On the Tractable Counting of Theory Models and its Application to Belief Revision and Truth Maintenance

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Abstract
We introduced decomposable negation normal form (DNNF) recently as a tractable form of propositional theories, and provided a number of powerful logical operations that can be performed on it in polynomial time. We also presented an algorithm for compiling any conjunctive normal form (CNF) into DNNF and provided a structure-based guarantee on its space and time complexity. We present in this paper a linear-time algorithm for converting an ordered binary decision diagram (OBDD) representation of a propositional theory into an equivalent DNNF, showing that DNNFs scale as well as OBDDs. We also identify a subclass of DNNF which we call deterministic DNNF, d-DNNF, and show that the previous complexity guarantees on compiling DNNF continue to hold for this stricter subclass, which has stronger properties. In particular, we present a new operation on d-DNNF which allows us to count its models under the assertion, retraction and flipping of every literal by traversing the d-DNNF twice. That is, after such traversal, we can test in constant-time: the entailment of any literal by the d-DNNF, and the consistency of the d-DNNF under the retraction or flipping of any literal. We demonstrate the significance of these new operations by showing how they allow us to implement linear-time, complete truth maintenance systems for two important classes of propositional theories.

Introduction
Knowledge compilation has been emerging recently as a new direction of research for dealing with the computational intractability of general propositional reasoning (Selman & Kautz March 1996; Cadoli & Donini 1997). According to this approach, the reasoning process is split into two phases: an off-line compilation phase and an on-line query-answering phase. In the off-line phase, the propositional theory is compiled into some target language, which is typically a tractable one. In the on-line phase, the compiled target is used to efficiently answer a (potentially) exponential number of queries. The main motivation behind knowledge compilation is to push as much of the computational overhead as possible into the off-line phase, in order to amortize that overhead over all on-line queries.

One of the key aspects of any compilation approach is the target language into which the propositional theory is compiled. Previous compilation approaches have proposed Horn theories, prime implicates/implicants, and ordered binary decision diagrams (OBDDs) as targets for such compilation (Selman & Kautz March 1996; Cadoli & Donini 1997). More recent compilation target language is decomposable negation normal form (DNNF) (Darwiche 1999a, 1999b). DNNF is universal, supports a rich set of polynomial-time operations; is more space-efficient than OBDDs (Darwiche 1999b); and is very simplistic as far as its structure and algorithms are concerned. Propositional theories in DNNF are highly tractable:

1. Deciding the satisfiability of a DNNF can be done in linear time.
2. Conjoining a DNNF with a set of literals can be done in linear time.
3. Projecting a DNNF on some atoms can be done in linear time. Intuitively, to project a theory on a set of atoms is to compute the strongest sentence entailed by the theory on these atoms.
4. Computing the minimum-cardinality of a DNNF can be done in linear time. The cardinality of a model is the number of atoms that are set to false in the model. The minimum-cardinality of a theory is the minimum-cardinality of any of its models.
5. Minimizing a DNNF can be done in quadratic time. To minimize a theory is to produce another theory which models are exactly the minimum-cardinality models of the original theory.
6. Enumerating the models of a DNNF can be done in time linear in its size and quadratic in the number of its models.

This paper rests on two key contributions. First, we show that DNNF representations scale as well as OBDD representations (Bryant 1992) by presenting a linear-time algorithm for converting an OBDD representation of a propositional theory into an equivalent DNNF representation. Second, we identify a subclass
of DNNF, which we call deterministic DNNF, d-DNNF, and present a new linear-time operation for counting its models under the assertion, retraction and flipping of literals. In particular, we show how to traverse a d-DNNF only twice and yet compute: its number of models under the assertion, retraction and flipping of each literal. This allows us to test in linear time: the consistency of the d-DNNF under the assertion, retraction and flipping of each literal, therefore, allowing us to implement linear-time, complete truth maintenance and belief revision systems.

What is interesting though is that two of our key complexity results with respect to DNNF compilations continue to hold with respect to the subclass of d-DNNF. This includes a structure-based algorithm which can compile any CNF into an equivalent DNNF in linear time given that the treewidth of the CNF is bounded [Darwiche 1999a]. It also includes the newly proposed algorithm for converting any OBDD into an equivalent DNNF in linear time.

