Chiral Radiation Transport Theory of Neutrinos

Naoki Yamamoto and Di-Lun Yang

Department of Physics, Keio University, Yokohama 223-8522, Japan

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Abstract

We construct the chiral radiation transport equation for left-handed neutrinos in the context of radiation hydrodynamics for core-collapse supernovae. Based on the chiral kinetic theory incorporating quantum corrections due to the chirality of fermions, we derive a general relativistic form of the chiral transfer equation with collisions. We show that such quantum corrections explicitly break the spherical symmetry and axisymmetry of the system. In the inertial frame, in particular, we find that the so-called side jump leads to quantum corrections in the collisions between neutrinos and matter. We also derive analytic forms of such corrections in the emission and absorption rates for the neutrino absorption process. These corrections result in the generation of kinetic helicity and cross helicity of matter, which should then modify the subsequent evolution of matter. This theoretical framework can be applied to investigate the impacts of the chirality of neutrinos on the evolution of core-collapse supernovae.

Unified Astronomy Thesaurus concepts: Supernova neutrinos (1666); Core-collapse supernovae (304); Radiative transfer equation (1336)

1. Introduction

Understanding the mechanism of core-collapse supernova explosions is one of the unsolved problems in astrophysics. When a massive star experiences collapse of the core, most of the gravitational binding energy is released in the form of neutrinos. For this reason, proper treatment of neutrino transport physics is Required to account for the core-collapse supernova explosions. Since neutrinos are mostly out of equilibrium and must be treated as radiation away from the dense core of supernovae, the theoretical formulations and numerical simulations for neutrino transport are based on the Boltzmann equation, or more precisely, the Einstein–Vlasov equation under certain approximations (Castor 1972; Bruenn 1985); see also O’Connor et al. (2018), O’Connor & Couch (2018), Summa et al. (2018), Richers et al. (2017), Vartanyan et al. (2018), Kotake et al. (2018), and Cabezón et al. (2018) for recent reviews and comparisons between numerical simulations. However, the most fundamental property of neutrinos—left-handedness—has been neglected in the conventional theoretical formulation and simulations of radiation hydrodynamics for neutrinos.

Recently, it has been shown in Yamamoto (2016a) that the parity violation by the chirality of neutrinos can affect the macroscopic hydrodynamic evolution of supernovae in a qualitative manner. In fact, there has been growing recent interest in the study of chiral transport phenomena that originate from chirality of (generally charged) particles not limited to neutrinos. The most renowned examples are the currents induced by magnetic fields and vorticity, dubbed the chiral magnetic effect (CME; Vilenkin 1980; Nielsen & Ninomiya 1983; Alekseev et al. 1998; Fukushima et al. 2008) and chiral vortical effect (CVE; Vilenkin 1979; Erdmenger et al. 2009; Son & Surowka 2009; Banerjee et al. 2011; Landsteiner et al. 2011), respectively. A remarkable aspect of these effects is their connection to the chiral anomaly, i.e., the quantum violation of the chiral symmetry in field theory (Adler 1969; Bell & Jackiw 1969). Such anomalous transport phenomena are relevant not only to neutrinos in core-collapse supernovae but also to a variety of physical systems such as hot electroweak plasmas in the early universe (Joyce & Shaposhnikov 1997; Boyarsky et al. 2012; Kamada & Long 2016), quark–gluon plasmas created in heavy ion collision experiments (Kharzeev et al. 2016), dense electromagnetic plasmas in neutron stars (Charbonneau & Zhitnitsky 2010; Akamatsu & Yamamoto 2013; Ohnishi & Yamamoto 2014; Kaminski et al. 2016), and emergent chiral matter near band crossing points of Weyl semimetals (Nielsen & Ninomiya 1983; Burkov & Balents 2011; Wan et al. 2011; Xu et al. 2011). However, the classical Boltzmann equation is unable to capture these chiral effects. To incorporate such quantum corrections, the so-called chiral kinetic theory (CKT) has been established. The pioneering construction started from a semiclassical derivation by introducing a Berry phase as the source of quantum corrections, which results in the modification on the free-streaming Boltzmann equation (Son & Yamamoto 2012; Stephanov & Yin 2012). Alternatively, a field-theoretic derivation known as the Wigner function approach was applied to derive CKT despite some limited conditions (Chen et al. 2013; Son & Yamamoto 2013). In

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In the context of core-collapse supernovae, there could also be prominent chirality imbalance of electrons produced by the electron capture process (Ohnishi & Yamamoto 2014; Dvornikov & Semikoz 2015). Although such chiral imbalance could be compensated by elastic electron scattering with the effect of nonzero electron mass (Grabowska et al. 2015; Kaplan et al. 2017), the remaining imbalance may still result in sizable chiral effects (Sigl & Leite 2016; Onishi & Maruyama 2020). To investigate the dynamics of the chiral matter near the dense core of supernovae in thermal equilibrium, one may resort to the chiral magnetohydrodynamics (ChMHD) as the modified magnetohydrodynamics involving the chiral anomaly (Yamamoto 2016a, 2016b; Rogachevskii et al. 2017; Hattori et al. 2019b). It has been demonstrated in Masada et al. (2018) that the ChMHD simulation reveals the dominance of inverse energy cascade as opposed to the direct energy cascade in conventional 3D neutrino radiation hydrodynamic simulations (for reviews, see Janka et al. 2016; Radice et al. 2018). In this paper, we will focus on the chiral effects of neutrinos.

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addition, the Lorentz invariance of the CKT was revealed and the modified frame transformation on distribution functions was introduced in relation to the so-called side-jump phenomenon stemming from the spin–orbit interaction (Chen et al. 2014, 2015; Hidaka et al. 2017; Yang 2018). More recently, through the Wigner function approach, a generic Lorentz-covariant CKT under background electromagnetic fields with systematic inclusion of collisions was obtained (Hidaka et al. 2017, 2018; Hidaka & Yang 2018); see also Mueller & Venugopalan (2017), Huang et al. (2018b), Carignano et al. (2018, 2019), Lin & Shukla (2019), and Lin & Yang (2020) for related recent developments. Furthermore, this derivation was generalized to curved spacetime in the case without collisions (Liu et al. 2019). The CKT has been widely applied to investigate anomalous transport pertinent to relativistic heavy ion collisions and Weyl semimetals (Gorbar et al. 2017a; Kharzeev et al. 2017; Hidaka & Yang 2018; Huang et al. 2018a; Sun & Ko 2018; Rybalka et al. 2019).

Given the established framework of CKT and Wigner functions with quantum corrections, in this paper we construct the radiation transport equation for left-handed neutrinos by incorporating the effects of the chirality, which we may call the chiral radiation transport (or transfer) equation. We first derive a general relativistic form of the chiral radiation transport equation with collisions (Equations 25 and 26), which shows that the quantum corrections explicitly break the spherical symmetry and axisymmetry of the system. We then focus on the inertial frame as one of widely used coordinate systems for numerical simulations of core-collapse supernovae. In this case, we find that, although the free-streaming part remains unchanged from the conventional transport equation, the so-called side-jump effects lead to quantum corrections between neutrinos and matter (Equations 32 and 33). As a demonstration, we analytically derive the quantum corrections involving the fluid vorticity and magnetic fields in the emission and absorption rates for the neutrino absorption process (Equations (63)–(65)). In addition, we also show that the side-jump effects modify the particle-number current and energy–momentum tensor of neutrinos through the Wigner functions (Equations 27 and 36) and that such quantum corrections affect the energy–momentum transfer between neutrinos and matter (Equations 29 and 30); see also Equation (F4).

The paper is organized as follows: In Section 2, we briefly review the derivation of 3D transfer equations from the Einstein–Vlasov equation mainly in the inertial frame. In Section 3, we then provide an introduction and generalization of CKT and the Wigner function formalism and present the quantum corrections on the energy–momentum transfer. In Section 4, we derive the chiral radiation transport equation and Wigner functions of neutrinos in the inertial frame. Section 5 is devoted to summary and outlook.

Throughout this work, we assume massless neutrinos. We use the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+,-,-,-)$. We define the Levi–Civita tensor $\epsilon^{\mu \nu \rho \sigma} = \frac{1}{6} \tilde{\epsilon}^{\mu \nu \rho \sigma} g^{0123}$, where $\tilde{\epsilon}^{\mu \nu \rho \sigma}$ denotes the permutation symbol and $g$ represents the determinant of the spacetime metric with the convention $\epsilon_{0123} = -\epsilon_{0123} = 1$. We absorb the electric charge $e$ into the definition of the gauge field $A_\mu$. We also introduce the notations $A_{[\mu, \nu]} = (A_{\mu} B_{\nu} - A_{\nu} B_{\mu})/2$ and $A_{(\mu, \nu)} = (A_{\mu} B_{\nu} - A_{\nu} B_{\mu})/2$. We will keep $\hbar$ only to indicate the $\hbar$ expansion, but we will suppress other $\hbar$’s except for our main results in Equations (63)–(65). We will take $c = 1$ after Section 4.2 and in Appendices D–F, except for Equations (63)–(65).

### 2. Classical Radiation Transport Equation

In this section, we review the derivation of the 3D classical transfer equation for delineating the neutrino radiation transport in the inertial frame. To make our discussion generic, we will first write down the Lorentz-covariant kinetic equation for charged particles in the presence of background electromagnetic fields in curvilinear coordinates. The kinetic equation for charge neutral neutrons can be obtained by turning off the electromagnetic fields later.

We start with the Einstein–Vlasov equation, which is a generalized Boltzmann equation in curved spacetime or in non-Cartesian coordinates (non-Minkowski spacetime). For massless fermions, the Einstein–Vlasov equation reads

$$\delta(q^2) q \cdot \Delta f = 0,$$

where $f(x, q)$ is the distribution function for a quasi-particle in phase space and

$$\Delta_a = \partial_a + (F^c_a - q^b \Gamma^c_{ab}) \partial_q^c$$

with $\partial_a \equiv \partial / \partial x^a$. $\Gamma^c_{ab}$ represents the Christoffel symbol, and $F_{ab}$ denotes the field strength for a $U(1)$ gauge field. For the moment, we ignore the collisions on the right-hand side of the kinetic equation, which can be further included later. Note that here $q^a$ and $x^a$ in $f$ are independent, which is generally held in the off-shell case. Nevertheless, when implementing the on-shell condition, the derivatives with $x^a$ and with $q^a$ become entangled. To avoid the complexity, an efficient way is to introduce an orthonormal frame of local coordinates such that the $x^a$ and $q^a$ are independent under the on-shell condition. One then performs the corresponding coordinate transformation to the coordinate system $(x^\mu, q^\alpha)$, e.g., $q^a = e^a_\alpha(x^\mu) q^\alpha(x^\nu)$ and $\eta_{ab} = e_a^\mu(x^\nu) e_b^\nu(x^\rho) g_{\mu \rho}(x^\sigma)$ via vierbeins (Lindquist 1966). Here the Roman and Greek indices run over $\{0, 1, 2, 3\}$ and $\{t, r, \theta, \phi\}$, respectively. Accordingly, for the Einstein–Vlasov equation, we have to apply the coordinate transformation on the Christoffel symbols,

$$\Gamma^c_{ab} = e^c_\gamma e^\alpha_a (e^b_\beta \Gamma^\gamma_{\alpha \beta} + \partial_\alpha e^b_\beta),$$

which are called the Ricci rotation coefficients. Note that $q^a \Delta_a$ is invariant under the coordinate transformation while the individual terms $q^a \partial_a$ and $q^a \partial^c \Gamma^c_{ab} \partial_q^c$ are not.

