A characteristic property of the space $s$

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Abstract

It is shown that under certain stability conditions a complemented subspace of the space $s$ of rapidly decreasing sequences is isomorphic to $s$ and this condition characterizes $s$. This result is used to show that for the classical Cantor set $X$ the space $C_\infty(X)$ of restrictions to $X$ of $C_\infty$-functions on $\mathbb{R}$ is isomorphic to $s$, so completing the theory developed in [7].

1 Introduction

In the present note we study the space $s$ of rapidly decreasing sequences, that is, the space

$$s = \{ x = (x_0, x_1, \ldots) : \lim n x_n n^k = 0 \text{ for all } k \in \mathbb{N} \}.$$

Equipped with the norms $\| x \|_k = \sup_n |x_n|(n + 1)^k$ it is a nuclear Fréchet space. It is isomorphic to many of the Fréchet spaces which occur in analysis, in particular, spaces of $C_\infty$-functions.

It is easily seen that instead of the sup-norms we might use the norms

$$|x|_k = \left( \sum_n |x_n|^2 (n + 1)^{2k} \right)^{1/2}$$

which makes $s$ a Fréchet-Hilbert space.

More generally, we define for any sequence $\alpha : 0 \leq \alpha_0 \leq \alpha_1 \leq \ldots + \infty$ the power series space of infinite type

$$\Lambda_\infty(\alpha) := \{ x = (x_0, x_1, \ldots) : |x|_t^2 = \sum_{n=0}^\infty |x_n|^2 e^{2t\alpha_n} < \infty \text{ for all } t > 0 \}.$$

Equipped with the hilbertian norms $\| \cdot \|_k$, $k \in \mathbb{N}_0$, it is a Fréchet-Hilbert space. It is nuclear if, and only if, $\limsup_n \log n/\alpha_n < \infty$. With this definition $s = \Lambda_\infty(\alpha)$ with $\alpha_n = \log(n + 1)$.

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A Fréchet space with the fundamental system of seminorms \( \| \cdot \|_0 \leq \| \cdot \|_1 \leq \ldots \) has property (DN) if
\[
\exists p \forall k \exists K, C > 0 : \| \cdot \|_k^2 \leq C \| \cdot \|_p \cdot \| \cdot \|_K.
\]
In this case \( \| \cdot \|_p \) is called a dominating norm.

\( E \) has property (Ω) if
\[
\forall p \exists q \forall m \exists 0 < \theta < 1, C > 0 : \| \cdot \|_q^* \leq C \| \cdot \|_p^* \| \cdot \|_m^{1-\theta}.
\]
Here we set for any continuous seminorm \( \| \cdot \| \) and \( y \in E' \) the dual, extended real valued, norm \( \| y \|_* = \sup \{|y(x)| : x \in E, \| x \| \leq 1\} \).

By Vogt-Wagner [8] a Fréchet space \( E \) is isomorphic to a complemented subspace of \( s \) if, and only if, it is nuclear and had properties (DN) and (Ω).

It is a long standing unsolved problem of the structure theory of nuclear Fréchet spaces, going back to Mityagin, whether every complemented subspace of \( s \) has a basis. If it has a basis then it is isomorphic to some power series space \( \Lambda_\infty(\alpha) \). The space \( \Lambda_\infty(\alpha) \) to which it is isomorphic, if it has a basis, can be calculated in advance by a method going back to Terzioğlu [4] which we describe now.

Let \( X \) be a vector space and \( A \subset B \) absolutely convex subsets of \( X \). We set
\[
\delta_n(A,B) := \inf \{ \delta > 0 : \text{exists linear subspace } F \subset X, \dim F \leq n \text{ with } A \subset \delta B + F \}.
\]
It is called the \( n \)-th Kolmogoroff diameter of \( A \) with respect to \( B \).

