Huygens’ Principle in Minkowski Spaces and Soliton Solutions of the Korteweg-de Vries Equation

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Abstract

A new class of linear second order hyperbolic partial differential operators satisfying Huygens’ principle in Minkowski spaces is presented. The construction reveals a direct connection between Huygens’ principle and the theory of solitary wave solutions of the Korteweg-de Vries equation.

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I. Introduction

The present paper deals with the problem of describing all linear second order partial differential operators for which Huygens’ principle is valid in the sense of “Hadamard’s minor premise”. Originally posed by J.Hadamard in his Yale lectures on hyperbolic equations \[26\], this problem is still far from being completely solved\[1\].

The simplest examples of Huygens’ operators are the ordinary wave operators

\[
\Box_{n+1} = \left( \frac{\partial}{\partial x^0} \right)^2 - \left( \frac{\partial}{\partial x^1} \right)^2 - \cdots - \left( \frac{\partial}{\partial x^n} \right)^2 \tag{1}
\]

in an odd number \( n \geq 3 \) of space dimensions and those ones reduced to (1) by means of elementary transformations, i.e. by local nondegenerate changes of coordinates \( x \mapsto f(x) \); gauge and conformal transformations of a given operator \( \mathcal{L} \mapsto \theta(x) \circ \mathcal{L} \circ \theta(x)^{-1} \), \( \mathcal{L} \mapsto \mu(x)\mathcal{L} \) with some locally smooth nonzero functions \( \theta(x) \) and \( \mu(x) \). These operators are usually called trivial Huygens’ operators, and the famous “Hadamard’s conjecture” claims that all Huygens’ operators are trivial.

Such a strong assertion turns out to be valid only for (real) Huygens’ operators with a constant principal symbol in \( n = 3 \) \[33\]. Stellmacher \[40\] found the first non-trivial examples of hyperbolic wave-type operators satisfying Huygens’ principle, and thereby disproved Hadamard’s conjecture in higher dimensional Minkowski spaces. Later Lagnese & Stellmacher \[31\] extended these examples and even solved \[32\] Hadamard’s problem for a restricted class of hyperbolic operators, namely

\[
\mathcal{L} = \Box_{n+1} + u(x^0) \tag{2}
\]

where \( u(x^0) \) is an analytic function (in its domain of definition) depending on a single variable only. It turns out that the potentials \( u(z) \) entering into (2) are rational functions which can be expressed explicitly in terms of some polynomials\[2\] \( \mathcal{P}_k(z) \):

\[
u(z) = 2 \left( \frac{d}{dz} \right)^2 \log \mathcal{P}_k(z), \quad k = 0, 1, 2, \ldots, \tag{3}\]

the latter being defined via the following differential-recurrence relation:

\[
\mathcal{P}_{k+1} \mathcal{P}_{k-1} - \mathcal{P}_k^2 = (2k+1) \mathcal{P}_k^2, \quad \mathcal{P}_0 = 1, \quad \mathcal{P}_1 = z. \tag{4}\]

Since the works of Moser et al. \[4\], \[3\] the potentials (3) are known as rational solutions of the Korteweg-de Vries equation decreasing at infinity\[3\].

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\[1\] Hadamard’s problem, or the problem of diffusion of waves, has received a good deal of attention and the literature is extensive (see, e.g., \[8\], \[12\], \[15\], \[21\], \[22\], \[27\], \[28\], \[35\], and references therein). For a historical account we refer the reader to the articles \[19\], \[25\].

\[2\] This remarkable class of polynomials seems to have been found for the first time by Burchnall and Chaundy \[11\].

\[3\] The coincidence of such rational solutions of the KdV-hierarchy with the Lagnese-Stellmacher potentials has been observed by Schimming \[38\], \[39\].
A wide class of Huygens’ operators in Minkowski spaces has been discovered recently by Veselov and one of the authors [9], [10] (see also the review article [8]). These operators can also be presented in a self-adjoint form

$$L = \Box_n + u(x)$$

with a locally analytic potential \( u(x) \) depending on several variables. More precisely, \( u(x) \) belongs to the class of so-called Calogero-Moser potentials associated with finite reflection groups (Coxeter groups):

$$u(x) = \sum_{\alpha \in \mathcal{R}_+} m_\alpha (m_\alpha + 1)(\alpha, \alpha) \cdot (\alpha, x)^2.$$  

In formula (6) \( \mathcal{R}_+ \equiv \mathcal{R}_+(\mathcal{G}) \) stands for a properly chosen and oriented subset of normals to reflection hyperplanes of a Coxeter group \( \mathcal{G} \). The group \( \mathcal{G} \) acts on \( M^{n+1} \) in such a way that the time direction is preserved. The set \( \{m_\alpha\} \) is a collection of non-negative integer labels attached to the normals \( \alpha \in \mathcal{R} \) so that \( m_{w(\alpha)} = m_\alpha \) for all \( w \in \mathcal{G} \). Huygens’ principle holds for (5), (6), provided \( n \) is odd, and

$$n \geq 3 + 2 \sum_{\alpha \in \mathcal{R}_+} m_\alpha.$$  

In the present work we construct a new class of self-adjoint wave-type operators (5) satisfying Huygens’ principle in Minkowski spaces. As we will see, this class provides a natural extension of the hierarchy of Huygens’ operators associated to Coxeter groups. On the other hand, it turns out to be related in a surprisingly simple and fundamental way to the theory of solitons.

To present the construction we consider a \((n+1)\)-dimensional Minkowski space \( M^{n+1} \cong \mathbb{R}^{1,n} \) with the metric signature \((+, -, -, \ldots, -)\) and fixed time direction \( \theta \in M^{n+1} \). We write \( \text{Gr}_{\perp}(n+1, 2) \subset \text{Gr}(n+1, 2) \) for a set of all 2-dimensional space-like linear subspaces in \( M^{n+1} \) orthogonal to \( \theta \). Every 2-plane \( E \in \text{Gr}_{\perp}(n+1, 2) \) is equipped with the usual Euclidean structure induced from \( M^{n+1} \). To define the potential \( u(x) \) we fix such a plane \( E \) and introduce polar coordinates \((r, \varphi)\) therein.

Let \( (k_i)_{i=1}^N \) be a strictly increasing sequence of integer positive numbers \( 0 \leq k_1 < k_2 < \ldots < k_{N-1} < k_N \), and let \( \{\Psi_i(\varphi)\} \) be a set of \( 2\pi \)-periodic functions on \( \mathbb{R}^1 \):

$$\Psi_i(\varphi) := \cos(k_i \varphi + \varphi_i), \quad \varphi_i \in \mathbb{R} ,$$  

associated to \((k_i)\). The Wronskian of this set

$$\mathcal{W} [\Psi_1, \Psi_2, \ldots, \Psi_N] := \det \begin{pmatrix} \Psi_1(\varphi) & \Psi_2(\varphi) & \cdots & \Psi_N(\varphi) \\ \Psi'_1(\varphi) & \Psi'_2(\varphi) & \cdots & \Psi'_N(\varphi) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_1^{(N-1)}(\varphi) & \Psi_2^{(N-1)}(\varphi) & \cdots & \Psi_N^{(N-1)}(\varphi) \end{pmatrix}$$  

Using the terminology adopted in the group representation theory we will call such integer monotonic sequences partitions.
does not vanish indentically since $\Psi_i(\varphi)$ are linearly independent.

Let 
\[
\Xi := \left\{ x \in \mathbb{M}^{n+1} \mid r^{|k|} \mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N] = 0, \ |k| := \sum_{i=1}^{N} k_i \right\}
\]
be an algebraic hypersurface of zeros of the Wronskian (9) in the Minkowski space $\mathbb{M}^{n+1}$, and let $\Omega \subset \mathbb{M}^{n+1} \setminus \Xi$ be an open connected part in its complement.

