ON THE MOORE-TACHIKAWA VARIETIES

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Abstract. Moore-Tachikawa varieties are certain Hamiltonian holomorphic symplectic varieties conjectured in the context of 2-dimensional topological quantum field theories. We discuss several constructions related to these varieties.

1. Introduction

Let $G_C$ be a simple and simply connected complex Lie group. In [10] Moore and Tachikawa conjectured the existence of a functor $\eta_{G_C}$ from the category of 2-bordisms to the category of holomorphic symplectic varieties with Hamiltonian action, such that gluing of boundaries corresponds to the holomorphic symplectic quotient with respect to the diagonal action $G_C$. If $W^b_{G_C}, b \in \mathbb{N}$, denotes the image of the basic cobordism with $b$ incoming circles, then the complex symplectic varieties are supposed to have the following properties:

(a) $W^b_{G_C}$ has a Hamiltonian action of $S^b \ltimes G^b_C$;
(b) the symplectic quotient of $W^b_{G_C} \times W^{b'}_{G_C}, b + b' \geq 3$, by $G_C$ embedded into $G^b_C \times G^{b'}_C$ via
g \mapsto \left(1, \ldots, 1, g, 1, \ldots, 1, 1, \ldots, 1\right)
is isomorphic to $W^{b+b'-2}_{G_C}$ (for any choice of positions of $g$);
(c) $W^2_{G_C} \simeq T^*G_C$ and $W^1_{G_C} \simeq G_C \times \mathcal{H}_G$, where $\mathcal{H}_G$ is a Slodowy slice to the regular nilpotent orbit;
(d) if $\mu_1, \mu_2, \mu_3$ denote the moment maps for the three $G_C$-actions on $W^3_{G_C}$, then $P(\mu_1) = P(\mu_2) = P(\mu_3)$ for any invariant polynomial $P \in \mathbb{C}[\mathfrak{g}^*_C]^{G_C}$.

We observe that these properties are actually contradictory. First of all, (b) implies that the symplectic quotient of $W^3_{G_C} \times W^1_{G_C}$ by $G_C$ is isomorphic to $W^2_{G_C} \simeq T^*G_C$ as a holomorphic $G_C \times G_C$-manifold. Property (d) implies now that the two moment maps on $T^*G_C$ in the trivialisation $T^*G_C \simeq G_C \times \mathfrak{g}^*_C$ given by right-invariant forms are $\nu_1(g, X) = X, \nu_2(g, X) = -\text{ad}(g^{-1})X$.

An ad hoc solution to this contradiction is to modify (b). Fix a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$ and let $\theta_{\mathfrak{h}_C}$ be the automorphism of $\mathfrak{g}_C$ determined by $h \mapsto -h$ on $\mathfrak{h}$ and $\alpha \mapsto -\alpha$ on roots $\mathfrak{h}_C^\perp$. For example, if $\mathfrak{g}_C = \mathfrak{sl}(k, \mathbb{C})$ and $\mathfrak{h}_C$ consists of diagonal matrices, then $\theta_{\mathfrak{h}_C}(A) = -A^T$. We denote the corresponding involution on $G_C$ by the same symbol.

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1Recall that a simple Lie algebra is determined by its root system, so that a vector space isomorphism of Cartan subalgebras of two such algebras $\mathfrak{g}, \mathfrak{g}'$, which induces an isomorphism of root systems, extends to an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$. 

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Axiom (b) can then be modified as follows:

(b') the symplectic quotient of $W_{G_c}^b \times W_{G_c}^{b'}$ by $G_C$ embedded into $G_{C}^b \times G_{C}^{b'}$ via

$$g \mapsto \left((1, \ldots, 1, \theta_{bc}(g), 1, \ldots, 1), (1, \ldots, 1, g, 1, \ldots, 1)\right)$$

is isomorphic to $W_{G_c}^{b+b'-2}$.

Simultaneously we replace the action of $G_C \times G_C$ on $W_{G_c}^2 \simeq T^*G_C$ given by left and right translations with the action given by (left translations, right translations composed with $\theta_{bc}$).

A more elegant solution is to consider oriented cobordisms and attach a holomorphic symplectic variety $W_{G^b_{G_c}}^c$ to any oriented pair of pants $X$ with the boundary consisting of $b$ incoming and $b'$ outgoing circles. We say that a boundary circle is “incoming” (resp. “outgoing”), if its orientation together with the inward (resp. outward) pointing normal is the orientation of $X$. Thus for two circles we shall have varieties $W_{G_c}^{2,0}$, $W_{G_c}^{1,1}$, and $W_{G_c}^{0,2}$. They are all isomorphic to each other as holomorphic symplectic varieties, but not as $G_C$-varieties. For instance, the isomorphism $\phi$ between $W_{G_c}^{2,0}$ and $W_{G_c}^{1,1}$ is equivariant with respect to one of the “incoming” copies of $G_C$, but it satisfies

$$\phi(g,m) = \theta_{bc}(g) \cdot \phi(m),$$

for the other incoming $G_C$ on $W_{G_c}^{2,0}$ and the outgoing $G_C$ on $W_{G_c}^{1,1}$. We can form a symplectic quotient only by matching an outgoing and an incoming circle.

