The Martin boundary of an extension by a hyperbolic group

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We prove uniform Ancona-Gouëzel-Lalley inequalities for an extension by a hyperbolic group \(G\) of a Markov map which allows to deduce that the visual boundary of the group and the Martin boundary are Hölder equivalent. As application, we identify the set of minimal conformal measures of a regular cover of a convex-cocompact CAT(-1)-manifold with the visual boundary of the covering group, provided that this group is hyperbolic.

1 Introduction and statement of main results

A standard approach for the encoding of the behaviour at infinity of a transient random walk is to analyse the positive harmonic functions of the associated Markov operator. As observed by Martin in the context of partial differential equations (\([23]\)), these harmonic functions also have a topological description through potential theory. Namely, the harmonic functions are related to possible limits of the Green kernel at infinity, and hence, to the boundary of the minimal compactification such that the Green kernel extends to a continuous function on this new domain. However, as the construction only requires a well-defined Green function, it is applicable in a wide range of situations, from elliptic equations and transient Markov processes to conformal densities on CAT(-1)-spaces (see, e.g., \([24, 26, 27]\)). However, due to the generality of the approach, the explicit description of this Martin boundary for specific situations is often non-trivial. To give two examples of those explicit characterizations in the context of simple random walks on discrete groups, the Martin boundary of \(\mathbb{Z}^d\) is a singleton whereas the one of the free group coincides with its visual boundary (see, e.g., \([33]\)).

The other fundamental objects of this article are metric spaces which are hyperbolic in the sense of Gromov. The abstract definition of these spaces requires a uniform estimate of the Gromov product (see Section\(^{[4]}\), which can be reduced to a simple geometric property if the space is a geodesic space. That is, a geodesic space is \(\delta\)-hyperbolic if each side of an arbitrary triangle is always contained in the \(\delta\)-neighbourhoods of the other two. Furthermore,
each Gromov hyperbolic space comes with its visual boundary, which abstractly is defined through the asymptotic behaviour of the Gromov product (see Section 5), but also can be defined through geometric data coming from Busemann’s function in case of a geodesic space (see [9]). Due to this second characterisation and the fact that the boundary comes with a natural metric, the visual boundary today is the standard tool for encoding and analysing the behaviour of geodesics and horospherical foliations.

However, even though these boundary constructions differ completely as the first is based on abstract potential theory whereas the second uses geometric phenomena, it is known since the works of Ancona in [4] on elliptic operators on negatively curved manifolds and Gouëzel-Lalley in [17, 15] for symmetric, simple random walks on hyperbolic groups that the Martin and the visual boundary coincide in these cases. Ancona’s contribution, from a general viewpoint, is a strategy of proof, which allows to deduce a geometric description of the Martin boundary through minimal harmonic functions from the so called Ancona inequalities, whereas the work of Gouëzel and Lalley provides us with an argument which allows to obtain uniform Ancona inequalities and a relation between the Green kernel and the Gromov product (see Theorems 1.6 and 2.2 for the setting in here). A further geometrization related to simple random walks is due to Kaimanovich (20), who proved that the visual boundary of a Gromov hyperbolic group coincides with the Poisson boundary, that is the set of bounded harmonic functions. However, it is worth noting that, even though the statements are similar, the method of proof in [20] is based on a submultiplicative ergodic theorem (see, also, [21]) and therefore is applicable also to random walks whose transitions neither have to be symmetric nor finitely supported (for transitions with exponential tails, see [16]). On the other hand, as the method of Gouëzel and Lalley allows to study $\rho$-harmonic functions, where $1/\rho \geq 1$ is the radius of convergence of the Green function, these seemingly weaker results (in the generality of the results in here) give rise to applications to regular covers of negatively curved manifolds at their intrinsic exponent of convergence as in Theorem 1.1 below.

We now proceed with the setting and the statement of our main results. Throughout, we assume that $\vartheta : (\Sigma, \mu) \to (\Sigma, \mu)$ is a probability preserving, topologically mixing and noninvertible Markov map of a standard probability space with respect to a finite partition such that $\log d\mu \circ \vartheta / d\mu$ has a Hölder continuous representative (see Definitions 2.1 and 2.2). Furthermore, we throughout assume that $G$ is a discrete group and that $\kappa : \Sigma \to G$ is constant on the atoms of the Markov partition. The extension of $\vartheta$ by $G$, or group extension for short, is then defined by

$$T : \Sigma \times G \to \Sigma \times G, (x, g) \to (\vartheta(x), g \kappa(x)),$$

and is the key object of this article. Observe that, as $\mu$ is $\vartheta$-invariant, the product of $\mu$ and the Haar measure on $G$ is $T$-invariant. In particular, the transfer operator associated to $T$ can be written as, for $f : \Sigma \times G \to \mathbb{R}$ in a suitable function space,

$$\mathcal{L}(f)(x, g) = \sum_{T(y, h) = (x, g)} \frac{d\mu}{d\mu \circ \vartheta}(y) f(y, h)$$

and satisfies $\mathcal{L}(1) = 1$. It is worth noting here that group extensions are random walks with dependent increments by identifying $\mu\left\{ x \in \Sigma : T^n(x, g) \in \Sigma \times \{ h \} \right\}$ with the probability of a
transition from \( g \in G \) to \( h \in G \) in time \( n \). Moreover, as each simple random walk can be identified with a group extension, group extensions of Markov maps with finite partitions generalise simple random walks with respect to a finitely supported transition rule.

We now return to the general group extensions of Markov maps. In this setting, the Green function is no longer a function but an operator defined by
\[
G_r(f) := \sum_{n=0}^{\infty} r^n \mathcal{L}^n(f),
\]
where \( f \) is an element of a suitable function space (see Proposition 3.1) and \( r < 1/\rho \), where \( 1/\rho \) is the radius of convergence of \( r \mapsto G_r(1_{\Sigma \times \{\text{id}\}})(x, \text{id}) \). Furthermore, observe that, by general ergodic theory, the operator \( G_r \) can be extended to \( r = 1/\rho \), provided that the map \( T \) is totally dissipative.

We are now in position to specify the main objective of this article. That is, we are interested in relating the Martin boundary, that is the possible limits of \( G_r(f)(z_n)/G_r(1_{\Sigma \times \{\text{id}\}})(z_n) \) with the visual boundary of \( G \), where \( G \) is Gromov hyperbolic and \( 1 \leq r \leq 1/\rho \). As a first step, we prove a uniform Ancona-Gouëzel-Lalley inequality. That is, by combining the first part of Theorem 4.6 with Remark 4.2 one obtains the first main result which states that the Green operator is almost multiplicative along geodesics. For ease of exposition, the statement here is formulated in the presence of geodesics.

**Theorem A.** Assume that \( G \) is a non-elementary and word hyperbolic group, that \( T \) is a topologically transitive and that \( G_T(1_{\Sigma \times \{\text{id}\}})(\cdot, \text{id}) \neq G_T(1_{\Sigma \times \{\text{id}\}})(\cdot, g) \). Then for any \( D > 0 \) and \( g, z, h \in G \) such that the distance between \( z \) and the geodesic segment \( [g, h] \) is smaller than \( D \), and any \( r \in [1, 1/\rho] \),
\[
G_r(1_{\Sigma \times \{h\}})(\cdot, g) \simeq G_r(1_{\Sigma \times \{h\}})(\cdot, z) G_r(1_{\Sigma \times \{z\}})(\cdot, g).
\]

The proof of this theorem makes use of the strategy of Gouëzel and Lalley in [17, 15] adapted to the setting of group extensions, which required to develop a potential theory for conformal and excessive measures in order to perform the necessary substitution of subharmonic functions by excessive measures (cf. Section 7). Moreover, again by following [17, 15], it is possible to obtain an estimate for the fluctuations of the Green operator through the Gromov product (see the second part of Theorem 4.6).

The multiplicative estimate in Theorem A is probably the key result in here, as it allows to employ Ancona’s argument in order to obtain a geometric characterisation of the Martin boundary and the arguments in [17] to prove Hölder continuity of the Green kernel. The Martin boundary associated to general transient Markov shifts was introduced by Shwartz in [30] and is similar to the well-known construction based on Markov operators from probability theory, even though the building blocks of the boundary are \( \sigma \)-finite conformal measures instead of positive harmonic functions. In the context of group extensions, the Martin boundary is defined by
\[
\mathcal{M}_T := \left\{ (x, g) \in \Sigma \times G : T^n(x, g) \to \infty, \lim_{n \to \infty} \frac{G_r(1_{\Sigma \times \{h\}})(T^n(x, g))}{G_r(1_{\Sigma \times \{\text{id}\}})(T^n(x, g))} \right\}/\sim
\]
where \( T^n(x, g) \to \infty \) means that \( (T^n(x, g)) \) leaves any compact set and \( (x, g) \sim (\tilde{x}, \tilde{g}) \) that the limits in the definition coincide for all \( h \in G \). We also remark that our definition differs slightly
from the ones by Shwartz in [30, 29] where the defining class of functions is a dense subset of the set of Hölder functions with compact support instead of \(\{\Sigma \times \{h\} : h \in G\}\) as in our definition. We also would like to point out that similar results to Theorems A and B recently and independently were obtained by Shwartz in [29] for the more general setting of locally finite shifts which carry the structure of a hyperbolic graph. However, the method in there does not allow to include the case \(r = 1/\rho\).

Our second principal result characterises \(\mathcal{M}_r\) geometrically and reveals that the local influence of \(\Sigma\) vanishes as \(T^n(x, g) \to \infty\).

**Theorem B.** Under the assumptions of Theorem A and for any \(1 \leq r \leq 1/\rho\), the following holds. For each sequence \((\xi_n)\) in \(G\) converging to \(\sigma\) in the visual boundary \(\partial G\), the limit

\[
\mu_\sigma(f) := \lim_{n \to \infty} \frac{G_r(f)(x, \xi_n)}{G_r(1_{\Sigma \times \{\text{id}\}})(x, \xi_n)}
\]

exists for each Hölder function \(f\) with compact support, the limit only depends on \(\sigma\) and \(\mu_\sigma\) is a minimal conformal measure. Furthermore, the map \(\sigma \to \mu_\sigma\) is a bijection from \(\partial G\) to the set of minimal conformal measures.

In fact, we prove more. Theorem 5.2 also gives important application of the above identification like a representation of any conformal measure as a convex combination of minimal ones and the exponential decay of \(G_r(1_{\Sigma \times \{\text{id}\}})\), that is

\[
\limsup_{n \to \infty} \max_{y \in \Sigma, |\gamma| = n} \sqrt{G_r(1_{\Sigma \times \{\text{id}\}})(y, \gamma)} < 1.
\]

Furthermore, it also follows easily Theorem 5.2 that there is a bijection from \(\partial G\) to \(\mathcal{M}_r\). However, by a refinement of Theorem A one obtains a Hölder continuous version of this statement (see Theorem 5.4).

**Theorem C.** Under the assumptions of Theorem A and for any \(1 \leq r \leq 1/\rho\), the map \(\partial G \to \mathcal{M}_r\) induced by \(\sigma \to \mu_\sigma\) is a Hölder continuous bijection with Hölder continuous inverse with respect to a different exponent.

The above results have the following, canonical application to regular covers of convex-cocompact geodesic spaces through the coding construction in [8]. We recall the definition of a regular cover in our setting. Assume that \(X\) is a CAT(-1)-space (see, e.g., [11]) and that \(\Gamma\) is a discrete subgroup of the isometries of \(X\) which acts convex-cocompactly on \(X\). Then, as it is well-known, \(X/\Gamma\) is a local CAT(-1)-space with compact convex core. Now assume that \(Y\) is a cover of \(X/\Gamma\). We then refer to \(Y\) as a regular cover if there exists a normal subgroup \(N\) of \(\Gamma\) such that \(X/N\) and \(Y\) are isometric. In this setting, the above provides a complete description of the space of \(\delta(N)\)-conformal measures for \(N\) (for details and the proof, see Theorem 6.2). The following theorem both complements and extends the recent result by Shwartz in [29] for cocompact Fuchsian groups and \(s\)-conformal measures with \(s > \delta(N)\).

**Theorem D.** Assume that \(N\) is non-elementary, that \(G := \Gamma/N\) is word hyperbolic and that the geodesic flow associated with \(X/N\) is topologically transitive. Then the set of minimal, \(\delta(N)\)-conformal measures and \(\partial G\) coincide.
This result might be seen as a further contribution to a list of analogies between the ergodic behaviour of the geodesic flow on regular covers and random walks on the covering group. Namely, even though there does not exist a complete dictionary, there are several parallel results, like Rees’ version of Polya’s result on the transience of the simple random walk for \( Z^d \)-covers (25), or Brooks’ amenability criterium (6) in the sprit of Kesten (22).

2 Group extensions of Markov maps

Recall that a Markov map (or Markov fibred systems) is defined as follows (see, e.g. [2, 1]).

**Definition 2.1.** Suppose that \((\Omega, \mathcal{B}, \mu)\) is a standard probability space and \(\alpha\) is an at most countable partition of \(\Omega\) into measurable sets of strictly positive measure. We refer to \((\Omega, \theta, \mu, \alpha)\) as a Markov map if, for all \(a, b \in \alpha\),

(i) \(\theta|_{\alpha}: a \to \theta(a)\) is invertible, bimeasurable and non-singular;

(ii) either \(\mu(a \cap \theta(b)) = 0\) or \(\mu((\theta(b))^c) = 0\),

(iii) and, for \(\alpha_n := \{a_1 \cap \theta^{-1}a_2 \cap \ldots \cap \theta^{-n+1}a_n: a_i \in \alpha, i = 1, \ldots, n\}\), the \(\sigma\)-algebra generated by \(\{\alpha_n: n > 0\}\) is equal to \(\mathcal{B}\) up to sets of measure 0.

Each Markov fibred system is a factor of a topological Markov chain \((\Sigma, \theta)\) where \(\theta\) is the left shift \(\sigma\) acting on

\[
\Sigma := \{(a_i: i \in \mathbb{N}) : a_i \in \alpha \text{ and } \mu(a_{i+1} \cap \theta(a_i)) > 0 \text{ } \forall i = 1, 2, \ldots\}
\]

which can be deduced from the following. Set \(\mathcal{W}^1 := \alpha\), and for \(w_i \in \mathcal{W}^1 (i = 1, \ldots, n)\) we say that \(w = (w_1 \ldots w_n)\) is an admissible word of length \(n\) if \(\theta(w_i) = w_{i+1}\) for \(i = 1, \ldots, n - 1\). The set of admissible words of length \(n\) will be denoted by \(\mathcal{W}^n\); the length of \(w \in \mathcal{W}^n\) by \(|w|\) and the set of all admissible words by \(\mathcal{W}^\infty = \bigcup_n \mathcal{W}^n\). Then

\[
\mathcal{W}^n \to \alpha_n, \quad (w_1 \ldots w_n) \to [w_1 \ldots w_n] := \prod_{k=1}^n \theta^{-k+1}(w_k) \quad (1)
\]
defines a bijection between \(\mathcal{W}^n\) and \(\alpha_n\). Moreover, \(\mu^*\) combined with (iii) in Definition 2.1 allows to lift \(\mu\) to a probability measure \(\mu^*\) on \(\Sigma\) such that the limit of \(\mu^*\) as \(n\) tends to infinity defines a measure theoretical isomorphism between \((\Sigma, \mu^*, \sigma)\) and \((\Omega, \mu, \theta)\). Therefore, by abuse of notation, we identify both systems with \((\Sigma, \mu, \theta, \alpha)\).

An important consequence of this identification is that it allows to effectively describe the preimage structure and induces a topology on \(\Sigma\) such that \(\theta\) is uniformly expanding. That is, each \(w \in \mathcal{W}^n\) can be identified with an inverse branch of \(\theta^n\) as follows. Since \(\theta^n\) maps \(|w|\) injectively onto its image, its inverse \(\tau_w : \theta^n(|w|) \to |w|\) is well defined and by (i) in Definition 2.1

\[
0 < \varphi_w(x) := \frac{d\mu \circ \tau_w}{d\mu}(x) < \infty
\]
for $\mu$-a.e. $x \in \theta^n([w])$. Furthermore, $\Sigma$ comes with a canonical topology generated by $\{[w]: w \in \mathcal{H}^\infty\}$ which coincides with the topology induced by the metric $d_r$ defined by, for any $r \in (0, 1)$,

$$d_r((x_i), (y_i)) := r^{\min\{|x_i - y_i|\}}.$$ 

The topology allows to define topological transitivity and mixing as follows. We refer to $(\Sigma, \theta)$ as topologically transitive if for all $a, b \in \alpha$, there exists $n_{a,b} \in \mathbb{N}$ such that $\mu(\theta^{n_{a,b}}(a) \cap b) > 0$ and as topologically mixing if for all $a, b \in \alpha$, there exists $N_{a,b} \in \mathbb{N}$ such that $\mu(\theta^n(a) \cap b) > 0$ for all $n \geq N_{a,b}$.

**Definition 2.2.** We refer to the Markov map $(\Sigma, \theta, \mu, \alpha)$ as a Gibbs-Markov map of finite type if $\alpha$ is finite, $(\Sigma, \theta)$ is topologically mixing and there exists $C > 0$, $r \in (0, 1)$ such that, for all $w \in \mathcal{H}^\infty$ and a.e. $x, y \in X$,

$$|\log \varphi_w(x) - \log \varphi_w(y)| \leq Cd_r(x, y).$$

The key feature of a the Gibbs-Markov property stems from the fact that the transfer operator $\mathcal{L}_0 : L^1(\mu) \rightarrow L^1(\mu)$ of $\theta$, which is defined as the dual of $U_0 : L^\infty(\mu) \rightarrow L^\infty(\mu)$, $f \mapsto f \circ \theta$ and can be written as

$$\mathcal{L}_0(f) := \sum_{w \in \mathcal{H}} \varphi_w f \circ \tau_w,$$

acts with a spectral gap on the space of Hölder continuous functions. This implies that there exists a unique invariant probability $\mu$ absolutely continuous with respect to $\mu$ and, in particular, that $\log dm/d\mu$ is Hölder continuous and $(\Sigma, \theta, \mu, \alpha)$ also has the Gibbs-Markov property (see [2][1][23]).

Now suppose that $G$ is a discrete group and that $\kappa : \Sigma \rightarrow G$ is a map such that $\kappa$ is measurable with respect to $\alpha$. We then refer to the skew product

$$T : \Sigma \times G \rightarrow \Sigma \times G, (x, g) \mapsto (\theta(x), g \kappa(x))$$

as a group extension. Furthermore, observe that $\kappa^n : \Sigma \rightarrow G, x \mapsto \kappa(x) \cdot \kappa(\theta(x)) \cdots \kappa(\theta^{n-1}(x))$ is measurable with respect to $\alpha_n$. Therefore, for $w \in \mathcal{H}^n$, we define $\kappa_w$ as $\kappa_w := \kappa^n(x)$ for some $x \in [w]$. Moreover, by a slight abuse of notation, let $\tau_w$ also refer to the inverse branch of $T^n$ on $[w, g]$. That is $\tau_w(x, g) := (\tau_w(x), g \kappa_{w^{-1}})$, whenever $\tau_w(x)$ is well defined (i.e., $x \in \theta^n([x]))$. Moreover, for $f : \Sigma \times G \rightarrow \mathbb{R}$, set $f_w(x, g) := f \circ \tau_w(x, g)$ for $x \in \theta^n([x])$ and $f_w(x, g) = 0$ otherwise. The iterates of the transfer operator associated with the group extensions now can be written in short form as

$$\mathcal{L}^n(f,x,g) := \sum_{w \in \mathcal{H}^n} \varphi_w(x) f_w(x,g).$$

The following Lemma provides an important estimate for the distortion of the iterates of $\mathcal{L}$.

