum measurements for the observers gives the quantum correlations.

The proof of the equivalence between Gleason and bipartite quantum correlations exploits the Choi-Jamiołkowski (CJ) isomorphism \([35]\) that relates maps to operators. In this case, the it says that any witness $W$ can be written as $(I \otimes \Gamma)(\Phi)$, where $\Gamma$ is a positive map and $\Phi$ is the projector onto the maximally entangled state $|\Phi\rangle = (1/\sqrt{d}) \sum_i |ii\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and $I$ stands for an identity map. With the aid of Ref. \([36]\), one can prove that any normalized witness can also be written as $(I \otimes \Lambda)(\Psi)$, where $\Lambda$ is now a positive and trace-preserving map, while $\Psi$ is a projector onto some pure bipartite state.\(^3\) It then follows that

$$\text{tr}(W M_{a_1}^\dagger \otimes M_{a_2}^\dagger) = \text{tr}[(I \otimes \Lambda)(\Psi) M_{a_1}^\dagger \otimes M_{a_2}^\dagger] = \text{tr}[\Psi M_{a_1}^\dagger \otimes \Lambda^\ast (M_{a_2}^\dagger)] = \text{tr}(\Psi M_{a_1}^\dagger \otimes M_{a_2}^\dagger),$$

(29)

where $\Lambda^\ast$ is the dual of $\Lambda$ and $\overline{M}_{a_2}^\dagger = \Lambda^\ast (M_{a_2}^\dagger)$ defines a valid quantum measurement because the dual of a positive trace-preserving map is positive and unital, that is, $\Lambda^\ast (I) = I$.

The next natural question is as to whether the equivalence between quantum and Gleason correlations holds for an arbitrary number of parties. As we show next, the answer to this question turns out to be negative, as there are local measurements acting on entanglement witnesses that produce supra-quantum correlations. Before proving this result, it is worth mentioning that local measurements on entanglement witnesses that can be written as

$$W = \sum_k (A_{a_1}^k \otimes \cdots \otimes A_{a_n}^k) (\rho_k),$$

(30)

\(^3\) To see this explicitly let us first notice that for a normalized witness $W$ it holds that $W = (I \otimes \Lambda)(\Phi)$ with trace-preserving $\Lambda$ iff $W_{a} = \text{tr}_b W = 1/d$. Then, if $W_{a} \neq 1/d$ but it is of full rank, one introduces another witness $\tilde{W} = (1/d)(W_{a}^{-1/2} \otimes I) W (W_{a}^{-1/2} \otimes I)$. Clearly, $\tilde{W}_a = 1/d$ and thus $\tilde{W}$ is isomorphic to a trace-preserving positive map $\tilde{\Lambda}$. Consequently,

$$W = d(\sqrt{W_a} \otimes 1) \tilde{W} (\sqrt{W_a} \otimes 1) = d(\sqrt{W_a} \otimes 1)(I \otimes \Lambda)(\Phi) (\sqrt{W_a} \otimes 1) = (I \otimes \Lambda)(\Psi),$$

(28)

where $\Psi$ denotes a projector onto some normalized pure state $|\Psi\rangle = \sqrt{d}(\sqrt{W_a} \otimes 1)|\Phi\rangle$ of full Schmidt rank. Finally, if $W_a$ is rank-deficient, one constructs yet another witness $W' = W + \mathcal{P} \otimes 1$, where $\mathcal{P} = 1 - \mathcal{P}_a$ with $\mathcal{P}_a$ denoting a projector onto the support of $W_a$. Then, $\mathcal{P} = 1$ is full-rank and therefore $W'$ admits the form \([28]\). To complete the proof, it suffices to notice that $W = (\mathcal{P} \otimes 1)W' (\mathcal{P} \otimes 1)$, and hence $W$ also assumes the form \([28]\) with a normalized pure state $|\Psi\rangle = \sqrt{d}(\mathcal{P}_a |W_a\rangle \otimes 1)|\Phi\rangle$, which is now not of full Schmidt rank.

\(^4\) The dual map $\Lambda^\ast$ of $\Lambda$ is the map such that $\text{tr}(\Lambda A) = \text{tr}(\Lambda^\ast A)$.\]
where $\rho_k$ are $N$-party quantum states, $\Lambda^k_{A_i}$ are positive trace preserving maps and the number of terms in the sum is arbitrary, do not lead to supra-quantum correlations. This is a rather straightforward generalization of the equivalence proof in the bipartite case.

In order to prove that in the multipartite case the set of Gleason correlations contains quantum correlations as a strict subset, we provide an example of entanglement witness and local measurements giving nonsignalling correlations which violate the three-party GYNI Bell inequality. Let us start by introducing the following set of four fully product vectors from the three-qubit Hilbert space:

$$
|\psi_1\rangle = |000\rangle, \quad |\psi_2\rangle = |1e^+e\rangle, \quad |\psi_3\rangle = |e1e^+\rangle, \quad |\psi_4\rangle = |e^+e1\rangle, (31)
$$

where $|e\rangle \in \mathbb{C}^2$ is an arbitrary vector different from $|0\rangle$ and $|1\rangle$, while $|e\rangle$ stands for a vector orthogonal to $|e\rangle$. One checks by hand that there is no other three-qubit fully product vector orthogonal to all $|\psi_i\rangle$s; such sets of product vectors are called unextendible product bases (UPBs) [27] (see section 5.1 for a detailed discussion on UPBs and more examples).

As noticed in [27], the set (31), called Shifts UPB, can be used for a simple construction of bound entangled state, i.e., an entangled state from which any type of maximally entangled state cannot be distilled [26]. The state is given by $\rho_{\text{UPB}} = (I - \Pi_{\text{UPB}})/4$ with $\Pi_{\text{UPB}}$ denotes the projector onto span$\{|\psi_i\rangle\}$.

Let us now consider the normalized entanglement witness detecting $\rho$:

$$
W = \frac{1}{4 - 8\varepsilon} (\Pi_{\text{UPB}} - \varepsilon I), \quad (32)
$$

where

$$
\varepsilon = \min_{|\alpha\beta\gamma\rangle} \langle \alpha\beta\gamma | \Pi_{\text{UPB}} | \alpha\beta\gamma \rangle. \quad (33)
$$

The fact that there is no fully product vector orthogonal to $|\psi_i\rangle$ implies that $\varepsilon > 0$, and, on the other hand, it is fairly easy show that $\varepsilon < 1/2$. One also notices that $\text{tr}(W \rho) = -\varepsilon/(1 - 2\varepsilon) < 0$.

Now, one can see that the witness $W$, when measured along the local bases in the definition of the UPB (31), leads to correlations that produce a value of GYNI game equal to $\beta = (1 - \varepsilon)/(1 - 2\varepsilon)$, which is larger than one for all positive $\varepsilon$ not larger than one-half. Thus, these correlations represent an example of Gleason correlations with no quantum analogue.