The varieties $W_{G_c}^{b,b'}$ should have the following properties:

(A) $W_{G_c}^{b,b'}$ has a Hamiltonian action of $(G_b \times G_b') \times G_{C}^{b+b'}$;

(B) the symplectic quotient of $W_{G_c}^{b,b'} \times W_{G_c}^{c,c'}$, $b+b' + c + c' \geq 3$, by $G_C$ embedded diagonally into one of the $b'$ $G_C$-factors on $W_{G_c}^{b,b'}$ and into one of the $c$ $G_C$-factors of $W_{G_c}^{c,c'}$ is isomorphic to $W_{G_c}^{b+c-c-1,b'+c'-1}$;

(C) $W_{G_c}^{2,0} \simeq G_C \times G_C$, and $W_{G_c}^{1,1} \simeq T^*G_C$;

(D) if $\mu_i$, $i = 1, \ldots, b$ (resp. $\mu'_i$, $i = 1, \ldots, b'$) denote the moment maps for the $b$ factors (resp. $b'$ factors), then $P(\mu_i) = P(\mu_j) = P(-\mu'_{k})$ for any $i, j, k$ and any invariant polynomial $P \in C[G_C^*]^{G_C}$;

(E) There exists an isomorphism of holomorphic symplectic varieties $\phi : W_{G_c}^{b,b'} \to W_{G_c}^{b-1,b'+1}$, which is equivariant with respect to the first $b-1$ $G_C$-factors on both varieties, and with respect to the first $b'$ $G_C$-factors on both varieties, while it satisfies (E) with respect to the $b$-th factor on $W_{G_c}^{b,b'}$ and $(b'+1)$-th factor on $W_{G_c}^{b-1,b'+1}$.

Remark 1.1. The existence of the action of $G_b \times G_b'$ implies then that there is an isomorphism $\phi_{ij} : W_{G_c}^{b,b'} \to W_{G_c}^{b-1,b'+1}$ having the properties in (E) for any $i = 1, \ldots, b$, and $j = 1, \ldots, b'$.

In the present work we construct an open dense subset $U_{G_c}^{b,b'}$ of $W_{G_c}^{b,b'}$ as a symplectic quotient of the product of $b$ copies of $W_{G_c}^{1,0}$ and $b'$ copies of $W_{G_c}^{0,1}$ by an
abelian group. This open subset consists of points of $W_{G_C}^{b,b'}$ such that the value of the moment map for each $G_C$-action is a regular element of $\mathfrak{g}_C$. Moreover, the manifolds $U_{G_C}^{b,b'}$ satisfy axioms (A), (B), (D), and (E). In other words, $U_{G_C}^{b,b'}$ satisfy all axioms (A)-(E), provided that we replace $W_{G_C}^{b,b'}$ with its open subset $G_C \times \mathfrak{g}_C^{\text{reg}}$.

In §3 we relate $U_{SL(k,C)}^{b,b'}$ to Hilbert schemes of points in $\mathbb{C} \times S_k^{b,b'}$, where $S_k^{b,b'}$ is the manifold of rank one tensors in $(\mathbb{C}^k)^{\otimes b} \otimes (\mathbb{C}^k)^{\otimes b'}$. As a byproduct, we find an interesting presymplectic manifold $F_{G_C}$ for any simple Lie group $G_C$ of type $A_{k-1}$. It contains $G_C \times \mathcal{S}_{G_C}$ as an open dense set and it has a locally free Hamiltonian action of $G_C$, such that the moment map induces a bijection between $G_C$-orbits in $F_{G_C}$ and adjoint orbits in $\mathfrak{sl}(k,\mathbb{C})$. It seems likely that such an $F_{G_C}$ exists for any simple Lie group. Indeed, a Lie-theoretic method of construction is indicated by the description in §3.

Remark 1. A very different construction of the Moore-Tachikawa varieties as Poisson varieties satisfying all the axioms has been given by Braverman, Finkelberg, and Nakajima [3] (see also [1]). It is expected that the two constructions produce varieties isomorphic on an open dense subset, but this does not seem to be easy to prove.

Remark 2. After this paper has been posted on the ArXiv, I have received a preprint from David Kazhdan [7]. In it he and Victor Ginzburg construct varieties $U_{SL(k,C)}^{b,0}$ (and much more) by what is essentially the same method, although the technical details are different.

