Noncommutative $\mathcal{N} = 2$ Supersymmetric Theories in Harmonic Superspace

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Abstract

We discuss a formulation of harmonic superspace approach for noncommutative $\mathcal{N} = 2$ supersymmetric field theories paying main attention on new features arising because of noncommutativity. We begin with the known notions of the harmonic superfield models and results obtained and consider how these notions and results are modified in the noncommutative case. The actions of basic $\mathcal{N} = 2$ models, like hypermultiplet and $\mathcal{N} = 2$ vector multiplet, are given in terms of unconstrained off-shell superfields on noncommutative harmonic superspace. We calculate the low-energy contributions to the effective actions of these models. The background field method in noncommutative harmonic superspace is developed and it is applied to study the 1-loop effective action in general noncommutative $\mathcal{N} = 2$ model and $\mathcal{N} = 4$ SYM theory.

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1 Introduction

Noncommutative field theory has recently attracted much attention due to its profound links with modern string/brane theory [1, 2] (see also the review [3]). However, the noncommutative field theory is very interesting itself, in particular, being non-local, it can be causal and its structure can be studied by standard method of quantum field theory.

Noncommutative field theories possess specific properties of UV- and IR-divergencies which distinguish them from ordinary theories very essentially. It was shown for noncommutative scalar field models [6] that the UV-divergencies lead to additional IR-singularities of effective action that was called the UV/IR-mixing. The change in the divergencie structure can break the renormalizability of theories that was pointed out for the models of matter and gauge fields in [7]. It is also interesting to note that the UV/IR-mixing in $U(N)$ gauge theories is related only to additional $U(1)$ phase while the $SU(N)$ part of effective action remains unchanged in low-energy limit [8].

Construction of noncommutative supersymmetric models is formally realized on the basis of a simple enough procedure. Noncommutativity usually concerns only the bosonic coordinates of superspace while the fermionic variables remain flat [9] (see an attempt to develop a superspace with noncommutativity both in bosonic sector and in fermionic one [10]). Quantum properties of $\mathcal{N} = 1$ supersymmetric theories of matter and gauge fields are discussed in many papers (see e.g. [11, 12, 13, 14]). A study of extended supersymmetric models deserves a special attention. For example, maximally extended supersymmetric noncommutative $\mathcal{N} = 4$ SYM is finite and has no UV/IR-mixing [15] what is convenient for investigating the low-energy behaviour [16, 17] and for constructing new UV-finite models [18]. Noncommutative $\mathcal{N} = 2$ SYM has also been studied in many details (see e.g. [19, 20, 21]). We point out that all actual calculations in noncommutative extended supersymmetric theories have been done in terms of component fields or using $\mathcal{N} = 1$ superfield technique.

The aim of this paper is to develop a formulation of noncommutative $\mathcal{N} = 2$ supersymmetric theories in harmonic superspace. A harmonic superspace approach [22] provides a unique possibility to formulate $\mathcal{N} = 2$ supersymmetric models in terms of off-shell $\mathcal{N} = 2$ superfields and therefore allows to preserve manifest $\mathcal{N} = 2$ supersymmetry what is especially important in quantum theory. We describe a construction of the basic noncommutative $\mathcal{N} = 2$ models within the harmonic superspace approach and investigate the new aspects stipulated by noncommutativity.

We begin with noncommutative generalization of hypermultiplet and $\mathcal{N} = 2$ vector multiplet models and then demonstrate the basic examples of one-loop calculations in these models and show how the effect of UV/IR-mixing arises in harmonic supergraphs. The effective action of Abelian $q$-hypermultiplet interacting with gauge superfield is found to be free of UV/IR-mixing and contributions of non-planar type. This allows us to calculate the holomorphic effective action in this theory in a gauge invariant form. We prove that it can be obtained from the holomorphic effective action of commutative $q$-hypermultiplet just by the insertion of the $\star$-product instead of usual multiplication. We consider also the selfinteracting $q$-hypermultiplet and find that it is nonrenormalizable because of wrong dimension of coupling constants as well as in commutative case. The selfinteraction can be induced by the quantum corrections from the four-point box diagrams as it was established for the commutative $q$-hypermultiplet model [23].

We investigate also the structure of low-energy effective action of general noncommutative $\mathcal{N} = 2$ model including the matter and gauge field. At low energies the effective action is determined by the holomorphic and non-holomorphic potentials in complete analogy with the commutative case. To study these potentials we use the $\mathcal{N} = 2$ background field method [24] adapted for noncommutative theories. The holomorphic effective action is found for the $q$-hypermultiplet model while the non-holomorphic potential us studied for the noncommutative $\mathcal{N} = 4$ SYM written in terms of $\mathcal{N} = 2$ superfields. It is shown that the non-holomorphic effective action of $\mathcal{N} = 4$ SYM consists of two parts.
related to the \( U(N)/[U(1)]^N \) and \([U(1)]^N \) sectors of \( U(N) \) gauge group. In the commutative limit the first part reproduces a standard non-holomorphic potential of commutative \( \mathcal{N} = 4 \) \( SU(N) \) SYM studied in \([24, 25]\) while the second one is responsible only for higher noncommutative corrections. We consider the above results as a starting point for detailed investigation of noncommutative corrections in the \( \mathcal{N} = 2, 4 \) supersymmetric theories.

The paper is organized as follows. In the second section we consider a generalization of harmonic superspace approach to the case of noncommutative geometry. The classical actions of noncommutative \( q \)-hypermultiplet and vector multiplet are given and corresponding Feynman rules are written down here. The third section is devoted to the examples of 1-loop quantum computations. In the fourth section we investigate the low-energy effective action of general noncommutative \( \mathcal{N} = 2 \) SYM model and in \( \mathcal{N} = 4 \) SYM on the basis of background field method. The structures of noncommutative holomorphic potential of \( q \)-hypermultiplet and non-holomorphic potential in \( \mathcal{N} = 4 \) SYM are discussed here. Finally, the section 5 is a summary.

## 2 Classical actions and Feynman rules

Standard harmonic superspace was introduced in the papers \([19]\) to develop the manifest \( \mathcal{N} = 2 \) superfield formulation of \( \mathcal{N} = 2 \) off-shell supersymmetric models. It is usually obtained from conventional \( \mathcal{N} = 2 \) superspace with the coordinates \( z^M = \{x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}\} \) by adding the harmonic variables \( u^\pm_i \) which parameterize the coset \( SU(2)/U(1) \). As a result, the \( \mathcal{N} = 2 \) harmonic superspace is represented by the following coordinates

\[
z^M_A = \{x^\mu_A, \theta^+_{\alpha A}, \bar{\theta}^+_{\dot{\alpha} A}, u^+_i\},
\]

where

\[
x^\mu_A = x^\mu - 2i\theta^{(i}(\sigma^\mu\bar{\theta}^j)}u^+_iu^-_j, \quad \theta^\pm_{\alpha A} = \theta^\pm_{\alpha}u^+_i, \quad \bar{\theta}^\pm_{\dot{\alpha} A} = \bar{\theta}^\pm_{\dot{\alpha}}u^+_i.
\]

The main conceptual advantage of harmonic superspace is that the \( \mathcal{N} = 2 \) vector multiplet and hypermultiplet can be described by unconstrained superfields over the so called analytic subspace parameterized by the coordinates

\[
\zeta_A = \{x^\mu_A, \theta^+_{\alpha A}, \bar{\theta}^+_{\dot{\alpha} A}, u^+_i\},
\]

which transform through each other under \( \mathcal{N} = 2 \) supersymmetric transformations.

In order to introduce the noncommutative geometry in \( \mathcal{N} = 2 \) harmonic superspace we perform the Moyal-Weyl deformations of bosonic coordinates

\[
[x_\mu, x_\nu] = i\theta_{\mu\nu},
\]

where \( \theta_{\mu\nu} \) is a constant antisymmetric tensor of dimension \(-2\). We consider only the case when fermionic variables remain flat. A general case of deformations of superspace is studied in ref. \([4]\). A natural multiplication of superfields on noncommutative plain is given by the \( \star \)-product

\[
(\phi \star \psi)(x) = \phi(x)e^{\frac{i}{2}\theta_{\mu\nu}\partial_\mu\partial_\nu}\psi(x),
\]

which is associative but noncommutative.

As a rule, noncommutative field theories can be obtained from conventional models of quantum field theory just by replacement of usual multiplication by \( \star \)-product. In this section we construct the classical actions of noncommutative hypermultiplet and \( \mathcal{N} = 2 \) vector multiplet in harmonic superspace. Some aspects of noncommutative \( \mathcal{N} = 2 \) SYM model in harmonic superspace such as

\footnote{We use the same symbol for the notation of fermionic variables and noncommutative parameter. This can not lead to misunderstanding since they appear in a completely different way and have nothing common.}
solitonic solutions were considered in ref. [12]. In this paper we study only quantum features of these theories. We use the notations for harmonic superspace objects accepted in refs. [19] and [24, 25, 26, 27].

### 2.1 Selfinteracting $q$-hypermultiplet

The classical action of selfinteracting $q$-hypermultiplet can be obtained from the action of usual $q$-hypermultiplet [19] by insertion of $\star$-product instead of usual multiplication

\[
S = S_0 + S_{int},
\]

\[
S_0 = \int d\zeta \langle\bar{q}^+\bar{\rho}(\zeta)D^{\alpha+}q^+\rangle,
\]

\[
S_{int} = \int d\zeta \langle\alpha \bar{q}^+ \star q^+ \star q^+ + \beta \bar{q}^+ \star q^+ \star \bar{q}^+ \star q^+\rangle.
\]  

Note that there are two coupling constants $\alpha, \beta$ due to two types of ordering of superfields. In commutative limit $\theta \to 0$ the two terms in the interaction (5) reduce to a single standard one $\lambda (q^+)/2(\bar{q}^+)/2$.

