Invariant Peano curves
of expanding Thurston maps

by

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1. Introduction

A Thurston map is a branched covering of the sphere \( f:S^2 \to S^2 \) that is \textit{post-critically finite}. A celebrated theorem of Thurston gives a \textit{topological characterization} of rational maps among Thurston maps (see [DH3]). In this paper we consider such maps that are \textit{expanding} (see §2 for precise definitions). In the case when \( f \) is a rational map this means that the Julia set of \( f \) is the whole sphere.

The following is the main theorem.

**Theorem 1.1.** Let \( f \) be an expanding Thurston map. Then for each sufficiently high iterate \( F = f^n \) there is a Peano curve \( \gamma:S^1 \to S^2 \) (onto) such that \( F(\gamma(z)) = \gamma(z^d) \) (for all \( z \in S^1 \)). Here \( d = \deg F \). This means that the following diagram commutes:

\[
\begin{array}{ccc}
S^1 & \xrightarrow{z^d} & S^1 \\
\gamma \downarrow & & \downarrow \gamma \\
S^2 & \xrightarrow{F} & S^2 \\
\end{array}
\]

Furthermore, we can approximate the Peano curve \( \gamma \) as follows. There is a homotopy \( \Gamma:S^2 \times [0,1] \to S^2 \), with \( \Gamma(z,0)=z \), such that

\[
\Gamma(z,1) = \gamma(z) \quad \text{for all } z \in S^1.
\]

Here we view \( S^1 \subset S^2 \) as the equator.

In fact \( \Gamma \) may be chosen to be a \textit{pseudo-isotopy}, meaning that it is an isotopy on \([0,1]\).

The result may be paraphrased as follows. Via \( \gamma \) we can view the sphere \( S^2 \) as a parameterized circle \( S^1 \). Wrapping this parameterized circle (which is \( S^2 \)) around itself \( d \) times yields the map \( F \).
The existence of such a semi-conjugacy $\gamma$ as above follows for many rational maps $F$ of degree 2 by work of Tan, Rees and Shishikura (see [Ta], [Re2] and [Sh]); the relevant construction of mating is reviewed in §1.2. Milnor constructs such a Peano curve $\gamma$ (i.e., a semi-conjugacy) for one specific example $F$ (see [Mi1]) in this setting. Kameyama gives a sufficient criterion for the existence of $\gamma$ (in [Ka, Theorem 3.5]).

Note that the result is purely topological, i.e., does not depend on $F$ being (equivalent to) a rational map or not.

We also prove the following converse statement to Theorem 1.1.

**Theorem 1.2.** Let $f: S^2 \to S^2$ be a Thurston map such that for some iterate $F = f^n$ there exists a Peano curve $\gamma: S^1 \to S^2$ (onto) satisfying $F(\gamma(z)) = \gamma(z^d)$ for all $z \in S^1$. Then $f$ is expanding.

According to Sullivan’s dictionary there is a close correspondence between the dynamics of rational maps and of Kleinian groups [Su]. Cannon and Thurston construct (in [CT]) an invariant Peano curve $\gamma: S^1 \to S^2$ for the fundamental group of a (hyperbolic) 3-manifold $M^3$ that fibers over the circle. Theorem 1.1 may be viewed as the corresponding result in the case of rational maps. Thus it provides another entry in Sullivan’s dictionary.

### 1.1. Group invariant Peano curves

We review the Cannon–Thurston construction from [CT]. The purpose is to put Theorem 1.1 into perspective.

Let $\Sigma$ be a compact hyperbolic 2-manifold, and $\varphi: \Sigma \to \Sigma$ be a pseudo-Anosov homeomorphism. Consider the equivalence relation on the product $\Sigma \times [0, 1]$ given by

$$(x, 0) \sim (\varphi(x), 1).$$

Then the 3-manifold $M^3 := \Sigma \times [0, 1]/\sim$ is called a manifold that fibers over the circle. Thurston has proved that $M^3$ admits a hyperbolic metric, see [Ot].

The fundamental groups $\pi_1(\Sigma)$ and $\pi_1(M^3)$ are Gromov hyperbolic, see [Gr] as well as [GH]. Thus they have boundaries at infinity, which in this case are $\partial_\infty \pi_1(\Sigma) = S^1$ and $\partial_\infty \pi_1(M^3) = S^2$.

This is seen by noting that $\pi_1(\Sigma)$ and hyperbolic 2-space $\mathbb{H}^2$, as well as $\pi_1(M^3)$ and hyperbolic 3-space $\mathbb{H}^3$, are quasi-isometric. The boundary at infinity of $\mathbb{H}^2$ is $S^1$, the boundary at infinity of $\mathbb{H}^3$ is $S^2$, the boundary of the disk, resp. the unit ball, in the Poincaré model of hyperbolic space.
The inclusion $\Sigma \to \Sigma \times \{0\} \to M^3$ induces an inclusion of the fundamental groups $\iota: \pi_1(\Sigma) \to \pi_1(M^3)$, which is a group homomorphism. In fact $\iota(\pi_1(\Sigma))$ is a normal subgroup of $\pi_1(M^3)$. The map $\iota$ extends to the boundaries at infinity $S^1 = \partial_\infty \pi_1(\Sigma)$ and $S^2 = \partial_\infty \pi_1(M^3)$ to a continuous map $\iota: S^1 \to S^2$.

It is well known (and not very hard to show), that a non-trivial normal subgroup $N \triangleleft G$ of a Gromov hyperbolic group $G$ has the same boundary at infinity as $G$. Thus $\partial_\infty \iota(\pi_1(\Sigma)) = \partial_\infty (\pi_1(M^3)) = S^2$. It follows that the map $\sigma$ is onto, i.e., a Peano curve.

Each element $g \in \pi_1(\Sigma)$ acts (by left-multiplication) on $\pi_1(\Sigma)$; this action extends to $S^1 = \partial_\infty \pi_1(\Sigma)$. Similarly each element $g \in \pi_1(M^3)$ acts on $\pi_1(M^3)$ and this action extends to $S^2 = \partial_\infty \pi_1(M^3)$. The map $\sigma$ is invariant with respect to this group action, meaning that for every $g \in \pi_1(\Sigma)$ one has $\iota(g)(\sigma(t)) = \sigma(g(t))$ for all $t \in S^1$. Thus the following diagram commutes:

\[
\begin{array}{ccc}
S^1 & \xrightarrow{g} & S^1 \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
S^2 & \xrightarrow{\iota(g)} & S^2
\end{array}
\]

The invariant Peano curve $\gamma$ from Theorem 1.1 is the object corresponding to the group invariant Peano curve $\sigma$ according to Sullivan’s dictionary.

The Cannon–Thurston construction has been extended by Minsky in [Min] and McMullen in [McM] to (some) cases where $\Sigma$ is not compact.

In [Th1] Thurston asked whether (in a sense) all hyperbolic 3-manifolds arise as manifolds that fiber over the circle. This has now become known as the virtual fibering conjecture. It stipulates that every hyperbolic 3-manifold has a finite cover which fibers over the circle. This would mean that we can understand every hyperbolic 3-manifold in terms of 2-manifolds. See [Ga] for more background on this conjecture, and [Ag] for recent progress.

Theorem 1.1 may be viewed as the solution of the problem corresponding to the virtual fibering conjecture according to Sullivan’s dictionary.

1.2. Consequences of Theorem 1.1

To not further increase the size of the present paper, we will develop the implications of the main theorem in a follow-up paper [Me1]. They are outlined here briefly to put the result into perspective.

Using the invariant Peano curve $\gamma: S^1 \to S^2$ from Theorem 1.1, an equivalence relation on $S^1$ is defined by

\[ s \sim t \iff \gamma(s) = \gamma(t) \]  

(1.1)
for all $s,t \in S^1$. Elementary topology yields that $S^1/\sim$ is homeomorphic to $S^2$ and that $z^d/\sim:S^1/\sim \rightarrow S^1/\sim$ is topologically conjugate to the map $F$.

**Theorem 1.3.** The following diagram commutes:

$$
\begin{array}{ccc}
S^1/\sim & \xrightarrow{z^d/\sim} & S^1/\sim \\
h \downarrow & & \downarrow h \\
S^2 & \xrightarrow{p} & S^2.
\end{array}
$$

Here the homeomorphism $h:S^1/\sim \rightarrow S^2$ is given by $h([s])=\gamma(s)$ for all $s \in S^1$.

The equivalence relation (1.1) may be constructed from finite data, more precisely from two finite families of finite sets of rational numbers.

The proper setting is as follows. For each $n \in \mathbb{N}$, two equivalence relations, $n,\sim^w$ and $n,\sim^b$, are defined. The equivalence relation $\sim$ defined in (1.1) is the closure of the union of all $n,\sim^w$ and $n,\sim^b$. Each $n,\sim^w$ is the pull-back of $n^{-1},\sim^w$ by $z^d$ (similarly $n,\sim^b$ is the pull-back of $n^{-1},\sim^b$). Thus $F$ can be recovered (up to topological conjugacy) from the equivalence relations $1,\sim^w$ and $1,\sim^b$.

This provides a way to describe expanding Thurston maps effectively.

The description above may be viewed as the two-sided version of the viewpoint introduced by Douady–Hubbard and Thurston ([DH1], [DH2], [Th2], [Th3], see also [Re2] and [Ke]), namely the combinatorial description of Julia sets in terms of external rays.

Recently (analogously defined) random laminations have been used to study the scaling limits of planar maps (see [Le] and [LP]).

The description of $F$ above yields in addition that $F$ arises as a mating of two polynomials. Mating of polynomials was introduced by Douady and Hubbard [Do] as a way to geometrically combine two polynomials to form a rational map. We recall the construction briefly.

Consider two monic polynomials $p_1$ and $p_2$ of the same degree with connected and locally connected Julia sets. Let $K_1$ and $K_2$ be their filled-in Julia sets. For $j=1, 2$ let

$$
\phi_j: \hat{\mathbb{C}} \setminus \overline{D} \rightarrow \hat{\mathbb{C}} \setminus K_j
$$

be the Riemann maps, normalized by $\phi_j(\infty)=\infty$ and

$$
\phi_j'(\infty)=\lim_{z \to \infty \phi_j(z)}>0
$$

(in fact then $\phi_j'(\infty)=1$). By Carathéodory’s theorem $\phi_j$ extends continuously to

$$
\sigma_j: S^1 = \partial \overline{D} \rightarrow \partial K_j,
$$
The topological mating of $K_1$ and $K_2$ is obtained by identifying $\sigma_1(z) \in \partial K_1$ with $\sigma_2(\bar{z}) \in \partial K_2$. More precisely, we consider the disjoint union of $K_1$ and $K_2$ and let $K_1 \sqcup K_2$ be the quotient obtained from the equivalence relation generated by $\sigma_1(z) \sim \sigma_2(\bar{z})$ (for all $z \in S^1 = \partial \mathbb{D}$). The map 

$$p_1 \sqcup p_2: K_1 \sqcup K_2 \to K_1 \sqcup K_2,$$

given by

$$(p_1 \sqcup p_2)|_{K_j} = p_j \quad \text{for} \quad j = 1, 2,$$

is well defined. If a map $f$ is topologically conjugate to $p_1 \sqcup p_2$, we say that $f$ is obtained as a (topological) mating. If both $K_1$ and $K_2$ have empty interior, each of the maps $\sigma_1$ and $\sigma_2$ descends to a Peano curve $\gamma: S^1 \to K_1 \sqcup K_2$ which provides a semi-conjugacy of $z^d: S^1 \to S^1$ to $p_1 \sqcup p_2$ (here $d = \deg p_1 = \deg p_2$).

In particular it is known (see [Ta], [Sh] and [Re2]) that the mating of two quadratic polynomials $p_1 = z^2 + c_1$ and $p_2 = z^2 + c_2$, where $c_1$ and $c_2$ are Misiurewicz points (i.e., the critical point 0 is strictly pre-periodic for $p_j$) not contained in conjugate limbs of the Mandelbrot set, results in a map that is topologically conjugate to a rational map $F$. The filled-in Julia sets of $p_1$ and $p_2$ have empty interior. The Julia set of $F$ is the whole sphere, and hence $F$ is expanding. Thus a Peano curve $\gamma$ as in Theorem 1.1 exists for such a map $F$.

Recall that a periodic critical point (of a Thurston map $f$) is a critical point $c$ such that $f^k(c) = c$ for some $k \geq 1$.

**Theorem 1.4. ([Me1])** Let $f: S^2 \to S^2$ be an expanding Thurston map without periodic critical points. Then every sufficiently high iterate $F = f^n$ is obtained as a topological mating of two polynomials.

If at least one of the filled-in Julia sets $K_1$ and $K_2$ has non-empty interior, we can take a further quotient of $K_1 \sqcup K_2$ by identifying the points of the closure of each bounded Fatou component. Technically we take the closure of the equivalence relation (on the disjoint union of $K_1$ and $K_2$) obtained from $\sigma_1(z) \sim \sigma_2(\bar{z})$ (for all $z \in S^1 = \partial \mathbb{D}$) as well as $x \sim y$ if $x$ and $y$ are in the closure of the same bounded Fatou component of $p_1$ or $p_2$.

The maps $p_1$ and $p_2$ descend to the quotient map $p_1 \sqcup p_2$.

**Theorem 1.5. ([Me1])** Let $f: S^2 \to S^2$ be an expanding Thurston map with (at least one) periodic critical point. Then every sufficiently high iterate $F = f^n$ is topologically conjugate to a map $p_1 \sqcup p_2$ as above.

The next theorem investigates the measure-theoretic mapping properties of $\gamma$.

**Theorem 1.6. ([Me1])** The Peano curve $\gamma$ maps Lebesgue measure of $S^1$ to the measure of maximal entropy (with respect to $F$) on $S^2$. 

The polynomials into which $F$ unmates, i.e., the polynomials $p_1$ and $p_2$ from Theorems 1.4 and 1.5, can be found by a simple explicit combinatorial algorithm. This is explained in [Me2].

As another application of Theorem 1.1 one obtains fractal tilings. Namely divide the circle $S^1 = \mathbb{R}/\mathbb{Z}$ into $d$ intervals $[j/d, (j+1)/d]$, $j=0,\ldots,d-1$. It follows from Theorem 1.1 that $F$ maps each set $\gamma([j/d, (j+1)/d])$ to the whole sphere. The tiling lifts to the orbifold covering, which is either the Euclidean or the hyperbolic plane.

1.3. Outline

The construction of the invariant Peano curve, i.e., the proof of Theorem 1.1, forms the core of this work.

In §1.5 an example is introduced that serves to illustrate the construction throughout the paper. §2 gives precise definitions of expanding Thurston maps, as well as gathers facts from [BM] relevant here.

We will fix a Jordan curve $C$ containing the set of all post-critical points ($= \text{post}(F)$). We construct approximations $\gamma^n : S^1 \to S^2$, that will go through $F^{-n}(C)$. The limit

$$\gamma = \lim_{n \to \infty} \gamma^n$$

will be the desired Peano curve.

The construction of $\gamma$ consists of two parts. In the first part (which is logically the second) we assume that we can deform $C$ to $\gamma^1 = F^{-1}(C)$ by a pseudo-isotopy relative to $\text{post}(F)$. The approximations $\gamma^n$ can then be constructed inductively by repeated lifts. This is done in §3. The correct parametrization of $\gamma^n$ is done in §4.

The second part is the construction of the pseudo-isotopy $H^0$ relative to $\text{post}(F)$, which deforms the Jordan curve $C$ to the first approximation $\gamma^1$.

We color one component of $S^2 \setminus C$ white, and the other black. Preimages of these Jordan domains by $F$ then form the white and black 1-tiles. At each vertex (of 1-tiles) we will declare which white and black 1-tiles are connected. These connections will be described by complementary non-crossing partitions.

Connections at all vertices will be defined in such a way that the white tile graph forms a spanning tree. The “outline” of this spanning tree forms the first approximation $\gamma^1$. The main work consists of making sure that $\gamma^1$ lies in the right homotopy class (that $C$ can be deformed to $\gamma^1$ by a pseudo-isotopy relative to $\text{post}(F)$).

§5 assembles some standard topological lemmas needed in the following. In §6 the necessary background about connections and complementary non-crossing partitions is developed.
The desired pseudo-isotopy $H^0$ (equivalently the spanning tree of white 1-tiles) is constructed in §7. It is here that we (possibly) need to take an iterate $F=f^n$ (in order to be in the right homotopy class).

In §8 an alternative combinatorial way to construct the approximations $\gamma^n$ is presented. An n-tile is the preimage of a component of $S^2 \setminus C$ by $F^n$. At each n-vertex of such an n-tile we define which n-tiles are connected. Following the “outline” of one connected component as before yields the approximation $\gamma^n$. These connections of n-tiles are constructed inductively in a purely combinatorial fashion.

Theorem 1.2 (existence of a Peano curve which semi-conjugates $z^d$ to $F$ implies expansion) is proved in §9.

The question arises whether it is necessary to take an iterate $F=f^n$ in Theorem 1.1. While we do not have a definite answer, we give an example in §10 which shows (in the opinion of the author) that the answer is likely yes. More precisely, for the considered example $h$, there exists no pseudo-isotopy $H^0$ as required (there is one for the second iterate $h^2$).

We finish with some open problems in §11.

1.4. Acknowledgments

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1.5. An example

We illustrate the proof using the following map $g$. It is a Lattès map (see [La] and [Mi3]).

Map the square $[0, \frac{1}{2}]^2 \subset \mathbb{C}$ to the upper half-plane by a Riemann map, normalized by mapping the vertices $0, \frac{1}{2}, \frac{1}{2} + \frac{1}{2}i$ and $\frac{1}{2}i$ to $0, 1, \infty$ and $-1$, respectively (here $i$ denotes the imaginary unit). By Schwarz reflection this map can be extended to a meromorphic function $\wp: \mathbb{C} \to \hat{\mathbb{C}}$. This is the Weierstraß $\wp$-function (up to a Möbius transformation), it is (doubly) periodic with respect to the lattice $L:=\mathbb{Z}^2$. Thus we may view $\wp$ as a (double) branched covering map of the sphere by the torus $\hat{T}^2:=\mathbb{C}/L$.

Color preimages of the upper half-plane by $\wp$ white, and preimages of the lower half-plane by $\wp$ black. The plane is then colored in a checkerboard fashion. Consider the map

$$\psi: \mathbb{C} \to \mathbb{C},$$
$$z \mapsto 2z.$$
We may view \( \psi \) as a self-map of the torus \( \mathbb{T}^2 \). One checks that there is a (unique and well defined) map \( g: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) such that the diagram

\[
\begin{array}{ccc}
\hat{\mathbb{C}} & \xrightarrow{\psi} & \hat{\mathbb{C}} \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
\hat{\mathbb{C}} & \xrightarrow{g} & \hat{\mathbb{C}}
\end{array}
\]

commutes. The map \( g \) is \textit{rational}, in fact

\[
g = 4 \frac{z(1-z^2)}{(z^2+1)^2}.
\]

The Julia set of \( g \) is the whole sphere.

One may describe \( g \) as follows. Push the Euclidean metric of \( \mathbb{C} \) to the (Riemann) sphere \( \hat{\mathbb{C}} \) by \( \varphi \). In this metric the sphere looks like a \textit{pillow} (technically this is an \textit{orbifold}, see for example \cite[Appendix E]{Mi2} and \cite[Appendix A]{Mc1}). Indeed, by construction, the upper and lower half-planes are then both isometric to the square \( [0, \frac{1}{2}]^2 \). Two such squares glued along their boundary form the sphere. We \textit{color} one of these squares (say the upper half-plane) \textit{white}, and the other square (the lower half-plane) \textit{black}. The map \( g \) is now given as follows. Divide each of the two squares into four small squares (of side-length \( \frac{1}{4} \)). Color these eight small squares in a checkerboard fashion white and black. Map one such small white square to the big white square. This extends by reflection to the whole pillow, which yields the map \( g \). There are obviously many different ways to color and map the small squares. The “right" way to do so (in order to obtain \( g \)) is indicated in Figure 1.
The six vertices of the small squares at which four small squares intersect are the critical points of $g$. They are mapped by $g$ to $\{1, \infty, -1\}$; these points in turn are mapped to 0, which is a fixed point. The set $\{0, 1, \infty, -1\} = \text{post}(g)$ is the set of all post-critical points.

The map $\wp$ is the orbifold covering map. The pictures explaining our construction will all be in the orbifold covering, i.e., in $\mathbb{C}$. For example, the Peano curve will be constructed by certain approximating curves. These are more easily visualized when lifted to $\mathbb{C}$.

1.6. The construction for the example

The construction is explained using the example $g$ defined in the last section.

The 0-th approximation $\gamma^0$ of the Peano curve is the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subset \hat{\mathbb{C}}$.

Note that $\hat{\mathbb{R}}$ contains all post-critical points of $g$. In the “pillow” model, $\hat{\mathbb{R}}$ is the common boundary of the two squares. The picture in the orbifold covering is shown in Figure 2.
in the lower left. The (lifts of the) post-critical points are the dots at the vertices.

The upper and lower half-planes (the two squares from which the “pillow” was constructed) are called the 0-tiles. Their preimages by \( g \) (the small squares to the left in Figure 1) are called the 1-tiles. We color them white if they are preimages of the upper half-plane, and black otherwise. There are four white as well as four black 1-tiles. The white 1-tiles intersect at the critical points, of which there are six. At each critical point (1-vertex) we define a connection. This is an assignment of which 1-tiles are connected and which are disconnected at this 1-vertex. Connections are defined in such a way that the resulting white tile graph is a spanning tree. This means that it contains all white 1-tiles and no loops. In our example the white 1-tiles are connected at the three critical points labeled by “\( \mapsto -1 \)” and “\( \mapsto \infty \)” in Figure 1, and disconnected at the others. The corresponding picture in the orbifold covering is shown in the lower right of Figure 2.

Following the boundary of this spanning tree gives the first approximation of the Peano curve \( \gamma^1 \) (again indicated in the lower right of Figure 2). To obtain the curve \( \gamma^1 \) on the pillow, one needs to “fold the two squares that are overlapping to the left and right on the back” (where they intersect in a critical point).

We will need the following additional assumption on the spanning tree. We have to be able to deform \( \gamma^0 \) to \( \gamma^1 \) by a pseudo-isotopy \( H^0 \) that keeps the post-critical points fixed. Recall that a pseudo-isotopy \( H^0: S^2 \times [0,1] \to S^2 \) is a homotopy that ceases to be an isotopy only at \( t=1 \).

The pseudo-isotopy is lifted to (pseudo-isotopies) \( H^n \) by iterates \( g^n \). The approximations of the Peano curve are constructed inductively. Namely \( \gamma^{n+1} \) is obtained as the deformation of \( \gamma^n \) by \( H^n \). Each curve \( \gamma^n \) goes through \( g^{-n}(\text{post}) \). The limiting curve \( \gamma \) is the desired Peano curve.

1.7. Notation

The Riemann sphere is denoted by \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). We denote the 2-sphere by \( S^2 \), when it is not assumed to be equipped with a conformal structure. By \( \text{int} \, U \) we denote the interior of a set \( U \). The cardinality of a (finite) set \( S \) is denoted by \( \#S \). The circle \( S^1 \) will often be identified with \( \mathbb{R}/\mathbb{Z} \) whenever convenient.

For two non-negative expressions \( A \) and \( B \) we write \( A \preceq B \) if there is a constant \( C > 0 \) such that \( A \leq CB \). We refer to \( C \) as \( C(\preceq) \). Similarly we write \( A \asymp B \) if \( A/C \leq B \leq CA \) for a constant \( C \geq 1 \).

- The \( n \)-iterate of a map \( f \) is denoted by \( f^n \); \( f^{-n}(A) \) denotes the preimage of a set \( A \) by the iterate \( f^n \).
- Upper indices indicate the order of an object, meaning that \( U^n \) is the preimage
of some object $U^0$ by $f^n$ or $F^n$.

- By $\text{crit} = \text{crit}(f)$ and $\text{post} = \text{post}(f)$ we denote the sets of critical and post-critical points, respectively (see the next section).
- The degree of $F$ is denoted by $d$, the number of post-critical points by $k$.
- $C$ is a Jordan curve containing all post-critical points.
- Lower indices $w$ and $b$ denote whether objects are colored white or black.
- $X^n_w$ and $X^n_b$ denote the white and black 0-tiles, respectively (§2).
- The sets of all $n$-tiles, $n$-edges and $n$-vertices are denoted by $X^n$, $E^n$ and $V^n$, respectively (§2).
- The expansion factor of a fixed visual metric for $F$ is denoted by $\Lambda$, see (2.3).
- $\gamma^n$ is the $n$-th approximation of the invariant Peano curve (§3).
- $H^0$ is the pseudo-isotopy that deforms $C$ to $\gamma^1$. $H^n$ is the lift of $H^0$ by $F^n$; it is a pseudo-isotopy that deforms $\gamma^n$ to $\gamma^{n+1}$ (see Definition 3.2 and Lemma 3.4).
- $\alpha^n_j \subset \mathbb{R}/\mathbb{Z}$ is a point that is mapped by $\gamma^n$ (and subsequently by $\gamma$) to an $n$-vertex (§4.2).
- $\pi_w \cup \pi_b$ is a complementary non-crossing partition. It describes which white and black 1-tiles are connected at some 1-vertex (§6.1).
- A lower index $\varepsilon$ indicates a geometric realization of an object, where in a small neighborhood of each 1-vertex we change tiles to “geometrically represent the connection” (Definition 6.8).

2. Expanding Thurston maps as subdivisions

**Definition 2.1.** A Thurston map is an orientation-preserving, post-critically finite, branched covering of the sphere 

$$f: S^2 \to S^2.$$

To elaborate:

(1) $f$ is a branched cover of the sphere $S^2$, meaning that locally we can write $f$ as $z \mapsto z^q$ after orientation-preserving homeomorphic changes of coordinates in domain and range.

More precisely, for each point $v \in S^2$ there exist $q \in \mathbb{N}$, (open) neighborhoods $V$ and $W$ of $v$ and $w = f(v)$, respectively, and orientation-preserving homeomorphisms $\varphi: V \to \mathbb{D}$ and $\psi: W \to \mathbb{D}$, with $\varphi(v) = 0$ and $\psi(w) = 0$, satisfying

$$\psi \circ f \circ \varphi^{-1}(z) = z^q$$
for all $z \in \mathbb{D}$. The integer $q = \deg f(v) \geq 1$ is called the \textit{local degree} of the map at $v$. A point $c$ at which the local degree $\deg f(c) \geq 2$ is called a \textit{critical point}. The set of all critical points is denoted by $\text{crit} = \text{crit}(f)$. There are only finitely many critical points, since $S^2$ is compact. Note that no assumptions about the smoothness of $f$ are made.

