First integrals of ordinary difference equations which do not possess a variational formulation

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Abstract

The paper presents a new method for finding first integrals of ordinary difference equations which do not possess Lagrangians, nor Hamiltonians. As an example we solve a third order nonlinear ordinary differential equation and its invariant discretization using three first integrals obtained using this method.

1 Introduction

Let us first consider a scalar $n$-th order PDE

$$F(x^1, ..., x^p, u, u_1, ..., u_n) = 0,$$

where

$$u_1 := \{u_i\} = \left\{ \frac{\partial u}{\partial x^i} \right\}, \quad ..., \quad u_k := \{u_{i_1...i_k}\} = \left\{ \frac{\partial^k u}{\partial x^{i_1}...\partial x^{i_k}} \right\}, \quad ..., \quad i = 1, ..., p.$$

Let $L$ be a linear operator

$$L = \sum_{k=0}^{\infty} F_{u_{i_1...i_k}} D_{i_1} \cdots D_{i_k},$$

where

$$F_{u_{i_1...i_k}} = \frac{\partial F}{\partial u_{i_1...i_k}}, \quad D_i = \frac{\partial}{\partial x^i} + u_{ij} \frac{\partial}{\partial u_j} + u_{ijl} \frac{\partial}{\partial u_{jl}} + ....,$$
then the adjoint operator is given by the relation
\[ L^* v = \frac{\delta}{\delta u} (vF) = \sum_{k=0}^{\infty} (-1)^k D_{i_1} \cdots D_{i_k} (vF_{u_{i_1} \cdots i_k}). \]  
(1.3)

It defines the adjoint equation
\[ F^* = \frac{\delta}{\delta u} (vF) = \sum_{k=0}^{\infty} (-1)^k D_{i_1} \cdots D_{i_k} (vF_{u_{i_1} \cdots i_k}) = 0. \]  
(1.4)

The basic operator identity (which is probably due to Lagrange, see for example [1], Eq. (2.75) on p. 80) is the following
\[ vLw - wL^* v = D^i C^i, \]  
(1.5)

where \( v \) and \( w \) are some functions of \( x = (x^1, \ldots, x^p) \), \( u \) and finite number of derivatives of \( u \). Here
\[ C^i = \sum_{k=0}^{\infty} D_{i_1} \cdots D_{i_k} (w) \frac{\delta}{\delta u_{i_1 \cdots i_k}} (vF), \]  
(1.6)

where
\[ \frac{\delta}{\delta u_{i_1 \cdots i_k}} = \sum_{s=0}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_1 \cdots i_s}} \]
are higher order Euler-Lagrange operators.

Consider Lie symmetries \([2, 3, 4]\)
\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} \zeta_{i_1 \cdots i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \]  
(1.7)

where \( \xi^i \) and \( \eta \) are some functions of \( x, u \) and finite number of derivatives of \( u \) and
\[ \zeta_{i_1 \cdots i_s} = D_{i_1} \cdots D_{i_s} (\eta - \xi^i u_i) + \xi^i u_{i_1 \cdots i_s}. \]
To each Lie symmetry \((1.7)\) there corresponds the canonical symmetry (evolutionary vector field)
\[ \bar{X} = \bar{\eta} \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} \bar{\zeta}_{i_1 \cdots i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \]  
(1.8)

where
\[ \bar{\eta} = \eta - \xi^i u_i, \quad \bar{\zeta}_{i_1 \cdots i_s} = D_{i_1} \cdots D_{i_s} (\bar{\eta}). \]

The identity \((1.5)\) can be used to link symmetries of the differential equation \((1.1)\), solutions of the corresponding adjoint equation \((1.4)\) and conservation laws. Notice that adjoint equation is always linear for \( v \) (if \( u \) is known).
Choosing \( w = \bar{\eta} = \eta - \xi^i u_i \) in (1.5), we obtain identities

\[
v\bar{X}F = \bar{\eta}F^* + D_i C^i
\]

and

\[
vXF = v\xi^i D_i(F) + \bar{\eta}F^* + D_i C^i, \tag{1.10}
\]

where

\[
C^i = \sum_{k=0}^\infty D_{i_1} \cdots D_{i_k}(\bar{\eta}) \frac{\delta}{\delta u_{i_1 \cdots i_k}}(vF). \tag{1.11}
\]

One can formulate the following theorem based on the Lagrange identity:

**Theorem 1.1** The system of equations (1.1), (1.4) possesses the following conservation law

\[
D_i C^i \big|_{(1.1), (1.4)} = 0 \tag{1.12}
\]

for each Lie symmetry (1.7) of the differential equation (1.1) and for each solution of the adjoint equation (1.4).

