TEASING: a fast and accurate approximation for the low multipole likelihood of the Cosmic Microwave Background temperature

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ABSTRACT

We explore the low-ℓ likelihood of the angular spectrum \( C_\ell \) of masked CMB temperature maps using an adaptive importance sampler. We find that, in spite of a partial sky coverage, the likelihood distribution of each \( C_\ell \) closely follows an inverse gamma distribution. Our exploration is accurate enough to measure the inverse gamma parameters along with the correlation between multipoles. Those quantities are used to build an approximation of the joint posterior distribution of the low-ℓ likelihood. The accuracy of the proposed approximation is established using both statistical criteria and a mock cosmological parameter fit. When applied to the WMAP5 data set, this approximation yields cosmological parameter estimates at the same level of accuracy as the best current techniques but with very significant speed gains.

Key words: cosmic microwave background – methods: data analysis – methods: statistical

1 INTRODUCTION

The CMB angular spectrum \( C = \{C_\ell\} \) is a central quantity for conducting statistical inference based on CMB observations (Bond & Efstathiou 1987). The high resolution of available (Hinshaw et al. 2009) and forthcoming CMB observations (Efstathiou et al. 2005) makes it necessary (at least in the case of partial sky coverage) to adopt a processing scheme in which the low-ℓ and high-ℓ parts of the data are processed independently (Efstathiou 2006). This paper addresses the large scale part of the problem: inference regarding low multipoles based on a partial low-resolution CMB map.

After defining the problem of low-ℓ pixel-based likelihood and introducing some notations (Sec. 2), we first show how to build a (large) set of \( N \) importance samples of the angular spectrum such that all integrals of interest for statistical inference can be approximated by Monte-Carlo estimates (Sec. 3). Based on those results, we propose in Sec. 4 a new approximation to the likelihood for partially observed low-resolution CMB maps. This approximation was initially built as part of the importance sampler but it turns out to be so accurate that it is of independent interest. This paper and the recent reference (Rudjord et al. 2008) are similar in spirit but differ in the sampling method and in the proposed likelihood approximation.

2 LIKELIHOOD

We recall some well-known facts about the likelihood of the angular spectrum of a CMB temperature map.

In the ideal case of noise-free, beam-free, full-sky map (represented by the vector \( x \) of pixels), one has direct access to the harmonic coefficients \( a_{\ell m} \) of the sky. Assuming an isotropic Gaussian field, the empirical angular spectrum \( \hat{C}_\ell = \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2 \) is a sufficient statistic for the data and their probability distribution takes the factorized form (Bond et al. 2000):

\[
p(x|C) \propto \prod_{\ell \geq 0} \exp \left( -\frac{2\ell + 1}{2} \left( \frac{\hat{C}_\ell}{\hat{C}_\ell} + \log C_\ell \right) \right). \tag{1}
\]

In the case of a flat prior \( p(C) \), expression (1) combined with Bayes rule \( p(C|x) = p(x|C)p(C)/p(x) \) reveals that, given \( x \), the angular spectrum \( C \) is distributed as a product of inverse gamma densities:

\[
p(C|x) = \prod_\ell i\Gamma(C_\ell; \alpha_\ell, \beta_\ell) \tag{2}
\]

\[
i\Gamma(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \tag{3}
\]

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with parameters $\alpha_\ell = (2\ell - 1)/2$ and $\beta_\ell = (2\ell + 1)C_\ell/2$.

Such a factorization does not hold when only a fraction of the sky is observed (or has to be ignored because of excessive contamination by foregrounds), or when the stationary CMB is contaminated by non-stationary noise (Gorski 1994; Tegmark 1997). However, for small sky masks and/or small deviations from stationarity, deviations from the factorized form (1) are expected to be small, suggesting the new likelihood approximation developed in Sec. 4.

Pixel-based likelihood. We turn to the actual case of interest: partial sky coverage, presence of independent additive Gaussian noise, low-pass effect of a beam. The data set, represented by an $N_{\text{pix}} \times 1$ vector $x$ of pixel values, can no longer be losslessly compressed into a sufficient spectral statistic $C_\ell$. Rather, one must use the plain Gaussian density:

$$p(x|R) = |2\pi R|^{-1/2} e^{-\frac{1}{2} x^T R^{-1} x}$$  (4)

where the covariance matrix $R$ of $x$ has contributions from the CMB signal and from noise. For two pixels $i$ and $j$ with angular separation $\theta_{ij}$, the CMB part of the covariance matrix has an $(i, j)$ entry given by (Bond et al. 2000)

$$\sum 2\ell + 1 \frac{2\ell + 1}{4\pi} W_\ell C_\ell P_k(\cos \theta_{ij})$$  (5)

where $P_k$ is the Legendre polynomial of order $\ell$ and where the window function $W_\ell$ can represent e.g. the spectral response of an azimuthally symmetric beam, or more generally the convolution of the signal with any azimuthally symmetric kernel. Hence, we ignore the complications due to an anisotropic beam as well as the presence of residual foreground contaminants.

