Proof of a Conjecture on the Wiener Index of Eulerian Graphs

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Abstract

The Wiener index of a connected graph is the sum of the distances between all unordered pairs of vertices. A connected graph is Eulerian if its vertex degrees are all even. In [Gutman, Cruz, Rada, Wiener index of Eulerian Graphs, Discrete Applied Mathematics 132 (2014), 247-250] the authors proved that the cycle is the unique graph maximising the Wiener index among all Eulerian graphs of given order. They also conjectured that for Eulerian graphs of order \( n \geq 26 \) the graph consisting of a cycle on \( n - 2 \) vertices and a triangle that share a vertex is the unique Eulerian graph with second largest Wiener index. The conjecture is known to hold for all \( n \leq 25 \) with exception of six values. In this paper we prove the conjecture.

Keywords: Wiener index; average distance; mean distance; total distance; Eulerian graph; degree

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1 Introduction

Let \( G = (V, E) \) be a finite, connected graph. The Wiener index of \( G \) is defined by

\[ W(G) = \sum_{\{u,v\} \subseteq V} d_G(u,v), \]

where \( d_G(u,v) \) denotes the usual distance between vertices \( u \) and \( v \) of \( G \), i.e., the minimum number of edges on a \( (u,v) \)-path in \( G \).

The Wiener index, originally conceived by the chemist Wiener \[25\], has been investigated extensively in the mathematical and chemical literature, often under different names, such as transmission, distance, total distance or gross status. Several of these results were originally obtained for the closely related average distance, also called mean distance, which is defined as \( \frac{1}{(n-2)} W(G) \), where \( n \) is the order of the graph \( G \). For some of its chemical applications see, for example, \[21\].

One of the most basic results on the Wiener index states that

\[ W(G) \leq \left( \frac{n+1}{3} \right) \]

for every connected graph on \( n \) vertices, and equality holds if and only if \( G \) is a path.

A path has only vertices of degree one and two, so it is reasonable to expect that better bounds can be obtained if restrictions are placed on the values of the degrees. Upper bounds on the Wiener index that take into account not only the order, but also the minimum degree were given, for example, in \[2, 7, 17\], and it was shown in \[1\] that stronger bounds hold in the presence of a vertex of large degree. The Wiener index in relation to the inverse degree, i.e., the sum of the inverses of all vertex degrees, was considered by Erdös, Pach, and Spencer \[11\].

Bounds on the Wiener index of trees in terms of vertex degree have also been considered extensively. Every tree has minimum degree 1, so it is natural to ask how large or small the Wiener index can be in trees of given maximum degree. Answering this question for the maximum value of the Wiener index is fairly straightforward (see \[20\] and \[23\]), however the determination of the minimum Wiener index by Fischermann, Hoffmann, Rautenbach, Székely and Volkmann \[12\] required much more effort. For the more general problem of determining the extremal values of the Wiener index of a tree with given degree sequence see, for example, \[6, 22\] and \[24\]. A good survey of results on the Wiener index of trees before 2000 was given in \[10\].

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Not the actual value, but the parity of the degrees has been used to bound the Wiener index. Trees in which all vertices have odd degree were considered by Lin [18], who determined their smallest and largest possible Wiener index of trees. This result was extended in [14] with the determination of all such trees of order \( n \) with the largest \( \left\lfloor \frac{n}{4} \right\rfloor + 1 \) values of the Wiener index, see also [13]. The smallest and largest Wiener index of a tree whose order and number of vertices of even degree are given, was determined in [19].

The Wiener index of connected graphs in which all vertices have even degrees, that is, Eulerian graphs, was considered by Gutman, Cruz and Rada [15], who obtained the following theorem.

**Theorem 1** (Gutman, Cruz and Rada [15]). Let \( G \) be an Eulerian graph of order \( n \). Then

\[
W(G) \leq W(C_n),
\]

where \( C_n \) is the cycle on \( n \) vertices. Equality holds if and only if \( G = C_n \).

The authors of Theorem 1 gave a direct proof of their result. However, since every Eulerian graph is 2-edge-connected, Theorem 1 can also be obtained as a consequence of Theorem 3(a) below, which states that the cycle is the unique graph maximising the Wiener index among all 2-edge-connected graphs of given order.

Gutman, Cruz and Rada [15] also presented a conjecture on the question which Eulerian graph of given order has the second largest Wiener index. For \( n \geq 5 \) let \( C_{n,3} \) be the graph of order \( n \) obtained from the disjoint union of two cycles on \( n-2 \) vertices and 3 vertices, respectively, by identifying two vertices, one from each cycle. Their conjecture states that \( C_{n,3} \) is the unique graph that has the second largest Wiener index among all Eulerian graphs of order \( n \) for \( n \geq 26 \). It is the aim of this paper to give a proof of this conjecture.

It was verified in [15] that the conjecture holds for all values of \( n \) up to 25 except \( n \in \{7,9\} \), for which there are other extremal graphs of larger Wiener index than \( C_{n,3} \), and \( n \in \{8,10,11,13\} \), for which \( C_{n,3} \) has second largest Wiener index, but there exists another graph of the same Wiener index. All Eulerian graphs of order 7, 8, 9, 10, 11, 13 that have the second largest Wiener index are shown in Figure 1.

The main result of this paper reads as follows.

**Theorem 2.** Let \( G \) be an Eulerian graph of order \( n \) with \( n \geq 26 \) that is not a cycle. Then

\[
W(G) \leq W(C_{n,3})
\]

Equality holds if and only if \( G = C_{n,3} \).

The determination of the unique Eulerian graph with second largest Wiener index is reminiscent of the corresponding problem for 2-connected graphs. A *cutvertex* of a connected graph is a vertex whose removal disconnects the graph. A connected graph with no cutvertex is said to be 2-connected, and a *block* of a graph is a maximal subgraph that is 2-connected. Plesnık [20] proved that among all 2-connected graphs of given order, the cycle is the unique graph maximising the Wiener index. While the proof of this result is relatively straightforward, determining the 2-connected graph of given order with the second largest Wiener index requires significantly more effort, see the paper by Bessy, Dross, Knor and Škrekovski [8]. The proof of Theorem 2 given in the present paper suggest that the situation is no different for Eulerian graphs. We note that Plesnık’s result on 2-connected graphs was, asymptotically, extended to \( k \)-connected graphs in [8].

