A recurrent formula of $A_\infty$-quasi inverses of dg-natural transformations between dg-lifts of derived functors

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November 19, 2019

Abstract

A dg-natural transformation between dg-functors is called an objectwise homotopy equivalence if its induced morphism on each object admits a homotopy inverse. In general an objectwise homotopy equivalence does not have a dg-inverse but has an $A_\infty$ quasi-inverse. In this note we give a recurrent formula of the $A_\infty$ quasi-inverse. This result is useful in studying the compositions of dg-lifts of derived functors of schemes.

1 Introduction

In [Sch18], Schnürer constructed Grothendieck six functor formalism of dg-enhancements for ringed spaces over a field $k$. In more details, for each $k$-ringed space $X$ we have a dg $k$-category $\mathbb{I}(X)$ which is a dg-enhancement of $D(X)$, the derived category of sheaves of $O_X$-modules. Moreover for a morphism $f : X \to Y$ of $k$-ringed space we have dg $k$-functors $f^* : \mathbb{I}(Y) \to \mathbb{I}(X)$ and $f_* : \mathbb{I}(X) \to \mathbb{I}(Y)$ which are dg-lifts of the derived functors $Lf^*$ and $Rf_*$, respectively. In addition, Schnürer showed that for two composable morphisms $f : X \to Y$ and $g : Y \to Z$, we have zig-zags of dg-natural transformations which are objectwise homotopy equivalences between $(gf)^*$ and $f^*g^*$ and $(gf)_*$ and $f_*(g_*)$.

We call a dg $k$-natural transformation $\Phi : F \to G$ an objectwise homotopy equivalence if for any object $E$, the induced morphism $\Phi_E : F(E) \to G(E)$ has a homotopy inverse. This does not mean that we could find a homotopy inverse of $\Phi$ because the objectwise homotopy inverses are not compatible with morphisms as illustrated in the following diagrams

Nevertheless, by [Lyu03, Proposition 7.15] we know that we can extend $\Phi^{-1}$ to an $A_\infty$ natural transformation $\Psi : G \Rightarrow F$ and $\Psi$ is an $A_\infty$ quasi-inverse of $\Phi$. In this note we give a detailed construction of the recurrent formula of $\Psi$ as suggested in [Lyu03] Appendix B]. In particular we show that we can construct $\Psi$ by compositions of objectwisely chosen homotopies. See Theorem 4.1 below. This formula will be used in [Wei19].
Acknowledgments

The author would like to thank Nick Gurski and Olaf Schnürer for very helpful discussions.

2 A review of dg-natural transformations and $A_{\infty}$-natural transformations

In this section we review some concepts around dg-functors, dg-natural transformations, and $A_{\infty}$-natural transformations.

Remark 1. We would like to point out that the best way to describe $A_{\infty}$-categories/functors/natural-transformations is in the framework of bar constructions and dg-cocategories, see [Lyu03]. In this note we just take the by-hand definition, which requires minimal amount of preparation but involves more complicated notations.

Definition 2.1 (dg-categories). Let $k$ be a commutative ring with unit. A differential graded or dg $k$-category is a category $C$ whose morphism spaces are cochain complexes of $k$-modules and whose compositions of morphisms

\[ C(Y, Z) \otimes_k C(X, Y) \to C(X, Z) \]

are morphisms of $k$-cochain complexes. Furthermore, there are obvious associativity and unit axioms.

Definition 2.2 (dg-functors). Let $k$ be a commutative ring with unit and $C$ and $D$ be two dg $k$-categories. A dg $k$-functor $F : C \to D$ consists of the following data:

1. A map $F : \text{obj}(C) \to \text{obj}(D)$;
2. For any objects $X, Y \in \text{obj}(C)$, a closed, degree 0 morphism of complexes of $k$-modules

\[ F(X, Y) : C(X, Y) \to D(FX, FY) \]

which is compatible with the composition and the units.

Definition 2.3 (dg-natural transformation). Let $k$ be a commutative ring with unit and $F, G : C \to D$ be two dg $k$-categories. A dg $k$-prenatural transformation $\Phi : F \Rightarrow G$ of degree $n$ consists of a morphism

\[ \Phi_X \in D^n(FX, GX) \]

for each object $X$ such that for any morphism $u \in C^m(X, Y)$ we have

\[ \Phi_Y Fu = (-1)^{mn}Gu \Phi_X. \]

The differential on $\Phi$ is defined objectwisely and it is clear that $d\Phi$ is a dg $k$-prenatural transformation of degree $n + 1$. We call $\Phi$ a dg $k$-natural transformation if $\Phi$ is closed and of degree 0.

