THE YOSIDA CLASS IS UNIVERSAL

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Abstract. We discuss families of meromorphic functions $f_h$ obtained from single functions $f$ by the re-scaling process $f_h(z) = h^{-\alpha} f(h + h^{-\beta} z)$ generalising Yosida’s process $f_h(z) = f(h + z)$. The main objective is to obtain information on the value distribution of the generating functions themselves. Among the most prominent generalised Yosida functions are first, second and fourth Painlevé transcendents. The Yosida class contains all limit functions of generalised Yosida functions—the Yosida class is universal.

Keywords. Normal family, Nevanlinna theory, spherical derivative, Painlevé transcendents, elliptic function, Yosida function, re-scaling.

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1. Introduction

Yosida functions. In [17] Yosida introduced the class $(A)$ of transcendental meromorphic in the complex plane having bounded spherical derivative

$$f# = \frac{|f'|}{1 + |f|^2}.$$ (1)

Then the translates $f_h(z) = f(z + h)$ of $f$ in the class $(A)$ form a normal family in $\mathbb{C}$, and vice versa; $f$ is called of the first category, if no limit function

$$f = \lim_{h_n \to \infty} f_{h_n}$$ (2)

is a constant (convergence is always understood with respect to the spherical metric). It is this additional condition that makes the class $A_0$ of Yosida functions so fascinating. The elementary functions (like $e^z$, tan $z$ etc) have bounded spherical derivatives, but are not Yosida functions. On the other hand $A_0$ contains the elliptic functions. A thorough investigation of the class $A_0$ was performed by Favorov [3], with emphasis on the distribution of zeros and poles.

Painlevé transcendents. The first Painlevé transcendents are the solutions to Painlevé’s first differential equation $w'' = z + 6w^2$; they are meromorphic in $\mathbb{C}$ (see [14]) and satisfy $w# = O(|z|^\frac{3}{2})$. More precisely, if $Q$ denotes the set of (non-zero) zeros of $w$ and $Q_\epsilon = \bigcup_{q \in Q} \{z : |z - q| < \epsilon|q|^{-\delta}\}$, then $w#(z) = O(|z|^{-\delta})$ holds outside $Q_\epsilon$, while $w#(q) \simeq |q|^{\frac{3}{2}}$. The family $(w_h)_{|h| \geq 1}$ with $w_h(z) = h^{-\frac{1}{2}} w(h + h^{-\frac{1}{3}} z)$ is normal in $\mathbb{C}$, and for “most” solutions the limit functions $\lim_{h_n \to \infty} w_{h_n}$ are non-constant. There exist, however, solutions with large zero- and pole-free regions, and in that case one
has constant solutions \( \not\equiv 0, \infty \), see \[15\]. We note, however, that in many applications it is only required that \( f_h \not\to 0, \infty \).

**Definition.** The class \( Y_{\alpha, \beta} \) \((\alpha \in \mathbb{R}, \beta > -1)\) consists of all in \( \mathbb{C} \) transcendental meromorphic functions \( f \) in \( \mathbb{C} \), such that the family \((f_h)_{|h| \geq 1}\) of functions

\[
(f_h(z) = h^{-\alpha} f(h + h^{-\beta} z))
\]

(for any or just one determination of \( h^{-\alpha} \) and \( h^{-\beta} \)) form a normal family in \( \mathbb{C} \), and no limit function \([2]\) is constant. Functions in \( Y_{\alpha, \beta} \) are called generalised Yosida functions. We also define the classes \( Y_{\alpha, 0} \) and \( Y_{0, \beta} \), such that the family \((f_h)_{|h| \geq 1}\) of functions

\[
(f_h(z) = h^{-\alpha} f(h + h^{-\beta} z))
\]

(only for \( \alpha \not\equiv 0 \)) has constant solutions of functions \( f \). We define the classes \( Y_{\alpha, \beta} \) and \( Y_{\alpha, 0} \) by postulating normality of the family of functions \((f_h)_{|h| \geq 1}\) of functions \( f \) only in \( \mathbb{C} \setminus \{ -1 \} \), and postpone the analysis of this class to the last section.

**Remarks and Examples.**

- Some of the results proved in this paper are not new. This, in particular, concerns theorems in the classes \( Y_{0, 0} \) and \( Y_{0, -1} \) (see the last section) and \( A_0 = Y_{0, 0} \). The proofs in this paper cover all parameters \( \alpha \in \mathbb{R} \) and \( \beta > -1 \).
- The re-scaling process in the definition of \( Y_{\alpha, \beta} \) is motivated by and formally related to the Pung-Zalcman process \([11][12][17][20]\). As far as I know, particular classes \( Y_{\alpha, \beta} \) with \( \alpha \not\equiv 0 \) occurred for the first time, although implicitly, in the paper \([15]\) on the Painlevé transcendents. Of course, this kind of rescaling is also not new. It goes back at least to Valiron, but was even used in Painlevé’s so-called \( \alpha \)-method.
- It will turn out that \( Y_{\alpha, \beta} \) is contained in the class \( W_{2+|\alpha|+\beta} \) discussed by Gavrilov [4]: \( f \in W_p \) \((p \geq 1)\) if and only if \( \sup_{|z| = r} |f(z)| \) \((r < \infty)\), see also Makhmudov [8]. \( W_2 \) is Yosida’s class (A). The class \( W_p^{(0)} \), also discussed by Gavrilov, coincides with \( Y_{0, p-2} \), while the same is true for \( A_0 \) and \( Y_{0, 0} \).
- \( f \in Y_{\alpha, \beta} \) implies \( 1/f \in Y_{-\alpha, -1} \), and \( f(z) = z^a f(z^b) \) \((a \in \mathbb{Z}, b \in \mathbb{N})\) belongs to \( Y_{a+b\alpha, b+b\beta-1} \); \( z^b \) and \( z^a \) may be replaced by a polynomial \( p \) and a rational function \( r \), respectively, with \( \deg p = b \) and \( r(z) \sim cz^a \) as \( z \to \infty \). We mention two simple corollaries:
  - If \( \alpha = -a/b \) is rational, then \( f \in Y_{0, b/|a|+1} \).
  - If \( 0 < \beta < 0 \) and \( b \) is sufficiently large, then \( b + b\beta - 1 \geq 0 \).
- To every \( n \in \{ 2, 3, 4, 6 \} \) there exists a meromorphic function \( f \) such that \( f(z^n) \) is an elliptic function (see Mues [9]). Thus \( f \in Y_{0, 1/n} \), and \( f(z) = z^n f(z^n) \) belongs to \( Y_{a+b/n-1} \) \((a \in \mathbb{Z}, b \in \mathbb{N})\).
- \( f' \in Y_{\alpha, \beta} \) implies \( f \in Y_{-\alpha, -\beta} \) for at least one primitive.
- “Most” of the first, second and fourth Painlevé transcendents belong to \( Y_{1/2, 1/2} \), \( Y_{1/4, 1/4} \) and \( Y_{1, 1} \), respectively (for “some” solutions the second condition is violated, namely those having large zero- and pole-free regions).
  - Any first Painlevé transcendent has a primitive \( W \) which also is a first integral: \( w'' = 2zw + 4w^3 - 2W \); in “most” cases \( W \in Y_{1/2, 1/2} \), although \( w'^2, zw, w^3 \in Y_{1/2, 1/2} \).
  - The second Painlevé equation \( w'' = a + zw + 2w^3 \) has a first integral \( W \): \( w'^2 = 2aw + zw^2 + w^4 - W \) with \( W' = w^2 \); since \( w^2 \in Y_{1/2} \) (in “most” cases), \( W \in Y_{1/2, 1/2} \) follows, although \( w'^2, zw, w^4 \in Y_{2, 2} \).
Painlevé's fourth equation $2ww'' = w^2 + 3w^4 + 8zw^3 + 4(z^2 - a)w^2 + 2b$ also has a first integral $W$: $w^2 = w^4 + 4zw^3 + 4(z^2 - a)w^2 - 2b - 4wW$ with $W' = w^2 + 2zw$ and $W \in \mathcal{Y}_{1,1}$, again only in “most” cases.

