Research Article

Fuzzy $\Gamma$-Hyperideals in $\Gamma$-Hypersemirings by Using Triangular Norms

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The concept of $\Gamma$-semihyperrings was introduced by Dehkordi and Davvaz as a generalization of semirings, semihyperrings, and $\Gamma$-semiring. In this paper, by using the notion of triangular norms, we define the concept of triangular fuzzy sub-$\Gamma$-semihyperrings as well as triangular fuzzy $\Gamma$-hyperideals of a $\Gamma$-semihyperring, and we study a few results in this respect.

1. Introduction

In [1], Nobusawa introduced $\Gamma$-rings as a generalization of ternary rings. Let $M$ be an additive group whose elements are denoted by $a, b, c, \ldots$ and $\Gamma$ another additive group whose elements are $\gamma, \beta, \alpha, \ldots$. Suppose that $a\gamma b$ is defined to be an element of $M$ and that $a\beta\alpha$ is defined to be an element of $\Gamma$ for every $a, b, \gamma, \beta$. If the products satisfy the following three conditions: (1) $(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b, a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b$, (2) $(a\gamma b)\beta c = a\gamma(b\beta c) = a(\gamma b\beta)c$; (3) if $a\gamma b = 0$ for any $a$ and $b$ in $M$, then $\gamma = 0$; then $M$ is called a $\Gamma$-ring in the sense of Nobusawa [1]. Barnes [2] weakened slightly the conditions in the definition of $\Gamma$-ring and gave a new definition of a $\Gamma$-ring. Let $M$ and $\Gamma$ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$ (sending $(a, \gamma, b)$ into $a\gamma b$ such that (1) $(a_1 + a_2)\gamma b = a_1\gamma b + a_2\gamma b, a(\gamma_1 + \gamma_2)b = a\gamma_1 b + a\gamma_2 b$, $a\gamma(b_1 + b_2) = a\gamma b_1 + a\gamma b_2$, (2) $(a\gamma b)\beta c = a\gamma(b\beta c)$; then $M$ is called a $\Gamma$-ring in the sense of Barnes [2]. Nowadays, $\Gamma$-rings mean the $\Gamma$-rings due to Barnes and other $\Gamma$-rings are known as $\Gamma_0$-rings, that is, gamma rings in the sense of Nobusawa. Barnes [2], Luh [3], and Kyuno [4] studied the structure of $\Gamma$-rings and obtained various generalization analogous to corresponding parts in ring theory. The notion of $\Gamma$-semirings was introduced by Rao [5] as a generalization of semirings as well as $\Gamma$-rings. Subsequently, by introducing the notion of operator semirings of a $\Gamma$-semiring, Dutta and Sardar [6] enriched the theory of $\Gamma$-semirings. Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician Marty [7]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic structure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, for example, see [8–10]. In [11, 12], Dehkordi and Davvaz studied the notion of a $\Gamma$-semihyperring as a generalization of semiring, semihyperring, and $\Gamma$-semiring.

Fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets have been introduced by Zadeh (1965) as an extension of the classical notion of sets [13]. Let $X$ be a set. A fuzzy subset $A$ of $X$ is characterized by a membership function $\mu_A : X \rightarrow [0, 1]$ which associates with each point $x \in X$ its grade or degree of membership $\mu_A(x) \in [0, 1]$. Fuzzy sets generalize classical sets since the characteristic functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values $0$ or $1$. After the introduction of fuzzy sets by Zadeh, reconsideration of the concept of classical mathematics began. In 1971, Rosenfeld [14] introduced fuzzy sets in the context of group theory and formulated...
the concept of a fuzzy subgroup of a group. Das characterized fuzzy subgroups by their level of subgroups in [15], since then many notions of fuzzy group theory can be equivalently characterized with the help of notion of level subgroups. The concept of a fuzzy ideal of a ring was introduced by Liu [16]. In 1992, Jun and Lee [17] introduced the notion of fuzzy ideals in \( \Gamma \)-rings and studied a few properties. In [6], Dutta and Sardar studied the structures of fuzzy ideals of \( \Gamma \)-rings. Also, see [18]. The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. There is a considerable amount of work on the connections between fuzzy sets and hyperstructures. In [19], Davvaz introduced the concept of fuzzy \( H \)-ideals of \( H \)-rings. Now, in this paper, we define the concept of triangular fuzzy sub-\( \Gamma \)-semihyperstructures and fuzzy \( \Gamma \)-hyperideals of a \( \Gamma \)-semihyperring by using triangular norms, and we study a few results in this respect.

