Abstract—This article studies social system inference from a single noisy trajectory of public evolving opinions, wherein observation noise leads to the statistical dependence of samples on time and coordinates. We first propose a cyber-social system that comprises individuals in a social network and a set of information sources in a cyber layer, whose opinion dynamics explicitly takes the asymmetric cognitive bias including confirmation bias and negativity bias and the process noise into account. Based on the proposed cyber-social model, we then study the sample complexity of least-square auto-regressive model estimation, which governs the length of a single observed trajectory that is sufficient for the identified model to achieve the prescribed levels of accuracy and confidence (PAC). Building on the identified social model, we then investigate social inference, with a particular focus on the weighted network topology and the model parameters of asymmetric cognitive bias. Finally, the theoretical results and the effectiveness of the proposed inference framework are validated by the U.S. Senate Member Ideology data.

Index Terms—Asymmetric confirmation bias, asymmetric negativity bias, network topology, sample complexity, social inference.

I. INTRODUCTION

Dynamical network identification from observed nodal states, with a particular focus on graph topology identification/reconstruction, has gained widespread attention in a wide variety of fields, ranging from power networks [1] to social networks [2]. However, in social networks, individual cognitive behaviors, e.g., confirmation bias [3], preferences for outlying content [4] and distancing and striving for uniqueness [5], in conjunction with process and observation noise, pose a formidable challenge to social network topology identification. Therefore, a fairly accurate social model which explicitly takes cognitive behaviors as well as process and observation noise into account is indispensable for the social topology identification with prescribed levels of accuracy and confidence (PAC).

The dynamics of the opinion formation or information spread in networks has been an important research subject for decades. A few well-known models include DeGroot model [6] that considers opinion evolution within a network in terms of the weighted average of individuals’ connections where weights are determined by influences. Friedkin-Johnsen model [7] incorporates individual innate opinion or subconscious bias, thereby making the model more suitable to several real-life scenarios, as well as real applications, e.g., debiasing social influence [8]. While the social information diffusion dynamics has always been an active research area, recently with the wide use of social media [9], in conjunction with automated news generation with the help of artificial intelligence technologies [10], [11], it has gained vital importance in studying misinformation spread and political polarization. In this regard cognitive bias, especially, the confirmation bias and novelty bias play a key role. Concretely, it is well understood that confirmation bias helps create “echo chambers” within online social networks [12], [13], in which misinformation and polarization thrive. Recently, Xu et al. [9] and Abdelzaher et al. [14] reveal the significant influence of consumer preferences for outlying content (due to novelty bias) on the opinion polarization in the modern era of information overload. Hence, the challenge in opinion dynamics moving forward is How to capture human cognitive bias in information consumption?

Though imposing bounded confidence on social influence, Hegselmann–Krause (HK) model [15] has the capability of capturing confirmation bias [16]. The HK model involves a discontinuity in the influence impact, i.e., an individual completely ignores the opinions that are “too far” from hers, which renders the steady-state analysis difficult. As a remedy, the continuous state-dependent social influence models proposed in [17], [18], [19], [20], and [21] to study the polarization and homogeneity are sensed in capturing confirmation bias as well, since both polarization and homogeneity are the results of the conjugate effect of confirmation bias and social influence [16], [22]. However, in many social problems, e.g., president election and product rating, humans hold the asymmetric cognitive bias, while the HK model with symmetric confidence (PAC).
Following observations on time and coordinates, we investigate the sample complexity of the proposed social system estimation.

3) Building on social system estimation, we drive a social-system inference procedure for weighted network topology and model parameters of cognitive bias.

4) We validate the theoretical results and the effectiveness of the proposed opinion evolution model by U.S. Senate Member Ideology data.

II. PRELIMINARIES

A. Notation

We let \( \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \) denote the set of \( n \)-dimensional real vectors and the set of \( m \times n \)-dimensional real matrices, respectively. \( \mathbb{N} \) stands for the set of natural numbers. We let \( \mathbf{0} \) denote the vector of all zeros, compatible dimensions. We define \( \mathbf{I}_n \) as \( n \)-dimension identity matrix. The superscript \( ^\top \) stands for the matrix transposition. For a matrix \( \mathbf{W} \), \( [\mathbf{W}]_{i,j} \) denotes the element in row \( i \) and column \( j \). For vectors \( x \) and \( y \), \( [x; y] = [x^\top, y^\top]^\top \). The \( \sigma \)-algebra is denoted by \( \sigma(\cdot) \).

Other important notations are highlighted as follows.

1) \( ||A|| \) denotes the spectral norm of matrix \( A \).
2) \( ||A||_F \) denotes the Frobenius norm of matrix \( A \).
3) \( ||x|| \) denotes the Euclidean norm of vector \( x \).
4) \( \mathbf{E} \) denotes the expectation operator.
5) \( \mathcal{S}^{n-1} \) denotes the unit sphere in \( \mathbb{R}^n \).
6) \( \Omega^c \) denotes the complement of event \( \Omega \).
7) \( \mathbf{P}(\Omega) \) denotes the probability of event \( \Omega \).
8) \( \lambda_{\min}(A) \) denotes the minimum eigenvalue of symmetric matrix \( A \).

The social system is composed of \( n \) individuals in social network and \( m \) information sources. The interaction among individuals is modeled by a digraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{v_1, \ldots, v_n\} \) is the set of vertices representing the individuals, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the set of edges of the digraph \( \mathcal{G} \) representing the influence structure. The communication from information sources to individuals is modeled by a bipartite digraph \( \mathcal{B} = (\mathcal{V} \cup \mathcal{K}, \mathcal{E}) \), where \( \mathcal{K} = \{\kappa_1, \ldots, \kappa_m\} \) is the set of vertices representing information sources, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{K} \) is the set of edges of the digraph.

B. Opinion Evolution With Cognitive Behavioral Asymmetry

We consider the following social-cognitive model which is adopted from [19] and [32] based on Friedkin-Johnsen model [7]:

\[
x_g(t+1) = a_g(t)x_g(t) + \sum_{j \in \mathcal{V}} w_{gj}x_j(t) + \sum_{d \in \mathcal{K}} c(\bar{x}_g(t), h_d(t))h_d(t) + p_g(t) \\
y_g(t) = x_g(t) + \alpha_g(t), \quad g \in \mathcal{V}, \ t \in \mathbb{N} \tag{1a}
\]

Here we clarify the notations and variables.

1) \( x_g(t) \in [-1, 1] \) is individual \( v_g \)’s opinion; \( y_g(t) \in [-1, 1] \) is observed her opinion for inference; \( h_d(t) \in [-1, 1] \) is information source \( u_d \)’s opinion at time \( t \).
2) $p_g(t)$ denotes process noise due to model error and uncertainty, $o_g(t)$ denotes observation noise.
3) $w_{gj}$ represents the influence of individual $v_j$ on $v_g$, and

$$w_{gj} = \begin{cases} > 0, & \text{if } (v_i, v_j) \in E \\ = 0, & \text{otherwise.} \end{cases}$$

We note the individual-individual influence weights $w_{gj}$ are cognition- or knowledge-trust based and thus fixed over time, since the cognitive factors that can influence trust decisions are founded on a deeper knowledge of the other person and the stability of the other’s behavior across time and contexts, which tends to vary little over a long period of time [33].

4) The state-dependent influence weight $c(x_g(t), h_d(t)) \geq 0$ models asymmetric cognitive bias as

$$c(x_g(t), h_d(t)) = 1 - v_g|x_g(t) - \sin\left(\frac{\pi}{2}h_d(t)\right) - \beta_x x_g(t) - \tan\left(\frac{\pi}{4}h_d(t)\right)$$

where $\beta_x$ and $v_g$ denote the parameters of cognitive bias and $x_g(t)$ denotes individual $v_g$’s sensed expectation from her neighbors, defined as the mean of neighbors’ opinions, i.e.,

$$x_g(t) = \frac{1}{\sum_{j \in V} w_{gj}} \sum_{j \in V} w_{gj} x_j(t).$$

5) $a_g(t) \geq 0$ is the “resistance parameter” of individual $v_g$.

To guarantee $x_g(t) \in [-1, 1]$ for all $t \in N$ and for all $g \in V$, it is determined in such a way that

$$a_g(t) + \sum_{j \in V} w_{gj} + \sum_{d \in \Delta} c(x_g(t), h_d(t)) + x_g = 1$$

where $x_g$ denotes the bound on process noise, i.e., $|p_g(t)| < x_g \leq 1$, for all $t \in N$.

