On the problem of processing data from a cross-sectional image of grained structure

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Abstract. The anisotropy of a grained structure is a consequence of non-uniform distribution of spatial orientation of grains. Conventional stereological methods allow determine spatial distribution of grain orientation from lines distribution observed on structure cross sections. The distribution of grain’s surface orientation in space is determined by measuring the total length of cross sections on a cutting plane. In this paper, we indicate partial but significant problem of stereology. A rectangular model of grained structure is considered. Then the kernel function, which connects the probability density function (distribution) of lengths of rectangles intersections visible on observation planes with size of rectangles sides, is calculated.

1. Introduction
During last decades a lot of sophisticated methods have been developed to study structures by 2D imaging. These methods can significantly help in precise investigation of grained structures. On the other hand these methods have limitations. They are neither always economically efficient nor can be applied for different situations. Mentioned methods certainly cannot be considered fast and cheap and therefore it makes sense to look for simpler alternatives. Stereological methods are promising, very simply and cheap for this application.

A growing literature is demonstrating that problems of geometrical orientation of structure and structural units distortion induced by deformation play important role in different areas. As it is clear from previous work, there are a few ways how to evaluate the grain orientation experimentally [1-6]. However, the most suitable way in practice is the scalar measurement of anisotropy based on stereological principle enabling to determine the degree of grains orientation. Using stereological method an oriented test lines enables to determine the degree of grains orientation in any area of polycrystalline pieces [7-10]. In addition, we suggest that results of such measurement, i.e. degree of grains orientation in volume of plastically deformed structure can be used to evaluation of local deformation parameters. In that case, it is sufficient measure the degree of grains orientation to obtain a value of local strain. Our work offers some details of statistical view of grained structures analysis using stereological concept.

2. The problem formulation
Let's look the grained structure as a set of grain boundaries. Consider rectangle model of grain boundary in bulk grained structure with shape of cube.
Arbitrary placed rectangle intersects the cross-sectional plane \( x = x_0 \) (Figure 1) if

\[
0 < x_0 - x \leq \frac{L}{2} \cos \alpha \quad \ldots \quad \alpha \in (0, 2\pi),
\]

where

\[
L = a \left| \cos (\beta) \right| + b \left| \sin (\beta) \right| \quad \ldots \quad \beta \in (0, 2\pi),
\]

\( a \) and \( b \) are length of rectangle sides. In the limit case it holds

\[
L_0 = u \sin (\varphi),
\]

where \( u = \sqrt{a^2 + b^2} \).

Then from the integral geometry technique results that the probability that the rectangle intersects the cross-sectional plane can be determined as

\[
P = \frac{M}{M_0},
\]

where

\[
M_0 = (2\pi \times 2\pi \times d) = (4\pi^2 d) \quad \text{and} \quad M = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left[ a \left| \cos (\beta) \right| + b \left| \sin (\beta) \right| \right] \cos \alpha \, d\beta \, d\alpha = 8(a + b).
\]

If the sample has the shape of cube with the side \( d \), mentioned probability can be calculated by simple formula

\[
P = \frac{2(a + b)}{\pi^2 d}.
\]

However if the rectangle intersects the cross-sectional plane, there can be various length of the intersection line. If the intersection shown in Figure 1 is already occurred we can expect next possible configurations of intersection line determined by points \( C_1 \) and \( C_2 \) positioned in the rectangle sides

\[
C_1 = [x_1 : y_1] \quad C_2 = [x_2 : y_2]
\]

1) \( P \left( x_1 \in (0,a) \wedge y_1 = 0 \wedge x_2 \in (0,a) \wedge y_2 = b \right) \)

2) \( P \left( x_1 \in (0,a) \wedge y_1 = 0 \wedge x_2 = 0 \wedge y_2 \in (0,b) \right) \)

3) \( P \left( x_1 \in (0,a) \wedge y_1 = 0 \wedge x_2 = a \wedge y_2 \in (0,b) \right) \)

4) \( P \left( x_1 = 0 \wedge y_1 \in (0,b) \wedge x_2 \in (0,a) \wedge y_2 = b \right) \)

5) \( P \left( x_1 = a \wedge y_1 \in (0,b) \wedge x_2 \in (0,a) \wedge y_2 = b \right) \)

6) \( P \left( x_1 = 0 \wedge y_1 \in (0,b) \wedge x_2 = a \wedge y_2 \in (0,b) \right) \)

The question is, what is probability, that length of the intersection line (i.e. distance of points \( C_1 \) and \( C_2 \)) is smaller than a value \( \xi \). All situations corresponding with configurations 1) to 6) shown in Figure 2 must be considered, i.e.

