ON THE REGULARITY OF PRIMES IN ARITHMETIC PROGRESSIONS

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Abstract. We prove that for a positive integer \(k\) the primes in certain kinds of intervals can not distribute too “uniformly” among the reduced residue classes modulo \(k\). Hereby, we prove a generalization of a conjecture of Recaman and establish our results in a much more general situation, in particular for prime ideals in number fields.

1. Introduction and Main Result

Let \(\omega(k)\) be the number of distinct prime factors of an integer \(k\), and let \(\varphi\) denote Euler’s totient function. We say that \(k\) is a \(P\)-integer if the first \(\varphi(k)\) primes which do not divide \(k\) form a complete residue system modulo \(k\). In 1978 Recaman [12] conjectured that there are only finitely many prime \(P\)-integers. In 1980 Pomerance [11] proved this by showing that there are in fact only finitely many \(P\)-integers, and conjectured moreover that every \(P\)-integer does not exceed 30. This was proved in special cases by Hajdu, Saradha, and Tijdeman [13, 16, 14]. In fact, they proved in [16] the conjecture of Pomerance under the assumption of the Riemann Hypothesis. Eventually, in a recent paper of Yang and Togbé [17] the conjecture was proven unconditionally.

However, one can rephrase the definition of \(P\)-integers as follows: Let, without further mentioning, \(p\) denote a prime, \(\mathbb{P}\) the set of primes, and \(p_n\) the \(n\)-th smallest prime. Then \(k\) is a \(P\)-integer if the block \(p_1, p_2, \ldots, p_{\varphi(k)+\omega(k)}\) of the first \(\varphi(k)+\omega(k)\) primes, lying in the closed interval \([p_1, p_{\varphi(k)+\omega(k)}]\), has precisely one element in each reduced residue class modulo \(k\), with the exception of \(\omega(k)\) primes (which lie in distinct, non-invertible residue classes). By viewing \(P\)-integers as instances of such distribution phenomena, there is an obvious and far more general notion for this.

Definition. Let \(\alpha, \beta, \gamma, \iota > 0\) denote integers, and \(G = (G, \cdot)\) an arithmetical semi-group with norm \(|\cdot|\), in the sense of Knopfmacher [6, p. 11], which takes only values in the positive integers. Consider for \(k \in G\) the equivalence relation \(a \sim b :\iff |a| \equiv |b| \mod |k|\) on \(G\) and let \(M\) denote the primes in \(G\) with norm in the interval \([\alpha, \beta]\). Then we say \(k \in G\) is a \(P(\alpha, \beta, \gamma, \iota)\)-integer if \(M\) has in each equivalence class corresponding to an invertible residue class modulo \(|k|\) at least \(\gamma\) elements, and the remaining \(\iota\) primes distribute in some arbitrary equivalence classes such that \(|M| = \gamma \varphi(k) + \iota\). (For ease of exposition we shall simply speak of \(P^*\)-integers if no confusion can arise.)

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1
A natural question is to estimate for a given $k \in G$ the smallest values of $\alpha, \beta$ such that $k$ is for the first time a $P^\ast$-integer. Let us simplify this question by considering the semi-group $G = \mathbb{N}$ of the natural numbers, endowed with its canonical norm, and by asking the following question: Fix $\alpha = 2$ and estimate for a given $k$ the smallest integer $\beta = \beta (k)$ such that $k$ is the first time a $P(2, \beta, 1, \iota)$-integer for some $\iota$. This problem is nothing but estimating Linnik’s constant which is widely open. Yet, the following probabilistic considerations suggest that $\beta$ should be in $O(k \log^2 k)$.

We start by estimating the probability $P(X)$ for a random set of $f(k) \geq \varphi(k)$ primes to not cover all of the $\varphi(k)$ reduced residue classes with at least one prime. We assume that a given prime $p$ is in an invertible residue class modulo $k$; for otherwise $p$ divides $k$ which can happen at most $\omega(k) \ll \log k$ times and picking such a prime out of $f(k)$ arbitrary primes almost never happens. In view of Dirichlet’s Theorem on arithmetic progressions, we assume moreover that a prime $p$ has about probability $\frac{1}{\varphi(k)}$ to be in a specific invertible residue class modulo $k$. Let $X_r$ denote the event that in the invertible residue class $r$ modulo $k$ none of the $f(k)$ primes occurs. Then, writing $f(k) = C(k) \varphi(k) \log k$, (say), the probability $P(X_r)$ of $X_r$ is

