Completeness and injectivity

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Abstract
We show that for any quantale \( \mathcal{Q} \), a \( \mathcal{Q} \)-category is skeletal and complete if and only if it is injective with respect to fully faithful \( \mathcal{Q} \)-functors. This is a special case of known theorems due to Hofmann and Stubbe, but we provide a different proof, using the characterisation of the MacNeille completion of a \( \mathcal{Q} \)-category as its injective envelope. For Lawvere metric spaces, our results yield those of Kemajou, Künnzi and Otafudu. We point out that their notion of Isbell convexity can be seen as a geometric formulation of categorical completeness for Lawvere metric spaces.

Keywords: Enriched category, quantale, injective object, injective envelope, MacNeille completion, hyperconvex hull, tight span

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1. Introduction

The main purpose of this paper is to present a theorem claiming the equivalence between completeness and injectivity in the context of quantale-enriched categories. In order to convey the idea of the theorem, we start with describing two classical theorems in order theory and metric space theory, and then a variant of the latter for directed metric spaces; the first and third of them are instances of our theorem (modulo minor modifications).

**Theorem 1.1** ([3]). A poset is a complete lattice if and only if it is injective.

Recall that a poset \( \mathcal{E} \) is a complete lattice if it has all suprema (or equivalently all infima) of subsets of \( \mathcal{E} \). On the other hand, a poset \( \mathcal{E} \) (whose partial order relation we denote by \( \preceq \)) is said to be injective if, whenever we have posets \( \mathcal{C} \) and \( \mathcal{D} \), a monotone map \( f: \mathcal{C} \to \mathcal{E} \) (i.e., a function such that for all \( c, c' \in \mathcal{C} \), \( c \preceq c' \Rightarrow f(c) \preceq f(c') \)), and an order embedding \( i: \mathcal{C} \to \mathcal{D} \) (i.e., a function such that \( c \preceq c' \) if and only if \( i(c) \preceq \mathcal{D} i(c') \)), there exists a (not necessarily unique) monotone map \( g: \mathcal{D} \to \mathcal{E} \) making the diagram

\[
\begin{array}{ccccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{E} \\
\downarrow{i} & & \downarrow{g} \\
\mathcal{D} & & \\
\end{array}
\]

commute. Notice that the property of being a complete lattice is described totally in terms of the internal structure of a poset, whereas that of being injective is formulated solely in terms of its external behaviour among all posets. The fascination of this theorem due to Banaschewski and Bruns lies in the fact that it connects an internal property with an external one.

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Theorem 1.2 (2). A metric space is hyperconvex if and only if it is injective.

A metric space $E$ (whose distance function we denote by $d_E$) is hyperconvex if, for any (possibly infinite) family $\{(e_i, r_i)\}_{i \in I}$ of pairs of a point $e_i \in E$ and a nonnegative real number $r_i$ satisfying $r_i + r_j \geq d_E(e_i, e_j)$ for all $i, j \in I$, there exists a point $e \in E$ such that $r_i \geq d_E(e_i, e)$ for all $i \in I$. Let us elaborate the definition. One can view each pair $(e_i, r_i)$ as the closed ball $B(e_i, r_i)$ in $E$ with centre $e_i$ and radius $r_i$. Then the condition that $r_i + r_j \geq d_E(e_i, e_j)$ for all $i, j \in I$ says that any two balls in the family potentially intersect; indeed, if $r_i + r_j < d_E(e_i, e_j)$ then $B(e_i, r_i) \cap B(e_j, r_j) = \emptyset$ by the triangle inequality. The existence of a point $e \in E$ such that $r_i \geq d_E(e_i, e)$ for all $i \in I$ means that the intersection $\bigcap_{i \in I} B(e_i, r_i)$ of all balls in the family is nonempty. For example, $\mathbb{R}^2$ with the Euclidean metric is not hyperconvex, but $\mathbb{R}^2$ with the maximum metric is; the following are some balls in these metric spaces.

The definition of injectivity for metric spaces parallels that for posets. A metric space $E$ is injective if, whenever we have metric spaces $C$ and $D$, a nonexpansive map $f: C \rightarrow E$ (i.e., a function such that $d_E(c, c') \geq d_D(f(c), f(c'))$), and an isometric embedding $i: C \rightarrow D$ (i.e., a function such that $d_C(c, c') = d_D(i(c), i(c'))$), there exists a (not necessarily unique) nonexpansive map $g: D \rightarrow E$ such that $f = g \circ i$. Again, this theorem of Aronszajn and Panitchpakdi is interesting in that it relates the internal property of hyperconvexity with the external one of injectivity.

The following theorem due to Kemajou, Künzi and Otafudu is the directed variant of Theorem 1.2. By a di-space we mean a possibly nonsymmetric (in the sense that $d_C(c, c')$ may be different from $d_C(c', c)$) generalisation of a metric space.

Theorem 1.3 ([16]). A di-space is Isbell convex if and only if it is injective.

The notion of Isbell convexity is a straightforward adaptation of hyperconvexity to the nonsymmetric setting. Precisely, a di-space $E$ is Isbell convex if, for any (possibly infinite) family $\{(e_i, x_i, y_i)\}_{i \in I}$ of triples of a point $e_i \in E$ and nonnegative real numbers $x_i$ and $y_i$ satisfying $x_i + y_j \geq d_E(e_i, e_j)$ for all $i, j \in I$, there exists a point $e \in E$ such that $x_i \geq d_E(e_i, e)$ and $y_i \geq d_E(e, e_i)$ for all $i \in I$. We shall say more about this notion in Section 7. The definition of injectivity for di-spaces is completely parallel to that for metric spaces.

Except that a precise relationship between the notions of complete lattice on the one hand, and of hyperconvex metric space and Isbell convex di-space on the other, is perhaps not apparent, Theorems 1.1–1.3 look quite similar. These results look even closer if one notes the fact that in all cases we have constructions of injective envelopes. Informally, an injective envelope of an object (i.e., poset, metric space or di-space) $C$ is the smallest injective object $C'$ to which $C$ embeds; we shall give a precise definition in an abstract setting in Section 5. The injective envelope of a poset, metric space and di-space is also known as its MacNeille completion [21], hyperconvex hull or tight span [13, 7, 10], and Isbell hull or directed tight span [16, 11], respectively.

In this paper we shall prove a generalisation of Theorems 1.1 and 1.3. (For a generalisation of Theorem 1.2, see [14].) We unify posets and di-spaces by categories enriched over a quantale. A quantale [22] $\mathcal{Q}$ is a complete lattice $(\mathcal{Q}, \preceq_{\mathcal{Q}})$ equipped with a compatible monoid structure $(\mathcal{Q}, I_{\mathcal{Q}}, \cdot_{\mathcal{Q}})$. Given any quantale $\mathcal{Q}$, one can consider categories enriched over $\mathcal{Q}$, or $\mathcal{Q}$-categories [15, 28, 29]. Informally, a $\mathcal{Q}$-category $\mathcal{C}$ is a set $\text{ob}(\mathcal{C})$ equipped with a $\mathcal{Q}$-valued pre order relation $\mathcal{C}(\cdot, \cdot): \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{C}) \rightarrow \mathcal{Q}$. Taking $\mathcal{Q} = 2$, the two-element quantale, we recover preordered sets as $2$-categories, whereas taking $\mathcal{Q} = \mathbb{R}^\geq$, the quantale of extended nonnegative real numbers with addition as the monoid structure, we obtain a mild generalisation of di-spaces (called Lawvere metric spaces) as $\mathbb{R}^\geq$-categories [19].

