THE QUANTIZATION OF A TORIC MANIFOLD IS GIVEN BY 
The INTEGER LATTICE POINTS IN THE MOMENT 
POLYTOPE

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Abstract. We describe a very nice argument, which we learned from Sue 
Tolman, that the dimension of the quantization space of a toric manifold, 
using a Kähler polarization, is given by the number of integer lattice points in 
the moment polytope.

1. Introduction

"The quantization of a toric manifold is given by the integer lattice points in the 
moment polytope."

In principle, this is a well-known result; nevertheless, it does not seem to be 
written down in exactly this language. Usually reference is made to the paper [Dn] 
of Danilov, where this result is phrased in algebro-geometric terms, about the sheaf 
cohomology of a manifold (compare (8)). Guillemin, Ginzburg, and Karshon de-
scribe it as a “folk theorem, usually attributed to Atiyah or Danilov.” ([GGK], p 
142)

A precise statement is as follows:

Theorem. Let $M$ be a toric $2n$-manifold, with moment polytope $\Delta \subset \mathbb{R}^n$. Then 
the dimension of the quantization is equal to the number of integer lattice points 
in $\Delta$, that is,

$$\dim H^0(M, \mathcal{O}_M) = \#(\Delta \cap \mathbb{Z}^n).$$

There is a lovely proof of this fact, which I learned from Sue Tolman, but as far 
as I know, it does not appear in the literature in this form. (In [GGK] they give a 
proof using these ideas (Proposition 8.4), but it is embedded in a much more general 
discussion and is not so easy to isolate.) The argument is so straightforward and 
accessible I thought it worth presenting on its own. I do not at all claim originality; 
rather, the aim of this paper is to present the argument simply and clearly. Thus, 
I have made no attempt to be as general as I can, but have chosen transparency 
over generality wherever possible.

Delzant’s construction of a toric manifold from its moment polytope is essential 
to this argument, and so we review it in Section 2. The proof of the Theorem ap-
ppears in Section 3 after stating a few facts about quantization and toric manifolds. 
Finally, in Section 4 we show how the concepts in this paper apply in a simple 
example.

Date: July 27, 2007.
2000 Mathematics Subject Classification. Primary 53D50.
Supported by a PIMS Postdoctoral Fellowship.
We assume the reader is familiar with the basic concepts of symplectic toric geometry, and thus we do not define terms like “toric manifold” and “moment map.” If these are unfamiliar, we recommend the introduction by Cannas da Silva (CdS).

Acknowledgements. I am grateful to Yael Karshon for explaining this argument to me in the first place several years ago, and for answering my questions more recently. I am also grateful to Sue Tolman for taking the time to answer my questions as well. Finally, I thank Paul Sloboda for his hospitality during the early stages of work on this paper.

2. Construction of toric manifolds

We present here two different, though related, constructions of a toric manifold from its moment polytope, which we call the “symplectic” and “complex” constructions. This is intended to be a review, and so we skip a number of details, including most of the proofs (and so, in particular, this is not a good place to learn the constructions for the first time. For that, the reader is directed to CdS for the symplectic construction, and [KT] and [A] for the complex construction).

2.1. Symplectic construction. This construction is due to Delzant (Dz). Given a convex polytope \( \Delta \subset \mathbb{R}^n \), it produces a symplectic manifold \( M^{2n} \), together with an effective action of the torus \( T^n \cong (S^1)^n \), whose moment map image is precisely \( \Delta \). The polytope is required to satisfy the condition that at each vertex there are \( n \) edges, generated by a \( \mathbb{Z} \)-basis for the lattice \( \mathbb{Z}^n \); such polytopes are often called Delzant polytopes. As shown in Dz (see the remark on p. 323), these are precisely the polytopes that appear as moment map images of toric manifolds. Cannas da Silva gives a lovely explanation of Delzant’s construction in CdS, which we follow to some extent, although we caution the reader that we use slightly different sign conventions than she does.