Remark 3. My motivation for this work is not so much the Moore-Tachikawa conjecture, as the hyperkähler geometry of $U_{G_C}^{b,b'}$. This will be presented in a future paper.

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2. An open dense subset of $W_{G_C}^{b,b'}$

Let $G_C$ be a complex semisimple Lie group of rank $r$ with Lie algebra $\mathfrak{g}_C$. We identify $\mathfrak{g}_C^*$ with $\mathfrak{g}_C$ using the Killing form $\langle , \rangle$. Let $\mathfrak{h}_C$ be a fixed Cartan subalgebra and let

$$\mathfrak{g}_C = \mathfrak{h}_C \oplus \bigoplus_{\alpha \in \Delta} \Phi_{\alpha}$$

be a root space decomposition. Let $\Delta$ be a set of simple roots. Let $\mathcal{S}_{G_C}$ be the Slodowy slice $e + Z(f)$ determined by a principal $\mathfrak{sl}(2,\mathbb{C})$-triple $(e,f,h)$ with $e \in \oplus_{\alpha \in \Delta} \Phi_{-\alpha}$.

Our starting point is the holomorphic symplectic manifold $W_{G_C}^{1,0} \simeq G_C \times \mathcal{S}_{G_C}$. The symplectic form at $(g,X)$ is given by

\begin{equation}
\langle dX \wedge g^{-1} dg \rangle + \{ X, g^{-1} dg \wedge g^{-1} dg \} = \{ dgg^{-1} \wedge d(\text{Ad}(g)X) \},
\end{equation}

where $\langle \phi \otimes \psi \rangle(u,v) = \langle \phi(u), \psi(v) \rangle - \langle \phi(v), \psi(u) \rangle$. The action of $G_C$ by left translations is Hamiltonian with the moment map given by $\mu(g,X) = \text{Ad}(g)X$.

\[ \text{We use "regular" to mean that the dimension of the centraliser equals the rank of } \mathfrak{g}_C. \]
We define analogously $\mathcal{W}_{G_{c}}^{0,1} \simeq G_{c} \times \mathcal{S}_{G_{c}}$, with the symplectic form
\[
\{g^{-1}dg \wedge d(\text{Ad}(g^{-1})X)\}
\]
The action of $G_{c}$ is now on the right and the moment map is $\mu'(g, X) = -\text{Ad}(g^{-1})X$. In particular, $\mathcal{W}_{G_{c}}^{1,0}$ and $\mathcal{W}_{G_{c}}^{0,1}$ satisfy (E) with
\[
\phi(X, g) = (-\text{Ad}(p) \circ \theta_{b_{c}}(X), p\theta_{b_{c}}(g)^{-1}p^{-1}),
\]
where $p \in G_{c}$ conjugates the opposite Slodowy slice (with $e \in \Phi_{\alpha} \Delta \Phi_{\alpha}$) to $\mathcal{S}_{G_{c}}$.

$\mathcal{W}_{G_{c}}^{1,0}$ and $\mathcal{W}_{G_{c}}^{0,1}$ also have Hamiltonian actions of the abelian group $A_{b_{c}} = \mathbb{C}[g_{c}]^{G_{c}} \simeq \mathbb{C}^{r}$ defined as follows. Write an invariant polynomial $P \in \mathbb{C}[g_{c}]^{G_{c}}$ as $P(X) = p(X, \ldots, X)$ for an invariant multilinear and symmetric form $p$ and define $C_{p}(X) \in g_{c}$ via
\[
\{C_{p}(X), Y\} = (\deg P) \cdot p(X, \ldots, X, Y) \quad \forall Y \in g_{c}.
\]
The elements $C_{p}(X), P \in \mathbb{C}[g_{c}]^{G_{c}}$, generate the centraliser $Z(X)$ of a regular $X$ (cf. [6, Lemma 6]). Now $P \in A_{b_{c}}$ acts on $\mathcal{W}_{G_{c}}^{1,0}$ via
\[
P(g, X) = (\exp(C_{p}(X)))g, X),
\]
and on $\mathcal{W}_{G_{c}}^{0,1}$ via
\[
P(g, X) = (\exp(C_{p}(X))g, X).
\]
The moment map $\nu$ for the action of $A_{b_{c}}$ is easily seen to be in both cases
\[
\nu(g, X)(P) = P(X), \quad P \in \text{Lie}(A_{b_{c}}) \simeq A_{b_{c}}.
\]

Moreover, the actions of $G_{c}$ and $A_{b_{c}}$ commute and the moment map for the $G_{c}$-action identifies the subset $g_{c}^{\text{reg}} \subset g_{c}$ of regular elements as the geometric quotient of $\mathcal{W}_{G_{c}}^{1,0}$ or $\mathcal{W}_{G_{c}}^{0,1}$ by $A_{b_{c}}$.