Free classical action $S_0$ is not modified by the introduction of noncommutativity, therefore, the two-point Green function has standard form [19]

\[
G^{(1,1)}(1|2) = -\frac{1}{\Box_1} \frac{(D_1^+)^4(D_2^+)^4\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3}.
\]  

The corresponding propagator (Fourier transformation of $G_0(1|2)$) is given by

\[
\frac{1}{p^2} = G_0(p) = \frac{1}{p^2} \frac{(D_1^+)^4(D_2^+)^4\delta^{8}(\theta_1 - \theta_2)}{(u_1^+ u_2^+)^3}.
\]  

The vertex can be found from the interaction $S_{int}$ as

\[
(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)[\alpha \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \beta \cos(\frac{\theta_1}{2} + \frac{\theta_2}{2})].
\]

Note that the noncommutative factor in (8) has the same form as in noncommutative $\phi^4$ model studied in ref. [3].

### 2.2 Noncommutative $\mathcal{N} = 2$ SYM action

The model of noncommutative $\mathcal{N} = 2$ vector multiplet was considered in refs. [13, 11] using the $\mathcal{N} = 1$ superfield approach. In terms of $\mathcal{N} = 2$ superfields it is written as

\[
S_{SYM} = \frac{1}{g^2} \text{tr} \int d^4xd^4\theta W \star W,
\]  

where $W$ is noncommutative strength superfield defined by the equation

\[
W = \frac{i}{4} \{\bar{\nabla}^+ \bar{\alpha}, \nabla^{-\alpha}\} \star.
\]  

The strength $W$ can be expressed through the prepotential $V^{++}$ in the same way as in commutative theory [21]

\[
W = \frac{1}{4} \tilde{D}^{\alpha}_+ \tilde{D}^{+\alpha}_+ \sum_{n=1}^\infty i(-ig)^n \int du_1 \ldots du_n \frac{V^{++}(z,u_1) \star V^{++}(z,u_2) \star \ldots \star V^{++}(z,u_n)}{(u^+ u_1^+)(u_1^+ u_2^+)(u_2^+ u_3^+) \ldots (u_n^+ u^+)}. 
\]
Note that the strength $W$ is a nonlinear function of $V^{++}$ even in Abelian case.

Using the relation (11) the classical action of noncommutative vector multiplet (9) can be rewritten in terms of unconstrained vector prepotential $V^{++}$ as

$$S_{SYM} = S_0 + S_{int},$$

where

$$S_0 = \text{tr} \int d^{12}z d u_1 d u_2 \frac{V^{++}(z, u_1) V^{++}(z, u_2)}{(u_1^+ u_2^+)^2},$$

$$S_{int} = \frac{1}{g^2} \text{tr} \int d^{12}z \sum_{n=3}^{\infty} \frac{(-i g)^n}{n!} \int d u_1 \ldots d u_n \frac{V^{++}(z, u_1) \ast \ldots \ast V^{++}(z, u_n)}{(u_1^+ u_2^+) \ldots (u_n^+ u_1^+)}.$$  \hspace{1cm} (12)

This action is gauge-invariant under noncommutative gauge transformations with gauge parameter $\lambda$

$$\delta V^{++} = -D^{++} \lambda - i[V^{++}, \lambda],$$  \hspace{1cm} (13)

where

$$[V^{++}, \lambda] = V^{++} \ast \lambda - \lambda \ast V^{++}.$$  \hspace{1cm} (14)

The vector superfield $V^{++}$ and the gauge parameter $\lambda$ are the Lie-algebra-valued analytic superfields

$$V^{++}(\zeta) = V^{++A}(\zeta) T^A, \quad \lambda(\zeta) = \lambda^A(\zeta) T^A,$$

where $T^A$ are the generators of $U(N)$ group. Note that the interaction in noncommutative vector superfield model (12) is non-polynomial even for an Abelian gauge group.

In the simplest case it is sufficient to consider only the cubic interaction

$$S_3 = \frac{i g}{3} \int d^{12}z d u_1 d u_2 d u_3 \frac{V^{++}(z, u_1) [V^{++}(z, u_2), V^{++}(z, u_3)]}{(u_1^+ u_2^+) (u_2^+ u_3^+) (u_3^+ u_1^+)},$$  \hspace{1cm} (15)

that defines the three-point vertex

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| \quad \psi_1 \\
| \quad f_1 \\
| \quad \psi_2 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| \quad \psi_3 \\
| \quad f_3 \\
| \quad \psi_4 \\
\end{array}
\end{array}
\end{array}
\end{array} = \frac{2 g (2 \pi)^4}{3 (u_1^+ u_2^+)(u_2^+ u_3^+)(u_3^+ u_1^+)} \delta^4(p_1 + p_2 + p_3) \sin \frac{p_1 \theta_{p_1}}{2}.$$  \hspace{1cm} (16)

The propagator of the vector superfield $V^{++}$ was found in the works (19) (in appropriate gauge fixing) in the form

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| \quad \zeta \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| \quad \zeta \\
\end{array}
\end{array}
\end{array}
\end{array} = \frac{1}{p^2} (D^+_1)^4 \delta^4(\theta_1 - \theta_2) \delta^{-2,2}(u_1, u_2).$$  \hspace{1cm} (17)

In order to obtain the full action of quantum SYM theory we have to add the ghost superfields $F, P$ like in (1) (with odd statistics and $U(1)$-charge to be zero)

$$S_{gh} = \int d \zeta \langle (-4) \rangle \text{tr} [F \ast D^{++}(D^{++} + ig V^{++}) \ast P].$$  \hspace{1cm} (18)

It can be shown that the ghost action (18) can be obtained on the basis of standard Faddeev-Popov prescription applied to the noncommutative vector superfield model (12), so the naive $\ast$-product generalization of conventional ghost action is valid here.

The corresponding Feynman rules for the ghost superfields are

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| \quad \zeta \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| \quad \zeta \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = - \frac{i}{p^2} (D^+_1)^4 (D^+_2)^4 \delta^4(\theta_1 - \theta_2) (u_1^+ u_2^+)/ (u_1^+ u_2^+)^2,$$

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
| \quad \zeta \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = g (2 \pi)^4 D^{++} \delta^4(p + k_1 + k_2) \sin \frac{k_1 \theta_{k_2}}{2}.$$  \hspace{1cm} (19)

The sine factor in eq. (19) is due to odd statistics of ghosts.
2.3 Interaction with matter

Let us consider the massive hypermultiplet model. Mass of \( q \)-hypermultiplet can be generated by central charge of \( \mathcal{N} = 2 \) superalgebra \[22, 23, 25\]. The corresponding free classical action reads

\[
S_0 = \int d\zeta^{-4}\bar{q}^+(D^{++} + iV_0^{++})q^+,
\]

with \( V_0^{++} = -\bar{W}_0\theta^2 - W_0\tilde{\theta}^2 \), \( W_0W_0 = m^2 \) is the mass. The propagator of this theory was found in the refs. \[20, 22, 23\] and can be represented in the form

\[
G_0^{(1,1)}(1|2) = \frac{1}{(p^2 + m^2)(D_1^+)^4(D_2^+)^4}\left\{ e^{i\Omega_0(1) - i\Omega_0(2)}\delta^2(z_1 - z_2) \right\},
\]

or in momentum space

\[
G^{(1,1)}(p) = \frac{1}{p^2 - m^2}(D_1^+)^4(D_2^+)^4\left\{ e^{i\Omega_0(1) - i\Omega_0(2)}\delta^2(\theta_1 - \theta_2) \right\},
\]

where \( \Omega_0 = -\bar{W}_0\theta + W_0\tilde{\theta} \) is a bridge superfield in the case under consideration \[23\].

Interaction of gauge superfields with matter can be introduced by a minimal way. We consider two types of representations of gauge group, fundamental and adjoint:

\[
S_{f,\text{int}} = \int d\zeta^{-4}\bar{q}^+ * V^{++} * Q^+,
\]

\[
S_{ad,\text{int}} = i\text{tr}\int d\zeta^{-4}\bar{q}^+ * [V^{++}, Q^+]_z,
\]

where \( Q^+ = q^+AT^A \), \( \bar{Q}^+ = \bar{q}^+AT^A \), \( V^{++} = V^{++AT^A} \), \( T^A \) are the generators of \( U(N) \) gauge group. The commutator in the action \[23\] is written as

\[
[V^{++}, Q^+]_z = V^{++} * Q^+ - Q^+ * V^{++} = \frac{1}{2}[V^{++AT^A}, q^{++B}]_z[T^A, T^B] + \frac{1}{2}[V^{++AT^A}, q^{+B}]_z[T^A, T^B]
\]

\[
- \frac{1}{2}[V^{++AT^A}, q^{+B}]_z d^{ABC}T^C + \frac{1}{2}[V^{++AT^A}, q^{+B}]_z f^{ABC}T^C,
\]

where we have used the relations

\[
[T^A, T^B] = f^{ABC}T^C, \quad \{T^A, T^B\} = -id^{ABC}T^C, \quad \text{tr}(T^AT^B) = -\frac{1}{2}\delta^{AB},
\]

\( f^{ABC} \) and \( d^{ABC} \) are the structure constants of \( U(N) \) group \[1\].