## 4 Application II: Quantum correlations and information principles

As mentioned in the introduction, an intense research effort has recently been devoted to understand why nonlocality appears to be limited in quantum mechanics. Information concepts have been advocated as the key missing ingredient needed to
single-out the set of quantum correlations \([8, 9, 10]\). The main idea is to identify ‘natural’ information principles, satisfied by quantum correlations, but violated by super-quantum correlations. The existence of the latter would then have implausible consequences from an information-theoretic point of view. Celebrated examples of these principles are information causality [14] or non-trivial communication complexity [11]. While the use of these information concepts has been successfully applied to specific scenarios [12, 13, 15, 37, 38], proving, or disproving, the validity of a principle for quantum correlations is extremely challenging. On the one hand, it is rather difficult to derive the Hilbert space structure needed for quantum correlations from information quantities. On the other hand, proving that some super-quantum correlations are fully compatible with an information principle seems out of reach, as one needs to consider all possible protocols using these correlations and show that none of them leads to a violation of the principle. Hence it is still unclear whether this approach is able to fully recover the set of quantum correlations.

Therefore it is relevant to derive general features of a principle that could potentially identify quantum correlations. Using GYNI, it was recently shown that such a principle must be genuinely multipartite. More specifically, no bipartite principle can characterize the set of quantum correlations when three of more observers are involved [21]. This rest of this section is devoted to this result.

Before discussing the result, it is worth recalling that, so far, most information-theoretic principles have been formulated in the bipartite scenario. Actually, even the general formulation of the no-signalling principle has a bipartite structure: correlations among \(N\) observers are compatible with the no-signalling principle whenever there exists no partition of the \(N\) parties into two groups such that the marginal probability distribution of one set of the parties depends on the measurements performed by the other set of parties (see (6)). Moving to information causality, it considers a scenario in which a first party, Alice, has a string of \(n_A\) bits. Alice is then allowed to send \(m\) classical bits to a second party, Bob. Information causality bounds the information Bob can gain on the \(n_A\) bits held by Alice whichever protocol they implement making use of the pre-established bipartite correlations and the message of \(m\) bits. Alice and Bob can violate this principle when they have access to some super-quantum correlations [13]. In the case \(m = 0\), information causality implies that in absence of a message, pre-established correlations do not allow Bob to gain any information about any of the bits held by Alice, which is nothing but the no-signaling principle. This suggests the following generalization of information causality to an arbitrary number of parties, mimicking what is done for the no-signalling principle: given some correlations \(P(a_1, \ldots, a_n | x_1, \ldots, x_N)\), they are said to be compatible with information causality whenever all bipartite correlations constructed from them satisfy this principle. This generalization ensures the correspondence between no-signaling and information causality when \(n = 0\) for an arbitrary number of parties. This generalization of information causality has recently been applied to the study of extremal tripartite non-signaling correlations [22].

Regarding non-trivial communication complexity, it studies how much communication is needed between two distant parties to compute probabilistically a function of some inputs in a distributed manner. It can also be interpreted as a generalization
of the no-signaling principle, as it imposes constraints on correlations when a finite amount of communication is allowed between parties. Different multipartite generalizations of the principle have been studied, see [4]. However, as for information causality, one can always consider the straightforward generalization in which the principle is applied to every partition of the $N$ parties in two groups.

We are now in position to review the proof of the impossibility of characterizing quantum correlations for an arbitrary number of parties using bipartite principles. For simplicity, we restrict the analysis to tripartite correlations.

4.1 Time-ordered-bilocal correlations and GYNI

The first ingredient in the proof is the characterization of multipartite correlations such that any bipartite correlations constructed from them have a classical local model. By definition, correlations satisfying this property do not violate any bipartite principle satisfied by classical correlations.

A priori, one would think that if the correlations $P(a_1, a_2, a_3|x_1, x_2, x_3)$ have a local model along all possible bipartitions, namely $A_1 - A_2 A_3$, $A_2 - A_1 A_3$ and $A_3 - A_1 A_2$, that is,

$$P(a_1, a_2, a_3|x_1, x_2, x_3) = \sum_\lambda P_1(\lambda) P_1(a_1|x_1, \lambda) P_1(a_2, a_3|x_2, x_3, \lambda) = \sum_\lambda P_2(\lambda) P_2(a_2|x_2, \lambda) P_2(a_1, a_3|x_1, x_3, \lambda) = \sum_\lambda P_3(\lambda) P_3(a_3|x_3, \lambda) P_1(a_1, a_2|x_1, x_2, \lambda),$$

(34)

then, any bipartite object constructed from it also has a local model. This intuition however has proven to be wrong in [39], where it was shown how non-local bipartite correlations can be derived from correlations having a decomposition of the form of (34). The characterization of multipartite correlations such that a local model exists for any bipartite correlations derived from it is then subtler than expected. Indeed, at the moment, it is unknown what is the largest set of correlations having this property [39]. It has however been shown in [21] that the set of time-ordered-bilocal correlations (TOBL) do fulfill this requirement. Tripartite correlations have a TOBL model whenever they can be written as

$$P(a_1, a_2, a_3|x_1, x_2, x_3) = \sum_\lambda P^{[ij]k}_\lambda P(a_i|x_i, \lambda) P_{j\rightarrow k}(a_j, a_k|x_j, x_k, \lambda) = \sum_\lambda P^{[jk]i}_\lambda P(a_i|x_i, \lambda) P_{j\leftarrow k}(a_j, a_k|x_j, x_k, \lambda)$$

(35)

for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, with the distributions $P_{j\rightarrow k}$ and $P_{j\leftarrow k}$ obeying the conditions
the following protocol: given $\lambda - P$ partition corresponding to a distribution which allows signaling in the two directions.

\[ P_{j\rightarrow k}(a_j|x_j, \lambda) = \sum_{a_k} P_{j\rightarrow k}(a_j, a_k|x_j, x_k, \lambda), \] (36)

\[ P_{j\leftarrow k}(a_k|x_k, \lambda) = \sum_{a_j} P_{j\leftarrow k}(a_j, a_k|x_j, x_k, \lambda). \] (37)

The notion of TOBL correlations first appeared in [29] (see [39] and [40] for a proper introduction and further motivation for such models). As can be seen from the relations (36) and (37), we impose the distributions $P_{j\rightarrow k}$ and $P_{j\leftarrow k}$ to allow for signaling at most in one direction, indicated by the arrow (see Table 1).

| $x_2$ | $x_3$ | $a_2$ | $a_3$ |
|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     |
| 0     | 1     | 0     | 1     |
| 1     | 1     | 1     | 1     |
| 1     | 1     | 1     | 0     |

Table 1 Different examples of deterministic bipartite probability distributions $P_{23}(a_2, a_3|x_2, x_3, \lambda)$ characterized by output assignments to the four possible combination of measurements. Left: inputs and outputs corresponding to a point $P_{2\rightarrow 3}(a_2, a_3|x_2, x_3, \lambda)$ in the decomposition (35). Center: inputs and outputs corresponding to a point $P_{2\leftarrow 3}(a_2, a_3|x_2, x_3, \lambda)$ in (38). Right: inputs and outputs corresponding to a distribution which allows signaling in the two directions.