$$
\left(1 - \frac{1}{\varphi(k)}\right)^{f(k)} \approx (1 + o(1)) k^{-C(k)}.
$$

By utilizing the inclusion-exclusion principle, we conclude that

$$
P(X) = P\left(\bigcup_r X_r\right) \approx \sum_r P(X_r) \approx \frac{\varphi(k)}{k^{C(k)}},
$$

whereas the union and the summation run through a complete residue system $r$ modulo $k$. Hence, if $C(k) > 1 + \varepsilon$ for some fixed $\varepsilon > 0$, we expect with a positive probability that our $f(k)$ primes cover all invertible residue classes at least once. On the other hand, if $C(k) < 1 - \varepsilon$ holds, we expect, by using the reversed Borel-Cantelli Lemma, that $X$ is likely to occur infinitely often. Since $p_n \sim n \log n$, the threshold $C = 1$ amounts to the estimate $\beta(k) \approx \varphi(k) \log k \log(\varphi(k) \log k) = O(k \log^2 k)$ for having about $\varphi(k) \log k$ primes in the interval $[2, \beta(k)]$. This approximation was suggested by a similar, but more complicated heuristic of Wagstaff [19] and is plausible in view of various results e.g. from Turán [18]. The latter showed, assuming the Extended Riemann Hypothesis, that for any $\delta > 0$ the smallest prime $P(k, l)$ in the invertible residue class $l$ modulo $k$ is exceeding the quantity $\varphi(k) \log^{2 + \delta}(k)$ for at most $o(\varphi(k))$ choices of $l$. There are other results of this kind [3] we refer the reader to [3] and the references therein. However, there is also reason to be cautious with respect to the above mentioned heuristic. In this direction there are, inter alia, the results of Maier [8], Rubinstein and Sarnak [7], or [10].

Let us stress that for $k \in G$, where $G$ is as in Definition [1] our heuristic suggests that one should need about $\varphi(|k|) \log |k|$ primes to cover the invertible residue classes modulo $|k|$ in $G$ at least once with primes and not just $\varphi(|k|) + \omega(|k|)$ as one

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1. By $\ll$ we denote the usual Vinogradov-symbol. The implied constant is absolute unless we specify a dependency of some variable by an appropriate subscript. We shall use the Landau-symbols in the same way.

2. Cf. [1].

3. The latest record for calculating Linnik’s constant, to the best of our knowledge, is due to T. Xylouris [21] who refined the previous work of Heath-Brown [3]. Moreover, Heath-Brown conjectured that $P(k, l) = O(k \log^2 k)$ holds for any coprime pairs $l, k$ and $k$ sufficiently large.
asks in Recaman’s conjecture. Our first result proves, under certain assumptions, that this is indeed the case. Furthermore, we say $G$ satisfies Axiom A (cf. [9] p. 75) with $\delta > 0$, if for some $0 < \eta < \delta$ the counting function $N_G(x) := \# \{ g \in G : |g| \leq x \}$ has the expansion $x^\delta + O(x^\eta)$ as $x \to \infty$. Thus we can state:

**Theorem 1.** Let $K := |k|$. Let $G$ as in Definition 1 satisfy Axiom A with some $\delta > 0$. Assume that numbers $\alpha = 1, \beta \ll K \log^a K$ and $\iota \ll \log^b K$ are given for some fixed $a, b > 0$. Then there are only finitely many such $P^\ast$-integers.

For instance, the assumptions (on the semi-group) above are satisfied if $G$ is the set of non-zero integral ideals of a number field $\mathbb{K}$ with the usual ideal norm. Moreover, one can also interpret the property to be a $P^\ast$-integer as the resolvability (in the set of primes) of a certain set of Diophantine equations and inequalities. For determining all such solutions, it is of interest to furnish Theorem 1 with explicit bounds on $k$ and it might be interesting in its own right to make a qualitative statement quantitative.

We shall do so only in the case $G = \mathbb{N}$ since one needs explicit bounds for the prime counting function $\pi_G(x) := \# \{ p \in G : p \text{ prime}, |p| \leq x \}$, for $x > 0$, of $G$ which are only known if one has sufficient arithmetic information about $G$. For instance, the error term in Landau’s prime ideal theorem naturally depends on the given number field. However, once these informations are given it is a straightforward task to extend our explicit results to more general cases.

Loosely speaking, our main result states, in a quantitative manner, that blocks of primes (in the natural numbers) of approximate length $\gamma \varphi(k)$ are, in general, not evenly distributed among the reduced residue classes modulo $k$. More precisely, we prove the following extension of Recaman’s conjecture:

**Theorem 2.** Let $\lambda \in \mathbb{N} \cup \{0\}$ and $d_1, d_2, d_3$ denote strictly positive real numbers. There are only finitely many $P(\alpha, \beta, \gamma, \iota)$-integers in $\mathbb{N}$ such that the growth restrictions $\alpha = \lambda k + O(k^{1-d_1}), \iota = O(k^{1-d_2})$ and $\beta = O(k \log^{d_3} k)$ are satisfied.

