Locally-Scalar
Representations of Graphs
in the Category of Hilbert Spaces

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Preface

In article [1] authors wrote: “Recently it became evident that a number of problems of linear
algebra allows common formulating and in this formulating common effective methods of inves-
tigations of such problems appear. It is interesting that these methods appear to be connected
with such concepts as Coxeter–Weyl group and Dynkin schemes.”

One of these problems was a problem of representations of quivers [2] (see also [3–5]). Ac-
cording to the Gabriel theorem a connected quiver has finite representative type if and only if
con相当ing nonoriented graph is a Dynkin scheme. Coxeter functors defined in [1] served to
explain this fact (which was obtained by P. Gabriel a posteriori).

Further the theory of representations of quivers (in the category of finite-dimensional vector
spaces) was widely developed [6–10].

One may consider representations of quivers also in metric and, in particular, Hilbert spaces
([11–13]). But if we use suggesting themselves natural definitions, the problem of classification
of representations of quivers (excluding a quiver each connected component of which consists
of one vertex or of two vertices and and arrow between them) in such categories will become
“wild” (in terms of [14], i. e. including a problem of unitary classification of arbitrary linear
operator or a pair of self-adjoint operators).

In the following we suggest a limitation of local scalarity to the representations of a graph
(or a quiver) in the category of Hilbert spaces, and after that a theory close to [1–10] arise,
though specific for Hilbert spaces aspects arise.

Note that in some particular case a problem of classification of locally-scalar representations
in other terms studied in [16–17] (see example 3).

In §§ 2, 3 for the category of locally-scalar representations of graph in the category of
Hilbert spaces functors of even and odd Coxeter reflections are defined. In § 4 with their help
an analogue of Gabriel theorem is proved.

1 Basic definitions

Let ℋ will be the category of Hilbert spaces, which objects are separable Hilbert spaces (finite-
dimensional or infinite-dimensional) and morphisms are bounded operators. To each ϕ ∈
$\mathcal{H}(A,B)$ uniquely corresponds $\varphi^* \in \mathcal{H}(B,A)$.

According to Gabriel [2], we will call oriented graphs as quivers, leaving the term “graph” for nonoriented graphs. In both cases we will admit loops, multiple arrows and edges.

Thus, a quiver $Q$ consists of two sets $Q_v$ (vertices) and $Q_a$ (arrows) and two maps $t$ and $h$ from $Q_a$ to $Q_v$, associated each arrow $\alpha \in Q_a$ with its tail $t(\alpha)$ and head $h(\alpha)$.

Graph $G$ consists of sets $G_v$ (vertices) and $G_e$ (edges) and a map $\varepsilon$ from $G_e$ to set of one- and two-element subsets $G_v$. To each quiver $Q$ in a natural way its graph $G = G(Q)$ corresponds.

Representation $\pi$ of a quiver $Q$ in an arbitrary category $K$ associates to each vertex $a \in Q_v$ an object $\pi(a) \in K$ and to each arrow $\alpha \in Q_a$ a morphism $\pi(\alpha) \in K(A,B)$, where $A = \pi(t(\alpha))$, $B = \pi(h(\alpha))$.

In $[1–10]$ $K = \text{mod}\, k$, where $k$ is a field. Representations of a quiver $Q$ over field $k$ form a category $\text{Rep}(Q,k)$.

If $\pi$ is a representation of $Q$ in $\mathcal{H}$ then to each arrow $\alpha : a \to b$, besides operator $\pi(\alpha) : A \to B$, an operator $\pi^*(\alpha) : B \to A$ corresponds. Thus, in the category $\mathcal{H}$ it is naturally to define a representation $\pi$ of a graph $G = G(Q)$ which associates to each arrow $a \in G_v$ an object $\pi(a) = H_a \in \mathcal{H}$, to each edge $\gamma \in G_e$ with $\{i,j\} = \varepsilon(\gamma)$ when $i \neq j$ a pair of interadjoint linear operators $\pi(\gamma) = \{\Gamma_{ij}, \tilde{\Gamma}_{ji}\}$ where $\Gamma_{ij} : H_j \to H_i$, and when $\varepsilon(\gamma) = \{i\}$ a pair of interadjoint operators $\pi(\gamma) = \{\Gamma_{ii}, \Gamma_{ii}^\ast\}$.

Consequently, we can identify representations of a quiver $Q$ and of a graph $G = G(Q)$ in the category $\mathcal{H}$. We will prefer in this paper to consider in Hilbert spaces representations of graphs, but all results can be naturally restated for quivers.

Representation $\pi$ of a graph $G$ is said to be finite-dimensional, if $\dim \pi(i) < \infty$ for all $i \in G_v$.

It is known that graph $G_0$, consisting of one loop, has infinite-dimensional indecomposable representations (in $\text{Rep}(G_0, \mathcal{H})$).

We will call representation $\pi$ of a graph $G$ discrete, if it decompose to direct sum (finite or infinite) of finite-dimensional representations.

Let us construct a category $\text{Rep}(G, \mathcal{H})$ of representations $G$ in $\mathcal{H}$.

Morphism $\pi : \to \tilde{\pi}$ is a family $\{C_i\}_{i \in G_v}$ of operators $C_i : \pi(i) \to \tilde{\pi}(i)$ such that diagrams

\[
\begin{array}{ccc}
H_i & \xrightarrow{\Gamma_{ji}} & H_j \\
\downarrow C_i & & \downarrow C_j \\
\tilde{H}_i & \xrightarrow{\tilde{\Gamma}_{ji}} & \tilde{H}_j
\end{array}
\]

are commutative, i. e.

\[
C_j \Gamma_{ji} = \tilde{\Gamma}_{ji} C_i. \tag{1}
\]

It can be shown that operators $C_i$ realizing an equivalence of two representations can be chosen as unitary (see, for instance, [15]), i. e. equivalent objects of the category $\text{Rep}(G, \mathcal{H})$ are unitary equivalent.

Support $G^n_v = G^n$ of a representation $\pi$ is a set $\{i \in G_v \mid \pi(i) \neq 0\}$.

Denote $M_i$ a set of vertices connected with vertex $i$ by edge; $\overline{M}_i = \{\gamma \in G_e \mid i \in \varepsilon(\gamma)\}$; if $X \subset G_v$ then $M(X) = \left( \bigcup_{i \in X} M_i \right) \setminus X$.

Representation $\pi$ faithful if $G^n_v = G_v$. 

2
Let $\pi \in \text{Rep} (G, \mathcal{H})$. Let $A(\gamma, i) = \Gamma^*_{ji} \Gamma_{ji}$ when $\varepsilon(\gamma) = \{i, j\}$, and $A(\gamma, i) = \Gamma^*_{ii} \Gamma_{ii} + \Gamma_{ii} \Gamma^*_{ii}$ when $\varepsilon(\gamma) = \{i\}$, and

$$A_i = \sum_{\gamma \in M_i} A(\gamma, i).$$

$A_i$ is a selfadjoint operator in space $H_i = \pi(i)$. If $M_i = \emptyset$ we will consider $A_i = 0$. Note that if $G$ does not consist loops and multiple edges then

$$A_i = \sum_{j \in M_i} \Gamma_{ij} \Gamma_{ji}.$$

A representation $\pi$ will be named as \emph{locally-scalar} if all operators $A_i$ are scalar, $A_i = \alpha_i I_{H_i}$, where $I_{H_i}$ is identical operator in space $H_i$; since $A_i$ are positive operators, $\alpha_i \geq 0$.

If $\pi(j) = 0$ for all $j \in M_i$ and $\pi(i) \neq 0$, then, obviously, $\alpha_i = 0$. We denote $\text{Rep} (G)$ full subcategory in $\text{Rep} (G, \mathcal{H})$, which objects are locally-scalar.

Denote by $V_G$ a linear real space, which consists of the collections $x = (x_i)$ of real numbers $x_i (i \in G_v)$; elements $x$ from $V_G$ we will call $G$-vectors. Vector $x = (x_i)$ we will call positive ($x > 0$) if $x \neq 0$ and $x_i \geq 0$ for $i \in G_v$. Let us denote $V^+_G = \{x \in V_G \mid x > 0\}$. Any function $f$ on $G_v$ with the real values can be identified with corresponding $G$-vector $(f(i))_{i \in G_v}$. $Z_G = \{d : G_v \to \mathbb{N}_0\}$, $Z^+_G = Z_G \cap V^+_G$.

