Mom technology and hyperbolic 3-manifolds

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1. Introduction

This is an expository paper on the work of the authors, found in [GMM2], [GMM3], and [Mil2], which proves that the Weeks manifold is the unique closed orientable 3-manifold of minimum volume and determines the ten one-cusped hyperbolic 3-manifolds of volume at most 2.848. Our work has focused on Mom technology, which has proven to be effective in determining these low-volume manifolds and has the potential for vast generalization.

This introduction will survey a selection of results on volumes of hyperbolic 3-manifolds. The body of the paper will outline the recent work of the authors mentioned above, and the last section will discuss outstanding open problems in this area.

Unless stated otherwise, all manifolds in this paper will be orientable.

The theory of volumes of hyperbolic 3-manifolds has received tremendous interest over the last 40 years. In 1968 Mostow proved via his famous Rigidity theorem in [Most] that volume is a topological invariant of closed manifolds. This was extended to finite-volume hyperbolic 3-manifolds by Marden [Ma] and Prasad [Pra]. In the mid 1970’s, Troels Jørgensen proved that for any constant $C$ the collection of all complete hyperbolic 3-manifolds of volume at most $C$ is obtained from a finite collection of cusped manifolds using the operation of Dehn filling; see [T1] or [Gr].

Using these ideas together with those of Gromov, Thurston showed in 1977 that the subset $V \subset \mathbb{R}$ of volumes of complete hyperbolic 3-manifolds is a closed well-ordered set of order type $\omega^n$. Furthermore, there are only finitely many manifolds of a given volume. Thurston also showed that every $n$-cusped hyperbolic 3-manifold is a limit volume of hyperbolic 3-manifolds with $n-1$ cusps and that any filling on an $n$-cusped manifold yields a manifold of lower volume. In particular, the smallest volume 3-manifold is closed.

We remind the reader that saying $V$ is of order type $\omega^n$ implies that there is a smallest element $v_1$, a next smallest element $v_2$, and so on, with $v_\omega$ being the
first limit element, then \( v_{2ω+1}, v_{2ω+2}, \ldots \) limiting on \( v_{2ω} \), the second limit volume. The limit volumes \( v_ω, v_{2ω}, v_{3ω}, \ldots \) limit on \( v_{ω\ ω} \), the first limit of limit volumes. The volumes \( v_{ω\ ω} \) represent the volumes of 1-cusped manifolds, \( v_{ω\ 2ω} \) the volumes of 2-cusped manifolds, and so on (although there may be compact manifolds or manifolds with fewer cusps with these volumes as well). In contrast, for \( n ≥ 4 \) the set of volumes of complete hyperbolic \( n \)-manifolds is discrete; see \([Wa]\).

**Theorem 1.1.** The Matveev-Fomenko-Weeks manifold is the unique smallest volume closed hyperbolic 3-manifold. In particular \( v_1 = 0.9427 \ldots \).

This manifold, commonly referred to as the Weeks manifold for short, can be constructed as the \((5, 1), (5, 2)\) Dehn filling on the complement of the Whitehead link. It was discovered and its volume was computed independently by Matveev and Fomenko in \( [MF] \) and Weeks using his SnapPea program \( [We] \). Matveev and Fomenko were motivated by Novikov who was interested in volumes for physical and dynamical reasons; see \( [NS] \). Independently, Przytycki asked whether this manifold was the smallest closed manifold. See \( [BPZ] \) for related questions.

There were many partial results towards this problem: \( [Me1], [Me2], [GM1], [GMT] \) (using \( [GM2] \)), \( [Pr2], [MM1], [Ag1], [Pr3] \), and \( (Agol - Dunfield) [AST] \). The ideas of many of these results were used in our proof of Theorem 1.1. Their role will be discussed further in the body of the paper. In addition, Weeks’ remarkable SnapPea program was indispensable for providing experimental data and geometric intuition.

Other results along these lines include: the smallest cusped hyperbolic orbifold in \( [Me3] \), the smallest cusped manifold (orientable or not) in \( [Ad] \), the smallest cusped orientable manifold in \( [CM] \), the smallest arithmetic hyperbolic 3-orbifold in \( [CF] \), the smallest compact manifold with totally geodesic boundary in \( [KoM] \), the smallest 2-cusped hyperbolic 3-manifolds in \( [Ag2] \), and via a tour de force the smallest hyperbolic orbifold in \( [GM3] \) and \( [MM3] \).

Underlying some of these results are the very useful packing results of \( [Bor], [Pr3], [Pr4], \) and \( [Miy] \). In particular Miyamoto \( [Miy] \) gives remarkable results on manifolds with totally geodesic boundary.

In other directions there is the long series of papers by Culler, Shalen and their co-authors which give lower bounds on volumes for hyperbolic 3-manifolds that satisfy certain topological constraints. See for example \( [CHS1], [CHS2], [ACS1], [ACS2] \), and their references.

See Milnor’s paper \( [Miln] \) for a detailed history of hyperbolic geometry through about 1980. Its appendix contains volume formulae for ideal tetrahedra and in particular a proof that the regular ideal simplex is the one of maximal volume. This last result was proven in higher dimensions by Haggerup and Munkholm \( [HM] \).

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### 2. Outline Of The Proof

For basic facts about hyperbolic 3-manifolds see \( [T1], [Ra], \) or \( [BP] \).

By a *cusped hyperbolic manifold* we mean a complete non-compact hyperbolic 3-manifold \( M \) with finite volume. By a *compact hyperbolic manifold* we mean a
compact manifold whose interior supports a complete hyperbolic metric. Such a manifold is either closed or its boundary is a union of tori. Any cusped hyperbolic 3-manifold \( M \) naturally compactifies to a compact hyperbolic 3-manifold \( \overline{M} \). A Dehn filling on the cusped manifold \( M \) is the interior of a manifold obtained by attaching solid tori to various components of \( \partial M \).

The basic idea behind our efforts to identify low-volume hyperbolic 3-manifolds, particularly the smallest closed manifold, is as follows. Given \( V \in \mathbb{R} \) we identify a finite, reasonable set of cusped manifolds \( M_1, \ldots, M_k \) such that every hyperbolic 3-manifold of volume at most \( V \) is obtained by filling at least one of the \( M_i \)'s. Then we identify all of the manifolds obtained by filling the \( M_i \)'s that have volume at most \( V \). When \( V = 2.848 \) and the manifolds in question have exactly one cusp, the first step is carried out in [GMM2] and [GMM3] and the last step is carried out in [Mill2]. We showed that exactly 10 one-cusped manifolds, the first 10 in the one-cusped SnapPea census, have volume at most 2.848. By Agol [ACS1] the smallest volume hyperbolic 3-manifold is obtained by filling a 1-cusped manifold of volume at most 2.848. This result makes crucial use of Agol-Dunfield [AST] which in turn makes crucial use of Perelman [P1, P2] and [GMT]. Further analysis in [Mill2] of these 10 manifolds identifies the Weeks manifold as the unique one with volume at most .9428.

Sections 3 through 6 discuss the work of [GMM2] and [GMM3]. Section 7 discusses [Mill2]. Section 8 describes previous work in identifying the smallest volume manifold as well as methods developed in those works that play an important role in the final resolution. Section 9 gives more detail on some final issues, in particular the completion of the proof of Theorem [1]. In section 10, problems and directions for future research are presented.

For now we describe, in the context of one-cusped manifolds, the idea behind finding the \( M_i \)'s cited above using the notion of Mom technology introduced in [GMM2]. A Mom-\( n \) manifold is a compact manifold with a particular type of handle structure, where \( n \) denotes the complexity of the structure. More formally, we have the following:

**Definition 2.1.** A Mom-\( n \) is a triple \( (M, T, \Delta) \) where

- \( M \) is a compact 3-manifold with boundary a non-empty disjoint union of tori,
- \( T \) is a component of \( \partial M \), a small neighborhood of which can be identified with \( T \times I \), and
- \( \Delta \) is a handle structure on \( M \) of the following type. Starting with \( T \times I \), where \( T \) is identified with \( T \times \{0\} \), attach \( n \) 1-handles to \( T \times \{1\} \) followed by \( n \) 2-handles on the “\( T \times 1 \)-side” according to the following rule: counting with multiplicity, each 1-handle meets at least two 2-handles and each 2-handle is attached to exactly three 1-handles, again counting with multiplicity.

We say that a Mom-\( n \) is hyperbolic if its interior is hyperbolic. Note that a Mom-\( n \) manifold has at least two boundary components. There are exactly 3 hyperbolic Mom-2 manifolds, 18 hyperbolic Mom-3 manifolds and conjecturally 117 hyperbolic Mom-4 manifolds; see [GMM2]. It is proven in [GMM3] that every 1-cusped hyperbolic 3-manifold of volume at most 2.848 is obtained by filling
a hyperbolic Mom-$n$ manifold, where $n \leq 3$, using the notion of an internal Mom-$n$ structure:

**Definition 2.2.** An internal Mom-$n$ structure on a (closed or cusped) hyperbolic 3-manifold $N$ consists of a Mom-$n$ $(M, T, \Delta)$ together with an embedding $i : M \to N$ such that the image of each component of $\partial M$ either cuts off a cusp neighbourhood or a solid torus to the outside of $M$. The Mom number of $N$ is the minimal $n$ such that $N$ has an internal Mom-$n$ structure with $M$ hyperbolic.

