AUTOMORPHISMS OF THE GENERALIZED QUOT SCHEMES

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Abstract. Given a compact connected Riemann surface $X$ of genus $g \geq 2$, and integers $r \geq 2$, $d_p > 0$ and $d_z > 0$, in [BDHW], a generalized quot scheme $Q_X(r, d_p, d_z)$ was introduced. Our aim here is to compute the holomorphic automorphism group of $Q_X(r, d_p, d_z)$. It is shown that the connected component of $\text{Aut}(Q_X(r, d_p, d_z))$ containing the identity automorphism is $\text{PGL}(r, \mathbb{C})$. As an application of it, we prove that if the generalized quot schemes of two Riemann surfaces are holomorphically isomorphic, then the two Riemann surfaces themselves are isomorphic.

1. Introduction

In [BDHW], a generalized quot scheme was defined; we quickly recall it. Let $X$ be a compact connected Riemann surface of genus $g \geq 2$, and $r \geq 2$, $d_p > 0$ and $d_z > 0$ are integers. Let $Q(O^\oplus_r X, d_p)$ be the quot scheme that parametrizes the coherent subsheaves of $O^\oplus_r X$ of rank $r$ and degree $-d_p$. This complex projective variety $Q(O^\oplus_r X, d_p)$ is a moduli space of vortices [BDW], [Br], [BR], [Ba], [EINOS]. This moduli space is extensively studied (cf. [Bif], [BGL]). The universal vector bundle over $X \times Q(O^\oplus_r X, d_p)$ will be denoted by $S$. The generalized quot scheme $Q_X(r, d_p, d_z)$ parametrizes torsionfree coherent sheaves $F$ on $X$ of rank $r$ and degree $d_z - d_p$ such that some member of the family $S$ is a subsheaf of $F$. In [BDHW], the fundamental group and the cohomology of $Q_X(r, d_p, d_z)$ were computed.

The natural action of $\text{GL}(r, \mathbb{C})$ on the trivial vector bundle $O^\oplus_r X$ produces a holomorphic action of $\text{PGL}(r, \mathbb{C})$ on $Q_X(r, d_p, d_z)$. The main result proved here says that $\text{PGL}(r, \mathbb{C})$ is the connected component, containing the identity element, of the group of holomorphic automorphisms of $Q_X(r, d_p, d_z)$; see Theorem 2.1.

Let $X'$ be a compact connected Riemann surface of genus at least two. Fix positive integers $r' \geq 2$, $d'_p$ and $d'_z$. Let $Q_X'(r', d'_p, d'_z)$ be the corresponding generalized quot scheme. As an application of Theorem 2.1, we prove the following (see Proposition 3.2):

**Proposition 1.1.** If the two varieties $Q_X(r', d'_p, d'_z)$ and $Q_X(r, d_p, d_z)$ are isomorphic, then $X$ is isomorphic to $X'$.

2. Holomorphic vector fields on $Q_X(r, d_p, d_z)$

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. Fix positive integers $r \geq 2$, $d_p$ and $d_z$. Let $Q(O^\oplus_r X, d_p)$ denote the quot scheme that parametrizes all the torsion quotients of $O^\oplus_r X$ of degree $d_p$. Therefore, elements of $Q(O^\oplus_r X, d_p)$ represent...
subsheaves $S$ of $\mathcal{O}_X^{\oplus r}$ such that rank$(S) = r$ and degree$(S) = -d_p$. These two conditions on the subsheaf $S$ are together equivalent to the condition that $\mathcal{O}_X^{\oplus r}/S$ is torsion of degree $d_p$. There is a universal short exact sequence of sheaves on $X \times \mathbb{Q}(\mathcal{O}_X^{\oplus r}, d_p)$

$$0 \rightarrow S \rightarrow p_X^* \mathcal{O}_X^{\oplus r} = \mathcal{O}_X^{\oplus r} \otimes \mathcal{O}(\mathcal{O}_X^{\oplus r}, d_p) \rightarrow T_1 \rightarrow 0,$$

(2.1)

where $p_X : X \times \mathbb{Q}(\mathcal{O}_X^{\oplus r}, d_p) \rightarrow X$ is the natural projection.

