Drift of scroll waves in thin layers caused by thickness features

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We show by formal asymptotic methods [7] that self-oscillatory Oregonator model, with kinetics $\alpha$ for use excitable FitzHugh-Nagumo system, with kinetics $\beta$. For $p = 3D$ are regimes of self-organization observed in physical, chemical and biological dissipative systems [1]. A particularly important example is re-entrant arrhythmias in the heart [2]. In nature, 2D systems are often thin layers of 3D media, and geometry of such layers affects the dynamics of spiral/scroll waves. Known phenomena include spiral filament tension [3] and layer curvature [4], which can cause spiral waves to drift to/from thinner regions and more curved regions of the 2D excitable sheet respectively. Here we consider effects caused by sharp features of the layer thickness. There is experimental evidence that such effects play significant role in atrial fibrillation [5, 6]. Here we analyse these effects theoretically by a combination of asymptotic and numerical methods, for two selected archetypical models.

We start from a generic homogeneous isotropic reaction-diffusion system in 3D:

\[ \mathbf{v}_t = f(\mathbf{v}) + D \nabla^2 \mathbf{v}, \tag{1} \]

where $\mathbf{v} = \mathbf{v}(\mathbf{r}, t) = (x, y, z)$. In numerical examples, we use excitatory FitzHugh-Nagumo system, with kinetics

\[ f : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} \alpha^{-1} \left( u - u^3/3 - v \right) \\ \alpha(u + \beta - \gamma v) \end{bmatrix} \tag{2} \]

for $\alpha = 0.3$, $\beta = 0.68$, $\gamma = 0.5$, and $D = \text{diag}(1, 0)$, and self-oscillatoryOregonator model, with kinetics

\[ f : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} p^{-1} \left( u(1 - u) - f(v u + q v) \right) \\ u - v \end{bmatrix} \tag{3} \]

for $p = 0.1$, $f = 1.5$, $q = 0.002$, and $D = \text{diag}(1, 0, 0.6)$ [7].

We consider system (1) in a thin layer, $z \in [z_{\text{min}}(x, y), z_{\text{max}}(x, y)]$, $(x, y) \in \mathbb{R}^2$, with no-flux boundaries at $z = z_{\text{min}}$ and $z = z_{\text{max}}$. Let $H(x, y) \equiv z_{\text{max}}(x, y) - z_{\text{min}}(x, y)$ be uniformly small, $0 < H(x, y) \leq \mu \ll 1$. Then we show by formal asymptotic methods [7] that

\[ \mathbf{v}(x, y, z, t) = \mathbf{u}(x, y, t) + O(\mu^2), \tag{4} \]

and Eq. (1) in the leading order in $\mu$ reduces to the following 2D approximation:

\[ \mathbf{u}_t = f(\mathbf{u}) + D \frac{1}{H(x, y)} \nabla \cdot (H(x, y) \nabla \mathbf{u}) + O(\mu^2) \tag{5} \]

We rewrite Eq. (5) in the form

\[ \mathbf{u}_t = f(\mathbf{u}) + D \nabla^2 \mathbf{u} + c \mathbf{h}(x, y, \nabla) \tag{6} \]

where

\[ c = \epsilon D K \cdot (\nabla \mathbf{u}) \tag{7} \]

We treat (6,7) as a perturbation problem with the formal small parameter $\epsilon$ (which is distinct from the small parameter $\mu$). We assume existence of a rigidly rotating spiral wave solution $\mathbf{U}$ at $\epsilon = 0$.

For $K = K(x)$, we have $c = \epsilon D K \xi u_x$. Let us consider first a step in thickness, i.e.

\[ H(x, y) = \begin{cases} H_+, \quad x > x_s, \\ H_-, \quad x < x_s. \end{cases} \tag{8} \]

Then we have $\epsilon K = \ln(H_+) + \epsilon \Theta(x-x_s)$, $\epsilon = \ln(H_+/H_-)$, and

\[ c = \epsilon \delta(x-x_s) D \xi u_x. \tag{9} \]

