Dynamical content of quantum diffeomorphisms in two-dimensional quantum gravity

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A model for 2D-quantum gravity from the Virasoro symmetry is studied. The notion of space-time naturally arises as a homogeneous space associated with the kinematical (non-dynamical) SL(2,R) symmetry in the kernel of the Lie-algebra central extension for the critical values of the conformal anomaly. The rest of the generators in the group, $L_{|n|\geq 2}$, mix space-times with different constant curvature. Only in the classical limit all space-times can be identified, defining a unique Minkowski space-time, and the operators $L_{|n|\geq 2}$ gauged away. This process entails a restriction to SL(2,R) subrepresentations, which creates a non-trivial two-dimensional symplectic classical phase space. The present model thus suggests that the role of general covariance in quantum gravity is different from that played in the classical limit.

1. Introduction

Our main goal is the construction of a space-time notion just from symmetry principles. Therefore, we consider space-time as a derived, or secondary, object in our theory. Assuming that the nature of the world is a quantum one, we want to start from the very beginning with a quantum theory in which, perhaps, there is not an explicit notion of space-time. We understand by such a quantum theory a unitary and irreducible representation of the algebra of physical operators and the space-time structure itself, in case such an structure makes any sense, should be built out of this quantum theory.

In order to fill the gap between the starting point (quantum theory) and the objective (space-time construction), we present a very simple model that we interpret as a quantum gravity (see [1]), in fact a two-dimensional one, even though the link with standard two-dimensional quantum gravity is not completely obvious.

The technical tool we are going to use is a Group Approach to Quantization (GAQ) (see [2] and references therein), which starts from a given group of symmetries and constructs a quantum theory in the sense mentioned above. Among the elements of this approach, one is specially important for this work as it will play the role of physical guide in our search of space-time. This element is group (pseudo-)cohomology, which is a mathematical concept related to central extensions of the considered group. Its importance follows from the fact that it allows the classification of the physical operators in two sets:

- Dynamical operators, which appear in conjugated pairs, and give a central term in their commutators: $[\cdot \cdot \cdot] \sim 1$.
- Kinematical operators, which, broadly speaking, do not give this central term (the distinction between dynamical and kinematical operators can be rigorously characterised in terms of a pre-symplectic form constructed from the group two-cocycle [2]). Space-time should appear within this second set which, in the language of GAQ, is called characteristic subalgebra and denoted by $\mathcal{G}_\Theta$.

2. The mathematical model

The first step in the construction of the model consists in selecting an appropriate starting group. We are choosing the Virasoro group, in abstract terms, for such a group. The reasons for this are, on the one hand, that it is a group simple enough in order to be handled and, at the same time,
rich enough for giving non-trivial answers. On the other hand, Polyakov’s action for 2D gravity can be derived from it.

The Virasoro algebra is defined \((h = \frac{c - c'}{24})\) by:

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n,-m}.
\]

A formal group law \((l^{nm} = F^m(l^n, l^m))\) can be derived from this algebra, and from this group law the rest of the elements of GAQ (quantization 1-form, left- and right-invariant infinitesimal generators...) can be computed. As we said before, we are particularly interested in the characteristic subalgebra, the kinematical degrees of freedom. Looking at the commutation relations, we find the two possible cases:

\[
i) \frac{c'}{c} \neq r^2, r \in \mathbb{Z} \Rightarrow \mathcal{G}_\Theta = \langle \vec{X}_L^{L} \rangle
\]

\[ii) \frac{c'}{c} = r^2, r \in \mathbb{Z} \Rightarrow \mathcal{G}_\Theta = \langle \vec{X}_L^{L}, \vec{X}_L^{L}, \vec{X}_L^{L} \rangle.
\]

Since we are looking for a space-time with one time dimension and at least one spatial dimension, the first case is excluded. Therefore, our first conclusion is that in order to find a space-time we must fall in the second critical case in which \(\mathcal{G}_\Theta\) closes an \(sl(2, R)\) algebra. In fact, unitarity imposes \(r = 1\), so that we have \(c = c'\), which is the only case we consider from now on.

