SOCLE OF A HAMILTONIAN GROUP

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ABSTRACT
The socle of a group $G$ is the subgroup generated by all minimal normal subgroups of $G$. In this short note, we determine the socle of a Hamiltonian group explicitly.

1. Introduction

A group is called a Dedekind group if all of its subgroups are normal (see [RD (1897)]). Therefore, all abelian groups are Dedekind groups. A non-abelian Dedekind group is called a Hamiltonian group. The smallest example of Hamiltonian group is $Q_8$, the quaternion group of order 8. In both finite and infinite order cases, Dedekind and Baer have shown that every Hamiltonian group is a direct product of $Q_8$, a group of exponent 2, and a torsion abelian group in which all elements have odd order (see [BR (1933)], [HM (1999)]). In particular, Hamiltonian groups are locally finite, nilpotent of class 2, and solvable of length 2. In the present paper we prove the following theorem:

**Theorem 1.1.** If $H = Q_8 \oplus B \oplus D$, where $B$ is an elementary abelian 2-group, and $D$ is a torsion abelian group with all elements of odd order, then $\text{Soc}(H) \simeq \mathbb{Z}_2 \oplus B \oplus \mathbb{P}(D)$, where $\mathbb{P}(D)$ is the group generated by all the elements of $D$ that have prime orders.

For a finite solvable group $G$, the socle is a product of elementary abelian $p$-groups for a collection of primes dividing the order of $G$. However, this may not include all primes that divide the order of $G$. This result follows from a well-known fact that the socle of a finite nilpotent group is a product of elementary abelian $p$-groups for the collection of primes dividing the order of the group. The main objective of above stated theorem is that it explicitly determines the socle of a Hamiltonian group. We prove theorem (1.1) shortly, before that we develop a few facts about proper essential subgroups (see [PG (1970)]).

**Definition 1.2.** A proper nontrivial subgroup $E$ of a group $G$ is said to be proper essential in $G$ if $E \cap H \neq \{1_G\}$ for every nontrivial subgroup $H$ of $G$.

Hereafter, we write $E$ is a proper essential subgroup of $G$ or $E$ is proper essential in $G$ if $E$ satisfies definition (1.2).
2. Proper essential subgroups

**Theorem 2.1.** Let \( \{G_\omega\}_{\omega \in \Lambda} \) be a family of groups indexed by a nonempty set \( \Lambda \). Then \( \bigoplus_{\omega \in \Lambda} G_\omega \) has a proper essential subgroup if and only if some \( G_\omega \) has a proper essential subgroup.

**Proof.** Let \( G = \bigoplus_{\omega \in \Lambda} G_\omega \) and \( \omega_0 \in \Lambda \) be such that \( G_{\omega_0} \) has a proper essential subgroup \( E_{\omega_0} \). Let \( X = \bigoplus_{\omega \in \Lambda} X_\omega \), where \( X_{\omega_0} = E_{\omega_0} \) and \( X_\omega = G_\omega \) if \( \omega \neq \omega_0 \). We claim that \( X \) is a proper essential subgroup of \( G \). Let \( N \) be any nontrivial subgroup of \( G \) and let \( (n_\omega)_{\omega \in \Lambda} \) be a nonzero element of \( N \). Clearly, \( (n_\omega)_{\omega \in \Lambda} \subseteq X \) if \( n_{\omega_0} = 1_{G_{\omega_0}} \). If \( n_{\omega_0} \neq 1_{G_{\omega_0}} \), consider the cyclic group \( \langle n_{\omega_0} \rangle \). As \( E_{\omega_0} \cap \langle n_{\omega_0} \rangle \neq \{1_{G_{\omega_0}}\} \), we conclude that \( X \) is a proper essential subgroup of \( G \).

For the converse, assume that \( G \) has a proper essential subgroup \( E \) but \( G_\omega \) does not have a proper essential subgroup for all \( \omega \in \Lambda \). Observe that, for all \( \omega \in \Lambda \), we have either \( E \cap G_\omega \) is proper essential in \( G_\omega \) or \( G_\omega \subseteq E \). As \( G_\omega \) does not have any proper essential subgroup, it must be that \( G_\omega \subseteq E \) for all \( \omega \in \Lambda \), that is, \( G \subseteq E \), a contradiction.