Eqs. (12,13,14) of [8] predict the drift velocity via

\[ c^{-1} \frac{d\mathbf{R}}{dt} = \mathbf{F} \left( \mathbf{R} \right) = (F_x, F_y), \tag{10} \]

\[ F(\mathbf{R}) = F_x + iF_y = \int_0^\infty \int_\theta_0 \mathbf{W} (r, \theta)^+ \alpha(r, \theta; \mathbf{R}) d\theta dr, \tag{11} \]

\[ \alpha(r, \theta; \mathbf{R}) = \delta x \phi \hat{h}(\mathbf{U}, r, \theta, \phi) \frac{d\phi}{2\pi}, \tag{12} \]

where $\hat{h}$ is the perturbation $\mathbf{h}$, calculated for $\mathbf{u} = \mathbf{U}$ and considered in the frame corotating with the spiral, and $\mathbf{W}$ are
(spatial) response functions of the spirals. The spirals and their response functions calculated for the two selected models using DXSpiral [7, 9] are illustrated in fig. 1. Evaluation of integral (12) with account of (9) and the coordinate transformations $h(\vec{r}, t) = h(r, \theta, \phi)$, $\vec{R} = (X, Y)$, $\vec{r} = (x, y)$, $d = X - x_s$, $\theta = \theta(\vec{r} - \vec{R}) + \phi$, $r = \rho(\vec{r} - \vec{R})$, $x + i y \equiv \rho(\vec{r}) \exp(i\theta(\vec{r}))$ gives

$$\alpha = \begin{cases} 0, & r \leq |d|, \\ \frac{D e^{-i\theta}}{\pi \sqrt{r^2 - d^2}} \left[ \frac{d^2}{r^2} U_r - \frac{i(r^2 - d^2)}{r^3} U_\theta \right], & r > |d|. \end{cases}$$

Eqs. (11) and (13) define the specific force produced by the thickness step which depends only on the distance between the current spiral centre and the step line and is an even function about the position of the step,

$$F(\vec{R}) = S(d), \quad d \equiv X - x_s,$$

$$S(-d) = S(d) = S_s(d) + iS_y(d).$$

The components of the function $S(d)$ for the two selected models are shown in fig. 2(b,e). The important feature are zeros of $S_s$ for $d = \pm d^*$ in both models. Assuming without loss of generality $x_s = 0$, the drift of a spiral is then described asymptotically by

$$\frac{dX}{dt} = \epsilon S_x(X), \quad \frac{dY}{dt} = \epsilon S_y(X), \quad \epsilon = \ln \left( \frac{H_+}{H_-} \right).$$

Fig. 2 illustrates predictions of the theory for the case of the stepwise thickness inhomogeneity and its comparison with the direct numerical simulations of both the 2D thickness-reduced system (5) and the full 3D system (1). Numerical simulations for both selected models were done with BeatBox [7, 10]. The relevant attractor for (16) is

$$X = -d^*, \quad Y = Y_0 + \epsilon S_y(-d^*)t,$$

where $S_x(-d^*) = 0$, $S_y(-d^*) < 0$. That is, in both models the spiral attaches to the step at its thinner side and drifts along with the speed $|S_y(-d)|0$. The speed of the drift is proportional to $\epsilon = \ln(H_+/H_-)$, and the direction of the drift depends on the spiral chirality: compare panels (a) and (d).

Now let us consider the following thickness profile: for some $x_{\xi} < x_r$,

$$H(x, y) = \begin{cases} H_0, & x < x_{\xi}, \\ H_1, & x_{\xi} < x < x_r, \\ H_2, & x_r < x, \end{cases}$$

which means a “ridge” for $H_1 > H_0$ and a “ditch” for $H_1 < H_0$. This case is easily reduced to the previous because

$$H(x, y) = H_1 + (H_0 - H_1)(\Theta(x - x_{\xi}) - \Theta(x - x_r)),$$

hence the formal perturbation is

$$ch = \epsilon \left[ \delta(x - x_{\xi}) - \delta(x - x_r) \right] Du_x$$

where $\epsilon = \ln(H_1/H_0)$. Let $x_{\xi} = x_s - w/2$, $x_r = x_s + w/2$. We use the linearity of (10)–(12) and the previous result to get the interaction force in the form

$$F(\vec{R}) = T(d; w) = -T(-d; w), \quad d \equiv X - x_s,$$

$$T = T_x + iT_y = S \left( d + \frac{w}{2} \right) - S \left( d - \frac{w}{2} \right).$$

