Cusps without chaos

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ABSTRACT

We present cuspy, non–axisymmetric, scale–free mass models of discs, whose gravitational potentials are of Stäckel form in parabolic coordinates. A black hole may be added at the centre, without in any way affecting the Stäckel form; the dynamics in these potentials is, of course, fully integrable. The surface density, $\Sigma_{\text{disc}} \propto 1/r^\gamma$, where $0 < \gamma < 1$ corresponds to steep cusps for which the central force diverges. Thus cusps, black holes, and non–axisymmetry are not a sure recipe for chaos, as is generally assumed. A new family of orbits, lens orbits, emerges to replace the box orbits of models of elliptical galaxies that have constant–density cores. Loop orbits are conspicuous by their absence. Both lenses and boxlets (the other family of orbits), can be elongated in the direction of the density distribution, a property that is favourable for the construction of non–axisymmetric, self–consistent equilibrium models of elliptical galaxies.

Key words: galaxies: elliptical and lenticular, cD—galaxies: kinematic and dynamics—galaxies: structure

1 INTRODUCTION

More than two decades ago, it became apparent that elliptical galaxies are not rotationally supported, oblate objects (Bertola & Cappaccioli 1975, Illingworth 1977). Subsequent dynamical investigations (c.f. Binney 1976, 1978, Schwarzschild 1979, 1982) led to the picture of ellipticals as slowly/non–rotating, triaxial objects with anisotropic velocity distribution, whose self–consistent gravitational potentials supported mostly regular orbits. The discovery of fully integrable, triaxial mass models by Kuzmin (1973) and de Zeeuw & Lynden–Bell (1985) provided analytic power, and opened up a new era of dynamical investigations. The key insight was the employment of “Stäckel potentials” in ellipsoidal coordinates; in addition to the energy, stellar orbits in Stäckel potentials respect two extra isolating integrals. A range of triaxial, self–consistent equilibrium configurations could now be constructed in an efficient manner (Statler 1987); a property common to all these models is a constant–density core in which stars execute near–harmonic oscillations (“box orbits”; see de Zeeuw 1985 for a thorough account of orbits).

Recent observations (c.f. Møller, Stiavelli, & Zeilinger 1995, Ferrarese et al 1994, Lauer et al 1995) of elliptical galaxies, however, do not support the notion of a constant–density core; indeed, the density of stars appears to rise toward the centre in a power–law cusp. It is also widely believed that these galaxies harbour dark objects, possibly supermassive black holes, at their centres (c.f. Kormendy & Richstone 1995). Gerhard & Binney (1985) showed that central black holes, or steep density cusps will destroy box orbits in triaxial configurations, and argued that the boxes will be replaced by chaotic orbits. They also noted that chaotic orbits fill a region of space that is too round for strongly non–axisymmetric, self–consistent equilibria to be constructed; thus it has been suggested that cuspy galaxies with black holes must necessarily be axisymmetric, at least near the centre (c.f. Merritt 1996 for an account based on recent numerical explorations). Although the link between cusps, non–axisymmetry, and chaos is supported by numerical results in model potentials, we question its generality. We construct separable mass models of cuspy, non–axisymmetric, scale–free discs, with central black holes in this paper. The potentials of these discs promote a new family of orbits, lens orbits, to replace the box orbits of galaxies with constant–density cores. We hope that our results will add fuel to the ongoing debate on the intrinsic shapes of elliptical galaxies.
$2$ POTENTIAL–SURFACE DENSITY PAIRS

We begin with an explicit formula for the potential (in the plane) of our discs, in the usual $(r, \theta)$ polar coordinates:

$$\Phi = K r^{(1-\gamma)} F(\theta) - \frac{GM}{r}, \quad \gamma > 0$$ \hspace{1cm} (1)

where $K > 0$ is a constant, and

$$F(\theta) = \left\{ (1 + \sin \theta)^{(2-\gamma)} + (1 - \sin \theta)^{(2-\gamma)} \right\}.$$ \hspace{1cm} (2)

