We study the non-Abelian topological vortices in condensed matter physics, whose topological flux quantum number is described by $\pi_2(S^2)$, not by $\pi_1(S^1)$. We present two examples, a magnetic vortex in two-gap superconductor and a vorticity vortex in two-component Bose-Einstein condensate. In both cases the condensates exhibit a global $SU(2)$ symmetry which allows the non-Abelian topology. We establish the non-Abelian flux quantization in two-gap superconductor by demonstrating the existence of non-Abelian magnetic vortex whose flux is quantized in the unit $4\pi/g$, not $2\pi/g$. We also discuss a genuine non-Abelian gauge theory of superconductivity which has a local $SU(2)$ gauge symmetry, and establish the non-Abelian Meissner effect in the non-Abelian superconductor. We compare the non-Abelian vortices with the well-known Abelian Abrikosov vortex, and discuss how these non-Abelian vortices could be observed experimentally in two-gap superconductor made of MgB$_2$ and spin-1/2 condensate of $^{87}$Rb atoms. Finally, we argue that the existence of the non-Abelian vortices provides a strong evidence for the existence of topological knots in these condensed matters whose topology is fixed by $\pi_3(S^2)$, which one can construct by twisting and connecting the periodic ends of the non-Abelian vortices.

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I. INTRODUCTION

Topological objects, in particular finite energy topological solitons, have played an important role in physics. In condensed matter the Abrikosov vortex in ordinary superconductors and similar ones in Bose-Einstein condensates (as well as in superfluids) are the well-known examples of the (1+2)-dimensional topological solitons, which have been studied extensively theoretically and experimentally. These vortices originate from the fact that the underlying theory has a $U(1)$ (i.e., an Abelian) symmetry, which provides a $\pi_1(S^1)$ topology to the vortices. On the other hand, recent experimental advances have enabled us to create far more complex condensed matters, such as two-gap superconductors and two-component Bose-Einstein condensates (BEC) which opens a new possibility for us to observe far more interesting topological objects in condensed matters.

This is because the new condensates, due to their multi-component structure, can have a complex topological structure which is absent in the single-component condensates. The purpose of this paper is to demonstrate the existence of non-Abelian vortices and topological knots in the new condensed matters. In the following we present two examples of novel non-Abelian vortex whose topology is described by $\pi_2(S^2)$, one in two-gap superconductor and one in two-component BEC, which are completely different from the well-known Abelian Abrikosov vortex. Furthermore we predict the existence of the totally new type of solitons, the topological knots, in these condensates. We argue that one can construct such a knot by smoothly twisting the non-Abelian vortex, bending and connecting the periodic ends of the vortex together. This opens a new possibility for us to observe far more interesting topological objects in condensed matters.

A most important difference between the non-Abelian magnetic vortex and Abrikosov vortex in ordinary superconductor is the non-Abelian flux quantization. In two-gap superconductor we show the existence of magnetic vortex whose flux is quantized in the unit $4\pi/g$, not $2\pi/g$. Mathematically this non-Abelian flux quanti-
zation has a deep reason. This follows from the fact that the volume of the $U(1)$ subgroup of $SU(2)$ is twice as big as the volume of the Abelian $U(1)$.

Since the superconductivity has always been described by an Abelian (i.e., electromagnetic) interaction one may assume that the effective theory of two-gap superconductor should be an Abelian gauge theory which has a global $SU(2)$ symmetry. But in this case the two condensates must carry the same charge, because there is no way that the Abelian gauge field can couple to a doublet condensate whose components have opposite charges. But this does not mean that a doublet condensate carrying opposite charges can not exhibit a superconductivity. In this paper we discuss an $SU(2)$ gauge gauge theory of two-gap superconductor made of an oppositely charged doublet condensate, which can exhibit a genuine non-Abelian superconductivity. We establish a non-Abelian Meissner effect in this theory showing that the theory allows a non-Abelian magnetic vortex. As far as we understand, this type of non-Abelian superconductivity has never been discussed before.

A similar non-Abelian vortex can also exist in two-component BEC. This is because two-component BEC also has a non-Abelian structure similar to two-gap superconductor. In this paper we consider two competing theories of two-component BEC, the Gross-Pitaevskii theory and the gauge theory of two-component BEC which has a vorticity interaction, and show that both theories allow non-Abelian vorticity vortices very similar to each other. We also show that the gauge theory of two-component BEC, with the vorticity interaction, is closely related to two-gap superconductor. The only difference is that in two-component BEC the gauge interaction is induced one, while in two-gap superconductor it is independent.

Finally we discuss the prototype non-Abelian vortex, the baby skyrmion in Skyrme theory, and show that the non-Abelian vortices in condensed matters are a direct generalization of the baby skyrmion. In fact we will see that all underlying theories of non-Abelian vortices discussed in this paper are closely related to the Skyrmie theory. This observation has an important implication, because this strongly implies the existence of topological knots in two-gap superconductor and two-component BEC. In Skyrmie theory it is well-known that one can construct a knot by twisting the baby skyrmion (making it periodic in $z$-coordinate) and connecting the periodic ends together. Exactly the same way we can construct a similar knot by twisting the non-Abelian vortex and connecting the periodic ends together in the new condensed matters. The twisted vortex ring made this way acquires the knot topology $\pi_3(S^2)$ and thus becomes a knot itself.

The paper is organized as follows. In Section II we present a non-Abelian magnetic vortex in two-gap superconductor in which the two condensates carry the same charge, and establish the non-Abelian magnetic flux quantization. In Section III we compare the non-Abelian vortex with the well-known Abelian Abrikosov vortex, and discuss the differences between the two vortices. In Section IV we establish a non-Abelian superconductivity presenting a genuine $SU(2)$ gauge theory which could describe a two-gap superconductor made of a doublet carrying opposite charges. With this we demonstrate the existence of a magnetic vortex in this non-Abelian superconductor which is identical to the vortex in two-gap superconductor based on Abelian gauge symmetry. In Section V we discuss a non-Abelian vortex in Gross-Pitaevskii theory of two-component BEC, and identify the vortex as a vorticity vortex. In Section VI we discuss a gauge theory of two-component BEC which has a vorticity interaction, and show that the theory has a non-Abelian vortex which is very similar to the one in Gross-Pitaevskii theory. In Section VII we show that the above non-Abelian vortices are a straightforward generalization of the prototype non-Abelian vortex, the baby skyrmion in Skyrme theory. Based on this we discuss how one can construct a topological knot from the non-Abelian vortices in the new condensates. Finally in Section VIII we discuss the physical implications of our results.

II. NON-ABELIAN MAGNETIC VORTEX IN TWO-GAP SUPERCONDUCTOR

A recent development in condensed matter physics is the discovery of two-gap superconductor made of MgB$_2$. The purpose of this section is to demonstrate the existence of a non-Abelian magnetic vortex in two-gap superconductors. The reason for this is that two-gap superconductor is made of a doublet which form an $SU(2)$ multiplet. In general, this type of topological vortex is possible when one has more than one condensate in the superconductor which has a symmetry group $G$ with a nontrivial $\pi_2(G/H)$ where $H$ is the Abelian subgroup, but in this paper we will concentrate on $SU(2)$ and two-gap superconductors.

Our non-Abelian vortex could be either relativistic or non-relativistic, and appear with both Abelian and non-Abelian gauge interaction. Let the complex doublet scalar field $\phi = (\phi_1, \phi_2)$ be the order parameter of a two-gap superconductor. In the mean field approximation the Landau-Ginzburg free energy of the two-gap superconductor could be expressed by

$$\mathcal{H} = \frac{\hbar^2}{2m_1} |(\nabla + igA)\phi_1|^2 + \frac{\hbar^2}{2m_2} |(\nabla + igA)\phi_2|^2 + \tilde{V}(\phi_1, \phi_2) + \frac{1}{2}(\nabla \times A)^2,$$

$$\tilde{V}(\phi_1, \phi_2) = -\tilde{\alpha}_1 \phi_1^\dagger \phi_1 - \tilde{\alpha}_2 \phi_2^\dagger \phi_2 - \tilde{\alpha}_3 (\phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1) + \frac{\tilde{\beta}_1}{2}(\phi_1^\dagger \phi_1)^2 + \frac{\tilde{\beta}_2}{2}(\phi_2^\dagger \phi_2)^2 + \tilde{\beta}_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2), \quad (1)$$
where \( \alpha_i \) and \( \beta_i \) are coupling constants. This is an obvious generalization of the Landau-Ginzburg free energy of ordinary superconductor, but one can simplify the above Hamiltonian with a proper normalization of \( \phi_1 \) and \( \phi_2 \) to 
\[
\mathcal{H} = \frac{1}{2}(\nabla + ig\mathbf{A})^2 + V(\phi_1, \phi_2) + \frac{1}{2}((\nabla \times \mathbf{A})^2),
\]
(2)
where \( V \) is the normalized potential. Furthermore one may assume that the normalized potential has the simplest form, assuming that the normalized coupling constants satisfy \( \alpha_1 \simeq \alpha_2, \alpha_3 \simeq 0, \) and \( \beta_1 \simeq \beta_2 \simeq \beta_3 \) for simplicity. In this case the potential reduces to
\[
V = -\alpha \phi^4 + \frac{\beta}{2}(\phi^4)^2.
\]
(3)
With this simplification the Hamiltonian acquires a global \( SU(2) \) symmetry (as well as the \( U(1) \) gauge symmetry). In general the \( SU(2) \) symmetry will be broken in real two-gap superconductors, but one may still regard the \( SU(2) \) symmetry as an approximate symmetry. In this sense it is worth studying the \( SU(2) \) symmetric potential first as a starting point. We will come back to a more general potential later.

With the above Hamiltonian one may try to obtain a non-Abelian vortex minimizing the free energy. On the other hand, to study a static vortex solution, one might as well start from the following relativistic Lagrangian
\[
\mathcal{L} = -\frac{1}{2}D_\mu \phi^2 + \mu^2 \phi^4 - \frac{\lambda}{2}(\phi^4)^2 - \frac{1}{4}F_{\mu\nu}^2, \\
D_\mu \phi = (\partial_\mu + igA_\mu)\phi.
\]
(4)
In the static limit the Lagrangian reproduces the above Hamiltonian, and gives us an identical equation of motion. So it must be clear that one can either start from the non-relativistic Hamiltonian (2) or the relativistic Lagrangian (4) to discuss the static solution. With this observation we will use the Lagrangian (4) to describe the two-gap superconductor in the following. The advantage of the Lagrangian approach, of course, is that it establishes the non-Abelian vortex of two-gap superconductor as a topological soliton of a renormalizable quantum field theory.