Fig. 3(a,b) shows the components of $T(d; w)$ for two selected values of the ridge width $w$, illustrating a pitchfork bifurcation of $T_x$ roots. The bifurcation condition $T_{x}(d; w) = \partial_d T_x(d; w) = 0$, observation that the bifurcation happens at $d^* = 0$ and evenness of $S(d)$ gives the critical value of width implicitly via $S_x'(w^* / 2) = 0$. For the FHN system, there are two positive roots for $S_x'(\cdot)$ (see fig. 2(b)), the smaller giving $w^* \approx 1.769$. Fig. 3(c,d) illustrates the drift along a cuneiform ditch, which may be in the first approximation considered a negative ditch with almost constant but slowly varying width. The bifurcation width $w^*$ is designated by the dashed horizontal line. We see that below this line the spiral wave drifts in accordance with the theory prediction but slows down markedly in the vicinity of this line. It does not stop completely but proceeds further, albeit at a much slower speed. This is due to the “wedging” effect of the varying width: at $w \geq w^*$, the forces from the two opposite steps making the banks of the ditch, do not compensate each other exactly due to the angle between them. To estimate roughly...
the associated correction, let the wedge angle be $\psi \ll 1$. Then the wedge-induced component of the drift speed at the bifurcation point is $2\epsilon \psi S_x(w^*) \sin(\psi/2) \approx \epsilon S_x(w^*) \psi$. For the simulation shown in fig. 3(c,d), we have $\psi \approx 0.03$, and $S_x(w^*) \approx 0.4142$, hence the drift speed $\epsilon \psi S_x(w^*) \approx 0.002366$. This drift speed is represented by the dotted line in fig. 3 and corresponds well with the simulations. Finally, let us consider the thickness perturbation of the form
\[ H(x,y) = H_0 \left(1 + \epsilon \Theta \left(R_d^2 - (x-x_d)^2 - (y-y_d)^2\right)\right) \]
i.e. thickening (for $\epsilon > 0$) or thinning (for $\epsilon < 0$) in a disk-
FIG. 4. Interaction of a spiral with a disk-shape bulge in Oregonator model. (a) Components of the interaction force calculated according to (11),(12),(19),(20), for $R_d = 225/1280 \approx 1.756$. (b) Tip trajectories in simulations of duration corresponding to half of predicted orbiting period (lines as indicated by the legend), together with initial transients (thin dotted lines). The green dashed circle: the theoretically predicted stationary orbit of the spiral centre drift. The black solid circle: the boundary of the bulge.

shaped area. Then we have

$$\alpha = \frac{e^{i\theta_0} e^{-i\phi} D}{2r \ell \sqrt{1 - r^2}} \left[ (l^2 - r^2) U_r - \frac{i(1 - r^2)}{r} U_\phi \right]$$

(19)

for $r \in (|R_d - \ell|, |R_d + \ell|)$, and $\alpha = 0$ otherwise. Here $\ell e^{i\theta_0} = (x_d - X) + i(y_d - Y)$ represents the vector from the current spiral centre to the feature centre, and $\kappa = (R_d^2 - \ell^2 - r^2) / (2r \ell)$. Hence the interaction force is

$$F_x + i F_y = e^{i\theta_0} (F_r(\ell) + i F_\phi(\ell)).$$

(20)

The radial $F_r(\ell)$ and the azimuthal $F_\phi(\ell)$ components calculated for the Oregonator model (3) for an arbitrarily chosen disk radius $R_d$ are shown in Fig. 4(a). We see there is a root of $F_r(\ell)$ at $\ell = \ell^* \approx 4.023$ and the corresponding value of $F^*_a = F_\phi(\ell^*) \approx 0.1055$ which predicts long-term behaviour of a spiral starting from an appropriate initial condition as “meander” or “orbital movement” along a circle of radius $\ell^*$ and linear speed $\epsilon F^*_a$, that is with the orbit period of $2\pi\ell^*/(\epsilon F^*_a) \approx 1314$. Fig. 4(b) compares these predictions with results of 2D and 3D numerical simulations at $\epsilon = \log(1.2)$. This example is similar to the case considered phenomenologically (simulations and experiment) in [6] and is analogous to “orbital motion” described in [11] for localized parametric heterogeneities.

