EXPONENTIAL STABILITY FOR A MULTI-PARTICLE SYSTEM WITH PIECEWISE INTERACTION FUNCTION AND STOCHASTIC DISTURBANCE

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Abstract. In this paper, a generalized Motsch-Tadmor model with piecewise interaction function is investigated, which can be viewed as a generalization of the model proposed in [9]. Our analysis bases on the connectedness of the underlying graph of the system. Some sufficient conditions are presented to guarantee the system to achieve flocking. Besides, we add a stochastic disturbance to the system and consider the flocking in the sense of expectation. As results, some criterions to the flocking solution with exponential convergent rate are established by the standard differential equations analysis.

1. Introduction. Collective phenomena of autonomous self-propelled agents are a universal phenomenon of nature, such as flocking of birds, swarming of bacteria, and the emergence of common language in primitive societies. It has appeared in many applications and theories, especially in artificial intelligence, physics and biology. In 2007, Cucker-Smale model [2, 3] was proposed, which is given by

\[
\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \varphi(|x_j - x_i|)(v_j - v_i).
\]

(1)

where \(\lambda (\lambda > 0)\) measures the interaction strength and \(\varphi(\cdot)\) represents the interaction. They gave sufficient frameworks of unconditional flocking and conditional flocking. In 2009, by using a Lyapunov function, Ha and Liu [7] presented a simple analysis about the system (1) and got the sufficient frameworks for flocking. Since then, several variants of Cucker-Smale model have been investigated, such as asymmetric interaction [15], hierarchical leadership [17, 10], the time lag arguments [11, 12, 13, 19] and randomly switching topologies [5], etc. In 2011, Motsch and Tadmor [15] proposed a asymmetric model:

\[
\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \frac{\lambda}{\sum_{j=1}^{N} \varphi(|x_j - x_i|)} \sum_{j=1}^{N} \varphi(|x_j - x_i|)(v_j - v_i).
\]

(2)

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Because of asymmetry, the momentum of the system (2) is not conserved. Motsch and Tadmor introduced the concept of active sets and showed that the diameters of the velocity and position satisfy a dissipative differential inequality. Then, they applied the Lyapunov functional approach to obtain the results. At the end of [15], they pointed out that the interaction decay rapidly or cut off at a finite distance is a more realistic situation.

In 2018, Chunyin Jin in [9] proposed a Motsch-Tadmor model with a cut-off interaction function:

\[
\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \frac{\lambda}{N_i(t)} \sum_{j=1}^{N} \chi_r(x_j - x_i)(v_j - v_i).
\] (3)

where \(N_i(t) = \text{card}\{j : l_{ij}(t) = |x_j(t) - x_i(t)| < r\}\) and \(\chi_r(s) = \begin{cases} 1, & |s| < r, \\ 0, & |s| \geq r. \end{cases}\)

Through the algebraic properties of the connected stochastic matrix, Jin got a sufficient condition that only depends on the model parameters and initial data to guarantee the connectedness of the neighbour graph related to the system. Then, he showed that the system achieves flocking at an exponential rate. Inspired by [9], we propose a generalization model of the above model:

\[
\begin{aligned}
\frac{dx_i}{dt} &= v_i, \\
\frac{dv_i}{dt} &= \frac{\lambda}{N_i(t)} \sum_{j \in N_i(t)} \chi_r(x_j - x_i)(v_j - v_i) \\
&\quad + \frac{\lambda}{N - N_i(t)} \sum_{j \notin N_i(t)} \chi_r(x_j - x_i)(v_j - v_i).
\end{aligned}
\] (4)

where

\[
N_i(t) = \{j : l_{ij}(t) = |x_j(t) - x_i(t)| < r\}, \quad N_i(t) = \text{card}(N_i(t)), \quad \chi_r(s) = \begin{cases} 1, & |s| < r, \\ \delta, & |s| \geq r. \end{cases}
\]

This is a linear model with piecewise interaction function, which can be further simplified to

\[
\begin{aligned}
\frac{dx_i}{dt} &= v_i, \\
\frac{dv_i}{dt} &= \frac{\lambda}{N_i(t)} \sum_{j \in N_i(t)} (v_j - v_i) + \frac{\lambda\delta}{N - N_i(t)} \sum_{j \notin N_i(t)} (v_j - v_i).
\end{aligned}
\] (5)

For any agents \(i, j (1 \leq i, j \leq N, i \neq j)\), if \(j \in N_i(t)\), we say \(j\) is the neighbour of \(i\), else we say \(j\) is the distant relative of \(i\) (i.e., \(j \notin N_i(t)\)). Thus, for each agent \(i\), the system (5) divides the remaining agents into two parts: neighbours and distant relatives.

**Remark 1.** The reason why we consider the above model is that we want to get a generalization result about [9]. In the above system, neighbours are always attracted to each other. However, distant relatives attract (\(\delta > 0\)) or repel (\(\delta < 0\)) each other. Thus, our research direction is to get the flocking condition of the system (5) with the changes of \(\delta\). In the subsequent proofs, we find a more general framework for
the system to achieve flocking. Specially, if $\delta \rightarrow 0$, our results become the classical results in [9].

The paper is organized as follows: In Section 2 we introduce the formulation of the model and some basic concepts. In Section 3 we present a sufficient conditions to ensure that system (5) achieves flocking. In Section 4 a stochastic disturbance is added to the system (5) and we consider the flocking in the sense of expectation. In Section 5, we give some numerical simulations to explain the importance of $\delta$. The conclusion is in the last.

2. Modelling Formulation and Preliminaries. Define the undirected graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where

$$\mathcal{V} = 1, 2, ..., N,$$

$$\mathcal{E}(t) = \{(i, j) : \l_{ij} := |x_i(t) - x_j(t)| < r, \ i, j \in \mathcal{V}\}.$$ 

We say $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ is the neighbour graph of the system (5). According to graph theory, a path in $\mathcal{G}(t)$ from $i$ to $j$ is a sequence of distinct vertexes $k_0 = i, k_1, ..., k_q = j \in \mathcal{V}$ such that $(k_{p-1}, k_p) \in \mathcal{E}(t)$ for every $1 \leq p \leq q$. $\mathcal{G}(t)$ is said to be connected at time $t$ if there is a path between any two vertexes of the graph at $t$. Then we denote the adjacency matrix, and the average matrix of the graph by

$$A(t) = (a_{ij}(t))_{N \times N}, \quad \text{and} \quad P(t) = (p_{ij}(t))_{N \times N},$$

where

$$a_{ij}(t) = \begin{cases} 1, & (i, j) \in \mathcal{E}(t), \\ 0, & (i, j) \notin \mathcal{E}(t), \end{cases} \quad \text{and} \quad p_{ij}(t) = \frac{a_{ij}(t)}{N_i(t)}.$$ 

(6)

Actually, $N_i(t) = \sum_{j=1}^{N} a_{ij}(t)$ is the number of neighbours. Define $N^c_i(t) = N - N_i(t)$, it is the number of distant relatives.

