Local uniqueness of multi-peak positive solutions to a class of fractional Kirchhoff equations

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Abstract. This paper is twofold. In the first part, combining the nondegeneracy result and Lyapunov-Schmidt reduction method, we derive the existence of multi-peak positive solutions to the singularly perturbation problem

\[
\left(\varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \lvert (-\Delta)^{s/2} u \rvert^2 \, dx \right) (-\Delta)^s u + V(x) u = u^p, \quad \text{in} \ \mathbb{R}^N,
\]

for \( \varepsilon > 0 \) sufficiently small, \( 2s < N < 4s \), \( 1 < p < 2^*_s - 1 \) and some mild assumptions on the function \( V \). The main difficulties are from the nonlocal operator mixed the nonlocal term, which cause the corresponding unperturbed problem turns out to be a system of partial differential equations, but not a single fractional Kirchhoff equation. In the second part, under some assumptions on \( V \), we show the local uniqueness of positive multi-peak solutions by using the local Pohožáev identity.

Keywords: Fractional Kirchhoff equations; Multi-peak solutions; Lyapunov-Schmidt reduction.

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1 Introduction and main results

In this paper, we are concerned with the following singularly perturbed fractional Kirchhoff problem

\[
\left(\varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \lvert (-\Delta)^{s/2} u \rvert^2 \, dx \right) (-\Delta)^s u + V(x) u = u^p, \quad \text{in} \ \mathbb{R}^N,
\]

where \( a, b, \varepsilon > 0 \) is a parameter, \( V : \mathbb{R}^N \to \mathbb{R} \) is a bounded continuous function, \((-\Delta)^s \) is the fractional Laplacian and \( p \) satisfies

\[
1 < p < 2^*_s - 1 = \begin{cases} \frac{N+2s}{N-2s}, & 0 < s < \frac{N}{2}, \\ +\infty, & s \geq \frac{N}{2}. \end{cases}
\]

where \( 2^*_s \) is the standard fractional Sobolev critical exponent.

Problem (1.1) and its variants have been studied extensively in the literature. The equation that goes under the name of Kirchhoff equation was proposed in [24] as a model for the transverse oscillation of a stretched string in the form

\[
\rho h \partial_t^2 u - \left( p_0 + \frac{E h}{2L} \int_0^L \lvert \partial_x u \rvert^2 \, dx \right) \partial_{xx} u = 0, \quad \text{for} \ t \geq 0 \ \text{and} \ 0 < x < L,
\]

for \( t \geq 0 \) and \( 0 < x < L \), where \( u = u(t, x) \) is the lateral displacement at time \( t \) and at position \( x \), \( E \) is the Young modulus, \( \rho \) is the mass density, \( h \) is the cross section area, \( L \) the length of the string, \( p_0 \) is the initial stress tension.

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Through the years, this model was generalized in several ways that can be collected in the form

$$\partial_t^2 u - M \left( \|u\|^2 \right) \Delta u = f(t, x, u), \quad x \in \Omega$$
or a suitable function $M : [0, \infty) \to \mathbb{R}$, called Kirchhoff function. The set $\Omega$ is a bounded domain of $\mathbb{R}^N$, and $\|u\|^2 = \|(-\Delta)u\|_2^2$ denotes the nonlocal Dirichlet norm of $u$. The basic case corresponds to the choice

$$M(t) = a + bt^{-\gamma}, \quad a \geq 0, b \geq 0, \gamma \geq 1.$$  

Problem (1.2) and its variants have been studied extensively in the literature. Bernstein obtains the global stability result in [5], which has been generalized to arbitrary dimension $N \geq 1$ by Pohozaev in [33]. We also point out that such problems may describe a process of some biological systems dependent on the average of itself, such as the density of population (see e.g. [4]). From a mathematical point of view, the interest of studying Kirchhoff equations comes from the nonlocality of Kirchhoff type equations. For instance, if we take $M(t) = a + bt$, the consideration of the stationary analogue of Kirchhoff’s wave equation leads to elliptic problems

$$\begin{cases}
(a + b \int_{\Omega} \|(-\Delta)u\|^2 dx)(-\Delta)^s u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.3)

for some nonlinear functions $f(x, u)$. Note that the term $(\int_{\Omega} \|(-\Delta)u\|^2 dx)(-\Delta)^s u$ depends not only on the nonlocal term $(-\Delta)^s u$, but also on the integral of $\|(-\Delta)u\|^2$ over the whole domain. In this sense, Eqs. (1.1) and (1.3) are no longer the usual pointwise equalities. This new feature brings new mathematical difficulties that make the study of Kirchhoff type equations particularly interesting.

When $s = 1$ and $N = 3$, (1.1) reduces to the following equation

$$\begin{cases}
- \left( \varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3 \\
u \in H^1(\mathbb{R}^3).
\end{cases}$$

(1.4)

The existence and multiplicity of solutions to (1.4) with $\varepsilon = 1$ were studied in some recent works. Li and Ye [28] obtained the existence of a positive ground state of (1.4) with $f(u) = |u|^{p-1}u$ for $2 < p < 5$. In [11], Deng, Peng and Shuai studied the existence and asymptotical behavior of nodal solutions of (1.4) with $V$ and $f$ is radially symmetric in $x$ as $b \to 0^+$. There are also some works concerning the concentrations behavior of solutions as $\varepsilon \to 0^+$. It seems that He and Zou [19] is the first to study singularly perturbed Kirchhoff equations. In [19], they considered problem (1.4) when $V$ is assumed to satisfy the global condition of Rabinowitz [35]

$$\liminf_{|x| \to \infty} V(x) < \inf_{x \in \mathbb{R}^3} V(x) > 0$$

and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function with subcritical growth of type $u^q$ for some $3 < q < 5$. They proved the existence of multiple positive solutions for $\varepsilon$ sufficiently small. A similar result for the critical case $f(u) = \lambda g(u) + |u|^4u$ was obtained separately in [20] and [39], where the subcritical term $g(u) \sim |u|^{p-2}u$ with $4 < p < 6$. In [22], He, Li and Peng constructed a family of positive solutions which concentrates around a local minimum of $V$ as $\varepsilon \to 0^+$ for a critical problem $f(u) = g(u) + |u|^4u$ with $g(u) \sim |u|^{p-2}u$ (4 < $p$ < 6). For the more delicate case that $f(u) = \lambda |u|^{p-2}u + |u|^4u$ with $2 < p \leq 4$ we refer to He and Li [21], where a family of positive solutions which concentrates around a local minimum of $V$ as $\varepsilon \to 0^+$ were obtained. We refer to e.g. [11, 13, 17, 18, 23, 26] for more mathematical researches on Kirchhoff type equations in the whole space. We also refer to [34] for a recent survey of the results connected to this model.

Very recently, Li et al. [26] proved that the positive ground state solution of (1.4) with $V \equiv 1$ and $f(u) = |u|^{p-1}u$ (1 < $p$ < 5) is unique and nondegenerate. Then, using the Lyapunov–Schmidt reduction method, they proved the existence and uniqueness of single peak solutions to equation (1.4) for all 1 < $p$ < 5. Under some mild conditions on $V$, Luo et al. [29] proved the existence of multi-peak solutions to (1.4). As a continuation of the work [29], Li et al. [27] establish a local uniqueness result.
for the multi-peak solutions by the technique of local Pohozaev identity from [10]. By local uniqueness, it means that if \( u_1^\varepsilon, u_2^\varepsilon \) are two \( k \)-peak solutions concentrating at the same \( k \) points, then \( u_1^\varepsilon \equiv u_2^\varepsilon \) for \( \varepsilon \) sufficiently small.

On the other hand, the interest in generalizing to the fractional case the model introduced by Kirchhoff does not arise only for mathematical purposes. In fact, following the ideas of [6] and the concept of fractional perimeter, Fiscella and Valdinoci proposed in [14] an equation describing the behaviour of a string constrained at the extrema in which appears the fractional length of the rope. Recently, problem similar to (1.1) has been extensively investigated by many authors using different techniques and producing several relevant results (see, e.g. [1, 2, 3, 15, 30, 31, 32, 41]).

Notice that even though it has been known that problem (1.1) has even multiple single peaks solutions, it is still an open problem whether there exist multi-peak solutions to the problem (1.1), which is in striking contrast to the extensive results on multi-peak solutions to singularly perturbed fractional Schrödinger equations. This motivates us to study multi-peak solutions to problem (1.1). To be precise, we give the definition of multi-peak solutions of equation (1.1) as usual.

**Definition 1.1.** Let \( k \in \{1, 2, \ldots\} \). We see that \( u_\varepsilon \) is a \( k \)-peak solution of (1.1) if \( u_\varepsilon \) satisfies

(i) \( u_\varepsilon \) has \( k \) local maximum points \( y_i^\varepsilon \in \mathbb{R}^N, i = 1, 2, \ldots, k \), satisfying

\[
y_i^\varepsilon \to a_i
\]

for some \( a_i \in \mathbb{R}^N \) as \( \varepsilon \to 0 \) for each \( i \).

(ii) For any given \( \tau > 0 \), there exists \( R \gg 1 \), such that

\[
|u_\varepsilon(x)| \leq \tau \quad \text{for} \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^{k} B_R a_i (y_i^\varepsilon).
\]

(iii) There exists \( C > 0 \) such that

\[
\int_{\mathbb{R}^N} (\varepsilon^{2s} a |(-\Delta)^{s} u_\varepsilon|^2 + u_\varepsilon^2) \, dx \leq C \varepsilon^N.
\]

In order to state our main results, we recall some preliminary results for the fractional Laplacian. For \( 0 < s < 1 \), the fractional Sobolev space \( H^s(\mathbb{R}^N) \) is defined by

\[
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x - y|^{\frac{N}{2} + s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},
\]

endowed with the natural norm

\[
\|u\|^2 = \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy.
\]

The fractional Laplacian \((-\Delta)^{s}\) is the pseudo-differential operator defined by

\[
\mathcal{F}((-\Delta)^{s} u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,
\]

where \( \mathcal{F} \) denotes the Fourier transform. It is also given by

\[
(-\Delta)^{s} u(x) = -\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{u(y) + u(x - y) - 2u(x)}{|y|^{N + 2s}} \, dy,
\]

where

\[
C(N, s) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N + 2s}} \, d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_N).
\]

From [12], we have

\[
\|(-\Delta)^{s} u\|^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)|^2 \, d\xi = \frac{1}{2} C(N, s) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy.
\]
From the viewpoint of calculus of variation, the fractional Kirchhoff problem (1.1) is much more complex and difficult than the classical fractional Laplacian equation as the appearance of the term \( b(\int_{\mathbb{R}^N}|(-\Delta)^s u|^2 dx) (\Delta)^s u \), which is of order four. This fact leads to difficulty in obtaining the boundedness of the corresponding energy functional if \( p \leq 3 \). Recently, Rădulescu and Yang \([37]\) established uniqueness and nondegeneracy for positive solutions to Kirchhoff equations with subcritical growth. More precisely, they proved that the following fractional Kirchhoff equation

\[
(a + b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx) (\Delta)^s u + u = u^p, \quad \text{in } \mathbb{R}^N, \tag{1.5}
\]

where \( a, b > 0, \frac{N}{2} < s < 1, 1 < p < 2^*_s - 1 \), has a unique nondegenerate positive radial solution. One of the main ideas is based on the scaling technique which allows us to find a relation between solutions of (1.5) and the following equation

\[
(\Delta)^s Q + Q = Q^p, \quad \text{in } \mathbb{R}^N \tag{1.6}
\]

where \( 0 < s < 1 \) and \( 1 < p < 2^*_s - 1 \). For high dimension and critical case we refer to \([16, 42, 43]\). We first summarize some results in \([37]\) for convenience.

**Proposition 1.1** Let \( a, b > 0 \) Assume that \( \frac{N}{2} < s < 1 \) and \( 1 < p < 2^*_s - 1 \). Then equation (1.5) has a ground state solution \( U \in H^s(\mathbb{R}^N) \) which is unique up to translation,

(i) \( U > 0 \) belongs to \( C^\infty(\mathbb{R}^N) \cap H^{2s+1}(\mathbb{R}^N) \);

(ii) there exist some \( x_0 \in \mathbb{R}^N \) such that \( U(\cdot - x_0) \) is radial and strictly decreasing in \( r = |x - x_0| \);

(iii) there exist constants \( C_1, C_2 > 0 \) such that

\[
\frac{C_1}{1 + |x|^{N+2s}} \leq U(x) \leq \frac{C_2}{1 + |x|^{N+2s}}, \quad \forall x \in \mathbb{R}^N.
\]

Moreover, \( U \) is nondegenerate in \( H^s(\mathbb{R}^N) \) in the sense that there holds

\[
\ker L_+ = \text{span}\{\partial_{x_1} U, \partial_{x_2} U, \ldots, \partial_{x_N} U\},
\]

where \( L_+ \) is defined as

\[
L_+ \varphi = \left(a + b \int_{\mathbb{R}^N} |(-\Delta)^s U|^2 dx\right) (\Delta)^s \varphi + \varphi - pU^{p-1} \varphi + 2b \left(\int_{\mathbb{R}^N} (-\Delta)^s U (-\Delta)^s \varphi dx\right) (\Delta)^s U
\]

acting on \( L^2(\mathbb{R}^N) \) with domain \( H^s(\mathbb{R}^N) \).

By Proposition 1.1, it is now possible that we apply Lyapunov-Schmidt reduction to study the perturbed fractional Kirchhoff equation (1.1). We want to look for multi-peak positive solutions of (1.1) in the Sobolev space \( H^s(\mathbb{R}^N) \) for sufficiently small \( \varepsilon \), which named semiclassical solutions. We also call such derived solutions as concentrating solutions since they will concentrate at certain point of the potential function \( V \).

Assume that \( V : \mathbb{R}^N \to \mathbb{R} \) satisfies the following conditions:

(V_1) \( V \in L^\infty(\mathbb{R}^N) \) and

\[
0 < \inf_{\mathbb{R}^N} V \leq \sup_{\mathbb{R}^N} V < \infty;
\]

(V_2) There exist \( k(k \geq 2) \) distinct points \( \{a_1, \ldots, a_k\} \subset \mathbb{R}^N \) such that for every \( 1 \leq i \leq k \), \( V \in C^\alpha(B_{\varepsilon_0}(a_i)) \) for some \( \alpha \in (0, \frac{N+4s}{2}) \), and

\[
V(a_i) < V(x) \quad \text{for } 0 < |x - a_i| < r
\]

holds for some \( r, 0 < r < r_0 = \frac{1}{2} \min_{1 \leq i, j \leq k, i \neq j} |a_i - a_j| \), where we assume \( a_i \neq a_j \) for \( i \neq j \);
(V₃) There exist $m > 1, \delta > 0, k \in \mathbb{N}, a_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,N}) \in \mathbb{R}^N, c_{i,j} \in \mathbb{R}$ with $c_{i,j} \neq 0$ for each $i = 1, 2, \ldots, k$ and $j = 1, \ldots, N$ such that $V \in C^1(B_\delta(a_i))$ and

$$
\begin{align*}
V(x) &= V(a_i) + \sum_{j=1}^{N} c_{i,j} |x_j - a_{i,j}|^m + O \left(|x - a_i|^{m+1}\right), \quad x \in B_\delta(a_i), \\
\partial_{x_j} V &= mc_{i,j} |x_j - a_{i,j}|^{m-2} (x_j - a_{i,j}) + O \left(|x - a_i|^m\right), \quad x \in B_\delta(a_j),
\end{align*}
$$

where $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$.

The assumption $(V_1)$ allows us to introduce the inner products

$$
\langle u, v \rangle_\varepsilon = \int_{\mathbb{R}^N} \left(\varepsilon^{2s} a(\Delta)^\frac{s}{2} u \cdot (-\Delta)^\frac{s}{2} v + V(x)uv\right) dx
$$

for $u, v \in H^s(\mathbb{R}^N)$. We also write

$$
H_\varepsilon = \left\{ u \in H^s(\mathbb{R}^N) : \|u\|_\varepsilon = \langle u, u \rangle_\varepsilon^{\frac{1}{2}} < \infty \right\}.
$$

Now we state our main results as follows.

**Theorem 1.1** Under the assumptions of Proposition 1.1 and assume that $V$ satisfies $(V_1)$ and $(V_2)$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) has a $k$-peak solution defined as in the Definition 1.1 concentrating around $a_i (1 \leq i \leq k)$.

**Theorem 1.2** Assume that $V$ satisfies $(V_1) - (V_3)$. If $u^{(1)}_\varepsilon, u^{(2)}_\varepsilon$ are two $k$-peak solutions concentrating at the set of $k$ different points $\{a_1, a_2, \ldots, a_k\}$, then

$$
u^{(1)}_\varepsilon \equiv u^{(2)}_\varepsilon
$$

holds for $\varepsilon$ sufficiently small. Moreover, let

$$
u_\varepsilon = \sum_{i=1}^{k} U^i \left(\frac{x - y^i_\varepsilon}{\varepsilon}\right) + \varphi_\varepsilon,
$$

be the unique solution, where $(U^1, U^2, \ldots, U^k)$ satisfy the system (2.5), then there hold

$$
|y^i_\varepsilon - a_i| = o(\varepsilon), \\
\|\varphi_\varepsilon\|_\varepsilon = O \left(\varepsilon^{\frac{k}{2} + m(1-\tau)}\right),
$$

for some $0 < \tau < 1$ sufficiently small.

To prove theorem 1.1, let us first recall that to construct multi-peak solutions to the Schrödinger equation

$$
-\varepsilon^2 \Delta u + V(x)u = u^q, \quad u > 0 \quad \text{in } \mathbb{R}^N,
$$

(1.7)

it is very important to understand the limiting equation as $\varepsilon \to 0$, which is known as the unperturbed Schrödinger equation

$$
-\Delta u + V(x)u = u^q, \quad u > 0 \quad \text{in } \mathbb{R}^N.
$$

Denote by $Q_i$ the unique (see [25]) positive radial solution to equation

$$
-\Delta Q_i + V(a_i) Q_i = Q^q_i \quad \text{in } \mathbb{R}^N.
$$

Then, to construct a $k$-peak solution to equation (1.7) concentrated at $\{a_1, \ldots, a_k\}$, natural candidates are functions of the form $u_\varepsilon = \sum_{i=1}^{k} Q_i \left(\frac{x - y^i_\varepsilon}{\varepsilon}\right) + \varphi_\varepsilon$, where $y^i_\varepsilon \to a_i$ and $\varphi_\varepsilon$ should be appropriately chosen such that $u_\varepsilon$ is indeed a solution to equation (1.7).

It seemed that the above idea should also work for problem (1.1) as well, with the unperturbed Kirchhoff equation (1.5) as the limiting equation. Indeed, to construct single peak solutions to problem (1.1),
this idea works, as can be seen in Rădulescu et al. [37]. However, as to construct multi-peak solutions, it turns out to be wrong. That is, there is no multi-peak solutions of the form  
\[ u_\varepsilon = \sum_{i=1}^{k} w_i (x - y_i\varepsilon) + \varphi_\varepsilon, \]
where \( w_i \) is the unique positive solution to
\[ \left( a + b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx \right) (-\Delta)^s u + V(a_i)u = u^p, \quad \text{in} \ \mathbb{R}^N. \]

To overcome this difficulty, we borrow some ideas from [29] for the Kirchhoff equation. By the definition of multi-peak solutions of problem (1.1), we first prove that if \( u_\varepsilon \) is a \( k \)-peak solution to (1.1), then \( u_\varepsilon \) must be of a particular form, and \( a_j \) must be critical points of \( V \) if \( V \) is continuously differentiable in a neighbourhood of \( a_i \). In fact, via this step, we prove that the right limiting equation of problem (1.1) is a system of partial differential equations, see Section 2. This reveals a new phenomenon of multi-peak solutions for singular perturbation problems, as which is quite different from the known knowledge on singularly perturbed elliptic equations.

To prove the local uniqueness, we will follow the idea of [7]. More precisely, if \( u^{(1)}_\varepsilon, u^{(2)}_\varepsilon \), are two distinct solutions, then it is clear that the function
\[ \xi_\varepsilon = \frac{u^{(1)}_\varepsilon - u^{(2)}_\varepsilon}{\|u^{(1)}_\varepsilon - u^{(2)}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}} \]
satisfies \( \|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 1 \). We will show, by using the equations satisfied by \( \xi_\varepsilon \), that \( \|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \to 0 \) as \( \varepsilon \to 0 \). This gives a contradiction, and thus follows the uniqueness. To deduce the contradiction, we will need quite delicate estimates on the asymptotic behaviors of solutions and the concentrating point \( y_\varepsilon \) due to the presence of the nonlocal term \( \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 \) \((-\Delta)^su \) and the fractional operator.

