Quartic and Octic Characters Modulo $n$

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Abstract. The average number of primitive quadratic Dirichlet characters of modulus $n$ tends to a constant as $n \to \infty$. The same is true for primitive cubic characters. It is therefore surprising that, as $n \to \infty$, the average number of primitive quartic characters of modulus $n$ grows with $\ln(n)$, and that the average number of primitive octic characters of modulus $n$ grows with $\ln(n)^2$. Leading coefficients in the asymptotic expressions are also computed.

Let $\mathbb{Z}_n^*$ denote the group (under multiplication modulo $n$) of integers relatively prime to $n$, and let $\mathbb{C}^*$ denote the group (under ordinary multiplication) of nonzero complex numbers. We wish to count homomorphisms $\chi: \mathbb{Z}_n^* \to \mathbb{C}^*$ satisfying certain requirements. A Dirichlet character $\chi$ is quadratic if $\chi(k)^2 = 1$ for every $k$ in $\mathbb{Z}_n^*$. It is well-known that

$$\text{# quadratic Dirichlet characters of modulus } \leq N = \sum_{n \leq N} a(n) \sim \frac{6}{\pi^2} N \ln(N)$$

as $N \to \infty$, where $a(n)$ is multiplicative with

$$a(2^r) = \begin{cases} 1 & \text{if } r = 1, \\ 2 & \text{if } r = 2, \\ 4 & \text{if } r \geq 3, \end{cases} \quad a(p^r) = 2$$

for prime $p \geq 3$ and $r \geq 1$; and

$$\text{# primitive quadratic Dirichlet characters of modulus } \leq N = \sum_{n \leq N} b(n) \sim \frac{6}{\pi^2} N$$

where $b(n)$ is multiplicative with

$$b(2^r) = \begin{cases} 0 & \text{if } r = 1, \\ 1 & \text{if } r = 2, \\ 2 & \text{if } r = 3, \\ 0 & \text{if } r \geq 4, \end{cases} \quad b(p^r) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \geq 2 \end{cases}$$
for prime $p \geq 3$. Is it surprising that the same constant $6/\pi^2$ appears in connection with averages of both $a(n)$ and $b(n)$? No. The underlying structure linking these two sequences is the Möbius inversion formula [1]:

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \quad \text{for } \text{Re}(s) > c \text{ for some } c < 1.$$  

This can be proved directly here. On the one hand [2],

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \left(1 + \frac{1}{2^s} + \frac{2}{2^{2s}} + \sum_{r=3}^{\infty} \frac{4}{2^r s}\right) \prod_{p^2>2} \left(1 + \sum_{r=1}^{\infty} \frac{2}{p^r s}\right) = \left(1 + \frac{1}{2^s} + \frac{2}{2^{2s}} + \frac{4}{2^s - 1} - \frac{4}{2^s} - \frac{4}{2^{2s}}\right) \prod_{p^2>2} \left(1 + \frac{2}{p^s - 1}\right) = \left(1 - \frac{3}{2^s} - \frac{2}{2^{2s}} + \frac{4}{2^s - 1}\right) \prod_{p^2>2} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{1}{p^{2s}}\right) = \left(1 - \frac{3}{2^s} - \frac{2}{2^{2s}} + \frac{4}{2^s - 1}\right) \zeta(s)^2 \left(1 - \frac{1}{2^2s}\right)^{-1} \zeta(2s)^{-1} = \left(1 + \frac{1}{2^2s} + \frac{2}{2^{3s}}\right) \left(1 - \frac{1}{2^s}\right) \zeta(s)^2 \left(1 - \frac{1}{2^{2s}}\right)^{-1} \zeta(2s)^{-1};$$

on the other hand,

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \left(1 + \frac{1}{2^2s} + \frac{2}{2^{3s}}\right) \prod_{p^2>2} \left(1 + \frac{1}{p^s}\right) = \left(1 + \frac{1}{2^2s} + \frac{2}{2^{3s}}\right) \prod_{p^2>2} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{1}{p^{2s}}\right) = \left(1 + \frac{1}{2^2s} + \frac{2}{2^{3s}}\right) \left(1 - \frac{1}{2^s}\right) \zeta(s) \left(1 - \frac{1}{2^{2s}}\right)^{-1} \zeta(2s)^{-1}.$$  

