IMAGES OF $\ell$-ADIC REPRESENTATIONS AND AUTOMORPHISMS OF ABELIAN VARIETIES

A. SILVERBERG AND YU. G. ZARHIN

1. INTRODUCTION

Suppose that $F$ is either a global field or a finitely generated extension of $\mathbb{Q}$, $A$ is an abelian variety over $F$, $\ell$ is a prime number, and $\ell \neq \text{char}(F)$. Let $\mathfrak{G}_\ell(F,A)$ denote the algebraic envelope of the image of the absolute Galois group of $F$ under the $\ell$-adic representation associated to $A$, and let $\mathfrak{G}_\ell(F,A)^0$ denote its identity connected component. In §3 we prove that the intersection of $\mathfrak{G}_\ell(F,A)^0$ with the torsion subgroup of the center of $\text{End}(A) \otimes \mathbb{Q}$ is independent of $\ell$. In the case where $F$ is a finitely generated extension of $\mathbb{Q}$, this would follow from the Mumford-Tate Conjecture. Our results do not assume the Mumford-Tate Conjecture, and apply even in the positive characteristic case, where there is no analogue of the Mumford-Tate Conjecture. The result in the characteristic zero case can therefore be viewed as providing evidence in the direction of the Mumford-Tate conjecture.

Let $F(\text{End}(A))$ denote the smallest extension of $F$ over which all the endomorphisms of $A$ are defined. Then (see Proposition 2.10 of [17]),

$$F(\text{End}(A)) \subseteq F_{\Phi,\ell}(A).$$

By enlarging the ground field, we may assume that $F = F(\text{End}(A)) = F_{\Phi,\ell}(A)$. We then consider twists of the $\ell$-adic representations associated to $A$. The results of this paper follow from the $\ell$-independence of the connectedness extensions associated to these twists.

See [18] for a study of the connectedness extensions attached to twists of abelian varieties. See [17] for conditions for the connectedness of $\mathfrak{G}_\ell(F,A)$.

2. DEFINITIONS, NOTATION, AND LEMMAS

Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ denote respectively the integers, rational numbers, and complex numbers. If $G$ is an algebraic group, let $G^0$ denote the identity connected component. If $F$ is a field, let $F^{\text{sep}}$ denote a separable closure of $F$ and let $\bar{F}$ denote an algebraic closure of $F$. If $A$ is an abelian variety over a field $F$, write $\text{End}_F(A)$ for the endomorphisms of $A$ which are defined over $F$, write $\text{Aut}_F(A)$ for the automorphisms of $A$ defined over $F$, let $\text{End}(A) = \text{End}_{F^{\text{sep}}}(A)$, let $\text{End}^0(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q},$
Proof. Since $w$ and does not depend on the choice of using Lemma 2.1. The result follows.

Lemma 2.2. Suppose $A$ and $B$ are abelian varieties over a field $F$, $L$ is a finite extension of $F$ in $F^s$, and $\ell$ is a prime number, then

$$\mathcal{G}_\ell(L, A) \subseteq \mathcal{G}_\ell(F, A)$$

In particular, if $\mathcal{G}_\ell(F, A)$ is connected, then $\mathcal{G}_\ell(F, A) = \mathcal{G}_\ell(L, A)$.

Lemma 2.3. Suppose $A$ is an abelian variety over a field $F$ and

$$c : \text{Gal}(F^s/F) \to \text{End}_F^0(A)$$

is a continuous homomorphism of finite order. For each prime $\ell \neq \text{char}(F)$ let

$$\rho_{\ell,c} : \text{Gal}(F^s/F) \to \text{Aut}(V_\ell(A)), \quad \sigma \mapsto c(\sigma)\rho_{A,\ell}(\sigma)$$

be the twist of $\rho_{A,\ell}$. Then $\{\rho_{\ell,c}\}$ constitutes a strictly compatible system of integral $\ell$-adic representations of $\text{Gal}(F^s/F)$. More precisely, suppose $M$ is a finite Galois extension of $F$ such that $c$ factors through $\text{Gal}(M/F)$. Let $S_\ell$ be the set of finite places $v$ of $F$ such that either $A$ has bad reduction at $v$, $v$ is ramified in $M/F$, or the residue characteristic of $v$ is $\ell$. Let $v$ be a finite place of $F$, $w$ a place of $F^s$ lying over $v$, and $\kappa_v$ and $\kappa_w$ the residue fields at $v$ and $w$, respectively. Let $\tau \in \text{Gal}(F^s/F)$ be an element that acts as the Frobenius automorphism of $\kappa_w/\kappa_v$. Suppose that $v \notin S_\ell$, and let