The noise part of the covariance matrix could take any form but, in this work, it is taken to correspond to an isotropic noise with angular spectrum $N_\ell$. We can thus define a total angular spectrum $D_\ell$

$$D_\ell = W_\ell C_\ell + N_\ell$$  (6)

which is unambiguously related to $C_\ell$ since the beam $B_\ell$ and the noise spectrum $N_\ell$ are assumed to be known.

Free parameters. In practice, we consider a more restricted model for the covariance matrix of the observed pixels. First, the adjustable multipoles are restricted to a range $\ell_{\text{min}} \leq \ell \leq \ell_{\text{max}}$ while other multipoles are kept at constant values. Second, we only consider uncorrelated noise with zero mean and variance $\sigma^2$ per pixel. It contributes a term $\sigma^2 \delta_{ij}$ to $R$ and corresponds to a flat angular spectrum $N_\ell = \sigma^2 / \Omega_{\text{pix}}$ if all pixels have the same area $\Omega_{\text{pix}}$. Then, the covariance matrix of $x$ as a function of $D = \{D_\ell\}_{\ell = \ell_{\text{min}}}^{\ell_{\text{max}}}$ is spilled out as $R(D) = R_{\text{var}}(D) + R^{\text{pix}}$ with

$$R_{ij}^{\text{var}}(D) = \sum_{\ell = \ell_{\text{min}}}^{\ell_{\text{max}}} 2\ell + 1 \frac{2\ell + 1}{4\pi} (D_\ell - N_\ell) P_k(\cos \theta_{ij})$$  (7)

$$R_{ij}^{\text{pix}} = \sum_{\ell \text{ fixed}} 2\ell + 1 \frac{2\ell + 1}{4\pi} W_\ell C_\ell P_k(\cos \theta_{ij}) + \sigma^2 \delta_{ij}$$  (8)

Priors and posterior distributions. In all the following, the prior distribution on $D$ is taken to be flat for $D_\ell \geq N_\ell$.

At all angular frequencies such that $W_\ell C_\ell \gg N_\ell$ (figure 1 illustrates the values used in this paper), this is almost identical to a flat prior on the positive values of $C_\ell$. The posterior distribution of $D$ given the data $x$ is

$$\pi(D) = p(D|x) \propto p(x|R(D)) \prod_{\ell = \ell_{\text{min}}}^{\ell = \ell_{\text{max}}} 1(D_\ell \geq N_\ell)$$

where $p(x|R(D))$ is evaluated using eqs (4), (7) and (8).

About noise and regularization. On a cut sky, the CMB part of the covariance matrix may be poorly conditioned with a trough in its eigenvalue spectrum corresponding to those modes which are mostly localized in the cut. In this case, it is customary (Erkens et al. 2007; Hinshaw et al. 2007) to add a very small amount of noise to the data and to add the corresponding contribution to the covariance matrix as in eq. (8). Another reason for adding uncorrelated noise is to cover spurious noise correlation possibly introduced when the observed sky map is downgraded and to simplify the noise structure (Dunkley et al. 2009). See figure 1 for the values used in our experiments. Another possibility is regularization by projection onto the most significant eigenvectors of the covariance matrix (Bond et al. 2000) but this possibility is not considered here.

3 BUILDING A SAMPLE OF THE LOW-\ell POSTERIOR WITH IMPORTANCE SAMPLING

This section reports on the construction of importance samples of the $C_\ell$ under their joint posterior for two data sets. The principle of importance sampling is first briefly recalled in section 3.1 our specific technique (an adaptive variant) is described in section 3.2 and applied to a synthetic CMB cut sky map (sec. 3.3) and to the official WMAP5 low resolution map (sec. 3.4).
3.1 Importance sampling

Importance sampling is a well established technique to explore a probability distribution when no method for directly sampling from it is available (the well known VEGAS algorithm [Lepage 1978] for instance, is based on importance sampling). Consider estimating the expectation $E_f(x) = \int f(x)\pi(x)dx$ of some function $f$ of $x$ when the random variable $x$ is distributed under $\pi$. If $x_i, i = 1, N$ are $N$ samples of $x$, then $E_f(x)$ can be estimated by the sample average $\frac{1}{N} \sum_i f(x_i)$. In contrast, importance sampling relies on samples $x_i$ distributed under a proposal distribution $g$ not necessarily equal to $\pi$. If the support of $g$ includes the support of $\pi$ then

$$E_f = \int f(x)\pi(x)dx = \int f(x)\frac{\pi(x)}{g(x)}g(x)dx$$

so that, if the samples $x_i$ are distributed under $g$, then $E_f$ is estimated without bias by

$$\frac{1}{N} \sum_{i=1}^{N} w_i f(x_i) \quad \text{where} \quad w_i = \frac{\pi(x_i)}{g(x_i)}$$

The factors $w_i$ are called importance weights.