We note also a certain analogy between our result, and the determination of the largest Wiener index of a connected graph with given order and number of cutvertices in [4] and [5]. If the number of cutvertices is sufficiently small relative to the order, then the extremal graph consists of a path, whose one end is attached to a cycle. So among all graphs with exactly one cutvertex, the extremal graph has two blocks whose order is as unequal as possible. Similarly, the extremal graph \( C_{n,3} \) has two blocks, which are as unequal as possible, given the restriction that every block of an Eulerian graph has at least three vertices.
The notation we use is as follows. If $G$ is a graph, then $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively, of $G$. The order $n(G)$ and the size $m(G)$ are the number of vertices and edges, respectively, of $G$. If $G$ and $H$ are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say that $H$ is a subgraph of $G$ and write $H \leq G$. If $A \subseteq V(G)$, then $G[A]$ denotes the subgraph of $G$ induced by $A$, i.e., the graph whose vertex set is $A$, and whose edges are exactly the edges of $G$ joining two vertices of $A$.

If $v$ is a vertex of $G$, then $N_G(v)$ denotes the neighbourhood of $v$, i.e., the set of all vertices of $G$ adjacent to $v$. For $i \in \mathbb{N}$ we define the $i$-th neighbourhood of $v$, $N_i(v)$, to be the set of vertices at distance exactly $i$ from $v$, and we let $n_i(v) = |N_i(v)|$. The degree of $v$ in $G$, i.e., the value $n_1(v)$, is denoted by $\text{deg}_G(v)$.

A cutset of $G$ is a set $S \subseteq V$ such that $G - S$, the graph obtained from deleting all vertices in $S$ and all edges incident with vertices in $S$ from $G$, is disconnected. An edge-cut of $G$ is a set $E_1 \subseteq E(G)$ such that $G - E_1$, the graph obtained from $G$ by deleting all edges in $E_1$, is disconnected. Let $k \in \mathbb{N}$. We say that $G$ is $k$-connected ($k$-edge-connected) if $G$ contains no cutset (no edge-cut) with fewer than $k$ elements. A cutvertex is a vertex $v$ with the property that $\{v\}$ is a cutset. An endblock a graph $G$ is a block of $G$ that contains only one cutvertex. It is known that every connected graph that is not 2-connected has at least two endblocks.

If $S$ is a cutset of $G$ and $H$ a component of $G - S$, then we say that $G[V(H) \cup S]$ is a branch of $G$ at $S$. If $S = \{v\}$, then we say that $H$ is a branch at $v$.

The total distance of a vertex $v$, $\sigma_G(v)$, is defined as the sum $\sum_{y \in V(G)} d_G(v, y)$. By $\sigma_G(A)$ we mean $\sum_{y \in V(G) - A} d_G(y, A)$, where $d_G(y, A)$ is defined as $\min_{a \in A} d_G(y, a)$.

The eccentricity $e(v)$ of a vertex $v$ of $G$ is the distance from $v$ to a vertex farthest from $v$ in $G$.

2 Preliminary Results

In this section we present definitions and results that will be used in the proof of Theorem 2. We begin with some bounds on the Wiener index and on the total distance of vertices in 2-connected and 2-edge-connected graphs.
Theorem 3 (Plesník [20]). (a) Let $G$ be a 2-edge-connected graph of order $n$. Then

$$W(G) \leq \begin{cases} \frac{n^2}{8} & \text{if } n \text{ is even,} \\ \frac{n^2-n}{8} & \text{if } n \text{ is odd.} \end{cases}$$

Equality holds if and only if $G$ is a cycle.

(b) Let $G$ be a 2-connected graph of order $n$ and $v$ a vertex of $G$. Then

$$\sigma_G(v) \leq \begin{cases} \frac{n^2}{3} & \text{if } n \text{ is even,} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Equality holds if $G$ is a cycle.

(c) Let $G$ be a 2-edge-connected graph of order $n$ and $v$ a vertex of $G$. Then

$$\sigma_G(v) \leq \frac{n(n-1)}{3}.$$

Corollary 1. Let $G$ be a 2-connected graph of order $n$ and $u, w$ two vertices of $G$. Let $u_1, u_2$ be two adjacent vertices of the cycle $C_n$. Then

$$\sigma_G(\{u, w\}) \leq \sigma_{C_n}(\{u_1, u_2\}).$$

**Proof:** Let $G'$ be the 2-connected graph obtained from $G$ by adding a new vertex $z$ and joining it to $u$ and $w$. Then

$$\sigma(z, G') = \sum_{x \in V(G)} (1 + d_G(x, \{u, w\})) = n + \sigma_G(\{u, w\}).$$

Let $C'_n$ be the graph obtained from $C_n$ by adding adding a new vertex $y$ and joining it to two adjacent vertices $u_1$ and $u_2$ of $C_n$. As above,

$$\sigma(y, C'_n) = \sum_{x \in V(C_n)} (1 + d_{C_n}(x, \{u_1, u_2\})) = n + \sigma_{C_n}(\{u_1, u_2\}).$$

Clearly, removing the edge $u_1u_2$ from $C'_n$ does not change $\sigma(y)$. But $C'_n - u_1u_2$ is $C_{n+1}$, so by Theorem 3(b), we have $\sigma(z, G') \leq \sigma(y, C'_n)$, which implies the statement of the lemma. \hfill \Box

![Figure 2: Graphs defined in Definitions 1](image-url)

**Definition 1.** (a) Let $n, a \in \mathbb{N}$ with $3 \leq a \leq n - 2$. Then $C_{n,a}$ denotes the graph of order $n$ obtained from two disjoint cycles $C_a$ and $C_{n+1-a}$ by identifying a vertex of $C_a$ with a vertex of $C_{n+1-a}$.

