The Communication Cost of Simulating Bell Correlations

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What classical resources are required to simulate quantum correlations? For the simplest and most important case of local projective measurements on an entangled Bell pair state, we show that exact simulation is possible using local hidden variables augmented by just one bit of classical communication. Certain quantum teleportation experiments, which teleport a single qubit, therefore admit a local hidden variables model.

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Recent theoretical research into quantum algorithms [1], quantum communication complexity [2], and quantum cryptography [3] has shown that quantum devices are more powerful than their classical counterparts. Indeed, the flourishing field of quantum information theory [4] aims to provide an information-theoretic quantification of the power underlying quantum resources. One important feature of quantum theory lies in the statistical correlations produced by measurements on local components of a quantum system. Almost forty years ago, John Bell showed that such correlations cannot be explained by descriptions based on realistic properties of local subsystems [5]. To experimentally distinguish quantum correlations from those produced by local hidden variables theories, Bell introduced the notion of Bell inequalities, with subsequent experimental evidence falling squarely on the side of quantum theory [6]. Bell inequalities, whilst usually considered relevant only to foundational studies of quantum theory, answer a fundamental information-theoretic question: what correlations can be produced between separate classical subsystems, which have interacted in the past, if no communication between the subsystems is allowed? Violation of a Bell inequality, however, does nothing to quantify what classical information-processing resources are required to simulate a particular set of quantum correlations.

The simplest and most important example of quantum correlations involves the correlations produced by projective measurements on a Bell pair. Bell pairs are the maximally-entangled states of two quantum bits (qubits) and are the basic resource currency of bipartite quantum information theory. Various equivalences are known: one shared Bell pair plus two bits of classical communication can be used to teleport one qubit [7] and, conversely, one shared Bell pair plus a single qubit of communication can be used to send two bits of classical communication via superdense coding [8].

Consider the gedanken experiment of Einstein, Podolsky, and Rosen [9] (EPR), as reformulated by Bohm [10]. Two spatially separate parties, Alice and Bob, each have a spin-1/2 particle, or qubit. The global spin wave function is the entangled Bell singlet state (also known as an EPR pair) \(|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)|.

Spin states |\uparrow\rangle, |\downarrow\rangle are defined with respect to a local set of coordinate axes: |\uparrow\rangle (resp. |\downarrow\rangle) corresponds to spin along the local +\hat{z} (resp. -\hat{z}) direction. Alice and Bob each measure their particle’s spin along a direction parameterized by a three-dimensional unit vector: Alice measures along \(\hat{a}\), Bob along \(\hat{b}\). Alice and Bob obtain results, \(\alpha \in \{+1,-1\}\) and \(\beta \in \{+1,-1\}\), respectively, which indicate whether the spin was pointing along \(+1\) or opposite \(-1\) the direction each party chose to measure. Locally, Alice and Bob’s outcomes appear random, with expectation values \(\langle \alpha \rangle = \langle \beta \rangle = 0\), but their joint probabilities are correlated such that that \(\langle \alpha \beta \rangle = -\hat{a} \cdot \hat{b}\). We refer to these correlations as Bell correlations.

It is not possible to reproduce these correlations using a protocol which draws on random variables shared between Alice and Bob, but does not allow communication after they have selected measurements [8]. So how much communication is required to exactly simulate them [11, 12, 13, 14, 15, 16, 17]? Naively, Alice can just tell Bob the direction of her measurement \(\hat{a}\) (or vice versa) but this requires an infinite amount of communication. The question of whether a simulation can be done with a finite amount of communication was raised independently by Maudlin [11], Brassard, Cleve, and Tapp [12], and Steiner [13]. Their approaches differ in how the communication cost of the simulation is defined: Brassard et al. take the cost to be the number of bits sent in the worst case; Steiner, the average. (Steiner’s model is weaker because the amount of communication in the worst case can be unbounded, although such cases occur with probability zero.) Brassard et al. present a protocol which simulates Bell correlations using exactly eight bits of communication (since improved to six bits [14]). Surprisingly [15], the only lower bound for the amount of communication is given by Bell’s theorem: at least some communication is needed. Here we present a simple protocol that uses just one bit of communication.