We will now apply the equation above to obtain the renowned kinetic equation with spherically symmetric metric shown in Lindquist (1966). For generality, we will lift the spherical symmetry for the distribution functions and consider the general expression of a spherical symmetric spacetime metric,

$$ds^2 = e^{2g_{\rho \phi}}(d \theta^2 - \sin^2 \theta d \phi^2),$$

which yields the following nonvanishing vierbeins, $e^r_0 = e^\phi$, $e^\rho_1 = e^\theta$, $e^\phi_0 = R$, and $e^\phi_\rho = R \sin \theta$. We also keep $\Phi$ and $\Lambda$ as arbitrary functions depending on $(t, r)$ for generality. The corresponding four-momentum satisfying the null on-shell
condition can be written as

\[ q^t = e^{-\Phi}E, \quad q^r = e^{-\lambda}\mu E, \quad q^\phi = \frac{\sqrt{1 - \mu^2}}{R} E \cos \phi, \]

\[ q^\theta = \frac{\sqrt{1 - \mu^2}}{R \sin \theta} E \sin \phi, \]

where \( \mu \equiv \cos \bar{\theta} \). Note that here we only need three extra variables \((E, \mu, \bar{\phi})\) to parameterize \( q^\phi \) owing to the on-shell condition. Considering a general case for the distribution functions \( f = f(t, r, \theta, \phi, E, \mu, \bar{\phi}) \), the on-shell kinetic equation for charge neutral particles (when \( F_{\mu\nu} = 0 \)) reads

\[ 0 = (q^a \partial_a - q^a \Gamma_{ba}^c \partial_c \partial_d) f \]

\[ = E \left( \frac{1}{c} \partial_t + \mu \bar{\partial}_t + \frac{\sqrt{1 - \mu^2}}{R} \cos \bar{\phi} \partial_\theta \right. \]

\[ + \frac{\sqrt{1 - \mu^2}}{R \sin \theta} \sin \bar{\phi} \partial_\phi \]

\[ - E (\mu \bar{\partial}_t \Phi + (1 - \mu^2) \bar{\partial}_t \ln R + \mu^2 \bar{\partial}_t \Lambda) \partial_E \]

\[ - (1 - \mu^2) (\bar{\partial}_t \Phi + \bar{\partial}_t \Lambda) \right) \partial_\mu \]

\[ \left. - \frac{1}{r} \left[ \right] \sin \bar{\phi} \partial_\phi \right) f, \]

where \( \bar{\partial}_t \equiv e^{-\Phi} \partial_t \) and \( \bar{\partial}_a \equiv e^{-\lambda} \partial_a \), and we used the relations in Equation (A7) shown in Appendix A. When further imposing the spherical symmetry for the distribution functions \( f(t, r, \Phi, \mu, \lambda) \), the kinetic equation reduces to the one found in Lindquist (1966).

We can directly implement Equation (6) to derive the kinetic equation in the inertial frame with the spacetime metric

\[ ds^2 = c^2 dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where the subscript “\( i \)” represents the inertial frame and \( c \) denotes the speed of light. By comparing Equations (4) and (7), we take

\[ t = t_i, \quad r = r, \quad \Phi = \ln c, \quad \Lambda = 0, \quad R = r, \]

and we define the corresponding on-shell momentum,

\[ q^i = \frac{E_i}{c}, \quad q^r = \mu_i E_i, \quad q^\theta = \frac{\sqrt{1 - \mu_i^2}}{r} E_i \cos \phi_i, \]

\[ q^\phi = \frac{\sqrt{1 - \mu_i^2}}{r \sin \theta} E_i \sin \phi_i. \]

Given \( f = f(t_i, r, \theta, \phi, E_i, \mu_i, \phi_i) \) in terms of the coordinates in the inertial frame, Equation (6) reduces to

\[ \left\{ \begin{array}{l}
\frac{1}{c} \partial_t + \mu_i \partial_r + \frac{\sqrt{1 - \mu_i^2}}{R} \cos \phi_i \partial_\theta \\
+ \frac{\sqrt{1 - \mu_i^2}}{r \sin \theta} \sin \phi_i \partial_\phi + \frac{1 - \mu_i^2}{r} \partial_\mu \\
- \frac{1 - \mu_i^2}{r} \sin \phi_i \cot \theta \partial_\phi \end{array} \right\} f = 0, \quad (10) \]

which does not depend on the fluid velocity and the energy derivative. For numerical calculations, it is practical to rewrite the transfer equations into a conservative form. The conservative form of Equation (10) becomes

\[ \left[ \begin{array}{l}
\partial_t + \mu_i \partial_r + \frac{\sqrt{1 - \mu_i^2}}{r} \cos \phi_i \partial_\theta \\
+ \frac{\sqrt{1 - \mu_i^2}}{r \sin \theta} \sin \phi_i \partial_\phi + \frac{1 - \mu_i^2}{r} \partial_\mu \\
- \frac{1 - \mu_i^2}{r} \sin \phi_i \cot \theta \partial_\phi \end{array} \right] f = 0. \quad (11) \]

This expression can also be found in, e.g., Sumiyoshi & Yamada (2012). Finally, one has to retrieve the collision terms responsible for radiation transfer in Equations (11), which will be discussed later with the inclusion of quantum corrections.

3. Chiral Kinetic Theory

3.1. Wigner Functions and Kinetic Theory

In this section, we shortly review and generalize the CKT for massless chiral fermions obtained from the Wigner function approach in curved spacetime. For generality, we will first consider charged particles in the presence of electromagnetic fields again, but we will focus on charge neutral neutrinos by turning off the electromagnetic fields later. As a starting point, we introduce the Wigner functions for left-handed fermions as the quantum expectation values of correlation functions in Minkowski spacetime,

\[ S_L^\psi(q, x) = \int d^4 y \ e^{ixq} S_L^\psi(x, y), \quad (12) \]

where \( S_L^\psi(x, y) \equiv \langle \psi_L^\dagger(x + y/2) \psi_L(x - y/2) \rangle \) and \( S_A^\psi(x, y) \equiv \langle \psi_L(x - y/2) \psi_L^\dagger(x + y/2) \rangle \) are the lesser and greater propagators for left-handed fermions, respectively (see, e.g., Blaizot & Iancu 2002, for a review). Here left- and right-handed fermions \( \psi_L/R \) are defined as \( \psi_{L/R} \equiv P_{L/R} \psi \) for a Dirac fermion \( \psi \), with the projection operators \( P_{L/R} \equiv (1 \mp \gamma^5)/2 \) and \( \gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). Between the field operators \( \psi_L \) and \( \psi_L \) in the expressions above, gauge links are implicitly embedded to preserve gauge invariance. The dynamics of Wigner functions in phase space are then dictated by Kadanoff–Baym equations derived from the Dirac equation. Nevertheless, in order to solve Kadanoff–Baym equations, one has to further perform the \( \hbar \) expansion that is equivalent to a gradient expansion. One then perturbatively solves the Kadanoff–Baym equations for Wigner functions with the \( \hbar \) expansion up to \( O(\hbar) \) to capture the leading-order quantum corrections and thereby derives the corresponding CKT as a modified Boltzmann equation (Hidaka et al. 2017).

In curved spacetime, the definition of phase space becomes more subtle owing to the lack of global momentum. Instead,
the phase space is defined on a tangent or cotangent bundle as applied in Liu et al. (2019) for the derivation of CKT in curved spacetime (or more precisely, non-Minkowski spacetime there). For convenience, we will choose the tangent bundle with the set \((x^\mu, p^\mu)\) as opposed to the choice in Liu et al. (2019). The Wigner functions and CKT may differ, but the physics remain unchanged when making different choices. Now, the definition of Wigner functions becomes

\[
S^\leq_L(q, x) = \int \frac{d^3y}{\sqrt{-g} (x)} e^{\frac{-i}{\hbar} S^\leq_L(y, x)}, \tag{13}
\]

where \(S^\leq_L(q, x) \equiv \langle \psi_L(x, y/2) \psi_L(x, -y/2) \rangle\), \(S^\geq_L(x, y) \equiv \langle \psi_L(x, -y/2) \psi_L(x, y/2) \rangle\), and \(g(x)\) denote the determinant of the spacetime metric. Here \(\psi_L(x, y) = e^{x^\mu \partial^\mu} \psi_L(x)\) and \(\psi^\dagger_L(x, y) = \psi^\dagger_L(x) e^{x^\mu \partial^\mu}\), where \(\tilde{D}_\mu = \nabla_\mu + i A_\mu / \hbar + \Gamma^\mu_{\nu\lambda} y^\nu \partial^\nu\) corresponds to the horizontal lift and \(\nabla_\mu\) denotes the covariant derivative with respect to \(x^\mu\). It turns out that the horizontal lift provides a proper covariant derivative on the phase space such that \(\tilde{D}_\mu y^\nu = 0\) and \(\tilde{D}_\mu g_{0,0}(x) = 0\) when \(A_\mu = 0\). With this definition, Equation (13) reduces to Equation (12) in Minkowski spacetime. Despite the technical subtleties, the strategy for the derivation of CKT in the Wigner function formalism in curved spacetime is the same as that in Minkowski spacetime. One may refer to Liu et al. (2019) for more details. The lesser propagator of left-handed fermions can be parameterized as \(S^\leq_L(q, x) = \sigma^\mu S^\leq_{\mu \nu}(q, x)\), where \(\sigma^\mu = (I, \sigma^1, \sigma^2, \sigma^3)\) with \(I\) being an identity matrix and \(\sigma^1, \sigma^2, \sigma^3\) the Pauli matrices.4

However, in order to construct the radiation hydrodynamic incorporating the energy–momentum transfer between neutrinos and matter, it is inevitable to include collision terms, which are not considered in Liu et al. (2019). Although a rigorous derivation of collisions in the CKT in curved spacetime might be technically more involved, we may generalize the derivation of the CKT with collisions in Minkowski spacetime shown in Hidaka et al. (2017, 2018) with proper modifications upon the Kadanoff–Baym equation to the case of curved spacetime. In light of the approach in Hidaka et al. (2017, 2018) and Liu et al. (2019), the Kadanoff–Baym equation with collisions for left-handed fermions leads to the following master equations up to \(O(\hbar)\):5

\[
\mathcal{D} \cdot \mathcal{L}^< = 0, \tag{14}
\]

\[
q \cdot \mathcal{L}^< = 0, \tag{15}
\]

\[
\hbar c (\mathcal{D}_\mu \mathcal{L}_\nu^L - \mathcal{D}_\nu \mathcal{L}_\mu^L) = - 2 \epsilon_{\mu
u\rho\sigma} q^\rho \mathcal{L}^< \sigma, \tag{16}
\]

where

\[
\mathcal{D}_\mu \mathcal{L}_\nu^L = \Delta_\mu \mathcal{L}_\nu^L - \Sigma^\mu_\nu \mathcal{L}_\nu^L + \Sigma^\nu_\mu \mathcal{L}_\mu^L \tag{17}
\]

\[\text{Note that the } n^\mu \text{ dependence of } S_{\mu \nu}(q) \text{ implies that } f^L_\mu \text{ is no longer invariant under the frame transformation. Given the fact that } L^\mu \text{ is frame independent, one can accordingly derive the modified frame transformation on } f^L_\mu \text{ between different frame choices, which is also related to the modified Lorentz transformation (Chen et al. 2014, 2015; Hidaka et al. 2017). More precisely, the distribution function } f^L_\mu \text{ in one frame with}\]

\[\text{denotes the spin tensor, which depends on a timelike frame vector } n^\mu(x) \text{ satisfying } n^2 = 1 \text{ and } \delta^\mu(q^2) = \delta(q^2) / \eta^2 \text{ and } F^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}/2. \text{ The frame vector } n^\mu(x) \text{ appears as a choice of the spin basis such that } n^\mu(\sigma_\mu = I) \text{ and that } \sigma^\mu \text{ perpendicular to } n^\mu \text{ becomes } \sigma^\mu(0, \sigma^2, \sigma^3). \text{ That is, we define } n^\mu = e^\mu_0 \text{ as the zeroth component of vierbeins (Hidaka & Yang 2018). However, one should note that } L^\mu \text{ is independent of the choice of } n^\mu. \text{ In addition, as discussed in Hidaka et al. (2017), the } O(\hbar) \text{ corrections proportional to } q^\mu \delta(q^2) \text{ as the trivial solutions for Equations (15) and (16) can be absorbed into } f^L_\mu. \text{ The quantum corrections at } O(\hbar) \text{ now incorporate two terms shown in Equation (18), in which the } \delta(q^2) \text{ term yields the modification on the on-shell condition due to the magnetic-moment coupling in the presence of background electromagnetic fields (Son & Yamamoto 2013; Chen et al. 2014). Although such a term vanishes for neutrinos, the other term in Equation (18) associated with the spin tensor } S_{\mu \nu}(q) \text{ called the side-jump term, exists even without background electromagnetic fields, which then influences the neutrino transport. This side-jump term stems from the spin-momentum locking of chiral fermions under the angular-momentum conservation, and it contributes to the so-called magnetization currents and CVE (Chen et al. 2014, 2015; Yang 2018).} \]