2. Simple Properties

**Theorem 1.** Every $f \in \mathcal{Y}_{\alpha,\beta}$ satisfies $f^\#(z) = O(|z|^{|\alpha|+\beta})$.

**Proof.** We may assume $\alpha \geq 0$, otherwise would replace $f$ by $1/f$, noting that $f^\# = (1/f)^\#$ and $1/f \in \mathcal{Y}_{-\alpha,\beta}$. For $|h| \geq 1$ we have

$$f_h^\#(0) = |h|^{-\alpha-\beta} f^\#(h) \frac{1 + |f(h)|^2}{1 + |h|^{-2\alpha}|f(h)|^2} \geq |h|^{-\alpha-\beta} f^\#(h),$$

while the left hand side is bounded by Marty's Criterion. q.e.d.

**Remarks.**

- The bound $|z|^{|\alpha|+\beta}$ is sharp (not only for the Painlevé transcendents).
- It is obvious that every limit function $f = \lim_{n \to \infty} f_{h_n}$ belongs to $\mathcal{W}_2$. More precisely, $f^\#$ is bounded by $m_f = \sup_{z \in \mathbb{C}, |h| > 1} f^\#(z)$: if $z_0$ is not a pole of $f$, then we have also $f_{h_n}' \to f'$ close to $z_0$, hence $f^\#(z_0) = \lim_{n \to \infty} f_{h_n}^\#(z_0) \leq m_f$. At a pole of $f$ we will consider $1/f$ instead of $f$ (more in Theorem 8).
- The limit functions of the Painlevé families $(w_h)$ are elliptic functions.

Yosida [17] has shown that given $f \in A_0$ and $\epsilon > 0$ there exists some $\delta > 0$, such that $\int_{|z-h| < \delta} f^\#(z) d(x,y) > \delta$ holds for every $h \in \mathbb{C}$. The analog for $\mathcal{Y}_{\alpha,\beta}$ is Theorem 2 below. For $\beta$ fixed, $|h| > 1$ and $\epsilon > 0$ we set

$$\Delta_\epsilon(h) = \{ z : |z - h| < \epsilon|h|^{-\beta} \}.$$

**Theorem 2.** For every $f \in \mathcal{Y}_{\alpha,\beta}$ and $\epsilon > 0$ we have

$$\inf_{|h| > 1} |h|^{2|\alpha|} \int_{\Delta_\epsilon(h)} f^\#(z)^2 d(x,y) > 0 \quad \text{and} \quad \inf_{|h| > 1} \sup_{z \in \Delta_\epsilon(h)} f^\#(z)|z|^{|\alpha|-\beta} > 0.$$

**Remark.** The second inequality was proved by Gavrilov [5] for the class $\mathcal{Y}_{0,-1}$ (which he denoted $\mathcal{W}^0_1$).

**Proof.** The integral in question is $I = \int_{|w| < \epsilon} f^\#(h + h^{-\beta}w)^2 |h|^{-2\beta} d(u,v)$. From

$$f^\#(\zeta)|h|^{-\beta} = |h|^\alpha f_h^\#(w) \frac{1 + |h|^{-2\alpha}|f(\zeta)|^2}{1 + |f(\zeta)|^2} \geq \min\{1, |h|^{-2\alpha}\}|h|^\alpha f_h^\#(w) = |h|^{-|\alpha|} f_h^\#(w)$$

($\zeta = h + h^{-\beta}w$, $|h| \geq 1$) follows $|h|^{2|\alpha|} I \geq \int_{|w| < \epsilon} f_h^\#(w)^2 d(u,v)$, and by definition of $\mathcal{Y}_{\alpha,\beta}$ the right hand side has a positive infimum with respect to $h$. q.e.d.
Theorem 3. Let $f$ be meromorphic in $\mathbb{C}$. Then in order that $f \in \mathcal{Y}_{\alpha,\beta}$ it is necessary and sufficient that

$$f^\#(z) = O(|z|^\beta) \quad \text{and} \quad \liminf_{|h| \to \infty} \sup_{z \in \Delta_c(h)} f^\#(z)|z|^{-\beta} > 0$$

for some [all] $\epsilon > 0$.

Proof. We just have to prove sufficiency. The first condition ensures that $(f_h)$ is a normal family in $\mathbb{C}$, and the second guarantees that the limit functions are non-constant: $\sup f^\#(z) > 0$. q.e.d.

Definition. Given $f \in \mathcal{Y}_{\alpha,\beta}$ we denote by $\mathcal{P}$ and $\mathcal{Q}$ the set of non-zero poles and zeros of $f$, respectively (if any), and set (for the definition of $\Delta_c$ see (5))

$$\mathcal{P}_c = \bigcup_{p \in \mathcal{P}} \Delta_c(p) \quad \text{and} \quad \mathcal{Q}_c = \bigcup_{q \in \mathcal{Q}} \Delta_c(q).$$

Theorem 4. For $f \in \mathcal{Y}_{\alpha,\beta}$ we have

$$\inf_{q \in \mathcal{Q}} \text{dist}(q, \mathcal{P})|q|^{\beta} > 0 \quad \text{and} \quad \inf_{p \in \mathcal{P}} \text{dist}(p, \mathcal{Q})|p|^{\beta} > 0.$$

Proof. Take any sequence $(q_n)$ of zeros such that $\text{dist}(q_n, \mathcal{P})|q_n|^{\beta} \to \inf_{q \in \mathcal{Q}} \text{dist}(q, \mathcal{P})|q|^{\beta}$ and $f_{q_n} \to f \neq \text{const}$, locally uniformly in $\mathbb{C}$. Then $f(0) = 0$ implies $|f(z)| < 1$ on some disc $|z| < \delta$, hence $\liminf \text{dist}(q_n, \mathcal{P})|q_n|^{\beta} > \delta$ by Hurwitz’ theorem, this showing that $\inf \text{dist}(q, \mathcal{P})|q|^{\beta} > \delta > 0$. Concerning the second assertion we just note that $1/f \in \mathcal{Y}_{-\alpha,\beta}$, so that the notions “pole” and “zero” may be interchanged. q.e.d.

Remark. We will say that the zeros and poles of $f$ are $\beta$-separated. From now on it will be tacitly assumed that $\mathcal{Q}_c \cap \mathcal{P}_c = \emptyset$.