This paper is organized as follows. We derive the form and location of multi-peak solutions to equation (1.1) in Section 2. In Section 3, we present some basic results which will be used later and explain the strategy of the proof of Theorem 1.1. Finally, we finish the proof of Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

**Notation.** Throughout this paper, we make use of the following notations.

- For any \( R > 0 \) and for any \( x \in \mathbb{R}^N, B_R(x) \) denotes the ball of radius \( R \) centered at \( x \);
- \( \| \cdot \|_q \) denotes the usual norm of the space \( L^q(\mathbb{R}^N), 1 \leq q \leq \infty \);
- \( o_n(1) \) denotes \( o_n(1) \to 0 \) as \( n \to \infty \);
- \( C \) or \( C_i (i = 1, 2, \cdots) \) are some positive constants may change from line to line.

**2 The form and locations of multi-peak solutions**

In this section, we first fix the form of multi-peak solutions of equation (1.1) and locate the related concentrating points. In particular, we prove that if equation (1.1) has a concentrating solution, then \( V \) must have at least one critical point. We will use the following inequality repeatedly.

**Lemma 2.1** For any \( 2 \leq q \leq 2^*_s \), there exists a constant \( C > 0 \) depending only on \( N, V, a \) and \( q \), but independent of \( \varepsilon \), such that
\[ \|\varphi\|_{L^q(\mathbb{R}^N)} \leq C \varepsilon^{N - \frac{N}{q}} \|\varphi\|_\varepsilon \]  
holds for all \( \varphi \in H_\varepsilon \).
\textbf{Proof:} By setting $\varphi(x) = \varphi(\varepsilon x)$ and using Sobolev inequality, we deduce

$$\int_{\mathbb{R}^N} |\varphi|^q = \varepsilon^N \int_{\mathbb{R}^N} |\varphi|^q$$

$$\leq C_1 \varepsilon^N \left( \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} \varphi|^2 + |\varphi|^2 \right) \right)^{q/2}$$

$$= C_1 \varepsilon^N \varepsilon^{2a} \left( \int_{\mathbb{R}^N} \left( \varepsilon^{2s} |(-\Delta)^{\frac{s}{2}} \varphi|^2 + |\varphi|^2 \right) \right)^{q/2}$$

$$\leq C_2 \varepsilon^N \varepsilon^{2a} \|\varphi\|_{L^q}^q$$

where $C_1$ is the best constant for the fractional Sobolev embedding $H^s(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$, and $C_2 > 0$ depends only on $n, a, q$ and $V$.

For convenience, we introduce the notation

$$u_{\varepsilon,y}(x) = u \left( \frac{x - y}{\varepsilon} \right)$$

for $\varepsilon > 0$ and $y \in \mathbb{R}^N$.

Denote by $u_{\varepsilon}^{(i)} \in H^s(\mathbb{R}^N)$ the unique positive radial solution (see [37, Theorem 1.1]) to equation

$$\left( a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_{\varepsilon}^{(i)}|^2 dx \right) (-\Delta)^s u + V(a_i)u = u^p, \quad \text{in } \mathbb{R}^N.$$ 

Then, for each $i = 1, \ldots, k$, $u_{\varepsilon,y_{\varepsilon}^i}^{(i)} = u_{\varepsilon}^{(i)} \left( \frac{x - y_{\varepsilon}^i}{\varepsilon} \right) > 0$ satisfies

$$\left( \varepsilon^{2s} a + \varepsilon^{4s-N} b \right) \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} u_{\varepsilon,y_{\varepsilon}^i}^{(i)}|^2 \right) dx \left( -\Delta \right)^s u_{\varepsilon,y_{\varepsilon}^i}^{(i)} + V(a_i) u_{\varepsilon,y_{\varepsilon}^i}^{(i)} = (u_{\varepsilon,y_{\varepsilon}^i}^{(i)})^p \quad \text{in } \mathbb{R}^N. \quad (2.2)$$

As aforementioned in the introduction, we have our first result as follows:

\textbf{Theorem 2.1} Let $u_{\varepsilon}$ be a k-peak solution of equation (1.1) defined as in the Definition 1.1, with local maximum points at $y_{\varepsilon}^l$ and $y_{\varepsilon}^l \to a_i$ as $\varepsilon \to 0$. Then, for $\varepsilon > 0$ sufficiently small, $u_{\varepsilon}$ is not of the form

$$u_{\varepsilon}(x) = \sum_{i=1}^{k} u_{\varepsilon,y_{\varepsilon}^l}^{(i)}(x) + \varphi_{\varepsilon}(x). \quad (2.3)$$

In fact, $u_{\varepsilon}$ must be of the form

$$u_{\varepsilon}(x) = \sum_{i=1}^{k} U^{(i)} \left( \frac{x - y_{\varepsilon}^l}{\varepsilon} \right) + \varphi_{\varepsilon}(x) \quad (2.4)$$

satisfying

(i) $(U^1, \ldots, U^k)$ is the unique positive radial solution to the system

$$\left\{ \begin{array}{l}
\left( a + b \sum_{i=1}^{k} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U^{(i)}|^2 dx \right) (-\Delta)^s U^{(i)} + V(a_i) U^{(i)} = (U^{(i)})^p \text{ in } \mathbb{R}^N, \\
U^{(i)} > 0 \text{ in } \mathbb{R}^N, \\
U^{(i)} \in H^s(\mathbb{R}^N).
\end{array} \right. \quad (2.5)$$

(ii) There holds

$$\|\varphi_{\varepsilon}\|_{L^p} = o \left( \varepsilon^\frac{N}{2} \right). \quad (2.6)$$

\textbf{Proof:} We first prove the negative case. It follows from Proposition 1.1 that

$$|u(x) + |(-\Delta)^{\frac{s}{2}} u(x)| | \leq \frac{C}{1 + |x|^{N+2s}}, \quad x \in \mathbb{R}^N$$

where $C > 0$ depends only on $n, a, q$ and $V$. \hfill \qed
for some $C > 0$. Note that $\frac{u_i - u_j}{\varepsilon} \to \infty$ since we assume $a_i \neq a_j$. Then for each $i \neq j$ there hold

$$
\int \left( \varepsilon^{2s} \left| (-\Delta) \hat{u}_{\varepsilon,y_i} + u_{\varepsilon,y_i} \right| + u_{\varepsilon,y_i} \right) dx = o(\varepsilon N) \quad \text{for} \quad i \neq j.
$$

(2.7)

If we write

$$
\mathcal{E}^{2s} = \varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta) \hat{u}_{\varepsilon} \right|^2 dx.
$$

Therefore,

$$
a \varepsilon^{2s} \leq \mathcal{E}^{2s} = \varepsilon^{2s} \left( a + b \sum_{i=1}^{k} \int_{\mathbb{R}^N} \left| (-\Delta) \hat{u}_{\varepsilon} \right|^2 dx + o(\varepsilon(1)) \right) \leq A \varepsilon^{2s}
$$

(2.8)

for some constant $A > a > 0$, where $o(1) \to 0$ as $\varepsilon \to 0$.

Assume that there exists a solution $u_\varepsilon$ to equation (1.1) is of the form (2.3), then we can write

$$
\left( \sum_{i=1}^{k} u_{\varepsilon,y_i} + \varphi \varepsilon \right)^p = \varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta) \hat{u}_{\varepsilon} \right|^2 dx \left( -\Delta \right)^s u_\varepsilon + V(x) u_\varepsilon
$$

(2.9)

Combining (2.2) and (2.9) we have

$$
\left( \sum_{i=1}^{k} u_{\varepsilon,y_i} + \varphi \varepsilon \right)^p - \sum_{i=1}^{k} \left( u_{\varepsilon,y_i} \right)^p = \sum_{i=1}^{k} \left( V(x) - V(a_i) \right) u_{\varepsilon,y_i} + \left( \mathcal{E}^{2s} \left( -\Delta \right)^s u_\varepsilon + V(x) \varphi \varepsilon \right)
$$

$$
+ \sum_{i=1}^{k} \left( \left( \mathcal{E}^{2s} - \left( \varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta) \hat{u}_{\varepsilon} \right|^2 dx \right) \left( -\Delta \right)^s u_{\varepsilon,y_i} \right) \right) \left( -\Delta \right)^s u_{\varepsilon,y_i}
$$

$$
= \sum_{i=1}^{k} \left( V(x) - V(a_i) \right) u_{\varepsilon,y_i} + \left( \mathcal{E}^{2s} \left( -\Delta \right)^s \varphi \varepsilon + V(x) \varphi \varepsilon \right)
$$

$$
+ \sum_{i=1}^{k} \left( \left( \mathcal{E}^{2s} - \left( \varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta) \hat{u}_{\varepsilon} \right|^2 dx \right) \left( -\Delta \right)^s u_{\varepsilon,y_i} \right) \right) \left( -\Delta \right)^s u_{\varepsilon,y_i}
$$

(2.10)

Let

$$
K_i = \sum_{j \neq i} \int_{\mathbb{R}^N} \left| (-\Delta) \hat{u}_{\varepsilon} \right|^2 dx > 0.
$$

The last term of (2.10) can be rewritten as

$$
\varepsilon^{2s} \sum_{i=1}^{k} \left( bK_i + o(1) \right) \left( -\Delta \right)^s u_{\varepsilon,y_i}.
$$

So regrouping (2.10) and multiply $u_{\varepsilon,y_i}^{(j)}$ on both sides of equation (2.10) and then integrate over $\mathbb{R}^N$. By integrating by parts, we obtain

$$
\sum_{i=1}^{k} \varepsilon^{2s} \left( bK_i + o(1) \right) \int_{\mathbb{R}^N} \left( -\Delta \right)^s u_{\varepsilon,y_i} + \left( -\Delta \right)^s u_{\varepsilon,y_i} dx
$$

$$
= -\int_{\mathbb{R}^N} \sum_{i=1}^{k} \left( V(x) - V(a_i) \right) u_{\varepsilon,y_i}^{(j)} dx - \int_{\mathbb{R}^N} \left( \mathcal{E}^{2s} \left( -\Delta \right)^s \varphi \varepsilon + V(x) \varphi \varepsilon \right) u_{\varepsilon,y_i}^{(j)} dx
$$

$$
+ \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} u_{\varepsilon,y_i} + \varphi \varepsilon \right)^p - \sum_{i=1}^{k} \left( u_{\varepsilon,y_i} \right)^p \right) u_{\varepsilon,y_i}^{(j)} dx
$$

$$
=: -T_1 - T_2 + T_3
$$

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It follows from (2.7) that
\[ \sum_{i=1}^{k} \varepsilon^{2a} (bK_i + o_\varepsilon(1)) \int_{\mathbb{R}^N} (-\Delta) u_v^{(i)} \cdot (-\Delta) u_v^{(i)} dx = \varepsilon^2 \left( bK_i \int_{\mathbb{R}^N} \left| (-\Delta) u_v^{(i)} \right|^2 dx + o_\varepsilon(1) \right) \quad (2.12) \]

Now we can rewrite $T_1$ into
\[ T_1 = \int_{\mathbb{R}^N} (V(x) - V(a_i)) \left( u_v^{(i)} \right)^2 dx + \sum_{i \neq j} \int_{\mathbb{R}^N} (V(x) - V(a_i)) u_v^{(i)} u_v^{(j)} dx \]
\[ =: T_{11} + T_{12}. \]

Since $V$ is bounded, (2.7) implies $T_{12} = o(\varepsilon^N)$. Decompose $T_{11}$ into
\[ T_{11} = \int_{\mathbb{R}^N} (V(x) - V(a_i)) \left( u_v^{(i)} \right)^2 dx + \int_{\mathbb{R}^N} (V(y_i^n) - V(a_i)) \left( u_v^{(i)} \right)^2 dx \]
\[ =: T_{111} + T_{112}. \]

By (V2), we have
\[ |T_{111}| \leq \int_{B_1(y_i^n)} |V(x) - V(a_i)| \left( u_v^{(i)} \right)^2 dx + \int_{B_1^c(y_i^n)} |V(x) - V(y_i^n)| \left( u_v^{(i)} \right)^2 dx \]
\[ \leq C \int_{B_1(y_i^n)} |x - y_i^n|^\alpha \left( u_v^{(i)} \right)^2 dx + 2\|V\|_{L^\infty(\mathbb{R}^N)} \int_{B_1^c(y_i^n)} \left( u_v^{(i)} \right)^2 dx \]
\[ = o(\varepsilon^N). \]

Since $y_i^n \to a_i$, we also have
\[ T_{112} = \int_{\mathbb{R}^N} (V(y_i^n) - V(a_i)) \left( u_v^{(i)} \right)^2 dx = o(\varepsilon^N). \]

Hence $T_{11} = o(\varepsilon^N)$, which together with the estimate of $T_{12}$ gives
\[ T_1 = o(\varepsilon^N). \quad (2.13) \]

The estimate of $T_2$ follows from (2.8) and Hölder’s inequality:
\[ T_2 = \int_{\mathbb{R}^N} (e^{2\varepsilon(-\Delta)^a} V(x) + V(x) \varphi_v) u_v^{(j)} dx = O\left( \left\| \varphi_v \right\|_\varepsilon \left\| u_v^{(j)} \right\|_\varepsilon \right) \]

Thus, by the assumption $\left\| \varphi_v \right\|_\varepsilon = o(\varepsilon^\frac{1}{2})$, we have
\[ T_2 = o(\varepsilon^N). \quad (2.14) \]

To estimate the last term $T_3$, we apply an elementary inequality to obtain
\[ |T_3| \leq C \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} u_v^{(i)} \right)^{p-1} |\varphi_v| + \sum_{i=1}^{k} u_v^{(i)} |\varphi_v|^{p-1} + |\varphi_v|^p \right) u_v^{(j)} dx \]
\[ \leq C \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} u_v^{(i)} \right)^{p-1} |\varphi_v| + \sum_{i=1}^{k} u_v^{(i)} |\varphi_v|^{p-1} + |\varphi_v|^p \right) u_v^{(j)} dx. \]

Using Hölder’s inequality, (2.1) and the assumption $\left\| \varphi_v \right\|_\varepsilon = o(\varepsilon^\frac{1}{2})$, we obtain
\[ \int_{\mathbb{R}^N} \left( u_v^{(i)} \right)^{p-1} u_v^{(j)} |\varphi_v| dx \leq \left\| u_v^{(i)} \right\|_{L^{p+1}(\mathbb{R}^N)} \left\| u_v^{(j)} \right\|_{L^{p+1}(\mathbb{R}^N)} \left\| \varphi_v \right\|_{L^{p+1}(\mathbb{R}^N)} = o(\varepsilon^N), \]
\[ \int_{\mathbb{R}^N} u_v^{(i)} u_v^{(j)} |\varphi_v|^{p-1} dx \leq \left\| u_v^{(i)} \right\|_{L^{p+1}(\mathbb{R}^N)} \left\| u_v^{(j)} \right\|_{L^{p+1}(\mathbb{R}^N)} \left\| \varphi_v \right\|_{L^{p+1}(\mathbb{R}^N)} = o(\varepsilon^N), \]
\[ \int_{\mathbb{R}^N} |\varphi_v|^p u_v^{(j)} dx \leq \left\| u_v^{(j)} \right\|_{L^{p+1}(\mathbb{R}^N)} \left\| \varphi_v \right\|_{L^{p+1}(\mathbb{R}^N)} = o(\varepsilon^N). \]
Therefore,
\[ T_3 = o (\varepsilon^N). \] (2.15)

Finally, combining (2.12)-(2.15) we get
\[ \varepsilon^N \left( b K_1 \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u^{(t_1)} \right|^2 \, dx + o(1) \right) = o (\varepsilon^N) \]
as \( \varepsilon \to 0. \) This is impossible since \( K_1 > 0. \) The first part of Theorem 2.1 is complete.

Now we start to prove the rest results. First recall that, in the case of fractional Schrödinger equations (i.e., \( b = 0 \)), if \( u_\varepsilon \) is a multi-peak solution, then \( u_\varepsilon \) must be of the form
\[ u_\varepsilon(x) = \sum_{i=1}^{k} u_{\varepsilon}^{i}(x) + \varphi_\varepsilon, \]
where \( u_{\varepsilon}^{i} \in H^{s}(\mathbb{R}^N) \) is the unique positive radial solution to the equation
\[ a(-\Delta)^s v + V(\varepsilon l)(a_i) v = v^p, v > 0 \text{ in } \mathbb{R}^N, \]
and \( \varphi_\varepsilon, \varphi_\varepsilon \) satisfy the listed properties in Theorem 2.1 (see [38]).

In our case, suppose \( u_\varepsilon \) is a multi-peak solution to equation (1.1) with local maximum points \( y_\varepsilon^i(1 \leq i \leq k) \). It is direct to verify that, for each \( 1 \leq i \leq k, \) \( \bar{u}_\varepsilon(x) \equiv u_\varepsilon(\varepsilon x + y_\varepsilon^i) \) is a uniformly bounded sequence in \( H_\varepsilon \) with respect to \( \varepsilon \) and satisfies
\[ (a + b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} \bar{u}_\varepsilon \right|^2 \, dx) (-\Delta)^s \bar{u}_\varepsilon + V(\varepsilon x + y_\varepsilon^i) \bar{u}_\varepsilon = \bar{u}_\varepsilon^p \text{ in } \mathbb{R}^N. \]

So, there exists a subsequence \( \varepsilon_l \to 0 \) such that \( \bar{u}_l(x) \equiv u_{\varepsilon_l}(\varepsilon_l x + y_\varepsilon^i_{\varepsilon_l}) \) converges weakly to a function \( U^i \) in \( H_\varepsilon \) and
\[ a + b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} \bar{u}_l \right|^2 \, dx \to A \]
as \( l \to \infty \) for some constant \( A > 0. \) It follows from Proposition 1.1 that: For every \( a, b > 0, \) there exists a unique solution \((U^1, \ldots, U^k)\) to the system (2.5) up to translations. Moreover, the constant
\[ a + b \sum_{i=1}^{k} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U^i \right|^2 \, dx \]
depends only on \( a, b, k \) and \( V(\varepsilon l)(a_i) \) (1 \( \leq i \leq k) \), but independent of the choice of the solutions \( U^i, 1 \leq i \leq k. \) Then, \( U^i \) must satisfy the fractional Schrödinger equation
\[ A(-\Delta)^s U^i + V(\varepsilon l)(a_i) U^i = (U^i)^p \text{ in } \mathbb{R}^N. \]
Note that \( x = 0 \) is a maximum point of \( U^i. \) Hence \( U^i(x) = U^i(|x|) \) must be the unique positive radial solution to the above equation. Moreover, it is well known that \( U^i(r) = U^i(|x|) \) is strictly decreasing as \( |x| \to \infty \) (see Proposition 1.1). So we can use the same concentrating compactness arguments as that of multi-peak solutions to fractional Schrödinger equations, to find that
\[ \bar{u}_l = \sum_{i=1}^{k} U^i \left( \frac{x - y_\varepsilon^i_{\varepsilon_l}}{\varepsilon_l} \right) + \varphi_\varepsilon, \]
with \( y_\varepsilon^i_{\varepsilon_l} \) and \( \varphi_\varepsilon, \) satisfying the properties mentioned above.

Finally, noting that \( \frac{|y_\varepsilon^i_{\varepsilon_l} - y_\varepsilon^j_{\varepsilon_l}|}{\varepsilon_l} \to \infty \) implies that \( U_{\varepsilon_l, y_\varepsilon^i_{\varepsilon_l}}, 1 \leq i \leq k, \) are mutually asymptotically orthogonal. That is,
\[ \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U_{\varepsilon_l, y_\varepsilon^i_{\varepsilon_l}} \cdot (-\Delta)^{\frac{s}{2}} U_{\varepsilon_l, y_\varepsilon^j_{\varepsilon_l}} + U_{\varepsilon_l, y_\varepsilon^i_{\varepsilon_l}} \cdot U_{\varepsilon_l, y_\varepsilon^j_{\varepsilon_l}} \right| \, dx \to 0 \]
as \( l \to \infty \) for \( i \neq j. \)

Hence, we deduce
\[ A = \lim_{l \to \infty} \left( a + b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} \bar{u}_l \right|^2 \, dx \right) = a + b \sum_{i=1}^{k} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U^i \right|^2 \, dx. \]
Thus, \( U^i (1 \leq i \leq k) \) satisfies the system (2.5). Then the constant \( A \) is independent of the choice of the weak convergent sequence \( \{ u_{\epsilon_i} \} \). This, in turn, means that the above analysis applies to the whole sequence \( \{ u_{\epsilon} \} \). The proof of Theorem 2.1 is complete. \( \square \)

In order to locate multi-peak solutions of equation (1.1), we recall the following local Pohožáev type identity from [36].

