Cosmological evolution of regularized branes in 6D warped flux compactifications

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(Dated: February 1, 2008)

We study the cosmological evolution of extended branes in 6D warped flux compactification models. The branes are endowed with the three ordinary spatial dimensions, which are assumed to be homogeneous and isotropic, as well as an internal extra dimension compactified on a circle. We embed these codimension 1 branes in a static bulk 6D spacetime, whose geometry is a solution of 6D Einstein-Maxwell or Einstein-Maxwell-dilaton theories, corresponding to a warped flux compactification. The brane matter consists of a complex scalar field which is coupled to the bulk $U(1)$ gauge field. In both models, we show that there is critical point which the brane cannot cross as it moves in the bulk. We study the cosmological behaviour, especially when the brane approaches this critical point or one of the two conical singularities. In the present setup where the bulk geometry is fixed, we find that the brane cosmology does not coincide with the standard one in the low energy limit.

PACS numbers: 04.50.+h, 98.80.Cq

I. INTRODUCTION

Braneworld models have been studied actively in the last decade (see e.g. \textsuperscript{1} for a few reviews on this topic). Among models which take explicitly into account the self-gravity of the branes, i.e. the backreaction on the geometry due to the presence of the brane, most efforts have been devoted to the study of gravity and cosmology in five dimensional (codimension 1) braneworld models, before (see \textsuperscript{2}) and especially after the stimulating proposals by Randall and Sundrum \textsuperscript{3}. However, if one is interested in a braneworld model in a six or higher dimensional bulk spacetime, one needs to consider a brane with codimension higher than 1.

Among higher codimensional braneworld models, some significant attention has been paid to codimension 2 branes (see e.g. Ref. \textsuperscript{4} and references therein), because they possess the property that, even with some non zero vacuum energy, the brane geometry can be flat, while the bulk geometry surrounding the brane is characterized by a deficit angle. Codimension 2 branes have thus been studied in the context of the cosmological constant problem \textsuperscript{5} \textsuperscript{6} \textsuperscript{7}, but they are also interesting on their own as the simplest case beyond codimension 1 branes.

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One difficulty however with codimension 2 brane is that the energy momentum tensor is in general restricted to that of a pure tension. To consider more general matter, such as would be relevant to study cosmology in the brane, one needs to regularize the brane. One well-motivated way of regularization is to replace the codimension 2 brane by a thin ring-like codimension 1 brane wrapped around the symmetry axis, as was firstly suggested in \textsuperscript{8} in the context of the ”rugby-ball” braneworld model. The rugby-ball model has a 2D compact bulk compactified by a magnetic flux and codimension two branes are located at poles. This model was originally introduced in order to resolve the cosmological constant problem \textsuperscript{9}, but it was also pointed out that the so-called self-tuning property did not actually work \textsuperscript{6,7} (see also \textsuperscript{9} for a review).

In the present work, we consider static warped 6D braneworld solutions \textsuperscript{10,11} with a $U(1)$ gauge field coupled to gravity, which are generalizations of the rugby-ball model. Only static regularized branes have been considered in these geometries so far as now \textsuperscript{12,13,14}. The goal of the present work is to study the cosmological behavior of branes in such 6D braneworld models, i.e. to consider the motion of the branes in these given bulk geometries. \textsuperscript{1} We basically follow the well-established method to discuss cosmology in 5D braneworld models, using the static bulk point of view \textsuperscript{15,16,17,18} rather than the brane-based approach \textsuperscript{2}. In order to satisfy all the junction conditions, we assume, like in \textsuperscript{8} that the branes contain a complex scalar field but we allow for a time evolution of its radial component.

The paper is organized as follows. In Sec. II, we give a general formalism to discuss cosmology on the brane. In Sec. III, we focus on the cosmological evolution of the brane in the models based on the 6D Einstein-Maxwell theory \textsuperscript{10}. In Sec. IV, we discuss the case of the 6D Einstein-Maxwell-dilaton model (more precisely the bosonic part of Nishino-Sezgin supergravity \textsuperscript{19}) \textsuperscript{11} as the same manner as the line of the previous section. In

\textsuperscript{1} In Ref. \textsuperscript{5}, cosmology in such models has been discussed by employing a different way of regularization.
given by the definition of the proper time, one has the following relation
given for example as functions of the proper time

The unit normal vector to the brane is defined by

where \( h_{ab} \) is the three-dimensional metric in the ordinary spatial directions and \( \theta_a \) is the unit vector along the internal \( \theta \) direction. \( \Sigma \) is the energy density and \( p \) and \( p_\theta \) are respectively the ordinary and internal pressures.

One consequence of the brane junction conditions is the generalized energy conservation law on the brane, which can be expressed as

where \( T_{ab} \) denotes the bulk energy-momentum tensor, and all quantities are evaluated at the brane location. The above expression differs from the usual cosmological conservation law in two respects: first, there is an additional term due the evolution of the internal spatial direction; second, one finds in general on the right hand side a bulk-energy exchange term, similar to what can be found in 5D brane cosmology (see e.g. [20]).

III. EINSTEIN-MAXWELL MODEL

A. Bulk spacetime in 6D Einstein-Maxwell theory

We now discuss the dynamics of an extended brane in the 6D Einstein-Maxwell theory. The action in the bulk is given by

where \( g_{AB} \) is the 6D metric, \( F_{AB} = 2\partial[A_B] \) is the field strength associated to the \( U(1) \) gauge vector \( A_B \), and \( \Lambda_0 \) is a cosmological constant.

By a double Wick rotation of the 6D Reissner-Nordstroem solution, one can obtain the solution [10]:

\[
\dot{\Sigma} + \frac{A}{4} (\Sigma + p) + \frac{C}{4} (\Sigma + p_\theta) = [T_{ab} n^a u_b],
\]

\[
[T_{ab} n^a u_b] = (T_{ab} n^a u_b)_{w = w_0} - (T_{ab} n^a u_b)_{w = w_0}(2.9)
\]

where \( T_{AB} \) denotes the bulk energy-momentum tensor, and all quantities are evaluated at the brane location. The above expression differs from the usual cosmological conservation law in two respects: first, there is an additional term due to the evolution of the internal spatial direction; second, one finds in general on the right hand side a bulk-energy exchange term, similar to what can be found in 5D brane cosmology (see e.g. [20]).
Assuming that $F$ has two positive roots $\rho_-$ and $\rho_+$ such that $0 < \rho_- < \rho_+$, one can replace the two parameters $b_0$ and $\mu_0$ by $\rho_+$ and $\alpha \equiv \rho_-/\rho_+$. Using $F(\rho_\pm) = 0$, $b_0$ and $\mu_0$ are given by

$$b_0^2 = \frac{6}{5} c_0^2 \Lambda_0 \rho_+^8 \alpha^3 - \frac{1 - \alpha^8}{1 - \alpha^3}, \quad \mu_0 = \frac{\Lambda_0}{10} \rho_+^5 \frac{1 - \alpha^8}{1 - \alpha^3}. \quad (3.4)$$

It is also convenient to replace the coordinates $\rho$ and $\varphi$ by the new coordinates $w$ and $\theta$, defined respectively by

$$\rho = \rho_+ w, \quad \varphi = \frac{1}{\Lambda_0 \rho_+ (1 - \alpha^3)} \theta, \quad (3.5)$$

so that the bulk metric now reads

$$ds_b^2 = R_0^2 \left[ \frac{4dw^2}{(1 - \alpha)^2 f(w)} + c_0^2 f(w)d\theta^2 \right] + \rho_+^2 w^2 \eta_{\mu\nu} dx^\mu dx^\nu, \quad (3.6)$$

with

$$f(w) = \frac{1}{5(1 - \alpha)^2} \left[ -w^2 - \frac{1}{w^6} \alpha^3(1 - \alpha^3) + \frac{1}{w^3} \left( 1 - \alpha^8 \right) \right],$$

and the field strength is now given by

$$F_{w\theta} = -\frac{2c_0 R_0 S}{(1 - \alpha)w^4}, \quad S = S(\alpha) \equiv \sqrt{3\alpha^3(1 - \alpha^3)} \frac{5(1 - \alpha^3)}{1 - \alpha^3}, \quad (3.7)$$

with $R_0^2 = (2\Lambda_0)^{-1}$.