We shall now want to define, for any $b, b' \geq 0$, a holomorphic symplectic variety $U_{G_{c}}^{b, b'}$ as the symplectic quotient (in appropriate sense) of the product of $b$ copies of $\mathcal{W}_{G_{c}}^{1,0}$ and $b'$ copies of $\mathcal{W}_{G_{c}}^{0,1}$ by the diagonal action of
\[
A_{0} = \{(P_{1}, \ldots, P_{b+b'}) \in (A_{b_{c}})^{b+b'}; \sum_{i=1}^{b+b'} P_{i} = 0\}.
\]
The level set of the moment map is chosen to be 0.

**Lemma 2.1.** The 0-level set of the moment map for the action of $A_{0}$ on $(\mathcal{W}_{G_{c}}^{1,0})^{b} \times (\mathcal{W}_{G_{c}}^{0,1})^{b'}$ is given by
\[X_{1} = \cdots = X_{b+b'} \in \mathcal{S}_{G_{c}}.
\]

**Proof.** Let $P \in \mathbb{C}[g_{c}]^{G_{c}}$ and consider a 1-dimensional subgroup of $A_{0}$ given by
\[\{0, \ldots, 0, z, 0, \ldots, 0, -z, 0, \ldots 0\},\]
where the nonzero entries are in the $i$-th and in the $j$-th place. The moment map is, according to (2.5), equal to
\[z = (g_{i}, X_{i})^{b+b'}_{i=1} \mapsto P(X_{i}) - P(X_{j}),\]
and so $z$ belongs to the zero set of the moment map for $A_{0}$ if and only if $X_{1}, \ldots, X_{b+b'}$ belong to the same adjoint orbit, i.e. $X_{1} = \cdots = X_{b+b'}$. \qed
Thus the 0-level set of the moment map is the manifold $Y = G_{C}^{b+b'} \times \mathcal{F}_{C}$. The action of $A_{0}$ on $Y$ is not, however, proper, due to the fact that the centralizer of a regular element may be disconnected. Fortunately, although $A_{0}$ is not reductive, the GIT issues are straightforward to resolve in this case.

We consider the closure $R \subset Y \times Y$ of the relation defined by $A_{0}$. Thus $R$ is the set

$$\{(g_{i}, X), (h_{i}, X) \in Y \times Y; \ Ad(u_{i})X = X, \ \prod u_{i} = 1\},$$

where $u_{i} = h_{i}^{-1}g_{i}$ if $i \leq b$ and $u_{i} = g_{i}h_{i}^{-1}$ if $i > b$. This is an analytic subset of $Y \times Y$ and the analytic relation $R$ satisfies the assumptions of a theorem of Grauert (see [8, Thm. 7.1] and the second paragraph on p. 203 there) which guarantees that the quotient of $Y$ by this relation, defined as the ringed space $(Y, O(Y))/R$, is a normal complex space. We define $U^{b,b'}_{G_{C}}$ to be this space. We have:

**Proposition 2.2.** $U^{b,b'}_{G_{C}}$ is smooth. The action of any $G_{C}$-factor on $U^{b,b'}_{G_{C}}$ is free and proper, and the quotient is biholomorphic to

$$Q = \{(X, y_{1}, \ldots, y_{b+b'-1}) \in \mathcal{F}_{G_{C}} \times g_{C}^{b+b'-1}; \ \ y_{i} \in O(X), \ i = 1, \ldots, b + b' - 1\},$$

where $O(X)$ is the adjoint orbit of $X$.

**Proof.** We first prove that the action of any $G_{C}$-factor is free and proper. This is a purely topological statement. Without loss of generality, we can consider the first factor $G_{C}$-factor. Suppose that $h \in G_{C}$ stabilises $[1, g_{2}, \ldots, g_{b+b'}, X] \in U^{b,b'}_{G_{C}}$. This means that there are $u_{1}, \ldots, u_{b+b'} \in Z_{G_{C}}(X)$ with $\prod u_{i} = 1$ such that $hu_{1} = 1$, $g_{i}u_{i} = g_{i}$ for $i \leq b$, and $u_{i}g_{i} = g_{i}$ for $i > b$. This implies that $h = 1$.

