SUBREARRANGEMENT-INVARIANT FUNCTION SPACES

BEN WALLIS

ABSTRACT. Rearrangement-invariance in function spaces can be viewed as a kind of generalization of 1-symmetry for Schauder bases. We define subrearrangement-invariance in function spaces as an analogous generalization of 1-subsymmetry. It is then shown that every rearrangement-invariant function space is also subrearrangement-invariant. Examples are given to demonstrate that not every function space on $(0, \infty)$ admits an equivalent subrearrangement-invariant norm, and that not every subrearrangement-invariant function space on $(0, \infty)$ admits an equivalent rearrangement-invariant norm. The latter involves constructing a family of function spaces inspired by D.J.H. Garling, and we further study them by showing that they contain copies of $\ell_p$.

1. Introduction

A Banach space with a 1-unconditional Schauder basis can be viewed as a function space, albeit over $\mathbb{N}$ equipped with the purely atomic counting measure. Rearrangement-invariance of function spaces in this case is equivalent to the notion of a 1-symmetry for the basis. Symmetry of bases, though, admits a natural weakening to the notion of a subsymmetric basis, which is not necessarily rearrangement-invariant in the fullest sense but nevertheless is invariant under rearrangements which preserve order. Such rearrangements can be generalized to other totally-ordered measure spaces to produce a kind of transformation which we shall call an MO-isomorphism. Armed with this concept, we may generalize 1-subsymmetry in sequence spaces with a 1-unconditional basis to the notion of subrearrangement-invariance, so as to make room for certain nonatomic measure spaces, namely $(0, \infty)$.

For several years in the mid-1960s, it was mistakenly thought that subsymmetric bases are always symmetric ([Si62]). This error was discovered by D.J.H. Garling, who published a counterexample in 1968 ([Ga68, §5]). Garling’s sequence space, then, already furnishes us with a subrearrangement-invariant function space on $\mathbb{N}$ which is not “essentially” rearrangement-invariant, that is, not rearrangement-invariant under any equivalent norm. It is also well-known that there exist 1-unconditional bases which are not subsymmetric, and hence function spaces on $\mathbb{N}$ which fail to admit an equivalent subrearrangement-invariant norm. In this paper, we extend these results to a purely nonatomic case, exhibiting an
example of a function space on \((0, \infty)\) which is not essentially subrearrangement-invariant, and another example which is subrearrangement-invariant but not essentially rearrangement-invariant.

The paper is organized as follows. In §2 we develop the notion of an \(\text{MO}\)-isomorphism, and then in §3 we use this concept to define subrearrangement-invariance for function spaces on infinite measure spaces. An example is then given to show that not all function spaces on \((0, \infty)\) are essentially subrearrangement-invariant. §4 is devoted to the exhibition of a function space on \((0, \infty)\), generalizing Garling’s construction, which is subrearrangement-invariant but not essentially rearrangement-invariant. Some geometric properties of these “Garling function spaces” are then explored in §5.

All Banach spaces and function spaces are taken over the real field \(\mathbb{R}\). If \(E \subseteq [-\infty, \infty]\) is Lebesgue-measurable, then \(\lambda\) shall denote the usual Lebesgue measure on \(E\) and \(\Lambda\) the respective \(\sigma\)-algebra of Lebesgue-measurable subsets. If \(I \subseteq [-\infty, \infty]\) is an interval, denote by \(\tau\) the usual metric topology on \(I\), and let \(\mathcal{B} = \sigma(\tau)\) denote its Borel \(\sigma\)-algebra. We denote by \(\beta\) the Borel measure on \(I\), i.e. the restriction of \(\lambda\) to \(\mathcal{B}\). All measure spaces we assume to be countably additive and \(\sigma\)-finite. For subsets \(A\) and \(B\) of the same totally-ordered superset, we write \(A < B\) whenever \(a < b\) for all \(a \in A\) and \(b \in B\). If \(F\) is a subset of some set \(E\) then \(1_{F} : E \to \{0, 1\}\) is the indicator function for \(F\). If \(\theta\) and \(\phi\) are real-valued functions, we use the symbolism \(\theta(x) \approx_{\varepsilon} \phi(y)\) to mean that for any \(\varepsilon > 0\), the arguments \(x\) and \(y\) can be chosen such that

\[
\phi(y) - \varepsilon < \theta(x) < \phi(y) + \varepsilon.
\]

Beyond that, all notation and terminology is either standard (such as appears, for instance, in \([LT77]\)) or defined as encountered.

2. \(\text{MO}\)-isomorphisms

The concept of rearrangement-invariance is based in large part on the theory of measure-preserving transformations and equimeasurability. Subsymmetry for Schauder bases, on the other hand, requires also preservation of order. To define subrearrangement-invariance, then, we will first need to impose an order structure on the underlying measure space. So it is that in this section, we concern ourselves with invertible maps between totally-ordered measure spaces which preserve both measure and order, and in both “directions.” Let us begin with some preliminaries.

\textbf{Definition 2.1.} Let \((E, \mu_{E})\) and \((F, \mu_{F})\) be measure spaces. A map \(m : E \to F\) is called a \textbf{measure-preserving transformation} if whenever \(A\) is a measurable subset of \(F\), the set \(m^{-1}(A)\) is measurable with \(\mu_{E}(m^{-1}(A)) = \mu_{F}(A)\). If furthermore \(m\) is bijective with \(m^{-1}\) also measure-preserving, we say that it is a \textbf{measure-isomorphism}.

When \(E\) and \(F\) are subspaces of \(\mathbb{R}\), the Lebesgue-to-Lebesgue measurability condition in the previous definition is stronger than the more common Lebesgue-to-Borel measurability condition. Let us make this clear in the following.
Definition 2.2. Let \((E, \mu_E)\) and \((F, \mu_F)\) be measure spaces. A map \(m : E \to F\) is called \((\mu_E, \mu_F)\)-measurable (or, when \(\mu_E\) and \(\mu_F\) are clear from context, simply, measurable) if whenever \(A\) is a measurable subset of \(F\), the set \(m^{-1}(A)\) is measurable in \(E\).

As mentioned above, we will also need the concept of equimeasurability.

Definition 2.3. Let \((\Omega, \mu)\) be a \(\sigma\)-finite measure space, and \(f : \Omega \to [-\infty, \infty]\) a \((\mu, \beta)\)-measurable function. The distribution function \(\text{dist}_f : [0, \infty) \to [0, \infty)\) of \(f\) is given by the rule
\[\text{dist}_f(s) = \mu\{x \in \Omega : |f(x)| > s\}.\]

Two such measurable functions \(f\) and \(g\) are said to be equimeasurable whenever \(\text{dist}_f = \text{dist}_g\). In this case we write \(f \sim g\).

Usually, maps between totally-ordered spaces which respect the order relations are called increasing or nondecreasing. However, it will be more appropriate for our purposes here to use the following, alternative terminology.

Definition 2.4. Let \(A\) and \(B\) be partially-ordered sets. We say that a map \(m : A \to B\) is order-preserving if and only if \(s \leq t\) in \(A\) implies \(m(s) \leq m(t)\) in \(B\). If furthermore \(m\) is bijective with \(m^{-1}\) also order-preserving, we say that it is an order-isomorphism.

Let’s now introduce the key concept for this section which we mentioned above.

Definition 2.5. If \(E\) and \(F\) are totally-ordered measure spaces, we denote by \(\text{MO}(E, F)\) the set of all maps \(m : E \to F\) such that \(m\) is both a measure-isomorphism and an order-isomorphism. Any such \(m \in \text{MO}(E, F)\) is called an \(\text{MO}\)-isomorphism between \(E\) and \(F\).

In other words, \(m : E \to F\) is an \(\text{MO}\)-isomorphism if and only if it is bijective, and \(m\) and \(m^{-1}\) are both measure-preserving and order-preserving. However, we can weaken these formal requirements somewhat if we like, per our next result.

Proposition 2.6. The following facts are both true.

(i) Suppose \((E, \mu_E)\) and \((F, \mu_F)\) are measure spaces and \(m : E \to F\) is a bijective measure-preserving transformation. For all measurable sets \(A \subseteq E\), if \(m(A)\) is also measurable then \(\mu_E(A) = \mu_F(m(A))\). In particular, if \(m\) has a measurable inverse then \(m^{-1} : F \to E\) is also measure-preserving.

(ii) If \(E\) and \(F\) are totally-ordered sets and \(m : E \to F\) is an order-preserving bijection, then \(m^{-1}\) is also order-preserving. In this case, they are both strictly order-preserving in the sense that \(x < y\) in \(E\) if and only if \(m(x) < m(y)\) in \(F\).

Proof. (i) We have
\[\mu_F[m(A)] = \mu_E(m^{-1}[m(A)]) = \mu_E(A).\]

(ii) Let \(s \leq t \in F\). If \(m^{-1}(s) \leq m^{-1}(t)\) then we are done. In case \(m^{-1}(t) \leq m^{-1}(s)\), we have
\[t = m(m^{-1}(t)) \leq m(m^{-1}(s)) = s,\]
and hence $s = t$. Then $m^{-1}(s) \leq m^{-1}(t)$ anyway. Strictness is now obvious from the fact that $m$ is bijective.  

We can improve the previous proposition in the special case where $E$ and $F$ are Lebesgue-measurable subspaces of $\mathbb{R}$ by eliminating the measurability condition on $m^{-1}$.

**Theorem 2.7.** Let $E$ and $F$ be Lebesgue-measurable subsets of $\mathbb{R}$, and let $m : E \to F$ be a bijection which is both order-preserving and measure-preserving. Then $m^{-1}$ is also order-preserving and measure-preserving, i.e. $m \in \mathcal{MO}(E, F)$.

**Proof.** By Proposition 2.6 it is enough to show that $m^{-1}$ is measurable. To that end, let us fix a measurable set $A \subseteq E$; we claim that $m(A)$ is also measurable, which will complete the proof.

