Fluids, superfluids and supersolids: dynamics and cosmology of self-gravitating media

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Abstract. We compute cosmological perturbations for a generic self-gravitating media described by four derivatively-coupled scalar fields. Depending on the internal symmetries of the action for the scalar fields, one can describe perfect fluids, superfluids, solids and supersolids media. Symmetries dictate both dynamical and thermodynamical properties of the media. Generically, scalar perturbations include, besides the gravitational potential, an additional non-adiabatic mode associated with the entropy per particle $\sigma$. While perfect fluids and solids are adiabatic with $\sigma$ constant in time, superfluids and supersolids feature a non-trivial dynamics for $\sigma$. Special classes of isentropic media with zero $\sigma$ can also be found. Tensor modes become massive for solids and supersolids. Such an effective approach can be used to give a very general and symmetry driven modelling of the dark sector.

Keywords: cosmology of theories beyond the SM, dark energy theory, cosmological perturbation theory, modified gravity

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1 Introduction

An impressive amount of data indicates that the Universe is accelerating [1] and a great effort is underway to understand what is driving such a phase [2]. Identifying the content of the dark sector is particularly challenging, thus it is very useful to classify the various alternatives by using symmetries. In our approach the dark sector is modelled as a generic self-gravitating medium with the only requirement to admit an isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) background solution. In the hydrodynamical approximation, it turns out that a generic medium can be effectively described by the theory of four derivatively coupled
scalar fields. The four scalar fields can be interpreted as comoving coordinates of the medium whose fluctuations represent the Goldstone modes for the broken spacetime translations. The very same scalar fields can be viewed as Stückelberg fields that allow to restore broken diffeomorphisms [3–8]. Such an effective field theory description has been already considered in [9–12] for particular type of media. From our analysis it turns out that the internal symmetries of the medium action are crucial. Indeed, we will show that dynamical and thermodynamical properties of the medium are determined by internal symmetries which reflect on the form of the energy momentum tensor (EMT) and give rise to conserved currents. Media can be conveniently classified, according to the internal symmetries of the scalar field theory [6, 9, 13, 14] in perfect fluids, superfluids, solids and supersolids. In the present unified approach, an important role is played by the entropy per particle $\sigma$ in the dynamics of the medium perturbations. Thermodynamical properties of a medium are studied by creating a dictionary among the operators of the effective field theory and the basic thermodynamical variables, extending to general media the analysis of [15], see also [10, 11].

The main dynamical features of linear cosmological perturbations can be conveniently analysed by introducing a set of five mass-like parameters $\{M_i\}$ related to first and second derivatives of the Lagrangian density $U$ which resemble the masses used in massive gravity theories [7, 12, 16–18]. This is not a coincidence: there is a close relationship between massive gravity theories and the physics of self-gravitating media [6, 9]. While the dynamical equations for cosmological perturbations are rather cumbersome when expressed in terms of the fluctuations of the scalar fields, they have a clear physical interpretation once the entropy per particle $\sigma$ is introduced. Generically, in the scalar sector, two dynamical modes exist: the fluctuation of the gravitational potential and the perturbation $\delta\sigma$ of the entropy per particle. There are media, like perfect fluids, where the dynamics of $\delta\sigma$ is very simple: $\delta\sigma$ is conserved in time. For superfluids and supersolids $\delta\sigma$ has a more complicated evolution.

The outline of the paper is the following. In section 2 it is introduced the effective description of a generic rotational invariant medium as the field theory of four derivatively coupled scalar fields. In section 3 we derive the correspondence between the operators in the effective theory and the basic thermodynamical variables of the medium. Section 4 is devoted to the study of scalar, vector and tensor cosmological perturbations around a FLRW spacetime. In section 5 the number of propagating degrees of freedom is related to the values of mass parameters in the effective theory. In section 6 we characterise adiabatic and isentropic media. In section 7 we study the class of Lagrangians that, at leading order, describes perfect fluids. In section 8 the same analysis is carried out for superfluids and in section 9 for solids. In section 10 supersolids are discussed, in particular we describe the dynamics of superhorizon scalar modes.

2 Action principle and symmetries for media Lagrangians

Non dissipative media can be described by using an effective field theory based on four Stückelberg scalar fields $\varphi^A$ ($A = 0, 1, 2, 3$), see for instance [6, 9, 10, 15] which can be related to the Goldstone bosons for the spontaneous breaking of spacetime translations [3, 4]. The medium physical properties are encoded in a set of symmetries of the scalar field action selecting, order by order in a derivative expansion, a finite number of operators (see [10, 19–21] for a next to leading study of the perfect fluid case). At the leading order the fundamental object is

$$C^{AB} = g^{\mu\nu} \partial_{\mu} \varphi^A \partial_{\nu} \varphi^B;$$

(2.1)
where $g_{\mu\nu}$ is the spacetime metric. The effective medium action is built assuming diffe invariance and internal rotational invariance, namely $\varphi^a \rightarrow R^b \varphi^b$, $a, b = 1, 2, 3$; with $R \in SO(3)$. In what follows, we will always use boldface capital letters for three-dimensional spatial matrices. Since the fields $\varphi^a$ can be interpreted as comoving coordinates, this allows to define a (unique) four-velocity $u^\mu$, through the conditions
\begin{equation}
 u^\mu \partial_\mu \varphi^a = 0, \quad u^\mu u_\nu g^{\mu\nu} = -1, \quad (2.2)
\end{equation}
whose only solution is
\begin{equation}
 u^\mu = -\frac{\epsilon^{\mu\nu\alpha\beta}}{6b\sqrt{-g}} \partial_\nu \varphi^a \partial_\alpha \varphi^b \partial_\beta \varphi^c, \quad (2.3)
\end{equation}
where
\begin{equation}
 b \equiv \sqrt{\det B}, \quad (2.4)
\end{equation}
and $B$ denotes the $3 \times 3$ matrix whose components are $B^{ab} \equiv C^{ab}$. When pulled-back into the spacetime, the medium metric $B_{ab}$ becomes a projector $h_{\mu\nu}$
\begin{equation}
 h_{\mu\nu} = B_{ab} \partial_\mu \varphi^a \partial_\nu \varphi^b \equiv g_{\mu\nu} + u_\mu u_\nu, \quad h_{\mu\nu} u^\nu = 0, \quad (2.5)
\end{equation}
and $B_{ab}$ indicates the matrix elements of $B^{-1}$. Moreover, we impose the condition
\begin{equation}
 X = C^{00} < 0 \quad (2.6)
\end{equation}
which allows to define another time-like four-velocity
\begin{equation}
 V^\mu = -\frac{\partial_\mu \varphi^0}{\sqrt{-X}}, \quad V^2 = -1. \quad (2.7)
\end{equation}

Following [6], the operators with definite transformation properties under internal rotation built from $C^{AB}$ are listed in table 1.

The most general action at LO for a medium described by the four $\varphi^A$ can be constructed in terms of the scalar invariants $X, Y, \tau_n$ and $y_n$ with $n = 1, 2, 3$ and $m = 0, \ldots, 3$ for a total of nine independent operators [6]
\begin{equation}
 S = M_{\text{Pl}}^2 \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} U(X, Y, \tau_n, y_n). \quad (2.8)
\end{equation}
3.3 superfluids

By imposing further symmetries besides internal rotational invariance, media can be characterised according to table 2. For a detailed analysis see [6]. One can also consider media with reduced internal dimensionality for which only the fields \( \varphi^a \) are relevant. For instance, \( \mathcal{U}(b) \) describes a perfect fluid while \( \mathcal{U}(\tau_1, \tau_2, \tau_3) \) is a solid.

3 Conserved currents, EMT and thermodynamics

As a consequence of the shift symmetry \( \varphi^A \to \varphi^A + c^A \), the equations of motion derived from (2.8) have the form

\[
2\nabla_\mu (U_{CA\mu} \partial^\mu \varphi^B) = 0 = \nabla_\mu J^A \mu \quad (3.1)
\]

where the \( J^A_\mu \) are the Noether currents for the shift symmetry

\[
J^A_\mu = \frac{\partial U}{\partial (\partial^\mu \varphi^A)} = 2U_{AB} \nabla^\mu \varphi^B, \quad U_{AB} = \frac{\partial U}{\partial C_{AB}}; \quad (3.2)
\]

the expression of \( J^A_\mu \) are given in appendix A. The energy-momentum tensor (EMT) derived from (2.8) (see appendix A for details) can be written in terms of the Noether currents as

\[
T_{\mu\nu} = -2\frac{\delta (\sqrt{-g} U)}{\sqrt{-g} \delta g^{\mu\nu}} = U g_{\mu\nu} - J_{A\mu} \partial_\nu \varphi^A. \quad (3.3)
\]

The conservation of the EMT, \( \nabla_\nu T^{\mu\nu} = 0 \), is equivalent to equations of motion of the scalar fields. It is convenient to define the following tetrad \( \{ \beta^A_\mu, A = 0, 1, 2, 3 \} \) defined by

\[
\beta^0_\mu = u_\mu, \quad h_{\mu\nu} = \beta^a_\mu \delta^a_{\nu}, \quad \text{with} \quad \beta^A_\mu \beta^B_\nu g^{\mu\nu} = \eta_{AB}, \quad (3.4)
\]

and the vector

\[
\xi_\mu = -h_{\mu\nu} \partial^\nu \varphi^0 = -Y u_\mu + \sqrt{-X} \mathcal{V}_\mu \quad (3.5)
\]

which is related to the relative velocity of the superfluid/supersolid component with respect to the normal component [11, 23]. Projecting the EMT (3.3) on \( \{ \beta^A_\mu \} \), we have

\[
T_{\mu\nu} = \rho u_\mu u_\nu + q_\mu u_\nu + q_\nu u_\mu + \mathcal{P}_{\mu\nu}; \quad \rho = T^{\mu\nu} u_\mu u_\nu, \quad q_\mu = -h_{\mu\alpha} T^{\alpha\beta} u_\beta, \quad \mathcal{P}_{\mu\nu} = h_{\mu\alpha} h_{\nu\beta} T^{\alpha\beta}. \quad (3.6)
\]
The EMT formally has the form of imperfect fluid with heat flow

\[ q_\mu = 2Y \left[ \sum_{m=0}^{3} U_{ym}(B^{m})^{ab}\nabla \varphi^{b}C^{a0} - U_{X}\xi_{\mu} \right]. \tag{3.7} \]

