Universal current fluctuations in the symmetric exclusion process and other diffusive systems

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Abstract – Using the macroscopic fluctuation theory of Bertini, De Sole, Gabrielli, Jona-Lasinio, and Landim, one can show that the statistics of the current of the symmetric simple exclusion process (SSEP) connected to two reservoirs on an arbitrary large finite domain in dimension \(d\) are the same as in the one-dimensional case. Numerical results on squares support this claim while results on cubes exhibit some discrepancy. We argue that the results of the macroscopic fluctuation theory should be recovered by increasing the size of the contacts. The generalization to other diffusive systems is straightforward.

Introduction. – When a system is connected for a long time to two heat baths at unequal temperatures or to two reservoirs of particles at different densities, it reaches a non-equilibrium steady state, with a non-vanishing current of heat or of particles. This current fluctuates with time and the study of its fluctuations and of its large deviations has become a central aspect in the theory of non-equilibrium systems\[1–14\]. A quantity which characterizes these fluctuations is the probability \(P(Q_t)\) of observing an energy or a number of particles \(Q_t\) flowing through the system during a time window \(t\). A notorious property of these fluctuations is known as the fluctuation theorem \[15–18\] which establishes a general relation between \(P(Q_t)\) and \(P(-Q_t)\), starting from some time reversal symmetry of the microscopic dynamics. Apart from simple models, however, it is usually difficult to predict the whole shape of the distribution \(P(Q_t)\).

The one-dimensional SSEP (symmetric simple exclusion process) describes a chain of \(L\) sites on which particles diffuse with hard core repulsion on the same site, connected at its two ends to two reservoirs at densities \(\rho_a\) and \(\rho_b\). It is one of the few examples for which it has been possible to determine the distribution \(P(Q_t)\) both by microscopic and macroscopic approaches\[1,2,19,20\]. This distribution reveals that, for a long chain of length \(L\) connected at its two ends to two reservoirs at densities \(\rho_a = 1\) and \(\rho_b = 0\), the Fano factor (the ratio of the first two cumulants) is given, in the long time limit, by

\[
\lim_{L \to \infty} \lim_{t \to \infty} \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{\langle Q_t \rangle} = \frac{1}{3}. \tag{1}
\]

In fact for the SSEP in one dimension, all the cumulants of the current are known\[19,20\] for arbitrary densities \(\rho_a\) and \(\rho_b\) of the left and of the right reservoirs, with their generating function given by

\[
\lim_{L \to \infty} \lim_{t \to \infty} \frac{L}{t} \log \langle \exp[\lambda Q_t] \rangle = \left(\text{Arcsinh}(\sqrt{\omega})\right)^2, \tag{2}
\]

where \(\omega\) is given by

\[
\omega = \rho_a (e^\lambda - 1) + \rho_b (e^{-\lambda} - 1) - \rho_a (e^\lambda - 1) \rho_b (e^{-\lambda} - 1). \tag{3}
\]

Note as a striking fact\[19,20\] that the generating function which is in principle a function of the three variables \(\rho_a, \rho_b\) and \(\lambda\) turns out to be a function of the single variable \(\omega\).

The cumulants which can be determined from (2), (3) are the same as those which have been computed for free fermions transmitted through multichannel disordered conductors\[21–23\]. Although most of the theoretical approaches in the classical (SSEP) case and in the quantum case are different, the description of the quantum problem based on the Boltzmann Langevin approach...
has similarities [24–28] with the macroscopic fluctuation theory [1,2,29,30] which is the appropriate theory to describe diffusive systems such as the SSEP on large scales. These similarities rely on the fact that the hard core repulsion of the exclusion process mimics properly the Pauli principle for fermions in the quantum problem [22]. In the quantum case, it has been shown [24,28] that the Fano factor as well as higher cumulants do not depend on the geometry.

For non-equilibrium diffusive systems one can also wonder how the cumulants of the integrated current are modified in higher dimension \((d > 1)\) and for more complicated graphs than linear chains. A numerical study of the Sierpinski gasket [31], with two corners connected to reservoirs at densities \(\rho_a = 1\) and \(\rho_b = 0\), has shown that in this case too, the Fano factor is still equal to 1/3 as in one dimension \((1)\). The question addressed here in this letter is the degree of generality of \((1)\) or \((2,3)\).