2. Basic Concepts

Let \( H \) be a nonempty set and let \( g^*(H) \) be the set of all nonempty subsets of \( H \). A hyperoperation on \( H \) is a map \( \ast : H \times H \rightarrow g^*(H) \) and the couple \((H, \ast)\) is called a hypergroupoid. If \( A \) and \( B \) are some properties of fuzzy \( \Gamma \)-hyperideals of \( \Gamma \)-semihyperring. Now, in this paper, we define the concept of triangular fuzzy sub-\( \Gamma \)-semihyperstructures and fuzzy \( \Gamma \)-hyperideals of a \( \Gamma \)-semihyperring by using triangular norms, and we study a few results in this respect.

3. 3-T-Fuzzy Sub-\( \Gamma \)-Semihyperstructures and \( T \)-Fuzzy \( \Gamma \)-Hyperideals

In this section, we define the notion of \( T \)-fuzzy sub-\( \Gamma \)-semihyperstructures and \( T \)-fuzzy \( \Gamma \)-hyperideals of a \( \Gamma \)-semihyperring and we study some of their properties. Let \( T \) be a \( t \)-norm and \( \mu \) be a fuzzy subset of a \( \Gamma \)-semihyperring \( R \). Then, we say \( \mu \) has imaginable property if \( \text{Im} \mu \subseteq \Delta_T \).
Definition 1. Let $R$ be a $\Gamma$-semihyperring, $T$ be a $t$-norm, and $\mu$ be a fuzzy subset of $R$. Then, $\mu$ is called a $T$-fuzzy sub-$\Gamma$-semihyperring of $R$ if

1. $T(\mu(x), \mu(y)) \leq \inf_{z \in x+y} \{\mu(z)\}$,
2. $T(\mu(x), \mu(y)) \leq \inf_{z \in x+y} \{\mu(z)\}$,

for all $x, y \in R$ and for all $y \in \Gamma$.

A $T$-fuzzy sub-$\Gamma$-semihyperring of $R$ is said to be imaginable if it satisfies the imaginable property. Clearly, if $R$ is a $\Gamma$-semiring, then $\mu$ is a $T$-fuzzy sub-$\Gamma$-semiring of $R$ when

1'. $T(\mu(x), \mu(y)) \leq \mu(x+y)$,
2'. $T(\mu(x), \mu(y)) \leq \mu(xy)$,

for all $x, y \in R$ and for all $y \in \Gamma$.

Example 2. Suppose that $R = \mathbb{N}$, the set of natural numbers, and $\Gamma$ is a nonempty subset of $R$. For any $x, y \in R$ and $y \in \Gamma$, we define $x + y = \{x, y\}$ and $xy = \{x, y\}$. Then, $R$ is a $\Gamma$-semihyperring. We define the fuzzy subset $\mu$ of $R$ by

$$\mu(x) = \begin{cases} 3/4 & \text{if } x \in \Gamma \\ 5/9 & \text{otherwise} \end{cases} \quad (1)$$

and we consider the $t$-norm $T(r,s) = rs/(2 - (r + s - rs))$, where $r, s \in [0, 1]$. Then, for any $x, y \in R$ and $y \in \Gamma$, we have

$$\inf_{z \in x+y} \{\mu(z)\} = \min \{\mu(x), \mu(y)\} = \begin{cases} 3/4 & \text{if } x, y \in \Gamma \\ 5/9 & \text{otherwise} \end{cases} \quad (2).$$

On the other hand, we have the following cases:

1. $x, y \in \Gamma$,
2. $x \notin \Gamma$ and $y \in \Gamma$ (or, $x \in \Gamma$ and $y \notin \Gamma$),
3. $x, y \notin \Gamma$.

Regarding the above cases, we have

$$T(\mu(x), \mu(y)) \leq \inf_{z \in x+y} \{\mu(z)\}, \quad (3)$$

Therefore, $\mu$ is a $T$-fuzzy sub-$\Gamma$-semihyperring of $R$.