(2) The map $f$ is \textit{post-critically finite}, meaning that the set of post-critical points

$$\text{post} = \text{post}(f) := \bigcup_{n \geq 1} \{ f^n(c) : c \in \text{crit}(f) \}$$

is finite. As usual $f^n$ denotes the $n$th iterate. We are only interested in the case when $\# \text{post}(f) \geq 3$.

(3) Consider a Jordan curve $C \supset \text{post}$. The Thurston map $f$ is called \textit{expanding} if

$$\text{mesh } f^{-n}(C) \to 0 \quad \text{as } n \to \infty.$$  

Here $\text{mesh } f^{-n}(C)$ is the maximal diameter of a component of $S^2 \setminus f^{-n}(C)$. It was shown in [BM, Lemma 6.1] that this definition is independent of the chosen curve $C$. This notion of “expansion” agrees with the one by H"{a}issinsky–Pilgrim in [HP] (see [BM, Proposition 6.2]).

Fix a Jordan curve $C \supset \text{post}$. Here and in the following, we always assume that such a curve $C$ is oriented. Let $U_w$ and $U_b$ be the two components of $S^2 \setminus C$, where $C$ is positively oriented as boundary of $U_w$. The closures of $U_w$ and $U_b$ are denoted by $X_w^0$ and $X_b^0$, respectively. We color $X_w^0$ white, and $X_b^0$ black. We refer to $X_w^0$ (resp. $X_b^0$) as the white (resp. black) $0$-tile.

The closure of one component of $f^{-n}(U_w)$ or of $f^{-n}(U_b)$ is called an $n$-tile. It was shown in [BM, Proposition 5.17] that for such an $n$-tile $X$ the map

$$f^n : X \to X_{w, b}^0$$

is a homeomorphism. \hfill (2.1)

This means in particular that each $n$-tile is a closed Jordan domain. The set of all $n$-tiles is denoted by $X^n$. The definition of “expansion” implies that $n$-tiles become arbitrarily small, this is the (only) reason we require expansion.

In [BM, Theorem 14.2] (see also [CFP3]) it was shown that if $f$ is expanding, then for every sufficiently high iterate $F = f^n$ we can choose $C$ to be \textit{invariant} with respect to $F$. This means that $F(C) \subseteq C$ ($\iff C \subseteq F^{-1}(C)$). It implies that each $n$-tile is contained in exactly one $(n-1)$-tile. Furthermore, $F$ may be represented as a \textit{subdivision} (see [BM, Chapter 12] as well as the ongoing work of Cannon, Floyd and Parry [CFP1], [CFP2]). We will require $C$ to be $F$-invariant only in \S 7. This is clearly a convenience in the proof, the author however feels that this assumption is not strictly necessary.
The set of all $n$-vertices is defined as

$$V^n = f^{-n}({\text{post}}).$$

(2.2)

Note that $\text{post} = V^0 \subset V^1 \subset \ldots$. Each point $v \in V^n$ is called an $n$-vertex.

The post-critical points (or 0-vertices) divide the curve $C$ into $k = \# \text{post}(f)$ closed Jordan arcs called 0-edges. The closure of one component of $f^{-n}(C) \setminus V^n$ is called an $n$-edge. For each $n$-edge $E^n$ there is a 0-edge $E^0$ such that $f^n(E^n) = E^0$. Furthermore the map $f^n : E^n \to E^0$ is a homeomorphism ([BM, Proposition 5.17]). The set of all $n$-edges is denoted by $E^n$, so that $f^{-n}(C) = \bigcup E^n$. There are $\#E^n = k \deg(f)^n$ $n$-edges.

Each $n$-edge will have an orientation, meaning that it has an initial and a terminal point. A 0-edge is positively oriented if its orientation agrees with the one of the Jordan curve $C$. Similarly, an $n$-edge $E^n$ is called positively oriented if $f^n$ maps the initial (resp. terminal) point of $E^n$ to the initial (resp. terminal) point of (the 0-edge) $f^n(E^n)$.

Each $n$-tile contains exactly $k = \#post$ $n$-edges and $k$ $n$-vertices in its boundary.

The $n$-tiles, $n$-edges and $n$-vertices form a cell complex when viewed as 2-, 1- and 0-cells, respectively (see [BM, Chapter 5]).

The $n$-edges and $n$-vertices form a graph in the natural way. Note that this graph may have multiple edges, but no loops.

We color the $n$-tiles white if they are preimages of $X^0_w$, and black if they are preimages of $X^0_b$. Each $n$-edge is shared by two $n$-tiles of different color. Thus $n$-tiles are colored in a “checkerboard fashion”. An oriented $n$-edge is positively oriented if and only if it is positively oriented as boundary of the white $n$-tile it is contained in (and negatively oriented as boundary of the black $n$-tile it is contained in). The set of white $n$-tiles is denoted by $X^n_w$, and the set of black $n$-tiles by $X^n_b$.

**Lemma 2.2.** The $n$-tiles of each color are connected, meaning that

$$\bigcup X^n_w \text{ and } \bigcup X^n_b \text{ are connected sets.}$$

**Proof.** Note that $\bigcup X^n_w$ (and $\bigcup X^n_b$) is connected if and only if $\bigcup E^n$ is connected.

If $\bigcup E^n$ is not connected, one component of $S^2 \setminus \bigcup E^n$ is not simply connected. This contradicts the fact that each such component is the interior of an $n$-tile, and thus simply connected.

In [BM, Chapter 8] visual metrics for an expanding Thurston map $f$ were considered. If $n$-tiles have been defined (in terms of a Jordan curve $C \supset \text{post}$), we define $m = m_{f,C}$ by

$$m(x, y) := \max\{n \in \mathbb{N} : \text{there exist non-disjoint } n\text{-tiles } X \ni x \text{ and } Y \ni y\}$$
for all \( x, y \in S^2 \), \( x \neq y \). We set \( m(x, x) = \infty \). A metric \( g \) on \( S^2 \) is called a visual metric for \( f \) if there is a constant \( \lambda > 1 \) (called the expansion factor of \( g \)) such that

\[
\varrho(x, y) \asymp \lambda^{-m(x, y)}
\]

for all \( x, y \in S^2 \) and a constant \( C = C(\asymp) \) independent of \( x \) and \( y \). Here it is understood that \( \lambda^{-\infty} = 0 \).

Visual metrics always exist, see [BM, Theorem 15.1], as well as [HP]. In fact \( \varrho \) can be chosen such that \( f \) is an expanding local similarity with respect to \( \varrho \). More precisely, for each \( x \in S^2 \) there exists a neighborhood \( U_x \ni x \) such that

\[
\frac{\varrho(f(x), f(y))}{\varrho(x, y)} = \lambda
\]

for all \( y \in U_x \setminus \{ x \} \). We do however not need this stronger form.

We fix a curve \( C \supset \text{post}(f) \) as well as an iterate \( F = f^n \) for now, assuming that they have certain properties (more precisely, that there is a pseudo-isotopy \( H^0 \) as in the next section). In \( \S 7 \) they will be chosen properly. Note that the post-critical set of \( F \) equals the post-critical set of \( f \), which is thus just denoted by “post”. Throughout the construction, we set

\[
d := \deg F = (\deg f)^n \quad \text{and} \quad k := \# \text{post}.
\]

From now on \( m \)-tiles, \( m \)-edges and \( m \)-vertices are understood to be with respect to \((F, C)\), meaning that they are \( mn \)-tiles, \( mn \)-edges and \( mn \)-vertices, respectively, with respect to \((f, C)\).

Clearly expansion of \( f \) implies expansion of \( F \). A visual metric for \( f \) with expansion factor \( \lambda \) is a visual metric for \( F \) with expansion factor \( \Lambda = \lambda^n \). Expression (2.4) continues to hold, where we have to replace \( \lambda \) by \( \Lambda := \lambda^n > 1 \).

**Lemma 2.3.** Let \( \varrho \) be a visual metric for \( F \) with expansion factor \( \Lambda \). Then there are \( \varepsilon_0 > 0 \) and a constant \( K \geq 1 \) such that the following holds: For any \( \varepsilon \in (0, \varepsilon_0) \) let \( \mathcal{N}(V^1, \varepsilon) \) be the \( \varepsilon \)-neighborhood of \( V^1 \) (defined in terms of \( \varrho \)). Then there is a neighborhood \( V^1_\varepsilon \) of \( V^1 \) such that

\[
\mathcal{N}\left(V^1, \frac{\varepsilon}{\Lambda} \right) \subset V^1_\varepsilon \subset \mathcal{N}(V^1, \varepsilon)
\]

and, for all \( n \in \mathbb{N} \), the set \( V = V^{n+1}_{\Lambda^{-n} \varepsilon} := F^{-n}(V^1_\varepsilon) \) satisfies

\[
\mathcal{N}\left(V^{n+1}, \Lambda^{-n} \frac{\varepsilon}{\Lambda} \right) \subset V \subset \mathcal{N}(V^{n+1}, \Lambda^{-n} \varepsilon).
\]

The proof of this lemma follows immediately from [BM, Lemmas 8.9 and 8.10].
3. The approximations $\gamma^n$

We begin the proof of Theorem 1.1. We assume (until the end of §7) that $F (= f^n$, the index “$n$” however will be “recycled”) is an expanding Thurston map and that $C \supset \text{post}$ is a fixed Jordan curve. The $n$-tiles and $n$-edges are defined in terms of $(F, C)$; see the previous section. Furthermore we fix a visual metric $\varrho$ for $F$ with expansion factor $\Lambda > 1$; see (2.3). Metrical properties and objects, such as the diameter and neighborhoods, will always be defined in terms of this metric.

The desired invariant Peano curve $\gamma$ will be constructed as the limit of approximations $\gamma^n$. Here $\gamma^0$ is the Jordan curve $C \supset \text{post}$. The first approximation $\gamma^1$ will be constructed in §7, more precisely a pseudo-isotopy $H^0$ (relative to post) that deforms $\gamma^0$ to $\gamma^1$ will be constructed.

In this section the approximations $\gamma^n$ of the invariant Peano curve will be constructed by repeated lifts of $H^0$. These curves are however not yet parameterized, they are Eulerian circuits.

3.1. Pseudo-isotopies

Definition 3.1. (Pseudo-isotopies) A homotopy $H: S^2 \times [0, 1] \to S^2$ is called a pseudo-isotopy if it is an isotopy on $S^2 \times [0, 1)$. We always require that $H(x, 0) = x$ on $S^2$. If $H(\cdot, t)$ is constant on a set $A \subset S^2$ it is a pseudo-isotopy relative to $A$ (from now on we will use the abbreviation “rel.” for “relative to”). Alternatively we then say that $H$ is supported on $S^2 \setminus A$. We interchangeably write $H_t(x) = H(x, t)$ to unclutter notation.

Remark. Given a pseudo-isotopy $H_t$ as above, it follows that $H_1$ is surjective (in fact $S^2 \setminus \{\text{point}\}$ has different homotopy type than $S^2$) and closed (since we are dealing with compact Hausdorff spaces). A pseudo-isotopy on a general space $S$ is required to end in a surjective, closed map.

Our starting point is a pseudo-isotopy $H^0 = H^0(x, t)$ as follows. This is the central object of the whole construction. In this and the following sections we show that such an $H^0$ is sufficient to construct the invariant Peano curve as desired. The construction of $H^0$ itself will be done in §7. In Lemma 7.2 an equivalent condition for the existence of $H^0$ will be given.

Definition 3.2. (Pseudo-isotopy $H^0$) We consider a pseudo-isotopy $H^0$ with the following properties:

(H^0 1) $H^0$ is a pseudo-isotopy rel. $V^0 = \text{post}$ (the set of all post-critical points).
(H\textsuperscript{0} 2) The set of all 0-edges $\bigcup E^0 = C$ is deformed by $H^0$ to $\bigcup E^1$, that is
\[ H^0 \left( \bigcup E^0 \right) = \bigcup E^1. \]

To simplify the discussion we require that $H^0$ deforms the 0-edges to 1-edges “as nicely as possible” (see Lemma 3.3 below). The construction would still work however, without imposing the following two properties:

(H\textsuperscript{0} 3) Let $\varepsilon_0 > 0$ be the constant from Lemma 2.3, $0 < \varepsilon < \min \{ \varepsilon_0, \frac{1}{2} \}$ and $V^1_{\varepsilon}$ be a neighborhood of $V^1$ as in Lemma 2.3; we require that
\[ H^0 : S^2 \times [1-\varepsilon, 1] \rightarrow S^2 \text{ is supported on } V^1_{\varepsilon}. \]
So $H^0$ “freezes” on $S^2 \setminus V^1_{\varepsilon}$.

(H\textsuperscript{0} 4) Consider a 1-vertex $v$. Only finitely many points of $C = \bigcup E^0$ are deformed by $H^0$ to $v$. In other words, we require that
\[ \{ x \in \bigcup E^0 : H^0_1(x) = v \} \text{ is a finite set.} \]

One final assumption will be made on $H^0$. However the precise meaning will only be explained in §3.4.

(H\textsuperscript{0} 5) View $\gamma^0 = C$ as a circuit of 0-edges. Let $\gamma^1$ be the Eulerian circuit obtained from $H^0$, see Definition 3.8 (iv). Then
\[ F : \gamma^1 \rightarrow \gamma^0, \]
is a $d$-fold cover; see Definition 3.10.

Consider $\{ x_j \} : = (H^0_1)^{-1}(V^1) \cap C$, the set of points on $C = \bigcup E^0$ that are mapped by $H^0_1$ to some 1-vertex (each $x_j$ possibly to a different one). Note that $\{ x_j \}$ is finite by (H\textsuperscript{0} 4) and $\{ x_j \} \supset \text{post} = V^0$ by (H\textsuperscript{0} 1). Thus the points $\{ x_j \}$ divide $C$ (and each 0-edge) into closed arcs $A_j$. Recall that $d = \text{deg } F$ and $k = \# \text{ post}.$

**Lemma 3.3.** There are $kd$ arcs $A_j$ as above. Furthermore
\[ E_j := H^0_1(A_j) \text{ is a 1-edge and } H^0_1 : A_j \rightarrow E^1_j \text{ is a homeomorphism,} \]
for each $j$. On the other hand
\[ \text{each 1-edge } E^1_j \text{ is the image of one such } A_j \text{ by } H^0_1. \]
Proof. Consider one arc $A_j$ as in the statement with endpoints $x_j$ and $x_{j+1}$. Note that $\bigcup E^1 \setminus V^1$ is disconnected, each component is the interior of a 1-edge. Thus

$$H^0_1(\text{int } A_j) \subset \text{int } E^1_j$$

for some 1-edge $E^1_j$. Assume that $H^0_1(A_j) \neq E^1_j$. Assume first that $H^0_1(A_j) \neq E^1_j$. Then $H^0_1(x_j) = H^0_1(x_{j+1})$ and there are distinct points $x, y \in \text{int } A_j$ mapped to the same point $z$ by $H^0_1$. But $z \in S^2 \setminus V^1$ for sufficiently small $\varepsilon$. Then $H^0_1 - \varepsilon(x) = H^0_1(y)$, which is a contradiction ($H^0_1 - \varepsilon$ is a homeomorphism). Thus $H^0_1(A_j) = E^1_j$. Exactly the same argument shows that distinct arcs $A_i$ and $A_j$ map to distinct 1-edges $E^1_i$ and $E^1_j$, respectively.

Finally, since $H^0_1(\bigcup E^0) = \bigcup E^1$ (by $(H^0_1 2)$), each 1-edge $E^1$ is the image of one such arc $A_j$ by $H^0_1$.

Thus, there is exactly one $A_j$ for each 1-edge, and so there are $kd$ such arcs.

\[ \square \]

3.2. Lifts of pseudo-isotopies

Lemma 3.4. (Lift of pseudo-isotopy) Let $H : S^2 \times [0, 1] \to S^2$ be a pseudo-isotopy rel. post=$V^0$. Then $H$ can be lifted uniquely by $F$ to a pseudo-isotopy $\tilde{H}$ rel. $V^1$. This means that $F(\tilde{H}(x, t)) = H(F(x), t)$ for all $x \in S^2$ and all $t \in [0, 1]$, i.e., the following diagram commutes:

\[
\begin{array}{ccc}
S^2 & \xrightarrow{\tilde{H}} & S^2 \\
F \downarrow & & \downarrow F \\
S^2 & \xrightarrow{H} & S^2.
\end{array}
\]

Furthermore, the following are true:

(1) If $H$ is a pseudo-isotopy rel. a set $S \subset S^2$, then the lift $\tilde{H}$ is a pseudo-isotopy rel. $F^{-1}(S)$.

(2) Let $H^n$ be the lift of $H$ by an iterate $F^n$. Then

$$\text{diam } H^n := \max_{x \in S^2} \text{diam } \{H^n(x, t) : t \in [0, 1]\} \lesssim \Lambda^{-n}.$$ 

Here the diameter is measured with respect to the fixed visual metric with expansion factor $\Lambda > 1$. The constant $C(\lesssim)$ is independent of $n$. 
The proof follows from the standard lifting of paths, see [BM, Proposition 11.1]. For property (2) see [BM, Lemma 11.3].

We now lift the pseudo-isotopy from the last subsection. Lifts retain the properties of \( H^0 \).

**Lemma 3.5. (Properties of \( H^n \))** Let \( H^0 \) be a pseudo-isotopy as in the last subsection. Let \( H^n \) be the lift of \( H^0 \) by \( F^n \) (equivalently the lift of \( H^{n-1} \) by \( F \)). The lifts satisfy the following properties:

1. \( H^n \) is a pseudo-isotopy rel. \( V^n \) (the set of all \( n \)-vertices).
2. The set of all \( n \)-edges \( \bigcup E^n \) is deformed by \( H^n \) to \( \bigcup E^{n+1} \), that is \( H^n \left( \bigcup E^n \right) = \bigcup E^{n+1} \).
3. Let \( V^1_\varepsilon \) be the neighborhood of \( V^1 \) as in (H^3), see also Lemma 2.3. The set \( V = V^1_{\varepsilon} = F^{-n}(V^1) \), which is a neighborhood of \( V^{n+1} \), is such that \( H^n : S^2 \times [1 - \varepsilon, 1] \to S^2 \) is supported on \( V \).
4. Consider an \((n+1)\)-vertex \( v \). Only finitely many points of \( \bigcup E^n \) are deformed by \( H^n \) to \( v \). In other words, \( \{ x \in \bigcup E^n : H^n(x) = v \} \) is a finite set.

We list the final property here. Again it will be explained and proved only in §3.4.

5. Let \( \gamma^n \) and \( \gamma^{n+1} \) be the Eulerian circuits from Definition 3.8 (iv). Then \( F : \gamma^{n+1} \to \gamma^n \) is a \( d \)-fold cover in the sense of Definition 3.10.

**Proof.** (H^n 1) is clear from Lemma 3.4 (1).

(H^n 3) follows directly from Lemma 2.3 and Lemma 3.4 (1).

(H^n 2) Since \( H^n \) is the lift of \( H^0 \) by \( F^n \), we have

\[
F^n \left( H^0 \left( \bigcup E^n \right) \right) = H^0 \left( F^n \left( \bigcup E^n \right) \right) = H^0 \left( \bigcup E^0 \right) = \bigcup E^1.
\]

Thus,

\[
H^0 \left( \bigcup E^n \right) \subset \bigcup E^{n+1}.
\]

To prove equality in the last expression consider \( \text{int} E^1 \), the interior of a 1-edge. Let \( U^0 = \text{int} A^0 = (H^0)^{-1}(\text{int} E^1) \cap \bigcup E^0 \) be the set in \( \bigcup E^0 \) that is deformed by \( H^0 \) to \( \text{int} E^1 \). This is an arc that does not contain a post-critical point (see Lemma 3.3).
Consider $U_1^n, \ldots, U_d^n \subset \bigcup E^n$, the preimages of $U^0$ by $F^n$; they are disjoint arcs. Each $U_j^n$ is deformed by $H_1^n$ to (the interior of) an $(n+1)$-edge (since $F^n(H_1^n(U_j^n)) = H_1^n(F^n(U_j^n)) = H_1^n(U^0) = \text{int } E^1$).

We remind the reader of the following elementary fact about lifts. Let $\sigma: [0, 1] \rightarrow S^2 \setminus \text{post}(F)$ be a path, and let $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ be two lifts by $F^n$ with distinct initial points. Then the endpoints of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are distinct. Indeed otherwise the lift of the reversed path $\sigma(1-t)$ would fail to be unique.

Therefore the $U_j^n$ are deformed by $H_1^n$ to (the interior of) $d^n$ distinct $(n+1)$-edges.

It follows that $\bigcup E^n$ is deformed by $H_1^n$ to $kd^{n+1}$ $(n+1)$-edges, i.e., all of them.

(H$n$ 4) Assume distinct points $\{x^n_j\}_{j \in \mathbb{N}} \subset \bigcup E^n$ are deformed to some $(n+1)$-vertex $v^{n+1}$ by $H_1^n$. Then the (infinitely many different) points $x^n_0 := F^n(x^n_j) \in \bigcup E^0$ are deformed by $H_1^n$ to the 1-vertex $v^1 := F^n(v^{n+1})$, contradicting property (H$n$ 4).

From now on we assume that the pseudo-isotopies $H^n$ are given as above.

Consider $\{x_j\}_j := (H_1^n)^{-1}(V^{n+1}) \cap \bigcup E^n$, the set of points on $\bigcup E^n$ that are mapped by $H_1^n$ to some $(n+1)$-vertex (each $x_j$ possibly to a different one). Note that $\{x_j\}_j$ is finite by (H$n$ 4) and $\{x_j\}_j \supset V^n$ by (H$n$ 1). Thus the points $\{x_j\}_j$ divide $\bigcup E^n$ (and each $n$-edge) into closed arcs $A_j$.

**Lemma 3.6.** There are $kd^{n+1}$ such arcs $A_j$ as above. Furthermore

$$E'_j := H_1^n(A_j) \text{ is an } (n+1)\text{-edge } \text{ and } H_1^n: A_j \rightarrow E'_j \text{ is a homeomorphism,}$$

for each $j$. On the other hand

each $(n+1)$-edge $E'$ is the image of one such $A_j$ by $H_1^n$.

**Proof.** This follows exactly as in Lemma 3.3. \hfill $\Box$

### 3.3. Eulerian circuits $\gamma^n$

We construct $\gamma^n$, the $n$th approximation of the invariant Peano curve, from the pseudo-isotopies $H^n$. The curves $\gamma^n$ however do not yet have the “right” parametrization. Thus $\gamma^n$ will for now be an Eulerian circuit in $\bigcup E^n$. However the parametrization of this Eulerian circuit will later still be denoted by $\gamma^n(t)$. 
**Definition 3.7.** An *Eulerian circuit* is a closed edge path that traverses each edge exactly once.

Consider now the graph of \( n \)-edges \( \bigcup E^n \), containing \( kdn \) \( n \)-edges. In this graph an Eulerian circuit is a finite sequence of oriented \( n \)-edges

\[
\gamma^n = E_0, \ldots, E_{kdn-1},
\]

such that the following holds (indices are taken mod \( kdn \)). Each \( n \)-edge appears exactly once, and the terminal point of \( E_j \) is the initial point of \( E_{j+1} \). In particular, the terminal point of \( E_{kdn-1} \) is the initial point of \( E_0 \). If \( v \) is the terminal point of \( E_j \) (the initial point of \( E_{j+1} \)), we say that \( E_{j+1} \) *succeeds* \( E_j \) in \( \gamma^n \) at \( v \).

Cyclical permutations of indices are not considered to change \( \gamma^n \), but orientation reversing does.

The approximations \( \gamma^n \) of the invariant Peano curve are defined as follows.

**Definition 3.8.** (Eulerian circuits \( \gamma^n \)) Recall that the Jordan curve \( C = \bigcup E^0 \) is positively oriented as boundary of the white 0-tile \( X^0_w \). Let

\[
\gamma^0 : S^1 \to C
\]

be an orientation-preserving homeomorphism. We define inductively

\[
\gamma^{n+1} : S^1 \to \bigcup E^{n+1},
\]

\[
t \mapsto H^n_1(\gamma^n(t)),
\]

for all \( n \geq 0 \). Let us note the following properties:

(i) The map is surjective by \((H^n 2)\).

(ii) The set \( W^n := (\gamma^n)^{-1}(V^n) \subset S^1 \) is finite by \((H^n 4)\).

(iii) For each \( n \)-edge \( E \) there is exactly one closed arc \([w_j, w_{j+1}] \subset \mathbb{R}/\mathbb{Z} = S^1\), formed by consecutive points \( w_j, w_{j+1} \in W^n \), such that

\[
\gamma^n : [w_j, w_{j+1}] \to E \quad \text{is a homeomorphism.}
\]

This follows directly from Lemma 3.6.

(iv) The map \( \gamma^n \) induces an Eulerian circuit (still denoted by \( \gamma^n \)) on \( \bigcup E^n \) in the obvious way, namely the \( n \)-edges are given the orientation and ordering induced by \( \gamma^n \).

We record how the Eulerian circuit \( \gamma^n \) is related to the Eulerian circuit \( \gamma^{n+1} \). Consider an \( n \)-edge \( E \), which is subdivided into arcs \( A_0, \ldots, A_m \) as in Lemma 3.6. An orientation of \( E \) induces an orientation of the arcs \( A_j \). As before we say that \( A_j \) succeeds \( A_i \) in \( E \) if the terminal point of \( A_i \) is the initial point of \( A_j \).
Lemma 3.9. Let $D'$ and $E'$ be two $(n+1)$-edges. Let $A', B' \subset \bigcup E^n$ be the two arcs that are mapped (homeomorphically) to $D'$ and $E'$, respectively, by $H^1_n$. Then $E'$ succeeds $D'$ in $\gamma^{n+1}$ if and only if either

(a) $A'$ and $B'$ are contained in the same $n$-edge $E$, and $B'$ succeeds $A'$ in $E$ (oriented by $\gamma^n$); or

(b) $A'$ and $B'$ are contained in different $n$-edges $E(A')$ and $E(B')$, the terminal point of $A'$ is the terminal point of $E(A')$, the initial point of $B'$ is the initial point of $E(B')$, and $E(B')$ succeeds $E(A')$ (in $\gamma^n$).

Proof. This is again obvious from the construction. 

3.4. $\gamma^{n+1}$ is a $d$-fold cover of $\gamma^n$

We are now ready to give the definition of properties ($H^0 5$) and ($H^n 5$).