Denote by $\mathcal{B} = \sigma(\tau)$ the Borel $\sigma$-algebra on $\mathbb{R}$, where $\tau$ denotes the usual metric topology $\mathbb{R}$. Let $\tau_E$ be the subspace topology on $E$, i.e. the topology defined by

$$\tau_E = E \cap \tau := \{E \cap U : U \in \tau\}.$$

Similarly, we denote by $\tau_F$ the subspace topology for $F$. It is well-known (and easy to see) that the set

$$E \cap \mathcal{B} := \{E \cap B : B \in \mathcal{B}\}$$

is a $\sigma$-algebra on $E$, called the *trace* $\sigma$-algebra. Since $E \cap \tau \subseteq E \cap \mathcal{B}$, we obtain

$$\sigma(\tau_E) = \sigma(E \cap \tau) \subseteq \sigma(E \cap \mathcal{B}) = E \cap \mathcal{B}.$$  

For the reverse inclusion, define

$$\Sigma := \{Y \subseteq \mathbb{R} : E \cap Y \in \sigma(\tau_E)\}.$$  

It is routine to verify that $\Sigma$ is a $\sigma$-algebra on $\mathbb{R}$. Also, it is clear that $\tau \subseteq \Sigma$, since for $U \in \tau$ we have $E \cap U \in \tau_E \subseteq \sigma(\tau_E)$. It follows that $\sigma(\tau) \subseteq \Sigma$, whence also by definition of $\Sigma$ we obtain $E \cap \sigma(\tau) \subseteq \sigma(\tau_E)$. This gives us the reverse inclusion as desired. We now have the identity

$$\sigma(\tau_E) = E \cap \mathcal{B},$$

and an identical argument shows that

$$\sigma(\tau_F) = F \cap \mathcal{B}.$$  

It’s a well-known fact in real analysis (cf. [RF10, Theorem 2.11(ii)]) that we can find $C \in \mathcal{B}$ such that $A \subseteq C$ and $\lambda(C \setminus A) = 0$. Now set $C' = E \cap C \in \sigma(\tau_E)$. Since $\lambda(C \setminus A) = 0$ there is a measure-zero set $D \in \mathcal{B}$ with $C' \setminus A \subseteq C \setminus A \subseteq D$. Set $D' := E \cap D \in \sigma(\tau_E)$ so that $C' \setminus A \subseteq D'$ and $\lambda(D') = 0$.

We claim that $m(B) \in \sigma(\tau_F)$ for all $B \in \sigma(\tau_E)$. To prove it, we follow the argument in [Bo07, Theorem 2.1.2]. Denote by $\mathcal{E}$ the collection of all sets $B \in \sigma(\tau_E)$ such that $m(B) \in \sigma(\tau_F)$. We show that $\mathcal{E}$ is a $\sigma$-algebra. Indeed, it is clear that $F = m(E) \in \sigma(\tau_F)$ so that $E \in \mathcal{E}$. Now, if $B_n \in \mathcal{E}$ then

$$m\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} m(B_n) \in \sigma(\tau_F), \quad \text{and} \quad m(E \setminus B_n) = F \setminus m(B_n) \in \sigma(\tau_F).$$
Observe that if \( c \in \mathbb{R} \) then \( (-\infty, c) \cap E \in \sigma(\tau_E) \), and since \( m \) is order-preserving then \( m[(\infty, c) \cap E] \) is an initial segment of \( F \). Such initial segments are either open or closed in the subspace topology, which means in particular that \( m[(-\infty, c)] \in \sigma(\tau_F) \). It follows that all sets of the form \( E \cap (-\infty, c) \) lie in \( \mathcal{E} \), and an analogous argument shows that all sets of the form \( E \cap (c, \infty) \) lie in \( \mathcal{E} \) as well. Now, it is well-known (and easy to see) that if \( S \) is any subbase for \( \tau \) then \( E \cap S \) is a subbase for \( \tau_E \). Since \( \mathcal{E} \) contains the rays \( E \cap (-\infty, c) \) and \( E \cap (c, \infty) \), and since those sets form a subbase for \( \tau_E \), we obtain that \( \sigma(\tau_E) \subseteq \mathcal{E} \subseteq \sigma(\tau_E) \), whence \( \sigma(\tau_E) = \mathcal{E} \).

In particular, we now have that \( m(B) \) is Lebesgue-measurable whenever \( B \in \sigma(\tau_E) \). Thus, along with Proposition 2.6, we have \( \lambda[m(D')] = 0 \). Observe

\[
m(C') \setminus m(A) = m(C' \setminus A) \subseteq m(D')
\]

so that (since subsets of measure-zero sets are themselves measure-zero) \( \lambda[m(C') \setminus m(A)] = 0 \) as well. Note also that since \( C' \in \sigma(\tau_E) \) we have \( m(C') \) measurable. Since \( A \subseteq C' \), we obtain

\[
m(A) = m(C') \setminus [m(C') \setminus m(A)]
\]

which shows that \( m(A) \) is measurable. \( \square \)

**Remark 2.8.** The previous result also holds if \( E \) and \( F \) are measurable subsets of \([-\infty, \infty]\) instead of just \( \mathbb{R} \). Indeed, if \(-\infty \) or \( \infty \) lie in \( E \), then we can cut them out to form the set \( E' \), and then also cut out \( m(-\infty) \) and \( m(\infty) \) from \( F \) as needed to form \( F' \). If necessary, we can also cut out \(-\infty \) and \( \infty \) from \( F' \) and \( m^{-1}(-\infty) \) and \( m^{-1}(\infty) \) from \( E' \), and relabel. The restriction \( m' \) of \( m \) to \( E' \) is now a bijection between \( E' \) and \( F' \) which is order-preserving and measure-preserving. Thus \( m' \) is an \( \mathfrak{MO} \)-isomorphism by Theorem 2.7. It follows that \( m \) is an \( \mathfrak{MO} \)-isomorphism as well.

**Proposition 2.9.** Let \( E \) and \( F \) be Lebesgue-measurable subspaces of \([-\infty, \infty]\), and let \( m : F \to E \) be a surjective measure-preserving transformation which is also order-preserving. Then there is a measure-zero subset \( F_0 \) of \( F \) such that \( m \) is a bijection between \( F \setminus F_0 \) and \( E \).

**Proof.** For each \( x \in E \), let \( I_x \) be an interval containing \( m^{-1}\{x\} \) which is minimal under the relation \( \subseteq \). Since \( m \) is order-preserving, the \( I_x \)'s are all disjoint, which means only countably many of them have positive measure. In particular, \( m^{-1}\{x\} \) is a singleton for all but countably many \( x \in E \). Set

\[
E_0 := \{ x \in E : m^{-1}\{x\} \text{ is not a singleton} \}
\]

For each \( x \in E_0 \), select some \( f_x \in m^{-1}\{x\} \). Now set

\[
F_0 := \bigcup_{x \in E_0} \left( m^{-1}\{x\} \setminus \{f_x\} \right)
\]

Clearly, \( m \) is a bijection between \( F \setminus F_0 \) and \( E \). Observe that each \( m^{-1}\{x\} \setminus \{f_x\} \) has measure zero and that \( E_0 \) is countable. It follows that \( F_0 \) has measure zero. \( \square \)
Definition 2.10. Let \( E \) be a totally-ordered set. An initial segment of \( E \) is any subset of \( E' \) of \( E \) such that \( E' < E \setminus E' \).

Remark 2.11. When \( E \) is a subset of \([ -\infty, \infty ] \), initial segments always have the form \( E \cap [ -\infty, x ] \) or \( E \cap [ -\infty, x ] \) for some \( x \in [ -\infty, \infty ] \).

The following is surely known, but we provide a proof for completeness.

Proposition 2.12. Let \( E \) be a Lebesgue-measurable subset of \([ -\infty, \infty ] \) with \( \lambda(E) < \infty \). Then for each \( t \in [ 0, \lambda(E) ] \) there is an initial segment \( E_t \) of \( E \) such that \( \lambda(E_t) = t \).

Proof. Note that if \( E_t \) is an initial segment of \( E \setminus \{ -\infty, \infty \} \) then \( E_t \cup \{ -\infty \} \) is an initial segment of \( E \) with the same measure as \( E_t \). Hence, without loss of generality, we may assume \( E \subseteq \mathbb{R} \).

Consider the case where \( E \) is bounded, say \( E \subseteq [ a, b ] \) for \( -\infty < a < b < \infty \). Define \( f : [ a, b ] \rightarrow [ 0, \lambda(E) ] \) by the rule

\[
f(x) = \lambda([ a, x ] \cap E).
\]

Observe that if \( y < x \in [ a, b ] \) then

\[
| f(x) - f(y) | = \lambda((y, x] \cap E) \leq | x - y |
\]

so that \( f \) is Lipschitz, in particular, continuous. As \( f(a) = 0 \) and \( f(b) = \lambda(E) \), we may now apply the Intermediate Value Theorem.

Next we consider the case where \( E \) is unbounded. Write

\[
E_n = [ -n, -n + 1 ) \cap E \cap ( n - 1, n ]
\]

and observe that

\[
E \setminus \{ 0 \} = \bigcup_{n=1}^{\infty} E_n.
\]

the union of disjoint measurable sets. By countable additivity, we now have

\[
\lambda(E) = \lambda(E \setminus \{ 0 \}) = \sum_{n=1}^{\infty} \lambda(E_n).
\]

Now fix \( t \in [ 0, \lambda(E) ] \). We will find an initial segment \( E_t \) of \( E \) with \( \lambda(E_t) = t \). If \( t = 0 \) just take \( E_t = \emptyset \). Otherwise, by the series identity above, we can find \( N \in \mathbb{N} \) with

\[
\sum_{n=N+1}^{\infty} \lambda(E_n) < t.
\]

This means in particular that

\[
\delta := \lambda(E \cap ( -\infty, -N ] ) < t.
\]

Also using the series identity above in a similar manner, we can find \( M \in \mathbb{N} \) so that

\[
\lambda(E \cap ( -N, M ] ) > t - \delta.
\]

Since \( E \cap ( -N, M ] \) is bounded, we can now find an initial segment with measure \( t - \delta \). Take its union with \( E \cap ( -\infty, -N ] \) to get an initial segment of \( E \) with measure \( t \).

\( \square \)
Remark 2.13. If $E$ is a subset of $[0, \infty]$ instead of just $[0, n]$, we may dispense with the condition that $\lambda(E) < K$. Indeed, in this case, $\lambda(E \cap [0, n]) \to K$ by countable additivity. Fix $t \in [0, \infty]$. If $t = \infty$ we can just take $E_t = E$. Otherwise there is $N \in \mathbb{N}$ such that $\lambda(E \cap [0, N]) > t$. Since an initial segment of $E \cap [0, N]$ is also an initial segment of $E$ in this case, we can apply Proposition 2.12.