To understand the operational meaning of these models, consider the bipartition $1 \mid 23$ for which systems 2 and 3 act together. In this situation, $P(a_1, a_2, a_3|x_1, x_2, x_3)$ can be simulated if a classical random variable $\lambda$ with probability distribution $P_\lambda^{123}$ is shared by parts 1 and the composite system $2 - 3$, and they implement the following protocol: given $\lambda$, 1 generates its output according to the distribution $P(a_1|x_1, \lambda)$; on the other side, and depending on which of the parties 2 and 3 measures first, 2 - 3 uses either $P_{2\rightarrow 3}(a_2, a_3|x_2, x_3, \lambda)$ or $P_{2\leftarrow 3}(a_2, a_3|x_2, x_3, \lambda)$ to produce the two measurement outcomes. Likewise, any other bipartition of the three systems admits a classical simulation.

By construction, the set of tripartite TOBL models is convex and is included (in fact, it is strictly included [39]) in the set of tripartite probability distributions of the form (34). Moreover, TOBL models always produce classical correlations under post-selection: indeed, suppose that we are given a tripartite distribution $P(a_1, a_2, a_3|x_1, x_2, x_3)$ satisfying condition (35), and a postselection is made on the outcome $\tilde{a}_3$ of measurement $\tilde{x}_3$ by party 3. Then, one has

\[ P(a_1, a_2|x_1, x_2, \tilde{a}_3, \tilde{x}_3) = \sum_{a_3} P'_\lambda P(a_1|x_1, \lambda) P'(2_2) \] (38)

with

\[ P'_\lambda = \frac{P^{123}_\lambda}{P(\tilde{a}_3|x_3)} P_{2\rightarrow 3}(\tilde{a}_3|\tilde{x}_3, \lambda), \quad P'(a_2|x_2, \lambda) = P_{2\leftarrow 3}(a_2|x_2, \tilde{x}_3, \tilde{a}_3, \lambda). \] (39)
Postselected tripartite TOBL boxes can thus be regarded as elements of the TOBL set with trivial outcomes for one of the parties.

We now demonstrate that any possible bipartite correlations derived from many uses of TOBL correlations have a local model and, thus, are compatible with any bipartite principle satisfied by classical (and obviously quantum) correlations. The most general protocol consists in distributing an arbitrary number of boxes described by $P^1, P^2, \ldots, P^N$ among three parties which are split into two groups, $A$ and $B$. Both groups can process the classical information provided by their share of the $N$ boxes. For instance, outputs generated by some of the boxes can be used as inputs for other boxes (see figure [2]). This local processing of classical information is usually referred to as wirings. Thus, in order to prove our result in full generality, we should consider all possible wirings of tripartite boxes. We show next that if $P^1, P^2, \ldots, P^N$ are in TOBL, then the resulting correlations $P_{\text{fin}}$ obtained after any wiring protocol have a local model with respect to the bipartition $A/B$, and therefore fulfill any bipartite information principle.

For simplicity, we illustrate our procedure for the wiring shown in figure [2] where boxes $P^1, P^2, P^3$ are distributed between two parties $A$ and $B$, and party $A$ only holds one subsystem of each box. The construction is nevertheless general: it applies to any wiring and also covers situations where for some TOBL boxes party $A$ holds two subsystems instead of just one (or even the whole box).

From (35) we have

$$P^i(a^i_1, a^i_2, a^i_3|x^i_1, x^i_2, x^i_3) = \sum_{\lambda^i} P^i_1(a^i_1|x^i_1, \lambda^i) P^i_2(a^i_2, a^i_3|x^i_2, x^i_3, \lambda^i)$$

$$= \sum_{\lambda^i} P^i_1(a^i_1|x^i_1, \lambda^i) P^i_2(a^i_2, a^i_3|x^i_2, x^i_3, \lambda^i),$$

for $i = 1, 2, 3$. Consider the first box that receives an input, in our case subsystem 2 of $P^1$. The first outcome $a^2_1$ can be generated by the probability distribution $P^i_2(a^i_1|x^i_1, \lambda^i)$ encoded in the hidden variable $\lambda^i$ that models these first correlations. This is possible because for this decomposition $a^2_1$ is defined independently of $x^i_3$, the input in subsystem 3. Then, the next input $x^2_3$, which is equal to $a^1_1$, generates the output $a^3_1$ according to the probability distribution $P^i_3(a^i_1,a^i_3|x^i_2,x^i_3,\lambda^i)$ encoded in $\lambda^i$. The subsequent outcomes $a^2_2$ and $a^1_2$ are generated in a similar way. The general idea is that outputs are generated sequentially using the local models according to the structure of the wiring on 2–3. Finally, subsystem 1 can generate its outputs $a^2_i$ by using the probability distribution $P^i_1(a^i_1|x^i, \lambda^i)$. This probability distribution is independent of the order in which parties 2 and 3 make their measurement choices for any of the boxes. Averaging over all hidden variables one obtains $P_{\text{fin}}$. This construction provides the desired local model for the final probability distribution.

The final step in the proof consists of showing that there exist correlations in the TOBL set that do not have a quantum realization. This was shown by means of the GYNI inequality. More precisely, it can be proven that, contrary to quantum correlations, this inequality is violated by TOBL correlations:
Fig. 2 Wiring of several tripartite correlations distributed among parties $A$ and $B$. The generated bipartite box accepts a bit $x$ (two bits $y_1, y_2$) as input on subsystem $A$ ($B$) and returns a bit $a$ (two bits $b_1, b_2$) as output. Relations (40) guarantee that the final bipartite distribution $P_{\text{fin}}(a, b | x, (y_1, y_2))$ admits a local model.

\[
\text{maximize } P(000|000) + P(110|011) + P(011|101) + P(101|110) \\
\text{subject to } P(a_1, a_2, a_3 | x_1, x_2, x_3) \in \text{TOBL.}
\] (42)

The maximization yields a value of $7/6$, implying the existence of supra-quantum correlations in TOBL. The form of the TOBL correlations leading to this violation can be found in [21]. Later, another example of supra-quantum correlations in TOBL was provided in [22], where the authors proved that an extremal point of the no-signalling polytope for three parties and two two-outcome measurements per party is also in TOBL and has no quantum realization.

5 Generalization of GYNI: Bell inequalities without quantum violation and unextendible product bases

The relation between GYNI’s Bell inequality and the three-qubit unextendible product basis (UPB) was used in the previous section to show that, contrary to the bipartite case [17] (see also [18]), in the three-partite scenario Gleason correlations make a larger set than the quantum ones. Actually, this link can be generalized and relates nontrivial Bell inequalities without violation to UPB, see Ref. [24]. All these Bell in-
equalities lack quantum violation, nevertheless, they are nontrivial in the sense that there exist some nonsignalling correlations violating them. They therefore complement the results of Ref. \[18\] providing new examples of multipartite scenarios where Gleason correlations are different from the quantum ones. More importantly, however, some of UPBs can lead to \textit{tight} Bell inequalities with no quantum violation, novel examples of which have recently been found \[25\].

