Iteratively reweighted adaptive lasso for conditional heteroscedastic time series with applications to AR-ARCH type processes

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Abstract
Due to the increasing impact of big data, shrinkage algorithms are of great importance in almost every area of statistics, as well as in time series analysis. In current literature the focus is on lasso type algorithms for autoregressive time series models with homoscedastic residuals. In this paper we present an iteratively reweighted adaptive lasso algorithm for the estimation of time series models under conditional heteroscedasticity in a high-dimensional setting. We analyse the asymptotic behaviour of the resulting estimator that turns out to be significantly better than its homoscedastic counterpart. Moreover, we discuss a special case of this algorithm that is suitable to estimate multivariate AR-ARCH type models in a very fast fashion. Several model extensions like periodic AR-ARCH or ARMA-GARCH are discussed. Finally, we show different simulation results and an application to electricity price and load data.

Keywords. High-dimensional time series; Lasso; Conditional Heteroscedasticity; AR-ARCH;

1 Introduction
High-dimensional shrinkage and parameter selection techniques are of increasing importance in statistics in the past years. In many statistical areas Lasso (least absolute shrinkage and selection operator) estimation methods introduced by Tibshirani (1996) are very popular. In time series analysis the influency of lasso type estimators is growing, especially as the asymptotic properties of stationary time series are usually very similar for stationary time series to the standard regression case, see e.g. Wang et al. (2007b), Nardi and Rinaldo (2011) and Yoon et al. (2013). The shrinkage property of the lasso make it attractive for subset selection in autoregressive models. In big data settings it provides efficient estimation technique, see Hsu et al. (2008), Ren and Zhang (2010), and Ren et al. (2013) for more details.

Unfortunately, almost the entire literature about $\ell_1$-penalised least square estimation, like the lasso, deals with homoscedastic models. The case of heteroscedasticity and conditional heteroscedasticity is rarely covered so far. Recently, Medeiros and Mendes (2012) show that the adaptive lasso estimator is consistent and asymptotic normal under very weak assumptions. They proof that the consistency and the asymptotic normality hold if the residuals follow a weak white noise process. This includes the case of conditional heteroscedastic ARCH and GARCH-type residuals. Nevertheless, their classical lasso approach do not make use of the structure of the conditional heteroscedasticity within the residuals. Without going into details, it is clear that the estimators might be improved if the structure of the conditional heteroscedasticity in the data is used. Further, Yoon et al. (2013) analysed the lasso estimator in an autoregressive regression model. Additionally, they formulate the lasso problem in an time series setting with ARCH errors. However, they do not come up with a solution to the estimation problem and left this for future research.

Recently, Wagener and Dette (2012) and Wagener and Dette (2013) analysed the properties of weighted lasso-type estimators in a classical heteroscedastic regression setting. They show that their estimators are consistent and asymptotically normal. In addition, they perform significantly better than applying their homoscedastic counterpart. Their results conditioned on the covariates can be used the construct a reweighted estimator that also works in time series settings.

In this paper we derive an iteratively reweighted adaptive lasso algorithm that tackles the mentioned problems. The algorithm is based on the results of Wagener and Dette (2013), as their results are suitable to generalise for models with conditional heteroscedasticity. We adduce sign consistency and asymptotic normality for the proposed estimator. Here, we consider a general high-dimensional setting where the
underlying process might have an infinite amount of parameters. Note that all the time series results hold in a classical regression as well.

In the second section we state the general problem. In the third one we motivate and provide the estimation algorithm. Subsequently we discuss its asymptotics in the next one. In Section 5 we consider an application to multivariate AR-ARCH type processes, including several extensions such as periodic AR-ARCH, AR-ARCH with structural breaks, threshold AR-ARCH and ARMA-GARCH models. The sixth section shows simulation results that underline results given above. It gives evidence that an incorporation of the heteroscedasticity in a high-dimensional setting is more important than in low dimensional problems. Finally, we employ the proposed algorithm in an application using electricity market data. We apply a two-dimensional AR-ARCH type model to the German/Austrian day-ahead electricity spot price of the European Power Exchange (EPEX), such as the electricity load for Germany. We end with some conclusions and final remarks.

2 The considered time series model

The model that we consider is basically similar to the one used by Yoon et al. (2013) or Medeiros and Mendes (2012). Let \((Y_t)_{t \in \mathbb{Z}}\) be the considered causal univariate time series. We assume that it follows the linear equation

\[ Y_t = X_{\infty,t}^0 + \varepsilon_t, \tag{1} \]

where \(X_{\infty,t} = (X_{1,t}, X_{2,t}, \ldots)\) is a possibly infinite vector of covariates of weakly stationary processes \((X_{i,t})_{t \in \mathbb{Z}}, (\varepsilon_t)_{t \in \mathbb{Z}}\) is an error process, and the parameter vector is \(\beta_\infty^0 = (\beta_1^0, \beta_2^0, \ldots)'\) with \(\sum_{i=1}^{\infty} |\beta_i^0| < \infty\). The covariates can also contain lagged versions of \(Y_t\), which allows flexible modelling of autoregressive processes.

A simple example of a process that helps for understanding this paper is an invertible seasonal MA(1) process. In particular, the AR(\(\infty\)) representation of a seasonal MA(1) with seasonality 2 is useful. It is given by \(Y_t = \varepsilon_t - \theta \varepsilon_{t-2} = \theta Y_{t-2} + \varepsilon_t\), where we have \(X_{\infty,t} = (Y_{t-2}, Y_{t-1}, \ldots)\) with \(\beta_\infty^0 = (0, \theta, 0, \theta, 0, \ldots)'\). The error process \((\varepsilon_t)_{t \in \mathbb{Z}}\) is assumed to follow a zero mean process, that \(\varepsilon_t\) is uncorrelated with the covariates \(X_{\infty,t}\). Hence we require \(\mathbb{E}(\varepsilon_t) = 0\) and \(\text{Cov}(\varepsilon_t, X_{i,t}) = 0\) for all \(i \in \mathbb{N}\). Moreover, we assume that \(\varepsilon_t\) is a weak white noise process, such that

\[ \varepsilon_t = \sigma_t Z_t \quad \text{where} \quad \sigma_t = g(\alpha^0_{\infty}; \infty, t)\] and \((Z_t)_{t \in \mathbb{Z}}\) is i.i.d. with \(\mathbb{E}(Z_t) = 0\) and \(\text{Var}(Z_t) = 1\). \tag{2}

Here, \(g\) is a positive function, \(L_{\infty,t}^0 = (L_{1,t}, L_{2,t}, \ldots)\) is a possibly infinite vector of covariates of weakly stationary processes \((L_{i,t})_{t \in \mathbb{Z}}, (\alpha^0_{\infty}) = (\alpha_0^0, \alpha_1^0, \ldots)'\) a parameter vector. Similarly to the covariates \(X_{\infty,t}^0\) in (1), \(L_{\infty,t}^0\) can also include lags of \(\sigma_t\) or \(\varepsilon_t\). This allows for a huge class of popular conditional variance models, like ARCH or GARCH type models. Choosing

\[ g(\alpha^0_{\infty}; \infty, t) = \sqrt{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-1}^2} \]

leads to the very popular GARCH(1,1) process. Note that the introduced setting is more general than the conditional heteroscedastic problem stated by Yoon et al. (2013) who mentioned only ARCH errors.

Below we assume that the time points 1 to \(n\) are observable for \(Y_t\). Thus, we denote by

\[
Y_n = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \quad X_n = \begin{pmatrix} X_{1,1} & \cdots & X_{1,p_n} \\ \vdots & \ddots & \vdots \\ X_{n,1} & \cdots & X_{n,p_n} \end{pmatrix}, \quad \beta_n = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{p_n} \end{pmatrix}, \quad \varepsilon_n = Y_n - X_n \beta_n \tag{4}
\]

the response vector, the \(n \times p_n\) matrix of the covariates, the parameter vector and the corresponding errors. Furthermore let \(X_{1,n}^0, \ldots, X_{n,n}^0\) be the rows of \(X_n^0\).

Since we deal with a high-dimensional setting we are interested in situations where the number of possible parameters \(p_n\) increases with sample size \(n\). Denote \(\beta_n^0 = (\beta_1^0, \ldots, \beta_{p_n}^0)'\) the restriction of \(\beta_\infty^0\) to its first \(p_n\) coordinates. Due to \(\sum_{i=1}^{\infty} |\beta_i^0| < \infty\) it follows for \(\varepsilon_n^0 = (\varepsilon_{n,1}, \ldots, \varepsilon_{n,n})' = Y_n - X_n \beta_n^0\) that there is a positive decreasing sequence \((\zeta_n)\) with \(\zeta_n \to 0\) such that \(\lim_{n \to \infty} P(\max_{1 \leq t \leq n} |\varepsilon_{n,t}^0 - \varepsilon_t| < \zeta_n) \to 1\) holds. Thus, for a sufficiently large \(n\) we can approximate \(Y_t\) by \(X_{n,t} \beta_n^0\) arbitrarily well.

However, in a lasso-type framework we assume that out of these \(p_n\) parameters only \(q_n\) with \(0 \leq q_n \leq p_n\) are non-zero. Hence, there are \(p_n - q_n\) parameters that are exactly zero. Without loss of generality we assume that \(X_n^0\) and \(\beta_n^0\) are arranged so that the first \(q_n\) components of \(\beta_n^0\) are non-zero, whereas the followings are zero. Obviously we have \(\beta_n^0 = (\beta_1^0, \ldots, \beta_{q_n}^0, 0, \ldots, 0)' = (\beta_n^0(1)', 0')'.\) Additionally
we introduce the naive partitioning of $X_n$ and $\beta_n$, in such a manner that $\beta_n = (\beta_n(1), \beta_n(2))'$ and $X_n = (X_n(1), X_n(2))$ holds.