To summarise, movement of transmural scroll waves through thin layers of excitable media of varying thickness can be approximately described by thickness-averaged two-dimensional equations, and a corresponding 2D perturbation theory can be successfully applied within its limits. This theory shows the propensity of scrolls to interact with sharp features of the layer geometry, which is distinct from and not reducible to previously known geometric effects such as filament tension or curvature-induced drift, and is completely independent from other factors that may cause drift such as parametric inhomogeneities. Interaction with sharp features can manifest nontrivial attractor structures, depending on the geometric parameters. These predictions should be immediately testable in experiments with the Belousov-Zhabotinsky reaction and may have important implications for understanding of evolution of re-entrant waves excitation in the heart, particularly in atria which have abundance of geometric features.

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[1] A. M. Zhabotinsky and A. N. Zaikin, in Oscillatory processes in biological and chemical systems, edited by E. E. Selkov, A. A. Zhabotinsky, and S. E. Shnol (Nauka, Pushchino, 1971) p. 279; M. A. Allessie, F. I. M. Bonke, and F. J. G. Schopman, Circ. Res. 33, 54 (1973); F. Alcantara and M. Monk, J. Gen. Microbiol. 85, 321 (1974); N. A. Gorelova and J. Bures, J. Neurobiol. 14, 353 (1983); B. F. Madore and W. L. Freedman, Am. Sci. 75, 252 (1987); S. Jakubith, H. H. Rotermund, W. Engel, A. von Oertzen, and G. Ertl, Phys. Rev. Lett. 65, 3013 (1990); J. Lechleiter, S. Girard, E. Peralta, and D. Clapham, Science 252 (1991); T. Frisch, S. Rica, P. Coullet, and J. M. Gilill, Phys. Rev. Lett. 72, 1471 (1994); M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).

[2] F. H. Fenton, E. M. Cherry, and L. Glass, Scholarpedia 3, 1665 (2008).

[3] V. N. Biktashev, A. V. Holden, and H. Zhang, Phil. Trans. Roy. Soc. Lond. ser. A 347, 611 (1994).

[4] H. Dierckx, E. Brisard, H. Verschelde, and A. V. Panfilov, Phys. Rev. E 88, 012908 (2013).

[5] T. J. Wu, M. Yashima, F. Xie, C. A. Athill, Y. H. Kim, M. C. Fishbein, Z. Qu, A. Garfinkel, J. N. Weiss, H. S. Karagueuzian, and P. S. Chen, Circulation Research 83, 448 (1998).

[6] M. Yamazaki, S. Mironov, C. Taravant, J. Brec, L. M. Vaquero, K. Bandaru, U. M. R. Avula, H. Honjo, I. Kodama, O. Berenfeld, and J. Kalifa, Cardiovascular Research 94, 48 (2012).

[7] See EPAPS Document No. [number will be inserted by publisher] for details of asymptotic and numerical procedures. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.

[8] I. V. Biktashev, D. Barkley, V. N. Biktashev, and A. J. Foulkes, Phys. Rev. E 81, 066202 (2010).

[9] I. V. Biktashev, D. Barkley, V. N. Biktashev, G. V. Bordyugov, and A. J. Foulkes, Phys. Rev. E 79, 056702 (2009), http://www.csc.liv.ac.uk/~ivb/software/DXSpiral.html.

[10] R. McFarlane and I. V. Biktashev, in BCS International Academic Conference “Visions of Computer Science” (Imperial College London, 2008) http://empslocal.ex.ac.uk/people/staff/vmb262/software/BeatBox/.

[11] V. N. Biktashev, D. Barkley, and I. V. Biktashev, Phys. Rev. Lett. 104, 058302 (2010).
Supplementary material:
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Thin layer asymptotics

When the thickness of the excitable medium layer is much smaller than the diffusion length \( \sqrt{||\mathbf{D}||/\max(||\partial f/\partial \mathbf{v}||)} \), then we can expect the concentration field \( \mathbf{v} \) to be nearly constant across the thickness of the layer, thus being effectively a two-dimensional field. To exclude the effects of the curvature we assume the layer to be flat on macroscopic scale. The formal setup is as follows:

\[
\partial_t \mathbf{v} = \mathbf{D} \nabla^2 \mathbf{v} + f(\mathbf{v}), \quad (x, y) \in \mathbb{R}^2, \quad z \in (z_{\min}(x, y; \mu), z_{\max}(x, y; \mu)),
\]