The vertex is found from the expressions \[23, 24\] with the help of relations \[25, 26\] in the form

\[
\frac{\bar{\psi}_k}{\psi_p} = (2\pi)^4\delta^4(p + k_1 + k_2)\left\{ e^{-it_1\theta k_2}, \text{fundamental representation}, \right.
\]

\[
F^{ABC}(k_1, k_2), \text{adjoint representation,}
\]

where

\[
F^{ABC}(k_1, k_2) = d^{ABC} \sin \frac{1}{2}k_1\theta k_2 + f^{ABC} \cos \frac{1}{2}k_1\theta k_2.
\]

Now we can apply the Feynman rules obtained for noncommutative models of hypermultiplet and vector multiplet to compute quantum corrections in these theories.
3 Examples of 1-loop computations

3.1 Noncommutative hypermultiplet four-point function

For the first example we consider the 1-loop four-point correction to the effective action of hypermultiplet

\[ \Gamma_4^{(1)}[\vec{q}^+, q^+] = \Gamma_s[\vec{q}^+, q^+] + \Gamma_t[\vec{q}^+, q^+], \]

where we assume that \[ n_{pl} \]

appearance of UV/IR-mixing.

This effective action differs from corresponding commutative one by the presence of two functions \[ F \]

Further computations are very similar to usual ones in commutative

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where \[ n_{pl} \]

and

\[ s, t \]

in the integrals (30). These functions change the UV-structure of integrals

To study the structure of momentum integrals (30) let us split the functions \[ F \]

Further computations are very similar to usual ones in commutative \[ q^4 \]-model considered in [19]. The resulting expression for the effective action \[ 23 \] can be rewritten as

\[ \Gamma_{s,t} = \int \frac{d^4 p_1 \ldots d^4 p_4}{(2\pi)^4} d^4 \theta_1 d^4 \theta_2 \delta^4(p_1 + p_2 + p_3 + p_4) \tilde{\phi}^+(1)\tilde{\phi}^+(2)\tilde{\phi}^+(3)\tilde{\phi}^+(4) \delta^4(p_1 + p_2 + p_3 + p_4) I_{s,t}(p_1, \ldots, p_4). \]

This effective action differs from corresponding commutative one by the presence of two functions \[ F_{s,t} \] in the integrals (30). These functions change the UV-structure of integrals \[ I_{s,t} \] what lead to the appearance of UV/IR-mixing.

To study the structure of momentum integrals (30) let us split the functions \[ F_{s,t} \] into planar (pl) and non-planar (npl) parts

\[ F_{s,t}(p_1, \ldots, p_4, k) = F_{s,t}^{pl}(p_1, \ldots, p_4) + F_{s,t}^{npl}(p_1, \ldots, p_4, k), \]

where we assume that \[ F_{s,t}^{pl} \] do not contain the terms depending on the momentum \[ k \]. The planar parts are calculated explicitly

\[ F_{s,t}^{pl} = \frac{\alpha^2}{4} \cos \frac{\theta p_1}{2} \cos \frac{\theta p_2}{2} + \frac{\beta}{2} \cos \frac{\theta p_3}{2} + \frac{\alpha \beta}{2} \cos \left( -\frac{\theta p_3}{2} + \frac{\theta p_4}{2} \right). \]
Substituting the expressions (34) into integrals (30)

\[ I^{pl}(p_1, \ldots, p_4) = 2F^{pl}(p_1, \ldots, p_4) \int \frac{d^4k}{k^2(p_1 + p_2 - k)^2}, \]
\[ I^{ps}(p_1, \ldots, p_4) = 8F^{ps}(p_1, \ldots, p_4) \int \frac{d^4k}{k^2(p_1 + p_3 - k)^2}, \]

we see that the planar contributions have the same divergencies as corresponding diagrams of commutative \( q \)-hypermultiplet. The momentum integrals in (35) have the IR-divergence (at low external momentum) which can be avoided in massive theory and UV-divergence which can not be renormalized since the coupling constants \( \alpha, \beta \) have the dimension \(-2\) as in commutative theory [19]. So, the self-interacting noncommutative \( q \)-hypermultiplet model is nonrenormalizable as well as the corresponding commutative theory.

Let us study now the structure of non-planar diagrams. One can show that all non-planar terms of functions (31) look like

\[ \cos \frac{p_1 \theta_{p_2}}{2} \cos \frac{p_3 \theta_{p_4}}{2} \cos(k\theta(p_1 + p_2)) \]

with various combinations of external momenta. They define the structure of momentum integrals of non-planar type

\[ I^{npl}_k(p_1, \ldots, p_4) \sim \cos \frac{p_1 \theta_{p_2}}{2} \cos \frac{p_3 \theta_{p_4}}{2} \int d^4k \frac{e^{ik\theta(p_1 + p_2)}}{k^2(p_1 + p_2 - k)^2} \]
\[ = \cos \frac{p_1 \theta_{p_2}}{2} \cos \frac{p_3 \theta_{p_4}}{2} \int_0^1 d\xi K_0(\sqrt{\xi(1 - \xi)P}), \]

where

\[ P = (p_1 + p_2)^2 \cdot (p_1 + p_2) \odot (p_1 + p_2), \]

\[ p_1 \odot p_2 = p_1^{\mu} (\theta_{\mu\nu} p_2^\nu), \]

\( K_0 \) is the modified Bessel function. This expression has no UV-divergence but it is singular at low momenta \( p_i \to 0 \) due to the asymptotics of Bessel function

\[ K_0(x) \xrightarrow{x \to 0} -\ln \frac{x}{2} + \text{finite}. \]

Such a singularity of effective action can not be avoided by the introduction of mass of hypermultiplet and defines the well known UV/IR-mixing [6]. At low momenta the leading part of momentum integral

\[ \int d^4k \frac{e^{ik\theta(p_1 + p_2)}}{k^2(p_1 + p_2 - k)^2} \sim \ln \frac{1}{P} \]

is singular in commutative limit \( \theta \to 0 \) what is responsible for the UV/IR-mixing. Therefore the effective action of noncommutative hypermultiplet can not be reduced to a standard one in commutative limit.

As a result, the model of noncommutative hypermultiplet is nonrenormalizable and has the UV/IR-mixing in the sector of non-planar diagrams.
3.2 1-loop two-point diagram in $\mathcal{N} = 2$ SYM

The next example is the 1-loop two-point function in noncommutative $\mathcal{N} = 2$ SYM. It consists of gauge and ghost loops

\[
\Gamma^2 = \Gamma^{pl}_2 + \Gamma^{npl}_2,
\]

which can be computed with the help of Feynman rules \cite{16,17,19}. We skip all the calculations because they are very similar to ones given in the previous section. The resulting expression for the effective action is represented as a sum of planar and nonplanar parts

\[
\Gamma^2 = \Gamma^{pl}_2 + \Gamma^{npl}_2,
\]

\[
\Gamma^{pl}_2 = \frac{-g^2}{(2\pi)^8} \int d^4p d^8\theta du_1 du_2 \int \frac{d^4k}{k^2(p - k)^2} \frac{V^{++}(p, \theta, u_1)V^{++}(-p, \theta, u_2)}{(u^+_1 u^+_2)^2},
\]

\[
\Gamma^{npl}_2 = \frac{-g^2}{(2\pi)^8} \int d^4p d^8\theta du_1 du_2 \int \frac{d^4k \cos k\theta p}{k^2(p - k)^2} \frac{V^{++}(p, \theta, u_1)V^{++}(-p, \theta, u_2)}{(u^+_1 u^+_2)^2} \int_0^1 d\xi K_0(\sqrt{\xi(1 - \xi)p^2} \cdot p \circ p).
\]

The momentum integral in the planar part of the effective action is UV-divergent. This divergence is eliminated by standard renormalization of coupling constant $g$. The nonplanar part $\Gamma^{npl}_2$ has no UV-divergence due to the $\cos k\theta p$ factor in momentum integral which is expressed through the Bessel function $K_0$. The Bessel function is finite at large momenta but it is singular at small $p$ \cite{37}. A leading term at low energies in $\Gamma^{npl}_2$ is of the form

\[
\Gamma^{npl}_2[V^{++}] = \frac{g^2}{2(2\pi)^8} \int d^4p d^8\theta du_1 du_2 \ln \frac{1}{p^2 \cdot p \circ p} \frac{V^{++}(p, \theta, u_1)V^{++}(-p, \theta, u_2)}{(u^+_1 u^+_2)^2}.
\]

This expression is singular in the limit $\theta \to 0$ and it defines the UV/IR-mixing in the $\mathcal{N} = 2$ SYM model.

3.3 Hypermultiplet loop

Let us consider the two-point diagram of massive q-hypermultiplet in external Abelian vector superfield

\[
\Gamma_2[V^{++}] = \Gamma^{fund}_2 + \Gamma^{adj}_2,
\]

where the Feynman rules are given by eqs. \cite{22,27}. We study both fundamental and adjoint representation of $U(1)$ gauge group.