\[\text{One can in fact construct the Wigner functions for Dirac fermions, } S^L(q, x), \text{ by replacing } S^L(q, x) \text{ in Equation (13) with } S^L(q, x) = \langle \psi(x, y/2) \psi(x, -y/2) \rangle. \text{ Based on the Clifford algebra, one may decompose the Wigner functions as, e.g., } S^L = S + i \Gamma^\mu_{\nu\lambda} \overline{\psi}^L \gamma^\nu \gamma^\lambda \psi^L + \overline{\psi}^L \gamma^\mu \psi^L + \overline{\psi}^L \gamma^\mu \gamma_5 \psi^L. \text{ In the massless limit, } \Gamma^\mu_{\nu\lambda} \text{ and } \overline{\psi}^L \gamma^\nu \gamma^\lambda \psi^L \text{ are decoupled from } S^L, \tilde{D}_\mu, \text{ and } S^\mu_{\nu}, \text{ because } S^\mu_{\nu} \text{ is the dual operator of the horizontal lift } \delta_\mu \text{ in the tangent space (} x^\mu, p^\mu) \text{ when neglecting gauge fields. Also, } \Sigma^\mu_{\nu} \text{ and } \Sigma^\nu_{\mu} \text{ correspond to lesser and greater self-energies depending on details of interactions in a given system. Here } \hbar \text{ can be regarded as an expansion parameter to track the quantum corrections. Equation (14) is constructed by replacing the spacetime derivatives } \delta_\mu \text{ by } \delta_\mu \text{ in the master equations in Hidaka et al. (2017, 2018). From Equations (15) and (16), the corresponding solution up to } O(\hbar) \text{ takes the form (see Appendix B) } \]

\[
\mathcal{L}^{\leq} = \frac{2 \pi (\delta(q^2) / \hbar c) S_{\mu \nu} D_{\mu \nu} - \hbar c F^{\mu \nu} q^\delta(q^2)}{2 \hbar c n^\mu n^\nu}, \tag{18}
\]

\[\text{where } D_{\mu \nu} = D_{\mu \nu} f^L_\mu, \text{ and } C_{\mu \nu} f^L_\mu = \Sigma^\mu_{\nu} f^L_\mu - \Sigma^\nu_{\mu} f^L_\mu, \text{ with } f^L_0 = f^L_0 \text{ and } f^L_1 = 1 - f^L_0 \text{ the distribution functions of incoming and outgoing fermions, respectively. Here } \]

\[
S_{\mu \nu} = \frac{\epsilon^{\mu \nu \rho \sigma} q^\rho n^\sigma}{2 q \cdot n}, \tag{19}
\]

\[\text{Hence and below, we ignore the one-particle potential denoted by } \Sigma^S \text{ in Hidaka et al. (2017) for simplicity, as it is irrelevant to the chiral effects that we are interested in. The inclusion of } \Sigma^S \text{ may modify the dispersion relation of the fermions.} \]

\[4 \text{ We here ignored the contribution of antiparticles, which can be included by multiplying the right-hand side of Equation (18) by the sign of } q \cdot n. \]
\( n^\mu \) is related to \( f_L^{(n)} \) in another frame with \( n^\nu \) by

\[
f_L^{(n)(\nu)} = f_L^{(n)} - \frac{\hbar c}{2} \sum_{\mu} \frac{\epsilon^{\mu\nu\alpha\beta} q_\alpha n_\beta n_\nu}{2 \langle q \cdot n \rangle (q \cdot n')} D_{\mu} f_L^{(n)}.
\] (20)

Note that the frame transformation between different frames is distinct from the usual coordinate transformation between the inertial frame and the comoving frame in radiation hydrodynamics. The two different notions of these “frame transformations” should not be confused with each other.

By plugging Equation (18) into Equation (14) and employing the relation

\[
\begin{aligned}
\Delta \mu, \Delta \nu \mathcal{f}_L &= \left\{ (\nabla_\mu + F_{\lambda \mu} \partial^\lambda - \Gamma^\lambda_{\mu \nu} q^\nu \partial_\lambda/q) \right. \\
&\quad \left. \nabla_\nu + F_{\lambda \nu} \partial^\lambda - \Gamma^\lambda_{\nu \mu} q^\mu \partial_\lambda/q \right\} f_L \\
&= 2 \left\{ (\nabla_\mu F_{\lambda \nu} \partial^\lambda - 2 q^\nu (\nabla_\nu F_{\alpha \mu} \partial_\alpha) \right\} f_L \\
&= 2 \left\{ (\nabla_\mu F_{\lambda \nu} \partial^\lambda - q^\nu R_{\alpha \mu} \partial_\alpha) \right\} f_L,
\end{aligned}
\] (21)

where we used \( R_{\alpha \beta} = 2 \partial_{[\alpha} F_{\beta]} + 2 \Gamma_{\alpha \beta}^\gamma \Gamma_{\gamma \nu} \). The CKT in curved spacetime as a modified Einstein–Vlasov equation up to \( \mathcal{O}(\hbar) \) is derived as

\[
\delta q^2 - \hbar c F_{\alpha \beta} S_{(0)}^{\mu \nu} \left[ q \cdot \partial - \hbar c \left( \frac{S_{(0)}^{\mu \nu} F_{\mu \nu}}{q \cdot n} + (D_{\mu} S_{(0)}^{\mu \nu}) \right) \right] \\
\times \partial_\nu - \hbar c S_{(0)}^{\mu \nu} (\nabla_\nu F_{\alpha \mu} - q^\mu R_{\alpha \mu} \partial_\mu) f_L = 0,
\] (22)

where \( \mathcal{D}_\mu f_L = \Delta \mu f_L - \bar{\mathcal{C}}_\mu [f_L] \) and

\[
\mathcal{C}^{\mu [f_L]} = \mathcal{C}[f_L] - \hbar c \epsilon^{\mu \nu \alpha \beta} q_\alpha n_\beta (1 - f_L) \Delta^\gamma \Sigma^\beta_\mu = f_L \Delta^\gamma \Sigma^\beta_\mu. 
\] (23)

with \( \Delta^\gamma_\mu = \Delta_\mu + \Sigma^\beta_\mu \). For right-handed fermions, the \( \mathcal{O}(\hbar) \) terms flip signs.

To delineate the transport for neutrinos, we can turn off the background electromagnetic fields. The dispersion relation for chiral fermions hence remains lightlike. In the flat spacetime such as in the inertial frame, we can further drop the term proportional to the Riemann tensor. It turns out that only the term associated with the horizontal lift acting on the spin tensor contributes to the quantum corrections. We can explicitly evaluate this term,

\[
D_{\mu} S_{(0)}^{\mu \nu} = \frac{\epsilon^{\mu \nu \alpha \beta} q_\alpha}{2q \cdot n} \left( \nabla_\mu n_\beta - n_\beta q^\mu \frac{\nabla_\mu n_\nu}{q \cdot n} \right),
\] (24)

where we applied the property of the Levi–Civita tensor, \( \nabla_\nu \epsilon^{\mu \nu \alpha \beta} = 0 \), and \( D_{\mu} q^\nu = 0 \). For generality, we will assume a nonvanishing Riemann tensor, in which case the CKT for left-handed neutrinos is given by

\[
\begin{aligned}
\frac{q^\mu (\partial_\mu - \Gamma^\lambda_{\mu \nu} q^\nu \partial_\lambda)}{2q \cdot n} &\times \left( \nabla_\mu n_\beta - n_\beta q^\mu \frac{\nabla_\mu n_\nu}{q \cdot n} \right) + \hbar c \frac{\epsilon^{\mu \nu \alpha \beta} q_\alpha}{2q \cdot n} q^\nu R^\lambda_{\mu \nu} \partial_\lambda \right\} f_L = (1 - f_L) \Gamma_{(\alpha}^{\gamma} - f_L \Gamma_{(\alpha}^{\gamma}),
\end{aligned}
\] (25)

under the on-shell condition \( q^2 = 0 \), where

\[
\Gamma_{(\alpha}^{\gamma} = q \cdot \Sigma^\gamma - \hbar c \frac{\epsilon^{\mu \nu \alpha \beta} q_\alpha}{2q \cdot n} q^\nu \frac{\nabla_\mu n_\nu}{q \cdot n} \Sigma_{\beta}^\nu + n_\beta D_{\nu} \Sigma^\beta_{\nu}.
\] (26)

are related to the emission and absorption rates via \( R_{\text{emis}} = c \Gamma^{\gamma} / \gamma^2 \) and \( R_{\text{abs}} = c \Gamma^{\gamma} / \gamma^2 \), respectively. Here we also dropped the nonlinear terms in self-energies due to the weakness of the weak interaction.

Note that all the quantum corrections in Equations (25) and (26) involve the Levi–Civita tensor \( \epsilon^{\mu \nu \alpha \beta} \), and hence they explicitly break the spherical symmetry and axisymmetry of the system. This consequence may simply be understood from the fact that the chirality related to the spin degree of freedom can only be defined in genuine 3D.

3.2. Conservative Equations and Energy–Momentum Transfer

After solving the CKT and obtaining \( f_L \), we have to insert \( f_L \) into the Wigner function in Equation (18) to obtain physical observables. For example, based on the definition in field theory, the particle-number current and symmetric energy–momentum tensor for left-handed fermions can be derived from the lesser propagators via

\[
J^\mu = 2 \int_q \mathcal{L}^{\mu, \nu}, \quad T^\mu = \int_q (\mathcal{L}^{\mu, \nu} q^\nu + \mathcal{L}^{\nu, \mu} q^\nu),
\] (27)

where we introduced the notation

\[
\int_q \equiv \int_q \frac{d^4 q}{(2\pi)^3} \sqrt{-g}.
\] (28)

From Equations (27) and (18), in the absence of \( \hbar \) corrections, one easily recognizes that \( J^\mu \) and \( T^{\mu \nu} \) simply reduce to taking the first and second moments of \( f_L \), respectively.

In fact, the quantum corrections further affect the conservative equations responsible for the energy–momentum transfer between neutrinos and matter. As shown in Appendix C, the conservative equation for the radiation energy–momentum tensor of neutrinos is given by

\[
\nabla_n T^{\mu \nu}_{\text{rad}} = 2 \int_q \left( q^\mu \mathcal{C}[f_L] + \frac{\pi}{2} \hbar c q^2 \epsilon^{\mu \nu \alpha \beta} q_\alpha D_{\beta} \mathcal{C}[f_L] \right).
\] (29)
where

\[
\tilde{\mathcal{C}}[f_q] \equiv \Sigma^\Sigma \cdot \mathcal{L}^\Sigma - \Sigma^\Sigma \cdot \mathcal{L}^\Sigma = 2\pi\delta(q^2) \\
\times \left[ q \cdot C - \hbar c \frac{\epsilon^{\mu\nu\rho\sigma} q_\mu n_\nu}{2q \cdot n} (\Sigma^\Sigma_{\mu\nu} D_{\rho\sigma}(1 - f_L) - \Sigma^\Sigma_{\rho\sigma} D_{\mu\nu} f_L(1 - f_L)) \right].
\]

(30)

Based on the energy–momentum conservation,

\[
\nabla_{\mu} T^\mu_{\text{rad}} + \nabla_{\mu} T^\mu_{\text{mat}} = 0,
\]

(31)

where \( T^\mu_{\text{mat}} \) denotes the energy–momentum tensor of matter, the right-handed side of Equation (29) will accordingly modify the transport of matter. Note that \( \Sigma^\Sigma \) can also incorporate quantum corrections.

### 4. Chiral Radiation Transport Equation for Neutrinos

#### 4.1. Transfer Equation in the Inertial Frame

In this section, we will further write down an explicit expression of the transfer equation for left-handed neutrinos including quantum corrections in the inertial frame with the spacetime metric in Equation (7). Thanks to the unchanging lightlike dispersion relation, we can still apply the same momentum parameterization as Equation (5). Nevertheless, we have to choose a proper “frame vector \( n^\mu \)” for computational convenience, yet the physics should be independent of the choice. In the inertial frame, we may take \( n^\mu = \xi^\mu \equiv (1/c, 0, 0, 0) \) such that \( q \cdot n = E_i \). One may in general choose an arbitrary timelike frame vector for \( n^\mu \), which, however, may cause unnecessary complication in calculations when solving for \( f_q^{(n)} \). Recall that one can always relate the distribution functions in different frames through the “modified frame transformation” in Equation (20). Moreover, since the collision terms involve incoming and outgoing particles with different momenta, we will hereafter use extra subscripts to characterize the momentum dependence of variables and operators (e.g., \( f^p = f^{(p)}(p, x) \) and \( D^p_{\mu\nu} = \nabla_{\mu} - \Gamma^\lambda_{\mu\nu} \partial^{\lambda} \rho \)).