Theorems 1.1 and 1.3 can be generalised as follows.
Theorem 1.4 ([12, 30]). Let \( Q \) be a quantale. A \( Q \)-category is skeletal and complete if and only if it is injective (with respect to fully faithful \( Q \)-functors).

The terms appearing in the above statement will be introduced in Section 3. Actually, this theorem is known in much more general settings. In [12, Theorem 2.7] it is proved for \( \mathcal{T} \)-categories for a topological theory \( \mathcal{T} \), and in [30] it is proved for categories enriched over a quantaloid, with attribution to Hofmann for private communication. See also [31, Proposition 5.2] and [25, Theorem 10.1].

The MacNeille completion can be generalised from posets to \( Q \)-categories; see [8, Definition 7.2] and [24, Definition 5.5.2]. It is also known that the MacNeille completion for \( \mathbb{R}^{\geq}_{+} \)-categories coincides with the Isbell hull [32]. We show that the abstract characterisation of the MacNeille completion of a poset as its injective envelope [3] also extends to \( Q \)-categories.

**Theorem 1.5.** Let \( Q \) be a quantale. For any \( Q \)-category \( C \), its MacNeille completion \( \mathcal{MC} \) is the injective envelope of \( C \).

In fact, we shall prove Theorem 1.4 using Theorem 1.5; our proof of Theorem 1.4 extends the proof of Theorem 1.1 in [3], and is different from those adopted in [12, 30, 31, 25]. If one assumes enough background on enriched categories, the latter proofs are arguably shorter, but we believe that our proof illuminating the role of the MacNeille completion is of independent interest.

The outline of this paper is as follows. In Sections 2 and 3 we introduce background materials on quantales and \( Q \)-categories respectively. In Section 4 we explain the MacNeille completion for \( Q \)-categories. Then, after a brief review of a formal theory of injective envelopes in Section 5, we prove Theorems 1.5 and 1.4 in Section 6. Finally, in Section 7, we revisit the notion of Isbell convexity of [16], and point out that it is equivalent to categorical completeness.

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2. Quantales

We first introduce quantales [22], also known as complete idempotent semirings [6, 20]. They are the enriching (or base) categories in the portion of enriched category theory we shall be concerned with.

**Definition 2.1.** A (unital) quantale \( Q \) is a complete lattice \((Q, \preceq_Q)\) equipped with a monoid structure \((Q, I_Q, \circ_Q)\) such that the multiplication \( \circ_Q \) preserves arbitrary suprema in each variable: \((\bigvee_{i \in I} y_i) \circ_Q x = \bigvee_{i \in I} (y_i \circ_Q x)\) and \(y \circ_Q (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (y \circ_Q x_i)\). We often omit the subscript \( Q \) from the data of a quantale, writing it simply as \( Q = (Q, \preceq, I, \circ) \).

Notice that in the definition of quantale, we do not assume commutativity of the multiplication \( \circ \) by default; quantales with commutative multiplication are said to be commutative.

The notion of adjunction is central to category theory. In this paper we shall only need the particularly simple case of adjunctions between posets, also known as Galois connections. Recall that given posets \((L, \preceq)\) and \((L', \preceq')\), two functions \( f : L \rightarrow L' \) and \( u : L' \rightarrow L \) are said to form an adjunction if, for any \( l \in L \) and \( l' \in L' \),

\[
f(l) \preceq' l' \iff l \preceq u(l')
\]

(1)

holds. We call \( f \) the left adjoint of \( u \) and \( u \) the right adjoint of \( f \), and write them as \( f \dashv u \). The adjointness relation (1) is powerful enough to determine each of the functions \( f \) and \( u \) from the other, and force both of them to be monotone functions [27].

We record the following well-known fact.
2.2. Let \((L, \preceq)\) be a complete lattice and \((L', \preceq')\) be a poset. A function \(f : L \rightarrow L'\) preserves arbitrary suprema if and only if there exists a function \(u : L' \rightarrow L\) such that \(f \circ u\).

As the first application of Proposition 2.2, observe that in any quantale \(Q = (Q, \preceq, I, \circ)\) there are two residuation operations: for any \(x \in Q\), the function \((-) \circ x : Q \rightarrow Q\) preserves arbitrary suprema, and hence has a right adjoint \((-) \not\succ x : Q \rightarrow Q\) called the right extension along \(x\); similarly, for any \(y \in Q\) the function \(y \circ (-)\) has a right adjoint \(y \not\preceq(-)\), called the right lifting along \(y\). Of course, in a commutative quantale the right extensions and right liftings coincide. The defining adjointness relations are:

\[
y \preceq z \not\succ x \iff y \circ x \preceq z \iff x \preceq y \not\preceq z.
\]  

(2)

Focusing on the leftmost and rightmost formulas of (2), we obtain

\[
z \not\succ x \preceq y \iff x \not\preceq y \not\preceq z.
\]

That is, \(z \not\succ (-) : Q \rightarrow Q\) (regarded as a function from the poset \(Q = (Q, \preceq)\) to its dual \(Q^{op} = (Q, \succeq)\)) is the left adjoint of \((-) \not\preceq z : Q \rightarrow Q\) (from \(Q^{op}\) to \(Q\)) for any \(z \in Q\). The three types of adjunctions

\[
\begin{align*}
Q & \xrightarrow{(-) \circ x} Q \quad Q & \xleftarrow{y \circ (-)} Q \quad Q & \xrightarrow{z \not\preceq (-)} Q^{op}
\end{align*}
\]

are fundamental in the theory of quantales.

We conclude this section with some examples of quantales.

Example 2.3 ([19]). The two-element quantale \(2 = (\{\bot, \top\}, \top, \bot, \land\) The underlying poset of this quantale consists of \(\top\) for “truth” and \(\bot\) for “falsity”, ordered by the entailment relation \(\top\), so that \(\bot \subseteq \top\). The monoid structure is given by conjunction \(\land\).

Example 2.4 ([19]). The Lawvere quantale \(\mathbb{R}^{\geq 0}_+ = ([0, \infty], \geq, 0, +)\). Here, \([0, \infty], \geq\) is the poset \(\{0, \infty\}, \geq\) of nonnegative real numbers ordered by the opposite \(\geq\) of the usual order \(\leq\), extended with the least element \(\infty\). The + operation is the extension of addition for nonnegative real numbers to \([0, \infty]\) so that \(x + \infty = \infty + x = \infty\) for all \(x \in [0, \infty]\). (This extension is forced by the axioms of quantale.) The right extension/right lifting is given by an extension of the truncated subtraction \(-\), defined by \(u - t = \max\{u - t, 0\}\) for nonnegative real numbers \(t\) and \(u\). Precisely, for \(y, z \in [0, \infty]\),

\[
z \not\succ y = y \not\preceq z = \begin{cases} 
z - y & \text{if } y, z \in [0, \infty) \\
0 & \text{if } y = \infty \\
\infty & \text{if } z = \infty \text{ and } y \in [0, \infty).\end{cases}
\]

This quantale is introduced in [19] for a categorical approach to the theory of metric spaces.

Example 2.5 ([19]). \(\mathbb{R}^{\geq 0}_\max = ([0, \infty], \geq, 0, \max)\). Its underlying poset \([0, \infty], \geq\) is the same as that of \(\mathbb{R}^{\geq 0}_+\). We take the binary max operation with respect to the usual ordering \(\leq\), namely the binary meet operation with respect to \(\geq\), as the multiplication. The right extension/right lifting is given by

\[
z \not\succ y = y \not\preceq z = \begin{cases} 
0 & \text{if } y \geq z \\
z & \text{otherwise.}\end{cases}
\]

This quantale is related to (a generalisation of) ultrametric spaces.