The leading coefficient $6/\pi^2$ arises because

$$\left. \left(1 + \frac{1}{2^2s} + \frac{2}{2^{3s}}\right) \left(1 - \frac{1}{2^s}\right) \left(1 - \frac{1}{2^{2s}}\right)^{-1} \zeta(2s)^{-1} \right|_{s=1} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$  

The Möbius inversion formula is valid, in fact, for arbitrary $\ell$th-order Dirichlet characters. From the preceding case $\ell = 2$, we observe the evaluation of $\sum_{n=1}^{\infty} b(n)n^{-s}$ to be slightly less complicated than that for $\sum_{n=1}^{\infty} a(n)n^{-s}$. Hence, to simplify calculations, our focus will be on $b(n)$ for the cases $\ell = 3, 4, 8$. 
1. Cubic Characters

1.1. General. A Dirichlet character $\chi$ is cubic if $\chi(k)^3 = 1$ for every $k \in \mathbb{Z}_n^*$. We have [3]

$$a(2^r) = 1, \quad a(3^r) = \begin{cases} 1 & \text{if } r = 1, \\ 3 & \text{if } r \geq 2, \end{cases} \quad a(p^r) = \begin{cases} 1 & \text{if } p \equiv 2 \text{ mod } 3, \\ 3 & \text{if } p \equiv 1 \text{ mod } 3, \end{cases}$$

for prime $p \geq 5$ and $r \geq 1$. The asymptotics for $\sum_{n \leq N} a(n)$, as well as the coefficient, were studied in [4] (via a different proof than the following).

1.2. Primitive. We have

$$b(2^r) = 0, \quad b(3^r) = \begin{cases} 2 & \text{if } r = 2, \\ 0 & \text{otherwise}, \end{cases} \quad b(p^r) = \begin{cases} 2 & \text{if } r = 1 \& p \equiv 1 \text{ mod } 3, \\ 0 & \text{otherwise} \end{cases}$$

for prime $p \geq 5$ and $r \geq 1$. The asymptotics for $\sum_{n \leq N} b(n)$, as well as the coefficient, were studied in [5]. We obtain

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \left(1 + \frac{2}{3^{2s}}\right) \prod_{p \equiv 1 \text{ mod } 3} \left(1 + \frac{2}{p^s}\right) \prod_{p \equiv 1 \text{ mod } 3} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{2}{p^{s(p^2+1)}}\right)$$

and it is known from section [4.1] that

$$\prod_{p \equiv 1 \text{ mod } 3} \left(1 - \frac{1}{p^s}\right)^{-2} \sim \zeta(s) \sqrt[3]{\frac{\sqrt{3}}{2\pi}} \prod_{p \equiv 1 \text{ mod } 3} \left(1 - \frac{1}{p^2}\right)^{-1} \quad (1)$$

as $s \to 1$; after cancellation, the coefficient becomes

$$\frac{11}{9} \frac{\sqrt{3}}{2\pi} \prod_{p \equiv 1 \text{ mod } 3} \left(1 - \frac{2}{p(p+1)}\right) = 0.317056516792841205670156... \quad (1)$$

Just as in the case of primitive quadratic characters, the function $\zeta(s)$ appears with exponent 1. Hence by the Selberg-Delange method [4], the average number of primitive cubic characters of modulus $n$ tends to a constant 0.317... as $n \to \infty$.