Now observe that a $G_{C}$-orbit over $X \in \mathcal{F}_{G_{C}}$ is the same thing as $(Z_{G_{C}}(X))^{b+b'-1}$, and therefore the (geometric) quotient is homeomorphic to $Q$. This is clearly Hausdorff, and hence the $G_{C}$-action is proper. It follows that the quotient by $G_{C}$ is biholomorphic to $Q$. $Q$ is smooth, since it is the fibre product of $g_{C}^{b+b'} \rightarrow \mathcal{F}_{G_{C}}$, which is a submersion. Therefore $U_{G_{C}}^{b,b'}$ is a principal bundle over $Q$, hence smooth. \hfill \Box

The symplectic form on $(W_{G_{C}}^{1,0})^{b} \times (W_{G_{C}}^{0,1})^{b'}$ is $A_{0}$-invariant, and hence it descends to $U_{G_{C}}^{b,b'}$. It is easily written down. For example, on $U_{G_{C}}^{b,0}$ it is given by:

$$-\left\langle dX \wedge \left(\sum_{i=1}^{b} g_{i}^{-1}dg_{i}\right) + \left\langle X, \sum_{i=1}^{b} g_{i}^{-1}dg_{i} \wedge g_{i}^{-1}dg_{i}\right\rangle. \right\rangle$$

**Remark 2.3.** The moment maps yield a holomorphic map:

$$\Phi : U_{G_{C}}^{b,b'} \rightarrow \{(X, y_{1}, \ldots, y_{b+b'}) \in \mathcal{F}_{G_{C}} \times g_{C}^{b+b'}; \ y_{i} \in O(X)\}.$$

The fibre of $\Phi$ at any point is $Z_{G_{C}}(X)$. If we represent a vertical vector field by a fundamental vector field generated by $\rho \in Z_{G_{C}}(X)$, then the symplectic form is $\langle \rho \wedge dX \rangle$ plus the sum of Kostant-Kirillov-Souriau forms on the first $b$ orbits minus the sum of Kostant-Kirillov-Souriau forms on the last $b'$ orbits. This follows easily from (2.8).

**Remark 2.4.** We can extend the construction to include the case $b = b' = 0$. The variety $W_{G_{C}}^{0,0} \cong U_{G_{C}}^{0,0}$ is the symplectic quotient of $W_{G_{C}}^{1,0} \times W_{G_{C}}^{0,1}$ by the diagonal action.
of \( G_C \). It follows that \( W_{G_C}^{0,0} \) is isomorphic to
\[
\{(g, X) \in G^C \times \mathcal{J}_{g_C}; \text{Ad}(g)X = X\}.
\]

We now have:

**Theorem 2.5.**  
(i) \( U_{G_C}^{1,1} \simeq G_C \times g_C^{\text{reg}} \subset W_{G_C}^{1,1} \).

(ii) The manifolds \( U_{G_C}^{b,b'} \) defined above satisfy axioms (A), (B), (D), and (E).

**Proof.** \( U_{G_C}^{1,1} \) is isomorphic to the quotient of
\[
\{(g_1, g_2, X_1, X_2); g_1, g_2 \in G_C, X_1 = X_2 \in \mathcal{J}_{g_C}\}
\]
by the relation
\[
(g_1, g_2, X) \sim (g_1u, u^{-1}g_2, X_1, X_2), \quad u \in Z_{G_C}(X).
\]
The map
\[
(g_1, g_2, X_1, X_2) \mapsto (g_1g_2, \text{Ad}(g_1)X_1)
\]
is constant on equivalence classes and induces an isomorphism \( U_{G_C}^{1,1} \simeq G_C \times g_C^{\text{reg}} \).

We now prove (ii). Axioms (A), (D), and (E) are obvious from the construction. Let us index the \( G_C \)-factors in \( U_{G_C}^{b,b'} \) by \( i, i' \), \( i = 1, \ldots, b \), \( i' = 1, \ldots, b' \), and similarly by \( j, j' \) on \( U_{G_C}^{b,c'} \). We consider the symplectic quotient of \( U_{G_C}^{b,b'} \times U_{G_C}^{b,c'} \) by \( G_C \) acting diagonally on the \( p' \)-factor on \( U_{G_C}^{b,b'} \) and on the \( q \)-th factor on \( U_{G_C}^{b,c'} \) (and trivially on all the other factors). The moment map at
\[
(m, \tilde{m}) = \left([g_1, \ldots, g_b, g_{b'}, \ldots, g_b, X], [h_1, \ldots, h_c, h_{c'}, \ldots, h_c, Y]\right) \in U_{G_C}^{b,b'} \times U_{G_C}^{c,c'}
\]
is \( \text{Ad}(g_{p'}^{-1})X + \text{Ad}(h_q)Y \), and so the level set of the moment map is given by \( X = \text{Ad}(g_{p'}h_q)Y \). Since \( X, Y \in \mathcal{J}_{g_C} \), we must have \( X = Y \) and \( g_{p'}h_q \in Z_{G_C}(Y) \). Quotienting by \( G_C \) allows us to make \( g_{p'} = 1 \), and then \( h_q \in Z_{G_C}(Y) \). Such an \( \tilde{m} \) is \( R \)-equivalent to one with \( h_q = 1 \). We send this pair of representatives to
\[
[g_1, \ldots, g_b, g_{b'}, \ldots, g_b, h_1, \ldots, h_c, h_{c'}, \ldots, h_c, Y] \in U_{G_C}^{b+b'-1,b'+c'-1}.
\]
It is straightforward to check that \( R \)-equivalence classes are mapped to \( R \)-equivalence classes, and so the symplectic quotient is isomorphic to \( U_{G_C}^{b+b'-1,b'+c'-1} \). \( \square \)

