Computing the intersection of two quadrics through projection and lifting

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Abstract
This paper is devoted to presenting a method to determine the intersection of two quadrics based on the detailed analysis of its projection in the plane (the so called cutcurve) allowing to perform the corresponding lifting correctly. This approach is based on a new computational characterisation of the singular points of the curve and on how this curve is located with respect to the projection of the considered quadrics (whose boundaries are the so called silhouette curves).

Keywords: Quadrics, Intersection curve, Cutcurve, Subresultants, Lifting.

1. Introduction

Quadrics are the simplest curved surfaces used in many areas and computing their intersection is a relevant problem. Algorithms dealing with this problem based on floating point arithmetic techniques are sensitive to rounding errors achieving a low running time to the detriment of their correctness. On the other hand, using symbolic methods guarantees the correctness of the results because they are based on exact arithmetic (if the considered quadrics are defined in exact terms) but their performance is typically and significantly lower than using methods based on numerically techniques (\cite{1}).

Levin (\cite{2},\cite{3}) developed a method to parameterize the intersection curve of two quadrics based on the analysis of the pencil generated by them. However, Levin’s method often fails to find the intersection curve when it is singular and generates a parameterization that involves the square root of some polynomial (\cite{4}). Also, when working with floating point numbers, sometimes Levin’s method outputs results that are topologically wrong and even fail to produce any parameterization (\cite{4}). Farouki \textit{et al.} (\cite{5}) made a complete study of the degenerated cases of quadric intersection by using factorization of multivariate polynomials

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and Segre characteristic. This method showed the exact parameterization of the intersection curve in many cases.

Later Wang et al. ([6]) improved Levin’s method making it capable of computing geometric and structural information - irreducibility, singularity and the number of connected components. Dupont et al. ([7]) presented an optimal algorithm for computing the explicit representation of the intersection of two arbitrary quadrics whose coefficients are rational numbers in the projective space by using the reduction of quadratic forms and producing new results characterizing the intersection curve of two quadrics. The performance of this algorithm was analyzed in [8].

Others have restricted the kind of quadrics to be considered and defined specific routines to each case ([9], [10], [11], [12], [13]) taking advantage of the fact that a geometric approach is typically more stable than the algebraic ones ([14]). However these approaches are limited to planar intersections and natural quadrics. Mourrain et al. ([15]) proposed an algorithm that reduces the intersection of two quadrics to a dynamic two-dimensional problem.

An alternative way to compute the intersection of two quadric surfaces in the three-dimensional space is based on analysing its projection onto one plane ([16]). The idea of this method is to reduce the three-dimensional problem to computing the arrangement of three plane algebraic curves defined implicitly. After determining and analysing the projection of the intersection curve onto a plane, the intersection curve can be recovered by determining the lifting of the projection curve. Implementation and theoretical aspects of this approach are also described in [1] and [17], respectively.

In this paper, a new method is presented to determine the intersection curve of two quadrics through projection onto a plane and lifting. In some cases, it will be possible to determine the exact parameterization of the intersection curve (involving radicals if needed) and, in others, the output (topologically correct) will be the lifting of the discretization of the branches of the projection curve once its singular points have been analysed. The way the lifting will be achieved is the main criteria followed to analyse the cutcurve.

This paper is organized as follows. In Section 2, we briefly review some preliminaries on conics and quadrics. In Section 3 some mathematical tools as resultants and subresultants are briefly presented for sake of completeness. Resultants are used in Section 4 to characterise the projection of the intersection curve (called, in what follows, the cutcurve) by using a bivariate polynomial of degree four, at most. Our approach is based on the analysis of the arrangement of the cutcurve and the silhouette of both quadrics, as in Figure 1 following [1] and [17]. Section 5 is devoted to introduce simpler methods to characterise the singular points of the cutcurve as well as its lifting by using the subresultants. Some examples are given in Section 6 and the conclusions are presented in Section 7.

2. Representing quadrics (and conics)

This section is devoted to introduce how quadrics will be represented when computing their intersection curve. Since we will project the considered quadrics onto the $xy$ plane and the boundary of this region will be a finite number of conic arcs we introduce here how these regions will be represented and manipulated.
2.1. Quadrics

Quadrics are the one of the simplest surfaces defined by degree two polynomials in $x$, $y$ and $z$. The equation of any quadric $A$ in $\mathbb{R}^3$ can be written as

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0$$

or in matricial form $(x \ y \ z \ 1) A (x \ y \ z \ 1)^T = 0$ where $A$ is the symmetric matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}.$$

2.2. Conics

The equation of any conic $A$ in $\mathbb{R}^2$ can be written as

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

or in matricial form $(x \ y \ 1) A (x \ y \ 1)^T = 0$ where $A$ is the symmetric matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Conics will be used later to define the boundary of regions in the plane, typically the intersection of the silhouette of the two quadrics whose intersection curve is to be computed.

**Example 2.1.** Let $f(x, y) = x^2 + y^2 - 7$ and $g(x, y) = x^2 - xy + y^2 - 2x$ be two bivariate polynomials defining two conics, circle and ellipse, respectively. We can define several regions in the plane bounded by the conics $f$ and $g$ by using the inequalities involving both polynomials (as in Figure 3).
In what follows we assume that it is easy to determine the intersection points of two conics and to manipulate the region of the plane defined by two inequalities involving degree two or one polynomials.

3. Mathematical tools

In this section we will make a brief introduction to resultants and subresultants and how they will applied later to compute the intersection curve between two quadrics.

3.1. Resultants

Resultants and subresultants will be the algebraic tool to use to determine, both, the projection of the intersection curve between the two considered quadrics and its lifting from the plane to the space since they allow a very easy and compact way to characterise the greatest common divisor of two polynomials \( f(x, y, z) \) and \( g(x, y, z) \) in our case when they involve parameters \((x\) and \(y\) in our case, since we are going to eliminate \(z\)).

**Definition 3.1.** Let

\[
P(T) = \sum_{i=0}^{m} a_{m-i}T^i \quad \text{and} \quad Q(T) = \sum_{i=0}^{n} b_{n-i}T^i
\]

be two polynomials in \(T\) with coefficients in a field \((\mathbb{Q} \text{ or } \mathbb{R} \text{ in our case})\). We define the \(j\)-th subresultant polynomial of \(P\) and \(Q\) with respect to the variable \(T\) in the following way (as

\[
P_j(T) = \det \begin{pmatrix} P(T) & P(T)Q(T) \\ Q(T) & P(T)Q(T) \end{pmatrix}
\]
in [18]):

\[
\text{Sres}_j(P,Q;T) = \begin{vmatrix}
  a_0 & a_1 & a_2 & \ldots & \ldots & a_m \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  a_0 & a_1 & a_2 & \ldots & \ldots & a_m \\
  1 & -T & \ddots & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  b_0 & b_1 & b_2 & \ldots & \ldots & b_n \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  b_0 & b_1 & b_2 & \ldots & \ldots & b_n \\
\end{vmatrix}_{n-j \times j} = \begin{vmatrix}
  b_0 & b_1 & b_2 & \ldots & \ldots & b_n \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  b_0 & b_1 & b_2 & \ldots & \ldots & b_n \\
\end{vmatrix}_{m-j}
\]

and we define the \(j\)-th subresultant coefficient of \(P\) and \(Q\) with respect to \(T\), \(\text{sres}_j(P,Q;T)\), as the coefficient of \(T^j\) in \(\text{Sres}_j(P,Q;T)\). The resultant of \(P\) and \(Q\) with respect to \(T\) is:

\[
\text{Resultant}(P,Q;T) = \text{Sres}_0(P,Q;T) = \text{sres}_0(P,Q;T).
\]

There are many different ways of defining and computing subresultants: see [19] for a short introduction and for a pointer to several references.

Subresultants allow to characterise easily the degree of the greatest common divisor of two univariate polynomials whose coefficients depend on one or several parameters. Since the resultant of \(P\) and \(Q\) is equal to the polynomial \(\text{sres}_0(P,Q;T)\), \(\text{sres}_0(P,Q;T) = 0\) if and only if there exists \(T_0\) such that \(P(T_0) = 0\) and \(Q(T_0) = 0\).

More generally, the determinants \(\text{sres}_j(P,Q;T)\), which are the formal leading coefficients of the subresultant sequence for \(P\) and \(Q\), can be used to compute the greatest common divisor of \(P\) and \(Q\) thanks to the following equivalence:

\[
\text{Sres}_i(P,Q;T) = \gcd(P,Q) \iff \left\{ \begin{array}{l}
\text{sres}_0(P,Q;T) = \ldots = \text{sres}_{i-1}(P,Q;T) = 0 \\
\text{sres}_i(P,Q;T) \neq 0
\end{array} \right. \tag{1}
\]

Let \(f\) and \(g\) be the two polynomials in \(\mathbb{R}[x,y,z]\)

\[
f(x,y,z) = z^2 + p_1(x,y)z + p_0(x,y) \quad g(x,y,z) = z^2 + q_1(x,y)z + q_0(x,y)
\]

\((\deg(p_1(x,y)) \leq 1, \deg(p_0(x,y)) \leq 2, \deg(q_1(x,y)) \leq 1 \text{ and } \deg(q_0(x,y)) \leq 2)\) defining the quadrics whose intersection curve is to be computed. Then the resultant of \(f\) and \(g\), with respect to \(z\), is equal to:

\[
\text{S}_0(x,y) \overset{\text{def}}{=} \text{Resultant}(f,g;z) = \begin{vmatrix}
  1 & p_1(x,y) & p_0(x,y) & 0 \\
  1 & p_1(x,y) & p_0(x,y) & 0 \\
  1 & q_1(x,y) & q_0(x,y) & 0 \\
  0 & 1 & q_1(x,y) & q_0(x,y)
\end{vmatrix} = 5
\]
\[(p_0(x, y) - q_0(x, y))^2 - (p_1(x, y) - q_1(x, y)) \begin{vmatrix} p_0(x, y) & p_1(x, y) \\ q_0(x, y) & q_1(x, y) \end{vmatrix}.\]

The degree of \(S_0(x, y)\) is at most four. The first subresultant of \(f\) and \(g\), with respect to \(z\), is equal to:

\[S_1(x, y; z) \overset{\text{def}}{=} S_{\text{res}}_1(f, g; z) = (q_1(x, y) - p_1(x, y))z + (q_0(x, y) - p_0(x, y)) = g(x, y, z) - f(x, y, z).\]

Computing the intersection of the two quadrics defined by \(f\) and \(g\) is equivalent to solving in \(\mathbb{R}^3\) the polynomial system of equations

\[f(x, y, z) = 0, \quad g(x, y, z) = 0.\]

The solution set to be computed, when non empty, may include curves and isolated points. We will use that the above polynomial system of equations, under some conditions, is equivalent to

\[S_0(x, y) = 0, \quad (q_1(x, y) - p_1(x, y))z + (q_0(x, y) - p_0(x, y)) = 0.\]

Analyzing \(S_0(x, y) = 0\) in \(\mathbb{R}^2\) will be called the projection step and moving the information obtained in \(\mathbb{R}^2\) to \(\mathbb{R}^3\) will be called the lifting step. We follow here the terminology used when computing the cylindrical algebraic decomposition of a finite set of multivariate polynomials (see [20], for example)

4. Projecting the intersection curve

In this section we will characterise the projection of the intersection curve of two quadrics

\[Q_1 : f(x, y, z) = 0 \quad Q_2 : g(x, y, z) = 0\]

onto the \((x, y)\)-plane. The usual way of dealing with projections of algebraic sets involves tools coming from the so called Elimination Theory. We start by analysing the well–known complex case (i.e. when we look for the intersection curve in \(\mathbb{C}^3\) and its projection on \(\mathbb{C}^2\)) to conclude with the characterisation of the projection over the reals (i.e. when we look for the intersection curve in \(\mathbb{R}^3\) and its projection on \(\mathbb{R}^2\)).