Definition 3.10. (Cover of Eulerian circuits) Let $\gamma^{n+1}$ and $\gamma^n$ be the Eulerian circuits constructed in Definition 3.8 (iv). We call

$$F: \gamma^{n+1} \longrightarrow \gamma^n$$

a $d$-fold cover if $F$ maps succeeding $(n+1)$-edges (in $\gamma^{n+1}$) to succeeding $n$-edges (in $\gamma^n$). An equivalent definition is as follows. Let

$$\gamma^n = E_0, ..., E_{dn-1} \quad \text{and} \quad \gamma^{n+1} = E'_0, ..., E'_{dn+1-1}$$

be two Eulerian circuits. Here each $E_j$ is an (oriented) $n$-edge and each $E'_j$ an (oriented) $(n+1)$-edge. Let $m$ be the index such that $F(E'_0) = E_m$. Then $\gamma^{n+1}$ is a $d$-fold cover of $\gamma^n$ by $F$ if

$$F(E'_j) = E_{m+j}$$

for all $j = 0, ..., dn+1-1$.

Convention. Indices of $n$-edges (and $n$-vertices) are taken mod $kd^n$ here and in the following.

Property ($H^0 5$) is equivalent to the following (seemingly weaker) condition. Recall that each 0-edge $E_j \subset C$ is positively oriented if its orientation agrees with the one induced by $C$. Similarly each $n$-edge $E^n$ is positively oriented if $F^n: E^n \rightarrow E_j$ preserves orientation. Recall furthermore that $n$-tiles are colored white or black if they are preimages by $F^n$ of the 0-tile $X^0_w$ or $X^0_b$, respectively. Each $n$-edge $E^n$ is contained in the boundary of exactly one white and one black $n$-tile. Then $E^n$ is positively oriented if it is positively oriented as boundary arc of the white $n$-tile in $X^n \supset E^n$. 


Lemma 3.11. Let $\gamma^1$ be an Eulerian circuit in $\bigcup E^1$. Then the following conditions are equivalent:

(1) $H^0 \ 5$: $F: \gamma^1 \to \gamma^0$ is a $d$-fold cover;

(2) Each 1-edge in $\gamma^1$ is positively oriented.

Proof. Let $p_0, \ldots, p_{k-1} \in C$ be the post-critical points, labeled positively on $C$. Consider an oriented 1-edge $E^1$ with initial point $v \in V^1$ and terminal point $v' \in V^1$. It is positively oriented if and only if $F(v')$ succeeds $F(v)$, i.e., if $F(v)=p_j$ and $F(v')=p_{j+1}$ for some $j$ (indices are taken mod $k$).

Let $\gamma^1$ go through 1-vertices $v_0, \ldots, v_{kd^n-1}$ in this order. Then $F: \gamma^1 \to \gamma^0$ is a $d$-fold cover if and only if $F(v_{j+1})$ succeeds $F(v_j)$ (for all $j$; indices are taken mod $kd^n$), which holds if and only if each edge in $\gamma^1$ is positively oriented.

Remark. It is not very hard to show that if $\gamma^1$ is obtained as in Definition 3.8 (without assuming $(H^0 \ 5)$), then either all 1-edges are positively oriented, or all 1-edges are negatively oriented in $\gamma^1$ (see [Me2, Lemma 6.7]). In the latter case our construction would result in a semi-conjugacy of $F$ to $z^{-d}$. Indeed a Peano curve $\gamma: S^1 \to S^2$ that semi-conjugates $F=f^n$ to $z^{-d}$ exists by a slight variation of the construction presented here. Namely in §7 the role of the white and black 1-tiles has to be reversed.

We now show how property $(H^0 \ 5)$ implies $(H^n \ 5)$, i.e., finish the proof of Lemma 3.5.

Lemma 3.12. Let $H^0$ be a pseudo-isotopy as in Definition 3.1 and $H^n$ be the lifts of $H^0$ by $F^n$. The Eulerian circuits $\gamma^n$ are the ones from Definition 3.8. Then

(3) $H^n \ 5$: $F: \gamma^{n+1} \to \gamma^n$ is a $d$-fold cover.

Proof. The reader is advised to consult Figure 3 for reference. Roughly speaking, by deforming $\bigcup E^0$ via $H^0$ and $\bigcup E^1$ via $H^1$, one can push the $d$-fold cover $F: \gamma^1 \to \gamma^0$ to a $d$-fold cover $F: \gamma^2 \to \gamma^1$. We give however a more pedestrian (combinatorial) proof.

The proof is by induction. Thus assume that $F: \gamma^n \to \gamma^{n-1}$ is a $d$-fold cover.

Assume that the $(n+1)$-edge $E'$ succeeds the $(n+1)$-edge $D'$ in $\gamma^{n+1}$. We need to show that the $n$-edge $E:=F(E')$ succeeds the $n$-edge $D:=F(D')$ in $\gamma^n$.

Let $A', B' \subset \bigcup E^n$ be the two arcs that are mapped by $H^n$ to $D'$ and $E'$, respectively, see Lemma 3.6. Let $A:=F(A'), B:=F(B') \subset \bigcup E^{n-1}$ be their images. Since $H^n$ is the lift of $H^{n-1}$ by $F$ (the diagram commutes), we have

$$H^n(A) = D \quad \text{and} \quad H^n(B) = E.$$  

There are two cases to consider, by Lemma 3.9.

Case 1. $A'$ and $B'$ are contained in the same $n$-edge $E^n$, and $B'$ succeeds $A'$ (given the orientation of $E^n$ by $\gamma^n$).
Figure 3. Commutative diagram for Lemma 3.12.

Note that since \( F: \gamma^n \to \gamma^{n-1} \) is a \( d \)-fold cover, \( F \) maps \( n \)-edges oriented by \( \gamma^n \) to \((n-1)\)-edges oriented by \( \gamma^{n-1} \).

Therefore \( A \) and \( B \) are contained in the same \((n-1)\)-edge \( E^{n-1} \) of \( F(E^n) \), and \( B \) succeeds \( A \) (given the orientation of \( E^{n-1} \) by \( \gamma^{n-1} \)). Thus \( E \) succeeds \( D \) in \( \gamma^n \).

**Case 2.** \( A' \) and \( B' \) are contained in different \( n \)-edges \( E(A') \) and \( E(B') \), such that \( A' \) and \( E(A') \) have the same terminal points, \( B' \) and \( E(B') \) have the same initial points, and \( E(A') \) and \( E(B') \) are succeeding in \( \gamma^n \).

Thus the \((n-1)\)-edge \( F(E(B')) \supset B \) succeeds \( F(E(A')) \supset A \) in \( \gamma^{n-1} \), since

\[
F: \gamma^n \to \gamma^{n-1}
\]

is a \( d \)-fold cover. Furthermore, the terminal point of \( A \) is the terminal point of \( F(E(A')) \), which is the initial point of both \( B \) and \( F(E(B')) \). Thus \( E \) succeeds \( D \) in \( \gamma^n \) by Lemma 3.9.

By repeating the argument in Lemma 3.11 we obtain inductively the following consequence.

**Corollary 3.13.** All \( n \)-edges in the Eulerian circuit \( \gamma^n \) are positively oriented (for each \( n \)).
4. Construction of $\gamma$

In this section we complete the construction of $\gamma$, i.e., the proof of Theorem 1.1, under the assumption of the existence of a pseudo-isotopy $H^0$ as in Definition 3.2.

**Lemma 4.1.** To construct $\gamma: S^1 \to S^2$ as in Theorem 1.1, it is enough to show that there is a Peano curve $\tilde{\gamma}: S^1 \to S^2$ such that the diagram

$$
\begin{array}{ccc}
S^1 & \xrightarrow{\tilde{\gamma}} & S^1 \\
\downarrow & & \downarrow \\
S^2 & \xrightarrow{F} & S^2
\end{array}
$$

commutes, where $\tilde{\gamma}(z) = e^{2\pi i \theta_0} z^d$.

(Here and in other powers of $e$, throughout the paper, $i$ denotes the imaginary unit.)

**Proof.** Let $\mu := e^{2\pi i \theta_0}/(1-d)$. This means that

$$e^{2\pi i \theta_0} \mu^d = e^{2\pi i \theta_0} \mu^{d-1} \mu = \mu.$$

Consider $\gamma(z) := \tilde{\gamma}(\mu z)$. Then

$$F(\gamma(z)) = F(\tilde{\gamma}(\mu z)) = \tilde{\gamma}(e^{2\pi i \theta_0} \mu^d z^d) = \tilde{\gamma}(\mu z^d) = \gamma(z^d).$$

In this section however we will drop the “$\tilde{}$” from the notation. This means that we will write $\gamma$, $\gamma^n$ and so on; when in fact we mean $\tilde{\gamma}$, $\tilde{\gamma}^n$, which become our desired objects by composing with a rotation as above.

### 4.1. The length of $n$-arcs

The circle $S^1$ will be divided into $n$-arcs, each of which will be mapped by $\gamma^n$ to an $n$-edge. We first need to find the right “length” of such $n$-arcs. It will be convenient to parameterize those lengths by the corresponding $n$-edges. Thus $l(E)$ will be the length of the $n$-arc (in $S^1$) that is mapped by $\gamma^n$ to the $n$-edge $E$. We require the following properties:

1. $l(E) > 0$ for every $n$-edge $E$.
2. For all $n$,

$$\sum_{E \in E^n} l(E) = 1.$$

3. Given an $(n+1)$-edge $E'$ let $E = F(E') \in E^n$. Then

$$l(E) = dl(E').$$
(4) Let $E$ be an $n$-edge. Then $H^1_n(E)$ is a chain $E'_1, \ldots, E'_N$ of $(n+1)$-edges. We require that
\[ l(E) = \sum_{i=1}^N l(E'_i). \]

To this end, consider (all) 0-edges $E_0, \ldots, E_{k-1}$ ordered by the first approximation $\gamma^0$ (positively on $C$). We say that an $n$-edge $E_n$ is of type $j$ if $F^n(E_n) = E_j$. Recall that $H^0$ deforms each 0-edge to several 1-edges. We define a matrix $M = (m_{ij})_{i,j}$, which keeps track of those deformations, by
\[ m_{ij} = \text{the number of 1-edges in } H^1_0(E_i) \text{ that are of type } j. \]

**Lemma 4.2.** Consider an $n$-edge $E^n_i$ of type $i$. Let $\tilde{m}_{ij}$ be the number of $(n+1)$-edges of type $j$ in $H^1_0(E^n_i)$. Then
\[ \tilde{m}_{ij} = m_{ij}. \]

Furthermore, let $m^n_{ij}$ be the number of $n$-edges of type $j$ contained in
\[ H^{n-1}_1 H^{n-2}_1 \ldots H^0_1(E_i). \]
Then
\[ (m^n_{ij})_{i,j} = M^n. \]

**Proof.** Let $E^{n+1}_1, \ldots, E^{n+1}_m$ be the $(n+1)$-edges in $H^1_0(E^n_i)$. Since $H^n$ is the lift of $H^0$ by $F^n$, it follows that $H^0$ deforms (the 0-edge) $E_i = F^n(E^n_i)$ to the 1-edges
\[ E^1_1 = F^n(E^{n+1}_1), \ldots, E^1_m = F^n(E^{n+1}_m). \]

The first statement follows, since $F^n$ preserves the type of edges.

The second statement follows immediately from the first. \hfill \Box

**Lemma 4.3.** The matrix $M$ is primitive, i.e., $M^n > 0$ for some $n$.

**Proof.** Recall from §3.4 that $F : \gamma^{n+1} \rightarrow \gamma^n$ is a $d$-fold cover. Thus, by induction, $F^n : \gamma^n \rightarrow \gamma^0$ is a $d^n$-fold cover. Therefore, along $\gamma^n$ the type of $n$-edges varies cyclically, in $\gamma^n$ an $n$-edge of type $j$ is succeeded by one of type $j + 1$. This means that every chain of $k$ $n$-edges in $\gamma^n$ contains exactly one $n$-edge of each type.

Fix a 0-edge $E_i$ connecting two post-critical points $p$ and $q$. Consider
\[ H^{n-1}_1 H^{n-2}_1 \ldots H^0_1(E_i). \]
This is a chain of $n$-edges in $\gamma^n$ that connects the points $p$ and $q$. Since $F$ is expanding (see Definition 2.1 (3)), the diameter of $n$-edges goes to 0 (uniformly) with $n$. Thus, by choosing $n$ large enough, our chain contains at least $k$ $n$-edges, and therefore at least one $n$-edge of each type.

With this choice of $n$ the claim follows from Lemma 4.2. \hfill \Box
Note that there are \( d \) 1-edges of each type, and thus \( \sum m_{ij} = d \). The Perron–Frobenius theorem (see, for example, [HJ, Theorems 8.2.11 and 8.1.21]) implies that \( d \) is a simple eigenvalue of \( M \) (in fact its spectral radius). Furthermore, there is a unique eigenvector \( t = \{ l_j \}_{j=0}^{k-1} \) to \( d \), such that \( l_j > 0 \) (for all \( j = 0, \ldots, k-1 \)) and \( \sum_{j=0}^{k-1} l_j = 1 \). We note that \( l_j \in \mathbb{Q} \) for all \( j = 0, \ldots, k-1 \). The length of (an \( n \)-arc in \( S^1 \) corresponding to) an \( n \)-edge \( E_j^n \) of type \( j \) is now defined as

\[
 l(E_j^n) := d^{-n} l_j. \tag{4.1}
\]

**Lemma 4.4.** The length defined above satisfies properties (l1)–(l4).

**Proof.** (l1) This follows immediately, since \( l_j > 0 \) for all \( j \).

There are \( d^n \) \( n \)-edges of each type. Thus

\[
 \sum_{E \in \mathbb{E}^n} l(E) = \sum_{j=0}^{k-1} l_j = 1,
\]

which is property (l2).

(l3) This is again clear, since \( F \) maps \( (n+1) \)-edges to \( n \)-edges of the same type.

Property (l4) follows from the fact that \( Ml = dl \). Let \( E_i^n \) be an \( n \)-edge of type \( i \), and \( E_1^{n+1}, \ldots, E_N^{n+1} \) be the \( (n+1) \)-edges contained in \( H(\mathbb{E}^n) \). Then, by Lemma 4.2,

\[
 \sum_{m} l(E_{m}^{n+1}) = d^{n-1} \sum_{j=0}^{k-1} m_{ij} l_j = d^{-n} l_i = l(E_i^n).
\]

Note that the lengths depend on the particular pseudo-isotopy \( H^0 \) chosen, it is not a property of the edges alone.

### 4.2. Parameterizing \( \gamma^n \)

Fix a post-critical point \( p_0 \). Consider the Eulerian circuit \( \gamma^0 = \mathcal{C} = \bigcup \mathbb{E}^0 \), that is

\[
 \gamma^0 = E_0, \ldots, E_{k-1}, \quad E_j \in \mathbb{E}^0.
\]

It is labeled in such a way that the initial point of \( E_0 \) is \( p_0 \). Recall that we want to parameterize \( \gamma \) so that \( \varphi = e^{2\pi i \theta_0} z^d \) is semi-conjugate to \( F \) (see Lemma 4.1). We now define \( \theta_0 \). If \( p_0 \) is a fixed point of \( F \) set \( \theta_0 := 0 \). Otherwise let \( E_0, \ldots, E_{m^0-1} \) be the (unique) positively oriented chain in \( \gamma^0 \) from \( p_0 \) to \( F(p_0) \). Then

\[
 \theta_0 := l(E_0) + \ldots + l(E_{m^0-1}). \tag{4.2}
\]
Label $\gamma^1 = E_0^1, \ldots, E_{kd-1}^1$ in such a way that $E_0^1$ is the initial 1-edge of the chain $H_1^0(E_0^1)$ in $\gamma^1$. In the same fashion label (the Eulerian circuit)

$$\gamma^n = E_0^n, \ldots, E_{kd^n-1}^n, \quad E_j^n \in E^n,$$

so that $E_0^n$ is the initial $n$-edge in $H_1^{n-1}(E_0^n)$ (for each $n$). Thus the initial point of each $E_0^n$ is $p_0$. Note, however, that $\gamma^n$ may go through $p_0$ several times.

It will be convenient to identify $S^1$ with $\mathbb{R}/\mathbb{Z}$. Divide the circle $\mathbb{R}/\mathbb{Z}$ into $k$ arcs $a_j$ as follows. Let

$$\alpha_0 := 0 \quad \text{and} \quad \alpha_j := l(E_0^1) + \ldots + l(E_{j-1}^1)$$

(4.3)

for $j = 1, \ldots, k-1$. Then $a_j := [\alpha_j, \alpha_{j+1}]$ (where indices are taken mod $k$).

**Convention.** When writing $[\alpha, \beta] \subset \mathbb{R}/\mathbb{Z}$ for an arc on the circle, we always mean the positively oriented arc from $\alpha$ to $\beta$. In particular $a_{k-1} = [\alpha_{k-1}, 0] = [\alpha_{k-1}, 1]$.

In the same fashion we divide the circle $\mathbb{R}/\mathbb{Z}$ into $kd^n$-arcs $a^n_j$ (for each $n$) by

$$\alpha_0^n := 0 \quad \text{and} \quad \alpha_j^n := l(E_0^n) + \ldots + l(E_{j-1}^n)$$

for $j = 1, \ldots, kd^n-1$. Then $a^n_j := [\alpha^n_j, \alpha^n_{j+1}]$.

**Convention.** The (lower) indices of points $\alpha^n_j$, $n$-arcs $a^n_j$ and $n$-edges $E^n_j$ are always taken mod $kd^n$. In particular, $a^n_{kd^n} = a^n_0$ and $a^n_{kd^n-1} = [\alpha^n_{kd^n-1}, 0] = [\alpha^n_{kd^n-1}, 1] \subset \mathbb{R}/\mathbb{Z}$.

We now define the approximations $\gamma^n$ on each $n$-arc $a^n_j \subset \mathbb{R}/\mathbb{Z}$ by

$$\gamma^n: a^n_j \longrightarrow E^n_j$$

is (any) orientation-preserving homeomorphism,

as parameterized curves. Thus initial and terminal points are mapped onto each other by $\gamma^n$. Note that $\gamma^n(0) = p_0$ for all $n$.

In $\mathbb{R}/\mathbb{Z}$ the map $\varphi(z) = e^{2\pi i \theta_0 z^d}$ is given by

$$\phi: \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}, \quad \phi(t) = dt + \theta_0 \mod 1.$$

**Lemma 4.5.** The parameterized curves $\gamma^n$ satisfy the following properties:

1. If $m \geq n$, then each point $\alpha^n_j$ is a point $\alpha^m_j$. Furthermore,

$$\gamma^m(\alpha^n_j) = \gamma^n(\alpha^m_j),$$

for all $j = 0, \ldots, kd^n-1$. Note that $\{\alpha^n_j\}_{j=0}^{kd^n-1} = (\gamma^n)^{-1}(V^n)$. So the $n$-th approximation determines the preimages (on the circle) of the $n$-vertices.
The map $\phi$ maps each point $\alpha_j^{n+1}$ to a point $\alpha_i^n$. For any point $\alpha_j^{n+1} \in \mathbb{R}/\mathbb{Z}$, 

$$F(\gamma^{n+1}(\alpha_j^{n+1})) = \gamma^n(\phi(\alpha_j^{n+1})).$$

Thus we have the following commutative diagram:

$$
\begin{array}{ccc}
\{\alpha_j^{n+1}\}_{j=0}^{kd^{n+1}-1} \subset \mathbb{R}/\mathbb{Z} & \xrightarrow{\phi} & \{\alpha_j^n\}_{j=0}^{kd^n-1} \subset \mathbb{R}/\mathbb{Z} \\
\gamma^{n+1} & & \gamma^n \\
V^{n+1} \subset S^2 & \xrightarrow{F} & V^n \subset S^2.
\end{array}
$$

This will imply the desired semi-conjugacy.

(3) The supremum norm is given in terms of the visual metric (2.3). Then 

$$\|\gamma^{n+1} - \gamma^n\|_\infty \lesssim \Lambda^{-n}$$

for all $n$. Here $C(\lesssim)$ does not depend on $n$.

Proof. (1) Consider $E_0$, the first 0-edge in $\gamma^0$. Then $H_1^0(E_0)$ is the chain $E_0^1, \ldots, E_{m-1}^1$ of 1-edges in $\gamma^1$. Note that the terminal point of $E_0$ is the terminal point of $E_{m-1}^1$. By property (l4), 

$$\alpha_1 = l(E_0) = l(E_0^1) + \ldots + l(E_{m-1}^1) = \alpha_m^1.$$ 

Thus 

$$\gamma^1(\alpha_1) = \gamma^1(\alpha_m^1) = \text{terminal point of } E_m^1 = \text{terminal point of } E_0 = \gamma^0(\alpha_1).$$

In the same fashion one shows that each $\alpha_j$ is a point $\alpha_i^1$, and $\gamma^1(\alpha_j) = \gamma^0(\alpha_j)$ for all $j = 0, \ldots, k-1$. The general statement follows by induction (see Lemma 4.2).

(2) Recall from the definitions of $\theta_0$ (4.2) and $\{\alpha_j\}_{j=0}^{k-1}$ (4.3) that $\alpha_m^0 = \theta_0$. Then, by (1) and the definition of $\theta_0$, we have 

$$\gamma^n(\theta_0) = \gamma^0(\theta_0) = F(\theta_0).$$

Let $m^n = m^n(\theta_0)$ be the index such that $\alpha_{m^n}^n = \theta_0$.

Consider $E_0^{n+1}$, the initial $(n+1)$-edge in $\gamma^{n+1}$. It is clear that $F(E_0^{n+1})$ is an $n$-edge with initial point $F(\theta_0)$ (by Corollary 3.13). There may be several such $n$-edges in general however. We next show that $F(E_0^{n+1})$ is in fact the “right” $n$-edge, namely the image (by $\gamma^n$) of the $n$-arc (on $\mathbb{R}/\mathbb{Z}$) with initial point $\theta_0$.

Claim 1. $F(E_0^{n+1}) = \gamma^n(\alpha_{m^n}^n) = E_{m^n}^n$. 


This is clear for \( n=0 \), since there is only one 0-edge with initial point \( F(p_0) \). To prove the claim by induction, we assume it is true for \( n-1 \).

Consider \( E_0^n \); by assumption, \( F(E_0^n) = \gamma^{n-1}(a^{n-1}_{m^{n-1}}) = E_{m^{n-1}}^{n-1} \). Let \( A^n \subset E_0^n \) be the (initial) \( n \)-arc that is deformed by \( H^n \) to \( E_0^n \). Let \( A^{n-1} := F(A^n) \subset E_{m^{n-1}}^{n-1} \); it is an \( n \)-arc that is deformed by \( H^{n-1} \) to an \( n \)-edge \( E_j^n \) (since \( H^n \) is the lift of \( H^{n-1} \) by \( F \)):

\[
\begin{array}{ccc}
A^n \subset E_0^n & \overset{H^n}{\longrightarrow} & E_0^{n+1} \\
F \downarrow & & \downarrow F \\
A^{n-1} \subset E_{m^{n-1}}^{n-1} & \overset{H^{n-1}}{\longrightarrow} & E_j^n.
\end{array}
\]

The crucial property is that by construction \( j = m^n \). This is seen as follows. By (l 4) the total length of the \((n-1)\)-edges preceding \( E_{m^{n-1}}^{n-1} \) (which is \( \theta_0 \)) is the same as the total length of all \( n \)-edges preceding \( E_j^n \):

\[
\theta_0 = l(E_0^{n-1}) + \ldots + l(E_{m^{n-1}-1}^{n-1}) = l(E_0^n) + \ldots + l(E_j^n),
\]

and thus \( j = m^n \).

Hence, \( F(E_0^{n+1}) = E_{m^n}^n \), since the diagram above commutes. This proves Claim 1.

**Claim 2.** \( F(E_j^{n+1}) = E_{m^n+j}^n \) for \( j = 0, \ldots, kd^{n+1}-1 \).

This follows from Claim 1, and the fact that \( F: \gamma^{n+1} \to \gamma^n \) is a \( d \)-fold covering in the sense of Definition 3.10. The reader is reminded (for the last time) that the index \( m^n+j \) is taken mod \( kd^n \).

**Claim 3.** The map \( \phi \) maps points \( \alpha_{j+1}^n \) to points \( \alpha_i^n \), in fact

\[
\phi(\alpha_{j+1}^n) = \alpha_{m^n+j}^n.
\]

To prove this claim note first that

\[
\phi(\alpha_0^{n+1}) = \phi(0) = \theta_0 = \alpha_m^n
\]

by definition. In the following we write \( \alpha \equiv \beta \) if \( \alpha \) and \( \beta \) represent the same point on the circle \( \mathbb{R}/\mathbb{Z} \), i.e., if \( \alpha - \beta \in \mathbb{Z} \).

By the previous claim, \( F(E_j^{n+1}) = E_{m^n+j}^n \), and thus

\[
l(E_{m^n+j}^n) = dl(E_j^{n+1})
\]
by Property (l3). Therefore
\[
\alpha_{m^n+j}^n = \alpha_{m^n}^n + l(E_{m^n}) + l(E_{m^n+1}) + \ldots + l(E_{m^n+j-1})
\]
\[
= \theta_0 + d(l(E_{0}^n) + \ldots + l(E_{j-1}^n)) = \theta_0 + da^n_{j+1} \equiv \phi(\alpha_{n+1}^n)
\]
for \( j=0, \ldots, kd^{n+1} - 1 \). Thus Claim 3 is proved.

It remains to show the semi-conjugacy. Note that, by construction, \( \gamma^n \) maps \( \alpha_{j}^n \) to the initial point of \( E_{j}^n \). Thus, by Claims 2 and 3,
\[
F(\gamma^{n+1}(\alpha_{j}^{n+1})) = F(\text{initial point of } E_{j}^{n+1}) = \text{initial point of } E_{m^n+j}^n
\]
\[
= \gamma^n(\alpha_{m^n+j}^n) = \gamma^n(\phi(\alpha_{n}^n)).
\]
This finishes the proof of property (2).

(3) The diameter of each \( n \)-edge \( E^n \) in the visual metric (2.3) is given by
\[
\text{diam } E^n \asymp \Lambda^{-n},
\]
see [BM, Lemma 8.4].