By Proposition 2.9, if $E \subseteq [-\infty, \infty]$ is measurable then we can find a measure-zero subset $E_0$ of $E$ such that a measure-preserving and order-preserving transformation from $E$ onto some other measurable subset $F \subseteq [-\infty, \infty]$ can be turned into a $\mathbb{M\mathcal{O}}$-isomorphism by restricting it to $E \setminus E_0$. In case $\lambda(E) < K$, we would like to take $F = [0, \lambda(E)]$, but this is not always possible. For instance, there is no $\mathbb{M\mathcal{O}}$-isomorphism between $[0, 1) \cup (1, 2]$ and $[0, 2]$. However, in our next result we show that we can do it up to a set of measure zero, i.e. by replacing $[0, \lambda(E)]$ with $[0, \lambda(E)] \setminus D_0$ for some measure-zero set $D_0$.

Let us give a brief preliminary.

Lemma 2.14. Let $E$ be a measurable subset of $\mathbb{R}$ with $\lambda(E) < K$. For each $t \in [0, \lambda(E)]$, let $E_t$ be an initial segment of $E$ (whose existence is guaranteed by Proposition 2.12). Define the map $m : E \to [0, \lambda(E)]$ by the rule

$$m(x) = \inf\{t \in [0, \lambda(E)] : x \in E_t\}.$$  

Then $m$ is both measure-preserving and order-preserving. Furthermore, $m$ can be extended to a map $m : \mathbb{R} \to [0, \lambda(E)]$ defined by

$$m(x) = \lambda((-\infty, x] \cap E).$$

Proof. It is obvious that $m$ is order-preserving, and it is explicitly proved in [BS88, Proposition 2.7.4] that it is also measure-preserving.

For $x \in E$ we set $E_{\leq x} := (-\infty, x] \cap E$, and observe that if $t > \lambda(E_{\leq x})$ then $x \in E_t$ and if $t < \lambda(E_{\leq x})$ then $x \notin E_t$. It follows that $m(x) = \lambda(E_{\leq x})$ for all $x \in E$. Thus, we can extend $m$ continuously to the function $M : \mathbb{R} \to [0, \lambda(E)]$ via the rule

$$M(x) = \lambda((-\infty, x] \cap E).$$

□

Theorem 2.15. Let $E$ be a measurable subset of $[-\infty, \infty]$ with $\lambda(E) < K$. Then there is a measure-zero subset $E_0$ of $E$, a measure-zero subset $D_0$ of $[0, \lambda(E)]$, and an $\mathbb{M\mathcal{O}}$-isomorphism between $E \setminus E_0$ and $[0, \lambda(E)] \setminus D_0$.

Proof. Since $\{-\infty, \infty\}$ has measure zero, we may assume without loss of generality that $E \subseteq \mathbb{R}$.

Let $m : \mathbb{R} \to E$ be as in Lemma 2.14. It is clear (as in, for instance, the proof of Proposition 2.12) that $m$ is Lipschitz, and hence continuous in the usual sense as well.

Since $\lambda$ is inner-regular, we can find a sequence $(K_n)_{n=1}^\infty$ of compact sets and a measure-zero set $L$ such that

$$E = L \cup \bigcup_{n=1}^\infty K_n.$$
It is known that the image of a bounded measure-zero set under a Lipschitz function is again measure-zero. Furthermore, the continuous image of a compact set is again compact, and in particular measurable. We now have

\[ m(E) = m\left(L \cup \bigcup_{n=1}^{\infty} K_n\right) = m(L) \cup \bigcup_{n=1}^{\infty} m(K_n). \]

It follows that \( m(E) \) is measurable.

We can now apply Proposition 2.9 to find a subset \( E_0 \) of measure zero such that \( m \) is a bijection between \( E \setminus E_0 \) and \( m(E) \). Set \( D_0 = [0, \lambda(E)] \setminus m(E) \).

By Proposition 2.6, \( \lambda(D_0) = 0 \), and by Theorem 2.7, \( m \) is an \( \mathcal{M}(\Omega) \)-isomorphism between \( E \setminus E_0 \) and \( [0, \lambda(E)] \setminus D_0 \).

\[ \square \]

3. Subrearrangement-invariant function spaces

For the following definition, recall that \( \beta \) denotes the Borel measure.

**Definition 3.1.** Let \( (\Omega, \mu) \) be a \( \sigma \)-finite measure space, and let \( \mathcal{M}_0^+(\Omega) \) denote the cone of (nonnegative) \((\mu, \beta)\)-measurable functions \( f: \Omega \to [0, \infty] \). Suppose \( \rho: \mathcal{M}_0^+(\Omega) \to [0, \infty] \) satisfies the following properties for all \( a \in (0, \infty) \) and all \( f, g \in \mathcal{M}_0^+(\Omega) \):

(i) \( \rho(f + g) \leq \rho(f) + \rho(g) \);
(ii) \( \rho(af) = a\rho(f) \); and
(iii) \( \rho(f) = 0 \) if and only if \( f \equiv 0 \) almost everywhere.

We can then define a normed linear space \( (X, \| \cdot \|_X) \) consisting the a.e.-equivalence classes of measurable functions \( f: \Omega \to [-\infty, \infty] \) satisfying \( \| f \|_X := \rho(|f|) < \infty \). In this case we say that \( \rho \) is a function norm on \( \Omega \), and \( X \) is a function space on \( \Omega \) with respect to \( \rho \).

Note that our definition differs from other classes of function spaces such as Banach function spaces defined in \cite{BS88} or Köthe function spaces. It is suitable for the present purposes, however.

**Remark 3.2.** If \( (e_i)_{i=1}^{\infty} \) is a 1-unconditional basis for a Banach space \( X \), we can define a function norm \( \rho_X \) by setting, for all \( f: \mathbb{N} \to [0, \infty] \),

\[ \rho_X(f) = \begin{cases} \left\| \sum_{i=1}^{\infty} f(i)e_i \right\|_X & \text{if } \sum_{i=1}^{\infty} f(i)e_i \text{ converges, and} \\ \infty & \text{otherwise.} \end{cases} \]

In this way, \( X \) can be viewed as a function space on \( \mathbb{N} \), with respect to \( (e_i)_{i=1}^{\infty} \).

**Definition 3.3.** Let \( (\Omega, \mu) \) be a \( \sigma \)-finite measure space. A function space \( X \) on \( \Omega \) is called rearrangement-invariant iff \( \| f \|_X = \| g \|_X \) for all equimeasurable functions \( f, g \in X \). It is essentially rearrangement-invariant provided it admits an equivalent rearrangement-invariant norm.

**Remark 3.4.** The above definition follows \cite{BS88} rather than the somewhat more stilted definition of rearrangement-invariance found, for instance, in \cite{LT79}.
We will need the following before we go on. It is a somewhat obvious fact, but we provide a proof for completeness.

**Proposition 3.5.** If \( f, g : \mathbb{N} \to [0, \infty) \) are equimeasurable with

\[
\lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = 0
\]

then either they are both identically zero or else there is a measure-isomorphism \( m : \text{supp}(f) \to \text{supp}(g) \) such that \( g \circ m = f \) on \( \text{supp}(f) \).

**Proof.** Obviously, if one of \( f \) and \( g \) is identically zero then, since they are equimeasurable, so is the other. So let us assume that neither is identically zero.

We are going to proceed with an inductive process, described as follows. Note that since \( f(n) \to 0 \) and \( f \) is not identically zero there are

\[
M_1 := \max_{n \in \mathbb{N}} f(n) \in (0, \infty) \quad \text{and} \quad S_1 := \{ n \in \mathbb{N} : f(n) = M_1 \}.
\]

If \( S_1 = \text{supp}(f) \) we halt the process. Otherwise we set

\[
M_2 := \max_{n \in \mathbb{N} \setminus S_1} f(n) \in (0, M_1) \quad \text{and} \quad S_2 := \{ n \in \mathbb{N} : f(n) = M_2 \}.
\]

If \( S_1 \cup S_2 = \text{supp}(f) \) we halt the process. Otherwise we set

\[
M_3 := \max_{n \in \mathbb{N} \setminus (S_1 \cup S_2)} f(n) \in (0, M_2) \quad \text{and} \quad S_3 := \{ n \in \mathbb{N} : f(n) = M_3 \}.
\]

Continue in this way to obtain a (possibly infinite) strictly decreasing sequence \((M_i)_{i=1}^N \subset (0, \infty)\), where \( N \in \mathbb{N} \cup \{\infty\} \), and a corresponding sequence of disjoint finite sets \((S_i)_{i=1}^N\). Observe that since \( f(n) \to 0 \) we must have \( \text{supp}(f) = \bigcup_{i=1}^N S_i \).

Let us perform the same procedure with \( g \), producing a strictly decreasing sequence \((M'_i)_{i=1}^{N'} \subset (0, \infty)\) and corresponding sequence of disjoint finite sets \((S'_i)_{i=1}^{N'}\).

Recall that \( \text{dist}_f(s) = \#\{ n \in \mathbb{N} : |f(n)| > s \} \). Then

\[
\#\text{supp}(f) = \text{dist}_f(0) = \text{dist}_g(0) = \#\text{supp}(g).
\]

Next, we claim that \( N = N' \), and that \( M_i = M'_i \) and \( \#S_i = \#S'_i \) for all \( i \). For proof, we shall perform another inductive procedure. Observe that \( \text{dist}_f(s) = 0 \) if and only if \( s \in [M_1, \infty) \). Also, \( \text{dist}_g(s) = 0 \) if and only if \( s \in [M'_1, \infty) \). Since \( \text{dist}_f(s) = 0 \) if and only if \( \text{dist}_g(s) = 0 \), it follows that \( M_1 = M'_1 \). Select \( s_1 \in [M_2 \vee M'_2, M_1) \) so that

\[
\#S_1 = \text{dist}_f(s_1) = \text{dist}_g(s_1) = \#S'_1.
\]

If \( S_1 = \text{supp}(f) \) we halt the procedure. In this case, since \( \#\text{supp}(f) = \#\text{supp}(g) \), we must have \( S'_1 = \text{supp}(g) \) so that \( N = N' \). Otherwise, observe that \( \text{dist}_f(s) = \#S_1 \) if and only if \( s \in [M_2, M_1) \) and \( \text{dist}_g(s) = \#S'_1 \) if and only if \( s \in [M'_2, M'_1) \).

Since \( \#S_1 = \#S'_1 \) and \( M_1 = M'_1 \), this means \( M_2 = M'_2 \). Select \( s_2 \in [M_3 \vee M'_3, M_2) \) so that

\[
\#(S_1 \cup S_2) = \text{dist}_f(s_2) = \text{dist}_g(s_2) = \#(S'_1 \cup S'_2).
\]

Since \( S_1 \) and \( S_2 \) are disjoint finite sets, and so are \( S'_1 \) and \( S'_2 \), with \( \#S_1 = \#S'_1 \), we must have \( \#S_2 = \#S'_2 \). If \( \text{supp}(f) = S_1 \cup S_2 \) we halt the procedure. In this
Proposition 3.7. A 1-unconditional basis \((e_i)_{i=1}^\infty\) for a real Banach space \(X\) is 1-symmetric if and only if \(X\) is rearrangement-invariant as a function space on \(\mathbb{N}\) with respect to \((e_i)_{i=1}^\infty\).