**Lemma 2.3.** Suppose that $T$ is a topologically transitive extension of a Gibbs-Markov map $\theta$ of finite type. Then, for each $h \in G$, there exist $K_h > 0$ and $N_h \in \mathbb{N}$ with the following property. For all $x, \tilde{x} \in \Sigma$, $g, \tilde{g} \in G$ with $h = g^{-1} \tilde{g}$, $m \in \mathbb{N}$, $L \geq 0$ and $f : \Sigma \times G \rightarrow [0, \infty)$ such that, for $z, \tilde{z}$ in the same cylinder of length $m$, either $f(z) = f(\tilde{z}) = 0$ or $|f(z)/f(\tilde{z}) - 1| \leq L$, there exists $k \leq N_h$ such that for all $n \geq m$

$$\mathcal{L}^n(f)(x,g) \leq (K_{\tilde{g}^{-1}g}(L+1)) \mathcal{L}^{n+k}(f)(\tilde{x}, \tilde{g}).$$
Proof. If \( g = \tilde{g} \) and \( x, \tilde{x} \) are in the same cylinder and \( n \geq m \), then

\[
\left| \mathcal{L}^n(f)(x, g) - \mathcal{L}^n(f)(\tilde{x}, \tilde{g}) \right| \\
\leq \sum_{v \in \mathcal{W}^n} \left| \left( \varphi_v(x) - \varphi_v(\tilde{x}) \right) f_v(x, g) \right| + \sum_{v \in \mathcal{W}^n} \varphi_v(\tilde{x}) \left| \left( f_v(x, g) - f_v(\tilde{x}, \tilde{g}) \right) \right| \\
\leq C_\varphi d_r(x, \tilde{x}) \sum_{v \in \mathcal{W}^n} \varphi_v(x) |f_v(x, g)| + C_\varphi \sum_{v \in \mathcal{W}^n} \varphi_v(x) |f_v(x, g)| \cdot \left| 1 - \frac{f_v(x, g)}{f_v(\tilde{x}, \tilde{g})} \right| \\
\leq C_\varphi \left( d_r(x, \tilde{x}) + \sup_{v \in \mathcal{W}^n} \left| 1 - \frac{f_v(x, g)}{f_v(\tilde{x}, \tilde{g})} \right| \right) \mathcal{L}^n(f)(x, g) \leq C_\varphi (1 + L) \mathcal{L}^n(f)(x, g).
\]

Now assume that \( x, \tilde{x} \) are not in the same cylinder. Then, by transitivity, there exists \( y \) in the same cylinder as \( x \) and \( k \in \mathbb{N} \) such that \( T^k(y, g) = (\tilde{x}, \tilde{g}) \), or, equivalently, there exists \( v \in \mathcal{W}^k \) with \( (y, g) = \tau_v(\tilde{x}, \tilde{g}) \). Hence, by the above,

\[
\mathcal{L}^{n+k}(f)(\tilde{x}, \tilde{g}) \geq \varphi_v(\tilde{x}) \mathcal{L}^n(f)(y, g) \geq \frac{\varphi_v(\tilde{x})}{1 + C_\varphi (L + 1)} \mathcal{L}^n(f)(x, g).
\]

Moreover, as \( \theta \) is of finite type, \( v \in \mathcal{W}^k \) can be chosen within a finite set, which proves the assertion for \( (x, g) \) and \( (\tilde{x}, \tilde{g}) \) with respect to \( K_{id} + K_{id} L \), for some \( K_{id} \) sufficiently large. The proof of the general case is almost the same: For \( (\tilde{x}, \tilde{g}) \), there exists by transitivity \( w \in \mathcal{W}^k \) such that \( g^{-1}x^k(w) = \tilde{g} \) and \( \tilde{x} \in \theta^k([w]) \). As \( \theta \) is of finite type and \( \kappa_w = g^{-1} \tilde{g} \), \( w \) again can be chosen from a finite set. Moreover, \( \tau_w(\tilde{x}, \tilde{g}) \in \Sigma \times \{g\} \). Hence, by the above, for some \( l \leq N_{id} \) and any \( n \geq m \),

\[
\mathcal{L}^{n+k+l}(f)(\tilde{x}, \tilde{g}) \geq \varphi_w(\tilde{x}) \mathcal{L}^{n+l}(f)(\tau_w(\tilde{x}, \tilde{g}) \geq \frac{\varphi_w(\tilde{x})}{K_{id} + K_{id} L} \mathcal{L}^n(f)(x, g).
\]

The assertion of the Lemma then follows from this. \( \square \)

3 The Green operator

In analogy to the Green functions in the theory of random walks, we formally define a family of operators by

\[
\mathcal{G}_r := \sum_{n=0}^{\infty} r^n \mathcal{L}^n.
\]

In order to be able to specify invariant function spaces, first observe that by invariance of \( \mu \), we have that \( \mathcal{L}(1) = 1 \) and hence, \( \mathcal{G}_r(1) = (1-r)^{-1} 1 \) for \( r \in (0, 1) \). However, \( \mathcal{G}_r \) might act on functions with compact support for some \( r \geq 1 \). In order to determine the critical value for this action, set \( R := 1/\rho \), where

\[
\rho := \limsup_n \sqrt[n]{\mathcal{L}^n(\chi_g)(x, h)} \text{ with } \chi_g := \mathbf{1}_{\Sigma \times \{g\}}.
\]

Observe that, by transitivity, \( \rho \) does not depend on \( g, h \in G \) and \( x \in \Sigma \). In particular, \( \mathcal{G}_r(\chi_g) \) is a finite function for each \( g \in G \) and \( 0 \leq r < R \) by Hadamard’s formula for the radius of
convergence of a power series. Hence, if \( \rho < 1 \) (which holds if \( G \) is nonamenable, see [31]), we have to consider values of \( r \) bigger than 1.

The main idea behind the construction of an invariant space is to consider functions, whose local Hölder coefficients are non-constant and dominated by positive eigenfunctions of \( \mathcal{L} \). That is, for \( \alpha > 0 \) and \( f : \Sigma \times G \to \mathbb{R} \), we define a function \( D_\alpha(f) : X \to [0, \infty] \) which is constant on cylinders of length 1 by

\[
D_\alpha(f)(z, g) := \sup_{x, y, [a, g]} \frac{|f(x) - f(y)|}{d(x, y)\alpha} \quad \text{for all } z \in [a].
\]

Furthermore, we refer to \( E_{\rho} \) as the set of all positive, Hölder continuous \( \rho \)-subharmonic functions, that is \( E_{\rho} := \{ h : \| D_\alpha(h) \|_\infty < \infty, h > 0, \mathcal{L}(h) \leq \rho h \} \), which is non-empty for \( \alpha \) equal to the Hölder exponent of \( \log \phi \) if \( |\mathcal{W}| < \infty \) by the main result in [32]. We are now in position to define the following spaces of Hölder continuous and locally Hölder continuous functions.

\[
\mathcal{H}_\alpha := \{ f : X \to \mathbb{R} | \| f \|_\infty < \infty, \| D_\alpha(f) \|_\infty < \infty \}
\]

\[
\mathcal{H}_{\alpha, \text{loc}} := \{ f : X \to \mathbb{R} | \exists h \in E_{\rho} \text{ s.t. } |f| \leq h, D_\alpha(f) \leq h \}
\]

**Proposition 3.1.** Suppose that \( \| D_\alpha(\log \phi) \|_\infty < \infty \) and that \( \mathcal{L}_\rho(1) = 1 \). Then \( \mathcal{G}_r \) acts on \( \mathcal{H}_\alpha \) as a bounded operator with respect to \( \| \cdot \|_\infty + \| D_\alpha(\cdot) \|_\infty \) for each \( r \in [0, 1] \). For \( r \in [0, R) \), \( \mathcal{G}_r \) acts on \( \mathcal{H}_{\alpha, \text{loc}} \) and there exists a constant \( C > 0 \) such that, for all \( f \in \mathcal{H}_{\alpha, \text{loc}} \) and \( h \in E_{\rho} \) with \( |f| \leq h \) and \( D_\alpha(f) \leq h \),

\[
|\mathcal{G}_r(f)| \leq \frac{R}{R - r} h, \quad D_\alpha(\mathcal{G}_r(f)) \leq \frac{CR}{R - r} h.
\]

**Proof.** For \( r < 1 \) and \( f \in \mathcal{H}_\alpha \), we have that

\[
\| \mathcal{G}_r(f) \|_\infty \leq \| f \|_\infty \| \mathcal{G}_r(1) \|_\infty = \| f \|_\infty \sum_n r^n = \| f \|_\infty (1 - r)^{-1}.
\]

It remains to show for the first part that \( \| D_\alpha(\mathcal{G}_r(f)) \|_\infty < \infty \). In order to do so, assume that \( x, y \in [v] \) for some \( v \in \mathcal{W} \) and recall that, by the Gibbs-Markov property, there exists \( C_{\phi, v} \), independent of \( x, y \) and \( w \in \mathcal{W} \) such that \( |1 - \varphi_v(x)/\varphi_v(y)| \leq C_{\phi, v} d(x, y)^{\alpha} \) and \( \varphi_v(x)/\varphi_v(y) \leq C_{\phi} \). Hence,

\[
|\mathcal{G}_r(f)(x, g) - \mathcal{G}_r(f)(y, g)| / d(x, y)^\alpha \leq \frac{1}{d(x, y)^\alpha} \sum_{n \in \mathbb{N}, w \in \mathcal{W}} r^n \left| \varphi_v(x) - \varphi_v(y) \right| \left| f_w(x, g) - f_w(y, g) \right| \leq \sum_{n \in \mathbb{N}, w \in \mathcal{W}} r^n C_{\phi} |\varphi_v(w)| \| f \|_\infty + \frac{r^n}{2\alpha} |\varphi_v(w)| \| D_\alpha(f) \|_\infty = \frac{C_{\phi}}{1 - r} \| f \|_\infty + \frac{1}{1 - \frac{r}{2\alpha}} \| D_\alpha(f) \|_\infty.
\]

Suppose that \( r < R \) and \( f \in \mathcal{H}_{\alpha, \text{loc}} \). Then there is \( h \in E_{\rho} \) with \( |f| \leq h \) and \( D_\alpha(f) \leq h \), and

\[
|\mathcal{G}_r(f)| \leq \mathcal{G}_r(1) \leq \sum_{n=0}^{\infty} r^n \mathcal{L}_\rho^n(h) = \sum_{n=0}^{\infty} (\rho r)^n h = \frac{R}{R - r} h.
\]
Hence, it remains to show that $D_\alpha(G_r(f)) \ll h$. By similar arguments, for $x, y \in [v]$ for some $v \in \mathcal{Y}$,

$$\left| G_r(f)(x, g) - G_r(f)(y, g) \right| / d(x, y)^\alpha \leq \frac{1}{d(x, y)^\alpha} \sum_{n \in \mathbb{N}, w \in \mathcal{Y}^n} r^n \left| \varphi_{w}(x) - \varphi_{w}(y) \right| \left| f_w(x, g) \right| + \varphi_{w}(y) \left| f_w(x, g) - f_w(y, g) \right|$$

$$\leq \sum_{n \in \mathbb{N}, w \in \mathcal{Y}^n} r^n C_q \varphi_{w}(y) \left| f_w(x, g) \right| + r^n \frac{2^\alpha}{2^\alpha} \varphi_{w}(y) D_a(f) \left( \tau_w(x) g \kappa^{-1}_w \right)$$

$$\leq C_q \left( G_r([f])(x, g) + G_{2^{-a}R}(D_a(f))(x, g) \right) \leq C_q \left( \frac{R}{R - r} + \frac{2^\alpha R}{2^\alpha R - r} \right) h(x, g).$$

In order to extend the action further to $r = R$, we introduce the following notion of transience in analogy to the theory of random walks. Now suppose that $\mu$ is a non-singular measure on $X$ with respect to $T$. We then refer to $T$ as transient or $\rho$-transient if $G_R(X_id)(x, id) < \infty$ for all $x \in \Sigma$.

**Proposition 3.2.** Assume that $T$ is a topologically transitive, $\rho$-transient extension of a Gibbs-Markov map of finite type. Then, for $f \in \mathcal{H}_{\infty}$ and $A \subset G$ finite, we have that $G_R(X_A) \in E_\rho$, $G_R(X_A \cdot f) \in \mathcal{H}_{\infty}$, and

$$\| G_r(X_A \cdot f) \| \leq \| X_A \cdot f \| \cdot \| G_R(X_A) \|,$$

$$D_a(G_R(X_A \cdot f)) \leq C_q \| X_A \cdot f \| \cdot \| G_R(X_A) \| + \| D_a(X_A \cdot f) \| \cdot \| G_{2^{-a}R}(X_A) \|.$$

**Proof.** It follows from Lemma 2.3 that $G_R(X_id)(x, g) < \infty$ for all $(x, g) \in G$. As $G_R(X_id)(x, g)$ is left invariant under multiplication by elements of $G$, it follows that $G_R(X_id)(x, h)$ is finite for all $h \in G$. Hence, $G_R(X_A)(x, h) < \infty$ and, as it easily can be verified, $G_R(X_A) \in E_\rho$. The remaining assertions follow as in the proof above.  

**Remark 3.3** Recall that a group is non-amenability if a strong isoperimetric inequality holds, that is

$$\inf \left\{ \frac{|gA \Delta A|}{|A|} : A \subset G, |A| < \infty \right\} > 0.$$ 

Moreover, non-amenability implies that $\rho < 1$ by [31] and $T : \Sigma \times G \to \Sigma \times G$ can not be conservative and ergodic for any measure by a result of Zimmer ([35]) which implies that $T$ is transient (see, e.g., [32, Prop. 2]). Hence, if $G$ is non-amenability, then $R > 1$ in Proposition 3.1 and the assertions of Proposition 3.2 hold.

As a consequence of Lemma 2.3, one immediately obtains the following independence of $G_r(f)(x, g)$ from $x$.

**Lemma 3.4.** Assume that $T$ is a topologically transitive, $\rho$-transient extension of a Gibbs-Markov map of finite type and that there are $m \in \mathbb{N}$, $L \geq 0$ and $f : \Sigma \times G \to [0, \infty)$ such that, for $z, \tilde{z}$ in the same cylinder of length $m$, either $f(z) = f(\tilde{z}) = 0$ or $|f(z) / f(\tilde{z}) - 1| \leq L$. Then, for $1 \leq r \leq R$ and $f$ such that $G_r(f)(x, id) < \infty$,

$$(K_{r^{-1}} L + 1)^{-1} G_r(f)(y, h) \leq G_r(f)(x, g) \leq K_{r^{-1}} L + 1) G_r(f)(y, h).$$

9
Proof. It follows from Lemma 2.3 that, from some \( k \in \mathbb{N} \),
\[
G_r(f)(x, g) = \sum_{n=0}^{\infty} r^n L^n(f)(x, g) \leq K_{h^{-1}}(L + 1) \sum_{n=0}^{\infty} r^n L^{n+k}(f)(y, h)
\]
\[
= \frac{K_{h^{-1}}(L + 1)}{r^k} \sum_{n=k}^{\infty} r^n L^n(f)(y, h) \leq K_{h^{-1}}(L + 1)G_r(f)(y, h).
\]
The second part follows from interchanging the roles of \((x, g)\) and \((y, h)\).

4 Ancona-Gouëzel inequalities for extensions by word hyperbolic groups

Hyperbolic groups were introduced by Gromov (18) in order to unify the theory of groups with a certain notion of negative curvature. In here, we exclusively consider the word metric on \( G \) which we recall now. For a fixed, finite set \( g \) of generators for \( G \), the word metric is defined by
\[
d(g, h) = \min\{k : g a_1 \ldots a_k = h \text{ or } h a_1 \ldots a_k = g, a_i \in g\}.
\]
In general, a metric space \((G, d)\) is referred to as Gromov hyperbolic or \( \delta \)-hyperbolic in the sense of Gromov if \((G, d)\) is a geodesic space and there exists \( \delta > 0 \) such that
\[
(x \cdot z)_o \geq \min\{(x \cdot y)_o, (y \cdot z)_o\} - \delta,
\]
for all \( x, y, z, o \in G \). In here, \((x \cdot y)_o \) refers to the Gromov product defined by
\[
(x \cdot y)_o := \frac{1}{2} (d(x, o) + d(y, o) - d(x, y)).
\]

In this situation, as \( d \) is the word metric, one refers to \( G \) as word hyperbolic. Important features of Gromov hyperbolic spaces are that triangles are \( 4\delta \)-thin and that the Cayley graph of \( G \) can be uniformly approximated by trees in the following sense (see Theorem 12 in [13]).

Lemma 4.1. Let \((M, d)\) be \( \delta \)-hyperbolic and \( F \subset M \) with \(|F| \leq 2^k + 2 \) and \( o \in M \). Then there exists a finite, rooted tree \( T \) and \( \Theta : F \rightarrow T \) such that

(i) \( d(x, o) = d(\Theta(x), \Theta(o)) \) for all \( x \in F \),

(ii) \( d(x, y) - 2k\delta \leq d(\Theta(x), \Theta(y)) \leq d(x, y) \) for all \( x, y \in F \).

Remark 4.2 A further important property of hyperbolic groups is related to non-amenability (see Rem. [33]). First recall that a group \( G \) is elementary if \( G \) has a cyclic subgroup of finite index. Then, a world hyperbolic group is either elementary (and therefore amenable) or non-elementary and non-amenable. This result is well known and can be deduced e.g. by combining Theorem A in [7] with the observation that the distance between two elements in \( G \) is uniformly bounded from below. Hence, Remark [33] implies that for any non-elementary, word hyperbolic group \( G \), the group extension \( T \) is transient and \( R > 1 \).
For extensions by hyperbolic groups, we now prove a strong version of Ancona's inequality as in [17] and [15] in the setting of random walks. Therefore, we first consider the operator \( \mathcal{H}_r \) defined by

\[
\mathcal{H}_r(f_1, f_2) := \mathcal{G}_r(f_1 \cdot \mathcal{G}_r(f_2)).
\]

**Lemma 4.3.** For \( r \in (0, R) \), \( \mathcal{H}_r : \mathcal{H} \times \mathcal{H}_{loc} \to \mathcal{H}_{loc} \). If \( T \) is transient, then \( \mathcal{H}_r(X_Af_1, X_Bf_2) \in \mathcal{H}_{loc} \) for all \( f_1 \in \mathcal{H} \) and \( A, B < G \) finite.

