Provably Efficient Reinforcement Learning for Discounted MDPs with Feature Mapping

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Abstract

Modern tasks in reinforcement learning are always with large state and action spaces. To deal with them efficiently, one often uses predefined feature mapping to represents states and actions in a low dimensional space. In this paper, we study reinforcement learning with feature mapping for discounted Markov Decision Processes (MDPs). We propose a novel algorithm which makes use of the feature mapping and obtains a $\tilde{O}(d\sqrt{T}/(1-\gamma)^2)$ regret, where $d$ is the dimension of the feature space, $T$ is the time horizon and $\gamma$ is the discount factor of the MDP. To the best of our knowledge, this is the first polynomial regret bound without accessing to a generative model or making strong assumptions such as ergodicity of the MDP. By constructing a special class of MDPs, we also show that for any algorithms, the regret is lower bounded by $\Omega(d\sqrt{T}/(1-\gamma)^{1.5})$. Our upper and lower bound results together suggest that the proposed reinforcement learning algorithm is near-optimal up to a $(1-\gamma)^{-0.5}$ factor.

1 Introduction

Designing efficient algorithms that learn and plan in sequential decision-making tasks with large state and action spaces has become the central goal of modern reinforcement learning (RL) in recent years. Due to numerous possible states and actions, traditional tabular reinforcement learning methods (Watkins, 1989; Jaksch et al., 2010; Azar et al., 2017) which directly access each state-action pair are computationally intractable. A common method to design reinforcement learning algorithms for large-scale state and action spaces is to make use of feature mappings such as linear functions or neural networks to map states and actions to a low-dimensional space and solve the decision-making problem in the feature space. Despite the empirical success of feature mapping based reinforcement learning methods (Singh et al., 1995; Kwok and Fox; Bertsekas, 2018), the theoretical understanding and the fundamental limits of these methods remain largely understudied.

In this paper, we aim to develop provable reinforcement learning algorithms with feature mapping for discounted Markov Decision Processes (MDPs). Discounted MDP is one of the most widely used models to formulate the modern reinforcement learning tasks such as Atari games (Mnih et al., 2015) and deep recommendation system (Zheng et al., 2018). With feature mapping, a series of recent work (Yang and Wang, 2019a; Lattimore and Szepesvari, 2019; Bhandari et al., 2018; Zou et al.,

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2019) have proposed provably efficient algorithms along with theoretical guarantees. However, these existing results either rely on a special oracle called generative model (Kakade et al., 2003) that allows an algorithm to query any possible state-action pairs and return both the reward and the next state (Yang and Wang, 2019a; Lattimore and Szepesvari, 2019), or needs strong assumptions such as uniform ergodicity (Bhandari et al., 2018; Zou et al., 2019) on the underlying MDP. A natural question arises:

*Can we design provably efficient RL algorithms with feature mapping under mild assumptions?*

We answer this question affirmatively. To be more specific, we consider a special class of discounted MDPs called linear kernel MDP, where the transition probability kernel can be represented as a linear function of a predefined $d$-dimensional feature mapping. We show that linear kernel MDP is a rich MDP class, which covers many classes of MDPs proposed in previous work (Yang and Wang, 2019b; Modi et al., 2019) as special cases. We propose a novel provably efficient algorithm namely Upper-Confidence Linear Kernel reinforcement learning (UCLK) to solve this MDP. We prove both upper and lower regret bounds and show that our algorithm is near-optimal under the linear kernel MDP setting. Our contributions are summarized as follows.

- We propose a novel algorithm UCLK to learn the optimal value function with the help of predefined feature mapping. We show that the regret\(^1\) for UCLK to learn the optimal value function is $\tilde{O}(d \sqrt{T}/(1 - \gamma)^2)$. It is worth noting that the regret is independent of the cardinality of the state and action spaces, which suggests that UCLK is efficient for large-scale RL problems. To the best of our knowledge, this is the first feature-based reinforcement learning algorithm that attains a polynomial regret bound for discounted MDPs without accessing the generative model or making strong assumptions on MDPs such as ergodicity.

- We also show that for any reinforcement learning algorithms, the regret to learn the optimal value function in linear kernel MDP is at least $\Omega(d \sqrt{T}/(1 - \gamma)^{1.5})$. This lower bound result suggests that UCLK is optimal concerning feature mapping dimension $d$ and time horizon $T$, and it is near-optimal concerning the discount factor up to $(1 - \gamma)^{-0.5}$. Our proof is based on a specially constructed linear kernel MDP, which could be of independent interest.

After the release of this paper on arXiv, we became aware that the linear kernel MDP setting is the same as the so-called parameterized transition mode or linear mixture model in earlier work (Jia et al., 2020; Ayoub et al., 2020). Jia et al. (2020) proposed a UCRL-VTR algorithm for finite-horizon MDPs which achieves a $\tilde{O}(d \sqrt{H^3 T})$ regret, where $H$ is the episode length. Ayoub et al. (2020) considered a more general function approximation setting called known transition model family and proved a similar regret bound. Both UCRL-VTR (Jia et al., 2020; Ayoub et al., 2020) and our UCLK build confidence sets to estimate the underlying unknown parameter vector of linear kernel MDPs. Nevertheless, our work differs from Jia et al. (2020); Ayoub et al. (2020) in the following aspects.

- UCRL-VTR is designed for finite-horizon MDPs while our UCLK is designed for discounted MDPs. Thus, to estimate the optimal value function, in each episode, UCRL-VTR constructs the upper confidence value functions from the end of the episode to the beginning, while UCLK uses both extended value iteration and a dynamic epoch length scheme to achieve this goal for discounted MDPs.

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\(^1\)The formal definition of the regret for discounted MDPs can be found in Definition 5.1.
• Jia et al. (2020); Ayoub et al. (2020) proposed a lower bound of regret by directly considering the hard tabular MDP firstly proposed in Jaksch et al. (2010). In contrast, our lower bound is based on a new hard MDP instance which is a generalization of the lower bound results in linear contextual bandits (Dani et al., 2008; Lattimore and Szepesvári, 2018). Our lower bound is tighter than that of Jia et al. (2020); Ayoub et al. (2020) in terms of the feature dimension $d$.

**Notation** We use lower case letters to denote scalars, and use lower and upper case bold face letters to denote vectors and matrices respectively. For a vector $x \in \mathbb{R}^d$ and matrix $\Sigma \in \mathbb{R}^{d \times d}$, we denote by $\|x\|_2$ the Euclidean norm and denote by $\|x\|_{\Sigma} = \sqrt{x^\top \Sigma x}$. For two sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n = O(b_n)$ if there exists an absolute constant $C$ such that $a_n \leq Cb_n$, and we write $a_n = \Omega(b_n)$ if there exists an absolute constant $C$ such that $a_n \geq Cb_n$. We use $\tilde{O}(\cdot)$ to further hide the logarithmic factors.

## 2 Related work

**Finite-horizon MDP with feature mappings.** There is a series of work focusing on solving finite-horizon MDP using RL with function approximation (Jin et al., 2019; Yang and Wang, 2019b; Wang et al., 2019; Modi et al., 2019; Jiang et al., 2017; Zanette et al., 2020; Du et al., 2019). For instance, Jin et al. (2019) assumed the underlying transition kernel and reward function are linear functions of a $d$-dimensional feature mapping and proposed an RL algorithm with $\tilde{O}(\sqrt{d^3H^3T})$ regret, where $H$ is the length of an episode. Yang and Wang (2019b) assumed the probability transition kernel is bilinear in two feature mappings in dimension $d$ and $d'$, and proposed an algorithm with $\tilde{O}(dH^2\sqrt{T})$ regret. Wang et al. (2019) assumed the Bellman backup of any value function is a generalized linear function of certain feature mapping and proposed an algorithm with a regret guarantee. Modi et al. (2019) assumed the underlying MDP can be represented as a linear combination of several base models and proposed an RL algorithm to solve it with a provable guarantee. Jiang et al. (2017) assumed the underlying MDP is of low inherent Bellman error and proposed an algorithm with polynomial PAC bounds. Zanette et al. (2020) studied a similar MDP as Jin et al. (2019) and proposed an algorithm with tighter regret bound. Du et al. (2019) suggested that the sample complexity to learn the optimal policy can be exponential if the approximation error to the value function is moderate. More discussions and insights regarding these negative results can be found in Van Roy and Dong (2019); Lattimore and Szepesvari (2019).

**Discounted MDP with a generative model.** For tabular discounted MDPs, many work focuses on RL with the help of a generative model (or called a simulator) (Kakade et al., 2003). To learn the optimal value function, Azar et al. (2013) proposed Empirical QVI, which learns an $\epsilon$-suboptimal value function with $\tilde{O}(|S||A|/((1-\gamma)^3\epsilon^2))$ optimal sample complexity. To learn the optimal policy, Kearns and Singh (1999) proposed Phased Q-Learning which learns an $\epsilon$-suboptimal policy with $\tilde{O}(|S||A|/((1-\gamma)^7\epsilon^2))$ sample complexity. Sidford et al. (2018b) proposed a Sublinear Randomized Value Iteration algorithm which achieves a $\tilde{O}(|S||A|/((1-\gamma)^4\epsilon^2))$ sample complexity. Sidford et al. (2018a) further proposed Variance-Reduced QVI algorithm which achieves the optimal $\tilde{O}(|S||A|/((1-\gamma)^3\epsilon^2))$ sample complexity. For discounted MDPs with function approximation, Yang and Wang (2019a) assumed the probability transition kernel can be parameterized by a $d$-dimensional feature mapping and proposed a Phased Parametric Q-Learning algorithm which learns an $\epsilon$-suboptimal policy with the optimal $\tilde{O}(d/((1-\gamma)^3\epsilon^2))$ sample complexity. Lattimore and
We consider infinite-horizon discounted Markov Decision Processes (MDP), which is denoted by a tuple $\langle S, A, \gamma, r, \mathbb{P} \rangle$. Here $S$ is the state space (may be infinite), $A$ is the action space, $\gamma : 0 \leq \gamma < 1$ is the discount factor, $r : S \times A \rightarrow [0, 1]$ is the reward function. For simplicity, we assume the reward function $r$ is deterministic and known. $\mathbb{P}(s'|s,a)$ is the transition probability function which denotes the probability for state $s$ to transfer to state $s'$ given action $a$. A policy $\pi : S \rightarrow A$ is a function which maps a state $s$ to an action $a$. For any algorithm $\mathcal{K}$, we denote the action-value function $Q^K_t(s,a)$ as follows, where $s_i,a_i$ are generated by $\mathcal{K}$:

$$Q^K_t(s,a) = r(s,a) + \mathbb{E} \left[ \sum_{i=1}^{\infty} \gamma^i r(s^{t+i},a^{t+i}) \middle| s^1 = s_1, \ldots, a^{t-1} = a_{t-1}, s^t = s, a^t = a \right],$$

where $s^{t+i} \sim \mathbb{P}(\cdot|s^{t+i-1},a^{t+i-1})$ and $a^{t+i} = \mathcal{K}(s^1,\ldots,a^{t-1},s^t)$. The value function $V^K_t(s)$ is defined as $V^K_t(s) = Q^K_t(s,\mathcal{K}(s_1,\ldots,a_{t-1},s))$. We define the optimal value function $V^*$ and the optimal action-value function $Q^*$ as $V^*(s) = \sup_{\mathcal{K}} V^K_t(s)$ and $Q^*(s,a) = \sup_{\mathcal{K}} Q^K_t(s,a)$. For simplicity, for any function $V : S \rightarrow \mathbb{R}$, we denote $[\mathbb{P}V](s,a) = \mathbb{E}_{s' \sim \mathbb{P}(\cdot|s,a)} V(s')$. Therefore we have the following Bellman equation, as well as the Bellman optimality equation:

$$Q^K_t(s_t,a_t) = r(s_t,a_t) + \gamma [\mathbb{P}V^K_{t+1}](s_t,a_t), \quad Q^*(s_t,a_t) = r(s_t,a_t) + \gamma [\mathbb{P}V^*](s_t,a_t).$$

In this work, we consider a special class of MDPs called linear kernel MDPs, where the transition probability function can be represented as a linear function of a given feature mapping $\phi : S \times A \times S \rightarrow \mathbb{R}^d$. It is worth noting that this is essentially the same MDP class as linear mixture model considered in Jia et al. (2020); Ayoub et al. (2020). Formally speaking, we have the following assumption for a linear kernel MDP.