Consider one \( n \)-arc \( a_{j}^n = [\alpha_{j}^n, \alpha_{j+1}^n] \). Then \( \gamma^n(a_{j}^n) = E_{j}^n \). The pseudo-isotopy \( H^n \) deforms \( E_{j}^n \) to a \((n+1)\)-chain \( E_{i+1}^{n+1}, \ldots, E_{i+m-1}^{n+1} \). The number \( m \) (of \((n+1)\)-edges in this chain) is uniformly bounded by Lemma 4.2. By (the proof of) property (1), we have \( \alpha_{j}^n = \alpha_{i+1}^n \) and \( \alpha_{j+1}^n = \alpha_{i+m}^n \), and so
\[
a_{j}^n = a_{i+1}^n \cup \ldots \cup a_{i+m-1}^n,
\]
where
\[
\gamma^{n+1}(a_{i+1}^{n+1}) = E_{i+1}^{n+1}, \ldots, \gamma^{n+1}(a_{i+m-1}^{n+1}) = E_{i+m-1}^{n+1}.
\]
Furthermore, the \((n+1)\)-chain \( E_{i+1}^{n+1}, \ldots, E_{i+m-1}^{n+1} \) and the \( n \)-edge \( E_{j}^{n} \) intersect in (the end-points of \( E_{j}^{n} \)) \( \gamma^n(\alpha_{j}^n) = \gamma^{n+1}(\alpha_{i+1}^{n+1}) \) and \( \gamma^n(\alpha_{j+1}^n) = \gamma^{n+1}(\alpha_{i+m}^{n+1}) \), again by property (1). Thus, on \( a_{j}^n \),
\[
\|\gamma^n - \gamma^{n+1}\|_{\infty} \leq \text{diam } E_{j}^{n} + \text{diam } E_{i+1}^{n+1} + \ldots + \text{diam } E_{i+m-1}^{n+1} \lesssim \Lambda^{-n} + m\Lambda^{-n-1} \lesssim \Lambda^{-n},
\]
as desired. \( \square \)

4.3. Construction of the invariant Peano curve \( \gamma \)

We now come to the proof of the main result, assuming the existence of a pseudo-isotopy \( H^0 \) as in Definition 3.2.
Define

\[ \gamma : \mathbb{R}/\mathbb{Z} \rightarrow S^2, \]
\[ t \mapsto \lim_{n \to \infty} \gamma^n(t). \]

Since the sequence \( \{\gamma^n\}_{n=1}^{\infty} \) converges uniformly by Lemma 4.5 (3), this is a parameterized curve.

Claim 1. \( \gamma \) is a Peano curve (onto).

This is clear since the curve \( \gamma \) contains by construction \( \bigcup_{n=1}^{\infty} V^n \) (all \( n \)-vertices). This set is dense in \( S^2 \).

Claim 2. \( F(\gamma(t)) = \gamma(\phi(t)) \) for all \( t \in \mathbb{R}/\mathbb{Z} \).

Note that, by properties (1) and (2) of Lemma 4.5, this is true for all \( t = \alpha^n_j \). The claim follows, since the set of all such points \( \alpha^n_j \) is dense in the circle \( \mathbb{R}/\mathbb{Z} \).

Thus we “just” need to construct the pseudo-isotopy \( H^0 \) (with properties (1)–(5)) to finish the proof of Theorem 1.1.

4.4. \( \gamma \) is the end of a pseudo-isotopy

The homotopy \( \Gamma : S^2 \times [0, 1] \rightarrow S^2 \) from Theorem 1.1 is constructed as follows. Roughly speaking we concatenate the homotopies \( H^n \). The precise definition is as follows. Break up the unit interval into intervals

\[ I = [0, 1] = [0, \frac{1}{2}] \cup \left[ \frac{1}{2}, \frac{3}{4} \right] \cup \ldots \cup \left[ 1 - 2^{-n}, 1 - 2^{-n-1} \right] \cup \ldots \cup \{1\}. \]

The \( n \)th interval in this union is denoted by \( I^n = [1 - 2^{-n}, 1 - 2^{-n-1}] \). Let \( s_n : I^n \rightarrow I \), \( s_n(t) = 2^{n+1}(t - (1 - 2^{-n})) \) for \( n \in \mathbb{N}_0 \). We define \( \Gamma : S^2 \times I \rightarrow S^2 \) by \( \Gamma(x, t) = H^0(x, s_0(t)) \) for \( t \in I^0 \) and \( \Gamma(x, t) = H^1(H^0(x), s_1(t)) \) for \( t \in I^1 \). In general,

\[ \Gamma(x, t) := H^n(H^{n-1}_1 \ldots H^0_1(x), s_n(t)) \]

for \( t \in I^n \) (for some \( n \in \mathbb{N}_0 \)) and all \( x \in S^2 \). Since the diameters of \( H^n \) tend to 0 exponentially (see Lemma 3.4 (2)), it follows that \( \Gamma \) extends to \( t = 1 \) by \( \Gamma(x, 1) := \lim_{t \to 1} \Gamma(x, t) \) continuously. This is the desired homotopy.

It is possible to choose \( \Gamma \) to be a pseudo-isotopy. This can be done explicitly by slightly altering the above construction. We do not work out the details here. It is however a direct consequence of the general theory of decomposition spaces. Namely it follows from the fact that every cell-like upper semi-continuous decomposition of a 2-manifold is shrinkable [Da, Theorem 25.1].
5. Some topological lemmas

Here we collect some topological theorems and lemmas for future reference. We first note the following form of the Jordan–Schönflies theorem.

**Theorem 5.1.** (Isotopic Schönflies theorem) Let $\gamma, \sigma \subset \mathbb{D}$ be two Jordan arcs with common endpoints $p, q \in \mathbb{D}$. Then there is an isotopy of $\mathbb{D}$ rel. $\partial \mathbb{D} \cup \{p, q\}$ that deforms $\gamma$ to $\sigma$.

We give a quick outline of how this form can be obtained from the standard Schönflies theorem.

**Theorem 5.2.** (Schönflies theorem [Mo, Theorem 10.4]) Let $h: J \subset \mathbb{R}^2 \to \tilde{J} \subset \mathbb{R}^2$ be a homeomorphism, where $J$ is a Jordan curve. Then $h$ may be extended to a homeomorphism $h: \mathbb{R}^2 \to \mathbb{R}^2$.

We remind the reader of the Alexander trick.

**Theorem 5.3.** (Alexander [Mo, Theorem 11.1]) Let $h: D \to D$ be a homeomorphism such that $h|_{S^1} = \text{id}_{S^1}$. Then the map $\phi: D \times [0, 1]$ defined by

$$
\phi(x, t) := \begin{cases} 
  th\left(\frac{x}{t}\right), & \text{if } 0 \leq |x| \leq t, \\
  x, & \text{if } t \leq |x| \leq 1,
\end{cases}
$$

is an isotopy with $\phi(\cdot, 0) = \text{id}_D$ and $\phi(\cdot, 1) = h$.

**Outline of proof of Theorem 5.1.** Consider first $p, q \in S^1 = \partial \mathbb{D}$. Let $C_1, C_2 \subset S^1$ be the two arcs bounded by $p$ and $q$. Let $h_j: \gamma \cup C_j \to \sigma \cup C_j$ be homeomorphisms which are constant on $S^1$, $j = 1, 2$. Using Theorem 5.2, they can be extended to a homeomorphism of $\mathbb{D}$. Theorem 5.3 gives the desired isotopy.

If $p = 0$ and $q \in S^1$, extend $\gamma$ and $\sigma$ to arcs with common endpoints $\tilde{p}, \tilde{q} \in S^1$. The previous procedure yields the isotopy.

If $p \in \mathbb{D}$ and $q \in S^1$, we use the same construction as before. Then we post-compose with the isotopy that maps the rays between $\phi(p, t)$ and $\zeta \in S^1$ to the rays between $p$ and $\zeta \in S^1$.

Finally let $p, q \in \mathbb{D}$. By the above, we may assume that $p = 0$. Extend $\gamma$ and $\sigma$ to curves $\tilde{\gamma}$ and $\tilde{\sigma}$, respectively, with common endpoints $\tilde{p}$ and $\tilde{q}$. As above we obtain an isotopy $\phi(x, t)$ rel. $S^1 \cup \{p\}$ deforming $\tilde{\gamma}$ to $\tilde{\sigma}$. We may assume that $\phi(q, 1) = q$ (choose the homeomorphisms $h_j$ such that $h_j(q) = q$). This means that $\phi$ deforms $\gamma$ to $\sigma$. Let $r_t := |\phi(q, t)|$ and $\alpha_t := \log r_0 / \log r_t$. Then post-composition with the radial stretch

$$
\psi(x, t) := |x|^{\alpha_t} \frac{x}{|x|}
$$
yields an isotopy \( \tilde{\phi} \) rel. \( S^1 \cup \{p\} \) which keeps \( |q| \) constant. Let \( \theta_t := \arg \tilde{\phi}(q,t) - \arg q \). Post-composing with 
\[ \varphi: r e^{i\theta_t} \mapsto re^{i(\theta - \theta_t(1-r)/(1-|q|))} \]
yields the desired isotopy. There is a tricky point hidden here: \( \theta_1 \) could be a multiple of \( 2\pi \). We can however always arrange that \( \theta_1 = 0 \) in the following way. Let \( \tilde{\gamma} |_{\tilde{q},q} \) and \( \tilde{\sigma} |_{\tilde{q},q} \) be the paths of the extensions from \( \tilde{q} \) to \( q \). By choosing the extensions \( \tilde{\gamma} \) and \( \tilde{\sigma} \) in such a way that the change of argument along \( \tilde{\gamma} |_{\tilde{q},q} \) and \( \tilde{\sigma} |_{\tilde{q},q} \) is equal, it follows that

\[ \theta_1 = 0. \]

The following is due to Epstein–Zieschang, see [Bu, Theorem A.5].

**Theorem 5.4.** (Isotopy rel. post) Let \( C, \gamma \subset S^2 \) be two Jordan curves going through the post-critical points \( p_0, ..., p_{k-1} \) in the same cyclical order. Let \( C_j \) and \( \gamma_j \) be the arcs on \( C \) and \( \gamma \), respectively, between \( p_j \) and \( p_{j+1} \) (indices are taken mod \( k \) here). Then the following conditions are equivalent:

1. \( C_j \) and \( \gamma_j \) are isotopic rel. post for all \( j = 0, ..., k-1 \);
2. \( C \) and \( \gamma \) are isotopic rel. post.

Combining the previous with Theorem 5.1, we obtain the following result.

**Theorem 5.5.** With notation as in the previous theorem assume that

\[ C_i \cap \gamma_j \neq \emptyset \quad \text{only for } j = i-1, i, i+1. \]

Then \( C \) and \( \gamma \) are isotopic rel. post.

6. Connections

In this and the following section the initial pseudo-isotopy \( H^0 \) is constructed. This was used to define the first approximation \( \gamma^1 \) of the Peano curve. Recall that \( \gamma^1 \) is an Eulerian circuit of 1-edges. Thus \( \gamma^1 \) is given by the following construction. For each 1-edge \( E \) ending at a 1-vertex \( v \) we have to define a succeeding 1-edge \( E' \ni v \). Since \( \gamma^1 \) will be non-crossing, there will be an even number of 1-edges in the sector between \( E \) and \( E' \) (as well as in the sector between \( E' \) and \( E \)). Let \( E \) be contained in the white 1-tile \( X \), and \( E' \) be contained in the white 1-tile \( X' \). From the above it follows that if \( \gamma^1 \) traverses \( E \) positively (as boundary of \( X \)), then it traverses \( X' \) positively (as boundary of \( X' \)).

Since \( \gamma^1 \) is non-crossing, it is possible to “distort the picture” in a neighborhood of \( v \) slightly, so that the resulting curves are simple. In this distorted picture the 1-tiles \( X \) and \( X' \) are connected at \( v \). See Figure 4 for an illustration.
Formally we will do the reverse of the description above. Namely at each 1-vertex
we will define a connection, which is an assignment of which 1-tiles are connected. This
will be done in a non-crossing manner. The approximation $\gamma^1$ and the pseudo-isotopy
$H^0$ are constructed from the connection of (all) 1-tiles.

6.1. Non-crossing partitions

Recall that a partition of the set $[n]:={0, \ldots, n-1}$ is a set $\pi={b_1, \ldots, b_N}$ of pairwise
disjoint subsets (called blocks) of $[n]$, whose union is $[n]$. It is crossing if and only if it
contains distinct blocks $b_i$ and $b_j$ with $a, c \in b_i$ and $b, d \in b_j$ such that

$$0 \leq a < b < c < d \leq n-1;$$

otherwise it is non-crossing.

It is easy to see that the partition $\pi={b_1, \ldots, b_N}$ of $[n]$ is non-crossing if and only if
the sets $B_{j'}:=\{e_m: m \in b_{j'}\}$, where $e_m:=e^{2\pi im/n}$, have the property that each $B_{j'}$ lies in
one component of $S^1 \setminus B_j$ (for $j' \neq j$).

With this description in mind let, for $i, j \in [n]$,

$$ [i, j] := \begin{cases} \{i, \ldots, j\}, & \text{if } i \leq j, \\ \{i, \ldots, n-1\} \cup \{0, \ldots, j\}, & \text{if } i > j, \end{cases} $$

and

$$ (i, j) := [i, j] \setminus \{i, j\}. $$

Let $b=\{j_0, \ldots, j_m\} \subset [n]$, where $j_0 < \ldots < j_m$. Then a component of $[n] \setminus b$ is defined to be
one of the sets

$$ (j_0, j_1), \ldots, (j_{m-1}, j_m), (j_m, j_0). $$
The partition \( \pi = \{b_1, \ldots, b_N\} \) is non-crossing if and only if each \( b_i \) lies in one component of \([n] \setminus b_j\) for all \( i \neq j \).

The set of non-crossing partitions of \([n]\) is partially ordered by refinement. Namely, for two partitions \( \pi \) and \( \sigma \), one defines \( \sigma \leq \pi \) if and only if every block in \( \pi \) is the union of blocks in \( \sigma \). Equipped with this partial ordering the non-crossing partitions (of \([n]\)) form a lattice, i.e., meet and join are well defined. The meet of (non-crossing) partitions \( \pi_1, \ldots, \pi_m \) is

\[
\bigwedge_{i=1}^m \pi_i := \{b_1 \cap \ldots \cap b_m : b_i \in \pi_i, i = 1, \ldots, m\}.
\]

(6.2)

It is the biggest (non-crossing) partition smaller than any \( \pi_i \). The join is the smallest non-crossing partition bigger than any \( \pi_i \) (the description is slightly more difficult).

Non-crossing partitions were introduced in [Kr], see [Si] for a recent survey. The number of non-crossing partitions of \([n]\) is equal to the \( n \)th Catalan number

\[
C_n := \frac{1}{n+1} \binom{2n}{n}.
\]

Consider now

\[
even = even_n = \{2m : m = 0, \ldots, n-1\} \quad \text{and} \quad \text{odd} = odd_n = \{2m+1 : m = 0, \ldots, n-1\},
\]

so that \([2n] = even \cup odd\).

Non-crossing partitions of even and odd are defined as before. We denote by \( \pi_w \) a non-crossing partition of even, and by \( \pi_b \) a non-crossing partition of odd. They will describe how white (black) tiles are connected at a vertex \( v \); see again Figure 4 for an illustration, and Figure 9 for a more complicated example.

**Lemma 6.1.** Let \( \pi_w \) be a non-crossing partition of \( even_n \). Then there is a unique maximal non-crossing partition \( \pi_b = \pi_b(\pi_w) \) of \( odd_n \) such that \( \pi_w \cup \pi_b \) is a non-crossing partition of \([2n]\).

**Proof.** Fix a block \( b_i \in \pi_w = \{b_1, \ldots, b_N\} \). Let \( c_1, \ldots, c_M \) be the components of \([2n] \setminus b_i\). Let

\[
a_j := \text{odd} \cap c_j, \quad j = 1, \ldots, M.
\]

Then \( \pi_b(b_i) := \{a_1, \ldots, a_M\} \). This is a non-crossing partition of odd. We now define (see (6.2))

\[
\pi_b := \bigwedge_{i=1}^N \pi_b(b_i);
\]

this is a non-crossing partition of odd. Also \( \pi_w \cup \pi_b \) is a non-crossing partition of \([2n]\).

Let \( \sigma_b \) be any non-crossing partition of odd such that \( \pi_w \cup \sigma_b \) is a non-crossing partition of \([2n]\). Then \( \sigma_b \leq \pi_b(b_i) \) for all \( i \). Thus \( \sigma_b \leq \pi_b \). \( \square \)
The partition $\pi_b=\pi_b(\pi_w)$ is called the partition complementary to $\pi_w$. We mention some more facts which can be found in [Kr, §3].

**Lemma 6.2. (Properties of complementary partitions)** Complementary partitions have the following properties:

- Two blocks $a$ and $b$ are called adjacent if there are $i \in a$ and $j \in b$ such that $i+1 \in b$ and $j+1 \in a$. The partition $\pi_w \cup \pi_b$ has the property that the two blocks containing $i$ and $i+1$ are adjacent for all $i$. This characterizes $\pi_b$, meaning that it is the unique non-crossing partition of odd, such that $\pi_w \cup \pi_b$ is non-crossing, with this property.
- One may define $\pi_w = \pi_w(\pi_b)$, the partition (of even) complementary to the partition $\pi_b$ (of odd) as before. Then the previous characterization shows that $\pi_w(\pi_b(\pi_w)) = \pi_w$.

Thus we simply say that the partitions $\pi_w$ and $\pi_b$ are complementary.

- It is possible to define a graph, where the vertices are the blocks of $\pi_w \cup \pi_b$, connected by edges if and only if they are adjacent. It is not very hard to show that this is a tree with $n$ edges. Thus $\pi_w \cup \pi_b$ contains exactly $n+1$ blocks.

From now on, we write $cnc$-partition for complementary non-crossing partitions $\pi_w \cup \pi_b$ as above.

We next proceed to construct a geometric realization of a given $cnc$-partition; see again Figure 4.

Divide the unit disk into $n+1$ (simply connected) domains $D_1, \ldots, D_{n+1}$ by $n$ disjoint Jordan arcs $g_1, \ldots, g_n \subset \partial \mathbb{D}$. More precisely, the (distinct) endpoints of each $g_j$ lie in $S^1 = \partial \mathbb{D}$, the interior of $g_j$ in $\mathbb{D}$. The arcs $g_m$ divide $S^1$ into $2n$ circular arcs $a_0, \ldots, a_{2n-1} \subset S^1$ (labeled positively on $S^1$). A partition $\pi(\{g_m\}_{m=1}^n)$ of $[2n]$ is obtained as follows:

$$i, j \in [2n] \text{ are in the same block of } \pi(\{g_m\}_{m=1}^n) \text{ if and only if } a_i \text{ and } a_j \text{ are in the boundary of the same component } D_l.$$  

(6.3)

So, for each component $D_l$ of $\mathbb{D} \setminus \bigcup_{j=1}^n g_j$, there is exactly one block $b_l \in \pi(\{g_m\}_{m=1}^n)$.

**Lemma 6.3.** The partition $\pi(\{g_m\}_{m=1}^n)$ is a $cnc$-partition. Conversely each $cnc$-partition of $[2n]$ is obtained in this way.

Furthermore $\overline{D}_k$ and $\overline{D}_l$ are not disjoint if and only if the (corresponding) blocks $b_k$ and $b_l$ are adjacent. In this case the intersection of $\overline{D}_k$ and $\overline{D}_l$ is one arc $g_m$. Conversely, each $g_m$ is the intersection of the closure of two components $\overline{D}_k$ and $\overline{D}_l$.

**Proof.** We first show that $\pi(\{g_m\}_{m=1}^n)$ is non-crossing. Consider distinct components $D_k$ and $D_l$. Then there is a Jordan arc $g_m \subset \partial D_k$ that separates $D_k$ from $D_l$. Let
α, β ∈ S^1 be the endpoints of g_m. Let a_i, a_i+1 ⊂ S^1 and a_j, a_j+1 ⊂ S^1 be the circular arcs containing α and β. We may assume that a_i ⊂ ∂D_k, and thus a_j+1 ⊂ ∂D_k. Then all arcs in the boundary of D_l are contained in a_{i+1}, ..., a_j. This means that b_l ⊂ [i+1, j], which is one component of [2n] \ b_k (recall that b_k is the block corresponding to D_k, while b_l is the block corresponding to D_l; see (6.1) for notation). This shows that π(\{g_m\}_{m=1}^n) is non-crossing.

If ∂D_l ∋ a_{i+1} (equivalently, i+1 ∈ b_l) it follows that g_m ⊂ ∂D_l. Thus a_j ⊂ ∂D_l (equivalently, j ∈ b_l). Hence i, j+1 ∈ b_k and i+1, j ∈ b_l, i.e., b_k and b_l are adjacent. This shows that the partition π(\{g_m\}_{m=1}^n) is a cnc-partition.

Furthermore, it is clear that b_k and b_l are adjacent if and only if ∂D_k and ∂D_l intersect.

It remains to show that each cnc-partition is obtained in this geometric fashion. Identify each j ∈ [2n] with the circular arc a_j = [e_j, e_{j+1}] ⊂ S^1 (where e_j = e^{2πij/2n}). For each block b_l ∈ π_w ∪ π_b the domain D_l is the hyperbolic polygon whose boundary intersects S^1 in \bigcup_{j \in b_l} a_j.

To be more precise, for each two adjacent blocks b ∋ i, j+1 and b' ∋ i+1, j, we connect e_{i+1} and e_{j+1} by a hyperbolic geodesic. Since every block distinct from b is contained in one component of [2n] \ \{i, j, j+1\}, the Jordan arcs g_m thus obtained are disjoint.

How 1-tiles are connected at a 1-vertex v will be described by complementary non-crossing partitions. Additional data is needed however, to make the construction well defined. Namely, if v = p is a post-critical point, we need to declare where p lies in the “distorted picture” (in the geometric representation of the complementary connections, see below).

Definition 6.4. (Marking) A cnc-partition π_w ∪ π_b is marked by singling out a pair of adjacent blocks b, c ∈ π_w ∪ π_b. Equivalently, this means that if the cnc-partition π_w ∪ π_b is given geometrically as above in Lemma 6.3, we mark one of the arcs g_m. In Figure 4 the marked arc g_m is indicated by the big dot.

Given a marked cnc-partition, we always assume that the geometric realization from Lemma 6.3 was chosen in such a way that the marked arc g_m contains the origin.

A third equivalent way to mark a connection is given in Corollary 6.14.

Assume now that the circular arcs from Lemma 6.3 are of the form a_j = [e_j, e_{j+1}] ⊂ S^1 (where e_j = e^{2πij/2n}). Color the set D_l white if the corresponding block b_l ∈ π_w, and black otherwise. Thus we obtain a “checkerboard tiling” of the unit disk, where sets which share a side g_m have different color.

Definition 6.5. (Geometric representation of cnc-partition) The decomposition of the closed unit disk into black and white sets as above is called a geometric representation
of the cnc-partition $\pi_w \cup \pi_b$, and is denoted by $\overline{\pi}(\pi_w \cup \pi_b)$. The union of the white sets $\overline{\pi}_i$ is denoted by $\overline{\pi}_w = \overline{\pi}_w(\pi_w \cup \pi_b)$, and the union of the black sets $\overline{\pi}_i$ by $\overline{\pi}_b = \overline{\pi}_b(\pi_w \cup \pi_b)$.

Denote by $S_j$, $j=0, \ldots, 2n-1$, a sector in $\overline{\pi}$, namely

$$S_j := \left\{ re^{2\pi i \theta} : \frac{j}{2n} \leq \theta \leq \frac{j+1}{2n} \text{ and } 0 \leq r \leq 1 \right\}. \quad (6.4)$$

**Lemma 6.6.** (Deforming $\overline{D}(\pi_w \cup \pi_b)$) Let the geometric representation $\overline{D}(\pi_w \cup \pi_b)$ be as above. Then there is a pseudo-isotopy $H$ of $\overline{D}$ rel. $\partial \overline{D} \cup \{0\}$ satisfying the following properties:

- $H$ deforms $\overline{D}(\pi_w \cup \pi_b)$ to sectors. More precisely,
  $$H_1(\overline{D}_w) = \bigcup_j S_j \text{ and } H_1(\overline{D}_b) = \bigcup_j S_j.$$

- The pseudo-isotopy $H$ “freezes” outside of a neighborhood of 0. By this we mean that, for $\varepsilon < \frac{1}{2}$,
  $$H : \overline{D} \times [1-\varepsilon, 1] \rightarrow \overline{D} \text{ is a pseudo-isotopy rel. } \overline{D} \setminus B_\varepsilon,$$
  where $B_\varepsilon = \{z : |z| < \varepsilon\}$.

- Only one point on each arc $g_m$ is deformed to 0 by $H$.

**Proof.** This follows from the Schönflies theorem (Theorem 5.1).

---

### 6.2. Connections

Let $v$ be a 1-vertex. A **connection** at $v$ consists of an assignment of which white and black 1-tiles are connected at $v$. The objective is to “cut” tiles at vertices, so that the boundary of the “white (or black) component” is a Jordan curve.

Let $n = \deg_F v$ be the degree of $F$ at $v$, let $X_0, \ldots, X_{2n-1}$ be the 1-tiles containing $v$, labeled positively around $v$, such that white 1-tiles have even index and black 1-tiles have odd index.

**Definition 6.7.** (Connection at a vertex) A **connection** at a 1-vertex $v$ consists of a labeling of 1-tiles containing $v$ as above and cnc-partitions $\pi_w = \pi(v)$ and $\pi_b = \pi(v)$ of even, odd (representing white 1-tiles) and odd, 1-tiles (representing black 1-tiles). The 1-tiles $X_i$ and $X_j$ (of the same color) are said to be **connected** at $v$ if $i$ and $j$ are contained in the same block of $\pi_w \cup \pi_b$, 1-tiles of different color are never connected. The 1-tile $X_i$ is **incident** (at $v$) to the block $b \in \pi(v)$ containing $i$. By Lemma 6.1, it is enough to define $\pi_w(v)$, then $\pi_b(v)$ will always be the complementary partition.
If \( v=p \) is a post-critical point the connection at \( p \) is *marked* in addition (see Definition 6.4). Recall that the *marked arc* of a geometric representation \( \mathcal{D}(\pi_w \cup \pi_b) \) (of the connection at the post-critical point \( p \), Definition 6.5) is assumed to *contain the origin*.

The connection illustrated in Figure 4 is given by \( \pi_w = \{\{0, 2, 6\}, \{4\}\} \) and \( \pi_b = \{\{1\}, \{3, 5\}, \{7\}\} \).

The marked arc is indicated by the dot.

When talking about 1-tiles \( X_j \) and cnc-partitions at the same time, it is always assumed without mention that the indices of the \( X_j \) are as above.

Let \( v \) be a 1-vertex, and \( n = \deg_v F \). Let \( X_0, \ldots, X_{2n-1} \) be the 1-tiles containing \( v \), labeled positively around \( v \) (white tiles have even index, black ones odd index as before). Every such 1-vertex \( v \) has arbitrarily small neighborhoods \( U = U(v) \), that are closed and homeomorphic to the closed disk \( \overline{D} \), such that there is a homeomorphism

\[
h = h_v : U \rightarrow \mathcal{D}
\]

that maps tiles to sectors (see (6.4)),

\[
h(X_j \cap U) = S_j
\]

for \( j = 0, \ldots, 2n-1 \). In particular, \( h(v) = 0 \). We require that the neighborhoods \( U(v) \) and \( U(v') \) have disjoint closures for distinct 1-vertices \( v \) and \( v' \). The reader should think of the neighborhood \( U \) as a “blowup” of the point \( v \).

