On the infrared behaviour of 3d Chern-Simons theories in $\mathcal{N} = 2$ superspace

Matías Leoni and Andrea Mauri

Dipartimento di Fisica, Università di Milano and INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy

E-mail: matias.leoni@mi.infn.it, andrea.mauri@mi.infn.it

Abstract: We discuss the problem of infrared divergences in the $\mathcal{N} = 2$ superspace approach to classically marginal three-dimensional Chern-Simons-matter theories. Considering the specific case of ABJM theory, we describe the origin of such divergences and offer a prescription to eliminate them by introducing non-trivial gauge-fixing terms in the action. We also comment on the extension of our procedure to higher loop order and to general three-dimensional Chern-Simons-matter models.

Keywords: Infrared divergences, Superspace, ABJM theory.
1. Introduction

The perturbative expansion of the off-shell amplitudes for a supersymmetric gauge theory may be plagued by infrared divergences for a number of different motivations. One possible source of infinities is given by the presence of positive mass-dimension couplings associated to massless fields in superrenormalizable theories. Canonical Yang-Mills theory coupled to massless matter in three dimensions turns out to be a good playground to study these phenomena [1]. In this case, going high in the order of the dimensionful coupling in the expansion of a given amplitude, for dimensional reasons one obtains
high powers of external momenta in the denominator. If the fields are massless, upon inserting those amplitudes in higher order graphs, IR divergences inevitably show up in the Euclidean integrals.

Considering theories with only dimensionless couplings greatly improves the situation. A direct power counting argument shows that in four-dimensions classical marginality is a sufficient condition to exclude the presence of IR infinities in amplitudes with generic external momenta [2]. Even if this argument may be generalized to marginal three dimensional theories, it only applies to component field formulations and cannot be applied to superspace formulations as we will show throughout this work.

In fact, a potential source of IR divergences has to be considered as soon as the computations are performed using supergraph techniques in supersymmetric gauge theories. While proving to be an efficient method to compute perturbative corrections, superspace algebra comes with additional infrared issues due to the peculiar nature of the gauge superfield propagator. As it is clearly described in [3] in the case of four-dimensional Super-Yang-Mills theories, the appearance of infrared infinities can be ascribed to the presence of (corrected) vector lines in loop diagrams. With a canonically gauge-fixed action and omitting the color structure, the gauge superfield propagator in momentum space can be written as

$$\frac{1 + (\alpha - 1)P_0}{p^2} \delta^4(\theta - \theta')$$

where $\alpha$ is the gauge-fixing parameter and $P_0 = -\frac{1}{p^2}(D^2\bar{D}^2 + \bar{D}^2D^2)$ is the superspin zero projector. Recall that this operator satisfies $P_0 + P_{1/2} = 1$, where $P_{1/2} = \frac{1}{p^2}D^\alpha \bar{D}^2D_\alpha$ is the superspin $1/2$ projector. It is clear that, already at the one-loop level, the Fermi-Feynman gauge $\alpha = 1$ is the only infrared safe choice. On the other hand radiative corrections, being governed by Slavnov-Taylor identities, come with the transverse structure $P_{1/2}$, thus reintroducing the infrared dangerous part in the propagator. Therefore, unless a way is found to perturbatively maintain the Fermi-Feynman form of the propagator, IR divergences will show up again starting from two-loop order.

An explicit prescription to cure the IR divergences in the case of four dimensional Super-Yang-Mills theory has been given in [4]. The main idea is to introduce a non-local gauge fixing term and renormalize the gauge fixing parameter to preserve the tree level structure of the Fermi-Feynman propagator. The prescription presented in [4] is strongly based on the nature of the model (gauge sector of Yang-Mills type) and on the space-time dimension.

The aim of this paper is to study the infrared behaviour of supergraph amplitudes in the case of marginal Chern-Simons-matter systems in three dimensions described in the $\mathcal{N} = 2$ superspace formalism. These models can be treated in strong analogy with 4d
Yang-Mills theories while exhibiting a completely different gauge structure. Due to the recent interest in CS-matter models in the context of AdS/CFT we start our analysis in the specific case of ABJM theory \([5]\). By directly computing Green’s functions up to two-loop order, we show that infrared divergences appear in the amplitudes but can be seen as a gauge artefact of the formalism. At first, we gauge fix the ABJM action in a canonical way (\(\alpha\)-gauge) and show that, in analogy with the four-dimensional SYM case, IR infinities show up when a corrected gauge vector propagator is inserted in loop amplitudes. By direct inspection of the dependence of the infrared singularities on the gauge fixing parameter \(\alpha\), we conclude that there is no suitable choice for the latter that both eliminate the divergences and preserve hermiticity of the action.

To solve this problem, we slightly revise the prescription of \([4]\) introducing a set of non-canonical gauge fixing terms (\(\eta\)-gauge). This new set of gauges has the virtue that it can be used, by perturbatively fine tuning the parameter \(\eta\), to complete loop by loop the transverse structure of the gauge vector propagator with the longitudinal part, thus improving its behaviour in the infrared. We will show how the infinities are canceled in this way by direct perturbative computations. The \(\eta\)-gauge can hence be considered as a tool to consistently study the perturbative expansion of the amplitudes of the model without the presence of IR divergences. It’s important to stress that infrared divergences, being an artefact of the superspace formalism, do not manifest themself in physical gauge invariant quantities. In this case the \(\alpha\)- and \(\eta\)-gauges produce coincident results.

As a byproduct of our analysis, we explicitly compute the finite expression in a general gauge for the two-loop propagator of the chiral superfield in ABJM. Moreover, we study a special vanishing external momenta limit of the two-loop vertex function of ABJM theory showing that it produces a finite result. Finally we comment on the extension of our results to a general perturbative order and to any classically marginal Chern-Simons-matter system.

2. ABJM action and gauge-fixing

To address the problem of IR divergences in three-dimensional Chern-Simons theories we restrict ourself to the specific case of ABJM model. This theory possesses remarkable properties such as extended supersymmetry and exact conformal invariance which will simplify the analysis of the infrared behaviour. We extend our results to more general CS theories in Section 4.

At first, we set up our notations and quantize the theory introducing the standard gauge fixing procedure and an alternative one that will ensure the cancelation of the IR divergences in loop amplitudes. We will use the \(\mathcal{N} = 2\) superspace formulation first
presented in [6] adapted to the notations of [7] (see Appendix A for further details).
In Euclidean space, we quantize the theory with a path integral measure of the form
\[ \int D\phi \, e^{i\phi} \]. ABJM theory can then be written as
\[ S = S_{CS} + S_{mat} + S_{pot}, \]
where
\[ S_{CS} = \frac{k}{4\pi} \int d^3x d^4\theta \int_0^1 dt \, \text{tr} \left[ V D^\alpha \left( e^{-itV} D_\alpha e^{itV} \right) - \bar{V} D^\alpha \left( e^{-it\bar{V}} D_\alpha e^{it\bar{V}} \right) \right] \]
\[ S_{mat} = \int d^3x d^4\theta \, (\bar{A}^A e^{i\bar{V}} A e^{-V} + \bar{B}_A e^{i\bar{V}} B^A e^{-V}) \]
\[ S_{pot} = \frac{2\pi i}{k} \int d^3x d^2\theta \, \epsilon_{AC} \epsilon_{BD} \text{tr}(B^A A_B B^C A_B) + \frac{2\pi i}{k} \int d^3x d^2\theta \, \epsilon_{AC} \epsilon_{BD} \text{tr}(\bar{B}_A \bar{A}^P \bar{B}_C \bar{A}^P). \]

The chiral superfields \( B^A \) and \( A_A \) (where \( A,B,C,D=1,2 \)) transform in the \((2,1)\) and \((1,2)\) of the global \( SU(2) \times SU(2) \) respectively. Moreover, they transform in the \((N,N)\) and \((\bar{N},\bar{N})\) of the gauge group \( U_k(N) \times U_{-k}(N) \), such that if explicit gauge group labeling is needed, the chiral superfields are \( B^a_A, \bar{B}^\dot{a}_A, \bar{A}^\dot{a}_a, \dot{A}^a_a \), with \( a, \dot{a}, \dot{b}, b = 1, \ldots, N \). The gauge vector superfields \( V \) and \( \bar{V} \) are in the adjoint representation of the groups and may be written either as \( V = T^i V_i \) with \( i = 1, \ldots, N^2 \) or with matrix labeling \( V^a_i, \bar{V}^\dot{a}_i \).