Taking \( n^\mu = \xi^\mu \) in the inertial frame, we have \( \nabla_{\mu} n_\mu = 0, D_{\mu\nu} S^{\mu\nu}_{\text{rad}} = 0, \) and \( R^{\mu\nu}_{\text{grav}} = 0, \) from which we find that the free-streaming part of the transfer equation remains unchanged from Equations (10) and (11). Moreover, the collision terms are simplified. The transfer equation in the conservative form with collisions is thus given by

\[
\frac{1}{c} \left[ \partial_t + \frac{\mu}{r^2} \partial_r r^2 + \sqrt{1 - \mu^2} \right] \frac{f_q^{(\xi)}}{E_i} \equiv \nabla_{\mu} T^\mu_{\text{rad}} + \nabla_{\mu} T^\mu_{\text{mat}} + \frac{1}{c} \partial_t \left( f_q^{(\xi)}(1 - \mu^2) \right)
\]

\[
\times \left( \cos \frac{\Theta}{\sin \theta} \partial_\theta \sin \theta + \sin \frac{\Theta}{\sin \theta} \partial_\phi \right) + \frac{1}{c} \partial_\mu \left( f_q^{(\xi)}(1 - \mu^2) \right)
\]

\[
- \sqrt{1 - \mu^2} \cot \theta \partial_\phi \sin \frac{\Theta}{\sin \theta} f_q^{(\xi)}(1 - f_q^{(\xi)}(1 - \mu^2))
\]

\[
= \frac{1}{c} \left[ (1 - f_q^{(\xi)}(1 - f_q^{(\xi)}(1 - \mu^2)) \Gamma^{\xi}_{\text{rad}}(1 - f_q^{(\xi)}(1 - \mu^2)) \right] E_i f_q^{(\xi)}.
\]

(32)

where, according to Equation (26),

\[
\Gamma^{\xi}_{\text{rad}}(1 - f_q^{(\xi)}(1 - \mu^2)) \equiv \frac{\epsilon^{\mu\nu\rho\sigma} q_\mu n_\nu}{2q \cdot n} (\Sigma^\Sigma_{\mu\nu} D_{\rho\sigma}(1 - f_L) - \Sigma^\Sigma_{\rho\sigma} D_{\mu\nu} f_L(1 - f_L)).
\]

(33)

and

\[
D^{(\xi)}_{q\lambda} = \nabla_{\mu} - (\Gamma^\lambda_{\mu\nu} + e^{\nu\sigma} \partial_\sigma e^{\lambda}_{\mu}) q^\nu q^\lambda
\]

\[
= \left\{ \nabla_{\mu} - (\Gamma^\lambda_{\mu\nu} - \frac{1}{\sqrt{1 - \mu^2}} \theta \partial_\theta \sin \frac{\Theta}{\sin \theta} (1 - \mu^2)) \right\} \frac{f_q^{(\xi)}}{E_i} \nabla_{\theta} f_L - \mu_1 (1 - \mu^2) \sin \phi f_L.
\]

(34)

Note again that the horizontal lifts here entail the proper coordinate transformation for Christoffel symbols to handle the mixing of spatial and momentum derivatives when working in the coordinates with on-shell momenta. Since \( \Sigma^\Sigma \) further contain quantum corrections, we have not written down the explicit expression for collisions above. The collision terms depend on microscopic theories or models characterizing the interactions between neutrinos and matter, which will be further discussed in the next subsection. For convenience, we will hereafter suppress the superscript “(\xi)” for \( f_q^{(\xi)} \).

Moreover, the side-jump term also affects the neutrino radiation through the radiative energy–momentum tensor via the Wigner functions in Equation (18). Unlike the classical case, in particular, the energy–momentum tensor no longer corresponds to the second moment of \( f_q \) in the presence of quantum corrections as mentioned in Section 3. After solving \( f_L \) from the transfer equations, we have to further employ the Wigner functions to evaluate physical observables for neutrino radiation. In the case of neutrinos, we can turn off the background electromagnetic fields. Similarly to the transfer equation, the proper coordinate transformation for Christoffel symbols has to be taken care of for \( D_{\xi} \). Following Section 2, we find

\[
\mathcal{L}_{\xi}^{\mu} = 2\pi\delta(q^2) e^{\nu}_{\mu} [q^2 D_{\xi q} - \hbar c S^{\mu\nu}_{\text{rad}}]
\]

\[
\times \left( (\partial_\beta - \Gamma^\lambda_{\beta\gamma} q^\gamma q^\delta ) f_{\xi q} + \hbar c S^{\mu\nu}_{\text{grav}} (\partial_\gamma f_{\xi q} - \hbar c S^{\mu\nu}_{\text{grav}} f_{\xi q}) \right)
\]

\[
= 2\pi\delta(q^2) [q^2 f_{\xi q} - \hbar c S^{\mu\nu}_{\text{grav}} (\partial_\gamma f_{\xi q} - \hbar c S^{\mu\nu}_{\text{grav}} f_{\xi q})]
\]

\[
= (1 - \mu^2) (\cos \phi f_{\xi q} - \hbar c S^{\mu\nu}_{\text{grav}} f_{\xi q}) + \hbar c S^{\mu\nu}_{\text{grav}} (\partial_\gamma f_{\xi q} - \hbar c S^{\mu\nu}_{\text{grav}} f_{\xi q}).
\]

(35)

Defining \( \mathcal{L}_{\xi}^{\mu} = 2\pi\delta(q^2) \tilde{\mathcal{L}}^{\mu}_{\xi q} \) and carrying out a straightforward computation, we obtain

\[
\tilde{\mathcal{L}}^{\mu}_{\xi q} f_{\xi q} = \frac{E_i}{c} f_{\xi q},
\]

\[
\tilde{\mathcal{L}}^{\mu}_{\xi q} f_{\xi q} = \mu_1 E_i f_{\xi q}
\]

\[
- \frac{\hbar c}{2r} \left[ \sqrt{1 - \mu^2} \left( \sin \frac{\Theta}{\sin \theta} (\partial_\theta f_{\xi q} - C_{\xi q}) \right) \right]
\]

\[
- \cos \frac{\Theta}{\sin \theta} (\partial_\phi f_{\xi q} - C_{\xi q}) \right) + \frac{\mu_1}{c} \partial_\phi f_{\xi q} \right]
\]

\[
= \frac{\sqrt{1 - \mu^2}}{E_i \cos \frac{\Theta}{\sin \theta}} f_{\xi q}.
\]

(36)

(37)

(38)

(39)

(40)

(41)
\[ + \frac{\hbar c}{2r^2} \left[ 1 - \mu_i^2 r \sin \phi_i \partial_{\phi_i} f_{Lq} - C_{q\phi} \right] \]

\[ - \frac{\mu_i}{\sin \theta} \left( \partial_{\phi_i} f_{Lq} - C_{q\phi} \right) - \mu_i \sqrt{1 - \mu_i^2} \sin \phi_i \partial_{\phi_i} f_{Lq} \]

\[ + \frac{\mu_i}{\sqrt{1 - \mu_i^2}} \sin \phi_i \partial_{\phi_i} f_{Lq} \]

\[ \mathcal{L}_{q}^{\phi_{Lq}} = \frac{1}{r \sin \theta} E_{i} \sin \phi_{i} f_{Lq} - \frac{\hbar c}{2r^2 \sin \theta} \]

\[ \times \left[ 1 - \mu_i \right] \cos \phi_{i} \partial_{\phi_i} f_{Lq} - \mu_i \left( \partial_{\phi_i} f_{Lq} - C_{q\phi} \right) \]

\[ - \mu_i \sqrt{1 - \mu_i^2} \left( \sin \phi_{i} \partial_{\phi_i} f_{Lq} \right) \]

(36)

One may implement the expressions for \( \mathcal{L}_{q}^{\phi_{Lq}} \) above to evaluate each component of \( T^\mu_{\nu} \) from Equation (27) with the input of \( f_{L} \).

Note that we only have to input the leading-order \( \mathcal{O}(\hbar^0) \) contributions of \( \mathcal{O}_{q}^{\phi_{Lq}} \) into Equation (36).

4.2. Details of Collisions

As mentioned previously, the collision terms depend on the underlying microscopic theory for the neutrino–matter interaction. To write down a more explicit expression of collisions and make a comparison with the classical collision terms widely applied in radiation hydrodynamics in core-collapse supernovae such as those in Bruenn (1985) and Reddy et al. (1998), we consider the weak interaction between neutrinos and nucleons as the matter sector. For simplicity and concreteness, we focus on the neutrino absorption on nucleons \((\nu_{L} + n \rightarrow e_{L} + p)\) and elastic neutrino–nucleon scattering \((\nu_{L} + N \rightarrow \nu_{L} + N, \text{where } N = n, p)\) based on the four-Fermi theory of the weak interaction. In addition to taking neutrinos as chiral fermions, we will assume that electrons are ultrarelativistic and treat them as approximate chiral fermions, which also incorporate quantum corrections due to the chirality. In the following, we will write down generic forms of lesser/greater self-energies and emission/absorption rates for each process above.

To simplify the expressions mostly concerned with the quantum field theory calculations, we will take \( c = 1 \) below and in Appendices D–F, except for the final results in Equations (63)–(65).

4.2.1. Neutrino Absorption on Nucleons

As the first example, we consider the neutrino absorption on nucleons

\[ \nu_{L}^{\mu} (q) + n (k) = e_{L}^{\nu} (q^\prime) + p (k^\prime), \]

where \( q^{\mu} \) and \( q^{\prime \nu} \) \((k^{\mu} \text{ and } k^{\prime \nu})\) correspond to the four-momenta of incoming or outgoing leptons (nucleons), respectively. This process is described by the four-Fermi theory of the weak interaction, expressed as the current–current interactions,

\[ \mathcal{L}_{\text{em}}^{\nu_{L}^{\mu} (q) + n (k) = e_{L}^{\nu} (q^\prime) + p (k^\prime)} = \frac{G_{F}}{\sqrt{2}} (j_{\nu_{L}}^{\mu})_{\nu} (j_{\nu_{L}}^{\mu})_{\nu} + \text{h.c.}, \]

where \( G_{F} \) is the Fermi constant and \( (j_{\nu_{L}}^{\mu})_{\nu} = \bar{\nu}_{\nu} \gamma_{\mu} (1 - \gamma^{5}) \nu_{\nu} \) and \( (j_{\nu_{L}}^{\mu})_{\nu} = \bar{\nu}_{\nu} \gamma_{\mu} (g_{\nu} - g_{\nu} \gamma^{5}) \nu_{\nu} \) are the lepton and nucleon charged currents, respectively, with \( g_{\nu} = 1 \) and \( g_{A} \approx 1.27 \); for recent calculations of \( g_{A} \) in lattice QCD, see Chang et al. (2018) and references therein.