Our aim in this section is to recall the method from Refs. \[24, 25\] and then discuss properties of the resulting Bell inequalities. We also provide some classes of nontrivial Bell inequalities with no quantum violation associated to UPBs. Finally, we go beyond UPB and show that there are also sets of orthogonal product vectors that are not UPBs but can be associated nontrivial Bell inequalities. Before that let us recall the notion of unextendible product bases and briefly review their properties.

5.1 \textit{Unextendible product bases}

We start by introducing an \(N\)-partite product Hilbert space
\[
H = \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_N},
\]
where \(d_i (i = 1, \ldots, N)\) denote, for the time being arbitrary, dimensions of the local Hilbert spaces. In what follows we will call an element \(|\psi\rangle\) of \(H\) \textit{fully product} if it assumes the form \(|\psi\rangle = \otimes_{i=1}^{N} |\psi_i\rangle \equiv |\psi_1, \ldots, \psi_N\rangle\) with \(|\psi_i\rangle \in \mathbb{C}^{d_i}\).

Then, let us consider a set of orthogonal product vectors
\[
S = \left\{ |\psi_m^{(1)}\rangle \otimes \ldots \otimes |\psi_m^{(N)}\rangle \right\}_{m=1}^{|S|},
\]
where \(|\psi_m^{(j)}\rangle (m = 1, \ldots, |S|)\) are local vectors belonging to \(\mathbb{C}^{d_i}\) and \(|S| \leq \dim H\). With this we have the following definition \[27\].

**Definition 1.** Let \(S\) be a set of orthogonal fully product vectors \[44\] from \(H\). We call \(S\) \textit{unextendible product basis} (UPB) if it spans a proper subspace \(\text{span} S \subset H\), i.e., \(|S| < \dim H\), and there is no product vector \(\otimes_{i=1}^{N} |\phi_i\rangle \in H\) orthogonal to \(\text{span} S\).

Fig. 3 Schematic definition of a UPB: a set of orthogonal product vectors \(S\) spanning a proper subspace \(\text{span} S \subset H\) such that there is no vector \(\otimes_{i=1}^{N} |\phi_i\rangle \in H\) orthogonal to \(S\). A normalized projector onto \((\text{span} S)^{\perp}\) \[45\] is a bound entangled state \[27\].
The notion of unextendible product bases reflects the peculiar feature of some of product Hilbert spaces $H$ in that they can be represented as direct sums of two orthogonal subspaces, of which one is spanned by product vectors, while the second does not contain any of them, i.e., is completely entangled (see Fig. 3). This has interesting consequences from the quantum information point of view. As it was first observed by Bennett and coworkers [27], UPBs can be used for a construction of bound entangled states, i.e., states that are entangled but nevertheless no entanglement can be distilled from them by means of local operations and classical communication [26].

To be more precise, following [27], let us consider a particular UPB $U$, and the normalized projector onto the subspace of $H$ orthogonal to $U$, i.e.,

$$\rho = \frac{1}{\dim H - |U|} (\mathbb{1} - \Pi).$$

By $\Pi$ and $\mathbb{1}$ we denoted, respectively, the projector onto the subspace spanned by $U$ and identity acting on $H$. Since there is not product vector orthogonal to $U$, the support of $\rho$ consists only of entangled states, implying that $\rho$ must be entangled. Then, it immediately follows from Eq. (45) that $\rho$ has all partial transpositions positive which, due to Ref. [26], justifies the statement that $\rho$ is bound entangled.

To illustrate the above definition we consider the following examples of UPBs.

**Example 1.** We start from the TILES UPB, one of the first bipartite UPBs introduced in Ref. [27]. It consists of five two-qutrit vectors of the form

$$U_{\text{TILES}} = \{|0\rangle(|0\rangle - |1\rangle), |2\rangle(|1\rangle - |2\rangle), (|0\rangle - |1\rangle)|2\rangle, (|1\rangle - |2\rangle)|0\rangle, (|0\rangle + |1\rangle + |2\rangle)^\otimes 2\}. \quad (46)$$

Notice that in two-qutrit Hilbert space there only exist five-elements UPBs and all of them are known [27, 41, 42, 43].

**Example 2.** Second, let us consider a general class of $N$-qubit unextendible product bases with odd $N = 2k - 1$ ($k \in \mathbb{N}; k \geq 2$) given by the following $2k$ vectors [41]:

$$U_{\text{GenShifts}} = \{|0\ldots 0\rangle, |1e_1 \ldots e_{k-1}|\bar{e}_{k-1} \ldots \bar{e}_1\rangle, |\bar{e}_1 e_1 \ldots e_{k-1}|\bar{e}_{k-1} \ldots \bar{e}_1\rangle, \ldots, |e_1 \ldots e_{k-1}|\bar{e}_{k-1} \ldots \bar{e}_1|\rangle\} \quad (47)$$

with $\{|0\rangle, |1\rangle\}$ and $\{|e_i\rangle, |\bar{e}_i\rangle\}$ ($i = 1, \ldots, k - 1$) being $k$ arbitrary but different bases in $\mathbb{C}^2$. The $i$th ($i \geq 2$) vector in (47), except for the first two ones, is obtained from the vector $i - 1$ by shifting all the local vectors by one to the right, and thus the name Generalized Shifts.

**Example 3.** Third, let us consider the general class of UPBs found by Niset and Cerf [44]. Here we take the Hilbert space $H = (\mathbb{C}^d)^\otimes N$, where $N \geq 3$ and $d \geq N - 1$, and the following set of $N(d - 1) + 1$ vectors:
\[ U_{\text{NC}} = \{ |e_{d-1}\rangle^{\otimes N} \} \cup \bigcup_{i=0}^{N-1} S_i, \]  
(48)

where

\[ S_0 = \{ |0, 1, \ldots, d-1, 0\rangle, \ldots, |0, 1, \ldots, d-1, d-2\rangle \} \]  
(49)

and \( S_i = V^i S_0 \) \((i = 1, \ldots, N-1)\) with \( V \) denoting a unitary permutation operator such that \( V |x_1 \ldots x_N\rangle = |x_N \ldots x_1\rangle \) for \( |x_i\rangle \in \mathbb{C}^d \), and \( \{ |e_i\rangle \}_{i=0}^{d-1} \) is any orthogonal basis in \( \mathbb{C}^d \) different from the standard one. Notice that \( U_{\text{NC}} \) can straightforwardly be generalized to an arbitrary local dimension \( d_i \geq N-1 \) \((i = 1, \ldots, N)\) just by adjusting both bases at each site to the respective dimension \( d_i \).