Lemma 3. Let $R$ be a $\Gamma$-semihyperring, $T$ be a $t$-norm, and $\mu$ be a $T$-fuzzy sub-$\Gamma$-semihyperring of $S$. Then

$$\inf_{z \in x_1+y_1} \{\mu(z)\}, \quad (4)$$

for all $x_1, \ldots, x_n \in R$ and $y \in \Gamma$, where

$$T_n(t_1, \ldots, t_n)$$

and so $T(\mu(x), \mu(y)) \in \Delta_T$. Assume that $a = T(\mu(x), \mu(y))$. If $a = 0$, then

$$T(\mu(x), \mu(y)) = 0 \leq \inf_{z \in x+y} \{\mu(z)\}, \quad (8)$$

Proof. The proof is straightforward by mathematical induction. $\square$

Lemma 4. Let $R$ be a $\Gamma$-semihyperring, $T$ be a $t$-norm, and $\mu$ be a $T$-fuzzy sub-$\Gamma$-semihyperring of $S$. Let $A$ and $B$ be nonempty subsets of $R$. Then

$$T\left(\inf_{a \in A} \{\mu(a)\}, \inf_{b \in B} \{\mu(b)\}\right) \leq \inf_{z \in \Delta} \{\mu(z)\}, \quad (6)$$

for all $y \in \Gamma$.

Proof. The proof is straightforward. $\square$

Theorem 5. Let $R$ be a $\Gamma$-semihyperring, $T$ be a $t$-norm, and $\mu$ be a fuzzy subset of $S$ with imaginable property and $b$ the maximum of $\Im \mu$. Then, the following two statements are equivalent:

1. $\mu$ is a $T$-fuzzy sub-$\Gamma$-semihyperring of $S$,
2. $\mu^{-1}[a,b]$ is a sub-$\Gamma$-semihyperring of $S$ whenever $a \in \Delta_T$ and $0 < a \leq b$.

Proof. (1) $\Rightarrow$ (2): Suppose that $a \in \Delta_T$ and $0 < a \leq b$. If $x, y \in \mu^{-1}[a,b]$, then $\inf_{z \in x+y} \{\mu(z)\} \geq T(\mu(x), \mu(y)) \geq T(a,a) = a$, which implies that $x + y \subseteq \mu^{-1}[a,b]$. Similarly, assume that $a \in \Delta_T$ and $0 < a \leq b$. If $x, y \in \mu^{-1}[a,b]$ and $y \in \Gamma$, then $\inf_{z \in x+y} \{\mu(z)\} \geq T(\mu(x), \mu(y)) \geq T(a,a) = a$. Then, we have $xy \subseteq \mu^{-1}[a,b]$, and so $\mu^{-1}[a,b]$ is a sub-$\Gamma$-semihyperring of $R$.

(2) $\Rightarrow$ (1): Suppose that $x, y \in S$ and $y \in \Gamma$. Since $\Im \mu \subseteq \Delta_T$, both $\mu(x)$ and $\mu(y)$ are in $\Delta_T$. Now, we have

$$T(\mu(x), \mu(y)) = T(\mu(x), \mu(y))$$

and so $T(\mu(x), \mu(y)) \in \Delta_T$. Assume that $a = T(\mu(x), \mu(y))$. If $a = 0$, then

$$T(\mu(x), \mu(y)) = 0 \leq \inf_{z \in x+y} \{\mu(z)\}, \quad (8)$$

where $T_n(t_1, \ldots, t_n)$, $T_n(t_1, \ldots, t_n)$, and $T_n(t_1, \ldots, t_n)$.
Now, let $0 < a = T(\mu(x), \mu(y)) \leq \mu(x) \land \mu(y) \leq \mu(x) \leq b$. Hence $x, y \in \mu^{-1}[a, b]$, which implies that $x + y \subseteq \mu^{-1}[a, b]$, and $xy \subseteq \mu^{-1}[a, b]$. Therefore $T(\mu(x), \mu(y)) \leq \inf_{z \in xy} \{\inf \mu(z)\}$ and $T(\mu(x), \mu(y)) \leq \inf_{z \in xy} \{\mu(z)\}$. \hfill \Box