It is known [32] that for an arbitrary finite graph connected to two reservoirs at densities \(\rho_a\) and \(\rho_b\), all the dependence of the generating function of the cumulants \(\log \langle \exp[\lambda Q_t] \rangle\) on \(\lambda\), \(\rho_a\) and \(\rho_b\) is always through the single parameter \(\omega\) defined in (3). This is proven [32] at an arbitrary time \(t\), by choosing the configuration at \(t = 0\) of the SSEP with the right measure. In the long time limit, this becomes true for any initial condition. Therefore for an arbitrary graph one knows in advance that

\[
\lim_{t \to \infty} \frac{1}{t} \log \langle \exp[\lambda Q_t] \rangle = G(\omega),
\]

where \(G\) is a function which depends on the characteristics of the considered graph, on the way it is connected to the reservoirs and on the parameter \(\omega\) defined in (3).

One purpose of this letter is to show that for a large class of graphs and geometries such as represented in fig. 1, when the distance \(L\) between the contacts is large, the function \(G(\omega)\) is the same as for a linear chain, up to a multiplicative factor \(\kappa(L)\), namely,

\[
G(\omega) \simeq \kappa(L)(\text{Arcsinh}(\sqrt{\omega}))^2.
\]

In (5), all the dependence on the shape of the graph, on the location of the points \(A\) and \(B\) connected to the reservoirs, on the nature of the connections, and on the system size are encoded in the factor \(\kappa(L)\). As a consequence, for all \(\rho_a\) and \(\rho_b\), the ratio between any pair of cumulants of \(Q_t\) is exactly the same as for the linear chain.

The term linear in \(\lambda\) in (3)–(5) gives the relation between the average current and the factor \(\kappa(L)\) in (5):

\[
\lim_{t \to \infty} \frac{Q_t}{t} = \kappa(L)(\rho_a - \rho_b).
\]

Our approach is based on the macroscopic fluctuation theory [1,2,29,30,33] which requires solving non-linear differential equations for the current and density profiles which produce a certain \(Q_t\). As the macroscopic fluctuation theory is formulated in the continuum, one needs the density and current profiles to vary slowly on the graph. Therefore our main assumption is that the graph is large, that one can define derivatives with respect to space on this graph, and that the contacts \(A\) and \(B\) with the reservoirs are large enough to avoid singular gradients of density or current profiles near these contacts.

**Computation on finite graphs.** – Let us consider the SSEP on a finite graph. Each site \(i\) on this graph is connected to its neighboring sites on the graph (this number of neighbors may depend on \(i\)). In addition, two particular sites (or two particular sets of points as in fig. 2) are connected to reservoirs as in fig. 1: site \(i = A\) is connected to a reservoir at density \(\rho_a\) and site \(B\) is connected to a reservoir at density \(\rho_b\). At a given time \(t\) each site \(i\) is either occupied \((\tau_i(t) = 1)\) or empty \((\tau_i(t) = 0)\). By definition of the dynamics, the occupation numbers of each pair of connected sites on the lattice are exchanged at rate 1. Moreover, the effect of reservoirs can be represented by the rates at which particles are injected on sites \(A\) and \(B\) when these sites are empty and the rates at which particles are removed from these two sites, when they are occupied. One simple way of updating sites \(A\) and \(B\) is for example to keep at any time \(t\) \(\tau_A(t) = 1\) with probability \(\rho_a\) and \(\tau_B(t) = 0\) with probability \(1 - \rho_a\), and similar probabilities for site \(B\) irrespective of the past history of the lattice. With this prescription, the occupation numbers of sites \(A\) and \(B\) are not correlated to the other sites of the lattice, they have no correlation at unequal times and on average one has \(\langle \tau_A(t) \rangle = \rho_a\) and \(\langle \tau_B(t) \rangle = \rho_b\).

