Schwinger pair production from Padé-Borel reconstruction

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In this work, we show how the knowledge of the first few terms of the Euler-Heisenberg Lagrangian’s weak-field expansion in a magnetic field background is enough to reconstruct the pair-production rate in a strong electric field background. To this end, we study its associated truncated Borel sum using Padé approximants, as advocated in a recent work by Costin and Dunne, J. Phys. A52, 445205 (2019).

INTRODUCTION

In recent years, the program of "resurgence" has started to collect a number of successes in quantum mechanics and field theory. The idea behind it is that the typical asymptotic expansions that are to be dealt with, for example usual weak coupling expansions, are to be understood as being part of a transseries. In simple terms, transseries are sums of asymptotic series weighted by non-perturbative factors such as exponentials and logarithms. A typical example is the semi-classical expansion, which is a sum of perturbative/asymptotic expansions around different saddle points. We refer the reader to [1] for a pedagogical introduction to the transseries.

The very analytic structure of transseries implies consistency relations between the different constituent asymptotic series. In particular, large order coefficients of a given expansion are known to be related to the small order coefficients of neighboring expansions. While being seemingly a mathematical curiosity, these relations have, for example, been used to predict the loop expansion around an instanton background for the quantum mechanical Sine-Gordon potential [2], predictions which have been explicitly verified up to three loops using diagrammatic methods [3]. For other interesting examples and reviews, we defer the reader to [4] and references therein.

An immediate complaint against the potential practical usefulness of such approaches is that the knowledge of large orders terms of realistic quantum field theories expansions is not necessarily available. In this spirit, reference [5] started to investigate the amount of "non-perturbative" information that can be extracted from a finite number of terms of an asymptotic expansion. Stunningly, using relatively few terms of the asymptotic series, they were able to reconstruct the knowledge from a few terms of the weak-field expansion of the Euler-Heisenberg effective Lagrangian in a background magnetic field and to extract its strong-field behavior. Then, and perhaps more interestingly, the same knowledge is enough to reconstruct the Euler-Heisenberg effective Lagrangian in a background electric field, for weak and strong fields, including its imaginary part. This means that this imaginary part, which gives the particle production rate in a constant electric field, can be inferred from a few terms of a perturbative expansion.

SCHWINGER EFFECT, GENERALITIES

Schwinger pair production is one of the most basic field-theoretic non-perturbative effect, see [6] for an extensive review. Its simplest realization is the vacuum emission of charged particles in the presence of strong electric fields. A way to study it is to compute the one-loop fermionic effective action in a background electromagnetic field. Then, the phenomenon of pair-productions is signaled by the appearance of an imaginary part in the effective action. For the sake of simplicity, we will hereafter restrict ourselves to the constant background case. There, one can explicitly write down the effective Lagrangian [7]. For a purely magnetic field, it admits the following closed-form [8]

\[ \mathcal{L}_{\text{eff}}(B) = \frac{(eB)^2}{2\pi^2} \left[ \zeta_H' \left( -1, \frac{m^2}{2eB} \right) \right. \]

\[ + \left. \zeta_H \left( -1, \frac{m^2}{2eB} \right) \ln \left( \frac{m^2}{2eB} \right) - \frac{1}{12} + \frac{1}{4} \left( \frac{m^2}{2eB} \right)^2 \right], \]

with \( \zeta_H(s, a) \) the Hurwitz zeta function and \( \zeta_H'(s, a) \) its derivative with respect to \( s \). The parameters \( m \) and \( e \)
are respectively the fermion’s mass and electric charge, while $B$ is the
strength of the constant background magnetic field. This expression is real and
there is no pair-production in a magnetic background, as there is apriori
no magnetically charged particle to be produced. The case of a pure electric
field background is recovered by analytically continuing $B \to \pm iE$ [15] (note
that in this sense [1] can also be understood as the Euclidean space
effective Lagrangian in an electric field background). Then, the effective Lagrangian
does develop an imaginary part, which can be written as [15]

$$\Im(\mathcal{L}_{\text{eff}}(E)) = \frac{m^4}{8\pi^3} \left(\frac{eE}{m^2}\right)^2 \text{Li}_2 \left(e^{-\frac{m^2}{m^2}}\right)$$  \hspace{0.5cm} (2)

$$= \frac{m^4}{8\pi^3} \left(\frac{eE}{m^2}\right)^2 \left[e^{-\frac{m^2}{m^2}} + \ldots\right],$$  \hspace{0.5cm} (3)

with $\text{Li}_2$ the second polylogarithm. From this expression, it is easy to see
the famous exponential suppression to the production rate $\Gamma_{\text{prod}}$, which by
definition is [13]

$$\Gamma_{\text{prod}} = 2\Im(\mathcal{L}_{\text{eff}}).$$  \hspace{0.5cm} (4)

Another representation of [1] that will be of use is the following
Laplace-type integral [14, 10]

$$\mathcal{L}_{\text{eff}}(B) = -\frac{e^2B^2}{8\pi^2} \int_0^\infty \frac{dp}{p^2} \left(c\text{th} p - \frac{1}{p} \frac{p}{3} \right) e^{-\frac{m^2}{m^2}p^2}. $$  \hspace{0.5cm} (5)

From this representation, it is clear that the imaginary part in the
electric case comes from the contribution to the integral of the poles of the
hyperbolic cotangent at integer multiples of $\pi i$.