Definition 2.4 ($A_{\infty}$-prenatural transformation). Let $k$ be a commutative ring with unit and $F, G : C \to D$ be two dg $k$-functors between dg $k$-categories. An $A_{\infty}$ $k$-prenatural transformation $\Phi : F \Rightarrow G$ of degree $n$ consists of the following data:

1. For any object $X \in \text{obj}(C)$, a morphism $\Phi^n_X \in D^n(FX, GX)$;
2. For any $l \geq 1$ and any objects $X_0, \ldots, X_l \in \text{obj}(C)$, a morphism

\[ \Phi^l_{X_0, \ldots, X_l} \in \text{Hom}_k^{n-l}(C(X_{l-1}, X_l) \otimes_k \cdots \otimes_k C(X_0, X_1), D(FX_0, GX_l)) \]
**Definition 2.5** (Differential of $A_\infty$-prenatural transformation). Let $k$ be a commutative ring with unit and $F, G : C \to D$ be two dg $k$-functors between dg $k$-categories. Let $\Phi : F \Rightarrow G$ be an $A_\infty$ $k$-natural transformation of degree $n$ as in Definition 2.4. Then the *differential* $d\Phi : F \Rightarrow G$ is an $A_\infty$ $k$-natural transformation of degree $n + 1$ whose components are given as follows:

1. For any object $X \in \text{obj}(C)$, $(d^\infty \Phi)^0_X = d(\Phi^0_X) \in D^{n+1}(FX, GX)$;

2. For any $l \geq 1$ and a collection of morphisms $u_i \in C(X_{i-1}, X_i)$ $i = 1, \ldots, l$,

   $$(d^\infty \Phi)^l(u_l \otimes \ldots \otimes u_1) =$$

   $$d(\Phi^l(u_l \otimes \ldots \otimes u_1)) + (-1)^{|u_l|}G(u_l)\Phi^{l-1}(u_{l-1} \otimes \ldots \otimes u_1) + (-1)^{|u_l|-|u_l|-|u_l|+l-1}\Phi^{l-1}(u_l \otimes \ldots \otimes u_2)F(u_1)$$

   $$+ \sum_{i=1}^l (-1)^{|u_l|+\ldots+|u_{i+1}|+l-i+1}\Phi^l(u_l \otimes \ldots du_i \otimes \ldots u_1)$$

   $$+ \sum_{i=1}^{l-1} (-1)^{|u_l|+\ldots+|u_{i+1}|+l-i+1}\Phi^{l-1}(u_l \otimes \ldots u_{i+1}u_i \otimes \ldots u_1) \quad (1)$$

We can check that $d^\infty \circ d^\infty = 0$ on $A_\infty$ $k$-prenatural transformations.

**Definition 2.6** ($A_\infty$-natural transformation). Let $k$ be a commutative ring with unit and $F, G : C \to D$ be two dg $k$-functors between dg $k$-categories. Let $\Phi : F \Rightarrow G$ be an $A_\infty$ $k$-natural transformation. We call $\Phi$ an $A_\infty$ $k$-natural transformation if $\Phi$ is of degree $0$ and closed under the differential $d^\infty$ in Definition 2.5.

It is clear that a dg $k$-natural transformation $\Phi$ can be considered as an $A_\infty$ $k$-natural transformation with $\Phi^0 = 0$ for all $l \geq 1$.

**Definition 2.7** (Compositions). Let $k$ be a commutative ring with unit and $F, G, H : C \to D$ be three dg $k$-functors between dg $k$-categories. Let $\Phi : F \Rightarrow G$ be a dg $k$-natural transformation and $\Psi : G \Rightarrow H$ be an $A_\infty$ $k$-natural transformation. Then the composition $\Psi \circ \Phi$ is defined as follows: For any object $X \in \text{obj}(C)$

   $$(\Psi \circ \Phi)^0_X := \Psi^0_X \Phi_X : FX \to GX \to HX$$

and for any $u_i \in C(X_{i-1}, X_i)$, $i = 1, \ldots, l$

   $$(\Psi \circ \Phi)^l(u_l \otimes \ldots u_1) := \Psi^l(u_l \otimes \ldots u_1)\Phi_{X_0}$$

We can check that $\Psi \circ \Phi$ is an $A_\infty$ $k$-natural transformation.

