Wick rotation for D-modules

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Abstract

We extend the classical Wick rotation to D-modules and higher codimensional submanifolds.

1 Introduction

Let $M$ be a real analytic manifold of the type $N \times \mathbb{R}$ and let $X = Y \times \mathbb{C}$ be a complexification of $M$. Consider a differential operator $P$ on $X$ such that $P$ is hyperbolic on $M$ with respect to the direction $N \times \{0\}$, a typical example being the wave operator on a spacetime. Denote by $L$ the real manifold $N \times \sqrt{-1}\mathbb{R}$. It may happen, and it happens for the wave operator, that $P$ is elliptic on $L$. Passing from $M$ to $L$ is called the Wick rotation by physicists who deduce interesting properties of $P$ on $M$ from the study of $P$ on $L$.

In the situation above, we had $\text{codim}_M N = \text{codim}_L N = 1$. In this paper, we treat the general case of two real analytic manifolds $M$ and $L$ in $X$, $X$ being a complexification of both $M$ and $L$, such that the intersection $N := M \cap L$ is clean, and we consider a coherent $\mathcal{D}_X$-module $\mathcal{M}$ which is hyperbolic with respect to $M$ on $N$ and elliptic on $L$. The main result is Theorem 3.10 which describes an isomorphism in a neighborhood of $N$ between the complex of hyperfunction solutions of $\mathcal{M}$ on $L$ defined in a given cone $\gamma \subset T_N L$ and the complex of hyperfunction solutions of $\mathcal{M}$ on $M$ with wave front set in a cone $\lambda \subset T_M^*X$ associated with $\gamma$. It is also proved that this isomorphism is compatible with the boundary values morphism from $M$ to $N$ and from $L$ to $N$.

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2 Sheaves, D-modules and wave front sets

2.1 Sheaves

We shall use the microlocal theory of sheaves of [KS90] and mainly follow its terminology. For the reader’s convenience, we recall a few notations and results.

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1
Geometry

Let $X$ be a real manifold of class $C^\infty$. For a subset $A \subset X$, we denote by $\overline{A}$ its closure and by $\text{Int}(A)$ its interior. We denote by

$$\tau_X : TX \to X, \quad \pi_X : T^*X \to X$$

the tangent bundle and the cotangent bundle to $X$. For a closed submanifold $M$ of $X$, we denote by $\tau_M : T_MX \to M$ and $\pi_M : T^*_MX \to M$ the normal bundle and the conormal bundle to $M$ in $X$. In particular, $T^*_X X$ is the zero-section of $T^*X$, that we identify with $X$.

For a vector bundle $\pi : E \to X$, we identify $X$ with the zero-section, we denote by $E_x$ the fiber of $E$ at $x \in X$, we set $\hat{E} = E \setminus X$ and we denote by $\hat{\pi} : \hat{E} \to X$ the projection. For a cone $\gamma$ in a vector bundle $E \to X$, we set $\gamma_x = \gamma \cap E_x$, we denote by $\gamma^a = -\gamma$ the opposite cone and by $\gamma^o$ the polar cone in the dual vector bundle $E^*$,

$$\gamma^o = \{(x, \xi) \in E^*; \langle \xi, v \rangle \geq 0 \text{ for all } x \in M, v \in \gamma_x\}.$$ 

For $A \subset X$, the Whitney normal cone of $A$ along $M$, $C_M(A) \subset T_MX$, is defined in [KS90, Def. 4.1.1].

To a morphism of manifolds $f : Y \to X$, one associates the maps:

$$\begin{align*}
T^*Y & \xrightarrow{f_d} Y \times_X T^*X \\
& \xrightarrow{\pi} Y \xrightarrow{f} X
\end{align*}$$

where $f_d$ is the transpose of the tangent map $Tf : TY \to Y \times_X TX$.

**Definition 2.1.** Let $\Lambda$ be a closed conic subset of $T^*X$. One says that $f$ is non characteristic for $\Lambda$ if the map $f_d$ is proper on $f^{-1}_\pi(\Lambda)$.