2. Quartic Characters

2.1. General. A Dirichlet character $\chi$ is quartic (biquadratic) if $\chi(k)^4 = 1$ for every $k \in \mathbb{Z}_n^*$. We have [6]

$$a(2^r) = \begin{cases} 1 & \text{if } r = 1, \\ 2 & \text{if } r = 2, \\ 4 & \text{if } r = 3, \\ 8 & \text{if } r \geq 4, \end{cases} \quad a(p^r) = \begin{cases} 2 & \text{if } p \equiv 3 \text{ mod } 4, \\ 4 & \text{if } p \equiv 1 \text{ mod } 4 \end{cases}$$
for prime $p \geq 3$ and $r \geq 1$.

2.2. Primitive. We have

$$b(2^r) = \begin{cases} 
1 & \text{if } r = 2, \\
2 & \text{if } r = 3, \\
4 & \text{if } r = 4, \\
0 & \text{otherwise},
\end{cases}$$

and

$$b(p^r) = \begin{cases} 
1 & \text{if } r = 1 \& p \equiv 3 \mod 4, \\
3 & \text{if } r = 1 \& p \equiv 1 \mod 4 \\
0 & \text{otherwise}
\end{cases}$$

for prime $p \geq 3$ and $r \geq 1$. We obtain

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \left(1 + \frac{1}{2^{2s}} + \frac{2}{2^{3s}} + \frac{4}{2^{4s}}\right) \prod_{p \equiv 3 \mod 4} \left(1 + \frac{1}{p^s}\right) \cdot \prod_{p \equiv 1 \mod 4} \left(1 + \frac{3}{p^s}\right)$$

$$= \left(1 + \frac{1}{2^{2s}} + \frac{2}{2^{3s}} + \frac{4}{2^{4s}}\right) \prod_{p \equiv 3 \mod 4} \left(1 - \frac{1}{p^s}\right)^{-3} \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{5p^s - 3}{p^{2s}(p^s + 1)}\right)$$

$$= \left(1 + \frac{1}{2^{2s}} + \frac{2}{2^{3s}} + \frac{4}{2^{4s}}\right) \left(1 - \frac{1}{2^{2s}}\right) \zeta(s) \left(1 - \frac{1}{2^{2s}}\right)^{-1} \zeta(2s)^{-1} \cdot \prod_{p \equiv 1 \mod 4} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{5p^s - 3}{p^{2s}(p^s + 1)}\right)$$

and, from section 4.2,

$$\prod_{p \equiv 1 \mod 4} \left(1 - \frac{1}{p^s}\right)^{-2} \sim \zeta(s) \frac{1}{\pi} \prod_{p \equiv 1 \mod 4} \left(1 - \frac{1}{p^2}\right)^{-1} \sim \zeta(s) \frac{\pi}{16K^2} \quad (2)$$

as $s \to 1$; thus the coefficient becomes

$$\frac{7}{\pi 16K^2} \prod_{p \equiv 1 \mod 4} \left(1 - \frac{5p - 3}{p^2(p + 1)}\right) = 0.190876721168528480112237...$$

where $K$ is the Landau-Ramanujan constant [7]. Unlike the preceding cases, the function $\zeta(s)$ appears with exponent 2. Hence by the Selberg-Delange method [4], the average number of primitive quartic characters of modulus $n$ is asymptotically (0.190...) $\ln(n)$ as $n \to \infty$. Is it surprising that the quartic case differs so dramatically from both the quadratic and cubic cases? We believe yes. There is no a priori reason for quartic characters to outnumber quadratic/cubic characters in such a manner.
3. Octic Characters

3.1. General. A Dirichlet character $\chi$ is octic if $\chi(k)^8 = 1$ for every $k$ in $\mathbb{Z}_n^*$. We have

$$a(2^r) = \begin{cases} 1 & \text{if } r = 1, \\ 2 & \text{if } r = 2, \\ 4 & \text{if } r = 3, \\ 8 & \text{if } r = 4, \\ 16 & \text{if } r \geq 5, \end{cases} \quad a(p^r) = \begin{cases} 2 & \text{if } p \equiv 3 \mod 4, \\ 4 & \text{if } p \equiv 5 \mod 8, \\ 8 & \text{if } p \equiv 1 \mod 8 \end{cases}$$

for prime $p \geq 3$ and $r \geq 1$.