Writing the current \( J^\mu_0 \) as

\[ J^\mu_0 = (U_{Y} - 2YU_{X})u^\mu + \frac{q^\mu}{Y} \tag{3.8} \]

it formally coincides with the entropy current \( s^\mu \) of an imperfect fluid \[ s^\mu = su^\mu + \frac{q^\mu}{T}. \tag{3.9} \]

Thus, it is natural to identify the temperature \( T \) with \( Y \) and the entropy density \( s \) with \( U_{Y} - 2YU_{X} \). Notice that of course no dissipation is present being \( \nabla_\mu J^\mu_0 = \nabla_\mu s^\mu = 0 \). The current

\[ J_\mu = bu_\mu, \quad \nabla^\mu J_\mu = 0 \tag{3.10} \]

is conserved off shell and can be identified with the particle density current. Splitting the tensor \( P_{\mu\nu} \) in a trace and a traceless part

\[ P_{\mu\nu} = h_{\mu\nu}P + P_{\mu\nu}^{tl}, \quad P = \frac{P_{\mu\nu}h_{\mu\nu}}{3}, \quad P_{\mu\nu}^{tl}h_{\mu\nu} = 0; \tag{3.11} \]

we have that (see eq. \((A.9)\)–\((A.10)\))

\[ P = p + \Pi, \quad p = U - \frac{1}{3}J_a^{\alpha}h_{\alpha\mu}\nabla_{\mu}\varphi^a, \quad \Pi = \frac{q_{\mu}q^\mu}{3Y}. \tag{3.12} \]

The “perfect” part \( p \) of \( P \) is identified with the thermodynamical pressure, while the viscosity \( \Pi \) is a component of the anisotropic stress. As a result

\[ \rho = -U + YU_Y - 2Y^2U_X; \tag{3.13} \]

\[ P = U - bU_b - \frac{2}{3} \sum_{n=1}^{3} n\tau_{n}U_{\tau_{n}} - \frac{2}{3} \sum_{n=0}^{3} n\varpi U_{\varpi} - \frac{2}{3}U_{X}(Y^2 + X) \tag{3.14} \]

\[ p = U - bU_b - \frac{2}{3} \sum_{n=1}^{3} n\tau_{n}U_{\tau_{n}} - \frac{2}{3} \sum_{n=0}^{3} (n + 1)\varpi U_{\varpi}. \tag{3.15} \]

A complete treatment of the thermodynamics of a self-gravitating medium will be given elsewhere. Here we simply outline general idea. Extending the very same reasoning in \[15\] to general media, the particle number density \( n \) and the entropy density \( s \) can be associated to the following projections of the conserved currents in the \( u^\mu \) frame

\[ n = -J_{\alpha}u^\alpha = b, \quad s = -J^0_{\alpha}u_\alpha = U_{Y} - 2YU_{X}. \tag{3.16} \]

An important quantity in the dynamics of self gravitating media is the entropy per particle \( \sigma = s/n \), whose evolution can be obtained from the conservation of \( J_\mu \) and \( s_\mu \)

\[ u^{\alpha}\nabla_{\alpha}\sigma = -\frac{2}{b}\nabla^{\alpha}\left[ -U_{X}\xi_{\alpha} + \sum_{n=0}^{3} U_{y_{n}}C^{0\alpha}(B^{n})^{ab}\partial_{\alpha}\varphi^{b} \right] = -\frac{1}{b}\nabla^{\alpha}\left( \frac{q_{\alpha}}{Y} \right). \tag{3.17} \]
While for perfect fluids, described by \( U(b, Y) \), \( \sigma \) is conserved as expected, for media in which the operators \( X \) and \( y_n \) are relevant (for instance superfluids and supersolids) \( \sigma \) is not conserved.

The presence of a superfluid/supersolid component requires at least an additional thermodynamical variable \([11, 25–28]\) \( \xi \), related to the space like vector \( \xi_\mu \), see eq. (3.5), namely

\[
\xi = \sqrt{\xi_\mu \xi^\mu} = \sqrt{X + Y^2}.
\]  

(3.18)

Indeed, by extending the reasoning in \([15]\) one can use as fundamental thermodynamical variables the densities \( n, s \), any function \( f(\xi) \) of \( \xi \) and the conjugate variables \( \mu, T \) and \( \eta \) \([23, 29]\). Setting for simplicity \( f(\xi) = \xi \), the thermodynamical dictionary is obtained by taking the operators \( b, Y, X, \tau_i \) and \( y_n \) as functions of the thermodynamical variables such that the first principle

\[
d\rho = T ds + \mu dn + \eta d\xi,
\]  

(3.19)

and the Euler relation in term of the thermodynamical pressure \( p \)

\[
\rho + p = Ts + \mu n
\]  

(3.20)

are satisfied. One can verify that (3.19)–(3.20) hold when

\[
Y = T, \quad s = U_Y - 2YU_X, \quad b = n, \quad \xi = (X + Y^2)^{1/2}, \quad \eta = -2UX\xi - 2\xi^{-1} \sum_{m=0}^{3} y_m U_{y_m}.
\]  

(3.21)

Thermodynamics of fluids and superfluids was considered also in \([10, 11, 13]\) and our results for such subcases are in agreement, except for exchange of \( s \) with \( n \) (and the corresponding conjugate variables \( T \) and \( \mu \)). Notice that the Euler relation (3.20) is somehow peculiar being independent from \( \xi \) while this is not the case for the first principle. Introducing the generalized pressure \( P \) by a Legendre transform of the thermodynamical pressure \( p \) in eq. (3.15) with respect to \( \xi \), namely

\[
P = p + \eta \xi = U - bU_b - \frac{2}{3} \sum_{n} n\tau_n U_{\tau_n} - 2(X + Y^2)U_X - \frac{2}{3} \sum_{n=0}^{3} (n + 4)y_n U_{y_n} ;
\]  

(3.22)

we have that the following improved Euler relation holds

\[
\rho + P = Ts + \mu \xi + \xi \eta.
\]  

(3.23)

As far as thermodynamics is concerned, all operators of the EFT can be considered as function of \( \rho, \sigma \) and \( \xi \), thus the 1-form \( dp \) can be written in terms of the 1-forms \( d\rho, d\sigma \) and \( d\xi \), namely

\[
dp = \left. \frac{\partial p}{\partial \rho} \right|_{\sigma, \xi} d\rho + \left. \frac{\partial p}{\partial \sigma} \right|_{\rho, \xi} d\sigma + \left. \frac{\partial p}{\partial \xi} \right|_{\rho, \sigma} d\xi
\]  

(3.24)

whose coefficients will be explicitly computed in section 4 at leading order in a FLRW space-time (being \( \xi = y_n = 0 \) on FRW, no difference at such order is present in between \( p \) and \( P \)).
4 Cosmology of self gravitating media

In this section we will study isotropic cosmological solutions and their linear perturbation dynamics. The Einstein equations are of the form

\[ R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \frac{T_{\mu\nu}}{2M_{P}^2}; \]  

(4.1)

where the EMT is given by (3.6). We consider the spatially flat Friedmann-Lemaitre-Robertson-Walker FLRW background

\[ ds^2 = a(t)^2 \eta_{\mu\nu} dx^\mu dx^\nu; \]  

(4.2)

where conformal time has been used and \( \eta_{\mu\nu} \) is the Minkowski metric. The scalar fields \( \varphi^A \) assume the following background value form compatible with isotropy and homogeneity

\[ \bar{\varphi}^0 = \phi(t), \quad \bar{\varphi}^a = x^a. \]  

(4.3)

It is very convenient to introduce a number of mass parameter s defined in the so called unitary gauge\(^1\) where the scalar fields fluctuations are gauged away and then their value coincides with the background form (4.3)

\[ \varphi_{(ug)}^A = \bar{\varphi}^A. \]  

(4.4)

Thus, all perturbations are pure metric perturbations

\[ g_{\mu\nu}^{(ug)}(t, \vec{x}) = a^2 (\eta_{\mu\nu} + h_{\mu\nu}^{(ug)}(t, \vec{x})). \]  

(4.5)

The Lagrangian \( U \) in the action (2.8) can be expanded up to second order as follows [7, 8, 16–18]

\[ \sqrt{-g} U \equiv \frac{\sqrt{-g}}{2} \tilde{T}^{\mu\nu} a^2 h_{\mu\nu}^{(ug)} + \frac{M_{P}^2}{4} [\lambda_0^2 h_{00}^{(ug)} + 2 \lambda_1^2 h_{0i}^{(ug)} h_{i0}^{(ug)} + \lambda_2^2 h_{ij}^{(ug)} - \lambda_3^2 h_{ij}^{(ug)}]; \]

\[ \tilde{T}^{\mu\nu} = \frac{1}{a^2} (\tilde{\rho} \delta^\mu_0 \delta^\nu_0 + \tilde{p} \delta^\mu_i \delta^\nu_j \delta^\nu_j). \]  

(4.6)

