HKT and OKT Geometries on Soliton Black Hole Moduli Spaces

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ABSTRACT

We consider Shiraishi’s metrics on the moduli space of extreme black holes. We interpret the simplification in the pattern of N-body interactions that he observed in terms of the recent picture of black holes in four and five dimensions as composites, made up of intersecting branes. We then show that the geometry of the moduli space of a class of black holes in five and nine dimensions is hyper-Kähler with torsion, and octonionic-Kähler with torsion, respectively. For this, we examine the geometry of point particle models with extended world-line supersymmetry and show that both of the above geometries arise naturally in this context. In addition, we construct a large class of hyper-Kähler with torsion and octonionic-Kähler with torsion geometries in various dimensions. We also present a brane interpretation of our results.
1. Introduction

Recent years have seen great progress in our understanding of string-theory and M-theory by considering the non-perturbative effects of classical solutions of the associated low energy supergravity theories representing $p$-branes. Supersymmetric (i.e. BPS) solutions describing $k$ parallel or intersecting $p$-branes typically depend on a number of harmonic functions on an $n$-dimensional Euclidean or conformally Euclidean transverse space. In the simplest case of just one harmonic function $H$ on $\mathbb{E}^n$, $H$ is taken to be one plus a sum of simple isolated poles located at positions $x_i \in \mathbb{E}^n$.

$$H = 1 + \sum_{i=1}^{k} \frac{\mu_i}{(n-2)|x-x_i|^{n-2}}$$ (1.1)

The residues, $\frac{\mu_i}{n-2}$, of the poles are fixed by quantization conditions but the locations may be freely specified and so, if the $p$-branes are distinct, the moduli space $\mathcal{M}_k$ of such solutions is the configuration space of $k$ particles moving on $\mathbb{E}^n$, that is $\tilde{C}_k(\mathbb{E}^n) \equiv (\mathbb{E}^n)^k \setminus \Delta$ where $\Delta$ is the diagonal set when two or more positions coincide. If the $p$-branes are identical, one should quotient by $S_k$, the permutation group on $k$ letters, to obtain $C_k(\mathbb{E}^n) = \tilde{C}_k(\mathbb{E}^n)/S_k$, but for the time being we shall ignore this point and shall always work on the covering space $\tilde{C}_k(\mathbb{E}^n)$. If $n > 2$, which is the case we are mainly interested in, this covering space is simply connected. If more than one harmonic function is involved, the moduli space will still be $\tilde{C}_k(\mathbb{E}^n)$ but now one must include the poles of all the harmonic functions; $k$ in this case is the number of different poles of all harmonic functions.

The metrics on the moduli spaces associated with extreme black holes have been known for sometime [1, 2, 3, 4, 5] but as yet their geometric significance has been obscure. The interactions of the black holes depend on the ‘dilaton’ coupling $a$ (whose precise definition is in the next section). One of Shiraishi’s observations was that although in general there are $N$-body interactions with arbitrary $N$, these simplify in the special cases $a = 0, 1/\sqrt{3}, 1, \sqrt{3}$ in four dimensions, and in the special cases $a = 0, 1, 2$ in five dimensions. Moreover in the case $a = 1$ in
all dimensions the metric dramatically simplifies and gives rise to just two-body interactions. In this paper, we give an interpretation of these observations using the recent picture of black holes as composites made up of intersecting branes [6]. The simplification in the \( a = 1 \) case arises because the solutions preserve \( 1/4 \) of maximal (eleven-dimensional) rigid supersymmetry. Another result of this paper is to show that the moduli space of five-dimensional \( (n = 4) \ a = 1 \) BPS black holes is a hyper-Kähler manifold with torsion (HKT) and that the moduli space of a class of nine-dimensional \( (n = 8) \ a = 1 \) BPS black holes is an octonionic-Kähler manifold with torsion (OKT). The relevance of torsion in black hole moduli spaces was first pointed out in [7].

The HKT geometry has been found in the context of two-dimensional \( (4,0) \) supersymmetric sigma models and it is a generalisation of the hyper-Kähler geometry [9, 8]. There is a close relation between hyper-Kähler and HKT geometries. For example, both admit a twistor construction and they can be reconstructed from data on their twistor spaces [10,11].

The OKT geometry will be found in the context of one-dimensional \( N=8 \) supersymmetric sigma models, i.e. of \( N=8 \) supersymmetric particle mechanics. The underlying algebraic structure of this geometry is that of the octonions. There is some similarity of this geometry to that of the octonionic string [12, 13, 14] and octonionic membrane [15] solutions of supergravity theories but we have not managed to establish a direct relation.

Our investigation of the geometry of the moduli space of black holes is guided by the relation between the number of supersymmetries of a sigma model and the geometry of its target space [16, 17, 9, 8]. For this we shall summarize the results of [18] on the geometry of one-dimensional \( N=1 \) supersymmetric sigma models. Then we shall examine the target space geometry of one-dimensional sigma models with extended supersymmetry generalizing some of the conditions found in [18]. In addition, we shall identify the geometry of the target space of a class of one-dimensional sigma models with eight supersymmetries as that of OKT
We shall also present a brane interpretation of our results. Following [6], we shall show that the black-hole solutions that we investigate have a ten-dimensional interpretation. In particular, the five-dimensional black hole solution can be lifted to the ten-dimensional solution of IIB supergravity having the interpretation of two three branes intersecting on a string, and the nine-dimensional black hole can be lifted to the ten-dimensional solution of IIA supergravity having the interpretation of a wave on a string. Then, using the ten-dimensional solutions, we shall specify the one-dimensional supersymmetry multiplet that describes the sector of the effective theory of the black holes that is related to the geometry of the associated moduli spaces.

This paper is organised as follows: in section two, we summarise the results of Shiraishi on the metrics on higher-dimensional black hole moduli spaces and then we explain some of his conclusions, using the fact that some of black hole solutions can be thought of as composite objects. In section 3, we investigate the various geometries that arise in the context of one-dimensional supersymmetric sigma models and define the OKT geometry. In section 4, we construct a number of examples of HKT geometries and show that the moduli space of a class of five-dimensional black holes is an HKT manifold. In section 5, we construct a number of examples of OKT geometries and show that the moduli space of a class of eight-dimensional black holes is an OKT manifold. In section 6, we use the interpretation of these black holes as intersecting-brane solutions of ten-dimensional supergravity theories to determine the nature of their effective theory. In section 7, we examine some of the brane probe geometries that arise in the context of five-dimensional black holes. In section 8, we remark on the structure of the geodesics of the HKT and OKT geometries that we have found and discuss the quantum behaviour of these metrics. In addition, we comment on the moduli spaces of other black hole solutions in various dimensions. Finally we include an appendix in which we present some results on harmonic forms and harmonic spinors on our moduli spaces.
2. Shiraishi metrics

The slow motion of parallel $p$-branes is expected to give rise to a classical particle motion on $\mathcal{M}_k$ and, quantum mechanically, to the quantization of that classical system. In a supersymmetric theory the classical system is extended to that of a supersymmetric spinning particle. The simplest action for the bosonic part of the system is based on geodesic motion with respect to an appropriate Riemannian metric $g_{\alpha\beta}$ on $\mathcal{M}_k$, but more elaborate possibilities could also be envisaged.

Rather than consider the motion of $p$-branes in $p+1+n$ spacetime dimensions one may, in view of the invariance of the setup under the action of $\mathbb{R}^p$ acting as translations on the p-brane world volume, dimensionally reduce and consider the equivalent problem of particles, i.e. $0-$branes, moving in $n+1$ spacetime dimensions. In the case that only one harmonic function is involved one may, possibly by making suitable duality transformations, regard the particles as carrying an electric charge associated to an abelian 2-form $F$ and also interacting via the exchange of a massless scalar $\phi$. If more than one harmonic function is involved then, in some cases, one needs to consider more than one 2-form and more than one scalar.

In the simple case of one harmonic function the equivalent Lagrangian in $n+1$ dimensions is

$$R - \frac{4}{n-1} (\nabla \phi)^2 - e^{-\frac{4a}{n-1} \phi} F^2$$  \hspace{1cm} (2.1)

The solutions are given by

$$ds^2 = -H^{-\frac{2(n-2)}{n-2+a^2}} dt^2 + H^{-\frac{2}{n-2+a^2}} dx^2$$  \hspace{1cm} (2.2)

$$F = \pm \sqrt{\frac{n-1}{2(n-2+a^2)}} d\left(\frac{dt}{H}\right),$$  \hspace{1cm} (2.3)

and

$$e^{-\frac{4a}{n-1} \phi} = H^{\frac{2a^2}{n-2+a^2}}.$$  \hspace{1cm} (2.4)
The dimensionless constant $a$ depends upon precisely what objects are being considered. For the moment we leave it unspecified. Note that

$(i)$ The correspondence between $p-$branes in $p+1+n$ dimensions and 0-branes in $n+1$ dimensions is not one-one. For example two different $p-$branes may reduce to give the same solution in $n+1$ spacetime dimensions.

$(ii)$ The solutions in $n+1$ dimensions are in general singular. Only the case $a = 0$ corresponds to a regular (Reissner-Nordström-Tangherlini) black hole with finite event horizon area. In general a solution in higher dimensions, singular or not, will reduce, in most cases, to a singular solution in $n+1$ spacetime dimensions. However, one expects that non-singular solutions in $n+1$ spacetime dimensions will lift to non-singular solutions in higher dimensions.

One may now compute the metric on $\mathcal{M}_k$ directly from the classical theory. This programme was initiated in [2] by calculating the asymptotic metrics at large separation in the case $n = 3$. The exact metric was calculated in the case $a = 0$ and $n = 3$ in [4] and the exact metrics for all $a$ and $n$ were worked out by Shiraishi [1] in terms of an integral over $\mathbb{R}^n$. His result, which includes all previous ones as special cases, is

$$ds^2 = \sum_{i=1}^{k} m_i dx_i^2 + \frac{(n-1)(n-a^2)}{8\pi(n-2+a^2)} \times$$

$$\int d^n x \frac{H^{2(1-a^2)}(n-2+a^2)}{x^{n-2+a^2}} \sum_{i<j} \frac{(x-x_i)(x-x_j)|dx_i-dx_j|^2\mu_i\mu_j}{|x-x_i|^n|x-x_j|^n},$$

where the mass $m_i$ of $i$'th particle is given by

$$m_i = \frac{\pi^{\frac{n}{2}-1}(n-1)}{4(n-2+a^2)\Gamma(\frac{n}{2})} \mu_i,$$

and $H$ is given in (1.1). The normalization of the metric is determined by the condition that the kinetic energy is $\frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha x^\beta$. 

6
Shiraishi noticed a number of striking features of his metrics, some of which had been noticed previously in special cases.

- In general the geodesic motion arises from $N$–body velocity-dependent forces for all values of $N$. However in special cases the forces simplify.
- If $a = 0$ and $n = 3$, there are only 2, 3 and 4 body forces.
- If $a = 0$ and $n = 4$, there are only 2 and 3 body forces.
- If $a = \frac{1}{\sqrt{3}}$ and $n = 3$, there are only 2 and 3 body forces.
- If $a = 1$ for all $n$, there are only 2-body forces. The asymptotic metric is exact in this case.
- If $a^2 = n$ then the metric is flat and the asymptotic metric is also exact in this case.

In retrospect these observations may be understood as follows.

In four spacetime dimensions there is a family of regular black hole solutions depending upon four independent harmonic functions $(H_1, H_2, H_3, H_4)$. These black-hole solutions can be lifted to solutions of eleven-dimensional supergravity which have the interpretation of intersecting branes preserving 1/8 of the spacetime supersymmetry [6, 20]. Each harmonic function is associated with a brane involved in the intersection [20, 21]. We have the following specializations and numbers of supersymmetries when only one harmonic function is involved:

$(i) \quad a = 0 \equiv (H, H, H, H) \leftrightarrow 4.$

$(ii) \quad a = \frac{1}{\sqrt{3}} \equiv (1, H, H, H) \leftrightarrow 4$

$(iii) \quad a = 1 \equiv (1, 1, H, H) \leftrightarrow 8.$

$(iv) \quad a = \sqrt{3} \equiv (1, 1, 1, H) \leftrightarrow 16.$

The N-body structure of the velocity dependent forces between the black holes is a reflection of their composite nature. It is rather striking that this reveals itself in this way. Put another way, by scattering black holes against one-another, one
could in principle unravel ‘experimentally’ their composite nature, and learn for example that if \( a = 1/\sqrt{3} \) then three basic objects are involved.