The singularities $\rho = \rho_-$ and $\rho = \rho_+$ now correspond to $w = \alpha$ and $w = 1$ in the new coordinate system. By expanding the above metric around these two singularities, one finds that they are in general conical singularities with some corresponding deficit angles. These conical singularities can be related to the presence of two codimension-2 branes. As first suggested in \cite{8}, one way to regularize these codimension 2 branes is to replace them by two codimension 1 branes, with one spatial dimension compactified on a circle at the respective positions $w_- > \alpha$ and $w_+ < 1$. These codimension 1 branes are the boundaries of regular spherical caps, which end smoothly the spacetime. The spherical caps can be described by solutions similar to that of the main bulk, i.e.

$$ds_{b(I)}^2 = \frac{R_0^2}{(1 - \alpha)^2 f(w)} \left[ \frac{4dw^2}{(1 - \alpha)^2 f(w)} + c_0^2 f(w)d\theta^2 \right] + \rho^2 w^2 \eta_{\mu\nu} dx^\mu dx^\nu, \quad (3.9)$$

where the index $I$ takes the values $I = +$ and $I = -$ to denote respectively the northern cap, $w_+ < w < 1$, and the southern cap, $0 < w < w_-$. Note that each bulk region is endowed with different cosmological constants $\Lambda_\pm$. The parameters $c_+$ and $c_-$ are chosen so that the geometry at the two poles $w = 1$ and $w = \alpha$ is now smooth, i.e. with no deficit angle. This implies \cite{12, 13}

$$c_+ = \frac{20(1 - \alpha)(1 - \alpha^3)}{5 - 8\alpha^3 + 3\alpha^8}, \quad c_- = \frac{20(1 - \alpha)(1 - \alpha^3)\alpha^4}{5\alpha^8 - 8\alpha^3 + 3}. \quad (3.10)$$

Moreover, the continuity of the $(\theta, \theta)$ component requires

$$\ell : R_0 c_0 = R_+ c_+ = R_- c_. \quad (3.11)$$

For a schematic picture of the bulk configuration, see e.g., Fig. 1 of Ref. \cite{18}.

The gauge field in the bulk is solved as

$$A^{(N)}_y := \frac{2S\ell}{3(1 - \alpha)} \left( \frac{1}{w^3} - 1 \right), \quad A^{(S)}_y := \frac{2S\ell}{3(1 - \alpha)} \left( \frac{1}{w^3} - \frac{1}{\alpha^3} \right), \quad (3.12)$$

expression valid respectively in the northern and southern regions. Note that in the region of the “0”-bulk the bulk gauge fields are doubly defined. This is consistent only if the difference is simply a pure gauge term, which implies the Dirac quantization condition

$$N = \frac{2S\ell(1 - \alpha^3)}{3(1 - \alpha)\alpha^3}, \quad N = 0, \pm 1, \pm 2, \cdots. \quad (3.13)$$

**B. Junction conditions**

The matter content of the brane must be compatible with the bulk solution, i.e. must satisfy the junction conditions for the metric and the gauge field. In our case, we consider a complex scalar field coupled to the bulk gauge field with the following action

$$S_{b, \pm} = -\int d^3x \sqrt{-q} \left( V(|\phi_\pm|) + \frac{1}{2} D_\alpha \phi_\pm (D^\alpha \phi_\pm)^* \right)$$

$$= -\int d^3x \sqrt{-q} \left( V(\Phi_\pm) + \frac{1}{2} \partial^\alpha \Phi_\pm \partial_\alpha \Phi_\pm + \frac{1}{2} \Phi_\pm^2 (\partial^\alpha \sigma_\pm - eA_\alpha)(\partial_\alpha \sigma_\pm - eA_\alpha) \right), \quad (3.14)$$

where $\Phi_\pm = \Phi_\pm(\tau) e^{i\sigma_\pm(\theta)}$. This is a natural extension of the brane action adopted in \cite{8, 12, 13} for static branes, with the difference that $\Phi_\pm$, which was frozen at a fixed value in the static case, must now be promoted to a time-dependent field. Hereafter, we will often omit the brane subscripts $\pm$ for brevity. The above action implies that the energy density, the external and internal pressures, defined earlier in \cite{2, 8}, are given explicitly by the following expressions:

$$\Sigma = V(\Phi) + \frac{1}{2} \Phi^2 C^{-2} (\partial_\sigma \sigma - eA_\sigma)^2 + \frac{1}{2} \Phi^2, \quad (3.15)$$

$$p = -V(\Phi) + \frac{1}{2} \Phi^2 C^{-2} (\partial_\sigma \sigma - eA_\sigma)^2 + \frac{1}{2} \Phi^2 \quad (3.16)$$

$$p_\theta = -V(\Phi) + \frac{1}{2} \Phi^2 C^{-2} (\partial_\sigma \sigma - eA_\sigma)^2 + \frac{1}{2} \Phi^2 \quad (3.17)$$
To discuss the dynamics of the (±)-branes, we can now apply the general formalism presented in the previous section. Comparing the metric (3.6) with Eq. (2.1), it is easy to identify

\[ A(w)^2 = \rho_w^2 w^2, \quad B(w)^2 = \frac{4R^2_\Sigma}{(1 - \alpha)^2 f(w)}, \]
\[ C(w)^2 = \ell^2 f(w). \]  

Note that the warp factor \( A \) is a monotonically increasing function with respect to \( w \) as assumed in the previous section. Since we are interested in the cosmological evolution of the branes, it is convenient to reexpress the various bulk components as functions of the brane scale factor

\[ a(\tau) \equiv A(a_b(\tau)), \]

instead of the radial coordinate \( w \). In the following, we will use the function \( C(a) \) defined by substituting in \( C(w) \) the expression of \( w \) as a function of \( a \), as well as

\[ \tilde{B}(a) = \left( \frac{dA}{dw} \right)^{-1} B = \frac{2}{(1 - \alpha)\rho_+ \sqrt{\ell}} \]  

where the right hand side is evaluated at the brane position.

Defining the jump across the branes as \([T]_{\pm} := \pm(T_\pm - T_0)\), the junction conditions for the metric \([K_{ab}] = -T_{ab}\) are given by

\[ \left( \frac{3}{a} + \frac{C'}{C} \right)X + \left( \frac{\dot{a}}{a} + \frac{\tilde{B}'}{B} \dot{\rho} \right) \left[ \frac{1}{X} \right] = p, \]  

\[ \frac{4}{a} \left[ X \right] + \left( \frac{\dot{a}}{a} + \frac{\tilde{B}'}{B} \dot{\rho} \right) \left[ \frac{1}{X} \right] = p_\theta, \]  

\[ \left( \frac{3}{a} + \frac{C'}{C} \right)X = -\Sigma, \]  

(3.21)

(3.22)

(3.23)

where \( X := \sqrt{\tilde{B}^2 - \dot{\rho}^2} \) and the prime ‘ denotes the derivative with respect to \( a \). These relations generalize the expressions given in [8] for a static brane. Note that in the static case, the left hand sides of (3.21) and (3.22) are identical, which implies \( p = -\Sigma \). Note also that in the cosmological case, the topological condition is the same as in the static case (see Appendix A-1 for details).

The last junction condition (3.23) implies the generalized Friedmann equation

\[ \frac{\ddot{a}^2}{a^2} - \frac{\ddot{\rho}_+^2}{16\pi^2\ell^2 f(3 + \frac{C'}{C}a)^2} + \frac{\pi^2\ell^2 f(3 + \frac{C'}{C}a)^2(\tilde{B}_\perp^2 - \tilde{B}_0^2)^2}{\dot{\rho}^2} - \frac{\tilde{B}_\perp^2 + \tilde{B}_0^2}{2a^4}, \]

(3.24)

where \( \ddot{\rho} := 2\pi\ell\sqrt{\Sigma} \), obtained by integrating \( \Sigma \) over the internal dimension, is the 4D effective energy density. On the right hand side, one finds a term quadratic in the energy density, which is familiar in brane cosmology [2], as well as a term which goes like \( 1/\rho^2 \) and which is characteristic of non \( Z_2 \)-symmetric brane cosmology [18], i.e. when the two bulk regions surrounding the brane have a different geometry.