(b) Let $n, a \in \mathbb{N}$ with $3 \leq a \leq n - 1$. Then $F_{n,a}$ denotes the graph of order $n$ obtained from two disjoint cycles $C_a$ and $C_{n+2-a}$ by choosing two adjacent vertices $u, v$ of $C_a$ and two adjacent vertices $u', v'$ of $C_{n+2-a}$ and identifying $u$ with $u'$ and $v$ with $v'$. 

![Graphs defined in Definitions 1](image-url)
The Wiener index of the graph $C_{n,a}$ was evaluated in [15]. Specifically for $C_{n,3}$ we have

$$W(C_{n,3}) = \begin{cases} \frac{1}{8}n^3 - \frac{1}{4}n^2 + \frac{3}{4}n - \frac{3}{4} & \text{if } n \text{ is even}, \\ \frac{3}{8}n^3 - \frac{1}{2}n^2 + \frac{3}{4}n - \frac{1}{4} & \text{if } n \text{ is odd}. \end{cases}$$ (1)

In our proofs below we make use of the fact that $\frac{1}{8}n^3 - \frac{1}{4}n^2 + \frac{11}{8}n - \frac{9}{4} \leq W(C_{n,3}) \leq \frac{1}{8}n^3 - \frac{1}{4}n^2 + \frac{3}{4}n - \frac{1}{4}$ for all $n \in \mathbb{N}$ with $n \geq 5$, irrespective of the parity of $n$.

**Lemma 1.** (Gutman, Cruz, Rada [15]) If $n \in \mathbb{N}$ is even, $n \geq 6$, then

$$W(C_{n,3}) > W(C_{n,4}) > \ldots \ldots > W(C_{n,2n-1}) > W(C_{n,n/2}).$$

If $n \in \mathbb{N}$ is odd, $n \geq 11$ and $n = 4k + 3$ for some $k \in \mathbb{N}$, then

$$W(C_{n,3}) > W(C_{n,4}) > \ldots \ldots > W(C_{n,2k}) > W(C_{n,2k+1}).$$

If $n \in \mathbb{N}$ is odd, $n \geq 11$ and $n = 4k + 1$ for some $k \in \mathbb{N}$, then

$$W(C_{n,3}) > W(C_{n,4}) > \ldots \ldots > W(C_{n,2k-2}) > W(C_{n,2k}) > W(C_{n,2k-1}) > W(C_{n,2k+1}).$$

For $n = 7, 9$ we have $W(C_{7,4}) > W(C_{7,3})$ and $W(C_{9,4}) > W(C_{9,3}) > W(C_{9,5})$.

**Lemma 2.** Let $n \geq 26$ and $4 \leq a \leq n - 2$. Then

$$W(F_{n,a}) \leq W(C_{n,3}),$$

with equality only if $a = 4$ or $a = n - 2$.

**Proof:** A tedious but straightforward calculation yields that

$$W(F_{n,a}) = \begin{cases} \frac{1}{8} \left[ (a(n-2)(a-n-2) + n(n^2 + 2n - 4) \right] & \text{if } n \text{ even, } a \text{ even,} \\ \frac{1}{8} \left[ (a(n-2)(a-n-2) + n(n^2 + 2n - 4) - 3n + 6 \right] & \text{if } n \text{ even, } a \text{ odd,} \\ \frac{1}{8} \left[ (a(n-2)(a-n-2) + n(n^2 + 2n - 4) - n - a + 2 \right] & \text{if } n \text{ odd, } a \text{ even,} \\ \frac{1}{8} \left[ (a(n-2)(a-n-2) + n(n^2 + 2n - 4) + a - 2n \right] & \text{if } n \text{ odd, } a \text{ odd.} \end{cases}$$

Since $F_{n,a} = F_{n,n+2-a}$, we may assume that $a \leq \lfloor \frac{n+2}{2} \rfloor$. The derivative with respect to $a$ of the four terms on the right hand side above equals $\frac{1}{8}((a(n-2)(2a-n-2))$ if $n$ is even, $\frac{1}{8}((a-2)(2a+n-2)-1)$ if $n$ is odd and $a$ is even, and $\frac{1}{8}((a-2)(2a+n-2)+1)$ if $n$ is even and $a$ is odd. Hence each of these four terms is strictly decreasing in $a$. It thus follows that $W(F_{n,a}) \leq W(F_{n,4})$ if $a$ is even, with equality only if $a = 4$, and $W(F_{n,a}) \leq W(F_{n,5})$ if $a$ is odd, with equality only if $a = 5$. Now an easy calculation shows that $W(F_{n,5}) < W(F_{n,5}) = W(C_{n,3})$. Hence the lemma follows. \hfill \Box

**Corollary 2.** Let $G$ be a graph of order $n \geq 26$ obtained from a cycle $C_n$ by adding three edges between vertices of $C_n$ that are not in $E(C_n)$ and that form a triangle. Then $W(G) < W(C_{n,3})$.

**Proof:** Let the three edges added to $C_n$ be $uv, uw, wu$. Then for at least one of these three edges, $w$, we have $C_n + w = F_{n,a}$ for some $a$ with $4 \leq a \leq \lfloor \frac{n+2}{2} \rfloor$. Applying Lemma[11] yields that $W(G) < W(F_{n,a}) \leq W(C_{n,3})$. \hfill \Box

### 3 Excluding 2-Connected Counterexamples

The goal of this section is to prove that an Eulerian graph of given order that is not a cycle, and which has maximum Wiener index among such graphs, cannot be 2-connected. Hence it will suffice to prove Theorem[2] for graphs that have a cutvertex.

We begin by showing that Theorem[2] holds for graphs that are obtained from two 2-connected graphs by gluing them together at two vertices, provided one of them contains a spanning cycle.
Lemma 3. (a) Let $G$ be a 2-connected graph of order $n$. If $G$ contains a cutset $\{u, w\}$ with the property that the union of some, but not all, branches of $G$ at $\{u, w\}$ has exactly a vertices and contains a spanning cycle, and the union of the remaining branches is 2-connected, then

$$W(G) \leq W(F_{n,a}).$$

Equality implies that $G = F_{n,a}$.
(b) If, in addition, $G$ is Eulerian, $n \geq 26$ and $4 \leq a \leq n - 2$, then $W(G) < W(C_{n,a})$.