Similarly, Let $\Phi : F \Rightarrow G$ be an $A_\infty$ $k$-natural transformation and $\Psi : G \Rightarrow H$ be a dg $k$-natural transformation. Then the composition $\Psi \circ \Phi$ is defined as follows: For any object $X \in \text{obj}(C)$

   $$(\Psi \circ \Phi)^0_X := \Psi_X \Phi^0_X : FX \to GX \to HX$$

and for any $u_i \in C(X_{i-1}, X_i)$, $i = 1, \ldots, l$

   $$(\Psi \circ \Phi)^l(u_l \otimes \ldots u_1) := \Psi_X \Phi^l(u_l \otimes \ldots u_1)$$

We can check that $\Psi \circ \Phi$ is an $A_\infty$ $k$-natural transformation.

**Remark 2.** We can define compositions for general $A_\infty$ $k$-prenatural transformations. See [Lyu03, Section 3] or [Sci08, Section I.1(d)].
Definition 2.8 ($A_\infty$ quasi-inverse). Let $k$ be a commutative ring with unit and $F$, $G : \mathcal{C} \to \mathcal{D}$ be two dg $k$-functors between dg $k$-categories. Let $\Phi : F \Rightarrow G$ be a dg $k$-natural transformation. We call an $A_\infty$ $k$-natural transformation $\Psi : G \Rightarrow F$ an $A_\infty$ quasi-inverse of $\Phi$ if there exists $A_\infty$ $k$-prenatural transformations $\eta : F \Rightarrow G$ and $\omega : G \Rightarrow F$ both of degree $-1$ such that

$$\Psi \circ \Phi - \text{id}_F = d^\infty \eta,$$

and

$$\Phi \circ \Psi - \text{id}_G = d^\infty \omega.$$

In more details, this means that we have

$$\Psi^0_X \Phi_X - \text{id}_{F^0_X} = d\eta^0_X,$$

and $\Phi_X \Psi^0_X - \text{id}_{G^0_X} = d\omega^0_X$ for any $X \in \text{obj}\mathcal{C}$

and for any $l \geq 1$ and any $u_i \in \mathcal{C}(X_{i-1}, X_i), i = 1, \ldots, l$, we have

$$\Psi^l(\Phi(u_1) \otimes \ldots \otimes \Phi(u_1)) =$$

$$d(\omega^l(u_1 \otimes \ldots \otimes u_1)) + (-1)^{|u_i|-1}G(u_i)\eta^{l-1}(u_{l-1} \otimes \ldots \otimes u_1) + (-1)^{|u_2| \ldots |u_i|+l-1}G(u_l \otimes \ldots \otimes u_2)F(u_1)$$

$$+ \sum_{i=1}^{l-1}(-1)^{|u_i|+\ldots+|u_{i+1}|+|l-i+1|}\eta^l(u_i \otimes \ldots u_i)$$

$$+ \sum_{i=1}^{l-1}(-1)^{|u_i|+\ldots+|u_{i+1}|+|l-i+1|}\omega^l(u_i \otimes \ldots u_i u_i \otimes \ldots u_1)$$

and

$$\Phi(\Psi^l(u_1 \otimes \ldots \otimes u_1)) =$$

$$d(\eta^l(u_1 \otimes \ldots \otimes u_1)) + (-1)^{|u_i|-1}G(u_i)\omega^{l-1}(u_{l-1} \otimes \ldots \otimes u_1) + (-1)^{|u_2| \ldots |u_i|+l-1}G(u_l \otimes \ldots \otimes u_2)F(u_1)$$

$$+ \sum_{i=1}^{l-1}(-1)^{|u_i|+\ldots+|u_{i+1}|+|l-i+1|}\omega^l(u_i \otimes \ldots u_i)$$

$$+ \sum_{i=1}^{l-1}(-1)^{|u_i|+\ldots+|u_{i+1}|+|l-i+1|}\omega^l(u_i \otimes \ldots u_i u_i \otimes \ldots u_1)$$

3 Review of functorial injective resolutions and a dg-lifts of derived functors

The main reference of this section is [Sch18].

