CORRECTIONS TO HYPERFINE SPLITTING AND LAMB SHIFT INDUCED BY THE OVERLAPPING TWO-LOOP ELECTRON SELF-ENERGY INSERTION IN THE ELECTRON LINE

Michael I. Eides
Petersburg Nuclear Physics Institute,
Gatchina, St.Petersburg 188350, Russia

and

Savely G. Karshenboim and Valery A. Shelyuto
D. I. Mendeleev Institute of Metrology,
St.Petersburg 198005, Russia

1E-mail address: eides@lnpi.spb.su
Abstract

Contributions to HFS and to the Lamb shift intervals of order $\alpha^2(Z\alpha)^5$ induced by the graph with the two-loop overlapping electron self-energy diagram inserted in the electron line are considered. Explicit expression for the overlapping two-loop self-energy diagram in the Fried-Yennie gauge is obtained. Contributions both to HFS and Lamb shift induced by the diagram containing such subgraph are calculated.
1 Introduction

Calculation of contributions of order $\alpha^2(Z\alpha)^5m$ to hyperfine splitting (HFS) and Lamb shift induced by the two-loop radiative photon insertions in the electron line was initiated in the previous paper [1]. We have calculated there the contributions induced by all graphs containing one-loop electron self-energy diagram as a subgraph, by the graph containing two one-loop vertices, and also the contribution induced by the rainbow two-loop electron self-energy insertion in the electron line\footnote{We use a chance to correct an error made in [1] in the calculation of the rainbow diagram contribution to the Lamb shift. Due to a normalization error the result cited in [1] is two times larger than the correct one.}

Calculation of the contribution induced by the overlapping two-loop electron self-energy insertion (see Fig.1) is presented below. Consideration of this contribution is impeded by all the usual difficulties which are connected with the presence of overlapping infinities in this case and the necessity to perform all necessary subtractions. In Section 6 we obtain explicit expressions for the renormalized overlapping diagram contribution to the self-energy operator in the Fried-Yennie (FY) gauge in the form of the Feynman parameter integral. These expressions are used later for calculation of the contributions to HFS and Lamb shift. It should be mentioned that taking into account also expressions for the rainbow and one-particle reducible\footnote{We are deeply grateful to Dr. Pachucki who attracted our attention to this error.} diagrams we have explicit general expression for the two-loop mass-operator in the FY gauge. We are aware of only one other calculation of the two-loop mass operator existing in the literature\footnote{We are deeply grateful to Dr. Pachucki who attracted our attention to this error.} which was performed in the Feynman gauge. We hope that the explicit expression in the FY gauge presented below may find applications also in other problems, e.g. in calculation of the positronium decay rate [3].
2 General Expression for the Overlapping Self-Energy Operator

Consider contribution to the two-loop self-energy operator induced by the diagram with overlapping photons. First of all one has to perform subtractions of potential infinities in internal vertices. Explicit expression for such internally subtracted contribution to the mass-operator in the FY gauge has the form

\[
\Sigma(p) = \left( \frac{\alpha}{4\pi} \right)^2 \int \frac{d^4 l}{i\pi^2} \int \frac{d^4 q}{i\pi^2} \frac{1}{l^2 q^2} (g^{\alpha\lambda} + 2 \frac{l^\alpha l^\lambda}{l^2}) (g^{\beta\sigma} + 2 \frac{q^\beta q^\sigma}{q^2})
\]

\[
\gamma_\alpha \left\{ \frac{\hat{p} + \hat{l} + m}{D(p + l)} \gamma_\beta \frac{\hat{p} + \hat{q} + m}{D(p + l + q)} \frac{\hat{p} + \hat{q} + m}{D(p + q)} \right. \\
\left. - \frac{\hat{p}_0 + \hat{l} + m}{D(p_0 + l)} \gamma_\beta \frac{\hat{p}_0 + \hat{q} + m}{D(p_0 + l + q)} \frac{\hat{p}_0 + \hat{q} + m}{D(p_0 + q)} \right. \\
\left. - \frac{\hat{p} + \hat{l} + m}{D(p + l)} \gamma_\beta \frac{\hat{p}_0 + \hat{q} + m}{D(p_0 + l + q)} \frac{\hat{p}_0 + \hat{q} + m}{D(p_0 + q)} \right\} \gamma_\sigma
\]

where

\[
D(p) = p^2 - m^2
\]

and \( \hat{p}_0 = m \) is the mass-shell momentum in subtracted terms.

There is a certain subtlety connected with this mass-shell limit. Although renormalization procedure in the FY gauge is performed below without introduction of the infrared photon mass (see, for more details [4]) there are spurious infrared divergences on the intermediate stages of calculations on the mass-shell which may produce finite but discontinuous on the mass shell results if the mass-shell limit is taken in the naive way prior to calculation of the integrals. To avoid these problems one has to perform calculation even of the subtracted terms with slightly off mass-shell external momentum (in this case with off mass-shell momentum \( p_0 \)) and only after calculation of all infrared unsafe integrals to take the mass-shell limit. In this way one preserves continuity of all physical results on the mass shell and obtains correct results.
We would like to obtain the expression for the overlapping diagram contribution to the self-energy operator in the form of the integral representation with minimal number of the Feynman parameters. To reduce the number of the parameters we transform the integrand in eq.\((1)\) with the help of trivial identity

\[
\hat{p} + \hat{l} + \hat{q} + m = \left(\frac{\hat{p} + m}{2} + \hat{l}\right) + \left(\frac{\hat{p} + m}{2} + \hat{q}\right)
\]  

and obtain

\[
\Sigma(p) = \left(\frac{\alpha}{4\pi}\right)^2 \int \frac{d^4l}{i\pi^2} \int \frac{d^4q}{i\pi^2} \frac{1}{l^2 q^2} \left(g^{\alpha\lambda} + 2 \frac{l^{\alpha} l^{\lambda}}{l^2}\right) \left(g^{\beta\sigma} + 2 \frac{q^{\beta} q^{\sigma}}{q^2}\right)
\]  

\[
\frac{1}{2} \left\{ \gamma_\sigma \frac{\hat{p} + \hat{q} + m}{D(p + q)} \gamma_\lambda R_\beta \gamma_\alpha + \gamma_\alpha R_\beta \gamma_\lambda \left(\frac{\hat{p} + \hat{q} + m}{D(p + q)}\right) \gamma_\sigma \right\},
\]

where

\[
R_\beta = \frac{(\hat{p} + \hat{l} + m) \gamma_\beta (\hat{p} + \hat{q} + 2\hat{l} + m)}{D(p + l)D(p + l + q)} - 2 \frac{(\hat{\rho} + \hat{l} + m) \gamma_\beta (\hat{\rho} + \hat{l} + m)}{D^2(p_0 + l)}.
\]  

One may check that both terms in the braces in eq.\((\text{3})\) produce after integration coinciding results, so we consider below only integral of the second term and simply double the result. Initial expression for further transformations has the form

\[
\Sigma(p) = \left(\frac{\alpha}{4\pi}\right)^2 \int \frac{d^4q}{i\pi^2} \frac{1}{q^2} \left(g^{\beta\sigma} + 2 \frac{q^{\beta} q^{\sigma}}{q^2}\right) R_\beta(p, q) \frac{\hat{p} + \hat{q} + m}{D(p + q)} \gamma_\sigma,
\]  

where

\[
R_\beta(p, q) = \int \frac{d^4l}{i\pi^2} \frac{1}{l^2} \left(g^{\alpha\lambda} + 2 \frac{l^{\alpha} l^{\lambda}}{l^2}\right) \gamma_\alpha R_\beta \gamma_\lambda.
\]  

Integration over \(l\) in eq.\((\text{6})\) is convergent but separate terms in the integrand produce logarithmically divergent results. We slightly rearrange integrand in order to separate compensating divergences

\[
R_\beta(p, q) = \int \frac{d^4l}{i\pi^2} \left(\frac{1}{l^2 D(p + l)D(p + l + q)}\right)
\]  

\[
3
\]
\[
\left( [g^{\alpha\lambda} + \frac{2l^{\alpha\lambda}}{l^2}] \right) \gamma_\alpha(\hat{p} + \hat{l} + m) \gamma_\beta(\hat{p} + 2\hat{l} + m) \gamma_\lambda - 6l^2 \gamma_\beta
\]

\[
-\frac{2}{l^2 D^2(p_0 + l)} \left[ (g^{\alpha\lambda} + \frac{2l^{\alpha\lambda}}{l^2}) \gamma_\alpha(\hat{p}_0 + \hat{l} + m) \gamma_\beta(\hat{p}_0 + \hat{l} + m) \gamma_\lambda - 3l^2 \gamma_\beta \right]
\]

\[
+6\gamma_\beta \left[ \frac{1}{D(p + l)D(p + l + q)} - \frac{1}{D^2(p_0 + l)} \right]
\]

\[
\equiv R_\beta^{(1)} + R_\beta^{(2)} + R_\beta^{(3)}.
\]

Representation in eq.(7) is very convenient for further transformations as each of the terms \(R_\beta^{(i)}\) (after substitution in eq.(5)) does not need more than four integration parameters to represent it in the form of the integral and each of these terms is a convergent integral in respect with integration over \(l\) which will be the first integral to perform. We will consider transformations of each of the terms in eq.(7) separately.