Subsequently, we concentrate on the estimation of $\beta_n^0$ by some $\beta_n$ based on a lasso type approach. Henceforth, we achieve never directly an estimate for $\beta_n^0$, but we can approximate it by $(\beta_n', \theta)'$.

3 Estimation algorithm

The proposed algorithm is based on the classical iteratively reweighted least squares procedure. It is used in the literature in a time series setting as e.g. by Mak et al. (1997). However, similar approaches are not popular in time series modelling, as there are usually better alternatives if the amount of parameters is small. In this case we can simply perform an estimation of the joint likelihood function of (1), see e.g. Bardet et al. (2009). In higher dimensions it is almost impossible to maximise the non-linear loss function with many parameters. In contrast, our algorithm can be based on the coordinate descent lasso estimation technique as suggested by Friedman et al. (2007) which provides a feasible and fast estimation technique.

For motivating the proposed algorithm we divide equation (1) by its volatility, resp. conditional standard deviation $\sigma_t$. Thus we receive

$$\tilde{Y}_t = \frac{1}{\tilde{\sigma}} Y_t$$

$$\tilde{X}_{\infty,t}^i = \frac{1}{\tilde{\sigma}} X_{\infty,t}^i.$$  (5)

where $\tilde{Y}_t = \frac{1}{\tilde{\sigma}} Y_t$ and $\tilde{X}_{\infty,t}^i = \frac{1}{\tilde{\sigma}} X_{\infty,t}^i$. Here, the noise $Z_t$ is homoscedastic with variance 1. So if the volatility $\sigma_t$ of the process $Y_t$ is known we can simply apply common lasso time series techniques under homoscedasticity. Unfortunately, this is never the case in practice. Now, the basic idea is to replace $\sigma_t$ by a suitable estimator $\tilde{\sigma}_t$, which allows us to perform a lasso estimate on a homoscedastic as in equation (5).

For estimating ARMA-GARCH processes, practitioners use sometimes a multiple step estimator. First the ARMA parameters are computed in an homoscedastic setting and then use the resulting estimated residuals to estimate the GARCH part in a second step, see e.g. Mak et al. (1997) or Ling (2007). We apply a similar step-wise estimation technique here.

In general we have no a priori information about $\sigma_t$, hence we should assume homoscedasticity in a first estimation step. We start with the estimation of the regression parameters $\beta_n^0$ resp. $\beta_n^0$ and receive the residuals $\tilde{\varepsilon}_{n,1}, \ldots, \tilde{\varepsilon}_{n,n}$. We use the residuals to estimate the conditional variance parameters $\alpha_n^\infty$ and thus $\sigma_1, \ldots, \sigma_n$ by $(\tilde{\sigma}_{n,1}, \ldots, \tilde{\sigma}_{n,n})$ afterwards. Next we reweight model (1) by $\tilde{\sigma}_t^{-1}$ to get a homoscedastic model version which we consider reestimate $\beta_n^0$ again. We can use this new estimate of $\beta_n^0$ to repeat this procedure. Thus we will end up in an iterative algorithm that hopefully converges in some sense to $\beta_n^0$ resp. $\beta_n^0$ with increasing sample size $n$.

We use an adaptive weighted lasso estimator to estimate $\beta_n^0$ within each iteration step. It is given by

$$\beta_{n,\text{lasso}}(\lambda_n, v_n, w_n) = \arg \min_{\beta} \sum_{t=1}^n w_{n,t}^2 \left( Y_t - \sum_{i=1}^p X_{t,i} \beta_i \right)^2 + \lambda_n \sum_{j=1}^m v_{n,j} |\beta_j|$$  (6)

or in vector notation

$$\beta_{n,\text{lasso}}(\lambda_n, v_n, w_n) = \arg \min_{\beta} (Y_n - X_n \beta)' W_n^2 (Y_n - X_n \beta) + \lambda_n v_n' |\beta|$$  (7)

where $W_n = \text{diag}(w_n)$, $w_n = (w_{n,1}, \ldots, w_{n,n})$ are the heteroscedasticity weights, $v_n = (v_{n,1}, \ldots, v_{n,m})$ are the penalty weights and $\lambda_n$ is a penalty tuning parameter. As described above, in the iteratively reweighted adaptive lasso algorithm we have the special choice $w_n = (w_{n,1}, \ldots, w_{n,n}) = (\tilde{\sigma}_{n,1}^{-1}, \ldots, \tilde{\sigma}_{n,n}^{-1})$ for the heteroscedasticity weights within each iteration step. We require $w_n = 1$ for the homoscedastic initial step.

Like Zou (2006) we consider for the tuning parameter $v_n$ the choice $v_n = \beta_n^{\tau\text{init}}$ for some $\tau > 0$ and some initial parameter estimate $\beta_{n,\text{init}}$. With $\tau = 0$ we obtain $v_n = 1$ which is the usual lasso estimator. Obviously, there is no initial estimator required in this case. However, it is worth considering the adaptive lasso as well as it showed a different performance in application.

The selection of the tuning parameters $\lambda_n$ and $\tau$ is crucial for the application and might demand some computational cost. For finding the optimal tuning parameters we suggest to use common time series methods that are based on information criteria. Zou et al. (2007), Wang et al. (2007b), Zhang et al. (2010) and Nardi and Rinaldo (2011) analyse information criteria in the lasso resp. adaptive lasso time series framework. Subsequently, let $\text{IC}(\beta_n)$ be an information criterion that corresponds to a model
fitted by $\beta_0$ and takes the minimal value in the optimum. We will assume that this information criterion is almost surely unique in its minimum. Possible options for this information criteria, are the Akaike information criterion (AIC), Bayes information criterion (BIC) or a cross-validation criterion. Here, it is worth mentioning that Kim et al. (2012) discuss the generalised information criterion (GIC) in a classical homoscedastic lasso framework where the amount of parameters $p_n$ depends on $n$. They establish that under some regularity conditions the GIC can be chosen so that a consistent model selection is possible.

For the initial estimate $\beta^{n,\text{init}}$ that is required for the penalty weights there are different options available. The simplest is the OLS estimator, which is available if $p_n < n$. Another alternatives are the elastic net or ridge regression estimator, see e.g. Zou and Hastie (2005).

Subsequently, we denote $\tilde{\alpha}_n = \tilde{\alpha}_n(\beta_0; X_n, Y_n)$ as a known plug-in estimator for $\alpha^0_n$, which is the projection of $\alpha^0_n$ to its first $l_n$ coordinates. We denote $g_n$ as restriction of $g$ that corresponds to $\alpha^0_n$. Thus $g_n$ is defined so that $\alpha^0_n$ is restricted to $\alpha^0_n$, but also so that $L^0_n$ is a restriction of $L^\infty_n = (L^\infty_{n,t})_{t \in \mathbb{Z}}$ to its first $m_n(l_n)$ coordinates. Similarly, let $\tilde{L}_n = \tilde{L}_n(\beta_0; X_n, Y_n)$ be an estimator for $(L^0_{n,1}, \ldots, L^0_{n,n})^T$.

For example, if $\varepsilon_t$ follows a GARCH(1,1) process we receive $\sigma_t = g_n(\alpha^0_n, L^0_{n,t})$ for all $n \in \mathbb{N}$, where $\alpha^0_n = (\alpha_0, \alpha_1, \alpha_2)$ with $l_n = 3$ and $L^0_{n,t} = (\varepsilon_{t-1}, \sigma_{t-1})$ with $m_n(l_n) = 2$ for all $n \in \mathbb{N}$. This is similarly feasible for every variance model with a finite amount of parameters. However, if $\sigma_t$ follows an infinite parameterised process e.g. through an ARCH($\infty$) process, $l_n$ and $m_n(l_n)$ should tend to infinity as $n \to \infty$.

The estimation scheme of the described iteratively reweighted adaptive lasso algorithm is given by:

1. initialise $v_n(\tau) = (v_{n,1}(\tau), \ldots, v_{n,m}(\tau)) = \beta^{n,\tau}_{n,\text{init}}$ with $\tau \geq 0$ and $w_n^{[0]} = 1$, $k = 1$
2. estimate by weighted lasso:

$$
\beta_n^{[k]} = \beta_n^{[k]}(w_n^{[k-1]}) = \text{IC}^{-1}\left(\min_{\lambda, \tau} \text{IC}\left(\beta_n, l_{\text{lasso}}(\lambda, v_n(\tau), w_n^{[k-1]})\right)\right)
$$

(8)

3. estimate the conditional variance model:

$$
\alpha_n^{[k]} = \tilde{\alpha}_n(\beta_n^{[k]}; X_n, Y_n) \text{ and } L_n^{[k]} = \tilde{L}_n(\beta_n^{[k]}; X_n, Y_n)
$$

(9)

4. compute new weights $w_n^{[k]} = (w_n^{[k,1]}, \ldots, w_n^{[k,n]})$ with $w_n^{[k]} = g_n(\alpha_n^{[k]}, L_n^{[k]})^{-1}$
5. if the stopping criterion is not met, $k = k + 1$ and back to 2. otherwise, return estimate $\beta_n^{[k]}$ and volatilities $\sigma_n^{[k]} = g_n(\alpha_n^{[k]}, L_n^{[k]})$

We can summarise that we have to specify the initial estimator $\beta^{n,\text{init}}$ with an inital value of $\tau$, the initial heteroscedasticity weights $w_n$ and the information criteria selection method represented by IC. To reduce the computation time it can be convenient in practice to choose $\tau = 0$ (lasso) or $\tau = 1$ (almost non-negative garotte) without searching for the optimal $\tau$ value within the adaptive lasso optimisation.

