Revisiting semistrong edge-coloring of graphs

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**Abstract**

A matching \( M \) in a graph \( G \) is semistrong if every edge of \( M \) has an endvertex of degree one in the subgraph induced by the vertices of \( M \). A semistrong edge-coloring of a graph \( G \) is a proper edge-coloring in which every color class induces a semistrong matching. In this paper, we continue investigation of properties of semistrong edge-colorings initiated by Gyárfás and Hubenko. We establish tight upper bounds for general graphs and for graphs with maximum degree 3. We also present bounds about semistrong edge-coloring which follow from results regarding other, at first sight nonrelated, problems. We conclude the paper with several open problems.

**KEYWORDS**

induced matching, semistrong chromatic index, semistrong edge-coloring, strong edge-coloring

1 | INTRODUCTION

A proper edge-coloring of a simple graph \( G \) is an assignment of colors to its edges such that adjacent edges receive distinct colors. A proper edge-coloring can also be seen as a decomposition of the edge set in a set of matchings in which every matching represents the edges of one color. The least integer \( k \) for which \( G \) admits a proper edge-coloring with \( k \) colors is called the chromatic index of \( G \) and denoted \( \chi'(G) \). On the other hand, an edge-coloring in which edges of the same color comprise an induced matching (also called a strong matching), that is, a set of edges such that the distance between endvertices of any two edges is at least 2, is a strong edge-coloring of \( G \). The least integer \( k \) for which \( G \) admits a strong edge-coloring with \( k \) colors is called the strong chromatic index of \( G \) and denoted \( \chi'_s(G) \).

While Vizing [17] proved that the chromatic index of a graph \( G \) attains one of just two values, \( \Delta(G) \) or \( \Delta(G) + 1 \), the interval of possible values for \( \chi'_s(G) \) is much wider; namely, between \( \Delta(G) \) and (at least) \( \frac{5}{4}\Delta(G)^2 \). Note that the lower bound increases to \( 2\Delta(G) - 1 \) if there
There are two vertices of maximum degree adjacent in $G$. The complete classification of regular graphs attaining this lower bound was just recently obtained in [14]. The upper bound $\frac{5}{4}\Delta(G)^2$ was conjectured by Erdős and Nešetřil in 1985 (see [5]), who also provided constructions of graphs attaining this upper bound. Despite many efforts (see, e.g., [2, 3, 15]), the best known upper bound to date is $1.772\Delta(G)^2$ for large enough $\Delta(G)$ due to Hurley et al. [13].

Apart from the general case, determining strong chromatic index for special classes of graphs received a lot of attention and therefore it is not surprising that relaxed variants of strong edge-colorings appeared. Roughly, we can divide them into three types:

(A) improper strong edge-colorings, where some edges at distance 1 or 2 from an edge can receive the same color (see, e.g., [11]);

(B) $(1,2)$-packing edge-colorings, where the set of edges is decomposed into at most $a$ matchings and at most $b$ induced matchings (see, e.g., [12]);

(C) edge-colorings in which edges of every color induce a matching with particular properties (see, e.g., [1, 7, 9]).

Although it might not seem so at first sight, the edge-colorings of all three types are very much related. In this paper, by continuing the work of Gyárfás and Hubenko initiated in [9], we consider properties of the semistrong edge-coloring, which can be classified as type C. A semistrong edge-coloring is a proper edge-coloring in which the edges of every color class induce a semistrong matching, where a matching $M$ of a graph $G$ is semistrong if every edge of $M$ has an end-vertex of degree one in the induced subgraph $G[V(M)]$. The least integer $k$ such that $G$ admits a semistrong edge-coloring with at most $k$ colors is the semistrong chromatic index of $G$ and denoted by $\chi'_{ss}(G)$.

The above definition does not allow existence of a semistrong edge-coloring in any graph with parallel edges, since both endvertices of any parallel edge $e$ have degree more than 1 in the graph induced by $e$. Therefore, in multigraphs, we only require, for a proper edge-coloring of a multigraph $G$ to be also semistrong, that the edges of every color class induce a semistrong matching in the underlying graph of $G$, that is, the graph, in which every two vertices adjacent in $G$ are connected by exactly one edge.

Clearly, every strong matching is also semistrong, and so, for every graph $G$, we have

$$\nu_s(G) \leq \nu_{ss}(G) \leq \nu(G),$$

where by $\nu(G)$, $\nu_s(G)$, and $\nu_{ss}(G)$ we denote the maximum size of a matching, strong matching, and semistrong matching of $G$, respectively. Consequently,

$$\chi'(G) \leq \chi'_{ss}(G) \leq \chi'_s(G).$$

Although the above series of inequalities is trivial, no better general upper bound than the bound for the strong edge-coloring has been known for the semistrong edge-coloring of general graphs. In this paper, we prove the following.

**Theorem 1.** For every graph $G$, we have

$$\chi'_{ss}(G) \leq \Delta(G)^2.$$
Let us note that there are graphs $G$ for which $\chi_{ss}'(G) = \chi_s'(G)$, for example, the complete and the complete bipartite graphs. The equality for the two classes is a simple consequence of the fact that in a semistrong edge-coloring the edges of every 4-cycle must be colored with four distinct colors. Therefore,

\[ \chi_{ss}'(K_n) = \chi_s'(K_n) = \binom{n}{2} \]

and

\[ \chi_{ss}'(K_{m,n}) = \chi_s'(K_{m,n}) = m \cdot n. \tag{1} \]

In [9], the authors found two additional families of graphs with the same value of strong and semistrong chromatic indices; namely the Kneser and the subset graphs. For two positive integers $n$ and $k$, with $k \leq n$, a Kneser graph $K(n, k)$ is a graph whose vertex set consists of all $k$-subsets of an $n$-element set, and two vertices are connected if and only if the corresponding sets are disjoint. A subset graph $S_n(k, \ell)$, for $0 \leq k \leq \ell \leq n$, is a bipartite graph where the partition vertex sets are the $k$- and $\ell$-subsets of the $n$ element set, and two vertices (subsets) are connected if and only if one of them is contained in the other. Extending the strong edge-coloring results for the two classes due to [6] and [16], Gyárfás and Hubenko proved the following.

**Theorem 2** (Gyárfás and Hubenko [9]).

(A) For every Kneser graph $K(n, k)$ it holds $\chi_{s}'(K(n, k)) = \chi_{ss}'(K(n, k)) = \binom{n}{2k}$.

(B) For every subset graph $S_n(k, \ell)$ it holds $\chi_{s}'(S_n(k, \ell)) = \chi_{ss}'(S_n(k, \ell)) = \binom{n}{\ell - k}$.

Additionally, the above authors considered the equality of the two invariant for the $n$-dimensional cubes and conjectured that $\nu(Q_n) = \nu_{ss}(Q_n) = 2^{n-2}$, for every $n \geq 2$. As noted by Gregor [8], the conjecture was established by Diwan [4], who in fact considered the problem of the minimum forcing number, most likely being unaware of resolving another conjecture.

**Theorem 3** (Diwan [4]). For every integer $n \geq 2$, we have

\[ \nu(Q_n) = \nu_{ss}(Q_n) = 2^{n-2}. \]

Furthermore, in [6] it was proved that $\nu(Q_n) = 2^{n-2}$ and $\chi_s'(Q_n) = 2n$, from which it can be concluded that the following holds.