3.2. Primitive. We have

$$b(2^r) = \begin{cases} 1 & \text{if } r = 2, \\ 2 & \text{if } r = 3, \\ 4 & \text{if } r = 4, \\ 8 & \text{if } r = 5, \\ 0 & \text{otherwise}, \end{cases} \quad b(p^r) = \begin{cases} 1 & \text{if } r = 1 & \text{and } p \equiv 3 \mod 4, \\ 3 & \text{if } r = 1 & \text{and } p \equiv 5 \mod 8, \\ 7 & \text{if } r = 1 & \text{and } p \equiv 1 \mod 8, \\ 0 & \text{otherwise} \end{cases}$$

for prime $p \geq 3$ and $r \geq 1$. We obtain

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = (1 + \frac{1}{2^s} + \frac{2}{2^s} + \frac{4}{2^s} + \frac{8}{2^s}) \prod_{p \equiv 3 \mod 4} \left(1 + \frac{1}{p^s}\right) \cdot \prod_{p \equiv 5 \mod 8} \left(1 + \frac{3}{p^s}\right) \cdot \prod_{p \equiv 1 \mod 8} \left(1 + \frac{7}{p^s}\right)$$

$$= (1 + \frac{1}{2^s} + \frac{2}{2^s} + \frac{4}{2^s} + \frac{8}{2^s}) \prod_{p \equiv 3 \mod 4} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{1}{p^{2s}}\right) \cdot \prod_{p \equiv 5 \mod 8} \left(1 - \frac{1}{p^s}\right)^{-3} \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{5p^s-3}{p^{2s}(p^s+1)}\right)$$

$$\cdot \prod_{p \equiv 1 \mod 8} \left(1 - \frac{1}{p^s}\right)^{-7} \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{27p^{5s} - 85p^{4s} + 125p^{3s} - 99p^{2s} + 41p^s - 7}{p^{6s}(p^s+1)}\right)$$

$$= (1 + \frac{1}{2^s} + \frac{2}{2^s} + \frac{4}{2^s} + \frac{8}{2^s}) \left(1 - \frac{1}{2^s}\right) \zeta(s) \left(1 - \frac{1}{2^s}\right)^{-1} \zeta(2s)^{-1} \cdot \prod_{p \equiv 5 \mod 8} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{5p^s-3}{p^{2s}(p^s+1)}\right) \cdot \prod_{p \equiv 1 \mod 8} \left(1 - \frac{1}{p^s}\right)^{-6} \left(1 - \frac{27p^{5s} - 85p^{4s} + 125p^{3s} - 99p^{2s} + 41p^s - 7}{p^{6s}(p^s+1)}\right)$$

and, from section [4.3],

$$\prod_{p \equiv 5 \mod 8} \left(1 - \frac{1}{p^s}\right)^{-4} \zeta(s) \frac{1}{2 \ln (1 + \sqrt{2})} \prod_{p \equiv 5 \mod 8} \left(1 - \frac{1}{p^{2s}}\right)^{-2}, \quad (3)$$
\[ \prod_{p \equiv 1 \text{ mod } 8} \left(1 - \frac{1}{p^s}\right)^{-4} \sim \zeta(s) \frac{2 \ln \left(1 + \sqrt{2}\right)}{\pi^2} \prod_{p \equiv 1 \text{ mod } 8} \left(1 - \frac{1}{p^2}\right)^{-2} \]  

(4)
as \( s \to 1 \). An expression for the coefficient becomes clear. More importantly, the function \( \zeta(s) \) appears with exponent \( 1 + 1 / 2 + 3 / 2 = 3 \). Hence by the Selberg-Delange method [4], the average number of primitive octic characters of modulus \( n \) has growth rate \( \approx \ln(n)^2 \) as \( n \to \infty \).

4. **Euler Product Residues**

Formulas (1), (2), (4) await proof, while the truth of (3) depends on both (2) and (4). Our approach uses the seemingly-unrelated method of Shanks & Schmid [8] for computing various generalized Landau-Ramanujan constants \( \kappa_m \).

