On the linear instability of the Ellis-Bronnikov-Morris-Thorne wormhole

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Abstract

We consider the wormhole of Ellis, Bronnikov, Morris and Thorne (EBMT), arising from Einstein’s equations in presence of a phantom scalar field. In this paper we propose a simple derivation of the linear instability of this system, making comparisons with previous works on this subject (and generalizations) by González, Guzmán, Sarbach, Bronnikov, Fabris and Zhidenko.

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1 Introduction

Throughout this paper, indicating with $c, h, G$ the speed of light, the reduced Planck constant and the gravitational constant, we stipulate

$$c = 1, \quad h = 1, \quad \kappa := 8\pi G. \quad (1.1)$$

We are interested in perturbations of a well known wormhole; this is described by the static spacetime metric

$$ds^2 = -dt^2 + d\ell^2 + (a^2 + \ell^2) d\Omega^2 \quad (-\infty < t, \ell < +\infty) \quad (1.2)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the line element of the unit spherical surface $S^2$ and $a$ is a positive constant, with the dimension of a length. For $\ell \to \pm\infty$, $ds^2$ approaches the flat Minkowski metric $-dt^2 + d\ell^2 + \ell^2 d\Omega^2$. The region with $\ell \simeq 0$ represents the wormhole throat, of size $a$; this connects the regions $\ell \gg a, \ell \ll -a$, representing two asymptotically flat universes. The spacetime geometry (1.2) received special attention in the classical 1988 paper by Morris and Thorne [1], considered as the origin of modern investigations on wormholes.

Indeed, the line element (1.2) had appeared in the literature before [1] (a fact on which Thorne apologized in [2]). This spacetime geometry was considered in a 1973 paper by Ellis [3], with the denomination of “drainhole” (and with some different motivation, namely, to model an elementary particle); here the metric (1.2) was derived solving Einstein’s equations in presence of a massless scalar field $\phi$ minimally coupled to gravity, after changing artificially the sign of the action functional for $\phi$. Again in [3], the scalar field was found to depend on $\ell$ with the law

$$\phi = \sqrt{\frac{2}{\kappa}} \arctan \frac{\ell}{a}. \quad (1.3)$$

Almost simultaneously to Ellis, Bronnikov [4] proposed a family of scalar field solutions of Einstein’s equations containing, as a special case, the solution (1.2) (1.3) (4). The scalar fields considered by Ellis and Bronnikov, with an anomalous sign in (the kinetic part of) of their action functional, are now referred to as phantom fields; their stress-energy violates the usual conditions of positivity of the energy, thus mimicking at the classical level a well known feature of quantum fields in their vacuum states [6] [7].

In the rest of this paper we refer to the names or initials of the previously mentioned authors and use the expressions “EBMT wormhole”, “EBMT solution” to indicate the phantom field solution (1.2) (1.3) of Einstein’s equations.

\textsuperscript{2} The family of Bronnikov solutions depends on a “mass” parameter, which is zero in the case (1.2) (1.3); see e.g. [3], Eqs. (15)(16) for a representation of this family very close to the notations of (1.2) (1.3). Such a mass-dependent generalization is never considered in the present work.
In this work we consider a perturbation of the solution (1.2) (1.3) of the form
\[ ds^2 = -dt^2 + \left(1 + \varepsilon \Omega \left( \frac{t}{a} \right) \right)^2 \, dt^2 + \left( \sqrt{a^2 + \ell^2} + \frac{\varepsilon a^3}{a^2 + \ell^2} R \left( \frac{t}{a}, \frac{\ell}{a} \right) \right)^2 \, d\Omega^2, \] (1.4)
\[ \phi = \sqrt{\frac{2}{\kappa}} \left( \arctan \frac{\ell}{a} + \varepsilon \Phi \left( \frac{t}{a}, \frac{\ell}{a} \right) \right) \left( -\infty < t, \ell < +\infty \right), \] (1.5)
where \( \varepsilon \) is a small real parameter and \( \Omega, R, \Phi \) are functions of the variables \( s := t/a, x := \ell/a \), to be determined; Einstein’s equations are expanded to the first order in \( \varepsilon \), giving rise to a system of linear equations for \( \Omega, R, \Phi \).

Our handling of this linear system produces in a simple way the general solution. As a matter of fact, \( \Omega \) and \( \Phi \) are represented explicitly as functions of \( R \) (and of their initial data), and a “master equation” is derived for \( R \); this has the form
\[ (\partial_{ss} - \partial_{xx} + V(x))R(s, x) = J_0(x) + s J_1(x), \quad V(x) := -\frac{3}{(1 + x^2)^2} \] (1.6)
where \( J_0, J_1 \) are source terms depending in the initial data for the system. Since the operator \(-\partial_{xx} + V(x)\) has a negative eigenvalue, Eq. (1.6) has solutions diverging exponentially for large times; this suffices to infer the linear instability of the EBMT solution. Admittedly, the linear instability of the EBMT system (and of more general wormhole type solutions) has been stated previously in the literature, on the grounds of suitably derived master equations for some recombination of the perturbation components; therefore, it is necessary to compare the present work with the previous papers on this subject.

This comparison is performed in the forthcoming subsections 1.a and 1.b; subsection 1.c concludes the present introduction describing the organization of our work.