The Lagrangian has the equation of motion
\[
D^2 \phi = \lambda(\phi^4 - \frac{\mu^2}{\lambda}\phi), \\
\partial_\mu F_{\mu\nu} = -j_\nu = ig\left[(D_\nu \phi)^4 - \phi^4(D_\nu \phi)\right].
\]
(5)
Now, with
\[
\phi = \frac{1}{\sqrt{2}}\alpha \zeta, \quad |\phi| = \frac{\rho}{\sqrt{2}}, \quad \zeta^\dagger \zeta = 1, \\
\hat{n} = \zeta^\dagger \sigma \zeta, \quad \hat{n}^2 = 1,
\]
(6)
we have the following identity
\[
\left(\partial_\mu - ig\hat{A}_\mu - \frac{1}{2}i\sigma \cdot \partial_\mu \hat{n}\right)\zeta = 0, \\
\hat{A}_\mu = -\frac{i}{g}\zeta^\dagger \partial_\mu \zeta.
\]
(7)
With this we can reduce (5) to
\[
\partial^2 \rho - \left(\frac{1}{4}(\partial_\mu \hat{n})^2 + g^2 B_\mu^2\right)\rho = \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho, \\
\partial_\mu F_{\mu\nu} = -f_\nu = g^2 \rho^2 B_\nu,
\]
(8)
which is equivalent to
\[
A + \vec{B} \cdot \sigma \zeta = 0, \\
\hat{n} \times \vec{B} - i\hat{n} \times (\hat{n} \times \vec{B}) = 0.
\]
(9)
So the second equation of (8) can be transformed to an equation for \( \hat{n} \). With this we can express (8) as
\[
\partial^2 \rho - \left(\frac{1}{4}(\partial_\mu \hat{n})^2 + g^2 B_\mu^2\right)\rho = \frac{\lambda}{2}(\rho^2 - \rho_0^2)\rho, \\
\hat{n} \times \partial^2 \hat{n} + 2 \frac{\partial^2 \rho}{\rho} \hat{n} \times \partial_\mu \hat{n} - \frac{g^2 \rho^2}{\rho} (\partial_\mu F_{\mu\nu}) \partial_\nu \hat{n} = 0, \\
\partial_\mu F_{\mu\nu} = g^2 \rho^2 B_\nu.
\]
(10)
This is the equation for two-gap superconductor (4). The second and third equations assure that the theory has two conserved currents, one \( SU(2) \) current and one \( U(1) \) current.

Notice that the equation (8) is written in terms of the \( SU(2) \) doublet \( \zeta \), whose target space is \( S^3 \). But the equation (10) is written completely in terms of \( \hat{n} \), whose target space is \( S^2 \). This is made possible because of the Abelian gauge invariance. The Abelian gauge invariance reduces the physical target space of \( \zeta \) to the gauge orbit space \( S^2 = S^3/S^1 \), which forms a \( CP^1 \) space which is identical to the target space of \( \hat{n} \). In fact with (8) the Lagrangian (4) can be expressed in terms of \( \hat{n} \) (with \( \rho \) and \( B_\mu \))
\[
\mathcal{L} = -\frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2 \left(\frac{1}{4}(\partial_\mu \hat{n})^2 + g^2 B_\mu^2\right) + \frac{1}{2} \rho^2 - \frac{\lambda}{8} \rho^4 - \frac{1}{4}(G_{\mu\nu} - \hat{F}_{\mu\nu})^2, \\
G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = F_{\mu\nu} + \hat{F}_{\mu\nu}, \\
\hat{F}_{\mu\nu} = \frac{1}{2g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}).
\]
(11)
Notice that the Lagrangian reproduces (11).

To obtain the vortex solution let \((g, \varphi, z)\) be the cylindrical coordinates and choose the following ansatz

\[
\zeta = \left( \frac{\cos \frac{f(g)}{2}}{\sin \frac{f(g)}{2}} \right),
\]

\[
A_{\mu} = \frac{m}{g} A(g) \partial_{\mu} \varphi.
\] (13)

With this we have

\[
\dot{\zeta} = \zeta \dot{\varphi} = \left( \frac{\sin f(g) \cos m \varphi}{\sin f(g) \sin m \varphi} \right),
\]

\[
\dot{A}_{\mu} = -\frac{m}{2g} (\cos f(g) + 1) \partial_{\mu} \varphi.
\] (14)

so that (11) is reduced to

\[
\ddot{\rho} + \frac{1}{g} \dot{\rho} - \frac{1}{4} \left( f^2 + \frac{m^2}{g^2} \sin^2 f \right)
\]

\[
+ \frac{m^2}{g^2} \left( A - \frac{\cos f + 1}{2} \right)^2 \rho = \frac{\lambda}{2} (\rho^2 - \rho^0) \rho,
\]

\[
\ddot{f} + \left( \frac{1}{g} + 2 \frac{\dot{\rho}}{\rho} \right) \dot{f} - \frac{2 m^2}{g^2} \left( A - \frac{1}{2} \right) \sin f = 0,
\]

\[
\ddot{A} - \frac{\dot{A}}{g} - g^2 \rho^2 \left( A - \frac{\cos f + 1}{2} \right) = 0.
\] (15)

In terms of

\[
B_{\mu} = \frac{m}{g} B \partial_{\mu} \varphi = \frac{m}{g} \left( A - \frac{\cos f + 1}{2} \right) \partial_{\mu} \varphi,
\]

this can be written as

\[
\ddot{\rho} + \frac{1}{g} \dot{\rho} - \left[ \frac{1}{4} \left( f^2 + \frac{m^2}{g^2} \sin^2 f \right) + \frac{m^2}{g^2} \right] \rho = \frac{\lambda}{2} (\rho^2 - \rho^0) \rho,
\]

\[
\ddot{f} + \left( \frac{1}{g} + 2 \frac{\dot{\rho}}{\rho} \right) \dot{f} - \frac{2 m^2}{g^2} \left( A - \frac{1}{2} \right) \sin f = 0,
\]

\[
\ddot{A} - \frac{\dot{A}}{g} - g^2 \rho^2 \left( A - \frac{\cos f + 1}{2} \right) = 0.
\] (16)

Now, we impose the following boundary condition for the non-Abelian vortex,

\[
\rho(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = \pi, \quad f(\infty) = 0,
\]

\[
A(0) = -1, \quad A(\infty) = 1.
\] (17)

This need some explanation, because the boundary value \(A(0)\) is chosen to be \(-1\), not 0. This is to assure the smoothness of the scalar field \(\rho(g)\) at the origin. Only with this boundary value \(\rho\), with \(\rho(0) = 0\), becomes analytic at the origin. At this point one might object the boundary condition, because it creates an apparent singularity in the gauge potential at the origin. But notice that this singularity is an unphysical (coordinate) singularity which can easily be removed by a gauge transformation. In fact the singularity disappears with the gauge transformation

\[
\phi \rightarrow \phi \exp(-im\varphi), \quad A_{\mu} \rightarrow A_{\mu} + \frac{m}{g} \partial_{\mu} \varphi,
\] (18)

which simultaneously changes the boundary condition \(A(0) = -1, A(\infty) = 1\) to \(A(0) = 0, A(\infty) = 2\). This boundary condition will have an important consequence in the following.

Notice that the magnetic field \(H\) of the vortex is expressed as

\[
H = \frac{m}{g} \dot{A}.
\] (19)

So we can deduce from (15) that asymptotically the scalar field \(\rho\) and the magnetic field \(H\) approach the asymptotic values \(\rho_0\) and zero with the following exponential damping

\[
\rho_0 - \rho \simeq \exp(-\sqrt{2\mu} \rho),
\]

\[
H \simeq \exp(-g \rho \theta),
\] (20)

respectively. This tells that, just like the single-component superconductor, the coherence length of the condensate \(\xi_C\) and the penetration length of the magnetic field \(\lambda_H\) are given by

\[
\xi_C = \frac{1}{\sqrt{2\mu}}, \quad \lambda_H = \frac{1}{\sqrt{2\mu}} \frac{\sqrt{\lambda}}{g},
\]

\[
\frac{\lambda_H}{\xi_C} = \frac{\sqrt{\lambda}}{g}.
\] (21)

So, when \(\sqrt{\lambda}\) is smaller (larger) than \(g\), the superconductor becomes type I (type II). Clearly, the condition

\[
\sqrt{\lambda} = g,
\] (22)

is the critical condition at which the superconductor changes its type from I to II.

With the boundary condition we can integrate (15) and obtain the non-Abelian vortex solution of the two-gap superconductor, which is shown in Fig[1]. Notice that the non-trivial profile of \(f(g)\) assures that the doublet \(\xi\) starts from the second component at the origin and ends up with the first component at the infinity. This assures that the vortex is essentially non-Abelian.
Clearly the magnetic field $H$ of the vortex has total flux given by

$$\dot{\phi} = \int H d^2x = \frac{2\pi m}{g} [A(\infty) - A(0)] = \frac{4\pi m}{g}. \quad (23)$$

Notice that the unit of the non-Abelian flux is $4\pi/g$, not $2\pi/g$. Obviously this is a direct consequence of the boundary condition $A(0) = -1$ (or more precisely $A(\infty) - A(0) = 2$) in (17) that we discussed before.

With the ansatz (13), one can express the Hamiltonian of the vortex as

$$\mathcal{H} = \frac{1}{2} \int \left[ (\dot{\rho} + \frac{m}{g} (A - \cos f + \frac{1}{2}) \rho)^2 + (\dot{f} + \frac{m}{g} \sin f)^2 \frac{\rho^2}{4} + (H + \sqrt{\lambda} (\rho^2 - \rho_0^2))^2 - \rho \right] d^2x, \quad (24)$$

so that the Hamiltonian has a minimum value when

$$\dot{\rho} + \frac{m}{g} (A - \cos f + \frac{1}{2}) \rho = 0,$$
$$\dot{f} + \frac{m}{g} \sin f = 0,$$
$$H + \sqrt{\lambda} (\rho^2 - \rho_0^2) = 0. \quad (25)$$

Furthermore, when the coupling constant $\lambda$ has the critical value (22), one can integrate the second order equation (16) to the above first order equation (24).