**Definition 3.1.** $M(S,A,\gamma,r,\mathbb{P})$ is called a linear kernel MDP if there exist a known feature mapping $\phi(s'|s,a) : S \times A \times S \rightarrow \mathbb{R}^d$ and an unknown vector $\theta \in \mathbb{R}^d$ with $\|\theta\|_2 \leq \sqrt{d}$, such that

- For any state-action-state triplet $(s,a,s') \in S \times A \times S$, we have $\mathbb{P}(s'|s,a) = \langle \phi(s'|s,a), \theta \rangle$;

- For any bounded function $V : S \rightarrow [0,R]$ and any tuple $(s,a) \in S \times A$, we have $\|\phi_V(s,a)\|_2 \leq \sqrt{dR}$, where $\phi_V(s,a) = \int_{s'} \phi(s'|s,a)V(s')ds' \in \mathbb{R}^d$. 

Szepesvári (2019) considered a similar setting to Yang and Wang (2019a) and proposed a Phased Elimination algorithm with $O(d/((1-\gamma)^4\epsilon^2))$ sample complexity.

**Discounted MDP without a generative model.** Another line of work aims at learning the discounted MDP without accessing to the generative model. Szita and Szepesvári (2010) proposed an MoRmax algorithm which achieves $O(|S||A|/((1-\gamma)^6\epsilon^2))$ sample complexity of exploration. Lattimore and Hutter (2012) proposed UCRL algorithm which achieves $O(|S|^2|A|/((1-\gamma)^3\epsilon^2))$ sample complexity of exploration. Strehl et al. (2006) proposed delay-Q-learning with $O(|S||A|/((1-\gamma)^5\epsilon^4))$ sample complexity of exploration. Dong et al. (2019) proposed Infinite Q-learning with UCB which achieves $O(|S||A|/((1-\gamma)^7\epsilon^2))$ sample complexity of exploration. Liu and Su (2020) proposed the regret definition for discounted MDPs and presented Double Q-Learning to achieve $O(\sqrt{|S||A|T}/(1-\gamma)^{2.5})$ regret. Our work falls into this category, and also uses regret to characterize the performance of RL.

## 3 Preliminaires

We consider infinite-horizon discounted Markov Decision Processes (MDP), which is denoted by a tuple $M(S,A,\gamma,r,\mathbb{P})$. Here $S$ is the state space (may be infinite), $A$ is the action space, $\gamma : 0 \leq \gamma < 1$ is the discount factor, $r : S \times A \rightarrow [0, 1]$ is the reward function. For simplicity, we assume the reward function $r$ is deterministic and known. $\mathbb{P}(s'|s,a)$ is the transition probability function which denotes the probability for state $s$ to transfer to state $s'$ given action $a$. A policy $\pi : S \rightarrow A$ is a function which maps a state $s$ to an action $a$. For any algorithm $\mathcal{K}$, we denote the action-value function $Q^K_t(s,a)$ as follows, where $s_i,a_i$ are generated by $\mathcal{K}$:
We denote the linear kernel MDP by $M_{\theta}$ for simplicity.

As we will show in the following examples, linear kernel MDPs cover several MDPs studied in previous work as special cases.

**Example 3.2** (Tabular MDPs). For an MDP $M(\mathcal{S}, \mathcal{A}, \gamma, r, \mathbb{P})$ with $|\mathcal{S}|, |\mathcal{A}| \leq \infty$, the transition probability function can be parameterized by $|\mathcal{S}|^2|\mathcal{A}|$ unknown parameters. The tabular MDP is a special case of linear kernel MDPs with the following feature mapping and parameter vector

$$d = |\mathcal{S}|^2|\mathcal{A}|, \quad \phi(s'|s, a) = e_{(s,a,s')} \in \mathbb{R}^d, \quad \theta = [\mathbb{P}(s'|s, a)] \in \mathbb{R}^d,$$

where $e_{(s,a,s')}$ denotes the corresponding natural basis in the $d$-dimensional Euclidean space.

**Example 3.3** (Linear combination of base models (Modi et al., 2019)). For an MDP $M(\mathcal{S}, \mathcal{A}, \gamma, r, \mathbb{P})$, suppose there exist $m$ base transition probability functions $\{p_i(s'|s, a)\}_{i=1}^m$, a feature mapping $\psi(s, a) : \mathcal{S} \times \mathcal{A} \rightarrow \Delta^d$ where $\Delta^d$ is a $(d' - 1)$-dimensional simplex, and an unknown matrix $W \in \mathbb{R}^{m \times d'} \in [0, 1]^{m \times d'}$ such that $\mathbb{P}(s'|s, a) = \sum_{k=1}^m [W \psi(s, a)]_k p_k(s'|s, a)$. Then it is a special case of linear kernel MDPs with feature mapping and parameter vector defined as follows

$$d = md', \quad \phi(s'|s, a) = \text{vec}(p(s'|s, a)\psi(s, a)^\top) \in \mathbb{R}^d, \quad \theta = \text{vec}(W) \in \mathbb{R}^d,$$

where vec$(\cdot)$ is the vectorization operator, and $p(s'|s, a) = [p_k(s'|s, a)] \in \mathbb{R}^m$.

**Example 3.4** (Feature embedding of a transition model (Yang and Wang, 2019b)). For an MDP $M(\mathcal{S}, \mathcal{A}, \gamma, r, \mathbb{P})$, suppose there exist feature mappings $\psi_1(s, a) : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^{d_1}$ satisfying $\|\psi_1(s, a)\|_2 \leq \sqrt{d_1}$, $\psi_2(s') : \mathcal{S} \rightarrow \mathbb{R}$ satisfying for any $V : \mathcal{S} \rightarrow [0, R], \|\int_{\mathcal{S}} V(s) \psi_2(s) ds\|_2 \leq R$ and an unknown matrix $M \in \mathbb{R}^{d_1 \times d_2}$ satisfying $\|M\|_F \leq \sqrt{d_1}$ such that $\mathbb{P}(s'|s, a) = \psi_1(s, a)^\top M \psi_2(s')$. Then it is a special case of linear kernel MDPs with the following feature mapping and parameter vector

$$d = d_1d_2, \quad \phi(s'|s, a) = \text{vec}(\psi_2(s')\psi_1(s, a)^\top) \in \mathbb{R}^d, \quad \theta = \text{vec}(M) \in \mathbb{R}^d.$$

Comparison with linear MDPs Yang and Wang (2019a); Jin et al. (2019) studied the so-called linear additive model or linear MDP, which assumes the probability transition function can be represented as $\mathbb{P}(:,s,a) = (\psi(s,a), \mu(\cdot))$, where $\psi(s,a)$ is a known feature mapping and $\mu(\cdot)$ is an unknown measure. It is worth noting that linear kernel MDPs studied in our paper and linear MDPs (Yang and Wang, 2019a; Jin et al., 2019) are two different classes of MDPs since they are based on different feature mappings, i.e., $\phi(s'|s, a)$ versus $\psi(s,a)$. One cannot be covered by the other.

In the rest of this paper, we assume the underlying linear kernel MDP is parameterized by $\theta^*$ and denote it by $M_{\theta^*}$.

### 4 The proposed algorithm

In this section, we propose an algorithm namely UCLK to learn the linear kernel MDP, which is illustrated in Algorithm 1. UCLK is essentially a multi-epoch algorithm inspired by Jaksch et al. (2010); Lattimore and Hutter (2012). In specific, the $k$-th epoch of Algorithm 1 starts at round $t_k$ and ends at round $t_{k+1} - 1$. The length of each epoch is not prefixed but depends on
Algorithm 1 Upper-Confidence Linear Kernel Reinforcement Learning (UCLK)

**Require:** Regularization parameter $\lambda$, exploration parameter $\beta$, number of value iteration rounds $U$

1: Receive $s_1$
2: Set $t \leftarrow 1$, $\Sigma_0 \leftarrow \lambda I$, $b_0 = 0$
3: for $k = 0, \ldots$ do
4: Set $t_k \leftarrow t$, $\hat{\theta}_k \leftarrow \Sigma_{t_k-1}^{-1}b_{t_k-1}$
5: Set $\mathcal{M}_k$ as the set of plausible MDPs

\[ \mathcal{M}_k \leftarrow \left\{ \hat{M} : \forall (s, a) \exists \theta_{s,a} \in \mathbb{R}^d, \ P^k(\cdot|s,a) = (\phi(\cdot|s,a), \theta_{s,a}), \left\| \Sigma_{t_k-1}^{1/2}(\theta_{s,a} - \hat{\theta}_k) \right\|_2 \leq \beta \right\} \]

6: Set $Q_k(s,a) \leftarrow \text{EVI}(\mathcal{M}_k, U)$, $\pi_k(s) \leftarrow \arg\max_a Q_k(s,a)$, $V_k(s) = \max_a Q_k(s,a)$. 
7: repeat
8: Take action $a_t = \pi_k(s_t)$, receive $s_{t+1} \sim P(\cdot|s_t, a_t)$
9: Set $\Sigma_t \leftarrow \Sigma_{t-1} + \phi V_k(s_t, a_t)\phi^\top$
10: Set $b_t \leftarrow b_{t-1} + \phi V_k(s_t, a_t)V_k(s_{t+1})$
11: $t \leftarrow t + 1$
12: until $\det(\Sigma_{t-1}) > 2\det(\Sigma_{t_k-1})$
end for

previous observations. In each epoch, UCLK maintains a set of statistically plausible MDPs based on previous observed states and actions, and it executes policy $\pi_k$ which is near-optimal among those plausible MDPs. The reason for using adaptive epoch length is that it can control the amount of “switching error” which occurs when policy $\pi_k$ is updated. Each epoch of UCLK can be divided into two phases, which we will discuss in detail in the sequel. Without confusion, for any MDP $\hat{M}$ we use $\hat{P}$ to denote its probability transition function.

**Planning phase (Line 4 to 6)** Planning phase is executed at the beginning of each epoch. In this phase, UCLK first computes $\hat{\theta}_k$ as the estimate of $\theta^*$, which is the minimizer of the following regularized least-square problem:

\[ \hat{\theta}_k \leftarrow \arg\min_{\theta \in \mathbb{R}^d} \sum_{j=0}^{k-1} \sum_{i=t_j}^{t_{j+1}-1} \left[ \langle \theta, \phi V_j(s_i, a_i) \rangle - V_j(s_{i+1}) \right]^2 + \lambda \|\theta\|_2^2, \tag{4.2} \]

which has a closed-form solution as shown in Line 4. Then Algorithm 1 computes the set of statistical plausible MDPs $\mathcal{M}_k$ and use it to compute the new policy $\pi_k$. As in Line 5, each MDP $\hat{M}$ in $\mathcal{M}_k$ is determined by a collection of probability transition functions $P(\cdot|s,a)$ over all state-action pairs $(s, a)$, satisfying that $P(\cdot|s,a) = (\phi(\cdot|s,a), \theta_{s,a})$, where $\phi(s'|s,a)$ is the predefined feature mapping and $\theta_{s,a}$ is a vector within the confidence ball $\|\Sigma_{t_k-1}^{1/2}(\theta_{s,a} - \hat{\theta}_k)\|_2 \leq \beta$ with radius $\beta$. Then in Line 6, Algorithm 1 uses the widely used Extended value iteration (EVI) (Dann and Brunskill, 2015; Jaksch et al., 2010; Lattimore and Hutter, 2012) to compute the action-value functions $Q_k$ and the policy $\pi_k$ corresponding to the near-optimal MDP among the set of plausible MDPs $\mathcal{M}_k$, which will be described in Algorithm 2 later.

