Fingerprints of closed trajectories of a strebel quadratic differential

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Accepted: 20 August 2021 / Published online: 30 August 2021
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Abstract
In this note, we study the fingerprints of closed smooth curves that are the trajectories of a particular Strebel quadratic differentials.

Keywords Strebel quadratic differentials · Lemniscates · Conformal maps · Fingerprints

Mathematics Subject Classification 30C10 · 30C15 · 34E05

1 A strebel quadratic differential

We consider a quadratic differential on the Riemann sphere \( \hat{\mathbb{C}} \) of the form:

\[
\sigma(z) = -\left( \sum_{i=1}^{n} \frac{\alpha_i}{z - a_i} \right)^2 dz^2,
\]

where the \( \alpha_i \)'s are pairwise distinct complex numbers, the \( \alpha_i \)'s are real numbers with non-negative sum \( \alpha \).

Finite critical points and infinite critical points of the quadratic differential \( \sigma \) are respectively its zero’s and poles \( \{ \infty \} \cup \{ a_j \}_{1 \leq j \leq n} \); all other points of \( \hat{\mathbb{C}} \) are called regular points of \( \sigma \).
**Horizontal trajectories** (or just trajectories) of the quadratic differential \( \varpi \) are trajectories of the live field given by

\[
\varpi(z) > 0.
\]

For our specific choice of \( \varpi \), they are given by

\[
\prod_{i=1}^{n} |z - a_i|^{a_i} = \text{const}.
\]

(2)

For a given non-negative real \( \lambda \), we define the lemniscate \( \Gamma_{\lambda} \) as the trajectory of the quadratic differential \( \varpi \) with \( \text{const} = \lambda \) in (2):

\[
\Gamma_{\lambda} = \left\{ z \in \mathbb{C} : |f(z)| = \prod_{i=1}^{n} |z - a_i|^{a_i} = \lambda \right\},
\]

where \( f(z) \) is the multi-valued function defined by

\[
f(z) = \prod_{i=1}^{n} (z - a_i)^{a_i}.
\]

Any connected component of \( \Gamma_{\lambda} \) will be called a sub-lemniscate.

The vertical (or, orthogonal) trajectories are obtained by replacing \( \Im \) by \( \Re \) in equation (2). The horizontal and vertical trajectories of the quadratic differential \( \varpi \) produce two pairwise orthogonal foliations of the Riemann sphere \( \hat{\mathbb{C}} \).

A trajectory passing through a critical point of \( \varpi \) is called critical. In particular, if it starts and ends at a finite critical point, it is called finite critical trajectory, otherwise, we call it an infinite critical trajectory. If two different trajectories are not disjoint, then their intersection must be a zero of the quadratic differential.

The closure of the set of finite and infinite critical trajectories is called the critical graph of \( \varpi \); we denote it by \( \Gamma_f \) (Fig. 1).

The local and global structures of the trajectories on \( \hat{\mathbb{C}} \) is well known (more details about the theory of quadratic differentials can be found in [1, 2], or [3]). The set \( \hat{\mathbb{C}} \setminus \Gamma_f \) consists of a finite number of domains called the Configurations Domain of \( \varpi \). For a general quadratic differential on a \( \hat{\mathbb{C}} \), there are five kind of Configuration Domains, see [2, Theorem 3.5]. In our cases, since all the infinite critical points of \( \varpi \) are poles of order 2 with negative residues, then there are three possible Domain Configurations:

- **The Circle domain** It is swept by closed trajectories and contains exactly one double pole that we will call the center of the domain. Its boundary is a closed critical trajectory. For a suitably chosen real constant \( c \) and some real number \( r > 0 \), the function \( z \mapsto r \exp(cf(t)) \) is a conformal map from the circle domain \( D \) onto the unit disk; it extends continuously to the boundary \( \partial D \), and sends the double pole to the origin.

- **The Ring domain** It is swept by closed trajectories. Its boundary consists of two connected components. For a suitably chosen real constant \( c \) and some real num-
bers $0 < r_1 < r_2$, the function $z \mapsto \exp(cf(t))$ is a conformal map from the circle domain $D$ onto the annulus $\{ z : r_1 < |z| < r_2 \}$ and it extends continuously to the boundary $\partial D$.

- The Dense domain It is swept by recurrent critical trajectory i.e., the interior of its closure is non-empty.

**Definition 1** A quadratic differential is *Strebel* if the set of its closed trajectories covers $\hat{\mathbb{C}}$ up to a zero Lebesgue measure set.

**Proposition 2** The quadratic differential $\varpi$ of the form (1) is Strebel.

**Proof** Since the infinite critical points of the quadratic differential $\varpi$ are only double poles with negative residues, it suffices to show that it has no recurrent trajectories. Indeed, if $\varpi$ has a recurrent trajectory, then, its Domain Configuration contains a dense domain $\mathcal{D}$, that is some trajectory

$$
\prod_{i=1}^{n} |z - a_i|^{\alpha_i} = \text{const}
$$

is dense in $\mathcal{D}$.