Integrating the second equation of (25) we have

$$\cos f(q) = \frac{q^{2m} - a^2}{q^{2m} + a^2}, \quad \sin f(q) = \frac{2aq^m}{q^{2m} + a^2}. \quad (26)$$

where $a$ is an integration constant. With this (25) is reduced to

$$\dot{\rho} + \frac{m}{g} (A - \frac{q^{2m}}{g} + a^2) \rho = 0,$$
$$\frac{m}{g} \dot{A} = \frac{q}{2} (\rho^2 - \rho_0^2) = 0. \quad (27)$$

In this case the Hamiltonian becomes

$$\mathcal{H} = \frac{g}{2} H \rho_0^2, \quad (28)$$

and the energy (per unit length) acquires the absolute minimum value

$$E = 2\pi m \rho_0^2 = \frac{g}{2} \rho_0^2 \dot{\phi}. \quad (29)$$

This tells that the minimum energy is fixed by the topological flux quantum number.

The magnetic vortex is topological. This is because the non-linear sigma field $\hat{n}$ defined in (6) naturally describes the mapping $\pi_2(S^2)$ from the compactified $xy$-plane $S^2$ to the physical target space $S^2$, whose quantum number $\pi_2(S^2)$ is given by

$$q = -\frac{1}{4\pi} \int \epsilon_{ij} \partial_i \zeta^j \partial_j \zeta d^2x = \frac{1}{8\pi} \int \epsilon_{ij} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}) d^2x = \frac{m}{2} \int \hat{f} \sin f d\phi = m. \quad (30)$$

Clearly this topology is due to the non-Abelian nature of two-gap superconductor. As we will see this is the topological quantum number that we will encounter repeatedly in the following.

We have shown that the above magnetic vortex has a non-Abelian flux quantization rule. So one might wonder whether the two-gap superconductor does not allow
a magnetic vortex which satisfies the ordinary flux quantization rule. It does. Indeed with a different boundary condition

\[ \dot{\rho}(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = \pi, \quad f(\infty) = 0, \]
\[ A(0) = 0, \quad A(\infty) = 1, \tag{31} \]

we obtain another magnetic vortex solution in two-gap superconductor which has $2\pi/\varrho$ flux. This is shown in Fig.2. The difference between the two solutions is that in Fig.2 both components of $\phi$ vanish at the core of the vortex, but in Fig.2 the second component has a finite condensation due to the boundary condition $\dot{\rho}(0) = 0$. Notice that this boundary condition would have been unacceptable in ordinary (one-gap) superconductor because it creates a physical singularity at the vortex core. But in two-gap superconductor this boundary condition is completely independent. In this case we may not be able to impose the boundary condition (17), in particular $f(\infty) = 0$, and will no longer have the non-Abelian flux quantization [9, 10]. This, together with the above result, tells that in two-gap superconductor different boundary conditions lead to different flux quantizations.

So far we have assumed the potential $\tilde{\phi}$ to obtain the vortex solution for simplicity. But as we have remarked the global $SU(2)$ symmetry of the potential will often be broken in reality [9]. For example the potential could be such that the vacuum density of two condensates is completely independent. In this case we may not be able to impose the boundary condition (17), in particular $f(\infty) = 0$, and will no longer have the non-Abelian flux quantization [9, 10]. This, together with the above result, tells that in two-gap superconductor different boundary conditions lead to different flux quantizations.

### III. COMPARISON WITH ABELIAN VORTEX

At this point it is important to understand exactly how different is the non-Abelian magnetic vortex from the well-known Abelian Abrikosov vortex [1]. To compare the non-Abelian vortex with the Abelian vortex, we let $\phi$ be the charged scalar field of the electron-pair condensate in ordinary superconductor. Then the Abrikosov vortex is described by the Abelian Landau-Ginzburg Lagrangian

\[ \mathcal{L} = -|D_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu}^2, \]
\[ D_\mu \phi = (\partial_\mu + ig A_\mu) \phi, \tag{32} \]

which has the equation of motion

\[ D^2 \phi = \lambda (\phi^\dagger \phi - \frac{\mu^2}{\lambda}) \phi, \]
\[ \partial_\mu F_{\mu\nu} = -j_\nu = ig \left[ (D_\nu \phi)^\dagger \phi - \phi^\dagger (D_\nu \phi) \right]. \tag{33} \]

With the ansatz

\[ \phi = \frac{1}{\sqrt{2}} \varrho(\varrho) \exp(-i m \varphi), \]
\[ A_\mu = \frac{m}{g} A(\varrho) \partial_\mu \varphi. \tag{34} \]

is reduced to

\[ \dot{\varrho} + \frac{1}{\varrho} \dot{\varrho} - \frac{m^2}{2} (A - 1)^2 \rho = \frac{\lambda}{2} (\varrho^2 - \rho_0^2) \rho, \]
\[ \dot{A} - \frac{1}{\varrho} \dot{A} - g^2 \rho^2 (A - 1) = 0. \tag{35} \]

Now, with the boundary condition

\[ \varrho(0) = 0, \quad \varrho(\infty) = \rho_0, \]
\[ A(0) = 0, \quad A(\infty) = 1, \tag{36} \]

one can easily obtain the well-known Abelian Abrikosov vortex solution [1], whose magnetic flux is given by

\[ \dot{\phi} = \int H d^2 x = \frac{2\pi m g}{g} [A(\infty) - A(0)] \]
\[ = \frac{2\pi m}{g}. \tag{37} \]

The ansatz (34) assures that the topological flux quantum number of the Abelian vortex is fixed by $\pi_1(S^1)$. Remember that here the penetration length of the magnetic field $\lambda_H$ and the coherence length of the condensate $\xi_C$ are also given by (24).

To compare the above Abelian vortex with the non-Abelian one, let us consider the critical case $\sqrt{\lambda} = g$. In this case (35) is reduced to the first order equations

\[ \dot{\varrho} \pm \frac{m}{\varrho} (A - 1) \rho = 0, \]
\[ \frac{m}{g} \dot{A} \pm \frac{g}{2} (\rho^2 - \rho_0^2) = 0. \tag{38} \]

This has to be compared with (26). When $f = 0$ (or $f = \pi$), the two sets of equations become identical to each other. This is not surprising because when $f = 0$ (or $f = \pi$) one component of the doublet becomes zero, which effectively reduces the doublet to a singlet. This tells two things. First, the two-gap superconductor can also (as it should) allow the Abelian vortex (with $f = 0$ or $f = \pi$), whose topology is fixed by $\pi_1(S^1)$. Notice, however, in this case the non-Abelian topological quantum number (as well as the non-Abelian magnetic flux) defined by (30) becomes identically zero. Secondly, when $f(\varrho)$ has a non-trivial profile, there is no way the non-Abelian vortex can be related to the Abelian one. This is because in this case there is no way (no gauge transformation) in which the doublet can be put into a singlet. Moreover, the non-trivial $f(\varrho)$ ensures that the topology of the non-Abelian vortex is $\pi_2(S^2)$, not $\pi_1(S^1)$. This distinguishes our non-Abelian vortex from the Abelian Abrikosov vortex.

Another important difference comes from the magnetic flux and the energy. In the Abelian case the Hamiltonian, with (22), becomes

\[ \mathcal{H} = \frac{g^2}{2} \mathcal{H}^2. \tag{39} \]
so that the energy of the vortex (per unit length) is given by

\[ E = \frac{g}{2} \int H \rho_0^2 d^2 x = m \pi \rho_0^2 = \frac{g}{2} \rho_0^2. \]  

(40)

This means that the non-Abelian vortex has twice as much magnetic flux and energy. This difference can be traced back to the difference of the boundary conditions (17) and (36). In the Abelian case (36) tells that \( A(\infty) - A(0) = 1 \), but in the non-Abelian case (17) tells that \( A(\infty) - A(0) = 2 \). Obviously this difference makes the difference in the magnetic flux and the energy. Mathematically this difference originates from the fact that the Abelian \( U(1) \) runs from 0 to 2\( \pi \), but the \( S^1 \) fiber of \( SU(2) \) runs from 0 to 4\( \pi \).

IV. NON-ABELIAN SUPERCONDUCTOR

So far we have discussed an Abelian gauge theory of two-gap superconductor. But notice that this type of superconductor must be made of the doublet whose components carry the same charge (e.g., a doublet made of two electron-electron pair condensates or two hole-hole pair condensates), because the doublet is coupled to the Abelian electromagnetic field. Obviously the above theory can not describe a doublet which is made of opposite charges (made of one electron-electron pair condensate and one hole-hole pair condensate). In this case an Abelian gauge theory can not explain the superconductivity. Now we show that this type of two-gap superconductor can be explained by a genuine non-Abelian \( SU(2) \) gauge theory, and show that this type of superconductor also allows a non-Abelian magnetic vortex identical to what we have discussed before.

To construct a theory of superconductor which has a genuine non-Abelian gauge symmetry, we need to understand the mathematical structure of the non-Abelian gauge potential. Consider \( SU(2) \) and let \( \hat{n} \) be a gauge covariant unit triplet which selects the charge direction of \( SU(2) \). In this case one can always decompose the non-Abelian gauge potential into the restricted potential \( \hat{A}_\mu \) and the valence potential \( \tilde{X}_\mu \):

\[ \hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \tilde{X}_\mu = \hat{A}_\mu + \tilde{X}_\mu, \]

(41)

where \( \hat{A}_\mu \) is the “electric” potential. Notice that the restricted potential is precisely the potential which leaves \( \hat{n} \) invariant under parallel transport,

\[ \hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + gA_\mu \times \hat{n} = 0. \]  

(42)

Under the infinitesimal gauge transformation

\[ \delta \hat{n} = -\tilde{\alpha} \times \hat{n}, \quad \delta \hat{A}_\mu = \frac{1}{g} D_\mu \tilde{\alpha}, \]  

(43)

one has

\[ \delta A_\mu = \frac{1}{g} \hat{n} \cdot \partial_\mu \tilde{\alpha}, \quad \delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \tilde{\alpha}, \quad \delta \tilde{X}_\mu = -\tilde{\alpha} \times \tilde{X}_\mu. \]  

(44)

This tells two things. First, \( \hat{A}_\mu \) by itself describes an \( SU(2) \) connection which enjoys the full \( SU(2) \) gauge degrees of freedom. Secondly, the valence potential \( \tilde{X}_\mu \) forms a gauge covariant vector field under the gauge transformation. Furthermore this tells that the decomposition is gauge-independent. Once the gauge covariant topological field \( \hat{n} \) is given, the decomposition follows automatically independent of the choice of a gauge \([11, 12]\).

The importance of the decomposition for our purpose is that one can construct a non-Abelian gauge theory, a restricted gauge theory which has a full non-Abelian gauge degrees of freedom, with the restricted potential \( \hat{A}_\mu \) alone \([11, 12]\). This is because the valence potential \( \tilde{X}_\mu \) can be treated as a gauge covariant source, so that one can exclude it from the theory without compromising the gauge invariance. Indeed we will see that it is this restricted gauge theory which describes the non-Abelian gauge theory of superconductivity.

