Transition redshift in $f(T)$ cosmology and observational constraints

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We extract constraints on the transition redshift, $z_{tr}$, determining the onset of cosmic acceleration, predicted by an effective cosmographic construction, in the framework of $f(T)$ gravity. In particular, employing cosmography we obtain bounds on the viable $f(T)$ forms and their derivatives. Since this procedure is model independent, as long as the scalar curvature is fixed, we are able to determine intervals for $z_{tr}$. In this way we guarantee that the Solar-System constraints are preserved and moreover we extract bounds on the transition time and the free parameters of the scenario. We find that the transition redshifts predicted by $f(T)$ cosmology, although compatible with the standard $\Lambda$CDM predictions, are slightly smaller. Finally, in order to obtain observational constraints on $f(T)$ cosmology, we perform a Monte Carlo fitting using supernova data, involving the most recent union 2.1 data set.

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I. INTRODUCTION

Observational evidences imply that the universe is undergoing a phase of anomalous acceleration after a precise time, usually named the transition time $z_{tr}$. In particular, the corresponding transition redshift $z_{tr}$ indicates at which stage the universe changed its dynamical properties and started accelerating after a phase of deceleration $\ddot{a}$. Recently, it has been argued that constraining $z_{tr}$ provides information on the form of the fluid responsible for the observed universe speeding up. Consequently, $z_{tr}$ may likely reveal possible new gravitational physics due to modifications of Einstein's gravity.

The fluid which triggers the current universe acceleration is often referred to as dark energy and fills more than the 70% of the whole universe energy budget. The standard cosmological model assumes that the dark energy source is supplied by the existence of a non-zero cosmological constant, $\Lambda$. The corresponding paradigm, named the $\Lambda$CDM model, is constructed by employing a net matter density composed by baryons and cold dark matter, with a constant dark energy term $\Omega_\Lambda \equiv \frac{3\Lambda}{8\pi G}$. Even though the model likely represents the simplest approach for describing universe’s dynamics, amongst others the cosmological constant does not furnish an explanation to the coincidence problem between matter and dark energy magnitudes. In other words, since the cosmological constant does not evolve in time, it is improbable that the ratio between matter and dark energy densities is so close today. Additionally, quantum field theory predictions forecast an enormous value for the cosmological constant if compared with the one measured by current cosmological observations. This issue is the well known fine-tuning problem and represents a challenge to understand the physical origin of the cosmological constant itself.

Due to the above caveats, one can modify the universe content, and attribute the dark energy sector to a canonical scalar field, a phantom field, to the combination of both fields in a unified model, or proceed to more complicated constructions (for reviews see [2, 9]).

An alternative way to reproduce the universe dynamics is by extensions of general relativity by means of additional degrees of freedom, which do not violate the equivalence principle, and represent a bid to formulate a semi-classical scheme for both late and early-time universe. In the usual approach to modify gravity, one starts by the usual curvature formulation of general relativity, and replaces the Ricci scalar $R$ in the Einstein-Hilbert action by arbitrary functions of it, or even more complicated curvature invariants. However, alternatively one can use as a base the torsional formulation of general relativity, namely the so called “teleparallel equivalent of general relativity” [11], and modify its action instead. In particular, in teleparallel gravity [12] the gravitational field is described not by the curvature tensor but by the torsion tensor, and thus the corresponding Lagrangian, namely the torsion scalar $T$, is constructed by contraction of the torsion tensor in a similar way that in usual general relativity the Lagrangian, namely the Ricci scalar $R$, is constructed by contractions of the curvature tensor. Hence, similarly to the $f(R)$ extension of general relativity, one can construct the $f(T)$ extension of teleparallel equivalent of general relativity [13, 14]. The interesting feature is that although general relativity coincides completely with teleparallel equivalent of general relativity, $f(T)$ gravity is different from $f(R)$ one, thus it is a novel gravitational modification with rich cosmological implications [14, 15].

With those considerations in mind, in this work we are interested in describing the dark energy effects, di-
rectly calculating the corresponding transition redshift $z_{tr}$ that is predicted in $f(T)$ cosmology. In order to do so, we only consider those $f(T)$ models which are consistent with present-time cosmographic constraints. Hence, we aim to obtain cosmographic bounds on the $f(T)$ scenarios, by considering the modified Friedmann equations, and then get the corresponding limits on $z_{tr}$. The main advantage of using cosmography is that the value of $z_{tr}$ is reconstructed by means of a model-independent procedure. Rephrasing it differently, we are able to distinguish which classes of $f(T)$ gravity pass the cosmographic requirements and thus are viable, by inferring the limits over $z_{tr}$ that those classes predict. In particular, we compare this transition epoch with the one determined by the standard cosmological paradigm, and we propose an effective cosmological model capable of reproducing the cosmographic constraints and compatible with the limits on $z_{tr}$. Additionally, to enable our treatment, we consider the use of the luminosity distance and we match cosmic union 2.1 supernova data [17] with the cosmographic expansions. Thus, we evaluate the corresponding deceleration parameter and we show that the transition redshift is effectively comparable to the one predicted by the ΛCDM approach.