Where \( \tilde{\rho} \) and \( \tilde{p} \) are the background values for the energy density and pressure. From the rotational invariance of the background, we see that the quadratic expansion of \( U \) is fixed by giving seven time dependent parameters: \( \tilde{\rho}, \tilde{\rho}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \). In particular

\[ \tilde{\rho} = \frac{U \phi'}{a} - \frac{2U \phi'^2}{a^2} - U; \]

\[ \tilde{\rho} = U - \frac{6U_t r_3}{a^6} - \frac{4U_t r_2}{a^4} - \frac{U_b}{a^3} - \frac{2U_t r_1}{a^2}. \]  

(4.7)

It is understood that \( U \) and its derivatives are computed at the background values of EFT operators which read

\[ \bar{\mathcal{Y}}^\mu = \bar{u}^\mu = \left( \frac{1}{a}, 0 \right), \quad \bar{b} = \frac{1}{a^3}, \quad \bar{\mathcal{Y}} = \phi'/a, \quad \bar{X} = -\left( \frac{\phi'}{a} \right)^2, \quad \bar{\tau}_n = \frac{3}{a^{2n}}, \quad \bar{y}_n = 0. \]  

(4.8)

Notice that at background level we have \( \bar{\xi} = \sqrt{\bar{X} + \bar{Y}^2} = 0. \)

\(^1\)In the Standard Model, the tree level masses of gauge bosons related to the breaking of SU(2) × U(1) are defined in the unitary gauge where the would-be Goldstone bosons are set to zero.
It is convenient to introduce the following combinations that will be instrumental to study the dynamics of perturbations

\begin{align}
M_0 &= \frac{\lambda_0^2}{\phi'^2} - \frac{a^4 \bar{\rho}}{2M_{Pl}^2}, \\
M_1 &= \frac{\lambda_1^2}{\phi'} - a^4 \frac{\bar{\rho}}{M_{Pl}^2}, \\
M_2 &= \frac{\lambda_2^2 \phi'}{2M_{Pl}^2}, \\
M_3 &= \lambda_3^2 \phi' - \frac{a^4 \bar{\rho}}{2M_{Pl}^2}, \\
M_4 &= \frac{\lambda_4^2}{\phi'^2}.
\end{align}

The parameters \( \{M_i\} \) have dimension 2 and their explicit value is given in appendix B. When the Lagrangian \( U \) has a global Lorentz invariance around Minkowski (where \( \phi' = 1 \)), at least at quadratic level, the masses \( \lambda_i \) are more constrained and can be expressed in terms of only two parameters \( A \) and \( B \) as \( \lambda_0^2 = A + B \) and \( \lambda_2^2 = -A, \lambda_3^2 = B \). The conservation of the background EMT is equivalent to the equation of motion for \( \phi^0 = \phi \)

\[ \bar{\rho}' = -3 \mathcal{H}(\bar{\rho} + \bar{p}) \Rightarrow \phi'' = (1 - 3c_s^2)\mathcal{H}\phi'. \]  

From the definitions (4.9) we can define two parameters \( c_b^2 \) and \( c_s^2 \) whose precise meaning and relation with thermodynamics will be given later (see eq. (4.20) for \( c_b^2 \) and eq. (4.25) for \( c_s^2 \))

\[ c_b^2 = -\frac{M_4}{M_0}, \quad c_s^2 = \frac{2M_{Pl}^2[3M_2^2 + M_0(M_2 - 3M_3)]}{3a^4M_0(\bar{p} + \bar{\rho})}. \]  

The background Einstein equations determine the evolution of the scale factor \( a \)

\[ \mathcal{H}^2 = \left(\frac{a'}{a}\right)^2 = \frac{a^2 \bar{\rho}}{6M_{Pl}^2}, \quad \mathcal{H}' = -\frac{a^2(\bar{p} + 3\bar{\rho})}{12M_{Pl}^2}. \]  

From the conservation of the currents (3.10) and from (3.17), evaluated at at background level, it follows that

\[ \bar{s}' + 3\mathcal{H}\bar{s} = 0, \quad \bar{n}' + 3\mathcal{H}\bar{n} = 0; \]

\[ \bar{\sigma} \equiv \frac{\bar{s}}{\bar{n}}, \quad \bar{\sigma}' = 0. \]  

Thus, the entropy per particle is always conserved on a FLRW background as a consequence of the perfect fluid form of the background EMT. Defining the equation of state parameter \( w = \bar{p}/\bar{\rho} \), we have that

\[ w' = -3\mathcal{H}(1 + w)(c_s^2 - w); \]

where \( c_s^2 = \bar{\rho}'/\bar{\rho}' \) is precisely the adiabatic sound speed. Notice that when \( w \) is constant in time, then \( c_s^2 = w \). In the following we will first study the dynamics of perturbations for generic values of \( M_i \). Typically, the vanishing of some of the \( M_i \) corresponds to an enhanced internal symmetry as summarized in table 3 where the mechanical properties (the structure of the EMT) are related to special values of \( \{M_i\} \), see (B.1).

### 4.1 Scalar perturbations

In the Newtonian gauge, at linear order, the perturbed metric in the scalar sector is given by

\[ ds^2 = a^2\eta_{\mu\nu}dx^\mu dx^\nu + 2a^2[\Psi(t, \vec{x})dt^2 + \Phi(t, \vec{x})d\vec{x}^2]; \]
Table 3. Media classification according to the EMT tensor properties and the relation with the values of $M_i$.

$$
\begin{array}{|c|c|c|c|}
\hline
\text{EMT} & \text{Lagrangian} & \text{Medium} & \text{Masses} \\
\hline
q_{\mu} = 0, \; P_{\mu\nu}^{\text{EMT}} = 0 & U(b, Y) & \text{Perfect Fluid} & M_{1,2} = 0 \\
q_{\mu} = 0, \; P_{\mu\nu}^{\text{EMT}} \neq 0 & U(b, \tau_n, Y) & \text{Solid} & M_1 = 0 \\
q_{\mu} \neq 0, \; P_{\mu\nu}^{\text{EMT}} \propto q_{\mu}q_{\nu} & U(b, Y, X) & \text{SuperFluid} & M_2 = 0 \\
q_{\mu} \neq 0, \; P_{\mu\nu}^{\text{EMT}} \neq 0 & U(b, Y, X, \tau_n, y_n) & \text{SuperSolid} & M_{1,2} \neq 0 \\
\hline
\end{array}
$$

where $\Phi$, $\Psi$ are the two Bardeen potentials [30, 31]. At the background level, the EMT of the medium in a homogeneous and isotropic universe has to take the form of a perfect fluid. A generic perturbed perfect fluid EMT can be written as

$$T_{\mu}^{\nu} = (\bar{\rho} + \bar{\rho} + \delta \rho + \delta \rho)(\bar{U}^{\mu} \bar{U}_{\nu} + (\bar{\rho} + \bar{\rho}) (\delta \bar{U}^{\mu} \bar{U}_{\nu} + \bar{U}^{\mu} \delta \bar{U}_{\nu}) + (\bar{\rho} + \delta \rho) \delta \rho^{\mu} + \Pi^{\mu}_{\nu},
$$

(4.16)

where $\bar{U}_\mu = (-a, \bar{0})$ is the background 4-velocity, $\delta \rho$, $\delta p$ are the perturbations of energy density and pressure. In the scalar sector, the velocity $\delta U^\mu$ and the anisotropic stress perturbations $\Pi^{\mu}_{\nu}$ can be written in terms of two extra scalars $v$ and $\Xi$ defined as

$$
\delta U_{\mu} = (a \Psi, a \partial_i v), \quad \Pi^{\mu}_{\nu} = (3 \partial^2 \delta \xi_{\mu}^{i} \delta_{i \nu}^{\xi} - \delta_{\mu}^{j} \partial_{\nu} \delta \xi_{\rho}^{j} \partial \xi_{\rho}^{j}) \Xi.
$$

(4.17)

Comparing eqs. (3.6) and (3.11) we obtain

$$p = \bar{\rho} + \delta \rho, \quad p + \Pi = \bar{\rho} + \delta \rho, \quad q_{\mu} = (\bar{\rho} + \bar{\rho}) \delta U_{\mu}, \quad P_{\mu\nu}^{\text{EMT}} = \Pi_{\mu\nu}
$$

(4.18)

In Fourier space, the linear perturbed Einstein equations then read

$$a^2 \delta \rho = 4M_P^2 [k^2 \Phi + 3H(\Phi' + H\Psi)]
$$

(4.19)

$$3H^2 (1 + w)v = 2(\Phi' + H\Psi)
$$

$$\Xi = 2M_P^2 (\Phi - \Psi)
$$

$$a^2 \delta p = - \frac{4M_P^2}{3} \{k^2 \Phi - \Psi [9wH^2 + k^2] + 3H(\Psi' + 2\Phi') + 3\Phi''\}
$$

where $k^2 = k^i k^j \delta_{ij}$, with $k^i$ is the comoving momentum.