We remark that more generally, any locale, i.e., a complete lattice in which the binary meet operation \(\land\) satisfies the infinitary distributive law \((\bigvee_{i \in I} y_i) \land x = \bigvee_{i \in I} (y_i \land x)\), acquires a quantale structure with \(\land\) as the multiplication; indeed quantales were first introduced as a quantum theoretic generalisation of locales [22]. The poset \([0, \infty], \geq\), or more generally any totally ordered complete lattice, is a locale.
Example 2.6. Let $\mathcal{M} = (M,e,\cdot)$ be a monoid. The **free quantale generated by** $\mathcal{M}$ is $\mathcal{P}M = (\mathcal{P}M,\subseteq,\{e\},\cdot)$, where $\mathcal{P}M$ is the power set of $M$ and the multiplication $\cdot$ on $\mathcal{P}M$ is the unique supremum-preserving extension of the original multiplication on $M$, which is given by

$$A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$$

for all $A,B \in \mathcal{P}M$. Unlike the previous examples, this quantale is not commutative unless $\mathcal{M}$ is.

Example 2.7. Let $A$ be a set. The poset $(\mathcal{P}(A \times A),\subseteq)$ of all binary relations on $A$ admits a quantale structure $(\mathcal{P}(A \times A),\subseteq,\sqcap,\circ)$, where $\sqcap$ denotes the diagonal relation on $A$ and $\circ$ denotes composition of relations. This quantale is not commutative in general.

3. **$\mathcal{Q}$-categories**

In this section, we introduce $\mathcal{Q}$-categories for a quantale $\mathcal{Q}$. They are instances of the well-established notion of enriched category [15]. **Throughout the rest of this paper**, $\mathcal{Q} = (Q,\leq_Q,I_Q,\circ_Q)$ denotes an arbitrary quantale, unless otherwise specified.

**Definition 3.1.** A **$\mathcal{Q}$-category** $\mathcal{C}$ consists of:

(\text{CD1}) a set $\text{ob}(\mathcal{C})$ of objects;

(\text{CD2}) for each $c,c' \in \text{ob}(\mathcal{C})$, an element $\mathcal{C}(c,c') \in Q$ satisfying the following axioms:

(\text{CA1}) for each $c \in \text{ob}(\mathcal{C})$, $I_Q \leq_Q \mathcal{C}(c,c)$;

(\text{CA2}) for each $c,c',c'' \in \text{ob}(\mathcal{C})$, $\mathcal{C}(c',c'') \circ_Q \mathcal{C}(c,c') \leq_Q \mathcal{C}(c,c'')$.

We also write $c \in \mathcal{C}$ for $c \in \text{ob}(\mathcal{C})$.

**Example 3.2.** In the case $\mathcal{Q} = 2$, we may identify the data of a 2-category $\mathcal{C} = (\text{ob}(\mathcal{C}),(\mathcal{C}(c,c'))_{c,c' \in \text{ob}(\mathcal{C})})$ with a set $\text{ob}(\mathcal{C})$ equipped with a binary relation $\leq_{\mathcal{C}}$ on it (defined as the set of all pairs $(c,c') \in \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{C})$ with $\mathcal{C}(c,c') = \top$). Axioms (CA1) and (CA2) for a 2-category then translate to reflexivity and transitivity of $\leq_{\mathcal{C}}$ respectively, hence a 2-category is nothing but a **preordered set**.

**Example 3.3.** In the case $\mathcal{Q} = \mathbb{R}^\geq_+\mathcal{Q}$, we may regard $\mathbb{R}^\geq_+\mathcal{Q}$-categories as generalised metric spaces [19]. Objects of an $\mathbb{R}^\geq_+\mathcal{Q}$-category $\mathcal{C}$ are thought of as points and the element $\mathcal{C}(c,c') \in [0,\infty]$ as the **distance from $c$ to $c'$**. Notice that the axioms for $\mathbb{R}^\geq_+\mathcal{Q}$-category indeed translate to some of the axioms for metric spaces:

(\text{CA1}) for each $c \in \text{ob}(\mathcal{C})$, $0 \geq \mathcal{C}(c,c)$ (that is, $\mathcal{C}(c,c) = 0$); and

(\text{CA2}) for each $c,c',c'' \in \text{ob}(\mathcal{C})$, $\mathcal{C}(c',c'') + \mathcal{C}(c,c') \geq \mathcal{C}(c,c'')$ (the triangle inequality).

We call $\mathbb{R}^\geq_+\mathcal{Q}$-categories **Lawvere metric spaces**. Every metric space is a Lawvere metric space, but not conversely. Lawvere metric spaces are more general than metric spaces in the following three aspects:

- distance may take $\infty$;
- distance is non-symmetric (or directed), i.e., $\mathcal{C}(c,c')$ may be different from $\mathcal{C}(c',c)$; and
- $\mathcal{C}(c,c') = \mathcal{C}(c',c) = 0$ does not necessarily imply $c = c'$.

**Example 3.4.** Similarly, $\mathbb{R}^\geq_+\mathcal{Q}$-categories may be regarded as generalised ultrametric spaces; note that axiom (CA2) now reads:
(CA2) for each \( c, c', c'' \in \text{ob}(\mathcal{C}) \), \( \max\{ \mathcal{C}(c', c''), \mathcal{C}(c, c') \} \geq \mathcal{C}(c, c''). \)

**Example 3.5.** Let \( \mathcal{M} = (M, e, \cdot) \) be a monoid. A \( \mathcal{P}\mathcal{M}\)-category \( \mathcal{C} \) has, for each pair \( c, c' \in \text{ob}(\mathcal{C}) \), a subset \( \mathcal{C}(c, c') \subseteq M \). These subsets must satisfy:

(\text{CA2}) for each \( c \in \text{ob}(\mathcal{C}) \), \( e \in \mathcal{C}(c, c) \); and

(\text{CA2}) for each \( c, c', c'' \in \text{ob}(\mathcal{C}) \), \( n \in \mathcal{C}(c', c'') \) and \( m \in \mathcal{C}(c, c') \), \( n \cdot m \in \mathcal{C}(c, c'') \).

It follows that a \( \mathcal{P}\mathcal{M}\)-category can be identified with an ordinary category \( \mathcal{C} \) equipped with a faithful functor \( \mathcal{C} \rightarrow \mathcal{M} \), where the monoid \( \mathcal{M} \) is regarded as a one-object category.

In fact, this example can be vastly generalised. For any (ordinary) category \( \mathcal{B} \), we can construct the free quantaloid \( \mathcal{P}\mathcal{B} \) over it: quantaloids [23] are a many-object version of quantales, just like categories can be seen as a many-object version of monoids. It turns out that a \( \mathcal{P}\mathcal{B}\)-category corresponds to a category \( \mathcal{C} \) equipped with a faithful functor \( \mathcal{C} \rightarrow \mathcal{B} \) [8]. Theorem 1.4 is known to generalise to quantaloid-enriched categories [30, 31, 25], and (skeletal and) complete/injective \( \mathcal{P}\mathcal{B}\)-categories correspond to topological functors over \( \mathcal{B} \); see [8] for a characterisation of topological functors in terms of completeness, and see [5, 9] for that in terms of injectivity. In order to keep the paper accessible to a wider audience, in this paper we shall not pursue quantaloid-enriched categories any further. On the other hand, although our main examples of base quantales are commutative, we shall not assume commutativity so that our arguments can be easily generalised to the case of quantaloids.