This paper is structured as follows. We first review DNNF, introduce the class of deterministic DNNF and then discuss the new operation for model counting. We follow that by discussing the application of this counting operation to truth maintenance, belief revision, and model-based diagnosis systems. We finally close with some concluding remarks. Proofs of all results are available in the full paper [Darwiche 2000b].

Deterministic DNNF

A propositional sentence is in negation normal form (NNF) if it is constructed from literals using only the conjoin and disjoin operators. Figure 1 shows a sentence in NNF depicted as a rooted, directed acyclic graph where the children of each node are shown below it in the graph. Each leaf node represents a literal and each non-leaf node represents a conjunction or a disjunction. We allow true and ¬false to appear as leaves in a DNNF to denote a conjunction with no conjuncts. Similarly, we allow false and ¬true as leaves to represent a disjunction with no disjuncts. The size of an NNF is measured by the number of edges in its graphical representation. Our concern here is mainly with a subclass of NNFs:

Definition 1 [Darwiche 1999a] A decomposable negation normal form (DNNF) is a negation normal form satisfying decomposability property: for any conjunction ∨₁ αᵢ appearing in the form, no atom is shared by any pair of conjuncts in ∨₁ αᵢ.

The NNF (A ∨ B) ∧ (¬A ∨ C) is not decomposable since atom A is shared by the two conjuncts. But the NNF in Figure 1 is decomposable. It has ten conjunctions and the conjuncts of each share no atoms. Decomposability is the property which makes DNNF tractable.

One of the key operations on DNNF is that of conditioning:

Definition 2 [Darwiche 1999a] Let α be a proposi-

Figure 2: A decomposition tree for the theory A ⊃ B, B ⊃ C, C ⊃ D.

1An instantiation of a set of atoms is a conjunction of literals, one literal for each atom in the set.
Figure 1: A theory in DNNF. The theory has eight models, representing the odd-cardinality models: \(\neg A \land B \land C \land D\); \(A \land \neg B \land C \land D\); \(A \land B \land \neg C \land D\); \(A \land B \land C \land \neg D\); \(\neg A \land \neg B \land \neg C \land D\); \(\neg A \land \neg B \land C \land \neg D\); \(\neg A \land B \land \neg C \land \neg D\); \(A \land \neg B \land \neg C \land \neg D\).

difference between this version and the one in (Darwiche 1999a) is that we have \(\text{cl2dDNNF}(\text{clause}(N) \mid \alpha)\) instead of \(\text{clause}(N) \mid \alpha\), therefore, converting a clause to a d-DNNF at the boundary condition.

**Theorem 1** (Darwiche 1999a) Let \(N\) be the root of decomposition tree \(T\) used in Figure 1. Then \(\text{cnf2ddnnf}(N, \alpha)\) will return \(\Delta \mid \alpha\) in DNNF, where \(\Delta\) contains the clauses attached to the leaves of \(T\). Moreover, the time and space complexity of the algorithm is \(O(nw2^n)\), where \(n\) is the number of clauses in \(\Delta\) and \(w\) is the width of decomposition tree \(T\).

**Theorem 2** The DNF returned by the algorithm in Figure 1 is deterministic.

The class of CNF theories with bounded treewidth is defined in (Darwiche 1999a), where it is shown that, for this class of theories, one can construct in linear time a decomposition tree of bounded width. Therefore, one can compile a d-DNNF of linear size for this class of theories.

Binary decision diagrams (BDDs) are among the most successful representations of propositional theories (Bryant 1992). Two special classes of BDDs, OBDDs and FBDDs, are especially popular given (Gerevich & Meinel 1994; Sieling & Wegener 1995):

- the number of linear-time operations they support and
- the number of real-world theories that admit efficient OBDD/FBDD representations.

We now present a linear-time algorithm for converting an FBDD into an equivalent d-DNNF, showing that d-DNNFs scale as well as FBDDs (which include OBDDs as a subclass). We start by the formal definitions of BDDs, OBDDs, and FBDDs.