In five spacetime dimensions there is a family of regular black hole solutions depending upon three independent harmonic functions \( (H_1, H_2, H_3) \) (see [22]). These black hole solutions again can be lifted to solutions of eleven-dimensional supergravity preserving \( 1/8 \) of the spacetime supersymmetry which have the interpretation of intersecting branes [6, 20]. Each harmonic function is associated with a brane involved in the intersection. We have the following specializations and numbers of supersymmetries when only one harmonic function is involved:

\[
(i) \quad a = 0 \equiv (H, H, H) \leftrightarrow 4 \\
(ii) \quad a = 1 \equiv (1, H, H) \leftrightarrow 8 \\
(iii) \quad a = \sqrt{3} \equiv (1, 1, H) \leftrightarrow 16
\]

The case \( a^2 = n \) arises if one dimensionally reduces a vacuum pp-wave from \( n + 2 \) spacetime dimensions, the electric charge arising as a Kaluza-Klein charge. The case \( n = 3 \) was noted in [19] and the electromagnetically dual case (Kaluza-Klein monopoles) directly verified to be flat in [3]. The case \( n = 4 \) corresponds to the motion of parallel five branes and was directly shown to be flat in [5]. The case \( n = 9 \) is the \( D \)-particle of type IIA string theory. It is clear that in all these cases the flatness of the moduli spaces arises as a consequence of the high degree of supersymmetry, i.e. 16.

One should be able to understand these metrics entirely from the point of view of supersymmetry. Naively one might have anticipated the following correspondences between the number of supersymmetries preserved by a solution and the geometry of its moduli space

\[
(i) \quad 4 \equiv \text{Complex or Hyper-complex Geometry.} \\
(ii) \quad 8 \equiv \text{Hyper-complex or “Octonionic” Geometry.} \\
(iii) \quad 16 \equiv \text{Flat Geometry.}
\]
However this immediately raises some puzzles because of the dimensionality, \( nk \), of the moduli spaces. This may indicate that there are some “missing” moduli which have not been taken into account.

Let us consider the case \( a^2 = 1 \). The metric dramatically simplifies to give

\[
ds^2 = \sum_i m_i dx_i dx_i + \frac{\pi^{n/2 - 1}}{4(n-2)\Gamma(n/2)} \sum_{i<j} |dx_i - dx_j|^2 \mu_i \mu_j / |x_i - x_j|^{n-2}.
\] (2.7)

It has already been observed [23] that if \( n = 3 \) the Shiraishi metric coincides with that on the quotient \( \mathcal{N}_{4k}/T^k \) of the Gibbons-Manton 4k-dimensional Hyper-Kähler manifold \( \mathcal{N}_{4k} \), admitting a triholomorphic \( T^k \) action, where \( T^k \) is the \( k \)-dimensional torus group. The Gibbons-Manton metric which is relevant for the black-hole moduli spaces has the opposite sign for the “mass parameters” from that relevant for BPS monopoles. The fact that the metric is hyper-Kähler is consistent with \( (4,4) \) supersymmetry (in the two-dimensional sense) but confirms the existence of missing moduli which have not yet been precisely identified. We remark that if \( a^2 = 1 \) and \( n = 8 \), then the metric on the moduli space of two black holes is similar to that one obtains for the metric on the moduli space of two heterotic strings studied by Gauntlett et al [24]. However, the latter metric has “mass parameters” with the opposite sign from those of the former. The effective theory associated with the heterotic strings has \( (8,0) \) supersymmetry (in the two-dimensional sense).

The main observations of the present paper concern the cases \( a = 1, n = 4 \) and \( a = 1, n = 8 \). If the \( a = 1, n = 4 \) solution is lifted up to six spacetime dimensions, it becomes the completely non-singular self-dual string solution [25]. If it further is lifted to ten dimensions, it becomes the solution of IIB supergravity with two 3-branes intersecting on a string, leading to a non-chiral effective theory on the string with eight real supercharges [26]. One of our claims is that the Shiraishi metric in this case is an example of what is called Hyper-Kähler Geometry with Torsion (HKT), which arises in the context of \( (4,4) \) supersymmetric two-dimensional sigma
models. At this stage, the significance of the torsion is not completely clear, but as we shall see, it appears naturally in the geometry of the target space of both one- and two-dimensional supersymmetric sigma models. The $a = 1$, $n = 8$ solution lifted to ten dimensions becomes the solution of IIA supergravity having the interpretation of a wave on a string. This leads to a chiral effective theory on the string, again with eight real supercharges. Our claim is that the Shiraishi metric in this case is an example of what we shall call Octonionic Kähler Geometry with Torsion (OKT). This geometry arises naturally in the context of one-dimensional supersymmetric sigma models but not in the context of two-dimensional ones.

3. The Supersymmetric Spinning Particle Revisited

The effective action of black holes that preserve a proportion of spacetime supersymmetry is described by that of a supersymmetric spinning particle propagating in a curved background. The background is determined by the geometry of the moduli space of black holes. Such an action, up to terms quadratic in velocities, is that of a one-dimensional supersymmetric sigma model with the black hole moduli space as target manifold.

It is well known that there is an interplay between the number of supersymmetries of a supersymmetric sigma model and the geometry of its target space. Therefore, knowing the amount of supersymmetry preserved by certain solutions of a supergravity theory, it is possible to impose strong restrictions on the geometry of their moduli space. For supersymmetric sigma models in one dimension, extended supersymmetry imposes weaker conditions on the geometry of the target space than the same amount of supersymmetry in dimensions two or higher. This is mainly due to the fact that more couplings amongst the fields are possible in one dimension, which in higher dimensions are ruled out by the world-volume Lorentz invariance. Therefore new geometries can arise on the target space of one-dimensional supersymmetric sigma models which do not have a direct analogue in one-dimensional supersymmetric sigma models with world-volume dimensions more than one. Since
we are mainly concerned with the applications of sigma models to black hole mod-
uli spaces, we shall describe the relation between the number of supersymmetries
in one-dimensional sigma models and the geometry of their target spaces. For this,
we shall begin with a summary of some of the results of [18] on one-dimensional
N=1 supersymmetric sigma models. Then we shall describe the models with ex-
tended supersymmetry generalizing some of the conditions found in [18]. For the
relation between the number of supersymmetries in two-dimensional sigma models
and the geometry of their target spaces see [17,8].

The supersymmetry algebra in one dimension is

\[ \{Q^I, Q^J\} = 2\delta^{IJ}H \]  

(3.1)

where \( \{Q^I; I = 1, \ldots, N\} \) are the supersymmetry charges and \( H \) is the Hamil-
tonian. In the following we shall describe the cases \( N = 1, 2, 4 \) and 8.

3.1. N=1 one-dimensional supersymmetry

The simplest case is that of N=1 supersymmetric sigma models, which have one
real supercharge \( Q \). There are several realisations of N=1 supersymmetry in one
dimension. For the purpose of this paper it will suffice to consider a special case of
[18] that consists of a multiplet with a real scalar \( X \) and its real fermionic partner
\( \lambda \). This is because this sector of the theory determines the geometry (metric and
complex structures) of the moduli space*. To describe the geometry associated
with the multiplet (\( X, \lambda \)), let the triplet (\( \mathcal{M}, g, c \)) be a Riemannian manifold \( \mathcal{M} \)
with metric \( g \) and a 3-form \( c \). The action of such a model is

\[ I = \frac{1}{2} \int dt \left( g_{ij} \frac{d}{dt} X^i \frac{d}{dt} X^j + ig_{ij} \lambda^i \nabla_t^{(+) \lambda^j} - \frac{1}{3!} \partial_{ijkl} \lambda^i \lambda^j \lambda^k \lambda^l \right) \]  

(3.2)

where \( X \) are the sigma model fields which are maps from the worldline to a manifold
\( \mathcal{M} \) and \( \lambda \) are worldline real one component fermions which are sections of the

* Note that for the complete description of the effective theory of a black hole solution the
other sectors in the action of [18] may have to be included.
bundle $X^*TM \otimes S$ ($S$ is the spin bundle over the worldline). The covariant derivative $\nabla_+^i$ is the pull back of the target space covariant derivative

$$\nabla^+ = \nabla + \frac{1}{2}c$$

(3.3)

with respect to the map $X$, where

$$\Gamma_+^{ij}_k = \Gamma^i_{jk} + \frac{1}{2}c^j_{jk},$$

(3.4)

$\Gamma$ is the Levi-Civita connection of the metric $g$ and the first index of $c$ is raised with the metric $g$. Therefore, $\nabla^+(+)$ is a metric connection with torsion $c$. This is reminiscent of the situation that arises in $(1,0)$ supersymmetric two-dimensional sigma models but there is an important difference: the torsion here is not necessarily a closed 3-form. In fact it turns out that if $c$ is closed then the action above can be obtained by reducing the action of $(1,0)$ supersymmetric two-dimensional sigma models.

For reasons that will become apparent later, it is convenient to give an alternative, but equivalent, description of one-dimensional $N=1$ sigma models in terms of superfields. For this, we introduce a real superfield $X$ which is a map from the $(1|1)$-dimensional real superspace $\Xi^{(1|1)}$ with coordinates $\{t; \theta\}$ into $M$ with components

$$X = X| \quad \lambda = DX|,$$

(3.5)

where the vertical line denotes the evaluation of the associated expression at $\theta = 0$ and $D$ is the supersymmetry derivative, i.e.

$$D^2 = i\frac{d}{dt}.$$  

(3.6)

The action (3.2) can now be rewritten in terms of the superfield $X$ as

$$I = -\frac{1}{2} \int dt d\theta \left( ig_{ij} DX^i \frac{d}{dt} X^j + \frac{1}{3!} c_{ijk} DX^i DX^j DX^k \right).$$

(3.7)

It is clear that this action is manifestly $N=1$ supersymmetric.
3.2. N=2 one-dimensional supersymmetry

Next, let us consider N=2 supersymmetric one-dimensional sigma models. As in the case of N=1 supersymmetry, there are many realizations of N=2 supersymmetry in one dimension [18]. However here we shall describe two special cases. To distinguish between them, we shall call the first one $N = 2a$ and the second one $N = 2b$. To describe the first realization, we introduce a real superfield $X$ which is a map from the real superspace $\Xi^{(1|2)}$ with coordinates $\{t; \theta^1, \theta^2\}$ into the manifold $\mathcal{M}$. The components of this superfield are

$$X = X| \quad \lambda = D_1 X| \quad \psi = D_2 X| \quad F = D_1 D_2 X|,$$

(3.8)

where $\lambda, \psi$ are the fermionic partners of the boson $X$, $F$ is an auxiliary field and $D_1, D_2$ are the supersymmetry derivatives, i.e.

$$D_1^2 = i \frac{d}{dt}, \quad D_2^2 = i \frac{d}{dt}, \quad D_1 D_2 + D_2 D_1 = 0.$$

(3.9)

The most general action of the $N = 2a$ multiplet is

$$I = \frac{1}{2} \int dt d^2\theta \left( (g + b)_{ij} D_1 X^i D_2 X^j + \ell_{ij} D_1 X^i D_1 X^j + m_{ij} D_2 X^i D_2 X^j \right),$$

(3.10)

where $b_{ij}, \ell_{ij}, m_{ij}$ are two-forms on $\mathcal{M}$. If the couplings $\ell, m$ vanish, then this action is the reduction of the usual two-dimensional (1,1)-supersymmetric sigma model; the couplings $\ell, m$ correspond to non-Lorentz invariant terms in the two-dimensional action. In particular, the torsion in this case is a closed three-form of the sigma model manifold. In what follows we shall assume that the couplings $\ell, m$ vanish since they do not enter in the applications to the black hole moduli spaces.

The $N = 2b$ one-dimensional supersymmetry is associated with a complex chiral superfield $Z$. These are most easily described by starting with a real superfield as
above and by imposing the condition

\[ D_2 X^i = I^i_j D_1 X^j , \]  (3.11)

where \( I \) is an endomorphism of the tangent bundle of \( \mathcal{M} \). Consistency of this constraint with the differential algebra of the supersymmetry operators (3.9) implies that

\[ I^2 = -1 \]
\[ N(I) = 0 \]  (3.12)

where \( N(I) \) is the Nijenhuis tensor of \( I \). Both these conditions imply that the endomorphism \( I \) is an (integrable) complex structure. Adopting complex coordinates on \( \mathcal{M} \) with respect to \( I \), we can write the above constraint as

\[ \bar{\Delta} Z = 0 , \]  (3.13)

where \( X = (Z, \bar{Z}) \) in complex coordinates and \( \Delta = D_2 + iD_1 \). The components of \( Z \) are

\[ Z = Z | \quad \lambda = \Delta Z | , \]  (3.14)

where \( Z \) is a complex boson and \( \lambda \) is a complex fermion. This multiplet is associated with the two-dimensional (2,0) multiplet. We remark that the corresponding sigma model target space of the N=2b multiplet is a complex manifold, unlike the sigma model target space of the N=2a multiplet described above which is real one.