We define $u(x)$ in terms of cylindrical coordinates in $\mathbb{M}^{n+1}$ with polar components in $E$:

\[
u = u_k(x) := -\frac{2\, r^2}{\partial^2} \left( \frac{\partial}{\partial \varphi} \right)^2 \log \mathcal{W}[\Psi_1(\varphi), \Psi_2(\varphi), \ldots, \Psi_N(\varphi)] . \tag{10}
\]

It is easy to see that in a standard Minkowskian coordinate chart $u(x)$ is a real rational function on $\mathbb{M}^{n+1}$ having its singularities on $\Xi$. In particular, it is locally analytic in $\Omega$.

Our main result reads as follows.

**Theorem.** Let $\mathbb{M}^{n+1} \cong \mathbb{R}^{1,n}$ be a Minkowski space, and let

\[
\mathcal{L}(k) := \Box_{n+1} + u_k(x) \tag{11}
\]

be a wave-type second order hyperbolic operator with the potential (10) associated to an arbitrary strictly monotonic partition $(k_i)$ of height $N$:

\[
0 \leq k_1 < k_2 < \ldots < k_N , \quad k_i \in \mathbb{Z}, \quad i = 1, 2, \ldots, N .
\]

Then operator $\mathcal{L}(k)$ satisfies Huygens’ principle at every point $\xi \in \Omega$, provided $n$ is odd, and

\[
n \geq 2 \, k_N + 3 . \tag{12}
\]

**Remark I.** A similar result is also valid if one takes an arbitrary Lorentzian 2-plane $H \in \mathcal{G}_{\|}(n+1,2)$ in the Minkowski space $\mathbb{M}^{n+1}$ containing the time-like vector $\theta$. More precisely, in this case the potential $u_k(x)$ associated to the partition $(k_i)$ is introduced in terms of pseudo-polar coordinates $(\varrho, \vartheta)$ in $H$:

\[
u = u_k(x) := -\frac{2\, \varrho}{\partial^2} \left( \frac{\partial}{\partial \vartheta} \right)^2 \log \mathcal{W}[\psi_1, \psi_2, \ldots, \psi_N] , \tag{13}
\]

where $x^0 = \varrho \sinh \vartheta$, and, say, $x^1 = \varrho \cosh \vartheta$. The functions $\psi_i$ involved in (13) are given by

\[
\psi_i = \cosh(k_i \vartheta + \vartheta_i) , \quad \vartheta_i \in \mathbb{R} . \tag{14}
\]

The theorem formulated above holds when the potential (10) is replaced by (13).

**Remark II.** The potentials (10) considerably extend the class of Calogero-Moser potentials (8) related to Coxeter groups of rank 2. Indeed, in $\mathbb{R}^2$ any Coxeter group $\mathcal{G}$ is a dihedral group
$I_2(q)$, i.e. the group of symmetries of a regular $2q$-polygon. It has one or two conjugacy classes of reflections according as $q$ is odd or even. The corresponding potential (10) can be rewritten in terms of polar coordinates as follows (see [36]):

$$u(r, \varphi) = \frac{m(m+1)q^2}{r^2 \sin^2(q \varphi)}, \quad \text{when } q \text{ odd},$$

and

$$u(r, \varphi) = \frac{m(m+1)(q/2)^2}{r^2 \sin^2(q/2) \varphi} + \frac{m_1(m_1+1)(q/2)^2}{r^2 \cos^2(q/2) \varphi}, \quad \text{when } q \text{ even}.$$  

It is easy to verify that formula (10) boils down to these forms if we fix $N := m; \varphi_i := (-1)^i \pi/2, i = 1, 2, \ldots, N$, and choose

$$k := (q, 2q, 3q, \ldots, mq),$$

when $q$ is odd, and

$$k := (q/2, q, 3q/2, \ldots, (m-m_1)q/2, q, (m-m_1)q/2, 2q, (m-m_1)q/2, \ldots, (m+m_1)q/2),$$

when $q$ is even and $m > m_1$, respectively.

**Remark III.** Let us set $\varphi_i = 4k^3\varphi + \varphi_0i$ and $\vartheta_i = -4k^3\vartheta + \vartheta_0i, \ i = 1, 2, \ldots, N; \varphi_0i, \vartheta_0i \in \mathbb{R}$. The angular parts of potentials (10), (13), i.e.

$$v(\varphi) = -2 \left( \frac{\partial}{\partial \varphi} \right)^2 \log W[\Psi_1(\varphi), \Psi_2(\varphi), \ldots, \Psi_N(\varphi)], \quad (15)$$

$$v(\vartheta) = -2 \left( \frac{\partial}{\partial \vartheta} \right)^2 \log W[\psi_1(\vartheta), \psi_2(\vartheta), \ldots, \psi_N(\vartheta)], \quad (16)$$

are known (see, e.g., [18], [34]) to be respectively singular periodic and proper $N$-soliton solutions of the Korteweg-de Vries equation

$$v_t = -v_{\varphi\varphi\varphi} + 6vv_{\varphi}. \quad (17)$$

It is also well-known that $N$-soliton potentials (16) constitute the whole class of so-called reflectionless real potentials for the one-dimensional Schrödinger operator $L = -\partial^2/\partial \vartheta^2 + v(\vartheta)$ (see, e.g., [1]).

In conclusion of this section we put forward the following conjecture.

**Conjecture.** The wave-type operators (11) with potentials of the form (10) give a complete solution of Hadamard’s problem in Minkowski spaces $\mathbb{M}^{n+1}$ within a restricted class of linear second order hyperbolic operators

$$\mathcal{L} = \left( \frac{\partial}{\partial x^3} \right)^2 - \left( \frac{\partial}{\partial x^1} \right)^2 - \left( \frac{\partial}{\partial x^2} \right)^2 - \ldots - \left( \frac{\partial}{\partial x^n} \right)^2 + u(x^1, x^2).$$

Note added in the proof. This conjecture has been proved recently by one of the authors in [6].
with real locally analytic potentials \( u = u(x^1, x^2) \) depending on two spatial variables and homogeneous of degree \((-2)\): \( u(\alpha x^1, \alpha x^2) = \alpha^{-2}u(x^1, x^2) , \alpha > 0 \).

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**II. Huygens’ principle and Hadamard-Riesz expansions**

The proof of the theorem stated above rests heavily on the Hadamard theory of Cauchy’s problem for linear second order hyperbolic partial differential equations. Here, we summarize briefly some necessary results from this theory following essentially M. Riesz’s approach \([37]\) (see also \([21]\), \([24]\)).

Let \( \mathbb{M}^{n+1} \cong \mathbb{R}^{1,n} \) be a Minkowski space, and let \( \Omega \) be an open connected part in \( \mathbb{M}^{n+1} \). We consider a (formally) self-adjoint scalar wave-type operator

\[
\mathcal{L} = \Box_{n+1} + u(x),
\]

defined in \( \Omega \), the scalar field (potential) \( u(x) \) being assumed to be in \( \mathcal{C}^\infty(\Omega) \). For any \( \xi \in \Omega \), we define a cone of isotropic (null) vectors in \( \mathbb{M}^{n+1} \) with its vertex at \( \xi \):

\[
\gamma(x, \xi) := (x^0 - \xi^0)^2 - (x^1 - \xi^1)^2 - \ldots - (x^n - \xi^n)^2 = 0 ,
\]

and single out the following sets :

\[
C_\pm(\xi) := \{ x \in \mathbb{M}^{n+1} | \gamma(x, \xi) = 0, \xi^0 \leq x^0 \},
\]

\[
J_\pm(\xi) := \{ x \in \mathbb{M}^{n+1} | \gamma(x, \xi) > 0, \xi^0 \leq x^0 \} .
\]