By a standard calculation, one finds the self-energies for this neutrino absorption process (see Appendix D for the details),

\[ \Sigma_{q \rightarrow q}^{(a b)} = \int_{p} \Pi_{p, \mu}^{(a b)} \mathcal{L}_{q}^{(a b)} f_{L q}^{(a b)}, \]

(39)

where

\[ \Pi_{p, \mu}^{(a b)} = 8G_{F}^{2} \int_{k} \left( g_{\mu}^{2} k_{q} + g_{q}^{2} k_{q} \right) \frac{1}{2} \delta (k^{2} - M_{h}^{2}) \]

\[ \times \delta (k^{2} - M_{h}^{2}) \delta \left( f_{L q}^{(a b)} \right) \left| \frac{1}{4} \right| k_{q} = k + k_{p}, \]

with \( p^{\mu} = (k' - k)^{\mu} = (q - q')^{\mu} \) being the four-momentum transfer in scattering, \( M_{h} \) the masses of neutrons/protons, \( f^{(N)} < \text{ and } f^{(N)} (N = n, p) \) the distribution functions of incoming and outgoing nucleons, respectively, and \( g_{\pm} \equiv g_{\nu} \pm g_{A} \). In general, the Wigner functions for left-handed electrons here also incorporate quantum corrections, which have to be solved from another quantum transport equation for chiral fermions. More precisely, we shall take

\[ \mathcal{L}_{q}^{(e) \mu} = 2\pi \left[ \delta (q^{2}) (q^{\mu} - \hbar \delta \left| \frac{q_{\nu} \delta \left( q^{\nu} \right) f_{L q}^{(e)}}{2q' \cdot \xi_{p}} \right|) f_{L q}^{(e)}, \]

(41)

where we retain the electromagnetic fields coupled to electrons and \( f_{L q}^{(e)} \) denote the distribution functions of left-handed incoming/outgoing electrons with \( n^{0} = \xi^{0} \). Generically, one has to solve for \( f_{L q}^{(e)} \) from a coupled CKT governing the dynamics of electrons. Then, we have

\[ \Sigma_{q \rightarrow q}^{(a b)} = \int_{p} \Pi_{p, \mu}^{(a b)} \left| \delta (q^{2}) \left( q^{\mu} - \hbar \frac{\epsilon^{(e) 3} \delta \left( \frac{q_{\nu} \delta \left( q^{\nu} \right) f_{L q}^{(e)}}{2q' \cdot \xi_{p}} \right)}{2q' \cdot \xi_{p}} \right| f_{L q}^{(e)}, \]

(42)

and accordingly,

\[ \Gamma_{(e) q}^{(a b)} = \int_{p} q^{\mu} \Pi_{p, \mu}^{(a b)} \left| \delta (q^{2}) \left( q^{\mu} - \hbar \frac{\epsilon^{(e) 3} \delta \left( \frac{q_{\nu} \delta \left( q^{\nu} \right) f_{L q}^{(e)}}{2q' \cdot \xi_{p}} \right)}{2q' \cdot \xi_{p}} \right| f_{L q}^{(e)}, \]

(43)

Regarding the effects of electromagnetic fields, one can further focus on an external magnetic field such that \( F^{\mu \nu} = B^{\nu} \xi^{\nu} - B^{\nu} \xi^{\mu} \) and \( F^{\mu \nu} = -\epsilon^{(e) 3} \delta \left( \frac{q_{\nu} \delta \left( q^{\nu} \right) f_{L q}^{(e)}}{2q' \cdot \xi_{p}} \right) \). In this case,
Equation (43) becomes
\[
\Gamma^{(ab)\xi}_{(\xi)q} = \int_p q^\mu \Pi^{(mp)\xi}_{p,\mu} \left\{ \delta(q^2) \left[ \left( q^\nu - \frac{\epsilon}{2q^2} \xi_\xi \right) D_{q^\nu} \right] \right. \\
+ \frac{\hbar}{2q^2} \left( B \cdot q^\nu \partial_{q^\nu} - B_\xi q^\nu \partial_{q^\nu} \right) f^{(\xi)\xi}_{L q^\nu} \\
- \frac{\hbar}{2q^2} \left( B \cdot q^\nu \xi_\xi - \xi_\xi B \cdot q^\nu \right) \delta(q^2) f^{(\xi)\xi}_{L q^\nu} \\
+ \frac{\epsilon}{2q^2} \xi_\xi D_{q^\nu} \int_p \Pi^{(mp)\xi}_{p,\mu} \delta(q^2) f^{(\xi)\xi}_{L q^\nu} \right\}_{q = q - p}, \tag{44}
\]
where \( V^\mu = (q^\mu - \xi^\mu \xi^\nu) V^\nu \) represents the component perpendicular to the frame vector \( \xi \) for an arbitrary vector \( V \). Despite the complexity of Equation (44), we can write down the following structure based on the symmetry of the system:
\[
\Gamma^{(ab)\xi}_{(\xi)q} = \Gamma^{(0)\xi}_{q} + \epsilon_{\xi} \Pi^{(mp)\xi}_{q} \delta(q^2) R_{q}^{(\xi)} \\
+ \left( \Gamma^{(1)\xi}_{q} \xi_\mu U_{\mu} + U_{\mu} \xi_\mu \Gamma^{(2)\xi}_{q} \right) \\
+ \hbar \left( \Gamma^{(3)\xi}_{q} \xi_\mu B + \Gamma^{(4)\xi}_{q} U - B \right), \tag{45}
\]
where \( \Gamma^{(0)\xi}_{q} \) is the classical collision term, \( \Gamma^{(1)\xi}_{q} \) is the quantum correction related to the chirality of fermions, and \( U_{\mu}(x) \) is a vector characterizing local properties of the matter. The detailed structure of \( U^\mu \) and the coefficients \( \Gamma^{(k)\xi}_{q} \) depend on the nucleon and electron distribution functions. For instance, assuming that the matter is in thermal equilibrium, \( U_\nu \) corresponds to the local fluid four-velocity and \( \partial_{\nu} \Gamma^{(2)\xi}_{q} \) contains the spatial derivatives of local temperature or chemical potentials.

4.2.2. Elastic Neutrino–Nucleon Scattering

We next consider the elastic neutrino–nucleon scattering
\[
\nu^L_1(q) + N(k) \rightarrow \nu^L_1(q') + N(k'). \tag{46}
\]
This is described by the current–current interactions of the form
\[
\mathcal{L}^{\nu_1^L} = \frac{G_F}{\sqrt{2}} (j_\nu)_1 (j_N)^\nu + \text{h.c.}, \tag{47}
\]
where \( (j_\nu)_1 = \frac{\bar{\nu}_\nu \gamma_\nu \bar{N}_{\gamma} (1 - \gamma^5) \psi_\nu \gamma^\nu (c_\nu - c_\nu \gamma^5) \gamma_\nu \) are the lepton and nucleon neutral currents, respectively. Here \( c_\nu = -1 \) and \( c_\nu = -g_\nu \) for \( N = n \), and \( c_\nu = -g_\nu \) for \( N = p \), where \( g_\nu \) is the Weinberg angle (see, e.g., Reddy et al. 1998).

The self-energies for this process are given by (see Appendix D)
\[
\Sigma^{(\nu_1^L)}_{q} = \int_p \Pi^{(NN)\lambda}_{p,\mu} \delta(q^2) f^{(\nu_1^L)\lambda}_{L q^\mu} \left[ q^\lambda - \frac{\epsilon_{\lambda\mu\beta\gamma}}{2q^2} \xi_\xi D_{q^\mu} \right] f^{(\nu_1^L)\lambda}_{L q^\mu} \right\}_{q = q - p}, \tag{48}
\]
where
\[
\Pi^{(NN)\lambda}_{p,\mu} = 8G_F^2 \int_k \left( c_k k_\nu k_\nu + c_k' k_\nu k_\nu - c_\nu c_\nu M_2^2 \right) \\
\times (2\pi)^2 \delta(k^2 - M_2^2) \delta(k^2 - M_3^2) f^{(\nu_1^L)\lambda}_{L q^\mu} \left| k' = p + k \right. \tag{49}
\]
with \( c_\pm = (c_\nu \pm c_\nu)/2 \), and
\[
L^{(\nu_1^L)\lambda}_{q} = 2\pi \left[ q^2 \delta(q^2) (2\pi) \delta(q^2) \right] f^{(\nu_1^L)\lambda}_{L q^\mu} \right\}_{q = q - p}, \tag{50}
\]
Consequently, we find
\[
\Gamma^{(\nu_1^L)}_{(\xi)q} = \int_p q^\mu \Pi^{(NN)\lambda}_{p,\mu} \left[ \delta(q^2) \\
\times \left( q^\lambda - \frac{\epsilon_{\lambda\mu\beta\gamma}}{2q^2} \xi_\xi D_{q^\mu} \right) f^{(\nu_1^L)\lambda}_{L q^\mu} \right] \right\}_{q = q - p}, \tag{51}
\]
which is similar to the form in Equation (44) without background fields. Nevertheless, because the neutrino distribution functions are involved in the integrand, one has to make further approximations to simplify the nonlinear terms in neutrino distribution functions. For example, the so-called isoenergetic approximation by assuming zero-energy transfer may be used (Bruenn 1985). In this approximation, one finds \( \Pi^{(NN)\nu_1^L} = \Pi^{(NN)\nu_1^L} = \Pi^{(NN)\nu_1^L} \), as can be easily shown in thermal equilibrium with detailed balance. Therefore, one can linearize the collision term in the kinetic theory with the isoenergetic approximation as
\[
1 - f^{(\nu_1^L)}_{L q^{\mu}} \Gamma^{(\nu_1^L)}_{(\xi)q} \rightarrow \int_p q^\mu \Pi^{(NN)\lambda}_{p,\mu} \left[ \delta(q^2) \\
\times \left( q^\lambda - \frac{\epsilon_{\lambda\mu\beta\gamma}}{2q^2} \xi_\xi D_{q^\mu} \right) f^{(\nu_1^L)\lambda}_{L q^\mu} \right] \right\}_{q = q - p}. \tag{52}
\]

When neglecting the quantum corrections involving the Levi–Civita tensor \( \epsilon_{\mu\nu\rho\sigma} \) or the magnetic field \( B^\nu \), our collision terms for both processes reduce to those presented in Reddy et al. (1998).

4.3. Collisions with Matter in Equilibrium

In the case of core-collapse supernovae, we can assume that the matter sector consisting of nucleons and electrons is in local thermal equilibrium and can be described by hydrodynamics, since the typical length scale of interest is much larger than their mean free paths. This allows us to derive an analytic form of the collision term by employing proper approximations, e.g., for the neutrino absorption process. The computations of the
quantum corrections in other processes will be reported elsewhere.

First, we assume that the fluid velocity is sufficiently small such that \( u^\mu = \gamma (1, v) \approx (1, 0) = \xi^\mu \), and hence \( F^{\mu\nu} \approx B^\mu u^\nu - B^\nu u^\mu \). Second, as we are here interested in the quantum corrections due to the vorticity \( \omega^{\mu} \), we will ignore the viscous corrections and the gradients of the temperature and chemical potentials. Under such assumptions, we can ignore the terms \( \nabla_{\mu} u_{\nu} \) and \( u^{\mu} \nabla_{\nu} u^{\mu} \), and as a result, we have \( \nabla_{\mu} u_{\nu} \approx - \epsilon_{\mu\nu \rho} \omega^{\rho} u^{\nu} \). According to Hidaka et al. (2018), the lesser/greater propagators for left-handed thermal electrons can then be written as

\[
\mathcal{L}^{(e)<\mu}_{\nu} = 2\pi \left[ \begin{array}{c} \delta(q^2) \left( q\mu f_{0,q} \right) \\
 - \frac{\hbar}{2}(\omega q \cdot u - u q \cdot \omega) f_{0,q} \left( 1 - f_{0,q} \right) \\
 - \hbar(B q \cdot u - u q \cdot B) \delta(q^2)f_{0,q} \end{array} \right],
\]

\[
\mathcal{L}^{(e)>\mu}_{\nu} = 2\pi \left[ \begin{array}{c} \delta(q^2) \left( q\mu \right) \left( 1 - f_{0,q} \right) \\
 + \frac{\hbar}{2}(\omega q \cdot u - u q \cdot \omega) f_{0,q} \left( 1 - f_{0,q} \right) \\
 - \hbar(B q \cdot u - u q \cdot B) \delta(q^2) \left( 1 - f_{0,q} \right) \end{array} \right],
\]

where

\[
f_{0,q} = \frac{1}{\exp\left(\beta(qa_{\mu} - \mu_i)\right)} - 1, \quad (i = n, p, e) \]

represent the Fermi–Dirac distribution functions with \( \beta = 1/(k_b T) \), with \( T, \mu_i \), and \( k_b \) being temperature, chemical potentials for \( i = n, p, e \), and Boltzmann constant, respectively. Here and below, \( \mathcal{O} \) stands for a quantity \( \mathcal{O} \) in local thermal equilibrium.