**C. Cognitive Behavioral Asymmetry**

The more than 40 years’ studies in cognitive and social psychology have revealed that the asymmetric effect/bias (i.e., the distance from X to Y may be estimated differently from Y to X) is a universal phenomenon, ranging from psychological similarity estimations [34] to social perception [35]. In this article, we refer the asymmetric bias to the cognitive behavior that an individual does not give identical influence weight to the opinions that have the same distance with hers. The cognitive bias considered in this article includes: 1) confirmation bias, which happens when a person gives more weight to evidence that confirms her beliefs and undervalues evidence that could disprove it [36], and 2) novelty bias, which refers to an individual’s preference of outlying information [4], [9], [14]. We next use a numerical example to describe how the model (2) can capture cognitive behavioral asymmetry.

For the sake of simplifying the presentation in this section, we refer $h_a$, $h_b$ and $x_g$ (dropping $t$ without loss of generality) to the opinions of the information sources Alex and Bob and the individual George, respectively, and we let $x_g = \bar{x}_g = \hat{x}_g$. We suppose the topic being discussed is “COVID-19 is a Hoax.” The hierarchy representations of $x_g \in [-1, 1]$ (also applied for $h_a$, $h_b$) are as follows.

1) $x_g = 0$: George’s opinion is neutral, i.e., not supporting or opposing.
2) $x_g \in (0, 1]$: George supports the statement with the supporting degree $|x_g|$.
3) $x_g \in [-1, 0)$: George opposes the statement with the opposing degree $|x_g|$.

To demonstrate the capability of the social influence model (2) in capturing the cognitive behavioral asymmetry due to confirmation bias and novelty bias, we consider the influence weights that George gives to the opinions of Alex and Bob in three scenarios as, respectively, shown in Fig. 1(a)–(c).

1) **Same Domain**: $h_a - \hat{x}_g = |\hat{x}_g - h_b| = 0.3$ and $h_a > \hat{x}_g > h_b > 0$.
2) **Crossing Domain**: $h_a - \hat{x}_g = |\hat{x}_g - h_b| = 0.3$ and $h_a > \hat{x}_g > 0$ and $h_b < 0$.
3) **Increasing Distance**: Given $\hat{x}_g = 0.6$, increase $|h_a - \hat{x}_g|$ in both directions of supporting and opposing.

**III. Social Inference: Problem Formulation**

In some social problems, e.g., the competing camps, information sources’ optimal seeding strategies need the knowledge of weighed network topology [23] and individuals’ cognitive bias parameters [19], which however are virtual but can be potentially inferred/estimated by observing a group of opinions evolution, and hence constitute inference objectives in this article. We thus denote the inference objectives by

$$S = (\hat{W}, \hat{\beta}, \hat{v})$$

where $\hat{W}$, $\hat{\beta}$ and $\hat{v}$ denote the inferred weighted adjacency matrix and cognitive bias parameter vectors that correspond to $[W]_{i,j} = w_{ij}$, $\beta = [\beta_1, \ldots, \beta_n]$ and $v = [v_1, \ldots, v_n]$.

Observing (1)–(4), we conclude that if the information sources express an extremal opinion 1 or −1, the social
model (1) transforms to a linear system, which can be leveraged to derive a simplified inference procedure of solution (5). Moreover, the proposed social influence model (2) implies that if the information sources can express an identical opinion, they are sensed as one information source from the perspective of an individual \( v_\ell \), since \( v_\ell \) gives the identical influence weight to them, which can be also leveraged to simplify the inference procedure. The scenario is formally described by

\[
h_d(t) = 1 \text{ or } -1, \quad \text{and } [K] = 1 \tag{6}
\]

considering which with the case of \( h_d(t) = -1 \), the dynamics in (1) transforms to

\[
x(t + 1) = a + W x(t) + p(t), \quad y(t) = x(t) + o(t) \tag{7}
\]

where we define

\[
\begin{align*}
a_g & \triangleq (a_g + \beta_x) (x_g(1) + 1) - \left( X_g + \sum_{j \in V} w_{gj} \right) x_g(1) - 1 \tag{8a} \\
W_{g,i} & \triangleq \begin{cases} 
\beta_x x_g(1) + \beta_y, & g = i \\
w_{gj} + \frac{(x_j(1) + 1) v_j x_g}{\sum_{i \in V} w_{ij}}, & g \neq i.
\end{cases} \tag{8b}
\end{align*}
\]

Remark 1: The substrategy (6) means the information sources cooperate to uniformly express one extremal opinion to be sensed as one unified source by their followers. Moreover, the proposed social influence model (2) implies that the inference framework first transforms to a linear system, which can be leveraged to simplify the dwell time of strategic extremal opinions to guarantee the inference solution (5) to be \((\phi, \delta)\)-PAC.

Given the estimation solution (9), inferring (5) constitutes the second problem.

Problem 2: Determine a social-system inference procedure that generates the inference solution (5).

We now present the social inference framework in Fig. 2, which builds on the answers to the Problems 1 and 2.

IV. PROBLEM 1: SOCIAL SYSTEM ESTIMATION

In this section, we first present the data processor of observations and the least-square model estimation. We then present the assumptions and investigate the sample complexity of estimation, leveraging which we finally derive the dwell time of strategic extremal opinions to guarantee the inference solution (5) to be \((\phi, \delta)\)-PAC.

A. Data Processor

We now present a data processor of observations of public evolving opinions, as shown in Fig. 2, which is a necessary step for the sample complexity analysis

\[
\begin{align*}
\bar{y}(t) & \triangleq y(t + 1) - y(t), \quad \bar{x}(t) \triangleq x(t + 1) - (t) \tag{12} \\
p(t) & \triangleq p(t + 1) - p(t), \quad \bar{o}(t) \triangleq o(t + 1) - o(t). \tag{13}
\end{align*}
\]

We construct a data matrix of processed observations

\[
X \triangleq [\bar{y}(1), \bar{y}(2), \ldots, \bar{y}(\tau - 3), \bar{y}(\tau - 2)] \tag{14}
\]

based on which, we introduce two matrices \( \Gamma \) and \( \Psi \) according to the following relation:

\[
E[XX^\top] = \sum_{i=1}^{\tau - 2} E[\bar{y}(i)\bar{y}^\top(i)] \triangleq \Gamma \triangleq \Psi^{-2}. \tag{15}
\]

B. Model Estimation

With the consideration of (12) and (13), we obtain the following dynamics from (7):

\[
\begin{align*}
\bar{x}(t + 1) &= W \bar{x}(t) + \bar{p}(t), \quad \bar{y}(t) = \bar{x}(t) + \bar{o}(t) \tag{16} \\
\bar{g}(t) &= \bar{p}(t) + \bar{o}(t + 1) - W \bar{o}(t). \tag{17b}
\end{align*}
\]

Remark 2: The relation (17b) explicitly shows the statistical dependence of random vector \( \bar{g}(t) \) on time indexed by \( t \) and \( t + 1 \). Equation (17b) also indicates the statistical dependence of \( \bar{g}(t) \) on its coordinates, i.e., the covariance matrix of \( \bar{g}(t) \) is not a diagonal matrix, which is due to the term \( \bar{o}(t + 1) - W \bar{o}(t) \).

Corresponding to (14), we construct the following matrices:

\[
Y \triangleq [\bar{y}(2), \bar{y}(3), \ldots, \bar{y}(\tau - 2), \bar{y}(\tau - 1)] \tag{18} \\
U \triangleq [\bar{g}(1), \bar{g}(2), \ldots, \bar{g}(\tau - 3), \bar{g}(\tau - 2)]. \tag{19}
\]

Noticing (14), (18), and (19), we verify from (17a) that

\[
Y = WX + U. \tag{20}
\]
We note that matrix $U$ is unknown. The relation (20) thus indicates the least-square optimal estimation of matrix is

$$\hat{Y} = Y X^T (X X^T)^{-1}.$$  \hfill (21)

Considering the dynamics (7) with obtained estimation (21), $a_{(k)}$ is estimated as

$$\hat{a} = \frac{1}{r-1} \sum_{q=1}^{r-1} (y(q + 1) - \hat{Y} y(q)).$$  \hfill (22)

C. Assumption

This section presents the assumptions on answering Problem 1. To simplify the presentation we define

$$s \triangleq [o; x; p]$$  \hfill (23)

where

$$o \triangleq [\tilde{o}(1); \tilde{o}(2); \ldots; \tilde{o}(r-2)] \in \mathbb{R}^{(r-2)n}$$  \hfill (24a)
$$x \triangleq [\tilde{x}(1); \tilde{x}(1); \ldots; \tilde{x}(1)] \in \mathbb{R}^{(r-2)n}$$  \hfill (24b)
$$p \triangleq [\tilde{p}(1); \tilde{p}(2); \ldots; \tilde{p}(r-3)] \in \mathbb{R}^{(r-2)n}.$$  \hfill (24c)

Finally, to introduce the assumption setting for our estimation and inference, we recall a definition.