Proof: Let $A$ be the vertex set of the union of the branches at $\{u, w\}$ that contains a spanning cycle, and let $B$ be the vertex set of the union of the remaining branches. Let $a = |A|$ and $b = |B|$. Then $A \cap B = \{u, w\}$. Let $H = G[B]$ and let $C$ be a spanning cycle of $G[\{u, w\}]$. We denote the set of vertices $x$ of $A - \{u, w\}$ for which $d_C(u, x) < d_C(w, x)$ ($d_C(u, x) > d_C(w, x)$, $d_C(u, x) = d_C(w, x)$) by $U(W, S)$. Then

$$W(G) = W(C) + W(B) - d_G(u, w) + \sum_{x \in U \cup W \cup S, y \in B - \{u, w\}} d_G(x, y)$$

$$\leq W(C) + W(H) - d_C(u, w) + \sum_{x \in U, y \in V(H) - \{u, w\}} (d_C(x, u) + d_H(u, y))$$

$$+ \sum_{x \in W, y \in V(H) - \{u, w\}} (d_C(x, w) + d_H(w, y)) + \sum_{x \in S, y \in V(H) - \{u, w\}} (d_C(x, \{u, w\}) + d_H(\{u, w\}, y))$$

$$= W(C) + W(H) - d_C(u, w) + (b - 2)\sigma_C(u, U) + |U|\sigma_H(u) - d_H(u, w))$$

$$+ (b - 2)\sigma_C(w, W) + |W|\sigma_H(w) - d_H(u, w)) + (b - 2)\sigma_C(\{u, w\}, S)$$

$$+ |S|\sigma_H(\{u, w\}). \tag{2}$$

Let $C_a$ and $C_b$ be the cycles of the graph $F_{n,a}$ defined above, and let $u'$ and $w'$ be the two adjacent vertices of $F_{n,a}$ shared by $C_a$ and $C_b$. Let $U'$ ($W'$, $S'$) be the set of vertices $x$ of $C_a - \{u, w\}$ with $d(u', x) < d(w', x)$, $d(u', x) > d(w', x)$, $d(u', x) = d(w', x)$. As above, we have

$$W(F_{n,a}) = W(C_a) + W(C_b) - d_{F_{n,a}}(u', w') + (b - 2)\sigma_{C_a}(u', U') + |U'|\sigma_{C_b}(u') - d_{F_{n,a}}(u', w')$$

$$+ (b - 2)\sigma_{C_b}(w', W') + |W'|\sigma_{C_a}(w') - 2d_{F_{n,a}}(u', w')$$

$$+ |S'|\sigma_{C_b}(\{u', w'\}) - |U'|\sigma_H(u') + |S'|\sigma_H(\{u', w'\}) - |S|\sigma_H(\{u, w\}) \tag{3}$$

Since $C_a$ and $C_b$ are cycles, we have $|U'| = |W'|$. Subtracting (3) from (2) yields thus

$$W(F_{n,a}) - W(G) \geq \frac{1}{2}[(W(C_a) - W(C)) + (W(C_b) - W(H)) + (d_G(u, w) - d_{F_{n,a}}(u', w')) - |U'|(\sigma_{C_a}(u') + \sigma_{C_b}(w') - 2d_{F_{n,a}}(u', w')) - |S'|\sigma_{C_b}(\{u', w'\}) - |S|\sigma_H(\{u, w\})$$

$$+ |S'|\sigma_{C_b}(\{u', w'\}) - |U'|\sigma_H(u') + |S'|\sigma_H(\{u', w'\}) - |S|\sigma_H(\{u, w\})] \nonumber$$

We now argue that the right hand side of the last inequality is nonnegative. Clearly, $C_a$ and $C_b$ are isomorphic, so $W(C) - W(C_a) = 0$. Since $H$ is 2-connected, and thus 2-edge-connected, we have $W(C_b) - W(H) \geq 0$ by Theorem 3(a). Since $d_{F_{n,a}}(u', w') = 1$ we have $d_G(u, w) - d_{F_{n,a}}(u', w') \geq 0$. Also $\sigma_{C_a}(u', U') + \sigma_{C_b}(w', W') + \sigma_{C_b}(\{u', w'\}) - \sigma_{C_a}(u, U) - \sigma_{C_b}(w, W) - \sigma_{C}(\{u, w\}, S) = \sigma_{C_a}(\{w', w''\}) - \sigma_{C}(\{u, w\})$, but $\sigma_{C_b}(\{w', w''\}) - \sigma_{C}(\{u, w\}) \geq 0$ by Lemma 4.

We now bound the remaining expression, $|U'|\sigma_{C_a}(u') + |S'|\sigma_{C_b}(u') - 2d_{F_{n,a}}(u', w') + |S'|\sigma_{C_b}(\{u', w'\}) - |U'|\sigma_H(u') + |S'|\sigma_{C_b}(\{u', w'\}) - |S|\sigma_H(\{u, w\})$, which we denote by $f$. In order to complete the proof of the lemma it remains to show that $f \geq 0$.