Now the self-energies in Equation (39) become

\[
\Sigma^{(0)}_{\mu} = \Sigma^{(0,0)}_{\mu} + \hbar \Sigma^{(0,1)}_{\mu} + \hbar \Sigma^{(B)}_{\mu},
\]

where

\[
\Sigma^{(0,0)}_{\mu} = 2\pi G F \int \frac{d^4 p}{(2\pi)^{4}} \frac{(g^2 \xi_k \xi_{k'} + g^2 \xi_k \xi_{k'} - g \xi_k M_n M_p \eta_{nm}}{1 - e^{\beta q u q}} \left[ \begin{array}{c} \delta(q^2) \left( q\mu \right) \\
 - \hbar(B q \cdot u - u q \cdot B) \delta(q^2) \left( 1 - f_{0,q} \right) \end{array} \right],
\]

\[
\Sigma^{(0,1)}_{\mu} = \frac{\hbar}{2\pi} \int \frac{d^4 p}{(2\pi)^{4}} \frac{g^2 \xi_k \eta_{nm}}{1 - e^{\beta q u q}} \left[ \begin{array}{c} \delta(q^2) \left( q\mu \right) \\
 - \hbar(B q \cdot u - u q \cdot B) \delta(q^2) \left( 1 - f_{0,q} \right) \end{array} \right],
\]

and

\[
\Sigma^{(B)}_{\mu} = 2\pi G F \int \frac{d^4 p}{(2\pi)^{4}} \frac{(g^2 \xi_k \xi_{k'} + g^2 \xi_k \xi_{k'} - g \xi_k M_n M_p \eta_{nm}}{1 - e^{\beta q u q}} \left[ \begin{array}{c} \delta(q^2) \left( q\mu \right) \\
 - \hbar(B q \cdot u - u q \cdot B) \delta(q^2) \left( 1 - f_{0,q} \right) \end{array} \right],
\]

In the following, we will set \( M_n \approx M_p \approx M \) and adopt the nonrelativistic approximation for nucleons. We will also use the “quasi-isothermal” approach that allows for the energy transfer up to \( O(1/M) \) (see Appendix E). One then finds

\[
\Sigma^{(0)}_{\mu} + \hbar \Sigma^{(B)}_{\mu} \approx \frac{1}{\pi} \left( g^2 \xi_k \eta_{nm} \right) \frac{G F}{M} \left[ \begin{array}{c} q\mu \\
 - \hbar B q u q \end{array} \right] \left( 1 - f_{0,q} \right),
\]

\[
\Sigma^{(0)}_{\mu} + \hbar \Sigma^{(B)}_{\mu} \approx \frac{1}{\pi} \left( g^2 \xi_k \eta_{nm} \right) \frac{G F}{M} \left[ \begin{array}{c} q\mu \\
 - \hbar B q u q \end{array} \right] \left( 1 - f_{0,q} \right),
\]

and

\[
\hbar(q \cdot \Sigma^{(0)}_{\mu} + S_{\mu}^{(0)} D_{\mu} \Sigma^{(0)}_{\mu} \approx \frac{\hbar}{2\pi} \left( g^2 \xi_k \eta_{nm} \right) \frac{G F}{M} \left[ \begin{array}{c} q\mu \\
 - \hbar B q u q \end{array} \right] \left( 1 - f_{0,q} \right),
\]

\[
\hbar(q \cdot \Sigma^{(0)}_{\mu} + S_{\mu}^{(0)} D_{\mu} \Sigma^{(0)}_{\mu} \approx \frac{\hbar}{2\pi} \left( g^2 \xi_k \eta_{nm} \right) \frac{G F}{M} \left[ \begin{array}{c} q\mu \\
 - \hbar B q u q \end{array} \right] \left( 1 - f_{0,q} \right),
\]

\[
\hbar(q \cdot \Sigma^{(0)}_{\mu} + S_{\mu}^{(0)} D_{\mu} \Sigma^{(0)}_{\mu} \approx \frac{\hbar}{2\pi} \left( g^2 \xi_k \eta_{nm} \right) \frac{G F}{M} \left[ \begin{array}{c} q\mu \\
 - \hbar B q u q \end{array} \right] \left( 1 - f_{0,q} \right),
\]

where \( \eta_{nm} = \frac{dM^2}{c^2} \) and \( n_{n/p} = \int \frac{d^3 k}{(2\pi)^3} \delta(n/p) \) are neutron/proton densities, and we decomposed the magnetic field into the longitudinal and transverse components with respect to the
momentum $q^\mu$ as

$$B^\mu = q^\mu |q| B_L + B_T^\mu, \quad q_\perp \cdot B_T = 0. \quad (62)$$

Note that while the quantum corrections due to magnetic fields are suppressed in the $M \to \infty$ limit, those corrections due to the fluid vorticity persist even in this limit. Assembling all pieces together, taking $|q| \approx E_i$ and restoring $\hbar$ and $c$, Equation (45) in thermal equilibrium reduces to (now $U^\mu = u^\mu$)

$$\tilde{\Gamma}^{\mu\nu}_{(\omega)} \approx \tilde{\Gamma}^{\mu\nu}_{(\omega)} + \hbar \tilde{\Gamma}^{(\omega)}_{\mu\nu}(q \cdot \omega) + \hbar \tilde{\Gamma}^{(R)}_{\mu\nu}(q \cdot B), \quad (63)$$

where

$$\tilde{\Gamma}^{(0)}_{\mu\nu} \approx \frac{1}{\pi \hbar^4 k^4} (g_{11}^2 + 3 g_{11}^2) G_T^2 E_i (1 - f_{0,q}^{(e)}) \times \left(1 - \frac{3 E_i}{M c^2} \right) \left(1 - e^{i \frac{\mu}{\hbar} \cdot \nu - \mu \cdot \rho} \right),$$

$$\tilde{\Gamma}^{(0)}_{\mu\nu} \approx \frac{1}{\pi \hbar^4 k^4} (g_{11}^2 + 3 g_{11}^2) G_T^2 E_i f_{0,q}^{(e)} \times \left(1 - \frac{3 E_i}{M c^2} \right) \left(1 - e^{i \frac{\mu}{\hbar} \cdot \nu - \mu \cdot \rho} \right),$$

and

$$\tilde{\Gamma}^{(\omega)}_{\mu\nu} \approx \frac{1}{2 \pi \hbar^4 k^4} (g_{11}^2 + 3 g_{11}^2) G_T^2 E_i (1 - f_{0,q}^{(e)}) \times \left(2 + \beta E_i f_{0,q}^{(e)} \right) \frac{n_p - n_n}{1 - e^{i \frac{\mu}{\hbar} \cdot \nu - \mu \cdot \rho}},$$

$$\tilde{\Gamma}^{(\omega)}_{\mu\nu} \approx \frac{1}{2 \pi \hbar^4 k^4} (g_{11}^2 + 3 g_{11}^2) G_T^2 E_i f_{0,q}^{(e)} \times \left(2 - \beta E_i (1 - f_{0,q}^{(e)}) \right) \frac{n_p - n_n}{1 - e^{i \frac{\mu}{\hbar} \cdot \nu - \mu \cdot \rho}}. \quad (64)$$

Consequently, the emission and absorption rates are obtained as $R_{\text{em}} = c \tilde{\Gamma}^{\mu\nu}_{\mu\nu} E_i$ and $R_{\text{abs}} = c \tilde{\Gamma}^{\mu\nu}_{\mu\nu} E_i$, respectively.

Finally, we discuss the physical consequences of these quantum corrections. First of all, both the $q \cdot \omega$ and $q \cdot B$ terms break the spherical symmetry and axisymmetry of the system, as we already argued in a generic frame. Note also that these terms break the parity symmetry, which is a feature specific to the parity-violating weak interaction. Moreover, an important feature of the $q \cdot \omega$ and $q \cdot B$ terms is that, for the neutrinos propagating collinear to the flow of matter, they can give leading-order contributions to the so-called kinetic helicity $v \cdot \omega$ and cross helicity $v \cdot B$ of the matter, respectively, where $\omega \equiv \nabla \times v$. The mechanism that chiral effects of neutrinos, combined with the neutrino–matter interaction, can generate the kinetic helicity and cross helicity of the matter was previously shown in Yamamoto (2016a) in the hydrodynamic regime of neutrinos. The new collision terms above provide its generalization to the case away from equilibrium, where hydrodynamics for neutrinos is not necessarily applicable. The presence of the kinetic helicity of the matter further induces magnetic helicity by the helical plasma instability (Yamamoto 2016a), and as a result, it gives the tendency toward the inverse energy cascade (Masada et al. 2018), which would be favorable for the supernova explosion.

In the presence of a background magnetic field $B_{\text{ex}}$ and/or a global rotation of the system characterized by the angular velocity $\Omega$, we have the collision terms of the form $q \cdot B_{\text{ex}}$ and/or $q \cdot \Omega$. These collision terms lead to the asymmetric neutrino emission with respect to the directions of $B_{\text{ex}}$ and $\Omega$, respectively, which may contribute to the pulsar kick.

5. Summary and Outlook

In this work, we have constructed the chiral radiation transport equation for left-handed neutrinos with the quantum corrections due to their chirality, mainly in the inertial frame. We have also shown the expression of the radiative energy–momentum tensor with quantum corrections via the Wigner functions. In particular, we derive the analytic forms of the emission and absorption rates including the quantum corrections for the neutrino absorption process. The formalism of neutrino chiral radiation hydrodynamics established in our work should be applied to perform numerical simulations for core-collapse supernova explosions and neutron star formation in future.

In principle, one can also develop the same formalism in the comoving frame, while a different frame vector may be chosen for computational convenience. However, even for the ordinary 3D Boltzmann equation without quantum corrections, the free-streaming part involving fluid velocity in such a coordinate system is rather complicated (Morita & Kaneko 1986; Castor 2009). The generalization to further include quantum corrections seems to be technically difficult in that direction.

On the other hand, in order to further explore nonequilibrium chiral transport of electrons in supernovae, one may employ a kinetic theory with quantum corrections for massive fermions that has been developed more recently (Gao & Liang 2019; Hattori et al. 2019a; Weickgenannt et al. 2019; Liu et al. 2020; Yang et al. 2020). In particular, the quantum kinetic theory developed in Yang et al. (2020) systematically includes collisional effects.

Finally, although we have focused on the chiral radiation transfer of neutrinos in this paper, our formulation here may also be extended to the radiative transfer of photons that incorporates the effects of their circular polarizations; see Yamamoto (2017) and Huang & Sadofyev (2019) for the CKT of photons. Such a formulation would be applicable to a variety of astrophysical systems involving photon radiation.

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Appendix A
Useful Relations for Coordinate Transformation

For a function \( f(q) \) with the on-shell condition \( q^2 = 0 \), we have

\[
df = \left( \frac{\partial f}{\partial q^r} \right) dq^r + \left( \frac{\partial f}{\partial q^\theta} \right) dq^\theta + \left( \frac{\partial f}{\partial q^\phi} \right) dq^\phi.
\]  
(A1)

Comparing Equations (A1) and (A2), we derive

\[
\frac{\partial f}{\partial q^r} = \left( \frac{\partial f}{\partial E} + \frac{1 - \mu^2}{E} \frac{\partial f}{\partial \mu} \right) e^{\lambda},
\]

\[
\frac{\partial f}{\partial q^\theta} = R \sqrt{1 - \mu^2} \left( \frac{\partial f}{\partial E} - \frac{\mu}{E} \frac{\partial f}{\partial \mu} \right) \cos \tilde{\phi} - \frac{\sin \tilde{\phi}}{E(1 - \mu^2)} \frac{\partial f}{\partial \tilde{\phi}},
\]

\[
\frac{\partial f}{\partial q^\phi} = R \sin \theta \sqrt{1 - \mu^2} \left( \frac{\partial f}{\partial E} - \frac{\mu}{E} \frac{\partial f}{\partial \mu} \right) \sin \tilde{\phi} + \frac{\cos \tilde{\phi}}{E(1 - \mu^2)} \frac{\partial f}{\partial \tilde{\phi}}.
\]  
(A7)

Appendix B
Perturbative Solution of Wigner Functions

We shall show that Equation (18) is the solution of Equation (16) up to \( O(\hbar) \). Taking

\[
\mathcal{L}^\nu = 2 \pi [\delta(q^2)(q^\mu - \hbar c S_{(\mu\nu)}^\delta) - \hbar E F^\nu q_\delta(q^2)] f_L, \quad (B1)
\]

we find

\[
\hbar c (D_\nu L_\nu - D_\nu L_\mu) = 2 \pi \hbar c [\delta(q^2)(2F_{q\nu} + 2q_{(\mu}D_{\nu)}) + 4q^\nu F_{q\mu}q_{\delta}(q^2)]
\]

\[
\times f_L + O(\hbar^2). \quad (B2)
\]

On the other hand, we have

\[
2 \epsilon_{\mu\lambda
\\nu}q^\nu L^\mu = -4 \pi \hbar c \epsilon_{\mu\lambda
\\nu} q^\nu
\]

\[
\times \left[ \delta(q^2) \frac{\epsilon_{\kappa\lambda\mu\beta}}{2q \cdot n} q_{\kappa} q_{\mu} D_{\alpha} + \tilde{F}_{\lambda\mu} q_{\delta}(q^2) \right] f_L
\]

\[
= -4 \pi \hbar c \left[ \delta(q^2) \left( q_{(\mu} D_{\nu)} + \frac{q^2}{q \cdot n} n_{(\mu} D_{\nu)} + \frac{q_{(\mu} n_{\nu)}}{q \cdot n} \cdot D \right)
\]

\[
+ (2q^\nu F_{q\mu}q_{\delta} + q^2 F_{\mu\nu}) \delta'(q^2) \right] f_L
\]

\[
= -2 \pi \hbar c [\delta(q^2)(2F_{q\nu} + 2q_{(\mu}D_{\nu)})
\]

\[
+ 4q^\nu F_{q\mu}q_{\delta}(q^2)] f_L + O(\hbar^2), \quad (B3)
\]

where we used \( q^2 \delta(q^2) = 0 \) and \( q^2 \delta'(q^2) = -\delta(q^2) \) in the third line. It is thus clear that Equation (18) satisfies Equation (16).
Appendix C