Under this assumption, the function

$$
z \mapsto \Re \left( \sum_{i=1}^{n} \alpha_i \log (z - a_i) \right)
$$

is continuous and constant on the open subset $\mathcal{D}$ of $\mathbb{C}$. Then it is constant everywhere in $\mathbb{C}$, which is clearly impossible by harmonicity. □
Let \( p(z) \) be the complex polynomial satisfying
\[
\sum_{i=1}^{n} \frac{\alpha_i}{z - a_i} = \frac{p(z)}{\prod_{i=1}^{n} (z - a_i)}.
\]
By assumption, the leading coefficient \( \alpha = \sum_{i=1}^{n} \alpha_i \neq 0 \) of \( p(z) \), implying that \( \deg p = n - 1 \). Let \( z_1, \ldots, z_{n-1} \) be the zeros (counted with multiplicities) of \( p(z) \). Define
\[
w_i = \prod_{i=1}^{n} |z_1 - a_i|^{a_i}, \quad w_{n-1} = \prod_{i=1}^{n} |z_{n-1} - a_i|^{a_i}.
\]

**Proposition 3** If for some \( 1 \leq i < j \leq n - 1 \), we have
\[
w_i = w_j = \max \{w_k; k = 1, \ldots, n-1\},
\]
then, there exists a finite critical trajectory connecting \( z_i \) and \( z_j \). In particular, the critical graph \( \Gamma_j \) is connected, if and only if \( w_1 = \cdots = w_{n-1} \).

**Proof** If there is no finite critical trajectory connecting \( z_i \) and \( z_j \), then a lemniscate \( \Gamma_c \), for some \( c > w_j \), is not connected. Namely, \( \Gamma_c \), is a disjoint union of \( s \geq 2 \) closed curves (sub-lemniscates) \( L_1, \ldots, L_s \), each of them encircles a part of the critical graph \( \Gamma_f \). Looking at each of these curves as a \( \sigma \)-polygon and applying the so-called Teichmüller Lemma (see [1, Thm.14.1]), we get for \( k = 1, \ldots, s \):
\[
0 = 2 + \sum n_k.
\]

Adding all these equalities, we obtain
\[
0 = 2s + 2(n - 1 - n) = 2s - 2;
\]
a contradiction. \( \square \)

## 2 Fingerprints of sub-lemniscates

Let \( \gamma \) be a Jordan \( C^\infty \)-curve in \( \mathbb{C} \); by Jordan’s theorem, \( \gamma \) splits \( \mathring{\mathbb{C}} \) into a bounded and an unbounded simply-connected components \( D_- \) and \( D_+ \). The Riemann mapping theorem asserts that there exist conformal maps \( \phi_- : \Delta \rightarrow D_- \), and \( \phi_+ : \mathring{\mathbb{C}} \setminus \Delta \rightarrow D_+ \), where \( \Delta \) is the unit disk. The map \( \phi_+ \) is uniquely determined by the normalization \( \phi_+(\infty) = \infty \) and \( \phi_+'(\infty) > 0 \). It is well-known that \( \phi_- \) and \( \phi_+ \) extend to \( C^\infty \)-diffeomorphisms on the closure of their respective domain. The **fingerprint** of \( \gamma \) is the the map \( k := \phi_+^{-1} \circ \phi_- : S^1 \rightarrow S^1 \) from the unit circle \( S^1 \) to itself. Note that \( k \) is uniquely determined up to a post-composition with an automorphism of \( D \) onto itself. Moreover, the fingerprint \( k \) is invariant under translations and scalings of the curve \( \gamma \). Fingerprints of “proper” polynomial and rational lemniscates have been studied [4, 5] and [6].
Let \( \lambda > 0, \lambda \not\in \{w_k, k = 1, \ldots, n - 1\} \). Since the quadratic differential \( \sigma \) is Strebel, the lemniscate \( \Gamma_{\lambda} \) is the union of a finite number of disjoint sub-lemniscates in \( \mathbb{C} \), each of them is either entirely included in a Circle or a Ring Domain of \( \sigma \).