### 3 Calculation of the Contribution to the Mass Operator Induced by the Term \(R_\beta^{(1)}\)

Let us consider first contribution to the mass operator induced by term \(R_\beta^{(1)}\) in eq.(7). After ordering over powers of the integration momentum \(l\) we obtain

\[
R_\beta^{(1)} = 2 \int \frac{d^4l}{i\pi^2 l^2} D(p + l)D(p + q + l) \left\{ \frac{1}{D(p + l)} \right\} \left\{ 2p_\beta(3m - \hat{p}) + \hat{l}p + (p^2 - m^2)(\gamma_\beta - \frac{\hat{l}\gamma_\beta p}{l^2}) \right\}
\]

\[
[6ml_\beta - 2\hat{l}\gamma_\beta \hat{p} - \hat{p}\gamma_\beta \hat{l} + 2\hat{l}(\hat{p} + m)\gamma_\beta + \gamma_\beta(\hat{p} + m)\hat{l}] - [2\hat{l}\gamma_\beta \hat{l} + l^2 \gamma_\beta] \}
\]

Next we combine denominators with the help of the identity

\[
(1 - x)^2 + x[(1 - z)D(p + l) + zD(p + l + q)] = (l + xQ)^2 - x\Delta,
\]

where

\[
Q = p + qz,
\]
\[
\Delta = m^2 - p^2(1 - x) - q^2z(1 - xz) - 2pqz(1 - x).
\]

After shift of the integration variable \(l \rightarrow l - xQ\) we perform momentum integration which leads to the expression

\[
R(1)\beta = \int_0^1 dx \int_0^1 dz \left\{ 2p\hat{\beta}(2 - x) - 3m \frac{\hat{Q}\hat{p}\hat{Q}}{\Delta^2} + x(1 - x) \frac{\hat{Q}\hat{p}\hat{Q}}{\Delta^2} \right\}
\]

\[
\equiv R^{(1a)}\beta + R^{(1b)}\beta + R^{(1c)}\beta.
\]

where

\[
C^{(1)}\beta = 6mQ\beta - 2\hat{Q}\gamma\hat{p} - \hat{p}\gamma\hat{Q} + 2\hat{Q}(\hat{p} + m)\gamma\beta + \gamma\beta(\hat{p} + m)\hat{Q},
\]

\[
C^{(2)}\beta = 2\hat{Q}\gamma\hat{p} + Q^2\gamma\beta.
\]

Each power of momentum \(l\) in the numerator of the integrand in eq.(8) produces respective power of factor \(x\) in the numerator of the integrand in eq.(10). We will consider calculation of the terms in different brackets on the right hand side (i.e. terms \(R^{(1i)}\beta\)) separately.

### 3.1 Contribution to the Self-Energy Operator Induced by the Term \(R^{(1a)}\beta\)

It is easy to see that the expression in eq.(10) leads to the contribution to the unrenormalized self-energy operator which remains finite even on the mass shell. However, we would like to obtain expression for the renormalized (doubly subtracted) self-energy operator and each subtraction inserts (after integration over \(q\)) additional power of \(x\) in the denominator, making integration over \(x\) potentially unsafe. Only term \(R^{(1a)}\beta\) in eq.(8) seems to produce infrared divergence on the mass shell after subtraction. To get rid of this apparent infrared divergence we first separate infrared unsafe terms in the expression for this term

\[
R^{(1a)}\beta = 4p\beta \int_0^1 dx \int_0^1 dz \left\{ \frac{\hat{p}(2 - x) - 3m}{\Delta} + x(1 - x) \frac{\hat{Q}Q^2}{\Delta^2} \right\}
\]
Next we use trivial identity which is the result of integration by parts and the relation \( \partial \Delta / \partial x = Q^2 \)

\[
\int_0^1 dx x(1-x) \frac{Q^2}{\Delta^2} = \int_0^1 dx \frac{1-2x}{\Delta}
\]

and obtain

\[
R^{(1a)}_{1\beta} = 4p_\beta \int_0^1 dx \int_0^1 dz \left\{ \frac{3 \hat{p}(1-x) - m}{\Delta} + x(1-x) \frac{\hat{Q} \hat{p} \hat{Q} - \hat{p} Q^2}{\Delta^2} \right\}.
\]

This new representation is completely infrared safe and admits subtraction on the mass-shell. Respective contribution to the mass operator has the form

\[
\Sigma^{(1a)}(p) = 4\left( \frac{\alpha}{4\pi} \right)^2 \int_0^1 dx \int_0^1 dz \int \frac{d^4q}{i\pi^2} \left( 3 \frac{\hat{p}(1-x) - m}{\Delta} + x(1-x) \frac{\hat{Q} \hat{p} \hat{Q} - \hat{p} Q^2}{\Delta^2} \right) \left( \hat{p} + \hat{q} + m \right) q^2 D(p+q)(2\hat{p} + \frac{\hat{q} \hat{p} \hat{q}}{q^2}).
\]

Next we introduce Feynman parameters with the help of the identity

\[
(1-t)q^2 + t[u(-\frac{\Delta}{z(1-xz)}) + (1-u)D(p+q)] = (q + p\eta t)^2 - t\Omega,
\]

where

\[
\Omega = p^2 \eta^2 t + (m^2 - p^2)(1-u) + \frac{m^2 - p^2(1-x)}{z(1-xz)} u \equiv \frac{m^2(\omega + \rho \xi)}{z(1-xz)};
\]

\[
\eta = 1 - \frac{x(1-z)u}{1-xz},
\]

\[
\rho = \frac{m^2 - p^2}{m^2},
\]

\[
\xi = (1-x)u + z(1-xz)(1-u - \eta^2 t),
\]
\[ \omega = ux + z(1 - xz)\eta^2 t. \]

Denominators in eq.(15) are combined as follows

\[
\frac{1}{\Delta^k D(p + q)q^{2n}} = \int_0^1 du \int_0^1 dt \frac{g(k, 1, n)}{[(q + p\eta t)^2 - t\Omega]^{k+1+n}} \equiv G(k, 1, n), \quad (18)
\]

where

\[
\begin{align*}
g(1, 1, 0) &= -\frac{1}{z(1 - xz)}, \
g(1, 1, 1) &= -\frac{2t}{z(1 - xz)}, \
g(1, 1, 2) &= -\frac{6t(1 - t)}{z(1 - xz)}, \\
g(2, 1, 1) &= \frac{6ut^2}{z^2(1 - xz)^2}, \
g(2, 1, 2) &= \frac{24ut^2(1 - t)}{z^2(1 - xz)^2}.
\end{align*}
\]

We displayed in eq.(17) and eq.(18) slightly more general formulae than those which are necessary to perform momentum integration in eq.(15). These auxiliary equations will be extensively used below for momentum integrations of other entries in eq.(15).