Remarkably the restricted potential \( \hat{A}_\mu \) retains all the essential topological characteristics of the original non-Abelian potential. First, \( \hat{n} \) defines \( \pi_3(S^2) \) which describes the non-Abelian monopoles \([11, 12]\). Secondly, it characterizes \( \pi_3(S^3) \) which describes not only the topologically distinct vacua but also the instanton numbers \([12, 14]\). Furthermore it has a dual structure,

\[ F_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu})\hat{n}, \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

\[ H_{\mu\nu} = -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu, \]  

(45)

where \( C_\mu \) is the “magnetic” potential \([11, 12]\).

With these preliminaries we now establish a non-Abelian superconductivity. Consider a \( SU(2) \) gauge theory described by the Lagrangian in which a doublet \( \Phi \) couples to the restricted \( SU(2) \) gauge potential,

\[ \mathcal{L} = -|\hat{D}_\mu \Phi|^2 + \mu^2 \Phi^\dagger \Phi - \frac{\lambda}{2} (\Phi^\dagger \Phi)^2 - \frac{1}{4} F_{\mu\nu}^2, \]

\[ \hat{D}_\mu \Phi = (\partial_\mu + \frac{g}{2\ell} \hat{\sigma} \cdot \hat{A}_\mu) \Phi. \]  

(46)

The equation of motion of the Lagrangian is given by

\[ \hat{D}^2 \Phi = \lambda (\Phi^\dagger \Phi - \frac{\mu^2}{\lambda}) \Phi, \]

\[ \hat{D}_\mu \hat{F}_{\mu\nu} = \partial_\nu \hat{n} = g \left( [\hat{D}_\nu \Phi]^\dagger \frac{\partial^2}{2} \Phi - \Phi^\dagger \frac{\partial^2}{2} \Phi \right) \cdot \hat{D}_\nu \Phi \cdot. \]  

(47)

Let \( \xi \) and \( \eta \) be two orthonormal doublets which form a basis,

\[ \xi \dagger \xi = 1, \quad \eta \dagger \eta = 1, \quad \xi \dagger \eta = \eta \dagger \xi = 0, \]
\[ \xi \overleftrightarrow{\partial} \xi = \hat{n}, \quad \eta \overleftrightarrow{\partial} \eta = - \hat{n}, \]
\[ (\hat{n} \cdot \overleftrightarrow{\partial}) \xi = \xi, \quad (\hat{n} \cdot \overleftrightarrow{\partial}) \eta = - \eta, \] (48)

and let
\[ \Phi = \phi_+ \xi + \phi_- \eta, \quad (\phi_+ = \xi \Phi, \quad \phi_- = \eta \Phi). \] (49)

With this we have the identity
\[ \left[ \partial_\mu - \frac{g}{2i} (C_\mu \hat{n} + \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}) \cdot \overleftrightarrow{\partial} \right] \xi = 0, \]
\[ \left[ \partial_\mu + \frac{g}{2i} (C_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}) \cdot \overleftrightarrow{\partial} \right] \eta = 0, \] (50)

and find
\[ \hat{D}_\mu \Phi = (D_\mu \phi_+) \xi + (D_\mu \phi_-) \eta, \] (51)

where
\[ D_\mu \phi_+ = (\partial_\mu + \frac{g}{2i} A_\mu) \phi_+, \quad D_\mu \phi_- = (\partial_\mu - \frac{g}{2i} A_\mu) \phi_-, \]
\[ A_\mu = A_\mu + C_\mu, \]
\[ C_\mu = \frac{2i}{g} \xi \overleftrightarrow{\partial} \xi = - \frac{2i}{g} \eta \overleftrightarrow{\partial} \eta. \]

From this we can express \[ (46) \] as
\[ L = -|D_\mu \phi_+|^2 - |D_\mu \phi_-|^2 + m^2 (\phi_+ \phi_+ + \phi_- \phi_-) \]
\[ -\frac{\lambda}{2} (\phi_+ \phi_+ + \phi_- \phi_-)^2 - \frac{1}{4} F_{\mu\nu}^2, \] (52)

where
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

This tells that the restricted \( SU(2) \) gauge theory \[ (46) \] is reduced to an Abelian gauge theory coupled to oppositely charged scalar fields \( \phi_+ \) and \( \phi_- \). We emphasize that this Abelianization is achieved without any gauge fixing.

The Abelianization assures that the non-Abelian theory is not different from the two-gap Abelian theory. Indeed with
\[ \chi = \left( \begin{array}{c} \phi_+ \\ \phi_- \end{array} \right), \] (53)
we can express the Lagrangian \[ (46) \] as
\[ L = -|D_\mu \chi|^2 + \mu^2 \chi \overleftrightarrow{\partial} \chi - \frac{\lambda}{2} (\chi \overleftrightarrow{\partial} \chi)^2 - \frac{1}{4} F_{\mu\nu}^2, \]
\[ D_\mu \chi = (\partial_\mu + igA_\mu) \chi, \] (54)

This is formally identical to the Lagrangian \[ (46) \] of two-gap Abelian superconductor discussed in Section II. The only difference is that here \( \chi \) and \( A_\mu \) are replaced by \( \phi \) and \( A_\mu \). This establishes that, with the proper redefinition of field variables \[ (49) \] and \[ (50) \], our non-Abelian restricted gauge theory \[ (46) \] can in fact be made identical to the Abelian gauge theory of two-gap superconductor. This proves the existence of non-Abelian superconductors made of the doublet consisting of oppositely charged condensates. As importantly our analysis tells that the two-gap Abelian superconductor has a hidden non-Abelian gauge symmetry because it can be transformed to the non-Abelian restricted gauge theory. This implies that the underlying dynamics of the topological superconductors is indeed the non-Abelian gauge symmetry. In non-Abelian superconductor it is explicit. But in two-gap Abelian superconductor it is hidden, where the full non-Abelian gauge symmetry only becomes transparent when one embeds the nontrivial topology properly into the non-Abelian symmetry \[ (50) \].

Once the equivalence of two Lagrangians \[ (46) \] and \[ (54) \] is established, it must be evident that the non-Abelian gauge theory of two-gap superconductor also admits a non-Abelian magnetic vortex. This proves the existence of a non-Abelian Meissner effect and non-Abelian superconductivity. All the results of Section II become equally valid here.

V. NON-ABELIAN VORTEX IN
GROSS-PITAEVSKII THEORY OF
TWO-COMPONENT BEC

The creation of the multi-component Bose-Einstein condensates of atomic gases \[ (3) \] has widely opened new opportunities for us to study the topological objects experimentally which so far have been only of theoretical interest. This is because the multi-component BEC could have a complex non-Abelian topological structure, and thus could have far more interesting topological vortices. Already new vortices have successfully been created with different methods in two-component Bose-Einstein condensates \[ (13, 16) \]. But surprisingly, there have been few theoretical study of these vortices. Indeed only recently the physical meaning of the vortices has been clarified as a vorticity vortex \[ (17) \]. In the following we discuss the non-Abelian vortex in Gross-Pitaevskii theory of two-component BEC in detail.

Let a complex doublet \( \phi = (\phi_1, \phi_2) \) be the two-component BEC, and consider the non-relativistic two-component Gross-Pitaevskii Lagrangian \[ (7) \]
\[ L = i \frac{\hbar}{2} \left[ (\partial_\mu \phi_1) \overleftrightarrow{\partial} (\partial_\nu \phi_1) - (\partial_\mu \phi_1) \overleftrightarrow{\partial} (\partial_\nu \phi_2) + (\phi_2 \overleftrightarrow{\partial}) (\partial_\nu \phi_2) \right] - \frac{\hbar^2}{2M} \left( |\partial_\mu \phi_1|^2 + |\partial_\mu \phi_2|^2 \right) + \mu_1 \phi_1 \overleftrightarrow{\partial} \phi_1 + \mu_2 \phi_2 \overleftrightarrow{\partial} \phi_2 - \frac{\lambda_{11}}{2} (\phi_1 \overleftrightarrow{\partial} \phi_1)^2 - \frac{\lambda_{12}}{2} (\phi_1 \overleftrightarrow{\partial} \phi_2)^2, \] (55)

where \( \mu_i \) are the chemical potentials and \( \lambda_{ij} \) are the quartic coupling constants which are determined by the scattering lengths \( a_{ij} \)
\[ \lambda_{ij} = \frac{4\pi \hbar^2}{M} a_{ij}. \] (56)
Notice that here we have neglected the trapping potential. This is justified if the range of the trapping potential is much larger than the size of topological objects we are interested in, and this is what we are assuming here.

The Lagrangian has the global $U(1) \times U(1)$ symmetry. But notice that for the spin $1/2$ condensate of $^{87}$Rb atoms, the scattering lengths $a_{ij}$ differ by only about 3% or so [17,16]. In this case one may safely assume

$$\lambda_{11} \simeq \lambda_{12} \simeq \lambda_{22} \simeq \tilde{\lambda}.$$  
(57)

With this (55) can be written as

$$\mathcal{L} = \frac{\hbar}{2} \left[ \phi^\dagger (\partial_t \phi) - (\partial_t \phi)^\dagger \phi \right] - \frac{\hbar^2}{2M} |\partial_t \phi|^2$$

$$-\frac{\lambda}{2} \left( \phi^\dagger \phi - \frac{\mu}{\lambda} \right)^2 - \delta \mu \phi^\dagger \phi_2,$$  
(58)

where

$$\mu = \mu_1, \quad \delta \mu = \mu_1 - \mu_2.$$  
(59)

Clearly the Lagrangian has a global $U(2)$ symmetry when $\delta \mu = 0$. So the $\delta \mu$ interaction can be understood to be the symmetry breaking term which breaks the global $U(2)$ symmetry to $U(1) \times U(1)$. Physically $\delta \mu$ represents the difference of the chemical potentials between $\phi_1$ and $\phi_2$ (Here one can always assume $\delta \mu \geq 0$ without loss of generality), so that it vanishes when the two condensates have the same chemical potential. Even when they differ the difference could be small, in which case the symmetry breaking interaction could be treated perturbatively. This tells that the theory has an approximate global $U(2)$ symmetry, even in the presence of the symmetry breaking term [17]. This confirms that the theory of two-component BEC is essentially non-Abelian.

