Research article

Nonexistence results for ultra-parabolic equations and systems involving fractional Laplacian operator in Heisenberg group

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A R T I C L E   I N F O

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A B S T R A C T

In this paper, sufficient conditions are obtained for the nonexistence of solutions to the Cauchy problems for ultra-parabolic equations involving fractional Laplacian operator. The nonexistence result is also extended to the corresponding system. The results to such problems are derived in the Heisenberg group. The proofs of our nonexistence theorems are based on the nonlinear capacity method. This method is based on the choice of a suitable test function in the weak formulation of the considered problems.

1. Introduction

Ultra-parabolic equations are multi-time parabolic equations and arise in the theory of Brownian motion (diffusion process) (Il’in and Has’minskii, 1964), transport theory (Fokker–Planck type equations) (Risken, 1984), statistics (stochastic process) (Chandrasekhar, 1943), biology (age structured population dynamics) (Iannelli, 1995), and have other practical applications in physics and engineering sciences. In recent years, some interesting results on applications of related problems for nonlinear Schrödinger equations (as a means to improve the communication capacity of optical fiber) can be found in Xiaoayon Liu et al., 2022; Qin Zhou, 2022) and for Ginzb erg-Landau equation (e.g. phenomena of soliton propagation & dissipative system, especially fiber laser), see the works (Wang L.L. and Liu W.J., 2020; Yuan-Yuan Yan and Wenjun Liu, 2021).

From the beginning of the study of nonlinear parabolic type equations with multi-dimensional time (Ugowski, 1974), many works on various aspects on nonlinear ultra-parabolic equations have been conducted by (Lanconelli et al., 2002; Lavre n yuk and Ofiškevič, 2007; Lavrenyuk and Prot sakh, 2007; Prot sakh, 2009; Tersenov, 2005) and the reference therein.

In particular, Kerbal and Kirane (2011) established nonexistence results to the ultra-parabolic equations and systems with Laplacian operator. Jelli et al. (2014) studied the nonexistence results of global solutions to ultra-parabolic equations and systems involving the fractional Laplacian operator in the Euclidean domain. No existence results for ultra-parabolic equations and systems were obtained in the Euclidean case. However, to the best of our knowledge, previously the nonexistence results for fractional ultra-parabolic equations and the corresponding system on the Heisenberg group were not considered.

In this study, we are concerned with the nonexistence results to the Cauchy problems for fractional ultra-parabolic equations and systems of the following types:

\[ u_t + u_{x_1} + (\Delta_H)^\frac{\alpha}{2} u = f(u), \quad (\xi, t_1, t_2) \in \mathbb{H} \times (0, \infty) \times (0, \infty), \]

subject to the initial conditions

\[ u(\xi, t_1, 0) = u_1(\xi, t_1), \quad (\xi, t_1) \in \mathbb{H} \times (0, \infty), \]

and the corresponding coupled system

\[
\begin{align*}
  u_{t_1} + u_{t_2} + (\Delta_H)^\frac{\alpha}{2} u &= f(u), & (\xi, t_1, t_2) &\in \mathbb{H} \times (0, \infty) \times (0, \infty), \\
  v_{t_1} + v_{t_2} + (\Delta_H)^\frac{\alpha}{2} v &= g(v), & (\xi, t_1, t_2) &\in \mathbb{H} \times (0, \infty) \times (0, \infty),
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
  u(\xi, t_1, 0) &= u_1(\xi, t_1), \quad u(\xi, 0, t_2) &= u_2(\xi, t_2), \quad (\xi, t_1, t_2) &\in \mathbb{H} \times (0, \infty), \\
  v(\xi, 0, t_2) &= v_1(\xi, t_1), \quad v(\xi, 0, t_2) &= v_2(\xi, t_2), \quad (\xi, t_1, t_2) &\in \mathbb{H} \times (0, \infty),
\end{align*}
\]

where \(\Delta_H\) is the Kohn–Laplacian operator, \((\Delta_H)^\frac{\alpha}{2}\) and \((\Delta_H)^\frac{\beta}{2}\), \(0 < \alpha, \beta < 2\) are fractional powers of the Kohn–Laplacian operator on \((2N + 1)\)-dimensional Heisenberg group \(\mathbb{H}\). The functions \(f\) and \(g\) are nonnegative functions satisfying

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\[ f(v) \geq c\|v\|^p, \quad g(u) \geq c\|u\|^p, \quad p, q > 1. \]

The objective of this paper is to obtain unsolvability conditions for the nonexistence of nontrivial global weak solution to the Cauchy problem of fractional ultra-parabolic equation (1.1)–(1.2) and the corresponding coupled system (1.3)–(1.4). Sufficient conditions for nonexistence results are obtained using the nonlinear capacity method which was proposed by (Pokhozhaev, 1997) and later developed in joint works (Mitidieri and Pokhozhaev, 2001). The considered equation and system have applications in diffusion theory in porous media and probability density system with 2d degrees of freedom.