It is instructive to study the low energy behaviour of the Friedmann equation by considering an expansion of Eq. (3.21) around a static brane configuration at \( a = a_0 \) such that \( \dot{a}\big|_{a=a_0} = 0 \). From Eq. (3.23), the energy density at \( a = a_0 \) is given by

\[ \Sigma_0 = -\left( \frac{3}{a} + \frac{C'}{C} \right)\left( \tilde{B}_\perp^{-1} - \tilde{B}_0^{-1} \right) \big|_{a=a_0}, \]

(3.25)

for the (+) brane and

\[ \Sigma_0 = -\left( \frac{3}{a} + \frac{C'}{C} \right)\left( \tilde{B}_\perp^{-1} - \tilde{B}_0^{-1} \right) \big|_{a=a_0}, \]

(3.26)

for the (−)-brane. For the (+)-brane, we obtain

\[ \frac{\ddot{a}_+^2}{a_+^2} \approx \frac{8\pi G_{\text{eff}}}{3} \ddot{\rho} + O(\dot{\rho}^2) \]

(3.27)

where the effective gravitational coupling is given by

\[ 8\pi G_{\text{eff}} = \frac{3(1 - \alpha)\rho_+}{2\pi(R_0 - R_+)\ell M_0^4 a(3 + \frac{a_+}{a_0})} \big|_{a=a_0}. \]

(3.28)

We observe that the gravitational coupling depends strongly on the scale factor and is even negative for \( a_0 > \rho_+ \left( \frac{3(1 - \alpha)^{1/5}}{8(1 - \alpha)} \right) \), namely in the vicinity of the northern pole. Therefore, one cannot expect the cosmology governed by the Friedmann equation (3.21) to coincide with standard cosmology in the low energy limit, in contrast to the 5D Randall-Sundrum cosmology. The conclusion here does not depend on the choice of brane matter. A similar expression for the effective gravitational coupling for the (−) brane is obtained by replacing \( R_0 \) and \( R_+ \) in Eq. (3.28) with, respectively, \( R_- \) and \( R_0 \).

Now, we consider more specifically the cosmological evolution when the brane matter consists of a complex scalar field. We must also take into account the junction condition for the gauge field which is given by

\[ F(a) \left[ X \right] = -e\Phi^2(\partial_\sigma - eA_\theta), \]

(3.29)

where we have defined the function \( F(a) \)

\[ F(a) \equiv \left( \frac{dA}{dw} \right)^{-1} F_{\omega} \big|_{A=a} = -\frac{2\varepsilon R\ell S(\alpha)}{(1 - \alpha)\rho_+} \left( \frac{\rho_+}{a} \right)^4. \]

(3.30)

Let us now combine the four junction conditions above. By subtracting Eq. (3.21) and Eq. (3.22), and then using Eqs. (3.10), (3.17) and (3.29), one gets the expression

\[ \left[ X \right] = -e^2\Phi^2 \frac{C_\alpha^2}{C^2} \left( \frac{C'}{C} - \frac{1}{a} \right), \]

(3.31)
where the right-hand side depends only on $a$ and $\Phi$ but not their derivatives. By substituting this expression back in the junction condition (3.23), one finds that the energy density of the brane $\Sigma$ is given explicitly in terms of $a$ and $\Phi$ only:

$$
\Sigma = e^2 \Phi^2 \frac{C'}{\mathcal{F}} \left( \frac{3}{a} \right) \left( \frac{C'}{C} - \frac{1}{a} \right) = \frac{e^2 \Phi^2 (1 - \alpha)^2}{4 S^2} \frac{\partial^2 f(a)}{\rho_\perp^2} \left( \frac{f'(a)}{2 f(a)} - \frac{1}{a} \right) \left( \frac{f'(a)}{2 f(a)} + \frac{3}{a} \right).
$$

(3.32)

One can even get a more precise constraint on the energy density in the brane by noting that the substitution of $w = \sqrt{C'/C - 1/a}$ into the gauge field junction condition (3.29) yields

$$
\partial_\theta \sigma - e A_\theta = e \frac{C'}{\mathcal{F}} \left( \frac{3}{a} \right) \left( \frac{C'}{C} - \frac{1}{a} \right).
$$

(3.33)

This implies that the part of the energy density that depends only on the scalar field $\Phi$ is constrained to be a specific function of $a$ and $\Phi$, namely

$$
V(\Phi) + \frac{1}{2} \Phi^2 = \frac{1}{2} e^2 \Phi^2 \frac{C'^2}{\mathcal{F}^2} \left( \frac{7}{a} + \frac{C'}{C} - \frac{1}{a} \right).
$$

(3.34)

By subtracting Eq. (3.24) from Eq. (3.29), we also obtain

$$
\left( \frac{\dot{a}}{a} + \frac{\tilde{B}'}{\tilde{B}} \dot{\tilde{a}}^2 \right) \left[ \frac{1}{X} \right] = \Phi^2
$$

(3.35)

which governs the brane acceleration.

The bulk $\alpha < w < 1$ can be divided into two regions, where the branes will be confined: a brane in a given region cannot move to the other one. This can be seen by noting that the sign of

$$
[X]_\pm = \pm \left( \sqrt{\frac{(1 - \alpha)^2 \rho_\perp^2 f(a)}{4 R_\perp^2}} + \dot{a}^2 - \sqrt{\frac{(1 - \alpha)^2 \rho_\perp^2 f(a)}{4 R_\perp^0}} + \dot{a}^2 \right)
$$

(3.36)

is determined by the relative values of $R_\perp$ on the two sides of the brane and thus cannot change. From Eq. (3.31), one sees that the sign of $[X]$ is negative for $\alpha < w < w_1$ and positive for $w_1 < w < 1$, where

$$
w_1 := \alpha \left( \frac{8(1 - \alpha^5)}{3(1 - \alpha^3)} \right)^{1/3},
$$

(3.37)

is the value at which $\dot{\tilde{a}}^2 = \frac{1}{\tilde{B}}$, which is plotted in Fig. 1, vanishes. Consequently, $w = w_1$ represents a limit that the branes cannot cross.

The existence of this critical value is a consequence of the junction conditions and our choice of brane matter. Indeed, on the one hand, we always have $p > p_0$ with the complex scalar field that was assumed. On the other hand, the extrinsic curvature tensor is dictated by the geometry and $w_1$ is the value where the sign of $K_1^1/3 - K_0^0$ changes. Since the sign of $p - p_0$ cannot change, we see that the brane cannot cross $w_1$ because the junction conditions cannot be satisfied on the other side. By noting that $R_+ < R_-$, which follows from the relations (3.11) and (3.10), one finds only two possibilities for the locations of the branes. The first possibility is to put the $(+)$-brane in the northern region, i.e. $w_+ > w_1$, and the $(-)$-brane in the southern region, i.e. $w_- < w_1$. Another possibility is to put both branes in the northern region, i.e. $w_+ > w_- > w_1$. It is however impossible to put both branes in the southern region.

Finally, it is worth noticing that the brane energy density $\Sigma$ is negative when $w_1 < w < w_2$, where

$$
w_2 := \left( \frac{3(1 - \alpha^5)}{8(1 - \alpha^3)} \right)^{1/5}
$$

(3.38)

is the value at which the function $\dot{\tilde{a}}^2 + \frac{1}{\tilde{B}}$ vanishes. Note that, whereas the coefficient in front of $\dot{\tilde{a}}^2$ in the Friedmann equation Eq. (3.24) as well as the effective gravitational coupling diverge when one approaches $w_2$, the energy density Eq. (3.32) tends to zero as well so that the overall term does not diverge: the brane can cross smoothly $w_2$, simply with a change of sign for the energy density.

