1. INTRODUCTION

An important industrial issue in the production, transport and packaging of granular materials is the identification of those factors that affect bulk density, including how the material has been handled throughout its processing history. A common phenomenon that is not well-understood is the increase in bulk density that takes place when a container of granular materials is shaken, tapped or vibrated. There is a rather extensive literature on this topic, ranging from very fundamental investigations on the development of packing algorithms and microstructural analysis to more practical experimental and numerical studies. While it is not possible in this paper to provide a complete review of the extensive literature on this topic, a few representative papers are discussed in an attempt to provide a rough sketch.

From a certain perspective, one may think of the phenomenon of density relaxation as having its foundations in the extensive literature on the packing of spheres. (See for example reference1-15). As early as 1611, Kepler explored the geometry of the snowflake while in 1665, Robert Hook investigated circle and sphere packings. In 1694, Gregory, a Scottish astronomer, suggested that thirteen rigid uniform spheres could be packed around a sphere of the same size - a hypothesis that was only disproved by Leech some 262 years later. In 1727, Hales examined the packing of dry peas pressed into a container - an experiment that was later known as the 'peas of Buffon' based on work in 1753 by Comte de Buffon. In 1887, Thompson[16] considered the question of how to fill Euclidean space using truncated octahedrons, while Slichter[17] was the first to attempt to find analytical expressions for the porosity in beds of uniform spheres. There has also been interest in the alternative question of what is the minimal solids fraction \( \nu_m \) of a rigid assembly of uniform spheres such that each must touch at least four others and the contact points must not lie all in one plane. Along these lines, in 1932 Hilbert[18] found a structure for which, \( \nu_m=0.123 \), while a year later, Heesch and Laves[19] created a stable arrangement of spheres such that \( \nu_m=0.056 \). It should

Abstract

The change in bulk density or solids fraction that occurs when a vessel of granular materials is vibrated is an important, industrially-relevant process. In this paper, we present findings from experiments and discrete element simulations on the relaxation behavior of assemblies of uniform spheres that are vertically oscillated. Physical measurements of the bulk solids fraction, qualitatively reproduced in the simulations, reveal noticeable trends in the data dependent on the vibration amplitude and frequency. By carrying out extended simulations, a 'phase' chart depicting the percentage improvement in bulk density in terms of amplitude and frequency is obtained. Our results suggest that the behavior revealed in this chart may be characteristic of density relaxation in bulk solids.

Keywords: Density relaxation, Discrete element simulation, Solids fraction evolution, Vibrated bulk solids
be pointed out, however, that these geometric structures cannot be created using standard experimental procedures.

The distinction between a 'loose' (or poured) and 'dense' random packing of spheres was conceived by Oman and Watson \(^{(20)}\) in 1944. This idea was quantified in experiments by Scott \(^{(21)}\), in which 3mm steel ball bearings poured into cylindrical containers were subjected to two minutes of shaking. Although the vibration parameters used were not reported, he found two values of the solids fraction, i.e., \(v_{\text{loose}}=0.59\) and \(v_{\text{dense}}=0.63\) corresponding to a random close and dense configuration. Improved experiments by Scott and Kilgour \(^{(22)}\) yielded a more precise value of the dense solids fraction \(v_{\text{dense}}=0.6366 \pm 0.0005\). We note that deposition experiments in viscous liquids to achieve stable loose structures have reported solids fraction values as low as \(v_{\text{loose}}=0.555 \pm 0.0005\) for glass spheres and \(v_{\text{loose}}=0.506 \pm 0.0005\) for toughened acrylic spheres \(^{(23)}\).