### 1.a Comparison with [8].
When the results of the present work were derived, we were not aware of the proceeding article [8] by González, Guzmán and Sarbach while we had knowledge of subsequent papers by the same authors, discussed hereafter [9] [10]; we were kindly informed about [8] by Professors González, Guzmán and Sarbach, when we mailed to them the first arXiv version of the present work.

Paper [8] projects an elegant setting for the linear analysis of the EBMT perturbed system, focusing on invariance features under spacetime coordinate changes (gauge transformations) infinitesimally close to the identity. The conclusion of the cited article is that a suitable recombination \( \chi \) of the perturbation components fulfills (in the notations of the present work) \( (\partial_{ss} - \partial_{xx} + V(x))\chi(s, x) = 0 \), with \( V \) as in Eq. (1.6); the same paper proves that \(-\partial_{xx} + V(x)\) has a negative eigenvalue, a fact yielding a virulent of linear instability. Unfortunately, the discussion of gauge transformations proposed in [8] contains some imprecision, which propagates to the formulation of the linearized Einstein equations. \(^3\)

\(^3\)We acknowledge the authors of [8] for an open and kind discussion on this subject. The analysis of infinitesimal gauge transformations in the cited paper fixes the attention on the radial coordinate (\( \ell \) in our notations), and does not consider changes of the time coordinate \( t \). The linearized Einstein equations of [8] and the subsequent stability analysis are correct under the condition (not stated explicitly) that the field perturbation is zero. It is easy to check that the field perturbation always vanishes in a suitable coordinate system; of course, the choice of these distinguished coordinates breaks the desired gauge invariance of the overall setting.
In view of this, we think that a reconsideration of the perturbed EBMT system in the linear approximation is not useless, even in the simple approach proposed in the present work. Our analysis is developed in a fixed gauge, defined requiring that the coefficient of $-dt^2$ in the spacetime line element be 1 (on this, see the comments accompanying our subsequent Eq. (2.6)). The substantial nature of the large time divergences arising from our computations is proved a posteriori, showing the impossibility to eliminate them via coordinate changes (see the discussion in the last lines of subsection 3.h).

1.b Comparison with [9] [10] [11] [12]. Paper [9] by González, Guzmán and Sarbach considers Bronnikov’s wormhole solution [4] of the Einstein-scalar equations; as already indicated, the EBMT system (1.2) (1.3) is a special case of this solution. In [9] the linear instability of the general Bronnikov solution is derived via a two-steps construction, that we now describe briefly. The first step is the reduction of the linearized Einstein equations to a scalar master equation where the unknown is a suitable recombination of the perturbation components, here indicated with $\chi_{\text{sing}}$. The potential $V_{\text{sing}}$ in this master equation is singular at the wormhole throat; other singularities, again located at the throat, affect a source term appearing in the same equation and the very definition of the recombination $\chi_{\text{sing}}$. The second step in the construction of [9] removes the singularities by a clever strategy: the idea is to apply to $\chi_{\text{sing}}$ a suitable first order differential operator, so as to obtain a function $\chi$ fulfilling a regular master equation. This is in fact possible if one knows a static solution of the singular master equation; the static solution determines the transformation relating $\chi_{\text{sing}}$ and $\chi$. In our notations the final, regular master equation reads $\left(\partial_{xx} - \partial_{xx} + V(x)\right)\chi(s, x) = 0$, where $V$ is a nowhere singular potential; the authors of [9] show that the operator $-\partial_{xx} + V(x)$ has a negative eigenvalue, a fact implying the linear instability of the Bronnikov solution.

In the special EBMT case, the regular potential $V$ coincides with the function in Eq. (1.6); however, as indicated before, in the EBMT case a regular master equation can be derived in a direct way with no need to use the previous two-steps construction.

Paper [9] has a companion work by the same authors [10] where the exact, nonlinear Einstein equations for the perturbed Bronnikov solution are treated numerically, providing evidence that the system is unstable even at the nonlinear level. (A numerical analysis of the same issue is also given in the second half of [8] for the special EBMT case). Admittedly, the subject of nonlinear instability is beyond the aims of our work.

Returning to the linear stability analysis, let us point out that the two-steps approach (a singular master equation, a subsequent regularization) has been extended by Bronnikov, Fabris and Zhidenko [11] to the whole class of static, radially symmetric scalar field solutions of Einstein’s equations with throats (including cases with an external potential for the scalar field). Let us also mention a very recent paper of Bronnikov [12], an excellent review about wormholes and black holes supported by scalar fields that considers, amongst else, the two-steps approach to linear stability problems.

1.c Organization of the paper. Making reference to subsection 1.a for the motivations of the present work, let us briefly outline its organization. Section 2 reviews some basic facts on (ordinary and) phantom scalar fields minimally coupled to gravity, and on Einstein’s equations for such systems with the assumption of radial symmetry; the EBMT solution (1.2) (1.3) is presented as a static solution of these equations.
Section 3 is the core of the paper. In subsections 3.a-3.b we perturb the EBMT solution as in Eqs. (1.4) (1.5), and linearize the corresponding Einstein equations. In subsections 3.c-3.e we express all perturbation components in terms of $R$, and derive a master equation for this component. In the final subsections 3.f-3.h we write down the general solution of our master equation (hence, of the linearized Einstein equations); we show that there are solutions diverging exponentially for large times, and that such divergences cannot be eliminated by coordinate changes. In 3.f-3.h we also take the occasion to set up a rigorous functional-analytic framework for the master equation, based on the language of Sobolev spaces.