Roughly speaking the Mom number of $N$ is the smallest $n$ such that there exists an essentially embedded hyperbolic Mom-$n$ manifold inside of $N$. So given a low-volume one-cusped 3-manifold, the goal is to find an internal Mom-$n$ structure with $n$ small. One intuitive reason for expecting the existence of such structures is as follows. Let $W$ be the maximal cusp of the one-cusped manifold $N$. Topologically, $W = T^2 \times [1, \infty)$ with a finite set of pairs of points of $\partial W$ identified with each other, and geometrically each $T^2 \times t$ is a horotorus. Further each $x \times [1, \infty)$ is an isometrically embedded geodesic. By straightforward geometric reasoning, $W$ (resp. $\partial W$) already has a decent amount of volume (resp. area). Now expand $W$ out in the standard Morse-theoretic way. If $N$ has low volume then this forces $W$ to rapidly encounter itself, leading to the creation of handles of index at least 1. If at some moment $n$ 1-handles, $n$ 2-handles, and no 3-handles have been created, the resulting manifold will have Euler characteristic 0 and hence is a candidate for being a hyperbolic Mom-$n$ manifold. See §3 for explicit examples.

Our motivation for the terminology *Mom manifold* is based on Thurston’s parent-child relationship between compact 3-manifolds. If $N$ is obtained by filling $M$, then Thurston called $M$ the parent and $N$ the child.

**Definition 2.3.** As per Shubert and Matveev, denote the intersections of the 1-handles and 2-handles with $T \times \{1\}$ by islands and bridges respectively, and the closed complement of the islands and bridges in $T \times \{1\}$ by lakes. A Mom-$n$ is full if the lakes are all simply connected.

Note that if $n$ is minimal, then being full is a necessary condition for being hyperbolic, as otherwise either $M$ has an embedded essential annulus joining $T \times \{1\}$ to $T \times \{0\}$ or the lake is compressible in $M$. In that case either $M$ is reducible, contradicting hyperbolicity, or an essential compressing disk for the lake is boundary parallel in $M$ and hence $n$ can be reduced, contradicting minimality of $n$.

Here is a very brief outline for finding a hyperbolic internal Mom-3 structure in a one-cusped hyperbolic 3-manifold $N$ of volume at most 2.848. The preimage $\{B_i\}$ in $\mathbb{H}^3$ of the maximal cusp is a union of horoballs. We consider $\pi_1(N)$-orbits of unordered pairs of such balls and $\pi_1(N)$-orbits of unordered triples of horoballs. Note that a triple of balls involves three pairs, however two such pairs may be in equivalent classes. A set of $n$ classes of triples which involve exactly $n$ classes of pairs is called a combinatorial Mom-$n$ structure. All one-cusped manifolds of volume 2.848 or less have a combinatorial Mom-$n$ structure with $n \leq 3$. See §4 for more details. A combinatorial Mom-$n$ structure gives rise to an immersed geometric Mom-$n$ structure, i.e. the cores of the 1 and 2 handles are totally geodesic. Here, pairs of balls give rise to 1-handles and triples of balls give rise to 2-handles. Unfortunately, the handles may intersect each other (or even the cusp) in undesirable ways. But under the 2.848 constraint a “controlled” combinatorial Mom-$n$
structure, \( n \leq 3 \), can be found, and after simplification this structure can be “promoted” to produce an embedded geometric Mom-\( k \) structure, \( k \leq n \). Then if the Mom-\( k \) manifold is not hyperbolic, geometric and topological arguments produce a hyperbolic internal Mom-\( p \) structure in \( N \) with \( p \leq k \). See §5 for more details. The issue of enumerating the Mom-\( n \) manifolds, \( n \leq 3 \) is discussed in §6.

3. Example: the figure-8 knot and the manifold \( m069 \).

As a first example, let \( N \) be the complement of the figure-8 knot in \( S^3 \); we will construct an internal Mom-2 structure. For the torus \( T \), we choose the boundary of a cusp neighborhood, i.e. the knot torus. Then to \( T \times I \) we add two 1-handles and two 2-handles as shown in Figure 1(a), to obtain a handle structure \( \Delta \) on a submanifold \( M \). Note that each 2-handle is connected to three 1-handles, counting multiplicity, and the boundary of \( M \) consists of two tori, one of which is \( T \times \{0\} \) and the other of which is on the “outside” of \( M \) as seen in the figure. Thus \( (M, T, \Delta) \) is a Mom-2. Furthermore, the component of \( \partial M \) which is not \( T \times \{0\} \) clearly bounds a solid torus in \( N \setminus M \), and \( M \) is hyperbolic as discussed below. Hence the embedding defines an internal Mom-2 structure on \( N \).

Readers familiar with the figure-8 knot complement will note immediately that the 1-handles and 2-handles in this example are neighborhoods of geodesic arcs and totally geodesic surfaces. In fact the 1-handles and 2-handles are neighborhoods of edges and faces in the canonical ideal triangulation of this manifold; this property is shared by all hyperbolic internal Mom-\( n \) structures in one-cusped manifolds as far as the authors are aware. This makes Mom-\( n \) structures easy to find in such manifolds once the canonical ideal triangulation of the manifold is known.

The interior of the submanifold \( M \) in this example is clearly homeomorphic to \( N \setminus \gamma \) where \( \gamma \) is the curve indicated in Figure 1(b). The complement of this link is homeomorphic to the complement of the Whitehead link after a Dehn twist along \( \gamma \); hence \( (M, T, \Delta) \) is a hyperbolic Mom-2. Another choice of internal Mom-2 structure on the figure-8 knot complement is shown in Figure 2(a). In this case we get a Mom-2 \( (M', T, \Delta') \) which is not hyperbolic because it is not full; the rightmost strand in the knot diagram forms an annular lake, and consequently \( M' \) contains an essential annulus.
Figure 2. (a) A non-hyperbolic internal Mom-2 structure on the figure-8 knot complement. (b) The corresponding non-hyperbolic link.

Figure 3. A cusp diagram (from data provided by SnapPea) for the manifold $m069$, with an internal Mom-3 structure highlighted.

The previous example, while very simple, doesn’t illuminate the Morse-theoretic nature of internal Mom-$n$ structures, nor does it illustrate the simplest way of finding internal Mom-$n$ structures in more general hyperbolic 3-manifolds. So consider the one-cusped manifold known as $m069$ in the SnapPea census. A cusp diagram for this manifold as produced by SnapPea is shown in Figure 3 including the link of the canonical ideal triangulation of $m069$ and the shadows of nearby horoballs in the universal cover. Not seen is the horoball $B_{\infty}$ which lies above a horizontal plane. The label on an edge in the diagram indicates the orthopair class of the pair of horoballs below the endpoints of the edge, while the label on a horoball indicates the orthopair class of the pair of horoballs consisting of that horoball and $B_{\infty}$. The notion of an orthopair class is defined in the next section; for now we merely note that the lower the label, the shorter (possibly equal) the distance between two horoballs of that pair.

There exists an internal Mom-3 structure on $m069$ where the three 1-handles correspond to the three shortest return paths from the cusp torus to itself. In other
words, the 1-handles are the first three 1-handles created by expanding the cusp neighborhood \( W \) as described in the previous section. These three 1-handles meet the cusp torus in six islands, which will occur at the centers of the horoball shadows labelled 1, 2, and 3 in Figure 3. The three 2-handles of the Mom-3 structure meet the cusp torus in a total of nine bridges, which correspond to the highlighted edges in Figure 3. One can confirm easily that the resulting lakes are simply connected, so this internal Mom-3 structure is full. Note that again the 1-handles and 2-handles of this Mom structure have totally geodesic cores. It is also true (if more difficult to confirm) that the submanifold corresponding to this Mom-3 structure is the manifold \( m202 \), and that the components of the complement of this submanifold consist of the original cusp neighborhood and a single additional solid torus.

4. Geometric and combinatorial Mom-\( n \) structures

As noted, the examples in the previous section have the property that the 1-handles and 2-handles of \( \Delta \) are all neighborhoods of geodesic arcs and totally geodesic surfaces respectively. We will call such an internal Mom-\( n \) structure a geometric Mom-\( n \) structure from now on to emphasize this fact. (Strictly speaking it would be more correct to say “geometric internal Mom-\( n \) structure” but for the sake of concision we will assume that the word “internal” is implied.) At the time of writing the authors know of no cusped manifold possessing an internal Mom-\( n \) structure which does not possess a geometric Mom-\( n \) structure.

In addition to its geometric description, we can also describe a geometric Mom-\( n \) structure in a one-cusped hyperbolic manifold \( N \) combinatorially as follows. Suppose \( W \) is a choice of cusp neighborhood in \( N \). Under the universal covering map \( \mathbb{H}^3 \to N \), the pre-image of \( W \) is a collection of horoballs \( \{B_i\} \), and a 1-handle in a geometric Mom-\( n \) structure can be lifted to a 1-handle in \( \mathbb{H}^3 \) joining two distinct horoballs \( B_i \) and \( B_j \). Hence each 1-handle in a geometric Mom-\( n \) structure corresponds to an element of the set of equivalence classes of unordered pairs \( \{(B_i,B_j)|i \neq j\}/\pi_1(N) \). Similarly each 2-handle in a geometric Mom-\( n \) structure corresponds to an equivalence class of unordered triples \( (B_i,B_j,B_k) \) under the action of \( \pi_1(N) \), where \( i, j, \) and \( k \) are all distinct.

For each equivalence class in the set \( \{(B_i,B_j)|i \neq j\}/\pi_1(N) \) the orthodistance \( d(B_i,B_j) \) is well-defined; the set of all such orthodistances counted with multiplicity forms the orthodistance spectrum \( o(1) \leq o(2) \leq \cdots \). By taking \( T \) to be the boundary of a maximal embedded cusp neighborhood we can assume that \( o(1) = 0 \). We order the elements of \( \{(B_i,B_j)\}/\pi_1(N) \) by their orthodistances, and say that the \( n \)-th orthopair class \( O(n) \) is the equivalence class with orthodistance \( o(n) \). Each unordered triple of horoballs \( (B_i,B_j,B_k) \) is “bounded” by three orthopair classes corresponding to the pairs \( (B_i,B_j) \), \( (B_j,B_k) \), and \( (B_k,B_i) \). We say a triple is a \((k,l,m)\)-triple, or is of type \((k,l,m)\), if the corresponding three orthopair classes are \( O(k), O(l), \) and \( O(m) \) in some order.