The notion that a consolidated state of optimal bulk density could be achieved through the use of high frequencies and relatively small displacement amplitudes was suggested by Stewart in 1951 - a claim that found support in the 1964 experiments of Evans and Millman \(^{(24)}\). Several years earlier, Macrae et al. \(^{(25)}\) proposed that bulk density was related to impact velocity, with a critical value producing optimal results. In 1967, D'Appolonia \(^{(26)}\) performed vibration tests on dry sand within a cylindrical vessel. A mechanical shaker was employed to produce unidirectional harmonic motion of the cylinder with displacement amplitudes up to 0.254 mm and frequencies \(10 \leq f \leq 60\) Hz. By measuring the volume change of the sand over a range of vibration parameters, a plot of bulk density versus dimensionless acceleration \(\Gamma=a\omega^2/g\) was generated, indicating that the greatest increase took place at \(\Gamma \approx 2\). It was unclear from their data why increasing the acceleration beyond an optimal value resulted in decrease in bulk density. Dobry and Whitman \(^{(27)}\), who extended D’Appolonia’s experiments, reported in 1973 that the most rapid compaction occurred when 0.9 < \(\Gamma\) < 1.1, while a maximum bulk density was achieved for 1.1 < \(\Gamma\) < 1.3. At higher accelerations between 1.3 and 2.0, the density either stabilized or continued to increase.

In addition to the use of continuous vibrations, there have been investigations on tapped systems, such as that by Takahashi and Suzuki \(^{(28)}\) who studied the evolution of the volume of real powders. They described the phenomenology via a first-order rate law whose solution yielded a solids fraction \(v(n)\) as a function of the number of taps \(n\) that evolved to an apparent final density \(v_\infty\) in accordance to \(v(n)=v_\infty+(v_{\text{loose}}-v_\infty)e^{-\Gamma n}\). The effect of detached, vertical sinusoidal taps applied to a tall cylindrical vessel filled with 2mm mono-disperse, soda-lime glass spheres was experimentally studied by Knight et al. \(^{(29)}\) using a noninvasive, capacitive technique to measure solids fraction. The evolution of the measured solids fraction was found to rely on the relative acceleration and on the number of taps. Experimental data was fit to a four-parameter phenomenological model of the form \(v(n)=v_\infty(1-e^{-Bn})\), where \(v_\infty\) is the steady-state density (dependent on the acceleration history), \(B\) is a relaxation time, and \(B\) is an undetermined constant that depends on \(\Gamma\). Linz \(^{(30)}\) proposed an explanation of the experimental model from an analysis of the stroboscopic decay law that he derived from his physical interpretation of the compaction process. Knight et al.’s experiments were extended by Nowak et al. \(^{(31,32)}\) to explore the frequency dependence and amplitude of the measured density fluctuations as a function of vibration intensity \(\Gamma\). They found that at certain intensities the system attained a well-defined average steady-state density with large fluctuations after extended tapping. The magnitude of these fluctuations depended not only on the depth at which measurements were made, but also on \(\Gamma\) (e.g., increasing \(\Gamma\) produced larger fluctuations about the mean density).

In addition to physical experiments, particle level simulations that offer insights into the micro-structural process taking place during the density relaxation process have been done. For example, Baker and Mehta \(^{(33)}\) studied the qualitative effects of vibrations on a system of mono-disperse spheres using a hybrid simulation technique in which a real ‘shake’ was modeled by a controlled volume expansion of the assembly, a hard-sphere Monte Carlo at a low temperature to reduce the system potential energy, followed by a modified sequential random-close-packing algorithm to achieve stability. The authors carried out a careful analysis of the solids fraction and coordination number distribution as a function of their simulated shaking intensity, as well as a study of contact networks that corroborated the roles of various identified relaxation mechanisms. Rosato and Yacoub \(^{(34)}\) carried out discrete element simulations to assess the effect low amplitude \((a/d<0.1)\), vertical oscillations applied to a vessel of frictional, inelastic spheres of diameter \(d\). They reported fairly good fits of the data with phenomenological predictions of Knight et al. \(^{(29)}\) and the exponential decay model of Takahashi and Suzuki \(^{(28)}\).
More recently, An et al. \cite{35} presented discrete element results that supported the earlier findings of Zhang et al. \cite{36,37} with regard to the effect of frequency and amplitude on solids fraction. They identified two respective densification mechanisms corresponding to low and high relative accelerations.

