INSTABILITY OF VORTEX SOLITONS FOR 2D FOCUSING NLS

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Abstract. We study instability of a vortex soliton \(e^{i(m\theta + \omega t)}\phi_{\omega,m}(r)\) to
\[ iu_t + \Delta u + |u|^{p-1}u = 0, \quad \text{for } x \in \mathbb{R}^n, \quad t > 0, \]
where \(n = 2\), \(m \in \mathbb{N}\) and \((r, \theta)\) are polar coordinates in \(\mathbb{R}^2\). Grillakis \cite{11} proved that every radially standing wave solutions are unstable if \(p > 1 + 4/n\). However, we do not have any examples of unstable standing wave solutions in the subcritical case \((p < 1 + n/4)\).

Suppose \(\phi_{\omega,m}\) is nonnegative. We investigate a limiting profile of \(\phi_{\omega,m}\) as \(m \to \infty\) and prove that for every \(p > 1\), there exists an \(m^* \in \mathbb{N}\) such that for \(m \geq m^*\), a vortex soliton \(e^{i(m\theta + \omega t)}\phi_{\omega,m}(r)\) becomes unstable to the perturbations of the form \(e^{i(m+j)\theta}v(r)\) with \(1 \ll j \ll m\).

1. Introduction

In the present paper, we consider instability of radially symmetric vortex solitons to 2-dimensional nonlinear Schrödinger equations
\[
\begin{cases}
iu_t + \Delta u + f(u) = 0 & \text{for } (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^2,
\end{cases}
\]
where \(n = 2\) and \(f(u) = |u|^{p-1}u\). Let \(\omega > 0\), \(m \in \mathbb{N} \cup \{0\}\), and let \(e^{i(\omega t + m\theta)}\phi_\omega(r)\) be a standing wave solution of (1) belonging to \(H^1(\mathbb{R}^2)\). Here \(r\) and \(\theta\) denote polar coordinates in \(\mathbb{R}^2\). Then \(\phi_\omega(r)\) is a solution to
\[
\begin{cases}
\phi'' + \frac{1}{r}\phi' - \left(\omega + \frac{m^2}{r^2}\right)\phi + f(\phi) = 0 & \text{for } r > 0, \\
\lim_{r \to 0} \frac{\phi(r)}{r^m} = \lim_{r \to 0} \frac{\phi'(r)}{mr^{m-1}}, \\
\lim_{r \to \infty} \phi(r) = 0.
\end{cases}
\]
We remark that \(e^{im\theta}\phi_\omega(r)\) is a solution to the scalar field equation
\[
\Delta \varphi - \omega \varphi + f(\varphi) = 0 \quad \text{for } x \in \mathbb{R}^2.
\]
A standing wave solution of the form \(e^{i(\omega t + m\theta)}\phi_\omega(r)\) appears in the study of nonlinear optics (see references in \cite{17}). If \(m = 0\) and \(\phi_\omega(r)\) is positive, then \(\phi_\omega\) is a ground state. Existence and uniqueness of the ground state are well known (see \cite{5}, \cite{6}, \cite{16} and reference therein).

If \(m \neq 0\), Iaia and Warchall proved the existence of smooth solutions to (2) with any prescribed number of zeroes. The uniqueness of positive

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solutions has been proved by [18] by using the classification theorem of positive solutions due to Yanagida and Yotsutani [29].

Let \( c > 0 \) and let \( Q_c \) be a positive solution to

\[
\begin{align*}
Q'' - cQ + f(Q) &= 0 \quad \text{for } x \in \mathbb{R}, \\
\lim_{x \to \pm \infty} Q(x) &= 0, \\
Q(0) &= \max_{x \in \mathbb{R}} Q(x).
\end{align*}
\]

Then

\[
Q_c(x) = \left( \frac{(p+1)c}{2} \right)^{\frac{1}{p-1}} \text{sech}^{p-1} \left( \frac{(p-1)\sqrt{c}}{2} x \right).
\]

In [17], Pego and Warchall numerically observe that as spin index \( m \) becomes larger, a solution \( \phi_{\omega,m} \) to (2) remains small initially and then is approximated by \( Q_c(r - \bar{r}) \) around \( r = \bar{r} \), where \( c = \omega + (m^2/\bar{r}^2) \) and \( \bar{r} \) is a positive number with \( \bar{r} = O(m) \) as \( m \to \infty \) (see also [21] and references in [17]). One of our goals in the present paper is to explain this phenomena.

Benci and D’Aprile [3] studied (2) in a general setting and locate the asymptotic peak of solutions (see also [10]). Recently, Ambrosetti, Malchiodi and Ni [2] have proved the existence of positive radial solutions concentrating on spheres to a class of singularly perturbed problem

\[
\varepsilon^2 \Delta u - V(u) + |u|^{p-1}u = 0,
\]

and obtain their asymptotic profile. Adopting the argument in [2], we obtain the following.

**Theorem 1.** Let \( p > 1 \) and let \( \phi_{\omega,m} \) be a positive solution to (2). Then there exists an \( m_* \in \mathbb{N} \) such that if \( m \geq m_* \),

\[
\| \phi_{\omega,m}(\cdot) - Q_c(\cdot - \bar{r}) \|_{H^2(\mathbb{R}^2)} = O(m^{-1/2}),
\]

\[
\| \phi_{\omega,m}(\cdot) - Q_c(\cdot - \bar{r}) \|_{L^\infty(\mathbb{R}^2)} = O(m^{-1}),
\]

where \( \bar{r} = 2m/\sqrt{(p-1)\omega} \) and \( c = (p+3)\omega/4 \).

**Remark 1.** Let \( r = ms, \varepsilon = 1/m \) and \( V(r) = \omega + r^{-2} \). Then (2) is transformed into

\[
\varepsilon^2 \Delta_r \phi - V(r)\phi + f(\phi) = 0.
\]

Though [2] assumes the boundedness of \( V(r) \) and cannot be applied directly to our problem, a maximum point of \( \phi_{\omega,m}(r) \) can be predicted from an auxiliary weighted potential \( rV(r) \) introduced by [2].

Let \( \varphi_\omega \) be a ground state to (3). As is well known, the standing wave solution \( e^{i\omega t} \varphi_\omega \) is stable if \( d\|\varphi_\omega\|_{L^2(\mathbb{R}^n)}^2/\omega > 0 \) and unstable if \( d\|\varphi_\omega\|_{L^2(\mathbb{R}^n)}^2/\omega < 0 \). See e.g. Berestycki-Cazenave [4], Cazenave-Lions [7], Grillakis-Shatah-Strauss [12], Shatah [23], Shatah-Strauss [24] and Weinstein [28]. Namely, the standing wave solution \( e^{i\omega t} \varphi_\omega \) is stable if \( 1 < p < 1 + 4/n \) and unstable if \( p \geq 1 + 4/n \). Grillakis [11] proved that every radially symmetric standing wave solution is linearly unstable if \( p > 1 + 4/n \). However, to the best our knowledge, it remains unknown whether there exists an unstable standing wave solution with higher energy in the subcritical case \((1 < p < 1 + 4/n)\).
Using Theorem 1, we find an unstable direction and prove $e^{i(\omega t + m\theta)} \phi_\omega(r)$ is unstable in $H^1(\mathbb{R}^2)$ if $p > 1$ and $m$ is sufficiently large.

**Theorem 2.** Let $p > 1$ and $\phi_{\omega,m}$ be as in Theorem 1. Then there exists an $m* \in \mathbb{N}$ such that if $m \geq m*$, a standing wave solution $e^{i(\omega t + m\theta)} \phi_\omega$ is linearly unstable.

**Remark 2.** By Shatah-Strauss Lemma (see [25, 26], see also [20]), we have orbital instability of the linearly unstable standing wave solutions.

**Remark 3.** If $p < 1 + 4/n$ and $u_0 \in H^1(\mathbb{R}^n)$, a solution to (1) exists globally in time and remains bounded in $H^1(\mathbb{R}^n)$. Thus the mechanism of instability shown in Theorem 2 is quite different from that of [4] where solutions around a standing wave solution blow up in finite time. The instability mechanism we find is close to transversal long-wave instability of 1-dimensional soliton (see Alexander-Pego-Sachs [1] for KP equation and Bridges [8, 9] for nonlinear Schrödinger equation). Theorem 1 shows that a profile of vortex soliton is close to 1D-soliton for large $m$ and thus it becomes possible to find unstable modes by using perturbation method.

**Remark 4.** If $\phi_{\omega,m}$ is nonnegative, then $e^{i(m\theta + \omega t)} \phi_\omega(r)$ is a ground state in the class $X_m = \{e^{imq} v(r) \mid v \in H^1_{rad}(\mathbb{R}^2), v \in L^2_{rad}(\mathbb{R}^2)\}$ and it follows from Grillakis et al. ([12]) that the standing wave solution $e^{i(m\theta + \omega t)} \phi_\omega(r)$ is stable in the class $X_m$ if $1 < p < 3$ ([10]). Thus the vortex soliton is stable to the symmetric perturbations in the subcritical case.