The quantum representation is achieved by taking the set of \(U(1)-\)complex functions defined on the group as the Hilbert space, and the group acting on it via the regular representation. The main problem then is that this representation is highly reducible. In principle, this can be solved by taking advantage of the trivial commutation among left- \((\vec{X}_L^L)\) and right-invariant \((\vec{X}_L^R)\) vector fields. This allows us to implement the representation with one set of vector fields (for instance, the right-invariant ones: \(L_n = i(\vec{X}_L^R)\)) and to reduce the representation in a consistent way by using the other set, via the introduction of polarization equations: \(\vec{X}_{L_{n<1}}^{R} = 0\).

However, in the general case for the Virasoro group, the resulting representation is still reducible. We have two solutions to this problem which prove to be equivalent. The first benefits from the existence of a vacuum state and consists in taking the orbit of the group through this vacuum. The resulting representation is an irreducible highest-weight representation, characterised by: \(L_n | 0 \rangle = 0\), \((n \geq -1)\) and \(| \Psi \rangle = L_{n_1} \ldots L_{n_1} | 0 \rangle\), \((n \leq -2)\). The second solution, which is more natural in the framework of GAQ, consists in imposing further polarizations conditions using higher-order differential operators. This makes pseudo-differential operators come on the scene, something which is a source of technical difficulties, however. For the sake of simplicity we are not going to consider those cases in which higher-order polarizations appear. The equivalence between the two solutions was proved in [7].

With regards to unitarity, the values of \(c\) and \(c'\) that make unitary the representation are [8]:

- \(c \geq 1\), with \(\frac{c - c'}{24} \geq 0\).
- \(0 < c < 1\) with: \(c = 1 - \frac{6}{m(m+1)}\) and \(\frac{c - c'}{24} = \frac{[(m+1)r - m]^2 - 1}{4m(m+1)}\), where \(1 \leq s \leq r \leq m - 1\), and \(m, r, s\) integers with \(m \geq 2\).

But the discrete values of \(c\) and \(c'\) for \(0 < c < 1\) are precisely those cases related with higher-order polarizations and are, therefore, disregarded.

Thus, what we have by now, is a representation of the Virasoro algebra in which \(c = c'\) (in order to find a space-time) and \(c \geq 1\) (to have unitarity). Furthermore, we have classified the operators in two sets: space-time operators \((L_{-1}, L_0, L_1)\), and dynamical operators \((L_n, | n | \geq 2)\), which we shall refer to as gravity operators in the sequel. This way, the starting point of the work, the quantum theory, is constructed and now we try to accomplish our main objective by identifying a space-time structure.

Firstly, we consider the reduction of the Virasoro Hilbert space, \(\mathcal{H}_{(c,c')}\), under the kinematical \(SL(2, R)\) subgroup, obtaining:

\[\mathcal{H}_{(c,c')} = \bigoplus_{N} (D^{(N)} - D^{(N-1)}) R^{(N)}_{S}\]

where,

- \(R^{(N)}_{S}\) is a maximal-weight irreducible representation of \(SL(2, R)\) with Casimir
\[ N(N-1). \] We denote the states in this representation by \(| N, n \rangle\).

- \( D^{(N)} \) is the dimension of the Virasoro level \( N \).

- The different representations \( R^{(N)}_A \) are orthogonal. This will be important in the physical interpretation because it permits a standard quantum mechanical interpretation.

Once the representation has been reduced, we proceed to associate a space-time with each \( SL(2, R) \) representation in the model; that is, as we have an infinite number of \( SL(2, R) \) representations, an infinite number of space-times are realised simultaneously in our model.

In order to make this association more concrete, we take an specific \( SL(2, R) \) representation and construct a \( C^* \)-algebra by considering all the products of the wave functions in the representation. At this point, we can apply a theorem by Gelfand and Naimark \([8]\), which allows the reconstruction of a manifold from the \( C^* \)-algebra. The problem with this approach is that the mentioned theorem is a rather abstract tool and the identification of the actual manifold under consideration is a difficult task. Fortunately, we can look at the problem in another way. For this, we consider again an isolated \( SL(2, R) \) representation and notice that we know another system with the same Hilbert space but for which the configuration space is explicitly known. This system is a particle moving on a two-dimensional AdS space-time. Thus, we associate with each \( SL(2, R) \) representation a one-sheet hyperboloid. In fact, this is what one expects to find from a presentation a one-sheet hyperboloid. In fact, this time. Thus, we associate with each particle moving on a two-dimensional AdS space-time. At this point, we can apply a theorem which is isomorphic to AdS group in two dimensions to the vector fields. In our case we need a scale. To solve this problem we introduce a constant \( a \), such that \([a] = \text{Length}\), and redefine \( n \to \frac{a}{c} \). Thus, the redefined integers are dimensionless.