3.1 Functorial injective resolutions

Let $k$ be a field and $X$ be a $k$-ringed space. Let $\mathcal{C}(X)$ be the dg $k$-category of complexes of sheaves on $X$ and $\mathcal{I}(X)$ its full dg $k$-subcategory of h-injective complexes of injective sheaves. Let $\mathcal{I}^-(X)$ and $\mathcal{I}^+(X)$ be the full subcategories of $\mathcal{I}(X)$ consisting of complexes with bounded or bounded below cohomology sheaves, respectively. See [Spa88] or [KS06, Chapter 14] for an introduction to h-injective complexes.

It is clear that $\mathcal{I}(X)$ is a strongly pretriangulated dg $k$-category hence its homotopy category $[\mathcal{I}(X)]$ is a triangulated $k$-category and the obvious functor $[\mathcal{I}(X)] \to D(X)$ is a triangulated equivalence.

We could construct an equivalence in the other direction.
Proposition 3.1. ([Sch18 Corollary 2.3]) Let $k$ be a field. Let $(X, \mathcal{O})$ be a $k$-ringed site and let $\mathcal{C}(X)_{h\text{flat}, \text{cwflat}}$ denote the full dg $k$-subcategory of $\mathcal{C}(X)$ of $h$-flat and componentwise flat objects. Then there exists dg $k$-functors

$$ i : \mathcal{C}(X) \to I(X) $$
$$ e : \mathcal{C}(X) \to \mathcal{C}(X)_{h\text{flat}, \text{cwflat}} \quad (4) $$

together with dg $k$-natural transformations

$$ \iota : \text{id} \to i : \mathcal{C}(X) \to \mathcal{C}(X) $$
$$ \varepsilon : e \to \text{id} : \mathcal{C}(X) \to \mathcal{C}(X) \quad (5) $$

whose evaluations $\iota_F : F \to iF$ and $\varepsilon_F : eF \to F$ at each object $F \in \mathcal{C}(X)$ are quasi-isomorphisms.

It is clear that the induced functor $[i] : [\mathcal{C}(X)] \to [I(X)]$ sends acyclic objects to zero, hence it factors to an equivalence

$$ [\tilde{i}] : D(X) \tilde{\to} [\mathbb{I}(X)] $$

of triangulated $k$-categories.

Intuitively the dg $k$-functor $i$ in Proposition 3.1 could be considered as a functorial injective resolution.

Remark 3. The result in Proposition 3.1 is an adaption of general results from enriched model category theory in [Rie14], in particular [Rie14] Corollary 13.2.4.

Remark 4. The assumption that $k$ is a field is essential for Proposition 3.1. Actually if $k = \mathbb{Z}$, then the pair $(i, \iota)$ in Proposition 3.1 does not exist. See [Sch18] Lemma 4.4 for a counterexample.

3.2 A dg-lift of the pull-back functor and push-forward functor

Definition 3.1. ([Sch18 2.3.4]) Let $k$ be a field. For a morphism of $k$-ringed spaces $f : X \to Y$, we define the injective pull back dg $k$-functor $f^*$ as

$$ f^* : \mathcal{C}(Y) \to \mathcal{C}(X) \quad (6) $$

Similarly we define the injective push forward dg $k$-functor $f_*$ as

$$ f_* : \mathcal{C}(X) \to \mathcal{C}(Y) \quad (7) $$

where $i$ and $e$ are defined in Proposition 3.1.

Remark 5. Actually in [Sch18] all Grothendieck’s six functors were lifted to dg $k$-functors.

Proposition 3.2. ([Sch18 Proposition 6.5]) Let $k$ be a field. For a morphism of $k$-ringed spaces $f : X \to Y$, the dg $k$-functors $f^*$ and $f_*$ in Definition 3.1 are dg-lifts of the derived pull back and derived push forward functors $Lf^* : D(Y) \to D(X)$ and $Rf_* : D(X) \to D(Y)$, respectively. More precisely, the diagrams

$$ \begin{array}{ccc}
D(Y) & \xrightarrow{Lf^*} & D(X) \\
\downarrow & & \downarrow \sim \\
[\mathbb{I}(Y)] & \xrightarrow{[f^*]} & [\mathbb{I}(X)]
\end{array} $$
and

\[
\begin{align*}
D(X) & \xrightarrow{Rf_*} D(Y) \\
\llbracket i \rrbracket & \sim \llbracket i \rrbracket \\
\llbracket I(X) \rrbracket & \xrightarrow{[\epsilon]} \llbracket I(Y) \rrbracket
\end{align*}
\]

commute up to a canonical 2-isomorphism.