The stopping criterion in step 5 has to be chosen as well such that the algorithm eventually stops. A plausible stopping criterion should measures the convergence of $w_n^{[k]}$ resp. $\sigma_n^{[k]}$. We suggest to stop the algorithm if $\|\sigma_n^{[k]} - \sigma_n^{[k-1]}\| < \epsilon$ for a selected vector norm $\|\cdot\|$ and some small $\epsilon > 0$. Nevertheless, in our simulation study we realised that the difference in the later steps are marginal, so that stopping at $k = 2$ or $k = 3$ seems to be reasonable for practice. This will be underlined by asymptotics of the algorithm analysed below, it can be shown that under certain conditions $k = 2$ is sufficient to get an optimal estimator if $n$ is large.

4 Asymptotics of the algorithm

For the general convergence analysis it is clear, that the asymptotic of the estimator $\beta_n^{[k]}$ will strongly depend on the (cond.) heteroscedasticity models (2) (esp. the formula for $g$) such as on the linked estimators $\tilde{\alpha}_n$ and $\tilde{L}_n$. Though, we consider the sign consistency as introduced by Zhao and Yu (2006) and asymptotic normality of the non-vanishing components of $\beta_n^{[k]}$.

If we assume, that the number of parameters $p_n$ does not depend on the sample size $n$, then we could investigate the results from Wagener and Dette (2012) for getting asymptotic properties. They prove sign consistency and asymptotic normality for the weighted lasso and weighted adaptive lasso estimator under some conditions.
The case where the number of parameters $p_n$ increases with $n$ is similarly analysed in a regression framework by Wagener and Dette (2013), but only for the adaptive lasso case with $\tau = 1$. They achieve basically the same asymptotic behaviour as for the fixed $p_n$ case, but it is clear that the conditions are more complicated than in the Wagener and Dette (2012) case.

The following assumptions will generalise the results of Wagener and Dette (2013). One crucial point is the assumption that the process $Y$ can be parameterised by infinitely many parameters. So error term $\varepsilon_n^0 = Y_n - X_n\beta_n^0$, based on the restriction $\beta_n^0$ of the true parameter vector $\beta_n^0$, is not identical to the true error restriction $\varepsilon_n^0$. In contrast to $\varepsilon_n^0$, the term is in general $\varepsilon_n^0$ correlated. This has to be taken into account for the proof concerning the asymptotic.

For the asymptotic properties we introduce a few more notations. Let $\tilde{X}^{[k]}_n = W_n^{[k]}X_n$ and $\tilde{Y}^{[k]}_n = W_n^{[k]}Y_n$ where $W_n^{[k]} = \text{diag}(w_n^{[k]})$. Let $\Sigma_n^0$ denote the true volatility matrix and $\Sigma_n^{[k]} = W_n^{[k]}$ its estimate in the $k$’s iteration. Additionally, we introduce $\tilde{\Gamma}_n^{[k]} = \frac{1}{n}(\tilde{X}^{[k]}_n)\tilde{X}^{[k]}_n$ as the scaled Gramian, where $\Gamma_n = \tilde{\Gamma}_n^{[1]} = \frac{1}{n}X'X_n$ is the unscaled Gramian. Furthermore, denote $W_n^0$ and $\tilde{\Gamma}_n^0$ the weight matrix and the Gramian that corresponds to the true matrix $\Sigma_n^0$. The submatrices to $\beta_n^0(1)$ are denoted by $\tilde{\Gamma}_n^{[k]}(1), \Gamma_n(1),$ and $\tilde{\Gamma}_n^0(1)$.

Similarly to Wagener and Dette (2013), we require the following additional assumptions:

(a) The process $(Y_t, Z_t, X_{1,t}, \ldots, X_{m,t}, L_{1,t}, \ldots, L_{m,t})_{t \in \mathbb{Z}}$ is weakly stationary with zero mean for all $m \in \mathbb{N}$.

(b) The covariates are standardized so that $\mathbb{E}(X_{i,t}^2) = 1$ for all $t \in \mathbb{Z}$ and $i \in \mathbb{N}$.

(c) For the minimum of the absolute non-zero parameters $b_n = \min\{|\beta_n^0(1)|\}$ and the initial estimator $\beta_n$ there exists a constant $b > 0$ so that

$$\lim_{n \to \infty} P\left(\min\{|\beta_n^0(1)|\} < b_n\right) = 0.$$ (10)

(d) There exists a sequence $(r_n)_{n \in \mathbb{N}}$ with $r_n \to \infty$ such that

$$\lim_{n \to \infty} P\left(\max\{|\beta_n^0(2)|\} < r_n^{-1}\right) = 0.$$ (11)

(e) There are positive constants $c_1, c_2$ and $d$ with $1 \leq d \leq 2$ such that

$$P(|\varepsilon| > x) \leq c_1 \exp(-c_2x^d).$$ (12)

(f) It holds that $n(p_n - q_n)(\lambda_n r_n)^{-2} \to 0$ as $n \to \infty$.

(g) There are constants $\lambda_{0,\min}, \lambda_{0,\max}$ and $\lambda_{1,\min}$ such that the eigenvalues satisfy

$$0 < \lambda_{0,\min} < \lambda_{0,\min}(\Gamma_n) < \lambda_{0,\max} < \infty,$$ (13)

$$0 < \lambda_{1,\min} < \lambda_{0,\min}(\tilde{\Gamma}_n^0) < \lambda_{0,\max}(\Gamma_n) < \infty.$$ (14)

(h) There are positive constants $\sigma_{\min}$ and $\sigma_{\max}$ such that

$$0 < \sigma_{\min} < q_n(\hat{\alpha}_n(\beta_n, X_n, Y_n), \hat{L}_{n,t}(\beta_n; X_n, Y_n)) < \sigma_{\max} < 0$$ (15)

for all $n \in \mathbb{N}$, $t \in \{1, \ldots, n\}$ and $\beta_n$ in an open neighbourhood of $\beta_n^0$.

(i) The functions $\beta_n \to q_n(\hat{\alpha}_n(\beta_n, X_n, Y_n), \hat{L}_{n,t}(\beta_n; X_n, Y_n))$ are twice differentiable in a neighbourhood of $\beta_n^0$ all $n \in \mathbb{N}$ and $t \in \{1, \ldots, n\}$. The corresponding partial derivatives are uniformly bounded on its first $q_n$ coordinates.

(j) For all $n \in \mathbb{N}$ the estimator $\hat{\alpha}_n$ and $\hat{L}_n$ are consistent for $\alpha_n^0$ and $L_n^0$, additionally $|g(\alpha_n^0, L_{\infty,t}) - g(\alpha_n(\beta_n^0; X_n, Y_n), \hat{L}_{n,t}(\beta_n^0; X_n, Y_n))| = O(\frac{\sqrt{\sigma_n^2}}{\sqrt{n}})$ as $n \to \infty$. 


As mentioned above, most of these assumptions are adjusted from the setting of Wagener and Dette (2013). Assumption (a) is standard in a time series setting. (b) is the scaling that is required in a lasso framework. (c) and (d) are usual assumptions in an adaptive lasso setting (see e.g. Zou (2006) or Huang et al. (2008)). (e) makes a statement about the tails of the errors. (f) states some convergence properties that make restrictions to the grow behaviour within the model, especially the number of parameters $p_n$ and the number of relevant parameters $q_n$. (g) gives bounds for the weighted and unweighted Gramian. (h), (i) and (j) state properties required for the heteroscedasticity in the model. The quite general formulation in (j) can be replaced by a more precise assumption when a variance model is specified.

Using the assumptions above we can state sign consistency and asymptotic normality.

**Theorem 1.** Under conditions (a) to (j), where either (e) or (e') holds, we it holds for all $k \geq 1$ that

$$\lim_{n \to \infty} P \left( \text{sign}(\beta_{n}^{|k|}) = \text{sign}(\beta_{0}) \right) = 1. \quad (16)$$

Moreover it holds for $\xi_n \in \mathbb{R}^{q_n}$ with $\|\xi_n\|_2 = 1$ that

$$\sqrt{n} s_n (k)^{-1} \xi_n^l \left( \beta_{n}^{|k|}(1) - \beta_{0}^{|k|}(1) \right) \to N(0, 1) \quad (17)$$

in distribution where $s_n^2 (1) = \xi_n^l (\Gamma_n (1))^{-1} \xi_n$ and $s_n^2 (k) = \xi_n^l (\Gamma_n (1))^{-1} \xi_n$ for $k \geq 2$.

The proof is given in the appendix. Note that the variance $s_n^2 (k)$ for $k \geq 2$ is substantially smaller than $s_n^2 (1)$. Hence the estimator $\beta_{n}^{|k|}$ has minimal asymptotic variance for all $k \geq 2$.

