AFFINE SLICE FOR THE COADJOINT ACTION OF A CLASS OF BIPARABOLIC SUBALGEBRAS OF A SEMISIMPLE LIE ALGEBRA

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ABSTRACT. In this article, we give a simple explicit construction of an affine slice for the coadjoint action of a certain class of biparabolic (also called seaweed) subalgebras of a semisimple Lie algebra over an algebraically closed field of characteristic zero. In particular, this class includes all Borel subalgebras.

1. INTRODUCTION

1.1. Throughout this paper, \( k \) is an algebraically closed field of characteristic zero, All vector spaces and Lie algebras considered are defined over \( k \). We consider the Zariski topology on these spaces. If \( X \) is an algebraic variety and \( x \in X \), we denote by \( T_x(X) \) the tangent space of \( X \) at \( x \).

Let \( g \) be a finite-dimensional Lie algebra over \( k \) and \( G \) its algebraic adjoint group. Recall that \( G \) and \( g \) act naturally on \( g^* \) via the coadjoint action. More precisely, for \( f \in g^* \), \( \sigma \in G \) and \( X, Y \in g \), we have

\[
(\sigma.f)(Y) = f(\sigma^{-1}(Y)) \quad (X.f)(Y) = f([Y, X]).
\]

An affine slice for the coadjoint action of \( g \) is an affine subspace \( V \) of \( g^* \) such that there exists an open subset \( O \) of \( V \) verifying the following conditions:

- (C1) The set \( G.O \) is dense in \( g^* \).
- (C2) For all \( f \in O \), we have \( T_f(G.f) \cap T_f(O) = \{0\} \).
- (C3) For all \( f \in O \), we have \( G.f \cap O = \{f\} \).

An affine slice may not exist, but when it does, we can deduce (using Rosenlicht’s Theorem for example, see Theorem 3.3.1) that the field of \( G \)-invariant rational functions on \( g^* \) is a purely transcendental extension of the ground field \( k \).

If \( g \) is abelian, then \( g^* \) is an affine slice. On the other hand, when \( g \) is a semisimple Lie algebra, we may identify \( g \) with \( g^* \) via the Killing form, and such a slice has been constructed by Kostant [4] by using a principal \( S \)-triple. Kostant [5] also constructed an affine slice for the nilpotent radical of a Borel subalgebra of a semisimple Lie algebra.

1.2. Let us assume from now on that \( g \) is a semisimple Lie algebra. A Lie subalgebra \( q \) of \( g \) is a biparabolic subalgebra or seaweed subalgebra if there exists a pair of parabolic subalgebras \( (p, p') \) such that \( q = p \cap p' \) and \( p + p' = g \).
such a pair of parabolic subalgebras is called **weakly opposite**, see [6] or [11, Chapter 40]).

We are interested in the following question:

**Question 1.2.1.** Does an affine slice exist for the coadjoint action of $q$?

Motivated by the study of semi-invariant polynomials on the dual, Joseph constructed in [2, 3], such a slice for certain “truncated” biparabolic subalgebras in a semisimple Lie algebra.

In this paper, we give a simple explicit construction of an affine slice for the coadjoint action of a class of (non truncated) biparabolic subalgebras of $g$ which includes all Borel subalgebras. The construction of the affine subspace, and the proof that it is indeed an affine slice, are pretty straightforward, and they rely on some rather nice properties of Kostant’s cascade construction of pairwise strongly orthogonal roots.

**1.3.** Let us first fix some notations. We shall assume from now on that $g$ is semisimple. Let $h$ be a Cartan subalgebra of $g$, $R$ the root system of $g$ relative to $h$, and $\Pi$ a set of simple roots of $R$. Denote by $R_+ (\text{resp. } R_-)$ the corresponding set of positive (resp. negative) roots. For $\alpha \in R$, we denote by $g_\alpha$ the corresponding root subspace, and $X_\alpha$ a non zero element of $g_\alpha$.

For any subset $E$ of $R$, we set $g^E = \sum_{\alpha \in E} g_\alpha$.

Denote by $\kappa$ the Killing form of $g$. This form induces a linear isomorphism $\varphi$ between $g$ and $g^*$. For any $X \in g$, the corresponding linear form, denoted by $\varphi_X$, verifies $\varphi_X(Y) = \kappa(X, Y)$ for all $Y \in g$.

Identifying $h^*$ as linear forms on $g$ which are zero on $g_R$, for any linear form $\lambda \in h^*$, we denote $h_\lambda$ the unique element in $h$ verifying $\lambda(H) = \kappa(h_\lambda, H)$ for all $H \in h$. We have

(i) $h_{a_1\lambda_1 + \cdots + a_r\lambda_r} = a_1h_{\lambda_1} + \cdots + a_rh_{\lambda_r}$ for any $\lambda_1, \ldots, \lambda_r \in h^*$ and $a_1, \ldots, a_r \in k$.

(ii) $kh_\alpha = [g^\alpha, g^{-\alpha}]$ for any $\alpha \in R$.