2 Some basic facts

2.a Gravitation and scalar fields. In a four-dimensional spacetime, we consider a gravitational field minimally coupled to a real scalar field $\phi$ with a vanishing field self-potential, i.e., a real scalar field with zero mass and no self-interaction. This system is described by the action functional

$$S[g_{\mu\nu}, \phi] := \int \left( \frac{R}{2\kappa} - \frac{\sigma}{2} \partial^\mu \phi \partial_\mu \phi \right) dv$$

where: $g_{\mu\nu}$ is the spacetime metric (of course used to raise and lower indices); $R$ and $dv$ are the scalar curvature and the volume element corresponding to this metric ($dv = \sqrt{\det(g_{\mu\nu})} \prod dx^\lambda$ in any spacetime coordinate system $(x^\lambda)$); $\sigma := 1$ for an ordinary field, $\sigma := -1$ for a phantom field. Both the metric and the scalar field are always assumed to be smooth.

The stationarity condition $\delta S/\delta g_{\mu\nu} = 0$ gives Einstein’s equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

where the right side contains the field stress-energy tensor

$$T_{\mu\nu} := \sigma \left( \partial_\nu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\lambda \phi \partial_\lambda \phi \right)$$

(2.3) The stationarity condition $\delta S/\delta \phi = 0$ gives the field equation

$$\Box \phi = 0$$

(2.4) where $\Box := \nabla_\mu \nabla^\mu$ and $\nabla_\mu$ is the covariant derivative induced by the metric $g_{\mu\nu}$ ($\nabla_\mu = \partial_\mu$ on scalar functions, like $\phi$).

As well known, Einstein’s equations have the equivalent form $R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) = \sigma \kappa \partial_\nu \phi \partial_\nu \phi$, used in many of the previously cited works. For our manipulations on the linearized equations, the form (2.2) is more convenient.
Indeed, Einstein’s equations \( (2.2) \) imply the field equation \( (2.4) \). In fact, Einstein’s equations and the contracted Bianchi identity give \( \nabla_\mu T^\mu_\nu = 0 \) and, on the other hand, the definition \( (2.3) \) implies \( \nabla_\mu T^\mu_\nu = \sigma (\Box \phi) \partial_\nu \phi; \) thus \( (2.2) \Rightarrow (\Box \phi) \partial_\nu \phi = 0 \). Refining these considerations, one obtains that \( (3) \)

\[
\Box \phi = 0. \tag{2.5}
\]

### 2.6 The radially symmetric case.

Now, let us consider a spacetime with line element \( ds^2 \) and a scalar field \( \phi \), where

\[
ds^2 = -dt^2 + q^2(t, \ell) d\ell^2 + r^2(t, \ell) d\Omega^2, \quad \phi = \phi(t, \ell) \quad (-\infty < t, \ell < +\infty). \tag{2.6}
\]

In the above \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \) denotes (again) the line element of the unit spherical surface \( S^2 \) \( (0 < \theta < \pi, 0 < \varphi < 2\pi) \) and \( q, \ell : \mathbb{R} \times \mathbb{R} \to (0, +\infty), \phi(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are smooth functions.

Let us mention that a line element of the seemingly more general form \( ds^2 = -h(t, \ell) dt^2 + q^2(t, \ell) d\ell^2 + r^2(t, \ell) d\Omega^2 \) can be reduced (at least locally) to the form in \( (2.6) \), with \( h = 1 \), performing a suitable coordinate change \( (t, \ell) \to (t', \ell') \).

From here to the end of the paper, we make systematic reference to Eq. \( (2.6) \) and to the coordinate system

\[
(t, \ell, \theta, \varphi) \equiv (x^\mu)_{\mu=t, t, \theta, \varphi}. \tag{2.7}
\]

The configuration that we are considering is radially symmetric configuration. For the metric \( g_{\mu\nu} \) and the field \( \phi \) described by \( (2.6) \), the only independent Einstein equations are those corresponding to the choices \( (\mu, \nu) = (t, t), (t, \ell), (\ell, \ell), (\theta, \theta) \) that read, respectively \( (4) \):

\[
\frac{1}{r^2} + \frac{2q_t r_t}{qr} + \frac{2q_\ell r_\ell}{q^2 r^2} - \frac{r_t^2}{q^2 r^2} - \frac{2r_\ell t}{q^2 r} = \frac{\sigma \kappa}{2} \left( \phi_t^2 + \phi_\ell^2 \right), \tag{2.8}
\]

\[
\frac{2q_t r_t}{qr} - \frac{2r_t t}{r} = \sigma \kappa \phi_t \phi_\ell, \tag{2.9}
\]

\[
-\frac{q_t^2}{r^2} - \frac{q_\ell^2}{r^2} + \frac{2r^2 q_t u}{r} = \frac{\sigma \kappa}{2} \left( q^2 \phi_t^2 + \phi_\ell^2 \right), \tag{2.10}
\]

\[
-\frac{q_t r_t}{q} - \frac{q_\ell r_\ell}{q} - \frac{q_t r_t}{q} - \frac{r r_t u}{q} + \frac{r r_\ell u}{q} = \frac{\sigma \kappa r^2}{2} \left( \phi_t^2 + \phi_\ell^2 \right). \tag{2.11}
\]