Let $f : \mathcal{Q} = \mathcal{Q}_X(r, d_p, d_z) \rightarrow \mathbb{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ be the relative quot scheme that parametrizes torsion quotients of $S^*$ of degree $d_z$ (see (2.1)). In other words, if $z$ is the point of $\mathbb{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ representing a subsheaf $S \subset \mathcal{O}_X^{\oplus r}$, then the fiber $f^{-1}(z)$ is the the space of all subsheaves of $S^*$ of rank $r$ and degree $d_p - d_z$.

Note that the degree of $S^*$ is $d_p$. Therefore, elements of $\mathcal{Q}$ parametrize diagrams of the form

$$0 \rightarrow S \rightarrow \mathcal{O}_X^{\oplus r} \rightarrow T_1 \rightarrow 0$$

(2.3)

where $T_1$ and $T_2$ are torsion sheaves of degree $d_p$ and $d_z$ respectively, and the subsheaf $S \subset \mathcal{O}_X^{\oplus r}$ corresponds to the image, under $f$, of the point of $\mathcal{Q}$ corresponding to $V$.

Let $\text{Aut}(\mathcal{Q})$ denote the group of all holomorphic automorphisms of $\mathcal{Q}$. Since $\mathcal{Q}$ is a smooth complex projective variety, its holomorphic automorphisms are automatically algebraic. Thus $\text{Aut}(\mathcal{Q})$ is a complex Lie group with Lie algebra $H^0(\mathcal{Q}, T\mathcal{Q})$, where $T\mathcal{Q}$ is the holomorphic tangent bundle of $\mathcal{Q}$; the Lie algebra structure is given by the Lie bracket operation of vector fields. Let

$$\text{Aut}^0(\mathcal{Q}) \subset \text{Aut}(\mathcal{Q})$$

be the connected component containing the identity element.

The standard action of $\text{GL}(r, \mathbb{C})$ on $\mathbb{C}^r$ produces an action of $\text{GL}(r, \mathbb{C})$ on $\mathcal{O}_X^{\oplus r}$, because the total space of $\mathcal{O}_X^{\oplus r}$ is identified with $X \times \mathbb{C}^r$. This action of $\text{GL}(r, \mathbb{C})$ on $\mathcal{O}_X^{\oplus r}$ defines an action of $\text{GL}(r, \mathbb{C})$ on $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$. This action of $\text{GL}(r, \mathbb{C})$ on $\mathcal{Q}(\mathcal{O}_X^{\oplus r}, d_p)$ evidently lifts to an action of $\text{GL}(r, \mathbb{C})$ on $\mathcal{Q}$ (see (2.2)). Indeed, $\text{GL}(r, \mathbb{C})$ acts on diagrams of type (2.3). Since $\text{GL}(r, \mathbb{C})$ is connected, we get a homomorphism

$$\text{GL}(r, \mathbb{C}) \rightarrow \text{Aut}^0(\mathcal{Q}).$$

The center $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of $\text{GL}(r, \mathbb{C})$ acts trivially on $\mathcal{Q}$. Hence the above homomorphism produces a homomorphism

$$\varphi : \text{PGL}(r, \mathbb{C}) \rightarrow \text{Aut}^0(\mathcal{Q}).$$

(2.4)

Theorem 2.1. The homomorphism $\varphi$ in (2.4) is an isomorphism.
Proof. Let
\[ p : \mathcal{Q} \longrightarrow \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X) \] (2.5)
be the morphism that sends any \( z \in \mathcal{Q} \) to the support of \( T_1 \) with multiplicity given by \( T_1 \) and the support of \( T_2 \) with multiplicity given by \( T_2 \), where \( T_1 \) and \( T_2 \) are the torsion sheaves in the diagram (2.3) corresponding to the point \( z \).