The rest of this paper is organized as follows: In section 2, we briefly recall some preliminary concepts on basic properties of Heisenberg group, fractional Kohn-Laplacian operator, definition of weak solution and some lemmas that we use in the proof of nonexistence theorems. The method, we employ for obtaining sufficient unsolvability conditions is described in Section 3. Section 4 is devoted to the statements and proofs of our main results. Finally, in the last section, we give general concluding remarks.

2. Preliminaries

For the convenience of the reader, we start by recalling some basic definitions, lemmas and properties which will be used throughout this paper.

Heisenberg group: The \((2N + 1)\)-dimensional Heisenberg group \(H\) is the Lie group denoted by \((\mathbb{R}^{2N + 1}, o)\) with the non-commutative group operation \(\circ\) and defined by
\[
\xi \circ \zeta = (x + x', y + y', t + t' + 2 \langle (x', y') - (x', y) \rangle),
\]
for all \(\xi = (x, y, t)\) and \(\zeta = (x', y', t') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}\) and usual inner product, \(\langle \cdot, \cdot \rangle\) in \(\mathbb{R}^N\). The natural distance on the Heisenberg group, \(H\) from the point \(\xi = (x, y, t)\) to the origin is introduced by (Folland and Stein, 1974) and defined as
\[
|\xi|^H = (\sum_{j=1}^{N} (x_j^2 + y_j^2) + t^2)^{\frac{1}{2}},
\]
where \(x = (x_1, x_2, \ldots, x_N)\) and \(y = (y_1, y_2, \ldots, y_N)\).

The Laplacian operator \(\Delta_H\) on the Heisenberg group, \(H\) is obtained from the vector fields, \(\xi_j\) and \(\eta_j\) which is defined by
\[
\Delta_H \equiv \sum_{j=1}^{N} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial t^2} \right)
\]
where
\[
\xi_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad \eta_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, N.
\]
After some computation, we get the following expressions
\[
\Delta_H = \sum_{j=1}^{N} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} + (x_j^2 + y_j^2) \frac{\partial^2}{\partial t^2} \right).
\]
The natural group of dilations, \(d_\lambda\) on the Heisenberg group with its Jacobian determinant \(\lambda^2\) is given by
\[
d_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t), \quad 0 < \lambda \in \mathbb{R}, \quad (x, y, t) \in H,
\]
where \(Q = 2N + 2\) is the homogeneous dimension of \(H\). The Laplacian operator, \(\Delta_H\) is homogeneous with respect to dilation and invariant from the left multiplication in \(H\). That is
\[
\Delta_H (\xi \circ \zeta) = \lambda^2 \Delta_H (\xi \circ \zeta), \quad \lambda > 0, \xi \in \mathbb{R}, \quad (x, y, t) \in H,
\]
\[
\Delta_H (v(\xi \circ \zeta)) = \lambda^2 \Delta_H (v(\xi \circ \zeta))\] .
Let \(v(\xi) = \omega(|\xi|^H)\), then we have
\[
\Delta_H (v(\xi \circ \zeta)) = \lambda^2 \Delta_H (v(\xi \circ \zeta)).
\]
\[
\int_\Omega u_1 \varphi (\xi, t_1, 0) \, d P_1 + \int_\Omega u_2 \varphi (\xi, 0, t_2) \, d P_2 + \int_\Omega f(u) \varphi d P > 0
\]

\[
= \int_\Omega u (\varphi_{t_1} - \varphi_{t_2} + (-\Delta_x)^{\frac{2}{p}} \varphi) \, d P,
\]

\[
\int_\Omega \int_\Omega \frac{u_1 (\varphi_{t_1} - \varphi_{t_2} + (-\Delta_x)^{\frac{2}{p}} \varphi)}{u_2} \, d P,
\]

\[
\int_\Omega \int_\Omega \frac{v (\varphi_{t_1} - \varphi_{t_2} + (-\Delta_x)^{\frac{2}{p}} \varphi)}{u_2} \, d P,
\]

\[
\int_\Omega \frac{(\varphi_{t_1} - \varphi_{t_2} + (-\Delta_x)^{\frac{2}{p}} \varphi)}{2} \, d P.
\]

\[
\int_\Omega \frac{|u|^q d P \leq \text{Cap}(L, \Omega).
\]

The well-known parametric Young’s inequality and Holder’s inequality (Brezis, 2011) are main tools to establish apriori estimates for the possible solutions to the given problems. The following Lemmas are also fundamental to obtain the optimal apriori estimates of the solutions of the problems.