Contribution to the mass operator in eq.(15) has a rather simple form in new notation

\[
\Sigma^{(1a)}(p) = \left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int \frac{d^q}{i\pi^2} \sum_{k,n=1,2} N_a(k, 1, n) G(k, 1, n), \quad (20)
\]

where

\[
\begin{align*}
N_a(1, 1, 1) &= 12[\hat{p}(1 - x) - m][2\hat{p}(\hat{p} + m) + (2\hat{q}\hat{p} + \hat{p}\hat{q})], \\
N_a(1, 1, 2) &= 12[\hat{p}(1 - x) - m](\hat{p} + m)\hat{q}\hat{p}, \\
N_a(2, 1, 1) &= 4x(1 - x)(\hat{Q}\hat{p}\hat{Q} - \hat{p}\hat{Q}^2)[2\hat{p}(\hat{p} + m) + (2\hat{q}\hat{p} + \hat{p}\hat{q})], \\
N_a(2, 1, 2) &= 4x(1 - x)(\hat{Q}\hat{p}\hat{Q} - \hat{p}\hat{Q}^2)(\hat{p} + m)\hat{q}\hat{p}.
\end{align*}
\]

We want to obtain after momentum integration as small powers of factor \( \Omega \) in the denominator as possible since in this case next integrations are more convergent and subtraction procedure is more accessible. Due to this reason we separated in the numerators characteristic structure \( \hat{Q}\hat{p}\hat{Q} - \hat{p}\hat{Q}^2 \)
which does not contain cubic in external momentum $p$ terms after shift of integration momentum and leads thus to smaller power of $\Omega$ after integration.

Next we shift integration momentum $q \rightarrow q - \eta t$ and obtain after straightforward integration in eq.(15)

$$\Sigma^{(1a)}(p) = \left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{12}{z(1-xz)} \left\{ \hat{p}(1-x) - m \right\}$$

$$\left\{ \frac{1}{\Omega} [2\hat{p}(\hat{p} + m) - 3p^2\eta t] - \hat{p}(\hat{p} + m)[\frac{1-t}{\Omega} + t(1-t)\frac{p^2\eta^2}{\Omega^2}] \right\}$$

$$+ \frac{x(1-x)u}{1-xz} \left\{ - \frac{p^2t}{\Omega} [\hat{p}(1 + 2\eta t) - 2(\hat{p} + m)z] - \frac{p^2zt(1-t)}{\Omega} (\hat{p} + 3m) \right.$$ 

$$\left. + \frac{p^4(1-t)}{\Omega^2} [2(\hat{p} - m)\eta t - (3\hat{p} - m)\eta^2t^2] \right\}.$$

One may get rid of denominator $\Omega^2$ with the help of integration by parts over $t$, taking into account that $\partial\Omega/\partial t = p^2\eta^2$ (compare eq.(13)). Integration by parts leads to substitutions in eq.(22)

$$\frac{t(1-t)p^2\eta^2}{\Omega^2} \rightarrow \frac{1-2t}{\Omega},$$

$$\frac{t^2(1-t)p^2\eta^2}{\Omega^2} \rightarrow \frac{t(2-3t)}{\Omega}.$$

Note that although the right hand side in eq.(22) is finite even on the mass shell, separate terms in the integrand may lead to infrared divergent contributions after subtraction. Integration by parts gave us the chance to get rid of these would be IR divergence which could become dangerous for the accessibility of the subtraction procedure.

Now expression in eq.(22) acquires the form

$$\Sigma^{(1a)}(p) = \left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{12}{m^2(\omega + \rho \xi)} \left\{ 3\hat{p}\hat{t}[\hat{p}(1-x) - m] \right.$$}

$$\left[ \hat{p}(1-\eta) + m \right] + \frac{x(1-x)u}{1-xz} p^2 \left[ 2(\hat{p} - m) \frac{1-2t}{\eta} - \hat{p}(1+2\eta t) + zt[m-5\hat{p}(1-2t)] \right].$$

This form is very suitable for subtraction which we postpone until all other contributions to the mass operator would be obtained.
3.2 Calculation of the Contribution to the Mass Operator Induced by the Term $R^{(1b)}_\beta$

Contribution to the self-energy operator corresponding to $R^{(1b)}_\beta$ has the form

$$\Sigma^{(1b)}(p) = -2\left(\frac{\alpha}{4\pi}\right)^2(p^2 - m^2) \int_0^1 dx \int_0^1 dz \int \frac{d^4q}{i\pi^2} \left\{(2 - x)\frac{\gamma^{\beta}}{\Delta} + x(1 - x)\frac{\hat{Q}_\gamma \hat{Q}}{\Delta^2} \right\} \hat{p} + \hat{q} + m \hat{q}^2 \Delta D(p + q) + 2\hat{q}q^2 \frac{\gamma^{\beta}}{q^2}.$$

(25)

Again as in the previous section we try to transform numerator of the integrand to the form containing minimal power of external momentum $p$. This goal is easily reached with the help of trivial identities

$$\hat{Q}g\hat{Q} = 2Q^2(\hat{p} + \hat{q} - m) + 2(1 - z)[\hat{Q}\hat{q}\hat{Q} - \hat{q}Q^2],$$

(26)

$$\hat{Q}\hat{q}\hat{Q} = \hat{q}Q^2 + [\hat{Q}\hat{q}\hat{Q} - \hat{q}Q^2].$$

Note that the term $\hat{Q}\hat{q}\hat{Q} - \hat{q}Q^2$ as well as the similar term $\hat{Q}\hat{p}\hat{Q} - \hat{p}Q^2$ in the previous section is free of cubic in the external momentum terms after shift of integration variable. We get rid of the terms containing $Q^2$ in the numerator with the help of integration by parts displayed in eq.(13). We then obtain

$$\Sigma^{(1b)}(p) = -2\left(\frac{\alpha}{4\pi}\right)^2(p^2 - m^2) \int_0^1 dx \int_0^1 dz \int \frac{d^4q}{i\pi^2} \sum_{k,n=1,2} N_b(k,1,n)G(k,1,n),$$

(27)

where

$$N_b(1,1,1) = 6[\hat{q}(1 - 2x) - \hat{p}x + m(2 - x)],$$

(28)

$$N_b(1,1,2) = 6(1 - x)\hat{q}\hat{p}q,$$

$$N_b(2,1,1) = 2x(1 - x)(2 - z)(\hat{Q}\hat{q}\hat{Q} - \hat{q}Q^2),$$

$$N_b(2,1,2) = 2x(1 - x)(\hat{Q}\hat{q}\hat{Q} - \hat{q}Q^2)(\hat{p} + m)\hat{q}.$$
\[
\Sigma^{(1b)}(p) = \left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{12(p^2 - m^2)}{z(1-xz)}
\]

\[
\left\{ \frac{1}{\Omega} \left[ \hat{p} \eta t (1-2x) + \hat{p} x - m (2-x) \right] + \hat{p} (1-x) \left[ \frac{1-t}{\Omega} + t (1-t) \frac{p^2 \eta^2}{\Omega^2} \right] \right\}
\]

\[
+ \frac{x(1-x)u}{1-xz} \left\{ \frac{\hat{p}(2-z)t}{\Omega} - \frac{p^2 (1-t)}{\Omega^2} \left[ \hat{p} - m \frac{1-2t}{\eta} \right] \right\}
\]

Next we integrate by parts to get rid of denominators \( \Omega^2 \) (compare previous section) and obtain

\[
\Sigma^{(1b)}(p) = \left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \left\{ \frac{12(p^2 - m^2)}{m^2(\omega + \rho \xi)} \left[ \hat{p} [x + (1-x)(2-3t) - \eta t (1-2x)] - m (2-x) + \frac{x(1-x)u}{1-xz} \left[ \frac{\hat{p}(2-z)t}{\Omega} + 2p \frac{1-2t}{\eta} \right] \right] \right\}
\]

\[
-(\hat{p} - m)^2 \frac{12p^2(\hat{p} + m)x(1-x)u(1-t)}{m^4(\omega + \rho \xi)^2}
\]

One term with denominator \( \Omega^2 \sim (\omega + \rho \xi)^2 \) survived integration by parts but it already contains explicit factor \( (\hat{p} - m)^2 \) and is completely safe under subtraction.

### 3.3 Contribution to the Self-Energy Operator Induced by the Term \( R^{(1c)}_\beta \)

Unlike previous terms the term \( R^{(1c)}_\beta \) is ultravioletly divergent and leads to a logarithmically divergent contribution to the self-energy operator

\[
\Sigma^{(1c)}(p) = 2 \left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int \frac{d^4q}{i\pi^2} \left[ \frac{x}{\Delta} C^{(1)}_\beta + \frac{x^2}{\Delta} C^{(2)}_\beta \right] \frac{\hat{p} + \hat{q} + m}{q^2 D(p + q)} \left( \gamma^\beta + \frac{2\hat{q}q^\beta}{q^2} \right)
\]

Since all divergent contributions contain extra powers of integration momentum \( q \) in the numerator we transform numerators prior to integration separating powers of \( q^2 \) explicitly. Then we cancel this factor \( q^2 \) over similar
denominator factor reducing thus the number of Feynman parameters in the divergent terms. In the end of calculations we will insert additional integration parameter in these terms in order to obtain integral representation for the divergent term which does not contain logs. As a result we will obtain integral representation for the contribution of the divergent terms to the self-energy operator which contains only four integration variables and contains denominators which are not too singular on the mass-shell.