As we will see, the limits on $z_{tr}$ show that the considered $f(T)$ classes reduce to the ΛCDM model in the lowest redshift domain, in agreement with [18, 19]. This feature indicates that the role played by the cosmological constant may be reinterpreted as a limiting case of a more general extension, and thus from those cosmographic corrections we show that small discrepancies occur at $z \leq 1$, whereas higher departures might be expected at high-redshift regimes. At this point, we involve a Monte Carlo fitting procedure based on the Metropolis algorithm, in order to compare our effective cosmological model with present-time data. Numerical limits, priors and final outcomes, testify the efficiency of our approach, showing that the effective torsional dark energy naturally satisfies the cosmographic requirements, and hence it may be a candidate as a valid alternative to describe the universe dynamics.

The paper is organized as follows. In Sec. [IV] we describe the techniques for recovering the cosmographic settings on the $f(T)$ classes of models and we moreover propose how to obtain $f(T)$ reconstructions. In Sec. [V] we enumerate the properties of the transition redshift and its important role in modern cosmology. Furthermore, we describe the main consequences in $f(T)$ gravity and we show how the modified Friedmann equations changed when the transition occurred. In Sec. [VI] we summarize the cosmographic results and we propose an effective reconstruction of $f(T)$ cosmology. To do so, we infer the deceleration parameter for the effective torsional dark energy models and finally we show the numerical priors on the transition redshift $z_{tr}$ predicted by our paradigm. In Sec. [VII] we compare the cosmological consequences of the examined models with modern data, employing the use of the union 2.1 supernova survey. We determine the free parameters of our approach and we show that the dark energy corrections are compatible with the bounds offered by alternative dark energy models. Finally, in Sec. [VIII] we summarize the conclusions and perspectives of our approach.

II. THE PROCEDURE FOR $f(T)$ RECONSTRUCTION FROM COSMOGRAPHY

In teleparallel formulation of gravity, as well as in its $f(T)$ extension, one uses the vierbein fields $e^A_\mu$, which form an orthonormal base for the tangent space at each point $x^\mu$ defined on a generic manifold, and thus the metric reads as $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$ (in the following greek indices and Latin indices span the coordinate and tangent spaces respectively). Additionally, instead of the torsionless Levi-Civita connection one uses the curvatureless Weyl–Zeebekö one $\Gamma^\lambda_{\mu\nu} = e^A_\lambda \partial_\mu e^A_\nu - e^A_\nu \partial_\mu e^A_\lambda + \kappa e^A_\lambda \gamma_{\mu\nu}$, and therefore the gravitational field is encoded in the torsion tensor

$$T^{\rho}_{\mu\nu} = e^A_\lambda \partial_\mu e^A_\nu - \partial_\mu e^A_\lambda e^A_\nu, \quad (1)$$

Hence, the Lagrangian of teleparallel gravity, namely the torsion scalar $T$, is constructed by contractions of the torsion tensor as [12]

$$T \equiv \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\rho\mu} - T_{\rho\mu} T^{\rho\mu\nu}. \quad (2)$$

Finally, one can extend teleparallel gravity and construct the action of $f(T)$ gravity as [13, 14]

$$S = \int d^4 x \left[ \frac{(T)}{2\kappa^2} \right], \quad (3)$$

where $e = \det (e^A_\mu) = \sqrt{-g}$ and $\kappa^2$ is the gravitational constant.

The general field equations of $f(T)$ gravity are obtained by varying the action $S + S_m$, with $S_m$ the matter action, in terms of the vierbein, and they read as

$$e^{-1} \partial_\rho (e^\rho_\alpha S^\mu_{\rho\alpha}) f' + e^\rho_\alpha S^\mu_{\rho\alpha} \partial_\mu (T) f'' - f e^\lambda_\alpha T^\rho_{\mu\lambda} S^\mu_{\rho\alpha} + \frac{1}{4} e^\alpha_\nu = \frac{\kappa^2}{2} e^A_\mu T^{(m)}_{\rho} \quad (4)$$

where the tensor $S^\mu_{\rho\alpha} = \frac{1}{2} \left( K^\mu_{\rho\alpha} + \epsilon_\mu^{\alpha\beta\gamma} T^{\alpha\omega\gamma} - \epsilon_\mu^{\alpha\beta\gamma} T^{\alpha\omega\gamma} \right)$ is defined in terms of the co-torsion $K^\mu_{\rho\alpha} = -\frac{1}{2} \left( T^{\mu\rho}_{\nu\alpha} - T_{\rho\nu\alpha} - T_{\rho\mu\alpha} \right)$, and where $T^{(m)}_{\rho}$ is the energy-momentum tensor corresponding to $S_m$. In (11) the primes denote derivatives with respect to $T$. Finally, since for $f(T) = T$ equations (11) provide exactly the same equations with general relativity, that is why the theory with $f(T) = T$ was named by Einstein “teleparallel equivalent of general relativity” [11].