**Remark 2.6.** We can rewrite the symplectic form \((2.8)\) as
\[
(2.9) \quad \sum_{i=1}^{b} \left(d_{q_i}g_i^{-1} \wedge d(\text{Ad}(g_i)X)\right).
\]
The \( g_C \)-valued functions \( \text{Ad}(g_i)X \) are the moment maps for the \( G_C \)-factors acting on \( U_{G_C}^{b,0} \). Thus, \( W_{G_C}^{0,0} \) (and similarly \( W_{G_C}^{b,b'} \)) must be an extension of \( U_{G_C}^{b,0} \) on which this form remains nondegenerate, but the values of the moment maps are no longer required to be regular. Moreover, for \( b = 2 \) the two moment maps are adjoints of each other, but for \( b \geq 3 \) this no longer can be the case, if axioms (B) and (C) are to be satisfied.
3. The case $G_C = SL(k, \mathbb{C})$

We shall now discuss in detail the $A_{k-1}$-case. It is actually simpler to describe the case when $G_C = GL(k, \mathbb{C})$; the manifolds $U_{C_r}^{b,b'}$ for a simple $G_C$ of type $A_{k-1}$ can be then obtained by taking submanifolds and finite quotients. One reason why $GL(k, \mathbb{C})$ is simpler is that the centraliser of any regular element is connected, so that there is no necessity to replace the group $A_0$ by the relation (2.7).

We therefore fix $G_C = GL(k, \mathbb{C})$ and write $U_{k}^{b,b'}$, $W_{k}^{b,b'}$, for the corresponding varieties. Let $S_{k}^{b,b'}$ be the manifold of rank 1 tensors in $(\mathbb{C}^k)^b \otimes (\mathbb{C}^k)^{*b'}$. It is biholomorphic to $(\mathbb{C}^k \setminus \{0\})^{b+b'}/T_0$, where

$$T_0 = \left\{ (\lambda_1, \ldots, \lambda_{b+b'}) \in (\mathbb{C}^*)^{b+b'} : \prod_{i=1}^{b} \lambda_i \prod_{i=b+1}^{b+b'} \lambda_i^{-1} = 1 \right\}.$$  

Consider now $X_{k}^{b,b'} = S_{k}^{b,b'} \times \mathbb{C}$ and denote by $\pi$ the projection onto the second factor. We recall [2, Ch. 6] the notion of a transverse Hilbert scheme of points. If $\pi : Z \rightarrow X$ is a surjective holomorphic map between complex manifolds or varieties, then the transverse Hilbert scheme $\text{Hilb}^{k}_{\pi}(Z)$ of $k$ points in $Z$ is an open subset of $\text{Hilb}^{k}(Z)$ consisting of those $D$ for which $\pi|_{D}$ is a scheme-theoretic isomorphism onto its image.

The $GL(k, \mathbb{C})^{b+b'}$-action on $S_{k}^{b,b'}$ induces a corresponding action on $\text{Hilb}^{k}(X_{k}^{b,b'})$. We shall say that a 0-dimensional subscheme $D$ of $X_{k}^{b,b'}$ is nondegenerate if the stabiliser of $D$ in each factor of $\prod_{i=1}^{b+b'} GL(k, \mathbb{C})$ is trivial.

**Theorem 3.1.** $U_{k}^{b,b'}$ is equivariantly biholomorphic to an open subset of the transverse Hilbert scheme $\text{Hilb}^{k}_{\pi}(X_{k}^{b,b'})$ consisting of nondegenerate subschemes.