**Lemma 2.1** (Ju’s inequality (Ju, 2005; Ahmad et al., 2015)). Let \( F(\varphi) \in C^2(\mathbb{R}) \) be a convex function. For a test function \( \varphi \) such that \( 0 \leq \varphi \in C_0^\infty (\mathbb{R}^{2N+1}) \), the following inequality holds pointwise:

\[
(\Delta_x)^{\frac{2}{p}} F(\varphi) \leq F'(\varphi) (-\Delta_x)^{\frac{2}{p}}, \quad 0 < a < 2.
\]

Moreover, if \( F'(0) = 0 \) and \( \varphi \in C_0^\infty (\mathbb{R}^{2N+1}) \), then we have

\[
\int_{\mathbb{R}^{2N+1}} F'(\varphi) (-\Delta_x)^{\frac{2}{p}} \varphi d \xi \geq 0.
\]

The inequality is also valid for \( a = 2 \) as

\[
\Delta_x \varphi^p = \beta \varphi^{p-1} \Delta_x \varphi + \beta (\beta - 1) \varphi^{p-2} \sum_{j=1}^{N} \left( |\varphi_{x_j}|^2 + |\varphi_{x_j}|^2 \right) \geq \beta \varphi^{p-2} \Delta_x \varphi,
\]

for \( F(\varphi) = \varphi^p \).

**Lemma 2.2** (Pokhozhaev, 2013). Suppose that \( u \in L^1(\Omega) \) and \( \int_x u d \xi \geq 0 \). Then, there exists a test function \( \varphi \) such that \( 0 \leq \varphi \leq 1 \) satisfying

\[
\int_\Omega u \varphi d \xi \geq 0, \quad \xi \in \Omega.
\]

### 3. Methodology

In this paper, we use the nonlinear algorithmic method which is based on obtaining maximally optimal apriori estimates of the solution by the algebraic analysis of the integral form of the problems under a special choice of test functions. This approach is very effective in the proof of theorems.

In the course of the proofs, we frequently use the fact that if \( \varphi \in C_0^\infty (\Omega) \) is a standard cut-off function and \( q > \lambda \) for sufficiently large \( \lambda \), then it is always possible to select the function \( \varphi \) in order to have

\[
\int_\Omega \frac{|L\varphi|^q}{q^{p-1}} d P < \infty,
\]

where \( L \) is the differential operator of the given equation.

To derive an optimal apriori estimate of the solution of the given problem, we introduce the quantity

\[
\text{Cap}(L, \Omega) := \inf \left\{ \int_\Omega \frac{|L\varphi|^q}{q^{p-1}} d P \right\},
\]

where the infimum is taken over all test functions \( \varphi \in C_0^\infty (\Omega) \). We call this quantity a nonlinear capacity induced by the given problem. Then, an optimal apriori estimate is given by

\[
\forall \mu, \lambda, \beta, \eta, \gamma, \varepsilon > 0, \quad \text{there exists } \varphi \in C_0^\infty (\Omega) \text{ such that }
\]

\[
\int_\Omega |u|^q d P \leq \text{Cap}(L, \Omega).
\]

Now, let us introduce the test function

\[
\varphi_R (\xi, t_1, t_2) = \varphi(t) \left( \frac{t_1^{\frac{\lambda}{\gamma}} + t_2^{\frac{\lambda}{\gamma}} + |x|^4 + r^2}{R^2} \right), \quad R > 0, \; \theta > 0,
\]

where \( \varphi \in C_0^\infty (\mathbb{R}; [0, 1]) \) is a nonincreasing standard cut-off function such that \( 0 \leq \varphi(s) \leq 1 \) and

\[
\varphi(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq 1, \\ 0, & \text{if } s \geq 2. \end{cases}
\]

for sufficiently large \( \lambda > 0 \) (which will be specified later according to the nature of the problem). Note that for a positive constant \( c \), we have

\[
-c \leq \frac{d \varphi}{ds} \leq 0, \quad \forall s \in [0, \infty).
\]