We first note three simple properties of Bell correlations: (i) if \(\hat{a} = \hat{b}\), then we must have \(\alpha = -\beta\): Alice and Bob must output perfectly anticorrelated bits; (ii) either party can reverse their measurement axis and flip their output bit; and (iii) the joint probability is only
dependent on $\hat{a}$ and $\hat{b}$ via the combination $\hat{a} \cdot \hat{b}$. In his original paper, Bell gave a local hidden variables model that reproduces these three properties for all possible axes, but his model fails to reproduce Bell correlations because the statistical correlations when $\hat{a} \neq \hat{b}$ are not as strong as those of quantum mechanics [3]. The protocol we describe below is inspired by Bell’s original protocol. Property (iii) implies that we may restrict attention to rotationally invariant protocols, for which all probabilities depend only on $\hat{a} \cdot \hat{b}$ and not $\hat{a}$ and $\hat{b}$ separately, by randomizing over all inputs with the same dot product.

More precisely, suppose $P$ is any protocol that simulates the hidden variables of quantum mechanics 

$\langle \alpha \beta \rangle = \sum_{d=\pm 1} \frac{1 + cd}{2} \sgn[\hat{b} \cdot (\hat{\lambda}_1 + d\hat{\lambda}_2)]$

where $E\{x\} = \int_{\text{SO}(3)} d\hat{\lambda}_1 d\hat{\lambda}_2 x$, $c = \sgn(\hat{a} \cdot \hat{\lambda}_1) \sgn(\hat{a} \cdot \hat{\lambda}_2)$ and we have used the trick that $(1+cd)/2 = 1$ if $c=d$ and 0 if $c \neq d$. After substituting for $c$ and expanding Eq. (1), we obtain the sum of four terms (because each term inside the summation sign is itself the sum of two terms) and, using $\sgn(\hat{a} \cdot \hat{\lambda}_1) c = \sgn(\hat{a} \cdot \hat{\lambda}_2)$, we note that the four terms are related by the symmetries $\hat{\lambda}_1 \leftrightarrow -\hat{\lambda}_2$ or $\hat{\lambda}_2 \leftrightarrow -\hat{\lambda}_2$, so each has the same expectation value. Hence

$\langle \alpha \beta \rangle = E\{2 \sgn(\hat{a} \cdot \hat{\lambda}_1) \sgn[\hat{b} \cdot (\hat{\lambda}_2 - \hat{\lambda}_1)]\}$. (2)

This integral may be evaluated with the help of the two diagrams shown in Fig. 2 with the result that $\langle \alpha \beta \rangle = -\hat{a} \cdot \hat{b}$, as required.

Our protocol exactly simulates the quantum mechanical probability distribution for projective measurements on the singlet Bell pair state. If a large number of simulations are performed in parallel, the communication

![Diagram](https://example.com/diagram.png)

**FIG. 1:** The protocol: The shared unit vectors $\hat{\lambda}_1$ and $\hat{\lambda}_2$ described in the text divide the Bloch sphere into four quadrants, as shown. Alice and Bob's actions depend on which quadrant their respective measurement axes lie in, and in Bob’s case, the bit he receives from Alice. (a) Alice’s output: if $\hat{a}$ lies in the shaded region, Alice outputs $+1$; in the unshaded region, she outputs $-1$. (b) The communication: Alice sends $c = +1$ if her measurement axis lies in the N or S quadrants, and $-1$ otherwise. (c) Bob’s output: this depends on the bit received from Alice. The shading is as for (a).
FIG. 2: Construction used to evaluate Eq. 4: (a) We first integrate over $\lambda_2$, taking $\hat{b}$ to point along the positive $z$-axis [18]. Observe that $\text{sgn} [\hat{b} \cdot (\lambda_2 - \lambda_1)]$ is positive in the top spherical cap (shaded) and negative otherwise. The area of the top spherical cap is $A_+ = 2\pi \int_0^\pi \sin \theta d\theta = 2\pi (1 - \cos t)$ where $\cos t = \hat{b} \cdot \lambda_1$, hence $\int d\lambda_2 \text{sgn} [\hat{b} \cdot (\lambda_2 - \lambda_1)] = A_+ - (4\pi - A_+) = -4\pi \cos t = -4\pi \hat{b} \cdot \lambda_1$. (b) We now take $\hat{a}$ to point along the positive $z$-axis [19], set $\hat{b} = (\sin r, 0, \cos r)$, and integrate over $\lambda_1$, obtaining $\int d\lambda_1 \text{sgn}(\hat{a} \cdot \lambda_1) \hat{b} \cdot \lambda_1 = \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \text{sgn}(\cos \theta) (\cos \theta \cos r + \sin \theta \cos \phi \sin r) = 2\pi \cos r = 2\pi \hat{a} \cdot \hat{b}$.
known to her, and (ii) Bob hand off the teleported state to Victor (or another party) to measure, rather than measuring it himself. Such a distinction is not important for the question we address, because the qubit transmitted from Victor to Alice, for example, can carry hidden variables describing its state. The point is that local hidden variables are hidden: although it is convenient to describe a local hidden variables model as if Alice and Bob had access to the hidden variables, the model still exists even if the hidden variables are inaccessable to them. There is no way for the experimenters to tell whether their experiment is described by quantum theory or by “gremlins” within their apparatus, executing the local hidden variables protocol described above.