**Definition.** A *(forward)* Riesz kernel of operator \( \mathcal{L} \) is a holomorphic (entire analytic) mapping \( \lambda \mapsto \Phi_\lambda^\Omega(x, \xi), \lambda \in \mathbb{C} \), with values in the space of distributions \( \mathcal{D}'(\Omega) \), such that for any \( \xi \in \Omega \):

\[
(i) \quad \text{supp } \Phi_\lambda^\Omega(x, \xi) \subseteq J_+ (\xi) ,
\]

\[
(ii) \quad \mathcal{L} [\Phi_\lambda^\Omega(x, \xi)] = \Phi_{\lambda-1}^\Omega(x, \xi) ,
\]

\[
(iii) \quad \Phi_0^\Omega(x, \xi) = \delta(x - \xi) .
\]

\textsuperscript{6}By a distribution \( f \in \mathcal{D}'(\Omega) \) we mean, as usual, a linear continuous form on the space \( \mathcal{D}(\Omega) \) of \( \mathcal{C}^\infty \)-functions with supports compactly imbedded in \( \Omega \) (cf., e.g., \([23]\)).
The value of the Riesz kernel $\Phi_+^\Omega(x, \xi) := \Phi_+(x, \xi)$ at $\lambda = 1$ is called a (forward) fundamental solution of the operator $L$:

$$L[\Phi_+(x, \xi)] = \delta(x - \xi), \quad \text{supp } \Phi_+(x, \xi) \subseteq \overline{J_+(\xi)}. \tag{22}$$

Such a solution is known to exist for any $u(x) \in C^\infty(\Omega)$, and it is uniquely determined.

**Definition.** The operator $L$ defined by (18) satisfies Huygens’ principle in a domain $\Omega_0 \subseteq \Omega$ in $M^{n+1}$ if

$$\text{supp } \Phi_+(x, \xi) \subseteq \overline{C_+(\xi)} = \partial J_+(\xi). \tag{23}$$

for every point $\xi \in \Omega_0$.

The analytic description of singularities of Riesz kernel distributions (and, in particular, fundamental solutions) for second order hyperbolic differential operators is given in terms of their asymptotic expansions in the vicinity of the characteristic cone by a graded scale of distributions with weaker and weaker singularities. Such “asymptotics in smoothness”, usually called Hadamard-Riesz expansions, turn out to be very important for testing Huygens’ principle for the operators under consideration.

In order to construct an appropriate scale of distributions (Riesz convolution algebra) in Minkowski space $M^{n+1}$ we consider (for a fixed $\xi \in M^{n+1}$) a holomorphic $D'$-valued mapping $C \rightarrow D'(M^{n+1})$, $\lambda \mapsto R_\lambda(x, \xi)$, such that $R_\lambda(x, \xi)$ is an analytic continuation (in $\lambda$) of the following (regular) distribution:

$$\langle R_\lambda(x, \xi), g(x) \rangle = \int_{J_+(\xi)} \frac{\gamma(x, \xi)^{\lambda - \frac{n+1}{2}}}{H_{n+1}(\lambda)} g(x) \, dx , \quad \text{Re } \lambda > \frac{n-1}{2}, \tag{24}$$

where $dx = dx^0 \wedge dx^1 \wedge \ldots \wedge dx^n$ is a volume form in $M^{n+1}$, $g(x) \in D(M^{n+1})$, and $H_{n+1}(\lambda)$ is a constant given by

$$H_{n+1}(\lambda) = 2\pi^{\frac{n-1}{2}} 4^{\lambda-1} \Gamma(\lambda) \Gamma(\lambda - (n-1)/2). \tag{25}$$

The following properties of this family of distributions are deduced directly from their definition.

For all $\lambda \in C$ and $\xi \in M^{n+1}$ we have

$$\text{supp } R_\lambda(x, \xi) \subseteq \overline{J_+(\xi)} \tag{26}$$

$$\Box_{n+1} R_\lambda = R_{\lambda-1}, \tag{27}$$

$$R_\lambda \ast R_\mu = R_{\lambda+\mu}, \quad \mu \in C, \tag{28}$$

$$(x - \xi, \partial_x) R_\lambda = (2\lambda - n + 1) R_\lambda, \tag{29}$$

$$\gamma'^\nu R_\lambda = 4^\nu (\lambda)_\nu (\lambda - (n-1)/2)_\nu R_{\lambda+\nu}, \quad \nu \in Z_{\geq 0}, \tag{30}$$

where $(\kappa)_\nu := \Gamma(\kappa + \nu)/\Gamma(\kappa)$ is Pochhammer’s symbol, and $\gamma = \gamma(x, \xi)$ is a square of the geodesic distance between $x$ and $\xi$ in $M^{n+1}$.
In addition, when \( n \) is odd, one can prove that
\[
R_\lambda(x, \xi) = \frac{1}{2\pi^{n-2}} \frac{\delta^{(\frac{n-1}{2}-\lambda)}(\gamma)}{4^{\lambda-1}(\lambda - 1)!} \quad \text{for} \quad \lambda = 1, 2, \ldots, (n-1)/2 ,
\]  
(31)

where \( \delta^{(m)}(\gamma) \) stands for the \( m \)-th derivative of Dirac’s delta-measure concentrated on the surface of the future-directed characteristic half-cone \( C_+^{(\xi)} \).

Another important property of Riesz distributions is that
\[
R_0(x, \xi) = \delta(x - \xi).
\]  
(32)

Formulas (26), (27), (32) show that \( R_\lambda(x, \xi) \) is a Riesz kernel for the ordinary wave operator \( \Box_{n+1} \). The property (31) means precisely that in even-dimensional Minkowski spaces \( \mathbb{M}^{n+1} \) (\( n \) is odd) Huygens’ principle holds for sufficiently low powers of the wave operator \( \Box^d \), \( d \leq (n-1)/2 \).

Now we are able to construct the Hadamard-Riesz expansion for the Riesz kernel of a general self-adjoint wave-type operator (18) on \( \mathbb{M}^{n+1} \).

First, we have to find a sequence of two-point smooth functions \( U_\nu := U_\nu(x, \xi) \in C^\infty(\Omega \times \Omega) \), \( \nu = 0, 1, 2, \ldots \), as a solution of the following transport equations:
\[
(x - \xi, \partial_x) U_\nu(x, \xi) + \nu U_\nu(x, \xi) = -\frac{1}{4} \mathcal{L} [U_{\nu-1}(x, \xi)] \quad \text{for} \quad \nu \geq 1.
\]  
(33)

It is well-known (essentially due to [26]) that the differential-recurrence system (33) has a unique solution provided each \( U_\nu \) is required to be bounded in the vicinity of the vertex of the characteristic cone and \( U_0(x, \xi) \) is fixed for a normalization, i.e.
\[
U_0(x, \xi) \equiv 1, \quad U_\nu(\xi, \xi) \sim O(1), \quad \forall \nu = 1, 2, 3, \ldots
\]

These functions \( U_\nu \) are called Hadamard’s coefficients of the operator \( \mathcal{L} \).

In terms of \( U_\nu \) the required asymptotic expansion can be presented as follows:
\[
\Phi_\lambda^\Omega(x, \xi) \sim \sum_{\nu=0}^{\infty} 4^\nu(\lambda)\nu U_\nu(x, \xi) R_{\lambda+\nu}(x, \xi).
\]  
(34)

One can prove that for a hyperbolic differential operator \( \mathcal{L} \) with locally analytic coefficients the Hadamard-Riesz expansion is locally uniformly convergent. From now on we will restrict our consideration to this case.