We have $a = 2|U'| + |S'| = 2|U| + |S|$. In $C_a$, vertices $u'$ and $w'$ are adjacent, so there is exactly one vertex equidistant from $u'$ and $w'$ if $a$ is odd, and there is no vertex equidistant from $u'$ and $w'$ if $a$ is even. Hence $|S'| = 1$ if $a$ is odd, and $|S'| = 0$ if $a$ is even. In $C$ the
vertices $u$ and $w$ are not necessarily adjacent, so we have $|S| = 1$ if $a$ is odd, and $|S| \in \{0,2\}$ if $a$ is even. We conclude that if $a$ is odd, then $|U| = |U'|$ and $|S| = |S'|$, and if $a$ is even then either $|U| = |U'|$ and $|S| = |S'|$, or $|U'| = |U| + 1$, $|S'| = 0$, and $|S| = 2$. If $|U'| = |U|$ and $|S| = |S|$, then $f = |U|(\sigma_C(u) - \sigma_H(u) + \sigma_C(w) - \sigma_H(w) + 2d_H(u,w) - 2d_C(u',w')) + |S|(\sigma_C(u,v,w') - \sigma_H(u,w))$. Each of the terms $|U|, |S|, \sigma_C(u,v) - \sigma_H(u), \sigma_C(w) - \sigma_H(w), 2d_H(u,w) - 2d_C(u',w'),$ and $\sigma_C(u',w') - \sigma_H(u,w)$ is nonnegative, hence $f \geq 0$ in this case. If $|U'| = |U| + 1$ and $|S| = 2$, $|S'| = 0$, then $f = |U|(\sigma_C(u') - \sigma_H(u) + \sigma_C(w') - \sigma_H(w) + 2d_H(u,w) - 2d_C(u',w') + \sigma_C(u') + \sigma_C(w') - \sigma_H(u,w) - 2d_C(u',w') - 2\sigma_H(u,w))$. As above, each of the terms $|U|, \sigma_C(u') - \sigma_H(u), \sigma_C(w') - \sigma_H(w), 2d_H(u,w) - 2d_C(u',w'),$ is nonnegative. We also have $\sigma_C(u') + \sigma_C(w') - 2d_C(u',w') - 2\sigma_H(u,w) \geq 0$ since $\sigma_C(u') + \sigma_C(w') - 2d_C(u',w') = \sum_{x \in \mathcal{V}(C)} (d_C(u',x) + d_C(w',x)) \geq 2 \min\{d_C(u',x), d_C(w',x)\} = 2\sigma_C\{{u',w'}\}$. Hence $f \geq 0$ also in this case. This proves the desired bound on $W(G)$.

Now assume that $W(G) = W(F_{n,a})$. Then we have equality between the corresponding terms in (2) and (3), in particular $W(G[A]) = W(C_a)$ and $W(H) = W(C_b)$. This implies by Theorem 1.2 a) that $G[A]$ and $H$ are cycles of length $a$ and $b$, respectively. We also have $d_C(u,w) = 1$. It follows that $G = F_{n,a}$.

(b) If $G$ is Eulerian, then $G \neq F_{n,a}$ and so $W(G) < W(F_{n,a})$. By Lemma 2 we have $W(F_{n,a}) \leq W(C_{n,a})$, and (b) follows.

**Lemma 4.** Let $n \in \mathbb{N}$ with $n \geq 26$. Among all Eulerian graphs of order $n$ that are not cycles, let $G$ be one that has maximum Wiener index. Then $G$ has a cutvertex.

**Proof:** Suppose to the contrary that $G$ is 2-connected. We first prove that every triangle of $G$ contains a vertex of degree 2.

Suppose to the contrary that $G$ contains a triangle $u_1u_2u_3$ with $\deg(u_i) > 2$ for $i = 1, 2, 3$. Let $E'$ be the edge set of this triangle. Then $G - E'$ is connected since otherwise, if $G - E'$ is disconnected, the vertices $u_1, u_2$ and $u_3$ are not all in the same component of $G - E'$, so there exists a component of $G - E'$ containing only one vertex, $u_1$ say, of the triangle. This implies that $u_1$ is a cutvertex of $G$, a contradiction to $G$ being 2-connected. Hence $G - E'$ is connected. Clearly, $G - E'$ is also Eulerian, and $W(G - E') > W(G)$. By our choice of $G$, the graph $G - E'$ is a cycle. But then $G$ is obtained from a cycle by adding the edges of a triangle, and so $W(G) < W(C_{n,a})$ by Corollary 2. This contradicts the choice of $G$ as having maximum Wiener index, and so (4) follows.

![Figure 3: Cases 1, 2A, 2B, and 2C in the proof of Lemma 4.](image)

Since $G$ is not a cycle, it has a vertex of degree greater than 2. For $v \in V(G)$ we define $\mathcal{V}$ to be the nearest vertex of degree greater than 2 (with ties broken arbitrarily) and let $f(v) = d(v, \mathcal{V})$. Note that $v = \mathcal{V}$ if and only if $\deg(v) > 2$. Since $G$ is Eulerian, we have $\deg(\mathcal{V}) \geq 4$ for every $v \in V$. For $v \in V$, we have $\mathcal{V} \in N_f(v)$, and thus $N(\mathcal{V}) \leq N_f(v) \cup N_f(v) \cup N_f(v)$. We claim that

$$\text{If } \deg(v) = 2, \text{ then } |N(\mathcal{V}) \cap N_{f(v)-(v)}| = 1.$$ (5)

Indeed, if $\deg(v) = 2$ then the neighbour, $w$, say, of $\mathcal{V}$ on a shortest $(v, \mathcal{V})$-path is in $N_{f(v)-(v)}$. If there was a second neighbour $w'$ of $\mathcal{V}$ in $N_{f(v)-(v)}$, then the vertices in $\bigcup_{i=0}^{f(v)-1} N_i(v) \cup \{\mathcal{V}\}$ would induce a cycle as a subgraph whose only vertex of degree greater than 2 in $G$ is $\mathcal{V}$, implying that $\mathcal{V}$ is a cutvertex of $G$, contradicting the 2-connectedness of $G$. This proves (5).
Let $v$ be a vertex of degree 2. By the definition of $f(v)$, all vertices in $\bigcup_{i=0}^{f(v)-1} N_i(v)$ have degree 2. It is easy to see that, since $G$ is 2-connected, this implies
\[
n_1(v) = n_2(v) = \cdots = n_{f(v)}(v) = 2. \tag{6}\]

Let $N_{f(v)}(v) = \{\pi, w\}$. It follows from (3) that $n_{f(v)}(v) + n_{f(v)+1}(v) \geq \deg(\pi) \geq 4$, so $n_{f(v)+1}(v) \geq 2$. We consider three cases, depending on the value $n_{f(v)+1}(v)$.

**Case 1**: There exists $v \in V(G)$ with $n_{f(v)+1}(v) = 2$.

Let $N_{f(v)+1}(v) = \{u_1, u_2\}$. Since $\deg_G(\pi) \geq 4$ it follows that $\pi$ is adjacent to $u_1, u_2, w$ and a vertex in $N_{f(v)-1}(v)$, we have $\deg_G(\pi) = 4$. Since $w$ is adjacent to a vertex in $N_{f(v)+1}(v)$, otherwise $\pi$ would be a cutvertex, to a vertex in $N_{f(v)-1}(v)$, and also to $\pi$, it follows that $\deg_G(w) > 2$, and thus $\deg_G(w) = 4$, so $w$ is also adjacent to $u_1$ and $u_2$.