**Proof.** We shall construct an isomorphism from $U_{\text{non-deg}}^{\text{non-deg}} \subset \text{Hilb}^{k}_{\pi}(X_{k}^{b,b'})$ to $U_{k}^{b,b'}$, where $U_{\text{non-deg}}^{\text{non-deg}}$ is the subset of nondegenerate subschemes. For the sake of transparency of the argument, we shall assume that $b' = 0$; the modifications needed for the general case are obvious. A point $D$ of $\text{Hilb}^{k}_{\pi}(X_{k}^{b,b'})$ consists of a divisor $\sum z_i k_i$ in $\mathbb{C}$, $\sum k_i = k$, together with a $(k_i - 1)$-jet of a section of $\pi$ at each $z_i$. We consider first the case when $\pi(D) = k0$. Then $D$ is a $(k-1)$-jet of a section of $\pi$ at 0, i.e., the truncation $p(\epsilon)$ of $x_1(\epsilon) \otimes \cdots \otimes x_k(\epsilon)$ to order $k-1$, where $x_j(\epsilon) = \sum_{m=0}^{k-1} x_j^m \epsilon^{m-1}$, $x_j^m \in \mathbb{C}^k$ and $x_j^0 \neq 0$. Furthermore, two collections of such $\mathbb{C}^k$-valued polynomials give the same $p(\epsilon)$ if and only if they differ by an element of the jet group

$$\left\{ \left( \lambda_1(\epsilon), \ldots, \lambda_k(\epsilon) \right) : \prod_{j=1}^{b} \lambda_j(\epsilon) = 1 \right\},$$

where each $\lambda_j(\epsilon)$ is a polynomial of degree at most $k-1$. This group acts by multiplying $x_j(\epsilon)$ on the right. Since we assume that $D$ is nondegenerate, the vectors $x_j^m$, $m = 0, \ldots, k-1$, are linearly independent. If we write $g_j$ for the invertible matrix with columns $x_j^m$, then action $x_j(\epsilon) \rightarrow x_j(\epsilon) \lambda(\epsilon)$ followed by truncation corresponds to the right multiplication of $g_j$ by $\lambda(J)$, where $J$ is the Jordan block with ones above the diagonal. Therefore the map

$$D \mapsto \left( J, \{g_1, \ldots, g_b\} \right) \in \left( \mathcal{V}(k, \mathbb{C}) \times GL(k, \mathbb{C})^b \right)/A_0$$
is well-defined and a biholomorphism between $\pi^{-1}(k0)$ and the subset of $U^{b,0}_k$ where $S = J$.

Now consider a general $D$, which can be written as $D = \sum D_i$ with $\pi(D_i) = k_i z_i$. In other words $D$ is the zero set of the polynomial $q(z) = \prod (z - z_i)^{k_i}$, which corresponds to an element $S$ of $\mathcal{F}_{\mathfrak{gl}(k,\mathbb{C})}$. Let $J(S)$ be a Jordan normal form of $S$. Since the stabiliser of $D$ is trivial in each factor $G_\mathbb{C}$, we have a decomposition $\mathbb{C}^k \cong \bigoplus V_{ji}$, where $V_{ji}$ is the space of based rational maps from $\mathbb{P}^1$ to $\mathbb{P}^k$ of degree $k$. These $G_i$ combine to give an element $G$ of $GL(k,\mathbb{C})^b_0/A_{0,b}$. Moreover, the choice of a Jordan normal form corresponds precisely to an ordered choice $(z - z_1)^{b_1}, \ldots, (z - z_s)^{b_s}$ of factors of $q(z)$. Thus we obtain a $GL(k,\mathbb{C})^b$-equivariant holomorphic bijection from $U_{\text{non-deg}} \subseteq \text{Hilb}_k(X_k^b)$ to $U_k^b$. Since both spaces are normal, Zariski’s main theorem implies that this map is a biholomorphism.

Example 3.2. For $b = 1$, $\text{Hilb}_k^b(X_k^{1,0})$ is the space of based rational maps from $\mathbb{P}^1$ to $\mathbb{P}^k$ of degree $k$, $U_k^1$ is then the subset of based full rational maps $\mathbb{P}^k$.

Remark 3.3. It is perhaps worth pointing out that $W_k^{1,1} \cong T^*GL(k,\mathbb{C})$ is the space of certain based rational maps from $\mathbb{P}^1$ to $Gr_k(\mathbb{C}^{2k})$. Thus $W_k^{1,0}, W_k^{0,1}$, and $W_k^{1,1}$ are certain strata of charge $k$ moduli spaces of Euclidean monopoles with minimal symmetry breaking, with gauge group $SU(3)$ in the case of $W_k^{1,0}$ and $W_k^{0,1}$, and $SU(4)$ in the case of $W_k^{1,1}$. Is there a similar interpretation of $W_k^{2,1}$ and $W_k^{1,2}$ as strata of charge $k$ $SU(5)$-monopoles with minimal symmetry breaking?

Remark 3.4. It follows that $U_{SL(k,\mathbb{C})}^{b,b'}$ is isomorphic to a subset of $U_{GL(k,\mathbb{C})}^{b,b'}$ consisting of “centred” $D$ (i.e. if $\pi(D) = \sum k_i z_i$ as a divisor, then $\sum k_i z_i = 0$ as the sum of complex numbers), and such that the element $G = \left[ g_1, \ldots, g_{b+b'} \right]$ of $GL(k,\mathbb{C})^{b+b'}/A_0$, obtained in the above proof, satisfies $\prod \det g_i = 1$.