$$\varphi_w = \rho_{\ell,c}(\tau) \in \text{Aut}(V_\ell(A)).$$

Then $\rho_{\ell,c}$ is unramified at $v$, the characteristic polynomial $P_v(t)$ of $\varphi_w$ lies in $\mathbb{Z}[t]$ and does not depend on the choice of $w$ and $\ell$, and the roots of $P_v(t)$ all have complex absolute value $\sqrt{|\kappa_v|}$. 

and let $\text{End}_F^0(A) = \text{End}_F(A) \otimes \mathbb{Z}_p$. If $\ell$ is a prime number and $\ell \neq \text{char}(F)$, let $T_\ell(A) = \lim A_\ell$ (the Tate module), let $V_\ell(A) = T_\ell(A) \otimes \mathbb{Z}_p$, and let $\rho_{A,\ell}$ denote the $\ell$-adic representation

$$\rho_{A,\ell} : \text{Gal}(F^s/F) \to \text{Aut}(T_\ell(A)) \subseteq \text{Aut}(V_\ell(A)).$$
Proof. We use that \( \{ \rho_{A, \ell} \} \) is a strictly compatible system of integral \( \ell \)-adic representations of \( \text{Gal}(\overline{F}^s/F) \) (see §3 and I.2 of [13]). Let \( A_v \) be the abelian variety over \( \kappa_v \) which is the reduction of \( A \) at \( v \). The choice of \( w \) allows us to identify the Tate modules \( V_f(A) \) and \( V_f(A_v) \), and this identification is compatible with the natural embedding \( \text{End}^0(A) \hookrightarrow \text{End}^0(A_v) \). Let

\[
Fr_w = \rho_{A, \ell}(\tau) \in \text{Aut}(V_f(A)).
\]

Then

\[
\varphi_w = c(\tau)Fr_w \in \text{Aut}(V_f(A)) \subseteq \text{Aut}(V_f(A_v)),
\]

and the identification of \( \text{Aut}(V_f(A)) \) with \( \text{Aut}(V_f(A_v)) \) identifies \( Fr_w \) with the Frobenius endomorphism of \( A_v \) inside \( \text{Aut}(V_f(A_v)) \). It follows from Weil’s results on endomorphisms of abelian varieties that \( P_\ell(v) \) has rational coefficients which do not depend on the choice of \( w \) and \( \ell \). If \( m = [M:F] \) then \( (c(\tau)Fr_w)^m = (Fr_w)^m \in \text{End}(A_v) \) and therefore all roots of \( P_\ell(v) \) are algebraic integers. Therefore, \( P_\ell(v) \in \mathbb{Z}[\ell] \). Further, Weil’s results imply that the eigenvalues of \( \varphi_w \) have absolute value \( \sqrt{\#\kappa_v} \). \( \square \)

**Theorem 2.4.** If \( F \) is either a finitely generated extension of \( \mathbb{Q} \) or a function field in one variable over a finite field, then every finite abelian group occurs as a Galois group over \( F \).

**Proof.** See Theorem 3.12c of [1], and IV.2.1 and IV.1.2 of [1]. \( \square \)

Next we define the Mumford-Tate group of a complex abelian variety \( A \) (see §2 of [10] or §6 of [22]). If \( A \) is a complex abelian variety, let \( V = H_1(A(\mathbb{C}), \mathbb{Q}) \) and consider the Hodge decomposition \( V \otimes \mathbb{C} = H_1(A(\mathbb{C}), \mathbb{C}) = H^{1,0} \oplus H^{0,-1} \). Define a homomorphism \( \mu : G_m \rightarrow GL(V) \) as follows. For \( z \in \mathbb{C} \), let \( \mu(z) \) be the automorphism of \( V \otimes \mathbb{C} \) which is multiplication by \( z \) on \( H^{1,0} \) and is the identity on \( H^{0,-1} \).