Similarly, we denote the distant relative graph of the system (5) by $\mathcal{G}^c(t) = (\mathcal{V}, \mathcal{E}^c(t))$, where

$$\mathcal{E}^c(t) = \{(i, j) : l_{ij} := |x_i(t) - x_j(t)| \geq r, \ i, j \in \mathcal{V}\}.$$ 

Its adjacency matrix and average matrix are

$$A^c(t) = (a_{ij}^c(t))_{N \times N}, \quad \text{and} \quad P^c(t) = (p_{ij}^c(t))_{N \times N},$$

respectively, where

$$a_{ij}^c(t) = \begin{cases} 0, & (i, j) \in \mathcal{E}(t), \\ 1, & (i, j) \notin \mathcal{E}(t), \end{cases} \quad \text{and} \quad p_{ij}^c(t) = \begin{cases} a_{ij}^c(t) \frac{N^c_i(t)}{N^c_i(t)}, & N^c_i(t) \neq 0, \\ 0, & N^c_i(t) = 0, \end{cases}$$ 

(7)

In the following content, we use $\mathcal{G}_0$ and $\mathcal{G}^c_0$ to represent the initial neighbour graph and initial distant relative graph associated with the system (5). And their average matrices are given by $P_0$ and $P^c_0$. Besides, we define

$$N_M = \max_i N_i(0), \quad N_m = \min_i N_i(0), \quad N^c_M = \max_i N^c_i(0), \quad N^c_m = \min_i N^c_i(0).$$

Following the matrix theory, $M = (m_{ij})_{N \times N}$ is called a stochastic matrix if

$$m_{ij} \geq 0 \quad \text{and} \quad \sum_{j=1}^{N} m_{ij} = 1, \quad i = 1, 2, ..., N.$$
For any $i,j (1 \leq i,j \leq N)$, there always exists a sequence of integers $k_1, k_2, \ldots, k_q (1 \leq q \leq N - 2)$, such that the entries of a matrix $m_{i,k_1}, m_{k_1,k_2}, \ldots, m_{k_q,j}$ are all non-zero, the matrix is called connected. From (6), we obtain

$$\sum_{j=1}^{N} p_{ij}(t) = 1, \quad \forall i = 1, 2, \ldots, N.$$  

So the average matrix $P(t)$ is a stochastic matrix at any time $t$. Hence, a neighbour graph $G(t)$ is connected at time $t$ if and only if its average matrix $P(t)$ is a connected stochastic matrix at time $t$. However, when $P(t)$ is a connected stochastic matrix, the average matrix $P^c(t)$ maybe neither connected nor a stochastic matrix. For instance, if there exists a agent $i_0$ such that $N_{c,i_0}^c(t) = 0$, then $\sum_{j=1}^{N} p_{i_0,j}^c(t) = 0$.

Define a new vector norm required in proofs. Let 

$$\|Z\| = \sup_{|\alpha| \neq 0} \frac{|Z\alpha|}{|\alpha|}, \quad \alpha \in \mathbb{R}^n.$$ 

be a norm of the real matrix $Z$. Then 

$$\|Z\|^2 = \sup_{|\alpha| \neq 0} \frac{\alpha^T Z^T Z \alpha}{|\alpha|^2}$$ 

is the largest eigenvalue of $Z^T Z$. Thus, If $Z$ is an orthogonal matrix $Z$, then $\|Z\| = 1$. Define 

$$V(t) = (v_1(t), v_2(t), \ldots, v_N(t))^T, \quad v_i(t) \in \mathbb{R}^n, 1 \leq i \leq N,$$ 

and its Euclid norm as 

$$\|V\|_2 = \left( \sum_{i=1}^{N} |v_i|^2 \right)^{\frac{1}{2}},$$ 

where $|v_i|$ is also the Euclid norm of $v_i$. From this definition and the Cauchy-Schwarz inequality, we can show that 

$$\|V\| \leq \|V\|_2 \leq \sqrt{n} \|V\|$$ 

Because the neighbour graph is sensitive to change in position, we introduce the following quantities, which play an important role in the analysis of neighbour graph. Define 

$$\Gamma := \min \left\{ r - \max_{(i,j) \in E(0)} l_{ij}(0), \min_{(i,j) \in E(0)} l_{ij}(0) - r \right\}, \quad (10)$$ 

and 

$$R_i := \operatorname{card} \{ j : (1 - \eta) r \leq l_{ij}(0) \leq (1 - \eta) r \}, \quad R_M := \max_i R_i,$$ 

where $l_{ij}(0) = |x_i(0) - x_j(0)|$ and $0 < \eta < 1$. Naturally, $\Gamma \geq 0$. If $\Gamma > 0$, this case is called non-critical neighbourhood situation. If $\Gamma \geq 0$, it is called general neighbourhood situation.

At the rest of this section, we list the definition of flocking and two lemmas.

**Definition 2.1.** [9] Suppose $(x_i(t), v_i(t)) \in \mathbb{R}^n \times \mathbb{R}^n (i = 1, 2, \ldots, N)$ is a solution to (5). The system (5) is said to achieve flocking if 

$$\lim_{t \to \infty} v_i(t) = \mathbf{v}, \quad i = 1, 2, \ldots, N,$$ 

where $\mathbf{v} \in \mathbb{R}^n$ is a constant vector.
Lemma 2.2. [9] If a stochastic matrix $Q = (q_{ij})_{N \times N}$ is connected, then $Q = TJT^{-1}$, where

$$J = \begin{pmatrix}
1 & J_2 & \cdots & J_k \\
J_2 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
J_k & \cdots & \cdots & 1
\end{pmatrix}, \quad \text{satisfies } J_i = \begin{pmatrix}
\mu_i & 1 & \cdots & 1 \\
0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \mu_i
\end{pmatrix}_{n_i \times n_i},$$

$s\sum_{i=2}^k n_i = N-1$, $\max_{2 \leq i \leq k} \Re \mu_i < 1$, and $T = (t_{ij})_{N \times N}$ satisfies $t_{i1} = a \neq 0$, $1 \leq i \leq N$.

Lemma 2.3. [9] Let $P = (p_{ij})_{N \times N}$ be the average matrix associated with an undirected graph. Then $P$ is similar to a diagonal matrix and all the eigenvalues of $P$ are real.

3. Analysis of the flocking behaviour. In this section, we consider two situations: the non-critical neighbourhood situation ($\Gamma > 0$) and the general neighbourhood situation ($\Gamma \geq 0$). With similar arguments in [9], we will show that our results hold in the interval $[0, t_0)$, and then extend $t_0$ to infinity.

3.1. Flocking behaviour in non-critical neighbourhood situation. Using Lemma 2.2 and Lemma 2.3, we achieve a key lemma given as follows.

Lemma 3.1. Let $G_0$ be the initial neighbour graph of the system (5), and its average matrix is $P_0 = (p_{ij})_{N \times N}$. $P_0^c = (p_{ij}^c)_{N \times N}$ is the average matrix of the distant relative graph. Assume $G_0$ is connected but not fully connected. Then there are matrices $S, U$ such that $P_0 = SJ_1S^{-1}, P_0^c = UJ_2U^{-1}$, where

$$J_1 = \begin{pmatrix}
1 & \mu_2I_{n_2} & \cdots & \mu_kI_{n_k} \\
0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
1 & \nu_2I_{m_2} & \cdots & \nu_kI_{m_k} \\
0 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{pmatrix},$$

satisfies $\sum_{i=2}^k n_i = N - 1$, $\sum_{j=2}^k m_j = N - 1$, $0 < |\mu_i| < 1$, $|\nu_j| < 1$, and $S = (s_{ij})_{N \times N}$ satisfies $\det S \neq 0, s_{i1} = a \neq 0, 1 \leq i \leq N$.