**Definition 6.** Let $R$ be a $\Gamma$-semihyperring, $T$ be a $t$-norm, and $\mu$ be a fuzzy subset of $R$. Then

1. $\mu$ is called a $T$-fuzzy left $\Gamma$-hyperideal of $R$ if
   \[ T(\mu(x), \mu(y)) \leq \inf_{z \in xy} \{\mu(z)\}, \quad \forall x, y \in R, \quad \mu(y) \leq \inf_{z \in xy} \{\mu(z)\}, \quad \forall x, y \in R, \quad \forall y \in \Gamma. \] (9)

2. $\mu$ is called a $T$-fuzzy right $\Gamma$-hyperideal of $R$ if
   \[ T(\mu(x), \mu(y)) \leq \inf_{z \in xy} \{\mu(z)\}, \quad \forall x, y \in R, \quad \mu(x) \leq \inf_{z \in xy} \{\mu(z)\}, \quad \forall x, y \in R, \quad \forall y \in \Gamma. \] (10)

3. $\mu$ is called a $T$-fuzzy $\Gamma$-hyperideal of $R$ if it is both a $T$-fuzzy left $\Gamma$-hyperideal and a $T$-fuzzy right $\Gamma$-hyperideal of $R$.

**Theorem 7.** Let $R$ be a $\Gamma$-semihyperring, $T$ be a $t$-norm, and $\mu$ be a fuzzy subset of $S$ with imaginable property and $b$ the maximum of $\operatorname{Im} \mu$. Then, the following two statements are equivalent:

1. $\mu$ is a $T$-fuzzy $\Gamma$-hyperideal of $R$,
2. $\mu^{-1}[a, b]$ is a $\Gamma$-hyperideal of $R$ whenever $a \in \Delta_T$ and $0 < a \leq b$.

**Proof.** The proof is similar to the proof of Theorem 5. \hfill \Box

Let $\mu$ be a fuzzy subset of $R$ and $t \in [0, 1]$. The set $U(\mu, t) = \{x \in R | \mu(x) \geq t\}$ is called a level subset of $\mu$. So, we obtain the following corollary.

**Corollary 8.** Let $R$ be a $\Gamma$-semihyperring and $\mu$ be a fuzzy subset of $R$. Then

1. $\mu$ is a Min-fuzzy sub-$\Gamma$-semihyperring of $R$ if and only if every nonempty level subset is a sub-$\Gamma$-semihyperring of $R$;
2. $\mu$ is a Min-fuzzy $\Gamma$-hyperideal of $R$ if and only if every nonempty level subset is a $\Gamma$-hyperideal of $R$.

**Corollary 9.** Let $A$ be a subset of $R$. Then

1. the characteristic function $\chi_A$ is a $T$-fuzzy sub-$\Gamma$-semihyperring of $R$ if and only if $A$ is a sub-$\Gamma$-semihyperring of $R$;
2. the characteristic function $\chi_A$ is a $T$-fuzzy $\Gamma$-hyperideal of $R$ if and only if $A$ is a $\Gamma$-hyperideal of $R$.

**Theorem 10.** Let $R$ be a $\Gamma$-semihyperring and $K$ be a sub-$\Gamma$-semihyperring of $R$. Let $T'$ be the $t$-norm defined by $T(a, b) = \max\{0, a + b - 1\}$ and $\mu$ be a fuzzy subset of $R$ defined by
   \[ \mu(x) = \begin{cases} r & \text{if } x \in K \\ s & \text{otherwise}, \end{cases} \] (11)

for all $a, b \in [0, 1]$ and $x \in R$, where $r, s \in [0, 1]$ such that $s < r$. Then, $\mu$ is a $T'$-fuzzy sub-$\Gamma$-semihyperring of $R$. In particular, if $r = 1$ and $s = 0$, then $\mu$ is imaginable.

**Proof.** The proof is similar to the proof of Theorem 2.6 in [29]. \hfill \Box

**Definition 11.** Let $R_1$ and $R_2$ be $\Gamma_1$ and $\Gamma_2$-semihyperrings, respectively. If there exists a map $\varphi : R_1 \rightarrow R_2$ and a bijection $f : \Gamma_1 \rightarrow \Gamma_2$ such that
   \[ \varphi(x + y) = \{ \varphi(z) \mid z \in x + y \} = \varphi(x) + \varphi(y), \] (12)

and
   \[ \varphi(xy) = \{ \varphi(z) \mid z \in xy \} = \varphi(x)f(y)\varphi(y), \] for all $x, y \in R_1$ and $y \in \Gamma$, then we say $(\varphi, f)$ is a homomorphism from $R_1$ to $R_2$. Also, if $\varphi$ is a bijection then $(\varphi, f)$ is called an isomorphism and $R_1$ and $R_2$ are isomorphic.