Furthermore, there is the option of (e) or (e') in the theorem. (e) restricts the residuals to have an exponential decay in the tail, like the normal or the Laplace distribution. However, this can be replaced by the stronger condition (e') in the theorem. In this situation, polynomially decaying tails in the residuals are possible, as long as they have finite variance. As discussed in Wagener and Dette (2013) this has an impact on the maximal possible growth of the amount of parameters $p_n$ in the estimation. There are situations where under assumption (e) $p_n$ can grow with every polynomial order, even slow exponential growth is possible. In contrast, given assumption (e') this is impossible. Here Wagener and Dette (2013) argued that for the sign consistency is possible for rates that increase slightly faster than linearly such as $p_n \sim n \log(n)$, but not for polynomial rates like $p_n \sim n^{1+\delta}$ for some $\delta > 0$. Wagener and Dette (2013) do not discuss this case for the asymptotic normality. In this situation, we can get an optimal rate of $n^{1/3-\delta}$ for the number of relevant parameters $q_n$ (having $b_n$ to be constant and $r_n \sim n^{1/2}$), when we have a growth of $n^{2/3-\delta}$ for $p_n$. The maximal possible growth in $p_n$ is of order $n^{1-\delta}$. However, this comes at the cost that $q_n = \mathcal{O}(n^{2\delta})$, so the amount of relevant parameters in the model is almost fixed for small $\delta$.

In assumption (f), the last convergence $\tilde{\beta}_{n} \to 0$ is relaxed in comparison to the analogous part in Wagener and Dette (2013). They required $\tilde{s}_{n}^2 \to 0$ to achieve asymptotic normality, in their settings the maximal rate for the number of relevant parameters $q_n$ in the model is consequently only $n^{1/5-\delta}$.

In empirical applications, practitioners often just want to apply a lasso type algorithm without caring much about the chosen size of $n$ and $p_n$. They tend to stick all available $n$ and $p_n$ into their model as long as it is computational feasible. In the empirical studies it helps to estimate the model for several sample sizes $n$ and specified growth rate for a selected number of possible covariates $p_n$. As we can observe the estimated values for $q_n$ and $\lambda_{n}$ of the model we can get clear indications for the asymptotic convergence properties. This helps to find the optimal tuning parameter $\lambda_n$, as well as the best information criterion.

5 Application to AR-ARCH type models

In the introduction we mention that one of the largest field of application might be high-dimensional AR-ARCH type processes. We discuss a standard multivariate AR-ARCH model in detail. Afterwards, we briefly deal with several extensions, the periodic AR-ARCH model, change-point AR-ARCH models, threshold AR-ARCH models, interaction models and ARMA-GARCH models.

Let $Y_t = (Y_{1,t}, \ldots, Y_{d,t})'$ be a $d$-dimensional multivariate process and denote $\mathcal{D} = \{1, \ldots, d\}$.
5.1 AR-ARCH model

The multivariate AR model is given by

\[ Y_{i,t} = φ_{i,0} + \sum_{j \in D} \sum_{k \in I_{i,j}} φ_{i,j,k} Y_{j,t-k} + ε_{i,t} \]  \hspace{1cm} (18)

for \( i \in D \), where \( φ_{i,j,k} \) are non-zero autoregressive coefficients, \( I_{i,j} \) are the index sets of the corresponding relevant lags and \( ε_{i,t} \) is the error term. The error processes \( (ε_{i,t})_{t \in Z} \) follow the same conditional variance structure as in (2), so

\[ ε_{i,t} = σ_{i,t} Z_{i,t} \]

where \( σ_{i,t} = g_i(α_i; L_i) \) and \( (Z_{i,t})_{t \in Z} \) is i.i.d. \hspace{1cm} (19)

with \( E(Z_{i,t}) = 0 \) and \( Var(Z_{i,t}) = 1 \).

Now, we define the representation (18) that matches the general representation (1) by

\[ Y_{i,t} = X_{i,t} β_i + ε_{i,t} \]

for \( i \in D \) where the parameter vector \( β_i = (φ_{i,0}, (φ_{i,1,k})_{k \in I_{i,1}}, \ldots, (φ_{i,d,k})_{k \in I_{i,d}})^T \) and corresponding regressor matrix \( X_{i,t} = (1, (X_{i,1,t-k})_{k \in I_{i,1}}, \ldots, (X_{i,d,t-k})_{k \in I_{i,d}})^T \).

Furthermore, we assume that \( ε_{i} = (ε_{1,t}, \ldots, ε_{d,t})^T \) follows an ARCH type model. In detail we consider multivariate power-ARCH process which generalises the common multivariate ARCH process slightly. Recently, Francq and Zakoïan (2013) discussed the estimation of such power-ARCH(∞) processes and showed applications to finance. It is given by

\[ σ_{i,t}^δ = α_{i,0} + \sum_{j \in D} \sum_{k \in J_{i,j}} α_{i,j,k} |ε_{j,t-k}|^{δ_{i}} \]  \hspace{1cm} (21)

with \( J_{i,j} \) as index set and \( δ_{i} \) as power of the corresponding \( σ_{i} \). The parameters satisfy the positivity restrictions, so \( α_{0,0} > 0 \) and \( α_{i,j,k} \geq 0 \). Moreover we require that the \( δ_{i} \)’s absolute moment \( E|Z_{i}|^{δ_{i}} \) exists. Obviously, we have

\[ g_i(α_i, L_i) = \left( α_{i,0} + \sum_{j \in D} \sum_{k \in J_{i,j}} α_{i,j,k} |ε_{j,t-k}|^{δ_{i}} \right)^{1/δ_{i}} \]  \hspace{1cm} (22)

where \( α_i = (α_{i,0}, (α_{i,1,k})_{k \in I_{i,1}}, \ldots, (α_{i,d,k})_{k \in I_{i,d}}) \) \hspace{1cm} (21)

\[ L_i = ((ε_{1,t-k})_{k \in J_{i,1}}, \ldots, (ε_{d,t-k})_{k \in J_{i,d}}) \]. The case \( δ_{1} = 2 \) leads to the well known ARCH process which turns into a multivariate ARCH(p) if \( J_{i,j} = \{1, \ldots, p\} \).

For estimating the ARCH part parameters we will make use of a recursion that holds on the residuals. This is given by

\[ |ε_{i,t}|^{δ_{i}} = \tilde{α}_{i,0} + \sum_{j \in D} \sum_{k \in J_{i,j}} \tilde{α}_{i,j,k} |ε_{j,t-k}|^{δ_{i}} + u_{i,t} \]  \hspace{1cm} (23)

where \( \tilde{α}_{i,0} = γ_i α_{i,0}, \tilde{α}_{i,j,k} = γ_i α_{i,j,k} \) and \( u_{i,t} = σ_{i,t}(|Z_{i,t}| - γ_i) \) with \( γ_i = γ_i(δ_{i}) = E|Z_{i}^{δ_{i}}| \). Here, \( u_{i,t} \) is a weak white noise process with \( E(u_{i,t}) = 0 \). The fitted values \( \tilde{δ}_{i} \) of equation (23) are proportional to the \( σ_{i}^{δ_{i}} \) up to the constant \( γ_i \). As \( γ_i \) is the \( δ_{i} \)’s absolute moment of \( Z_{i,t} \), it holds \( γ_i = 2 \) if \( δ_{i} = 2 \). If \( δ_{i} = 1 \) and \( ε_{i,t} \) follows a normal distribution \( γ_i \) it is exactly \( \sqrt{2π^{-1}} \approx 0.798 \), whereas e.g. the standardised t-distributions have a larger first absolute moment.

Clearly, the true index sets \( I_{i,j} \) and \( J_{i,j} \) are unknown in practice. Thus we fix some index sets \( I_{i,j}(n) \) and \( J_{i,j}(n) \) for the estimation that can depend on the underlying sample size \( n \). If the true index sets \( I_{i,j} \) and \( J_{i,j} \) are finite, then the choices \( I_{i,j}(n) = \{1, \ldots, \max(I_{i,j})\} \) and \( J_{i,j}(n) = \{1, \ldots, \max(J_{i,j})\} \) are obvious selections. If \( I_{i,j} \) and \( J_{i,j} \) are infinite, \( I_{i,j}(n) \) and \( J_{i,j}(n) \) should be chosen so that they are monotonic increasing in the sense that \( I_{i,j}(n-1) \subseteq I_{i,j}(n) \) and \( J_{i,j}(n-1) \subseteq J_{i,j}(n) \) with \( \bigcup_{n \in N} I_{i,j}(n) = N \) and \( \bigcup_{n \in N} J_{i,j}(n) = N \). The size of \( I_{i,j}(n) \) and \( J_{i,j}(n) \) is directly related to the size of the estimated parameters \( p_{i,n} \) for \( β_{i,n} \) and \( l_{i,n} \) for \( α_{i,n} \). It holds \( p_{i,n} = 1 + \sum_{j \in D} J_{i,j}(n) \) and \( l_{i,n} = 1 + \sum_{j \in D} I_{i,j}(n) \).

Here, \( β_{i,n} \) and \( α_{i,n} \) are the restrictions of \( β \) and \( α \) to their first \( p_{i,n} \) and \( l_{i,n} \) coordinates.