If now \( E \) is a complemented subspace of \( s \), that is, \( E \) is nuclear and has properties (DN) and (Ω), then we choose \( p \) such that \( \| \cdot \|_p \) is a dominating norm and for \( p \) we choose \( q > p \) according to property (Ω). We set
\[
\alpha_n = -\log \delta_n(U_q,U_p)
\]
where \( U_k = \{ x \in E : \| x \|_k \leq 1 \} \). The space \( \Lambda_\infty(\alpha) \) is called the associated power series space and \( E \cong \Lambda_\infty(\alpha) \) if it has a basis.

If \( \limsup_n \alpha_{2n}/\alpha_n < \infty \) then, by Aytuna-Krone-Terzioğlu [2, Theorem 2.2], \( E \cong \Lambda_\infty(\alpha) \). This is, in particular, the case if \( E \) is stable, that is, if \( E \oplus E \cong E \).

For all that and further results of structure theory of infinite type power series spaces see [6], for results and unexplained notation of general functional analysis see [3].

2 Main result

**Lemma 2.1** Let \( E \) be a complemented subspace of \( s \), \( \| \cdot \|_0 \) a dominating hilbertian norm and \( \| \cdot \|_1 \) a hilbertian norm chosen for \( \| \cdot \|_0 \) according to (Ω). If there is a linear isomorphism \( \psi : E \oplus E \to E \) such that
\[
\| x \|_0 + \| y \|_0 \leq C_0 \| \psi(x \oplus y) \|_0 \\
\| \psi(x \oplus y) \|_1 \leq C_1 (\| x \|_1 + \| y \|_1)
\]
then \( E \cong s \).
Proof. For $x \oplus y \in E \oplus E$ we set $|||(x, y)|||_0 := (\|x\|_0^2 + \|y\|_0^2)^{1/2}$ and $|||(x, y)|||_1 := (\|x\|_1^2 + \|y\|_1^2)^{1/2}$. With new constants $C_k$ we have

$$||| x \oplus y |||_0 \leq C_0 \| \psi(x \oplus y) \|_0 \text{ and } \| \psi(x \oplus y) \|_1 \leq C_1 ||| x \oplus y |||_1.$$  (1)

To calculate the associated power series space for $E$ we set:

$$\alpha_n = - \log \delta_n(U_1, U_0) \text{ where } U_k = \{x \in E : \|x\|_k \leq 1\}, \quad \beta_n = - \log \delta_n(V_1, V_0) \text{ where } V_k = \{x \oplus y \in E \oplus E : ||| x \oplus y |||_k \leq 1\}.$$

Due to the estimates (1) we have

$$\frac{1}{C_1} \psi(V_1) \subset U_1 \subset U_0 \subset C_0 \psi(V_0)$$

and therefore

$$\delta_n(V_1, V_0) = \delta_n(\psi V_1, \psi V_0) \leq C_0 C_1 \delta_n(U_1, U_0)$$

which implies

$$\alpha_n \leq \beta_n + d$$

with $d = \log C_0 C_1$.

By explicit calculation of the Schmidt expansion of the canonical map $j^0_1$ between the local Hilbert spaces of $||| \cdot |||_1$ and $||| \cdot |||_0$ and by use of the fact that singular numbers and Kolmogoroff diameters coincide, we obtain that $\beta_{2n} = \beta_{2n+1} = \alpha_n$ for all $n \in \mathbb{N}_0$.

Therefore we have $\alpha_{2n} \leq \beta_{2n} + d = \alpha_n + d$ for all $n \in \mathbb{N}_0$ and this implies $\alpha_{2k} \leq \alpha_1 + k d$ for all $k \in \mathbb{N}_0$. For $n \in \mathbb{N}$ we find $k \in \mathbb{N}$ such that $2^{k-1} \leq n \leq 2^k$ and we obtain $\alpha_n \leq \alpha_{2k} \leq \alpha_1 + k d \leq (\alpha_1 + d) + d \log n$.