In order to apply $f(T)$ gravity in a cosmological framework we assume a spatially-flat Friedmann-Robertson-Walker metric $ds^2 = dt^2 - a(t)^2 (dx^2 + r^2 d\Omega^2)$, with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, which can arise from the vierbein
\( e^a_{\mu} = \text{diag}(1, a, a, a) \). In this case, the field equations give rise to the modified Friedmann equation

\[
H^2 = \frac{1}{3} (\rho_m + \rho_T) ,
\]

\[
\dot{H} = -\frac{1}{2} (\rho_m + p_m + \rho_T + P_T) ,
\]

with \( H \equiv \frac{\dot{a}}{a} \) the Hubble parameter and dots indicating derivatives with respect to the cosmic time. In the above expressions \( \rho_m \) and \( p_m \) are the energy density and pressure of the matter sector considered to correspond to a perfect fluid, and moreover from now on we use units in which \( \kappa^2 = 1 \). Furthermore, we have introduced the energy density and pressure of the effective dark energy sector, which incorporates the torsional modifications, as

\[
\rho_T \equiv -\frac{f}{2} - \frac{T}{2} + Tf',
\]

\[
P_T \equiv \frac{1}{2} \left[ \frac{f - f'T + 2T^2 f''}{f + 2T f''} \right] .
\]

Thus, the dark energy equation-of-state parameter writes as \( w_{DE} \equiv w_T \equiv P_T/\rho_T \). Finally, note that for the FRW geometry, the calculation of the torsion scalar \( T \) leads to the useful relation

\[
T = -6H^2.
\]

The issue of finding out a form for the dark energy equation of state passes through the determination of the most viable forms of \( f(T) \). If one knows the \( f(T) \) form, it is possible to infer the interpretations of \( \rho_T \) and \( P_T \) as terms associated to torsional dark energy, i.e. a torsional contribution driving the observed cosmic acceleration. The idea to get a viable \( f(T) \) function lies on requiring that at small redshift the \( f(T) \) model reproduces the observational data and predicts a compatible transition redshift. Hence, the strategy of this manuscript is to frame a phenomenological reconstruction of \( f(T) \) and its derivatives in terms of cosmography, which becomes a sort of initial settings for \( f(T) \) models. Having in mind these cosmographic requirements, we simply impose the validity of the cosmological principle, the geometrical setting of the scalar curvature, and the possibility of expanding \( f(T) \) and its derivatives around present time in Taylor series. We discuss below each of those three requirements, in order to define the cosmographic series and its application in \( f(T) \) cosmology.

- First, employing the cosmological principle permits to frame the universe expansion history in terms of a single parameter, namely the scale factor \( a(t) \) which enters the Friedmann-Robertson-Walker metric as function of the cosmic time only. One gathers viable outcomes imposing that this function may be expanded in Taylor series around present time, and constraining the corresponding Taylor coefficients associated to the scale-factor derivatives. This strategy compares \( a(t) \)'s derivatives directly with cosmic data and may be used as a reconstruction for the \( a(t) \) shape. This benefit allows one to distinguish among all paradigms, derived from imposing the form of \( f(T) \), the ones whose cosmographic requirements better match with data.

- The second caveat is the issue of spatial curvature which leads to a degeneracy problem between its value and the variation of the acceleration. It has been proved that photon geodesics change their paths according to its value. Thus, expanding a physical quantity into a cosmographic series needs to fix somehow the value of spatial curvature, in order to allow cosmography to be as model-independent as possible. According to previous approaches, one imposes geometrical bounds on \( \Omega_k \) by assuming the matching between early and late time observations. We therefore assume that the universe is spatially flat, with possible small deviations which do not influence the whole dynamics.

- Finally, since all observable quantities of interest are assumed to smoothly evolve as the universe expands, it is licit to assume that Taylor expansions may be easily accounted and no saddle points or poles occur. It follows that all functions are analytic and the cosmographic treatment is perfectly plausible. After those properties, one soon expands in Taylor series the scale factor \( a(t) \) as

\[
a(t) \equiv \sum_{l=0}^{\infty} \frac{1}{l!} a_p \bar{t}^p \approx a_0 + a_1 \bar{t} + a_2 \bar{t}^2 + \ldots ,
\]

where \( a_p \equiv \frac{d^p a}{dt^p} \), with \( \bar{t} \equiv t - t_0 \) and \( t_0 \) the present time. Finally, it proves convenient to express the observable quantities under interest in terms of the redshift \( z = -1 + a_0/a \).