**Definition 2.2.** Let \( G \) be a group. If \( G \) has a proper essential subgroup, we define \( \delta(G) \) to be the intersection of all proper essential subgroup of \( G \), and \( G \) otherwise.

Observe that \( \delta(G) \) is a characteristic subgroup of \( G \) and hence is normal in \( G \). Also, we have \( \delta(G) = G \) if \( G \) is a finite simple group, because, if \( G \) has a proper essential subgroup, then \( \delta(G) \) is a proper essential, therefore, proper non-trivial characteristic subgroup of \( G \), a contradiction. Now, if \( G \) is a finite group such that \( \text{Soc}(G) = G \), then \( G = \bigoplus_{\alpha=1}^n S_\alpha \), where each \( S_\alpha \) is a simple group for \( 1 \leq \alpha \leq n \). Thus, applying theorem 2.1 we can conclude that \( G \) does not contain proper essential subgroup. In the following theorem, we establish that \( \delta \) commutes with the direct sum.

**Theorem 2.3.** If \( \{G_\omega\}_{\omega \in \Lambda} \) be a family of groups indexed by a nonempty set \( \Lambda \), then \( \delta(\bigoplus_{\omega \in \Lambda} G_\omega) = \bigoplus_{\omega \in \Lambda} \delta(G_\omega) \).

**Proof.** Let \( G = \bigoplus_{\omega \in \Lambda} G_\omega \), \( G' = \bigoplus_{\omega \in \Lambda} \delta(G_\omega) \), \( T = \{\omega \in \Lambda \mid \delta(G_\omega) = G_\omega\} \) and \( T' = \{\omega \in \Lambda \mid \delta(G_\omega) \neq G_\omega\} \). Clearly the sets \( T \) and \( T' \) are disjoint and \( T \cup T' = \Lambda \).

We have the following cases: (a) \( T \neq \emptyset \), \( T' = \emptyset \); (b) \( T = \emptyset \), \( T' \neq \emptyset \); (c) \( T \neq \emptyset \), \( T' \neq \emptyset \).

(a) : Since \( G \) does not have any proper essential subgroup (see theorem 2.1), we conclude that \( \delta(G) = G' \).

(b) : Let \( (g_\omega)_{\omega \in \Lambda} \in \delta(G) \), \( \tau \in \Lambda \) be any element and \( E_\tau \) be any proper essential subgroup of \( G_\tau \). Consider \( X = \bigoplus_{\omega \in \Lambda} X_\omega \), where \( X_\omega = G_\omega \) for \( \omega \neq \tau \) and \( X_\tau = E_\tau \). Since \( X \) is proper essential in \( G \), we have \( g_\tau \in E_\tau \). As \( E_\tau \) was arbitrary we get \( g_\tau \in \delta(G_\tau) \). But \( \tau \) was arbitrarily chosen as well. Hence, \( (g_\omega)_{\omega \in \Lambda} \in G' \), that is, \( \delta(G) \subseteq G' \). Now, let \( E \) be any proper essential subgroup of \( G \). Observe that for any \( \omega \in \Lambda \) we have either \( E \cap G_\omega \) is proper essential in \( G_\omega \) or \( G_\omega \subseteq E \). But either of the cases give us \( \delta(G_\omega) \subseteq E \). As \( \omega \) and \( E \) were arbitrary, we get that \( G' \subseteq \delta(G) \).

(c) : Let \( (g_\omega)_{\omega \in \Lambda} \in \delta(G) \), \( \tau \in T' \) and \( E_\tau \) be any proper essential subgroup of \( G_\tau \). Consider \( X = \bigoplus_{\omega \in \Lambda} X_\omega \), where \( X_\omega = G_\omega \) for \( \omega \in \Lambda \setminus \{\tau\} \) and \( X_\tau = E_\tau \). Since \( X \) is proper essential in \( G \), we have \( g_\tau \in E_\tau \). As \( \tau \) was arbitrary in \( T' \), we get that \( g_\tau \in \delta(G_\tau) \) for all \( \tau \in T' \), that is, \( \delta(G) \subseteq G' \). Now, let \( E \) be any proper essential subgroup of \( G \). Observe that for any \( \omega \in T' \) we have either \( E \cap G_\omega \) is proper essential in \( G_\omega \) or \( G_\omega \subseteq E \). But either of the cases give us \( \delta(G_\omega) \subseteq E \). As \( \omega \) was arbitrary in \( T' \) and \( G_\lambda \subseteq E \) for all \( \lambda \in T \), we get that \( G' \subseteq \delta(G) \).
Lemma 2.4. If a torsion group $G$ has a proper essential subgroup, then $\delta(G)$ is proper essential in $G$. Moreover, we have $\delta(G) = \mathbb{P}(G)$, where $\mathbb{P}(G)$ is the group generated by all the elements of $G$ that have prime orders.