Both classes of multipartite UPBs from examples 2 and 3 (here up to local unitary operations) recover, for \( N = 3 \), the already introduced Shifts UPB \((31)\), i.e., \( U_{\text{Shifts}} = \{ |000\rangle, |1e_2\rangle, |e_1e_3\rangle, |\bar{e}_1\bar{e}_2\rangle \} \) with \( \{ |e_i\rangle, |\bar{e_i}\rangle \} \) being an arbitrary basis of \( \mathbb{C}^2 \) different from the standard one. Clearly, this set can be slightly generalized by taking the second basis different at each site, that is, \( \{ |000\rangle, |1\bar{e}_2e_3\rangle, |e_1\bar{e}_3\rangle, |\bar{e}_1e_2\rangle \} \) (the first basis can be fix to the standard one by a local unitary operation). Then, as it was shown by Bravyi \[(45)\], any three-qubit UPB is equivalent to this one up to local unitary operations and permutations of the parties.

### 5.2 Constructing Bell inequalities with no quantum violation from unextendible product bases

We are now ready to recall the method from \([24, 25]\) allowing to associate a non-trivial Bell inequality with no quantum violation to a UPB having certain property.

#### 5.2.1 The construction

To begin, consider again the product Hilbert space \( H \) and the set of vectors \( S \). For the time being, we do not assume \( S \) to be a UPB, keeping, however, the assumption that elements of \( S \) are orthogonal product vectors from \( H \). Then, let us collect all different local vectors appearing in all vectors \( |\Psi_m\rangle \) at the \( i \)th site in the local sets

\[ S^{(i)} = \{ |\Psi_m^{(i)}\rangle \}_{m=1}^{s_i} \quad (i = 1, \ldots, N), \]  
(50)

where \( s_i \leq |S| \). Subsequently, among elements of \( S^{(i)} \) we search for mutually orthogonal vectors and collect them in separate subsets \( S_{n}^{(i)} \) \((n = 0, \ldots, k_i)\) such that \( S_{0}^{(i)} \cup \ldots \cup S_{k_i}^{(i)} = S^{(i)} \) for any \( i \) (see Fig. 4). Notice that these subsets may, but do not have to, span the corresponding Hilbert space \( \mathbb{C}^{d_i} \).

It should be emphasized that there exist sets \( S \) for which the local subsets cannot be unambiguously defined. This is, for instance, the case for vectors \( U_{\text{TILES}} \) [cf. Eq.
Fig. 4 Schematic description of our construction. From the set $S$ having the local independence property, one constructs the local sets $S^{(i)}$ ($i = 1, \ldots, N$) by collecting different local vectors in $|\Psi_m\rangle$. Then, one distinguishes local subsets $S_k^{(i)}$ of mutually orthogonal vectors among elements of each local set $S^{(i)}$.

In other words, what we need is that the local subsets are constructed in the way that by replacing one of them by another one of the same size, we keep the orthogonality of elements of $S$. In yet another words, the above property guarantees that the orthogonality of $S$ is preserved under any unitary rotation of elements of any local subset $S_k^{(i)}$, which, in a sense, makes them independent. Hence, for the purposes of the present framework, we propose to call it local independence property.

A particular example of a set having the above property is the already introduced Shifts UPB (31). At each site there are four different vectors $|0\rangle$, $|1\rangle$, $|e\rangle$, and $|\overline{e}\rangle$, which can be grouped in two distinct sets $S_0 = \{|0\rangle, |1\rangle\}$ and $S_1 = \{|e\rangle, |\overline{e}\rangle\}$. Since, by the very assumption, $|e\rangle \neq |0\rangle, |1\rangle$, none of the vectors from $S_0$ is orthogonal to none of elements of $S_1$, and hence $U_{\text{Shifts}}$ has the local independence property.

Interestingly, as it can easily be checked, all sets of orthogonal vectors in multiqubit Hilbert spaces have the above property and all local subsets contain at most two elements. On the other hand, the example of TILES UPB shows that this in general is not the case when local dimensions are larger than two.

Let us now pass to our construction of Bell inequalities. To every vector $|\Psi_m\rangle$ from $S$ [cf. Eq. (44)] we can associate a conditional probability $P(a_m|x_m)$, or, strictly speaking, vectors of measurements settings and outcomes

$$a_m = (a^{(1)}_m, \ldots, a^{(N)}_m) \quad \text{and} \quad x_m = (x^{(1)}_m, \ldots, x^{(N)}_m) \quad (51)$$

in the following way:

- the measurement setting $x^{(i)}_m$ of the observer $i$ is given by the index $k$ enumerating the subset $S_k^{(i)}$ containing $|\psi^{(i)}_m\rangle$. 



the measurement outcome $a_m^{(i)}$ corresponds to the position of $|\psi_{m}^{(i)}\rangle$ in the set $S_k^{(i)}$.

Eventually, we simply add the obtained conditional probabilities and maximize the resulting expression over all classical correlations, which leads us to the following Bell inequality

$$\sum_{m=1}^{|S|} P(a_m|x_m) \leq 1.$$  \hspace{1cm} (52)

The value of the right-hand side of the above, the so-called classical bound, directly follows from the orthogonality of elements of $S$. Since the latter are product, for each pair of vectors $|\psi_n \rangle$, $|\psi_m \rangle$ ($m \neq n$), there exists site, say $i$, such that $|\psi_m^{(i)}\rangle \perp |\psi_n^{(i)}\rangle$, and so the local independence property says $|\psi_m^{(i)}\rangle$, $|\psi_n^{(i)}\rangle$ are distinct elements of the same local subset $S_k^{(i)}$. Consequently, the associated conditional probabilities $P(a_m|x_m)$ and $P(a_n|x_n)$ have at site $i$ the same measurement settings but different outcomes. This means that for any deterministic local model, if one of these two probabilities is one, the other one equals zero. Let us further call such probabilities orthogonal. Since the above holds for any pair of conditional probabilities, the right-hand side of (52) clearly amounts to one.

Notice then that, in principle, we can consider a more general inequality by combining the conditional probabilities $P(a_m|x_m)$ ($m = 1, \ldots, |S|$) with arbitrary positive weights $q_m$. However, we always get in this way a Bell inequality which is weaker than the one above and certainly cannot be tight (see below).

### 5.2.2 Properties

Let us now shortly characterize the obtained Bell inequalities (52). We collect their most important properties in the following theorem [24, 25].

**Theorem 1.** Let $S$ be a set of orthogonal product vectors from $H$ having the local independence property. Then the following implications are true:

(i) the associated Bell inequality (52) is not violated by quantum correlations

(ii) if $S$ is a UPB in $H$, then the Bell inequality (52) is nontrivial in the sense that it is violated by some nonsignalling correlations,

(iii) if $S$ is a full basis in $H$ or can be completed to one in such a way that it maintains the local independence property, the associated Bell inequality (52) is not violated by any nonsignalling correlations.

**Proof.** (i): Let us, in contrary, assume that indeed the Bell inequality (52) associated to $S$ is violated by a quantum state $\rho$. Then, there exist local measurement operators and the resulting Bell operator, denoted $B$, such that $tr(B\rho) > 1$. This means that at least one of the eigenvalues of $B$ has to exceed one. On the other hand, it is clear that the local measurement operators can be assumed to be projective; if $\rho$ violates (52)
with POVM, one is able to find another quantum state $\rho'$ acting on a larger Hilbert space violating the same Bell inequality with projective measurements.