**1.4.** For any subset $S$ of $\Pi$, we set

$$R^S = R \cap ZS, \quad R^+_S = R \cap NS = R_+ \cap ZS, \quad R^-_S = R^S \setminus R^+_S = R_- \cap ZS$$

respectively the set of roots, positive roots and negative roots of the subroot system generated by $S$.

Let $S$ and $T$ be subsets of $\Pi$. Set $\Delta_{S,T} = R^+_S \cup R^-_T$, and

$$q_{S,T} = h \oplus g^{\Delta_{S,T}}.$$ 

Since $\Delta_{S,T} = (R_- \cup R^+_S) \cap (R_+ \cup R^-_T)$, we see easily that $q_{S,T}$ is the biparabolic subalgebra of $g$ associated to the pair of weakly opposite parabolic subalgebras $(h \oplus g^{R_- \cup R^+_S}, h \oplus g^{R_+ \cup R^-_T})$.

It is well-known (see [6] or [11, Chapter 40]) that if $q$ is a biparabolic subalgebra of $g$, then there exist $S, T$ such that $q$ is conjugated to $q_{S,T}$.
We shall therefore fix two subsets $S$ and $T$ of $\Pi$, and consider the bi-parabolic subalgebra $q_{S,T}$. Denote by $Q_{S,T}$ the connected algebraic subgroup of $G$ whose Lie algebra is $q_{S,T}$.

We shall conserve the above notations in the rest of this paper.

2. Properties of Kostant’s cascade construction

2.1. In this section, we recall some basic properties of Kostant’s cascade construction of pairwise strongly orthogonal roots, and prove a technical lemma related to biparabolic subalgebras.

Let $S \subset \Pi$. We define a set $K(S)$ by induction on the cardinal of $S$ as follows:

(i) $K(\emptyset) = \emptyset$.

(ii) If $S_1, \ldots, S_r$ are connected components (of the Dynkin diagram) of $S$, then

$$K(S) = K(S_1) \cup \cdots \cup K(S_r).$$

(iii) If $S$ is connected, then there is a unique largest positive root $\varepsilon_S$ in $R^+_S$ and

$$K(S) = \{S\} \cup K(\{\alpha \in S; \varepsilon_S(h_\alpha) = 0\}).$$

It is an immediate consequence of the definition that if $K$ and $L$ are distinct elements of $K(S)$, then $\varepsilon_K$ and $\varepsilon_L$ are strongly orthogonal. In particular, we have $\varepsilon_K(h_{\varepsilon_L}) = 0$.

The following properties for $K, L \in K(S)$ are direct consequences of the definition (see also [11, Chapter 40]):

(P1) $K$ and $L$ are connected, and either $K \subset L$, $L \subset K$ or $K \cap L = \emptyset$.

Moreover, if $K \cap L = \emptyset$, then $K$ is strongly orthogonal to $L$.

(P2) The set $K^* = \{\alpha \in K; \varepsilon_K(h_\alpha) \neq 0\}$ is of cardinal 2 if $K$ is of $A_\ell$, $\ell \geq 2$, and is of cardinal 1 otherwise. Note also that if $K$ is of type $A$, then $K^*$ is exactly the set of endpoints of the Dynkin diagram of $K$.

(P3) The connected components of $K \setminus K^*$ are elements of $K(S)$. In particular, $K^* \cap L^* = \emptyset$ if $K \neq L$.

Lemma 2.1.1. Let $S, T \subset \Pi$.

a) If $K \in K(S) \cap K(T)$ and $L \in K(S) \cup K(T)$, then $\varepsilon_K \pm \varepsilon_L \notin R$.

b) Suppose that the following conditions are verified:

i) $S \cap T \neq \emptyset$, $K(S) \cap K(T) = \emptyset$;

ii) $\{\varepsilon_E; E \in K(S) \cup K(T)\}$ is a linearly independent set of roots.

For any non zero element $h \in \text{Vect}(h_\alpha; \alpha \in S \cap T)$, there exists $E \in K(S) \cup K(T)$ such that $\varepsilon_E(h) \neq 0$.

Proof. Part a) is a direct consequence of the definition of the cascade construction. Part b) requires more work, and we shall prove it in several steps.
Step 1. Let us fix a non-zero element $h \in \text{Vect}(h_\alpha; \alpha \in S \cap T)$. Since $S \cap T \subset \Pi$, we have the unique decomposition

$$h = \sum_{\alpha \in S \cap T} \lambda_\alpha h_\alpha$$

where $\lambda_\alpha \in k$. Since $h \neq 0$, the set $C = \{\alpha \in S \cap T; \lambda_\alpha \neq 0\}$ is non-empty. Let us denote by $C_1, \ldots, C_r$ the connected components of $C$.

For $1 \leq i \leq r$, we have $C_i \subset S \cap T \subset S$. By (P1), there is a unique minimal (by inclusion) element $S_i$ of $K(S)$ containing $C_i$. It follows from (P3) and the fact that $S_i$ is minimal that the intersection $C_i \cap S_i^\bullet$ is non-empty. Similarly, there is a unique minimal element $T_i$ of $K(T)$ containing $C_i$, and $C_i \cap T_i^\bullet$ is non-empty.