To quantize the theory we re-scale the vector gauge fields \( V \to \sqrt{\frac{4\pi}{k}} V \) and we choose in each gauge sector the gauge fixing functions \( F = \bar{D}^2 V, \bar{F} = D^2 V \). The standard procedure in \( d = 3 \) is to introduce in the functional integral the factor:
\[ \int \mathcal{D}f \mathcal{D}\bar{f} \Delta(V)\Delta^{-1}(V) \exp \left( \frac{1}{2\alpha} \int d^3x d^2\theta \, \text{tr} (f \bar{f}) \right) \exp \left( \frac{1}{2\alpha} \int d^3x d^2\theta \, \text{tr} (\bar{f} f) \right), \]
where
\[ \Delta(V) = \int d\Lambda d\bar{\Lambda} \, \delta(F(V, \Lambda \bar{\Lambda}) - f) \, \delta(\bar{F}(V, \Lambda \bar{\Lambda}) - \bar{f}), \]
with \( \Lambda \) the chiral superfield of the gauge transformation \( e^V \to e^{i\Lambda} e^V e^{-i\Lambda} \) and \( \alpha \) a dimensionless parameter. Notice that, following [10], we are introducing a gauge averaging given by gaussian weights with chiral integrals of the form \(~ e^{\int \bar{f} f} e^{\int \bar{f} f} \). This is to be contrasted with the standard \( d = 4, \mathcal{N} = 1 \) superspace procedure where one averages with a non-chiral (whole superspace) gaussian weight of the form \( e^{\int \bar{f} f} \).

The average produces the canonical quadratic gauge fixed action [8]
\[ S^{(\alpha)}_{gf} = \frac{1}{2} \int d^3x d^4\theta \, \text{tr} \left[ V \left( \bar{D}^\gamma D_\gamma + \frac{1}{\alpha} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) \right] - \text{tr} \left[ \bar{V} \left( \bar{D}^\gamma D_\gamma + \frac{1}{\alpha} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) \right], \]

\[ (2.6) \]
such that, after inverting the kinetic operator we obtain the gauge field propagators in momentum space

\[
V^a_b V^c_d = \frac{1}{p^2} (\dd^\alpha D_\alpha + \alpha D^2 + \dd \dd^2) \delta^i_j \delta^k_l,
\]

\[
\hat{V}^a_b \hat{V}^c_d = -\frac{1}{p^2} (\dd^\alpha D_\alpha + \alpha D^2 + \dd \dd^2) \delta^i_j \delta^k_l,
\]

with \(\delta^i_j(\theta,\theta') = \delta^i_j(\theta - \theta').\) From now on we shall call this gauge fixing procedure as the "\(\alpha\)-gauge". This propagator simplifies greatly when \(\alpha \to 0\) (Landau Gauge). In the next Section we will see that, if we want to preserve the hermiticity of the gauge fixed action considering \(\alpha\) and \(\dd \alpha\) as complex conjugates, then the infrared divergences cannot be canceled by a simple fine tuning of the gauge fixing parameter.

To solve this problem, in analogy with [4], we propose a different gauge averaging procedure. We choose the same gauge fixing functions as before, but this time we introduce the following term in the functional integral:

\[
\det \hat{\mathcal{M}} \int Df D\bar{f} \Delta(V) \Delta^{-1}(V) \exp \left( \int d^3x d^4\theta \text{tr} \left[ \hat{V}(p) \left( \dd^\gamma D_\gamma - \frac{|p|}{\eta^{(p)}_0} P_0 \right) V(p) \right] - \text{tr} \left[ \hat{\hat{V}}(p) \left( \dd^\gamma D_\gamma + \frac{|p|}{\eta^{(p)}_0} P_0 \right) \hat{V}(p) \right] \right).
\]

(2.8)

In this way we allow for a non-trivial gauge averaging by the insertion of the operator \(\hat{\mathcal{M}}\). It is important to stress that as long as \(\hat{\mathcal{M}}\) is field independent, the \(\det \hat{\mathcal{M}}\) factor appearing in the functional integral is irrelevant and there is no need to introduce Nielsen-Kallosh ghosts [9] in the action. In \(d = 4\) the \(\hat{\mathcal{M}}\) operator is dimensionless and one can simply choose \(\hat{\mathcal{M}} = \text{constant}\). In three dimensions it has dimensions of length so that we may choose either a dimensionful constant or a non-local gauge fixing term. Our choice of this operator in momentum space is \(\hat{\mathcal{M}}(p) = \frac{1}{\eta^{(p)}_0 |p|}\), where \(\eta^{(p)}_0\) is a dimensionless function that contains \(\epsilon\) powers of \(p\). The early introduction of the \(\epsilon = \frac{3}{2} - \frac{d}{2}\) regulator parameter has to be understood formally in the sense of dimensional reduction, that is, we will still perform D-algebra calculations in three dimensions and only at the end we will regularize Feynman integrals. More specifically, we will define \(\eta\) as an odd power series in the 't Hooft coupling \(\lambda = \frac{N}{k}\) with coefficients that we will conveniently choose. By choosing the same Gaussian measure (2.8) for both gauge sectors we obtain the gauge fixed action in momentum space

\[
S^{(\eta)}_{gf} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} d^4\theta \text{tr} \left[ V(-p) \left( \dd^\gamma D_\gamma - \frac{|p|}{\eta^{(p)}_0} P_0 \right) V(p) \right] - \text{tr} \left[ \hat{V}(-p) \left( \dd^\gamma D_\gamma + \frac{|p|}{\eta^{(p)}_0} P_0 \right) \hat{V}(p) \right].
\]

(2.9)

\(^1\)This choice would introduce Nielsen-Kallosh ghosts in the action if the computations were performed using the background field method as in [4].
Inverting the operators of the quadratic part of the gauge fixed action we obtain the gauge field propagators

\[
V^a_b \equiv \frac{1}{p^2} \left( \bar{D}^a D^b + \frac{\eta(p)}{|p|} \mathcal{P}_0 \right) \delta^b_{a'} \delta^c_{d'},
\]

\[
\hat{V}^\hat{a}_\hat{b} \equiv -\frac{1}{p^2} \left( \bar{D}^\hat{a} D^\hat{b} + \frac{\eta(p)}{|p|} \mathcal{P}_0 \right) \delta^\hat{b}_{\hat{a}'} \delta^\hat{c}_{\hat{d}'}.
\]

(2.10)

We shall call this gauge fixing procedure as the “\(\eta\)-gauge”. We will show that allowing \(\eta\) to be corrected order by order in the 't Hooft coupling \(\frac{N}{k}\), we will be able to cancel the infrared divergent parts in the amplitudes. To complete the gauge fixing procedure, we rewrite the \(\Delta^{-1}(V)\) factor in the path integral using the usual Fadeev-Popov \(b-c\) and \(\hat{b}-\hat{c}\) system of Grassmanian chiral superfield ghosts. In both of our gauge fixing choices the ghost action reads

\[
S_{fp} = \int d^3 x d^4 \theta \left[ \bar{b} c + \bar{c} b + \frac{1}{2} \sqrt{\frac{4\pi}{k}} (b + \bar{b}) [V, c + \bar{c}] 
- \bar{\hat{b}} \hat{c} - \hat{c} \bar{\hat{b}} - \frac{1}{2} \sqrt{\frac{4\pi}{k}} (\hat{b} + \bar{\hat{b}}) [\hat{V}, \hat{c} + \bar{\hat{c}}] + \mathcal{O}(1/k) \right].
\]

(2.11)

In Appendix B we detail some of the relevant Feynman rules for ABJM theory.

3. Infrared behaviour of amplitudes in ABJM theory

We would like now to understand the origin of the IR divergences in the perturbative expansion of the off-shell amplitudes. In order to do so, we compute the one-loop correction to the vector gauge superfield \(V\) in the \(\alpha\)- and \(\eta\)-gauges. Performing a direct two-loop computation of the (finite) corrections to the self-energy of the matter superfield and to the superpotential, we will see that IR infinities only arise when the one-loop corrected gauge propagator is inserted in loop diagrams. A suitable choice of the \(\eta\) gauge fixing parameter will then cancel the divergences. Our explicit examples will be completed with an all loop analysis in Section 4.