Now $\pi$, $w$ and $u_i$ form a triangle for $i = 1, 2$. Since $\deg_G(\pi) = \deg_G(w) = 4$, it follows by (4) that $\deg_G(u_1) = \deg_G(u_2) = 2$. So the vertices in $N_{f(v)+1}(v)$ have only neighbours in $N_{f(v)+1}(v) \cup N_{f(v)}(v)$. This implies that $e_G(v) = f(v) + 1$, and so $V(G) = \bigcup_{i=0}^{f(v)+1} N_i(v)$. It follows that $G$ consists of the cycle induced by $\bigcup_{i=0}^{f(v)} N_i(v)$ and the two additional vertices $u_1$ and $u_2$ of degree two, both adjacent to $\pi$ and $w$. Hence $G$ is the first graph depicted in Figure 3.

Applying Lemma 3 to the cutset $\{\pi, w\}$ now yields that $W(G) < W(C_{n,3})$. This contradiction to the maximality of $W(G)$ proves the lemma in Case 1.

**Case 2**: There exists $v \in V(G)$ with $n_{f(v)+1}(v) = 3$.

Let $N_{f(v)+1}(v) = \{u_1, u_2, u_3\}$. We consider subcases as follows.

**Case 2a**: $\pi w \in E(G)$ and $\deg(w) = 2$.

Then $w$ has a unique neighbour, $u_3$ say, in $N_{f(v)+1}(v)$. The set $\bigcup_{i=0}^{f(v)} N_i(v) \cup \{u_3\}$ induces a cycle in $G$ in which only $\pi$ and possibly $u_3$ have degree greater than 2. The situation is depicted in the third graph of Figure 3. The set $\{\pi, u_3\}$ is a cutset. The branch containing $v$ and $w$ induces a cycle of length $2f(v) + 2$, and the union of the remaining branches is 2-connected since $\pi$ and $u_3$ are adjacent. Since $4 \leq 2f(v) + 2 \leq n - 2$, we have $W(G) < W(C_{n,3})$ by Lemma 3, a contradiction to the maximality of $W(G)$.

**Case 2b**: $\pi w \notin E(G)$ and $\deg(w) > 2$.

By (3) and $n_{f(v)+1}(v) = 3$ it follows that $\pi$ and $w$ are both adjacent to $u_1, u_2$ and $u_3$. If at least one vertex in $\{u_1, u_2, u_3\}$, $u_1$ say, has degree 2, then the union of the two branches at the cutset $\{\pi, w\}$ containing $v$ and $u_1$ has at least four vertices and a spanning cycle, while the union of the remaining branches is 2-connected. Hence it follows from Lemma 3 that $W(G) < W(C_{n,3})$, contradicting the maximality of $W(G)$. So we may assume that $u_1, u_2$ and $u_3$ all have degree greater than 2. The situation is depicted in the fourth graph of Figure 3. Let $E'$ be the edge set of the 4-cycle $\pi, u_1, w, u_2, \pi$. Then $G - E'$ is connected since otherwise, similarly to the proof of (4), one of the vertices $u_1, u_2$ or $u_3$ would be a cutvertex of $G$. Since all vertices in $G - E'$ have even
degree, it follows that \( G - E' \) is Eulerian. Since at least one vertex of \( G - E' \) has degree greater than 2, viz \( v_3 \), we conclude that \( G - E' \) is not a cycle. But \( W(G - E') > W(G) \), a contradiction to the maximality of \( W(G) \).

**Case 3:** \( n_{f(v)+1}(v) \geq 4 \) for all \( v \in V \).

Let \( v \in V(G) \) be fixed. We first show that

\[
\sigma_G(v) \leq \begin{cases} 
\frac{1}{4}n^2 - n + \frac{11}{4} + 2f(v) & \text{if } n \text{ is odd,} \\
\frac{1}{4}n^2 - n + 3 + 2f(v) & \text{if } n \text{ is even.} 
\end{cases}
\tag{7}
\]

We note that \( n_0(v) = 1, n_1(v) = n_2(v) = \ldots = n_{f(v)}(v) = 2 \), and \( n_{f(v)+1}(v) \geq 4 \) imply that \( n \geq 5 + 2f(v) \), so \( f(v) \leq \left\lfloor \frac{n - 5}{2} \right\rfloor \). Let \( k = e(v) \). Then \( \sigma_G(v) = \sum_{i=0}^k in_i(v) \). The values \( n_i(v) \) satisfy the following conditions: (i) \( n_0(v) = 1 \) and (ii) \( \sum_{i=0}^k n_i(v) = n \). Since \( G \) is 2-connected, we have (iii) \( n_i(v) \geq 2 \) for \( i = 1, 2, \ldots, k-1 \), and (iv) \( n_{f(v)+1}(v) \geq 4 \) by the defining condition of Case 3.

In order to bound \( \sum_{i=1}^k in_i(v) \) from above, assume that \( n \) and \( f(v) \) are fixed, and that integers \( k, n_0, n_1, \ldots, n_k \) are chosen to maximise \( \sum_{i=1}^k in_i \) subject to conditions (i)-(iv). Then \( n_0 = 1 \), and \( n_i = 2 \) for all \( i \in \{1, 2, \ldots, k-1\} - \{f(v) + 1\} \), since otherwise, if \( n_t > 2 \), we can modify the sequence \( n_0, \ldots, n_k \) by decreasing \( n_t \) by 1 and increasing \( n_{t+1} \) by 1 to obtain a new sequence \( n'_0, \ldots, n'_k \) which satisfies (i)-(iv), but for which \( \sum_{i=0}^k in'_i > \sum_{i=0}^k in_i \), a contradiction. The same argument yields that \( n_{f(v)+1} = 4 \), and also that \( n_i \in \{1, 2\} \) if \( k > f(v) + 1 \). Therefore, if \( n \) is odd we have \( k = \frac{n - 5}{2} \) and \( \sum_{i=0}^k in_i = \frac{1}{4}n^2 - n + \frac{11}{4} + 2f(v) \), and if \( n \) is even we have \( k = \frac{n - 5}{2} + 1 \), \( n_k = 1 \) and \( \sum_{i=0}^k in_i = \frac{1}{4}n^2 - n + 30 + 2f(v) \), which is (7).