**Proof.** As \( \mathcal{G}_r \) acts on \( \mathcal{H}_{loc} \) by Proposition 3.1, it remains to observe that \( f \mathcal{G}_r \in \mathcal{H}_{loc} \) for \( f \in \mathcal{H} \) and \( g \in \mathcal{H}_{loc} \) in order to obtain that \( \mathcal{H}_r \) is well defined. For \( r = R \), Proposition 5.2 implies that \( \mathcal{G}_R(X_Bf_2) \in \mathcal{H}_{loc} \). Hence, as \( X_Af_1 \mathcal{G}_R(X_Bf_2) \in \mathcal{H} \) by finiteness of \( A \), another application of Proposition 5.2 shows that \( \mathcal{H}_R(X_Af_1, X_Bf_2) \in \mathcal{H}_{loc} \). \( \square \)

**Lemma 4.4.** Assume that \( G \) is a hyperbolic group. If \( T \) is transient, then

\[
\sup \left\{ \sum_{|g| = k} |\mathcal{H}_r(X_g, X_{id})(x, id)| \mid k \in \mathbb{N}, x \in \Sigma \right\} < \infty.
\]

**Proof.** The proof reads in verbatim as the one of Lemma 2.5 in [15]. The only differences are that \( \mathcal{H}_r \) now is an operator, well defined by Lemma 4.3 and that one has to apply once Lemma 3.4 in the estimates up to a constant. \( \square \)

We now adapt the principal estimate for obtaining the strong version of Ancona’s inequalities in [15] to our operator setting. In order to do so, for \( A < G \), set

\[
\mathcal{G}_r(f | A) = \sum_{n=0}^{\infty} r^n \mathcal{L}^n \left( f \cdot \prod_{k=1}^{n-1} X_A \circ T^k \right).
\]

That is, \( \mathcal{G}_r(f | A) \) corresponds to the sum of those paths, which stay in \( \Sigma \times A \). In analogy to [15], the following estimate holds.

**Lemma 4.5.** Assume that \( G \) is hyperbolic, \( T \) is a topologically transitive, transient extension of a Gibbs-Markov map of finite type and that \( \mathcal{G}_R(X_g)(x, id) = \mathcal{G}_R(X_{id})(x, g) \) independent of \( (x, g) \in \Sigma \times G \). Then there exists \( n_0 \in \mathbb{N} \) and \( \lambda > 1 \) such that for any \( n > n_0, g, id, h \in G \) on a geodesic segment (in this order) with \( d(g, id) > n, d(id, h) > n \), we have that

\[
\mathcal{G}_R(X_h|B(id, n)\mathcal{L}^c)(x, g) \leq 2^{-\lambda n} \text{ for all } x \in \Sigma.
\]

**Proof.** As the proof reads in almost all parts in verbatim as the one of Lemma 2.6 in [15], we again only indicate the necessary adaptations. The proof is based on a sequence of barriers \( A_i \) such that the operator norm of \( L_i : \ell^2(A_{i+1}) \to \ell^2(A_i) \) is smaller than \( 1/2 \). These operators in the context considered in here have to be defined by, for \( a \in A_i \) and \( f \in \ell^2(A_{i+1}) \),

\[
L_i(f)(a) = \sum_{b \in A_{i+1}} \left( \int \chi_a \mathcal{G}_R(X_b) d\mu \right) f(b).
\]

Furthermore, it follows from \( \mathcal{G}_R(X_g)(x, id) = \mathcal{G}_R(X_{id})(g, x) \) and Lemma 3.4 that

\[
\sum_{|g| = k} (\mathcal{G}_R(X_g)(x, id))^2 = C^{|e|} \sum_{|g| = k} \mathcal{H}_R(X_g, X_{id})(x, id).
\]
As the method of proof in [15] allows to construct barriers the \( A_i \) such that \( \|L_i\|_{\ell^2} \) is arbitrary small, it is possible to absorb the constants \( C \) and the one arising from a further application of Lemma 3.4.

We are now in position to prove the main result of this section which generalizes the results for random walks with independent increments for cocompact Fuchsian groups in [17, Th. 4.6] and hyperbolic groups in [15, Th. 2.9] to group extensions. The proof is an adaption of the arguments in [17] to the setting of Green operators. In particular, as the proof of the exponential decay in the second part relies on a potential theoretic argument, it turns out to be necessary to replace the concept of minimal harmonic functions by its dual, that is by minimal conformal measures as defined in Section 7.

**Theorem 4.6.** Assume that \( G \) is hyperbolic, \( T \) is a topologically transitive, transient extension of a Gibbs-Markov map of finite type such that \( G_R(\chi_g)(x, \text{id}) = G_R(\chi_{\text{id}})(x, g) \), independent of \((x, g) \in \Sigma \times G\).

1. **Uniform Ancona inequality.** For any \( D > 0 \), there exists \( C > 0 \) such that for any \( g, h \in G \), \( x \in \Sigma \) and any \( z \in G \) such that the distance between \( z \) and a path from \( g \) to \( h \) of length \( d(g, h) \) is smaller than \( D \), and any \( r \in [1, R] \),

\[
G_r(\chi_z)(x, g) \leq CG_r(\chi_z G_r(\chi_h))(x, g). \tag{3}
\]

2. **Gouëzel-Lalley inequality.** There exist \( C > 0 \) and \( \lambda \in (0, 1) \) such that for any \( r \in [1, R] \), for any \( x, y \in \Sigma \) and for any \( g, g', h, h' \in G \) in a configuration approximated by a tree as shown below, then

\[
\left| \frac{G_r(\chi_h)(x, g)/G_r(\chi_h)(y, g')}{G_r(\chi_{\text{id}})(x, g)/G_r(\chi_{\text{id}})(y, g')} - 1 \right| \leq C\lambda^n.
\]

![Figure 1: Configuration of \( g, g', h, h' \in G \)](image)

**Proof.** We closely follow the proofs of Theorems 4.1, 4.3 and 4.6 in [17]. In here, we also make use of the following notational convention as in [13]. Even \( G \) is not necessarily a geodesic space, there is, for any pair \( g, h \in G \) with distance \( d = d(g, h) \), a path of length \( d \) from \( g \) to \( h \) in the Cayley graph. By identifying the edges with copies of \([0,1]\), one obtains a continuous path \( \gamma : [0, d] \to G \) from \( g \) to \( h \) which is an isometry. As \( \gamma \) not necessarily is uniquely determined, we refer to \( \gamma \) as a geodesic from \( g \) to \( h \) and denote it by \([g, h]\). Furthermore, in order to slightly simplify the parameters, we assume that \( G \) is \( \delta/4 \)-hyperbolic in order to guarantee that triangles are \( \delta \)-thin, that is \([g, h]\) is always contained in a \( \delta \)-neighbourhood of \([g, w] \cup [h, w]\), for any configuration of \( g, h, w \in G\).
Part (i). For the proof of part (i), assume that that \(|g, h|\) is a geodesic segment in \(G\), that \(z \in [g, h] \setminus [g, h]\) and set \(d := d(g, h)\). Furthermore, let \(\gamma: [0, d] \to [g, h]\) refer to the isometry obtained by identifying the edges with copies of \([0, 1]\) such that \(\gamma(0) = g \) and \(\gamma(d) = h\). We now construct finite sequences of \(t_i, s_i \in [0, d]\) and balls \(B_i\) as follows. To begin, set \(s_0 = 0\) and \(t_0 = d\). The \(s_i, t_i\) are then inductively constructed as follows (see Figure 2).

(i) If \(t_i - s_i \leq 16\) then the induction stops. In fact, Figure 2 illustrates a possible last step in the iteration.

(ii) If \(t_i - s_i > 16\) and \(d(\gamma(s_i), z) \leq d(\gamma(t_i), z)\), then \(s_{i+1} = s_i + (t_i - s_i)/4\) and \(t_{i+1} = t_i\). As \(s_{i+1} - s_i = (t_i - s_i)/4 > 4\), there exists a ball \(B_{i+1}\) is ball with center in \(\gamma((s_i, s_{i+1})) \cap G\) and radius in \(\mathbb{N}\) such that \(B_{i+1}\) covers \(\gamma((s_i, s_{i+1}))\) up to two segments of total length at most 3.

(iii) If \(t_i - s_i > 16\) and \(d(\gamma(s_i), z) > d(\gamma(t_i), z)\), then \(s_{i+1} = s_i\) and \(t_{i+1} = t_i - (t_i - s_i)/4\). As above, there exists a ball \(B_{i+1}\) with center in \(\gamma((t_{i+1}, t_i)) \cap G\) and radius \(r_i \in \mathbb{N}\) such that \(B_{i+1}\) covers \(\gamma((t_{i+1}, t_i))\) up to two segments of total length at most 3.

Now assume that the induction stopped at step \(n\). Then it is straightforward to see that \(s_0 \leq s_1 \cdots \leq s_n < t_0 \leq t_{i+1} \cdots \leq t_0, t_i - s_i = d(3/4)^i\), that \((3/4)^i d \leq 16\text{diam}B_{i+1} \leq 4(3/4)^i d\) and that the distance between two adjacent balls is at most 4.

![Figure 2: The construction of \(B_{i+1}\)](image)

Now assume that \(f \in \mathcal{H}\) and that \(B_i, B_j, B_k\) are three of these balls in this order from the left to the right with respect to \(\gamma\). We now expand \(X_{B_i} G_r(f \chi_{B_j})\) according to the position of \(B_j\) relative to \(z\) as follows.

\[
X_{B_i} G_r(f \chi_{B_k}) = X_{B_i} G_r(f \chi_{B_j}|B_j^c) + \left\{ \begin{array}{ll}
X_{B_i} G_r(X_{B_j} G_r(f \chi_{B_k}|B_j^c)) & : B_j \text{ on the left of } z \\
X_{B_i} G_r(X_{B_j} G_r(f \chi_{B_k}|B_j^c)) & : B_j \text{ on the right of } z 
\end{array} \right.
\]

That is, in the first case, we separate the orbits starting in \(B_k\) and ending in \(B_i\) at their last visit to \(B_j\) whereas in the second case at their first visit to \(B_j\). We now apply this expansion inductively as follows. In the first case, we apply \(4\) to \(X_{B_i} G_r(f \chi_{B_j})\) with respect to \(B_j\) between \(B_j\) and \(B_k\), and in the second case to \(X_{B_i} G_r(f^* \chi_{B_k})\) with respect to \(B_j\) between \(B_i\) and \(B_j\), where \(f^* = G_r(f\chi_{B_k}|B_j^c)\). In order to obtain a manageable expression, set

\[
G = G_r, \quad G_j = G_r(|B_j^c), \quad L_j(f) := G_r(\chi_{B_j} \cdot f|B_j^c), \quad R_j(f) := \chi_{B_j} \cdot G_r(f|B_j^c).
\]

Furthermore, for \(k \leq n\) assume that the \(a_k(i) = 1, \ldots, k\) \((i = 1, \ldots, k)\) are determined by the order of the \(B_j\) along the path \(\gamma\) in the sense that \(B_{a_k(i)}\) is followed by \(B_{a_k(i+1)}\) etc. and that \(\ell_k\) is
given by $B_{\alpha_k(\ell_i)} < z < B_{\alpha_k(\ell_i+1)}$. Set

$$E_k := \chi_{[g]} \cdot L_{\alpha_k(1)} \cdots L_{\alpha_k(\ell_i)} \circ G \circ R_{\alpha_k(\ell_i+1)} \cdots R_{\alpha_k(k)}(X_{[h]}),$$

$$D_k := \chi_{[g]} \cdot L_{\alpha_k(1)} \cdots L_{\alpha_k(\ell_i)} \circ G_{k+1} \circ R_{\alpha_k(\ell_i+1)} \cdots R_{\alpha_k(k)}(X_{[h]}),$$

$$D_0 := \chi_{[g]} \cdot G_1(X_{[h]}).$$

In Figure 3, typical orbits related to $D_3$ and $E_4$ are illustrated. That is, in the first case, the orbit is stopped at the first visit to $B_1$, then passes without hitting $B_4$ to the last visit to $B_2$ and through the last visit to $B_2$ to $g$, whereas in the second case, the orbit has to pass through $B_4$. We now show by induction that $\chi_{[g]} G_r(\chi_{[h]}) = E_k + \sum_{i=0}^{k-1} D_i$. If $k = 1$, then $\alpha_1(1) = 1$ and $\ell_1 \in \{0,1\}$. In particular, $E_1 = \begin{cases} \chi_{[g]} \cdot G \cdot (\chi_{B_1} \cdot G(\chi_{[h]})) & : \ell_1 = 1 \\ \chi_{[g]} \cdot G \cdot (\chi_{B_1} \cdot G(\chi_{[h]})) & : \ell_1 = 0 \end{cases}$

Hence, $\chi_{[g]} G_r(\chi_{[h]}) = E_1 + D_0$ by [4]. In order to extend the result to any $k \leq n$, it suffices to apply [4] to

$$\chi_{B_{\alpha_k(\ell_i)}} \left( G \left( \chi_{B_{\alpha_k(\ell_i+1)}} \right) - G \left( \chi_{B_{\alpha_k(\ell_i+1)}} \right) \right)$$

in order to show that $E_k = E_{k+1} + D_k$, and, in particular, $\chi_{[g]} G_r(\chi_{[h]}) = E_k + \sum_{i=0}^{k-1} D_i$ for all $k \leq n$ by induction. Now assume that $u \in B_{\alpha_k(\ell_i)}$ and $v \in B_{\alpha_k(\ell_i+1)}$. It follows from $\delta$-hyperbolicity that the distance from a geodesic segment $[u, v]$ to the center of $B_{k+1}$ is at most $\delta$. In particular, there is ball of radius $\text{diam} B_{k+1}/2 - (\delta + 1)$ with center in $[u, v] \cap G$ which is contained in $B_{k+1}$. It now follows from Lemma 4.5 and the construction that

$$\chi_{[u]} G_r(1_{[a,v]} | B_{k+1}^c) \leq \chi_{[u]} G_r(\chi_{[u]} | B_{k+1}^c) \leq 2^{-\lambda \text{diam} B_{k+1}/2 - (\delta + 1)} \leq 2^{-\lambda \frac{d}{(3/4)^k} - (\delta + 1)}$$

for any $a \in W\cdot i$, provided that $d(3/4)^k \geq 32(n_0 + 1 + \delta)$. As $d \mu / d \mu \circ \theta$ is bounded away from zero, there exists $p \in (0,1)$ such that $G_r(1_{[a,v]})(x, u) \gg p^{d(u,v)}$ for all $x \in \Sigma$. Again by construction and the triangle inequality, it follows that $d(u, v) \leq |s_{k-1} - s_{k-1}| = d(3/4)^{k-1}$. Hence, there exist $\alpha > 1, \beta > 1$ such that

$$\chi_{[u]} G_r(1_{[a,v]} | B_{k+1}^c) \leq \alpha^{-\beta \frac{d(3/4)^k}{d(3/4)^k} \chi_{[u]} G_r(1_{[a,v])}. \quad (5)$$
Now set $\alpha_{d,k} := \alpha^{-\beta d(3/4)^k} + d(3/4)^k$ and suppose that $f > 0$ satisfies $\sup \{D_\alpha (\log f)(z, v) : z \in \Sigma \} \leq \log C$. Then

$$X_{[u]} G_r \{ (X_{[v]} f) | B_{k+1}^c \} \leq \sum_{a \in \mathcal{W}^1} \sup_{z \in [a, v]} f(x, v) X_{[u]} G_r \{ 1_{[a, v]} | B_{k+1}^c \} \leq \sum_{a \in \mathcal{W}^1} \sup_{x \in [a]} f(x, v) \alpha_{d,k} X_{[u]} G_r \{ 1_{[a, v]} \} \leq C \alpha_{d,k} X_{[u]} G_r \{ X_{[u]} f \}$$

By the Gibbs-Markov property of $\theta$, it therefore follows that

$$\sum_{i=1}^{k-1} D_i = \sum_{i=1}^{k-1} \sum_{a \in B_{\ell_i}(\ell_i+1), \forall n \notin B_{\ell_i}(\ell_i+1)} X_{[g]} \cdot L_{\alpha_1} \cdots L_{\alpha_{i}} \{ X_{[u]} G_r \{ X_{[v]} R_{\alpha_i} (\ell_i+1) \cdots (X_{[h]}) \} | B_i \} \leq C \alpha \sum_{i=1}^{k-1} \sum_{a \in B_{\ell_i}(\ell_i+1), \forall n \notin B_{\ell_i}(\ell_i+1)} \alpha_{d, i} X_{[g]} \cdot L_{\alpha_1} \cdots L_{\alpha_{i}} \{ X_{[u]} G_r \{ X_{[v]} R_{\alpha_i} (\ell_i+1) \cdots (X_{[h]}) \} \} \leq \sum_{i=1}^{k-1} \alpha_{d, i} X_{[g]} G_r \{ X_{[h]} \}.$$

The next step relies on the fact that $\alpha_{d,k}$ and $t_k - s_k$ are functions of $d(3/4)^k$, which allows to choose $M$ such that $t_k - s_k \geq M$ implies that $\sum_{i=1}^{k-1} \alpha_{d, i} \leq \frac{1}{2}$. For $k$ maximal with this property, it also follows that $B_{\alpha_i}(\ell_i)$ and $B_{\alpha_i}(\ell_i+1)$ are contained in a ball with center $z$ and radius $t_k - s_k + \text{diam} B_{k-1}$. As the radius is uniformly bounded by a multiple of $M$, there exists $C > 0$ such that

$$X_{[g]} \cdots X_{[u]} G_r \{ X_{[v]} \cdots G_r (X_{[h]}) \} \leq C X_{[g]} \cdots X_{[u]} G_r \{ X_{[z]} G_r (X_{[v]} \cdots G_r (X_{[h]}) \})$$

for all $u \in B_{\alpha_i}(\ell_i)$ and $v \in B_{\alpha_j}(\ell_j+1)$. Putting these estimates together yields

$$X_{[g]} G_r \{ X_{[h]} \} = E_k + \sum_{i=0}^{k-1} D_i \leq E_k + \sum_{i=0}^{k-1} \alpha_{d,k} E_i \leq E_k + \frac{1}{2} X_{[g]} G_r \{ X_{[h]} \} \leq C \sum_{u \in B_{\alpha_i}(\ell_i), v \in B_{\alpha_j}(\ell_j+1)} X_{[g]} \cdots X_{[u]} G_r \{ X_{[v]} \cdots G_r (X_{[h]}) \} \leq \frac{1}{2} X_{[g]} G_r \{ X_{[h]} \} \leq C X_{[g]} G_r \{ X_{[z]} G_r (X_{[v]} \cdots G_r (X_{[h]}) \}) + \frac{1}{2} X_{[g]} G_r \{ X_{[h]} \} \leq C X_{[g]} G_r \{ X_{[z]} G_r (X_{[v]} \cdots G_r (X_{[h]}) \}) + \frac{1}{2} X_{[g]} G_r \{ X_{[h]} \}.$$

Hence, $X_{[g]} G_r \{ X_{[h]} \} \leq 2 C X_{[g]} G_r \{ X_{[z]} G_r (X_{[v]} \cdots G_r (X_{[h]}) \})$, proving (3) for $z \in [g, h]$.

Now assume that $z$ is $D$-close to the geodesic segment $[g, h]$. In particular, there exists $z' \in [g, h]$ with $d(z, z') < D$ and (3) holds with respect to $z'$. Furthermore, note that $|g : d(g, \text{id}) \leq D|$ is a finite set as $G$ is finitely generated. As $z^{-1} z' \in [g : d(g, \text{id}) \leq D]$, an application of Lemma 3.4 gives a uniform bound for $|\log G_r (X_{z'} G_r (X_{h})) (x, g) / G_r (X_{z} G_r (X_{h})) (x, g)|$ which implies that (3) holds with respect to a different constant.