Let \( \Delta \) be a convex polytope in \( \mathbb{R}^n \) with \( N \) facets (codimension-1 faces), satisfying Delzant’s condition. For each facet of \( \Delta \), let \( v_j \in \mathbb{Z}^n \) be the primitive inward-pointing vector normal to the facet. Define a projection \( \pi \) from \( \mathbb{R}^N \) to \( \mathbb{R}^n \) by taking the \( j \)th basis vector in \( \mathbb{R}^N \) to \( v_j \in \mathbb{R}^n \):

\[
\pi : \mathbb{R}^N \to \mathbb{R}^n \\
e_j \mapsto v_j
\]

(1)

Delzant’s condition on \( \Delta \) implies that the \( v_j \) span \( \mathbb{R}^n \); in fact, the \( n \) vectors normal to the facets meeting at any one vertex form a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^n \) (this is left as a linear algebra exercise for the reader, although we note that this is how a Delzant polytope is defined in [GGK]). Thus \( \pi \) maps \( \mathbb{Z}^N \) onto \( \mathbb{Z}^n \) and so induces a map (which we also call \( \pi \)) between tori,

\[
\pi : \mathbb{R}^N / \mathbb{Z}^N \to \mathbb{R}^n / \mathbb{Z}^n .
\]

Let \( K \) be the kernel of this map, and \( k \) the kernel of the map (1), which will in fact be the Lie algebra of \( K \). We then get two exact sequences

\[
1 \longrightarrow K \overset{i}{\longrightarrow} T^N \overset{\pi}{\longrightarrow} T^n \longrightarrow 1 \\
0 \longrightarrow k \overset{i}{\longrightarrow} \mathbb{R}^N \overset{\pi}{\longrightarrow} \mathbb{R}^n \longrightarrow 0
\]

(2a) (2b)

\footnote{\( a \) vector \( v \in \mathbb{Z}^n \) is primitive if its coordinates have no common factor}
and the dual exact sequence

\[
\begin{array}{c}
0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^N)^* \xrightarrow{i^*} \mathbb{R}^* \longrightarrow 0
\end{array}
\]

with induced maps as shown. Since we will be working a lot with \(i^*\), we will denote it by \(L\), both for ease of notation and to emphasize that it is a linear map between vector spaces.

Using the vectors \(v_j\), we can write the polytope as

\[\Delta = \{ x \in \mathbb{R}^n \mid \langle x, v_j \rangle \geq \lambda_j, \ 1 \leq j \leq N \}\]

for some real numbers \(\lambda_j\). We assume that the \(\lambda_j\) are all integers; this will ensure that \(M\) is pre-quantizable (see Fact \(\ [*]\) in \(\ [3]\)). This gives us a vector \(\lambda \in \mathbb{Z}^N\).

Let \(\nu = L(-\lambda) \in \mathbb{R}^*\) (identifying \((\mathbb{R}^N)^*\) with \(\mathbb{R}^N\)). Since the sequence \(\ [3]\) is exact, \(L^{-1}(0) = \text{im} \pi^*\), so \(L^{-1}(\nu) = \text{im}(\pi^* - \lambda)\) and, since \(L\) is a linear map between vector spaces, \(L^{-1}(\nu)\) is an affine subspace of \(\mathbb{R}^N\). The intersection of this affine subspace with \(\mathbb{R}^N_+\), the positive quadrant in \(\mathbb{R}^N\), can be identified with the polytope \(\Delta\). More precisely,

**Claim 1.** Let \(\Delta' = L^{-1}(\nu) \cap \mathbb{R}^N_+\). Then the map \(\pi^* - \lambda\) restricts to an affine bijection from \(\Delta\) to \(\Delta'\), such that the integer lattice points in \(\Delta\) correspond to \(\Delta' \cap \mathbb{Z}^N_+\).