**Execution phase (Line 7 to 12)** Execution phase is used to execute the policy in each epoch, collect observations, and update parameters. At round $t$ in the $k$-th epoch, Algorithm 1 follows the
Algorithm 2 Extended Value Iteration: EVI($\mathcal{M}, U$)

**Require:** Set of plausible MDPs $\mathcal{M}$, number of value iteration rounds $U$

1: Let $Q^{(0)}(s, a) = 1/(1 - \gamma)$ for any $(s, a) \in S \times A$.
2: for $u = 1, \ldots, U$ do
3:     Let $V^{(u-1)}(s) = \max_{a \in A} Q^{(u-1)}(s, a)$ and
4:          $Q^{(u)}(s, a) \leftarrow r(s, a) + \gamma \max_{\tilde{M} \in \mathcal{M}} [\tilde{P} V^{(u-1)}](s, a)$
5:     end for

**Ensure:** $Q^{(U)}(s, a)$

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policy $\pi_k$ to take the action $\pi_k(s_t)$ and observes the new state $s_{t+1}$. Algorithm 1 then computes vector $\phi_{V_k}(s_t, a_t)$ according to Definition 3.1 and the value function at $s_{t+1}$, i.e., $V_k(s_{t+1})$. Next, Algorithm 1 updates parameters $\Sigma_t$ and $b_t$ by $\phi_{V_k}(s_t, a_t)$. The loop repeats until $\det(\Sigma_{t-1}) > 2\det(\Sigma_{t_k-1})$. This is the same as the stopping criterion used by Rarely Switching OFUL in Abbasi-Yadkori et al. (2011).

**Implementation of Algorithm 1** UCLK is an online reinforcement learning algorithm as it does not need to store all the past observations. Instead, UCLK only needs to maintain a vector $b_t$ and a matrix $\Sigma_t$, which costs $O(d^2)$ space complexity. To compute the integration $\phi_{V_k}(s, a)$ for some $(s, a)$, we could either compute it exactly if the integration is tractable or compute it approximately using Monte Carlo integration.

**Extended value iteration** As we mentioned before, Algorithm 1 makes use of EVI in Algorithm 2 to compute the action-value function corresponding to the near-optimal MDP among all the plausible MDPs $\mathcal{M}_k$ described in the planning phase. At each iteration, to obtain the new action-value function $Q^{(u)}$, Algorithm 2 performs one-step optimal value iteration (4.1) by selecting the best possible MDP $\tilde{M}$ among $\mathcal{M}$ to maximize the Bellman backup over the previous value function $V^{(u-1)}$. Algorithm 2 returns the last action-value function as its output. To implement EVI, we need to compute the maximization problem in (4.1). Define the confidence set of $\theta$, i.e., $B_k(s, a)$, as follows:

$$B_k(s, a) \leftarrow \left\{ \theta : \langle \phi(|s, a), \theta \rangle \text{ is a probability distribution, } \left\| \Sigma_t^{1/2}(\theta - \hat{\theta}_k) \right\|_2 \leq \beta \right\}. \quad (4.3)$$

Then (4.1) can be rewritten as

$$Q^{(u)}(s, a) \leftarrow r(s, a) + \gamma \max_{\tilde{M} \in \mathcal{M}} [\tilde{P} V^{(u-1)}](s, a) = r(s, a) + \gamma \max_{\theta \in B(s, a)} \langle \theta, \phi_{V^{(u-1)}}(s, a) \rangle.$$ 

Therefore, the maximization problem in (4.1) is reduced to a constrained maximization problem over the convex set $B_k(s, a)$, which can be solved by projected gradient methods (Boyd et al., 2004) efficiently in practice.

## 5 Main theory

In this section, we provide the theoretical analysis of Algorithm 1. Following Liu and Su (2020), for any reinforce learning algorithm $\mathcal{K}$, we define its regret in the first $T$ rounds as follows.
Definition 5.1. (Liu and Su, 2020) Given a (randomized) RL algorithm $K$, we define its regret on MDP $M(S, A, \gamma, r, \mathbb{P})$ in the first $T$ rounds as the sum of the suboptimality $\Delta_t$ for $t = 1, \ldots, T$, i.e.,

$$\text{Regret}(K, M, T) = \sum_{t=1}^{T} \Delta_t,$$

where $\Delta_t = V^*(s_t) - \sum_{t' = 0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'})$.

We introduce a shorthand notation $\text{Regret}(T)$ for $\text{Regret}(UCLK_\theta^*, M, T)$, when there is no confusion. Due to the optimality of the optimal value function $V^*$, we know that $\Delta_t \geq 0$ for any algorithm $K$. This fact suggests that $\text{Regret}(T)$ can be regarded as a cumulative error for $K$ to learn the optimal value function of MDP $M$.

Remark 5.2. A related quantity widely used for discounted MDPs is called the sample complexity of exploration $N(\epsilon, \delta)$ (Szita and Szepesvári, 2010; Lattimore and Hutter, 2012; Dong et al., 2019), which is defined as the number of rounds $t$ where $\Delta_t$ is greater than $\epsilon$ with probability at least $1 - \delta$.

Remark 5.4. Several aspects of Theorem 5.3 are worth to comment. Thanks to the feature mapping $\phi$ and the multi-epoch nature of Algorithm 1, the regret bound (5.2) in Theorem 5.3 is independent of $|S|$ and $|A|$, which suggests that UCLK is sample efficient even for MDPs with large state and action spaces. This is in sharp contrast to the tabular RL algorithms, whose regret bound or sample complexity depends on $|S|$ and $|A|$ polynomially. Moreover, the exploration parameter $\beta$ and the number of extended value iteration rounds $U$ depend on $T$ logarithmically. For the case where $T$ is unknown, we can use the “doubling trick” (Besson and Kaufmann, 2018) to learn $T$ adaptively, and it will only increase the regret (5.2) by a constant factor.
In addition to the upper bound result, we also prove the lower bound result. The following theorem shows a lower bound for any algorithm to learn a linear kernel MDP.

**Theorem 5.5.** Suppose $\gamma \geq 2/3, d \geq 2$ and $T \geq \max\{d^2/225, 5\gamma\}/(1 - \gamma)$. Then for any algorithm $\mathcal{K}$, there exists a linear kernel MDP $M_{\tilde{\theta}}$ such that

$$E[\text{Regret}(\mathcal{K}, M_{\tilde{\theta}}, T)] \geq \frac{\gamma d \sqrt{T}}{6400 \sqrt{2(1 - \gamma)^{1.5}}} - \frac{\gamma}{(1 - \gamma)^2}. \tag{5.3}$$

**Remark 5.6.** Theorem 5.5 suggests that when $T$ is large enough, the lower bound of regret (5.3) is $\Omega(d \sqrt{T}/(1 - \gamma)^{1.5})$. Compared with the upper regret bound $O(d \sqrt{T}/(1 - \gamma)^2)$, we can conclude that UCLK has an optimal dependence on the feature mapping dimension $d$ and the time horizon $T$, and the dependence on the discount factor is only worse than the lower bound by a $(1 - \gamma)^{-0.5}$ factor.

![Figure 1](image.png)

**Figure 1:** Class of hard-to-learn linear kernel MDPs considered in Theorem 5.5. The left figure demonstrates the state transition probability starting from $x_0$ with different action $a_i$. The right figure demonstrates the state transition probability starting from $x_1$ with any action.

At the core of the proof of Theorem 5.5 is to construct a class of hard-to-learn MDP instances. We show the construction of these instances here and defer the detailed proof to Appendix A. Let $M(S, A, \gamma, r, P_\theta)$ denote these hard MDPs. The state space $S$ consists of two states $x_0, x_1$. The action space $A$ consists of $2d - 1$ vectors $a \in \mathbb{R}^{d-1}$ whose coordinates are 1 or $-1$. The reward function $r$ satisfies that $r(x_0, a) = 0$ and $r(x_1, a) = 1$ for any $a \in A$. The probability transition function $P_\theta$ is parameterized by a ($d - 1$)-dimensional vector $\theta \in \Theta$, which is defined as

$$P_\theta(x_0|x_0, a) = 1 - \delta - \langle a, \theta \rangle, \quad P_\theta(x_1|x_0, a) = \delta + \langle a, \theta \rangle,$$

$$P_\theta(x_0|x_1, a) = \delta, \quad P_\theta(x_1|x_1, a) = 1 - \delta,$$

$$\Theta = \left\{-\Delta/(d - 1), \Delta/(d - 1)\right\}^{d-1},$$

where $\delta$ and $\Delta$ are positive parameters that need to be determined in later proof. It can be verified that $M$ is indeed a linear kernel MDP with the vector $\tilde{\theta} = (\theta^T, 1)^T \in \mathbb{R}^d$ while $\Delta \leq d - 1$ and the feature mapping $\phi(s', s, a)$ defined as follows:

$$\phi(x_0|x_0, a) = \left(\begin{array}{c} -a \\ 1 - \delta \end{array}\right), \phi(x_1|x_0, a) = \left(\begin{array}{c} a \\ \delta \end{array}\right), \phi(x_0|x_1, a) = \left(\begin{array}{c} 0 \\ \delta \end{array}\right), \phi(x_1|x_1, a) = \left(\begin{array}{c} 0 \\ 1 - \delta \end{array}\right).$$

Several comments are made as follows. The class of hard-to-learn linear kernel MDPs can be regarded as an extension of the hard instance in linear bandits literature (Dani et al., 2008; Lattimore and
Szepesvári (2018) to MDPs. Our constructed MDPs are similar to those in Jaksch et al. (2010); Osband and Van Roy (2016) for the average-reward MDPs and Lattimore and Hutter (2012) for the discounted MDPs. By Example 3.2, we know that tabular MDPs can be regarded as specialized linear kernel MDPs with a \(|S|^2|A|\)-dimensional feature mapping. However, simply applying the MDPs in Jaksch et al. (2010); Osband and Van Roy (2016); Lattimore and Hutter (2012) to our setting would yield a \(\Omega(\sqrt{|S||A|T/(1-\gamma)^{1.5}})\) lower bound for regret, which is looser than our result because \(\sqrt{|S||A|} \leq |S|^2|A| = d\).

6 Conclusion

We proposed a novel algorithm for solving linear kernel MDPs called UCLK. We prove that the regret of UCLK can be upper bounded by \(\tilde{O}(d\sqrt{T/(1-\gamma)^2})\), which is the first result of its kind for learning discounted MDPs without accessing the generative model or making strong assumptions like uniform ergodicity. We also proved a lower bound \(\Omega(d\sqrt{T/(1-\gamma)^{1.5}})\) which holds for any algorithm. There still exists a gap of \((1-\gamma)^{-0.5}\) between the upper and lower bounds, and we leave it as an open problem for future work.