**Proposition 12.** Let $R_1$ and $R_2$ be $\Gamma_1$ and $\Gamma_2$-semihyperrings, respectively. Let $(\varphi, f)$ be a homomorphism from $R_1$ to $R_2$. If $\lambda$ is a $T'$-fuzzy sub-$\Gamma$-semihyperring of $R_2$, then $\varphi^{-1}(\lambda)$ is a $T'$-fuzzy sub-$\Gamma$-semihyperring of $R_1$.

**Proof.** Suppose that $x, y \in R_1$ and $y \in \Gamma$. Then, we have
   \[ \inf_{z \in xy} \{ \varphi^{-1}(\lambda)(z) \} = \inf_{z \in xy} \{ \lambda(\varphi(z)) \} \] (13)

Therefore, $\varphi^{-1}(\lambda)$ is a $T'$-fuzzy sub-$\Gamma$-semihyperring of $R_1$. \hfill \Box

**Proposition 13.** Let $R_1$ and $R_2$ be $\Gamma_1$ and $\Gamma_2$-semihyperrings, respectively. Let $(\varphi, f)$ be a homomorphism from $R_1$ to $R_2$. If $\lambda$ is a $T'$-fuzzy $\Gamma$-hyperideal of $R_2$, then $\varphi^{-1}(\lambda)$ is a $T'$-fuzzy $\Gamma$-hyperideal of $R_1$.

**Proof.** The proof is similar to the proof of Proposition 12. \hfill \Box
Let \( \{a_i\}_{i \in I} \) and \( \{b_j\}_{j \in J} \) be two sets of real numbers in \([0, 1]\). Then, we say \( T \) is infinitely distributive if
\[
T \left( \sup_{i \in I} \{a_i\}, \sup_{j \in J} \{b_j\} \right) = \sup_{i \in I} \{ T(a_i, b_j) \}. \tag{14}
\]
If \( T \) is continuous, then \( T \) is infinitely distributive [30].

**Lemma 14.** Let \( T \) be a continuous \( t \)-norm and \( \{\mu_i\}_{i \in I} \) be a family of \( T \)-fuzzy sub-\( \Gamma \)-semihyperpings of \( R \). Then, \( \bigcap_{i \in I} \mu_i \) is a \( T \)-fuzzy sub-\( \Gamma \)-semihyperring of \( R \).

**Proof.** For any \( x, y \in R \) and \( y \in \Gamma \), we have
\[
\inf_{z \in x \times y} \left\{ \left( \bigcap_{i \in I} \mu_i \right)(z) \right\} = \inf_{i \in I} \left\{ \inf_{z \in x \times y} \{ \mu_i(z) \} \right\}
\[
= \inf_{i \in I} \left\{ \left( \bigcap_{i \in I} \mu_i \right)(z) \right\}
\[
\geq \inf_{i \in I} \left\{ T(\mu_i(x), \mu_i(y)) \right\}
\[
= T \left( \inf_{i \in I} \{ \mu_i(x) \}, \inf_{i \in I} \{ \mu_i(y) \} \right)
\[
= T \left( \left( \bigcap_{i \in I} \mu_i(x) \right), \left( \bigcap_{i \in I} \mu_i(y) \right) \right).
\]

**Proposition 18.** Let \( R_1 \) and \( R_2 \) be two \( \Gamma \)-semihyperpings and let \( \mu \) and \( \lambda \) be fuzzy subsets of \( R_1 \) and \( R_2 \), respectively. Then
\[
1. \text{ if } \mu \text{ and } \lambda \text{ are } T \text{-fuzzy sub-} \Gamma \text{-semihyperpings of } R_1 \text{ and } R_2, \text{ respectively, then } \mu \times \lambda \text{ is a } T \text{-fuzzy sub-} \Gamma \text{-semihyperring of } R_1 \times R_2;
\]
\[
2. \text{ if } \mu \text{ and } \lambda \text{ are } T \text{-fuzzy } \Gamma \text{-hyperideals of } R_1 \text{ and } R_2, \text{ respectively, then } \mu \times \lambda \text{ is a } T \text{-fuzzy } \Gamma \text{-hyperideal of } R_1 \times R_2.
\]