$G$-vector $d(i) = \dim \pi(i)$ is \emph{dimension} of a finite-dimensional representation $\pi \in \text{Rep} (G, \mathcal{H})$; if $A_i = f(i) I_{H_i}$ for $i \in G_v$, then $f(i)$ we will call as \emph{character} of locally-scalar representation $\pi$, and $\pi$ will be $f$-representation. Character is determined uniquely on the support $G^\pi$ of the representation $\pi$ (and ambiguously determined outside of the support). If $G_v = G^\pi$ then character of the representation is determined uniquely and is denoted by $f_\pi$. In a common case let us denote by $\{f_\pi\}$ the set of characters of the representation $\pi$.

For graphs there may exist both infinite-dimensional and finite-dimensional indecomposable representations and the dimensions of the latter may not be bounded. At the same time, even in the elementary cases, there exist infinitely many indecomposable representations in the fixed dimension corresponding to different characters.

We will say that $G$ is (locally-scalar) \emph{finitely representable} in $\mathcal{H}$, if all its locally-scalar representation are discrete, dimensions of its indecomposable locally-scalar representations are bounded in the whole and in each dimension the number of indecomposable representations with given character is finite.

In what follows we will prove that connected finite graph is finitely representable if and only if it is Dynkin scheme $(A_n, D_n, E_6, E_7, E_8)$, and in this case its indecomposable representations are uniquely determined by dimension and a value of character defined on the support of the representation.

**Example 1.** An arbitrary graph $G$ has a locally-scalar representation $\pi$ in the dimension $d(i) \equiv 1$. Let $\pi(i) = \mathbb{C} e_i$, $(e_i, e_i) = 1$ and $\pi(\gamma_{ij}) = \{\Gamma_{ji}, \Gamma_{ij}\}$, where $\Gamma_{ji} e_i = e_j$ and $\Gamma_{ij} e_j = e_i$ ($\varepsilon(\gamma_{ij}) = \{i, j\}$). Then character $f(i) = |M_i|$. $\pi$ indecomposable if and only if $G$ is connected.

**Example 2.** $G_0$ consists of one vertex $a$ and one loop $\gamma$. Let $\pi$ be a locally-scalar representation of a graph $G_0$, for which $\pi(\gamma) = \{\Gamma, \Gamma^*\}$, $f(a) = \alpha$, i. e. $\Gamma \Gamma^* + \Gamma^* \Gamma = \alpha I_H$. In [14] it was shown that all indecomposable locally-scalar representations no more than two-dimensional; given fixed positive $\alpha$ in the dimension 2 indecomposable representations depend on two continuous parameters, so graph $G_0$ is not finitely representable.
Let us instance nondiscrete representation of the graph $G_0$ which is decomposable, though it can not be decomposed into direct sum of the indecomposable representations (but only into their integral). Consider a Hilbert space $H$ with (orthonormal) basis $\{e_i\}$, $i \in \mathbb{Z}$, and an operator $\Gamma(e_i) = e_{i+1}$ in it ($\Gamma^*\Gamma + \Gamma\Gamma^* = 2I_H$). Hence $H$ does not contain finite-dimensional invariant subspaces.

Example 3. A problem of (unitary) classification of locally-scalar representations of the graph $G_n$ was considered in fact (in other terms) in the works [16–19]. In [16–17] it was studied a problem of classification up to unitary equivalence of collections of orthoprojectors $P_1, P_2, \ldots, P_n$ (in the separable Hilbert space $H$) such that $\sum_{k=1}^n P_k = \alpha I_H$ and, in particular, a problem of description of the set $\Sigma_n$ of that real numbers $\alpha$, for which there exists at least one collection of such orthoprojectors (in the nonzero space).

Corresponding to the orthoprojector $P_i$ a space $H_i = \text{Im} P_i$ and a natural enclosure (isometry) $\Gamma_i : H_i \to H$ we will get a locally-scalar representation $\pi$ of the graph $G_n$ if we put $\pi(i) = H_i$, $\pi(0) = H$, $\pi(\gamma_i, \Gamma_i^*)$. At that the character $f(i) = 1$ at $i \neq 0$ and $f(0) = \alpha (\Gamma_i^*\Gamma_i = I_{H_i}, P_i = \Gamma_i\Gamma_i^*$ and $\sum_{k=1}^n \Gamma_k\Gamma_k^* = \alpha I_H$).

In [18–19] for representations of the graph $G_n$ were constructed functors which structure and role during the description of representations is the same as one of Coxeter functors in [1]. Using these functors, in [17] full description of the set $\Sigma_n$ and new results concerning the collections of orthoprojectors were obtained.

As it was mentioned above, the graph $G_0$ (a loop) has infinite-dimensional indecomposable representations, but they are not locally-scalar. As it follows from [17], the graph $G_n$ has infinite-dimensional indecomposable locally-scalar representations if and only if $n > 4$.

2 Functors of reflections

Example 2 implies if $G$ contains a cycle then $G$ is not finitely representable in $\mathcal{H}$. We will study a question of finite representability and thus we restrict ourselves to graphs not containing cycles: everywhere below $G$ is a finite connected graph without cycles (a wood).

Let us fix a decomposition of the set $G_v$ as $\tilde{G} \bigsqcup \tilde{G} (\text{univocal up to permutation } \tilde{G} \text{ and } \tilde{G})$ such that for each $\alpha \in G_e$ one of the vertices from $\varepsilon(\alpha)$ is situated in $\tilde{G}$ and the other in $\tilde{G}$. Vertexes of the set $\tilde{G}$ is said to be even and of $\tilde{G}$ — odd.

For each vertex $i \in G_v$ denote as $\sigma_i$ a linear transformation in the space $V_G$ which is defined by formulas $(\sigma_i x)_j = x_j$ with $j \neq i$, $(\sigma_i x)_i = -x_i + \sum_{j \in M_i} x_j$. We will call $\sigma_i$ a reflection in the vertex $i$. $W$ is a group of transformations of the space $V_G$ generated by the reflections $\sigma_i$.

Let us fix a numeration of the vertices of the graph $G$, numbering first odd and then even vertices. The product of the reflections in all odd vertices we denote as $\tilde{e}$, and in the even —
as \( \overset{\circ}{c} \) (remark that reflections in odd (even) vertices commute). Coxeter transformation on \( V_G \)

is \( c = \overset{\circ}{c} c \); \( c^{-1} = \overset{\circ}{c} \). \( \overset{\circ}{c} \) are said to be odd (even) Coxeter transformations.

Let us provide a denotation for the composition of Coxeter transformations: 

\[
\overset{\circ}{c}_k = \cdots \overset{\circ}{c} \overset{\circ}{c}, \quad k \text{ times}
\]

Let \( X \subset G_\circ \). \( \text{Rep}_o (G, X) \) — full subcategory of locally-scalar representations \( \pi \) in the category \( \text{Rep} (G) \) for which \( G^\pi = X \) and the character is positive on \( G^\pi \cap \overset{\circ}{\delta} \). Let \( (M(X))^\circ = M(X) \cap \overset{\circ}{G} \), \( \delta : (M(X))^\circ \to \mathbb{R}^+ \). A case when \( (M(X))^\circ = \emptyset \) is not excluded.

Let us construct a functor

\[
\overset{\circ}{F}_{X, \delta} : \text{Rep}_o (G, X) \to \text{Rep} (G).
\]

In what follows implies that local scalarity of representations is essential for constructing functor \( \overset{\circ}{F}_{X, \delta} \) (if \( (M(X))^\circ = \emptyset \), we will denote it \( \overset{\circ}{F}_X \), and if \( X = G_v \) then \( \overset{\circ}{F}_G \)) and under action of \( \overset{\circ}{F}_{X, \delta} \) locally-scalar representation \( \pi \) turn to locally-scalar again.

If Hilbert spaces \( H \) and \( \hat{H} \) are decomposed into orthogonal sum of subspaces: 

\[
H = H^{(1)} \oplus H^{(2)} \oplus \cdots \oplus H^{(m)}, \quad \hat{H} = \hat{H}^{(1)} \oplus \hat{H}^{(2)} \oplus \cdots \oplus \hat{H}^{(m)}
\]

then arbitrary linear operator \( A : H \to \hat{H} \) can be written as a matrix \( A = [A_{ij}]_{i,j=1}^{m,n} \) where operators \( A_{ij} \) act from the space \( H^{(j)} \) to the space \( \hat{H}^{(i)} \). Such matrices multiplies according to usual rules of multiplying the block matrices.

