Transition from a strong-coupling fixed point to an intermediate-coupling fixed point in a single-channel SU(N) Kondo model: role of the filling and two-stage screening

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(Dated: March 22, 2022)

We study an extended SU(N) single-impurity Kondo model in which the impurity spin is described by a combination of Abrikosov fermions and Schwinger bosons. Our aim is to describe both the quasiparticle-like excitations and the locally critical modes observed in various physical situations, including non-Fermi liquid behavior in heavy fermion systems in the vicinity of a quantum critical point. We identify the strong coupling fixed point of the model and study its stability within second order perturbation theory. Already in the single channel case and in contrast with either the pure bosonic or the pure fermionic case, the strong coupling fixed point is unstable against the conduction electron kinetic term as soon as the amount of Abrikosov fermions reaches a critical value. In the stability region, the partially screened, dressed impurity at site 0 repels the conduction electrons on adjacent sites. In the instability region, the impurity tends to attract (N − 1) conduction electrons to the neighboring sites, giving rise to a two-stage Kondo effect with additional screening. This result opens the route to the existence of an intermediate coupling fixed point, characterized by non-Fermi liquid behavior.

I. INTRODUCTION

Recent experiments in Heavy-Fermion compounds have shown the existence of a quantum phase transition from a magnetically disordered to a long-range magnetic ordered phase, driven by change in chemical composition, pressure or magnetic field. For an extensive survey of the experimental situation we refer the reader to the review article of Stewart. In a very unusual way, the behavior of the system in the disordered phase close to the quantum critical point (QCP) differs from that of a Fermi-liquid. For example CeCu$_{2}$−$_{x}$Au$_{x}$ and (Ce$_{1−x}$La$_{x}$)Ru$_{2}$Si$_{2}$ present an antiferromagnetic transition, respectively, at $x_c = 0.1$ and at $x_c = 0.08$. While far from the QCP, the magnetically disordered phase is a Fermi liquid with a large effective mass, the temperature dependence of the physical quantities in the disordered phase in the vicinity of the QCP is of non-Fermi-liquid like type. Typically, in CeCu$_{2}$Au$_{0}$ the specific heat C depends on T as $C/T \sim -ln(T/T_0)$, the magnetic susceptibility $\chi = 1 - \alpha\sqrt{T}$, and the T-dependent part of the resistivity as $\Delta \rho \sim T$ instead of $C/T \sim \chi \sim Const$ and $\Delta \rho \sim T^2$ as in the Fermi-liquid state. Once a long-range magnetic order is set up, the effect of a pressure or of a magnetic field is to drive the system back to a magnetically disordered phase with a Fermi-liquid behavior. The same type of behavior has been observed in other systems such as YbRh$_2$Si$_2$, CeNi$_2$Ge$_2$, CeCu$_{0.5}$Si$_{1-x}$Ge$_2$, CeIr$_3$, CePd$_2$Si$_2$, and U$_{1−x}$Y$_x$Pd$_3$. The associated breakdown of the Fermi-liquid theory poses fundamental questions about the possible formation of novel electronic states of matter with new types of elementary excitations resulting from the presence of strong correlations among electrons.

On the theoretical side, two scenarios are in competition to describe quantum phase transitions: either the itinerant magnetism scenario (i), or more recently proposed, the locally critical picture (ii).

In the former case, (i), the quasiparticles still exist at the QCP and the theory focuses on the study of the low-lying, large-wavelength (low-$\omega$, low-$q$) fluctuations of the order parameter close to the transition. The calculations have been performed within the renormalization group scheme, or in the self-consistent spin-fluctuation theory, and have been recently extended to the microscopic model which is believed to describe the Heavy Fermions, the Kondo lattice. In all the cases, they lead to a $\Phi^4$-theory with an effective dimension $d_{eff} = d + z$ where $d$ is the spatial dimension and $z$ is the dynamic exponent. In the experimental situations, $d_{eff}$ is above its upper critical value equal to 4, since $d$ is equal to 3 or 2, and $z$ varies from 2 to 3 depending whether the spin fluctuations are staggered or uniform. Hence the system is described by a Gaussian fixed point with anomalous temperature dependence of $C/T$ and $\alpha = \Delta \rho/T$ but with predictions which cannot account for the non-Fermi-liquid behavior observed experimentally.

The second scenario, (ii), has been motivated by the recent results obtained by inelastic neutron scattering experiments performed on CeCu$_{2}$Au$_{0.1}$. The dynamical spin susceptibility $\chi''(Q, \omega)$ near the magnetic instability wavevector Q has been found to obey an anomalous $\omega/T$ scaling law as a function of temperature: $\chi''(Q, \omega) \sim T^{-\alpha}(\omega/T)$ with an exponent $\alpha$ of order 0.75. Moreover, such a $\omega$ and $T$ dependence appear to stand over the entire Brillouin zone revealing in the bulk susceptibility too. This fact strongly suggests that the spin dynamics are critical not only at large length scales but also at atomic length scales contrary to what happens in the traditional itinerant magnetism pic-
ture, (i). From these results, one can deduce that local critical modes coexist with large-wavelength fluctuations of the order parameter implying a non-Gaussian fixed point beyond the $\Phi^4$-theory. Alternative theories to the spin-fluctuation scheme are needed to describe the local feature of the quantum critical point characterized by the simultaneous disappearance of the quasiparticles and the formation of local moments. In this direction, we will mention recent calculations based on a dynamical mean field theory (DMFT) which seem to lead to encouraging results concerning the scaling law variation of $\chi''(q, \omega)$.

A recent approach which has been developed in order to describe the local QCP, is based on a supersymmetric theory in which the spin is described in a mixed fermionic-bosonic representation. The interest of the supersymmetric approach is to describe the quasiparticles and the local moments on an equal footing through the fermionic and the bosonic part of the spin, respectively. It appears to be specially well-indicated in the case of the locally critical scenario in which the magnetic temperature scale $T_K$, and the Fermi scale $T_F$ (the Kondo temperature) below which the quasiparticles die, vanish at the same point, $\delta_C$.

An important aspect in the discussion of the breakdown of the Fermi-liquid theory is related to the question of the stability of the strong coupling (SC) fixed point. Whereas all the issues presented previously concerning Heavy fermion systems have to do with properties of the lattice, the instability of the SC fixed point can be regarded already by studying the single impurity problem.

The traditional source of instability in the single impurity Kondo model is the presence of several channels for the conduction electrons with the existence of two regimes, underscreened and the overscreened, with very different behaviors as we are about to recall. Indeed we will see that this is not the only possible source of instability of the strong coupling fixed point. Recent works have shown that more general Kondo impurities of symmetry group $SU(N)$ may also lead to an instability of the SC fixed point already with one channel of conduction electrons.

In order to fix ideas, let us start with the antiferromagnetic single-channel Kondo impurity model. It is well known that within a renormalization group (RG) analysis the flow takes the Kondo coupling $J$ all the way to strong coupling. The weak-coupling beta function follows the renormalization group equation

$$\beta(g) = \frac{dg(\Lambda)}{d\Lambda} = -g^2$$

where $g = \rho_0 J$ and $\rho_0$ is the density of states of conduction electrons. The system flows a to strong coupling fixed point which is stable and the associated behavior of the system is that of a local Fermi liquid.

The situation is rather different when one considers several channels for the conduction electrons. In the case of a spin $S$ of symmetry group $SU(2)$ in Kondo interaction with conduction electrons belonging to $K$ different channels, Blandin and Nozières have shown that the multichannel Kondo model can lead to two very different situations depending how $K$ compares to $2S$. Their calculation corresponds to a second order perturbation theory in the hopping amplitude, $t$, of the conduction electrons, around the strong coupling fixed point. They analyze their results by deriving an effective coupling, $J_{eff}$, between the spin of the composite formed by the impurity dressed by the conduction electrons in the strong coupling limit, and the spin of the conduction electron on the neighboring sites. They are then able to apply the same RG analysis to $J_{eff}$ as indicated in Eq. (1). In the underscreened regime, when $K < 2S$, the effective coupling is found to be ferromagnetic and the strong coupling fixed point is stable. In the overscreened regime when $K > 2S$, the effective coupling is found to be antiferromagnetic and hence the strong coupling fixed point is unstable. The former $K < 2S$ regime corresponds to the one-stage Kondo effect with the formation of an effective spin, $(S-1/2)$, resulting from the screening of the impurity spin by the conduction electrons located on the same site. The system described by the strong coupling fixed point, behaves as a local Fermi liquid. The instability of the strong coupling fixed point obtained in the latter regime, $K > 2S$, is associated with a multi-stage Kondo effect in which successively the impurity spin is screened by conduction electrons on the same site, and then the resulting dressed impurity is screened again by conduction electrons on the neighboring site and so forth. The instability of the strong coupling fixed point in the overscreened regime is an indication of the existence of an intermediate coupling fixed point which has been then investigated by means of other methods. As it is well established now, the intermediate coupling fixed point leads to non-Fermi-liquid excitation spectrum with an anomalous residual entropy at zero temperature.

It has recently put forward that other sources of instability of the SC fixed point may exist else but the multiplicity of the conduction electron channels. Recent works have shown that the presence of a more general Kondo impurity where the spin symmetry is extended from $SU(2)$ to $SU(N)$, and the representation is given by a L-shaped Young tableau, may also lead to an instability of the SC fixed point already in the one-channel case. In the large $N$ limit, Coleman et al. have found that the SC fixed point becomes unstable as soon as $q$ (the number of boxes in the Young tableau along the first column), is larger than $N/2$ whatever the value of $2S$ (the number of boxes in the Young tableau along the first row) is. This result opens the route to the existence of an intermediate coupling fixed point with presumably non-Fermi-liquid excitation spectrum. The consideration of a L-shaped Kondo impurity fits in with the supersymmetry approach that we have evoked before since both spin operators and states can be expressed in terms of bosons and fermions.
At that point, it is worth noting that the supersymmetry theory, or specifically taking into consideration more general L-shaped Kondo impurities appears to offer valuable insights into the two issues raised by the breakdown of the Fermi liquid theory that we have summarized above, i.e. both the existence of locally critical modes and the question of the instability of the SC fixed point. Somehow it seems that the choice of L-shaped Kondo impurities captures the physics present in real systems, with the coexistence of the screening of the spin by conduction electrons responsible for the formation of quasiparticles, and the formation of a localized magnetic moment that persists and eventually leads to a phase transition as the coupling to other impurities becomes dominant. In the same way as large N expansions may provide insights into real systems even at finite value of the degeneracy, the study of more general impurities may enlighten the understanding of experimental situations.

The aim of the paper is to study the extended SU(N) L-shaped single-impurity Kondo model in the one-channel case. We want to understand how the system behaves, not only as a function of the impurity parameters, (2S,q), but also as a function of the number of electrons, nd, available on neighboring sites, that is to say, of the filling. We find that as long as the bosonic component of spin is of order N, there is a transition around the point where the fermionic component of the impurity is q = N/2. At this particular point, the energy shift is, to lowest order in perturbation theory around the strong-coupling fixed point, equal to (−2t^2/J), independently of the impurity parameters, q, S and N. When q < N/2, the low-energy physics corresponds to a stable strong-coupling fixed point. For q > N/2 the strong-coupling fixed point is unstable and anomalous behavior is expected, in particular, a two-stage quenching effect, as predicted by Ref. 21. This phase diagram is not accidental, but is due to the relation of the effective dressed impurity in the strong-coupling regime to the conduction electrons in neighboring sites, as our study of the dependence of the energy shifts on nd reveals. If q < N/2, the energy is minimized when the dressed impurity repels the electrons on the next site. That is, when nd = 1. At q = N/2, the energy shift is independent of nd. Finally, the lowest energy shift for q > N/2 corresponds to a maximal nd, indicating the accumulation of conduction electrons on neighboring sites, leading to a two-stage Kondo quenching.

The rest of the paper is organized as follows. In Section II, we introduce the model and the main features of the strong-coupling limit, where the electron kinetic term is neglected. In this limit the model is reduced to a single site problem, where the impurity is coupled to nc conduction electrons. We identify the ground state and the energies of the excited states with one more or one less conduction electron, which will play a role in the lowest order in perturbation theory. In section II, we discuss the stability of the strong coupling fixed point as resulting from the sign of the effective coupling J_{eff} between the spin of the strong coupling composite and the spin of the conduction electrons on the neighboring site. The calculation is based on a second order perturbation theory in t and is performed for an arbitrary number nd of conduction electrons on the neighboring site. Section IV contains the discussion of the results. In the large N limit, we show how J_{eff} is derived from the energy shift difference between the symmetric and the antisymmetric configurations, and how the analysis of the nd dependence of the energy shift provides information on the nature of Kondo screening, with the realization of a two-stage Kondo effect when the SC fixed point becomes unstable. When the behavior of the system is controlled by the strong coupling fixed point, i.e. when q < N/2, the impurity in the ground state tends to repel electrons on neighboring sites. Once q > N/2, the repulsion becomes attraction. We show how this feature is already present in the purely fermionic case, and is a consequence of the particle-hole symmetry (more details are contained in Appendix B). The fact that there is extra degeneracy in the supersymmetric impurity, due to the bosonic contribution, leads to the instability of the strong coupling fixed point as soon as q > N/2. We finish the section with a short discussion on the behavior of physical quantities in the different regimes.