\(^6\)Here is a derivation of \( (2.6) \). Let us assume Einstein’s equations \( (2.2) \); then \( (\Box \phi) \partial_\nu \phi = 0 \) or, in index-free notation, \( (\Box \phi) d\phi = 0 \) where \( d \) is the usual differential. Denoting with \( M \) the spacetime, let us introduce the open set \( D := \{ x \in M \mid (\Box \phi)(x) \neq 0 \} \). Of course \( \Box \phi = 0 \) on \( D \); hereafter we show that \( \Box \phi = 0 \) is even on the complementary set \( M \setminus D \). In fact, let \( x \in M \setminus D \); then \( x \) belongs to the frontier \( \partial D \), or \( x \) is an inner point of \( M \setminus D \). If \( x \in \partial D \) each neighborhood of \( x \) contains a point \( x' \in D, \) for which \( (\Box \phi)(x') = 0 \); so, by continuity, \( (\Box \phi)(x) = 0 \). If \( x \) is an inner point of \( M \setminus D \), let us choose an open connected neighborhood \( U \) of \( x \) such that \( U \subset M \setminus D \); then \( d\phi = 0 \) on \( U \), whence \( \phi = \text{constant} \) on \( U \) and, consequently, \( \Box \phi = 0 \) on \( U \).

\(^7\)This follows, e.g., from the general discussion of \( [13] \), §97 on synchronous coordinate systems on arbitrary spacetimes.

\(^7\)Refining these considerations, one obtains that \( (3) \)
(here and in the sequel, subscripts like \( \ell \) or \( t \) are used to indicate derivatives). The field equation \( \Box \phi = 0 \) will not even be written since, according to (2.5), it is a consequence of Eqs. (2.8-2.11). For future use we record the explicit expression of the scalar curvature for the metric (2.6), which is as follows:

\[
R = \frac{2}{r^2} + \frac{4qr\ell}{qr} + \frac{4qr\ell}{q^3r} + \frac{2r^2}{r^2} - \frac{2r^2}{q^2r^2} + \frac{2qtt}{q} + \frac{4r\ell}{r} - \frac{4r\ell}{q^2r} .
\]

(2.12)

2.c The EBMT wormhole [1, 3, 4]. This corresponds to the following static solution of the Einstein equations (2.8-2.11):

\[
q(\ell) := 1, \quad r(\ell) := \sqrt{a^2 + \ell^2},
\]

(2.13)

\[
\sigma := -1 \ (\text{phantom field}) , \quad \phi(\ell) := \sqrt{\frac{2}{\kappa}} \ arctan \left( \frac{\ell}{a} \right) \ (-\infty < \ell < +\infty),
\]

where \( a > 0 \) is a parameter, with the dimension of a length. The line element \( ds^2 \) corresponding to (2.13) has the form (1.2); it describes a traversable wormhole with a throat of size \( \inf \ell r(\ell) = a \). In the present case, Eq. (2.12) for the scalar curvature gives

\[
R = -\frac{2a^2}{(a^2 + \ell^2)^2} .
\]

(2.14)

3 Linear instability of the EBMT wormhole: a simplified derivation.

From here to the end of the paper \( \phi \) is a phantom scalar field, i.e.,

\[
\sigma := -1 .
\]

(3.1)

3.a Radial perturbations of the EBMT solution. We consider a line element \( ds^2 \) and a scalar field \( \phi \) as in (2.6), with

\[
q(t, \ell) := 1 + \varepsilon Q \left( \frac{t}{a}, \frac{\ell}{a} \right), \quad r(t, \ell) := \sqrt{a^2 + \ell^2} + \frac{\varepsilon a^3}{a^2 + \ell^2} R \left( \frac{t}{a}, \frac{\ell}{a} \right),
\]

(3.2)

\[
\phi(t, \ell) := \sqrt{\frac{2}{\kappa}} \ arctan \left( \frac{\ell}{a} \right) + \varepsilon \Phi \left( \frac{t}{a}, \frac{\ell}{a} \right) ;
\]

here \( \varepsilon \in \mathbb{R} \) is a small dimensionless parameter (that we ultimately send to zero) and \( Q, R, \Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are smooth dimensionless functions, to be determined; these depend on the variables

\[
s := \frac{t}{a} , \quad x := \frac{\ell}{a}
\]

(3.3)

which are dimensionless in our units with \( c = 1 \). The factor \( a^3/(a^2 + \ell^2) \) multiplying \( R \) in Eq. (3.2) will simplify our subsequent calculations. (Note the equivalence between Eq. (3.2) and Eqs. (1.3) (1.5) of the Introduction).
3.b **Linearizing Einstein’s equations (and the scalar curvature).** Let us substitute the expressions (3.2) into Einstein’s equations (2.8–2.11) and expand them up to the first order in \( \varepsilon \). Of course, these equations are satisfied to the zeroth order in \( \varepsilon \), corresponding to the EBMT solution; moreover, Eqs. (2.8–2.11) hold to the first order in \( \varepsilon \) if and only if we have, respectively:

\[
Q + xQ_x + \frac{2(1 - 2x^2)R}{(1 + x^2)^{3/2}} + \frac{3xR_x}{(1 + x^2)^{3/2}} - \frac{R_{xx}}{\sqrt{1 + x^2}} + \Phi_x = 0 ,
\]

(3.4)

\[
xQ_s + \frac{2xR_s}{(1 + x^2)^{3/2}} - \frac{R_{sx}}{\sqrt{1 + x^2}} + \Phi_s = 0 ,
\]