Monte-Carlo integration reaches its maximum efficiency when the samples are drawn independently under a proposal distribution $g$ which is identical to the target distribution $\pi$. While MCMC methods try to draw from the target distribution $\pi$, they do not build independent samples; in contrast, importance sampling (usually) relies on independent draws from an approximate distribution $g$ and corrects the discrepancy using importance weights $w_i$. Therefore, importance sampling should outperform MCMC methods whenever independent samples can be drawn from a proposal distribution which is “close enough” to the target.

The agreement between target and proposal distributions can be measured by the Kullback-Leibler divergence

$$K(\pi|g) \equiv \int \log \frac{\pi(x)}{g(x)} \pi(x)dx,$$

which is often remapped as the so-called perplexity criterion: $P(\pi|g) \equiv \exp-K(\pi|g)$ so that perfect agreement is reached when $P = 1$. Another criterion is the effective sample size (ESS) of an importance sample:

$$ESS = \frac{\left(\sum_i w_i\right)^2}{\sum_i w_i^2}$$

If the proposal matches the target perfectly, then $ESS = N$, otherwise it is smaller than the number of importance samples. The effective sample size is directly related to the variance of the MC estimates.

Importance sampling is well fitted to the problem at hand for at least two reasons: ease of parallelization and availability of a good proposal distribution.

Parallelization is a strong requirement due to the high computational cost of CMB studies. We are planning to sample a 30- to 40-dimensional space, and the computation of the likelihood for a given angular spectrum costs about 5 seconds for $\ell_{\text{max}} = 48$ and $N_{\text{pix}} = 3072$ on a typical 2GHz CPU. Since importance sampling can be trivially parallelized, it makes it straightforward to take full advantage of CPU clusters. For instance, computing $10^5$ samples would take about 4 days on a single CPU but is reduced to mere hours on a cluster. The Markov-Chain Monte-Carlo algorithm cannot be parallelized as easily. Indeed, to be able to mix different parallel chains, one has to ensure that they have correctly converged [Rosenthal 2000], which can be a difficult task in 30 to 40 dimensions.

Regarding the proposal distribution, one can draw inspiration from the noise-free, full-sky case [2] since a mask hiding less than 20% of the sky and a high signal to noise situation are expected to modify it only slightly [3]. Indeed, as demonstrated below, a product of independent inverse gamma distributions turn out to be a very efficient proposal distribution, provided it is correctly tuned. Such a tuning is achieved via an adaptive importance sampling, as explained next.

3.2 An adaptive importance sampling algorithm

Importance sampling is efficient only if the proposal distribution is close enough to the target, an objective which may be difficult to reach in large dimensions (sampling angular spectra in the range $0 \leq \ell \leq 40$ qualifies as large problem). To tackle this complexity, we resort to adaptive importance sampling which consists in running a sequence of importance runs in which the proposal distribution is improved at each run based on the results of previous runs. A more detailed description of adaptive importance sampling (based upon the PMC algorithm from Cappé et al. [2008]) in the context of cosmology can be found in Wraith & et al. [2009].

General scheme. The general scheme, based on a parametric family of proposal distributions $g(y; \theta)$, is as follows:

(i) Start with the best available guess of $\theta$ for the parameters of the proposal distribution.

(ii) Sample under $g(y; \theta)$. Compute and store the importance weights.

(iii) Re-estimate $\theta$ so that $g(y; \theta)$ best matches the current sample set.

(iv) If the (estimated) perplexity $P(\pi(y)|g(y; \theta))$ is high enough (e.g. above 0.5) or if it has not changed significantly during the last iterations, exit to (v). Otherwise, go to (ii) for another importance run with the re-estimated parameters.

(v) Use the last value of $\theta$ for a large final importance sampling run.

Sampling angular spectra. In our experiments, we sample the total angular spectrum, that is, $y = D = \{D_\ell\}_{\ell=\ell_{\text{min}}}^{\ell=\ell_{\text{max}}}$ and use independent inverse gamma distributions for the proposal:

$$g(D; \theta) = \prod_{\ell=\ell_{\text{min}}}^{\ell=\ell_{\text{max}}} \Gamma(D_\ell; \alpha_\ell, \beta_\ell).$$

Hence we must adapt a vector $\theta = \{\alpha_\ell, \beta_\ell\}_{\ell=\ell_{\text{min}}}^{\ell=\ell_{\text{max}}}$ of $2(\ell_{\text{max}}-\ell_{\text{min}}+1)$ parameters. As a starting point at step (i) we use

$$\alpha_\ell = \frac{(2\ell+1)}{2} f_{\text{sky}} - 1, \quad \beta_\ell = \frac{(2\ell+1)}{2} f_{\text{sky}} D_\ell^{ML}.$$
where $D^\text{ML}_{\ell}$ is the maximum likelihood estimate of the angular spectrum. At step (iii) parameters $\alpha_{\ell}$ and $\beta_{\ell}$ are re-estimated at their maximum likelihood values (see appendix).