Our plan of the present paper is as follows. In Section 2, we specify a solution to (2) which is expected to become close to a solution to (1) as $m$ tends to infinity. In Section 3, we investigate some properties of the linearized operator around an approximate solution constructed is Section 2. In Section 4, we prove Theorem 1 following the lines of [2] and using Linearized operator around an approximate solution constructed is Section 2. Finally, we introduce several notations. For Banach spaces $X$ and $Y$, let $B(X,Y)$ be the space of all bounded linear operators from $X$ to $Y$ and let $\|A\|_{B(X,Y)}$ be the operator norm of an operator $A$: $X \to Y$. We abbreviate $B(X,X)$ as $B(X)$. We denote by $D(A)$ and $R(A)$ the domain and the range of the operator $A$, respectively. We use notations $\|f\|_{L^p_2(\mathbb{R}^2)} = (\int_0^\infty |f(r)|^2 2r dr)^{1/2}$, $\|f\|_{H^1_2(\mathbb{R}^2)} = (\int_0^\infty (|f'(r)|^2 + |f(r)|^2) 2r dr)^{1/2}$, $\Delta_r = \partial_r^2 + r^{-1} \partial_r$ and $\|f\|_{H^2_2(\mathbb{R}^2)} = \|(1 - \Delta_r) f\|_{L^2_2(\mathbb{R}^2)}$. Various constants will be simply denoted by $C$ and $C_i$ ($i \in \mathbb{N}$) in the course of calculations.

2. AN APPROXIMATION

In this section, we will construct an approximate solution to (2) for large $m$. Suppose that a positive solution to (2) is approximated by $Q_c(r - \bar{r})$ around $r = \bar{r}$ for large $m$. Let $\alpha_0 = \bar{r}/m$, $\varepsilon = m^{-1}$, $s = r - \bar{r}$ and
\( v(s) = \phi_\omega(r) \). Then (2) transforms into (8)
\[
\begin{cases}
\left\{ \begin{array}{l}
v_{ss} + \frac{\varepsilon}{\alpha_0 + \varepsilon s} v_s - \left( \omega + \frac{1}{(\alpha_0 + \varepsilon s)^2} \right) v + f(v) = 0 \quad \text{for } s \in (-\bar{r}, \infty), \\
\lim_{s \to -\bar{r}} \frac{v(s)}{(s+\bar{r})^m} = \lim_{s \to -\bar{r}} \frac{v_s}{m(s+\bar{r})^{m-1}}, \\
\lim_{s \to +\infty} v(s) = 0.
\end{array} \right.
\]

Substituting \( v(s) = v_0(s) + \varepsilon v_1(s) + O(\varepsilon^2) \) into (2) and formally equating the power of \( \varepsilon \), we obtain
\[
\begin{cases}
v''_0 - cv_0 + f(v_0) = 0, \\
\lim_{s \to \pm\infty} v_0(s) = 0,
\end{cases}
\]
and
\[
\begin{cases}
v''_1 - cv_1 + f'(v_0)v_1 = -\alpha_0^{-1}v'_0 - 2\alpha_0^{-3}sv_0, \\
\lim_{s \to \pm\infty} v_1(s) = 0,
\end{cases}
\]
where \( c = \omega + \alpha_0^{-2} \). Let \( v_0(s) = Q_c(s), \ L_c := \partial^2_s - c + f'(Q_c) \) and \( D(L_c) = H^2(\mathbb{R}) \). Since \( \ker(L_c) = \operatorname{span}\{Q_c'\} \), the Fredholm alternative implies that (10) has a solution \( v_1 \in L^2(\mathbb{R}) \) if and only if
\[
\int_\mathbb{R} Q'_c(s) \left( Q'_c(s) + \frac{2s}{\alpha_0^2} Q_c(s) \right) ds = \int_\mathbb{R} \left( Q'_c(s)^2 - \frac{1}{\alpha_0^2} Q_c(s)^2 \right) ds = 0.
\]

**Lemma 3.** Let \( c = \omega + \alpha_0^{-2} \) and let \( Q_c \) be a solution to (2). If (11) holds, then \( c = (p + 3)\omega/4 \) and \( \alpha_0 = 2/\sqrt{(p - 1)\omega} \).

**Proof.** By (11),
\[
\left( \frac{dQ_c}{dx} \right)^2 = cQ_c^2 \left( 1 - \left( \frac{Q_c}{A} \right)^{p-1} \right),
\]
where \( A^{p-1} = (p + 1)c/2 \). We compute
\[
\int_{-\infty}^{\infty} Q'_c(x)^2 dx = 2 \int_0^A \left( \frac{dQ_c}{dx} \right)^2 \left( -\frac{dx}{dQ_c} \right) dQ_c = 2\sqrt{c} \int_0^A u \sqrt{1 - \left( \frac{u}{A} \right)^{p-1}} du = \frac{2}{p - 1} \sqrt{c} A^2 B \left( \frac{2}{p - 1}, \frac{3}{2} \right),
\]
and
\[
\int_{-\infty}^{\infty} Q_c(x)^2 dx = 2 \int_0^A Q_c^2 \left( -\frac{dx}{dQ_c} \right) dQ_c = \frac{2}{\sqrt{c}} \int_0^A \frac{u}{\sqrt{1 - \left( \frac{u}{A} \right)^{p-1}}} du = \frac{2A^2}{(p - 1)^{3/2}} B \left( \frac{2}{p - 1}, \frac{1}{2} \right).
\]
Combining the above, we have \( c = (p + 3)\omega/4 \) and \( \alpha_0 = 2/\sqrt{(p - 1)\omega} \). \( \square \)
Let $\chi(s)$ be smooth nonnegative functions on $\mathbb{R}$ satisfying $0 \leq \chi(r) \leq 1$ and
\[
\chi(r) = \begin{cases} 
1 & \text{if } |r| \leq 2, \\
0 & \text{if } |r| \geq 3,
\end{cases}
\]
and let $\chi_l(s) = \chi(s/l)$, where $l = -\frac{2}{\sqrt{c}} \max(1, \frac{1}{p-1}) \log \epsilon$. Following [2], we put
\begin{align}
(12) \quad & \Phi(\epsilon, \rho)(r) = \chi_l(r - \rho)Q_c(r - \rho), \quad c = \omega + (\epsilon \rho)^{-2}, \\
(13) \quad & \phi_{\omega,m} = \Phi(\epsilon, \rho) + w,
\end{align}
and search for a positive solution to (2) for large $m$. To fix the decomposition (14), we assume
\begin{equation}
(14) \quad (w, \partial_{\rho} \Phi)_{L^2(\mathbb{R}^2)} = 0.
\end{equation}
Substituting (12) into (2), we obtain
\begin{equation}
(15) \quad \mathcal{L}(\epsilon, \rho)w + R_1(\epsilon, \rho, w) + R_2(\epsilon, \rho) = 0,
\end{equation}
where $R_2 = R_{21} + R_{22} + R_{23}$ and
\[
\mathcal{L}(\epsilon, \rho) = \Delta_r - \omega - \frac{m^2}{r^2} + f'(\Phi(\epsilon, \rho)), \\
R_1 = f(\Phi(\epsilon, \rho) + w) - f(\Phi(\epsilon, \rho)) - f'(\Phi(\epsilon, \rho))w, \\
R_{21} = f(\Phi(\epsilon, \rho)) - \tau_{\rho}(\chi_l f'(Q_c)) \\
R_{22} = \left( c - \omega - \frac{m^2}{r^2} \right) \Phi(\epsilon, \rho) + \frac{1}{r} \tau_{\rho}(\chi_l Q'_c) \\
R_{23} = \tau_{\rho}(\chi''_l Q_c + 2\chi'_l Q'_c) + \frac{1}{r} \tau_{\rho}(\chi'_l Q_c).
\]
Here $\tau_l$ denotes the translation, that is, $(\tau_l f)(x) = f(x - h)$. We will search a solution $(\rho, w)$ to (14) and (15) with $\rho \in (\alpha_0/(2\epsilon), 2\alpha_0/\epsilon)$ for large $m \in \mathbb{N}$.

3. Spectrum of the linearized operator $\mathcal{L}(\epsilon, \rho)$

In this section, we examine spectral properties of the linearized operator $\mathcal{L}(\epsilon, \rho)$. To begin with, we recall some properties of the operator $\Delta_r - \omega - \frac{m^2}{r^2}$.

Lemma 4. Let $0 < \epsilon < 1/2$ and $\mathcal{L}_0(\epsilon) : L^2_r(\mathbb{R}^2) \rightarrow L^2_r(\mathbb{R}^2)$ be a closed operator such that
\[
\mathcal{L}_0(\epsilon)u = \Delta_r u - \omega u - (\epsilon r)^{-2} u
\]
for $u \in C_0^\infty(\mathbb{R}_+)$. Then $\mathcal{L}_0(\epsilon)$ is a self-adjoint operator with
\[
D(\mathcal{L}_0(\epsilon)) = \{ u \in H^2_r(\mathbb{R}^2) \mid r^{-2} u \in L^2_r(\mathbb{R}^2) \} \quad \text{and} \quad R(\mathcal{L}_0(\epsilon)) = L^2_r(\mathbb{R}^2).
\]
Proof. Let $X = \{ H^2_r(\mathbb{R}^2) \mid r^{-2} u \in L^2_r(\mathbb{R}^2) \}$ be a Hilbert space equipped with the norm $\| u \|_X = (\| u \|_{H^2_r(\mathbb{R}^2)}^2 + \| r^{-2} u \|_{L^2_r(\mathbb{R}^2)}^2)^{1/2}$.