The appendices contain the technical details of the calculations. In Appendix A we outline the construction of three particle states with SU(3) symmetry, as an introduction to the group theoretical formalism used. Explicit expressions for the impurity states and the eigenstates of the model in the strong coupling limit are derived in Appendix B. We also include a general presentation of the different representations of the spin, either bosonic, fermionic or L-shaped, as considered in the paper. We will show how in the latter case the spin operators and the impurity states are expressed in terms of fermion and boson creation and annihilation operators within two constraints. Appendix C contains a calculation of the energy shift to the strong coupling fixed point to lowest order in perturbation theory, for the completely antisymmetric impurity. Since the ground state is a singlet, there is no splitting of levels. Nevertheless, the behavior of the energy with the filling, nd, on the neighboring site shares many common features with the problem that we have studied. Finally, we include the details of the calculation of the matrix elements needed in the second order perturbation theory calculation in Appendix D.

II. THE MODEL AND ITS STRONG-COUPLING LIMIT

A. SU(N) single-impurity Kondo model

We consider a generalized, single-impurity, Kondo model with one channel of conduction electrons and a spin symmetry group extended from SU(2) to SU(N).
An impurity spin, $\mathbf{S}$, is placed at the origin (site 0). In this article we will deal with impurities that can be realized by a combination of bosonic and fermionic operators, and are thus described by a L-shaped representation in the language of Young tableaux, as illustrated in Fig. 1 (for details, see Appendix B).

If $2S$ and $q$ are the numbers of boxes along the first row and the first column, respectively, the representation is denoted by $[2S, q^{N-1}]$. Its degeneracy is reported in Table 1. The conduction electrons transform under the fundamental representation of $SU(N)$ and can be represented by Young tableaux made out of single boxes. The dimension of the fundamental representation is $N$ which just means that each electron can be in one of $N$ states of spin.

The Hamiltonian describing the model is written as

$$H = \sum_{k,\alpha} \varepsilon_{k} c_{k,\alpha}^\dagger c_{k,\alpha} + J \sum_{A} S^{A} \sum_{\alpha,\beta} c_{\alpha}^\dagger(0) \tau_{A,\alpha,\beta} c_{\beta}(0),$$

where $c_{k,\alpha}^\dagger$ is the creation operator of a conduction electron with momentum $k$, and $SU(N)$ spin index $\alpha = a, b, \ldots, r_{N}$. $c_{\alpha}(0) = \frac{1}{\sqrt{N}} \sum_{k} c_{k,\alpha}$ is the creation operator of a conduction electron at the origin, $N_{S}$ is the number of sites, and $\tau_{A,\alpha,\beta} = (A = 1, \ldots, N^{2} - 1)$ are the generators of the $SU(N)$ group in the fundamental representation, with $Tr[\tau_{A} \tau_{B}] = \delta_{AB} / 2$. In the $SU(2)$ case, $\tau^{A} = \sigma^{A} / 2$, where $\{\sigma^{A}\}$ are the Pauli matrices. The conduction electrons interact with the impurity spin $S^{A} (A = 1, \ldots, N^{2} - 1)$, placed at the origin, via Kondo coupling, $J > 0$. When the impurity is in the fundamental representation, we recover the Coqblin-Schrieffer model describing conduction electrons in interaction with an impurity spin of angular momentum $j > 1$, resulting of the combined spin and orbit exchange scattering. In our notation, $a = j, b = j - 1, \ldots, r_{N} = -j$.

**B. Strong-Coupling fixed point**

In the strong-coupling limit, the Hamiltonian reduces to the local Kondo interaction term at site 0

$$H = J \sum_{A} S^{A} \sum_{\alpha,\beta} c_{\alpha}^\dagger(0) \tau_{A,\alpha,\beta} c_{\beta}(0),$$

where $\tau_{A,\alpha,\beta} = (A = 1, \ldots, N^{2} - 1)$ are the generators of the $SU(N)$ group in the fundamental representation.

The ground state, $|GS\rangle$, is formed by binding the right amount of conduction electrons to the impurity in order to minimize the Kondo energy. Let us denote by $Y$ (Fig. 2) the representation of the $n_{c}$ conduction electrons coupled to the impurity, $R$ that of the free impurity (Fig. 1), and $R_{SC}$ the representation of one of the strong-coupling states resulting of the direct product $R \otimes Y$ (cf. Fig. 3) (see Appendix B for details).

When $N = 2$, the Kondo energy can be written in terms of conserved quantities

$$J \tilde{S} \cdot \sum_{\alpha,\beta} c_{\alpha}^\dagger(0) \tau_{\alpha,\beta} c_{\beta}(0) |GS\rangle =$$

$$\frac{J}{2} \left[ S^{SC}(S^{SC} + 1) - S^{R}(S^{R} + 1) - S^{Y}(S^{Y} + 1) \right] |GS\rangle,$$

where $S(S + 1)$ is the eigenvalue of the Casimir operator $\tilde{S}^{2}$ for $N = 2$. The generalization to $SU(N)$ is given by

$$J \sum_{A} S^{A} \sum_{\alpha,\beta} c_{\alpha}^\dagger(0) \tau_{A,\alpha,\beta} c_{\beta}(0) |GS\rangle =$$

$$\frac{J}{2} \left[ \hat{C}_{2}(R_{SC}) - \hat{C}_{2}(R) - \hat{C}_{2}(Y) \right] |GS\rangle,$$

where $\hat{C}_{2}(\hat{R})$ is the quadratic Casimir operator of the representation $\hat{R}$, defined earlier, which commutes with all the generators of the group. For a representation given by a Young Tableau with $m_{j}$ boxes in the $j$-th row until the row $j = h$, the eigenvalue $\hat{C}_{2}(\{m_{j}\})$ of the quadratic Casimir operator is

$$\hat{C}_{2}(\{m_{j}\}) = \frac{1}{2} \left[ \frac{Q(N^{2} - Q)}{N} + \sum_{j=1}^{h} m_{j}(m_{j} + 1 - 2j) \right].$$

**FIG. 1:** Young Tableau description of an impurity with mixed symmetry, $[2S, q^{N-1}]$, realized by a combination of fermions and bosons.

**FIG. 2:** Young tableau description of $n_{c}$ conduction electrons, localized at the impurity site.

**FIG. 3:** Young tableau description of the formation of the strong coupling ground state. We denote the presence of conduction electrons at site 0 by $c$. Notice that the first column in the Young tableau for $R_{SC}$ is a singlet and can be removed.
where \( Q = \sum_{j=1}^{N} n_{ij} \) is the total number of boxes. Table I summarizes the expressions of the Casimir eigenvalues for the impurities described in this work and for the conduction electrons, as well as the dimension of their spin representations.

Minimization of the energy Eq. (4), leads to a ground state with \( n_c = (N - q) \) conduction electrons coupled to the L-shaped Kondo impurity ensuring partial screening. The resulting composite at site 0, with energy \( E_0 \), is made out of the impurity dressed by the conduction electrons in order to form a singlet along the first column. The associated Young tableau in the strong coupling regime is given in Fig. 3. Note that the first column of length \( N \) can be removed without changing the representation since it is a singlet. When the strong-coupling fixed point is stable, this corresponds to a one-stage Kondo effect in which the impurity is screened by the conduction electrons to form a bosonic \((S - 1/2)\) impurity.

C. Ground state

Let us now write the expression of the fundamental state associated with this strong-coupling fixed point. The ground state is degenerate. The states in the multiplet transform as a completely symmetric representation of \( SU(N) \), described by a Young tableau with \((2S - 1)\) boxes, denoted by \([2S - 1]\), (Fig. 3). We choose a realization of the impurity in terms of \(2S\) bosonic operators and \((q - 1)\) fermionic operators, which happens to be more convenient. We could have constructed impurity states with the same \( SU(N) \) symmetry using \((2S - 1)\) bosons and \( q \) fermions (see Appendix B). We would like to emphasize that all the results that we establish in this paper are independent of the operator representation which we choose to work with. The highest weight state is then written as

\[
|GS\rangle_{[a]_{aa}}^{2S-1} = \frac{1}{\sqrt{(2S-1)!}} (b_0^c)^{2S-1} |\Delta\rangle,
\]

with

\[
|\Delta\rangle \equiv \frac{1}{\gamma} A(b_1^c (\prod_{\alpha=2}^{i_2} f_\alpha^c)(\prod_{\beta=q+1}^{i_{q+1}} c_\beta^c))|0\rangle,
\]

\[
\gamma \equiv \sqrt{(2S + N - 1)C_{N-1}^{q-1}}.
\]

Here, \(|\Delta\rangle\) transforms itself as a \( SU(N) \) singlet and it will be annihilated by any of the raising and lowering operators, \( T^\pm |\Delta\rangle = U^\pm |\Delta\rangle = \cdots = 0 \). This “state” would describe the strong coupling ground state for a purely fermionic impurity.

D. Excited States

There are two types of excited states in the strong-coupling regime. Either the degenerate ground state acquires an additional conduction electron at the impurity site, \(|GS+1\rangle\), or it loses one conduction electron, \(|GS-1\rangle\). In the former case, \(|GS+1\rangle\), the spin of the additional conduction electron can be either symmetrically or antisymmetrically correlated with the spin of the impurity as schematized in Fig. 4.

In the limiting case of \( SU(2) \) spin, these two configurations correspond to a spin of the conduction electron that is either parallel or antiparallel to the impurity spin. In the general \( SU(N) \) case, we will keep on speaking of symmetric and antisymmetric configurations respectively.

States with one less electron will be denoted by \(|GS-1\rangle\) and are represented by the Young tableau in Fig. 5. Let us denote by \( \Delta E_1^S = E_1^S - E_0 \), \( \Delta E_1^A = E_1^A - E_0 \) and \( \Delta E_2 = E_2 - E_0 \) the energy differences, with respect to the ground state energy, associated with these three excited states \(|GS+1)^S\rangle\), \(|GS+1)^A\rangle\) and \(|GS-1\rangle\). Using the same casimirology method as presented at the beginning of this section for the determination of the ground state energy, we have summarized our results in Table II respectively for arbitrary \( N \) and in the large-\( N \) limit with \((2S + q - 1)/N \) finite.

One can check that the results in Table II coincide with (Eqs.[25-26]) in ref. 27, within a \( N/2 \) factor stemming from a different definition of the Kondo coupling \( J \) (cf. Eq.(1) of ref. 25) and of the Casimir (cf. Eq.(17) of ref. 25), and a change in the notations \( n_j^c = q \) and \( n_b = 2S \).
TABLE I: Dimension, $d$, and eigenvalues of the casimir operator, $C_2$ for the symmetric, antisymmetric, L-shaped, and fundamental representations studied in this paper. In the L-shaped case, $Q = (2S + q - 1)$ is the total number of boxes in the Young tableau, and $Y' = (q - 2S)$ measures the row-column asymmetry.

|         | Symmetric $[2S]$ | Antisymmetric $[1^q]$ | L-shaped $[2S, 1^{q-1}]$ | Fundamental $[1]$ |
|---------|------------------|-----------------------|--------------------------|------------------|
| $d$     | $C^{2S}_{N+2S-1}$ | $C^0_N$               | $\left(\frac{2S}{2S+q-1}\right)C^{2S}_{N+2S-1}C^{q-1}_{N-1}$ | $N$              |
| $C_2$   | $\frac{1}{2N}(2S(2S + N)(N - 1))$ | $\frac{1}{2N}(q(N - q)(N + 1))$ | $\frac{Q}{2}(N - Y' - Q/N)$ | $\frac{1}{2N}(N^2 - 1)$ |
TABLE II: Strong coupling excitation energies, \( \Delta E^S \), \( \Delta E^A \), and \( \Delta E_z \), in the case of an L-shaped impurity, measured with respect to the ground state, of the states with one more conduction electrons on site 0, coupled symmetrically and anti-symmetrically, respectively, to the dressed impurity on site 0, and of the state with one less electron.

|                  | \( \Delta E^S \) | \( \Delta E^A \) | \( \Delta E_z \)       |
|------------------|------------------|------------------|------------------------|
| Arbitrary \( N \) | \( \Delta(2S + N - q - Q/N) \) | \( \Delta(N - q) \) | \( \Delta(2S - N) \) |
| Large \( N \) limit (\( Q/N \) finite) | \( \Delta(N - q) \) | \( \Delta(N - q) \) | \( \Delta(q + Q/N) \) |

III. STABILITY OF THE STRONG COUPLING FIXED POINT

We have identified the strong coupling fixed point in the previous section. For \( J \to \infty \), the lowest energy state corresponds to \( n_s = (N - q) \) electrons partially screening the impurity at the origin, and free electrons in the other sites, unable to hop to the impurity site.