3.1 One-loop vector propagator

The one loop corrected gauge vector field receives contributions from matter, ghost and gauge vector fields as we show in Figure 1. In the \(\alpha\)-gauge these evaluate
The following ideas may be also worked out at subleading order in $N$ but we would have to add a mixed $V - V$ gauge fixing term.

\[ \eta^c_q = \frac{1}{2} \eta^c_{(k)} \lambda + O(\lambda^3), \]  

[3.3]
such that to order $\lambda$ we obtain the total correction of the propagator:

$$
\lambda \left( \frac{-6\pi G^\epsilon_{(1,1)}(k^2)^{-\epsilon} \mathcal{P}_{1/2} + \eta^{(k)}_1 \mathcal{P}_0}{(k^2)^{1/2}} \right) \delta(\theta,\theta').
$$

(3.4)

It’s easy to see that if we choose $\eta^{(k)}_1 = -6\pi G^\epsilon_{(1,1)}(k^2)^{-\epsilon}$ we exactly complete the transverse structure $\mathcal{P}_{1/2}$ with the longitudinal part $\mathcal{P}_0$ to obtain

$$
-6\pi G^\epsilon_{(1,1)} \lambda \frac{\delta(\theta,\theta')}{(k^2)^{1/2+\epsilon}}.
$$

(3.5)

In the next Section we compute two-loop Green’s functions and show that the improved IR behaviour of the $\eta$-gauge propagator in (3.5) is enough to cure the problem of IR infinities.

3.2 Matter self-energy at two-loop order

3.2.1 Landau gauge

Working in the Landau gauge simplifies greatly the calculation since many diagrams can be discarded due to the form of the gauge vector propagator. It is easy to see that all one-loop corrections vanish with standard gauge averaging. In Figure 2 we display all non-vanishing self energy two loop quantum corrections of matter fields in this gauge. The blob in diagrams (c) and (d) represents the insertion of the full one loop correction to the gauge propagator. Any other potentially contributing diagram is zero either by D-algebra or by color symmetry.

[Figure 2: Two loop self-energy quantum corrections.]

Let us for example calculate with detail diagram $b$ of Figure 2. Taking into account the possibility of having $V-\bar{V}$, $V-V$ and $\bar{V}-\bar{V}$ internal lines and using color vertex factors as (B.7), it evaluates to

$$
\Pi_b = -\frac{1}{2} \left( N^2 - 1 \right) \left( \frac{4\pi}{k} \right)^2 \int d^4\theta d^4\theta' \int \frac{d^3p}{(2\pi)^3} \left( \mathcal{B}(\bar{A}(-p) \mathcal{B}^A(p)) \right) D_b(\theta, \theta')
$$

(3.6)
with
\[
D_b(\theta, \theta') = \int \frac{d^3k \, d^3l}{(2\pi)^3(2\pi)^3} \frac{\bar{D}^\alpha D_\alpha \delta^4(\theta, \theta') \bar{D}^2 D^2 \delta^4(\theta, \theta') \bar{D}^\beta D_\beta \delta^4(\theta, \theta')}{l^2 k^2 (k + l + p)^2},
\] (3.7)
the D-algebra factor of the supergraph. As we mentioned before, we perform all D-algebra manipulations in three dimensions and we calculate the final Feynman integral in d dimensions. After the usual integration by parts we obtain an ultraviolet divergent contribution
\[
\Pi_b = (N^2 - 1) \left( \frac{4\pi}{k} \right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p) B^A(p) \right) G_{(1,1)}^\epsilon G_{(1,1/2+\epsilon)}^\epsilon (p^2)^{-\epsilon}. \] (3.8)
To obtain the contribution a from Figure 2 we need vertex factors (B.5) and (B.6). We get the UV divergent contribution \( \Pi_a = 2\Pi_b \).

Using the corrected vector propagator we obtain for graph d of Figure 2 an UV/IR divergent tadpole
\[
\Pi_d = -3 \left( N^2 - 1 \right) \left( \frac{4\pi}{k} \right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p) B^A(p) \right) G_{(1,1)}^\epsilon \int \frac{d^4k}{(2\pi)^d} \frac{1}{(k^2)^{\frac{d}{2}+\epsilon}}. \] (3.9)
and for graph c an infrared divergent contribution
\[
\Pi_c = 3 \left( N^2 - 1 \right) \left( \frac{4\pi}{k} \right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p) B^A(p) \right) G_{(1,1)}^\epsilon \int \frac{d^4k}{(2\pi)^d} \frac{2p \cdot (p + k)}{(k^2)^{\frac{d}{2}+\epsilon}(k + p)^2}. \] (3.10)
Summing up a, b, c and d we obtain the cancelation of all UV divergent contributions and we are left with an IR divergent piece
\[
\Pi_{a+b+c+d} = 3 \left( N^2 - 1 \right) \left( \frac{4\pi}{k} \right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p) B^A(p) \right) G_d(p), \] (3.11)
with \( G_d(p) \) as given in the appendix. Finally, diagram e produces a finite correction
\[
\Pi_e = -2 \left( N^2 - 1 \right) \left( \frac{4\pi}{k} \right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p) B^A(p) \right) I_e \] (3.12)
where \( I_e \) is the factor obtained after closing the D-Algebra:
\[
I_e = \int \frac{d^3k \, d^3l}{(2\pi)^3(2\pi)^3} \frac{(k + p)^2(l + p)^2 - k^2l^2 + p^2(k + l + p)^2}{k^2(k + p)^2(k + l + p)^2 l^2(l + p)^2} = \frac{1}{64}. \] (3.13)
To conclude, the sum of all contributions gives a finite and an infrared divergent piece
\[
\Pi = \left( N^2 - 1 \right) \left( \frac{4\pi}{k} \right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p) B^A(p) \right) \left( 3G_d(p) - \frac{1}{32} \right). \] (3.14)
Working in the Landau gauge, we explicitly see that infrared infinities are only given by graphs c and d, which correspond to insertion of the 1-loop corrected vector propagator.
3.2.2 $\alpha$-gauge

We now take the more general case $\alpha \neq 0$. Once again there are no one-loop matter corrections. The list of two-loop self energy contributions gets larger. Apart from those already displayed in Figure 2, which are modified by the more general $\alpha$-dependent propagator, we have some additional contributions displayed in Figure 3. The new contributions produce additional UV, IR divergences and finite pieces. The sum of the original diagrams we had, with the modified $\alpha$-dependent propagator gives

$$\Pi^\alpha_{\alpha+...+e} = (N^2 - 1) \left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \mathcal{B}_A(-p) \mathcal{B}_A(p) \right) \times \left( -4\alpha \bar{\alpha} G^e_{(1,1)} G^e_{(1,1/2+\epsilon)} (p^2)^{-2\epsilon} + (3 + \alpha \bar{\alpha}) \mathcal{G}_d(p) - \frac{1}{32} (1 + \alpha \bar{\alpha}) \right).$$  \hspace{1cm} (3.15)

And the contributions from the additional diagrams of Figure 3

$$\Pi^\alpha_{f+g+h} = (N^2 - 1) \left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \mathcal{B}_A(-p) \mathcal{B}_A(p) \right) \times \left( 4\alpha \bar{\alpha} G^e_{(1,1)} G^e_{(1,1/2+\epsilon)} (p^2)^{-2\epsilon} + \frac{1}{32} \alpha \bar{\alpha} \right).$$ \hspace{1cm} (3.16)

By summing up all the contributions, we find as expected that all UV $\alpha$-dependent divergences cancel out. The finite piece we had already encountered in the Landau gauge is not modified, and the IR divergent piece gets shifted:

$$\Pi^\alpha = (N^2 - 1) \left(\frac{4\pi}{k}\right)^2 \int \frac{d^3p \, d^4\theta}{(2\pi)^3} \text{tr} \left( \mathcal{B}_A(-p) \mathcal{B}_A(p) \right) \left( (3 + \alpha \bar{\alpha}) \mathcal{G}_d(p) - \frac{1}{32} \right).$$ \hspace{1cm} (3.17)