Hereafter, we denote \( c \) is a generic positive constant that may change from line to line. According to the test function defined in (3.1), we can introduce and set the new independent variables

\[
(\xi, t) = (x, y, t_1, t_2) \rightarrow (\bar{x}, \bar{y}, \bar{t}, \bar{t}_1, \bar{t}_2) = (\bar{x}, \bar{y}) \text{ by the formula}
\]

\[
x = R \bar{x}, \quad y = R \bar{y}, \quad t = R^2 \bar{t}, \quad t_1 = R^4 \bar{t}_1, \quad t_2 = R^4 \bar{t}_2.
\]

Consequently,

\[
\varphi_R (\xi, t_1, t_2) = \varphi'(s), \quad s = \frac{x^4 + t_1^4 + |x|^4 + |t_1|^4 + r^2}{R^4}.
\]

Now, let us denote

\[
\varphi_R \in \mathbb{R}_R := \left\{ (x, y, t, t_1, t_2) \in \mathbb{R} \times (0, \infty) \times (0, \infty) : 0 \leq t_1^4 + t_2^4 + |x|^4 + |t_1|^4 + r^2 \leq R^4 \right\},
\]

\[
\Gamma_R := \left\{ (x, y, t, t_1, t_2) \in \mathbb{R} \times (0, \infty) : 0 \leq t_1^4 + t_2^4 + |x|^4 + |t_1|^4 + r^2 \leq R^4 \right\}, \quad j = 1, 2.
\]

From definition of \( \varphi_R \) in (3.1), we observe that

\[
\sup \{ \varphi_R \} \subset \Omega_R, \quad \sup \left\{ \frac{\partial \varphi_R}{\partial \xi} \right\}, \quad \sup \{ (-\Delta_x)^{\frac{2}{p}} \varphi_R \} \subset \overline{B_2}(\Omega_R).
\]

### 4. Results and proofs

In this section, the nonexistence theorems on the Cauchy problems for two-time ultraparabolic equation and corresponding coupled system will be stated and proved. Sufficient conditions for the nonexistence of weak solutions to these problems are obtained.

Our main first nonexistence result for equation (1.1) subject to (1.2) is as follows.

**Theorem 4.1.** Assume that \( f \) satisfies condition (1.5), \( q > 1 \) and \( 0 \leq u_1, u_2 \in L^q_{\text{loc}} (\Gamma) \) such that

\[
\int_\Gamma u_1 d P_1 + \int_\Gamma u_2 d P_2 > 0.
\]

If

\[
1 < q \leq 1 + \frac{a}{Q + a} \quad Q := 2N + 2,
\]

then there is no nontrivial global weak solution of problem (1.1)–(1.2).

**Proof of Theorem 4.1.** The proof is carried out by contradiction. Suppose that \( u(x, t) \) is a nontrivial global weak solution of problem (1.1)–(1.2) and take sufficiently smooth nonnegative test function \( \varphi_R \) (as defined in (3.1) and (3.2)) for sufficiently large parameter \( \lambda > 0 \) (which will be specified later).
Using (1.5), from (2.1) we have

\[ \int_{\Gamma} u_{1}\phi^{t}(\xi, t_{1}, 0) dP_{1} + \int_{\Gamma} u_{2}\phi^{t}(\xi, 0, t_{2}) dP_{2} + c \int_{\Omega} |u|^q \phi^t dP \]

\[ \leq \int_{\Omega} \left( \left| \frac{\partial}{\partial t_1} \phi^t \right| + \left| \frac{\partial}{\partial t_2} \phi^t \right| + \left| (-\Delta_{\Omega})^{\frac{3}{2}} \phi^t \right| \right) dP. \]  

(4.1)

By the parametric Young’s inequality with conjugate exponents \( q \) and \( q' = q/(q-1) \), from the first two integrals on the right-hand side of (4.1) we get

\[ \int_{\Omega} \left| (-\Delta_{\Omega})^{\frac{3}{2}} \phi^t \right| dP \leq \lambda \int_{\Omega} |u|^q \phi^t dP + \lambda c_i \int_{\Omega} \left| (-\Delta_{\Omega})^{\frac{3}{2}} \phi^t \right| \phi^{t-j} dP, \quad j = 1, 2, \]  

(4.2)

Similarly, using Ji’s inequality (see Lemma 2.1) and parametric Young’s inequality, from the third integral on the right-hand side of (4.1) we obtain

\[ \int_{\Omega} \left| \frac{\partial \phi}{\partial t_j} \right| dP \leq \lambda \int_{\Omega} |u|^q \phi^t dP + \lambda c_i \int_{\Omega} \left| (-\Delta_{\Omega})^{\frac{3}{2}} \phi^t \right| \phi^{t-j} dP, \]  

(4.3)

for the parameter

\[ \lambda > q' = \frac{q}{q-1}. \]