\[
z_{\min}(x, y; \mu) = 0, \quad z_{\max}(x, y; \mu) = H(x, y; \mu) = \mu \bar{H}(x, y), \quad \mu \ll 1,
\]

\[
\bar{n}(z_{\min}) \cdot \mathbf{D} \nabla \mathbf{v}(x, y, z_{\min}) = 0,
\]

\[
\bar{n}(z_{\max}) \cdot \mathbf{D} \nabla \mathbf{v}(x, y, z_{\max}) = 0,
\]

where \( \bar{n}(\cdot) \) is the normal vector at the corresponding surface. The shape of the layer is asymmetric and it may seem that variations of thickness may introduce small curvature effects; however it is easy to see that the above formulation is exactly equivalent to a symmetric one,

\[
z_{\min}(x, y; \mu) = -\mu \bar{H}(x, y), \quad z_{\max}(x, y; \mu) = \mu \bar{H}(x, y).
\]

The boundary conditions at \( z = z_{\min}, z = z_{\max} \) mean that the isosurfaces for the \( \mathbf{v} \) concentrations need to intersect the domain boundary perpendicularly. To accommodate this property in our asymptotic solution, we change from the original Cartesian coordinates \( \mathbf{r} = (x^3) = (x, y, z) \) to new curvilinear coordinates \((\rho^j) = (\xi, \eta, \zeta), j = 1, 2, 3\), in the following way:

- Coordinate \( \zeta \) is “transmural”, that is
  \[
z(\xi, \eta, 0) = z_{\min}(x(\xi, \eta, 0), y(\xi, \eta, 0)), \quad z(\xi, \eta, 1) = z_{\max}(x(\xi, \eta, 1), y(\xi, \eta, 1)). \tag{21}\]

- The other two “intramural” coordinates \((\xi, \eta)\) are chosen locally orthogonal to \( \zeta \), i.e.
  \[
  \frac{\partial \mathbf{r}}{\partial \xi} \cdot \frac{\partial \mathbf{r}}{\partial \xi} = \frac{\partial \mathbf{r}}{\partial \eta} \cdot \frac{\partial \mathbf{r}}{\partial \eta} = 0. \tag{22}\]

- The intramural coordinates match the horizontal Cartesian coordinates in the sense that
  \[
x(\xi, \eta, 0) = \xi, \quad y(\xi, \eta, 0) = \eta. \tag{23}\]

Thus the choice of the curvilinear coordinates is fully determined by the choice of function \( \zeta(\mathbf{r}) \). A convenient choice is “heat coordinates”, when \( \zeta(\mathbf{r}) = T(\mathbf{r}; \mu) \) which is a solution of the boundary-value problem

\[
\nabla^2 T(x, y, z) = 0, \quad z \in (0, \mu \bar{H}(x, y));
\]

\[
T(x, y, 0) = 0,
\]

\[
T(x, y, \mu \bar{H}(x, y)) = 1, \tag{24}\]

i.e. is identified with an established temperature distribution when a unit temperature drop is imposed across the layer. Then the lines \( \xi = \text{const}, \eta = \text{const} \) can be interpreted as the lines of heat flux, and in the new coordinates, the boundary conditions become simply condition of zero derivative in \( \zeta \), as the vectors \( \bar{n} \) are tangent to these flux lines. The two leading terms of the asymptotic of the solution of (24) are

\[
T = \frac{z}{\mu \bar{H}} + \frac{\mu^2 \bar{H}^2}{6} \left(1 - \frac{z^2}{\mu^2 \bar{H}^2}\right) \frac{z}{\mu \bar{H}} \left( (\nabla L)^2 - \nabla^2 L \right) + O(\mu^4); \quad (z \in [0, \mu \bar{H}])
\]

where

\[
\bar{H} = \bar{H}(x, y), \quad L = L(x, y) = \ln \bar{H}(x, y).
\]
The above derivation was done in the assumption of smoothness of the thickness profile $\tilde{H}(x,y)$. Hence the applications considered in the main paper will be formally covered by this approximation and the 2D spiral perturbation theory, if the “sharp features” considered there are smooth on the scale of $\tilde{H}$ but sharper than the typical scale of the effective response functions’ support. Deviation from this condition in actual 3D simulations may account for some of the discrepancies between the 3D and 2D simulations.