Summation of (7) over all \( v \in V(G) \) yields

\[
2W(G) = \sum_{v \in V(G)} \sigma_G(v) \leq \begin{cases} 
\frac{1}{4}n^3 - n^2 + \frac{11}{4}n + 2 \sum_{v \in V(G)} f(v) & \text{if } n \text{ is odd,} \\
\frac{1}{4}n^3 - n^2 + 3n + 2 \sum_{v \in V(G)} f(v) & \text{if } n \text{ is even.} 
\end{cases}
\tag{8}
\]

We now bound \( \sum_{v \in V(G)} f(v) \). Since \( G \) is an Eulerian graph but not a cycle, \( G \) contains a vertex \( w \) of degree at least 4. Since for \( i \in \{0, 1, \ldots, e(w)\} \) every vertex \( v \in N_i(w) \) satisfies \( f(v) \leq d(v, w) \), we have

\[
\sum_{v \in V(G)} f(v) \leq \sum_{v \in V(G)} d(v, w) = \sigma_G(w).
\]

Now \( G \) has more than one vertex of degree greater than two, since otherwise such a vertex would be a cutvertex, contradicting the 2-connectedness of \( G \). That implies that the strict inequality \( \sum_{v \in V(G)} f(v) < \sigma_G(w) \) holds. Noting that \( f(w) = 0 \), we obtain by (7) that

\[
\sum_{v \in V(G)} f(v) < \sigma_G(w) \leq \begin{cases} 
\frac{1}{4}n^2 - n + \frac{11}{4} & \text{if } n \text{ is odd,} \\
\frac{1}{4}n^2 - n + 3 & \text{if } n \text{ is even.} 
\end{cases}
\tag{9}
\]

From (8) and (9) we get

\[
W(G) < \begin{cases} 
\frac{1}{5}n^3 - \frac{1}{4}n^2 + \frac{3}{4}n + \frac{11}{4} & \text{if } n \text{ is odd,} \\
\frac{1}{5}n^3 - \frac{1}{4}n^2 + \frac{3}{4}n + 3 & \text{if } n \text{ is even.} 
\end{cases}
\]

But the right hand side of the last inequality equals \( W(C_{n,3}) \). This contradiction to the maximality of \( W(G) \) completes the proof. \( \Box \)

### 4 Completing the proof of Theorem [2]

**Proof of Theorem [2]** Suppose to the contrary that the theorem is false, and let \( n \) be the smallest value with \( n \geq 26 \) for which the theorem fails. Let \( G \) be an Eulerian graph of order \( n \) that is not
a cycle, and that has maximum Wiener index among all such graphs. By Lemma 4, G has a cutvertex, so G is not 2-connected. Then G has at least two endblocks. Let H be a smallest endblock of G, let v be the cutvertex of G contained in H, and let K be the union of the branches at v distinct from H. Let A and B be the vertex set of H and K, respectively, and let a = |A| and b = |B|. Then b = n − a + 1, and since H is a smallest endblock we have a ≤ \( \frac{n+1}{2} \). We have

\[
W(G) = \sum_{(x,y) \subseteq A} d_H(x, y) + \sum_{(x,y) \subseteq B} d_K(x, y) + \sum_{x \in A - \{v\}} \sum_{y \in B - \{v\}} (d_H(x, v) + d_K(v, y))
\]

\[
= W(H) + W(K) + (a - 1)\sigma_K(v) + (b - 1)\sigma_H(v).
\]

(10)

Since H is an endblock, H is 2-connected, but K may or may not be 2-connected.

Case 1: K is 2-connected. Similarly to (10) we obtain for the graph \( C_{n,a} \) and its two blocks \( C_a \) and \( C_b \) that

\[
W(C_{n,a}) = W(C_a) + W(C_b) + (a - 1)\sigma_{C_a}(w) + (b - 1)\sigma_{C_b}(w),
\]

where \( w \) is the cutvertex of \( C_{n,a} \). Since H and K are 2-connected, we have by Theorem 3 that \( W(H) \leq W(C_a) \), W(K) \leq W(C_b), \( \sigma_K(v) \leq \sigma_{C_b}(w) \) and \( \sigma_H(v) \leq \sigma_{C_a}(w) \). Hence we have \( W(G) \leq W(C_{n,a}) \). By Lemma 1 we have \( W(C_{n,a}) \leq W(C_{n,3}) \), and so we have \( W(G) \leq W(C_{n,3}) \), as desired.

Assume that \( W(G) = W(C_{n,3}) \). Then \( W(K) = W(C_b) \), and so \( K = C_b \), and similarly \( H = C_a \) by Theorem 3. Now Lemma 1 implies that \( a = 3 \). It follows that \( G = C_{n,3} \), and so the theorem holds in Case 1.

Case 2: K is not 2-connected. We now bound each term on the right hand side of (10) separately. Clearly, K is an Eulerian graph of order \( n + 1 \) but not a cycle. Since G is a smallest counterexample to Theorem 4, the bound in Theorem 3 holds for \( K \) unless \( b < 5 \) or \( b \in \{7, 9\} \). However, since \( b \geq \frac{2n+1}{3} \) and \( n \geq 26 \), b is not one of these exceptional values and Theorem 4 holds for \( K \). Therefore,

\[
W(K) \leq W(C_{n-a+1,3}) \leq \frac{1}{8}(n - a + 1)^3 - \frac{1}{4}(n - a + 1)^2 + \frac{3}{2}(n - a + 1) - 2.
\]

(11)

It follows from Theorem 3(c) that

\[
\sigma_K(v) \leq \frac{1}{3}(n - a + 1)(n - a) - \frac{1}{2}(n - a + 1) - 2.
\]

(12)