Since we are interested in solving (1.1) we need conservation laws for this equation alone, without using solutions of the adjoint equation (1.4). There is a way to get rid of the adjoint variable \( v \) as suggested by Ibragimov [5],[6]:

**Theorem 1.2** Let the adjoint equation (1.4) for \( v \) be satisfied for all solutions \( u \) of the differential equation (1.1) upon a substitution

\[
v = \varphi(x^1, \ldots, x^p, u, u_1, u_2, \ldots), \quad \varphi \neq 0. \tag{1.13}
\]

Then, any Lie symmetry (1.7) of the equation (1.1) leads to the conservation law (1.12), where \( v \) and its derivatives should be eliminated via equation (1.13) and its differential consequences.

The purpose of this note is to present discrete counterparts of these results for ordinary difference equations. We do not assume Lagrangian or Hamiltonian formulation of the equation (1.1). A discrete analog of the Noether theorem [7], which provides conservation laws for difference ODEs and PDEs, was developed in [8, 9, 10, 11]. Discrete Hamiltonian equations were considered in [12, 13, 14].

## 2 Scalar ODEs

In this section we specify the results of Section 1 for scalar ordinary differential equations (ODEs) of order \( n \)

\[
F(x, u, \dot{u}, \ddot{u}, \ldots, u^{(n)}) = 0, \tag{2.1}
\]
which possesses Lie point symmetries

\[X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} + \sum_{s=1}^{\infty} \zeta_s \frac{\partial}{\partial u^{(s)}}. \tag{2.2}\]

In this case we have total differentiation

\[D = \frac{\partial}{\partial x} + \dot{u} \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}} + ... + u^{(k+1)} \frac{\partial}{\partial u^{(k)}} + ....\]

The variational operator is

\[\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D \frac{\partial}{\partial \dot{u}} + D^2 \frac{\partial}{\partial \ddot{u}} + ... + (-1)^k D^k \frac{\partial}{\partial u^{(k)}} + ...\]

and higher Euler–Lagrange operators are

\[\frac{\delta}{\delta u^{(i)}} = \frac{\partial}{\partial u^{(i)}} - D \frac{\partial}{\partial u^{(i+1)}} + D^2 \frac{\partial}{\partial u^{(i+2)}} + ... + (-1)^k D^k \frac{\partial}{\partial u^{(i+k)}} + ...\]

In this case identity (1.10) takes the form

\[vX(F) = v\xi D(F) + \bar{\eta}F^* + D(I), \tag{2.3}\]

where

\[I = \sum_{i=0}^{n-1} D^{i}(\bar{\eta}) \frac{\delta}{\delta u^{(i+1)}}(vF), \quad \bar{\eta} = \eta - \xi \dot{u}. \tag{2.4}\]

Theorem 1.2 takes the following form.

**Theorem 2.1** Let the adjoint equation

\[F^* = \frac{\delta}{\delta u}(vF) = v \frac{\partial F}{\partial u} - D \left( v \frac{\partial F}{\partial \dot{u}} \right) + ... + (-1)^n D^n \left( v \frac{\partial F}{\partial u^{(n)}} \right) = 0 \tag{2.5}\]

be satisfied for all solutions of the original ODE (2.1) upon a substitution

\[v = \varphi(x, u, \dot{u}, \ddot{u}, ..., u^{(n-1)}), \quad \varphi \neq 0. \tag{2.6}\]

Then, any Lie point symmetry (2.2) of the equation (2.1) leads to a first integral (2.4), where v and its derivatives should be eliminated via equation (2.6) and its differential consequences.

First integrals I, given by (2.4), can depend on \(u^{(n)}\) as well as higher derivatives. We will call such expressions higher first integrals. It is reasonable to use the ODE (2.1) and its differential consequences to express these first integrals as functions of the minimal set of variables, i.e., in the form \(\tilde{I}(x, u, \dot{u}, ..., u^{(n-1)})\).
Example 2.1 Let us consider the ODE \[ F = \frac{1}{u^2} \left( \ddot{u} \dddot{u} - \frac{3}{2} \dot{u}^2 \right) = 0, \] \hfill (2.7)

which admits symmetries

\[
X_1 = \frac{\partial}{\partial u}, \quad X_2 = u \frac{\partial}{\partial u}, \quad X_3 = u^2 \frac{\partial}{\partial u},
\]
\hfill (2.8)

\[
X_4 = \frac{\partial}{\partial x}, \quad X_5 = x \frac{\partial}{\partial x}, \quad X_6 = x^2 \frac{\partial}{\partial x}.
\]
\hfill (2.9)

