Discrete dynamics of energy and momentum transfer

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Abstract. In the present paper we introduce a hypothesis that represents the processes of energy and momentum transfer between interacting particles as a discrete over time. The discrete-time framework is grounded by a fundamental constant that give rise to unique quantum impulse equation and relations that the corresponding mechanical power, force and torque satisfy. In addition to the energy-frequency and momentum-wavelength relations, the theory generate power-frequency and force-wavelength analogs, respectively. Operator representations of the mechanical power, force and torque in the position space are discussed.

1. Introduction
Quantum mechanics is considered as one of the greatest achievements in theoretical physics [1–5]. Describing phenomena governed by principles beyond those defining the macroscopic world, it would always raise discussions on a possible interdependence with classical mechanics [6–10]. The quantum theory appears as an invaluable tool for interpreting experimental findings from the nano-scale world and thus being a ground for vast number of researches accounting for different atomic and subatomic processes. Some prominent examples are the photon-electron interactions [11–13], the electrons’ interactions that underpin the magnetic and conductive properties of matter [14–21] and different transport phenomena [22–27]. Furthermore, determining the time response of the electrons [28] and the time that it takes for quantum jumps to occur [29] are really fascinating and promising achievements of quantum physics nowadays. From the perspective of present technological progress, the possibility of observing and controlling quantum effects such as tunneling [30–32] and entanglement [33–37] promises a number of future applications in the field of logical devices [38–43].

Whether one studies problems in the field of fundamental or applied quantum mechanics, defining the Hamiltonian is a key point in understanding the system’s dynamics, characterized by all processes of energy and momentum transfer. However, within its continuous time framework, the Hamiltonian approach does not provide enough information for the rate at which these processes take place over time, a knowledge that at some point may strongly enrich the study. In order to explore the dynamics of energy and momentum transfer more closer and search for a possible discreteness in time an additional investigation about the observables such as the mechanical power that a certain interaction generate and the applied forces is required.

In the present paper, we use the first quantization formalism to introduce the transfer of energy and momentum among interacting particles as a discrete over time process. The underlying hypothesis points out to the existence of a lower limit for the rate at which these processes take place over time and leads to unique relations that the relevant dynamical quantities such as the mechanical power, force and torque satisfy. In particular, we represent the Planck–Einstein and de Broglie relations in terms of the mechanical
power and force, respectively. We demonstrate that the impulse-momentum theorem for quantum-scale objects is grounded by an impulse equation that has no classical analogue. Further, we introduce the position representation of the operators associated to the respective power, force and torque, establish the commutation relations these observables obey and study the uncertainty. A key role within all results plays a constant that has a unit measure of energy. It emerges naturally in the theory and has as fundamental part in all equations as the Planck’s constant. The Planck’s constant, however, remains unique in terms of the study discussed in Refs. \[44–46\].

2. Generalities

2.1. Classical observables

The Lagrangian and Hamiltonian mechanics are among the most successfully applied approaches when studying the dynamics of an arbitrary mechanical system. Although their consideration usually includes generalized variables, henceforth, with respect to the objective to be followed, all functions and relations are given in terms of Cartesian coordinates.

Consider an isolated system consisting of a collection of point particles. The Lagrangian \[L = L(r, \dot{r})\], where \[r = (r_1, r_2, \ldots)\] are the position vectors of the particles and \[\dot{r} = (\dot{r}_1, \dot{r}_2, \ldots)\] are the respective velocities, collects all the knowledge needed to determine how the considered system evolves over time. Respectively, predicting the particles configuration at any given moment of time, one has to solve the Euler–Lagrange equations. On the other hand, knowing the relevant momenta \[p = (p_1, p_2, \ldots)\], the evolution in time can be studied with the aid of the Hamiltonian mechanics by using the Hamiltonian \[H = H(r, p)\] and solving the Hamiltonian equations. Both, the Lagrangian and Hamiltonian approaches are related, such that one can determine the total energy of the system by accounting for the relation

\[H = \sum_i \dot{r}_i \cdot p_i - L.\] (1)

Although knowing the total energy is usually sufficient in determining the dynamical state of a mechanical system, studying the rate at which the transfer of energy and momentum between the interacting particles takes place with time may strongly enrich the study. To this end, in addition to (1) and under the condition \[\dot{H} = 0\], we account for the mechanical power \[P\] and the respective generalized forces \[F_i = \partial_i \partial r L\], satisfying the relation

\[P = \sum_i \dot{r}_i \cdot F_i,\] (2)

where \[P = \sum_i P_i\], with \[P_i\] denoting the mechanical power generated within the work done by \[F_i\]. Further, the impulse acquired by the \(i\)-th particle in the process of interaction with the remaining particles reads

\[J_i = \int_{\bar{T}_i} F_i dt,\] (3)

where the domain \(\bar{T}_i\) defines the time span within the \(i\)-th force acts. In the processes of elastic collision, for all \(i\), the impulse is distributed over an infinitesimal short time-interval \(\lambda_i \in \bar{T}_i\). Accordingly, \(F_i\) represents an impact force given by \(F_i = J_i \delta(t - \lambda_i)\).