With this, we can construct a combinatorial analogue to the idea of a geometric Mom-\( n \) structure:

**Definition 4.1.** A combinatorial Mom-\( n \) structure in a one-cusped hyperbolic manifold \( N \) is a collection of \( n \) distinct equivalence classes of triples of horoballs in the universal cover of \( N \), of types \((k_1,l_1,m_1)\), \ldots, \((k_n,l_n,m_n)\), such that the integers \( k_1, l_1, m_1, \ldots, k_n, l_n, m_n \), are all elements of the same \( n \)-element subset of \( \mathbb{Z}_+ \).
We can say trivially that a manifold with a geometric Mom-$n$ structure must also possess a combinatorial Mom-$n$ structure. (For example, the geometric Mom-3 structure on $n069$ described in the previous section corresponds to a combinatorial Mom-3 structure with triples of type $(1,1,2), (1,3,3),$ and $(2,2,3)$.) Proving the implication in the other direction is more difficult, as we will see. But a motivation for doing so comes from the following:

**Theorem 4.2.** If $N$ is a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$ then $N$ possesses a combinatorial Mom-$n$ structure with $n = 2$ or $3$.

The complete proof of the above theorem is one of the key results of [GMM3]; we give a sketch of the result here. As with previous results in this area the result comes from studying the maximal cusp diagram of the manifold $N$.

Let $T$ be a cusp torus in $N$ such that the restriction of the metric to $T$ is flat, and such that $T$ bounds a maximal cusp neighborhood in $N$. Then $T$ will lift to a collection of horospheres in the universal cover $\mathbb{H}^3$, and the cusp neighborhood will lift to a collection of horoballs $\{B_i\}$. Choose one of these horoballs and call it $B_\infty$; then $B_\infty$ is fixed by a subgroup $H$ of $\pi_1(N)$ isomorphic to $\mathbb{Z} + \mathbb{Z}$, and the quotient of $\partial B_\infty$ by $H$ is just $T$. The cusp diagram of $N$ consists of $T$ together with the shadows of all of the other horoballs $\{B_i\}$ on $\partial B_\infty$, modulo the action of $H$. These shadows form a collection of circular disks on $T$ of varying radii, many of which will overlap. We are interested in using these shadows to obtain a lower bound on the area of $T$, for it is a simple matter to prove that $\text{Vol}(B_\infty/H) = \frac{1}{2} \text{Area}(T)$, and $\text{int}(B_\infty)/H$ is embedded in $N$.

This result is used in [Ad], for example, to find the minimum-volume cusped non-orientable manifold as follows. Consider all of those horoballs which are tangent to $B_\infty$. For example, any horoball $B_1$ such that $(B_1, B_\infty)$ is an element of $O(1)$ would be such a horoball. Adams noted that there must be at least two such horoballs modulo the action of $H$; call them $B_1$ and $B_1'$. Specifically one can choose $B_1'$ to equal $g(B_\infty)$, where $g \in \pi_1(N)$ is any element such that $g^{-1}(B_\infty) = B_1$. (The fact that there is no element of $\pi_1(N)$ which exchanges $B_1$ and $B_\infty$ implies that $B_1$ and $B_1'$ will not be in the same $H$-orbit.) Since $B_1$ and $B_1'$ are disjoint horoballs both tangent to $B_\infty$, their shadows on $\partial B_\infty$ must be disjoint disks of radius $1/2$, which implies (using known bounds on circle packing in the plane) that $\text{Area}(T) \geq 2\pi(1/2)^2(\sqrt{2}/\pi) = \sqrt{3}$. Hence from above we have $\text{Vol}(B_\infty/H) \geq \sqrt{3}/2$. Applying the bound on horoball packing in $\mathbb{H}^3$ due to Boroczky ([Bor]) proves that $N$ must have volume at least as large as that of a regular ideal simplex. Showing that the Gieseking manifold is the unique non-orientable manifold which achieves this minimum completes the proof in [Ad].

Note that the above argument uses no facts at all about $N$ other than the fact that it has a non-trivial cusp neighborhood. By assuming orientability and carefully studying the possible arrangements of horoball shadows on $T$, Cao and Meyerhoff significantly extended Adams’s argument, proving in [CM] that the figure-8 knot complement and its sibling were the smallest orientable cusped hyperbolic 3-manifolds. This result differs from the result of [Ad] in an interesting respect. In [Ad], Adams computed a lower bound which turned out to be realized by a particular manifold (similarly, in [Mc3] Meyerhoff computed a lower bound on the volume of hyperbolic orbifolds that was realized by a particular orbifold). In contrast, [CM] sets up a dichotomy. In [CM], maximal cusp diagrams are sorted according to the following question: are there abutting full-sized disks (that is, shadows of
horoballs with $o(k) = 0$) in the diagram? If not then there is enough space to get good bounds on the area of the cusp torus and hence the volume of the manifold. On the other hand, the presence of such abutting disks is a special situation with group-theoretic implications for the Kleinian group $\pi_1(N)$ that can be analyzed to get specific worst cases, namely the two lowest-volume orientable cusped hyperbolic 3-manifolds.

There is a similar dichotomy at work in Mom technology. Loosely speaking, in the maximal cusp diagram either the disks are not close, which leads to good area and hence volume bounds, or else the disks are close, leading to topological implications about the presence of an internal Mom structure. The key insight here is that the presence or absence of a $(k,m,n)$-triple of horoballs in $\mathcal{N}$ provides geometric information about the arrangement of the corresponding shadows. In particular suppose $(B_1, B_j, B_\infty)$ forms a triple of type $(k,m,n)$, with the pair $(B_1, B_\infty)$ belonging to $\mathcal{O}(k)$ and the pair $(B_j, B_\infty)$ belonging to $\mathcal{O}(m)$. (Note that by transitivity of the group action, for any triple of horoballs we can always assume that $B_\infty$ is one of the elements of the triple and there are three ways to do so.) Then we can prove (and do so in [GMM3]) that the shadows of $B_1$ and $B_j$ on $\partial B_\infty$ have radii $\frac{1}{2}e^{-o(k)}$ and $\frac{1}{2}e^{-o(m)}$ respectively, and that there is a path along $T$ joining the centers of these two shadows of length $e^{o(n)−o(m)−o(k)/2}$. Let $e_n = e^{o(n)/2}$ for all $n$ for ease of notation; then the shadows of $B_1$ and $B_j$ have radii $\frac{1}{2}e_i^{−2}$ and $\frac{1}{2}e_j^{−2}$ respectively and the path between their centers has length $e_n/(e_m e_k)$.

Now suppose in particular that $k = m = 1$. Then $e_k = e_m = 1$ since $T$ was chosen to bound a maximal cusp neighborhood, so the shadows of $B_1$ and $B_j$ each have radius $\frac{1}{2}$, and the distance between their centers is $e_n \geq 1$. This together with the obvious circle-packing argument immediately gives a lower bound of $\sqrt{3}$ for the area of $T$, as in [Ad]. However, we can say more. We show in [GMM3] that there are no $(1,1,1)$-triples of horoballs (or indeed any $(m,m,m)$-triples for any $m$) in any orientable manifold $\mathcal{N}$, and hence $n \geq 2$ in the above computation. This implies that the distance between the centers of these two shadows is at least $e_2 \geq 1$, and this in turn improves our bound on the area of $T$ to $\sqrt{3}e_2^2$, a significant improvement if $e_2$ is large. Whereas if $e_2$ is small, we can find at least two additional shadows of radius $\frac{1}{2}e_2^{−2}$ which contribute area to $T$ and improve our lower bound in another way.

Suppose we make a further assumption, namely that there are no $(1,1,2)$-triples in our manifold $\mathcal{N}$. Then the distance between the centers of the first two shadows is at least $e_3$, improving our area bound still further. If we relax our assumption, and assume that there is at most one $(1,1,2)$-triple up to the action of $\pi_1(N)$, then the distance between the centers may be as low as $e_2$, but only along at most one path in $T$. In other words, if we construct disks of radius $e_3/2$ about the centers of the first two shadows, then these new larger disks will overlap at most once and we can still obtain a lower bound on the area of their union and hence on the area of $T$. And if $\mathcal{N}$ has two or more $(1,1,2)$-triples of horoballs, then $\mathcal{N}$ has a combinatorial Mom-2 structure.

Arguing in this fashion in [GMM3] we show that a manifold which does not possess a combinatorial Mom-2 or Mom-3 structure must either have one of the parameters $e_2$ or $e_3$ be sufficiently large (for example, if $e_3 > 1.5152$); the significance of this number is explained in the next section) or else it falls into one of 18 cases enumerated by the presence or absence of certain types of triples of horoballs. If
If $e_2$ or $e_3$ is sufficiently large, an argument similar to the one above shows that $\text{Vol}(M) > 2.848$, while in the remaining 18 cases a rigorous computer-assisted version of the above argument also shows that $\text{Vol}(M) > 2.848$, completing the proof of Theorem 4.2.