Let \( \mathcal{C} \) be a \( \mathcal{Q}\)-category. We define a preorder relation \( \preceq_{\mathcal{C}} \) on \( \mathcal{C} \) as

\[
eq \preceq_{\mathcal{C}} \iff 1_{\mathcal{Q}} \preceq_{\mathcal{Q}} \mathcal{C}(c, c').
\]

(Note that the notation \( \preceq_{\mathcal{C}} \) agrees with the one introduced in Example 3.2.) Two objects \( c, c' \in \mathcal{C} \) are said to be isomorphic if both \( c \preceq_{\mathcal{C}} c' \) and \( c' \preceq_{\mathcal{C}} c \) hold. Isomorphic objects behave exactly in the same manner: if \( c, c' \in \mathcal{C} \) are isomorphic then for every object \( d \in \mathcal{C} \), we have \( \mathcal{C}(c, d) = \mathcal{C}(c', d) \) and \( \mathcal{C}(d, c) = \mathcal{C}(d, c') \). We call \( \mathcal{C} \) skeletal if isomorphic objects in \( \mathcal{C} \) are equal, i.e., if the induced preorder relation \( \preceq_{\mathcal{C}} \) on \( \text{ob}(\mathcal{C}) \) is actually a partial order relation.

A skeletal 2-category is a poset, and a skeletal \( \mathbb{R}_{\min}^{\geq 0} \) or \( \mathbb{R}_{\max}^{\leq 0} \)-category \( \mathcal{C} \) satisfies the condition that for all \( c, c' \in \mathcal{C} \), \( \mathcal{C}(c, c') = \mathcal{C}(c', c) = 0 \) implies \( c = c' \).

Let us move on to define completeness of \( \mathcal{Q}\)-categories. Given a \( \mathcal{Q}\)-category \( \mathcal{C} \), an object \( c \in \mathcal{C} \) and an element \( x \in Q \), an object \( c' \in \mathcal{C} \) is said to be a power of \( c \) by \( x \) if for any \( d \in \mathcal{C} \), the equation

\[
\mathcal{C}(d, c') = x \cdot_{\mathcal{C}} \mathcal{C}(d, c)
\]

holds [15, Section 3.7]. Powers of \( c \) by \( x \) may or may not exist in \( \mathcal{C} \), but when they exist they are unique up to isomorphism: if \( c' \) is a power of \( c \) by \( x \), then an object \( c'' \in \mathcal{C} \) is also a power of \( c \) by \( x \) if and only if \( c' \) and \( c'' \) are isomorphic. In particular, in a skeletal \( \mathcal{Q}\)-category powers are unique. We denote the power of \( c \) by \( x \) by \( x \pitchfork c \).

There is also a dual notion of copower of \( c \in \mathcal{C} \) by \( x \in Q \), which is defined as an object \( c' \in \mathcal{C} \) such that for any \( d \in \mathcal{C} \), the equation

\[
\mathcal{C}(c', d) = \mathcal{C}(c, d) \cdot_{\mathcal{C}} x
\]

holds. The copower of \( c \) by \( x \) is denoted by \( x * c \).

**Definition 3.6 ([29, Section 2]).** A \( \mathcal{Q}\)-category \( \mathcal{C} \) is said to be:

- powered if for any \( c \in \mathcal{C} \) and \( x \in Q \), the power \( x \pitchfork c \) exists in \( \mathcal{C} \);
- copowered if for any \( c \in \mathcal{C} \) and \( x \in Q \), the copower \( x * c \) exists in \( \mathcal{C} \);
• order-complete if the preordered set \((\text{ob}(C), \preceq_C)\) is complete (i.e., if its poset reflection\(^2\) is a complete lattice); and

• complete if it is powered, copowered and order-complete.

Next we define morphisms between \(Q\)-categories, called \(Q\)-functors.

**Definition 3.7.** Let \(C\) and \(D\) be \(Q\)-categories.

1. A \(Q\)-functor \(f: C \rightarrow D\) is a function \(f: \text{ob}(C) \rightarrow \text{ob}(D)\) such that for each \(c, c' \in C\),

\[
C(c, c') \preceq_Q D(f(c), f(c'))
\}

holds.

2. A \(Q\)-functor \(f: C \rightarrow D\) is **fully faithful** if for each \(c, c' \in C\), \((3)\) is satisfied with equality. We call fully faithful \(Q\)-functors embedding for short.

For any \(Q\)-category \(C\) we have the identity \(Q\)-functor \(\text{id}_C: C \rightarrow C\) (given by the identity function on \(\text{ob}(C)\)), and \(Q\)-functors are closed under composition. So \(Q\)-categories and \(Q\)-functors form an (ordinary) category \(Q\text{-Cat}\). Note that an embedding \(f: C \rightarrow D\) of \(Q\)-categories need not be injective as a function \(f: \text{ob}(C) \rightarrow \text{ob}(D)\), though embeddings out of a skeletal \(C\) are injective.

For example, a \(2\)-functor \(f: C \rightarrow D\) is a monotone map, and an \(R^0_{\geq}\) or \(R^0_{\max}\)-functor \(f: C \rightarrow D\) is a nonexpansive map. Embeddings specialise to order embeddings and isometric embeddings respectively.

We say that a \(Q\)-category \(E\) is **injective** (with respect to embeddings) if, whenever we have \(Q\)-categories \(C\) and \(D\), a \(Q\)-functor \(f: C \rightarrow E\), and an embedding \(i: C \rightarrow D\), there exists a (not necessarily unique) \(Q\)-functor \(g: D \rightarrow E\) such that \(f = g \circ i\).

Thus we have defined all terms appearing in Theorem 1.4. In fact, we can already prove the easier direction.

**Lemma 3.8.** A skeletal and complete \(Q\)-category is injective.

**Proof.** Let \(E\) be a skeletal and complete \(Q\)-category. Given a diagram as in

\[
\begin{array}{ccc}
C & \xrightarrow{f} & E \\
\downarrow{\wr} & & \downarrow{\wr} \\
D & \xrightarrow{i} & E
\end{array}
\]

we may define a \(Q\)-functor \(g: D \rightarrow E\) as the (pointwise) left (or right) Kan extension of \(f\) along \(i\), namely

\[
g(d) = \text{Lan}_i f(d) = \bigvee_{c \in C} D(i(c), d) \cdot f(c)
\]

\[
\text{or } g(d) = \text{Ran}_i f(d) = \bigwedge_{c \in C} D(d, i(c)) \cdot f(c).
\]

Then, provided that \(i\) is an embedding, \(f(c)\) and \((g \circ i)(c)\) are isomorphic for all \(c \in C\) [28, Proposition 6.7]; but since \(E\) is skeletal, isomorphic objects are necessarily equal, so \(f = g \circ i\). (Incidentally, \(\text{Lan}_i f\) and \(\text{Ran}_i f\) are respectively the least and greatest \(g\) such that \(f = g \circ i\); that is, a \(Q\)-functor \(g: D \rightarrow E\) satisfies \(f = g \circ i\) if and only if \(\text{Lan}_i f \preceq g \preceq \text{Ran}_i f\) with respect to the pointwise order \(\preceq\) induced from \(\preceq_E\).) \(\square\)

\(^2\)The poset reflection of a preordered set \((P, \preceq)\) is the quotient of it by the equivalence relation \(\preceq \cap \succeq\).
4. The MacNeille completion of a \( Q \)-category

In this section we explain the MacNeille completion of a \( Q \)-category. We start with some preparation.