Proof. (⇒): Let \((e_i)_{i=1}^\infty\) be 1-symmetric, and suppose \(f\) and \(g\) are equimeasurable sequences in \(X\). Then \(\|f\|\) and \(\|g\|\). If \(f\) and \(g\) are identically zero then \(\|f\|_X = 0 = \|g\|_X\) and we are done. Otherwise by Proposition 3.5 there is a bijection \(m : \text{supp}(f) \to \text{supp}(g)\) with \(f = g \circ m\) on \(\text{supp}(f)\). Now we have, by 1-symmetry and 1-unconditionality

\[
\|f\|_X = \left\| \sum_{i=1}^{\infty} f(i)e_i \right\|_X = \left\| \sum_{i \in \text{supp}(f)} |f(i)|e_i \right\|_X = \left\| \sum_{i \in \text{supp}(g)} |g(m(i))|e_i \right\|_X = \left\| \sum_{i \in \text{supp}(g)} |g(i)|e_i \right\|_X = \left\| \sum_{i=1}^{\infty} g(i)e_i \right\|_X = \|g\|_X.
\]

(⇐): Suppose that \(X\) is rearrangement-invariant with respect to \((e_i)_{i=1}^\infty\), and select a permutation \(\pi\) of \(\mathbb{N}\). Then its inverse \(\pi^{-1}\) exists and is a measure-preserving transformation. Select any \(f \in X\), and note that \(|f(i)| < \infty\) for all
\( i \in \mathbb{N} \). By Proposition 3.6, \(|f|\) and \(|f| \circ \pi^{-1}\) are equimeasurable. Now we have, by 1-unconditionality and rearrangement-invariance

\[
\left\| \sum_{i=1}^{\infty} f(i)e_{\pi(i)} \right\|_X = \left\| \sum_{i=1}^{\infty} |f(i)|e_{\pi(i)} \right\|_X \\
= \left\| \sum_{i=1}^{\infty} (|f| \circ \pi^{-1})(i)e_i \right\|_X \\
= \| |f| \circ \pi^{-1} \|_X \\
= \| |f| \|_X \\
= \left\| \sum_{i=1}^{\infty} |f(i)|e_i \right\|_X \\
= \left\| \sum_{i=1}^{\infty} f(i)e_i \right\|_X.
\]

\[\Box\]

At last, we are ready to introduce the main subject under study.

**Definition 3.8.** Let \((\Omega, \mu)\) be a totally-ordered \(\sigma\)-finite measure space satisfying \(\mu(\Omega) = \infty\). We say that a function space \(X\) on \(\Omega\) is subrearrangement-invariant if for every measurable \(F \subseteq \Omega\), every \(m \in \mathcal{MO}(\Omega, F)\), and every \(f \in X\), we have \(\|f \circ m\|_X = \|f1_F\|_X\). We say that \(X\) is essentially subrearrangement-invariant whenever it admits an equivalent subrearrangement-invariant norm.

Here, the restriction \(\mu(\Omega) = \infty\) has been included since \(\mathcal{MO}(\Omega, F)\) would be empty otherwise, whenever \(\mu(F) \neq \mu(\Omega)\), and that would make every function space on \(\Omega\) trivially subrearrangement-invariant.

The following result in some sense justifies our definition of subrearrangement-invariance.

**Proposition 3.9.** A 1-unconditional basis \((e_i)_{i=1}^{\infty}\) for a real Banach space \(X\) is 1-subsymmetric if and only if \(X\) is subrearrangement-invariant as a function space on \(\mathbb{N}\) with respect to \((e_i)_{i=1}^{\infty}\).

**Proof.** (\(\Rightarrow\)): Suppose \((e_i)_{i=1}^{\infty}\) is 1-subsymmetric. Let \(F \subseteq \mathbb{N}\) and \(m \in \mathcal{MO}(\mathbb{N}, F)\), and select any \(f \in X\). By 1-subsymmetry of \((e_i)_{i=1}^{\infty}\) we have

\[
\|f \circ m\|_X = \left\| \sum_{i=1}^{\infty} (f(m(i)))e_i \right\|_X \\
= \left\| \sum_{i \in F} f(i)e_{m^{-1}(i)} \right\|_X \\
= \left\| \sum_{i \in F} f(i)1_F(i)e_{m^{-1}(i)} \right\|_X.
\]
\[ \sum_{i=1}^{\infty} f(i) 1_F(i) e_i \]
\[ \| f 1_F \|_X \]

Hence, \( X \) is subrearrangement-invariant with respect to \((e_i)_{i=1}^{\infty}\).

\( \Leftarrow \): Suppose now that \( X \) is subrearrangement-invariant with respect to \((e_i)_{i=1}^{\infty}\). Let \((e_{i_k})_{k=1}^{\infty}\) be a subsequence and \( f \in X \). Define \( m(k) = i_k \) for \( k \in \mathbb{N} \), and \( F := (i_k)_{k=1}^{\infty} \). Clearly, \( m \in \mathbb{M}(\mathbb{N}, F) \). Define \( g : \mathbb{N} \to [0, \infty] \) by letting \( g(i) = (|f| \circ m^{-1})(i) \) if \( i \in F \) and \( g(i) = 0 \) otherwise. We will need to check that \( g \in X \), but this follows from the facts below, together with the identity \( g 1_F = g \).

Now, by subrearrangement-invariance and \( 1 \)-unconditionality we have

\[ \left\| \sum_{k=1}^{\infty} f(k) e_{i_k} \right\|_X = \left\| \sum_{k=1}^{\infty} (f \circ m^{-1})(i_k) e_{i_k} \right\|_X \]
\[ = \left\| \sum_{i \in F} (f \circ m^{-1})(i) e_i \right\|_X \]
\[ = \left\| \sum_{i=1}^{\infty} g(i) 1_F(i) e_i \right\|_X \]
\[ = \left\| g 1_F \right\|_X \]
\[ = \left\| g \circ m \right\|_X \]
\[ = \left\| f \circ m^{-1} \circ m \right\|_X \]
\[ = \left\| f \right\|_X \]
\[ = \left\| \sum_{i=1}^{\infty} f(i) e_i \right\|_X \].

\( \square \)

It is well-known that every \( 1 \)-symmetric basis is \( 1 \)-subsymmetric. Similarly, it is easy to show that rearrangement-invariance implies subrearrangement-invariance.

**Proposition 3.10.** Let \((\Omega, \mu)\) be a totally-ordered \( \sigma \)-finite measure space satisfying \( \mu(\Omega) = \infty \). If \( X \) is a rearrangement-invariant function space on \( \Omega \), then it is also subrearrangement-invariant.

*Proof.* Select any \( f \in X \), measurable \( F \subseteq \Omega \), and \( m \in \mathbb{M}(\Omega, F) \). Notice that \( f|_F \circ m = f \circ m \) so that, by Proposition 3.6, \( f \circ m \sim f|_F \). We also clearly have \( f 1_F \sim f|_F \), and hence \( f \circ m \sim f 1_F \). By rearrangement-invariance this means \( \| f \circ m \|_X = \| f 1_F \|_X \).

Let us close this section by discussing the nontriviality of essential-subrearrangement invariance. There are, after all, well-known examples of \( 1 \)-unconditional bases which are not subsymmetric under any renorming, for instance the basis for the Tsirelson space. This furnishes us with examples of function spaces on \( \mathbb{N} \) which are not essentially subrearrangement-invariant. The
following example gives us a function space on the purely nonatomic measure space $(0, \infty)$ which fails to be essentially subarrangement-invariant. Note that it can be viewed as a kind of “spinoff” of the Schreier sequence space.

**Example 3.11.** Denote by $\mathcal{A}$ the family of all subsets $A$ of $(0, \infty)$ satisfying $\lambda(A) \leq \sqrt{\text{inf} A}$. For a nonnegative $(\lambda, \beta)$-measurable function $f : (0, \infty) \to [0, \infty]$, we set

$$\rho_Y(f) = \sup_{A \in \mathcal{A}} \int_A f(t) \, dt.$$ 

Then $\rho_Y$ is a function norm, and we can denote by $Y$ the function space it generates. Furthermore, $Y$ is a Banach space which fails to be essentially subarrangement-invariant.

**Proof.** That $\rho_Y$ is a function norm is clear from the definition.

Let us show that $Y$ is complete. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $Y$. As such, we can find $M \in (0, \infty)$ with $\|f_n\|_Y \leq M$ for all $n \in \mathbb{N}$.

We claim that there is a $(\lambda, \beta)$-measurable function $f : (0, \infty) \to [-\infty, \infty]$ such that $f_n \to f$ a.e.-pointwise. To see this, select any $\varepsilon \in (0, 1)$, and for each $k \in \mathbb{N}$, set $I_k := k \varepsilon + (0, \varepsilon]$. Observe that $I_1 < I_2 < I_3 < \cdots$ is a sequence of disjoint intervals satisfying $I_k \in \mathcal{A}$ for each $k \in \mathbb{N}$, and $\bigcup_{k=1}^\infty I_k = (\varepsilon, \infty)$. Now for each $k, m, n \in \mathbb{N}$ we have

$$\|f_m|_{I_k} - f_n|_{I_k}\|_{L_1(I_k)} \leq \|f_m - f_n\|_Y$$

so that $(f_n|_{I_k})_{n=1}^\infty$ is Cauchy in $L_1(I_k)$. Hence, there are measurable $f^{(k)} : I_k \to [-\infty, \infty]$ such that $f_n|_{I_k} \to f^{(k)}$ a.e.-pointwise. Now we write $f_\varepsilon := \bigcup_{k=1}^\infty f^{(k)}$ so that $f_\varepsilon : (\varepsilon, \infty) \to [-\infty, \infty]$ is measurable with $f_n|_{(\varepsilon, \infty)} \to f_\varepsilon$ a.e.-pointwise. Since $\varepsilon \in (0, 1)$ was arbitrary, we can find a sequence $1 > \varepsilon_i \downarrow 0$ so that for each $i$ there is a measurable function $f_{\varepsilon_i} : (\varepsilon_i, \infty) \to [-\infty, \infty]$ such that $f_n|_{(\varepsilon_i, \infty)} \to f_{\varepsilon_i}$ a.e.-pointwise. It is clear that all the $f_{\varepsilon_i}$’s a.e-agree on their domains, and this gives us a measurable $f : (0, \infty) \to [-\infty, \infty]$ such that $f_n \to f$ a.e.-pointwise as claimed.