5. Upgrading a combinatorial structure

We now wish to take the data associated to a combinatorial Mom-2 or Mom-3 structure in a manifold $N$ and use it to construct a hyperbolic internal Mom structure. In short, we want to upgrade our combinatorial structure to a topological one. The principle is straightforward. If $(B_i, B_j, B_k)$ is a triple of horoballs in the universal cover of $N$ which realizes a triple of type $(l, m, n)$ in a combinatorial Mom-2 or Mom-3 structure, then there is a totally geodesic 2-cell $\sigma$ in $\mathbb{H}^3$ bounded by the shortest geodesic arcs from $B_i$ to $B_j$, $B_j$ to $B_k$, and $B_k$ to $B_i$, along with arcs in the boundaries of all three horoballs. We wish to use the projection of $\sigma$ to $N$ as the core of a 2-handle in a hyperbolic internal Mom-2 or Mom-3 structure for $N$. We do the same for every triple in the combinatorial Mom-$n$ structure. Similarly the geodesic arcs in the boundary of $\sigma$ project to geodesic arcs in $N$, and we wish to use all such arcs as the cores of 1-handles in a hyperbolic internal Mom-$n$ structure.

There are numerous obstacles to this straightforward idea, however:

- The resulting handle structure may not be embedded in $N$.
- Even if it is embedded, the boundary of the resulting submanifold $M$ may not be a collection of tori.
- Even if $\partial M$ is a collection of tori, $M$ may not be a hyperbolic submanifold of $N$, i.e. $M$ may have an essential embedded sphere, annulus, or torus.

We now discuss how each of these obstacles can be overcome.

First there is the question of embeddedness. Here we make use of the fact that the structure we are attempting to construct is a geometric Mom-$n$ structure. For example, suppose a 1-handle in our putative Mom structure intersects itself, because the core geodesic arc of the 1-handle intersects its elf. This implies that the structure we are attempting to construct is a geometric Mom-structure. In short, we want to upgrade our combinatorial structure to a topological one. The principle is straightforward. If $(B_i, B_j, B_k)$ is a triple of horoballs in the universal cover of $N$ which realizes a triple of type $(l, m, n)$ in a combinatorial Mom-2 or Mom-3 structure, then there is a totally geodesic 2-cell $\sigma$ in $\mathbb{H}^3$ bounded by the shortest geodesic arcs from $B_i$ to $B_j$, $B_j$ to $B_k$, and $B_k$ to $B_i$, along with arcs in the boundaries of all three horoballs. We wish to use the projection of $\sigma$ to $N$ as the core of a 2-handle in a hyperbolic internal Mom-2 or Mom-3 structure for $N$. We do the same for every triple in the combinatorial Mom-$n$ structure. Similarly the geodesic arcs in the boundary of $\sigma$ project to geodesic arcs in $N$, and we wish to use all such arcs as the cores of 1-handles in a hyperbolic internal Mom-$n$ structure.

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First there is the question of embeddedness. Here we make use of the fact that the structure we are attempting to construct is a geometric Mom-$n$ structure. For example, suppose a 1-handle in our putative Mom structure intersects itself, because the core geodesic arc of the 1-handle intersects itself. This implies that we have two horoballs $B_i$ and $B_j$ in the universal cover such that the arc $\lambda$ from $B_i$ to $B_j$ intersects the arc $g(\lambda)$ for some $g \in \pi_1(N)$. It can be shown that if $\lambda$ is sufficiently short (specifically if $e_n \leq \sqrt{2}$, where $O(n)$ is the corresponding orthopair class) then the interior of one of $B_i$ or $B_j$ must intersect one of $g(B_i)$ or $g(B_j)$, a contradiction. For slightly longer arcs (specifically if $e_n \leq 1.5152$) it is shown in [GMM3] that this implies that the interior of one of $B_i$, $B_j$ intersects one of $g^k(B_i)$, $g^k(B_j)$ for some $k \leq 4$. Note that as mentioned previously it is also shown in [GMM3] that Vol$(N) > 2.848$ if $e_2 > 1.5152$ or $e_3 > 1.5152$, which implies that longer arcs need not be considered.

Another possibility is that two different 1-handles intersect, in which case there exists four horoballs $B_i$, $B_j$, $B_k$, and $B_l$ in the universal cover such that the arc from $B_i$ to $B_j$ intersects the arc from $B_k$ to $B_l$. In this case it is shown in [GMM3] that if the arcs are sufficiently short (again meaning that $e_m \leq 1.5152$ and $e_n \leq 1.5152$ where $O(n)$ and $O(m)$ are the appropriate orthopair classes) then a new, simpler combinatorial Mom structure can be constructed which excludes one of the two troublesome 1-handles.

Other types of intersections may occur between the various handles of our putative handle structure, but in each case one of these three things occurs: either we obtain a geometric contradiction, or the lengths of the 1-handles in the structure
assume values which imply that $\text{Vol}(N) > 2.848$, or else we can construct a strictly simpler combinatorial Mom structure and start again. Therefore by induction we have the following:

**Theorem 5.1.** If $N$ is a one-cusped hyperbolic 3-manifold and $\text{Vol}(N) \leq 2.848$ then $N$ has a combinatorial Mom-$n$ structure with $n = 2$ or $3$ corresponding to a handle structure on an embedded submanifold $M$.

The arguments appear in full detail in [GMM3]; they are lengthy but elementary.

The next concern is the topology of the boundary components of the resulting manifold $M$. This turns out to be the simplest problem to overcome. By construction the Euler characteristic of $M$ is 0, and hence if the boundary components are not all tori then one of them must be a sphere. Here again we can take advantage of the fact that our putative Mom structure is a geometric one. Given a handle structure of the type described up to this point, with only two 2-handles whose cores are totally geodesic 2-cells, it is in fact impossible to construct a submanifold $M$ of $N$ with at least two boundary components, one of them a sphere. Hence if we start with a combinatorial Mom-2 structure then the boundary components of $M$ are automatically tori. With a combinatorial Mom-3 structure there is one way to construct a submanifold $M$ with a spherical boundary component, but that way requires that the combinatorial structure have exactly two triples of type $(k, l, m)$ where $k$, $l$, and $m$ are three distinct integers. We say that such a combinatorial structure is not torus friendly. The same analysis described in the previous section shows that a manifold with such a combinatorial Mom-3 structure, and no torus friendly combinatorial Mom structures satisfying the previous constraints on $e_2$ and $e_3$, must have volume greater than 2.848, thus dealing with this one exceptional case.

Now we have a submanifold $M$ embedded in $N$, such that the boundary of $M$ is a union of tori, together with a handle structure of the appropriate type. If we can show that $M$ can be assumed to be hyperbolic then we will have the hyperbolic internal Mom-$n$ structure that we desire. (Astute readers may have noticed that we haven’t shown that each component of $\partial M$ is either the cusp torus $T$ or else bounds a solid torus in $N \setminus i(M)$. However if $M$ is hyperbolic then this condition will be satisfied automatically: given the way that $M$ was constructed, the only other possibility is that some boundary component of $M$ bounds a “tube with knotted hole” in $N$, and if this is the case then $M$ would contain an embedded essential sphere.)

As with embeddedness, the idea is to show that if $M$ is not hyperbolic then we can find a strictly simpler internal Mom structure, and hence we can assume that $M$ is hyperbolic by induction. For example, suppose $M$ contains an embedded essential sphere. Following the ideas of Matveev, we assume the sphere is in normal position with respect to the handle structure of $M$ and then split both $M$ and the handle structure along the surface. After throwing away the component of the split manifold which does not contain $T$, the result is a submanifold of $N$ with torus boundary components and one spherical boundary component, which must bound a ball in $N$. Adding that ball as a 3-handle and cancelling it with a 2-handle results in a new submanifold $M_1$ with torus boundary components, and a new handle structure $\Delta_1$ which has strictly lower complexity (in the sense of Matveev) than the handle structure we started with. The new handle structure is
not necessarily in the form of a Mom-$n$, because the 2-handles of the structure may not be attached to the correct number of 1-handles. But 2-handles which attach to four or more 1-handles can be split into 2-handles of valence three. And 2-handles which attach to two or fewer 1-handles can be eliminated from the handle structure, usually by cancelling them with a 1-handle. Furthermore, these operations do not increase Matveev complexity. The end result is a handle structure for $M_1$ in the form of a Mom-$k$, resulting in an internal Mom-$k$ structure on $N$ where the new Mom number $k$ is strictly less than the Mom number we started with.

Since $M$ is compact and $\partial M$ is a non-empty union of tori, $M$ is either hyperbolic or else contains an essential embedded 2-sphere, annulus, or torus. In each case we can construct a new manifold $M_1 \subset N$ with a simpler handle structure as measured by Matveev complexity; the spherical case is outlined in the previous paragraph, but the other cases are more complicated. A technical point is that the new handle structure might not have simply connected lakes (i.e. is not full) and hence this proof requires $n \leq 4$ since it relies on a solution to Problem 10.26 discussed in §10 which is only known for $m \leq 2$. See [GMM2] for more details.

Hence we conclude:

**Theorem 5.2.** If $N$ is a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$ then $N$ has an internal Mom-$n$ structure $(M, T, \Delta)$ with $n = 2$ or $3$ such that $M$ is hyperbolic, and hence $N$ is obtainable by a Dehn filling on $M$.

### 6. Classification of Mom-$n$’s, $n < 4$

Theorem 5.2 allows us to enumerate all one-cusped manifolds with $\text{Vol}(N) \leq 2.848$ in two steps. First, we need to enumerate all hyperbolic Mom-2’s and Mom-3’s, and second, we need to enumerate all Dehn fillings on those manifolds which produce one-cusped manifolds with volume less than or equal 2.848.