Let then $P_m = \otimes_{i=1}^N P_m(i)$ denote a product projective measurement operator corresponding to $P(a_m|x_m)$, which, in general, may be different from the corresponding vectors $|\Psi_m\rangle \in S$. Clearly, orthogonality of the conditional probabilities $P(a_m|x_m)$ is translated to the orthogonality of the corresponding $P_m$. Precisely, as already stated, any pair of probabilities $P(a_m|x_m)$ and $P(a_n|x_n)$ has at some site, say $i$, the same settings but different outcomes, implying that $P_i^m \perp P_i^n$ and hence $P_m \perp P_n$. As a result all $P_m (m = 1, \ldots, |S|)$ are orthogonal and the Bell operator $B = \sum_m P_m$ is again a projector contradicting the fact that for some $\rho$, $\text{tr}(B\rho) > 1$.

(ii): Our proof is constructive, that is, for any Bell inequality associated to a UPB we will provide particular NC violating it. We denote by $\Pi$ the projector onto span$S$, and introduce, in a full analogy to (32), the following indecomposable witness

$$W = \frac{1}{|S| - \text{dim}H} (\Pi - \varepsilon \mathbb{1})$$

with $\varepsilon$ being a positive number defined as

$$\varepsilon = \min(x_1, \ldots, x_N|\Pi|x_1, \ldots, x_N),$$

where minimum is taken over all fully product vectors from $H$. One directly checks that this witness detects entanglement of the state (45) constructed from the UPB $S$, i.e., $\text{tr}(W\rho) < 0$. This, after substituting the exact (52) of $\rho$, can be rewritten as

$$\text{tr}(W\Pi) > 1.$$  

Clearly, $\Pi$ can be seen as a Bell operator corresponding to our Bell inequality associated to $S$. To complete the proof of (ii) it suffices then to notice any local measurements performed on any entanglement witness, in particular (53), give nonsignalling correlations (see e.g. [18, 17]).

(iii): Let us start from the case when $|S| < \text{dim}H$ and assume that $S$ can be completed to a basis of $H$ maintaining the local independence property (if $H = (\mathbb{C}^2)^\otimes N$ one can always do that provided $S$ is completable). Let then $|\Psi_m\rangle (m = |S| + 1, \ldots, \text{dim}H)$ denote product orthogonal vectors completing $S$, i.e., span$(S \cup \{|\Psi_m\rangle\}) = H$. Consequently, one can associate a Bell inequality (52) to the set $S$ and conditional probabilities $P(a_m|x_m)$ to the new vectors $|\Psi_m\rangle (m = |S| + 1, \ldots, \text{dim}H)$ in an unambiguous way. Then

$$\sum_{m=1}^{|S|} P(a_m|x_m) \leq \sum_{m=1}^{\text{dim}H} P(a_m|x_m) \leq 1,$$

meaning that it suffices to prove that the Bell inequality appearing on the right-hand side (the one constructed from a full basis in $H$) is trivial. For this purpose, we note that the latter is saturated by the uniform probability distribution $P(a|x) = 1/\text{dim}H$ for any $a$ and $x$, which is an interior point of the corresponding polytope of clas-
sical correlations. Consequently, this Bell inequality is saturated by all vertices of the polytope, and hence by any affine combination thereof, in particular, all nonsignalling correlations.

It is illuminating to see how the properties of $S$ determine the properties of the associated Bell inequality. Orthogonality of elements of $S$ implies that it lacks quantum violation. If $S$ is additionally a UPB, then the Bell inequality is nontrivial because it detects some nonsignalling correlations. On the other hand, pit is trivial if $S$ is a full basis in $H$ or can be completed to one maintaining the local independence property. In the case of $H = (\mathbb{C}^2)^\otimes N$, up to sets that can only be completed to UPBs, the implication (iii) becomes equivalence [25]. In the higher-dimensional case, however, there are sets having local independence property which are not UPBs but cannot be extended maintaining the local independence property (see Sec. 5.3).

The more important and interesting question concerns the tightness of these Bell inequalities. As shown in Refs. [24, 25] there exist example of both tight and non-tight Bell inequalities associated to UPBs (see Sec. 5.2.3 for examples) and, so far, it remains unclear what decides on tightness.

### 5.2.3 Examples

Just to get a better insight into the construction let us apply to it to some particular examples of sets $S$, in particular those presented in Sec. 5.1.

**Example 4.** Using the already exploited relation between the GYNI Bell inequality (17) and Shifts UPB let us show how the above construction works in practice. As already noticed, $U_{\text{Shifts}}$ has two different bases at each site $S_0 = \{|0\rangle, |1\rangle\}$ and $S_1 = \{|e\rangle, |\overline{e}\rangle\}$. The vector $|e\rangle \in \mathbb{C}^2$ is, by assumption, different than $|0\rangle$ and $|1\rangle$, and hence $U_{\text{Shifts}}$ has the local independence property. We then associate a conditional probability to every vector in $U_{\text{Shifts}}$:

\[
|000\rangle \mapsto P(000|000), \quad |1e\rangle \mapsto P(110|011), \\
|e1\rangle \mapsto P(011|101), \quad |\overline{e}1\rangle \mapsto P(101|110).
\]  

Simply by adding the above probabilities we get (17). In exactly the same was one shows that GYNI$_N$ can be associated to a certain $N$-qubit UPB [24, 25]. Moreover, it was recently shown in Ref. [25] that GYNI$_N$ is tight for odd $N$.

Interestingly, the GYNI$_3$ is the only tight three-partite Bell inequality with no quantum violation in the scenario of two dichotomic measurements per site, and it is associated to the only class of UPB in $(\mathbb{C}^2)^\otimes 3$ characterized in Ref. [45].

**Example 5.** Second, let us consider the Generalized Shifts UPB (47). The corresponding Hilbert space is $H = (\mathbb{C}^2)^\otimes N$ with $N = 2k - 1$ for integer $k \geq 2$. Following the above rules, at each site one can define $k$ local subsets $S_0 = \{|0\rangle, |1\rangle\}$ and $S_i = \{|e_i\rangle, |\overline{e}_i\rangle\} (i = 1, \ldots, k - 1)$, which will later define $k$ observables. We then associate a conditional probability to every element of $U_{\text{GenShifts}}$:
\( |0\ldots0\rangle \mapsto P(0\ldots0|0\ldots0) \)
\( |1e_1\ldots e_{k-1}\bar{e}_{k-1}\ldots\bar{e}_1\rangle \mapsto P(10\ldots01\ldots1|01\ldots0k, k-1\ldots1) \)
\[
\vdots
\]
\( |e_1\ldots e_{k-1}\bar{e}_{k-1}\ldots\bar{e}_{k-11}\rangle \mapsto P(0\ldots01\ldots11|01\ldots0k-1, k-1\ldots11). \tag{58} \)

Summing all these probabilities up, we get the \( N \)-partite Bell inequality with odd \( N \):
\[
P(0\ldots0|0\ldots0) + \sum_{i=1}^{2k-1} D^i P(10\ldots01\ldots1|01\ldots0k-1, k-1\ldots1) \leq 1, \tag{59} \]
where \( D \) denotes an operation shifting the input and output vectors by one to the right, i.e., \( D(x_1, \ldots, x_N) = (x_N, x_1, \ldots, x_{N-1}) \). Notice that since at each site one has \( k \) two-element local subsets \( S_i \), the Bell inequality (59) corresponds to the scenario with \( k \) dichotomic observables per site.