Acknowledgement

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A Proof of main theory

In this section we provide the proof of main theory.

A.1 Proof of Theorem 5.3

In this section we prove Theorem 5.3. Let \(K(T) - 1\) be the number of epochs when Algorithm 1 executes \(t = T\) rounds, and \(t_{K(T)} = T + 1\). Let \(\mathcal{K}\) denote UCLK. We have the following technical lemmas.

Lemma A.1. With probability at least \(1 - \delta\), Algorithm 1 satisfies

\[
\text{Regret}(T) \leq \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left[ V^*(s_t) - V^K_t(s_t) \right] + \frac{K(T)}{(1-\gamma)^2} + \frac{2\sqrt{T\log 1/\delta}}{(1-\gamma)^2} + \frac{4\gamma}{(1-\gamma)^2}.
\]

Lemma A.1 suggests that the total regret \(\text{Regret}(T)\) of Algorithm 1 can be bounded by an error term \(E\) and the sum of error term \(E_k\) for each epoch. Note that \(E_k\) itself is the sum of the difference between the optimal value function \(V^*\) and the value function \(V^K_t\) over all states in epoch \(k\).

Lemma A.2. Let \(\beta\) be defined in (5.1). Then with probability at least \(1 - \delta\), for all \(0 \leq k \leq K(T) - 1\), we have \(\theta^* \in B_k(s,a)\) for any \((s,a) \in S \times A\).
Lemma A.2 suggests that in every epoch of Algorithm 1, \( \theta^* \) is contained in the confidence sets \( \{B_k(s,a)\} \) defined in (4.3) for all \((s,a)\) with high probability.

**Lemma A.3.** With probability at least \( 1 - \delta \), for all \( 0 \leq k \leq K(T) - 1 \), we have \( 1/(1 - \gamma) \geq Q_k(s,a) \geq Q^*(s,a) \) for any \((s,a)\) \( \in S \times A \).

**Lemma A.3** suggests that in every epoch of Algorithm 1, \( Q_k(s,a) \) found by EVI is an upper bound for the optimal action-value function \( Q^*(s,a) \).

Recall that the goal of EVI is to find the action-value function \( Q_k \) corresponding to the optimal MDP in \( \mathcal{M}_k \), which should satisfy the following optimality condition

\[
Q_k(s_t, a_t) = r(s_t, a_t) + \gamma \max_{M \in \mathcal{M}_k} \mathbb{E} Q_k(M) | s_t, a_t = \langle \theta, \phi V_k(s_t, a_t) \rangle.
\]

However, it is impossible to find the exactly optimal value function since EVI only performs finite number of iterations. The following lemma characterizes the error of EVI after \( U \) iterations.

**Lemma A.4.** For any \( 0 \leq k \leq K(T) - 1 \) and \( t_k \leq t \leq t_{k+1} - 1 \), there exists \( \theta_t \in B_k(s_t, a_t) \) such that \( Q_k(s_t, a_t) \leq r(s_t, a_t) + \gamma \langle \theta_t, \phi V_k(s_t, a_t) \rangle + 2\epsilon U \).

**Lemma A.4** suggests that in every epoch of Algorithm 1, \( \theta_t \) must be updated for \( \epsilon \) suboptimal action-value function.

**Lemma A.5.** We have \( K(T) \leq d \log[(2\lambda + dT)/(\lambda(1 - \gamma)^2)] \).

**Lemma A.5** suggests that Algorithm 1 only needs to update its policy for \( K(T) = \tilde{O}(d) \) times, which is almost independent of the time horizon \( T \). In sharp contrast, RL algorithms with feature mapping in the finite-horizon setting need to update their policy every \( H \) steps (Jin et al., 2019; Modi et al., 2019), which leads to \( O(T/H) \) number of updates.

We also need the following three additional lemmas.

**Lemma A.6** (Azuma-Hoeffding inequality). Let \( \{X_k\}_{k=0}^{\infty} \) be a discrete-parameter real-valued martingale sequence such that for every \( k \in \mathbb{N} \), the condition \( |X_k - X_{k-1}| \leq \mu \) holds for some non-negative constant \( \mu \). Then with probability at least \( 1 - \delta \), we have

\[
|X_n - X_0| \leq 2\mu \sqrt{n \log 1/\delta}.
\]

**Lemma A.7** (Lemma 11 in Abbasi-Yadkori et al. (2011)). For any \( \{x_t\}_{t=1}^{T} \subset \mathbb{R}^d \) satisfying that \( \|x_t\|_2 \leq L \), let \( A_0 = \lambda I \) and \( A_t = A_0 + \sum_{i=1}^{t-1} x_i x_i^\top \), then we have

\[
\sum_{t=1}^{T} \min\{1, \|x_t\|_{A_t^{-1}}^2\} \leq 2d \log \frac{d\lambda + TL^2}{d\lambda}.
\]

**Lemma A.8** (Lemma 12 in Abbasi-Yadkori et al. (2011)). Suppose \( A, B \in \mathbb{R}^{d \times d} \) are two positive definite matrices satisfying that \( A \succeq B \), then for any \( x \in \mathbb{R}^d \), \( \|x\|_A \leq \|x\|_B \cdot \sqrt{\det(A)/\det(B)} \).

**Proof of Theorem 5.3.** Therefore, by Lemma A.1, with probability \( 1 - \delta \), we have

\[
\text{Regret}(T) \leq \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{K(T)-1} \left[ V^*(s_t) - V_k^T(s_t) \right] + \frac{K(T)}{(1 - \gamma)^2} + \frac{2\sqrt{T \log 1/\delta}}{(1 - \gamma)^2} + \frac{4\gamma}{(1 - \gamma)^2}
\]

\[
\leq \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{K(T)-1} \left[ V_k(s_t) - V_k^T(s_t) \right] + \frac{K(T)}{(1 - \gamma)^2} + \frac{2\sqrt{T \log 1/\delta}}{(1 - \gamma)^2} + \frac{4\gamma}{(1 - \gamma)^2}.
\]  

(A.1)
where the last inequality holds on because of Lemma A.3 and for Regret'\( (T) \), we have

\[
\text{Regret}'(T) = \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left[ Q_k(s_t, a_t) - V^K_{t+1}(s_{t+1}) \right],
\]

(A.2)

where the equality holds because of the Assumption 3.1. Substituting (A.3) and (A.4) into we have

\[
\text{Regret}'(T) - 2T \gamma^U \leq \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left( \langle \theta_t, \phi_{V_k}(s_t, a_t) \rangle - \langle \theta^*, \phi_{V_{t+1}^K}(s_t, a_t) \rangle \right)
\]

where the inequality holds due to Lemma A.4. By the Bellman equation and the fact that \( a_t = \pi_k(s_t) \), we have

\[
V^K_t(s_t) = r(s_t, a_t) + \gamma \langle \theta^*, \phi_{V_{t+1}^K}(s_t, a_t) \rangle,
\]

(A.4)

where the second equality holds on because Assumption 3.1. Substituting (A.3) and (A.4) into (A.2), we have

\[
\text{Regret}'(T) - 2T \gamma^U \\
\leq \gamma \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left( \langle \theta_t, \phi_{V_k}(s_t, a_t) \rangle - \langle \theta^*, \phi_{V_{t+1}^K}(s_t, a_t) \rangle \right)
\]

\[
= \gamma \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left( \langle \theta_t - \theta^*, \phi_{V_k}(s_t, a_t) \rangle + \gamma \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \langle \theta^*, \phi_{V_{t+1}^K}(s_t, a_t) \rangle \right)
\]

\[
= I_1 + \gamma \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left[ \mathbb{P}(V_k - V_{t+1}^K)(s_t, a_t) \right]
\]

\[
= I_1 + \gamma \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left\{ \mathbb{P}(V_k - V_{t+1}^K)(s_t, a_t) - (V_k(s_{t+1}) - V_{t+1}^K(s_{t+1})) \right\}
\]

\[
I_2
\]

\[
+ \gamma \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left( V_k(s_{t+1}) - V_{t+1}^K(s_{t+1}) \right).
\]

(A.5)
Next we bound $I_1, I_2$ and $I_3$ separately. For term $I_1$, we have

$$I_1 = \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \langle V \theta_t - \theta^*, \phi V_k(s_t, a_t) \rangle$$

$$= \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \langle \theta_t - \theta^*, \phi V_k(s_t, a_t) \rangle$$

$$\leq \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left( \| \theta_t - \hat{\theta}_k \| \Sigma_{i-1} + \| \hat{\theta}_k - \theta^* \| \Sigma_{i-1} \right) \| \phi V_k(s_t, a_t) \| \Sigma_{i-1}^{-1}$$

$$\leq 2 \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \left( \| \theta_t - \hat{\theta}_k \| \Sigma_{i-1} + \| \hat{\theta}_k - \theta^* \| \Sigma_{i-1} \right) \| \phi V_k(s_t, a_t) \| \Sigma_{i-1}^{-1}$$

$$\leq 4\beta \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \| \phi V_k(s_t, a_t) \| \Sigma_{i-1}^{-1},$$

where the first inequality holds due to the Cauchy-Schwarz inequality and triangle inequality, the second inequality holds due to Lemma A.8 with the fact that $\text{det}(\Sigma_{i-1}) \leq 2 \text{det}(\Sigma_{t_k-1})$, the third inequality holds due to Lemma A.2 and the fact that $\theta_t \in B_k(s_t, a_t)$ from Lemma A.4. Meanwhile, denote $P_t(s'|s_t, a_t) = \langle \phi(s'|s_t, a_t), \theta_t \rangle$, then for $t_k \leq t \leq t_{k+1} - 1$, we have

$$\langle \theta_t - \theta^*, \phi V_k(s_t, a_t) \rangle = [P_t V_k](s_t, a_t) - [P V_k](s_t, a_t) \leq \frac{2}{1 - \gamma},$$

then $I_1$ can be further bounded as

$$I_1 \leq \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \min \left\{ \frac{2}{1 - \gamma}, 4\beta \| \phi V_k(s_t, a_t) \| \Sigma_{i-1}^{-1} \right\}$$

$$\leq 4\beta \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \min \left\{ 1, \| \phi V_k(s_t, a_t) \| \Sigma_{i-1}^{-1} \right\}$$

$$\leq 4\beta \sqrt{T} \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \min \left\{ 1, \| \phi V_k(s_t, a_t) \|^2 \Sigma_{i-1}^{-1} \right\},$$

(A.6)

where the second inequality holds because $2/(1 - \gamma) \leq \beta$, the last inequality holds due to Cauchy-Schwarz inequality. To further bound (A.6), we have

$$\sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} \min \left\{ 1, \| \phi V_k(s_t, a_t) \|^2 \Sigma_{i-1} \right\} \leq 2d \log \frac{\lambda + T/(1 - \gamma)^2}{\lambda},$$