Let \(f\) and \(g\) be the two polynomials in \(\mathbb{R}[x, y, z]\)

\[f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = z^2 + q_1(x, y)z + q_0(x, y)\]

(2)

with \(\deg(p_1) \leq 1, \deg(p_0) \leq 2, \deg(q_1) \leq 1\) and \(\deg(q_0) \leq 2\).

Let \(Q_1\) and \(Q_2\) be the two sets defined as:

\[Q_1 : \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = 0\} \quad Q_2 : \{(x, y, z) \in \mathbb{C}^3 : g(x, y, z) = 0\},\]

and \(\pi\) be the projection :

\[
\pi : \mathbb{C}^3 \rightarrow \mathbb{C}^2 \quad (x, y, z) \mapsto (x, y)
\]

Next (well known) theorem characterises the set \(\pi(Q_1 \cap Q_2)\).
Theorem 4.1.
\[ \pi (\mathcal{Q}_1 \cap \mathcal{Q}_2) = \{ (x, y) \in \mathbb{C}^2 : S_0(x, y) = 0 = 0 \} \]

Proof. Let \((a, b) \in \mathbb{C}^2\) such that \(S_0(a, b) = 0\). Then

\textbf{Resultant}(f(a, b, z), g(a, b, z); z) = 0

and, according to [1], there exists \(c \in \mathbb{C}\) such that \(f(a, b, c) = 0\) and \(g(a, b, c) = 0\). Thus \((a, b, c) \in \mathcal{Q}_1 \cap \mathcal{Q}_2\) and \((a, b) \in \pi (\mathcal{Q}_1 \cap \mathcal{Q}_2)\).

On the other hand, if \((a, b) \in \pi (\mathcal{Q}_1 \cap \mathcal{Q}_2)\) then there exists \(c \in \mathbb{C}\) such that \((a, b, c) \in \mathcal{Q}_1 \cap \mathcal{Q}_2\). This implies \(f(a, b, c) = 0\) and \(g(a, b, c) = 0\) and that the polynomials \(f(a, b, z)\) and \(g(a, b, z)\) have a common root \((c \in \mathbb{C})\). Thus we can conclude, according to [1], that \(\textbf{Resultant}(f, g; z) = S_0(a, b) = 0\).

Example 4.2. If \(f(x, y, z) = z^2 + x\) and \(g(x, y, z) = z^2 + y\) then \(S_0(x, y) = (x - y)^2\). In this case the point \((1, 1)\) verifies \(S_0(1, 1) = 0\) but does not belongs to \(\pi (\mathcal{E}_1 \cap \mathcal{E}_2)\) since \(f(1, 1, z) = z^2 + 1\) and \(g(1, 1, z) = z^2 + 1\) do not have (common) real roots.

Theorem 4.3.
\[ \pi (\mathcal{E}_1 \cap \mathcal{E}_2) = \left\{ (x, y) \in \mathbb{R}^2 : \begin{array}{l} S_0(x, y) = 0 \\ p_1(x, y)^2 - 4p_0(x, y) \geq 0 \\ q_1(x, y)^2 - 4q_0(x, y) \geq 0 \end{array} \right\} \]

Proof. Let \(\Delta_{\mathcal{E}_1}(x, y) = p_1(x, y)^2 - 4p_0(x, y)\) and \(\Delta_{\mathcal{E}_2}(x, y) = q_1(x, y)^2 - 4q_0(x, y)\) be the discriminants of \(f(x, y, z)\) and \(g(x, y, z)\) (respectively) with respect to \(z\).

If \((a, b) \in \pi (\mathcal{E}_1 \cap \mathcal{E}_2)\) then there exists \(c \in \mathbb{R}\) such that \((a, b, c) \in \mathcal{E}_1 \cap \mathcal{E}_2\). Thus \(f(a, b, c) = 0\) and \(g(a, b, c) = 0\) and \(f(a, b, z)\) and \(g(a, b, z)\) have a common root \((c \in \mathbb{R})\) and we conclude that \(S_0(a, b) = 0\). As \(c\) is a real root of \(f(a, b, z)\) and \(g(a, b, z)\) we also have \(\Delta_{\mathcal{E}_1}(a, b) \geq 0\) and \(\Delta_{\mathcal{E}_2}(a, b) \geq 0\).
On the other hand, if \((a, b) \in \mathbb{R}^2\) verifies \(S_0(a, b) = 0\) then \(f(a, b, z)\) and \(g(a, b, z)\) have a common root \(c \in \mathbb{C}\): \(f(a, b, c) = 0\) and \(g(a, b, c) = 0\) (according to \(\text{(1)}\)). However, if \(\Delta_{E_1}(a, b) \geq 0\) and \(\Delta_{E_2}(x, y) \geq 0\) then \(c\) must be a real solution of \(f(a, b, z) = 0\) and \(g(a, b, z) = 0\), concluding that \((a, b) \in \Pi(\mathcal{E}_1 \cap \mathcal{E}_2)\).

Previous theorem gives a precise description for the projection of the intersection curve of two quadrics when their defining equations have the structure introduced in \(\text{(2)}\). It corresponds to the part of the curve
\[
\{(x, y) \in \mathbb{R}^2: S_0(x, y) = 0\}
\]
inside the region
\[
\mathcal{A}_{\mathcal{E}_1, \mathcal{E}_2} \overset{\text{def}}{=} \{(x, y) \in \mathbb{R}^2: \Delta_{\mathcal{E}_1}(x, y) \geq 0, \Delta_{\mathcal{E}_2}(x, y) \geq 0\}.
\]
As expected we find a semialgebraic set in \(\mathbb{R}^2\) since, according to Tarski Principle (see [20]), the projection of any semialgebraic set is a semialgebraic set too. The region where lives the projection of the intersection curve of the two considered quadrics is bounded by a finite set of conic arcs since any \(\Delta_{\mathcal{E}_i}(x, y)\) is a polynomial in \(\mathbb{R}[x, y]\) of total degree equal to 2.

In \([1]\), the curve in \(\mathbb{R}^2\) defined by \(S_0(x, y) = 0\) is called the cutcurve of \(\mathcal{E}_1\) and \(\mathcal{E}_2\) and the curve in \(\mathbb{R}^2\) defined by \(\Delta_{\mathcal{E}_i}(x, y) = 0\) the silhouette of \(\mathcal{E}_i\). We modify slightly this definition to make the definition of the cutcurve more suitable for our purposes.

**Definition 4.4.** Let \(\mathcal{E}_1\) and \(\mathcal{E}_2\) be two quadrics in \(\mathbb{R}^3\) defined by \(f(x, y, z) = 0\) and \(g(x, y, z) = 0\) respectively. The cutcurve of \(\mathcal{E}_1\) and \(\mathcal{E}_2\) is the set
\[
\left\{ (x, y) \in \mathbb{R}^2: S_0(x, y) = 0, p_1(x, y)^2 - 4p_0(x, y) \geq 0, q_1(x, y)^2 - 4q_0(x, y) \geq 0 \right\}
\]

According to Theorem 4.3 the cutcurve of \(\mathcal{E}_1\) and \(\mathcal{E}_2\) is equal to the projection of \(\mathcal{E}_1 \cap \mathcal{E}_2\) onto the \(xy\) plane, \(\Pi(\mathcal{E}_1 \cap \mathcal{E}_2)\). Next three examples show that the cutcurve of \(\mathcal{E}_1\) and \(\mathcal{E}_2\) can be a curve, part of a curve (i.e. a semialgebraic set) or even a single point, but always a semialgebraic set.

**Example 4.5.** Let \(f\) and \(g\) be the polynomials
\[
f(x, y, z) = 3z^2 + x^2 + y^2 - 1 \quad g(x, y, z) = z^2 + 3x^2 + y^2 - 1
\]
defining the two ellipsoids \(\mathcal{E}_1\) and \(\mathcal{E}_2\) whose intersection curve is to be computed. In this case we have:
\[
S_0(x, y) = 64x^4 + 32x^2y^2 - 32x^2 + 4y^4 - 8y^2 + 4 = 4(4x^2 + y^2 - 1)^2
\]
\[
\Delta_{\mathcal{E}_1}(x, y) = -12 \left(x^2 + y^2 - 1\right) \quad \Delta_{\mathcal{E}_2}(x, y) = -4 \left(3x^2 + y^2 - 1\right)
\]
and
\[ \mathcal{A}_{\mathcal{E}_1, \mathcal{E}_2} = \{(x, y) \in \mathbb{R}^2: \Delta_{\mathcal{E}_1}(x, y) \geq 0, \Delta_{\mathcal{E}_2}(x, y) \geq 0\} = \{(x, y) \in \mathbb{R}^2: 3x^2 + y^2 - 1 \leq 0\}. \]

Since the curve defined by \( S_0(x, y) = 0 \) is contained completely in \( \mathcal{A}_{\mathcal{E}_1, \mathcal{E}_2} \) (see Figure 4) we conclude that
\[ \Pi(\mathcal{E}_1 \cap \mathcal{E}_2) = \{(x, y) \in \mathbb{R}^2: S_0(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2: 4x^2 + y^2 - 1 = 0\} \]

Figure 4: A whole curve as a cutcurve.