**Definition 6.8.** (Geometric representation of a connection) Let a connection at \( v \) be given, with cnc-partition \( \pi_w \cup \pi_b \), geometrically represented by \( \mathcal{D}(\pi_w \cup \pi_b) \) as in Definition 6.5. Let \( h \) and \( U = U(v) \) be as above. A *geometric representation of the connection at \( v \)* is given by replacing \( U \) by \( h^{-1}(\mathcal{D}(\pi_w \cup \pi_b)) \).

More precisely, the white 1-tiles in \( U \), i.e. \( (X_0 \cup X_2 \cup \ldots X_{2n-2}) \cap U \), are replaced by \( h^{-1}(\mathcal{D}_w) \) (see Definition 6.5). Note that this set is colored white. Similarly we replace the black 1-tiles in \( U \), i.e. \( (X_1 \cup X_3 \cup \ldots X_{2n-1}) \cap U \), by \( h^{-1}(\mathcal{D}_b) \). This set is colored black.

Let \( \nu = p \) be a post-critical point and the connection at \( p \) be marked by the arc \( g_m \).

More precisely, in the geometric representation \( \mathcal{D}(\pi_w \cup \pi_b) \) of the connection \( \pi_w \cup \pi_b \) at \( p \), the marking corresponds to the arc \( g_m \subset \mathcal{D}(\pi_w \cup \pi_b) \). Since the marked arc was chosen to contain 0, it follows that in this case \( p \in h^{-1}(g_m) \), and thus the geometric representation of the marked arc contains \( p \). This is the purpose of the marking, namely to keep track of where in the geometric representation of the connection the post-critical point is located.
Definition 6.9. (Connection) A connection of 1-tiles is an assignment of a connection at every 1-vertex. Representing the connection at each 1-vertex geometrically as above gives a geometric representation of this connection of 1-tiles. Objects arising from a geometric representation will be denoted with an ε-subscript.

Assume a geometric representation of a connection of 1-tiles is given. From the construction it follows that each boundary component of some white or black component is a Jordan curve. Let \( X \) be a 1-tile with 1-vertices \( v_0, \ldots, v_{k-1} \). Then the geometric representation of \( X \) is
\[
X_\varepsilon := X \setminus \bigcup_{j=0}^{k-1} U(v_j),
\]
where the neighborhood \( U(v_j) \) of \( v_j \) is as in (6.5). Note that by construction two 1-tiles \( X_\varepsilon \) and \( Y_\varepsilon \) (of the same color) are connected at a 1-vertex \( v \) if and only if their geometric representations \( X_\varepsilon \) and \( Y_\varepsilon \) are connected in \( U(v) \). This means that \( X_\varepsilon \) and \( Y_\varepsilon \) can be joined by a path in \( U(v) \) that does not intersect any boundary of some white or black component.

6.3. The connection graph

Given a connection of 1-tiles, we construct the white and black connection graphs.

Definition 6.10. (Connection graph) The white connection graph is constructed as follows. For each white 1-tile \( X \) there is a vertex \( c(X) \) (thought of as the center of the 1-tile \( X \)). For each 1-vertex \( v \) and block \( b \in \pi_w(v) \) there is a vertex \( c(v,b) \). The vertex \( c(X) \) is connected to \( c(v,b) \) by an edge if and only if \( X \) is incident to \( b \) at \( v \).

The black connection graph is constructed in the same manner from black 1-tiles and their connections.

We will identify a 1-tile \( X \) with (the vertex of the white connection graph) \( c(X) \). For example we will say that two white 1-tiles \( X \) and \( Y \) are connected (given a connection of 1-tiles) if \( c(X) \) and \( c(Y) \) lie in the same component of the white connection graph.

Definition 6.11. (Cluster) A white (black) cluster \( K \) is one component of the white (black) connection graph. Using the previous identification, we say that \( K \) contains a 1-tile \( X \) (and write \( X \subset K \)), if \( c(X) \in K \). This means that we identify \( K \) with the union of 1-tiles “contained” in it. Similarly, a 1-edge \( E \) (resp. 1-vertex \( v \)) is said to be contained in \( K \) if \( E \subset X \subset K \) (resp. \( v \in X \subset K \)) for some 1-tile \( X \). Each 1-tile is contained in exactly one cluster (of the same color), each 1-edge is contained in exactly two clusters (one black and one white). A 1-vertex \( v \) may be contained in several clusters (in fact at most \( n+1 \), where \( n = \deg_F v \)).

Assume a geometric representation of the connection has been given. Let \( X \) be a 1-tile contained in the cluster \( K \). Then there is a unique component \( K_\varepsilon \) (of the same color as \( X \)) containing (the geometric representation) \( X_\varepsilon \). Recall that some 1-tile \( Y \) is connected
to $X$ at a 1-vertex $v$ if and only if they are connected at $v$ in a geometric representation of the connection. Thus one obtains inductively that any 1-tile $Z$ is contained in $K$ if and only if $Z_c \subseteq K_c$. Thus each white (black) cluster $K$ corresponds to one white (black) component $K_c$ (of a geometric representation of the connection) and vice versa. We call $K_c$ a geometric representation of the cluster $K$.

A cluster $K$ is a tree if the underlying component of the connection graph is a tree, i.e., contains no cycles. The white cluster $K$ is a spanning tree, if it is a tree and contains all white 1-tiles.

In the next section the connection of 1-tiles will be constructed such that the white 1-tiles form a spanning tree in “the right homotopy class”.

Remark. Assume that all white 1-tiles are connected at each 1-vertex. Of course we can extract a spanning tree (in the standard sense) from the resulting white connection graph. This spanning tree however will have only one vertex for each 1-vertex $v$. Thus not all spanning trees in the sense of the previous definition can be obtained in this way. See Corollary 6.20 for an inductive way to construct trees in the connection graph.

The first approximation of the Peano curve $\gamma^1$ will be constructed as “the outline” of the spanning tree. One should think of the construction as follows. A geometric representation of this (white) spanning tree will be a Jordan domain. The positively oriented boundary of this domain “is” the first approximation $\gamma^1$.

6.4. Succeeding edges

Let a connection of 1-tiles be given. Let $E$ be a 1-edge contained in the white 1-tile $X_i$, positively oriented (as boundary of $X_i$) with terminal point $v$.

Since 1-tiles are cyclically ordered around $v$, the 1-tiles that are connected at $v$ with $X_i$ are cyclically ordered as well.

Let $X_j$ be the cyclical successor (in positive order around $v$) of $X_i$ among 1-tiles connected to $X_i$ at $v$. If no other 1-tile is connected to $X_i$ at $v$, we let $X_j = X_i$.

Formally $i$ and $j$ are contained in the same block of $\pi_w$, and none of the numbers in $[i+1, j-1]$ are contained in this block.

Note that $X_j$ is a white 1-tile. Thus an oriented 1-edge $E' \subseteq X_j$ is positively oriented if and only if it is positively oriented as boundary of $X_j$.

Definition 6.12. (Successor) Let $v$ and $E$, as well as $X_i$ and $X_j$, be as above. The successor to $E$ (at $v$) is the positively oriented 1-edge $E' \subseteq X_j$ with initial point $v$. Note that each 1-edge $E'$ is the successor to exactly one 1-edge $E$. 
See Figure 4 for an illustration. For each 1-edge \( E \) with initial point \( v \) and terminal point \( w \), let \( E_\varepsilon := E \setminus ((U(v) \cup U(w)) \). Here \( U(v) \) and \( U(w) \) are the neighborhoods of \( v \) and \( w \) from (6.5). Recall from Lemma 6.3 how a cnc-partition was geometrically represented by dividing the disk by arcs \( g_m \). We call such an arc \( g_m \) positively oriented if it is positively oriented as boundary arc of a white set \( D_l \).

**Lemma 6.13.** (Equivalent formulations for succeeding edges) Consider white 1-tiles \( X_i \supset E \) and \( X_j \supset E' \), where \( E \) and \( E' \) are positively oriented 1-edges containing a 1-vertex \( v \). The following statements are equivalent:

1. \( E' \) is the successor to \( E \) at \( v \).
2. \( E' \) is succeeding \( E \) on \( \partial K_\varepsilon \), where \( K_\varepsilon \) is a geometric representation of the white cluster \( K \) containing \( E \). This means that, when \( \partial K_\varepsilon \) is positively oriented (as boundary of \( K_\varepsilon \)) there is no (geometric representation of a 1-edge \( \tilde{E} \)) \( \tilde{E} \subset \partial K_\varepsilon \) on the positively oriented arc from \( E_\varepsilon \) to \( E'_\varepsilon \).
3. Represent the connection at \( v \) geometrically as in Lemma 6.3. Using the notation from this lemma, there is a (positively oriented) arc \( g_m \) that connects the right endpoint of the arc \( a_i \subset S^1 \) to the left endpoint of the arc \( a_j \subset S^1 \).
4. There are adjacent blocks \( b \in \pi_w(v) \) and \( c \in \pi_b(v) \) such that \( i, j \in b \) and \( i + 1, j - 1 \in c \).

The proof is clear from the proof of Lemma 6.3.

**Corollary 6.14.** (Marked connection) A marking of a connection at a post-critical point \( p \) may be given

1. by marking an arc \( g_m \) from a geometric representation of the connection at \( p \);
2. or equivalently by marking a pair of succeeding 1-edges \( E \) and \( E' \) at \( p \);
3. or equivalently by marking a pair of adjacent blocks \( b \in \pi_w(p) \) and \( c \in \pi_b(p) \).

The precise correspondences (i.e., which marked arc corresponds to which marked pair of succeeding edges, corresponds to which marked pair of adjacent blocks) is given by Lemma 6.13.

The 1-tiles containing the successors \( E \) and \( E' \) are connected at \( v \). If on the other hand the 1-tiles \( X \) and \( Y \) are connected at \( v \), we can find a chain of succeeding 1-edges.

**Lemma 6.15.** Two 1-tiles \( X \) and \( Y \) (of the same color) are connected at the 1-vertex \( v \) if and only if there is a chain

\[
X = X_1, E_1, E'_2, X_2, ..., X_{m-1}, E_{m-1}, E'_m, X_m = Y.
\]

Here \( X_j \supset v \) are 1-tiles of the same color as \( X \) and \( Y \), \( E_j, E'_j \subset X_j \) are 1-edges and \( E'_{j+1} \) succeeds \( E_j \) at \( v \).
Note that, above, the labeling of the white 1-tiles is not the one used in the definition of the connection at \( v \) (there are some white 1-tiles with odd index).

Proof. If the 1-tiles in the lemma are white, the cyclical order of 1-tiles connected to \( X \) at \( v \) from \( X=X_1 \) to \( Y=X_m \) is given by \( X_1,...,X_m \). If the 1-tiles are black this gives the anti-cyclical order. Clearly going (anti-)cyclically around \( v \) among 1-tiles connected to \( X \) gives all such 1-tiles.

6.5. Adding clusters

The spanning tree will be built successively by adding more “secondary clusters” to a “main cluster”.

Let the connection at a 1-vertex \( v \) be given by the cnc-partition \( \pi_w \cup \pi_b \) (of \( [2n] \)), where \( n=\deg_F(v) \), and \( K \) and \( K' \) be two white clusters containing \( v \). Let \( b\in \pi_w \) be a block with indices of 1-tiles in \( K \) (\( X_j \subset K \) if \( j \in b \)), and \( b'\in \pi_w \) be a block with indices of 1-tiles in \( K' \). We add the cluster \( K' \) to \( K \) at \( v \) by replacing \( b \) and \( b' \) in \( \pi_w \) by \( \tilde{b}=b\cup b' \). The resulting partition \( \tilde{\pi}_w \) may however not be non-crossing anymore.

**Lemma 6.16.** (Adding clusters) The partition \( \tilde{\pi}_w \) is non-crossing if and only if there is a block \( c\in \pi_b \) that is adjacent to both \( b \) and \( b' \) (see Lemma 6.2).

In this case, let \( \tilde{K} \) be the cluster in the new connection graph that contains \( K \) and \( K' \). If \( K \) and \( K' \) are trees, then \( \tilde{K} \) is a tree as well.

The situation is illustrated in Figure 5.

Proof. We show the equivalence first.

\((\Leftarrow)\) Assume that \( \tilde{\pi}_w \) is crossing. Then there is a block \( \tilde{b}\in \pi_w \) such that there are
\( a, a' \in b, d \in b \) and \( d' \in b' \) satisfying
\[
a < d < a' < d'.
\]

This means that \( b \) and \( b' \) have to be contained in different components of \([2n] \setminus \{a, a'\}\). Thus every block \( c \in \pi_b \) adjacent to \( b \) has to be in a different component of \([2n] \setminus \{a, a'\}\) than every block \( c' \in \pi_b \) adjacent to \( b' \). Thus there is no block \( c \in \pi_b \) adjacent to both \( b \) and \( b' \).

\((\Rightarrow)\) Assume now that there is no \( c \in \pi_b \) adjacent to both \( b \) and \( b' \). Let
\[
b = \{b_1, \ldots, b_N\} \quad \text{and} \quad b' = \{b'_1, \ldots, b'_M\},
\]
where \( b_1 < \cdots < b_N \) and \( b'_1 < \cdots < b'_M \). Since \( \pi_w \) is non-crossing, \( b \) and \( b' \) are in disjoint intervals, meaning that we may assume that, for some \( j \),
\[
b_j < b'_1 < b'_M < b_{j+1}.
\]
Since \( \pi_b \) is complementary to \( \pi_w \), there are blocks \( c, c' \in \pi_b \) such that
\[
b_j + 1, b_{j+1} - 1 \in c \quad \text{and} \quad b'_1 - 1, b'_M + 1 \in c',
\]
by Lemma 6.2. The blocks \( c \) and \( c' \) are distinct by assumption. Let \( c'_1 := \min\{c'_j \in c'\} \) and \( c'_2 := \max\{c'_j \in c'\} \). The numbers \( c'_1 - 1 \) and \( c'_2 + 1 \) are in the same block \( \tilde{b} \in \pi_w \) (since \( \pi_w \) and \( \pi_b \) are complementary). Thus we have the following ordering:
\[
\begin{align*}
&b_j < b_j + 1 < c'_1 \quad \text{in} \quad b \quad \text{in} \quad c \quad \text{in} \quad b', \quad b'_1 < b'_M < c'_1 < c'_2 < c'_2 + 1 < b_{j+1} - 1 < b_{j+1}.
\end{align*}
\]
Clearly \( b \cup b' \) and \( \tilde{b} \) are crossing, which finishes this implication.

We now show the second statement. Recall that in the white connection graph the block \( b \in \pi_w \) is represented by a vertex \( c(v, b) \) and \( b' \in \pi_w \) is represented by a (different) vertex \( c(v, b') \). The new white connection graph (where the connection at \( v \) is given by \( \tilde{\pi}_w \)) is obtained by identifying \( c(v, b) \) and \( c(v, b') \); this yields the vertex \( c(v, \tilde{b}) \). Then \( \tilde{K} \) is the component (of the new white connection graph) containing \( c(v, \tilde{b}) \). If \( K \) and \( K' \) are trees, then clearly \( \tilde{K} \) is a tree as well.

Assume that \( c \) is adjacent to both \( b \) and \( b' \), i.e., that we can add \( K' \) to \( K \) at \( v \) in this fashion. Let the notation be as in the previous proof, i.e., \( b = \{b_1, \ldots, b_N\} \) and \( b' = \{b'_1, \ldots, b'_M\} \), where
\[
b_1 < \cdots < b_N, \quad b'_1 < \cdots < b'_M \quad \text{and} \quad b_j < b'_1 < b'_M < b_{j+1}.
\]
(6.6)
Then the complementary partition \( \tilde{\pi}_b \) to \( \tilde{\pi}_w \) is given by replacing \( c \in \pi_b \) by the two blocks

\[
\tilde{c} = c \cap [b_j, b'_j] \quad \text{and} \quad \tilde{c}' = c \cap [b'_M, b_{j+1}].
\]

(6.7)

These two blocks are both adjacent to \( \tilde{b} = b \cup b' \in \tilde{\pi}_b \).

If we add a cluster \( K' \) to a cluster \( K \) as above at a post-critical point \( p \), we need to specify the marking (see Definition 6.4) of the new connection at \( p \).

**Definition 6.17. (Marking of new connection)** Let \( \pi_w \cup \pi_b \) be a marked cnc-partition, i.e., a connection at a post-critical point \( p \). Then the marking of the cnc-partition \( \tilde{\pi}_w \cup \tilde{\pi}_b \) from the previous lemma is given as follows (notation is as before). Let the marked adjacent blocks in \( \pi_w \cup \pi_b \) be

- \( b, c \) or \( b', c \); then we can pick \( \tilde{b}, \tilde{c} \) as the marked adjacent blocks in \( \tilde{\pi}_w \cup \tilde{\pi}_b \);
- \( d, c \), where \( d \in \pi_c \setminus \{b, b'\} \); then \( \tilde{b}, c \) is adjacent to either \( \tilde{c} \) or \( \tilde{c}' \), which are the marked adjacent blocks in \( \tilde{\pi}_w \cup \tilde{\pi}_b \);
- \( b, e \) or \( b', e \), where \( e \in \pi_b \setminus \{c\} \); then \( \tilde{b}, e \) are the marked adjacent blocks in \( \tilde{\pi}_w \cup \tilde{\pi}_b \);
- \( d, e \), where \( d \in \pi_c \setminus \{b, b'\} \) and \( e \in \pi_b \setminus \{c\} \); then \( d, e \) are the marked adjacent blocks in \( \tilde{\pi}_w \cup \tilde{\pi}_b \).

**Lemma 6.18.** Assume that a white cluster \( K' \) can be added to a white cluster \( K \) at a 1-vertex \( v \) as in Lemma 6.16 to form a cluster \( \tilde{K} \). Then there exist (uniquely) succeeding 1-edges at \( v \)

\[
E, E' \subset K \quad \text{as well as} \quad D, D' \subset K'
\]

such that

\[
E \text{ and } D' \quad \text{as well as} \quad D \text{ and } E'
\]

are succeeding in \( \tilde{K} \).

The situation is again illustrated in Figure 5.

**Proof.** Consider the blocks \( b, b' \in \pi_w(v) \) which are both adjacent to the block \( c \in \pi_b(v) \) as in Lemma 6.16 (here \( b \) contains indices of 1-tiles in \( K \), while \( b' \) contains indices of 1-tiles in \( K' \)). The succeeding 1-edges \( E, E' \subset K \) and \( D, D' \subset K' \) are the ones corresponding to these adjacencies according to Lemma 6.13. Using the notation from (6.6), we obtain that these 1-edges are contained in the following (white) 1-tiles. In \( K \) and \( K' \), we have

\[
E \subset X_{b_j}, \quad E' \subset X_{b_{j+1}}, \quad D \subset X_{b'_M} \quad \text{and} \quad D' \subset X_{b'_j}.
\]

Recall the description of the blocks \( \tilde{c}, \tilde{c}' \in \tilde{\pi}_b \) from (6.7). They are both adjacent to \( \tilde{b} = b \cup b' \in \tilde{\pi}_w \). Then \( b_j + 1, b'_j - 1 \in c \) and \( b'_M + 1, b_{j+1} - 1 \in c' \). Therefore (using Lemma 6.13 again) we obtain that \( E, D' \) and \( D, E' \) are succeeding in \( \tilde{K} \). \( \square \)
We will often be in the following specific situation. Consider a white cluster $K$. Assume that the only white 1-tiles that are possibly connected at a 1-vertex $v$ are in $K$. Put differently, this means that all distinct white 1-tiles $Y,Y' \ni v$ not in $K$ are disconnected at $v$. Let $X_i \ni v$ be a white 1-tile not contained in $K$. The following lemma means that we can add $X_i$, or the cluster containing $X_i$, to $K$ at $v$.

**Lemma 6.19.** In the situation as above, there is a block $b \in \pi_w$ containing indices of white 1-tiles in $K$ ($X_j \subset K$ if $j \in b$), such that the partition $\tilde{\pi}_w$, obtained by replacing $b, \{i\} \in \pi_w$ by $\tilde{b} = b \cup \{i\}$, is non-crossing.

Furthermore, if $K$ and the cluster containing $X_i$ are trees, the resulting cluster $\tilde{K}$ ($\supset K \cup X_i$) is a tree as well.

**Proof.** Consider the graph $\Gamma$ representing $\pi_w \cup \pi_b$ from Lemma 6.2 (this is neither the white connection graph nor the graph $\bigcup E^1$).

Let $X_j \ni v$ be a white 1-tile not contained in $K$. Since $X_j$ is not connected to any other 1-tile at $v$ the singleton $\{j\}$ is a block of $\pi_w$. This block is adjacent to a single block (in $\pi_b$), and thus $\{j\}$ is a leaf of $\Gamma$ (incident to a single edge).

Consider the block $c \in \pi_b$ adjacent to $\{i\} \in \pi_w$. Since $\Gamma$ is connected, $c$ has to be connected to a block $b \in \pi_w$ containing indices corresponding to 1-tiles in $K$. This means that $b$ and $c$ are adjacent blocks. The result now follows from Lemma 6.16. \hfill \Box

We record the following corollary (see also Lemma 2.2).

**Corollary 6.20.** (Trees in connection graphs) A (cluster that is a) tree in the white (black) connection graph may be constructed inductively by adding one 1-tile to a cluster at a time. Every tree in the white (black) connection graph (in a cluster) is obtained in such a way.

### 6.6. Boundary circuits

The first approximation of the Peano curve $\gamma^1$ will be given as the boundary circuit of a (cluster that is a) spanning tree (in the white connection graph).

**Definition 6.21.** (Boundary circuit of a cluster) Consider a cluster $K$. A boundary circuit $\mathcal{E}$ of $K$ is a circuit $E_0, \ldots, E_{M-1}$ of positively oriented 1-edges in $K$ such that $E_{j+1}$ is the successor of $E_j$ for each $j$ (indices are taken mod $M$, in particular $E_0$ succeeds $E_{M-1}$); furthermore, no 1-edge appears twice in $\mathcal{E}$.

Recall that every 1-edge has exactly one successor and one predecessor. Thus it is clear that starting from any 1-edge $E_0 \subset K$ and following succeeding 1-edges will yield a boundary circuit.
We note the following, which is an immediate consequence of Lemma 6.13 and Corollary 6.14, see also the discussion after Definition 6.8.

**Lemma 6.22.** \((K_\varepsilon \text{ contains } p)\) Let \(K\) be a cluster and \(p\) be a post-critical point. A boundary circuit of \(K\) contains the marked succeeding 1-edges at \(p\) if and only if \(p \in K_\varepsilon\) for any geometric representation \(K_\varepsilon\) of \(K\).

**Lemma 6.23.** Consider a cluster \(K\). The following are equivalent:
1. the cluster \(K\) is a tree;
2. \(K\) has only a single boundary circuit;
3. each geometric representation \(K_\varepsilon\) of \(K\) is a Jordan domain.

In this case the single boundary circuit \(E\) of \(K\) is an Eulerian circuit in \(K\). This means that each of the \(km\) 1-edges in \(K\) appears exactly once in \(E\). Here \(m\) is the number of 1-tiles in \(K\) \(k=\# \text{ post-} \#0\)-edges).

**Proof.** Assume without loss of generality that the cluster \(K\) is white.

\((1) \Rightarrow (2)\) Recall from Corollary 6.20 that every tree can be obtained inductively by adding more 1-tiles to one cluster in the connection graph. Start with a white tile graph that is totally disconnected, meaning that no two white 1-tiles are connected (at any 1-vertex). Consider one white 1-tile \(X_0\) and a 1-edge \(E_0 \subset X_0\). Clearly \(E_0\) is contained in an Eulerian circuit in \(X_0\) of length \(k\) (containing all 1-edges in \(\partial X_0\)). So the statement is true for the cluster \(K_0=X_0\) (consisting of a single 1-tile).

Let the white connection graph be given such that all clusters except one cluster \(K_{j-1}\) contain a single 1-tile, i.e., as in Lemma 6.19. Assume that \(E_0,\ldots, E_{kj-1}\) is an Eulerian circuit in \(K_{j-1}\) containing all 1-edges in \(K_{j-1}\), where \(j\) is the number of 1-tiles in \(K_{j-1}\).

Add a 1-tile \(X\) to \(K_{j-1}\) at a 1-vertex \(v \in K_{j-1}\) as in Lemma 6.19 to form a new component \(K_j\). The above procedure then yields as a path

\[E_0, \ldots, E_i, E_i^X, \ldots, E_{k_1}^X, E_{i+1}, \ldots, E_{kj-1},\]

see Lemma 6.18. Here \(E_i^X, \ldots, E_{k_1}^X\) are the 1-edges in \(X\), positively oriented, starting at \(v\).

This is an Eulerian circuit in \(K_j\). The construction ends when \(K=K_j\). Since the constructed circuit contains all 1-edges in \(K\), there is only a single boundary circuit.

\((2) \Rightarrow (3)\) Consider a neighborhood \(U\) of a 1-vertex \(v \in K\) as in Definition 6.8. The boundary of \(K_\varepsilon\) is constructed from boundary circuits by replacing \(E_j\) and \(E_{j+1} \cap U\) by \(h^{-1}(g_m)\). Thus \(\partial K_\varepsilon\) is a single Jordan curve.

\((3) \Rightarrow (1)\) Assume that \(K\) is not a tree. Then there exists a circuit in \(K\). This means there are 1-tiles \(X_0,\ldots,X_{N-1}\) in \(K\) such that \(X_j\) is connected to \(X_{j+1}\) at a 1-vertex \(v_j\)
invariant Peano curves of expanding Thurston maps

(indices mod $N$), where all 1-vertices $v_j$ are distinct. Then in the interior of any geometric representation $K_\varepsilon$ we can find a Jordan curve following this circuit (connecting $X_{0,\varepsilon}$ to $X_{1,\varepsilon}$ at $v_{0,\varepsilon}$ and so on). This Jordan curve divides $K_\varepsilon$ into two components. Note that both components contain boundary of $K_\varepsilon$, namely the (geometric representations of the) two arcs on $\partial X_j$ between $v_{j-1}$ and $v_j$ lie in different components. Thus $K_\varepsilon$ is not a Jordan domain.

We record the following, which is an easy corollary.

**Lemma 6.24.** (Boundary circuit of added trees) Consider trees $K$ and $K'$ with boundary circuits $E = E_0, ..., E_{N-1}$ and $E' = D_0, ..., D_{M-1}$. Assume that we can add them at a 1-vertex $v$ as in §6.5 to form a tree $\tilde{K}$. Then the boundary circuit $\tilde{E}$ of $\tilde{K}$ is

$$E_0, ..., E_i, D_{i+1}, ..., D_{M-1}, D_0, ..., D_j, E_{i+1}, ..., E_{N-1}.$$ 

**Proof.** This is clear from Lemma 6.18, where $E_i, E_{i+1} \subset K$ and $D_j, D_{j+1} \subset K'$ are the succeeding 1-edges associated with adding $K$ to $K'$.