More precisely, if \( \Gamma_a \) is a sub-lemniscate of \( \Gamma_{\lambda} \), then it’s obvious (by definition of the Circle Domains) that, \( \Gamma_a \) is entirely included in a Circle Domain of \( \Gamma_{\lambda} \), if and only if

\[
\text{card} \left( D_- \cap \{a_j\}_{1 \leq j \leq n} \right) = 1, \quad \text{or} \quad D_+ \cap \left( \{\infty\} \cup \{a_j\}_{1 \leq j \leq n} \right) = \{\infty\}.
\]

### 2.1 Lemniscates in a circle domain

**Theorem 4** Let be a a pole \( \sigma \) ( \( a \in \{\infty\} \cup \{a_j\}_{1 \leq j \leq n} \) ) and \( \gamma_a \) a sub-lemniscate of \( \sigma \). Then, the fingerprint \( k : S^1 \to S^1 \) of \( \gamma_a \) is given by

\[
\begin{align*}
  k(z) &= B_{\infty}(z)^{1/a}, \quad \text{if} \ a = \infty \\
  k^{-1}(z) &= z^{a/a} B_j(z)^{1/a}, \quad \text{if} \ a = a_j.
\end{align*}
\]

with

\[
B_{\infty}(z) = e^{i\theta_{\infty}} \prod_{i=1}^{n} \left( \frac{z - \phi_{-1}(a_i)}{1 - \phi_{-1}(a_i)z} \right)^{a_i},
\]

and

\[
B_j(z) = e^{i\theta_j} \prod_{i\neq j} \left( \frac{z - \phi_{-1}(a_i)}{1 - \phi_{-1}(a_i)z} \right)^{a_i}.
\]

for some real numbers \( \theta_{\infty} \) and \( \theta_j \).

**Proof** Jenkins’s Theorem on the Configuration Domains of the quadratic differential \( \sigma \) asserts that there exists a connected neighborhood \( \mathcal{U}_a \) of \( a \) (a Circle Domain of \( \sigma \)) bounded by finite critical trajectories of \( \omega \), such that all trajectories of \( \sigma \) (lemniscates of \( p \)) inside \( \mathcal{U}_a \) are closed smooth curves encircling \( a \). Moreover, for a suitably chosen non-vanishing real constant \( c \), the function

\[
\psi_a : z \mapsto \exp \left( c \sum_{i=1}^{n} \int_{z}^{\frac{a_i}{z-a_i}} dt \right)
\]

is a conformal map from \( \mathcal{U}_a \) onto a certain disk centered in \( z = 0 \).
A more explicit form of it is

$$\psi_a(z) = \beta \prod_{i=1}^{n} (z - a_i)^{c_{a_i}}$$

for some complex number $\beta$. Having in mind that $\psi$ is univalent near $a$, we get

$$c = \begin{cases} 
\frac{1}{q}, & \text{if } a = \infty \\
\frac{1}{a_j}, & \text{if } a = a_j.
\end{cases}$$

It follows that the function

$$z \mapsto \begin{cases} 
\prod_{i=1}^{n} (z - a_i)^{a_i a/a}, & \text{if } a = \infty, \\
\prod_{i=1}^{n} (z - a_i)^{a_{i/a_j}}, & \text{if } a = a_j.
\end{cases}$$

is a conformal map from $\mathcal{U}_a$ onto a certain disk $\Delta_a$ centered in $z = 0$. For the sake of simplicity, we may assume that $\Delta_a$ with a radius $R > 1$. For the given sub-lemniscate $\gamma_a$ in $\mathcal{U}_a$ (see Fig. 2), it is straightforward that the previous function maps $\Omega_-$ conformally onto the unit disk $\Delta$. Thus,

$$\begin{cases} 
\phi_+^{-1}(z) = f(z)^{1/a}, & \text{if } a = \infty, \\
\phi_-^{-1}(z) = f(z)^{1/a_i}, & \text{if } a = a_j.
\end{cases} \quad (3)$$

In the case $a = \infty$, the functions

$$z \mapsto \frac{\phi_-^{-1}(z) - a_i}{|z \phi_+^{-1}(a_i)|} |z| \leq 1, \quad i = 1, \ldots, n$$

Fig. 2 Critical graph of $-\left(\frac{1}{z+1} - \frac{1}{z+1} + \frac{\sqrt{3}}{z}\right)^2 dz^2$, and lemniscates in Circle Domains: $a = \infty$, and $a = -1$
are holomorphic and non-vanishing in the simply-connected unit disk $\overline{\Delta}$. It follows that the function

$$ z \mapsto \frac{f \circ \phi_-(z)}{z^a \prod_{i=1}^n \left( \frac{z - \phi_i^{-1}(a_i)}{1 - \phi_i^{-1}(a_i)z} \right)^{a_i}} ; |z| \leq 1 $$

is well defined, holomorphic and non-vanishing in $\overline{\Delta}$, and has modulus one on $S^1$. Thus,

$$ \frac{f \circ \phi_-(z)}{\prod_{i=1}^n \left( \frac{z - \phi_i^{-1}(a_i)}{1 - \phi_i^{-1}(a_i)z} \right)^{a_i}} = e^{i\theta} ; |z| \leq 1, $$

for some real $\theta$. From (3), we get

$$ \phi_+^{-1} \circ \phi_- = B_\infty(z)^{1/a} ; |z| = 1. $$

In the case $a = a_j$, with the normalization $\frac{\phi_j(z)}{z} \sim b > 0$, as $z \to \infty$, the functions