Fix any $A \in \mathcal{A}$, and note that $(f_n|_A)_{n=1}^\infty$ is a Cauchy sequence when viewed as functions in $L_1(A)$. For convenience, as an abuse of notation we will write $f_n|_A = f$ when context is clear. Hence, the sequence converges in $L_1(A)$ to some $g_A \in L_1(A)$, where $g_A$ must be a.e.-identical to $f$ on $A$. Find $n \in \mathbb{N}$ so that $\|g_A - f_n\|_{L_1(A)} \leq 1$. We now have

$$\int_A |f(t)| \, dt \leq \int_A |f - f_n|(t) \, dt + \int_A |f_n(t)| \, dt \leq \|g_A - f_n\|_{L_1(A)} + \|f_n\|_Y \leq 1 + M.$$ 

It follows that $\rho_Y(|f|) \leq 1 + M$ and hence $f \in Y$. Next, we select $\varepsilon > 0$ and find $N \in \mathbb{N}$ so that $\|f_\ell - f_n\|_Y < \varepsilon/2$ for all $\ell, n \geq N$. Select in particular $\ell \geq N$ so that $\|g_A - f_\ell\|_{L_1(A)} < \varepsilon/2$ as well. Then for $n \geq N$ we have

$$\int_A |f - f_n|(t) \, dt \leq \int_A |f - f_\ell|(t) \, dt + \int_A |f_\ell - f_n|(t) \, dt$$
\[ \|g_A - f_\ell\|_{L_1(A)} + \|f_\ell - f_n\|_Y < \varepsilon. \]

It follows that \( \|f - f_n\|_Y < \varepsilon \) for all \( n \geq N \), and hence that \( f_n \to f \) in \( Y \). Therefore, \( Y \) is complete.

Now we will show that it fails to be essentially subrearrangement-invariant. Select any \( b \in (0, \infty) \). When selecting \( A \in \mathcal{A} \) to estimate \( \|1_{(0,b]}\|_Y \), we may assume without loss of generality that \( \inf A \leq b \), else \( \int_A 1_{(0,b]}(t) \, dt = 0 \). Hence, \( \int_A 1_{(0,b]}(t) \, dt \leq \lambda(A) \leq \sqrt{\inf A} \leq \sqrt{b} \) so that \( \|1_{(0,b]}\|_Y \leq \sqrt{b} \). On the other hand, if \( c \geq \sqrt{b} \) then \( \|1_{(c,c+b]}\|_Y = b \).

It is clear that \( 1_{(c,c+b]} \odot m = 1_{(0,b]} \) for the shift map \( m \in \mathcal{M}\mathcal{O}((0, \infty), (c, \infty)) \) defined by \( m(t) = t + c \). Hence,

\[ \frac{\|1_{(c,c+b]}\|_Y}{\|1_{(c,c+b]} \odot m\|_Y} = \frac{\|1_{(c,c+b]}\|_Y}{\|1_{(0,b]}\|_Y} \geq \frac{b}{\sqrt{b}} = \sqrt{b}. \]

As \( b \in (0, \infty) \) was arbitrary, it follows that \( Y \) is not essentially subrearrangement-invariant. \( \square \)

4. Garling function spaces

The converse of Proposition 3.10 fails to hold in general, as can be seen from the following example. If \( 1 \leq p < \infty \) and \( w = (w(k))_{k=1}^\infty \) is a nonincreasing sequence of positive real numbers satisfying \( w \in c_0 \setminus \ell_1 \), then we can define the Garling sequence space \( g(w, p) \) as the space of all scalar sequences \( f : \mathbb{N} \to [-\infty, \infty] \) satisfying

\[ \|f\|_g := \sup_{(i(k))_{k=1}^\infty \in \mathbb{N}^\mathbb{N}} \left( \sum_{k=1}^\infty |f(i(k))|^p w(k) \right)^{1/p} < \infty, \]

where \( \mathbb{N}^\mathbb{N} \) denotes the family of all increasing sequences in \( \mathbb{N} \). (We usually also impose the condition that \( w(1) = 1 \) but this is not always necessary.) It is known from [AAW18, Proposition 2.4] and [AALW18, Lemma 3.1] that the unit vectors in \( g(w, p) \) form a 1-unconditional basis which is 1-subsymmetric but not symmetric. In particular, thusly viewed as a function space on \( \mathbb{N} \), by Propositions 3.7 and 3.9, it is subrearrangement-invariant but fails to be rearrangement-invariant, or even just essentially rearrangement-invariant. Nevertheless, it remains to be seen whether essential subrearrangement-invariance is a strictly weaker condition than essential rearrangement-invariance in the nonatomic setting. We devote this section, therefore, to exhibiting a function space on \( (0, \infty) \) which is subrearrangement-invariant but fails to be essentially rearrangement-invariant.

To accomplish this, we shall simply generalize Garling’s construction. In fact, we will use the very same “split into two sums” trick that Garling did in his original paper [Ga68, §5]. However, in order for this strategy to work, we need
to make some adaptations. Part of that will involve the using the measure-theoretic results from §2 of the present paper. Also, we need to characterize Garling sequence spaces slightly differently.

**Proposition 4.1.** Fix a nonincreasing function \( w : \mathbb{N} \to (0, \infty) \) with \( w \in c_0 \setminus \ell_1 \). For each function \( f : \mathbb{N} \to [0, \infty] \) we define

\[
\rho_g(f) = \sup_{E,F \subseteq \mathbb{N} \in \mathcal{MO}(E,F)} \left( \sum_{k \in E}(f \circ m(k))^p w(k) \right)^{1/p}.
\]

Then \( \rho_g \) is a function norm generating the space \( g(w, p) \).

**Proof.** Let \( (i(k))_{k=1}^{\infty} \in \mathbb{N}^\uparrow \). By taking \( E = \mathbb{N} \), \( F = (i(k))_{k=1}^{\infty} \), and \( m(k) = i(k) \), it is clear that \( \rho_g(f) \geq \|f\|_g \). For the reverse inequality, let \( E, F \subseteq \mathbb{N} \) and \( m \in \mathcal{MO}(E,F) \). We may assume without loss of generality that \( E \) and \( F \) are both infinite. Thus, there is a unique \( n \in \mathcal{MO}(\mathbb{N}, E) \), and this satisfies \( m \circ n \in \mathcal{MO}(\mathbb{N}, F) \). Since \( w \) is nonincreasing, we have

\[
\sum_{k \in E}(f \circ m(k))^p w(k) = \sum_{j=1}^{\infty}(f \circ m \circ n)(j)^p w(n(j)) \\
\leq \sum_{j=1}^{\infty}(f \circ m \circ n)(j)^p w(j) \\
\leq \|f\|^p_g.
\]

That \( \rho_g \) is a function norm generating \( g(w, p) \) follows immediately. \( \square \)

**Definition 4.2.** Let \( W \) denote the set of all nonincreasing \((\lambda, \beta)\)-measurable functions \( W : (0, \infty) \to (0, \infty) \) satisfying the following conditions:

(W1) \( \lim_{t \to \infty} W(t) = 0 \),

(W2) \( \int_{0}^{\infty} W(t) \, dt = \infty \), and

(W3) \( \int_{0}^{1} W(t) \, dt < \infty \).

For each \((\lambda, \beta)\)-measurable \( f : (0, \infty) \to [0, \infty] \), set

\[
\rho_G(f) = \sup_{E,F \in A \setminus \ell_1 \mathcal{M}(E,F)} \left( \int_{E}(f \circ m)(t)^p W(t) \, dt \right)^{1/p},
\]

where \( W \in W \) and \( 1 \leq p < \infty \). We then define a **Garling function space**, denoted \( G_{W,p}(0, \infty) \), as the space of all a.e.-equivalence classes of measurable functions \( f : (0, \infty) \to [-\infty, \infty] \) satisfying \( \|f\|_G := \rho_G(|f|) < \infty \).

**Remark 4.3.** Conditions (W1) and (W2) are the only ones we use in §4 and the proof of Proposition 5.1. However, for the other results in §5, condition (W3) is essential.

It is clear that \( \rho_G \) is a function norm, and hence \( G_{W,p}(0, \infty) \) is a function space on \((0, \infty)\). We will show later in §5 that it is in fact a Banach space, i.e. that it is complete.
Proposition 4.4. Fix $1 \leq p < \infty$, and let $W \in \mathcal{W}$. Then $G_{W,p}(0,\infty)$ is subrearrangement-invariant.

Proof. Fix $D \in \Lambda$ and $n \in \mathbb{MO}((0,\infty), D)$, and $f \in G_{W,p}(0,\infty)$. Observe that there are $E, F \in \Lambda$ and $m \in \mathbb{MO}(E, F)$ such that

$$
\|f 1_D\|_G^p \approx \varepsilon \int_E ((f 1_D) \circ m)(t)^p W(t) \, dt \\
= \int_E (f \circ m)(t)^p (1_D \circ m)(t) W(t) \, dt \\
= \int_{m^{-1}(D) \cap E} (f \circ m)(t)^p W(t) \, dt \\
\leq \int_{m^{-1}(D)} (f \circ m)(t)^p W(t) \, dt \\
= \int_{m^{-1}(D)} ((f \circ n) \circ (n^{-1} \circ m))(t)^p W(t) \, dt \\
\leq \|f \circ n\|_G^p,
$$

where the last inequality follows from the fact that $n^{-1} \circ m$ is an $\mathbb{MO}$-isomorphism from $m^{-1}(D)$ onto its image. On the other hand, there are $A, B \in \Lambda$ and $\ell \in \mathbb{MO}(A, B)$ such that

$$
\|f \circ n\|_G^p \approx \varepsilon \int_A (f \circ n \circ \ell)(t)^p W(t) \, dt \\
= \int_A (f \circ n \circ \ell)(t)^p (1_D \circ n \circ \ell)(t) W(t) \, dt \\
= \int_A ((f 1_D) \circ (n \circ \ell))(t)^p W(t) \, dt \\
\leq \|f 1_D\|_G^p,
$$

where the first equality follows due to the fact that $1_D \circ n \circ \ell$ is the identity function on $A$, and the final inequality follows from the fact that $n \circ \ell$ is an $\mathbb{MO}$-isomorphism from $A$ onto its image. \qed

To show that a Garling function space fails to admit an equivalent rearrangement-invariant norm, we need the following intuitively obvious lemma.