Amongst all observables, we are much interested in the use of the luminosity distance, since we will use supernovae Ia type to fix our cosmological bounds. Hence, imposing \( a_0 = 1 \) and inserting it in the definition of the luminosity distance:

\[
D_L = (1 + z) \int_0^z \frac{dz}{H(z)} ,
\]

we write down the Taylor series around \( z = 0 \) as

\[
\left\{ \begin{array}{l}
D_L = \frac{1}{H_0} \bar{d}_L(z; \theta) , \\
\bar{d}_L(z; \theta) = 1 + d_{L1} z + d_{L2} z^2 + \ldots ,
\end{array} \right.
\]

truncated at the third order. Moreover, applying also the definitions

\[
\dot{H} = -H^2 (1 + q) ,
\]

\[
\ddot{H} = H^3 (j + 3q + 2) ,
\]
we obtain the corresponding coefficients of the Taylor expansion in terms of the cosmographic series:

\[ d_{L1} = 1 - \frac{q_0}{2}, \quad (12a) \]
\[ d_{L2} = 1 - \frac{q_0(1 + 3q_0) + j_0}{6}. \quad (12b) \]

In these expressions we have introduced the deceleration and jerk parameters as

\[ q = \frac{1}{aH^2} \frac{d^2a}{dt^2}, \quad (13a) \]
\[ j = \frac{1}{aH^3} \frac{d^3a}{dt^3}, \quad (13b) \]

indicating respectively whether the universe is accelerating or not and how the acceleration changed sign, with the subscript “0” denoting the value of a quantity at present. Observations indicate \( j_0 > 0 \) and then testify that the transition time occurred. However, there still exists a tension between the possibilities \( 0 < j_0 < 1 \) and \( j_0 > 1 \). \[29\].

From the above analysis it follows that the cosmographic series is the set of coefficients evaluated at present time. The cosmographic series has been built up in function of a Taylor series expanded around \( z = 0 \), albeit it is possible to handle it also in terms of the cosmic time. This is clearly possible involving the definition of the redshift in terms of the cosmic time as:

\[ \frac{dz}{1 + z} = -H(z)dt. \quad (14) \]

Since the cosmographic series may be expressed in terms of the scale-factor derivatives, it does not depend upon the particular choice of the cosmological model. This property represents a key to conclude that imposing a cosmological model \( a \) priori is unnecessary and any modified gravity may be limited by assuming the cosmographic requirements. We here follow the technique of reconstructing the \( f(T) \) models by means of late-time cosmography. To do so, we rewrite the luminosity distance \[23\] in terms of \( f(T) \) derivatives. This is possible since there exists a direct correspondence between \( q_0 \) and \( j_0 \) with the \( f(T) \) form and its derivatives. In particular, rewriting \[12\] as function of \( f(T) \), we frame the effective model derived from the torsional dark energy by directly comparing \( D_L \) with data. Rephrasing it differently, instead of using \( q, j, \ldots \) we consider \( f(T), f'(T), f''(T), \ldots \) and we obtain the numerical outcomes on those quantities. The cosmographic constraints on \( f(T) \) and its derivatives, point out the numerical priors that we use as initial settings for reconstructing the shape of viable effective \( f(T) \) models, which reproduce dark energy at small redshift. Afterwards, we predict the transition redshift from our effective model and we understand whether our model indicates a viable \( z_{tr} \) if compared with the \( \Lambda CDM \) predictions. We will report the connections between the cosmographic series and the \( f(T) \) derivatives in Sec. IV.

### III. The \( f(T) \) Transition Redshift

In the scenario at hand, the \( f(T) \) term drives the dark energy contribution, interpreting the dark sector as due to a torsional dark energy. It is widely believed that the dark energy contribution dominates over matter at our time, while it appears negligible at higher redshift regimes. The type of the transition and the time at which it occurs are extremely relevant, since they indicate the dark energy nature and may also provide information on how the dark energy evolves in time. In particular, the transition time and correspondingly the transition redshift emphasize the change from decelerated to accelerated cosmological expansion, and represent a prediction of any particular model involved to describe the universe expansion history. In other words, direct measurements of the transition redshift provide direct information on both the deceleration and acceleration epochs.

To show how to investigate \( z_{tr} \) in the framework of \( f(T) \) gravity, let us consider the definition of the transition redshift, which occurs at a zero of the deceleration parameter \( q \). We will find out the transition redshift \( z_{tr} \) for a class of cosmographic \( f(T) \) models, and we will also compare it with standard model predictions and with recent bounds on \( z_{tr} \) itself.

Passing through the phase of transition between matter and dark energy dominance, and assuming the matter to be dust (i.e. \( P_m = 0 \)), it is useful to combine the two Friedmann equations \[13\] and \[14\] to obtain the torsional pressure in terms of the deceleration parameter as:

\[ P_T = H^2(2q - 1), \quad (15) \]

where we made use of relation \[13\]. At the transition time we therefore obtain the value of the torsional pressure as

\[ P_T(z_{tr}) = -H_{tr}^2, \quad (16) \]

which corresponds to \( q = 0 \) at the transition redshift \( z_{tr} \), with Hubble rate \( H_{tr} \). This expression is equivalent to the standard barotropic dark-energy pressure in the framework of general relativity given by \[31\]:

\[ P_{tr} = -H_{tr}^2. \quad (17) \]

In particular, the two results, \[14\] and \[17\], lead to the same formal outcome. In fact, assuming \( H_{tr} \) to be positive definite, both the torsional and standard dark-energy pressures are negative at the transition. However, the physical meaning behind \[14\] and \[17\] is different, in the sense that in the first case the transition is induced by the torsional terms, while in the standard approach the transition is realized due to the dark energy or curvature terms.