Another important point is that the Virasoro algebra has a natural notion of classical limit which is obtained by making \( c \to \infty \), or in redefined terms, \( \frac{a}{c} \to \infty \). In this case, the constant \( \frac{a}{c} \) behaves as a parameter of a perturbative series, playing the role of the Planck constant in our model. This suggests to redefine the Virasoro generators: \( H_n \equiv \frac{a}{c} L_n \).

The previous introduction of a hyperboloid associated with a given representation does not provide a metric structure for space-time. The only natural metric we can find inside the model is the one induced from the Killing metric of \( SL(2, R) \), thus providing an AdS space-time. In order to fully determine it, we have to introduce a scale: the radius of the hyperboloid. It is defined from the Casimir in terms of the redefined Virasoro generators:

\[
\frac{1}{R^2} \equiv H_0^2 - \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1) = \left(\frac{a}{c}\right)^2 \frac{N(N-1)}{a^2} = \frac{N(N-1)}{c^2}. \tag{1}
\]

### 3. Physical interpretation

Now we can provide a physical interpretation. As we have stressed, we associate an AdS space-time of radius \( R = \frac{\sqrt{N(N-1)}}{\sqrt{\text{Length}}} \) with each \( SL(2, R) \) representation.

Given an specific \( SL(2, R) \) representation, we interpret each vector in it \(| N, n \rangle\) as a state of the corresponding space-time. Since we are dealing with maximal-weight representations, a tower of states of the space-time is found, where the maximal-weight vector plays the role of space-time ground state. \( H_0 \) is interpreted as the energy and \( H_{-1} (H_1) \) as raising (lowering) space-time operators. When we consider the dynamical operators, \( H_{|n| \geq 2} \), we notice that they do not leave invariant the \( SL(2, R) \) representations and therefore they produce the effect of mixing the different space-times.
The whole picture provided by the model is as follows: we understand by Universe the entire ensemble of different space-times which are realised at the same time. A state of the Universe is a particular state in the Virasoro representation, that is, an specific linear combination of states in different \( SL(2, R) \) representations or, in other words, a quantum superposition of different space-times. The coefficients of the linear combination define a weight distribution of space-times. The question about the radius of the Universe makes no real sense, since we have space-times of different radii. The meaningful question is about the probability for the Universe to have a certain radius, and then it is essential the orthogonality of the \( SL(2, R) \) representations, which allows the definition of orthogonal projectors. Finally, the effect of gravity is that of mixing the different space-times, thus changing the weight distribution of space-times.

As far as the classical limit is concerned, we can consider the limit \( c \to \infty \) and find that for every space-time the energy of the ground state tends to zero (\( \text{Energy}(|N, 0\rangle) = \frac{N^2}{2c} \to 0 \)), and the radius to infinity (\( R = \frac{c}{\sqrt{N(N-1)}} \to \infty \)). There is no physical way to distinguish between the different space-times in this limit and it makes sense, accordingly, to identify them. In order to do that, we define the equivalence relation, \( |N, n\rangle \sim |N', n\rangle \), and take the quotient \( H = H_{(c,c)}/\sim \). The problem with this quotient is that the dynamical operators are ill-behaved in it; in fact, they are multivalued. To avoid this trouble, we impose to these operators to act trivially on the states (this is only consistent with the commutation relations in the classical limit, as pointed out by S. Carlip after the talk): \( H_{|n| \geq 2} | \Psi \rangle = 0 \).

Hitherto, we have considered the Virasoro group as an abstract one, but at this point we can look at it as a diffeomorphism group, so that the previous constraints are precisely the classical diffeomorphisms constraints. What we find is that diffeomorphism invariance is only recovered in the classical limit, while diffeomorphisms have a dynamical content at the quantum level.

4. Conclusions

- We have constructed a space-time notion from a quantum theory but only for the critical value of the Virasoro anomaly (\( c = c' \)). Out of this value space-time makes no sense, even though the quantum theory does exist.

- The model presents the Universe as a quantum superposition of different space-times which are mixed by gravity modes.

- We have implemented a model which exhibits a quantum breakdown of diffeomorphism invariance through the appearance of physical (non-gauge) degrees of freedom through an anomaly process. General covariance is recovered only in the classical limit.

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