The first step is the more straightforward one. If $(M, T, \Delta)$ is a hyperbolic Mom-$n$, then in particular it must be full. This implies that $\Delta$ retracts to a spine for $M$, i.e. a cellular complex which intersects every homotopically non-trivial simple closed curve in $M$. This spine consists of:

- a 0-cell corresponding to every island,
- a 1-cell corresponding to the core of every 1-handle of $\Delta$, and a 1-cell corresponding to every bridge,
- a 2-cell corresponding to the core of every 2-handle of $\Delta$, and a 2-cell corresponding to every lake.

Note that if $(M, T, \Delta)$ were not full, then there might be lakes which are not disks and this construction would not work. We will abuse notation and let $\Delta$ refer to the spine of $M$ from this point forward.

Being a spine of $M$, $\Delta$ is dual to an ideal polyhedral cellulation of $M$ with one 3-cell for every 0-cell of $\Delta$ and an ideal vertex for every boundary component of $M$; clearly there are $2n$ such 3-cells, two for each 1-handle in the Mom-$n$. If a 1-handle has valence two, then the corresponding dual 3-cells will be pyramids built on a digonal base; we ignore such 3-cells after flattening them down to a face. If a 1-handle has valence $v > 2$, then the two corresponding dual 3-cells are each pyramids built on a $v$-sided polygon as base; see Figure 4. Furthermore, the bases are each dual to the 1-handle itself, and hence are glued together in the resulting dual cellulation, resulting in a $v$-sided dipyramid. Also, the sums of the valences of
the 1-handles in the Mom-$n$ equals the sums of the valences of the 2-handles, which is $3n$ by definition.

This generates a finite number of possible polyhedral decompositions for a hyperbolic Mom-2 or Mom-3. Specifically, a hyperbolic Mom-2 must be composed of either a single ideal octahedron, or two ideal three-sided dipyramids. A hyperbolic Mom-3 must consist of either a single ideal five-sided dipyramid, an ideal octahedron and an ideal three-sided dipyramid, or three ideal three-sided dipyramids. And furthermore, the “polar” ideal vertices of these dipyramids, i.e. the vertices not adjacent to the bases of the original pyramids, are all dual to the $T \times \{0\}$ boundary component of $M$ and hence must be identified to other “polar” vertices by the face-pairing identifications. This is enough information to enumerate all of the possible hyperbolic Mom-2’s and Mom-3’s.

Computer assistance was used here, to generate a candidate list of polyhedral gluings. The number of such gluings is a factorial function of the total number of faces; to shrink this list, at this stage the computer was programmed to check that the links of ideal vertices had Euler characteristic zero, and to use the obvious symmetries of the dipyramids to eliminate redundant gluings whenever possible. This resulted in 44 candidate polyhedral gluing descriptions for Mom-2’s and 4187 such descriptions for Mom-3’s. Then SnapPea was used to make a preliminary identification of which gluing descriptions corresponded to hyperbolic manifolds, and to eliminate duplicates among the hyperbolic manifolds.

In each case where SnapPea found a hyperbolic structure it also found a triangulation of the manifold identical to a triangulation of a manifold in the SnapPea census. Since Harriet Moser has confirmed the hyperbolicity of all the manifolds in the census ([Mose]) this confirms the hyperbolicity of our Mom-2’s and Mom-3’s; i.e., SnapPea did not make any false positive errors. For those manifolds for which SnapPea failed to find a hyperbolic structure, the fundamental group of the manifold was computed and examined by hand, and it was shown that none of these groups was the fundamental group of a finite-volume hyperbolic manifold. (In most cases, this was accomplished by showing that the group had a non-trivial center. A few cases required a more detailed analysis, usually involving finding distinct maximal abelian subgroups with non-trivial intersection, which is impossible in a discrete co-finite volume group of hyperbolic isometries.) In this way, we confirmed that SnapPea did not make any false negative errors. The results of this analysis are as follows:
Theorem 6.1. If \((M, T, \Delta)\) is a hyperbolic Mom-2 then \(M\) is homeomorphic to one of the manifolds known in the SnapPea census as \(m_{125}, m_{129}\), or \(m_{203}\). If \((M, T, \Delta)\) is a hyperbolic Mom-3 then \(M\) is homeomorphic to one of the preceding three manifolds, or to one of \(m_{202}, m_{292}, m_{295}, m_{328}, m_{329}, m_{359}, m_{366}, m_{367}, m_{391}, m_{412}, s_{596}, s_{647}, s_{774}, s_{776}, s_{780}, s_{785}, s_{786}, s_{788}, s_{898}, \) or \(s_{959}\). □

Some comments about the above list: \(m_{129}\) is the complement of the Whitehead link, and \(m_{203}\) is the complement of the link known as \(6^2_2\) in standard link tables. Among the Mom-3’s, \(s_{776}\) is the complement of the alternating three-element chain link, sometimes known as the “magic manifold” and extensively analyzed in [MP]; it is also the only three-cusped manifold in the above list, and the manifolds from \(m_{125}\) to \(m_{391}\) in the above list can all be obtained by a Dehn filling on \(s_{776}\).

A similar analysis was performed for hyperbolic Mom-4’s, and resulted in a list of 138 hyperbolic Mom-4’s which included each of the 21 manifolds listed above. Some aspects of this analysis, however, are still conjectural. In particular we have not yet analyzed the fundamental groups of all the manifolds for which SnapPea failed to find a hyperbolic structure. The list of Mom-4’s includes some manifolds which do not appear in the SnapPea census; descriptions of these manifolds can be found in [GMM2]. See the Mom-4 Conjecture in §10.

7. Enumeration of filled manifolds

After identifying all hyperbolic Mom-2’s and Mom-3’s, the next step is to identify all Dehn fillings on those manifolds which might result in one-cusped hyperbolic manifolds of volume less than or equal to 2.848. We turn to [FKP], which says the following:

Theorem 7.1. (Futer, Kalfagianni, and Purcell) Let \(M\) be a complete, finite-volume hyperbolic 3-manifold with cusps. Suppose \(C_1, \ldots, C_k\) are disjoint horoball neighborhoods of some subset of the cusps. Let \(s_1, \ldots, s_k\) be slopes on \(\partial C_1, \ldots, \partial C_k\), each with length greater than \(2\pi\). Denote the minimal slope length by \(l_{\text{min}}\). If \(M(s_1, \ldots, s_k)\) satisfies the Geometrization Conjecture, then it is a hyperbolic manifold, and

\[
\text{Vol}(M(s_1, \ldots, s_k)) \geq \left(1 - \left(\frac{2\pi}{l_{\text{min}}}\right)^2\right)^{\frac{3}{2}} \text{Vol}(M).
\]

Note that a \textit{slope} here just denotes the homotopy class of curves on the boundary of \(M\) which end up glued to the boundary of a disk after Dehn filling, and \(M(s_1, \ldots, s_k)\) denotes the filled manifold. Assuming the Geometrization Conjecture is true, then the above theorem has the following simple reformulation, as noted in [Mill2]:

Corollary 7.2. Suppose \(M, s_1, \ldots, s_k,\) and \(l_{\text{min}}\) are defined as above, and let \(N = M(s_1, \ldots, s_k)\). If \(N\) is hyperbolic we have

\[
l_{\text{min}} \leq 2\pi \left(\sqrt{1 - \left(\frac{\text{Vol}(N)}{\text{Vol}(M)}\right)^{\frac{2}{3}}}\right)^{-1}.
\]
(Note that the right-hand side of the above inequality is always greater than or equal to $2\pi$, so that condition on the length of the slopes can dropped; if the boundary slopes do not all have length greater than $2\pi$ then the inequality is trivially true.)

Applying Corollary 7.2 to each of the manifolds in Theorem 6.1, and assuming that $\text{Vol}(N) \leq 2.848$, we see that there are a finite number of Dehn fillings on each manifold that need to be considered. Furthermore these fillings are easily enumerated; see [Mill2] for details. It is then necessary to determine which of these fillings result in hyperbolic manifolds and which do not, and for those manifolds which are hyperbolic it is necessary to rigorously prove which filled manifolds actually have volume no greater than 2.848. Again, these steps were completed with computer assistance; first, Snap (see [Goo]) was used to make a preliminary determination, and then Snap’s conclusions were confirmed by more rigorous means.

We used Snap rather than SnapPea at this step to make more extensive use of Harriet Moser’s algorithm in [Mose]. That is, rather than attempt to locate each hyperbolic manifold in SnapPea’s census (a more daunting task here than in the proof of Theorem 6.1 due to the larger number of cases to consider) we instead applied Moser’s algorithm directly to the filled manifolds under consideration to prove their hyperbolicity. Moser’s algorithm is designed to use data produced by Snap as input, hence it was more convenient here to use Snap rather than SnapPea. In this way we confirmed the hyperbolicity of those one-cusped filled manifolds for which Snap claimed to find a hyperbolic structure.

For those manifolds for which Snap failed to find a hyperbolic structure, we confirmed non-hyperbolicity by examining the manifolds’ fundamental groups and checking for non-trivial centers and the like, as in the proof of Theorem 6.1.

Next, for those filled manifolds which are hyperbolic it is necessary to rigorously establish which manifolds satisfy $\text{Vol}(N) \leq 2.848$. Here again we turn to computer assistance. At this point in the process Snap has already produced an ideal triangulation of all of the manifolds in question, and has computed values for the shape parameters of each of the tetrahedra involved. From this information, the volume of each manifold can be computed via the Lobachevsky function $\mathcal{L}(\theta)$ (see, for example, [Miln]); however the question of floating-point error must be addressed. Fortunately, one of the intermediate steps in Moser’s algorithm is to compute an error bound $\delta$ on the distance in $\mathbb{C}^k$ between the shape parameters computed by Snap and the actual shape parameters associated to the hyperbolic metric. With this information and using affine 1-jets as in [GMM3] we can rigorously compute an upper and lower bound on the volume of each hyperbolic manifold under consideration.