In the standard \( \Lambda CDM \) cosmological model, the cosmological constant contributes about 70% of the present cosmological energy budget and the consequence on cosmology lies on an evolving deceleration parameter \( q \) of
and moreover $f'(T_0) = 1$ to guarantee the Solar-System constraints in order to preserve the value of $G$ at our time.

Thus, using expressions (21) and (26), we acquire the priors on $T$, $f(T)$ and $f''(T)$ as

$$f(T_0) \in [-5.23; -4.79],$$
$$f''(T_0) \in [-0.19; 0.01],$$
$$T_0 \in [-3.11; -2.77],$$
$$T \in [0.75; 2.24],$$
$$T \in [-7.26; -0.36],$$

which have been obtained assuming a normalized Hubble rate $H_0 \in [0.68; 0.72]$ and a mass density $\Omega_{m,0} \in [0.274; 0.318]$, with $q_0 \in [-0.8, -0.5]$, $j_0 \in [0.5, 1.5]$ and $j_{tr} \approx j_0$ [21]. The last condition has been imposed assuming that the universe is slightly evolving in the redshift domain $z \leq 1$. We stress that the priors [20] are the requirements that determine whether a specific $f(T)$ form is viable or not.

The strategy is the following: we assume the validity of the cosmographic series as the initial conditions of the modified Friedmann equations, and then we integrate the first Friedmann equation. Thus, we infer the numerical values of $H(z)$ for different redshifts, and we separately extrapolate those points, determining a list of numbers for $H(z)$ and $z$. Finally, through the use of testing functions, we reconstruct an effective $f(T)$ which reproduces the numerical limits. Hence, from this function one obtains a parameterized cosmological model, which departs from the ΛCDM scenario, corresponding to a varying dark-energy sector.

Our treatment suggests that a possible approximation of the dark-energy density term $\rho_{DE}$ may be

$$\rho_{DE} \approx \log \left[ \alpha + \beta \sum_{i=0}^{N} a^i \right].$$

Truncating at the second order in $a$, we obtain the Hubble rate as

$$\frac{H^2}{H_0^2} = \Omega_m(z) + \log[\alpha + \beta(2 - 3a + a^2)],$$

where we considered $\Omega_m \equiv \Omega_{m,0}(1 + z)^3$. The parameter $\alpha$ is fixed in order to guarantee that at $z = 0$ the Hubble rate is identically $H = H_0$. Therefore, we have

$$\alpha = e^{1 - \Omega_{m,0}}.$$

Hence, the cosmographic reconstruction of torsional dark energy provides a deceleration parameter of the form

$$q = \frac{3\Omega_{m,0}(1 + z)^5}{2(1 + z)^3} \frac{\alpha + \beta + 3z[1 + \Omega_m(z)(1 + 2z)]\beta}{\left[\Omega_m(z) + \log \left( \frac{\alpha + 2\beta (1 + 2z)}{(1 + z)^3} \right) \right] - 1}.$$
In Fig. 1 we depict the behaviors of \( H(z)/H_0 \) and \( q(z) \), given in \((25)\) and \((30)\) respectively. We deduce that up to the redshift domain \( z \leq 2 \), our approach is compatible with the standard cosmological model, and in fact only small differences occur between our predictions and the \( \Lambda \)CDM ones, which are slightly larger. This is due to the fact that our \( H(z) \) parameter indicates a dark-energy evolution which does not depart significantly from the case of a constant dark-energy term at small redshifts. Hence, our Hubble rate well approximates the standard \( \Lambda \)CDM contribution, slightly evolving as the redshift increases. This is more evident in Fig. 2, in which we plot the dark-energy term \((27)\) normalized by means of the standard critical density \( \rho_c \equiv 3H_0^2/8\pi G \).

![Figure 1: The evolution of \( H(z)/H_0 \) (upper graph) and \( q(z) \) (lower graph), according to viable \( f(T) \) cosmology (blue-solid curves) versus the \( \Lambda \)CDM predictions (red-dashed curves). We employed the indicative values \( \Omega_{m,0} = 0.27 \) and \( \beta = 1 \) and we normalized through \( H_0 = 100 \text{Km Mpc}^{-1} \text{s} \).](image)

![Figure 2: The evolution of the dark-energy density according to viable \( f(T) \) cosmology (blue-solid curves) versus the \( \Lambda \)CDM predictions (red-dashed curves). The two cosmological paradigms exhibit very similar behaviors, and thus the \( \Lambda \)CDM curve is almost indistinguishable from the viable \( f(T) \) one. We employed the indicative values \( \Omega_{m,0} = 0.27 \) and \( \beta = 1 \) and we normalized through \( H_0 = 100 \text{Km Mpc}^{-1} \text{s} \).](image)

Afterwards, linearizing the deceleration parameter around \( z = 0 \), keeping first-order terms, we find the transition redshift as

\[
 z_{tr} \simeq \frac{\alpha^2(\Omega_{m,0} + \log \alpha)(2 \log \alpha - \Omega_{m,0} - \beta)}{[9\Omega_{m,0} \alpha^2 + (\alpha - \beta) \beta \log \alpha - \beta \beta + \Omega_{m,0}(5\alpha + \beta)]}.
\]

We mention that this approximation is efficient, since one expects a transition at \( z \leq 1 \) and therefore the linearized \( q \) does not substantially differ from the exact value.