(A.7)

where the inequality holds due to Lemma A.7 with the fact $\| \phi V_k(s_t, a_t) \|_2 \leq \sqrt{d}/(1 - \gamma)$ deduced by Definition 3.1 and $|V_k| \leq 1/(1 - \gamma)$ from Lemma A.3. Substituting (A.7) into (A.6), we have

$$I_1 \leq 6\beta \sqrt{dT \log \frac{\lambda + T/(1 - \gamma)^2}{\lambda}}.$$

(A.8)
For the term $I_2$, since the policy $\pi_k$ is determined before step $t$, for $t_k \leq t \leq t_{k+1} - 1$, we have
\[
\mathbb{E} \left[ \mathbb{P}(V_k - V_{t+1}^K) (s_t, a_t) - (V_k(s_{t+1}) - V_{t+1}^K(s_{t+1})) \right] = 0. \tag{A.9}
\]
Furthermore, we have $0 \leq V^k(s) - V_{t+1}^K(s) \leq 1/(1 - \gamma)$, which implies that
\[
\left| \mathbb{P}(V_k - V_{t+1}^K) (s_t, a_t) - (V_k(s_{t+1}) - V_{t+1}^K(s_{t+1})) \right| \leq \frac{1}{1 - \gamma}. \tag{A.10}
\]
Thus by (A.9) and (A.10), using Lemma A.6, we have
\[
I_2 = \gamma \sum_{k=0}^{(K(T)-1)t_{k+1}-1} \sum_{t=t_k}^{t_{k+1}-1} \mathbb{P}(V_k - V_{t+1}^K) (s_t, a_t) - (V_k(s_{t+1}) - V_{t+1}^K(s_{t+1})) \leq \frac{2\gamma}{1 - \gamma} \sqrt{T \ln \frac{1}{\delta}}. \tag{A.11}
\]
For the term $I_3$, we have
\[
I_3 = \gamma \sum_{k=0}^{K(T)-1} \sum_{t=t_k}^{t_{k+1}-1} (V_k(s_{t+1}) - V_{t+1}^K(s_{t+1}))
\]
\[
= \gamma \sum_{k=0}^{K(T)-1} \left[ \sum_{t=t_k}^{t_{k+1}-1} (V_k(s_t) - V_{t+1}^K(s_t)) - (V_k(s_{t_k}) - V_{t_k}^K(s_{t_k})) + (V_k(s_{t_{k+1}}) - V_{t_{k+1}}^K(s_{t_{k+1}})) \right]
\]
\[
\leq \gamma \sum_{k=0}^{K(T)-1} \left[ \sum_{t=t_k}^{t_{k+1}-1} (V_k(s_t) - V_{t+1}^K(s_t)) + \frac{2}{1 - \gamma} \right]
\]
\[
= \gamma \text{Regret}'(T) + \frac{2K(T)\gamma}{1 - \gamma}, \tag{A.12}
\]
where the first inequality holds on due to $0 \leq V^k(s) - V_{t+1}^K(s) \leq 1/(1 - \gamma)$. Finally, substituting (A.8), (A.11) and (A.12) into (A.5), we have
\[
\text{Regret}'(T) - 2\gamma^U T 
\leq 6\beta \sqrt{dT \log \frac{\lambda + T/(1 - \gamma)^2}{\delta}} + \frac{2\gamma}{1 - \gamma} \sqrt{T \ln \frac{1}{\delta}} + \gamma \text{Regret}'(T) + \frac{2K(T)\gamma}{1 - \gamma}. \tag{A.13}
\]
Thus, we have
\[
\text{Regret}'(T) 
\leq \frac{6\beta}{1 - \gamma} \sqrt{dT \log \frac{\lambda + T/(1 - \gamma)^2}{\delta}} + \frac{2\gamma}{(1 - \gamma)^2} \sqrt{T \ln \frac{1}{\delta}} + \frac{2K(T)\gamma}{(1 - \gamma)^2} + \frac{2\gamma^U T}{1 - \gamma}. \tag{A.14}
\]
Substituting $\beta$, (A.14) into (A.1) and rearranging it, we have

$$\text{Regret}(T) \leq \frac{6}{1 - \gamma} \sqrt{dT \log \frac{\lambda + T/(1 - \gamma)^2}{\lambda}} \left(1 + \frac{1}{1 - \gamma} \sqrt{d \log \frac{\lambda(1 - \gamma)^2 + T d}{\delta \lambda(1 - \gamma)^2} + \sqrt{\lambda d}}\right) + \frac{2\gamma}{(1 - \gamma)^2} \sqrt{T \ln \frac{1}{\delta} + 2\gamma^{UT}} + \frac{2K(T)\gamma}{(1 - \gamma)^2} + \frac{2\sqrt{T \log 1/\delta}}{(1 - \gamma)^2} + \frac{4\gamma}{(1 - \gamma)^2} \leq \frac{6}{1 - \gamma} \sqrt{dT \log \frac{\lambda + T/(1 - \gamma)^2}{\lambda}} \left(1 + \frac{1}{1 - \gamma} \sqrt{d \log \frac{\lambda(1 - \gamma)^2 + T d}{\delta \lambda(1 - \gamma)^2} + \sqrt{\lambda d}}\right) + \frac{3\sqrt{T \log 1/\delta}}{(1 - \gamma)^2} + 1 + \frac{3d}{(1 - \gamma)^2} \log \frac{2\lambda + T d}{\lambda(1 - \gamma)^2} + \frac{4}{(1 - \gamma)^2},$$

where the last inequality holds due to Lemma A.5 and the fact that $U = \lfloor \log(T/(1 - \gamma)) / (1 - \gamma) \rfloor$. Taking an union bound of Lemma A.6, Lemma A.1, Lemma A.2 and Lemma A.3, we conclude the proof.

\[\square\]

### A.2 Proof of Theorem 5.5

In this section we prove Theorem 5.5. We set $\delta = 1 - \gamma$, $\Delta = d \sqrt{1 - \gamma} / (90\sqrt{2T})$. We only consider the case where $\mathcal{K}$ is a deterministic algorithm, since the regret result of the case where $\mathcal{K}$ is stochastic is lower bounded by that of the deterministic one. Let $N_0$ denote the total visit number to state $x_0$. Similarly, let $N_1$ denote the total visit number to state $x_1$, $N_0^a$ denote the total visit number to state $x_0$ with action $a$ and $N_0^{\tilde{A}}$ denote the total visit number to state $x_0$ with actions in a subset $\tilde{A}$ of the state space. Let $P_{\theta}(\cdot)$ denote the distribution over $\mathcal{S}^T$, where $s_1 = x_0$, $s_{t+1} \sim P_{\theta}(\cdot | s_t, a_t)$, $a_t$ is decided by $\mathcal{K}$. Let $E_{\theta}$ denote the expectation w.r.t. distribution $P_{\theta}$. We need the following lemmas. The first lemma suggests upper bounds for $E_{\theta}N_0$ and $E_{\theta}N_1$.

**Lemma A.9.** Suppose $2\Delta < \delta$ and $(1 - \delta) / \delta < T / 5$, then for $E_{\theta}N_1$ and $E_{\theta}N_0$, we have

$$E_{\theta}N_1 \leq \frac{T}{2} + \frac{1}{2\delta} \sum_a \langle a, \theta \rangle E_{\theta}N_0^a, \quad E_{\theta}N_0 \leq 4T/5.$$

The next lemma is a version of Pinsker’s inequality adapted in Jaksch et al. (2010), which upper bounds the total variation distance between two signed measure in terms of the Kullback-Leibler (KL) divergence.

**Lemma A.10** (Pinsker’s inequality). Denote $s = \{s_1, \ldots, s_T\} \in \mathcal{S}^T$ as the observed states from step 1 to $T$. Then for any two distributions $P_1$ and $P_2$ over $\mathcal{S}^T$ and any bounded function $f : \mathcal{S}^T \to [0, B]$, we have

$$E_1 f(s) - E_2 f(s) \leq \sqrt{\log 2 / 2B \sqrt{\text{KL}(P_2 \| P_1)}},$$

where $E_1$ and $E_2$ are expectations with respect to $P_1$ and $P_2$.

The next lemma gives the bound for KL divergence.
Lemma A.11. Suppose that $\theta$ and $\theta'$ only differs from $j$-th coordinate, $2\Delta < \delta \leq 1/3$. Then we have the following bound for the KL divergence between $P_\theta$ and $P_{\theta'}$:

$$KL(P_\theta \| P_{\theta'}) \leq \frac{16\Delta^2}{(d-1)^2}\mathbb{E}_{\theta}N_0.$$  

Now we begin our proof. The proof roadmap is similar to that in Jaksch et al. (2010) which aims to prove lower bound for tabular MDPs.

Proof of Theorem 5.5. First, we can verify that all assumptions in Lemma A.9 to Lemma A.11 are satisfied with the assumptions on $\gamma$ and $T$ and the choice of $\delta$ and $\Delta$. For a given $\theta$, the optimal policy for $M(S, A, \gamma, r, P_\theta)$ is to choose action $a = (\text{sgn}(\theta_i))$ at $x_0$ and to choose any action $a$ at state $x_1$. Therefore by the optimality Bellman equation, we know that $V^*(x_0)$ and $V^*(x_1)$ can be represented as follows

$$V^*(x_0) = \frac{\gamma(\Delta + \delta)}{(1-\gamma)(\gamma(2\delta + \Delta - 1) + 1)}, \quad V^*(x_1) = \frac{\gamma(\Delta + \delta) + 1 - \gamma}{(1-\gamma)(\gamma(2\delta + \Delta - 1) + 1)}. \quad (A.15)$$

Suppose we have an MDP $M(S, A, \gamma, r, P_\theta)$. During this proof the starting state $s_1$ is set to be $x_0$. For simplicity, let $\text{Regret}(\theta)$ denote $\text{Regret}(\mathcal{K}, M(S, A, \gamma, r, P_\theta), T)$ without confusion. Then recall the definition of regret, we have

$$\text{Regret}(\theta) = \sum_{t=1}^{T} \left[ V^*(x_t) - \sum_{t'=t}^{\infty} \gamma^{t'-t} r(x_{t'}, a_{t'}) \right]$$

$$= \sum_{t=1}^{T} V^*(x_t) - \sum_{t'=1}^{T} \gamma^{t'} r(x_{t'}, a_{t'}) \sum_{t=1}^{t'} \gamma^{-t} - \sum_{t'=T+1}^{\infty} \gamma^{t'} r(x_{t'}, a_{t'}) \sum_{t=1}^{T} \gamma^{-t}$$

$$\geq \sum_{t=1}^{T} V^*(x_t) - \frac{1}{1 - \gamma} \sum_{t'=1}^{T} \gamma^{-t} - \frac{1}{1 - \gamma} \sum_{t'=1}^{T} r(x_{t'}, a_{t'}) - \frac{\gamma}{(1 - \gamma)^2}, \quad (A.16)$$

where the last inequality holds due to the facts that $\sum_{t=1}^{T} \gamma^{-t} \leq (\gamma^{t'} - \gamma^{t'+1})^{-1}$ and $r(s, a) \leq 1$. Now we do the summation over $2^d-1$ possible $\theta$, then the expectation of regret can be written as follows:

$$\frac{1}{|\Theta|} \sum_{\theta} \left[ \mathbb{E}_\theta \text{Regret}(\theta) + \frac{\gamma}{(1 - \gamma)^2} \right]$$

$$\geq \frac{1}{|\Theta|} \sum_{\theta} \mathbb{E}_\theta \left[ N_0 V^*(x_0) + N_1 V^*(x_1) - \frac{1}{1 - \gamma} \sum_{t=1}^{T} r(s_1, a_t) \right]$$

$$= \frac{1}{|\Theta|} \sum_{\theta} \mathbb{E}_\theta \left[ N_0 \frac{\gamma(\Delta + \delta)}{\gamma(2\delta + \Delta - 1) + 1} + N_1 \left( \frac{\gamma(\Delta + \delta) + 1 - \gamma}{\gamma(2\delta + \Delta - 1) + 1} \right) \right]$$

$$= \frac{1}{|\Theta|} \sum_{\theta} \mathbb{E}_\theta \left[ N_0 \frac{\gamma(\Delta + \delta)}{\gamma(2\delta + \Delta - 1) + 1} + N_1 \left( \frac{\gamma(\Delta + \delta)}{\gamma(2\delta + \Delta - 1) + 1} \right) \right]$$

$$- \frac{1}{|\Theta|} \sum_{\theta} \mathbb{E}_\theta \left[ N_1 \frac{\gamma(\Delta + \delta)}{\gamma(2\delta + \Delta - 1) + 1} - N_1 \frac{\gamma(\Delta + 2\delta)}{\gamma(2\delta + \Delta - 1) + 1} \right]$$

$$= \frac{1}{1 - \gamma} \left[ \frac{\gamma(\Delta + \delta)}{\gamma(2\delta + \Delta - 1) + 1} - \frac{1}{1 - \gamma} \frac{\gamma(\Delta + \delta)}{\gamma(2\delta + \Delta - 1) + 1} \right] \sum_{\theta} \mathbb{E}_\theta N_1, \quad (A.17)$$