1) \( P(A_1) = P \left( \sqrt{(x_1 - x_2)^2 + b^2} < \xi \right), \)

2) \( P(A_2) = P \left( \sqrt{y_1^2 + y_2^2} < \xi \right), \)

3) \( P(A_3) = P \left( \sqrt{(x_1 - a)^2 + y_2^2} < \xi \right), \)

4) \( P(A_4) = P \left( \sqrt{x_2^2 + (y_1 - b)^2} < \xi \right), \)

5) \( P(A_5) = P \left( \sqrt{(x_2 - a)^2 + (y_2 - b)^2} < \xi \right), \)

6) \( P(A_6) = P \left( \sqrt{a^2 + (y_1 - y_2)^2} < \xi \right) \quad \ldots \quad \xi \in \left(0, \sqrt{a^2 + b^2}\right). \)
It is clear, that
\[ P(A_2) = P(A_3) = P(A_4) = \Pi(\gamma) \ldots \Pi(\gamma) = \Pi(\sqrt{x^2 + y^2 < \xi}) \]
and the probability is
\[ P(\xi) = C_1 P(A_1) + C_2 \Pi(\gamma) + C_3 P(A_6), \]
where \( C_1, C_2 \) and \( C_3 \) are constants resulting from normalization of distribution function \( f(\xi) \) determined by derivative of \( P \).

### 3. The length of line visible in the cross-section

We briefly demonstrate analysis of configuration considering 1) in which \( \xi > b \). Let \( a \) is the length of the interval at which the cross-section points in structure is carrying out. Consider two different randomly chosen cross-section points \( C_1, C_2 \) formed in the sides with length \( a \). Let \( x_1 \) and \( x_2 \) denote positions where cross-section \( C_1 \) and \( C_2 \) are formed. Then the length of the cross-section line is valuable as
\[ L = \sqrt{(x_1 - x_2)^2 + b^2} \]

Actually, we are interested in the probability \( P(A_1) \) that the interval \( L \) is shorten then arbitrarily chosen value \( \xi \), i.e. \( P(A_1) \) denotes the expectation of the cross-section length between \( C_1 \) and \( C_2 \) is not longer then \( \xi \)
\[ \sqrt{(x_1 - x_2)^2 + b^2} < \xi \ldots \Delta x = |x_1 - x_2| < \sqrt{\xi^2 - b^2}. \]

Problem of the probability \( P(A_1) \) can be solved by computations with conditional probability. It is, however, more instructive to solve it by another method, one that minimizes the amount of computation and maximizes the role of probabilistic reasoning. Application of integral geometry technique is appropriate for the probability \( P(A_1) \) evaluation in this case.

Like combinatorial enumeration, where sequences of objects bearing a common feature are unified by the idea of a generating function, integral geometry studies sets of geometric objects bearing a common feature, which are unified by the idea of an invariant measure. The basic idea is extremely simple. With integral geometry, we search for the likelihood that we will "hit" a particular area of a figure. To calculate the probability \( P(A_1) \), we will need to find 2D areas of the shapes involved in the problem. We will need to know the total area, which means the biggest area in the diagram, like the entire dartboard. We will also need to know the desired area, which is the part we are trying to "hit".

2D metric space can be identified to explain interval scaling of the cross-section points \( C_1, C_2 \) formation. We can define axes that allow to illustrate positions where both of \( C_1 \) and \( C_2 \) are created (Figure 3). Intervals with lengths \( a \) can be easily found on both of these axes. Every point on the surface of square shown in 3 represent possible events of the \( C_1, C_2 \) positions. Then the number of all possible events is represented by area of this square \( S_0 \). Number of favourable formation events (i.e. events when
the time interval $\Delta x$ is shorten then $\sqrt{\frac{x^2}{\xi} - b^2}$ is represented by the grey area $S$ shown in Figure 1. This is the desired stands for the area that we want to "hit". Once we can easy calculate both of these areas and the formula for the probability $P(A_1)$ calculation is simply

$$P(A_1) = 1 - \left(\frac{a - \sqrt{x^2 - b^2}}{a}\right)^2.$$ 

The configuration 6) ($\xi > a$) can be analyzed by the same way and the result is analogous

$$P(A_6) = 1 - \left(1 - \frac{\sqrt{x^2 - a^2}}{b}\right)^2.$$ 

If the configuration corresponding to the probability $\Pi(\gamma)$ is analyzed using integral geometry technique described above, obtained result shows that three situations must be considered $\xi < b$, $b < \xi < a$ and $a < \xi$.