Fix an integer \( m \neq 0 \). Define

\[ L_d(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{1}{n^s} \]

where \( d = -m \) if \( 4 \mid m \) and \( d = -4m \) otherwise; \( (\cdot/\cdot) \) is the Kronecker-Jacobi-Legendre symbol. Define also

\[ \Lambda_m(s) = \sum_{n=1}^{\infty} f(n) \frac{n}{n^s} \]

where \( f(n) = 1 \) if there exist integers \( x, y \) such that \( n = x^2 + m y^2 \) and \( f(n) = 0 \) otherwise. Let

\[ \kappa_m = \delta_m \sqrt{L_d(1) \frac{2|m|}{\pi} \varphi(2|m|)} \prod_{\text{odd } p: \ (p/m) = -1} \left(1 - \frac{1}{p^2}\right)^{-1/2} \]

where the rational number \( \delta_m \) is unspecified for the moment, and \( \varphi \) denotes the Euler totient function. It turns out that \( \sum_{n \leq N} f(n) \sim \kappa_m N / \sqrt{\ln(N)} \) as \( N \to \infty \), although this fact is not directly relevant to our purposes.

4.1. **Case when \( \ell = 3 \).** Let \( m = 3 \) and \( \delta_m = 2/3 \). It follows that

\[ \kappa_3 = \frac{2}{3} \sqrt{L_{-12}(1) \frac{1}{\pi} \varphi(6)} \prod_{\text{odd } p: \ p \equiv 2 \text{ mod } 3} \left(1 - \frac{1}{p^2}\right)^{-1/2} \]

\[ = \frac{2}{3} \sqrt{\frac{\pi}{2\sqrt{3}}} \frac{1}{2} \prod_{p \equiv 2 \text{ mod } 3} \left(1 - \frac{1}{p^2}\right)^{-1/2} \cdot \left(1 - \frac{1}{2^2}\right)^{1/2} \]

\[ = \frac{1}{\sqrt{2\sqrt{3}}} \prod_{p \equiv 2 \text{ mod } 3} \left(1 - \frac{1}{p^2}\right)^{-1/2} = 2^{1/2} 3^{-7/4} \pi \prod_{p \equiv 1 \text{ mod } 3} \left(1 - \frac{1}{p^2}\right)^{1/2}. \]
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Now,

$$\Lambda_3(s) = (1 - 3^{-s})^{-1} \prod_{p \equiv 1 \text{ mod } 3} (1 - p^{-s})^{-1} \cdot \prod_{p \equiv 2 \text{ mod } 3} (1 - p^{-2s})^{-1}$$

by definition and

$$\Lambda_3(s)^2 = \zeta(s)L_{-3}(s) (1 - 3^{-s})^{-1} \prod_{p \equiv 2 \text{ mod } 3} (1 - p^{-2s})^{-1}$$

by elementary considerations [9]. Dividing the second by the first, we have

$$\Lambda_3(s) = \zeta(s)L_{-3}(s) \prod_{p \equiv 1 \text{ mod } 3} (1 - p^{-s})$$

and thus

$$\Lambda_3(s)(s - 1)^{1/2} = \zeta(s)(s - 1)L_{-3}(s) \prod_{p \equiv 1 \text{ mod } 3} (1 - p^{-s}) \cdot (s - 1)^{-1/2}$$

as $s \to 1$. Hence formula (11) is true.