As in Case 1, Theorem 3 yields the following bounds for \( W(H) \) and \( \sigma_H(v) \)

\[
W(H) \leq W(C_a) \leq \frac{1}{3}a^3 \quad \text{and} \quad \sigma_H(v) \leq \frac{1}{4}a^2.
\]

(13)

Substituting (13), (11) and (12) into (10) yields that

\[
W(G) \leq \frac{1}{8}a^3 + \frac{1}{8}(n - a + 1)^3 - \frac{1}{4}(n - a + 1)^2 + \frac{3}{2}(n - a + 1) - 2 + \frac{1}{3}(a - 1)(n - a + 1)(n - a) + \frac{1}{4}(n - a)a^2.
\]

From equation (11) and the remark following it, we have

\[
W(C_{n,3}) \geq \frac{1}{8}n^3 - \frac{1}{4}n^2 + \frac{11}{8}n - \frac{9}{4}.
\]

Subtracting these two bounds we obtain, after simplification,

\[
W(C_{n,3}) - W(G) \geq \frac{1}{24}\left((a - 1)n^2 + (a^2 - 18a + 8)n - 2a^3 + 13a^2 + 25a - 39\right).
\]

10
Denote the right hand side of the above inequality by \( f(n, a) \). To complete the proof of the Lemma it suffices to show that \( f(n, a) > 0 \) for \( n \geq 26 \) and \( 3 \leq a \leq \frac{n+1}{2} \). Now \( \frac{\partial f}{\partial a} = \frac{1}{24} \left\{ n^2 + (2a - 18)n - 6a^2 + 26a + 25 \right\} \). For constant \( n \), this is a quadratic function of \( a \) which is concave down and thus it attains its minimum for \( a \in \left[ 3, \frac{n+1}{2} \right] \) at \( a = 3 \) or \( a = \frac{n+1}{2} \). Since for \( a = 3 \) we have \( \frac{\partial f}{\partial a} = \frac{1}{24} (n^2 - 12n + 49) > 0 \), and for \( a = \frac{n+1}{2} \) we have \( \frac{\partial f}{\partial a} = \frac{1}{18} (n^2 - 14n + 73) > 0 \), the derivative \( \frac{\partial f}{\partial a} \) is positive for \( 3 \leq a \leq \frac{n+1}{2} \). It follows that the function \( f \) is increasing in \( a \) for constant \( n \), and thus \( W(C_{n, 3}) - W(G) \geq f(3) = \frac{1}{24} (2n^2 - 37n + 99) \), which is greater than 0 for \( n \geq 26 \). This completes the proof of Theorem 2 \( \square \)

5 Eulerian Graphs with Small Wiener Index

A natural question that arises in the context of the Wiener index of Eulerian graphs is how small the Wiener index of an Eulerian graph can be. For Eulerian graphs of given order, this was answered in [15].

**Proposition 1** (Gutman, Cruz and Rada [15]). Let \( G \) be an Eulerian graph of order \( n \), where \( n \geq 3 \). Then

\[
W(G) \geq \left\{ \begin{array}{ll} \binom{n}{2} & \text{if } n \text{ is odd,} \\ \binom{n}{2} + \frac{n}{2} & \text{if } n \text{ is even.} \end{array} \right.
\]

Equality holds if and only if \( G \) is complete (for odd \( n \)), or \( G \) is obtained from the complete graph by removing the edges of a perfect matching (for even \( n \)).

Finding the minimum value of the Wiener index of Eulerian graphs becomes more challenging if not only the order, but also the size of the graph is considered. We have the following lower bound on the Wiener index due to Plesnık [20].

**Proposition 2.** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Then

\[
W(G) \geq 2 \left( \binom{n}{2} \right) - m.
\]

Equality holds if and only if the diameter of \( G \) is at most 2.

Proposition 2 yields a lower bound on the Wiener index of Eulerian graphs of given order and size. However, if \( m \) is so small relative to \( n \) that there is no Eulerian graph of diameter two of order \( n \) and size \( m \), then this bound is not sharp. The following result determines the minimum size of an Eulerian graph of order \( n \) and diameter 2. In the proof we use the fact that the minimum size of a graph of order \( n \) and diameter 2 not containing a vertex of degree \( n - 1 \) is \( 2n - 5 \), which was proved by Erdős and Rényi [9], see also [10].

**Proposition 3.** Let \( G \) be an Eulerian graph of order \( n \) and diameter two. Then

\[
m(G) \geq \left\{ \begin{array}{ll} \frac{4}{7} (n - 1) & \text{if } n \text{ is odd,} \\ 2n - 5 & \text{if } n \text{ is even.} \end{array} \right.
\]

This bound is sharp for \( n \geq 9 \).

**Proof:** First let \( n \) be even. Since \( G \) contains only vertices of even degree, \( G \) has no vertex of degree \( n - 1 \). The above-mentioned result by Erdős and Rényi [9] now proves that \( m(G) \geq 2n - 5 \), as desired.

To see that the bound is sharp consider the graph obtained from a triangle with vertices \( a, b \) and \( c \) and a star \( K_{1, n-4} \) by joining two of the leaves of the star to \( a \), joining two other leaves to \( b \), and joining the remaining \( n - 8 \) leaves to \( c \). (We note that this graph was already described in [10].)
Now let $n$ be odd. If $G$ contains no vertex of degree $n - 1$, then we have $m \geq 2n - 5$ as above, and the result follows. If $G$ contains a vertex of degree $n - 1$, then all other vertices have degree at least 2, and so the degree sum of $G$ is at least $n - 1 + 2(n - 1)$, and so $m \geq \frac{3}{2}(n - 1)$, as desired. The graph obtained from $\frac{2}{2}$ copies of the graph $K_2$ by adding an new vertex and joining it to each of the $n - 1$ vertices shows that the bound is sharp.

This leads naturally to the following question which we pose as a problem.

**Question 1.** Given $n$ with $n \geq 9$, and $m$ with $m < 2n - 5$ if $n$ is even and $m < \frac{3}{2}(n - 1)$ if $n$ is odd. What is the minimum Wiener index of an Eulerian graph of order $n$ and size $m$ and which graphs attain it?

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