Let \( \pi \in \text{Ob Rep}_o (G, X) \) and \( i \in X^\circ \cup (M(X))^\circ \). Let us fix \( f \in \{f_\pi\} \) considering \( f(j) = \delta(j) \) with \( j \in (M(X))^\circ \).

Conditions of local scalarity in the vertex \( i \) imply that operator

\[
\Gamma_i^* = \left[ \frac{1}{\sqrt{\alpha_i}} \Gamma_{ig_1}, \frac{1}{\sqrt{\alpha_i}} \Gamma_{ig_2}, \cdots \frac{1}{\sqrt{\alpha_i}} \Gamma_{ig_k} \right],
\]

acting from the space \( H^{(i)} = H_{g_1} \oplus H_{g_2} \oplus \cdots \oplus H_{g_k} \) (the sum is orthogonal, \( \alpha_i = f(i) \), \( \{g_1, g_2, \ldots, g_k\} = M_i \)) to the space \( H_i \), has a property \( \Gamma_i^* \Gamma_i = I_{H_i} \) (i. e. operator \( \Gamma_i \) is isometry from the space \( H_i \) to the space \( H^{(i)} \)). Let \( \hat{H}_i \) — orthogonal supplement to \( \text{Im} \Gamma_i \). Then, if \( \Delta_i \) is a natural enclosure \( \hat{H}_i \) to \( H^{(i)} \) and

\[
\Delta_i^* = \left[ \frac{1}{\sqrt{\alpha_i}} \Delta_{ig_1}, \frac{1}{\sqrt{\alpha_i}} \Delta_{ig_2}, \cdots \frac{1}{\sqrt{\alpha_i}} \Delta_{ig_k} \right],
\]

then operator

\[
U_i^* = \left[ \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}} \Gamma_{ig_1}, \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}} \Gamma_{ig_2}, \cdots \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_i}} \Gamma_{ig_k} \right]
\]

(2)

is a unitary operator, \( U_i^* : H^{(i)} \to H_i \oplus \hat{H}_i \), \( U_i^* U_i = I_{H_i} \oplus \hat{H}_i \), \( U_i U_i^* = I_{H^{(i)}} \).

Consider operators \( \Delta_{g_1,i} : \hat{H}_i \to H_{g_1} \) and \( \Delta_{i,g_1} = \Delta_{g_1,i}^* : H_{g_1} \to \hat{H}_i \).

For \( i \in X^\circ \cup (M(X))^\circ \) let \( \overset{\circ}{\pi} (i) = \hat{H}_i \) and \( \overset{\circ}{\pi} (j) = H_j \) for other vertices \( j \). \( \overset{\circ}{\pi} (\gamma_{ij}) = \{\Delta_{ji}, \Delta_{ij}\} \)

for \( i \in X^\circ \cup (M(X))^\circ \), \( j \in M_i \) and \( \overset{\circ}{\pi} (\gamma_{kl}) = \pi (\gamma_{kl}) \) for other edges. As a result we obtain representation \( \overset{\circ}{\pi} \in \text{Ob Rep} (G, H) \).
Let us show that the representation \( \hat{\pi} \) will be locally-scalar.

Since \( U_i \) are unitary operators then
\[
\sum_{j \in M_i} \Delta_{ij} \Delta_{ji} = \alpha_i I_{H_i} \quad \text{and} \quad \Gamma_{ji} \Gamma_{ij} + \Delta_{ji} \Delta_{ij} = \alpha_i I_{H_j}
\]
with \( i \in X^0 \bigcup (M(X))^0, \ j \in M_j \).

Statings (3) for vertices \( j \in X^* = X \cap \mathcal{G} \) imply
\[
\sum_{i \in M_j} \Delta_{ji} \Delta_{ij} = ( -\alpha_j + \sum_{i \in M_j} \alpha_i ) I_{H_j}
\]

Thus, if we assume that values of \( \hat{\pi} \) matches with values of \( f \) in vertices \( i \in X^0 \), and for \( j \in X^* \) \( \hat{f}(j) \) is equal to the sum of values of \( f \) in the vertices adjacent to \( j \) minus value of \( f \) in the vertex \( j \), then \( \hat{f} \in \{ f_\hat{\pi} \} \).

Let \( \hat{F}_{X,\delta} (\pi) = \hat{\pi} \). Note that with different \( \delta \) we will obtain different locally-scalar representation.

Define action of \( \hat{F}_{X,\delta} \) on the morphisms \( \text{Rep}_o(G, X) \). Let \( \pi, \ \pi' \in \text{Ob} \text{Rep}_o(G, X) \). \( C = \{ C_k \}_{k \in G} : \pi \rightarrow \pi' \) — morphism in \( \text{Rep}_o(G, X) \). If \( j \not\in X^0 \bigcup (M(X))^0 \) let \( \tilde{C}_j = C_j \).

Let \( i \in (M(X))^0 \bigcup X^0 \) and \( \alpha_i = \hat{f}(i) > 0, \ \tilde{\alpha}_i = \hat{f}(i) > 0 \). Equalities (4) for \( j \in M_i \) imply
\[
\sum_{j \in M_i} \Delta_{ij} \Delta_{ji} = \sum_{j \in M_i} \tilde{\Delta}_{ij} \tilde{\Delta}_{ji} = ( \sum_{j \in M_i} C_i \Delta_{ji} ) = ( \sum_{j \in M_i} \tilde{C}_i \tilde{\Delta}_{ji} )
\]

On the other hand (5) for \( j \in M_i \) implies
\[
\sum_{j \in M_i} \Delta_{ij} \Delta_{ji} = \sum_{j \in M_i} \tilde{\Delta}_{ij} \tilde{\Delta}_{ji} = ( \sum_{j \in M_i} \tilde{C}_i ) C_i = \tilde{\alpha}_i C_i \quad \text{and}
\]
\[
\tilde{C}_i = \frac{1}{\tilde{\alpha}_i} \sum_{j \in M_i} \tilde{\Delta}_{ij} \tilde{\Delta}_{ji}.
\]

Thus, if \( \alpha_i \neq \tilde{\alpha}_i \), \( C_i = 0 \).

Let if \( \alpha_i \neq \tilde{\alpha}_i \), \( C_i = 0 \) and if \( \alpha_i = \tilde{\alpha}_i \)
\[
\hat{C}_i = \frac{1}{\alpha_i} \sum_{j \in M_i} \hat{\Delta}_{ij} \hat{\Delta}_{ji}.
\]

Let us show that the diagram is commutative
\[
\begin{array}{ccc}
\hat{H}_i & \xrightarrow{\hat{\Delta}_{ji}} & \hat{H}_j = H_j \\
\hat{C}_i \downarrow & & \downarrow \hat{\alpha}_j = C_j \\
\hat{H}_i & \xrightarrow{\hat{\Delta}_{ji}} & \hat{H}_j = H_j
\end{array}
\]

where \( j \in M_i, \ i \in \mathcal{G} \), i. e. \( C_j \hat{\Delta}_{ji} = \hat{\Delta}_{ji} \hat{C}_i \).
From (7) follows

\[ \Delta_{ji} \circ \tilde{C}_i = \Delta_{ji} \cdot \frac{1}{\alpha_i} \sum_{k \in M_i \atop k \neq j} \Delta_{ik} C_k \Delta_{ki} \]

From the unitarity of the operator \( U_i^* \) (written in form (2)) and, analogously, of the operator \( \tilde{U}_i^* \) it is follows that

\[ \tilde{\Gamma}_{ji} \tilde{\Gamma}_{ik} + \tilde{\Delta}_{ji} \tilde{\Delta}_{ik} = 0 \text{ when } j \neq k \text{ and } \tilde{\Gamma}_{ji} \tilde{\Gamma}_{ij} + \tilde{\Delta}_{ji} \tilde{\Delta}_{ij} = \alpha_i \tilde{H}_{ij}, \]

and, consequently,

\[ \tilde{\Delta}_{ji} \circ \tilde{C}_i = \frac{1}{\alpha_i} \sum_{k \in M_i \atop k \neq j} \left[ -\tilde{\Gamma}_{ji} \tilde{\Gamma}_{ik} C_k \Delta_{ki} + \left( I_{\tilde{H}_{ij}} - \frac{1}{\alpha_i} \tilde{\Gamma}_{ji} \tilde{\Gamma}_{ij} \right) \cdot C_j \Delta_{ji} - \frac{1}{\alpha_i} \sum_{k \in M_i \atop k \neq j} \tilde{\Gamma}_{ji} \tilde{\Gamma}_{ik} C_k \Delta_{ki} - \frac{1}{\alpha_i} \tilde{\Gamma}_{ji} \tilde{\Gamma}_{ij} C_j \Delta_{ji} \right]. \]

Because of the statings (1) we obtain that

\[ \tilde{\Delta}_{ji} \circ \tilde{C}_i = C_j \Delta_{ji} \]

and the diagram (8) is commutative.