By virtue of (4.2) and (4.3), for sufficiently small parameter \( \varepsilon > 0 \) (take \( \varepsilon = c/4\lambda \) and constant \( c = \varepsilon^{\frac{q}{q-1}} \)) we obtain

\[ \int_{\Gamma} u_{1}\phi_{R}(\xi, t_{1}, 0) dP_{1} + \int_{\Gamma} u_{2}\phi_{R}(\xi, 0, t_{2}) dP_{2} + \frac{c}{2} \int_{\Omega} |u|^q \phi_{R} dP \]

\[ \leq \lambda c_i \int_{\Omega} \left( \left| \frac{\partial \phi}{\partial t_j} \right| + \left| \frac{\partial \phi}{\partial t_1} \right| + \left| (-\Delta_{\Omega})^{\frac{3}{2}} \phi \right| \right) \phi^{t-j} dP. \]  

(4.4)

In view of (3.1) and (3.2), we have

\[ A_{\eta,j}(\phi) := \int_{\supp \left( \frac{\partial \phi}{\partial t_j} \right)} \left| \frac{\partial \phi}{\partial t_j} \right|^{q'} \phi^{t-j} dP \leq c R^{-\frac{1}{q} + \frac{1}{q'} + \frac{1}{2}}, \quad j = 1, 2, \]

(4.5)

\[ B_{\gamma,a}(\phi) := \int_{\supp \left( (-\Delta_{\Omega})^{\frac{3}{2}} \phi \right)} \left| (-\Delta_{\Omega})^{\frac{3}{2}} \phi \right|^{q'} \phi^{t-j} dP \leq c R^{-\frac{3}{q_2} + \frac{1}{q'} + \frac{1}{2}}. \]

(4.6)

Then, from (4.4), it follows that

\[ \int_{\Gamma} u_{1}\phi_{R}(\xi, t_{1}, 0) dP_{1} + \int_{\Gamma} u_{2}\phi_{R}(\xi, 0, t_{2}) dP_{2} + \frac{c}{2} \int_{\Omega} |u|^q \phi_{R} dP \]

\[ \leq c \left( 2R_{\ell_{1}(t_{1})} + R_{\ell_{2}(t_{2})} \right) \leq 2c \left( 2R_{\ell_{1}(t_{1})} + R_{\ell_{2}(t_{2})} \right), \]  

(4.6)

where

\[ \gamma_{1}(\phi) = -\frac{4}{9} q' + \frac{8}{9}, \quad \gamma_{2}(q, a) = -aq' + \frac{8}{9}. \]

We can easily verify that the right-hand side of inequality (4.6) attain minimum value if and only if both terms have the same exponent. Thus, the optimal choice of \( \theta \) is \( 4/a \).

Consequently, from (4.6) it follows that

\[ \int_{\Gamma} u_{1}\phi_{R}(\xi, t_{1}, 0) dP_{1} + \int_{\Gamma} u_{2}\phi_{R}(\xi, 0, t_{2}) dP_{2} + \frac{c}{2} \int_{\Omega} |u|^q \phi_{R} dP \leq c R^{\ell_{1}(t_{1})} \]

where

\[ \sigma(q, a) = a \left( \frac{q-2}{q-1} \right) + Q. \]

Next, we impose that \( \sigma(q, a) \leq 0 \). This is equivalent to the following inequality:

\[ 1 < q \leq q' := 1 + \frac{a}{Q + a}. \]

Since the test function \( \varphi_{R} \equiv 1 \) on \( \Omega_{R} \), for sufficiently large \( R \) it follows that

\[ 0 < \int_{\Gamma} u_{1} dP_{1} + \int_{\Gamma} u_{2} dP_{2} + \frac{c}{2} \int_{\Omega} |u|^q dP \leq c R^{\ell_{1}(t_{1})}. \]  

(4.7)

Here, there are two cases.

Case 1: If \( q < q' \), then passing to the limit in (4.7) as \( R \to \infty \), we obtain

\[ 0 < \int_{\Gamma} u_{1} dP_{1} + \int_{\Gamma} u_{2} dP_{2} + \frac{c}{2} \int_{\Omega} |u|^q dP = 0, \]

which leads to a contradiction.

