Direct Mott Insulator-to-Superfluid Transition in the Presence of Disorder

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We introduce a new renormalization group theory to examine the quantum phase transitions upon exiting the insulating phase of a disordered, strongly interacting boson system. For weak disorder we find a direct transition from this Mott insulator to the Superfluid phase. In \( d > 4 \) a finite region around the particle-hole symmetric point supports this direct transition, whereas for \( 2 \leq d < 4 \) perturbative arguments suggest that the direct transition survives only precisely at commensurate filling. For strong disorder the renormalization trajectories pass next to two fixed points, describing a pair of distinct transitions; first from the Mott insulator to the Bose glass, and then from the Bose glass to the Superfluid. The latter fixed point possesses statistical particle-hole symmetry and a dynamical exponent \( z \), equal to the dimension \( d \).

PACS numbers: 72.15.Rn, 67.40.Yv, 74.20.Mn, 75.10.Nr

The field of quantum phase transitions in disordered systems experienced remarkable growth recently. Its experimental relevance comprises the superconductor-insulator transition in thin films \[1\]. \(^4\)He in disordered media \[2\], as well as Quantum Hall systems, and many types of disordered magnets \[3\]. Quantum criticality may lie at the heart of the anomalous normal state properties of high \( T_c \) superconductors, and it can describe the characteristics of vortex systems with correlated disorder \[4\].

The intensely studied localization transition of interacting, disordered bosons has served as a useful paradigm for quantum phase transitions. Most of the originally proposed theoretical picture \[5\] was subsequently confirmed numerically \[6\]. However a key issue remains controversial. Analytic theories suggest, that the localized, “Bose Glass” phase intervenes everywhere between the interaction driven (Mott) insulator and the superfluid \[7\]. However, none of the numerical studies of the problem \[7,9\] found the Bose glass in 2d at commensurate densities. Experimental tests in Josephson junction arrays also analyzed the critical behaviour \[10\]. A physical motivation for such a choice is that spatial fluctuations of the field are likely to renormalize the customary uniform distribution to a smooth function in finite dimensions \[11\].

We first present a mean field treatment of the problem by chosing \( J_{ij} = J/N \), connecting all sites. In this formulation the asymptotics of the distribution of the site disorder plays a crucial role. It has recently been shown that for \( P(\epsilon) \sim (D - |\epsilon|)^\alpha \), with \( \alpha > 0 \), there is indeed a localization transition on the mean field level \[12\]. A possible motivation for such a choice is that spatial fluctuations of the field are likely to renormalize the site-energy distribution.

In this letter we present a novel renormalization group analysis of the problem. We find that in dimensions \( d \geq 2 \) a direct insulator-superfluid transition takes place for weak disorder in these systems. Furthermore the dynamical critical exponent \( z = d \) at the Bose Glass - Superfluid transition, where a statistical version of particle-hole symmetry develops.

We consider the Hamiltonian:

\[
H = -\sum_{i,j} J_{ij} a_i^\dagger a_j - \sum_i \mu_i \hat{n}_i + U \sum_i \hat{n}_i (\hat{n}_i - 1)
\]

where \( a_i^\dagger, a_i \) annihilate (create) a boson at site \( i \) and \( \hat{n}_i = a_i^\dagger a_i \) is the number operator, and \( \mu_i = \mu + \epsilon_i \), where the chemical potential \( \mu \) controls the density of the bosons, and \( \epsilon_i \) is a random site energy, with distribution \( P(\epsilon) \) over the finite support \([-D,D]\).

For clean systems the phase diagram consist of a Superfluid phase (SF) and lobe-like Mott insulating phases (MI) with integer boson densities \[13\]. In the presence of disorder some site energies fall within the clean gap, shrinking the Mott lobes. It was proposed that a new, Bose Glass (BG) phase appears between the superfluid and the insulator, characterized by the absence of a SF order parameter, and a Mott gap. The excitations are presumed localized, with a finite density of states (DOS) at low energies and thus with a vanishing compressibility \[5\].