Let $S = \bigcup_{i=1}^r S_i$ and $T = \bigcup_{i=1}^r T_i$. By (P1), we have

$$K(S) = \bigcup_{i=1}^r K(S_i) \subset K(S), \quad K(T) = \bigcup_{i=1}^r K(T_i) \subset K(T)$$

and $C \subset S \cap T \subset S \cap T$. Moreover, $S$ and $T$ verify the hypotheses of the lemma.

We may therefore assume that $S = S$, $T = T$. Furthermore, we may clearly also assume that $S \cup T = \Pi$ and $\Pi$ is connected.

Under these assumptions, $S = \bigcup_{i=1}^r S_i$, and it follows from (P1) that connected components of $S$ are exactly the maximal elements (by inclusion) of $\{S_1, \ldots, S_r\}$. Again, the same applies for $T$.

Step 2. Suppose that $S_i$ is a connected component of $S$. Then

$$\varepsilon_{S_i}(h) = \sum_{\alpha \in C} \lambda_\alpha \varepsilon_{S_i}(h_\alpha) = \sum_{\alpha \in C \cap S_i} \lambda_\alpha \varepsilon_{S_i}(h_\alpha) = \sum_{\alpha \in C \cap S_i^\bullet} \lambda_\alpha \varepsilon_{S_i}(h_\alpha).$$

If $C \cap S_i^\bullet = \{\alpha\}$, then

$$\varepsilon_{S_i}(h) = \lambda_\alpha \varepsilon_{S_i}(h_\alpha) \neq 0,$$

and we have the result. We saw in 1) that $C_i \cap S_i^\bullet$ is non-empty. So we are reduced to the case where $C \cap S_i^\bullet$ is of cardinal 2.

Of course, this applies to all connected components of $S$, and similarly for those of $T$. It follows from (P2) that we may assume that:

i) Any connected component of $S$ (resp. $T$) is of type $A$ and has rank at least 2.

ii) If $S_i$ (resp. $T_i$) is a connected component of $S$ (resp. $T$), then $S_i^\bullet \subset C$ (resp. $T_i^\bullet \subset C$).

We shall show that assumptions i) and ii) imply that $\Pi$ is of type $A$. Since $\Pi = S \cup T$, any element of $\Pi$ is contained in a connected component of $S$ or $T$. We shall consider a connected component of $S$ or $T$ containing
α specified below according to the Dynkin type of Π.

\[
\begin{align*}
\text{Types } B & \text{ or } C & \text{Type } F_4 & \text{Type } G_2 \\
\alpha & & \alpha & \alpha
\end{align*}
\]

If Π is of type B, C or G₂, then it is not possible to have a connected component of S or T containing α specified below verifying the conditions of assumption i). If Π is of type F₄, then the only possible connected component of S or T containing α verifying the conditions of assumption i) is \{α, β\}. Now assumption ii) and (P2) say that \{α, β\} ⊂ C ⊂ S ∩ T. It follows that \{α, β\} is a connected component for both S and T, but this is impossible since \(\mathcal{K}(S) \cap \mathcal{K}(T) = \emptyset\).

Now if Π is of type D or E or F, we consider the connected components of S or T containing α and β specified below. As in the case of type F, assumption i) implies that they must both be the line joining α and β, and assumption ii) implies that S and T has a common connected component. Again this is impossible since \(\mathcal{K}(S) \cap \mathcal{K}(T) = \emptyset\).

\[
\begin{align*}
\text{Type } D & \text{ Type } E \\
\alpha & \beta & \alpha & \beta
\end{align*}
\]

We are therefore reduced to the case where Π is of type A.

**Step 3.** Let us suppose that \(\varepsilon_E(h) = 0\) for all \(E \in \mathcal{K}(S) \cup \mathcal{K}(T)\). We shall show that this contradicts with the linear independence of the set \{\varepsilon_E; E \in \mathcal{K}(S) \cup \mathcal{K}(T)\}.

Let \(S_k = \{β_1, \ldots, β_ℓ\}\) be a connected component of S where the numbering of the β’s follows the numbering of the Dynkin diagram in [11, Chapter 18]. We have (see for example [11, Chapter 40])

\[
\{\varepsilon_E; E \in \mathcal{K}(S_k)\} = \{\varepsilon_i = β_i + \cdots + β_{ℓ+1-i}; i = 1, \ldots, [\ell/2]\}
\]

where \([\ell/2]\) is the largest integer less than or equal to \(\ell/2\). For \(1 \leq i \leq [\ell/2]\), we have by (P1) that

\[
0 = \varepsilon_i(h) = \sum_{j=1}^{\ell} \lambda_{β_j} \varepsilon_i(h_{β_j}) = \begin{cases} 
\frac{c(λ_{β_1} + λ_{β_ℓ})}{c(-λ_{β_{i-1}} + λ_{β_i} + λ_{β_{i+1-i}} - λ_{β_{i+2-i}})} & \text{if } i = 1; \\
0 & \text{if } i > 1 
\end{cases}
\]

where \(c\) is a non zero constant because all the roots have the same length in type A. It follows that \(λ_{β_j} = -λ_{β_{j+1-j}}\) for all \(j = 1, \ldots, ℓ\). In particular, \(λ_{β_j} = 0\) if and only if \(λ_{β_{j+1-j}} = 0\).