2.2. Quantum observables

In quantum mechanics, the information about the dynamics of a multi particle system, with position vectors \(r = (r_1, r_2, \ldots)\), is embodied in a state function \(\Psi = \Psi(r, t)\), where \[||\Psi||^2 = 1\]. Each state function is related to the probability of observing the system in a particular dynamical state characterized by a number of physical quantities with a given uncertainty in their values. Satisfying the Schrodinger equation

\[i\hbar \partial_t \Psi(r, t) = \hat{H} \Psi(r, t),\] (4)
the explicit representation of $\Psi(r, t)$ is related to the Hamiltonian’s representation, which in turn depends on the introduced interactions and hence the way the system is being manipulated, i.e. observed. Respectively, depending on how one interacts with the system’s constituents, the latter may demonstrate either their wave-like or point-like character.

The wave-like character of the collective point-like particles’ dynamics described by (1) and (2) become apparent on the very frontier between the classical and quantum theoretical approaches of representing observables. One prominent example that represents the wave-particle duality concept is a system composed of free particles. In this case, the representation of the Hamiltonian in (4) maps its classical counterpart in (1) accordingly. Thus, we have

$$\hat{H}\Psi_p(r, t) = \sum_i \dot{r}_i \cdot \hat{p}_i \Psi_p(r, t) - \hat{L}\Psi_p(r, t),$$

where in the absence of spin degrees of freedom, the state function $\Psi_p(r, t)$ is represented as a product of plane waves each with phase given by

$$\phi_i = \hbar^{-1}(p_i \cdot r_i - E_i t),$$

$E_i$—denotes the $i$-th particle’s kinetic energy. Respectively, we have

$$\hat{H}\Psi_p(r, t) = \sum_i E_i \Psi_p(r, t), \quad \hat{p}_i \Psi_p(r, t) = p_i \Psi_p(r, t), \quad \hat{L}\Psi_p(r, t) = L\Psi_p(r, t),$$

where the Lagrangian $L$ and the associated operator are functions only of kinetic terms and refer to both relativistic and non-relativistic cases. As an example, for a system composed of massless particles one has $\hat{L}\Psi_p(r, t) = 0$ and $|\dot{r}_i| = c$ for all $i$, where $c$ is the light speed in vacuum.

The phase associated to the $i$-th particle is further given by

$$\phi_i = k_i \cdot r_i - \omega_i t,$$

where $\omega_i$ and $k_i$ are the corresponding angular frequency and wave vector, respectively. Both (6) and (8) satisfy equation (5) simultaneously and stand as a ground for the Planck–Einstein and de Broglie relations given by

$$E_i = \hbar\omega_i, \quad p_i = \hbar k_i,$$

respectively, that marked the origin of quantum theory. It is essential to emphasize that since $\Psi_p(r, t)$ represents a plane wave, the transformation of equation (5) into (1) holds under no semi-classical approximation.

3. Mechanical power, force and torque as a quantum observables

3.1. The hypothesis of discreteness in time

In addition to the representation of energy, momentum and orbital angular momentum in quantum mechanics, equation (2) can be used as a ground for deriving the relevant representations of the mechanical power, force and torque. Since the power-force relation in (2) does not depend on the explicit representation of the Lagrangian in (1), its quantum mechanical counterpart will be adequate to any type of interactions. Therefore, for an arbitrary with respect to the considered system Hamiltonian representation in (4), the quantum mechanical analogue of (2) reads

$$\hat{P}\Psi_F(r, t) = \sum_i \dot{r}_i \cdot \hat{F}_i \Psi_F(r, t),$$

where the state function satisfies the following equations

$$\hat{P}\Psi_F(r, t) = \sum_i P_i \Psi_F(r, t), \quad \hat{F}_i \Psi_F(r, t) = F_i \Psi_F(r, t).$$
Figure 1. Sketch of accelerating indistinguishable and non-interacting quantum-objects due to the action of a central repulsion force. The colors indicate the change in momentum from a macroscale perspective. The red color marks a lower momentum than the blue one, where for \( k \in \mathbb{N} \), \( N_k \) is larger than the Avogadro constant and \( \delta^{(k)} p \) is given in (15) and (16).

