Born’s rule from almost nothing

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Abstract

We here put forward a simple argument for Born’s rule based on the requirement that the probability distribution should not be a function of the number of degrees of freedom.

1 Introduction

Quantum mechanics does not make definite predictions but only predicts probabilities for measurement outcomes. One calculates these probabilities from the wave-function using Born’s rule. In axiomatic formulations of quantum mechanics, Born’s rule is usually added as an axiom on its own right. However, it seems the kind of assumption that should not require a postulate, but that should instead follow from the physical properties of the theory.

Since the early days of quantum mechanics, there have thus been many attempts to derive Born’s rule from other assumptions: non-contextuality [1], decision theory [2], non-triviality [3], and the number of degrees of freedom in composite systems [4, 5, 6]. The argument discussed here is most similar to the ones for many worlds presented in [8, 9] and the one using environment-assisted invariance [7]. However, as will become clear shortly, the ontological baggage of these arguments is unnecessary.

2 Argument

To get started, let $|\Psi\rangle$ and $|\Phi\rangle$ be elements of a vector space over the complex numbers which denote points on the complex sphere of dimension $N \in \mathbb{N}^+$, i.e. $|\langle \Psi | \Psi \rangle| = 1$ and $|\langle \Phi | \Phi \rangle| = 1$.

Claim: The only well-defined and consistent distribution for transition probabilities $P_N(|\Psi\rangle \rightarrow |\Phi\rangle)$ on the complex sphere of dimension $N$ which is continuous, independent of $N$, and invariant under unitary operations is $P_N(|\Psi\rangle \rightarrow |\Phi\rangle) = |\langle \Psi | \Phi \rangle|^2$. The continuity assumption is unnecessary if one restricts the original space to states of norm $K/N$ or, correspondingly, to rational-valued probabilities as a frequentist interpretation would suggest.

By $P_N$ being a well-defined probability distribution we mean

$$P_N(|\Psi\rangle \rightarrow |\Phi\rangle) \in \mathbb{R} \land 0 \leq P_N(|\Psi\rangle \rightarrow |\Phi\rangle) \leq 1 \ \forall \ |\Psi\rangle, |\Phi\rangle.$$ (1)
By $P_N$ being consistent, we mean
\begin{equation}
\sum_{i=1}^{N} P_N(|\Psi^i\rangle \rightarrow |\Psi^j\rangle) = 1 \quad \forall \ |\Psi\rangle \ , \ \langle \Psi^i|\Psi^j\rangle = \delta^{ij} , \quad (2)
\end{equation}
\begin{equation}
P_N(|\Psi^i\rangle \rightarrow |\Psi^j\rangle) = \delta^{ij} \quad \text{for} \quad \langle \Psi^i|\Psi^j\rangle = \delta^{ij} . \quad (3)
\end{equation}

**Proof:** Since $P_N$ is invariant under unitary operations, transition probabilities can only be functions of scalar products, i.e $P_N(|\Psi\rangle \rightarrow |\Phi\rangle) = P_N(|\Psi|\Phi\rangle) \forall |\Psi\rangle, |\Phi\rangle$. This means that $P_N$ is actually a map from the complex unit sphere of dimension 1 to the interval $[0,1] \in \mathbb{R}$. It follows from (3) that $P(|\Psi|\Phi\rangle) = 0$ whenever $\langle \Psi|\Phi\rangle = 0$, because if $|\Psi\rangle$ and $|\Phi\rangle$ are orthogonal, one could construct a basis of which they are elements. This means $P(0) = 0$. That $P_N$ is independent of $N$ then just means that this function is the same for all $N$, so we will from now on omit the index $N$.