Extraction of the explicit factors \( q^2 \) is performed with the help of the identities

\[
C^{(1)}_\beta (\hat{p} + m) (\gamma^\beta + \frac{2 \hat{q} q^\beta}{q^2}) = C^{(1)}_\beta (\hat{p} + m) \gamma^\beta + 3\hat{p}(-3\hat{p}^2 + 4m\hat{p} + 2m^2)
\]

\[
+ 6z[m(\hat{q}\hat{p} - \hat{p}\hat{q}) + 3m(\hat{p} + m)\hat{q} - 2p^2\hat{q}]
\]

\[
+(3\hat{p}^2 + 2m\hat{p} + 4m^2)\frac{2\hat{q}\hat{p} + \hat{p}q^2}{q^2} + 12z \frac{(pq)\hat{q}\hat{p}\hat{q}}{q^2},
\]

\[
C^{(1)}_\beta (\hat{q}) (\gamma^\beta + \frac{2 \hat{q} q^\beta}{q^2}) = q^2 3z(8m - 5\hat{p}) + 2(4m - 3\hat{p})(\hat{p}\hat{q} + 2\hat{q}\hat{p}) - z(2\hat{q}\hat{p}\hat{q} + \hat{p}q^2),
\]

\[
C^{(2)}_\beta (\hat{p} + m) (\gamma^\beta + \frac{2 \hat{q} q^\beta}{q^2}) = C^{(2)}_\beta (\hat{p} + m) \gamma^\beta + q^2 3z^2(2m - \hat{p}) + 3\hat{p}\hat{q}^2
\]

\[
+ 2z(4p^2\hat{q} + 5m\hat{p}\hat{q} + m\hat{q}\hat{p}) + 3z^2 (2\hat{q}\hat{p}\hat{q} + \hat{p}q^2) + \hat{p}(\hat{p} + 2m) \frac{2\hat{q}\hat{p} + \hat{p}q^2}{q^2} + 4z \frac{(pq)\hat{q}\hat{p}\hat{q}}{q^2},
\]

\[
C^{(2)}_\beta (\hat{q}) (\gamma^\beta + \frac{2 \hat{q} q^\beta}{q^2}) = q^2 (12z^2\hat{q} + 18z\hat{p}) + 2z(2\hat{q}\hat{p}\hat{q} + q^2\hat{p}) + 4(2\hat{p}\hat{q}\hat{p} + p^2\hat{q}).
\]

Then we obtain contribution to the self-energy operator in the form

\[
\Sigma^{(1)}(p) = 2(\frac{\alpha}{4\pi})^2 \int_0^1 dx \int_0^1 dz \int \frac{d^4q}{i\pi^2} \sum_{n=0}^2 N_c(1,1,n)G(1,1,n),
\]

where

\[
N_c(1,1,0) = 3xz[4\hat{q}xz + \hat{p}(-5 + 6x - xz) + 2m(4 + xz)],
\]

\[
N_c(1,1,1) = [xC^{(1)}_\beta + x^2C^{(2)}_\beta](\hat{p} + m) \gamma^\beta + 3\hat{p}x(-3\hat{p}^2 + 4m\hat{p} + 2m^2)
\]

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\[ +2x(4m - 3\hat{p})(\hat{p}q + 2\hat{q}\hat{p}) + 3\hat{p}\hat{p}^2x^2 + 4x^2(2\hat{p}\hat{p}\hat{q} + p^2\hat{q}) + 6xz[m(\hat{q}\hat{q} - \hat{p}\hat{q}) + 3m(\hat{p} + m)\hat{q} - 2p^2\hat{q}] + 2x^2z(4p^2q + 5m\hat{p}\hat{q} + m\hat{q}\hat{p}) + xz(-1 + 2x + 3xz)(2\hat{q}\hat{p}\hat{q} + \hat{p}^2q), \]

\[ N_c(1, 1, 2) = [p^2x(3 + x) + 2m\hat{p}(1 + x) + 4m^2x](2\hat{q}\hat{p}\hat{q} + \hat{p}^2q)^2 + 4xz(3 + x)(p\hat{q})\hat{q}\hat{p}. \]

Performing next momentum integration in eq. (33) (note that in the term with numerator \( N_c(1, 1, 0) \) integration momentum is shifted as \( q \to q - p\eta \) unlike the standard shift \( q \to q - p\eta \) in all other terms) we obtain

\[ \Sigma^{(1c)}(p) = -2\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{1}{z(1 - xz)} \]

\[ \{N_c(1, 1, 0)(\log \frac{\Lambda^2}{\Omega_{t=1}} - 1) + 6\hat{p}x^2z^2\eta \frac{N_c(1, 1, 1)}{\Omega} + (1 - t)N_c(1, 1, 2)\} \]

where

\[ N_c(1, 1, 0) = 3xz[\hat{p}(-5 + 6x - xz - 4xz\eta) + 2m(4 + xz)], \]

\[ N_c(1, 1, 1) = \hat{p}p^2[9x(-3 + x) + 6x(3 - 2x)\eta t + 10xz(3 - 2x)\eta t + 3xz(-1 + 2x + 5xz)\eta^2t^2 + m^26x(6 - (4 + 7z + 2xz)\eta t) + m^2\hat{p}\hat{q}(3 - 5z\eta t)], \]

\[ N_c(1, 1, 2) = \hat{p}p^2\eta^2t^2[2p^2x(3 + x)(3 - 4z\eta t) + 6m\hat{p}x(1 + x) + 12m^2x] \]

and

\[ \Omega_{t=1} = \Omega(t = 1). \]

Once again one may easily get rid of denominator \( \Omega^2 \) with the help of integration by parts over \( t \). We then obtain

\[ \Sigma^{(1c)}(p) = -2\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \{\frac{N_c(1, 1, 0)}{z(1 - xz)}(\log \frac{\Lambda^2}{\Omega_{t=1}} - 1) \]

\[ + \frac{6\hat{p}x^2z^2\eta}{z(1 - xz)} + \frac{N_c'(1, 1, 2) - N_c(1, 1, 1)}{m^2(\omega + p\xi)}\} \]

where

\[ N_c'(1, 1, 2) = 3\hat{p}(1 - 2t)x[p^2(3 + x) + 2m\hat{p}(1 + x) + 4m^2] - 4\hat{p}p^2(2 - 3t)x(3 + x)\eta t. \]
4 Calculation of the Contribution to the Mass Operator Induced by the Term $R^{(2)}_{\beta}$

Contribution to the mass operator connected with the term $R^{(2)}_{\beta}$ in eq.(7) is the simplest one to calculate. Let us start with convergent integration over $l$

$$R^{(2)}_{\beta} = -2 \int \frac{d^4l}{i\pi^2 l^2 D^2(p_0 + l)} \left\{ 4\gamma_\beta [m^2 + 2 \left( \frac{p_0 l}{l^2} \right)^2] + [4ml_\beta + 8(p_0 l)\gamma_\beta] - [2l\gamma_\beta l + l^2\gamma_\beta] \right\}. \tag{40}$$

We combine denominators in the same way as in eq.(9) ($z = 0$ now since there are only two different factors in the denominator)

$$(1 - x)l^2 + xD(p_0 + l) = (l + xp_0)^2 - x\Delta_0, \tag{41}$$

where $\Delta_0 = \Delta(z = 0)$.

Integrand on the right hand side in eq.(40) is a sum of terms containing increasing powers of integration momentum $l$ (or powers of the Feynman parameter $x$ after integration over momentum). Here we encounter the problem concerning the term with the lowest power of integration momentum. It is easy to see that if one puts momentum $p_0$ to be exactly on the mass-shell then respective integral turns out to be the sum of two infrared divergent terms. These divergences, of course, cancel leading to a finite result, however, one has to be extremely careful performing this calculation. Correct way to perform integration here is to preserve small nonzero virtuality $\rho_0 = (m^2 - p_0^2)/m^2$ connected with momentum $p_0$ in the denominator $\Delta_0 = m^2[x + \rho_0(1 - x)]$ at intermediate stages of integration. We thus regularize would be infrared divergences, perform then necessary integrations and go on the mass-shell only in the end of all calculations

$$R^{(2)}_{\beta} = -2m^2\gamma_\beta \int_0^1 dx \left\{ -4 \left[ \frac{2 - x}{\Delta_0} - 2m^2 \frac{x(1 - x)}{\Delta_0^2} \right] + 12 \frac{x}{\Delta_0} + 3 \frac{x^2}{\Delta_0} \right\} + 3\gamma_\beta. \tag{42}$$

This simple integration nicely illustrates the necessity to be extremely careful about going to the mass-shell limit in the presence of the would be
infrared divergences. Really, if we naively put \( p_0 = (m, 0) \) prior to integration over \( x \) in eq.(42) we would obtain a finite result (potentially infrared divergent terms cancel in the integrand even in this case) but continuity in the external momentum would be lost and, hence, that result would be wrong.