### 3.3 Objectwise homotopy equivalences

By Definition 3.1 it is clear that we do not have \((gf)^* = f^*g^*\). Actually \((gf)^*\) and \(f^*g^*\) are connected by a zig-zag of dg natural transformations. To describe this relation more clearly, we introduce the following definitions.

**Definition 3.2.** [[Sch18, 2.1.3]] Let \(k\) be a field and \(X, Y\) be \(k\)-ringed spaces. Let \(F, G : I(X) \to I(Y)\) be \(k\)-functors. A \(k\)-natural transformation \(\Phi : F \Rightarrow G\) is called an objectwise homotopy equivalence if for any object \(E \in \text{obj}(I(X))\), the morphism \(\Phi_E : F_E \to G_E\) has a homotopic inverse.

**Proposition 3.3.** [[Sch18, Proposition 6.17, Lemma 6.21]] Let \(k\) be a field and \(X \xrightarrow{f} Y \xrightarrow{g} Z\) be morphisms of \(k\)-ringed spaces. Then there exist zig-zags of objectwise homotopy equivalences

\[
T_{f,g} : (gf)^* \sim f^*g^* \\
T_{f,g} : (gf) \sim g \cdot f \\
T_{id} : id^* \sim id \\
T_{id} : id \sim id
\]

(8)

**Proof.** We give the relation between \((gf)^*\) and \(f^*g^*\) to illustrate the idea. We use the \(k\)-natural transformations \(\iota : id \to i\) and \(\epsilon : e \to id\) in Proposition 3.1 and have the following objectwise homotopy equivalences

\[
(gf)^* = \iota(gf)^* \epsilon \sim if^*g^*e \\
\llbracket e \rrbracket \sim if^*eg^*e \sim if^*eig^*e \\
= f^*g^*.
\]

\[\Box\]

### 4 Objectwise homotopy equivalences and \(A_\infty\) quasi-inverses

**Definition 4.1.** Let \(k\) be a field and \(X, Y\) be \(k\)-ringed spaces. Let \(F, G : I(X) \to I(Y)\) be two \(k\)-functors and \(\Phi : F \Rightarrow G\) be a \(k\)-natural transformation which is an objectwise homotopy equivalence. For each object \(E \in \text{obj}(I(X))\) we can choose \(\Psi_E \in \llbracket 0 \rrbracket(Y)(G_E, F_E), h_E \in \llbracket -1 \rrbracket(Y)(F_E, F_E), p_E \in \llbracket -1 \rrbracket(Y)(G_E, G_E)\), such that

\[
\Psi_E \Phi_E - id_{F_E} = dh_E, \quad \Phi_E \Psi_E - id_{G_E} = dp_E.
\]

We call such a choice an objectwise homotopy inverse system of \(\Phi\).

For a objectwise homotopy equivalence \(\Phi\), its homotopy invese system always exists. The following theorem is the main result of this note.
Theorem 4.1. Let $k$ be a field and $X, Y$ be $k$-ringed spaces. Let $F, G : \mathbb{L}(X) \to \mathbb{L}(Y)$ be two dg $k$-functors and $\Phi : F \to G$ be a dg $k$-natural transformation which is an objectwise homotopy equivalence. Then there exists an $A_\infty$ quasi-inverse of $\Phi$. More precisely, we choose and fix an objectwise homotopy inverse system $\mathcal{H}$ of $\Phi$ as in Definition 4.1 and there exist an $A_\infty$ $k$-natural transformation $\Psi : G \Rightarrow F$ and $A_\infty$ $k$-prenatural transformations $\eta : F \Rightarrow F$ and $\omega : G \Rightarrow G$ of degree $-1$ such that

$$\Psi \circ \Phi - \text{id}_F = d\eta, \text{ and } \Phi \circ \Psi - \text{id}_G = d\omega.$$

Moreover, $\Psi, \eta$, and $\omega$ are defined by compositions of $F, G, \Phi,$ and $\mathcal{H}$.