For the estimation of \( β \), resp. \( β_{i,n} \), we can apply the iteratively reweighted adaptive lasso algorithm as described in the previous section. However, we have to specify an estimation method for the variance

\[ \frac{1}{\sqrt{n}} \text{This definition of } β \text{ is only well defined if all } I_{i,j} \text{ for } j \in D \text{ are finite. If one is infinite we can consider another enumeration. Everything holds in the same way.} \]

\[ \frac{2}{\sqrt{n}} \text{This definition of } α \text{ is only well defined if all } J_{i,j} \text{ for } j \in D \text{ are finite. If one is infinite we can consider another enumeration. Everything holds in the same way.} \]
part. In particular we require the estimators $\hat{\alpha}_i$ and $\hat{L}_i$, more precisely their restrictions $\hat{\alpha}_{i,n}$ and $\hat{L}_{i,n}$ to its $l_{i,n}$ resp. $m_{i,n}(l_{i,n})$ coordinates. For $\hat{L}_{i,n}(\beta_{i,n}; X_{i,n}, Y_{i,n})$ we have the estimator

$$\hat{L}_{i,n,t} = \hat{L}_{i,n,t}(\beta_{i,n}; X_{i,n,t}, Y_{i,t}) = |Y_{i,t} - X_{i,n,t}\beta_{i,n}|^{\delta_i}$$

which provides an estimate for $|\varepsilon_{i,t}|^{\delta_i}$ resp. $|\varepsilon_{i,n,t}|^{\delta_i}$. For the estimation of $\hat{\alpha}_{i,n}$ we suggest to minimise the problem

$$\|\hat{L}_{i,n,t} - A_{i,t}\alpha_i\|_2,$$

where $A_{i,t} = (1, (\hat{L}_{1,n,t-k})_{k \in J_{i,1}}, \ldots, (\hat{L}_{d,n,t-k})_{k \in J_{i,d}})$, which corresponds to the plug-in version of equation (23). For the estimation of (25), a common non-negative least squares (NNLS) estimation technique can be considered. If the variance is high-dimensional, approaches like the positive lasso are suitable as well. Hence high-dimensional lasso type algorithms with positivity constraint can be used to the parameters. But as the residuals in (23) follow only a weakly white noise process, for the asymptotic of this procedure slightly more advanced results are required. For the non-restricted adaptive lasso Medeiros and Mendes (2012) show sign consistency and asymptotic normality under certain conditions for such a situation with a weakly stationary error process.

However, the simple NNLS estimation procedure can act as a shrinkage procedure as well, as some parameters can be estimated to be 0. This well known sparsity effect of NNLS settings is recently analysed by Meinshausen et al. (2013) and Slawski et al. (2013). Slawski et al. (2013) gave evidence that the NNLS parameters can be estimated to be 0. This well known sparsity effect of NNLS settings is recently analysed by Lawson and Hanson (1995).

5.2 Periodic AR-ARCH model

Another class of models where we can apply the proposed estimation technique is the class of periodic AR-ARCH models. Here, we assume a model as the aforementioned model, but all parameters are allowed to vary periodically over time. This is very suitable for modelling seasonal effects in high-dimensional data.

Thus, the model for the conditional mean equation is given by

$$Y_{i,t} = \phi_{i,0}(t) + \sum_{j \in D} \sum_{k \in J_{i,j}} \phi_{i,j,k}(t)Y_{j,t-k} + \varepsilon_{i,t}$$

and for the conditional variance equation

$$\sigma_{i,t}^{\delta_i} = \alpha_{i,0}(t) + \sum_{j \in D} \sum_{k \in J_{i,j}} \alpha_{i,j,k}(t)|\varepsilon_{j,t-k}|^{\delta_i}.$$  

As mentioned, the time dependent parameters vary periodically over time. Assuming a periodicity of $S$ we have $\phi_{i,0}(t) = \sum_l B_{i,0,l}(t)\phi_{i,0,l}, \phi_{i,j,k}(t) = \sum_l B_{i,j,k,l}(t)\phi_{i,j,k,l}, \alpha_{i,0}(t) = \sum_l B_{i,0,l}(t)\alpha_{i,0,l}$, and $\alpha_{i,j,k}(t) = \sum_l B_{i,j,k,l}(t)\alpha_{i,j,k,l}$, where $B_{i,0,l}$ and $B_{i,j,k,l}$ are $S$-periodic basis functions.

Note that the processes is in general not weakly stationary anymore. However, they are periodically weakly stationary (also known as weakly cyclo-stationary). So if $S \in \mathbb{N}$ then the subsequences $(Y_{S(t+s)})_{t \in \mathbb{Z}}$ follow a weakly stationary process. For more details see e.g. Aknouche and Al-Eid (2012).

As choice for the periodic basis functions, periodic indicator functions are suitable if $S$ is small, as the parameter space will be blown up by a factor of $S$. If $S$ is large, a Fourier approximation, periodic B-splines or periodic wavelets, might be a good choice as basis to keep the parameter space reasonable.

As mentioned, the process $Y_t$ is not stationary in general, so the asymptotic theory given above can not be applied. Nevertheless, a similar theorem will hold likely for periodic stationary processes as well. In the proof we have to work one a level of the mentioned weakly stationary subsequences, similarly as in Ziel (2015). The estimation procedure can be performed as in the AR-ARCH model part.

5.3 AR-ARCH with structural breaks

Another field of possible applications is the one of change point models, i.e. models where we have at least one structural break. Here, the basic model is a time-varying AR-ARCH model as defined in equations (26) and (27) for the periodic AR-ARCH model. The basis functions are defined so that they can capture structural breaks instead of periodic effects. The resulting model is of the same structure as the change-point model used by Chan et al. (2013).
If we have a priori information about the change point we can take this into account. If we have no information, some clever segmentation of the time should be considered. One option is to allow a change in every parameter (especially $\phi_i,0$) and every time point. This can be handled by choosing $n$ basis functions for each parameter so that they build a triangular matrix. The resulting model is a special case of the so called fused lasso (see e.g. Tibshirani et al. (2005)) and suitable for change-point analysis. This particular mentioned approach of modelling change-points is analysed in Levy-leduc and Harchaoui (2008) and Harchaoui and Lévy-Leduc (2010). However, as this blows up the parameter space enormously. In every case we receive $p_n > n$.

A general problem of the change point model is that the theorem above cannot be applied due to the structural breaks. Even though the proposed algorithm might be a powerful tool to solve the problem, we have to use it carefully. Any inference after estimating the model, should be backed up by some Monte-Carlo studies.

5.4 Threshold AR-ARCH model

Threshold AR-ARCH models are popular when the mean or variance reversion properties change dependent on the past of the process. Threshold AR models are popular as they are simple but powerful examples for regime switching models. Threshold ARCH processes have many applications in finance, because they are suitable to capture the so called leverage effect.

The general model is given by

$$Y_{i,t} = \phi_{i,0} + \sum_{j \in D} \sum_{k \in I_{i,j}} \phi_{i,j,k,t} \mathbb{1}_{\{Y_{j, t-k} > a_{k,t}\}} Y_{j, t-k} + \varepsilon_{i,t} \tag{28}$$

with thresholds $a_{k,t}$ and

$$\sigma_{i,t}^\delta = \alpha_{i,0} + \sum_{j \in D} \sum_{k \in I_{i,j}} \alpha_{i,j,k,t} \mathbb{1}_{\{\varepsilon_{j, t-k} > b_{k,t}\}} |\varepsilon_{j, t-k}|^{\delta_i} + \varepsilon_{i,t} \tag{29}$$

with thresholds $b_{k,t}$. The option of one threshold at $b_{1,k} = 0$ in the conditional variance model is very popular. This leads to the well known TARCH model, introduced by Rabemananjara and Zakoian (1993). Ziel et al. (2015) applied the proposed algorithm to a similar multivariate AR-TARCH type model successfully to electricity market data.

Here, we can use the algorithm proposed above, as all covariate processes and $Y_t$ can be weakly stationary. The mentioned zero-threshold option is often suitable in practice as it only doubles the volatility parameter space.

5.5 AR-ARCH model with quadratic interactions

Interaction models are very popular in classical regression settings, especially in medicine. This type of model was e.g. analysed by Choi et al. (2010) or Bien et al. (2013), but not in a time series context. In general we can apply the theorem for these models as well, as the interactions are in general weakly stationary processes, if they have still a finite second moment. The full quadratic interaction model is given by

$$Y_{i,t} = \phi_{i,0} + \sum_{j \in D} \sum_{k \in I_{i,j}} \phi_{i,j,k,t} Y_{j, t-k} + \sum_{j \in D} \sum_{k \in I_{i,j}} \sum_{m \in I_{i,j}} \phi_{i,j,k,l,m} Y_{j, t-k} Y_{l, t-m} + \varepsilon_{i,t} \tag{30}$$

A problem that arises is the size of the parameter space which is $p_n(p_n + 1)/2$, where the standard AR-ARCH model has $p_n$ parameters.

5.6 ARMA-GARCH model

The last extension considers a very popular class of models. We know that every ARMA($p$, $q$) model can be rewritten as an AR($\infty$). Similarly a univariate GARCH($p$, $q$) can be expressed as an ARCH($\infty$). Hence, it is clear that every ARMA-GARCH model can be written as an AR($\infty$)-ARCH($\infty$). This AR($\infty$)-ARCH($\infty$) can be well approximated by an AR($\tilde{p}$)-ARCH($\tilde{q}$) for large $\tilde{p}$ and $\tilde{q}$. However, this gives an approximation and will likely include more parameters than the original ARMA-GARCH model.
Recently, Chen and Chan (2011) proposed a method of how to estimate ARMA processes in a lasso framework, using this kind of approximation. The idea is simple: Given the ARMA model
\[ Y_{i,t} = \phi_{i,0} + \sum_{j \in D} \sum_{k \in I} \phi_{i,j,k} Y_{j,t-k} + \sum_{k \in K} \theta_{i,j,k} \varepsilon_{j,t-k} + \varepsilon_{i,t} \] (31)
we consider first an AR(\(\hat{p}\))-model with large \(\hat{p}\) that can approximate the true ARMA model sufficiently well. The residuals of this fitted model are used for constructing the regressor matrix that contains the lagged autoregressive part and moving average part. We repeat the lasso estimation with this regressor matrix. So this procedure leads automatically to a two step approach. Clearly, we can iterate this more often to receive better stability, similarly to the algorithm we presented in this paper. Chen and Chan (2011) showed that under certain conditions this estimation principle based on the adaptive lasso can lead to consistent estimates.