(3.5)

\[
Q - \frac{(1 - 2x^2)R}{(1 + x^2)^{3/2}} - xR_x + \frac{R_{ss}}{(1 + x^2)^{3/2}} - \Phi_x = 0 ,
\]

(3.6)

\[
xQ_x + (1 + x^2)Q_{ss} + \frac{3(1 - 2x^2)R}{(1 + x^2)^{3/2}} + \frac{4xR_x}{(1 + x^2)^{3/2}} + \frac{R_{ss} - R_{xx}}{\sqrt{1 + x^2}} + 2\Phi_x = 0 .
\]

(3.7)

By obvious considerations based on (2.5), the linearized Einstein’s equations (3.4–3.7) ensure the field equation \( \Box \phi = 0 \) to hold as well up to the first order in \( \varepsilon \). For future use we also write down the first order expansion of the scalar curvature (2.12), which is as follows:

\[
R = -\frac{2a^2}{(a^2 + \ell^2)^2} + \frac{2\varepsilon}{a^2} \mathcal{K} \left( \frac{t}{a}, \frac{\ell}{a} \right) + O(\varepsilon^2) ,
\]

(3.8)

\[
\mathcal{K} := \frac{2(2 + x^2)Q}{(1 + x^2)^{3/2}} + \frac{xQ_x}{1 + x^2} + Q_{ss} + \frac{4(1 - 2x^2)R}{(1 + x^2)^{5/2}} + \frac{6xR_x}{(1 + x^2)^{5/2}} + \frac{2R_{ss} - R_{xx}}{(1 + x^2)^{3/2}} .
\]

3.c **Finding the field perturbation \( \Phi \).** Integrating with respect to \( s \), we see that Eq. (3.5) holds if and only if \( \Phi(s, x) = -xQ(s, x) - 2xR(s, x)/(1 + x^2)^{3/2} + R_x(s, x)/\sqrt{1 + x^2} + \mathcal{C}(x) \) where \( \mathcal{C} : \mathbb{R} \to \mathbb{R} \) is a smooth function. Inserting this expression for \( \Phi \) into Eq. (3.4), we see that the latter holds if and only if \( \Phi \) is constant. Summing up: Eqs. (3.4) (3.5) hold if and only if

\[
\Phi(s, x) = -xQ(s, x) - \frac{2xR(s, x)}{(1 + x^2)^{3/2}} + \frac{R_x(s, x)}{\sqrt{1 + x^2}} + C_0
\]

(3.9)

where \( C_0 \in \mathbb{R} \) is a constant. The value of \( C_0 \) is immaterial (note that \( \Phi \) appears in the linearized equations (3.4–3.7) only through its derivatives; the same can be said for \( \phi \) in the exact equations (2.8–2.11)).
3.4 \( Q \) and \( \Phi \) as functions of \( R \). Now we are left with Eqs. (3.6) (3.7); evidently, this pair is equivalent to the pair formed by Eq (3.6) and by Eq. (3.6) + Eq. (3.7) (side by side sum). The combination (3.6) + (3.7) is reduced, after substituting the expression (3.9) for \( \Phi \), to the equation

\[
Q_{ss} + \frac{2R_{ss}}{(1 + x^2)^\frac{3}{2}} = 0 ;
\]  

(3.10)

this holds if and only if \( Q(s, x) = -2R(s, x)/\left(1 + x^2\right)^{\frac{3}{2}} \) + a function with vanishing \( ss \)-derivative, i.e.,

\[
Q(s, x) = -\frac{2R(s, x)}{(1 + x^2)^{\frac{3}{2}}} + P_0(x) + sP_1(x) \quad (i = 0, 1)
\]  

(3.11)

where \( P_0, P_1 : \mathbb{R} \to \mathbb{R} \) are smooth functions; these are closely related to the set of initial data

\[
Q_0(x) := Q(0, x), \quad R_0(x) := R(0, x), \quad Q_1(x) := Q_s(0, x), \quad R_1(x) := R_s(0, x),
\]  

(3.12)

since (3.11) implies

\[
P_i(x) = Q_i(x) + \frac{2R_i(x)}{(1 + x^2)^{\frac{3}{2}}} \quad (i = 0, 1).
\]  

(3.13)

Returning to Eq. (3.9) for \( \Phi \), and substituting therein Eq. (3.11) for \( Q \), we obtain for the field perturbation the final expression

\[
\Phi(s, x) = \frac{R_x(s, x)}{\sqrt{1 + x^2}} - x \left( P_0(x) + sP_1(x) \right) + C_0 \quad (P_i \text{ as in } (3.13)).
\]  

(3.14)

3.e A master equation for \( R \). We finally substitute the expression (3.11) (3.14) for \( Q, \Phi \) into (3.6); the equation obtained in this way holds if and only if

\[
(R_{ss} + \mathcal{H} R)(s, x) = J_0(x) + sJ_1(x),
\]  

(3.15)

where

\[
\mathcal{H} := -\frac{d}{dx^2} + \mathcal{V}, \quad \mathcal{V}(x) := -\frac{3}{(1 + x^2)^2} \quad (x \in \mathbb{R}),
\]  

(3.16)

\[
J_i(x) := -\sqrt{1 + x^2} \left( 2P_i(x) + xP_{i,x}(x) \right) \quad (i = 0, 1, \ P_i \text{ as in } (3.13)).
\]  