Given some random initial condition, the evolution of the average density \(\rho_i = \langle \tau_i(t) \rangle\) on site \(i\) is given for all \(i \neq A, B\) by

\[
\frac{d\rho_i}{dt} = \sum_{j \neq i} \rho_j - \rho_i,
\]

where the sum is over the neighbors \(j\) of site \(i\) on the graph. At sites \(A\) and \(B\), one has \(\rho_A = \rho_a\) and \(\rho_B = \rho_b\). The average profile \(\bar{\rho}_i\), in the non-equilibrium steady state, satisfies everywhere the lattice version of the Laplace equation

\[
\Delta \bar{\rho}_i = \sum_{j \neq i} \bar{\rho}_j - \bar{\rho}_i = 0,
\]

except in \(i = A\) and \(i = B\). Denoting \(Q_t\) the number of particles flowing through the system (see just below for a

Fig. 1: Finite graph with sites \(A\) and \(B\) maintained at densities \(\rho_a\) and \(\rho_b\) by reservoirs.
Fig. 2: The Fano factors have been determined numerically for squares and cubes of linear size $L$. In each case the black dots indicate the points of the lattice which are maintained at density $1$ (sites $A$) and $0$ (sites $B$) by the reservoirs. A single site is connected to each reservoir in cases $a,b,c$ while in case $d$, there are 4 sites connected to each reservoir.

precise definition) during time $t$, our goal is to determine $\mu(\lambda)$ defined by

$$\langle e^{\lambda Q_t} \rangle \sim e^{\mu(\lambda) t} \quad \text{for large } t. \quad (8)$$

The knowledge of $\mu(\lambda)$ allows to determine all the cumulants of $Q_t$

$$\lim_{t \to \infty} \langle Q_t^n \rangle \left. \frac{d^n \mu(\lambda)}{d\lambda^n} \right|_{\lambda=0}.$$

We now need to define precisely the number $Q_t$ of particles flowing through the system. One could count the number $Q^A_t$ of particles flowing between the reservoir at density $\rho_A$ and site $A$ (i.e., the total number of particles injected in site $A$). One could alternatively count the number $Q^B_t$ of particles flowing from site $B$ to the reservoir at density $\rho_B$. Here, based on [34], we choose the following definition of $Q_t$:

$$Q_t = \frac{1}{2} \sum_{i,j} (V_i - V_j) q_{i,j}(t), \quad (9)$$

where $q_{i,j}(t) = -q_{j,i}(t)$ is the number of particles transferred from site $i$ to site $j$ during time $t$ and $V_i$ is an arbitrary function defined on each site $i$ except for $V_A$ and $V_B$ which are given by

$$V_A = 1; \quad V_B = 0 \quad (10)$$

and the factor $1/2$ results from double counting of each bond.

Using that $\tau_i(0) - \tau_i(t) = \sum_{j \sim i} q_{i,j}(t)$ (where $\tau_i(t)$ is the number of particles at site $i$ and time $t$), one can rewrite (9) as

$$Q_t = \sum_i V_i \sum_{j \sim i} q_{i,j}(t) = V_A \sum_{j \sim A} q_{A,j}(t)$$

$$+ V_B \sum_{j \sim B} q_{B,j}(t) + \sum_{i \neq A,B} V_i (\tau_i(0) - \tau_i(t)).$$

Fig. 3: The Fano factor as a function of $1/L$ is displayed for the two geometries (see fig. 2) $a$ (diamonds) and $b$ (squares) of squares of size up to $27 \times 27$. The results indicate convergence to $1/3$ for both geometries.

which, using (10), becomes

$$Q_t = \sum_{j \sim A} q_{A,j}(t) + \sum_{i \neq A,B} V_i (\tau_i(0) - \tau_i(t)). \quad (11)$$

Since the graph is finite, particles cannot accumulate on the graph (because the number of particles cannot exceed the number $N$ of lattice sites). Clearly the first term on the r.h.s. of (11) can become arbitrary large with time while the second term remains bounded ($< \sum_i |V_i|$). Therefore changing the $V_i$’s changes $Q_t$ by an amount which cannot grow with time, so that the value of $\mu(\lambda)$ in (8) is the same for any choice of the $V_i$’s. Similarly one can show that the differences $Q_t - Q^A_t$ or $Q_t - Q^B_t$ remain bounded in time (for example by choosing all $V_i = 0$ for $i \neq A$, one has $Q_t - Q^A_t = \tau_A(0) - \tau_A(t)$ which obviously remains bounded). Therefore all the definitions of $Q_t$ one can think of lead in the long time limit to the same $\mu(\lambda)$.