The same principle can be applied to the GARCH model as well. So we first estimate a high dimensional ARCH model and take the estimated conditional variances for constructing the response matrix required for the GARCH model. This method opens a lot of possibilities for applications in financial frameworks. In multivariate settings, we have to specify a special GARCH model. In fact we can use every GARCH model that we can express in regression form, so even the BEKK-GARCH is possible.

6 Simulation study

In this section we perform Monte-Carlo simulations to learn about the finite sample properties of the model algorithm. We restrict ourselves to a univariate settings where \(p_n\) is fixed and does not depend on \(n\). To analyse the effect of the reweighted algorithm we select models that are close to mean and covariance stationarity.

For all simulation studies we consider a one-dimensional AR-ARCH-type process
\[ Y_t = \sum_{k \in I} \phi_k Y_{t-k} + \varepsilon_t. \] (32)

In all adaptive lasso estimation procedures we choose only the lasso itself, so \(\tau = 0\).

Subsequently, we want to focus on the impact of different information criteria and the sample size. The information criterions that we consider are the Akaike information criteria (AIC), the Hannan-Quinn criterion (HQC) and the Bayesian information criterion (BIC). These are all special cases of the generalised information criterion (see e.g. Kim et al. (2012)) that is given by \(\text{GIC}(\kappa_n) = \log(\hat{\sigma}_1^2) + \kappa_n K/n\), where \(K\) represents the number of parameters in the model. We get the AIC, HQC and BIC by choosing either \(\kappa_0 = 2\), or \(\kappa_n = 2 \log(\log(n))\) or \(\kappa_n = \log(n)\), respectively.

Furthermore we choose the following setting for this simulation study:

- \(\varepsilon_t = \sigma_t Z_t\) with \(\sigma_t = \alpha_0 + \sum_{i=1}^{10} \alpha_i |\varepsilon_i|\) with \(\alpha_0 = 0.01\), \(\alpha_i = 0.09\) for \(i \in \{1, \ldots, 10\}\) and
  (i) \(Z_t \sim N(0, 1)\) resp. (ii) \(Z_t \sim t_3(0, 1)\) which represents the standardised t-distribution with 3 degrees of freedom
- proposed superset \(I_{1,1} = \{1, \ldots, 100\}\)
- true subset \(I_{1,1}\) is generated by sampling \(M\) times uniformly without replacement from \(I_{1,1}\)
  1. \(M = 20\) with \(\phi_i = 0.049\) for \(i \in I_{1,1}\)
  2. \(M = 25\) with \(\phi_i = 0.039\) for \(i \in I_{1,1}\)
- simulated for \(n \in \{1000, 2000, 4000\}\) with a Monte Carlo sample size \(N = 200\)

To get more realistic results, we take models where the relevant lags in \(I_{1,1}\) are not fixed for all simulation, but uniformly sampled without replacement from \(\{1, \ldots, 100\}\). After simulating the process, we estimate the process ignoring the heteroscedasticity structure and use the proposed iteratively reweighted algorithm. The simulation results are visualised in Figure 1. There we see the proportion of the irrelevant included parameters (PIIP) and the proportion of the relevant included parameters (PRIP). Additionally the graphs show the corresponding 2-sigma ranges for the estimates, represented by box. The closer the PIIP to 0 and PRIP to 1 the better is the model. We observe that for all models the different information criteria show the expected behaviour. So the conservative BIC includes the least amount of parameters. Hence, the PIIP is the smallest for BIC in comparison to HQC and AIC, but the PRIP is smaller as well. The PRIP is always larger for the proposed algorithm that takes the conditional heteroscedasticity
into account as the homoscedastic analog. This seems to be independent of the sample size, the chosen information criteria and residual distribution. For the PIIP this situation in not that clear, but for larger \( n \) we have smaller PIIP for the proposed model which matches our expectations. Moreover, the results seem to be very robust with regards to the tails of the errors. So the results for the t-distribution with 3 degrees of freedom are still satisfying in comparison to their normally distributed analog.

Nevertheless, it is not clear how the behaviour is in an out-of-sample forecasting study. Therefore, we conduct another simulation study where we focus on the out-of-sample forecasting error. We select a model with \( M = 20, \tau = 0, \) BIC as information criterion, \( Z_t \sim N(0, 1) \) and \( n = 2000. \) We take the bootstrap sample size \( N = 10000 \) and compute the \( h \)-step ahead mean absolute forecast error (MAE\(_h\)) and the mean of \( j \)-step ahead mean absolute forecast error until step \( h \) (MMAE\(_h\)). They are defined by \( \text{MAE}_h = \frac{1}{N} \sum_{i=1}^{N} | \hat{Y}_{n+h} - Y_{n+h} | \) and \( \text{MMAE}_h = \frac{1}{N} \sum_{i=1}^{h} \text{MAE}_h, \) where \( \hat{Y}_{n+h} \) denotes the forecast of \( Y_{n+h} \). We consider a forecast horizons of \( h \in \{1, \ldots, 100\}. \) We compute the forecasting errors for the homoscedastic and iteratively reweighted heteroscedastic method. Additionally, we calculate the forecasting error for the corresponding oracle model. For the oracle we assume that the the underlying lag structure of the autoregressive model is known. The simulation results are given in Figure 2. Here, we observe additionally to the estimated MAE\(_h\) and MMAE\(_h\) their estimated 2-\( \sigma \) ranges, namely \( \text{MAE}_h \pm 2\sigma(\text{MAE}_h) \) resp. \( \text{MMAE}_h \pm 2\sigma(\text{MMAE}_h). \) We observe basically observe the same relationships for MAE\(_h\) and MMAE\(_h\), but the MMAE\(_h\) is better to interpret due to smaller confidence bands. We see that the homoscedastic algorithms performs significantly worse than the heteroscedastic one. The same fact can be observed, within the oracle procedures, but the improvement is not that significant. This is a
remarkable fact for applications. It indicated that we can benefit more from taking the heteroscedasticity into account in settings with unknown model structure than in a setting where the underlying structure is known. However, we usually do not know the true underlying model as the oracle does, especially in high-dimensional settings. It shows that the proposed estimation algorithm can lead to crucial improvements in a high-dimensional setting. This is also observed by Ziel et al. (2015) in applications of the proposed estimation algorithm to electricity market data.

7 Application to electricity market data

In this section we shortly show an application of the proposed model to a two-dimensional AR-ARCH model, to the two-dimensional process \((Y_{1,t})_{t \in \mathbb{Z}} = (Y_{1,t}, Y_{2,t})_{t \in \mathbb{Z}}\). Here we consider the hourly day-ahead electricity spot price for Germany/Austria at the European Power exchange (EPEX) as one process \((Y_{1,t})_{t \in \mathbb{Z}}\) and the hourly electricity load of Germany as \((Y_{2,t})_{t \in \mathbb{Z}}\). The considered time range is from 28.09.2010 to 17.04.2014, so we have \(n = 31128\) observations. We suppose that \(Y_{1,t}\) follows an AR-ARCH model as given in (18) and (21). For the autoregressive parameters we propose the lags \(I_{i,j} = \{1, \ldots, 2500\}\) for \(i,j \in \{1, 2\}\) and for the ARCH part we suppose \(J_{i,j} = \{1, \ldots, 700\}\). We consider the AIC as information criteria and for the adaption parameter \(\tau\) we take the lasso case with \(\tau = 0\). Then we apply the iteratively reweighted algorithm and stop after \(R_{\text{max}} = 3\) iterations. So we have to solve \(dR_{\text{max}} = 2 \times 3\) lasso problems of size \(n \times p = 28628 \times 5001\) and an NNLS problems of size \(n \times 1401\).

The estimated \(\hat{\beta}_{i,n}\) for \(i \in \{1, 2\}\) are given in Figure 3. Here we also see that most of the parameters are not included in the model. The proportion of included parameters is only 3.3% for the price process, whereas for the load about 13.5% of all parameters are included into the model. Using a estimated model we can perform asymptotic inference using theorem 1 and forecast the time series. A 120-step ahead forecast is given in Figure 4. There we can see how the conditional mean and variance structure influences the forecasts.

8 Summary and Conclusion

We described an iterative algorithm to solve adaptive lasso problems with conditionally heteroscedastic residuals. The sign consistency and asymptotic normality in a quite general time series setting is shown. The asymptotic theory shows that a significant estimation improvement possible if the conditional heteroscedasticity is considered. We discuss the application to AR-ARCH type models and show an application to electricity market data.
Another very important issue is the identification of the optimal penalty parameter $\lambda_n$ in high-dimensional time series settings. A different direction of further research might concern the robustness of the algorithm. The simulation study carried out showed that the algorithm work well in a finite sample setting and even with quite heavy tailed residuals. However, sometimes in such a heavy tailed situation it might be worth to consider LAD-lasso (see e.g. Wang et al. (2007a)) that minimises the sum of the absolute residuals, instead of their squares as in the lasso type algorithms.

Our simulation studies underline the asymptotic results. Additionally, we show that considering the heteroscedasticity in high-dimensional settings with unknown parameter specification is more important than in cases where the true underlying model is known, as it can substantially improve the forecasting performance. This observation will likely have a strong impact to high-dimensional time series modelling, as almost every time series exhibits conditional heteroscedasticity, especially in economics and finance.

The asymptotic theory shows that only 2 iterations are required for receiving optimal asymptotic behaviour. Thus the algorithm is suitable for applications, as the computational effort is only doubled in this situation.