Proof. The proof is a refinement of [Lyu03, Proposition 7.15]. We construct $\Psi, \eta,$ and $\omega$ by induction. First we construct the left inverse. Let $\Psi^0 = \Psi$ and $\eta^0 = p_\mathcal{E}$ as in Definition 4.1. Now suppose that for an $m \geq 1$ we have constructed $\Psi^i$ and $\eta^i, i = 1, \ldots, m - 1$ by compositions of $F, G, \Phi,$ and $\mathcal{H}$ such that the auxiliary $A_\infty$ $k$-prenatural transformations

$$\tilde{\Psi} = (\Psi^0, \Psi^1, \ldots, \Psi^{m-1}, 0, \ldots)$$
$$\tilde{\eta} = (\eta^0, \eta^1, \ldots, \eta^{m-1}, 0, \ldots)$$

satisfy

$$(d^\infty \tilde{\Psi})^l = 0, \text{ and } (\tilde{\Psi} \circ \Phi - \text{id}_F)^l = (d^\infty \tilde{\eta})^l$$

hold for $l = 1, \ldots, m - 1$. Now we introduce

We denote $(d^\infty \tilde{\Psi})^m$ by $\lambda^m$. For objects $\mathcal{E}_0, \ldots, \mathcal{E}_m, \lambda_m$ can by considered as a degree $1 - m$ map

$$\mathbb{L}(X)(\mathcal{E}_{m-1}, \mathcal{E}_m) \otimes \ldots \otimes \mathbb{L}(X)(\mathcal{E}_0, \mathcal{E}_1) \to \mathbb{L}(Y)(G\mathcal{E}_0, F\mathcal{E}_m).$$

For later applications we consider $\lambda^m$ as a degree $0$ map

$$\lambda^m \in \text{Hom}^0(\mathbb{L}(X)(\mathcal{E}_{m-1}, \mathcal{E}_m)[1] \otimes \ldots \otimes \mathbb{L}(X)(\mathcal{E}_0, \mathcal{E}_1)[1], \mathbb{L}(Y)(G\mathcal{E}_0, F\mathcal{E}_m)[1])$$

Moreover we denote $(\tilde{\Psi} \circ \Phi - \text{id}_F - d^\infty \tilde{\eta})^m$ by $\mu^m$. As before we have

$$\mu^m \in \text{Hom}^1(\mathbb{L}(X)(\mathcal{E}_{m-1}, \mathcal{E}_m)[1] \otimes \ldots \otimes \mathbb{L}(X)(\mathcal{E}_0, \mathcal{E}_1)[1], \mathbb{L}(Y)(F\mathcal{E}_0, F\mathcal{E}_m)[1])$$

Lemma 4.2. In the above notation, $\lambda^m$ and $\mu^m$ are defined by compositions of $F, G, \Phi,$ and $\mathcal{H}$.

Proof. By Definition 4.1 we have

$$\lambda^m(u_m \otimes \ldots \otimes u_1) = (d^\infty \tilde{\Psi})^m(u_m \otimes \ldots \otimes u_1) =$$

$$( -1)^{|u_{m-1}|} G(u_1) \Psi^{m-1}(u_{m-1} \otimes \ldots \otimes u_1)$$

$$+ \sum_{i=1}^{m-1} (-1)^{|u_m| + \ldots + |u_{i+1}|} \Psi^{m-1}(u_m \otimes \ldots \otimes u_{i+1} \otimes u_i \otimes \ldots \otimes u_1)$$

The claim for $\lambda^m$ is clear by the induction hypothesis. We can prove the claim for $\mu^m$ in the same way.

Lemma 4.3. In the above notation, we have

$$d\lambda^m = 0, \text{ and } d\mu^m = \lambda^m \Phi \mathcal{E}_0.$$
Proof. By the induction hypothesis we have \((d^\infty \Psi)^l = 0\) for \(l = 1, \ldots, m - 1\). Then \(d^\infty d^\infty \Psi = d[(d^\infty \Psi)^m] = d\lambda^m\). But we have \(d^\infty d^\infty \Psi = 0\) hence \(d\lambda^m = 0\).