Let $|\Psi^i\rangle, i \in \{1...N\}$ be an arbitrary orthonormal basis and
\begin{equation}
|\Psi^r\rangle := e^{i\theta} \sqrt{\frac{1}{N}} \sum_{i=1}^{N} |\Psi^i\rangle , \quad (4)
\end{equation}
where $\theta$ is a real number in $[0,2\pi]$. Because of (2) we then have
\begin{equation}
\sum_{j=1}^{N} P(\langle \Psi^j|\Psi^r\rangle) = NP(e^{i\theta}/\sqrt{N}) = 1 , \quad (5)
\end{equation}
which means
\begin{equation}
P(e^{i\theta}/\sqrt{N}) = 1/N . \quad (6)
\end{equation}

Next we use a new basis, $|\bar{\Psi}^i\rangle$ with $i \in \{1...N\}$. With $K \in \mathbb{N}^+, K < N$ we define
\begin{equation}
|\bar{\Psi}^j\rangle := \sqrt{\frac{1}{K}} \sum_{l=1}^{K} \exp\left(-2\pi i \frac{(l-1)(j-1)}{K}\right) |\Psi^l\rangle \quad \text{for} \quad 1 \leq j \leq K ,
\end{equation}
\begin{equation}
|\bar{\Psi}^j\rangle := |\Psi^j\rangle \quad \text{for} \quad K + 1 \leq j \leq N . \quad (7)
\end{equation}

The first $K$ basis-vectors are obviously orthogonal to the last $N-K$ and we already know that the last $N-K$ are orthonormal, so it remains to show that the first $K$ basis vectors are orthogonal. For this we note that for $j, m \leq K$
\begin{equation}
\langle \bar{\Psi}^j|\bar{\Psi}^m\rangle = \frac{1}{K} \sum_{l=1}^{K} \exp\left(-2\pi i \frac{m-j}{K}\right)^{l-1} . \quad (8)
\end{equation}
For $m = j$ the sum gives $= K$, so the norm is $= 1$ as it should be. For $m \neq j$ the sum is a geometric series and we have
\begin{equation}
\langle \bar{\Psi}^j|\bar{\Psi}^m\rangle = \frac{1 - \exp(-2\pi i (m-j))}{1 - \exp(-2\pi i (m-j)/K)} = 0 . \quad (9)
\end{equation}
The first basis vector $|\tilde{\Psi}^1\rangle$ is parallel to $|\Psi^*\rangle$ in the subspace spanned by the first $K$ basis vectors $|\Psi^i\rangle$, therefore the $|\tilde{\Psi}^j\rangle$'s for $2 \leq j \leq K$ are also orthogonal to $|\Psi^*\rangle$. With this, and using $P(0) = 0$, we get

$$\sum_{j=1}^{N} P(\langle \tilde{\Psi}^j | \Psi^* \rangle) = P(e^{i\theta} \sqrt{K/N}) + \sum_{j=K+1}^{N} P(e^{i\theta} / \sqrt{N}) = 1 .$$ (10)

And by using (6) we arrive at

$$P(e^{i\theta} \sqrt{K/N}) = K/N \quad \forall K, N \in \mathbb{N}^+, K \leq N, \theta \in [0, 2\pi[ .$$ (11)

If one was dealing with probabilities that only took on rational values, then one can stop here. For real-valued probabilities one merely notes that the only continuous function on the complex unit sphere with property (11) is $P(x) = |x|^2$, hence

$$P_N(|\Psi\rangle \rightarrow |\Phi\rangle) = |\langle \Psi | \Phi \rangle|^2 \quad \square$$ (12)

3 Discussion

The verbal summary of this argument is that since the probability distribution has to be invariant under unitary operations, we can permute all basis elements, which means that the probabilities of a symmetric state like $|\Psi^*\rangle$ have to be evenly distributed, which fixes them. Then we can combine any $K$-dimensional subset of these basis elements to one whose probability is just the sum of probabilities assigned to the previous $K$ basis elements. Again we can do that in arbitrary permutations and this fixes the probabilities for all $K/N$. The assumption that the distribution is independent of $N$ is essential; it is what allows us to fill in the sphere densely.

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