Normalizing $\phi$ to $(\sqrt{2M}/\hbar)\phi$ and putting

$$\phi = \frac{1}{\sqrt{2}} \rho \zeta, \quad (\zeta^\dagger \zeta = 1)$$  
(60)

we have the following Hamiltonian in the static limit (in the natural unit $c = \hbar = 1$) from the Lagrangian [88],

$$\mathcal{H} = \frac{1}{2} (\partial_t \rho)^2 + \frac{1}{2} \rho^2 |\partial_t \zeta|^2 + \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2$$

$$+ \frac{\delta \mu^2}{2} \rho^2 \zeta^\dagger \zeta_2,$$  
(61)

where

$$\lambda = 4M^2 \tilde{\lambda}, \quad \rho_0^2 = \frac{4\mu M}{\lambda}, \quad \delta \mu^2 = 2M \delta \mu.$$  

This can be expressed as

$$\mathcal{H} = \lambda \rho_0^2 \hat{\mathcal{H}},$$

$$\hat{\mathcal{H}} = \frac{1}{2} (\hat{\partial}_t \rho)^2 + \frac{1}{2} \rho^2 |\partial_t \zeta|^2 + \frac{1}{8} (\rho^2 - 1)^2$$

$$+ \frac{\delta \mu}{4\mu} \rho^2 \zeta^\dagger \zeta_2,$$  
(62)

where

$$\hat{\rho} = \frac{\rho}{\rho_0}, \quad \hat{\partial}_t = \kappa \partial_t, \quad \kappa = \frac{1}{\sqrt{\lambda \rho_0}}.$$  

Notice that $\hat{\mathcal{H}}$ is completely dimensionless, with only one dimensionless coupling constant $\delta \mu/\mu$. This tells that the physical unit of the Hamiltonian is $\lambda \rho_0$, and the physical scale of the coordinates is $\kappa$. Since the correlation length $\xi$ is given by $\xi = 1/\sqrt{2\mu M}$, $\kappa$ is comparable to the correlation length ($\xi = \sqrt{2} \kappa$).

Minimizing the Hamiltonian (61) we have

$$\partial^2 \rho - |\partial_t \zeta|^2 \rho = \left( \frac{\lambda}{2} (\rho^2 - \rho_0^2) + \delta \mu^2 (\zeta^\dagger \zeta_2) \right) \rho,$$

$$\left\{ (\partial^2 - \zeta^\dagger \partial^2 \zeta) + \frac{2}{\rho} (\partial_t - \zeta \partial_t \zeta) \right\} \zeta_2 = 0,$$

$$\left\{ (\partial^2 - \zeta^\dagger \partial^2 \zeta) + \frac{2}{\rho} (\partial_t - \zeta \partial_t \zeta) \right\} \zeta_3 = 0,$$

$$\zeta^\dagger \partial_t (\rho^2 \partial_t \zeta) - \partial_t (\rho^2 \partial_t \zeta^\dagger) = 0.$$  
(63)

The equation is almost identical to the equation (11) of two-gap superconductor, although on the surface it appears totally different from (11). To show this we let

$$\hat{n} = \zeta^\dagger \partial_t \zeta, \quad V_\mu = -i \zeta^\dagger \partial_\mu \zeta,$$  
(64)

and find the following identities

$$(\partial_\mu \hat{n})^2 = 4 (|\partial_\mu \zeta|^2 - |\zeta \partial_\mu \zeta|^2) = 4 (|\partial_\mu \zeta|^2 - V_\mu^2),$$

$$\hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = -2i (\partial_\mu \zeta^\dagger \partial_\nu \zeta - \partial_\nu \zeta^\dagger \partial_\mu \zeta)$$

$$= 2 (\partial_\mu V_\nu - \partial_\nu V_\mu).$$  
(65)

Moreover, from (17) we have

$$(\partial_\mu - i V_\mu - \frac{1}{2} \vec{\sigma} \cdot \partial_\nu \hat{n}) \zeta = 0.$$  
(66)

With these identities we can rewrite the equation (63) into a completely different form. Indeed with (65) the first equation of (63) can be written as

$$\partial^2 \rho - \left[ \frac{1}{4} (\partial_t \hat{n})^2 + V_\mu^2 \right] \rho = \left( \frac{\lambda}{2} (\rho^2 - \rho_0^2) \right.$$ 

$$+ \delta \mu^2 (\zeta^\dagger \zeta_2) \left. \right\} \rho.$$  
(67)

Moreover, with (66) the second and third equations of (63) can be expressed as

$$\frac{1}{2} \left( A + \vec{B} \cdot \vec{\sigma} \right) \zeta = 0,$$

$$A = \partial_\mu \hat{n}^2 + i (2 \zeta^\dagger \zeta_2 - 1) \delta \mu^2,$$

$$\vec{B} = \partial^2 \hat{n} + 2 \frac{\partial \rho}{\rho} \partial_\mu \hat{n} + 2i V_\mu \hat{n} \times \partial_\nu \hat{n}$$

$$+ i \delta \mu^2 \vec{k},$$  
(68)
where \( h = (0, 0, 1) \). Thus we can write \((68)\) as
\[
\hat{n} \times \partial^2 \hat{n} + \frac{2}{\rho} \frac{\partial \rho}{\partial \hat{n}} \times \partial_1 \hat{n} - 2 V_i \partial_i \hat{n} = \delta \mu^2 \hat{k} \times \hat{n}.
\]
Finally, the last equation of \((68)\) is written as
\[
\partial_i (\rho^2 V_i) = 0,
\]
which tells that \( \rho^2 V_i \) is solenoidal (i.e., divergenceless). So we can always replace \( V_i \) with another field \( B_i \)
\[
V_i = \frac{1}{\rho^2} \epsilon_{ijk} \partial_j B_k = \frac{1}{\rho^2} \partial_j G_{ij},
\]
and express \((69)\) as
\[
\hat{n} \times \partial^2 \hat{n} + 2 \frac{\partial \rho}{\rho} \hat{n} \times \partial_1 \hat{n} + \frac{2}{\rho^2} \partial_i G_{ij} \partial_j \hat{n} = \delta \mu^2 \hat{k} \times \hat{n}.
\]
With this \((69)\) can now be written as
\[
\partial^2 \rho - \frac{1}{4} \left( \partial_i \hat{n} \right)^2 + V_i^2 \rho = \frac{\lambda}{2} \left( \rho^2 - \rho_0^2 \right) + \delta \mu^2 \left( \zeta_1 \zeta_2 \right) \rho,
\]
\[
\hat{n} \times \partial^2 \hat{n} + 2 \frac{\partial \rho}{\rho} \hat{n} \times \partial_1 \hat{n} + \frac{2}{\rho^2} \partial_i G_{ij} \partial_j \hat{n} = \delta \mu^2 \hat{k} \times \hat{n},
\]
\[
\partial_i G_{ij} = -\rho^2 V_j.
\]
This tells that \((69)\) can be transformed to a completely different form which has a clear physical meaning. The last equation tells that the theory has a conserved \( U(1) \) current \( j_\mu \),
\[
j_\mu = \rho^2 V_\mu,
\]
which is nothing but the Noether current of the global \( U(1) \) symmetry of the Lagrangian \((63)\). The second equation tells that the theory has another partially conserved \( SU(2) \) Noether current \( \bar{j}_\mu \),
\[
\bar{j}_\mu = \rho^2 \hat{n} \times \partial_\mu \hat{n} - 2 \rho^2 V_\mu \hat{n},
\]
which comes from the approximate \( SU(2) \) symmetry of the theory broken by the \( \delta \mu \) term. It also tells that there is one more \( U(1) \) current
\[
k_\mu = \hat{k} \cdot \bar{j}_\mu,
\]
which is conserved even when \( \delta \mu \) is not zero. This is because the \( SU(2) \) symmetry is broken down to \( U(1) \) when \( \delta \mu \) is not zero.

More importantly this reveals that the Gross-Pitaevskii theory of two-component BEC is closely related to the Landau-Ginzburg theory of two-gap superconductor. Indeed, the equation \((11)\) of two-gap superconductor and the equation \((68)\) of two-component BEC acquire formally an identical form when \( \delta \mu = 0 \), except that here \( gB_\mu \) and \( F_{ij} \) is replaced by \( V_i \) and \(-G_{ij} \). This is really remarkable, but actually is not surprising. This is because, when the electromagnetic interaction is switched off, the Landau-Ginzburg Lagrangian \((4)\) reduces to the Gross-Pitaevskii Lagrangian \((63)\) in the limit \( \delta \mu = 0 \) (in non-relativistic limit). In this case the two theories really become identical.

To obtain the vortex solution in two-component BEC, we choose the ansatz
\[
\rho = \rho(\varphi),
\]
\[
\zeta = \left( \frac{\cos f(\varphi)}{2} \exp(-im\varphi) \right).
\]
With the ansatz \((69)\) is reduced to
\[
\dot{\rho} + \frac{1}{\rho} \dot{\rho} - \left( \frac{1}{4} f^2 + \frac{m^2}{\varrho^2} \cos^2 \frac{f}{2} + \delta \mu^2 \sin^2 \frac{f}{2} \right) \rho
\]
\[
= \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho,
\]
\[
\dot{f} + \left( \frac{1}{\varrho} + 2 \frac{\dot{\rho}}{\rho} \right) \dot{f} + \left( \frac{m^2}{\varrho^2} - \delta \mu^2 \right) \sin f = 0.
\]
Now with the boundary condition
\[
\rho(0) = 0, \quad \rho(\infty) = \rho_0, \quad f(0) = \pi, \quad f(\infty) = 0,
\]
we can solve the equation and obtain the non-Abelian vortex solution in two-component BEC shown in Fig. 3. Notice that \((68)\) also admits the well-known Abelian vortex solution with \( \zeta_1 = 0 \) or \( \zeta_2 = 0 \) (or equivalently \( f = 0 \) or \( f = \pi \)). But obviously they are different from the non-Abelian vortex, which has a non-trivial profile of \( f(\varphi) \).
One can show that this vortex is topological, which carries a topological quantum number. In fact it can be viewed as a quantized vorticity flux \[\Phi_z = -\pi \Phi_1\] along the z-axis. However, it does not change the physical nature of the vortex. In particular, the vortex can be interpreted as a vorticity flux. Finally, the vortex is topological in origin. It has a well-defined non-Abelian topology \(\pi_2(S^2)\), even when \(\delta \mu\) is not zero. This is unexpected, because in this case we have only \(U(1) \times U(1)\) symmetry which cannot provide the non-Abelian topology. Moreover the topological quantum number has a clear physical meaning. It represents the vorticity quantum number.