Since \(S_k\) is a connected component of S, and \(C ⊂ S ∩ T\), any \(C_i\) is either contained in \(S_k\) or is strongly orthogonal to \(S_k\). If \(C_i\) is contained in \(S_k\), then there exist \(1 ≤ p ≤ q ≤ ℓ\) such that \(C_i = \{β_p, \ldots, β_q\}\). It follows from the discussion of the preceding paragraph that \(C_i^p = \{β_{ℓ+1-q}, \ldots, β_{ℓ+1-p}\} ⊂ C\). Moreover, since \(β_{p-1}\) and \(β_{q+1}\) are not in \(C_i\), we deduce that \(C_i^p\) is a
connected component of $C$. We have therefore a symmetry on the set of connected components of $C$ in $S_k$.

Since each connected component of $C$ is contained in a unique connected component of $S$, we obtain a permutation $\sigma$ on the set $C = \{C_1, \ldots, C_r\}$ by defining $\sigma(C_i) = C_i'$. If $C_i \in C$ is contained in $S_k$. By symmetry, we must have one of the following 2 configurations:

(I) : $\beta_1, \ldots, \beta_p, \beta_{p+1}, \ldots, \beta_q, \beta_{q+1}, \ldots, \beta_{\ell-q}, \beta_{\ell+1-q}, \ldots, \beta_{\ell-p}, \beta_{\ell+1-p}, \ldots, \beta_{\ell}$

(II) : $\beta_1, \ldots, \beta_p, \beta_{p+1}, \ldots, \beta_{\ell-p}, \beta_{\ell+1-p}, \ldots, \beta_{\ell}$

where $(C, C') = (C_i, \sigma(C_i))$ or $(\sigma(C_i), C_i)$, and

\[ \sum_{\alpha \in C_i} \alpha + \sum_{\alpha \in \sigma(C_i)} \alpha = \varepsilon_{p+1} - \varepsilon_{q+1} = \varepsilon_{S_i} - \varepsilon_{q+1} \]

in configuration (I), and

\[ \sum_{\alpha \in C_i} \alpha = \varepsilon_{p+1} = \varepsilon_{S_i} \]

in configuration (II). Observe that

(O1) In configuration (I), by definition $\varepsilon_{q+1} = \varepsilon_E$ for some $E \in K(S)$. Moreover, since neither $\beta_{p+1}$ nor $\beta_{\ell-q}$ can be in $\text{supp}(h)$, we have that $E \neq \varepsilon_{S_m}$ for any $m$.

(O2) Since we are in the type $A$ case, we see from the above that if $S_j = S_i$, then $C_j = C_i$ on $\sigma(C_i)$.

The same argument applied to $T$ gives another permutation $\tau$ and the properties above. Observe that $\sigma$ and $\tau$ are both of order 2.

Step 4. We define a coloured graph $\mathcal{G}$ whose vertices are the elements of $C$, and we have an edge coloured $\sigma$ (resp. $\tau$) between $C$ and $C'$ if $C' = \sigma(C)$ (resp. $C' = \tau(C)$). So loops in $\mathcal{G}$ correspond exactly to elements of $C$ fixed by either $\sigma$ or $\tau$.

The connected components of $\mathcal{G}$ correspond therefore to the orbits of $C$ under the action of the group generated by $\sigma$ and $\tau$.

Since $\sigma$ and $\tau$ are both of order 2, any vertex $C$ is the endpoint of exactly one $\sigma$-coloured edge and one $\tau$-coloured edge. It follows that a connected
component of $G$ has to be in one of the following forms (after renumbering the $C_i$'s):

\[
\begin{align*}
G_1 &: \sigma \rightsquigarrow C_1 \quad \tau \quad C_2 \quad \sigma \quad \ldots \quad \sigma \quad C_{k-1} \quad \tau \quad C_k \quad \sigma \\
G_2 &: \tau \quad \sigma \quad C_1 \quad \sigma \quad C_2 \quad \tau \quad \ldots \quad \tau \quad C_{k-1} \quad \sigma \quad C_k \quad \tau \\
G_3 &: \sigma \quad \tau \quad C_1 \quad \tau \quad C_2 \quad \sigma \quad \ldots \quad \sigma \quad C_{k-1} \quad \sigma \quad C_k \quad \tau \\
G_4 &: C_2 \quad \tau \quad C_3 \quad \sigma \quad \ldots \quad \tau \quad C_{2k-1} \quad \sigma \quad C_{2k} \\
\end{align*}
\]

We now associated to an edge $e$, a weight $\gamma_e$ as follows:

\[
\gamma_e = \begin{cases} \\
\sum_{\alpha \in C_j} \alpha + \sum_{\alpha \in C_l} \alpha & \text{if $e$ is an edge of the form } C_j \quad \text{---} \quad C_l, j \neq l \\
\sum_{\alpha \in C_j} \alpha & \text{if $e$ is an edge of the form } \quad \quad \quad \quad \text{---} \quad C_j. \\
\end{cases}
\]