The physical system of interest in this paper is a model granular material comprised of acrylic spheres housed in a vertically oscillated cylindrical container. We propose a basis for trends observed in our experiments (Section 2) on the bulk solids fraction as a function of the applied vibration amplitude and frequency. Our explanation hinges on a detailed series of discrete element simulations (Sections 3 and 4) that qualitatively reproduce the experimental behavior. While the parameter space examined in the simulations is rather extensive (e.g., see reference \cite{36}), in this paper we report on a subset of our studies that is relevant to the experiments. A phase diagram portraying ‘improvement’ in bulk density reveals distinct regions in the frequency-amplitude space which correlate with the experiments.

2. Experiments and Results

The containment vessel is an acrylic cylinder of diameter \(D\) formed from stacked rings that are mounted onto a B&K shaker. Fig. 1 shows a schematic of the experimental system. An accelerometer attached to the piston provides feedback control through which precise adjustments to the frequency and amplitude could be made. The first part of the experiment involved measurements of the initial (before vibrations are applied to the piston) bulk solids fraction. Particles (acrylic spheres, \(d = 3.175\) mm) were slowly poured into the cylinder and then the top layer was removed by sliding the top ring across. This was done to ensure that the cylinder was filled with a level surface to a height of 30\(d\). The poured bulk solids fraction \(v_0\) could then be computed from the volume of the cylinder and weight of its contents. For the aspect ratio \(D/d=20\) used in the experiments, we obtained an average value \(v_0 \approx 0.604\) that is typically associated with a loose random packing \cite{6}.

The vibration experiments were carried out by filling the cylinder with mono-disperse acrylic spheres \((d = 3.175\) mm) to an undisturbed bed depth \(H \approx 95.3\) mm using the method previously described. The filled cylinder was rigidly mounted onto the shaker head, which was sinusoidally oscillated over a range of amplitudes \(0.04 < a/d < 0.24\) and frequencies \(\omega\) between 25Hz–100Hz, (corresponding to relative accelerations \(\Gamma \equiv a\omega^2/g\) between 0.94 and 11.54). For each selected frequency and amplitude, the shaker was run for ten minutes, with each experiment repeated several times to confirm the results. In order to reduce the buildup of static charge, the inside tube wall and particles were treated with a household antistatic agent. In measuring the bulk solids fraction \(v_1\) after the vibrations were stopped, a level particle surface was formed using the same procedure as described in the pouring experiments. Figs. 2-6 show the improvement in solids fraction, \((\frac{v_1 - v_0}{v_1} \times 100)\) versus \(\Gamma\) for increasing values of \(a/d\).

Our experiments reveal four trends in the behavior of \(v\) versus \(\Gamma\) that depend on the level of the displacement amplitude. Fig. 2 shows that for \(a/d = 0.04\), the solids fraction increases with \(\Gamma\), but the improvement is only about 3.3% at \(\Gamma = 5.1\). Observations of the vibrated bed indicated that there was little or no bulk motion with the exception of some activity of the particles at the top surface for \(\Gamma > 2\). We conjecture that at these excitation levels the bulk density increases through slow, cooperative particle rearrangements throughout the assembly, analogous to the ‘push filling’ mechanism suggested by An et al. \cite{35}. When \(a/d\) is between 0.06 and 0.10, results in Fig. 3a show a maximum in bulk solids fraction \((v_m \approx 0.636\), cor-

![Fig. 1 Schematic of experimental apparatus.](image-url)
responding to an improvement $>5\%$) when $\Gamma$ is between 5 and 7, followed by a slight expansion of the bed (or a reduction in $\nu$) with a further increase of $\Gamma$. Near the peaks (at $\Gamma=\Gamma_c$), particles near the cylinder walls were seen to remain in contact with each other and to travel downwards in a collective manner, a phenomenon indicative of convective motion that has been reported in other studies. The movement of these wall particles became more pronounced as $\Gamma$ was increased to $\Gamma_c$. Further observations made near the walls and on the surface at $\Gamma_c$ revealed closed-packed arrangements of the particles as illustrated in Fig. 3b. We believe that the influence of the cylinder walls in the presence of the slow convection in this region at $\Gamma_c$ is responsible for the creation of these structures, which in turn contribute to the overall increase in bulk solids fraction. It is worthwhile noting that Nowak et al. also pointed out that the walls of the cylindrical container used in their experiments (aspect ratio $D/d \sim 9.4$) may have influenced the compaction process. In fact, they attained a substantially larger mean solids fraction ($\nu = 0.656$) than what is known as random close packing of spheres ($\nu = 0.6366$).