4.2. Case when $\ell = 4$. Let $m = 1$ and $\delta_m = 1$. It follows that

$$\kappa_1 = 1 \sqrt{L_{-4}(1) \frac{2}{\pi \varphi(2)}} \prod_{\text{odd } p: p \equiv 3 \text{ mod } 4} \left(1 - \frac{1}{p^2}\right)^{-1/2}$$

$$= \sqrt{\frac{\pi}{4}} \frac{1}{\pi} \prod_{p \equiv 3 \text{ mod } 4} \left(1 - \frac{1}{p^2}\right)^{-1/2}$$

$$= \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \text{ mod } 4} \left(1 - \frac{1}{p^2}\right)^{-1/2} = K = \frac{\pi}{4} \prod_{p \equiv 1 \text{ mod } 4} \left(1 - \frac{1}{p^2}\right)^{1/2}.$$  

Now,

$$\Lambda_1(s) = (1 - 2^{-s})^{-1} \prod_{p \equiv 1 \text{ mod } 4} (1 - p^{-s})^{-1} \cdot \prod_{p \equiv 3 \text{ mod } 4} (1 - p^{-2s})^{-1}$$

by definition and

$$\Lambda_1(s)^2 = \zeta(s)L_{-4}(s) (1 - 2^{-s})^{-1} \prod_{p \equiv 3 \text{ mod } 4} (1 - p^{-2s})^{-1}$$
by elementary considerations [10]. Dividing the second by the first, we obtain

\[
\frac{\Lambda_1(s)(s - 1)^{1/2}}{\sqrt{\pi \kappa_1}} = \frac{\zeta(s)(s - 1)}{1} \frac{L_{-4}(s)}{\frac{\pi}{4}} \prod_{p \equiv 1 \mod 4} (1 - p^{-s}) \cdot (s - 1)^{-1/2}
\]

as \( s \to 1 \). Hence formula (2) is true.

4.3. Case when \( \ell = 8 \). The argument here seems fairly roundabout and we wonder if a simpler approach is possible. Two values of \( m \) require consideration here. First, let \( m = 2 \) and \( \delta_m = 1 \). It follows that

\[
\kappa_2 = 1 \frac{L_{-8}(1)}{\pi} \frac{4}{\varphi(4)} \prod_{\text{odd } p: \atop p \equiv 5,7 \mod 8} \left(1 - \frac{1}{p^2}\right)^{-1/2} = \frac{4}{\sqrt{2}} \prod_{p \equiv 5,7 \mod 8} \left(1 - \frac{1}{p^2}\right)^{-1/2} = 2^{-7/4}\pi \prod_{p \equiv 1,3 \mod 8} \left(1 - \frac{1}{p^2}\right)^{1/2}.
\]

As before,

\[
\Lambda_2(s) = (1 - 2^{-s})^{-1} \prod_{p \equiv 1,3 \mod 8} (1 - p^{-s})^{-1} \cdot \prod_{p \equiv 5,7 \mod 8} (1 - p^{-2s})^{-1},
\]

\[
\Lambda_2(s)^2 = \zeta(s)L_{-8}(s) \prod_{p \equiv 5,7 \mod 8} (1 - p^{-2s})^{-1}
\]

and, upon division, we obtain

\[
\frac{\Lambda_2(s)(s - 1)^{1/2}}{\sqrt{\pi \kappa_2}} = \frac{\zeta(s)(s - 1)}{1} \frac{L_{-8}(s)}{\frac{\pi}{2\sqrt{2}}} \prod_{p \equiv 1,3 \mod 8} (1 - p^{-s}) \cdot (s - 1)^{-1/2}.
\]
Second, let \( m = -2 \) and \( \delta_m = 1 \). It follows that

\[
\kappa_{-2} = 1 \sqrt{L_8(1) \frac{1}{\pi \varphi(4)}} \prod_{p \equiv 3,5 \text{ mod } 8} \left( 1 - \frac{1}{p^2} \right)^{-1/2} = \sqrt{2 \ln (1 + \sqrt{2})} \prod_{p \equiv 3,5 \text{ mod } 8} \left( 1 - \frac{1}{p^2} \right)^{-1/2} = 2^{-5/4} \ln \left( 1 + \sqrt{2} \right)^{1/2} \prod_{p \equiv 1,7 \text{ mod } 8} \left( 1 - \frac{1}{p^2} \right)^{1/2}.
\]