The last sum equals 0 because of the orthogonality of the block-rows of the unitary operator (2). Therefore, \( \tilde{\Delta}_{ji} \circ \tilde{C}_i = C_j \Delta_{ji} \)

Commutativity of the dual diagram can be checked analogously, consequently \( \hat{C} = \{ \hat{C}_k \}_{k \in G_v} \) is really a morphism of the category \( \text{Rep}(G, \mathcal{H}) \).

Let us show that \( \hat{F}_{X, \delta} \) retains products of morphisms and identical morphisms. Let \( C = \{ C_k \} : \pi \to \tilde{\pi} \) and \( D = \{ D_k \} : \tilde{\pi} \to \tilde{\pi} \) are morphisms in the category \( \text{Rep}_o (G, X) \). Commutativity of the diagrams

\[
\begin{array}{ccc} 
\tilde{H}_i & \xrightarrow{\Delta_{ji}} & \tilde{H}_j \\
\downarrow \hat{C}_i & & \downarrow \hat{C}_i \\
\tilde{H}_i & \xrightarrow{\tilde{\Delta}_{ji}} & \tilde{H}_j
\end{array}
\begin{array}{ccc} 
\tilde{H}_i & \xrightarrow{\Delta_{ij}} & \tilde{H}_j \\
\downarrow \hat{C}_i & & \downarrow \hat{C}_i \\
\tilde{H}_i & \xrightarrow{\tilde{\Delta}_{ij}} & \tilde{H}_j
\end{array}
\]

imply commutativity of the diagram

\[
\begin{array}{ccc} 
\tilde{H}_i & \longrightarrow & \tilde{H}_j \\
\downarrow \hat{D}_i \hat{C}_i & & \downarrow \hat{D}_j \hat{C}_j \\
\tilde{H}_i & \longrightarrow & \tilde{H}_j
\end{array}
\begin{array}{ccc} 
\tilde{H}_i & \longrightarrow & \tilde{H}_j \\
\downarrow \hat{D}_i \hat{C}_i & & \downarrow \hat{D}_j \hat{C}_j \\
\tilde{H}_i & \longrightarrow & \tilde{H}_j
\end{array}
\]

Thus, \( \hat{D}_i \hat{C}_i = \frac{1}{\alpha_i} \sum_{j \in M_i} \tilde{\Delta}_{ij} D_j C_j \Delta_{ji} = (D_i \hat{C}_i), \) i. e. \( \hat{F}_{X, \delta} (DC) = \hat{F}_{X, \delta} (D) \cdot \hat{F}_{X, \delta} (C). \)
If $C = \{I_{H_k}\}_{k \in G_v}$ then with $j \not\in (M(X))^c \cup X^c \overset{\circ}{C}_{j} = C_j = I_{H_j}$, and with $i \in (M(X))^c \cup X^c \overset{\circ}{C}_i = \frac{1}{a_i} \sum_{j \in M_i} \Delta_{ij} I_{H_j} \Delta_{ji} = I_{H_i}$, i.e. $\overset{\circ}{F}_{X,\delta}(\{I_{H_k}\}) = \{I_{H_k}\}$.

So, we have constructed a functor $\overset{\circ}{F}_{X,\delta}$.

Functor

$$
\overset{\circ}{F}_{X,\delta} : \text{Rep}_*(G, X) \rightarrow \text{Rep}_*(G).
$$

is constructed in analogous way.

Let now $\pi$ — finite-dimensional representation of the graph $G$. It is easy to count that under the action of the functor $\overset{\circ}{F}_{X,\delta}$ dimensional function $d(i)$ changes in the following way: representation $\overset{\circ}{F}_{X,\delta}(\pi)$ has dimension $\overset{\circ}{c}(d)$, the character of representation $\overset{\circ}{F}_G(\pi)$ is $\overset{\circ}{c}(f)$.

$$
\begin{align*}
\overset{\circ}{c}(d)(i) &= \begin{cases} 
-d(i) + \sum_{j \in M_i} d(j) & \text{ when } i \in \overset{\circ}{G} \\
 d(i) & \text{ when } i \in \overset{\circ}{G} 
\end{cases} \\
\overset{\circ}{c}(f)(i) &= \begin{cases} 
-f(i) + \sum_{j \in M_i} f(j) & \text{ when } i \in \overset{\circ}{G} \\
 f(i) & \text{ when } i \in \overset{\circ}{G} 
\end{cases}
\end{align*}
$$ (9)

Analogously the dimension $\overset{\circ}{c}(d)$ of representation $\overset{\circ}{F}_{X,\delta}(\pi)$ and the character of representation $\overset{\circ}{F}_G(\pi)$ can be found.

Given $d \in \overset{\circ}{Z}_G^{+}$, $f \in \overset{\circ}{V}_G^{+}$ consider a full subcategory $\text{Rep}(G, d, f)$ in $\text{Rep}(G)$, $\text{Ob}\text{Rep}(G, d, f) = \{\pi \mid d = d(\pi), f \in \{f_\pi\}\}$. All representations $\pi$ have common support $X_d = G^\pi = \{i \in G_v \mid d(i) \neq 0\}$.

We will consider these categories under condition $(d, f) \in S = \{(d, f) \in \overset{\circ}{Z}_G^{+} \times \overset{\circ}{V}_G^{+}, |d(i) + f(i)| > 0, i \in G_v\}$.

Let us mention that the dimension of a representation is determined uniquely, but the character, in general, is not, so the objects of $\text{Rep}(G, d, f)$ can be considered as pairs $(\pi, f)$, $f \in \{f_\pi\}$.

$\text{Rep}(G, d, f) \subset \text{Rep}(G, d, f)$ — full subcategory which objects are pairs $(\pi, f)$ when representations $\pi$ is indecomposable in $\text{Rep}(G, \mathcal{H})$. $\text{Rep}_*(G, d, f) \subset \text{Rep}(G, d, f)$ ($\text{Rep}_*(G, d, f) \subset \text{Rep}(G, d, f)$) — full subcategory, which object are pairs $(\pi, f)$ with $f(i) > 0$ when $i \in X^\circ = X \bigcap \overset{\circ}{G}$ ($f(i) > 0$ with $i \in X^\circ = X \bigcap \overset{\circ}{G}$). $\text{Rep}_*(G, d, f)$ and $\text{Rep}_*(G, d, f)$ are defined in a natural way.

Let $S_\circ = \{(d, f) \in S \mid f(i) > 0 \text{ when } i \in \overset{\circ}{X}_d\}$, $S_* = \{(d, f) \in S \mid f(i) > 0 \text{ when } i \in \overset{\bullet}{X}_d\}$

With $(d, f) \in S_\circ$ ($(d, f) \in S_*$) let us construct a functor of even and odd reflections

$$
F_{d} : \text{Rep}_*(G, d, f) \rightarrow \text{Rep}_*(G, \overset{\circ}{c}(d), \overset{\circ}{f}_d),
$$

$$
\overset{\circ}{f}_d(i) = \begin{cases} 
\overset{\circ}{c}(f)(i) & \text{ when } i \in \overset{\bullet}{X}_d \\
f(i) & \text{ when } i \notin \overset{\bullet}{X}_d 
\end{cases}
$$
\[ F_{df}: \text{Rep}_\bullet (G, d, f) \to \text{Rep}_\bullet (G, \dot{c} (d), \dot{f}_d), \]

\[ \dot{f}_d (i) = \begin{cases} \dot{c} (f) (i) & \text{when } i \in \tilde{X}_d \\ f(i) & \text{when } i \notin \tilde{X}_d \end{cases} \]

let \( F_{df} (\pi, f) = (\tilde{\pi}, \tilde{f}_d) \) where \( \tilde{\pi} = F_{X_d,\delta} (\pi) \) and \( F_{df} (\pi, f) = F_{X_d,\delta} (\pi). \)

Note that if \( X_d = G \) (\( X_d = G \)) then \( f = \tilde{c} (f) (f_d = \tilde{c} (d)). \)

If \( \{C_k\}_{k \in G_v} \) is a morphism from the representation \( \pi \) to the representation \( \tilde{\pi} \) then \( F_{df} (\{C_k\}) = F_{X_d,\delta} (\{C_k\}), F_{df} (\{C_k\}) = F_{X_d,\delta} (\{C_k\}). \)

Formulas 10, 11, and 17 imply that functor \( F_{df} \) is faithful and full. Besides, it is easy to see that \( F_{\tilde{c}(d), f_d} \circ F_{df} \simeq \text{Id} \) and \( F_{df} F_{\tilde{c}(d), f_d} \simeq \text{Id} \) and functor \( F_{df} \) realizes the equivalence of categories.