Conservative Equations

We derive the conservative equation for the energy–momentum tensor in the curve spacetime. For simplicity, we consider the case without electromagnetic fields. We find

\[
2 \int q D \cdot \mathcal{L}^{\mu} = 2 \int_q (\partial_\mu + \Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\mu\nu}) \mathcal{L}^{\mu} = 2 \int_q \mathcal{L}^{\mu} - \int_q (\Gamma^\rho_{\mu\nu} \mathcal{L}^{\mu} - \Gamma^\lambda_{\mu\nu} \mathcal{L}^{\mu}) = 2 (\partial_\mu + \Gamma^\rho_{\mu\nu}) \int \frac{d^4q}{(2\pi)^4} \mathcal{L}^{\mu} = \nabla_\mu J^\mu,
\]

(C1)

where we used integration by parts and dropped the surface terms, and employed the relation

\[
\partial_\mu \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \partial_\mu g_{\alpha\beta} = \sqrt{-g} \Gamma^\rho_{\mu\nu}.
\]

(C2)

Note that \( \nabla_\mu \int d^4q \neq 0 \). Recall that the master equation with the collisions reads \( D \cdot \mathcal{L}^\mu = \tilde{C}[f] \), where \( \tilde{C}[f] \) is defined in Equation (30). We hence obtain the conservative equation for the particle-number current,

\[
\nabla \cdot J = 2 \int \tilde{C}[f].
\]

(C3)

For the energy–momentum tensor, on the other hand, we start with

\[
2 \int q^{\mu \nu} D_\mu \mathcal{L}^{<\nu \rho} = 2 \int q^{\mu \nu} (\partial_\mu \mathcal{L}^{<\nu \rho} + \Gamma^{\mu}_{\nu \rho} \mathcal{L}^{<\nu \rho} - \Gamma^{\lambda}_{\nu \rho} q^{\rho \lambda} \partial_\lambda \mathcal{L}^{<\nu \rho}) = 2 \int \frac{d^4q}{(2\pi)^4} \left[ (\partial_\mu q^{\nu \rho} + \Gamma^{\mu}_{\nu \rho} q^{\nu \rho}) \sqrt{-g} \mathcal{L}^{<\nu \rho} \right] + q^{\nu \rho} \Gamma^{\mu}_{\nu \rho} \sqrt{-g} \mathcal{L}^{<\nu \rho} = \nabla_\mu T^{\mu \nu} + \Gamma^{\mu}_{\nu \rho} T^{\mu \rho} + \Gamma^{\nu}_{\nu \rho} T^{\nu \rho} = \nabla_\mu T^{\mu \nu},
\]

(C4)

where we utilized a similar trick to that in the case for the particle-number current. Following the trick in Gorbar et al. (2017b) and Hidaka & Yang (2018), we find

\[
\nabla_\mu T^{\mu \nu} = \int \left( 2 q^{\nu \rho} D_\rho \mathcal{L}^\nu - q^{\nu \rho} \mathcal{L}^\nu - q^{\nu \rho} D_\rho \mathcal{L}^\nu \right) = \int \left( 2 q^{\nu \rho} D_\rho \mathcal{L}^\nu - \frac{1}{2} \epsilon^{\nu \rho \sigma \lambda} \epsilon_{\mu \lambda \sigma \rho} q^{\lambda} D_\mu \mathcal{L}^\nu \right).
\]

(C5)

By using the second line of Equation (B3) without electromagnetic fields, one obtains

\[
\epsilon_{\mu \lambda \sigma \rho} q^{\lambda} D_\mu \mathcal{L}^\nu = 2 \pi h c \delta(q^2) D_\nu q^{\mu} \cdot n \frac{q_{\nu} n_{\mu} \cdot D_\nu f_L}{q \cdot n} = 2 \pi h c \delta(q^2) D_\nu \left( q_{\nu} D_\nu f_L \right) + O(h^2),
\]

(C6)

and consequently,

\[
- \frac{1}{2} \epsilon^{\nu \rho \sigma \lambda} \epsilon_{\mu \lambda \sigma \rho} q^{\lambda} D_\mu \mathcal{L}^\nu \approx - \pi h c \delta(q^2) q^{\nu} \epsilon^{\nu \rho \sigma \lambda} \epsilon_{\mu \lambda \sigma \rho} q^{\lambda} D_\mu D_\rho f_L
\]

\[
= \frac{\pi}{2} h c \delta(q^2) q^{\nu} \epsilon^{\nu \rho \sigma \lambda} \epsilon_{\mu \lambda \sigma \rho} q^{\lambda} D_\mu D_\rho f_L = 2 D_\mu \mathcal{L}_{\nu \mu} f_L = 0.
\]

(C7)

Nonetheless, the term involving the Riemann curvature tensor will vanish when integrating over momentum as

\[
\int \delta(q^2) q^{\nu} \epsilon^{\nu \rho \sigma \lambda} \epsilon_{\mu \lambda \sigma \rho} q^{\lambda} D_\mu D_\rho f_L = \int \delta(q^2) q^{\nu} \epsilon^{\nu \rho \sigma \lambda} \epsilon_{\mu \lambda \sigma \rho} q^{\lambda} D_\mu D_\rho f_L = 0,
\]

(C8)

where we used integration by parts and dropped the surface terms again, and we implemented the following properties for Riemann curvature tensor:

\[
R_{\alpha \rho \sigma \epsilon} = -R_{\rho \sigma \alpha \epsilon} = -R_{\rho \epsilon \alpha \sigma},
\]

(C9)

and the Bianchi identity

\[
R_{\alpha \rho \sigma \epsilon} + R_{\epsilon \sigma \rho \alpha} + R_{\rho \alpha \epsilon \sigma} = 0.
\]

(C10)

Similar results can be found for right-handed fermions by flipping the sign of the \( O(h) \) term. Therefore, from Equation (C5), we derive

\[
\nabla_\mu T^{\mu \nu} = 2 \int \left( q^{\nu} \tilde{C}[f_L] + \frac{\pi}{2} h c \delta(q^2) \epsilon^{\nu \rho \sigma \lambda} q^{\lambda} D_\nu \mathcal{C}_{\mu} \right)
\]

(C11)

as the conservative equation for the energy–momentum tensor.

One may sometimes write the conservative equation in an alternative form. By using the Schouten identity, which states that an antisymmetric tensor of rank 5 vanishes in four spacetime dimensions, i.e.,

\[
0 = 5 \delta^{[\nu}_{\rho} \epsilon_{\mu \nu \rho \sigma \lambda]} = 3 \delta^{[\nu}_{\rho} \epsilon_{\mu \nu \rho \sigma \lambda]} + \delta^{[\nu}_{\rho} \epsilon_{\mu \nu \rho \sigma \lambda]} + \delta^{[\nu}_{\rho} \epsilon_{\mu \nu \rho \sigma \lambda]}
\]

(C12)

one finds

\[
q^{\nu} \tilde{C}[f_L] = 2 \pi \delta(q^2) \left( q^{\nu} (q \cdot C) + \frac{h c}{2 q \cdot n} \epsilon^{\nu \rho \sigma \lambda} q^{\lambda} D_\nu \mathcal{C}_{\mu} \right)
\]

(C13)

which leads to

\[
\nabla_\mu T^{\mu \nu} = 4 \pi \int q \delta(q^2) \left( q^{\nu} (q \cdot C) + \frac{1}{4} h c \epsilon^{\nu \rho \sigma \lambda} q^{\lambda} D_\mu D_\rho f_L \right)
\]

(C14)
Appendix D  
Collision Terms in the Four-Fermi Theory

We can write down an explicit form of the collision terms in \( \mathcal{C} \) for the neutrino absorption process in Equation (37) as
\[
\mathcal{L}^{(\nu)\leq}_{\nu} = \frac{G F}{2} \int \frac{d^4k d^4q d^4k'}{2^8(\pi)^3} \delta(k^2 - M^2)(\kappa - M_N) f^0_{k'} \leq \leq \, \, \, ,
\]
and similarly for \( \mathcal{L}^{(\bar{\nu})\leq}_{\bar{\nu}} \), and we ignored the antineutrinos. Here we also defined
\[
\int_{k',k} = \frac{d^4k d^4q d^4k'}{2^8(\pi)^3} \delta(k^2 + M^2)(k' - q - q').
\]
The self-energies then read
\[
\Sigma^{\leq}_{\nu} = 8G F^2 \int \frac{d^4k d^4q d^4k'}{2^8(\pi)^3} \delta(k^2 - M^2)(k' - q - q').
\]
and similarly for \( \mathcal{L}^{(\nu)\leq}_{\nu} \).

The energy of the elastic scattering in Equation (46) can be obtained by following replacements in Equation (D4): \( n \rightarrow N, p \rightarrow N, g_{\nu, A} \rightarrow g_{\nu, A}/2, \) and \( \mathcal{L}^{(\nu)\leq}_{\nu} \rightarrow \mathcal{L}^{(\nu)\leq}_{\nu} \).

Appendix E  
Self-energies in Equilibrium

We describe the details of the calculations of the self-energies \( \Sigma^{(0)}_{\nu}, \Sigma^{(\pi)}_{\nu}, \) and \( \Sigma^{(B)}_{\nu} \) under certain approximations. 

We first consider the nonrelativistic limit for nucleons, where \( k_{\nu} = M_w + k_{\nu}^w \). Here we introduced \( V_0 \equiv V \cdot u \) and \( V_0^w \equiv V^w \cdot (V \cdot u)u^w \) for an arbitrary four-vector \( V^w \). In addition, we will use \( |V| \) to represent the norm of \( V^w \). Then, we can approximate the following factors appearing in

Equations (56)-(58) as
\[
\begin{align}
(g^2_{\nu} k_{\nu} k_{\nu}^w + g^2_{\nu} k_{\nu} k_{\nu}^w + g_{\nu} g_{\nu} M_{\nu} \eta_{\nu} \eta_{\nu}) q_{\nu} q_{\nu}^w & \approx \left[ g^2_{\nu} + g^2_{\nu} \right] M_{\nu}^2 + g_{\nu} g_{\nu} M_{\nu} p_{\nu} \\
& \times (q_0 - p_0) u_{\nu} - g_{\nu} g_{\nu} M_{\nu} (q_0 - p_0) \eta_{\nu},
\end{align}
\]

where we used \( k' = p + k \) and \( q' = q - p \). Following Reddy et al. (1998), we may drop all the last terms above owing to the small prefactors proportional to \( g_{\nu} g_{\nu} - g_{\nu} g_{\nu} \), while we still keep the \( g_{\nu} g_{\nu} \) terms proportional to \( u_{\nu} \). Also, we will ignore the difference of neutron and proton masses and set \( M_{\nu} \approx M_{\nu} \approx M \). Then, one may combine \( \Sigma^{(0)}_{\nu} \) and \( \Sigma^{(B)}_{\nu} \) as
\[
\Sigma^{(0)}_{\nu} + \Sigma^{(B)}_{\nu} \approx 8u_{\nu}(g^2_{\nu} + g^2_{\nu} - g_{\nu} g_{\nu} M^2 G^2_F)
\]
\[
\times \left( \int_{k'} (q_0 - p_0) (2\pi)^3 \right)
\]
\[
\times \delta(k^2 - M^2 + B_{\nu} q_{\nu} q_{\nu} - B_{\nu} q_{\nu} - B_{\nu} q_{\nu} - B_{\nu} q_{\nu})
\]
\[
\times \delta((k + p)^2 - M^2) f^p_{k^p} f^p_{k^p} f^p_{k^p} f^p_{k^p} - k^p - k^p - k^p - k^p.
\]