Definition 1 (Convex Concentration Property [37]): Let $z$ be a random vector in $\mathbb{R}^n$. $z$ has the convex concentration property with constant $\kappa$ if for every $1$-Lipschitz convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, we have $\mathbf{E}[|\varphi(z)|] < \infty$ and for every time $t > 0$

$$\mathbf{P}[|\varphi(z) - \mathbf{E}[\varphi(z)]| \geq t] \leq 2e^{-\frac{t^2}{\kappa}}.$$  \hfill (25)

With the definitions at hand, we make the following assumptions for solving Problem 1.

Assumption 1: Consider the social dynamics (1) with (17b), the stacked vector $s$ (23) and the initial opinion-distance vector $\tilde{x}(1)$ in (12),

1) $p(k) \overset{\text{i.i.d.}}{\sim} \mathcal{D}_p(\mu_p, \sigma_p^2 I_n)$, $o(k) \overset{\text{i.i.d.}}{\sim} \mathcal{D}_o(\mu_o, \sigma_o^2 I_n)$, $\tilde{x}(1) \overset{\text{i.i.d.}}{\sim} \mathcal{D}_o(0, \sigma_o^2 I_n)$.
2) $s$ has the convex concentration property with constant $\kappa > 0$.
3) $[\tilde{g}(t + 1)], i \in \mathbb{V}$, is $\mathcal{F}_t$-measurable (i.e., measurable with respect to the filtration $\mathcal{F}_t \triangleq \sigma([\tilde{g}(s)], s \leq t)$ and conditionally $\gamma$-sub-Gaussian for some $\gamma > 0$, i.e., $\mathbf{E}[e^{\lambda \tilde{g}(t+1)}|\mathcal{F}_t], \forall \lambda \in \mathbb{R}$.

Remark 3: The subscripts $p$, $o$ and $i$ in Assumptions 1-1) are used to indicate that the distributions of process noise $p(k)$, observation noise $o(k)$ and initial opinion distance $\tilde{x}(1)$ can be different. Examples under Assumption 1-2), as summarized in [38], include any random vector $\eta \in \mathbb{R}^s$ with independent coordinates and almost sure $[|\eta_i|] \leq 1$ for any $i \in \{1, \ldots, s\}$, random vectors obtained via sampling without replacement [39], vectors with bounded coordinates satisfying some uniform mixing conditions or Dobrushin type criteria. Examples of $[\tilde{g}(t)]$, under Assumption 1-3) include a bounded zero-mean noise lying in an interval of length at most $2\gamma$, a zero-mean Gaussian noise with variance at most $\gamma^2$ [40]. Under Assumption 1-2), Lemma 1 in Appendix A is employed to derive (61) and (62) in Appendix B.

Remark 4 (Nonzero Mean): If $p(k) \overset{\text{i.i.d.}}{\sim} \mathcal{D}_p(\mu_p, \sigma_p^2 I_n)$, it can be rewritten as $p(k) = \mu_p + \tilde{p}(k)$, with $\tilde{p}(k) \overset{\text{i.i.d.}}{\sim} \mathcal{D}_p(0, \sigma_p^2 I_n)$. In this scenario, model (7), as an example, can be written as $x(k+1) = (a + \mu_p) I + \mathcal{C}x(k) + \tilde{p}(k)$. Thus, the Assumption 1-1) on process noise holds in general.

Remark 5 (Existing Results): In recent years, the sample complexity of an ordinary least-square estimator in identifying system matrix has been studied. We summarize the studied models with the associated assumptions of the seminal work [28], [41] as follows.

The Hanson-Wright inequality [30] constitutes one backbone of sample complexity study in [28], leveraging which the obtained Theorem 5.1, Propositions 5.1, 11.2 and 11.3 therein rely on the assumptions included in (m1c). We note that for our studied system (17a), due to (17b), the term $\tilde{g}(t)$ in (17a) that corresponds to $\eta(k+1)$ in (m1a) is not isotropic and $[\tilde{g}(k)]_{k=1}^\infty$ are not i.i.d.. Meanwhile, we do not assume the system (17a) has the zero initial condition and isotropic control. Through introducing the isotropic control inputs to the observations indicated by (m2b), the work [41] proposes a different analysis of sample complexity, which however still relies on the Hanson-Wright inequality that requires the
assumptions (m2c) which however does not hold in this article. Furthermore, noticing Assumption 1, we here can conclude that nearly all of the assumptions made in [28] and [41] do not hold in this article, which will hinder the application of sample complexity results to the social inference procedure proposed in this article.

Under Assumption 1-1), we obtain the covariance matrix of vector $s$ given in (23) as

\[
\mathbf{C} \triangleq \mathbb{E}[ss^T] = \begin{bmatrix}
2 \kappa_n^2 \mathbf{I}_{(t-2)\times n} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \kappa_n^2 \mathbf{I}_{(t-2)\times n} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & 2 \kappa_n^2 \mathbf{I}_{(t-3)\times n}
\end{bmatrix}
\]

where $\mathbf{O}$ denotes zero matrix with compatible dimensions.

D. Sample Complexity

This section investigates the sample complexity of estimation (21), whose associated complexity bounds will answer Problem 1. Before processing the analysis, we define a set of matrices

\[
\begin{align*}
\mathbf{\Upsilon} & \triangleq \text{diag}\{\Psi, \Psi, \ldots, \Psi\} \in \mathbb{R}^{t \times n(t-2)} \\
\Pi & \triangleq [\mathbf{I}_{(t-2)\times n}, \mathbf{\tilde{W}}, \mathbf{\tilde{W}}]
\end{align*}
\]

where $\Psi$ is defined in (15), and

\[
\begin{align*}
\mathbf{\tilde{W}} & \triangleq \text{diag}\{\mathbf{I}_n, \mathbf{W}, \mathbf{W}^2, \ldots, \mathbf{W}^{t-3}\} \\
\mathbf{\tilde{W}} & \triangleq \text{diag}\{\mathbf{O}, \mathbf{I}_n, W(1), W(2), \ldots, W(t-4)\} \\
W(t) & \triangleq [\mathbf{I}_n, \mathbf{W}, \mathbf{W}^2, \ldots, \mathbf{W}^t]
\end{align*}
\]

We now present an auxiliary proposition for the sample complexity analysis, whose proof appears in Appendix B.

Proposition 1: Under Assumption 1, we have

\[
\mathbf{P}\left[\| \mathbf{\hat{X}X^T\Psi - \mathbf{I}_n \| > \rho \right] \leq 2 \left(2 + e\right)^n \cdot \epsilon \cdot e^{-\frac{1}{2} \rho^2 \min\left[\frac{(t-2)\kappa_n^2}{\|\mathbf{\hat{X}X^T\Psi}\|}, \frac{(t-2)\kappa_n^2}{\|\mathbf{\hat{X}X^T\Psi}\|}\right]}
\]

for $\epsilon \in [0, (1/2)]$ and some universal constant $c > 0$.

Leveraging Proposition 1, the sample complexity is presented in the following theorem, whose proof is presented in Appendix C.