**Definition 2.5.** The Mumford-Tate group \( MT_A \) of \( A \) is the smallest algebraic subgroup of \( GL(V) \), defined over \( \mathbb{Q} \), which after extension of scalars to \( \mathbb{C} \) contains the image of \( \mu \).

If \( A \) is an abelian variety over a subfield \( F \) of \( \mathbb{C} \), we fix an embedding of \( \mathbb{F} \) in \( \mathbb{C} \). This gives an identification of \( V_f(A) \) with \( H_1(A, \mathbb{Q}) \otimes \mathbb{Q}_\ell \), and allows us to view \( MT_A \times \mathbb{Q}_\ell \) as a linear \( \mathbb{Q}_\ell \)-algebraic subgroup of \( GL(V_f(A)) \). Let

\[
MT_{A, \ell} = MT_A \times \mathbb{Q}_\ell.
\]

Then \( MT_A(\mathbb{Q}_\ell) = MT_{A, \ell}(\mathbb{Q}_\ell) \). The Mumford-Tate conjecture for abelian varieties (see [12]) may be reformulated as the equality of \( \mathbb{Q}_\ell \)-algebraic groups, \( \mathcal{G}_\ell(F, A)^0 = MT_{A, \ell} \).

**Conjecture 2.6** (Mumford-Tate Conjecture). If \( A \) is an abelian variety over a finitely generated extension \( F \) of \( \mathbb{Q} \), then \( \mathcal{G}_\ell(F, A)^0 = MT_{A, \ell} \).

The inclusion \( \mathcal{G}_\ell(F, A)^0 \subseteq MT_{A, \ell} \) was proved by Piatetski-Shapiro [1], Deligne [2], and Borovoi [6].

It is well-known that \( MT_A \) contains the homotheties \( G_m \) and that the centralizer of \( MT_A \) in \( \text{End}(V) \) is \( \text{End}^0(A) \). Therefore, the center of \( MT_A(\mathbb{Q}) \) contains \(-1\) and is contained in the center of \( \text{End}^0(A) \).
3. $\ell$-INDEPENDENCE

Suppose that $F$ is either a finitely generated extension of $\mathbb{Q}$ or a global field. Suppose $F = F_0(A)$, so that $\mathfrak{G}_\ell(F, A) = \mathfrak{G}_\ell(F, A)^0 = \mathfrak{G}_\ell(L, A)$ for all finite extensions $L$ of $F$. It follows from [13], [14], [20], and [21], and VI.5 and XII.2 of [2], that $\mathfrak{G}_\ell(F, A)$ is a reductive $\mathbb{Q}_\ell$-algebraic group, whose centralizer in $\text{End}(V_\ell(A))$ is $\text{End}(A) \otimes \mathbb{Q}_\ell$. This implies that the center of $\mathfrak{G}_\ell(F, A)(\mathbb{Q}_\ell)$ is isomorphic to $\mu_{\mathfrak{G}_\ell}(\mathbb{Q}_\ell)$, and therefore is independent of $\ell$. In the following two results we prove that $\mu_A \cap \mathfrak{G}_\ell(F, A)(\mathbb{Q}_\ell)$ is independent of $\ell$ (without assuming the Mumford-Tate Conjecture).

It follows from Weil’s results on abelian varieties [19] (as was pointed out by Deligne; see 2.3 of [12]) that $\mathfrak{G}_\ell(F, A)$ contains the homotheties $G_m$. In particular, $-1 \in \mathfrak{G}_\ell(F, A)(\mathbb{Q}_\ell)$.

**Theorem 3.1.** Suppose $A$ is an abelian variety over a finitely generated extension $F$ of $\mathbb{Q}$, and $F = F_\ell(A)$. Let $\mu_A$ denote the group of elements of finite order in the center of $\text{End}^0(A)$. Then $\mu_A \cap \mathfrak{G}_\ell(F, A)(\mathbb{Q}_\ell)$ is independent of the prime $\ell$.