To determine the conditions on the couplings of the action (3.7) required by N=2 supersymmetry, we follow [8] and express the second supersymmetry transformation in terms of the N=1 superfield \( X \) as

\[ \delta X^i = \eta I^i_j DX^j , \]  (3.15)

where \( \eta \) is the parameter of the transformation. A straightforward computation
reveals that the action (3.7) is invariant under this transformation provided\(^*\) that

\[
g_{k\ell}I_k^j I^\ell_j = g_{ij} \\
\nabla_{(i}^{(+)} I_{j)}^{k} = 0 \\
\partial_{[i} \left( I^m_{\ j} c_{m|k\ell]} \right) - 2I^m_{[i} \partial_{[m} c_{j\ell]} \right) = 0.
\]

The first condition is the usual hermiticity condition of the metric \(g\) with respect to the complex structure \(I\). An alternative way to write the second condition is

\[
\nabla_\iota_{(+)} I_{jk} = \nabla_\iota_{[i}^{(+)} I_{j]k}\, ,
\]

lowering the index of the complex structure with the metric \(g\). If the torsion \(c\) vanishes, then this condition becomes the Yano tensor condition which has already appeared in the context of one-dimensional supersymmetric sigma models in [32]. Therefore, the condition (3.17) is a generalisation of the Yano tensor condition for a connection with torsion. The last condition in (3.16) does not have a direct geometrical interpretation. For convenience, we shall use the form notation\(^\dagger\)

\[
\iota_I dc - \frac{2}{3} d\iota_I c = 0
\]

for this relation, where \(\iota_I\) is the inner derivation with respect to the the complex structure \(I\).

It is instructive to compare the conditions that we have found on the geometry of the target space of one-dimensional sigma models with \(N=2b\) supersymmetry with those of two-dimensional sigma models with (2,0) supersymmetry. The first condition in (3.16) also arises in the context of two-dimensional (2,0)-supersymmetric sigma models. The last two conditions in (3.16) do not have a

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\(^*\) For another derivation of the invariance of the action conditions see [35].

\(^\dagger\) Our normalization convention for a p-form, \(\omega\), is \(\omega = \frac{1}{p!} \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}\).
direct two-dimensional interpretation. In fact in two dimensions, both conditions are replaced by the covariant constancy condition of $I$,

$$\nabla_i^{(+)} P_k = 0,$$

(3.19)

with respect to the $\nabla^{(+)}$ connection. It turns out this is a much stronger condition and any solution of the covariant constancy condition solves the last two conditions in (3.16). However the converse is not true. Finally, we remark that it is straightforward to write an off-shell superfield action for the N=2b multiplet using the method of [8, 18] but we shall not present it here.

### 3.3. N=4 one-dimensional supersymmetry

Next, let us turn to investigate the one-dimensional N=4 supersymmetric sigma models. We shall again describe two special multiplets that we shall call N=4a and N=4b, respectively. The former is the reduction of the two-dimensional (2,2) supersymmetry multiplet and the latter is associated with the two-dimensional (4,0) supersymmetry multiplet. Since the N=4a multiplet is the reduction of the two-dimensional (2,2) one, the geometry of the target manifold of the one-dimensional sigma model in this case is the same as that of the two-dimensional one; we shall not repeat the analysis here (see for example [8]). The geometry associated with the N=4b multiplet is not necessarily the hyper-Kähler with torsion (HKT) geometry of the two-dimensional (4,0) multiplet. To find the conditions on the geometry of the target space required by the N=4b multiplet, we use N=1 superfields to write the extended supersymmetry transformations as

$$\delta X^i = \eta^r I_r^j DX^j,$$

(3.20)

where $\{\eta^r; r = 1, 2, 3\}$ are the supersymmetry parameters and $\{I_r; r = 1, 2, 3\}$ are endomorphisms of the tangent bundle of the sigma model manifold. The conditions
from the closure of the N=4 supersymmetry algebra are
\[ I_r I_s + I_s I_r = -2\delta_{rs} \]
\[ N(I_r, I_s) = 0 \]  

(3.21)

and the conditions from the invariance of the action are

\[ g_{k\ell} I_r^k I_r^\ell j = g_{ij} \]
\[ \nabla^{(+)}_{i} I_r^k j = 0 \]
\[ \iota_r dc - \frac{2}{3} d_I r c = 0 \],

(3.22)

where \( N(I_r, I_s) \) is the Nijenhuis tensor for the pair of endomorphisms \((I_r, I_s)\) (see for example [27]) and \( \iota_r \) denotes inner derivation with respect to the endomorphism \( I_r \). Therefore, the target manifold admits three complex structures which have vanishing mixed Nijenhuis tensors and obey the algebra of a basis in Cliff(\(E^3\)) equipped with a negative definite inner product. In fact, the three complex structures in this case obey the algebra of imaginary unit quaternions. This is because if one is given two anticommuting complex structures \( I_1, I_2 \), one can construct a third one \( I_3 \) by multiplying the two together, \( i.e. I_3 = I_1 I_2 \). In addition, the metric is hermitian with respect to all complex structures. The last two conditions in (3.22) are the analogues of the last two conditions (3.16) but now for each complex structure.

It is instructive to compare these conditions with those of HKT manifolds [11]. A weak HKT manifold is a Riemannian manifold \( \{M, g, c\} \) equipped with a metric \( g \), a three-form \( c \) and three complex structures \( \{I_r; r = 1, 2, 3\} \) that obey the following compatibility conditions:

\( i \) The complex structures obey the algebra of imaginary unit quaternions

\[ I_r I_s = -\delta_{rs} + \epsilon_{rst} I_t \],

(3.23)
(ii) the metric is hermitian with respect to all complex structures

\[ g_{k\ell} I_r^k I_r^\ell = g_{ij} \]  \hspace{1cm} (3.24)

(no summation over the index \( r \)) and

(iii) the complex structures are covariantly constant with respect to the \( \nabla^{(+)} \) covariant derivative

\[ \nabla_k^{(+)} I_r^i = 0 . \]  \hspace{1cm} (3.25)

If in addition the three-form \( c \) is closed, then \( \mathcal{M} \) has a strong HKT structure. In the classical theory, the target space of two-dimensional (4,0)-supersymmetric sigma models has a strong HKT structure\(^\dagger\).

The main difference between HKT manifolds and those arising in the context of one-dimensional N=4b supersymmetric sigma models is that the covariant constancy condition of the complex structures in the weak HKT geometry is replaced by the last two conditions in (3.22). It turns out that the covariant constancy condition implies those of (3.22). Therefore any weak HKT manifold solves all the conditions required by N=4b one-dimensional supersymmetry.

We can write a one-dimensional N=4b supersymmetry multiplet in superspace as

\[ D_r X^i = I_r^i j D_0 X^j \]  \hspace{1cm} (3.26)

where \( X^i \) are maps from the superspace \( \Xi^{(1|4)} \) with coordinates \( \{ t, \theta^0, \theta^r ; r = 1, 2, 3 \} \) into the sigma model target space \( \mathcal{M} \) and \( \{ D_0, D_r ; r = 1, 2, 3 \} \) are the

\[ \dagger \] It was observed in [11], though, that in the quantum theory the target space becomes a weak HKT manifold due to the anomaly cancellation mechanism.
supersymmetry derivatives obeying the algebra

\[ D^2_0 = i \frac{d}{dt} \]

\[ D_0 D_r + D_r D_0 = 0 \]

\[ D_s D_r + D_r D_s = 2i \delta_{rs} \frac{d}{dt}. \]  

(3.27)

An action for this multiplet is

\[ I = -\frac{1}{2} \int dtd\theta^0 \left( ig_{ij} D_0 X^i \frac{d}{dt} X^j + \frac{1}{3!} c_{ijk} D_0 X^i D_0 X^j D_0 X^k \right). \]

(3.28)

Note that although this action is not a full superspace integral, it is $N=4b$ supersymmetric provided that the couplings satisfy the conditions (3.21) and (3.22).

### 3.4. $N=8$ one-dimensional supersymmetry

Finally, let us consider the one-dimensional sigma models with $N=8$ supersymmetry. Again we shall describe two special $N=8$ multiplets which we shall call $N=8a$ and $N=8b$. The former multiplet is the reduction of the two-dimensional $(4,4)$ supersymmetry multiplet and the latter is associated with the two-dimensional $(8,0)$ supersymmetry multiplet. Since the $N=8a$ multiplet is the reduction of the two-dimensional $(4,4)$ one, the geometry of the target manifold of the one-dimensional sigma model in this case is the same as that of the two-dimensional one; we shall not repeat the full analysis here (see for example [8]). In particular, if the supersymmetry algebra closes off-shell, then the sigma model target space $\mathcal{M}$ admits two commuting strong HKT structures. One of the HKT structures is with respect to the connection

\[ \nabla^{(+)} = \nabla + \frac{1}{2} c \]

and the other is with respect to the connection

\[ \nabla^{(-)} = \nabla - \frac{1}{2} c. \]

(3.29)

(3.30)

An action for the $N=8a$ multiplet can be written in a way similar to that for the $(4,4)$ multiplet in two dimensions [8].
Next, let us turn to examine the N=8b case. To find the conditions required by N=8b supersymmetry, we use N=1 superfields to write the extended supersymmetry transformations as

$$\delta X^i = \eta^a I_a{}^i_J DX^j$$  \hspace{1cm} (3.31)

where \(\{\eta^a; a = 1, \ldots, 7\}\) are the supersymmetry parameters. The conditions required by the closure of the supersymmetry algebra are

$$I_a I_b + I_b I_a = -2\delta_{ab}$$
$$N(I_a, I_b) = 0$$  \hspace{1cm} (3.32)

and the conditions required by the invariance of the action are

$$g_{k\ell} I_a^k I_a^\ell_{ij} = g_{ij}$$
$$\nabla^{(+)} I_a^k_{(i} = 0$$
$$\iota_a dc - \frac{2}{3} dt_a c = 0 ,$$  \hspace{1cm} (3.33)

where \(\iota_a\) denotes inner derivation with respect to the endomorphism \(I_a\). The endomorphisms \(\{I_a\}\) are complex structures that obey the algebra of a basis in \(\text{Cliff}(E^7)\) equipped with a negative definite inner product; the underlying algebraic structure is naturally associated with that of octonions. The remaining conditions are similar to those of the N=4b multiplet but this time they apply to seven complex structures instead of three. We shall call a Riemannian manifold \(\{\mathcal{M}, g, c\}\) equipped with metric \(g\), antisymmetric tensor \(c\), and complex structures \(\{I_a\}\) that obey the compatibility conditions (3.32) and (3.33) an \textit{Octonionic Kähler with Torsion} manifold, or OKT for short.

This appears to be the appropriate generalisation of the HKT structure in the (8,0)-supersymmetric context. To see this, observe that in the HKT structure, the last two conditions in (3.33) are replaced with the covariant constancy condition

$$\nabla^{(+)} (I_a)^j_{i} = 0 ,$$  \hspace{1cm} (3.34)

of the complex structures. Now if the manifold is eight dimensional, then the cur-
vature of $\nabla^{(\pm)}$ must vanish because the complex structures form an irreducible representation of Cliff($E_7$). This is very restrictive and one of the reasons behind the absence of known examples of interacting two-dimensional $(8,0)$-supersymmetric sigma models. However as we shall see, there are examples of non-trivial OKT manifolds, which moreover serve as moduli spaces of $n=8$ $a=1$ black holes preserving $1/4$ of the supersymmetry of IIA supergravity.

To describe an off-shell $N=8b$ superspace multiplet, let $X$ be a map from the $\Xi^{(1|8)}$ superspace with coordinates $\{t; \theta^0, \theta^a, a = 1, \ldots, 7\}$ into an OKT manifold $\mathcal{M}$. Then we impose the constraints

$$D_a X^i = I_a^{\ i} D_0 X^j$$

where $\{D_0, D_a; a = 1, \ldots, 7\}$ are the supersymmetry derivatives along the corresponding Grassmann directions in $\Xi^{(1|8)}$. The algebra of $\{D_0, D_a; a = 1, \ldots, 7\}$ is a direct generalization of that in (3.27) and we shall not present it here. An action for this multiplet is

$$I = -\frac{1}{2} \int dt d\theta^0 \left( i g_{ij} D_0 X^i \frac{d}{dt} X^j + \frac{1}{3!} c_{ijk} D_0 X^i D_0 X^j D_0 X^k \right).$$

Note that although this action is not a full superspace integral, it is $N=8b$ supersymmetric provided that the couplings in the action satisfy the conditions (3.32) and (3.33). We remark that this action is similar to that of the $N=4b$ multiplet above but that the coupling constants in the action of the $N=8b$ multiplet obey different conditions from those of the $N=4b$ case.
4. Hyper-Kähler Geometry with Torsion

A strong Hyper-Kähler Geometry with Torsion \(\{M, g, c\}\) may be described in a number of ways. An equivalent but perhaps more concise description than the one of the previous section is to say that it consists of

(i) a 4\(k\)-dimensional hypercomplex manifold \(M\), a compatible metric \(g\) with associated Levi-Civita connection \(\nabla\) and

(ii) a closed 3-form \(c\) such that if we use the inverse metric to convert \(c\) to a vector valued 2-form, which we also call \(c\), then the metric preserving affine connection:

\[
\nabla^{(+)} = \nabla + \frac{1}{2} c
\]

(4.1)

preserves the hypercomplex structure.