(3.17)

\( \mathcal{H} \) is, formally, a Schrödinger type operator in space dimension 1 with potential \( \mathcal{V} \); the functions \( J_i \) are fully determined by the functions \( P_i \) or, due to (3.13), by the initial data \( Q_i, R_i \ (i = 0, 1) \). Eq. (3.15) is our master equation; it is a wave-type equation for \( R \) with a source term \( J_0(x) + sJ_1(x) \).
3.f Spectral analysis tools to solve the master equation for $\mathcal{R}$. The solution of Eq. (3.15) is reduced to the spectral analysis of the operator $\hat{\mathcal{F}}$ defined by (3.16), in a convenient Hilbertian framework; in view of this, from now on the derivative $d^2/dx^2$ appearing therein will be intended in the most general sense, i.e., in the sense of the Schwartz distributions theory [14]. From the general theory of Schrödinger operators on the real line with smooth potentials vanishing at infinity [15], one infers the following statements (i-iii):

(i) Consider the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ of complex valued, square integrable functions on $\mathbb{R}$; let $\mathcal{F}$ denote the restriction of $\hat{\mathcal{F}}$ to the domain $\{ \mathcal{F} \in L^2(\mathbb{R}, \mathbb{C}) \mid \mathcal{F} \in L^2(\mathbb{R}, \mathbb{C}) \}$; then, $\mathcal{F}$ is a selfadjoint operator in $L^2(\mathbb{R}, \mathbb{C})$.

(ii) The discrete spectrum $\sigma_d(\mathcal{F})$ consists of finitely many, negative eigenvalues; the continuous spectrum $\sigma_c(\mathcal{F})$ coincides with $[0, +\infty)$. Any eigenvalue $-E \in \sigma_d(\mathcal{F})$ has an associated space of square integrable eigenfunctions, of dimension 1. Every point $W \in (0, +\infty)$ has an associated, 2-dimensional space of “generalized” eigenfunctions: these are functions $\mathcal{F}$ which fulfill $\mathcal{F}\mathcal{F} = W\mathcal{F}$ but do not belong to $L^2(\mathbb{R}, \mathbb{C})$.

(iii) Choosing appropriately a normalized eigenfunction $\mathcal{F}_E$ for each eigenvalue $-E \in \sigma_d(\mathcal{F})$ and two generalized eigenfunctions $\mathcal{F}_W^j (j = 1, 2)$ for each $W \in (0, +\infty)$, one can build a “generalized” orthonormal basis for $L^2(\mathbb{R}, \mathbb{C})$. These choices can be made so that all the previous eigenfunctions are real valued.

We will now profit from the analysis already performed in [8] [9] for the operator $\hat{\mathcal{F}}$, resting on specific features of its potential $\mathcal{V}$. In [8], it is shown that $\hat{\mathcal{F}}$ has at least one (necessarily negative) eigenvalue; in [9] it is proved that the discrete spectrum of $\hat{\mathcal{F}}$ consists of exactly one eigenvalue, and a numerical estimate is given for it: [9]

$$\sigma_d(\mathcal{F}) = \{-E\} \ , \hspace{10mm} E \simeq 1.40 \ . \ (3.18)$$

According to (iii), we have a generalized orthonormal basis formed by a normalized, real valued eigenfunction $\mathcal{F}_E$ and by a pair of generalized, non square integrable real valued eigenfunctions $\mathcal{F}_W^j (j = 1, 2)$ for each $W \in (0, +\infty)$. It should be noted that $\mathcal{F}_E$ is an even function: $\mathcal{F}_E(-x) = \mathcal{F}_E(x)$; this reflects a general result on the eigenfunction for the minimum eigenvalue of a Schrödinger operator $-d^2/dx^2 + \mathcal{V}$ with an even potential $\mathcal{V}$.

From now on on $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$; we consider the space $L^2(\mathbb{R}, \mathbb{K})$ of square integrable functions from $\mathbb{R}$ to $\mathbb{K}$. For each $\mathcal{F} \in L^2(\mathbb{R}, \mathbb{K})$ we have (intending suitably all the integrals that follow [15])

$$\mathcal{F}(x) = \langle \mathcal{F} | \mathcal{F} \rangle \mathcal{F}_E(x) + \sum_{j=1,2} \int_0^{+\infty} dW \langle \mathcal{F}_W^j | \mathcal{F} \rangle \mathcal{F}_W^j(x) \ , \quad (3.19)$$

where $\langle \mathcal{F} | \mathcal{F} \rangle := \int dV \mathcal{F}(x) \mathcal{F}(x) \mathcal{F}(x) \mathcal{F}(x) \in \mathbb{K}$ for $\mathcal{F} = \mathcal{F}_E, \mathcal{F}_W^j$. Moreover, let $\| \cdot \|$ denote the norm of $L^2(\mathbb{R}, \mathbb{K})$ defined by $\| \mathcal{F} \|^2 = \int |\mathcal{F}(x)|^2 dx$; then, we have the representation $\| \mathcal{F} \|^2 = |\langle \mathcal{F} | \mathcal{F} \rangle|^2 + \sum_{j=1,2} \int_0^{+\infty} dW |\langle \mathcal{F}_W^j | \mathcal{F} \rangle|^2$. If $\mathcal{F}$ and $\mathcal{F}$ are both in $L^2(\mathbb{R}, \mathbb{C})$, one also has $\langle \mathcal{F}_E | \mathcal{F} \rangle = -E \langle \mathcal{F}_E | \mathcal{F} \rangle$ and $\langle \mathcal{F}_W^j | \mathcal{F} \rangle = W \langle \mathcal{F}_W^j | \mathcal{F} \rangle$ for $W > 0$, $j = 1, 2$.