The homomorphism \( \varphi \) is injective because the homomorphism
\[ \text{PGL}(r, \mathbb{C}) \longrightarrow \text{Aut}(\mathbb{C}P^{r-1}) \]
given by the standard action of \( \text{GL}(r, \mathbb{C}) \) on \( \mathbb{C}^r \) is injective. Indeed, for any
\[ x = ((x_1, \cdots, x_{d_p}),(y_1, \cdots, y_{d_z})) \in \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X), \]
where all the above \( d_p + d_z \) points are distinct, the fiber \( p^{-1}(x) \) is \((\mathbb{C}P^{r-1})^{d_p} \times (\mathbb{C}P^{r-1})^{d_z} \) (see (2.5)), and the action of \( \text{PGL}(r, \mathbb{C}) \) on \( p^{-1}(x) \) coincides with the diagonal action of \( \text{PGL}(r, \mathbb{C}) \) on the factors in the above Cartesian product.

We need to prove that \( \varphi \) is surjective.

The Lie algebra of \( \text{PGL}(r, \mathbb{C}) \) will be denoted by \( \mathfrak{g} \); it is the Lie algebra structure on trace zero \( r \times r \)-matrices with complex entries given by commutator. Let
\[ \theta : \mathfrak{g} \longrightarrow H^0(\mathcal{Q}, T \mathcal{Q}) \] (2.6)
be the homomorphism of Lie algebras corresponding to the homomorphism \( \varphi \) in (2.4). To prove that \( \varphi \) is surjective, it suffices to show that \( \theta \) is surjective.

The following lemma is a key step in the computation of \( H^0(\mathcal{Q}, T \mathcal{Q}) \).

**Lemma 2.2.** The Lie algebra \( H^0(\mathcal{Q}, T \mathcal{Q}) \) has a natural injective homomorphism to \( \mathfrak{g} \oplus \mathfrak{g} \).

**Proof of Lemma 2.2.** For any positive integer \( k \), let \( P(k) \) denote the group of all permutations of \( \{1, \cdots, k\} \). Consider the action of \( P(d_p) \times P(d_z) \) on \( X^{d_p + d_z} \) that permutes the first \( d_p \) factors and the last \( d_z \) factors. The quotient \( X^{d_p + d_z}/(P(d_p) \times P(d_z)) \) is \( \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X) \). Let
\[ q : X^{d_p + d_z} \longrightarrow \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X) \]
be the corresponding quotient map.

Let
\[ \widetilde{U} \subset X^{d_p + d_z} \] (2.7)
be the complement of the big diagonal, so \( \widetilde{U} \) parametrizes all possible distinct \( d_p + d_z \) ordered points of \( X \). The image \( q(\widetilde{U}) \subset \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X) \) will be denoted by \( U \). Since \( p^{-1}(U) \) is a Zariski dense open subset of \( \mathcal{Q} \), where \( p \) is defined in (2.5), we have
\[ H^0(\mathcal{Q}, T \mathcal{Q}) \subset H^0(p^{-1}(U), T(p^{-1}(U))). \] (2.8)

The Galois group \( \Gamma := P(d_p) \times P(d_z) \) for the étale covering
\[ q|_{\widetilde{U}} : \widetilde{U} \longrightarrow U, \]
where \( \widetilde{U} \) is defined in (2.7), acts on the fiber product \( \mathcal{Z} := p^{-1}(U) \times_U \widetilde{U} \). We have
\[ H^0(p^{-1}(U), T \mathcal{Q}) = H^0(\mathcal{Z}, T \mathcal{Z})^\Gamma, \] (2.9)
because the projection $Z \to p^{-1}(U)$ to the first factor of the fiber product is an étale Galois covering with Galois group $\Gamma$.