Distinct oscillations (Fig. 4) in the solids fraction versus $\Gamma$ were found when $a/d = 0.16$, where the improvement in the bulk density was generally less than 3.6%. At this amplitude level, convection could still be observed as particles adjacent to the walls moved downward. At the highest amplitude level used ($a/d = 0.24$), the solids fraction versus $\Gamma$ remains almost constant and little improvement (approximately 0.5%) was attained (Fig. 5). At these conditions, a marked expansion of the bed depth occurred (analogous to other experiments reported in the literature) and the system resembled a dense, energetic “gas” of...
particles (it required relatively little effort to insert a rigid bar into the vibrating mass). We were unable to see any particle convection near the cylinder walls in this case.

3. Discrete Element Simulations

The discrete element simulations reported in this paper employ the soft “partially latching spring model” developed by Walton et al. 48-50) for elastic-plastic collisions. This model features mechanisms of energy dissipation for both the normal (i.e., along the line of centers of the pair of colliding particles) and tangential directions. Energy loss in the normal direction is accomplished using linear springs that are activated as particles overlap and then move apart. Because the loading spring constant $K_1$ is smaller than the unloading (or restituting) spring of constant $K_2$, the normal relative separation velocity is smaller than the relative approach velocity, and this produces a constant effective restitution coefficient $e = K_1/K_2$. For the flows simulated in this study, collision velocities were not of sufficient magnitude to warrant the use of a velocity-dependent restitution that is appropriate for particle velocities of the order of 1 m/s and greater.

In the tangential direction the Walton-Braun model approximates Mindlin’s and Deresiewicz’s theory 51) for elastic spheres subjected to tangential loading. Dissipation is achieved through a tangential stiffness that decreases with tangential displacement until it is zero, at which point full sliding occurs at the friction limit $\mu$. Although contacts that experience rotation coupled with tangential sliding are not a feature of this model, particles can rotate due to the transmission of tangential impulse.

The time step $\Delta t$ through which the particle equations of motion are integrated is approximated from the normal force model by dividing twice the time spent in unloading period during a particle collision into $n$ steps. It can be shown that $\Delta t = \frac{\pi e}{m} \sqrt{2m_i K_1}$ where $m$ is the particle mass and $e$ is the restitution coefficient. The value selected for $K_1$ ensured that overlaps were less than approximately 0.01 in accordance with the behavior of real colliding particles, so that $\Delta t \sim 10^{-9}$ s. The mass density of the simulated particles ($\rho = 1200$ kg/m$^3$) corresponded to acrylic plastic to match the experimental material. The equations of motion of the $N$ particles are integrated using a leap-frog method with a backward Euler approximation at $t = 0$. For the translational motion (rotation equations are analogous), the discretization is given by

$$v_i^{t+1/2} = v_i^{t+1/2} + \frac{F_i}{2m_i} \Delta t, \quad t > 0, \quad i = 1, 2, \ldots N$$

$$x_i^{t+1} = x_i^{t} + v_i^{t+1/2} \Delta t$$

$$v_i^{t-1/2} = v_i^{t} - \frac{F_i}{4m_i}, \quad i = 1, 2, \ldots N$$

where $F$ is the net force on the $i^{th}$ particle.

In the results of our study described next, the computational cell was a rectangular box having a solid side walls and a floor whose motion was governed by $s(t) = \text{asin}(2\pi ft)$.

