Magnetism induced by electric impurities: A one-body problem with half the physics of the fractional quantum Hall effect

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Abstract

We study spectra for a 2D electron in the lowest Landau level with randomly distributed, repulsively correlated electric impurities. The lowest energy band reflects an effective magnetic field downshifted by an integer multiple of the impurity density. The downshift is precisely half that for the corresponding electron density in the composite-fermion picture of the fractional quantum Hall regime.

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1 Introduction

In a famous paper on the relation between mathematics and physics [1], Eugene Wigner speculated on the possibility of phenomena that cannot be
built from something ‘more fundamental’. Within physics, reduction to fundaments has generally taken the form of going to smaller distances and higher energies, hoping to find fewer and simpler rules. Phenomena at large distances or low energies, even some once regarded as fundamental, then become consequences of the small-distance or high-energy behavior. In practice, even though experimental and numerical evidence may give a link great conviction, rigorous proofs are quite rare. This is one reason why experiment is essential to the progress of physics, and also why successful prediction of ‘less fundamental’ physics also is rare.

Anderson [2] and Laughlin [3] introduced an appealing perspective on such questions, defining “emergent phenomena” that catch physicists by surprise, even though they might in principle have been deduced from more microscopic knowledge. Their articles emphasize that such occurrences are associated especially with systems including large numbers of components, leading to the phrase “more is different.”

The fractional quantum Hall effect (FQHE) is a prime example of such unpredicted physics. Following the experimental discovery, Laughlin [4] gave an explanation, providing an ansatz for the many-body wave functions relevant to the simple FQHE fractions. In a further advance Jain [5, 6, 7, 8] introduced ‘composite fermions’ to account for general fractions. The CF description works well even for the compressible regime where the ratio of electron density to magnetic flux density is around $\nu = \frac{1}{2}$. The evidence justifying these approaches comes from experiment, from exact numerical computations for small systems, and from self-consistent computations for larger systems.

These successes leave little doubt that we have a valid theory for FQHE, but it is tempting to ask for more: One may assume that only electrons in the lowest Landau level need be considered, very nearly reducing a problem in two space dimensions to a problem in one space dimension. Thus this might be one of those rare cases where rigorous deduction of the phenomena from the (very simple) fundamental dynamics is within reach.

We aim to establish a pathway towards understanding the FQHE as a deducible phenomenon. In terms of one quantitative measure, we shall see that the first steps described below come exactly halfway towards that goal, although more by numerical than by analytic methods. The starting point should be a two-dimensional gas of electrons whose screened electric interaction gives a short-range repulsion, with a magnetic field in the direction normal to the plane of electron motion so strong that only the lowest Landau
level for one orientation of electron spin need be considered.

In this work we make a drastic additional simplification, by treating all electrons except one as fixed in position, and consider the resulting one-body problem of an electron moving in a magnetic field under the influence of an array of electric impurities mimicking the effect of all the other electrons. Because these electrons have strong repulsive correlations, we assume such correlations for the impurities also. Studying a system with tens to thousands of impurities spread throughout a region of uniform magnetic field, we find that the one-body spectrum shows a characteristic level clustering which implies an effective magnetic field shifted down from the applied field by an integer number $n$ of flux quanta per impurity. In Jain’s composite fermion model, a field shift specified by an even integer $2q$ is found, where the value of $q$ increases as the impurity density decreases. We find the remarkable result for densities in a range from $\nu = \frac{1}{3}$ towards $\nu = \frac{1}{9}$, $q = n$, i.e., the two integers track together. Thus by this measure our one-body problem accounts for exactly half the physics in the fractional quantum Hall regime.