Analogous statement about \( \tilde{F}_{df} \) is valid too.

Let us define a category \( \text{Rep} (G, \underline{[]} ) \). Let \( \text{Ob Rep} (G, \underline{[]} ) = \bigcup_{(d, f) \in S} \text{Ob Rep} (G, d, f) \), morphisms between objects from \( \text{Ob Rep} (G, d_1, f_1) \) and \( \text{Ob Rep} (G, d_2, f_2) \) match with morphisms in \( \text{Rep} (G, d_1, f_1) \) when \( (d_1, f_1) = (d_2, f_2) \) are missing when \( (d_1, f_1) \neq (d_2, f_2) \).

Define following full subcategories in \( \text{Rep} (G, \underline{[]} ) \): \( \text{Rep}_\circ (G, \underline{[]} ) \) has as a set of objects

\[ \bigsqcup_{(d, f) \in S_\circ} \text{Ob Rep} (G, d, f), \]

\( \text{Rep}_\bullet (G, \underline{[]} ) \) — the set \( \bigsqcup_{(d, f) \in S_\bullet} \text{Ob Rep} (G, d, f) \). Functors \( \tilde{F}_{df} (\tilde{F}_{df}) \)

in a natural way generate a functor \( \tilde{F} (\tilde{F}) \) on the category \( \text{Rep}_\circ (G, \underline{[]} ) \) \( \text{Rep}_\bullet (G, \underline{[]} ) \) which realizes the equivalence of the category with itself.

So, we proved:

**Theorem 2.1.** Functors of even and odd reflections

\[ \tilde{F}: \text{Rep}_\circ (G, \underline{[]} ) \to \text{Rep}_\circ (G, \underline{[]} ) \]

\[ \tilde{F}: \text{Rep}_\bullet (G, \underline{[]} ) \to \text{Rep}_\bullet (G, \underline{[]} ), \]

are defined; they realize the equivalence of the category with itself and the equivalence of following full subcategories:

\[ \tilde{F}: \text{Rep}_\circ (G, d, f) \to \text{Rep}_\circ (G, \dot{c} (d), \dot{f}_d) \text{ when } (d, f) \in S_\circ \]

\[ \tilde{F}: \text{Rep}_\bullet (G, d, f) \to \text{Rep}_\bullet (G, \dot{c} (d), \dot{f}_d) \text{ when } (d, f) \in S_\bullet. \]

At that \( (\tilde{F})^2 \simeq \text{Id}, (\tilde{F})^2 \simeq \text{Id}. \)

Let \( g \in G_v, \Pi_g \) — simplest representation of a graph \( G \): \( \Pi_g (g) = C, \Pi_g (i) = 0 \) when \( i \neq g, i \in G_v \). The characters of the representations \( \Pi_g \) we will denote as \( f_g: f_g (g) = 0, \) at the same time assuming that \( f_g (i) > 0 \) when \( i \neq g. \)

For the simplest representation \( \Pi_g \) the dimension \( d_g (g) = 1, d_g (i) = 0 \) when \( i \neq g. \)

Objects \( (\Pi_g, f_g) \) are said to be simplest objects of the category \( \text{Rep} (G, \underline{[]} ) \). If \( g \in G (g \in G) \)

then \( (\Pi_g, f_g) \in \text{Ob Rep}_\circ (G, \underline{[]} ) \) \( (\Pi_g, f_g) \in \text{Ob Rep}_\bullet (G, \underline{[]} ) \).
3 Coxeter transformations

Denote as $B$ a quadratic form on the space $V_G$ defined by formula $B(x) = \sum_{i \in G_v} x_i^2 - \sum_{\gamma_{ij} \in G_e} x_i x_j$, and as $\langle , \rangle$ — corresponding symmetric bilinear form. Form $B(x)$ is named the Tits form of the graph $G$.

In propositions 3.1–3.3 we collect well-known results from [1] which we will use further.

Proposition 3.1.
1. Let $i \in G_v$, then $\sigma_i(x) = x - 2 < \bar{i}, x > \bar{i}$, $\sigma_i^2 = 1$.
2. Group $W$, generated by reflections $\sigma_i$, retains the integral lattice in $V_G$ and retains the quadratic form $B$.
3. If the form $B$ is positive defined then group $W$ is finite.
4. Form $B$ is positive defined for graphs $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ (Dynkin graphs) and only for them.

For each $i \in G_v$ we denote as $\bar{i}$ a vector in $V_G$ such that $(\bar{i})_j = 0$ when $i \neq j$ and $(\bar{i})_i = 1$. Vector $x \in V_G$ is said to be a root if for some $i \in G_v$ and $w \in W$ we have $x = w \bar{i}$. Vectors $\bar{i}$ are simple roots. Root $x$ is positive if $x > 0$.

Proposition 3.2.
1. If $x$ — root then $x$ — integral vector and $B(x) = 1$.
2. If $x$ — root then $-x$ — root.
3. If $x$ — root then either $x > 0$ or $(-x) > 0$.

Proposition 3.3. If the form $B$ for a graph $G$ is positive defined then:
1) transformation $c$ in the space $V_G$ has no nonzero invariant vectors;
2) if $x \in V_G$, $x \neq 0$ then for some natural $k$ vector $c^k x$ is not positive.

Let us return to locally-scalar representations of a graph $G$ in the category of Hilbert spaces.

Lemma 3.4. If $G$ is Dynkin graph and $d = (d_i)$ — its positive not simple root then $d$ results from the simple root by the sequence of even and odd Coxeter transformations.

Proof. Let $m$ — the minimal natural number with the property: $c^m(d) \notin G^+_V$ (such $m$ can be found by the proposition [3.3]). Applying to $d$ sequentially transformations of odd and even reflections we will obtain positive root $\tilde{d}$ such that the next vector will be negative. Then, as is well-known, $\tilde{d}$ will be simple root (transfer from the positive root to the negative one is made only through the simple root). Let $\tilde{d} = c_k^i (d)$ (or $c_k^o (d)$), then $d = c_k^{-1} (\tilde{d})$ (or $c_k^{-1} (\tilde{d})$) and the lemma is proved.

Lemma 3.5. Let $\pi \in \text{Rep}(G)$, $i \in G_v$, $d(i) \neq 0$, $f(i) = 0$. Then $\pi = \pi_i \bigoplus \pi'$, where $G^{\pi_i} = i$ and $G^{\pi'} \neq i$ (a case $\pi' = 0$ is not excluded).

Indeed, $f(i) = 0$ implies $\Gamma_{ij} = 0$ when $\gamma_{ij} \in M(i)$, and $d(i) \neq 0$ implies $\pi = \pi_i \bigoplus \pi'$.

Proposition 3.6. All locally-scalar representation of a Dynkin graph $G$ in the category of Hilbert spaces are discrete.
Proof will be carried out by the induction on $n = |G_v|$. When $n = 1$ the statement is trivial (separable Hilbert space is orthogonal direct sum, finite or infinite, of the one-dimensional spaces).

Let the statement is proved for Dynkin graphs with the number of vertices, which is $\leq n - 1$. Let $n > 1$ and $\pi \in \text{Ob Rep}(G)$. If $\pi^* \neq \pi_v$ then it is possible to turn from $\pi$ to the representation $\pi$ of the graph $\tilde{G}$ with the less number of vertices ($\tilde{G}_w = \pi^*$) and to make use of the presumption of induction.