Here we find convenient to take for $V_i$ a solution of the Laplace equation

$$\Delta V_i \equiv \sum_{j \sim i} V_j - V_i = 0 \quad (12)$$

with boundary conditions (10). (Thus when $\rho_A \neq \rho_B$, one has simply $V_i = (\rho_A - \rho_B)/(\rho_A - \rho_B)$).

Using a method similar to the one used in [31] (the method consists essentially in calculating the steady state densities and two point correlations as in [20]) we have determined numerically the Fano factor

$$F = \lim_{t \to \infty} \frac{\langle Q^2_t \rangle - \langle Q_t \rangle^2}{\langle Q_t \rangle}$$

for finite squares of size $L \times L$ and cubes of size $L \times L \times L$ with open boundary conditions. The systems are connected to the reservoirs at densities $\rho_A = 1$ and $\rho_B = 0$ as in fig. 2.

The results are shown in fig. 3 for the square. For both geometries $a$ and $b$ of fig. 2, the results seem to converge
showing that there is a rescaling factor $L^2$ between the microscopic time $t$ and the macroscopic time $\tau = t/L^2$. Therefore on a macroscopic scale the density $\rho(\vec{r}, \tau)$ is a function of the macroscopic variables $\vec{r}$ and $\tau$ with the boundary conditions

$$\rho(\partial A, \tau) = \rho_a \quad \text{and} \quad \rho(\partial B, \tau) = \rho_b \quad (13)$$

at the contacts $\partial A$ and $\partial B$ with the reservoirs.

Associated with the density $\rho(\vec{r}, \tau)$ the current $\vec{j}(\vec{r}, \tau)$ is a vector field which satisfies the conservation law

$$\frac{d \rho(\vec{r}, \tau)}{d\tau} + \nabla \cdot \vec{j}(\vec{r}, \tau) = 0. \quad (14)$$

The potential $v(\vec{r})$, which is the continuous version of the potential $V_i$ introduced in (9) to measure the current, satisfies (see (12))

$$\Delta v(\vec{r}) = 0, \quad (15)$$

with

$$v(\partial A) = 1 \quad \text{and} \quad v(\partial B) = 0 \quad (16)$$

and Neumann boundary conditions otherwise.

Then (9) becomes

$$Q_i = -L^d \int_0^{\tau} d\tau \int d\vec{r} \nabla v(\vec{r}) \cdot \vec{j}(\vec{r}, \tau), \quad (17)$$

where the factor $L^d$ comes from the change of scale.

According to the macroscopic fluctuation theory [1,2,29,33], the probability of obtaining time dependent profiles for both density and current over a long time interval $t$, is

$$P(\{\rho(\vec{r}, \tau), \vec{j}(\vec{r}, \tau)\}) \sim \exp \left[-L^d I_t\right]$$

with the action $I_t$ given by

$$I_t = \int_0^{t/L^2} d\tau \left[\vec{j}(\vec{r}, \tau) \cdot \nabla \rho(\vec{r}, \tau) + D(\rho(\vec{r}, \tau)) \nabla \rho(\vec{r}, \tau)\right]^2. \quad (18)$$

This action simply means that the current is the sum of a Fick’s law contribution $-D(\rho)\nabla \rho$ and of a Gaussian noise $\vec{\eta}$ of variance $\sigma$, $\delta$-correlated both in time and space

$$\vec{j}(\vec{r}, \tau) = -D(\rho(\vec{r}, \tau)) \nabla \rho(\vec{r}, \tau) + \vec{\eta}(\vec{r}, \tau).$$

A diffusive system is characterized by the two functions $D(\rho)$ and $\sigma(\rho)$ which show up in (18). For the SSEP, one has [19]

$$D(\rho) = 1; \quad \sigma(\rho) = 2\rho(1-\rho). \quad (19)$$

Here we keep $D$ and $\sigma$ arbitrary as our calculation below is valid for more general diffusive systems. Then, the generating function $\mu(\lambda)$ defined in (8) is

$$\mu(\lambda) = \lim_{t \to \infty} \max_{\{\rho\}} \left[\frac{\lambda Q_t - L^d I_t}{t}\right] \quad (20)$$

with $Q_t$ and $I_t$ given by (17), (18) and where we have to optimize over all the time dependent density and current profiles $\{\vec{j}(\vec{r}, \tau), \rho(\vec{r}, \tau)\}$ which satisfy the conservation law (14).