Proof. From Lemma 2.2 and Lemma 2.3, we know that there exists a non-degenerate matrix $S = (s_{ij})_{N \times N}$ with $s_{i1} = a \neq 0 (1 \leq i \leq N)$ and $P_0 = SJ_1S^{-1}$. Next we consider $P_0^c = (p_{ij}^c)_{N \times N}$.

If $N_m^c > 0$, then $P_0^c$ is a connected stochastic matrix. By using Lemma 2.2 and Lemma 2.3, there exists a non-degenerate matrix $U = (u_{ij})_{N \times N}$ satisfying $u_{i1} = b \neq 0, 1 \leq i \leq N$ and $P_0^c = UJ_2U^{-1}$.

If $N_m^c = 0$, then $P_0^c$ is neither connected nor a stochastic matrix. Let $E_{ij}$ is an elementary matrix obtained by swapping the $i$th row and the $j$th row of the identity matrix $E$. It is easy to know that $E_{ij}^TE_{ij} = I$. From elementary transformation, there exists a non-degenerate matrix $F$, which is the product of a finite number of $E_{ij}$, such that $P_0^c = FQF^{-1}$, where

$$Q = \begin{pmatrix}
Q_1 & O \\
O & O_1
\end{pmatrix}, \quad Q_1 = (q_{ij})_{N_1 \times N_1}, \quad O = \begin{pmatrix}0\end{pmatrix}_{N_2 \times N_2}, \quad N_1 + N_2 = N,$$
and $Q_1$ is a connected stochastic matrix. By Lemma 2.2 and Lemma 2.3, there exists $S_1$ such that $Q_1 = S_1J_3S_1^{-1}$, where

$$J_3 = \begin{pmatrix} 1 & \nu_2 I_{m_2} & \cdots & \nu_{s-1} I_{m_{s-1}} \\ & \ddots & \ddots & \ddots \\ & & 1 & \nu_{s-1} I_{m_{s-1}} \end{pmatrix}, \quad \sum_{i=2}^{s-1} m_i = N_1 - 1.$$

and $S_1 = (s_{ij})_{N_1 \times N_1}$ satisfies $s_{11} = b \neq 0, 1 \leq i \leq N_1$. According to the block matrix theory, we have

$$P_0^c = F \left( \begin{pmatrix} S_1 \\ I \end{pmatrix} \right) \left( \begin{pmatrix} J_3 & O \\ O & \end{pmatrix} \right) \left( \begin{pmatrix} S_1^{-1} \\ I \end{pmatrix} \right) F^{-1}.$$

Let $U = F \left( \begin{pmatrix} S_1 \\ I \end{pmatrix} \right) J_2 = \left( \begin{pmatrix} J_3 & O \\ O & \end{pmatrix} \right)$, then $P_0^c = UJ_2U^{-1}$. \qed

**Remark 2.** According to the connectedness of $G_0$, there are four situations:

- (1) $G_0$ is fully connected. In this case, $P_0$ is a connected stochastic matrix and $P_0^c = \{0\}^N_{N \times N}$. There exists a non-degenerate matrix $S$ such that $P_0 = S\text{diag}(1, 0, ..., 0)S^{-1}$.
- (2) $G_0$ is connected but not fully connected and $N_M = N$. Then $P_0$ is a connected stochastic matrix, $P_0^c$ is neither connected nor a stochastic matrix.
- (3) $G_0$ is connected and $N_M < N$. Then $P_0$ and $P_0^c$ are both connected stochastic matrices.
- (4) $G_0$ is not connected. In this case, $P_0$ is not connected and $P_0^c$ is a connected stochastic matrix. There exist non-degenerate matrices $S$ and $U$ such that $P_0 = S J_1^t S^{-1}$, $P_0^c = U J_2 U^{-1}$, where

$$J_1^t = \begin{pmatrix} I_{n_1} \\ \mu_2 I_{n_2} \\ \ddots \\ \mu_k I_{n_k} \end{pmatrix}, \quad \sum_{i=1}^{k} n_i = N, \quad 0 < |\mu_i| < 1.$$

and $U = (u_{ij})_{N \times N}$ satisfies $u_{1i} = b \neq 0, 1 \leq i \leq N$.

**Proposition 1.** Assume the neighbour graph $G(t)$ remains unchanged all the time, the initial graph $G_0$ is connected and $N_M < N$. Then

$$\|V(t) - \nabla_\infty\| \leq \frac{(1 - \mu_0)K}{1 - \mu_0 - |\delta|(1 - \nu_0)K K_e} \|V_0 - \nabla_0\| e^{-\lambda(1 - \mu_0 - |\delta|(1 - \nu_0)K K_e)t},$$

where

$$\nabla_0 = S\text{diag}(1, 0, ..., 0)S^{-1}V_0,$$

$$\nabla_\infty = \nabla_0 - \lambda \delta \int_0^\infty S\text{diag}(1, 0, ..., 0)S^{-1}(I - P_0^c)V(s)ds,$$

$$\mu_0 = \max_{2 \leq i \leq k} \mu_i, \quad \nu_0 = \min_{2 \leq j \leq s} \nu_j, \quad K = \sqrt{\frac{N_M}{N_M}}, \quad K_e = \sqrt{\frac{N_e}{N_M}}.$$

**Proof.** Since $G(t)$ remains unchanged all the time, the average matrices of the neighbour graph and the distant relative graph are constant matrices $P_0$ and $P_0^c$. What's
more, they are both connected stochastic matrices. Then the second equation of
(5) becomes
\[
\frac{dV}{dt} = -\lambda (I - P_0) V - \lambda \delta (I - P^c_0) V, \quad V(0) = V_0.
\]
(12)

By using the variation-of-constant formula, we obtain
\[
V(t) = \exp[-\lambda (I - P_0)t] V_0 - \lambda \delta \int_0^t \exp[-\lambda (I - P_0)(t - s)] (I - P^c_0)V(s)ds.
\]
(13)

According to Lemma 3.1, we have
\[
\exp[-\lambda (I - P_0)t] = S \begin{pmatrix}
1 & e^{-\lambda(1 - \mu_2)t} I_{n_2} & \cdots & e^{-\lambda(1 - \mu_k)t} I_{n_k}
\end{pmatrix} S^{-1}
\]
(14)

where 0 < |\mu_i| < 1. Set \( \mu_0 = \max_{2 \leq i \leq k} \mu_i \) and
\[
\nabla_0 : = S \text{diag}(1, 0, \ldots, 0) S^{-1} V_0,
\]
(15)
\[
\nabla(t) : = \nabla_0 - \lambda \delta \int_0^t S \text{diag}(1, 0, \ldots, 0) S^{-1} (I - P^c_0)V(s)ds.
\]
(16)