At background level, by using the fact that $\xi = 0$ and $\bar{\sigma}' = 0$, eq. (3.24) gives

$$\bar{\rho}' = \frac{\partial \bar{\rho}}{\partial \bar{\rho}} \bigg|_{\bar{\sigma}} \bar{\rho}' + \frac{\partial \bar{\rho}}{\partial \bar{\sigma}} \bigg|_{\bar{\rho}} \bar{\sigma}' = c_s^2 \bar{\rho}', \quad \frac{\partial \bar{\rho}}{\partial \bar{\rho}} \bigg|_{\bar{\sigma}} = \bar{\rho}' = c_s^2.
$$

(4.20)

Notice that, in the expansion of (3.24) at linear order, no contribution from $\xi$ is present: $\delta \xi$ is of order two (see eq. (C.1)) being all spatial velocities zero on a FLRW by isotropy. This allows us to write the total pressure variation as an adiabatic contribution proportional to the energy density perturbation $\delta \rho$ and a non adiabatic part proportional to the entropy per particle perturbation $\delta \sigma$:

$$\delta p \equiv c_s^2 \delta \rho + \frac{\partial \bar{\rho}}{\partial \bar{\sigma}} \bigg|_{\bar{\rho}} \delta \sigma \equiv c_s^2 \delta \rho + \Gamma, \quad \Gamma = \frac{\partial \bar{\rho}}{\partial \bar{\sigma}} \bigg|_{\bar{\rho}} \delta \sigma.
$$

(4.21)

\[2\]Note that the dimension of these two extra scalars is $[v] = -1$ and $[\Xi] = 2$. 

\[\]
For barotropic fluids with $\bar{p} = \bar{p}(\bar{\rho})$, we have that $\frac{\partial \bar{p}}{\partial \bar{\rho}} = 0$. As a consequence, $\Gamma$ is zero also when $\delta \sigma \neq 0$, i.e. the entropic fluctuations do not back react on the evolution of the Bardeen potentials. Combining (4.19) and (3.24), one gets a second order ODE for the Bardeen potential $\Phi$

\[ \Phi'' + 3H(c_s^2 + 1)\Phi' + \Phi[k^2c_s^2 + 3H^2(c_s^2 - w)] = -\frac{\hat{\Xi}}{6}[k^2 - 9H^2(c_s^2 - w)] + \frac{H\hat{\Xi}'}{2} - \frac{a^2\hat{\Gamma}}{4} ; \]

we have defined $\hat{\Xi} = M_{Pl}^{-2}\Xi$ and $\hat{\Gamma} = M_{Pl}^{-2}\Gamma$.

Given $\Gamma$ and $\Xi$, the perturbed Einstein equations allow to determine $\Phi$ through eq. (4.22) and then in turn $\delta \rho$ and $v$ (4.19). Actually, the EFT formulation and its thermodynamical interpretation determine uniquely both $\Gamma$ and $\Xi$. Indeed, given $U$, the masses $M_i$ are defined and $\delta \rho$, $\delta p$ and $v$ can be expressed in terms of scalar fields $\varphi^A$ and the Bardeen scalars. At linear level, by using the rotational invariance of the background, the scalar part of $\varphi^A$ can be written as

\[ \varphi^0 = \phi(t) + \pi_0(t, \vec{x}) , \quad \varphi^a = x^a + \partial_a \pi_L(t, \vec{x}) . \]

From the expressions (3.13), (3.17) we get

\[ \delta \rho = -\frac{6M_{Pl}^2H^2}{a^2}(3\Phi + k^2\pi_L)(1 + w) + \frac{\phi'}{a^4}\delta \sigma ; \]
\[ \delta p = -\frac{6M_{Pl}^2H^2}{a^2}c_s^2(3\Phi + k^2\pi_L)(1 + w) + \frac{c_s^2\phi'}{a^4}\delta \sigma ; \]
\[ \delta \sigma = 2M_{Pl}^2\frac{M_0}{\phi'}\left[ \Psi + \frac{\pi_0}{\phi'} + c_s^2(3\Phi + k^2\pi_L) \right] ; \]
\[ -v = \pi_L + \frac{M_1\phi'}{6a^2H^2(1 + w)}(\phi'\pi_L' - \pi_0) , \quad \hat{\Xi} = -\frac{2M_2}{a^2}\pi_L . \]

Then, comparing the perturbed Einstein equations (4.19) with the expansion of the medium’s perturbations in terms of the scalar fields (4.24), we can read out directly the expression of the intrinsic entropic perturbations $\Gamma$ as a function of the entropy per particle fluctuation $\delta \sigma$

\[ \Gamma = \delta \rho - c_s^2\delta p \quad \Rightarrow \quad \Gamma = \frac{\phi'(c_s^2 - c_s^2)}{a^4}\delta \sigma . \]

We stress that for barotropic fluids $c_s^2 = c_b^2$ and $\Gamma = 0$. The evolution equation (3.17) for $\sigma$, expanded at linear order, reads

\[ \delta \sigma' = \frac{M_1M_{Pl}^2}{\phi'2\pi}k^2(\pi_0 - \phi'\pi_L') . \]

Thus, in the EFT approach for any medium, we have the following superhorizon limit $k/H \to 0$

\[ \lim_{k \to 0} \delta \sigma = \delta \sigma_0(k) = \text{constant in time} \]

Note that the above limit has to be taken carefully and, at the end, we have always to check that the equations for the background $\phi'$ and the perturbations for $\pi_{0,L}$ do not spoil such a statement.\(^3\)

\(^3\)Recently the violation of the Weinberg Theorem appears has been discussed in a number of papers, see for instance [32-35]. The naive superhorizon limit $k \to 0$ actually means that the adimensional ratio $k/H$ goes to zero. So, for a given $k$ mode, we have always to check how fast such a ratio $k/H$ decrease in time with respect to other characteristic time scales of the problem.
Similarly, for media such that $M_1 = 0$ we have $\delta \sigma = \delta \sigma_0(k) = \text{const.}$ for any $k$. Referring to table 3, this is the case for perfect fluids and solids. On the other hand, when $M_1 \neq 0$ subhorizon entropic perturbations are dynamically generated even if they were zero at superhorizon scales.

By using (4.22) together with (4.24), (4.25) and (4.26), we arrive at the following coupled system of equations for $\Phi$ and $\delta \sigma$

$$
\Phi'' + [3(1 + c_h^2)H + F_1] \Phi' + \frac{M_2 H \phi'}{6M_0^2 a^4 k^2 (w + 1) H^2} \delta \sigma' + \left[ \frac{\phi'(c_h^2 - c_s^2)}{4M_0^2 a^2} + F_2 \right] \delta \sigma
$$

$$
+ [3H^2(c_s^2 - w) + k^2 c_s^2 + F_3] \Phi = 0; \tag{4.28}
$$

$$
\frac{\phi^2 [a^2 H^2(w + 1) + M_1]}{a^2 M_1 M_0^2 H^2 (w + 1)} \delta \sigma' + \left[ \frac{4k^4 M_2 \phi'}{9H^3(w + 1)[a^2 k^2(w + 1) + 2M_2]} \right] \Phi' \tag{4.29}
$$

$$
- \left[ \frac{2k^4 \phi'(c_h^2 - c_s^2)}{3H^2 (w + 1)} + F_4 \right] \Phi - \left[ \frac{\phi'^2 k^2 [3a^2 H^2(w + 1) + M_0^2(c_s^2 - 2c_h^2)]}{6a^2 M_0 M_1^2 H^2(w + 1)} + F_5 \right] \delta \sigma = 0; \tag{4.29}
$$

where the $F_i$ are functions of $k^2$, $M_i$ and such that

$$
\lim_{M_i \to 0} F_i = 0, \quad i = 1, 2, 3, 4, 5. \tag{4.30}
$$

The explicit form of $F_i$ are given in appendix D. Equations (4.28)–(4.29) capture in a closed form the dynamics of a general medium with $M_{1,0} \neq 0$. When $M_1 \to 0$, $M_0 \to 0$ eq. (4.29) is singular. The cases where $M_1$ and $M_0$ are zero deserve special scrutiny and will be discussed in section 5. The equation for the Bardeen potential (4.28) has an apparent singularity for $k \to 0$. However, eq. (4.26) shows that $\delta \sigma' \propto M_1 k^2$, as a result, the limit of $M_2 \frac{\delta \sigma'}{k^2} \propto M_1 M_2$ for $k \to 0$ exists and it is finite.

The source terms to the right-hand side of (4.22) are important for the superhorizon evolution of the comoving curvature perturbation $\mathcal{R}$ defined as

$$
\mathcal{R} = -\Phi - \frac{2(\Phi' + H\Psi)}{3H(1 + w)}. \tag{4.31}
$$

Indeed, by using (4.22), we get that

$$
6(1 + w) H \mathcal{R}' = a^2 \tilde{T} + k^2 \left( \frac{2\tilde{\rho}}{3} + 4c_s^2 \Phi \right). \tag{4.32}
$$

A similar analysis applies for the perturbation $\zeta$ of the curvature of constant energy density hypersurfaces given by

$$
\zeta = -\Phi - \frac{\delta \rho}{3(\bar{\rho} + \bar{p})} = \mathcal{R} - \frac{2k^2 \Phi}{9(1 + w)H^2} = -\Phi - \frac{a^2 \delta \rho}{18M_0^2 H^2 (1 + w)}. \tag{4.33}
$$

We get for the evolution of $\zeta$

$$
6(1 + w) H \zeta' = a^2 \tilde{T} + 2k^2 (1 + w) \zeta + \frac{2k^2}{9H^2}[2k^2 + 9(1 + w)H^2] \Phi. \tag{4.34}
$$

One of the virtues of our effective field theory analysis is that it provides $\Gamma$ and $\Xi$ as a function of the Bardeen potential and the entropic perturbations, namely

$$
\Gamma = \frac{\phi'(c_h^2 - c_s^2)}{a^4} \delta \sigma; \tag{4.35}
$$

$$
\hat{\Xi} = -\frac{2M_2}{a^2}\pi L = M_2 \frac{2a^2 [2k^2 + 3(3w + 5)H^2] \Phi + 6H\Phi']}{3a^2 H^2[a^2 k^2(w + 1) + 2M_2]} - \delta \sigma \phi'. \tag{4.36}
$$
Thus, for a general medium, there are two sources which can trigger a non-trivial dynamics for superhorizon perturbations [36]: intrinsic entropic perturbations $\Gamma$ and anisotropic stress $\Xi$ when non-vanishing for $k/\mathcal{H} \to 0$.