**Definition 4** A binary decision diagram (BDD) over a set of binary variables \(X = \{x_1, \ldots, x_n\}\) is a directed acyclic graph with one root and at most two leaves labeled 0 and 1. Each non-leaf node \(m\) is labeled by a variable \(\text{var}(m) \in X\) and has two outgoing edges labeled 0 and 1, where \(\text{low}(m)\) and \(\text{high}(m)\) denote the nodes pointed to by these edges, respectively. The computation path for input \((a_1, \ldots, a_n)\), where \(a_i \in \{0, 1\}\), is defined as follows. One starts at the root. At inner node \(m\), where \(\text{var}(m) = x_i\), one moves to node \(\text{low}(m)\) if \(a_i = 0\) and to node \(\text{high}(m)\) otherwise. The BDD represents the Boolean function \(f\) if the computation path for each input \((a_1, \ldots, a_n)\) leads to the leaf node labeled with \(f(a_1, \ldots, a_n)\).

**Algorithm cnf2ddnnf**

```
/* N is a node in a decomposition tree */
/* \alpha is an instantiation */

\text{cnf2ddnnf}(N, \alpha)
\begin{align*}
\psi &\leftarrow \text{project}(\alpha, \text{atoms}(N)) \\
\text{if} \, \text{CACHE}_N(\psi) \neq \text{NIL}, \text{return} \text{CACHE}_N(\psi) \\
\text{if} \, N \text{ is a leaf node,} \\
\text{then} \gamma &\leftarrow \text{cl2dDNNF}(\text{clause}(N) \mid \alpha) \\
\text{else} \gamma &\leftarrow \bigvee_{\beta} \text{cnf2ddnnf}(\text{N}_r, \alpha \land \beta) \land \\
\text{cnf2ddnnf}(\text{N}_l, \alpha \land \beta) \land \beta \\
\text{where} \beta \text{ ranges over all instantiations} \\
\text{of} \, \text{atoms(}N_l\text{)} \cap \text{atoms(}N_r\text{)} - \text{atoms(}\alpha\text{)} \\
\text{CACHE}_N(\psi) &\leftarrow \gamma \\
\text{return} \gamma
\end{align*}
```

Figure 3: Compiling a CNF into d-DNNF. Each node \(N\) in the decomposition tree has an associated cache \(\text{CACHE}_N\); \(N_l\) and \(N_r\) are the left and right children of node \(N\), respectively; \(\text{clause}(N)\) returns the clause attached to leaf node \(N\); \(\text{atoms}(N)\) are the atoms of clauses appearing under node \(N\); \(\text{atoms}(\alpha)\) returns the atoms appearing in instantiation \(\alpha\); \(\text{project}(\alpha, A)\) returns the subset of instantiation \(\alpha\) pertaining to atoms \(A\); and \(\text{cl2dDNNF}(\beta)\) returns a d-DNNF of clause \(\beta\).
Algorithm fbd2ddnnf
/* m is a node in an FBDD */
FBD2DDNNF(m)
if CACHE(m) = NIL, return CACHE(m)
if m is a leaf node labeled with 1, then γ ← true
else if m is a leaf node labeled with 0, then γ ← false
else γ ← FBD2DDNNF(low(m)) ∧ ¬xi
   ∨ FBD2DDNNF(high(m)) ∧ xi
where var(m) = xi
CACHE(m) ← γ
return γ

Figure 4: Converting an FBDD into a d-DNNF. CACHE(m) stores the d-DNNF computed for the FBDD rooted at node m.

The size of a BDD is measured by the number of nodes it contains.

Definition 5 A binary decision diagram is called a free BDD (FBDD) if on each computation path each variable is tested at most once. A free BDD is called an ordered BDD (OBDD) if on each computation path the variables are tested in the same order.

OBDDs are a strict subclass of FBDDs [Bryant 1991] and have received much consideration in the verification literature, where they are used to test the equivalence between the specs of a Boolean function and its circuit implementation. OBDDs/FBDDs permit such a test to be performed in polynomial time. Their popularity stems from the existence of efficient OBDD/FBDD representations of many complex, real-world propositional theories. DNNF is more space-efficient than FBDDs [Darwiche 1999b], but this should not be surprising as FBDDs admit more linear-time operations than does DNNF. For example, the equivalence of two DNNFs cannot be decided in polynomial time while it can for FBDDs.

Figure 5 depicts a recursive algorithm for converting an FBDD into a DNNF, showing that DNNFs scale as well as FBDDs. The algorithm should be called on the root of given FBDD.

Theorem 3 (Darwiche 1999b) The algorithm of Figure 5 takes time linear in the size of given FBDD and returns a DNNF of the theory represented by the given FBDD.