The paper is organized as follows: Firstly, we deduce a necessary condition for $g \in G$, where $G$ is always assumed to be as in Definition 1 to be a $P^\ast$-integer and prove Theorem 1. This will be done via a combinatorial argument which leads to inequalities involving sums over the prime counting function $\pi$ evaluated at certain points. Secondly, we will remove $\pi$ from these inequalities by approximating it and then deal with the sums in such a manner that we receive explicit formulas for seeing which large $k$ violate the arising inequalities.

## 2. Preliminaries and Proof of Theorem 1

We first collect some results which we will need in the proofs.

**Lemma 3** (Cf. [9] Thm. 1). Let $\theta(x) := \sum_{p \leq x} \log p$ denote the Chebyshev function where the summation runs through all primes $p \leq x$. With

$$
\varepsilon(x) := \sqrt{\frac{8 \log x}{17 \pi \cdot \eta}} e^{-\sqrt{n^{-1} \log x}} \quad \text{for } x \geq 149, \eta := 6.455
$$

we have

$$
|\theta(x) - x| < x\varepsilon(x), \quad \text{for } x \geq 149.
$$
Remark 4. We recall that

1. \( p_n \geq n \log n \) for any \( n \geq 1 \), see [15] p. 69, and
2. for \( k \geq 2 \) we have, cf. [2] Thm. 6.9, \[
E_{0,-} := 2\pi(0.5k) - \pi(k) > \frac{k}{\log(0.5k)} \left( 1 + \frac{1}{\log(0.5k)} + \frac{2}{\log^2(0.5k)} \right)
- \frac{k}{\log(k)} \left( 1 + \frac{1}{\log(k)} + \frac{2.334}{\log^2(k)} \right).
\]

3. Moreover, we need the estimates

\[
(2.1) \quad \sqrt{\frac{2}{\pi}} \left( 2S + 1 \right) \leq \prod_{s=1}^{S} \frac{2s + 1}{2s} \leq \frac{2S + 1}{\sqrt{3\pi}}
\]

4. The following estimate holds, cf. [15] p. 72:

\[
\varphi(k) \geq \frac{k}{1.7811 \log \log k + \frac{2.81}{\log \log k}}, \quad k \geq 3.
\]

5. Let \( \text{li}(x) \) denote the integral \( \int_{2}^{x} \frac{dt}{\log t} \) for \( x > 0 \). If for an arithmetical semigroup \( G \) the counting functions \( g(x) := \# \{ g \in G : |g| \leq x \} \) takes the form

\[
g(x) = Ax^\beta + O(x^\delta \log^{-\beta} x), \quad \beta > 3, \delta > 0, x \to \infty,
\]

then the prime counting function of \( G \) can be written as

\[
\pi_G(x) = \text{li}(x^\beta) + O(x^\delta \log^{-c} x) \quad \text{for any } c < \frac{\beta}{3}.
\]

This is due to Wegmann [20]. In particular, the conclusion is true, if \( G \) satisfies Axiom \( A \).

Our method to detect \( P^* \)-integers originates from [16], which we shall describe in the following. We write \( \pi_G(x) = \pi(x) \) and denote for natural numbers \( x, K \) by \( x \mod K \) the unique remainder \( r \in \{0, \ldots, K - 1\} \) such that \( x = qK + r \) holds for some \( q \in \mathbb{N} \). Let us assume that \( K \) is a \( P^* \)-integer and put \( K := |K| \). Then, by the symmetry of coprime residue classes modulo \( K \) about \( 0.5K \), the cardinalities of the sets

\[
A_1 := \{ p \in G : \alpha \leq |p| \leq \beta, p \text{ prime, } |p| \mod K \leq 0.5K \},
A_2 := \{ p \in G : \alpha \leq |p| \leq \beta, p \text{ prime, } |p| \mod K > 0.5K \},
\]

differ by at most \( \iota \) elements. For checking this condition, we count the size of \( A_i \) which is done by the following lemma:

Lemma 5. Let

\[
E_{j,1}(k) := \pi((j + 0.5)K) - \pi(jK - 1), \quad E_{j,2}(k) := \pi((j + 1)K) - \pi((j + 0.5)K)
\]

for \( j \geq 0, i = 1, 2 \). If \( \lambda, \Lambda \) denote integers such that \( \lambda K \leq \alpha < (\lambda + 1)K \), and \( \lambda K \leq \beta < (\lambda + 1)K \) hold, then we have

\[
(2.2) \quad |A_i| = M_i(k) + \sum_{j \in I} E_{j,i}(k), \quad I := I_{\lambda, \Lambda} := \{ \lambda + 1, \lambda + 2, \ldots, \Lambda - 1 \},
\]

whereas \( M_i(k) \) is defined in [25].
Proof. We partition the set $A_1$ into subsets $A_{1,j}$ of primes having norm in $[jK, (j + 0.5)K]$ and $A_2$ into subsets of primes having norm in $[(j + 0.5)K, (j + 1)K]$ where $\lambda \leq j \leq \Lambda$. Note that $E_{j,i}(k)$ counts how many primes are located in $A_{1,j}$ for $\lambda < j < \Lambda$ and $i = 1, 2$. This gives rise to the term $\sum_{j \in \mathbb{I}} E_{j,i}(k)$. Counting the primes near the end-points $j = \lambda$ and $\Lambda$ demands more care because one needs to distinguish whether $\alpha - \lambda K \leq 0.5K$ holds or not and whether $\beta - \lambda K \leq 0.5K$ holds or not in order to start or stop counting with the suitable $A_{1,\lambda}$ or $A_{i,\Lambda}$. Thus, we get four cases to which we shall refer to in the following manner:

| $\beta - \lambda K \leq 0.5K$ | $\alpha - \lambda K \leq 0.5K$ | $\alpha - \lambda K > 0.5K$ |
|-----------------------------|-----------------------------|-----------------------------|
| case (i)                    | case (iii)                  | case (iv)                   |
| case (ii)                   |                            |                             |

In view of equation (2.3), we can define the proclaimed functions $M_i$ by using (henceforth) the short hand notation $x_j := jK$, $\pi_j := \frac{\pi(x_j) + x_j}{j}$ via

\[
M_1(k) := \begin{cases} 
\pi(\pi_K) - \pi(\alpha - 1) + \pi(\beta) - \pi(x_\Lambda) & \text{in case (i)} \\
\pi(\pi_K) - \pi(\alpha - 1) + E_{\Lambda,1}(k) & \text{in case (ii)} \\
\pi(\beta) - \pi(x_\Lambda - 1) & \text{in case (iii)} \\
E_{\Lambda,1}(k) & \text{in case (iv)} 
\end{cases} \]

\[
M_2(k) := \begin{cases} 
E_{\Lambda,2}(k) & \text{in case (i)} \\
E_{\Lambda,2}(k) + \pi(\beta) - \pi(\pi_K) & \text{in case (ii)} \\
\pi(x_{\Lambda+1}) - \pi(\alpha - 1) & \text{in case (iii)} \\
\pi(x_{\Lambda+1}) - \pi(\alpha - 1) + \pi(\beta) - \pi(\pi_K) & \text{in case (iv)} 
\end{cases} \]

It is useful to put $E_j(k) := E_{j,1}(k) - E_{j,2}(k)$, $M(k) := M_1(k) - M_2(k)$, for writing

\[
|A_1| - |A_2| = M(k) + \sum_{j \in \mathbb{I}} E_j(k). 
\]

Moreover, we say an assertion $A(k)$ concerning natural numbers is eventually true if there exists a $k_0 \in \mathbb{N}$ such that $A(k)$ holds true for all $k \geq k_0$.

Proof of Theorem 1. Since $\alpha = 1$ we may assume $\lambda = 0$, and that either case (i) or (ii) of Table 1 occurs. Let $0 < \delta \leq 1$ for the moment. Remark 4 gives an approximation for the prime counting function from which we infer

\[
M(k) \geq 2\text{li}((0.5K)^{\delta}) - \text{li}(K^{\delta}) + E_\Lambda + O(K^{\delta} \log^{-\eta}(0.5K)), \quad \eta > 0.
\]

Moreover, we have

\[
2\text{li}((0.5K)^{\delta}) - \text{li}(K^{\delta}) = \int_2^{K^{\delta}} 2^{1-\delta} - 1 + \frac{\delta \log 2}{\log(2^{\delta} \tau)} \frac{d\tau}{\log(2^{\delta} \tau)}, \quad \delta > 0.
\]

Since the derivative of $x \mapsto \text{li}(x^{\delta})$ is eventually decreasing, it follows from the mean value theorem that $2\text{li}(\pi_j^{\delta}) - \text{li}(x_j^{\delta}) - \text{li}(x_{j+1}^{\delta})$ is eventually positive for any $j \geq 1$. 