For $1 \leq i \leq 4$, let $G_i^\sigma$ (resp. $G_i^\tau$) denote the set of $\sigma$-coloured edges (resp. $\tau$-coloured edges) in $G_i$. Since each vertex is the endpoint of exactly one $\sigma$-coloured edge and one $\tau$-coloured edge, we check easily from the forms above that

\[
\sum_{e \in G_i^\sigma} \gamma_e - \sum_{e \in G_i^\tau} \gamma_e = 0.
\]

In view of Step 3, the weight of an edge can be expressed by a linear combination of elements of $\varepsilon_E$, $E \in \mathcal{K}(S) \cup \mathcal{K}(T)$. The above equality therefore gives a linear relation

\[
\sum_{E \in \mathcal{K}(S) \cup \mathcal{K}(T)} \mu_E \varepsilon_E = 0.
\]

By (O1), (O2) of Step 3 and the fact that $\mathcal{K}(S) \cap \mathcal{K}(T) = \emptyset$, the element $\varepsilon_{S_1}$ appears only in the expression of the weight in terms of $\varepsilon_E$, $E \in \mathcal{K}(S) \cup \mathcal{K}(T)$, of the edge

\[
\sigma \quad \rightsquigarrow \quad C_1 \quad \text{for } G_1 \text{ and } G_3,
\]

and the edge

\[
C_1 \quad \rightsquigarrow \quad \sigma \quad \text{for } G_2 \text{ and } G_4.
\]

We conclude that this relation between the $\varepsilon_E$’s is not trivial. This contradicts our hypothesis on the linear independence of the set $\{\varepsilon_E; E \in \mathcal{K}(S) \cup \mathcal{K}(T)\}$. The proof is now complete. \(\square\)

2.2. Condition ii) in part b) of Lemma 2.1.1 can not be dropped. For example, if we take $g$ simple of type $A_5$, $S = \Pi$ and $T = \Pi \setminus \{\alpha_3\}$ where the numbering of the simple roots is as in [11] Chapter 18. Then condition i) is verified while condition ii) is not verified, and the element

\[
h = h_{\alpha_1} - h_{\alpha_2} + h_{\alpha_4} - h_{\alpha_5}
\]

verifies $\varepsilon_E(h) = 0$ for all $E \in \mathcal{K}(S) \cup \mathcal{K}(T)$. 

3. Slices for the coadjoint action of biparabolic subalgebras

3.1. In the section, we fix subsets S, T of Π. We conserve the notations of paragraphs 1.4 and 2.1.

Set

\[ \Gamma = \{ \varepsilon_K; K \in \mathcal{K}(S) \} \cup \{ -\varepsilon_L; L \in \mathcal{K}(T) \}, \]

\[ \Gamma_0 = R_+ \cap (\Gamma \cap \Gamma''), \Gamma_1 = \Gamma \setminus (\Gamma_0 \cup \Gamma_0). \]

Let \( h_1^+ \) be the subspace of \( h^* \) spanned by \( \Gamma, m = g^{\Delta S,T \cap h_1^+}, n = g^{\Delta S,T \setminus h_1^+}. \)

From 1.3, we deduce immediately that

(1) \( q_{S,T} = h \oplus m \oplus n, [h,m] \subset m, [h + m,n] \subset n, [m,m] \subset m \oplus \sum_{\alpha \in \Gamma} k h_\alpha. \)

In particular, \( h \oplus m \) is a Lie subalgebra of \( q_{S,T}. \)

Let us identify \( q_{S,T}^* \) with \( h^* \oplus m^* \oplus n^* \) via the linear isomorphism \( \varphi \) in paragraph 1.3. In particular, \( m^* \) (resp. \( n^* \)) is the vector subspace spanned by \( \varphi X_\alpha, \alpha \in \Delta_{S,T} \cap h_1^+ \) (resp. \( \alpha \in \Delta_{S,T} \setminus h_1^+. \))

We deduce from the identities (1) above that

(2) \( h \cdot h^* = \{0\}, h \cdot m^* \subset m^*, h \cdot n^* \subset n^*, \)

\[ m \cdot m^* \subset m^* \oplus h_1^+, m \cdot n^* \subset n^*, n \cdot h^* \subset n^*, n \cdot m^* \subset n^*. \)

Lemma 3.1.1. Let \( h_1^+ \) be the set of elements \( \lambda \) of \( h^* \) such that \( \lambda(h_\alpha) = 0 \) for all \( \alpha \in \Gamma. \) Then

\[ m \cdot h_1^+ = \{0\} \text{ and } h^* = h_1^+ \oplus h_1^- \]

Proof. The first equality follows from (1), (2) and the definition of \( h_1^+. \) The second is a special case of [11], proposition 4, p.145].