By Theorem 10.10 and Example 4 in [22, Appendix to X.1], the operator $\mathcal{L}_0(\epsilon)$ is essentially self-adjoint in $C_0^\infty(\mathbb{R}_+)$. Thus for any $u \in D(\mathcal{L}_0(\epsilon))$, there exist $u_n \in C_0^\infty(\mathbb{R}_+) \ (n \in \mathbb{N})$ such that $\mathcal{L}_0(\epsilon)u_n \rightarrow \mathcal{L}_0(\epsilon)u$ and $u_n \rightarrow u$ in $L^2_r(\mathbb{R}^2)$ as $n \rightarrow \infty$. 
Integrating by parts, we have
\begin{equation}
\| \mathcal{L}_0(\varepsilon) w \|_{L^2_\varepsilon(\mathbb{R}^2)}^2
= \| (\omega - \Delta_r) w \|_{L^2_\varepsilon(\mathbb{R}^2)}^2 + 2 \mathbb{R} \left( (\omega - \Delta_r) w, (\varepsilon r)^{-2} w \right)_{L^2_\varepsilon(\mathbb{R}^2)} + \| (\varepsilon r)^{-2} w \|_{L^2_\varepsilon(\mathbb{R}^2)}^2
\geq \| (\omega - \Delta_r) w \|_{L^2_\varepsilon(\mathbb{R}^2)}^2 + (\varepsilon^{-4} - 4\varepsilon^{-2}) \| r^{-2} w \|_{L^2_\varepsilon(\mathbb{R}^2)}^2
\end{equation}
for every $w \in C_0^\infty(\mathbb{R}_+)$.

Next we prove $D(\mathcal{L}_0(\varepsilon)) \supset X$. For every $u \in X$, there exist $u_n \in C_0^\infty(\mathbb{R}_+)$ ($n = 1, 2, \ldots$) such that $\lim_{n \to \infty} \| u_n - u \|_X = 0$. Since
\[ \| \mathcal{L}_0(\varepsilon) w \|_{L^2_\varepsilon(\mathbb{R}^2)} \leq \max(1, \omega) \| w \|_{H^2_\varepsilon(\mathbb{R}^2)} + \varepsilon^{-2} \| r^{-2} w \|_{L^2_\varepsilon(\mathbb{R}^2)}, \]
we see that $\{ \mathcal{L}_0(\varepsilon) u_n \}_{n=1}^\infty$ and $\{ u_n \}_{n=1}^\infty$ are Cauchy sequences in $L^2_\varepsilon(\mathbb{R}^2)$ and that there exists a $v \in L^2_\varepsilon(\mathbb{R}^2)$ such that $\mathcal{L}_0(\varepsilon) u_n \to v$ as $n \to \infty$. Since $\mathcal{L}_0(\varepsilon)$ is closed, it follows that $v = \mathcal{L} u$ and $u \in D(\mathcal{L}_0(\varepsilon))$. Thus we prove $D(\mathcal{L}_0(\varepsilon)) = X$.

Finally, we will show that $R(\mathcal{L}_0(\varepsilon)) = L^2_\varepsilon(\mathbb{R}^2)$. The self-adjointness of $\mathcal{L}_0(\varepsilon)$ and (16) implies
\[ R(\mathcal{L}_0(\varepsilon))^\perp = \ker(\mathcal{L}_0(\varepsilon)) = \{ 0 \}. \]
Hence it follows that $R(\mathcal{L}_0(\varepsilon)) = L^2_\varepsilon(\mathbb{R}^2)$ and that for every $v \in L^2_\varepsilon(\mathbb{R}^2)$, there exist $u_n \in X$ and $v_n \in L^2_\varepsilon(\mathbb{R}^2)$ ($n \in \mathbb{N}$) such that
\[ \mathcal{L}_0(\varepsilon) u_n = v_n \to v \quad \text{in} \quad L^2_\varepsilon(\mathbb{R}^2) \quad \text{as} \quad n \to \infty. \]
By (16), there exists $u \in X$ such that $\lim_{n \to \infty} u_n = u$ in $X$. Since $\mathcal{L}_0(\varepsilon)$ is closed we have $v = \mathcal{L}_0(\varepsilon) u \in R(\mathcal{L}_0(\varepsilon))$. This completes the proof of Lemma [1].
Proof. The former part of the lemma can be obtained by a simple computation. Let \( c = 1 \). Weyl’s essential spectrum theorem tells us that the spectrum of \( L_1 \) consists of essential spectrum \((-\infty, -1]\) and discrete eigenvalues. Since \( Q_1' \) has exactly one zero and \( L_1 Q_1' = 0 \), it follows from Strum’s comparison theorem that 0 is a second eigenvalue of \( L_1 \) and that \( \ker(L_1) \) is spanned by \( Q_1' \). Since \( L_1(u(c^{1/2}x)) = c(L_1u)(c^{1/2}x) \) for every \( u \in H^2(\mathbb{R}) \), we have \( \sigma(L_c) = \{c\lambda | \lambda \in \sigma(L_1)\} \). Thus we prove Lemma 5.

Proof of Lemma 5. Let \( \chi_0(s) = 1 - \chi_1(s) \) and \( \chi_1(s) = \chi_1(s) \). By (5) and the fact that \( \text{supp} \chi_0 \subset \{r \in \mathbb{R} | |r| \geq 2l \} \),

\[
(L(\varepsilon, \rho)w, w)_{L^2_\omega(\mathbb{R}^2)} = (L(\varepsilon, \rho)\chi_1 w, \chi_1 w)_{L^2_\omega(\mathbb{R}^2)} + 2(L(\varepsilon)\chi_0 w, \chi_1 w)_{L^2_\omega(\mathbb{R}^2)} + O(\varepsilon^{-2(p-1)}\sqrt{\varepsilon})||w||_{L^2_\omega(\mathbb{R}^2)}^2.
\]

Integrating by parts and substituting \(|\chi_0'(r)| + |\chi_1'(r)| = O(l^{-1})\) into the resulting equation, we have

\[
-(L(\varepsilon)\chi_0 w, \chi_0 w)_{L^2_\omega(\mathbb{R}^2)} = \int_0^\infty \chi_0 \left( \chi_0 w \right)^2 + (\omega + (\varepsilon r)^{-2})\left(\chi_0 w\right)^2 \, dr = \int_0^\infty \chi_0 \chi_1 \left( w_r^2 + \omega w^2 + (\varepsilon r)^{-2}w^2 \right) \, dr + O(l^{-1}\|w\|^2_{H^1(\mathbb{R}^2)}).
\]

and

\[
-(L(\varepsilon)\chi_0 w, \chi_1 w)_{L^2_\omega(\mathbb{R}^2)} = \int_0^\infty \chi_0 \chi_1 \left( w_r^2 + \omega w^2 + (\varepsilon r)^{-2}w^2 \right) \, dr + O(l^{-1}\|w\|^2_{H^1(\mathbb{R}^2)}).
\]

Let \( U : L^2_\omega(\mathbb{R}^2) \to L^2(\mathbb{R}_+) \) be the unitary operator defined by \( U \phi(r) = r^\frac{1}{2} \phi(r) \). Then

\[
\tau_{-\rho} U L(\varepsilon) U^{-1} = \partial_r^2 - \omega - \frac{1 - \frac{1}{4} \varepsilon^2}{(\alpha + \varepsilon r)^2} + f'(\chi_1 Q_c)
\]

\[
= L_c + \left( \frac{1}{\alpha^2} - \frac{1 - \frac{1}{4} \varepsilon^2}{(\alpha + \varepsilon r)^2} \right) + f'(\chi_1 Q_c) - f'(Q_c),
\]

where \( \alpha = \rho/m \) and \( c = \omega + \alpha^{-2} \). Let \( \check{\chi}_1 \) and \( \check{\chi}_2 \) be smooth nonnegative functions on \( \mathbb{R} \) satisfying

\[
\sup_{r \in \mathbb{R}} |\chi_i'(r)| = O(l^{-1}) \quad \text{for } i = 0, 1,
\]

\[
\check{\chi}_0(r) = \begin{cases} 0 & \text{if } |r| \leq l, \\ 1 & \text{if } |r| \geq 2l \end{cases}, \quad \check{\chi}_1(r) = \begin{cases} 1 & \text{if } |r| \leq 3l, \\ 0 & \text{if } |r| \geq 4l \end{cases}.
\]

Put \( \check{w}(r) = (r + \rho)^{1/2} \chi_1(r)(w(r + \rho)). \) Using \( w \perp \partial_r \Phi(\varepsilon, \rho) \) and

\[
\partial_r \Phi(\varepsilon, \rho) = -\tau_\rho (\chi_1 Q_c)' - \frac{2\varepsilon}{\alpha^3} \tau_\rho (\chi_1 \partial_c Q_c),
\]

(20)
we have
\[ 0 = (w, \partial_p \Phi(\epsilon, \rho))_{L^2(\mathbb{R}^2)} \]
\[ = -\int_{-\rho}^{\infty} (\chi_1 Q_2)'(r)w(r + \rho)(r + \rho)dr + O(\epsilon^{1/2} \|w\|_{L^2(\mathbb{R}^2)}) \]
\[ = -\int_{\mathbb{R}} (\rho + r)^{1/2} \chi_1 \tilde{w} Q'_1 dr + O \left( (\rho^{1/2} e^{-2\sqrt{\epsilon}} + \epsilon^{1/2}) \|w\|_{L^2(\mathbb{R}^2)} \right) \]
\[ = -\rho^{1/2} \int_{\mathbb{R}} \tilde{w} Q'_1 dr + O(\epsilon^{1/2} \|w\|_{L^2(\mathbb{R}^2)}). \]
Hence it follows that
\[ (\tilde{w}, Q'_1)_{L^2(\mathbb{R})} = O(\epsilon \log \epsilon \|w\|_{L^2(\mathbb{R}^2)}). \]
Similarly, we have
\[ (\tilde{w}, Q_{2t}^{\rho,t})_{L^2(\mathbb{R}^2)} = O(\epsilon \log \epsilon \|w\|_{L^2(\mathbb{R}^2)}). \]
Combining Lemma 5 with (19), (21) and (22), we see that there exist positive constants \( C_1 \) and \( C_2 \) such that
\[ -(\mathcal{L}(\epsilon, \rho)w, w)_{L^2(\mathbb{R}^2)} \geq C_1 \|\tilde{w}\|_{H^1(\mathbb{R})}^2 \geq C_2 \|\chi_1 w\|_{H^1(\mathbb{R}^2)}^2. \]
Thus by (17), (18) and (23), there exist positive numbers \( c_1 \) and \( \varepsilon_* \) such that
\[ -(\mathcal{L}(\epsilon, \rho)w, w)_{L^2(\mathbb{R}^2)} \geq c_1 \|w\|_{H^1(\mathbb{R}^2)}^2 \]
for every \( \varepsilon \in (0, \varepsilon_*) \), \( \rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon) \).