Theorem 1: Consider the estimated matrix $\mathbf{\hat{W}}$ in (21). Under Assumption 1, for any $\epsilon \in [0, (1/2)]$, $\delta \in (0, 1)$, and $\phi > 0$, we have

\[
\mathbf{P}\left[\left|\mathbf{\hat{W}} - \mathbf{W}\right| > \phi \right] \leq \delta
\]

if the following hold:

\[
\begin{align*}
\min\left[\frac{(1-2\epsilon)^2}{4(\tau - 2)\|\mathbf{\Upsilon}\|\|\mathbf{\Pi}\|\|\mathbf{C}\|}, \frac{(1-2\epsilon)^2}{2\|\mathbf{\Pi^T}\mathbf{\Gamma}\|}\right] & \geq \frac{\gamma^2}{2} \ln \frac{(\frac{1}{\delta} + 1)^n}{\delta} \\
\Gamma & \geq \frac{64k^2}{\phi^2} \ln \left(\frac{2^{1-0.5a} \cdot s^2}{\delta}\right) I_n > 0.
\end{align*}
\]

E. Dwell Time Computation

The dwell time of strategic extremal opinion can be computed from the conditions (32) and (33). The current forms are not ready for the computation, which is due to the unknown $\mathbf{W}$ included in $\mathbf{\Pi}$. With the consideration of $x(t) \in [-1, 1]$, $\forall t \in \mathbb{V}$, $\forall t \in \mathbb{N}$, the system (7) indicates that the matrix $\mathbf{\hat{W}}$ defined in (8b) is Schur stable. It is thus practical to assume we know matrix-norm bounds $\mathbf{\breve{h}}$ and $\mathbf{\breve{b}}$ such that

\[
0 \leq \mathbf{\breve{h}} \leq \|\mathbf{\hat{W}}\| \leq \mathbf{\breve{b}} < 1.
\]

The bounds $\mathbf{\breve{h}}$ and $\mathbf{\breve{b}}$ can be leveraged to estimate the bounds on the matrix norms of $\mathbf{\Pi}$ and $\mathbf{\Upsilon}$ to compute dwell time $\tau$.

With the bounds at hand, we present a corollary of Theorem 32, whose proof is given in Appendix D.

Corollary 1: The conditions (32) and (33) hold if

\[
\min\left[\frac{(1-2\epsilon)^2}{4(\tau - 2)\|\mathbf{\Upsilon}\|\|\mathbf{\Pi}\|\|\mathbf{C}\|}, \frac{(1-2\epsilon)^2}{2\|\mathbf{\Pi^T}\mathbf{\Gamma}\|}\right] \geq \frac{\gamma^2}{2} \ln \frac{(\frac{1}{\delta} + 1)^n}{\delta} \\
f \geq \frac{64k^2}{\phi^2} \ln \left(\frac{2^{1-0.5a} \cdot s^2}{\delta}\right) > 0
\]

where

\[
\begin{align*}
f & \triangleq \frac{2(\tau - 2)\kappa_n^2 + \sigma_\tau^2 \frac{b_{i}^{1-2\epsilon} - 1}{b_{i} - 1} - 2\sigma_\tau^2 \tau - 2 \frac{2}{b_{i} - 1} + 2\sigma_\tau^2 \frac{b_{i}^{1-2\epsilon} - 1}{(b_{i} - 1)^2}}{\frac{1}{(1-b_{i})}} \\
g & \triangleq \frac{1 - b_{i}^{1-2\epsilon} - 1}{(1-b_{i})^2}
\end{align*}
\]

Remark 6 (Dwell Time Computation): According to (10) and Corollary 1, the dwell time $\tau$ is computed such that (35) and (36) hold simultaneously.

Remark 7: The conditions (35) and (36) straightforwardly imply that they are more likely to hold in the case of larger $\delta$ or $\phi$, i.e., the smaller prescribed levels of accuracy or confidence, which can lead to smaller dwell time. The conditions (35) and (36) also implies given dwell times, the larger size of social network $n$ can require larger $\delta$ or $\phi$, which can further results in larger model inferences error.
Algorithm 1: Inference From Estimation

Input: Matrix $\mathbf{W}$ (21) and vector $\mathbf{a}$ (22).

1. Cognitive bias parameter: $\tilde{\beta}_g \leftarrow \frac{\mathbf{W}_{g,1}}{\chi_g(1)+1}$;
2. Influence sum: $\sum_{i \in \mathcal{V}} \tilde{w}_{gi} \leftarrow \frac{\mathbf{W}_{g,i} - \mathbf{W}_{g,1}}{\chi_g(1)+1} + \sum_{i \in \mathcal{V}} \tilde{w}_{gi}$;
3. Cognitive bias parameter: $\tilde{\nu}_g \leftarrow \sum_{i \in \mathcal{V}} \tilde{w}_{gi} - \tilde{w}_{g,1}$;
4. Weighted topology denoted by $[\tilde{w}_{gi}]$ are obtained through solving:
   \[
   \tilde{w}_{g1} + \ldots + \tilde{w}_{g(g-1)} + \tilde{w}_{g(g+1)} + \ldots + \tilde{w}_{g,1} = \sum_{i \in \mathcal{V}} \tilde{w}_{gi},
   \]
   \[
   \tilde{w}_{gi} - \tilde{w}_{kj} = \left( \frac{[\mathbf{W}]_{g,j} - [\mathbf{W}]_{g,i}}{\sum_{i \in \mathcal{V}} \tilde{w}_{gi} + (\chi_g(1)+1)\tilde{\nu}_g} \right) \tilde{w}_{gi}, \quad i, j \in \mathcal{V}.
   \]

V. PROBLEM 2: SOCIAL SYSTEM INFERENCE

With the obtained estimation (9), we investigate the computation of (5). Considering the structures of real vector and matrix in (8), we write estimations (21) and (22) in the following forms:

\[
\tilde{a}_g \triangleq (\tilde{\nu}_g + \tilde{\beta}_g)(\chi_g(1)+1) - \sum_{i \in \mathcal{V}} \tilde{w}_{gi} x_g(1) - 1 \quad (39a)
\]

\[
[\tilde{\mathbf{W}}]_{g,i} \triangleq \begin{cases} 
\tilde{\beta}_g x_g(1) + \tilde{\beta}_g, & g = i \\
\tilde{\nu}_g + \frac{(\chi_g(1)+1)\nu_{wg}}{\sum_{i \in \mathcal{V}} \tilde{w}_{gi}}, & g \neq i.
\end{cases} \quad (39b)
\]

based on which the inference procedure is described by Algorithm 1. The associated analysis is presented in the following theorem, whose proof appears in Appendix E.

Theorem 2: Consider the inference procedure in Algorithm 1. If the initial opinion $x_g(1) \neq -1$ for $\forall g \in \mathcal{V}$, Algorithm 1 generates inference solution (5).

VI. EMPIRICAL VALIDATION

In this section, we use U.S. Senate Member Ideology data [42] to validate the theoretical results and model from perspectives of generalization error and model error. Since a senator member usually retires after (at most) 12 congresses, it is not practical to model the state member as an individual in our proposed opinion evolution model. Alternatively, an individual in our model represents one U.S. state, and her opinion corresponds to the average ideological data of senate members from the same state. Meanwhile, we model U.S. President as information source in our model. To perform validation, we use the first-dimension ideological data obtained via Nokken-Poole estimation, which describes the economic liberalism-conservatism of a member. We consider the data from the 37th Congress to the 116th Congress, during which U.S. President is from the Republican Party or Democratic Party Presidents of the United States. However, the ideology of U.S. President is not estimated [42]. As an alternate, we set the default ideology of the president as +1 if the president is from the Republican Party, and −1 if the president is from the Democratic Party.

A. Prediction Error

Following the same logic to derive (7), if $h_g(t) = 1$, the dynamics in (1) transforms to

\[
x(t + 1) = \tilde{p} + \mathbf{W} x(t) + p(t), \quad y(t) = x(t) + o(t) \quad (42)
\]

where we define

\[
\tilde{p}_g \triangleq (a_g + \beta_g)(\chi_g(1)+1) - \left( \chi_t + \sum_{j \in \mathcal{V}} w_{gj} \right) x_g(1) + 1
\]

\[
[\tilde{\mathbf{W}}]_{g,i} \triangleq \begin{cases} 
\beta_g - \beta_g x_g(1), & g = i \\
\frac{(1-\chi_g(1))\nu_{wg}}{\sum_{i \in \mathcal{V}} \tilde{w}_{gi}}, & g \neq i
\end{cases} \quad (43b)
\]

Fig. 3. (a) 19 states’ model-based prediction errors. (b)–(e) Four states’ real and predicted trajectories of ideology.
The dynamics (7) with (8) and the dynamics (42) with (43a) show that when the information sources hold extremal opinions, the proposed model (1) transforms to linear stochastic systems but have distinctive differences. This observation indicates that if the inferred social systems are leveraged for prediction, the inferred switching model that corresponds to (7) and (42), i.e.,

\[ y(t + 1) = \frac{\hat{\alpha}}{\alpha_x} + \frac{\hat{\alpha_y}}{\alpha_y} x(t), \quad \text{Democratic Party} \]
\[ y(t + 1) = \frac{\hat{\alpha}}{\alpha_x} + \frac{\hat{\alpha_y}}{\alpha_y} y(t), \quad \text{Republican Party} \]

would have a smaller prediction error than that of a fixed social model, e.g., DeGroot model [6] or Friedkin-Johnsen model [7]. For the comparison, the fixed model via estimations (21) and (22) is Friedkin-Johnsen model based, i.e.,

\[ y(t + 1) = \frac{\hat{\alpha}}{\alpha_x} + \hat{\alpha_y} x(t). \]  

We use the ideological data of the 40th Congress to 106th Congress to infer models and save the rest of the data (i.e., 107th–116th) to measure prediction error. Meanwhile, we assume we know that in the 107th–110th, 115th and 116th Congresses, U.S. Presidents are from Democratic Party, while in the 111th–114th Congresses, U.S. Presidents are from Democratic Party. We follow the following procedure to perform the model-based prediction.