For \( \lambda = 1 \) formula (34) provides an expansion of the fundamental solution of the operator \( \mathcal{L} \) in a neighborhood of the vertex \( x = \xi \) of the characteristic cone:
\[
\Phi_+(x, \xi) = \sum_{\nu=0}^{\infty} 4^\nu\nu! U_\nu(x, \xi) R_{\nu+1}(x, \xi).
\]  
(35)

When \( n \) is even, we have \( \supp R_{\nu+1}(x, \xi) = J_+^{(\xi)} \) for all \( \nu = 0, 1, 2, \ldots \), and therefore Huygens’ principle never occurs in odd-dimensional Minkowski spaces \( \mathbb{M}^{2l+1} \).
On the other hand, in the case of an odd number of space dimensions $n \geq 3$, we know due to (31) that for $\nu = 0, 1, 2, \ldots, (n - 3)/2$, $\text{supp} R_{\nu+1}(x, \xi) = C_+(\xi)$. Hence, using (30), we can rewrite the series (35) in following form:

$$
\Phi_+(x, \xi) = \frac{1}{2\pi p} \left( V(x, \xi) \delta^{(p-1)}(\gamma) + W(x, \xi) \eta_+(\gamma) \right),
$$

where $p := (n - 1)/2$, $\eta_+(\gamma)$ is a regular distribution characteristic for the region $J_+(\xi)$:

$$
\langle \eta_+(\gamma), g(x) \rangle = \int_{J_+(\xi)} g(x) \, dx, \quad g(x) \in \mathcal{D}(M^{n+1}),
$$

and $V(x, \xi), W(x, \xi)$ are analytic functions in a neighborhood of the vertex $x = \xi$ which admit the following expansions therein:

$$
V(x, \xi) = \sum_{\nu=0}^{p-1} \frac{1}{(1-p) \cdots (\nu - p)} U_{\nu}(x, \xi) \gamma^\nu,
$$

$$
W(x, \xi) = \sum_{\nu=p}^{\infty} \frac{1}{(\nu - p)!} U_{\nu}(x, \xi) \gamma^{\nu-p}, \quad p = \frac{n - 1}{2}.
$$

The function $W(x, \xi)$ is usually called a logarithmic term of the fundamental solution.

It follows directly from the representation formula (36) that operator $\mathcal{L}$ satisfies Huygens’ principle in a neighborhood of the point $\xi$, if and only if, the logarithmic term $W(x, \xi)$ of its fundamental solution vanishes in this neighborhood identically in $x$:

$$
W(x, \xi) \equiv 0.
$$

The function $W(x, \xi)$ is known to be a regular solution of the characteristic Goursat problem for the operator $\mathcal{L}$:

$$
\mathcal{L} [W(x, \xi)] = 0, \quad \text{with a boundary value given on the cone surface } C_+(\xi).
$$

Such a boundary problem has a unique solution, and hence, the necessary and sufficient condition for $\mathcal{L}$ to be Huygens’ operator becomes

$$
W(x, \xi) \triangleq 0,
$$

where the symbol $\triangleq$ implies that the equation in hand is satisfied only on $C_+(\xi)$. By definition (38), the latter condition is equivalent to the following one

$$
U_p(x, \xi) \triangleq 0, \quad p = \frac{n - 1}{2}.
$$

In this way, we arrive at the important criterion for the validity of Huygens’ principle in terms of coefficients of the Hadamard-Riesz expansion (34). Equation (41) is essentially due to Hadamard [26]. It will play a central role in the proof of our main theorem.

\footnote{Such a terminology goes back to Hadamard’s book [26], where the function $W(x, \xi)$ is introduced as a coefficient under the logarithmic singularity of an elementary solution (see for details [15], pp. 740–743).}
III. Proof of the main theorem

We start with some remarks concerning the properties of the one-dimensional Schrödinger operator

\[ L_{(k)} := -\left( \frac{\partial}{\partial \varphi} \right)^2 + v_k(\varphi) \]  

(42)

with a general periodic soliton potential

\[ v_k(\varphi) := -2 \left( \frac{\partial}{\partial \varphi} \right)^2 \log W[\Psi_1, \Psi_2, \ldots, \Psi_N] . \]  

(43)

Here, as already discussed in the Introduction, \( W[\Psi_1, \Psi_2, \ldots, \Psi_N] \) stands for a Wronskian of the set of periodic functions on \( \mathbb{R}^1 \):

\[ \Psi_i(\varphi) := \cos(k_i \varphi + \varphi_i) , \quad \varphi_i \in \mathbb{R} , \]  

(44)

associated to an arbitrary strictly monotonic sequence of real positive numbers ("soliton amplitudes"): \( 0 \leq k_1 < \ldots < k_{N-1} < k_N \).

It is well-known (see, e.g., [34]) that any such operator \( L_{(k)} \) (as well as its proper solitonic counterpart ([16])) can be constructed by a successive application of Darboux-Crum factorization transformations ([17], [16]) to the Schrödinger operator with the identically zero potential:

\[ L_0 := -\left( \frac{\partial}{\partial \varphi} \right)^2 . \]  

(45)

To be precise, let \( L \) be a second order ordinary differential operator with a sufficiently smooth potential:

\[ L := -\left( \frac{\partial}{\partial \varphi} \right)^2 + v(\varphi) . \]  

(46)

We ask for formal factorizations of the operator

\[ L - \lambda I = A^* \circ A , \]  

(47)

where \( I \) is an identity operator, \( \lambda \) is a (real) constant, and \( A, A^* \) are the first order operators adjoint to each other in a formal sense.

According to Frobenius’ theorem (see, e.g., [29]), the most general factorization ([17]) is obtained if we take \( \chi(\varphi) \) as a generic element in \( \text{Ker}(L - \lambda I) \setminus \{0\} \) and set

\[ A := \chi \circ \left( \frac{\partial}{\partial \varphi} \right) \circ \chi^{-1} , \quad A^* := -\chi^{-1} \circ \left( \frac{\partial}{\partial \varphi} \right) \circ \chi . \]  

(48)

Indeed, \( A^* \circ A \) is obviously self-adjoint second order operator with the principal part \( -\partial^2/\partial \varphi^2 \). Hence, it is of the form ([16]). Moreover, since \( A[\chi] = 0 \), we have \( \chi \in \text{Ker} A^* \circ A \), so that ([17]) becomes evident.
Note that for every $\lambda \in \mathbb{R}$ we actually get a one-parameter family of factorizations of $L - \lambda I$. This follows from the fact that $\dim \ker(L - \lambda I) = 2$, whereas $\chi(\varphi)$ and $C \chi(\varphi)$ give rise to the same factorization pair $(A, A^*)$.

By definition, the Darboux-Crum transformation maps an operator $L = \lambda I + A^* \circ A$ into the operator

$$
\tilde{L} := \lambda I + A \circ A^*,
$$

(49)
in which $A$ and $A^*$ are interchanged. The operator $\tilde{L}$ is also a (formally) self-adjoint second-order differential operator

$$
\tilde{L} := -\left( \frac{\partial}{\partial \varphi} \right)^2 + \tilde{v}(\varphi),
$$

(50)

where $\tilde{v}(\varphi)$ is given explicitly by

$$
\tilde{v}(\varphi) = v(\varphi) - 2 \left( \frac{\partial}{\partial \varphi} \right)^2 \log \chi(\varphi).
$$

(51)

The initial operator $L$ and its Darboux-Crum transform $\tilde{L}$ are obviously related to each other via the following intertwining identities:

$$
\tilde{L} \circ A = A \circ L, \quad L \circ A^* = A^* \circ \tilde{L}.
$$

(52)

The Darboux-Crum transformation has a lot of important applications in the spectral theory of Sturm-Liouville operators and related problems of quantum mechanics [30]. In particular, it is used to insert or remove one eigenvalue without changing the rest of the spectrum of a Schrödinger operator (for details see the monograph [34] and references therein).