**Proof.** For \(x \in \mathbb{R}^n\), \((\pi^* - \lambda)(x) \in \mathbb{R}^N_+\) iff \(x \in \Delta\), as follows:

\[
(\pi^* - \lambda)(x) \in \mathbb{R}^N_+ \iff \langle \pi^*(x) - \lambda, e_j \rangle \geq 0 \quad \forall j
\]

\[
\iff \langle \pi^*(x), e_j \rangle - \lambda_j \geq 0
\]

\[
\iff \langle x, \pi(e_j) \rangle \geq \lambda_j
\]

\[
\iff \langle x, v_j \rangle \geq \lambda_j \quad \forall j
\]

\[
\iff x \in \Delta.
\]

Since \((\pi^* - \lambda)\) is an affine injection \(\mathbb{R}^n \rightarrow \mathbb{R}^N\), it is a bijection onto its image, and so it is a bijection from \(\Delta\) onto \(\text{im}(\pi^* - \lambda) \cap \mathbb{R}^N_+\), which is \(L^{-1}(\nu) \cap \mathbb{R}^N_+\) as argued above.

Finally, since all of the coordinates of each of the \(v_j\) and \(\lambda\) are integers, \((\pi^* - \lambda)\) maps \(\mathbb{Z}^n\) into \(\mathbb{Z}^N\). It only remains to see that if a point in \(\Delta'\) has integer coordinates, then it is the image under \((\pi^* - \lambda)\) of a point in \(\mathbb{Z}^n\), for which it suffices to show that if \(\pi^*(x) = y \in \mathbb{Z}^N\), then \(x \in \mathbb{Z}^n\).

The map \(\pi^*\) can be written as \(y = Vx\) where \(y\) and \(x\) are the variables in \(\mathbb{R}^N\) and \(\mathbb{R}^n\), written as column vectors, and \(V\) is the \(N \times n\) matrix whose rows are the vectors \(v_j\).

As noted in the previous section, \(n\) of the \(v_j\)s corresponding to one vertex form a \(\mathbb{Z}\)-basis of \(\mathbb{Z}^n\); wolog suppose \(v_1, \ldots, v_n\) form such a basis. Let \(\tilde{V}\) be the \(n \times n\) matrix whose rows are \(v_1, \ldots, v_n\), so that the equation \(Y = \tilde{V}x\) defines \(y_1, \ldots, y_n\) from the \(x\)s (where \(Y\) is the column vector of \(y_1, \ldots, y_n\)).

Since \(v_1, \ldots, v_n\) form a \(\mathbb{Z}\)-basis for \(\mathbb{Z}^n\), the determinant of \(\tilde{V}\) is \(\pm 1\), so \(\tilde{V}\) is invertible and its inverse has integer entries. Thus, given a \(Y\) with integer entries, \(x = \tilde{V}^{-1}Y\) will also have integer entries, and so integer lattice points in the image of \(\pi^*\) come from integer lattice points in \(\mathbb{R}^n\).

\[\square\]
The torus $T^N$ acts on $\mathbb{C}^N$ in the standard way, by componentwise multiplication; this action is Hamiltonian with moment map $\phi(z_1, \ldots, z_N) = (|z_1|^2, \ldots, |z_N|^2)$ (where here we mean the number $\pi$, not the map from (2)). The inclusion $i: K \hookrightarrow T^N$ induces a Hamiltonian action of $K$ on $\mathbb{C}^N$ with moment map $\mu = i^* \circ \phi$ from $\mathbb{C}^N \to k^*$.

Let $M = \mu^{-1}(\nu)/K$. The action of $T^N$ on $\mathbb{C}^N$ commutes with the action of $K$ and thus descends to a Hamiltonian action on the quotient $M$. This action is not effective; however, the quotient torus $T_n = T^N/K$ acts effectively. It is a theorem of Delzant that $M$ with this action is a smooth toric manifold, with moment polytope $\Delta$. (See [Dz], p 329 or [CdS], Claim 2 in section I.2.5.)