For future research it might be important to analyse the mentioned model extensions more carefully. Another very important issue is the identification of the optimal penalty parameter $\lambda_n$ in high-dimensional time series settings. A different direction of further research might concern the robustness of the algorithm. The simulation study carried out showed that the algorithm work well in a finite sample setting and even with quite heavy tailed residuals. However, sometimes in such a heavy tailed situation it might be worth to consider LAD-lasso (see e.g. Wang et al. (2007a)) that minimises the sum of the absolute residuals, instead of their squares as in the lasso type algorithms.

Figure 3: estimated parameters $\hat{\beta}_{i,n}$ for $i \in \{1, 2\}$

Figure 4: Observed sample with forecasts and their prediction region given with dotted and dashed lines.
where we assume that (e) is true. If (e') is true, the same follows by arguments illustrated in Wagener and Dette (2013).

Let \( k > 1 \) and assume that the theorem holds for \( k - 1 \) so that \( \|\beta_n^{(k-1)} - \beta_n^0\|_2 = O\left(\frac{1}{\sqrt{n}}\right) \). Following the Karush-Kuhn-Tucker conditions we have that

\[
(Y_n - X_n\beta)'(W_n^{[k-1]} - \beta_n^0) + \lambda_n v_n'\beta \]

is minimised by \( \beta = (\beta(1)', \beta(0)')' \in \mathbb{R}^{p_n} \) if and only if

\[
X_j'(W_n^{[k-1]})^2(Y_n - X_n\beta) = \frac{\lambda_n}{2} v_j \text{sign}(\beta_j) \text{ if } \beta_j \neq 0
\]

and

\[
|X_j'(W_n^{[k-1]})^2(Y_n - X_n\beta)| < \frac{\lambda_n}{2} v_j \text{ if } \beta_j = 0
\]

holds. Thus we have the estimator \( \beta_n^{(k)} = (\beta_n^{(k)}(1)', \beta_n^{(k)}(0)')' \in \mathbb{R}^{p_n} \) where

\[
\beta_n^{(k)}(1) = \beta_n^0(1) + \frac{1}{n} (\hat{\Gamma}_n^{(k)})^{-1} X_n(1) (W_n^{[k-1]})^2 \varepsilon_n^0 - \frac{\lambda_n}{2n} (\hat{\Gamma}_n^{(k)})^{-1} s_n(1)
\]

where \( s_n(1) = (v_1, \ldots, v_{q_1})' \text{sign}(\beta_n^0(1)) \).

Analogue to Wagener and Dette (2013) we receive that

\[
P(\beta_n^{(k)} \neq \beta_n^0) \leq P(A_1) + P(A_2) + P(A_3) + P(A_4)
\]

with \( A_1 = \{ \frac{1}{n} |\eta_{1,j}| \geq \frac{1}{4} |\beta_n^0(j)| \text{ for some } j \leq q_1 \} \), \( A_2 = \{ \frac{\lambda_n}{n} |\eta_{2,j}| \geq |\beta_n^0(j)| \text{ for some } j \leq q_1 \} \), \( A_3 = \{ |\eta_{3,j}| \geq \frac{\lambda_n}{n} v_j \text{ for some } j > q_1 \} \) and \( A_4 = \{ |\eta_{4,j}| \geq \frac{\lambda_n}{n} v_j \text{ for some } j > q_1 \} \) where

\[
\eta_{1,j} = \epsilon_{n,j}(\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n^0 \\
\eta_{2,j} = \epsilon_{n,j}(\hat{\Gamma}_n^{(k)})^{-1} s_n(1) \\
\eta_{3,j} = X_j'(W_n^{[k-1]})^2 (I_n - n^{-1} X_n(1)(\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n^0 \\
\eta_{4,j} = \lambda_n (2n)^{-1} X_j'(W_n^{[k-1]})^2 X_n(1)(\hat{\Gamma}_n^{(k)})^{-1} s_n(1).
\]

Hence we only need to show \( P(A_1) \rightarrow 0 \) as \( n \rightarrow \infty \). Regarding \( P(A_1) \) we have with definition of \( b_n \) that

\[
P(A_1) \leq P\left( \max_{1 \leq j \leq q_1} |\eta_{0,j}| \geq \frac{b_n}{4} \right) + P\left( \max_{1 \leq j \leq q_1} |\eta_{1,j}| - |\eta_{1,j}| \geq \frac{b_n}{4} \right)
\]

where \( \eta_{0,j} = \epsilon_{n,j}(\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n. \) Following steps as in Wagener and Dette (2013) we can show that \( P\left( \frac{1}{n} \max_{1 \leq j \leq q_1} |\eta_{0,j}| \geq \frac{b_n}{4} \right) \rightarrow 0. \) Further, with Cauchy Schwarz we get

\[
|\eta_{1,j} - \eta_{0,j}|| = \| (W_n^{[k-1]})^2 - (\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n^0 \|
\]

where \( \eta_{1,j} = \epsilon_{n,j}(\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n. \) Following steps as in Wagener and Dette (2013) we can show that \( P\left( \frac{1}{n} \max_{1 \leq j \leq q_1} |\eta_{0,j}| \geq \frac{b_n}{4} \right) \rightarrow 0. \) Further, with Cauchy Schwarz we get

\[
|\eta_{1,j} - \eta_{0,j}|| = \| (W_n^{[k-1]})^2 - (\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n^0 \|
\]

and

\[
\leq \| (W_n^{[k-1]})^2 - (\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n^0 \|
\]

by triangle inequality. Using assumption (j) we get directly that \( \| (W_n^{[k-1]})^2 - (\hat{\Gamma}_n^{(k)})^{-1} X_n(1)'(W_n^{[k-1]})^2 \varepsilon_n^0 \|_2 = O\left(\frac{1}{\sqrt{n}}\right) \).
Following similar steps as in Wagener and Dette (2013) we receive with Taylor expansion and Cauchy
Schwarz that
\[
\| (\hat{W}_n - W_n^{(k-1)})^2 \|_2 \leq c_1 \sqrt{n} \| \beta_n^{(k-1)} - \beta_n^0 \|_2 + c_2 q_n \| \beta_n^{(k-1)} - \beta_n^0 \|_2 = O \left( \frac{\sqrt{q_n}}{\sqrt{n}} + \frac{q_n}{n} \right) = O \left( \frac{\sqrt{q_n}}{\sqrt{n}} \right)
\]
for some \( c_1, c_2 > 0 \). Moreover, we receive \( \| n\Gamma_n^0 \|_2^{1/2} = O(\sqrt{n}) \) and \( \| \Gamma_n^{0(1)} \|_2 = O(1) \). For estimating
\[
\| e_0^0 \|_2 \text{ we remember that the two vectors } e_0^0 = Y_n - X_n \beta_n^0 \text{ and } e_0^\infty = (e_1, \ldots, e_n) \text{ with } \epsilon_i = Y_i - X_i \beta_0 \text{ for } i \in \{1, \ldots, n\} \text{ are in general different from each other.}
\]
Then the triangle inequality yields \( \| e_0^0 \|_2 \leq \| e_0^0 - e_0^\infty \|_2 + \| e_0^\infty \|_2 \). We have \( \| e_0^0 - e_0^\infty \|_2 = \sum_{j=p+1}^\infty \beta_j X_{n,k} \|_2 = O(1) \) with \( \sum_{j=1}^\infty \| \beta_k \| < \infty \) and
\[
\| e_0^\infty \|_2 = O(\sqrt{n}).
\]
Further we have \( \| W_n^{(k-1)} \|_2 = O(1) \) and \( \| (\Gamma_n^{0(1)})^{-1} - (\Gamma_n^{-1})^{-1} \|_2 \leq \| (\Gamma_n^{0(1)} - (\Gamma_n^{-1}))^{-1} \|_2 = O \left( \frac{\sqrt{n}}{\sqrt{n}} \right) \) which leads to \( \| (\Gamma_n^{0(1)})^{-1} - (\Gamma_n^{-1})^{-1} \|_2 = O \left( \frac{\sqrt{n}}{\sqrt{n}} \right) \) by using the same arguments as Wagener and Dette (2013). Using all these estimates we receive with (34) that
\[
\frac{1}{\sqrt{n}} \max_{1 \leq j \leq q_n} | \eta_{n,j} - \eta_{0,j}^0 | = O \left( \frac{\sqrt{n}}{\sqrt{n}} \right)
\]
which yields \( P \left( \frac{1}{\sqrt{n}} \max_{1 \leq j \leq q_n} | \eta_{n,j} - \eta_{0,j}^0 | > \frac{1}{\sqrt{n}} \right) \to 0 \) as \( n \to \infty \). Thus we have \( P(A_1) \to 0 \).

Now we consider \( P(A_2) \leq P \left( n^{-1} \frac{1}{\max_{1 \leq j \leq q_n} | \eta_{n,j} |} > b_n \right) \). We have \( | \eta_{n,j} | \leq \| (\Gamma_n^{0(1)})^{-1} \|_2 \| s_n(1) \|_2 \).

Arguments as in Wagener and Dette (2013) give directly \( \| (\Gamma_n^{0(1)})^{-1} \|_2 \leq \lambda_{1,\min}^{-1} + c_4 \) and \( \| s_n(1) \|_2 \leq \frac{\sqrt{n}}{\sqrt{n}} \) with a probability of at least \( 1 - c_3 \) for all \( c_3, c_4 > 0 \) for a sufficiently large \( n \). Hence we have \( P(A_2) \to 0 \) as \( n \to \infty \).