One can show that for the fundamental representation this diagram has no nonplanar contributions and it is equal to the corresponding diagram in commutative theory (we omit all the calculations and give only the final results):

\[
\Gamma^{fund}_2 = \frac{2}{(2\pi)^8} \int d^4p d^8\theta du_1 du_2 \frac{V^{++}(p, \theta, u_1)V^{++}(-p, \theta, u_2)}{(u^+_1 u^+_2)^2} \int \frac{d^4k}{(k^2 + m^2)((p - k)^2 + m^2)}.
\]
This expression has a standard UV-divergence which has to be renormalized as usual. There is no UV/IR-mixing since no nonplanar contributions appear.

In the adjoint representation the two-point function (43) receives both planar and nonplanar contributions:

\[ \Gamma^{npl} = \Gamma^{pl}(s,t) = \frac{V^{++}(-1)V^{++}(2)}{(p_1 + p_2)^2} \int \frac{d^4k}{(k^2 + m^2)((k - p)^2 + m^2)} \]

Using the asymptotics of Bessel function (37) one can find that the leading low-energy contribution is of the form

\[ \Gamma^{npl}(s,t) = \frac{1}{2} \int \frac{d^4p}{(2\pi)^8} i^8\theta du_1 du_2 \ln \frac{1}{m^2 p \circ p} \frac{V^{++}(p,\theta,u_1)V^{++}(-p,\theta,u_2)}{(u_1^+ u_2^+)^2}. \]

The equation (46) is responsible for the UV/IR-mixing in this diagram.

### 3.4 Induced hypermultiplet selfinteraction

The last example is the four-point box diagram with hypermultiplet external lines of the type

\[ \Gamma_4 = \Gamma_s + \Gamma_t, \]

\[ \Gamma_s = \]

\[ \Gamma_t = \]

In the low energy approximation the loops shrink down to points and the finite parts of these diagrams give the coupling constants \( \alpha, \beta \) of selfinteracting \( q \)-hypermultiplet (8) induced only by quantum corrections. Such an induced selfinteraction in the commutative hypermultiplet model was considered in [23].

To compute these diagrams we apply the Feynman rules obtained (17, 22, 27) and, after simple transformations, we obtain the following expressions

\[ \Gamma_s[p_1, \ldots, p_4] = -\frac{i a}{(2\pi)^3} \int d^4 p_1 d^4 p_2 d^4 p_3 d^4 p_4 F_{s,t}(p_1, \ldots, p_4) I_s(t), \]

\[ \times \int d^8 \theta \int du_1 du_2 q^+(p_1) q^-(p_2) q^+(p_3) q^-(p_4) \varepsilon^{i\Omega_0(2) - i\Omega_0(1)} \varepsilon^{i\Omega_0(2) - i\Omega_0(1)} \]

where

\[ I_s(p_1, \ldots, p_4) = \frac{1}{(2\pi)^7} \int \frac{d^4 k}{k^2 (k + p_1 + p_2)^2 ((k + p_1)^2 + m^2)((k - p_1)^2 + m^2)} \]

\[ I_t(p_1, \ldots, p_4) = \frac{1}{(2\pi)^7} \int \frac{d^4 k}{k^2 (k + p_1 + p_2)^2 ((k + p_1)^2 + m^2)((k - p_1)^2 + m^2)} \]

\[ F_s(p_1, \ldots, p_4) = e^{-i p_1 \theta p_3} e^{-i p_2 \theta p_4}, \]

\[ F_t(p_1, \ldots, p_4) = e^{-i p_2 \theta p_1} e^{-i p_3 \theta p_4}. \]

It should be noted that the momentum integrals (44) are UV-finite. Therefore there is no UV/IR-mixing and these diagrams are smooth in the limits \( \theta \to 0 \) and \( p_i \to 0 \). At zero external momenta
both these integrals are computed exactly:

\[ I_{s,t} = \frac{1}{(4\pi)^2 m^2} \left( \frac{1}{m^2} \ln(1 + \frac{m^2}{\Lambda^2_{s,t}}) - \frac{1}{\Lambda^2_{s,t} + m^2} \right), \]  

(51)

where \( \Lambda_{s,t} \) are the parameters of infrared cutoff which do not coincide generally speaking. The values of \( \Lambda_{s,t} \) should be taken less than the mass of the lightest particle what corresponds to the Wilsonian low-energy effective action.

In order to extract the relevant low-energy contribution from eq. (48) it is necessary to employ the covariant derivatives algebra [19]

\[ [D^{++}, D^{--}] = D^0, \]  

(52)

where

\[ D^{\pm\pm} = D^{\pm\pm} + iV^{\pm\pm}_0, \quad V^{\pm\pm}_0 = D^{\pm\pm} \Omega_0. \]  

(53)

The relation (52) is used to prove the identity

\[ q^+ \ast q^+ \ast \tilde{q}^+ \ast \tilde{q}^+ e^{i\Omega_0(2) - i\Omega_0(1)} e^{i\Omega_0(2) - i\Omega_0(1)} = \frac{1}{2} [D^{++}_1, D^{--}_1] q^+ \ast q^+ \ast \tilde{q}^+ \ast \tilde{q}^+ e^{i\Omega_0(2) - i\Omega_0(1)} e^{i\Omega_0(2) - i\Omega_0(1)}. \]  

(54)

Then it can be shown that only the first term \( \sim D^{++} D^{--} \) in this identity gives the leading low-energy contribution. Integrating by parts one can cancel the harmonic distribution \( 1/(u_1 + u_2)^2 \) by the use of another identity [19]

\[ D^{++}_1 \frac{1}{(u_1 + u_2)^2} = D^{--}_1 \delta^{(2,-2)}(u_1, u_2). \]  

(55)

The harmonic \( \delta \)-function on the right hand side of eq. (55) removes one of the two remaining harmonic integrals and allows us to rewrite eq. (48) to the form

\[ \Gamma_{s,t}[\bar{q}^+, q^+] = -ig^4 I_{s,t} \cdot S_{s,t}[\bar{q}^+, q^+], \]  

(56)

where

\[ S_s = \int d^4 x d^8 \theta d u D^{--} D^{--} q^+ \ast \tilde{q}^+ \ast q^+ \ast \tilde{q}^+, \]  

\[ S_t = \int d^4 x d^8 \theta d u D^{--} D^{--} q^+ \ast q^+ \ast \tilde{q}^+ \ast \tilde{q}^+. \]  

(57)

The integrals over full \( N = 2 \) superspace in eqs. (57) can be transformed to ones over the analytic subspace

\[ S_s = -2m^2 \int d\zeta^{(-4)} q^+ \ast \tilde{q}^+ \ast q^+ \ast \tilde{q}^+, \]  

\[ S_t = -2m^2 \int d\zeta^{(-4)} q^+ \ast q^+ \ast \tilde{q}^+ \ast \tilde{q}^+. \]  

(58)

where we have used the explicit form for \( v^{--} \) given by eq. (13) and relation \( \bar{W} = m^2 \). As a result, we find the low-energy part of the effective action (56) in the form of four-point \( q \)-hypermultiplet interaction [3]

\[ \Gamma_4 = \int d\zeta^{(-4)} (\lambda_s \tilde{q}^+ \ast q^+ \ast q^+ \ast \tilde{q}^+ + \lambda_t q^+ \ast q^+ \ast \tilde{q}^+ \ast \tilde{q}^+ + \lambda_t q^+ \ast q^+ \ast \tilde{q}^+ \ast \tilde{q}^+ + \lambda_t q^+ \ast q^+ \ast \tilde{q}^+ \ast \tilde{q}^+), \]  

(59)

where the coupling constants \( \lambda_{s,t} \) appear due to quantum corrections (52) as

\[ \lambda_{s,t} = \frac{g^2}{(4\pi)^2} \left( \frac{1}{m^2} \ln(1 + \frac{m^2}{\Lambda^2_{s,t}}) - \frac{1}{\Lambda^2_{s,t} + m^2} \right). \]  

(60)
Note that the effective action (59) is smooth in the limit $\theta \to 0$ and both induced coupling constants $\lambda_s, t (60)$ reduce to a single coupling constant obtained in ref. [23] for the commutative hypermultiplet model.

4 Structure of low-energy effective action of general noncommutative $\mathcal{N} = 2$ theory

The total $\mathcal{N} = 2$ theory consists of the actions of vector superfield (9) and hypermultiplets in fundamental and adjoint representations (23, 24)

$$S = S_{SYM} + S^{f}_{HP} + S^{ad}_{HP},$$

$$S_{SYM} = \frac{1}{g^2} \text{tr} \int d^4xd^4\theta W \star W,$$

$$S^{f}_{HP} = \int d\zeta (-4) [\bar{q}^+ D^{++} q^+ + ig\bar{q}^+ \star V^{++} \star q^+],$$

$$S^{ad}_{HP} = \text{tr} \int d\zeta (-4) (\bar{Q}^+ D^{++} Q^+ + ig\bar{Q}^+ \star [V^{++}, Q^+]_\star).$$

The action (61) is gauge invariant with respect to the following gauge transformations

$$\delta q^+ = i\lambda \star q^+, \quad \delta \bar{q}^+ = -i\bar{q}^+ \star \lambda,$$

$$\delta Q^+ = i[\lambda, Q^+]_\star, \quad \delta \bar{Q}^+ = -i[\bar{Q}^+, \lambda]_\star,$$

$$\delta V^{++} = -D^{++} \lambda + i[\lambda, V^{++}]_\star.$$  

We are interested only in the part of effective action depending on the gauge superfields but not hypermultiplets, therefore in the most general form it reads

$$\Gamma[W, \bar{W}] = \left( \int d^4xd^4\theta \mathcal{F}(W) + \text{c.c.} \right) + \int d^4xd^4\theta \mathcal{H}(W, \bar{W})$$

$$+ \text{terms depending on covariant derivatives of } W, \bar{W},$$

where, as usual, the complex chiral superfield $\mathcal{F}$ is called a holomorphic potential and the real superfield $\mathcal{H}$ is a non-holomorphic potential. If we suppose that $\mathcal{F}$ and $\mathcal{H}$ are smooth functions of strengths and they do not contain any derivatives of $W, \bar{W}$ (as in commutative case) then these functions do not receive any corrections depending on the parameter of noncommutativity $\theta$. However, we loose the manifest gauge invariance in this case since the gauge invariance require the $\star$-product multiplication of superfields what involves the spatial derivatives. To provide the manifest gauge invariance of low-energy effective action one should take into account the noncommutative corrections to $\mathcal{F}$ and $\mathcal{H}$ (corrections depending on $\theta$). In general, the problem of finding the noncommutative analogs of holomorphic and non-holomorphic potentials is open so far and the question of gauge invariance of effective action in the $\mathcal{N} = 2, 4$ noncommutative SYM is still under discussion [3, 10, 13, 17]. But in some specific cases (the model of $q$-hypermultiplet in fundamental representation) the holomorphic prepotential is written in a manifestly supersymmetric and gauge invariant form what will be shown in this section.