**Definition 4.1 (Cf. [28, Proposition 6.1]).** Let \( C \) be a \( Q \)-category. The \( Q \)-category \( \mathcal{P}C \) of presheaves over \( C \) is defined as follows.

- An object is a **presheaf over** \( C \), that is a family \( P = (P_c)_{c \in C} \) of elements of \( Q \) satisfying the inequality \( P_{c'} \circ C(c, c') \leq_Q P_c \) for each \( c, c' \in C \).
- The element \( \mathcal{P}C(P, P') \) of \( Q \) is given by \( \bigwedge_{c \in C} P'_c \not\leq P_c \).

Dually, the \( Q \)-category \( \mathcal{P}^1C \) of copresheaves over \( C \) is defined as follows.

- An object is a **copresheaf over** \( C \), that is a family \( R = (R_c)_{c \in C} \) of elements of \( Q \) satisfying the inequality \( C(c, c') \circ Rc \leq_Q Rc' \) for each \( c, c' \in C \).
- The element \( \mathcal{P}^1C(R, R') \) of \( Q \) is given by \( \bigwedge_{c \in C} R'_c \not\leq R_c \).

For any \( Q \)-category \( C \), there are well-known embeddings \( y_C : C \to \mathcal{P}C \) and \( y^1_C : C \to \mathcal{P}^1C \) called the **Yoneda and co-Yoneda embeddings** respectively; they are defined as \( y_C(c) = (C(c, c'))_{c' \in C} \) and \( y^1_C(c) = (C(c, c'), c')_{c' \in C} \).

For example, when \( Q = 2 \), \( \mathcal{P}C \) can be understood as the poset of all **lower sets** of \( C \) (i.e., subsets \( P \subseteq \text{ob}(C) \) such that \( c' \in P \) and \( c \leq c' \) imply \( c \in P \)), ordered by inclusion. The Yoneda embedding maps an element \( c \in C \) to the principal lower set \( \downarrow c = \{ c' \in C | c' \leq c \} \) generated by it. Dually, \( \mathcal{P}^1C \) is the poset of all **upper sets** of \( C \) ordered by the opposite of inclusion, and the co-Yoneda embedding maps \( c \in C \) to the principal upper set \( \uparrow c \).

There exists a pair of canonical \( Q \)-functors

\[
\mathcal{C} \triangleright (\cdot) : \mathcal{P}C \to \mathcal{P}^1C \quad \text{and} \quad (\cdot) \lhd \mathcal{C} : \mathcal{P}^1C \to \mathcal{P}C.
\]

The functor \( \mathcal{C} \triangleright (\cdot) : \mathcal{P}C \to \mathcal{P}^1C \) maps a presheaf \( P = (P_c)_{c \in C} \) to the copresheaf \( \mathcal{C} \triangleright P = ((C \triangleright P)_c)_{c \in C} \) defined as

\[
(C \triangleright P)_c = \bigwedge_{c' \in C} C(c', c) \triangleright P_{c'};
\]

the functor \( (\cdot) \lhd \mathcal{C} \) maps \( R \in \mathcal{P}^1C \) to \( R \lhd \mathcal{C} \in \mathcal{P}C \) defined as

\[
(R \lhd \mathcal{C})_c = \bigwedge_{c' \in C} Rc' \lhd C(c, c').
\]

The functors (4) form a **\( Q \)-adjunction** \( C \triangleright (\cdot) \vdash (\cdot) \lhd \mathcal{C} \), in the sense that \( \mathcal{P}C(P, R \lhd \mathcal{C}) = \mathcal{P}^1C(C \triangleright P, R) \) for all \( P \in \mathcal{P}C \) and \( R \in \mathcal{P}^1C \). This can be checked as follows:

\[
\mathcal{P}C(P, R \lhd \mathcal{C}) = \bigwedge_{c \in C} \left( Rc' \lhd C(c, c') \right) \triangleright P_c
= \bigwedge_{c, c' \in C} Rc' \lhd C(c, c') \triangleright P_c
= \mathcal{P}^1C(C \triangleright P, R).
\]

This adjunction is called the **Isbell adjunction** [26, 8].

The **MacNeille completion** [8, 24] \( \mathcal{M}C \) of \( C \) is the \( Q \)-category defined as follows.

**CD1** An object is a pair \((P, R)\) of a presheaf \( P \in \mathcal{P}C \) and a copresheaf \( R \in \mathcal{P}^1C \) such that \( P = R \lhd \mathcal{C} \) and \( R = C \triangleright P \) hold.
Given two objects \((P, R)\) and \((P', R')\), the element
\[ MC((P, R), (P', R')) \in Q \]
is defined as \(PC(P, P')\), or equivalently as \(PC(R, R')\); indeed, we have
\[ PC(P, P') = PC(R, R') \supset C \] 
Hence we have natural embeddings \(p_C: MC \rightarrow PC\) and \(p_C': MC \rightarrow PC'\) defined by projections. The Yoneda (resp. co-Yoneda) embedding factors through \(p_C\) (resp. \(p_C'\)), so we have a canonical embedding \(i_C: C \rightarrow MC\) which maps each \(c \in C\) to \((C(-, c), C(c, -)) \in MC\). We summarise the situation in the diagram below.

\[ \begin{array}{ccc}
C & \xrightarrow{ic} & MC \\
\downarrow & & \uparrow \\
PC & \xrightarrow{pc} & \downarrow \\
\downarrow & & \downarrow \\
P'C & \xrightarrow{pc'} & PC'
\end{array} \]

Proposition 4.2 ([24, 8]). Let \(C\) be a \(Q\)-category. The MacNeille completion \(MC\) is skeletal and complete.

Proof. This is an immediate consequence of the fact that \(PC\) is skeletal and complete, and that \(p_C\) has a left adjoint, thus making \(MC\) a full reflective subcategory of \(PC\). See e.g., [8, Proposition 7.6 (a)]. \(\square\)

5. A formal theory of injective envelopes

In this section, we recall the notion of injective envelope and its basic properties [1]. Throughout this section, let \(\mathcal{X}\) be an (ordinary) category and \(\mathcal{H}\) be a class of morphisms in \(\mathcal{X}\), whose elements are called embeddings. We make no assumptions on \(\mathcal{X}\) and \(\mathcal{H}\), unless otherwise specified. An example to bear in mind is the case where \(\mathcal{X} = \mathcal{Q}\)-Cat and \(\mathcal{H}\) is the class of all fully faithful \(\mathcal{Q}\)-functors.

Definition 5.1 ([1, Definitions 2.1]).
1. An object \(E\) of \(\mathcal{X}\) is injective if, whenever we have objects \(C\) and \(D\), a morphism \(f: C \rightarrow D\), and an embedding \(i: C \rightarrow D\), there exists a (not necessarily unique) morphism \(g: D \rightarrow E\) such that \(f = g \circ i\).
2. A morphism \(f: C \rightarrow D\) in \(\mathcal{X}\) is called an essential embedding if: (i) \(f\) is an embedding, and (ii) for any object \(E\) and morphism \(g: D \rightarrow E\), if \(g \circ f\) is an embedding then so is \(g\).
3. An injective envelope of an object \(C\) of \(\mathcal{X}\) is a pair \((D, f)\) consisting of an injective object \(D\) and an essential embedding \(f: C \rightarrow D\).