The target density $\pi(D)$ is the posterior distribution of $D$ when the prior distribution of $D$ is flat. Hence, it is proportional to the likelihood.

In the two examples presented below, this iterative algorithm reached a perplexity above 0.6 after the first step of 50k samples and a 500k samples set was produced during the final sampling phase.

### 3.3 Synthetic map

We first describe the results of adaptive importance sampling runs on a synthetic CMB map. The map is prepared at resolution $N_{\text{side}} = 16$ from the WMAP5 best fit power spectrum [Dunkley et al. (2009)] using HEALPix (Górski et al. 2005). To avoid aliasing small scale power into large scale modes, the map is smoothed prior to down-sampling using a synthetic window function $w_{\ell}$:

$$W_\ell = \begin{cases} 1 & 0 \leq \ell \leq 40 \\ \frac{1 + \cos((\ell-40) \times \frac{\pi}{8})}{2} & 40 \leq \ell \leq 48 \\ 0 & 48 \leq \ell \end{cases}$$

which is used to explore the posterior of $C_\ell$ up to $\ell = 40$.

The posterior of the power spectrum is given by the likelihood described in Eq. (1), with a flat prior. The Galactic region is excluded using the WMAP5 mask, hiding 18% of the sky. The map is shown in figure 2. A 1µK/pixel noise is taken into account in the likelihood, but no noise is actually added to the map. This level should not affect our results as it is much lower than $\Omega_{\text{pix}}C_{40}$ (see figure 1). We build a sample of the posterior of the masked map using the adaptive importance sampling algorithm described above. We only explore $\ell = 2$ to 40, the other modes ($\ell = 0, 1$ and $41 \leq \ell \leq 48$) being held constant to the ML estimate.

The initial proposal is given by the product of independent inverse gamma distributions, as described in 3.2 centered at $D^\text{ML}_{\ell}$ with a width given by an effective sky coverage equal to $f_{\text{sky}} \times 0.98$ to ensure that the initial proposal is wide enough.

Only one adaptation step was needed. It took about 58min on 80 2 GHz CPUs to produce the first 50k samples (about 6sec for each likelihood evaluation, taking into account all overheads). The final 500k samples run took 6 hours and 21 minutes on 120 2 GHz CPU (about 5.5sec for each likelihood evaluation, taking into account all overheads). The adaptive algorithm behaved very well: the first step reached $P = 0.68$ while the second run hit $P = 0.93$. This last run had an effective sample size $\text{ESS} = 437029$, i.e. a ratio $\text{ESS}/N = 0.874$.

Figures. (3), (4) give an overview of the results. First, looking at the 1D marginal distributions, figure (3) shows a few marginals ($\pi_\ell$) and their best inverse gamma fits. The inverse gamma model is seen to account very well for both the tails and the mode of the distribution, in line with the high perplexity reached in the last iteration. This agreement validates a posteriori the adaptive approach. On this synthetic map, at least, the marginals follow closely an inverse gamma distribution.

The peaks of the marginals and an effective sky coverage at multipole $\ell$, denoted $f_\ell$, are obtained by inverting

$$\alpha_{\ell} = \frac{(2\ell + 1)}{2} f_\ell - 1$$

$$\beta_{\ell} = \frac{(2\ell + 1)}{2} f_\ell \left( W_\ell C^\text{peak}_{\ell} + N_\ell \right).$$

Both quantities are shown in figure (4). The $C^\text{peak}_{\ell}$ and $C^\text{ML}_{\ell}$ discrepancy is small; it is below the percent order, albeit with a few modes disagreeing by at most 3%. The effective sky coverage, however, is quite different from $f_{\text{sky}}$. Its behaviour indicates a transition between scales that are not affected significantly by the cut, and scales that are smaller than the cut, so that their deficit of modes is given by $f_{\text{sky}}$. Our resolution is probably not good enough to reach this regime.

One would expect some discrepancy between the $C^\text{peak}_{\ell}$ and the ML estimate. Indeed, since the cut induces correlation between scales, there is no reason for the peak of the posterior to be identical to the peak of the marginals in each direction. The small discrepancy can only be explained by a low level of correlation between the $C_\ell$s, so that the peak of the marginals is close to the joint peak. As a first estimate of the correlation, figure (5) shows the correlation matrix measured on our sample

$$[V]_{\ell,\ell'} \equiv \text{Corr} (C_\ell, C_{\ell'}).$$

In this figure, the diagonal of the matrix is removed so as not to dominate the off diagonal terms. Those exhibit a pattern below the 6% level. Most of the correlation is located around $\ell = 12$, and the correlation seems to extend significantly for about 6 modes off the diagonal.