**Proof.** It is straightforward. 

In [12], Dehkordi and Davvaz studied Noetherian and Artinian \( \Gamma \)-semihyperpings in crisp case. A collection \( \mathcal{A} \) of subsets of a \( \Gamma \)-semihyperring \( R \) satisfies the ascending chain condition (or Acc) if there does not exist a properly ascending infinite chain \( A_1 \subset A_2 \subset \cdots \) of subsets from \( \mathcal{A} \). Recall that a subset \( B \in \mathcal{A} \) is a maximal element of \( \mathcal{A} \) if there does not exist a subset in \( \mathcal{A} \) that properly contains \( B \). Similar to [18], in the following, we obtain some results related to fuzzy sets and Noetherian \( \Gamma \)-semihyperpings.

**Proposition 19 (see [12]).** Let \( R \) be a \( \Gamma \)-semihyperring. Then, the following conditions are equivalent:
\[
1. \text{ } R \text{ satisfying the Acc condition on right (left) } \Gamma \text{-hyperideals},
\]
\[
2. \text{ every nonempty family of right (left) } \Gamma \text{-hyperideals has a maximal element},
\]
\[
3. \text{ every right (left) } \Gamma \text{-hyperideals is finitely generated}.
\]

**Definition 20 (see [12]).** A \( \Gamma \)-semihyperring \( R \) is right (left) Noetherian if the equivalent conditions of the above proposition are satisfied. In the same way, we can define an Artinian \( \Gamma \)-semihyperring. Let \( I \) be a \( \Gamma \)-hyperideal of a \( \Gamma \)-semihyperring \( R \) and \( I \) be a Noetherian \( \Gamma \)-semihyperring. Then, \( I \) is called a Noetherian \( \Gamma \)-hyperideal of \( R \).

**Example 21 (see [12]).** Let \( A_n = [n, n+1] \) for every \( n \in \mathbb{Z} \), \( R = \bigcup_{n \in \mathbb{Z}} A_n \), and \( \Gamma = \mathbb{Z} \). Then, \( R \) is a Noetherian \( \Gamma \)-semihyperring with respect to the following hyperoperations:
\[
x \oplus y = A_{n+m}, \quad x \otimes y = A_{nm}, \quad \text{where } x \in A_n \text{ and } y \in A_m.
\]
Theorem 22. Let \( \{ A_k \mid k \in \mathbb{N} \} \) be a family of \( \Gamma \)-ideals of a \( \Gamma \)-semiring \( R \), where \( A_1 \supset A_2 \supset A_3 \cdots \).

Let \( \mu \) be a fuzzy subset of \( R \) defined by

\[
\mu(x) = \begin{cases} 
    k+1 & \text{if } x \in A_k \setminus A_{k+1}, \ k = 0, 1, 2, \ldots \\
    1 & \text{if } x \in \bigcap_{k=0}^{\infty} A_k,
\end{cases}
\]

for all \( x \in R \), where \( A_0 \) stands for \( R \). Let \( T \) be a \( t \)-norm with \( \text{Im} \mu \subseteq \Delta_T \).

Then, \( \mu \) is a \( T \)-fuzzy \( \Gamma \)-ideal of \( R \).

Proof.
Let \( x, y \in R \). Suppose that \( x \in A_k \setminus A_{k+1} \) and \( y \in A_r \setminus A_{r+1} \) for \( k = 0, 1, 2, \ldots \) and \( r = 0, 1, 2, \ldots \). Without loss of generality, we may assume that \( k \leq r \). Then, obviously \( y \in A_k \).

Since \( A_k \) is a \( \Gamma \)-ideal of \( R \), it follows that \( x + y \subseteq A_k \) and \( xy \subseteq A_k \), which imply that \( \inf_{x \in A_k, y \in A_k} (z) \geq k/(k + 1) \) implies that \( \inf_{x \in A_k, y \in A_k} (z) \geq k/(k + 1) = \mu(y) \) for all \( \alpha \in \Gamma \).

If \( x \in \bigcap_{k=0}^{\infty} A_k \) and \( y \in \bigcap_{k=0}^{\infty} A_k \), then \( x + y \subseteq \bigcap_{k=0}^{\infty} A_k \).