In order to better understand the low-energy physics of the system, we should consider the finite Kondo coupling, allowing virtual hopping from and to the impurity site. These processes generate interactions between the composite at site 0 and the conduction electrons on neighboring sites, that can be treated as perturbations of the strong coupling fixed point. Applying an analysis similar to that of Nozières and Blandin, to the nature of the excitations, we can argue whether or not the strong coupling fixed point remains stable once virtual hopping is allowed.

We consider a system with an additional site next to the dressed impurity, filled with \( n_d \) electrons. The ground state consists of two multiplets, with different symmetry properties. Once the hopping is turned on, the degeneracy is lifted, and each multiplet acquires a different energy shift denoted by \( \Delta E^S \) and \( \Delta E^A \), respectively (see Fig. 6). We can reproduce this spectrum by considering an effective coupling between the spin of the dressed spin at site 0, and the spin of the \( n_d \) electrons on site 1. If \( E^S \) lies (above)below \( E^A \), the effective coupling is (anti)ferromagnetic.

Thus, if the coupling between the effective spin at the impurity site and that of the electrons on site 1 is ferromagnetic we know, from the scaling analysis at weak coupling, that the perturbation is irrelevant, and the low energy physics is described by the strong coupling fixed point. That is, an underscreened, completely symmetric, effective impurity weakly coupled to a gas of free electrons with a phase shift indicating that there are already \( (N - q) \) electrons screening the original impurity. In the completely antisymmetric case (\( 2S = 1 \)), the phase shift corresponds to the unitary limit, \( \delta = \pi/2 \), for \( SU(2) \), and is a function of \( q/N \) for \( SU(N) \), reaching the unitary limit for \( q = N/2 \) (see appendix C).

If, on the contrary, the effective coupling is antiferromagnetic, the perturbation is relevant, the strong coupling fixed point is unstable and the low-energy physics of the model corresponds to some intermediate coupling fixed point, to be identified. The authors in Refs. 24 and 25, have claimed that the flow away from the strong coupling fixed point is characterized by a two-stage quenching of the impurity. After the impurity has been partially screened by the \( (N - q) \) conduction electrons on site 0, \((N - 1)\) additional conduction electrons tend to accumulate around the impurity, leading to further screening.

In this section we explicitly calculate the effects of hopping on the strong coupling fixed point to the lowest order in perturbation theory, that is, second order in \( t \). We will consider the case with an arbitrary number \( n_d \) of conduction electrons in site 1 generalizing the case \( n_d = 1 \) considered in Ref. 25. This will allow us to understand the origin of the instability of the strong coupling fixed point and eventually enlighten the question about the nature of the Kondo screening: either one-stage or two-stage Kondo effect depending on which regime is considered.

Before turning on the hopping term, let us consider ground states of the form \( |GS, n_d \rangle = \sum |GS_0 \rangle |n_d \rangle \), with \( n_d \) electrons on site 1. According to the \( SU(N) \) symme-
try there are two possible configurations, depending on whether the \( n_d \) electrons are coupled symmetrically or antisymmetrically to the composite on site 0. This corresponds to the Clebsch-Gordan series \([2S - 1] \otimes [n_d] \rightarrow [2S, 1^{n_d - 1}] \oplus [2S - 1, 1^{n_d}]\). We denote the states by \( |GS, n_d\rangle^S \), and \( |GS, n_d\rangle^A \), respectively (Fig. 7).

The \( SU(N) \) symmetry is preserved by the hopping. That means that the perturbation will shift the energies \( |GS, n_d\rangle^S \) and \( |GS, n_d\rangle^A \) separately, without mixing the states. We will thus denote the shifts by \( \Delta E_0^S \) and \( \Delta E_0^A \), respectively.

The hopping term is of the form

\[
H_h = H_1 + H_2 = \sum_{\alpha} c_{\alpha}^d d_{\alpha} + \sum_{\alpha} d_{\alpha}^\dagger c_{\alpha}, \quad (H_1)^\dagger = H_2
\]

where \( d_{\alpha} \) creates an electron on site 1. We can distinguish two types of processes, corresponding to different intermediate states. The first type, which we denote process 1, corresponds to an electron hopping from site 1 into site 0 first, probing excited states \( |GS + 1\rangle^S,A \), and then hopping back to site 1. The indices \( S, A \) correspond to the two possible intermediate states depending whether the conduction electron which hops to site 0 is symmetrically or antisymmetrically correlated with the dressed impurity as we will see in details in next section. The contribution of the process 1 to the energy shift is the following

\[
t^2 \sum_{\alpha, \beta} \frac{\langle GS, n_d | d_{\beta}^\dagger c_{\beta} | GS + 1, n_d - 1 \rangle^i}{(E_0 - E_1)} (E_0 - E_1),
\]

with \( i = S, A \) (see Appendix B).

In process 2, the electron hops from site 0 to site 1 first and then back to site 0, probing \( |GS - 1\rangle \), leading to a contribution to the energy shift of the form

\[
t^2 \sum_{\alpha, \beta} \frac{\langle GS, n_d | c_{\beta}^d d_{\beta} | GS - 1, n_d + 1 \rangle}{(E_0 - E_2)} (E_0 - E_2),
\]

Hence, the energy shifts for the symmetric and antisymmetric configurations are, respectively,

\[
\Delta E_0^S = \frac{M_1^S}{E_0 - E_1} + \frac{M_2^S}{E_0 - E_2},
\]

\[
\Delta E_0^A = \frac{M_2^A}{E_0 - E_1} + \frac{M_1^A}{E_0 - E_2},
\]

where the expressions in the denominators, \( (E_0 - E_1) = -\Delta E_1^S \), \( (E_0 - E_1) = -\Delta E_1^A \), and \( (E_0 - E_2) = -\Delta E_2 \) measuring the energy of the excited states compared to the energy of the ground state are given in Table I. The matrix elements, \( M_i \), will be introduced below as we will study the contribution of each process. The energy difference between the two states,

\[
\Delta E_0^S - \Delta E_0^A = \frac{M_2^S}{E_0 - E_2} + \frac{M_2^A}{E_0 - E_2} - \frac{M_1^S}{E_0 - E_1} - \frac{M_1^A}{E_0 - E_1},
\]

determines the sign of the effective interaction and the stability of the strong coupling fixed point.

### A. Process 1, symmetric configuration

We consider first the case where the \( n_d \) electrons in the site-1 are coupled to the site-0 state in the most symmetric configuration. In the shorthand notation that we use for the Young Tableaux, it corresponds to the state \([2S - 1] \otimes [1^{n_d}] \rightarrow [2S, 1^{n_d - 1}]\). Here, as opposed to the case \( n_d = 1 \), the hamiltonian transforms the ground state

\[
|GS, n_d\rangle^S = (d_{a}^\dagger d_{b}^\dagger \cdots d_{u}^\dagger) |GS\rangle,
\]

into a linear combination of two excited states : \( |GS + 1, n_d - 1\rangle^S \), with energy \( E_S \), and \( |GS + 1, n_d - 1\rangle^S \), with energy \( E_A \) depending on whether the additional conduction electron in site 0 is coupled symmetrically or antisymmetrically to the dressed impurity. The state ob-
and then we have normalized the resulting state. The normalization coefficients \( \Omega \) and \( \Lambda \) are independent of \( n_d \), as we are considering hopping of a single electron. From here, we obtain the following matrix elements:

\[
M_1^S = |^S(GS+1,n_d-1)|H_1|GS,n_d)^S|^2 = t^2 \left( \frac{2S + n_d - 1}{2S} \right) \left( \frac{2S + q - 1}{2S + N - 1} \right), \tag{12}
\]

\[
\overline{M}_1^S = |^S(GS+1,n_d-1)|H_1|GS,n_d)^S|^2 = t^2(n_d-1) \left( \frac{2S - 1}{2S} \right) \left( \frac{q - 1}{N - 1} \right). \tag{13}
\]

We see right away that \( \overline{M}_1^S \) is proportional to \( (n_d - 1) \), and vanishes for \( n_d = 1 \), whereas \( M_1^S \) depends noticeably on \( n_d \) only for \( 2S \ll n_d < N \).

### B. Process 1, antisymmetric configuration

Next, we consider the case where the electrons on site 1 are coupled to the effective spin on site 0 according to \([2S - 1] \otimes [1^{n_d}] \rightarrow [2S - 1, 1^{n_d}]\). In the previous section it was easy to write down the strong coupling ground state by just putting together the effective spin and the \( n_d \) electrons in the highest weight state possible, to obtain Eq. \((10)\). Here we have to work out the necessary Clebsch-Gordan coefficients. We present some of these coefficients in Table \((V)\) of Appendix \((\text{E})\). The explicit form of the ground state is

\[
|GS,n_d\rangle_{abc...v} = \frac{1}{\sqrt{2S + n_d - 1}} \left( \sqrt{2S - 1} \prod_{i=2}^{n_d+1} d_{y_i}|GS\rangle_{aa} + \sum_{j=2}^{n_d+1} (-1)^{j-1} \prod_{i=1,i\neq j}^{n_d+1} d_{y_i}|GS\rangle_{ay_j} \right), \tag{14}
\]

and the matrix element

\[
M_1^A = |^A(GS + 1,n_d - 1)|H_1|GS,n_d)^A|^2 = t^2 n_d \left( \frac{q - 1}{N - 1} \right). \tag{15}
\]

Notice the dependence on \( n_d \), and the fact that the matrix element does not depend on \( 2S \). Combining together \( M_1^S \) and \( M_1^A \) as it appears in Eq. \((10)\), we have

\[
\overline{M}_1^S - M_1^A = -t^2 \left( \frac{2S + n_d - 1}{2S} \right) \left( \frac{q - 1}{N - 1} \right). \tag{16}
\]

This is a term with the same \( n_d \) dependence as \( M_1^S \) but with the opposite sign.
excited state with the same symmetry as
these processes. The evaluation of the two remaining
processes, on the other hand, is simplified by using the following trick connecting the
impurity site, coupled symmetrically (a), or antisymmetrically (b), to the impurity. The result is a state
proportional to a given excited state, with additional conduction electron

\[ |GS, n_d\rangle^A \]

\[ \langle GS, n_d| d_{\sigma}^\dagger c_{\sigma'} c_{\sigma} d_{\sigma'}^\dagger |GS, n_d\rangle^S \]

\[ \delta M^S = t^2 \sum_{\sigma \sigma'} S_{\sigma \sigma'} \langle c_{\sigma'} d_{\sigma}^\dagger d_{\sigma} c_{\sigma} \rangle^S \]

where \( \delta M^S \) should be considered over the ground state, \( |GS, n_d\rangle^S \). On the other hand, the following property holds for the matrix elements of process 2

\[ M_2^A = t^2 \sum_{\sigma} A_{\sigma}^A (d_{\sigma}^\dagger d_{\sigma})^A - \delta M^A, \]

\[ M_2^S = t^2 \sum_{\sigma} S_{\sigma}^A (d_{\sigma}^\dagger d_{\sigma}^\dagger c_{\sigma} c_{\sigma})^A - \delta M^A, \]

\[ M_2^S - M_2^A = M_2^S + M_1^A - M_1^S - M_1^A \]

IV. DISCUSSION

Once we have computed all the matrix elements, we can evaluate the difference in energy shifts between the symmetric and the antisymmetric configurations, \( \Delta E_0^S - \Delta E_0^A \), induced by the perturbation \( H_b \), according to Equation (18). We can then compare the result to an effective spin-spin interaction, with coupling \( J_{c,eff} \), between the spin \( S_0 \) of the dressed impurity at site 0, and the spin \( S_1 \) of the \( n_d \) electrons at site 1. If the symmetric configuration is the lowest in energy (Fig. 10(a)), it means that the effective coupling is ferromagnetic and the perturbation is relevant. Thus, the strong coupling fixed point is stable and allows to describe the low-energy behavior of the system. If, on the contrary, it is the antisymmetric configuration the lowest in energy (Fig. 10(b)), the effective coupling is antiferromagnetic, and we know from scaling arguments that the perturbation is relevant.
The strong coupling fixed point is then unstable with respect to hopping, and the behavior of the system is described by an intermediate coupling fixed point. Incorporating the expressions of the matrix elements, Eqs. [12][13][19], and those of the excitation energies (Table I) into Eq. (11), one finds

\[
\Delta E^S_0 - \Delta E^A_0 = -(2S + n_d - 1) \left( \frac{2t^2}{J} \right) \times \left\{ \frac{2S + q - 1}{2S(2S + N - 1)(2S + N - q - (2S + q - 1)/N)} \right. \\
+ \left. \frac{(N - q)(q + (2S + q - 1)/N)}{2S(N - 1)(N - q - (2S + q - 1)/N)} \right\}. 
\]

(20)

As the bosonic part increases, the energy difference becomes smaller. The dependence of the r.h.s. of Eq. (20) with \(2S + n_d - 1\) is linear and appears factored out. The effect of \(n_d\) on \(\Delta E^S_0 - \Delta E^A_0\) is weak as long as \(n_d \ll 2S\).

\[
\Delta E^S_{[2S,1^n_d]} - \Delta E^A_{[2S,1^n_d]} = \frac{J_{eff}}{2} \left[ C_2([2S,1^n_d]) - C_2([2S,1,1^n_d]) \right] \\
= -\frac{J_{eff}}{4} (2S + n_d - 1) \left[ Y'_{[2S,1^n_d]} - Y'_{[2S,1,1^n_d]} \right] \\
= \frac{J_{eff}}{2} (2S + n_d - 1),
\]

where we have used the results of Table I. Since both states have Young Tableaux with the same number of boxes, \(Q_{eff} = 2S + n_d - 1\), the energy difference depends only on the second constraint \(B7\), \(Y'_{eff} = Y'\) (see Appendix B). As a consequence, the dependence on \(n_d\) is factorized in the same way as in Eq. (20). Thus, we can identify the effective coupling, \(J_{eff}\), with the term in curly brackets in Eq. (20) times \(4t^2/J\). In Fig. 11 we plot this energy difference for several values of \(S\).

