On simply connected $K$-contact non-Sasakian manifolds

Bogusław Hajduk and Aleksy Tralle

Abstract. We solve the problem posed by Boyer and Galicki about the existence of simply connected $K$-contact manifolds with no Sasakian structure. We prove that such manifolds do exist using the method of fat bundles developed in the framework of symplectic and contact geometry by Sternberg, Weinstein and Lerman.

Mathematics Subject Classification. 53D05.

Keywords. Fat bundle, $K$-contact manifold, Sasakian manifold, symplectic manifold, contact manifold.

1. Introduction

In [BG] Boyer and Galicki asked the following question (see Open Problem 7.4 on page 235 in this book).

Problem 1.1. Do there exist simply connected closed $K$-contact manifolds with no Sasakian structure?

In this work, we answer this question positively.

Theorem 1.2. There exist simply connected $K$-contact manifolds which do not carry any Sasakian structure.

Let $(M, \eta)$ be a co-oriented contact manifold with a contact form $\eta$. We say that $(M, \eta)$ is $K$-contact if there is an endomorphism $\Phi$ of $TM$ such that the following conditions are satisfied:

1. $\Phi^2 = -\text{Id} + \xi \otimes \eta$, where $\xi$ is the Reeb vector field of $\eta$;
2. the contact form $\eta$ is compatible with $\Phi$ in the sense that

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$$

for all $X, Y$, and $d\eta(\Phi X, X) > 0$ for all nonzero $X \in \text{Ker} \eta$;
the Reeb field of $\eta$ is a Killing vector field with respect to the Riemannian metric defined by the formula
\[
g(X, Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y).
\]
In other words, the endomorphism $\Phi$ defines a complex structure on $\text{Ker}\, \eta$ compatible with $d\eta$, hence orthogonal with respect to the metric
\[
g = d\eta \circ (\Phi \otimes \text{Id}).
\]
By definition, the Reeb field $\xi$ is orthogonal to $\text{Ker}\, \eta$.

For a contact manifold $(M, \eta)$ define the metric cone or the symplectization as
\[
\mathcal{C}(M) = (M \times \mathbb{R}^+ > 0, t^2 \eta + dt^2).
\]
Given a $K$-contact manifold $(M, \eta, \Phi, g)$, the almost complex structure $I$ on $\mathcal{C}(M)$ is defined by
\[
I(X) = \Phi(X) \quad \text{on} \, \text{Ker}\, \eta; \quad (1.1)
\]
\[
I(\xi) = t \frac{\partial}{\partial t}, \quad I \left( t \frac{\partial}{\partial t} \right) = -\xi. \quad (1.2)
\]

A $K$-contact manifold is called Sasakian if the almost complex structure $I$ is integrable, hence defines a dilatation-invariant complex structure on $\mathcal{C}(M)$, endowing $\mathcal{C}(M)$ with a Kähler structure.

Geometry of metric contact manifolds is important because of their applications, for instance in the theory of Einstein metrics \cite{BG, BGM}. $K$-contact manifolds have nice topological properties (see \cite[Chapter 7]{BG} and \cite{GNT}). For example, they admit Cohen–Macauley torus actions (this is an analogue of the equivariant formality). Most of the known examples of $K$-contact manifolds are Sasakian, although examples of nonsimply connected $K$-contact manifolds with no Sasakian structure are known \cite{BG}. This rises the following general problem: \textit{given a contact manifold $M$, find conditions which ensure that there exists a Sasakian metric compatible with the contact structure.} This was posed by Ornea and Verbitsky in \cite{OV}.

Although the main result of this paper belongs to the framework of metric contact geometry, our methods come from symplectic and contact geometry and are based on the notions of symplectic and contact fatness developed by Sternberg \cite{S} and Weinstein \cite{W} in the symplectic setting and by Lerman in the contact case \cite{L1, L3}. Let $G \to P \to B$ be a principal bundle with a connection. Let $\theta$ and $\Theta$ be the connection one-form and the curvature form of the connection, respectively. Both forms have values in the Lie algebra $\mathfrak{g}$ of the group $G$. Denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by $\langle \cdot, \cdot \rangle$. By definition, a vector $u \in \mathfrak{g}^*$ is fat if the two-form
\[
(X, Y) \to \langle \Theta(X, Y), u \rangle
\]
is nondegenerate for all horizontal vectors $X, Y$. Note that if a connection admits at least one fat vector, then it admits the whole coadjoint orbit of fat vectors. It is important to notice that in this work we consider manifolds with