AN EXTENSION OF PART (I). In order to deduce Part (ii) from Ancona's inequality, it is necessary to extend part (i). In order to do so, observe that the induction relies on (3), which is
obtained through a decomposition of orbits. Hence, provided that \( \Omega \) is a set which contains \( \bigcup_k B_k \), equation 4 generalizes to

\[
\chi_{B_j} G_r (f \chi_{B_k} | \Omega) = \chi_{B_j} G_r (f \chi_{B_k} | B_j^c \cap \Omega) + \begin{cases} 
\chi_{B_j} G_r (\chi_{B_j} G_r (f \chi_{B_k} | \Omega) | B_j^c \cap \Omega) : & B_j \text{ left of } z \\
\chi_{B_k} G_r (\chi_{B_j} G_r (f \chi_{B_k} | B_j^c \cap \Omega) | \Omega) : & B_j \text{ right of } z,
\end{cases}
\]

which then implies that a version of the induction \( E_k + \sum_{j=0}^{k-1} D_j \) holds with respect to orbits which never leave \( \Omega \). Moreover, this generalisation does not cause any problem with the application of Lemma 4.5 as the estimates in there only might get better. However, the estimate 5 relies on the fact that there exists an orbit connecting \( u \) and \( v \). Therefore, it is also required that \( \Omega \) contains a \( M \)-neighbourhood of the convex hull of \( \bigcup_k B_k \), where \( M \) depends on the topological transitivity of \( T \). That is, \( M \) has to be chosen such that for any \( a \in W \), \( u \in B_k \), \( v \in B_j \), there exists \( x \in \{a\} \) and \( n \in \mathbb{N} \) such that the geodesic from \( u \) to \( v \) is contained in \( \{uk^j(x) : 0 \leq j \leq n\} \cap \Omega \) and \( \log n \ll d(u, v) \). As the remaining assertions follow in verbatim, we obtain the following relative version of Part (i) by adding the trivial estimate, provided that \( \Omega \) contains the \( M \)-neighbourhood of the convex hull of \( \bigcup_k B_k \).

\[
G_r (\chi_{B_j} G_r (\chi_{B_k} | \Omega) | \Omega) (x, g) \leq G_r (\chi_{B_k} | \Omega) (x, g) \leq CG_r (\chi_{B_j} G_r (\chi_{B_k} | \Omega) | \Omega) (x, g).
\]

**Part (ii).** The adaption of the arguments in in \([17]\) for the proof of (ii) depends on the potential theory of conformal and excessive measures as developed in the appendix (Section 7) of this article. In particular, it is necessary to anticipate the following notion from Section 5 which also is the central object in Theorem 5.2 below. We refer to a Radon measure \( m \) as \( \lambda \)-excessive or excessive if \( \mathcal{L}^* (m) \leq \lambda m \), that is \( \mathcal{L}^* (m) \) is absolutely continuous with respect to \( m \) and \( d \mathcal{L}^* (m) / dm \leq \lambda \). Moreover, we say that \( m \) is conformal on \( B \) if \( \mathcal{L}^* (m)|_{B} = \lambda m|_{B} \).

We begin with an argument from geometry. For \( \xi, \eta \in G \) choose \( k \in \mathbb{N} \) such that

\[
D := d(\xi, \eta) / k \geq 2 \max(d(id, \kappa(x)) : x \in \Sigma).
\]

For \( 1 \leq j \leq k \), let \( z_j \in G \) refer the closest point on the geodesic arc from \( \xi \) to \( \eta \) with \( (z_j \cdot \xi) \eta > jD + D/2 \) and set

\[
\Omega_j := \{ h \in G : (h \cdot \xi) \eta \geq jD \}, \quad \Lambda_j := \{ h \in \Omega_j : (h \cdot \xi) \eta \leq D/2 + jD \}.
\]

Observe that \( \Omega_j \supset \Omega_{j+1} \) and that for \( h \in \Omega_j \), the geodesic from \( h \) to \( \eta \) passes through \( B(z_j, \delta) \) by the thin triangle property. For \( h \in \Omega_j \) and \( g \in \Omega_j^c \), it follows from the construction that \( (h \cdot \xi) \eta > (g \cdot \xi) \eta \). By approximation by a tree, the geodesic from \( h \) to \( g \) has to pass through \( B(z_j, 4\delta) \). If, in addition, \( h \in \Omega_{j+1} \), it follows from the choice of \( D \) that any orbit from \( h \) to \( g \) has to pass through \( \Lambda_j \), say at \( z \). By a further approximation by a tree, also the geodesic from \( h \) to \( z \) visits \( B(z_j, 4\delta) \).

This geometrical construction now allows to deduce the following estimates. By decomposing orbits with respect to the last visit to \( \Lambda_j \), it follows from 6 and an extension of Lemma 4.5 that the estimate 5 carries over to \( \Omega \).
to $G_r(\Omega_j)$ that there exists $c \geq 1$ such that, for any $\omega \in \Sigma$,

$$X_g G_r(X_h | \Omega_j) = X_g \sum_{z \in A_j} G_r(X_z G_r(X_h | \Omega_j) | \Omega_j \setminus \Lambda_j)$$

$$= c^{k+1} X_g \sum_{z \in A_j} G_r(X_z G_r(X_h | \Omega_j) | \Omega_j \setminus \Lambda_j)$$

$$= c^{k+2} G_r(X_h | \Omega_j)(\omega, z_j) \cdot X_g \sum_{z \in A_j} G_r(X_z G_r(X_h | \Omega_j) | \Omega_j \setminus \Lambda_j)$$

$$= G_r(X_h | \Omega_j)(\omega, z_j) \cdot X_g G_r(X_z | \Omega_j),$$

Given $A \subset G$, set $X_A := \{(x, g) : x \in \Sigma, g \in A \}$. Now assume that, for some $1 \leq j \leq k$, $m$ is a Radon measure which is $1/r$-conformal and non-trivial on $\mathcal{X}_{\Omega_j}$ such that $m(\bigcap_{n=0}^{\infty} T^{-n}(\mathcal{X}_{\Omega_j})) = 0$. In particular, we have that

$$A_k := T^{-k}(\mathcal{X}_{\Omega_j}) \cap \bigcap_{n=0}^{k-1} T^{-n}(\mathcal{X}_{\Omega_j}), \quad k = 1, 2, \ldots$$

is a partition of $\mathcal{X}_{\Omega_j}$ up to a set of measure zero. Hence, for $h \in \Omega_{j+1}$,

$$m(\mathcal{X}_h) = \sum_{k=1}^{\infty} \int_{A_k} X_h d m = \sum_{k=1}^{\infty} \int_{A_k} X_{\Omega_j} \cdot \mathcal{L}(X_{\Omega_j}, \mathcal{L}(\cdots \mathcal{L}(X_h) \cdots)) \cdot dm$$

$$= \frac{1}{r} \int_{A_k} X_{\Omega_j} G_r(X_h | \Omega_j) d m = \frac{c^{k+2}}{r} G_r(X_h | \Omega_j)(\omega, z_j) \int_{A_k} X_{\Omega_j} G_r(X_z | \Omega_j) d m$$

$$= c^{k+2} G_r(X_h | \Omega_j)(\omega, z_j) \cdot m(\mathcal{X}_z).$$

Setting $h = \xi$, it follows that $m(\mathcal{X}_z) = c^{k+2} m(\mathcal{X}_z)/G_r(X_\xi | \Omega_j)(\omega, z_j)$. Hence, for $v_j$ defined through

$$\int f d v_j := c^{-4} \frac{G_r(f | \Omega_j)(\omega, z_j)}{G_r(X_\xi | \Omega_j)(\omega, z_j)},$$

we have that, for any $h \in \Omega_{j+1}$,

$$c^{-4} m(\mathcal{X}_h) \leq m(\mathcal{X}_z) v_j(\mathcal{X}_h) \leq c^4 m(\mathcal{X}_h). \quad (7)$$

Observe that in most cases, $m$ and $v_j$ are non-singular with respect to each other. In order to apply (7) also to $m = v_{j-1}$, note that $v_j$ is $1/r$-conformal on $(T^{-1}(\mathcal{X}_{\Omega_j}) \cap \mathcal{X}_{\Omega_j}) \setminus \{(\omega, z_j)\} \supset \mathcal{X}_{\Omega_{j+1}}$ as

$$G_r(\mathcal{L}(f) | \Omega_j) = \frac{1}{r} \left( G_r(f | \Omega_j) - f \right) + G_r(X_\xi \cdot \mathcal{L}(f) | \Omega_j) - X_\xi \cdot \mathcal{L}(f),$$

and that, by construction, $v_j \left(\bigcap_{n=0}^{\infty} T^{-n}(\mathcal{X}_{\Omega_{j+1}})\right) = 0$. It is worth noting that these two properties are needed for the lower bound of $(\mu_j - \nu)(\mathcal{X}_h)$ below whereas the upper bound is independent from this.

After these preparations, we are now in position to prove part (ii). In order to do so, for $\alpha := 1 - c^{-4}$, $x_1, x_2 \in \Sigma$ and $g_1, g_2 \in \Omega_1$, let $\mu_1, \mu_2, v$ refer to the Radon measures defined by

$$\mu_1(f) := \frac{G_r(f)(x_1, g_1)}{G_r(X_\xi)(x_1, g_1)}, \quad v := \sum_{j=1}^{k-1} \alpha^{j-1} v_j.$$
By inductively applying (7) to \( m = \mu \) for the estimate from above and \( m = v_j \) for the estimate from below, it follows that, for \( h \in \Omega_k \),

\[
\begin{split}
(\mu - v)(\mathcal{X}_h) &= (\mu - v_1)(\mathcal{X}_h) - \frac{1}{j-1} \sum_{j' = 2}^{k-1} a^{j'-1} v_{j'}(\mathcal{X}_h) \\
&\leq a^{k-1} \mu_1(\mathcal{X}_h),
\end{split}
\]

\[
(\mu - v)(\mathcal{X}_h) = (\mu - v_1)(\mathcal{X}_h) - \frac{1}{j-1} \sum_{j' = 2}^{k-1} a^{j'-1} v_{j'}(\mathcal{X}_h) \geq a \left( v_1(\mathcal{X}_h) - \frac{1}{j-1} \sum_{j' = 2}^{k-1} a^{j'-2} v_{j'}(\mathcal{X}_h) \right) \\
&\geq a^{k-1} v_{k-1}(\mathcal{X}_h) \geq 0.
\]

Moreover, note that (7) implies that \( \mu_1(\mathcal{X}_h) \approx \mu_2(\mathcal{X}_h) \). Hence,

\[
\left| \frac{\mu_1(\mathcal{X}_h)}{\mu_2(\mathcal{X}_h)} - 1 \right| = \left| \frac{\mu_1(\mathcal{X}_h) - \mu_2(\mathcal{X}_h)}{\mu_2(\mathcal{X}_h)} \right| = \left| \frac{(\mu_1 - v)(\mathcal{X}_h) + (\mu_2 - v)(\mathcal{X}_h)}{\mu_2(\mathcal{X}_h)} \right| \\
\leq a^{k-1} \frac{\mu_1(\mathcal{X}_h) + \mu_2(\mathcal{X}_h)}{\mu_2(\mathcal{X}_h)} << a^{k-1} \frac{\mu_2(\mathcal{X}_h)}{\mu_2(\mathcal{X}_h)} << a^k,
\]

which is part (ii) of the theorem for \( h' = \xi \) and \( \lambda := a^{1/D} \). The remaining assertion, that is \( h' \in \Omega_k \) easily follows from this.

5 Geometry of the Martin boundary

Theorem 4.6 has immediate implications for a boundary theory of group extensions as it indicates what might be the canonical notion of a Martin boundary through a geometrization. Namely, the second estimate in Theorem 4.6 allows to obtain a geometrization by a bi-Hölder equivalence of the Martin boundary with the visual boundary of \( G \).

The boundary of a hyperbolic group \( G \) is defined as follows (see, e.g. [14]). A sequence \((g_n)\) is said to converge at infinity if \( \lim_{n, m \to \infty} (g_n \cdot g_m)_p = \infty \) for some \( p \in G \), and we say that \((g_n)\) and \((h_n)\) converge to the same limit at infinity if \( \lim_{n \to \infty} (g_n, h_n)_p = \infty \) for some \( p \in G \). The boundary \( \partial G \) of \( G \) is then defined as the set of equivalence classes of this relation, and, in particular, for a convergent sequence \((g_n)\), the limit is defined as its associated equivalence class. Moreover, these definitions do not depend on the choice of \( p \).

In order to define the visual metric on \( \partial G \), for \( \xi, \eta \in \partial G \), let

\[
(\xi \cdot \eta) := \sup \left\{ \liminf_{m, n \to \infty} (g_m \cdot h_n)_\alpha : g_n \to \xi, h_m \to \eta \right\}.
\]

As shown in [14], if \( G \) is \( \delta \)-hyperbolic, then \( (\xi \cdot \eta) - 2\delta \leq \liminf_{m, n} (g_m \cdot h_n)_\alpha \leq (\xi \cdot \eta) \), for any approximating sequences \((g_n)\) and \((h_m)\). Furthermore, for \( \lambda_\text{visual} \in (\sqrt{1/2}, 1) \), it is shown in [14] that

\[
r(\xi, \eta) := \lambda_\text{visual}^{(\xi \cdot \eta)}, d_\text{visual}(\xi, \eta) := \inf \left\{ \sum_{k=1}^{n-1} r(x_k, x_{k+1}) : n \in \mathbb{N}, x_k \in \partial G, x_1 = \xi, x_n = \eta \right\}, \Xi : \partial G \to \mathcal{M}_r
\]

(8)
defines a metric on $\partial G$ and that there exists $C \in (0, 1)$ such that $Cr(\xi, \eta) \leq d(\xi, \eta) \leq r(\xi, \eta)$ for all $\xi, \eta \in \partial G$. Moreover, $\partial G$ is compact with respect to this metric. For this choice of $\lambda$, we refer to $d_{\text{visual}}$ as the visual metric on $\partial G$.

The following definition is inspired by the classical construction of the Martin boundary as it gives rise to a continuous extension of the Green operators. For $h \in G$ and $r < R$, let

$$k_r(h, \cdot) : \Sigma \times G \to \mathbb{R}, \ (x, g) \mapsto \frac{G_r(\chi_h)(x, g)}{G_r(\chi_{id})(x, g)}$$

and note that $k_r(h, \cdot)$ is a bounded function by Lemma 3.4 for each $h \in G$. Now assume that $(g_n)$ is a sequence in $G$. We refer to $(g_n)$ as unbounded, written as $|g_n| \to \infty$, if $(g_n)$ leaves any finite subset of $G$ infinitely often. Furthermore, let

$$M_r := \{(x, g) \in \Sigma \times G : |\kappa^n(x)| \to \infty \text{ and } \lim_{n \to \infty} k_r(h, T^n(x, g)) \text{ exists for all } h \in G\}$$

and $(x, g) \sim (\tilde{x}, \tilde{g})$ if and only if $\lim_{n \to \infty} k_r(h, T^n(x, g)) = \lim_{n \to \infty} k_r(h, T^n(\tilde{x}, \tilde{g}))$ for all $h \in G$.

In analogy to the theory known from random walks, we refer to $\mathcal{M}_r := M_r / \sim$ as the Martin boundary of the group extension $(X, T)$. As a consequence of Theorem 4.6, one obtains the following relation of $\partial G$ and $\mathcal{M}_r$.

**Proposition 5.1.** Assume that $G$ is hyperbolic, $T$ is a topologically transitive, transient extension of a Gibbs-Markov map of finite type and that $G_R(\chi_{g})(x, \text{id}) \simeq G_R(\chi_{id})(x, g)$, independent of $(x, g) \in \Sigma \times G$. For $r \leq R$, the following holds.

(i) For $(x, g) \in \Sigma \times G$ such that $(\kappa^n(x))$ converges at infinity, we have that $(x, g) \in M_r$. Moreover, for $(\tilde{x}, \tilde{g}) \in \Sigma \times G$ such that $(\kappa^n(x))$ and $(\tilde{g}_n)(\tilde{x})$ converge to the same limit at infinity in the sense of Gromov, it follows that $(x, g) \sim (\tilde{x}, \tilde{g})$.

(ii) For each sequence $(g_n)$ which converges at infinity, there exists $x \in \Sigma$ such that $(\kappa^n(x))$ and $(g_n)$ converge to the same limit at infinity.

**Proof.** For the proof of (i), observe that, for $h \in G$ and $N$ sufficiently large, the second part of Theorem 4.6 is applicable to $\text{id}, h, \ g \kappa^k(x)$ and $g \kappa^l(x)$, for $k, l \geq N$. As $n$ in the statement of the theorem can be written as

$$n = \frac{1}{2} \left[ (g \kappa^k(x) \cdot g \kappa^l(x))_{\text{id}} + (g \kappa^k(x) \cdot g \kappa^l(x))_h - d(h, \text{id}) \right], \quad (9)$$

with $(g \cdot \tilde{g})_h$ referring to the Gromov product with base $h$ and $d$ the word metric on $G$, it immediately follows that $((g \kappa^k(x) \cdot g \kappa^l(x)) \to \infty$ implies that $\log k_r(h, T^n(x, g))$ is a Cauchy sequence for all $r \leq R$. The second part follows by substituting $g \kappa^k(x)$ with $\tilde{g}_n \kappa(x)$ in (9).

Assertion (ii) follows from the fact that the transitivity of $T$ allows to construct $x \in \Sigma$ such that $(\kappa^n(x))$ stays uniformly close to the piecewise geodesic arc with vertices $(g_n)$. It is then well known that $(\kappa^n(x))$ and $(g_n)$ have the same limit. □

As an immediate corollary of the result, the application

$$\Xi : \partial G \to \mathcal{M}_r, \ \eta \to \left\{ (x, g) \in M_r : \lim_{n \to \infty} g \kappa^n(x) = \eta \right\} / \sim$$
is well defined. In order to show that the map is invertible, we apply ideas by Ancona and Shwartz in [4, 30] to our setting as follows. First observe that, for \( \sigma \in \mathcal{M} \) and \((x, g)\) in the equivalence class \( \sigma \),

\[
\kappa_f : G \times \mathcal{M} \rightarrow \mathbb{R}, \quad (h, \sigma) \mapsto \lim_{n \rightarrow -\infty} \kappa_f(h, T^n(x, g))
\]

is well defined and extends the definition of \( \kappa_f \), but, in contrast to the setting of Markov chains, the function \( h \mapsto \kappa_f(h, \sigma) \) is not related to an \( r \)-harmonic function. However, by assuming transience, the definition of \( \kappa_f \) extends to \( \mathcal{H}_\infty \times X \) for \( r \leq R \) (see Prop. [3.2]). In particular, as \( h \) is identified with \( X_h \), a calculation shows that

\[
\kappa_f(L(X_h), \sigma) = \lim_n \kappa_f(L(X_h), T^n(x, g)) = \lim_n \frac{1}{r} \left( \kappa_f(X_h, T^n(x, g)) - \frac{X_h(T^n(x, g))}{G_f(X_{id}, T^n(x, g))} \right)
\]

\[
= \frac{1}{r} \kappa_f(X_h, \sigma), \quad (10)
\]

where the last equality follows from the fact that \((g \kappa_n(x))\) leaves every finite subset of \( G \). This identity implies that the canonical approach in here is to consider conformal and excessive measures as developed in the section on potential theory below. In here, we refer to a Radon measure \( m \) as \( \lambda \)-excessive if \( L^*(m) \leq \lambda m \), and as \( \lambda \)-conformal if \( L^*(m) = \lambda m \). Moreover, a conformal measure \( \mu \) is referred to as minimal if any conformal measure \( m \) with \( m \leq v \) is a multiple of \( \mu \). The following theorem identifies \( \partial G \) with minimal, conformal measures.