FIG. 11: Effective coupling, \(J_{eff}\), as a function of \(q/N\), for different values of \(2S\), in the large-N limit.

We now calculate explicitly the interaction energy of two spins, in the representations \([2S - 1]\) and \([1^n_d]\), respectively, and identify the effective coupling \(J_{eff}\) from

\[
J_{eff} \sum_A \left( S_0^{[2S-1]} \right)^A \left( S_1^{[1^n_d]} \right)^A,
\]

where \(A = 1, N^2 - 1\). The spectrum consists of two multiplets, according to the Clebsh-Gordan Series \([2S - 1] \otimes [1^n_d] \to [2S,1^n_d-1] \otimes [2S - 1,1^n_d]\). We have added superindices to indicate which is the symmetric state and which is the antisymmetric one. As before, the energy is given in terms of Casimir operators, Eq. (11), and the energy difference between the states is

\[
\Delta E^S_{[2S,1^n_d-1]} - \Delta E^A_{[2S,1^n_d-1]} = \frac{J_{eff}}{2} \left[ C_2([2S,1^n_d-1]) - C_2([2S,1,1^n_d]) \right] \\
= \frac{J_{eff}}{4} (2S + n_d - 1) \left[ Y'_{[2S,1^n_d-1]} - Y'_{[2S,1,1^n_d]} \right] \\
= \frac{J_{eff}}{2} (2S + n_d - 1),
\]

(21)

\(J_{eff}\) is of order \((1/N^2)\). The result expressed in Eq. (21) means that the value of the effective coupling \(J_{eff}\) is independent of the number of conduction electrons, \(n_d\), on site 1 and coincides with the result obtained in Ref. 27 in the case \(n_d = 1\). This is due to the cancellation of the \((2S + n_d - 1)\) factor that we have mentioned before. The coupling remains ferromagnetic as long as \(q < N/2\), as can be seen in Fig. 12 and by inspection of the numerator in the r.h.s. of Eq. (21). As soon as \(q > N/2\), the strong coupling fixed point becomes unstable.

The case \(q = N/2\) requires particular attention, since the leading contribution to \(J_{eff}\) vanishes. Taking into account the whole expression for the effective coupling, we
dependence of $\Delta$ is smaller than a critical value with

find that the strong coupling fixed point for an impurity

separately. It is easy to show that

$13)$. We see that the strong coupling fixed point at $q = N/2$ becomes unstable already for moderate values of $2S/N$ (short line ending in a point).

It is worth studying the energy shift for each state

separately. It is easy to show that

$\Delta E^A_0 = \Delta E^S_0 = \Delta E^A_0 + (2S + n_d - 1) J_{eff} / 2$. (23)

Since $J_{eff}$ is $O(1/N^2)$ in the large-N limit, the energy difference ($\Delta E^S_0 - \Delta E^A_0$) is $O(1/N)$, and both energy levels ($\Delta E^S_0$ and $\Delta E^A_0$) have the same leading term in $O(1)$. Notice that when we consider the large-N limit in which $Q/N$ is finite, the antisymmetric energy shift, $\Delta E^S_0$ is almost independent of $2S$. The dependence of

$\Delta E^S_0$ on $S$ is mainly contained in the $J_{eff}$ term, which will eventually determine the stability of the system.

The most important property of Eq. (22) is its behavior as a function of $n_d$. To leading order in $1/N$, we can write

$\Delta E^A_0 = - \left( \frac{2t^2}{J} \right) \left[ \left( \frac{n_d}{N-1} \right) \left( \frac{q-1}{N-q-Q/N} \right) \right. \left. + \left( 1 - \frac{n_d}{N-1} \right) \left( \frac{N-q}{q+Q/N} \right) \right]$. (24)

Notice that the factor multiplying $n_d/N$ in the r.h.s. of

Eq. (24) also appears in $J_{eff}$ (see Eq. (21)). This result has the immediate following physical consequence. The change of sign of $J_{eff}$, and hence of the stability of the strong coupling fixed point, is directly connected to the change in the behavior of $\Delta E^A_0 \sim \Delta E^S_0$ with $n_d$. In particular, when $J_{eff} = 0$

$\Delta E^A_0 (q = N/2) = - \frac{2t^2}{J}$,

for any value of $n_d$. The value of $n_d$ that minimizes the energy given by Eq. (24) at arbitrary $q$ will depend on the sign of $(N - 2q)$. If the regime where the strong coupling fixed point is stable, $q/N < 1/2$, $J_{eff} < 0$, the lowest energy corresponds to $n_d = 1$, whereas for $q/N > 1/2$, $J_{eff} > 0$, the energy expressed in Eq. (24) is minimized for $n_d = (N - 1)$. We have plotted $\Delta E^S_0$ in Fig. 13 (compare to Fig. 11). The shaded region corresponds to the possible values of $\Delta E^A_0$ for the whole range of $n_d$, bounded by the limiting cases, $n_d = 1$, and $n_d = (N - 1)$. In the region where the strong coupling fixed point is stable, the electrons on site 1 are repelled by the effective impurity (repulsive regime), whereas in the unstable, $q/N > 1/2$ regime, the lowest energy in second order perturbation theory corresponds to a state where the effective impurity attracts as many electrons as possible on site 1 (attractive regime). This is precisely the mechanism behind the two-stage quenching.

FIG. 12: Proportionality factor $\lambda = N(N - 2q)/q(N - q)$ as a function of $q/N$. $\lambda$ determines both the sign of $J_{eff}$ and the dependence of $\Delta E^A$ on $n_d$.

FIG. 13: Phase diagram of the model (for $2S/N$ finite), as a function of the impurity parameters, $2S$, and $q$. As soon as $q > N/2$ the strong coupling fixed point becomes unstable. For $q = N/2$, the strong coupling fixed point remains stable only for moderate values of $2S/N$ (short line ending in a point).
The accumulation of electrons on site 1 is not related to $J_{\text{eff}}$ which is independent of $n_d$, but results from the dependence of $\Delta E_0^A \sim \Delta E_0^S$ with $n_d$. Moreover, as we show in Appendix C, $\Delta E_0^A$ coincides with the energy shift for a fermionic impurity (completely antisymmetric representation), in the large-N limit. When the impurity is fermionic, there is no degeneracy of the strong coupling fixed point, which is always stable, and the lowest order perturbation theory just shifts the ground state energy. Nevertheless, there are two regimes, repulsive and attractive, depending on the value of $q$, and characterized by the value of $n_d$ that minimizes the energy. This behavior is a consequence of the particle-hole symmetry in the fermionic case, given by the transformations $q \to (N-q)$ and $n_d \to (N-n_d)$ (cf. Appendix C). The behavior of a fermionic impurity with $q$ is the same as in the case $(N-q)$, if we reinterpret the electrons as holes and the impurity as made out of holes. Therefore, if $n_d = 1$ minimizes the energy for $q < N/2$ (electron repulsion), then the energy for a hole impurity, made out of $(N-q)$ fermions, is minimized by the state that repels the holes, $(N-n_d) = 1$, implying an attraction of electrons, $n_d = (N-1)$.

The addition of a bosonic component to the impurity, leading to the formation of a row in the L-shaped Young tableau representing the impurity, breaks this particle-hole symmetry. Whereas the two regimes described above are still present, due to the fermionic component, the degeneracy of the states due to the bosonic component leads to the instability of the strong coupling fixed point at the same point as where the dressed impurity starts attracting the conduction electrons on site 1.

We finish by making some remarks on the physical properties of the model in the different regimes. As is common to all models with an antiferromagnetic Kondo coupling, there will be a crossover from weak coupling above a given Kondo scale, $T_K$, to a low-energy regime. When the strong coupling fixed point is stable, we should expect for $T \ll T_K$ a weak coupling of the effective impurity at site-0 with the rest of the electrons. The physical properties at low temperature are controlled by the degeneracy of the effective impurity, $d([2S-1]) = C_{N+2S-2}^{N-1}$. Thus, we should expect a residual entropy $S^i \sim \ln C_{N+2S-2}^{N-1}$ and a Curie susceptibility, $\chi^i \sim C_{N+2S-2}^{N-1}/N$, with logarithmic corrections.

This is the result that we would expect for a purely symmetric impurity. The difference with respect to the case at hand is that in the L-shaped impurity model only $(N-q)$ electrons are allowed at the origin, instead of $(N-1)$. Thus, we would expect to find different results for quantities that involve the scattering phase shift of electrons off the effective impurity (see Appendix C).

In the $q > N/2$, we do not have access to the intermediate coupling fixed point that determines the low-energy behavior. Nevertheless, it is reasonable to think that there would be a magnetic contribution to the entropy, and a Curie-like contribution to the susceptibility, since the impurity remains unscreened. This behavior is different from that of the multichannel Kondo model, which is also characterized by an intermediate coupling fixed point, but where the impurity magnetic degrees of freedom are completely quenched. The degeneracy of the true ground state is an open question, but we can assume that the entropy will be smaller than that of the strongly coupled fixed point. It is in the scattering properties that we might be able to see the anomalous features of this new fixed point more clearly.

V. CONCLUSIONS

In this paper, we have studied the $SU(N)$, single-channel Kondo model, with a general impurity spin, involving both bosonic and fermionic degrees of freedom (corresponding, respectively, to the horizontal and vertical directions in a L-shaped Young tableau). This model shows a transition from a strong coupled fixed point to an intermediate coupling fixed point, when the amount of fermionic degrees of freedom, $q$, becomes larger than $N/2$. We have identified the origin of this instability as due to the change, from repulsive to attractive, of the effective interaction between the dressed impurity and the conduction electrons in the neighboring sites. This change is already present in the purely fermionic case, where it happens at the particle-hole symmetry point, $q = N/2$. 

![FIG. 14: Leading order term in the energy shift, $\Delta E_0^A \sim \Delta E_0^S$, as a function of $q/N$, for $1 < n_d < (N-1)$ (shaded region), and in the limiting cases $n_d = 1$ (dashed line), and $n_d = (N-1)$ (straight line). Notice that the value at $q/N = 1/2$ is equal to $-2t^2/J$, for any $n_d$. Note that for $q/N < 0.5$, the energy is minimized for $n_d = 1$ while for $q/N > 0.5$, the minimization is obtained for $n_d = (N-1)$](image-url)
These results shed light on the nature of the Kondo screening. Whereas the screening is restricted to a one-stage process in the strong-coupling regime, involving mainly the fermionic part of the impurity, the accumulated screening of conduction electrons on neighboring sites in the attractive regime gives rise to a two-stage Kondo effect. The latter case corresponds to a regime where the low-energy physics is controlled by an intermediate-coupling fixed point. Obviously, the interesting open problem now is to understand the physics associated with the this new intermediate coupling fixed point. This issue might have important future applications for the lattice problem, with potential consequences for the understanding of non-Fermi liquid behaviour observed in heavy-fermion systems.