Several checks can be performed to assess the accuracy of this matrix. First, the effective sample size allows us to estimate the error on the matrix measurement to be of the order of 0.15%, which is well below the observed correlation pattern. One can also measure the correlation matrix on the results of the first iteration of the adaptive algorithm, which provides us with an independent exploration of the posterior. The noise was much higher (with a level, according to the $\text{ESS}$ of this run of about 0.6%), but the pattern observed on figure (3) is easily recovered. Finally, we checked on a full sky run that no correlation pattern is visible.
3.4 WMAP5 map

We perform a similar experiment using the WMAP map distributed along with the five year WMAP likelihood code found on the Lambda website\(^2\). The setting is slightly different, since the window function is a 9.18° Gaussian beam, cutting much more high frequency power than the window function (11) (see figure 1). Therefore, only the range \(2 \leq \ell \leq 30\) is explored here, with the other multipole powers held constant at their ML values. As done in the WMAP likelihood code, a 1\(\mu\)K/pixel noise is added to the data and to the model. We take care of adding the specific noise realization used in the likelihood code. Indeed, with the beam used, the signal to noise at \(\ell = 30\) is only \(\sim 14\) and our tests have shown a small dependency of the value of the higher \(C_\ell\)s on the noise realization.

As in the previous run, only one adaptation step turns out to be needed. It took 32 minutes on 120 CPUS for 50k samples, while the second and final run produced 500k samples in 5 hours and 19 minutes. The first iteration reached \(P = 0.48\), the second one \(P = 0.96\) and an effective sample size \(ESS = 457600\) (\(ESS/N = 0.92\)).

The results are generally similar to those reported in section 3.3. We do not show more 1D marginal plots, but present the recovered \(C_\ell^{\text{peak}}\) and \(f_\ell\) (figure 6), as well as the correlation matrix (figure 7). The \(C_\ell^{\text{peak}}\) and the ML estimates are somewhat similar to the WMAP5 power spectrum, with a small discrepancy also observed by Eriksen et al. (2007) using Gibbs sampling and in Rudjord et al. (2008) (zooming on their figure 5). At any rate, the discrepancy is always within the \(C_\ell\) error bars.

The effective coverage \(f_\ell\) is similar to the one reported in section 3.3 with a transition from 1 to \(f_{\text{sky}}\) but differs in some details, indicating that it is not only a function of the mask, but also of the actual data set.

Finally, figure 7 shows the correlation matrix. It exhibits structures similar to those in figure 5. As for the \(f_\ell\), the differences between figures 7 and 5 indicate that the correlation matrix does not depend only on the mask.

\(^2\)\url{http://lambda.gsfc.nasa.gov/}
4 APPROXIMATING THE LOW-$\ell$ LIKELIHOOD

For both data sets considered in previous section, the posterior distribution of the total angular spectrum $D$ revealed similar and striking features: the marginals are very well approximated by inverse gamma distributions and there is a weak correlation between multipoles (below the 10% level). Since we used a flat prior, these findings suggest that a copula approximation to the likelihood should be quite accurate (in addition to being fast, by design). This approach is somewhat similar to what has been proposed by [Bond et al. (2000)] and implemented at low-$\ell$ in [Rudjord et al. (2008)] and at high-$\ell$ in [Hamimeche & Lewis (2008)]. It differs in that, instead of offset log normal (as in [Bond et al. (2000)]), spline approximation ([Rudjord et al. 2008]) or Taylor expansion inspired approximation ([Hamimeche & Lewis 2008]), we use inverse gamma cumulative functions for Gaussianization.

4.1 Copula approximation

A good approximation formula must at least reproduce the inverse gamma marginals, and the observed level of correlation. A generic way of building multivariate distributions with specified marginals and some correlation is provided by copula models ([Sklar 1959]).