We now expand a little on this definition. Firstly a hypercomplex manifold is one admitting three integrable complex structures, \(\{I_r; r = 1, 2, 3\} = \{I, J, K\}\) satisfying the algebra of the imaginary quaternions. Alternatively one may say that the structural group of the tangent bundle \(T\mathcal{M}\) may be reduced from \(GL(4k; \mathbb{R})\) to \(GL(k; \mathbb{H})\). A compatible metric is one for which \(I, J, K\) are isometries. In other words \(g\) is Hermitian with respect to all three complex structures. Given \(g\) we may construct three 2-forms \((\omega_I, \omega_J, \omega_K)\) from \(I, J, K\) by index lowering. If we were dealing with a Hyper-Kähler structure, all three 2-forms would be closed and the holonomy of the Levi-Civita connection \(\nabla\) would lie in \(Sp(k) \subset SO(4k; \mathbb{R})\). For a Hyper-Kähler Geometry with torsion we demand something weaker: merely that the holonomy of the metric preserving affine connection \(\nabla^{(+)} = \nabla + \frac{1}{2} c\) lies in \(Sp(k) \subset SO(4k; \mathbb{R})\). Equivalently one demands that

\[
d\omega_I - \iota_I c = 0 , \quad d\omega_J - \iota_J c = 0 , \quad d\omega_K - \iota_K c = 0 .
\]

(4.2)

Evidently one may construct products of HKT structures so as to obtain HKT structures on the product. One also has a natural notion of a group \(G\) of symmetries
of an HKT structure. One may also restrict an HKT structure to a totally geodesic hyper-complex submanifold \( \Sigma \) of an HKT manifold. This requires that all vectors tangent to \( \Sigma \) remain tangent to \( \Sigma \) when acted upon by \( I, J \) and \( K \). One also requires that the connection \( \nabla^{(+)} \) is used to propagate vectors initially parallel to \( \Sigma \) such that they remain parallel to \( \Sigma \). A convenient way to identify a totally geodesic hyper-complex submanifold is as the fixed point set of a group \( G \) of symmetries. In terms of sigma models, totally geodesic submanifolds arise by imposing constraints on the sigma model which commute with the action of supersymmetry and allow its consistent truncation to a model with fewer fields.

There are two basic examples of HKT structures, the flat structure with zero torsion on \( \mathbb{H} \) and the Wess-Zumino-Witten model on \( \mathbb{H} \setminus \{0\} \equiv \mathbb{H}^* \). Both admit \((4,4)\) supersymmetry (for the latter case see [34]). In the natural quaternionic notation, adapted to one hyper-complex structure, the metric in the first case may be expressed as

\[
d s^2 = dq d\bar{q}
\]

with vanishing torsion and in the second by

\[
d s^2 = \frac{dqd\bar{q}}{qq}.
\]

while the torsion three-form corresponds to the volume form on the unit three sphere. To pass to the other hypercomplex structure, one takes the quaternionic conjugate. Topologically the Wess-Zumino-Witten model is defined on \( \mathbb{R} \times S^3 \), the universal covering space of what mathematicians call the Hopf surface \( S^1 \times S^3 \). This is a well known example of a complex manifold which does not admit a Kähler structure. The metric is the product metric on \( \mathbb{R} \times S^3 \) and the 3-from \( c \) is the volume form on \( S^3 \). If one identifies \( S^3 \) with \( SU(2) \) and defines \( 2t = \log(q\bar{q}) \), then the connection \( \nabla^{(+)} \) is given by

\[
\frac{\partial}{\partial t} + \nabla^{(+)}_{SU(2)}
\]

where \( \nabla^{(+)}_{SU(2)} \) is the standard connection on the Lie group \( SU(2) \) defined using, say,
the right translations. The three two forms obtained from the complex structures by index lowering are

$$\omega_r = dt \wedge \sigma_r + d\sigma_r, \quad \text{(4.6)}$$

where $r = 1, 2, 3$ and $\sigma_r$ are left-invariant one-forms on $SU(2)$, i.e. $d\sigma_r = \frac{1}{2} \epsilon^{rst} \sigma_s \wedge \sigma_t$.

The Wess-Zumino-Witten model admits the orientation preserving symmetry

$$R : \quad q \to -q \quad \text{(4.7)}$$

preserving both $HKT$ structures. In effect of $R$ is the antipodal map on the 3-sphere factor which is orientation preserving. It therefore leaves the volume form and hence the torsion invariant.

### 4.1. Multi-Models

In order to obtain more complicated models, including the Shiraishi metric for the relative moduli of the n=4, a=1 black holes, we take products of the above two basic models:

$$(\mathbb{H}^v \times (\mathbb{H}^e)^e) \quad \text{(4.8)}$$

with coordinates $w_a$ and $q_i$ respectively, with metric

$$ds^2 = \sum^v dw_a d\bar{w}_a + \sum^e \frac{dq_i d\bar{q}_i}{q_i \bar{q}_i} \quad \text{(4.9)}$$

and we impose some constraints. Consider for example the case $v = e = 1$. Dropping the indices on $w$ and $q$ we impose

$$w - q = 0 \quad \text{(4.10)}$$

and recognize the well known metric on $\mathbb{H}^+$, the transverse space of a single solitonic 5-brane [28]. This corresponds to the Shiraishi metric on the relative moduli space.
of two \( a = 1, n = 1 \) 0-branes, or by lifting to six dimensions to obtain two self-dual strings. Note that the constraint restricts us to the fixed point set of the \( \mathbb{Z}_2 \) action:

\[
(w + q, w - q) \rightarrow (w + q, -w + q). \tag{4.11}
\]

Geometrically the \( \mathbb{Z}_2 \) action is reflection in the hyperplane defined by the constraint.

To get the transverse metric of the \( k \)-5-brane 4-metric one takes \( \nu = 1 \) and \( e = k \) and imposes the \( k \) constraints

\[
w + a_i - q_i = 0, \tag{4.12}
\]

where the \( a_i \) are \( k \) constant quaternions. These complete asymptotically Euclidean 4-metrics on \( \mathbb{H} \setminus \cup_i \{a_i\} \) were proposed as gravitational instantons by D’Aurilia and Regge [29] and dubbed axionic instantons by Rey [30].

To get the Shiraishi metrics we take \( \nu = k \) and \( e = \frac{1}{2}k(k - 1) \) and replace the index \( i \) on \( q_i \) by the compound index \( ab \) with \( 0 < a < b \leq k \). The constraints are

\[
w_a - w_b - q_{ab} = 0. \tag{4.13}
\]

To be more precise, agreement with the Shiraishi metrics requires an appropriate rescaling of the coordinates so as to introduce the various “mass parameters” that appear in the black hole moduli metrics. This completes our demonstration that the moduli space of self-dual strings in six dimensions admits \((4, 4)\) supersymmetry.

More general metrics will be studied later. Before doing so we want to comment on the similarity between the construction just given for HKT geometries associated to the Shiraishi metrics with \( a = 1 \) in \( n+1 = 5 \) spacetime dimensions and the Hitchin-Karlhede-Lindstrom-Rocek quotient construction of the Hyper-Kähler geometries associated to the Shiraishi metrics with \( a = 1 \) in \( n+1 = 4 \) spacetime
dimensions [10]. The moment map constraints used in the latter construction involve 3-vectors instead of 4-vectors, but are otherwise identical. It seems that this is just a particular case of T-duality. The 4-dimensional metrics lift to 6-branes and the 5-dimensional metrics lift to five branes in 10 spacetime dimensions. From the point of view of the moduli space the relative moduli space in the case of two of 6-branes is the 3-metric obtained by taking the ordinary quotient of the Taub-NUT metric by a triholomorphic $U(1)$ and in the case of the 5-branes it is the axionic instanton 4-metric.

Note the amusing fact that taking the moduli space of the axionic instantons, regarded as Neveu-Schwartz 5-branes, leads to a flat metric.

4.2. Sigma Model Duality

An alternative method of constructing HKT geometries is to use the close relationship between hyper-Kähler and HKT geometries. This is most easily demonstrated in four dimensions where starting from a hyper-Kähler geometry with metric $ds_{hk}^2$, one can obtain a weak HKT one using the Callan-Harvey-Strominger ansatz \[^{†}[28]\]

\[
ds^2 = H ds_{hk}^2 \\
c = 3 \ast dH ,
\]

(4.14)

where $H$ is a function on the hyper-Kähler manifold and the Hodge duality operation has been taken with respect to the hyper-Kähler metric. The Callan-Harvey-Strominger ansatz gives a strong HKT geometry provided that $H$ is a harmonic function with respect to the hyper-Kähler metric, \textit{i.e.}

\[
\nabla_{hk}^2 H = 0 .
\]

(4.15)

The complex structures of the HKT geometry are those of the hyper-Kähler geometry. For example, if we choose as a hyper-Kähler manifold $E^4$ then the associated

\[^{†}\] Not all 4-dimensional HKT geometries can constructed in this way [45, 46].
HKT geometry, with

\[ H = 1 + \sum_i \frac{\mu_i}{2|x - x_i|^2} \quad (4.16) \]

is that of the solitonic five-brane or that of the relative moduli space of two \( n = 4, a = 1 \) black holes described in the previous sections. The HKT metric is complete and asymptotically flat. At the centres of the harmonic function there are infinite throats isometric to \( \mathbb{R} \times S^3 \). It is worth pointing out that the associated one- or two-dimensional sigma models may have \( N=8a \) or \( (4,4) \) off-shell supersymmetry, respectively. This is because in \( \mathbb{E}^4 \), we can introduce two commuting triplets of complex structures with each triplet obeying the algebra of imaginary unit quaternions. The associated Kähler 2-forms of the first triplet is a basis of the self-dual two forms in \( \mathbb{E}^4 \) and, similarly, the associated Kähler 2-forms of the second triplet is a basis of the anti-self-dual two forms in \( \mathbb{E}^4 \).

This relation between hyper-Kähler and HKT geometries can be extended beyond four dimensions using sigma-model duality [39]. Sigma-model duality is an operation which is applied to the metric, torsion and dilaton couplings of a two-dimensional sigma model admitting an \( U(1) \) isometry. The effect of the operation is to give another quantum-mechanically equivalent sigma model with a \( U(1) \) isometry but with couplings different from those of the original model. For what follows, it is sufficient to investigate the effect of sigma-model duality on a bosonic sigma model with just a metric coupling. For this, let us suppose that the sigma model metric

\[ ds^2 = V^{-1}(d\tau + \omega)^2 + V \gamma_{ij} dx^i dx^j \quad (4.17) \]

admits a Killing vector \( X = \partial/\partial \tau \), where \( \gamma \) is the metric on the space of orbits of the isometry. Performing sigma model duality along \( X \), we find that the couplings of the dual model are

\[ ds^2 = V(d^2 \tau + \gamma_{ij} dx^i dx^j) \]

\[ c = -3 d\tau \wedge d\omega \]

\[ e^{2\Phi} = V \quad (4.18) \]
where $\Phi$ is the ‘dilaton’. If the dimension of the target space of the sigma model is less than or equal to nine, then sigma model duality coincides with the T-duality of the common Neveu-Schwartz⊗Neveu-Schwartz sector of the various string theories. The main point to observe is that, although the original sigma model has just a metric coupling, after T-duality we find that the dual model has, apart from the metric coupling, a non-zero Wess-Zumino term and a ‘dilaton’. T-duality under certain conditions preserves supersymmetry. Since two-dimensional sigma models with hyper-Kähler metrics admit $(4,4)$ supersymmetry, the dual models also admit $(4,4)$ supersymmetry, and hence their target space has two copies of a strong HKT structure. In what follows, we shall neglect the ‘dilaton’ because it is not necessary for determining the geometry of the moduli space of black holes.

As an example, let us find the HKT structure associated to the Gibbons-Hawking hyper-Kähler metric \[47, 48\]

\[
ds^2 = H^{-1}(d\tau + \omega)^2 + Hds^2(\mathbb{E}^3) ,
\]

(4.19)

where

\[
\star dH = -d\omega ,
\]

(4.20)

i.e. $H$ is a harmonic function on three-dimensional Euclidean space $\mathbb{E}^3$. This metric admits a tri-holomorphic Killing vector field $X = \partial/\partial \tau$. After an appropriate identification of the $\tau$ coordinate, the metric is geodesically complete. Applying T-duality to this metric leads to

\[
ds^2 = Hds^2(\mathbb{E}^4)
\]

\[
c = 3 \star dH ,
\]

(4.21)

where the Hodge duality operation has been taken with respect to the flat metric on $\mathbb{E}^4$, (see also [40, 41]). This HKT geometry is similar to that derived above using the ansatz (4.14) and with the four-dimensional Euclidean space $\mathbb{E}^4$ as a starting hyper-Kähler manifold. However, there is a difference in that the conformal factor
$H$ in (4.21) is a harmonic function on $\mathbb{E}^3$ rather than a harmonic function on $\mathbb{E}^4$ which is the case in (4.14). A consequence of this is that the HKT geometry (4.21) is incomplete. However, it is clear that we can find a complete geometry associated to (4.21) by allowing the harmonic function $H$ to be harmonic on $\mathbb{E}^4$.