Sheaves

Let $k$ be a field. One denotes by $\mathcal{D}^b(k_X)$ the bounded derived category of sheaves of $k$-vector spaces on $X$. We simply call an object of this category “a sheaf”. For a closed subset $A$ of a manifold we denote by $k_A$ the constant sheaf on $A$ with stalk $k$ extended by $0$ outside of $A$. More generally, we shall identify a sheaf on $A$ and its extension by $0$ outside of $A$. If $A$ is locally closed, we keep the notation $k_A$ as far as there is no risk of confusion. We denote by $\omega_X$ the dualizing complex on $X$. Recall that $\omega_X \simeq \text{or}_X [\text{dim } X]$ where $\text{or}_X$ is the orientation sheaf and $\text{dim } X$ is the dimension of $X$. More generally, we consider the relative dualizing complex associated with a morphism $f : Y \to X$, $\omega_{Y/X} = \omega_Y \otimes f^{-1}(\omega_X^{\otimes -1})$ and its inverse, $\omega_{X/Y} = \omega_{Y/X}^{\otimes -1}$. We denote by $\mathcal{D}_X^\bullet (\bullet) = \mathcal{R}\mathcal{H}\text{om} (\bullet, k_X)$ the duality functor on $X$.

We shall use freely the six Grothendieck operations on sheaves.

Microlocalization

For a closed submanifold $M$ of $X$, we have the functors

$$\begin{align*}
\nu_M : \mathcal{D}^b(k_X) & \to \mathcal{D}^b_{\mathbb{R}^+}(k_{TMX}) \text{ specialization along } M, \\
\mu_M : \mathcal{D}^b(k_X) & \to \mathcal{D}^b_{\mathbb{R}^+}(k_{TM^*X}) \text{ microlocalization along } M, \\
\mu\text{hom} : \mathcal{D}^b(k_X) \times \mathcal{D}^b(k_X)^{\text{op}} & \to \mathcal{D}^b_{\mathbb{R}^+}(k_{TM^*X}).
\end{align*}$$
Here, for a vector bundle $E \rightarrow M$ or $E \rightarrow X$, $\text{D}^b_{\mathbb{R}^+}(k_E)$ is the full subcategory of $\text{D}^b(k_E)$ consisting of conic sheaves, that is, sheaves locally constant under the $\mathbb{R}^+$-action.

The functor $\mu_M$, called Sato's microlocalization functor, is the Fourier–Sato transform of the specialization functor $\nu_M$. The bifunctor $\mu_{\text{hom}}$ of [KS90] is a slight generalization of $\mu_M$.

Recall that $\mu_M(\bullet) = \mu_{\text{hom}}(k_M, \bullet)$.

Let $\lambda$ be a closed convex proper cone of $T^*_M X$ containing the zero-section $M$. For $F \in \text{D}^b(k_X)$, we have an isomorphism (see [KS90, Th. 4.3.2]):

$$R \pi_M^* R \Gamma_{\lambda}(\mu_M(F)) \otimes \omega_{X/M} \simeq R \tau_M^* R \Gamma_{\lambda^\circ}(\nu_M(F)).$$

(Recall that $\lambda^\circ$ is the opposite of the polar cone $\lambda$.)

**Microsupport**

To a sheaf $F$ is associated its microsupport $\mu_{\text{supp}}(F)^4$, a closed $\mathbb{R}^+$-conic co-isotropic subset of $T^*X$.

Let us recall some results that we shall use.

**Theorem 2.2.** Let $f : Y \rightarrow X$ be a morphism of real manifolds and let $F \in \text{D}^b(k_X)$. Assume that $f$ is non characteristic for $F$, that is, for $\mu_{\text{supp}}(F)$. Then the morphism $f^{-1}F \otimes \omega_{Y/X} \rightarrow f^!F$ is an isomorphism.

As a particular case of this result, we get a kind of Petrowski theorem for sheaves (see Theorem 2.11 below):

**Corollary 2.3.** Let $M$ be a closed submanifold of $X$ and let $F \in \text{D}^b(k_X)$. Assume that $T^*_M X \cap \mu_{\text{supp}}(F) \subset T^*_X X$. Then $F \otimes k_M \simeq R \Gamma_{M} F \otimes \omega_{M/X} [\text{codim}_X M]$.