**Proof.** Over $\mathbb{C}$, we can view $A$ as $C^d/L$ with $L$ a lattice in $C^d$. Then $L' = \sum_{\gamma \in \mu_A} \gamma(L)$ is a $\mu_A$-invariant lattice in $C^d$ that contains $L$ as a subgroup of finite index. The complex abelian variety $C^d/L'$ has a model $A'$ defined over a finite extension $F'$ of $F$ such that $A$ and $A'$ are $F'$-isogenous and $\mu_A$ coincides with the set of elements of finite order in the center of $\text{End}(A')$. Since $\mathfrak{G}_\ell(F', A') = \mathfrak{G}_\ell(F', A) = \mathfrak{G}_\ell(F, A)$, we may assume without loss of generality that $\mu_A$ coincides with the set of elements of finite order in the center of $\text{End}(A)$. By Theorem 2.4, we can choose an abelian extension $M$ of $F$ such that $\text{Gal}(M/F)$ is isomorphic to $\mu_A$. Let

$$\chi: \text{Gal}(M/F) \rightarrow \mu_A$$

be an isomorphism, let $c: \text{Gal}(F^s/F) \rightarrow \mu_A$ be the composition of $\chi$ with the projection $\text{Gal}(F^s/F) \rightarrow \text{Gal}(M/F)$, and let $B$ denote the twist of $A$ by the cocycle induced by $c$. By Lemma 2.3, $\mathfrak{G}_\ell(F, B)^0 = \mathfrak{G}_\ell(F, A)$. The character $c$ induces an isomorphism

$$\text{Gal}(M/F_\ell(B)) \cong \mu_A \cap \mathfrak{G}_\ell(F, B)^0(\mathbb{Q}_\ell) = \mu_A \cap \mathfrak{G}_\ell(F, A)(\mathbb{Q}_\ell).$$

Since $\text{Gal}(M/F_\ell(B))$ is independent of $\ell$, we are done. \quad \square

**Theorem 3.2.** Suppose $F$ is a function field in one variable over a finite field, $A$ is an abelian variety over $F$, and $\ell$ is a prime number not equal to $\text{char}(F)$. Suppose $F = F_\ell(A)$, and let $\mu_A$ denote the group of elements of finite order in the center of $\text{End}^0(A)$. Then $\mu_A \cap \mathfrak{G}_\ell(F, A)(\mathbb{Q}_\ell)$ is independent of $\ell$.

**Proof.** By Theorem 2.4 we can choose an abelian extension $M$ of $F$ such that $\text{Gal}(M/F)$ is isomorphic to $\mu_A$. Let

$$\chi: \text{Gal}(M/F) \rightarrow \mu_A$$

be an isomorphism, let $c: \text{Gal}(F^s/F) \rightarrow \mu_A$ be the composition of $\chi$ with the projection $\text{Gal}(F^s/F) \rightarrow \text{Gal}(M/F)$, and define $\rho_{\ell,c}: \text{Gal}(F^s/F) \rightarrow \text{Aut}(V_\ell(A))$
by \( \rho_{\ell,c}(\sigma) = c(\sigma)\rho_{A,\ell}(\sigma) \). For \( F \subseteq F' \subseteq F^* \), let \( \Phi_{\ell,c}(F') \) denote the Zariski closure of \( \rho_{\ell,c}(\text{Gal}(F^*/F')) \). Let \( F_{\Phi,c} \) denote the smallest extension \( F' \) of \( F \) in \( F^* \) such that \( \Phi_{\ell,c}(F') \) is connected. By Lemma 2.3, \( \{ \rho_{\ell,c} \} \) is a strictly compatible system of integral \( \ell \)-adic representations. Therefore by Proposition 6.14 of [5], \( F_{\Phi,c} \) is independent of \( \ell \). By definition,

\[
\text{Gal}(M/F_{\Phi,c}) \cong \mu_A \cap \Phi_{\ell,c}(F)^0(\mathbb{Q}_\ell) \quad \text{and} \
\Phi_{\ell,c}(M) = \Phi_{\ell}(M, A).
\]

Lemma 2.1 is valid with \( \Phi_{\ell,c}(F') \) in place of \( \Phi(F', A) \); the proof remains unchanged. Therefore,

\[
\Phi_{\ell,c}(F)^0 = \Phi_{\ell,c}(M)^0 = \Phi_{\ell}(M, A)^0 = \Phi_{\ell}(F, A).
\]

Since \( \text{Gal}(M/F_{\Phi,c}) \) is independent of \( \ell \), we are done. \( \square \)

References

[1] M. Borovoi, The action of the Galois group on the rational cohomology classes of type \((p, p)\) of abelian varieties (Russian), Mat. Sbornik (N. S.) 94 (136) (1974) 649–652 = Math. USSR Sbornik 23 (1974) 613–616.