Because the entries in the first column of \( S \) are the same, it is easy to show that all
the rows of \( \nabla_0 \) and \( \nabla(t) \) are the same. Then the following two equations hold:
\[
\{ \exp[-\lambda (I - P_0)t] - S \text{diag}(1, 0, \ldots, 0) S^{-1} \} \nabla_0 = 0,
\]
\[
(I - P^c_0) \nabla(s) = 0.
\]

Combining (13) and (16), we get
\[
V(t) - \nabla(t)
= \{ \exp[-\lambda (I - P_0)t] - S \text{diag}(1, 0, \ldots, 0) S^{-1} \} V_0
- \lambda \delta \int_0^t \{ \exp[-\lambda (I - P_0)(t - s)] - S \text{diag}(1, 0, \ldots, 0) S^{-1} \} (I - P^c_0)V(s)ds
= \{ \exp[-\lambda (I - P_0)t] - S \text{diag}(1, 0, \ldots, 0) S^{-1} \} (V_0 - \nabla_0)
- \lambda \delta \int_0^t \{ \exp[-\lambda (I - P_0)(t - s)] - S \text{diag}(1, 0, \ldots, 0) S^{-1} \} (I - P^c_0)(V(s)
- \nabla(s))ds.
\]
(17)

Direct calculation gives
\[
\| \exp[-\lambda (I - P_0)t] - S \text{diag}(1, 0, \ldots, 0) S^{-1} \| \leq \| S \| \cdot \| S^{-1} \| e^{-\lambda(1 - \mu_0)t}.
\]
(18)

Applying Lemma 3.1 again, we have
\[
I - P^c_0 = U \begin{pmatrix}
0 & (1 - \nu_2) I_{m_2} & \cdots & (1 - \nu_k) I_{m_k}
\end{pmatrix} U^{-1}
\]
(19)

where 0 < |\nu_j| < 1. Let \( \nu_0 = \min_{2 \leq j \leq s} \nu_j \), then
\[
\| I - P^c_0 \| \leq (1 - \nu_0) \| U \| \cdot \| U^{-1} \|.
\]
(20)
Using (24) and (25), we conclude that
\[
\|S\| \cdot \|S^{-1}\| \leq \frac{1}{\sqrt{N_M}} \|Z\| \cdot \sqrt{N_M} \|Z^{-1}\| \leq \sqrt{\frac{N_M}{N_m}} =: K,
\] (21)
where \(Z\) is the orthogonal matrix to diagonalize the symmetric matrix
\[
\left(\sqrt{N_i(0)N_j(0)}\right)_{N \times N}.
\]
Similarly, we have
\[
\|U\| \cdot \|U^{-1}\| \leq \sqrt{\frac{N_M}{N_m}} =: K^c.
\] (22)
Thus, combining (17)-(22) yields
\[
e^{\lambda(1-\mu_0)t}\|V(t) - \overline{V}(t)\| \leq K\|V_0 - \overline{V}_0\| + \lambda|\delta|(1-\nu_0)KK^c \int_0^t e^{\lambda(1-\mu_0)s}\|V(s) - \overline{V}(s)\|ds.
\] (23)
By applying Gronwall’s inequality (23), we obtain
\[
\|V(t) - \overline{V}(t)\| \leq K\|V_0 - \overline{V}_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0)KK^c)t}.
\] (24)
Combining
\[
\overline{V}(t) = \overline{V}_0 - \lambda \delta \int_0^t S\text{diag}(1,0,...,0)S^{-1}(I - P_0^c)V(s)ds
\]
\[
= \overline{V}_0 - \lambda \delta \int_0^t S\text{diag}(1,0,...,0)S^{-1}(I - P_0^c)(V(s) - \overline{V}(s))ds.
\]
with (24), we deduce that there is
\[
\overline{V}_\infty := \overline{V}_0 - \lambda \delta \int_0^\infty S\text{diag}(1,0,...,0)S^{-1}(I - P_0^c)V(s)ds,
\]
such that
\[
\|\overline{V}(t) - \overline{V}_\infty\| \leq \lambda|\delta|(1-\nu_0)KK^c \int_t^\infty \|V(s) - \overline{V}(s)\|ds
\]
\[
\leq \lambda|\delta|(1-\nu_0)KK^c \int_t^\infty K\|V_0 - \overline{V}_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0)KK^c)t}ds
\]
\[
= \frac{|\delta|(1-\nu_0)KK^c}{1 - \mu_0 - |\delta|(1-\nu_0)KK^c} K\|V_0 - \overline{V}_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0)KK^c)t}.
\] (25)
Using (24) and (25), we conclude
\[
\|V(t) - \overline{V}_\infty\| \leq \frac{(1-\mu_0)K}{1 - \mu_0 - |\delta|(1-\nu_0)KK^c} \|V_0 - \overline{V}_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0)KK^c)t}.
\] (26)
This completes the proof. \(\square\)

**Remark 3.** In the above proposition, if \(1 - \mu_0 > |\delta|(1-\nu_0)KK^c\), then \(V(t)\) converges to \(\overline{V}_\infty\) at an exponential rate. Additionally, from Remark 3.1, we know that the above result belongs to the situation (3). In case (1), it is easy to show that \(V(t)\) also converges to \(\overline{V}_\infty\) at an exponential rate. However, in case (2) and case (4), our method is not working.
Using the above proposition, we obtain the sufficient conditions for the system (5) to achieve flocking in the non-critical neighbour situation.

**Theorem 3.2.** Let $G(t)$ be the neighbour graph related to the system (5). Assume $\Gamma > 0$, the initial neighbour graph $G_0$ is connected and the initial data satisfies

$$
||V_0 - \overline{V}_0|| \leq \frac{\lambda(1 - \mu_0 - \delta(1 - \nu_0)KK^c)^2}{\sqrt{2n(1 - \mu_0)K}} \Gamma, \quad N_M < N
$$

and 

$$
1 - \mu_0 - \delta(1 - \nu_0)KK^c > 0.
$$

Then the system (5) will achieve flocking with the convergent rate

$$
||V(t) - \overline{V}_\infty|| \leq \frac{(1 - \mu_0)K}{1 - \mu_0 - \delta(1 - \nu_0)KK^c} ||V_0 - \overline{V}_0|| e^{-\lambda(1 - \mu_0 - \delta(1 - \nu_0)KK^c)t}
$$

where

$$
V_\infty = V_0 - \lambda \delta \int_0^\infty \text{Sdiag}(1,0,...,0)S^{-1}(I - P^c_0)V(s)ds
$$

and $\mu_0, \nu_0, V_0, K, K^c$ are defined in (14), (19), (15), (21), (22).