Now we have $Z = \tilde{U} \times (\mathbb{C}P^{r-1})^{d_p} \times (\mathbb{C}P^{r-1})^{d_z}$, where $\mathbb{C}P^{r-1}$ is the projective space parametrizing the lines in $\mathbb{C}^r$. Note that

$$H^0(\mathbb{C}P^{r-1}, T\mathbb{C}P^{r-1}) = \mathfrak{g} = H^0(\mathbb{C}P^{r-1}, T\mathbb{C}P^{r-1}).$$

It is known that $H^0(\tilde{U}, T\tilde{U}) = 0$ [BDH, p. 1452, Proposition 2.3]. Also, we have $H^0(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0$ [BDH, p. 1449, Lemma 2.2]. These together imply that

$$H^0(Z, TZ) = \mathfrak{g}^d \oplus \mathfrak{g}^d.$$  

The action of $\Gamma = P(d_p) \times P(d_z)$ on $\mathfrak{g}^d \oplus \mathfrak{g}^d = H^0(Z, TZ)$ (see (2.9)) is the one that permutes first $d_p$ factors and the last $d_z$ factors. Hence we have $H^0(Z, TZ)^\Gamma = \mathfrak{g} \oplus \mathfrak{g}$. Therefore,

$$H^0(p^{-1}(U), (T(p^{-1}(U)))) = \mathfrak{g} \oplus \mathfrak{g} \quad (2.10)$$

by (2.9). Now the lemma follow from (2.3). □

Next we will need a property of the Hecke transformations.

Let $Y$ be a smooth complex algebraic curve and $y_0 \in Y$ a point; the curve $Y$ need not be projective. Fix a linear nonzero proper subspace $0 \neq S \subset \mathbb{C}^r$. Consider the short exact sequence of sheaves on $Y$

$$0 \to V \to \mathcal{O}_Y^\oplus \to \mathcal{O}_Y^\oplus \to \mathcal{O}_{Y_0^\oplus} / S = \mathbb{C}^r / S \to 0. \quad (2.11)$$

Let $P(V)$ denote the projective bundle over $Y$ that parametrizes the lines in the fibers of $V$. Take any $A \in \text{GL}(r, \mathbb{C})$. Let $\hat{A}$ be the automorphism of $P(\mathcal{O}_Y^\oplus) = Y \times \mathbb{C}P^{r-1}$ given by $A$; this automorphism acts trivially on $Y$ and has the standard action on $\mathbb{C}P^{r-1}$. Since $V$ and $\mathcal{O}_Y^\oplus$ are identified over $Y \setminus \{y_0\}$, the above automorphism $\hat{A}$ produces an automorphism of $P(V)|_{Y \setminus \{y_0\}}$. This automorphism of $P(V)|_{Y \setminus \{y_0\}}$ will be denoted by $\hat{A}'$.

**Lemma 2.3.** The above automorphism $\hat{A}'$ of $P(V)|_{Y \setminus \{y_0\}}$ extends to a self-map of $P(V)\ if \ and \ only \ if \ A(S) = S$.

**Proof of Lemma 2.3.** Let $\text{GL}(V)$ be the Zariski open subset of the total space of $\text{End}(V) = V \otimes V^*$ parametrizing endomorphisms of fibers that are automorphisms. The quotient $\text{PGL}(V) = \text{GL}(V)/\mathbb{C}^*$ is a group–scheme over $Y$ with fibers isomorphic to the group $\text{PGL}(r, \mathbb{C})$. If an algebraic map of the total space

$$\tau: P(V) \to P(V)$$

is an automorphism satisfies the condition that there is a nonempty Zariski open subset $U_\tau \subset Y$ such that $\tau$ restricts to an automorphism of $P(V)|_{U_\tau}$ over the identity map of $U_\tau$, then $\tau$ is actually an automorphism over the identity map of $Y$. We note that the group of automorphisms of $P(V)$ over the identity map of $Y$ is precisely the group of sections, over $Y$, of the group–scheme $\text{PGL}(V)$.

Fix a subspace $S' \subset \mathbb{C}^r$ complementary to $S$, so $\mathbb{C}^r = S \oplus S'$. Let $E_S := Y \times S$ and $E_{S'} := Y \times S'$ be the trivial algebraic vector bundles over $Y$ with fibers $S$ and $S'$ respectively. Then we have

$$\mathcal{O}_Y^\oplus = E_S \oplus E_{S'} \quad \text{and} \quad V = E_S \oplus (E_{S'} \otimes \mathcal{O}_Y(-y_0)). \quad (2.12)$$
From the above decompositions it follows immediately that if $A(S) = S$, then $\hat{\mathcal{A}}'$ extends to an automorphism of $P(V)$. 