Let \( X_1 = Q(\epsilon, \rho)X, Y_1 = Q(\epsilon, \rho)L^2(\mathbb{R}^2) \) and \( \mathcal{A}(\epsilon, \rho) = Q(\epsilon, \rho)\mathcal{L}(\epsilon, \rho)Q(\epsilon, \rho) \).
Lemma 6 yields that \( \mathcal{A}(\epsilon, \rho) : X_1 \to Y_1 \) is isomorphic.

**Corollary 7.** There exist positive numbers \( \varepsilon_* \) and \( \nu \) such that
\[ \|\mathcal{A}(\epsilon, \rho)^{-1}u\|_X \leq \nu \|u\|_{L^2(\mathbb{R}^2)} \]
for every \( u \in Y_1, \varepsilon \in (0, \varepsilon_*) \) and \( \rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon) \).

**Proof.** Let \( Q_1 \) and \( Q_2 \) be orthogonal projections such that
\[ Q_1u = \frac{(u, Q(\epsilon, \rho)\Phi(\epsilon, \rho)^{p+1/2})_{L^2(\mathbb{R}^2)}}{\|Q(\epsilon, \rho)\Phi(\epsilon, \rho)^{p+1/2}\|_{L^2(\mathbb{R}^2)}} Q(\epsilon, \rho)^{p+1/2}, \]
\[ Q_2 = Q(\epsilon, \rho) - Q_1. \]
Then \( \mathcal{A}(\epsilon, \rho) \) can be written as
\[ \mathcal{A}(\epsilon, \rho) = \begin{pmatrix} Q_1 \mathcal{L}(\epsilon, \rho)Q_1 & Q_1 \mathcal{L}(\epsilon, \rho)Q_2 \\ Q_2 \mathcal{L}(\epsilon, \rho)Q_1 & Q_2 \mathcal{L}(\epsilon, \rho)Q_2 \end{pmatrix}. \]
In view of Lemma 6 we see that there exists a \( c_2 > 0 \) such that
\[ (\mathcal{L}(\epsilon, \rho)Q(\epsilon, \rho)\Phi(\epsilon, \rho)^{p+1/2}, Q(\epsilon, \rho)\Phi(\epsilon, \rho)^{p+1/2})_{L^2(\mathbb{R}^2)} \geq c_2 \|Q(\epsilon, \rho)\Phi(\epsilon, \rho)^{p+1/2}\|_{L^2(\mathbb{R}^2)}^2. \]
Furthermore, we see that
\[ \lim_{\epsilon \downarrow 0} \left( \|Q_1 \mathcal{L}(\epsilon, \rho)Q_2\|_{B(L^2(\mathbb{R}^2))} + \|Q_2 \mathcal{L}(\epsilon, \rho)Q_1\|_{B(L^2(\mathbb{R}^2))} \right) = 0. \]
Combining the above with Lemma 5, we obtain

\[(25) \quad \sup_{\varepsilon \in (0, \varepsilon_*)} \sup_{\rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon)} \|A(\varepsilon, \rho)^{-1}\|_{B(L^2(\mathbb{R}^2))} < \infty.\]

Let

\[B(\varepsilon, \rho) = \mathcal{P}(\varepsilon, \rho) \mathcal{L}(\varepsilon, \rho) + \mathcal{L}(\varepsilon, \rho) \mathcal{P}(\varepsilon, \rho) - \mathcal{P}(\varepsilon, \rho) \mathcal{L}(\varepsilon, \rho) \mathcal{P}(\varepsilon, \rho) - f'(\Phi(\varepsilon, \rho)).\]

Then

\[(26) \quad \mathcal{L}_0(\varepsilon) = A(\varepsilon, \rho) + B(\varepsilon, \rho).\]

Using (25), (26) and the fact that

\[\sup_{\varepsilon \in (0, \varepsilon_*)} \sup_{\rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon)} \|B(\varepsilon, \rho)\|_{B(L^2(\mathbb{R}^2))} < \infty,\]

we have

\[(27) \quad \|\mathcal{L}_0(\varepsilon) A(\varepsilon, \rho)^{-1} u\|_{L^2(\mathbb{R}^2)} \leq C \|u\|_{L^2(\mathbb{R}^2)}\]

for every \(u \in Y_1, \varepsilon \in (0, \varepsilon_*)\) and \(\rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon)\). Combining (16) and (27), we obtain (24).

We will use the lemma below to estimate \(L^\infty\)-norm of \(w\) in the following section.

**Corollary 8.** Let \(p > 1\). Then there exist positive numbers \(\varepsilon_*\) and \(C\) such that

\[(28) \quad \|A(\varepsilon, \rho)^{-1} u\|_{L^\infty(\mathbb{R}^2)} \leq C \|u\|_{L^\infty(\mathbb{R}^2)}\]

for every \(u \in L^\infty_1(\mathbb{R}^2) \cap Y_1, \varepsilon \in (0, \varepsilon_*)\) with \(\varepsilon^{-1} \in \mathbb{N}\) and \(\rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon)\).

**Proof.** Let \(m = \varepsilon^{-1} \in \mathbb{N}\) and

\[P^\perp u = u - \frac{Q_{c}'}{L^2(\mathbb{R})} (u, Q_{c}')_{L^2(\mathbb{R})} Q_{c}'.\]

\[K(\varepsilon, \rho) = Q(\varepsilon, \rho) \left\{ (\tau_{\rho} \tilde{\chi}_0) \mathcal{L}_0(\varepsilon)^{-1}(\tau_{\rho} \chi_0) + U^{-1} \tau_{\rho} \tilde{\chi}_1 P^\perp L^\perp_{\varepsilon} \chi_{1} \tau_{-\rho} U \right\} Q(\varepsilon, \rho).\]

Noting that \(e^{im\theta} \mathcal{L}_0(\varepsilon) u(r) = (\Delta - \omega)(e^{im\theta} u(r))\), we have

\[\sup_{m \in \mathbb{N}} \|\mathcal{L}_0(\varepsilon)^{-1}\|_{B(L^\infty(\mathbb{R}^2))} < \infty.\]

Furthermore, \(L_{\varepsilon} : P^\perp L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\) has a bounded inverse. Hence it follows that

\[(29) \quad \sup_{m \geq \varepsilon^{-1}} \sup_{\rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon)} \|K(\varepsilon, \rho)\|_{B(L^\infty(\mathbb{R}^2))} < \infty.\]

We compute

\[A(\varepsilon, \rho) K(\varepsilon, \rho) = A(\varepsilon, \rho)(\tau_{\rho} \tilde{\chi}_0) \mathcal{L}_0(\varepsilon)^{-1}(\tau_{\rho} \chi_0) + A(\varepsilon, \rho) U^{-1} \tau_{\rho} \tilde{\chi}_1 P^\perp L_{\varepsilon}^{-1} P^\perp \chi_{1} \tau_{-\rho} U.\]

\[= I + II.\]
Since $A(\varepsilon, \rho) = L_0(\varepsilon) - B(\varepsilon, \rho)$ and $\|B(\varepsilon, \rho)\tau_\rho \tilde{\chi}_0\|_{B(L^\infty)} = O(e^{-\bar{p} \sqrt{\varepsilon}})$, where $\bar{p} = \min(1, p - 1)$, we have

$$1 = \mathcal{Q}(\varepsilon, \rho) L_0(\varepsilon)(\tau_\rho \tilde{\chi}_0) L_0(\varepsilon)^{-1}(\tau_\rho \chi_0) - B(\varepsilon, \rho)(\tau_\rho \tilde{\chi}_0) L_0(\varepsilon)^{-1}(\tau_\rho \chi_0)$$

$$= \mathcal{Q}(\varepsilon, \rho) \left\{ \tau_\rho (\chi_0 \varepsilon) + [\Delta, \tau_\rho \tilde{\chi}_0] L_0(\varepsilon)^{-1} \tau_\rho \chi_0 - B(\varepsilon, \rho)(\tau_\rho \tilde{\chi}_0) L_0(\varepsilon)^{-1}(\tau_\rho \chi_0) \right\}$$

$$= \tau_\rho \chi_0 + O(l^{-1}) \text{ in } B(\mathcal{Q}(\varepsilon, \rho) L^\infty_r(\mathbb{R}^2)).$$

Let $B_1 = \mathcal{P}(\varepsilon, \rho) \mathcal{L}(\varepsilon, \rho) \mathcal{P}(\varepsilon, \rho) - \mathcal{L}(\varepsilon, \rho) \mathcal{P}(\varepsilon, \rho)$. Then

$$A(\varepsilon, \rho) = \mathcal{Q}(\varepsilon, \rho) \mathcal{L}(\varepsilon, \rho) + B_1(\varepsilon, \rho).$$