---

$^a$Paper [9] does not report directly the value of $E$ but, rather, the dimensionless “unstability time” $T := 1/\sqrt{E}$. For this quantity it is stated that $T \simeq 0.846$ (see Table 1 of the cited work); this implies for $E$ the estimate in [3.18].
To go on, one can introduce the function space
\[ E(\mathbb{R}, \mathbb{K}) := \{ \mathcal{F} \mid \mathcal{F}, \mathcal{F}^t, \mathcal{F}^2 \mathcal{F} \ldots \in L^2(\mathbb{R}, \mathbb{K}) \} , \tag{3.20} \]
which is a Fréchet space \cite{14} with the countably many norms \( \mathcal{F} \mapsto \| \mathcal{F} \|, \| \mathcal{F} \|, \| \mathcal{F}^2 \|, \ldots ; \) note that \( \mathcal{Y} \in E(\mathbb{R}, \mathbb{K}) \). By means of some Sobolev imbeddings (see again \cite{14}, Theorem 7.25), one shows that \( E(\mathbb{R}, \mathbb{K}) = \{ \mathcal{F} \in C^\infty(\mathbb{R}, \mathbb{K}) \mid \mathcal{F}, \mathcal{F}_x, \mathcal{F}_{xx}, \ldots \in L^2(\mathbb{R}, \mathbb{K}) \} \) and that the previous family of norms is topologically equivalent to the family of (semi-)norms \( \mathcal{F} \mapsto \| \mathcal{F} \|, \| \mathcal{F}_x \|, \| \mathcal{F}_{xx} \|, \ldots \).

3.g Solving the master equation for \( \mathcal{R} \); conclusions for the linearized Einstein equations. Let us keep all notations of subsection 3.f in particular, \( \mathcal{Y}_E \) and \( \mathcal{Y}_W \) are the real valued eigenfunctions in item (iii) therein.

Assume that \( \mathcal{Q}, \mathcal{R}, \Phi \in C^\infty(\mathbb{R}^2, \mathbb{R}) \) are solutions of the linearized Einstein equations (3.3)-(3.7); then, we have Eqs. (3.11)-(3.14) for \( \mathcal{Q} \) (addition, assume that:
\[(d) \ \mathcal{Q}(\alpha) \in E(\mathbb{R}, \mathbb{R}) \quad \text{for} \quad i = 1, 2 \ (\text{this is, in fact, a condition about the data \( \mathcal{Q}_i \) and defining } \mathcal{Q}_i \text{ via Eqs. } (3.13)-(3.17)). \)
\[(d) \ \mathcal{Q}(\beta) \text{ for each } s \in \mathbb{R}, \text{ the function } \mathcal{R}(s, \cdot) : x \mapsto \mathcal{R}(s, x) \text{ is in } E(\mathbb{R}, \mathbb{R}) \text{ and the mapping } s \mapsto \mathcal{R}(s, \cdot) \text{ is } C^\infty \text{ from } \mathbb{R} \ni s \text{ to the space } E(\mathbb{R}, \mathbb{R}). \]

Then, at each “time” \( s \), we have an expansion of the form (3.19) for \( \mathcal{F} := \mathcal{R}(s, \cdot) \). It is inferred from (3.15) that \( (d^2/ds^2 - E)(\mathcal{Y}_E|\mathcal{R}(s, \cdot)) = (\mathcal{Y}_E|\mathcal{R}_0) + s(\mathcal{Y}_E|\mathcal{R}_1) \) and \( (d^2/ds^2 + W)(\mathcal{Y}_W|\mathcal{R}(s, \cdot)) = (\mathcal{Y}_W|\mathcal{R}_0) + s(\mathcal{Y}_W|\mathcal{R}_1) \) for all \( W > 0 \); these ODEs for the components of \( \mathcal{R}(s, \cdot) \) are solved by elementary means, and one obtains:
\[ \mathcal{R}(s, x) = \left[ (\mathcal{Y}_E|\mathcal{R}_0) \cosh(\sqrt{E}s) + (\mathcal{Y}_E|\mathcal{R}_1) \frac{\sinh(\sqrt{E}s)}{\sqrt{E}} \right] \mathcal{Y}_E(x) \]
\[ + \frac{\cosh(\sqrt{E}s)}{E} - \frac{1}{E^3/2} \] \[ + \sum_{j=1,2} \int_0^{+\infty} dW \left[ (\mathcal{Y}_W|\mathcal{R}_0) \cos(\sqrt{W}s) + (\mathcal{Y}_W|\mathcal{R}_1) \frac{\sin(\sqrt{W}s)}{\sqrt{W}} \right] \mathcal{Y}_W(x) \]
\[ + \frac{1 - \cos(\sqrt{W}s)}{W} + \frac{\sqrt{W}s - \sin(\sqrt{W}s)}{W^{3/2}} \]

This equation determines the function \( \mathcal{R} \), which in turn appears in the expressions \( (3.11)-(3.14) \) for \( \mathcal{Q}, \Phi \).