In what follows, we will assume, as with the additivity principle [8,19,30], that the optimum in (20) is

Fig. 4: The Fano factor as a function of $1/L^2$ is displayed for the two geometries (see fig. 2) $c$ (diamonds) and $d$ (squares) of cubes of size up to $10 \times 10 \times 10$. The results do not seem to converge to $1/3$, but for the geometry $d$ the extrapolated value is closer to $1$. As the size of the contacts increases, keeping it small compared to the system size, the Fano factor becomes closer to $1/3$. to $F = 1/3$ as $L \to \infty$. In fig. 3 the results are plotted versus $1/L$. We have no particular reason to think that this is the right convergence law. Trying other power law extrapolations show equally well the convergence to $1/3$.

In the case of the cube, our data in fig. 4 indicate a large $L$ limit distinct from $1/3$ for both geometries $c$ and $d$ of fig. 2. For geometry $d$, however, where the contacts with the reservoirs are larger, the limiting value is closer to $1/3$. We will argue below, that if the size of the contacts is macroscopic (for example if the region in contact with each reservoir contains $(\epsilon L)^d$ sites, no matter how small $\epsilon$ is provided that it remains finite as $L \to \infty$), the Fano factor should be $1/3$ in any dimension and for all geometries, in agreement with the trend we see going from geometry $c$ to geometry $d$ in fig. 4.

The macroscopic fluctuation theory. – We consider now a continuous theory, on a $d$-dimensional domain of arbitrary shape. The typical size of the domain is $L$ (with $L$ large compared to the distance between any pair of connected sites on the lattice). Each reservoir maintains at a fixed density (respectively $\rho_a$ and $\rho_b$ for reservoirs $A$ and $B$) all the sites at a distance less than $L\epsilon$ from the center of the connection. Here $\epsilon$ is small but remains fixed as $L$ increases, so the number of sites in contact with the reservoirs increases with $L$ but remains a small fraction of the total number of sites in the graph.

If we introduce a macroscopic coordinate $\vec{r}$ on the domain, such that $|\vec{r}_A - \vec{r}_B| = O(1)$ (so that there is a rescaling by a factor $L$ between the microscopic coordinates on the graph and the macroscopic coordinate $\vec{r}$), the densities $\rho_i$ become a scalar field $\rho(\vec{r})$ whose time evolution is the continuous version of (6)

$$\frac{d\rho}{dt} = \frac{1}{L^2} \Delta \rho \quad (6)$$

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achieved by *time-independent density and current profiles* \( \{ \vec{j}(\vec{r}), \rho(\vec{r}) \} \). In [1,2] it was proven that this assumption is valid for the SSEP as well as for a large class of other diffusive systems (see [1,2] for precise conditions on \( D \) and \( \sigma \)). Then (20) can be rewritten as

\[
\mu(\lambda) = -L^{d-2} \min_{\{ \vec{j}(\vec{r}), \rho(\vec{r}) \}} \int d\vec{r} \left( \lambda \vec{\nabla} v(\vec{r}) \cdot \vec{j}(\vec{r}) + \frac{[\vec{j}(\vec{r}) + D(\rho(\vec{r})) \vec{\nabla} \rho(\vec{r})]^2}{2\sigma(\rho(\vec{r}))} \right),
\]

and the conservation law (14) becomes

\[
\vec{\nabla} \cdot \vec{j}(\vec{r}) = 0. \tag{22}
\]

By optimizing (21) over \( \rho(\vec{r}) \) and \( \vec{j}(\vec{r}) \), given the conservation law (22), we deduce,

\[
\begin{aligned}
D'(\rho) \frac{\vec{j} + D(\rho) \vec{\nabla} \rho}{\sigma} - \sigma'(\rho) \frac{[\vec{j} + D(\rho) \vec{\nabla} \rho]^2}{2\sigma(\rho)^2} &\quad - \vec{\nabla} \cdot \left( D(\rho) \frac{\vec{j} + D(\rho) \vec{\nabla} \rho}{\sigma(\rho)} \right) = 0, \tag{23}