### 4.3. HKT geometry in various dimensions

To find more general HKT geometries in $4k$ dimensions, $k > 1$, from those in section (4.1), we shall begin with a class of hyper-Kähler geometries that admit a $U(1)^k$ group of triholomorphic isometries. To describe these metrics, let us decompose the $4k$ coordinates $\{y^M; M = 1, \ldots, 4k\}$ of the hyper-Kähler manifold as

$$y^M = (\tau_i, x^r_i); \quad r = 1, 2, 3, \quad i = 1, \ldots, k,$$

(4.22)

where $\tau_i$ are the coordinates adapted to the isometries, i.e. the Killing vector fields are $X^i = \partial/\partial \tau_i$. Then the hyper-Kähler metric can be written as follows:

$$ds^2 = U^{ij}(d\tau_i + \omega_{ik} \cdot dx^k)(d\tau_j + \omega_{jl} \cdot dx^l) + U_{ij}dx^i \cdot dx^j$$

(4.23)

where $(\cdot)$ denotes the inner product with respect to the flat metric in $\mathbb{E}^3$ and the coefficients $U$ are functions only of $x$; $U = U(x)$. Moreover, they satisfy

$$U_{ij} = U_{ji}$$

$$\ast dU_{ij} = -d\omega_{ij},$$

(4.24)

where the Hodge star operation is taken with respect to the flat metric in $\mathbb{E}^3$. To find explicit solutions to the above conditions (4.24), we follow [42] and write

$$U_{ij} = U_{ij}^\infty + \Delta U_{ij},$$

(4.25)

where $U^\infty$ is the asymptotic value of $U$ as $|x^k|$ goes to infinity. Then $\Delta U_{ij}$ is the
sum of terms of the form
\[ \frac{p_ip_j}{|p_kx^k - a|}, \]  
for different values of \( p \) and different centres \( a \). Therefore, the most general form of \( \Delta U_{ij} \) is
\[ \Delta U_{ij} = \sum_{\{p\}} \sum_a \frac{p_ip_j}{|p_kx^k - a(\{p\})|} \]  
(4.27)

It is well-known that this geometry can be characterised by the properties of the co-dimension three planes
\[ p_kx^k - a = 0, \]  
(4.28)
in \( \mathbb{E}^{3k} \). In particular, the metric is non-singular provided that \( \{p_1, \ldots, p_k\} \) are co-prime integers, and the various \( (3k - 3) \)-planes do not coincide and intersect only pairwise.

There is a chain of strong HKT structures associated with the above hyper-Kähler geometry. Each HKT structure in the chain can be found by T-dualizing the hyper-Kähler geometry along one or more Killing vector field directions \( \tau_i \). The case that we shall present here is the one that arises after dualizing all Killing vector directions once. The resulting strong HKT geometry is
\[ ds^2 = U_{ij} \left( d\tau^i d\tau^j + dx^i \cdot dx^j \right) \]
\[ c = \frac{3}{2} \epsilon_{rs}^t \partial_t U_{jk} d\tau^i \wedge dx^r \wedge dx^s. \]  
(4.29)

This metric is singular. However as in the 4-dimensional case, we can extend the dependence of the coefficient \( U_{ij} \) of the metric. For this, we decompose the \( 4k \)-coordinates \( \{y^M, M = 1, \ldots, 4k\} \) of the HKT manifold as
\[ y^M = x^{\mu i} \quad i = 1, \ldots, k \quad \mu = 0, \ldots, 3. \]  
(4.30)
The new metric and torsion can be written as
\[ ds^2 = U_{ij} \, dx^i \cdot dx^j \]
\[ c = \frac{1}{2} \epsilon_{\mu\nu\lambda} \partial_{\mu} U_{jk} \, dx^\mu \wedge dx^\nu \wedge dx^\lambda , \]
where \( U \) is a function of \( x^k \) and \( \epsilon \) is the Levi-Civita tensor with respect to the flat metric on \( \mathbb{E}^4 \). For the torsion \( c \) to be a 3-form, we require
\[ U_{ij} = U_{ji} \]
\[ \partial_{i\mu} U_{jk} = \partial_{j\mu} U_{ik} , \]
where Hodge star operation is with respect to the flat metric in \( \mathbb{E}^4 \). If we further require \( c \) to be a closed 3-form, then
\[ \partial_i \cdot \partial_j U_{kl} = 0 . \]

To find explicit examples of strong HKT geometries, we write
\[ U_{ij} = U_{ij}^\infty + \Delta U_{ij} , \]
where \( U_{ij}^\infty \) is the asymptotic value of \( U_{ij} \) as \( |x^k| \) goes to infinity. Then, \( \Delta U_{ij} \) is a sum of terms of the form
\[ \frac{p_i p_j}{|p_k x^k - a|^2} = \frac{1}{4} \partial_i \cdot \partial_j \log|p_k x^k - a|^2 \]
for different choices for the real numbers \( p \) and for the centres \( a \). Therefore, the most general form for \( \Delta U_{ij} \) is
\[ \Delta U_{ij} = \frac{1}{4} \sum_{\{p\}} \sum_{a} \partial_i \cdot \partial_j \log|p_k x^k - a(\{p\})|^2 . \]

In direct correspondence with the associated hyper-Kähler geometries, these HKT geometries are naturally associated with the co-dimension four planes
\[ p_k x^k - a = 0 \]
in \( \mathbb{E}^{4k} \). It appears that these metrics are non-singular on the complement of these planes in \( \mathbb{E}^{4k} \) provided that the planes, if they intersect, intersect only pairwise.
It remains to give the complex structures of the above HKT geometry. These are

\[ \mathbf{I}_r = 1 \otimes I_r , \]  
(4.38)

where \( \{ I_r; r = 1, 2, 3 \} \) are the three complex structures in \( \mathbb{E}^4 \) associated with, say, a basis in the space of constant self-dual 2-forms in \( \mathbb{E}^4 \). Then it is straightforward to verify that the covariant constancy condition of the complex structures with respect to the \( \nabla^{(+)} \) covariant derivative and the closure of \( c \) imply only the conditions (4.32) and (4.33). Thus we have directly shown that \( (\mathcal{M}, g, c) \) of (4.31) with coefficients given in (4.36) has a strong HKT structure. In fact there is another HKT structure on \( (\mathcal{M}, g, c) \) associated with the connection \( \nabla^{(-)} \) and with complex structures

\[ \mathbf{J}_r = 1 \otimes J_r , \]  
(4.39)

where in this case \( \{ J_r; r = 1, 2, 3 \} \) are the three complex structures in \( \mathbb{E}^4 \) associated with a basis in the space of constant anti-self-dual 2-forms in \( \mathbb{E}^4 \). It turns out that the two HKT structures commute. Therefore, the corresponding one- or two-dimensional sigma model whose target space is the strong HKT manifold \( (\mathcal{M}, g, c) \) may admit off-shell \( \text{N}=8 \) or (4,4) supersymmetry, respectively.

5. Octonionic Kähler Geometry with Torsion

The algebraic structure underlying the OKT geometry is that of the octonions, \( \mathbb{O} \). Let \( \{ e_0, e_a; a = 1, \ldots, 7 \} \) be a basis in \( \mathbb{O} \) consisting of the unit octonions. In this basis, we choose \( e_0 \) to be the identity, so it commutes with all the other elements of the basis. The rest of the basis elements satisfy

\[ e_a e_b = -\delta_{ab} + \varphi_{ab}^c e_c \]  
(5.1)

where \( \varphi_{abc} = \varphi_{ab}^d \delta_{dc} \) are the structure constants of the octonions; \( \varphi \) is antisym-
metric in all its indices. Next, let $*\varphi$ be the Poincaré dual of $\varphi$,

$$
*\varphi_{abcd} = \frac{1}{3!} \epsilon_{abcd} \varphi^{pqr} \varphi_{pqr},
$$

then

$$
\varphi_{abf} \varphi_{cdf} \mathcal{f} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} - *\varphi_{abcd},
$$

and

$$
*\varphi_{f abc} \varphi_{f de} = 6 \varphi_{[d} \delta_{e]}^{[ab} \delta_{c]}.
$$

We can use the structure constants of the octonions to introduce seven complex structures in $\mathbb{E}^8$ as follows:

$$(I_a)_{b}^{0} = \delta_{ab}$$

$$(I_a)^{b}_{0} = -\delta_{a}^{b}$$

$$(I_a)^{b}_{c} = \varphi_{a}^{b} c.$$

(5.5)

It is straightforward to see (i) that the flat metric in $\mathbb{E}^8$ is hermitian with respect to all the complex structures and (ii) that these complex structures obey the gamma-matrix relations of a basis in Cliff($\mathbb{E}^7$) equipped with a negative-definite inner product. It is then clear that $\mathbb{E}^8$ equipped with the above complex structures and the flat metric is an OKT manifold.

To find non-trivial examples of eight-dimensional OKT geometries, we use the ansatz

$$
ds^2 = H \, ds^2(\mathbb{E}^8)$$

$$
ce_{\mu\nu\rho} = \Omega_{\mu\nu\rho} \lambda \partial_{\lambda} H,$$

(5.6)

where $\mu, \nu, \rho, \lambda = 0, \ldots, 7$, $H$ is a function on $\mathbb{E}^8$ and $\Omega$ is a four-form on $\mathbb{E}^8$ (We have raised the index of $\Omega$ with the flat metric on $\mathbb{E}^8$). The choice of a conformally-flat metric in the ansatz is motivated by the form of the moduli metric of the $n = 8$, $a = 1$ black holes. In addition, we choose as complex structures those of (5.5). The integrability of $I_a$ follows immediately because they are constant tensors and so
all Nijenhuis conditions are satisfied. The metric in (5.6) is clearly hermitian with respect to all complex structures. So it remains to solve the last two conditions in (3.33). The second condition in (3.33) implies that

\[
\Omega_{0abc} = \varphi_{abc} \\
\Omega_{abcd} = -^* \varphi_{abcd}.
\]

Therefore \(\Omega\) is the \(\text{Spin}(7)\) invariant anti-self-dual 4-form in \(E^8\) associated with the above octonionic structure. After some computation, the third condition in (3.33) implies that

\[
\delta^{\mu\nu} \partial_\mu \partial_\nu H = 0.
\]

Hence, \(H\) is a harmonic function on \(E^8\). For this computation, we have used the following identities for the octonionic structure constants:

\[
\varphi_{[ab}(f \varphi_{cd]} e) = \frac{4}{3}^* \varphi_{[abc}(f \delta_{d]} e) - \frac{1}{3}^* \varphi_{abcd} \delta^{fe} \\
\varphi_{p[a}(f^* \varphi_{bcd]} e) = -\delta_{p[a} \varphi_{bcd]} \delta^{fe} + 3\delta_{p[a} \varphi_{bc]}(f \delta_{d]} e) - \varphi_{[abc}(f \delta_{d]} e) \delta_{p}.
\]

Note that \(c\) is not a closed three form.

Apart from the flat OKT structure in \(E^8\) for which \(H = 1\), the simplest non-trivial OKT structure arises when

\[
H = \frac{1}{|x|^6}.
\]

Setting

\[
\rho = \frac{1}{2|x|^2},
\]

the metric becomes

\[
ds^2 = d\rho^2 + 4\rho^2 d\Omega^2_{(7)}
\]

where \(d\Omega^2_{(7)}\) is the standard round metric on \(S^7\). Geometrically, the metric is a cone over the round 7-sphere of radius 2. The metric (5.12) is defined on \(\mathbb{O}^* = E^8\setminus\{0\}\).
and is complete as \( \rho \to \infty \), i.e. as \(|x| \to 0\), but has a conical singularity at finite distance as \( \rho \to 0 \), i.e. \(|x| \to \infty\). To cure this problem, we set

\[
H = 1 + \frac{1}{|x|^6}.
\] (5.13)

and we get a complete metric on \( \mathbb{O}^* \) which interpolates between the flat metric near \(|x| \to 0\) and the conical metric near \(|x| \to \infty\).

Note that by taking the octonionic conjugate of the above construction, we obtain another OKT structure. Each of these structures is invariant under the freely acting involution

\[
x \to -x.
\] (5.14)

5.1. Octonionic Multi-Models

In order to obtain more complicated models, including the Shiraishi metric for the relative moduli of the \( n = 8, a = 1 \) black holes, we begin with the metric

\[
ds^2 = \sum_{a=1}^{v} du_a d\bar{u}_a + \sum_{i=1}^{c} d\omega_i d\bar{\omega}_i \frac{(\omega_i \bar{\omega}_i)^3}{(o_i \bar{o}_i)^3},
\] (5.15)

on

\[
(\mathbb{O})^v \times (\mathbb{O}^*)^c,
\] (5.16)

where \( \mathbb{O}^* = \mathbb{E}^8 \setminus \{0\} \) and \( u_a, o_i \) are octonions. This metric is an OKT metric because it is a sum of OKT metrics. Using these we can obtain more complicated OKT geometries by imposing suitable constraints. For example, let \( v = c = 1 \).