Let $\pi^* = \pi_v$. Assume that $f \in \{f_v\}, f(g) = 0, g \in G_v$, then by the lemma $\pi = \pi_\delta \oplus \pi'$ and we can use the presumption of induction again.

Let $\pi^* = \pi_v$ and the character $f(i) > 0$ when $i \in G_v$. Applying to $(\pi, f)$ by turns functors $F^\circ, F_G$ we will obtain, subject to (9):

a) either $c^k(f)$ is positive for all $k$, which contradicts to proposition 3.3

b) or after applying functor one of the conditions $\pi^* = \pi_v, f(i) > 0$ when $i \in G_v$ will fail (and in this case we will return to the situation considered earlier).

Proposition 3.7. Any locally-scalar indecomposable representation $\pi$ of a Dynkin graph $G$ results from the simplest representation $\Pi_\delta$ by the functors of even and odd reflections. More precisely: if $(\pi, f) \in \hat{\text{Rep}}(G, \underline{1})$ then there exists a sequence $(\pi(k), f(k))$ of objects of the category $\hat{\text{Rep}}(G, \underline{1})$, $k \in \overline{0, n}$ such that $(\pi(0), f(0)) = (\Pi_\delta, f_\delta)$ (let, for definiteness, $g \in \tilde{G}$) and with odd $k^\circ F^\circ (\pi(k), f(k))$ and even $k^\circ F (\pi(k), f(k))$.

Proof.

Let $\pi$ has a dimension $d$ and a character $f$. Let $n$ — minimal number such that $\{c_n(d), c_n(d)\} \not\subseteq V_G^\perp$. We will name $n$ as growth of the locally-scalar representation $\pi$.

Proposition will be proved by induction on $n$. Let $n = 1$, then $f(g) = 0$ for some $g \in \pi^* \cap \tilde{G}$ (in the contrary case we will apply to $\pi$ the functor $\hat{F}_{X, \delta}$, where $X = \pi^*$, and obtain $d(\hat{F}_{X, \delta}(\pi)) = \hat{c}(d(\pi)) \in V_G^\perp$). Then from indecomposability $\pi$ and the lemma 3.5 it is follows that $\pi^* = \{g\}$, i.e. $\pi = \Pi_\delta$, $(\pi, f)$ is a simplest object.

Let the statement is proved for locally-scalar representations of the growth $\leq n - 1$ and $\pi$ has a growth $n \geq 1$.

By the lemma 3.5 $(\pi, f) \in \text{Rep}_\circ(G, \underline{1})$ and $(\pi, f) \in \text{Rep}_\circ(G, \underline{1})$ and to the pair $(\pi, f)$ both functors $\hat{F}$ and $F$ may be applied. We will apply that functor, which will entail a new pair with representation of lower growth. Then we make use of the presumption of induction.

Theorem 2.1 and the proposition 3.7 imply

Corollary 3.8. In the category $\hat{\text{Rep}}(G, d, f)$ all objects are equivalent.
a) $h(i) = 2l(i)$. In this case vertex $j$ is odd ($\bullet$). In the series of roots $\bar{i}, \bar{c}(\bar{i}), \bar{c}_2(\bar{i}), \ldots, \bar{c}_{2l(i)}(\bar{i}) (\bar{i} = \bar{j})$ any root results from the simple root $(\bar{i} \text{ or } \bar{j})$ by transformations $\bar{c}, \bar{c}$ for the number of steps $\leq \frac{h(i)}{2}$. Middle root in series we will obtain, for definiteness, from odd simple root (in particular, it is possible that $\bar{i} = \bar{j}$). Any root in series $\bar{i}, \bar{c}(\bar{i}), \ldots, \bar{c}_{2l(i)+1}(\bar{i}) = \bar{j}$ results from the simple root $(\bar{i} \text{ or } \bar{j})$ by transformations $\bar{c}, \bar{c}$ in $< \frac{h(i)}{2}$ number of steps.

b) $h(i) = 2l(i) + 1, \bar{c}_{2l(i)+1}(\bar{i}) = \bar{j}$. In this case $\bar{j}$ has the same parity as $\bar{i}$ (in particular, it is possible that $\bar{i} = \bar{j}$). Any root in series $\bar{i}, \bar{c}(\bar{i}), \ldots, \bar{c}_{2l(i)+1}(\bar{i}) = \bar{j}$ results from the simple root $(\bar{i} \text{ or } \bar{j})$ by transformations $\bar{c}, \bar{c}$.

Let $S_G$ — set of simple roots of a graph $G$, $u_g(i) \in (\mathbb{R}^+)^{G_\setminus\{g\}}$ (we will construct a character $f_g(i)$ by function $u_g$ assuming $f_g(i) = u_g(i)$ when $i \in G_\setminus\{g\}$ and $f_g(g) = 0$). Let $N_G = \{\bar{i}, k, u_i\} \in S_G; k \leq \frac{h(i)}{2}$ for $i \in G$ and $k < \frac{h(i)}{2}$ for $i \in G_\setminus\{g\}$, and $\text{Ind } G$ — set of indecomposable locally-scalar representations of a graph $G$ defined up to unitary equivalence.

By the simple root $\bar{g}$ and function $u_g(i)$ by the sequence of Coxeter reflections in $k$ steps we will obtain from the simplest object $(\Pi_\bar{g}, f_\bar{g})$ an object $(\pi, f)$ of the category $\text{Rep } G (\Pi)$ ($\pi$ is indecomposable in $\text{Rep } G$ representation from $\text{Ind } G$).

So, we have defined a map

$$\varphi : N_G \to \text{Ind } G.$$ 

All indecomposable locally-scalar representations of Dynkin graph are obtained in this way (see proposition 3.7).

Let $\text{Ind } G$ is a subset in the set $\text{Ind } G$ of faithful representations, $T_G$ — subset of the simple roots $\bar{i}$ and $L_i$ — set of that values $k$, for which triples $(\bar{i}, k, u_i) \in \varphi^{-1}(\text{Ind } G)$.

Following statement holds:

**Theorem 3.9.** The map $\varphi : N_G \to \text{Ind } G$ is surjection, at that each faithful representation $\pi$ from $\text{Ind } G$ has unique $\varphi$-inverse image; for nonfaithful $\pi$ vector $\bar{i}$ and the number $k$ in $\varphi^{-1}(\pi)$ are determined uniquely, and function $u_i$ — ambiguously.

**Proof** we will obtain by direct count for Dynkin graphs $E_6$, $E_7$, $E_8$ and by induction for graphs $A_n$ and $D_n$.

1. For graph $A_n$

   $$h(i) = n \text{ for } i = \overline{1, n};$$
   when $n = 2m + 1$, or $n = 2m$ with even $m$: $T_G = \{m + 1\}, L_{m+1} = \{m\};$
   when $n = 2m$ with odd $m$: $T_G = \{m\}, L_m = \{m\}.$

2. For graph $D_n$ ($n \geq 4$)

   $$h(i) = 2n - 3, \ i = \overline{1, n};$$
when \( n = 2m + 1 \): \( T_G = \{3, 4, \ldots, m + 2\} \), \( L_3 = \{2m - 2, 2m - 1\} \), \( L_4 = \{2m - 3, 2m - 2\} \), 
\( \ldots \), \( L_{m+1} = \{m, m + 1\} \), \( L_{m+2} = \{m\} \);

when \( n = 2m \): \( T_G = \{3, 4, \ldots, m + 1\} \), \( L_3 = \{2m - 3, 2m - 2\} \), 
\( \ldots \), \( L_{m+1} = \{m - 1, m\} \).

3. For graph \( E_6 \)

\[
h(i) = 11, \ i = \overline{1, 6}; \ T_G = \{2, 3, 4, 6\}, \ L_2 = \{5\}, \ L_3 = \{2, 3, 4, 5\}, \ L_4 = \{5\}, \ L_6 = \{4\}.
\]

4. For graph \( E_7 \)

\[
h(i) = 17, \ i = \overline{1, 7}; \ T_G = \{2, 3, 4, 5, 7\}, \ L_2 = \{5, 6, 7\}, \ L_3 = \{3, 4, 5, 6, 7, 8\}, \ L_4 = \{3, 6, 7, 8\}, \ L_5 = \{7\}, \ L_7 = \{4, 8\}.
\]

5. For graph \( E_8 \)

\[
h(i) = 29, \ i = \overline{1, 8}; \ T_G = \{1, 2, 3, 4, 5, 6, 8\}, \ L_1 = \{7, 12, 13\}, \ L_2 = \{5, 8, 11, 14\}, \ L_3 = \{4, 14\}, \ L_4 = \{3, 4, 7, 14\}, \ L_5 = \{8, 13\}, \ L_6 = \{11, 12\}, \ L_8 = \{9, 10, 13, 14\}.
\]

Hence, at the time of proving the theorem 3.9, we proved

**Corollary 3.10.** For any Dynkin graph \( G \) there exists one-to-one correspondence between faithful indecomposable locally-scalar representations of graph \( G \) and triples \((\bar{i}, k, u_i)\) \( \in N_G \) where \( \bar{i} \in T_G \), \( k \in L_i \).