Let $M$ be a closed submanifold of $X$. If $\Lambda \subset T^*X$ is a closed conic subset, its Whitney normal cone along $T^*_M X$ is a closed biconic subset of $T^*_T M T^*X \simeq T^*_T T^*_M X$. Moreover, there exists a natural embedding

$$T^*M \hookrightarrow T^*T^*_M X \simeq T^*_T M T^*X.
$$

(2.3)

Now we consider a morphism of manifolds $g : L \rightarrow X$ and let $M \subset X$ and $N \subset L$ be two closed submanifolds such that the map $g$ induces a closed embedding $g|_N : N \hookrightarrow M$. One gets the maps

(2.4)

$$
\begin{array}{c}
T^*L \xrightarrow{g_4} L \times_X T^*X \xrightarrow{g_6} T^*X \\
T^*_N L \xrightarrow{g_{Nd}} N \times_M T^*_M X \xrightarrow{g_{Nd}} T^*_M X.
\end{array}
$$

The next result is a particular case of [KS90, Th. 6.7.1] in which we choose $V = T^*_X L$ and write $g : L \rightarrow X$ instead of $f : Y \rightarrow X$. (The reason of this change of notations is that we need to consider the complexification of the embedding $N \hookrightarrow M$ that we shall denote by $f : Y \hookrightarrow X$.)

**Theorem 2.4.** Let $F \in \text{D}^b(k_X)$ and assume

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4. $\mu_{\text{supp}}(F)$ was denoted $\text{SS}(F)$ in [KS90].
(a) $g$ is non characteristic for $F$,

(b) the map $g_{N \pi}$ is non characteristic for $C_{T_M X}(\mu \text{supp}(F))$,

(c) $g_d^{-1}T_N^* L \cap g_{\pi}^{-1} \text{supp}(F) \subset L \times X T_M^* X$.

Then one has the commutative diagram of natural isomorphisms on $T_L^* X$:

$$\begin{align*}
\begin{array}{c}
\text{RG}_{N \| \cdot} (\omega_{N/M} \otimes g_{\pi}^{-1} \mu_M(F)) \xrightarrow{\sim} \mu_N (\omega_{L/X} \otimes g^{-1} F) \\
\text{RG}_{N \| \cdot} (g_i^1 \mu_M(F)) \xleftarrow{\sim} \mu_N (g^1 F).
\end{array}
\end{align*}$$

**Notation 2.5.** As usual, we have simply written $\omega_M$ instead of $\pi^{-1} \omega_M$ and similarly with other locally constant sheaves.

Consider the projections

$$\begin{align*}
T_N^* L \xrightarrow{\text{g}_{\pi}} N \times M T_M^* X \xrightarrow{\text{g}_{\pi}} T_M^* X
\end{align*}$$

One has the isomorphisms

$$\begin{align*}
\text{R} \pi_{N*} \text{RG}_{N \| \cdot} (g_{N \pi}^i \mu_M(F)) & \simeq \text{R} \pi_* (g_{N \pi}^i \mu_M(F)) \\
& \simeq g^i \text{R} \pi_{M*} \mu_M(F) \simeq \text{R} \Gamma_N F,
\end{align*}$$

and

$$\begin{align*}
\text{R} \pi_{N*} \mu_N (g^1 F) \simeq \text{R} \Gamma_N g^1 F \simeq \text{R} \Gamma_N F.
\end{align*}$$

Moreover, one easily proves:

**Lemma 2.6.** The isomorphisms (2.7) and (2.8) are compatible with the morphisms obtained by applying $\text{R} \pi_{N*}$ to (2.5).

**Lemma 2.7.** In the situation of Theorem 2.4 assume moreover that $g : L \to X$ is a closed embedding, $N = L \cap M$ and the intersection is clean (that is, $TN = N \times_M TM \cap N \times_L TL$). Then condition (c) follows from (b).

**Proof.** Let us choose a local coordinate system $(x', x'', y', y'')$ on $X$ such that $M = \{ y' = y'' = 0 \}$ and $L = \{ x'' = y'' = 0 \}$. Denote by $(x', x'', y', y''; \xi', \xi'', \eta', \eta'')$ the coordinates on $T^* X$ and by $(x', x''; \xi', \xi'')$ the coordinates on $T^* M$. Then

$$\begin{align*}
M & = \{ y' = y'' = 0 \}, & T_M^* X & = \{ y' = y'' = \xi' = \xi'' = 0 \}, \\
L & = \{ x'' = y'' = 0 \}, & T_L^* X & = \{ x'' = y'' = \xi' = \eta' = 0 \}, \\
N & = \{ x'' = y' = y'' = 0 \}, & T_N^* X & = \{ x'' = y' = y'' = \xi' = 0 \},
\end{align*}$$