Dropping the indices on \( u, o \), we impose

\[
u - o = 0
\] (5.17)

and recognize the OKT metric on \( \mathbb{E}^8 \setminus \{0\} \), corresponding to the harmonic function

\[
H = 1 + \frac{1}{|x|^6}.
\] (5.18)

This corresponds to the Shiraishi metric on the relative moduli space of two \( n = 8 \),
\( a = 1 \) black holes. Note that the constraint restricts us to the fixed point set of the \( \mathbb{Z}_2 \) action

\[
(u + o, u - o) \rightarrow (u + o, -u + o) .
\]  

(5.19)

The multi-centre OKT metric on \( E_8 \setminus \cup_{i=1}^{k} \{a_i\} \) with centres \( \{a_i; i = 1, \ldots, k\} \) can be found by choosing \( v = 1 \) and \( e = k \), and then by imposing the conditions

\[
u + a_i - o_i = 0 ,
\]  

(5.20)

where \( a_i \) are \( k \) constant octonions.

Finally, in order to derive the Shiraishi metrics for the relative moduli of \( n = 8, a = 1 \) black holes, we can take \( v = k \) and \( e = \frac{1}{2}k(k - 1) \) and replace the index \( i \) on \( o_i \) by the compound index \( ab \) with \( 0 < a < b \leq k \). The constraints are

\[
u_a - u_b - o_{ab} = 0 .
\]  

(5.21)

To be more precise, agreement with the Shiraishi metrics requires an appropriate rescaling the coordinates to introduce the various “mass parameters” that appear in the black hole moduli metrics. This completes our demonstration that the moduli space of \( n = 8, a = 1 \) black holes admits \( N=8 \) supersymmetry.

5.2. OKT geometry in various dimensions

To find more general examples of OKT geometries in \( 8k \) dimensions, \( k > 1 \), from those of the previous section, we first write the coordinates of \( \{x^M; M = 1, \ldots, 8k\} \) of \( E^{8k} \) as

\[
x^M = x^{\mu i} ; \quad \mu = 1, \ldots 8 , \quad i = 1, \ldots, k .
\]  

(5.22)

Then we use the ansatz

\[
ds^2 = U_{ij} dx^i \cdot dx^j
\]

\[
c = \frac{1}{3!} \Omega_{\mu
\nu \lambda}^p \partial_i^p U_{jk} dx^{\mu i} \wedge dx^{\nu j} \wedge dx^{\lambda k} ,
\]  

(5.23)
where $U$ is a function of $\mathbb{E}^{8k}$. For $c$ to be a three-form, the matrix $U$ must satisfy

$$
U_{ij} = U_{ji} \quad \text{(5.24)}
$$

$$
\partial_{\rho i} U_{jk} = \partial_{\rho j} U_{ik} .
$$

To complete the ansatz, we choose as complex structures

$$
I_a = 1 \otimes I_a , \quad \text{(5.25)}
$$

where $\{I_a; a = 1, \ldots, 7\}$ are the complex structures on $\mathbb{E}^8$ associated with the octonions as in (5.5). After some computation, we find that the above ansatz satisfies the conditions of an OKT geometry provided that

$$
\partial_i \cdot \partial_j U_{kl} = 0 , \quad \text{(5.26)}
$$

where ($\cdot$) denotes the inner product with respect to the Euclidean 8-metric.

It remains for us to find examples of such geometries. For this let us write

$$
U_{ij} = U_{ij}^\infty + \Delta U_{ij} , \quad \text{(5.27)}
$$

where $U_{ij}^\infty$ is a constant matrix which can be thought as the asymptotic value of $U_{ij}$ as $|x^i|$ goes to infinity. Solving (5.24) and (5.26) for $\Delta U_{ij}$, we find that it is a linear combination of

$$
\frac{p_ip_j}{|p_kx^k - a|^6} , \quad \text{(5.28)}
$$

for different choices of k-vectors $\{p_1, \ldots, p_k\}$ and different choices of centres $a$. Therefore the most general expression for $\Delta U_{ij}$ is

$$
\Delta U_{ij} = \sum_{\{p\}} \sum_a \frac{p_ip_j}{|p_kx^k - a(\{p\})|^6} . \quad \text{(5.29)}
$$

It is clear that this OKT geometry is associated with the co-dimension eight planes

$$
p_kx^k - a = 0 , \quad \text{(5.30)}
$$

in $\mathbb{E}^{8k}$. It appears that the metric (5.23) with $\det U^\infty \neq 0$ and coefficients given
in (5.29) is non-singular on the complement of these planes in $E^{8k}$ provided that the planes, if they intersect, intersect only pairwise.

6. Moduli Space Geometries for Black Holes from Intersecting branes

It is well-known by now that the various black hole solutions of supergravities in various dimensions have a ten- or eleven-dimensional interpretation as IIA, IIB or M-theory intersecting branes [6]. This has led to the better understanding of the black hole solutions preserving less than half of the eleven-dimensional supersymmetry. As we shall explain, this interpretation is also helpful to determine some of the structure of the moduli spaces of $n = 4, a = 1$ and $n = 8, a = 1$ black holes. In particular, the ten-dimensional interpretation that we shall present below allows us (i) to determine the underlying worldvolume Lorentz invariance and (ii) to find the appropriate one-dimensional supersymmetry multiplet of the effective theory of the black holes. Both points require some explanation. To explain the former, we remark that the dimensionality of the effective theory of the ten-dimensional solution is determined by the residual Lorentz invariance that remains unbroken by the solution. If it turns out that the Lorentz invariance group of the ten-dimensional solution is that of two or more dimensions, then the effective action will resemble that of an extended object, i.e. that of a string or a brane. This implies that the effective theory of the corresponding black holes is obtained by reducing the effective theory of the extended object to one dimension. However, as we have seen in section 2, the supersymmetry multiplets in one dimension that can be obtained as reductions of the two-dimensional ones have a geometry that is more restrictive than that of the generic one-dimensional supersymmetry multiplets. This results in stronger constraints on the geometry of the moduli of the black hole solution. To explain the latter point, we note that there are supersymmetry projection operators associated with the ten-dimensional solution. A close investigation of these operators reveals the type of one-dimensional supersymmetry
multiplet which should be used to describe the effective theory of the associated black holes.

Let us begin with the \( n = 4, a = 1 \) black hole solution. The most symmetric ten-dimensional lifting of this solution is that of the solution of IIB supergravity having the interpretation of two 3-branes intersecting on a string. This solution is known to preserve \( 1/4 \) of the supersymmetry of IIB theory. The explicit form of the solution in the string frame is

\[
ds^2 = H_1^{-\frac{1}{2}} H_2^{-\frac{1}{2}} ds^2(E^{(1,1)}) + H_1^{\frac{1}{2}} H_2^{\frac{1}{2}} ds^2(E^2) + H_1^{\frac{1}{2}} H_2^{\frac{1}{2}} ds^2(E^4)
\]

\[
G_5 = F_5 + *F_5 ,
\]

where

\[
F_5 = \omega_1(E^{(1,3)}) \wedge dH_1^{-1} + \omega_2(E^{(1,3)}) \wedge dH_2^{-1} ,
\]

(6.1)

\[
H_1, H_2 \text{ are two harmonic functions on } E^4 \text{ associated with the two 3-branes, } \omega_1(E^{(1,3)}) \text{ and } \omega_2(E^{(1,3)}) \text{ are the volume forms along the world-volume directions of the two 3-branes and the Hodge duality operation is taken with respect to the metric (6.1). If we reduce this solution along the relative transverse directions, then we get a self-dual string solution in six dimensions which is determined by two harmonic functions [26]. (For } H_1 = H_2, \text{ we recover the self-dual string solution of [25].) Now if we further reduce along the string direction and set } H_1 = H_2, \text{ we find the } n = 4, a = 1 \text{ black hole solution of section 2.}
\]

To continue, let us suppose that the string lies in the directions 0, 1, the first 3-brane lies in the directions 0, 1, 2, 3 and the second 3-brane lies in the directions 0, 1, 4, 5. The supersymmetry projections associated to the solution (6.1) are

\[
\Gamma_{0123} \eta^1 = \eta^2
\]

\[
\Gamma_{0145} \eta^1 = \eta^2 ,
\]

(6.3)

where \( \eta^1, \eta^2 \) are Majorana-Weyl spinors and \( \{ \Gamma_a; a = 0, \ldots, 9 \} \) are the ten-dimensional Gamma-matrices. Since the solution has a two-dimensional Lorentz in-
variance and preserves 1/4 of the IIB supersymmetry, the effective theory must be two-dimensional with eight supersymmetry charges. However as we have seen there are at least two different two-dimensional supersymmetry multiplets with eight supersymmetry charges, i.e. the (4,4) multiplet and the (8,0) one. An examination of the supersymmetry conditions (6.3) reveals that the two-dimensional chirality operator $\Gamma_{01}$ does not have a definite sign when acting on the Killing spinors $\eta$. This leads to a non-chiral effective theory on the string with (4,4) supersymmetry and therefore to a geometry on the target space with two copies of a strong HKT structure. The effective theory of the associated black hole is just the reduction of the two-dimensional effective theory along the spatial world-sheet direction. This of course leads to a one-dimensional effective theory based on the N=8a multiplet for which, as we have seen, its geometry is entirely determined by that of the two-dimensional (4,4) multiplet. It is remarkable that this is exactly what we have found by studying the Shiraishi metric on the moduli space of $n = 4, a = 1$ black holes.

Next let us turn to examine the $n = 8, a = 1$ black hole case. The most symmetric description of this black hole solution in ten dimensions is the IIA supergravity solution with the interpretation of a wave on a string. The explicit ten-dimensional solution in the string frame is

$$ds^2 = H_1^{-1}(du dv + (H_2 - 1)du^2) + ds^2(\mathbb{E}^8)$$

$$G_3 = dt \wedge dy \wedge dH_1^{-1}$$

$$e^{2\Phi} = H_1^{-1},$$

where $u = t + y$ and $v = -t + y$, and $H_1, H_2$ are the harmonic functions on $\mathbb{E}^8$ associated with the string and the pp-wave, respectively. This solution preserves 1/4 of the IIA supersymmetry and reduces along the string direction $y$ to a nine-dimensional black-hole solution with two harmonic functions $H_1, H_2$. If we next set $H_1 = H_2$, then we recover the $n = 8, a = 1$ black hole solution of section 2.

As in the previous case, let us consider the supersymmetry projections associated with the wave-on-a-string solution of IIA supergravity above. Assuming that
the string lies in the directions 0, 1, we find
\[ \Gamma_{01} \Gamma_{11} \eta = \eta \]
\[ \Gamma_{01} \eta = \eta , \]  
(6.5)

where $\eta$ is a ten-dimensional Majorana spinor. The solution (6.4) has two commuting Killing vectors along the the $u, v$ directions but no two-dimensional Lorentz invariance. From this, we conclude that the effective theory is best described by a one-dimensional sigma model. Since the ten-dimensional solution preserves 1/4 of the supersymmetry, the associated effective theory must have $N=8$ one-dimensional supersymmetry. So it remains to be determined whether the appropriate multiplet is the $N=8a$ or the $N=8b$. One way to find out which of the two multiplets is the relevant one is to observe that, from the string perspective, the supersymmetry preserved is chiral, because in (6.5) the two-dimensional chirality operator $\Gamma_{01}$ acts on $\eta$ with a definite sign. But as we have explained in section 3, the one-dimensional multiplet that is associated with chiral two-dimensional supersymmetry with eight supercharges is $N=8b$. It is remarkable that this is exactly what we have found by analysing the Shiraishi geometry of the moduli space of $n = 8, a = 1$ black holes.

7. Brane probes and black holes

The occurrence of non-trivial metrics on the moduli spaces for multiple $p$-branes may easily be seen using “test-brane” probes, i.e. actions for $p$-branes situated in backgrounds corresponding to other $p$-branes, which may be considered to be “heavy” and for which one may ignore, in leading order, the back-reaction of the probe on the background metric. In all the cases that we consider here, there is a “no-force” phenomenon for such probes: they see no potential arising from the background, as a result of a cancellation of forces due to gravity and scalars versus those from the antisymmetric tensor fields. The general action for a $p$-brane probe
with tension $T_\alpha$ and charge $Q_\alpha$ may be written

$$I_{\text{probe}} = -T_\alpha \int d^{p+1}\xi (-\det \partial_{\mu}x^m \partial_{\nu}x^n g_{mn})^{\frac{1}{2}} e^{\frac{1}{2}\xi^\mu i a_\alpha \phi/2} + Q_\alpha \int \tilde{A}_\alpha^{[p+1]}, \tag{7.1}$$

where

$$\tilde{A}_\alpha^{[p+1]} = (p+1)^{-1} \partial_{\mu_1} x^{m_1} \cdots \partial_{\mu_{p+1}} x^{m_{p+1}} A^{\alpha}_{m_1 \cdots m_{p+1}} d\xi_{\mu_1} \wedge \cdots d\xi_{\mu_{p+1}} \tag{7.2}$$

is the pull-back to the worldvolume of the gauge potential $A^\alpha$ that couples to the probe brane, $\tilde{a}_\alpha$ are the “dilaton vector” coupling parameters for the dilatonic scalars $\phi$ occurring in the kinetic-term exponential prefactor for $F^\alpha = dA^\alpha_{[p+1]}$, and $\zeta^{pr} = \pm 1$ according to whether the probe is of electric or magnetic type.