Lemma 4.5. Fix $p \in [1, \infty)$ and $r \in (0,\infty)$. Let $W \in \mathcal{W}$ and $f : (0,\infty) \to [0,\infty]$ a measurable function which is nondecreasing on $(0,r)$ and zero everywhere else. Then there is $s \in [0,r]$ so that

$$
\|f\|_G = \left( \int_0^r f(t+r-s)^p W(t) \, dt \right)^{1/p}
$$

Unfortunately, it requires a somewhat technical proof. We begin with some preliminaries.
Proposition 4.6 ([Bo07, Theorem 2.9.3]). Let $(\Omega, \mu)$ be a measure space and $f : \Omega \to [-\infty, \infty]$ a $(\mu, \beta)$-measurable function. Then the $\mu$-integrability of $f$ is equivalent to the Lebesgue integrability of the function $t \mapsto \text{dist}_f(t)$, and
\[
\int_\Omega |f| \, d\mu = \int_0^\infty \text{dist}_f(t) \, dt.
\]

Corollary 4.7. If $E$ and $F$ are measurable subsets of $(0, \infty)$, and $f : (0, \infty) \to [0, \infty]$ is a (nonnegative) $(\lambda, \beta)$-measurable function, then for any measure-preserving transformation $m : E \to F$ we have
\[
\int_E (f \circ m)(t) \, dt = \int_F f(t) \, dt.
\]

Proof. By Proposition 3.6, $f$ and $f \circ m$ are equimeasurable, which is to say that $\text{dist}_f = \text{dist}_{f \circ m}$. Now by Proposition 4.6 we have
\[
\int_E (f \circ m)(t) \, dt = \int_0^\infty \text{dist}_{f \circ m}(t) \, dt = \int_0^\infty \text{dist}_f(t) \, dt = \int_F f(t) \, dt.
\]

\[\Box\]

Proof of Lemma 4.5. First, observe that since the map
\[b \mapsto \int_0^b f(t + r - b)^p W(t) \, dt\]
is continuous on the compact set $[0, r]$, we can find $s \in [0, r]$ so that
\[
\int_0^s f(t + r - s)^p W(t) \, dt = \sup_{b \in [0, r]} \int_0^b f(t + r - b)^p W(t) \, dt. \tag{4.1}
\]

Let $E, F \in \Lambda$ and $m \in \mathcal{MO}(E, F)$ be such that
\[
\|f\|_G^p \approx \int_E (f \circ m)(t)^p W(t) \, dt. \tag{4.2}
\]

Without loss of generality we may assume that $F \subseteq (0, r)$, and set $b := \lambda(F) \leq r$. By Theorem 2.15 we can find measure-zero subsets $E_0$ of $E$ and $D_0$ of $(0, r)$, and an $\mathcal{MO}$-isomorphism
\[n : (0, b) \setminus D_0 \to E \setminus E_0.\]

We claim that
\[
(f \circ m \circ n)(t)^p \leq f(t + r - b)^p, \tag{4.3}
\]
or, equivalently, $b - t \leq r - (m \circ n)(t)$, for each $t \in (0, b) \setminus D_0$. Indeed, as $m \circ n$ is order-preserving, we have, for $c \in (t, b) \setminus D_0$,
\[
(m \circ n)((t, c) \setminus D_0) \subseteq [(m \circ n)(t), (m \circ n)(c)]
\]
and since $m \circ n$ is a measure isomorphism from $(0, b) \setminus D_0$ onto its image, we also have
\[
c - t = \lambda((t, c) \setminus D_0)
= \lambda((m \circ n)(((t, c) \setminus D_0)))
\]
\[ \begin{align*}
\leq & \lambda[(m \circ n)(t), (m \circ n)(c)] \\
= & (m \circ n)(c) - (m \circ n)(t).
\end{align*} \]

For \( \varepsilon > 0 \) we are free to choose \( c \in (t, b) \setminus D_0 \) so that \( b - c < \varepsilon \). Hence,
\[ b - t < c - t + \varepsilon \leq (m \circ n)(c) - (m \circ n)(t) + \varepsilon \leq r - (m \circ n)(t) + \varepsilon. \]
As \( \varepsilon > 0 \) was arbitrary, this means \( b - t \leq r - (m \circ n)(t) \) as desired.

Next we claim that
\[ (W \circ n)(t) \leq W(t), \quad (4.4) \]
or, equivalently, \( t \leq n(t) \), for each \( t \in (0, b) \setminus D_0 \). Indeed, for \( \delta \in (0, t) \setminus D_0 \) we have
\[ n((\delta, t) \setminus D_0) \subseteq [n(\delta), n(t)] \]
and hence
\[ t - \delta = \lambda(n((\delta, t) \setminus D_0)) \leq \lambda[n(\delta), n(t)] = n(t) - n(\delta) \leq n(t). \]
As \( \delta \in (0, t) \setminus D_0 \) can be chosen arbitrarily close to zero, this means \( t \leq n(t) \) as claimed.

From (4.3) and (4.4) we obtain that
\[ (f \circ m \circ n)(t)^p(W \circ n)(t) \leq f(t - r + b)^pW(t) \]
for all \( t \in (0, b) \setminus D_0 \), and hence, by the above together with (4.1), (4.2), and Corollary 4.7, we have
\[ \|f\|_G^p \approx \varepsilon \int_{E} (f \circ m)(t)^pW(t) \, dt \\
= \int_{E \setminus E_0} (f \circ m)(t)^pW(t) \, dt \\
= \int_{(0, b) \setminus D_0} (f \circ m \circ n)(t)^p(W \circ n)(t) \, dt \\
\leq \int_{(0, b) \setminus D_0} f(t + r - b)^pW(t) \, dt \\
= \int_{0}^{b} f(t + r - b)^pW(t) \, dt \\
\leq \int_{0}^{s} f(t + r - s)^pW(t) \, dt \\
\leq \|f\|_G^p. \]

We are now set to prove the main result of this section.

**Theorem 4.8.** If \( W(t) = (t + 1)^{-1/2} \) then \( G_{W,1}(0, \infty) \) fails to admit an equivalent rearrangement-invariant norm.
Proof. Fix $r \in (0, \infty)$ and let $f_r : (0, \infty) \to [0, \infty]$ and $f^*_r : (0, \infty) \to [0, \infty]$ be defined by

$$f_r(t) = \begin{cases} (r + 1 - t)^{-1/2} & \text{if } 0 < t < r, \\ 0 & \text{if } r \leq t < \infty \end{cases}$$

and

$$f^*_r(t) = \begin{cases} (t + 1)^{-1/2} & \text{if } 0 < t < r, \\ 0 & \text{if } r \leq t < \infty. \end{cases}$$

We claim that $f_r$ and $f^*_r$ are equimeasurable. Indeed, it is clear that $\text{dist}_{f_r}(s) = \text{dist}_{f^*_r}(s) = r$ for all $0 \leq s \leq (1 + r)^{-1/2}$ and $\text{dist}_{f_r}(s) = \text{dist}_{f^*_r}(s) = 0$ for all $1 \leq s \leq \infty$. Now select $(1 + r)^{-1/2} < s < 1$. We have $f_r(t) > s$ if and only if both $0 < t < r$ and $(r + 1 - t)^{-1/2} > s$, or, equivalently, $r + 1 - s^{-2} < t < r$. In this case we have

$$\text{dist}_{f_r}(s) = \lambda\{t \in (0, \infty) : f_r(t) > s\} = \lambda(r + 1 - s^{-2}, r) = s^{-2} - 1.$$ 

Similarly, $f^*_r(t) > s$ if and only if both $0 < t < r$ and $(t + 1)^{-1/2} > s$, or, equivalently, $0 < t < s^{-2} - 1$. This gives us

$$\text{dist}_{f^*_r}(s) = \lambda\{t \in (0, \infty) : f^*_r(t) > s\} = \lambda(0, s^{-2} - 1) = s^{-2} - 1$$

so that $f_r$ and $f^*_r$ are equimeasurable as claimed.

Note that

$$\|f^*_r\|_G \geq \int_0^r (t + 1)^{-1} dt = \log(r + 1) \to \infty$$

as $r \to \infty$. Thus, to complete the proof, it is enough to show that $\|f_r\|_G$ is bounded by a number not depending on $r$.

Now we apply Garling’s own “split into two sums” trick, except in our case the “sums” are actually integrals. Since $f_r$ is increasing on its support $(0, r)$, and $W \in \mathcal{W}$, by Lemma 4.5 we must have $s \in [0, r]$ so that

$$\|f_r\|_G = \int_0^s f_r(t - s + r)W(t) \, dt.$$ 

$$= \int_0^s (1 - t + s)^{-1/2}(t + 1)^{-1/2} \, dt$$

$$= \int_0^{s/2} (1 - t + s)^{-1/2}(t + 1)^{-1/2} \, dt + \int_{s/2}^s (1 - t + s)^{-1/2}(t + 1)^{-1/2} \, dt.$$ 

Hence, it suffices to show that each of these pieces is bounded by a number not depending on $s$. For the first piece, note that if $t \in (0, s/2]$ then $(1 - t + s)^{-1/2} \leq (s/2 + 1)^{-1/2}$. Hence,

$$\int_0^{s/2} (1 - t + s)^{-1/2}(t + 1)^{-1/2} \, dt \leq (s/2 + 1)^{-1/2} \int_0^{s/2} (t + 1)^{-1/2} \, dt$$

$$= (s/2 + 1)^{-1/2} \cdot 2 \left[(s/2 + 1)^{1/2} - 2\right] \leq 2.$$
For the second piece, note that if \( t \in [s/2, s] \) then \((t + 1)^{-1/2} \leq (s/2 + 1)^{-1/2} \), so that
\[
\int_{s/2}^{s} (1 - t + s)^{-1/2}(t + 1)^{-1/2} \, dt \leq (s/2 + 1)^{-1/2} \int_{s/2}^{s} (1 - t + s)^{-1/2} \, dt
= (s/2 + 1)^{-1/2} \cdot 2 \left[(s/2 + 1)^{1/2} - 2\right]
\leq 2.
\]

\[\square\]

5. Geometric properties of Garling function spaces

As Garling function spaces are interesting in their own right, they deserve some additional attention. It turns out that they are complete, i.e. they form Banach spaces, and contain almost-isometric copies of \( \ell_p \). As a consequence, the space \( G_{W,1}(0, \infty) \) is nonreflexive. It remains an open question as to whether \( G_{W,p}(0, \infty) \) is reflexive when \( 1 < p < \infty \).