In view of the definition of $\mathcal{P}(\varepsilon, \rho)$, [118] and the fact that $L_c Q'_c = 0$, we have

$$\|B_1(\varepsilon, \rho)\|_{B(L^\infty_r(\mathbb{R}^2))} = O(\varepsilon l)$$

for $\varepsilon \in (0, \varepsilon_* )$ and $\rho \in (\alpha_0 / (2 \varepsilon), 2 \alpha_0 / \varepsilon)$. Furthermore [118] implies

$$\left\| \mathcal{Q}(\varepsilon, \rho) - P_1 \right\|_{B(L^\infty_r(\mathbb{R}^2))} = O(\varepsilon).$$

Let

$$\mathcal{R} = \alpha^{-2} - (\alpha + \varepsilon r)^{-2} + f'(\chi_1 Q_c) - f'(Q_c).$$

Then we have $\mathcal{L}(\varepsilon, \rho) = U^{-1}(\tau_\rho L_c) U + \tau_\rho \mathcal{R}$ and

$$\|((\tau_\rho \mathcal{R}) \tilde{\chi}_1)(1 - \Delta_r)^{-1}\|_{B(L^\infty_r(\mathbb{R}^2))} = O(\varepsilon l + e^{-2(p-1)\sqrt{\varepsilon}}).$$

Combining the above, we have

$$\mathcal{I} = \mathcal{Q}(\varepsilon, \rho) U^{-1} \tau_\rho (P_1 \tilde{\chi}_1 L_c) \tilde{\chi}_1 P_1 L_c^{-1} P_1 \chi_1 \tau_\rho \mathcal{R} U + O(\varepsilon l)$$

$$= \mathcal{Q}(\varepsilon, \rho) \tau_\rho (\tilde{\chi}_1 \chi_1) + O(l^{-1})$$

$$= \tau_\rho \chi_0 + O(l^{-1}) \text{ in } B(\mathcal{Q}(\varepsilon, \rho) L^\infty_r(\mathbb{R}^2)).$$

From [118][119], we deduce [118].

4. The method of Liapunov-Schmidt

In this section, we use the method of Liapunov-Schmidt to obtain a solution to [118][119]. Let us translate [118][119] into a system

$$A(\varepsilon, \rho)w + Q(\varepsilon, \rho) R_1(w, \varepsilon, \rho) + Q(\varepsilon, \rho) R_2(\varepsilon, \rho) = 0,$$

$$\mathcal{P}(\varepsilon, \rho)(\mathcal{L}(\varepsilon, \rho)w + R_1(w, \varepsilon, \rho) + R_2(\varepsilon, \rho)) = 0.$$

**Lemma 9.** Let $p > 1$. Then there exist an $\varepsilon_0 > 0$ and a $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $\rho \in (\alpha_0 / (2 \varepsilon), 2 \alpha_0 / \varepsilon)$, Eq. [118] has a unique solution $w(\varepsilon, \rho)$ that is continuous in $\varepsilon$ and $\rho$ and satisfies

$$\|w(\varepsilon, \rho)\|_X \leq C \varepsilon^{\frac{1}{2}} \text{ as } \varepsilon \downarrow 0.$$

**Proof.** Let $T: X_1 \times (0, \varepsilon_0] \times (\alpha_0 / (2 \varepsilon), 2 \alpha_0 / \varepsilon) \to X_1$ be a continuous mapping defined by

$$T(w, \varepsilon, \rho) = -A(\varepsilon, \rho)^{-1} Q(\varepsilon, \rho) \left\{ R_1(w, \varepsilon, \rho) + R_2(\varepsilon, \rho) \right\},$$

and let $\tilde{X} = \{ w \in X_1 \| \|w\|_X \leq r_0 \}$, where $r_0$ is a positive number to be fixed later.
To begin with, we will show that $T$ maps $\tilde{X}$ into itself. We compute

$$
\|R_1\|_{L^2_x(\mathbb{R}^2)} = \left\| \int_0^1 \left\{ f'(\Phi(\varepsilon, \rho) + \theta w) - f'((\Phi(\varepsilon, \rho)) \right\} d\theta \right\|_{L^2_x(\mathbb{R}^2)}
$$

$$
\leq \delta(r_0) \|w\|_{L^2_x(\mathbb{R}^2)},
$$

where $\delta(r_0)$ is a positive constant with $\lim_{r_0 \downarrow 0} \delta(r_0) = 0$. Eq. (5) and the definition of $\chi_1$ imply

$$
\|R_{21}\|_{L^2_x(\mathbb{R}^2)} = \left\| \tau_{\rho} \left\{ (\chi_1^{-1} - \chi_1) Q_c^{p-1} \right\} \right\|_{L^2_x(\mathbb{R}^2)}
$$

$$
\leq C \rho^{1/2} e^{-2(p-1)\sqrt{\varepsilon}},
$$

and

$$
\|R_{23}\|_{L^2_x(\mathbb{R}^2)} \leq C \rho^{1/2} e^{-2\sqrt{\varepsilon}}.
$$

Since $\rho^{-1} = O(\varepsilon)$ and $l = -\frac{2}{\sqrt{\varepsilon}} \max(1, \frac{1}{p-1}) \log \varepsilon$,

$$
\|R_{21}\|_{L^2_x(\mathbb{R}^2)} + \|R_{23}\|_{L^2_x(\mathbb{R}^2)} \leq C_1 \varepsilon^{\frac{7}{2}}.
$$

Using (5) and $\alpha^{-2} - (\alpha + \varepsilon s)^{-2} = \frac{2}{\alpha^2} \varepsilon + O(\varepsilon^2 s^2)$, we have

$$
\| \left( c - \omega - \frac{m^2}{r^2} \right) \tau_{\rho}(\chi_1 Q_c') \|_{L^2_x(\mathbb{R}^2)}^2
$$

$$
= \varepsilon^{-1} \int_{-3\varepsilon}^{\infty} (\alpha + \varepsilon s) \left\{ \left( \frac{1}{\alpha^2} - \frac{1}{(\alpha + \varepsilon s)^2} \right) \chi_1(s) Q_c(s) \right\}^2 ds
$$

$$
\leq C \varepsilon
$$

for every $\alpha = \varepsilon \rho \in (\alpha_0 / 2, 2\alpha_0)$. Similarly, we have

$$
\left\| \frac{1}{r} \chi_1 Q_c' \right\|_{L^2_x(\mathbb{R}^2)} \leq C \varepsilon^{1/2}.
$$

Thus we obtain

$$
\|R_{22}\|_{L^2_x(\mathbb{R}^2)} \leq C_2 \varepsilon^{1/2}.
$$

Combining (36) and (38), with Corollary 7, we have

$$
\|T(w, \varepsilon, \rho)\|_X \leq \nu(\delta(r_0)) \|w\|_X + C_1 \varepsilon^{7/2} + C_2 \varepsilon^{1/2}).
$$

Put $r_0 = 2\nu C_2 \varepsilon^{1/2}$. Then $T(\cdot, \varepsilon, \rho)$ maps $\tilde{X}$ into itself if $\varepsilon_0$ is sufficiently small.

Next, we will show that $T(\cdot, \varepsilon, \rho)$ is a contraction mapping. For $w_1, w_2 \in \tilde{X}$,

$$
\|T(w_1, \varepsilon, \rho) - T(w_2, \varepsilon, \rho)\|_X
$$

$$
\leq \nu \|R_1(w_1, \varepsilon, \rho) - R_1(w_2, \varepsilon, \rho)\|_{L^2_x(\mathbb{R}^2)}
$$

$$
= \nu \left\| \int_0^1 \left\{ f'(\Phi + \theta w_1 + (1 - \theta)w_2) - f'(\Phi) \right\} d\theta (w_1 - w_2) \right\|_{L^2_x(\mathbb{R}^2)}
$$

$$
\leq \Lambda \|w_2 - w_1\|_{L^2_x(\mathbb{R}^2)}
$$

where $\Lambda = \nu r_0^2 \sup_{\eta \in \tilde{X}} \|f'(\Phi + \eta)\|_{C^0}$ and $\tilde{p} = \min(1, p-1)$. Taking $\varepsilon_0$ smaller if necessary, we see that $T(\cdot, \varepsilon, \rho) : \tilde{X} \to \tilde{X}$ is a contraction mapping. Thus
we prove that there exists a solution \( w(\varepsilon, \rho) \) to (33) with \( \|w\|_\infty \leq 2\nu C_2 \varepsilon^{1/2} \) that is continuous in \( \varepsilon \in (0, \varepsilon_0) \) and \( \rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon) \). \( \square \)

**Corollary 10.** Let \( p > 1 \). Then there exist an \( \varepsilon_0 > 0 \) and a \( C > 0 \) such that if \( \varepsilon \in (0, \varepsilon_0) \), \( \varepsilon^{-1} \in \mathbb{N} \) and \( \rho \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon) \), a solution \( w(\varepsilon, \rho) \) to (33) satisfies

\[
\|w(\varepsilon, \rho)\|_{L^\infty} \leq C\varepsilon. \tag{40}
\]