Theorem 4 The DNNF returned by the algorithm of Figure 5 is deterministic.

This has major implications on our reported results regarding truth maintenance and belief revision systems, as it proves, constructively, that we can build efficient truth maintenance and belief revision systems for any propositional theory which has an efficient FBDD representation. Figure 6 depicts an FBDD and its corresponding d-DNNF as generated by the algorithm of Figure 5.

Counting Models of d-DNNF
We now turn to an operation on d-DNNF which is of major significance to a number of AI applications, including belief revision, truth maintenance and diagnosis. Specifically, given a d-DNNF ∆ and a set of literals S, we describe two traversal operations each taking linear time. By the end of the first traversal, we will be able to count the models of ∆ ∪ S. By the end of the second traversal, we will be able to count the models of:

1. ∆ ∪ S ∪ {l} for every literal l ∉ S;
2. ∆ ∪ S \ {l} for every literal l ∈ S;
3. ∆ ∪ S \ {l} ∪ {¬l} for every literal l ∈ S.

That is, once we traverse the d-DNNF twice, we will be able to obtain each of these counts using constant-time, lookup operations. As we shall see in the following section, these counts are all we need to implement an interesting number of AI applications.

The traversal will not take place on the d-DNNF itself, but on a secondary structure that we call the counting graph. Without loss of generality, we will assume from here on that the d-DNNF is smooth:

Definition 6 A DNNF is smooth iff
1. every literal and its negation appear in the DNNF;
2. for any disjunction ∨i αi in the DNNF, we have atoms(∨i αi) = atoms(αi) for every αi.

The d-DNNF in Figure 7 is smooth as it satisfies these two conditions. We can easily make a DNNF smooth using two operations:

1. If the negation of literal l does not appear in the DNNF, replace the occurrence of l with l ∨ (¬l ∧ false);
2. For each disjunction ∨i αi with αi ∧ ∩Σ (A ∨ ¬A), where Σ are the atoms appearing in ∨i αi but not in αi.

These operations preserve both the decomposability and determinism of a DNNF. They may increase the size of given DNNF but only by a factor of O(n), where n is the number of atoms in the DNNF. This increase is quite minimal in practice though. Note that the d-DNNFs generated by the algorithm of Figure 5 satisfy the first condition; and the d-DNNFs generated by the algorithm of Figure 7 satisfy the second condition as long as cl2dnnf(clause(N) | α) satisfies some simple conditions; see (Darwiche 2000b).

The counting graph of a d-DNNF is a function of many variables represented as a rooted DAG.

Definition 7 The counting graph of a smooth d-DNNF is a labeled, rooted DAG. It contains a node labeled with l for each literal l, a node labeled with + for each or-node, and a node labeled with ∗ for each and-node in the d-DNNF. There is an edge between two nodes in the counting graph if there is an edge between their corresponding nodes in the d-DNNF.
Figure 5: An FBDD and its corresponding DNNF, which are equivalent to \((x_2 \land x_3) \lor (x_1 \land \neg x_2 \land \neg x_3)\).

Figure 6 depicts the counting graph of the \(d\)-DNNF in Figure 1. The size of a counting graph is therefore equal to the size of its corresponding \(d\)-DNNF. We will see now how such a graph can be used to perform the counting operations we are interested in.

**Definition 8** The value of a node \(N\) in a counting graph under a set of literals \(S\) is defined as follows:
- \(\text{val}(N) = 0\) if \(N\) is labeled with literal \(l\) and \(-l \notin S\);
- \(\text{val}(N) = 1\) if \(N\) is labeled with literal \(l\) and \(-l \notin S\);
- \(\text{val}(N) = \prod_i \text{val}(N_i)\) if \(N\) is labeled with \(*\), where \(N_i\) are the children of \(N\);
- \(\text{val}(N) = \sum_i \text{val}(N_i)\) if \(N\) is labeled with \(+\), where \(N_i\) are the children of \(N\).

The value of a counting graph \(G\) under literals \(S\), written \(G(S)\), is the value of its root under \(S\).

Here’s our first counting result.