From this we may conclude that if we only allow a hermitian gauge-fixed action, such that $\bar{\alpha}$ is literally the complex conjugate of $\alpha$, then it is not possible to choose a value of the gauge fixing parameter $\alpha$ such that the self-energy corrections are infra-red safe. This is a direct consequence of the fact that IR divergences are eventually produced only by corrected vector propagators. It’s also important to notice that the finite correction to the propagator turns out to be gauge independent, even if the propagator itself is not a physical quantity.
3.2.3 \( \eta \)-gauge

In the \( \eta \)-gauge the vector superfield propagator is written as:

\[
\langle V^c_{-\vec{p}}(p) V^a_{\vec{p}}(p) \rangle = \left( \frac{\bar{D}^\alpha D_\alpha}{p^2} + \frac{\eta^{\alpha}_p}{|p|} P_0 \right) \delta^4(\theta, \theta') \delta^b \delta^d.
\]

(3.18)

where \( \eta^{\alpha}_p \) is expanded as in (3.3). The first piece of the propagator gives rise to matter self energy diagrams starting from two loops with the same contributions as in the Landau gauge (see fig. 2) such that, for \( N \gg 1 \), it gives a finite and an infrared divergent piece of order \( \left( \frac{N}{k} \right)^2 \) given by

\[
\Pi^{\eta}_{a^+...+e} = \left( \frac{4\pi N}{k^2} \right)^2 \int \frac{d^3p \ d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p)B^A(p) \right) \left( 3G_d(p) - \frac{1}{32} \right).
\]

(3.19)

The second part of the propagator produces one-loop corrections to matter self energy such that, if the gauge parameter is of order \( \frac{N}{k} \), the contribution is of order \( \left( \frac{N}{k} \right)^2 \). In this case, two loop and higher corrections will contribute beyond \( \left( \frac{N}{k} \right)^2 \) so we do not consider them. The one-loop contributions are displayed in Figure 4.

**Figure 4:** One loop corrections in the \( \eta \)-gauge. The small black squares in the gauge vector propagators should be intended as the \( \eta \) dependent piece of the propagator.

After straightforward D-algebra, and with the choice \( \eta^{(k)}_p = -6\pi G^\epsilon_{(1,1)}(k^2)^{-\epsilon} \lambda + O(\lambda^3) \) we obtain

\[
\Pi^\eta_{1\text{-loop}} = - \left( \frac{4\pi N}{k^2} \right)^2 \int \frac{d^3p \ d^4\theta}{(2\pi)^3} \text{tr} \left( \bar{B}_A(-p)B^A(p) \right) 3G_d(p).
\]

(3.20)

From this we see that with the choice of \( \eta^{(p)}_p \) that produces the IR improved gauge propagator, we cancel the infrared divergent part obtaining only the universal finite piece already computed in the \( \alpha \)-gauge:

\[
\Pi^\eta = \Pi^\eta_{a^+...+e} + \Pi^\eta_{1\text{-loop}} = -\frac{1}{2} \pi^2 \frac{N^2}{k^2} \int d^3x \ d^4\theta \text{ tr} \left( \bar{B}_A B^A \right).
\]

(3.21)

We therefore conclude that the improved IR behaviour of the gauge propagator is sufficient to eliminate the presence of the unwanted divergences. In the next Section we further check this assertion computing at two-loop order the matter four-point Green’s function.
3.3 Superpotential vertex corrections

The set of all two-loop graphs which contribute to superpotential corrections to leading order in $N$ in the Landau gauge are depicted in Figure 5; any other potentially contributing 2-loop graph is zero due to color symmetry, supersymmetry or particular symmetries of the Feynman integrals involved. Notice that, since in the Landau gauge the one loop correction to the vertex is exactly zero, we can discard many diagrams at 2-loops that contain the 1-loop diagram as a subdiagram.

![Figure 5: All two loop quantum corrections of the superpotential.](image)

To simplify notation, whenever we put a $\mathcal{D}(\cdots)$ in front of the graph inside an equation, we mean the scalar graph with all the momenta in the numerator generated after closing the D-algebra (we put on equal foot the $V$ and $\bar{V}$ lines and only in color/flavor vertex factors will we consider the sign difference between their propagators and couplings to matter). Else, in the absence of $\mathcal{D}$ in front of the graph, we just mean the corresponding scalar Feynman integral.

To leading order in $N$, all the two loop contributions produce a term proportional to the classical superpotential (no double traces are generated) given by

$$
\Gamma_i[A,B] = \left(\frac{4\pi N}{k}\right)^2 C_i \int d^2\theta \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_4}{(2\pi)^3} (2\pi)^3 \delta(p_1 + p_2 + p_3 + p_4) \frac{2\pi i}{k} \epsilon_{AC} \epsilon^{BD} \text{tr}(B^A(p_1)A_D(p_2)B^C(p_3)A_B(p_4)) \mathcal{D}_i(p_1,\cdots,p_4) \quad i = a,\cdots,f, \quad (3.22)
$$

where $C_i$ is the vertex factor of graph $i$, $\mathcal{D}_i(p_1,\cdots,p_4)$ is the Feynman integral which results after performing the D-algebra so as to eliminate all the $d^4\theta$ integrals except for the last one which is used to transform the D-operators applied on the fields into external momenta by using that $\int d^4\theta = \int d^4\theta \bar{D}^2(\cdots)$. The vertex factors for all graphs are $C_a = \frac{1}{2}, C_b = \frac{1}{4}, C_c = -3, C_d = 1, C_e = -1, C_f = 2$. We will always consider
Let us start the computation of graph $c$ of Figure 5 which is the one we expect to give an IR divergence. A not so straightforward calculation of the $D$-algebra gives

$$D\left(\begin{array}{c}\end{array}\right) = \frac{G_{(1,1)}^e}{2} \int \frac{d^dk}{(2\pi)^d} \frac{k^2 (p_3 + p_4)^2 - p_3^2 (k + p_4)^2 - p_4^2 (k - p_3)^2}{(k^2)^{3/2+\epsilon}(k + p_4)^2(k - p_3)^2}$$

$$= \frac{1}{2}(p_3 + p_4)^2 \bigg[ - \frac{1}{2}G_d(p_3) - \frac{1}{2}G_d(p_4) \bigg],$$

where we have written the Feynman integral in terms of a finite scalar integral and infrared divergent contributions. Once again we obtain infrared divergences when we attach a one-loop corrected gauge vector inside a loop. We expect that IR divergences in the superpotential are canceled by the exact same choice for $\eta$ we found to improve the gauge vector propagator infrared behaviour. This is in fact true: the one-loop graph with the $\eta$-dependent part of the gauge vector propagator gives

$$D\left(\begin{array}{c}\end{array}\right) = -6\pi \lambda (G_d(p_3) + G_d(p_4)).$$

(3.24)

With the value of the gauge parameter we made before $\eta = -6\pi G_{(1,1)}^e (p^2)^{-\epsilon} \lambda + O[\lambda^3]$, this insertion produces a superpotential correction with the same structure as in (3.22). In this way, the sum of this graph with graph $d$ which was also IR divergent gives

$$-3(4\pi \lambda)^2 D\left(\begin{array}{c}\end{array}\right) + 4\pi \lambda D\left(\begin{array}{c}\end{array}\right) = -\frac{3}{2}(4\pi \lambda)^2 (p_3 + p_4)^2.$$

(3.25)

which is finite. These are the only dangerous IR graphs contributing to the superpotential; the graphs which remain to be analyzed are all finite. To show this, we list the integrals resulting from D-algebra computations.