**Lemma 2.2** Let \( u \) be a positive solution of (1.1). Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \). Then, for each \( j \), 1, 2, \cdots, \( N \), there hold

\[
\int_{\Omega} \frac{\partial V}{\partial x_j} u^2 dx = \left( \varepsilon^2 a + \varepsilon^{4s-N} b \right) \int_{\Omega^N} |(-\Delta) \hat{u} u^2| dx \int_{\Omega^2} \left( |(-\Delta) \hat{u} u^2| - 2 \frac{\partial u}{\partial v} \frac{\partial u}{\partial x_j} \right) d\sigma \\
+ \int_{\Omega} V u^2 \nu_j d\sigma - \frac{2}{p+1} \int_{\Omega} u^{p+1} \nu_j d\sigma.
\]

(2.16)

Here \( \nu = (\nu_1, \nu_2, \cdots, \nu_N) \) is the unit outward normal of \( \partial \Omega \).

**Lemma 2.3** Suppose \( V \) satisfies (V1) – (V3). Let \( u_{\epsilon} = k_{i=1}^k u_{\epsilon, y_i} + \varphi_{\epsilon} \) be a multi-peak solution to equation (1.1) as in Theorem 2.1. Then \( \nabla V(a_i) = 0 \) for each \( i = 1, \ldots, k \).

**Proof:** Without loss of generality, we only prove the result for \( i = 1 \). Assume that

\[ |V_{z_1}(a_1)| = C_0 > 0. \]

We will apply Lemma 2.2 to \( u_{\epsilon} \) with \( \Omega = B_r (a_1) \) to deduce the contradiction.

We choose the radius \( r \) as follows. Let \( r_0 \equiv \min \left\{ 1, \frac{|y_i-y_j|}{10} \right\} \). By (2.1) and the assumption \( \| \varphi \|_\varepsilon = o \left( \varepsilon^\frac{N}{2} \right) \), we have

\[ \| \varphi \|_{L^{p+1}(\mathbb{R}^N)} \leq C \varepsilon^{\left( \frac{N(p+1)}{2} - 1 \right)} \| \varphi \|_\varepsilon = o \left( \varepsilon^\frac{N}{2} \right). \]

Set \( f = \varepsilon^2 \left( |(-\Delta) \hat{u} \varphi| + |\varphi| + |\varphi|^{p+1} \right) \). Using polar coordinates, \( \int_{\partial B_r(a_1)} f = \int_{B_{r_0}(a_1)} f \), we can choose \( r \in (0, r_0) \) such that

\[ \int_{\partial B_r(a_1)} \left( \varepsilon^2 |(-\Delta) \hat{u} \varphi|^2 + |\varphi|^2 + |\varphi|^{p+1} \right) d\sigma = o \left( \varepsilon^N \right). \]

(2.17)

Now we apply the Pohožáev identity to \( u_{\epsilon} \) with \( \Omega = B_r (a_1) \) with \( r \) being chosen in the above. We obtain

\[
\int_{B_r(a_1)} \frac{\partial V}{\partial x_1} u_{\epsilon}^2 dx = \varepsilon^2 s \int_{\partial B_r(a_1)} \left( |(-\Delta) \hat{u} u_{\epsilon}|^2 \nu_1 - 2 \frac{\partial u}{\partial v} \frac{\partial u}{\partial x_1} \right) d\sigma \\
+ \int_{\partial B_r(a_1)} V u_{\epsilon}^2 \nu_1 d\sigma - \frac{2}{p+1} \int_{\partial B_r(a_1)} u_{\epsilon}^{p+1} \nu_1 d\sigma.
\]

(2.18)

where

\[ \varepsilon^2 s = \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta) \hat{u} u_{\epsilon}|^2 dx = O \left( \varepsilon^2 s \right). \]

We estimate (2.18) term by term. To estimate \( \int_{B_r(a_1)} \frac{\partial V}{\partial x_1} u_{\epsilon}^2 dx \), split into

\[
\int_{B_r(a_1)} \frac{\partial V}{\partial x_1} u_{\epsilon}^2 = \int_{B_r(a_1)} (V_{z_1}(x) - V_{z_1}(a_1)) u_{\epsilon}^2 dx + V_{z_1}(a_1) \int_{B_r(a_1)} u_{\epsilon}^2 dx.
\]

(2.19)

By continuity, we have

\[
\left| \int_{B_r(a_1)} (V_{z_1}(x) - V_{z_1}(a_1)) u_{\epsilon}^2 dx \right| \leq \max_{x \in B_r(a_1)} |V_{z_1}(x) - V_{z_1}(a_1)| \int_{B_r(a_1)} u_{\epsilon}^2 dx.
\]

\[ \square \]
Noting that $|a_i - a_1| > 2\varepsilon$ for each $i \neq 1$, using Proposition 1.1 (iii) and the assumption $||\varphi_\varepsilon||_\varepsilon = o \left(\varepsilon^{\frac{N}{2}}\right)$, we deduce

$$C_1 \varepsilon^N \leq \int_{B_r(a_1)} u^2 \, dx = \int_{B_r(a_1)} \left(\frac{1}{r} \right)^2 + o \left(\varepsilon^N\right) \, dx \leq C_2 \varepsilon^N$$

for $\varepsilon$ sufficiently small, where $C_1, C_2 > 0$ are independent of $\varepsilon$. Hence, for $\varepsilon$ sufficiently small, there holds

$$\left| \int_{B_r(a_1)} (V_{x_2} - V_{x_1})(a_1) u^2 \, dx \right| \leq C_2 \max_{x \in B_r(a_1)} |V_{x_2}(x) - V_{x_1}(a_1)| \varepsilon^N$$

and

$$|V_{x_1}(a_1)| \int_{B_r(a_1)} u^2 \geq C_0 C_1 \varepsilon^3.$$

Combining the above two estimates and choosing $r$ sufficiently small, we obtain

$$\left| \int_{B_r(a_1)} \frac{\partial V}{\partial x_1} u^2 \, dx \right| \geq \left( C_0 C_1 - C_2 \max_{x \in B_r(a_1)} |V_{x_2}(x) - V_{x_1}(a_1)| \right) \varepsilon^N \geq \frac{C_0 C_1}{2} \varepsilon^N. \quad (2.20)$$

On the contrary, we have

$$\varepsilon^{2s} \int_{\partial B_r(a_1)} \left| \left( (-\Delta)^s u \right) \nu \nu - 2 \frac{\partial u}{\partial r} \frac{\partial u}{\partial x_1} \right| \, d\sigma \leq C \varepsilon^{2s} \int_{\partial B_r(a_1)} \left( \sum_{i=1}^k \left| (-\Delta)^s w \right|^2 + \left| (-\Delta)^s \varphi \right|^2 \right) \, d\sigma \quad (2.21)$$

and

$$\left| \int_{\partial B_r(a_1)} V u^p \nu_1 \, d\sigma - \frac{2}{p+1} \int_{\partial B_r(a_1)} u^{p+1} \nu_1 \, d\sigma \right| \leq C \int_{\partial B_r(a_1)} \left( \sum_{i=1}^k \left| w \right|^2 + |\varphi|^2 + \sum_{i=1}^k \left| w \right|^{p+1} + |\varphi|^{p+1} \right) \, d\sigma \quad (2.22)$$

where we have used (2.17) and the polynomial decay of $U^i$ at infinity. Finally, combining (2.19)-(2.22), we obtain

$$\frac{C_0 C_1}{2} \varepsilon^N \leq o \left(\varepsilon^N\right), \quad \text{as } \varepsilon \to 0.$$

We reach a contradiction. The proof is complete. 

3 Some preliminaries

In this section, we introduce Lyapunov-Schmidt reduction method of the proof of Theorem 1.2 and present some elementary estimates for later use.

It is known that every solution to Eq. (1.1) is a critical point of the energy functional $I_{\varepsilon} : H_{\varepsilon} \to \mathbb{R}$, given by

$$I_{\varepsilon}(u) = \frac{1}{2} \|u\|^2 + \frac{b \varepsilon^{4s-N}}{4} \left( \int_{\mathbb{R}^N} \left| (-\Delta)^s u \right|^2 \right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1} \, dx$$

for $u \in H_{\varepsilon}$. It is standard to verify that $I_{\varepsilon} \in C^2 (H_{\varepsilon})$. So we are left to find a critical point of $I_{\varepsilon}$. However, due to the presence of the double nonlocal terms $(-\Delta)^s$ and $\left( \int_{\mathbb{R}^N} \left| (-\Delta)^s u \right|^2 \right)^2$, it requires more careful estimates on the orders of $\varepsilon$ in the procedure. In particular, the nonlocal terms brings new difficulties in the higher order remainder term, which is more complicated than the case of the fractional Schrödinger equation (1.5) or usual Kirchhoff equation (1.4).

To obtain multi-peak solutions to equation (1.1), Theorem 2.1 inspires us to construct solutions of the form (2.4). Following the idea from Cao and Peng [9] (see also [37]), we will use the unique ground state

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\((U^1, \ldots, U^k)\) to the system \((2.5)\) to build the solutions of \((1.1)\). Since the \(\varepsilon\)-scaling makes it concentrate around \(\xi\), this function constitutes a good approximate solution to \((1.1)\).

Let \(k\) be any positive integer and for \(\delta > 0\) such that \([a_i - \delta, a_i + \delta]\) are disjoint for all \(i = 1, \ldots, k\), we define

\[
D_{\varepsilon, \delta} = \{y = (y^1, \ldots, y^k) \in (\mathbb{R}^N)^k : |y^i - a_i| < \delta, \frac{|y^i - y^j|}{\varepsilon} \geq \varepsilon^{-1}, i = 1, \ldots, k\},
\]

where \(\frac{N+2\varepsilon}{N+2\varepsilon+\alpha} < \theta < 1\). Fixing \(y \in D_{\varepsilon, \delta}\), we define

\[
U_{\varepsilon, y} = \sum_{i=1}^{k} U_{\varepsilon, y}^i,
\]

and

\[
M_{\varepsilon, \eta} = \{(y, \varphi) : y \in D_{\varepsilon, \delta}, \varphi \in E_{\varepsilon, y}, \|\varphi\|^2 \leq \eta \varepsilon^N\}
\]

where we denote \(E_{\varepsilon, y}\) by

\[
E_{\varepsilon, y} := \left\{ \varphi \in H^s(\mathbb{R}^N) : \left\langle \frac{\partial U_{\varepsilon, y}^i}{\partial y^i}, \varphi \right\rangle_\varepsilon = 0, i = 1, \ldots, k \right\}.
\]

We will restrict our argument to the existence of a critical point of \(I_{\varepsilon}\) that concentrates, as \(\varepsilon\) small enough, near the spheres with radii \(a_1/\varepsilon, \ldots, a_k/\varepsilon\). Thus we are looking for a critical point of the form

\[
u_{\varepsilon} = U_{\varepsilon, y} + \varphi_{\varepsilon}
\]

where \(\varphi_{\varepsilon} \in E_{\varepsilon, y}\), and \(y^i_\varepsilon \to r_i, \|\varphi_{\varepsilon}\|^2 = o(\varepsilon^N)\) as \(\varepsilon \to 0\). For this we introduce a new functional \(J_{\varepsilon} : M_{\varepsilon, \eta} \to \mathbb{R}\) defined by

\[
J_{\varepsilon}(y, \varphi) = I_{\varepsilon}(U_{\varepsilon, y} + \varphi), \quad \varphi \in E_{\varepsilon, y}.
\]

In fact, we divide the proof of Theorem 1.2 into two steps:

**Step 1:** for each \(\varepsilon, \delta\) sufficiently small and for each \(y \in D_{\varepsilon, \delta}\), we will find a critical point \(\varphi_{\varepsilon, y}\) for \(J_{\varepsilon}(y, \cdot)\) (the function \(y \mapsto \varphi_{\varepsilon, y}\) also belongs to the class \(C^1(I_{\varepsilon})\));

**Step 2:** for each \(\varepsilon, \delta\) sufficiently small, we will find a critical point \(y_{\varepsilon}\) for the function \(j_{\varepsilon} : D_{\varepsilon, \delta} \to \mathbb{R}\) induced by

\[
y \mapsto j_{\varepsilon}(y) \equiv J(y, \varphi_{\varepsilon, y}).
\]

That is, we will find a critical point \(y_{\varepsilon}\) in the interior of \(D_{\varepsilon, \delta}\).

It is standard to verify that \((y_{\varepsilon}, \varphi_{\varepsilon, y})\) is a critical point of \(J_{\varepsilon}\) for \(\varepsilon\) sufficiently small by the chain rule. This gives a solution \(u_{\varepsilon} = U_{\varepsilon, y} + \varphi_{\varepsilon, y}\) to Eq. (1.1) for \(\varepsilon\) sufficiently small in virtue of the following lemma.

**Lemma 3.1** [37] There exist \(\varepsilon_0, \eta_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0], \eta \in (0, \eta_0]\), and \((y, \varphi) \in M_{\varepsilon, \eta}\) the following are equivalent:

(i) \(u_{\varepsilon} = U_{\varepsilon, y} + \varphi_{\varepsilon, y}\) is a critical point of \(I_{\varepsilon}\) in \(H_{\varepsilon}\).

(ii) \((y, \varphi)\) is a critical point of \(J_{\varepsilon}\).

Now, in order to realize **Step 1**, we expand \(J_{\varepsilon}(y, \cdot)\) near \(\varphi = 0\) for each fixed \(y\) as follows:

\[
J_{\varepsilon}(y, \varphi) = J_{\varepsilon}(y, 0) + l_{\varepsilon}(\varphi) + \frac{1}{2} \langle L_{\varepsilon} \varphi, \varphi \rangle + R_{\varepsilon}(\varphi)
\]

where \(J_{\varepsilon}(y, 0) = I_{\varepsilon}(U_{\varepsilon, y})\), and \(l_{\varepsilon}, L_{\varepsilon}\) and \(R_{\varepsilon}\) are defined for \(\varphi, \psi \in H_{\varepsilon}\) as follows:

\[
l_{\varepsilon}(\varphi) = \langle I_{\varepsilon}^\prime(U_{\varepsilon, y}), \varphi \rangle = (U_{\varepsilon, y}, \varphi) + b \varepsilon^{4-N} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\varepsilon}{2}} U_{\varepsilon, y}|^2 dx \int_{\mathbb{R}^N} (-\Delta)^{\frac{\varepsilon}{2}} U_{\varepsilon, y} \cdot (-\Delta)^{\frac{\varepsilon}{2}} \varphi dx - \int_{\mathbb{R}^N} U_{\varepsilon, y}^p \varphi dx\]

(3.2)
and $\mathcal{L}_\varepsilon : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is the bilinear form around $U_{\varepsilon,y}$ defined by

$$
\langle \mathcal{L}_\varepsilon \varphi, \psi \rangle = \langle l'(U_{\varepsilon,y}) [\varphi], \psi \rangle + b e^{4\varepsilon-N} \left( \int_{\mathbb{R}^N} |(-\Delta)^s U_{\varepsilon,y}|^2 \, dx \right) \int_{\mathbb{R}^N} (-\Delta)^s \varphi \cdot (-\Delta)^s \psi \, dx + 2 e^{4\varepsilon-N} b \left( \int_{\mathbb{R}^N} (-\Delta)^s U_{\varepsilon,y} \cdot (-\Delta)^s \varphi \, dx \right) \left( \int_{\mathbb{R}^N} (-\Delta)^s U_{\varepsilon,y} \cdot (-\Delta)^s \psi \, dx \right) - p \int_{\mathbb{R}^N} U_{\varepsilon,y}^{p-1} \psi \, dx
$$

and $R_\varepsilon$ denotes the second order remainder term given by

$$
R_\varepsilon(\varphi) = J_\varepsilon(y, \varphi) - J_\varepsilon(y,0) - l_\varepsilon(\varphi) - \frac{1}{2} \langle \mathcal{L}_\varepsilon \varphi, \varphi \rangle.
$$

We remark that $R_\varepsilon$ belongs to $C^2(H_\varepsilon)$ since so is every term in the right hand side of (3.3). In the rest of this section, we consider $l_\varepsilon : H_\varepsilon \to \mathbb{R}$ and $R_\varepsilon : H_\varepsilon \to \mathbb{R}$ and give some elementary estimates.

**Lemma 3.2** [40] *For any constant $0 < \sigma \leq \min\{\alpha, \beta\}$, there is a constant $C > 0$, such that*

$$
\frac{1}{(1 + |y - x^i|)^\alpha} \frac{1}{(1 + |y - x^j|)^\beta} \leq \frac{C}{|x^i - x^j|^{\alpha+\beta-\sigma}} \left( \frac{1}{(1 + |y - x^i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1 + |y - x^j|)^{\alpha+\beta-\sigma}} \right)
$$

*where $\alpha, \beta > 0$ are two constants.*

**Lemma 3.3** *Assume that $V$ satisfies $(V_1)$ and $(V_2)$. Then, there exists a constant $C > 0$, independent of $\varepsilon$, such that for any $y \in B_1(0)$, there holds*

$$
|l_\varepsilon(\varphi)| \leq C e^{\frac{\beta}{2}} \left( \varepsilon^\alpha + \sum_{i=1}^k \| (V(y^i) - V(a_i)) \| \right) \| \varphi \| \varepsilon + \varepsilon^\frac{\beta}{2} \| \varphi \| \varepsilon \left( \frac{|y - x^i|}{1 + |y - x^i|} \right)^{\alpha+\beta+\sigma}, \quad \text{if } p > 2
$$

$$
|l_\varepsilon(\varphi)| \leq C e^{\frac{\beta}{2}} \left( \varepsilon^\alpha + \sum_{i=1}^k \| (V(y^i) - V(a_i)) \| \right) \| \varphi \| \varepsilon + \varepsilon^\frac{\beta}{2} \| \varphi \| \varepsilon \left( \frac{|y - x^i|}{1 + |y - x^i|} \right)^{\alpha+\beta+\sigma}, \quad \text{if } 1 < p \leq 2
$$

*for $\varphi \in H_\varepsilon$. Here $\alpha$ denotes the order of the Hölder continuity of $V$ in $B_{\alpha}(0)$.*

**Proof:** Since $(U^1, \cdots, U^k)$ are the unique positive solutions to the system (2.5) and the definition of $U_{\varepsilon,y}$, we can write

$$
l_\varepsilon(\varphi) = \int_{\mathbb{R}^N} \sum_{i=1}^k (V(x) - V(a_i)) U_{\varepsilon,y}^i \varphi \, dx + \int_{\mathbb{R}^N} \left( \sum_{i=1}^k U_{\varepsilon,y}^i \varphi - (U_{\varepsilon,y})^p \varphi \right) \, dx.
$$

Using the conditions imposed on $V(x)$ and $\alpha < \frac{N+1}{2}$, we have

$$
\left| \int_{\mathbb{R}^N} \sum_{i=1}^k (V(x) - V(y^i)) U_{\varepsilon,y}^i \varphi \, dx \right| \leq \left( \int_{\mathbb{R}^N} \left( \sum_{i=1}^k (V(x) - V(y^i)) U_{\varepsilon,y}^i \right)^2 \, dx \right)^{\frac{1}{2}} \| \varphi \| \varepsilon
$$

$$
\leq \left( \int_{\mathbb{R}^N} \left( \sum_{i=1}^k V(x + y^i) - V(y^i) U^i \right)^2 \, dx \right)^{\frac{1}{2}} \| \varphi \| \varepsilon
$$

$$
\leq \varepsilon^{\frac{\beta}{2} + \alpha} \left( \int_{\mathbb{R}^N} |y|^{2\alpha} |U^i|^2 \, dy \right)^{\frac{1}{2}} \| \varphi \| \varepsilon
$$

$$
\leq C e^{\frac{\beta}{2} + \alpha} \left( C + \left( \int_{\mathbb{R}^N \setminus B_{\varepsilon}(0)} \frac{|y|^{2\alpha}}{1 + |y|^{2(N+2\alpha)}} \, dy \right)^{\frac{1}{2}} \| \varphi \| \varepsilon
$$

$$
\leq C e^{\frac{\beta}{2} + \alpha} \| \varphi \| \varepsilon
$$
and
\[
\left| \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(y_i') - V(a_i)) U_{\varepsilon,y_i'} \varphi dx \right| \\
\leq C \varepsilon^{\frac{k}{2}} \sum_{i=1}^{k} |V(y_i') - V(a_i)| \left( \int_{\mathbb{R}^N} |U_i'|^2 dx \right)^{\frac{1}{2}} \| \varphi \|_\varepsilon
\]
(3.5)

Combining (3.4) and (3.5), we obtain
\[
\left| \int_{\mathbb{R}^N} (V(x) - V(a_i)) U_{\varepsilon,y} \varphi \right| \\
= \left| \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(x) - V(y_i')) U_{\varepsilon,y_i'} \varphi dx + \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(y_i') - V(a_i)) U_{\varepsilon,y_i'} \varphi dx \right| \\
\leq C \varepsilon^{\frac{k}{2} + \alpha} \| \varphi \|_\varepsilon + C \varepsilon^{\frac{k}{2}} \sum_{i=1}^{k} |V(y_i') - V(a_i)| \| \varphi \|_\varepsilon.
\]

On the other hand, we know that
\[
\left| \int_{\mathbb{R}^N} \sum_{i \neq j} (U_{\varepsilon,y_i'}^{i-j})^p \varphi - (U_{\varepsilon,y}^p) \varphi dx \right| = \begin{cases} \displaystyle \int_{\mathbb{R}^N} \sum_{i \neq j} \left( (U_{\varepsilon,y_i'}^{i-j})^{p-1} U_{\varepsilon,y_i'}^{i-j} + U_{\varepsilon,y_i'}^{i-j} (U_{\varepsilon,y_i'}^{i-j})^{p-1} \right) \varphi dx, & \text{if } p > 2. \\ \displaystyle \int_{\mathbb{R}^N} \sum_{i \neq j} \left( (U_{\varepsilon,y_i'}^{i-j})^{\frac{2}{p-1}} (U_{\varepsilon,y_i'}^{i-j})^{\frac{2}{p-1}} \right) \varphi dx, & \text{if } 1 < p \leq 2. \end{cases}
\]

By Lemma 3.2 and direct computation, if \( p > 2, \)
\[
\left| \int_{\mathbb{R}^N} \sum_{i \neq j} \left( (U_{\varepsilon,y_i'}^{i-j})^{p-1} U_{\varepsilon,y_i'}^{i-j} + U_{\varepsilon,y_i'}^{i-j} (U_{\varepsilon,y_i'}^{i-j})^{p-1} \right) \varphi dx \right| \\
\leq C \left( \int_{\mathbb{R}^N} \sum_{i \neq j} (U_{\varepsilon,y_i'}^{i-j})^{2(p-1)} U_{\varepsilon,y_i'}^{i-j} \varphi dx \right)^{\frac{1}{2}} \| \varphi \|_\varepsilon \\
\leq C \varepsilon^{\frac{k}{2}} \sum_{i \neq j} \left( \int_{\mathbb{R}^N} \frac{1}{(1 + |y - y_i'|^2)^{(N+2s)(p-1)}} \frac{1}{(1 + |y - y_i'|^2)^{2(N+2s)}} dy \right)^{\frac{1}{2}} \| \varphi \|_\varepsilon \\
\leq C \varepsilon^{\frac{k}{2}} \sum_{i \neq j} \frac{1}{|y_i' - y_j'|^{N+2s}} \| \varphi \|_\varepsilon
\]
(3.7)

and if \( 1 < p \leq 2, \)
\[
\left| \int_{\mathbb{R}^N} \sum_{i \neq j} (U_{\varepsilon,y_i'}^{i-j})^p \varphi dx \right| \\
\leq C \sum_{i \neq j} \left( \int_{\mathbb{R}^N} (U_{\varepsilon,y_i'}^{i-j})^p \varphi dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \varphi^2 dx \right)^{\frac{1}{2}}
\]
(3.8)

It follows form (3.6)-(3.8) that the result has been proved.