**Proof.** From the continuity of $l_{ij}$, there is $t_0 > 0$ such that $G(t)$ keeps unchanged on $[0,t_0]$. Let $t^*$ be the supremum among all the above $t_0$. Then we claim $t^* = \infty$. Otherwise $t^* < \infty$. By the definition of supremum, there is $(i_0,j_0)$ such that $l_{i_0j_0}(t^*) = r$. According to the first equation of (5), we obtain

$$
x_i(t) - x_j(t) = x_i(0) - x_j(0) + \int_0^t [v_i(\tau) - v_j(\tau)]d\tau, \quad \forall t \in [0,t^*]. \quad (27)
$$

Combining (9) with Proposition 1, we infer that

$$
|v_i(\tau) - v_j(\tau)| \leq \sqrt{2}||V(\tau) - V_\infty||_2 \\
\leq \sqrt{2n}||V(\tau) - V_\infty|| \\
\leq \frac{\sqrt{2n}(1 - \mu_0)K}{1 - \mu_0 - \delta(1 - \nu_0)KK^c} ||V_0 - \overline{V}_0|| e^{-\lambda(1 - \mu_0 - \delta(1 - \nu_0)KK^c)t}.
$$

Then, using the equations (27), (28), (10) and the initial condition

$$
||V_0 - \overline{V}_0|| \leq \frac{\lambda(1 - \mu_0 - \delta(1 - \nu_0)KK^c)^2}{\sqrt{2n(1 - \mu_0)K}} \Gamma,
$$

we have

$$
l_{ij}(t^*) = |x_i(t^*) - x_j(t^*)| \\
\leq l_{ij}(0) + \frac{\sqrt{2n}(1 - \mu_0)K||V_0 - \overline{V}_0||}{1 - \mu_0 - \delta(1 - \nu_0)KK^c} \int_0^\infty e^{-\lambda(1 - \mu_0 - \delta(1 - \nu_0)KK^c)t}d\tau \\
< l_{ij}(0) + \Gamma \\
\leq r, \quad \forall (i,j) \in \mathcal{E}(0),
$$

and

$$
l_{ij}(t^*) = |x_i(t^*) - x_j(t^*)| \\
\geq l_{ij}(0) - \frac{\sqrt{2n}(1 - \mu_0)K||V_0 - \overline{V}_0||}{1 - \mu_0 - \delta(1 - \nu_0)KK^c} \int_0^\infty e^{-\lambda(1 - \mu_0 - \delta(1 - \nu_0)KK^c)t}d\tau \\
> l_{ij}(0) - \Gamma \\
\geq r, \quad \forall (i,j) \notin \mathcal{E}(0).
$$
Proof. Rewrite the second equation of the system (5) as

\[ (19) \]

Then the system (5) will achieve flocking and there exists

\[ \mathbf{V}_\infty = \mathbf{V}_0 + \lambda \int_0^\infty \text{diag}(1,0,...,0)S^{-1}(P - P_0)V(s)ds \]

\[
= -\lambda \delta \int_0^\infty \text{diag}(1,0,...,0)S^{-1}(I - P^c)V(s)ds.
\]

such that

\[ ||V(t) - \mathbf{V}_\infty|| \leq 2K||V_0 - \mathbf{V}_0||e^{-\frac{\lambda(1-\mu_0)}{2}} \]

where \( L = \frac{R_M(N_m + N_M)}{N_m(N_m - R_M)}, \quad L^c = \frac{R_M(N_m^c + N_M^c)}{N_m^c(N_m^c - R_M^c)}, \quad \mu_0, \nu_0, \mathbf{V}_0, K, K^c \) are defined in (14), (19), (15), (21), (22).

3.2. Flocking behaviour in general neighbourhood situation. In the case of general neighbourhood situation, we assume that the adjacency matrix does not change frequently and sharply, and consider the following condition.

Lemma 3.3. [9] Let \( G \) be an undirected graph, and \( \hat{G} \) is another undirected graph formed by changing the neighbours of \( G \). If for any vertex \( i(1 \leq i \leq N) \), the number of its neighbours changed \( \Delta N_i \) satisfies \( \Delta N_i \leq R_M < N_m \), then the corresponding average matrices \( P \) and \( \hat{P} \) satisfy

\[ ||P - \hat{P}|| \leq \frac{R_M(N_m + N_M)}{N_m(N_m - R_M)} \]

Theorem 3.4. Let \( G(t) \) be the neighbour graph of the system (5). Assume the initial neighbour graph \( G_0 \) is connected and the initial data satisfies

\[ (H_1) : K(L + |\delta|(1 - \nu_0)K^c + |\delta|L^c) < \frac{1 - \mu_0}{2}, \quad \min\{N_m, N_m^c\} > R_M; \]

\[ (H_2) : ||V_0 - \mathbf{V}_0|| < \frac{\lambda(1 - \mu_0)\eta}{2\sqrt{2\mu K}}, \quad 0 < \eta < 1. \]

Then the system (5) will achieve flocking and there exists

\[ \mathbf{V}_\infty = \mathbf{V}_0 + \lambda \int_0^\infty \text{diag}(1,0,...,0)S^{-1}(P - P_0)V(s)ds \]

\[ -\lambda \delta \int_0^\infty \text{diag}(1,0,...,0)S^{-1}(I - P^c)V(s)ds. \]

Proof. Rewrite the second equation of the system (5) as

\[
\begin{align*}
\frac{dV}{dt} &= -\lambda(I - P_0)V + \lambda(P - P_0)V - \lambda \delta(I - P^c)V, \\
V(0) &= V_0.
\end{align*}
\]

By the variation-of-constant formula, we obtain

\[
V(t) = \exp[-\lambda(I - P_0)t]V_0 + \lambda \int_0^t \exp[-\lambda(I - P_0)(t - s)](P - P_0)V(s)ds \]

\[ -\lambda \delta \int_0^t \exp[-\lambda(I - P_0)(t - s)](I - P^c)V(s)ds. \]

Define

\[
\hat{V}(t) := \mathbf{V}_0 + \lambda \int_0^t \text{diag}(1,0,...,0)S^{-1}(P - P_0)V(s)ds \]

\[ -\lambda \delta \int_0^t \text{diag}(1,0,...,0)S^{-1}(I - P^c)V(s)ds. \]
Using (32) and (33), we have
\[
V(t) - \nabla(t) = \{\exp[-\lambda(I - P_0)t] - \text{diag}(1, 0, ..., 0)S^{-1}\} V_0
\]
\[
+ \lambda \int_0^t \{\exp[-\lambda(I - P_0)(t-s)] - \text{diag}(1, 0, ..., 0)S^{-1}\} (P - P_0)V(s)ds
\]
\[
- \lambda \delta \int_0^t \{\exp[-\lambda(I - P_0)(t-s)] - \text{diag}(1, 0, ..., 0)S^{-1}\} (I - P^c)V(s)ds
\]
\[
= \{\exp[-\lambda(I - P_0)t] - \text{diag}(1, 0, ..., 0)S^{-1}\} (V_0 - \nabla_0)
\]
\[
+ \lambda \int_0^t \{\exp[-\lambda(I - P_0)(t-s)] - \text{diag}(1, 0, ..., 0)S^{-1}\} (P - P_0)(V(s) - \nabla(s))ds
\]
\[
- \nabla(s)ds - \lambda \delta \int_0^t \{\exp[-\lambda(I - P_0)(t-s)] - \text{diag}(1, 0, ..., 0)S^{-1}\} (I - P^c)
\]
\[
(V(s) - \nabla(s))ds. \tag{34}
\]
Combining (34) with (18) yields
\[
e^{\lambda(1-\mu_0)t}\|V(t) - \nabla(t)\| \leq K\|V_0 - \nabla_0\|
\]
\[
+ \lambda K \int_0^t e^{\lambda(1-\mu_0)s}\|V(s) - \nabla(s)\| \cdot \|P - P_0\|ds
\]
\[
+ \lambda \delta K \int_0^t e^{\lambda(1-\mu_0)s}\|V(s) - \nabla(s)\| \cdot \|I - P^c\|ds. \tag{35}
\]