The explicit construction of the family of operators (42) with periodic soliton potentials (43) is based on the following Crum’s lemma:

**Lemma** ([12]). Let $L$ be a given second order Sturm-Liouville operator (46) with a sufficiently smooth potential, and let $\{\Psi_1, \Psi_2, \ldots, \Psi_N\}$ be its eigenfunctions corresponding to arbitrarily fixed pairwise different eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, i.e. $\Psi_i \in \ker(L - \lambda_i I), i = 1, 2, \ldots, N$. Then, for arbitrary $\Psi \in \ker(L - \lambda I)$, $\lambda \in \mathbb{R}$, the function

$$
\chi_N(\varphi) := \frac{\mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N, \Psi]}{\mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N]},
$$

(53)
satisfies the differential equation

$$
\left[ -\left( \frac{\partial}{\partial \varphi} \right)^2 + v_N(\varphi) \right] \chi_N(\varphi) = \lambda \chi_N(\varphi)
$$

(54)

with the potential

$$
v_N(\varphi) := v(\varphi) - 2 \left( \frac{\partial}{\partial \varphi} \right)^2 \log \mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N].
$$

(55)
Given a sequence of real positive numbers \((k_i)_{i=1}^N: 0 \leq k_1 < k_2 < \ldots < k_N\), the Darboux-Crum factorization scheme:

\[
L_i := A_{i-1} \circ A_{i-1}^* + k_i^2 I = A_i \circ A_i + k_{i+1}^2 I \quad \rightarrow \quad L_{i+1} := A_i \circ A_i^* + k_{i+1}^2 I,
\]

starting from the Schrödinger operator (45) with a zero potential

\[
L_0 \equiv -\left( \frac{\partial}{\partial \varphi} \right)^2 = A_0^* \circ A_0 + k_1^2 I,
\]

produces the required operator \(L_{(k)} \equiv L_N\) with the general periodic potential (13).

Now we proceed to the proof of our main theorem formulated in the Introduction.

When \(N = 0\), the statement of the theorem is evident, since the operator \(L_0\) is just the ordinary wave operator in an odd number \(n\) of spatial variables.

Using the Darboux-Crum scheme as outlined above we will carry out the proof by induction in \(N\).

Suppose that the statement of the theorem is valid for all \(m = 0, 1, 2, \ldots, N\). Consider an arbitrary integer monotonic partition \((k_i)\) of height \(N\): \(0 < k_1 < k_2 < \ldots < k_N\), \(k_i \in \mathbb{Z}\).

By our assumption, the wave-type operator \(L_N := L_{(k)} = \Box_{n+1} + u_{k}(x)\), (57)

associated to this partition, satisfies Huygens’ principle in the \((n+1)\)-dimensional Minkowski space \(M^{n+1}\) with \(n\) odd, and \(n \geq 2k_N + 3\). We fix the minimal admissible number of space variables, i.e. \(n = 2k_N + 3\), and denote

\[
p := \frac{n-1}{2} = k_N + 1 .
\]

By construction, the operator \(L_N\) can be written explicitly in terms of suitably chosen cylindrical coordinates in \(M^{n+1}\):

\[
L_N = \Box_{n-1} - \left[ \left( \frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( -\left( \frac{\partial}{\partial \varphi} \right)^2 + v_N(\varphi) \right) \right],
\]

(59)

where \((r, \varphi)\) are the polar coordinates in some Euclidean 2-plane \(E\) orthogonal to the time direction in \(M^{n+1}\), i.e. \(E \in \text{Gr}_1(n+1, 2)\); \(\Box_{n-1}\) is a wave operator in the orthogonal complement \(E^\perp \cong M^{n-1}\) of \(E\) in \(M^{n+1}\); and \(v_N(\varphi)\) is a 2\(\pi\)-periodic potential given by (13).

Let \(k := k_{N+1}\) be an arbitrary positive integer such that

\[
k > k_N .
\]

(60)

We apply the Darboux-Crum transformation (56) with the spectral parameter \(k\) to the angular part of the Laplacian in \(E\). For this we rewrite \(L_N\) in the form

\[
L_N = \Box_{n-1} - \left[ \left( \frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( A_N^* \circ A_N + k^2 \right) \right],
\]

(61)
and set
\[ \mathcal{L}_{n+1} := \Box_{n-1} - \left[ \left( \frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( A_N \circ A_N^* + k^2 \right) \right], \tag{62} \]
where \( A_N := A_N(\varphi) \) and \( A_N^* := A_N^*(\varphi) \) are the first order ordinary differential operators of the form \( \langle 18 \rangle \).

According to \( \langle 52 \rangle \), we have
\[ \mathcal{L}_{n+1} \circ A_N = A_N \circ \mathcal{L}_n, \quad \mathcal{L}_n \circ A_N^* = A_N^* \circ \mathcal{L}_{n+1}. \tag{63} \]

Let \( \Phi^N_\lambda(x, \xi) \) and \( \Phi^{N+1}_\lambda(x, \xi) \) be the Riesz kernels of hyperbolic operators \( \mathcal{L}_n \) and \( \mathcal{L}_{n+1} \) respectively. Then, by virtue of \( \langle 63 \rangle \) we must have the relation
\[ A_N^*(\varphi) \left[ \Phi^{N+1}_\lambda \right] - A_N(\varphi) \left[ \Phi^N_\lambda \right] = 0 \text{ for all } \lambda \in \mathbb{C}, \tag{64} \]
where \( A_N(\varphi) \) is the differential operator \( A_N \) written in terms of the variable \( \phi \) conjugated to \( \varphi \). Indeed, if identity \( \langle 64 \rangle \) were not valid, one could define a holomorphic mapping \( \tilde{\Phi}^N : \mathbb{C} \to \mathcal{D}', \lambda \mapsto \tilde{\Phi}^N_\lambda(x, \xi) \), such that
\[ \tilde{\Phi}^N_\lambda(x, \xi) := \Phi^N_\lambda(x, \xi) + a \left( A_N^*(\varphi) \left[ \Phi^{N+1}_\lambda \right] - A_N(\varphi) \left[ \Phi^N_\lambda \right] \right). \tag{65} \]
The distribution \( \tilde{\Phi}^N_\lambda(x, \xi) \), depending on an arbitrary complex parameter \( a \in \mathbb{C} \), would also satisfy all the axioms \( \langle 21 \rangle \) in the definition of a Riesz kernel for the operator \( \mathcal{L}_n \). In this way, we would arrive at the contradiction with the uniqueness of such a kernel.

In particular, when \( \lambda = 1 \), the identity \( \langle 64 \rangle \) gives the relation between the fundamental solutions \( \Phi^N_\lambda(x, \xi) \equiv \Phi^N_1(x, \xi) \) and \( \Phi^{N+1}_\lambda(x, \xi) \equiv \Phi^{N+1}_1(x, \xi) \) of operators \( \mathcal{L}_n \) and \( \mathcal{L}_{n+1} \). In accordance with \( \langle 30 \rangle \), we have
\[ \Phi^N_1(x, \xi) = \frac{1}{2\pi^p} \left( V_N(x, \xi) \delta^{(p-1)}(\gamma) + W_N(x, \xi) \eta_+(\gamma) \right) \tag{66} \]
and
\[ \Phi^{N+1}_1(x, \xi) = \frac{1}{2\pi^p} \left( V_{N+1}(x, \xi) \delta^{(p-1)}(\gamma) + W_{N+1}(x, \xi) \eta_+(\gamma) \right), \tag{67} \]
where \( \gamma \) is a square of the geodesic distance between the points \( x \) and \( \xi \) in \( \mathbb{M}^{n+1} \). Substituting \( \langle 30 \rangle, \langle 31 \rangle \) into \( \langle 32 \rangle \), we get the relation between the logarithmic terms \( W_N(x, \xi) \) and \( W_{N+1}(x, \xi) \) of operators \( \mathcal{L}_n \) and \( \mathcal{L}_{n+1} \)
\[ A_N^*(\varphi) \left[ W_{N+1}(x, \xi) \right] - A_N(\varphi) \left[ W_N(x, \xi) \right] = 0. \tag{68} \]
By our assumption, \( \mathcal{L}_n \) is a Huygens’ operator in \( \mathbb{M}^{n+1} \), so that \( W_N(x, \xi) \equiv 0 \). Hence, equation \( \langle 38 \rangle \) implies \( A_N^*(\varphi) \left[ W_{N+1}(x, \xi) \right] = 0 \). On the other hand, as discussed in Sect. II, the logarithmic term \( W_{N+1}(x, \xi) \) is a regular solution of the characteristic Goursat problem for \( \mathcal{L}_{n+1} \), i.e. in particular,
\[ \mathcal{L}_{n+1} \left[ W_{N+1}(x, \xi) \right] = 0. \tag{69} \]
Taking into account definition (62) of the operator \( L_{N+1} \), we arrive at the following equation for \( W_{N+1}(x, \xi) \):

\[
\Box_{n-1} W_{N+1}(x, \xi) = \left( \left( \frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} \right) W_{N+1}(x, \xi) . \tag{70}
\]