We next show that adding a tree $K'$ which “does not contain a post-critical point” to another tree $K$ does not change the “homotopy type” of $\partial K_\varepsilon$.

**Definition 6.25.** (Trivial tree) A cluster $K'$ that is a tree is called trivial if a (and thus any) geometric representation $K'_\varepsilon$ does not contain a post-critical point. Equivalently the boundary circuit of $K'$ does not contain the marked successors $E = E(p)$ and $E' = E'(p)$ at $p$ for any post-critical point $p$ (see Corollary 6.14).

**Lemma 6.26.** (Adding a trivial tree does not change homotopy type) Consider a cluster $K$ that is a tree, and a trivial tree $K'$ as above. Assume that it is possible to add $K'$ to $K$ at some 1-vertex $v$ as in Lemma 6.16, to obtain the tree $\tilde{K}$.

Then, if $\partial K_\varepsilon$ is isotopic to a Jordan curve $C$ rel. post, we have that $\partial \tilde{K}_\varepsilon$ is isotopic to $C$ rel. post as well (for any geometric representations $K_\varepsilon$ and $\tilde{K}_\varepsilon$ of $K$ and $\tilde{K}$).

**Proof.** Let $U = U(v)$ be as in Definition 6.8. We consider a neighborhood $V$ of “$K'_\varepsilon \subset \tilde{K}_\varepsilon$”. More precisely, $V$ satisfies the following:

- $V$ is a Jordan domain;
- $V$ contains no post-critical point;
- $V$ is a neighborhood of $K'_\varepsilon \setminus U$;
- $\partial V$ intersects $\partial \tilde{K}_\varepsilon$ exactly twice, where $\partial V \cap \partial \tilde{K}_\varepsilon = \{w_1, w_2\} \subset U$.

The arc $\tilde{K}_\varepsilon \setminus \{w_1, w_2\}$ contained in $V$ is now deformed to one contained in $U$ by an isotopy rel. $\partial V$ as in Theorem 5.1. This isotopy deforms $\tilde{K}_\varepsilon$ to $K_\varepsilon$. 

\[\square\]
7. Construction of $H^0$

The 0-th pseudo-isotopy $H^0$ as required in §3 is constructed here, and thus the first approximation $\gamma^1$ of the Peano curve.

Consider two oriented Jordan curves $\mathcal{C}, \mathcal{C}' \subset S^2$. We say that $\mathcal{C}$ and $\mathcal{C}'$ are orientation-preserving isotopic rel. $A$ if there is an isotopy $H: S^2 \times [0, 1] \to S^2$ rel. $A$, with $H_0 = \text{id}_{S^2}$, such that $H_1$ maps $\mathcal{C}$ orientation preserving to $\mathcal{C}'$.

We construct a connection of 1-tiles with the following properties.

Definition 7.1. (Properties of connections)  
(C1) The associated white connection graph (§6.3) is a spanning tree $K$.  
(C2) The Jordan curve $\partial K_\varepsilon$ is orientation-preserving isotopic to $\mathcal{C} = \gamma^0$ rel. post, where $K_\varepsilon$ is a geometric representation of $K$, see Lemma 6.23.

Here $\partial K_\varepsilon$ is positively oriented as boundary of $K_\varepsilon$, recall that $\mathcal{C}$ is positively oriented as boundary of the white 0-tile $X_w^0$.

Lemma 7.2. A connection of 1-tiles satisfies properties (C1) and (C2) if and only if there exists a pseudo-isotopy $H^0$ as in Definition 3.2.

Proof. $(\Rightarrow)$ Concatenate an isotopy $\tilde{H}$ rel. post that deforms $\mathcal{C}$ to $\partial K_\varepsilon$ (orientation preserving) with a pseudo-isotopy rel. post that deforms $\partial K_\varepsilon$ in a neighborhood $U(v)$ (as in (6.5)) of each 1-vertex as in Lemma 6.6. This yields a pseudo-isotopy rel. post that clearly satisfies $(H^0 1)–(H^0 4)$. Since $\tilde{H}_1$ maps $\mathcal{C}$ orientation preserving to $\partial K_\varepsilon$, it follows that every 1-edge in the first approximation $\gamma^1$ (constructed via $H^0$ as in §3.3) is positively oriented. It follows from Lemma 3.11 that $(H^0 5)$ is satisfied.

$(\Leftarrow)$ Let $\gamma^1 = H^0_1(\gamma^0)$ be the Eulerian circuit constructed from $H^0$ as in §3.3. By Lemma 6.15 we can reconstruct the connection at each 1-vertex from $\gamma^1$. It is a cnc-partition by Lemma 6.3. Since $\gamma^1$ contains all 1-edges, all white 1-tiles are connected. Furthermore $\gamma^1_\varepsilon := H^0_{1-\varepsilon}(\gamma^0)$ is a Jordan curve, and thus it follows from Lemma 6.23 that the white connection graph is a spanning tree, i.e., (C1). Finally $\gamma^1_\varepsilon$ is clearly isotopic to $\gamma^0$ rel. post. From $(H^0 5)$ and Lemma 3.11 it follows that the orientation on $\gamma^1_\varepsilon$ induced by $\mathcal{C}$ and $H^0_{1-\varepsilon}$ agrees with the orientation of $\gamma^1_\varepsilon$ as boundary of (a geometric representation of the white spanning tree) $K_\varepsilon$. Thus (C2) holds.

Let us note the following immediate consequence.

Theorem 7.3. Let $F: S^2 \to S^2$ be an expanding Thurston map. The following two equivalent conditions are sufficient for the existence of an (onto) invariant Peano curve $\gamma: S^1 \to S^2$ as in Theorem 1.1:  
(1) There is a Jordan curve $\mathcal{C} \supset \text{post}$ and a pseudo-isotopy $H^0$ in Definition 3.2.
(2) There is a Jordan curve $C \supset \text{post}$ and a connection of 1-tiles satisfying the properties from Definition 7.1.

In [Me1] it will be shown that the same conditions are sufficient to ensure that $F$ arises as a mating. Furthermore, the polynomials $p_1$ and $p_2$ into which $F$ unmates, may then be obtained by an explicit algorithm. More precisely, the critical portraits of $p_1$ and $p_2$ may be obtained from the vector $l$ considered in §4.1, see [Me2].

The proof of Theorem 1.1 will be finished by constructing the white connection as in Definition 7.1.

Let us first note the following, which is an immediate consequence of the proof of the previous lemma. Assume that a connection of 1-tiles satisfying (C1) and (C2) is given. Let $H^0$ be a corresponding pseudo-isotopy from Lemma 7.2.

**Lemma 7.4.** The first approximation $\gamma_1$ (viewed as an Eulerian circuit) constructed from $H^0$ as in §3.3 is equal to the boundary circuit of the (white) spanning tree $K$ (see Lemma 6.23).

The main work in constructing the connection as desired lies in ensuring property (C2).

The starting point is to take a sufficiently high iterate $F = f^n$ such that there is an $F$-invariant Jordan curve $C \supset \text{post}$ and the 1-tiles defined in terms of $(F, C)$ (i.e., closures of components of $S^2 \setminus F^{-1}(C)$) are sufficiently small. We require two separate conditions, since they are needed in distinct parts of the construction; they could be expressed as a single one. In fact, the second condition is only given later, when the suitable description becomes available.

**Lemma 7.5.** For each sufficiently high $n \in \mathbb{N}$ there is a Jordan curve $C$ with $\text{post} \subset C$ satisfying the following condition:

- $C$ is invariant for the iterate $F = f^n$. This means that $F(C) \subset C$.

The 1-tiles for $(F, C)$ satisfy the following conditions:

- There is no 1-tile $X$ that joins opposite sides of $C$. This means that no 1-tile $X$ meets disjoint 0-edges in the case $\# \text{post} \geq 4$, and no 1-tile $X$ intersects all three 0-edges in the case $\# \text{post} = 3$.

- The 1-tiles do not form a link in the sense of Definition 7.12.

This is essentially [BM, Theorem 14.2], see also [CFP3]. A proof of this lemma is given in §7.3, here we show how the arguments in [BM] are slightly adjusted to obtain the statement in the above form.

The iterate $F = f^n$ as well as the $F$-invariant Jordan curve $C$ as above will be fixed from now on, tiles are defined in terms of $(F, C)$. 
Let us first give a slightly incomplete outline of the construction. Recall that $X_0^w$ and $X_0^b$ are the white and black 0-tiles, respectively; they are both bounded by the invariant curve $C$. We consider a spanning tree of white 1-tiles in $X_0^w$. Then we consider a spanning tree of black 1-tiles in $X_0^b$, the complementary white 1-tiles in $X_0^b$ form ("homotopically") trivial trees in the sense of Definition 6.25. These (white) trivial trees (in $X_0^b$) are then attached to the white spanning tree in $X_0^w$.

This construction has to be slightly adjusted for the following reason: the white 1-tiles in $X_0^w$ (as well as the black 1-tiles in $X_0^b$) need not be connected. So there are no spanning trees as described before.

7.1. Decomposing $X_0^w$

Here we decompose the white 0-tile $X_0^w$ into white trees.

Consider the white 1-tiles in $X_0^w$. We assume in the next lemma that they are all connected at all 1-vertices $v$ in the interior of $X_0^w$, and disconnected at all 1-vertices on $C$. The resulting white connection graph may not be connected.

**Lemma 7.6.** The white connection graph in $X_0^w$ as above has exactly one (white) cluster that intersects all sides (0-edges).

**Proof.** Let $K$ be a (white) cluster in $X_0^w$ as above. Consider one component $B$ (in the standard topological sense) of $X_0^w \setminus K$. We call the set $a:= \partial B \cap K$ a boundary arc of $K$.

**Claim 1.** Every boundary arc $a$ as above is contained in a single black 1-tile.

Clearly $a$ is a union of 1-edges. Either $a$ starts and ends at two distinct 1-vertices $v, w \in C$, or $a$ is a closed curve. Let $E, E' \ni v$ be two 1-edges contained in $a$ which are consecutive in $\partial B$, where $v \notin C$ is a 1-vertex. Note that by construction all white 1-tiles $X_j \ni v$ are connected at $v$. Thus $E$ and $E'$ are contained in the same black 1-tile. The claim follows.

Assume now that $a$ is a closed curve. Then $a$ is a Jordan curve in the boundary of a single black 1-tile. Thus the corresponding component $B$ is the interior of a single black 1-tile. Hence $a$ does not separate $K$ from any other distinct white cluster $K'$ in $X_0^w$.

We call a black 1-tile $Y \subset X_0^w$ non-trivial if $Y \cap C$ contains at least two 1-vertices. A **complementary component** of $Y$ is the closure of a component $X_0^w \setminus Y$.

**Claim 2.** Let $X, X' \subset X_0^w$ be two distinct white 1-tiles. Then $X$ and $X'$ are contained in distinct white clusters $K, K' \subset X_0^w$ if and only if there is a black 1-tile $Y \subset X_0^w$ such that $X$ and $X'$ are contained in complementary components of $Y$. 
The implication \((\Leftarrow)\) is clear. To see the other implication we note that if \(X'\) is contained in a cluster distinct from the cluster \(K \supset X\), then \(X'\) has to be contained in the closure of one component of \(X_0^w \setminus K\). Such a component is separated from \(K\) by a boundary arc \(a\). However, if \(a\) does not contain two 1-vertices \(v, w \in C\), this component is a single black 1-tile, meaning that it does not contain \(X'\). Otherwise \(X'\) is separated from \(X\) by the black 1-tile \(Y\) containing \(a\), proving the claim.

Recall that we assumed that no 1-tile joins opposite sides of \(C\) (see Lemma 7.5). Thus for every non-trivial black 1-tile \(Y\) there is a complementary component of \(Y\), denoted by \(K_Y\), that intersects all 0-edges.

We now define \(K := \bigcap_Y K_Y\), where the intersection is taken over all non-trivial black 1-tiles \(Y \subset X_0^w\). Since two non-trivial black 1-tiles \(Y, Y' \subset X_0^w\) do not cross, it follows that \(K\) intersects all 0-edges.

By Claim 2 it follows that all white 1-tiles contained in \(K\) are connected, i.e., belong to the same cluster denoted by \(K\).

Assume that \(K\) intersects a given 0-edge \(E^0\) in a 1-edge \(E\). This cannot happen if \(E\) is contained in a black 1-tile \(Y \subset X_0^w\), since \(Y\) would be non-trivial, and the corresponding set \(K_Y\) does not contain \(E\). Thus \(E\) is contained in a white 1-tile, which is in \(K\).

If \(K\) intersects \(E^0\) only in a 1-vertex \(v\), there is a boundary arc \(a \subset \partial K\) containing \(v\). Let \(Y \subset X_0^w\) be the corresponding non-trivial black 1-tile containing \(a\). Let \(E \subset a\) be the 1-edge containing \(v\). Since \(E\) is not in \(C\), the white 1-tile containing \(E\) is in \(K\).

This means that there is a white 1-tile in \(K\) that intersects \(E^0\).

In each white cluster in \(X_0^w\) define a spanning tree (see Definition 6.11). The spanning tree in the cluster from Lemma 7.6 is called the main tree \(K_M\), the spanning trees in the other clusters are called the secondary trees in \(X_0^w\). The connections at all 1-vertices \(v \in X_0^w \setminus C\) are thus defined, they will not be changed any more in the construction.

Let \(E\) be the boundary circuit of the main tree \(K_M\) (see Definition 6.21 and Lemma 6.23). Let \(v_0, \ldots, v_{N-1}\) be the 1-vertices on \(C\) that \(E\) visits (in this order). Note that a 1-vertex \(v\) may appear several times in this list.

Notation. Given points \(v, w \in C\), denote by \([v, w]\) (resp. \((v, w)\)), the closed (resp. open) positively oriented arc on \(C\) from \(v\) to \(w\). Note that \((v, v) = \emptyset\).

Lemma 7.7. The points \(\{v_i\}_{i=0}^{N-1}\) satisfy the following conditions (indices are taken mod \(N\) here):

1. Each (open) arc \((v_i, v_{i+1})\) contains no point \(v_i\). This means that the points \(\{v_i\}_{i=0}^{N-1}\) are positively oriented on \(C\).

2. The points \(v_i\) and \(v_{i+1}\) are not contained in disjoint 0-edges, in particular each 0-edge contains at least one point \(v_i\).
(3) For all $i$ there is a black 1-tile $Y \ni v_i, v_{i+1}$.

(4) Let $K$ be a secondary tree in $X^0_w$. Then there is an arc $[v_i, v_{i+1}]$ such that

$$K \cap C \subset [v_i, v_{i+1}].$$

Proof. (1) Let $K_{M, \varepsilon}$ be a geometric representation of $K_M$ as in Lemma 6.23 (3). The path $\gamma_i$ on $E$ between $v_i$ and $v_{i+1}$ is then represented by a Jordan arc $\gamma_{i, \varepsilon}$ with endpoints $v_{i, \varepsilon}$ and $v_{i+1, \varepsilon}$ such that $|v_i - v_{i, \varepsilon}|$ and $|v_{i+1} - v_{i+1, \varepsilon}|$ are arbitrarily small. Since all white 1-tiles are disconnected at every 1-vertex $v \in C$, we may assume that $v_{i, \varepsilon} \in C$ and $\gamma_{i, \varepsilon} \subset X^0_w$ for all $i$.

The arcs $\gamma_{i, \varepsilon}$ are non-crossing, and thus the points $\{v_{i, \varepsilon}\}_{i=0}^{N-1}$ are ordered cyclically or anti-cyclically on $C$. Hence the points $\{v_{i, \varepsilon}\}_{i=0}^{N-1}$ are ordered cyclically or anti-cyclically on $C$.

The winding number of $E$ around $x \notin E$ is 1 if and only if $x$ is in the interior of a white 1-tile of the main tree. This follows from an inductive argument as in Corollary 6.20.

Assume that the points $\{v_{i, \varepsilon}\}_{i=0}^{N-1}$ are ordered anti-cyclically on $C$. Let $C_i$ be the (positively oriented) arc on $C$ between $v_i$ and $v_{i+1}$. Then $\gamma_i + C_i$ has winding number 0 around any point $x$ in the interior of a 1-tile of the main tree. Thus $E + C$ has winding number 0 around such an $x$. This is a contradiction.

(3) Consider $v_i$ and $v_{i+1}$. Then

- either $v_i = v_{i+1}$ in which case the statement is trivial;
- or $[v_i, v_{i+1}]$ is a 1-edge, property (3) is then clear again;
- or $v_i$ and $v_{i+1}$ are the boundary points of a boundary arc $a$ of $K_M$, as in Claim 1 from the proof of Lemma 7.6. In this case there is a black 1-tile $Y \ni a$.

(2) This follows immediately from (3) and the assumption that no 1-tile intersects disjoint 0-edges. Furthermore $K_M$ intersects a 0-edge $E$ if and only if it intersects it in some 1-vertex. The set of all 1-vertices in which $K_M$ intersects $C$ is equal to the set $\{v_i\}_{i=0}^{N-1}$. Thus, since $K_M$ intersects each 0-edge, it follows that each 0-edge contains one point $v_i$.

(4) The reader is reminded of Claims 1 and 2 in the proof of Lemma 7.6. For every secondary component $K$ there is an arc $a$ contained in a (non-trivial) black 1-tile $Y$ such that $\text{int} \ K$ is in the component of $X^0_w \setminus a$ not intersecting all 0-edges. Let $v_i$ and $v_{i+1}$ be the endpoints of $a$ (see the discussion from (3)), then

$$K \cap C \subset [v_i, v_{i+1}].$$
7.2. Decomposing $X^0_b$

We now decompose the black 0-tile $X^0_b$. Consider the black 1-tiles in $X^0_b$. Construct clusters of black 1-tiles as before. Namely assume that all black 1-tiles are connected at each 1-vertex $v \in X^0_b \setminus C$. All (black and white) 1-tiles in $X^0_b$ are disconnected at each 1-vertex $v \in C$. Pick a spanning tree in each cluster (of black 1-tiles in $X^0_b$). This defines the connections at all 1-vertices $v \in X^0_b \setminus C$, they will not be changed anymore in the construction. As in Lemma 7.6, there is exactly one such tree (of black 1-tiles in $X^0_b$) that intersects all 0-edges.

Consider now the white 1-tiles in $X^0_b$. The connections at 1-vertices $v \in X^0_b \setminus C$ are already given (they are all disconnected at each 1-vertex $v \in C$).

**Lemma 7.8.** Every white cluster $K$ in $X^0_b$ as above

- is a tree;
- furthermore

$$K \cap C \subset [v, w],$$

where $v, w \in C$ are 1-vertices contained in a single white 1-tile.

**Proof.** Assume that $K$ is not a tree. Then $K$ has at least two distinct boundary circuits (see Lemma 6.23).

**Claim.** There is a (white) 1-tile $X \subset K$ and a 1-vertex $v \in X$ at which 1-edges $E, E' \subset X$, from distinct boundary circuits, intersect.

If the claim were not true we could partition $K$ into 1-tiles containing 1-edges from distinct boundary circuits. These partitions, and therefore $K$, would not be connected by Lemma 6.15.

Let $v, E$ and $E'$ be as in the claim. Note that $v \notin C$, since all 1-tiles are disconnected at $C$.

Consider the black 1-tiles $Y, Y' \subset X^0_b$ that contain $E$ and $E'$. Let $K_b, K'_b \subset X^0_b$ be the black clusters containing $Y$ and $Y'$, respectively. Since they are by assumption trees, they are distinct (again by Lemma 6.23).

On the other hand the (black) 1-tiles $Y$ and $Y'$ were connected at $v$, before spanning trees were picked. This means that they are in the same tree ($K_b = K'_b$), which is a contradiction.

The arguments from Lemmas 7.6 and 7.7 apply verbatim to $X^0_b$. Thus there is a unique black tree $K_{M,b} \subset X^0_b$ that intersects each 0-edge. Let $w_0, \ldots, w_N$ be the 1-vertices that the boundary circuit of $K_{M,b}$ visits (in this order); note that these points are ordered positively on $C$ (recall that 1-edges in a boundary circuit of a cluster were always positively oriented as boundary of the white 1-tiles they are contained in, regardless of the color of
the cluster). As in Lemma 7.7 one obtains that the endpoints \( w_i \) and \( w_{i+1} \) of each arc \([w_i, w_{i+1}]\) are contained in a single white 1-tile. Each set \( K \cap C \) is contained in one such arc \([w_i, w_{i+1}]\).

We call the (white) trees from the previous lemma the secondary trees in \( X_0^b \). Let us record the following immediate consequence of Lemmas 7.7 and 7.8.

**Lemma 7.9.** No secondary tree \((\text{in } X_0^w \text{ or } X_0^b)\) intersects disjoint 0-edges.

We will need to break up boundary circuits.

**Definition 7.10.** (Subpaths of boundary circuits) Let \( E \) be a boundary circuit and \( D, E \subset E \) be two 1-edges. Then \( \mathcal{E}(D, E) \) is the positively oriented subpath (of 1-edges) of \( E \) with initial 1-edge \( D \) and terminal 1-edge \( E \). Note that \( \mathcal{E}(E, E) = E \).

In the next lemma we consider a secondary tree \( K \subset X_0^b \) with boundary circuit \( E \). Consider two distinct 1-vertices \( v, w \in (E \cap C) \). Let \( E_v, E'_v \subset E \) and \( E_w, E'_w \subset E \) be succeeding 1-edges at \( v \) and \( w \).

Given \( x, y \in C \), in the following we write \([x, y]_b\) for the boundary arc on \( C = \partial X_0^b \) between \( x \) and \( y \) that is positively oriented with respect to \( X_0^b \) (and thus negatively oriented on \( C \)).

**Lemma 7.11.** The subpath \( \mathcal{E}(E'_w, E_v) \) does not intersect \([v, w]_b \setminus \{v, w\} \).

**Proof.** The situation is illustrated in Figure 6. Assume that the statement is false, i.e., that \( \mathcal{E}(E'_w, E_v) \) intersects \([v, w]_b \setminus \{v, w\}\) in a 1-vertex \( u \in \mathcal{C} \). Let \( E_u, E'_u \subset \mathcal{E}(E'_w, E_v) \) be the succeeding 1-vertices at \( u \). Then \( \text{int } K \) is divided into points bounded by (having winding number 1) \( \mathcal{E}(E'_u, E_u) \cup [u, w]_b \) and \( \mathcal{E}(E'_u, E_u) \cup [v, u]_b \).

Thus \( E_u \) and \( E'_u \) are contained in different white 1-tiles \( X, X' \subset K \). Hence \( X \) and \( X' \) are connected at \( u \). This contradicts the construction of \( K \), where no 1-tiles are connected at any 1-vertex in \( C \).

### 7.3. Connecting the trees

The secondary trees are attached to the main tree at the 1-vertices on \( C \).
Initially all white 1-tiles are disconnected at each 1-vertex $v \in C$. To use the results from §6.5 we want the connections at all 1-vertices $v \in C$ to be cnc-partitions. Thus we now assume that all black 1-tiles are connected at each 1-vertex $v \in C$, and hence the connections form cnc-partitions as desired.

We first add secondary trees to ensure that all points of post are contained in the main tree. Consider the main tree $K_M$ (in $X^0$) from §7.1. Let $v_0, \ldots, v_{N-1}$ be the 1-vertices on $C$ along the boundary circuit $E$ of $K_M$, see Lemma 7.7.

We first add secondary trees to ensure that all points of post are contained in the main tree. Consider the main tree $K_M$ (in $X^0$) from §7.1. Let $v_0, \ldots, v_{N-1}$ be the 1-vertices on $C$ along the boundary circuit $E$ of $K_M$, see Lemma 7.7.

Consider one (positively oriented) 0-edge $E_0$ with terminal point $p \in \text{post}$, let $v_i$ be the last of the 1-vertices as above on $E_0$. Then either $v_i = p$ or $v_i \notin \text{post}$.

- If $v_i = p$, let $E_j \subset E_0$ be the last 1-edge with terminal point $v_i$, and $E_{j+1} \subset E_0$ be the succeeding 1-edge. The connection at $p$ is now marked by $E_j$ and $E_{j+1}$, see Corollary 6.14.
- If $v_i \notin \text{post}$, consider the 1-edge $E = [v_i, w] \subset E_0$ succeeding $v_i$ in $C$. Let $K$ be the secondary cluster containing $E$. This means that $K$ contains the (unique) white 1-tile containing $E$. Add $K$ to the main tree $K_M$ at $v_i$. Note that no white 1-tile is connected at $v_i$, so this is possible by Lemma 6.19. We obtain a new main tree, still denoted by $K_M$.

- Repeat the above procedure till the main tree contains $p$.

The added secondary components will only intersect the 0-edges preceding and succeeding $E_0$. Then we want to use the same procedure on the other 0-edges. There is one problem however: we may encounter a 1-edge $E$ as above that belongs to a secondary component already added before (when the above procedure was applied to a different 0-edge $E'$). This may lead to a boundary circuit of $K_M$ in which the post-critical points are traversed not in the same order as in $C$, violating (C2).

To elaborate, let $E_1 = E_0$, $E_2$ and $E_3$ be the 0-edges succeeding $E_0$. Let $q$ be the terminal point of $E_2$, and $v_j$ be the last of the points $\{v_i\}_{i=0}^{N-1}$ on $E_2$. The described problem occurs if there is a secondary component $K$ containing a 1-edge in $[v_i, p] \subset E_1$ and a 1-edge in $[v_j, q] \subset E_2$. By Lemma 7.7 (3) and (4), as well as Lemma 7.8, this can only happen if there are white and black 1-tiles linked in a certain way, see Figure 7.

Definition 7.12. (Link) A link means that the following exist:

- a (black) 1-tile $X_1$ containing $v_i \in E_0$ and intersecting $E_2$;
- a (black) 1-tile $X_2$ containing $v_j \in E_0$ and intersecting $E_3$;
- a (white) 1-tile $Y$ intersecting $[v_i, p] \subset E_0$ and $[v_j, q] \subset E_0$.

Thus we have given the description of the last property in Lemma 7.5.

Proof of Lemma 7.5. We essentially recall the proof of [BM, Theorem 14.2], see also [BM, Theorem 14.3] and its proof.

More precisely, we break up each 0-edge into two 0-arcs and use the same arguments as in [BM] to show that there is an $f^n$-invariant curve $\tilde{C}$, such that no $n$-tile connects
disjoint 0-arcs.