Knowledge of the potential in the plane of the disc suffices to determine the surface density through the integral (c.f. Binney & Tremaine 1987),

$$\Sigma(r', \theta') = \frac{1}{4\pi^2 G} \int \frac{(\nabla^2 \Phi) r \, dr \, d\theta}{|r^2 + r'^2 - 2rr' \cos(\theta - \theta')|^{3/2}}.$$ \hspace{1cm} (3)

It is evident, from dimensional considerations, that $\Sigma(r, \theta) = S(\theta)/r^{\gamma} + M\delta(r)$ is the form of the surface density arising from a $\Phi$ given by equation (1). For $0 < \gamma < 1$, straightforward manipulations of equation (3), using Legendre functions $P_{\pm \gamma}$, demonstrate that

$$S(\theta') = \frac{K}{4\pi G \sin(\pi \gamma)} \int_{-\pi}^{\pi} d\theta \left\{ (1-\gamma)^2 F(\theta) + F''(\theta) \right\} P_{-\gamma}[-\cos(\theta - \theta')] + \frac{K}{8\pi G \sin(\pi \gamma)} \int_{-\pi}^{\pi} d\theta \left\{ \sin^2 \theta P_{\gamma}'\gamma(-\cos \theta) + (1-\gamma)(2-\gamma)P_{-\gamma}(-\cos \theta) \right\} F(\theta + \theta'),$$ \hspace{1cm} (4)

is finite and positive. Finiteness is guaranteed by the singularity of $P_{-\gamma}(\mu)$ at $\mu = -1$ being integrable, and positivity simply because $F(\theta)$, $P_{-\gamma}(\mu)$ and $P_{\gamma}'\gamma(\mu)$ are all positive quantities. If we neglect the contribution from the black hole, $0 < \gamma < 1$ for all our models implies that the force on a test particle diverges, whereas its circular speed goes to zero for small $r$. Isocontours of surface density, and potential for $\gamma$ equal to 0.1 and 0.5 are displayed in Figure 1. The contours (Figures 1a and 1c) are guitar–shaped, with the depression near the ordinate increasing with $\gamma$; of course, the potential isocontours are much rounder than the density isocontours.

The location of the black hole—the origin—is indicated by the solid dot. While drawing the potential isocontours (Figures 1b and 1d), we have excluded the contribution of the black hole, to better display the potentials of the discs themselves. The major axes of the potential isocontours are exactly twice as long as the minor axes, a property that is readily verified using equation (2).

A remarkable property of this family of potentials is that all stellar orbits are non chaotic. To make this explicit, we will rewrite it in St"ackel form in parabolic coordinates.

\begin{figure}
\begin{center}
\includegraphics[width=0.8\textwidth]{fig1}
\end{center}
\caption{Isocontours of Surface Density and Potential: Figures (a) and (c) show isocontours of the surface density for two different values of $\gamma$. The corresponding potentials are displayed in the panels on the right in (b) and (d). Successive isocontours in (a) have density ratios of 1.1, whereas the ratio is 1.5 for the other three figures. The location of the central black hole is shown as a solid dot in all four figures, although the contribution to the potentials is not included in (b) and (d).}
\end{figure}

3 DYNAMICS IN PARABOLIC COORDINATES

Let $\xi = r(\sin \theta + 1)$ and $\eta = r(\sin \theta - 1)$ be parabolic coordinates in the plane. Lines of constant $\xi$ and $\eta$ intersect the $y$–axis at $\xi/2$ and $\eta/2$ respectively, and the $x$–axis at $\pm \xi$ and $\pm \eta$ respectively. Denoting the canonical momenta by $p_\xi$ and $p_\eta$, the Hamiltonian,

$$H(p_\xi, p_\eta, \xi, \eta) = \left( \frac{2\xi}{\xi - \eta} \right) p_\xi^2 + \left( \frac{2\eta}{\eta - \xi} \right) p_\eta^2 + \Phi,$$ \hspace{1cm} (5)

governs dynamics in the $(\xi, \eta)$ plane. Since $\Phi$ is independent of time, $H = E$ is a conserved quantity. The potential of equation (5) may be written in the St"ackel form,

$$\Phi = \frac{F_+ (\xi)}{\xi - \eta} + \frac{F_- (\eta)}{\eta - \xi},$$ \hspace{1cm} (6)