According to (38), the logarithmic term \( W_{N+1} \) admits the following expansion

\[
W_{N+1}(x, \xi) = \sum_{\nu=p}^{\infty} U_{\nu}(x, \xi) \frac{\gamma^{\nu-p}}{(\nu-p)!}, \quad p = \frac{n-1}{2}, \tag{71}
\]

where \( U_{\nu}(x, \xi) \) are the Hadamard coefficients of the operator \( L_{N+1} \). Since the potential of the wave-type operator \( L_{N+1} \) depends only on the variables \( r, \varphi \), its Hadamard coefficients \( U_{\nu} \) must depend on the same variables \( r, \varphi \) and their conjugates \( \rho, \phi \) only:

\[
U_{\nu} = U_{\nu}(r, \varphi, \rho, \phi) \quad \text{for all} \quad \nu = 0, 1, 2, \ldots \tag{72}
\]

This follows immediately from the uniqueness of solution of Hadamard’s transport equations (33).

On the other hand, since

\[
\gamma = s^2 - r^2 - \rho^2 + 2r \rho \cos(\varphi - \phi) , \tag{73}
\]

where \( s \) is a geodesic distance in the space \( E^\perp \cong M^{n-1} \) orthogonally complementary to the 2-plane \( E \), we conclude that \( W_{N+1} \) is actually a function of five variables: \( W_{N+1} = W_{N+1}(s, r, \rho, \varphi, \phi) \).

On the space of such functions the wave operator \( \Box_{n-1} \) in \( E^\perp \) acts in the same way as its “radial part”, i.e.

\[
\Box_{n-1} W_{N+1} = \left( \left( \frac{\partial}{\partial s} \right)^2 + \frac{n-2}{s} \frac{\partial}{\partial s} \right) W_{N+1} .
\]

Hence, equation (70) becomes

\[
\left( \left( \frac{\partial}{\partial r} \right)^2 - \left( \frac{\partial}{\partial s} \right)^2 - \frac{n-2}{s} \frac{\partial}{\partial s} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} \right) W_{N+1} = 0 . \tag{74}
\]

Now we substitute the expansion (71) into the left-hand side of the latter equation and develop the result into the similar power series in \( \gamma \), taking into account formula (73). After simple calculations we obtain

\[
\sum_{\nu=p}^{\infty} \left[ \left( U_{\nu}'' + \frac{1}{r} U_{\nu}' - \frac{k^2}{r^2} U_{\nu} \right) - 4 \left( r - \rho \cos(\varphi - \phi) \right) U_{\nu+1}' \right]
\]

\[
14
\]
\[-2 \left( 2(\nu + 1) - \frac{\rho}{r} \cos(\varphi - \phi) \right) U_{\nu+1} - 4 \rho^2 \sin^2(\varphi - \phi) U_{\nu+2} \right] \frac{\gamma^\nu - p}{(\nu - p)!} = 0 ,
\]

where the prime means differentiation with respect to \( r \).

Since the functions \( U_{\nu} \) do not depend explicitly on \( \gamma \), equation (76) can be satisfied only if each coefficient under the powers of \( \gamma \) vanishes separately. In this way we arrive at the following differential-recurrence relation for the Hadamard coefficients of the operator \( \mathcal{L}_{N+1} \):

\[
4 \rho^2 \sin^2(\varphi - \phi) U_{\nu+2} = \left( U''_{\nu} + \frac{1}{r} U'_\nu - \frac{k^2}{r^2} U_{\nu} \right) + \frac{2\rho}{r} \cos(\varphi - \phi) \left( 2r U'_{\nu+1} + U_{\nu+1} \right) - 4 \left( r U''_{\nu+1} + (\nu + 1) U_{\nu+1} \right) ,
\]

where \( \nu \) runs from \( p \): \( \nu = p, p + 1, p + 2, \ldots \)

To get a further simplification of equation (77) we notice that all the Hadamard coefficients of the operators under consideration \((11), (10)\) are homogeneous functions of appropriate degrees. More precisely, they have the following specific form

\[
U_{\nu}(r, \varphi, \rho, \phi) = \frac{1}{(r \rho)^\nu} \sigma_{\nu}(\varphi, \phi) , \quad \nu = 0, 1, 2, \ldots ,
\]

where \( \sigma_{\nu}(\varphi, \phi) = \sigma_{\nu}(\phi, \varphi) \) are symmetric \( 2\pi \)-periodic functions depending on the angular variables only.

In order to prove Ansatz (78) we have to go back to the relation (74) between the Riesz kernels of operators \( \mathcal{L}_N \) and \( \mathcal{L}_{N+1} \):

\[
A^*_N(\varphi) \left[ \Phi^{N+1}_\lambda (x, \xi) \right] - A_N(\lambda) \left[ \Phi^N_\lambda (x, \xi) \right] = 0 , \quad \lambda \in \mathbb{C}
\]

If we substitute the Hadamard-Riesz expansions (34) of the kernels \( \Phi^N_\lambda (x, \xi) \) and \( \Phi^{N+1}_\lambda (x, \xi) \) into (79) directly and take into account that \( A_N \) and its adjoint \( A_N^* \) are the first order ordinary differential operators of the following form (cf. (81)):

\[
A_N(\varphi) = \frac{\partial}{\partial \varphi} - f_N(\varphi) , \quad A_N^*(\varphi) = - \frac{\partial}{\partial \varphi} - f_N(\varphi) ,
\]

where \( f_N(\varphi) = (\partial/\partial \varphi) \log \chi_N(\varphi) \), we obtain

\[
\sum_{\nu=0}^{\infty} 4^{\nu}(\lambda)_\nu \left[ 2r \rho \sin(\varphi - \phi) \left( U^{N+1}_{\nu+1} - U^N_{\nu+1} \right) - \left( \frac{\partial}{\partial \varphi} + f_N(\varphi) \right) U^N_{\nu} \right] R_{\lambda + \nu} = 0 ,
\]

where \( U^N_{\nu}(r, \varphi, \rho, \phi) \) and \( U^{N+1}_{\nu}(r, \varphi, \rho, \phi) \) are the Hadamard coefficients of operators \( \mathcal{L}_N \) and \( \mathcal{L}_{N+1} \) respectively; \( R_\lambda := R_\lambda(x, \xi) \) is the family of Riesz distributions in \( \mathbb{M}^{n+1} \).
The same argument as above (see the remark before formula (77)) shows that all the coefficients of the series \( \mathcal{R} \) under the Riesz distributions of different weights must vanish separately. So we arrive at the recurrence relation between the sequences of Hadamard’s coefficients of operators \( L_N \) and \( L_{N+1} \):

\[
U_{N+1}^{N+1} = U_{N+1}^N + \frac{1}{2r\rho \sin(\varphi - \phi)} \left[ \left( \frac{\partial}{\partial \varphi} + f_N(\varphi) \right) U_{N+1}^N + \left( \frac{\partial}{\partial \phi} - f_N(\phi) \right) U_N^N \right],
\]

where \( U_0^{N+1} = U_0^N \equiv 1 \) and \( \nu = 0, 1, 2, \ldots \) Now it is easy to conclude from (82) by induction in \( N \) that the Ansatz (78) really holds for Hadamard’s coefficients of all wave-type operators (11) with potentials (10).