**Theorem 5** Let \(\Delta\) be a smooth \(d\)-DNNF, \(S\) be a set of literals, and let \(G\) be the counting graph of \(\Delta\). The value of \(G\) under \(S\) is the number of models of \(\Delta \cup S\):

\[
G(S) = \text{Models}((\Delta \cup S)).
\]

Note that \(G(S) > 0\) iff \(\Delta \cup S\) is consistent. Therefore, by traversing the counting graph once we can test the consistency of \(d\)-DNNF \(\Delta\) conjoined with any set of literals \(S\). Figure 6 depicts the counting graph of \(d\)-DNNF \(\Delta\) in Figure 1, evaluated under the literals \(S = A, \neg B\). This indicates that \(\Delta \cup \{A, \neg B\}\) has two models.

We now present the central result in this paper. First, we note that when viewing a counting graph \(G\) as a function of many variables, we will use \(V_l\) to denote the variable (node) which corresponds to literal \(l\). Second, we can talk about the partial derivative of \(G(S)\) with respect to any of its variables \(V_l\), \(\partial G(S)/\partial V_l\). Due to the decomposability of \(d\)-DNNF, the function \(G(S)\) is linear in each of its variables. Therefore, the change to the count \(G(S)\) as a result of adding, removing or flipping a literal in \(S\) can be obtained from the partial derivatives, without having to re-evaluate the counting graph \(G\).

This leads to the following consequential result:

**Theorem 6** Let \(\Delta\) be a smooth \(d\)-DNNF, \(S\) be a set of literals, and let \(G\) be the counting graph of \(\Delta\). We have:

**Assertion:** When \(l, \neg l \notin S\):

\[
\text{Models}((\Delta \cup S \cup \{l\}) = \partial G(S)/\partial V_l.
\]

**Retraction:** When \(l \in S\):

\[
\text{Models}((\Delta \cup S \setminus \{l\}) = \partial G(S)/\partial V_l + \partial G(S)/\partial V_{\neg l}.
\]

**Flipping:** When \(l \in S\):

\[
\text{Models}((\Delta \cup S \setminus \{l\} \cup \{-l\}) = G(S) - \partial G(S)/\partial V_l + \partial G(S)/\partial V_{\neg l}.
\]

Therefore, if we can compute the partial derivative of \(G(S)\) with respect to each of its variables, then we can count the models of \(\Delta \cup S\) under the assertion of new literals not in \(S\), and under the retraction or flipping of literals in \(S\). Figure 6 depicts the value of each of these partial derivatives for the \(d\)-DNNF in Figure 1. The counting graph is evaluated under literals \(S = A, \neg B, C\) and the partial derivatives are shown below each variable. According to these derivatives, we have:

**Assertion:** \(\text{Models}((\Delta \cup S \setminus \{D\}) = 1 \text{ and } \text{Models}((\Delta \cup S \cup \{\neg D\}) = 0\). This immediately tells us that \(\Delta \cup S = D\).

**Retraction:** \(\text{Models}((\Delta \cup S \setminus \{A\}) = 1 + 1 = 2; \text{Models}((\Delta \cup S \setminus \{B\}) = 1 + 1 = 2; \text{and Models}((\Delta \cup S \setminus \{C\}) = 1 + 1 = 2\). Therefore, retracting any literal in \(S\) increases the number of models to 2.

**Flipping:** \(\text{Models}((\Delta \cup S \setminus \{A\} \cup \{\neg A\}) = 1 - 1 + 1 = 1; \text{Models}((\Delta \cup S \setminus \{B\} \cup \{B\}) = 1 - 1 + 1 = 1; \text{and Models}((\Delta \cup S \setminus \{C\} \cup \{\neg C\}) = 1 - 1 + 1 = 1\). Therefore, flipping any literal in \(S\) will not change the number of models (although it does change the model itself).
Figure 6: A counting graph of the DNNF $\Delta$ in Figure 1 evaluated under $S = A, \neg B$. The evaluation indicates that $\Delta \cup S$ has two models ($A \land \neg B \land C \land D$ and $A \land \neg B \land \neg C \land \neg D$ in this case).

Figure 7: A counting graph of the DNNF $\Delta$ in Figure 1 evaluated under literals $S = A, \neg B, C$, indicating one model of $\Delta \cup S$ ($A \land \neg B \land C \land D$ in this case). Partial derivatives are shown below the leaves.