Hence, \( \inf_{x \in A_k, y \in A_k} (z) = 1 = \mu(y) \).

Thus, \( \mu(x) \) is a \( T \)-fuzzy \( \Gamma \)-ideal of \( R \).

Theorem 23. Let \( R \) be a \( \Gamma \)-semiring satisfying descending chain condition, \( \mu \) be a fuzzy subset of \( R \), and let \( T \) be a \( t \)-norm with \( \text{Im} \mu \subseteq \Delta_T \).

Then, \( \mu \) is a \( T \)-fuzzy \( \Gamma \)-ideal of \( R \). If \( \mu \) is a strictly decreasing function, then \( \mu \) has finite number of values.

Proof.
Let \( \{ t_k \} \) be a strictly decreasing sequence of elements of \( \text{Im} \mu \).

Then \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t \). Then, \( U(\mu(t)) \) is an ideal of \( M \) for all \( r = 2, 3, \ldots \).

Let \( x \in U(\mu(t)) \).

Then \( \mu(x) \geq t_r \geq t_{r-1} \), so \( x \in U(\mu(t_{r-1})) \).

Hence \( U(\mu(t)) \subseteq U(\mu(t_{r-1})) \).

Since \( t_{r-1} \in \text{Im} \mu \), there exists \( x_{r-1} \in M \) such that \( \mu(x_{r-1}) = t_{r-1} \).

It follows that \( x_{r-1} \in U(\mu(t_{r-1})) \).

Thus, \( U(\mu(t)) \subseteq U(\mu(t_{r-1})) \) and so we obtain a strictly decreasing sequence \( U(\mu(t)) \supset U(\mu(t_{r-1})) \supset U(\mu(t_{r-2})) \supset \cdots \) of \( \Gamma \)-ideals of \( M \) which is not terminating. This contradicts the assumption that \( M \) satisfies the descending chain condition. Consequently, \( \mu \) has finite number of values.

Theorem 24. Let \( R \) be a \( \Gamma \)-semiring, \( \mu \) be a fuzzy subset of \( R \), and \( T \) be a \( t \)-norm with \( \text{Im} \mu \subseteq \Delta_T \).

Then, the following conditions are equivalent:

1. \( R \) is a Noetherian \( \Gamma \)-semiring,
2. the set of values of any \( \Gamma \)-ideal of \( R \) is a well-ordered subset of \( [0, 1] \).

Proof.
(1) \( \Rightarrow \) (2): Suppose that the set of values of \( \mu \) is not a well-ordered subset of \( [0, 1] \). Then, there exists a strictly decreasing sequence \( \{ t_k \} \) such that \( \mu(x) = t_k \). It follows that \( U(\mu(t_1)) \subset U(\mu(t_2)) \subset \cdots \) is a strictly ascending chain of \( \Gamma \)-ideals of \( M \), where \( U(\mu(t_r)) = \{ x \in M \mid \mu(x) \geq t_r \} \), for every \( r = 1, 2, \ldots \).

This contradicts the assumption that \( R \) is a Noetherian \( \Gamma \)-semiring.

(2) \( \Rightarrow \) (1): Suppose that the condition (2) is satisfied and \( R \) is not a Noetherian \( \Gamma \)-semiring. There exists a strictly ascending chain

\[
A_1 \subset A_2 \subset A_3 \subset \cdots
\]

of \( \Gamma \)-ideals of \( R \). Note that \( A = \bigcap_{k=0}^{\infty} A_k \) is a \( \Gamma \)-ideal of \( R \). Define a fuzzy subset in \( R \) by

\[
\mu(x) = \begin{cases} 
    \alpha & \text{if } x \notin A, \\
    1 & \text{if } x \in [k, \infty),
\end{cases}
\]

for all \( x \in R \), where \( \alpha \) is a strictly decreasing sequence of elements of \( \text{Im} \mu \).

We claim that \( \mu \) is a \( T \)-fuzzy \( \Gamma \)-ideal of \( R \).

If \( x \in A_k \setminus A_{k-1} \) and \( y \in A_k \setminus A_{k-1} \), then \( x - y \in A_k \).

It follows that \( \inf_{x \in A_k, y \in A_k} (z) \geq 1/k = \mu(y) \) for all \( \alpha \in \Gamma \).

Since \( A_k \) is a \( \Gamma \)-ideal of \( R \), it follows that \( x + y \in A_k \).