Theorem 5. Every $f \in \mathcal{Y}_{\alpha,\beta}$ satisfies

\begin{align*}
(i) \quad |f(z)| &= O(|z|^\alpha) \quad (z \notin \mathcal{P}_c); \\
(ii) \quad |1/f(z)| &= O(|z|^{-\alpha}) \quad (z \notin \mathcal{Q}_c); \\
(iii) \quad |f(z)| &\asymp |z|^{\alpha} \quad (z \notin \mathcal{P}_c \cup \mathcal{Q}_c); \\
(iv) \quad |f'(z)/f(z)| &= O(|z|^\beta) \quad (z \notin \mathcal{P}_c \cup \mathcal{Q}_c); \\
v) \quad f^\#(z) &= O(|z|^{\beta-|\alpha|}) \quad (z \notin \mathcal{P}_c \cup \mathcal{Q}_c).
\end{align*}

Proof. Let $(h_n)$ be any sequence outside $\mathcal{P}_c$, such that $f_h$ tends to $f \neq \text{const}$, locally uniformly in $\mathbb{C}$, and $|f(h_n)||h_n|^{\alpha}$ tends to $M_c = \sup_{z \in \mathcal{P}_c} |f(z)||z|^{-\alpha}$. Then $M_c = |f(0)|$ is finite. The second assertion follows from $1/f \in \mathcal{Y}_{-\alpha,\beta}$, and together we obtain $|f(z)| \asymp |z|^{\alpha}$ $(z \notin \mathcal{P}_c \cup \mathcal{Q}_c)$. (iv) follows from $\frac{h_n^{-\beta}f'(h_n)}{f(h_n)} \to \frac{f'(0)}{f(0)} \neq \infty$, and from (iii) and (iv) follows $f^\#(z) = \frac{|f'(z)|}{|f(z)|} \frac{1}{|f(z)|} = O(|z|^{\beta-|\alpha|})$, hence (v). q.e.d.

Remark. The symbol $\asymp$ has proved very useful: $\phi(z) \asymp \psi(z)$ in some real or complex region means $|\phi(z)| = O(|\psi(z)|)$ and $|\psi(z)| = O(|\phi(z)|)$.

Corollary 1. Every function $f \in \mathcal{Y}_{\alpha,\beta}$ has infinitely many zeros and poles.
Proof. If \( f \) had only finitely many poles, then \( f \) were rational as follows from \( f(z) = O(|z|^\alpha) \) outside \( P \), hence in \( |z| > R \). \ q.e.d.

Theorem 6. For \( f \in \mathcal{Y}_{\alpha,\beta} \) and \( \tilde{f} \in \mathcal{Y}_{\tilde{\alpha},\tilde{\beta}} \) with sets of poles and zeros \( Q \) and \( \tilde{Q} \), and \( P \) and \( \tilde{P} \), respectively, the product \( f \tilde{f} \) belongs to \( \mathcal{Y}_{\alpha+\tilde{\alpha},\beta+\tilde{\beta}} \) if \( Q \cup \tilde{Q} \) and \( P \cup \tilde{P} \) are \( \beta \)-separated. In particular, \( f^m \) belongs to \( \mathcal{Y}_{m\alpha,\beta} \).

Proof. The hypotheses ensure that zeros [poles] of \( f \) cannot collide with poles [zeros] of \( \tilde{f} \), hence \( f_n \tilde{f}_n \to \tilde{f} \). \ q.e.d.

By Theorem 6 the zeros and poles of \( f \) are \( \beta \)-distributed in the following sense:

Theorem 7. Given \( f \in \mathcal{Y}_{\alpha,\beta} \) there exist positive numbers \( \epsilon_0, \eta_0, \) and \( M \), such that

(i) every disc \( \Delta_{\epsilon_0}(z_0) \) contains at least one zero and one pole;

(ii) every disc \( \Delta_{\epsilon_0}(z_0) \) contains at most \( M \) zeros (counted by multiplicities) and no pole, or at most \( M \) poles of \( f \) and no zeros.

In particular, the zeros and poles of \( f \) have bounded multiplicities.

Proof. Suppose there exist sequences \( h_n \to \infty \) and \( \eta_n \to \infty \), such that \( \Delta_{\eta_n}(h_n) \) contains no poles (the same for zeros), while \( f_n \to \tilde{f} \neq \text{const} \), locally uniformly in \( \mathbb{C} \). Then by Hurwitz’ Theorem, \( \tilde{f} \) is finite in every euclidian disc \( |z| < \eta_n \), hence is an entire function, this contradicting Corollary 6. Similarly, if we assume that the pair \((\epsilon_0, M)\) does not exist, then there exist sequences \( \epsilon_n \to 0 \) and \( h_n \to \infty \), such that \( f \) has at least \( n \) zeros (say) in \( \Delta_{\epsilon_n}(h_n) \), while \( f_n \) tends to some non-constant function \( \tilde{f} \). By Hurwitz’ theorem, \( \tilde{f} \) has a zero at the origin of order \( \geq n \) for every \( n \), which is absurd. Thus there exists \( \epsilon > 0 \), such that the number of zeros in \( \Delta_{\epsilon_0}(z_0) \) is bounded, uniformly with respect to \( z_0 \). Diminishing \( \epsilon_0 \), if necessary, it we may achieve by Theorem 6 that none of the discs \( \Delta_{\epsilon_0}(z_0) \) contains a pole. \ q.e.d.

Remark. It is not hard to prove that there also exists some \( \theta_0 > 0 \), such that \( f \) assumes every value in every disc \( \Delta_{\theta_0}(z_0) \).

Theorem 8. [The Yosida Class is Universal] For \( f \in \mathcal{Y}_{\alpha,\beta} \) (\( \beta > -1 \)) the limit functions \( \tilde{f} = \lim_{h_n \to \infty} f_n \) belong to the Yosida class \( A_0 = \mathcal{Y}_{0,0} \).

Proof. First of all \( \tilde{f} \) has bounded spherical derivative, hence \( \tilde{f} \in W_2 \) and the family \((f_h)_{h \in \mathbb{C}} \) of translations \( f_h(z) = f(z + h) \) is normal in \( \mathbb{C} \). Also the corresponding sets \( \mathcal{P} \) and \( \mathcal{Q} \) are \( 0 \)-distributed (euclidian distance between \( \mathcal{P} \) and \( \mathcal{Q} \) is positive), and equally \( 0 \)-separated: there exist positive numbers \( \epsilon_0, \eta_0 \) and \( M \), such that every disc \( |z - h| < \eta_0 \) contains at least one zero and one pole, while every disc \( |z - h| < \epsilon_0 \) contains at most \( M \) poles [zeros], and no zeros [poles]. If \( (h_n) \) is any sequence tending to \( \infty \), then the disc \( |z - h_n| < \eta_0 \) contains at least one zero \( q_n \) and one pole \( p_n \). Since \( |p_n - q_n| \geq 2\epsilon_0 \), all limit functions of \((f_{h_n})\) also have at least one zero and one pole in \( |z| \leq 2\eta_0 \), and therefore are non-constant. \ q.e.d.