### C. Brane dynamics

To summarize, we have shown in the previous subsection that, after eliminating the explicit dependence on the brane Goldstone mode $\sigma$ and vector potential $A_\theta$ via the Maxwell junction condition, the other three junction conditions yield

$$
\sqrt{B_2^{-2} + \dot{a}^2} - \sqrt{B_1^{-2} + \dot{a}^2} = \Phi^2 G(a),
$$

(3.39)

$$
\frac{1}{\Phi^2} \left( \frac{1}{2} \dot{\Phi}^2 + V(\Phi) \right) = K(a),
$$

(3.40)

$$
\left( \frac{\dot{a} - \partial_a f(a)}{2 f(a)} \right) \left( \frac{1}{\sqrt{B_2^{-2} + \dot{a}^2}} - \frac{1}{\sqrt{B_1^{-2} + \dot{a}^2}} \right) = \Phi^2,
$$

(3.41)
where

\[ K(a) := \frac{e^2(1-\alpha)^2}{8\pi^2\rho_+^4} f(a) a^8 \left( \frac{\partial_a f}{2f} + \frac{7}{a} \right) \left( \frac{\partial_a f}{2f} - \frac{1}{a} \right), \]

\[ G(a) := -\frac{e^2(1-\alpha)^2}{4\pi^2\rho_+^4} f(a) a^8 \left( \frac{\partial_a f}{2f} - \frac{1}{a} \right). \]  

(3.42)

The above equations apply to both branes: for the northern (+)-brane, the subscripts "1" and "2" correspond to the "0" and "+" bulk solutions while for the (-)-brane, they correspond to the "−" and "0" bulk solutions, respectively.

The first equation (3.39) implies

\[ \frac{1}{4\Phi^4G^2} \left[ \Phi^4G^2 - (\tilde{B}_1^{-1} + \tilde{B}_2^{-1})^2 \right] \left[ \Phi^4G^2 - (\tilde{B}_1^{-1} - \tilde{B}_2^{-1})^2 \right] \]

\[ = \dot{a}^2 = H(a, \Phi)^2. \]

By noting that

\[ \sqrt{\tilde{B}_1^{-2} + \mathcal{H}^2} = -\frac{\Phi^4G^4 + \tilde{B}_1^{-2} - \tilde{B}_2^{-2}}{2\Phi^2G}, \]

\[ \sqrt{\tilde{B}_2^{-2} + \mathcal{H}^2} = \frac{\Phi^4G^2 + \tilde{B}_2^{-2} - \tilde{B}_2^{-2}}{2\Phi^2G} \]

(3.44)

and that the acceleration of the scale factor can be reexpressed as

\[ \dot{a} = \frac{d}{dt} \dot{a} = \frac{d}{da} \mathcal{H} \left( \frac{\partial \mathcal{H}}{\partial a} + \frac{\partial \mathcal{H}}{\partial \Phi} \frac{d\Phi}{da} \right), \]

we find, after some manipulations, that (3.41), divided by \( \dot{a}^2 \), can be reduced to

\[ \left( \frac{d\Phi}{da} \right)^2 = \frac{4\Phi^6G^3}{\Phi^4G^2 - (\tilde{B}_1^{-1} + \tilde{B}_2^{-1})^2} \left( \frac{\Phi^4G^2 - (\tilde{B}_1^{-1} - \tilde{B}_2^{-1})^2}{\Phi^4G^2 - (\tilde{B}_1^{-1} + \tilde{B}_2^{-1})^2} \right) \]

\[ \times \left( \frac{2\frac{d\Phi}{da}}{\Phi^4G^2 - (\tilde{B}_1^{-1} + \tilde{B}_2^{-1})^2} \right)^2 \left( \frac{\partial_a G}{G} - \frac{\partial_a f}{2f} \right) \].

(3.46)

This equation can be seen as a first order differential equation for the function \( \Phi(a) \). Given some initial condition \( \Phi_* = \Phi(a_*) \) at some initial point \( a_* \), one can integrate this differential equation and thus obtain \( \Phi(a) \). This solution \( \Phi(a) \) can then be substituted into (3.33), from which one can extract the evolution of the scale factor as a function of the cosmic time \( \tau \). So far, we have not used the second equation (3.40). Since the evolution of \( \Phi \) and of \( a \) can be determined solely from the two other equations, this additional equation plays the role of a constraint. This means that the potential cannot be arbitrarily chosen but is instead dictated by the geometry. Note that this is a consequence of requiring that the bulk is described by a given static solution. Thus, once the cosmological evolution is known, the potential for the radial scalar field is determined by

\[ V = -\frac{1}{2} \mathcal{H}^2 \left( \frac{d\Phi}{da} \right)^2 + \Phi^2 K(a), \]

(3.47)

which can be expressed in terms of \( \Phi \) by inverting the function \( \Phi = \Phi(a) \).

D. Behaviour near the "barrier"

As we saw earlier, \( w = w_1 \) represents an uncrossable barrier for the branes. It is thus interesting to study the cosmological behaviour of a brane that approaches this critical value. In the limit \( a \rightarrow a_1 := w_1 \rho_+ \), the function \( G \) vanishes. Keeping in the differential equation (3.40) the terms that dominate in this limit, one obtains

\[ \left( \frac{d\Phi}{da} \right)^2 \simeq \frac{8\Phi^6G^3}{(B_2^{-2} - B_1^{-2})^2} \left( \frac{1}{\Phi} \frac{d\Phi}{da} + \frac{1}{2} \left( \frac{\partial_a G}{G} - \frac{\partial_a f}{2f} \right) \right). \]

(3.48)

where one can substitute \( G(a) = G'(a_1)(a - a_1) \).

One can then easily find the approximate solution

\[ \Phi \simeq \Phi_1 + \Phi_1 (a - a_1)^2, \quad \Phi^2 = \Phi_1^6 \left( \frac{G'^3}{(B_2^{-2} - B_1^{-2})^2} \right)_{a = a_1}. \]

The potential Eq. (3.47) approaches the constant value

\[ V \simeq -\frac{1}{2} \Phi_1^2 G'(a_1), \]

(3.50)

where we have used Eq. (3.49). The above analysis applies for the (+)-brane (and possibly for the (−)-brane) in the northern region when the brane approaches \( a_1 \) from above. It is also valid in the southern region when the (−) brane approaches \( a_1 \) from below (the (+)-brane cannot be located in the southern region \( a < a_1 \) as discussed earlier in subsection B).

Before closing this subsection, we give the scale factor and the scalar field in terms of the cosmic time. In the northern region (i.e. \( a > a_1 \)), we find

\[ a - a_1 \simeq \frac{\sqrt{B_2^{-2} - B_1^{-2}}}{\Phi_1 \sqrt{2G'}} (\tau_1 - \tau)^{1/2} \]

(3.51)

where \( \tau_1 \) the time when the brane reaches \( w_1 \). Note that \( B_2^{-2} > B_1^{-2} \) in the northern region \( w_1 < a/\rho_+ < 1 \) because \( R_2 < R_1 \) \( R_+ < R_0 \) for the (+) brane or \( R_0 < R_- \) for the (−) brane.

The scalar field evolution is then given by

\[ \Phi \simeq \Phi_1 \left( 1 + \sqrt{G'(a_1)(\tau_1 - \tau)} \right). \]

(3.52)

This shows that the scalar field velocity approaches a constant and that the ratio \( \Phi/\Phi \), in this limit, is independent of the initial conditions. This analysis is similar in the limit \( a \rightarrow a_1^- \) on the southern side: one just replaces \( a - a_1 \) by \( a_1 - a \) and \( B_2^{-2} - B_1^{-2} \) by \( B_1^{-2} - B_2^{-2} \).

E. Behavior around the poles

1. (+)-brane

In order to analyse the behaviour of the differential equation for \( \Phi \), Eq. (3.46), it is convenient to introduce
the function $C_+(a)$ defined by

$$\Phi^2 = C_+^{1/2} f^{1/2}G^{-1}.$$  

(3.53)

If one assumes that $C_+$ is smooth and non-vanishing in the limit $a \to \rho_+$, one finds from Eq. (3.10) the following behaviour:

$$\frac{1}{q_+} - C_+ \simeq A_+ \left( \rho_+ - a \right)^{\Gamma_+},$$  

(3.54)

where $A_+$ is an integration constant and where we have introduced the notation

$$\frac{1}{q_+} : = \frac{(1 - \alpha)^2 \rho_+^2}{4} \left( \frac{1}{R_+} - \frac{1}{R_0} \right)^2,$$

$$\frac{1}{r_+} : = \frac{(1 - \alpha)^2 \rho_+^2}{4} \left( \frac{1}{R_+} + \frac{1}{R_0} \right)^2,$$  

(3.55)

and

$$\Gamma_+ : = \frac{1}{32} \frac{f' (\rho_+) \left( \frac{1}{r_+} - \frac{1}{q_+} \right)}{G (\rho_+)} = \frac{3}{20} \frac{1}{e^2 R_+ R_0} \frac{a^2 (1 - \alpha^5)}{1 - \alpha^8}$$

$$= \frac{3c_0}{e^2 \ell^2} \frac{(1 - \alpha) \alpha^3 (1 - \alpha^5)}{5 - 8 \alpha^3 + 3 \alpha^8},$$

(3.56)

using (3.11) in the last equality.