4. Simulation Results and Discussion

A poured assembly of spheres was obtained by
initially positioning particles randomly within the computational cell, after which they were allowed to fall under the action of gravity for 1.5 seconds. This generated a stable configuration with a bulk solids fraction that did not change appreciably for runs of longer duration. We remark that the results of our pouring simulations\textsuperscript{36} produced bulk solid fractions that were consistent with experimental measurements in the literature. The poured assembly was then used as the starting point for the vibration simulations. The assembly was energized by applying vertical, sinusoidal oscillations to the floor of the computational cell for three seconds, followed by a relaxation phase during which particles fell under gravity to a stable state. Although not reported in this paper, we found that the three seconds of vibration was sufficient for the system to reach steady-state conditions from time-averaged depth profiles of ‘granular temperature’ and solids fraction.

In what follows, the behavior of the bulk solids fraction as a function of amplitude and frequency is presented. In all cases, spheres were assigned a normal restitution coefficient $e = 0.9$, and friction coefficient $\mu_p = 0.1$. As will be seen, the results were in reasonable qualitative agreement with the trends observed in our previously described experiments. In Section 4.2, a much larger system whose aspect ratio more closely matches the experiments is considered. Here, the simulated results are presented in the form of a chart that shows the improvement in density as a function of frequency and amplitude.

4.1 Vibration Frequency and Amplitude

Four amplitude ratios were considered ($a/d = 0.02$, $0.08$, $0.24$, $0.48$) over frequencies ranging from 5Hz to 90Hz. Spheres were assigned a normal restitution coefficient $e = 0.9$, and friction coefficient $\mu_p = 0.1$. These values were obtained from experiments of Louge \textsuperscript{52} for acrylic spheres (i.e., $\mu_p = 0.096 \pm 0.006$, $e=0.934 \pm 0.009$). Although we reduced the value of $e$ somewhat to improve computational efficiency, this did not produce any significant changes in the qualitative trends reported in this paper. The selection of a shallow configuration (i.e., poured fill height of approximately $7d$ and $v_0 \approx 0.577$) and an aspect ratio $L/d = 9.4$ minimized the system size ($N = 600$), so that greater computational efficiency was obtained. Despite the small system size, the qualitative trends obtained were consistent with those found in the physical experiments. Although not reported here, we also carried out studies in deeper beds and found similar behavior of the system’s ‘dynamic’ state\textsuperscript{36}, which in turn impacts the densification upon relaxation.

The general trend in Fig. 6 when $a/d = 0.02$ is very much the same as what took place in the physical experiments (see Fig. 2). Over the frequency range tested at this amplitude, there is a continual improvement in bulk solids fraction. When $a/d = 0.08$, the data shown in Fig. 7 suggests a peak around 6% near $f \approx 50$Hz; for greater frequencies, a reduction in solids fraction takes place. When the amplitude $a/d$ is $0.24$ (Fig. 8), the peak value occurs at approximately 40 Hz, and the curve decays thereafter until, near 80 Hz, no improvement in bulk density is possible at higher frequencies. The occurrence of the peak and decay afterwards is consistent with the experimental observations of Appolonia et al.\textsuperscript{26}. Finally, at $a/d= 0.48$, the improvement in solids fraction is minimal (Fig. 9), and after a frequency of approximately 35 Hz, the system does not experience any densification upon relaxation. This trend is analogous to the experiments reported in Fig. 5.
A comparison between the experiments (Fig. 2-5) and the simulation (Fig. 6-9) shows reasonable qualitative agreement. Furthermore, the maximum improvement in solids fraction of approximately 6% agrees with the experimental measurements. Although there are quantitative differences between the simulated and experimental results (possibly attributed to boundary conditions and aspect ratio), the simulation does generate all of the important critical phenomena observed in the experiments.

### 4.2 Densification Phase Chart

We studied a system of $N = 8000$ spheres in a computational cell having an aspect ratio $L/d \approx 25$. In so doing, we were able to further validate the simulations in a system whose cross-sectional dimension was comparable to our experiments. The procedure previously described was followed to generate the initial poured assembly ($\nu_0 \approx 0.604$), which filled the cell to a depth of approximately $11d$. Due to limited computational resources, we were unable to carry out studies for deeper beds at this aspect ratio, which would have permitted real quantitative comparisons with our experiments.