Having in mind the form of the Hubble rate, we can infer limits over \( \Omega_{m,0} \) and \( \beta \). This permits one to determine \( z_{tr} \) from expression \((31)\). In the next section we describe the fitting procedure using supernova data, and we extract numerical bounds on the free parameters of our cosmographic torsional dark-energy scenario.

**V. THE MATCHING WITH OBSERVATIONS**

The above approach provided a particular set of cosmographic quantities related to the \( f(T) \) form. Correspondingly, the effective Hubble rate was built in terms of corrections to the simple teleparallel gravity, that is to general relativity. Hence, these corrections are due to the difference between \( f(T) \) cosmology and the standard paradigm of \( \Lambda \)CDM cosmology. These terms may be compared with the cosmological constant value without showing great departures at small redshift regimes, as we discussed in the above section. The contribution due to the \( f(T) \) sector needs to match an adequate convergence at small and high redshift domains. Hence, it is required to obtain bounds on the observational parameters, in order to understand whether the cosmological model at hand passes or not the observational constraints.

In order to proceed, we perform the analysis by involving the Monte Carlo technique with the use of the union 2.1 supernova compilation \([17]\). This survey is built up by 580 measurements of apparent magnitudes, with the corresponding redshifts and magnitude errors. Assuming a Gaussian distribution, one acquires the relevant fact that the luminosity distance may be rewritten in terms of the cosmographic series itself. Thus, all observations may be performed by directly fitting \( D_L \) with union 2.1 data.

We employ type Ia supernova observations since they probably represent the most suitable cosmic compilation. The role of supernovae has been crucial for cosmological parameter-fittings, since supernovae are considered standard candles. It follows that their luminosity curves are easily related to distances themselves.

The union 2.1 data set is capable of reducing previous
In particular, having a spatially flat universe, the luminosity distance simply reduces to

\[d_L = \frac{1}{a} \int_0^\psi \frac{d\psi}{H(\psi)},\]  

(35)

The Hubble derivatives, evaluated as a function of the redshift \(z\) at our time \((z = 0)\), give us

\[
\frac{dH}{dz} = H_0 \left(1 + q_0\right), \quad \frac{d^2H}{dz^2} = H_0 \left(j_0 - q_0^2\right),
\]

(36a, 36b)

providing a third order Taylor series for the luminosity distance of the form: \(d_L = \eta_1 z + \eta_2 z^2 + \eta_3 z^3 + \ldots\), where

\[
\eta_1 = 1, \quad \eta_2 = 2 - \frac{3\Omega_{m,0} + \beta \exp(\Omega_{m,0} - 1)}{2}, \quad \eta_3 = \frac{1}{8} \left[3\Omega_{m,0} + \beta \exp(\Omega_{m,0} - 1)\right] \cdot \left[3\Omega_{m,0} - 2 + \beta \exp(\Omega_{m,0} - 1)\right].
\]

(37a, 37b, 37c)

It is useful to stress here that expression (10) represents a general Taylor expansion and may be applied to any cosmological model. The advantage of passing through it is that one directly fits a particular model of interest, weighting the coefficients directly with the most recent data. A simple strategy for definitively alleviating the problem of matching data with our model, is to assume \textit{a priori} compatible cosmographic priors. To do so, we employ the theoretical bounds given by [20].

This treatment represents the key to obtain suitable cosmographic intervals, in which the free parameters of our model, i.e. \(\Omega_{m,0}\) and \(\beta\), do not violate the cosmological limits. Our numerical outcomes also need to be compatible with the ones already proposed in the literature and do not have to influence the analyses themselves. In addition, we aim at finding out numerical outcomes over \(z_{tr}\), which can be indirectly derived from the experimental analysis by using [31].

Hence, the Bayesian technique provides the likelihood function:

\[\mathcal{L} \propto \exp(-\chi^2/2),\]  

(38)

whose maximum corresponds to the minimum of the \(\chi^2\). We obtain our numerical results performing a test with the free available code ROOT and the additional package BAT [35]. Our analyses are based on two statistical treatments, characterized by different maximum order of parameters. We perform such a procedure in order to provide a hierarchy among all parameters. Firstly, we allow all parameters to freely vary (Fit1), and secondly we fix the mass density parameter through values compatible with the most recent Planck measurements [31] (Fit2).