The evaluation of the free energy is achieved by performing a Hubbard-Stratonovich transformation on the kinetic term. In the \( N \to \infty \) limit the dynamic fluctuations are suppressed, thus the saddle point approximation becomes exact, yielding for the free energy

\[
\beta f = -N \beta J |m|^2 + \sum_i N \ln \text{Tr} \exp[\beta S_i(m)]
\]
where $m$ takes its saddle-point value. $m$ is proportional to the superfluid order parameter so it is zero in both the MI and BG phases. The MI phase is distinguished from the BG by the presence of a gap in its spectrum. The MI-BG transition is driven by the local collapse of this gap at some sites, which induces singular contributions to $f$. While this is the dominant scenario at small $J$, for stronger kinetic couplings a direct MI-SF transition is possible: long range fluctuations might generate a nonzero value for $m$ before local instabilities would take place. In this case the free energy can be expanded in powers of $m$ and a Landau-type action describes the MI-SF transition yielding the usual mean-field exponents.

The advantage of employing a smooth distribution function of the disorder becomes apparent here. For the uniform distribution the MI-BG transition is absent for $J > 0$, whereas it is manifestly present at some finite $J$ for positive $\alpha$’s. We give credence to the above physical picture by developing a new renormalization group (RG) scheme, similar to that of Ref. [13]. As will be shown below, the soft core problem always scales to large interaction strengths, therefore we first present the procedure for hard core bosons and will discuss the effect of finite $U$ later. Minimizing the free energy yields the saddle-point equation at $T \to 0$

$$m = m \frac{J}{N} \sum_i \frac{1}{\sqrt{\mu_i^2 + J^2 m^2}} .$$

This sum is unfortunately plagued with singular terms with $\mu_i \sim 0$. The key concept of our RG technique is to avoid these singular terms by performing the sum step by step, integrating out those sites which have the highest excitation energies: $E = \max_i |\mu_i|$. The contribution of these sites is then exactly absorbed into the renormalization of the parameters as:

$$dm/\alpha = m \frac{J}{N} \sum_i \frac{1}{\sqrt{\mu_i^2 + J^2 m^2}} ,$$

$$dJ/\alpha = J \frac{\tilde{z} - 1 + J/\sqrt{E^2 + J^2 m^2}}{\sqrt{E^2 + J^2 m^2}} ,$$

$$dD/\alpha = D \frac{\tilde{z} - 1/(\alpha + 1)}{\alpha + 1} ,$$

$$dE/\alpha = \tilde{z} E - D/(\alpha + 1) .$$

Here $dx = -dmN$ parametrizes the change in the number of sites, and the dynamic critical exponent $\tilde{z}$ relates this change to the rescaling of the unit of energy $\Omega$: $d\Omega/\Omega = \tilde{z} dN/N$. These equations apply in the asymptotic region, where $P(\epsilon) \sim (D + \epsilon)^\alpha$. This region is reached by integrating out the high energy sites with $\epsilon_i > 0$ in one step. Keeping the total bandwidth fixed, $E = 1$, requires $\tilde{z} = D/(\alpha + 1)$.

The flow trajectories of the scaling equations are displayed in Fig.1. for the case of $m = 0$. We find an attractive critical fixed point at $D = J = 0$. In the absence of disorder and hopping this point is naturally identified as the Mott Insulator. The critical fixed point at $D = 0$, $J = 1$ separates from the MI phase a region with runaway trajectories towards large $J$ and vanishing disorder $D$. These trajectories are regularized by inducing finite values of $m$: thus they characterize a superfluid. This identification of a direct Mott-Superfluid transition for weak disorder is the central result of our paper.

![FIG. 1. The renormalization group trajectories for $\alpha = 1$.](image_url)
ries. For disorder values $D > 1$ the system first scales close to the MI-BG fixed point at $D = 1$, $J = 0$, which is characterized by the exponents, such as $\tilde{z} = 1/(\alpha + 1)$. But the flows continue towards the BG-SF fixed point at $D = 2$, $J = 0$. The corresponding critical behaviour becomes independent of $\alpha$ as the disorder distribution scales to a uniform function ($\alpha = 0$). In particular, this restores the statistical particle-hole symmetry at the BG-SF fixed point, with $\tilde{z} = 1$. Here let us recall that our infinite-range hopping model is equivalent to the mean-field treatment of a finite-dimensional system. In a finite dimension $d$ the dynamical exponent $z$ relates the rescaling of the linear size, $L$, of the system to the energy-scale change $\Omega' / \Omega = (L' / L)^{\tilde{z}}$. By writing $(N' / N) = (L' / L)^d$, from $\tilde{z} = 1$ we obtain $z = d$ for the BG-SF transition. This result was first derived from the scaling of the compressibility in Ref. [3], but was recently debated [13]. Finally, the special, multicritical fixed point at $D = 1$, $J = \alpha/(\alpha + 1)$ will be analyzed in a subsequent publication.