The adjoint equation (2.5) takes the form

\[
F^* = -\frac{\ddot{v}}{\ddot{u}} = 0.
\]
\hfill (2.10)

This linear adjoint equation has three independent solutions of the form \( v = v(x) \):

\[
v_a = 1, \quad v_b = x, \quad v_c = x^2. \quad \hfill (2.11)
\]

Using these three solutions and six symmetries (2.8),(2.9), one can find \( 3 \times 6 = 18 \) first integrals, some of which can be trivial. Among non-trivial first integrals we chose three independent ones:

\[
\tilde{I}_{1a} = \frac{\dddot{u}^2}{2 \ddot{u}^3}, \quad \tilde{I}_{2a} = \frac{\dddot{u} \dddot{u}}{2 \ddot{u}^3} - \frac{\dddot{u}}{\ddot{u}}, \quad \tilde{I}_{1b} = \frac{x \dddot{u}^2}{2 \ddot{u}^3} + \frac{\dddot{u}}{\ddot{u}^2}.
\]
\hfill (2.12)

The notation \( \tilde{I}_{ja} \) means that this integral corresponds to symmetry \( X_j \) and solution \( v_a \) of the adjoint equation (2.10). Setting these integrals equal to constants and eliminating \( \dot{u} \) and \( \ddot{u} \) from (2.12), we obtain two families of solutions (generic and degenerate)

\[
u(x) = \frac{1}{C_1 x + C_2} + C_3 \quad \text{and} \quad u(x) = C_1 x + C_2, \quad \hfill (2.13)
\]

where \( C_1 \neq 0, C_2 \) and \( C_3 \) are constants expressed in terms of the first integrals.

\[ \diamond \]

3 Symmetry–preserving discretization of scalar ODEs and first integrals of the difference schemes

In this section we are interested in discretizations of the scalar ODE (2.1). For the discretization of an ODE of order \( n \) we need a difference stencil with at least \( n + 1 \) points. We will use precisely \( n + 1 \) points, namely, points \( x_m, \ldots, x_{m+n} \).
These points are not specified in advance and will be defined by an additional mesh equation \[11\].

As a discretization we will consider a discrete equation on \(n + 1\) points

\[
F(x_m, u_m, x_{m+1}, u_{m+1}, \ldots, x_{m+n}, u_{m+n}) = 0, \tag{3.1}
\]

which is considered on the mesh

\[
\Omega(x_m, u_m, x_{m+1}, u_{m+1}, \ldots, x_{m+n}, u_{m+n}) = 0. \tag{3.2}
\]

These two equations form the difference system to be used. In the continuous limit the first equation goes into the original ODE and the second equation turns into an identity (for example, \(0 = 0\)).

The Lie point symmetry generator is the same as in the continuous case

\[
X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u}, \tag{3.3}
\]

but its prolongation to the points of the difference stencil is

\[
X = \xi_m \frac{\partial}{\partial x_m} + \eta_m \frac{\partial}{\partial u_m} + \ldots + \xi_{m+n} \frac{\partial}{\partial x_{m+n}} + \eta_{m+n} \frac{\partial}{\partial u_{m+n}}, \tag{3.4}
\]

where \(\xi_k = \xi(x_k, u_k)\) and \(\eta_k = \eta(x_k, u_k)\).

It is helpful to introduce backwards (left) shift operator \(S_-:\)

\[
S_-(m) = m - 1, \quad S_-(u_m) = u_{m-1}, \quad S_-(x_m) = x_{m-1}.
\]

Discrete variational operators are defined by the relation

\[
\delta \sum_m F(m, x_m, u_m, x_{m+1}, u_{m+1}, \ldots, x_{m+n}, u_{m+n})
= \sum_m \left( \delta u_m \sum_{k=0}^{\infty} S_k^- \frac{\partial}{\partial u_{m+k}} + \delta x_m \sum_{k=0}^{\infty} S_k^- \frac{\partial}{\partial x_{m+k}} \right) F(m, x_m, u_m, x_{m+1}, u_{m+1}, \ldots, x_{m+n}, u_{m+n}).
\]

This provides us with two operators

\[
\frac{\delta}{\delta u_m} = \sum_{k=0}^{\infty} S_k^- \frac{\partial}{\partial u_{m+k}}, \quad \frac{\delta}{\delta x_m} = \sum_{k=0}^{\infty} S_k^- \frac{\partial}{\partial x_{m+k}}. \tag{3.5}
\]