Let \( k_0 \) be a fixed integer such that there are at least twice as many \( k_0 \)-vertices as post-critical points (recall that the number of \( n \)-vertices grows exponentially). Fix a Jordan curve \( \mathcal{C} \subset \mathbb{S}^2 \) such that \( \text{post} \subset \mathcal{C} \); additionally \( \mathcal{C} \) has the property that each arc on \( \mathcal{C} \) between two consecutive post-critical points \( p \) and \( q \) contains a \( k_0 \)-vertex distinct from \( p \) and \( q \). Let \( P \) be the set of all such \( k_0 \)-vertices and post-critical points. The points in \( P \) divide \( \mathcal{C} \) into 0-arcs. Each 0-edge on \( \mathcal{C} \) is divided into two 0-arcs.

Consider the \( n \)-tiles given in terms of \((f, \mathcal{C})\), where \( n \geq k_0 \). Since \( f \) is expanding, \( n \)-tiles get arbitrarily small, meaning that \( \max_{X \in X^n} \text{diam } X \to 0 \) as \( n \to \infty \). This implies, by [BM, Lemma 11.17], that there is an \( n_0 \geq k_0 \) such for all \( n \geq n_0 \) there is a Jordan curve \( \mathcal{C}' \subset f^{-n}(\mathcal{C}) \) isotopic to \( \mathcal{C} \) rel. \( P \) (and thus \( P \subset \mathcal{C}' \)). Furthermore, no \( n \)-tile joins opposite sides of \((\mathcal{C}', P)\). This means that there is no \( n \)-tile that intersects disjoint closed 0-arcs of \( \mathcal{C}' \).

Let \( H: \mathbb{S}^2 \times [0, 1] \to \mathbb{S}^2 \) be an isotopy rel. \( P \) that deforms \( \mathcal{C} \) to \( \mathcal{C}' \), i.e., \( H_1(\mathcal{C}) = \mathcal{C}' \). Then \( \hat{F} := H_1 \circ f^n \) is a Thurston map, such that \( \mathcal{C}' \) is \( \hat{F} \)-invariant, since

\[
\hat{F}(\mathcal{C}') = H_1(f^n(\mathcal{C}')) \subset H_1(\mathcal{C}) = \mathcal{C}'.
\]

The 1-tiles for \((\hat{F}, \mathcal{C}')\) are exactly the \( n \)-tiles for \((f, \mathcal{C})\). Since no 1-tile for \((\hat{F}, \mathcal{C}')\) joins opposite sides of \( \mathcal{C}' \), we can choose \( \hat{F} \) to be expanding, see [BM, Corollary 13.18]. Furthermore, no 1-tile for \((\hat{F}, \mathcal{C}')\) intersects disjoint 0-arcs of \( \mathcal{C}' \).

The map \( \hat{F} \) is Thurston equivalent to \( f^n \). Since they are both expanding, they are actually topologically conjugate, i.e., there is a homeomorphism \( h: \mathbb{S}^2 \to \mathbb{S}^2 \), such that \( h \circ \hat{F} \circ h^{-1} = f^n \) (see [BM, Theorem 11.4]). Let \( \hat{C} := h(\mathcal{C}') \). Note that \( \hat{C} \) is \( f^n \)-invariant,
since
\[ f^n(\tilde{C}) = h \circ \tilde{F} \circ h^{-1}(\tilde{C}) = h \circ \tilde{F}(C') \subset h(C') = \tilde{C}. \]

We call the images of 0-arcs on \( C' \) by \( h \) the 0-arcs of \( \tilde{C} \). The images of 1-tiles for \((\tilde{F}, C')\) by \( h \) are the \( n \)-tiles for \((f, \tilde{C})\). It follows that no \( n \)-tile (for \((f, \tilde{C})\)) intersects disjoint 0-arcs of \( \tilde{C} \). Recall that each 0-edge of \( \tilde{C} \) contains exactly two 0-arcs.

With this choice of \( F = f^n \) and \( \tilde{C} \), we will show that a link as in Definition 7.12 cannot occur. Let \( A_j^- \) and \( A_j^+ \) be the two 0-arcs in \( E^0_j \), where \( A_j^+ \) succeeds \( A_j^- \) in \( C \). Then the white 1-tile \( Y \) has to intersect \( E^0_j \) in \( \text{int} A_j^- \), while the black 1-tile \( X \) has to intersect \( E^0_j \) in \( \text{int} A_j^+ \). The claim follows.

Since we assumed that \( F = f^n \) and \( C \) were chosen to satisfy the properties from Lemma 7.5, there are no links. Thus the following holds. Let \( K \) be a secondary cluster added (to the main tree) when considering the 0-edge \( E^0 \), and let \( \tilde{K} \) be a secondary cluster added when considering a distinct 0-edge \( \tilde{E}^0 \).

**Corollary 7.13.** The secondary clusters \( K \) and \( \tilde{K} \), given as above, are distinct.

Thus we can apply the above procedure to each 0-edge. This yields the (new) main tree (still denoted by \( K_M \)). Note that \( K_M \supset \text{post} \) by construction. More precisely, \( K_M \) contains the marked succeeding 1-edges \( E(p) \) and \( E'(p) \) at each post-critical point \( p \). This means that \( K_{M, \varepsilon} \supset \text{post} \) (for any geometric representation \( K_{M, \varepsilon} \) of \( K_M \)), see Lemma 6.22.

### 7.4. The main tree is in the right homotopy class

Recall from Definition 7.10 how a boundary circuit \( \mathcal{E} \) was broken up into subpaths. Assume that \( \mathcal{E} \) contains the marked succeeding 1-edges \( E(p) \) and \( E'(p) \) at \( p \in \text{post} \), as well as the marked succeeding 1-edges \( E(q) \) and \( E'(q) \) at \( q \in \text{post} \). Then

\[ \mathcal{E}(p, q) := \mathcal{E}(E'(p), E(q)) \]

and, for any 1-edge \( E \subset \mathcal{E} \),

\[ \mathcal{E}(p, E) := \mathcal{E}(E'(p), E) \quad \text{and} \quad \mathcal{E}(E, q) := \mathcal{E}(E, E(q)). \]

Furthermore, if \( E \) and \( E' \) are succeeding in \( \mathcal{E} \), we define

\[ \mathcal{E}(E', E) = \emptyset. \]

We are now ready to finish the proof of Theorem 1.1. \( K_M \) is the main tree as constructed in §7.3.
Lemma 7.14. The main tree $K_M$ is in the right homotopy class, i.e., satisfies (C2).

Proof. Let $E$ be the boundary circuit of $K_M$. Consider a 0-edge $E^0$ with initial and terminal points $p, q \in \text{post}$; and the subpath $E(p, q) \subset E$ as defined above. We will prove the following statement.

Claim 1. $E(p, q)$ does not intersect any 0-edge disjoint with $E^0$.

The statement of the lemma follows quickly from this claim. Namely consider a geometric representation $K_{M,\epsilon}$ of $K_M$, where the neighborhoods $U(v)$ from (6.5) were chosen in such a way that $U(v) \cap \mathcal{C} = \emptyset$ whenever $v \notin \mathcal{C}$. It follows from Claim 1 that the (positively oriented) arc on $\partial K_{M,\epsilon}$ from $p$ to $q$ does not intersect 0-edges disjoint from $E^0$. Theorem 5.5 now finishes the proof.

To prove Claim 1 we go through the construction of $K_M$. Consider $K_{M,0}$, the main tree from §7.1 (before any secondary tree was added), with boundary circuit $E_0$. Let $w_0, w_1 \in E^0$ be the first and last 1-vertices on $E^0$ that $E_0$ visits; and $E_0, E'_0 \subset E_0$, as well as $E_1, E'_1 \subset E_0$, be the first and last succeeding 1-edges at $w_0$ and $w_1$. Consider $E_0(E'_0, E_1)$, and note that $E(E'_0, E_1) = E(E'_0, E_0) = \emptyset$ in the case when $E_0$ intersects $E^0$ only once. This subpath does not intersect any 0-edge disjoint from $E^0$ by Lemma 7.7 (in fact it may only intersect adjacent 0-edges if $w_0 = p$ or $w_1 = q$).

Note that $E_0(E'_0, E_1)$ is a subpath of $E(p, q)$, or $E(E'_0, E_1) = E_0(E'_0, E_1)$, which we call the middle subpath of $E(p, q)$. The remaining subpaths of $E(p, q)$ are given as follows. Let $D_0$ be the 1-edge preceding $E'_0$ in $E$ and $D_1$ be the 1-edge succeeding $E_1$ in $E$. Then the initial subpath of $E(p, q)$ is $E(p, D_0)$ (connecting $p$ to $E(E'_0, E_1)$), and the terminal subpath of $E(p, q)$ is $E(D'_1, q)$ (connecting $E(E'_0, E_1)$ to $q$). Note that the initial and/or the terminal subpath may be empty. We focus our attention for now on the terminal subpath.

Let $K_1, ..., K_m$ be the secondary trees that were added in §7.3 to “reach” the post-critical point $q$. The last secondary tree $K_m$ contains the post-critical point $q$ by construction.

Let $K_{M,j}$ be the main tree obtained when the secondary tree $K_j$ was added to $K_{M,j-1}$ at the 1-vertex $w_j \in E^0$. Let $E_j, E'_j \subset K_{M,j-1}$ and $D_j, D'_j \subset K_j$ be the succeeding 1-edges associated with adding $K_j$ to $K_{M,j-1}$ by Lemma 6.18. Note that by construction the 1-vertices of $K_{M,j}$ closest to $q$ on the 0-edge $E^0$ are contained in $K_j \subset K_{M,j}$. Hence $K_{j+1}$ is attached to $K_{M,j}$ at 1-edges contained in $K_j$.

Thus, if we denote by $E_j$ the boundary circuit of the secondary tree $K_j$, we have $D_j, D'_j, E_{j+1}, E'_{j+1} \subset E_j$, and $E_j$ consists of the two (non-empty) subpaths $E_j(D'_j, E_{j+1})$ and $E_j(E'_{j+1}, D_j)$ for $j = 1, ..., m - 1$. We break $E_m$ up into the (non-empty) subpaths $E_m(D'_m, q)$ and $E_m(q, D_m)$.
Lemma 6.18 implies that the terminal subpath $\mathcal{E}(D'_1, q)$ is given as the concatenation of (subpaths from the boundary circuits from the secondary trees $K_j$)

$$\mathcal{E}_1(D'_1, E_2), \mathcal{E}_2(D'_2, E_3), \ldots, \mathcal{E}_m(D'_m, q),$$

(7.1)

see Figure 8. It follows from Lemma 7.9 that $\mathcal{E}(D'_1, q)$ does not intersect any 0-edge disjoint from $E^0$.

It remains to show that the initial subpath does not intersect a 0-edge disjoint from $E^0$.

Instead of looking at the initial subpath of $\mathcal{E}(p, q)$, we consider the initial subpath of $\mathcal{E}(q, r)$. Here $r$ is the terminal point of the 0-edge $(E^0)'$ succeeding $E^0$. Let $E_N \subset \mathcal{E}_0$ be the first 1-edge intersecting $(E^0)'$ in a 1-vertex $w_N$. The initial subpath of $\mathcal{E}(q, r)$ is $\mathcal{E}(q, E_N)$; it is given as the concatenation of

$$\mathcal{E}_m(q, D_m), \mathcal{E}_{m-1}(E'_m, D_{m-1}), \ldots, \mathcal{E}_1(E'_1, D_1), \mathcal{E}_0(E'_1, E_N),$$

where $D_j$ and $E'_j$ are as above. These are the “complementary subpaths” to the ones in (7.1) (of the boundary circuits of the secondary trees $K_j$). See again Figure 8.
It remains to show that this path does not intersect a 0-edge disjoint from \((E^0)\). Clearly \(E_0(E_1, E_N)\) intersects \(C\) only at the endpoints, which are in \(E^0\) and \((E^0)\).

Recall that \(E_j(E_{j+1}, D_j) \subset K_j\), where \(K_j\) does not intersect disjoint 0-edges. Thus \(E_j(E_{j+1}, D_j)\) may only intersect \(E^0\), \((E^0)\), or the \(E^0\)-preceding 0-edge \(E_0\).

**Claim 2.** The subpath \(E_j(E_{j+1}, D_j)\) does not intersect \(E_0\).

This is clear if \(K_j \subset X_0\), since then \(K_j \cap C \subset [w_1, q] \cup [q, w_N]\) by Lemma 7.7 (4).

Assume now that \(K_j \subset X_0\). Let \(w\) be the initial point of \(E_j(E_{j+1}, D_j)\) and \(v\) be its terminal point. Note that, by construction, \(w \in E^0\) is closer to \(q\) on \(E^0\) than \(v \in E^0\). From Lemma 7.11 it follows that \(E_j(E_{j+1}, D_j) \subset [v, w] \subset E^0\setminus\{p\}\). Claim 2 follows.

The argument that the initial subpath \(E(p, D_0)\) does not intersect 0-edges disjoint from \(E^0\) is completely analogous. This completes the proof of Claim 1, and thus the proof of the lemma.

We finish the construction of the main tree, i.e., of the connection of 1-tiles by adding the remaining secondary trees to the main tree arbitrarily, to form the spanning tree \(K_M\).

The previous lemma, together with Lemma 6.26, implies that \(K_M\) satisfies properties (C1) and (C2). Thus there is a pseudo-isotopy \(H^0\) as required in Definition 3.2, by Lemma 7.2. This yields the invariant Peano curve by \S\,3 and \S\,4. The proof of Theorem 1.1 is thus finished.

### 8. Combinatorial construction of \(\gamma^n\)

The \((n+1)\)-th approximation \(\gamma^{n+1}\) of the invariant Peano curve \(\gamma\) was constructed as a deformation of \(\gamma^n\) by \(H^n\). Here \(H^n\) was the lift of the “initial pseudo-isotopy” \(H^0\) by \(F^n\). In this section we give an alternative way to construct \(\gamma^{n+1}\) from \(\gamma^n\), namely in a purely combinatorial fashion.

Recall from Lemma 7.4 that the first approximation \(\gamma^1\) may be obtained as the boundary circuit of the white spanning tree, defined via the connection of 1-tiles. Here we construct the connection of \(n\)-tiles (which will again satisfy (C1) and (C2)) in such a way that \(\gamma^n\) is the boundary circuit of the white tree of \(n\)-tiles. See Figure 2 for an illustration of the desired connections of \(n\)-tiles.

The connections of \(n\)-tiles could be constructed from the approximations \(\gamma^n\) (using Lemma 6.15). We do however take the opposite route here, namely we construct the connections inductively and show that their boundary circuits are the approximations as defined before.
8.1. Connection of \( n \)-tiles

We give the (inductive) description of the connection of \( n \)-tiles first, before showing that it has the desired properties.

Fix \( n \geq 1 \). Assume that the connection of \( n \)-tiles is given. This means that at each \( n \)-vertex \( v \) a \( \text{cnc} \)-partition \( \pi_0^n(v) \cup \pi_1^n(v) \) is defined; if \( v=p \in \text{post} \), it is marked (see Definition 6.7). The connection satisfies properties (C1) and (C2), and the (single) boundary circuit is equal to the \( n \)th approximation \( \gamma^n \) (viewed as an Eulerian circuit).

Consider now an \((n+1)\)-vertex \( v \). The connection of \((n+1)\)-tiles at \( v \) is defined as follows.

**Case 1.** \( v \) is not an \( n \)-vertex.

Note that this implies that \( v \) is not a critical point. Thus we can define the connection at \( v \) as the “pull-back” of the connection at \( F(v) \).

More precisely, let \( w:=F(v) \in \text{post} \). Let \( X_0^n, ..., X_{2m-1}^n \) be the \( n \)-tiles around \( w \) (labeled positively around \( w \)). Label the \((n+1)\)-tiles \( X_0^{n+1}, ..., X_{2m-1}^{n+1} \) around \( v \) in such a way that \( F(X_j^{n+1})=X_j^n, j=0, ..., 2m-1 \). Then

\[
X_i^{n+1} \text{ and } X_j^{n+1} \text{ are connected at } v \iff X_i^n \text{ and } X_j^n \text{ are connected at } w. \tag{8.1}
\]

In other words, the connection (of \((n+1)\)-tiles) at \( v \) is defined by

\[
\pi^{n+1}_w(v) \cup \pi^{n+1}_b(v) := \pi^n_w(w) \cup \pi^n_b(w).
\]

**Case 2.** \( v \) is an \( n \)-vertex (\( v \in \text{post} \cap \text{V}^n \)).

Then \( p:=F^n(v) \in \text{post} \). Consider two white \((n+1)\)-tiles \( X^{n+1}, Y^{n+1} \ni v \). They are connected (at \( v \)) if and only if

- either they are contained in the image of the same (white) \( n \)-tile \( X^n \) by the pseudo-isotopy \( H^n \), namely
  \[
  X^{n+1}, Y^{n+1} \subset H_1^n(X^n),
  \]
  and their images by \( F^n \) are connected, i.e., the 1-tiles \( F^n(X^{n+1}) \) and \( F^n(Y^{n+1}) \) are connected at \( p \);
- or \( X^{n+1} \) and \( Y^{n+1} \) are contained in the images of connected \( n \)-tiles \( X^n, Y^n \ni v \),
  \[
  X^{n+1} \subset H_1^n(X^n) \quad \text{and} \quad Y^{n+1} \subset H_1^n(Y^n),
  \]
  \( X^n \) and \( Y^n \) are connected at \( v \), and \( X^{n+1} \) and \( Y^{n+1} \) both map to 1-tiles that are “connected to the marked succeeding 1-edges”, i.e., the 1-tiles \( F^n(X^{n+1}) \) and \( F^n(Y^{n+1}) \) are connected at \( p \) to the white 1-tiles \( X^1 \) and \( \tilde{X}^1 \) that contain the marked succeeding 1-edges \( E^1 \) and \( \tilde{E}^1 \).
Figure 9. Inductive construction of connections.

The connection of black \((n+1)\)-tiles at \(v\) is defined analogously to the above.

We will formalize the description above. To do this, we will first have to label the involved 1-tiles, \(n\)-tiles and \((n+1)\)-tiles in a consistent manner. See Figure 9 for an illustration.

Recall from Lemma 3.6 that for each \((j+1)\)-edge \(E^{j+1}\) there is a unique arc \(A^j\) contained in a \(j\)-edge \(E^j\) that is deformed by the pseudo-isotopy \(H^j\) to \(E^{j+1}\). Since we will often want to keep track of where such an \(E^{j+1}\)-edge “comes from”, in this case we use the notation

\[
H^j: A^j \subset E^j \rightarrow E^{j+1}.
\]

We will single out one 0-, 1-, \(n\)- and \((n+1)\)-edge. Let \(E^0\) be the 0-edge with initial point \(p\) (\(E^0\) is positively oriented as boundary of the white 0-tile \(X_0^0\)). The 1-edge \(E^1\) is the marked one with initial point \(p\). Thus, there is an arc \(A^0 \ni p\) such that

\[
H^0: A^0 \subset E^0 \rightarrow E^1.
\]
We choose (arbitrarily) one $n$-edge $\hat{E}^n \ni v$ such that $F^n(\hat{E}^n) = \hat{E}^0$. Finally we choose the $(n+1)$-edge $\hat{E}^{n+1} \ni v$ such that there is an $n$-arc $\hat{A}^n \ni v$ satisfying

$$H^n: \hat{A}^n \subset \hat{E}^n \to \hat{E}^{n+1}.$$

Let $2m$ be the number of $n$-tiles containing $v$ (this means that $m = \deg_{F^m}(v)$) and $2k$ be the number of 1-tiles containing $p$. Then the number of $(n+1)$-tiles containing $v$ is $2km$.

The 1-tiles $X_{b_1}^1, \ldots, X_{b_{2k-1}}^1$ around $p$, the $n$-tiles $X_{b_0}^n, \ldots, X_{b_{2m-1}}^n$ around $v$ and the $(n+1)$-tiles $X_{b_0}^{n+1}, \ldots, X_{b_{2m+1}}^{n+1}$ around $v$ are labeled positively (around $p$ and $v$, respectively) and such that $\hat{E}^1 \subset X_{b_1}^1$, $\hat{E}^n \subset X_{b_0}^n$ and $\hat{E}^{n+1} \subset X_{b_0}^{n+1}$.

Recall that white tiles are always labeled by even, and black tiles by odd indices. Thus $X_{b_0}^1, X_{b_0}^n$ and $X_{b_0}^{n+1}$ are all white tiles. This finishes the labeling.

The blocks $b^{n+1}$ of the cnc-partition $\pi_w^{n+1}(v) \cup \pi_b^{n+1}(v)$ are defined as follows. For each block $b^i \in \pi_w^n(v) \cup \pi_b^n(v)$ and each $j = 0, \ldots, m-1$ there is a block

$$b^{n+1} = b_j^{n+1}(b^i) = b^i + 2kj = \{i + 2kj : i \in b^i\}. \quad (8.2)$$

This corresponds to the first part of the description above.

Now let $b_1^j \in \pi_w^1(p)$ be the block containing 0; it contains indices of white 1-tiles that are connected to the marked succeeding 1-edges at $p$. The sets $b_j^{n+1}(b_1^i) = b_1^i + 2kj$ are defined as in $(8.2)$, they contain indices of $(n+1)$-tiles that are mapped to (1-tiles with indices in) $b_1^i$ by $F^n$. For each block $b^n \in \pi_w^n(v)$ there is a block $b_j^{n+1} \in \pi_w^{n+1}(v)$ given by

$$b_j^{n+1} = b_j^{n+1}(b^n) := \bigcup \{b_1^i + 2kj : 2j \in b^n\}. \quad (8.3)$$

This is the formal description of the second part described above.

In the same fashion, let $c_1^j \in \pi_b^1(p)$ be the block containing $2k-1$. It contains indices of black 1-tiles connected to the marked succeeding 1-edges at $p$. For each block $c^n \in \pi_b^n(v)$ there is a block $c_j^{n+1} \in \pi_b^{n+1}(v)$ given by

$$c_j^{n+1} = c_j^{n+1}(c^n) := \bigcup \{c_1^i + 2kj : 2j+1 \in c^n\}. \quad (8.4)$$

The cnc-partition $\pi_w^{n+1}(v) \cup \pi_b^{n+1}(v)$ consists of all blocks $b_j^{n+1}(b^i)$ as in $(8.2)$, with $b^i \neq b_1^i$ and $b^i \neq c_1^j$, as well as all blocks $b_j^{n+1} = b^{n+1}(b^n)$ and $c_j^{n+1} = c^{n+1}(c^n)$ as above.

**Case 3.** $v \in \text{post.}$

Note that $\text{post} = V^0 \subset V^n$. This case is thus a subcase of case 2. The cnc-partition $\pi_w^{n+1}(v) \cup \pi_b^{n+1}(v)$ is thus already constructed in case 2. It remains to mark it. Recall
that in case 2 the \( n \)-edge \( \tilde{E}^n \) with \( F^n(\tilde{E}^n) = \tilde{E}^0 \) was chosen arbitrarily. Now, however, we let \( \tilde{E}^n \) be the marked \( n \)-edge with initial point \( v \).

The marked \((n+1)\)-edge with initial point \( v \) is \( \tilde{E}^{n+1} \) (recall that there is an arc \( \tilde{A}^n \ni v \) such that \( H^n: \tilde{A}^n \subset \tilde{E}^n \to \tilde{E}^{n+1} \)).

Alternatively, consider the blocks

\[
b^{n+1} = b^{n+1}(0) \in \pi_{w}^{n+1}(v) \quad \text{and} \quad c^{n+1} = c^{n+1}(2km - 1) \in \pi_{v}^{n+1}(v)
\]

such that \( 0 \in b^{n+1} \) and \( 2km - 1 \in c^{n+1} \). These two adjacent blocks mark the connection of \((n+1)\)-tiles at \( p \) (see Corollary 6.14).

### 8.2. Properties of connections

Here we prove that the connections of \( n \)-tiles defined above have the desired properties.

**Proposition 8.1.** The connection of \( n \)-tiles as defined in §8.1 satisfies the following properties:

1. each \( \pi_{w}^{n}(w) \cup \pi_{v}^{n}(v) \) is a cnc-partition;
2. the connection of \( n \)-tiles satisfies properties (C1) and (C2) from Definition 7.1;
3. the (single) boundary circuit of the cluster of white \( n \)-tiles is equal to the \( n \)th approximation \( \gamma^n \) (viewed as an Eulerian circuit).

**Proof.** To be able to keep the notation from §8.1, we will prove the statements for the connection of \((n+1)\)-tiles.

(1) The statement will be proved by induction. Thus we assume that \( \pi_{w}^{n}(w) \cup \pi_{v}^{n}(v) \) is a cnc-partition for each \( n \)-vertex \( w \). Consider now an arbitrary \((n+1)\)-vertex \( v \). We want to show that \( \pi_{w}^{n+1}(v) \cup \pi_{v}^{n+1}(v) \) is a cnc-partition. This is trivial in case 1 (i.e., if \( v \) is not an \( n \)-vertex). Thus assume that we are in case 2, i.e., that \( v \in V^{n+1} \cap V^n \).

(1a) We first prove that \( \pi_{w}^{n+1}(v) \cup \pi_{v}^{n+1}(v) \) is non-crossing. Consider first two blocks

\[
b^{n+1} = b^{n+1}(b^1), \quad c^{n+1} = c^{n+1}(c^1) \in \pi_{w}^{n+1}(v) \cup \pi_{v}^{n+1}(v)
\]

as in (8.2), where \( i, j = 0, \ldots, m - 1 \) and \( b^1, c^1 \in \pi_{w}^{1}(p) \cup \pi_{v}^{1}(p) \setminus \{b^1, c^1\} \). If \( i \neq j \), the blocks \( b^{n+1} \) and \( c^{n+1} \) are non-crossing, since \( b^{n+1} \) and \( c^{n+1} \) are contained in disjoint intervals, namely \( b^{n+1} \subset [2ki, 2k(i+1) - 1] \) and \( c^{n+1} \subset [2kj, 2k(j+1) - 1] \).

If \( i = j \), the blocks \( b^{n+1} \) and \( c^{n+1} \) are non-crossing, since the blocks \( b^1 \) and \( c^1 \) are.

(1b) Now let \( b^{n+1} = b^{n+1}(b^n) \) be as before, and

\[
b^{n+1} = b^{n+1}(b^n) = \bigcup \{b^1 + 2kj : 2j \in b^n \}
\]
be as in (8.3) (where \( b^n \in \pi_w^n(v) \)). Assume without loss of generality that \( i=0 \). Then \( b^{n+1} \) is contained in one component of \([0,2k-1) \setminus b_1 \). Each set \( b_1^{\uparrow} + 2kj \) distinct from \( b_1 \) is contained in an interval distinct from \([0,2k-1) \). It follows that \( b^{n+1} \) and \( b_1^{n+1} \) are non-crossing.

That \( b^{n+1} \) and \( c_1^{n+1} \) (as in (8.4)) are non-crossing is shown by the same argument.