Injective envelopes of an object are unique up to isomorphisms.\(^3\)

Lemma 5.2 ([1, Remarks 2.2 (2)]). Let \(C\) be an object of \(\mathcal{X}\), and \((D, f)\) and \((D', f')\) be injective envelopes of \(C\). Then there exists an isomorphism \(g: D \rightarrow D'\) such that \(g \circ f = f'\).

\(^3\)However, note that the nature of this “uniqueness” is quite different from that for usual categorical notions determined by their universal properties. We also remark that the operation of taking the injective envelopes of objects does not easily extend to morphisms [1]; cf. [26] and [8, Section 7].
Proof. By the injectivity of $D'$, we obtain a morphism $g$ as in the following commutative diagram.

\[
\begin{array}{ccc}
C & \xrightarrow{f'} & D' \\
\downarrow{f} & & \downarrow{g} \\
D & \xrightarrow{h} & D'
\end{array}
\]

We claim that any morphism $g$ between injective envelopes as above (i.e., commuting with the essential embeddings) is an isomorphism. Since $f$ is an essential embedding and $f' = g \circ f$ is an embedding, it follows that $g$ is also an embedding. Using the injectivity of $D$, we obtain a morphism $h$ as below.

\[
\begin{array}{ccc}
D & \xrightarrow{id_D} & D \\
\downarrow{g} & \xrightarrow{h} \downarrow{g} \\
D' & & D'
\end{array}
\]

So $g$ is a section (split monomorphism) whereas $h$ is a retraction (split epimorphism). Precomposing $f$ with the above diagram, we obtain the following.

\[
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow{f'} & \xrightarrow{h} \downarrow{h} \\
D' & & D'
\end{array}
\]

So $h$ is also a morphism between injective envelopes. Iterating the same argument as above, we see that $h$ is a section; hence $h$ is an isomorphism and so is its section, $g$.

Corollary 5.3. Suppose that the class $H$ of embeddings contains all identity morphisms of $X$. Let $C$ be an object of $X$ and $(D, f)$ be an injective envelope of $C$. Then $C$ is injective if and only if $f$ is an isomorphism.

Proof. If $C$ is injective, then by the assumption, $(C, \text{id}_C)$ is an injective envelope of $C$. So by Lemma 5.2 there exists an isomorphism $g: C \to D$ such that $g \circ \text{id}_C = f$. Hence $g = f$ and $f$ is an isomorphism.

Conversely, the class of all injective objects is clearly closed under isomorphism.

The injective envelope is defined by the complementary properties of essentialness of the embedding and injectivity of the codomain. In fact it is “extremal” with respect to these two properties, in the following sense.

Proposition 5.4 (Cf. [3, Proposition 2]). Let $C$ be an object of $X$ and $(E, g)$ be its injective envelope.

1. For any essential embedding $f: C \to D$, there exists a (not necessarily unique) embedding $i: D \to E$ with $i \circ f = g$.
2. For any embedding $h: C \to F$ into an injective $F$, there exists a (not necessarily unique) embedding $k: E \to F$ with $k \circ g = h$.
6. The MacNeille completion is the injective envelope

In this section we prove Theorem 1.5 and, using that, Theorem 1.4. Whenever we use the notions introduced in the previous section, we take \( \mathcal{X} = \mathcal{Q}hd \text{Cat} \) and \( \mathcal{H} \) to be the class of all fully faithful \( \mathcal{Q} \)-functors.

The key step is to give an intrinsic characterisation of essential embeddings. For each \( \mathcal{Q} \)-functor \( f : \mathcal{C} \rightarrow \mathcal{D} \), we have \( \mathcal{Q} \)-functors \( f^* : \mathcal{D} \rightarrow \mathcal{P} \mathcal{C} \) and \( f_* : \mathcal{D} \rightarrow \mathcal{P}^! \mathcal{C} \), defined as \( f^*(d) = (\mathcal{D}(f(c), d))_{c \in \mathcal{C}} \) and \( f_*(d) = (\mathcal{D}(d, f(c)))_{c \in \mathcal{C}} \) respectively. We call \( f \) dense if \( f^* \) is an embedding, and codense if \( f_* \) is an embedding [15, Chapter 5].

For any \( \mathcal{Q} \)-category \( \mathcal{C} \), the Yoneda embedding \( y_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{P} \mathcal{C} \) is \((\text{id}_{\mathcal{C}})^{*}\), whereas the co-Yoneda embedding \( y^!_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{P}^! \mathcal{C} \) is \((\text{id}_{\mathcal{C}})^{*}_{\mathcal{C}}\). In particular, \( \text{id}_{\mathcal{C}}^* \) is both dense and codense. Moreover, \((y_{\mathcal{C}})^*: \mathcal{P} \mathcal{C} \rightarrow \mathcal{P} \mathcal{C} \) is the identity \( \mathcal{Q} \)-functor \( \text{id}_{\mathcal{P} \mathcal{C}} \) (the Yoneda lemma), hence \( y_{\mathcal{C}} \) is dense (but in general not codense). Dually, \( y^!_{\mathcal{C}} \) is codense (but in general not dense). (Incidentally, \((y_{\mathcal{C}})^* = \mathcal{C} \not\subset (-) : \mathcal{P} \mathcal{C} \rightarrow \mathcal{P}^! \mathcal{C} \) and \((y^!_{\mathcal{C}})^* = (\mathcal{C} \not\subset (-) \mathcal{P}^! \mathcal{C} \rightarrow \mathcal{P} \mathcal{C} [8, \text{Remark } 6.7].)

The canonical embedding \( y_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{M} \mathcal{C} \) of \( \mathcal{C} \) into its MacNeille completion is both dense and codense ([26, Theorem 4.16], [8, Proposition 7.6] and [18, Theorem 6.5]), because \((y_{\mathcal{C}})^*: \mathcal{M} \mathcal{C} \rightarrow \mathcal{P} \mathcal{C} \) and \((y^!_{\mathcal{C}})^*: \mathcal{M} \mathcal{C} \rightarrow \mathcal{P}^! \mathcal{C} \) coincide with the (fully faithful) projections \( p_{\mathcal{C}} \) and \( p^!_{\mathcal{C}} \) respectively.

**Proposition 6.1** (Cf. [3, Lemma 3]). A \( \mathcal{Q} \)-functor is an essential embedding if and only if it is a dense and codense embedding.

**Proof.** Suppose that \( f : \mathcal{C} \rightarrow \mathcal{D} \) is an essential embedding. Then the composite \( f^* \circ f : \mathcal{C} \rightarrow \mathcal{P} \mathcal{C} \) maps each \( c \in \mathcal{C} \) to \( (\mathcal{D}(f(c), f(c)), c \in \mathcal{C}) \), i.e., it is the Yoneda embedding \( y_{\mathcal{C}} \). In particular, \( f^* \circ f \) is an embedding and hence so is \( f^* \), showing that \( f \) is dense. A similar argument shows that \( f \) is codense.