There is one missing link now: How do we compute the partial derivatives of a counting graph with respect to each of its variables? This actually turns out to be easy due to results in (Masao 1984; Gunter Rote 1990) which show how to evaluate and simultaneously compute all partial derivatives of a function by simply traversing its computation graph in linear time. Although (Gunter Rote 1990) casts such computation in terms of summing weights of paths in such a graph, we present a more direct implementation here. In particular, if we let $\text{PD}(N)$ denote the partial derivative of $G(S)$ with respect to a node $N$ in the counting graph, then $\text{PD}(N)$ is the summation of contributions made by parents $M$ of $N$:

$$\text{PD}(N) = \begin{cases} 1, & \text{N is the root node;} \\ \sum_{M} \text{CPD}(M, N), & \text{otherwise;} \end{cases}$$

where the contribution of parent $M$ to its child $N$ is computed as follows:

$$\text{CPD}(M, N) = \begin{cases} \text{PD}(M), & \text{M is a +node;} \\ \text{PD}(M) \prod_{K \neq N} \text{VAL}(K), & \text{M is a *node;} \end{cases}$$

where $K$ is a child of $M$. This computation can be performed by first traversing the counting graph once to evaluate it, assigning $\text{VAL}$ to each node $N$, and then traversing it a second time, assigning $\text{PD}$ for each node $N$. We are then mainly interested in $\text{VAL}(N)$ where $N$ is the root node, and $\text{PD}(N)$ where $N$ is a leaf node.

Therefore, both the value of a counting graph under some literals $S$ and the values of each of its partial derivatives under $S$ can be computed by traversing the graph twice. Once to compute the values, and another to compute the partial derivatives. Note that such traversal needs to be redone once the set of literals $S$ changes.

We close this section by pointing out that partial differentiation turns out to play a key role in probabilistic
reasoning as well. Specifically, we present a comprehensive framework for probabilistic reasoning in (Darwiche 2000a) based on compiling a Bayesian network into a polynomial and then reducing a large number of probabilistic queries into the computation of partial derivatives of the compiled polynomial.

Complete, Linear-Time Truth Maintenance

We now turn to some applications of the results in the previous section. That is, what can we do once we are able to count models under the conditions stated above?

We first consider truth maintenance systems and show how our results allow us to implement complete truth maintenance systems which take linear time on two important classes of propositional theories: those with bounded treewidth, and those admitting a linear FBDD representation. For each class of such theories, we can compile a smooth d-DNNF \( \Delta \) in linear time and then use it for truth maintenance as follows.

A truth maintenance system takes a set of clauses \( \Gamma \) and a set of literals \( S \) and tries to determine for each literal \( l \) whether \( \Gamma \cup S \models l \). The most common truth maintenance system is the one based on closing \( \Gamma \cup S \) under unit resolution (Forbus & de Kleer 1993). Such a system takes linear time, but is incomplete. Given that the set of literals in \( S \) changes to \( S' \), the goal of a truth maintenance system is then to update the truth of each literal under the new “context” \( S' \). Sometimes, clauses in \( S \) are retracted and/or asserted. A truth maintenance system is expected to update the truth of literals under such clause changes too.

Our model-counting results allow us to implement a complete truth maintenance system as follows. We compile the theory \( \Gamma \) into a smooth d-DNNF \( \Delta \) and construct the counting graph \( G \) of \( \Delta \). Given any set of literal \( S \), we evaluate \( G \) under \( S \) and compute its partial derivatives also under \( S \). This can be done in time linear in the size of \( \Delta \). We are now ready to answer all queries expected from a truth maintenance system by simple, constant-time, look-up operations:

 Literal \( l \) is entailed by \( \Delta \cup S \) iff \( \Delta \cup S \cup \{ \neg l \} \) has no models: \( \partial G(S)/\partial V_{\neg l} = 0 \).

 Retracting literal \( l \) from \( S \) will render \( \Delta \cup S \) consistent iff \( \Delta \cup S \setminus \{ l \} \) has at least one model: \( \partial G(S)/\partial V_l + \partial G(S)/\partial V_{\neg l} > 0 \).

 Flipping literal \( l \) in \( S \) will render \( \Delta \cup S \) consistent iff \( \Delta \cup S \setminus \{ l \} \cup \{ \neg l \} \) has at least one model: \( G(S) - \partial G(S)/\partial V_l + \partial G(S)/\partial V_{\neg l} > 0 \).