**Theorem 5.2.** Assume that \( G \) is hyperbolic, \( T \) is a topologically transitive, transient extension of a Gibbs-Markov map of finite type and that \( G_R(X_g)(x, id) \approx G_R(X_{id})(x, g) \), independent of \((x, g)\) in \( \Sigma \times G \) and that \( r \leq R \). Then the following holds.

(i) If \( \sigma \in \mathcal{M} \) and \((x, g)\) is an element of the equivalence class \( \sigma \) and \( f \in \mathcal{H}_\infty \), \( f \geq 0 \), then

\[
\mu_\sigma(f) := \lim_{n \rightarrow -\infty} \kappa_f(f, T^n(x, g)),
\]

always exists and defines a \( 1/r \)-conformal, minimal measure. Moreover, any \( 1/r \)-conformal, minimal measure is obtained in this way.

(ii) For any conformal measure \( \mu \), there exists a uniquely defined finite measure \( \nu \) on \( \partial G \) such that \( d\mu = d\mu_\sigma d\nu(\sigma) \), that is, for any \( f \in \mathcal{H}_\infty \),

\[
\mu(f) = \int_{\partial G} \kappa_f(f, \sigma) d\nu(\sigma).
\]

(iii) If \( \tilde{\sigma} \neq \sigma \), then \( \lim_{\gamma \rightarrow \tilde{\sigma}} \mu_\sigma(\gamma) = \infty \) and \( \lim_{\gamma \rightarrow \sigma} \mu_\sigma(\gamma) = 0 \). In particular, \( \mu_\sigma \neq \mu_\sigma \).

(iv) If \( \tilde{\sigma} \neq \sigma \), then \( g \rightarrow \log \mu_\sigma(X_g)/\mu_\tilde{\sigma}(X_g) \) extends to a continuous function on \( \bar{G} \setminus \{\tilde{\sigma}, \sigma\} \). Furthermore, if \( g, h \in G \) and \( \tilde{\sigma}, \sigma \in \partial G \) are in configuration as in figure [1], then, with \( C, \lambda \) as in Theorem [4.6],

\[
\left| \frac{\mu_\sigma(X_g)}{\mu_\sigma(X_h)} - \frac{\mu_\tilde{\sigma}(X_g)}{\mu_\tilde{\sigma}(X_h)} \right| \leq CA^n.
\]
(v) The Green operator $G_r(X_g)$ converges to 0 uniformly and exponentially fast, that is

$$\limsup_{n \to \infty} \max_{y \in \Sigma} \sqrt[n]{G_r(X_g)}(y, y) < 1.$$ 

Proof. The strategy is as follows. We begin with the construction of an accumulation point of $\mu_n(f) := \mathbb{I}_r(f, T^n(x, g))$ with respect to a particular $x$ (Step 1) and then apply the Ancona-Gouëzel inequality in order to identify a region where the the limit is comparable to a reduced measure for a given conformal measure (Step 2). We then conclude from this description that the accumulation point is minimal (Step 3) and therefore unique, which implies convergence for each $x$ in the equivalence class of $\sigma$ (Step 4). An application of the argument in Step 3 then allows to prove assertions (iii-v) of the theorem (Steps 5 and 6). In Step 7, it is then shown how to deduce the remaining assertion from the work of Shwartz in [30].

Step 1. Accumulation points. Assume that $(x, \text{id})$ is as in Proposition [31] that is there exists a subsequence $(n_k)$ and $x \in \Sigma$ such that $(\kappa^{n_k}(x)) \to \sigma$ and $\kappa^{n_k}(x)$ stays within a bounded distance to a geodesic half ray $[\text{id}, \sigma]$.

In order to construct a limit measure, observe that $\mu_{n_k}(f)$ defines a measure for each $k \in \mathbb{N}$ and moreover, as $\lim_{k \to \infty} \mu_{n}(X_{h_k}) = \mathbb{K}_r(X_{h_k}, \sigma)$, for all $h \in G$. Hence, by compactness of $\Sigma \times \{h\}$ and a diagonal argument, there exists a further subsequence, also denoted by $(n_k)$, such that $\mu := \lim_{k \to \infty} \mu_{n_k}$ converges weakly on compact sets. Moreover, (10) implies that $\mu$ is conformal. In particular, it follows from bounded distortion that, for each $w \in \mathbb{H}^n$,

$$\mu([w, h]) = \varphi_w r^n \mu(X_{h\kappa^n(w)}) = \varphi_w r^n \lim_{k \to \infty} \mathbb{K}_r(X_{h_k}, T^n(x, g)).$$ (11)

Step 2. Reduced measures and the Ancona inequality. Fix $g \in G$. By construction, $\kappa^{n_k}(x) \to \sigma$ and $\kappa^{n_k}(x)$ stays within a bounded distance to the geodesic half ray $[\text{id}, \sigma]$. Hence, there exists $K$ such that $\kappa^{n_k}(x)$ stays within a bounded distance to the geodesic half ray $[g, \sigma]$ for any $k \geq K$. Hence, (3) of Theorem [4.6] is applicable to $g, \kappa^{n_k}(x), \kappa^{n_k}(x)$ for $K \leq k < l$. This implies after dividing by $G_r(X_{\text{id}})(T^{n_k}(x, \text{id})$ and applying Lemma [3.4] for any $y \in \Sigma$ that

$$\mu(X_g) = \lim_{l \to \infty} \mathbb{K}_r(X_g, T^{n_k}(x, g)) = \lim_{l \to \infty} \frac{G_r(X_g)(T^{n_k}(x, \text{id}))}{G_r(X_{\text{id}})(T^{n_k}(x, \text{id}))} \mathbb{K}_r(X_{\text{id}})(T^{n_k}(x, \text{id})) = \int Y_{\kappa^{n_k}(x)} G_r(X_g) d\mu$$

$$= G_r(X_g) \{y, \kappa^{n_k}(x)\} \mu(X_{\kappa^{n_k}(x)}).$$ (12)

Set $h = \kappa^{n_k}(x)$. As $y$ is arbitrary, integrating with $v(X_{h_k})^{-1} v|_{X_h}$ for some measure $v$ gives

$$\mu(X_g) = \frac{\mu(X_h)}{v(X_h)} \int G_r(X_g) d\nu|_{X_h} = \frac{\mu(X_h)}{v(X_h)} (G^*_r(v|_{X_h}))(X_g).$$ (13)

Now assume that $a \in \mathbb{H}^1$. By Theorem [7.6] $G^*_r(v|_{a, b})$ is already reduced. If, in addition, $v$ is a conformal measure, then Theorem [7.5] implies that the reduced measure is obtained by
applying the operator $\mathcal{F}^{*}_{(a, h)}$. Hence,

$$
G_{r}^{*}(v|_{(a, h)})(X) = R_{(a, h)}([G_{r}^{*}(v|_{(a, h)}))](X) = \mathcal{F}^{*}_{(a, h)} \circ G_{r}^{*}(v|_{(a, h)})(X)
$$

$$
= \int \mathbf{1}_{[a, h]} G_{r}^{*}(\mathbf{1}_{[a, h]} \mathfrak{F}|_{[a, h]} (X)) d\nu
\leq \sup_{z \in [a, h]} G_{r}^{*}(\mathbf{1}_{[a, h]})(z) \sup_{z \in [a, h]} \mathfrak{F}|_{[a, h]}(X) d\nu([a, h])
\leq C_{\varphi} \sup_{z \in [a, h]} G_{r}^{*}(\mathbf{1}_{[a, h]})(z) \int \mathfrak{F}|_{[a, h]}(X) d\nu
\leq C_{\varphi} \sup_{z \in [a, h]} G_{r}^{*}(\mathbf{1}_{[a, id]})(z) \cdot R_{[a, h]}(v)(X) \ll R_{[a, h]}(v)(X),
$$

where (†) follows from bounded distortion of $\varphi$ and (‡) from the fact that $\varphi(x, g)$ does not depend on the second coordinate. In particular, by construction of $\mathfrak{F}|_{[a, h]}$,

$$
G_{r}^{*}(v|_{x_{h}})(X) = \sum_{a \in \# a} G_{r}^{*}(v|_{(a, h)})(X) \ll \sum_{a \in \# a} G_{r}^{*}(v)(X) \ll \mathfrak{F}_{x_{h}}^{*}(v)(X) = R_{x_{h}}(v)(X).
$$

Hence, $R_{x_{h}}(v)(X) \ll G_{r}^{*}(v|_{x_{h}})(X)$. Combining the estimate with (13) then implies that

$$
R_{x_{h}}(\mu)(X) \ll \mu(X) \ll \frac{\mu(x_{h})}{v(x_{h})} R_{x_{h}}(v)(X) \ll \frac{\mu(x_{h})}{v(x_{h})} v(X),
$$

(14)

provided that $h$ is sufficiently close to $(g, \sigma)$. Now assume that $w \in \mathcal{H}^{n}$ for some $n \in \mathbb{N}$ and $g \in G$. It follows from conformality as in [11] that $v([w, g]) = \varphi_{w} r^{-n} v(X_{g^{n}(w)})$. However, by the choice of $x$, there exists $K(g, w) \in \mathbb{N}$ such that $h = \kappa^{n}_{k}(x)$ is sufficiently close to $g^{n}(w), \sigma$ for all $k \geq K(g, w)$. This proves that

$$
\mu([w, h]) \ll \frac{\mu(X^{n,k}_{x}(x))}{v(X^{n,k}_{x}(x))} v([w, h]) \quad \forall k \geq K(g, w).
$$

(15)

**Step 3. Minimality.** Assume that $v \leq \mu$. In order to show that $v$ is proportional to $\mu$, set $b := \text{ess inf } d\nu/d\mu$ and $a := \text{ess sup } d\nu/d\mu$. If $a = 0$, then $v = b \mu$ and there is nothing left to show. If $a > 0$, consider $v_{1} := a^{-1} (v - b \mu)$. Then

$$
\text{ess inf } \frac{d v_{1}}{d \mu} = a \left( \text{ess inf } \frac{d \nu}{d \mu} - b \right) = 0, \quad \text{ess sup } \frac{d v_{1}}{d \mu} = a^{-1} \left( \text{ess sup } \frac{d \nu}{d \mu} - b \right) = 1.
$$

(16)

Moreover, it follows from construction that $v_{2} := \mu - v_{1}$ has the same property. Hence, for each $\epsilon > 0$, there exists $A$ of positive measure such that $v_{2}(A) < \epsilon \mu(A)$. Through approximation by cylinder sets, we may suppose without loss of generality that $A$ is a cylinder set. It follows from (15) for $k$ sufficiently large that

$$
\epsilon \geq \frac{v_{2}(A)}{\mu(A)} = \frac{v_{2}(X_{k^{n,k}(x)})}{\mu(X_{k^{n,k}(x)})}.
$$

Hence, as $\epsilon$ is arbitrary,

$$
1 \geq \lim_{k \to \infty} \frac{v_{1}(X_{k^{n,k}(x)})}{\mu(X_{k^{n,k}(x)})} = 1 - \lim_{k \to \infty} \frac{v_{2}(X_{k^{n,k}(x)})}{\mu(X_{k^{n,k}(x)})} = 1.
$$

22
Equation \([15]\) now implies that \(\mu \ll \nu_1\), contradicting \([16]\). Hence \(a = 0\) and \(\nu = b\mu\).

**Step 4. Existence of the Limit.** Assume that \(\hat{\mu}\) is given by a converging subsequence with respect to some arbitrary \((x, g)\) in the equivalence class of \(\sigma\). It follows from \([11]\) that \(d\hat{\mu}/d\mu \leq C\) for some \(C > 0\). Then \(C^{-1}\hat{\mu} \leq \mu\). It follows from minimality that \(\mu\) and \(\hat{\mu}\) are colinear. As \(\mu(X_g) = \hat{\mu}(X_g)\), it follows that \(\hat{\mu} = \mu\).

**Step 5. Uniqueness.** Assume that \(\mu_\sigma(X_g) \approx \mu_\bar{\sigma}(X_g)\) with respect to a constant which is independent from \(g \in G\). Furthermore, assume that \(h\) and \(\tilde{h}\) are elements of the geodesic from \(\sigma\) to \(\bar{\sigma}\). By choosing \(h\) and \(\tilde{h}\) sufficiently distant from each other, it follows that for each \(g \in G\), either \(h\) is sufficiently close to \([g, \sigma]\) or \(\tilde{h}\) is sufficiently close to \([g, \bar{\sigma}]\). Hence, by \([14]\) applied simultaneously to \(\mu_\sigma\) and \(\mu_\bar{\sigma}\),

\[
\mu_\sigma(X_g) \approx \mu_\sigma(X_g) \ll R_{X_h}(\mu_\sigma)(X_g) + R_{\tilde{X}_h}(\mu_\sigma)(X_g) =: v(X_g),
\]

Moreover, as \(v\) is excessive but not conformal, it follows for \(w \in \mathcal{W}^n\) that

\[
v([w, g]) = \int 1_{[w, g]} d\nu \geq r^n \int 1_{[w, g]} d(\mathcal{L}_n)^* (v) = r^n \int \mathcal{L}_n(1_{[w, g]}) d\nu = r^n \varphi_w v(T^n([w, g])).
\]

By repeating the argument for a finite collections of disjoint words \((u_i)\) contained in \(\theta^n([w])\) such that \(x^{[u_i]} = \text{id}\) and \(\bigcup_i \theta^{[u_i]}(\{u_i\}) = \Sigma\),

\[
v([w, g]) \gg r^n \varphi_w v(T^n([w, g])) \geq r^n \varphi_w \sum_i v([u_i, g \kappa^n(w)])
\]

\[
\gg r^n \varphi_w \sum_i r^{[u_i]} \varphi_w v(X_{g \kappa^n(w)}) \gg r^n \varphi_w v(X_{g \kappa^n(w)}).
\]

Hence, by combining this estimate with \([11]\) and \([17]\), there exists \(C > 0\) such that \(\mu_\sigma([w, g]) \ll Cv([w, g])\) for all \(w \in \mathcal{W}^n, n \in \mathbb{N}\) and \(g \in G\). Hence, \(\mu_\sigma \ll C\nu\). However, as \(\nu\) is a potential, that is, it can be written as \(\nu = G_\sigma^*(m)\), the Riesz decomposition implies that \(\mu_\sigma = 0\), which is a contradiction.

**Step 6. Limits of \(\mu_\sigma(X_g), \mathcal{G}_\tau(X_g)\) and \(\mu_\sigma(X_g)/\mu_\bar{\sigma}(X_g)\).** As above, assume that \(\sigma\) and \(\bar{\sigma}\) are in \(\partial G\), and that \(h\) and \(\tilde{h}\) are elements of a geodesic from \(\sigma\) to \(\bar{\sigma}\) passes first through \(h\) and then through \(\tilde{h}\). Then, by \([14]\),

\[
\frac{\mu_\sigma(X_h)}{\mu_\sigma(X_\tilde{h})} \ll \frac{\mu_\sigma(X_g)}{\mu_\sigma(X_\tilde{h})} \ll \frac{\mu_\sigma(X_h)}{\mu_\sigma(X_\tilde{h})},
\]

for all \(g\) such that the geodesic rays \([g, \sigma]\) and \([g, \bar{\sigma}]\) are sufficiently close to \(h\) and \(\tilde{h}\), respectively. This is illustrated in figure \([3]\) for the case of \(G\) acting isometrically on the Poincaré disc. In there, the grey part stands for the possible locations of \(g\). Moreover, for \(\gamma\) such that \([\gamma, \bar{\sigma}]\) passes sufficiently close to \(h\), the same argument shows that \(\mu_\sigma(X_g)/\mu_\sigma(X_\tilde{h}) \gg \mu_\sigma(X_h)/\mu_\sigma(X_\tilde{h})\). As \(\mu_\sigma(X_\tau) \approx \mu_\sigma(X_\tau)\) for \(\tau\) in a subsequence converging to \(\sigma\) would imply that \(\mu_\sigma = \mu_\bar{\sigma}\), it follows that \(\lim_{\gamma \to \bar{\sigma}} \mu_\sigma(X_\gamma)/\mu_\bar{\sigma}(X_\gamma) = \infty\). By repeating the argument for \(\gamma \to \bar{\sigma}\), one obtains that

\[
0 \; \overset{\gamma \to \sigma}{\longrightarrow} \frac{\mu_\sigma(X_\gamma)}{\mu_\bar{\sigma}(X_\gamma)} \ll \frac{\mu_\sigma(X_h)}{\mu_\bar{\sigma}(X_h)} \ll \frac{\mu_\sigma(X_\gamma)}{\mu_\bar{\sigma}(X_\gamma)} \overset{\gamma \to \sigma}{\longrightarrow} \infty.
\]
If, in addition, \( g \) is an element of a geodesic from \( \sigma \) to \( \bar{\sigma} \), then (12) and symmetry imply that

\[
\frac{\mu_\sigma(X_\gamma)}{\mu_\tau(X_\gamma)} \geq \frac{G_r(X_g)(y,\gamma)^{-1}}{G_r(X_g)(y,g)\mu_\sigma(X_g)} \geq G_r(X_g)(y,\gamma)^{-2} \frac{\mu_\sigma(X_g)}{\mu_\bar{\sigma}(X_g)}.
\]

Hence, \( G_r(X_g)(y,\gamma) \to 0 \) as \( \gamma \to \sigma \) and, by compactness of \( \partial G \), \( G_r(X_g)(y,\gamma) \to 0 \) uniformly as \( |\gamma| \to \infty \). Therefore, a further application of part (i) of Theorem 4.6 implies that the convergence is exponential, that is

\[
\limsup_{n \to \infty} \max_{y \in \Sigma, |\gamma|=n} \sqrt[n]{G_r(X_g)(y,\gamma)} < 1.
\]

Furthermore, it follows from (12) that \( \lim_{\gamma \to \sigma} \mu_\sigma(X_\gamma) = \infty \) and \( \lim_{\gamma \to \sigma} \mu_\bar{\sigma}(X_\gamma) = 0 \) if \( \bar{\sigma} \neq \sigma \). In order to analyse the behaviour of \( \mu_\sigma(X_g) / \mu_\bar{\sigma}(X_g) \) for \( g \) distant from the geodesic from \( \sigma \) to \( \bar{\sigma} \) fix \( g_n \to g_\infty \in \partial G \setminus \{\sigma, \bar{\sigma}\} \). Then part (ii) of Theorem 4.6 implies as in Proposition 5.1 that \( \log(\mu_\sigma(X_g) / \mu_\bar{\sigma}(X_g)) \) is a Cauchy sequence and that the function \( g \to \log(\mu_\sigma(X_g) / \mu_\bar{\sigma}(X_g)) \) extends continuously to \( \partial G \setminus \{\sigma, \bar{\sigma}\} \). The remaining assertion is an immediate corollary of part (ii) of Theorem 4.6.