As before,

\[
\Lambda_{-2}(s) = (1 - 2^{-s})^{-1} \prod_{p \equiv 1,7 \text{ mod } 8} (1 - p^{-s})^{-1} \cdot \prod_{p \equiv 3,5 \text{ mod } 8} (1 - p^{-2s})^{-1},
\]

\[
\Lambda_{-2}(s)^2 = \zeta(s) L_8(s) (1 - 2^{-s})^{-1} \prod_{p \equiv 3,5 \text{ mod } 8} (1 - p^{-2s})^{-1}
\]

and, upon division, we obtain

\[
\underline{\Lambda_{-2}(s)(s - 1)^{1/2} = \zeta(s)(s - 1) L_8(s) \prod_{p \equiv 1,7 \text{ mod } 8} (1 - p^{-s}) \cdot (s - 1)^{-1/2}}.
\]

Therefore

\[
\lim_{s \to 1} \prod_{p \equiv 1,3 \text{ mod } 8} \left( 1 - \frac{1}{p^s} \right)^{-2} \cdot (s - 1) = \frac{\pi}{8 \kappa_{-2}^2} = \frac{2^{1/2}}{\pi} \prod_{p \equiv 1,3 \text{ mod } 8} \left( 1 - \frac{1}{p^2} \right)^{-1},
\]

\[
\lim_{s \to 1} \prod_{p \equiv 1,7 \text{ mod } 8} \left( 1 - \frac{1}{p^s} \right)^{-2} \cdot (s - 1) = \frac{\ln (1 + \sqrt{2})^2 \cdot 1}{2\pi} \kappa_{-2}^2 = \frac{2^{3/2} \ln (1 + \sqrt{2})}{\pi^2} \prod_{p \equiv 1,7 \text{ mod } 8} \left( 1 - \frac{1}{p^2} \right)^{-1},
\]

but these alone do not go far enough. A slightly revised formula (2):

\[
\lim_{s \to 1} \prod_{p \equiv 1,5 \text{ mod } 8} \left( 1 - \frac{1}{p^s} \right)^{-2} \cdot (s - 1) = \frac{\pi}{16 \kappa_1^2} = \frac{1}{\pi} \prod_{p \equiv 1,5 \text{ mod } 8} \left( 1 - \frac{1}{p^2} \right)^{-1},
\]
when coupled with the preceding two limits, make possible the isolation of \( p \equiv 1 \mod 8 \) as follows:

\[
\frac{4}{\pi^4} \ln \left(1 + \sqrt{2}\right) \prod_{1,3,5,7} (1 - p^{-2})^{-1} \cdot \prod_1 (1 - p^{-2})^{-2} = 2^{1/2} \pi \prod_{1,3} (1 - p^{-2})^{-1} \cdot \frac{2^{3/2} \ln \left(1 + \sqrt{2}\right)}{\pi^2} \prod_{1,7} (1 - p^{-2})^{-1} = \lim_{s \to 1} \left( \prod_{1,3} (1 - p^{-s})^{-2} \cdot (s - 1) \right) \left( \prod_{1,5} (1 - p^{-s})^{-2} \cdot (s - 1) \right) \left( \prod_{1,7} (1 - p^{-s})^{-2} \cdot (s - 1) \right) = \lim_{s \to 1} \left( \prod_{1,3,5,7} (1 - p^{-s})^{-2} \cdot (s - 1)^2 \right) \cdot \lim_{s \to 1} \left( \prod_1 (1 - p^{-s})^{-4} \cdot (s - 1) \right).
\]

Hence formula (1) is true.