**Corollary 1** (Diwan [4] and Faudree et al. [6]). For every integer $n \geq 2$, we have

\[ \chi_{ss}'(Q_n) = \chi_s'(Q_n) = 2n. \]

Another graph family for which the semistrong chromatic index is completely determined are trees. The strong chromatic index of a tree $T$ is at most $2\Delta(T) - 1$ [6]. In the semistrong
setting, the bound is much lower as follows from the result due to He and Lin [11] who considered the \((s, t)\)-relaxed strong edge-coloring, that is, an edge-coloring, in which, for every edge \(e\) of a graph \(G\), the number of edges adjacent to \(e\) having the same color as \(e\) is at most \(s\), and the number of edges at distance 2 from \(e\) having the same color as \(e\) is at most \(t\). The corresponding chromatic index is \((s, t)\)-relaxed strong chromatic index, denoted by \(\chi^{(s,t)}(G)\). They proved that, for every tree \(T\), if \(s = 0\) and \(t = \Delta(T) - 1\), then \(\chi^{(s,t)}(T) \leq \Delta(T) + 1\) [11, Lemma 5.1]. For the proof, they provided a construction [11, Algorithm 2] of an edge-coloring using at most \(\Delta(T) + 1\) colors that is also a semistrong edge-coloring, and thus they also proved the following.

**Corollary 2** (He and Lin [11]). For every tree \(T\) it holds

\[ \chi^{ss}(T) \leq \Delta(T) + 1. \]

Moreover, if \(T\) has diameter at most 4, then

\[ \chi^{ss}(T) = \Delta(T). \]

In this paper, we also present a tight result for graphs with maximum degree 3. From Theorem 1 it already follows that at most 9 colors are needed and that the bound is attained by \(K_{3,3}\). We improve this bound as follows.

**Theorem 4.** For every connected graph \(G\) with maximum degree 3, distinct from \(K_{3,3}\), we have

\[ \chi^{ss}(G) \leq 8. \]

Note that the bound in Theorem 4 is tight, since the semistrong chromatic index of the 5-prism is 8 (see Figure 1). This follows from the fact that the size of any maximum semistrong matching in the 5-prism is 2, while it has 15 edges.

The structure of the paper is the following. We begin by presenting notation, definitions and auxiliary results in Section 2. In Sections 3 and 4, we prove Theorems 1 and 4, respectively, and
finally, in Section 5, we discuss several additional edge-colorings related to the semistrong edge-coloring and conclude with some open problems and suggestions for further work.

2 | PRELIMINARIES

In this section, we introduce terminologies, notation, and some auxiliary results.

When constructing a semistrong edge-coloring with at most \( k \) colors, we always assume that the colors are taken from the set of the first \( k \) positive integers; as customary, we write \([k] = \{1, \ldots, k\}\). We also abuse the notation and denote the set of colors appearing on the edges from a set \( X \) in a semistrong edge-coloring \( \sigma \) as \( \sigma(X) \).

Given a partial semistrong edge-coloring \( \sigma \), a color \( \alpha \) is available for an edge \( e \) if there is no edge at distance at most 2 from \( e \) colored with \( \alpha \). The set of available colors for an edge \( e \) is denoted \( A_e(\sigma) \), or simply, \( A_e \). A color \( \alpha \) is forbidden for an edge \( e \) if coloring \( e \) with \( \alpha \) violates the assumptions of the semistrong edge-coloring.

With sets of available colors defined, in several cases, we will use the following application of Hall's Marriage Theorem [10].

**Theorem 5.** Let \( G \) be a graph and \( \sigma \) a partial semistrong edge-coloring of \( G \). Let \( X = \{e_1, \ldots, e_k\} \) be the set of noncolored edges of \( G \). Let \( \mathcal{F} = \{A(e_1), \ldots, A(e_k)\} \). If for every subset \( \mathcal{X} \subseteq \mathcal{F} \) it holds that

\[
|\mathcal{X}| \leq |\bigcup_{\mathcal{X} \in \mathcal{X}}|,
\]

then one can choose an available color for every edge in \( X \) such that all the edges receive distinct colors.

By \( C(v) \) we denote the set of colors of the edges incident with \( v \). Throughout the paper, we sometimes simply write “coloring” instead of “semistrong edge-coloring.”

By \( N(uv) \), we denote the set of edges adjacent to \( uv \), that is, the edge-neighborhood of \( uv \), and by \( N^2(uv) \), we denote the set of edges at distance 1 or 2 from \( uv \), that is, the 2-edge-neighborhood of \( uv \). Similarly, by \( N_u(uv) \), we denote the set of edges adjacent to \( uv \) with \( u \) being one of their endvertices, and by \( N^2_u(uv) \), we denote the set of edges at distance 1 or 2 from \( uv \) such that every edge \( e \) in \( N^2_u(uv) \) has \( u \) as an endvertex or there is an edge having \( u \) as endvertex, connecting \( e \) and \( uv \).

3 | PROOF OF THEOREM 1

In this section, we prove a stronger result than Theorem 1; namely, we establish the following.

**Theorem 6.** For every graph \( G \), we have

\[
\chi'_{(0,1)}(G) \leq \Delta(G)^2.
\]

Moreover, there is always a \((0, 1)\)-relaxed strong edge-coloring of \( G \) using at most \( \Delta(G)^2 \) colors such that the edges of every color induce a semistrong matching.
Clearly, Theorem 1 is a direct corollary of Theorem 6. Note however that (0, 1)-relaxed strong edge-coloring without additional conditions is not equivalent to the semistrong edge-coloring, since in the former, for example, nonrainbow 4-cycles can appear.

Proof. For every edge $e = uv \in E(G)$, let $L(e) = (N_u^2(e) \cap N_v^2(e)) \setminus N(e)$. Additionally, let $k(e) = |N(e)|$ and $\ell(e) = |L(e)|$. Observe that $k(e) \le 2(\Delta(G) - 1)$, $\ell(e) \le (\Delta(G) - 1)^2$ and $|N^2(e)| \le 2(\Delta(G) - 1)\Delta(G) - \ell(e)$.

Let $\sigma(G)$ be a proper edge-coloring of $G$ with at most $\Delta(G)^2$ colors such that every edge $e$ receives a color distinct from all colors of edges in $L(e)$. Note that such a coloring always exists, since there are at most $k(e) + \ell(e) \le \Delta(G)^2 - 1$ conflicts for every edge $e$, and therefore one can, for example, simply take a greedy approach to find it.

We now proceed by a contradiction. Among all possible above described colorings, let $\sigma(G)$ be a coloring with the minimum number of distance-2 conflicts, that is, pairs of edges at distance 2 receiving the same colors. Denote the number of distance-2 conflicts for the coloring $\sigma(G)$ by $\iota(\sigma)$. Observe that if every edge $e$ has at most one distance-2 conflict, then $\sigma(G)$ is a semistrong edge-coloring. Thus, we may assume that there exists an edge $e'$ having at least two distance-2 conflicts.