**STEP 7. INTEGRAL REPRESENTATION** Now assume that \( \mu \) is a minimal conformal measure. It then follows from Corollary 3.9 in [30] that \( \mu \) can be represented by \( \int f \, d\mu = cK_r(f, \omega) \), for some \( c > 0 \) and \( \omega \in \mathcal{M}_r \). Now let \( (x, g) \) be such that \( T^n(x, g) \) for \( g \in \partial G \setminus \{\sigma, \bar{\sigma}\} \). Then, by the first assertion in (i), \( \mu = c\mu_\sigma \). In particular, \( \partial G \) can be identified with the set of minimal conformal measures, which proves the second assertion of (i). The representation of arbitrary conformal measures then is a corollary of Theorem 3.12 in [30].

Now assume that \( z \in M_r \) and define \( m_z(X_h) := \lim_{n \to \infty} K_r(h, T^n(z)) \). It now follows as in the above proof of (i) of Theorem 5.2 that \( m_z \) extends uniquely to a minimal conformal measure. Therefore, there exists a unique \( \sigma \in \partial G \) such that \( \mu_\sigma = m_z \), which proves the following.
Corollary 5.3. The map $\Xi : \partial G \to \mathcal{M}_r$ is a bijection.

In order to analyse the topological properties of $\Xi$, we start with the construction of a metric which is compatible with the Martin compactification of $\Sigma \times G$. However, in order to obtain Hölder continuity, it will turn out that we have to modify the classical definition slightly by taking logarithms as follows. By Lemma 3.4, $\| \log \kappa_r (h, \cdot) \|_\infty < \infty$. Hence, there exists $\{ c_h : 0 : h \in G \}$ such that

$$\Delta_r ((x, g), (\tilde{x}, \tilde{g})) := \sum_{h \in G} c_h \| \log \kappa_r (h, (x, g)) - \log \kappa_r (h, (\tilde{x}, \tilde{g})) \| < 1$$

for all $(x, g), (\tilde{x}, \tilde{g}) \in \Sigma \times G$. Furthermore, if $(\tilde{x}, \tilde{g}) \in M_r$ then

$$d_{\text{visual}}((x, g), (\tilde{x}, \tilde{g})) := \lim_{n \to \infty} \Delta_r (T^n (x, g), T^n (\tilde{x}, \tilde{g})), \quad \text{for } (x, g), (\tilde{x}, \tilde{g}) \in M_r$$

is well defined and, as it easily can be shown, defines a metric on $\mathcal{M}_r$. It is worth noting that it follows from general topology that $\Sigma \times G \cup \mathcal{M}_r$ is the unique compactification of $\Sigma \times G$ such that each $\kappa_r (h, \cdot)$ extends to a continuous function, and that $d'_{\text{visual}}$ is a metric for this topology (see, e.g., [24]). In particular, this topology is independent from the parameters $\{ c_h \}$ and from considering $\log \kappa_r$ instead of $\kappa_r$. However, Theorems 4.6 and 5.2 allow to obtain precise estimates for $| \log \kappa_r (h, \cdot) |$ and $| \log \kappa_r (h, \cdot) - \log \kappa_r (h, \cdot) |$ which gives rise to the following definition of $c_h$, for $h \in G$ and $\lambda$ as in Theorem 4.6.

$$c_h := \frac{\# \{ g \in G : |g| = |h| \}}{\log |G_h(x, \text{id})|}, \quad \lambda^{2|h|}$$

Furthermore, observe that the sequence $(\log \# \{ g \in G : |g| = n \})$ is sub-additive, which implies that the following exponential growth rate $\hat{r}$ exists.

$$\hat{r} := \lim_{n \to \infty} \sqrt[n]{\# \{ g \in G : |g| = n \}}.$$

Theorem 5.4. Assume that $G$ is hyperbolic, $T$ is a topologically transitive extension of a Gibbs-Markov map of finite type and that $G_{R}(\Sigma \times G)(x, \text{id}) \equiv G_{R}(\Sigma \times G)$ and that $r \leq R$. Then, the map $\Xi : (\partial G, d_{\text{visual}}) \to (\mathcal{M}_r, d'_{\text{visual}})$ is a homeomorphism and, if $d'_{\text{visual}}$ is defined through $(c_h)$ as in (18), then

$$d_{\text{visual}}(\sigma, \tilde{\sigma})^\beta \ll d'_{\text{visual}}(\Xi(\sigma), \Xi(\tilde{\sigma})) \ll d_{\text{visual}}(\sigma, \tilde{\sigma})^\alpha,$$

for $\alpha = \log \lambda / \log \lambda_{\text{visual}}$, $\beta = (2 \log \lambda - \log \hat{r} + \epsilon) / \log \lambda_{\text{visual}}$ and an arbitrary $\epsilon > 0$. In particular, $\Xi$ and $\Xi^{-1}$ are Hölder continuous with exponents $\alpha$ and $1 / \beta$, respectively.

Proof. In order to deduce Hölder continuity, we begin with a geometric description of the visual metric. Suppose that $\sigma, \tilde{\sigma} \in \partial G$. By Proposition 5.1, there exist $x, \tilde{x} \in \Sigma$ such that $\lim \kappa^\infty (x) = \sigma$ and $\lim \kappa^\infty (\tilde{x}) = \tilde{\sigma}$. Furthermore, it follows from the above estimates for the visual metric that

$$d_{\text{visual}}(\sigma, \tilde{\sigma}) \geq C \lambda_{\text{visual}}^{\alpha - \beta} \geq C \lambda_{\text{visual}}^{\alpha - (\log \kappa^\infty (x) - \kappa^\infty (\tilde{x})) + 2\varepsilon} \geq \left( \lambda_{\text{visual}}^{2\varepsilon} \right) \lambda_{\text{visual}}^{\alpha - (\log \kappa^\infty (x) - \kappa^\infty (\tilde{x}))}.$$
Moreover, as $(\kappa^n(x))$ and $(\kappa^n(\bar{x}))$ converge at infinity, it also follows that these sequences leave any finite subset of $G$. Now suppose that $\liminf_{m,n \to \infty} (\kappa^n(x) \cdot \kappa^n(\bar{x})) = N$. Then there exists $m,n$ arbitrary large such that $(g \cdot \tilde{g}) = N$, for $g := \kappa^m(x)$ and $\tilde{g} := \kappa^n(\bar{x})$. In particular, $d_{\text{visual}}(\sigma, \tilde{\sigma}) \approx N^{\frac{1}{\text{const}}}$. This geometric characterization now allows to employ Theorems 4.6 and Theorem 5.2 in order to obtain refined estimates for $|\log \kappa_r(h, \cdot)|$ and $|\log \kappa_r(h, \cdot) - \log \kappa_r(h, \cdot)|$. For $g, \tilde{g}, \in G$ and let $\zeta \in G$ refer to the element in the geodesic arc $[id, h]$ closest to $g$, and $\zeta_1, \zeta_2 \in G$ refer the elements in a geodesic arc $[g, \tilde{g}]$ closest to $h$ and id, respectively (see Figure 5).

![Figure 5: A configuration of $g, \tilde{g}$ and $h, \text{id}$ with $\zeta_1 \neq \zeta_2$](image_url)

(i) By applying part (i) of Theorem 4.6 it follows that, for some uniform constant $C > 0$,
\[
|\log \kappa_r(h, (x, g))| = |\log \kappa_r(h, (x, \zeta))| \pm C.
\]
It follows from the symmetry $G_r(X_g)(x, \text{id}) \approx G_r(X_{\text{id}})(x, g)$ and a further application of part (i) of Theorem 4.6 that $\kappa_r(h, (x, \zeta)) = G_r(X_{\bar{h}})(x, \text{id}) / G_r(X_{\tilde{\zeta}})(x, \text{id})$. As $G_r(X_{\bar{h}})(x, \text{id}) \gg G_r(X_{\tilde{\zeta}})(x, \text{id})$, we obtain that
\[
|\log \kappa_r(h, (x, g))| = |\log G_r(X_{\bar{h}})(x, \text{id}) - 2 \log G_r(X_{\tilde{\zeta}})(x, \text{id})| \pm C \leq |\log G_r(X_{\bar{h}})(x, \text{id})| + C.
\]
Moreover, part (v) of Theorem 5.2 implies that this bound grows linearly in $|h|$.

(ii) If $g, \tilde{g}$ and $h, \text{id}$ are in configuration as in part (ii) of Theorem 4.6 then $\zeta_1 = \zeta_2$ and
\[
|\log \kappa_r(h, (x, g)) - \log \kappa_r(h, (\bar{x}, \tilde{g}))| \ll \lambda^{(g, \tilde{g})^{-|h|}}.
\]

(iii) If $g, \tilde{g}$ and $h, \text{id}$ are not necessarily in this configuration (see Figure 5), then part (i) above implies that
\[
|\log \kappa_r(h, (x, g)) - \log \kappa_r(h, (\bar{x}, \tilde{g}))| = 2|\log G_r(X_{\bar{h}})(x, \zeta_2)| \pm C.
\]

We begin with the estimate from above. In order to do so, observe that $|\log G_r(h, (x, \text{id}))|^{-1}$ tends to 0 as $|h| \to \infty$ by (v) of Theorem 5.2. This implies that $\sum_{h \in G} c_h < \infty$. Moreover, by
approximation by a graph, we may assume that $|\xi_2| = (g \cdot \tilde{g}) =: N$. Furthermore, we have that $|h| \leq N/2$ implies that $h, \text{id}$ and $g, \tilde{g}$ are in a configuration as in (ii) of Theorem 4.6. Hence

$$d_{\text{Martin}}^r(\sigma, \tilde{\sigma}) = \lim_{g \to \sigma, \tilde{g} \to \sigma} g, \tilde{g} \to \sigma \sum_{|h| \leq N/2} c_h |\log K_r(h, (x, g)) - \log K_r(h, (\tilde{x}, \tilde{g}))|$$

$$+ \lim_{g \to \sigma, \tilde{g} \to \sigma} g, \tilde{g} \to \sigma \sum_{|h| > N/2} c_h |\log K_r(h, (x, g)) - \log K_r(h, (\tilde{x}, \tilde{g}))|$$

$$\ll \sum_{h=1}^{N/2} c_h \lambda^{|(g \cdot \tilde{g}) - |h|} + \sum_{|h| > N/2} c_h \ll \lambda_N \sum_{n=1}^{\infty} \lambda^n + \sum_{n=N/2}^{\infty} \lambda^{2n} \ll \lambda^N.$$

For the estimate from below, it suffices to consider the term with $h = \xi_1 = \xi_2$. Namely, it follows from (iii) that

$$d_{\text{Martin}}^r(\sigma, \tilde{\sigma}) \gg \lim_{g \to \sigma, \tilde{g} \to \sigma} g, \tilde{g} \to \sigma c_{|\xi_1|} |\log K_r(\xi_2, (x, g)) - \log K_r(\xi_2, (\tilde{x}, \tilde{g}))|$$

$$\gg \frac{\lambda^N}{\kappa N} |\log G_r(X, x, (\sigma, \tilde{\sigma}))|^{-1}.$$

As $|\log G_r(X, x, (\sigma, \tilde{\sigma}))| \geq |\log G_r(X, x, (\sigma, \tilde{\sigma}))|$ grows linearly, it follows that $d_{\text{Martin}}^r(\sigma, \tilde{\sigma}) \geq C(\lambda^2/(h + \varepsilon))^N$, for any $\varepsilon > 0$ and a constant $C > 0$. Hence, we have shown that

$$d_{\text{visual}}(\sigma, \tilde{\sigma})^\beta \asymp \lambda_N \frac{\lambda^N}{(h + \varepsilon)^N} \ll d_{\text{Martin}}^r(\Xi(\sigma), \Xi(\tilde{\sigma})), \lambda_N = \lambda_N \ll d_{\text{visual}}(\sigma, \tilde{\sigma})^\alpha,$$

for $\alpha = \log \lambda / \log \lambda_{\text{visual}}$ and $\beta = (2 \log \lambda - \log (h + \varepsilon)) / \log \lambda_{\text{visual}}$. 

\section{An application to regular covers of convex-cocompact CAT(-1) metric spaces}

We give an application of our main theorem in order to characterize the set of $\delta$-conformal measures of a regular cover of a convex-cocompact CAT(-1) metric space, where we assume that $\delta$ is the abscissa of convergence of the cover and that the covering group is word hyperbolic.

We now recall the basic definitions and refer for the details to [11]. A CAT(-1) space $X$ is a geodesic space such that each geodesic triangle is thinner than a comparison triangle in the hyperbolic plane with constant curvature $-1$. An important feature of CAT(-1) spaces is that they are strongly hyperbolic which implies, in particular, that their visual boundary $\partial X$ coincides with the topological boundary, the visual metric on $\partial X$ simplifies to $D_{\partial X}(\xi, \eta) = e^{-\langle \xi, \eta \rangle}$, and, for any isometry $g$ of $X$, the conformal derivative $g'$ exists and satisfies, for any $\xi \in \partial X$

$$g'(\xi) := \lim_{\eta \to \xi, \eta \in \partial X} \frac{D_{\partial X}(g^2(\xi), g^2(\eta))}{D_{\partial X}(\xi, \eta)} = e^{-B_{\partial X}(g^{-1}(\eta), o)}.$$

In here, $B_{\partial X}(x, y) := \langle y \cdot \xi \rangle - \langle x \cdot \xi \rangle$ is the Busemann function, and $o$ is some point in $X$.

As we are interested in quotients of $X$, assume that $\Gamma$ is a discrete subgroup of the isometry group $\text{Isom}(X)$ of $X$ which acts freely and properly discontinuously on $X$. Furthermore,
as it is well known, these assumptions imply that \(X/\Gamma\) inherits several properties of \(X\), but only locally. For example, each element in \(X/\Gamma\) has a neighbourhood which is a CAT(-1) metric space. A basic object in the analysis of \(\Gamma\) is its limit set \(\Lambda(\Gamma)\) defined by \(\Lambda(\Gamma) := \overline{\Gamma(0)} \cap \partial X\) and the class of \(s\)-conformal measures (for \(s > 0\)), where a Borel probability measure on \(\partial X\) is referred to as an \(s\)-conformal measure if

\[
\mu(g(A)) = \int_A (g'(\xi))^s \, d\mu(\xi)
\]

for any Borel subset \(A\) of \(\partial X\). The relevance of these measures stems from the fact that under reasonable assumptions on \(\Gamma\) and \(s\), there is only one conformal measure, and this measure gives rise to canonical measures for the geodesic flow and the horocyclic foliation. As a standing assumption, we assume from now on that any subgroup of \(\text{Isom}(X)\) is discrete, and acts freely and properly discontinuously on \(X\). To each such \(\Gamma\), the Poincaré exponent is defined by

\[
\delta(\Gamma) := \sup \left\{ s \geq 0 : \sum_{g \in \Gamma} e^{-sd(0, g(0))} = \infty \right\}.
\]

Moreover, \(\Gamma\) is referred to as of divergence type if \(\sum_{g \in \Gamma} e^{-\delta d(0, g(0))} = \infty\). The relevance of this is a consequence of the Hopf-Tsuji theorem (for the setting of CAT(-1) spaces, see [11]), which states that \(\Gamma\) is of divergence type if and only if, for any \(\delta\)-conformal measure \(\mu\), the action of \(\Gamma\) on \((\partial X, \mu)\) is ergodic. Moreover, being of divergence type implies uniqueness of \(\mu\).

In fact, we are interested in the interplay of the following type of groups.

**Definition 6.1.** We refer to a cover \(Y\) of \(X/\Gamma\) as a regular cover (or periodic cover) if there exists a normal subgroup \(N\) of \(\Gamma\) such that \(Y = X/N\). In this situation, \(G := \Gamma/N\) refers to the covering group (or period) of the cover. Moreover, we refer to \(\Gamma\) as convex-cocompact if the convex hull of \(\Lambda(\Gamma)\) in \(X\) is compact.

Observe that the class of convex-cocompact groups acting on a CAT(-1) metric space is a flexible object as manifolds of pinched negative curvature with compact convex core as well as the action of a word hyperbolic group on its Cayley graph are in this class. Furthermore, due to the close connection to the basic example of an Anosov flow, the geodesic flow on a closed manifold of constant negative curvature, the action of this class of groups is well studied. For example, the Poincaré exponent of a convex-cocompact group his finite and the group is of divergence type ([11]) and, in particular, the geodesic flow is ergodic with respect to the Liouville-Patterson-Sullivan measure constructed from the unique \(\delta\)-conformal measure.

The geodesic flow on regular covers, on the other hand, in many cases is totally dissipative as the dynamics somehow behave like a random walk on the covering group. That is, even though there does not exist a complete dictionary between random walks and the regular covers, there are several parallel results, like Rees’ version of Polya’s result on the transience of the simple random walk for \(\mathbb{Z}^d\)-covers ([25]), or Brooks’ amenability criterion ([6], see also [31, 12, 10]) in the spirit of Kesten ([22]). Our application of Theorem 5.2 adds a further item to this list as it provides a complete description of the \(\delta(N)\)-conformal minimal measures in analogy to Ancona’s result on the geometric realization of minimal harmonic functions.
now on, we refer to $\mathcal{G}(X)$, $\mathcal{G}(X/G)$ and $\mathcal{G}(X/N)$ as the space of geodesics of the local CAT(-1) spaces $X$, $X/G$ and $X/N$.

**Theorem 6.2.** Assume that $X$ is a CAT(-1) space, that $\Gamma$ is a convex-cocompact, discrete subgroup of Isom($X$) and that $N$ is a normal, non-elementary subgroup of $\Gamma$ such that $G := \Gamma/N$ is hyperbolic and such that the geodesic flow on $\mathcal{G}(X/N)$ is topologically transitive. Then the set of minimal, $\delta(N)$-conformal measures can be identified with $\partial G$.

**Proof.** The strategy of proof is as follows. In Step 1, we construct the Markov map $(\Sigma, \theta)$ and its group extension by $G$. In Step 2, we then specify the associated potential and show that the reversibility condition $\mathbb{G}_B^G(\chi_g)(x, \text{id}) = \mathbb{G}_B^G(\chi_{\text{id}})(x, g)$ holds. Finally, in Steps 3 and 4, we identify the $\delta(N)$-conformal measures on $\partial X$ with the conformal measures on $\Sigma \times G$. The theorem then follows from Theorem 5.2.

**Step 1. The Markov map $\theta$.** The first part of proof makes use of coding of the geodesic flow on $\mathcal{G}(X/G)$ as constructed in [8]. In there, the authors construct a Poincaré section such that

(i) the first return map $H$ is coded by a topologically transitive bilateral subshift of finite type,

(ii) the atoms $\{A_1, \ldots, A_n\}$ of the Markov partition of the section are of the form $A_i = \pi(R_i)$, where the $\{R_i\}$ are, in Hopf coordinates, of the form

$$R_i = \{ (\zeta, \eta, t_i(\zeta, \eta)) : \zeta \in U_i, \eta \in V_i \},$$

for some disjoint open subsets $U_i, V_i$ of $\partial X$ and functions $t_i(\zeta, \eta) : U_i \times V_i \to \mathbb{R}$,

(iii) the return time $h : \bigcup_{i=1}^n A_i \to (0, \infty)$ to the section is Hölder continuous.