Anisotropic stress $\Xi$ is absent when $M_2 = 0$, while intrinsic entropic pressure perturbations $\Gamma$ are ineffective, see eq. (4.21), either for $c_s^2 - c_b^2 \propto \frac{\partial \Phi}{\partial a} |\rho| = 0$ or $\delta \sigma = 0$. For barotropic fluids where $p = p(\rho)$, we have $c_s^2 - c_b^2 = 0$. Generically, entropy perturbations are absent either when $M_0$ or when $M_1 = 0$ with a vanishing initial value for $\delta \sigma$ (see section 6). For the anisotropic stress we note that, taking the superhorizon limit, one should specify the relative size of the different scales entering in the game; namely we encounter in the dynamical equations (4.28)–(4.29) contributions of the form

$$k^2 \Xi \simeq \begin{cases} 
\frac{M_2}{\mathcal{H}^2(1+w)} \left( \frac{2(3w+5)}{a} \mathcal{H}^2 \delta \Phi + \frac{4 \mathcal{H}}{a^{3/2}} \Phi' - \frac{\delta \Phi'}{3a^{1/2}} \right) & M_2 \ll a^2 \mathcal{H}^2 (1+w) \\
\frac{k^2}{\mathcal{H}^2} \left( 3w + 5 \right) \mathcal{H}^2 \Phi + 2 \mathcal{H} \Phi' - \frac{\delta \Phi'}{3a^{1/2}} & a^2 \mathcal{H}^2 (1+w) \ll \{M_2, a^2 \mathcal{H}^2 (1+w)\} .
\end{cases}$$

(4.37)

This shows concretely how, in presence of different scales, sending to zero different dimensional quantities ($k$ and $M_2$) do not commute: $[\lim_{k \to 0}, \lim_{M_2 \to 0}] \neq 0$. We will defer the detailed study of such limits, related to the violation of Weinberg theorem on the existence of adiabatic modes [36, 37], to a dedicated paper.

### 4.2 Tensor perturbations

Tensor perturbations are particularly simple, in fact the transverse and traceless spin two part $\chi_{ij}$ of the metric perturbations

$$ds^2 = a^2 \left[ -dt^2 + (\delta_{ij} + \chi_{ij}(t, \vec{x})) dx^i dx^j \right]$$

are gauge invariant. The quadratic Lagrangian for tensor perturbations in the Fourier basis is [6, 7, 17, 18]

$$L_{(2)}^t = \frac{M_2^2}{2} \left[ a^2 \chi_{ij}^2 - \chi_{ij}^2 (k^2 a^2 + M_2) \right].$$

(4.39)

Thus, the linearised Einstein equations for the tensor modes reads

$$\chi''_{ij} + 2 \mathcal{H} \chi'_{ij} + \left( k^2 + \frac{M_2}{a^2} \right) \chi_{ij} = 0 .$$

(4.40)

For fluids and superfluids where $M_2 = 0$, the dynamics of spin 2 modes is standard. This is not the case for solids and supersolids where $M_2 \neq 0$ and it can trigger a enhancement/suppression of $\chi_{ij}$ depending on its sign. This could induce, for instance, observable effects on the propagation and lensing of CMB B-modes if the continuous medium is relevant at sufficiently early times. Remarkably the mass parameter, $M_2$, responsible for the gravitational slip $\Phi - \Psi$, also enters in the propagation of gravitational waves.

### 4.3 Vector perturbations

Vector perturbations of the metric have the form

$$ds^2 = a^2 \left[ -dt^2 + (\delta_{ij} + \partial_i s_j(t, \vec{x}) + \partial_j s_i(t, \vec{x})) dx^i dx^j + 2 \nu_i(t, \vec{x}) dt dx^i \right] ;$$

(4.41)

with $\partial_i s_i = \partial_i \nu_i = 0$. The quadratic Lagrangian reads [6, 7, 17, 18]

$$L_{(2)}^v = \frac{M_2^2}{2} \left[ k^2 a^2 (\nu_i - s'_i)^2 - k^2 M_2 s_i^2 + M_1^{\text{eff}} s_i^2 \right];$$

(4.42)
where

\[ M_1^{\text{eff}} = M_1 + \frac{a^4(\bar{\rho} + \bar{p})}{M_{Pl}^2} = M_1 + 6a^2\mathcal{H}^2(1 + w) = \frac{\lambda^2}{\varphi'} + \frac{a^4\bar{p}}{M_{Pl}^2} \]

\[ = \frac{1}{M_{Pl}^2} \left( 2\varphi'^2 \sum_{n=0}^3 a^{-2n}U_{y_n} + \varphi' a^3U_Y - aU_b - 2 \sum_{n=1}^3 na^{-2n}U_{\tau_n} \right). \]  

(4.43)

The fields \( \nu_i \) have a purely algebraic equations of motion

\[ \nu_i = \frac{k^2a^2s_i'}{k^2a^2 + M_1^{\text{eff}}}, \]

(4.44)

and thus they can be integrated out, giving the Lagrangian

\[ L_v^{(2)} = \frac{M_{Pl}^2k^2}{2} \left( \frac{a^2M_1^{\text{eff}}}{k^2a^2 + M_1^{\text{eff}}} s_i'^2 - M_2s_i^2 \right). \]

(4.45)

The vector \( s_i \) propagate only if \( M_1^{\text{eff}} \neq 0 \). The dispersion relation is not trivial only when \( M_2 \neq 0 \). Thus, \( M_2 \), besides controlling the dispersion relation of tensors, also determines the dynamics of vectors.

5 Masses and degrees of freedom

In order to disentangle the two \( M_{1,0} = 0 \) it is convenient to examine the structure of the equations of motion retaining all the original fields, though some of them can be integrate out. From \( \delta \rho \) in (4.19), (4.24) and (4.36), we get the relation

\[ \pi_L = M_2(\#\delta\sigma + \#\Phi + \#\Phi'). \]  

(5.1)

From (4.26) and the derivative of (5.2) we obtain

\[ \delta\sigma' = M_1(\#\pi_0 + \#M_2\delta\sigma + \#\Phi + \#\Phi'). \]

(5.2)

Finally, the definition of \( \delta\sigma \) in (4.24), together with (5.2), give

\[ \delta\sigma = [\#M_0\pi_0' + (\#M_4 + \#M_0)(\Phi + \#\Phi')]. \]  

(5.3)

We denote with \# a generic functions of \( k, \mathcal{H}, a, M_i \) whose detailed form is not relevant for us. The above set of equations is equivalent to the coupled system of second order differential equations for \( \Phi \) and \( \delta\sigma \) given in (4.28)–(4.29) and in appendix D. Let us now examine the following degenerate cases.

• \( M_1 = 0 \).

From (5.2) we have that

\[ \delta\sigma' = 0 \quad \Rightarrow \quad \delta\sigma = \delta\sigma_0(k). \]  

(5.4)

Then (5.3) becomes a second order equation for \( \pi_0 \). On the contrary, (5.1) shows that \( \pi_L \) is an auxiliary field. Thus, there are two propagating fields: \( \pi_0 \) and \( \Phi \).
Table 4. Structure of the scalar equations of motion and degrees of freedom (DoF) in terms of the masses.

| $M_0$ | $M_1$ | $M_{1\text{eff}}$ | Propagating DoF | Eqs for $\Phi$ and $\delta\sigma$ |
|-------|-------|-------------------|------------------|---------------------------------|
| $\neq 0$ | $\neq 0$ | $\neq 0$ | $\Phi, \pi_0$ (or $\pi_L$) | $\Phi'' + \ldots = 0, \delta\sigma'' + \ldots = 0$ |
| $0$ | $\neq 0$ | $\neq 0$ | $\Phi$ | $\Phi'' + \ldots = 0, \delta\sigma = 0$ |
| $\neq 0$ | $0$ | $\neq 0$ | $\Phi, \pi_0$ | $\Phi'' + \ldots = 0, \delta\sigma' = 0$ |
| $\neq 0$ | $\neq 0$ | $0$ | $\Phi$ | $\Phi'' + \ldots = 0, \delta\sigma + \ldots = 0$ |

- $M_0 = 0$
  
  Notice that (4.11) implies that also $M_1 = 0$. From (5.3) it follows that
  \[
  \delta\sigma = 0. \tag{5.5}
  \]

  As a result, (5.1)–(5.2) imply that both $\pi_0$ and $\pi_L$ are auxiliary fields. Thus the only propagating field is $\Phi$ [38–41].

Thus, irrespective of the values of $M_{0,1}$, $\pi_L$ is an auxiliary field (5.1) and can be always integrated out. Moreover, $\pi_0$ can be traded for the gauge invariant entropy per particle perturbation $\delta\sigma$. Let us consider the dynamics of $\delta\sigma$. The coefficient of $\delta\sigma'$ in (4.29) is proportional to $M_{1\text{eff}}^4$. The following case is possible.

- $M_{1\text{eff}} = 0$ is special: $\delta\sigma$ has to satisfy (4.29)
  \[
  \# \delta\sigma + \# \Phi' + \# \Phi = 0; \tag{5.6}
  \]
  i.e. $\delta\sigma$ is determined by $\Phi$ and from (5.1), (5.2) we see that both $\pi_L$ and $\pi_0$ are auxiliary fields. Thus, again only $\Phi$ propagates. In such a case also vectors do not propagate and we have a total of three degrees of freedom (one scalar and two tensors) [42]. Note the difference with the case $M_1 = 0$, where though still $\delta\sigma' = 0$ and there is an extra propagating scalar mode.

A summary of the above results is given in table 4.

Let us briefly discuss the connections between massive gravity theories and self gravitating media. In [6] it was shown that rotational invariant massive gravity theories, described by the potential [7, 8, 42, 43] $V(g^{00}, g^{0i}, g^{ij})$, are equivalent, up to a gauge transformation, to a medium described by the Lagrangian $U(b, Y, X, \tau_n, y_n)$.