In this paper we have adopted the \(SU(2)\) symmetric quartic interaction with \(\mu = 1\) for simplicity. But in reality one may have to face a more complicated quartic interaction in two-component BEC. Nevertheless many of our results would undoubtedly survive with this complication. For other quartic potentials we refer the readers to existing literature \[7, 18\].

VI. GAUGE THEORY OF TWO-COMPONENT BEC

An important difference between two-component BEC and ordinary (one-component) BEC is the vorticity. The ordinary BEC has no vorticity but two-component BEC has a non-trivial vorticity. This is because the velocity field of one-component BEC is given by the gradient of the phase angle of the condensate, so that it has a vanishing vorticity. Due to the doublet structure, however, the velocity field of two-component BEC is not given by the gradient of the phase angle of the condensate. So a two-component BEC has a non-vanishing vorticity which plays an important role as we have seen in the above analysis.

In general creating vorticity costs energy. But the above theory of two-component BEC does not reflects this point properly, because the Gross-Pitaevskii Lagrangian \[3\] has no vorticity interaction. But we can construct a gauge theory of two-component BEC which can naturally accommodate the vorticity interaction \[4\]. In the following we show that the gauge theory of two-component BEC also allows a vorticity vortex very similar to the one we have in Gross-Pitaevskii theory.

Consider a “charged” two-component condensate \(\phi\) interacting “electromagnetically”, which can be described by the gauged Gross-Pitaevskii Lagrangian

\[
\mathcal{L} = i\frac{\hbar}{2} \left[ \phi^\dagger (D_\mu \phi) - (D_\mu \phi)^\dagger \phi \right] - \frac{\hbar^2}{2M} |D_\mu \phi|^2 + \mu \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu}^2, \\
D_\mu \phi = (\partial_\mu - i g A_\mu) \phi.
\]
izing $\phi$ to $(\sqrt{2M/\hbar})\phi$ and putting

$$\phi = \frac{1}{\sqrt{2}} \alpha \zeta, \quad (\zeta^\dagger \zeta = 1)$$

we have the following Hamiltonian from (85) in the static limit

$$\mathcal{H} = \frac{1}{2} (\partial_i \rho)^2 + \frac{1}{2} \rho^2 |\tilde{D}_i \zeta|^2 + \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2,$$

$$+ \frac{1}{4} \tilde{F}_{ij}^2,$$  

where $\rho_0^2 = 2 \mu/\lambda$, and we have again rescaled $\mu^2$ and $\lambda$.

Obviously the Lagrangian (85) can be identified as a non-relativistic Landau-Ginzburg Lagrangian of two-gap superconductor. But, of course, here we are dealing with the neutral condensates, so that the “electromagnetic” interaction should be treated not as independent but as self-induced. This means that the gauge potential has to be a composite field of the condensate, and we may identify the “electromagnetic” potential by the velocity field of $\zeta^\dagger \zeta$.

$$g \tilde{A}_\mu = V_\mu = -i \xi^\dagger \partial_\mu \zeta. \quad (87)$$

A justification for this is that we can actually derive this from the Hamiltonian (85) if we neglect the Maxwell term. Indeed (87) becomes nothing but the Euler-Lagrange equation of the Hamiltonian for the potential in the absence of the Maxwell term.

Now, introducing a $CP^1$ field $\xi$ by

$$\zeta = \exp(i \gamma) \xi, \quad \xi^\dagger \xi = 1, \quad (88)$$

we have

$$\tilde{A}_\mu = -\frac{i}{g} \xi^\dagger \partial_\mu \xi + \partial_\mu \gamma,$$

$$\tilde{F}_{\mu \nu} = -\frac{i}{g} (\partial_\mu \xi^\dagger \partial_\nu \xi - \partial_\nu \xi^\dagger \partial_\mu \xi)$$

$$= \frac{1}{g} \tilde{H}_{\mu \nu}, \quad (89)$$

where $\tilde{H}_{\mu \nu}$ is the vorticity of the velocity potential $V_\mu$. With this the Hamiltonian (85) is written as

$$\mathcal{H} = \frac{1}{2} (\partial_i \rho)^2 + \frac{1}{2} \rho^2 \left(|\partial_i \xi|^2 - |\xi^\dagger \partial_i \xi|^2 \right) + \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2$$

$$+ \frac{1}{4g} (\partial_i \xi^\dagger \partial_j \xi - \partial_j \xi^\dagger \partial_i \xi)^2. \quad (90)$$

This tells two things. First the Hamiltonian naturally accommodates the vorticity interaction. Secondly the doublet $\zeta$ is completely replaced by the $CP^1$ field $\xi$ in the Hamiltonian, so that the theory becomes a theory of $CP^1$ field (coupled to a scalar field $\rho$).

One might wonder why we need to include the vorticity interaction in the Hamiltonian, when we do not have such interaction in one-component BEC. The reason is that creating a vorticity costs energy. So physically it makes sense to keep the vorticity interaction in the Hamiltonian. Moreover, here the coupling constant $g$ now represents the strength of the vorticity interaction. So we can always remove the vorticity interaction if necessary, by putting $g = \infty$. This justifies the vorticity interaction in the Hamiltonian (4).

Minimizing the Hamiltonian (with the constraint $\zeta^\dagger \zeta = 1$) we have the following equation of motion

$$\partial^2 \rho - \left(|\partial_i \xi|^2 - |\xi^\dagger \partial_i \xi|^2 \right) \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho,$$

$$\left\{ (\partial^2 - \xi^\dagger \partial^2 \xi) + 2 \frac{\partial_i \rho}{\rho} - \xi^\dagger \partial_i \xi \right\} (\partial_i - \xi^\dagger \partial_i \xi)$$

$$- \frac{2}{g^2 \rho^2} (\partial_i (\partial_i \xi^\dagger \partial_j \xi - \partial_j \xi^\dagger \partial_i \xi)) (\partial_j - \xi^\dagger \partial_j \xi) \right\} \xi = 0. \quad (91)$$

To understand the meaning of (91) notice that we can rewrite the identities (85) and (63) as

$$|\partial_i \xi|^2 - |\xi^\dagger \partial_i \xi|^2 = \frac{1}{4} (\partial_i \tilde{n})^2,$$

$$- i (\partial_i \xi^\dagger \partial_j \xi - \partial_j \xi^\dagger \partial_i \xi) = \frac{1}{2} \tilde{n} \cdot (\partial_i \tilde{n} \times \partial_j \tilde{n})$$

$$= \tilde{H}_{ij}, \quad (92)$$

and

$$(\partial_i - ig \tilde{A}_i - \frac{1}{2g} \sigma \cdot \partial_i \rho) \tilde{n} = 0,$$

$$\tilde{A}_i = -\frac{i}{g} \xi^\dagger \partial_i \zeta. \quad (93)$$

With this (following the same procedure we adopted in the above section) we can rewrite (41) as

$$\partial^2 \rho - \frac{1}{4} (\partial_i \tilde{n})^2 \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho,$$

$$\tilde{n} \times \partial^2 \tilde{n} + 2 \frac{\partial_i \rho}{\rho} \tilde{n} \times \partial_i \tilde{n} + \frac{1}{g^2 \rho^2} \partial_i \tilde{H}_{ij} \partial_j \tilde{n} = 0. \quad (94)$$

This is the equation of two-component BEC that we are looking for. Notice that the second equation assures that the theory has a conserved $\mathbf{2}$ current, which is a direct consequence of the global $SU(2)$ symmetry of the Lagrangian (85).

The equation (94) should be compared with the equation (41) of two-gap superconductor and the equation (63) of two-component BEC, which are very similar to each other. In particular the similarity between (41) and (63) is unmistakable. Indeed, when $B_\mu = 0$, the first two equations of (41) reduce exactly the equation (63) of two-component BEC. This tells that the gauge theory of two-component BEC is almost identical to the Landau-Ginzburg theory of two-gap superconductor. The only difference is that in BEC the electromagnetic interaction
becomes induced, because here the condensate is neutral.

With (22) the Hamiltonian \( \mathcal{H} \) acquires a remarkable form

\[
\mathcal{H} = \frac{1}{2} (\partial_i \rho)^2 + \frac{\rho^2}{8} (\partial_i \hat{n})^2 + \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2
  + \frac{1}{16g^2} (\partial_i \hat{n} \times \partial_j \hat{n})^2
  = \lambda \rho_0^4 \mathcal{H},
\]

where

\[
\dot{\hat{\rho}} = \frac{\rho}{\rho_0}, \quad \dot{\hat{n}} = \kappa \partial_i n, \quad \kappa = \frac{1}{\sqrt{\lambda \rho_0}}.
\]

So the theory can be written completely in terms of the non-linear sigma field \( \hat{n} \) (and the scalar field \( \rho \)). The reason for this is the Abelian gauge invariance of \( \mathcal{H} \), which remains intact with the self-induced interaction introduced by (22). And, just as in two-gap superconductor, this Abelian gauge invariance reduces the physical target space of \( \zeta \) to the gauge orbit space \( S^2 = S^3/S^1 \) of the \( CP^1 \) field \( \xi \), which is identical to the target space of the non-linear sigma field \( \hat{n} \). This is why we could transform the equation of motion (94) completely into the equation for \( \hat{n} \) in (94). This means that we can describe the theory as a self interacting \( CP^1 \) model, or equivalently a self interacting non-linear sigma model (coupled to a scalar field \( \rho \)) [1].

To construct the desired vortex solution in this theory we choose the ansatz

\[
\rho = \rho(\varphi), \quad \zeta = \left( \frac{\cos f(\varphi)}{2}, \frac{\sin f(\varphi)}{2} \right), \quad \hat{n} = \left( \frac{\sin f(\varphi) \cos m \varphi}{\cos f(\varphi)}, \frac{\sin f(\varphi) \sin m \varphi}{\cos f(\varphi)} \right), \quad \dot{A}_\mu = -\frac{m}{2g} \cos f(\varphi) + 1) \partial_\mu \varphi,
\]

and reduce the equation to

\[
\ddot{\rho} + \frac{1}{\rho} \dot{\rho} - \frac{1}{4} \left( f^2 + \frac{m^2}{\rho^2} \sin^2 f \right) \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho,
\]

\[
\hat{\rho} = \frac{\rho}{\rho_0}, \quad \dot{\hat{n}} = \kappa \partial_i n, \quad \kappa = \frac{1}{\sqrt{\lambda \rho_0}}.
\]

Notice the remarkable similarity between the above equation and the magnetic vortex equation (10) in two-gap superconductor. Indeed without the electromagnetic field (i.e., with \( B = 0 \)) the first two equations of (10) becomes identical to the above equation. Now, with the boundary condition

\[
\dot{\rho}(0) = 0, \quad \rho(\infty) = \rho_0,
\]

\[
f(0) = \pi, \quad f(\infty) = 0,
\]

we obtain the the non-Abelian vortex solution in gauge theory of two-component BEC shown in Fig 4. Notice again that the non-trivial profile of \( f \) assures that the vortex is non-Abelian.