It is a simple exercise now to obtain respective contribution to the mass operator

\[
\Sigma^{(2)}(p) = -3\left(\frac{\alpha}{4\pi}\right)^2 \int \frac{d^4 q}{i\pi^2} \frac{\hat{p} + \hat{q} + m}{q^2 D(p + q)}(\gamma^\beta + 2 \hat{q} \gamma^\beta). \tag{43}
\]

It is easy to see that integral in eq.(43) only by numerical factor differs from the expression for the unrenormalized one-loop self-energy operator in the FY gauge (see, e.g [4]). Hence, we may use results of one-loop calculations and immediately put down contribution to the renormalized self-energy operator induced by the term under consideration

\[
\Sigma_R^{(2)}(p) = (\frac{\alpha}{4\pi})^2 (\hat{p} - m)^2 \frac{9 \hat{p}}{m^2} \int_0^1 dx \frac{x}{m^2} \frac{x}{x + \rho(1 - x)}, \tag{44}
\]

and we remind that virtuality \( \rho \) was defined in eq.(17).

5 Calculation of the Contribution to the Mass Operator Induced by the Term \( R^{(3)}_\beta \)

Consider contribution to the mass operator induced by the term \( R^{(3)}_\beta \) in eq.(4). Integration over \( l \) is again convergent and one easily obtains

\[
R^{(3)}_\beta = 6\gamma_\beta \int \frac{d^4 l}{i\pi^2} \left[ \frac{1}{D(p + l)} D(p + l + q) - \frac{1}{D^2(p_0 + l)} \right] \tag{45}
\]

\[
= 6\gamma_\beta \int_0^1 dz \log \frac{m^2}{\Delta_1} = -6\gamma_\beta \int_0^1 dz z(1 - 2z) \frac{1}{\Delta_1},
\]

where

\[
\Delta_1 = \Delta(x = 1) \equiv m^2 - q^2 z(1 - z). \tag{46}
\]

Respective contribution to the self-energy operator has now the form
\[
\Sigma^{(3)}(p) = -6\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dz \left(1 - 2z\right) \int \frac{d^4q}{i\pi^2} \frac{\gamma_\beta(p + \hat{q} + m)}{\Delta_1 D(p + q)} (\gamma^\beta + 2\hat{q}\hat{q}^\beta) \quad (47)
\]
\[
= -6\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dz \left(1 - 2z\right) \int \frac{d^4q}{i\pi^2} (3(2m - \hat{p})G_1(1, 1, 0)
+ (2\hat{q}\hat{q} + \hat{p}\hat{q})G_1(1, 1, 1)),
\]

where subscript accompanying function \(G_1(1, 1, 0)\) means that one has to substitute in the respective definition in eq.(18) \(\Delta_1\) from eq.(46).

One may easily perform momentum integration now and obtain

\[
\Sigma^{(3)}(p) = 18\left(\frac{\alpha}{4\pi}\right)^2 \int_0^1 dz \frac{1 - 2z}{1 - z} \int_0^1 du \int_0^1 dt \left\{ (2m - \hat{p})(\log \Lambda_2 + \Omega_{11}\frac{\Lambda^2}{\Omega_{11}})
- \frac{\Lambda^2}{\Lambda^2 + \Omega_{11}} - \hat{p}p^2(1 - u)^2t^2
\right\},
\]

where

\[
\Omega_{|x=1} = \Omega(x = 1) \equiv p^2(1 - u)^2t + (m^2 - p^2)(1 - u) + \frac{m^2u}{z(1 - z)},
\]
\[
\Omega_{11} = \Omega(x = 1, t = 1) \equiv p^2(1 - u)^2 + (m^2 - p^2)(1 - u) + \frac{m^2u}{z(1 - z)}.
\]

Note that one has to preserve function \(\Omega_{11}\) even on the background of the infinite cutoff \(\Lambda^2\) to facilitate convergence of integration over Feynman parameters in eq.(18).

6 Total Expression for the Overlapping Diagram Contribution to the Self-Energy Operator

\[2\text{footnote}{\text{Compare detailed discussion on this point in [1].}}\]
6.1 General Representation of the Overlapping Diagram Contribution to the Self-Energy Operator

Next task is to perform double subtraction in all contributions to the self-energy operator obtained above in eq.(24), eq.(30), eq.(38) and eq.(48). Contribution in eq.(44) is already subtracted and does not need any further transformations. The term containing factor $(\hat{p} - m)^2$ in eq.(30) is also already subtracted. All other contributions to the self-energy operator either contain in the integrand denominator $\omega + \rho \xi$ or logarithm of the cutoff momentum. Consider first logarithmically divergent contributions.

Divergent contribution in eq.(38) has the form

$$\Sigma^{(1c)}(p) = -(\frac{\alpha}{4\pi})^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \frac{6x}{1-xz} [(\hat{p} - m)h_1 + mh_2][\log \frac{\Lambda^2}{\Omega_{t=1}} - 1],$$

(50)

where

$$h_1 = -5 + 6x - xz - 4xz\eta, \quad h_2 = 3 + 6x + xz - 4xz\eta.$$  (51)

Momentum cutoff disappears after subtraction and emerging logarithm may be put down with the help of integral representation

$$\log \frac{\omega|_{t=1} + \rho \xi|_{t=1}}{\omega|_{t=1}} = \rho \xi|_{t=1} \int_0^1 \frac{dv}{\omega|_{t=1} + \rho \xi|_{t=1}v}.$$  (52)

This parametric representation of the logarithm gives one a chance to gain an additional apparent factor $\hat{p} - m$ in the numerator which is very convenient for the subtraction procedure.

Subtracted expression in eq.(51) acquires then the form

$$\Sigma^{(1c),R}(p) = -(\frac{\alpha}{4\pi})^2 \frac{(\hat{p} - m)^2}{m^2} \int_0^1 dx \int_0^1 dz \int_0^1 du \frac{6x\xi|_{t=1}}{1-xz} (\hat{p}(h_1 + 2h_2 \frac{\xi|_{t=1}v}{\omega|_{t=1}}) + m(h_1 + h_2 + 2h_2 \frac{\xi|_{t=1}v}{\omega|_{t=1}})).$$

(53)

Consider next divergent contribution in eq.(48). It looks like exactly as the previous divergent contribution with $h_1 = 1$ and $h_2 = -1$
\[ \Sigma_{\text{div}}^{(3)}(p) = -18\left( \frac{\alpha}{4\pi} \right)^2 \int_0^1 dz \frac{1-2z}{1-z} \int_0^1 du [\hat{p} - m - m] \]  

One easily obtains doubly subtracted contribution to the mass operator

\[ \Sigma_{\text{div,R}}^{(3)}(p) = -18\left( \frac{\alpha}{4\pi} \right)^2 \frac{\hat{p} - m}{m^2} \int_0^1 dz \frac{1-2z}{1-z} \int_0^1 du \xi_{11} \int_0^1 dv \frac{dv}{\omega_{11} + \rho \xi_{11} v} \]  

where

\[ \xi_{11} = \xi(x = 1, t = 1) = z(1-z)u(1-u), \]

\[ \omega_{11} = \omega(x = 1, t = 1) = u + z(1-z)(1-u)^2. \]