The structure of functions $\mathcal{F}$ and $\mathcal{H}$ in (66) and all quantum contributions to them can be defined on the basis of perturbative quantum computations. The most natural tool for this is the background field method which we consider below.
The background field method for the commutative $N = 2$ SYM theory in harmonic superspace was developed in the paper [24]. In this section we generalize these results to noncommutative SYM model (61).

Let us start with the classical action of noncommutative vector multiplet (12)

\[ S_{SYM}[V^{++}] = \frac{1}{g^2} \text{tr} \int d^{12}z \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \int du_1 \ldots du_n \frac{V^{++}(z,u_1) \star V^{++}(z,u_2) \star \cdots \star V^{++}(z,u_n)}{(u_1^1 u_2^1)(u_2^1 u_3^1) \ldots (u_n^1 u_1^1)} \] (67)

and perform a splitting of gauge superfield into background $V^{++}$ and quantum $v^{++}$ parts

\[ V^{++} \rightarrow V^{++} + gv^{++}. \] (68)

There are two types of gauge transformations:

i) background

\[ \delta V^{++} = -D^{++} \lambda - i[V^{++},\lambda], \quad \delta v^{++} = i[\lambda,v^{++}]; \] (69)

ii) quantum transformations

\[ \delta V^{++} = 0, \quad \delta v^{++} = -\frac{1}{g} \nabla^{++} \star \lambda - i[v^{++},\lambda]. \] (70)

It can be shown that upon the splitting (68) the classical action (67) can be rewritten in the form

\[ S_{SYM}[V^{++} + gv^{++}] = S_{SYM}[V^{++}] + \frac{1}{4g} \text{tr} \int d\zeta (-4) b \star \nabla^{++} \star (\nabla^{++} \star c + ig[v^{++},c]) \] (71)

where

\[ \Delta S_{SYM}[v^{++},V^{++}] = -\text{tr} \int d^{12}z \sum_{n=2}^{\infty} \frac{(-ig)^{n-2}}{n} \int du_1 du_2 \ldots du_n \frac{v^{++}(z,u_1) \star v^{++}(z,u_2) \star \cdots \star v^{++}(z,u_n)}{(u_1^1 u_2^1)(u_2^1 u_3^1) \ldots (u_n^1 u_1^1)}. \] (72)

Here we introduced the notations

\[ v^{++} = e^{-i\Omega} \star v \star e^{i\Omega}, \quad W_{\lambda} = e^{i\Omega} \star W \star e^{-i\Omega}, \] (73)

where $\Omega$ is a "bridge" superfield corresponding to the background field $V^{++}$ [24]. The classical action (71) is manifestly gauge invariant with respect to the background gauge transformations (69).

To fix the gauge degrees of freedom in the effective action we follow a Faddeev-Popov anzatz. The gauge-fixing condition can be chosen in the same form as in commutative SYM theory

\[ D^{++} v^{++} = 0. \] (74)

Following the same steps as in commutative case [24], one obtains that the ghost and gauge-fixing actions look like this

\[ S_{FP} = \text{tr} \int d\zeta (-4) b \star \nabla^{++} \star (\nabla^{++} \star c + ig[v^{++},c]), \]

\[ S_{fg} = \frac{1}{2\alpha} \text{tr} \int d^{12}z du_1 du_2 \frac{v^{++}(1) \star v^{++}(2)}{(u_1^1 u_2^1)} - \frac{1}{4\alpha} \text{tr} \int d^{12}z d\zeta v^{++}(D^{-})^2 v^{++}, \] (75)
where $b, c$ are the ghosts, $\alpha$ is an arbitrary parameter, we set $\alpha = -1$. Taking all these expressions together we obtain the following expression for the effective action

$$
e^{iS_{\text{SYM}}[V^{++}]} = e^{iS_{\text{SYM}}[V^{++}]} \int Dv^{++} D\bar{b} Dc D\phi D\chi^{(4)} D\sigma e^{iS_q[v^{++}, b, c, \phi, \chi^{(4)}, \sigma]},$$

(76)

where

$$S_q[v^{++}, b, c, \phi, \chi^{(4)}, \sigma] = S_2 + S_{\text{int}},$$

(77)

$$S_2 = -\frac{i}{2} \text{tr} \left( d\zeta^{(-4)} v^{++} \star \hat{\Box}_* \star v^{++} + \frac{1}{2} \text{tr} \left( d\zeta^{(-4)} \chi^{(4)} \star \hat{\Box}_* \star \sigma \right) + i \text{tr} \left( d\zeta^{(-4)} b \star \nabla^{++} \star \nabla^{++} \star c \right) + \frac{1}{2} \text{tr} \left( d\zeta^{(-4)} \phi \star \nabla^{++} \star \nabla^{++} \star \phi \right) \right)$$

and

$$S_{\text{int}} = -\text{tr} \left( d^{12} z \sum_{n=3}^\infty d u_1 d u_2 \ldots d u_n \frac{(-ig)^{n-2} v^{++}_n(z, u_1) \star \ldots \star v^{++}_n(z, u_n)}{(u_1 u_2) \ldots (u_n u_1)} \right) + i g \text{tr} \left( d\zeta^{(-4)} d a b \star \nabla^{++} \star [v^{++}, c] \right).$$

Here $\phi$ is a Nielsen-Kallosh ghost, $\chi^{(4)}$ and $\sigma$ are the auxiliary superfields which are necessary for correct counting of degrees of freedom [24]. The action $S_2$ is defined by the operator

$$\hat{\Box}_* = -\frac{1}{2} (D^+)^4 \nabla^- \star \nabla^- = D^m \star D_m + \frac{i}{2} (\hat{W} \hat{W}) \star \frac{i}{2} (D^{+\alpha} W) \star D^{-\alpha} - \frac{i}{2} (D^{-\alpha} D^\alpha W) \star D^- + \frac{i}{2} (\hat{D}^{+\alpha} \hat{D}^{-\alpha} \hat{W}) \star D^-$$

(78)

which is a noncommutative analog of covariant analytic d’Alambertian [24].

The quadratic part of the action (77) defines the structure of 1-loop effective action as

$$\Gamma^{(1)}[V^{++}] = \frac{i}{2} \text{Tr}_{(2,2)} \ln \hat{\Box}_* - \frac{i}{2} \text{Tr}_{(4,0)} \ln \hat{\Box}_* + i \text{Tr}_q \ln \nabla^{++} - \frac{i}{2} \text{Tr}_{ad} \ln (\nabla^{++}_*)^2,$$

(79)

where the formal expressions of $\text{Tr}$ in eq. (77) are given by the following functional integrals

$$\text{Det}_{(2,2)} \hat{\Box}_*^{-1} = \int Dv^{++} D\bar{u}^{++} \exp \left\{ -i \text{tr} \left( d\zeta^{(-4)} v^{++} \star \hat{\Box}_* \star u^{++} \right) \right\},$$

$$\text{Det}_{(4,0)} \hat{\Box}_*^{-1} = \int D\rho^{(4)} D\sigma \exp \left\{ -i \text{tr} \left( d\zeta^{(-4)} \rho^{(4)} \star \hat{\Box}_* \star \sigma \right) \right\},$$

$$\text{Det}_{\nabla^{++}_*^{-1}} = \int Dq^{+} Dq^{+} \exp \left\{ -i \text{tr} \left( d\zeta^{(-4)} q^+ \star \nabla^{++}_* \star q^+ \right) \right\}. $$

The terms in the first line of eq. (80) come from the actions of vector superfield $v^{++}$ and auxiliary superfields $\chi^{(4)}, \sigma$. The term $\text{Tr}_{ad} \ln (\nabla^{++}_*)^2$ corresponds to the contributions from ghosts $b, c, \phi$. The third term in eq. (77) is responsible for the contributions from matter superfields [6, 14, 6].

The formal expression (77) determines the low-energy structure of $\mathcal{N} = 2$ SYM model (6) and provides a base for perturbative quantum computations in manifestly supersymmetric and gauge invariant way in noncommutative theory.