As a converse of the above statements, we consider functions \( \mathcal{Q}_i, \mathcal{R}_i \in C^\infty(\mathbb{R}, \mathbb{R}) \) such that \( \mathcal{R}_i, \mathcal{J}_i \in E(\mathbb{R}, \mathbb{R}) \) for \( i = 0, 1 \), where the \( \mathcal{J}_i \)'s are defined by Eqs. (3.13)-(3.14). Defining \( \mathcal{R} \) and, subsequently, \( \mathcal{Q}, \Phi \) via Eqs. (3.21) \( (3.11)-(3.13) \), one can show the following:
\[(a) \text{ For each } s \in \mathbb{R}, \text{ the map } \mathcal{R}(s, \cdot) : x \mapsto \mathcal{R}(s, x) \text{ is in } E(\mathbb{R}, \mathbb{R}); \text{ the map } s \mapsto \mathcal{R}(s, \cdot) \text{ is } C^\infty \text{ from } \mathbb{R} \to E(\mathbb{R}, \mathbb{R}). \]
\[(b) \mathcal{Q}, \mathcal{R}, \Phi \in C^\infty(\mathbb{R}^2, \mathbb{R}). \]
\[(c) \mathcal{Q}_i \text{ and } \mathcal{R}_i \text{ are the initial data for } \mathcal{Q} \text{ and } \mathcal{R}, \text{ in the sense of Eq. (3.12)}. \]
\[(d) \mathcal{Q}, \mathcal{R}, \Phi \text{ fulfill the linearized Einstein equations (3.3)-(3.7)}. \]
3.6 Linear instability of the EBMT wormhole. This is proved showing that the linearized Einstein equations have solutions diverging in the large $s$ limit. The simplest solution of this kind is obtained choosing the initial data

$$R_0(x) := y_{-E}(x), \quad R_1(x) := 0, \quad Q_0(x) = -\frac{2y_{-E}(x)}{(1 + x^2)^{3/2}}, \quad Q_1(x) := 0.$$  \hspace{1cm} (3.22)

Then $\langle y_{-E} | R_0 \rangle = 1$, $\langle y_{-E} | R_0 \rangle = 0$ and Eqs. (3.13) (3.17) give $P_i = 0, \partial_i = 0$ for $i = 0, 1$. From here and from Eqs. (3.21) (3.11) (3.14) we get

$$R(s, x) = y_{-E}(x) \cosh(\sqrt{E}s), \hspace{1cm} (3.23)$$

Clearly, this solution diverges exponentially for $s \rightarrow \pm \infty$; the same feature appears in many associated geometrical objects. Let us consider, for example, the scalar curvature $R$ of the spacetime metric; substituting Eqs. (3.23) into Eq. (3.8) (and using the relation $Ry_{-E} = -Ey_{-E}$, i.e., $y_{-E,xx} = (E - 3/(1 + x^2)^2)y_{-E}$) we get

$$R = -\frac{2a^2}{(a^2 + \ell^2)^2} + \frac{4\varepsilon}{a^2}K \left( \frac{\ell}{a} \right) \cosh \left( \sqrt{E} \frac{t}{a} \right) + O(\varepsilon^2), \hspace{1cm} (3.24)$$

$$K(x) := \left( \frac{1}{(1 + x^2)^{7/2}} - \frac{E}{(1 + x^2)^{3/2}} \right)y_{-E}(x) + \frac{x}{(1 + x^2)^{1/2}}y_{-E,x}(x).$$

We remark that the above function $K$ is not identically zero; in particular,

$$K(0) = (1 - E)y_{-E}(0) \neq 0$$  \hspace{1cm} (3.25)

Let us also stress that the divergence for $t \rightarrow \pm \infty$ of the coefficient of $\varepsilon$ in Eq. (3.24) is not an artifact that one could eliminate by an everywhere smooth coordinate change $(t, \ell) \mapsto (\tau, \lambda)$, $\varepsilon$-close to the identity. In fact, let us consider any coordinate change of the form

$$t = \tau + \varepsilon a\mathcal{T} \left( \frac{\tau}{a}, \frac{\lambda}{a} \right), \quad \ell = \lambda + \varepsilon a\mathcal{L} \left( \frac{\tau}{a}, \frac{\lambda}{a} \right)$$  \hspace{1cm} (3.26)

where $\mathcal{T}, \mathcal{L} : \mathbb{R}^2 \rightarrow \mathbb{R}$ are smooth (dimensionless) functions; then, Eq. (3.21) gives

$$R = -\frac{2a^2}{(a^2 + \lambda^2)^2} + \frac{4\varepsilon}{a^2} \left[ K \left( \frac{\lambda}{a} \right) \cosh \left( \sqrt{E} \frac{\tau}{a} \right) + \frac{2(\lambda/a)\mathcal{L}(\tau/a, \lambda/a)}{(1 + \lambda^2/a^2)^3} \right] + O(\varepsilon^2). \hspace{1cm} (3.27)$$

In particular, at spacetime points with $\lambda = 0$ we have

$$R = -\frac{2a^2}{(a^2 + \lambda^2)^2} + \frac{4\varepsilon}{a^2} \cosh \left( \sqrt{E} \frac{\tau}{a} \right) + O(\varepsilon^2), \hspace{1cm} (3.28)$$

and the coefficient of $\varepsilon$ in the above equation diverges (again exponentially) for $\tau \rightarrow \pm \infty$, due to the previous remark $K(0) \neq 0.$
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