Are there quantum teleportation experiments which do not have such a local hidden variables description? One obvious possibility is an experiment that teleports entanglement itself. But there is a more subtle possibility. If we allow Bob to measure the qubit using elements of a positive operator-valued measure, then there may not be a local hidden variables description which respects the two bit classical communication bound. More generally, if Alice teleports n qubits (which requires 2n bits of communication) and Bob makes a joint measurements on them, then it is known that any exact local hidden variables theory requires that Alice send at least a constant times 2^n bits of communication in the worst case. Whether this holds for protocols with bounded error is an important open question.

Finally, using the classical teleportation protocol, we obtain a (not necessarily optimal) protocol to simulate joint projective measurements on partially entangled states of two qubits, which uses two bits of communication: Alice first simulates her measurement and determines the post-measurement state of Bob’s qubit; Alice and Bob then execute the classical teleportation protocol.

The results presented here offer an intriguing glimpse into the nature of correlations produced in quantum theory. If we interpret Bell inequality violation to mean that some communication is necessary to simulate Bell correlations, then our results prove that the minimal amount, one bit, is all that is necessary for projective measurements on Bell pairs. Is our straightforward protocol an indication of a deep structure in quantum correlations? We hope that our protocol and the development of a general theory of the communication cost of simulating quantum correlations will help shed light on this fundamental question.

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[1] P. W. Shor, in Proceedings of the 35th Annual Symposium on the Foundations of Computer Science, edited by S. Goldwasser (IEEE Computer Society, Los Alamitos, CA, 1994), pp. 124–134; L. Grover, in Proceedings of the 28th Annual ACM Symposium on the Theory of Computation (ACM Press, New York, 1996), pp. 212–219.
[2] R. Raz, in Proceedings of the 31st ACM Symposium on Theory of Computing (ACM Press, New York, 1999), pp. 358–367.
[3] C. H. Bennett and G. Brassard, in Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing (IEEE Computer Society, Los Alamitos, CA, 1984), pp. 175–179.
[4] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, New York, 2000).
[5] J. S. Bell, Physics 1, 195 (1964).
[6] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. 49, 91 (1982); A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982); G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 81, 5039 (1998); W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, Phys. Rev. Lett. 81, 3563 (1998).
[7] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[8] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[9] A. Einstein, P. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[10] D. Bohm, Quantum Theory (Prentice-Hall, New York, 1951).
[11] T. Maudlin, in PSA 1992, Volume 1, edited by D. Hull, M. Forbes, and K. Okruhlik (Philosophy of Science Association, East Lansing, 1992), pp. 404–417.
[12] G. Brassard, R. Cleve, and A. Tapp, Phys. Rev. Lett. 83, 1874 (1999).
[13] M. Steiner, Phys. Lett. A 270, 239 (2000).
[14] J. A. Csisz´ ak, Phys. Rev. A 66, 014302 (2002).
[15] D. Bacon and B. F. Toner, Phys. Rev. Lett. 90, 157904 (2003).
[16] N. J. Cerf, N. Gisin, and S. Massar, Phys. Rev. Lett. 84, 2521 (2000).
[17] S. Massar, D. Bacon, N. Cerf, and R. Cleve, Phys. Rev. A 63, 052305 (2001); A. Coates (2002), quant-ph/0203112.
[18] K. H. Schatten, Phys. Rev. A 48, 103 (1993).
[19] N. Gisin and B. Gisin, Phys. Lett. A 260, 323 (1999).
[20] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, Phys. Rev. Lett. 80, 1121 (1998).
[21] S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and P. van Loock, Phys. Rev. A 64, 022321 (2001).
[22] L. Hardy, quant-ph/9906123 (1999).