Using the continuity of \(l_{ij}(t)\), there is \(t_0 > 0\) such that
\[
|l_{ij}(t) - l_{ij}(0)| < \eta r, \quad \forall i, j \quad t \in [0, t_0).
\]
Let \(t^*\) be the supremum among all the above \(t_0\). Then we claim \(t^* = \infty\). If not, \(t^* < \infty\). From the definition of supremum, there is \((i_0, j_0)\) such that
\[
|l_{i_0j_0}(t^*) - l_{i_0j_0}(0)| = \eta r. \tag{36}
\]
From (11) and Lemma 3.3, we get
\[
\|P(s) - P_0\| \leq \frac{R_M(N_m + d_M)}{N_m(N_m - R_M)} =: L, \quad s \in [0, t^*].
\]
\[
\|P^c(s) - P^c_0\| \leq \frac{R_M(N_m^c + N_M^c)}{N_m^c(N_m^c - R_M)} =: L^c, \quad s \in [0, t^*]. \tag{37}
\]
Combining (20) with (37) yields
\[
\|I - P^c(s)\| \leq \|I - P^c_0\| + \|P^c(s) - P^c_0\| \leq (1 - \nu_0)K^c + L^c, \quad s \in [0, t^*]. \tag{38}
\]
Hence, using (37), (38), initial condition \((H_1)\) and applying the Gronwall Inequality (35) yields
\[
\|V(t) - \nabla(t)\| \leq K\|V_0 - \nabla_0\|e^{-\frac{\lambda(1-\mu_0)t}{2}}, \quad t \in [0, t^*]. \tag{39}
\]
According to the first equation of the system (5), we obtain
\[
x_i(t) - x_j(t) - (x_i(0) - x_j(0)) = \int_0^t [v_i(\tau) - v_j(\tau)]d\tau, \quad \forall t \in [0, t^*].
\]
Combining (10), (37), and the initial condition (H2), we know
\[ |l_{ij}(t^*) - l_{ij}(0)| \leq |x_i(t^*) - x_j(t^*) - (x_i(0) - x_j(0))| \]
\[ \leq \int_0^{t^*} |v_i(\tau) - v_j(\tau)|d\tau \]
\[ \leq \sqrt{2n} \int_0^{\infty} K\|V_0 - V_0\|e^{-\frac{\lambda(1-\mu_0)}{2}t}d\tau \]
(40)
which contradicts (36). Hence \( t^* = \infty \). Then
\[ \|V(t) - V(t)\| \leq K\|V_0 - V_0\|e^{-\frac{\lambda(1-\mu_0)t}{2}}, \ t \in [0, \infty). \] (41)
Combining
\[ V(t) = V_0 + \lambda \int_0^t S\text{diag}(1, 0, ..., 0)S^{-1}(P - P_0)V(s)ds \\
- \lambda\delta \int_0^t S\text{diag}(1, 0, ..., 0)S^{-1}(I - P^c)V(s)ds \]
\[ = V_0 + \lambda \int_0^t S\text{diag}(1, 0, ..., 0)S^{-1}(P - P_0)(V(s) - V(s))ds \\
- \lambda\delta \int_0^t S\text{diag}(1, 0, ..., 0)S^{-1}(I - P^c)(V(s) - V(s))ds. \]
with (41), we deduce that there is
\[ V_\infty := V_0 + \lambda \int_0^\infty S\text{diag}(1, 0, ..., 0)S^{-1}(P - P_0)V(s)ds \\
- \lambda\delta \int_0^\infty S\text{diag}(1, 0, ..., 0)S^{-1}(I - P^c)V(s)ds. \]
satisfying
\[ \|V(t) - V_\infty\| \leq \lambda K(L + |\delta|)\|V_0 - V_0\|e^{-\frac{\lambda(1-\mu_0)t}{2}}ds \]
\[ \leq \frac{\lambda(1-\mu_0)}{2} \int_0^\infty K\|V_0 - V_0\|e^{-\frac{\lambda(1-\mu_0)t}{2}}ds \]
(42)
Using (41) and (42), we conclude
\[ \|V(t) - V_\infty\| \leq 2K\|V_0 - V_0\|e^{-\frac{\lambda(1-\mu_0)t}{2}}. \] (43)

4. Flocking behaviour with stochastic disturbance. In this section, we focus on the effect of the noise on flocking behaviour. In this field, many scholars study the influence of stochastic disturbances that are typically modelled in terms of white noise, see [4, 8, 1, 18, 16, 6]. In current work, we consider a noise in the measurement [6]. Assume the velocity of agent \( j \) measured by agent \( i \) is given by the expression
\[ \omega_{ij} = v_j + \kappa_i(v_j - v_i)dw_i, \ i, j = 1, 2, ..., N, \]
where $W(t) = (w_1(t), w_2(t), ..., w_N(t))$ is the $N$-dimensional Brownian motions, and $\kappa_i \geq 0$ is the imprecision of $i$’s measurement device. Adding this stochastic disturbance to the system (5), we obtain follow model:

$$
\begin{align*}
\text{d}x_i &= v_i \text{d}t, \\
\text{d}v_i &= \left[ \frac{\lambda}{\tilde{N}_i(t)} \sum_{j \in \bar{N}_i(t)} (v_j - v_i) + \frac{\lambda \delta}{N - \tilde{N}_i(t)} \sum_{j \notin \bar{N}_i(t)} (v_j - v_i) \right] \text{d}t \\
&\quad + \left[ \frac{\sigma_i}{\tilde{N}_i(t)} \sum_{j \in \bar{N}_i(t)} (v_j - v_i) + \frac{\sigma_i \delta}{N - \tilde{N}_i(t)} \sum_{j \notin \bar{N}_i(t)} (v_j - v_i) \right] \text{d}w_i,
\end{align*}
$$

(44)

where $\tilde{N}_i(t) = \{ j : l_{ij}(t) := |\mathbb{E} [x_j(t)] - \mathbb{E} [x_i(t)] | < \delta \} \text{ and } \tilde{N}_i(t) = \text{card}(\bar{N}_i(t))$.

Note that if $\sigma_i = 0, \forall i$, the system (44) is reduced to the system (5).