Example 4.6. Let \( f \) and \( g \) be the polynomials
\[ f(x, y, z) = z^2 + x^2 + y^2 - 7 \quad g(x, y, z) = z^2 - x^2 + xy + 2x - y^2 \]

defining the two quadrics \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) whose intersection curve is to be computed. In this case we have:
\[ S_0(x, y) = (-2x^2 + xy - 2y^2 + 2x + 7)^2 \]
\[ \Delta_{\mathcal{G}_1}(x, y) = -4x^2 - 4y^2 + 28 \quad \Delta_{\mathcal{G}_2}(x, y) = 4x^2 - 4xy + 4y^2 - 8x \]

and
\[ \mathcal{A}_{\mathcal{G}_1, \mathcal{G}_2} = \{(x, y) \in \mathbb{R}^2: \Delta_{\mathcal{G}_1}(x, y) \geq 0, \Delta_{\mathcal{G}_2}(x, y) \geq 0\} = \{(x, y) \in \mathbb{R}^2: -x^2 - y^2 + 7 \geq 0, x^2 - xy + y^2 - 2x \geq 0\}. \]

In this case the curve in \( \mathbb{R}^2 \) defined by \( S_0(x, y) = 0 \) is not contained completely in \( \mathcal{A}_{\mathcal{G}_1, \mathcal{G}_2}: \) the projection of \( \mathcal{G}_1 \cap \mathcal{G}_2 \) is equal to the portion of the ellipse \(-2x^2 + xy - 2y^2 + 2x + 7 = 0 \) inside the circle \( x^2 + y^2 \leq 7 \) (see Figure 5).

Example 4.7. Let \( f \) and \( g \) be the polynomials
\[ f(x, y, z) = z^2 + x^2 \quad g(x, y, z) = z^2 + y^2 \]

defining the two quadrics \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) whose intersection curve is to be computed. In this case we have:
\[ S_0(x, y) = (-x^2 + y^2)^2 \quad \Delta_{\mathcal{B}_1}(x, y) = -4x^2 \quad \Delta_{\mathcal{B}_2}(x, y) = -4y^2 \]
Figure 5: The cutcurve is not the whole curve $S_0(x,y) = 0$.

and

$$A_{B_1,B_2} = \{(x,y) \in \mathbb{R}^2: \Delta_{B_1}(x,y) \geq 0, \ \Delta_{B_2}(x,y) \geq 0\} = \{(0,0)\}.$$  

In this case the only point of the curve in $\mathbb{R}^2$ defined by $S_0(x,y) = 0$ contained in $A_{B_1,B_2}$ is the point $(0,0)$ (see Figure 6).

Figure 6: Single point as a cutcurve.

Next results analyze all possible cases concerning the projection of the intersection curve of two quadrics in terms of the structure of the polynomials (with at most degree two) defining the considered quadrics. Their proof is similar to the introduced for proving Theorem 4.3. As before $E_1$ and $E_2$ will denote the quadrics defined by the degree 2 polynomials $f(x,y,z)$ and $g(x,y,z)$ in $\mathbb{R}[x,y,z]$ and $\Pi$ the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$ with eliminates the last variable. The region of $\mathbb{R}^2$ where the projection of the intersection curve is located will be denoted by $A_{E_1,E_2}$.

**Corollary 4.8.** Let $f$ and $g$ be the polynomials in $\mathbb{R}[x,y,z]$ defined by:

$$f(x,y,z) = z^2 + p_1(x,y)z + p_0(x,y) \quad \text{and} \quad g(x,y,z) = q_1(x,y)z + q_0(x,y)$$

with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, $\deg(q_1) \leq 1$, $\deg(q_0) \leq 2$ and $q_1(x,y) \neq 0$. Then

$$\Pi (E_1 \cap E_2) = \{(x,y) \in \mathbb{R}^2: S_0(x,y) = 0, p_1(x,y)^2 - 4p_0(x,y) \geq 0\}$$

where

$$S_0(x,y) = q_0(x,y)^2 + q_1(x,y)^2p_0(x,y) - p_1(x,y)q_1(x,y)p_0(x,y).$$
Example 4.9. Let \( f \) and \( g \) be the polynomials
\[
f(x, y, z) = z^2 - x^2 + y^2 + 3z - 3 \quad g(x, y, z) = (x + y)z - 2x
\]
defining the two quadrics \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) whose intersection curve is to be computed. In this case we have:
\[
S_0(x, y) = -x^4 - 2x^3y + 7x^2 + 2xy^3 + y^4 - 3y^2
\]
and
\[
\Delta_{\mathcal{H}_1}(x, y) = 4x^2 - 4y^2 + 21
\]
In this case
\[
\mathcal{A}_{\mathcal{H}_1, \mathcal{H}_2} = \{(x, y) \in \mathbb{R}^2 : \Delta_{\mathcal{H}_1}(x, y) \geq 0\} = \{(x, y) \in \mathbb{R}^2 : 4x^2 - 4y^2 + 21 \geq 0\}.
\]
In this case the curve in \( \mathbb{R}^2 \) defined by \( S_0(x, y) = 0 \) is a quartic curve with three connected components contained completely in \( \mathcal{A}_{\mathcal{H}_1, \mathcal{H}_2} \):

Corollary 4.10. Let \( f \) and \( g \) be two polynomials in \( \mathbb{R}[x, y, z] \) defined by:
\[
f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = q_0(x, y)
\]
with \( \deg(p_1) \leq 1 \), \( \deg(p_0) \leq 2 \), \( \deg(q_0) \leq 2 \) and \( q_0(x, y) \neq 0 \). Then
\[
\Pi(\mathcal{E}_1 \cap \mathcal{E}_2) = \{(x, y) \in \mathbb{R}^2 : q_0(x, y) = 0, p_1(x, y)^2 - 4p_0(x, y) \geq 0\}.
\]
If two quadrics have no points in common, computing the projection curve based only in the resultant can lead to wrong conclusions.

Example 4.11. Let \( f \) and \( g \) be the polynomials
\[
f(x, y, z) = z^2 + (x - 3)^2 \quad g(x, y, z) = (x - 1)^2 + y^2 - 1
\]
defining the two quadrics \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) whose intersection curve is to be computed. In this case we have:
\[
S_0(x, y) = (x - 1)^2 + y^2 - 1
\]
and
\[ \Delta_{I_1}(x, y) = -4x^2 + 24x - 36 = -4(x - 3)^2. \]

In this case
\[ A_{I_1, I_2} = \{(x, y) \in \mathbb{R}^2 : \Delta_{I_1}(x, y) \geq 0\} = \{(x, y) \in \mathbb{R}^2 : -(x-3)^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 : x = 3\}. \]

The curve in \( \mathbb{R}^2 \) defined by \( S_0(x, y) = 0 \) has no points in common with the line defined by \( A_{I_1, I_2} \):

**Corollary 4.12.** Let \( f \) and \( g \) be the polynomials \( \mathbb{R}[x, y, z] \) defined by:
\[
\begin{align*}
 f(x, y, z) &= p_1(x, y)z + p_0(x, y) \\
 g(x, y, z) &= q_1(x, y)z + q_0(x, y)
\end{align*}
\]
with \( \operatorname{deg}(p_1) \leq 1 \), \( \operatorname{deg}(p_0) \leq 2 \), \( \operatorname{deg}(q_1) \leq 1 \), \( \operatorname{deg}(q_0) \leq 2 \) and \( p_1(x, y) \neq 0 \), \( q_1(x, y) \neq 0 \).
Then
\[ \Pi(\mathcal{E}_1 \cap \mathcal{E}_2) = \{(x, y) \in \mathbb{R}^2 : S_0(x, y) = 0\} \]

**Example 4.13.** Let \( f \) and \( g \) be the polynomials
\[
\begin{align*}
 f(x, y, z) &= xz - x - y \\
 g(x, y, z) &= z - x^2 - y^2
\end{align*}
\]
defining the two quadrics \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) whose intersection curve is to be computed. In this case we have:
\[ S_0(x, y) = x(-x^2 - y^2) + x + y. \]
The cut curve is defined by \( S_0(x, y) = 0 \):
Corollary 4.14. Let \( f \) and \( g \) the polynomials in \( \mathbb{R}[x, y, z] \) defined by:
\[
 f(x, y, z) = p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = q_0(x, y)
\]
with \( \deg(p_1) \leq 1, \deg(p_0) \leq 2, \deg(q_0) \leq 2 \) and \( q_0(x, y) \neq 0, p_1(x, y) \neq 0 \). Then the cutcurve is defined by:
\[
 \Pi(E_1 \cap E_2) = \{(x, y) \in \mathbb{R}^2 : q_0(x, y) = 0\}
\]

Example 4.15. Let \( f \) and \( g \) be the polynomials
\[
 f(x, y, z) = (5x - 10y - 5)z + 2x + 3y - 4 \quad g(x, y, z) = -3x^2 - xy + y^2 + 3x - 2y + 2
\]
defining two quadrics \( K_1 \) and \( K_2 \). In this case we have:
\[
 S_0(x, y) = -3x^2 - xy + y^2 + 3x - 2y + 2
\]
and the cutcurve is defined only by \( S_0(x, y) = 0 \):

---

Corollary 4.16. Let \( f \) and \( g \) be two polynomials in \( \mathbb{R}[x, y, z] \) defined by:
\[
 f(x, y, z) = p_0(x, y) \quad g(x, y, z) = q_0(x, y)
\]
with \( \deg(p_0) \leq 2, \deg(q_0) \leq 2 \) and \( p_0(x, y) \neq 0, q_0(x, y) \neq 0 \). Then \( S_0(x, y) = 1 \) and both quadrics define, in the \((x, y)\)-plane, two conics. The projection of the intersection curve is given by the common points to this two conics.

Example 4.17. Let \( f \) and \( g \) be the polynomials
\[
 f(x, y, z) = xy \quad g(x, y, z) = x^2 + y^2 - 1
\]
defining two quadrics \( L_1 \) and \( L_2 \). These two quadrics define in the \((x, y)\)-plane two lines and a circle, respectively. In this case, the projection of the intersection curve is given by four points:
5. Lifting to $\mathbb{R}^3$ the cutcurve in $\mathbb{R}^2$

In this section we study the lifting to $\mathbb{R}^3$ of the cutcurve. We will pay special attention to the singular points of the cutcurve since they are the points where more complicated situations we must deal with when lifting the cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$ to $\mathcal{E}_1 \cap \mathcal{E}_2$. This means that, first, we must be able of isolating them in order to achieve its lifting in the easiest and most efficient possible way.