Consider in space \( V_G \) a scalar product \( xy \), meaning vectors \( \bar{i} (i \in G_v) \) to form an orthonormal basis in \( V_G \). So, if \( x = (x_i) \), \( y = (y_i) \) then \( xy = \sum_{i \in G_v} x_i y_i \), \( x = \sum x_i \bar{i} = \sum_{a \in G} x_a \bar{a} + \sum_{b \in \hat{G}} x_b \bar{b} \).

Let us provide denotations \( x^* = \sum_{a \in G} x_a \bar{a} \), \( x^0 = \sum_{b \in \hat{G}} x_b \bar{b} \) \((x = x^* + x^0)\).

Making allowed item 1) of the proposition 3.11 it is easy to obtain that \( \bar{c}(x) = x - 2 \sum_{a \in G} < \bar{a}, x > \bar{a} = x - 2 \sum_{a \in G} < \bar{a}, x^* + x^0 > \bar{a} = x - 2 \sum_{a \in G} < \bar{a}, x^* > \bar{a} - 2 \sum_{a \in G} < \bar{a}, x^0 > \bar{a} = x - 2x^* - 2 \sum_{a \in G} < \bar{a}, x^0 > \bar{a} = x^0 - x^* - 2 \sum_{a \in G} < \bar{a}, x^0 > \bar{a} \) (Here we uses that fact that \( < \bar{a}, \bar{a} > = 1 \) and \( < \bar{a}, \bar{a'} > = 0 \) when \( a \neq a', a, a' \in \hat{G} \)).

Finally, \( \bar{c}(x) = x^0 - x^* - 2 \sum_{a \in G} < \bar{a}, x^0 > \bar{a} \).
Analogously,
\[
\hat{c}(x) = x^\ast - x^\circ - 2 \sum_{b \in G} < \hat{b}, x^\ast > \hat{b}.
\] (10)

Let us prove

**Proposition 3.11.**
1. If vectors \(x, y \in V_G\) and \(x^\ast y^\ast = x^\circ y^\circ\) then \([\hat{c}(x)]^\ast \cdot \hat{c}(y)]^\ast = [\hat{c}(x)]^\circ \cdot \hat{c}(y)]^\circ\);

2. If \(\pi\) — finite-dimensional \(f\)-representation of a graph \(G\) with dimension \(d(i), i \in G_v\) then \(f^\ast \cdot d^\ast = f^\circ \cdot d^\circ\)

**Proof.**
1. Equalities (10) imply

\[
[\hat{c}(x)]^\ast \cdot \hat{c}(y)]^\ast = (-x^\ast - 2 \sum_{a \in G} < \hat{a}, x^\circ > \hat{a})y^\ast = -x^\ast y^\ast - 2 \sum_{a \in G} < \hat{a}, x^\circ > y^\ast_a = -x^\ast y^\ast - 2 < y^\ast, x^\circ >,
\]

\[
[\hat{c}(x)]^\circ \cdot \hat{c}(y)]^\circ = x^\circ \cdot (-y^\circ - 2 \sum_{b \in G} < \hat{b}, y^\ast > \hat{b}) = -x^\circ y^\circ - 2 \sum_{b \in G} < \hat{b}, y^\ast > x^\circ_b = -x^\circ y^\circ - 2 < x^\circ, y^\ast >,
\]

and since \(x^\ast y^\ast = x^\circ y^\circ\) and \(< y^\ast, x^\circ > = < x^\circ, y^\ast >\) then \([\hat{c}(x)]^\ast \cdot \hat{c}(y)]^\ast = [\hat{c}(x)]^\circ \cdot \hat{c}(y)]^\circ\).

2. The condition of the local scalarity \(\sum_{j \in M_i} \Gamma_{ij} \Gamma_{ji} = \alpha_i I_{H_i}\) for finite-dimensional representation implies \(\sum_{j \in M_i} \text{Tr}(\Gamma_{ij} \Gamma_{ji}) = \alpha_id_i (\alpha_i = f(i), d_i = d(i), i \in G_v)\).

Summing such equalities at first by odd vertices and then by even ones, we will obtain

\[
f^\ast d^\ast = \sum_{\{\gamma_{ij}, \gamma_{ji}\} \subset G_v} \text{Tr}(\Gamma_{ij} \Gamma_{ji}) = \sum_{i \in G_v} \alpha_i d_i = f^\circ d^\circ.
\]

**4 Analogue of the Gabriel theorem**

**Proposition 4.1.** If a graph \(G\) is one of the extended Dynkin graphs \(\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8\) then dimensions of the indecomposable locally-scalar representations of the graph \(G\) are not bounded in a whole, so that \(G\) is not finitely representable in \(H\).

**Proof.**

From the root theory of extended Dynkin graphs it is follows that if \(g \in G^\circ\) then when \(t > 1\)
\(c^t(g) \in V_G^+\) and it is not a simple root (and if \(g \in G^\circ\) then when \(t > 1\)
\(c^{-t}(g) \in V_G^+\) and it is not a simple root) (see also [21]), hence \(c^{t_1}(g) \neq c^{t_2}(g)\) when \(t_1 \neq t_2\).

Then if we fix \(g \in G^\circ\) and simplest object \((\Pi, f) \in \text{Rep}(G, [I])\) we will obtain an infinite sequence of objects \(\cdots \tilde{F} \tilde{F} \cdots (\Pi, f)\) in which dimensions of representations are different.

Let us point to a connection between indecomposable representations of quivers which graph is a Dynkin scheme and locally-scalar representations of these graphs.
Let $\pi^A$ is a finite-dimensional representation of a quiver $Q$ over the field $\mathbb{C}$. If spaces of the representation are Hilbert (i.e. unitary, since they are finite-dimensional) then the representation $\pi^A$ can be naturally continued to the representation $\pi$ of the graph $G = G(Q)$ in the category of Hilbert spaces.

If $\pi^A$ is equivalent in $\text{Rep}(Q, \mathbb{C})$ to the representation $\widetilde{\pi}^A$ such that the continued representation $\widetilde{\pi}$ of a graph $G$ is locally-scalar, we will say that the representation $\pi^A$ of a quiver $Q$ is \textit{unitarizable}.

Proposition 4.2. \textit{Graph $G$ is a Dynkin graph if and only if with} $G = G(Q)$ \textit{any finite-dimensional indecomposable representation $\pi^A$ of a quiver $Q$ is unitarizable.}

\textbf{Proof.} If a graph $G$ is a Dynkin graph then dimension $d(i)$ of the representation $\pi^A$ is a root which can be obtained from a some simple root $\bar{g}$ by the transformations $\bullet$ and $\circ$. From the simplest representation $\Pi_g$ in this dimension by functors $\check{F}$ and $\hat{F}$ a locally-scalar representation $\widetilde{\pi}$ can be obtained. Representations $\pi^A$ and $\widetilde{\pi}^A$ of a quiver $Q$ are equivalent in $\text{Rep}(Q, \mathbb{C})$ since for Dynkin graphs in the dimension $d(i)$ there exist a unique up to the equivalence indecomposable representation ([1,2]).

The condition of unitarizability can be restated also in such a way: for the representation $\pi^A$ of a quiver $Q$ it is possible to define a scalar product in the representation spaces so as the continued representation of a graph $G$ in the category of Hilbert spaces will be locally-scalar.

We will prove the second part of the statement of the proposition if we show that for any extended Dynkin graph $G$ (and accordingly for $G' \supset G$) and any quiver $Q$, such that $G(Q) = G$, there exists indecomposable representation $\pi^A$ of the quiver $Q$ which will not be unitarizable.