Since $E \subset s$, which implies the left inequality below, we have shown that there is a constant $D > 0$ such that

$$\frac{1}{D} \log n \leq \alpha_n \leq D \log n$$

for large $n \in \mathbb{N}$. This implies that $\Lambda_\infty(\alpha) = s$. \square

A Fréchet-Hilbert space $E$ is called **normwise stable** if it admits a fundamental system of hilbertian seminorms for which there is an isomorphism $\psi : E \oplus E \to E$ such that

$$\frac{1}{C_k}(\|x\|_k + \|y\|_k) \leq \|\psi(x \oplus y)\|_k \leq C_k(\|x\|_k + \|y\|_k)$$

for all $k$. Since, clearly, $s$ is normwise stable we have shown.

**Theorem 2.2** $E \cong s$ if, and only if, $E$ is isomorphic to a complemented subspace of $s$ and normwise stable.

We may express Lemma 2.1 also in the following way:
**Theorem 2.3** Let the Fréchet-Hilbert space $E$ be a complemented subspace of $s$, $\| \cdot \|_0$ a dominating norm and $\| \cdot \|_1$ be a norm chosen according to $(\Omega)$. Let $P$ be a linear projection in $E$, continuous with respect to $\| \cdot \|_0$. We set $E_1 = R(P)$, $E_2 = N(P)$ and assume that there are linear isomorphisms $\psi_j : E \to E_j$, $j = 1, 2$, continuous with respect to $\| \cdot \|_1$ such that $\psi^{-1}$ is continuous with respect to $\| \cdot \|_0$. Then $E \cong s$.

**Proof.** We set $\psi(x \oplus y) := \psi_1(x) + \psi_2(y)$ and obtain with suitable constants:

$$
\|x\|_0 + \|y\|_0 \leq C'(\|\psi_1(x)\|_0 + \|\psi_2(y)\|_0) \leq C_0\|\psi_1(x) + \psi_2(y)\|_0 = C_0\|\psi(x \oplus y)\|_0
$$

$$
\|\psi(x \oplus y)\|_1 = \|\psi_1(x) + \psi_2(y)\|_1 \leq \|\psi_1(x)\|_1 + \|\psi_2(y)\|_1 \leq C_0(\|x\|_1 + \|y\|_1).
$$

Lemma 2.1 yields the result. □

**3 Application**

An interesting application of this result is the following. Let $X \subset [0, 1]$ be the classical Cantor set and $C_{\infty}(X) := \{f|_X : f \in C_{\infty}[0, 1]\} = \{f|_E : f \in C_{\infty}(\mathbb{R})\}$. The space $C_{\infty}(X)$ equipped with the quotient topology is a nuclear Fréchet space and, since $C_{\infty}[0, 1] \cong s$ isomorphic to a quotient of $s$, hence has property $(\Omega)$. By a theorem of Tidten [5] it has also property (DN). Therefore it is isomorphic to a complemented subspace of $s$ (see [8]).

We should remark that, due to the fact that $X$ is perfect, we have $C_{\infty}(X) = \mathcal{E}(X)$ where $\mathcal{E}(X)$ denotes the space of Whitney jets on $X$, for which Tidten’s result is formulated.

By obvious identifications we have

$$
C_{\infty}(X) \cong C_{\infty}(X \cap [0, 1/3]) \oplus C_{\infty}(X \cap [2/3, 1]) \cong C_{\infty}(X) \oplus C_{\infty}(X)
$$

and it is easily seen that this establishes normwise stability. Therefore we have shown

**Theorem 3.1** If $X$ is the classical Cantor set, then $C_{\infty}(X) \cong s$.

It should be remarked that in [1] it has been shown that for the Cantor set $X$ the diametral dimensions of $\mathcal{E}(X)$ and $s$ coincide, from where, by means of the Aytuna-Krone-Terzioglu Theorem, on can derive the same result.

Referring to the terminology of [7] we have also shown that $A_{\infty}(X) \cong s$ which completes the theory developed in [7].

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