The simplest graph is $g$: it has three possible channels of which two contribute to leading order in $N$. This factor of 2 is already taken into account in the vertex factor $C_g$. The D-algebra of this graph is simply

$$D\left(\begin{array}{c}\end{array}\right) = \int \frac{d^3k}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} \frac{-(p_1 + p_2)^2}{k^2(k + p_1 + p_2)^2l^2(l - p_3 - p_4)^2} = \frac{1}{64}.$$

(3.26)

In all other diagrams, a sum over different distributions of internal lines has to be taken into account such that diagrams $b$, $c$ and $e$ appear four times with different momentum distribution, while diagrams $a$, $d$ appear eight times.
A straightforward calculation shows that

\[ \mathcal{D} \left( \begin{array}{c}
\end{array} \right) = 2(p_3 + p_4)^2 \left( \begin{array}{c}
\end{array} \right) . \]  

(3.27)

As mentioned before, this scalar integral is finite in three dimensions. For graph \( a \) we obtain the finite result

\[ \mathcal{D} \left( \begin{array}{c}
\end{array} \right) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) p_4^\mu (p_3 + p_4)^\nu (k + p_4)^\rho (l - p_4)^\sigma}{k^2 (k + p_4)^2 (k - p_3)^2 (k + l)^2 (l - p_4)^2 l^2} . \]  

(3.28)

Notice that the presence of a three lined vertex is potentially dangerous, but the momenta in the numerator of the Feynman integral that we obtain through D-algebra guarantees finiteness. The same is true for graph \( d \)

\[ \mathcal{D} \left( \begin{array}{c}
\end{array} \right) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{-Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) p_4^\mu (k + l)^\rho (k - p_3)^\alpha p_3^\beta p_4^\gamma}{k^2 (k + p_4)^2 (k - p_3)^2 (k + l)^2 (l + p_3)^2 l^2} , \]  

(3.29)

and also for graph \( e \)

\[ \mathcal{D} \left( \begin{array}{c}
\end{array} \right) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{-Tr(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) p_4^\mu p_2^\rho k^\sigma}{k^2 (k - p_2)^2 (k + l + p_3)^2 (l - p_4)^2 l^2} , \]  

(3.30)

which are once again finite in three dimensions.

3.3.1 A particular exceptional momenta configuration

By using the \( \eta \) gauge fixing, we showed in the last Section that it was possible to obtain an infrared safe function of the external momenta for the superpotential corrections. Moreover, it is clear that the sum of 1PI graphs plus four-legged graphs with corrected legs (using self-energy corrections we derived before), is a physical gauge invariant quantity. Having found an universal finite value for the matter propagator correction we conclude that also the correction to the superpotential is (at least at two loops) gauge independent. We would like now to compute it for a special external momenta configuration \(^3\).

The calculation we are going to present here should be interpreted along the lines of [19]. In these papers, by means of direct computation of specific diagrams in four-dimensional supersymmetric models, it was shown that finite contributions may survive the limit of vanishing external momenta for the 1PI vertex function as soon as massless

\(^3\)See [17] for the calculation of the effective action on a vector superfield background.

- 14 -
particles were present. Moreover, these contributions could break holomorphy in the coupling constants or supersymmetries of the action if they were to be interpreted as a "finite renormalization" of the superpotential. It then became clear (see [20] for a review and references therein) that the correct interpretation of these contributions was to consider them as IR singular D-terms in superspace, which are absent for instance in the more suitable Wilsonian definition of the effective superpotential. In what follows we would like to show that also in the case of ABJM theories does exist a special limit of vanishing external momenta for the vertex function which gives rise to a finite result.

The vertex function, with the IR safe gauge choice, is guaranteed to be finite as long as the momenta are non-exceptional. By exceptional we mean when there exists at least one equation of the form \( \sum_i \rho_i p_i = 0 \) with \( \rho_i \) either 0 or 1 and not all 0 nor all 1. In our case, it is easy to see that many exceptional configurations produce spurious IR divergences, for example if we choose any of the four momenta, say \( p_1 \) to be zero.

If we were interested in finding an exceptional configuration which is IR safe and which leads to a constant, we would need at least two supplementary "exceptional" equations. In fact, we found that modulo equivalent choices, there is only one such choice of exceptional momenta which is IR finite. This is given by choosing \( p_1 + p_2 = 0 \) and \( p_1 + p_4 = 0 \). We proceed to evaluate the graphs for this choice.

For graph \( b \) we obtain

\[
D \left( \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \right) = 2(p_3 + p_4)^2 \rightarrow 0. \tag{3.31}
\]

The reader might be worried that we put this graph to zero in the exceptional configuration because of the \( (p_3 + p_4)^2 = (p_1 + p_2)^2 \) numerator without taking into account that the integral multiplying it is infrared divergent when \( p_3 + p_4 = 0 \). A careful power expansion in \( |p_3 + p_4| \) gives

\[
2(p_3 + p_4)^2 \rightarrow \left( \frac{1}{16\pi} \frac{K \left[ \sqrt{1 - \frac{p_3^2}{p_4^2}} \right]}{|p_4|} \right) |p_3 + p_4| + \left( \frac{1}{16\pi^2} \log \left( \frac{p_3^2}{p_4^2} \right) \right) (p_3 + p_4)^2 + \cdots, \tag{3.32}
\]

where \( K(z) \) is the complete elliptic integral of the first kind\(^4\) and the ellipsis are for higher orders in \( |p_3 + p_4| \). From this equation we see that \( p_3 + p_4 \rightarrow 0 \) is well defined and zero since all the coefficients in the expansion are finite in this limit.

\(^4\)Notice that the coefficient in front of \( |p_3 + p_4| \) is implicitly symmetric under \( p_3 \leftrightarrow p_4 \) due to the property \( K \left[ \sqrt{1 - \frac{p_3^2}{p_4^2}} \right] = \frac{p_4}{|p_4|} K \left[ \sqrt{1 - \frac{p_4^2}{p_3^2}} \right] \). In fact, all the coefficients of the expansion have this symmetry.
Graphs $a$ may be represented in terms of elementary and Mellin-Barnes integral functions of the Lorentz invariants $x = \frac{p_3^2}{(p_3+p_4)^2}$ and $y = \frac{p_4^2}{(p_3+p_4)^2}$ given by

\[
D\left(\begin{array}{c}
\includegraphics{graph1}
\end{array}\right) + D\left(\begin{array}{c}
\includegraphics{graph2}
\end{array}\right) = -(p_3 + p_4)^2 \left(\begin{array}{c}
\includegraphics{graph3}
\end{array}\right)
\]
\[
+ \frac{1}{64\pi^2} \left(\frac{2\pi^2}{3} - Li_2(1-x) - Li_2(1-y) - \log(x) \log(y)\right)
\]
\[
+ \frac{\sqrt{\pi}}{64\pi^3} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds dt \Gamma^*(s) \Gamma\left(\frac{1}{2} - s\right) \Gamma(-t) \Gamma(t + s) \Gamma(1 + t + s)(x^s y^t + x^t y^s).
\]

(3.33)

This expression admits a well defined limit for $(p_3 + p_4)^2 \to 0$ given by

\[
\to \frac{1}{32\pi^2} \left[ \arccos^2\left(\frac{|p_3|}{|p_4|}\right) + \arccos^2\left(\frac{|p_4|}{|p_3|}\right) + \frac{1}{4} \log^2\left(\frac{p_3^2}{p_4^2}\right) \right].
\]

(3.34)

If we consider more in particular that $p_3 + p_4 = 0$, then not only $(p_3 + p_4)^2 = 0$ but also $p_3^2 = p_4^2$, we find

\[
D\left(\begin{array}{c}
\includegraphics{graph1}
\end{array}\right) + D\left(\begin{array}{c}
\includegraphics{graph2}
\end{array}\right) \to 0.
\]

(3.35)

Now we move on to graphs $d$. By expanding the products of momenta in the numerator of the integrals and properly completing squares (see trace properties of $\gamma^\mu$ matrices in the appendix), one can compare the resulting expression with the squared-completed expression of graphs $a$ to conclude that

\[
D\left(\begin{array}{c}
\includegraphics{graph1}
\end{array}\right) + D\left(\begin{array}{c}
\includegraphics{graph2}
\end{array}\right) = D\left(\begin{array}{c}
\includegraphics{graph3}
\end{array}\right) + D\left(\begin{array}{c}
\includegraphics{graph4}
\end{array}\right)
\]
\[
+ 2(p_3 + p_4)^2 \left(\begin{array}{c}
\includegraphics{graph5}
\end{array}\right) - \frac{1}{32}.
\]

(3.36)