We suppose \(F \to 0\) sufficiently fast when \(m \to \pm\infty\) so that the difference functional is well defined. Note that these operators are given for the scheme \(3.1, 3.2\) with arbitrary \(n\). To the system of difference equations \(3.1, 3.2\) there correspond the adjoint equations

\[
F^* = \frac{\delta}{\delta u_m} (v_m F + w_m \Omega) = 0, \quad \Omega^* = \frac{\delta}{\delta x_m} (v_m F + w_m \Omega) = 0, \tag{3.6}
\]
which are always linear for the adjoint variables $v_m$ and $w_m$. Let us fix the value of index $m$, which corresponds to the left point in the equations (3.1), (3.2), and define higher discrete Euler–Lagrange operators

$$
\frac{\delta}{\delta u_{m(j)}} = \sum_{k=0}^{\infty} S^k \frac{\partial}{\partial u_{m+j+k}}, \quad \frac{\delta}{\delta x_{m(j)}} = \sum_{k=0}^{\infty} S^k \frac{\partial}{\partial x_{m+j+k}}. \quad (3.7)
$$

**Lemma 3.1 (Main identity)** The following operator identity holds

$$
v_mX(F) + w_mX(\Omega) = \eta_m F^* + \xi_m \Omega^* + (1 - S)J, \quad (3.8)
$$

where

$$
J = \sum_{j=1}^{n} (\xi_{m+j} \frac{\delta}{\delta x_{m(j)}} + \eta_{m+j} \frac{\delta}{\delta u_{m(j)}}) (v_m F + w_m \Omega). \quad (3.9)
$$

The identity can be proven by direct verification. From the identity we obtain the following result.

**Theorem 3.2 (Main result for discretized ODEs)** Let the adjoint equations (3.6) be satisfied for all solutions of the original equations (3.1), (3.2) upon a substitution

$$
v_m = \varphi_1(m, x_m, u_m, \ldots, x_{m+n-1}, u_{m+n-1}), \quad \varphi_1 \neq 0 \quad \text{or} \quad \varphi_2 \neq 0. \quad (3.10)
$$

$$
w_m = \varphi_2(m, x_m, u_m, \ldots, x_{m+n-1}, u_{m+n-1}),
$$

Then, any Lie point symmetry (3.3) of the equations (3.1), (3.2) leads to first integral

$$
J = \sum_{j=1}^{n} (\xi_{m+j} \frac{\delta}{\delta x_{m(j)}} + \eta_{m+j} \frac{\delta}{\delta u_{m(j)}}) (v_m F + w_m \Omega), \quad (3.11)
$$

where $v_m, w_m, \ldots, v_{m-n}, w_{m-n}$ should be eliminated by means of Eqs. (3.10) and their shifts to the left.

First integrals $J$ which depend on more than $n$ points can always be expressed as $\tilde{J}(m, x_m, u_m, \ldots, x_{m+n-1}, u_{m+n-1})$ with the help of the equations (3.1), (3.2).

**Example 3.1** Let us return to the ODE (2.7). As a discretization we consider an invariant scheme which consists of invariant discretization of the ODE

$$
F = \frac{u_{m+3} - u_{m+1}}{x_{m+3} - x_{m+1}} \frac{u_{m+2} - u_m}{x_{m+2} - x_m} - \frac{u_{m+3} - u_{m+2}}{x_{m+3} - x_{m+2}} \frac{u_{m+1} - u_m}{x_{m+1} - x_m} = 0 \quad (3.12)
$$

and invariant mesh

$$
\Omega = \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)} - K = 0, \quad K \neq 0, \quad (3.13)
$$
which was introduced in [15]. The scheme was constructed so as to admit all six symmetries (2.8), (2.9).

It is convenient to rewrite the scheme as

\[
\tilde{F} = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - K = 0, \tag{3.14}
\]

\[
\Omega = \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)} - K = 0. \tag{3.15}
\]

In this specific example the difference system splits into two similar independent equations, which can be considered separately.

The adjoint equations (3.6) take the form

\[
\tilde{F}^* = -v_m + (K - 1)v_{m-1} + (1 - K)v_{m-2} + v_{m-3} = 0, \tag{3.16}
\]

\[
\Omega^* = -w_m + (K - 1)w_{m-1} + (1 - K)w_{m-2} + w_{m-3} = 0. \tag{3.17}
\]

1. It is easy to find solutions \(v_m = v_m(m), w_m = 0\). We restrict ourselves to the simplest case \(K = 4\) (the other cases will be considered elsewhere).