Hence, we conclude that

\[(2.6) \quad \sum_{j=1}^{\Lambda} E_j(k) > (\Lambda - 1) O(K^{\delta} \log^{-\eta}(0.5K)), \quad \eta > 0.\]

Using Equation (2.4) and the above estimate we find that

\[|A_1| - |A_2| > \frac{K^{\delta} \log 2}{\log(K^\delta) \log(0.5K)} + (\Lambda - 1) O(K^{\delta} \log^{-\eta}(0.5K)),\]

which proves the claim in the case \(0 < \delta \leq 1\). Now let \(\delta > 1\). Then the difference \(2 \text{li}(x_j^\delta) - \text{li}(x_j^\delta) - \text{li}(x_j^\delta+1)\) is negative for any \(j \geq 1\). We note that \(M(k)\) is bounded from above by \(2 \text{li}((0.5K)^\delta) - \text{li}(K^\delta)\) up to an error term

\[O(K^{\delta} \log^{-\eta}(0.5K)), \quad \text{in case (i)} \]

\[\log(0.5K) + \frac{\text{li}(\beta^\delta) - \text{li}(x_\Lambda^\delta)}{\log(2x_\Lambda^\delta)} \quad \text{in case (ii)}.

The assumption on \(\beta\) implies that the expressions in the brackets are in \(O(K^{\delta-\epsilon})\) for some \(\epsilon > 0\) and hence \(O(K^{\delta} \log^{-\eta}(0.5K))\). Therefore, we obtain from (2.6) that for some suitable constant \(c > 0\) the estimate

\[M(k) < -\frac{cK^{\delta}}{\delta \log(K^\delta)} + O(K^{\delta} \log^{-\eta}(0.5K))\]

holds. Because the left hand side of (2.6) is bounded by \((\Lambda - 1) O(K^{\delta} \log^{-\eta}(0.5K))\), we conclude from (2.4) that \(-\epsilon < |A_1| - |A_2|\) is eventually violated. \(\square\)

3. Auxiliary Results

In what follows we investigate conditions for a natural number \(k\) to be a \(P^*\)-integer. It is important to notice, that \(M\) is strictly positive in case (i) and (can be) strictly negative in case (iv) of table (1). Therefore, upper and lower bounds are needed, in order to derive the asymptotic of the difference in (2.4). In order to prove Theorem 2, it suffices to derive lower a bound, though upper bounds can be derived in the same way. This is done by the following two results.

**Lemma 6.** Let \(k \geq 2953 652 287\), \(\varepsilon\) as in Lemma 3, \(x_j = kj\), and \(j\) be a natural number. Define the functions

\[E_j, -(k) := 2\pi_j \frac{1 - \varepsilon(\pi_j)}{\log x_j} - x_j \frac{1 + \varepsilon(x_j)}{\log x_j} - x_{j+1} \frac{1 + \varepsilon(x_{j+1})}{\log x_{j+1}},\]

and

\[r_j(k) := \frac{k\varepsilon(\pi_j)}{\log^2 x_j}, \quad r_0(k) := 0.\]

Then the inequality

\[(3.1) \quad E_j, -(k) < r_j(k) < E_j(k)\]

holds for \(j \geq 0\).

**Proof.** We apply the well-known formula

\[(3.2) \quad \pi(x) = \frac{\theta(x)}{\log(x)} + \int_2^x \frac{\theta(\tau)}{\tau \log^2 \tau} d\tau\]
to see that \( E_j(k) \) equals the sum
\[
\frac{2\theta(x_j)}{\log x_j} - \frac{\theta(x_{j+1})}{\log x_{j+1}} + \int_{x_j}^{x_{j+1}} \frac{\theta(\tau)}{\tau \log^2 \tau} \, d\tau - \int_{x_j}^{x_{j+1}} \frac{\theta(\tau)}{\tau \log^2 \tau} \, d\tau.
\]

Lemma 3 for \( j \geq 1 \) and Remark 4 for \( j = 0 \) yield that the first three terms above exceed \( E_{j,-}(k) \) for \( j \geq 0 \). By using Lemma 3, we infer
\[
\int_{x_j}^{x_{j+1}} \frac{\theta(\tau)}{\tau \log^2 \tau} \, d\tau - \int_{x_j}^{x_{j+1}} \frac{\theta(\tau)}{\tau \log^2 \tau} \, d\tau > \frac{k}{2} \frac{1 + \varepsilon(x_j)}{\log^2 x_j} - \frac{k}{2} \frac{1 + \varepsilon(x_j)}{\log^2 x_j} = r_j(k)
\]
which implies \( 3.1 \). □

Observing that
\[
M(k) = \begin{cases} E_\lambda(k) + \pi(x_\lambda) - \pi(\alpha - 1) + \pi(\beta) - \pi(x_\lambda) & \text{in case (i)} \\ E_\lambda(k) + \pi(x_\lambda) - \pi(\alpha - 1) + 2\pi(\overline{x}_\lambda) - \pi(x_\lambda) - \pi(\beta) & \text{in case (ii)} \end{cases}
\]
we derive the following technical but crucial corollary.