Note that the condition $C_+ = 1/q_+$ is equivalent to the codimension 2 limit

$$2 \pi \ell \sqrt{f(a)} \Sigma_+ = \delta_N,$$  

(3.57)

where $\delta_N$ is the deficit angle at the northern pole $\rho = \rho_+$, which can be related to the coefficients $c_0$ and $c_+$ [12, 13]

$$\frac{c_0}{c_+} = \frac{1}{2 \pi} \frac{\delta_N}{3}.$$  

(3.58)

In order to express the scale factor in terms of the cosmic time $\tau$, the simplest way is to use Eq. (3.43), which yields

$$a^2 = \left( \frac{C_+ - \frac{1}{q_+}}{4 C_+} \right) \left( \frac{C_+ - \frac{1}{r_+}}{4 C_+} \right) f$$

$$\simeq - \frac{f' (\rho_+)}{4} q_+ \left( \frac{1}{r_+} - \frac{1}{q_+} \right) A_+ (\rho_+ - a)^{\Gamma_+ + 1}.$$  

(3.59)

Then, expanding around $C_+ \simeq 1/q_+$ and substituting the behaviour Eq. (3.54) into the above expression, we obtain

$$\rho_+ - a \simeq B_+ (\tau_+ - \tau)^{\beta_+}$$  

(3.60)

where

$$\beta_+ = \frac{2}{1 - \Gamma_+},$$

$$B_+ = \frac{q_+ \left( \frac{1}{r_+} - \frac{1}{q_+} \right) (- f' (\rho_+)) A_+}{4 \beta_+^2}$$  

(3.61)

Thus for $\Gamma_+ < 1$, we obtain $\beta_+ > 2$, namely the velocity and acceleration are vanishing at the position of the conical singularity. For the $(+)$-brane, we have the condition $c_+ > c_0$ ($R_+ < R_0$) and thus

$$e^2 \ell^2 \Gamma_+ < \frac{60 \alpha^3 (1 - \alpha^2)^2 (1 - \alpha^3) (1 - \alpha^5)}{(5 - 8 \alpha^3 + 3 \alpha^8)^2} \leq \frac{1}{4}.$$  

(3.62)

One can thus get $\Gamma_+ < 1$ for sufficiently large $e^2 \ell^2$. Noting that

$$\mathcal{H} \simeq \frac{2}{1 - \Gamma_+} B_+ (1 - \Gamma_+)^{1/2} (\rho_+ - a)^{(1 + \Gamma_+)^{1/2}},$$

we find that, if $\Gamma_+ < 1$, the brane reaches the northern pole in a finite time. Indeed,

$$\langle \Delta \tau \rangle_+ = \int_{\alpha_0}^{\rho_+} \frac{da}{\mathcal{H}(a, \Phi(a))},$$

(3.64)

is finite if $(1 + \Gamma_+)^2/2 < 1$ ($\alpha_0$ is the initial position of the brane). Thus, it is impossible to obtain an ever expanding brane Universe.

2. $(-)$-brane

The analysis of the opposite limit $a \to \rho_- = \alpha \rho_+$ is quite similar to the previous case. In the region $\rho_- < a < w_1 \rho_+$, we always have $G(a) < 0$, and it is now convenient to introduce $C_- (a)$ defined by

$$\Phi^2 = C_- (a)^{1/2} f^{1/2} (-G)^{-1}.$$  

(3.65)

Assuming $C_-$ is non-vanishing and regular in the limit $a = \rho_-$, one can easily find the behaviour

$$\frac{1}{q_-} - C_- \simeq A_- (a - \rho_-)^{\Gamma_-}$$

(3.66)

where $A_-$ is an integration constant and

$$\Gamma_- : = \frac{1}{32} \frac{f' (\rho_+) \left( \frac{1}{r_-} - \frac{1}{q_-} \right)}{G (\rho_+ \alpha)} = \frac{3}{20} \frac{1}{e^2 R_- R_0} \frac{1}{\alpha^5}$$

$$= \frac{3c_0}{e^2 \ell^2} \frac{(1 - \alpha) (1 - \alpha^3)}{5 \alpha^8 - 8 \alpha^3 + 3},$$

(3.67)

with $q_-$ and $r_-$ defined as in (3.55) by replacing $R_+$ by $R_-$. Note that the condition $C_- = 1/q_-$ is equivalent to the conical limit

$$2 \pi \ell \sqrt{f(a)} \Sigma_- = \delta_S,$$  

(3.68)

where $\delta_S$ is the deficit angle at the southern pole and we have used the relation [12, 13]

$$\frac{c_0}{c_-} = 1 - \frac{\delta_S}{2 \pi}.$$  

(3.69)
The expression of the scale factor in terms of the cosmic time is given by
\[ a - \rho_+ \simeq B_-(\tau_- - \tau)^{\beta_-} \]  \hspace{1cm} (3.70)

with
\[ \beta_- = \frac{2}{1 - \Gamma_-}, \]
\[ B_- = \left( \frac{q_-(\frac{1}{q_+} - \frac{1}{q_-})}{4\beta_-^2} \right)^{1/(1 - \Gamma_-)} \] \hspace{1cm} (3.71)

and we find, as near the northern conical singularity, that the brane approaches \( \rho_- \) in a finite time if \( \Gamma_- < 1 \).

F. Numerical example

In order to illustrate the cosmological behaviour of the branes and to confirm our analytical estimates in the various limits discussed above, we have integrated numerically (3.46) for the (+)-brane by starting from some initial condition \( \Phi^\ast = \Phi(a^\ast) \). Our choice of parameters is the following:
\[ \alpha = 0.1, \quad e\rho_+ = 1, \quad R_+/\rho_+ = 4.0, \quad R_0/\rho_+ = 5.0 \] \hspace{1cm} (3.72)

This gives \( w_1 = 0.11696 \). The initial position and initial amplitude are set to be
\[ a^\ast = 0.2\rho_+, \quad \Phi = 0.01 \frac{M_6^2}{e\sqrt{\rho_+}}. \] \hspace{1cm} (3.73)

From this initial position, we have integrated for decreasing values of \( a \) until reaching \( w_1 \) as well as for increasing values of \( a \) up to the northern conical singularity.

We have plotted the profile of \( \Phi(a) \) in Fig. 2. We observe that the scalar field vanishes at the conical singularity and approaches a constant value, with vanishing speed, at \( w = w_1 \), which is consistent with our analytical solutions.

In Fig. 3 we have plotted the effective potential and total energy densities, respectively \( \sqrt{T}V(\Phi) \) (blue, dashed curve) and \( \sqrt{T}\Sigma \) (red, solid curve) as functions of \( a \) (in units of \( \rho_+ \)) for the solution of Fig. 2. Note that the total energy density is vanishing at \( a = \rho_+ w_1 \) while the potential \( V(\Phi) \) is negative at the same point.

Our numerical example corresponds to the case where the brane reaches the conical singularity in a finite time as discussed earlier.