Next we give estimates for \( R_\varepsilon \) and its derivatives \( R_\varepsilon^{(i)} \) for \( i = 1, 2. \)
Lemma 3.4 There exists a constant $C > 0$, independent of $\varepsilon$ and $b$, such that for $i \in \{0, 1, 2\}$, there hold
\[
\left\| R^{(i)}_\varepsilon(\varphi) \right\| \leq C \varepsilon^{-\frac{N(p-1)}{2}} \left\| \varphi \right\|_p^{p+1-i} + C(b+1)\varepsilon^{-\frac{N}{2}} \left( 1 + \varepsilon^{-\frac{N}{2}} \left\| \varphi \right\|_2 \right) \left\| \varphi \right\|_2^{N-i}
\]
for all $\varphi \in H_\varepsilon$.

Proof: By the definition of $R_\varepsilon$ in (3.3), we have
\[
R_\varepsilon(\varphi) = A_1(\varphi) - A_2(\varphi)
\]
where
\[
A_1(\varphi) = \frac{b \varepsilon^{4s-N}}{4} \left( \int_{\mathbb{R}^N} |(-\Delta)^s \varphi|^2 dx \right) - 4 \int_{\mathbb{R}^N} |(-\Delta)^s \varphi|^2 dx \int_{\mathbb{R}^N} (-\Delta)^s U_{\varepsilon,y} \cdot (-\Delta)^s \varphi dx
\]
and
\[
A_2(\varphi) = \frac{1}{p+1} \int_{\mathbb{R}^N} \left( u_{\varepsilon,y} + \varphi \right)^{p+1} - u_{\varepsilon,y}^{p+1} - (p+1)u_{\varepsilon,y}^p \varphi \frac{p(p+1)}{2} \left( u_{\varepsilon,y}^{p-1} \varphi \right) dx.
\]
Use $R^{(i)}_\varepsilon$ to denote the $i$ th derivative of $R_\varepsilon$, and also use similar notations for $A_1$ and $A_2$. By direct computations, we deduce that, for any $\varphi, \psi \in H_\varepsilon$,
\[
\left\langle R^{(i)}_\varepsilon(\varphi), \psi \right\rangle = \left\langle A_1^{(i)}(\varphi), \psi \right\rangle - \left\langle A_2^{(i)}(\varphi), \psi \right\rangle
\]
where
\[
\left\langle A_1^{(1)}(\varphi), \psi \right\rangle = b \varepsilon^{4s-N} \left( \int_{\mathbb{R}^N} |(-\Delta)^s \varphi|^2 dx \int_{\mathbb{R}^N} (-\Delta)^s \varphi \cdot (-\Delta)^s \psi dx + \int_{\mathbb{R}^N} |(-\Delta)^s \varphi|^2 dx \int_{\mathbb{R}^N} (-\Delta)^s U_{\varepsilon,y} \cdot (-\Delta)^s \varphi \psi dx \right)
\]
and
\[
\left\langle A_2^{(1)}(\varphi), \psi \right\rangle = \int_{\mathbb{R}^N} \left( (u_{\varepsilon,y} + \varphi)^p \psi - u_{\varepsilon,y}^p \psi \right) dx.
\]
We also deduce, for any $\varphi, \psi, \xi \in H_\varepsilon$, that
\[
\left\langle R^{(2)}_\varepsilon(\varphi)[\psi], \xi \right\rangle = \left\langle A_1^{(2)}(\varphi)[\psi], \xi \right\rangle - \left\langle A_2^{(2)}(\varphi)[\psi], \xi \right\rangle
\]
where
\[
\left\langle A_1^{(2)}(\varphi)[\psi], \xi \right\rangle = 2b \varepsilon^{4s-N} \left( \int_{\mathbb{R}^N} (-\Delta)^s \varphi \cdot (-\Delta)^s \psi dx \right) \left( \int_{\mathbb{R}^N} (-\Delta)^s \varphi \cdot (-\Delta)^s \xi dx \right)
\]
and
\[
\left\langle A_2^{(2)}(\varphi)[\psi], \xi \right\rangle = \int_{\mathbb{R}^N} \left( p(u_{\varepsilon,y} + \varphi)^{p-1} \xi \psi - p u_{\varepsilon,y}^{p-1} \xi \psi \right) dx.
\]
First, we estimate $A_1(\varphi), A_1^{(1)}(\varphi)$ and $A_1^{(2)}(\varphi)$. Notice that
\[
\left\| (-\Delta)^s U_{\varepsilon,y} \right\|_{L^2(\mathbb{R}^N)} = C_0 \varepsilon^{\frac{N-2s}{2}}
\]
with \( C_0 = \|(-\Delta)\tilde{\varphi}\|_{L^2(\mathbb{R}^N)}\), and that
\[
\|(-\Delta)\tilde{\varphi}\|_{L^2(\mathbb{R}^N)} \leq C_1 \varepsilon^{2s-N} \|\varphi\|_\varepsilon, \quad \varphi \in H_\varepsilon
\]
holds for some \( C_1 > 0 \) independent of \( \varepsilon \). Combining above two estimates together with Hölder’s inequality yields
\[
\int_{\mathbb{R}^N} |(-\Delta)\tilde{\varphi} \cdot (-\Delta)\tilde{\psi}|dx \int_{\mathbb{R}^N} |(-\Delta)\tilde{\varphi} \cdot (-\Delta)\tilde{\xi}|dx \leq C\varepsilon^{-\frac{N+2s}{2}}
\]
and that
\[
\int_{\mathbb{R}^N} |(-\Delta)\tilde{\varphi} \cdot (-\Delta)\tilde{\psi}|dx \int_{\mathbb{R}^N} |(-\Delta)\tilde{\varphi} \cdot (-\Delta)\tilde{\xi}|dx \leq C\varepsilon^{-2N}
\]
for all \( \varphi, \psi, \eta, \xi \in H_\varepsilon \). These estimates imply that
\[
|A_1^{(i)}(\varphi)| \leq C\varepsilon^{-\frac{N}{2}} \left(1 + \varepsilon^{-\frac{N}{2}}\|\varphi\|_\varepsilon\right) \|\varphi\|_{\varepsilon}^{N-i}
\]
for some constant \( C > 0 \) independent of \( \varepsilon \).

Next we estimate \( A_2^{(i)}(\varphi) \) (the \( i \) th derivative of \( A_2(\varphi) \)) for \( i = 0, 1, 2 \). We consider the case \( 1 < p < 2 \) first.

On one hand, if \( 1 < p < 2 \), from (2.1), we find
\[
|A_2(\varphi)| \leq C \left(\int_{\mathbb{R}^N} \varphi^{(p+1)}dx\right) \leq C\varepsilon^{\frac{1-p}{2}N} \|\varphi\|_{\varepsilon}^{(p+1)},
\]
\[
|\langle A_2^{(1)}(\varphi), \psi \rangle| \leq C \left(\int_{\mathbb{R}^N} \varphi^p \psi dx\right)
\]
\[
\leq C \left(\int_{\mathbb{R}^N} |\varphi|^{p+1}dx\right)^{\frac{1}{p+1}} \left(\int_{\mathbb{R}^N} |\psi|^{p+1}dx\right)^{\frac{1}{p+1}}
\]
\[
\leq C \left(\varepsilon^{\frac{1-p}{2}N} \|\varphi\|_{\varepsilon}^{p+1}\right)^{\frac{1}{p+1}} \left(\varepsilon^{\frac{1-p}{2}N} \|\psi\|_{\varepsilon}^{p+1}\right)^{\frac{1}{p+1}}
\]
\[
\leq C\varepsilon^{\frac{1-p}{2}N} \|\varphi\|_{\varepsilon}^{p+1} \|\psi\|_{\varepsilon}
\]
and
\[
|\langle A_2^{(2)}(\varphi), \psi, \xi \rangle| \leq C \left(\int_{\mathbb{R}^N} \varphi^{p-1} \psi \xi dx\right)
\]
\[
\leq C \left(\int_{\mathbb{R}^N} |\varphi|^{(p-1)+1}dx\right)^{\frac{1}{(p-1)+1}} \left(\int_{\mathbb{R}^N} |\psi|^{p+1}dx\right)^{\frac{1}{p+1}} \left(\int_{\mathbb{R}^N} |\xi|^{p+1}dx\right)^{\frac{1}{p+1}}
\]
\[
\leq C \varepsilon^{\frac{1-p}{2}N} \|\varphi\|_{\varepsilon}^{p+1} \|\psi\|_{\varepsilon} \|\xi\|_{\varepsilon}
\]
On the other hand, for the case \( p > 2 \), using (2.1) again, we also can obtain that
\[
|R_\varepsilon(\varphi)| \leq C \int_{\mathbb{R}^N} U_{\varepsilon,y}^{p-2} \varphi^3 \leq C\varepsilon^{-\frac{N}{2}} \|\varphi\|_\varepsilon^3,
\]
\[
|\langle R_\varepsilon(\varphi), \psi \rangle| \leq C\varepsilon^{-\frac{N}{2}} \|\varphi\|_\varepsilon^2 \|\psi\|_\varepsilon,
\]
and
\[
|\langle R_\varepsilon''(\varphi), \psi, \xi \rangle| \leq C\varepsilon^{-\frac{N}{2}} \|\varphi\|_\varepsilon \|\psi\|_\varepsilon \|\xi\|_\varepsilon.
\]
So the results follow.

Now we will give the energy expansion for the approximate solutions.

**Lemma 3.5** Assume that \( V \) satisfies (V1) and (V2). Then, for \( \varepsilon > 0 \) sufficiently small, there is a small constant \( \tau > 0 \) and \( C > 0 \) such that,
\[
I_\varepsilon(U_{\varepsilon,y}) = A\varepsilon^N + \varepsilon^N \left(\sum_{i=1}^{k} B_i (V(y_i) - V(a_i))\right) - C\varepsilon^N \sum_{i \neq j} \frac{1}{|y_i - y_j|^{N+2s}}
\]
\[
+ O \left(\varepsilon^{N+\alpha} + \varepsilon^{N} \sum_{i \neq j} \frac{1}{|y_i - y_j|^{N+2s+\tau}}\right)
\]

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where
\[ A = \left( \frac{1}{2} - \frac{1}{p+1} \right) \sum_{i=1}^{k} \int_{\mathbb{R}^N} |U_i|^{p+1} \, dx + \frac{k}{4} \left( \sum_{i=1}^{k} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{3}{4}} U_i|^2 \, dx \right)^2 \]
and
\[ B_i = \frac{1}{2} \int_{\mathbb{R}^N} |U_i|^2 \, dx. \]

**Proof:** By direct computation, we can know

\[ I_\varepsilon(U_{\varepsilon,y}) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \varepsilon^2 a \left| (-\Delta)^{\frac{3}{4}} U_{\varepsilon,y} \right|^2 + V(x) U_{\varepsilon,y}^2 \right) \, dx + \frac{\varepsilon^4 \alpha - N b}{4} \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{3}{4}} U_{\varepsilon,y} \right|^2 \, dx \right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} U_{\varepsilon,y}^{p+1} \, dx \]

\[ = \frac{1}{2} \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(x) - V(a_i)) U_{\varepsilon,y}^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i=1}^{k} \sum_{j=1}^{k} U_{\varepsilon,y}^p U_{\varepsilon,y}^p \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} U_{\varepsilon,y}^{p+1} \, dx \]

Now, we discuss each term in the right hand of (3.9). Firstly, by direct calculation, we have

\[ \int_{\mathbb{R}^N} (V(x) - V(a_i)) U_{\varepsilon,y}^2 \, dx = \int_{\mathbb{R}^N} (V(x) - V(a_i)) \left( \sum_{i=1}^{k} U_{\varepsilon,y}^2 + \sum_{i \neq j} U_{\varepsilon,y} U_{\varepsilon,y} \right) \, dx \]

\[ = \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(x) - V(y)) + V(y) - V(a_i) \, U_{\varepsilon,y}^2 \, dx \]

\[ + \int_{\mathbb{R}^N} \sum_{i \neq j} (V(x) - V(y)) + V(y) - V(a_i) \, U_{\varepsilon,y} \, dx \]

\[ = -\varepsilon^2 \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(a_i) - V(y)) (U^i)^2 \, dx + \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(x) - V(y_i)) U_{\varepsilon,y}^2 \, dx \]

\[ + \int_{\mathbb{R}^N} \sum_{i \neq j} (V(x) - V(y)) + V(y) - V(a_i) \, U_{\varepsilon,y} \, dx. \]

Secondly, we have

\[ \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(x) - V(y)) U_{\varepsilon,y}^2 \, dx = \varepsilon^2 \int_{\mathbb{R}^N} \sum_{i=1}^{k} (V(x) - V(y_i)) (U^i)^2 \, dy \]

\[ \leq C \varepsilon^2 \int_{\mathbb{R}^N} \frac{|y|^\alpha}{1 + |y|^{2(N+2\alpha)}} \, dy \]

\[ \leq C \varepsilon^{N+\alpha} \left( C + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{|y|^{(2N+4\alpha-\alpha)}} \, dy \right) \]

\[ \leq C \varepsilon^{N+\alpha} \]

and
\[
\int_{\mathbb{R}^N} (V(x) - V(y)) U_{\varepsilon,y} U_{\varepsilon,y'} dx
\]
\[
= \varepsilon^N \int_{\mathbb{R}^N} (V(\varepsilon y + y') - V(y')) U^i U^j \left( y - \frac{y' - y}{\varepsilon} \right) dy
\]
\[
\leq C \varepsilon^N \int_{\mathbb{R}^N} \frac{|y'|^{\alpha}}{1 + |y|^{(N+2)s} + |y - \frac{y' - y}{\varepsilon}|^{N+2s}} dy
\]
\[
= C \varepsilon^N \left( \int_{B_{\varepsilon^{-s-1}(0)}} \frac{|y'|^{\alpha}}{1 + |y|^{(N+2)s}} dy + \int_{B_{\varepsilon^{-s+1}(0)} \setminus B_{\varepsilon^{-s+1}(\frac{N}{2} \varepsilon)}} \frac{|y'|^{\alpha}}{1 + |y|^{(N+2)s} + |y - \frac{y' - y}{\varepsilon}|^{2(N+2s) - \alpha}} dy \right)
\]
\[
\leq C \varepsilon^{N+\alpha} \int_{\mathbb{R}^N \setminus \left( B_{\varepsilon^{-s+1}(0)} \cup B_{\varepsilon^{-s+1}(\frac{N}{2} \varepsilon)} \right)} \frac{|y'|^{\alpha}}{1 + |y|^{(N+2)s} + |y - \frac{y' - y}{\varepsilon}|^{N+2s}} dy
\]
where \( R \) is a fixed constant and \( \tau > 0 \) is a small constant. Thirdly,
\[
\int_{\mathbb{R}^N} (V(a_t) - V(y')) U_{\varepsilon,y'} dy
\]
\[
\leq C (V(a_t) - V(y')) \varepsilon^N \int_{\mathbb{R}^N} \frac{1}{1 + |y|^{(N+2)s} + |y - \frac{y' - y}{\varepsilon}|^{N+2s}} dy
\]
\[
\leq C (V(a_t) - V(y')) \varepsilon^N \int_{\mathbb{R}^N} \left( \frac{1}{1 + |y|^{(N+2)s}} + \frac{1}{1 + |y - \frac{y' - y}{\varepsilon}|^{N+2s}} \right) dy
\]
\[
\leq C \varepsilon^{N+\tau} (V(a_t) - V(y'))
\]
\[
\text{since } \frac{|y' - y|}{\varepsilon} \geq \varepsilon^{\theta-1}, \theta < 1.
\]

So, combining (3.10)-(3.13), we obtain
\[
\int_{\mathbb{R}^N} \sum_{i=1}^k (V(x) - V(a_i)) U_{\varepsilon,y}^2 dx = -\varepsilon^N \int_{\mathbb{R}^N} \sum_{i=1}^k (V(a_i) - V(y')) (U^i)^2 dx + O(\varepsilon^{N+\alpha})
\]
(3.14)

Next, we will estimate the last two terms on RHS of (3.9). Since
\[
\int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon,y}^i U_{\varepsilon,y}^j dx \leq \int_{\mathbb{R}^N} \left| (U^i)^p \left( \frac{x - y^i}{\varepsilon} \right) U^j \left( \frac{x - y^j}{\varepsilon} \right) dx \right|
\]
\[
\leq C \varepsilon^N \sum_{i \neq j} \frac{1}{|y^i - y^j|^{(N+2s)}} \int_{\mathbb{R}^N} \left( \frac{1}{1 + |x - \frac{y^i}{\varepsilon}|^{p(N+2s)}} + \frac{1}{1 + |x - \frac{y^j}{\varepsilon}|^{p(N+2s)}} \right) dx
\]
\[
\leq C \varepsilon^N \sum_{i \neq j} \frac{1}{|y^i - y^j|^{(N+2s)}}
\]

\[
\int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon, y, j}^p U_{\varepsilon, y, i} \, dx = \int_{\mathbb{R}^N} \sum_{i \neq j} \left( \frac{1}{1 + \left| \frac{y - y^j}{\varepsilon} \right|^{N+2s}} \right)^p \frac{1}{\left( 1 + \left| \frac{x - y^i}{\varepsilon} \right|^{N+2s} \right)^p} \, dx
\]
\[
\geq \varepsilon^N \int_{B_{\varepsilon^{-1/2}}} \sum_{i \neq j} \left( \frac{1}{1 + \left| \frac{y - y^j}{\varepsilon} \right|^{N+2s}} \right)^p \frac{1}{\left( 1 + \left| \frac{x - y^i}{\varepsilon} \right|^{N+2s} \right)^p} \, dx
\]
\[
+ \varepsilon^N \int_{B_{\varepsilon^{-1/2}}} \sum_{i \neq j} \left( \frac{1}{1 + \left| \frac{y - y^j}{\varepsilon} \right|^{N+2s}} \right)^p \frac{1}{\left( 1 + \left| \frac{x - y^i}{\varepsilon} \right|^{N+2s} \right)^p} \, dx
\]
\[
\geq C\varepsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\varepsilon} \right|^{N+2s}}
\]

we have
\[
\int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon, y, j}^p U_{\varepsilon, y, i} \, dx = C\varepsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\varepsilon} \right|^{N+2s}} \quad (3.15)
\]