We denote the neighbour graph of the system (44) by $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where

$$
\mathcal{V} = 1, 2, ..., N,
\mathcal{E}(t) = \{(i, j) : l_{ij}(t) = |\mathbb{E} [x_j(t)] - \mathbb{E} [x_i(t)] | < \delta \}, \ i, j \in \mathcal{V}.
$$

Similarly, let $\mathcal{G}^e(t) = (\mathcal{V}, \mathcal{E}^e(t))$ be the distant relative graph of the system (44)

$$
\mathcal{E}^e(t) = \{(i, j) : l_{ij}(t) = |\mathbb{E} [x_j(t)] - \mathbb{E} [x_i(t)] | \geq \delta \}, \ i, j \in \mathcal{V}.
$$

Besides, the definitions of other notations are same as those in section 2 and 3.

Finally, we give the definition of flocking in sense of expectation.

**Definition 4.1.** Assume $(x_i(t), v_i(t)) \in \mathbb{R}^n \times \mathbb{R}^n (i = 1, 2, ..., N)$ is the solution of the system (44). We say the system (44) achieve flocking if

$$
\lim_{t \to \infty} \mathbb{E} [v_i(t)] = \overline{v}, \quad i = 1, 2, ..., N,
$$

where $\overline{v} \in \mathbb{R}^n$ is a constant vector, $\mathbb{E} [\cdot]$ denotes the expectation of a stochastic process.

In the following, we obtain the sufficient condition to guarantee the system (44) achieve flocking by using the same strategy in section 3.

**Proposition 2.** Assume the neighbour graph $\mathcal{G}(t)$ remains unchanged all the time, the initial graph $\mathcal{G}_0$ is connected and $N_M < N$. Then there exists

$$
\mathcal{V}_\infty = \mathcal{V}_0 - \lambda \delta \int_0^\infty \text{Sdiag}(1, 0, ..., 0) S^{-1} (I - P_0^e) \mathbb{E} [V(s)] \text{d}s
$$

such that

$$
\| \mathbb{E} [V(t)] - \mathcal{V}_\infty \| \leq \frac{(1 - \mu_0) K}{1 - \mu_0 - |\delta(1 - \nu_0) K K^c|} \| \mathcal{V}_0 - \mathcal{V}_0 \| e^{-\lambda \mu_0 - |\delta(1 - \nu_0) K K^c| t},
$$

where $\mu_0, \nu_0, \overline{v}_0, K, K^c$ are defined in (14), (19), (15), (21), (22).

**Proof.** From assumptions, the average matrices of the neighbour graph and the distant relative graph are constant matrices $P_0$ and $P_0^e$. Besides, they are both connected stochastic matrices. Then the second equation of (44) becomes

$$
\begin{align*}
\text{d}V &= -[\lambda (I - P_0) V + \lambda \delta (I - P_0^e) V] \text{d}t \\
&\quad - [\Theta (I - P_0) V + \delta \Theta (I - P_0^e) V] \text{d}W,
\end{align*}
$$

(45)

\begin{align*}
V(0) &= V_0.
\end{align*}
where $\Theta = diag(\sigma_1, \sigma_2, ..., \sigma_N)$. According to the properties of Itô formula:

$$dt \cdot dt = 0, \quad dt \cdot dW = 0 \quad \text{and} \quad dW \cdot dW = dt,$$

we have

$$d(\exp[-\lambda(I - P_0)t]V) = -\lambda(I - P_0) \exp[-\lambda(I - P_0)t]V dt + \exp[-\lambda(I - P_0)t]dV$$

$$- \lambda(I - P_0) \exp[-\lambda(I - P_0)t]dt \cdot dV$$

$$= -\lambda(I - P_0) \exp[-\lambda(I - P_0)t]V dt + \exp[-\lambda(I - P_0)t]dV.$$

Solving (45) by method of integrating factor yields

$$V(t) = \exp[-\lambda(I - P_0)t]V_0 - \lambda \delta \int_0^t \exp[-\lambda(I - P_0)(t - s)](I - P_0^c)V(s)ds$$

$$- \int_0^t \exp[-\lambda(I - P_0)(t - s)] [\Theta(I - P_0)V + \delta \Theta(I - P_0)V]dW. \tag{46}$$

Using $E \left[ \int_0^t f(s)dW \right] = 0$ (see [14]), we obtain

$$E[V(t)] = \exp[-\lambda(I - P_0)t]V_0 - \lambda \delta \int_0^t \exp[-\lambda(I - P_0)(t - s)](I - P_0^c)E[V(s)]ds. \tag{47}$$

By (14), let

$$\nabla(t) := \nabla_0 - \lambda \delta \int_0^t Sdiag(1, 0, ..., 0)s^{-1}(I - P_0^c)V(s)ds.$$

then

$$E[\nabla(t)] = \nabla_0 - \lambda \delta \int_0^t Sdiag(1, 0, ..., 0)s^{-1}(I - P_0^c)E[V(s)]ds. \tag{48}$$

Thus, combining (47) and (48) yields

$$E[V(t)] - E[\nabla(t)]$$

$$= \left\{ \exp[-\lambda(I - P_0)t] - Sdiag(1, 0, ..., 0)S^{-1} \right\} V_0$$

$$- \lambda \delta \int_0^t \left\{ \exp[-\lambda(I - P_0)(t - s)] - Sdiag(1, 0, ..., 0)S^{-1} \right\} (I - P_0^c)E[V(s)]ds$$

$$= \left\{ \exp[-\lambda(I - P_0)t] - Sdiag(1, 0, ..., 0)S^{-1} \right\} (V_0 - \nabla_0)$$

$$- \lambda \delta \int_0^t \left\{ \exp[-\lambda(I - P_0)(t - s)] - Sdiag(1, 0, ..., 0)S^{-1} \right\} (I - P_0^c)$$

$$(E[V(s)] - E[\nabla(s)])ds. \tag{49}$$

Using (18), (20) and (49), we have

$$e^{\lambda(1-\mu_0)t}\|E[V(t)] - E[\nabla(t)]\|$$

$$\leq K\|V_0 - \nabla_0\| + \lambda \delta(1 - \nu_0)KK^c \int_0^t e^{\lambda(1-\mu_0)s}\|E[V(s)] - E[\nabla(s)]\|ds. \tag{50}$$

Applying Gronwall’s Inequality (50) yields

$$\|E[V(t)] - E[\nabla(t)]\| \leq K\|V_0 - \nabla_0\|e^{-\lambda(1-\mu_0) - \delta(1-\nu_0)KK^c)t}. \tag{51}$$
Then, combining
\[ E[V(t)] = \nabla_0 - \lambda \delta \int_0^t \text{Sdiag}(1,0,...,0)S^{-1}(I-P_0^{c})E[V(s)]\, ds \]
\[ = \nabla_0 - \lambda \delta \int_0^t \text{Sdiag}(1,0,...,0)S^{-1}(I-P_0^{c})(E[V(s)] - E[\nabla(s)])\, ds, \]
with (51), we infer that there is
\[ \nabla_\infty := \nabla_0 - \lambda \delta \int_0^\infty \text{Sdiag}(1,0,...,0)S^{-1}(I-P_0^{c})E[V(s)]\, ds. \]
such that
\[ \|E[V(t)] - \nabla_\infty\| \leq \lambda|\delta|(1 - \nu_0)KK^c \int_t^\infty \|E[V(s)] - E[\nabla(s)]\|\, ds \]
\[ \leq \lambda|\delta|(1 - \nu_0)KK^c \int_t^\infty K\|V_0 - \nabla_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0))K^c} \, ds \]
\[ = \frac{|\delta|(1 - \nu_0)KK^c}{1 - \mu_0 - |\delta|(1 - \nu_0)K^c} K\|V_0 - \nabla_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0))K^c} t. \]  
(52)

From (51) and (52), we conclude
\[ \|E[V(t)] - \nabla_\infty\| \leq \frac{(1 - \mu_0)K}{1 - \mu_0 - |\delta|(1 - \nu_0)K^c} \|V_0 - \nabla_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0))K^c} t. \]  
(53)

This completes the proof. \ \Box

From the proof of the above proposition, it is easy to know that the expectation of system (44) is similar to system (5). Thus, we give sufficient conditions to achieve flocking without proofs.