Case 2: If \( q = q' \), then the right-hand side of (4.7) is bounded as \( R \to \infty \). Thus, \( |u|^q \) is integrable. Using Holder’s inequality with conjugate exponents \( q \) and \( q' = q/(q-1) \) from inequality (4.1), we obtain

\[ 0 < \int_{\Gamma} u_{1} \varphi_{R}(\xi, t_{1}, 0) dP_{1} + \int_{\Gamma} u_{2} \varphi_{R}(\xi, 0, t_{2}) dP_{2} + \frac{c}{2} \int_{\Omega} |u|^q \varphi_{R} dP \]

\[ \leq \lambda \left( (A_{\eta,1}(\phi))^{\frac{1}{q}} + (A_{\eta,2}(\phi))^{\frac{1}{q}} + (B_{\gamma,a}(\phi))^{\frac{1}{q}} \right) \int_{\Omega} |u|^q \varphi_{R} dP \]

\[ \leq c R^{-\frac{1}{q} + \frac{1}{q'}} \int_{\Omega} |u|^q \varphi_{R} dP \leq c \int_{\Omega} |u|^q \varphi_{R} dP \]

(4.8)

From integrability of \( |u|^q \), it follows that

\[ \int_{\Omega} |u|^q \varphi_{R} dP \to 0, \quad \text{as} \quad R \to \infty. \]

Hence, letting \( R \to \infty \) in (4.8) yields

\[ 0 < \int_{\Gamma} u_{1} dP_{1} + \int_{\Gamma} u_{2} dP_{2} + \frac{c}{2} \int_{\Omega} |u|^q dP = 0, \]

which leads to a contradiction. As a result, \( u \equiv 0 \) in both cases. Hence, we conclude that no global weak solution other than the trivial one, which ends the proof of Theorem 4.1. \( \square \)

Next, we extend the analysis of single ultraparabolic equation to the case of a coupled system of two-time ultraparabolic equations. The result concerning the Cauchy problem (1.3)–(1.4) is given by the following theorem.

**Theorem 4.2.** Assume that the functions \( f \) and \( g \) satisfy the condition (1.5), \( p, q > 1, u_{j}, v_{j} > 0, j = 1, 2 \) and \( (u_{1}, v_{1}) \cdot (u_{2}, v_{2}) \in L_{\infty}^{1}(\Gamma) \times L_{\infty}^{1}(\Gamma) \) such that

\[ \int_{\Gamma} u_{1} dP_{1} + \int_{\Gamma} u_{2} dP_{2} > 0, \quad \int_{\Gamma} v_{1} dP_{1} + \int_{\Gamma} v_{2} dP_{2} > 0. \]

If

\[ 1 < p \leq \min \left\{ 1 + \frac{\rho(a + b)}{2 + Q \min \left\{ \frac{a}{a + b} \right\}}, 1 + \frac{Q \min \left\{ \frac{a}{a + b} \right\}}{2 + Q \min \left\{ \frac{a}{a + b} \right\}} \right\}. \]