There are at most $k(e') + \ell(e') \le \Delta(G)^2 - 1$ colors that cannot be used for $e'$. Moreover, if there is some other color $\alpha$ used at most once on the edges from $N^2(e') \setminus (N(e') \cup L(e'))$, then we recolor $e'$ with $\alpha$ and so decrease $\iota(\sigma)$, a contradiction. Therefore, the number of colors in $N^2(e')$ is at most

$$k(e') + \ell(e') + \frac{1}{2}(|N^2(e')| - k(e') - \ell(e')) \le \Delta(G)^2 - 1.$$

So, we can recolor the edge $e'$ with an available color and decrease $\iota(\sigma)$, a contradiction. \hfill $\Box$

Let us note here that the above proof implies that less than $\Delta(G)^2$ colors suffice for every graph in which no edge has both endvertices of maximum degree.

4 | PROOF OF THEOREM 4

In this section, we improve the upper bound obtained in the previous section for the class of graphs with maximum degree 3.

Proof of Theorem 4. We prove the theorem by a contradiction. Suppose that $G$ is the minimal counterexample, that is, a connected subcubic graph distinct from $K_{3,3}$ with $\chi_{ss}'(G) = 9$ and minimum number of edges among all such graphs; this means that $G$ has at least 9 edges. We continue by establishing additional structural properties of $G$.

Claim 1. $G$ does not contain parallel edges.

Proof. Suppose to the contrary that there are two edges, $e_1$ and $e_2$, both connecting vertices $u$ and $v$ in $G$. By the minimality of $G$, there exists a semistrong edge-coloring $\sigma$ of $G' = G \setminus \{e_1\}$ with at most 8 colors. Note that $\sigma$ is a partial semistrong edge-coloring of $G$.
(with only $e_1$ being noncolored), since the distances between the edges in $G'$ are the same as the distances between the edges in $G$. Since there are at most 7 edges in $N^2(e_1)$, there is at least 1 available color for $e_1$. Therefore, we can color $e_1$, and hence extend $\sigma$ to all edges of $G$, a contradiction.

\begin{claim}
$G$ is 2-connected.
\end{claim}

\begin{proof}
Suppose to the contrary that there is a cut-vertex $v$ in $G$. In the setting of subcubic graphs this also means that there is a bridge $uv$ in $G$. Let $G_u$ (resp., $G_v$) be the component of $G \setminus \{uv\}$ containing $u$ (resp., $v$). By the minimality, there is a semistrong edge-coloring $\sigma_u$ of $G_u$ with at most 8 colors, and similarly, there is a semistrong edge-coloring $\sigma_v$ of $G_v$ with at most 8 colors.

The colorings $\sigma_u$ and $\sigma_v$ induce a partial proper edge-coloring $\sigma$ of $G$ with only $uv$ being noncolored. Note that $\sigma$ might not be semistrong, since the colors of the edges incident with $u$ and $v$ may be in conflict. Therefore, we permute the colors of $\sigma_v$ in such a way that $\sigma(N_v(\{u\})) \cap \sigma(N_u(\{v\})) = \emptyset$ and $|\sigma(N^2(\{u\}))| \leq 6$. Note that this can be done, since there are at most four edges in $N^2_v(\{u\}) \setminus N_v(\{uv\})$. This means that there is an available color for the edge $uv$, and hence we can extend $\sigma$ to all the edges of $G$, a contradiction.
\end{proof}

\begin{claim}
There are no adjacent triangles in $G$.
\end{claim}

\begin{proof}
Clearly, $G$ is not isomorphic to $K_4$ as it admits a strong edge-coloring with 6 colors. Therefore, we may assume, for a contradiction, that there are two adjacent triangles in $G$, $(u, w_1, w_2)$ and $(v, w_1, w_2)$, where $u$ and $v$ are not adjacent. Let $u_1$ and $v_1$ be the neighbors of $u$ and $v$, respectively, distinct from $w_1$ and $w_2$. Note that $u_1$ and $v_1$ exist and are distinct, since $G$ has at least 9 edges and is 2-connected.

Now, consider the graph $G' = G \setminus \{w_1, w_2\}$. Note first that by Claim 2, $G'$ is connected. It is not isomorphic to $K_{3, 3}$, and thus admits a semistrong edge-coloring $\sigma$ with at most 8 colors due to the minimality of $G$. The coloring $\sigma$ induces a partial semistrong edge-coloring of $G$, in which only the edges incident with the two adjacent triangles are noncolored. Observe that $w_1w_2$ has at least 6 available colors, while the other four noncolored edges have at least 4 available colors each. This means that by Theorem 5, we can color them and hence extend $\sigma$ to all the edges of $G$, a contradiction.
\end{proof}

\begin{claim}
There is no 4-cycle adjacent to a triangle in $G$.
\end{claim}

\begin{proof}
Suppose to the contrary that there is a 3-cycle $T = (u, w_1, w_2)$ and a 4-cycle $F = (v, v', w_1, w_2)$ in $G$. By Claims 1 and 3, the vertex $u$ is distinct from the vertices $v$ and $v'$, and $uv$, $uv'$ are not the edges of $G$. Consequently, $vv'$ and $uw_1$ may receive the same color in a semistrong edge-coloring.

Consider the graph $G' = G \setminus \{w_1, w_2\}$. By Claim 2, $G'$ is connected, not isomorphic to $K_{3, 3}$, and therefore, by the minimality of $G$, it admits a semistrong edge-coloring $\sigma$ using at most 8 colors. Consider the partial edge-coloring of $G$ induced by $\sigma$. First, we uncolor the edge $vv'$. We infer that $vv'$ has at least 2 available colors, $v'w_1$ and $vw_2$ have at least 3 each, $uw_1$ and $uw_2$ have at least 4 each, and $w_1w_2$ has at least 5 available colors. If $A(vv') \cap A(uw_1) \neq \emptyset$, then we can color the edges $vv'$ and $uw_1$ with the same color, and the remaining four noncolored edges are colorable by Theorem 5, a contradiction. Thus, we may assume that
Proof. By Claim 2, every triangle in $G$ is incident with at most one 2-vertex. Now, suppose the contrary and let $T = (v_1, v_2, v_3)$ be a triangle incident with a 2-vertex $v_i$. Then, by the minimality, $G' = G \setminus \{v_1\}$ admits a semistrong edge-coloring $\sigma$ using at most 8 colors. The coloring $\sigma$ induces a partial semistrong edge-coloring of $G$ with only the edges $v_1v_2$ and $v_1v_3$ being noncolored. Since both noncolored edges have at least 3 available colors, we can extend $\sigma$ to all edges of $G$, hence obtaining a contradiction. □

Claim 6. There is no triangle in $G$.

Proof. Suppose to the contrary that $T = (v_1, v_2, v_3)$ is a triangle in $G$. Let $u_1, u_2, u_3$ be the third neighbors of $v_1, v_2,$ and $v_3,$ respectively. By Claims 5, 3, and 4, we have that all three vertices $u_1, u_2,$ and $u_3$ exist, are distinct, and pairwise nonadjacent. We call an edge $u_iv_i,$ for every $i \in [3],$ an incoming edge, and every edge in $E(T)$ a triangle edge. By the minimality, $G \setminus \{v_1, v_2, v_3\}$ admits a semistrong edge-coloring $\sigma$ with at most 8 colors. We show that $\sigma$ can be extended to all edges of $G$.