In summary, then, $M$ is the symplectic reduction of $\mathbb{C}^N$ by $K$ at $\nu \in k^*$, where both $\nu$ and $K$ are determined by the polytope (using the sequences (2) and (3)).

2.2. Complex construction. The above construction produces $M$ as a symplectic manifold with a torus action, but does not give the Kähler structure. For that, we use a different construction that realizes $M$ as the quotient of an open set $U_\mathcal{F}$ in $\mathbb{C}^N$ by the action of a complex torus $K_\mathbb{C}$. This is usually (eg in [A], [KT]) described in terms of a fan. However, it can also be done directly from the polytope, and we have chosen this approach to avoid having to explain the notion of fans. Note that, even phrased in the language of fans, this approach is not the same as the complex construction that appears in [CdS]. The latter is the algebraic geometry construction of $M$ as a toric variety, which is the same as given for example in [F] and [O].

Begin with a Delzant polytope $\Delta$ as in the previous section, and construct the map $\pi$ and the exact sequences (2) and (3) as described there. If we complexify the sequence (2), we get the following exact sequence

$$1 \longrightarrow K_\mathbb{C} \stackrel{i}{\longrightarrow} T^N_\mathbb{C} \stackrel{\pi}{\longrightarrow} T^N \longrightarrow 1$$

of complex tori, where $T^N_\mathbb{C}$ denotes the complex torus $(\mathbb{C}^*)^N$, and $K_\mathbb{C} = \ker i$ is the complexification of $K$.

Let $F_1, F_2, \ldots, F_N$ be the facets of $\Delta$. Define a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, N\}$ as follows:

- $\emptyset \in \mathcal{F}$
- $I \in \mathcal{F} \iff \bigcap_{j \in I} F_j \neq \emptyset$

i.e., $I = \{j_1, \ldots, j_k\}$ is in $\mathcal{F}$ iff the intersection $F_{j_1} \cap \cdots \cap F_{j_k}$ is non-empty.

The open set $U_\mathcal{F} \subset \mathbb{C}^N$ is constructed as follows. Given a point $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, let its zero-index set be the set

$$I_z = \{j \mid z_j = 0\}.$$
Then $U_{\mathcal{F}}$ is defined to be the set of $z$ in $\mathbb{C}^N$ whose zero-index sets are in $\mathcal{F}$,

$$z \in U_{\mathcal{F}} \iff I_z \in \mathcal{F}.$$  

Then it is a theorem that $M = U_{\mathcal{F}}/K_{\mathbb{C}}$, where $K_{\mathbb{C}}$ acts via the inclusion $i: K_{\mathbb{C}} \hookrightarrow T_{\mathbb{C}}^N$, is a smooth toric manifold. (See for example [A], Proposition VII.1.14 and surrounding discussion. Her $U_{\Sigma}$ is the same as our $U_{\mathcal{F}}$, although it is constructed differently.)

**Example.** Suppose $\Delta$ is the unit square in $\mathbb{R}^2$, with facets numbered as shown in Figure 1. Then

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}.$$

The only points not in $U_{\mathcal{F}}$ are those with first and third coordinate zero, or second and fourth coordinate zero, and so

$$U_{\mathcal{F}} = \mathbb{C}^4 \setminus (\{z_1 = 0 = z_3\} \cup \{z_2 = 0 = z_4\}).$$

**Remark.** Note that $\mathbb{C}^N \setminus U_{\mathcal{F}}$ is the union of submanifolds of (complex) codimension at least 2, for the following reason: from (6), $\mathbb{C}^N \setminus U_{\mathcal{F}}$ will be the set of points whose zero-index set is not in $\mathcal{F}$. Since each of the singletons $\{j\}$ are in $\mathcal{F}$, all points in $\mathbb{C}^N \setminus U_{\mathcal{F}}$ have at least two coordinates which are zero. Thus $\mathbb{C}^N \setminus U_{\mathcal{F}}$ is the union of sets of codimension at least 2.