IV. EINSTEIN-MAXWELL-DILATON MODEL

A. Bulk spacetime in 6D Einstein-Maxwell-dilaton model

In this section, we consider a slightly different model based on solutions of the 6D Einstein-Maxwell-dilaton theory, which now include a dilaton denoted \( \varphi \). Our bulk action is based on the bosonic part of the Nishino-Sezgin supergravity [11]:
\[ S_B = \int d^6x\sqrt{-g}\left( \frac{1}{2}R - \frac{1}{2}(\partial \varphi)^2 - \frac{1}{4}e^{-\varphi}F_{AB}F^{AB} - 4g_0^2e^\varphi \right). \] \hspace{1cm} (4.1)
where the other fields can be consistently set to zero. It is possible to find exact solutions, given by [1]

\[ ds^2 = (2\rho + w)\eta_{\mu \nu}dx^\mu dx^\nu + \frac{\rho_+}{2g_0} \left( \frac{4dw^2}{(1 - \alpha)^2f(w)} + f(w)c_0^2d\theta^2 \right), \quad (4.2) \]

\[ F_{\theta \phi} = -\frac{\sqrt{2c_0}}{(1 - \alpha)g_0w}, \quad \varphi = -\ln(2\rho + w), \quad (4.3) \]

where

\[ f(w) = \frac{2(1 - w^2)(w^2 - \alpha^2)}{w^3(1 - \alpha)^2}. \quad (4.4) \]

The geometry possesses two conical singularities at \( w = \alpha \) and \( w = 1 \), respectively. As in the non-dilatonic case, we regularize the corresponding codimension 2 branes by introducing two codimension-1 branes at the respective positions \( w_+ > \alpha \) and \( w_- < 1 \), delimiting two regular caps. The northern cap \((w_+ < w < 1)\) and the southern cap \((\alpha < w < w_-)\), can be described by solutions similar to that of the main bulk, i.e. of the form \((4.2) - (4.3)\), but with parameters \( g_\pm \) and \( c_\pm \) that differ from \( g_0 \) and \( c_0 \).

The regularity of the caps imposes [12]

\[ c_+ = \frac{1}{1 + \alpha}; \quad c_- = \frac{\alpha^2}{1 + \alpha}, \quad (4.5) \]

while the continuity of the \((\theta, \theta)\) component requires

\[ \ell^2 = \frac{e^2}{2g_+^2} = \frac{e^2}{2g_-^2} = \frac{e^2}{2g_0^2}. \quad (4.6) \]

The solutions for the bulk gauge field around the north and south poles are given by

\[ A_0^{(N)} = \frac{\alpha \ell}{1 - \alpha}(\frac{1}{w^2} - 1), \quad A_0^{(S)} = \frac{\alpha \ell}{1 - \alpha}(\frac{1}{w^2} - \frac{1}{\alpha^2}) \quad (4.7) \]

The difference should be a pure gauge term, which implies the Dirac quantization condition

\[ N = \frac{\ell(1 + \alpha)}{\alpha}. \quad (4.8) \]

### B. Junction conditions and brane dynamics

We describe the brane matter by an action of the form

\[ S_{b, \pm} = -\int d^5x\sqrt{-g}\left[ V(\Phi_{\pm}, \varphi) + \frac{1}{2} \xi(\varphi) \left( (\partial_\alpha \Phi_{\pm})^2 + \Phi_{\pm}^2 \left( \partial_\alpha \sigma_{\pm} - eA_0 \right)^2 \right) \right]. \quad (4.9) \]

It is similar to the non-dilatonic case with two differences: the potential \( V(\Phi_{\pm}, \varphi) \) now depends on the dilaton, and we have introduced a coupling \( \xi(\varphi) \), assumed to be strictly positive, between the kinetic term of the scalar field and the bulk dilaton. The energy density \( \Sigma \) and the pressures along the external and internal directions, respectively \( p \) and \( p_\theta \), are defined like in \((3.15)-(3.17)\), with the only difference that the potential now depends on the dilaton \( \varphi \), in addition to the scalar field \( \Phi \). The comparison of the metric \((4.2)\) with Eq. \((2.1)\) yields the identification

\[ A^2 = 2\rho + w, \quad B^2 = \frac{2\rho_+}{g_1^2(1 - \alpha)^2f(w)}, \]

\[ C^2 = \frac{\rho_+}{2g_1^2}f(w) = \rho_+\ell^2f(w), \quad (4.10) \]

and the scale factor of the brane is, as before, \( a(\tau) \equiv A(w_b(\tau)) \). As in the previous section, we also define the rescaled metric component

\[ \tilde{B}(A) = B\frac{dw}{dA} = \left( \frac{2\rho + w}{g_1^2(1 - \alpha)} \right)^{1/2}. \quad (4.11) \]

The junction conditions for the metric are exactly the same as in the previous section, and are thus given by \((3.15)-(3.17)\). They imply in particular the generalized Friedmann equation

\[ \frac{\dot{a}^2}{a^2} = \frac{\tilde{\rho}^2}{16\pi^2\ell^2 \rho_+ f(3 + \frac{\ell}{\alpha^2})^2} + \frac{\pi^2\ell^2 f(\rho_+ (3 + \frac{\ell}{\alpha^2})^2 (\tilde{B}_- - \tilde{B}_0)^2)^2 - \tilde{B}_+ + \tilde{B}_0}{2a^2}, \]

\[ a^4 \quad (4.12) \]

where \( \tilde{\rho} := 2\ell\sqrt{\rho_+ \sqrt{\Sigma}} \). The Friedmann equation is very similar to the one obtained in the Einstein-Maxwell case and one would find, by repeating the same procedure as in the previous section, that the low-energy behaviour differs from standard cosmology.

The Maxwell junction condition, however, is modified because the dilaton is explicitly coupled to the electromagnetic field. It now reads

\[ e^{-\varphi} \mathcal{F}(a) \left[ \tilde{X} \right] = -e\xi(\varphi)\Phi^2(\partial_\theta \sigma - eA_0), \quad (4.13) \]

where we have defined

\[ \mathcal{F}(a) := \left( \frac{dA}{dw} \right)^{-1} |_{F_{\theta \phi}} = -\frac{16\alpha \ell \rho_+^2}{1 - \alpha^2 a^5}. \quad (4.14) \]

Moreover, we have an additional junction condition, that of the dilaton, which is given by

\[ \tilde{\varphi}(a) \left[ \tilde{X} \right] = \frac{\partial V}{\partial \varphi} + \frac{1}{2} \xi(\varphi) \left( -\Phi^2 + \Phi^2 C^{-2} \left( \partial_\theta \sigma - eA_0 \right)^2 \right), \]

\[ \tilde{\varphi}(a) := \left( \frac{dA}{dw} \right)^{-1} \varphi, w = -\frac{2}{a}. \quad (4.15) \]

As in the Einstein-Maxwell model, the bulk is divided in two by a limit that cannot be crossed by the branes. In the present case, the position of this limit is given by

\[ w_1 := \sqrt{\frac{2}{1 + \alpha^2} \alpha}. \quad (4.16) \]
Similarly, the total energy density in the brane is negative when
\[ w_1 < \frac{a^2}{2\rho_+} < w_2 := \sqrt{\frac{1 + \alpha^2}{2}}, \]  
(4.17)
and positive otherwise. As in the case of the Einstein-Maxwell model, the (+)-brane cannot be in the region \((2\rho_-)^{1/2} < a < (2\rho_+ w_1)^{1/2}\), where \(\rho_- := \rho_+ \alpha\). The brane can also cross \(w_2\) simply with a change of the sign of the energy density.