5.1. Determining the singular points of the cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$

Let $f$ and $g$ be the polynomials in $\mathbb{R}[x, y, z]$ defined by:

$$f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y)$$
$$g(x, y, z) = z^2 + q_1(x, y)z + q_0(x, y)$$

with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, $\deg(q_1) \leq 1$, $\deg(q_0) \leq 2$. We restrict our attention to this case because this is the most complicated situation we must deal with: for those quadrics whose equations have a different (and simpler) structure, the singular points of the cutcurve are easier to manipulate since their lifting will be given automatically by one of the equations (being of degree smaller than or equal to 1).

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the corresponding quadrics defined by $f$ and $g$ and $S_0(x, y) = 0$ the implicit equation of the cutcurve to $\mathcal{E}_1 \cap \mathcal{E}_2$. Next two theorems show how to determine easily the singular points of the cutcurve. The first one tells that those points in the cutcurve $S_0(x_0, y_0) = 0$ and in the line $p_1(x, y) = q_1(x, y)$ are always singular points of the cutcurve.

**Theorem 5.1.** If $S_0(x_0, y_0) = 0$ and $p_1(x_0, y_0) = q_1(x_0, y_0)$ then

$$\frac{\partial S_0}{\partial x}(x_0, y_0) = 0 \quad \frac{\partial S_0}{\partial y}(x_0, y_0) = 0$$

**Proof.** Since

$$S_0(x, y) = \text{Resultant}(f, g; z) = \begin{vmatrix} 1 & p_1 & p_0 & 0 \\ 0 & 1 & p_1 & p_0 \\ 1 & q_1 & q_0 & 0 \\ 0 & 1 & q_1 & q_0 \end{vmatrix} = \begin{vmatrix} 1 & p_1 & p_0 & zf \\ 0 & 1 & p_1 & f \\ 1 & q_1 & q_0 & zg \\ 0 & 1 & q_1 & g \end{vmatrix}$$
Let \((\text{cutcurve})\) come from two different sources: we have

\[
\frac{\partial S_0}{\partial x} = \begin{vmatrix}
0 & p_1 & p_0 & zf \\
0 & 1 & p_1 & f \\
0 & q_1 & q_0 & zg \\
0 & 1 & q_1 & g \\
\end{vmatrix} + \begin{vmatrix}
1 & p_{1x} & p_0 & zf \\
0 & 0 & p_1 & f \\
0 & q_{1x} & q_0 & zg \\
0 & 0 & q_1 & g \\
\end{vmatrix} + \begin{vmatrix}
1 & p_1 & p_0x & zf \\
0 & 1 & p_{1x} & f \\
1 & q_1 & q_0x & zg \\
0 & 1 & q_{1x} & g \\
\end{vmatrix} + \begin{vmatrix}
1 & p_1 & p_0 & zf \\
0 & 1 & p_1 & f_x \\
0 & 1 & q_0 & zg_x \\
0 & 1 & q_1 & g_x \\
\end{vmatrix} \tag{3}
\]

Let \((x_0, y_0)\) be such that \(S_0(x_0, y_0) = 0\), \(z_0\) such that:

\[
f(x_0, y_0, z_0) = z_0^2 + p_1(x_0, y_0)z_0 + p_0(x_0, y_0) = 0 \quad g(x_0, y_0, z_0) = z_0^2 + q_1(x_0, y_0)z_0 + q_0(x_0, y_0) = 0
\]

and \(p_1(x_0, y_0) = q_1(x_0, y_0)\): then \(p_0(x_0, y_0) = q_0(x_0, y_0)\). Replacing in (3), \(x = x_0, y = y_0\) and \(z = z_0\) we get

\[
\frac{\partial S_0}{\partial x}(x_0, y_0) = \begin{vmatrix}
1 & p_1(x_0, y_0) & p_0(x_0, y_0) & z_0f_x(x_0, y_0, z_0) \\
0 & 1 & p_1(x_0, y_0) & f_x(x_0, y_0, z_0) \\
1 & q_1(x_0, y_0) & q_0(x_0, y_0) & z_0g_x(x_0, y_0, z_0) \\
0 & 1 & q_1(x_0, y_0) & g_x(x_0, y_0, z_0) \\
\end{vmatrix} = \begin{vmatrix}
1 & p_1 & p_0 & z_0f_x \\
0 & 1 & p_1 & f_x \\
1 & q_1 & q_0 & z_0g_x \\
0 & 1 & q_1 & g_x \\
\end{vmatrix} = \begin{vmatrix}
1 & p_1 & p_0 & z_0f_x \\
0 & 0 & q_0 - q_1 & z_0(f_x - g_x) \\
0 & 0 & f_x - g_x & f_x - g_x \\
\end{vmatrix} = 0
\]

Replacing \(x\) by \(y\) we prove also that \(\frac{\partial S_0}{\partial y}(x_0, y_0) = 0\). \(\square\)

Second theorem in this section allow us to conclude that the singular points of the cutcurve come from two different sources:

- either they are in the line \(p_1(x, y) = q_1(x, y)\), or
- they are not in the line \(p_1(x, y) = q_1(x, y)\) and they are the projection of a tangential intersection point between \(\mathcal{E}_1\) and \(\mathcal{E}_2\).

Note that not all tangential intersection points between \(\mathcal{E}_1\) and \(\mathcal{E}_2\) come from the second option.

**Theorem 5.2.** If \((x_0, y_0)\) is a singular point of the cutcurve and \(p_1(x_0, y_0) \neq q_1(x_0, y_0)\) then

\[
\frac{\partial f}{\partial x}(x_0, y_0, z_0) \frac{\partial g}{\partial z}(x_0, y_0, z_0) - \frac{\partial f}{\partial z}(x_0, y_0, z_0) \frac{\partial g}{\partial x}(x_0, y_0, z_0) = 0
\]

\[
\frac{\partial f}{\partial y}(x_0, y_0, z_0) \frac{\partial g}{\partial z}(x_0, y_0, z_0) - \frac{\partial f}{\partial z}(x_0, y_0, z_0) \frac{\partial g}{\partial y}(x_0, y_0, z_0) = 0
\]

where

\[
z_0 = -\frac{q_0(x_0, y_0) - p_0(x_0, y_0)}{q_1(x_0, y_0) - p_1(x_0, y_0)}.
\]
Proof. It is enough to check that:

I. \( \frac{\partial S_0}{\partial x} (x_0, y_0) = (q_1 (x_0, y_0) - p_1 (x_0, y_0)) \). 

II. \( \frac{\partial S_0}{\partial y} (x_0, y_0) = (q_1 (x_0, y_0) - p_1 (x_0, y_0)) \).

Then

\[
\frac{\partial S_0}{\partial x} (x_0, y_0) = \begin{vmatrix} 1 & q_1 (x_0, y_0) & q_0 (x_0, y_0) & z_0 f_x (x_0, y_0, z_0) \\ 1 & p_1 (x_0, y_0) & p_0 (x_0, y_0) & f_x (x_0, y_0, z_0) \\ 0 & 1 & q_1 (x_0, y_0) & g_x (x_0, y_0, z_0) \\ 0 & 1 & q_1 (x_0, y_0) & g_x (x_0, y_0, z_0) \end{vmatrix} = \begin{vmatrix} 1 & p_1 & p_0 & z_0 f_x \\ 0 & 1 & p_1 & f_x \\ 1 & q_1 & q_0 & z_0 g_x \\ 0 & 1 & p_1 & g_x \end{vmatrix}
\]

Then

\[
\frac{\partial S_0}{\partial x} (x_0, y_0) = \begin{vmatrix} 1 & p_1 & p_0 & z_0 f_x \\ 0 & 1 & p_1 & f_x \\ 0 & q_0 - p_1 & g_x - f_x \\ 0 & q_1 - p_1 & G_x - f_x \end{vmatrix} = \begin{vmatrix} 1 & p_1 & p_0 & z_0 f_x \\ 0 & 1 & p_1 & f_x \\ q_0 - p_1 & q_1 - p_1 & z_0 (g_x - f_x) \\ 0 & 1 & q_1 - p_1 & g_x - f_x \end{vmatrix} = \begin{vmatrix} 1 & p_1 & p_0 & z_0 f_x \\ 0 & 1 & p_1 & f_x \\ q_0 - p_1 & q_1 - p_1 & z_0 (g_x - f_x) - (q_1 - p_1) & p_1 & f_x \\ 0 & 1 & q_1 - p_1 & g_x - f_x \end{vmatrix}
\]

Since \( p_1 (x_0, y_0) \neq q_1 (x_0, y_0) \) then

\(-z_0 (q_1 (x_0, y_0) - p_1 (x_0, y_0)) = q_0 (x_0, y_0) - p_0 (x_0, y_0)\)

and

\[
\frac{\partial S_0}{\partial x} (x_0, y_0) = \begin{vmatrix} -z_0 (q_1 - p_1) & z_0 (g_x - f_x) \\ q_1 - p_1 & g_x - f_x \end{vmatrix} = \begin{vmatrix} -z_0 & z_0 \\ q_1 - p_1 & 1 \end{vmatrix} (q_1 - p_1) (g_x - f_x) - (q_1 - p_1) p_1 \\ q_1 & g_x \end{vmatrix} = \begin{vmatrix} -2z_0 (g_x - f_x) - p_1 f_x \\ q_1 & g_x \end{vmatrix}
\]

Since \( f_x (x_0, y_0, z_0) = 2z_0 + p_1 (x_0, y_0) \) and \( g_x (x_0, y_0, z_0) = 2z_0 + q_1 (x_0, y_0) \) we have:

\[
p_1 = f_x - 2z_0 \quad q_1 = g_x - 2z_0
\]
Then we have

\[ \Delta p \]

Lemma 5.4.

and \( \Delta \)

Proof. The tangent planes to \( f \) and \( g \) at \((x, y, z)\) are given by:

\[
\frac{\partial S_0}{\partial x} (x_0, y_0) = \left( q_1 - p_1 \right) \begin{pmatrix} -2z_0 (g_x - f_x) - f_z - 2z_0 f_x \\ g_z - 2z_0 g_x \end{pmatrix} = \left( q_1 - p_1 \right) \begin{pmatrix} -2z_0 g_x + 2z_0 f_x - g_x (f_z - 2z_0) + f_x (g_z - 2z_0) \\ \end{pmatrix} = \left( q_1 - p_1 \right) (f_x g_z - f_z g_x).
\]