By considering the $p$-th iterate of $H$, where $p$ refers to the period of $\theta$, the subshift of finite type $\Theta^p$ decomposes into its topological mixing components. However, as $H$ is the first return map, any of these components provides us with a Markov coding for the geodesic flow whose associated subshift of finite type is topological mixing. Hence, we may assume without loss of generality that the shift is topologically mixing.

In order to associate elements of $\Gamma$ to $H$, we proceed as follows. The possible transitions of $\Sigma$ define a connected graph $\mathfrak{G}$. Now choose a subgraph $\mathfrak{T}$ which is connected, has no loops and has the same set of vertices as $\mathfrak{G}$, that is, $\mathfrak{T}$ is a minimal spanning tree of $\mathfrak{G}$. We now construct a lift of the atoms $\{A_1, \ldots, A_n\}$ to $\mathcal{G}(X)$ based on the choice of $\mathfrak{T}$. Suppose that the lift $\hat{A}_i \subset \mathcal{G}(X)$ of $A_i$ already was constructed. Furthermore, suppose that $A_j$ is a neighbour of $A_i$ in $\mathfrak{T}$.

Then $H(A_j) \cap (A_j) \neq \emptyset$ or $H(A_j) \cap A_j \neq \emptyset$. In the first case, there exists a unique lift $\hat{A}_j \subset \mathcal{G}(X)$ such that $\{ g_{h(\pi(x))} : x \in \hat{A}_j \} \cap \hat{A}_j \neq \emptyset$. And, if $H(A_j) \cap A_j = \emptyset$, the same argument gives rise to a unique $\hat{A}_j \subset \mathcal{G}(X)$ with $\{ g_{h(\pi(x))} : x \in \hat{A}_j \} \cap \hat{A}_j \neq \emptyset$. As $\mathfrak{T}$ is a minimal spanning tree, this construction provides a construction of lifts $\{ \hat{A}_1, \ldots, \hat{A}_n \}$. Moreover, as $\pi \circ g = \pi$ for all $g \in \Gamma$, we may identify $R_i := \hat{A}_i$, for $i = 1, \ldots, n$ in the property of the above coding.

The elements of $\Gamma$ associated to $H$ are now constructed as follows. Suppose that $x = (\zeta, \eta, t_i(\zeta, \eta)) \in R_i$ and that $g_{h(\pi(x))}(\pi(x)) \in A_j$. Then there exists a unique $\kappa_x \in \Gamma$, depending only on $i$ and $j$, such that $g_{h(\pi(x))}(\pi(x)) \in \kappa_x(R_j)$ and, in Hopf coordinates,

$$g_{h(\pi(x))}(\pi(\zeta, \eta, t_i(\zeta, \eta))) = \pi (\zeta, \eta, t_i(\zeta, \eta) + h(\pi(x))) = \pi (\kappa_x^{-1}(\zeta), \kappa_x^{-1}(\eta), t_j(\kappa_x^{-1}(\zeta), \kappa_x^{-1}(\eta))).$$
However, as the functions $t_i$ can be recovered from $\xi, \eta$ and $i$, we identify $H$ with
\[
\bigcup_{i=1}^n U_i \times V_i \to \bigcup_{i=1}^n U_i \times V_i, \quad (\xi, \eta) \mapsto \left( \kappa_{(\xi, \eta, t_i, \xi, \eta)}^{-1}(\xi), \kappa_{(\xi, \eta, t_i, \xi, \eta)}^{-1}(\eta) \right), \text{ for } (\xi, \eta) \in U_i \times V_i, \quad (20)
\]
where $\bigcup$ stands for the disjoint union. The following observation is crucial and follows immediately from the definition of a Markov partition (cf. Def. 3.8 in [8]). Namely, $\kappa_{(\xi, \eta, t_i, \xi, \eta)}$ in fact only depends on $\eta$ and $i$. Hence, with $\kappa_{\eta, i} := \kappa_{(\xi, \eta, t_i, \xi, \eta)}$, for $\eta \in V_i$, we obtain that
\[
\begin{align*}
H : \bigcup_{i=1}^n U_i \times V_i & \to \bigcup_{i=1}^n U_i \times V_i \\
(\xi, \eta) & \mapsto \left( \kappa_{\eta, i}^{-1}(\xi), \kappa_{\eta, i}^{-1}(\eta) \right)
\end{align*}
\]
where $\bigcup$ stands for the disjoint union. The following observation is crucial and follows immediately from the definition of a Markov partition (cf. Def. 3.8 in [8]). Namely, $\kappa_{(\xi, \eta, t_i, \xi, \eta)}$ in fact only depends on $\eta$ and $i$. Hence, with $\kappa_{\eta, i} := \kappa_{(\xi, \eta, t_i, \xi, \eta)}$, for $\eta \in V_i$, we obtain that
\[
\begin{align*}
\theta : \bigcup_{i=1}^n V_i & \to \bigcup_{i=1}^n V_i \\
\eta & \mapsto \kappa_{\eta, i}^{-1}(\eta)
\end{align*}
\]
commutes. In particular, by setting $\Sigma := \{ [V] \}$ and $\theta_{[V]} := \kappa_{\eta, i}^{-1}(\eta)$, we obtain a non-invertible, surjective Markov map which is coded by a one-sided topological mixing subshift of finite type and which, up to points on the boundaries $\{ \partial V_i \}$, is a factor of $H$.

**Step 2. Associated Measures and Reversibility of the Extension.** We now analyse the regularity of the potential function
\[
\varphi(\eta, i) := \delta(N) \log \left( (\kappa_{\eta, i}^{-1})'(\eta, i) \right) = -\delta(N) B_\eta \left( \kappa_{\eta, i}(\eta), \eta \right). \quad (21)
\]
As it is well known, the Busemann function is 1-Lipschitz continuous. Furthermore, it follows from the expansivity of the geodesic flow, that the map $\theta$ is eventually uniformly expanding. By combining these two observations, it follows that $\varphi : \Sigma \to \mathbb{R}$ is Hölder continuous with respect to the shift metric on $\Sigma$. Therefore, by application of Ruelle’s operator theorem, there exists a unique equilibrium state $hd\mu$ for $\varphi$, where $h$ is a Hölder continuous function which is bounded away from 0, and $\mu$ is a conformal measure, which means in the setting of Markov maps or shift spaces that $dv \circ \theta = \lambda e^{-\varphi} dv$, for some $\lambda > 0$.

The potential $\varphi$ is related to the Poincaré series of $N$ through the group extension
\[
T : \Sigma \times G \to \Sigma \times G, (x, gN) \mapsto (\theta(x), gxN)
\]
as follows. As $G$ is non-amenable, observe that it follows from the main result in [31] that $\lambda > 1$. In order to determine $R$, we make use of the fact that $R$ is the radius of convergence of the series $G_r(\chi_{id})(x, id)$, seen as a function of $r$. By conformality and by Lemma 2.3,
\[
G_r(\chi_{id})(x, id) = \int_{\chi_{id}} G_r(\chi_{id}) h d\mu = \sum_{n=0}^{\infty} r^n \int_{\chi_{id}} T^n \chi_{id} h d\mu = \sum_{n=0}^{\infty} r^n \sum_{w \in \mathcal{W}^n : \kappa w \in N} \mu([w])
\]
Now set, for $w \in \mathcal{W}^n$, $\kappa_w := \kappa_{x \kappa \theta(x) \cdots \kappa \theta^{-1}(x)}$ for some $x \in [w]$. As the Busemann function is a cocycle, it follows for $x = (\eta, i) \in [w]$ that
\[
S_n(\varphi)(x) := \sum_{k=0}^{n-1} \varphi \circ \theta^k(x) = -\delta(N) B_\eta (\kappa_w(\eta), \eta).
\]
Furthermore, a well known geometric argument for convex-cocompact groups shows that there exists a constant $K$, independent from $x = (\eta, i)$ and $w$ such that $\eta$ is in $[\eta \in \partial X : (\eta \cdot o) \leq K]$, known as the $K$-shadow of $\kappa_w(o)$ from $o$ of parameter $K$, which then implies that $B_\eta(\kappa_w(o), o) = d(\kappa_w(o), o) \pm C$, for some $C > 0$ (see, e.g., Observation 4.5.3 in [11]). Hence,

$$\mathcal{G}_r(\mathcal{X}_{id})(x, id) = \sum_{n=0}^{\infty} (\lambda r)^n \sum_{w : \kappa(w) \in N} e^{-\delta(N)d(\kappa_w(o), o)} = \sum_{w : \kappa(w) \in N} e^{-d(\kappa_w(o), o)(\delta(N) - C)}.$$  

As the return time to the section is bounded from away from 0 and infinity by construction, it follows that $C^{-1} |w| < d(\kappa_w(o), o) < C |w|$ for some $C > 0$. By combining this estimate with the observation that $\{w : \kappa(w) = g\} \neq \emptyset$ for all $g \in N$ as $X$ is a geodesic space and the $\bigcup A_i$ are a Poincaré section for the flow on $X/\Gamma$, one obtains the bound

$$\mathcal{G}_r(\mathcal{X}_{id})(x, id) \gg \sum_{g \in N} e^{-d(g(o), o)(\delta(N) - C)} $$

provided that $\lambda r \geq 1$. Hence, $\mathcal{G}_r(\mathcal{X}_{id})(x, id) = \infty$ for $\lambda r > 1$ as $\delta(N)$ is the Poincaré exponent of $N$. On the other hand, as $G$ is non-amenable, it follows from an application of a result by Zimmer in [35] (see also [19]) that the product of $\mu$ on $\Sigma$ and the counting measure on $G$ is not conservative. Hence, Proposition 5.3 in [32] implies that $\mathcal{G}_{\lambda^{-1}}(\mathcal{X}_{id})(x, id) < \infty$. Therefore, $R = 1/\lambda$.

We now verify the reversibility condition. In order to do so, we make use of the generalisation by Adachi in [13] of Rees’ refinement ([25]) and obtain that we may in fact assume that there is an involution $\iota$ on the elements of the partition which corresponds to the time reversal of the flow. This involution extends to finite words and, by a simple geometric argument, we have that $\kappa_w^{-1} = \kappa_w$ (see, e.g., the construction of the coding in [12]). Hence, for $g \in \Gamma$, this implies that

$$\mathcal{G}_R(\mathcal{X}_{gN})(x, id) \approx \sum_{w : \kappa(w) \in g^{-1}N} e^{-\delta(N)d(\kappa_w(o), o)} = \sum_{w : \kappa(w) \in gN} e^{-\delta(N)d(\kappa_w(o), o)} \approx \mathcal{G}_R(\mathcal{X}_{id})(x, gN).$$

**STEP 3. IDENTIFICATION OF CONFORMAL MEASURES.** We now show that there is a canonical bijection between $\delta(N)$-conformal measures on $\partial X$ with respect to $N$ and $\varphi$-conformal measures on $\Sigma \times G$. Furthermore, as the definition of conformality in both cases precisely describes the behaviour along $G$ and $T$-orbits, respectively, the topological transitivity implies that a conformal measure is uniquely determined by its action on the $V_i$.

For a given $\delta(N)$-conformal measures on $\partial X$ with respect to $N$ the identity [19] holds by definition for $s = \delta(N)$, any $g \in N$ and each Borel set $A \subset \partial X$. Define $\tilde{m} \chi_V \chi_{\{id\}} := m$. In particular it follows for any finite word $w$ with $\kappa_w \in N$ and each Borel subset $(A, i) \subset w$ with $T^{\kappa_w^{-1}}(A, i) \subset V_j$ from [21] that

$$\tilde{m} \left( T^{\kappa_w^{-1}}(A, i) \times \{id\} \right) = \tilde{m} \left( (\kappa_w^{-1}(A), j) \times \{id\} \right) = \tilde{m} \left( \kappa_w^{-1}(A) \right)$$

$$= \int_A (\kappa_w^{-1}(A))^{\delta(N)} d\tilde{m} = \int_{(A, i)} e^{\varphi_{(A, i)}} d\tilde{m}. $$
Hence, \( \tilde{m} \) is conformal with respect to those branches of \( T \) which start and end in \( \Sigma \times \{\text{id}\} \). Now assume that \( [w, g] \) is a cylinder in \( \Sigma \times G \). By topologically transitivity, there exists a cylinder \( [v, \text{id}] \) and \( n \in \mathbb{N} \) such that \( T^n([v, \text{id}]) \supset [w, g] \) and \( T^n|_{[v, \text{id}]} \) is injective. Then, as it easily can be verified (see, e.g., [32]),

\[
d\tilde{m}(A \cap [w, g]) := \int_{[v, \text{id}] \cap T^{-n}(A \cap [w, g])} e^{-S_n(x)} d\tilde{m}(x, \text{id})
\]

extends \( \tilde{m} \) to a well-defined and conformal measure on \( \Sigma \times G \).

We now show the reverse direction. On order to do so, fix a conformal measure \( m \) on \( \Sigma \times G \) and set, for each \( V_i, m_i := m|_{V_i \times \{\text{id}\}} \). Moreover, assume that \( g \in N \) and that \( A \) is an open subset of \( \partial X \) such that \( A \subset V_i \) and \( g(A) \subset V_j \) for some \( 1 \leq i, j \leq n \). Hence, \( A \subset V_i \cap g^{-1}(V_j) \) and, as the coding is topologically transitive, there exist \( h \in \Gamma \) and \( 1 \leq k \leq n \) with

\[
B := h(V_k) \subset A, \quad h(U_k) \supset U_l \cup g^{-1}(U_l).
\]

That is, there are open subset of \( U_l \times B \) and \( U_j \times g(B) \) which eventually flow into \( h^{-1}(U_k) \times h^{-1}(V_k) \) and \( (gh)^{-1}(U_k) \times (gh)^{-1}(V_k) \), respectively. Hence, there exist \( s, t \in \mathbb{N} \) such that locally \( T^s \) and \( T^t \) are of the form

\[
T^s : (B, i) \times \{\text{id}\} \to V_k \times \{hN\}, \quad ((x, i), \{\text{id}\}) \to ((h^{-1}(x), k), hN),
\]

\[
T^t : (gB, j) \times \{\text{id}\} \to V_k \times \{hN\}, \quad ((x, j), \{\text{id}\}) \to ((h^{-1}g^{-1}(x), k), hN),
\]

where we have used that \( ghN = hN \). Hence, by conformality of \( m \) and the cocycle property of the Busemann function, it follows for any integrable function \( f : B \to \mathbb{R} \) that

\[
\int f \, dm = \int \mathcal{L}^f(1_{B} \times \{\text{id}\}) \, dm = \int_{V_k \times \{hN\}} f(hx)e^{-\delta(N)B_h(h(o), o)} \, dm,
\]

\[
\int f \circ g^{-1} \, dm = \int \mathcal{L}^f(1_{gB} \times \{\text{id}\}) \, dm = \int_{V_k \times \{hN\}} f(hx)e^{-\delta(N)B_{gh}(gh(o), o)} \, dm
\]

\[
= \int_{V_k \times \{hN\}} f(hx)e^{-\delta(N)B_{gh}(gh(o), o) + B_{gh}(g(o), o)} \, dm.
\]

By combining (22) with (23), one then obtains that \( dm_j(x) \circ g = e^{-\delta(N)B_{gh}(gh(o), o)} \, dm_j(x) \) for \( x \in B \). By transitivity of \( T \) we may assume, without loss of generality, that \( h \in N \) and \( k = i \). Then, as \( h(A) \subset B \) and \( gh(A) = gh^{-1}(gA) \subset gB \), we have that the restriction to \( gB \) of some power of \( \theta \) is given by \( gh^{-1}g^{-1} \). Hence, for an integrable function \( f : A \to \mathbb{R} \), (22) this representation of \( m_j|_{gB} \) with respect to \( m_i|_B \) and the argument in (22) applied to \( g(B) \) and \( m_j \) imply that

\[
\int f \, dm_i = \int_B f \circ h^{-1} e^{\delta(N)B_{h}(h(o), o)} \, dm_i = \int_B f \circ h^{-1} e^{\delta(N)(B_{h}|_{[h(o), o]} + B_{g}(g(o), o))} \, dm_j \circ g
\]

\[
= \int_{g(B)} f \circ (gh)^{-1} e^{\delta(N)B_{gh}(gh(o), o)} \, dm_j
\]

\[
= \int_{g(A)} f \circ g^{-1} e^{\delta(N)(B_{gh^{-1}g^{-1}g(h(o), o) - B_{gh^{-1}g^{-1}g(h(o), o)})} \, dm_j
\]

\[
= \int_{g(A)} f \circ g^{-1} e^{\delta(N)B_{gh^{-1}g^{-1}}(o)} \, dm_j.
\]

32
In particular, if $g = \text{id}$, this implies that $m_i|_{V_i \cap V_j} = m_j|_{V_i \cap V_j}$. Hence, $dm_i(x) := dm_i\big|_{V_i \setminus V_j}$ for $x \in V_i$ is a well defined measure on $\partial X$. Furthermore, by applying the above identity for arbitrary $g \in N$, (21) shows that $m^1$ is a $\delta(N)$-conformal measure for $N$ as defined in (19).

Hence, we have shown that $m \rightarrow (\tilde{m}(\Sigma \times [\text{id}]))^{-1} \tilde{m}$ is a bijection from the set of $\delta(N)$-conformal measures for $N$ on $\partial X$ to the set of $\varphi$-conformal measures on $\Sigma \times G$ which are normalised by giving measure 1 to $\Sigma \times [\text{id}]$, and that the inverse of this map is given by $m \rightarrow (m^1(\partial X))^{-1} m^1$.

**STEP 4. RELATING $\varphi$- AND $\varphi'$-CONFORMAL MEASURES.** Recall that the reference measure on $\Sigma$ is given by $dv = h d\mu$, where $h d\mu$ is the equilibrium state for the potential $\varphi$. As it is well known, $\mu$ is the unique conformal measure with respect to potential $\tilde{\varphi}(\eta, i) = \varphi(\eta, i) + \log h(\eta, i) - \log g(\theta(\eta, i)) - \log \lambda$, and the transfer operators $L_{\mu}$ and $L_{\nu}$ of $\mu$ and $\nu$, respectively, are related through $\lambda h L_{\nu}(f) = L_{\mu}(h f)$. It then follows immediately from the definitions, that this relation extends to $\lambda h L_{\varphi}(f) = L_{\psi}(h f)$ on the level of group extensions where we silently extended $h$ to a function on $\Sigma \times G$. Hence, as

$$\int f \mathcal{L}_\varphi^\mu (dm) = \int \mathcal{L}_\mu (f h) \frac{L_{\mu}(h f \mu)}{\lambda h} \lambda h dm = \lambda \int \mathcal{L}_\psi (h f) \lambda h dm$$

for any continuous function $f$ with compact support and each $\sigma$-finite measure $m$, one obtains that $dm \rightarrow hdm$ defines a bijection between the space of conformal measures with respect to $\varphi$ and $\varphi'$, respectively.