Following the same method as in the Einstein-Maxwell case, one can rewrite the junction conditions as
\[
\frac{1}{\phi^2} \left[ \frac{1}{2} \xi(\phi) \phi^2 + V(\phi, \varphi) \right] = \xi(\phi) K(a), 
\]
(4.18)
where
\[
\xi(\phi) = \sqrt{\phi^2 + \xi(\phi) \phi^2}, 
\]
(4.19)
and the differential equation for \(\phi\) is now
\[
\left( \frac{d\phi}{da} \right)^2 = \frac{4\phi^4 G^2 \xi(\phi)^3}{(\phi^4 G^2 \xi(\phi)^2 - (\tilde{B}_1^{-1} + \tilde{B}_2^{-1})^2)(\phi^4 G^2 \xi(\phi)^2 - (\tilde{B}_1^{-1} - \tilde{B}_2^{-1})^2)} \times \left( 2 \frac{d\phi}{da} + \frac{\partial_\phi G}{G} + \partial_\phi \ln \left( \frac{a \xi(\phi)}{\sqrt{f}} \right) \right). 
\]
(4.20)

The junction condition (4.19) implies
\[
V = \xi(\phi) \left( -\frac{1}{2} \mathcal{H}^2 \left( \frac{d\phi}{da} \right)^2 + \phi^2 K(a) \right), 
\]
(4.21)
which provides an expression for the potential \(V\) as a function of \(a\). The potential depends on both the scalar field \(\phi\) and the dilaton \(\varphi\), and it is possible to disentangle the dependence on \(\phi\) and \(\varphi\) by using the dilaton junction condition Eq. (4.15) with Eq. (4.18), which yields the following expression for the partial derivative of the potential with respect to the dilaton:
\[
\frac{\partial V}{\partial \varphi} = -\Phi(a)^2 G(a) \left( \frac{2}{a} \xi(\phi(a)) - \frac{1}{2} \xi'(\phi(a)) \left( \frac{f'}{2f} - \frac{1}{a} \right) \right) 
+ \frac{1}{2} \xi'(\phi(a)) \left( \frac{d\phi}{da} \right)^2 \mathcal{H}(a, \Phi(a))^2. 
\]
(4.22)

C. Behaviour near the ”barrier”

We now analyze the behavior near the critical point \(a \to a_1 := (2\rho_+ w_1)^{1/2}\). The differential equation of the scalar field Eq. (4.24) becomes in the limit \(\Phi^4 G^2 \xi^2 \to 0\)
\[
\xi(\phi) \left( \frac{d\phi}{da} \right)^2 
\approx \frac{8\Phi^4 G^3 \xi^3}{(B_2^{-2} - B_1^{-2})^2} \left[ \frac{1}{\Phi} \frac{d\phi}{da} + \frac{1}{2} \frac{\partial_\phi G}{G} + \partial_\phi \ln \left( \frac{a \xi(\phi)}{\sqrt{f}} \right) \right]. 
\]
(4.23)
We expand \(G(a) = G'(a_1)(a-a_1)\) and \(K(a) = K'(a_1)(a-a_1)\), with \(G'(a_1) > 0\) and \(K'(a_1) < 0\). Then, assuming that \(\xi\) tends smoothly towards a finite value \(\xi_1 := \xi(\varphi(a_1))\), we find the following approximate behaviour
\[
\Phi \approx \Phi_0 + \Phi_1 (a-a_1)^2, \quad \Phi_1 = \frac{\xi(\varphi) \Phi_0 G'(a_1)}{(B_2^{-2} - B_1^{-2})^2} |_{a=a_1}, 
\]
(4.24)
The potential is given by
\[
V \approx -\Phi_0^2 \xi(\varphi) G'(a_1)^2 |_{a=a_1} < 0, 
\]
(4.25)
where we have used Eq. (4.28). Thus, the potential is finite and negative in the limit \(a \to a_1\).
We now derive the expression for the scale factor in terms of the brane proper time. In the case $a \to a_+^+$, we obtain

$$a - (2\rho_+ w_1)^{1/2} = \frac{\sqrt{B_2^2 - B_1^2}}{\Phi_0 \sqrt{G'\xi(\varphi)}} \big|_{a = (2\rho_+ w_1)^{1/2}} (\tau - \tau)^{1/2}$$

(4.30)

where $\tau_1$ represents the time when the brane reaches $a_1$. Note that $B_2^2 > B_1^2$ for $a_1 < a < (2\rho_+)^{1/2}$ because $g_2 > g_1$ ($g_+ > g_0$). The scalar field configuration is given by

$$\Phi = \Phi_0 \left( 1 + \sqrt{G'\xi(\varphi)} \right) \big|_{a = (2\rho_+ w_1)^{1/2}} (\tau_1 - \tau).$$

(4.31)

Thus the scalar field has a constant velocity. The brane terminates its motion at the point. The analysis $a \to a_-$ can be done in the similar manner: one just replaces $a - a_1$ by $a_1 - a$ and $B_2^{-2} - B_1^{-2}$ by $B_1^{-2} - B_2^{-2}$.

D. Behaviour around the poles

In the limit $a \to (2\rho_+)^{1/2}$, it is useful to rewrite the scalar field as

$$\Phi^2 = C_+(a)^{1/2} \frac{f^{1/2}}{a\xi(\varphi)} G^{-1}.$$  

(4.32)

If $C_+(a)$ is a smooth and non-vanishing function near $a = (2\rho_+)^{1/2}$, an approximate solution of the differential equation is given by

$$\frac{1}{q_+} - C_+ \simeq A_+ ((2\rho_+)^{1/2} - a)^{\Gamma_+},$$

(4.33)

where

$$\frac{1}{q_+} := \frac{(1 - \alpha)^2 \rho_+}{2} (g_+ - g_0)^2,$$

$$\frac{1}{r_+} := \frac{(1 - \alpha)^2 \rho_+}{2} (g_+ + g_0)^2,$$

and

$$\Gamma_+ = \frac{1}{32} a^2 \xi G (2\rho_+)^{1/2} \left( \frac{1}{r_+} - \frac{1}{q_+} \right) = \frac{\alpha^2}{c_0 e^2 \xi \ell^2} \frac{c_0}{1 + \alpha}.$$  

(4.34)

Note that the condition $C_+ = 1/q_+$ is equivalent to the conical limit

$$2\pi \sqrt{\rho_+ \ell} \sqrt{f(a) G} \big|_{a \to \sqrt{2\rho_+}} = \delta_N,$$  

(4.35)

where $\delta_N$ represents the deficit angle at the northern pole. The expression for the scalar field in terms of $\tau$ is given by

$$(2\rho_+)^{1/2} - a \simeq B_+ (\tau_+ - \tau)^{\beta_+},$$

(4.36)

where

$$\beta_+ = \frac{2}{1 - \Gamma_+},$$

$$B_+ = \left( \frac{q_+ (1 - \frac{1}{q_+}) A_+ - f'(2\rho_+)^{1/2}}{4\beta_+^2} \right)^{1/(1 - \Gamma_+)}.$$  

(4.37)

For the (+)-brane, we have the condition $c_+ > c_0$ ($g_+ > g_0$) and a discussion similar to that of the Einstein-Maxwell model tells us that it is possible, with a sufficiently large charge for the brane, to have $\Gamma_+ < 1$, in which case the brane reaches the northern pole in a finite time. Thus, one does find an ever expanding brane Universe.

The results for the opposite limit $a \to (2\rho_-)^{1/2}$ are quite similar to the above results. Using the decomposition,

$$\Phi^2 = C_- (a)^{1/2} \frac{f^{1/2}}{a\xi(\varphi)} (-G)^{-1},$$

(4.38)

one can identify the behaviour

$$\frac{1}{q_-} - C_- \simeq A_- (a - (2\rho_-)^{1/2})^{\Gamma_-},$$

(4.39)

where

$$\Gamma_- = \frac{1}{32} a^2 \xi G (2\rho_-)^{1/2} \left( \frac{1}{r_-} - \frac{1}{q_-} \right) = \frac{1}{e^2 \xi \ell^2} \frac{c_0}{1 + \alpha}$$

(4.40)

and $q_-$ and $r_-$ are defined as in (4.34) by replacing $g_+$ with $g_-$. Note that the condition $C_- = 1/q_-$ corresponds to

$$2\pi \sqrt{\rho_- \ell} \sqrt{f(a) G} \big|_{a \to -\sqrt{2\rho_-}} = \delta_S,$$  

(4.41)

where $\delta_S$ denotes the deficit angle at the southern pole. In terms of the cosmic proper time $\tau$, we find

$$(a - (2\rho_-)^{1/2})^{1/2} = B_- (\tau_- - \tau)^{\beta_-},$$

(4.42)

where

$$\beta_- = \frac{2}{1 - \Gamma_-},$$

$$B_- = \left( \frac{q_- (1 - \frac{1}{q_-}) A_- - f'((2\rho_-)^{1/2})}{4\beta_-^2} \right)^{1/(1 - \Gamma_-)}.$$  

(4.43)

For the (-)-brane, $c_- > c_0$ ($g_- > g_0$) and one can have $\Gamma_- < 1$, in which case the brane reaches the south pole in a finite time.
V. DISCUSSIONS

In this work, we have investigated the dynamics of the regularized branes in a 6D bulk. We considered models based on either the Einstein-Maxwell theory or Einstein-Maxwell-dilaton theory (more precisely a bosonic part of Nishino-Sezgin supergravity). In both models, the original systems are composed of a warped bulk bounded by codimension 2 branes located at the poles. In order to introduce non-trivial matter on the branes we have regularized the codimension 2 branes, located at the poles, by replacing them with ring-like codimension 1 branes.