**Corollary 7.** The term \( M(k) \) is bounded from below in the cases (i) – (ii) by \( E_{\lambda,-}(k) - r_\lambda(k) - \Delta(\lambda,k) + R(k) \) whereas we put
\[
\Delta(\lambda,k) := \begin{cases} 0 \quad & \text{if } \lambda = 0 \\ -\frac{\pi(\alpha - 1)}{\log x_\lambda} (1 + \Delta(x_\lambda,\alpha)) \quad & \text{if } \lambda > 0 \end{cases}
\]
\[
R(k) := 0 \text{ in case (i) and } R(k) := E_{\lambda,-}(k) - r_\lambda(k) \text{ in case (ii) and define}
\]
\[
\tilde{\Delta}(x_-,x_+) := \left(1 - \frac{x_-}{x_+}\right) \frac{1 + \varepsilon(x_-)}{\log^2 x_-} - \frac{x_-}{x_+} + 2\varepsilon(x_-), \quad 0 < x_- \leq x_+.
\]

**Proof.** The inequality
\[
(3.3) \quad \pi(x_+) - \pi(x_-) < \frac{x_+}{\log x_-} \left(1 + \tilde{\Delta}(x_-,x_+)\right)
\]
can be deduced from Equation 3.2 via
\[
\pi(x_+) - \pi(x_-) < x_+ \frac{1 + \varepsilon(x_+)}{\log x_+} - x_- \frac{1 - \varepsilon(x_-)}{\log x_-} + \int_{x_-}^{x_+} \frac{x_-}{\log^2 t} \, dt
\]
and bracketing out the term \( x_+/\log x_- \) on the right hand side. Let \( \lambda \geq 1 \). Using the Estimate 3.3 with \( x_+ := \alpha \) and \( x_- := x_\lambda \), we get
\[
(3.4) \quad \pi(\alpha - 1) - \pi(x_\lambda) < \frac{\alpha}{\log x_\lambda} \left(1 + \Delta(x_\lambda,\alpha)\right).
\]

In the cases (i), (ii) the claim follows now by
\[
(3.5) \quad E_{\lambda,-}(k) + \pi(x_\lambda) = 2\pi(\overline{x}_\lambda) - \pi(x_{\lambda+1}), \quad E_{\lambda,-}(k) < 2\pi(\overline{x}_\lambda) - \pi(x_\lambda) - \pi(\beta),
\]
and applying Lemma 3. If \( \lambda = 0 \), then the claim follows in the cases (i), and (ii) directly from the estimate 3.5 and Remark 4.
Since we know explicit bounds for the growth of the term $M$, we need to derive explicit bounds for
\[ \sum_{j \in \mathcal{I}} E_j. \]
In view of Lemma 6, we can concentrate on dealing with sums
\[ \sum_{j=a}^{b} E_{j-}(k). \]
To this end, we define $f(x) := x(\log x)^{-1}$, and note that $E_{j-}(k)$ splits into
\[ 2f(\tau_j) - f(x_j) - f(x_{j+1}) - 2\varepsilon(\tau_j)f(\tau_j) - \varepsilon(x_j)f(x_j) - \varepsilon(x_{j+1})f(x_{j+1}). \]
Let $E'_j(k)$ denote the first three terms above, and let $E''_j(k)$ denote the remaining three. For deriving explicit lower and upper bounds for sums over $E_{j-}(k)$, it suffices to deal with the (slightly easier) sums over $E'_j(k)$ and $E''_j(k)$. This will be done in the following.