The fact that these two constructions yield the same manifolds is Remark 2.6 in [KT]. The reason is the following: The sets $\mu^{-1}(\nu)$ and $U_{\mathcal{F}}$ are related as follows:

$$U_{\mathcal{F}} = K_{\mathbb{C}} \cdot \mu^{-1}(\nu)$$

(see [GGK], section 5.5.). Thus it is no surprise that $U_{\mathcal{F}}/K_{\mathbb{C}} = \mu^{-1}(\nu)/K$.

### 3. Quantization

The purpose of geometric quantization is to associate, to a symplectic manifold $(M, \omega)$, a Hilbert space (or a vector space) $\mathcal{Q}(M)$. (The terminology “quantization” comes from physics, where we think of $M$ as a classical mechanical system, and $\mathcal{Q}(M)$ as the space of wave functions of the corresponding quantum system.) Much of the motivation for geometric quantization in mathematics comes from representation theory.

The basic building block in the geometric quantization of $(M, \omega)$ is a complex line bundle $L \rightarrow M$ with a connection whose curvature form is $\omega$, called a *prequantum line bundle*. If such an $L$ exists, $M$ is called *prequantizable*, which will be the case...
if $[\omega] \in H^2(M, \mathbb{R})$ is integral. The quantization space $Q(M)$ is constructed from sections of $L$.

The space of all sections is in general “too big,” and so a further structure, called a polarization, is used to cut down on the number of sections. One of the most common is a Kähler polarization, which is a complex structure on $M$ compatible with the symplectic form. In this case, we take $L$ to be a holomorphic line bundle, and the quantization space is the space of holomorphic sections of $L$ over $M$: $Q(M) = \Gamma_\mathcal{O}(M, L) = H^0(M, \mathcal{O}_L)$. This is what we use in the case of toric manifolds.

**Remark.** The quantization is often defined to be the virtual vector space

$$Q(M) = \sum_{j \geq 0} (-1)^j H^j(M, \mathcal{O}_L).$$

When $M$ is a toric manifold, all these cohomology groups with $j > 0$ are zero (see e.g. [Dn], Cor. 7.4, or [O], Cor. 2.8), and so the quantization is just $H^0(M, \mathcal{O}_L)$, namely the space of holomorphic sections.

(There are many sources for geometric quantization, for the reader who wishes more than these very sketchy details. The books [W] and [S] are classic references, if both rather technical; [P] is perhaps easier as an introduction, though still very complete. The referee pointed me to [E], available on the arXiv. John Baez has a good brief introduction on the Web at [B]. [GGK] also has a good, brief introduction at the beginning of Chapter 6, and refer to numerous other sources. There are of course many other references.)

We first state some facts about quantization applied to toric manifolds:

**Fact 1.** The toric manifold $M$ constructed from $\Delta$ is pre-quantizable if the $\lambda \in \mathbb{R}^N$ appearing in the symplectic construction is in $\mathbb{Z}^N$.

See [GGK], Example 6.10 on p. 93; see also [Dz], p 327.

**Fact 2.** If the toric manifold $M$ is presented as $U_F/K_C$ as in §2.2 then

$$L = U_F \times_{K_C} \mathbb{C}$$

is a prequantum line bundle, where $K_C$ acts on $\mathbb{C}$ with weight $\nu = L(-\lambda) \in k^*$ ($\lambda$ as in §2.1).

Now we come to the main point of the paper, which is to give a proof of the following theorem:

**Theorem.** Let $M_\Delta$ be a toric manifold, with moment polytope $\Delta \subset \mathbb{R}^n$. Then the dimension of the quantization space is equal to the number of integer lattice points in $\Delta$,

$$\dim H^0(M, \mathcal{O}_L) = \#(\Delta \cap \mathbb{Z}^n).$$

This “if” is actually an “iff,” modulo some subtleties about equivariance. If $M$ is equivariantly prequantizable, in the sense of [GGK] chapter 6, then it is necessary that $\lambda$ be in $\mathbb{Z}^N$. If $M$ is “non-equivariantly” prequantizable, then the polytope (and thus $\lambda$) still satisfies an integrality condition. Since the moment map is defined only up to a constant, we can add a constant to $\lambda$ without changing the construction of $M$; this corresponds to translating $\Delta$ without changing its shape. The integrality condition implies that we can choose the constant so that $\lambda$ lies in $\mathbb{Z}^N$. 