For \( A_3 \) we receive as in Wagener and Dette (2013) that
\[
P(A_3) \leq P \left( \max_{q_n+1 \leq j \leq p_n} | \eta_{n,j}^0 | \geq \frac{\lambda_n r_n}{8} \right) + P \left( \max_{q_n+1 \leq j \leq p_n} | \eta_{n,j}^0 - \eta_{3,n}^j | \geq \frac{\lambda_n r_n}{8} \right) + P \left( \max_{q_n+1 \leq j \leq p_n} \beta_{j,\text{init}} > r_n^{-1} \right)
\]
where
\[
\eta_{n,j}^0 = X_j' \left( \hat{W}_n^{(k-2)} \right)^2 (I_n - n^{-1} X_n(1)(\Gamma_n^{0(1)})^{-1} X_n(1)(\hat{W}_n^{(k-2)})^2) e_0^0.
\]

Following the arguments as in Wagener and Dette (2013) we can show with \( H(\beta_n) = X_j' \left( \hat{W}_n^{(k-2)} \right)^2 (I_n - n^{-1} X_n(1)(\Gamma_n^{0(1)})^{-1} X_n(1)(\hat{W}_n^{(k-2)})^2) \) that \( \| H(\beta_n) \|_F \leq c_5 \sqrt{n} \) for some \( c_5 > 0 \). Triangle inequality gives
\[
\| \frac{1}{\sqrt{n}} H(\beta_n) e_0^0 \|_{\psi_d} \leq \| \frac{1}{\sqrt{n}} H(\beta_n) e_0^\infty \|_{\psi_d} + \| \frac{1}{\sqrt{n}} H(\beta_n)(e_0^0 - e_0^\infty) \|_{\psi_d} \text{ where } \| \cdot \|_{\psi_d} \text{ denotes Orlicz norm with } \psi_d(x) = \exp(x^d) - 1.
\]

As in Wagener and Dette (2013) we receive that for \( \| \frac{1}{\sqrt{n}} H(\beta_n) e_0^\infty \|_{\psi_d} \leq c_6 \log(n) \) with \( d = 1 \). For the second term we have \( \| \frac{1}{\sqrt{n}} H(\beta_n)(e_0^0 - e_0^\infty) \|_{\psi_d} \leq \frac{c_6}{\sqrt{n}} \| (H(\beta_n)) \|_F \| e_0^0 - e_0^\infty \|_2 = O(1) \). Thus we get \( \max_{q_n+1 \leq j \leq p_n} | \eta_{n,j}^0 | \geq \frac{\lambda_n r_n}{8} \) \to 0 as \( n \to \infty \). Moreover, \( \max_{q_n+1 \leq j \leq p_n} | \eta_{n,j}^0 - \eta_{3,n}^j | = O \left( \frac{\sqrt{n}}{\sqrt{n}} \right) \) using estimates derived for \( A_1 \). Hence we have \( P(\max_{q_n+1 \leq j \leq p_n} | \eta_{n,j}^0 - \eta_{3,n}^j | > \frac{\lambda_n r_n}{8} ) \to 0 \) which gives \( P(A_3) \to 0 \).

For \( A_4 \) the situation is similar, we have as in Wagener and Dette (2013) that \( P(A_4) \leq P(\max_{q_n+1 \leq j \leq p_n} | \eta_{n,j} | \geq \frac{\lambda_n r_n}{8} ) + P(\max_{q_n+1 \leq j \leq p_n} \beta_{j,\text{init}} > r_n^{-1} ) \). As it holds
\[
| \eta_{n,j} | \leq \frac{\lambda_n}{2} \left( \| (\Gamma_n^{0(1)})^{-1} \|_2 \| (\hat{W}_n^{(k-1)})^2 \|_2 \| X_n(1) \|_2 \| s_n(1) \|_2 \right) \leq O \left( \frac{\lambda_n r_n}{\sqrt{n}} \right)
\]
we directly get \( P(A_4) \to 0 \) as \( n \to \infty \). Hence, \( \beta_n^{(k)} \) is sign consistent.

For the asymptotic normality we use similar concepts as in Wagener and Dette (2013). So given sign consistency of \( \beta_n^{(k)} \) we have
\[
\beta_n^{(k)(1)} = \beta_n^{(0)(1)} + \frac{1}{n} (\Gamma_n^{0(1)})^{-1} X_n(1)(\hat{W}_n^{(k-1)})^2 e_0^0 - \frac{\lambda_n}{2n} (\Gamma_n^{0(1)})^{-1} s_n(1)
\]
which directly gives
\[
\sqrt{n} \frac{s_n(k)}{\sqrt{\lambda_n \gamma_n \beta_n^{(k)(1)}}} \left( \beta_n^{(k)(1)} - \beta_n^0 \left( 1 \right) \right) = \frac{1}{\sqrt{n s_n(k)}} e_0^0 (\Gamma_n^{0(1)})^{-1} X_n(1)(\hat{W}_n^{(k-1)})^2 e_0^0 - \frac{1}{\sqrt{n s_n(k)}} s_n(1) \left( \frac{\lambda_n}{2n} (\Gamma_n^{0(1)})^{-1} s_n(1) \right)
\]
As in Wagener and Dette (2013), we get for the second term with estimates shown before, that
\[
\left| \frac{1}{\sqrt{n s_n(k)}} e_0^0 (\Gamma_n^{0(1)})^{-1} s_n(1) \right| \leq \frac{\lambda_n \gamma_n \beta_n^{(k)(1)}}{2n s_n(k) \sqrt{n b_n}} \left( \frac{\lambda_n^{-1} | \lambda_n^{-1} \left( \lambda_n^{-1} + c_4 \right) }{\sqrt{n b_n}} \right).
\]
For estimating the first term we use the decomposition $(\hat{F}_n^{[k]})^{-1}X_n(1)(W_n^{[k]-1})^2 = B_1 + B_2 + B_3$ where $B_1 = (\hat{F}_n^{[k]} - 1)X_n(1)(W_n^{[k]-1})^2, B_2 = ((\hat{F}_n^{[k]})^{-1} - \hat{F}_n^{[k]} - 1)X_n(1)(W_n^{[k]-1})^2, B_3 = (\hat{F}_n^{[k]})^{-1}X_n(1)(W_n^{[k]-1})^2 - W_n^{[k]-2}$.

Now we decompose $1 - \frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k e_n^0 = \frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k e_n^0 + \frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k (e_n^0 - e_n^0)$. For the first term we have $\frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k e_n^0 = \frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 \sum_{t=1}^n a_t Z_t$ with $a_t = \frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k X_n(1)$. So we can calculate $E \sum_{t=1}^n a_t Z_t = 0$ and $E(\sum_{t=1}^n a_t Z_t)^2 = \sum_{t=1}^n E(a_t)^2 = 1$. Then we get with the central limit theorem $\frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k e_n^0 \rightarrow N(0,1)$ in distribution as $n \rightarrow \infty$. Moreover we obtain

$$\left| \frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k (e_n^0 - e_n^0) \right| \leq \frac{1}{\sqrt{n}n_n(k)} \| (\hat{F}_n^{[k]})^{-1} - \hat{F}_n^{[k]} - 1 \|_2 \| X_n(1)(W_n^{[k]-1})^2 e_n^0 \|_2$$

as $\| e_n^0 - e_n^0 \|_2 \rightarrow 0$ as $n \rightarrow \infty$).

Regarding $B_2$ we have as in Wagener and Dette (2013) that

$$\left| \frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k e_n^0 \right| \leq \frac{\sqrt{\lambda_{max}}}{\sigma_{min} \sqrt{n}} \| (\hat{F}_n^{[k]})^{-1} - \hat{F}_n^{[k]} - 1 \|_2 \| X_n(1)(W_n^{[k]-1})^2 e_n^0 \|_2$$

Using the estimate $\| X_n(1)(W_n^{[k]-1})^2 e_n^0 \|_2 \leq \| X_n(1)(W_n^{[k]-1})^2(e_n^0 - e_n^0) \|_2 + \| X_n(1)(W_n^{[k]-1})^2 e_n^0 \|_F$ we have as above $\| X_n(1)(W_n^{[k]-1})^2(e_n^0 - e_n^0) \|_2 \|_2 + \| X_n(1)(W_n^{[k]-1})^2 e_n^0 \|_F$ we have and the previous estimates it follows that $\frac{1}{\sqrt{n}n_n(k)}c_n(k)^2 B_k e_n^0 \rightarrow N(O(\sqrt{n}n_n(k)))$. With $\| (\hat{F}_n^{[k]})^{-1} - \hat{F}_n^{[k]} - 1 \|_2 = O(\sqrt{n})$ and Slutsky’s theorem that $\frac{\sqrt{n}}{\sqrt{n_n(k)}}c_n(k)^2 (\hat{\beta}_n(k) - \beta_n^0(k)) \rightarrow N(0,1)$.

At the beginning we assumed that $\| \hat{\beta}_n(k) - \beta_n^0(k) \|_2 = O(\sqrt{n})$ holds for $k > 1$. Indeed for $k > 1$ we even have $\| \hat{\beta}_n(k) - \beta_n^0(k) \|_2 = O(\frac{1}{\sqrt{n}})$. In the case $k = 1$ we do not have this property, so the basis case of the induction is missing. However, the proof is similar to the sign consistency and asymptotic normality proof with $k > 1$, as Wagener and Dette (2013) explained it for the unconstrained weighted adaptive lasso. Note that the proof itself is less complex than the case $k > 1$, but involves the assumptions $\lambda_{min} \leq \lambda_{min}(\hat{F}_n^{[k]})$ and $g_n(\hat{\alpha}_n(\beta_n; X_n, Y_n), \hat{L}_n, (\beta_n; X_n, Y_n)) < \sigma_{max}$ for all $\beta_n$ in a neighbourhood of $\beta_n^0$ that were not used in the previous part.

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