Since \((\tilde{\Psi} \circ \Phi - \text{id}_F - d^\infty \tilde{\eta})^l = 0\) for \(l = 1, \ldots, m - 1\), it is clear that
\[
(d^\infty (\tilde{\Psi} \circ \Phi - \text{id}_F - d^\infty \tilde{\eta}))^m = d\mu^m
\]
On the other hand since \(\Phi\) and \(\text{id}_F\) are \(dg\) \(k\)-natural transformations, we have \(d^\infty \Phi = 0\) and \(d^\infty \text{id}_F = 0\). Therefore
\[
d^\infty (\tilde{\Psi} \circ \Phi - \text{id}_F - d^\infty \tilde{\eta}) = (d^\infty \tilde{\Psi}) \circ \Phi.
\]

Compare the degree \(m\) component we have \(d\mu^m = \lambda^m \Phi E_0\).

As suggested by [Lyu03] Appendix B we let
\[
\Psi^m = \lambda^m p E_0 - \mu^m \Psi E_0
\]
and
\[
\eta^m = -\mu^m h E_0 + \mu^m \Psi E_0 \Phi E_0 h E_0 - \lambda^m p E_0 \Phi E_0 h E_0 - \mu^m E_0 p E_0 \Phi E_0 + \lambda^m p E_0 p E_0 p E_0 \Phi E_0.
\]

It is clear that
\[
d(\Psi^m) = -\lambda^m, \quad \text{and} \quad d(\eta^m) = \Psi^m \Phi E_0 + \mu^m.
\]

Let
\[
\tilde{\Psi} = (\Psi^0, \Psi^1, \ldots, \Psi^{m-1}, 0, \ldots) \quad \tilde{\eta} = (\eta^0, \eta^1, \ldots, \eta^{m-1}, 0, \ldots)
\]

It is clear by Equation (11) and the induction hypothesis that
\[
(d^\infty \tilde{\Psi})^l = 0, \quad \text{and} \quad (\tilde{\Psi} \circ \Phi - \text{id}_F)^l = (d^\infty \tilde{\eta})^l
\]
hold for \(l = 1, \ldots, m\). Then by induction we construct an \(A^\infty\) \(k\)-natural transformation \(\Psi : G \Rightarrow F\) of degree 0 and an \(A^\infty\) \(k\)-prenatural transformation \(\eta : F \Rightarrow F\) of degree 1 such that
\[
d^\infty \Psi = 0, \quad \text{and} \quad \Psi \circ \Phi - \text{id}_F = d^\infty \eta.
\]

Notice that \(\Psi\) and \(\eta\) are defined by compositions of \(F, G, \Phi, \) and \(H\).

We need to construct the homotopy of the other composition. Actually in the same way we can construct an \(A^\infty\) \(k\)-natural transformation \(\Psi' : G \Rightarrow F\) of degree 0 and an \(A^\infty\) \(k\)-prenatural transformation \(\omega' : G \Rightarrow G\) of degree 1 such that
\[
d^\infty \Psi' = 0, \quad \text{and} \quad \Phi \circ \Psi' - \text{id}_G = d^\infty \omega'.
\]

where \(\Psi'\) and \(\omega'\) are also defined by compositions of \(F, G, \Phi, \) and \(H\). It is clear that
\[
\Psi' = \Psi + d^\infty (\Psi \omega' - \eta \Psi').
\]

Therefore let
\[
\omega := \omega' + \Phi \eta \Psi' - \Phi \Psi \omega'
\]
then we have
\[
\Phi \circ \Psi - \text{id}_G = d^\infty \omega.
\]

and \(\omega\) is also defined by compositions of \(F, G, \Phi, \) and \(H\).
Remark 6. The recurrent definition of $\Psi$ and $\eta$ is given by Equation (9) and (10).

**Corollary 4.4.** Let $k$ be a field and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of $k$-ringed spaces. Then there exist $A_\infty$ $k$-natural transformations

$$T_{f,g} : (gf)^* \sim f^* g^*$$
$$T_{f,g} : (gf) \sim g f$$
$$T_{id} : \text{id}^* \sim \text{id}$$
$$T_{id} : \text{id} \sim \text{id}$$

(12)

which only depend on the choice of objectwise homotopy inverse systems as in Definition 4.1.

**Proof.** It is a direct corollary of Proposition 3.3 and Theorem 4.1.

Remark 7. In [Wei19] the formula in this note is used to construct $A_\infty$-inverse of dg-natural transformations between twisted complexes. Moreover in an upcoming work [Wei], the formula in this note, together with the method in [Wei16], [Wei18], [BHW17], and [AØ18], can be used to obtain an injective dg-resolution of the equivariant derived category [BL94].

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