Thus, according to the analysis we made before, in the limit $p_1 + p_2 = -p_3 - p_4 \to 0$ we obtain

\[
D\left(\begin{array}{c}
\includegraphics{graph1}
\end{array}\right) + D\left(\begin{array}{c}
\includegraphics{graph2}
\end{array}\right) \to -\frac{1}{32}.
\]

(3.37)

---

Definitions, properties and relevant references of Mellin-Barnes representation are given in the Appendix C.
Finally, it is possible to calculate the Feynman integral of graph $e$ when $p_1 + p_2 = 0$ and $p_1 + p_4 = 0$ by substituting $p_1 = p, p_2 = -p, p_4 = -p$ and from momentum conservation $p_3 = p$ to obtain

$$D \left( \begin{array}{c} \downarrow \\ \downarrow \end{array} \right) \rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{-2p^2 k.l}{k^2 (k + p)^2 (k + l + p)^2 (l + p)^2 l^2} = \frac{1}{8\pi^2} - \frac{1}{64}. \quad (3.38)$$

With all these elements we may make the sum to find the finite two-loop contribution

$$\Gamma^{(2)}[A, B] = \lambda^2 (-8 - \frac{3}{2} \pi^2) \int d^2\theta d^3x \frac{2\pi i}{k} \epsilon_{AC} \epsilon^{BD} \text{tr} (B^A A_D B^C A_B). \quad (3.39)$$

As mentioned before we expect this to be a well defined and gauge invariant result. Nevertheless, it’s easy to show that if such contribution had to be interpreted as a “finite renormalization” of the superpotential it would inevitably break extended supersymmetry.

4. General analysis

We would like now to make some comments on the generality of our results. In a three dimensional theory there are two sources of infrared divergences in Feynman integrals. On the one hand we have the insertion of self energy corrected lines which may produce high powers of the propagators $\frac{1}{(k^2)^a}$ with $a \geq \frac{3}{2}$. On the other hand the presence of a three-lined vertex interaction with no external legs is potentially dangerous since, if there are only scalar propagators attached to it (no momenta in the numerator), an IR divergence is produced after loop integration.

In general $\mathcal{N} = 2$ Chern-Simons-Matter theories there are three-lined vertexes that couple chiral fields with the gauge vector and there is also the three gluon vertex. Consider the matter-vector coupling as shown in figure (6).

![Figure 6: Matter-Gluon coupling](image_url)
If we integrate by parts on vertex 4 at least one of the $D$-operators of the gluon propagator, we get
\[
\sim \bar{D}^\alpha D_\alpha \delta_{(1,4)} \frac{D^2 \bar{D}^2}{k^2 \delta_{(4,2)}} \bar{D}^2 D^2 \delta_{(4,3)} = \frac{k^{\alpha \beta}}{k^2 l^2 (k - l)^2} D_\alpha \delta_{(1,4)} D_\beta \bar{D}^2 \delta_{(4,2)} \bar{D}^2 D^2 \delta_{(4,3)}.
\]
(4.1)

The appearance in the numerator of one of the momenta carried by the lines eliminates the IR threat as long as no self energy corrections are involved in the full graph (we deal with them in what follows). A similar analysis can be done for the three gluon vertex.

It is quite obvious that the insertion of self-energy matter corrected lines inside any given graph, does not lower the scaling of the propagator thus not leading to IR issues. Then we conclude that IR problems are only generated by the insertion of self-energy corrected gluon lines: with the aid of the modified propagator we proposed in the introduction, it seems plausible that IR divergences can in principle be cured to all loop orders.

To leading order we have shown that the key in the elimination of IR divergences was the completion of the 1-loop corrected gauge vector by adding the longitudinal part with the $\eta$ piece of the propagator. Having understood this mechanism that improves the IR behaviour of the gauge propagator correction at 1-loop, we may generalize this notion to all orders in $\lambda$. Due to gauge invariance and parity we know [12] that the all order 1PI vector self-energy calculated with the ordinary piece of the propagator is given by
\[
\Delta V = \frac{1}{2} \int \frac{d^3k d^4\theta}{(2\pi)^3} \text{Tr} \left( V(-k) \left( \sum_{l=1} \lambda^{2l+1} \bar{D}^\alpha D_\alpha + \sum_{l=0} \lambda^{2l+1} |k| P_{1/2} \right) V(k) \right),
\]
(4.2)

where the coefficients $A^\epsilon_i(k)$ and $B^\epsilon_i(k)$ are functions that contain $\epsilon$-powers of the momentum. That is, odd loop corrections contain the superspin 1/2 projector, and even loop corrections reproduce the original structure of the action. Any odd-loop correction from (4.2), when attached inside a graph produces a propagator given by
\[
- \sum_{l=0} B_{1/2}^\epsilon(k) \lambda^{2l+1} \frac{P_{1/2}}{|k|} \delta_{(\theta,\theta')},
\]
(4.3)

which, as noted before, will produce an IR divergence. On the other hand even-loop corrections, when attached inside a graph, produce a term $\sim D^\alpha D_\alpha/k^2$ which behaves in the same way as the basic propagator, thus not leading to IR issues. These formulas can be readily derived using (A.8).
Having understood the effect of corrected vector propagator insertions in graphs, we can now proceed to fix the \( \eta \) parameter perturbatively as an odd power series in \( \lambda \). After fixing it to order one, \( \eta^{(p)}_\lambda = -6\pi G_{1(1)}^\epsilon (p^2)^{-\epsilon} \lambda + O(\lambda^3) \), one calculates every connected (not only 1PI) self energy vector correction at order \( \lambda^3 \), including lower loop \( O(\lambda^3) \) corrections with the \( \eta^{(p)}_\lambda \) piece of the propagator. With this result we fix the next coefficient \( 3 \eta^{(p)}_\lambda \) such that we complete the transverse projector with the longitudinal one effectively removing the source of infrared divergence at order \( \lambda^3 \). This process may be continued recursively thus improving the IR behavior of the propagator to all loops.

In this way, if one considers a given graph which contains an L-loop-dressed gauge vector, then if \( L \) is odd there will always be a complementary graph in which we substitute that dressed line with the \( \eta \) piece of the propagator at the corresponding order in \( \lambda \), such that the whole line will behave as \( \sim \frac{\delta(\theta,\theta')}{|k|} \); instead, when \( L \) is even, the line behaves as the ordinary propagator \( \sim \frac{D^\alpha D_\alpha \delta(\theta,\theta')}{k^2} \) and needs no modifications. In both cases the graph will be IR safe.

Some comments are in order about the ultraviolet behaviour of the non-locally gauged fixed theory. When we studied the effect of the \( \eta \) insertion in the case of matter self-energy graphs and superpotential corrections, we saw that the renormalization properties were not modified up to two loops. This makes this alternative gauge fixing procedure consistent, since we expect any gauge independent quantity of the theory to be independent from the procedure. As a further non trivial check, we have also verified that the renormalization properties of the theory in the gauge vector sector are also not modified up to two-loop order.

5. Conclusions

We studied the infrared behaviour of the off-shell amplitudes in three-dimensional Chern-Simons-matter theories with specific attention to the ABJM model. In \( \mathcal{N} = 2 \) superspace IR divergences show up in a very similar way as in four-dimensional Super-Yang-Mills theory, being related to the corrected vector superfield propagator insertions. At first, we showed that if the theory is gauge fixed in a standard fashion there is no way to get rid of the divergent integrals without losing the hermiticity of the action. Then we introduced a non-local gauge fixing procedure which leads to divergences cancelation without spoiling the renormalizability of the theory. In order to do so, the gauge-fixing parameter had to be perturbatively fine tuned. Moreover, we found in our computations that infrared infinities seem to be always associated to gauge dependent parts in the amplitudes, thus not affecting the physical quantities of the theory. As a
non-trivial output of our calculations we provided the two-loop finite correction to the 1PI vertex function for ABJM theory in equation (3.39).

It would be interesting to address the same problems in the case of Chern-Simons-matter theories described in $\mathcal{N} = 1$ superspace, starting for instance from the formulation of BLG theory [13] given in [14]. In this case the analogy with the four-dimensional case is lost and one might expect a different infrared behaviour of superspace propagators. It would be also interesting to perform a similar analysis in the $\mathcal{N} = 3$ harmonic superspace formulation of [18].