Due to theorem [1] all the Bell inequalities (59) are nontrivial. However, it is unclear whether they are tight. For \( N = 3 \) the above class recovers the GYNI which is tight, while already for \( N = 5 \) the corresponding Bell inequality is not tight.

**Example 6.** Consider now the class of UPBs provided in Ref. [44], i.e., \( U_{NC} \) presented in example 3. Here \( H = (\mathbb{C}^d)^\otimes N \) with \( d \geq N - 1 \). From Eqs. (48) and (49) it follows that at each site one can distinguish two local subsets \( S_0 = \{|i\rangle\}_{i=0}^{d-1} \), i.e., the standard basis, and \( S_1 = \{|e_i\rangle\}_{i=0}^{d-1} \). Since the elements of \( U_{NC} \) are orthogonal irrespectively of the choice of the second basis, \( U_{NC} \) has the local independence property. Associating conditional probabilities to elements of \( U_{NC} \) and summing them up, one gets the \( N \)-partite Bell inequality:
\[
P(d-1, \ldots, d-1|1, \ldots, 1) + \sum_{i=0}^{N-1} \sum_{j=0}^{d-2} D^i P(0, 1, \ldots, d-1, j|0, \ldots, 0, 1) \leq 1, \tag{60} \]
where \( D \) is defined as before and \( D^0 \) is an identity.

Theorem [1] says that all the Bell inequalities (60) are nontrivial, however, it is not clear whether they are tight in general. For \( N = 3 \) and \( d = 2 \), this class gives GYNI3, but for \( N = 4 \) and \( d = 3 \) one checks that the resulting Bell inequality is not tight.

Let us notice that within the above framework one can also obtain tight Bell inequalities with no quantum violation from UPBs that are independent of GYNI. Some new examples as for instance the following four-partite Bell inequality
\[
p(0000|0000) + p(1000|0111) + p(0110|1012) + p(0001|0110) \\
+ p(1011|0001) + p(1101|0102) + p(1110|1101) \leq 1 \tag{61} \]
were found recently in Ref. [25].
5.3 Further generalizations

We will show now that not only UPBs lead to nontrivial Bell inequalities with no quantum violation. If at least one \( d_i \) in \( H \) is larger than two, there exist sets of orthogonal product vectors that are not UPBs in the sense of definition 1 but still, the associated Bell inequalities \( (52) \) lack quantum violation and are violated by nonsignalling correlations.

To be more precise, let us consider again set of orthogonal product vectors \( S \) and let us split local sets \( \overline{S}^{(i)} \) [cf. Eq. (50)] into subsets \( \overline{S}^{(i)}_k \) following the same rules as before. Then, we have the definition.

**Definition 2.** Let \( S \) be a set of orthogonal fully product vectors from \( H \) having the local independence property. Then, if \( |S| < \dim H \) and there does not exist a product vector \( \otimes_{i=1}^N |\phi_i⟩ \in \mathcal{H} \) with \( |\phi_i⟩ \in \overline{S}^{(i)} \) which is orthogonal to all vectors from \( S \), we call \( S \) a weak unextendible product basis (wUPB).

Clearly, any UPB is also a wUPB. Also, if all \( d_i = 2 \) in eq. (43), these two notions are equivalent. If, however, at least one of the local dimensions \( d_i \) is larger than two, there exist wUPB that are not UPB. As a particular example consider the following.

**Example 7.** Consider the following set of vectors from \( H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \):

\[
S = \{ |000⟩, |1\overline{1}f⟩, |e1\overline{1}⟩, |\overline{e}e1⟩, |\overline{e}e2⟩, |e1\overline{f}⟩ \},
\]

where \( |f⟩, |\overline{f}⟩ \), and \( |\overline{f}⟩ \) are three orthogonal vectors from \( \mathbb{C}^3 \). At the first two sites one distinguishes two local sets \( S^{(1)}_0 = S^{(2)}_0 = \{ |0⟩, |1⟩ \} \) and \( S^{(1)}_1 = S^{(2)}_1 = \{ |e⟩, |\overline{e}⟩ \} \), while at the third site \( S^{(3)}_0 = \{ |0⟩, |1⟩, |2⟩ \} \) and \( S^{(3)}_1 = \{ |f⟩, |\overline{f}⟩, |\overline{f}⟩ \} \).

The set \( S \) has the local independence property because irrespectively of the choice of all these subsets, all its elements are orthogonal. However, it is clearly not a UPB because \( |e0g⟩ \) and \( |\overline{e}g⟩ \) with \( \mathbb{C}^3 \ni |g⟩ \perp |0⟩, |f⟩ \) are orthogonal to \( S \). Still, \( S \) is a wUPB; there is no product vector \( |\phi_1⟩⟨\phi_2| |\phi_3⟩ \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \) with \( |\phi_i⟩ \in \overline{S}^{(i)}_j \) \((i = 1, 2, 3; j = 1, 2)\), which is orthogonal to \( S \).

It remains an open question as to whether the quantum state constructed from a wUPB, i.e., the state \( (45) \) with \( \Pi \) denoting now a projector onto the subspace spanned by the wUPB, is entangled. It is, nevertheless, still a PPT state.

Following the rules given in Sec. 5.2.1 any wUPB can be associated a Bell inequality \( (52) \) with no quantum violation which is violated by some nonsignalling correlations. In fact, we have the following theorem.

**Theorem 2.** If \( S \) is a wUPB, the associated Bell inequality \( (52) \) is violated by some nonsignalling correlations.

**Proof.** The proof goes along the same lines as the one of (ii) of theorem 1. It suffices to consider the same operator as in Eq. (53) with \( \Pi \) denoting now a projector onto the subspace spanned by the wUPB \( S \) and the minimum in Eq. (54) taken over product vectors \( \otimes_{i=1}^N |\phi_i⟩ \in \mathcal{H} \) with local vectors \( |\phi_i⟩ \in \overline{S}^{(i)} \).
A measurements of such $W$ along the settings corresponding to the local sets $S^{(i)}_{k}$ produce the value of the Bell inequality \((52)\) constructed from the wUPB $S$ given by $|S|(1 - \varepsilon)/(|S| - \varepsilon \dim H)$. This, due to the fact that $|S| < \dim H$ and $\varepsilon < |S|/\dim H$, is always larger than one.

Notice that if $S$ is a wUPB but not UPB, the operator $W$ is no longer an entanglement witness, but still it is a Hermitian operator. It therefore represents nonsignalling correlations \([18]\), which are not Gleason correlations. It is then an open question as to whether Bell inequalities associated to wUPBs "detect" Gleason correlations.