Acknowledgments

The authors would like to thank N. Andrei for his continuous encouragement and discussions, and for a critical reading of this paper. We would also like to thank S. Burdin, P. Coleman, Ph. Nozières, et C. Pépin for very helpful discussions. We are grateful to two different Programs, **Strongly Correlated Electron Systems** at the ICTP, Trieste in 1999, and at the Isaac Newton Institute for Mathematical Sciences, Cambridge in 2000 where this work was initiated and further developed. We would like to take this opportunity to thank the organizers for creating a stimulating environment for scientific exchange.

**APPENDIX A: COMPOSITION OF THREE FUNDAMENTAL REPRESENTATIONS OF SU(3)**

Before dealing with the general problem of constructing the highest weight impurity states in SU(N) in Appendix B, we will write down in detail all the three-particle states with SU(3) symmetry. This will allow us to see how the states constructed with different numbers of bosons and fermions can be basis for representations with the same Young tableau. We will also see the role of the SU(3|3) and SU(1|1) supersymmetry groups induced by the realization in terms of bosons and fermions.

The direct product $3 \otimes 3 \otimes 3$ of three fundamental representations of SU(3) gives the following Clebsh-Gordan result from the composition of two fundamental representations

$$3 \otimes 3 = 6 \oplus \bar{3}.$$  \hfill (A2)

In addition to Young Tableaux, we can use weight diagrams to describe the states in the representation, Fig. 15. In SU(3), we associate a triangle to the fundamental representation, 3. Each vertex corresponds to a particular state of the multiplet, and the different states are related by the action of the lowering operators. In Fig.

![Weight Diagram](image)

FIG. 15: Weight diagram for the fundamental representation of SU(3), with the states and the relevant lowering operators.

We include the weight diagrams associated with Eq. (A2).

The representation 6 is completely symmetric. Its states are realized in terms of Schwinger bosons. For instance its highest weight state can be written as

$$|aa\rangle_6^f = \frac{1}{\sqrt{2!}} |b\rangle^1_a |0\rangle ,$$

Here, and in the following, the values of SU(3) spin are denoted by a, b, and c. Likewise, the representation $\bar{3}$ is completely antisymmetric, and its states are more conveniently expressed in terms of fermions. For the highest weight state we have

$$|ab\rangle_3^f = f_a^f f_b^f |0\rangle .$$

There is another way of realize both 6 and $\bar{3}$ using one boson and a fermion. Being symmetric, the highest weight of 6 is easy to write, since

$$|aa\rangle_6^f = f_a^f b_a^f |0\rangle ,$$

is already symmetrized. To obtain the highest weight of $\bar{3}$, we first have to find a state with the same quantum numbers in 6, by acting with $T_a^b = (f_b^f b_a^f + f_a^f b_b^f)$ on $|aa\rangle_6^f$, to get

$$|ab\rangle_3^f = \frac{1}{\sqrt{2}} (f_a^f b_a^f + f_b^f b_b^f) |0\rangle ,$$
FIG. 16: Weight diagram for the Clebsch-Gordan series of the product of two fundamental representations of $SU(3)$. 
and find a state orthogonal to $|ab\rangle^6$

$$|ab\rangle^3 = \frac{1}{\sqrt{2}} (f_a^b b_a^b - f_a^b b_b^b) |0\rangle .$$

This process is described in Fig. 16. It is easy to see that the states with the subindex $f$ are related to those with the subindex $b$ by the SU(1|1) supersymmetric operator $\theta = \sum_a b_a^b f_a$, so that $\theta (\ldots) f = \sqrt{2} (\ldots) b$.

Consider now the three-particle states. The easiest state to write is the highest weight state in the most symmetric representation, $10$ (cf. Fig 17)

$$|aaa\rangle_{10} = |a\rangle^3 |aa\rangle^6 .$$

It can be easily expressed in terms of bosons in agreement with Eq. (B3)

$$|aaa\rangle_{b} = \frac{1}{\sqrt{3}} (b^a\rangle^3 |0\rangle .$$

Alternatively, we can use a realization with two bosons (from 6) and a fermion (from 3)

$$|aaa\rangle_{f} = \frac{1}{\sqrt{2}} f_a^f (b_a^b)^2 |0\rangle .$$

Other states of the representation are obtained by the repeated action of lowering operators. For instance,

$$|aab\rangle_{10} = \frac{1}{\sqrt{3}} (\sqrt{2} |a\rangle^3 |ab\rangle^6 + |b\rangle^3 |aa\rangle^6 ) ,$$

which leads to

$$|aab\rangle_{b} = \frac{1}{\sqrt{2}} (b_a^f b_b^a |0\rangle ,

$$|aab\rangle_{f} = \frac{1}{\sqrt{3}} (2 f_a^b b_a^b f_b^b + f_b^f (b_b^f)^2 ) .$$

The octets, 8 are mixed symmetry representations that have to be built by a combination of fermions and bosons. The Clebsch-Gordan series (A1) indicates that $8^1$, built from the product $3 \otimes 6$ is naturally realized by states with one fermion and two bosons, Fig. 17, whereas $8^2$ is realized by the product of one boson and two fermions $(3 \otimes 3)$, Fig. 18. The highest weight state, $|aab\rangle^8$, of the octet $8^1$ is orthogonal to $|aab\rangle^16$ defined in Eq. (A3)

$$|aab\rangle^8_b = \frac{1}{\sqrt{3}} (|a\rangle^3 |ab\rangle^6 - \sqrt{2} |b\rangle^3 |aa\rangle^6 )$$

$$= \frac{1}{\sqrt{3}} b_a^b (f_a^b b_b^f - f_a^f b_b^b) |0\rangle ,$$

in agreement with the general expression of the state $\psi_b$ given in (B3).

As usual, the other states of the octet are built by the repeated action of generators $T^- = (f_a^b f_a + b_a^b b_a)$,

$$U^- = (f_a^b f_a + b_a^b b_a),$$

and $V^- = (f_a^b f_a + b_a^b b_a)$, of SU(3). For instance

$$|abc\rangle^8_b = \frac{1}{\sqrt{2}} T^- |aac\rangle^8$$

$$= \frac{1}{\sqrt{6}} (|a\rangle^3 |bc\rangle^6 + |b\rangle^3 |ac\rangle^6 - 2 |c\rangle^3 |ab\rangle^6 )$$

$$= \frac{1}{\sqrt{6}} (f_a^b b_a^b f_c^b + f_c^b b_c^b f_a^b - 2 f_a^b b_a^b f_c^b) |0\rangle .$$

This state is degenerate, since there is another state in the multiplet with the same quantum numbers. In order to find this last state, $|abc\rangle^8$, we have to combine the action of lowering operators with orthogonality with respect to $|abc\rangle^8$. Acting with $U^- |abc\rangle^8$ leads to a state that is not orthogonal to $|abc\rangle^8$. Therefore, we write

$$U^- |aac\rangle^8 = \sqrt{2} (\alpha |abc\rangle^8 + \beta |abc\rangle^8) .$$

There are two ways of reaching states with quantum numbers $|abc\rangle$ starting from $|aab\rangle^8$. We use this and the fact that $[T^+, U^-] = 0$, $(T^+ = (T^+)^*)$, to derive

$$2 \alpha = \sqrt{2} \langle abc | U^- | abc \rangle^8 = \langle aac | T^+ U^- T^- | aac \rangle^8$$

$$= \langle aab | U^+ T^+ U^- T^- | aab \rangle^8$$

$$= \langle aab | U^+ T^- U^- T^- | aab \rangle^8 = 1 .$$

Hence one deduces $\alpha = 1/2$ and $\beta = \sqrt{3}/2$ and one gets the expression of the last state $|abc\rangle^8$ of the octet

$$|abc\rangle^8_b = \frac{1}{\sqrt{2}} (|a\rangle^3 |bc\rangle^6 - |b\rangle^3 |ac\rangle^6 )$$

$$= \frac{1}{\sqrt{2}} (f_a^b b_a^b f_c^b - f_a^f b_a^b f_b^f) |0\rangle .$$

To summarize, the rule is to describe the highest weight state of the octet $|aab\rangle_b$ using the bosonic representation, $\psi_b$, and hence to derive the other states of the octet by the above construction allowing to recover the Clebsch-Gordan coefficients involved in the spin composition related to the direct product $3 \otimes 6$.

The highest weight state of the octet $8^2$ is also the highest weight state of $3 \otimes 3$

$$|aab\rangle_f^2 = |a\rangle^3 |ab\rangle^3 = b_a^f f_a^f b_b^f |0\rangle ,$$

in agreement with the general expression of the state $\psi_f$, (F3). The construction of the other states follow the same lines as in the case of the $8^1$ octet. For instance,

$$|abc\rangle_f^2 = \frac{1}{\sqrt{2}} (|a\rangle^3 |bc\rangle^3 - |b\rangle^3 |ca\rangle^3 )$$

$$= \frac{1}{\sqrt{2}} (b_a^f f_a^f f_b^b f_c^f - f_a^f b_a^f f_b^f f_c^f) |0\rangle ,$$

$$|acb\rangle_f^2 = \frac{1}{\sqrt{6}} (|a\rangle^3 |bc\rangle^3 + |b\rangle^3 |ca\rangle^3 - 2 |c\rangle^3 |ab\rangle^3 )$$

$$= \frac{1}{\sqrt{6}} (b_a^f f_a^f f_b^b f_c^f + b_a^f f_a^f f_b^f f_a^f - 2 f_a^f b_a^f f_b^f f_a^f) |0\rangle .$$
FIG. 17: Weight diagram for the Clebsch-Gordan series of the product $3 \otimes 6$. We indicate some of the states and outline the process of obtaining Clebsch-Gordan coefficients.

FIG. 18: Weight diagram for the Clebsch-Gordan series of the product $3 \otimes \bar{6}$. 
Once again, the two basis states corresponding to the same Young tableau, are related by the $SU(1|1)$ operators, $\theta^\dagger$, and $\theta$.

Finally, the singlet state, $|abc\rangle$, is built by orthogonality with the states $|abc\rangle^{8^2}$ and $|acb\rangle^{8^2}$, from the octet in the product $3 \otimes 3$ (Fig. 18),

$$|abc\rangle = \frac{1}{\sqrt{3}}(|a\rangle^3|bc\rangle^3 + |b\rangle^3|ca\rangle^3 + |c\rangle^3|ab\rangle^3), \quad (A4)$$

in agreement with the expression of the states in the completely antisymmetric representation of the spin $\binom{3}{3}$. The simplest way to realize this state is with three fermions. Then

$$|abc\rangle = f^\dagger_a f^\dagger_b f^\dagger_c |0\rangle. \quad (A5)$$

But it can also be written with one boson and two fermions, either by acting with $\theta$ on $\binom{1}{3}$ or by substituting on $\binom{1}{1}$

$$|abc\rangle = \frac{1}{\sqrt{3}}(b^\dagger_a f^\dagger_b f^\dagger_c + b^\dagger_b f^\dagger_c f^\dagger_a + b^\dagger_c f^\dagger_a f^\dagger_b).$$

We have shown how to construct the states for different representations of $SU(3)$. These results can be summarized as tables of Clebsch-Gordan coefficients like the ones that we presented in Tables $\|V\|V\|$ for $SU(N)$.

Let us make some comments about the number of states. The direct product of three fundamental representations of $SU(3)$ generates a space of dimension 27, which breaks down as a direct sum of irreducible representations according to the Clebsch-Gordan series $\binom{3}{3}$. By considering all the realizations of these states in terms of bosons and fermions subject only to the constraint $Q = n_f + n_b = 3$, we are working on the higher dimensional space of a representation of $SU(3)$ with a total of 38 states, as represented in Fig 19. The figure also reports the relation between these states with the introduction of an additional supersymmetric operator, $Z_{abc} = f^\dagger_a b^\dagger_b c^\dagger_c$, acting on the highest weight states. The states $10$ and $10'$ respectively are identical (as well as $1$ and $1'$) as far as the $SU(3)$ symmetry is concerned. This is not the case of the two states $8^1$ and $8^2$ which correspond to different spin representations even if the associated Young tableaux are the same. Altogether, one recovers the expected total of 27 different states.

**APPENDIX B: IMPURITY STATES AND LOW-LYING STATES IN THE STRONG-COUPLING LIMIT**

We present in this appendix the explicit form of the highest weight (spin) states for the impurity and for the low-lying states in the $J \to \infty$ limit. We discuss in detail the use of Young Tableaux both to describe the symmetry properties of the states and to study de $SU(N)$ generalization of the composition of several spins.