To prove the converse, assume that $\hat{\mathcal{A}}'$ extends to an automorphism of $P(V)$. It suffices to show that the subbundle $E_S \subset V$ in (2.12) is preserved by the section of $\text{PGL}(V)$ corresponding to the automorphism of $P(V)$. Note that the restriction of this section of $\text{PGL}(V)$ to $Y \setminus \{y_0\}$ is given by $A$. There is no nonzero homomorphism from $E_S$ to $E_{S'} \otimes O_Y(-y_0)$ which is given by a constant homomorphism $B : S \to S'$ on $Y \setminus \{y_0\}$ because such a homomorphism over $Y \setminus \{y_0\}$ extends to a homomorphism from $(E_S)_{y_0}$ to $(E_{S'})_{y_0}$ and this homomorphism $(E_S)_{y_0} \to (E_{S'})_{y_0}$ coincides with $B$. On the other hand, the image of $(E_{S'} \otimes O_Y(-y_0))_{y_0}$ in $(E_{S'})_{y_0}$ is the zero subspace. So if $B \neq 0$, then the homomorphism over $Y \setminus \{y_0\}$ does not extend to a homomorphism from $E_S$ to $E_{S'} \otimes O_Y(-y_0)$ over $Y$. Therefore, we conclude that $A(S) = S$. □

Fix distinct $d_p - 1$ points $x_1, \cdots, x_{d_p - 1}$ on $X$. For each $x_i$, fix a hyperplane $H_i$ in $(\mathcal{O}^r_X)_{x_i} = \mathbb{C}^r$. Also, fix distinct $d_z - 1$ points $y_1, \cdots, y_{d_z - 1}$ on $X$ such that $x_i \neq y_j$ for all $1 \leq i \leq d_p - 1$ and $1 \leq j \leq d_z - 1$. Fix a line $L_j$ in $(\mathcal{O}^r_X)_{y_j} = \mathbb{C}^r$ for each $j$.

Now take the complement $Y = X \setminus \{x_1, \cdots, x_{d_p - 1}, y_1, \cdots, y_{d_z - 1}\}$. Take any nontrivial element

$$\text{Id} \neq A \in \text{PGL}(r, \mathbb{C}). \tag{2.13}$$

Fix a point $x_0 \in Y$ and also fix a hyperplane

$$S \subset (\mathcal{O}^r_X)_{x_0} = \mathbb{C}^r$$

such that

$$A(S) \neq S; \tag{2.14}$$

since $A \neq \text{Id}$, such a subspace exists. Consider the vector bundle $V$ on $Y$ constructed in (2.11) using $S$. As before, $P(V)$ denotes the projective bundle over $Y$ parametrizing the lines in the fibers of $V$.

There is an embedding

$$\delta : P(V) \to Q \tag{2.15}$$

which we will now describe. For the map $f$ in (2.2), the image $f \circ \delta(P(V))$ is the point given by the quotient

$$0 \to \hat{V} \to \mathcal{O}^r_X \to (\oplus_{i=1}^{d_p - 1} (\mathcal{O}^r_X)_{x_i}/H_i) \oplus (\mathcal{O}^r_X)_{x_0}/S \to 0,$$

where $H_i$ are the hyperplanes fixed above; in particular, $f \circ \delta$ is a constant map. Note that $\hat{V}$ is an extension of the vector bundle $V$ to $X$. For any point $y \in Y$ and any point in the fiber $y' \in P(V)_y$, consider the short exact sequence on $X$

$$0 \to E \to \hat{V}^* \to (\oplus_{j=1}^{d_z - 1} (\hat{V}_{y_j}^*/L_j^+) \oplus (\hat{V}_{y_j}^*/L(y')^+)) \to 0,$$

where $L(y') \subset V_y$ is the line in $V_y$ corresponding to the above point $y'$, and $L_j^+ \subset \hat{V}_{y_j}^*$ (respectively, $L(y')^+ \subset \hat{V}_{y_j}^*$) is the annihilator of $L_j$ (respectively, $L(y')$); note that $\hat{V}_{y_j}$ is identified with $\mathbb{C}^r$ and $\hat{V}_y$ is identified with $V_y$. Therefore, we have