Three cases will suffice to illustrate the general phenomenon and make contact with the discussions of the preceding sections. In nine spacetime dimensions, one has an electrically-charged black hole solution, in whose background one may place an electrically-charged probe particle, which however is coupled to a different field strength (i.e. orthogonal to that of the background in charge space). Despite the fact that the background and the probe couple to different field strengths $F_\alpha$, a no-force condition is nevertheless obtained [38, 31]. For a probe coupling to a field strength $F^\alpha$ orthogonal to that excited in the background, the second term in (7.1) vanishes, and so the “potential” felt by the probe arises only from the first term. The dilatonic factor in (7.1) is $\exp\{\frac{1}{2}(a_{\text{probe}} \cdot a_{\text{back}}/|a_{\text{back}}|)\phi_{\text{back}}\}$. Specifically, in the $N = 2$ nine-dimensional maximal supergravity theory, the dot product of the dilaton vectors corresponding to the background and to the probe is $a_{\text{probe}} \cdot a_{\text{back}} = -\frac{12}{7}$, while the diagonal dot products are $a_\alpha \cdot a_\alpha = \frac{16}{7}$. For the electrically-charged black hole solution, the nine-dimensional metric $ds^2 = e^{2A}dx^\mu dx^\nu \eta_{\mu \nu} + e^{2B}dy^m dy^m$ has $e^A = H^{\frac{3}{7}}$, while the background dilatonic scalar is given by $e^{\phi_{\text{back}}} = H^{\frac{2}{\sqrt{7}}}$, where $H$ is the harmonic function governing the solution. Putting together the metric and dilatonic factors, one finds a potential $V_{\text{probe}} = e^A e^{-\frac{3}{2\sqrt{7}} \phi_{\text{back}}} = 1$, demonstrating the zero-force condition for this probe-background configuration.
Continuing on to the next order in the velocities, one finds a kinetic term for the moduli $-\frac{1}{2}T_\alpha e^{-2A}e^{2B}\partial^\mu y^m\partial_\mu y^m$, giving a moduli-space metric $H\delta_{mn}$. Similar analysis shows that the same result is found in the two other test-brane cases related to our earlier discussions: a nine-dimensional magnetic 5-brane in the background of another magnetic 5-brane, again corresponding to orthogonal field strengths, and similarly a six-dimensional electrically-charged string probe in a magnetic string background, once again corresponding to orthogonal field strengths. In all three of these cases, a constant potential is obtained, and in all three cases the moduli-space metric is given directly by the background’s harmonic function: $H\delta_{mn}$.

In order to compare these brane-probe results with the exact metrics given in (2.7) (noting that all three cases correspond to $a^2 = 1$), one should separate the center-of-mass and relative moduli in (2.7), giving a relative moduli metric for the two-center case

$$ds^2_{rel} = \frac{\pi^{\frac{n}{2} - 1}}{4(n - 2)\Gamma(n/2)} \left[ \frac{\mu_1\mu_2}{\mu_1 + \mu_2} + \frac{\mu_1\mu_2}{n - 2} \right] \left| d(x_1 - x_2) \right|^2 .$$  \hfill (7.3)

Taking the limit $\mu_1/\mu_2 \to \infty$ and dropping an overall factor $\frac{\pi^{\frac{n}{2} - 1}}{4(n - 2)\Gamma(n/2)}\mu_2$, one obtains $H\left| d(x_1 - x_2) \right|^2$, with

$$H = 1 + \frac{\mu_1}{(n - 2)|x_1 - x_2|^{n-2}} ,$$ \hfill (7.4)

in agreement with the brane-probe results.

The above test-brane discussion ignores back reaction effects of the probe on the underlying spacetime, so the result for the moduli space metric is not exact, i.e. it is only “asymptotically” valid. But for the three “crossed field strength” cases considered, this discussion is enough to establish the existence of a non-trivial moduli-space metric. This should be contrasted with the moduli-space metric for similarly-oriented parallel 2-branes in eleven dimensions [36] or the analogous interaction between two parallel 1-branes (strings) in ten dimensions [37]. In these
latter cases, there is a zero-force condition arising because of a cancellation between the second and first terms in (7.1). The moduli-space metric in these latter cases, however, proves to be flat. This flat metric may be understood as a consequence of the high degree of unbroken supersymmetry respected by the probe-background system. For similarly-oriented parallel eleven-dimensional 2-branes or ten-dimensional strings, the probe-background system leaves unbroken a full $\frac{1}{2}$ of the original rigid supersymmetry of the theory, which on a two-dimensional worldsheet corresponds to (8,8) supersymmetry. This degree of unbroken supersymmetry is too restrictive to allow anything other than a flat moduli-space geometry. The crossed-field-strength configurations, however, leave unbroken only $\frac{1}{4}$ of the supersymmetry, corresponding either to worldsheet supersymmetry (8,0) (for the nine-dimensional particles) or to (4,4) (for the nine-dimensional 5-branes or the six-dimensional strings).

The test-brane analysis captures some of the general features of the exact metrics that we have discussed in previous sections. In particular, the moduli-space metric seen by a test-brane probe is of the same geometrical class as the exact metric, although it represents only an asymptotic limit of the exact metric. This follows the pattern of the analogous discussion for moduli-space metrics for magnetic monopoles [23].

7.1. Other five-dimensional black holes

Another well known example of a five-dimensional supersymmetric black hole is that which arises in the computation of Bekenstein-Hawking entropy by counting string states [49]. This solution has the ten-dimensional interpretation of a D-string within a D-5-brane and a wave propagating along the D-string. The explicit solution in the string frame is

$$ds^2 = H_1^{-\frac{1}{2}}H_2^{\frac{1}{2}}(dudv + (H_3 - 1)du^2) + H_1^{\frac{1}{2}}H_2^{-\frac{1}{2}}dS^2(E^4) + H_1^{\frac{1}{2}}H_2^{\frac{1}{2}}dS^2(E^4)$$

$$G_3 = dt \wedge dy \wedge dH_1^{-1} + \ast dH_2$$

$$e^\Phi = H_1^{-\frac{3}{2}}H_2^{\frac{3}{2}}$$

(7.5)
where $u = t + y$ and $v = -t + y$, the Hodge duality operation is taken with respect to the overall transverse space $\mathbb{E}^4$, and $H_1$, $H_2$ and $H_3$ are the harmonic functions associated to the string, 5-brane and pp-wave respectively. This solutions preserves 1/8 of the supersymmetry of the IIB supergravity. Setting all the positions of the harmonic functions to be the same and then compactifying the solution to five dimensions along all the worldvolume directions of the D-5-brane, we find a black hole solution with finite horizon area. In the following we shall take $H_1$, $H_2$ and $H_3$ to have one centre.

Due to the interpretation of this black hole solution as a bound state of branes and a pp-wave, one can compute the metric induced on one of the branes if it is considered as a probe in the background generated by the remaining components. Since there are two different branes and a pp-wave in the bound state several possibilities exist and the corresponding probe metrics were computed in [43, 44]. In particular, if we consider D-5-brane probes in the background generated by a wave on the string, the induced metric on the D-5-brane is

$$ds^2 = H_3 H_1 ds^2(\mathbb{E}^4).$$  \hspace{1cm} (7.6)

If $H_3 = 1$, then the theory, from the string perspective, has (4,4) supersymmetry and the geometry on the moduli space is two commuting copies of strong HKT structures. This metric is the same as that on the relative moduli space of two $n = 4$, $a = 1$ black holes. Therefore its multi-black hole generalisations have already been discussed in the previous sections. The most interesting case arises when both $H_3$ and $H_1$ are not constant. In this case, from the two-dimensional perspective, the amount of supersymmetry preserved is (4,0). This can be easily seen from an analysis similar to that done for the black holes in the previous section. This leads to a one-dimensional effective theory with an N=4b supersymmetry multiplet. As we have already mentioned, one of the geometries compatible with this multiplet is the weak HKT geometry. In fact, it turns out that the relative moduli space of two such black holes admits a weak HKT structure. The metric
is as in (7.6) and the torsion is

\[ c = 3 \ast d(H_3 H_1) \]  

(7.7)

where the Hodge operation has been taken with respect to the flat \( \mathbb{E}^4 \) metric. The complex structures are those of \( \mathbb{E}^4 \).

It is straightforward to construct generalisation of the above geometry similar to that for the moduli metric of \( k \) black holes. For this we write the ansatz

\[ ds^2 = U_{ij} dx^i \cdot dx^j \]

\[ c = \frac{1}{2} \epsilon_{\mu \nu \lambda \rho} \partial_\rho U_{ij} dx^\mu i \wedge dx^\nu j \wedge dx^\lambda k , \]

(7.8)

where the coordinate is \( \{ x^\mu; i = 1, \ldots, k; \mu = 0, \ldots, 3 \} \) and \( U \) is a function of \( x \). This geometry is a weak HKT provided that

\[ U_{ij} = U_{ji} \]

\[ \partial_\mu U_{jk} = \partial_\mu U_{ik} , \]

(7.9)

with complex structures

\[ I_r = 1 \otimes I_r \]

(7.10)

where \( I_r \) are three complex structures on \( \mathbb{E}^4 \). A large class of asymptotically flat weak HKT geometries can be obtained by choosing

\[ U_{ij} = U^\infty_{ij} + \Delta U_{ij} \]

(7.11)

where \( U^\infty \) is the asymptotic value of \( U \) as \( |x^k| \to \infty \),

\[ \Delta U_{ij} = \sum_{\{p\}} \sum_a p_i p_j f_a(\{p\})(|p_k x^k - a(\{p\})|) \]

(7.12)

and \( f \) is any function that vanishes as \( |x^k| \to \infty \).

\* After this paper appeared, the metric on the moduli space of these five-dimensional black holes was obtained in [50].
8. Concluding Remarks

One motivation for obtaining the moduli space metrics is to find the geodesics and hence to study the scattering of solitons in the low velocity limit. We shall not attempt to find all the geodesics in all cases, but rather limit ourselves to making some general remarks in the cases with $a = 1$. The Lagrangian is given entirely by the kinetic energy $T$ which may be expressed as

$$T = T_0 + T_{n-2},$$

where both terms are positive and quadratic in velocities. The first is independent of the positions $x_i$ while the second is homogeneous of degree $n-2$ in the position variables.

The equations of motion are

$$\frac{d}{dt}p_i = \frac{\partial T_{n-2}}{\partial x_i},$$

where $p_i$ are the canonical momenta. Taking the dot product with $x_i$ and summing over $i$ gives

$$\frac{d}{dt}\left( \sum x_i \cdot p_i \right) = 2T - (n - 2)T_{n-2} = 2T_0 + (4 - n)T_{n-2},$$

where we have used Euler’s theorem on homogeneous functions.

Now assume that there are bound geodesics, i.e. geodesics that are confined inside a compact set for all time. A special case would be closed geodesics. We can then average this equation over a large time to obtain a virial type relation:

$$2\langle T_0 \rangle = (n - 4)\langle T_{n-2} \rangle,$$

where $\langle \ldots \rangle$ denotes a time average. If $n \leq 4$ we obtain an immediate contradiction since the left hand side is positive while the right hand side is negative or zero.
Thus if \( n = 3 \) or \( n = 4 \) there can be no bound geodesics for any number of solitons. By contrast if \( n > 4 \) no contradiction results, merely a statement about the ratio of the two contributions to the energy. Thus one might anticipate the existence of bound geodesics if \( n > 4 \). In the case of two solitons it is easy to see that there are (unstable) closed geodesics on the relative moduli space. This is because the asymptotically flat outer infinity (for which \( |x_1 - x_2| \to \infty \) is separated by a totally geodesic \((n-1)\)-sphere from the asymptotically conical infinity for which \( |x_1 - x_2| \to 0 \).

For more than two solitons it seems rather likely that one could rigorously establish the existence of closed geodesics using the Benci-Giannoni theorem [51] but we will not pursue this here. Geometrically it seems rather plausible that every closed geodesic will be unstable. One approach to studying this might be to examine the curvature of the metrics. It also seems rather likely that the geodesic motion and scattering will exhibit some of the chaotic features encountered in the closely related three-dimensional metrics studied in [53].