Note that there are at least 2 available colors for every incoming edge and at least 4 available colors for every triangle edge. Moreover, note also that an incoming edge must have a color distinct from the color of the opposite triangle edge. On the other hand, incoming edges may receive the same colors, since they do not belong to a common 4-cycle as $u_i$'s are not adjacent.

Suppose first that \( \bigcap_{i=1}^{3} A(v_i) \bigcap (u_i) \geq 1 \) and let 1 be the color available for all the three edges $u_i v_i$. In this case, color all three edges by 1. There remain at least 3 available colors for each triangle edge, so we can complete the coloring.

Hence, we may assume that \( \bigcap_{i=1}^{3} A(v_i) \bigcap (u_i) = 0 \). We consider two subcases.

(i) Two incoming edges have a common available color, say 1 \( \in A(u_1v_1) \cap A(u_2v_2) \). In this case, we may assume that \( 2, 3 \subseteq A(u_3v_3) \). Color the edges $u_1v_1$ and $u_2v_2$ with 1. Next, consider the available colors of the triangle edges (after coloring the two incoming edges). If \( \bigcup_{e \in E(T)} A(e) \bigcap (2, 3) \cap 2 \geq 2 \) then there exists a coloring of the four noncolored edges by Theorem 5. Thus, we may assume that $A(v_1v_2) = A(v_2v_3) = A(v_1v_3) = \{2, 3, 4\}$. This in particular means that the edges incident with the vertices $u_1, u_2,$ and $u_3$ distinct from the incoming edges, are colored with the colors \{5, 6, 7, 8\}. Moreover, for every pair of vertices $u_i$ and $u_j, i \neq j, i, j \in [3]$, the union of colors on their incident edges (without the color 1) has cardinality 4. As this is not possible, we reached a contradiction.

(ii) All the available colors of the incoming edges are distinct (see Figure 2 for an illustration). Suppose first that there exists an available color, say 1, of some incoming edge such that $A(e) \bigcap \{1\} \geq 4$ for every triangle edge $e$. If

\[
\left( \bigcup_{e \in E(T)} A(e) \bigcup \{3, 4, 5, 6\}\right) \bigcap \{1\} \geq 5,
\]


then we color $u_1v_1$ by 1 and extend the coloring to the remaining five edges by Theorem 5. Thus, we may assume that

$$A(v_1v_2) \setminus \{1\} = A(v_2v_3) \setminus \{1\} = A(v_1v_3) \setminus \{1\} = \{3, 4, 5, 6\}.$$ 

Note that in this case $C(u_1) = \{7, 8\}$, $C(u_2) = \{2, x\}$, and $C(u_3) = \{2, y\}$, where $\{x, y\} = \{7, 8\}$. This in particular means that the color 1 is available for all three triangle edges. Hence, we color $u_1v_1$ by 2, and color the remaining five edges by Theorem 5.

Therefore, by symmetry, we may assume that every color from $[6]$ is available for some triangle edge. Hence, we infer that $A(v_1v_2) \cup A(v_2v_3) \cup A(v_1v_3) \mid \geq 6$ and we can color all the six edges by Theorem 5, a contradiction.

This establishes the claim. □

Claim 7. No 4-cycle in $G$ is incident with a 2-vertex.

Proof. By Claim 2, we have that there are at most two 2-vertices incident with any 4-cycle. We again proceed by contradiction. Let $F = (v_1, v_2, v_3, v_4)$ be a 4-cycle in $G$. Suppose first that the vertices $v_1$ and $v_2$ are 2-vertices (and thus $v_3$ and $v_4$ are 3-vertices). By the minimality of $G$, the graph $G' = G \setminus \{v_1, v_2\}$ admits a semistrong edge-coloring $\sigma$ using at most 8 colors. The coloring $\sigma$ induces a partial coloring of $G$ with only the three edges being noncolored. Since each of the three edges has at least 3 available colors, we can extend $\sigma$ to $G$, a contradiction.

Thus, we may assume that in $F$, there is a pair of opposite 3-vertices, say $v_1$ and $v_3$, and at least one 2-vertex, say $v_2$. However, if $v_4$ is a 2-vertex also, then we proceed as in the proof of Claim 3. Thus, we may assume that $d(v_4) = 3$. Let $u_1$ and $u_3$ be the neighbors of $v_1$ and $v_3$, respectively, not incident with $F$. In the case when $u_1 = u_3$, we consider a partial coloring of $G$ obtained from a coloring of $G' = G \setminus \{v_2\}$. There are at least 2 available colors for the noncolored edges $v_1v_2$ and $v_2v_3$, and so we can extend the coloring to all edges of $G$, a contradiction.

Therefore, we may assume that $u_1 \neq u_3$. Suppose first that $u_1$ and $u_3$ are adjacent. Let $G' = G \setminus \{v_1, v_2, v_3\}$. Clearly, $G'$ is not isomorphic to $K_{3,3}$ and so it admits a semistrong edge-coloring $\sigma$ using at most 8 colors. In the coloring of $G$ induced by $\sigma$, we uncolor the

\[ \text{Figure 2} \quad \text{A triangle with all the available colors of the incoming edges distinct.} \]
edge $u_1u_3$, obtaining seven noncolored edges. The edge $u_1u_3$ has at least 2 available colors, $u_1v_1$ and $u_3v_3$ have at least 3, $v_1v_2$ and $v_2v_3$ have at least 6, and $v_1u_4$ and $v_3u_4$ have at least 4. Now, we color $u_1u_3$ with an available color $\alpha$. Since any edge of $F$ may receive the same color as $u_1u_4$, we remove $\alpha$ from $A(e)$ for no edge of $F$, while the number of available colors for $u_1v_1$ and $u_3v_3$ may decrease by 1. But now, it is easy to see that Theorem 5 applies to the remaining six noncolored edges, and thus $\sigma$ can be extended to all the edges of $G$.

Finally, we may assume that $u_1$ and $u_3$ are not adjacent. In this case, by the minimality of $G$, the graph $G' = (G \setminus V(F)) \cup \{u_1u_3\}$ admits a semistrong edge-coloring $\sigma$ using at most 8 colors. In the coloring of $G$ induced by $\sigma$, we color the edges $u_1v_1$ and $u_3v_3$ with the color $\sigma(u_1u_3)$. There remain five noncolored edges: the edge $e$ incident with $u_4$ not on $F$ and the four edges of $F$; $e$ has at least 1 available color, $v_1v_2$ and $v_2v_3$ have at least 3, and $v_1u_4$ and $v_3u_4$ have at least 5. We can color them by Theorem 5, a contradiction.

Claim 8. No subgraph of $G$ is isomorphic to $K_{2,3}$.

Proof. Suppose to the contrary that $G$ contains a subgraph isomorphic to $K_{2,3}$ with bipartition sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. By Claim 6, no two vertices of $Y$ are adjacent. From Claim 7 it follows that every vertex $y \in Y$ has a neighbor $u_i$ distinct from $x_1$ and $x_2$. We consider three cases.