The equation (77) is convenient for studying the structure of effective action of noncommutative $\mathcal{N} = 4$ SYM. The classical action of $\mathcal{N} = 4$ SYM model written in terms of $\mathcal{N} = 2$ superfields in harmonic superspace reads (see [24, 27] for commutative case)

$$S_{\mathcal{N}=4}[V^{++}, q^+, \hat{q}^+] = \frac{1}{g^2} \text{tr} \int d^4 x d^4 \theta \hat{W}^2 - \frac{1}{g^2} \text{tr} \int d\zeta^{(-4)} q^+ i \nabla^{++}_* \star q^+,$$

(81)

where $q^+_1 = (q^+_1, \hat{q}^+_1)$, $q^+_1 = e^{-i/2} q^+_1 = (\hat{q}^+_1, -q^+_1)$ are the hypermultiplets in adjoint representation. It can be shown that the action (81) possess two more hidden supersymmetries and therefore corresponds to the $\mathcal{N} = 4$ SYM model.
For the $\mathcal{N} = 4$ SYM theory \cite{31}, the terms in the second line of eq. \cite{79} cancel each other and the 1-loop effective action reads
\[
\Gamma_{\mathcal{N}=4}^{(1)}[V^{++}] = \frac{i}{2} \text{Tr}_{(2,2)} \ln \hat{\square}_* - \frac{i}{2} \text{Tr}_{(4,0)} \ln \hat{\square}_*.
\]
(82)
The perturbative expansion of $\text{Tr} \ln$ of noncommutative d'Alambertian $\hat{\square}_*$ in eq. \cite{82} defines the low-energy effective action of noncommutative $\mathcal{N} = 4$ SYM which is given by non-holomorphic potential \cite{66}. The structure of non-holomorphic potential in this theory will be discussed further.

4.2 Holomorphic potential of $q$-hypermultiplet

Let us consider the model of massive single $q$-hypermultiplet in fundamental representation of $U(1)$ gauge group with the classical action
\[
S = \int d\zeta(-4) \tilde{q}^+ \star (D^{++} + iV_0^{++} + iV^{++}) \star q^+ = \int d\zeta(-4) \tilde{q}^+ \star \nabla^{++} \star q^+,
\]
(83)
where $V_0^{++} = -\bar{W}_0(\theta^+)^2 - W_0(\theta^+)^2$, $W_0\bar{W}_0 = m^2$. The propagator and the vertex of the model are given by eqs. \cite{22,27}.

The 1-loop effective action of the model can be formally written as
\[
\Gamma^{(1)}[V^{++}] = \text{itr} \ln(\nabla^{++} \star) = \sum_{n=1}^{\infty} \Gamma_n,
\]
(84)
\[
\Gamma_n = \frac{i(-1)^n}{n} \int d\zeta(-4) \ldots d\zeta(-4) G_0(\zeta, \zeta_2) \star V^{++}(\zeta_2) \ldots G_0(\zeta_n, \zeta_1) \star V^{++}(\zeta_1).
\]
Two-point function was calculated in the section \cite{33} where it was found that this model has no UV/IR-mixing. Now we compute the contributions to the holomorphic effective action from arbitrary $n$-point function with noncommutative corrections. In momentum space the $n$-point function $\Gamma_n$ reads
\[
\Gamma_n = \frac{i(-1)^n}{n(2\pi)^n} \int d^4p_1 d^4\theta^1 d\zeta_1 \ldots d^4p_n d^4\theta^n d\zeta_n \int d^4k \delta^4(\sum p_i) e^{-\frac{i}{2} k \theta(\sum p_i)} \times G_0(k + p_2 + p_3 + \ldots + p_n) G_0(k + p_3 + \ldots + p_n) \ldots G_0(k + p_n) G_0(k)
\]
(85)
\times e^{-\frac{i}{2} \sum \theta p_i V^{++}(1) \ldots V^{++}(n)}.

The first exponent in eq. \cite{85} reduces to unity due to the relation
\[
e^{-\frac{i}{2} \theta \sum p_i} \delta^4(\sum p_i) = \delta^4(\sum p_i).
\]
This equation guarantees the absence of nonplanar diagrams and UV/IR-mixing. Therefore, the limit $\theta \to 0$ is smooth and in low-energy approximation (when we neglect all derivatives of $V^{++}$) the factor $e^{-\frac{i}{2} \sum \theta p_i}$ can be dropped. As a result, we obtain usual expression for the holomorphic effective action of commutative $q$-hypermultiplet, calculated in the ref. \cite{25}
\[
\Gamma[V^{++}] = -\frac{1}{64\pi^2} \int d^4x d^4\theta W^2 \ln \frac{W^2}{\Lambda^2} + \text{c.c.}
\]
(86)
The absence of $\star$-product in eq. \cite{86} is evident since we consider an approximation when $\star$-product reduces to ordinary multiplication. A non-trivial result here is the absence of non-planar diagrams and UV/IR-mixing.
Now consider the next order in the approximation when all derivatives related to noncommutativity are kept. In this case we can not drop the factor $e^{-\frac{i}{2}\sum p_\alpha \theta p_\alpha}$ in eq. (84) but use it to restore the $\star$-product of superfields due to the identity

$$\int d^4p_1 \ldots d^4p_n e^{-\frac{i}{2}\sum n_{i<j}p_\alpha \theta p_\alpha} V_1(p_1) \ldots V_n^{++}(p_n) \delta^4(\sum p_i) = \int d^4x V_1^{++}(x) \star \ldots \star V_n^{++}(x).$$

(87)

It should be noted that we have to express the resulting effective action in terms of strength superfield (11) which is a nonlinear function of $V^{++}$ even in the Abelian case. To simplify this problem we choose the external gauge superfield in the form

$$\tilde{W} = -\frac{1}{4} \int du \bar{D}_\alpha \bar{D}^{-\dot{\alpha}} \tilde{V}^{++}(z,u),$$

(88)

what corresponds to the first term in series (11). With such a special background $\tilde{W}$ one obtains the standard expression for holomorphic effective action [25] written in terms of strength superfields $\tilde{W}$ where conventional multiplication should be replaced by $\star$-product due to the relation (87). To return to arbitrary background it is necessary to restore the full strength $W$ from $\tilde{W}$ with the help of gauge transformations

$$W = e^{\lambda} \star \tilde{W} \star e^{-\lambda}$$

with a special $\lambda = \lambda(W)$. Therefore, for arbitrary $W$ the holomorphic effective action of noncommutative $q$-hypermultiplet reads

$$\Gamma[V^{++}] = -\frac{1}{64\pi^2} \int d^4x d^4\theta W \star W \star \ln \frac{W}{\Lambda} + c.c.$$  

(89)

Note that this expression is manifestly gauge invariant and have correct commutative limit [86].

A generalization of this result to the case of $U(N)$ gauge group broken down to $\{U(1)\}^N$ is rather trivial since the strength $W$ belonging to $\{U(1)\}^N$ can be chosen as $W = \text{diag}(W_1, \ldots, W_N)$. Therefore the effective action of such theory is a sum of actions (89)

$$\Gamma_{\{U(1)\}^N}[V^{++}] = \sum_{i=1}^{N} \Gamma_{U(1)}[V_i^{++}],$$

(90)

where $\Gamma_{U(1)}[V_i^{++}]$ is given by eq. (89).

In the case of adjoint representation of gauge group the situation becomes not so simple. Let us consider the classical action of $q$-hypermultiplet in adjoint representation [24] when the vector superfield belongs to the Cartan subalgebra of $u(N)$, i.e.

$$Q^+ = \sum_{i,j=1}^{N} q_{ij} e_{ij}, \quad V^{++} = \sum_{k=1}^{N} V_k^{++} e_{kk},$$

(91)

where

$$(e_{ij})_{kl} = \delta_{ik} \delta_{jl}, \quad i,j,k,l = 1, \ldots, n$$

(92)

is a Cartan-Weyl basis of $u(N)$ algebra [24]. One can easy check that the interaction [24] now reads

$$\text{tr} \int d\zeta^{(4)} Q^+ \star [V^{++}, Q^+] = \sum_{k,l=1}^{N} \int d\zeta^{(4)} (\bar{q}_{kl} \star V_k^{++} \star q_{kl} - q_{kl} \star \bar{q}_{kl} \star V_l^{++}).$$

(93)

15
Therefore the contributions from hypermultiplets $q_{ij}^+$, decouple and the formal expression for the 1-loop effective action reads

$$\Gamma[V^{++}] = \sum_{k,l=1}^{N} \text{Tr} \ln \nabla_{kl}^{++},$$

(94)

where the operators $\nabla_{kl}^{++}$ act on hypermultiplets by the rule

$$\nabla_{kl}^{++} q_{kl}^+ = D^{++} q_{kl}^+ + iV_{kl}^{++} \star q_{kl}^+ - iq_{kl}^+ \star V_{kl}^{++}.$$ 

Each term in the sum (94) can be calculated in the same way as it was done in sect. 3.3. A leading contribution to the effective action is given by the term which is responsible for the UV/IR-mixing

$$\Gamma[V^{++}] = \frac{1}{32\pi^2} \int d^4p d^8\theta du_1 du_2 \ln \frac{4}{m^2p \circ p} \frac{V^{++}(p,\theta,u_1)V^{++}(-p,\theta,u_2)}{(u_1^2u_2^2)^2},$$

(95)

where $V^{++} = \sum_k V_{k}^{++}$ is the gauge superfield related to the $U(1)$ subgroup of $U(N)$ gauge group. The expression (95) is singular in commutative limit. We see that the UV/IR-mixing appear only in the $U(1)$ sector what was firstly obtained for noncommutative $U(N)$ Yang-Mills model in [9] and for supersymmetric gauge theories in [10, 11].