Finally, for each hyperbolic filled manifold of sufficiently small volume an isometry was found between that manifold and a manifold in the SnapPea census. By this process, the following theorem was proved in [Mill2]:

**Theorem 7.3.** If $N$ is a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$ and $N$ is obtainable by filling one of the manifolds in Theorem 6.1 then $N$ is one of the first ten orientable one-cusped manifolds in the SnapPea census; that is, $N$ is one of $m003$, $m004$, $m006$, $m007$, $m009$, $m010$, $m011$, $m015$, $m016$, or $m017$.

□

This together with Theorem 5.2 establishes the following:
Theorem 7.4. There are only ten orientable one-cusped hyperbolic manifolds \( N \) with \( \text{Vol}(N) \leq 2.848 \), and these are exactly the first ten orientable one-cusped manifolds in the SnapPea census.

8. History of the compact case

Before completing our discussion of the proof of Theorem 1.1, we take a moment to discuss the history of the search for the minimum-volume compact hyperbolic 3-manifold.

The Kazhdan-Margulis theorem when applied to hyperbolic 3-manifolds establishes the existence of a constant \( \epsilon \) and a natural decomposition of any hyperbolic 3-manifold into an \( \epsilon \)-thick part and an \( \epsilon \)-thin part; see [KM]. This decomposition can be used to show that there is a positive lower bound to the volume of hyperbolic 3-manifolds. This approach is carried out in Section 4 of [Bu], where Buser and Karcher implemented an idea of Gromov.

Virtually all low-volume bounds arise out of analyses of embedded solid tubes. The first such analysis was in [Me1], where Meyerhoff used Jorgensen’s trace inequality (see [Jo]) to construct solid tubes around short geodesics in hyperbolic 3-manifolds. The shorter the geodesic, the larger the solid tube is, not only in radius but in volume. From this construction follows a trade-off argument: either a hyperbolic 3-manifold has a short geodesic and hence a solid tube with decent volume, or it doesn’t, in which case there must be an embedded ball, again of decent volume. The resulting bound is roughly 0.0006.

Subsequently, sphere-packing in \( H^3 \) was used to gain some control over the volume of a hyperbolic 3-manifold outside of an embedded ball. This improved volume bound in the embedded ball case can be used to make an improved trade-off in the tube-versus-ball argument of [Me1]. The necessary sphere-packing results had been produced earlier by Boroczky and Florian (see [BF], or [Bor] for an English version), and the volume bound was pushed up to roughly 0.0008 in [Me2]. Gehring and Martin noted that the calculations in [Me1] could be made a bit finer and produced a bound of roughly 0.001 in [GM1].

For a number of years no further progress on lower bounds was made in the closed case. However in the cusped case, Adams was able to use the maximal cusp diagram to get improved bounds for \( v_c \). As previously noted in §4, Adams showed in [Ad] that the volumes of orientable and non-orientable cusped hyperbolic 3-manifolds are bounded below by \( V = 1.01 \ldots \), the volume of the ideal regular simplex; he further showed that in the non-orientable case this volume is realized uniquely by the Gieseking manifold.

In the mid-1990’s Gabai, Meyerhoff and N. Thurston needed to greatly improve known solid-tube radius bounds so as to apply Gabai’s Rigidity Theorem and prove that homotopy hyperbolic 3-manifolds are hyperbolic. To do this they analyzed 2-generator groups naturally associated to the shortest geodesic in a hyperbolic 3-manifold. The space of such groups is determined by three complex parameters, two of which are the complex length of the shortest geodesic and the complex distance between the two nearest lifts of that geodesic to \( H^3 \). (A third parameter is necessary to actually determine the isometry between those two lifts.) With some work, the relevant 3-parameter space was shown to be compact and a rigorous computer analysis of this space was carried out in [GMT]. One crucial tool
needed to obtain compactness of the parameter space is the solid-tube construction described above; in particular, according to [Mc1], shortest geodesics of length less than 0.0979 automatically have big enough tubes. The result of the [GMT] analysis shows that, with seven manageable families of exceptions, the shortest geodesic in a hyperbolic 3-manifold has tube radius greater than \( \log(3)/2 \). This is a considerable improvement on previous tube bounds, and directly leads to a manifold volume bound of roughly 0.1, a hundred-fold improvement (the exceptions all must have volume above 1.0).

Coupling this \( \log(3)/2 \) result with a (slightly earlier) remarkable theorem of Gehring and Martin produced a volume bound of 0.166. See [GM2] and [GMT]. Gehring and Martin showed how to generalize the maximal cusp diagram approach (e.g., in [Ad]) to the case of closed manifolds and maximal solid tubes. In the cusped case the maximal cusp diagram is formed by the shadows of horoballs \( \{ B_i \} \) on the boundary of a base horoball \( B_\infty \); the resulting shadows are disks. In the closed case, instead of horoballs a maximal solid tube around a short geodesic is lifted to \( \mathbb{H}^3 \) to get a collection of solid tubes. One of these tubes is designated as the base, and then the other tubes are projected to the boundary of the base tube thereby creating a collection of shadows on that boundary. Arguments similar to those in the cusped case should work here as well, but there are daunting technical problems involved in projecting tubes onto the tube boundary, as the shapes of the shadows are now much more complicated. Gehring and Martin were able to sidestep many of these problems by embedding a ball within each tube and projecting that ball to the base tube instead, although this sacrifices some volume. Their approach results in a formula for volume of the solid tube in terms of the radius of the tube alone; surprisingly the length of the core geodesic is not needed.

Przeworski then improved on the Gehring-Martin maximal tube diagram approach by analyzing the shadows of the solid tubes themselves rather than the shadows of balls, and by analyzing some of the volume outside the maximal tube. He produced a volume lower bound of 0.28, in [Pr1] and [Pr2].

Given the success of the [GMT] method, it seemed natural to try to extend the parameter space argument of [GMT] and to focus it more strictly on volume questions. The first step in such a procedure would be to produce an appropriate compact parameter space to analyze. Because the lowest known volume manifold was the Weeks manifold with volume 0.942\ldots and because the volume bound produced by Adams in the cusped case is 1.01\ldots it would seem clear that very short geodesics (nearly cusps) could be eliminated from the parameter space argument. The problem was that Adams’s bound utilized horoball-packing results, and these packing results could not be generalized at the time to the case of short geodesics and big tubes. Thus the volume bound of [GM2] in the case of very short geodesics (or tubes of very large radius) approaches \( \sqrt{3}/2 = 0.866\ldots \) in the limit. This is Adams’s result when horoball-packing is ignored, and not adequate for attaining the low-volume manifold.

The authors were able to produce the desired compact parameter space without use of tube-packing in [GMM1]. We introduced a simple method for improving on the cusped volume bound of Adams, and then perturbed the method to the closed case thereby producing a compact length bound for the parameter space. Our method was to look at the next-largest disks in the maximal cusp diagram, i.e. the shadows of the horoballs which are at a distance of \( o(2) \) from \( B_\infty \), to
use the language of this paper. What makes this argument work is the following dichotomy, similar to the arguments of §4: either these horoballs are close to $B_\infty$ and the associated shadows produce substantial extra area and hence volume, or they are not close in which case the centers of the full-sized disk shadows must be far from each other, again producing substantial extra area and hence volume. This approach carries over to the closed case as well.

Note that despite having an appropriate compact parameter space to analyze, the [GMT] approach has so far proven difficult to extend beyond the original bounds. Note also that the implicit improvement on Adams’s cusp bound in [GMM1] was considerably less than the improvement that had already been attained by Cao and Meyerhoff. In fact, Cao and Meyerhoff had doubled Adams’s bound and this turned out to be precisely $v_\omega$; see [CM]. One could try to perturb the Cao-Meyerhoff methods to the closed case, but they are quite intricate and this approach was sidelined as other events moved to the fore; specifically, [GMM1] seemed to spark a flurry of activity.

First Marshall and Martin showed how to rethink certain aspects of the Gabai-Meyerhoff-Milley argument and were able to improve the volume bound for closed manifolds to 0.2855, in [MM1]. In separate research, Marshall and Martin developed tube-packing methods in $\mathbb{H}^3$ in the large-tube-radius setting. That is, if the radius of a tube is sufficiently large (roughly radius 5) in a hyperbolic 3-manifold, then the lifts of the tube are geometrically sufficiently similar to a horoball that packing bounds can be successfully obtained; see [MM2].

Then by an elegant argument in [Pr3], Przeworski obtained tube-packing results in broader generality than [MM2]. Przeworski’s results are typically applied whenever any new volume result is established, via a tube volume argument.

In a major development, Agol studied the relationship between a closed hyperbolic 3-manifold with an embedded geodesic and the associated cusped manifold obtained by removing that geodesic. Using delicate geometric constructions and applying a result of Boland-Connell-Souto [BCS], Agol was able to bound the volume of the closed manifold in terms of the volume of the cusped manifold and the radius of the maximal tube around the geodesic. Using the volume bounds of [CM], Agol produced a volume lower bound of 0.32 for closed manifolds in [Ag1]. Przeworski then followed this result with [Pr4], which further improved bounds for the density of tube packings. Combining these results with the results in [Ag1], Przeworski obtained a volume bound of 0.3324.

Finally, Agol and Dunfield realized that Perelman’s work on Ricci curvature (en route to Perelman’s proof of Thurston’s Geometrization Conjecture) substantially improves the results of [Ag1], which involved Ricci curvature arguments. The volume bound so produced is 0.66, and Agol, Storm, and Thurston re-proved this result in [AST]. (Both the 0.3324 and 0.66 results use the Log(3)/2 theorem of [GMT].) Of course, this is close to the hoped-for bound of 0.942....