To conclude, let us notice that the Bell inequality corresponding to the set \((62)\):

\[
p(000|000) + p(110|011) + p(011|101) + p(101|110) \\
+ p(012|101) + p(102|110) \leq 1,
\]

which has two three-outcome observables at the third site, is tight. This is because it is lifted three-partite GYNI Bell inequality \([17], [46]\).

6 Conclusions

'Guess you neighbour’s input' is a multipartite nonlocal game that, despite its simplicity, captures important features of multipartite correlations. Moreover, it has unexpected connections to topics in quantum foundations and quantum information theory. In particular, it shows that the natural multipartite generalization of Gleason’s Theorem fails for more than two parties, that intrinsically multipartite principles are needed to characterize quantum correlations and that there exists a link between unextendible orthogonal product bases and Bell inequalities with no quantum violation.

From a speculative point of view, GYNI suggests that we are lacking an intrinsically multipartite principle in our understanding of correlations. Indeed, the most interesting feature of the game is that it represents a multipartite strengthening of the no-signaling principle, which is by construction a bipartite principle, that is obeyed by quantum correlations. This naturally raises the question of what physical or information-theoretic principles lie behind GYNI. We hope our work stimulatates further research in this direction.

Acknowledgements Discussions with T. Fritz are acknowledged. This work was supported by the ERC starting grant PERCENT, the EU AQUTE and QCS projects, the Spanish CHIST-ERA DIQIP, FIS2008-00784 and FIS2010-14830 projects, and the UK EPSRC. R. A. is supported by the Spanish MINECIN through the Juan de la Cierva program.
References

1. L. Hardy, quant-ph/0101012v4.
2. J. S. Bell, Physics 1, 195 (1964).
3. A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
4. H. Buhrman, R. Cleve, S. Massar and R. de Wolf, Rev. Mod. Phys. 82, 665 (2010).
5. J. Barrett L. Hardy and A. Kent, Phys. Rev. Lett. 95, 010503 (2005); A. Acín et al., Phys. Rev. Lett. 98, 230501 (2007); L. Masanes, S. Pironio and A. Acín, Nature Comm. 2, 238 (2011).
6. S. Pironio et al., Nature 464, 1021 (2010); R. Colbeck, PhD Thesis, University of Cambridge; R. Colbeck and A. Kent, J. Phys. A: Math. and Theor. 44 (9), 095305 (2011).
7. S. Popescu and R. Rohrlich, Found. Phys. 24, 379 (1994).
8. G. Brassard, Nat. Phys. 1, 2 (2005).
9. S. Popescu, Nat. Phys. 2, 507 (2006).
10. R. Clifton, J. Bub and H. Halvorson, Found. Phys. 33, 1561 (2003).
11. W. van Dam, Nonlocality & Communication complexity, Ph.D. thesis, University of Oxford (2000); see also quant-ph/0501159v1.
12. G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger, Phys. Rev. Lett. 96, 250401 (2006).
13. N. Brunner and P. Skrzypczyk, Phys. Rev. Lett. 102, 160403 (2009).
14. M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żakowski, Nature 461, 1101 (2009).
15. J. Allcock, N. Brunner, M. Pawłowski, and V. Scarani, Phys. Rev. A 80, 040103(R) (2009).
16. M. Navascues and H. Wunderlich, Proc. Roy. Soc. Lond. A 466, 881 (2009).
17. H. Barnum, S. Beigi, S. Boixo, M. B. Elliott, and S. Wehner, Phys. Rev. Lett. 104, 140401 (2010).
18. A. Acín, R. Augusiak, D. Cavalcanti, C. Hadley, J. K. Korbicz, M. Lewenstein, and M. Piani, Phys. Rev. Lett. 104, 140404 (2010).
19. M. L. Almeida, J.-D. Bancal, N. Brunner, A. Acín, N. Gisin, and S. Pironio, Phys. Rev. Lett. 104, 230404 (2010).
20. A. Winter, Nature 466, 1053 (2010).
21. R. Gallego, L. Würflinger, A. Acín and M. Navascués, Phys. Rev. Lett. 107, 210403 (2011).
22. T. H. Yang, D. Cavalcanti, M. Almeida, C. Teo and V. Scarani, New J. Phys. 14, 013061 (2012).
23. A. Gleason, J. Math. Mech. 6, 885 (1957).
24. R. Augusiak, J. Stasińska, C. Hadley, J. K. Korbicz, M. Lewenstein, and A. Acín, Phys. Rev. Lett. 107, 070401 (2011).
25. R. Augusiak, T. Fritz, M. Kotowski, M. Kotowski, M. Pawłowski, M. Lewenstein, and A. Acín, Phys. Rev. A 85, 042113 (2012).
26. M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).
27. C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. 82, 5385 (1999).
28. J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Phys. Rev. A 71, 022101 (2005).
29. S. Pironio, J.-D. Bancal and V. Scarani, J. Phys. A: Math. Theor. 44 065303 (2011).
30. C. Śliwa, Phys. Lett. A 317, 165 (2003).
31. P. Busch, Phys. Rev. Lett. 91, 120403 (2003).
32. D. Foulis and C. Randall, Interpretations and Foundations of Quantum Theory 5, 920 (1979); M. Kläy, C. Randall and D. Foulis, Int. J. Theor. Phys. 26, 199 (1987); H. Barnum, C. A. Fuchs, J. M. Renes and A. Wilce, arXiv:quant-ph/0507108.
33. N. R. Wallach, arXiv:quant-ph/0002058.
34. M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996); B. M. Terhal, Phys. Lett. A 271, 319 (2000).
35. A. Jamiołkowski, Rep. Math. Phys. 3, 275 (1972); M.-D. Choi, Linear Algebra Appl. 10, 285 (1975).
36. M. Horodecki, P. Horodecki, and R. Horodecki, Open Syst. Inf. Dyn. 13, 103 (2006).
37. A. Ahanj, S. Kunkri, A. Rai, R. Rahaman, and P. S. Joag, Phys. Rev. A 81, 032103 (2010).
38. D. Cavalcanti, A. Salles and V. Scarani, Nat. Comm. 1, 136 (2010).
39. R. Gallego, L. Würflinger, A. Acín and M. Navascués, [arXiv:1112.2647].
40. J. Barrett, S. Pironio, J.-D. Bancal and N. Gisin, [arXiv:1112.2626].
41. D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Commun. Math. Phys. 238, 379 (2003).
42. J. M. Leinaas, J. Myrheim, and P. Ø. Sollid, Phys. Rev. A 81, 062330 (2010).
43. Ł. Skowronek, J. Math. Phys. 52, 122202 (2011).
44. J. Niset and N. J. Cerf, Phys. Rev. A 74, 052103 (2006).
45. S. B. Bravyi, Quant. Info. Proc. 3, 309 (2004).
46. S. Pironio, J. Math. Phys. 46, 062112 (2005).