Just like in the Gross-Pitaevskii theory the vortex here is topological, whose topological quantum number is expressed by \( \pi_2(S^2) \) of the condensate \( \xi \) (or by the non-linear sigma field \( \hat{n} \)),

\[
q = -\frac{i}{4\pi} \int \epsilon_{ij} \partial_i \xi^j \partial_j \xi d^2 x = \frac{1}{8\pi} \int \epsilon_{ij} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}) d^2 x = m.
\]

Moreover, our analysis tells that the non-Abelian vortex is nothing but the quantized vorticity flux \( \hat{H}_\mu \), which is confined by a Meissner effect. Again this is because the \( U(1) \) gauge symmetry of \( \mathcal{H} \) assures the existence of a conserved supercurrent

\[
\dot{\mu} = \partial_\mu \hat{H}_\mu, \quad \partial_\mu \dot{\mu} = 0,
\]

which generates and confines the vorticity flux. This tells that two vortex solutions in this and the last section are almost identical. Both describe a quantized vorticity and have identical topology. The only difference is the dynamics. In the first case the vorticity interaction is absent, but in the second case it plays an important role. So we have two competing theories of two-component BEC. Which describes the real two-component BEC can only be answered by experiments.
VII. A PROTOTYPE NON-ABELIAN VORTEX: BABY SKYRMION

There is a well-known prototype non-Abelian vortex, the baby skyrmion in Skyrme theory, which is very similar to the non-Abelian vortices in condensed matters. In fact all non-Abelian vortices that we discussed in this paper originate from the baby skyrmion. This is because Skyrme theory itself is closely related to the above theories of two-gap superconductor and two-component BEC. In this sense it is worth reviewing the Skyrme theory and clarify the connection between the Skyrme theory and above theories of condensed matters.

The Skyrme theory has a rich topological structure. The theory allows not only the original skyrmion \[19\], but also a prototype knot known as Faddeev-Niemi knot \[8\], \[20\]. Furthermore, it allows a non-Abelian vortex called the baby skyrmion \[21\]. Since the Skyrme theory can be viewed as a non-linear sigma model, it has a global \(SU(2)\) symmetry which allows a non-trivial topology \(\pi_3(S^3)\). In \((1+3)\)-dimension this \(\pi_3(S^3)\) is responsible for the skyrmion. But with the Hopf fibering of \(S^3\) to \(S^2 \times S^1\), this \(\pi_3(S^3)\) can be reduced to \(\pi_3(S^2)\) which provides the knot topology. Moreover in \((1+2)\)-dimension the theory allows \(\pi_2(S^2)\), which is responsible for the baby skyrmion. Among these topological objects the baby skyrmion and the knot are of particular relevance to us.

The importance of the baby skyrmion follows from the fact that it can give rise to the Faddeev-Niemi knot. In fact one can construct a twisted vortex ring by twisting it (making it periodic in \(z\)-coordinate) and connecting the periodic ends together. The twisted vortex ring acquires the knot topology \(\pi_3(S^2)\), and becomes the Faddeev-Niemi knot \[8\], \[21\]. This observation strongly implies the existence of topological knots in two-gap superconductor and two-component BEC, because one could also construct a twisted vortex ring with the above non-Abelian vortices. So it is crucial to understand the baby skyrmion for us to construct the non-Abelian knots in condensed matters. For this reason we briefly review the baby skyrmion in this section.

Let \(\omega\) and \(\hat{n}\) be the massless scalar field and non-linear sigma field in Skyrme theory. With

\[
U = \exp(\omega \hat{\sigma} \cdot \hat{n})
\]

\[
= \cos \frac{\omega}{2} - i(\hat{\sigma} \cdot \hat{n}) \sin \frac{\omega}{2}
\]

\[
(\hat{n}^2 = 1),
\]

\[
L_\mu = U \partial_\mu U^\dagger,
\]

one can write the Skyrme Lagrangian as

\[
\mathcal{L} = \frac{\mu_\nu^2}{4} \text{tr} L_\mu^2 + \frac{\alpha}{32} \text{tr} ([L_\mu, L_\nu])^2.
\]

(102)

A remarkable feature of the Skyrme Lagrangian is that

\[
\omega = \pi,
\]

(103)

is a solution of the equation of motion, independent of \(\hat{n}\). This means that, as far as we are concerned with the classical solutions, we can assume \(\omega = \pi\). In this case the Lagrangian (102) is reduced to the Skyrme-Faddeev Lagrangian \[8\],

\[
\mathcal{L} \rightarrow -\frac{\mu_\nu^2}{2}(\partial_\mu \hat{n})^2 - \frac{\alpha}{4}(\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2,
\]

(104)

whose equation of motion is given by

\[
\hat{n} \times \partial^2 \hat{n} + \frac{\alpha}{\mu_\nu^2}(\partial_\mu \hat{H}_{\mu\nu}) \partial_\nu \hat{n} = 0,
\]

\[
\hat{H}_{\mu\nu} = \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}).
\]

(105)

This is the equation which describes not only the baby skyrmion but also the Faddeev-Niemi knot \[8\]. Notice that here \(\hat{H}_{\mu\nu}\) is mathematically identical to the vorticity \(\hat{H}_{\mu\nu}\) in BEC (up to the overall factor two), so that it can be expressed as a field strength of potential \(2V_{\mu}\).

Now it must be clear that the equation (105) is very similar to the equation (11) of two-gap superconductor and the equations (79) and (93) of two-component BEC. In fact all three equations of two-gap superconductor and two-component BEC can be viewed as straightforward generalizations of (105). This implies that the Skyrme theory and the above theories of condensed matters are closely related.

The Skyrme-Faddeev theory allows (not only the Faddeev-Niemi knot but also) a vortex solution called the baby skyrmion \[21\]. To be specific, let

\[
\hat{n} = \left(\begin{array}{c}
\sin f(\varphi) \
\cos f(\varphi)
\end{array}\right).
\]

(106)

With this the equation (105) becomes

\[
\left(1 + \frac{\alpha}{\mu_\nu^2} \frac{m^2}{\varphi^2} \sin^2 f(\varphi)\right)\ddot{f}
\]

FIG. 5: The baby skyrmion with \(m = 1\) in Skyrme theory, where we have put \(\alpha/\mu^2 = 1\) for simplicity. Notice that \(\varphi\) is in the unit of \(\sqrt{\alpha}/\mu\).
\[
\begin{align*}
+ \left( \frac{1}{\varrho} + \frac{\alpha m^2}{\mu^2 \varrho^2} \hat{f} \sin f \cos f - \frac{\alpha m^2}{\mu^2 \varrho^2} \sin^2 f \right) \hat{f} \\
- \frac{m^2}{\varrho^2} \sin f \cos f = 0.
\end{align*}
\]

Now, with the boundary condition
\[ f(0) = \pi, \quad f(\infty) = 0, \]

one can obtain a (massless) baby skyrmion solution shown in Fig. 5. Notice the unmistakable similarity between the equation (107) and the vortex equations (106) and (97) of two-gap superconductor and two-component BEC. With \( \rho = \rho_0 \) the second equation of (97) becomes identical to (107). Furthermore with \( \rho = \rho_0 \) and \( B = 0 \) the second equation of (106) becomes identical to (107).

In particular, when \( \alpha = 0 \), the above equation has the well-known analytic solution
\[
\cos f(\varrho) = \frac{\varrho^2 - a^2}{\varrho^2 + a^2}, \quad \sin f(\varrho) = \frac{2\varrho a}{\varrho^2 + a^2},
\]

which is precisely the solution (20) that we have in two-gap superconductor.

This implies that the baby skyrmion can actually be viewed as a confined magnetic vortex in condensed matter, confined by the Meissner effect. Indeed we can view that \( H_{\mu \nu} \) in (105) is generated by the conserved supercurrent
\[
\tilde{J}_\mu = \partial_\nu H_{\mu \nu}, \quad \partial_\mu \tilde{J}_\mu = 0.
\]

This means that we can interpret \( H_{\mu \nu} \) as a magnetic field created by the supercurrent which confines the magnetic flux. This demonstrates the existence of the Meissner effect in Skyrme theory. This tells that the baby skyrmion is very much like a magnetic vortex in two-gap superconductor or a vorticity field in two-component BEC.

Clearly the baby skyrmion is topological. Again with the one-point compactification of the \( xy \)-plane \( R^2 \) to \( S^2 \), \( \hat{u} \) defines the homotopy \( \pi_2(S^2) \) of the mapping from the \( S^2 \) to the target space \( S^2 = SU(2)/U(1) \) with the integral topological quantum number \( m \),
\[
q = \frac{1}{8\pi} \int \epsilon_{ijk} \hat{n} \cdot (\partial_i \hat{n} \times \partial_j \hat{n}) \, d^2x = m.
\]

Obviously this is identical to the topological quantum number of the non-Abelian vortices in condensed matters. This confirms that indeed the baby skyrmion is a prototype non-Abelian vortex which is very similar to all non-Abelian vortices in condensed matters.

The baby skyrmion is unstable in the sense that, by enlarging the size of the baby skyrmion, one can lower the energy of the baby skyrmion. So it could not be thought to represent a realistic physical object. Nevertheless it plays a very important role because it can give rise to the Faddeev-Niemi knot [5]. Indeed one can view the knot as a twisted magnetic flux ring, which one obtains by twisting the baby skyrmion (making it periodic in \( z \)-coordinate) and connecting the periodic ends together. The identification of the Faddeev-Niemi knot as a twisted vortex ring made of baby skyrmion strongly indicates the existence of a similar knot in two-gap superconductor and two-component BEC. This is because we can also construct a similar knot by twisting the magnetic vortex or the vorticity vortex to make a twisted vortex ring [4, 5, 17].

The knot quantum number of Faddeev-Niemi knot is given by
\[
\begin{align*}
\kappa &= \frac{1}{16\pi^2} \int \epsilon_{ijk} V_i \tilde{H}_{jk} d^3x \\
&= \frac{1}{4\pi^2} \int \epsilon_{ijk} \xi_i \partial_j (\xi_j \xi_k \partial_l) \xi^l d^3x.
\end{align*}
\]

Exactly the same topology should describe the non-Abelian knots in two-gap superconductor and two-component BEC.