**Lemma 8.** For natural numbers $a \leq b$ and $k \geq e^4$ we have the following estimate
\[ \frac{8}{k} \sum_{j=a}^{b} E'_j(k) > \frac{\log \frac{4b+6}{9a}}{\log^2(x_{b+1})}. \]

**Proof.** Let us note that
\[ E'_j(k) = \int_{x_j}^{x_{j+1}} f'(x) - f'(x + 0.5k) \, dx. \]
Observing that $f'(x) - f'(x + 0.5k)$ equals
\[ \left( \frac{1}{\log x} - \frac{1}{\log (x + 0.5k)} \right) \left( 1 - \left( \frac{1}{\log x} + \frac{1}{\log (x + 0.5k)} \right) \right), \]
we infer, since $k \geq e^4$, the inequality
\[ \frac{1}{2} \frac{\log(1 + \frac{1}{2j+2})}{\log(x) \log(x + 0.5k)} < f'(x) - f'(x + 0.5k), \quad x \in [x_j, \tau_j], \quad j \geq 1. \]
Integrating with respect to $x$ from $x_j$ to $\tau_j$, in view of (3.8), and summing over $j$ yields
\[ \frac{k}{4} \sum_{j=a}^{b} \frac{\log(1 + \frac{1}{2j+2})}{\log(\tau_j) \log(x_{j+1})} < \sum_{j=a}^{b} E'_j(k). \]
By using partial summation, we obtain
\[ \sum_{j=a}^{b} \frac{\log(1 + \frac{1}{2j+2})}{\log^2(x_{j+1})} > \frac{\log \prod_{s=a}^{b} \frac{2(s+1)+1}{2(s+1)}}{\log^2(x_{b+1})}. \]
The estimates (2.1) imply that the product in the numerator above can be bounded from below by $(4b+6)^{0.5}(9a)^{-0.5}$. Therefore, we obtain (3.7) from (3.9).

With the above estimates at hand, we can derive lower bounds on (3.6).
Corollary 9. Let \( j \geq 1, a \leq b \) denote natural numbers and \( \sigma_{a,b} := \sum_{j=a}^{b} \). Then
\[
\sum_{j=a}^{b} \frac{E_{j,-} (k) - r_j (k)}{k} > \frac{\log \frac{4b+6}{9a}}{8 \log^2(x_{b+1})} - 5 \frac{\varepsilon(x_a)}{\log(x_a)} \sigma_{a+1,b+1}
\]
holds.

Proof. Let us note that
\[
\sum_{j=a}^{b} \frac{\varepsilon(x_j) j}{\log(x_j)} < \frac{\varepsilon(x_a)}{\log(x_a)} \sigma_{a,b} \quad \text{and} \quad \sum_{j=a}^{b} \frac{\varepsilon(x_j) (j + 0.5)}{\log(x_j)} < \frac{\varepsilon(x_a)}{\log(x_a)} \sigma_{a+1,b+1}
\]
hold. Observing \( \sigma_{a+1,b+1} \geq \sigma_{a,b} \) implies
\[
(3.10) \quad \frac{1}{k} \sum_{j=a}^{b} E''_j (k) < 4 \frac{\varepsilon(x_a)}{\log(x_a)} \sigma_{a+1,b+1}.
\]
By using (3.7) and (3.10), we deduce
\[
\frac{1}{k} \sum_{j=a}^{b} (E'_j (k) - E''_j (k)) > \frac{\log \frac{4b+6}{9a}}{8 \log^2(x_{b+1})} - 4 \frac{\varepsilon(x_a)}{\log(x_a)} \sigma_{a+1,b+1}.
\]
Combining this inequality with the obvious upper bounds for \( \frac{1}{k} \sum_{j=a}^{b} r_j (k) \) while using \( \sigma_{a+1,b+1} \geq (b - a + 1) \) yields the claim. \( \square \)