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where the first equality holds due to the representation of $V^*(x_0)$, $V^*(x_1)$ in (A.15) and the fact that $r(s_t, a_t) = 1$ for $s_t = x_1$ and $r(s_t, a_t) = 0$ for $s_t = x_0$. Next we are going to bound $|\Theta|^{-1} \sum_{\theta} \mathbb{E}_{\theta} N_1$. We have

$$
\frac{1}{|\Theta|} \sum_{\theta} \mathbb{E}_{\theta} N_1 \leq \frac{T}{2} + \frac{1}{2\delta|\Theta|} \sum_{\theta} \sum_{a} \langle a, \theta \rangle \mathbb{E}_{\theta} N_0^a
$$

$$
= \frac{T}{2} + \frac{1}{2\delta(d-1)|\Theta|} \sum_{j=1}^{d-1} \sum_{a} \sum_{\theta} \mathbb{E}_{\theta}(2 \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} - 1)N_0^a
$$

$$
\leq \frac{T}{2} + \frac{1}{2\delta(d-1)|\Theta|} \sum_{j=1}^{d-1} \sum_{a} \sum_{\theta} \mathbb{E}_{\theta} \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} N_0^a,
$$

(A.18)

where the first inequality holds due to Lemma A.9, the second inequality holds since $(2 \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} - 1) \leq 1\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\}$. From now on we focus on some specific $j \in [d-1]$. Taking $\theta'$ to be the only $j$-th coordinate differs from $\theta$, then

$$
\mathbb{E}_{\theta} \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} N_0^a + \mathbb{E}_{\theta'} \mathbb{1}\{\text{sgn}(\theta_j') = \text{sgn}(a_j)\} N_0^a
$$

$$
= \mathbb{E}_{\theta'} N_0^a + \mathbb{E}_{\theta} \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} N_0^a - \mathbb{E}_{\theta'} \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} N_0^a.
$$

(A.19)

Thus taking summation of (A.19) for all $a \in \mathcal{A}$ and $\theta \in \Theta$, we have

$$
2 \sum_{a} \sum_{\theta} \mathbb{E}_{\theta} \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} N_0^a
$$

$$
= \sum_{\theta} \sum_{a} \left[ \mathbb{E}_{\theta'} N_0^a + \mathbb{E}_{\theta} \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} N_0^a - \mathbb{E}_{\theta'} \mathbb{1}\{\text{sgn}(\theta_j) = \text{sgn}(a_j)\} N_0^a \right]
$$

$$
= \sum_{\theta} \left[ \mathbb{E}_{\theta'} N_0 + \mathbb{E}_{\theta} N_0^{A^0_j} - \mathbb{E}_{\theta'} N_0^{A^0_j} \right]
$$

$$
\leq \sum_{\theta} \left[ \mathbb{E}_{\theta'} N_0 + \frac{cT}{8} \sqrt{\text{KL}(P_{\theta'}||P_{\theta})} \right]
$$

$$
\leq \sum_{\theta} \left[ \mathbb{E}_{\theta'} N_0 + \frac{cT\Delta}{d\sqrt{\delta}} \sqrt{\mathbb{E}_{\theta} N_0} \right],
$$

(A.20)

where $\mathcal{A}^0_j$ is the set of $a$ which satisfies that $\text{sgn}(\theta_j) = \text{sgn}(a_j)$, $c = 4\sqrt{\log 2}$. The first inequality holds due to Lemma A.10 with the fact that $N_0^{A^0_j}$ is a function of $s_1, \ldots, s_T$ and $N_0^{A^0_j} \leq T$, the second inequality holds due to Lemma A.11. Substituting (A.20) into (A.18), we have

$$
\frac{1}{|\Theta|} \sum_{\theta} \mathbb{E}_{\theta} N_1 \leq \frac{T}{2} + \frac{\Delta}{4\delta(d-1)|\Theta|} \sum_{j=1}^{d-1} \sum_{\theta} \left[ \mathbb{E}_{\theta'} N_0 + cT \frac{\Delta}{d\sqrt{\delta}} \sqrt{\mathbb{E}_{\theta} N_0} \right]
$$

$$
= \frac{T}{2} + \frac{\Delta}{4\delta|\Theta|} \sum_{\theta} \left[ \mathbb{E}_{\theta'} N_0 + cT \frac{\Delta}{d\sqrt{\delta}} \sqrt{\mathbb{E}_{\theta} N_0} \right]
$$

$$
\leq \frac{T}{2} + \frac{\Delta T}{5\delta} + \frac{cT^3/2\Delta^2}{4d\delta^3/2},
$$

(A.21)
where the last inequality holds due to $\mathbb{E}_{\theta}N_0, \mathbb{E}_{\theta'}N_0 \leq 4T/5$ from Lemma A.9. Substituting (A.21) into (A.17), we have

$$
\frac{1}{|\Theta|} \sum_{\theta} \left[ \mathbb{E}_{\theta} \text{Regret}(\theta) + \frac{\gamma}{(1-\gamma)^2} \right]
\geq \frac{1}{1-\gamma} \frac{\gamma(\Delta + \delta)}{(2\delta + \Delta - 1) + 1} - \frac{1}{1-\gamma} \frac{\gamma(\Delta + 2\delta)}{\gamma(2\delta + \Delta - 1) + 1} \cdot \left( \frac{T}{2} + \frac{\Delta T}{5\delta} + \frac{cT^{3/2}\Delta^2}{4d\delta^{3/2}} \right)
$$

$$
= \frac{1}{(1-\gamma)(\gamma(2\delta + \Delta - 1) + 1)} \left[ \frac{\gamma \Delta T}{2} - \gamma(\Delta + 2\delta) \frac{\Delta T}{5\delta} - \gamma(\Delta + 2\delta) \frac{cT^{3/2}\Delta^2}{4d\delta^{3/2}} \right]
$$

$$
\geq \frac{1}{4(1-\gamma)^2} \left[ \frac{\gamma \Delta T}{2} - \gamma(\Delta + 2\delta) \frac{9\delta \Delta T}{4\delta^2} - \gamma \frac{9\delta cT^{3/2}\Delta^2}{4d\delta^{3/2}} \right]
$$

$$
= \frac{1}{4(1-\gamma)^2} \left[ \frac{1}{20} \gamma \Delta T - \gamma \frac{9cT^{3/2}\Delta^2}{16d\delta} \right]
$$

$$
= \frac{\gamma d\sqrt{T}}{1600c(1-\gamma)^{1.5}},
$$

where the second inequality holds since $\delta = 1 - \gamma$ and $\gamma(2\delta + \Delta - 1) + 1 \leq 1 - \gamma + 3\delta \gamma = 1 - \gamma + 3(1 - \gamma) \gamma \leq 4(1 - \gamma)$, the third inequality holds due to the fact that $4\Delta < \delta \leq 1/3$, the last inequality holds due to the choice of $\Delta$ and $\delta$. Therefore, there exists $\theta \in \Theta$ such that

$$
\mathbb{E}_{\theta} \text{Regret}(\theta) \geq \frac{\gamma d\sqrt{T}}{1600c(1-\gamma)^{1.5}} - \frac{\gamma}{(1-\gamma)^2}. \tag{A.22}
$$

Setting $\tilde{\theta} = (\theta^\top, 1)^\top \in \mathbb{R}^d$ completes our proof. \hfill \Box

## B Proof of lemmas in Section A.1

### B.1 Proof of Lemma A.1

**Proof of Lemma A.1.** For simplicity, let $A_t$ denote $V^K_t(s_t)$. Then it is easy to verify that $A_t \leq 1/(1-\gamma)$ for any $t$, and

$$
A_t - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'})
$$

$$
= r(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1}} [A_{t+1}] - \left( r(s_t, a_t) + \gamma \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{(t+1)+t'}, a_{(t+1)+t'}) \right)
$$

$$
= \gamma \mathbb{E}_{s_{t+1}} [A_{t+1}] - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{(t+1)+t'}, a_{(t+1)+t'}) \tag{B.1}
$$
where $\mathbb{E}_{s_{t+1}}[\cdot]$ denotes the expectation over all randomness starting from $s_{t+1}$. Therefore, taking summation of (B.1) from $t = 1$ to $T$, we have

$$
\sum_{t=1}^{T} \left( A_t - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \right)
= \gamma \sum_{t=2}^{T+1} \left( \mathbb{E}_{s_t} [A_t] - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \right)
\leq \gamma \sum_{t=1}^{T} \left( \mathbb{E}_{s_t} [A_t] - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \right) + \frac{4\gamma}{1-\gamma},
$$

(B.2)

where the inequality holds due to $0 \leq A_t, \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \leq 1/(1-\gamma)$. Note that $\{A_t - \mathbb{E}_{s_t} [A_t]\}$ forms a martingale difference sequence, then by Lemma A.6 and the fact that $0 \leq A_t \leq 1/(1-\gamma)$, we have that with probability $1-\delta$,

$$
\sum_{t=1}^{T} \left( \mathbb{E}_{s_t} [A_t] - A_t \right) \leq \frac{2\sqrt{T \log 1/\delta}}{1-\gamma}.
$$

(B.3)

Substituting (B.3) into (B.2), we have

$$
\sum_{t=1}^{T} \left( A_t - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \right)
\leq \gamma \sum_{t=1}^{T} \left( \mathbb{E}_{s_t} [A_t] - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \right) + \frac{4\gamma}{1-\gamma} + \frac{4T \log 1/\delta}{(1-\gamma)^2}.
$$

(B.4)

Rearranging (B.4), we have

$$
\sum_{t=1}^{T} \left( A_t - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \right) \leq \frac{2\sqrt{T \log 1/\delta}}{(1-\gamma)^2} + \frac{4\gamma}{(1-\gamma)^2}.
$$

(B.5)

Therefore, by (B.5) we have

$$
\text{Regret}(T) - \sum_{k=0}^{K(T)-1} \sum_{t_{k+1} = t_k}^{t_{k+1} - 1} [V^*(s_t) - V^K(s_t)]
= \sum_{k=0}^{K(T)-1} \sum_{t_{k+1} = t_k}^{t_{k+1} - 1} \left[ A_t - \sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t+t'}, a_{t+t'}) \right]
\leq \frac{K(T)}{(1-\gamma)^2} + \frac{2\sqrt{T \log 1/\delta}}{(1-\gamma)^2} + \frac{4\gamma}{(1-\gamma)^2}.
$$
B.2 Proof of Lemma A.2

Proof of Lemma A.2. Recall the definition of \( \hat{\theta}_k \) in Algorithm 1, we have

\[
\hat{\theta}_k = \left( \lambda I + \sum_{j=0}^{k-1} \sum_{i=t_j}^{t_{j+1}-1} \phi V_j(s_i, a_i) \phi V_j(s_i, a_i)^\top \right)^{-1} \left( \sum_{j=0}^{k-1} \sum_{i=t_j}^{t_{j+1}-1} \phi V_j(s_i, a_i) V_j(s_{i+1}) \right).
\]