The copula model. Denote $N^{(d)}(\cdot; \mu, M)$ the $d$-variate Gaussian density with mean $\mu$ and covariance matrix $M$. Consider a set of zero-mean unit-variance Gaussian variables $G_\ell$ with density $N^{(d)}(G_\ell; 0, M_G)$ where $M_G$ has only 1’s on the diagonal and possibly non-zero off diagonal terms. Consider those transformed variables $D_\ell = D(G_\ell)$ which have an inverse gamma distribution with parameters $\alpha_\ell$ and $\beta_\ell$. 
that is, $G_\ell$ and $D_\ell$ are related by
\[ N(G_\ell; 0, 1) \ dG_\ell = i\Gamma(D_\ell; \alpha_\ell, \beta_\ell) \ dD_\ell. \] (15)
The distribution of $D_\ell$ is then easily seen to be
\[ \tilde{\pi}(D_\ell) \equiv \prod_k \iGamma(D_k; \alpha_k, \beta_k). \ N^{(d)}(G_\ell; 0, M_D) \prod_k C^{(1)}(G_k; 0, 1). \] (16)
Distribution (16) is called the copula approximation. It belongs to a parametric model with $2d + d(d - 1)/2$ parameters: each of the $d$ multipoles requires a pair $(\alpha_\ell, \beta_\ell)$ for the marginal distribution and the correlation matrix $M_D$ depends on $d(d - 1)/2$ free parameters.

**Two properties.** Probability distributions of the form (16) enjoy two nice properties which readily follow from their construction. First, the marginal distribution of each $D_\ell$ remains an inverse Gamma regardless of the correlation level (which is independently controlled by the matrix $M_D$). Second, marginalization over any subset of $D_\ell$ is readily achieved by removing the corresponding rows and columns of matrix $M_D$.

**Gaussianization.** Evaluating the copula density (16) requires explicit Gaussianization, that is mapping $D_\ell$ to $G_\ell$. This is easy since relation (15) implies that
\[ G_\ell \equiv c N^{-1}(ci\Gamma(D_\ell; \alpha_\ell, \beta_\ell)), \] (17)
where $ci\Gamma(\cdot; \alpha, \beta)$ denotes the cumulative distribution function (CDF) of the inverse gamma distribution and $c N^{-1}$ is the inverse CDF (or quantile function) of the standard normal distribution, sometimes called the probit function. The former is
\[ ci\Gamma(x; \alpha, \beta) \equiv \int_0^x \Gamma(t; \alpha, \beta) \ dt = \Gamma (\alpha, \beta/x) / \Gamma (\alpha). \]
while the latter, if missing from a statistical library, can be computed as $c N(x)^{-1} = \sqrt{2\pi} \text{erf}^{-1}(2x - 1)$ with $\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-t^2) \ dt$.

**Speed.** Copula evaluation is very fast. Using a custom code to compute the inverse error function, and the free GSL library\footnote{http://www.gnu.org/software/gsl/} for the gamma and cumulative gamma distribution, we can compute about 18000 samples per second while the pixel-based likelihood needs about 5.5 seconds per sample on the same computer within the same setting (i.e. same overheads). Moreover, one can also sample directly from the copula by first drawing $G_\ell$ under to their multivariate Gaussian distribution and then invert Eq. (17) to get the $D_\ell$ values.

**Learning the copula model.** Learning the $2d + d(d - 1)/2$ parameters of a copula models from (importance) samples of $D_\ell$ is straightforward. In a first step, one estimates for each $\ell$, the inverse gamma parameters $(\alpha_\ell, \beta_\ell)$ by maximum likelihood (see appendix\footnote{http://www.gnu.org/software/gsl/}). In a second step, the samples are Gaussianized via eq. (17) using the estimated values of $(\alpha_\ell, \beta_\ell)$. Finally, matrix $M_D$ is plainly estimated as the sample correlation matrix of the Gaussianized samples.

**Significance of correlation.** Given a copula model $\tilde{\pi}$ with correlation matrix $M_D$, there is a simpler copula model with the same marginals but without correlation, that is, with $M_D = I$. This model is denoted $\tilde{\pi}_0$ and called the uncorrelated model which, of course, is not as accurate as $\tilde{\pi}$. Since $\tilde{\pi}_0$ and $\tilde{\pi}$ are Gaussian distributions, the loss can be quantified exactly thanks to a Pythagorean property of the KLD which yields
\[ K(\pi|\tilde{\pi}_0) = K(\pi|\tilde{\pi}) + K(\pi|\tilde{\pi}_0). \] (18)
It shows that the mismatch $K(\pi|\tilde{\pi}_0)$ of the uncorrelated approximation to the posterior is larger than the mismatch $K(\pi|\tilde{\pi})$ of the regular copula by a positive term $K(\pi|\tilde{\pi}_0)$. This term can be computed in closed form:
\[ K(\pi|\tilde{\pi}_0) = -\frac{1}{2} \log \det M_D \] (19)
which is positive unless $M_D = I$ and readily gives a measure of the price to pay for ignoring correlation.

**4.2 Validation : first results**

We first look at some self-consistency results when learning a copula model from the importance samples obtained from the WMAP data set discussed in section 3.3.

**High perplexity.** The first important thing to report is that, on the perplexity scale, the copula approximation is remarkably good: we reach $P(\pi|\tilde{\pi}) = 0.99$ on a 500k simulation sample using estimates of the $C_D^{\text{peak}}$, $f_k$ and $M_D$ obtained on the same sample. As a simple cross-validation test, we split the 500k sample into two subsets of equal size, re-estimate the copula parameters on the first subset and compute the perplexity using the second subset. We find a negligible decrease in perplexity of about $5 \times 10^{-4}$.