Now we have to consider subtraction of all other entries in eq. (24), eq. (30), eq. (38) and eq. (48) which contain factor \( \omega + \rho \xi \) in the denominator. Collecting all such terms we obtain the expression

\[ \Sigma_{\omega+\rho \xi}(p) = \left( \frac{\alpha}{4\pi} \right)^2 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{12}{\omega + \rho \xi} \left( \frac{\hat{p} - m}{m^2} \right)^2 [\hat{p} f_{1p} + m f_{1m}] \]

\[ + (\hat{p} - m) f_2 + m f_3, \]  

where

\[ f_i = f'_i + \frac{x(1-x)u}{1-xz} f''_i, \]

\[ f'_{1p} = 2 + x(-7 + x + 3t + xt) + [-2 + x(4-2x + 9z - 2xz - 6zt - 2xzt)]\eta t + S, \]

\[ f'_{1m} = 2 + x(-8 + x + 5t + 4xt) + [-1 + x(4-4x + 11z - 6xz - 12zt - 4xzt)]\eta t + 2S, \]

\[ f'_2 = x(-7 + x + 8t + 7xt) + [-1 + x(6 - 6x + 8z - 10xz - 18zt - 6xzt)]\eta t + 3S, \]

\[ f'_3 = 3x(1 + x)t + x(2 - 2x - 3z - 4xz - 6zt - 2xzt)\eta t + S, \]
\[ f''_1 = t(1 - 6z + 10zt - 2z\eta t) + 4\frac{1 - 2t}{\eta}, \]

\[ f''_1m = t(2 - 11z + 20zt - 4z\eta t) + 6\frac{1 - 2t}{\eta}, \]

\[ f''_2 = t(1 - 15z + 30zt - 6z\eta t) + 6\frac{1 - 2t}{\eta}, \]

\[ f''_3 = -t(1 + 4z - 10zt + 2z\eta t), \]

\[ S = \frac{1}{2}xz(-1 + 2x + 5xz)\eta^2t^2. \]

After subtractions we obtain

\[
\Sigma_{|\omega + \rho \xi, R(p)} = \left(\frac{\alpha}{4\pi}\right)^2 \frac{(\hat{p} - m)^2}{m^2} \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{1}{\omega + \rho \xi} \left\{ \hat{p}[f_1 + f_2 \frac{\xi}{\omega}] + 2f_3(\frac{\xi}{\omega})^2 \right\} + m[f_1 + (f_2 + f_3) \frac{\xi}{\omega} + 2f_3(\frac{\xi}{\omega})^2] - \frac{p^2(\hat{p} + m)x(1 - x)u(1 - t)}{m^2(\omega + \rho \xi)} \right\}
\]

Subtraction in the last term in eq. 48 may be easily performed finishing thus calculation of all contributions to the subtracted self-energy operator induced by the diagram with overlapping photons.

Collecting all entries to the self-energy we obtain

\[
\Sigma_R(p) = \left(\frac{\alpha}{4\pi}\right)^2 \frac{(\hat{p} - m)^2}{m^2} \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^1 dv \left\{ \frac{12}{\omega + \rho \xi} \left\{ \hat{p}[f_1 + f_2 \frac{\xi}{\omega}] + 2f_3(\frac{\xi}{\omega})^2 \right\} - \frac{p^2(\hat{p} + m)x(1 - x)u(1 - t)}{m^2(\omega + \rho \xi)} \right\}
\]

\[
- \frac{6x}{1 - xz} \frac{\xi_{|t=1}}{\omega_{|t=1} + \rho \xi_{|t=1}} \left\{ \hat{p}(h_1 + 2h_2 \frac{\xi_{|t=1}}{\omega_{|t=1}}) + m(h_1 + h_2 + 2h_2 \frac{\xi_{|t=1}}{\omega_{|t=1}}) \right\}
\]

\[
- 18z(1 - 2z)(1 - u)^2 \frac{t^2}{\omega_{|x=1} + \rho \xi_{|x=1}} \left\{ \hat{p}[1 + 3 \frac{\xi_{|x=1}}{\omega_{|x=1}} + 2(\frac{\xi_{|x=1}}{\omega_{|x=1}})^2] + 2m(1 + \frac{\xi_{|x=1}}{\omega_{|x=1}})^2 \right\}
\]

\[
- 18 \frac{1 - 2z}{1 - z} \frac{\xi_{11}}{\omega_{11} + \rho \xi_{11}v} \left\{ \hat{p}(1 - 2 \frac{\xi_{11}v}{\omega_{11}}) - 2m \frac{\xi_{11}v}{\omega_{11}} \right\} + 9 \frac{\hat{p}x}{x + \rho(1 - x)}
\]
\[ \equiv \frac{\alpha^2}{4\pi} \frac{\hat{\rho} - m}{m^2} [\hat{\rho}\sigma_p(\rho) + m\sigma_m(\rho)]. \]

Note that despite the appearance each term on the right hand side is at most a four dimensional integral.

### 6.2 Infrared Safe Representation of the Overlapping Diagram Contribution to the Self-Energy Operator

The representation for the overlapping diagram contribution in eq.(59) is quite suitable for calculation of the respective contribution to HFS but needs further transformations for the Lamb shift calculation. The problem emerges when one tries to calculate values of the subtraction constants \(\sigma_p(0)\) and \(\sigma_m(0)\) which enter the expression for the Lamb shift (see below). Integrals containing terms \(f_2\xi/\omega\) and \(2f_3\xi^2/\omega^2\) in eq.(59) diverge as \(\log \rho\) when \(\rho\) goes to zero (we remind that according to eq.(17) \(\omega = xu + zt(1-xz) + O(xuzt)\)) and this divergence disappears only in the sum

\[ M(\rho) \equiv \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{1}{\omega + \rho\xi} [f_2\xi/\omega + 2(f_3\xi^2/\omega)] \] (60)

which enters eq.(59).

Throughout investigation of the integrand in eq.(60) with the help of the definitions in eq.(17) and eq.(57) shows that only terms

\[ (f_2\xi)_{IR} = x(-7 + x)z(1-xz) - tu, \] (61)

\[ (f_3\xi^2)_{IR} = xtl(5 + x - 3z - 4xz)z^2(1-xz)^2 + u^2(5 - u) \]

lead to divergencies at vanishing virtuality. Each term in the subtracted numerator \(f_2\xi - (f_2\xi)_{IR}\) contains either factor \(xu\) or \(zt\) which suppress divergence at vanishing virtuality. In the same way each term in the other subtracted numerator \(f_3\xi^2 - (f_3\xi^2)_{IR}\) contains one of the factors \((xu)^2\), \((zt)^2\) or \((xu)(zt)\) and respective integrals are also infrared safe at vanishing virtuality.

We are going to transform integral representation in eq.(59) to a form containing integrands which lead only to the integrals finite at \(\rho = 0\). Cancellation of infrared divergent terms may be greatly facilitated by reducing
the multiplicity of integration. To reduce the number of integrations we note that infrared properties of the integrands does not change if one substitutes in the numerators in eq.(61) $x \to \partial \omega / \partial u$ and $tu \to u \partial \omega / \partial z$. Using this fact we separate infrared divergent terms by substituting these derivatives in eq.(61) and define new infrared safe functions

$$\tilde{f}_2 \xi \equiv f_2 \xi - D_2 \omega,$$

$$\tilde{f}_3 \xi^2 \equiv f_3 \xi^2 - D_3 \omega,$$

where

$$D_2 = (-7 + x)z(1 - xz) \frac{\partial}{\partial u} - u \frac{\partial}{\partial z},$$

$$D_3 = t(5 + x - 3z - 4xz)z^2(1 - xz)^2 \frac{\partial}{\partial u} + xu^2(5 - u) \frac{\partial}{\partial z}.$$}

Now $M(\rho)$ in eq.(64) acquires the form

$$M(\rho) \equiv \tilde{M}(\rho) + M_1(\rho),$$

where $\tilde{M}(\rho)$ differs from $M(\rho)$ only due to substitutions $f_2 \xi \to \tilde{f}_2 \xi$ and $f_3 \xi^2 \to \tilde{f}_3 \xi^2$. Term $\tilde{M}(\rho)$ is infrared safe and logarithmic infrared divergency is connected only with the term

$$M_1(\rho) = \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{1}{\omega} \left[ \frac{D_2 \omega}{\omega^2} + \frac{2D_3 \omega}{\omega^2} \right]$$

$$= \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dv \left[ \frac{D_2 \omega}{(\omega + \rho v \xi)^2} + 2(1 - v) \frac{2D_3 \omega}{(\omega + \rho v \xi)^3} \right].$$

We easily extract infrared safe part $M_\xi(\rho)$ from the term $M_1(\rho)$ with the help of the trivial substitution

$$\omega = -\rho v \xi + (\omega + \rho v \xi),$$

which is valid according to the definition in eq.(17).