The crucial idea behind our study is that an electron in the presence of electric impurities and a strong magnetic field feels an induced or effective magnetic field transmuting the electric repulsion into magnetic torque. To motivate this idea, consider a two-dimensional Aharonov atom in a uniform, perpendicular magnetic field. Let the attractive potential binding the electron to the (uncharged) ‘nucleus’ constrain the electron to occupy only degenerate states centered on the nucleus with azimuthal quantum numbers $m = 0$ or $m = 1$ of the lowest Landau level. The degeneracy will be broken by interactions of the atom with electric impurities. Imagine that the nucleus is allowed to move towards a concentrated charge at point $r$. For large separations, the $m = 0$ state of the atom will give the least overlap of the electron and charge, and thus the lowest energy. However, at zero separation the $m = 1$ state of the atom will give the lowest energy.

Because there is mixing between the two states, the atom will shift from the $m = 0$ to the $m = 1$ state in the course of an adiabatic journey from large to zero separation between the nucleus and the fixed electron. In the limit of weak mixing the transition from $m = 0$ to $m = 1$ takes place on a circle specified by a precise separation $R$ between the charge and the nucleus of the Aharonov atom. Therefore, when the nucleus of the atom sits at that separation, the two wavefunctions must match everywhere on the transition circle. This can be achieved provided there is a gauge transformation matching the “atomic” wave functions across the transition circle, given by the phase fac-
tor \( e^{i\phi} \). The result of this gauge transformation is equivalent to an additional effective magnetic flux of one quantum, opposing the uniform magnetic field. Because the mixing by the Coulomb field is not infinitely weak, the transition circle becomes a diffuse ring, in which the effective flux is distributed. If the nucleus moves slowly in a closed curve, the effective or induced magnetic flux enclosed is an example of Berry’s ‘geometric magnetism’ [10]. Clearly, this effect could not occur if the given uniform magnetic field were not present, because there would be no way for the Coulomb repulsion to “choose” the sign of the effective magnetic field.

The fictitious Aharonov atom suggests a more realistic example. Let us choose as our slow variable, in place of the heavy nucleus, the guiding-center coordinate for the electron, allowing in principle the full (degenerate) set of azimuthal \( m \) values with respect to this center. This would specify a complete set of states for any location of the guiding center, but the requirement of minimum energy in the presence of electric impurities should remove the consequent redundancy.

Now imagine some density of distributed electric impurities. If the density is low enough, then when the guiding center is exactly in a space between impurities, as far as possible from each of the nearest neighbors, it is energetically favorable to have \( m = 0 \). On the other hand, if the guiding center coincides with one of the impurities, a higher \( m \) value minimizes the energy, by keeping the electron away from the central impurity. The new \( m \) value cannot be too high, however, to avoid overlap with other impurities. Thus the qualitative expectation is that the optimum \( m \) value for this position will increase as the impurity density decreases. The consequence will be an effective magnetic field, always of one sign, distributed throughout the plane, and on average an integer multiple of the impurity density, with the multiple being larger as the density becomes smaller.

We shall see that such behavior indeed occurs for the lowest-energy part of the spectrum in the presence of randomly distributed, repulsively correlated impurities. It doesn’t happen for a regular lattice, most likely because some higher \( m \) values then can be degenerate with the two specified in the above discussion.

Our approach superficially resembles earlier models for genuine impurities, related to the integer quantum Hall effect. The main differences between this analysis and those studies [11, 12, 13, 14, 15, 16, 17, 18] are: a) impurity density of less than one per flux quantum for our analysis, rather than five or more per flux quantum for the earlier studies, and b) our identification of
the impurities as other electrons, distributed with repulsive correlations, as opposed to uncorrelated lattice and surface irregularities and atomic impurities.