Proof. Assume on the contrary that $\delta(G)$ is not proper essential in $G$. Therefore, there exists a subgroup $H \neq \{1_G\}$ of $G$ such that $H \cap \delta(G) = \{1_G\}$. If $h \in H \setminus \{1_G\}$ is such that $|h| = n$, then $\langle h \rangle \cap \delta(G) = \{1_G\}$. Thus, for each $i$ with $1 \leq i \leq n - 1$, there exists at least one proper essential subgroup $E_i$ of $G$ such that $h^i \in E_i$. But this shows that $\langle h \rangle \setminus \{1_G\} \subseteq \bigcup_{i=1}^{n-1} E_i$, that is, $\left( \bigcap_{i=1}^{n-1} E_i \right) \cap \langle h \rangle = \{1_G\}$. As finite intersection of proper essential subgroups of $G$ is again proper essential in $G$, we get that $\bigcap_{i=1}^{n-1} E_i$ is a proper essential subgroup of $G$ which intersects $\langle h \rangle$ trivially, a contradiction.

Finally, if $g \in G$ be such that $g^p = 1_G$ for some prime $p$, then $g \in \delta(G)$. This shows that $\mathbb{P}(G) \subseteq \delta(G)$. Now, as $\mathbb{P}(G)$ is also proper essential in $G$ and $\delta(G)$ is the intersection of all proper essential subgroups of $G$, we get that $\delta(G) \subseteq \mathbb{P}(G)$.

We now show that for a Hamiltonian group $\mathbb{H}$, $\text{Soc}(\mathbb{H})$ is the intersection of all proper essential subgroups of $\mathbb{H}$.

Theorem 2.5. If $\mathbb{H}$ is a Hamiltonian group, then $\delta(\mathbb{H}) = \text{Soc}(\mathbb{H})$.

Proof. Let $\mathbb{H} = Q_8 \oplus B \oplus D$, where $B$ is an elementary abelian 2-group, and $D$ is a torsion abelian group with all elements of odd order. Now, applying theorem (2.1) we get that $\mathbb{H}$ has a proper essential subgroup. Since $\mathbb{H}$ is a torsion group, applying lemma (2.4) we conclude that $\delta(\mathbb{H})$ is a proper essential subgroup of $\mathbb{H}$. If $N$ is any minimal normal subgroup of $\mathbb{H}$, then $N \cap \delta(\mathbb{H}) \neq \{1_H\}$ shows that $N \subseteq \delta(\mathbb{H})$. As $N$ was arbitrary, we conclude that $\text{Soc}(\mathbb{H}) \subseteq \delta(\mathbb{H})$. As any nontrivial subgroup $K$ of $\mathbb{H}$ contains a minimal normal subgroup, we conclude that $\text{Soc}(\mathbb{H})$ is a proper essential subgroup of $\mathbb{H}$. Hence, we get that $\delta(\mathbb{H}) \subseteq \text{Soc}(\mathbb{H})$.

3. Proof of theorem 1.1.

Applying theorem (2.3) and (2.5), we conclude that $\text{Soc}(\mathbb{H}) = \delta(Q_8) \oplus \delta(B) \oplus \delta(D)$. We claim that $\delta(B) = B$. Because, if $B$ possess a proper essential subgroup $E$, then, as all elements of $B$ have order 2, we must have $b \in E$ for all $b \in B$. But this implies $E = B$. This shows that $B$ does not contain any proper essential subgroup. Since, we have $\delta(Q_8) \simeq \mathbb{Z}_2$, so, applying lemma (2.4) we conclude that $\text{Soc}(\mathbb{H}) \simeq \mathbb{Z}_2 \oplus B \oplus \mathbb{P}(D)$.

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