**Proof.** Analogously to (36)–(38), we have

\[
\|R_1\|_{L^\infty} \leq \delta(r_0)\|w\|_{L^\infty}, \tag{41}
\]

\[
\|R_2\|_{L^2(\mathbb{R}^2)} \leq \|R_{21}\|_{L^\infty} + \|R_{22}\|_{L^\infty} + \|R_{23}\|_{L^\infty} = O(\varepsilon), \tag{42}
\]

where \( \delta(r_0) \) is a positive number with \( \lim_{r_0\to0} \delta(r_0) = 0 \). Thus by Corollary 10,

\[
\|w(\varepsilon, \rho)\|_{L^\infty} \leq C (\|R_1\|_{L^\infty} + \|R_2\|_{L^\infty}) = C\delta(r_0)\|w\|_{L^\infty} + O(\varepsilon). \tag{43}
\]

Thus we have (40). \( \square \)

Let

\[
F(\varepsilon, \rho) = (\mathcal{L}(\varepsilon, \rho)w(\varepsilon, \rho) + R_1(\varepsilon, \rho), \varepsilon, \rho) + R_2(\varepsilon, \rho), \partial_\rho \Phi(\varepsilon, \rho))_{L^2(\mathbb{R}^2)}. \tag{44}
\]

By Lemma 11, the system of (33) and (34) is reduced to an equation

\[
F(\varepsilon, \rho) = 0. \tag{45}
\]

**Lemma 11.** Let \( p > 1 \) and let \( \varepsilon_0 > 0 \) be a sufficiently small number. If \( \varepsilon \in (0, \varepsilon_0) \), there exists a \( \rho = \rho(\varepsilon) \in (\alpha_0/(2\varepsilon), 2\alpha_0/\varepsilon) \) satisfying (11). \( \square \)

**Proof.** Let \( R_c = \frac{\varepsilon}{\alpha + \varepsilon} \partial_\nu + \left(\frac{1}{\alpha} - \frac{1}{(\alpha + \varepsilon)^2}\right) + f'(\chi_1 Q_c) - f'(Q_c) \). Using (3), the definition of \( \chi_1 \) and the fact that \( L_c Q_c = 0 \) and \( \rho = O(\varepsilon^{-1}) \), we compute

\[
\|\mathcal{L}(\varepsilon, \rho)\tau_\rho(\chi_1 Q_c')\|_{L^2(\mathbb{R}^2)} \leq \|\tau_\rho \chi_1 L_c Q_c'\|_{L^2(\mathbb{R}^2)} + \||\partial_\rho^2, \tau_\rho \chi_1|\tau_\rho Q_c'\|_{L^2(\mathbb{R}^2)} + \|\tau_\rho(\mathcal{R}_c(\chi_1 Q_c'))\|_{L^2(\mathbb{R}^2)} = O(\varepsilon^{1/2}). \tag{46}
\]

Similarly, we have

\[
\|\mathcal{L}(\varepsilon, \rho)(\partial_\rho \Phi(\varepsilon, \rho) + \tau_\rho(\chi_1 Q_c'))\|_{L^2(\mathbb{R}^2)} = O(\varepsilon^{1/2}). \tag{47}
\]

By Lemma 11, (46), and (47),

\[
\|\mathcal{L}(\varepsilon, \rho)w, \partial_\rho \Phi(\varepsilon, \rho))_{L^2(\mathbb{R}^2)}\| \leq C\varepsilon^{1/2}\|w\|_{L^2(\mathbb{R}^2)} = O(\varepsilon). \tag{48}
\]

Lemma 9 and Corollary 10 yield

\[
\|R_1\|_{L^2(\mathbb{R}^2)} \leq C\|w\|_{L^\infty} \|w\|_{L^2(\mathbb{R}^2)} = O(\varepsilon^{p+\frac{1}{2}}). \tag{49}
\]
where $\tilde{p} = \min(p - 1, 1)$. Combining (37) and (45) with
\[
\| \partial_\rho \Phi(\varepsilon, \rho) \|_{L^2_\rho(\mathbb{R}^2)} = O(\varepsilon^{-1/2}),
\]
we have
\[
(R_1 + R_{21} + R_{23}, \partial_\rho \Phi)_{L^2_\rho(\mathbb{R}^2)} = O(\varepsilon). \tag{46}
\]
In view of (38) and the fact that
\[
\| \partial_\rho \Phi(\varepsilon, \rho) + \tau_\rho(\chi_1 Q'_c) \|_{L^2_\rho(\mathbb{R}^2)} = O(\varepsilon^{1/2}),
\]
(47) \((R_{22}, \partial_\rho \Phi(\varepsilon, \rho) + \tau_\rho(\chi_1 Q'_c))_{L^2_\rho(\mathbb{R}^2)} = O(\varepsilon)\).

By (44), (46) and (47),
\[
F(\varepsilon, \rho) = -(R_{22}, \tau_\rho(\chi_1 Q'_c))_{L^2_\rho(\mathbb{R}^2)} + O(\varepsilon). \tag{47}
\]
Substituting
\[
1 - \frac{1}{(\alpha + \varepsilon s)^2} = \frac{2\varepsilon}{\alpha^3} s + O(\varepsilon^2 s^2) \quad \text{as } \varepsilon \downarrow 0,
\]
and integrating by parts, we have
\[
(R_{22}, \tau_\rho(\chi_1 Q'_c))_{L^2_\rho(\mathbb{R}^2)} = \frac{1}{\varepsilon} \int_{-\rho}^{\infty} \left( \frac{1}{\alpha^2} - \frac{1}{(\alpha + \varepsilon s)^2} \right) \chi_1(s)^2 Q_c(s) Q'_c(s) (\alpha + \varepsilon s) ds
\]
\[
+ \int_{-\rho}^{\infty} \chi_1(s)^2 Q'_c(s)^2 ds
\]
\[
= \int_{-\rho}^{\infty} \chi_1(s)^2 \left( \frac{2s}{\alpha^2} Q_c Q'_c + Q'_c^2 \right) ds + O(\varepsilon)
\]
\[
= \int_{\mathbb{R}} \left\{ Q'_c^2 - \frac{1}{\alpha^2} Q_c^2 \right\} ds + O(\varepsilon).
\]
Combining the above, we see that
\[
F(\varepsilon, \rho) = \int_{\mathbb{R}} \left( Q'_c(s)^2 - (\varepsilon\rho)^{-2} Q_c(s)^2 \right) ds + O(\varepsilon),
\]
where $c = \omega + (\varepsilon \rho)^{-2}$. Hence it follows from Lemma 9 and the intermediate value theorem that (41) has a solution $\rho = \rho(\varepsilon)$ satisfying
\[
\rho = (\alpha_0 + o(1))\varepsilon^{-1} \quad \text{as } \varepsilon \downarrow 0.
\]
Thus we complete the proof of Lemma 11. \hfill \Box

Now, we are in position to prove Theorem 1.

Proof of Theorem 1. Lemmas 9 and 11 and Corollary 10 imply that there exists a solution $\phi_{\omega}$ to (2) satisfying (6) and (7). Suppose that $\phi_{\omega}$ is a sign-changing solution. Since $\phi''_{\omega} \geq 0$ and $\phi'_{\omega} = 0$ at the minimum point, it follows from (2) that
\[
\min_{r > 0} \phi_{\omega}(r) < -\omega^{1/(p-1)}.
\]
But this contracts to (17) if $\varepsilon > 0$ is sufficiently small. Thus the solution $\phi_{\omega}$ to (2) is nonnegative. Since a nonnegative solution is unique (see [18]), we obtain Theorem 1. \hfill \Box
5. Instability of vortex solitons

In this section, we will prove Theorem 2. Let \( u(x, t) = e^{iωt}(e^{imθ}φ_ω(r) + e^{λt}v) \) and linearize (41) around \( v = 0 \) and \( t = 0 \). Then

\[
iλv + (Δ - ω + β_1(r))v + e^{2imθ}β_2(r)v = 0,
\]

where

\[
β_1(r) = \frac{p + 1}{2}φ_ω(r)p^{-1}, \quad β_2(r) = \frac{p - 1}{2}φ_ω(r)p^{-1}.
\]

Put \( v = e^{i(j+mθ}y_+, \bar{v} = e^{i(j-mθ}y_- \) and complexify (48) into a system

\[
\begin{cases}
\left(Δ_r - ω - \frac{(m+j)^2}{r^2} + iλ + β_1(r)\right)y_+ + β_2(r)y_- = 0, \\
\left(Δ_r - ω - \frac{(m-j)^2}{r^2} - iλ + β_1(r)\right)y_- + β_2(r)y_+ = 0.
\end{cases}
\]

If \( λ \) is an eigenvalue of the linearized operator, there exist a \( j ∈ \mathbb{Z} \) and a solution \((y_+, y_-)\) to (49) that satisfy \( (e^{i(j+mθ}y_+, e^{i(j-mθ}y_-) \in H^1(\mathbb{R}^2, \mathbb{C}^2)\). We will show the existence of unstable eigenvalues for \( j \) with \( 1 ≪ j ≪ m \).