In massive gravity the existence of a scalar sixth mode is typically associated to a ghost mode which generates instabilities at any scale. It is worth to stress that even media like perfect fluids have six degrees of freedom, supporting a second scalar mode without any apparent instability [42]. Let us consider for instance the perfect fluid $U(b, Y)$. Once the variables $\Phi, \delta\sigma$ are used, the equations of motions (7.1) take a very simple form (we set $c_s^2 = w$ for simplicity)
  \[
  \Phi'' = -k^2 w \Phi - 3H(1+w)\Phi' + \frac{w-c_s^2}{4\alpha^2 M_{1\text{eff}}^4} \delta\sigma_0 \tag{5.7}
  \]
  and the conservation of entropy per particle $\delta\sigma' = 0$ implies $\delta\sigma = \delta\sigma_0$. No sign of instabilities is present, as it should be, being a perfect fluid. The matter will be studied in a future dedicated paper.
6 Adiabatic and isentropic media

Adiabatic media feature a constant in time entropy per particle $\sigma(\vec{x})$. From (5.4), we see that this happens whenever $M_1 = 0$. Thus, at least at the linearised order, adiabaticity is equivalent to

$$M_1 \equiv \frac{2\phi'^2}{M^2_{Pl}} \left( \sum_{n=0}^{3} a^{-2n} U_{y_n} + a^2 U_X \right) = 0.$$  \hspace{1cm} (6.1)

Clearly, a sufficient condition for adiabaticity is the absence in the effective action (2.8) of $X$ and $y_n$. As will see later this is the case for perfect fluids and solids.

A stronger thermodynamical requirement is that the medium is isentropic, namely $\sigma$ is strictly a constant and thus no temporal or spatial variations are allowed. This implies that $\delta\sigma$ should vanish identically. From (5.5) this is the case when

$$M_0 \equiv \frac{\phi'^2 a^2}{2 M^2_{Pl}} \left[ (U_{Y^2} - 2U_X) - 4 \phi' a U_{YX} + 4 \phi'^2 a^2 U_{X^2} \right] = 0.$$  \hspace{1cm} (6.2)

A sufficient condition for an isentropic medium is that the function $U$ entering the effective action (2.8) does not depend from the operators $Y$ and $X$. From our thermodynamical dictionary, see (3.21), it is clear that the presence of the operator $Y$ turns on the entropy density $s$, while $X$ can be related to a superfluid component.

A medium can be isentropic even when $U$ depends on $Y$ and $X$ for a suitable choice of $U$. Interpreting (6.2) as a differential equation for $U$ in the $X$ and $Y$ variable, one can verify that for instance the Lagrangian $\sqrt{-X} U_1(b, \tau_n, y_n) + Y U_2(b, \tau_n, y_n)$ is isentropic. This is the case also for $U(X + Y^2, b, \tau_n, y_n)$ where $Y$ and $X$ appears in the special combination $X^2$ typical of the operators $O_\alpha$ entering in a subclass of superfluids and supersolids with $U(O_{\alpha\beta n}, \tau_n)$, see (E.9). Finally, also $U(X + Y^2, b, \tau_n, y_n)$ forms a rather general class of isentropic media. The combination $X + Y^2$ is precisely the thermodynamical variable entering in the description of superfluids and supersolids.

7 Perfect fluids

Perfect fluids are probably the simplest media one can think of and are ubiquitous in cosmology. They are characterised by an EMT (see (3.6)) with vanishing heat flow $q_\mu$ and anisotropic stress $P_{\mu\nu}$ [25, 37]. Thus the pressure is isotropic and $M_2 = 0$, being $\Xi = 0$, see (4.24). The internal symmetry for perfect fluids corresponds to spatial volume preserving diffeomorphisms and, as shown in table 2, this select at leading order a small number of operators: $b$, $Y$ and $X$. From the symmetry requirements and by inspection of the general form of the EMT (A.2), one can deduce that $U(b, Y)$ describes a perfect fluid [10, 12, 13, 19, 20, 44].
Having $M_1 = 0$, the entropy per particle is conserved. The equations of motion for $\Phi$ and $\delta \sigma$ read

$$
\Phi'' = \frac{(c_s^2 - c_b^2) \phi'}{4a^2 M_{Pl}^2} \delta \sigma_0 + [3H^2(w - c_s^2) - k^2 c_s^2] \Phi - 3(c_s^2 + 1)H\Phi';
$$

(7.1)

$$
\delta \sigma = \delta \sigma_0(k) .
$$

(7.2)

Note that for barotropic fluids described by $U(b)$ and $U(Y)$ we have $c_b^2 = c_s^2$; thus no entropic source is present in the evolution equation for the Bardeen potential. A perfect fluid can be also described by $U(X)$ [45]; however, in this case $M_{1\text{eff}} = 0$, and, as shown in section 5, the field $\delta \sigma$ is non dynamical. Moreover, from (4.29), taking the limits: $M_{1\text{eff}} \rightarrow 0$, $M_2 \rightarrow 0$ and $c_b^2 = c_s^2$, we get that $\delta \sigma = 0$, so the fluid is also isentropic.

The main features of perfect fluids are summarised in table 5.

### 8 Superfluids

Superfluids are characterised by the EMT in (3.6) with $q_\mu = 2Y U_X \xi_\mu$ and $P_{\mu \nu} = -2U_X \xi_\mu \xi_\nu$. Moreover, the anisotropic perturbation $\Pi_{\mu \nu}$ assumes the specific form $\Pi_{\mu \nu} = -2U_X \xi_\mu \xi_\nu$. A superfluid can be roughly thought as a mixture of a perfect fluid plus a superfluid irrotational component and can be described by the Lagrangians $U(b, Y, X)$ or $U(b, O_a)$ [11, 13, 43, 44]. Superfluids, besides spatial volume preserving diffs, support a temporal shift symmetry $\varphi^0 \rightarrow \varphi^0 + c$, where $c$ is constant. Around a FLRW background, being the relative velocity $\xi_\mu$ of order one, it implies that $P_{\mu \nu} \propto \xi_\mu \xi_\nu$ is at least second order in cosmological perturbation theory. Indeed, we always have $M_2 = 0$. Then, the equation for the Bardeen potential $\Phi$ is (see the limits (4.30))

$$
\Phi'' = \frac{(c_s^2 - c_b^2) \phi'}{4a^2 M_{Pl}^2} \delta \sigma + [3H^2(w - c_s^2) - k^2 c_s^2] \Phi - 3(c_s^2 + 1)H\Phi'.
$$

(8.1)

The entropy per particle is not conserved ($M_1 \neq 0$) and satisfies the following equation

$$
\left[ \frac{\phi^2 M_{1\text{eff}}}{6a^2 M_1 (w + 1) H^2} \delta \sigma' \right]' = k^2 \left[ \frac{\phi^2 (c_s^2 - 2c_b^2)}{6a^2 (w + 1) H^2} + \frac{\phi'^2}{2M_0} \right] \delta \sigma + \left[ \frac{2M_{Pl}^2 k^2 \phi' (c_b^2 - c_s^2)}{3(1 + w) H^2} \right] \Phi .
$$

(8.2)

The main features of Superfluids are summarised in table 6.

Isentropic superfluids with Lagrangian $U(b, O_a)$ are rather peculiar. The form of $U$ is protected by symmetry, see table 2, and only a single scalar degree of freedom is present. Indeed, in the combination $X/Y^2$ the $\pi_0$ field doesn’t enter at the first order, see (C.1). Finally eq. (7.1) with $\delta \sigma_0 = 0$ shows that such class of media behaves more like isentropic perfect fluids rather then superfluids. Superfluids in cosmology are typically associated to the Lagrangian $U(X)$ with possible shift symmetry breaking [47, 48] with possible connections with Mond [49]; for a recent analysis, also with other operators, see [50].

| Lagrangian | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_{1\text{eff}}$ | DoF | Features |
|------------|-------|-------|-------|-------|-------|----------------|-----|----------|
| $U(b)$     | 0     | 0     | 0     | $\neq 0$ | 0     | $\neq 0$ | 1   | Barotropic, Isentropic |
| $U(Y)$     | $\neq 0$ | 0     | 0     | 0     | $\neq 0$ | $\neq 0$ | 2   | Barotropic, Adiabatic |
| $U(b, Y)$  | $\neq 0$ | 0     | 0     | $\neq 0$ | 0     | $\neq 0$ | 2   | Adiabatic |
| $U(X)$     | $\neq 0$ | $\neq 0$ | 0     | 0     | 0     | 0     | 1   | Barotropic, Isent., Irrot. |

Table 5. Masses and thermodynamical classification of Perfect Fluids.
Table 6. Masses and features of superfluids.

| Lagrangian | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_1^{\text{eff}}$ | DoF | Feature |
|------------|-------|-------|-------|-------|-------|-----------------|-----|---------|
| $U(b, X)$  | $\neq 0$ | $\neq 0$ | $0$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | 2  |         |
| $U(Y, X)$  | $\neq 0$ | $\neq 0$ | $0$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | 2  |         |
| $U(b, Y, X)$ | $\neq 0$ | $\neq 0$ | $0$   | $\neq 0$ | $\neq 0$ | $\neq 0$ | 2  |         |
| $U(O_n)$   | 0     | $\neq 0$ | 0     | 0     | $\neq 0$ | 1   | Isentropic  |
| $U(b, O_n)$| 0     | $\neq 0$ | 0     | 0     | $\neq 0$ | 1   | Isentropic  |

Table 7. Mass spectrum and thermodynamical classification of Solids.

| Lagrangian | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_1^{\text{eff}}$ | DoF | Features |
|------------|-------|-------|-------|-------|-------|-----------------|-----|----------|
| $U(\tau_n)$ | 0     | 0     | $\neq 0$ | $\neq 0$ | 0     | $\neq 0$ | 1   | Isentropic  |
| $U(\tau_n, Y)$ | $\neq 0$ | 0     | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | 2   | Adiabatic  |

9 Solids

Solids are described by the Lagrangian $U(\tau_n)$ [22, 51] or, for finite temperature solids, by $U(Y, \tau_n)$. Differently from fluids, the anisotropic stress $\Pi_{\mu\nu}$ is not vanishing which implies that the two Bardeen potentials, $\Psi$ and $\Phi$, are not equal (4.19)

$$\Phi - \Psi = \frac{\Xi}{2M_{Pl}^2} = -\frac{M_2}{a^2} \pi L. \quad (9.1)$$

Being $M_1 = 0$, solids are adiabatic, thus $\delta \sigma = \delta \sigma_0(k)$. The evolution equation for $\Phi$ reads

$$\Phi'' + [3(1 + c_s^2)\mathcal{H} + \mathcal{F}_1] \Phi' + \left[\frac{\phi'(c_b^2 - c_s^2)}{4M_{Pl}^2a^2} + \mathcal{F}_2\right] \delta \sigma_0 + [3\mathcal{H}^2(c_s^2 - w) + k^2c_s^4 + \mathcal{F}_3] \Phi = 0. \quad (9.2)$$

Notice that for solids described by $U(\tau_n)$ we also have $M_{0,4} = 0$ so that $\delta \sigma = 0$, $c_s^2$ is not defined while for the speed of sound $c_s^2$ we have that (6.3) is still valid.