Our numerical results are summarized in Table I, where we separately report the obtained and the inferred parameters. We perform such a procedure in order to provide a hierarchy among all parameters. Firstly, we allow all parameters to freely vary (Fit1), and secondly we fix the mass density parameter through values compatible with the most recent Planck measurements [31] (Fit2).

| Parameter | Fit1                    | Fit2                    |
|-----------|-------------------------|-------------------------|
| \(H_0\)   | 69.490 \(\pm\) 0.366   | 69.450 \(\pm\) 0.342   |
| \(\alpha\) | 1.367 \(\pm\) 0.286    | 2.067 \(\pm\) ...     |
| \(\beta\)  | \(-1.147\) \(\pm\) 0.969 | \(0.834\) \(\pm\) 0.186 |
| \(\Omega_{m,0}\) | 0.687 \(\pm\) 0.217 | 0.274 \(\pm\) ... |
| \(z_{tr}\)  | \(0.247\) \(\pm\) 0.345 | \(0.643\) \(\pm\) 0.034 |
| \(\Delta z_{tr}\)  | \(0.385\) \(\pm\) ... | \(0.011\) \(\pm\) ... |
| \(P_{r,tr}\)  | \(-0.687\) \(\pm\) 0.060 | \(-1.032\) \(\pm\) 0.070 |

Table I: Table of our experimental and \textit{a posteriori}-derived results. We report the 1σ confidence level errors for our fitting procedure, performed through the Metropolis algorithm. The associated errors on derived quantities have been obtained through the logarithmic rule [1, 36]. To evaluate the transition redshift in the \(\Lambda\)CDM model we considered \(\Omega_{m,0} = 0.315\) from the Planck measurements. Finally, \(H_0\) is given in Kms/Mpc.

The cosmological results show that the first fit (Fit1), in which all coefficients are taken free, does not give conclusive results. In this case, in fact, the mass density is overestimated probably due to the strong multiplicative degeneracy between the coefficients \(\Omega_{m,0}\) and \(\beta\), as one can see from [47]. The cosmographic analysis suffers from this kind of degeneracy and shows the same inefficiency in bounding \(z_{tr}\), which seems to significantly depart from the \(\Lambda\)CDM predictions, as shown by look-
Figure 3: Contour plots for our observational analyses in the case where all parameters are free to vary (Fit1): $\Omega_{m,0}$ versus $H_0$ (upper graph), $\beta$ parameter versus $H_0$ (middle graph) and $\beta$ versus $\Omega_{m,0}$ (lower graph).

In the second fit (Fit2), where we fix the matter density parameter to a value compatible with the Planck measurements, namely $\Omega_{m,0} = 0.274$ [31], the results are mostly accurate. This fixing enables to get refined limits even on the other two free coefficients of our model, as can be seen in Fig. 4 (compare with the middle graph of Fig. 3). As a consequence, we obtain more precise bounds on $z_{tr}$, which becomes perfectly compatible with the constraints predicted by the ΛCDM model, at the $1 - \sigma$ confidence level. Our value however seems to be slightly smaller than theoretical expectations ($z_{tr} = 0.74$ according to [31]).

Hence, we conclude that combined observational tests will represent a landscape to better fix constraints over the involved quantities, showing more accurate limits on the transition predicted by $f(T)$ gravity. Further, this seems to be evident by looking at the contour plots of Figs. 3 and 4 in which a delineated curve corresponds to the second fit. In other words, from these Figures it is clear that the first fit shows higher errors since the contours are larger than the one inferred from the second fit. Nevertheless, in all cases the predicted transition time occurs at $z < 1$, in agreement with the standard theoretical framework.

VI. CONCLUSIONS AND PERSPECTIVES

In this paper we investigated the transition redshift derived in an effective model inferred from $f(T)$ gravity. In order to do so, we extracted an approximate reconstruction of the $f(T)$ dark-energy term. The effective dark energy contribution has been obtained by numerically solving the Friedmann equations, employing as initial conditions the numerical outcomes obtained from cosmographic bounds. In this way, we defined a set of numerical constraints on $f(T)$ and its derivatives in a model-independent way, and we were able to fix the evolving dark-energy term through a logarithmic correction.
Our cosmographic model well adapts to the late-time constraints, and it reproduces a cosmological model which smoothly departs from the standard $\Lambda$CDM paradigm. The corresponding limits on the free parameters of the model have been obtained by directly fitting the luminosity distance with supernova data, using the most recent union 2.1 compilation. We extracted viable constraints on the free parameters of the scenario, in two distinct fits with different hierarchy between coefficients. We first considered all parameters free to vary and afterwards we fixed the value of the matter density consistently with current Planck results. All predictions provided intervals for the transition redshift which are compatible with present expectations, although the numerical outcomes are slightly smaller than the ones predicted by the standard cosmological model. Departures have been encountered in the case where we leave all parameters free to vary, due to the degeneracy problem between coefficients in the luminosity distance definition. Possible approaches will be devoted to better fix those constraints by means of combined cosmological tests. Moreover, we could mostly investigate the properties of our logarithmic corrections, studying their consequences in the early phases of the universe evolution.

Finally, it would be interesting to extend the above analysis in the case of higher-order torsional cosmology, and in particular in the case where the teleparallel equivalent of the Gauss-Bonnet combination is used in the action, as in $f(T,\mathcal{G})$ cosmology. The corresponding results could be compared with both $\Lambda$CDM cosmology, as well as with the $f(R,\mathcal{G})$ cosmology. Such an analysis could provide more information on the possible distinguishability of curvature and torsional gravity using cosmographic methods.

