A note on quaternionic Kähler manifolds with ends of finite volume

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Abstract

We prove that complete non-locally symmetric quaternionic Kähler manifolds with an end of finite volume exist in all dimensions $4m \geq 4$.

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1 Introduction

Quaternionic Kähler manifolds constitute one of the most interesting classes of Einstein manifolds in Riemannian geometry [Bes]. They occur naturally in the context of the classification of Riemannian holonomy groups [B]. One of the long-standing conjectures in differential geometry is that complete quaternionic Kähler manifolds of positive scalar curvature are symmetric [LS]. At the time of writing, it has been proven in dimensions 4 [H, FK], 8 [PS], 12 and 16 [BWW]. The first examples of complete quaternionic Kähler manifolds of negative scalar curvature which are not locally symmetric were found in [A2]. However, until today, no such examples of finite volume are known.

It was shown in [CRT] that complete non-locally symmetric quaternionic Kähler manifolds of negative scalar curvature with an end of finite volume exist in dimensions 4 and 8.

The construction was based on the existence in all dimensions $4m \geq 4$ of complete non-locally symmetric quaternionic Kähler manifolds $(M, g)$ of negative scalar curvature with a cohomogeneity one action by a Lie group $G$ of isometries which admits a lattice $\Gamma \subset G$ acting freely an properly discontinuously on $M$. In fact, the quotients $X = M/\Gamma$, with the induced metric, were shown to have the topological structure of a cylinder $X = \mathbb{R} \times G/\Gamma$ fibering over the line with fibers of finite volume. Moreover, the volume of the half-cylinder $\{(t, x) \in X \mid t > 0\} \subset X$, was shown to be finite. In particular, if $\Gamma \subset G$ is cocompact, then $X$ has two ends and one of them is of finite volume. However, cocompact lattices in $G$ were only shown to exist in dimensions $\leq 8$. This left the following problem open.

**Problem.** Do there exist complete non-locally symmetric quaternionic Kähler manifolds of negative scalar curvature with an end of finite volume in all dimensions $4m \geq 4$?

In this note we give a positive answer to this problem.

**Theorem 1.1.** For all $4m \geq 4$ there exists a complete non-locally symmetric quaternionic Kähler manifold $(M, g)$ of negative scalar curvature with two ends such that one end is of finite volume and the other of infinite volume.

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2 Proof of Theorem 1.1

The following lemma is a slight refinement of the statements about the metric (1.8) in [F, Theorem 1.1]. For convenience we have redefined the connection forms. For expository reasons we include a proof.
Lemma 2.1. Let \((N, g_N)\) be a hyper-Kähler manifold with integral Kähler forms \(\sigma_1, \sigma_2, \sigma_3\), that is the corresponding de Rham classes \([\sigma_i]\) belong to the image of the natural map \(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})\). Let \(P \to N\) be a \(T^3\)-principal bundle with connection \((\alpha_1, \alpha_2, \alpha_3) \in \Omega^1(P, \mathbb{R}^3)\) and curvature forms \(d\alpha_1 = \sigma_1, d\alpha_2 = \sigma_2, d\alpha_3 = \sigma_3\), where \(T^3 = \mathbb{R}^3/\mathbb{Z}^3\). Then

(i) \(g := dt^2 + 4\rho^2\sum \alpha_i^2 + \rho g_N, \quad \rho = e^{2t}\) is a quaternionic Kähler metric on \(M := \mathbb{R} \times P\).

(ii) The scalar curvature of \((M, g)\) is negative.

(iii) \((M, g)\) is complete if and only if \((N, g_N)\) is.

Proof. (i) Consider the two-forms

\[
\omega_i := 2\rho dt \wedge \alpha_i + 4\rho^2 \alpha_j \wedge \alpha_k + \rho \sigma_i, \quad i = 1, 2, 3,
\]

where \((i, j, k)\) is a cyclic permutation of \(\{1, 2, 3\}\). Together with the metric \(g\) they define an almost hyper-Hermitian structure \((g, J_1, J_2, J_3)\) on \(M\) with the fundamental forms \(\omega_i = g \circ J_i = g(J_i \cdot , \cdot )\), \(i = 1, 2, 3\). To prove that the metric is quaternionic Kähler with the quaternionic structure \(Q = \text{span}\{J_1, J_2, J_3\}\) it suffices to check that the ideal of the exterior algebra generated by the forms \(\omega_1, \omega_2, \omega_3\) is a differential ideal [S]. This follows immediately by calculating the differentials of (1), as done in [F].

(ii) To see that the scalar curvature is negative it suffices to remark that the fibers of the projection \(M \to N\) are quaternionic submanifolds and of constant negative sectional curvature. Since a quaternionic submanifold has the same reduced scalar curvature as the ambient quaternionic Kähler manifold [A1], the claim follows.