Response functions quadratures

**Straight step**

The function (13) has a singularity at $r = |d|$, so the resulting integral by $r$ cannot be adequately evaluated by the usual trapezoidal rule. So we proceed instead in the following way. Let the radii grid be $r \in \{j\Delta \rho \mid j = 0, 1, 2 \ldots\}$, and $|d| = k\Delta \rho$ for some $k \in \mathbb{Z}_+$. Then we can write, for a regular function $f(r)$ and a $\sigma > -1$,

$$\int_{|d|}^{\infty} f(r)(r^2 - d^2)^\sigma \, dr = \int_{|d|}^{\infty} F(r)(r - d)^\sigma \, dr = \sum_{j=k}^{\infty} \int_{j\Delta \rho}^{(j+1)\Delta \rho} F(r)(r - d)^\sigma \, dr \approx \sum_{j=k}^{\infty} C_j f_j \Delta \rho,$$

where $F(r) = f(r)(r + d)^\sigma$, $f_j = f(j\Delta \rho)$, and linear interpolation of $F(r)$ within each subinterval gives

$$C_j = \begin{cases} \frac{(j-k+1)^{\sigma+1} - (j-k-1)^{\sigma+1}}{j^{\sigma+1}} (j+k)^\sigma, & j > k, \\ \frac{(2k)^\sigma}{(2\sigma+1)} \frac{1}{\Delta \rho^2}, & j = k, \\ 0, & j < k. \end{cases}$$

For $\sigma = -1/2$,

$$C_j = \begin{cases} \frac{1}{\Delta \rho \sqrt{j+k}}, & j < k, \\ \frac{1}{\Delta \rho \sqrt{j+k}} \left[ \sqrt{j+k+1} - \sqrt{j-k-1} \right], & j > k. \end{cases}$$

In other words, we can use the usual trapezoidal formula, but should multiply $f_j = f(j\Delta \rho)$ by coefficients $C_j$ given above instead of $(r^2 - d^2)^{-1/2} = (r^2 - d^2)^\sigma = (j^2 - k^2)^{\sigma/2} \Delta \rho^{2\sigma} = (j^2 - k^2)^{-1/2} \Delta \rho^{-1}$.

**Circular step**

Similarly, the quadrature for interaction with a disk involves $\alpha$ described by (19), and so is also singular, as it contains denominator $\sqrt{1 - \kappa^2}$ which becomes zero at both ends of the integration interval:

$$\sqrt{1 - \kappa^2} = \frac{1}{2\pi \ell} \left[ (r - r_{\min})(r_{\max} - r)(r + r_{\min})(r + r_{\max}) \right]^{1/2}$$

where $r_{\min} = |R_d - \ell|$, $r_{\max} = |R_d + \ell|$. Doing as before, we get

$$\int_{r_{\min}}^{r_{\max}} \frac{F(r)}{\sqrt{(r - r_{\min})(r_{\max} - r)}} \, dr = \sum_{j=0}^{N} C_j F(r_j) \Delta \rho,$$
where
\[
C_0 = \frac{1}{\Delta \rho^2} \left[ (A_0 - A_1) \left( r_1 - r_{\text{mid}} \right) + R_0 - R_1 \right],
\]
\[
C_j = \frac{1}{\Delta \rho^2} \left[ (A_j - A_{j+1}) \left( r_{j+1} - r_{\text{mid}} \right) + (A_j - A_{j-1}) \left( r_{j-1} - r_{\text{mid}} \right) + 2R_j - R_{j+1} - R_{j-1} \right], \quad j = 1, \ldots, N - 1,
\]
\[
C_N = \frac{1}{\Delta \rho^2} \left[ (A_N - A_{N-1}) \left( r_{N-1} - r_{\text{mid}} \right) - R_{N-1} + R_N \right],
\]
\[
r_{\text{mid}} = \frac{1}{2} (r_{\text{min}} + r_{\text{max}}),
\]
\[
\Delta \rho = (r_{\text{max}} - r_{\text{min}})/N,
\]
\[
r_j = r_{\text{min}} + j \Delta \rho, \quad j = 0, \ldots, N,
\]
\[
R_j = \sqrt{(r_{\text{max}} - r_j)(r_j - r_{\text{min}})},
\]
\[
A_j = \arcsin \left( \frac{2(r_j - r_{\text{mid}})}{r_{\text{max}} - r_{\text{min}}} \right).
\]