$$\hat{V} \hookrightarrow E^*.$$

The map $\delta$ in (2.15) sends any $y'$ to the above extension $E^*$ of $\hat{V}$ constructed from $y'$. 
From (2.10) we know that $\text{PGL}(r, \mathbb{C})$ is contained in $\text{Aut}(p^{-1}(U))$ with

$$0 \oplus \mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g} = H^0(p^{-1}(U), T(p^{-1}(U)))$$

as its Lie algebra. This action of $\text{PGL}(r, \mathbb{C})$ on $p^{-1}(U)$ clearly preserves the intersection $\delta(P(V)) \cap p^{-1}(U)$. Therefore, if the action of the element $A$ in (2.13) extends to $\mathcal{Q}$, then the extended action must preserve the image $\delta(P(V))$.

On the other hand, from Lemma 2.3 we know that the action of $A$ on $\text{P}(V)|_{Y \setminus \{p_0\}}$ does not extend to $P(V)$ because (2.14) holds. This completes the proof of the theorem. □

3. Holomorphic maps from a symmetric product

Proposition 3.1. Let $X$ and $Y$ be compact connected Riemann surface with

$$\text{genus}(X) \geq \text{genus}(Y) \geq 2.$$ 

If there is a nonconstant holomorphic map $\beta : \text{Sym}^d(Y) \rightarrow X$, then $d = 1$, and $\beta$ is an isomorphism.

Proof. Let $\beta : \text{Sym}^d(Y) \rightarrow X$ be a nonconstant holomorphic map. Let

$$\beta^* : H^0(X, \Omega^1_X) \rightarrow H^0(\text{Sym}^d(Y), \Omega^1_{\text{Sym}^d(Y)})$$

be the pull-back of $1$–forms defined by $\omega \mapsto \beta^* \omega$. This homomorphism $\beta^*$ is injective, because $\beta$ is surjective. Since

$$\dim H^0(\text{Sym}^d(Y), \Omega^1_{\text{Sym}^d(Y)}) = \text{genus}(Y)$$

[Ma, p. 322, (4.3)], the injectivity of $\beta^*$ implies that $\text{genus}(Y) \geq \text{genus}(X)$. Therefore, the given condition $\text{genus}(X) \geq \text{genus}(Y)$ implies that

- $\text{genus}(X) = \text{genus}(Y)$, and
- the above homomorphism $\beta^*$ is an isomorphism.

If $d \geq 2$, the wedge product

$$\wedge^2 H^0(\text{Sym}^d(Y), \Omega^1_{\text{Sym}^d(Y)}) \rightarrow H^0(\text{Sym}^d(Y), \Omega^2_{\text{Sym}^d(Y)})$$

is a nonzero homomorphism [Ma, p. 325, (6.3)]. On the other hand, the wedge product on $H^0(X, \Omega^1_X)$ is the zero homomorphism because $H^0(X, \Omega^2_X) = 0$. In other words, $\beta^*$ is not compatible with the wedge product operation on holomorphic $1$–forms if $d \geq 2$. So we conclude that $d = 1$.

Since $\text{genus}(X) = \text{genus}(Y)$, from Riemann–Hurwitz formula for Euler characteristic if follows that $\text{degree}(\beta) = 1$. In other words, $\beta$ is an isomorphism. □

Let $X'$ be a compact connected Riemann surface of genus at least two. Fix positive integers $r' \geq 2$, $d'_p$ and $d'_z$. Let

$$\mathcal{Q}' = \mathcal{Q}'_{X'}(r', d'_p, d'_z)$$

be the corresponding generalized quot scheme (see (2.2)).

Proposition 3.2. If the two varieties $\mathcal{Q}'$ and $\mathcal{Q}$ (constructed in (2.2)) are isomorphic, then $X$ is isomorphic to $X'$. 
Proof. Assume that \( Q' \) and \( Q \) are isomorphic. We will show that \( X \) and \( X' \) are isomorphic.