On the other hand, by Lemma 3.2 and (3.15), one can obtain

\[
\int_{\mathbb{R}^N} U_{\varepsilon, y, j}^{p+1} \, dx = \int_{\mathbb{R}^N} \sum_{i = 1}^k U_{\varepsilon, y, i}^{p+1} \, dx + (p + 1) \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon, y, i}^p U_{\varepsilon, y, j} \, dx
\]
\[
+ \begin{cases} 
O \left( \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon, y, i}^{p-1} U_{\varepsilon, y, j}^2 \, dx \right), & (p > 2) \\
O \left( \int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon, y, i} \ U_{\varepsilon, y, j}^2 \, dx \right), & (1 < p \leq 2)
\end{cases}
\]
\[
= \varepsilon^N \int_{\mathbb{R}^N} \sum_{i = 1}^k (U^j)^{p+1} \, dx + (p + 1)C\varepsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\varepsilon} \right|^{N+2s}}
\]
\[
+ O \left( \varepsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\varepsilon} \right|^{N+2s+\tau}} \right)
\]

where we use the facts that if \( p > 2 \),

\[
\int_{\mathbb{R}^N} \sum_{i \neq j} U_{\varepsilon, y, i}^{p-1} U_{\varepsilon, y, j}^2 \, dx
\]
\[
= C \int_{\mathbb{R}^N} \varepsilon^N \sum_{i \neq j} (U^j)^{p-1} \left( y - \frac{y^j}{\varepsilon} \right) \left( y - \frac{y^j}{\varepsilon} \right)^2 \, dy
\]
\[
\leq C\varepsilon^N \int_{\mathbb{R}^N} \sum_{i \neq j} \left( \frac{1}{1 + \left| \frac{y - y^j}{\varepsilon} \right|^{N+2s}} \right)^{p-1} \left( \frac{1}{1 + \left| \frac{y - y^j}{\varepsilon} \right|^{N+2s}} \right)^2 \, dy
\]
\[
\leq C\varepsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\varepsilon} \right|^{N+2s+\tau}} \int_{\mathbb{R}^N} \frac{1}{1 + \left| \frac{y - y^j}{\varepsilon} \right|^{p(N+2s)-\tau}} + \frac{1}{1 + \left| \frac{y - y^j}{\varepsilon} \right|^{p(N+2s)-\tau}} \, dy
\]
\[
\leq C\varepsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\varepsilon} \right|^{N+2s+\tau}}
\]
and if \( 1 < p \leq 2 \)
\[
\int_{\mathbb{R}^N} \sum_{i \neq j} u_{\epsilon,y_i}^* u_{\epsilon,y_j}^* \, dx \\
\leq C \epsilon^N \sum_{i \neq j} \int_{\mathbb{R}^N} U^{p+1} \left( y - \frac{y^j - y^i}{\epsilon} \right)^{p+1} \, dy \\
\leq C \epsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\epsilon} \right|^{N+2s+\tau}} \int_{\mathbb{R}^N} \frac{1}{1 + |y|^{p(N+2s)-\tau}} + \frac{1}{1 + |y - y^j - y^i|^{p(N+2s)-\tau}} \, dy \\
\leq C \epsilon^N \sum_{i \neq j} \frac{1}{\left| \frac{y^j - y^i}{\epsilon} \right|^{N+2s+\tau}}
\]

So, the result follows from (3.14)-(3.16).

4 Semiclassical solutions for the fractional Kirchhoff equation

4.1 Finite dimensional reduction

In this subsection we complete Step 1 for the Lyapunov-Schmidt reduction method as in Section 4.

We first consider the operator \( L \),
\[
\langle L \varphi, \psi \rangle = \langle \varphi, \psi \rangle + \epsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} U_{\epsilon,y} \right|^2 \, dx \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \varphi \cdot (-\Delta)^{\frac{s}{2}} \psi \, dx \\
+ 2 \epsilon^{4s-N} b \left( \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\epsilon,y} \cdot (-\Delta)^{\frac{s}{2}} \varphi \, dx \right) \left( \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} U_{\epsilon,y} \cdot (-\Delta)^{\frac{s}{2}} \psi \, dx \right) - p \int_{\mathbb{R}^N} U_{\epsilon,y}^{p-1} \varphi \psi \, dx
\]
for \( \varphi, \psi \in H_\epsilon \). The following result shows that \( L \) is invertible when restricted on \( E_{\epsilon,y} \).

Lemma 4.1 There exist \( \epsilon_1 > 0, \delta_1 > 0 \) and \( \rho > 0 \) sufficiently small, such that for every \( \epsilon \in (0, \epsilon_1), \delta \in (0, \delta_1) \), there holds
\[
\| L \varphi \|_\epsilon \geq \rho \| \varphi \|_\epsilon, \quad \forall \varphi \in E_{\epsilon,y}
\]
uniformly with respect to \( y \in D_{\epsilon,\delta} \).

Proof: The proof is depend on each \( i = 1, 2, \cdots, k \), we omit it for convenience. We use a contradiction argument. Suppose that there exist \( \epsilon_n, \delta_n \to 0, y_n \in D_{\epsilon,\delta} \) and \( \varphi_n \in E_{\epsilon_n,y_n} \) satisfying
\[
\langle L \varphi_n, g \rangle \leq \frac{1}{n} \| \varphi_n \|_{\epsilon_n} \| g \|_{\epsilon_n}, \quad \forall g \in E_{\epsilon_n,y_n}.
\]
(4.1)

Since this inequality is homogeneous with respect to \( \varphi_n \), we can assume that
\[
\| \varphi_n \|_{\epsilon_n}^2 = \epsilon_n^N \text{ for all } n.
\]

Denote \( \tilde{\varphi}_n(x) = \varphi_n (\epsilon_n x + y_n) \). Then
\[
\int_{\mathbb{R}^N} \left( a \left| (-\Delta)^{\frac{s}{2}} \tilde{\varphi}_n \right|^2 + V(\epsilon_n x + y_n) \tilde{\varphi}_n^2 \right) = 1.
\]
As \( V \) is bounded and \( \inf V > 0 \), we infer that \( \{ \tilde{\varphi}_n \} \) is a bounded sequence in \( H_\epsilon \). Hence, up to a subsequence, we may assume that
\[
\tilde{\varphi}_n \to \varphi \text{ in } H_\epsilon, \\
\tilde{\varphi}_n \to \varphi \text{ in } L_{\text{loc}}^{p+1}(\mathbb{R}^N), \\
\tilde{\varphi}_n \to \varphi \text{ a.e. in } \mathbb{R}^N,
\]
for some \( \varphi \in H_\epsilon \). We will prove that \( \varphi \equiv 0 \).
First we prove that $\varphi = \sum_{l=1}^{N} c^l \partial_{x_l} U$ for some $c^l \in \mathbb{R}$. To this end, let $\tilde{E}_n = \{ \tilde{g} \in H_{\varepsilon} : \tilde{g}_{x_n, y_n} \in E_{\varepsilon_n, y_n} \}$, that is,

$$\tilde{E}_n = \left\{ \tilde{g} \in H_{\varepsilon} : \int_{\mathbb{R}^N} \left( a(-\Delta)^{\frac{1}{2}} \partial_{x_j} U \cdot (-\Delta)^{\frac{1}{2}} \tilde{g} + V(\varepsilon_n x + y_n) \partial_{x_j} U \tilde{g} \right) = 0 \text{ for } j = 1, 2, \cdots, N \right\}.$$  

For convenience, denote at the moment

$$\langle u, v \rangle_{*n} = \int_{\mathbb{R}^N} \left( a(-\Delta)^{\frac{1}{2}} u \cdot (-\Delta)^{\frac{1}{2}} v + V(\varepsilon_n x + y_n) uv \right) \quad \text{and} \quad \| u \|^2_{*, n} = \langle u, u \rangle_{*, n}$$

Then (4.1) can be rewritten in terms of $\tilde{\varphi}_n$ as follows:

$$\langle \tilde{\varphi}_n, \tilde{g} \rangle_{*, n} + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} U|^2 \int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} \tilde{\varphi}_n \cdot (-\Delta)^{\frac{1}{2}} \tilde{g}$$

$$+ 2b \int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} U \cdot (-\Delta)^{\frac{1}{2}} \tilde{\varphi}_n \int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} \tilde{g} - p \int_{\mathbb{R}^N} U^{p-1} \tilde{\varphi}_n \tilde{g}$$

$$\leq n^{-1} \| \tilde{g}_n \|_{*, n}$$

where $\tilde{g}_n(x) = g(\varepsilon_n x + y_n) \in \tilde{E}_n$.

Now, for any $g \in C_0^\infty(\mathbb{R}^N)$, define $a^l_n \in \mathbb{R}(1 \leq l \leq N)$ by

$$a^l_n = \frac{\langle \partial_{x_l} U, g \rangle_{*, n}}{\| \partial_{x_l} U \|^2_{*, n}}$$

and let $\tilde{g}_n = g - \sum_{l=1}^{N} a^l_n \partial_{x_l} U$. Note that

$$\| \partial_{x_l} U \|^2_{*, n} \to \int_{\mathbb{R}^N} \left( a |(-\Delta)^{\frac{1}{2}} \partial_{x_l} U|^2 + (\partial_{x_l} U)^2 \right) > 0$$

and for $l \neq j$

$$\langle \partial_{x_l} U, \partial_{x_j} U \rangle_{*, n} = \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \partial_{x_l} U \partial_{x_j} U \to \int_{\mathbb{R}^N} \partial_{x_l} U \partial_{x_j} U = 0$$

Hence the dominated convergence theorem implies that

$$a^l_n \to a^l = \frac{\int_{\mathbb{R}^N} \left( a(-\Delta)^{\frac{1}{2}} \partial_{x_l} U \cdot (-\Delta)^{\frac{1}{2}} g + \partial_{x_l} U g \right)}{\int_{\mathbb{R}^N} \left( a |(-\Delta)^{\frac{1}{2}} \partial_{x_l} U|^2 + (\partial_{x_l} U)^2 \right)}$$

and

$$\langle \partial_{x_l} U, \tilde{g}_n \rangle_{*, n} \to 0$$

as $n \to \infty$. Moreover, we infer that

$$\| \tilde{g}_n \|_{*, n} = O(1)$$

Now substituting $\tilde{g}_n$ into (4.2) and letting $n \to \infty$, we find that

$$\langle \mathcal{L}_+ \varphi, g \rangle = \sum_{l=1}^{N} a^l \langle \mathcal{L}_+ \varphi, \partial_{x_l} U \rangle = 0$$

where $\mathcal{L}_+$ is defined as in Proposition 1.1. Since $U_{x_l} \in \text{Ker } \mathcal{L}_+$ by Proposition 1.1, we have $\langle \mathcal{L}_+ \varphi, \partial_{x_l} U \rangle = 0$. Thus

$$\langle \mathcal{L}_+ \varphi, g \rangle = 0, \quad \forall g \in C_0^\infty(\mathbb{R}^n).$$

This implies that $\varphi \in \text{Ker } \mathcal{L}_+$. Applying Proposition 1.1 again gives $c^l \in \mathbb{R}$ ($1 \leq l \leq N$) such that

$$\varphi = \sum_{l=1}^{N} c^l \partial_{x_l} U.$$
Next we prove \( \varphi \equiv 0 \). Note that \( \tilde{\varphi}_n \in \tilde{E}_n \), that is,

\[
\int_{\mathbb{R}^N} (a(-\Delta)\tilde{\varphi}_n \cdot (-\Delta)\tilde{\varphi}_n + V(\varepsilon_n x + y_n)\tilde{\varphi}_n \partial_x U) = 0
\]

for each \( l = 1, 2, \ldots, N \). By sending \( n \to \infty \), we derive

\[
c^l \int_{\mathbb{R}^N} (a |(-\Delta)\tilde{\varphi}_n \cdot (-\Delta)\tilde{\varphi}_n + (\partial_x U) |) = 0
\]

which implies \( c^l = 0 \). Hence

\( \varphi \equiv 0 \) in \( \mathbb{R}^N \).

Now we can complete the proof. We have proved that \( \tilde{\varphi}_n \to 0 \) in \( H_\varepsilon \) and \( \tilde{\varphi}_n \to 0 \) in \( L^{p+1}_{\text{loc}}(\mathbb{R}^N) \). As a result we obtain

\[
p \int_{\mathbb{R}^N} U^p \varepsilon_n y_n \tilde{\varphi}_n^2 = p\varepsilon_n \int_{\mathbb{R}^N} U^p \tilde{\varphi}_n^2
\]

where \( \varepsilon_n \to (1) \) and there exists a constant \( \delta < \delta_0 \) such that

\[
\int_{\mathbb{R}^N} (-\Delta)\tilde{\varphi}_n \cdot (-\Delta)\tilde{\varphi}_n \leq \frac{c_n}{2}
\]

for \( n \) sufficiently large. However, this implies that \( \varphi \equiv 0 \) in \( \mathbb{R}^N \).

We reach a contradiction. The proof is complete.

Lemma 4.1 implies that by restricting on \( E_{\varepsilon,y} \), the quadratic form \( \mathcal{L}_\varepsilon : E_{\varepsilon,y} \to E_{\varepsilon,y} \) has a bounded inverse, with \( \|\mathcal{L}_\varepsilon^{-1}\| \leq \rho^{-1} \) uniformly with respect to \( y \in D_{\varepsilon,\delta} \). This further implies the following reduction map.

**Lemma 4.2** There exist \( \varepsilon_0 > 0, \delta_0 > 0 \) sufficiently small such that for all \( \varepsilon \in (0, \varepsilon_0), \delta \in (0, \delta_0) \), there exists a \( C^1 \) map \( \varphi_{\varepsilon,\delta} : D_{\varepsilon,\delta} \to H_\varepsilon \) with \( y \mapsto \varphi_{\varepsilon,\delta} \in E_{\varepsilon,y} \) satisfying

\[
\left\langle \frac{\partial J_\varepsilon(y, \varphi_{\varepsilon,y})}{\partial \varphi}, \psi \right\rangle = 0, \quad \forall \psi \in E_{\varepsilon,y}.
\]

Moreover, there exists a constant \( C > 0 \) independent of \( \varepsilon \) small enough and \( \kappa \in (0, \frac{d}{2}) \) such that

\[
\|\varphi_{\varepsilon,y}\| \leq C \varepsilon^{\frac{d}{2} + \alpha + \kappa} + C \varepsilon^{\frac{d}{2}} \sum_{i=1}^{k} (V(y_i) - V(a_i))^{1-\kappa} + \varepsilon^{\frac{d}{2}} \left\{ \begin{array}{ll}
\frac{1}{(1 + 2\varepsilon + \varepsilon^{\delta})^{\frac{d}{2}}} & \text{if } p > 2, \\
\frac{1}{(1 + 2\varepsilon + \varepsilon^{\delta})^{\frac{1}{2}}} & \text{if } 1 < p \leq 2.
\end{array} \right.
\]

**Proof:** This existence of the mapping \( y \mapsto \varphi_{\varepsilon,y} \) follows from the contraction mapping theorem. We construct a contraction map as follows. Let \( \varepsilon_1 \) and \( \delta_1 \) be defined as in Lemma 4.1. Let \( \varepsilon_0 \leq \varepsilon_1 \) and \( \delta_0 \leq \delta_1 \). We will choose \( \varepsilon_0 \) and \( \delta_0 \) later. Fix \( y \in D_{\varepsilon,\delta} \) for \( \delta < \delta_0 \). Recall that \( J_\varepsilon(y, \varphi) = I_\varepsilon(U_{\varepsilon,y}) + l_\varepsilon(\varphi) + \frac{1}{2} \mathcal{L}_\varepsilon \varphi + R_\varepsilon(\varphi) \).
So we have
\[ \frac{\partial J_\varepsilon(\phi)}{\partial \phi} = l_\varepsilon + L_\varepsilon \phi + R'_\varepsilon(\phi). \]

Since \( E_{\varepsilon,y} \) is a closed subspace of \( H_\varepsilon \), Lemma 3.3 and 3.4 implies that \( l_\varepsilon \) and \( R'_\varepsilon(\phi) \) are bounded linear operators when restricted on \( E_{\varepsilon,y} \). So we can identify \( l_\varepsilon \) and \( R'_\varepsilon(\phi) \) with their representatives in \( E_{\varepsilon,y} \). Then, to prove Lemma 4.2, it is equivalent to find \( \phi \in E_{\varepsilon,y} \) that satisfies
\[ \varphi = A_\varepsilon(\varphi) \equiv -L_\varepsilon^{-1} (l_\varepsilon + R'_\varepsilon(\phi)). \] (4.3)

In fact, if \( p > 2 \), we set
\[ S_\varepsilon := \left\{ \varphi \in E_{\varepsilon,y} : \| \varphi \|_\varepsilon \leq \varepsilon^{\frac{2}{N} + a - \kappa} + \varepsilon^{\frac{2}{N}} \sum_{i=1}^{k} |V(y^i) - V(a_i)|^{1-\kappa} + \varepsilon^{\frac{2}{N}} \sum_{i \neq j} \frac{1}{\| y^i - y^j \|^{(N+2s-\kappa)}} \right\} \]
for any small \( \kappa > 0 \). We shall verify that \( A_\varepsilon \) is a contraction mapping from \( S_\varepsilon \) to itself. For \( \varphi \in S_\varepsilon \), by Lemma 3.3 and 3.4, we obtain
\[ \| A_\varepsilon(\varphi) \| \leq C(\| l_\varepsilon \| + \| R'_\varepsilon(\omega) \|) \]
\[ \leq C \| l_\varepsilon \| + C \varepsilon^{\frac{2}{N}} \| \varphi \|_\varepsilon^2 \]
\[ \leq C \left( \varepsilon^{\frac{2}{N} + a} + \varepsilon^{\frac{2}{N}} \sum_{i=1}^{k} |V(y^i) - V(a_i)| + \varepsilon^{\frac{2}{N}} \sum_{i \neq j} \frac{1}{\| y^i - y^j \|^{(N+2s-\kappa)}} \right) \]
\[ + \varepsilon^{\frac{2}{N}} \left( \varepsilon^{-N+2a-2\kappa} + \varepsilon^{N} \sum_{i=1}^{k} |V(y^i) - V(a_i)|^{2(1-\kappa)} + \varepsilon^{N} \sum_{i \neq j} \frac{1}{\| y^i - y^j \|^{(2(N+2s-\kappa))}} \right) \]
\[ \leq C \left( \varepsilon^{\frac{2}{N} + a - \kappa} + \varepsilon^{\frac{2}{N}} \sum_{i=1}^{k} |V(y^i) - V(a_i)|^{1-\kappa} + \varepsilon^{\frac{2}{N}} \sum_{i \neq j} \frac{1}{\| y^i - y^j \|^{(N+2s-\kappa)}} \right). \]

Then, we get \( A_\varepsilon \) maps \( S_\varepsilon \) to \( S_\varepsilon \). On the other hand, for any \( \varphi_1, \varphi_2 \in S_\varepsilon \)
\[ \| A_\varepsilon(\varphi_1) - A_\varepsilon(\varphi_2) \| = \| L_\varepsilon^{-1} R'_\varepsilon(\varphi_1) - L_\varepsilon^{-1} R'_\varepsilon(\varphi_2) \| \]
\[ \leq C \| R'_\varepsilon(\varphi_1) - R'_\varepsilon(\varphi_2) \| \]
\[ \leq C \| R''_\varepsilon(\theta \varphi_1 + (1 - \theta) \varphi_2) \| \| \varphi_1 - \varphi_2 \|_\varepsilon \]
\[ \leq \frac{1}{2} \| \varphi_1 - \varphi_2 \|_\varepsilon. \]

So, \( A_\varepsilon \) is a contraction map from \( S_\varepsilon \) to \( S_\varepsilon \). If \( 1 < p \leq 2 \), we will use
\[ \| R'_\varepsilon \| \leq C \varepsilon^{\frac{2-N}{2}} \| \varphi \|_p, \quad \| R''_\varepsilon \| \leq C \varepsilon^{\frac{2-N}{2}} \| \varphi \|^{p-1} \]
instead of
\[ \| R'_\varepsilon \| \leq C \varepsilon^{-\frac{N}{2}} \| \varphi \|_p^2, \quad \| R''_\varepsilon \| \leq C \varepsilon^{-\frac{N}{2}} \| \varphi \|_p \]
in the above process. We set
\[ S_\varepsilon := \left\{ \varphi \in E_{\varepsilon,y} : \| \varphi \|_\varepsilon \leq \varepsilon^{\frac{2}{N} + a - \kappa} + \varepsilon^{\frac{2}{N}} \sum_{i=1}^{k} (V(y^i) - V(a_i))^{1-\kappa} + \varepsilon^{\frac{2}{N}} \sum_{i \neq j} \frac{1}{\| y^i - y^j \|^{(N+2s-\kappa)}} \right\}. \]

Similarly as above, we also see \( A_\varepsilon \) is a contraction map from \( S_\varepsilon \) to \( S_\varepsilon \). Thus, there exists a contraction map \( y \mapsto \varphi_{\varepsilon,y} \) such that (5.3) holds.