Let \( w_1 = y_+ - y_- \), \( w_2 = y_+ + y_- \), \( ε = m^{-1} \) and \( δ = jε \). Let \( s = r - α_0m \). Then (49) can be rewritten as

\[
H(ε, δ)w = λw,
\]

where \( w = (w_1, w_2) \),

\[
H(ε, δ) = i\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},
\]

and

\[
h_{11} = h_{22} = \frac{-2mj}{r^2},
\]

\[
h_{12} = Δ_r - ω - \frac{m^2 + j^2}{r^2} + φ_ω^{p^{-1}}
\]

\[
h_{21} = Δ_r - ω - \frac{m^2 + j^2}{r^2} + pφ_ω^{p^{-1}}.
\]

We remark that

\[
τ_- h_{11} = τ_- h_{22} = \frac{2δ}{(α_0 + εr)^2}
\]

\[
τ_- h_{12} = ετ_0^2 + \frac{ε}{α_0 + εr}τ_r - ω - \frac{1 + δ^2}{(α_0 + εr)^2} + φ_ω^{p^{-1}}
\]

\[
τ_- h_{21} = ετ_0^2 + \frac{ε}{α_0 + εr}τ_r - ω - \frac{1 + δ^2}{(α_0 + εr)^2} + pφ_ω^{p^{-1}}.
\]

Before we investigate the spectrum of \( H(ε, δ) \), let us consider the spectrum of a linear operator

\[
H(δ) := i\begin{pmatrix} -2α_0^{-2}δ & L_+ - α_0^{-2}δ^3 \\ L_- - α_0^{-2}δ^3 & -2α_0^{-2}δ \end{pmatrix}
\]

where \( L_+ = τ_0^2 - c + pφ_ω^{p^{-1}} \), \( L_- = τ_0^2 - c + Q_c^{p^{-1}} \), \( D(L_+) = D(L_-) = H^2(\mathbb{R}) \) and \( c = ω + α_0^{-2} \).
To begin with, we recall some spectral properties of $H(0)$. Let

$$\Phi_1 = \begin{pmatrix} 0 \\ Q_c \end{pmatrix}, \quad \Phi_2 = -i \begin{pmatrix} \partial_t Q_c \\ 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} Q_c' \\ 0 \end{pmatrix}, \quad \Phi_4 = -\frac{i}{2} \begin{pmatrix} 0 \\ sQ_c \end{pmatrix},$$

and

$$\Phi_1^* = \theta_1 \sigma_2 \Phi_2, \quad \Phi_2^* = \theta_1 \sigma_2 \Phi_1, \quad \Phi_3^* = \theta_2 \sigma_2 \Phi_4, \quad \Phi_4^* = \theta_2 \sigma_2 \Phi_3,$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \theta_1 = 2 \left( \frac{d}{dc} \|Q_c\|^2_{L^2(\mathbb{R})} \right)^{-1}, \quad \theta_2 = 4 \|Q_c\|_{L^2(\mathbb{R})}^{-2}.$$

Then we have

$$H(0)\Phi_1 = 0, \quad H(0)\Phi_2 = \Phi_1, \quad H(0)\Phi_3 = 0, \quad H(0)\Phi_4 = \Phi_3,$$

$$H(0)^*\Phi_1^* = \Phi_1^*, \quad H(0)^*\Phi_2^* = 0, \quad H(0)^*\Phi_3^* = \Phi_3^*, \quad H(0)^*\Phi_4^* = 0,$$

and $\langle \Phi_i, \Phi_j^* \rangle = \delta_{ij}$ for $i, j = 1, 2, 3, 4$. Here we denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(\mathbb{R}, \mathbb{C}^2)$.

**Proposition 12** (see [27]). Let $p > 1$ and $p \neq 5$. Then $\lambda = 0$ is a discrete eigenvalue of $H(0)$ with algebraic multiplicity 4.

Using Proposition 12, we investigate the spectrum of $H(\delta)$.

**Lemma 13.** Let $1 < p < 5$. Then there exist a positive number $\rho_0$ and a neighborhood $U \subset \mathbb{C}$ of 0 such that for every $\delta \in (0, \rho_0)$, $\sigma(H(\delta)) \cap U$ consists of algebraically simple eigenvalues $\lambda_i(\delta)$ ($i = 1, 2, 3, 4$) satisfying

$$|\Re \lambda_i(\delta) - \alpha_0^{-1} \gamma \delta| \leq \alpha_0^{-1} \gamma \delta / 4, \quad \lim_{\delta \to 0} \inf_{\delta, \delta} \left( \min_{1 \leq i, j \leq 4} \frac{1}{|i - j|} \left| \lambda_i(\delta) - \lambda_j(\delta) \right| \right) > 0,$$

where

$$\gamma = \left( 2 \frac{\|Q_c\|_{L^2(\mathbb{R})}^2}{\frac{d}{dc}\|Q_c\|^2_{L^2(\mathbb{R})}} \right)^{1/2}.$$

**Proof.** Let $P_H(\delta)$ be a projection defined by

$$P_H(\delta) = \frac{1}{2\pi i} \int_{|\lambda| = \rho_0} (\lambda - H(\delta))^{-1} d\lambda,$$

and let $Q_H(\delta) = I - P_H(\delta)$. In view of Proposition 12, there exist positive numbers $\rho_0$ and $\delta_0$ such that $X_0 := R(P_H(\delta))$ is 4-dimensional for every $\delta \in (0, \delta_0)$.

Let $X_0$ be a linear subspace whose basis is $\langle \Phi_1, \Phi_2, \Phi_3, \Phi_4 \rangle$. We decompose $H^2(\mathbb{R}; \mathbb{C}^2)$ and $L^2(\mathbb{R}; \mathbb{C}^2)$ as

$$H^2(\mathbb{R}; \mathbb{C}^2) = X_0 \oplus Q_H(0)H^2(\mathbb{R}; \mathbb{C}^2), \quad L^2(\mathbb{R}; \mathbb{C}^2) = X_0 \oplus Q_H(0)L^2(\mathbb{R}; \mathbb{C}^2).$$

Then

$$H(\delta) = \begin{pmatrix} H_{11}(\delta) & H_{12}(\delta) \\ H_{21}(\delta) & H_{22}(\delta) \end{pmatrix},$$

where

$$H_{11}(\delta) = P_H(0)H(\delta)P_H(0), \quad H_{12}(\delta) = P_H(0)H(\delta)Q_H(0)$$

$$H_{21}(\delta) = Q_H(0)H(\delta)P_H(0), \quad H_{22}(\delta) = Q_H(0)H(\delta)Q_H(0).$$
By a simple computation, we have

\[ H_{11}(\delta) = -2i\alpha_0^{-2}\delta I + \begin{pmatrix} 0 & 1 + b_2\delta^2 & 0 & 0 \\ b_1\delta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 + b_4\delta^2 \\ 0 & 0 & b_3\delta^2 & 0 \end{pmatrix}, \]

\[ H_{12}(\delta) = -i\alpha_0^{-2}\delta^2 P_H(0)\sigma_1 Q_H(0), \quad H_{21}(\delta) = -i\alpha_0^{-2}\delta^2 Q_H(0)\sigma_1 P_H(0), \]

where

\[ b_1 = \alpha_0^{-2}\theta_1\|Q_c\|_{L^2(\mathbb{R})}^2, \quad b_2 = -\alpha_0^{-2}\theta_1\|\partial_c Q_c\|_{L^2(\mathbb{R})}^2, \]

\[ b_3 = -4\alpha_0^{-4}, \quad b_4 = \alpha_0^{-2}\|s Q_c\|_{L^2(\mathbb{R})}^2\|Q_c\|_{L^2(\mathbb{R})}^{-2}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

First, we investigate the spectrum of \( H_{11}(\delta) \). Suppose \( \lambda \) is an eigenvalue of the matrix \( H_{11}(\delta) \). Then

\[
\det(\lambda I - H_{11}(\delta)) = \{ (\lambda + 2i\alpha_0^{-2}\delta)^2 - b_1\delta^2 - b_2b_4\delta^4 \} \{ (\lambda + 2i\alpha_0^{-2}\delta)^2 - b_3\delta^2 - b_3b_4\delta^4 \} = 0.
\]

Hence there exist eigenvalues \( \hat{\lambda}_i \) (\( i = 1, 2, 3, 4 \)) of \( H_{11}(\delta) \) satisfying

\[ \hat{\lambda}_1 = -\delta \left( 2i\alpha_0^{-2} - \alpha_0^{-1}\gamma + O(\delta^2) \right), \quad \hat{\lambda}_2 = -\delta \left( 2i\alpha_0^{-2} + \alpha_0^{-1}\gamma + O(\delta^2) \right), \]

\[ \hat{\lambda}_3 = -4i\alpha_0^{-2}\delta \left( 1 + O(\delta^2) \right), \quad \hat{\lambda}_4 = O(\delta^3). \]

Let \( R_{ii}(\lambda, \delta) = (\lambda - H_{ii}(\delta))^{-1} \) for \( i = 1, 2 \) and let

\[
R_0(\lambda, \delta) = \begin{pmatrix} R_{11}(\lambda, \delta) & 0 \\ 0 & R_{22}(\lambda, \delta) \end{pmatrix}, \quad V_0(\lambda, \delta) = \begin{pmatrix} 0 & H_{12}(\lambda, \delta)R_{22}(\lambda, \delta) \\ H_{21}(\lambda, \delta)R_{11}(\lambda, \delta) & 0 \end{pmatrix}.
\]

We remark that \( R_{22}(\lambda, \delta) \) is uniformly bounded for \( \lambda \in U \) and \( \delta \in (0, \delta_0) \). Suppose that \( |\lambda - \hat{\lambda}_i| = c_1\delta \), where \( c_1 \in (0, \alpha_0^{-1}|\gamma|\delta/4) \) is a constant such that \( |\hat{\lambda}_j - \hat{\lambda}_k| \geq c_1\delta \) for every \( j, k = 1, 2, 3, 4 \) with \( j \neq k \). Then in view of the definitions of \( H_{12}(\lambda, \delta) \) and \( H_{21}(\lambda, \delta) \), we have

\[ \|V_0(\lambda, \delta)\|_{B(L^2(\mathbb{R}))} = O(\delta), \]

and

\[ (\lambda - H(\delta))^{-1} = R_0(\lambda, \delta) \sum_{i=0}^{\infty} V_0(\lambda, \delta)^i. \]