Conversely, suppose that \( f : \mathcal{C} \rightarrow \mathcal{D} \) is a dense and codense embedding. Take any \( \mathcal{Q} \)-functor \( g : \mathcal{D} \rightarrow \mathcal{E} \) such that \( g \circ f \) is an embedding. We aim to show that \( g \) is also an embedding, namely that for each \( d, d' \in \mathcal{D} \) we have \( \mathcal{D}(d, d') = \mathcal{E}(gd, gd') \). Since \( g \) is a \( \mathcal{Q} \)-functor, it suffices to show the inequality

\[
\mathcal{E}(gd, gd') \preceq \mathcal{D}(d, d').
\]  

(5)

Since \( f \) is dense, we have

\[
\mathcal{D}(d, d') = \mathcal{P} \mathcal{C}(f^*d, f^*d') = \bigwedge_{c \in \mathcal{C}} \mathcal{D}(fc, d') \not\subset \mathcal{D}(fc, d).
\]  

(6)

Since \( f \) is codense, we have

\[
\mathcal{D}(fc, d') = \mathcal{P}^! \mathcal{C}(f_*c, f_*d', f_*d') = \bigwedge_{c' \in \mathcal{C}} \mathcal{D}(d', f'c') \not\subset \mathcal{D}(fc, f'c').
\]  

(7)

Substituting (7) into (6), we obtain

\[
\mathcal{D}(d, d') = \bigwedge_{c \in \mathcal{C}} \left( \bigwedge_{c' \in \mathcal{C}} \mathcal{D}(d', f'c') \not\subset \mathcal{D}(fc, f'c') \right) \not\subset \mathcal{D}(fc, d)
\]

\[
= \bigwedge_{c, c' \in \mathcal{C}} \left( \mathcal{D}(d', f'c') \not\subset \mathcal{D}(fc, f'c') \right) \not\subset \mathcal{D}(fc, d).
\]

(cf. [7, Theorem 1]). Hence to show (5) it suffices to show, for each \( c, c' \in \mathcal{C} \),

\[
\mathcal{E}(gd, gd') \preceq \mathcal{D}(d', f'c') \not\subset \mathcal{D}(fc, f'c').
\]

Using the adjointness relation (2) twice, this is equivalent to

\[
\mathcal{D}(d', f'c') \circ \mathcal{E}(gd, gd') \circ \mathcal{D}(fc, d) \preceq \mathcal{D}(fc, f'c'),
\]

11
which can be checked easily as follows:

\[
\mathcal{D}(d', fc') \circ \mathcal{E}(gd, gd') \circ \mathcal{D}(fc, d) \leq_{\mathcal{Q}} \mathcal{E}(gd', gfc') \circ \mathcal{E}(gd, gd') \circ \mathcal{E}(gfc, gd)
\]

\[
\leq_{\mathcal{Q}} \mathcal{E}(gfc, gfc') = \mathcal{C}(c, c') = \mathcal{D}(fc, fc').
\]

Now we can show Theorem 1.5 claiming that for any \(\mathcal{Q}\)-category \(\mathcal{C}\), \((\mathcal{MC}, ic)\) is its injective envelope. The \(\mathcal{Q}\)-category \(\mathcal{MC}\) is skeletal and complete by Proposition 4.2, hence injective by Lemma 3.8. Since the \(\mathcal{Q}\)-functor \(ic\) is a dense and codense embedding, it is an essential embedding by Proposition 6.1.

Theorem 1.4 follows at once. Since all identity \(\mathcal{Q}\)-functors are essential embeddings, by Corollary 5.3, a \(\mathcal{Q}\)-category \(\mathcal{C}\) is injective if and only if the embedding \(ic: \mathcal{C} \rightarrow \mathcal{MC}\) is an isomorphism. In particular, if \(\mathcal{C}\) is injective, then it is isomorphic to the skeletal and complete \(\mathcal{MC}\), so \(\mathcal{C}\) is also skeletal and complete. The converse has already been shown in Lemma 3.8.

We remark that, being the injective envelope, the MacNeille completion enjoys the extremal properties described in Proposition 5.4; cf. [24, Proposition 5.5.5] and [8, Proposition 7.6 (e)].

7. Isbell convexity as categorical completeness

Finally, in this section we clarify the relationship of Theorem 1.3 due to Kemajou, Künzi and Otafudu [16, Theorem 1], and the \(\mathcal{Q} = \mathbb{R}^{\geq0}\) case of our theorem.

We first work over a general quantale \(\mathcal{Q}\) and provide an alternative description of objects of the MacNeille completion. Given (possibly infinite) sets \(A\) and \(B\), a \(\mathcal{Q}\)-matrix from \(A\) to \(B\) is simply a function \(X: A \times B \rightarrow Q\) [4]. We denote such a \(\mathcal{Q}\)-matrix as \(X: A \rightarrow B\). Given \(\mathcal{Q}\)-matrices \(X: A \rightarrow B, Y: B \rightarrow C\) and \(Z: A \rightarrow C\), define the \(\mathcal{Q}\)-matrices:

- \(Y \circ X: A \rightarrow C\) as \((Y \circ X)(a, c) = \bigvee_{b \in B} Y(b, c) \circ X(a, b)\);
- \(Z \not\leq X: B \rightarrow C\) as \((Z \not\leq X)(b, c) = \bigwedge_{a \in A} Z(a, c) \not\leq X(a, b)\); and
- \(Y \not\geq Z: A \rightarrow B\) as \((Y \not\geq Z)(a, b) = \bigwedge_{c \in C} Y(b, c) \not\geq Z(a, c)\).

It is routine to check that these operations on \(\mathcal{Q}\)-matrices satisfy the adjointness relations analogous to (2), namely

\[ Y \leq Z \not\leq X \iff Y \circ X \preceq Z \iff X \preceq Y \not\geq Z, \]

where we order \(\mathcal{Q}\)-matrices of a fixed domain and codomain by the pointwise order induced from \(\leq_{\mathcal{Q}}\). Also, for any set \(A\) we have the diagonal \(\mathcal{Q}\)-matrix \(I_A: A \rightarrow A\) defined as \(I_A(a, a) = 1_{\mathcal{Q}}\) and \(I_A(a, a') = 1_{\mathcal{Q}}\) (the least element of \(\mathcal{Q}\)) whenever \(a \neq a'\). Note that any \(\mathcal{Q}\)-category \(\mathcal{C}\) can be seen as a \(\mathcal{Q}\)-matrix \(C: \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})\) satisfying \(I_{\text{ob}(\mathcal{C})} \preceq C\) and \(C \circ C \preceq C\). We can view a presheaf \(P\) over \(\mathcal{C}\) as a \(\mathcal{Q}\)-matrix \(P: \text{ob}(\mathcal{C}) \rightarrow 1\) satisfying \(P \circ C \preceq P\), where 1 denotes a singleton. Dually, a copresheaf \(R\) over \(\mathcal{C}\) is a \(\mathcal{Q}\)-matrix \(R: 1 \rightarrow \text{ob}(\mathcal{C})\) satisfying \(C \circ R \preceq R\).

The notations for the \(\mathcal{Q}\)-functors (4) constituting the Isbell adjunction agree with those for the above operations on \(\mathcal{Q}\)-matrices.

**Proposition 7.1.** For a \(\mathcal{Q}\)-category \(\mathcal{C}\), define the set

\[ \mathcal{U}_C = \{ (X, Y) \mid X: \text{ob}(\mathcal{C}) \rightarrow 1, Y: 1 \rightarrow \text{ob}(\mathcal{C}) \text{ and } Y \circ X \preceq C \}. \]

A pair \((X, Y)\) of \(\mathcal{Q}\)-matrices \(X: \text{ob}(\mathcal{C}) \rightarrow 1\) and \(Y: 1 \rightarrow \text{ob}(\mathcal{C})\) belongs to \(\mathcal{MC}\) if and only if it is maximal in \(\mathcal{U}_C\), in the sense that (i) \((X, Y) \in \mathcal{U}_C\), and (ii) for any pair \((X', Y')\) \(\in \mathcal{U}_C\), if \(X \preceq X'\) and \(Y \preceq Y'\) then \(X = X'\) and \(Y = Y'\).
Proof. Suppose \((P, R) \in \mathcal{MC}\). Then \(R \circ P = (\mathcal{C} \not\supset P) \circ P \not\subseteq \mathcal{C}\), so \((P, R) \in \mathcal{UC}\). Given any \((X', Y') \in \mathcal{UC}\), \(P \not\subseteq X'\) implies \(Y' \not\subseteq \mathcal{C} \not\supset X' \subseteq \mathcal{C} \not\supset P = R\); similarly, \(R \not\subseteq Y'\) implies \(X' \not\subseteq P\).