\end{aligned}
\]

\[
\vec{j} = -D(\rho) \vec{\nabla} \rho + \sigma(\rho)(\vec{\nabla} h - \lambda \vec{\nabla} v), \tag{24}
\]

where \( \rho, v, \vec{j} \) and \( h \) are functions of \( \vec{r} \). (Here \( h(\vec{r}) \) is a Lagrange multiplier associated to the constraint (22).)

Since the conservation law (22) is not satisfied at the boundaries \( A \) and \( B \), there is no Lagrange parameters at the boundaries and

\[
h(\partial A) = h(\partial B) = 0. \tag{25}
\]

Upon defining a function \( H(\vec{r}) \) by

\[
H(\vec{r}) = h(\vec{r}) - \lambda v(\vec{r})
\]

instead of \( h(\vec{r}) \), we see that (24) and (25) become

\[
\vec{j} = -D(\rho(\vec{r})) \vec{\nabla} \rho(\vec{r}) + \sigma(\rho(\vec{r})) \vec{\nabla} H(\vec{r}) \tag{26}
\]

and

\[
H(\partial A) = -\lambda; \quad H(\partial B) = 0. \tag{27}
\]

Inserting the expression (26) of \( \vec{j}(\vec{r}) \) into (22) and (23), we deduce that \( H \) and \( \rho \) should satisfy

\[
\vec{\nabla} \cdot (D(\rho(\vec{r})) \vec{\nabla} \rho(\vec{r})) = \vec{\nabla} \cdot (\sigma(\rho(\vec{r})) \vec{\nabla} H(\vec{r})) \tag{28}
\]

and

\[
D(\rho(\vec{r})) \Delta H(\vec{r}) = -\frac{\sigma'(\rho(\vec{r}))}{2} (\vec{\nabla} \cdot H(\vec{r}))^2. \tag{29}
\]

So the problem of computing \( \mu(\lambda) \) in (21) is reduced to finding \( H \) and \( \rho \) solutions of (28), (29) with the boundary conditions (27)

\[
\rho(\partial A) = \rho_a, \quad \rho(\partial B) = \rho_b \tag{30}
\]

and \( \vec{j}(\vec{r}) \) given by (26).

The link between the one-dimensional case and higher dimensions. – We are now going to argue that if one knows the solutions of (28), (29) in one dimension, then one can obtain the solutions for an arbitrary domain in any dimension. Consider the case of a one-dimensional chain in contact with reservoirs at its boundary [19]. Let \( \rho_1(x) \) and \( H_1(x) \) be the solutions of eqs. (28), (29), in one dimension, when point \( A \) is at position \( x = 1 \) and point \( B \) is at position \( x = 0 \). In this case, the solution of (15) is simply

\[
v_1(x) = x \quad \text{and} \quad \rho_1 \text{ and } H_1 \text{ satisfy } (28), (29)
\]

\[
(D(\rho_1(x)) \rho'_1(x))' = \left( \sigma(\rho_1(x)) H'_1(x) \right)' \tag{31}
\]

\[
D(\rho_1(x)) H''_1(x) = -\frac{\sigma(\rho_1(x))}{2} H'_1(x)^2. \tag{32}
\]

Then, using that \( v(\vec{r}) \) is solution of the Laplace equation (15), it is easy to check that

\[
H(\vec{r}) = H_1(v(\vec{r})); \quad \rho(\vec{r}) = \rho_1(v(\vec{r})) \tag{33}
\]

solve eqs. (28), (29) with the boundary conditions (27), (30) in arbitrary dimension and for an arbitrary domain. The same relation between the \( d \)-dimensional and the one-dimensional problems was already used [24] in the quantum case. Replacing \( H \) and \( \rho \) by their expressions (33) into (26) and subsequently into (21) leads to

\[
\mu(\lambda) = L^{d-2} \int d\vec{r} \left( \vec{\nabla} v(\vec{r}) \right)^2 \left( D(\rho_1) \lambda \rho'_1 - \lambda \sigma(\rho_1) H'_1 - \frac{\sigma(\rho_1)}{2} H'_1^2 \right), \tag{34}
\]

where \( H_1 \) and \( \rho_1 \) stand, respectively, for \( H_1(v(\vec{r})) \) and \( \rho_1(v(\vec{r})) \).