then problem (1.3)–(1.4) does not admit nontrivial global weak solution.
Proof of Theorem 4.2. Assume to the contrary. Let \((u, v)\) be a nontrivial weak solution to problem \((1.3)\)–\((1.4)\). Using the test function \(\varphi_R\) defined in \((3.1)\), from \((2.2)\), we have
\[
\int_{\Omega} u_1 \varphi_R (\xi, t_1, 0) \, dP_1 + \int_{\Omega} u_2 \varphi_R (\xi, 0, t_2) \, dP_2 + c \int_{\Omega} |v|^q \varphi_R \, dP \\
\leq \lambda \int_{\Omega} |u| \left( \frac{\partial \varphi}{\partial t_1} + \frac{\partial \varphi}{\partial t_2} + \left| -\Delta \varphi \right| \right)^\gamma \phi^{1-\gamma} \, dP. \tag{4.9}
\]
By Holder’s inequality with conjugate exponents \(p\) and \(p’ = p/(p - 1)\), from \((4.9)\) we obtain
\[
\int_{\Omega} u_1 \varphi_R (\xi, t_1, 0) \, dP_1 + \int_{\Omega} u_2 \varphi_R (\xi, 0, t_2) \, dP_2 + c \int_{\Omega} |v|^q \varphi_R \, dP \\
\leq \lambda \left[ (A_{p_1}(\phi))^{\gamma} + (A_{p_2}(\phi))^{\gamma} + (B_{p\alpha}(\phi))^{\gamma} \right] \left( \int_{\Omega} |u|^p \varphi_R \, dP \right)^\frac{1}{p’}. \tag{4.10}
\]
By using \((4.5)\) and Lemma 2.2, for \(R\) large enough, we get
\[
\int_{\Omega} |v|^q \varphi_R \, dP \leq c \left( \frac{R^{\frac{1}{p}}}{R^{p\alpha}} + \frac{R^{\frac{1}{q}}}{R^{q\beta}} \right) \left( \int_{\Omega} |u|^p \varphi_R \, dP \right)^\frac{1}{p’}. \tag{4.11}
\]
Now, denote
\[
X := \left( \int_{\Omega} |u|^p \varphi_R \, dP \right)^\frac{1}{p’}, \quad Y := \left( \int_{\Omega} |v|^q \varphi_R \, dP \right)^\frac{1}{q’}.
\]
Then, from \((4.10)\) and \((4.11)\) we get
\[
Y^q \leq c \left( R^{\frac{1}{p}} + R^{\frac{1}{q}} \right) X, \tag{4.12}
\]
\[
X^p \leq c \left( R^{\frac{1}{p}} + R^{\frac{1}{q}} \right) Y. \tag{4.13}
\]
Combining \((4.12)\) and \((4.13)\) we obtain
\[
Y^{pq-1} \leq c \left( R^{\frac{1}{p}} + R^{\frac{1}{q}} \right)^p \left( R^{\frac{1}{p}} + R^{\frac{1}{q}} \right)^p, \tag{4.14}
\]
\[
X^{pq-1} \leq c \left( R^{\frac{1}{p}} + R^{\frac{1}{q}} \right)^q \left( R^{\frac{1}{p}} + R^{\frac{1}{q}} \right)^q. \tag{4.15}
\]
Next, let us introduce the following notations
\[
\delta_1 := \frac{p}{p’} \max \left\{ \gamma_1(p, \gamma_2(q, \beta)), \frac{1}{\beta} \right\}, \quad \delta_2 := \frac{q}{q’} \max \left\{ \gamma_1(q, \gamma_2(p, \beta)), \frac{1}{\alpha} \right\},
\]
We impose that \(\delta_1 \leq 0\) and \(\delta_2 \leq 0\), from which we obtain
\[
\delta_1 = \left( Q + \frac{8}{\beta} \right) \frac{q}{q’} - p \min \left\{ \alpha, \frac{4}{\beta} \right\} - \min \left\{ \beta, \frac{4}{\alpha} \right\} \leq 0, \\
\delta_2 = \left( Q + \frac{8}{\alpha} \right) \frac{p}{p’} - q \min \left\{ \beta, \frac{4}{\alpha} \right\} - \min \left\{ \alpha, \frac{4}{\beta} \right\} \leq 0.
\]
By similar argument, we choose optimal value of \(\theta = \min \{4/\alpha, 4/\beta\}\). Then
\[
\min \left\{ a, \frac{4}{\theta} \right\} = a, \quad \min \left\{ \beta, \frac{4}{\theta} \right\} = \beta.
\]
Hence
\[
\begin{cases}
\delta_1 \leq 0 \\
\delta_2 \leq 0
\end{cases} \Rightarrow \left\{ 
\begin{array}{c}
pq \leq 1 + \frac{q(\alpha + \beta)}{2 + \Omega \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}} \min \left\{ \frac{1}{\alpha}, \frac{1}{\beta} \right\} \\
pq \leq 1 + \frac{p(q\alpha + \beta) + \alpha \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}}{2 + \Omega \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}} \end{array} \right.
\]
which is equivalent to
\[
1 < pq \leq (pq)^r \Rightarrow \min \left\{ 1 + \frac{q(\alpha + \beta)}{2 + \Omega \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}}, 1 + \frac{p(q\alpha + \beta) + \alpha \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}}{2 + \Omega \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}} \right.
\]
Here, we consider the following two cases.
Case 1: When \(pq < (pq)^r\), which is equivalent to \(\delta_1 < 0\) and \(\delta_2 < 0\), then the right-hand sides of \((4.14)\) and \((4.15)\) converges to zero as \(R \to \infty\). Consequently, both \(X\) and \(Y\) vanishes as \(R \to \infty\). This implies that
\[
\int_{\Omega} \|u\|^p \, dP \to 0, \quad \int_{\Omega} \|v\|^q \, dP \to 0, \quad R \to \infty.
\]
As a result, \(u \equiv 0\) and \(v \equiv 0\), which contradicts to the nontriviality of \((u, v)\).