1) We consider three group data: Republican Data (extracted if the president is from Republican Party), Democratic Data (extracted if the president is from Democratic Party) and Mixed Data (no separation).
2) We use the Democratic Data, Republican Data, and Mixed Data to, respectively, infer the submodels (44a) and (44b) and the fixed model (45), via estimations (21) and (22).
3) For the prediction, we input the ideological data of the 107th Congress as the same initial condition for the switching model (44) and the fixed model (45).
4) From the 107th to 110th Congresses, we use model (44b) for prediction, from the 111th to 114th Congresses, we switch to model (44a) for prediction, in the 115th and 116th Congresses, we switch back to model (44b) for prediction.

Hence, we consider a network with 19 U.S. states. We denote individual \( v_i \)'s predicted ideology at congress number \( t \) by \( \hat{y}_i(t) \). We define the following metric to measure prediction error:

\[ e_i = \frac{1}{10} \sum_{t=107}^{116} |y_i(t) - \hat{y}_i(t)|, \quad i \in \{\text{NC, MS,}, \ldots, \text{NE, KS}\}. \]

The 19 states’ prediction errors and the picked four states’ real ideology and predicted trajectories are, respectively, shown in Fig. 3 (a)–(e), observing which we discover that switching social model (44) has more accurate prediction than a single fixed model (45).

B. Model Error and Fitting Error

Hereafter, we investigate the influence of dwell time on the model error and fitting Error. We make the worst case assumptions on noise, i.e., \( \sigma_o = \sigma_p = \sigma_i = 1 \). Under Assumption 1, we set \( \gamma = 1.59 \), for matrix-norm bounds, we choose \( \hat{\gamma} = 0.9 \) and \( \hat{h} = 0.1 \), we set other parameters as \( \varepsilon = 0.056 \), \( \rho = \left(\frac{\gamma}{1.06}\right) \), \( \epsilon = 9.5 \) and \( \kappa = 2\sqrt{2} \). According to the set parameters, the dwell times can be computed from (35) and (36) for the given \( (\phi, \delta) \)--PAC.

Differentiating from numerical examples and man-made systems, we do not have real exact model parameters as references to straightforwardly measure model error pertaining to \( (\phi, \delta) \)--PAC. Observing the matrix and vector in (8) and recalling the convex combination (4), we can perform model validation from the following social-system properties.

1) \( 0 \leq \alpha_x + \sum_{j \in V} |W|_{i,j} \leq 1, \quad \forall i \in V \).
2) The magnitudes of all entries of \( W \) are smaller than one.
3) The fitting curve and the trajectories of the inferred model under an arbitrary initial condition in \([-1, 1]\) are all constrained into \([-1, 1]\) for any time.
We now consider the ideological data of the 40th–116th Congresses. The size of Republican Data is 44, which means \( \tau = 44 + 1 = 45 \). Following Corollary 1, for the \((1, 0.1)\)-PAC, the allowed network size is 6. Then, by (21) and (22), we have \( \hat{\theta} = \begin{bmatrix} 0.0156, -0.0461, 0.0985, 0.1476, 0.0012, -0.0679 \end{bmatrix}^T \), which is due to the missing or un-embedding social communication detection and classification.

We next increase the size of the social network to include the five trajectories under random initial conditions in \([-1, 1]\), which shows that all of the trajectories are constrained into \([-1, 1]\) (see the range of light blue area).

We also consider the learning of social dynamics via deep neural networks, whose architecture is shown in Fig. 5, where “act” denotes the activation function. The trajectories of the trained DNN model (training loss: 1.1118374\( \times \)10^(-3)) obtained via Monte Carlo simulations with 500 randomly generated initial conditions in \([-1, 1]\) are shown in Fig. 4 (a)–(f), which shows that all of the trajectories are constrained into \([-1, 1]\) (see the range of light blue area).

We next increase the size of the social network to include all of the 30 states in the 40th Congress, such that the \((\phi, \delta)\)-PAC cannot be guaranteed. In this setting, the fitting curve and the five trajectories under random initial conditions in \([-1, 1]\) are shown in Fig. 4(g) and (h), which show that although the inferred model fits the real data well, without satisfying high PAC, the inferred model has a larger model error such that its evolving ideologies under some initial conditions exceed the range \([-1, 1]\) and the inferred model can be unstable.

VII. CONCLUSION

In this article, we have proposed an opinion evolution model which explicitly takes the asymmetric cognitive bias including confirmation bias and negativity bias and the process noise into account. Based on the proposed model, we have studied the problem of social system inference from network topology and model parameters of asymmetric cognitive bias. We have analyzed the sample complexity of the proposed inference procedure in the presence of observation noise, which leads to the statistical dependence of observed public evolving opinions on time and coordinates. Real data validations suggest the effectiveness of the obtained theoretical results and the proposed opinion evolution model. In the future, we will investigate the generalization of the social-system inference framework for the large-scale social networks with the incorporation of social communication detection and classification.

APPENDIX A: AUXILIARY LEMMAS

**Lemma 1** ([38]): Let \( f \) be a mean zero random vector in \( \mathbb{R}^n \), whose covariance matrix is denoted by \( \text{Cov}(f) \). If \( f \) has the convex concentration property with constant \( \kappa \), then for any \( A \in \mathbb{R}^{n \times n} \) and every \( t > 0 \), we have

\[
\mathbb{P} \left( \left| \frac{1}{t} \mathbf{W} f - \mathbb{E}[\mathbf{W} f] \right| \geq t \right) \leq 2e^{-\frac{\kappa^2}{t^2 \delta^2 \text{Cov}(f)_{\text{min}}}}
\]

for some universal constant \( \delta \).

**Lemma 2** ([43, Ch. 4]): Let \( W \) be a \( d \times d \) symmetric random matrix. Furthermore, let \( N \) be an \( \varepsilon \)-net of \( S^{d-1} \) with minimal cardinality. Then for all \( \rho > 0 \), we have

\[
\mathbb{P} \left( \| W \| > \rho \right) \\
\leq \left( \frac{2}{\varepsilon + 1} \right) \max_{\| u \|_2 > (1 - \varepsilon)\rho, \ v \in [0, 1]} \mathbb{P} \left( \| u^\top W u \| > (1 - 2\varepsilon)\rho \right)
\]

(46)

\[
\mathbb{P} \left( \| W \| > \rho \right) \\
\leq \left( \frac{2}{\varepsilon + 1} \right) \max_{\| u \|_2 > (1 - \varepsilon)\rho, \ v \in [0, 1]} \mathbb{P} \left( \| u^\top W u \| > (1 - 2\varepsilon)\rho \right)
\]

(47)

**Lemma 3** ([40]): Let \( \{ \eta_t \}_{t \geq 1} \) be a filtration. Let \( \{ \eta_t \}_{t \geq 1} \) be a stochastic process adapted to \( \{ \mathcal{F}_t \}_{t \geq 1} \) and taking values in \( \mathbb{R} \). Let \( \{ \eta_t \}_{t \geq 1} \) be a predictable stochastic process with respect to \( \{ \mathcal{F}_t \}_{t \geq 1} \), taking values in \( \mathbb{R}^d \). Furthermore, assume that \( \eta_t \) is \( \mathcal{F}_t \)-measurable and conditionally \( \gamma \)-sub-Gaussian for some \( \gamma > 0 \). Let \( S > 0 \), \( \eta^\top = [\eta_2, \eta_3, \ldots, \eta_{t+1}] \), and \( X^\top = [x_1, x_2, \ldots, x_t] \). The following

\[
\left\| (X^\top X + S)^{-0.5} X^\top \eta \right\|_2 \leq 2\gamma \ln \left( \frac{\text{det}((X^\top X + S)^{-1})^{0.5}}{\delta} \right)
\]

holds with the probability of at least \( 1 - \delta \).