We begin by establishing that Garling function spaces are in fact Banach spaces.

**Proposition 5.1.** Fix \( 1 \leq p < \infty \) and \( W \in \mathbb{W} \). Then space \( G_{W,p}(0, \infty) \) is complete.

**Proof.** Let \( (f_i)_{i=1}^{\infty} \) be a Cauchy sequence in \( G_{W,p} \). Let \( E, F \in \Lambda \) and \( m \in \mathbb{MO}(E, F) \). Observe that
\[
\|f_i - f_j\|_G^p \geq \int_0^\infty |f_i(t) - f_j(t)|^pW(t) \, dt \geq \|f_i\|_{W^1/p}^p - \|f_j\|_{W^1/p}^p\|_{L_p(0, \infty)}^p
\]
so that \( (|f_i|_{W^1/p})_{i=1}^{\infty} \) is Cauchy in \( L_p(0, \infty) \). As such, it converges a.e.-pointwise to \( g \in L_p(0, \infty) \). Similarly,
\[
\|f_i - f_j\|_G^p \geq \int_E \| |f_i \circ m |(t) - (f_j \circ m)(t)|^pW(t) \, dt
\geq \| |f_i \circ m |W^1/p - |f_j \circ m |W^1/p |_{L_p(E)}^p
\]
so that \( (|f_i \circ m |W^1/p)_{i=1}^{\infty} \) converges both in \( L_p(E) \) and a.e.-pointwise to some \( g_E \in L_p(E) \). Set \( f := gW^{-1/p} \) so that \( (|f_i|)_{i=1}^{\infty} \) converges a.e.-pointwise to \( f \). As \( (|f_i |W^{1/p})_{i=1}^{\infty} \) now converges a.e.-pointwise to \( |f |W^{1/p} \), it follows that \( |f |\mathbb{M}|W^{1/p} \) and \( g_E \) are a.e.-identical.

Since \( (f_i)_{i=1}^{\infty} \) is Cauchy, we can find \( M \in (0, \infty) \) so that \( \|f_i\|_G^p \leq M \) for all \( i \in \mathbb{N} \). Furthermore, we can find \( i_0 \in \mathbb{N} \) so that \( \|g_E - |f_{i_0} |W^{1/p} |_{L_p(E)}^p \leq 1 \). Then
\[
\int_E |f \circ m |(t)^pW(t) \, dt \leq \int_E \| |f | - |f_{i_0} | | \circ m |(t)^pW(t) \, dt + \int_E |f_{i_0} \circ m |(t)^pW(t) \, dt
\leq \|g_E - |f_{i_0} |W^{1/p} |_{L_p(E)}^p + \|f_{i_0} |_G^p
\leq 1 + M.
\]
As \( E, F, m \) were arbitrary, we have \( \rho_G(|f|) \leq (1 + M)^{1/p} < \infty \) so that \( f \in G_{W,p}(0, \infty) \).

Next, select \( \varepsilon > 0 \) and find \( N \in \mathbb{N} \) so that \( \|f_i - f_j\|_G < \varepsilon/2 \) for all \( i, j \geq N \). Select \( j_0 \geq N \) so that \( \|g_E - f_{j_0} \circ m\|_{L_p(E)} < \varepsilon/2 \). Then for \( i \geq N \) we have

\[
\int_E \|f - f_i \circ m\| |t|^p W(t) \, dt \leq \int_E \|f - f_{j_0}\| |t|^p W(t) \, dt + \int_E \|f_i - f_{j_0}\| |t|^p W(t) \, dt
\]

\[
\leq \|g_E - f_{j_0} \circ m\|_{L_p(E)}^p + \|f_i - f_{j_0}\|_G^p < \varepsilon.
\]

Again as \( E, F, m \) were arbitrary and independent of \( N \), it follows that \( \|f - f_i\|_G^p < \varepsilon \) for all \( i \geq N \). As \( \varepsilon > 0 \) was also arbitrary, \( f_i \to f \) in \( G_{W,p}(0, \infty) \).

To close, we will show that when \( 1 \leq p < \infty \) and \( W \in \mathbb{W} \), the space \( G_{W,p}(0, \infty) \) contains a copy of \( \ell_p \). To do this, we will use a basic sequence of characteristic functions as an auxiliary structure. Let us gather some facts about it in the next lemma. In what follows, we denote \( 1_i = 1_{(i-1,i]} \) for each \( i \in \mathbb{N} \).

**Lemma 5.2.** Fix \( 1 \leq p < \infty \) and \( W \in \mathbb{W} \), and set \( K = \int_0^1 W(t) \, dt \). Then the sequence \( (1_i/K)_{i=1}^\infty \) is a normalized, monotone, 1-unconditional and 1-subsymmetric basic sequence in \( G_{W,p}(0, \infty) \) which 1-dominates the unit vector basis \( (g_i)_{i=1}^\infty \) of the Garling sequence space \( g(w, p) \), where \( w = (w(i))_{i=1}^\infty \) is formed by letting \( w(i) = K^{-1} \int_{i-1}^i W(t) \, dt \) for each \( i \in \mathbb{N} \). Furthermore, they are isometrically equivalent for constant coefficients.

**Proof.** By replacing \( W \) with \( K^{-1}W \) if necessary, we may assume without loss of generality that \( K = 1 \).

It’s clear that \( (1_i)_{i=1}^\infty \) is normalized. It is also clear that if \( M < N \in \mathbb{N} \) and \( (a_i)_{i=1}^\infty \) is any sequence of scalars then we have

\[
\left\| \sum_{i=1}^M a_i 1_i \right\|_G \leq \left\| \sum_{i=1}^N a_i 1_i \right\|_G,
\]

which is precisely the criterion for forming a monotone basic sequence.

Next we show that it is 1-unconditional. Let \( (a_i)_{i=1}^\infty, (b_i)_{i=1}^\infty \in c_{00} \) and satisfy \( |a_i| \leq |b_i| \) for all \( i \in \mathbb{N} \). Then we can find \( E, F \in \Lambda \) and \( m \in \mathbb{M\bar{O}}(E, F) \) such that, setting \( U_j = m^{-1}(F \cap (j - 1, j]) \) for each \( j \in \mathbb{N} \) so that \( U_1 < U_2 < \cdots \) with \( E = \bigcup_{j=1}^\infty U_j \) and each \( 1_j \circ m = 1_{U_j} \restriction E \),

\[
\left\| \sum_{i=1}^\infty a_i 1_i \right\|_G^p \approx \varepsilon \int_E \left\| \sum_{i=1}^\infty a_i 1_i(m(t)) \right\| W(t) \, dt
\]

\[
= \int_E \sum_{i=1}^\infty |a_i|^p 1_i(m(t)) W(t) \, dt
\]

\[
= \sum_{j=1}^\infty \int_{U_j} \sum_{i=1}^\infty |a_i|^p 1_i(m(t)) W(t) \, dt
\]
By an analogous argument we have
\[ \sum_{j=1}^{\infty} \int_{U_j} |a_j|^p 1_{U_j}(t) W(t) \, dt = \sum_{j=1}^{\infty} \int_{U_j} |a_j|^p 1_{U_j}(t) W(t) \, dt \]
whence
\[ \left\| \sum_{i=1}^{\infty} a_i 1_i \right\|_G^p \leq \left\| \sum_{i=1}^{\infty} b_i 1_i \right\|_G^p \]
so that \((1_i)_{i=1}^{\infty}\) is 1-unconditional.

Let us show that it is 1-subsymmetric. Indeed, if \((a_i)_{i=1}^{\infty} \in c_{00}\) and \((1_{i_k})_{k=1}^{\infty}\) is some subsequence, then we can find \(E, F \in \Lambda\) and \(m \in \mathcal{MO}(E, F)\) such that
\[ \left\| \sum_{k=1}^{\infty} a_{i_k} 1_{i_k} \right\|_G^p \approx \varepsilon \int_E \left\| \sum_{k=1}^{\infty} a_{i_k} 1_{i_k}(m(t)) \right\|_W^p W(t) \, dt. \]
Set \(E' = \bigcup_{k=1}^{\infty} m^{-1}(F \cap (i_k - 1, i_k])\), and define an \(\mathcal{MO}\)-isomorphism \(\ell : (0, \infty) \to \bigcup_{k=1}^{\infty} (i_k - 1, i_k]\) by gluing together the shift maps \((k-1, k) \mapsto (i_k - 1, i_k]\). Then \(\ell^{-1} \circ m\) is an \(\mathcal{MO}\)-isomorphism between \(E'\) and its image, and for each \(k \in \mathbb{N}\) and \(t \in E'\) we have \(1_{i_k}(m(t)) = 1_{k}(\ell^{-1}(m(t))).\) Furthermore, \(1_{i_k}(m(t)) = 0\) for each \(k \in \mathbb{N}\) and \(t \in E \setminus E'\). Hence,
\[ \left\| \sum_{k=1}^{\infty} a_{i_k} 1_{i_k} \right\|_G^p \approx \varepsilon \int_{E'} \left\| \sum_{k=1}^{\infty} a_{i_k} 1_{i_k}(\ell^{-1}(m(t))) \right\|_W^p W(t) \, dt \]
\[ = \int_{E'} \left\| \sum_{k=1}^{\infty} a_{i_k} 1_{i_k}(m(t)) \right\|_W^p W(t) \, dt \]
\[ = \int_{E'} \left\| \sum_{k=1}^{\infty} a_{i_k} \ell^{-1}(m(t)) \right\|_W^p W(t) \, dt \]
\[ \leq \left\| \sum_{k=1}^{\infty} a_{i_k} \right\|_G^p. \]

To show the reverse inequality, we instead choose \(E, F, m\) so that
\[ \left\| \sum_{k=1}^{\infty} a_{i_k} \right\|_G^p \approx \varepsilon \int_{E'} \left\| \sum_{k=1}^{\infty} a_{i_k} 1_{i_k}(m(t)) \right\|_W^p W(t) \, dt. \]
Define $\ell$ as before so that $\ell \circ m$ is an $\MO$-isomorphism between $E$ and its image, and $1_k(m(t)) = 1_k(\ell(m(t)))$ for each $k \in \mathbb{N}$ and $t \in E$. Then

$$\int_E \left| \sum_{k=1}^{\infty} a_k 1_k(m(t)) \right|^p W(t) \, dt = \int_E \left| \sum_{k=1}^{\infty} a_k (1_k(\ell(m(t)))) \right|^p W(t) \, dt$$

$$\leq \left\| \sum_{k=1}^{\infty} a_k 1_k \right\|_G^p.$$ 

It follows that $(1_i)_{i=1}^{\infty}$ is 1-subsymmetric.