It is worth noting that for any \( 0 \leq j \leq k-1 \) and \( t_j \leq i \leq t_{j+1} - 1 \),

\[
[\mathbb{P} V_j](s_i, a_i) = \int_{s'} \mathbb{P}(s'|s_i, a_i) V_j(s_i, a_i) ds'
= \int_{s'} \langle \phi(s'|s_i, a_i), \theta^* \rangle V_i(s') ds'
= \left\langle \int_{s'} \phi(s'|s_i, a_i) V_i(s'), \theta^* \right\rangle
= \langle \phi V_j(s_i, a_i), \theta^* \rangle,
\]

thus \( \{V_j(s_{i+1}) - \langle \phi V_j(s_i, a_i), \theta^* \rangle\} \) forms a martingale difference sequence. Besides, since \( V_j(s) \leq 1/(1 - \gamma) \) for any \( s \), then \( V_j(s_{i+1}) - \langle \phi V_j(s_i, a_i), \theta^* \rangle \) is a sequence of \( 1/(1 - \gamma) \)-subgaussian random variables with zero means. Meanwhile, we have \( \| \phi V_j(s_i, a_i) \|_2 \leq \sqrt{d}/(1 - \gamma) \) and \( \| \theta^* \|_2 \leq S \) by Definition 3.1. By Theorem 2 in Abbasi-Yadkori et al. (2011), we have that with probability at least \( 1 - \delta \), \( \theta^* \) belongs to the following set for all \( 1 \leq k \leq K \):

\[
\left\{ \theta : \left\| \Sigma_{t_k-1}^{1/2} (\theta - \hat{\theta}_k) \right\|_2 \leq \frac{1}{1 - \gamma} \sqrt{\frac{d \log \lambda(1 - \gamma)^2 + t_k d}{\delta \lambda(1 - \gamma)^2}} + \sqrt{\lambda S} \right\}.
\]

Finally, by the definition of \( \beta_k \) and the fact that \( \langle \theta^*, \phi(s'|s, a) \rangle = \mathbb{P}(s'|s, a) \) for all \( (s, a) \), we draw the conclusion that \( \theta^* \in B_k(s, a) \) for \( 1 \leq k \leq K \) and \( (s, a) \in S \times A \).

B.3 Proof of Lemma A.3

Proof of Lemma A.3. We use induction to prove this lemma. When \( u = 0 \), we have

\[
\frac{1}{1 - \gamma} = Q^{(0)}(s, a) \geq Q^*(s, a),
\]

where the inequality holds due to the fact that \( Q^*(s, a) \leq 1/(1 - \gamma) \) caused by \( 0 \leq r(s, a) \leq 1 \). Assume that Lemma A.3 holds for \( u \), then \( Q^{(u)}(s, a) \geq Q^*(s, a) \), which leads to \( V^{(u)}(s) \geq V^*(s) \). Furthermore, we have

\[
Q^{(u+1)}(s, a) = r(s, a) + \gamma \max_{\theta \in B(s, a)} \langle \theta, \phi V(s, a) \rangle = r(s, a) + \gamma \mathbb{P} V^{(u)}(s, a) \leq 1 + \frac{\gamma}{1 - \gamma} = \frac{1}{1 - \gamma},
\]
where the second inequality holds due to the definition of $B(s, a)$, the inequality holds due to the fact that $V^{(u)}(s) \leq 1/(1 - \gamma)$. We also have
\[
Q^{(u+1)}(s, a) = r(s, a) + \gamma \max_{\theta \in B(s, a)} \langle \theta, \phi_{V^{(u)}}(s, a) \rangle \\
\geq r(s, a) + \gamma \langle \theta^*, \phi_{V^{(u)}}(s, a) \rangle \\
= r(s, a) + \gamma [PV^{(u)}](s, a) \\
\geq r(s, a) + \gamma [PV^*](s, a) \\
= Q^*(s, a),
\]
where the first inequality holds because $\theta^* \in B(s, a)$ for any $(s, a) \in S \times A$ due to Lemma A.2, and the second inequality holds on because the induction assumption. Thus we have that Lemma A.3 holds for $u + 1$. Therefore, our conclusion holds.

\[\square\]

B.4 Proof of Lemma A.4

Proof of Lemma A.4. We first prove the following inequality:
\[
Q^{(U)}(s, a) - Q^{(U-1)}(s, a) \leq 2\gamma^{U-1}. \tag{B.9}
\]
By the update rule in Algorithm 2, for any $u \geq 2$, we have
\[
Q^{(u)}(s, a) = r(s, a) + \gamma \max_{\theta \in B(s, a)} \langle \theta, \phi_{V^{(u-1)}}(s, a) \rangle, \\
Q^{(u-1)}(s, a) = r(s, a) + \gamma \max_{\theta \in B(s, a)} \langle \theta, \phi_{V^{(u-2)}}(s, a) \rangle.
\]
Thus for any $(s, a) \in S \times A$, we have
\[
\left| Q^{(u)}(s, a) - Q^{(u-1)}(s, a) \right| = \gamma \max_{\theta \in B(s, a)} \left| \langle \theta, \phi_{V^{(u-1)}}(s, a) \rangle - \max_{\theta \in B(s, a)} \langle \theta, \phi_{V^{(u-2)}}(s, a) \rangle \right| \\
\leq \gamma \max_{\theta \in B(s, a)} \left| \langle \theta, \phi_{V^{(u-1)}}(s, a) \rangle - \phi_{V^{(u-2)}}(s, a) \rangle \right| \tag{B.10} \\
= \gamma \left| \tilde{\theta}, \phi_{V^{(u-1)}}(s, a) - \phi_{V^{(u-2)}}(s, a) \rangle \right| \\
= \gamma \left| \tilde{\mathbb{P}}[V^{(u-1)} - V^{(u-2)}](s, a) \right| \tag{B.11}
\]
where $\tilde{\theta}$ is the $\theta$ which attains the maximum of (B.10), $\tilde{\mathbb{P}}(s'|s, a) = \langle \tilde{\theta}, \phi(s'|s, a) \rangle$. The inequality holds due to the contraction property of max function. Then (B.11) can be further bounded as follows:
\[
\gamma \left| \tilde{\mathbb{P}}[V^{(u-1)} - V^{(u-2)}](s, a) \right| \leq \gamma \max_{s' \in S} \left| V^{(u-1)}(s') - V^{(u-2)}(s') \right| \\
= \gamma \max_{s' \in S} \max_{a' \in A} Q^{(u-1)}(s', a') - \max_{a' \in A} Q^{(u-2)}(s', a') \\
\leq \gamma \max_{(s', a') \in S \times A} \left| Q^{(u-1)}(s', a') - Q^{(u-2)}(s', a') \right|, \tag{B.12}
\]
where the first inequality holds due to the fact that $|\bar{P}f(s,a)| \leq \max_{s'\in S} |f(s')|$ for any $(s,a,s')$, the second inequality holds due to the contraction property of max function. Substituting (B.12) into (B.11) and taking the maximum over $(s,a)$, we have

$$\max_{(s,a)\in S\times A} \left| Q^{(u)}(s,a) - Q^{(u-1)}(s,a) \right| \leq \gamma \max_{(s,a)\in S\times A} \left| Q^{(u-1)}(s,a) - Q^{(u-2)}(s,a) \right|.$$  

Therefore, we have

$$\max_{(s,a)\in S\times A} \left| Q^{(U)}(s,a) - Q^{(U-1)}(s,a) \right| \leq \gamma^{U-1} \max_{(s,a)\in S\times A} \left| Q^{(1)}(s,a) - Q^{(0)}(s,a) \right| = \gamma^{U-1} \max_{(s,a)\in S\times A} \left| r(s,a) + \frac{\gamma}{1-\gamma} - \frac{1}{1-\gamma} \right| \leq 2\gamma^{U-1},$$

where the last inequality holds due to the fact that $0 \leq r(s,a) \leq 1$ for any $(s,a)$. Therefore we prove (B.9). To prove the original statement, we have

$$Q_k(s_t,a_t) = Q^{(U)}(s_t,a_t)$$

$$= r(s_t,a_t) + \gamma \max_{\theta \in B_k(s_t,a_t)} \langle \theta, \phi_{V_k(U-1)}(s_t,a_t) \rangle$$

$$= r(s_t,a_t) + \gamma \bar{P}V^{(U-1)}(s_t,a_t)$$

$$= r(s_t,a_t) + \gamma \bar{P}V^{(U)}(s_t,a_t) + \gamma \bar{P}[V^{(U-1)} - V^{(U)}](s_t,a_t)$$

$$\leq r(s_t,a_t) + \gamma \bar{P}V^{(U)}(s_t,a_t) + \gamma \max_{(s,a)\in S\times A} \left| Q^{(U)}(s,a) - Q^{(U-1)}(s,a) \right|$$

$$\leq r(s_t,a_t) + \gamma \bar{P}V^{(U)}(s_t,a_t) + 2\gamma^{U}$$

$$= r(s_t,a_t) + \gamma \langle \hat{\theta}, \phi_{V(U)}(s_t,a_t) \rangle + 2\gamma^{U},$$

(B.14)

where $\hat{\theta}$ is the $\theta$ which attains the maximum of (B.13), $\bar{P}(s'|s_t,a_t) = \langle \hat{\theta}, \phi(s'|s_t,a_t) \rangle$. The first inequality holds due to the fact that $|\bar{P}f(s_t,a_t)| \leq \max_{s'\in S} |f(s')|$ and $\max_s |V^{(U-1)}(s) - V^{(U)}(s)| \leq \max_{s,a} |Q^{(U-1)}(s,a) - Q^{(U)}(s,a)|$, the second inequality holds due to (B.9). Taking $\theta_t = \hat{\theta}$, our conclusion holds.

\[\square\]

**B.5 Proof of Lemma A.5**

Proof of Lemma A.5. For simplicity, we denote $K = K(T)$. Note that $\det(\Sigma_0) = \lambda^d$. We further have

$$\|\Sigma_T\|_2 = \left\| \lambda \mathbf{I} + \sum_{k=0}^{K-1} \sum_{t=t_k}^{t_{k+1}-1} \phi_{V_k}(s_t,a_t) \phi_{V_k}(s_t,a_t)^\top \right\|_2$$

$$\leq \lambda + \sum_{k=0}^{K-1} \sum_{t=t_k}^{t_{k+1}-1} \|\phi_{V_k}(s_t,a_t)\|_2^2$$

$$\leq \lambda + \frac{Td}{(1-\gamma)^2},$$

(B.15)
where the first inequality holds due to the triangle inequality, the second inequality holds due to the fact $V_k \leq 1/(1 - \gamma)$ from Lemma A.3 and Definition 3.1. (B.15) suggests that $\det(\Sigma_T) \leq (\lambda + Td/(1 - \gamma)^2)^d$. Therefore, we have

$$\left(\lambda + \frac{Td}{(1 - \gamma)^2}\right)^d \geq \det(\Sigma_T) \geq \det(\Sigma_{t_{K-1} - 1}) \geq 2^{K-1} \det(\Sigma_{t_0 - 1}) = 2^{K-1} \lambda^d,$$