Hence, \( \Psi_F(r, t) \) is given as a direct product of plane waves, such that the \( i \)-th one is characterized by the phase

\[
\varphi_i = \varepsilon^{-1} \left( F_i \cdot r_i - P_i t \right),
\]

where \( \varepsilon \) is a constant that has the unit measure of energy. From (10) follows that, for the considered system, in addition to the eigenstates in (4), there exists a state function \( \Psi_F(r_i, t) \) associating to each individual process of energy and momentum transfer a plane wave. As a result, the phase in (12) is further given by

\[
\varphi_i = \hbar^{-1} \left( \rho_i \cdot r_i - \omega_i t \right) = \left( \kappa_i \cdot r_i - \varpi_i t \right),
\]

where \( \omega_i = \varepsilon \kappa_i \) and \( \rho_i = \hbar \kappa_i \) represent the energy and momentum obtained by the \( i \)-th particle in the process of interaction, \( \omega_i \) and \( \kappa_i \) are the respective angular frequency and wave vector. Therefore, within the time-interval that the \( i \)-th net force acts, the energy and momentum transferred to the \( i \)-th particle are equal to the change in its kinetic energy \( \delta E_i \equiv \delta \varepsilon_i \) and momentum \( \delta p_i \equiv \delta \rho_i \), respectively. The total energy of the system is conserved and hence the change in system’s momentum is zero \( \sum_i \rho_i = 0 \).

3.2. Power-frequency and force-momentum relations

Representing the de Broglie hypothesis, equation (10) give rise to additional Planck–Einstein relation. Taking into account (12) and (13), for all \( i \), we obtain two fundamental relations representing the mechanical power \( P_i \) and force \( F_i \) as a function of the frequency \( \omega_i \) and the vector \( \kappa_i \), respectively. Thus, we have

\[
P_i = \varepsilon \omega_i, \quad F_i = \varepsilon \kappa_i.
\]

The last relations show that the greater the frequency \( \omega_i \) the greater the mechanical power generated in the relevant process of energy transfer. Accordingly, the greater the change in momentum \( \delta p_i = \hbar \kappa_i \) the greater the net force acting on the \( i \)-th particle.

Equations (14) lead to alternative with respect to the classical dynamics force-momentum relation and formulation of the mechanical power. Taking into account (13) and (14), we get

\[
P_i = \frac{\varepsilon}{\hbar} \delta E_i, \quad F_i = \frac{\varepsilon}{\hbar} \delta p_i,
\]

where the constant \( \hbar/\varepsilon \) has a unit measure of time. In contrast to the classical dynamics and the Ehrenfest’s theorem, relations (15) give a fundamental limit for the rate at which the processes of energy
and momentum transfer may happen under the discussed conditions. In other words, on a quantum level the time-interval for transferring energy and momentum appears as a fundamental constant equal to $\hbar/\varepsilon$. Now, since the upper limit of the speed at which some quantity of energy is transferred through the three-dimensional space is also restricted by the light speed in vacuum $c$, we obtain the minimal spatial distance, $\rho = c\hbar/\varepsilon$, within which the energy may be transferred. Thus, an energy gap in the energy spectrum of a quantum system is always associated to a spatial gap with value no smaller than $\rho$. Hence, the existence of both boundaries, $\hbar/\varepsilon$ and $c$, generates a discreteness in the energy spectrum and spatial architecture of any physical system, like we observe in atoms.

We would like to point out that the absolute value of $\delta p_i$ for all $i$ is not restricted. Therefore, in the case $i$-th particle’s momentum changes $N \gg N_A$ times, such that $N = \sum_{l\in \mathbb{N}} N_l$ and $N_A$ is the Avogadro constant, one has

$$\Delta p_i = \sum_{l\in \mathbb{N}} N_l \delta^{(l)} p_i, \quad \Delta t = \frac{\hbar}{\varepsilon} N,$$

(16)

where for $k \neq n \in \mathbb{N}$ the change in momentum $|\delta^{(k)} p_i| \neq |\delta^{(n)} p_i|$. As a result, the force-momentum relation can be rewritten in the continuous time framework by using the substitutions $\Delta p_i \to \int \! dp_i$ and $\Delta t \to \int \! dt$. Thus, the processes of interactions can be interpret in term of the Ehrenfest’s theorem.