The properties of solids are summarised in table 7. Superhorizon perturbations for solids are similar to supersolids due to the fact that in such a limit all fluids allow adiabatic solutions. Solids received lot of attentions in many cosmological contests, see [35, 52–58].

10 Supersolids

Supersolids are characterised by the presence of unrelated relative velocity $\xi_{\mu}$ and anisotropic perturbation tensor $\Pi_{\mu\nu}$ [13, 44, 59]. General supersolid have two scalar degrees of freedom described by the coupled set of equations (4.28) and (4.29). Let us discuss briefly the superhorizon ($k \rightarrow 0$) regime. The limit for small $k/\mathcal{H}$ can be implemented in two different ways as show in (4.37) depending on the relative size between $k^2$ and $M_2$. For simplicity we set here $c_s^2 = w$. For $M_2 \ll a^2k^2(1 + w) \ll a^2\mathcal{H}^2(1 + w)$ we get

$$\Phi'' = -3(w + 1)\mathcal{H}\Phi' + \frac{\delta \sigma \phi'(w - c_b^2)}{4a^2M_{Pl}^2} \quad (10.1)$$

and in absence of entropy perturbations, the Bardeen potential $\Phi$ at leading order is constant.
In the second case, $a^2k^2(1+w) \ll \{a^2\mathcal{H}^2(1+w)M_2\}$, setting $M'_2 = \kappa_2 \mathcal{H} M_2$, we have

$$
\Phi'' = -\delta \sigma \left( \frac{\phi'(3c_s^2 - w + \kappa_2 - 2)}{12a^2M_P^2} - \frac{M_2 \phi'}{18M_P^2(1+w)a^4}\mathcal{H}^2 \right) + \mathcal{H}(\kappa_2 - 3w - 5)\Phi'
+ \left( \frac{1}{2}(\kappa_2 - 2)(3w + 5)\mathcal{H}^2 - \frac{M_2}{a^2} \right)\Phi
$$

(10.2)

This time the Bardeen potential is not constant also in absence of entropy perturbations. The above observation has interesting implications for inflationary models where superhorizon perturbations violate the adiabatic Weinberg theorem \cite{57, 60–64}. In a dedicate paper we will analyse the behaviour of the various gauge invariant scalars in the superhorizon limit.

There are also some special supersolids that deserve a mention as the subclasses $U(X, w_n)$, $U(w_n)$ and $U(O_{\alpha\beta n})$ that result symmetry protected, see table 2. The first two Lagrangians have $M_{1\text{eff}} = 0$ and, from the analysis of section 5, it follows that $\delta \sigma$ is an auxiliary field and only a single scalar degree of freedom is present. The supersolids $U(O_{\alpha\beta n})$ have $M_0 = 0$ (E.9) and so they are isentropic.

11 Conclusions

We have studied the dynamics of cosmological perturbations around a FLRW Universe in the presence of a generic self-gravitating medium by using an effective field theory approach. The low energy modes of the medium (phonons) are related to four scalar fields $\phi^A$ corresponding to the Goldstone modes of the spontaneously broken spacetime translations. We provide a complete classification of the medium both from a dynamical than from a dynamical and thermodynamical point of view. The Lagrangian density $U$ describing the medium depends on a set of scalar operators built from the scalar fields $\phi^A$ according to prescribed internal symmetries. The dynamics of scalar perturbations is generically described by two coupled second order differential equations, one for the Bardeen potential $\Phi$ and one for the fluctuations of the entropy per particle $\delta \sigma$, that turns out to be a combination of Goldstone fields. Besides the background pressure and energy density, the dynamics of linear cosmological perturbations is encoded in a set of five masses $\{M_i\}$ derived from the medium’s Lagrangian. Media are classified according to the internal symmetries of the EFT and the structure of the EMT which impact on the values of $\{M_i\}$. Remarkably, we find that also simple thermodynamical properties of a medium correspond to specific values of the mass parameters $\{M_i\}$. We have found that media can be classified according to the following scheme:

- **Adiabatic media**, with $\delta \sigma(\vec{x})$ time independent, have $M_1 = 0$:
  - Perfect Fluids at finite Temperature: $U(b, Y)$
  - Solids at finite Temperature: $U(\tau_n, Y)$

- **Isentropic media** with $\delta \sigma = 0$ are characterised by $M_0 = 0$:
  - Perfect Fluid: $U(b)$
  - Solids: $U(\tau_n)$
  - Superfluid: $U(O_{\alpha}), U(X + Y^2)$
  - Supersolids: $U(O_{\alpha}, \tau_n, y_n), U(O_{\alpha\beta n}), U(X + Y^2, \tau_n, y_n), \sqrt{-X} U_1(\tau_n, y_n) + YU_2(\tau_n, y_n)$
• The Lagrangian \( U(X) \) describes an irrotational isentropic perfect fluid with \( \delta \sigma = 0 \); indeed \( M_1^{\text{eff}} = M_2 = 0 \) and \( c_v^2 = c_s^2 \).

• Isotropic media have zero anisotropic stress \( \Pi_{\mu\nu} = 0 \) and thus the two Bardeen potentials \( \Phi \) and \( \Psi \) are equal. Such a media are characterised by \( M_2 = 0 \) and are:
  - Perfect Fluids;
  - Superfluids.

• Generically, superhorizon perturbations for all media admit adiabatic solutions (\( \lim_{k/H \to 0} \delta \sigma(k,t) = \delta \sigma(k) \)) despite, in superfluids and supersolids, entropy perturbations have a non trivial dynamics.

• Media which are non adiabatic but with still non-dynamical entropy perturbations are characterised by \( M_1^{\text{eff}} = 0 \) with Lagrangian \( U(bY, X, w_n) \). For such media the Bardeen potential \( \Phi \) determines completely \( \delta \sigma \), namely \( \delta \sigma = f(\Phi) \).

As shown by (10.1)–(10.2), the superhorizon evolution of the Bardeen potential can be not trivial. Recently in a number of models \([33, 34, 60, 65–67]\) it has been reported violations of the adiabatic Weinberg theorem \([36, 37]\) whereby curvature perturbations are constant on super-horizon scales (4.32). The violation of the theorem, which allows superhorizon modes to grow, involves the analyticity properties of the suitably normalized Goldstone fields \( \pi_0, L \) in the limit \( k \to 0 \) and the existence of peculiar backgrounds (see for instance fluid inflation \([32, 34]\)) for which the would be decreasing mode is turned into a growing one. In our approach, we trade the Goldstone fields for the Bardeen potentials and the entropy perturbation \( \delta \sigma \). In such a way, the sources of the violation of the adiabatic Weinberg theorem become evident from the equations (10.1) and (10.2). A complete analysis of such a matter, including the inflationary backgrounds, for generic media deserves a dedicated study and it will be given elsewhere.

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**A EMT and currents**

The EMT derived from the action (2.8) is \([6]\)

\[
T_{\mu\nu} = U g_{\mu\nu} - 2 \frac{\partial U}{\partial C_{AB}} \partial_\mu \varphi^A \partial_\nu \varphi^B = U g_{\mu\nu} - 2 \sum_k U \mathcal{O}_k \frac{\partial \mathcal{O}_k}{\partial g^{\mu\nu}},
\]

(A.1)

where \( \mathcal{O}_k \) are the nine scalar LO operators appearing in (2.8) and we use the notation \( U \mathcal{O}_k = \partial U / \partial \mathcal{O}_k \). Their partial derivatives with respect to \( g^{\mu\nu} \) are

\[
\begin{align*}
\frac{\partial Y}{\partial g^{\mu\nu}} &= -\frac{Y}{2} u_\mu u_\nu, \\
\frac{\partial X}{\partial g^{\mu\nu}} &= -X \nu_\mu \nu_\nu, \\
\frac{\partial \tau_n}{\partial g^{\mu\nu}} &= n \partial_\mu \varphi \cdot B^{n-1} \cdot \partial_\nu \varphi, \\
\frac{\partial b}{\partial g^{\mu\nu}} &= b \frac{h_{\mu\nu}}{2}, \\
\frac{\partial y_n}{\partial g^{\mu\nu}} &= \sum_{m=1}^n \partial_\mu \varphi \cdot B^{n-m} \cdot Z \cdot B^{m-1} \cdot \partial_\nu \varphi - 2 \sqrt{-X} C \cdot B^n \cdot \partial_\mu \varphi \nu_\nu, \\
\end{align*}
\]