The next order contribution to the effective action which is finite in the limit $\theta \to 0$ corresponds to the holomorphic potential of commutative $q$-hypermultiplet in adjoint representation of $SU(N)$ group calculated in [28].

To summarize, the effective action of noncommutative $q$-hypermultiplet in adjoint representation of $U(N)$ group is singular in the commutative limit and contains the terms which are responsible for the UV/IR-mixing. The finite contribution in the limit $\theta \to 0$ is given by the standard holomorphic potential of $q$-hypermultiplet in adjoint representation of $SU(N)$ gauge group.

### 4.3 Non-holomorphic potential in noncommutative $\mathcal{N} = 4$ SYM

Let us consider the $\mathcal{N} = 4$ SYM model written in terms of $\mathcal{N} = 2$ superfields [61]. The effective action of noncommutative $\mathcal{N} = 4$ SYM (82) is determined by the functional integrals (80) in terms of noncommutative covariant d’Alambertian (78). In commutative case it was shown in refs. [26, 27] that the effective action can be represented in the form of functional integral over unconstrained $\mathcal{N} = 1$ superfields what allows to restore the non-holomorphic potential very easy. We will see that the effective action of noncommutative $\mathcal{N} = 4$ SYM can also be represented as a functional integral over unconstrained $\mathcal{N} = 1$ superfields what gives a starting point for calculating the noncommutative corrections to the holomorphic potential. This is done only with the help of special constraints on the background strength superfield $W$. The first requirement is the on-shell constraint

$$\mathcal{D}^{\alpha}(i \mathcal{D}_\alpha) \star W = 0$$

(96)

that simplifies the noncommutative covariant d’Alambertian to the form

$$\hat{\Box}_* = \mathcal{D}^m \star \mathcal{D}_m + \frac{1}{2} \{W, W\}_* + \frac{i}{2} (D^{+\alpha} W) \star \mathcal{D}_\alpha + \frac{i}{2} (\bar{D}_{\dot{\alpha}}^+ \bar{W}) \star \bar{D}^{-\dot{\alpha}}.$$ 

(97)

Moreover, we consider only the case when the gauge symmetry $U(N)$ is broken down to $[U(1)]^N$ and the strength $W$ belongs to the Cartan subalgebra of $u(N)$. Further calculations are very similar to ones made for conventional $\mathcal{N} = 4$ SYM in harmonic superspace given in refs. [26, 27].
At first, we make a replacement of functional variables in the integrals (98)

\[ u^{++} = F^{++} + \nabla^{++} \star \sigma \]
\[ u^{+} = G^{++} + \nabla^{++} \star \int d\tilde{\zeta}(-4)G^{(0,0)}(\zeta, \tilde{\zeta})\rho^{(+1)}(\tilde{\zeta}), \]

where \( F^{++} \) and \( G^{++} \) are constrained \( \mathcal{N} = 2 \) superfields

\[ \nabla^{++} \star F^{++} = 0, \quad \nabla^{++} \star G^{++} = 0. \]

This allows us to rewrite the effective action in the form of single functional integral but over the constrained superfield \( F^{++} \):

\[ \exp(i\Gamma_{\mathcal{N}=4}^{(1)}) = \frac{\int D\mathcal{F}^{++} \exp\{-\frac{i}{2} \text{tr} \int d\zeta(-4) \mathcal{F}^{++} \star \hat{\Box} \star \mathcal{F}^{++}\}}{\int D\mathcal{F}^{++} \exp\{\frac{i}{2} \text{tr} \int d\zeta(-4) \mathcal{F}^{++} \star \mathcal{F}^{++}\}. \]

The superfield \( \mathcal{F}^{++} \) can be expanded over harmonic variables as

\[ \mathcal{F}^{++}(z, u) = \mathcal{F}^{ij}(z)u_i^+ u_j^+, \]

where the superfields \( \mathcal{F}^{ij} \) satisfy the constraints

\[ D^{(i} \star \mathcal{F}^{jk)} = \mathcal{D}^{(i} \star \mathcal{F}^{jk)} = 0, \quad \mathcal{F}^{ij} = \mathcal{F}_{ij}. \]

The operator \( \hat{\Box} \) acts on \( \mathcal{F}^{ij} \) as follows

\[ \hat{\Box} \mathcal{F}^{ij} = (D^m \star D_m + \frac{1}{2} \{W, W\}) \star \mathcal{F}^{ij} + \frac{i}{3} (D^{(ij} \star W) \star D_{ijk} \star \mathcal{F}^{k)} + \frac{i}{3} (D^{(ij} \star \tilde{W}) \star \bar{D}_{ijk} \star \mathcal{F}^{k}). \]

The next step is to go to \( \mathcal{N} = 1 \) projections of \( \mathcal{N} = 2 \) superfield \( \mathcal{F}^{++} \) in order to reduce the functional integral (100) to one over unconstrained superfields. We introduce \( \mathcal{N} = 1 \) Grassman coordinates \( (\theta^{\alpha}, \tilde{\theta}_{\dot{\alpha}}) \) by the rule \( \theta^{\alpha} = \theta^{\alpha 1}, \tilde{\theta}_{\dot{\alpha}} = \tilde{\theta}_{\dot{\alpha} 1} \), the corresponding gauge covariant derivatives \( \mathcal{D}^{\alpha} = \mathcal{D}^{\alpha 1}, \tilde{\mathcal{D}}_{\dot{\alpha}} = \tilde{\mathcal{D}}_{\dot{\alpha} 1} \) and then define the \( \mathcal{N} = 1 \) projections of an arbitrary \( \mathcal{N} = 2 \) superfield as \( f = f(x^m, \theta^\alpha, \tilde{\theta}_{\dot{\alpha}}) \theta_{\alpha} = \tilde{\theta}_{\dot{\alpha}} = 0 \). The \( \mathcal{N} = 1 \) projections of \( \mathcal{N} = 2 \) strength \( W \) and \( \mathcal{F}^{ij} \) read

\[ \phi = W|, \quad 2iW_\alpha = \mathcal{D}^2 \star W|, \]
\[ \Psi = \mathcal{F}^{11}|, \quad \bar{\Psi} = \tilde{\mathcal{F}}^{11}|, \quad F = \tilde{F} = -2i\mathcal{F}^{12}. \]

The equations (102) reduce to the following constraints on \( \Psi, \bar{\Psi}, F \)

\[ \mathcal{D}_\alpha \star \Psi = 0, \quad -\frac{1}{4} (\mathcal{D})^2 \star F + [\phi, \bar{\Psi}] = 0. \]

At this point we require the \( \mathcal{N} = 1 \) components of background strength superfield \( W \) to be covariantly constant

\[ \mathcal{D}_\alpha \star \phi = 0, \quad \mathcal{D}_\alpha \star W_\beta = 0. \]

In commutative case such constraints were sufficient to calculate a non-holomorphic potential. We are interested in noncommutative corrections to this result, therefore we also require such constraints in noncommutative case. For such a background the operator \( \hat{\Box} \) does not mix the superfields \( \Psi, \bar{\Psi} \) and \( F \)

\[ (\hat{\Box} \star \mathcal{F}^{ij}) = \Delta \star (\mathcal{F}^{ij}). \]
where
\[ \Delta = D^m \star D_m + \frac{1}{2} \{ \phi, \bar{\phi} \}_\star - W^\alpha \star D_\alpha + \bar{W}^\alpha \star \bar{D}^\alpha. \] (108)

This allows us to rewrite the functional integral in the form
\[
\exp(i \Gamma^{(1)}_{N=4}) = \frac{\int D\bar{\Psi} D\Psi D\bar{F} \exp\{i \text{tr} \int d^8 z(-\bar{\Psi} \star \Delta \star \Psi + \frac{1}{2} F \star \Delta \star F)\}}{\int D\bar{\Psi} D\Psi D\bar{F} \exp\{i \text{tr} \int d^8 z(-\bar{\Psi} \bar{\Psi} + \frac{1}{2} F^2)\}}. \] (109)

The superfields \( \Psi, \bar{\Psi}, F \) in the expression are the Lie algebra valued superfields of \( U(N) \) gauge group while the background strength \( W_\alpha \) and \( \phi \) belong to the Cartan subalgebra. They can be written in the Cartan-Weyl basis of \( U(N) \) as
\[
F = \sum_{k<l} F^{kl} e_{kl} + \sum_{k=1} F^k e_{kk}, \quad \phi = \sum_{k=1} \phi^k e_{kk}. \] (110)

Given an element \( \phi \) in the Cartan subalgebra, one finds
\[
[\phi, F]_\star = \sum_{k<l} (\phi^k \ast F^{kl} - F^{kl} \ast \phi^l) e_{kl} + \sum_{k=1} (\phi^k \ast F^k - F^k \ast \phi^k) e_{kk}. \] (111)

Therefore, the second equation of constraints can be written as
\[
-\frac{1}{4} (\bar{D}^2 \star F^{kl}) + (\phi^k \ast \Psi^{kl} - \Psi^{kl} \ast \phi^l) = 0, \quad -\frac{1}{4} (\bar{D}^2 \star F^k) + (\phi^k \ast \Psi^k - \Psi^k \ast \phi^k) = 0. \] (112)