In the rest of this work, we will be concerned with the
weak-field expansion of [1]. It is given as [13, 17]

$$\mathcal{L}_{\text{eff}}(B) \sim \frac{m^4}{4\pi^2} \sum_{n=0}^\infty (-1)^n(2n+1)\zeta(2n+4) \left(\frac{2eB}{m^2}\right)^{2n+4}, $$  \hspace{0.5cm} (6)

with $\zeta(x)$ the Riemann zeta function. For the electric field, the expansion reads

$$\mathcal{L}_{\text{eff}}(E) \sim \frac{m^4}{4\pi^2} \sum_{n=0}^\infty (2n+1)\zeta(2n+4) \left(\frac{2eE}{m^2}\right)^{2n+4}. $$  \hspace{0.5cm} (7)

Both series are asymptotic because of the factorial growth of their coefficients. They are also both real
to all orders. It is in this sense that the rate [4] is

a non-perturbative quantity; at any given order in [7],
$\Gamma_{\text{prod}} = 0$.

Actually, these weak-field expansions can be resummed
to [5] using Borel summation. In this language, again,
the imaginary part appears because of the presence of
the poles in the Laplace transform; [6] is "Borel summable" while [7] is not.

**STRONG-FIELD REGIME FROM WEAK-FIELD
EXPANSION**

Following [10], we want to understand how much of the
full Lagrangian [1] we can reconstruct using a finite number of terms in [6]. To this purpose, again as in [10], we construct the corresponding truncated Borel sum. From it, we build Padé approximants, which are then used to compute a resummed Lagrangian $\mathcal{L}_{\text{res}}$ through a Laplace transform. The idea behind this procedure is to try to exploit the fact that, while the original expansion is only asymptotic, its Borel transform is convergent.

To keep notations clear, we set $x = \frac{m^2}{2eB}$ and write the
asymptotic expansion [8], truncated at order $N$, as

$$\frac{\mathcal{L}_{\text{eff}}(x, N)}{m^4} \sim \frac{1}{64\pi^6} \frac{1}{x^4} \sum_{n=0}^N (-1)^n(2n+1)\zeta(2n+4) \left(\frac{1}{x}\right)^{2n} $$

$$= \frac{1}{64\pi^6} \frac{1}{x^4} \sum_{n=0}^N a_{2n} \left(\frac{1}{x}\right)^{2n}, $$  \hspace{0.5cm} (8)

with $a_{2n} = (-1)^n(2n+1)\zeta(2n+4) \left(\frac{x}{2}\right)^{2n}$. We also define a truncated
Borel sum

$$\mathcal{B}\mathcal{L}_{\text{eff}}(p, N) = \sum_{n=1}^N \frac{a_{2n}}{(2n-1)!} p^{2n-1}. $$  \hspace{0.5cm} (9)

With these definitions, we construct a Padé approximant of [10]. To have easy access to the poles of the
Padé function and have good control over the numerical Laplace transform, we use rational Padé approximants of the type

$$\mathcal{P}^{2N}\mathcal{B}\mathcal{L}_{\text{eff}}(p, N) = \sum_{n=1}^N \frac{c_n}{1 + b_np}. $$  \hspace{0.5cm} (10)

The coefficients $c_n, b_n$, which are in principle complex
numbers, are computed by matching this expression to [10] around $p = 0$, see [18] for an explicit algorithm.

Finally, we compute our resummed Lagrangian as follows

$$\frac{\mathcal{L}_{\text{res}}(x, N)}{m^4} = \frac{1}{64\pi^6} \frac{1}{x^4} \left(a_0 + \int_0^\infty dp e^{-px} \mathcal{P}^{2N}\mathcal{B}\mathcal{L}_{\text{eff}}(p, N)\right). $$ \hspace{0.5cm} (11)
Note in particular that without the Padé interpolation, we would have achieved nothing, as in this case [12] would literally be equal to [9].

We show the result of this procedure, which from now on we will refer to as Padé-Borel reconstruction, in figure 1. The plain black line is the closed-form [1]. The dotted lines are the truncated weak-field expansions, for different truncation order $N$. The dashed lines are the Padé-Borel reconstructed expressions for the same $N$. Note that $x \to \infty$ resp. $x \to 0$ corresponds to the weak resp. strong-field regime, the goal being to be able to extrapolate from the former to the latter. Being an asymptotic expansion, every order makes it break down faster. The Padé-Borel reconstruction takes advantage of the fact that the Borel sum is convergent; every order improves the answer.

This is our first result. With the knowledge of only the first few terms of the weak-field expansion [6], it is possible to explore the regime of strong magnetic fields by first constructing the corresponding truncated Borel sum, Padé approximating it and computing its Laplace transform.