(1c) Now let \( b_1^{n+1} = b_1^{n+1}(b^n) \) be as before and \( b_1^{n+1} = b_1^{n+1}(\tilde{b}^n) \) be a distinct set as in (8.3), meaning that the block \( b^n \in \pi_w^n(v) \) is distinct from \( b^n \). Since \( \tilde{b}^n \) and \( \tilde{b}^n \) are non-crossing, it follows that \( b_1^{n+1} \) and \( b_1^{n+1} \) are non-crossing. The same argument shows that distinct \( c_1^{n+1} \) and \( \tilde{c}_1^{n+1} \) as in (8.4) are non-crossing.

(1d) Consider now two sets \( b_1^{n+1} = b_1^{n+1}(b^n) \) and \( c_1^{n+1} = c_1^{n+1}(c^n) \) as in (8.3) and (8.4) \( \lambda = \pi_w^n(v) \) and \( \lambda = \pi_b^n(v) \). Recall that \( \pi_w^n(v) \cup \pi_b^n(v) \) is a cnc-partition, by inductive hypothesis. Assume first that \( b^n \) and \( c^n \) are non-adjacent (see Lemma 6.2), i.e., they do not contain indices \( i \) and \( i+1 \), respectively. Then, from the fact that \( b^n \) and \( c^n \) are non-crossing, it follows that \( b_1^{n+1} \) and \( c_1^{n+1} \) are non-crossing.

(1c) Now let \( b^n \) and \( c^n \) be adjacent. Recall that \( 0 \in b_1 \) and \( 2k-1 \in c_1 \). Thus, there is an index \( i^1 \in b_1 \) such that \( i^1 + 1 \in c_1 \), since \( \pi_b^1(p) \cup \pi_b^1(p) \) is a cnc-partition. This means that

\[
\begin{align*}
\text{Similarly, since } b^n \text{ and } c^n \text{ are adjacent, there are indices } i^n, j^n \in b^n \text{ such that } \iota^n + 1, j^n - 1 \in c^n; \\
\text{meaning that } \quad b^n \subset [j^n, i^n] \quad \text{and} \quad c^n \subset [i^n + 1, j^n - 1].
\end{align*}
\]

Here we are using the notation from (6.1). From this we obtain the smallest and biggest elements in \( b_1^{n+1} = b_1^{n+1}(b^n) \) and \( c_1^{n+1} = c_1^{n+1}(c^n) \) according to (8.3) and (8.4), namely

\[
\begin{align*}
\text{Thus, } b_1^{n+1} \text{ and } c_1^{n+1} \text{ are non-crossing.}
\end{align*}
\]

We now prove that \( \pi_w^{n+1}(v) \) and \( \pi_b^{n+1}(v) \) are complementary. Let \( i^{n+1} = 0, \ldots, 2km - 1 \) be arbitrary. We have to show that the two blocks of \( \pi_w^{n+1}(v) \cup \pi_b^{n+1}(v) \) containing \( i^{n+1} \) and \( i^{n+1} + 1 \) are adjacent.

If we are in case (1a), i.e., if \( i^{n+1} + 1 \in b^{n+1} = b_1^{n+1}(b^1) \) and \( i^{n+1} + 1 \in c^{n+1} = b_1^{n+1}(c^1) \), where \( b^1, c^1 \in \pi_b^1(p) \cup \pi_b^1(p) \setminus \{b_1, c_1\} \), it follows that \( i = j \). Then \( b^1 \) and \( c^1 \) are adjacent, which implies that \( b^{n+1} \) and \( c^{n+1} \) are adjacent.

When we are in case (1b), it follows that \( b^1 \) and \( b_1^1 \) are adjacent. This implies that \( b^{n+1} \) and \( b_1^{n+1} \) are adjacent.
Cases (1c) and (1d) cannot happen.

In case (1e), it is clear from the description that

\[ j^n k, j^{i+1} k \in b^{n+1}_s \quad \text{and} \quad i^1 + i^n k + 1, j^n k - 1 \in c^{n+1}_s. \]

Thus, \( b^{n+1}_s \) and \( c^{n+1}_s \) are adjacent.

(3) Let \( D^{n+1} \) and \( \bar{D}^{n+1} \) be two \((n+1)\)-edges. We have to show that \( D^{n+1} \) and \( \bar{D}^{n+1} \) are succeeding in \( \gamma^{n+1} \) if and only if they are succeeding with respect to the connection of \((n+1)\)-tiles.

We keep the notation from §8.1. Case (1) is again clear. Thus we assume that we are in case (2), meaning that \( v \in \mathbb{V}^{n+1} \cap \mathbb{V}^n \). Recall that \( \tilde{E}^0 \) is the 0-edge with initial point \( p = F^n(v) \) and \( \tilde{E}^1 \supseteq p \) is the marked 1-edge (some arc \( \tilde{A}^0 \subseteq \tilde{E}^0 \) containing \( p \) is deformed by \( H^0 \) to \( \tilde{E}^1 \)).

Let \( \tilde{E}^0 = \tilde{E}^0_n, \ldots, \tilde{E}^0_{n-1} \supseteq v \) be all \( n \)-edges such that \( F^n(\tilde{E}^0_j) = \tilde{E}^0 \) (labeled positively around \( v \)).

Consider the \((n+1)\)-edges \( \tilde{E}^{n+1}_j \) such that \( H^n: \tilde{A}_j^0 \subseteq \tilde{E}^{n+1}_j \to \tilde{E}^{n+1}_j \), for some arc \( \tilde{A}_j^0 \supseteq v \). These \((n+1)\)-edges \( \tilde{E}^{n+1}_0, \ldots, \tilde{E}^{n+1}_{m-1} \) are again labeled positively around \( v \). Note that these are not all of the \((n+1)\)-edges containing \( v \).

Claim. \( F^n(\tilde{E}^{n+1}_j) = \tilde{E}^1 \) for all \( j = 0, \ldots, m-1 \).

To prove the claim, we first note that \( F^n(\tilde{E}^{n+1}_j) \) is a 1-edge which we denote by \( \tilde{D}^1 \). Since \( \tilde{A}^0_j \subseteq \tilde{E}^0_n \), the arc \( \tilde{B}^0 := F^n(\tilde{A}^0_j) \) is contained in \( \tilde{E}^0_n = F^n(\tilde{E}^0_n) \), with initial point \( p = F^n(v) \). As \( H^n \) is the lift of \( H^0 \) by \( F^n \), we have

\[ \tilde{D}^1 = F^n(\tilde{E}^{n+1}_j) = F^n(H^n_1(\tilde{A}^0_j)) = H^n_0(F^n(\tilde{A}^0_j)) = H^n_1(\tilde{B}^0). \]

The unique arc in \( \tilde{E}^0 \) with initial point \( p \) that is deformed to a 1-edge is \( \tilde{A}^0 \). Therefore \( \tilde{B}^0 = \tilde{A}^0 \), and hence \( \tilde{D}^1 = \tilde{E}^1 \), proving the claim.

Note that a sector of sufficiently small radius between \( \tilde{E}^{n+1}_j \) and \( \tilde{E}^{n+1}_{j+1} \) is mapped bijectively by \( F^n \) to some neighborhood of \( p \) with \( \tilde{E}^1 \) removed.

Assume now that the \((n+1)\)-edges \( D^{n+1} \) and \( \bar{D}^{n+1} \) are succeeding in \( \gamma^{n+1} \) at the \((n+1)\)-vertex \( v \). This is the case if and only if there are distinct arcs \( \tilde{A}^n, \bar{A}^n \supseteq x \) such that

\[ H^n: \tilde{A}^n \subseteq D^n \to D^{n+1} \quad \text{and} \quad \tilde{A}^n \subseteq \bar{D}^n \to \bar{D}^{n+1} \]

(where \( D^n, \bar{D}^n \in \mathbb{E}^n \)).

Case 1. \( \tilde{A}^n \) and \( \bar{A}^n \) are contained in the same \( n \)-edge, or equivalently \( x \notin \mathbb{V}^n \). Note that

\[ \tilde{D}^{n+1} \neq \tilde{E}^{n+1}_j \quad \text{for all} \quad j = 0, \ldots, m-1, \]
and that $H^n(x) = v$. If $D^{n+1} = \tilde{E}^{n+1}_j$ for a $j = 0, \ldots, m - 1$, it would follow that both endpoints of $\tilde{E}^{n+1}_j$ are equal to $v$, which is impossible.

It follows that $D^{n+1}$ and $\tilde{D}^{n+1}$ are contained in one sector between $\tilde{E}^{n+1}_j$ and $\tilde{E}^{n+1}_{j+1}$, since $H^n$ is a pseudo-isotopy.

Case 2. $x = v$, and thus $A^n$ and $\tilde{A}^n$ are contained in $n$-edges that succeed at $v$. Then

$$\tilde{D}^{n+1} = \tilde{E}^{n+1}_j$$

for some $j = 0, \ldots, m - 1$ in this case.

Consider two $(n+1)$-edges $D^{n+1}$, $\tilde{D}^{n+1} \ni v$, with $\tilde{D}^{n+1} \neq \tilde{E}^{n+1}_j$ (for all $j = 0, \ldots, m - 1$). They are succeeding in $\gamma^{n+1}$ at $v$ if and only if they are contained in one sector between $\tilde{E}^{n+1}_j$ and $\tilde{E}^{n+1}_{j+1}$, and the 1-edges $F^n(D^{n+1})$ and $F^n(\tilde{D}^{n+1})$ are succeeding in $\gamma^1$ (since $F^n$ is bijective on this sector). This happens if and only if $D^{n+1}$ and $\tilde{D}^{n+1}$ are succeeding with respect to $\pi^{n+1}_w(v)$ by definition (see (8.2)).

Let $E^0 \ni p$ be the 0-edge with terminal point $p$, i.e., the one preceding $\tilde{E}^0$. Let $E^n_0, \ldots, E^n_{m-1}$ be all $n$-edges such that $F^n(E^n_j) = E^0_0$, labeled such that $E^n_j$ lies between $\tilde{E}^n_j$ and $\tilde{E}^n_{j+1}$. Then $E^n_j$ and $E^n_{j+1}$ are both contained in the same white $n$-tile $X^n_j$. Thus $E^n_j$ and $E^n_{j+1}$ are succeeding (at $v$) if and only if $i$ and $j$ are succeeding indices of a block $b^n \in \pi^n_w(v)$.

Consider the 1-edge $E^1$ such that $H^0: A^0 \ni E^0 \to E^1$ for an arc $A^0 \ni p$. Let $X^n_1$ be the white 1-tile containing $E^1$. Now consider the $(n+1)$-edge $E^{n+1}_j$ such that

$$H^n: A^n_j \ni E^n_j \to E^{n+1}_j$$

for an arc $A^n_j \ni v$. Since $H^n$ is a pseudo-isotopy, it follows that $E^{n+1}_j$ is in the sector between $\tilde{E}^{n+1}_j$ and $\tilde{E}^{n+1}_{j+1}$; indeed it follows that $E^{n+1}_j \subset X^{n+1}_{2j+1}$, since the diagram in Figure 9 commutes (recall that $\tilde{E}^{n+1}_j \subset X^{n+1}_{2j}$).

Consider now two $(n+1)$-edges $D^{n+1} \ni v$ and $\tilde{D}^{n+1} = \tilde{E}^{n+1}_j \ni v$. They are succeeding in $\gamma^{n+1}$ if and only if $D^{n+1} = E^{n+1}_j \subset X^{n+1}_{2j+i}$, where $i$ and $j$ are succeeding indices of a block $b^n \in \pi^n_w(v)$. This happens if and only if they are succeeding with respect to $\pi^{n+1}_w(v) \cup \pi^{n+1}_b(v)$ by definition (see (8.3)) (in the notation from (1e), $i = i^n$, $j = j^n$ and $l = i^1$).

(2) follows as in §4.4.

9. Invariant Peano curve implies expansion

In this section we prove Theorem 1.2. Thus we assume that for some iterate $F = f^n$ there is a Peano curve $\gamma: S^1 \to S^2$ (onto) such that $F(\gamma(z)) = \gamma(z^d)$ for all $z \in S^1$ (where $d = \deg F$). We want to show that $f$ is expanding.
The following is [BM, Lemma 6.3].

**Lemma 9.1.** Let \( f \) be a Thurston map and \( F = f^n \), where \( n \in \mathbb{N} \). Then \( f \) is expanding if and only if \( F \) is expanding.

We will use the following equivalent formulation of “expanding” due to Hâissinsky–Pilgrim [HP]. For a proof of the following lemma we refer the reader to [BM, Proposition 6.2].

**Lemma 9.2.** A Thurston map \( F \) is expanding if and only if there exists a finite open cover \( U^0 \) of \( S^2 \) by connected sets such that the following holds.

Denote by \( U^n \) the set of connected components of \( F^{-n}(U) \) for all \( U \in U^0 \). Then
\[
\text{mesh} U^n \to 0 \quad \text{as} \quad n \to \infty.
\]
Here \( \text{mesh} U^n \) denotes the biggest diameter of a set in \( U^n \).

**Proof of Theorem 1.2.** Let \( \gamma : S^1 \to S^2 \) be a Peano curve (onto) such that
\[
F(\gamma(z)) = \gamma(z^d) \quad \text{for all} \quad z \in S^1 \quad (\text{where} \quad d = \deg F). \tag{9.1}
\]

Fix a point \( x^0 \in S^2 \). Let \( W(x^0) \subset S^2 \) be an open neighborhood of \( x^0 \) that is a Jordan domain. Furthermore we assume that \( W(x^0) \) is so small that each component of \( F^{-1}(W(x^0)) \) contains exactly one point of \( F^{-1}(x^0) \).

Consider \( \gamma^{-1}(W(x^0)) =: \mathcal{J}(x^0) = \bigcup_j I_j \subset S^1 \); this is a (countable) union of open arcs \( I_j \). Let
\[
\mathcal{J}(x^0) := \bigcup \{ I_j : \gamma(I_j) \ni x^0 \} \subset S^1,
\]
\[
V(x^0) := \gamma(\mathcal{J}(x^0)) \subset S^2.
\]

Note that \( \gamma(S^1 \setminus \mathcal{J}(x^0)) \) is a compact set that does not contain \( x^0 \). Thus \( V(x^0) \) is a neighborhood of \( x^0 \).

Fix an \( x^n \in F^{-n}(x^0) \). Let \( V^n(x^n) \subset S^2 \) be the path component of \( F^{-n}(V(x^0)) \) containing \( x^n \).

As before we view the circle as \( \mathbb{R}/\mathbb{Z} \), the map \( z \mapsto z^d \) is then given as
\[
\phi_d : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z},
\]
\[
t \mapsto dt \pmod{1}.
\]

Let \( \mathcal{J}^n := \phi_d^{-1}(\mathcal{J}(x^0)) \). Note that \( \mathcal{J}^n = \bigcup J^n_i \) is a (countable) union of open intervals, each of which has length \( \leq d^{-n} \). Thus uniform continuity of \( \gamma \) implies that
\[
\text{diam} \gamma(J^n_i) \leq \omega(d^{-n}) \to 0 \quad \text{as} \quad n \to \infty,
\]

where $\omega$ is the modulus of continuity of $\gamma$.

From (9.1) it follows that each set $\gamma(J^n_j)$ contains a point $x^n_j \in F^{-n}(x^0)$. If $x^n_j \neq x^n$ then $\gamma(J^n_j)$ is contained in a component of $F^{-n}(W(x^0))$ distinct from the one containing $x^n$, and thus $\gamma(J^n_j) \cap V^n(x^n) = \emptyset$. It follows that

$$\gamma^{-1}(V^n(x^n)) = \bigcup \{ J^n_j : \gamma(J^n_j) \ni x^n \} =: J^n(x^n).$$

Since $\gamma(J^n_i) \cap \gamma(J^n_j) \ni x^n$ for $J^n_i, J^n_j \subset J^n(x^n)$, it follows that

$$\operatorname{diam} V^n(x^n) \leq 2\omega(d^n).$$

The sets $V^0(x^0)$ are not necessarily open, and int $V^0(x^0)$ is not necessarily connected. Let $U(x^0) \subset V^0(x^0)$ be an open connected set containing $x^0$. Pick a finite subcover $\mathcal{U}^0$ of $\{ U(x^0) : x^0 \in S^2 \}$. From the above it follows that $\operatorname{mesh} \mathcal{U}^n \to 0$ as $n \to \infty$. Thus $F$ is expanding by Lemma 9.2, and hence $f$ is expanding by Lemma 9.1.

10. An example

The obvious question to ask is whether an iterate $F = f^n$ is necessary in Theorem 1.1 (or whether one may choose $n = 1$). None of the assumptions in §7 seem to be necessary. It is possible to show (similarly as in [BM, Example 14.12]) that the map $f$ for which Milnor constructs an invariant Peano curve in [Mi1] does not have an invariant Jordan curve $C \supset \text{post}$; also the 1-tiles do intersect disjoint 0-edges.

In this section we consider an example of an expanding Thurston map $h$, where no pseudo-isotopy $H^0$ as desired exists. This means that for any Jordan curve $C \supset \text{post}$ (not necessarily invariant) there is no pseudo-isotopy $H^0$ rel. $\text{post}(h)$ as in Definition 3.2 such that $H^0_1(C) = \bigcup E^1 = h^{-1}(C)$.

Thus, one has to take an iterate (in fact $h^2$ will do) in our construction. Of course there could be a Peano curve $\gamma$ which semi-conjugates $z^d$ to $h$, but a substantially different proof would be required.

The map $h$ is a Lattès map as the map $g$ from §1.5. Start with the square

$$[0, \frac{1}{2}\sqrt{2}] \times [0, 1],$$

which is mapped by a Riemann map to the upper half-plane. This extends to a meromorphic map $\psi = \psi_L : \mathbb{C} \to \mathbb{C}$, which is periodic with respect to the lattice $L = \sqrt{2}\mathbb{Z} \times 2\mathbb{Z}$. Consider the map (here $i$ denotes the imaginary unit)

$$\psi : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sqrt{2}iz$$

(10.1)
Figure 10. The map $h$.

Note that $\psi(L) \subset L$. The map $h$ is the one that makes the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\psi} & C \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
S^2 & \xrightarrow{h} & S^2.
\end{array}
$$

The degree of $h$ is 2. Again one may use $\varphi$ to push the Euclidean metric from $C$ to $S^2$. In this metric the upper and lower half-planes are both isometric to the rectangle $[0, \frac{1}{2}\sqrt{2}] \times [0, 1]$. Two such rectangles glued together along their boundaries form a pillow as before. Divide each rectangle horizontally in two. The small rectangles are similar to the big ones. The map $h$ is given by mapping each small rectangle (they are the 1-tiles) to big ones (the 0-tiles) as indicated in Figure 10. The critical points are $c_1$ and $c_2$, the post-critical points are $p_0$, $p_1$, $p_2$ and $p_3$; they are mapped as follows (this is known as the ramification portrait):

$$
\begin{align*}
&c_1 \xrightarrow{2:1} p_1 \\
&c_2 \xrightarrow{2:1} p_2
\end{align*}
$$

(10.2)

**Lemma 10.1.** Let $\gamma^0 = C \supset \text{post}(h)$ be (any such) Jordan curve and $\gamma^1$ be an Eulerian circuit in $h^{-1}(C)$ such that $h: \gamma^1 \to \gamma^0$ is a $d$-fold cover. Then there is no pseudo-isotopy $H^0$ rel. $\text{post}(h)$ as in Definition 3.2 that deforms $\gamma^0$ to $\gamma^1$. 

Sketch of proof. The proof is a (rather tedious) case-by-case analysis. There are however only two cases that are essentially different. One of each is presented.

Case 1. The curve $C$ goes through $p_0$, $p_1$, $p_2$ and $p_3$ (in this cyclical order).

We fix an orientation of $C$. Let $U_w$ and $U_b$ be the two components of $S^2 \setminus C$, where the positively oriented boundary of $U_w$ is $C$. The closures of $U_w$ and $U_b$ are the white and black 0-tiles $X_0^w = U_w \cup C$ and $X_b = U_b \cup C$ as before. Similarly, we define the (white) 1-tiles as closures of components of $h^{-1}(U_w)$.

Since the degree of $h$ is 2, there are two white 1-tiles. They intersect at the critical points $c_1$ and $c_2$. The boundary of each 1-tile contains four points that are mapped to $p_0$, $p_1$, $p_2$ and $p_3$ (in this cyclical order). There are two different Eulerian circuits $\gamma^1$ in $h^{-1}(C)$ such that $h: \gamma^1 \to \gamma^0$ is a 2-fold cover. They correspond to connecting the two 1-tiles either at $c_1$ or at $c_2$. One situation (connection at $c_2$) is shown in Figure 11. Note that the cyclical ordering of the post-critical points (shown as dots) is different from the one on $C$. Thus there is no pseudo-isotopy $H^0$ as desired that deforms $C = \gamma^0$ to $\gamma^1$.

When $C$ goes through the post-critical points in the order

$$(p_0, p_2, p_1, p_3), \quad (p_0, p_3, p_1, p_2), \quad \text{or} \quad (p_0, p_3, p_2, p_1),$$

the same argument works.

Case 2. The curve $C$ goes through $p_0$, $p_1$, $p_3$ and $p_2$ (in this cyclical order). The 0- and 1-tiles are defined and colored as before (see §2).

As before, there are two different Eulerian circuits $\gamma^1$ in $h^{-1}(C)$, such that $h: \gamma^1 \to \gamma^0$ is a 2-fold cover. They correspond to whether the white 1-tiles are connected at $c_1$ or $c_2$. Assume that they are connected at $c_2$. The argument when they are connected at $c_1$ is again completely analogous.
Assume that the pseudo-isotopy $H^0$ is as in Definition 3.2. Then $H^0$ deforms (the white 0-tile) $X^0_w$ into the two 1-tiles.

In the following we work in the (orbifold) covering. Recall that $X^0_w, X^0_b \subset S^2$ are the white and black 0-tiles (given by $C$). Pull this tiling back by $\wp$ to a tiling of $C$. More precisely, a 0-tile $\tilde{X} \subset C$ is the closure of one component of $\wp^{-1}(U_{w,b})$. Similarly as in the proof of (2.1) one shows that $\wp : \tilde{X} \to X^0_{w,b}$ is a homeomorphism. We color one such 0-tile $\tilde{X}$ white or black if it is the preimage of $X^0_w$ or $X^0_b$, respectively. This gives a tiling of the plane $C$ into white and black 0-tiles.

Recall that the ramification points of $\wp$ are the points in $\frac{1}{2}\sqrt{2}Z \times Z$. At each such ramified point $c \in \frac{1}{2}\sqrt{2}Z \times Z$, two white and two black tiles intersect. Furthermore, the map $\wp$ is symmetric with respect to each such point. This means that $\wp(c+z) = \wp(c-z)$ for all $z \in C$. Thus the tiling of $C$ is pointwise symmetric with respect to each such point $c$.

We now define the 1-tiles in $C$. They may be obtained in two different ways; either as preimages of 1-tiles in $S^2$ by $\wp$, or as preimages of 0-tiles $\tilde{X} \subset C$ by $\psi$ (10.1).

Fix one white 0-tile $\tilde{X} \subset C$. Note that $\tilde{X}$ has four vertices $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \in \frac{1}{2}\sqrt{2}Z \times Z$, they are mapped by $\wp$ to $p_0, p_1, p_2$ and $p_3$, respectively. We may assume that $\tilde{p}_0=0$.

As in Lemma 3.4, the pseudo-isotopy $H^0$ lifts to a pseudo-isotopy (rel. $\frac{1}{2}\sqrt{2}Z \times Z$) $\tilde{H}^0 : C \times [0,1] \to C$. Note that $\tilde{H}^0$ deforms $\tilde{X}$ into two 1-tiles (in $C$) connected at a point $\tilde{c}_2$. Here $\wp(\tilde{c}_2) = c_2$.

The ordering of the post-critical points along $C$ together with (10.2) implies that the situation looks as in Figure 12. Here “$\mapsto \tilde{p}_j$” labels a point $\tilde{z}$ that satisfies $h(\wp(\tilde{z})) = p_j$.

The symmetry of the 1-tiles with respect to the point $\tilde{c}_2$ implies that

$$2\tilde{c}_2 = \tilde{p}_3 = \tilde{p}_1 + \tilde{p}_2.$$ 

Note that $\tilde{c}_2$ and $\tilde{p}_1$ are contained in the same 1-tile $\tilde{X}^1$, which contains $\tilde{p}_0=0$. There are two 0-tiles containing $\tilde{p}_0$, symmetric with respect to the origin. Thus (here $i$ denotes...
the imaginary unit)
\[ \pm \psi(\tilde{X}^1) = \pm \sqrt{2i} \tilde{X}^1 = \tilde{X}. \]

Therefore
\[ \pm \sqrt{2i} \tilde{c}_2 = \tilde{p}_2 \quad \text{and} \quad \pm \sqrt{2i} \tilde{p}_1 = \tilde{p}_3. \]

Combining these three equations yields
\[ \tilde{p}_2 = \pm \sqrt{2i} \tilde{c}_2 = \pm \frac{1}{2} \sqrt{2i} \tilde{p}_3 = \pm \frac{1}{2} \sqrt{2i} (\pm \sqrt{2i} \tilde{p}_1) = -\tilde{p}_1. \]

Thus,
[\[ \tilde{p}_3 = \tilde{p}_1 + \tilde{p}_2 = 0, \]
which is a contradiction.

If \( \mathcal{C} \) goes through the post-critical points in the cyclical order \( p_0, p_2, p_3 \) and \( p_1 \), the argument is completely analogous to the one above.

\[ \Box \]

11. Open problems and concluding remarks

A rational map of degree \( d \) can naturally be viewed as a point in \( \mathbb{C}^{2d+1} \) via its coefficients. Consider a post-critically finite rational map \( f \) without periodic critical points. This is an expanding Thurston map in our sense, the Julia set is all of \( S^2 \). Rees [Re1] has shown that such a map can be disturbed in a set of positive measure (in \( \mathbb{C}^{2d+1} \)) such that the Julia set stays \( S^2 \).

Open problem 1. Let \( f \) be a rational map with Julia set \( S^2 \). Does Theorem 1.1 hold in this case?

On the other hand one may ask if the theorem continues to hold if the Julia set is not the whole sphere. This however is false. Namely, Kameyama [Ka, §4] gives an example of a post-critically finite rational map where no such semi-conjugacy exists.

Finally one can ask if a corresponding result holds in the group case.

Open problem 2. Let \( \Gamma \) be a Gromov-hyperbolic group whose boundary at infinity is \( S^2 \). Is there a Peano curve \( \gamma: S^1 \rightarrow S^2 \) which is invariant under a non-trivial normal subgroup of \( \Gamma \)?

A positive answer might conceivably open another line of attack on Cannon’s conjecture.
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