The equations can be resolved with respect to the superfields \( \Psi^{kl} \) and \( \Psi^k \) as a series in \( \theta^2 \). A formal solution is written as
\[
\Psi^{kl} = B^{kl} F^{kl}, \quad \Psi^k = B^k F^k, \] (113)

where \( B^{kl}, B^k \) are some operators which we do not specify. Taking into account the eqs. one can transform the integral in the denominator of eq. \( \text{109} \) as follows
\[
\text{tr} \int d^8 z(-\bar{\Psi} \star \Psi + \frac{1}{2} F \star F) = \int d^8 z \sum_{k<l} (B^k F^{kl} \star B^k F^{kl} + \frac{1}{2} F^{kl} \star F^{kl}) + \int d^8 z \sum_{k=1} (B^k F^k \star B^k F^k + \frac{1}{2} F^k \star F^k). \] (114)

As a result, we obtain the following formal expression for the functional integral in the denominator of eq. \( \text{109} \)
\[
\int D\bar{\Psi} D\Psi D\bar{F} \exp\{i \text{tr} \int d^8 z(-\bar{\Psi} \star \Psi + \frac{1}{2} F \star F)\} = \prod_{k<l} \text{Det}^{-1}(C_{kl}) \prod_{k=1} \text{Det}^{-1}(C_k), \] (115)

A solution to the equation in the second line of is not unique since in the limit \( \theta \to 0 \) this eq. becomes the identity, but it does not spoil the picture.
where we introduced the superfields \( V^{kl} = F^{kl}, \bar{V}^{kl} = F^{lk}, \) \( k < l, \) and the operators \( C_{kl}, C_k \) which act as
\[
\int d^8 z \bar{B}^{kl} F^{kl} = \int d^8 z F^{kl} \ast \bar{C}^{kl} F^{kl},
\]
\[
\int d^8 z B^k F^k = \int d^8 z F^k \ast C^k F^k.
\]
(116)

Following the same steps we represent the numerator of eq. (109) in the form
\[
\int \mathcal{D} \bar{\Psi} \mathcal{D} \Psi \mathcal{D} F \exp \{ i \text{tr} \int d^8 z (-\bar{\Psi} \ast \Delta \ast \Psi + \frac{1}{2} F \ast \Delta \ast F) \}
= \int \mathcal{D} \bar{V}^{kl} \mathcal{D} V^{kl} \mathcal{D} V^k \exp \{ i \int d^8 z \left( \sum_{k<l} \bar{V}^{kl} \ast C_{kl} \Delta_{kl} \ast V^{kl} + \sum_{k=1}^N V^k \ast C_k \Delta_k \ast V^k \right) \}
= \prod_{k<l} \text{Det}^{-1}(C_{kl}) \prod_{k=1}^N \text{Det}^{-1}(C_k) \prod_{k<l} \text{Det}^{-1}(\Delta_{kl}) \prod_{k=1}^N \text{Det}^{-1}(\Delta_k),
\]
(117)

where the operators \( \Delta_{kl}, \Delta_k \) act on the superfields \( V^{kl}, V^k \) by the rules:
\[
\Delta_{kl} \ast V^{kl} = D^m \ast D^m \ast V^{kl}
- (W^{ka} \ast (D_\alpha \ast V^{kl}) \ast W^{\alpha} + (\bar{W}^{\bar{k}}_\bar{a} \ast (\bar{D}_{\bar{\alpha}} \ast V^{kl}) \ast \bar{W}_{\bar{\alpha}}))
+ \phi^k \ast \phi^l \ast V^{kl} - 2 \phi^k \ast V^{kl} \ast \bar{\phi}^l + 2 \phi^k \ast V^{kl} \ast \bar{\phi}^l
+ V^{kl} \ast \phi^l \ast \phi^l + \phi^k \ast \phi^l \ast V^{kl} + V^{kl} \ast \phi^l \ast \phi^l,
\]
\[
\Delta_k \ast V^k = D^m \ast D^m \ast V^k
- (W^{ka} \ast (D_\alpha \ast V^k) \ast W^{\alpha} + (\bar{W}^{\bar{k}}_\bar{a} \ast (\bar{D}_{\bar{\alpha}} \ast V^k) \ast \bar{W}_{\bar{\alpha}}))
+ \phi^k \ast \phi^k \ast V^k - 2 \phi^k \ast V^k \ast \bar{\phi}^k + 2 \phi^k \ast V^k \ast \bar{\phi}^k
+ V^k \ast \phi^k \ast \phi^k + \phi^k \ast \phi^k \ast V^k + V^k \ast \phi^k \ast \phi^k.
\]
(118)

(119)

Substituting the functional integrals (115,117) into eq. (109) we see that the operators \( C^k \) and \( C^{kl} \) in numerator and denominator cancel each other and the resulting expression for the effective action is represented in the following form
\[
\Gamma_{\mathcal{N}=4}^{(1)} = \sum_{k<l}^N \Gamma_{kl} + \sum_{k=1}^N \Gamma_k,
\]
(120)

\[
\Gamma_{kl} = i \text{Tr} \ln \Delta_{kl}, \quad \Gamma_k = i \text{Tr} \ln \Delta_k.
\]
The formal expressions of \( \text{Tr} \ln \) of the operators \( \Delta_{kl}, \Delta_k \) should be understood in the sense of the functional integrals over the \( \mathcal{N} = 1 \) superfields \( V^{kl}, V^k \)
\[
\text{Det}^{-1}(\Delta_{kl}) = \int \mathcal{D} \bar{V}^{kl} \mathcal{D} V^{kl} \exp \{ i \int d^8 z (\bar{V}^{kl} \ast \Delta_{kl} \ast V^{kl}) \},
\]
\[
\text{Det}^{-1/2}(\Delta_k) = \int \mathcal{D} V^k \exp \{ i \int d^8 z (V^k \ast \Delta_k \ast V^k) \}.
\]
(121)
The first term \( \sum_{k<l}^N \Gamma_{kl} \) corresponds to \( U(N)/[U(1)]^N \) sector of \( U(N) \) group while the second term is defined by the integrals over superfields belonging to the \( [U(1)]^N \) subgroup.

It should be noted that the commutative limit is smooth since \( \mathcal{N} = 4 \) SYM has no divergences. Taking the limit \( \theta \to 0 \) in the expressions (118,119) we see that the operator \( \Delta_{kl} \) reproduces the standard non-holomorphic potential of commutative \( \mathcal{N} = 4 \) \( SU(N) \) SYM studied in [27], while the contributions from the operator \( \Delta_k \) vanish. Therefore, the second term of effective action \( \sum_{k=1}^N \Gamma_k \) is responsible for higher noncommutative corrections only.
Thus, we have obtained here a useful representation \(^{120}\) of 1-loop effective action of noncommutative \(\mathcal{N} = 4\) SYM in terms of functional integral over unconstrained \(\mathcal{N} = 1\) superfields what provides a basis for perturbative calculations of 1-loop effective action and studying the noncommutative corrections.

5 Summary

We have developed a formulation of the basic \(\mathcal{N} = 2\) supersymmetric field models in noncommutative harmonic superspace. The classical actions of noncommutative hypermultiplet and \(\mathcal{N} = 2\) vector multiplet can be easily constructed in such a superspace in terms of unconstrained \(\mathcal{N} = 2\) superfields. We formulated the Feynman rules for these theories, considered the calculations of various one-loop harmonic supergraphs and pointed out the new aspects arising in comparison with corresponding commutative theories.

We observed that the \(\mathcal{N} = 2\) SYM model has a standard (for noncommutative theories) UV/IR-mixing and its effective action is singular in the limit of small noncommutativity \(\theta \to 0\). The model of Abelian noncommutative \(q\)-hypermultiplet interacting with external vector superfield is free of UV/IR-mixing and has no non-planar diagram contributions. The model of selfinteracting \(q\)-hypermultiplet is nonrenormalizable but the quartic interaction vertex can be induced by 1-loop quantum corrections as it is in the commutative case.

We formulated a background field method in noncommutative harmonic superspace and considered a structure of low-energy effective action of general noncommutative \(\mathcal{N} = 2\) model. It is shown that the one-loop effective action is represented in terms of superfield functional determinants of the operators \(\hat{\Box}_\ast\) and \(\nabla^{++}\) described in the paper. We found that the holomorphic effective action of noncommutative \(q\)-hypermultiplet in the fundamental representation is free of non-planar contributions and it can be written in manifestly supersymmetric and gauge invariant form as in commutative case. The effective action of \(q\)-hypermultiplet in adjoint representation has UV/IR-mixing due to the nonplanar contributions what does not allow us to represent the effective action in a gauge invariant form.

We have considered the one-loop effective action of noncommutative \(\mathcal{N} = 4\) SYM theory. This theory has been studied within the \(\mathcal{N} = 2\) background field method and it has been shown that the effective action consists of two contributions associated with \(U(\mathcal{N})/U(1)^\mathcal{N}\) and \(U(1)^\mathcal{N}\) sectors of the \(U(\mathcal{N})\) gauge group. The first term reproduces a standard non-holomorphic potential in \(\theta \to 0\) limit while the second one is responsible for higher noncommutative corrections only.

The constructions developed in the paper and the results concerning the formulation of \(\mathcal{N} = 2\) background field method in noncommutative harmonic superspace and the structure of low-energy effective action can be applied for the study of various quantum aspects in noncommutative \(\mathcal{N} = 2\) supersymmetric field models.

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