APPENDIX B: PROOF OF PROPOSITION 1

It follows from (15) that

\[
\mathbb{E}[\Psi^\top XX^\top \Psi] = I_n
\]

by which, we obtain that

\[
\left\| (X^\top \Psi)^\top XX^\top \Psi - I_n \right\| \\
= \left\| (X^\top \Psi)^\top XX^\top \Psi - I_n \right\| \\
= \sup_{u \in S^{d-1}} \| u^\top (XX^\top \Psi)^\top XX^\top \Psi u - u^\top \mathbb{E}[\Psi^\top XX^\top \Psi] u \| \\
= \sup_{u \in S^{d-1}} \| \left\| (XX^\top \Psi) u \right\|_2 - \mathbb{E}[\| XX^\top \Psi u \|_2] \|.
\]

(49)
We obtain from (16) that
\[ \bar{y}(t+1) = \mathcal{W}_t \bar{x}(1) + \sum_{l=0}^{t-1} \mathcal{W}_l \bar{p}(t-l) + \bar{e}(t). \] (50)

We let \( u \in S^{n-1} \), and define
\[ U \triangleq \text{diag}(u, u, \ldots, u) \in \mathbb{R}^{n(t-2) \times (t-2)} \] (51)
observing which, (50), \( \Upsilon \) in (27), \( X \) in (14), \( \Pi \) in (28) with (29a), and \( s \) in (23) with (24a) we have
\[ X^T \Psi u = U^T \Upsilon^T \Pi s \] (52)
substituting which into (49), we arrive at
\[ \|(X^T \Psi)^T X^T \Psi - I_n\| = \sup_{u \in S^{n-1}} \|(U^T \Upsilon^T \Pi s \|_2^2 - E(U^T \Upsilon^T \Pi s \|_2^2). \] (53)

Let us define
\[ \Delta \triangleq \Pi^T \Upsilon U^T \Upsilon^T \Pi \] (54)
by which, we obtain
\[ \|\Delta\|_F^2 \leq \|\Pi^T \Upsilon U\|_2^2 \|\Pi^T \Upsilon^T \Pi\|_F^2 \] (55)
\[ = \|\Pi^T \Upsilon U\|_2^2 \|\Pi^T \Upsilon^T \Pi\|_F^2 \] (56)
\[ \leq \|\Pi^T \Upsilon^2\|_F \|\Pi^T \Upsilon^T \Pi\|_2^2 \|U\|_F^2 \] (57)
\[ \leq \|\Pi^T \Upsilon^4\| \|U\|_F^2 \] (58)
\[ \|\Delta\| \leq \|\Pi^T \Upsilon^2\| \|U\|_F^2 = \|\Pi^T \Upsilon\|_2^2 \] (59)

where (55)–(57) are obtained via considering the well-known inequalities \( \|AB\| \leq \|A\| \|B\| \), \( \|AB\|_F \leq \|A\| \|B\|_F \) and \( \|A\|_F = \|A^T\|_F \), (58) from (57) is obtained via considering \( \|U\|_2^2 = 1 \) and \( \|U\|_F^2 = \tau - 2 \) and \( \tau \) by which we have \( \|U\|_F^2 \) (51), and (59) follows from \( \|AB\| \leq \|A\| \|B\|, \|A\| = \|A^T\| \) and \( \|U\| = 1 \).

Under Assumption 1-1), we verify from (23) with (24a) that \( s \) has zero mean. Since \( \epsilon > 0 \) and \( \rho > 0 \), under Assumption 1-2), applying Lemma 1 (in Appendix A) with (60) and (54), we conclude that
\[ \|\Pi^T \Upsilon^T \Pi s \|_2^2 - E\|\Pi^T \Upsilon^T \Pi s \|_2^2 > \rho \] (61)
holds with probability at most
\[ 2e^{-\rho^2 \min \left\{ \frac{1}{\tau - 2 \|\Pi^T \Upsilon^4\|} \|U\|_F^2 \right\}} \] (62)
which, in conjunction with (53), implies that
\[ \|(X^T \Psi)^T X^T \Psi - I_n\| = \sup_{u \in S^{n-1}} \|(X^T \Psi)^T X^T \Psi - I_n\| \geq (1 - 2\epsilon) \rho \] holds with probability at most
\[ 2e^{-\rho^2 \min \left\{ \frac{1}{\tau - 2 \|\Pi^T \Upsilon^4\|} \|U\|_F^2 \right\}}. \] (63)

Then, applying (47) in Lemma 2 leads to Proposition 1.

\section*{Appendix C: Proof of Theorem 32}

Observing the relation (20) and the optimal estimation (21), we obtain \( \hat{\mathcal{V}} - \mathcal{V} = UX^T (XX^T)^{-1} \). Thus,
\[ \|\hat{\mathcal{V}} - \mathcal{V}\| = \|UX^T (XX^T)^{-1}\| \leq \|UX^T (XX^T)^{-0.5}\| \cdot \|(XX^T)^{-0.5}\| \] for which we define two events
\[ \mathcal{E}_1 \triangleq \left\{ \|UX^T (XX^T)^{-0.5}\| \cdot \|(XX^T)^{-0.5}\| > \phi \right\} \] (63)
\[ \mathcal{E}_2 \triangleq \left\{ \|UX^T (XX^T)^{-0.5}\| \cdot \|(XX^T)^{-0.5}\| \leq \frac{1}{2} \right\} \] (64)
from which we have
\[ P[\|\hat{\mathcal{V}} - \mathcal{V}\| > \phi] \leq P[\mathcal{E}_1 \cap \mathcal{E}_2] + P[\mathcal{E}_2^c]. \] (65)

We next derive the upper bounds on \( P[\mathcal{E}_2^c] \) and \( P[\mathcal{E}_1 \cap \mathcal{E}_2] \). \textbf{Upper Bound on} \( P[\mathcal{E}_2^c] \): Let us set \( \gamma = \sqrt{2\epsilon} \), inserting which into (32) results in
\[ \min \left\{ \frac{(1 - 2\epsilon)^2}{4(\tau - 2)\|\Pi^T \Upsilon^4\|^2} \right\} \geq \frac{\epsilon^2}{\delta^2} \ln \frac{4 \cdot \frac{1}{2} + 1}{n} \]
which is equivalent to
\[ 2 \cdot \left( \frac{2}{\epsilon} + 1 \right) = e^{\epsilon^2 \min \left\{ \frac{1}{\tau - 2 \|\Pi^T \Upsilon^4\|} \|U\|_F^2 \frac{(1 - 2\epsilon)}{2} \|\Pi^T \Upsilon^4\|^2 \right\}} \leq \frac{\delta}{2} \] (66)
which together with Proposition 1 with the setting of \( \rho = \frac{1}{2} \) imply that when (32) holds
\[ P[\mathcal{E}_2^c] \leq \frac{\delta}{2}. \] (66)

\textbf{Upper Bound on} \( P[\mathcal{E}_1 \cap \mathcal{E}_2] \): When \( \mathcal{E}_2 \) in (64) occurs, we have
\[ \frac{1}{2} I_n \leq \Psi X X^T \Psi \leq \frac{3}{2} I_n \] (67)
which means that
\[ 0 < \frac{1}{2} \Psi^{-2} \leq XX^T \leq \frac{3}{2} \Psi^{-2} \] (68)
by which we have
\[ \lambda_{\min}^{0.5} (XX^T) \geq \frac{1}{2} \Psi^{-2} \leq \frac{1}{2} \Psi^{-2} \] (69)
by which we obtain \( (1/\beta) \geq \|(XX^T)^{-0.5}\| \). We then conclude from (63) and (64) that
\[ \mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \left\{ \|UX^T (XX^T)^{-0.5}\| > \beta \phi \right\} \cap \mathcal{E}_2. \] (70)
From the left-hand inequality of (68) we have
\[ 2XX^T \geq \frac{1}{2} \Psi^{-2} + XX^T \] which means that
\[ (XX^T)^{-1} \leq 2 \left( \frac{1}{2} \Psi^{-2} + XX^T \right)^{-1} \]
which, in conjunction with (70), leads to
\[ \mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \left\{ \sqrt{2}\|UX^T (S + XX^T)^{-0.5}\| > \beta \phi \right\} \cap \mathcal{E}_2. \] (71)
where we denote
\[ S \triangleq \frac{1}{2} \Psi^{-2}. \tag{72} \]