(B.16)

where the second inequality holds since $\Sigma_T \succeq \Sigma_{t_{K-1} - 1}$, the third inequality holds due to the fact that $\det(\Sigma_{t_{k-1}}) \geq 2 \det(\Sigma_{t_{k-1} - 1})$ by the update rule in Algorithm 1. (B.16) suggests

$$K \leq d \log \frac{2\lambda + Td}{\lambda(1 - \gamma)^2}.$$

C Proof of lemmas in Section A.2

C.1 Proof of Lemma A.9

Proof of Lemma A.9. We have

$$\mathbb{E}_\theta N_1 = \sum_{t=2}^T \mathcal{P}_\theta(s_t = x_1)$$

$$= \sum_{t=2}^T \mathcal{P}_\theta(s_t = x_1 | s_{t-1} = x_1) \mathcal{P}_\theta(s_{t-1} = x_1) + \sum_{t=2}^T \mathcal{P}_\theta(s_t = x_1, s_{t-1} = x_0).$$

(C.1)

For $I_1$, since $\mathcal{P}_\theta(s_t = x_1 | s_{t-1} = x_1) = 1 - \delta$ no matter which action is taken, thus we have

$$I_1 = (1 - \delta) \sum_{t=2}^T \mathcal{P}_\theta(s_{t-1} = x_1) = (1 - \delta) \mathbb{E}_\theta N_1 - (1 - \delta) \mathcal{P}_\theta(s_T = x_1).$$

(C.2)

Next we bound $I_2$. We can further decompose $I_2$ as follows.

$$I_2 = \sum_{t=2}^T \sum_a \mathcal{P}_\theta(s_t = x_1 | s_{t-1} = x_0, a_{t-1} = a) \mathcal{P}_\theta(s_{t-1} = x_0, a_{t-1} = a)$$

$$= \sum_{t=2}^T \sum_a (\delta + \langle a, \theta \rangle) \mathcal{P}_\theta(s_{t-1} = x_0, a_{t-1} = a)$$

$$= \sum_a (\delta + \langle a, \theta \rangle) \left[ \mathbb{E}_\theta N_0^a - \mathcal{P}_\theta(s_T = x_0, a_T = a) \right].$$

(C.3)
Substituting (C.2) and (C.3) into (C.1) and rearranging it, we have

$$\mathbb{E}_\theta N_1 = \sum_a (1 + \langle a, \theta \rangle / \delta) \mathbb{E}_\theta N_0^a - \sum_a \left(1 - \frac{\delta}{\delta} \mathcal{P}_\theta(s_T = x_1) + \langle a, \theta \rangle / \delta \mathcal{P}_\theta(s_T = x, a_T = a)\right)$$

$$= \mathbb{E}_\theta N_0 + \delta^{-1} \sum_a \langle a, \theta \rangle \mathbb{E}_\theta N_0^a - \psi_\theta,$$

which suggests that

$$\mathbb{E}_\theta N_1 \leq T/2 + \delta^{-1} \sum_a \langle a, \theta \rangle \mathbb{E}_\theta N_0^a / 2. \quad \text{(C.4)}$$

We now bound \( \mathbb{E}_\theta N_0 \). By (C.4), we have

$$\mathbb{E}_\theta N_1 \geq \mathbb{E}_\theta N_0 + \delta^{-1} \sum_a \langle a, \theta \rangle \mathbb{E}_\theta N_0^a - \psi_\theta$$

$$\geq \mathbb{E}_\theta N_0 - \frac{\Delta}{\delta} \mathbb{E}_\theta N_0 - \frac{1 - \delta}{\delta} \mathcal{P}_\theta(s_T = x_1) - \mathcal{P}_\theta(s_T = x_0) - \frac{\Delta}{\delta} \mathcal{P}_\theta(s_T = x_0)$$

$$= (1 - \Delta/\delta) \mathbb{E}_\theta N_0 - (1 - \delta)/\delta + \frac{1 - \Delta}{\delta} \mathcal{P}_\theta(s_T = x_0)$$

$$\geq (1 - \Delta/\delta) \mathbb{E}_\theta N_0 - (1 - \delta)/\delta,$$

where the first inequality holds due to (C.4), the second inequality holds due to the fact that \( \langle a, \theta \rangle \leq \Delta \), the last inequality holds since \( \mathcal{P}_\theta(s_T = x_0) > 0 \). (C.6) suggests that

$$\mathbb{E}_\theta N_0 \leq \frac{T}{2} \left(1 - \frac{\delta}{\Delta} \right) \leq \frac{4}{5} T,$$

where the last inequality holds due to the fact that \( 2\Delta \leq \delta \) and \( (1 - \delta)/\delta < T/5 \).

### C.2 Proof of Lemma A.11

We need the following lemma:

**Lemma C.1 (Lemma 20 in Jaksch et al. (2010)).** Suppose \( 0 \leq \delta' \leq 1/2 \) and \( \epsilon' \leq 1 - 2\delta' \), then

$$\delta' \log \frac{\delta'}{\delta' + \epsilon'} + (1 - \delta') \log \left(\frac{1 - \delta'}{1 - \delta' - \epsilon'}\right) \leq \frac{2(\epsilon')^2}{\delta'}.$$

**Proof of Lemma A.11.** Let \( s_t \) denote \( \{s_1, \ldots, s_t\} \). By the Markovian property of MDP, we can first decompose the KL divergence as follows:

$$\text{KL}(\mathcal{P}_\theta || \mathcal{P}_\theta) = \sum_{t=1}^{T-1} \text{KL} \left[ \mathcal{P}_\theta(s_{t+1} | s_t) || \mathcal{P}_\theta(s_{t+1} | s_t) \right],$$

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where the KL divergence between $\mathcal{P}_{\theta'}(s_{t+1}|s_t), \mathcal{P}_{\theta}(s_{t+1}|s_t)$ is defined as follows:

$$\text{KL}\left[\mathcal{P}_{\theta'}(s_{t+1}|s_t)\left|\mathcal{P}_{\theta}(s_{t+1}|s_t)\right.\right] = \sum_{s_{t+1}\in S^{t+1}} \mathcal{P}_{\theta'}(s_{t+1}) \log \frac{\mathcal{P}_{\theta'}(s_{t+1})}{\mathcal{P}_{\theta}(s_{t+1})}.$$

Now we further bound the above terms as follows:

$$\sum_{s_{t+1}\in S^{t+1}} \mathcal{P}_{\theta'}(s_{t+1}) \log \frac{\mathcal{P}_{\theta'}(s_{t+1})}{\mathcal{P}_{\theta}(s_{t+1})}$$

$$= \sum_{s_t\in S^t} \mathcal{P}_{\theta'}(s_t) \sum_{x\in S} \mathcal{P}_{\theta'}(s_{t+1} = x|s_t) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = x|s_t)}{\mathcal{P}_{\theta}(s_{t+1} = x|s_t)}$$

$$= \sum_{s_{t-1}\in S^{t-1}} \mathcal{P}_{\theta'}(s_{t-1}) \sum_{x\in S, a\in A} \mathcal{P}_{\theta'}(s_t = x', a_t = a|s_{t-1})$$

$$\cdot \sum_{x\in S} \mathcal{P}_{\theta'}(s_{t+1} = x|s_{t-1}, s_t = x', a_t = a) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = x|s_{t-1}, s_t = x', a_t = a)}{\mathcal{P}_{\theta}(s_{t+1} = x|s_{t-1}, s_t = x', a_t = a)}.$$

When $x' = x_1$, $\mathcal{P}_{\theta'}(s_{t+1} = x|s_{t-1}, s_t = x', a_t = a) = \mathcal{P}_{\theta}(s_{t+1} = x|s_{t-1}, s_t = x', a_t = a)$ for all $\theta', \theta$ since the transition probability at $x_1$ is irrelevant to $\theta$ due to the MDP we choose. That implies when $x' = x_1$, $I_1 = 0$. Therefore,

$$\sum_{s_{t+1}\in S^{t+1}} \mathcal{P}_{\theta'}(s_{t+1}) \log \frac{\mathcal{P}_{\theta'}(s_{t+1})}{\mathcal{P}_{\theta}(s_{t+1})}$$

$$= \sum_{s_{t-1}\in S^{t-1}} \mathcal{P}_{\theta'}(s_{t-1}) \sum_{a} \mathcal{P}_{\theta'}(s_t = x_0, a_t = a|s_{t-1})$$

$$\cdot \sum_{x\in S} \mathcal{P}_{\theta'}(s_{t+1} = x|s_{t-1}, s_t = x_0, a_t = a) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = s|s_{t-1}, s_t = x_0, a_t = a)}{\mathcal{P}_{\theta}(s_{t+1} = s|s_{t-1}, s_t = x_0, a_t = a)}$$

$$= \sum_{a} \mathcal{P}_{\theta'}(s_t = x_0, a_t = a)$$

$$\cdot \sum_{x\in S} \mathcal{P}_{\theta'}(s_{t+1} = s|s_t = x_0, a_t = a) \log \frac{\mathcal{P}_{\theta'}(s_{t+1} = x|s_t = x_0, a_t = a)}{\mathcal{P}_{\theta}(s_{t+1} = x|s_t = x_0, a_t = a)}.$$

(C.7)

To bound $I_2$, due to the structure of the MDP, we know that $s_{t+1}$ follows the Bernoulli distribution over $x_0$ and $x_1$ with probability $1 - \delta - \langle a, \theta' \rangle$ and $\delta + \langle a, \theta' \rangle$, then we have

$$I_2 = (1 - \langle \theta', a \rangle - \delta) \log \frac{1 - \langle \theta', a \rangle - \delta}{1 - \langle \theta, a \rangle - \delta} + (\langle \theta', a \rangle + \delta) \log \frac{\langle \theta', a \rangle + \delta}{\langle \theta, a \rangle + \delta} \leq \frac{2(\langle \theta', a \rangle)^2}{(\langle \theta', a \rangle) + \delta},$$

(C.8)

where the inequality holds due to Lemma C.1 with $\delta' = \langle \theta', a \rangle + \delta$ and $\epsilon' = \langle \theta - \theta', a \rangle$. It can be verified that

$$\delta' = \langle \theta', a \rangle + \delta \leq \Delta + \delta \leq 1/2,$$

(C.9)

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where the first inequality holds due to the definition of $\theta'$, the second inequality holds since $\Delta < \delta/4 \leq 1/12$, the last inequality holds since $\delta' = (\theta', a) + \delta \leq \Delta + \delta$ due to the definition of $\theta'$.

(C.9) and (C.10) suggests that we can apply Lemma C.1 onto (C.8). $I_2$ can be further bounded as follows:

$$I_2 \leq \frac{4(\theta' - \theta, a)^2}{\delta} = \frac{16\Delta^2}{(d - 1)^2\delta},$$

(C.11)

where the inequality holds due to (C.8) and the fact that $\delta + (\theta', a) \geq \delta - \Delta \geq \delta/2$. Substituting (C.11) into (C.7), taking summation from $t = 1$ to $T - 1$, we have

$$\text{KL}(P_{\theta'} \| P_\theta) = \sum_{t=1}^{T-1} \sum_{s_{t+1} \in S_{t+1}} P_{\theta'}(s_{t+1}|s_t) \text{log} \frac{P_{\theta'}(s_{t+1}|s_t)}{P_\theta(s_{t+1}|s_t)} \leq \frac{16\Delta^2}{(d - 1)^2\delta} \sum_{t=1}^{T-1} \sum_{a} P_{\theta'}(s_t = x_0, a_t = a) = \frac{16\Delta^2}{(d - 1)^2\delta} \sum_{t=1}^{T-1} P_{\theta'}(s_t = x_0) \leq \frac{16\Delta^2}{(d - 1)^2\delta} \mathbb{E}_{\theta'} N_0,$$

where the last inequality holds due to the definition of $N_0$. \hfill \Box

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