(iii) The completeness of \((N, g_N)\) implies that of \((M, g)\) by observing that \(g\) is of the form \(g = dt^2 + g_t\), where \(g_t\) is a family of metrics on \(P\) which over compact subsets \(K \subset \mathbb{R}\) is uniformly bounded from below by a complete metric \(g_K\) on \(P\) and applying [CHM, Lemma 2]. In fact, we can simply take the (product) metric \(g_K = 4\rho_0^2 \sum \alpha_i^2 + \rho_0 g_N\), where \(\rho_0 = \min_K \rho\).

For the converse, we note first that any embedded submanifold of a complete Riemannian manifold is complete. Hence, the completeness of \((M, g)\) implies that of \((P, g_t)\). Since, for fixed \(t\), \(g_t\) is simply a product metric, the completeness of the factor \((N, \rho(t)g_N)\) and hence of \((N, g_N)\) follows.

\(\square\)

Lemma 2.2. There exists a K3 surface \(S\) which admits a hyper-Kähler structure \((g_S, J_1, J_2, J_3)\) with integral Kähler forms.

Proof. This follows from [D, Table 1]. In fact, from the table we see that there exists a K3 surface of degree 10 with transcendental lattice of rank two and intersection form represented by \(10 \cdot \text{id}\) with respect to a basis of the lattice. Complementing such a basis by the Fubiny study class of the projective embedding, we obtain three integral Kähler classes. Choosing a representative in each of the three classes we obtain a hyper-Kähler triple on \(S\) defining the desired hyper-Kähler structure. \(\square\)
Lemma 2.3. For every compact hyper-Kähler manifold \((N, g_N)\) with integral Kähler forms the quaternionic Kähler manifold \((M, g)\) of Lemma 2.1 has two ends, one of finite volume and the other of infinite volume.

Proof. The (metric) volume form of \((M, g)\) is related to the volume form of \((P, g_0)\) by

\[\text{vol}_g = 8\rho^{2n+3} \, dt \wedge \text{vol}_{g_0}.\]

Thus for all \(t_0 \leq 0 \leq t_1\) we have

\[f(t_0, t_1) := \text{vol}\left(\{(t, p) \in M \mid t_0 < t < t_1\}\right) = C \int_{t_0}^{t_1} \rho(t)^{2n+3} \, dt, \quad C = 8 \int_P \text{vol}_{g_0}.\]

Inserting \(\rho(t) = e^{2t}\) we obtain that

\[f(0, t) = \frac{C}{4n + 6} (e^{(4n+6)t} - 1) \to \infty \quad (t \to \infty)\]

\[f(-t, 0) = \frac{C}{4n + 6} (1 - e^{-(4n+6)t}) \to \frac{C}{4n + 6} \quad (t \to \infty).\]

This proves the claim. \(\square\)

Proof. (of Theorem 1.1) By [CRT] we can assume that \(m \geq 3\). In fact, it suffices to assume \(m \geq 2\). Let \((S, g_S)\) be any hyper-Kähler manifold as in Lemma 2.2. The product \((N, g_N) = (S, g_S)^{(m-1)}\) is a hyper-Kähler manifold of dimension \(4m - 4\) with integral Kähler forms. By Lemma 2.1 we can associate a complete quaternionic Kähler manifold \((M, g)\) of dimension \(4m\) and by Lemma 2.3 it has an end of finite volume. Now it suffices to show that \((M, g)\) is not locally symmetric. Since \((M, g)\) is a complete quaternionic Kähler manifold of negative scalar curvature, if it were locally symmetric, its universal covering would be a symmetric space of non-compact type and therefore contractible. It would follow that \(\pi_2(M) = 0\) and the homotopy sequence of the fibration \(M \to N\) would then yield

\[\pi_2(M) = 0 \to \pi_2(N) \to \pi_1(\mathbb{R} \times T^3) = \mathbb{Z}^3,\]

but there is no injective homomorphism of \(\pi_2(N)\) into \(\mathbb{Z}^3\), since \(\pi_2(N) = \pi_2(S)^{m-1} = \mathbb{Z}^{22(m-1)}\). This proves that \((M, g)\) is not locally symmetric, establishing Theorem 1.1. Here we have used that \(S\) is simply connected and therefore, by Hurewicz's theorem, \(\pi_2(S) \cong H_2(S, \mathbb{Z})\). The latter group is torsion-free\(^1\) and its rank is well known to be 22, see [Hu, Ch. 1]. So \(H_2(S, \mathbb{Z}) \cong \mathbb{Z}^{22}\). \(\square\)

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\(^1\)In fact, the torsion of \(H_2(S, \mathbb{Z})\) coincides with that of \(H^3(S, \mathbb{Z})\) by the universal coefficient theorem and the latter is trivial, see [Hu, Ch. 1].
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