(i) Some of $u_i$’s are the same. Since $G$ is not isomorphic to $K_{3,3}$, we may assume that $u_1 = u_2 \neq u_3$. Let $G' = G \setminus \{x_1, x_2, y_1, y_2\}$. Clearly, $G'$ is distinct from $K_{3,3}$ and, by Claim 2, it is connected. Thus, by the minimality, $G'$ admits a semistrong edge-coloring $\sigma$ using at most 8 colors. In $G$, the coloring $\sigma$ induces a partial semistrong edge-coloring with eight noncolored edges; each of the edges $x_1y_1$, $x_1y_2$, $x_2y_1$, and $x_2y_2$ have at least 6 available colors, and $u_1y_1$, $u_1y_2$, $x_1y_3$, and $x_2y_3$ have at least 5. Thus, $|A(u_1y_1) \cap A(x_1y_3)| \geq 2$ and $|A(u_1y_2) \cap A(x_2y_3)| \geq 2$, which means that we can color $u_1y_1$ and $x_1y_3$ with a common available color $\alpha$, and $u_1y_2$ and $x_2y_3$ with a common available color distinct from $\alpha$. This is possible, since $u_1$ and $y_3$ are not adjacent. After that, the remaining four noncolored edges still have at least 4 available colors each, and so we can color them by Theorem 5, hence extending $\sigma$ to all edges of $G$, a contradiction.

(ii) All $u_i$’s are distinct, but some of them are adjacent. By the symmetry, we may assume that $u_1u_2 \in E(G)$. Again, let $G' = G \setminus \{x_1, x_2, y_1, y_2\}$, which is connected by Claim 2 and by the minimality admits a semistrong edge-coloring $\sigma$ using at most 8 colors. In the corresponding partial coloring of $G$ with eight noncolored edges, we additionally recolor $u_1u_2$ with a color that does not appear in its 2-edge-neighborhood (there are at least two such colors) and is distinct from the color of $u_3y_3$. Now, we have that $x_1y_1$, $x_1y_2$, $x_1y_3$, $x_2y_1$, $x_2y_2$, and $x_2y_3$ each have at least 5 available colors, and $u_1y_1$ and $u_2y_2$ have at least 3. Moreover, $|A(u_1y_1) \cap A(x_1y_3)| \geq 2$, since the two colors from $C(u_2)$ are forbidden for both edges. Analogously, $|A(u_2y_3) \cap A(x_2y_1)| \geq 2$. Next, we color $u_1y_1$ and $x_1y_2$ with a common available color $\alpha$, and $u_2y_3$ and $x_2y_1$ with a common available color distinct from $\alpha$. After that, the remaining four noncolored edges have at least 3 available colors each. If the union of their available colors is of size at least 4, then we apply Theorem 5, and so extend $\sigma$ to all the edges of $G$, a contradiction. Thus, we may
assume that all four edges have the same set of 3 available colors. But then, we can color $x_2 y_2$ with the color of $u_1 u_2$, while the three remaining noncolored edges still have 3 available colors each. Now, we can again apply Theorem 5 to obtain a semistrong edge-coloring of all edges of $G$, a contradiction.

(iii) All $u_i$’s are distinct and pairwise nonadjacent. In this case, we use $G’ = G \setminus (X \cup Y)$, which admits a semistrong edge-coloring $\sigma$ using at most 8 colors. The three edges $u_i y_i$ (call them incoming edges) each have at least 2 available colors, and all the remaining noncolored edges have at least 6 available colors. To simplify our argument, we reduce the number of available colors for the incoming edges; in particular, if there are more than 2 available colors for an incoming edge $e$, then we delete some of them from $A(e)$ to obtain $|A(e)| = 2$. Now, we proceed as follows. We first color $x_1 y_1$ with a color $\alpha_1 \not\in A(u_1 y_1)$. If $\alpha_1 \in A(u_2 y_2)$, then we also color $u_2 y_2$ with it, and, similarly, if $\alpha_1 \in A(u_3 y_3)$, then we also color $u_3 y_3$ with it. Next, we color $x_2 y_1$ with a color $\alpha_2 \not\in A(u_1 y_1)$, and as above, we color any noncolored incoming edge having $\alpha_2$ as an available color (note that such as edge is clearly distinct from $u_1 y_1$). In this way, every noncolored incoming edge retains 2 available colors, and the remaining noncolored edges have at least 4 available colors each. We continue by coloring $x_1 y_2$ with a color $\alpha_3 \not\in A(u_2 y_2)$, and coloring any noncolored incoming edge having $\alpha_3$ as an available color. Finally, we color $x_2 y_3$ with a color $\alpha_4 \not\in A(u_3 y_3)$, and color any noncolored incoming edge having $\alpha_4$ as an available color. At this point, all the noncolored edges have at least 2 available colors. We finish by coloring the edge $x_1 y_3$ and $x_2 y_2$ by distinct available colors, where we use their colors to color any noncolored incoming edge not adjacent to them, and complete the coloring by coloring the remaining noncolored incoming edges by their available colors. Thus, we colored all the edges of $G$, a contradiction. \hfill \Box

Claim 9. There is no 4-cycle in $G$.

Proof. Suppose to the contrary that $F = (v_1, v_2, v_3, v_4)$ is a 4-cycle in $G$. We denote the third neighbor of $v_i$ by $u_i$, for every $i \in [4]$ (see Figure 3), and call the edges $u_i v_i$ incoming.

![Figure 3](image_url)
The edges of $F$ are cycle edges. From Claim 7 we infer that all $u_i$’s exist, and by Claims 6 and 8, they are all distinct.

Let $G'$ be the graph obtained from $G$ by removing the vertices of $F$, and adding the edges $u_1u_3$ and $u_2u_4$ (it is possible that parallel edges are introduced). Note that $G'$ is not necessarily connected, but in such a case, by Claim 8, none of its components is isomorphic to $K_{3,3}$. Thus, by the minimality, either $G'$ is isomorphic to $K_{3,3}$, meaning that $G$ is a 5-prism, which admits a coloring with 8 colors, or there exists a semistrong edge-coloring $\sigma$ of $G'$ using at most 8 colors. Without loss of generality, we may assume that $\sigma(u_1u_3) = 1$ and $\sigma(u_2u_4) = \alpha$, where $\alpha \in \{1, 2\}$. The coloring $\sigma$ induces a partial coloring of $G$ with the incoming edges and the cycle edges of $F$ being noncolored. We show that $\sigma$ can be extended to all the edges of $G$.

First, observe the following. By the construction, if there is a color $\beta \notin C(u_i)$, then $\beta$ can be used for at least one cycle edge incident with $v_i$. In particular, if $\beta \notin C(u_{i+1})$, then $\beta$ can be used for $v_i v_{i+1}$. On the other hand, if $\beta \in C(u_{i-1}) \cap C(u_{i+1})$, then there are two possibilities.

(i) $u_{i-1}u_{i+1} \notin E(G)$ or $e' = u_{i-1}u_{i+1} \in E(G)$ and $\sigma(e') \neq \beta$. Since $u_{i-1}u_{i+1} \in E(G')$, it follows that any of $v_i v_{i-1}$ and $v_i v_{i+1}$ can be colored with $\beta$.