$\begin{array}{llll}
 n_b = 3, \quad n_f = 0 & 10 & b & b \quad b \\
 Z_{ab} \downarrow & \bigwedge \theta^\dagger & \\
 n_b = 2, \quad n_f = 1 & 8^1 & 0 & \bar{b} \\
 Z_{ac} \downarrow & \bigwedge \theta^\dagger & \\
 n_b = 1, \quad n_f = 2 & 1^\prime & \bar{b} & \bar{f} \\
 8^2 & \bar{f} & \bar{f} & \bar{f} \\
 n_b = 0, \quad n_f = 3 & 1 & \bar{f} & \bar{f} & \bar{f}
\end{array}$

FIG. 19: States with $Q = (n_b + n_f)$ grouped according to $SU(3)$ representations

1. Impurity State

Before studying the general case of a L-shaped Young tableau representation of spin, we will consider the two limiting cases of a completely symmetric (bosonic) and antisymmetric (fermionic) representations of the spin in $SU(N)$.

The case of a completely symmetric representation of the spin is equivalent to a system of $2S$ identical particles symmetric under the permutation of two of them. The associated Young tableau is made of a single line of $2S$ boxes

expressed in shorthand notation as $[\bar{2}S]$. Associated with the Young tableau, there is a symmetrizer operator made out of the sum of all the permutations of $2S$ elements. It is convenient to use an explicit representation of the localized spin in terms of $N$ species of Schwinger bosons $b_\alpha$, $(\alpha = a, b, ..., r_N)$ subject to the constraint

$$\vec{n}_b = \sum_\alpha b^\dagger_\alpha b_\alpha = 2S.$$

The $(N^2 - 1)$ components of the spin operator can be represented as $S^A = \sum_\alpha b^\dagger_\alpha \tau_{\alpha \beta} b_\beta$, where the highest weight state (the analog of the state with the largest value of $S^z$ in $SU(2)$), can be written as

$$|(a)^{2S}\rangle = \frac{1}{\sqrt{(2S)!}}(b^\dagger_a)^{2S}|0\rangle, \quad (B1)$$
where $|0\rangle$ denotes the vacuum state for the bosons. Other states of the representation can be obtained from this one by the repeated action of lowering operators, taking advantage of the underlying $SU(2)$ subalgebras within $SU(N)$. Take, for instance, the $(2S + 1)$ states $\{|(a^\dagger)^x(b^\dagger)^y\rangle^{[2S]}\}$ with $x + y = 2S$. They transform as a regular, $SU(2)$, spin-$S$ multiset under the action of the $SU(N)$ operators $T_{ab} = b_a^\dagger b_a$, $T_{ab}^+ = b_a^\dagger b_b$, and $T_{ab}^- = (b_a^\dagger b_a - b_b^\dagger b_b)/2$. In particular,

$$T_{ab}^\dagger |(a)^{2S}\rangle^{[2S]} = \sqrt{2S} \frac{1}{\sqrt{(2S - 1)!}} (b_a^\dagger)^{2S-1} b_a^\dagger |0\rangle .$$

Note that each index in the set of quantum numbers, $\{\alpha, \beta, \ldots, \rho, 2S\}$, describing the states of the representation, can take $N$ values independently of the rest of the set, and that for each set of values there is only one state. The dimension of the representation is thus given by $C_{N+2S-1}^{2S}$, corresponding to the number of ways of choosing $2S$ elements out of a group of $(N + 2S - 1)$.

The other limiting case corresponds to a completely antisymmetric representation of the spin. It is equivalent to the case of $q$ identical antisymmetric particles under the permutation of two of them. The associated Young tableau is made out of a single column of $q$ boxes

$$q \left\{ \begin{array}{c} \ \ \ \ \ \ \ \ \end{array} \right\} \leftarrow \sum_{P \in S_q} \delta_P P ,$$

and expressed in shorthand notation as $|1^q\rangle$ with $q < N$. Associated with the Young tableau, there is an antisymmetrizer operator made out of the sum of all the permutations of $q$ elements weighted by a $\delta_P = \pm 1$ factor as for antisymmetric identical particles. It is convenient to use an explicit representation of the localized spin in terms of $N$ species of Abrikosov pseudo-fermions $f_\alpha$ ($\alpha = a, b, \ldots, f_N$) subject to the constraint

$$\hat{n}_f = \sum_\alpha f_\alpha^\dagger f_\alpha = q .$$

The generators of $SU(N)$ in this realization are $S^\alpha = \sum_{\alpha\beta} f_\alpha^\dagger \tau_{\alpha\beta} f_\beta$, and the highest weight state of the representation can be written as

$$|(a^\dagger b^\dagger)_{r_1} \cdots r_q\rangle^{[1^q]} = f^\dagger_{a_1} f^\dagger_{b_1} \cdots f^\dagger_{r_q} |0\rangle .$$

involving a set $\{a, b, \ldots \}$ of $q$ different indices. Other states of the representation can be obtained from this one by the repeated action of lowering operators such as $T^\dagger_{ab} = f^\dagger_a f_b$, taking advantage of the underlying $SU(2)$ subalgebras. Note that the states are $SU(2)$ doublets with respect to these subalgebras. The dimension of the representation is given by $C^q_N$.

An important property of both kinds of representations is that the states are non-degenerate. That is, each set of allowed quantum numbers completely determine the state. This is not the case for mixed symmetry representations, as we are about to see.

Let us now consider the general L-shaped representation of spin, Fig. 20, which interpolates between the previous two limits. Its dimension can be easily obtained using Robinson’s formula: the result is $C_{N+2S-1}^{2S}$, (Table IV). The L-shaped representation is the result of the direct product of a symmetric and an antisymmetric representation. This can be done in two, non-equivalent ways (Fig. 21) : either as the $[1^{q-1}] \otimes [2S] \to [2S, 1^{q-1}] \oplus \cdots$, Clebsch-Gordan series, or as $[1^q] \otimes [2S-1] \to [2S, 1^{q-1}] \oplus \cdots$. The construction of the highest weight states for each of the cases is detailed at the end of this subsection (cf. Eqs. (B9) and (B12)) and leads to

$$\psi_b = |(a)^{2S} b_{r_1} \cdots r_q\rangle^{[2S, 1^{q-1}]} = \frac{1}{\sqrt{2S + q - 1}} \frac{(b_a^\dagger)^{2S-1}}{\sqrt{(2S - 1)!}} A(b_a^\dagger f^\dagger_{a_1} f^\dagger_{b_1} \cdots f^\dagger_{r_q}) |0\rangle ,$$

and

$$\psi_f = |(a)^{2S} b_{r_1} \cdots r_q\rangle^{[2S, 1^{q-1}]} = \frac{(b_a^\dagger)^{2S-1}}{\sqrt{(2S - 1)!}} (f^\dagger_{a_1} f^\dagger_{b_1} \cdots f^\dagger_{r_q}) |0\rangle ,$$

where $A(\cdots)$ is the antisymmetrizer. The impurity spin operator has a form which is independent of either $2S$ or $q$ and is given by $S = \sum_{\alpha\beta} (b_a^\dagger \tau_{\alpha\beta} b_\beta + f_a^\dagger \tau_{\alpha\beta} f_\beta)$. The first constraint has to do with the conservation of the number of particles

$$\hat{Q} = \hat{n}_f + \hat{n}_b = (2S + q - 1) ,$$

where $(2S + q - 1)$ denotes the number of boxes in the L-shaped Young tableau. In the limiting cases discussed previously, once the value of $Q$ is fixed, the representation is completely determined. Here, however, it is necessary to add a second constraint to identify states with
the right symmetry. To that end, it is worth noticing that the set of states $\psi_b$ and $\psi_f$ form a basis for a representation of the larger, supersymmetric group $SU(N|N)$, with generators given as linear combinations of the operators $b_\alpha^\dagger b_\beta$, $f_\alpha^\dagger f_\beta$, $b_\alpha^\dagger f_\beta$, $f_\alpha^\dagger b_\beta$. Thus, all the L-shaped impurities that interpolate between the symmetric and the antisymmetric case are related by the supersymmetric group. As a matter of fact, the constraints that fix the antisymmetric case are related by the supersymmetry that interpolate between the symmetric and antisymmetric case are related by the supersymmetric group. 

Consider for instance, the action of $\theta_1^\dagger$ on $\psi_b$, 

$$
\begin{align*}
\theta_1^\dagger (b_\alpha^\dagger)^{2S-1}A(b_\alpha^\dagger f_\beta^\dagger f_\gamma^\dagger \cdots f_{r_q}^\dagger) |0\rangle &= (2S + q - 1)(b_\alpha^\dagger)^{2S-1}(f_\beta^\dagger f_\gamma^\dagger \cdots f_{r_q}^\dagger) |0\rangle.
\end{align*}
$$

where the right hand side corresponds to $\psi_f$. This leads to the relations

$$
\begin{align*}
\theta_1^\dagger \psi_b &= \sqrt{2S + q - 1} \psi_f,
\psi_f &= \sqrt{2S + q - 1} \psi_b.
\end{align*}
$$

Notice also that $\theta_1^\dagger \psi_f = \theta_1 \psi_b = 0$. The operators $\theta$ and $\theta_1^\dagger$ relate states that transform under a representation of $SU(N)$ given by the same Young Tableau. Together with $\hat{Q}$, they form the $SU(1|1)$ supersymmetric algebra \cite{24}. Furthermore, the operators $P_b = \frac{1}{2} \theta \theta_1^\dagger$ and $P_f = \frac{1}{2} \theta_1^\dagger \theta$ are the projectors out of the bosonic $\psi_b$ and the fermionic $\psi_f$ states, respectively. The states $\psi_f$ and $\psi_b$ are the exact analog of the familiar example of the formation of the two octets, $8^1$ and $8^2$, out of the composition of three fundamental representations in $SU(3)$. In appendix A we construct the states explicitly in this example and derive the corresponding Clebsch-Gordan coefficients.

The second constraint is then given by $\hat{Y}$, a bilinear combination of the operators \{$\hat{n}_f, \hat{n}_b, \theta, \theta_1^\dagger$\} since it is a consequence of the invariance of the Casimir operator, $\hat{C}_2$, (the $SU(N)$ generalization of $S^2 = S(S + 1)$), which for a L-shaped representation is given by

$$
\hat{C}_2(\hat{R}) = \sum_A S^A S^A = \frac{1}{2} \left[ \hat{Q}(N - \frac{\hat{Q}}{N}) - \hat{Y} \right].
$$

Here, $\hat{Y} = \hat{Q}(\hat{n}_f - \hat{n}_b) + [\theta, \theta_1^\dagger]$. Once the first constraint, (B5) is fulfilled, the invariance of the Casimir $\hat{C}_2$ is ensured provided that the operator $\hat{Y}$ is invariant too. This leads to the second constraint

$$
\hat{Y} = Q(q - 2S)
$$

It is easy to check that the operators $\theta$ and $\theta_1^\dagger$ commute with $\hat{Q}$ and $\hat{Y}$, which implies that the constraints are also compatible with the $SU(1|1)$ supersymmetry. Note that this was not the case for the operator $\hat{Y}' = \hat{n}_f - \hat{n}_b + \sqrt{\theta, \theta_1^\dagger}$ defined in Eq. 10 of ref. 24.

The constraints completely determine the representation, but they cannot distinguish between the states $\psi_f$ and $\psi_b$. The physical properties of the system depend only on the Young tableau associated with the Kondo impurity, and not on the particular way the representation basis is constructed.

We finish this section by describing in detail the construction of the relevant states of the impurity multiplet.

The direct product of irreducible representations of $SU(N)$ decomposes into a direct sum of irreducible representations (Clebsch-Gordan series). A well-known example is the addition of angular momentum in $SU(2)$. In order to find the states in the new basis (leading to the Clebsch-Gordan coefficients (CGC)), we follow a similar procedure to the one used for angular momentum (see also Appendix A). That is, we first identify the highest weight state of the product of representations, which is always non-degenerate, with the highest weight state of the most symmetric of the representations in the Clebsch-Gordan series. Then, we use lowering operators (particular combinations of the generators of $SU(N)$) to generate other states in the same representation. For arbitrary $N$ there might be more than one state with the same quantum numbers in the same representation (as in the $SU(3)$ octet, $8^1$ and $8^2$), and we will have to find orthogonal combinations. Finally, we find states in the next representation by looking for additional orthogonal states with the same quantum numbers, and acting on them with lowering operators.