Now let

\[
P_{H,i}(\delta) = \frac{1}{2\pi i} \oint_{|\lambda - \hat{\lambda}_i| = c_1\delta} (\lambda - H(\delta))^{-1} d\lambda, \quad \tilde{P}_{H,i}(\delta) = \frac{1}{2\pi i} \oint_{|\lambda - \hat{\lambda}_i| = c_1\delta} R_0(\lambda, \delta) d\lambda.
\]

Combining (54) and (55) with the fact that

\[
\|R_0(\lambda, \delta)V_0(\lambda, \delta)\|_{B(L^2(\mathbb{R}))} = \left\| \begin{pmatrix} 0 & R_{11}H_{12}R_{22} \\ R_{22}H_{21}R_{11} & 0 \end{pmatrix} \right\|_{B(L^2(\mathbb{R}))} = O(\delta),
\]

We have
we have

\[ \| P_{H,i}(\delta) - \hat{P}_{H,i}(\delta) \| = O(\delta) \quad \text{for every } i = 1, 2, 3, 4. \]

Hence it follows that \( R(\hat{P}_{H,i}(\delta)) \) is isomorphic to \( R(P_{H,i}(\delta)) \) and that \( R(P_{H,i}(\delta)) \) is 1-dimensional for \( i = 1, 2, 3, 4 \). Furthermore, we see that eigenvalues of \( H(\delta) \) which lie in \( U \) satisfy \( |\lambda - \hat{\lambda}_i| < c_1 \delta \) for an \( i \in \mathbb{N} \) with \( 1 \leq i \leq 4 \).

Since \( d\|Q^c\|_{L^2(R^d)}^2/\delta > 0 \) for \( p \in (1, 5) \), we see that \( \gamma \) is a positive number and that there exist eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfying

\[ \alpha_0^{-1} \gamma \delta / 2 < \Re \lambda_1 < 3\alpha_0^{-1} \gamma \delta / 2, \quad -3\alpha_0^{-1} \gamma \delta / 2 < \Re \lambda_2 < -\alpha_0^{-1} \gamma \delta / 2. \]

Thus we complete the proof of Lemma 13. \( \square \)

**Proposition 14.** Let \( j, m \in \mathbb{N}, \varepsilon = m^{-1} \) and \( \delta = j \varepsilon \). Let \( \beta = \min(p - 1, 1)/6 \). Then there exists an \( m_* \in \mathbb{N} \) such that if \( m \geq m_* \), the linearized operator \( \mathcal{H}(\varepsilon, \delta) \) with \( j = [m^{\beta}] \) has an unstable eigenvalue.

**Proof.** In order to prove Proposition 14 we will show the spectrum of \( \mathcal{H}(\varepsilon, \delta) \) becomes close to the spectrum of \( H(\delta) \) as \( \varepsilon \downarrow 0 \). Let

\[ \mathcal{H}_0 = \imath \left( \begin{array}{cc} -2jm / r^2 & \Delta_r - \omega - m^2 / r^2 \\ \Delta_r - \omega - m^2 / r^2 & -2jm / r^2 \end{array} \right), \]

and \( H_0 = U \mathcal{H}_0 U^{-1} \). Let

\[ \mathcal{D}(\lambda) = (\tau_r \tilde{\chi}_0)(\lambda - H(\varepsilon, \delta))^{-1} (\tau_r \chi_0) + \tau_r \tilde{\chi}_1 (\lambda - H(\delta))^{-1} \chi_1 \tau_r. \]

Then we have

\[ \mathcal{D}(\lambda) U(\lambda - \mathcal{H}(\varepsilon, \delta)) U^{-1} = I + R_3 + R_4, \]

where

\[ R_3 = \imath (\tau_r \tilde{\chi}_0)(\lambda - H_0)^{-1} \left( \begin{array}{cc} 0 & \left[ \partial_{\tau_r}^2, \tau_r \chi_0 \right] \\ \left[ \partial_{\tau_r}^2, \tau_r \chi_0 \right] & 0 \end{array} \right) - (\tau_r \chi_0) \phi_{\omega}^{p-1} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \]

\[ R_4 = \imath \tau_r \tilde{\chi}_1 (\lambda - H(\delta))^{-1} \left( \begin{array}{cc} 0 & \left[ \partial_{\tau_r}^2, \chi_1 \right] \\ \left[ \partial_{\tau_r}^2, \chi_1 \right] & 0 \end{array} \right) - \chi_1 (R_{41} + R_{42}) \]

\[ R_{41} = \left( \begin{array}{cc} -2\delta / (\alpha_0 + \varepsilon) & 2\delta / (\alpha_0 + \varepsilon) \\ 1 + \delta^2 / (\alpha_0 + \varepsilon)^2 & 1 + \delta^2 / (\alpha_0 + \varepsilon)^2 \\ 0 & 1 + \delta^2 / (\alpha_0 + \varepsilon)^2 \\ \end{array} \right) \]

\[ R_{42} = \left( \begin{array}{cc} 0 & f'(\phi_{\omega}) - f'(Q_c) \\ f'(\phi_{\omega}) - f'(Q_c) & 0 \end{array} \right). \]

We remark that

\[ \| \left[ \partial_{\tau_r}^2, \chi_i \right] \|_{B(L^2(\mathbb{R}), H^{-1}(\mathbb{R}))} = O(l^{-1}) \quad \text{for } i = 0, 1, \]

\[ \| \chi_1 R_{41} \|_{B(L^2(\mathbb{R}))} + \| R_{42} \|_{B(L^2(\mathbb{R}))} = O(\varepsilon^{6\beta}). \]

We have

\[ \sup_{\lambda \in \mathbb{C}, |\lambda| \leq \omega/2} \| (\lambda - \mathcal{H}_0)^{-1} \|_{B(H^{-2}(\mathbb{R}), L^2(\mathbb{R}))} < \infty, \]

since

\[ \imath \mathcal{H}_0 O = \imath \left( \begin{array}{cc} \Delta_r - \omega - (m+j)^2 / r^2 & 0 \\ 0 & -\Delta_r + \omega + (m-j)^2 / r^2 \end{array} \right), \]
where

\[ O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]

Lemma 13 yields that for \( \delta \in (0, \delta_0) \), there exists a \( c > 0 \) such that

\[ \| (\lambda - H(\delta))^{-1} \|_{B(L_2^2(\mathbb{R}^2))} \leq C\delta^{-1} \]

for every \( \lambda \in U \) with \( \min_{1 \leq i \leq 4} |\lambda - \lambda_i(\delta)| \geq c\delta \) and that \( \Re(\lambda_1(\delta) - c\delta) > 0 \).

Let \( l = \delta^{-3} \). Then it follows from the above that

\begin{align*}
\| R_3 \|_{B(L_2^2(\mathbb{R}^2))} &= O(\delta^3 + e^{-2\sqrt{c}\delta^{-3}}), \\
\| R_4 \|_{B(L_2^2(\mathbb{R}^2))} &= O(\delta^2 + \varepsilon^6\delta^{-4}).
\end{align*}

Put

\[ \mathcal{P}_{H,1}(\varepsilon, \delta) = \frac{1}{2\pi i} \oint_{|\lambda| = \varepsilon\delta} (\lambda - \mathcal{H}(\varepsilon, \delta))^{-1} d\lambda, \]

\[ \mathcal{P}_{H,1}(\varepsilon, \delta) = U^{-1}\tau_r\chi_1 P_{H,1}(\delta)\chi_1 \tau_r U. \]

Making use of Cauchy’s theorem and noting that \( \delta \sim \varepsilon^3 \), we have

\[ \| \mathcal{P}_{H,1}(\varepsilon, \delta) - \mathcal{P}_{H,1}(\varepsilon, \delta) \|_{B(L_2^2(\mathbb{R}^2))} \]

\[ \leq C\delta^{-1} \sup_{|\lambda| = \varepsilon\delta} \left( \| R_3 \|_{B(L^2(\varepsilon\delta, \infty))} + \| R_4 \|_{B(L^2(\varepsilon\delta, \infty))} \right) \]

\[ \leq C(\delta + \varepsilon^6\delta^{-5}) \]

\[ = O(\delta). \]

From the above, we conclude that the range of \( \mathcal{P}_{H,1}(\varepsilon, \delta) \) is isomorphic to the range of \( P_{H,1}(\delta) \) and that there exists an eigenvalue \( \lambda \) of \( \mathcal{H}(\varepsilon, \delta) \) with \( \Re \lambda > 0 \). Thus we complete the proof of Proposition 14. \( \square \)

Now we are in position to prove Theorem 2.

**Proof of Theorem 2**. Let \( \mathcal{L} \) be the linearized operator of \( \mathcal{H} \) around \( e^{i(\omega t + m\theta)}\phi_\omega \).

Then

\[ \mathcal{L} = i \begin{pmatrix} \Delta - \omega + \beta_1(r) & e^{2im\theta} \beta_2(r) \\ -e^{-2im\theta} \beta_2(r) & -\Delta + \omega - \beta_1(r) \end{pmatrix}. \]

Proposition 14 tells us that \( \mathcal{L} \) has unstable eigenvalues if \( m \in \mathbb{N} \) is large and \( p \in (1, 5) \). On the other hand, Proposition 15 tells us that \( \mathcal{L} \) has an unstable eigenvalue if \( p > 3 \). Hence it follows that \( \mathcal{L} \) has an unstable eigenvalue if \( p > 1 \) and \( m \in \mathbb{N} \) is sufficiently large. \( \square \)

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