2 Methods

To describe a single electron moving in two dimensions in a magnetic field, we use the Hamiltonian

\[ H = \frac{1}{2} [\sigma \cdot (p - eA)]^2, \tag{1} \]

in units where \( \hbar = c = m = 1 \) and with \( \sigma_i \) the Pauli spin matrices. This Hamiltonian includes the interaction of the electron’s spin with the magnetic field, and implies that the lowest Landau level states are spin polarized, with energy \( E = 0 \). We choose the cylindrical gauge and consider the magnetic field \( B \) to be strong enough that we can consider only the lowest Landau level and neglect mixing. In keeping with the cylindrically symmetric gauge, we choose our basis states to be concentric orthonormal eigenfunctions of the form

\[ \phi_m(z) = \frac{1}{\sqrt{\pi m!}} z^m e^{-z^2/2} \tag{2} \]

where \( z = x + iy = re^{i\theta} \) has been expressed in units of the magnetic length

\[ \ell_B = \sqrt{2/eB}. \tag{3} \]

The magnetic length is both the length scale of the root-mean-square radius of the \( m = 0 \) basis state and the radius of the area through which one quantum of magnetic flux, \( \Phi_o = 2\pi/e \), passes. The total number of flux quanta passing through the system is then given by the squared radius of the system. That radius we take as the root-mean-square radius of the largest basis state. We construct our potential out of sums of impurities where each impurity is a pair of Gaussians,

\[ V_n(z) = V_0 \left[ \frac{1}{\alpha^2} e^{-|z-z_n|^2/\alpha^2} - \frac{1}{\alpha^2 \beta^2} e^{-|z-z_n|^2/\alpha^2 \beta^2} \right]. \tag{4} \]

The use of gaussians or other short-range potentials has been common in numerical FQHE studies, e.g., [4]. The paired Gaussians give a system with
zero average electrical potential and therefore a total energy $E = 0$, and approximate a repulsive charge screened by a compensating cloud. For moderate impurity densities, we set $\beta^2 = 2$ so that the area computed from the root-mean-square radius of the negative shell would be twice that of the positive core, and then choose $\alpha$ so that plots of the density of states, participation ratio, and root-mean-square radius as a function of energy are as symmetric with respect to energy as possible for a density of one impurity for every two flux quanta. The widths of the Gaussians then scale inversely with the square root of the impurity density, as we expect increased impurity density to increase localization of the individual impurities to minimize their overlap and hence their interaction energy. The potentials are constructed so that their space integrals are unchanged by this scaling. The impurities are randomly distributed, or randomly distributed with a minimum separation between adjacent impurity centers (hard-shell constraint), or placed on a hexagonal lattice. The centers of the impurities are required to lie within the disk, but their shape is unaffected by the disk boundary. The impurity density $\nu$ is the number of impurities per flux quantum.

### 3 Results

We consider energy spectra and compactness, $C$, of states. The latter is defined by

$$C = \sqrt{\frac{\cal P}{R_{\text{rms}}}}. \tag{5}$$

The compactness measure uses the participation ratio, which measures the area of a state, [19, 20, 21],

$$P_\alpha = \left( \frac{\int d^2 r |\Psi_\alpha|^2}{\int d^2 r |\Psi_\alpha|^4} \right)^2 = \left( \frac{\int d^2 r |\Psi_\alpha|^4}{\int d^2 r |\Psi_\alpha|^4} \right)^{-1}. \tag{6}$$

and the root-mean-square radius, which measures the width of the state,

$$R_{\text{rms}}^{(\alpha)} = \sqrt{\int d^2 r r^2 |\Psi_\alpha|^2}. \tag{7}$$

Compactness measures the degree to which a state resembles a disk or a ring and has an intrinsic scale of $\sqrt{2\pi}$. For a uniform disk and densities corresponding to the $m = 0$ and $m = 1$ basis states, the ratio has exactly
Figure 1: Summed energy histograms for ten different random distributions of impurities for the case of parameters set to $\alpha^2 = \frac{0.35}{\nu}$, $\beta^2 = 2$, minimum distance between centers of impurities is $0.70 \frac{\text{SystemSize}}{\sqrt{\# \text{Impurities}}}$, using matrix elements up to 201 places from the diagonal. The filling factor, $\nu$, is indicated at the top of each subfigure. Shading groups states into bands containing $\nu$ of the states counting from the highest energy.