3.2. For \( a = (a_\alpha)_{\alpha \in \Gamma} \in (k \setminus \{0\})^\Gamma, \) set

\[ f_a = \begin{cases} a_\alpha \varphi_{X-a} & \text{if } \alpha \in \Gamma_1, \\ a_\alpha \varphi_{X-a} + a_\alpha \varphi_{X-a} & \text{if } \alpha \in \Gamma_0, \end{cases} \]

and \( f_a = \sum_{\alpha \in \Gamma_0 \cup \Gamma_1} f_a, \) all considered as linear forms on \( q_{S,T}. \)

Let

\[ m_0^* = \sum_{\alpha \in \Gamma_0} k f_a, W_a = h_1^+ \oplus m_0^* \text{ and } V_a = f_a + W_a. \]

Note that \( f_a \in m^* \) and \( W_a \subset h^* \oplus m_0^*. \)

Lemma 3.2.1. For all \( f \in V_a, \) we have \( q_{S,T} \cdot f \cap W_a = \{0\}. \) In particular, \( T_f \{ q_{S,T} \cdot f \} \cap T_f (V_a) = \{0\} \) for all \( f \in V. \)

Proof. Let \( (H,X,Y) \in h \times m \times n \) and \( (t,m) \in h_1^+ \times m_0^* \) be such that

\( (H + X + Y). (f_a + t + m) = w \in W_a \subset h^* \oplus m_0^*. \)

From the identities (2), we deduce that \( Y.(f_a + t + m) = 0, H.t = X.t = 0, \)

\( H.(f_a + m) \in m^*, X.(f_a + m) \in m^* \oplus h_1^+, \) and therefore

\( w = (H + X). (f_a + m) \in (m^* \oplus h_1^+) \cap W_a = m_0^*. \)
by Lemma 3.1.1.

Since $H.(f_a + m)$ is a linear combination of $\varphi_{X_{\alpha}}, \alpha \in \Gamma$, we deduce that

$$X.(f_a + m) = m_0 + m_1$$

where $m_0$ is a linear combination of $\varphi_{X_{\alpha}}, \alpha \in \Gamma_0 \cup -\Gamma_0$, and $m_1$ is a linear combination of $\varphi_{X_{\alpha}}, \alpha \in \Gamma_1$.

On the other hand, $X.(f_a + m)$ is a linear combination of elements of the form $X_\alpha \varphi_{X_{-\beta}} \in \mathfrak{K} \varphi_{X_{-\alpha}}$ with $\alpha \in \Delta \cap h_1^*$ and $\beta \in \Gamma$. If $m_0$ is non-zero, then we can find $\alpha \in \Delta \cap h_1^*$ and $\beta \in \Gamma$ such that $\alpha - \beta \in \Gamma_0 \cup -\Gamma_0$. This is impossible by Lemma 2.1.1 a). Thus $m_0 = 0$, and we have

$$w = (H + X).(f_a + m) = H.(f_a + m) + m_1.$$  

Finally, for $\alpha \in \Gamma_0$, we have

$$H.f^\alpha_a = \alpha(H)(a_\alpha \varphi_{X_\alpha} - a_{-\alpha} \varphi_{X_{-\alpha}}).$$

The elements $f^\alpha_a, H.f^\alpha_a, \alpha \in \Gamma_0$, are linearly independent since $a$ has non-zero entries. It follows that $\alpha(H) = 0$ for all $\alpha \in \Gamma_0$, and so $H.(f_a + m)$ is a linear combination of $\varphi_{X_{-\alpha}}, \alpha \in \Gamma_1$. We deduce that $w$ is also a linear combination of $\varphi_{X_{-\alpha}}, \alpha \in \Gamma_1$. Since $w \in m^*_0$, we have $w = 0$ as required.  

For $\alpha \in \Gamma_0$, set

$$Z_\alpha = a_\alpha X_\alpha + a_{-\alpha} X_{-\alpha}.$$  

Let $t = \{H \in \mathfrak{h}; \alpha(H) = 0 \text{ for all } \alpha \in \Gamma\}$, and

$$\mathfrak{r}_a = t \oplus \bigoplus_{\alpha \in \Gamma_0} \mathbb{K} Z_\alpha \subset t \oplus \mathfrak{g}^{\mathfrak{g}_0 \cup -\Gamma_0}.$$  

**Lemma 3.2.2.** We have $\mathfrak{r}_a \cdot W_a = \{0\}$, and $\mathfrak{r}_a \subset q^f_{S,T}$ for all $f \in V_a$ where $q^f_{S,T} = \{X \in q_{S,T}; X.f = 0\}$.

**Proof.** Observe that for $\alpha \in \Gamma_0$, we have $f^\alpha_a = \varphi_{Z_\alpha}$, and one verifies easily using Lemma 2.1.1 a) that

$$Z_\alpha.f^\beta_a = 0$$

for any $\beta \in \Gamma$. In particular, $\mathfrak{r}_a \subset q^f_{S,T}$.

By Lemma 3.1.1, $m.h^\perp_1 = \{0\}$, and $t.h^\perp_1 = \{0\}$ by (2), the result follows easily.  

**Lemma 3.2.3.** Suppose that $\mathfrak{r}_a = q^f_{S,T}$. Then there exists an open subset $O_a$ of $V_a$ such that $Q_{S,T}.O_a$ is dense in $q^f_{S,T}$.