Conversely, suppose that \((X, Y)\) satisfies conditions (i) and (ii). Then the pair \((X, \mathcal{C} \not\supset X)\) satisfies \((\mathcal{C} \not\supset X) \circ X \not\subseteq \mathcal{C}, X \not\subseteq X\) and, by (i), \(Y \not\subseteq \mathcal{C} \not\supset X\). So by (ii) we conclude \(Y = \mathcal{C} \not\supset X\). Similarly, using the pair \((Y' \setminus \mathcal{C}, Y)\) we see that \(X = Y' \setminus \mathcal{C}\). In order to show that \(X\) is a presheaf over \(\mathcal{C}\), it suffices to show \(X \circ \mathcal{C} \not\subseteq X\). Since \(X = Y' \setminus \mathcal{C}\), it suffices to show \(X \circ \mathcal{C} \not\subseteq Y' \setminus \mathcal{C}\), which is equivalent to \(Y \circ X \circ \mathcal{C} \subseteq \mathcal{C}\). Using \(Y \circ X \subseteq \mathcal{C}\) and \(C \circ \mathcal{C} \subseteq \mathcal{C}\), we obtain the desired result. Similarly, \(Y\) is a copresheaf over \(\mathcal{C}\).

\(\square\)

Proposition 7.2. Let \(\mathcal{C}\) be a \(\mathcal{Q}\)-category. For each \((X, Y) \in \mathcal{UC}\), there exists \((P, R) \in \mathcal{MC}\) such that \(X \not\subseteq P\) and \(Y \not\subseteq R\).

Proof. This is immediate from Zorn’s lemma, but a more explicit proof is also possible.

We claim that \((P, R) = ((\mathcal{C} \not\supset X) \setminus \mathcal{C}, \mathcal{C} \not\supset X)\) has the desired properties.

First, we have \(X \not\subseteq P\) and \(Y \not\subseteq R\), since the former is equivalent to \((\mathcal{C} \not\supset X) \circ X \not\subseteq \mathcal{C}\), which in turn is equivalent to \(\mathcal{C} \not\supset X \not\subseteq \mathcal{C}\), whereas the latter is equivalent to \(Y \circ X \not\subseteq \mathcal{C}\).

We show \((P, R) \in \mathcal{MC}\) using Proposition 7.1. \((P, R)\) is in \(\mathcal{UC}\) because \(R \circ P = (\mathcal{C} \not\supset X) \circ ((\mathcal{C} \not\supset X) \setminus \mathcal{C}) \subseteq \mathcal{C}\) is equivalent to \((\mathcal{C} \not\supset X) \setminus \mathcal{C} \subseteq (\mathcal{C} \not\supset X) \setminus \mathcal{C}\). To show \((P, R)\) is a maximal element in \(\mathcal{UC}\), suppose we are given any \((X', Y') \in \mathcal{UC}\) with \(P \not\subseteq X'\) and \(R \not\subseteq Y'\). \(R \not\subseteq Y'\) implies \(X' \not\subseteq Y' \setminus \mathcal{C} \subseteq R \setminus \mathcal{C} = P\); so we have \(P = X'\). Using \(X \not\subseteq P = X'\), we have \(Y \circ X \not\subseteq Y' \circ X' \subseteq \mathcal{C}\), which is equivalent to \(Y' \circ X \not\subseteq X = R\). So we also have \(R = Y'\).

\(\square\)

Remark 7.3. Jawhari, Misane and Pouzet define an analogue of the MacNeille completion for symmetric\(^4\) \(\mathcal{Q}\)-categories over a commutative (or more generally involutive) integral\(^5\) quantale \(\mathcal{Q}\), and show that it is the injective envelope [14]. Their definition is a variant of the alternative description of the MacNeille completion given in Proposition 7.1. As mentioned in Section 1, their result generalises Theorem 1.2.

We can characterise complete \(\mathcal{Q}\)-categories in terms of the canonical embedding to the MacNeille completion.

Proposition 7.4. A \(\mathcal{Q}\)-category \(\mathcal{C}\) is complete if and only if the embedding \(i_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{MC}\) is surjective (as a function between the sets of objects).

Proof. Since \(i_\mathcal{C}\) is always fully faithful and \(\mathcal{MC}\) skeletal, \(i_\mathcal{C}\) is surjective if and only if it is an equivalence of \(\mathcal{Q}\)-categories [28, Proposition 4.4]. Since completeness is invariant under equivalence, if \(i_\mathcal{C}\) is surjective then \(\mathcal{C}\) is complete. For the converse, see e.g., [8, Proposition 7.6].

\(\square\)

Corollary 7.5. A \(\mathcal{Q}\)-category \(\mathcal{C}\) is complete if and only if, given any \((X, Y) \in \mathcal{UC}\), there exists \(c \in \mathcal{C}\) such that \(X \not\subseteq \mathcal{C}(\cdot, c)\) and \(Y \not\subseteq \mathcal{C}(c, \cdot)\).

Now let us specialise to the case \(\mathcal{Q} = \mathbb{R}^{\geq 0}_+\). Extending the notion of Isbell convexity for di-spaces slightly, let us say a Lawvere metric space (= \(\mathbb{R}^{\geq 0}_+\)-category) \(\mathcal{C}\) is \textit{Isbell convex} if, for any family \(((c_i, x_i, y_i))_{i \in I}\) where \(c_i \in \mathcal{C}\) and \(x_i, y_i \in [0, \infty]\), if \(x_i + y_j \geq \mathcal{C}(c_i, c_j)\) holds for each \(i, j \in I\), then there exists \(c \in \mathcal{C}\) such that \(x_i \geq \mathcal{C}(c, c_i)\) and \(y_j \geq \mathcal{C}(c, c_j)\) for all \(i \in I\).

Proposition 7.6. A Lawvere metric space is Isbell convex if and only if it is complete (in the sense of Definition 3.6).

The above proposition is immediate from Corollary 7.5. So, modulo the above discussion (and the difference between Lawvere metric spaces and di-spaces), our Theorem 1.4 yields Theorem 1.3 when \(\mathcal{Q} = \mathbb{R}^{\geq 0}_+\). We remark that the proof of Theorem 1.3 in [16], however, relies heavily on Zorn’s lemma and is quite different from our proof.

A completely parallel comment applies to the relationship of our theorem when \(\mathcal{Q} = \mathbb{R}^{\geq 0}_{\text{max}}\) and Künzi and Otafudu’s characterisation of injective \(\mathbb{R}^{\geq 0}_{\text{max}}\)-categories by \(q\)-spherical completeness in [17, Theorem 2].

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\(^4\) A \(\mathcal{Q}\)-category \(\mathcal{C}\) over a commutative \(\mathcal{Q}\) is symmetric if \(\mathcal{C}(c, c') = \mathcal{C}(c', c)\) for all \(c, c' \in \mathcal{C}\).

\(^5\) A quantale \(\mathcal{Q}\) is integral if the unit \(I_\mathcal{Q}\) is the greatest element in \((\mathcal{Q}, \preceq_\mathcal{Q})\).
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