With \( u \in S^{n-1} \), we now define two additional events
\[ \mathcal{A}_1 \triangleq \left\{ ||(S + XX^T)^{-0.5} UX^T||^2 \right\} \]
\[ > 16\epsilon\sigma^2 \ln((\det((S + XX^T)^{-1}))^{0.5} \delta_0^{-1}) \tag{73} \]
\[ \mathcal{A}_2(u) \triangleq \left\{ ||(S + XX^T)^{-0.5} UX^T u||_2^2 \right\} \]
\[ > 4\epsilon\sigma^2 \ln((\det((S + XX^T)^{-1}))^{0.5} \delta_0^{-1}) \tag{74} \]

We note that under Assumption 1-3, \( u \in S^{n-1} \) implies that \( (g(t + 1))^T u \) is \( F_t \)-measurable and conditionally \( \gamma \)-sub-Gaussian for some \( \gamma > 0 \). Meanwhile, we note that \( 4\epsilon\sigma^2 = 2\gamma^2 \). In light of Lemma 3 in Appendix A, we then have \( P[\mathcal{A}_2(u)] \leq \delta_0 \). Furthermore, applying (46) with the setting of \( \epsilon = \frac{1}{4} \) in Lemma 2, we obtain
\[ P[\mathcal{A}_1] = \delta_0 \max_{W \in \mathcal{W}} P[\mathcal{A}_2(u)] \leq \delta_0^\alpha \tag{75} \]

We let \( \delta_0 = \frac{\delta}{2\epsilon} \), such that
\[ \bar{\beta} \geq \frac{4\sqrt{2\epsilon}\sigma}{\phi} \sqrt{\ln \left( \frac{2 \cdot 5^{1.5n}}{\delta} \left( \frac{1}{10} \right)^{0.5n} \right)} \]
\[ = \frac{4\sqrt{2\epsilon}\sigma}{\phi} \sqrt{\ln \left( \frac{1}{\delta} \left( \frac{1}{\delta} \right)^{0.5} \right)} \]
\[ \geq \frac{4\sqrt{2\epsilon}\sigma}{\phi} \sqrt{\ln \left( \frac{1}{\delta} \left( \frac{1}{\delta} \right)^{0.5} \right)} \tag{76} \]

where the last inequality from its previous step is obtained via considering the inequality \( XX^T \geq S \) that follows from (68) and (72). Combining the inequality in (71) with (76) yields
\[ ||U XX^T (S + XX^T)^{-0.5}|| \]
\[ > \frac{\bar{\beta}}{\alpha} \geq 4\sqrt{\epsilon} \sigma \sqrt{\ln \left( \frac{1}{\delta} \left( \frac{1}{\delta} \right)^{0.5} \right)} \]

by which, and considering (71) and (73), we deduce that under condition (76), if the event \( E_1 \) occurs, the event \( \mathcal{A}_1 \) occurs consequently. We thus obtain
\[ P[E_1 \cap E_2] \leq P[\mathcal{A}_1 \cap E_2]. \tag{77} \]

We note that (33) is equivalent to
\[ \lambda_{\min} \left( \frac{1}{2} \right) \geq \frac{4\sqrt{2\epsilon}\sigma}{\phi} \sqrt{\ln \left( \frac{2 \cdot 5^{1.5n}}{\delta} \left( \frac{1}{10} \right)^{0.5n} \right)} \]

inserting \( \beta \) in (69) with \( \Gamma \) in (15) into which yields
\[ \beta \geq \frac{4\sqrt{2\epsilon}\sigma}{\phi} \sqrt{\ln \left( \frac{2 \cdot 5^{1.5n}}{\delta} \left( \frac{1}{10} \right)^{0.5n} \right)} \]

by which we conclude that (76) holds if (33) is satisfied. Moreover, recalling that the event \( E_2 \) always occurs under the condition (32) [proved in Upper Bound on \( P(E_2) \)], we conclude from (75) and (77) that
\[ P[\mathcal{A}_1 \cap E_2] \leq P[\mathcal{A}_1] \leq \delta_0 \]

holds as long as both (32) and (33) hold. In addition, due to \( \delta_0 = (\delta/2 \cdot 5^n) \), we have
\[ P[\mathcal{A}_1 \cap E_2] \leq \frac{\delta}{2} \tag{78} \]

Finally, combining (65) with (66) and (78) yields (31).

**APPENDIX D: PROOF OF COROLLARY 1**

**Condition (36):** Under Assumption 1, it follows from (50) with (12) and (13) that
\[ E[\bar{y}(t + 1) \bar{y}^T (t + 1)] = E[\bar{o}(t) \bar{a}(t)^T] + \bar{w}^T E[\bar{x}(1) \bar{x}^T (1)] (\bar{w}^T)^T \]
\[ + \sum_{i=0}^{t-1} \bar{w}^T E[\bar{p}(t - i) \bar{p}^T (t - i)] (\bar{w}^T)^T \]
\[ = 2\sigma_1^2 I_n + \sigma_2^2 \bar{w}^T (\bar{w}^T)^T + 2\sigma_5^2 \sum_{i=0}^{t-1} \bar{w}^T (\bar{w}^T)^T \]

which together with (34) lead to
\[ E[\bar{y}(t + 1) \bar{y}^T (t + 1)] \geq 2\sigma_1^2 I_n + \sigma_2^2 \bar{w}^T I_n + 2\sigma_5^2 \bar{w}^T I_n. \tag{79} \]

As a consequence, we obtain from (15) and (79) that \( \lambda_{\min}(\Gamma) \geq \frac{1}{\bar{\rho}} \), where \( \bar{\rho} \) is given in (37). We thus can conclude that (33) holds if (36) is satisfied.

**Condition (35):** It follows from (27) that \( ||\Upsilon||^2 = ||\Psi||^2 \). Meanwhile, we obtain from (15) and (79) that \( \Psi^2 \leq \bar{\rho}^{-1} I_n \), where \( \bar{\rho} \) is given in (37). We then have
\[ ||\Upsilon||^2 \leq \frac{1}{\bar{\rho}}. \tag{80} \]

We verify from (28) with (29a) that
\[ ||\Pi||^2 \leq \frac{1 - \bar{\rho}^{-3}}{1 - \bar{\rho}} \tag{81} \]

considering which and recalling the well-known inequality \( ||AB|| \leq \|A\| ||B|| \), we then arrive at
\[ ||\Pi^T \Upsilon||^2 \leq ||\Pi||^2 ||\Upsilon||^2 \leq g \tag{82} \]

where \( g \) is given in (38). We note that the inequality (82) implies
\[ \min \left\{ \frac{(1 - 2\epsilon)^2 \rho^2}{4(\tau - 2)||\Pi^T \Upsilon||^2 ||\Pi^T \Upsilon||^2}, \frac{(1 - 2\epsilon)^2 \rho^2}{2||\Pi^T \Upsilon||^2} \right\} \]
\[ \geq \min \left\{ \frac{(1 - 2\epsilon)^2 \rho^2}{4(\tau - 2)g^2 ||\Pi^T \Upsilon||^2}, \frac{(1 - 2\epsilon)^2 \rho^2}{2g} \right\} \]
indicating that if (35) is satisfied, (32) holds.
We obtain from (39b) that
\[
\sum_{i \in V} \hat{\nu}_{g,i} + \hat{\nu}_g (x_g(1) + 1) = \sum_{g \in V} \hat{\nu}^\top_{g,i} \hat{\nu}_g
\]
which results in \(\hat{\nu}_g\) computation in Line 1 of Algorithm 1. Summing (39b) over \(i \neq g \in V\) yields
\[
\sum_{i \in V} \hat{\nu}_{g,i} + \hat{\nu}_g (x_g(1) + 1) = \sum_{g \in V} \hat{\nu}^\top_{g,i} \hat{\nu}_g
\]
substituting which from (39a) leads to
\[
\hat{\nu}_g (x_g(1) + 1) - \sum_{g \neq g \in V} \hat{\nu}_{g,i} (x_g(1) + 1) - \sum_{g \in V} \hat{\nu}_{g,i}
\]
\[
= \hat{\alpha}_g - \sum_{g \neq g \in V} \hat{\nu}^\top_{g,i} \hat{\nu}_g
\]
which is expressed as (41). Meanwhile, the available \(\sum_{g \neq g \in V} \hat{\nu}_{g,i}\) means (40). Hence, the weighted network topology can be obtained by solving (40) and (41).