**Theorem 4.2.** Let \( \mathcal{G}(t) \) be the neighbour graph related to the system (44). Assume \( \Gamma > 0 \), the initial neighbour graph \( \mathcal{G}_0 \) is connected and the initial data satisfies
\[ \|V_0 - \nabla_0\| \leq \lambda(1-\mu_0-|\delta|(1-\nu_0)K^c)^2 \sqrt{\frac{2n(1 - \mu_0)}{1 - \mu_0 - |\delta|(1 - \nu_0)K^c}} \Gamma, \quad N_M < N \]
and \( 1 - \mu_0 - |\delta|(1 - \nu_0)K^c > 0. \) Then the system (44) will achieve flocking with the convergent rate
\[ \|E[V(t)] - \nabla_\infty\| \leq \frac{(1 - \mu_0)K}{1 - \mu_0 - |\delta|(1 - \nu_0)K^c} \|V_0 - \nabla_0\|e^{-\lambda(1-\mu_0-|\delta|(1-\nu_0))K^c} t \]
where
\[ \nabla_\infty = \nabla_0 - \lambda \delta \int_0^\infty \text{Sdiag}(1,0,...,0)S^{-1}(I-P_0^{c})E[V(s)]\, ds \]
and \( \mu_0, \nu_0, \nabla_0, K, K^c \) are defined in (14), (19), (15), (21), (22).

**Proof.** Refer to the proof of Theorem 3.2. \ \Box

**Theorem 4.3.** Let \( \mathcal{G}(t) \) be the neighbour graph of the system (44). Assume the initial neighbour graph \( \mathcal{G}_0 \) is connected and the initial data satisfies
\[ (L_1) : K(L + |\delta|(1 - \nu_0)K^c + |\delta|L^c) < \frac{1 - \mu_0}{2}, \quad \min \{N_m, N_m^c\} > R_M; \]
\[ (L_2) : \|V_0 - \nabla_0\| < \frac{\lambda(1-\mu_0)\eta^p}{2\sqrt{2nK}}, \quad 0 < \eta < 1. \]
Then the system (44) will achieve flocking with the convergent rate
\[
\|\mathbb{E}[V(t)] - V_\infty\| \leq 2K\|V_0 - V_0\|e^{-\frac{\lambda(1-\mu_0)t}{2}}
\]
where
\[
V_\infty = V_0 + \lambda \int_0^\infty S\text{diag}(1, 0, ..., 0)S^{-1}(P - P_0)\mathbb{E}[V(s)] \, ds
\]
\[
- \lambda \delta \int_0^\infty S\text{diag}(1, 0, ..., 0)S^{-1}(I - P^c)\mathbb{E}[V(s)] \, ds.
\]
and $\mu_0, \nu_0, V_0, K, K^c, L, L^c$ are defined in (14), (19), (15), (21), (22), (37).

Proof. Refer to the proof of Theorem 3.4.

5. Numerical simulation. In this section, we consider 10-agents in the system (5) and the system (44) respectively. The initial data is as follows:

| $x_1(0)$ | $x_2(0)$ | $x_3(0)$ | $x_4(0)$ | $x_5(0)$ | $x_6(0)$ | $x_7(0)$ | $x_8(0)$ | $x_9(0)$ | $x_{10}(0)$ |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2.9144  | 4.1151  | 5.2503  | 5.5936  | 3.2668  | 8.3814  | 5.6265  | 9.8483  | 0.9512  | 8.1604  |
| $v_1(0)$ | $v_2(0)$ | $v_3(0)$ | $v_4(0)$ | $v_5(0)$ | $v_6(0)$ | $v_7(0)$ | $v_8(0)$ | $v_9(0)$ | $v_{10}(0)$ |
| 0.2760  | 0.6797  | 0.6551  | 0.1626  | 0.1190  | 0.4984  | 0.9597  | 0.3404  | 0.5853  | 0.2238  |

where the positions are randomly selected in (0,10), the velocities are randomly selected in (0,1).

For the system (5) and (44), we have

![Figure 1](image)

**Figure 1.** $\lambda = 3$, $\delta = 0$ and $r = 1$. In this case, only the neighbours attract each other. Because the value of $r$ is small, the system does not achieve flocking.

**Remark 4.** From Figs.1-2, when we increase the value of $\delta \geq 0$, the system changes from non-flocking to flocking. Actually, if $\delta > 0$, the flocking behaviour will emerge unconditional to initial data. By Fig.1 and Fig.3, we can also augment the radius $r$ to make the system achieve flocking. According to Figs.3,4,5,6, as $\delta \leq 0$ decreases, the flocking system becomes non-flocking. Even more interesting is that if $|\delta|$ is small enough, the system maintains flocking behaviour (see Fig.7), which is consistent with our conclusions.
Figure 2. $\lambda = 3$, $\delta = 0.1$ and $r = 1$. In this case, not only the neighbours attract each other but also the distant relatives attract each other in a weak force. Hence unconditional flocking occurs.

Figure 3. $\lambda = 3$, $\delta = 0$ and $r = 2$. The system achieves flocking because of a bigger value of $r$ than Fig. 1.

6. Conclusion. We analyse the flocking behaviour of a generalized Motsch-Tadmor model with piecewise interaction function. Some sufficient conditions achieving flocking are obtained. Our results are extensional versions related to [9]. They both depend only on the model parameters and initial date, which are easy to verify. Additionally, we consider the influence of stochastic disturbance and get a sufficient condition ensuring flocking in the sense of expectation. In follow-up work, we will discuss the influence of time delay on the system (5), and study the critical delay that ensures the system achieving flocking and the emergence of periodic flocking.

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Figure 4. $\lambda = 3$, $\delta = -0.1$ and $r = 2$. In this case, the neighbours attract each other however the distant relatives repel each other in a weak force. The system also achieves flocking.

Figure 5. $\lambda = 3$, $\delta = -0.3$ and $r = 3$. By increasing the repulsive force between distant relatives, the system does not achieve flocking.

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Figure 6. $\lambda = 3$, $\delta = -0.5$ and $r = 3$. Continue to increase the repulsive force between distant relatives, the system still does not achieve flocking.

Figure 7. $\lambda = 3$, $\delta = -0.1$, $\sigma_i = 2(i = 1, 2, \ldots, n)$ and $r = 2$. The system achieves flocking under a moderate stochastic disturbance.

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