Observe also that $U_{PGL(k,\mathbb{C})}^{b,b'}$ is a symplectic quotient of $U_{GL(k,\mathbb{C})}^{b,b'}$ by the diagonal action of $\mathbb{C}^* = \{ (\lambda_1, \ldots, \lambda_l) \in GL(k,\mathbb{C})^{b+b'} \}$.

3.1. The Fitting-transverse Hilbert scheme. We now consider a larger subset of $\text{Hilb}_k^b(X_k^{b,b'})$, consisting of subschemes $D$ such that its Fitting image $\pi_{\text{fit}}(D)$ has length $k$. This means that if $D$ has locally length $l$ and is set-theoretically supported in a single fibre $\pi^{-1}(z)$, then (locally) $\pi_* \mathcal{O}_D \cong \bigoplus_{i=1}^l \mathcal{O}_{l_i,z}$ with $\sum l_i = l$. Yet another way of saying this is that $D = \bigsqcup_{i=1}^l D_i$ with each $D_i$ (set-theoretically) supported at a single point and transverse to $\pi$. Such subschemes are l.c.i., and therefore this open subset is smooth. We define $F_k^{b,b'}$ to be the subset of this open set consisting of locally nondegenerate subschemes, i.e. such that the stabiliser of $D$ in each factor of $\prod_{i=1}^{b+b'} GL(k,\mathbb{C})$ is finite.

We shall now show that $F_k^{1,0}$ (and of course $F_k^{0,1}$) is a Hamiltonian presymplectic manifold, i.e. it has a closed 2-form and a $G$-equivariant moment map. For an introduction to presymplectic manifolds see [3].

Proposition 3.5. The symplectic form on $W_k^{1,0}$ extends to a closed form on $F_k^{1,0}$. The action of $GL(k,\mathbb{C})$ is locally free and Hamiltonian, and the moment map induces a bijection between $GL(k,\mathbb{C})$-orbits in $F_k^{1,0}$ and adjoint orbits in $\mathfrak{gl}(k,\mathbb{C})$. 
Remark 3.7. The above proof also indicates how to construct an analogous space $F_{k}^{b,b'}$ for any reductive Lie group by attaching $G_{\mathbb{C}}$ to adjoint orbits in certain combinatorial way. We leave the details to the reader.

Remark 3.8. The closed form $\omega$ on $F_{k}^{1,0}$ can be written locally in the form (2.1) where $X = J(D)$.

The manifolds $F_{k}^{b,b'}$ are presymplectic and Hamiltonian for any $b,b'$. However $F_{k}^{1,1} \neq T^{*}GL(k,\mathbb{C})$ and $F_{k}^{b,b'}$ do not seem well behaved with respect to symplectic quotients (even apart from the fact that the action is not proper).

References

[1] T. Arakawa, ‘Chiral algebras of class $S$ and Moore-Tachikawa symplectic varieties’, arXiv preprint, https://arxiv.org/pdf/1811.01577.pdf

[2] M.F. Atiyah and N.J. Hitchin, The geometry and dynamics of magnetic monopoles, Princeton University Press, Princeton, 1988.

[3] F. Bottacin, ‘A Marsden-Weinstein Reduction Theorem for Presymplectic Manifolds’, unpublished preprint (2005), https://www.math.unipd.it/~bottacin/papers/presympred.pdf

[4] A. Braverman, M. Finkelberg, and H. Nakajima, ‘Ring objects in the equivariant derived Satake category arising from Coulomb branches (with an appendix by Gus Lonergan)’, arXiv preprint, https://arxiv.org/pdf/1706.02112.pdf

[5] T.A. Crawford, ‘Full holomorphic maps from the Riemann sphere to complex projective spaces’, J. Differential Geom. 38 (1993), 161–189.

[6] S. D’Amorim Santa-Cruz, ‘Twistor geometry for hyper-Kähler metrics on complex adjoint orbits’, Ann. Global Anal. Geom. 15 (1997), 361–377.
[7] V. Ginzburg and D. Kazhdan, ‘A class of symplectic varieties associated with the space $G/U$’, preprint.
[8] H. Grauert, Th. Peternell, and R. Remmert (eds.), Several complex variables VII. Sheaf-theoretical methods in complex analysis, Springer-Verlag, Berlin 1994.
[9] T. Maszczyk, ‘On special rational curves in Grassmannians’, Arch. Math. 88 (2007), 323–332.
[10] G.W. Moore and Y. Tachikawa, ‘On 2d TQFTs whose values are holomorphic symplectic varieties’, in: String-Math 2011, 191–207, Proc. Sympos. Pure Math., 85, AMS, Providence RI, 2012.

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