\begin{figure}[h]
\centering
\includegraphics[width=0.9\linewidth]{figure2}
\caption{The phase boundary of the first Mott lobe for $\alpha = 1$, for disorder values $D = 0$, 0.5, 0.9. The thick line represents the Bose-Glass.}
\end{figure}

Now we return to the case of soft core bosons. The novelty is that from the start particle and hole excitations at the same site have to be considered simultaneously. Formally this is implemented by expanding the saddle point equation in $m$, and then replacing $1/|\mu_i|$ by $(n_i / E_{h,i} + (n_i + 1) / E_{p,i})$, where $n_i$ is the number of bosons at site $i$ in the ground state. Its value is determined from $2U(n_i - 1) < \mu_i < 2Un_i$. The excitation energy of a hole is $E_{h,i} = \mu_i - 2U(n_i - 1)$, and of a particle $E_{p,i} = 2U(n_i - \mu_i)$. Finally, $U$ renormalizes, as $dU / dx = 3U$. Let us observe that as long as the disorder does not scale to zero, $\tilde{z} = D/(\alpha + 1)$ is positive, driving $U$ to infinity. Thus for the finite $D$ transitions the trajectories indeed scale to $U = \infty$, thus we find the same fixed points, as in the hard core case. When the system scales towards small disorder, $U$ increases, but saturates at a nonuniversal finite value. This stretches the hard core MI and the MI-SF fixed points into fixed lines. Therefore the fixed point structure of the soft core case is equivalent to that of the hard core case.

We explicitly calculated the phase boundary of the Mott Insulator for soft core interactions by requiring that the coefficient of $m$ on the rhs. of the appropriately modified saddle point equation equal 1. Since at the MI-BG phase boundary $\mu$ is either $D$ or $2U - D$, the straight separatrix at $D = 1$ in the $D - J$ plane translates into straight sides of the Mott lobes, as shown in Fig.2. It is worth noting that with increasing disorder the Mott lobe shrinks much faster along the $\mu$ axis, than along the $J$ axis, in contrast to previous suggestions [14].

Now we address the question of what happens on finite dimensional lattices. First the pure case will be reviewed, and then a small disorder turned on. We concentrate on the regions around the tip of the Mott lobes, where the transition happens at finite hopping $J$, so $D$ can be regarded as a small parameter.

In the pure case there are two types of transitions: exiting the MI by changing the chemical potential $\mu$ (generic case) and by increasing the hopping $J$ at the tip of the lobe (multicritical point) [3]. The generic transition is driven by particle or hole excitations. The dynamical exponent is $z = 2$ and the upper critical dimension is $d_c = 2$. The gap linearly disappears with $\delta \mu = \mu - \mu_c$, requiring $\nu z = 1$, and $\nu = 1/2$ for any dimension $d \geq 1$. However the correlations inside the ground state do not diverge upon approaching the transition. On the other hand, the multicritical point possesses particle-hole symmetry, setting the value of $z$ to 1; the transition is of the type of a $d + 1$ dimensional XY model, hence $d_c = 3$. The correlation length of the MI ground state does diverge upon approaching the transition.