**Discretization**

**Two-dimensional simulations**

We use explicit Euler timestepping with time step $\Delta t$ and central differencing for the diffusion term in (5), with the following discretization scheme
\[
\left[ \frac{1}{H} \nabla \cdot \left( H \nabla u \right) \right]_{i,j} = \frac{1}{2\Delta x^2} \frac{1}{H_{i,j}} \sum_{(i',j') \in I} \left( H_{i+i',j+j'} + \tilde{H}_{i,j} \right) \left( u_{i+i',j+j'} - u_{i,j} \right)
\]
where $(i, j)$ are 2D indices of the regular space grid of the size $N_x \times N_y$ with step $\Delta x$ and $I = \{(-1, 0), (1, 0), (0, -1), (0, 1)\}$. We employ standard non-flux boundary conditions. The discretization parameters used for different results are described in Table I.

| Figure | $\Delta t$ | $\Delta x$ | $N_x$ | $N_y$ |
|--------|-------------|-------------|------|------|
| 2(a,c) | $6.4 \times 10^{-4}$ | $8 \times 10^{-2}$ | 400  | 400  |
| 2(d,f) | 0.25        | $1.5 \times 10^{-3}$ | 200  | 200  |
| 3(c,d) | $6.4 \times 10^{-4}$ | $8 \times 10^{-2}$ | 400  | 800  |
| 4(b)   | 0.25        | $1.5 \times 10^{-3}$ | 200  | 200  |

**Three-dimensional simulations**

The discretization in 3D is a natural extension of the 2D scheme, except instead of a fancy diffusion operator of (5) we now have the plain diffusion of (1). The complication now comes from the more complicated geometry of the domain, which requires special attention to the boundary conditions. We have employed the following discretization:
\[
\left[ \nabla^2 v \right]_{i,j,k} = \frac{1}{\Delta x^2} \sum_{(i',j',k') \in I} \chi_{i+i',j+j',k+k'} \left( v_{i+i',j+j',k+k'} - v_{i,j,k} \right)
\]
where $\chi_{i,j,k} = 1$ if the grid point $(i, j, k)$ is within the domain and 0 otherwise, and the neighbourhood template is $I = \{(-1, 0, 0), (1, 0, 0), (0, -1, 0), (0, 1, 0), (0, 0, -1), (0, 0, 1)\}$. The space grid is regular with step $\Delta x$, rectangular $N_x \times N_y$ in the horizontal direction, and with $k \in \{1, \ldots, N_z(i,j)\}$ where $N_z(i,j)$ represents the thickness profile. In all our examples, $N_z(i,j)$ takes only two values, denoted as $N_{z,1}$ and $N_{z,2}$. The discretization parameters used for different results are described in Table II.
TABLE II. Discretization parameters in 3D simulations

Response function computations

For DXSpiral computations of the FitzHugh-Nagumo model, we use disk radius $\rho_{\text{max}} = 25$, number of radial intervals $N_\rho = 1280$ and number of azimuthal intervals $N_\theta = 64$. For the Oregonator model, we have correspondingly $\rho_{\text{max}} = 15$, $N_\rho = 128$ and $N_\theta = 64$. 

| Figure | $\Delta t$ | $\Delta x$ | $N_\rho$ | $N_\theta$ | $N_{z,1}/N_{z,2}$ |
|--------|------------|-------------|----------|------------|-------------------|
| 2(c)   | $6.4 \times 10^{-4}$ | $8 \times 10^{-2}$ | 400      | 400        | 1/2, 2/3, 3/4, 3/5, 4/6, 6/7, 9/10, 14/15, 18/20, 19/20, 38/40, 40/80, 76/80 |
| 2(f)   | 0.25       | $1.5 \times 10^{-3}$ | 200      | 200        | 1/2, 2/3, 3/4, 3/5, 4/6, 6/7, 9/10, 14/15, 18/20, 19/20, 38/40, 40/80, 76/80 |
| 3(d)   | $6.4 \times 10^{-4}$ | $8 \times 10^{-2}$ | 400      | 800        | 5/6               |
| 4(b)   | 0.25       | $1.5 \times 10^{-3}$ | 200      | 200        | 5/6               |