3. Value Distribution

In this section we are concerned with the value distribution of functions \( f \in \mathcal{Y}_{\alpha,\beta} \). For the definition of the Nevanlinna functions \( T(r, f) \), \( m(r, f) \) and \( N(r, f) \), and for
basic results in Nevanlinna Theory the reader is referred to Hayman \[7\] and Nevanlinna \[10\]. From the Ahlfors-Shimizu formula

\[ T(r, f) = \frac{1}{\pi} \int_0^r \int_{|z| < t} f^#(z)^2 \, d(x, y) \, dt + O(1) \]

and Theorem 4 follows \( T(r, f) = O(r^{2(|\alpha| + \beta + 1)}) \), hence \( f \in \mathcal{Y}_{\alpha, \beta} \) has order of growth

\[ \mathbf{g}(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \]

at most \( 2(|\alpha| + \beta + 1) \). Replacing \( f \) by \( \tilde{f}(z) = z^\alpha f(z^b) \) with \( \tilde{f} \in \mathcal{Y}_{a + b\alpha, b + \beta - 1} \) yields \( \mathbf{g}(f) = \mathbf{g}(\tilde{f})/b \leq 2(|a + b\alpha| + \beta + 1) \), and since

\[ \inf_{a \in \mathbb{Z}, b \in \mathbb{N}} |a/b + \alpha| = 0, \]

we obtain in any case:

**Theorem 9.** Every \( f \in \mathcal{Y}_{\alpha, \beta} \) has order of growth \( \mathbf{g}(f) \leq 2\beta + 2 \).

**Remark.** For the first Painlevé transcendents the first estimate yields \( \mathbf{g}(w) \leq \frac{3}{2} \), while the order is \( \mathbf{g}(w) = \frac{5}{2} = 2\left(\frac{1}{2} + 1\right) \). Similarly we have the (sharp) estimates \( \mathbf{g}(u) \leq 3 = 2\left(\frac{1}{2} + 1\right) \) and \( \mathbf{g}(w) \leq 4 = 2(1 + 1) \) for the second and fourth Painlevé transcendents, respectively (see [15]).

**Theorem 10.** Every \( f \in \mathcal{Y}_{\alpha, \beta} (\beta > -1) \) has \( r^{2\beta + 2} \) zeros and poles in \( |z| < r \) :

\[ n(r, 0) \asymp r^{2\beta + 2} \quad \text{and} \quad n(r, \infty) \asymp r^{2\beta + 2}. \]

In particular, \( f \) has order of growth \( \mathbf{g}(f) = 2\beta + 2 \).

**Remark.** We remind the reader that \( \phi(r) \asymp \psi(r) \) means \( \phi(r) = O(\psi(r)) \) and \( \psi(r) = O(\phi(r)) \) as \( r \to \infty \).

**Proof.** With every pole \( p \) in \( |p| < r \) we associate the disc \( \Delta_{\alpha_0}(p) \); by Theorem 7 it contains at most \( M \) poles. Starting with \( p_1 (|p_1| < r) \), let \( p_2 (|p_2| < r) \) be any of the poles not contained in \( \Delta_{\alpha_0}(p_1) \), \( p_3 (|p_3| < r) \) not contained in \( \Delta_{\alpha_0}(p_1) \cup \Delta_{\alpha_0}(p_2) \), and so forth; we may arrange that \( |p_{n+1}|^{-\beta} \geq |p_{n+1}|^{-\beta} \) holds. Then obviously \( n(r, \infty) = O(\phi(r)) \), where \( \phi(r) \) counts how many mutually disjoint discs \( \Delta_{\alpha_0/2}(p) \) may be placed in a large euclidian disc \( |z| < r + \epsilon_0 r^{-\beta} \). The geometric answer is \( \phi(r) = O(r^{2\beta + 2}) \), if \( \beta > 0 \), and \( \phi(r) \leq \phi(r/2) + O(r^{2\beta + 2}) \) if \( -1 < \beta < 0 \), which also implies \( \phi(r) = O(r^{2\beta + 2}) \) (consider the radii \( r = 2^k \)). Thus

\[ n(r, \infty) = O(r^{2\beta + 2}) \]

holds in any case. To prove the converse, we note that for \( r \) sufficiently large the annulus \( |z| - r| < \eta_0 r^{-\beta} \) contains at least \( c r^{\beta + 1} \) mutually disjoint discs of radius \( \eta_0 r^{-\beta} \), hence also at least \( c r^{\beta + 1} \) poles. Again we have to distinguish the cases (i) \( \beta > 0 \) and (ii) \( -1 < \beta < 0 \). Starting with \( r_1 \) sufficiently large we define in case (i) \( r_k = r_{k-1} + 2\eta_0 r_{k-1}^{-\beta} \), while in case (ii) \( r_k \) denotes the unique solution to the equation \( r_k = r_{k-1} + 2\eta_0 r_{k-1}^{-\beta} \) \( (k = 2, 3, \ldots) \); note that \( r \to r - 2\eta_0 r^{-\beta} \) is increasing on \( r^{-\beta - 1} < 1/(2|\beta|\eta_0) \) if \( -1 < \beta < 0 \). Then the annuli \( |z| - r_k| < \eta_0 r_{k}^{-\beta} \) are mutually disjoint, and each contains at least \( c r_k^{\beta + 1} \) poles of \( f \). We claim

\[ \nu_k = n(r_k, \infty) \geq 2c r_k^{2\beta + 2}, \]

\[ ^1 \text{There is a misprint in [15]: } "T(r, f) = 2T(r, w) + O(\log r)" \text{ for } f(z) = \frac{1}{z} w(z^2), \] of course, has to be replaced by "\( T(r, f) = T(r^2, w) + O(\log r)". \]
provided \( c \) is sufficiently small, this implying \( n(r, \infty) \geq cr^{2\beta+2} \) for \( r \) sufficiently large (note that \( r_k \to \infty \)). Assuming \( \nu_k \geq 2cr^{2\beta+2}_k \) to be true, we obtain

\[
\nu_k \geq \nu_{k-1} + c' r_k^{\beta+1} \geq 2cr_k^{2\beta+2} + c' r_k^{\beta+1} \\
\geq 2cr_k^{2\beta+2} - 2c(r_k - r_{k-1})(2\beta + 2)r_k^{\beta+1} + c' r_k^{\beta+1}
\]

by the Mean Value Theorem. In case (ii) we have \( r_k - r_{k-1} = 2\eta_0 r_k^{-\beta} \), while in case (i) \( r_k - r_{k-1} = 2\eta_0 r_k^{-\beta} \leq 3\eta_0 r_k^{-\beta} \) holds (assuming \( r_1 \) sufficiently large). We thus obtain

\[
\nu_k \geq 2cr_k^{2\beta+2} + c r_k^{\beta+1} |c' - 2c3\eta_0(2\beta + 2)| = 2cr_k^{2\beta+2}
\]

if \( c \) is chosen to satisfy \( c' - 2c3\eta_0(2\beta + 2) = 0 \). Finally from \( r_k = O(r_{k-1}) \) follows

\[
r^{2\beta+2} = O(n(r, \infty))
\]

in all cases \( \beta > -1 \). The assertion about the order of growth now follows from \( g(f) \leq 2\beta + 2 \) on one hand, and \( T(r, f) \geq N(r, f) \asymp r^{2\beta+2} \) on the other. \textit{q.e.d.}

From the proof we obtain:

**Corollary 2.** For \( \beta > -1 \) and \( c > \eta_0 \), every annulus \( ||z| - r|| < cr^{-\beta} \) contains \( \asymp r^{\beta+1} \) zeros [poles] of \( f \in \mathcal{Y}_{\alpha, \beta} \).

**Theorem 11.** For every \( f \in \mathcal{Y}_{\alpha, \beta} \) holds

\[
m(r, f) + m(r, 1/f) = \frac{1}{2\pi} \int_0^{2\pi} |\log |f(re^{i\theta})|| \, d\theta = O(\log r), \tag{8}
\]

and, in particular,

\[
T(r, f) \sim N(r, f) \sim N(r, 1/f) \asymp r^{2\beta+2}. \tag{9}
\]

**Remark.** For the class \( \mathcal{Y}_{0, \beta} \) the first assumption was proved by Favorov \( \mathbf{3} \), even with \( O(1) \) instead of \( O(\log r) \).