We remark that Theorem 6.2 is related to conformal measures associated to ends of hyperbolic $n$-manifolds as introduced in [5]. In there, the authors construct for an arbitrary hyperbolic $n$-manifold a finite family of open sets such that each $\alpha$-conformal measure can be represented as a sum of conformal measures, where each of these measures is associated to one of these open sets. Hence, our result in here might be seen as a refinement of the above for regular covers as we obtain a complete description of the set of $\delta(N)$-conformal measures. In particular, the above shows that these ends could be replaced through an iterative construction by $\partial G$, provided that $\Gamma/N$ is word hyperbolic.

Moreover, Shwartz recently obtained a similar result for regular covers of cocompact Fuchsian groups and with respect to $\alpha$-conformal measures with $\alpha > \delta(N)$ (see [29]). This restriction is a consequence of the version of Shwartz of Ancona’s inequality which does not allow to include the critical parameter. On the other hand, by applying Theorem 4.1 in [30] to our setting, we obtain as by Shwartz in [29] an ergodic theoretic description of these minimal measures as a corollary. In order to do so, we recall the notions of limit sets and uniform approximating sequences. The limit set of a group of isometries $\Gamma$ is defined by $\Lambda(\Gamma) := \Gamma(\langle \text{o} \rangle) \cap \partial X$, that is $\Lambda(\Gamma)$ is the set of accumulation points of the orbit $\Gamma(\langle \text{o} \rangle)$ in $\partial X$. Moreover, we say that the sequence $(g_n)$ in $\Gamma$ uniformly approximates $\eta \in \partial X$ if $\lim_{n \to \infty} g_n(\text{o}) = \eta$ and there exists $C > 0$, depending on $(g_n)$, such that the distance between the geodesic ray from $\text{o}$ to $\eta$ is bounded from above by $C$.

**Corollary 6.3.** Under the assumptions of Theorem 6.2 the following holds.

(i) Assume that $\mu$ is a minimal, $\delta(N)$-conformal measure for $N$. Then $\mu$ is ergodic for the action of $N$ on $\partial X$ and there exists $\sigma \in \partial(\Gamma/N)$ such that for $\mu$-a.e. $\eta \in \Lambda(N)$, and for every $(g_n)$ in $\Gamma$ which uniformly approximates $\eta$, we have that $\lim_{n \to \infty} g_n N = \sigma$.  

33
(ii) Assume that \( \sigma \in \partial(\Gamma/N) \). Then there exists a unique \( \delta(N) \)-conformal measure \( \mu \) for \( N \) such that for \( \mu \)-a.e. \( \eta \in \Lambda(N) \), and for every \( (g_n) \in \Gamma \) which uniformly approximates \( \eta \), we have that \( \lim_{n \to \infty} g_n N = \sigma \).

**Proof.** We begin with the proof of the first part. The ergodicity of \( \mu \) is an immediate consequence of minimality as any \( G \)-invariant set \( A \subset \partial X \) defines a \( \delta(N) \)-conformal measure \( dm := 1_A d\mu \) with \( m \leq \mu \). Hence, \( \mu(A) = 0 \) or \( \mu(A) = 1 \).

In order to show convergence, we employ Theorems 5.2 and 6.2 as they imply that there exists \( \sigma \in \partial(\Gamma/N) \) such that \( \mu = \mu_\sigma \). By Theorem 4.1 in [30] and the coding constructed in the proof of the theorem, it then follows for almost every element in \( ((\eta,i),id) \in V_i \times \{id\} \) that the orbit \( (T^n((\eta,i),id)) \) converges to the element \( \Xi^{-1}(\sigma) \) in the Martin boundary (cf. Theorem 5.4 for the definition of \( \Xi \)). However, by Theorem 5.2 above, this implies that the second coordinate of \( (T^n((\eta,i),id)) \) converges to \( \sigma \in \partial(\Gamma/N) \). It hence remains to relate this convergence with uniform approximation.

In order to do so, set \( g_n := \kappa_{\theta L^{-1}(\eta,i)} \circ \cdots \circ \kappa_{(\eta,i)} \) and choose an element \( \xi \in U_i \). Then, by the coding construction, the geodesic \( (\xi,\eta) \) from \( \xi \) to \( \eta \) passes through the closure of \( g_n(\bigcup_j R_j) \), where the \( R_j \) refer to the atoms of the coding construction in the proof above. However, as \( \Gamma \) is convex-cocompact, the diameter of the projection of \( \bigcup_j R_j \) to \( X \) is finite. Therefore, \( (g_n(\mathbf{o})) \) stays within a bounded distance from \( (\xi,\eta) \) and converges to \( \eta \). By combining this observation with the above convergence, one then obtains that \( (g_n(\mathbf{o})) \) uniformly approximates \( \eta \) and that \( g_n N \to \sigma \) almost surely.

It is left to prove the claim for an arbitrary sequence \( (h_n) \) which uniformly approximates \( \eta \). As the return time to \( \bigcup_j R_j \) is bounded from above, it follows that \( \sup_{j} d_X(g_n(\mathbf{o}),g_{n+1}(\mathbf{o})) < \infty \). In particular, there exists a sequence \( (n_k) \) such that \( \sup_{j} d_X(h_k(\mathbf{o}),g_{n_k}(\mathbf{o})) < \infty \). Hence, as \( \Gamma \) acts discontinuously on \( X \), the set \( \{h_k^{-1}g_{n_k} : k \in \mathbb{N} \} \) is finite. Therefore, \( h_k N \) and \( g_{n_k} N \) stay within a bounded distance with respect to the theorem.

The second part of the theorem is consequence of Part (i) combined with the fact that there is a bijection between \( \partial(\Gamma/N) \) and the set of minimal, \( \delta(N) \)-conformal measures.

\[ \square \]

7 Appendix: Reduced measures and the domination principle

In this part, following the exposition in [34], well-known ideas from potential theory for Markov operators are adapted to the setting of Ruelle operators on locally compact shift spaces (see also [30]). We begin with the following version of the Riesz decomposition theorem. Throughout this part, we assume that the potential is transient.

**Proposition 7.1** (Lemma 3.2 in [30]). Let \( \mu \) be a 1/r-excessive measure for \( 0 < r \leq R \). Then there exists a unique pair of Radon measures \( \mu_0 \) and \( \nu \) such that \( \mu_0 \) is 1/r-conformal and \( \mu = \mu_0 + G_r^\ast(\nu) \) in the sense that

\[ \int f d\mu = \int f d\mu_0 + \int G_r(f)(z) d\nu(z) \]

for any continuous \( f \) with compact support. Moreover, \( \nu = \mu - r^{-1} \mathcal{L}_r^\ast(\mu) \).

34
Proof. The existence follows from Lemma 3.2 in [30]. Now assume that \( \mu_0 + G_\tau^*(v) = \tilde{\mu}_0 + G_\tau^*(\tilde{v}) \). By applying \( r^{-1}L^* \) to both sides, it follows that

\[
\mu_0 + G_\tau^*(v) - v = \tilde{\mu}_0 + G_\tau^*(\tilde{v}) - \tilde{v}.
\]

Hence, \( v = \tilde{v} \). \( \square \)

Recall that \( \mu \leq v \) if \( \mu(A) \leq v(A) \) for all \( A \in \mathcal{B} \). The Riesz decomposition theorem has the following useful consequences.

Lemma 7.2. Assume that \( \mu \) is \( 1/r \)-excessive and that \( \mu \leq G_\tau^*(v) \) for a measure \( v \) such that \( G_\tau^*(v) \) is \( \sigma \)-finite. Then there exists \( v_0 \) such that \( \mu = G_\tau^*(v_0) \). In particular, if \( \mu \) is harmonic, then \( \mu = 0 \).

Proof. By the above, \( \mu = \mu_0 + G_\tau^*(v_0) \). Therefore, \( \mu_0 \leq G_\tau^*(v) \) and \( \mu_0 = 0 \) since

\[
\mu_0 = r^{-n}(L^*)^*(\mu_0) \leq r^{-n}(L^*)^*(G_\tau^*(v)) \rightarrow 0.
\]

The assertion then follows from the uniqueness of the Riesz decomposition. \( \square \)

For a family of \( \sigma \)-finite measures \( \{\mu_i : i \in I\} \), define \( \mu^I(A) := \inf_{i \in I} \mu_i(A) \) and

\[
\bigwedge_{i \in I} \mu_i(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu^I(\{B_j \} : \bigcap_{j=1}^{\infty} B_j = A, B_j \in \mathcal{B}) \right\}, \quad \text{for } A \in \mathcal{B}.
\]

We then refer to \( \bigwedge_{i \in I} \mu_i(A) \) as the infimum of the family \( \{\mu_i : i \in I\} \).

Proposition 7.3. The infimum of a family of Radon measures is a Radon measure. Moreover, the infimum of a family of \( \lambda \)-excessive measures is \( \lambda \)-excessive.

Proof. We begin showing that \( \mu := \bigwedge_{i \in I} \mu_i \) is a measure. In order to do so, first observe that for a partition of \( A \in \mathcal{B} \) into a \( \{A_i \in \mathcal{B} : i \in \mathbb{N}\} \) that \( \mu(A) \leq \sum_i \mu(A_i) \) by construction. On the other hand, observe that for partitions \( \{B_j : j \in \mathbb{N}\}, \{B_k^I : k \in \mathbb{N}\} \) of \( A \) into Borel sets such that the second is finer than the first,

\[
\sum_{j=1}^{\infty} \mu^I(\{B_j \} \leq \sum_{k=1}^{\infty} \mu^I(\{B_k^I \). \tag{24}
\]

In particular, this implies that \( \mu(A) \geq \sum_i \mu(A_i) \), as each partition of \( A \) has a refinement which is measurable with respect to \( \sigma \)-finite \( \mu \). Hence, \( \mu \) is \( \sigma \)-additive and therefore a measure. Moreover, as \( \mu(A) \leq \mu_i(A) \) for all \( i \), \( \mu \) is \( \sigma \)-finite. The Radon property follows immediately from \( \mu(A) \leq \mu_i(A) \) for all \( i \in I \) and \( A \in \mathcal{B} \), which proves the first assertion.

Now assume that \( f : X \to [0, \infty) \) is uniformly continuous and bounded, that \( A \in \mathcal{B} \) with \( \mu(A) < \infty \) and that \( \epsilon \) is arbitrary. By applying (24), we may suppose that \( \mu(A) \leq \sum_j \mu^I(\{B_j \) \leq \mu(A) + \epsilon \), where the \( \{B_j : j \in \mathbb{N}\) is a partition of \( A \) into sets of diameter \( \delta \). If \( \delta \) is chosen sufficiently small, uniform continuity implies that

\[
\int_A f \ d\mu = \sum_{j=1}^{\infty} \int_{B_j} f \ d\mu \leq \sum_{j=1}^{\infty} \left( \inf_{x \in B_j} f(x) + \epsilon \right) \mu^I(\{B_j \) \leq \sum_{j=1}^{\infty} \int_{B_j} f \ d\mu_i + \epsilon(\mu(A) + \epsilon)
\leq \inf_{i \in I} \int_A f \ d\mu_i + \epsilon \mu(A) + \epsilon^2.
\]

35
As \( \epsilon \) is arbitrary, it follows that \( \int_A f \, d\mu \leq \inf I \int_A f \, d\mu_i \). The remaining assertion easily follows from this.

Now assume that \( \mu \) is a \( \lambda \)-excessive measure and that \( A \in \mathcal{B} \). Then we refer to

\[
R_A(\mu) := \bigwedge \{ \nu \in \mathcal{B} : \mathcal{L}^\nu(A) \leq \lambda \nu, \nu|_A \geq \mu|_A \}
\]

as the reduced measure associated with \( \mu \) on \( A \), which is a well-defined, \( \lambda \)-excessive Radon measure by the above Proposition. Furthermore, if \( A = \Sigma \times K \), for some finite \( K \subset G \), then \( \mathcal{G}^*_\mu(A) \) is well-defined, \( \lambda \)-excessive and \( \mathcal{G}^*_\mu(A) \geq \mu|_A \). In particular, \( R_A(\mu) \leq \mathcal{G}^*_\mu(A) \).

Hence, Lemma 7.2 implies that there exists \( \nu_0 \) such that \( R_A(\mu) = \mathcal{G}^*_\mu(\nu_0) \) and, as \( \nu_0 = R_A(\mu) - r^{-1} \mathcal{L}^\nu(R_A(\mu)) \), \( \nu_0 \) is a Radon measure. Also note that \( R_A(\mu)|_A = \mu|_A \) by construction. Hence, by letting \( K \to G \), one immediately obtains the following.

**Proposition 7.4.** Assume that \( \mu \) is \( 1/r \)-excessive. Then there exists an increasing sequence of Radon measures \( (\nu_n) \) such that \( \mu(A) = \lim_{n \to \infty} \mathcal{G}^*_\mu(\nu_n)(A) \), for all \( A \in \mathcal{B} \).

Now assume that \( A \) is measurable with respect to the partition into \( n \)-cylinders, for some \( n \in \mathbb{N} \). Then \( 1_A \) is Lipschitz continuous, and, in particular, \( \mathcal{L}_A(f) := \mathcal{L}(1_A f) \) acts on continuous functions with compact support. Furthermore, consider

\[
\mathcal{G}^A := 1_A \sum_{n=0}^\infty r^n(\mathcal{L}_A)^n, \quad \mathcal{F}_A := 1_A \sum_{n=0}^\infty r^n(\mathcal{L}_A)^n.
\]

Observe that this choice of \( A \) implies by Proposition 3.2 that \( \mathcal{G}^A \) and \( \mathcal{F}_A \) act on Lipschitz functions with support on \( \Sigma \times K \), for \( K \subset G \) finite, for \( 0 < r \leq R \). In particular, the actions of \( (\mathcal{G}^A)^* \) and \( \mathcal{F}_A^* \) on \( \sigma \)-finite measures are well defined.

**Theorem 7.5.** Assume that \( \mu \) is \( 1/r \)-excessive and that \( A \) is measurable with respect to the partition into \( n \)-cylinders, for some \( 0 < r \leq R \) and \( n \in \mathbb{N} \). Then the reduced measure on \( A \) is equal to \( \mathcal{F}_A^*(\mu) \).

**Proof.** Set \( B := A' \). By Proposition 7.4, there exists a monotone sequence of Radon measures \( (\nu_n) \) such that \( \mathcal{G}^*_\mu(\nu_n) \to \mu \). Moreover, by decomposing orbits with respect to the first entry to \( A \), one obtains that \( \mathcal{G}_r = \mathcal{G}^B_r + \mathcal{G}_r \circ \mathcal{F}_A \), which then implies that

\[
\mathcal{F}_A^* \circ \mathcal{G}^*_\mu(\nu_n) = (\mathcal{G}_r - \mathcal{G}^B_r)^* (\nu_n) \leq \mathcal{G}^*_\mu(\nu_n).
\]

By taking the limit as \( n \to \infty \), one obtains that \( \mathcal{F}_A^*(\mu)(A) \leq \mu(A) \). As, by construction, \( \mathcal{F}_A^*(\mu)|_A = \mu|_A \), it follows that \( \mu|_A \leq \nu|_A \) implies that \( \mathcal{F}_A^*(\mu) \leq \mathcal{F}_A^*(\nu) \) globally. Hence, if \( \mathcal{F}_A^*(\mu) \) is excessive, then \( \mathcal{F}_A^*(\mu) = R_A(\mu) \).

It remains to show that \( \mathcal{F}_A^*(\mu) \) is excessive. By iterated application of \( \mathcal{L}^\nu(A) \leq r^{-1} \mu \) one obtains for a test function \( h \geq 0 \) that

\[
\int 1_A h \, d\mu \geq r \int \mathcal{L}(1_A h) \, d\mu = r \int 1_A \mathcal{L}_A(h) \, d\mu + r \int 1_B \mathcal{L}_A(h) \, d\mu \geq r \int 1_A \mathcal{L}_A(h) \, d\mu + r^2 \int 1_A \mathcal{L}_B \mathcal{L}_A(h) \, d\mu + r^2 \int 1_B \mathcal{L}_B \mathcal{L}_A(h) \, d\mu \geq \int 1_A \sum_{k=0}^n r^{k+1} \mathcal{L}_B^k \mathcal{L}_A(h) \, d\mu + r^{n+1} \int 1_B \mathcal{L}_B^n \mathcal{L}_A(h) \, d\mu.
\]
Hence, by monotone convergence,
\[
\int 1_A h d\mu \geq \int 1_A \sum_{n=0}^{\infty} r^{n+1} \mathcal{L}_B^n \mathcal{L}_A(h) d\mu.
\]

On the other hand,
\[
r \mathcal{F}_A \circ \mathcal{L} = 1_A \sum_{n=0}^{\infty} r^{n+1} \mathcal{L}_B^n \circ (\mathcal{L}_B + \mathcal{L}_A) = \mathcal{F}_A - 1_A \sum_{n=0}^{\infty} r^{n+1} \mathcal{L}_B^n \circ \mathcal{L}_A,
\]
which implies that
\[
r \int \mathcal{L} h d\mathcal{F}_A^*(\mu) = \int h d\mathcal{F}_A^*(\mu) - \int h - \sum_{n=0}^{\infty} r^{n+1} \mathcal{L}_B^n \mathcal{L}_A(h) d\mu \leq \int h d\mathcal{F}_A^*(\mu),
\]
proving that \( \mathcal{F}_A^*(\mu) \) is \( 1/r \)-excessive.

As \( \mathcal{F}_A^*(\mu) \) is dominated by \( G_r^*(\mu|A) \), it follows from Lemma 7.2 that there exists a unique \( \nu \) with \( \mathcal{F}_A^*(\mu) = G_r^*(\nu_A) \). If \( \mu = G_r^*(\nu) \) for some \( \nu \), this gives rise to a map \( \nu \mapsto \nu_A \), where \( \nu_A \) is referred to as the balayée of \( \nu \) and can be constructed explicitly as follows. Set
\[
\mathcal{R}_A(f) := \sum_{n=0}^{\infty} r^n (1_A \mathcal{L})^n (1_A f).
\]
As each orbit with at least one visit to \( A \) can be decomposed either with respect to the last or the first visit to \( A \), it follows that \( G_r \circ \mathcal{F}_A = \mathcal{R}_A \circ G_r \). Therefore, the balayée of \( \nu \) is given by \( \nu_A = \mathcal{R}_A^*(\nu) \) as, for \( \mu = G_r^*(\nu) \),
\[
R_A(\mu) = \mathcal{F}_A^*(\mu) = \mathcal{F}_A^* \circ G_r^*(\nu) = G_r^* \circ \mathcal{F}_A^*(\nu) = G_r^*(\nu_A).
\]
In particular, if \( \nu \) is supported on \( A \), then \( \nu_A = \mathcal{R}_A^*(\nu) = \nu \). This proves the following result, known as domination principle.

**Theorem 7.6.** Assume that \( A \) is measurable with respect to the partition into \( n \)-cylinders for some \( n \in \mathbb{N} \) and that \( \nu \) is a finite Radon measure whose support is contained in \( A \) such that \( G_r^*(\nu) \) is well defined \((0 < r \leq R)\). If \( \mu \) is \( 1/r \)-excessive and \( \mu|A \geq G_r^*(\nu)|A \), then \( \mu \geq G_r^*(\nu) \).

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