Then we have

\[
\frac{\partial S_0}{\partial x} (x_0, y_0) = \left( q_1 (x_0, y_0) - p_1 (x_0, y_0) \right) \begin{vmatrix} f_x (x_0, y_0, z_0) & f_z (x_0, y_0, z_0) \\ g_x (x_0, y_0, z_0) & g_z (x_0, y_0, z_0) \end{vmatrix}.
\]

Since \((x_0, y_0)\) is a singular point of the curve and \(p_1 (x_0, y_0) \neq q_1 (x_0, y_0)\) we conclude that

\[
\begin{vmatrix} f_x (x_0, y_0, z_0) & f_z (x_0, y_0, z_0) \\ g_x (x_0, y_0, z_0) & g_z (x_0, y_0, z_0) \end{vmatrix} = 0
\]

as desired.

\[ \square \]

Corollary 5.3. If \((x_0, y_0)\) is a singular point of the cutcurve, \(p_1 (x_0, y_0) \neq q_1 (x_0, y_0)\) and

\[
z_0 = -\frac{q_0 (x_0, y_0) - p_0 (x_0, y_0)}{q_1 (x_0, y_0) - p_1 (x_0, y_0)}
\]

then \(f(x_0, y_0, z_0) = 0, g(x_0, y_0, z_0) = 0\) and the quadrics defined by \(f\) and \(g\) have the same tangent plane at \((x_0, y_0, z_0)\).

Proof. The tangent planes to \(f = 0\) and \(g = 0\) at \((x_0, y_0, z_0)\) are given by:

\[
\frac{\partial f}{\partial x} (x_0, y_0, z_0) (x - x_0) + \frac{\partial f}{\partial y} (x_0, y_0, z_0) (y - y_0) + \frac{\partial f}{\partial z} (x_0, y_0, z_0) (z - z_0) = 0
\]

\[
\frac{\partial g}{\partial x} (x_0, y_0, z_0) (x - x_0) + \frac{\partial g}{\partial y} (x_0, y_0, z_0) (y - y_0) + \frac{\partial g}{\partial z} (x_0, y_0, z_0) (z - z_0) = 0
\]

Previous theorem implies that the quadrics \(f = 0\) and \(g = 0\) have the same tangent plane at \((x_0, y_0, z_0)\).

\[ \square \]

Next we show how to compute the singular points of the cutcurve when they are in the line \(p_1 (x, y) = q_1 (x, y)\). Next lemma will allow us to connect the equation of the cutcurve with the line \(p_1 (x, y) = q_1 (x, y)\) and with the discriminants of \(E_1\) and \(E_2\) (denoted by \(\Delta_{E_1} (x, y)\) and \(\Delta_{E_2} (x, y)\), respectively).

Lemma 5.4.

\[
S_0 (x, y) = \frac{1}{16} \left[ (p_1 - q_1)^4 + (\Delta_{E_1} - \Delta_{E_2})^2 - 2 (p_1 - q_1)^2 (\Delta_{E_1} + \Delta_{E_2}) \right]
\]

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Proof. Since \( \Delta_{\varepsilon_1} = p_1^2 - 4p_0 \) and \( \Delta_{\varepsilon_2} = q_1^2 - 4q_0 \), we have
\[
\frac{1}{16} \left[ (p_1 - q_1)^4 + (\Delta_{\varepsilon_1} - \Delta_{\varepsilon_2})^2 - 2(p_1 - q_1)^2 (\Delta_{\varepsilon_1} + \Delta_{\varepsilon_2}) \right] = 0
\]
\[
= p_0^2 - 2p_0q_0 + q_0^2 + p_1^2q_0 + q_1^2p_0 - p_1q_0q_1 - p_0p_1q_1 = 0
\]
\[
= (p_0 - q_0)^2 - (p_1 - q_1) (p_0q_1 - q_0p_1) = S_0(x, y)
\]
as desired. \(\square\)

As a consequence of the equality in the previous lemma, we conclude that we can always compute the intersection points between the cutcurve and the line \( p_1(x, y) = q_1(x, y) \) by just solving a degree two equation.

**Corollary 5.5.** The intersection points between the cutcurve and the line \( p_1(x, y) = q_1(x, y) \) are the same than the intersection points between the curve
\[
\Delta_{\varepsilon_1}(x, y) - \Delta_{\varepsilon_2}(x, y) = 0
\]
and the line \( p_1(x, y) = q_1(x, y) \). The polynomial obtained after replacing \( y \) in \( \Delta_{\varepsilon_1}(x, y) - \Delta_{\varepsilon_2}(x, y) \) by the value obtained by solving \( p_1(x, y) - q_1(x, y) = 0 \) in terms of \( y \) will be denoted by \( \tau_{\varepsilon_1, \varepsilon_2}(x) \).

In some cases the line \( p_1(x, y) = q_1(x, y) \) (or part of it) is contained completely in the cutcurve: this can be checked directly by performing the corresponding substitution on \( S_0(x, y) \) or by using the previous corollary.

**Proposition 5.6.** If \((\alpha, \beta)\) is a point such that
\[
S_0(\alpha, \beta) = 0, \Delta_{\varepsilon_1}(\alpha, \beta) = 0, \Delta_{\varepsilon_2}(\alpha, \beta) = 0
\]
then \( p_1(\alpha, \beta) = q_1(\alpha, \beta) \) and \((\alpha, \beta)\) is a singular point of the cutcurve.

Proof. From Lemma 5.4 we have
\[
S_0(\alpha, \beta) = 0 \iff \left( \frac{(p_1 - q_1)^2}{4} \right)^2 = 0 \iff p_1 = q_1
\]
as desired. \(\square\)

As a consequence we have that those points \((x, y)\) verifying \( S_0(x, y) = 0, \Delta_{\varepsilon_1}(x, y) = 0 \) and \( \Delta_{\varepsilon_2}(x, y) = 0 \) are singular points of the cutcurve.

In the particular case when \( p_1(x, y) \equiv q_1(x, y) \) we have that
\[
S_0(x, y) = (p_0(x, y) - q_0(x, y))^2
\]
and, as expected, all points in the cutcurve are singular. But this implies that the cutcurve is a conic \( (p_0(x, y) - q_0(x, y) = 0) \) and all further computations (including the lifting) are greatly simplified (see Example 6.1 for a particular case where this situation arises).
Finally, next theorem shows how to determine those singular points of the cutcurve not belonging to the line \( p_1(x, y) = q_1(x, y) \). These points, according to Theorem 5.2, come from tangential intersection points of the two considered quadrics and they are very easily lifted but more difficult to determine than those in the line \( p_1(x, y) = q_1(x, y) \). To determine these points we have to solve the system of equations

\[
S_0(x, y) = 0, \quad \frac{\partial S_0}{\partial x}(x, y) = 0, \quad \frac{\partial S_0}{\partial y}(x, y) = 0, \quad p_1(x, y) \neq q_1(x, y)
\]

inside \( \mathcal{A}_{\varepsilon_1, \varepsilon_2} \).

**Theorem 5.7.** Let \((x_0, y_0)\) be a singular point of the cutcurve such that \( p_1(x_0, y_0) \neq q_1(x_0, y_0) \). If

- \( U_0(x) = \text{Sres}_0 \left( S_0, \frac{\partial S_0}{\partial x}; y \right) \).
- \( U_1(x, y) = \text{Sres}_1 \left( S_0, \frac{\partial S_0}{\partial x}; y \right) = U_{11}(x)y + U_{10}(x) \).
- \( V_0(x) = \text{Sres}_0 \left( S_0, \frac{\partial S_0}{\partial y}; y \right) \).
- \( V_1(x, y) = \text{Sres}_1 \left( S_0, \frac{\partial S_0}{\partial y}; y \right) = V_{11}(x)y + V_{10}(x) \).

then \( x_0 \) is a real root of the polynomial

\[
\Omega_{\varepsilon_1, \varepsilon_2}(x) = \gcd(U_{10}(x)V_{11}(x) - V_{10}(x)U_{11}(x), U_0(x), V_0(x)) / \gcd(U_{10}(x)V_{11}(x) - V_{10}(x)U_{11}(x), \varepsilon_1, \varepsilon_2(x))
\]

and

\[
y_0 = -U_{10}(x_0)/U_{00}(x_0) = -V_{10}(x_0)/V_{00}(x_0).
\]

**Proof.** Since \((x_0, y_0)\) is a singular point of the cutcurve, we have the following equalities:

\[
U_0(x_0) = 0, U_1(x_0, y_0) = 0, V_0(x_0) = 0, V_1(x_0, y_0) = 0.
\]

By using \( U_1(x_0, y_0) = 0 \) and \( V_1(x_0, y_0) = 0 \) we have

\[
U_{10}(x_0)V_{11}(x_0) - V_{10}(x_0)U_{11}(x_0), U_0(x_0), V_0(x_0) = 0
\]

and that \( x_0 \) is a real root of

\[
\gcd(U_{10}(x)V_{11}(x) - V_{10}(x)U_{11}(x), U_0(x), V_0(x)).
\]

According to Corollary 5.5, we have that \( p_1(x_0, y_0) \neq q_1(x_0, y_0) \) implies that \( \tau_{\varepsilon_1, \varepsilon_2}(x_0) \neq 0 \) and this allows to conclude that \( \Omega_{\varepsilon_1, \varepsilon_2}(x_0) = 0 \) as desired.
5.2. Determining the regular points of the cutcurve of \( E_1 \) and \( E_2 \) inside their silhouette curves

Let \( f \) and \( g \) be the polynomials in \( \mathbb{R}[x, y, z] \) defined by:
\[
f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \quad \quad g(x, y, z) = z^2 + q_1(x, y)z + q_0(x, y)
\]
with \( \deg(p_1) \leq 1, \deg(p_0) \leq 2, \deg(q_1) \leq 1, \deg(q_0) \leq 2 \). Next propositions help to determine in a simpler way the points in the cutcurve which belong to each silhouette curve.