We ran a test to identify a specific value of the amplitude and frequency at which the relaxed bulk density was largest; this occurred at $a/d = 0.16$ and $f = 40$ Hz. With these parameters, the system was vibrated until the computed solids fraction curve flattened out (Fig. 10). An extrapolation of the data as $1/t \to 0$ yielded a solids fraction $\nu = 0.658$, in close agreement with the experimental results of Nowak et al.\textsuperscript{53}. Arrangements of the particles adjacent to the side walls at $t = 13s$ revealed the formation of hexagonally packed structures analogous to what was observed in
our experiments

A series of case studies was carried out spanning vibration amplitudes and frequencies $0.02 < a/d < 0.48$ and $10\text{Hz} < f < 90\text{Hz}$. Our results are summarized in Fig. 11 in the form of a chart showing the improvement in bulk solids fraction as a function of amplitude and frequency. Four distinct regions are visible corresponding to various levels of improvement as indicated by the gray scale. A region of optimal improvement (in the amplitude-frequency space) is clearly visible, a trend that is in agreement with our experiments. Although not presented in this paper, we also examined profiles of the system’s granular temperature, solids fraction and ratio of lateral to vertical kinetic energy, which revealed common characteristics for each region\(^3\). At the poorest levels of improvement, the vibrated system was found to be in a relatively high energetic state which settled to a
5. Conclusions

The densification behavior that takes place when a vertically oscillated system of uniform spheres is allowed to relax under gravity was investigated. The work finds its motivation in the industrial sector involved with the packaging of bulk solids. Experiments were carried out that exhibited clear trends regarding the influence of amplitude and frequency in achieving a state of optimal density. These trends were qualitatively reproduced in discrete element experiments. The results of the latter study were summarized in a ‘phase’ chart that revealed distinct frequency-amplitude regions which were characterized by various levels of the ‘improvement’ in the bulk density. Except for quantitative differences that depend on particle properties and mass overburden, we expect that similar patterns for the improvement would emerge for other materials.

6. Acknowledgements.

We acknowledge Union Carbide Corporation for the financial support of the Granular Science Laboratory and the research activities related to this paper. The authors also thank P. Singh, D. Blackmore, and J. Luke for their interest in this work and discussions, and O. Dybenko for help with the plots.

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Author’s short biography

Anthony D. Rosato, Ph.D.

Anthony Rosato received his PhD in Mechanical Engineering from Carnegie Mellon University in 1985. He has been at the New Jersey Institute of Technology since 1987 where he holds the rank of Professor of Mechanical Engineering. Dr. Rosato’s research interests are in the broad field of particle technology, with a focus on computational modeling and experimental studies of granular flows related to the solids handling and processing industries. From 1995-1999, he served as the founding director of NJIT’s Particle Technology Center, and he has been the director of the Granular Science Laboratory in the ME Department at NJIT since 1999. Dr. Rosato has held visiting appointments at Lawrence Livermore National Laboratory, Worcester Polytechnic Institute, the Lovelace Institutes (Albuquerque, NM), ESPCI in Paris and Stanford University. He is a Fellow of the ASME, Co-editor-in-chief of Mechanics Research Communications, and a member of the Academy of Mechanics, and Sigma Xi.

Dr. Ninghua Zhang

Ninghua Zhang received his PhD in 2004 from the New Jersey Institute of Technology, where he carried out experiments and particle-level simulations on density relaxation induced by vibrations. Before he began his doctoral studies in the United States, Dr. Zhang was employed as a project engineer in the glass and fiberglass industry in China. Here, he was involved in research and development related to manufacturing processes, and he also served as a liaison to several US companies. After 2004, he worked in the medical device field and currently, he is a design engineer for Oxford Plastics in New Jersey. Dr. Zhang is an affiliate of the Granular Science Laboratory in the Department of Mechanical Engineering at NJIT, where he is involved with several ongoing research projects.