Thus, the copula approximation appears to work extremely well on this data set. Still, one should look further than a single number. This section looks into more details of the approximation.

**Gaussianization.** Even though the marginals were found to be well approximated by inverse gamma distributions, the Gaussianized importance samples may show some small hints of... non Gaussianity. Indeed, for each $G_\ell$, we computed the skewness, the kurtosis and the Kullback divergence to a standard Gaussian. See figure\footnote{http://www.gnu.org/software/gsl/} for the WMAP data set (a similar plot can be obtained on the other data set). The plot shows a small deviation from Gaussianity showing that the target densities are not exactly inverse gamma distributed. In addition, those non Gaussian indicators degrade with $\ell$ and are correlated with $f_k$. Since the latter measures deviation from the full-sky case, this is not unexpected.

**Correlation matrices.** By design, the copula correctly predicts the correlation matrix of the Gaussianized variables but it is not necessarily accurate as a predictor of the correlation matrix $V$ of $G_\ell$. Here, we check that $V$ is well predicted by the covariance matrix of the copula model, denoted $\hat{V}$. Matrix $\hat{V}$ is estimated as described before (based on an importance sample); matrix $\hat{V}$ is obtained from the same importance samples, re-weighted by $\tilde{\pi}/\pi$. The results are displayed on figure\footnote{http://www.gnu.org/software/gsl/} and show an excellent agreement, with small and evenly distributed errors.
4.3 Perplexities.

We briefly report on the relative perplexity and Kullback divergence between the posterior and its approximations on the WMAP5 data set. Some results are reported in Table 1. Since the Gaussianized variables were found to be weakly correlated, it may be tempting simply to ignore this correlation and to resort to the \textit{uncorrelated} approximation \( \tilde{\pi}_0 \) defined at sec. 1.4. In this case, the fit is slightly degraded: we measure \( \mathcal{P}(\pi|\tilde{\pi}_0) = 0.97 \), in line with the perplexity obtained after the last step of the adaptive importance run \( \mathcal{P} = 0.96 \), sec. 3.4 showing that the determination of \( C^\text{peak} \) and \( f_\ell \) is only marginally improved by the 500k simulation. The contribution of correlation to the quality of the fit is given on the Kullback scale by the Pythagorean decomposition (18). Numerical evaluation by Monte Carlo integration gives, term-to-term:

\[
35.18 \times 10^{-3} \approx 8.61 \times 10^{-3} + 27.3 \times 10^{-3}.
\]

This is only an approximate equality because of MC errors. The last term was also evaluated using eq. (19), yielding 27.5 \( \times 10^{-3} \). These values show that correlation accounts for most part of the mismatch in the sense that \( K(\pi|\tilde{\pi}) \approx \frac{1}{2} K(\tilde{\pi}|\pi_0) \).

Those results can be compared to the naive approximation used as the initial proposal in our adaptive importance sampling runs, that is, the copula approximation \( \tilde{\pi}_0 \) with \( C^\text{ML} \), \( f_\text{sky} \) and ignoring the correlation. It gives a perplexity of \( \mathcal{P}(\pi|\tilde{\pi}_\text{naive}) = 0.76 \) corresponding to a huge increase in Kullback divergence.

Finally, we compute, as a comparison baseline, the perplexity of the classical offset log-normal approximation (Bond et al. 2000). The estimation of the curvature at the peak is easily derived from \( f_\ell \). The perplexity goes down to \( \mathcal{P} = 0.2 \) for that approximation.

4.4 Validation: pseudo-cosmological parameters

We now compare several likelihood functions via their impact on estimation of (pseudo) cosmological parameters from WMAP data. Since only the low \( \ell \) part of the spectrum is considered, only a few cosmological parameters can be fitted. We choose to perform our comparisons using a simple model with only two parameters, amplitude and spectral index, that is, we consider

\[
\hat{C}_\ell \equiv C^\text{ref}_\ell \times A \left( \frac{\ell}{\ell_0} \right)^n,
\]

where \( C^\text{ref}_\ell \) is a reference angular spectrum (here the WMAPI best fit spectrum) and where the relative amplitude \( A \) and the relative spectral index \( n \) are our pseudo-cosmological parameters. The reference power spectrum be-
ing a fit on a broader range of multipoles, the posterior of \((A, n)\) is not centered at \((1, 0)\).