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[1] S. Capozziello, M. De Laurentis, O. Luongo, A. C. Ruggeri, Galaxies, 1, 216-260, (2013).
[2] E. J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D, 15, 1753-1936, (2006).
[3] P. J. E. Peebles, B. Ratra, Rev. Mod. Phys., 75, 559-606, (2003).
[4] S. Capozziello, M. De Laurentis and O. Luongo, arXiv:1411.2822 [gr-qc].
[5] P. Wu, H. W. Yu, Phys. Lett. B, 693, 415, (2010); S. H. Chen, J. B. Dent, S. Dutta, E. N. Saridakis, Phys. Rev. D 83, 023508 (2011); J. B. Dent, S. Dutta, E. N. Saridakis, JCAP 1101, 009 (2011); K. Bamba, R. Myrzakulov, S. ’i. Nojiri and S. D. Odintsov, Phys. Rev. D 85, 104036 (2012); G. Otalora, JCAP 1307, 044 (2013); K. Izumi and Y. C. Ong, JCAP 1306, 029 (2013).
[6] E. N. Saridakis, JCAP 1101, 021 (2011); M. Sharif, S. Rani, Mod. Phys. Lett. A26, 1657 (2011); M. R. Setare, M. J. S. Houndjo, Can. J. Phys., 91, 260-267, (2012); J. Amoros, J. de Haro and S. D. Odintsov, Phys. Rev. D 87, 104037 (2013); G. G. L. Nashd and W. El Hanafy, Eur. Phys. J. C 74, no. 10, 3099 (2014); V. Fayaz, H. Hossienkhani, A. Farmany, M. Amirabadi and N. Azimi, Astrophys. Space Sci. 351, 299 (2014).
[7] S. Capozziello, M. De Laurentis, O. Luongo, Phys. Rept., 509, 167-321, (2011).
[8] A. Einstein 1928, Sitz. Preuss. Akad. Wiss. p. 217; ibid p. 224; A. Unzicker and T. Case, physics/0503046.
[9] R. Aldrovandi and J. G. Pereira, Teleparallel Gravity: An Introduction (Springer, Dordrecht, 2013); J. W. Maluf, Annalen Phys. 525, 339 (2013).
[10] R. Ferraro, F. Fiorini, Phys. Rev. D, 75, 084031, (2007); Phys. Rev. D, 78, 124019, (2008).
[11] E. V. Linder, Phys. Rev. D, 81, 127301, (2010) [Erratum-ibid. D, 82, 109902, (2010)].
[12] R. Ferraro, F. Fiorini, Phys. Rev. D, 75, 084031, (2007); Phys. Rev. D, 78, 124019, (2008).
[13] E. V. Linder, Phys. Rev. D, 81, 127301, (2010) [Erratum-ibid. D, 82, 109902, (2010)].
Quantum Field Theory and Gravity, Tomsk, Russia, (2014), arXiv: 1411.2350; S. Capozziello, O. Farooq, O. Luongo, B. Ratra, Phys. Rev. D, 90, 044016, (2014).

[25] A. Aviles, A. Bravetti, S. Capozziello, O. Luongo, Phys. Rev. D, 87, 044012, (2013); B. Bochner, D. Pappas and M. Dong, arXiv:1308.6050 [astro-ph.CO].

[26] S. Nesseris, J. Garcia-Bellido, Phys. Rev. D, 88, 063521, (2013).

[27] M. Visser, Gen. Rel. Grav., 37, 1541-1548, (2005).

[28] A. R. Neben, M. S. Turner, ApJ, 769, 133, (2013).

[29] O. Luongo, Mod. Phys. Lett. A, 28, 1350080, (2013).

[30] O. Farooq, S. Crandall, B. Ratra, Phys. Lett. B, 726, 72-82, (2013).

[31] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 571, A16 (2014).

[32] R. Amanullah, C. Lidman, D. Rubin, G. Aldering, P. Astier, et al., Astrophys. J. 716, 712-738, (2010).

[33] Supernova Cosmology Project Collaboration, M. Kowalski et al., Astrophys. J. 686, 749-778, (2008).

[34] M. Goliath et al., Astron. Astrophys. 380, 6, (2008).

[35] http://root.cern.ch/drupal; https://www.mppmu.mpg.de/bat

[36] L. Verde, Lect. Not. Phys., 800, 147-177, (2010); S. Weinberg, Cosmology, Oxford Univ. Press, Oxford, (2008).

[37] O. Farooq, B. Ratra, Astroph. J., 766, L7, (2013).

[38] G. Kofinas and E. N. Saridakis, Phys. Rev. D 90, no. 8, 084044 (2014); G. Kofinas, G. Leon and E. N. Saridakis, Class. Quant. Grav. 31, 175011 (2014); G. Kofinas and E. N. Saridakis, Phys. Rev. D 90, no. 8, 084045 (2014).

[39] A. De Felice, J. M. Gerard and T. Suyama, Phys. Rev. D 82, 063526 (2010).

[40] M. De Laurentis, arXiv:1411.7001 [gr-qc] (2014) , to appear in Mod. Phys. Lett. A.

[41] M. De Laurentis and A.J. Lopez-Revelles, Int.J.Geom. Meth. Mod. Phys. 11 (2014) 1450082 .