$$ z \mapsto \begin{cases} \phi_j(z) - a_i, & \text{if } i \neq j \\
\frac{z - \phi_i^{-1}(a_i)}{1 - \phi_i^{-1}(a_i)z}, & \text{if } i = j 
\end{cases} ; |z| \geq 1 $$

are holomorphic and non-vanishing in the simply-connected $\mathbb{C} \setminus \overline{\Delta}$, it follows that the function

$$ z \mapsto \frac{f \circ \phi_+(z)}{z^a \prod_{i \neq j} \left( \frac{z - \phi_i^{-1}(a_i)}{1 - \phi_i^{-1}(a_i)z} \right)^{a_i}} ; |z| \geq 1 $$

is well defined and holomorphic in $\mathbb{C} \setminus \overline{\Delta}$, does not vanish there, is continuous in $\mathbb{C} \setminus \Delta$, and has modulus one on $\partial \Delta = S^1$. We deduce the existence of $\theta' \in \mathbb{R}$ such that

$$ f \circ \phi_+(z) = e^{i\theta'} z^a \prod_{i \neq j} \left( \frac{z - \phi_i^{-1}(a_i)}{1 - \phi_i^{-1}(a_i)z} \right)^{a_i} ; \|z\| \geq 1. $$

Combining with (3), we get

$$ \phi_+^{-1} \circ \phi_- = z^{a/a} B_j(z)^{1/a} ; |z| = 1. $$

$\square$
2.2 Lemniscates in a ring domain

In this section, let $\mathcal{U}$ be a Ring Domain of the quadratic differential $\varpi_p$. It is bounded by two sublemniscates $\Gamma_{p,r}$ and $\Gamma_{p,R}$. We may assume that

$$0 < r < 1 < R.$$

Suppose that $a_1, \ldots, a_s (1 < s < n)$ are in the bounded domain of $\mathbb{C}$ with the boundary $\Gamma_{p,r}$. We consider the lemniscate $\Gamma_{p,1}$ of $p$ in $\mathcal{U}$ (see Fig. 3).

Since the function

$$z \mapsto \frac{f \circ \phi(z)}{\prod_{i=1}^{s} \left( \frac{z - \phi^{-1}(a_i)}{1 - \phi^{-1}(a_i)z} \right)}$$

is holomorphic in $\Delta$, continuous in $\overline{\Delta}$, does not vanish in $\Delta$, and has modulus one on $\partial \Delta$, we deduce that there exists $\theta_1 \in \mathbb{R}$ such that

$$\frac{f \circ \phi(z)}{\prod_{i=1}^{s} \left( \frac{z - \phi^{-1}(a_i)}{1 - \phi^{-1}(a_i)z} \right)} = e^{i\theta_1} |z| \leq 1,$$

and then

$$f \circ \phi(z) = e^{i\theta_1} \prod_{i=1}^{s} \left( \frac{z - \phi^{-1}(a_i)}{1 - \phi^{-1}(a_i)z} \right)^{a_i} |z| \leq 1.$$

![Fig. 3 Critical graph of $-\left( \frac{1}{z+1} - \frac{1}{z+1} + \frac{\sqrt{2}}{z} \right) dz^2$ with a lemniscate in a Ring Domain ($a = 1, b = 0$)](image-url)
Reasoning like in the previous subsection on \( \phi_+(z) \) in the case \( a = a_j \), we get for some \( \theta_2 \in \mathbb{R} \)

\[
 f \circ \phi_+(z) = e^{i\theta_2} z^a \prod_{i=s+1}^n \frac{z - \phi_+^{-1}(a_i)}{1 - \phi_+^{-1}(a_i) z}; |z| \geq 1.
\]

Combining the last two equalities for \( |z| = 1 \), with \( \theta_1 - \theta_2 = \theta \), we obtain the following

**Theorem 5** Let \( \Gamma_{p,1} \) be a smooth connected lemniscate such that \( \Omega_- \) contains exactly two different zeros \( a \) and \( b \) of \( p \) with respective multiplicities \( \alpha \) and \( \beta \). The fingerprint \( k : S^1 \to S^1 \) of \( \Gamma_{p,1} \) satisfies the functional equation

\[
 (B \circ k)(z) = e^{i\theta} A(z); |z| = 1.
\]

where \( A \) and \( B \) are given by

\[
 B(z) = z^a \prod_{i=s+1}^n \frac{z - \phi_+^{-1}(a_i)}{1 - \phi_+^{-1}(a_i) z},
\]

\[
 A(z) = \prod_{i=1}^s \left( \frac{z - \phi_+^{-1}(a_i)}{1 - \phi_+^{-1}(a_i) z} \right)^{a_i}.
\]

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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