The brane matter is composed of a complex scalar field coupled to the bulk $U(1)$ gauge field (and to the bulk dilaton in the dilatonic model). Such matter was used for a static (fixed) brane, where the radial mode of the complex scalar field was assumed to be stabilized at a local minimum of its potential and thus frozen at a fixed vacuum expectation value $\bar{\rho}$. For moving branes, one must allow for a time dependence of the radial mode and this radial mode controls the dynamics of the brane in the bulk.

We have studied the dynamics of such branes in the two types of models mentioned above. In both models, branes exhibit very similar behaviors and their essential features can be summarized as follows. First, there is a critical radius, translated into a critical scale factor $a_1$ given by $w_1\rho_+$ in the Einstein-Maxwell model (see Eq. (3.37)) and $\sqrt{2\rho_+\rho_1}$ in the Einstein-Maxwell-dilaton model (see Eq. (4.10)), which cannot be crossed by a brane: an expanding brane with $a < a_1$ at some time cannot expand beyond $a_1$; conversely, a contracting brane with $a > a_1$ cannot shrink below $a_1$. Investigating the motion of a brane close to this critical point, we have found that a brane reaches the point $a_1$, within a finite time, with a divergent velocity ($\dot{a} \propto |\tau - \tau_1|^{-1/2}$, where $\tau$ is the proper time on the brane). Therefore, the Hubble parameter of the brane diverges at $a = a_1$. Second, near the poles, we have found behaviours where the brane reaches the conical singularity within a finite time with vanishing velocity and acceleration. This occurs if the brane charge is sufficiently large.

For the Einstein-Maxwell model, we have solved the brane motion numerically and confirmed our analytic results in both limits. In both types of models, the size of the internal dimension goes to zero as the brane approaches a conical singularity or to a constant as the brane approaches the critical value $a_1$. These conclusions were obtained by assuming a complex scalar field as brane matter and it would be worthwhile investigating the cosmological behaviour for other kinds of matter.

More generally, i.e. independently of the specific brane matter content, we have seen that the Friedmann equations Eqs. (3.24) and (4.12), in the Einstein-Maxwell and Einstein-Maxwell-dilaton models, respectively, do not behave like standard cosmology in the low energy limit, because the effective gravitational coupling, Eq. (4.7), is strongly time-dependent (and even negative in the vicinity of the pole). Thus, our results show that, in contrast with the 5D Randall-Sundrum cosmology in the low-energy limit, the dynamics of the brane in the present setup is not compatible with the usual cosmological evolution and modifications are necessary to obtain a sensible cosmology. One possible modification would be to consider time-dependent bulk solutions $\bar{\rho}$ instead of the static geometries considered here. \(^2\) We will report these investigations in future publications.

VI. ADDENDUM

While this paper was being completed, we became aware of a related work \(^{23}\), which studies the cosmological behaviour in the same type of models as those discussed here. As far as we can see, our results agree with those of \(^{23}\). The two works are also complementary in many respects.

Acknowledgements

We would like to thank A. Papazoglou for instructive discussions in the latest stage of this work. DL and MM were partially supported by the CNRS-JSPS exchange programme, and the work of MM was also supported in part by the project "Dark Universe" at the ASC. MM is grateful for the warm hospitality at IAP and APC. DL would like to thank the Yukawa Institute for Theoretical Physics, Kyoto University, for their warm hospitality.

APPENDIX A: BRANE EQUATIONS AND QUANTUM NUMBERS

1. Einstein-Maxwell model

The dynamics of the radial scalar field on the brane is given by

$$\ddot{\Phi} + \dot{A} \left( \frac{3}{A} + \frac{C'(A)}{C} \right) \dot{\Phi} + V_\Phi + \frac{1}{C^2} \left( \partial_\theta \sigma - e A_\theta \right)^2 \Phi = 0.$$  \hspace{1cm} (A1)

The equation of motion of $\Phi$ field is consistent with the energy momentum conservation law on the brane.

$$\ddot{\Sigma} + \frac{3 \dot{A}}{A} \left( \Sigma + p \right) + \frac{C' \dot{A}}{C} \left( \Sigma + p_\theta \right) = \left[ T_{AB} u^A n^B \right].$$  \hspace{1cm} (A2)

with

$$\left[ T_{AB} u^A n^B \right] = \dot{A} \left( F_{AB} \right)^2 \left[ X \right] C^{-2}$$

$$= \dot{A} C^{-2} e^{2\Phi} \left( \partial_\theta \sigma - e A_\theta \right) F_{AB},$$  \hspace{1cm} (A3)

\(^2\) Low energy behaviour of brane gravity in the Einstein-Maxwell model was discussed recently in Ref. \(^{22}\).
The equation of motion for the Goldstone modes on the branes is given by
\[ \partial_a \left( \sqrt{-g} \Phi^2 q^{ab} (\partial_b \sigma - e A_b) \right) = 0 , \] (A4)

which implies \( \partial_\theta^2 \sigma_\pm = 0 \). The equation of motion for the axial mode \( \sigma \) can be solved as \( \sigma_\pm = n_\pm \theta \). From Eq. (3.12), (3.29), (3.31), one finds that the brane quantum numbers are given by
\[ n_+ = \frac{N}{2} \left( 5 - 8 \alpha^3 + 3 \alpha^8 \right) , \]
\[ n_- = -\frac{N}{2} \left( 5 - 8 \alpha^3 + 3 \alpha^8 \right) , \] (A5)

where we have also used Eq. (3.13). Thus, between quantum numbers, we have the following relations:
\[ n_+ - n_- = N . \] (A6)

Note that the same relations as Eq. (A5) are obtained in the case of the static brane. Due to the redundancy of the junction condition, they appear as a constraint between the model parameters.

2. Einstein-Maxwell-dilaton model

The dynamics of the radial scalar field on the brane is given by
\[ \xi \dot{\Phi} + \xi A \left( \frac{3}{A} + \frac{C'(A)}{C} \right) \dot{\Phi} + \xi \dot{\varphi} \dot{\Phi} + V_\Phi + \xi \frac{C}{C^2} (\partial_b \sigma - e A_b)^2 \Phi^2 = 0 . \] (A7)

The equation of motion of \( \Phi \) field is consistent with the energy momentum conservation law on the brane:
\[ \Sigma + \frac{3 A}{2} \left( \Sigma + p \right) + \frac{C' A}{C} \left( \Sigma + p_\theta \right) = \left[ T_{AB} u^A n^B \right] . \] (A8)

where
\[ \left[ T_{AB} u^A n^B \right] = A \left\{ e^{-\varphi} (F_{\alpha \beta})^2 C^{-2} + (\varphi_A)^2 \right\} \left[ X \right] = -\varepsilon \xi \dot{A} C^{-2} \Phi^2 (\partial_\theta \sigma - e A_\theta) e^{-\varphi} F_{\alpha \beta} + \varphi_{,A} V_{,A} + \frac{1}{2} \zeta (\varphi) \varphi_A A \left( -\dot{\Phi}^2 + \Phi^2 C^{-2} (\partial_\theta \varphi - e A_\theta) \right) , \] (A9)

and at the final step we used the Maxwell and dilaton junction conditions.

The equation of motion for the Goldstone modes on the brane is again given by
\[ \partial_a \left( \sqrt{-g} \xi^2 q^{ab} (\partial_b \sigma - e A_b) \right) = 0 , \] (A10)

which implies \( \partial_\theta^2 \sigma = 0 \). The solution is also given by \( \sigma = n_\pm \theta \). The quantum numbers \( n_\pm \) are found to be given by
\[ n_\pm = \pm \frac{N}{2} , n_+ - n_- = N . \] (A11)
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