 If we want to reason about the assertion/retraction of clauses in theory \( \Gamma \), we can replace each clause \( \alpha \) in \( \Gamma \) by \( C_{\alpha} \equiv \alpha \), where \( C_{\alpha} \) is a new atom that represents the truth of clause \( \alpha \). We then compile the extended theory \( \Gamma \) into d-DNNF \( \Delta \). To assert all clauses initially, we have to include all atoms \( C_{\alpha} \) in the set of literals \( S \). The assertion/retraction of clauses can then be emulated by the assertion/retraction of atoms \( C_{\alpha} \). For example, in case of a contradiction, we can ask whether removing a clause \( \alpha \) will resolve the contradiction by asking whether \( \Delta \cup S \setminus \{ C_{\alpha} \} \) has more than one model:

\[
\partial G(S)/\partial V_{C_{\alpha}} + \partial G(S)/\partial V_{\neg C_{\alpha}} > 0.
\]

Complete, Linear-Time Belief Revision

We now turn to a second major application of model counting on d-DNNF: the implementation of a very common class of belief revision systems, which is adopted in model-based diagnosis and in certain forms of default reasoning. The problem here is as follows. We have a set of special atoms \( D = \{ d_1, \ldots, d_n \} \) in the theory \( \Gamma \) which represent defaults. Typically, we assume that all of these defaults are true, allowing us to draw some default conclusions. We then receive some evidence \( S \) (a set of literals) which is inconsistent with \( \Gamma \cup D \). We therefore know that not all defaults are true and some must be retracted—that is, some \( d_i \)'s will have to be replaced by \( \neg d_i \) in \( D \). Our goal then is to identify a set of literals \( D' \) such that

1. \( d_i \in D' \) or \( \neg d_i \in D' \) for all \( i \);
2. \( \Gamma \cup D' \cup S \) is consistent;
3. the number of negative literals in \( D' \) is minimized;

and then report the truth of every literal under the new set of defaults \( D' \). Note that there may be more than one set of defaults \( D' \) that satisfies the above properties. In such a case, a literal holds after the revision process only if it holds under \( \Gamma \cup D' \cup S \) for every \( D' \).

How can we do this? As we shall see now, if \( \Gamma \) is a smooth d-DNNF, then all of this can be done in time linear in the size of \( \Gamma \).

This works exactly as in the previous section, except that we have to minimize the d-DNNF first, a process which eliminates some of the d-DNNF models (Darwiche 1999a). To define this minimization process more precisely, we need the following definition first:

Definition 9 If \( \Sigma \) is a set of atoms, then the \( \Sigma \)-cardinality of a model is the number of atoms in \( \Sigma \) that the model sets to false.

To \( \Sigma \)-minimize a theory \( \Gamma \) is to convert it into another theory whose models are exactly the models of \( \Gamma \) having a minimum \( \Sigma \)-cardinality. Consider the d-DNNF \( \Gamma \) in Figure 4 for an example and suppose that \( \Sigma = \{ A, B, C, D \} \); that is, we want to minimize the d-DNNF with respect to each of its atoms. This theory has eight models, each having an odd cardinality (one or three). If we \( \Sigma \)-minimize this d-DNNF, we obtain another with four models, shown in Figure 13.

\[\text{[Footnotes]}\]

2We are assuming that smoothing a d-DNNF does not increase its size by more than a constant factor.

3Note that the flipping of literals is outside the scope of classical truth maintenance systems in the sense that they must retract \( l \) and then assert \( \neg l \), taking linear time, to perform the above operation.
Given these definitions, we can re-phrase the problem of belief revision (stated above) as follows. Let Σ be a set of atoms representing defaults, and let Γ be a smooth d-DNNF. Given observation S, Σ-minimize the theory Γ ∪ S to yield ∆ and then report on the truth of each literal under the minimized theory ∆.

As it turns out, one can minimize a smooth d-DNNF in linear time, to yield another smooth d-DNNF to which we can apply the techniques of the previous section. We now describe the process of minimizing a DNNF which is described in more details in (Darwiche 1999). We do this in a two-step process:

1. We assign a cardinality to every node in the d-DNNF as follows:
   (a) each literal whose atom is not in Σ gets cardinality zero;
   (b) each positive literal whose atom is in Σ gets cardinality one;
   (c) each negative literals whose atom is in Σ gets cardinality zero;
   (d) the cardinality of a disjunction is the min of its disjuncts’ cardinalities;
   (e) the cardinality of a conjunction is the summation of its conjuncts’ cardinalities.