Now the last step comes from the fact that for any function \( \Phi \) (any here means that \( \Phi \) is an arbitrary polynomial, and by extension any continuous function) the following identity holds:

\[
\int d\vec{r} \Phi(v(\vec{r})) (\vec{\nabla} v(\vec{r}))^2 = \int_0^1 \Phi(x) dx \times \int d\vec{r} (\vec{\nabla} v(\vec{r}))^2, \tag{35}
\]

whenever \( v(\vec{r}) \) satisfies (15) and (16). This identity is straightforward when \( \Phi(x) \) is constant. For a monomial \( \Phi(x) = x^n \), it can be obtained using an integration by parts and (15), (16), namely,

\[
\int d\vec{r} v(\vec{r})^n (\vec{\nabla} v(\vec{r}))^2 = - \int d\vec{r} v(\vec{r})^n (\vec{\nabla} (1 - v(\vec{r}))). \vec{\nabla} v(\vec{r}) = n \int d\vec{r} (v(\vec{r})^{n-1} - v(\vec{r})^n)(\vec{\nabla} v(\vec{r}))^2,
\]

so that

\[
\int d\vec{r} v(\vec{r})^n (\vec{\nabla} v(\vec{r}))^2 = \frac{n}{n+1} \int d\vec{r} v(\vec{r})^{n-1}(\vec{\nabla} v(\vec{r}))^2 = \cdots = \frac{1}{n+1} \int d\vec{r} (\vec{\nabla} v(\vec{r}))^2,
\]

thus showing that (35) should be true for any continuous function \( \Phi(x) \) as it can be approximated by a sum of monomials.
Applying the identity (35) to (34) we deduce that

$$\mu(\lambda) = \left[ L^{d-2} \int d\vec{r} \left( \nabla \cdot \mathbf{v}(\vec{r}) \right)^2 \right] \times \int dx \left( \lambda D(\rho_1(x)) \rho'_1(x) \right. $$

$$- \sigma(\rho_1(x)) \left( H'_1(x) - \frac{\sigma(\rho_1(x)) (H'_1(x))^2}{2} \right) $$

$$= \left[ L^{d-2} \int d\vec{r} \left( \nabla \cdot \mathbf{v}(\vec{r}) \right)^2 \right] \times \left[ L\mu_1(\lambda) \right]$$

implying that $\mu(\lambda)$ for an arbitrary domain is the same as the one-dimensional generating function $\mu_1(\lambda)$, up to a multiplicative factor function independent of $\lambda, \rho_1, \rho_0$ and of the functions $D$ and $\sigma$. Therefore the ratio between any pair of cumulants is the same in one dimension.

**Conclusion.** – In the present work we have seen that whenever the continuous macroscopic fluctuation theory can be applied, the cumulants of the integrated current for the SSEP on an arbitrary large finite domain in dimension $d$ are the same as in one dimension. Our numerical results of fig. 3 for squares confirm this claim in dimension $d = 2$. In dimension $d = 3$, the discrepancy in fig. 4 decreases with the size of the contacts and indicate that the continuous theory should become applicable for large enough contacts (the importance of the nature of connections was already pointed out in [35] in the quantum case). The approach works as well for more general diffusive systems: when $D(\rho)$ and $\sigma(\rho)$ are such that the optimal profiles are time independent the generating function $\mu(\lambda)$ should be the same as in the one-dimensional case. Let us conclude by briefly mentioning some open questions related to the present work. 1) Is it possible to generalize our results to the case where the optimal profiles become time dependent [1,2,30,36]? 2) What happens for an infinite graph with a fixed density at infinity? 3) Is there an extension to random graphs? 4) Is it possible to understand finite-size corrections as in one dimension [37]? 5) Is there a generalization to systems in contact with more than 2 reservoirs at unequal densities?

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