(ii) $e' = u_{i-1}u_{i+1} \in E(G)$ and $\sigma(e') = \beta$. In this case, $\beta \notin \sigma(N^2_{u_{i-1}}(u_{i-1}u_{i+1}))$ or $\beta \notin \sigma(N^2_{u_{i+1}}(u_{i-1}u_{i+1}))$, say $\beta \notin \sigma(N^2_{u_{i+1}}(u_{i-1}u_{i+1}))$, and thus $\beta$ can be used for $v_i v_{i-1}$.

Now, we consider two cases. Suppose first that $\alpha = 2$. We color $u_1v_1$ and $u_3v_3$ with 1 (this can be done, since $\sigma(u_1u_3) = 1$ and $u_3v_3 \notin N^2_{v_1}(u_1v_1)$, $u_1v_1 \notin N^2_{v_3}(u_3v_3)$), and $u_2v_2$ and $u_4v_4$ with 2 (with an analogous reasoning). Now, let $A^*(e)$, for $e \in E(F)$, denote the set of colors that can be used to color the edge $e$ without violating the assumptions of the semistrong edge-coloring. Clearly, $A(e) \subseteq A^*(e)$ and so $|A^*(e)| \geq |A(e)| \geq 2$. Moreover, by the observation above, we also infer that $|A^*(v_{i+1}) \cup A^*(v_{i-1})| \geq 4$. Thus, the sets $A^*(e)$ fulfill conditions of Theorem 5, and we can extend $\sigma$ to all edges of $G$, a contradiction.

Next, suppose that $\alpha = 1$. We again color $u_1v_1$ and $u_3v_3$ with 1, but in this case, we cannot necessarily color $u_2v_2$ and $u_4v_4$ with the 1 as the semistrong condition might be violated in the case if 1 appears also in the 2-edge-neighborhood of $u_1u_3$ in $G'$. However, there is at least 1 available color distinct from 1 for each of $u_2v_2$ and $u_4v_4$. If there is the same available color for both edges, we color them with it and proceed as in the previous case. Hence, we may assume that they are different, say 2 and 3, respectively, and so we color $u_2v_2$ with 2 and $u_4v_4$ with 3. We have that $|A^*(v_{i+1}) \cup A^*(v_2v_3)| \geq 4$, $|A^*(v_3v_4) \cup A^*(v_1v_4)| \geq 4$, and $|A^*(e)| \geq 2$ for every $e \in E(F)$. Thus, we can again apply Theorem 5 and extend $\sigma$ to all edges of $G$, a contradiction. This establishes the claim.

From the above claims it follows that the graph $G$ is a bridgeless subcubic graph with girth at least 5.

Now, let $\sigma$ be a proper edge-coloring of $G$ with the minimum number of edges $uv$ having an edge of color $\sigma(\{uv\}) \in \sigma(N^2_{u}(uv) \cap N^2_{v}(uv))$; in other words, $uv$ is the middle edge of some path $P_6$ that is contained in the subgraph of $G$ induced by the endvertices of edges colored with $\sigma(\{uv\})$. We denote the number of such edges in $G$ by $\ell_6(\sigma)$ and we will
refer to them as the bad middle edges. Additionally, among all such colorings $\sigma$, we choose a coloring with the minimum number of pairs of edges of the same color at distance 2. We denote the number of such pairs in $G$ by $\iota_4(\sigma)$ and we refer to such pairs as the bad pairs. Clearly, $\iota_6(\sigma) > 0$ and every bad middle edge is involved in at least two bad pairs. Note also that an edge can be the bad middle edge of several $P_6$'s, but we count it only once.

Let $uv$ be a bad middle edge in $G$. We may assume that $\sigma(uv) = 1$. Consider the 2-edge-neighborhood of the edge $uv$ and label the neighboring vertices as in Figure 4. By Claims 6 and 9, all the neighbors of $u$ and $v$ are distinct and nonadjacent. There are at most 8 edges at distance 2 from $uw$, where at least two of them are colored with 1. By the minimality of $\iota_6(\sigma)$, we cannot recolor $uv$ without introducing at least one new bad middle edge.

Clearly, each of the seven remaining colors must appear in $N^2(uv)$, otherwise we recolor $uv$ with the one not appearing in $N^2(uv)$ and thus decrease $\iota_6(\sigma)$. Moreover, if some color $\alpha$ appears at most once at distance 2 from $uv$, then we recolor $uv$ with $\alpha$, decrease the number of bad middle edges of color 1 by at least 1 and increase the number of bad middle edges of color $\alpha$ by at most 1. However, while we decrease the number of bad pairs of color 1 by at least 2, we only increase the number of bad pairs of color $\alpha$ by 1, hence violating the minimality of $\iota_6(\sigma)$ and $\iota_4(\sigma)$. Therefore, by a simple counting argument, we infer that the four edges at distance 1 from $uv$ must all obtain distinct colors, and the colors on the edges at distance 2 from $uv$ are different from colors at distance 1 and appear in pairs; consequently, $N^2(uv) \setminus N(uv)! = 8$.

Suppose now that there is a color, say 6, on the edges at distance 2 from $uv$ appearing at the vertices $u_1$ and $u_2$. By the above argument, we may assume that the edges are colored as in Figure 5 and $uv$ cannot be recolored with another color without increasing $\iota_6(\sigma)$ or retaining the value of $\iota_6(\sigma)$ and increasing $\iota_4(\sigma)$.

This means that there are two edges of color 6 in $N^2(vv_1)$, otherwise we can recolor $vv_1$ with 6 and $uv$ with 4 obtaining a coloring $\sigma'$ with either $\iota_6(\sigma') < \iota_6(\sigma)$ or $\iota_6(\sigma') = \iota_6(\sigma)$ and $\iota_4(\sigma') < \iota_4(\sigma)$. Analogously, there are edges of colors 2 and 3 in $N^2(v_1v)$. But then, we can recolor $vv_1$ with 7 and $uv$ with 4, again decreasing $\iota_6(\sigma)$ or retaining the value of $\iota_6(\sigma)$ and decreasing $\iota_4(\sigma)$, a contradiction.

Therefore, we may assume that there are three colors, which, by recoloring $uv$, induce a $P_6$ with $uv$ being the bad middle edge, as depicted in Figure 6. Note that there are
precisely two non-isomorphic colorings of the 2-edge-neighborhood of $uv$, but our argument is analogous for both of them.

We continue by considering the colors of the edges at distance 3 from $uv$.

**Claim 10.** For every edge $xy$ adjacent to $uv$, where $x \in \{u, v\}$, we have

$$|\sigma([xy] \cup N^2_1(xy))| = 7$$

and

$$\sigma(xz) \notin \sigma([xy] \cup N^2_1(xy)),$$

where $z \notin \{u, v, y\}$.

**Proof.** We start by considering the edge $uu_2$ (we label the vertices as depicted in Figure 7). There already are colors 7 and 8 in $N^2_{u_2}(uu_2)$.

If no edge from $N^2_{u_2}(uu_2)$ is colored with 1, then we can set $\sigma(uu_2) = 1$ and $\sigma(uv) = 3$, hence decreasing $\iota_6(\sigma)$ by at least 1. So, we may assume that there is an edge colored with 1 in $N^2_{u_2}(uu_2)$. Similarly, there is an edge colored with 6 in $N^2_{u_2}(uu_2)$, for otherwise we set...
\(\sigma(uu_2) = 6\) and \(\sigma(\text{uv}) = 3\), decreasing \(\iota_4(\sigma)\) by 1, and retaining or decreasing \(\iota_6(\sigma)\) as we lose one bad middle edge of color 1 and introduce at most one bad middle edge of color 6.