**Proof.** Let \( C_r \) denote the circle \( |z| = r \). For \( 0 < \epsilon < \epsilon_0 \) fixed, the contribution of \( C_r \setminus (\mathcal{Q}_r \cup \mathcal{P}_r) \) to the integral is \( O(\log r) \) by Theorem \( \mathbf{4} \) (it says, among others, that \( |f(z)| \asymp |z|^\alpha \)). Let \( K \) be a component of \( \mathcal{Q}_r \) [or \( \mathcal{P}_r \)] that intersects the circle \( C_r \). If \( K \) contains the zeros \( q_\mu \) (\( 1 \leq \mu \leq m \leq M \)) [or the poles \( p_\nu \) (\( 1 \leq \nu \leq n \leq M \)], but not zeros and poles simultaneously], then \( \Phi(z) = f(z) \prod_{\mu=1}^m (z - q_\mu)^{-1} \) is zero- and pole-free in \( K \) and satisfies \( |\Phi(z)| \asymp r^{\alpha} r^{-m\beta} \) on \( \partial K \) by Theorem \( \mathbf{4} \) and also in \( K \) by the Maximum-Minimum Principle. Thus the contribution of \( C_r \cap K \) to the integral is

\[
\sum_{\mu=1}^m \int_{I_r} |\log |re^{i\theta} - q_\mu|| \, d\theta + O(|I_r| \log r),
\]

where \( I_r = \{ \theta \in [0, 2\pi) : re^{i\theta} \in K \} \) and \( |I_r| \) is its linear measure. From Lemma \( \mathbf{2} \) at the end of section \( \mathbf{5} \) follows \( |I_r| = O(r^{\beta+1}) \), hence

\[
\int_{I_r} |\log |re^{i\theta} - q_\mu|| \, d\theta = O\left( \int_0^{r^{-\beta-1}} |\log(r\theta)|| \, d\theta \right) = O(r^{\beta+1} \log r).
\]

The assertion follows from the fact, that by virtue of Corollary \( \mathbf{2} \) there are at most \( O(r^{\beta+1}) \) components \( K \) intersecting \( C_r \). \textit{q.e.d.}
THEOREM 12. For every \( f \in \mathcal{Y}_{\alpha, \beta} \) and \( c \in \mathbb{C} \) we have \( m \left( r, \frac{1}{f - c} \right) = O(\log r) \).

REMARK. Yosida [17] proved \( m(r, 1/(f - c)) = O(r) \) for \( f \in A_0 \).

PROOF. We just note that Theorem 11 also holds for \( f - c \) instead of \( f \). For \( \alpha \geq 0 \) we have \( f - c \in \mathcal{Y}_{\alpha, \beta} \), while \( 1/f \in \mathcal{Y}_{-\alpha, \beta} \) if \( \alpha < 0 \) and

\[ |\log |f - c|| \leq |\log |c|| + |\log |f|| + |\log |1/f - 1/c||. \]

q.e.d.

THEOREM 13. The c-points \((c \neq 0)\) of \( f \in \mathcal{Y}_{\alpha, \beta} \) are \( \beta \)-close to the zeros, and \( \beta \)-separated from the poles if \( \alpha > 0 \), and vice versa if \( \alpha < 0 \):

\[ \lim_{\zeta \rightarrow \infty, f(\zeta) = c} |\zeta|^\beta \text{dist}(\zeta, \mathcal{Q}) = 0 \quad \text{and} \quad \inf_{f(\zeta) = c} |\zeta|^\beta \text{dist}(\zeta, \mathcal{P}) > 0. \]

For \( \alpha = 0 \) and any pair \((a, b)\) the sets of \( a \)- and \( b \)-points are \( \beta \)-separated.

PROOF. The first assertion \((\alpha > 0)\) follows from \( f - c \in \mathcal{Y}_{\alpha, \beta} \) and Theorem \( \alpha \). If \((\zeta_n)\) denotes any sequence of c-points such that \( f(\zeta_n) \rightarrow f \neq \text{const} \), then we have also \( \zeta_n^0(f(\zeta_n + \zeta_n^{-\beta}z) - c) \rightarrow f(z) \) and \( f(0) = 0 \). From Hurwitz’ Theorem then follows \( |\zeta_n|^\beta \text{dist}(\zeta_n, \mathcal{Q}) \rightarrow 0 \) \((n \rightarrow \infty)\). Finally, since \( \mathcal{Y}_{0, \beta} \) is Möbius invariant, every pair \((a, b)\) can play the role of \((0, \infty)\). q.e.d.

4. DERIVATIVES

The derivative of \( f_h \) is \( f'_h(z) = h^{-\alpha - \beta}f'(h + h^{-\beta}z) \), and since the limit functions of the family \((f_h)\) are non-rational, one might expect that \( f' \in \mathcal{Y}_{\alpha + \beta, \beta} \). Now a trivial necessary condition for \( \phi_n \rightarrow \phi \neq \text{const} \), locally uniformly in some domain \( D \), is that the a-points and b-points of \( \phi_n \) are locally uniformly 0-separated (separated with respect to euclidian metric in any compact subset of \( D \)). In general, \( \phi_n \rightarrow \phi \) does not imply \( \phi'_n \rightarrow \phi' \) if \( \phi_n \) has poles, in other word, there is no Weierstrass Convergence Theorem for meromorphic functions (while the converse is true: \( \phi'_n \rightarrow \psi \) implies that \( \psi \) has a primitive \( \phi \), and \( \phi_n \rightarrow \phi + \text{const} \)). The obstacle that prevents \( \phi'_n \) from converging to \( \phi' \) is the existence of colliding poles of \( \phi_n \) and/or of zeros of \( \phi'_n \) colliding with poles.

LEMMA 1. Suppose that \( \phi_n \) converges to \( \phi \), locally uniformly in \(|z| < r \), and \( \phi \) has a pole of order \( m \) at \( z = 0 \). Then \( \phi'_n \rightarrow \phi' \), locally uniformly in some neighbourhood of \( z = 0 \), each of the following conditions is necessary and sufficient: there exist \( \rho > 0 \) and \( n_0 \), such that for \( n \geq n_0 \)

(i) \( \phi_n \) has only one pole (of order \( m \)) in \(|z| < \rho \);

(ii) \( \phi'_n \) has no zeros in \(|z| < \rho \).

PROOF. Since \( \phi' \) has a pole of order \( m + 1 \) at \( z = 0 \), and no other pole and also no zero in \(|z| < 2\rho \), it is necessary for \( \phi'_n \rightarrow \phi' \), uniformly in some neighbourhood of \( z = 0 \), that \( \phi'_n \) (\( n \geq n_0 \)) has \( m + 1 \) poles (counted with multiplicities) and no zero in \(|z| < \rho \), say. Since every pole of \( \phi_n \) of order \( \ell \) is a pole of order \( \ell + 1 \) of \( \phi'_n \), this means that \( \phi_n \) has only one pole in \(|z| < \rho \). Conversely, if \( \phi_n \) has only one pole \( b_n \) (of order \( m \)) with \( b_n \rightarrow 0 \), then we have \( \phi_n(z) = \frac{\psi_n(z)}{(z - b_n)^m}, \psi_n \rightarrow \psi, \psi(0) \neq 0, \psi'_n \rightarrow \psi' \),

\[ |z\psi'(z) - m\psi(z)|_{z=0} \neq 0, \quad \text{and} \]

\[ \phi'_n(z) = \frac{(z - b_n)\psi'_n(z) - m\psi_n(z)}{(z - b_n)^{m+1}} \rightarrow \frac{z\psi'(z) - m\psi(z)}{z^{m+1}} = \phi'(z), \]
uniformly in some neighbourhood of \( z = 0 \). It remains to show that (ii) implies (i). If \( \phi_n \) has \( p > 1 \) different poles in \( |z| < \rho \) of total multiplicity \( m \), then by the Riemann-Hurwitz formula \( \phi \) has \( m - 1 \) critical points close to \( z = 0 \), only \( m - p \) of them arising from multiple poles. Thus \( \phi'_n \) has \( p - 1 \) zeros close to \( z = 0 \). q.e.d.