**Proof.** By definition, $\mathfrak{r}_a$ is a commutative Lie subalgebra whose elements are all semisimple. It follows from [11, 40.1.3, 40.1.5, 40.1.6] that $f_a$ is a stable, and hence regular, element of $q^f_{S,T}$. In particular, the $Q_{S,T}$-orbit of $f_a$ has maximal dimension, or equivalently the dimension of its stabilizer $Q_{S,T}^a$ is minimal. Moreover, by our hypothesis,

$$\dim Q_{S,T}^a = \dim q^f_{S,T} = \dim \mathfrak{r}_a = \dim t + 2\Gamma_0 = \dim h^\perp_1 + \mathfrak{r}_a = \dim W_a.$$
Let \( O_a \) be the set of regular elements \( q_{S,T}^* \) contained in \( V_a \). It is a non-empty open subset of \( V_a \).

Consider the \( Q_{S,T} \)-equivariant morphism

\[
\Phi : Q_{S,T} \times O_a \rightarrow q_{S,T}^*, \ (\sigma, f) \mapsto \sigma(f).
\]

Let \( f \in O_a \). Then \( \Phi^{-1}(f) = \{ (\sigma, \sigma^{-1}(f)) ; \sigma^{-1}(f) \in O_a \} \). By Lemma \[3.2.1\] \( Q_{S,T} \cdot f \cap O_a \) is a finite set. It follows that

\[
\dim \Phi^{-1}(f) = \dim Q_{S,T}^f = \dim Q_{S,T}^{f_a} = \dim W_a,
\]

and hence

\[
\dim q_{S,T} + \dim W_a = \dim (Q_{S,T} \times O_a) = \dim \Phi^{-1}(f) + \dim \im(\Phi).
\]

We deduce that \( \dim \im(\Phi) = \dim q_{S,T} = \dim q_{S,T}^* \), thus \( \Phi \) is a dominant morphism, and the result follows. \( \square \)

**Theorem 3.2.4.** Suppose that \( \Gamma = \Gamma_1 \) is a linearly independent subset of roots. Then there exists \( a \in (k \setminus \{0\})^\Gamma \) such that \( V_a \) is a slice for the coadjoint action of \( q_{S,T} \).

**Proof.** By [11] 40.9.4], there exists \( a \in (k \setminus \{0\})^\Gamma \) such that \( f_a \) is a stable element of \( q_{S,T}^* \). Moreover, since \( \Gamma_0 \) is empty, we have \( q_{S,T}^{f_a} = t = t_a \). By Lemmas \[3.2.3\] and \[3.2.1\] we only need to show that condition (C\( _3 \)) is verified for some open subset of \( O_a \) where \( O_a \) is the open subset of \( V_a \) defined in the proof of Lemma \[3.2.3\].

In view of Lemma \[3.2.2\] \( O_a \) is the set of elements \( f \) in \( V_a \) verifying \( q_{S,T}^f = t \). Now suppose that \( f \in O_a \) and \( \sigma \in Q_{S,T} \) be such that \( \sigma(f) \in O_a \). Since \( q_{S,T}^f = q_{S,T}^{\sigma(f)} = \sigma(q_{S,T}^f) \), we deduce that \( \sigma \in N_{Q_{S,T}}(t) \).

Denote \( \Delta_0 = \Delta \cap -\Delta, s = h \oplus q^{\Delta_0}, l = [s, s], u = q^{\Delta \setminus \Delta_0} \), and \( a \) the centre of the reductive subalgebra \( s \). Then \( q_{S,T} = l \oplus a \oplus u \) is a (refined) Levi decomposition of \( q_{S,T} \), where \( f = a \oplus u \) is the radical of \( q_{S,T} \), and \( l \) is a Levi subalgebra of \( q_{S,T} \).

In particular, we have \( t \subset h \subset l \oplus a, \) and \( [q_{S,T}, q_{S,T}] = l \oplus u \). We deduce easily that

\[
t \cap [q_{S,T}, q_{S,T}] = t \cap l = t \cap \Vect(h_\alpha; \alpha \in S \cap T).
\]

Thus \( t \cap l = \{0\} \) if \( S \cap T = \emptyset \). If \( S \cap T \neq \emptyset \), then the fact that \( \Gamma = \Gamma_1 \) is a linearly independent subset implies that conditions i) and ii) of Lemma \[2.1.1\] b) are verified, and it follows immediately from the conclusion of Lemma \[2.1.1\] b) and the definition of \( t \) that

\[
t \cap l = t \cap \Vect(h_\alpha; \alpha \in S \cap T) = \{0\}.
\]

So we have \( t \cap l = \{0\} \) in both cases.

Let \( L \) and \( K \) be the connected algebraic subgroups of \( Q_{S,T} \) whose Lie algebras are \( l \) and \( t \) respectively. Then \( Q_{S,T} = KL \). Let us write \( \sigma = \sigma_K \sigma_L \) where \( \sigma_K \in K \) and \( \sigma_L \in L \).
Let \( x \in t \). Then \( x = x_l + x_a \) where \( x_l \in l \) and \( x_a \in a \). Since \([t, a] = \{0\}\) and \([q_{S,T}, q_{S,T}] = l \oplus u\), we have \( \sigma_L(x_a) = x_a \), and

\[
\sigma(x) = \sigma_K(\sigma_L(x_l + x_a)) = \sigma_K(\sigma_L(x_l)) + \sigma_K(x_a) = \sigma_L(x_l) + y + x_a + z
\]

where \( y, z \in u \). But \( \sigma(x) \in t \), and so \( \sigma(x) = \sigma_L(x_l) + x_a \) and \( \sigma(x) - x = \sigma_L(x_l) - x_l \in t \cap l = \{0\} \)

by (3). Thus \( \sigma(x) = x \).