An analogous argument implies that also \(4, 5 \in N^2_{\text{uu}_2}(uu_2)\) and hence

\[
\sigma\left( N^2_{\text{uu}_2}(uu_2) \setminus N_{\text{uu}_2}(uu_2) \right) = \{1, 4, 5, 6\}. 
\]

Note that the above argumentation works regardless if all the edges in \(N^2_{\text{uu}_2}(uu_2)\) are distinct from the edges in \(N^2(\text{uv})\) or not (none of them is in \(N(\text{uv})\) due to the girth condition), since we only recolor the edge \(uv\) with 3, which does not appear in \(N^2(\text{uv})\), and recolor the edge \(uu_2\) with a color which does not appear in \(N^2_{\text{uu}_2}(uu_2)\).

Furthermore, by the symmetry, we also have that

\[
\sigma\left( N^2_{\text{uu}_2}(uu_l) \setminus N_{\text{uu}_2}(uu_l) \right) = \{4, 5, 7, 8\},
\]

\[
\sigma\left( N^2_{\text{vv}_2}(vv_1) \setminus N_{\text{vv}_2}(vv_1) \right) = \{2, 3, 6, 7\},
\]

\[
\sigma\left( N^2_{\text{vv}_2}(vv_2) \setminus N_{\text{vv}_2}(vv_2) \right) = \{1, 2, 3, 8\}. 
\]

This establishes the claim.

We continue with an analysis of a possible arrangement of colors also on the edges at distance 3 from \(\text{uv}\) (depicted in Figure 8). Note that the colors need not be all distinct.

In the next claim, we show that using a similar argument as in the proof of Claim 10, we can determine the colors in 2-edge-neighborhoods of the edges at distance more than 2 from \(\text{uv}\).

**Claim 11.** Let \(w_0w_1...w_k\) be an induced path on \(k + 1\) vertices in \(G\), for some integer \(k \geq 2\), with \(w_0 = v\) and \(w_1 = u\) (or \(w_0 = u\) and \(w_1 = v\)). Then for every \(j, 2 \leq j \leq k\), we have that \(d(w_j) = 3\) and \(\sigma(N^2_{w_j}(w_{j-1}w_j)) = [8] \setminus \{\sigma(w_{j-1}w_j), \sigma(w_{j-1}x_{j-1})\}\), where \(x_{j-1}\) is the neighbor of \(w_{j-1}\) distinct from \(w_{j-2}\) and \(w_j\).
Proof. Note that the case for \( k = 2 \) already follows from Claim 10. We proceed by contradiction. Let \( k \geq 3 \) be the least integer such that there is an induced path \( P = w_0w_1\ldots w_k \) in \( G \), for which the claim does not hold. Hence, for every \( j \), \( 2 \leq j \leq k - 1 \), we have \( d(w_{j-1}) = d(w_j) = 3 \) and \( \sigma(N_{w_j}^2(w_{j-1}w_j)) = [8] \setminus \{ \sigma(w_{j-1}w_j), \sigma(w_{j-1}x_{j-1}) \} \).

Suppose now that there is a color \( \alpha \in [8] \setminus \{ \sigma(w_{k-1}w_k), \sigma(w_{k-1}x_{k-1}) \} \) which is not in \( \sigma(N_{w_k}^2(w_{k-1}w_k)) \). Then, for every \( j \in \{1, \ldots, k - 1\} \), we recolor \( w_{j-1}w_j \) with \( \sigma(w_jw_{j+1}) \), and finally we recolor \( w_{k-1}w_k \) with \( \alpha \), obtaining a proper edge-coloring \( \sigma' \). As the path \( P \) is induced, there is no edge outside of \( P \) joining two vertices of \( P \). If after recoloring, we introduce a new pair of edges at distance 2 colored with the same color (note that at most one such pair may appear), then exactly one of these edges belongs to \( P \) or it is the pair \( w_{k-1}w_k, w_{k-3}w_{k-2} \).

After the recoloring, the edge \( w_0w_1 \) is not a bad middle edge anymore, and at most one new bad middle edge \( e' \) was created; \( e' \) (if it exists) is either incident with \( x_{k-1} \) or \( w_{k-2} \).
Thus, \( t_6(\sigma') \leq t_6(\sigma) \). Moreover, \( t_4(\sigma') < t_4(\sigma) \), since exactly one new pair \((w_k w_{k-1} \text{ and } e')\) was created, and two pairs (both with the edge \(w_0 w_1\)) were destroyed, a contradiction. \(\square\)

We finalize the proof of the theorem by recoloring some of the edges of \(G\) to obtain a contradiction in terms of the assumptions on \(\sigma\). Consider the labeling of the vertices as depicted in Figure 9.

Without loss of generality, we may also assume the coloring of the edges as given in the figure, where \([\alpha, \beta] = [6, 8]\).

We first establish some additional properties of \(G\).

Claim 12. \( d(u, v_{22}) \geq 3, d(u, v_{21}) \geq 3, \) and \( d(v, u_{22}) \geq 3.\)

**Proof.** Suppose, to the contrary, that \(d(u, v_{22}) < 3.\) Then, since \(G\) has girth at least 5, we have \(d(u, v_{22}) = 2\), meaning that \(v_{22}\) is adjacent to \(u_1\) or \(u_0\); namely \(v_{22} = u_{11}, v_{22} = u_{12}, v_{22} = u_{21}, \) or \(v_{22} = u_{22}.\) We consider each of the four cases separately.

Suppose first that \(v_{22} = u_{11}.\) Then, by Claim 10 with \(w_k = v_2,\) the vertex \(v_{22}\) is not incident with edges of color 4 or 5. Thus, by Claim 10 with \(w_k = u_1, C(v_{22}) = \{6, 7, 8\}.\) But this is a contradiction with Claim 10 with \(w_k = v_2,\) since \(\beta \in [6, 8].\)

Next, suppose that \(v_{22} = u_{12}.\) Let \(v'_{22}\) be the neighbor of \(v_{22}\) distinct from \(u_1 \) and \(v_2.\) By Claim 10 applied twice, with \(w_k = v_2 \) and \(w_k = u_1,\) we infer that \(\sigma(v_{22}, v'_{22}) = 8\) and consequently \(\beta = 6.\) If \(3 \notin C(v'_{22}),\) then by Claim 10 with \(w_k = u_2,\) inferring that there is no edge of color 3 in \(N^2(u_{11})\), we may recolor \(u_1 v_{22}\) with 3 and so decrease \(t_6(\sigma)\) by 1. Therefore, \(3 \in C(v'_{22}),\) meaning that at least one of the colors 4 and 5 is not in \(C(v'_{22}).\) Without loss of generality, we may assume \(5 \notin C(v'_{22})\) (otherwise we swap the colors of \(v_1\) and \(v_2\)). Now, we set \(\sigma(u) = \sigma(v_{22}) = 5, \sigma(v_{22}) = 1, \) and \(\sigma(u_{11} v_{22}) = 7.\) Note that such a recoloring reduces \(t_6(\sigma)\) by 1, since colors 5 and 7 do not introduce any new \(P_5s,\) a contradiction.