First term on the right hand side in eq.(66) produces infrared safe contribution to $M_1(\rho)$.
\[ \rho M_\xi(\rho) = -\rho \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^1 dv v \left\{ \frac{D_2 \xi}{(\omega + \rho v \xi)^2} + 4(1-v) \frac{D_3 \xi}{(\omega + \rho v \xi)^3} \right\} \]

while the second term on the right hand side contains complete derivatives over one of the Feynman parameters which may be easily integrated. Hence, we obtain the following representation for the \( M_1(\rho) \) term

\[ M_1(\rho) = \rho M_\xi(\rho) + M_{up}(\rho) + M_{down}(\rho), \]

where

\[ M_{up}(\rho) = \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^1 dv \left\{ \frac{(7-x)z(1-xz)}{\omega|_{u=1} + \rho v \xi|_{u=1}} \right\}, \]

\[ u \frac{t(5+x-3z-4xz)z^2(1-xz)^2}{(\omega|_{u=1} + \rho v \xi|_{u=1})^2} + \left\{ \frac{xu^2(5-u)}{(\omega|_{z=1} + \rho v \xi|_{z=1})^2} \right\}; \]

\[ M_{down}(\rho) = \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^1 dv \left\{ \frac{-7+x}{t + \rho v(1-t)} - \frac{1}{x + \rho v(1-x)} + 2(1-v) \left\{ \frac{t(5+x-3z-4xz)}{[t + \rho v(1-t)]^2} + \frac{x(5-u)}{[x + \rho v(1-x)]^2} \right\} \right\}; \]

It is easy to check that all integrands in \( M_{up}(\rho) \) are infrared safe even at vanishing virtuality, while each integrand in the expression for the term \( M_{down}(\rho) \) produces \( \log \rho \) in the small \( \rho \) limit. Happily due to simplicity of parametric integrals in the expression for \( M_{down}(\rho) \) one may easily obtain its parametric representation in the form of one-dimensional integral

\[ M_{down}(\rho) = \frac{15}{2} \int_0^1 dv \left\{ -\frac{1}{2} - \frac{\rho v}{1-\rho v} \ln \frac{1}{\rho v} \right\} \]

+ 2(1-v)\left\{ \frac{\rho v}{1-\rho v} \ln \frac{1}{\rho v} + \frac{\rho v}{1-\rho v} \left( \frac{1}{\rho v} \ln \frac{1}{\rho v} - 1 \right) \right\}.

It is easy to see that total expression for \( M_{down}(\rho) \) at \( \rho \to 0 \) is free of logs and one may easily obtain
Thus, we have obtained representation for the contribution of the overlapping diagram to the self-energy operator in the form

\[
\Sigma_R(p) = \left(\frac{\alpha}{4\pi}\right)^2 \left(\hat{p} - m\right)^2 \{\hat{p} \tilde{\sigma}_p(\rho) + m \tilde{\sigma}_m(\rho)\} 
+ 12(\hat{p} + m)[\rho \xi(\rho) + M_{up}(\rho) + M_{down}(\rho)]
\]

where \(\tilde{\sigma}_p(\rho)\) and \(\tilde{\sigma}_m(\rho)\) differ from the respective expressions in eq.(59) only by substitutions \(f_2 \xi \rightarrow \tilde{f}_2 \xi\) and \(f_3 \xi^2 \rightarrow \tilde{f}_3 \xi^2\) (compare eq.(64)).

Each term on the right hand side in eq.(72) is explicitly finite at vanishing virtuality. Hence, this representation for the self-energy operator is very convenient for subtraction of its value on the mass shell and for calculation of the respective contribution to the Lamb shift.

### 7 Contributions to the Energy Splittings induced by the Overlapping Mass Operator

#### 7.1 Contribution to HFS Splitting

Radiative correction to hyperfine splitting induced by the graph in Fig.1 is given by the matrix element of this diagram calculated with on mass-shell external electron lines and projected on respective spin states and multiplied by the square in the origin of the Schrödinger-Coulomb wave function (for more details see, e.g. [1] and [4]). It is not difficult to obtain explicit expression to the energy splitting having formula for the mass operator in eq.(59)

\[
\Delta E_{HFS} = \frac{\alpha^2(Z\alpha)}{\pi n^3} E_F \left(-\frac{1}{2\pi^2}\right) \int_0^\infty d|k| \frac{9\pi^2}{4} + \pi \mathcal{H},
\]

where\(^3\)

\(^3\)Integration momentum \(k\) is measured in the units of electron mass in this section.
\[ \mathcal{H} = \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \frac{6}{\sqrt{\omega \xi}} [f_1 + f_2 \xi] \\
+ 2f_3 \left( \frac{\xi}{\omega} \right)^2 + \frac{3x(1-x)u(1-t)}{\sqrt{\omega \xi}} \frac{1}{\xi - 1} (\xi - 1) \]
\[ - \frac{2x}{1-xz} \frac{\xi|_{t=1}}{\omega|_{t=1}} (3h_1 + 2h_2 \frac{\xi|_{t=1}}{\omega|_{t=1}}) \]
\[ -9 \frac{1}{\omega|_{x=1}} \frac{\xi|_{x=1}}{\omega|_{x=1}} [1 + 3 \frac{\xi|_{x=1}}{\omega|_{x=1}} + 2 (\frac{\xi|_{x=1}}{\omega|_{x=1}})^2] \]
\[ -6 \frac{1}{1-z} \frac{\xi|_{11}}{\omega|_{11}} (3 - 2 \frac{\xi|_{11}}{\omega|_{11}}) \].

Numerically we obtain
\[ \Delta E_{HFS} = -1.984(1) \frac{\alpha^2 (Z\alpha)}{\pi n^3} E_F. \]  

### 7.2 Contribution to the Lamb Shift

Contribution to the Lamb shift induced by the diagram in Fig.1 is given by the matrix element similar to the one in the case of HFS, the only difference being in the spin structure and the necessity to perform an additional subtraction on the mass shell, already mentioned above (see, e.g. [4] and [5]). This subtraction is greatly facilitated by the representation in eq. (72). Using this representation we obtain

\[ \Delta E_L = \frac{8(Z\alpha)^5}{\pi n^3} \left( \frac{\alpha}{4\pi} \right)^2 \frac{m}{m_r} 3 \int_0^\infty d|k| \frac{\{[\tilde{\sigma}_p(k^2) + \tilde{\sigma}_m(k^2)] - [\tilde{\sigma}_p(0) + \tilde{\sigma}_m(0)]\}}{k^2} \]
\[ + 24k^2 M_L(k^2) + 24[M_{up}(k^2) - M_{up}(0)] + 24[M_{down}(k^2) - M_{down}(0)] \]
\[ \equiv \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} \frac{m}{m_r} 3 \frac{1}{2\pi^2} \int_0^\infty d|k| [\Delta \mathcal{E}_{L1}(k^2) + \Delta \mathcal{E}_{L2}(k^2) + \Delta \mathcal{E}_{L3}(k^2) + \Delta \mathcal{E}_{L4}(k^2)]. \]

According to definitions in eq. (59) and eq. (72)
\[ \bar{\sigma}_p(k^2) + \bar{\sigma}_m(k^2) = \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^1 dv \{ \frac{12}{\omega + k^2 \xi} \} \tag{77} \]

\[ [f_{1p} + f_{1m} + f_3 \frac{\xi}{\omega} + 2(\frac{f_2 \xi}{\omega} + \frac{2f_3 \xi^2}{\omega^2})] - \frac{24(1 - k^2)x(1 - x)u(1 - t)}{(\omega + k^2 \xi)^2} \]

\[ - \frac{6x}{1 - xz} \frac{\xi_{|t=1}}{\omega_{|t=1} + k^2 \xi_{|t=1}v} (2h_1 + h_2 + 4h_2 \frac{\xi_{|t=1}v}{\omega_{|t=1}}) \]

\[ -18 \frac{z(1 - 2z)(1 - u)^2}{\omega_{|x=1} + k^2 \xi_{|x=1}v} \left[ 3 + 7 \frac{\xi_{|x=1}}{\omega_{|x=1}} + 4 \left( \frac{\xi_{|x=1}}{\omega_{|x=1}} \right)^2 \right] \]

\[ -18 \frac{1 - 2z}{1 - z} \frac{\xi_{11}}{\omega_{11} + k^2 \xi_{11}v} \left( 1 - 4 \frac{\xi_{11}v}{\omega_{11}} \right) + 9 \frac{x}{x + k^2(1 - x)} \}, \]