This being true for any \( x \in t \), we deduce that \( \sigma \in C_{Q_{S,T}}(t) \) which is the connected algebraic subgroup of \( Q_{S,T} \) whose Lie algebra is \( C_{q_{S,T}}(t) \).

By definition, we have \( C_{q_{S,T}}(t) = h \oplus m \).

Let \( X \in C_{q_{S,T}}(t) \), then by Lemma 3.1.1 and the identities (2), we have

\[
X.h^\perp = \{0\}, \quad X.f_a \in m^* \oplus h^\perp.
\]

It follows that \( \sigma(g) = g \) for all \( g \in h^\perp \), and \( \sigma(f_a) - f_a \in m^* \oplus h^\perp \).

Writing \( f = f_a + g \) where \( g \in W_a = h^\perp \) (since \( \Gamma_0 \) is empty), we have

\[
\sigma(f) - f = \sigma(f_a) + g - f = \sigma(f_a) - f_a \in (m^* \oplus h^\perp) \cap h^\perp = \{0\}.
\]

Hence \( \sigma(f_a) = f_a \), and \( \sigma(f) = f \). So condition (C3) is verified by \( O_a \). \( \square \)

The hypothesis of Theorem 3.2.4 is clearly satisfied when \( S \) or \( T \) is empty. We have the following result.

**Corollary 3.2.5.** An affine slice exists for the coadjoint action of a Borel subalgebra.

**Remark 3.2.6.** When \( g \) is simple of type \( A_\ell \), then any (except one when \( \ell \) is odd) minimal parabolic subalgebra of \( g \) verifies the hypotheses of Theorem 3.2.4. So an affine slice exists for the coadjoint action of these minimal parabolic subalgebras.

**Remark 3.2.7.** When \( \Gamma \) is a linearly independent set with \( \Gamma_0 \) non empty, \( V_a \) is not in general an affine slice for any \( a \). Take for example \( S = T = \Pi \), then \( V_a \) is a Cartan subalgebra and therefore condition (C3) is not verified.

### 3.3.

We finish the paper by establishing the claim in the introduction that the existence of an affine slice for the coadjoint action of a Lie algebra \( g \) implies that the field of \( G \)-invariant rational functions on \( g^* \) is a purely transcendental extension of \( k \).

**Theorem 3.3.1.** Let \( S \) be an affine slice for the coadjoint action of a Lie algebra \( g \), and denote by \( G \) the algebraic adjoint group of \( g \).

a) The field of \( G \)-invariant rational functions on \( g^* \) is a purely transcendental extension of \( k \).
b) There exists an open subset $U$ of $S$ such that the ring of regular functions on $U$ is isomorphic to the ring of $G$-invariant regular functions on $G.U$.

Proof. By Rosenlicht’s Theorem [3], there exists a non-empty $G$-stable open subset $U$ of $g^*$ such that a geometric quotient $U/G$ exists. Let us denote

$$\pi : U \rightarrow U/G$$

this quotient morphism. Recall that $\pi$ is an open morphism.

Let $\mathcal{O}$ be an open subset of $S$ verifying the conditions (C$_1$), (C$_2$), (C$_3$). In particular, $G.\mathcal{O} = g^*$. So $G.\mathcal{O}$ contains a non-empty $G$-stable open subset $\mathcal{W}$ of $V$.

Set $\Omega = U \cap \mathcal{W}$. Then $\Omega$ is a non-empty $G$-stable open subset of $g^*$, and $\Omega \cap S$ is a non-empty open subset of $S$ verifying

$$G.(\Omega \cap S) = \Omega.$$

Consider the morphism

$$\Phi : \Omega \cap S \rightarrow U/G, \ x \mapsto \pi(x).$$

By our construction of $\Omega$, $\Phi$ is injective and its image is $\pi(\Omega)$. It follows that $\Phi$ is dominant ($\pi$ being open).

Being a non-empty open subset of an affine space, $\Omega \cap S$ is normal, and hence by [11, Corollary 17.4.4] and the injectivity of $\Phi$, we deduce that $\Phi$ is a birational equivalence. Thus we have the following isomorphism of rational functions

$$\mathbf{R}(\Omega \cap S) \simeq \mathbf{R}(U/G) \simeq \mathbf{R}(U)^G = \mathbf{R}(g^*)^G$$

by [11, Proposition 25.3.6].

Since $\mathbf{R}(\Omega \cap S) = \mathbf{R}(S)$ is the field of fractions of a polynomial algebra over $\mathbb{k}$, we have part a).

Part b) is a direct consequence of the fact that $\Phi$ is a birational equivalence and $\pi$ is a geometric quotient. □

Remark 3.3.2. Of course, we may generalize the notion of an affine slice to any finite-dimensional $g$-module, and Theorem 3.3.1 remains valid for any $g$-module admitting an affine slice.

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