I. If $\xi < b$ then

$$P(\xi) = C_1\pi\xi^2.$$ 

II. If $b < \xi < a$ then

$$P(\xi) = C_1\{1 - e_1\}\pi\xi^2 + C_2\left[1 - \left(1 - \frac{\sqrt{x^2 - b^2}}{a}\right)^2\right].$$

III. If $a < \xi$ then

$$P(\xi) = C_1\{1 - e_1 - e_2\}\pi\xi^2 + C_2\left[1 - \left(1 - \frac{\sqrt{x^2 - b^2}}{a}\right)^2\right] + C_3\left[1 - \left(1 - \frac{\sqrt{x^2 - a^2}}{b}\right)^2\right].$$

where

$$e_1 = e_1(\xi) = \frac{2}{\pi}\left[\arcsin\left(\frac{b}{\xi}\right) - \frac{b}{\xi}\sqrt{1 - \left(\frac{b}{\xi}\right)^2}\right], \quad e_2 = e_2(\xi) = \frac{2}{\pi}\left[\frac{\pi}{2} - \arcsin\left(\frac{a}{\xi}\right) - \frac{a}{\xi}\sqrt{1 - \left(\frac{a}{\xi}\right)^2}\right].$$

It is evident, that derivative of $P(\xi)$ shows singularities for $\xi = a$ and $\xi = b$. To avoid this fact the generalized function describing the distribution of length of line elements visible in the cross-section was evaluated numerically as $f(\xi) = P(<\xi, \xi+d\xi>)$. Calculated data for the $a = 10$ and $b = 5$ is shown in Figure 4.

**Figure 4.** Distribution function for length of the line element visible in cross-section.
4. Conclusion

It is clear that real grain boundaries have not a rectangular form. Boundaries of various shape are in the grained structure. Therefore obtained analytical formulas must be generalized to suitable forms. However our result document that there is a distribution of length of line elements visible in the cross-section that would be considered when the conventional stereological techniques are applied.

Finally, results from all conventional 2D imaging techniques using a cross-sectional image of the structure would be analyzed with the consideration of this fact.

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References

[1] Sutton M A, Orteu J-J and Schreier H W 2009 Image Correlation for Shape, Motion and Deformation Measurements (Hardcover ISBN 978-0-387-78746-6)
[2] Sack R A 1961 Indirect Evaluation of Orientation in Polycrystalline Materials J. Polym. Sci. 51 543-560
[3] Keating T J, Wolf P R and Scarpas F L 1975 An Improved Method of Digital Image Correlation Photogramm. Eng. Rem. S. 41 993–1002
[4] Masteghin M G and Orlandi M O 2016 Grain-Boundary Resistance and Nonlinear Coefficient Correlation for SnO₂-Based Varistors Mat. Res. 19 (São Carlos Epub 26 Sep 2016)
[5] Bunkholt S and Knut M and Nes E 2014 Subgrain Structures Characterized by Electron Backscatter Diffraction Mater. Sci. Forum 794-796 3-8
[6] Zou H F and Zhang Z F 2010 Application of electron backscatter diffraction to the study on orientation distribution of intermetallic compounds at heterogeneous interfaces (Sn/Ag and Sn/Cu) J. App. Phys. 108 103518
[7] Saltykov S A 1970 Stereometric metallography (Metallurgia, Moscow, in Russian)
[8] Lewis H D, Walters K L and Johnson K A 1973 Particle size distribution by area analysis: Modifications and extensions of the Saltykov method Metallography 6 93–101
[9] Xu Y H and Pitot H C 2003 An improved stereologic method for three-dimensional estimation of particle size distribution from observations in two dimensions and its application Comput. Methods Programs Biomed. 72 1-20
[10] Rayaprolu D B and Jaffrey D 1982 Comparison of discrete particle sectioning correction methods based on section diameter and area Metallography 15 193–202