Examining the energy spectra for the different kinds of distributions at impurity densities between 7 and 500 impurities per 1000 magnetic flux quanta, we find that for double Gaussian impurities randomly placed with a hard-shell constraint, the energy spectrum consists of a fraction $1 - n\nu$ of the states in the lowest energy band, and $n$ higher energy bands each containing a fraction $\nu$ of the states. This occurs only for hard-shell repulsion with packing fractions approaching hexagonal close packing. Neither a random distribution nor a precise hexagonal close packing produces this type of energy spectrum. We also find that compactness as a function of energy for the lowest energy band is qualitatively similar to that for the single band that exists for one impurity per flux quantum. Both of these observations suggest that the lowest band is a filled Landau level in a reduced magnetic
Figure 2: Combined distributions of compactness as a function of energy for ten different random distributions of impurities for the case of parameters set to $\alpha^2 = 0.35$, $\beta^2 = 2$, minimum distance between centers of impurities is $0.70 \frac{\text{SystemSize}}{\sqrt{\#\text{Impurities}}}$, using matrix elements up to 201 places from the diagonal and cross-terms in the participation function no more than 100 apart. The filling factor, $\nu$ is indicated at the top of each subfigure.
field. According to [8], when the filling factor is given by

$$\nu = \frac{p}{2pq + 1},$$

(8)

then the effective magnetic field is given by

$$B^* = B(1 - 2q\nu).$$

For $\nu = \frac{1}{3}$ and $\nu = \frac{1}{5}$, we see our effective field shifted from the applied field by exactly half the amount found in the composite fermion picture. At impurity densities falling between the discrete electron filling factors permitted in the composite fermion picture, there is a smooth transition in the numbers of upper bands split off from the lowest band; only a limited number of bands will split off. In the vicinity of $\nu = \frac{1}{9}$, the upper bands gradually lose their definition and are reabsorbed into the lowest energy band: At $\nu = \frac{1}{8}$ we have $n = 4$, consistent with the value of $q$ in CF theory for $\nu = \frac{1}{9}$, while for $\nu = \frac{1}{10}$ all the bands have merged.

The appearance and the disappearance of banding with decreasing impurity density implies the presence of an effective magnetic field which eventually dissolves at still lower impurity densities. Comparison with single Gaussians and unconstrained random and crystalline distributions shows that the separation induced by the combination of the double Gaussian and the hard-shell constraint is critical to the phenomena described above. This tells us that the energy banding is linked to a moderate amount of order in the system - neither too much nor too little should be present.

4 Conclusions

We find that in the case of hard-shell repulsion the number and character of the single-electron states in the lowest band correspond to the lowest Landau level in an effective magnetic field. Interpreting the impurities as the other electrons of a fractional quantum Hall system, the effective field is shifted down from the applied field by one half the amount expected from CF theory. Seeing the effect only for hard-shell repulsion makes sense, because this case best represents the known behavior of electrons in the FQHE. The fact that the bands are diffuse, rather than sharp as one would expect in a complete theory, is inevitable for fixed impurities.
Our strongest constraint is the restriction to just one dynamical electron, clearly losing much of the electron-electron dynamics crucial for the FQHE. This means that we give up any representation of Fermi-Dirac statistics, surely crucial to a complete theory. Partially relaxing the one-particle constraint, by considering simultaneously motion of each of a pair of electrons (in the spirit of Cooper’s consideration \[22\] of an electron pair in the background of an electron Fermi sea as a way of understanding superconductivity), might perhaps give the full reduction of the magnetic field seen in CF theory, with wave-function antisymmetrization selecting odd-denominator rather than even-denominator filling factors. In any case, completing the project of deducing FQHE still would require treating all the electrons as dynamical, with the guiding-center coordinates as the adiabatic variables.

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