**Proposition 5.8.** If \( S_0(\alpha, \beta) = 0 \) then
\[
\Delta_{E_1}(\alpha, \beta) = 0 \iff \Delta_{E_2}(\alpha, \beta) = (p_1(\alpha, \beta) - q_1(\alpha, \beta))^2 \iff 2(q_0(\alpha, \beta) + p_0(\alpha, \beta)) = p_1(\alpha, \beta)q_1(\alpha, \beta)
\]
\[
\Delta_{E_2}(\alpha, \beta) = 0 \iff \Delta_{E_1}(\alpha, \beta) = (p_1(\alpha, \beta) - q_1(\alpha, \beta))^2 \iff 2(q_0(\alpha, \beta) + p_0(\alpha, \beta)) = p_1(\alpha, \beta)q_1(\alpha, \beta)
\]

**Proof.** Using Lemma 5.4, we have
\[
\Delta_{E_2}^2 - 2(p_1 - q_1)^2 \Delta_{E_2} + (p_1 - q_1)^4 = 0 \iff \Delta_{E_2} = (p_1 - q_1)^2
\]
and we get the first equivalence. Using this one and \( \Delta_{E_2} = q_1^2 - 4q_0 \), we conclude \( p_1q_1 = 2(p_0 + q_0) \). The second one is similar. \( \square \)

As a consequence we have that solving the system
\[
S_0(x, y) = 0, \quad \Delta_{E_1}(x, y) = 0
\]
is the same than solving the simpler system
\[
2(p_0(x, y) + q_0(x, y)) = p_1(x, y)q_1(x, y), \quad \Delta_{E_1}(x, y) = 0.
\]
And solving the system
\[
S_0(x, y) = 0, \quad \Delta_{E_2}(x, y) = 0
\]
is the same than solving the simpler system
\[
2(p_0(x, y) + q_0(x, y)) = p_1(x, y)q_1(x, y), \quad \Delta_{E_2}(x, y) = 0.
\]

**Proposition 5.9.** The system
\[
S_0(x, y) = 0, \quad p_1(x, y) \neq q_1(x, y), \quad \Delta_{E_1}(x, y) = 0, \quad \Delta_{E_2}(x, y) = 0
\]
has no real solutions.

**Proof.** Since
\[
\Delta_{E_1} = 0 \Rightarrow p_1^2 - 4p_0 = 0 \Rightarrow p_0 = \frac{p_1^2}{4},
\]
\[
\Delta_{E_2} = 0 \Rightarrow q_1^2 - 4q_0 = 0 \Rightarrow q_0 = \frac{q_1^2}{4},
\]
by using Proposition 5.8, we have
\[
2(q_0 + p_0) = p_1q_1 \Rightarrow 2 \left( \frac{p_1^2}{4} + \frac{q_1^2}{4} \right) = p_1q_1 \Rightarrow (p_1 - q_1)^2 = 0.
\]

However, \( (p_1 - q_1)^2 = 0 \) has no solutions because \( p_1 \neq q_1 \). \( \square \)
5.3. Lifting the points of the cutcurve of $E_1$ and $E_2$

Following results provide a way of performing the lifting of the points of the cutcurve depending on the different possibilities that can arise in terms of the equations defining the quadrics.

**Theorem 5.10.** Let $f$ and $g$ be two polynomials in $\mathbb{R}[x, y, z]$:

$$f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = z^2 + q_1(x, y)z + q_0(x, y) \quad (4)$$

with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, $\deg(q_1) \leq 1$ and $\deg(q_0) \leq 2$, defining the two quadrics $E_1$ and $E_2$. If $(\alpha, \beta)$ is a point in the cutcurve such that $q_1(\alpha, \beta) \neq p_1(\alpha, \beta)$ then the $z$-coordinate of the point in the intersection curve is given by:

$$z = \frac{p_0(\alpha, \beta) - q_0(\alpha, \beta)}{q_1(\alpha, \beta) - p_1(\alpha, \beta)}.$$

If $(\alpha, \beta)$ is a point in the cutcurve such that $q_1(\alpha, \beta) = p_1(\alpha, \beta)$, the lifting of this singular point can be made by using $g(\alpha, \beta, z) = 0$ or $f(\alpha, \beta, z) = 0$.

**Proof.** As we have seen

$$S_1(x, y; z) = S_{\text{res}_1}(f, g; z) = (q_1(x, y) - p_1(x, y))z + (q_0(x, y) - p_0(x, y)).$$

If $(\alpha, \beta) \in \Pi(E_1 \cap E_1)$ then

$$S_1(\alpha, \beta; z) = 0 \iff z = \frac{p_0(\alpha, \beta) - q_0(\alpha, \beta)}{q_1(\alpha, \beta) - p_1(\alpha, \beta)}$$

as desired. \qed

The proof of next corollaries is trivial.

**Corollary 5.11.** Let $f$ and $g$ be the polynomials

$$f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = q_1(x, y)z + q_0(x, y) \quad (5)$$

with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, $\deg(q_1) \leq 1$ and $\deg(q_0) \leq 2$. Let $(\alpha, \beta)$ be a point of the cutcurve. If $q_1(\alpha, \beta) \neq 0$ then the lifting of $(\alpha, \beta)$ is given by:

$$z = -\frac{q_0(\alpha, \beta)}{q_1(\alpha, \beta)}.$$

If $q_1(\alpha, \beta) = 0$ then $q_0(\alpha, \beta) = 0$ and the lifting of $(\alpha, \beta)$ is given by:

$$z = -\frac{p_1(\alpha, \beta) \pm \sqrt{p_1(\alpha, \beta)^2 - 4p_0(\alpha, \beta)}}{2}.$$
Corollary 5.12. Let $f$ and $g$ be the polynomials
\[ f(x, y, z) = p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = q_1(x, y)z + q_0(x, y) \]
with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, $\deg(q_1) \leq 1$ and $\deg(q_0) \leq 2$. Let $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that $f(\alpha, \beta, \gamma) = g(\alpha, \beta, \gamma) = 0$ and $(\alpha, \beta)$ be a point in the cutcurve
\[ p_1(x, y)q_0(x, y) - p_0(x, y)q_1(x, y) = 0. \]
If $p_1(\alpha, \beta) \neq 0$ or $q_1(\alpha, \beta) \neq 0$ then the lifting of $(\alpha, \beta)$ is given by:
\[ z = -\frac{p_0(\alpha, \beta)}{p_1(\alpha, \beta)} \quad \text{or} \quad z = -\frac{q_0(\alpha, \beta)}{q_1(\alpha, \beta)} \]
respectively.

If $p_1(\alpha, \beta) = q_1(\alpha, \beta) = 0$ then $p_0(\alpha, \beta) = q_0(\alpha, \beta) = 0$ and the line $\{(\alpha, \beta, z) : z \in \mathbb{R}\}$ is in the intersection curve of $f(x, y, z) = 0$ and $g(x, y, z) = 0$.

Corollary 5.13. Let $f$ and $g$ be the polynomials
\[ f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = q_0(x, y) \]
with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, and $\deg(q_0) \leq 2$. The lifting of a point $(\alpha, \beta)$ of the cutcurve is given by:
\[ z = -\frac{p_0(\alpha, \beta) \pm \sqrt{p_1(\alpha, \beta)^2 - 4p_0(\alpha, \beta)}}{2}. \]

Corollary 5.14. Let $f$ and $g$ be the polynomials
\[ f(x, y, z) = p_1(x, y)z + p_0(x, y) \quad g(x, y, z) = q_0(x, y) \]
with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, and $\deg(q_0) \leq 2$. Let $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that $f(\alpha, \beta, \gamma) = g(\alpha, \beta, \gamma) = 0$ and $(\alpha, \beta)$ a point in the cutcurve
\[ q_0(x, y) = 0. \]
If $p_1(\alpha, \beta) \neq 0$ then the lifting of $(\alpha, \beta)$ is given by:
\[ z = -\frac{p_0(x, y)}{p_1(x, y)}. \]
If $p_1(\alpha, \beta) = 0$ then $p_0(\alpha, \beta) = 0$ and the line $\{(\alpha, \beta, z) : z \in \mathbb{R}\}$ is in the intersection curve of $f(x, y, z) = 0$ and $g(x, y, z) = 0$.

Corollary 5.15. Let $f$ and $g$ be the polynomials
\[ f(x, y, z) = p_0(x, y) \quad g(x, y, z) = q_0(x, y) \]
with $\deg(p_0) \leq 2$, and $\deg(q_0) \leq 2$. Common points to both quadrics are given by:
\[ \{(\alpha, \beta, z) : p_0(\alpha, \beta) = 0, q_0(\alpha, \beta) = 0, z \in \mathbb{R}\}. \]
6. Examples

This section will present some examples showing how to compute the intersection curve of two quadrics by using the results presented in the previous sections. In all examples two polynomials $f$ and $g$ define two quadrics $E_1$ and $E_2$ respectively. In these examples

$$\mathcal{A}_{E_1,E_2} = \{(x,y) \in \mathbb{R}^2 : \Delta_{E_1}(x,y) \geq 0, \Delta_{E_2}(x,y) \geq 0\}$$

defines the region where the cutcurve, defined by $S_0(x,y) = 0$, lives. Typically the lifting of the cutcurve will be made by using $S_1(x,y,z)$. When one of the singular points of the cutcurve can not be lifted by using $S_1(x,y,z)$, we will use $g(x,y,z)$ or $f(x,y,z)$ for that purpose (for those singular points outside $p_1(x,y) = q_1(x,y)$ we can use $S_1(x,y,z)$ for performing the lifting too).

The way of proceeding will be always the following one:

1. Compute the implicit equation fo the cutcurve $S_0(x,y)$.
2. Compute the region $\mathcal{A}_{E_1,E_2}$ where the cutcurve lives: requires to compute the region defined by the two silhouette curves which are two conics.
3. Compute the singular points of the cutcurve $S_0(x,y)$ in the line $p_1(x,y) = q_1(x,y)$: just solving a degree two univariate equation by Corollary 5.5. Their lifting is made by using $f$ or $g$.
4. Compute the singular points of the cutcurve $S_0(x,y)$ outside the line $p_1(x,y) = q_1(x,y)$: points coming from the projection of a tangential intersection whose lifting is made by using $S_1(x,y,z)$.
5. Compute the regular points of the cutcurve which are in the silhouette curves by using the Proposition 5.8 and their lifting by using $S_1(x,y;z)$.
6. Compute the branches of the cutcurve (always inside $\mathcal{A}_{E_1,E_2}$) by closed formulae involving radicals or discretising them and their lifting by using $S_1(x,y;z)$.