At last, we claim that the map \( y \mapsto \varphi_{\varepsilon,y} \) belongs to \( C^1 \). Indeed, by similar arguments as that of Cao, Nousair and Yan [8], we can deduce a unique \( C^1 \)-map \( \tilde{\varphi}_{\varepsilon,y} : D_{\varepsilon,\delta} \to E_{\varepsilon,y} \) which satisfies (5.3). Therefore, by the uniqueness \( \varphi_{\varepsilon,y} = \tilde{\varphi}_{\varepsilon,y} \), and hence the claim follows. \( \square \)
4.2 Proof of Theorem 1.1

Let $\varepsilon_0$ and $\delta_0$ be defined as in Lemma 4.2 and let $\varepsilon < \varepsilon_0$. Fix $0 < \delta < \delta_0$. Let $y \mapsto \varphi_{\varepsilon,y}$ for $y \in D_{\varepsilon,\delta}$ be the map obtained in Lemma 4.2. As aforementioned in Step 2 in Section 3, it is equivalent to find a critical point for the function $j_\varepsilon$ defined as in (3.1) by Lemma 3.1. By the Taylor expansion, we have

$$j_\varepsilon(y) = I_\varepsilon(U_{\varepsilon,y}) + I_\varepsilon(\varphi_{\varepsilon,y}) + \frac{1}{2} \langle \mathcal{L}_\varepsilon \varphi_{\varepsilon,y}, \varphi_{\varepsilon,y} \rangle + R_\varepsilon(\varphi_{\varepsilon,y}).$$

We analyze the asymptotic behavior of $j_\varepsilon$ with respect to $\varepsilon$ first.

By Lemma 3.3, Lemma 3.4, Lemma 3.5 and Lemma 4.2, we have

$$j_\varepsilon(y) = I_\varepsilon(U_{\varepsilon,y}) + O \left( \|I_\varepsilon\| \|\varphi_{\varepsilon}\| + \|\varphi_{\varepsilon}\|^2 \right)$$

$$= A \varepsilon^N + O \left( \sum_{i=1}^{k} B_i(V(y^i) - V(a_i)) \right) - C_1 \varepsilon^N \sum_{i \neq j} \left| \frac{\epsilon^i - \epsilon^j}{\epsilon} \right|^{N+2s}$$

$$+ \varepsilon^{N+2a} + \varepsilon^N \sum_{i=1}^{k} |V(a_i) - V(y^i)|^2 + \varepsilon^N \sum_{i \neq j} \left| \frac{\epsilon^i - \epsilon^j}{\epsilon} \right|^{N+2s}, \quad \text{if } p > 2$$

$$+ O \left( \varepsilon^{N+\theta} + \varepsilon^N \sum_{i \neq j} \left| \frac{\epsilon^i - \epsilon^j}{\epsilon} \right|^{N+2s+\tau} \right)$$

(4.4)

$$= A \varepsilon^N - C_1 \varepsilon^N \left( \sum_{i=1}^{k} B_i(V(a_i) - V(y^i)) \right) - C_2 \varepsilon^N \sum_{i \neq j} \left| \frac{\epsilon^i - \epsilon^j}{\epsilon} \right|^{N+2s}$$

$$+ O \left( \varepsilon^N \sum_{i=1}^{k} |V(a_i) - V(y^i)|^2 + \varepsilon^{N+\alpha} + \varepsilon^N \sum_{i \neq j} \left| \frac{\epsilon^i - \epsilon^j}{\epsilon} \right|^{N+2s+\tau} \right).$$

Now consider the minimizing problem

$$j_\varepsilon(y_{\varepsilon}) \equiv \inf_{y \in D_{\varepsilon,\delta}} j_\varepsilon(y).$$

Assume that $j_\varepsilon$ is achieved by some $y_{\varepsilon}$ in $D_{\varepsilon,\delta}$. We will prove that $y_{\varepsilon}$ is an interior point of $D_{\varepsilon,\delta}$.

Let $y^i = a_i + L \varepsilon^i e_i$, for any $\theta < 1$ and $L > 0$, vectors $e_1, \ldots, e_k$ with $|e_i - e_j| = 1$ for $i \neq j$. Thus, for $L > 0$ large, $\varepsilon$ small enough and $y \in D_{\varepsilon,\delta}$, since $(1 - \theta)(N + 2s) < \alpha \varepsilon$ there exists $C_1 > 0$ such that

$$A \varepsilon^N - C_1 \varepsilon^{N+(1-\theta)(N+2s)} \leq A \varepsilon^N - C_\varepsilon^{N+(1-\theta)(N+2s)} - C_\varepsilon^{N+\alpha \varepsilon}$$

$$\leq j_\varepsilon(y_{\varepsilon})$$

$$\leq j_\varepsilon(y)$$

$$\leq A \varepsilon^N - C_1 \varepsilon^N \sum_{i=1}^{k} B_i \left( V(a_i) - V(y^i) \right) - C_2 \varepsilon^N \sum_{i \neq j} \left| \frac{\epsilon^i - \epsilon^j}{\epsilon} \right|^{N+2s}$$

where $y_{\varepsilon} = (y_{\varepsilon}^1, \ldots, y_{\varepsilon}^k)$.

Employing (4.4), we deduce that for $\varepsilon$ sufficient small,

$$\varepsilon^N \sum_{i=1}^{k} B_i \left( V(a_i) - V(y^i) \right) + \varepsilon^N \sum_{i \neq j} \left| \frac{\epsilon^i - \epsilon^j}{\epsilon} \right|^{N+2s} \leq C \varepsilon^{N+(1-\theta)(N+2s)}$$

$$25$$
that is,
\[
\sum_{i=1}^{k} B_i \left( V \left( y^i \right) - V \left( a_i \right) \right) \leq C \varepsilon^{\left( 1 - \theta \right) \left( N + 2s \right)},
\]
\[
\sum_{i \neq j} \left| y^i - y^j \right| \leq \frac{\varepsilon}{\varepsilon - 1}.
\]
Thus, \( y^i \) is an interior point of \( D_{\varepsilon, \delta} \) and hence, a critical point of \( j^i \) for \( \varepsilon \) sufficiently small. Then Theorem 1.1 now follows from the claim and Lemma 3.1.

5 Uniqueness of semiclassical bounded states

Now, let \( u^i = U_{\varepsilon, y^i} + \varphi_{\varepsilon, y^i} \) be an arbitrary solution of (1.1) derived as in Section 4. We know \( y^i = o(1) \) as \( \varepsilon \to 0 \). Before proving Theorem 1.2, we first collect some useful facts. And then we will improve this asymptotics estimate by assuming that \( V \) satisfies the additional assumption \( (V_1) \), and by means of the above Pohozaev type identity. We first recall some useful estimates.

**Lemma 5.1** Suppose that \( V(x) \) satisfies \( (V_1) \), then we have

\[
\int_{\mathbb{R}^N} \sum_{i=1}^{k} \left( V(a_i) - V(x) \right) U_{\varepsilon, y^i}^i(x) u(x) dx = O \left( \varepsilon^{\frac{N}{2} + m} + \varepsilon^{\frac{N}{2}} |y^i - a_i|^m \right) \|u\|_{\varepsilon},
\]

and

\[
\int_{B_d(y^i)} \frac{\partial V(x)}{\partial x_i} U_{\varepsilon, y^i}^i(x) u(x) dx = O \left( \varepsilon^{\frac{N}{2} + m - 1} + \varepsilon^{\frac{N}{2}} |y^i - a_i|^{m-1} \right) \|u\|_{\varepsilon},
\]

for any \( d \in (0, \delta) \), where \( u(x) \in H_{\varepsilon} \).

**Proof:** First, from \( (V_3) \) and Hölder's inequality, for a small constant \( d \), we have

\[
\left| \int_{B_d(y^i)} \sum_{i=1}^{k} \left( V(a_i) - V(x) \right) U_{\varepsilon, y^i}^i(x) u(x) dx \right| \\
\leq C \int_{B_d(y^i)} |x - a_i|^m U_{\varepsilon, y^i}^i(x) |u(x)| dx \\
\leq C \left( \int_{B_d(y^i)} |x - a_i|^{2m} |U_{\varepsilon, y^i}^i(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_d(y^i)} |u(x)|^2 dx \right)^{\frac{1}{2}} \\
\leq C \varepsilon^{\frac{N}{2}} \left( \varepsilon^m + |y^i - a_i|^m \right) \|u\|_{\varepsilon}.
\]

Also, by the polynomial decay of \( U_{\varepsilon, y^i}^i(x) \) in \( \mathbb{R}^N \setminus B_d(y^i) \), we can deduce that, for any \( \gamma > 0 \),

\[
\left| \int_{\mathbb{R}^N \setminus B_d(y^i)} \sum_{i=1}^{k} \left( V(a_i) - V(x) \right) U_{\varepsilon, y^i}^i(x) u(x) dx \right| \leq C \varepsilon^\gamma \|u\|_{\varepsilon}.
\]

Then, taking suitable \( \gamma > 0 \), from (5.2) and (5.3) we get (5.1).

Next, from \( (V_3) \) and Hölder’s inequality, for any \( d \in (0, \delta) \), we have

\[
\left| \int_{B_d(y^i)} \frac{\partial V(x)}{\partial x_i} U_{\varepsilon, y^i}^i(x) u(x) dx \right| \\
\leq C \int_{B_d(y^i)} |x - a_i|^{m-1} U_{\varepsilon, y^i}^i(x) |u(x)| dx \\
\leq C \varepsilon^{\frac{N}{2}} \left( \varepsilon^{m-1} + |y^i - a_i|^{m-1} \right) \|u\|_{\varepsilon}.
\]
Lemma 5.2 For any fixed number \( l \in \mathbb{N}^+ \), suppose that \( \{u_i(x)\}_{i=1}^l \) satisfies

\[
\int_{\mathbb{R}^N} |u_i(x)| \, dx < +\infty, \quad i = 1, \ldots, l.
\]

Then for any \( x_0 \), there exist a small constant \( d \) and another constant \( C \) such that

\[
\int_{\partial B_d(x_0)} |u_i(x)| \, d\sigma \leq C \int_{\mathbb{R}^N} |u_i(x)| \, dx, \quad \text{for all } i = 1, \ldots, l.
\]  \hspace{1cm} (5.4)

Proof: Let \( M_i = \int_{\mathbb{R}^N} |u_i(x)| \, dx \), for \( i = 1, \ldots, l \). Then for a fixed small \( r_0 > 0 \),

\[
\int_{B_{r_0}(x_0)} \left( \sum_{i=1}^l |u_i(x)| \right) \, dx \leq \sum_{i=1}^l M_i, \quad \text{for all } i = 1, \ldots, l.
\]  \hspace{1cm} (5.5)

On the other hand,

\[
\int_{B_{r_0}(x_0)} \left( \sum_{i=1}^l |u_i(x)| \right) \, dx \geq \int_0^{r_0} \int_{\partial B_r(x_0)} \left( \sum_{i=1}^l |u_i(x)| \right) \, d\sigma \, dr.
\]  \hspace{1cm} (5.6)

Then (5.5) and (5.6) imply that there exists a constant \( d < r_0 \) such that

\[
\int_{\partial B_r(x_0)} |u_i(x)| \, d\sigma \leq \sum_{i=1}^l M_i, \quad \text{for all } i = 1, \ldots, l.
\]  \hspace{1cm} (5.7)

So taking \( C = \max_{1 \leq i \leq l} \frac{\sum_{i=1}^l M_i}{r_0 M_i} \), we can obtain (5.4) from (5.7).

Applying Lemma 5.2 to \( \varepsilon^{2s} \left| (-\Delta)^{\frac{\gamma}{2}} \varphi_\varepsilon \right|^2 + \varphi_\varepsilon^2 \), there exists a constant \( d = d_\varepsilon \in (1, 2) \) such that

\[
\int_{\partial B_d(y_\varepsilon)} \left( \varepsilon^{2s} \left| (-\Delta)^{\frac{\gamma}{2}} \varphi_\varepsilon \right|^2 + \varphi_\varepsilon^2 \right) \, d\sigma \leq \| \varphi_\varepsilon \|_\varepsilon^2
\]  \hspace{1cm} (5.8)

By an elementary inequality, we have

\[
\int_{\partial B_d(y_\varepsilon)} \left| (-\Delta)^{\frac{\gamma}{2}} u_\varepsilon \right|^2 \, d\sigma \leq 2 \int_{\partial B_d(y_\varepsilon)} \left| (-\Delta)^{\frac{\gamma}{2}} U_{\varepsilon, y_\varepsilon} \right|^2 \, d\sigma + 2 \int_{\partial B_d(y_\varepsilon)} \left| (-\Delta)^{\frac{\gamma}{2}} \varphi_\varepsilon \right|^2 \, d\sigma.
\]

By Proposition 1.1 we can know that there exists a small constant \( d_1 \), such that for any \( \gamma > 0 \) and \( 0 < d < d_1 \), we have

\[
U_{\varepsilon, y_\varepsilon} + \left| (-\Delta)^{\frac{\gamma}{2}} U_{\varepsilon, y_\varepsilon} \right| = O(\varepsilon^\gamma), \quad \text{for } x \in B_d(x),
\]  \hspace{1cm} (5.9)

and

\[
U_{\varepsilon, y_\varepsilon} + \left| (-\Delta)^{\frac{\gamma}{2}} U_{\varepsilon, y_\varepsilon} \right| = o(\varepsilon^\gamma), \quad \text{for } x \in \partial B_d(x).
\]  \hspace{1cm} (5.10)

Hence, for the constant \( d \) chosen as above, we deduce

\[
\varepsilon^{2s} \int_{\partial B_d(y_\varepsilon)} \left| (-\Delta)^{\frac{\gamma}{2}} u_\varepsilon \right|^2 \, d\sigma = O\left( \| \varphi_\varepsilon \|_\varepsilon^2 + \varepsilon^\gamma \right).
\]  \hspace{1cm} (5.11)

In particularly, it follows from (5.9) and (5.10) that for any \( \gamma > 0 \), it holds

\[
\int_{\mathbb{R}^N} U_{\varepsilon, y_\varepsilon}^{q_1} U_{\varepsilon, y_\varepsilon}^{q_2} \, dx = O(\varepsilon^\gamma),
\]  \hspace{1cm} (5.12)

and

\[
\int_{\mathbb{R}^N} \varepsilon^{2s} (-\Delta)^{\frac{\gamma}{2}} U_{\varepsilon, y_\varepsilon} (-\Delta)^{\frac{\gamma}{2}} U_{\varepsilon, y_\varepsilon} \, dx = O(\varepsilon^\gamma),
\]  \hspace{1cm} (5.13)

where \( q_1, q_2 > 0 \).

Now we can improve the estimate for the asymptotic behavior of \( y_\varepsilon \) with respect to \( \varepsilon \).
Lemma 5.3 Assume that $V$ satisfies (V1)–(V3). Let $u_\varepsilon = u_{\varepsilon,y,\varepsilon} + \varphi_\varepsilon$ be a solution derived as in Theorem 2.1. Then

$$|y_\varepsilon^i - a_i| = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.$$  

Proof: The proof is similar to [36] for the single-peak case, we give it here for convenience. To analyze the asymptotic behavior of $y_\varepsilon^i$ with respect to $\varepsilon$, we apply the Pohozˇ aev-type identity (3.1) to $u = u_\varepsilon$ with $\Omega = B_d(y_\varepsilon^i)$, where $d \in (1, 2)$ is chosen as in (5.8). Note that $d$ is possibly dependent on $\varepsilon$. We get

$$\int_{B_d(y_\varepsilon^i)} \frac{\partial V}{\partial x_j} (U_{\varepsilon,y,\varepsilon} + \varphi_\varepsilon)^2 \, dx = \sum_{i=1}^3 I_i$$

with

$$I_1 = \left(\varepsilon^{2a} + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(\Delta)^{\frac{p}{2}} u_\varepsilon|^2 \, dx \right) \int_{\partial B_d(y_\varepsilon^i)} \left[|(-\Delta)^{\frac{p}{2}} u_\varepsilon|^2 \nu_j - 2 \frac{\partial u_\varepsilon}{\partial x_j} \right] \, d\sigma,$$

and

$$I_2 = \int_{\partial B_d(y_\varepsilon^i)} V(x) u_\varepsilon^{p+1} u_\varepsilon \nu_j \, d\sigma,$$

and

$$I_3 = -\frac{2}{p+1} \int_{\partial B_d(y_\varepsilon^i)} u_\varepsilon^{p+1} u_\varepsilon \nu_j \, d\sigma.$$

It follows from Theorem 2.1 that

$$\varepsilon^{2a} + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(\Delta)^{\frac{p}{2}} u_\varepsilon|^2 \, dx = O(\varepsilon^{2a}).$$

Thus, from (5.11) we deduce $I_1 = O\left(\|\varphi_\varepsilon\|^2_{L^p} + \varepsilon^\gamma\right)$. Using similar arguments and choosing a suitable $d$ if necessary, we also get $I_2 = O\left(\|\varphi_\varepsilon\|^2_{L^p} + \varepsilon^\gamma\right)$. For $I_3$, by Lemma 5.2 we have

$$I_3 \leq C\left(\int_{\partial B_d(y_\varepsilon^i)} |\varphi_\varepsilon(x)|^{p+1} \, d\sigma + \varepsilon^\gamma\right) \leq C\left(\int_{\mathbb{R}^N} |\varphi_\varepsilon(x)|^{p+1} \, dx + \varepsilon^\gamma\right) \leq C(\|\varphi_\varepsilon\|^2_{L^p} + \varepsilon^\gamma).$$

Hence

$$\sum_{i=1}^3 I_i = O(\|\varphi_\varepsilon\|^2_{L^p} + \varepsilon^\gamma).$$

(5.15)

To estimate the left hand side of (5.14), notice that from Lemma 5.1

$$\int_{B_d(y_\varepsilon^i)} \frac{\partial V}{\partial x_j} (U_{\varepsilon,y,\varepsilon} + \varphi_\varepsilon)^2 \, dx = \int_{B_d(y_\varepsilon^i)} \frac{\partial V}{\partial x_j} \left[|U_{\varepsilon,y,\varepsilon}^i|^2 + \varphi_\varepsilon^2\right] \, dx + 2 \int_{B_d(y_\varepsilon^i)} \frac{\partial V}{\partial x_j} U_{\varepsilon,y,\varepsilon}^i \varphi_\varepsilon(x) \, dx$$

$$= \int_{B_d(y_\varepsilon^i)} \frac{\partial V}{\partial x_j} |U_{\varepsilon,y,\varepsilon}^i|^2 \, dx + O\left(\|\varphi_\varepsilon\|^2_{L^p} + \varepsilon^{N+2m-2} + \varepsilon^N |y_\varepsilon^i - a_i|^{2m-2}\right).$$

(5.16)

By the assumption (V3), we deduce, for each $i = 1, 2, \cdots, N$,

$$\int_{B_d(y_\varepsilon^i)} \frac{\partial V}{\partial x_j} (U_{\varepsilon,y,\varepsilon}^i)^2 \, dx = m\varepsilon^{N} \int_{B_d(y_\varepsilon^i)} |x_j - a_{i,j}|^{m-2} (x_j - a_{i,j}) (U_{\varepsilon,y,\varepsilon}^i)^2 \, dx + O\left(\int_{B_d(y_\varepsilon^i)} |x_j - a_{i,j}|^m (U_{\varepsilon,y,\varepsilon}^i)^2 \, dx\right)$$

$$= m\varepsilon^{N} \int_{B_d^c(0)} |x_j - a_{i,j} + y_{\varepsilon,i}|^{m-2} (x_j - a_{i,j} + y_{\varepsilon,i}) (U^i)^2 \, dx + O\left(\varepsilon^N (\varepsilon^m + |y_{\varepsilon,i}|^m)\right).$$
Which gives
\[
\int_{B_d(y_i^ε)} \frac{\partial V}{\partial x_j}(U, y_i + \varphi_ε)\, dx = mε_{i,j} ε^m \int_{\mathbb{R}^N} \varepsilon x_j - a_{i,j} + y_{ε,i} \, dx \leq \varepsilon^m \left( mε_{ij} + \varepsilon x_j - a_{i,j} + y_{ε,i} \right) (U^i)^2 \, dx + O \left( \frac{\varepsilon^m}{\varepsilon} \right) + O \left( \varepsilon \| \varphi_ε \|_ε + \varepsilon \| \varphi_ε \|_ε^2 + ε^N \left( ε^m + |y_i^ε - a_i|^m \right) \right) .
\]
(5.17)