Similarly, \( \mu(x) \) is a \( T \)-fuzzy \( \Gamma \)-ideal of \( R \).

Therefore, \( \mu \) is a \( T \)-fuzzy \( \Gamma \)-ideal of \( R \).

For a family \( \{ \mu_\alpha \mid \alpha \in \Lambda \} \) of fuzzy subsets in \( R \), we define the join \( \vee_{\alpha \in \Lambda} \mu_\alpha \) and the meet \( \wedge_{\alpha \in \Lambda} \mu_\alpha \) as follows:

\[
\bigvee_{\alpha \in \Lambda} \mu_\alpha (x) = \sup \{ \mu_\alpha (x) \mid \alpha \in \Lambda \},
\]

\[
\bigwedge_{\alpha \in \Lambda} \mu_\alpha (x) = \inf \{ \mu_\alpha (x) \mid \alpha \in \Lambda \},
\]

for all \( x \in R \), where \( \Lambda \) is any index set.

Theorem 25. The family of \( T \)-fuzzy \( \Gamma \)-ideals of \( R \) is a completely distributive lattice with respect to meet \( \cap \) and join \( \vee \).

Proof.
Since \([0, 1] \) is a completely distributive lattice with respect to the usual ordering in \([0, 1]\), it is sufficient to show that \( \vee_{\alpha \in \Lambda} \mu_\alpha \) and \( \wedge_{\alpha \in \Lambda} \mu_\alpha \) are \( T \)-fuzzy \( \Gamma \)-ideals of \( R \) for family \( \{ \mu_\alpha \mid \alpha \in \Lambda \} \) of \( T \)-fuzzy \( \Gamma \)-ideals of \( R \).
For any $x, y \in R$, we have

$$\inf_{z \in x \vee y} \left( \bigvee_{\alpha \in \Lambda} \mu_{\alpha}(z) \right) \geq \sup_{\alpha \in \Lambda} \left\{ T \left( \mu_{\alpha}(x), \mu_{\alpha}(y) \right) \mid \alpha \in \Lambda \right\}$$

$$\geq T \left( \sup_{\alpha \in \Lambda} \left\{ \mu_{\alpha}(x) \mid \alpha \in \Lambda \right\}, \sup_{\alpha \in \Lambda} \left\{ \mu_{\alpha}(y) \mid \alpha \in \Lambda \right\} \right)$$

$$= T \left( \left( \bigvee_{\alpha \in \Lambda} \mu_{\alpha}(x) \right), \left( \bigvee_{\alpha \in \Lambda} \mu_{\alpha}(y) \right) \right),$$

$$\inf_{z \in x \wedge y} \left( \bigwedge_{\alpha \in \Lambda} \mu_{\alpha}(z) \right) \geq \inf_{\alpha \in \Lambda} \left\{ T \left( \mu_{\alpha}(x), \mu_{\alpha}(y) \right) \mid \alpha \in \Lambda \right\}$$

$$= T \left( \left( \bigwedge_{\alpha \in \Lambda} \mu_{\alpha}(x) \right), \left( \bigwedge_{\alpha \in \Lambda} \mu_{\alpha}(y) \right) \right).$$

(21)

Now, let $x, y \in M$ and $\beta \in \Gamma$. Then

$$\inf_{z \in x \wedge y} \left( \bigvee_{\alpha \in \Lambda} \mu_{\alpha}(z) \right) \geq \sup_{\alpha \in \Lambda} \left\{ \mu_{\alpha}(y) \mid \alpha \in \Lambda \right\}$$

$$= \left( \bigvee_{\alpha \in \Lambda} \mu_{\alpha}(y) \right),$$

(22)

$$\inf_{z \in x \wedge y} \left( \bigwedge_{\alpha \in \Lambda} \mu_{\alpha}(z) \right) \geq \inf_{\alpha \in \Lambda} \left\{ \mu_{\alpha}(y) \mid \alpha \in \Lambda \right\}$$

$$= \left( \bigwedge_{\alpha \in \Lambda} \mu_{\alpha}(y) \right).$$

(23)

Hence, $\vee_{\alpha \in \Lambda} \mu_{\alpha}$ and $\wedge_{\alpha \in \Lambda} \mu_{\alpha}$ are $T$-fuzzy $\Gamma$-hyperideals of $R$. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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