To verify this, we choose

$$F_+ (\xi) = 2K\xi^{(2-\gamma)} - GM,$$

$$F_- (\eta) = -2K\eta^{(2-\gamma)} + GM,$$ \hspace{1cm} (7)

and substitute for $(\xi, \eta)$ in terms of $(r, \theta)$. It is well–known that, for such potentials, the Hamilton–Jacobi equation separates, yielding an extra conserved quantity (c.f. Landau & Lifshitz 1976),

$$I = 2\xi p_\xi^2 - E\xi + F_+ (\xi)$$
\[ 2np^2_\eta - E\eta + \mathcal{F}_-(\eta), \]  

(8)

The isolating integrals, \( E \) and \( I \) may be used to classify orbits. For the generic case of a black hole, plus a cuspy density distribution, \( E \) can assume both positive and negative values. For fixed \( E \), the "second" integral, \( I \), will determine the excursions in \( \xi \) and \( \eta \). The requirement that \( p^2_\eta \) and \( p^2_\xi \) be non negative, forces

\[ I \geq g(\xi), \quad -I \geq g(\eta), \]  

(9)

where

\[ g(s) = 2Ks^{(2-\gamma)} - Es - GM; \quad s \geq 0. \]  

(10)

For fixed \( E \), the range of \( I \) is determined by the minimum of the function \( g(s) \), which is necessarily non–positive. If we denote this minimum value by \( -I_m(E) \), we obtain the condition \(-I_m < I < I_m \). Since \( s > 0 \),

\[ I_m(E) = \begin{cases} CE^{(2-\gamma)/2} + GM & \text{if } E \geq 0 \\ GM & \text{if } E \leq 0, \end{cases} \]  

(11)

where \( C = (1 - \gamma)/[2K(2 - \gamma)]^{1/(\gamma - 1)/(2 - \gamma)} \) is a positive constant. Let us consider two cases (the illustrative figures for which, given in Figures 2a–2d, are drawn for \( \gamma = 0.5 \)):

(i) \( E < 0 \) : We find that \( g(s) \) is a monotonically increasing function of \( s \), attaining a minimum value of \(-GM\) for \( s = 0 \) (see Figure 2a). This implies that \(-GM < I < GM \). For a fixed (say, positive) value of \( I \), the motion is bounded by the coordinate curves \( 0 \leq \xi \leq \xi_m \), and \( 0 \leq |\eta| \leq \eta_m \), where \( \xi_m(I, E) > \eta_m(E, I) > 0 \) are the two roots of \( g(s) = \pm I \); the intersections of the dashed lines with the solid curve, in Figure 2a, gives the location of \( \xi_m \) and \( \eta_m \). The orbits fill a lenticular region, bounded by the parabolas \( \xi = \xi_m \) and \( \eta = -\eta_m \); we call these lens orbits. As the shaded region of Figure 2b indicates, the lens orbits can visit the origin; they are to cusps with black holes, what box orbits are to analytic, triaxial cores.

(ii) \( E > 0 \) : As seen in Figure 2c, \( g(s) \) is no longer a monotonic function of \( s \); a minimum, \(-I_m < 0 \), is attained at \( s = [E/2K(2 - \gamma)]^{1/(\gamma - 1)} \). For \(|I| < GM\), we again obtain lens orbits. But a new type of orbit appears for \( GM < |I| \leq I_m \); the shaded region of Figure 2d shows the region filled by one such orbit for a typical value of \( I \), which is taken to be positive. These are centrophic box–like orbits; we call these boxlets, borrowing the term from Miralda-Escudé & Schwarzschild (1989). When \( I \) takes its maximum value equal to \( I_m \), the orbit collapses to one section of a constant–\( \eta \) curve lying between two halves of the \( \xi = \xi_m \) curve. This is a 2 : 1 resonant banana orbit, which now emerges naturally as a section of a paraboloid.