We next exploit the isovector approximation in Bruenn (1985) by taking \( p_0 \rightarrow 0 \), which in fact can be realized under the nonrelativistic approximation for nucleons. In the nonrelativistic limit, one can rewrite the following delta function as
\[
\delta((k + p)^2 - M^2) = \delta(p^2 + 2k \cdot p)
\]
\[
\approx \frac{1}{2M} \delta \left( p_0 - \frac{|p|^2}{2M} - \frac{|p||k| \cos \theta_{pk}}{M} \right)
\]
by dropping higher-order terms suppressed by \( 1/M \), where \( \theta_{pk} \) denotes the angle between \( k \) and \( p \). When further neglecting the \( O(|p|/M) \) and \( O(|k|/M) \) terms, the delta function can be approximated as \( \delta((k + p)^2 - M^2) \approx \delta(p_0)/(2M) \), which explicitly yields \( p_0 \rightarrow 0 \). Nonetheless, in order to include the quantum corrections from magnetic fields, we should retain at least the \( O(|p|/M) \) terms in Equation (E5). In contrast, we may omit the \( |k| \cos \theta_{pk} |M/2 | \) term by symmetry when assuming that \( f^p_{k^p} \) only depends on \( p_0 + k_0 \) given that the nucleons are near thermal equilibrium. Physically, when \( M \rightarrow \infty \), the momentum conservation, it is hence necessary to include at least \( |p|/M \) terms for preserving the magnetic field contribution albeit
the suppression in the nonrelativistic limit. We will accordingly
apply δ((k + p)^2 − M^2) = δ(p^2 + 2p·k) ≈ (2M)^{-1}δ(p_0 - |p|^2/(2M))
as a “quasi-isoenenergetic” approximation. On the other hand, for an
arbitrary integrand G(p, k), one can write the integral as
\[ \int_p \int_k \delta((q - p)^2 + \frac{hB \cdot (q - p)}{q_0 - p_0}) \delta(k^2 - M^2) G(p, k) \]
\[ \approx \int_p \int_k \frac{dp_0 dp_d |d(cos \theta_{pq})| |p|}{(2\pi)^2} \left[ 1 - \frac{hB_{L}p}{2|q|(q_0 - p_0)} \right] \times \delta(cos \theta_{pq} - \cos \theta_{B}) \int \frac{d^2k}{(2\pi)^2} G(p, k) \bigg|_{k_0 = E_k}, \]
(E6)

where we decomposed the magnetic field as Equation (62) and
\[ \cos \theta_B = \frac{1}{2|p||q|} \left[ \left[ 1 - \frac{hB_{L}p}{2|q|(q_0 - p_0)} \right] \cos \theta_{pq} - \cos \theta_{B} \right]. \]
(E7)

Here we applied
\[ \delta((q - p)^2 + \frac{hB \cdot (q - p)}{q_0 - p_0}) \]
\[ = \delta\left(2|q| + \frac{hB_{L}p}{q_0 - p_0}\right) |p| \cos \theta_{pq} + p^2 - 2q_0p_0 - \frac{hB_{L}|p|}{q_0 - p_0}B_T \cdot p \]
\[ \approx \frac{1}{2|p||q|} \left[ 1 - \frac{hB_{L}p}{2|q|(q_0 - p_0)} \right] \delta(cos \theta_{pq} - \cos \theta_{B}). \]
(E8)

where \( B_T \cdot p = -B_T|p| \sin \theta_{pq} \cos \phi_{pq} \) is also dropped by assuming \( |p| \gg M \) in the derivation.\(^{10}\) In the following, we will set \( q_0 = |q| \) based on the on-shell condition for neutrinos.

Based on the quasi-isoenenergetic approximation, further assuming that \( f_{k_{0}}^{(\pm)} \) and \( f_{p+k_0}^{(\pm)} \) only depend on \( k_0 \) and \( p_0 + k_0 \), we find
\[ \Sigma_{\mu_\nu}^{(0)\pm} + 2 \Sigma_{\mu_\nu}^{(0)\mp} \approx 4\pi u_{\nu}(g_\sigma^2 + 3g_{\lambda}^2) G_{\sigma}^{2} f_{0,q}^{(\pm)} \]
\[ \times \int_{p_{\min}}^{p_{\max}} \frac{dp_0}{(2\pi)^2} |d| \left[ \frac{|p|^2}{2M} - \frac{hB_{L}p}{2|q|} \right] \int \frac{d^2k}{(2\pi)^2} f_{0,q}^{(\pm)} f_{0,k}^{(\mp)} \],
(E9)

where we rewrote \( g_{\pm} \) in terms of \( g_{V,A} \). Here \( p_{\max} \) and \( p_{\min} \) are determined by the dispersion relation in Equation (E8) as
\[ p_{\max} = |q| \left(2 - \frac{2M}{|q|} + \frac{hB_{L}}{2|q|^2}\right), \]
\[ p_{\min} = \frac{hB_{L}}{2|q|^2}. \]
(E10)

\( ^{10} \) Although this assumption is not rigorously justified, the contribution from \( B_T \) will eventually be irrelevant regardless of this assumption, since the term involving the magnetic field after the integral must be proportional to \( q \cdot B \) from the symmetry, which can only incorporate the contribution from \( B_L \).

Using the following relations for the nucleon Fermi–Dirac distribution,
\[ f_{0,q}^{(\pm)} - f_{0,k}^{(\pm)} = \frac{f_{0,k}^{(\pm)} - f_{0,q}^{(\pm)}}{1 - e^{\beta(p_\mu - p_{\nu})}}, \quad f_{0,q}^{(\pm)} - f_{0,k}^{(\pm)} = \frac{f_{0,k}^{(\pm)} - f_{0,q}^{(\pm)}}{1 - e^{\beta(p_\mu - p_{\nu})}}, \]
(E11)

one finds Equation (60).

Additionally, we also have the collision terms associated with the fluid vorticity,
\[ hq \cdot \tilde{\Sigma}^{(\nu)\pm} \approx \pm 2\pi h(g_\sigma^2 + 3g_{\lambda}^2) \]
\[ \times G_{\sigma}^{2} \beta(q \cdot \omega) f_{0,q}^{(\pm)} (1 - f_{0,q}^{(\pm)}) \]
\[ \times \int_{0}^{2|q|} \frac{dp_0}{(2\pi)^2} |p| \int \frac{d^2k}{(2\pi)^2} f_{0,q}^{(\pm)} f_{0,k}^{(\mp)} \]
\[ \approx 4\pi h(g_\sigma^2 + 3g_{\lambda}^2) G_{\sigma}^{2} \int |q| \beta(q \cdot \omega) f_{0,q}^{(\pm)} \]
\[ \times \int_{0}^{2|q|} \frac{dp_0}{(2\pi)^2} |p| \int \frac{d^2k}{(2\pi)^2} f_{0,q}^{(\pm)} f_{0,k}^{(\mp)} \],
(E12)

where we used \( \xi_{\nu} \approx u_{\nu} \) and dropped the higher-order terms in |\nu|.

We hence arrive at Equation (61).

Appendix F
Conservative Equation in Equilibrium

Let us consider the conservative equation for the energy–momentum tensor in Equation (29) when the matter sector is in thermal equilibrium. When taking \( \eta_{\mu} = \xi_{\mu} \approx u_{\mu} \), the h term in \( \tilde{C}^{f_{0,q}^{(\pm)}} \) vanishes since now \( \Sigma^{\mu}_{\pm} \approx u_{\mu} \). We thus focus on the second term on the right-hand side of Equation (29). By using
\[ \nu_{\eta_{\mu}}(q \cdot u)^2 = 2(q \cdot u)[2(q \cdot u)^2 \omega^\nu - (q \cdot u)(q \cdot \omega)u^\nu + (q \cdot \omega)q_{\mu}^\nu] \]
(F1)

and
\[ \nu_{\eta_{\mu}}(q \cdot u) \nabla_{\mu} f_{0,q}^{(\pm)} = -\beta f_{0,q}^{(\pm)} (1 - f_{0,q}^{(\pm)})(q \cdot u)^2 \omega^\nu + (q \cdot \omega)q_{\mu}^\nu \]
(F2)
under the on-shell condition $q^2 = 0$, we have

$$e^{\nu\mu \beta \gamma} q_{\nu} D_{\alpha} \tilde{S}^{(0)\alpha}_{\mu} = \left( q^2 (4 + \beta \alpha) - 2 |q\omega|^2 \right) \frac{\tilde{S}^{(0)\alpha}_{\mu}}{|q|^2},$$

$$e^{\nu\mu \beta \gamma} q_{\nu} D_{\alpha} \tilde{S}^{(0)\alpha}_{\mu} = \left( q^2 (4 - \beta \alpha) - 2 |q\omega|^2 \right) \frac{\tilde{S}^{(0)\alpha}_{\mu}}{|q|^2},$$

from which we derive

$$- h e^{\nu\mu \beta \gamma} q_{\nu} \tilde{S}^{(0)\alpha}_{\mu} \xi_{\beta\gamma} \approx \frac{\tilde{S}^{(0)\alpha}_{\mu}}{|q|^2} - h f_{\nu\omega} (q^2 (4 + \beta \alpha) - 2 |q\omega|^2) \frac{\tilde{S}^{(0)\alpha}_{\mu}}{|q|^2},$$

where the explicit expression of $\tilde{S}^{(0)\alpha}_{\mu}$ can be found in Equation (60) by taking $B_{\eta} = 0$.

ORCID iDs
Naoki Yamamoto @ https://orcid.org/0000-0003-2021-0104
Di-Lun Yang @ https://orcid.org/0000-0002-7379-2577

References
Adler, S. L. 1969, PhRv, 177, 2426
Akamatsu, Y., & Yamamoto, N. 2013, PhRvL, 111, 052002
Alekseev, A. Yu., Cheianov, V. V., & Frohlich, J. 1998, PhRvL, 81, 3503
Banerjee, N., Bhattacharyya, J., Bhattacharyya, S., et al. 2011, JHEP, 01, 094
Bell, I. S., & Jackiw, R. 1969, NCimA, 60, 47
Blajot, J.-P., & Iancu, E. 2002, PhR, 359, 355
Blaizot, J.-P., & Iancu, E. 2002, PhR, 359, 355
Blaschke, D. N., & Cirigliano, V. 2016, PhRvD, 94, 033009
Boyarsky, A., Frohlich, J., & Ruchayskiy, O. 2012, PhRvL, 108, 031301
Bruenn, S. W. 1985, ApJS, 58, 771
Burkov, A. A., & Balents, L. 2011, PhRvL, 107, 127205
Cabezon, R. M., Pan, K.-C., Liebendorfer, M., et al. 2018, A&A, 619, A118
Carignano, S., Manuel, C., & Torres-Rincon, J. M. 2018, PhRvD, 98, 076005
Carignano, S., Manuel, C., & Torres-Rincon, J. M. 2019, arXiv:1908.00561
Castor, J. I. 1972, ApJ, 178, 779

11 Here the (four-)momentum derivatives in the horizontal lifts acting on $\tilde{S}^{(0)\alpha}_{\mu}$ do not contribute, as is consistent with the physical expectation that the $h$ corrections will depend on either $\omega_\eta$ or $B_\eta$ in local thermal equilibrium. Technically, we find that $\frac{\partial}{\partial x_\alpha}$ is a $\omega_\alpha(q \cdot \omega)$, with $\omega_\alpha(q \cdot \omega)$ being functions of $q_\alpha$ alone, and combined with the relations $\Gamma_\mu^\alpha = \Gamma_\mu^\alpha = \Gamma_{0\mu} = 0$ in the inertial frame, the (four-)momentum derivatives lead to vanishing results after contracting with the Christoffel symbols. A similar argument is applied to $\omega_\alpha \tilde{S}^{(0)\alpha}_{\mu}$.
Vilenkin, A. 1980, Phys. Rev. D, 22, 3080
Vlasenko, A., Fuller, G. M., & Cirigliano, V. 2014, Phys. Rev. D, 89, 105004
Wan, X., Turner, A. M., Vishwanath, A., & Savrasov, S. Y. 2011, Phys. Rev. B, 83, 205101
Weickgenannt, N., Sheng, X.-L., Speranza, E., Wang, Q., & Rischke, D. H. 2019, Phys. Rev. D, 100, 056018
Xu, G., Weng, H., Wang, Z., Dai, X., & Fang, Z. 2011, Phys. Rev. Lett., 107, 186806
Yamamoto, N. 2016a, Phys. Rev. D, 93, 065017
Yamamoto, N. 2016b, Phys. Rev. D, 93, 125016
Yamamoto, N. 2017, Phys. Rev. D, 96, 051902
Yang, D.-L. 2018, Phys. Rev. D, 98, 076019
Yang, D.-L., Hattori, K., & Hidaka, Y. 2020, arXiv:2002.02612