(A.2)
where

\[ C^i \equiv C_0^i, \quad (A.3) \]

and \( \mathbf{C} \) denotes the vector with components \( C^i \), in these expressions \( \partial_\mu \varphi \) has to be understood as a \( 1 \times 3 \) matrix of components \( \partial_\mu \varphi^i \). We have used that

\[ \partial u^\alpha = - \frac{u^\alpha}{2} u_\mu u_\nu, \quad \frac{\partial V^\mu}{\partial g^{\alpha \beta}} = - \frac{V^\mu}{2} V_\alpha V_\beta. \quad (A.4) \]

The dot (\( \cdot \)) represents the standard three-dimensional matrix product. Notice that for convenience we have included \( b \), though it can be written as a combination of the three \( \tau_n \). The operators \( w_n \) are not independent; they are non-linear combinations of the scalars \( X, \tau_n \) and \( y_n \):

\[ w_1 = \tau_1 - \frac{y_0}{X}, \quad w_2 = \tau_2 - 2 \frac{y_1}{X^2}, \quad w_3 = \tau_3 - 3 \frac{y_2}{X^3} + \frac{3 y_0 y_1}{X^2} - \frac{y_3}{X^3}. \quad (A.5) \]

The four Noether currents \( J^\mu_A \) for shift symmetry can be written as

\[
J^0_\mu = 2 U_X \nabla_\mu \varphi^0 + U_Y u_\mu + 2 \sum_{m=0}^{3} (B^m)^{ab} \nabla_\mu \varphi^b C^0_a; \quad (A.6)
\]

\[
J^a_\mu = (b U_b - Y U_Y)(B^{-1})^{ac} \nabla_\mu \varphi^c + U_Y \ell_\mu^a + 2 \sum_{m=1}^{3} m U_{\tau_m} (B^{m-1})^{ac} \nabla_\mu \varphi^c \\
+ 2 \sum_{m=0}^{3} U_{y_m} \sum_{n=1}^{3} (B^{m-n} Z B^{n-1})^{ac} \nabla_\mu \varphi^c + 2 \sum_{m=0}^{3} U_{y_m} C^{db} (B^n)^{ba} \nabla_\mu \varphi^0 \quad (A.7)
\]

where

\[
\ell_\mu^a = \frac{\epsilon^{\mu \nu \alpha \beta}}{2b \sqrt{g}} \nabla_\nu \varphi^0 \nabla_\alpha \varphi^a \nabla_\beta \varphi^b \epsilon_{abc}. \quad (A.8)
\]

It is also of particular interest the projection of the currents orthogonal to \( u^\mu \), namely \( \sigma^\mu_A = u^\mu \ell^\nu_A \); we have

\[
\sigma^0_\mu = -2 U_X \xi_\mu + 2 \sum_{m=0}^{3} (B^m)^{ab} \nabla_\mu \varphi^b C^0_a; \quad (A.9)
\]

\[
\sigma^a_\mu = (b U_b - Y U_Y)(B^{-1})^{ac} \nabla_\mu \varphi^c + U_Y h_\mu \ell_\nu^a + 2 \sum_{m=1}^{3} m U_{\tau_m} (B^{m-1})^{ac} \nabla_\mu \varphi^c \\
- 2 \ell_\mu^a \sum_{m=0}^{3} U_{y_m} (B^m)^{ac} C^0_a + 2 \sum_{m=0}^{3} U_{y_m} \sum_{n=1}^{3} (B^{m+n} Z B^{n-1})^{ac} \nabla_\mu \varphi^c. \quad (A.10)
\]
where (4.7) have been used.

C Operators expansion around FRW

First order expansion of the fundamental operators

\[ \begin{align*}
  b &= \frac{1}{a^3} \left(1 - 3\Phi + \partial^2\pi_L\right), \\
  Y &= \frac{\phi'}{a} \left(1 + \Psi + \frac{\pi_0'}{\phi}\right), \\
  X &= -\left(\frac{\phi'}{a}\right)^2 \left(1 + 2\Psi + 2\frac{\pi_0'}{\phi}\right), \\
  \tau_n &= \frac{1}{a^{2n}} \left[3 + 2n(-3\Phi + \partial^2\pi_L)\right], \\
  u_\mu &= (-a + a\Psi, -a\partial_\mu\pi_L), \\
  \nu_\mu &= \left(-a + a\Psi, -\frac{a}{\phi'}\partial_\mu\pi_0\right)
\end{align*} \]

while the \(y_n\) are all of order two.

D Equations of motion

The functions \(F_i\) entering in the equations for \(\Phi\) and \(\delta\sigma\) are given by

\[ \begin{align*}
  F_1 &= 2M_2^2 \mathcal{H} [(3w - c_0^2) + 2] - 2M_2^2 ; \\
  F_2 &= \frac{\phi' [3a^2(w + 1)\mathcal{H}M_2 - 2M_2^2] - a^2 M_2 (w + 1)\phi' [3\mathcal{H}^2 (3(w - c_0^2) + 2) + k^2]}{18a^4 M_2^2 \mathcal{H}^2 (w + 1) \mathbf{D}} ; \\
  F_3 &= \frac{M_2 f_2 + M_2 f_1 + M_2 f_0}{36a^2 M_2^2 \mathcal{H}^2 (w + 1) [a^2 k^2 (w + 1) + 2M_2]} ; \\
  F_4 &= -\frac{2k^4 M_2 \phi [3\mathcal{H}^2 (3w + 5) + 2k^2]}{27\mathcal{H}^4 (w + 1) \mathbf{D}} ; \\
  F_5 &= \frac{k^4 M_2 \phi \sigma^2}{27a^2 M_2^2 \mathcal{H}^4 (w + 1) \mathbf{D}} .
\end{align*} \]
where

\[ f_2 = 72 M_P^4 \mathcal{H}^2 (w + 1) + 16 M_P^4 k^2 ; \]
\[ f_1 = 4 a^2 (w + 1) M_P^4 \{ 3 \mathcal{H}^2 [3 \mathcal{H}^2 (3w + 5) (3w - 3c_s^2 + 2) + k^2 (9w + 7 - 6c_s^2) ] \} + 2k^4 \} ; \] (D.6)
\[ f_0 = -12 a^2 M_P^4 (w + 1) \mathcal{H} [3 \mathcal{H}^2 (3w + 5) + 2k^2] ; \]

with \( D \equiv [a^2 k^2 (w + 1) + 2M_0^2] \). The equation for \( \Phi \) has a smooth limit when some of the \( M_i \) are sent to zero. On the contrary, the equation for \( \delta \sigma \) is singular when \( M_1 \) or \( M_0 \) are sent to zero, see section 5 for such a cases.

E  Masses for special supersolids

Consider a Lagrangian that depends the operators \( w_n \); see (A.5). The parameters \( \{ M_i \} \) can be derived by the following identification in the formulas (B.1)

\[ U_{\tau_n} \rightarrow U_{w_n}, \quad U_{y_n} \rightarrow \frac{(n + 1)a^2}{(\phi')^2} U_{w_n+1} . \] (E.1)

For special supersolids described by \( U(b, Y, X, w_n) \), the expression (4.43) for \( M_{1 \text{eff}} \) takes the form

\[ M_{1 \text{eff}} = a \frac{a^2 \phi' U_Y - U_b}{M_P^2} . \] (E.2)

In particular, for the Lagrangian of the form \( U(bY, X, w_n) \) we have that \( M_{1 \text{eff}} = 0 \) and then only 3 degrees of freedom propagate \([39–42, 68]\), see table 4. In presence of the \( O_{\alpha \beta n} \) operators, the \( \{ M_i \} \) can be computed by the following substitutions

\[ U_X \rightarrow (-1)^{\alpha+1} \alpha Y^{-2(\beta+1)} y_n^3 U_O ; \] (E.3)
\[ U_Y \rightarrow 2(-1)^{\alpha+1} (\alpha + \beta) Y^{-2\beta-1} y_n^2 U_O ; \] (E.4)
\[ U_{y_n} \rightarrow (-1)^{\alpha} \beta Y^{-2\beta} y_n^{-1} U_O ; \] (E.5)
\[ U_{XX} \rightarrow (-1)^{\alpha} \alpha Y^{-4\beta-4} y_n^4 \left[ (-1)^{\alpha} \alpha U_O y_n^2 + (\alpha - 1) U_O Y^{2\beta} \right] ; \] (E.6)
\[ U_{XY} \rightarrow 2(-1)^{\alpha} \alpha (\alpha + \beta) Y^{-4\beta-3} y_n^4 \left[ (-1)^{\alpha} \alpha U_O y_n^2 + U_O Y^{2\beta} \right] ; \] (E.7)
\[ U_{YY} \rightarrow 2(-1)^{\alpha} (\alpha + \beta) Y^{-4\beta-2} y_n^3 \left[ 2(1)^{\alpha} (\alpha + \beta) U_O y_n^2 + (2\alpha + 2\beta + 1) U_O Y^{2\beta} \right] ; \] (E.8)

where we have used only that \( X = -Y^2 \) but not he fact that \( \bar{y}_n = 0 \). In the case of the Lagrangian \( U(O_{\alpha \beta n}, \tau_n) \) we have

\[ M_0 = \frac{2}{M_P^2} (-1)^\alpha a^2 \beta Y^{-4} y_n^3 \left[ 2(-1)^\alpha \beta U_O y_n^2 + (2\beta + 1) U_O Y^{2\beta} \right] = 0 ; \] (E.9)

irrespective of the value of \( \beta \). Thus, both when \( \beta = 0 \) and only the operators \( O_{\alpha} = O_{\alpha 0n} \) are present (see section 8 and section 10) and also for \( \beta > 0 \), due to the fact that \( \bar{y}_n = 0 \).

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