Returning to equation (77) and substituting (78) therein, we obtain the following three-term recurrence relation for the angular functions \( \sigma_\nu(\varphi, \phi) \):

\[
4 \sin^2(\varphi - \phi) \sigma_{\nu+2} = (\nu^2 - k^2) \sigma_{\nu} - 2(2\nu + 1) \cos(\varphi - \phi) \sigma_{\nu+1},
\]

where \( \nu = p, p + 1, p + 2, \ldots \).

In order to analyze equation (83) it is convenient to introduce a formal generating function for the quantities \( \{ \sigma_\nu \} \):

\[
F(t) := \sum_{\nu=p}^{\infty} \sigma_\nu(\varphi, \phi) \frac{t^{\nu-p}}{(\nu-p)!}.
\]

The recurrence relation (83) turns out to be equivalent to the classical hypergeometric differential equation for the function \( F(t) \)

\[
\left( 4 (1 - \omega^2) + 4\omega t - t^2 \right) \frac{d^2 F}{dt^2} + (2p + 1)(2\omega - t) \frac{dF}{dt} + (k^2 - p^2) F = 0,
\]

where \( \omega := \cos(\varphi - \phi) \). The general solution to (85) is given in terms of Gauss’ hypergeometric series:

\[
F(t) = C_2 \, \, _2F_1(p-k; p+k; p+1/2 \mid z) + C_1 \, \, z^{-p+1/2} \, \, _2F_1(1/2 - k; 1/2 + k; 3/2 - p \mid z),
\]

where \( z := (t - 2\omega + 2)/4 \) and \( _2F_1 \) is defined by

\[
_2F_1(a; b; c \mid z) := \sum_{\mu=0}^{\infty} \frac{(a)_\mu (b)_\mu}{(c)_\mu} \frac{z^\mu}{\mu!}.
\]

As discussed in Sect.II, the Hadamard coefficients \( U_\nu(x, \xi) \) must be regular in a neighborhood of the vertex of the characteristic cone \( x = \xi \). When \( x \to \xi \), we have \( \omega \to 1 \) and \( U_p(\xi, \xi) \propto \sigma_\nu(\phi, \phi) = F(0)|_{\omega=1} \) is not bounded unless \( C_1 = 0 \).

In this way, setting \( C_1 = 0 \) in (83), we obtain

\[
\sum_{\nu=p}^{\infty} \sigma_\nu(\varphi, \phi) \frac{t^{\nu-p}}{(\nu-p)!} = C_2 \, \, _2F_1(p-k; p+k; p+1/2 \mid (t - 2\omega + 2)/4).
\]

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Now it remains to recall that by our assumption (60) $k \in \mathbb{Z}$ and $k > k_N$. Since $p = (n - 1)/2 = k_N + 1$, we have $k \geq p$. So the hypergeometric series in the right-hand side of equation (88) is truncated. In fact, the generating function (84) is expressed in terms of the classical Jacobi polynomial $P_{k-p-1/2, k+p+1/2}^{(p-1/2, p+1/2)}(\omega - t/2)$ of degree $k - p$. Hence, $\sigma_{k+1}(\varphi, \phi) \equiv 0$, and the $(k + 1)$-th Hadamard coefficient of the operator $L_{N+1}$ vanishes identically:

$$U_{k+1}(x, \xi) \equiv 0.$$  

(89)

According to Hadamard’s criterion (41), it means that the operator $L_{N+1}$ satisfies Huygens’ principle in Minkowski space $M^{n+1}$, if $n$ is odd and

$$n \geq 2k + 3,$$

Thus, the proof of the theorem is completed.

**IV. Concluding remarks and examples**

In the present paper we have constructed a new hierarchy of Huygens’ operators in higher dimensional Minkowski spaces $M^{n+1}, n > 3$. However, the problem of complete description of the whole class of such operators for arbitrary $n$ still remains open. As mentioned in the Introduction, the famous Hadamard’s conjecture claiming that any Huygens’ operator $L$ can be reduced to the ordinary d’Alembertian $\Box_{n+1}$ with the help of trivial transformations is valid only in $M^{3+1}$. Recently, in the work [4] one of the authors put forward the relevant modification of Hadamard’s conjecture for Minkowski spaces of arbitrary dimensions. Here we recall and discuss briefly this statement.

Let $\Omega$ be an open set in Minkowski space $M^{n+1} \cong \mathbb{R}^{n+1}$, and let $\mathcal{F}(\Omega)$ be a ring of partial differential operators defined over the function space $C^\infty(\Omega)$. For a fixed pair of operators $L_0, L \in \mathcal{F}(\Omega)$ we introduce the map

$$\text{ad}_{L, L_0} : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega), \quad A \mapsto \text{ad}_{L, L_0}[A],$$

such that

$$\text{ad}_{L, L_0}[A] := L \circ A - A \circ L_0.$$  

(91)

Then, given $M \in \mathbb{Z}_{>0}$, the iterated $\text{ad}_{L, L_0}$-map is determined by

$$\text{ad}_{L, L_0}^M[A] := \text{ad}_{L, L_0} \left[ \text{ad}_{L, L_0} \left[ \ldots \text{ad}_{L, L_0}[A] \ldots \right] \right] = \sum_{k=0}^{M} (-1)^k \binom{M}{k} L^{M-k} \circ A \circ L_0^k.$$  

(92)

**Definition.** The operator $L \in \mathcal{F}(\Omega)$ is called $M$-gauge related to the operator $L_0 \in \mathcal{F}(\Omega)$, if there exists a smooth function $\theta(x) \in C^\infty(\Omega)$ non-vanishing in $\Omega$, and an integer positive number $M \in \mathbb{Z}_{>0}$, such that

$$\text{ad}_{L, L_0}^M[\theta(x)] \equiv 0 \quad \text{identically in} \ \mathcal{F}(\Omega).$$  

(93)
In particular, when $M = 1$, the operators $L$ and $L_0$ are connected just by the trivial gauge transformation $L = \theta(x) \circ L_0 \circ \theta(x)^{-1}$.

The modified Hadamard’s conjecture claims:

*Any Huygens’ operator $L$ of the general form

$$L = \Box_{n+1} + (a(x), \partial) + u(x),$$

in a Minkowski space $M^{n+1}$ ($n$ is odd, $n \geq 3$) is $M$-gauge related to the ordinary wave operator $\Box_{n+1}$ in $M^{n+1}$.*

For Huygens’ operators associated to the rational solutions of the KdV-equation (2), (3) and to Coxeter groups (5), (6) this conjecture has been proved in [4] and [7]. In these cases the required identities (93) are the following

$$\text{ad}_{L_k, L_0}^{M_k+1}[\mathcal{P}_k(x^0)] = 0, \quad M_k := \frac{k(k+1)}{2},$$

where $L_k$ is given by (2) with the potential (3) for $k = 0, 1, 2, \ldots$ and

$$\text{ad}_{L_m, L_0}^{M_m+1}[\pi_m(x)] = 0, \quad M_m := \sum_{\alpha \in \mathbb{R}^+} m_{\alpha},$$

where $L_m$ is defined by (5), (6) and $\pi_m(x) := \prod_{\alpha \in \mathbb{R}^+} (\alpha, x)^{m_{\alpha}}$.