Since $c_{i,j} \neq 0$ by assumption $(V_3)$, combining (5.14)-(5.17) we deduce
\[
\varepsilon^N \int_{\mathbb{R}^N} \varepsilon x_j - a_{i,j} + y_{ε,i} \, dx \leq \varepsilon^m \left( mε_{ij} + \varepsilon x_j - a_{i,j} + y_{ε,i} \right) (U^i)^2 \, dx = O \left( \frac{\varepsilon^m}{\varepsilon} \right) + O \left( \varepsilon \| \varphi_ε \|_ε + \varepsilon \| \varphi_ε \|_ε^2 + ε^N \left( ε^m + |y_i^ε - a_i|^m \right) \right) .
\]
By Lemma 5.1 and $(V_3)$,
\[
\| \varphi_ε \|_ε = O \left( \frac{\varepsilon^m}{\varepsilon} \left( ε^{m−τ} + |y_i^ε - a_i|^{m(1−τ)} \right) \right) .
\]
Thus,
\[
\int_{\mathbb{R}^N} \varepsilon x_j - a_{i,j} + y_{ε,i} \, dx \leq \varepsilon^m \left( mε_{ij} + \varepsilon x_j - a_{i,j} + y_{ε,i} \right) U^2 \, dx = O \left( \frac{\varepsilon^m}{\varepsilon} + |y_i^ε - a_i|^{m(1−τ)} \right) .
\]
(5.18)

On the other hand, let $m^* = \min(m, 2)$. We have
\[
|y_{ε,i} - a_{i,j}|^m \leq \varepsilon x_j - a_{i,j} + y_{ε,i} \, dx \leq \varepsilon m \left( mε_{ij} + \varepsilon x_j - a_{i,j} + y_{ε,i} \right) (U^i)^2 \, dx + O \left( \frac{\varepsilon^m}{\varepsilon} + |y_i^ε - a_i|^m \right) \]
by the following elementary inequality: for any $e, f \in \mathbb{R}$ and $m > 1$, there holds
\[
||e + f|^m - |e|^m - m|e|^{m−2}e|f| \leq O \left( \frac{|e|^{m−m^*}|f|^m^* + |f|^m} \right)
\]
for some $C > 0$ depending only on $m$. So, multiplying (5.19) by $U^2$ on both sides and integrate over $\mathbb{R}^N$. We get
\[
|y_{ε,i} - a_{i,j}|^m \int_{\mathbb{R}^N} (U^i)^2 \, dx \leq m \int_{\mathbb{R}^N} \varepsilon x_j - a_{i,j} + y_{ε,i} \, dx \leq \varepsilon m \left( mε_{ij} + \varepsilon x_j - a_{i,j} + y_{ε,i} \right) (U^i)^2 \, dx + O \left( \frac{\varepsilon^m}{\varepsilon} + |y_i^ε - a_i|^m \right) \]
for each $i$. Applying (5.18) to the above estimate yields
\[
|y_i^ε - a_i|^m = O \left( \varepsilon^{m−τ} + |y_i^ε - a_i|^{m(1−τ)} \right) |y_i^ε - a_i| + ε^m + |y_i^ε - a_i|^{m−m^*} \varepsilon^{m^*} .
\]
Recall that $mτ < 1$. Using $ε$-Young inequality
\[
XY \leq δX^m + \frac{δ}{m−m^*}Y^{m−m^*}, \quad ∀δ, X, Y > 0
\]
we deduce
\[
|y_i^ε - a_i| = O(ε).
\]

We have to prove that $|y_i^ε - a_i| = o(ε)$. Assume, on the contrary, that there exist $ε_k → 0$ and $y_{ε_k,i} → 0$ such that $y_{ε_k,i}/ε_k → A \in \mathbb{R}^N$ with $A = (A_1, A_2, \cdots, A_N) \neq 0$. Then (5.18) gives
\[
\int_{\mathbb{R}^N} \left| x_j + \frac{y_{ε_k,i} - a_{i,j}}{ε_k} \right|^{m−2} \left( x_j + \frac{y_{ε_k,i} - a_{i,j}}{ε_k} \right) U^2 \, dx = O \left( \frac{\varepsilon^m}{\varepsilon} \right) .
\]
Taking limit in the above gives
\[
\int_{\mathbb{R}^N} \left| x_j + A_i \right|^{m−2} \left( x_j + A_i \right) (U^i)^2 \, dx = 0 .
\]
However, since $U = U(|x|)$ is strictly decreasing with respect to $|x|$, we infer that $A = 0$. We reach a contradiction. The proof is complete. 

}\]
As a consequence of Lemma 5.3 and the assumption \((V_3)\), we infer that
\[
\|\varphi_\varepsilon\|_\varepsilon = O \left( \frac{1}{\varepsilon^{m(1-\tau)}} \right).
\] (5.20)

Here we can take \(\tau\) so small that \(m(1-\tau) > 1\) since \(m > 1\).

In the following we prove the local uniqueness of semiclassical bounded states obtained before. We use a contradiction argument as that of [7, 26]. Assume \(u^{(i)}_\varepsilon = U_{\varepsilon} \psi^{(i)}_\varepsilon + \varphi_\varepsilon^{(i)}\), \(i = 1, 2\), are two distinct solutions derived as in Section 3. By the argument in Section 3, \(u^{(i)}_\varepsilon\) are bounded functions in \(\mathbb{R}^N\), \(i = 1, 2\). Set
\[
\xi_\varepsilon = \frac{u^{(1)}_\varepsilon - u^{(2)}_\varepsilon}{\|u^{(1)}_\varepsilon - u^{(2)}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}}
\]
and set
\[
\tilde{\xi}_\varepsilon(x) = \xi_\varepsilon \left( \varepsilon x + \bar{y}^{(1)}_\varepsilon \right).
\]

It is clear that
\[
\|\tilde{\xi}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 1.
\]
Moreover, by the Claim 3 in Section 3, there holds
\[
\tilde{\xi}_\varepsilon(x) \to 0 \quad \text{as} \quad |x| \to \infty
\] (5.21)
uniformly with respect to sufficiently small \(\varepsilon > 0\). We will reach a contradiction by showing that \(\|\tilde{\xi}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \to 0\) as \(\varepsilon \to 0\). In view of (5.21), it suffices to show that for any fixed \(R > 0\),
\[
\|\tilde{\xi}_\varepsilon\|_{L^\infty(B_R(0))} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\] (5.22)

First we have

**Lemma 5.4** There holds
\[
\|\xi_\varepsilon\|_\varepsilon = O \left( \frac{1}{\varepsilon^{m}} \right).
\]

**Proof:** Recall the inner product on \(H^s(\mathbb{R}^N)\), we can compute that
\[
\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1}{2}} u \right|^2 - \left| (-\Delta)^{\frac{1}{2}} v \right|^2 \, dx
= \int_{\mathbb{R}^N} \left( (-\Delta)^{\frac{1}{2}} u + (-\Delta)^{\frac{1}{2}} v \right) \left( (-\Delta)^{\frac{1}{2}} u - (-\Delta)^{\frac{1}{2}} v \right) \, dx
= \int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} (u + v) (-\Delta)^{\frac{1}{2}} (u - v) \, dx.
\]

Then assume that \(u^{(i)}_\varepsilon\), \(i = 1, 2\), are two solutions to (1.1), we obtain that
\[
\left( \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1}{2}} u^{(1)}_\varepsilon \right|^2 \, dx \right) (-\Delta)^s \xi_\varepsilon + V \xi_\varepsilon
= \varepsilon^{4s-N} b \left( \int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} (u^{(1)}_\varepsilon + u^{(2)}_\varepsilon) (-\Delta)^{\frac{1}{2}} \xi_\varepsilon \, dx \right) (-\Delta)^s u^{(2)}_\varepsilon + C_\varepsilon(x) \xi_\varepsilon
\] (5.23)

and that
\[
\left( \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1}{2}} u^{(2)}_\varepsilon \right|^2 \, dx \right) (-\Delta)^s \xi_\varepsilon + V \xi_\varepsilon
= \varepsilon^{4s-N} b \left( \int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} (u^{(1)}_\varepsilon + u^{(2)}_\varepsilon) (-\Delta)^{\frac{1}{2}} \xi_\varepsilon \, dx \right) (-\Delta)^s u^{(1)}_\varepsilon + C_\varepsilon(x) \xi_\varepsilon
\] (5.24)
where
\[
C_\varepsilon(x) = p \int_0^1 \left( t u^{(1)}_\varepsilon(x) + (1 - t) u^{(2)}_\varepsilon(x) \right)^{p-1} \, dt.
\]
It is straightforward to deduce from (Proof as $\varepsilon \to 0$), using the fact that

$$
\int_{\mathbb{R}^N} \frac{1}{2} \frac{1}{\varepsilon^2} + \left| \frac{1}{\varepsilon^2} \right|^2 \frac{1}{\varepsilon^2} dx \leq \int_{\mathbb{R}^N} C \varepsilon^2 dx.
$$

On the other hand, letting $\varphi^{(i)} \varepsilon$ be the small perturbation term corresponding to $u^{(i)} \varepsilon$, then we have

$$
|C \varepsilon(x)| \leq C \left( \int_{\mathbb{R}^N} \frac{1}{\varepsilon M^{(i)}} + \int_{\mathbb{R}^N} \varphi^{(i)} \varepsilon(x) \int_{\mathbb{R}^N} \left| \varphi^{(i)} \varepsilon \right|^{(i+1)} dx \right).
$$

Since $|\varepsilon(x)| \leq 1$, for $i = 1, 2$, we have

$$
\int_{\mathbb{R}^N} \frac{1}{\varepsilon M^{(i)}} \varphi^{(i)} \varepsilon(x) dx \leq C \varepsilon^N,
$$

and

$$
\int_{\mathbb{R}^N} \left| \varphi^{(i)} \varepsilon \right|^{(i+1)} dx \leq C \left( \int_{\mathbb{R}^N} \varphi^{(i)} \varepsilon \right)^2 \left( \int_{\mathbb{R}^N} \varphi^{(i)} \varepsilon \right) \left( \int_{\mathbb{R}^N} \varphi^{(i)} \varepsilon \right)^2 \left( \int_{\mathbb{R}^N} \varphi^{(i)} \varepsilon \right)^2 dx
$$

In the last inequality we have used the fact that $\|\varphi^{(i)} \varepsilon \|_{\varepsilon M^N} = O(\varepsilon \frac{\varepsilon}{\varepsilon})$. Therefore,

$$
\|\varepsilon^{(i)} \varepsilon \|_{\varepsilon M^N} \| \varphi^{(i)} \varepsilon \|_{\varepsilon M^N} \left( \int_{\mathbb{R}^N} \left| \varphi^{(i)} \varepsilon \right|^{(i+1)} dx \right)^{\frac{1}{2}}.
$$

which implies the desired estimate. The proof is complete.

Next we study the asymptotic behavior of $\varphi^{(i)}, \bar{\varepsilon} \varepsilon$.

**Lemma 5.5** Let $\bar{\varepsilon} = \frac{\varepsilon}{\varepsilon} + \varepsilon y^{(i)}$ for an arbitrary $i_0 \in \{1, 2, \cdots, k\}$. There exist $d_i \in \mathbb{R}, i = 1, 2, \cdots, N$, such that (up to a subsequence)

$$
\bar{\varepsilon} \varepsilon \to \sum_{i=1}^{N} d_i \partial x_i U^{i_0} \quad \text{in} \quad C_{\text{loc}}^1 (\mathbb{R}^N)
$$

as $\varepsilon \to 0$.

**Proof:** It is straightforward to deduce from (5.3) that $\varphi^{(i)} \varepsilon$ solves

$$
\left( a + \varepsilon \frac{1}{\varepsilon^2} \right) \left( \int_{\mathbb{R}^N} \left| (-\Delta) \varphi^{(i)} \varepsilon \right|^2 dx \right) \left( \int_{\mathbb{R}^N} \left| \varphi^{(i)} \varepsilon \right|^2 dx \right) \left( \int_{\mathbb{R}^N} \left| \varphi^{(i)} \varepsilon \right|^2 dx \right) \left( \int_{\mathbb{R}^N} \left| \varphi^{(i)} \varepsilon \right|^2 dx \right)
$$

For convenience, we introduce

$$
\bar{u}^{(i)} \varepsilon(x) = \left( \varepsilon x + \varepsilon y^{(i)} \right) \left( \varepsilon x + \varepsilon y^{(i)} \right)
$$

and

$$
\varphi^{(i)} \varepsilon(x) = \left( \varepsilon x + \varepsilon y^{(i)} \right) \left( \varepsilon x + \varepsilon y^{(i)} \right)
$$

(5.26)
for $i = 1, 2$. Then, we have
\[
\varepsilon^{N-4s} \int_{\mathbb{R}^N} \left| (-\Delta)^s u^{(1)}_\varepsilon \right|^2 \, dx = \int_{\mathbb{R}^N} \left| (-\Delta)^s \bar{u}^{(1)}_\varepsilon \right|^2 \, dx
\]
(5.27)
and
\[
\varepsilon^{N-4s} b \left( \int_{\mathbb{R}^N} \left| (-\Delta)^s u^{(1)}_\varepsilon \right|^2 + \left| (-\Delta)^s u^{(2)}_\varepsilon \right|^2 \right) = b \left( \int_{\mathbb{R}^N} \left| (-\Delta)^s \bar{u}^{(1)}_\varepsilon \right|^2 + \left| (-\Delta)^s \bar{u}^{(2)}_\varepsilon \right|^2 \right)
\]
(5.28)
which are uniformly bounded. Moreover, we have
\[
\int_{\mathbb{R}^N} \left| (-\Delta)^s \varphi^{(1)}_\varepsilon \right|^2 \, dx = \epsilon^{-N} O \left( \left\| \varphi^{(1)}_\varepsilon \right\|_{L^2}^2 \right) = O \left( \epsilon^{2m(1-\tau)} \right)
\]
(5.29)
by (5.20), and
\[
\int_{\mathbb{R}^N} \left| (-\Delta)^s \bar{\xi} \right|^2 \, dx = \varepsilon^{N-4s} \int_{\mathbb{R}^N} \left| (-\Delta)^s \bar{\xi} \right|^2 \, dx = O(1)
\]
(5.30)
by Lemma 5.4.

Thus, in view of $\| \bar{\xi}_\varepsilon \|_{L^\infty(\mathbb{R}^N)} = 1$ and estimates in the above, the elliptic regularity theory implies that $\bar{\xi}_\varepsilon$ is locally uniformly bounded with respect to $\varepsilon$ in $C^{1,\beta}_{\text{loc}}(\mathbb{R}^N)$ for some $\beta \in (0, 1)$. As a consequence, we assume (up to a subsequence) that
\[
\bar{\xi}_\varepsilon \to \bar{\xi} \quad \text{in} \quad C^{1}_{\text{loc}}(\mathbb{R}^N)
\]
We claim that $\bar{\xi} \in \text{Ker} \mathcal{L}$, that is,
\[
\left( a + b \sum_{j=1}^k \int_{\mathbb{R}^N} \left| (-\Delta)^s U_j \right|^2 \right) \left( -\Delta \right)^s \bar{\xi} + 2b \left( \int_{\mathbb{R}^N} \left( -\Delta \right)^s U \cdot \left( -\Delta \right)^s \bar{\xi} \right) + V(a(\bar{\xi}) \bar{\xi} = p(U) p(U) p(U) \bar{\xi},
\]
(5.31)
which can be seen as the limiting equation of (5.26). It follows from (5.27) and (5.29) that
\[
\varepsilon^{N-4s} b \int_{\mathbb{R}^N} \left| (-\Delta)^s u^{(1)}_\varepsilon \right|^2 \, dx = b \sum_{j=1}^k \int_{\mathbb{R}^N} \left| (-\Delta)^s U_j \right|^2 \, dx
\]
\[
= b \int_{\mathbb{R}^N} \left( \left| (-\Delta)^s u^{(1)}_\varepsilon \right|^2 - \sum_{j=1}^k \left| (-\Delta)^s U_j \right|^2 \right) \, dx
\]
\[
= b \sum_{j=1}^k \int_{\mathbb{R}^N} \left( \left| (-\Delta)^s U_j \right|^2 + \left| (-\Delta)^s \varphi^{(1)}_\varepsilon \right|^2 - \left| (-\Delta)^s U_j \right|^2 \right) \, dx
\]
\[
= O \left( \varepsilon^{m(1-\tau)} \right) \to 0.
\]
Similarly, we deduce from (5.28)-(5.30) that
\[
\int_{\mathbb{R}^N} (-\Delta)^s \left( \bar{u}^{(1)}_\varepsilon + \bar{u}^{(2)}_\varepsilon - 2U \right) \cdot (-\Delta)^s \bar{\xi}_\varepsilon \, dx
\]
\[
= \int_{\mathbb{R}^N} (-\Delta)^s \left( \sum_{j=1}^k U_j \left( x + \left( y_j^{(1)} - y_j^{(2)} \right) \varepsilon \right) - U \right) \cdot (-\Delta)^s \bar{\xi}_\varepsilon \, dx
\]
+ \int_{\mathbb{R}^N} (-\Delta)^s \left( \varphi^{(1)}_\varepsilon + \varphi^{(2)}_\varepsilon \right) \cdot (-\Delta)^s \bar{\xi}_\varepsilon \, dx
\]
\[
\to 0.
\]
It follows from Lemma 5.3 that, for any $\Phi \in C_0^\infty (\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (-\Delta) \tilde{\Phi} \left( u^{(2)} - U^{(1)} \right) \cdot (-\Delta) \tilde{\Phi} \, dx$$

$$= \int_{\mathbb{R}^N} (-\Delta) \tilde{\Phi} \left( \sum_{j=1}^k U^j \left( x + \left( y^{(1)}_{\varepsilon} - y^{(2)}_{\varepsilon} \right) / \varepsilon \right) - U^j \right) \cdot (-\Delta) \tilde{\Phi} \, dx$$

$$+ \int_{\mathbb{R}^N} (-\Delta) \tilde{\Phi} \varphi^{(2)}_{\varepsilon} \cdot (-\Delta) \tilde{\Phi} \, dx$$

$$\to 0.$$  

Combining the above formulas and $\xi_{\varepsilon} \to \xi$ in $C^1_{\text{loc}} (\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N} (-\Delta) \tilde{\Phi} \left( u^{(1)}_{\varepsilon} + u^{(2)}_{\varepsilon} \right) \cdot (-\Delta) \tilde{\Phi} \, dx$$

$$\to 2b \left( \int_{\mathbb{R}^N} (-\Delta) \tilde{\Phi} U^{(1)} \cdot (-\Delta) \tilde{\Phi} \, dx \right) \left( -\Delta \right)^s U^{(1)}$$

in $H^{-s} (\mathbb{R}^N)$.