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A. Superspace Notations

We use three-dimensional superspace notations adapted from [7]. We work in Euclidean space with a trivial metric $\eta_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, 1, 1)$ and with Dirac Matrices $(\gamma^\mu)_{\alpha}^{\beta} = i(\sigma_1, \sigma_2, \sigma_3)$. We raise and lower spinor indexes through $\psi^\alpha = C^\alpha\beta \psi_\beta$ and $\psi_\alpha = \bar{\psi}\beta C^\beta\alpha$, where the antisymmetric symbols $C^\alpha\beta$ and $C_\alpha\beta$ are defined by $C_{12} = -C_{12} = i$. Notice that with this convention contractions are always made going from the upper left corner to the lower right corner such that Grassmannian bilinears do not pick a sign after hermitian conjugation: $(\psi^2)\dagger = (\psi^\alpha\psi_\alpha)^\dagger = \bar{\psi}\alpha \psi_\alpha = \bar{\psi}\alpha$. Gamma matrices satisfy

$$\gamma^\mu \gamma^\nu = -\eta^{\mu\nu} - \varepsilon^{\mu\nu\rho\sigma} \gamma^\rho \gamma^\sigma, \quad \varepsilon^{123} = 1,$$

from which one can derive many useful trace properties such as

$$\text{Tr}(\gamma^\mu \gamma^\nu) = -2\eta^{\mu\nu}, \quad \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 2\varepsilon^{\mu\nu\rho\sigma},$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 2\eta^{\mu\nu} \eta^{\rho\sigma} - 2\eta^{\mu\rho} \eta^{\nu\sigma} + 2\eta^{\mu\sigma} \eta^{\nu\rho}.$$  \hfill (A.2)

The similarity between $N=1 d=4$ superspace and $N=2 d=3$ superspace is based on the isometry between both isospin groups: $U(1)$ for $d=4$ and $SO(2)$ for $d=3$. $N=2$ superspace has two 2-component anticommuting coordinates $\theta^\alpha$ such that if one defines complex coordinates $\theta^\alpha = \theta^\alpha_1 - i\theta^\alpha_2$, $\bar{\theta}^\alpha = \theta^\alpha_1 + i\theta^\alpha_2$ and complex spinor derivatives

$$\partial_\alpha = \frac{1}{2}(\partial^{(1)}_\alpha + i\partial^{(2)}_\alpha), \quad \bar{\partial}_\alpha = \frac{1}{2}(\partial^{(1)}_\alpha - i\partial^{(2)}_\alpha),$$ \hfill (A.3)

they satisfy $\partial_\alpha \theta^\beta = \delta^\beta_\alpha$, $\bar{\partial}_\alpha \bar{\theta}^\beta = \delta^\beta_\alpha$, $\bar{\partial}_\alpha \theta^\beta = 0$, $\partial_\alpha \bar{\theta}^\beta = 0$. Covariant derivatives are

$$D_\alpha = \frac{1}{2}(D^{(1)}_\alpha + iD^{(2)}_\alpha) = \partial_\alpha + \frac{1}{2}\bar{\theta}^\beta i\partial_\beta, \quad \bar{D}_\alpha = \frac{1}{2}(D^{(1)}_\alpha - iD^{(2)}_\alpha) = \bar{\partial}_\alpha + \frac{1}{2}\theta^\beta i\partial_\beta, \hfill (A.4)$$

such that they carry a representation of the super-algebra

$$\{D_\alpha, D_\beta\} = i(\gamma^\mu)_{\alpha\beta} \partial_\mu \equiv i\partial_{\alpha\beta}, \quad \{D_\alpha, D_\beta\} = 0, \quad \{D_\alpha, \bar{D}_\beta\} = 0.$$ \hfill (A.5)

Apart from the fact that one does not make distinctions between dotted and un-dotted spinor indexes, this is the same algebra of covariant derivatives of $N=1 d=4$ superspace thus making Feynman supergraph rules very similar to the known rules. On the other hand, one may construct contractions that were not allowed in four dimensions such as $D^\alpha D_\alpha$ or $\bar{D}^\alpha \theta_\alpha$. An important property of the vector representation is that it is symmetric: $C^\alpha\beta p_\alpha\beta = 0$ and $C^\alpha\beta \partial_\alpha\beta = 0$, which is evident after one realizes that $\gamma^\mu$ matrices with both spinor indexes up or down are symmetric with respect to those
Finally, we define the indexes. Defining \( \Box = \partial^\mu \partial_\mu = \frac{1}{2} \partial^{\alpha \beta} \partial_{\alpha \beta} \), \( D^2 = \frac{1}{2} D^\alpha D_\alpha \) and \( \bar{D}^2 = \frac{1}{2} \bar{D}^\alpha \bar{D}_\alpha \) the following properties hold

\[
D_\alpha D^2 = 0, \quad D^\alpha D^2 = 0, \quad [D^\alpha, D^2] = i \partial^{\alpha \beta} \partial_\beta, \quad [D^\beta, D^2] = i \partial^{\alpha \beta} D_\alpha
\]

\[
\tag{A.6}
D^2 \bar{D}^2 D^2 = \Box D^2, \quad D^\alpha D_\beta = \delta^\alpha_\beta D^2, \quad \bar{D}^\alpha \bar{D}_\beta = \delta^\alpha_\beta \bar{D}^2.
\]

Superspin projectors are defined as

\[
\mathcal{P}_0 = \frac{1}{\Box} (D^2 \bar{D}^2 + \bar{D}^2 D^2), \quad \mathcal{P}_{1/2} = -\frac{1}{\Box} D^\alpha \bar{D}^2 D_\alpha,
\]

\[
\tag{A.7}
\text{and together with } \bar{D}^\alpha D_\alpha \text{ operator, they satisfy the useful properties}
\]

\[
\mathcal{P}_0^2 = \mathcal{P}_0, \quad \mathcal{P}_{1/2}^2 = \mathcal{P}_{1/2}, \quad \mathcal{P}_0 + \mathcal{P}_{1/2} = 1, \quad \mathcal{P}_0 \mathcal{P}_{1/2} = 0,
\]

\[
\tag{A.8}
(\bar{D}^\alpha D_\alpha)^2 = \Box \mathcal{P}_{1/2}, \quad \mathcal{P}_{1/2} \bar{D}^\alpha D_\alpha = \bar{D}^\alpha D_\alpha, \quad \mathcal{P}_0 \bar{D}^\alpha D_\alpha = 0.
\]

Our conventions for integration are \( \int d^2 \theta = \frac{1}{2} \int d\theta^a d\bar{\theta}_a, \int d^2 \bar{\theta} = \frac{1}{2} \int d\bar{\theta}^a d\theta_a \) and \( \int d^4 \theta = \int d^2 \theta d^2 \bar{\theta} \), such that up to a total space-time derivative

\[
\int d^2 \theta \ldots = D^2 \ldots |_{\theta = \bar{\theta} = 0} \quad \text{and} \quad \int d^2 \bar{\theta} \ldots = \bar{D}^2 \ldots |_{\theta = \bar{\theta} = 0}.
\]

Finally, we define the \( \theta \)-space \( \delta \)-function as \( \delta^4(\theta - \theta') = (\theta - \theta')^2 (\bar{\theta} - \bar{\theta}')^2 \).