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Proof. A holomorphic section of $L = (U_F \times \mathbb{C})/K_C$ over $M = U_F/K_C$ corresponds to a $K_C$-equivariant, holomorphic function $s: U_F \to \mathbb{C}$. Because $\mathbb{C}^N \times U_F$ is the union of submanifolds of codimension greater than or equal to 2 (see the Remark in §2.2), $s$ extends to a holomorphic function on all of $\mathbb{C}^N$, which we still call $s$. (This follows from Hartogs’ theorem, for example [GH] p. 7 — a holomorphic function on $\mathbb{C}^N$ for $N > 1$ cannot have an isolated singularity, and therefore cannot have a singularity on a submanifold of codimension $\geq 2$.)

Thus we are looking for the $K_C$-equivariant, holomorphic functions $s: \mathbb{C}^N \to \mathbb{C}$, where the action of $K_C$ on $\mathbb{C}$ is with weight $\nu$, and the action on $\mathbb{C}^N$ is via the inclusion $i: K_C \hookrightarrow T^N_C$ and the standard action of $T^N_C$ on $\mathbb{C}^N$. Write such a function $s$ as its Taylor series, so that

$$s = \sum_{I \in \mathbb{Z}^N_+} a_I z^I$$

(where $I = \{j_1, \ldots, j_N\}$ is a multi-index, $a_I$ is a complex number, and as usual in complex variables $z^I$ means $z_1^{j_1}z_2^{j_2} \cdots z_N^{j_N}$). Consider one term $z^I$ in this sum at a time.

First note that, for $t \in T^N_C$ and $z \in \mathbb{C}^N$,

$$(t \cdot z)^I = (t_1 z_1, \ldots, t_N z_N)^I = ((t_1 z_1)^{j_1} \cdots (t_N z_N)^{j_N}) = t^I z^I.$$  

Now suppose $s(z) = z^I$, and see when it is equivariant. First,

$$s(k \cdot z) = s(i(k) \cdot z) = (i(k) \cdot z)^I = i(k)^I z^I = k^\nu(I) z^I.$$  

On the other hand,

$$k \cdot s(z) = k^\nu \cdot z^I.$$  

Thus $s(k \cdot z) = k \cdot s(z)$ when $i^*(I) = \nu$, i.e. $L(I) = \nu$.

Therefore, a basis for the space of equivariant sections, and thus for $H^0(M, \mathcal{O}_L)$, is

$$\{z^I \mid L(I) = \nu, \quad I \in \mathbb{Z}^N_+\}.$$  

The set of such $I$ is $\mathbb{Z}^N_+ \cap L^{-1}(\nu)$, which, as noted in Claim[1], corresponds precisely with the set of integer lattice points in the moment polytope $\Delta$.

4. Example

To see how all of these constructions play out, we will go through one example in detail (with some calculations left as exercises). Take the polytope to be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, m)$, and $(m, 0)$, for $m \in \mathbb{Z}_+$, as shown. Here $N = 3$ and $n = 2$, so we will be constructing a 4-dimensional manifold as a quotient of $\mathbb{C}^3$, with an action of $T^2$.

Figure 2. The polytope $\Delta$ for this example
The three normal vectors are
\[ v_1 = (0, 1) \quad v_2 = (1, 0) \quad v_3 = (-1, -1) \]
and \( \lambda = (0, 0, -m) \). Therefore the map \( \pi: \mathbb{R}^3 \to \mathbb{R}^2 \) in (1) can be written as the matrix
\[
\begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & -1
\end{bmatrix}
\]
or, writing the coordinates in \( \mathbb{R}^3 \) as \((x, y, z)\),
\[
\pi(x, y, z) = (y - z, x - z).
\]
The kernel of this map is \( \{x = y = z\} \) in \( \mathbb{R}^3 \), which is \( k \), which we identify with \( \mathbb{R} \) by \( i: t \mapsto (t, t, t) \).