This is the point where Mom technology re-enters the narrative, for it is Agol and Dunfield’s result together with the classification of low-volume cusped manifolds due to Mom technology that allow us to prove that the Weeks manifold is volume-minimizing, as described in the next section.
9. The minimum-volume closed manifold

Starting with Theorem 7.4 we want to apply the results of Agol and Dunfield to tackle the closed case. Although these results are documented in [AST], for our purposes it was convenient to use a slightly different formulation of the same result which appears as Lemma 3.1 of [ACS1]:

**Lemma 9.1.** Suppose that $N$ is a closed orientable hyperbolic 3-manifold and that $C$ is a shortest geodesic in $N$ with an embedded tubular neighborhood of radius at least $\log(3)/2$. Set $M = N \setminus C$, equipped with a hyperbolic metric. Then $\operatorname{Vol}(M) < 3.02 \operatorname{Vol}(N)$.

Recall that the Weeks manifold $W$ satisfies $\operatorname{Vol}(W) = 0.9427 \ldots$, and furthermore note that in the above theorem if $\operatorname{Vol}(M) > 2.848$ then $\operatorname{Vol}(N) > 0.943 > \operatorname{Vol}(W)$. This is the reason behind the choice of 2.848 as a volume bound in Theorem 7.4. Also note that if the shortest geodesic in $N$ does not have an embedded tubular neighborhood of radius at least $\log(3)/2$ then $\operatorname{Vol}(N) > \operatorname{Vol}(W)$ according to [GMT]. Hence combining Theorem 7.4, Lemma 9.1, and [GMT] yields the following:

**Theorem 9.2.** Suppose that $N$ is a closed orientable hyperbolic 3-manifold with $\operatorname{Vol}(N) \leq \operatorname{Vol}(W)$. Then $N$ is obtained by Dehn filling on one of the first ten orientable one-cusped manifolds in the SnapPea census.

Clearly we now need to enumerate all Dehn fillings of those ten manifolds which can result in a closed manifold with volume no greater than 0.943. This analysis, performed in [Mill2], is similar enough to the proof of Theorem 7.3 that we will not repeat the details here. We will mention a complication which did not occur in the proof of Theorem 7.3 however. One of the closed manifolds that needs to be examined for this analysis is the manifold $\operatorname{Vol}3$, the third-smallest known closed hyperbolic manifold. As the name suggests, $\operatorname{Vol}3$ does not have volume smaller than the Weeks manifold. However proving this using the techniques used in the proof of Theorem 7.3 is complicated by the fact that $\operatorname{Vol}3$ is the only known hyperbolic manifold for which a non-negatively oriented ideal triangulation has not been found. Moser’s algorithm and the standard formula for hyperbolic volume of manifolds both depend on having an ideal triangulation without negatively oriented tetrahedra. Hence in this one case we considered not $\operatorname{Vol}3$ but the unique double cover of $\operatorname{Vol}3$, and showed that its double cover had volume no less than 1.886. (See also the discussion following Problem 10.34 in the next section.) All other closed fillings of the manifolds listed in Theorem 7.3, except for the Weeks manifold, have volume greater than 0.943. This completes the proof of Theorem 1.1.

10. Problems and Directions

We close this paper with some open problems and possible future directions for research. To begin with, the authors view the work of [GMM1], [GMM2], [GMM3] and [Mill2] as steps in addressing the following:

**Hyperbolic Complexity Conjecture 10.1.** (Thurston, Hodgson-Weeks, and Matveev-Fomenko) The complete low-volume hyperbolic 3-manifolds can be obtained by filling cusped hyperbolic 3-manifolds of small topological complexity.
Remark 10.2. A detailed discussion of this conjecture can be found in the introduction to [GMM2]. In particular, one of the challenges is to quantify the adjectives low and small. Our point of view is that, at least for low-volume manifolds, the Mom number is an excellent topological measure compatible with volume.

The experimental evidence provided by SnapPea is compelling. Among the 1-cusped manifolds in the SnapPea census, experimental evidence suggests that all such manifolds with volume at most 3.18 (resp. 4.05, resp. 5.33) have Mom number two (resp. at most three, resp. at most four).

Among the 117 smallest closed orientable 3-manifolds in the census, i.e. the manifolds with volume less than 2.5, all but 5 have internal Mom-2 structures which are based on a shortest closed geodesic. This means that some boundary component of the Mom manifold bounds a solid torus whose core is a shortest geodesic. Four of the remaining 5 have internal Mom-3 structures based on a shortest geodesic; they are $m038(-1,2)$, $m038(1,2)$, $m038(4,1)$, and $m038(-5,1)$. The remaining manifold, $m207(1,2)$ in the SnapPea census, has volume approximately equal to 2.468 and an internal Mom-4 structure based on a shortest geodesic (specifically, the core geodesic is a shortest geodesic and $m207$ itself has a Mom-4 structure).

Problem 10.3. Develop a Mom technology theory for closed orientable hyperbolic 3-manifolds and directly prove that the Weeks manifold is the lowest volume closed hyperbolic 3-manifold.

That is, generalize the methods of [GMM3] to directly address closed 3-manifolds. For example, let $\gamma$ be a shortest geodesic in the 3-manifold $N$ and $V$ a maximal solid tube about $\gamma$. By passing to the universal covering of $M$, fixing one preimage $V_0$ of $V$ and considering the other preimages $\{V_i\}$ it makes sense to talk about orthoclasses, triples and hence the notion of combinatorial Mom-$n$ structure. As noted in §8, Prezowski has [Pr2] developed the theory of shadows of projections of one solid tube onto another in [Pr2]. With a generalization of the “lessvol” function which appears in [CM] and [GMM3] to the closed case, one could directly generalize [GMM3].

If $V$ is the maximal solid tube described above, by [GMT] we know that either $V$ has tube radius at least $\log(3)/2$ or $M$ lies in one of seven exceptional families of 3-manifolds. By [JR] two of these families are isomorphic, and by [GMT], [Ly] and [CLLM] associated to each of these families is a unique closed orientable hyperbolic 3-manifold.

Problem 10.4. Complete the proof of Conjecture 1.31 of [GMT] by showing that each of these six manifolds $N_0, N_1, \ldots, N_5$ cover only themselves.

Remark 10.5. Jones and Reid showed in [JR] that $N_3$ (also known as Vol3) nontrivially covers no 3-manifold and Reid showed in an appendix to [CLLM] that $N_1$ and $N_5$ nontrivially cover no 3-manifold.

See Corollary 1.29, Remarks 1.32, and Theorem 4.1 of [GMT] for more information about these exceptional manifolds.

Problem 10.6. Find the lowest volume closed nonorientable 3-manifolds.

One difficulty with nonorientable 3-manifolds is that one of the hypotheses of the $\log(3)/2$ theorem of [GMT] is orientability. In particular it is not applicable to orientation reversing curves. Nevertheless, by Milley some information carries over; see [Mill].
Problem 10.7. Find a version of the \( \log(3)/2 \) theorem for shortest geodesics in closed nonorientable hyperbolic 3-manifolds and for cusped hyperbolic 3-manifolds. Improve the value of \( \log(3)/2 \) for closed orientable 3-manifolds.

Problem 10.8. Find the lowest volume cusped nonorientable 3-manifolds with torus cusps.

Remark 10.9. [Ad] showed that the Gieseking manifold is the unique lowest volume cusped, nonorientable hyperbolic 3-manifold, but this manifold has a Klein bottle cusp. The manifold known as \( m131 \) in the SnapPea census is a nonorientable manifold with a single torus cusp and has volume equal to the volume of the Whitehead link complement, which is approximately 3.663; it is the smallest such manifold known to the authors. The filled nonorientable manifold \( m131(3, 1) \) has volume equal to the volume of the figure-eight knot complement, i.e. approximately 2.029, and is the smallest compact non-orientable manifold known to the authors. Nathan Dunfield [D2] points out that this manifold fibers over \( S^1 \) with orientation-reversing monodromy, with fiber a surface of genus 2.

Problem 10.10. Find the lowest volume closed Haken 3-manifolds.

Remark 10.11. As of this writing the smallest known Haken manifold is the manifold obtained by \( (\frac{14}{3}, \frac{3}{2}) \) Dehn filling on the Whitehead link complement. It was discovered by Nathan Dunfield [D1] and has volume volume approximately 2.207.

Problem 10.12. Find the lowest volume closed orientable fibered 3-manifolds.

Remark 10.13. The smallest known closed fibered manifold is \( m011(9, 13) = v0073(3, -1) \). It has volume 2.7317\( \cdots \) and is a genus-5 bundle. It was discovered by Saul Schleimer and described to the authors by Nathan Dunfield [D2].

Problem 10.14. Find the lowest volume closed orientable 3-manifolds with \( \beta_1 = n \).

Remark 10.15. Manifolds with \( \beta_1 = 0 \) are orientable, hence the Weeks manifold is the smallest manifold, orientable or not, with \( n = 0 \) [GMM3]. For \( n = 1 \), the manifold of the previous remark is the smallest known manifold. It experimentally minimizes all longitudinal fillings of the 1-cusped census manifolds and minimizes fillings of longitudinal surgery on knots of 13 crossings or less; see [D2]. For \( n = 2 \) the smallest known example is \( (0/1, 0/1) \)-surgery on link \( 9_2^3 \) also known as \( v1539(5, 1) \). This manifold, discovered by Dunfield, has volume 4.7135\( \cdots \) and is a genus-2 fiber bundle. Experimental work of Dunfield [D2] shows that this minimizes volume among fillings of 2-cusped census manifolds and 0-surgery on homologically split 2-component links with at most 14 crossings.