There are three independent solutions of the adjoint equation

\[
v^a_m = 1, \quad v^b_m = m, \quad v^c_m = m^2.
\]

Applying Theorem 3.2 for these solutions and symmetries (2.8), we get \(3 \times 3 = 9\) first integrals. Here we present three independent ones:

\[
\tilde{J}_{1a} = 2 \left( \frac{4}{u_{m+2} - u_m} - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right),
\]

\[
\tilde{J}_{2a} = 2 \left( \frac{4u_{m+2}}{u_{m+2} - u_m} - \frac{u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}}{u_{m+1} - u_m} - 2 \right),
\]

\[
\tilde{J}_{1b} = 2m \left( \frac{4}{u_{m+2} - u_m} - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right)
\]

\[
+ \left( -\frac{4}{u_{m+2} - u_m} + \frac{3}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right).
\]

2. Since the equations (3.14) and (3.15) have the same form for \(u\) and for \(x\), we can consider solutions \(v_m = 0, w_m = w_m(m)\) in the same manner and obtain similar first integrals for the variable \(x_m\).

Finally, from six independent first integrals we obtain the solution of the scheme as

\[
u_m = \frac{1}{C_1m + C_2} + C_3 \quad \text{or} \quad u_m = C_1m + C_2 \quad (3.18)
\]
and for the mesh points

\[ x_m = \frac{1}{C_4m + C_5} + C_6 \quad \text{or} \quad x_m = C_4m + C_5, \quad (3.19) \]

where \( C_1 \neq 0, C_2, C_3, C_4 \neq 0, C_5 \) and \( C_6 \) are constants related to the first integrals.

**Remark 3.3** Let us note that the solution (3.18) on the mesh (3.19) can be expressed as

\[ u_m(x_m) = \frac{1}{\alpha x_m + \beta} + \gamma \quad \text{or} \quad u_m = \alpha x_m + \beta, \quad (3.20) \]

where \( \alpha \neq 0, \beta \) and \( \gamma \) are constants. We note that this solution is exactly the same as solution (2.13) of the ODE (2.7), i.e., the scheme (3.12),(3.13) is exact.

\[ \Diamond \]

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**References**

[1] R. Dennemeyer (1968) *Introduction to partial differential equations and boundary value problems* (New York: McGraw-Hill)

[2] L.V. Ovsyannikov (1982) *Group analysis of differential equations* (New York: Academic)

[3] N.H. Ibragimov (1985) *Transformation Groups Applied to Mathematical Physics* (Dordrecht: Reidel)

[4] P.J. Olver (1993) *Applications of Lie groups to differential equations* Second edition (New York: Springer–Verlag)

[5] N.H. Ibragimov (2011) Nonlinear self-adjointness and conservation laws, *J. Phys A: Math. Gen.* 44, 432002.

[6] N. Ibragimov (2010–2011) Nonlinear self-adjointness in constructing conservation laws, *Archives of ALGA* 7/8, ALGA Publications, Karlskrona, Sweden.
[7] E. Noether (1918) Invariante Variationsprobleme, Nachr. Konig. Gesell. Wis- 

sen., Gottingen, Math.-Phys. Kl., 2, 235–257.

[8] V. Dorodnitsyn (2001) Noether–type theorems for difference equations, Applied Numerical Mathematics 39, 307–321.

[9] V. Dorodnitsyn, R. Kozlov and P. Winternitz (2003) Symmetries, La- 
grangian formalism and integration of second order ODEs, J. of Nonlinear Math. Phys 10(2), 41–56.

[10] V. Dorodnitsyn, R. Kozlov and P. Winternitz (2004) Continuous symmtries of Lagrangians and exact solutions of discrete equations J. Math. Phys. 45(1), 336–359.

[11] V. Dorodnitsyn (2011) Applications of Lie Groups to Difference Equations Chapman & Hall/CRC differential and integral equations series.

[12] V. Dorodnitsyn and R. Kozlov (2009) First integrals of difference Hamiltonian equations J. Phys. A: Math. Theor. 42, 454007.

[13] V. Dorodnitsyn and R. Kozlov (2010) Invariance and first integrals of con- 
tinuous and discrete Hamiltonian equations Journal of Engineering Mathe- 
matics 66, 253–270.

[14] V. Dorodnitsyn and R. Kozlov (2011) Lagrangian and Hamiltonian for- 
malism for discrete equations: symmetries and first integrals, SMS Lecture Notes, Symmetries and Integrability of Difference Equations, Edited by D.Levi et al., Cambridge University Press, 7–49.

[15] A. Bourlioux, C. Cyr-Gagnon and P. Winternitz (2006) Difference schemes with point symmetries and their numerical tests J. Phys A: Math. Gen. 39(22), 6877–6896.

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