The direct product of a symmetric and an antisymmetric representation of $SU(N)$ can be expressed as a direct sum of two L-shaped representations, according to the Clebsch-Gordan series

$$
\begin{align*}
\frac{[2S - 1]}{2S - 1} \otimes [1^q] &= \frac{[2S, 1^{q-1}]}{q} \oplus \frac{[2S - 1, 1^q]}{q + 1}.
\end{align*}
$$
with dimensions

$$C_{N}^{q} = \frac{2S}{2S + q - 1} C_{N+2S-1}^{q-1} + \frac{2S - 1}{2S + q - 1} C_{N+2S-2}^{q-1} .$$

We proceed now to write the states in each L-shaped representation. The most symmetric (highest weight state) of the product $[2S - 1] \otimes [1^q]$ is

$$|(a)^{2S-1}|(\delta)^{2S-1}|abc \ldots r_q|^{1^q} ,$$

where $2S$ particles have the same quantum number $a$. This state is also the highest weight state of the most symmetric of the representations, $[2S, 1^q-1]$, except that $a$ is replaced by $\gamma$.

Notice that the lowering operators that transform the value $a$ into $\delta \in [b, r_q]$ affect the $[2S - 1]$ state only, since the $[1^q]$ term is completely antisymmetric. Thus, these states are also non-degenerate.

$$|(a)^{2S-1}|(\delta)^{2S-1}|abc \ldots r_q|^{1^q} = \frac{\delta q}{|a\rangle} .$$

(B8)

Other states with $2S$ of the $\alpha_j$ equal to $a$, can be obtained from Eq. (B8). They are also nondegenerate. For instance, the state with the value $\gamma$ replaced by $r_{q+1}$ is

$$|(a)^{2S-bc \ldots r_q r_{q+1}}|^{2S,1^{q-1}}$$

$$= \frac{1}{\sqrt{2S+q-1}} (\delta)^{2S-1}|abc \ldots r_q|^{1^q} .$$

(B9)

Altogether, there are $q$ states of this type (if we restrict the $\alpha_j$ to the range $[a, r_{q+1}]$).

Next, we construct the states in $[2S, 1^{q-1}]$ where $(2S-1)$ of the $\alpha_j$ are equal to $a$ and the rest of labels are different and take values in the range $[b, r_{q+1}]$. There are $q$ linearly independent states of this kind. For instance, acting with $T_{ac}$ on (B10) we get

$$T_{ac} (|(a)^{2S-bcd \ldots r_q r_{q+1}}(no \gamma)|^{2S,1^{q-1}}$$

$$= \frac{\delta q}{|a\rangle} |(a)^{2S-1}|(bc \ldots r_q r_{q+1}(no \gamma)|^{1^q} .$$

(B10)

$$\frac{\delta q}{|a\rangle} .$$

(B11)

The orthogonalization of these $q$ states leads to the first $q$ rows of Table VII of CGC. The last line in the table correspond to a state with the same quantum numbers $|(a)^{2S-1}|bcd \ldots r_q r_{q+1}|$, and orthogonal to all the states in $[2S, 1^q]$. This state is the highest weight of $[2S - 1, 1^q]$.

It is easy to see that it must be of the form

$$|\langle(a)^{2S-1}bcd \ldots r_q r_{q+1}|^{2S-1, 1^q}$$

$$= \frac{1}{\sqrt{2S+q-1}} (\delta)^{2S-1}|bcd \ldots r_q|^{1^q}$$

$$+ \sum_{\beta=b}^{r_q} (-1)^{\beta a} |(a)^{2S-2}(\beta)|{2S-1}|bcd \ldots r_q(\beta)\rangle^{1^q} .$$

(B12)

2.  Ground state

The explicit form of the highest weight state for the ground state corresponds to Eq. (3),

$$|GS\rangle_{(a)_{\alpha a}}^{[2S-1]} = \frac{1}{\sqrt{2S-1}} (b^\dagger_{a})^{2S-1} |\Delta\rangle ,$$

with

$$|\Delta\rangle \equiv \frac{1}{\gamma} A(b^\dagger_{a} (\prod_{\alpha=a}^{i_q} f^i_{\alpha}) (\prod_{\beta=b}^{i_{q+1}} c^i_{\beta}))[0] ,$$

$$\gamma \equiv \sqrt{(2S+N-1)C_{N-1}} .$$

Notice the additional term $C_{N-1}$ in the normalization factor $\gamma$, as compared to Eq. (13), due to the presence of two kinds of fermions $f^i_{\alpha}$ and $c^i_{\beta}$. We adopt the bosonic realization of the impurity $\psi_b$, which simplifies the calculations. We would like to emphasize that all the results are independent of the realization chosen, since only the form of the Young tableau is relevant to the interaction.

Other states in the same $[2S - 1]$ multiplet can be obtained by just acting on the $(b^\dagger_{a})^{2S-1}$ term. For instance

$$T^{-} |GS\rangle_{(a)_{aa}} = \sqrt{2S-1} |GS\rangle_{(a)ab}$$

$$= (2S-1) \frac{1}{\sqrt{2S-1}} b^\dagger_{a} (b^\dagger_{a})^{2S-2} |\Delta\rangle ,$$

(B13)

$$U^{-} |GS\rangle_{(a)ab} = |GS\rangle_{(a)ac}$$

$$= \frac{1}{\sqrt{2S-2}} b^\dagger_{a} (b^\dagger_{a})^{2S-2} |\Delta\rangle ,$$

(B14)
\[ |GS\rangle_{(a)ab} = \frac{1}{\sqrt{2S-2}} b^\dagger_a (b^\dagger_b) 2S-2 |\Delta\rangle , \]
\[ |GS\rangle_{(a)bb} = \frac{1}{\sqrt{2(2S-3)}} (b^\dagger_a)^3 (b^\dagger_b) 2S-3 |\Delta\rangle , \]
\[ |GS\rangle_{(a)ac} = \frac{1}{\sqrt{2S-2}} b^\dagger_a c^\dagger_b (b^\dagger_b) 2S-2 |\Delta\rangle , \]
\[ |GS\rangle_{(a)bc} = \frac{1}{\sqrt{2S-3}} b^\dagger_a b^\dagger_c (b^\dagger_b) 2S-3 |\Delta\rangle . \] (B15)

3. Excited states

Let us now write the expression of the excited states of the strong-coupling fixed point. The states in \(|GS + 1\rangle^S\) transform as the completely symmetric representation \([2S]\). The highest weight state can be obtained by acting with \(c^\dagger_a\) on the ground state,

\[ |GS + 1\rangle^S_{(a)aaa} = \frac{1}{\Omega} c^\dagger_a |GS\rangle_{(a)aa} , \]

where the normalization factor, \(\Omega = \sqrt{\frac{2S^2 + q - 1}{2S + N - 1}}\), appears because the additional c-electron has to be antisymmetrized with respect to the \((N - q)\) electrons already present on site-0. Other states in the multiplet can be obtained by repeated action of the lowering operators, as in Eqs. (B13, B15). For instance

\[ |GS + 1\rangle^S_{(a)aab} = \frac{1}{\Omega \sqrt{2S}} \left[ \sqrt{2S - 1} c^\dagger_a |GS\rangle_{(a)ab} + c^\dagger_b |GS\rangle_{(a)aa} \right] \] (B16)

States in \(|GS + 1\rangle^A\) transform as \([2S - 1, 1]\). The highest weight state is \(|GS + 1\rangle^A_{(a)abb}\), and it will be orthogonal to the state defined in Eq. (B16). Thus

\[ |GS + 1\rangle^A_{(a)aab} = \frac{1}{\Lambda \sqrt{2S}} \left[ c^\dagger_a |GS\rangle_{(a)ab} - \sqrt{2S - 1} c^\dagger_b |GS\rangle_{(a)aa} \right] , \]

with \(\Lambda = \sqrt{\frac{1}{2S-1}}\). Notice the difference with respect to the normalization factor. Only \(\Omega\) depends on \(2S\).

Other states in \(|GS + 1\rangle^A\) are obtained in a very similar way to the construction of the octets, \(8^1\) and \(8^2\) in \(SU(3)\) (see Appendix A). For instance

\[ |GS + 1\rangle^A_{(a)abc} = \frac{1}{\Lambda \sqrt{2S(2S-1)}} \left[ \sqrt{2S - 2} c^\dagger_a |GS\rangle_{bc} + c^\dagger_b |GS\rangle_{ac} - (2S - 1) c^\dagger_c |GS\rangle_{ab} \right] , \]
The hopping hamiltonian leads to two types of processes, as described earlier, where the intermediate states have either one more or one less conduction electron. With the help of Table I we can write the excitation energies

\[ \Delta E_1 = \frac{J}{2} \left( \frac{N + 1}{N} \right)(N - q), \quad \Delta E_2 = \frac{J}{2} \left( \frac{N + 1}{N} \right)q. \]

It is clear that \( \Delta E_1 \) and \( \Delta E_2 \) are related by the particle-hole transformation \( q \leftrightarrow (N - q) \).

The ground state with \( n_d \) electrons on site 1 is

\[ |GS, n_d\rangle = (d^\dagger_{1, \sigma} \cdots d^\dagger_{N, \sigma})|GS\rangle. \]

Since there is only one intermediate state for each process, we can use the trick described in section IIIA, and write

\[ M_1 = t^2 \left( n_d - \sum_{\sigma, \sigma'} \langle GS, n_d| c^\dagger_{\sigma} d^\dagger_{\sigma', \sigma} c_{\sigma'}|GS, n_d\rangle \right), \quad (C2) \]

As long as \( n_d > 1 \), the second term in Eq. (C2) vanishes for \( \sigma \neq \sigma' \), \( c^\dagger_{\sigma} \) acting on \( |GS, n_d\rangle \) just counts the number of terms where there is a \( c^\dagger_{\sigma} \). There are \( C_N \) such terms. Thus,

\[ \sum_{\sigma, \sigma'} \langle GS, n_d| c^\dagger_{\sigma} d^\dagger_{\sigma', \sigma} c_{\sigma'}|GS, n_d\rangle = n_d \left( \frac{C_N^{\sigma}}{C_N} \right) \]

\[ = n_d \left( \frac{N - q}{N} \right), \]

and

\[ M_1 = t^2 n_d \frac{q}{N}. \]

Finally, following Eq. (8),

\[ M_2 = t^2 (n_c - n_d) + M_1 = t^2 (N - n_d) \left( \frac{N - q}{N} \right). \]

\( M_1 \) and \( M_2 \) are related by the same transformations \( (q \leftrightarrow N - q, \ n_d \leftrightarrow N - n_d) \) as the excitation energies. That means that the energy shift is invariant under these transformations

\[ \Delta E_f = -\left( \frac{2t^2}{J} \right) \times \left( \frac{n_d}{N + 1} \left( \frac{q}{N - q} \right) + \frac{N - n_d}{N + 1} \left( \frac{N - q}{q} \right) \right). \]

In the large-N limit, this result is equivalent to Eq. (24). To leading order in \( 1/N \), the energy shift of the strong coupling fixed point is determined by the fermionic component, and the behavior under the particle-hole transformation.

We end this appendix by studying the phase shift \( \delta \), of the conduction electrons scattered off the impurity site. This quantity characterizes the impurity contribution to the resistivity \( \rho' \). At zero temperature and magnetic field, we have

\[ \rho' \propto \sin^2 \delta. \quad (C3) \]

The phase shift for antisymmetric impurities in \( SU(N) \), was computed in ref. 28. In the completely screened case it reads

\[ e^{2i\delta} = -e^{-i\pi(1-\frac{2}{N})}. \quad (C4) \]

If we choose the phase shift so that \( |\delta| \leq \frac{\pi}{2} \), we have

\[ \delta = \begin{cases} \pi \left( \frac{2}{N} \right), & q < N/2 \\ -\pi \left( \frac{N-q}{N} \right), & q > N/2 \end{cases} \quad (C5) \]

The unitary limit, \( |\delta| = \frac{\pi}{2} \) is reached in the particle-hole symmetric case, \( q = N/2 \). We see that this corresponds to the point where \( \Delta E_f \) is independent of \( n_d \), indicating the change from the attractive to the repulsive regime.

**APPENDIX D: DETAILS OF THE CALCULATIONS OF THE MATRIX ELEMENTS**

Here we construct explicitly the excited states that are involved in the second order perturbation theory, and then we compare them to the action of the hopping term, \( (c^\dagger_{\sigma} d_{\sigma}) \), on the ground state. First, we add a c-electron to the ground state, and then we combine it with \( (n_d - 1) \) electrons from site 1.