4. Proof of the Main Theorem

Proof of Theorem 3. It suffices to establish that
\[
S (k) := |A_1| - |A_2| - \ell
\]
is eventually strictly positive. Assume for the moment that we are in the cases (i) or (ii) of Table II. Equation (2.4) and Lemma 6 imply
\[
S (k) > M (k) - \ell + \sum_{j \in \mathbb{Z}} (E_{j,-} (k) - r_j (k)).
\]
By using Corollary 4 we deduce that \( S (k) \) exceeds
\[
(4.1) \quad R(k) - \Delta (\lambda, k) - \ell + \sum_{j=\lambda}^{\Lambda-1} (E_{j,-} (k) - r_j (k)).
\]
Let \( \lambda \geq 1 \) and define \( b = \Lambda - 1 \) in case (i) and \( b = \Lambda \) in case (ii). Then applying Corollary 4 with \( a = \lambda, b \) yields that it suffices to check whether
\[
(4.2) \quad -\frac{\alpha 1 + \Delta (x_\lambda, \alpha)}{k} \log(x_\lambda) - \frac{\ell}{k} + \frac{\log \frac{4b+6}{9a}}{8 \log^2(x_{b+1})} - 5 \frac{\varepsilon(k)}{\log(k)} \sigma_{1,b+1} > 0.
\]
As \( x_\lambda \alpha^{-1} - 1 < C k^{-d_1} \) holds for some \( C > 0 \), there is an explicitly computable \( C_1 > 0 \) such that \( 1 + \Delta (x_\lambda, \alpha) < C_1 \varepsilon(k) \). Hence, we can estimate the left hand side of (4.2) from below by
\[
-C \varepsilon(k) - \frac{\ell}{k} + \frac{\log \frac{4b+6}{9a}}{8 \log^2(x_{b+1})} - 5 \frac{\varepsilon(k)}{\log(k)} \sigma_{1,b+1}.
\]
Using the bounds $b + 2 \leq C_3 \log^{d_3} k$, $\iota < C_2 k^{1 - d_2}$ with some $C_2, C_3 > 0$ yields that it suffices to prove that
\begin{equation}
\frac{\log \frac{4b+6}{9\lambda}}{8 \log^2 (x_{b+1})} - \frac{5}{4} C_3 k^{d_3 - 1} (k) - C_1 \varepsilon (k) - \frac{C_2}{k^{d_2}} > 0
\end{equation}
is positive. This is certainly true for sufficiently large $k$ if we can establish that $\frac{4b+6}{9\lambda}$ exceeds 1 eventually. Since for a $P^*$-integer $\gamma \geq 1$ implies $\beta \geq p_{\varphi(k)}$, we conclude from Remark 4 that
\[ \beta > \varphi (k) \log \varphi (k) \gg k \frac{\log k}{\log \log k}. \]

Hence, $b$ can be assumed to be arbitrarily large, as desired. Now let $\lambda = 0$. Applying Corollary 9 with $a = 1$, and $b$ as before, we deduce from (4.1) that it suffices to check whether
\[ E_{0,-} (k) - \frac{\pi (\alpha)}{k} + \frac{\log \frac{4b+6}{9\lambda}}{8 \log^2 (x_{b+1})} - \frac{5}{4} \frac{\varepsilon (k)}{\log (k)} \sigma_{1,b+1} > 0. \]

Since $\pi (\alpha) < C_1 k^{1 - d_1}$, $\iota < C_2 k^{1 - d_2}$ and $\sigma_{1,b+1} \leq C_3^2 \log^{2d_3 - 1} k$ we see that we need to check
\begin{equation}
E_{0,-} (k) + \frac{\log \frac{4b+6}{9\lambda}}{8 \log^2 (x_{b+1})} - 5 \varepsilon (k) C_3^2 \log^{2d_3 - 1} k - C_1 k^{-d_1} - C_2 k^{-d_2} > 0,
\end{equation}
which is satisfied for sufficiently large $k$. This proves the claim in the cases (i) or (ii). In the case (iii) or (iv), we write $\alpha = x_\lambda - \Delta$ for some $0 < \Delta = O(k^{1 - d_1})$. In comparison to $S(k)$ in the cases (i) and (ii), we have to add the additional expression $E = \pi (x_\lambda + \Delta) - \pi (x_\lambda) - (\pi (x_\lambda) - \pi (x_\lambda - \Delta))$ to the former $S(k)$. One checks easily that $E = O(x_\lambda \varepsilon (x_\lambda))$. Hence, $E$ can not effect the sign of $S(k)$ for large $k$ in the cases (i) and (ii) since its order is lower than the order of $S(k)$, as we see by considering the terms in (4.3) and (4.4). This completes the proof. \hfill \square

Using the above proof we can state explicit bounds on certain kinds of $P^*$-integers.

**Corollary 10.** Let $b + 2 \leq C_3 \log^{d_3} k$, $\iota < C_2 k^{1 - d_2}$ with some $C_2, C_3 > 0$. Under the assumptions of Theorem 2 there is an effectively computable number $C_0 > 0$ such that every natural number $k \geq C_0$ satisfying (4.3) if $\lambda \geq 1$, or (4.4) if $\lambda = 0$ is not such a $P(\alpha, \beta, \gamma, \iota)$-integer.

**Remark 11.** Let us add some further comments:

- It poses no general problem to modify our arguments to study the distribution of other sequences in residue classes, since we essentially employed the euclidean structure, properties of the norm function, and the growth properties of the prime counting function. E.g. one can derive similar results about the distribution of numbers or elements with $s$-prime factors where $s$ is a fixed natural number, while considering semi-groups with the just mentioned properties.

- Moreover, one could slightly relax the growth restriction in Theorem 2 and still conclude finiteness of such $P^*$-integers. However, this would only complicate the technical aspects of the proof and bring no deeper insight.
ON THE REGULARITY OF PRIMES IN ARITHMETIC PROGRESSIONS

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