Problem 10.16. Determine the first infinite stem of closed and/or cusped hyperbolic 3-manifolds.

Remark 10.17. Agol has shown in [Ag2] that the complements of the Whitehead link and the pretzel link \( (-2,3,8) \) are the two lowest volume 2-cusped hyperbolic 3-manifolds. Their volumes are 3.663\( \ldots \). Thus we need to determine all the 1-cusped hyperbolic 3-manifolds with volume at most 3.663\( \ldots \).

Problem 10.18. Find the \( n \)-cusped complete finite volume hyperbolic 3-manifolds of least volume.
Remark 10.19. As previously mention this was solved in the cases n=1,2 respectively by Cao - Meyerhoff [CM] and Agol [Ag2].

Problem 10.20. Develop a theory of low-volume manifolds via minimal surfaces.

Remark 10.21. For example if N is nonorientable, then N contains an embedded \(\pi_1\)-injective surface. Such a surface can be represented by a stable minimal surface \(T\). By Uhlenbeck, \(\text{Area}(T_0) \geq -\pi(\chi(T_0)) \geq \pi\); see [Ha]. Thus thickening and expanding the surface creates lots of volume unless it rapidly crashes into itself, i.e. encounters handles of index \(\geq 1\). A careful analysis should yield lower bounds on volumes and a suitable understanding of the geometry should yield a finite set of manifolds which contains the lowest volume one.

If N is orientable, then as announced by Pitts - Rubinstein in [PR] one can find an index-1 minimal Heegaard surface or a stable 1-sided Heegaard surface \(T\). Again careful estimates should yield lower bound estimates on volume. Note that if \(T\) is a 1-sided Heegaard splitting, then \(N\) will also contain an index-1 minimal surface disjoint from \(T\).

Mom-4 Conjecture 10.22. The collection of hyperbolic Mom-4 manifolds is exactly the set of 117 manifolds enumerated in [GMM2].

Problem 10.23. Let \(N\) be a hyperbolic 3-manifold possessing an internal Mom-\(n\) structure. Does \(N\) necessarily possess a geometric Mom-\(n\) structure, i.e. a structure where the 1-handles and 2-handles of \(\Delta\) are neighborhoods of geodesic arcs and totally geodesic surfaces respectively?

Problem 10.24. Let \(N\) be as above. Does \(N\) necessarily possess a geometric Mom-\(n\) structure where the 1-handles and 2-handles of \(\Delta\) are neighborhoods of edges and faces in a canonical ideal cellulation of \(N\)?

Problem 10.25. Let \(N\) be a hyperbolic 3-manifold possessing a full internal Mom-\(n\) structure. Does \(N\) possess an hyperbolic full internal Mom-\(k\) structure for some \(k \leq n\)?

Problem 10.26. Let \(N\) be a hyperbolic 3-manifold possessing a general based internal Mom-\(m\) structure. Does \(N\) possess an internal full hyperbolic Mom-\(k\) structure for some \(k \leq m\)?

Remarks 10.27. (1) We conjecture that the answer to each of the last four problems is “no” for \(n\) sufficiently large.

(2) See [GMM2] for the definition of a general based structure. Such a structure may arise if the internal Mom structure has annular lakes, the simplest example being the figure-8 knot as shown in Figure 2.

(3) A positive solution to Problem 10.26 for \(n = 2\) is given in Lemma 4.5 of [GMM2]. A positive solution for \(n = 3\) under the additional hypothesis that the Mom-3 structure is geometric with the base torus cutting off a cusp is given in [GMM3].

(4) A positive solution to Problem 10.25 for \(n = 4\) is given in Theorem 4.1 of [GMM2].

(5) Versions of the Mom-4 Conjecture, Problem 10.25, and Problem 10.26 are needed to extend the Mom-3 technology of [GMM3] to Mom-\(n\) technology. Developing a suitable enumeration of general based Mom-\(n\) manifolds would enable one to get around these issues.
A striking application of the Mom technology in combination with other geometric and topological arguments is the Lackenby-Meyerhoff solution [LM] of the long-standing Gordon conjecture.

**Theorem 10.28.** (Lackenby - Meyerhoff) Let $M$ be a compact 3-manifold with boundary a torus, with interior admitting a complete finite-volume hyperbolic structure. Then the number of non-hyperbolic Dehn fillings on $M$ is at most 10.

**Problem 10.29.** Find all 1-cusped hyperbolic 3-manifolds with low-volume maximal cusps. In particular enumerate all such manifolds with cusp volume at most 2.5.

Since the volume of a cusp is one half the area of its boundary torus this question is equivalent to enumerating all 1-cusp manifolds with maximal tori of area at most 5.

A solution to this problem would have the following two applications. First, much of the work in [GMM3] is involved with getting estimates on the volume of the maximal cusp. Hence a solution to Problem 10.29 would provide a significant weapon for attacking Problem 10.16.

Second, [LM] makes vital use of cusp area bounds and thus Problem 10.29 would provide an avenue towards establishing the strong form of the Gordon Conjecture:

**Conjecture 10.30.** The figure-8 knot complement is the unique 1-cusped hyperbolic 3-manifold which realizes the maximal number of non-hyperbolic Dehn fillings.

Of course, the dichotomy inherent in Mom technology makes it a natural tool for working on generalizations of Conjecture 10.30. For example:

**Problem 10.31.** List all 1-cusped hyperbolic 3-manifolds that have 6 or more non-hyperbolic Dehn fillings.

There are infinitely many 1-cusped manifolds arising from filling one component of the Whitehead link or its sister that have six exceptional surgeries; see [Gor]. Thus a satisfactory solution to Problem 10.31 would be to list finitely many multi-cusped manifolds (preferably 2-cusped or 3-cusped) and all their fillings that yield a 1-cusped manifold that has six or more exceptional fillings.

The following problems and discussion on number-theoretic issues were generously provided by Walter Neumann and Alan Reid.

**Problem 10.32.** Suppose that $M$ is a finite-volume hyperbolic 3-manifold. Is the volume of $M$ irrational?

There seems to be no explicit reference for this “folklore” question. It is worth remarking that by Apery’s proof that $\zeta(3)$ is irrational it is known that there are finite volume hyperbolic 5-manifolds with irrational volumes (see [Ker]).

In fact, there are much more far-reaching questions than this, namely explicit conjectures about when volumes are linearly dependent over $\mathbb{Q}$, and the same for Chern Simons (which can be rational) [NY]. This is discussed briefly below.

As described in [N], these conjectures have appeared in different forms in the literature. For volume they are equivalent to the sufficiency of the Dehn invariant conjecture for $\mathbb{H}^3$ scissors congruence.
The following conjecture due to Milnor appears in [Miln]; see also [N] and [NY]. Here \( \mathcal{L}(\theta) \) denotes the Lobachevsky function.

**Conjecture 10.33. (Milnor, [Miln])** If we consider only angles \( \theta \) which are rational multiples of \( \pi \), then every \( \mathbb{Q} \)-linear relation

\[
q_1 \mathcal{L}(\theta_1) + \cdots + q_n \mathcal{L}(\theta_n) = 0
\]

is a consequence of the relations

\[
\begin{align*}
\mathcal{L}(\pi + \theta) &= \mathcal{L}(\theta), \\
\mathcal{L}(-\theta) &= -\mathcal{L}(\theta), \\
\mathcal{L}(n\theta) &= n \sum_{k \mod n} \mathcal{L}(\theta + k\pi/n).
\end{align*}
\]

In a similar vein, the following appears as question 23 of [T2].

**Problem 10.34. (Thurston, [T2])** Show that the volumes of hyperbolic 3-manifolds are not all rationally related.

By work of Borel [Bo], it is known that given an arithmetic hyperbolic 3-manifold \( M \) with invariant trace-field \( k \) there is a real number \( v_k \) such that \( \text{Vol}(M) \) is an integral multiple of \( v_k \).

This has the practical application that it can be used to prove that an approximation to the volume of an arithmetic manifold can be made exact. For example, in [JR] it is proved (see Lemma 3.2 and the proof of Theorem 3.1 of [JR]) that \( \text{Vol}3 \) has volume \( v_0 \) (the volume of the regular ideal tetrahedron in \( \mathbb{H}^3 \)).

For hyperbolic 3-manifolds, arithmetic or otherwise, another result of Borel [Bo2] (see also [NY]) says:

**Theorem 10.35.** For any number field \( k \) with \( r \) complex places, there are real numbers \( v_1, v_2, \ldots, v_r \) such that for any finite-volume hyperbolic 3-manifold \( M \) whose invariant trace-field is \( k \) there are \( r \) integers \( \alpha_1, \ldots, \alpha_r \) such that

\[
\text{Vol}(M) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r.
\]

**Problem 10.36. (Neumann-Reid) For a number field \( k \) as above, identify the real numbers \( v_1, \ldots, v_r \).**

Even when \( r = 1 \), so that we are in the situation of the invariant trace-field of an arithmetic hyperbolic 3-manifold, there could be non-arithmetic hyperbolic 3-manifolds with the same invariant trace-field.

Given this, and the discussion for the arithmetic case, some basic questions arise: For example, if one knows the invariant trace-field of a hyperbolic manifold is \( \mathbb{Q}(\sqrt{-3}) \), is the volume then an integer multiple of \( v_0 \)? This would be implied by the Lichtenbaum conjecture, which would more generally imply a best value for \( v_1 \) in Theorem 10.35 whenever \( r = 1 \). There appears to be no good reference for this, but Gangl has one in process [Ga] and [Gra] is relevant to this particular case.
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