[2] P. Deligne (notes by J. Milne), Hodge cycles on abelian varieties, in Hodge cycles, motives, and Shimura varieties (P. Deligne, J. Milne, A. Ogus, K.-y. Shih), Lecture Notes in Mathematics 900, Springer-Verlag, Berlin-Heidelberg-New York, 1982, pp. 9–100.

[3] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, Invent. Math. 73 (1983) 349–366.

[4] G. Faltings, Complements to Mordell, Chapter VI of Rational Points (G. Faltings, G. Wüstholz, et al.), Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1984.

[5] M. Larsen, R. Pink, On \( \ell \)-independence of algebraic monodromy groups in compatible systems of representations, Invent. math. 107 (1992) 603–636.

[6] M. Larsen, R. Pink, Abelian varieties, \( \ell \)-adic representations, and \( \ell \)-independence, Math. Ann. 302 (1995) 561–579.

[7] B. H. Matzat, Konstruktive Galoistheorie, Lecture Notes in Math. 1284, Springer-Verlag, Berlin-Heidelberg, 1987.

[8] L. Moret-Bailly, Pinceaux de Variétés Abéliennes, Astérisque 129 (1985).

[9] I. I. Piatetski-Shapiro, Interrelations between the Tate and Hodge conjectures for abelian varieties (Russian), Mat. Sbornik 85 (1971) 610–620 = Math. USSR Sbornik 14 (1971) 615–625.

[10] K. Ribet, Hodge classes on certain types of abelian varieties, Amer. J. Math. 105 (1983) 523–538.

[11] D. J. Saltman, Generic Galois Extensions and Problems in Field Theory, Adv. in Math. 43 (1982) 250–283.

[12] J.-P. Serre, Représentations \( \ell \)-adiques, in Algebraic Number Theory (Proceedings of the International Taniguchi Symposium, Kyoto, 1976) (S. Iyanaga, ed.), Japan Society for the Promotion of Science, Tokyo, 1977, pp. 177–193 = # 112 of Œuvres, Vol. III, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1986, pp. 384–400.

[13] J.-P. Serre, Lettres to K. Ribet, Jan. 1, 1981 and Jan. 29, 1981.

[14] J.-P. Serre, Résumé des cours de 1984–1985, Résumé des cours de 1985–1986, Collège de France.

[15] J.-P. Serre, Abelian \( \ell \)-adic representations and elliptic curves, Second edition, Addison-Wesley, Redwood City, CA, 1989.

[16] J.-P. Serre, Propriétés conjecturales des groupes de Galois motiviques et des représentations \( \ell \)-adiques, in Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.), Proc. Symp. Pure Math. 55 (1994), Part 1, pp. 377–400.

[17] A. Silverberg, Yu. G. Zarhin, Connectedness results for \( \ell \)-adic representations associated to abelian varieties, Comp. math. 97 (1995) 273–284.

[18] A. Silverberg, Yu. G. Zarhin, Connectedness extensions for abelian varieties, preprint.

[19] A. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948.
[20] Yu. G. Zarhin, *Endomorphisms of abelian varieties over fields of finite characteristic* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975) 272–277 = Math. USSR - Izv. 9 (1975) 255–260.

[21] Yu. G. Zarhin, *Abelian varieties in characteristic p* (Russian), Mat. Zametki 19 (1976) 393–400 = Math. Notes 19 (1976) 240–244.

[22] Yu. G. Zarhin, *Weights of simple Lie algebras in the cohomology of algebraic varieties* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984) 264–304 = Math. USSR - Izv. 24 (1985) 245–282.

**Department of Mathematics, Ohio State University, 231 W. 18 Avenue, Columbus, Ohio 43210–1174, USA**

*E-mail address:* silver@math.ohio-state.edu

**Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA.**

**Institute for Mathematical Problems in Biology, Russian Academy of Sciences, Pushchino, Moscow Region, 142292, Russia**

*E-mail address:* zarhin@math.psu.edu