REMARK. In any case the sequence \( \phi'_n \) tends to \( \phi' \), locally uniformly in \( 0 < |z| < \rho \). If (i) or (ii) is violated, then some of the poles of \( \phi'_n \) collide with zeros of \( \phi'_n \), and in the limit multiplicities disappear as do the zeros of \( \phi'_n \). If \( \phi_n = 1/P_n \), \( P_n \) a polynomial of degree \( m \), the equivalence of (i) and (ii) follows from the Gauß-Lucas Theorem.

**Theorem 14.** In order that for \( f \in \mathcal{Y}_{\alpha, \beta} \) the derivative \( f' \) belongs to \( \mathcal{Y}_{\alpha + \beta, \beta} \), each of the following conditions is necessary and sufficient:

(i) \( \inf_{p \in \mathcal{P}} |p|^{-\beta} \text{dist}(p, \mathcal{P} \setminus \{p\}) > 0 \);

(ii) \( \inf_{f'(c)=0} |c|^{-\beta} \text{dist}(c, \mathcal{P}) > 0 \).

**Corollary 3.** If the poles of \( f \in A_0 \) are 0-separated from each other, then every derivative of \( f \) also belongs to \( A_0 \).

**Example.** We construct \( f \in \mathcal{Y}_{0,0} \) such that \( f' \notin \mathcal{Y}_{0,0} \): \( \phi(z) = \sum_{k=1}^{\infty} \frac{1}{(z-k^2)^2-k^2} \) is meromorphic in \( \mathbb{C} \). If \( |z-k^2| \geq k/2 \) holds for every \( k \), then \( \sum_{k=2}^{\infty} Mk^{-2} \) is a convergent majorant, hence \( f(z) = o(1) \) as \( z \to \infty \) outside \( \bigcup_{k \geq 1} \{z : |z-k^2| < k/2\} \), while in case \( |z-\ell^2| < \ell/2 \) for some \( \ell \) we have \( |z-k^2| \geq k/2 \) for \( k \neq \ell \) and \( f(z) = \frac{1}{(z-\ell^2)^2-\ell^2} + o(1) \) as \( z \to \infty \) by the same reason. Thus the limit functions \( \lim_{h_n \to \infty} \phi_{h_n} \) are either constants or else have the form \( (z-z_0)^{-2} \). Then for \( f_0 \in \mathcal{Y}_{0,0} \) we have also \( f = f_0 + \phi \in \mathcal{Y}_{0,0} \), but \( f' \notin \mathcal{Y}_{0,0} \).

From \( f' \in \mathcal{Y}_{\alpha+\beta, \beta} \) would follow \( m(r, 1/f') = O(\log r) \). This, however, is true anyway and provides a new proof of Theorem 12.

**Theorem 15.** Every \( f \in \mathcal{Y}_{\alpha, \beta} \) satisfies \( m(r, 1/f') = O(\log r) \).

**Remark.** This was proved by Yosida [17] for \( f \in A_0 \) with \( O(\log r) \) replaced by \( O(r) \).

**Proof.** Taking into account that \( m(r, 1/f) = O(\log r) \) and \( m(r, f) = O(\log r) \), hence also \( m(r, f') \leq m(r, f) + O(\log r) = O(\log r) \) holds,

\[
(10) \quad m(r, 1/f') = -\frac{1}{2\pi} \int_0^{2\pi} \log f^\#(re^{i\theta}) \, d\theta + O(\log r)
\]

follows. We claim that the right hand side of (10) is \( O(\log r) \). The lower estimate follows from Theorem 11 — \( \log f^\#(z) \geq -(|\alpha|+\beta) \log |z| + O(1) \). It remains to prove

\[2\text{More generally, Yosida [17] proved that } 2T(r, f) - N_1(r) = -\frac{1}{2\pi} \int_0^{2\pi} \log f^\#(re^{i\theta}) \, d\theta + O(1) \text{ holds, where } N_1(r) \text{ "counts" the critical points of } f. \text{ The following proof is straight forward:}
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} \log f^\#(re^{i\theta}) \, d\theta = m(r, f') - m(r, 1/f') - 2m(r, f) + O(1)
\]

\[
= -[N(r, f) + N(r, f)] + N(r, 1/f') - 2T(r, f) + 2N(r, f) + O(1)
\]

\[
= [N(r, f) - N(r, f)] + N(r, 1/f') - 2T(r, f) + O(1)
\]

\[
= N_1(r) - 2T(r, f) + O(1).
\]
that
\[
-\frac{1}{2\pi} \int_0^{2\pi} \log[r^{\alpha-\beta} f^\#(re^{i\theta})] \, d\theta \leq C
\]
holds. To this end we divide \([0, 2\pi]\) into \(\approx r^{\beta+1}\) intervals of length \(\approx r^{-\beta-1}\). If (11) is not true, then there exists a sequence \(r_n \to \infty\) and intervals \(I_n\) of length \(\approx r_n^{-\beta-1}\), such that
\[
J_n = -r_n^{\beta+1} \int_{I_n} \log[r^{\alpha-\beta} f^\#(r_n e^{i\theta})] \, d\theta \to \infty.
\]

We may assume that \(I_n = [-r_n^{-\beta-1}, r_n^{-\beta-1}]\) and \(f_n \to \not\equiv\) const. From
\[
fr_n^\#(z) = r_n^{-\alpha-\beta} f^\#(z) \frac{1 + |f(z)|^2}{1 + r_n^{-2\alpha}|f(z)|^2} \leq r_n^{-\alpha-\beta} f^\#(z) \max\{1, r_n^{2\alpha}\} = r_n^{\alpha-\beta} f^\#(z)
\]
\((\zeta = r_n e^{i\theta} = r_n + r_n^{-\beta} z, r_n d\theta = |d\zeta| = r_n^{-\beta}|dz|)\) then follows
\[
-\log[r_n^{\alpha-\beta} f^\#(z)] \leq -\log fr_n^\#(z) \text{ and } \limsup_{n \to \infty} J_n \leq -\int_{[-i, i]} \log f^\#(z) |dz|, \text{ q.e.d.}
\]

5. Series And Product Developments

Since the series \(\sum_{p \in \mathcal{P}} |p|^{-s-1}\) and \(\sum_{q \in \mathcal{Q}} |q|^{-s-1}\) diverge if \(s = 2\beta + 2 = g(f)\), and converge if \(s > 2\beta+2\), the canonical products \(\prod_{q \in \mathcal{Q}} E(z, q, [2\beta]+2)\) and \(\prod_{p \in \mathcal{P}} E(z, p, [2\beta]+2)\)
converge absolutely and locally uniformly; \(E(u, g) = (1 - u)e^{u + u^2/2 + \cdots + u^g/g}\) denotes the Weierstrass prime factor of genus \(g\). Hence any \(f \in \mathcal{Y}_{\alpha,\beta}\) has the

HADAMARD PRODUCT REPRESENTATION
\[
f(z) = z^s e^{S(z)} \prod_{q \in \mathcal{Q}} E\left(\frac{z}{q}, q^m\right) \prod_{p \in \mathcal{P}} E\left(\frac{z}{p}, p^m\right)
\]
\((s \in \mathbb{Z} \text{ and } S \text{ a polynomial with } \deg S \leq m = 2 + [2\beta])\), and differentiation yields the

MITTAG-LEFFLER EXPANSION
\[
\frac{f'(z)}{f(z)} = \frac{s}{z} + S'(z) + \sum_{q \in \mathcal{Q}} \frac{z^m}{(z-q)q^m} - \sum_{p \in \mathcal{P}} \frac{z^m}{(z-p)p^m}.
\]

If we do not insist in absolute convergence, then much more can be said.