Now, we estimate $C_{\varepsilon} \left( \varepsilon x + y^{(1)}_{\varepsilon} \right)$

$$U \left( \frac{x - y^{(1)}_{\varepsilon}}{\varepsilon} \right) - U \left( \frac{x - y^{(2)}_{\varepsilon}}{\varepsilon} \right) = O \left( \frac{x - y^{(1)}_{\varepsilon} - y^{(2)}_{\varepsilon}}{\varepsilon} \nabla U \left( \frac{x - y^{(1)}_{\varepsilon} + \theta \left( y^{(1)}_{\varepsilon} - y^{(2)}_{\varepsilon} \right)}{\varepsilon} \right) \right),$$

where $0 < \theta < 1$. Then

$$u^{(1)}_{\varepsilon} (x) - u^{(2)}_{\varepsilon} (x) = o(1) \nabla U \left( \frac{x - y^{(1)}_{\varepsilon} + \theta \left( y^{(1)}_{\varepsilon} - y^{(2)}_{\varepsilon} \right)}{\varepsilon} \right) + O \left( |\varphi^{(1)}_{\varepsilon} (x)| + |\varphi^{(2)}_{\varepsilon} (x)| \right).$$

So, from (5.9), for any $\gamma > 0$, we have

$$C_{\varepsilon} (x) = \left( o(1) \nabla U \left( \frac{x - y^{(1)}_{\varepsilon} + \theta \left( y^{(1)}_{\varepsilon} - y^{(2)}_{\varepsilon} \right)}{\varepsilon} \right) + O \left( |\varphi^{(1)}_{\varepsilon} (x)| + |\varphi^{(2)}_{\varepsilon} (x)| \right) \right)^{p-1}$$

$$+ pU^{p-1} \left( \frac{x - y^{(1)}_{\varepsilon}}{\varepsilon} \right) + o(\varepsilon^\gamma), \quad x \in B_d \left( y^{(1)}_{\varepsilon} \right).$$

Then using Lemma 5.3, we can obtain

$$C_{\varepsilon} \left( \varepsilon x + y^{(1)}_{\varepsilon} \right) = pU^{p-1} (x) + O \left( |\varphi^{(1)}_{\varepsilon} (x)| + |\varphi^{(2)}_{\varepsilon} (x)| \right)^{p-2} + o(1), \quad x \in B_d \left( y^{(1)}_{\varepsilon} \right).$$

Also, for $i = 1, 2$, we know

$$\int_{\mathbb{R}^N} |\varphi^{(i)}_{\varepsilon} (x)|^2 \, dx \leq C \left( \int_{\mathbb{R}^N} \left| \varphi^{(i)}_{\varepsilon} (x + y_{\varepsilon}) \right|^2 \, dx \right)^{\frac{1}{2}} \|\Phi\|_{L^2 (\mathbb{R}^N)}$$

$$\leq C \left( \varepsilon^{-1} \|\varphi^{(i)}_{\varepsilon}\|_{L^2 (\mathbb{R}^N)} \right)^{p-1} \varepsilon^{\frac{2^* (p-1)}{2^*} - \frac{N}{2^*}} \|\Phi\|_{L^2 (\mathbb{R}^N)}$$

$$\leq C \left( \varepsilon^{-\frac{N}{2^*}} \|\varphi^{(i)}_{\varepsilon}\|_{L^2 (\mathbb{R}^N)} \right)^{p-1} \|\Phi\|_{H^s (\mathbb{R}^N)}$$

Then for any $\Phi \in C_0^\infty (\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} C_{\varepsilon} \left( \varepsilon x + y^{(1)}_{\varepsilon} \right) \xi_{\varepsilon} \Phi \to p \int_{\mathbb{R}^N} U^{p-1} \xi_{\varepsilon} \Phi = o(1) \|\Phi\|_{H^s (\mathbb{R}^N)}.$$

Therefore, we obtain (5.31). Then $\tilde{\xi} = \sum_{i=1}^N d_i \partial_{x_i} U$ follows from Proposition 1.1 for some $d_i \in \mathbb{R} (i = 1, 2, \cdots, N)$, and thus the Lemma is proved.

Now we prove (5.22) by showing the following lemma.
Lemma 5.6 Let \( d_i \) be defined as in Lemma 5.5. Then
\[
d_i = 0 \quad \text{for } i = 1, 2, \ldots, N.
\]

**Proof:** We use the Pohozaev-type identity (3.1) to prove this lemma. Apply (3.1) to \( u^{(1)}_\varepsilon \) and \( u^{(2)}_\varepsilon \) with \( \Omega = B_d \left( \bar{y}^{(1)}_\varepsilon \right) \), where \( d \) is chosen in the same way as that of Lemma 5.5. We obtain
\[
\int_{B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \frac{\partial V}{\partial x_i} \left( u^{(1)}_\varepsilon - u^{(2)}_\varepsilon \right)^2 \, dx = \left( \varepsilon^2 a + \varepsilon^4 s - N \right) b \int_{\mathbb{R}^N} \left| (-\Delta)^{s} u^{(1)}_\varepsilon \right|^2 \, dx \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \left( (-\Delta)^{s} u^{(1)}_\varepsilon \right) \left( (-\Delta)^{s} u^{(2)}_\varepsilon \right) \nu_i \, d\sigma
\]
\[
- \left( \varepsilon^2 a + \varepsilon^4 s - N \right) b \int_{\mathbb{R}^N} \left| (-\Delta)^{s} u^{(2)}_\varepsilon \right|^2 \, dx \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \left( (-\Delta)^{s} u^{(2)}_\varepsilon \right) \left( (-\Delta)^{s} \xi_{\varepsilon} \right) \nu_i \, d\sigma
\]
\[
+ \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} V(x) \left( u^{(1)}_\varepsilon \right)^2 \left( u^{(2)}_\varepsilon \right)^2 \nu_i \, d\sigma - \frac{2}{p+1} \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \left( u^{(1)}_\varepsilon \right)^{p+1} \left( u^{(2)}_\varepsilon \right)^{p+1} \nu_i \, d\sigma.
\]

In terms of \( \xi_{\varepsilon} \), we get
\[
\int_{B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \frac{\partial V}{\partial x_i} \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_{\varepsilon} \, dx = \sum_{i=1}^{4} I_i
\]

with
\[
I_1 = \left( \varepsilon^2 a + \varepsilon^4 s - N \right) b \int_{\mathbb{R}^N} \left| (-\Delta)^{s} u^{(1)}_\varepsilon \right|^2 \, dx \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \left( (-\Delta)^{s} u^{(1)}_\varepsilon \right) \left( (-\Delta)^{s} u^{(2)}_\varepsilon \right) \nu_i \, d\sigma;
\]
\[
I_2 = -2 \left( \varepsilon^2 a + \varepsilon^4 s - N \right) b \int_{\mathbb{R}^N} \left| (-\Delta)^{s} u^{(2)}_\varepsilon \right|^2 \, dx \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \left( \frac{\partial u^{(1)}_\varepsilon}{\partial v} \frac{\partial u^{(1)}_\varepsilon}{\partial x_i} - \frac{\partial u^{(2)}_\varepsilon}{\partial v} \frac{\partial u^{(2)}_\varepsilon}{\partial x_i} \right) \nu_i \, d\sigma;
\]
\[
I_3 = \varepsilon^4 s - N \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \left( (-\Delta)^{s} u^{(2)}_\varepsilon \right)^2 \nu_i \, d\sigma \int_{\mathbb{R}^N} \left( (-\Delta)^{s} u^{(1)}_\varepsilon \right) \left( (-\Delta)^{s} \xi_{\varepsilon} \right) \, dx;
\]
\[
I_4 = \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} V \left( u^{(1)}_\varepsilon \right) \left( u^{(2)}_\varepsilon \right) \xi_{\varepsilon} \nu_i \, d\sigma - 2 \int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} A_{\varepsilon} \xi_{\varepsilon} \nu_i \, d\sigma,
\]

where \( A_{\varepsilon}(x) = \int_0^1 \left( t u^{(1)}_\varepsilon(x) + (1-t) u^{(2)}_\varepsilon(x) \right) \, dt \).

We estimate (5.34) term by term. Note that
\[
\varepsilon^2 a + \varepsilon^4 s - N \int_{\mathbb{R}^N} \left| (-\Delta)^{s} u^{(i)}_\varepsilon \right|^2 \, dx = O \left( \varepsilon^{2a} \right)
\]
for each \( i = 1, 2 \). Moreover, by similar arguments as that of Lemma 5.3, we have
\[
\int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} \left| (-\Delta)^{s} u^{(i)}_\varepsilon \right|^2 \, d\sigma = O \left( \left\| (-\Delta)^{s} \varphi^{(i)}_\varepsilon \right\|_{L^2(\mathbb{R}^N)}^2 \right).
\]

Thus, by (5.22) and Lemma 5.1-5.3, we deduce
\[
\sum_{i=1}^{3} I_i = \sum_{i=1}^{2} O \left( \varepsilon^2 \left\| (-\Delta)^{s} \varphi^{(i)}_\varepsilon \right\|_{L^2(\mathbb{R}^N)}^2 \right) = O \left( \varepsilon^{N + 2m(1-\tau)} \right)
\]
and
\[
\int_{\partial B_d \left( \bar{y}^{(1)}_\varepsilon \right)} V(x) \left( u^{(1)}_\varepsilon \right) \left( u^{(2)}_\varepsilon \right) \xi_{\varepsilon} \nu_i \, d\sigma = O \left( \varepsilon^{N + m(1-\tau)} \right)
\]
and
\[ \int_{\partial B_r(y^{(n)}_\varepsilon)} A_{x} \xi \, d\sigma = O \left( \varepsilon^{(N+m(1-\tau))}p \right). \]

Hence we conclude that
\[ \text{the RHS of (5.34)} = O \left( \varepsilon^{N+m(1-\tau)} \right). \tag{5.35} \]

Next we estimate the left hand side of (5.34). We have
\[ \int_{B_r(y^{(n)}_\varepsilon)} \frac{\partial V}{\partial x_j} \left( u^{(1)}_{x\varepsilon} + u^{(2)}_{x\varepsilon} \right) \xi(x) \, dx = mc_{a_1, j} \int_{B_r(y^{(n)}_\varepsilon)} |x_j - a_{i_0, j}|^{m-2} (x_j - a_{i_0, j}) \left( u^{(1)}_{x\varepsilon} + u^{(2)}_{x\varepsilon} \right) \xi \, dx \]
\[ + O \left( \int_{B_r(y^{(n)}_\varepsilon)} |x_j - a_{i_0, j}|^{m} \left( u^{(1)}_{\varepsilon} + u^{(2)}_{\varepsilon} \right) \xi(x) \, dx \right). \]

Observe that
\[ mc_{a_1, j} \int_{B_r(y^{(n)}_\varepsilon)} |x_j - a_{i_0, j}|^{m-2} (x_j - a_{i_0, j}) \left( u^{(1)}_{x\varepsilon} + u^{(2)}_{x\varepsilon} \right) \xi \, dx \]
\[ = mc_{a_1, j} \int_{B_r(y^{(n)}_\varepsilon)} |x_j + y^{(1)}_{x\varepsilon, i} - a_{i_0, j}|^{m-2} \left( x_j + y^{(1)}_{x\varepsilon, i} - a_{i_0, j} \right) \left( U^{i_0}(y) + U^{i_0} \left( y + \frac{y^{i_0(1)} - y^{i_0(2)}}{\varepsilon} \right) \right) \xi \, dx \]
\[ + mc_{a_1, j} \int_{B_r(y^{(n)}_\varepsilon)} |x_j - a_{i_0, j}|^{m-2} (x_j - a_{i_0, j}) \left( \varphi^{(1)}_{x\varepsilon} + \varphi^{(2)}_{x\varepsilon} \right) \xi \, dx. \]

Since \( U \) decays at infinity (see Section 3) and \( |y^{(1)}_{x\varepsilon, i} - a_i| = o(\varepsilon) \), using Lemma 5.5 we deduce
\[ mc_{a_1, j} \int_{B_r(y^{(n)}_\varepsilon)} |x_j + y^{(1)}_{x\varepsilon, i} - a_{i_0, j}|^{m-2} \left( x_j + y^{(1)}_{x\varepsilon, i} - a_{i_0, j} \right) \left( U^{i_0}(y) + U^{i_0} \left( y + \frac{y^{i_0(1)} - y^{i_0(2)}}{\varepsilon} \right) \right) \xi \, dx \]
\[ = 2mc_{a_1, j} \varepsilon^{m+2} \sum_{j=1}^{N} d_j \int_{\mathbb{R}^N} |x_j|^{m-2} x_j U^{i_0}(x) \partial_{x_j} U^{i_0} \, dx + o(\varepsilon^{m+2}) \]
\[ = D_{1}d_{1}\varepsilon^{m+2} + o(\varepsilon^{m+2}) \tag{5.36} \]

where
\[ D_{1} = 2mc_{i} \int_{\mathbb{R}^N} |x_j|^{m-2} x_i U^{i_0}(x) \partial_{x_j} U^{i_0} \neq 0. \]

On the other hand, by H"older's inequality, (5.20) and Lemma 5.4, we have
\[ mc_{a_1, j} \int_{B_r(y^{(n)}_\varepsilon)} |x_j - a_{i_0, j}|^{m-2} (x_j - a_{i_0, j}) \left( \varphi^{(1)}_{x\varepsilon} + \varphi^{(2)}_{x\varepsilon} \right) \xi \, dx = 2 \sum_{i=1}^{O} \left( \int_{\mathbb{R}^N} |\varphi^{(i)}_{x\varepsilon}||\xi| \right) \]
\[ = 2 \sum_{i=1}^{O} \left( ||\varphi^{(i)}_{x\varepsilon}|| \, ||\xi|| \right) \]
\[ = O \left( \varepsilon^{N+m(1-\tau)} \right). \tag{5.37} \]

Therefore, from (5.36) and (5.37), we deduce
\[ mc_{a_1, j} \int_{B_r(y^{(n)}_\varepsilon)} |x_j - a_{i_0, j}|^{m-2} x_i \left( u^{(1)}_{x\varepsilon} + u^{(2)}_{x\varepsilon} \right) \xi \, dx = D_{1}d_{1}\varepsilon^{N+m-1} + o(\varepsilon^{N+m-1}). \tag{5.38} \]

Similar arguments give
\[ O \left( \int_{B_r(y^{(n)}_\varepsilon)} |x_j - a_{i_0, j}|^{m} \left( u^{(1)}_{x\varepsilon} + u^{(2)}_{x\varepsilon} \right) \xi \, dx \right) = O \left( \varepsilon^{N+m} \right). \tag{5.39} \]
Hence, combining (5.38) and (5.39), we obtain

$$\text{the LHS of } (5.34) = D_i \varepsilon^{N+m-1} + o(\varepsilon^{N+m-1}).$$

(5.40)

So (5.35) and (5.40) imply that

$$d_i = 0.$$

The proof is complete.

Proof of Theorem 1.2: If there exist two distinct solutions $u_i^{(i)}, i = 1, 2$, then by setting $\xi_\varepsilon$ and $\bar{\xi}_\varepsilon$ as above, we find that

$$\|\xi_\varepsilon\|_{L^\infty(B_R(0))} = o(1)$$

by assumption, and that

$$\|\bar{\xi}_\varepsilon\|_{L^\infty(\cup B_R(y_i^\varepsilon))} = o(1) \quad \text{as } \varepsilon \to 0$$

by (5.21) and (5.22). In the domain $\mathbb{R}^N \setminus \bigcup_{i=1}^k B_R(y_i^\varepsilon)$, we can apply the same argument as that of [36] to conclude that

$$\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N \setminus \bigcup_{i=1}^k B_R(y_i^\varepsilon))} = o(1).$$

Thus $\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = o(1)$ holds, which is in contradiction with $\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 1$. The proof is complete.

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References

[1] V. Ambrosio. Concentrating solutions for a fractional Kirchhoff equation with critical growth. *Asymptot. Anal.*, 116(3-4):249–278, 2020.

[2] V. Ambrosio. Concentration phenomena for a class of fractional Kirchhoff equations in $\mathbb{R}^N$ with general nonlinearities. *Nonlinear Anal.*, 195:111761, 39, 2020.

[3] L. Appolloni, G. Molica Bisci, and S. Secchi. On critical Kirchhoff problems driven by the fractional Laplacian. *Calc. Var. Partial Differential Equations*, 60(6):Paper No. 209, 2021.

[4] A. Arosio and S. Panizzi. On the well-posedness of the Kirchhoff string. *Trans. Amer. Math. Soc.*, 348(1):305–330, 1996.

[5] S. Bernstein. Sur une classe d’equations fonctionnelles aux derivees partielles. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]*, 4:17–26, 1940.

[6] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010.

[7] D. Cao, S. Li, and P. Luo. Uniqueness of positive bound states with multi-bump for nonlinear Schrödinger equations. *Calc. Var. Partial Differential Equations*, 54(4):4037–4063, 2015.

[8] D. Cao, E.S. Noussair, and S. Yan. Solutions with multiple peaks for nonlinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(2):235–264, 1999.

[9] D. Cao and S. Peng. Semi-classical bound states for Schrödinger equations with potentials vanishing or unbounded at infinity. *Comm. Partial Differential Equations*, 34(10-12):1566–1591, 2009.

[10] Y. Deng, C. Lin, and S. Yan. On the prescribed scalar curvature problem in $\mathbb{R}^N$, local uniqueness and periodicity. *J. Math. Pures Appl. (9)*, 104(6):1013–1044, 2015.
[11] Y. Deng, S. Peng, and W. Shuai. Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. *J. Funct. Anal.*, 269(11):3500–3527, 2015.

[12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[13] G. M. Figueiredo, N. Ikoma, and J. R. Santos Júnior. Existence and concentration result for the Kirchhoff type equations with general nonlinearities. *Arch. Ration. Mech. Anal.*, 213(3):931–979, 2014.

[14] A. Fiscella and E. Valdinoci. A critical Kirchhoff type problem involving a nonlocal operator. *Nonlinear Anal.*, 94:156–170, 2014.

[15] G. Gu, X. Yang, and Z. Yang. Infinitely many sign-changing solutions for nonlinear fractional Kirchhoff equations. *Applicable Analysis*, 10.1080/00036811.2021.1909722, 2021.

[16] G. Gu and Z. Yang. On the singularly perturbation fractional Kirchhoff equations: Critical case. *Adv. Nonlinear Anal.*, 11(1):1097–1116, 2022.

[17] Z. Guo. Ground states for Kirchhoff equations without compact condition. *J. Differential Equations*, 259(7):2884–2902, 2015.

[18] F. He, D. Qin, and X. Tang. Existence of ground states for Kirchhoff-type problems with general potentials. *J. Geom. Anal.*, 31(8):7709–7725, 2021.

[19] X. He and W. Zou. Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. *J. Differential Equations*, 252(2):1813–1834, 2012.

[20] X. He and W. Zou. Ground states for nonlinear Kirchhoff equations with critical growth. *Ann. Mat. Pura Appl. (4)*, 193(2):473–500, 2014.

[21] Y. He and G. Li. Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. *Calc. Var. Partial Differential Equations*, 54(3):3067–3106, 2015.

[22] Y. He, G. Li, and S. Peng. Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. *Adv. Nonlinear Studies*, 14(2):483–510, 2014.

[23] T. Hu and C. Tang. Limiting behavior and local uniqueness of normalized solutions for mass critical Kirchhoff equations. *Calc. Var. Partial Differential Equations*, 60(6):Paper No. 210, 2021.

[24] G. Kirchhoff. *Vorlesungen über Mathematische Physik, Mechanik*. Lecture 19. Leipzig: Teubner., 1877.

[25] M.K. Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.*, 105(3):243–266, 1989.

[26] G. Li, P. Luo, S. Peng, C. Wang, and C. Xiang. A singularly perturbed Kirchhoff problem revisited. *J. Differential Equations*, 268(2):541–589, 2020.

[27] G. Li, Y. Niu, and C. Xiang. Local uniqueness of multi-peak solutions to a class of Kirchhoff equations. *Ann. Acad. Sci. Fenn. Math.*, 45(1):121–137, 2020.

[28] G. Li and H. Ye. Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$. *J. Differential Equations*, 257(2):566–600, 2014.

[29] P. Luo, S. Peng, C. Wang, and C. Xiang. Multi-peak positive solutions to a class of Kirchhoff equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 149(4):1097–1122, 2019.

[30] X. Mingqi, V. D. Rădulescu, and B. Zhang. Combined effects for fractional Schrödinger-Kirchhoff systems with critical nonlinearities. *ESAIM Control Optim. Calc. Var.*, 24(3):1249–1273, 2018.
[31] X. Mingqi, V. D. Rădulescu, and B. Zhang. A critical fractional Choquard-Kirchhoff problem with magnetic field. Commun. Contemp. Math., 21(4):1850004, 36, 2019.

[32] X. Mingqi, V. D. Rădulescu, and B. Zhang. Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity. Calc. Var. Partial Differential Equations, 58(2):Paper No. 57, 27, 2019.

[33] S. I. Pohozaev. A certain class of quasilinear hyperbolic equations. Mat. Sb. (N.S.), 96(138):152–166, 168, 1975.

[34] P. Pucci and V. D. Rădulescu. Progress in nonlinear Kirchhoff problems [Editorial]. Nonlinear Anal., 186:1–5, 2019.

[35] P. H. Rabinowitz. On a class of nonlinear Schrödinger equations. Z. Angew. Math. Phys., 43(2):270–291, 1992.

[36] V. D. Rădulescu and Z. Yang. Local uniqueness of semiclassical bounded states for a singularly perturbed fractional kirchhoff problem. prepared, 2021.

[37] V. D. Rădulescu and Z. Yang. A singularly perturbed fractional Kirchhoff problem. prepared, 2021.

[38] X. Shang and J. Zhang. Multi-peak positive solutions for a fractional nonlinear elliptic equation. Discrete Contin. Dyn. Syst., 35(7):3183–3201, 2015.

[39] J. Wang, L. Tian, J. Xu, and F. Zhang. Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth. J. Differential Equations, 253(7):2314–2351, 2012.

[40] J. Wei and S. Yan. Infinitely many solutions for the prescribed scalar curvature problem on $S^N$. J. Funct. Anal., 258(9):3048–3081, 2010.

[41] M. Xiang, B. Zhang, and V. D. Rădulescu. Superlinear Schrödinger-Kirchhoff type problems involving the fractional $p$-Laplacian and critical exponent. Adv. Nonlinear Anal., 9(1):690–709, 2020.

[42] Z. Yang. Non-degeneracy of positive solutions for fractional Kirchhoff problems: high dimensional cases. J. Geom. Anal., 32(4):Paper No. 139, 24, 2022.

[43] Z. Yang and Y. Yu. Critical fractional Kirchhoff problems: uniqueness and nondegeneracy. prepared, 2021.