Now we include site disorder $D$ in the model. The relevancy of the disorder was extensively studied in past work. In classical models the perturbative Harris criterion is generally accepted to signal the irrelevancy of disorder [10]. In their non-perturbative study, Chayes et al. proved that if the transition can be tuned by changing the disorder around a finite disorder fixed point, then the correlation length exponent must obey $\nu > 2/d$ [17]. This coincides with the Harris criterion for classical systems, but is expected to apply for quantum transitions as well. In the quantum case, however, perturbative considerations may yield further bounds on exponents. Imagine for instance of tuning the system to criticality. Here $\omega(k) \sim k^z$, by the definition of the dynamical critical exponent. In a finite size system this yields a level spacing for the first excited states of the clean Hamiltonian $\delta \omega \sim L^{-z}$. On the other hand, disorder introduces a mixing between these states, of the order of $V_{kk'} \sim L^{-d/2}$. Second order perturbation theory shows that the disorder - induced shift of the energy levels is much less than the level spacing for $d \geq 2$. Also, the perturbative change in the ground state wavefunction remains small, thus disor-
order is irrelevant. All of the above arguments suggest that at the generic transition, where $\nu = 1/2$ and $z = 2$, weak disorder is irrelevant for $d > d_c = 4$. Thus the Bose-glass covers the sides of the Mott lobe only partially, leaving a finite region around the tip, where a direct MI-SF transition occurs, as in mean field. On the other hand, for $d < 4$ disorder becomes relevant, suggesting that the Bose-glass covers the sides of the Mott lobe completely.

At the tip the Chayes criterion gives $d_c = 4$ again, however the perturbative criterion yields $d_c = 2$, since $z = 1$. These two results may be reconciled by considering the following possibility. For $D > 0$ one could argue that the randomness hinders ordering tendencies, thus the ordered superfluid phase should appear only at larger hoppings $J$. On the other hand, site - disorder is moving some energy levels inside the Mott gap. These sites may be able to support a superfluid even for hoppings smaller than $J_c$, the critical value for the clean system. Thus it is not inconceivable that these two competing influences cancel each other, and the phase boundary remains independent of the disorder up to some finite $D$ value.

In this case the Chayes et al. bound does not apply, as its proof requires that the transition be driven by tuning the disorder. According to the other inequality then, the disorder is relevant only for $d < 2$. Remarkably, all numerical studies, performed at the tip of the Mott lobes, find a direct MI-SF transition. In the recent 2d work of Kisker and Rieger it was found that for $D < 0.4J$ the $\nu$ exponent at the tip of the lobe approximately equaled its clean value of $\nu_{3d,XY}$, violating the Chayes et al. bound. Correspondingly, the locus of the phase transition was consistent with the clean value, allowing for the possibility of systematic errors. Even more convincingly, the transition was characterized by genuinely new exponents around $D \approx 0.4$, such as $z = 1.4$, corresponding neither to the generic, nor to the XY criticality. Such a new multicritical behaviour is indeed expected, if a critical value of the disorder, $D_c$ exists, such that for $D > D_c$ the disorder is relevant.

This scenario suggests that there can be a direct MI-SF transition even for finite disorder at the tip of the Mott lobe, for $d \geq 2$. One can rationalize this result by observing that the correlation length diverges upon approaching the critical point from either side. One expects the disorder to be screened and smoothed quite effectively by fluctuations of such large spatial extent.

We wish to end on cautionary notes. Obviously these same numerical results can be in accord with the Chayes et al. bound, if the Bose glass phase is extremely slim, or if it manifests itself only on length scales, exceeding the biggest accessible system size. Such an anomalous behaviour however lacks a theoretical explanation to date. Additional subtleties associated with applying the Chayes et al. criterion around multicritical points were emphasized in Ref. 18.

In sum, we studied interacting bosons in the presence of disorder on a lattice. We constructed the phase diagram on the mean field level using an unconventional renormalization group scheme. At weak disorder a direct Mott Insulator-Superfluid transition takes place. We argued that this transition is present for $d > 4$ in a finite region around the lobe-tips, whereas it survives down to two dimensions at the tip of the Mott lobes. Several numerical studies are consistent with this picture, as well as the limited experimental evidence on Josephson junction arrays. We proposed a possible reconciliation of these results with the well-known bound for $\nu$ of Chayes et al. At strong disorder, we found that the Bose glass - Superfluid transition is characterized by statistical particle - hole symmetry and the exponent relation $z = d$ on the mean field level.

We acknowledge useful discussions with R. Scalettar, L. Chayes, T. Giamarchi, H. Rieger, S. Sachdev, R. Singh, P. Weichman, and P. Young. This research was supported by NSF DMR-95-28535.