The corresponding map on tori is
\[
(e^{2\pi ix}, e^{2\pi iy}, e^{2\pi iz}) \mapsto (e^{2\pi i(y-z)}, e^{2\pi i(x-z)})
\]
with kernel \( K = (e^{2\pi it}, e^{2\pi it}, e^{2\pi it}) \), which is \( S^1 \) embedded into \( T^3 \) as the diagonal subtorus.

For the dual sequence, the map \( \pi^* \) will be given by the transpose matrix
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & -1 & -1
\end{bmatrix}
\]
or, writing coordinates in \( \mathbb{R}^2 \) as \((a, b)\),
\[
\pi^*(a, b) = (a, b, (-a - b)).
\]
Similarly, the map \( L = i^* \) is \( L(x, y, z) = x + y + z \).

From this, we get that \( \nu = L((-0, 0, -m)) = m \). Therefore, the affine space \( L^{-1}(\nu) \) is the space \( \{x + y + z = m\} \) lying in \( \mathbb{R}^3 \); the intersection of this space with the positive orthant \( \mathbb{R}^3_+ \) is a triangle, whose identification with \( \Delta \) is easy to see. (See Figure 3.)

![Figure 3. \( L^{-1}(\nu) \) with integer lattice points](image)

Pulling this intersection \( L^{-1}(\nu) \cap \mathbb{R}^3_+ \) back by the map \( \phi: \mathbb{C}^3 \to \mathbb{R}^3 \) gives us
\[
\mu^{-1}(\nu) = \{z \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = m/\pi \} \cong S^5;
\]
the reduction of this by the diagonal action of \( S^1 \) is \( \mathbb{C}P^2 \).

Note that the integer points in \( \Delta \) will be the set \( \{(x, y, z) \in \mathbb{Z}^3 \mid x, y, z \geq 0, \ x + y + z = m\} \), which is in one-to-one correspondence with the set
\[
\{(x, y) \in \mathbb{Z}^2 \mid x, y \geq 0, \ x + y \leq m\} \]
integer points in $\Delta$.

For the complex construction, labelling the facets using the same numbering as we used for the normal vectors, the collection of subsets $\mathcal{F}$ corresponding to this polytope is

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}.$$ 

Thus $U_{\mathcal{F}}$ is the set of points in $\mathbb{C}^3$ which have either zero, one, or two coordinates zero, that is, $U_{\mathcal{F}} = \mathbb{C}^3 \setminus \{0\}$.

The complex torus here is the complexification of $K$, i.e. $K_C = \mathbb{C}^*$, acting on $\mathbb{C}^3$ by the diagonal action. The quotient of $\mathbb{C}^3 \setminus \{0\}$ by the diagonal action of $\mathbb{C}^*$ is $\mathbb{C}P^2$.

(Notice that in passing to the complex construction we lose the information about the “size” of the reduced space. This is a general phenomenon — $U_{\mathcal{F}}$ “remembers” the directions of the faces of the polytope, but not their sizes.)

Finally, the prequantum line bundle will be $L = U_{\mathcal{F}} \times_{K_C} \mathbb{C}^*$, where $K_C$ acts on $\mathbb{C}^3$ by the diagonal action. The quotient of $\mathbb{C}^3 \setminus \{0\}$ by the diagonal action of $\mathbb{C}^*$ is $\mathbb{C}P^2$.

Looking at the polytope in Figure 2, we can see that there will be $(m + 1) + m + \cdots + 1$ points with integer coordinates, and so the quantization will have dimension $m(m+1)/2$.

Exercise: Repeat the above procedure when the polytope is the square considered in the example in §2.2.

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