Figurative representation is depicted in figure 1, where $N \gg N_k$. According to the illustrated example, for all $i$, the corresponding average force is given by $f_i = \Delta p_i/\Delta t$.

3.3. Impulse-momentum theorem and torque

The power and force given in (15) are time independent. As a result, for all $i$, the impulse equation associated to the relation on the right hand side in (15) reads

$$J_i = \frac{\hbar}{\varepsilon} F_i.$$

(17)

The last equation represents the Impulse-momentum theorem within the concept of wave-particle duality. In contrast to (3), the impulse in (17) is defined for a discrete time-interval $\hbar/\varepsilon$.

In addition to relations (15) we can further obtain an expression for the torque applied to the $i$-th particle. Denoting its orbital angular momentum by $l_i$, and the applied torque by $\tau_i$, from the right hand side of equation (15) we have

$$\tau_i = \frac{\varepsilon}{\hbar} \delta l_i.$$

(18)

Since the transfer of momentum between the particles is restricted in time, the time that it takes for the relevant orbital angular momentum to change is also a constant equals $\hbar/\varepsilon$.

3.4. Impact force: Zero initial or final momentum

Consider a system exhibiting a lack of non-contact forces. In this case, for all $i$, the observable $F_i$ in (14) represents the net impact force occurring due to an incident collisions. Therefore, interacting once the $i$-th particle’s initial energy and momentum change. Suggesting a zero initial or final momentum, depending on whether the particle starts from rest or transfer all of its momentum, respectively, equations (14) reduce to

$$P_i = \varepsilon \omega_i, \quad F_i = \varepsilon k_i.$$

(19)

Moreover, equations (15) and (18) read

$$P_i = \frac{\varepsilon}{\hbar} E_i, \quad F_i = \frac{\varepsilon}{\hbar} p_i, \quad \tau_i = \frac{\varepsilon}{\hbar} l_i.$$

(20)

One then may ask is there any relation that for all $i$, $P_i$ satisfies in the case of zero momentum. Answering that question, we rewrite the representations (6), (8) and (12) in terms of the four-momentum,
4. Operators and uncertainty

4.1. Position representation

The Hamiltonian associated to the transferred energy and momentum satisfies the equation \( \hat{H} \Psi_\rho(r, t) = \sum_\beta \delta_\beta \Psi_\rho(r, t) \), where regarding the identity between (12) and (13), we have \( \Psi_\rho(r, t) \equiv \Psi_F(r, t) \). The discrete character of the studied dynamics, however, become apparent by taking into account equation

\[
\iota \epsilon \partial_t \Psi_\rho(r, t) = \hat{P} \Psi_\rho(r, t),
\]

where the operator of mechanical power commute with the respective Hamiltonian, such that for all \( i \) the relation on the left hand side in (15) takes place. For example, let the \( i \)-th particle represents an independent quantum harmonic oscillator with frequency \( \omega_i \) and energy \( E_n_i \). Now, consider a hypothetical interaction in which the \( i \)-th particle transfers all of its energy within the time-interval \( \hbar/\epsilon \). Then, the maximal mechanical power generated is given by \( P_{n_i} = \epsilon \omega_i (2n_i + 1)/2 \) with \( n_i \in \mathbb{N}_0 \). Another example includes the relation on the left hand side in (20). In the non-relativistic limit, one gets the following position representation \( \hat{P}_l \equiv -\iota \hbar \Delta_i/2m_i \).

For \( \mathcal{K} = \{x, y, z\} \), the net force applied to the \( i \)-th particle is associated to the three component operator \( \hat{F}_i = (\hat{F}_i^\nu)_{\nu \in \mathcal{K}} \). Its position representation reads

\[
\hat{F}_i = -\iota \epsilon \partial_i.
\]

For \( \beta, \nu \in \mathcal{K} \) the components of the operator in (24) obey the commutation relation \( \{\hat{B}_i, \hat{F}_i^\nu\} = \iota \epsilon \delta_{\beta \nu} \) and commute with each one component of the relevant momentum operator \( \hat{P}_l \equiv -\iota \hbar \nabla_l \), where \( \delta_{\beta \nu} = \{1 \text{ if } \beta = \nu; 0 \text{ if } \beta \neq \nu \} \) is the Kronecker delta. In the relativistic case, the expression for the four-force operator reads \( \hat{F}_i^\mu = \iota \epsilon \partial_i^\mu \), where the derivative \( \partial_i^\mu = (\partial_\alpha - \nabla) \).