### B. Feynman Rules

We list some of the Feynman rules for ABJM theory. The vector superfield propagators are given in the \( \alpha \)-gauge by:

\[
V^a_b V^c_d = \frac{1}{p^2} \left( \bar{D}^\alpha D_\alpha + \alpha D^2 + \bar{\alpha} \bar{D}^2 \right) \delta^4_{(\theta, \theta')} \delta^b_c \delta^d_a,
\]

\[
\tag{B.1}
\hat{V}^a_b \hat{V}^c_d = -\frac{1}{p^2} \left( \bar{D}^\alpha D_\alpha + \alpha D^2 + \bar{\alpha} \bar{D}^2 \right) \delta^4_{(\theta, \theta')} \delta^b_c \delta^d_a,
\]

while using the \( \eta \)-gauge we obtain:

\[
V^a_b V^c_d = \left( \frac{\bar{D}^\alpha D_\alpha}{p^2} + \frac{\eta(p)}{|p|} \mathcal{P}_0 \right) \delta^4_{(\theta, \theta')} \delta^b_c \delta^d_a,
\]

\[
\hat{V}^a_b \hat{V}^c_d = \left( -\frac{\bar{D}^\alpha D_\alpha}{p^2} + \frac{\eta(p)}{|p|} \mathcal{P}_0 \right) \delta^4_{(\theta, \theta')} \delta^b_c \delta^d_a.
\]

\[
\tag{B.2}
\]
From the ghost action in (2.11) we find the ghost propagators:

\[ b^a_b \cdots c^c_d = -\frac{1}{p^2}\delta^4(\theta - \theta')\delta^b_d\delta^c_a, \quad \bar{b}^a_b \cdots \bar{c}^c_d = \frac{1}{p^2}\delta^4(\theta - \theta')\delta^\bar{b}_\bar{d}\delta^\bar{c}_\bar{a}, \]  

(B.3)

and from \( S_{mat} \) we obtain the matter field propagators

\[ \bar{B}^A_A \rightarrow B^B = \frac{1}{p^2}\delta^4(\theta - \theta')\delta^A_B\delta^a_d\delta^b_c, \quad \bar{A}^B_A \rightarrow A^A = \frac{1}{p^2}\delta^4(\theta - \theta')\delta^A_B\delta^a_d\delta^b_c, \]  

(B.4)

where lowercase gauge indexes are omitted if they correspond to the same uppercase flavor index (e.g. \( \bar{B}^A_A \equiv (\bar{B}^A_A)^{\bar{a}}_a \), \( \bar{A}^B_B \equiv (\bar{A}^B_B)^{\bar{b}}_b \)).

Some of the non trivial color structures of the vertexes we will need are

\[ B^A_A A^D = i\frac{4\pi}{k}\epsilon_{ACD} \left( \delta^\bar{b}_\bar{b}\delta^\bar{d}_\bar{d}\delta^a_a - \delta^\bar{d}_\bar{d}\delta^a_a \delta^\bar{b}_\bar{b} \right), \]  

(B.5)

\[ \bar{B}^A_A \bar{A}^D = i\frac{4\pi}{k}\epsilon_{ACD} \left( \delta^\bar{b}_\bar{b}\delta^\bar{d}_\bar{d}\delta^a_a - \delta^\bar{d}_\bar{d}\delta^a_a \delta^\bar{b}_\bar{b} \right), \]  

(B.6)

\[ V^b_c V^d_f = \frac{1}{2}\delta^a_d\delta^a_c \left[ D^a(q)\bar{D}_a(p) - \bar{D}^a(q)D_a(p) \right], \]  

(B.7)

and similarly for other vertexes involving \( A \) and \( \hat{V} \) fields. We apply the usual D-algebra rules and regularize integrals when needed using dimensional reduction prescriptions.

C. Relevant Integrals

In the computation of two-point functions we introduced the function \( G_{(a,b)}^{\alpha} \) defined as

\[ \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2a} \frac{1}{(k + p)^2b} = G_{(a,b)}^{\alpha} = \frac{\Gamma(a + b - d/2)\Gamma(d/2 - a)\Gamma(d/2 - b)}{(4\pi)^d/2\Gamma(a)\Gamma(b)\Gamma(d - a - b)(p^2)^{a+b-d/2}}, \]  

(C.1)
and a particular two loop infrared divergent integral

\[ G_d(p) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{p^2}{(l + k)^2 (k + p)^2 k^2} = G^e_{(1, 1)} G^e_{(1, 3/2 + \epsilon)} (p^2)^{-2\epsilon} \]  \hspace{1cm} (C.2)

= (p^2)^{-2\epsilon} \left( -\frac{1}{64\pi^2} \epsilon + \frac{1 + \gamma - \log(4\pi)}{32\pi^2} + O(\epsilon) \right). \hspace{1cm} (C.3)

All the relevant integrals in the calculation of the propagators can be reduced to the \( G \)-function form.

In the computation of four-point integrals in the exceptional configuration, we found it necessary to expand Feynman integrals in powers of the kinematic invariants in order to carefully take the appropriate limits. In the end, the correct limit through this analysis became a confirmation of the “naive” result since it coincides with the limit taken directly on the integrand. To deal with the expansions we used multiple Mellin-Barnes contour representation of vertex integrals [15, 16]. These representations are based on the identity

\[ \frac{1}{(k^2 + M^2)^a} = \frac{1}{(M^2)^a} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \Gamma(-s) \Gamma(s + a) \left( \frac{k^2}{M^2} \right)^s, \]  \hspace{1cm} (C.4)

where the contour is given by a straight line along the imaginary axis such that indentations are used if necessary in order to leave the series of poles \( s = 0, 1, \cdots, n \) to the right of the contour and the series \( s = -a, -a - 1, \cdots, -a - n \) to the left of the contour. After Feynman-parametrizing a triangle integral and using (C.4), the following formula holds

\[ \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^{2\mu_1} (k - p)^{2\mu_2} (k + q)^{2\mu_3}} = \frac{(4\pi)^{-d/2} (p + q)^{2(d/2 - \sum \mu_i)}}{\prod \Gamma(\mu_i) \Gamma(d - \sum \mu_i)} \]

\[ \times \left( \frac{1}{2\pi i} \right)^2 \int_{-i\infty}^{i\infty} ds dt \Gamma(-s) \Gamma(-t) \Gamma(\frac{d}{2} - \mu_1 - \mu_2 - s) \Gamma(\frac{d}{2} - \mu_1 - \mu_3 - t) \times \]

\[ \Gamma(\mu_1 + s + t) \Gamma(\sum \mu_i - \frac{d}{2} + s + t) \left( \frac{p^2}{(p + q)^2} \right)^s \left( \frac{q^2}{(p + q)^2} \right)^t; \]  \hspace{1cm} (C.5)

and for a vector-like triangle we have

\[ \int \frac{d^d k}{(2\pi)^d} \frac{k^{\nu}}{k^{2\mu_1} (k - p)^{2\mu_2} (k + q)^{2\mu_3}} = \frac{(4\pi)^{-d/2} (p + q)^{2(d/2 - \sum \mu_i)}}{\prod \Gamma(\mu_i) \Gamma(d - \sum \mu_i + 1) \left( \frac{1}{2\pi i} \right)^2 \times}

\times \int_{-i\infty}^{i\infty} ds dt \Gamma(-s) \Gamma(-t) \Gamma(\mu_1 + s + t) \Gamma(\sum \mu_i - \frac{d}{2} + s + t) \left( \frac{p^2}{(p + q)^2} \right)^s \left( \frac{q^2}{(p + q)^2} \right)^t \times \]
\[
\left[ \Gamma\left( \frac{d}{2} - \mu_1 - \mu_2 - s \right) \Gamma\left( \frac{d}{2} - \mu_1 - \mu_3 - t + 1 \right) p^n - \Gamma\left( \frac{d}{2} - \mu_1 - \mu_2 - s + 1 \right) \Gamma\left( \frac{d}{2} - \mu_1 - \mu_3 - t \right) q^n \right],
\]

where the multiple contours are taken using the same convention as the first definition unless otherwise indicated. It is customary to indicate with a * over the \( \Gamma(z) \) function the case where one leaves a pole to the other of the conventional side of the contour. With these representations, among with Barnes 1\textsuperscript{st} and 2\textsuperscript{nd} lemmas

\[
\frac{1}{2\pi i} \int ds \ 2 \Gamma(a + s) \Gamma(b + s) \Gamma(c - s) \Gamma(d - s) = \frac{\Gamma(a + c) \Gamma(a + d) \Gamma(b + c) \Gamma(b + d)}{\Gamma(a + b + c + d)} ,
\]

\[
= \frac{\Gamma(a) \Gamma(b) \Gamma(c) \Gamma(a + d) \Gamma(b + d) \Gamma(c + d)}{\Gamma(e - a) \Gamma(e - b) \Gamma(e - c)} , \quad \text{with} \quad e = a + b + c + d ,
\]

and their multiple corollaries, we were able to carefully expand the 2-loop four-point integrals in the relevant kinematic invariants in order to take the limit of exceptional momenta.

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