In the examples (6.3), (6.5) and (6.6), the lifting of the cutcurve will be made after discretising the regular branches of the cutcurve (always inside $\mathcal{A}_{E_1,E_2}$). All examples include computations made with Maple and GeoGebra.

In the next figures the cutcurve is drawn in black. Silhouette curves are drawn in blue (and in the last example in red and blue). The line $p_1(x,y) = q_1(x,y)$ will be always dotted.

**Example 6.1.** Let $f$ and $g$ be the polynomials

$$f(x,y,z) = z^2 - 5x^2 - 3xy - 7y^2 - x + 10y + 8 \quad g(x,y,z) = z^2 - 3x^2 - y^2 - 10x + 4y - 3$$

defining two hyperboloids of one sheet, $\mathcal{E}_1$ and $\mathcal{E}_2$, whose intersection curve is to be computed. In this case we have:

$$S_0(x,y) = (-2x^2 - 3xy - 6y^2 + 9x + 6y + 11)^2$$

$$\mathcal{A}_{E_1,E_2} = \{(x,y) \in \mathbb{R}^2 : 20x^2+12xy+4x+28y^2-40y-32 \geq 0, 12x^2+40x+4y^2-16y+12 \geq 0\}.$$
In this example we have that all points in the cutcurve are singular which is motivated by the fact that, in this case, the line $p_1(x, y) = q_1(x, y)$ is not a true line since all points in $\mathbb{R}^2$ satisfies that equation. When this happens the cutcurve is a conic and all further computations are greatly simplified. Since

$$S_1(x, y; z) = -2x^2 - 3xy - 6y^2 + 9x + 6y + 11$$

and $sres_1(f, g; z) = 0$, the lifting of $\Pi(\mathcal{E}_1 \cap \mathcal{E}_2)$ will be made using

$$g(x, y, z) = z^2 - 3x^2 - y^2 - 10x + 4y - 3$$

$$g(x, y, z) = 0 \Leftrightarrow z = \sqrt{3x^2 + y^2 + 10x - 4y + 3} \lor z = -\sqrt{3x^2 + y^2 + 10x - 4y + 3}$$

To parameterize the intersection curve, we must determine the cutcurve $S_0(x, y) = 0$ and its lifting:

- For $x \in \left[-10 \frac{\sqrt{22} + 30}{13}, 10 \frac{\sqrt{22} + 30}{13}\right]$ we have:

$$h_1(x) = \frac{1}{12} \left(\sqrt{3} \sqrt{-13 \ x^2 + 60 \ x + 100} - 3 \ x + 6\right)$$

$$h_2(x) = \frac{1}{12} \left(-\sqrt{3} \sqrt{-13 \ x^2 + 60 \ x + 100} - 3 \ x + 6\right)$$

- Let $e_1$ and $e_2$ be the functions defined by:

$$e_1(x, y) = \sqrt{3x^2 + y^2 + 10x - 4y + 3} \quad e_2(x, y) = -\sqrt{3x^2 + y^2 + 10x - 4y + 3}$$

Then the parameterization of the intersection curve is given by:

$$(x, h_1(x), e_1(x, h_1(x))), (x, h_2(x), e_1(x, h_2(x))), (x, h_1(x), e_2(x, h_1(x))), (x, h_2(x), e_2(x, h_2(x)))$$

with $\left[-10 \frac{\sqrt{22} + 30}{13}, 10 \frac{\sqrt{22} + 30}{13}\right]$. 
**Example 6.2.** Let \( f \) and \( g \) be the polynomials

\[
f(x, y, z) = z^2 - 4z + x^2 + 2xy + 3y^2 + x - 10y + 2 \quad g(x, y, z) = z^2 - 4z - 9x + 9y + 4
\]

defining one ellipsoid and one parabolic cylinder, \( E_1 \) and \( E_2 \), whose intersection curve is to be computed. In this case we have:

\[
S_0(x, y) = (x^2 + 2xy + 3y^2 + 10x - 19y - 2)^2
\]

\[
\mathcal{A}_{E_1,E_2} = \{(x, y) \in \mathbb{R}^2: -4x^2 - 8xy - 12y^2 - 4x + 40y + 8 \geq 0, \ x - y \geq 0\}
\]

As in the previous example, all the points in the cutcurve are singular. Since

\[
S_1(x, y; z) = -x^2 - 2xy - 3y^2 - 10x + 19y + 2
\]

and \( \text{sres}_1(f, g; z) = 0 \), the lifting of \( \Pi(E_1 \cap E_2) \) will be made using

\[
g(x, y, z) = z^2 - 4z - 9x + 9y + 4
\]

\[
g(x, y, z) = 0 \Leftrightarrow z = 2 + 3\sqrt{x - y} \vee z = 2 - 3\sqrt{x - y}
\]

To parameterise the intersection curve, we must determine the cutcurve \( S_0(x, y) = 0 \) and its lifting:

- For \( x \in \left[-\frac{49}{4} - \frac{\sqrt{3171}}{4}, -\frac{49}{4} + \frac{\sqrt{3171}}{4}\right] \) we have:

\[
h_1(x) = -\frac{x}{3} + \frac{19}{6} + \frac{\sqrt{-8x^2 - 196x + 385}}{6} \quad h_2(x) = -\frac{x}{3} + \frac{19}{6} - \frac{\sqrt{-8x^2 - 196x + 385}}{6}
\]

- Let \( e_1 \) and \( e_2 \) be the functions defined by:

\[
e_1(x, y) = 2 + 3\sqrt{x - y} \quad e_2(x, y) = 2 - 3\sqrt{x - y}
\]

The parameterization of the intersection curve is given by:

\[
(x, h_2(x), e_1(x, h_2(x))), \quad (x, h_2(x), e_2(x, h_2(x))
\]

with \( x \in \left[9 - \sqrt{129}/12, 9 + \sqrt{129}/12\right] \).
Example 6.3. Let $f$ and $g$ be the polynomials
\[ f(x, y, z) = z^2 + (-6x - y - 1)z - 9x^2 - 3xy + 4y^2 + 9x - 9y - 2 \]
\[ g(x, y, z) = z^2 - 2z + x^2 - 3y^2 + 9x - 2y + 6 \]

defining two ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$, whose intersection curve is to be computed. In this case we have:
\[ S_0(x, y) = 136x^4 + 72x^3y - 238x^2y^2 - 78xy^3 + 46y^4 + 432x^2y - 15xy^2 - 108y^3 \]
\[ + 249x^2 + 204xy - 28y^2 + 33x + 100y + 54 \]
\[ \mathcal{A}_{\mathcal{E}_1, \mathcal{E}_2} = \left\{ (x, y) \in \mathbb{R}^2 : \begin{array}{l} 27x^2 + 24xy - 15y^2 - 24x + 38y + 9 \geq 0, \\ -4x^2 + 12y^2 - 36x + 8y - 20 \geq 0 \end{array} \right\}. \]

From Corollary 5.5, the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$ are determined by solving:
\[ 76x^2 + 24xy - 27y^2 + 12x + 30y + 29 = 0 \land 6x + y + 1 = 2. \]

In this way we get the singular points
\[ A = \left( \frac{9}{104} + \frac{\sqrt{10345}}{520}, \frac{25}{52} - \frac{3\sqrt{10355}}{260} \right) \]
\[ B = \left( \frac{9}{104} - \frac{\sqrt{10345}}{520}, \frac{25}{52} + \frac{3\sqrt{10355}}{260} \right). \]

The first one, $A$, is an isolated point of the cutcurve but outside $\mathcal{A}_{\mathcal{E}_1, \mathcal{E}_2}$ and will not be lifted. The second one, $B$, will be lifted by using $f(x, y, z) = 0$ or $g(x, y, z) = 0$.

Point $C = (-0.5989698028, -0.6502822952)$ is common to $S_0(x, y) = 0$ and $\Delta_{\mathcal{E}_1}(x, y) = 0$ and, by using Proposition 5.8, was determined by solving:
\[ -16x^2 - 6xy + 2y^2 + 24x - 24y + 6 = 0 \land -4x^2 + 12y^2 - 36x + 8y - 20 = 0 \]

Points $D = (-2.336955328, -6.163216205)$ and $E = (21.765280490, -32.199082657)$ are common to $S_0(x, y) = 0$ and $\Delta_{\mathcal{E}_1}(x, y) = 0$ and, from Proposition 5.8, were determined by solving:
\[ -16x^2 - 6xy + 2y^2 + 24x - 24y + 6 = 0 \land 72x^2 + 24xy - 15y^2 - 24x + 38y + 9 = 0 \]

The lifting of $\Pi(\mathcal{E}_1 \cap \mathcal{E}_2)$, outside the singular points of the cutcurve, will be made by using:
\[ S_1(x, y; z) = 10x^2 + 3xy + 6xz - 7y^2 + yz + 7y - z + 8 \]
\[ S_1(x, y; z) = 0 \iff z = \frac{10x^2 + 3xy - 7y^2 + 7y + 8}{-6x - y + 1}. \]
Example 6.4. Let $f$ and $g$ be the polynomials

$$f(x, y, z) = z^2 + xz + y \quad g(x, y, z) = z^2 + yz + x$$

defining two hyperbolic paraboloids, $\mathcal{E}_1$ and $\mathcal{E}_2$, whose intersection curve is to be computed. In this case we have

$$S_0(x, y) = (x - y)^2(x + y + 1)$$

and

$$A_{\mathcal{E}_1, \mathcal{E}_2} = \{(x, y) \in \mathbb{R}^2: x^2 - 4y \geq 0, \ y^2 - 4x \geq 0\}.$$ 

All the points in the line $p_1(x, y) = q_1(x, y)$ belong to the cut curve and they are singular: this can be checked by solving (according to Corollary 5.5):

$$x^2 - y^2 + 4x - 4y = 0 \land x - y = 0$$

In this case, there is one special singular point, $D = (-1/2, -1/2)$, that has been already classified since it belongs to the line $p_1(x, y) = q_1(x, y)$.

Points $O = (0, 0)$, $B = (1, -2)$ and $C = (4, 4)$ are common to $S_0(x, y) = 0$ and $\Delta_{\mathcal{E}_1}(x, y) = 0$ and, from Proposition 5.8, were determined by solving

$$-xy + 2x + 2y = 0 \land y^2 - 4x = 0.$$ 

On the other hand, points $A = (-2, 1)$, $O = (0, 0)$ and $C = (4, 4)$ are common to $S(x, y) = 0$ and $\Delta_{\mathcal{E}_2}(x, y) = 0$ and, from Proposition 5.8, were determined by solving

$$-xy + 2x + 2y = 0 \land x^2 - 4y = 0.$$ 

Note that points $O$ and $C$ are common to $S_0 = 0$, $\Delta_{\mathcal{E}_1} = 0$, $\Delta_{\mathcal{E}_2} = 0$ and, as seen in Proposition 5.6, belong to the line $x - y = 0$.