1. **Symmetric process, \( |GS, n_d\rangle^S \)**

The strong coupling excited state \( |GS + 1\rangle^S \) is easy to compute, since it is the highest weight state in the product \( |2S - 1\rangle \otimes |1\rangle \rightarrow |2S\rangle \otimes |2S - 1, 1\rangle \). We have

\[ |GS + 1\rangle^S_{aaa} = \frac{1}{\Omega} c_{\sigma}^\dagger |GS\rangle_{aa} \]

with the normalization factor \( \Omega^2 = \frac{2S+q-1}{2S+1} \). Other states within the same representation, which transform as \( |2S\rangle \), can be obtained by acting with the corresponding lowering operators. For instance,

\[ |GS + 1\rangle^S_{aab} = \frac{1}{\Omega \sqrt{2S}} \left( \sqrt{2S - 1} c_{\sigma}^\dagger |GS\rangle_{ab} + c_{\sigma}^\dagger |GS\rangle_{aa} \right). \]

The highest weight state of the antisymmetric multiplet \( |GS + 1\rangle^A \), which transforms as \( |2S - 1\rangle \) is a state orthogonal to \( |GS + 1\rangle^S_{aab} \) that is

\[ |GS + 1\rangle^A_{aab} = \frac{1}{\Lambda \sqrt{2S}} \left( c_{\sigma}^\dagger |GS\rangle_{ab} - \sqrt{2S - 1} c_{\sigma}^\dagger |GS\rangle_{aa} \right), \]

with a different normalization factor, \( \Lambda^2 = \frac{q-1}{q+1} \). Other states can be obtained from these three. The results are
summarized in the Tables IV-VI of Clebsch-Gordan coefficients.

Next, we have to add the \((n_d - 1)\) electrons on site 1 to get the symmetric state in the symmetric configuration, and after some algebra, we get

\[
|GS + 1, n_d - 1\rangle_{aab\ldots u} = \frac{1}{\sqrt{2S + n_d - 1}} \left( \sqrt{2S} \left( \prod_{i=2}^{n_d} d_{x_i}^\dagger \right) |GS + 1\rangle_{aaa} + \sum_{j=2}^{n_d} (-1)^{j-1} \left( \prod_{i=1, i \neq j}^{n_d} d_{x_i}^\dagger \right) |GS + 1\rangle_{aax_j} \right)
\]

for the symmetric state in the symmetric configuration, and

\[
|GS + 1, n_d - 1\rangle_{aab\ldots u} = \frac{1}{\sqrt{n_d - 1}} \left( \sum_{j=2}^{n_d} (-1)^{j-1} \left( \prod_{i=1, i \neq j}^{n_d} d_{x_i}^\dagger \right) |GS + 1\rangle_{aax_j} \right)
\]

for the antisymmetric state in the symmetric configuration, and

which is proportional to a given strong coupling excited state. Since we can write \((D1)\) as

\[
\left( \sum_{\sigma} c_{\sigma}^\dagger d_{\sigma} \right) |GS, n_d\rangle^A_{aab\ldots u} = \frac{(-1)^{n_d-1}}{\sqrt{2S + n_d - 1}} \left( \sqrt{2S - 1} \sum_{l=2}^{n_d} (-1)^l \left( \prod_{i=2, i \neq l}^{n_d} d_{x_i}^\dagger \right) c_{x_l}^\dagger |GS\rangle_{aa} \right)
\]

which is proportional to a given strong coupling excited state. Since we can write \((D1)\) as

\[
\left( \sum_{\sigma} c_{\sigma}^\dagger d_{\sigma} \right) |GS, n_d\rangle^A_{aab\ldots u} = \frac{(-1)^{n_d-1}}{\sqrt{2S + n_d - 1}} \left( \sum_{l=2}^{n_d} (-1)^l \left( \prod_{i=2, i \neq l}^{n_d} d_{x_i}^\dagger \right) c_{x_l}^\dagger |GS\rangle_{aa} \right)
\]

2. Antisymmetric process, \(|GS, n_d\rangle^A\)

The action of \((c_{d, \sigma}^\dagger d_{\sigma})\) on the ground state \(|GS, n_d\rangle^A\), with \(n_d\) electrons on site 1 coupled antisymmetrically, produces

\[
\left( \sum_{\sigma} c_{\sigma}^\dagger d_{\sigma} \right) |GS, n_d\rangle^A_{aab\ldots u} = \frac{(-1)^{n_d-1}}{\sqrt{2S + n_d - 1}} \left( \sum_{l=2}^{n_d} (-1)^l \left( \prod_{i=2, i \neq l}^{n_d} d_{x_i}^\dagger \right) c_{x_l}^\dagger |GS\rangle_{aa} \right)
\]

which is proportional to a given strong coupling excited state. Since we can write \((D1)\) as

\[
\left( \sum_{\sigma} c_{\sigma}^\dagger d_{\sigma} \right) |GS, n_d\rangle^A_{aab\ldots u} = \frac{(-1)^{n_d-1}}{\sqrt{2S + n_d - 1}} \left( \sum_{l=2}^{n_d} (-1)^l \left( \prod_{i=2, i \neq l}^{n_d} d_{x_i}^\dagger \right) c_{x_l}^\dagger |GS\rangle_{aa} \right)
\]

after some algebra, we get

\[
\left( \sum_{\sigma} c_{\sigma}^\dagger d_{\sigma} \right) |GS, n_d\rangle^A_{aab\ldots u} = \frac{(-1)^{n_d-1}}{\sqrt{2S + n_d - 1}} \sum_{l=2}^{n_d} (-1)^l \left( \prod_{i=2, i \neq l}^{n_d} d_{x_i}^\dagger \right) c_{x_l}^\dagger |GS\rangle_{aa} \right)
\]

1. The result of the calculations are summarized in the Tables VII. We have
This expression can be written using the antisymmetric states $|GS + 1\rangle^A$, with the help of the following relations
\[
\Lambda(\sqrt{2S}|GS + 1\rangle^A_{ax,ix}) - \sqrt{2S - 2}|GS + 1\rangle^A_{ax,ix}) = \sqrt{2S - 1}(c^\dagger_{x_i}|GS\rangle_{ax} - c^\dagger_{x_j}|GS\rangle_{ax}) ,
\]
\[
\Lambda\sqrt{2S}|GS + 1\rangle^A_{ax,ix} = c^\dagger_{ax}|GS\rangle_{ax} - \sqrt{2S - 1}c^\dagger_{x_i}|GS\rangle_{ax} ,
\]
to obtain
\[
\left(\sum_{\sigma} c^\dagger_{\sigma}d_{\sigma}\right)|GS, n_d\rangle^A_{ab...v} = \frac{(-1)^{n_d-1}\Lambda}{\sqrt{(2S + n_d - 1)(2S - 1)}}
\times \sum_{l=1}^{n_d+1} (-1)^{l+1} \left\{ \left( \prod_{i=2,i\neq l}^{n_d+1} d^\dagger_{x_i}\right) \sqrt{2S(2S - 1)} |GS + 1\rangle^A_{ax,ix} \right.
\right. \\
- \left. \sum_{j=2}^{l-1} (-1)^j \left( \prod_{i=1,i\neq j,l}^{n_d+1} d^\dagger_{x_i}\right) \left( \sqrt{2S} |GS + 1\rangle^A_{ax,ix} - \sqrt{2S - 2} |GS + 1\rangle^A_{ax,ix} \right) \right\} .
\]
Finally, since
\[
\left(\sum_{\sigma} c^\dagger_{\sigma}d_{\sigma}\right)|GS, n_d\rangle^A_{ab...v} \propto |GS + 1, n_d - 1\rangle^A_{ab...v}
\]
we just have to normalize the previous state in order to obtain the excited state $|GS + 1, n_d - 1\rangle^A_{ab...v}$. Up to a sign, we have
\[
|GS + 1, n_d - 1\rangle^A_{ab...v} = \frac{1}{\sqrt{n_d(2S + n_d - 1)(2S - 1)}}
\times \sum_{l=1}^{n_d+1} (-1)^{l+1} \left\{ \left( \prod_{i=2,i\neq l}^{n_d+1} d^\dagger_{x_i}\right) \sqrt{2S(2S - 1)} |GS + 1\rangle^A_{ax,ix} \right.
\right. \\
- \left. \sum_{j=2}^{l-1} (-1)^j \left( \prod_{i=1,i\neq j,l}^{n_d+1} d^\dagger_{x_i}\right) \left( \sqrt{2S} |GS + 1\rangle^A_{ax,ix} - \sqrt{2S - 2} |GS + 1\rangle^A_{ax,ix} \right) \right\} .
\]
and
\[
\left(\sum_{\sigma} c^\dagger_{\sigma}d_{\sigma}\right)|GS, n_d\rangle^A_{ab...v} = (\Lambda\sqrt{n_d})|GS + 1, n_d - 1\rangle^A_{ab...v} .
\]

3. Tables of Clebsch-Gordan coefficients

The calculation of the excited states involves the use of some Clebsch-Gordan coefficients. We have evaluated these quantities explicitly for arbitrary $2S$, $n_d$ and $N$, following the steps outlined in Appendix B. We summarize our results in Tables IV-VII. In the tables, all the coefficients are assumed to be under the sign of the square root. For instance, $-(2S - 1)$ corresponds to $-\sqrt{2S - 1}$. The states should be divided by the normalization factor $\sqrt{N}$. Tables IV-V are necessary when one electron is added to the effective impurity, $(|1\rangle \otimes |2S - 1\rangle)$. Table VII corresponds to the addition of $n_d$ electrons.
TABLE IV: Clebsch-Gordan coefficients for the process \([1] \otimes [2S - 1] \rightarrow [2S] \oplus [2S - 1,1]\). The label \(|\{a\}aab\rangle\) indicates a state of [2S], whereas \(|\{a\}aa,b\rangle\) denotes the highest weight state in [2S - 1,1].

| abb | N | |a\rangle|\{a\}ab\rangle | |b\rangle|\{a\}aa\rangle |
|-----|---|-----------------|-----------------|
| | 2S | 2S - 1 | 1 |
| | 2S | 1 | -(2S - 1) |

TABLE V: Same as Table IV but for states with quantum numbers \((a)^{2S-2}bb\)

| abb | N | |a\rangle|\{a\}ab\rangle | |b\rangle|\{a\}ab\rangle |
|-----|---|-----------------|-----------------|
| | 2S | 2S - 2 | 2 |
| | 2S | 2 | -(2S - 2) |

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TABLE VII: Some of the CG coefficients for the product of \( k = n_d \) electrons, and an effective impurity \([2S - 1], ([1^k] \otimes [2S - 1] \rightarrow [2S, 1^{k-1}] \oplus [2S - 1, 1^k] \oplus \cdots)\). Here we only keep the coefficients for the representations \([2S, 1^{k-1}]\), and \([2S - 1, 1^k]\) (last row).

States from \([1^k]\) are denoted by a column of labels, \( |\cdot\rangle\); those from \([2S - 1]\) are denoted \(|a\rangle\), and are states close to the highest weight state, denoted \(|a\rangle\). The states in \([2S, 1^{k-1}]\) are labeled \(|ab, c, \ldots\rangle\) and those in \([2S - 1, 1^k]\), \(|a, b, c, \ldots\rangle\).

| \([1^k] \otimes [2S - 1]\) | \(N\) | \(|[2S - 1]\) \(\mid a\rangle\) | \(|\cdot\rangle b\rangle\) | \(|\cdot\rangle c\rangle\) | \(\cdots\) | \(|\cdot\rangle u\rangle\) | \(|\cdot\rangle v\rangle\) |
|---|---|---|---|---|---|---|---|
| \(|a\rangle abc \cdots u v\) | \(2S(2S - 1)\) | \(2S - 1\) | \((2S - 1)^2\) | 0 | \(\cdots\) | 0 | 0 |
| \(|a, c, \ldots, u, v\) \(2S(2S + 1)\) | \((2S - 1)^2\) | 1 | \((2S)^2\) | \(\cdots\) | 0 | 0 |
| \(|a, c, \ldots, u, v\) \(2S(2S + 1)\) | \((2S - 1)^2\) | 0 | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) |
| \(|a, c, \ldots, u, v\) \(2S(2S + 1)\) | \(2S - 1\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) |
| \(|a, c, \ldots, u, v\) \(2S(2S + 1)\) | \(2S - 1\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) |
| \(|a, c, \ldots, u, v\) \(2S(2S + 1)\) | \(2S - 1\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) | \(\cdots\) |

| \(|a, b, \ldots, u, v\) \(2S - 1\) | \((2S + k - 2)(2S + k - 3)\) | \((2S - 1)^2\) | \((-1)^k(2S - 1)\) | \((-1)^k\) | \((-1)^k\) | \(\cdots\) | \((2S + k - 3)^2\) | 0 |
| \(|a, b, \ldots, t, u\) \(2S + k - 1\) \((2S + k - 2)\) | \((-1)^{k+1}(2S - 1)\) | \((-1)^{k+1}\) | \((-1)^{k+1}\) | \(\cdots\) | 1 | \((2S + k - 2)^2\) |
| \(|a, b, \ldots, u, v\) \(2S + k - 1\) \((2S + k - 2)\) | \((-1)^{k+1}(2S - 1)\) | \((-1)^{k+1}\) | \((-1)^{k+1}\) | \(\cdots\) | 1 | \((2S + k - 2)^2\) |