**VIII. DISCUSSIONS**

In this paper we have presented a convincing evidence how the new condensed matters, in particular the two-gap superconductor and two-component BEC, can have novel non-Abelian vortices. A characteristic feature of the non-Abelian vortices is the non-trivial profile of the doublet. At the core the vortex is made of only the second component, but as we move away from the core the first component takes over and fills the space completely at infinity.

Our analysis tells that at the center of all these non-Abelian vortices lies the baby skyrmion. Indeed all these non-Abelian vortices stem from the baby skyrmion. This suggests that the Skyrme theory could also play an important role in condensed matter physics. Ever since Skyrme proposed his theory, the Skyrme theory has always been associated to nuclear and/or high energy physics. This has lead people to believe that the topological objects in Skyrme theory can only be realized at high energy, at the GeV scale. But our analysis opens up a new possibility for us to construct them in a completely different environment at much lower energy scale, in the new condensed matters [4, 5]. This is really remarkable.

Another important lesson from our analysis is that the non-Abelian dynamics could play a crucial role in condensed matter, in particular in multi-component condensed matter. Perhaps this might not be so surprising, given the fact that the multi-component condensed matters can only (and naturally) be identified as non-Abelian multiplets. Nevertheless, it is really remarkable that one can actually construct a non-Abelian gauge theory of superconductivity. Moreover the gauge theory of non-Abelian superconductivity suggests that, implicitly or
explicitly, the underlying dynamics of multi-component condensed matters can ultimately be related to a non-Abelian dynamics. This suggests that the non-Abelian gauge theory could play an important role in condensed matter physics in the future.

Perhaps a most immediate outcome of our analysis is the existence of topological knots in two-component condensed matters \[1, 2\]. From our analysis it must become clear that we can also construct a helical vortex by twisting the non-Abelian vortex along the \(z\)-axis and making it periodic in \(z\)-coordinate. Of course such a helical vortex by itself may be unstable and likely to unwind itself to a straight non-Abelian vortex, unless the periodicity condition is enforced by hand. However, we can make it stable by making it a vortex ring by smoothly connecting two periodic ends. The stability follows from the fact that due to the twist the vortex ring can not collapse dynamically. For example the twisted vorticity ring has a velocity current along the knot. This in turn generates a non-vanishing angular momentum around the \(z\)-axis, which provides a centrifugal repulsive force to prevent the collapse of the vortex ring. So the twist which creates the instability in the helical vortex now serves to provide the dynamical stability of the knot \[1, 2\].

Furthermore, this dynamical stability of the knot can be backed up by the topological stability. This is because mathematically the non-linear sigma field \(\hat{n}\), after forming a knot, acquires a non-trivial topology \(\tau_3(S^2)\), which can not be changed by a smooth deformation of the field. This endorses our earlier claim that the new condensates allow not only the non-Abelian vortices but also stable knots \[1, 2\].

From our analysis there should be no doubt that the non-Abelian vortices and the topological knots must exist in the new condensed matters. If so, the challenge now is to verify the existence of these topological objects experimentally. Constructing the knots might not be a simple task at present moment. But the construction of the non-Abelian vortices could be rather straightforward (at least in principle), which might have already been done \[15, 16\]. To identify the non-Abelian vortices, there are two points one has to keep in mind. First, the (magnetic) flux of the non-Abelian vortices is twice as much as that of the Abelian counterparts. Secondly, the non-Abelian vortices must have a non-trivial profile of \(f(\phi)\). This is a crucial point which distinguishes them from the Abelian vortices. With this in mind, one should be able to construct and identify the non-Abelian vortices in the new condensates without much difficulty.

We conclude with the following remarks:

1. We have emphasized the potential importance of vorticity interaction in two-component BEC which is different from the polynomial Gross-Pitaevskii interaction which one has in single-component BEC. The advantage of the vorticity interaction is that it is a gauge interaction, except that here the gauge potential is given by the velocity field of doublet \(\zeta\). This makes the gauge theory very similar to the Landau-Ginzburg theory of two-gap superconductor. In spite of the apparent similarity the vorticities in single-component BEC and in one-gap superconductor have been thought to be based on seemingly different dynamics, the one on the polynomial interaction the other on the gauge interaction. Our self-interaction restores their similarity at the theoretical level. We propose that the self-induced gauge theory of two-component BEC could also play a fundamental role in studying the non-Abelian superfluidity in multi-component superfluid. In fact we believe that the vorticity vortex in two-component BEC could also describe the vortex in \(^3\text{He}\) superfluid \[22\].

2. As we have pointed out, the two-component BEC and two-gap superconductor can also admit the Abelian vortex as a solution when \(f = 0\) or \(f = \pi\). Moreover, in the critical case when the coherence length and the penetration length are the same, the magnetic flux and the energy of the non-Abelian vortex with \(q = m\) and those of Abelian vortex with \(q = 2m\) becomes degenerate. This raises an intriguing possibility, the possibility of tunneling between the Abelian vortex with \(q = 2m\) and the non-Abelian vortex with \(q = m\). This is an interesting question deserved to be studied further.

3. In this paper we have concentrated on the condensed matter physics. But it must be clear that our results should also have important implications in high energy physics and cosmology. Indeed the existence of a chromoelectric knot in QCD \[23\] and an electroweak knot in Weinberg-Salam model \[24\] which are very similar to the knots discussed here have already been proposed. Moreover the existence of a cosmic string in cosmology has been speculated by many authors \[25\]. We believe that the above theories of condensed matters, in particular the \(SU(2)\) gauge theory of superconductor can easily be embedded in any standard model of grand unification, and be a realistic model for such a cosmic string.

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[1] A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957); H. Nielsen and P. Olesen, Nucl. Phys. 61, 45 (1973).

[2] J. Nagamatsu et al., Nature 410, 63 (2001); S. L. Bud’ko et al., Phys. Rev. Lett. 86, 1877 (2001); C. U. Jung et
al., Appl. Phys. Lett. **78**, 4157 (2001).

[3] C. Myatt *et al.*, Phys. Rev. Lett. **78**, 586 (1997); D. Stamper-Kurn, *et al.*, Phys. Rev. Lett. **80**, 2027 (1998); J. Stenger *et al.*, Nature **396**, 345 (1998).

[4] Y. M. Cho, cond-mat/0112325, IJPAP in press.

[5] Y. M. Cho, cond-mat/0112498.

[6] J. Ruostekoski and J. Anglin, Phys. Rev. Lett. **86**, 3934 (2001); U. Al Khawaja and H. Stoof, Nature **411**, 818 (2001); H. Stoof *et al.*, Phys. Rev. Lett. **87**, 120407 (2001).

[7] R. Battye, N. Cooper, and P. Sutcliffe, Phys. Rev. Lett. **88**, 080401 (2002); C. Savage and J. Ruostekoski, Phys. Rev. Lett. **91**, 101403 (2003); M. Metlitski and A. Zhitnitsky, JHEP **0406**, 017 (2004).

[8] Y. M. Cho, Phys. Rev. Lett. **87**, 252001-1 (2001); Y. M. Cho, Phys. Lett. **B603**, 88 (2004); Y. M. Cho, P. S. Park, and N. S. Yong, hep-th/0404181.

[9] I. Mazin, O. Anderson, O. Jepsen, O. Dolgov, J. Kortus, A. Golubov, A. Kuzmenko, and D. van der Marel, Phys. Rev. Lett. **89**, 107002 (2002); A. Koshelev and A. Golubov, Phys. Rev. Lett. **90**, 177002 (2003); T. Dahn and N. Schopohl, Phys. Rev. Lett. **91**, 017001 (2003); M. Zhitomirsky and V. Dao, Phys. Rev. **B69**, 054508 (2004).

[10] E. Babaev, Phys. Rev. Lett. **89**, 067001 (2002); E. Babaev, L. Faddeev, and A. Niemi, Phys. Rev. **B65**, 100512 (2002).

[11] Y. M. Cho, Phys. Rev. **D21**, 1080 (1980); Y. M. Cho, Phys. Rev. **D62**, 074009 (2000).

[12] Y. M. Cho, Phys. Rev. Lett. **46**, 302 (1981); Phys. Rev. **D23**, 2415 (1981); W. S. Bae, Y. M. Cho, and S. W. Kimm, Phys. Rev. **D65**, 025005 (2002).

[13] A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. **B59**, 85 (1975); G. ’t Hooft, Phys. Rev. Lett. **37**, 8 (1976).

[14] Y. M. Cho, Phys. Lett. **B81**, 25 (1979).

[15] D. Hall *et al.*, Phys. Rev. Lett. **81**, 1536 (1998); C. Law *et al.*, Phys. Rev. Lett. **81**, 5215 (1998); J. Williams and M. Holland, Nature **401**, 568 (1999).

[16] M. Matthews *et al.*, Phys. Rev. Lett. **83**, 2498 (1999); K. Madison *et al.*, Phys. Rev. Lett. **84**, 806 (2000).

[17] Y. M. Cho, cond-mat/0409636.

[18] J. Garcia-Ripoll and V. Perez-Garcia, Phys. Rev. Lett. **84**, 4264 (2000); Phys. Rev. **A62**, 033601 (2000); D. Skryabin, Phys. Rev. **A63**, 013602 (2000); S. Chui, V. Ryzhov, and E. Tareyeva, Phys. Rev. **A63**, 023605 (2001); D. Jepek, P. Capuzzi, and H. Cataldo, Phys. Rev. **A64**, 023605 (2001).

[19] T. H. R. Skyrme, Proc. Roy. Soc. (London) **260**, 127 (1961); **262**, 237 (1961); Nucl. Phys. **31**, 556 (1962).

[20] L. Faddeev and A. Niemi, Nature **387**, 58 (1997); J. Gladikowski and M. Hellmund, Phys. Rev. **D56**, 5194 (1997); R. Battye and P. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998).

[21] B. Piette, H. Muller-Kristen, D. Tchirakian, and W. Zakrzewski, Phys. Lett. **B320**, 294 (1994); B. Piette, B. Schroers, and W. Zakrzewski, Nucl. Phys. **A643**, 205 (1995).

[22] For the non-Abelian vortex in $^3$He superfluid see, for example, G. Volovik, *The Universe in a Helium Droplet*, Clarendon Press, Oxford (2003).

[23] Y. M. Cho, Phys. Lett. **B616**, 101 (2005).

[24] Y. M. Cho, hep-th/0110076.

[25] E. Witten, Nucl. Phys. **B249**, 557 (1985); A. Everett, Phys. Rev. Lett. **61**, 1807 (1988).