Figure 1. Magnetic field effective Lagrangian. Closed-form (plain line), weak-field expansion (dotted lines) and Padé-Borel reconstruction (dashed lines) for different truncation order $N$. The weak-field expansion has a typical asymptotic behavior; every order makes it break down faster. The Padé-Borel reconstruction takes advantage of the fact that the Borel sum is convergent; every order improves the answer.

Figure 2. Real part of electric field effective Lagrangian. Closed-form (plain line) and Padé-Borel reconstruction (dashed lines) for different truncation order $N$. The Padé-Borel reconstruction leads to correct and convergent results even after analytic continuation.

SCHWINGER EFFECT RECONSTRUCTED

Now, we will show that this method, using the same data, actually also gives access to the regime of strong electric fields. In particular, we will see that we can use it to recover the Schwinger pair production rate.

To consider an electric field, we proceed with the analytic continuation $x \to \mp ix$. This leads us to study

\[
\frac{\mathcal{L}_{\text{eff}}^\alpha(\mp ix, N)}{m^4} = \frac{1}{64\pi^6 x^4} \left[ a_0 + \int_0^\infty dp e^{\pm ipx} \mathcal{P}^{2N} B_{\text{eff}}(p, N) \right].
\] (13)

Technically, to compute this Laplace transform, we consider all the different fractions of (11) separately. We then rotate the integration contour in the complex plane by some angle and take into account any poles we might have crossed in the process.

Let us first look at what we obtain for the real part of the resummed electric field effective Lagrangian obtained through this analytic continuation, figure 2. As in the magnetic case, few terms of the weak Lagrangian allows for a precise extrapolation up into the strong-field regime. In particular, the reconstruction is able to predict correctly non-trivial features such as the change of signs which happens around $x = 0.1$ (note that we are plotting the absolute value).

More interesting are the results for the imaginary part of the effective Lagrangian, i.e. the pair-production rate.
They are shown in figure [3]. They behave in exactly the same way; few terms of the weak-field expansion still give a quantitatively correct prediction of the rate. As little as the first two terms are required to reconstruct an imaginary part which is qualitatively correct at weak-field. With only the first six terms one can make quantitative predictions up to strong fields. This has to be contrasted again with the original asymptotic series, which uses the same data but is real to all orders.

The perhaps surprising capability of the Padé-Borel reconstruction to recover the pair-production rate is due to the fact that the Padé approximants of the truncated Borel sums are able to reproduce the correct analytic structure of the Borel sum. In terms of our variable $x$, the actual Borel sum \[ \Gamma_{\text{prod}} \] is a meromorphic function with single poles at $x = 2\pi in$ for $n \in \mathbb{Z}, n \neq 0$. As already mentioned, the imaginary part \[ \text{Im}(\Gamma_{\text{prod}}) \] can be understood as coming from the contribution of every single pole. It is dominated by the lowest-lying ones at $x = \pm 2\pi i$

\[
\Gamma_{\text{prod}}^{\text{lead}} = \frac{1}{2\pi e^{-2\pi x}}, \quad (14)
\]

which we also show in figure [3]

As the Padé-Borel approximants are constructed only from an asymptotic expansion around the real axis it is, however, a non-trivial fact that they are able to mimic correctly this analytic structure. We show it occurring in figure [4] where we display the poles of our Padé approximants. As the truncation order $N$ is taken to be larger, they accumulate around $x = 2\pi i$. Note that to approximate the correct prefactors, a single pole is replaced by a combination of different ones centered around $x = 2\pi in$. The leading poles at $\pm 2\pi i$ are first reproduced accurately by the truncation order $N = 6$, which is consistent with the behavior of the results presented in figure [3].

This is our second and most important result. The knowledge of a few terms of the weak-field expansion of the effective Euler-Heisenberg Lagrangian in a magnetic field background is enough to reconstruct the particle production rate in a strong electric field.

**CONCLUSION**

This work can be summarized as follows: using only the truncated weak-field asymptotic expansion of the Euler-Heisenberg effective Lagrangian, we were able to reconstruct the full Euler-Heisenberg Lagrangian, including its imaginary part which gives the Schwinger pair production rate. This may come as a surprise, as this rate is predicted to be zero at all-orders of the weak-field asymptotic expansion.

What this result suggests, as already realized in [10], is that all coefficients in such asymptotic expansions con-
tain information about the analytic structure of the underlying transseries in the whole complex plane. This information can be extracted by studying the associated Borel sum even upon truncation to a finite number of terms, by taking advantage of the fact that the Borel sum is a convergent series. Another remarkable fact is that the knowledge of the truncated Borel sum along the real axis is enough to gain information about its analytic structure, using Padé approximants, throughout the complex plane. In particular, this means that the underlying transseries is constrained enough to force the Padé approximants to develop poles at the correct locations.

The precise mechanism behind this phenomenon remains to be better understood; this will be essential to apply this method to unsolved problems and obtain trustworthy predictions. The aim of this work was however to illustrate its potential use in a simple physical problem.

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