The observable \( \tau_i \) gives the acquired by the \( i \)-th particle orbital angular momentum due to the applied force \( F_i \). It is represented by a three component operator \( \hat{\tau}_i = (\hat{\tau}_i^\gamma)_{\gamma \in \mathcal{K}} \) and similar to the orbital and spin angular momenta, its components satisfy the algebra

\[
\{\hat{r}_i^\gamma, \hat{r}_i^\beta\} = \iota \epsilon \gamma \delta_{\beta \nu} \hat{r}_i^\nu, \quad \gamma, \beta, \nu \in \mathcal{K},
\]

where \( \epsilon_{\gamma \beta \nu} \) is the Levi-Civita symbol. In coordinate space the torque has the following representation \( \hat{\tau}_i \equiv -\iota \{\hat{F}_i \times \nabla_i\} \), where \( \hat{r}_i \) is the position operator associated with the \( i \)-th particle space coordinates. The magnitude of the torque applied to the \( i \)-th particle reads \( |\hat{\tau}_i| = \epsilon \sqrt{T_i^2 (T_i + 1)} \). Therefore, if the \( i \)-th particle acquires an orbital angular momentum with quantum number \( l_i \), then \( \tau_i \equiv l_i \).

It is essential to remark that the domain of all operators discussed in this section differs from the domain of all operators associated to the constituent particles.
4.2. Uncertainty relations
According to (10) and all relations that follow, the exchange of energy and momentum between any two interacting particles has a wave character and the region in space the corresponding wave propagate is restricted by the distance between the particles. In that respect, the Heisenberg uncertainty principle applies and for all \( i \) and \( \nu \in K \), one obtains the following inequalities

\[
\Delta \nu_i \Delta F^i_{\nu} \geq \frac{\epsilon}{2}, \quad \Delta t \Delta P_i \geq \frac{\epsilon}{2}.
\]  

Therefore, the more localized the transfer of momentum in space is the more the lack of information for the applied force is and vice versa. Respectively, the greater the uncertainty in energy that a particle obtain in a process of interaction the greater the uncertainty in generated mechanical power.

4.3. Electromagnetic field example
Expressing the free-space electromagnetic field operators in terms of (12), in addition to the Hamiltonian and total momentum operators of the electromagnetic field, we obtain the operators

\[
\hat{P} = \sum_{k,\sigma} \epsilon \omega_k \hat{n}_{k,\sigma}, \quad \hat{F} = \sum_{k,\sigma} \epsilon k \hat{n}_{k,\sigma},
\]  

where the number operator \( \hat{n}_{k,\sigma} \) gives the number of photons in a mode determined by the wave vector \( k \) and the polarization index \( \sigma \). The operator on the left hand side in (27) gives the maximal mechanical power generated when creating the photons \( |(k,\sigma)^n,\ldots,(k',\sigma')^n\rangle \) form the vacuum \( |0\rangle \). Respectively, the operator \( \hat{F} \) describes the same processes in terms of applied force. In other words, it gives the amount of momentum transferred during the creation of the aforementioned number of photons within the time-interval \( \hbar / \epsilon \).

5. Conclusion
To summarize, we introduce a hypothesis that describes the energy and momentum transfer between interacting particles as a discrete over time process. The used discrete-time framework is grounded by a constant, denoted accordingly by \( \epsilon \), that has a unit measure of energy and appears as fundamental as the Planck’s constant, see (13). According to the proposed theory the energy spectrum of an arbitrary physical system is always discrete, since any exchange of energy with its surrounding medium is restricted over time. In other words, a system can not acquire or radiate energy continuously due to the lower boundary “\( \hbar / \epsilon \)” of the time-interval within which these processes may happened. The quantity of transferred energy and momentum, however, may vary according to the number and type of constituent particles and the way they interact. Another peculiar feature is the wave-like propagation of the exchanged energy. The waves of individual processes of energy transfer may superpose, so that the particle absorbing the energy follows a path defined by their constructive interference. To describe the dynamics of these processes one may use the mechanical power (14), force (15), impulse-momentum theorem (17) and torque (18). Followed by representations in Minkowski space, operator representations and uncertainty relations, see sections 3.4 and 4, respectively.

Although the role of \( \epsilon \) in physics may remain a peculiar attribute to the theory, it is worth exploring all possibilities and make efforts in determining its value.

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