Suppose now that \(v_{22} = u_{21}.\) Then, by Claim 10 with \(w_k = u_2, \sigma(N^2(u_{22}) \setminus N_0(u_{22})) = \{1, 4, 5, 6\},\) but this is not possible, since \(\sigma(u_{22} v_{22}) = 7 \in C(N^2_0(u_{22}) \setminus N_0(u_{22})),\) a contradiction.

Finally, suppose that \(v_{22} = u_{22}.\) In this case, \(\sigma(u_{22} v_{22}) = \sigma(v_{22} v_{22})\) violating the assumption that \(\sigma\) is a proper edge-coloring, a contradiction.

An analogous reasoning can be used to prove that \(d(u, v_{21}) \geq 3\) and \(d(v, u_{22}) \geq 3.\) This establishes the claim. \(\square\)

Now, recolor \(uv\) with \(7, u_2 u_{22}\) and \(v_2 u_{22}\) with 3, and \(u u_2\) with 5. There is one edge, \(v_2 v_{22},\) colored with 7 in \(\text{N}^2(uv),\) and, by Claim 11, the only edge colored with 7 in \(\text{N}^2(v_2 v_{22})\) is \(uv.\) By Claim 10, there is one edge \(e_1 = xy\) colored with 5 in \(\text{N}^2(uu_2),\) where \(x \in \{u_{22}, u_{22}\}.\) Note that \(y \notin \{v_{11}, v_{12}, v_{21}, v_{22}\},\) by Claim 10 applied twice, with \(w_k = v_1\) and \(w_k = v_2.\) This means that \(d(v, y) \geq 3,\) and thus, by Claim 11, the only edge of color 5 in \(\text{N}^2(e_1)\) is \(uu_2.\) Finally, by Claim 10 with \(w_k = v_2,\) there is one edge \(e_2\) colored with 3 in \(\text{N}^2(v_2 v_{22}),\) and by Claims 11 and 12 there is one edge \(e_3\) colored with 3 in \(\text{N}^2_{u_{22}}(u_2 u_{22}).\) By Claim 11, none of \(e_2\) and \(e_3\) is a bad middle edge (and we are done), unless \(e_2 = e_3.\) Thus, for the rest of the proof, assume that \(e_2 = e_3\) and \(\sigma(e_2) = 3.\) We may assume that \(e_2\) is incident with \(v_{22}\) (the case when \(e_2\) is incident with \(v_{21}\) proceeds with an analogous reasoning), and so let \(e_2 = v_{22} z_2\) (see Figure 10).

In this case, we additionally recolor \(u_2 u_{21}\) with 3 and \(u_2 u_{22}\) with 8. Note that by Claim 11, there is no edge of color 8 at distance 2 from \(u_2 u_{22},\) and there is an edge \(e_4\) of color 3 in
By Claim 10 and since \( e_2 \) is incident with \( v_2 \), \( 3 \not\in C(v_2) \), and so \( e_4 \neq v_2 \). Therefore, we are done, unless \( e_4 = x'y' \) is a bad middle edge. But in this case, the path \( vuu_2u_21x'y' \) is induced, meaning that by Claim 11 with \( w_k = y' \), there is no edge of color 3 in \( N^2_{y'}(e_4) \).

This establishes the proof. \( \Box \)

5 | CONCLUSION

We believe that the bounds presented in this paper, although tight, can be improved. In particular, we are not aware of any family of connected graphs \( G \), other than the complete bipartite graphs \( K_{n,n} \), that would attain the bound \( \chi^{s'}_s(G) = \Delta(G)^2 \). Therefore, we propose the following.

**Conjecture 1.** For every connected \( G \), distinct from \( K_{n,n} \), it holds that

\[
\chi^{s'}_s(G) \leq \Delta(G)^2 - 1.
\]

Similarly, the 5-prism is, up to our knowledge, the only connected subcubic graph with the semistrong chromatic index 8. Based on that and computational verification of subcubic graphs on small number of vertices, we also propose the next conjecture.

**Conjecture 2.** For every connected graph \( G \) with maximum degree 3, distinct from \( K_{3,3} \) and the 5-prism, we have

\[
\chi^{s'}_s(G) \leq 7.
\]

On the other hand, there are infinitely many bridgeless subcubic graphs with the semistrong chromatic index at least 7. Namely, consider the graph \( H \) obtained by taking two
copies of $K_{2,3}$ and adding two edges connecting distinct 2-vertices from each of two copies (see Figure 11). This graph contains only one semistrong matching of size at least 3 consisting of two edges added between the copies of $K_{2,3}$ and one nonadjacent edge. For the remaining 11 edges we need 6 additional colors. Since $H$ contains two 2-vertices, we can append it to any other bridgeless subcubic graph with at least two 2-vertices, hence obtaining an infinite number of graphs with the semistrong chromatic index at least 7.

There are many other important graph classes for which the semistrong edge-coloring has not been studied specifically yet, for example, planar graphs. In [6], it is proved that for a strong edge coloring of a planar graph $G$ at most $4\Delta(G) + 4$ colors are needed, and there are planar graphs $G$ with $\chi'_s(G) = 4\Delta(G) - 4$ for any $\Delta(G) \geq 2$. Note that these examples have $\chi''_s(G) \leq 2\Delta(G)$. But as we showed with $K_{3,3}$ and the 5-prism, for cubic graphs, we need 9 and 8 colors. So, one may ask, what is happening when the maximum degree is larger. We propose the following.

**Problem 1.** For a given maximum degree, determine the tight upper bound for the semistrong chromatic index of planar graphs.

**Conjecture 3.** There is a (small) constant $C$ such that for any planar graph $G$, it holds that

$$\chi''_s(G) \leq 2\Delta(G) + C.$$ 

Finally, let us briefly mention another edge-coloring variation, which is closely related to the semistrong edge-coloring, yet different. Baste and Rautenbach [1], motivated by the results of Goddard et al. [7], introduced the $r$-chromatic index $\chi'_r(G)$ as the minimum number of $r$-
degenerate matchings into which the edge set of a graph $G$ can be decomposed. An $r$-degenerate matching is a matching $M$ such that the induced graph $G[V(M)]$ is $r$-degenerate. Clearly,

$$\chi'(G) \leq \chi'_r(G) \leq \chi'_s(G).$$

Since semistrong matchings are not necessarily 1-degenerate and 1-degenerate matchings are not necessarily semistrong, there is no direct correspondence between the two edge-colorings. In particular, for the 5-path $P_6$, we have $\chi'_1(P_6) = 2 < \chi'_{ss}(P_6) = 3$, and for the graph $H_1$ being a triangle with a pending edge incident with every vertex (see Figure 12), we have $\chi'_{ss}(H_1) = 4 < \chi'_1(H_1) = 5$. Namely, while in a semistrong edge-coloring all pendant edges can receive the same color, this is not the case for 1-degenerate matching coloring, since their endvertices induce the whole graph, containing a cycle.

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FINAL REMARK
Before we finished preparation of this manuscript, Martina passed away, taken by a treacherous disease. We would like to dedicate this work to her memory.

DATA AVAILABILITY STATEMENT
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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