2. For each or-node N and its child M, we delete the edge connecting N and M if the cardinality of N is smaller than the cardinality of M.

Figure 4 depicts the result of assigning cardinalities to the d-DNNF of Figure 1 and Figure 3 depicts the result of deleting some of its edges. This is the minimized d-DNNF and it has four models:

\[ \neg A \land B \land C \land D; \]
\[ A \land \neg B \land C \land D; \]
\[ A \land B \land \neg C \land D; \]
\[ A \land B \land C \land \neg D. \]

Once we have minimized the d-DNNF, we can apply the results of the previous section to obtain the answers we want.

As an example, Figure 4 depicts the counting graph of the minimized d-DNNF ∆ in Figure 3, with its value and partial derivatives computed under the observation S = \{¬A\}. From these partial derivatives and Theorem 3 we immediately get:

**Assertion:** Models\#(∆ ∪ S ∪ {¬B}) = 0; Models\#(∆ ∪ S ∪ {¬C}) = 0; Models\#(∆ ∪ S ∪ {¬D}) = 0. That is, B, C and D are all entailed by ∆ ∪ S.

**Retraction:** Models\#(∆ ∪ S \{¬A\}) = 1 + 3 = 4.
That is, we have four models if we retract ¬A.

**Flipping:** Models\#(∆ ∪ S \{¬A\} ∪ \{A\}) = 1−1+3 = 3.
That is, we have three models if we flip ¬A to A.

### Predicting the Behavior of Broken Devices

The above results have direct application to model-based diagnosis, where ∆ is the device description, S is the device observation and D contains the health modes \(ok_1, \ldots, ok_n\). Initially, we assume that all device components are working normally, but then find some observation S such that ∆ ∪ D = \{ok_1, \ldots, ok_n\} ∪ S is inconsistent.

To regain consistency we must postulate that some of the components are not healthy, therefore, flipping some of the \(ok_i\)s into \(\neg ok_i\) in the set D. Assuming a smallest number of faults, we want to minimize the number of unhealthy components needed to regain consistency. A set \(D'\) such that:

1. \(ok_i \in D'\) or \(\neg ok_i \in D'\) for all i;
2. \(\Delta \cup D' \cup S\) is consistent;
3. the number of negative literals in \(D'\) is minimized;

is called a minimum-cardinality diagnosis and one goal of model-based diagnosis to enumerate such diagnoses (Darwiche 1998; de Kleer, Mackworth, & Reiter 1992).

Another practical problem, however, which has received much less attention in model-based diagnosis is the following: Assuming a smallest number of faults, what is the truth value of every literal appearing in the device description ∆? That is, we do not want to know what the minimum-cardinality diagnoses are. Instead — and under the assumption that one of them has materialized — we want to predict the behavior of the given faulty device. But this is exactly the problem we treated in the previous section.

Therefore, our results allow us to predict the value of each device port (literal l) in a broken device, assuming that the number of broken components is minimal, in time linear in the size of device description ∆, as long as ∆ is a smooth d-DNNF. We are unaware of any similar complexity result for model-based diagnosis.

### Conclusion

We have identified two classes of propositional theories, those which have a bounded treewidth, and those which have a linear-sized FBDD representation. We have shown that each of these classes of theories can be converted in linear time into a tractable form that we called deterministic DNNF, d-DNNF. We have also defined linear-time, model-counting operations on d-DNNF, allowing us to implement (a) linear-time, complete truth maintenance systems and (b) linear-time, complete belief revision systems for the two identified classes of propositional theories. Our results also have major implications on the practice of model-based diagnosis as they allow us to efficiently predict the behavior of a broken device, assuming a smallest number of broken components.
Figure 8: Assigning a cardinality to each node in a d-DNNF with \( \Sigma = \{A, B, C, D\} \).

Figure 9: A minimized d-DNNF.

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Figure 10: The counting graph of the minimized d-DNNF in Figure 9 evaluated under $S = \{\neg A\}$. Partial derivatives are shown below leaves.

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