The lifting of $\Pi(\mathcal{E}_1 \cap \mathcal{E}_2)$, outside the singular points of the cut curve, will be made by using:

$$S_1(x, y; z) = (y - x)z + x - y$$

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\[ S_1(x, y; z) = 0 \iff z = \frac{x - y}{x - y} = 1 \]

Singular points of the cutcurve will be lifted by using:

\[ g(x, y, z) = z^2 + yz + x \]

\[ g(x, y, z) = 0 \iff z = \frac{y}{2} + \frac{\sqrt{y^2 - 4x}}{2} \quad \forall z = -\frac{y}{2} - \frac{\sqrt{y^2 - 4x}}{2} \]

To characterise the intersection curve of \( E_1 \) and \( E_2 \) we must determine the cutcurve \( S_0(x, y) = 0 \) and its lifting. We will use the following functions:

- For \( x \in \mathbb{R} \), we define:
  \[ h_1(x) = x \quad h_2(x) = -x - 1 \]

- Let \( e_1 \) and \( e_2 \) be the functions defined by:
  \[ e_1(x, y) = \frac{y}{2} + \frac{\sqrt{y^2 - 4x}}{2} \quad e_2(x, y) = -\frac{y}{2} - \frac{\sqrt{y^2 - 4x}}{2} \]

The parameterisation of the intersection curve is given by the following three components:

- For \( x \in ]-\infty, 0] \cup [4, +\infty[\): \( (x, h_1(x), e_1(x, h_1(x)) \)

- For \( x \in ]-\infty, 0] \cup [4, +\infty[\): \( (x, h_1(x), e_2(x, h_1(x)) \).

- For \( x \in \mathbb{R} \): \( (x, h_2(x), 1) \).
Example 6.5. Let \( f \) and \( g \) be the polynomials

\[
f(x, y, z) = z^2 + (y - 2x - 1)z - x^2 - y^2 - xy + x - y + 1
\]
\[
g(x, y, z) = z^2 + (x - y)z + x^2 + y^2 - xy - 2x + y - 5
\]
defining one hyperboloid of one sheet and one ellipsoid, \( E_1 \) and \( E_2 \), whose intersection curve is to be computed. In this case we have

\[
S_0(x, y) = 7x^4 - 11x^3y + 23x^2y^2 - 6xy^3 + 4y^4 - 17x^3 + 12x^2y - 8xy^2 + 6y^3 - 50x^2 + 26xy - 29y^2 + 10x - 9y + 31
\]
\[
A_{E_1, E_2} = \{(x, y) \in \mathbb{R}^2: 8x^2 + 5y^2 + 2y - 3 \geq 0, -3x^2 + 2xy - 3y^2 + 8x - 4y + 20 \geq 0\}
\]

From Corollary 5.5, singular points of the cutcurve in the line \( p_1(x, y) = q_1(x, y) \) were determined by solving:

\[
11x^2 - 2xy - 8x + 8y^2 + 6y - 23 \land -3x + 2y - 1 = 0
\]

They are

\[
A = \left( -\frac{3}{13}, \frac{3\sqrt{14}}{13}, \frac{2}{13}, -\frac{9\sqrt{14}}{26} \right) \quad B = \left( -\frac{3}{13}, \frac{3\sqrt{14}}{13}, \frac{2}{13}, \frac{9\sqrt{14}}{26} \right)
\]

and can be lifted by using \( g(x, y, z) = 0 \) or \( f(x, y, z) = 0 \).

Common points to \( S_0(x, y) = 0 \) and \( \Delta_{E_2}(x, y) = 0 \) can be determined, according to Proposition 5.8, by solving

\[
2x^2 - 7xy + y^2 - x - y - 8 = 0 \land -3x^2 + 2xy - 3y^2 + 8x - 4y + 20 = 0.
\]

These points are \( C = (-1.468654, 0.233082), D = (-0.575622, 1.494633), E = (-0.285566, -3.292475) \) and \( F = (4.341889, 0.829820) \). The lifting of \( \Pi(E_1 \cap E_2) \), outside the singular curves of the cutcurve, will be made by using:

\[
S_1(x, y; z) = (3x - 2y + 1)z + 2x^2 + 2y^2 - 3x + 2y - 6
\]
\[
S_1(x, y; z) = 0 \iff z = -\frac{2x^2 + 2y^2 - 3x + 2y - 6}{3x - 2y + 1}
\]
Last example shows a situation where the introduced technics are specially useful to determine the intersection curve of the two considered quadrics.

**Example 6.6.** Let $f$ and $g$ be the polynomials

\[ f(x, y, z) = z^2 + \left( -\frac{2}{3}x + \frac{2}{3}y \right) z + \frac{x^2}{3} + \frac{y^2}{3} - \frac{1}{3} \]

\[ g(x, y, z) = z^2 + \left( \frac{-2x + 24y - 2}{17} \right) z + \frac{x^2}{17} + \frac{2x}{17} - \frac{3}{17} + \frac{12y^2}{17} \]

defining two ellipsoids, $E_1$ and $E_2$, whose intersection curve is to be computed. In this case we have:

\[ S_0(x, y) = \frac{196}{2601}x^4 + \frac{616}{2601}x^3y + \frac{920}{2601}x^2y^2 + \frac{836}{2601}xy^3 + \frac{361}{2601}y^4 - \frac{112}{2601}x^3 - \frac{56}{867}x^2y - \frac{112}{2601}xy^2 \]

\[ -\frac{76}{2601}y^3 - \frac{104}{867}x^2 - \frac{632}{2601}xy - \frac{368}{2601}y^2 + \frac{176}{2601}x + \frac{184}{2601}y + \frac{52}{2601} \]

\[ \mathcal{A}_{E_1, E_2} = \{(x, y) \in \mathbb{R}^2 : -2x^2 - 2xy - 2y^2 + 3 \geq 0, -4x^2 - 6xy - 15y^2 - 8x - 6y + 13 \geq 0\} \]

Singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$ can be determined, as seen in Corollary 5.5 by solving

\[ -434x^2 - 362xy - 38y^2 + 288x + 216y + 399 = 0 \land -14x - 19y + 3 = 0 \]

They are

\[ A = \left( \frac{3}{14}, -\frac{11\sqrt{95}}{70}, -\frac{11\sqrt{95}}{95} \right) \quad C = \left( \frac{3}{14}, \frac{11\sqrt{95}}{70}, -\frac{11\sqrt{95}}{95} \right) \]

and are outside $\mathcal{A}_{E_1, E_2}$; therefore they will not be lifted. Moreover, the cutcurve has a third singular (and isolated) point $B = (1, 0)$ which is inside $\mathcal{A}_{E_1, E_2}$ but not in the line $p_1(x, y) = q_1(x, y)$. This means that $B$ is the projection of a tangential intersection point of $E_1$ and $E_2$ and it has been computed by using Theorem 5.7.

Points

\[ D = (-1.310086292, 1.116297338) \quad E = (-1.032926046, -0.320076179) \]

are common to $S_0(x, y) = 0$ and $\Delta_{E_1}(x, y) = 0$ and, from Proposition 5.8 were determined by solving

\[ 18x^2 + 26xy + 29y^2 + 4x + 2y - 26 = 0 \land -2x^2 - 2xy - 2y^2 + 3 = 0. \]

On the other hand, the points

\[ F = (-1.310059433, 1.116297338) \quad G = (-1.032926046, -0.320076179) \]
Figure 7: Cutcurve and silhouette curves showing the region where points A, F and D are.

are common to $S_0(x,y) = 0$ and $\Delta_{\varepsilon_2}(x,y) = 0$ and, from Proposition 5.8, were determined by solving

$$18x^2 + 26xy + 29y^2 + 4x + 2y - 26 = 0 \land -4x^2 - 6xy - 15y^2 - 8x - 6y + 13 = 0.$$  

Figure 7 shows the location of all these points with respect to the cutcurve and the silhouette curves.

In order to determine what is the relative position of the points A, F and D with respect to this three curves we need to use the results introduced in the previous section. Figure 8 shows in detail what is happening in that area.

Figure 8: Location of points A, F and D with respect to the cutcurve and the silhouette curves.

Point A is outside the region $A_{\varepsilon_1,\varepsilon_2}$ and does not play any role when computing the intersection curve between the two considered quadrics. Points F and D does belong to the intersection between the cutcurve and one of the silhouette curves. Proposition 5.6 helps to conclude that the intersection between the cutcurve and the two silhouette curves is empty and not A as Figure 7 could suggest.
The lifting of the regular points of the cutcurve and of the point $B$ will be made by using:

$$S_1(x, y; z) = (28x + 38y - 6)z - 14x^2 + 19y^2 + 6x + 8$$

$$S_1(x, y; z) = 0 \iff z = \frac{14x^2 - 19y^2 - 6x - 8}{28x + 38y - 6}$$

Note that $B$ is the projection of the point $(1, 0, 0)$ where the ellipsoids are tangent and this is the reason why it can be lifted by using $S_1(x, y; z)$ (as seen in Theorem 5.2). The two quadrics together with their intersection curve can be found in Figure 9.

![Figure 9: Two ellipsoids with a curve and an isolated point (tangent to both ellipsoids) in common.](image)

7. Conclusions

We have introduced a new approach to deal with the computation of the intersection curve of two quadrics. The main ingredients of this approach are a detailed analysis of the cutcurve and of its relation with the silhouette curves together with the using of an uniform way to perform the lifting of the cutcurve to the intersection curve of the two considered quadrics.

Concerning the analysis of the cutcurve we classify its singular points in two different types depending on how they will be lifted. Those belonging to the line $p_1(x, y) = q_1(x, y)$ are easy to compute and difficult to lift and those not in that line are more complicated to be determined but easier to lift.

This approach is not intended to classify the intersection curve between the two considered quadrics. Its main goal is to produce in a very efficient way a description of the intersection curve topologically correct. This is the reason why we allow in the lifting of the cutcurve, when possible, the use of radicals or we rely on the discretisation of the branches of the cutcurve (uniquely determined by the points computed in that curve).

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