**Theorem 16.** Suppose \(f \in \mathcal{Y}_{\alpha,\beta}\) and \(f(0) \neq 0, \infty\). Then
\[
\frac{f'(z)}{f(z)} = T_{m-1}(z) + \lim_{r \to \infty} \left[ \sum_{|q| < r} \frac{z^m}{(z-q)q^m} - \sum_{|p| < r} \frac{z^m}{(z-p)p^m} \right]
\]
holds, locally uniformly in \(\mathbb{C} \setminus (\mathcal{P} \cup \mathcal{Q})\); \(m\) is any integer \(> \beta\), and \(T_{m-1}\) is the \((m-1)\)-th Taylor polynomial for \(f'/f\) at \(z = 0\). Each zero \(q\) and pole \(p\) in the sum occurs according to its multiplicity.
functions I means that in (12) and (16) we may replace \( \mu > \beta \) exist for any integer \( r \)

\[
\text{Remark. There are, of course, also more or less complicated modifications of Theorem (13) with (14) Φ(\( z \)) having simple poles and pole } q \text{.}
\]

Proof. The following technique is well-known. Let \( \Phi \) be meromorphic in the plane having simple poles \( ξ \) with residues \( ρ(\xi) \), and assume that \( \Phi(0) \neq 0, ∞ \) and \( |\Phi(z)| = O(|z|^β) \) holds on the circles \( |z| = r_k \to ∞ \). Then

\[
I_k(z) = \frac{1}{2\pi i} \int_{|\xi|=r_k} \frac{\Phi(\xi)z^m}{(\xi - z)^m} d\xi = O(r_k^{β-m}) \to 0 \quad (k \to ∞),
\]

provided \( m > β \). On the other hand, the Residue Theorem yields

\[
I_k(z) = \Phi(z) + \sum_{|\xi|<r_k} \frac{ρ(\xi)z^m}{(z - ξ)^m} - T_{m-1}(z),
\]

with \( T_{m-1} \) the \( (m - 1) \)-th Taylor polynomial of \( Φ \) at \( z = 0 \), hence

\[
\Phi(z) = T_{m-1}(z) + \lim_{k \to ∞} \sum_{|\xi|<r_k} \frac{ρ(\xi)z^m}{(z - ξ)^m}.
\]

This applies to \( Φ = f'/f \) with poles \( p \) and \( q \), if \( r_k \) can be chosen to lie outside \( P_r \cup Q_r \). If, however, \( |z| = r_k \) intersects some connected component \( C \) of \( P_r \) and/or \( Q_r \), we may by virtue of Lemma 2 (see the end of this section) replace the intersection \( C \cap \{ z : |z| = r_k \} \) by one or more subarcs of \( \partial C \) of total length \( O(r_k^{-β}) \). This way we obtain the Jordan curve \( Γ_k \); it is contained in the annulus \( A_k : ||z| - r_k| < cr_k^{-β} \) and since there are at most \( r_k^{β+1} \) such components, the length of \( Γ_k \) is \( O(r_k) \). To get rid of \( Γ_k \) and even \( r_k \) we just remark that for \( |z| < R \) and \( r_k \to ∞ \) we have

\[
\sum_{q \in A_k \cap Q} \left| \frac{z^m}{(z - q)^m} \right| + \sum_{p \in A_k \cap P} \left| \frac{z^m}{(z - p)^m} \right| = O(r_k^{-m-1}r_k^{β+1}) \to 0. \quad \text{q.e.d.}
\]

Noting that

\[
\frac{z^m}{(z - ξ)^m} = \frac{1}{z - ξ} + \sum_{j=0}^{m-1} \frac{z^j}{ξ^{j+1}} = \frac{d}{dz} \log E\left(\frac{z}{ξ}, m\right),
\]

we obtain:

**Theorem 17.** Every \( f \in \mathcal{Y}_{α, β} \) may be written as

\[
f(z) = z^s e^{S(z)} \lim_{r \to ∞} \prod_{|q|<r} E\left(\frac{z}{q}, m\right),
\]

where \( m \) is any integer \( > β \), \( s \in \mathbb{Z} \), and \( S \) is a polynomial with \( \deg S \leq m \). Each zero \( q \) and pole \( p \) in the products occurs according to its multiplicity.

**Remark.** There are, of course, also more or less complicated modifications of Theorem 16 if \( f \) has multiple poles. Since \( m > β \) is arbitrary, the limits

\[
\lim_{r \to ∞} \left[ \sum_{|p|<r} p^{-μ} - \sum_{|q|<r} q^{-μ} \right]
\]

exist for any integer \( μ > m > β \).

For \( β \) an integer, the term in brackets, the sums in 16, and also the sequence of functions \( I_k(z) \) in 13, remain uniformly bounded if we choose \( μ = m = β \), which means that in 12 and 16 we may replace \( m \) by \( β \) if we simultaneously replace \( r \to ∞ \) by \( r_k \to ∞ \) for some suitably chosen sequence \( (r_k) \).
THEOREM 17. If $\beta > -1$ is an integer, then every $f \in \mathcal{Y}_{\alpha, \beta}$ may be written as

$$f(z) = z^s e^{S(z)} \lim_{k \to \infty} \prod_{|q| < r_k} E\left(\frac{z}{q}, \beta\right) \prod_{|p| < r_k} E\left(\frac{z}{p}, \beta\right),$$

for some suitably chosen sequence $r_k \to \infty$; $s$ is an integer and $S$ is a polynomial with deg $S \leq \beta + 1$. In particular, for $f$ in the Yosida class $A_0$ and also in $\mathcal{Y}_{\alpha, 0}$ this means

$$f(z) = z^s e^{az+b} \lim_{k \to \infty} \prod_{|q| < r_k} \left(1 - \frac{z}{q}\right) \prod_{|p| < r_k} \left(1 - \frac{z}{p}\right)$$

and

$$f'(z) = \frac{a + s}{z} + \lim_{k \to \infty} \left[ \sum_{|q| < r_k} \frac{1}{z - q} - \sum_{|p| < r_k} \frac{1}{z - p} \right].$$