It is remarkable that for the operators constructed in the present work the modified Hadamard’s conjecture is also verified. More precisely, for a given wave-type operator

$$L_{(k)} = \Box_{n+1} - \frac{2}{r^2} \left( \frac{\partial}{\partial \varphi} \right)^2 \log \mathcal{W}[\Psi_1(\varphi), \Psi_2(\varphi), \ldots, \Psi_N(\varphi)],$$

associated to a positive integer partition $(k_i) : 0 \leq k_1 < k_2 < \ldots < k_N$, we have the identity

$$\text{ad}_{L_{(k)}, L_0}^{|k|+1}[\Theta_{(k)}(x)] = 0,$$

where $\Theta_{(k)}(x) := r^{|k|} \mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N]$ and $|k| := \sum_{i=1}^N k_i$ is a weight of the partition $(k_i)$.

We are not going to prove (98) in the present paper. A more detailed discussion of this identity and associated algebraic structures will be the subject of our subsequent work. Here, we only mention that such type identities naturally appear [4]–[5] in connection with a classification of overcomplete commutative rings of partial differential operators [13], [14], [41], and with the bispectral problem [20].

We conclude the paper with several concrete examples illustrating our main theorem.

1. As a first example we consider the dihedral group $I_2(q)$, $q \in \mathbb{Z}_{>0}$, acting on the Euclidean plane $E \cong \mathbb{R}^2 \subset \text{Gr}_1(n+1,2)$ and fix the simplest partition $k = (q)$ and the phase $\varphi = \pi/2$.  

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According to Remark II, in this case our theorem gives the wave-type operator with the Calogero-Moser potential related to the Coxeter group $I_2(q)$ with $m = 1$:

$$\mathcal{L}_{(k)} = \Box_{n+1} + \frac{2q^2}{r^2 \sin^2(q \varphi)}.$$  

This operator satisfies Huygens’ principle in $M^{n+1}$ if $n$ is odd and $n \geq 2q + 3$. The Hadamard coefficients of $\mathcal{L}_{(k)}$ can be presented in a simple closed form in terms of polar coordinates on $E$:

$$U_0 = 1,$$

$$U_\nu = \frac{1}{(2\rho)^\nu} \frac{T_{q}^{(\nu)}(\cos(\varphi - \phi))}{\sin(q \varphi) \sin(q \phi)}, \quad \nu \geq 1,$$

where $T_q(z) := \cos(q \arccos(z))$, $z \in [-1, 1]$, is the $q$-th Chebyshev polynomial, and $T_{q}^{(\nu)}(z)$ is its derivative of order $\nu$ with respect to $z$. These formulas are easily obtained with the help of recurrence relation (82).

2. Now we fix $N = 2$, $k_1 = 2$, $k_2 = 3$ and $\varphi_1 = \pi/2$, $\varphi_2 = 0$. The corresponding wave-type operator

$$\mathcal{L}_{(k)} = \Box_{n+1} + \frac{10 \left( x_1^3 + x_2^3 \right) (15x_2^2 - x_1^2)}{(5x_2^2 + x_1^2)^2 x_1^2},$$

satisfies Huygens’ principle for odd $n \geq 9$. The nonzero Hadamard coefficients of this operator are given explicitly by the formulas:

$$U_0 = 1,$$

$$U_1 = \frac{40 x_2 x_1 x_2 x_1 + 15 x_1^2 x_2^2 + 75 x_2^2 x_1^2 + 15 x_2^2 x_1^2 - 5 x_1^2 x_1^2}{2 x_1 x_2 (5 x_2^2 + x_1^2)(5 x_2^2 + x_1^2)},$$

$$U_2 = \frac{120 x_2 x_1 x_2 x_1 + 15 x_1^2 x_1^2 - 5 x_1^2 x_1^2 + 15 x_1^2 x_1^2 + 75 x_2^2 x_2^2}{4 x_1 x_2 (5 x_2^2 + x_1^2)(5 x_2^2 + x_1^2)},$$

$$U_3 = -\frac{15 x_2 x_2}{x_1^2 x_1^2 (5 x_2^2 + x_1^2)(5 x_2^2 + x_1^2)}.$$  

3. Now we take $N = 3$, the partition $k = (1, 3, 4)$, and the phases $\varphi_1 = \varphi_2 = \varphi_3 = \pi/2$. The corresponding operator

$$\mathcal{L}_{(k)} = \Box_{n+1} + \frac{12 \left( 49x_1^4 + 28x_1^2 x_2^2 - x_1^2 \right)}{x_2^2 (7x_1^2 + x_2^2)^2},$$

is a Huygens operator in $M^{n+1}$ when $n$ is odd and $n \geq 11$. The nonzero Hadamard’s coefficients are

$$U_0 = 1,$$

$$U_1 = \frac{-21 x_1^2 x_1^2 - 42 x_2 x_1 x_1^2 - 21 x_1^2 x_2^2 + 3 x_2^2 x_2^2 - 147 x_1^2 x_1^2}{x_2^2 (7x_1^2 + x_2^2)(7 x_1^2 + x_2^2)},$$

$$19$$
According to the theorem, it is huygensian provided coefficients are given by the following formulas:

\[
U_2 = \frac{735 \xi_1^2 x_1^2 + 504 x_2 \xi_1 \xi_2 x_1 + 105 \xi_1^2 x_2^2 - 21 \xi_2^2 x_2^2 + 105 \xi_2^2 x_1^2}{4 \xi_2^2 x_2^2 (7 x_1^2 + x_2^2) (7 \xi_1^2 + \xi_2^2)},
\]

\[
U_3 = \frac{-1260 x_2 \xi_1 \xi_2 x_1 + 21 \xi_2^2 x_2^2 - 105 \xi_1^2 x_2^2 - 105 \xi_2^2 x_1^2 - 735 \xi_1^2 x_1^2}{8 \xi_2^3 x_2^3 (7 x_1^2 + x_2^2) (7 \xi_1^2 + \xi_2^2)},
\]

\[
U_4 = \frac{315 x_1 \xi_1}{4 \xi_2^3 x_2^3 (7 x_1^2 + x_2^2) (7 \xi_1^2 + \xi_2^2)}.
\]

4. The last example illustrates Remark I following the theorem (see Introduction). In this case we consider the operator (11) with the potential (13) associated with the proper \( N \)-soliton solution of the KdV equation. We take \( N = 2 \) and fix \( k_1 = 1, k_2 = 2. \) The real phases are chosen as follows \( \vartheta_1 = \arctanh (1/2), \vartheta_2 = \arctanh (1/4) . \) The corresponding operator \( \mathcal{L}_k \) reads

\[
\mathcal{L}_k = \Box_{n+1} + \frac{2 (2x_0 - 3x_1) (3x_1^3 - 6x_0 x_1^2 + 4x_1 x_0^2 + 8x_0^3)}{x_1^2 (4x_0^2 - 2x_0 x_1 - x_1^2)^2}.
\]

According to the theorem, it is huygensian provided \( n \) is odd and \( n \geq 7. \) The nonzero Hadamard coefficients are given by the following formulas:

\[
U_0 = 1,
\]

\[
U_1 = \frac{4 \xi_0^2 x_1^2 + 9 \xi_1^2 x_1^2 - 16 \xi_0^2 x_0^2 + 8 \xi_0^2 x_0 x_1}{2x_1 (4x_0^2 - 2x_0 x_1 - x_1^2) \xi_1 (4\xi_0^2 - 2\xi_0 \xi_1 - \xi_1^2)} + \frac{8 \xi_0 \xi_1 x_0^2 - 12 \xi_0^2 x_0 x_1 + 4 \xi_1^2 x_0^2 - 12 \xi_0 \xi_1 x_0^2 + 16 \xi_0 \xi_1 x_0 x_1}{2x_1 (4x_0^2 - 2x_0 x_1 - x_1^2) \xi_1 (4\xi_0^2 - 2\xi_0 \xi_1 - \xi_1^2)},
\]

\[
U_2 = -\frac{5 (2 \xi_0 - \xi_1) (2x_0 - x_1)}{4 x_1 (4x_0^2 - 2x_0 x_1 - x_1^2) \xi_1 (4\xi_0^2 - 2\xi_0 \xi_1 - \xi_1^2)}.
\]

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