Case 2: When \(pq = (pq)^r\), then it is equivalent to
\[
\delta_1 \leq 0 \quad \delta_2 \leq 0 \Rightarrow \left\{ \begin{array}{c}
pq \leq 1 + \frac{q(\alpha + \beta)}{2 + \Omega \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}} \min \left\{ \frac{1}{\alpha}, \frac{1}{\beta} \right\} \\
pq \leq 1 + \frac{p(q\alpha + \beta) + \alpha \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}}{2 + \Omega \min \left\{ \frac{1}{2}, \frac{1}{\gamma_1(q, \gamma_2(p, \beta))} \right\}} \end{array} \right.
\]
Similarly, from \((4.15)\), we have
\[
X^{pq-1} \leq c^{pq} \Rightarrow c^{pq} \leq c^{2pq-1} \quad R \to \infty,
\]
Hence, \(X \leq +\infty\) and \(Y \leq +\infty\) and thus
\[
\int_{\Omega} |u|^p \, dP \leq +\infty, \quad \int_{\Omega} |v|^q \, dP \leq +\infty.
\]
By Holder’s inequality, from \((4.9)\) we have
\[
0 < \int_{\Gamma_{R,1}} u_1 \varphi_R (\xi, t_1, 0) \, dP_1 + \int_{\Gamma_{R,2}} u_2 \varphi_R (\xi, 0, t_2) \, dP_2 + c \int_{\Omega} \|v\|^q \varphi_R \, dP \\
\leq c \left( R^{\frac{1}{p}} + R^{\frac{1}{q}} \right) \int_{\Omega} \|u\|^p \varphi_R \, dP \tag{4.16}
\]
Since \(\delta_1 \leq 0\) and \(\delta_2 = 0\), we have
\[
\max \left\{ \gamma_1(p, \gamma_2(q, \beta)), \frac{1}{\beta} \right\} = \frac{p}{p’} \max \left\{ \gamma_1(p, \gamma_2(q, \beta)), \frac{1}{\alpha} \right\}, \quad \frac{pq}{q’} \max \left\{ \gamma_1(p, \gamma_2(q, \beta)), \frac{1}{\alpha} \right\} \leq 0,
\]
which implies that
max \( \left\{ r_1(p), r_2(p, a) \right\} \leq 0. \)

Then (4.16) becomes

\[
0 < \int_{\Gamma_{R_1}} u_1 \varphi_R (\xi, t_1, 0) \, dP_1 + \int_{\Gamma_{R_2}} u_2 \varphi_R (\xi, t_2, 0) \, dP_2 + c \int_{\Omega_R} |v|^q \varphi_R \, dP
\]

\[
\leq c \left( \int_{\Omega_{R'(U \Omega_R)}} |u|^p \, dP \right)^{\frac{1}{p}}. \tag{4.17}
\]

Similarly, we have

\[
0 < \int_{\Gamma_{R_1}} v_1 \varphi_R (\xi, t_1, 0) \, dP_1 + \int_{\Gamma_{R_2}} v_2 \varphi_R (\xi, t_2, 0) \, dP_2 + c \int_{\Omega_R} |v|^q \varphi_R \, dP
\]

\[
\leq c \left( \int_{\Omega_{R'(U \Omega_R)}} |v|^q \, dP \right)^{\frac{1}{q}}. \tag{4.18}
\]

From the integrability property of \(|u|^p\) and \(|v|^q\), the right-hand sides of (4.17) and (4.18) tends zero as \(R \to \infty\).

Thus, letting \(R \to \infty\) in (4.17) and (4.18), we obtain

\[
0 < \int_{\Gamma} u_1 \, dP_1 + \int_{\Gamma} u_2 \, dP_2 + c \int_{\Omega} |v|^q \, dP = 0,
\]

and

\[
0 < \int_{\Gamma} v_1 \, dP_1 + \int_{\Gamma} v_2 \, dP_2 + c \int_{\Omega} |u|^p \, dP = 0.
\]

Then, we immediately obtain a contradiction. Therefore, in both cases no nontrivial global weak solution \((u, v)\) to the problem. This completes the proof of Theorem 4.2. \(\square\)

5. Conclusion

The study of nonexistence theory for differential equations is as important as the study of existence theory. Indeed, the sufficient conditions for nonexistence of solutions provide the necessary conditions for the existence of solutions. Results on nonexistence of solutions provide important and necessary information about the limiting behaviours of practical problems governed by differential equations.

The unsolvability conditions, which propose the necessary condition for the existence of solutions to Cauchy problems for multi-time ultra-parabolic equations and coupled systems have been obtained in this study. Such conditions have been established using the test function method. The well-known inequalities such as Young’s inequality, Holder’s inequality and Ju’s inequality have been also used to prove the nonexistence theorems.

The results obtained in this study have applications in determining unsolvability criteria for various practical problems which can be modeled by multi-time parabolic equations and systems. It provides essential information about the range of parameters and initial datum that arise in the governing equations for which their solutions do not exist.

Declarations

Author contribution statement

Birilew Belayneh Tsegaw: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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