REMARK. The method of proof of Theorem 16 applies also to $f$ itself. If $f$ has only simple poles $p$ with residues $\rho(p)$ and if $f(0) \neq \infty$, then

$$(17) \quad f(z) = T(z) + \lim_{k \to \infty} \sum_{p \in D_k} \frac{\rho(p)z^m}{(z - p)p^m}$$

holds, locally uniformly in $\mathbb{C} \setminus P$; $m$ is any integer $> \alpha$, and $T$ is the $(m-1)$-th Taylor polynomial of $f$ at $z = 0$. To get rid of $D_k$ we need information about the residues. Let $C_k$ be any component of $P$, that intersects the circle $|z| = r_k$. We may assume that $C_k$ contains the poles $p_k^{(\nu)}$ ($1 \leq \nu \leq n$), for $f_k = f_k^{(1)}$, and also that $f_k \to f \neq const$. Then the contribution of the poles in the sequence $(C_k)$ to $f$ is

$$\lim_{k \to \infty} \sum_{\nu=1}^{\infty} \frac{\rho(p_k^{(\nu)})z^{a_\nu}}{z - (p_k^{(\nu)} - p_k^{(\nu)} - a_\nu)^{-1}} = P(z) \prod_{\nu=1}^{n} (z - a_\nu)^{-1};$$

$P$ is a polynomial of degree $< n$, and the numbers $a_\nu = \lim_{k \to \infty} (p_k^{(\nu)} - p_k^{(\nu)})$ are not necessarily distinct, since $f$ may have multiple poles. If $a_\nu = \cdots = a_\alpha$ and $\neq a_\mu$ else, then $\left| \sum_{\nu=1}^{\infty} \rho(p_k^{(\nu)}) \right| = O(|p_k|^{\alpha-\beta})$ holds. Since there are at most $O(r_k^{\beta+1})$ components $C_k$, the contribution of the annulus $A_k$ to the sum in (17) is $O(r_k^{\alpha-m}) \to 0$ as $k \to \infty$, and again we obtain

$$(18) \quad f(z) = T(z) + \lim_{r \to \infty} \sum_{|p| < r} \frac{\rho(p)z^m}{(z - p)p^m}.$$

In the particular case $f \in A_0 = \mathcal{Y}_{0,0}$ with simple poles only, (18) holds with $m = \alpha = 0$ and $r = r_k$, hence

$$f(z) = a + \lim_{k \to \infty} \sum_{|p| < r_k} \frac{\rho(p)}{z - p}.$$

We finish this section by proving a technical lemma as follows:

LEMMA 2. Let $C$ be any domain that consists of $n$ discs $\Delta_r(\nu)$ and intersects $|z| = r$. Then for $\epsilon$ sufficiently small and $r$ sufficiently large, $C$ has diameter and boundary curve length $\leq K_n r^{-\beta}$; the constant $K_n$ only depends on $n$. 

Proof. We will prove by induction that there exists some $c > 0$, such that for $\epsilon$ sufficiently small any domain $C_k = \bigcup_{j=1}^k \Delta_j(h_k)$ $(1 \leq k \leq n)$ that intersects $|z| = r$ is contained in the annulus $A_k : |z| - r < 2\epsilon r \omega$ with diam $C_k \leq 2\epsilon r \omega - \beta$.

This is obviously true if $k = 1$. Assuming the assertion to be true for $k$ discs, we consider the domain $C_{k+1} = C_k \cup \Delta_j(h)$ satisfying diam $C_{k+1} \leq 2\epsilon r \omega - \beta$. From $\Delta_j(h) \cap A_k \neq \emptyset$ then follows $|h|^{-\beta} < cr^{-\beta}$, $C_{k+1} \subset A_{k+1}$ and diam $C_{k+1} \leq 2c(k+1)r^{-\beta}$. The limitations imposed on $\epsilon$ and $c$ are $(1 \pm 2\epsilon c)^{-\beta} \leq c$ and $2\epsilon c < 1$. Thus the diameter of $C$ and the length of the boundary curve of $C$ is $O(\epsilon r^{-\beta})$.

6. The Case $\beta = -1$

The limit functions of the family of functions $f_h(z) = h^{-\alpha} f(h + h z)$ have an essential singularity at $z = -1$, since zeros and poles accumulate there. Hence we postulate normality only in $C \setminus \{-1\}$ to define the class $\mathcal{Y}_{\alpha, -1}$. Apart from this it is not hard to verify that Theorems 1 and $f^#(z) = O(|z|^{[\alpha+1]})$, 2 and $f^#(z) = 2\beta + 2$, 3 and $m(r, f) = O(\log r)$, 4 and $m(r, f) = O(\log r)$ as well as Corollary 1 remain true also if $\beta = -1$. Beyond the fact $g(f) = 0$ we are looking for more detailed information on the growth of $T(r, f)$ and $n(r, c)$. The analog to Theorem 1 in connection with Theorems 1 and 2 is as follows:

**Theorem 18.** Suppose $f \in \mathcal{Y}_{\alpha, -1}$. Then $f^#(z) = O(|z|^{-[\alpha+1]})$,

$$n(r, \infty) \asymp \log r \quad \text{and} \quad m(r, f) = O(\log r)$$

hold; the same is true for every $c \in \mathbb{C}$ instead of $\infty$. In particular we have

$$T(r, f) = N(r, f) + O(\log r) \asymp \log^2 r.$$ 

Proof. For $\lambda > 1$ we consider the annuli $A_n : \lambda^{n-1} \leq |z| < \lambda^n$. By Theorem 7 each $A_n$ contains at most $O((\lambda - 1)^{-1})$ poles if $\lambda$ is sufficiently close to 1 (according to $\eta_0$ in Theorem 7). Thus $n(r, \infty) \asymp \log r$ follows, and the same is true for any value $c \in \mathbb{C}$ instead of $c = \infty$. q.e.d.

**Example.** Transcendental meromorphic solutions to algebraic differential equations $w' = R(z, w)$ have order of growth $\varrho \geq 1/3$ or else $\varrho = 0$ (Bank and Kaufman 2, the author 13). An example for the latter case is due to Bank and Kaufman 11 (slightly modified): $w^2 = \frac{4w(w^2 - g_2/4)}{1 - z^2}$. One of its solutions

satisfies $f(z) = \varphi(z)$, where $\varphi$ is the Weierstrass $\wp$-function to the differential equation $\wp'^2 = 4\wp(\wp^2 - g_2/4)$, and $g_2 > 0$ is chosen in order that $\varphi$ has period lattice $\pi (\mathbb{Z} + i\mathbb{Z})$. The zeros $\pm \cosh \pi (k + \frac{1}{2})$ and $\pm \sqrt{g_2}/2$-points $\pm \sinh \pi k$ are real, and the poles $\pm i \sinh (\pi k)$ and $-\sqrt{g_2}/2$-points $\pm i \sin \pi (k + \frac{1}{2})$ are purely imaginary.

From $f^#(z) = \frac{4|f(z)||f(z)^2 - g_2/4|}{|z^2 - 1|(|z^2 + |f(z)|^2)|^2}$ and the distribution of critical points follows $f^#(z) = O(|z|^{-1})$, and $f^#(z) \asymp |z|^{-1}$ in $|z| \leq \pi^{-1} \epsilon$ and in $|z| \geq \frac{3}{4} \pi \epsilon$, hence $f \in \mathcal{Y}_{\alpha, 0}$ by Theorem 3. The $k$-th derivative of $f$ belongs to $f \in \mathcal{Y}_{-k, -1}$.

Any limit function satisfies $f(z)^2 = \frac{4f(z)(f(z)^2 - g_2/4)}{(z + 1)^2}$, hence has the form $f(z) = \frac{\text{Malmquist-Yosida Theorem 14, 15.}}{3}$
\( \wp(c + i \log(z + 1)) \) – it is, of course, single-valued since \( \wp \) has period \( \pi \), with essential singularity at \( z = -1 \).

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