It is worth noting that loop orbits are completely absent. In the absence of the black hole \((M = 0)\), the lens orbits disappear, leaving the boxlets as the only family of orbits for cusps without black holes.

Figure 2. Orbital Structure: In all figures \( \gamma = 0.5 \), and \( I \) has been assumed to be positive. The solid vertical and horizontal lines in (a) and (c) correspond to the axes \( s = 0 \) and \( g(s) = 0 \) respectively. Similar lines in (b) and (d) are the axes \( x = 0 \) and \( y = 0 \) respectively. (a) \( E < 0 \). The solid curve is \( g(s) \), which intersects the dashed lines \( g = \pm I \) at \( s = \xi_m \) and \( s = \eta_m \) respectively. (b) Lens orbits are restricted to the shaded region of the \( x \)–\( y \) plane, which is bounded by the curves \( \xi = \xi_m \) and \( \eta = -\eta_m \). (c) When \( E > 0 \), the minimum of \( g(s) \) is shifted away from \( s = 0 \). For \( GM < I < I_m \), the dashed line at \(-I \) intersects the \( g(s) \) curve at two non–zero values of \( \eta \). This property forces the boxlets to be centrophic, as shown in (d).

4 CONCLUSIONS

The potential–density pairs presented in this paper are the only examples known to us, of non–axisymmetric distributions of matter that have density cusps giving rise to separable motion. The potentials are of Stäckel form, and owe their existence to the geometric properties of the parabolic coordinate curves. As a bonus we can also include a central black hole of arbitrary mass, while maintaining the separable nature. The discs are significantly non–axisymmetric, with the major axes being more than twice as long as the minor axes. The cusps can be steep, with logarithmic slopes of the surface density lying between \(-1\) and \( 0 \). Different discs may be superposed to obtain mass models that are not scale–free, yet whose potentials are of Stäckel form. It is a surprising fact that the presence of a black hole stabilises a family of orbits, the lenses, which can approach arbitrarily close to the centre. The lenses do not exist in the absence of the black hole—the centrophic boxlets are the only stable family of orbits for non–axisymmetric cusps, without a central black

\footnote{Stäckel potentials in elliptic coordinates do not appear to describe density cusps of interest to dynamics at the centres of elliptical galaxies!}

\footnote{Of course, a star that approaches too close to the black hole will be tidally shredded.}
hole. From the geometry of the constant $\xi$ and $\eta$ curves, it is evident that the lenses and boxlets can be elongated in the same sense as the mass distribution, a property that is favourable to the construction of non axisymmetric, self-consistent equilibria. Although we have limited discussion to discs that are symmetric with respect to reflections about both $x$ and $y$ axes, it is a simple matter to break the symmetry with respect to the $x$ axis; weighting the functions $\mathcal{F}_+ (\xi)$ and $\mathcal{F}_- (\eta)$ differently in equations (1) will produce a family of lopsided, cuspy discs. In fact, choosing functional forms for $\mathcal{F}_+ (\xi)$ and $\mathcal{F}_- (\eta)$ different from the ones given in equations (1) presents us with a variety of separable potentials (which, however, may not always be derivable from positive densities). The separable potentials we have presented might also be a suitable point of departure for perturbative analyses of dynamics in more realistic models of non-axisymmetric, lopsided discs. The scale–free nature simplifies the problem, but we clearly need to construct more realistic, finite mass configurations. Generalization to triaxial cusps, and construction of self–consistent equilibria remain as some of the outstanding questions. The restoration of integrability, and the existence of favourably oriented orbits in the idealized problem we have considered suggests that it is perhaps premature to conclude that elliptical galaxies with central density cusps, and black holes cannot be triaxial.

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