Figure 10 shows the 1, 2 and 3 \(\sigma\) contours and the peak position for different likelihood approximations. The top panel presents a comparison between the WMAP5 likelihood code, used both in pixel based and Gibbs mode \cite{Dunkley2009}, and copula approximations with or without correlations (i.e. \(\pi\) and \(\tilde{\pi}\)). They all appear to be in remarkably good agreement. The small discrepancies in the contour curves (which are smaller than the grid step size) are much smaller than the width of the mode. The peaks of the copula approximations and of the Gibbs approximation are very slightly displaced compared to the official WMAP5 results, at a distance of the order of the step size of the grid on which likelihoods are evaluated. The bottom panel presents a comparison with the log-normal approximation described in previous chapter. As expected, the quality of that last approximation is poor, with a deviation of the best fit \((A, n)\) of the order of \(\sigma/4\). Nonetheless, the areas of the 1, 2 and 3 \(\sigma\) regions are similar, probably because these areas are mostly controlled by the values of \(f_t\).

5 CONCLUSION

Using an adaptive importance sampling algorithm, we explored the low-\(\ell\) posterior of partially observed CMB maps, both synthetic and real. From this exploration, we built a copula-based approximation for that posterior distribution. Numerical evaluation of that approximation is much faster than the pixel-based computation. We showed that the approximation is very close to the actual posterior with an accuracy which is probably sufficient for most cosmological applications. For example, on a simple two-parameter pseudo cosmological model, we found a discrepancy which is negligible with respect to the width of the posterior mode (figure 10).

The copula approximation uses two ingredients: a model of marginal distributions and a correlation matrix. The marginals are mostly distributed as inverse gammas, as in the full-sky case, but with different parameters. Maybe surprisingly, the correlations between (Gaussianized) multipoles are found to be quite low (< 10\%). Ignoring them in the toy cosmological model illustrated by figure 10 does not change significantly the posterior. However, when considering the full joint distribution of the multipoles (as opposed to its projection onto the two-parameter toy model), the correlation is significant: the Kullback divergence from the true posterior to its copula approximation quadruples if the correlation is left out. In both cases however, the Kullback divergence remains small.

The main limitation of the proposed approximation is that it requires an exploration of the posterior to measure the parameters of the approximation. We used an adaptive importance sampling algorithm, but a MCMC algorithm, Gibbs-based \cite{Wandelt2004} or Hybrid MC-based \cite{Taylor2008} can also be used. Both methods exhibit good scaling properties thanks to a smart re-writing of the posterior and could, if convergence is well controlled, provide estimates at higher \(\ell\). Indeed, a very recent work, published at the time we were finishing this paper follows a similar path and demonstrate a Gaussianization technique based on splines rather than on inverse gamma models \cite{Rudjord2008}.

Another approach would be to determine the parameters of the marginals directly from the likelihood, without resorting to a sampling-based exploration. We are currently working on an analytical derivation of the approximation which would make it possible to build an approximation valid for higher \(\ell\) at low computational cost. Being able to reach smaller scales is also important to explore the transition between low \(\ell\) estimates and high-\(\ell\) ones. Indeed, at very small scales, the problem becomes intractable and requires the use of asymptotic approximations to the likelihood \cite{Perival2006,Smith2006}.

Finally, it is not clear yet whether the same kind of approximation can be built for polarized fields. In the temperature case addressed here, we took advantage of a low correlation situation, thanks to a high signal to noise ratio and relatively small masked area. Polarized observations will be noisier and it remains to be seen if copula approxi-
imations are up to the task. This is the subject of current investigations.

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APPENDIX A: ML ESTIMATION OF INVERSE GAMMA PARAMETERS

The log-likelihood \( \log L(\alpha, \beta) \) for a sample of \( N \) independent realizations \( X_i \) under an inverse gamma density is

\[
\log L = \sum_{i=1}^{N} \left( \alpha \log \beta - \log \Gamma(\alpha) - (\alpha + 1) \log X_i - \frac{\beta}{X_i} \right)
\]

as seen from eq. (3). The ML estimate for \((\alpha, \beta)\) is the solution of \( \frac{\partial \log L}{\partial \alpha} = 0 \) and \( \frac{\partial \log L}{\partial \beta} = 0 \) leading to the two estimating equations:

\[
\log \beta - \psi(\alpha) = \frac{1}{N} \sum_{i=1}^{N} \log X_i, \quad \frac{\alpha}{\beta} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{X_i}
\]

where \( \psi(u) \) is the log-derivative of the gamma function, also known as the digamma function. Using the last equation to express \( \beta \) in terms of \( \alpha \), the ML estimate can be obtained by solving

\[
\log \alpha - \psi(\alpha) = \frac{1}{N} \sum_{i=1}^{N} \log X_i - \log \left( \frac{1}{N} \sum_{i=1}^{N} \frac{1}{X_i} \right). \tag{A1}
\]

This is quickly done numerically in a few steps of a Newton algorithm; both the digamma function and its derivative being available in the GSL package.

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