If $(d(i))_{i \in G_v} = \text{minimal imaginary root of the graph } G$ (i.e. minimal positive root of the equation $B_G(x) = 0$) then such representation exists in the dimension $2d(i)$.

Let, for definiteness, orientation of $Q$ will be such that for any even vertex $i$ all incident with $i$ arrows are “come into” vertex, and for odd — “come out” it.

If all incident with $i$ arrows “come into” vertex $i$ then there exists (see, for instance, [10]) a representation $\pi^A$, for which

$$
\pi^A(\gamma_{ij}) = \begin{bmatrix}
\pi^A(\gamma_{ij}) & X(\gamma_{ij}) \\
0 & \pi^A(\gamma_{ij})
\end{bmatrix},
$$

(11)

$$
\pi^A(i) = H_1^{(i)} \bigoplus H_2^{(i)}, \pi^A(j) = H_1^{(j)} \bigoplus H_2^{(j)}, \pi^A(\gamma_{ij}) : H_1^{(j)} \rightarrow H_1^{(i)}, \pi^A(\gamma_{ij}) : H_2^{(j)} \rightarrow H_2^{(i)},
$$

$$
d_i = \text{dim } H_1^{(i)} = \text{dim } H_2^{(i)} \text{ for } i \in G_v.
$$

Assume that representation $\pi^A$ is unitarizable, i.e. in the spaces of representation we can define a scalar product so as the continued representation $\pi$ of the graph $G$ will become a locally-scalar.

Selecting an orthogonal basis in spaces $H_1^{(i)}$ and supplementing it to an orthogonal basis in $H_1^{(i)} \bigoplus H_2^{(i)}$ we will obtain new matrices of the representation $\pi^A$, remaining reduced and satisfying to the condition of the local scalarity. Saving generality, we will consider that the representation $\pi^A$ itself has this property.

Let $a_{kl}^{(ij)}$ — matric elements of the matrix $\pi^A(\gamma_{ij})$, hence sum $\sum_{k,l} a_{kl}^{(ij)} a_{kl}^{(ij)}$ is a sum of squares of the column lengths of the matrix $\pi^A(\gamma_{ij})$ and, therefore,

$$
A \equiv \sum_{\gamma_{ij}} \sum_{k,l} a_{kl}^{(ij)} a_{kl}^{(ij)} = \sum_{j \in G} d_j \alpha_j.
$$
If \( x_{kl}^{(ij)} \) and \( b_{kl}^{(ij)} \) are matrix elements of the matrices \( X(\gamma_{ij}) \) and, respectively, \( \pi_2^*(\gamma_{ij}) \), then for residuary columns of the matrices of the representation we have

\[
C + B \equiv \sum_{\gamma_{ij}} \sum_{kl} x_{kl}^{(ij)} d_{kl}^{(ij)} + \sum_{\gamma_{ij}} \sum_{kl} b_{kl}^{(ij)} b_{kl}^{(ij)} = \sum_{j \in G} d_j \alpha_j,
\]

where \( C > 0; \) thus \( A > B \).

Analogously actions with rows (summing by even vertices) gives us an inequality \( B > A \). Thus, assumption about local scalarity of the representation \( \pi \) leads us to contradiction.

Above we defined the sets \( S, S_0, S_* \). Let us define a map \( \tilde{\mathcal{F}}: S_0 \rightarrow S_0 \) for which \( \tilde{\mathcal{F}}(d, f) = (\tilde{\mathcal{F}}(d), \tilde{\mathcal{F}}(f)) \), and a map \( \mathcal{F}: S_* \rightarrow S_* \) for which \( \mathcal{F}(d, f) = (\mathcal{F}(d), \mathcal{F}(f)) \). If the pair \( \cdots \mathcal{F}(d, f) \) is defined then we will denote it as \( \mathcal{F}_k(d, f) \), and the pair \( \cdots \tilde{\mathcal{F}}(d, f) \) as \( \cdots \tilde{\mathcal{F}}(d, f) \) with \( k \) times.

(\( \tilde{\mathcal{F}}(d, f), k = 0, 1, 2, \ldots \); with \( k = 0 \) we get the pair \( (d, f) \) itself.

(Rigorously speaking, it is necessary to define subsets \( S_i, i \in \mathbb{N} \) in the following way: \( S_1 = S_0 \cap S_* \), \( S \supset S_1 \supset \ldots \supset S_k \supset \ldots \) so as \( S_{i+1} = \{(d, f) | \mathcal{F}(d, f) \in S_i \} \) and \( \mathcal{F}(d, f) \in S_1 \). So, \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) are defined on \( S_k \) with values in \( S_{k-1} \). Then we have \( \mathcal{F}_k \equiv \cdots \mathcal{F}(d, f) \) and \( \mathcal{F}_k \equiv \cdots \mathcal{F}(d, f) \) defined on \( S_k \) with values in \( S \).

A pair \( (d, f) \) is said to be root on \( G \), if for some \( k \in \mathbb{N}_0 \) \( (d, f) = \mathcal{F}_k(g, f_g), \) if \( g \in \mathcal{G} \), or \( (d, f) = \mathcal{F}_k(g, f_g), \) if \( g \in \tilde{\mathcal{G}}. \) Two root pairs \( (d, f_1) \) and \( (d, f_2) \) are said to be equivalent, if \( f_1 | x_d \equiv f_2 | x_d. \) A class of equivalence of the root pair \( (d, f) \) we will denote as \( [d, f] \).

**Theorem 4.3.** Let \( G \) — connected finite graph without cycles.

1. Following conditions are equivalent:
   a) graph \( G \) is one of the Dynkin graphs \( A_n, D_n, E_6, E_7, E_8; \)
   b) graph \( G \) is finitely representable in \( \mathcal{H}; \)
   c) any finite-dimensional indecomposable representation of a quiver \( Q \) is unitarizable.

2. A map \( \pi \rightarrow (d(i), f(i)) \) where \( d(i) \) is a dimension of \( \pi \) and \( f(i) \) is a character of \( \pi \) establishes a one-to-one correspondence between classes of equivalent of the indecomposable locally-scalar representations of a graph \( G \) and classes of equivalent root pairs of a graph \( G \).

**Proof.**

1. a) implies b). Indeed, if \( G \) is a Dynkin graph then all representations of \( G \) are discrete according to the proposition 3.4. All dimensions of indecomposable representations are bounded in a whole since there a finite number of them (dimensions of indecomposable representations are roots of a graph \( G \) by the proposition 3.7). In order to prove a finite representability of \( G \) in \( \mathcal{H}, \) obviously, it is enough to show that in \( \tilde{\text{Rep}}(G, d, f) \) it is contained exactly one, up to equivalence, object (corollary 3.8), and that two finite-dimensional representations \( \pi \) and \( \tilde{\pi} \) from \( \text{Rep}(G) \) are unitary equivalent if and only if they have equal dimension \( d, \) common character \( f \) and \( (\pi, f) \) is equivalent to \( (\tilde{\pi}, f) \) in \( \text{Rep}(G, d, f). \)

b) implies a), since if \( G \) is not a Dynkin graph then it is not finitely representable by the proposition 4.4 (it contains an extended Dynkin graph).

a) and c) are equivalent by the proposition 4.2.
2. If two locally-scalar representations $\pi_1$ and $\pi_2$ of a graph $G$ are unitary equivalent and indecomposable (and, consequently, finite-dimensional), then it is obvious that their dimensions are equal, and their characters (uniquely determined) on the common support are equal too.

Let locally-scalar indecomposable representations $\pi_1$ and $\pi_2$ have equal dimension $d$ and equal character $f$ (on the common support characters are match, and out of the support we will define them as equal). Then $\pi_1$ and $\pi_2$ have equal growth $d = \circ t (g)$ or $\circ t (\bar{g})$. Thus, pairs $(\pi_1, f)$, $(\pi_2, f)$ as the objects of the category $\text{Rep}(G, \mathbb{I})$ can be obtained from the simplest object $(\Pi_g, f_g)$ by the functors of the even and odd reflections (see the proof of the proposition 3.7) and, therefore, match.

**Remark 1.** Even for Dynkin graphs indecomposable locally-scalar representations are not determined uniquely by the character: there exist examples of indecomposable representations with different dimensions but equal characters.

**Remark 2.** It is follows from our constructions that for Dynkin graphs (and, seemingly, only for them) all locally-scalar representations realize over the field of the real numbers.

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