The classical moduli geometry of black holes may receive quantum corrections. Some of these corrections are due to the short-distance renormalisation effects of the associated effective theory. The cases that we have studied involve effective theories with \((4,4)\), \((8,0)\) and \((4,0)\) supersymmetries (in the two-dimensional sense). The moduli geometries with \((4,4)\) off-shell supersymmetry are protected against quantum corrections because of the non-renormalisation theorem of [8]. In fact, since these moduli geometries have constant complex structures, the Obata connection vanishes and the superfield constraints can be solved exactly in terms of prepotentials allowing for a manifest \((4,4)\)-supersymmetric perturbation theory. So the moduli metric of \( n = 4, a = 1 \) black holes and some of the probe metrics of section 7 are expected to be exact in all orders of perturbation theory. The same appears to apply for the moduli metric of \( n = 8, a = 1 \) black holes. Finally, the probe metrics in section 7 with \((4,0)\) supersymmetry may receive corrections. However these corrections are not due to ultra-violet divergences but rather to finite local counterterms that are necessary for the cancellation of anomalies in extended
supersymmetry transformations (for more details see [33]).

So far we have investigated the geometry on the moduli space of \( n = 4, a = 1 \) and \( n = 8, a = 1 \) black holes. The geometry of moduli space of the rest of the \( a = 1 \) black holes is rather unclear. It is likely that in the Shiraishi description of the moduli space of all \( 4 < n < 8, a = 1 \) black holes there are missing moduli. This is in direct analogy to the missing moduli in the case of \( n = 3, a = 1 \) black holes [23] (see also section 2). However, it is worth mentioning that all Shiraishi metrics on the moduli space of \( a = 1 \) black holes can be constructed by taking the quotient of the moduli space of \( n = 8, a = 1 \) black holes with a suitable group of translations. To see this let us first consider the case of the relative moduli, \( \mathcal{M}^{(\ell)}_4 \), of two \( n = \ell, a = 1 \) black holes. It is clear that these moduli can be identified with the quotient space \( \mathcal{N}_4/\mathbb{R}^{8-\ell} \), where \( \mathcal{N}_4 \) is the relative moduli space of two \( n = 8, a = 1 \) black holes and \( \mathbb{R}^{8-\ell} \) acts with translations on the first \( 8 - \ell \) coordinates of \( \mathcal{N}_4 \). More generally the moduli space of \( k n = \ell, a = 1 \) black holes, \( \mathcal{M}^{(\ell)}_{4k} \), can be identified with the quotient \( \mathcal{N}_{4k}/\mathbb{R}^{(8-\ell)k} \), where \( \mathcal{N}_{4k} \) is the relative moduli space of \( k n = 8, a = 1 \) black holes.

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APPENDIX

A.1. Harmonic Forms

The quantization of point particle mechanics may involve differential forms. Special interest is attached to $L^2$ harmonic forms. For simplicity, consider the relative moduli space of two solitons. By suitable rescalings the metric may be brought to the form

$$ds^2 = (1 + \frac{1}{r^{n-2}})(dr^2 + r^2d\Omega_{n-1}^2),$$

where $d\Omega_{n-1}^2$ is the standard round metric on $S^{n-1}$. As noted in the previous section, if $n > 4$ there is a totally geodesic $(n - 1)$-sphere located at that value of $r$ for which $r^2 + \frac{1}{r^{n-2}}$ is least. This suggest that if $n > 4$ there is an associated harmonic form $(n - 1)$-form equal to the volume form $\eta_{n-1}$ on $S^{n-1}$. As we shall now show, this is indeed true and moreover the form is in $L^2$. Obviously $\eta_{n-1}$ is closed:

$$d\eta_{n-1} = 0.$$

The Hodge dual form is given by

$$\star\eta_{n-1} = f(r)dr$$

for some function $f(r)$ whose precise form we don’t need. Evidently $\eta_{n-1}$ is co-closed and therefore harmonic. The $L^2$ norm of $\eta_{n-1}$ depends on the finiteness of the radial integral

$$\int_0^\infty \frac{dr}{r^{n-1}(1 + \frac{1}{r^{n-2}})^{(\frac{n}{2}-1)}}.$$

The integral is convergent at infinity as long as $n > 3$ For small $r$ the integrand goes like $r^{\frac{1}{2}(n^2-6n+6)}$. Thus as long as $n > 4$, $\eta_{n-1}$ is indeed an $L^2$ harmonic form.
A.2. **Harmonic Spinors**

A similar analysis, but now with a negative result, may be given for $L^2$ harmonic spinors. We use the conformal invariance of the massless Dirac equation. If $\psi_0$ is a solution of the massless Dirac equation in the flat metric

$$ ds = dr^2 + r^2 d\Omega_{n-1}^2 $$

then

$$ \psi = H^{-\frac{n+1}{2}} \psi_0 $$

is a solution on the conformally-rescaled metric

$$ ds = H\left(dr^2 + r^2 d\Omega_{n-1}^2\right). $$

The $L^2$ norm of $\psi$ depends on the radial integral:

$$ \int_0^{\infty} drr^{n-1} H^{\frac{1}{2}} \bar{\psi}_0 \psi_0. $$

This becomes

$$ \int_0^{\infty} drr^{n-1}(1 + \frac{1}{r^{n-2}})^{\frac{1}{2}} \bar{\psi}_0 \psi_0. $$

To obtain convergence at the upper end we must choose $\psi_0$ to decay at large $r$. It follows that $\psi_0$ must be a linear combination of terms of the form

$$ \frac{1}{r^{l+n-1}} \chi_l $$

where $l = 0, 1, \ldots$ and $\chi_l$ is an appropriate spinor harmonic on $S^{n-1}$ [52]. The
spinor field $\psi$ will be in $L^2$ if
\[
\frac{1}{r^{2l+2n-2}} r^2 \psi
\]
is integrable as $r \to 0$. This requires:
\[
2l + \frac{3}{2} n - 2 < 1
\]
which can never be satisfied for any $n$. Given that one cannot find an $L^2$ harmonic spinor on the relative moduli space of two solitons it seems rather unlikely that one can find one on the higher dimensional moduli spaces but we have no general proof. Actually the previous result about the relative moduli space of two solitons may be obtained more simply by invoking Lichnerowicz’s well known result concerning harmonic spinors on spaces with non-negative Ricci scalar in the cases $n = 4$ and $n = 5$. The point is that since the function $H$ is harmonic, the Ricci scalar $R$ of the relative moduli space is given by
\[
R = -(n - 1) \frac{(n - 6)}{4} \frac{(\nabla H)^2}{H^3}.
\]
Thus if $n = 4$ or $n = 5$ the Ricci scalar is positive and Lichnerowicz’s argument applies.
REFERENCES

1. K Shiraishi, Nucl. Phys. B402 (1993) 399-410.
2. G W Gibbons & P J Ruback, Phys. Rev. Lett. 57 (1986) 1492.
3. P J Ruback, Commun. Math. Phys. 107 (1986) 93.
4. R C Ferrell & D M Eardley, Phys. Rev. Lett. 59 (1987) 1617.
5. A G Felce & T M Samols, Phys. Letts. B308 (1993) 30: hep-th/921118.
6. G. Papadopoulos & P.K. Townsend, Phys. Lett. B380 (1996) 273.
7. G. W. Gibbons & R. Kallosh, Phys. Rev. D51 (1995) 2839.
8. P.S. Howe & G. Papadopoulos, Nucl. Phys. B289 (1987) 264; Class. Quantum Grav. 5 (1988) 1647.
9. C.M. Hull, Lectures on Nonlinear Sigma Models and Strings, Lectures given in the Super Field Theories workshop, Vancouver Canada (1986), published in Vancouver Theory Workshop.
10. N.J. Hitchin, A. Karlhede, U. Lindström & M. Roček, Commun. Math. Phys.108 (1987) 535.
11. P.S. Howe & G. Papadopoulos, Phys. Lett. B379 (1996) 80.
12. J. Harvey & A. Strominger, Phys. Rev. Let. 5 (1991) 549.
13. T.A. Ivanova, Phys. Lett. B315 (1993) 277.
14. M. Günaydin & H. Nicolai, Phys. Lett. B351 (1995) 169: hep-th/9502009; Addendum-ibid B376 (1996) 329.
15. M.J. Duff, J.M. Evans, R.R. Khuri, J.X. Lu, & R. Minasian, The Octonionic Membrane hep-th/9706124.
16. B. Zumino, Phys. Lett. B87 (1979) 203.
17. S.J. Gates, C.M. Hull & M. Roček, Nucl. Phys. B248 (1984) 157.
18. R. Coles & G. Papadopoulos, Class. Quantum Grav. 7 (1990) 427.
19. G. Gibbons, *Nucl. Phys.* **B207** (1982) 337.

20. A.A. Tseytlin, *Nucl. Phys.* **B475** (1996) 149: hep-th/9604033.

21. J.P. Gauntlett, D.A. Kastor & J. Traschen, *Nucl. Phys.* **B478** (1996) 544: hep-th/9604179.

22. A.A. Tseytlin, *Mod. Phys. Lett.* **A111** (1996) 689.

23. G.W. Gibbons & N. Manton, *Phys. Lett.* **B356** (1995) 32: hep-th/9506052.

24. J. P. Gauntlett, J. A. Harvey, M. M. Robinson & D. Waldram *Nucl Phys B411* (1994) 461.

25. M.J. Duff & J.X. Lu, *Nucl. Phys.* **B416** (1994) 301.

26. G. Papadopoulos, *The Universality of M-branes*, Talk given at the Imperial College Workshop on *Gauge Theories, Applied Supersymmetry and Quantum Gravity* (1996), hep-th/9611029.

27. P.S. Howe & G. Papadopoulos, *Commun. Math. Phys.* **151** (1993) 467.

28. C. G. Callan, J. A. Harvey & A. Strominger, *Nucl. Phys.* **B359** (1991) 611.

29. R. D’Auria & T. Regge, *Nucl. Phys.* **B195** (1982) 308.

30. S. J. Rey, *Phys. Rev.* **D43** (1991) 526.

31. J. Rhamfeld, *Phys. Lett.* **B372** (1996) 198.

32. G.W. Gibbons, R.H. Rietdijk & J.W. van Holten, *Nucl. Phys.* **B404** (1993) 42.

33. P.S. Howe & G. Papadopoulos, *Nucl. Phys.* **B381** (1992) 360.

34. M. Roˇcek, K. Schoutens & A. Sevrin, *Phys. Lett.* **B265** (1991) 303.

35. F. de Jonghe, K. Peeters & K. Sfetsos, *Class. Quantum Grav.* **14** (1997) 35.

36. M.J. Duff & K.S. Stelle, *Phys. Lett.* **B253** (1991) 113.

37. A. Dabholkar, G.W. Gibbons, J.A. Harvey & F. Ruiz-Ruiz, *Nucl. Phys.* **B340** (1990) 33.
38. N. Khviengia, Z. Khviengia, H. Lü & C.N. Pope, *Phys. Lett.* **B388** (1996) 21;
M.J. Duff & J. Rahmfeld, *Nucl. Phys.* **B481** (1996) 332.

39. T. Buscher, *Phys. Lett.* **B201** (1988) 466; **B194** (1987) 59.

40. I. Bakas & K. Sfetsos, *Phys. Lett.* **B349** (1995) 448.

41. E. Kiritsis, C. Coumans & D. Lüst *Int. Journ. Mod. Phys.* **A9** (1994) 1361.

42. J.P. Gauntlett, G.W. Gibbons, G. Papadopoulos & P.K. Townsend, *Hyper-Kähler manifolds and multiply intersecting branes* Nucl. Phys. (1997), to appear: hep-th/9702202.

43. M. R. Douglas, D. Kabat, P. Pouliot & S.H. Shenker, *D-branes and Short Distances in String Theory* hep-th/9608024.

44. M. Douglas, J. Polchinski & A. Strominger, *Probing Five-Dimensional Black Holes with D-branes*, hep-th/9703031.

45. G. Bonneau & G. Valent, *Local heterotic geometry in holomorphic coordinates*, hep-th/9401003.

46. G. Papadopoulos, *Phys. Lett.* **B356** (1995) 249.

47. G.W. Gibbons & S.W. Hawking, *Phys. Lett.* **78B** (1978) 430.

48. G.W. Gibbons & P. J. Ruback, *Commun. Math. Phys.* **115** (1988) 267.

49. C.G. Callan & J. Maldacena, *Nucl. Phys.* **B472** (1996) 591: hep-th/9602043.

50. D.M. Kaplan and J. Michelson, *Scattering of several multiply charged extremal D=5 black holes*, hep-th/9707021.

51. V. Benci & F. Giannoni, *Duke Math. Journ.* **68** (1992) 195.

52. S.R. Das, G.W. Gibbons & S.D. Mathur, *Phys. Rev. Letts.* **78** (1997) 417: hep-th/9609052.

53. N. Cornish & G.W. Gibbons, *Class. Quantum Grav.* **14** (1997) 1865: gr-qc/9612060.