REFERENCES

[1] Y. Weng, Y. Liao, and R. Rajagopalan, “Distributed energy resources topology identification via graphical modeling,” IEEE Trans. Power Syst., vol. 32, no. 4, pp. 2682–2694, Jul. 2017.
[2] H.-T. Wu, A. Scaglione, and A. Leishem, “The social system identification problem,” in Proc. 54th IEEE Conf. Decis. Control, Dec. 2015, pp. 406–411.
[3] R. S. Nickerson, “Confirmation bias: A ubiquitous phenomenon in many guises,” Rev. Gen. Psychol., vol. 2, no. 2, pp. 175–220, 1998.
[4] P. Lamberson and S. Soroa, “A model of attentiveness to outlying news,” J. Commun., vol. 68, no. 5, pp. 342–364, 2018.
[5] M. Maś, A. Flache, and J. A. Kitts, “Cultural integration and differentiation in groups and organizations,” in Perspectives on Culture and Agent-Based Simulations. Cham, Switzerland: Springer, 2014, pp. 71–90.
[6] M. H. DeGroot, “Reaching a consensus,” J. Amer. Statist. Assoc., vol. 69, no. 345, pp. 118–121, Mar. 1974.
[7] N. E. Friedkin and E. C. Johnsen, “Social influence and opinions,” J. Math. Sociol., vol. 15, nos. 3–4, pp. 193–206, 1990.
[8] A. Das, S. Gollapudi, R. Panigrahy, and M. Salek, “Debiasing social wisdom,” in Proc. 19th ACM SIGKDD Int. Conf. Knowl. Discovery Data Mining, 2013, pp. 500–508.
[9] C. Xu, J. Li, T. Abdelzaher, H. Ji, B. K. Szymanski, and J. Dellarocas, “The paradox of information access: On modeling social-media-induced polarization,” 2020, arXiv:2004.01106.
[10] P. Giridhar and T. Abdelzaher, “Social media signal processing,” in Social-Behavioral Modeling for Complex Systems. Hoboken, NJ, USA: Wiley, 2019, ch. 20, pp. 477–493.
[11] H. Cui, T. Abdelzaher, and L. Kaplan, “A semi-supervised active-learning truth estimator for social networks,” in Proc. World Wide Web Conf., 2019, pp. 296–306.
[12] D. M. J. Lazer et al., “The science of fake news,” Science, vol. 359, no. 6380, pp. 1094–1096, 2018.
[13] M. D. Vicario, W. Quattrociocchi, A. Scala, and F. Zollo, “Polarization and fake news: Early warning of potential misinformation targets,” ACM Trans. Web, vol. 13, no. 2, pp. 1–22, 2019.
[14] T. Abdelzaher et al., “The paradox of information access: Growing isolation in the age of sharing,” 2020, arXiv:2004.01967.
[15] R. Hegselmann and U. Krause, “Opinion dynamics and bounded confidence models, analysis, and simulation,” J. Artif. Societies Social Simul., vol. 5, no. 3, pp. 1–33, 2002.
[16] M. Del Vicario, A. Scala, G. Caldarelli, H. E. Stanley, and W. Quattrociocchi, “Modeling confirmation bias and polarization,” Sci. Rep., vol. 7, no. 1, pp. 1–9, 2017.
[17] P. E. Kahn and S. Motsch, “Clustering and asymptotic behavior in opinion formation,” J. Differ. Equat., vol. 257, no. 11, pp. 4165–4187, 2014.
[18] S. Motsch and E. Tadmor, “Heterophilous dynamics enhances consensus,” SIAM Rev., vol. 56, no. 4, pp. 577–621, Nov. 2014.
[19] Y. Mao, E. Akyol, and N. Hovakimyan, “Impact of confirmation bias on competitive information spread in social networks,” IEEE Trans. Control Netw. Syst., vol. 8, no. 2, pp. 816–827, Jun. 2021, doi: 10.1109/TCNS.2021.3050117.
[20] Y. Mao, N. Hovakimyan, and T. Abdelzaher, “Dynamics of public opinion evolution with asymmetric cognitive bias,” 2021, arXiv:2105.11569.
[21] Y. Mao and E. Akyol, “On inference of network topology and confirmation bias in cyber-social networks,” IEEE Trans. Signal Inf. Process. Over Netw., vol. 6, pp. 633–644, 2020.
[22] M. Del Vicario et al., “The spreading of misinformation online,” Proc. Nat. Acad. Sci. USA, vol. 113, no. 5, pp. 554–559, 2016.
[23] S. Dhamal, W. Ben-Ameur, T. Chahed, and E. Altman, “Optimal investment strategies for competing camps in a social network: A broad framework,” IEEE Trans. Netw. Sci. Eng., vol. 6, no. 4, pp. 628–645, Oct. 2018.
[24] Y. Mao and E. Akyol, “On inference of network topology and confirmation bias in cyber-social networks,” IEEE Trans. Signal Inf. Process. Over Netw., vol. 6, pp. 633–644, 2020.
[25] Y. Jedda and A. Privotiere, “Finite-time identification of stable linear systems: Optimality of the least-squares estimator,” 2020, arXiv:2003.07937.
[26] M. Simchowitz, H. Mania, S. Tu, M. I. Jordan, and B. Recht, “Learning without mixing: Towards a sharp analysis of linear system identification,” in Proc. Conf. Learn. Theory, 2018, pp. 1–35.
[27] T. Sarkar and A. Rakshit, “Near optimal finite time identification of arbitrary linear dynamical systems,” in Proc. Int. Conf. Mach. Learn., 2019, pp. 5610–5618.
[28] T. Sarkar, A. Rakshit, and M. A. Dahleh, “Nonparametric finite time LTI system identification,” 2019, arXiv:1902.01848.
[29] S. Oymak and N. Ozay, “Non-asymptotic identification of LTI systems from a single trajectory,” in Proc. Amer. Control Conf., 2019, pp. 5655–5661.
[30] M. Rudelson and R. Vershynin, “Hanson-wright inequality and sub-Gaussian concentration,” Electron. Commun. Probab., vol. 18, pp. 1–9, Jan. 2013.
[31] A. Banerjee, Q. Gu, V. Sivakumar, and S. Z. Wu, “Random quadratic forms with dependence: Applications to restricted isometry and beyond,” in Proc. Adv. Neural Inf. Process. Syst., 2019, pp. 12599–12609.
[32] Y. Mao, J. Li, N. Hovakimyan, T. Abdelzaher, and C. Lebiere, “Cost function learning in memorized social networks with cognitive behavioral asymmetry,” IEEE Trans. Computat. Social Syst., early access, Nov. 10, 2022, doi: 10.1109/TCSST.2022.3218485.
[33] R. Borum, The Science of Intercpersonal Trust, Washington, DC, USA: Mental Health Law & Policy Faculty Publications, 2010, p. 574.
[34] J.-P. Codol, M. Jarymowicz, M. Kaminska-Feldman, and A. Szuster-Zbrojewicz, “Asymmetry in the estimation of interpersonal distance and identity affirmation,” Eur. J. Social Psychol., vol. 84, no. 4, pp. 1189–1204, 2014.
[35] A. Voversky, “Features of similarity,” Psychol. Rev., vol. 84, no. 4, pp. 327–352, 1977.
[36] I. Noor. (2020). Confirmation Bias. Simply Psychology. Accessed: Nov. 20, 2020. [Online]. Available: https://www.simplypsychology.org/confiration-bias.html
[37] M. Ledoux, The Concentration Measure Phenomenon, no. 89, Providence, RI USA: American Mathematical Society, 2001.
[38] R. Adamczak, “A note on the Hanson–Wright inequality for random vec- tors with dependencies,” Electron. Commun. Probab., vol. 20, pp. 1–13, Jan. 2015.
[39] D. Paulin et al., “The convex distance inequality for dependent random variables, with applications to the stochastic travelling salesman and other problems,” Electron. J. Probab., vol. 19, pp. 1–34, Jan. 2014.
[40] Y. Abbasi-Yadkori, D. Pál, and C. Szepesvári, “Improved algorithms for linear stochastic bandits,” in Proc. Adv. Neural Inf. Process. Syst., 2011, pp. 2312–2320.
[41] S. Oymak and N. Ozay, “Non-asymptotic identification of LTI systems from a single trajectory,” in Proc. Amer. Control Conf., 2019, pp. 5655–5661.
[42] J. B. Lewis, K. Poole, H. Rosenthal, A. Boche, A. Rudkin, and L. Sonnet. (2018). Voteview: Congressional Roll-Call Votes Database (2018). [Online]. Available: https://voteview.com
[43] R. Vershynin, High-Dimensional Probability: An Introduction With Applications in Data Science. Cambridge, U.K.: Cambridge Univ. Press, 2018.

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