Let \( \eta : \text{Sym}^{dp}(X) \times \text{Sym}^{dz}(X) \to \text{Pic}^{dp}(X) \times \text{Pic}^{dz}(X) \) be the morphism defined by
\[
((x_1, \cdots, x_{dp}), (y_1, \cdots, y_{dz})) \mapsto (\mathcal{O}_X(x_1 + \cdots + x_{dp}), \mathcal{O}_X(y_1 + \cdots + y_{dz})).
\]

Since the general fiber of the map \( p \) in (2.5) is a product of copies of projective spaces, the composition
\[
\eta \circ p : Q \to \text{Pic}^{dp}(X) \times \text{Pic}^{dz}(X)
\]
is the Albanese map for \( Q \), as there is no nonconstant holomorphic map from a projective space to an abelian variety. In particular, the Albanese variety of \( Q \) is the Albanese map for \( Q \times X \) that genus \( = 2g \) is of dimension \( 2g = 2 \cdot \text{genus}(X) \). Therefore, comparing the Albanese varieties of \( Q \) and \( Q' \) we conclude that genus \( X = g = \text{genus}(X') \).

Fix a maximal torus \( T \) in \( \text{Aut}^0(Q) \). In view of Theorem 2.1, this amounts to choosing a trivialization of \( \mathcal{O}_X^{\oplus r} \), with two trivializations being identified if they differ by multiplication with a constant nonzero scalar. The fixed–point locus
\[
Q^T \subset Q,
\]
for the action of \( T \) on \( Q \), is a disjoint union of copies of
\[
(\text{Sym}^{a_1}(X) \times \cdots \times \text{Sym}^{a_r}(X)) \times (\text{Sym}^{b_1}(X) \times \cdots \times \text{Sym}^{b_r}(X))
\]
with \( \sum_{i=1}^r a_i = dp \) and \( \sum_{i=1}^r b_i = dz \).

Take a component
\[
Z = (\text{Sym}^{a_1}(X) \times \cdots \times \text{Sym}^{a_r}(X)) \times (\text{Sym}^{b_1}(X) \times \cdots \times \text{Sym}^{b_r}(X))
\]
of \( Q^T \) such that at least one of the \( 2r \) integers \( \{ a_1, \cdots, a_r, b_1, \cdots, b_r \} \) is one.

We will first show that \( Z \) is not holomorphically isomorphic to \( \text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y) \), where \( Y \) is compact connected Riemann surface of genus \( g \) and \( c_j \geq 2 \) for all \( 1 \leq j \leq n \). To prove this, assume that \( Z \) is isomorphic to \( \text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y) \), where \( Y \) and \( c_i \) are as above. Consider the composition
\[
\text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y) \overset{\sim}{\to} Z \overset{q}{\to} X,
\]
where \( q \) is the projection to a factor of \( Z \) which is the first symmetric power of \( X \) (it is assumed that such a factor exists). Since all \( c_j \geq 2 \), from Proposition 3.1 it follows that there is no nonconstant map from \( \text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y) \) to \( X \). Therefore, we conclude that \( Z \) is not holomorphically isomorphic to \( \text{Sym}^{c_1}(Y) \times \cdots \times \text{Sym}^{c_n}(Y) \).

Fix a maximal torus \( T' \subset \text{Aut}(Q')^0 \). Since \( Q' \) is isomorphic to \( Q \), there is a component
\[
(\text{Sym}^{a'_1}(X') \times \cdots \times \text{Sym}^{a'_r}(X')) \times (\text{Sym}^{b'_1}(X') \times \cdots \times \text{Sym}^{b'_r}(X'))
\]
of the fixed point locus \( (Q')^T \subset Q' \) which is isomorphic to \( Z \). Now from Proposition 3.1 it follows that
\[
\bullet \text{ at least one of the } 2r \text{ integers } \{ a'_1, \cdots, a'_r, b'_1, \cdots, b'_r \} \text{ is one, and}
\bullet X \text{ is isomorphic to } X'.
\]
This completes the proof. \( \square \)
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