To show that it 1-dominates $g(w, p)$, we again let $(a_i)_{i=1}^{\infty} \in c_{00}$. Select any subsequence $(a_{i_k})_{k=1}^{\infty}$. As before, there is an $\MO$-isomorphism $\ell : (0, \infty) \to \bigcup_{k=1}^{\infty}(i_k-1, i_k]$ defined by gluing together the shift maps $(k-1, k] \mapsto (i_k-1, i_k]$. Note that $1_{i_k} \circ \ell = 1_k$ for each $k \in \mathbb{N}$. We now have

$$\sum_{k=1}^{\infty} |a_{i_k}|^p w(k) = \sum_{k=1}^{\infty} |a_{i_k}|^p \int_{k-1}^{k} W(t) \, dt$$

$$= \sum_{k=1}^{\infty} |a_{i_k}|^p \int_{0}^{\infty} 1_k(t) W(t) \, dt$$

$$= \int_{0}^{\infty} \sum_{k=1}^{\infty} |a_{i_k}|^p 1_k(t) W(t) \, dt$$

$$= \int_{0}^{\infty} \sum_{k=1}^{\infty} |a_{i_k}|^p 1_{i_k}(\ell(t)) W(t) \, dt$$

$$= \int_{0}^{\infty} \left| \sum_{i=1}^{\infty} a_i 1_i(\ell(t)) \right|^p W(t) \, dt$$

$$\leq \left\| \sum_{i=1}^{\infty} a_i 1_i \right\|_G^p.$$ 

By taking the supremum over all subsequences we obtain

$$\| (a_i)_{i=1}^{\infty} \|_G \leq \left\| \sum_{i=1}^{\infty} a_i 1_i \right\|_G.$$ 

Finally, we consider the last part of the lemma, about being isometrically equivalent for constant coefficients to $(g_i)_{i=1}^{\infty}$. Indeed, as $(1_i)_{i=1}^{\infty}$ already 1-dominates it as shown above, we need only show the reverse inequality, i.e. that $(g_i)_{i=1}^{\infty}$ 1-dominates $(1_i)_{i=1}^{\infty}$ for constant coefficients. To that end, fix $N \in \mathbb{N}$ and let $E, F \in \Lambda$ and $m \in \MO(E, F)$ be such that

$$\| |1_i|_{\infty} \|_G^p \approx \int_E \sum_{i=1}^{N} 1_i(m(t)) W(t) \, dt.$$ 

For each $i = 1, \cdots, N$, define $A_i := m^{-1}(F \cap (i-1, i))$, and then set $A := \bigcup_{i=1}^{N} A_i$. It is clear that $\lambda(A) < \infty$, so by Theorem 2.15 we can find measure-zero subsets
$D_0$ of $[0, \lambda(A)]$ and $A_0$ of $A$, and an $\mathbb{M}_0$-isomorphism $n$ from $D := [0, \lambda(A)] \setminus D_0$ onto $A \setminus A_0$. We claim that $t \leq n(t)$ for all $t \in D$. Indeed, if we set $b := \inf n(D)$ then since $b \geq 0$ and $n(D) \subseteq [b, n(t)]$ we have

$$t = \lambda[0, t] = \lambda(D) = \lambda(n(D)) \leq \lambda[b, n(t)] = n(t) - b \leq n(t).$$

As $W$ is nonincreasing it follows that $W(n(t)) \leq W(t)$ for all $t \in D$. Note also that $\lambda(A) \leq N$ so that $D \subseteq [0, N]$. Furthermore, it is clear that $1_i(m(t)) = 0$ for all $i = 1, \cdots, N$ and all $t \in E \setminus A$. Together with Corollary 4.7 we now obtain

$$\left\| \sum_{i=1}^N 1_i \right\|_G^p \approx \epsilon \int_E \sum_{i=1}^N 1_i(m(t))W(t) \, dt$$

$$= \int_A \sum_{i=1}^N 1_i(m(t))W(t) \, dt$$

$$= \sum_{j=1}^N \int_{A_j} \sum_{i=1}^N 1_i(m(t))W(t) \, dt$$

$$= \sum_{j=1}^N \int_{A_j} 1_j(m(t))W(t) \, dt$$

$$= \sum_{j=1}^N \int_{A_j} W(t) \, dt$$

$$= \int_A W(t) \, dt$$

$$\leq \int_D W(n(t)) \, dt$$

$$\leq \int_D W(t) \, dt$$

$$\leq \int_0^N W(t) \, dt$$

$$= \sum_{k=1}^N w(k)$$

$$= \left\| \sum_{i=1}^N g_i \right\|_g^p .$$

As $(g_i)_{i=1}^\infty$ and $(1_i)_{i=1}^\infty$ are both 1-subsymmetric, we are done.

**Theorem 5.3.** Fix $1 \leq p < \infty$ and let $W \in \mathcal{W}$. Then for any $\epsilon > 0$ the basic sequence $(1_i)_{i=1}^\infty$ admits a normalized constant coefficient block basic sequence which is $(1 + \epsilon)$-equivalent to $\ell_p$, and which is 2-complemented in $[1_i]_{i=1}^\infty$.

**Proof.** Let $g(w, p), (g_i)_{i=1}^\infty$, and $K$ be as in Lemma 5.2, so that $(1_i/K)_{i=1}^\infty$ is isometrically equivalent to $(g_i)_{i=1}^\infty$ for constant coefficients. It was shown in $[\text{AAW18},$
§3] that there exists a constant coefficient block basic sequence of \((g_n)_{n=1}^\infty\) which is \((1+\varepsilon)\)-equivalent to \(\ell_p\), for any \(\varepsilon > 0\). In particular, we can select

\[ y'_i = \sum_{n=k_i}^{k_{i+1}-1} g_n \quad \text{and} \quad y_i = \frac{y'_i}{\|y'_i\|_G} \quad \text{for each } i \in \mathbb{N}, \]

where \(1 = k_1 < k_2 < k_3 < \cdots \in \mathbb{N}\), so that \((y_i)_{i=1}^\infty\) is \((1+\varepsilon)\)-equivalent to \(\ell_p\).

Next, write

\[ x'_i = \sum_{n=k_i}^{k_{i+1}-1} 1_n/K \quad \text{and} \quad x_i := \frac{x'_i}{\|x'_i\|_G} \quad \text{for each } i \in \mathbb{N}, \]

where \(1 = k_1 < k_2 < k_3 < \cdots \in \mathbb{N}\).

We claim that \((x_i)_{i=1}^\infty\) is \(1\)-dominated by the unit vector basis of \(\ell_p\). Indeed, if \((a_i)_{i=1}^\infty \in c_{00}\) then we can find \(E, F \in \Lambda\) and \(m \in \text{MO}(E, F)\) such that

\[
\left\| \sum_{i=1}^\infty a_i x_i \right\|_G^p \lesssim \int_E \left( \sum_{i=1}^\infty a_i |x_i(m(t))| \right)^p W(t) \, dt \\
= \int_E \sum_{i=1}^\infty |a_i|^p |x_i(m(t))| W(t) \, dt \\
= \sum_{i=1}^\infty |a_i|^p \int_E |x_i(m(t))| W(t) \, dt \\
\leq \sum_{i=1}^\infty |a_i|^p \|x_i\|_G^p \\
= \sum_{i=1}^\infty |a_i|^p
\]

so that \((x_i)_{i=1}^\infty \lesssim_1 \ell_p\) as claimed.

By Lemma 5.2, \((1/K)_{i=1}^\infty\) is isometrically equivalent to \((g_i)_{i=1}^\infty\) for constant coefficients, and so \(\|y'_i\|_g = \|x'_i\|_G\) for each \(i \in \mathbb{N}\). Again from Lemma 5.2, we know that \((g_i)_{i=1}^\infty\) is \(1\)-dominated by \((1/K)_{i=1}^\infty\). It follows that

\[
\ell_p \approx_1 (y_i)_{i=1}^\infty \lesssim_1 (x_i)_{i=1}^\infty \lesssim_1 \ell_p.
\]

That \((x_i)_{i=1}^\infty\) spans a 2-complemented subspace of \([1]_{i=1}^\infty\) follows from the fact that constant-coefficient block basic sequences of a 1-subsymmetric basis are always 2-complemented (see, for instance, [LT77, Proposition 3.a.4]).

**Remark 5.4.** Although \(\ell_p\) is complemented in \([1]_{i=1}^\infty\), we do not yet know if it is complemented in \(G_{W,p}(0, \infty)\).

**Corollary 5.5.** Fix \(1 \leq p < \infty\) and \(W \in \mathcal{W}\). Then for every \(\varepsilon > 0\), the space \(G_{W,p}(0, \infty)\) contains a subspace which is \((1+\varepsilon)\)-isomorphic to \(\ell_p\). Hence, in particular, the space \(G(W, 1)\) is nonreflexive.

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References

[AALW18] Fernando Albiac, José L. Ansorena, Denny Leung, and Ben Wallis. “Optimality of the rearrangement inequality with applications to Lorentz-type sequence spaces,” Mathematical Inequalities and Applications 21 (2018), pp127–132.

[AAW18] Fernando Albiac, José L. Ansorena, and Ben Wallis. “Garling sequence spaces,” Journal of the London Mathematical Society 98:1 (2018 Apr 13), pp204–222.

[Bo07] Valdimir I. Bogachev. Measure Theory, Volume I (2007), ISBN 978-3-540-34513-8.

[BS88] Colin Bennett and Robert Sharpley. Interpolation of Operators (1988). ISBN 978-0-120-88730-9.

[Ga68] D.J.H. Garling. “Symmetric bases of locally convex spaces,” Studia Math. 30 (1968), pp163–181.

[LT77] Joram Lindenstrauss and Lior Tzafriri. Classical Banach Spaces I. (1977), ISBN 3-540-60628-9.

[LT79] Joram Lindenstrauss and Lior Tzafriri. Classical Banach Spaces II. (1977), ISBN 3-540-08888-1.

[RF10] H.L. Royden and P.M. Fitzpatrick. Real Analysis, fourth edition (2010), ISBN 978-0-13-511355-4.

[Si62] Ivan Singer. “Some characterizations of symmetric bases in Banach spaces,” Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. et Phys. 10 (1962), pp185–192.

Division of Math/Science/Business, Kishwaukee College, Malta, IL 60150, United States.

E-mail address: benwallis@live.com