Subtraction may be easily performed with the help of identities

\[ \frac{1}{k^2 \xi} \frac{1}{\omega + k^2 \xi} - \frac{1}{\omega} = - \frac{\xi}{\omega (\omega + k^2 \xi)}, \tag{78} \]

\[ \frac{1}{k^2 \xi} \left( \frac{1}{\omega + k^2 \xi} \right)^2 - \frac{1}{\omega^2} = - \frac{\xi}{\omega (\omega + k^2 \xi)^2} + \frac{1}{\omega (\omega + k^2 \xi)}, \]

and we obtain

\[ \Delta \mathcal{E}_{L1}(k^2) = \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^1 dv \]

\[ \left\{ - \frac{12}{\omega + k^2 \xi \omega} [f_{1p} + f_{1m} + f_3 \frac{\xi}{\omega} + 2(\frac{f_2 \xi}{\omega} + \frac{2f_3 \xi^2}{\omega^2})] \right\} \]

\[ + \frac{24x(1 - x)u(1 - t)}{(\omega + k^2 \xi)^2} \left[ 1 + 2 \frac{\xi}{\omega^2} + k^2 \left( \frac{\xi}{\omega} \right)^2 \right] \]

\[ + \frac{6x}{1 - xz} \frac{\xi_{|t=1}}{\omega_{|t=1} + k^2 \xi_{|t=1}v} (2h_1 + h_2 + 4h_2 \frac{\xi_{|t=1}v}{\omega_{|t=1}}) \]

\[ + 18 \frac{z(1 - 2z)(1 - u)^2}{\omega_{|x=1} + k^2 \xi_{|x=1}v} \left[ 3 + 7 \frac{\xi_{|x=1}}{\omega_{|x=1}} + 4 \left( \frac{\xi_{|x=1}}{\omega_{|x=1}} \right)^2 \right] \]

\[ + 18 \frac{1 - 2z}{1 - z} \frac{\xi_{11}}{\omega_{11} + k^2 \xi_{11}v} \left( 1 - 4 \frac{\xi_{11}v}{\omega_{11}} \right) + 9 \frac{x}{x + k^2(1 - x)} \}, \]

Integration over \( v \) is greatly facilitated by the obvious relations
\begin{equation}
\int_0^1 dv \int_0^\infty \frac{d|k|}{k^2} F(k^2 v) = \frac{2}{3} \int_0^\infty \frac{d|k|}{k^2} F(k^2),
\end{equation}
\begin{equation}
\int_0^1 dv dv \int_0^\infty \frac{d|k|}{k^2} F(k^2 v) = \frac{2}{5} \int_0^\infty \frac{d|k|}{k^2} F(k^2),
\end{equation}
which are valid for arbitrary function F. After integration over \( v \) and \( k \) we obtain
\begin{equation}
\int_0^\infty d|k| \Delta \mathcal{E}_{L1}(k^2) = -\frac{9\pi^2}{4} + \pi L,
\end{equation}
where
\begin{equation}
L = \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^\infty d|k| \left\{- \frac{6}{\sqrt{\omega^2 k^2}} [f_{1p} + f_{1m} + f_3 \frac{\xi}{\omega}] + 2\left( \frac{f_2 \xi}{\omega} + \frac{2f_3 \xi^2}{\omega^2} \right) \right\} + 6x(1-x)u(1-t) \left( 1 + 3 \frac{\xi}{\omega} \right)
\end{equation}
\begin{equation}
+ \frac{2x}{1-xz} \left( \frac{\xi}{\omega} \right) \left( \frac{\xi}{\omega} \right) \left( 2h_1 + h_2 + \frac{12}{5} h_2 \frac{\xi}{\omega} - \frac{\xi}{\omega} \right) + 9 \xi \left( 1 - z \right) \left( 1 - u \right)^2 \left( \frac{\xi}{\omega} \right) \left( \frac{\xi}{\omega} \right) \left( 3 + 7 \frac{\xi}{\omega} + 4 \left( \frac{\xi}{\omega} \right)^2 \right) + 6 \left( \frac{1 - 2z}{1 - z} \right) \left( \frac{\xi}{\omega} \right) \left( 1 - \frac{12}{5} \xi \right).
\end{equation}

Next we perform integration over \( v \) and \( |k| \) in the second term on the right hand side in eq. (76)
\begin{equation}
\int_0^\infty d|k| \Delta \mathcal{E}_{L2}(k^2) = 24 \int_0^\infty d|k| M_\xi(k^2)
\end{equation}
\begin{equation}
= -24 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^\infty dv dv \int_0^\infty d|k| \left[ \frac{D_2 \xi}{(\omega + k^2 v \xi)^2} \right] \left[ \frac{D_3 \xi}{(\omega + k^2 v \xi)^3} \right] + 4(1-v) \left( \frac{\omega + k^2 v \xi}{\sqrt{\omega^2 \xi^2}} \right) \left( \frac{D_2 \xi}{5 \xi} + \frac{6D_3 \xi}{\omega} \right) \equiv \pi L_\xi.
\end{equation}
Contribution of the third term on the right hand side in eq.(76) is equal to

\[ \int_0^\infty d|k| \Delta \mathcal{E}_{L3}(k^2) = 24 \int_0^\infty \frac{d|k|}{k^2} [M_{up}(k^2) - M_{up}(0)] \tag{84} \]

\[ = 24 \int_0^1 dx \int_0^1 dz \int_0^1 du \int_0^1 dt \int_0^\infty d|k| \left\{ -\frac{v \xi |_{u=1}}{\omega |_{u=1}} \frac{(7 - x)z(1 - xz)}{\omega |_{u=1} + k^2 v \xi |_{u=1}} + 2(1 - v)\frac{v \xi |_{u=1}}{\omega |_{u=1}} \left[ \frac{1}{(\omega |_{u=1} + k^2 v \xi |_{u=1})^2} \right. \right. \\
\left. + \left. \frac{1}{(\omega |_{u=1} + k^2 v \xi |_{u=1})^2} \right] t(5 + x - 3z - 4xz)z^2(1 - xz)^2 \right. \left. + \frac{v \xi |_{z=1}}{\omega |_{z=1}} \left[ \frac{1}{(\omega |_{z=1} + k^2 v \xi |_{z=1})^2} \right. \right. \right. \\
\left. + \left. \frac{1}{(\omega |_{z=1} + k^2 v \xi |_{z=1})^2} \right] xu^2(5 - u) \right\} \\
\]

\[ = 8\pi \int_0^1 dx \int_0^1 dz \int_0^1 dt \frac{z(1 - xz)}{\omega |_{u=1}^2} \left\{ -7 + x + \frac{6}{5} t(5 + x - 3z - 4xz)z(1 - xz) \right\} \]

\[ + 8\pi \int_0^1 dx \int_0^1 du \int_0^1 dt \frac{u \xi |_{z=1}}{\omega |_{z=1}^2} \left\{ -1 + \frac{6}{5} xu(5 - u) \right\} \]

\[ \equiv \pi L_{up}. \]

Last term on the right hand side in eq.(76) gives

\[ \int_0^\infty d|k| \Delta \mathcal{E}_{L4}(k^2) = 24 \int_0^\infty \frac{d|k|}{k^2} [M_{down}(k^2) - M_{down}(0)] \tag{85} \]

\[ = 180 \int_0^1 dv v \int_0^\infty \frac{d|k|}{1 - k^2 v} \left\{ \ln(k^2 v) - 2(1 - v)\ln(k^2 v) + 1 + \ln(k^2 v) \right\} \\
\]

\[ = 24 \int_0^\infty d|k| \left\{ \frac{\ln k^2}{1 - k^2} - \frac{4}{1 - k^2} \right\} \\
\]

\[ = 12\pi^2. \]

Combining expressions in eq.(81), eq.(83), eq.(84) and eq.(85) we obtain final expression for the contribution to the Lamb shift in the form
\[ \Delta E_L = m \left( \frac{m_r}{m} \right)^3 \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} \frac{1}{2\pi^2} \left\{ \frac{39\pi^2}{4} + \pi (L + L_\xi + L_{ap}) \right\}, \quad (86) \]

or numerically

\[ \Delta E_L = 1.749(2) \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} \frac{m_r}{m}^3 m. \quad (87) \]

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Figure Caption

Fig.1. Diagram with two external photons and overlapping two-loop electron self-energy insertion in the electron line.
Fig. 1