5. Unanswered Questions

For the cases \( \ell = 5, 6, 7 \), the function \( \zeta(s) \) appears with exponents 1, 3, 1 respectively and [5]

\[
\kappa_{-3} = \frac{1}{2} \sqrt{\frac{3 \ln (2 + \sqrt{3})}{\pi}} \prod_{p \equiv 5,7 \mod 12} \left(1 - \frac{1}{p^2}\right)^{-1/2}, \quad \kappa_5 = \frac{1}{2} \sqrt{\frac{3 \ln (2 + \sqrt{3})}{\pi}} \prod_{p \equiv 11,13,17,19 \mod 20} \left(1 - \frac{1}{p^2}\right)^{-1/2}, \quad \kappa_6 = \frac{1}{2} \sqrt{\frac{3 \ln (2 + \sqrt{3})}{\pi}} \prod_{p \equiv 13,17,19,23 \mod 24} \left(1 - \frac{1}{p^2}\right)^{-1/2}, \quad \kappa_7 = \frac{1}{2} \sqrt{\frac{3 \ln (2 + \sqrt{3})}{\pi}} \prod_{p \equiv 5,11,13,17 \mod 28} \left(1 - \frac{1}{p^2}\right)^{-1/2}.
\]

It does not seem to be possible to isolate \( p \equiv 1 \mod \ell \) beyond the following partial results:

\[
\lim_{s \to 1} \prod_{p \equiv 1,4 \mod 5} \left(1 - \frac{1}{p^s}\right)^{-2} \cdot (s - 1) = \frac{\ln \left(9 + 4 \sqrt{5}\right)}{4 \pi} \kappa_{-5} = \frac{3 \sqrt{7}}{4 \pi} \prod_{p \equiv 1,4 \mod 5} \left(1 - \frac{1}{p^2}\right)^{-1}, \quad \lim_{s \to 1} \prod_{p \equiv 1,2,4 \mod 7} \left(1 - \frac{1}{p^s}\right)^{-2} \cdot (s - 1) = \frac{9 \pi}{112 \pi \kappa^2_7} \prod_{p \equiv 1,2,4 \mod 7} \left(1 - \frac{1}{p^2}\right)^{-1}.
\]
for \( \ell = 5, 7 \), which are deduced from

\[
\Lambda_{-5}(s) = (1 - 5^{-s})^{-1} \prod_{p \equiv 1, 4 \mod 5} (1 - p^{-s})^{-1} \cdot \prod_{p \equiv 2, 3 \mod 5} (1 - p^{-2s})^{-1},
\]

\[
\Lambda_{-5}(s)^2 = \zeta(s)L_{20}(s) (1 - 5^{-s})^{-1} \prod_{p \equiv 2, 3 \mod 5} (1 - p^{-2s})^{-1} \cdot (1 + 2^{-s})^{-1}
\]

and

\[
\Lambda_{7}(s) = (1 - 7^{-s})^{-1} \prod_{2 < p \equiv 1, 2, 4 \mod 7} (1 - p^{-s})^{-1} \cdot \prod_{p \equiv 3, 5, 6 \mod 7} (1 - p^{-2s})^{-1} \cdot \left[ (1 - 2^{-s})^{-1} - 2^{-s} \right],
\]

\[
\Lambda_{7}(s)^2 = \zeta(s)L_{-28}(s) (1 - 7^{-s})^{-1} \cdot \prod_{p \equiv 3, 5, 6 \mod 7} (1 - p^{-2s})^{-1} \cdot \left[ (1 - 2^{-s})^{-1} - 2^{-s} \right]^2 (1 - 2^{-s})
\]

as before. The L-series \( \Lambda_{-3}(s), \Lambda_{5}(s), \Lambda_{6}(s), \Lambda_{-6}(s), \Lambda_{-7}(s) \) might be difficult to study, due to a failure of \( f(n) \)-multiplicativity. When \( m = 6 \), for example, \( f(10) = 1 \) since \( 10 = 2^2 + 6 \cdot 1^2 \), but \( f(2) = f(5) = 0 \). When \( m = -6 \), as another example, \( f(10) = 1 \) since \( 10 = 4^2 - 6 \cdot 1^2 \), but again \( f(2) = f(5) = 0 \). Although the asymptotics of \( \sum_{n \leq N} b(n) \) for \( \ell = 5, 6, 7 \) are understood, expressions for leading coefficients (analogous to those for \( \ell = 3, 4, 8 \)) remain open.

6. Acknowledgement

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