$$g_d : (x', x''; \xi', \xi'', \eta', \eta'') \mapsto (x', y'; \xi', \eta').$$

Therefore $g_d^{-1}T_N^* L = \{ (x', y'; \xi', \xi'', \eta', \eta'') \in L \times_X T^* X ; y' = \xi' = 0 \} = T_N^* X$. Let $\theta \in T_{T_N^* X}^* T_N^* X$ with $\theta \notin C_{\pi}^* T_M^* (\mu \text{supp}(F))$. Then $(x', x''; \eta', \eta'') + \theta \notin \mu \text{supp}(F)$. Choosing $\theta \in T_N^* M$, $\theta \neq 0$, we get that $(x', 0; 0, \xi'', \eta', \eta'') \in \mu \text{supp}(F)$ implies $\xi'' = 0$. Q.E.D.
2.2 Analytic wave front set

From now on and until the end of this paper, unless otherwise specified, all manifolds are (real or complex) analytic and the base field $k$ is $\mathbb{C}$.

Let $M$ be a real manifold of dimension $n$ and let $X$ be a complexification of $M$. One denotes by $\mathcal{A}_M$ the sheaf of complex valued real analytic functions on $M$, that is, $\mathcal{A}_M = \mathcal{O}_X|_M$.

One denotes by $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaves on $M$ and $T^*_M X$ of Sato’s hyperfunctions and microfunctions, respectively. Recall that these sheaves are defined by

$$\mathcal{A}_M := \mathcal{O}_X \otimes \mathcal{C}_M, \quad \mathcal{B}_M := \text{RHom}(\mathcal{D}'_X \mathcal{C}_M, \mathcal{O}_X), \quad \mathcal{C}_M := \mu\text{hom}(\mathcal{D}'_X \mathcal{C}_M, \mathcal{O}_X).$$

In particular, $\text{RHom}(\mathcal{D}'_X \mathcal{C}_M, \mathcal{O}_X)$ and $\mu\text{hom}(\mathcal{D}'_X \mathcal{C}_M, \mathcal{O}_X)$ are concentrated in degree 0.

Since $\mathcal{D}'_X \mathcal{C}_M \cong \mathcal{O}_M[-n] \cong \omega_{M/X} \cong \omega_M^{-1}$, we get that

$$\mathcal{B}_M \cong \text{R}\Gamma_M(\mathcal{O}_X) \otimes \omega_M \cong H^n_M(\mathcal{O}_X) \otimes \mathcal{O}_M,$$

$$\mathcal{C}_M \cong \mu_M(\mathcal{O}_X) \otimes \omega_M \cong H^n(\mu_M(\mathcal{O}_X)) \otimes \mathcal{O}_M.$$

The sheaf $\mathcal{B}_M$ is flabby and the sheaf $\mathcal{C}_M$ is conically flabby.

Moreover, since $\text{R}\pi_* \circ \mu\text{hom} \cong \text{RHom}$, we have the isomorphism $\mathcal{B}_M \cong \pi_\ast \mathcal{C}_M$. One deduces the isomorphism:

$$\text{spec}: \Gamma(M; \mathcal{B}_M) \cong \Gamma(T^*_M X; \mathcal{C}_M).$$

Definition 2.8 ([Sat70]). The analytic wave front set of a hyperfunction $u \in \Gamma(M; \mathcal{B}_M)$, denoted $WF(u)$, is the support of $\text{spec}(u)$, a closed conic subset of $T^*_M X$.

The next result is well-known to the specialists. Let $M$ be a real analytic manifold, $X$ a complexification of $M$ and let $\lambda$ be a closed convex proper cone in $T^*_M X$.

Theorem 2.9. Let $u \in \Gamma(M; \mathcal{B}_M)$ with $WF(u) \subset \lambda$. Assume that $M$ is connected and that $u \equiv 0$ on an open subset $U \subset M$, $U \neq \emptyset$. Then $u \equiv 0$ on $M$.

Proof. Let $S = \text{supp}(u)$ and let $x \in \partial S$. Choosing a local chart in a neighborhood of $x$, we may assume from the beginning that $M$ is open in $\mathbb{R}^n$ and that $\lambda \subset M \times \sqrt{-1}\gamma^\circ$ where $\gamma$ is a non empty open convex cone of $\mathbb{R}^n$. Then there exists a holomorphic function $f \in \Gamma(\langle M \times \sqrt{-1}\gamma \rangle \cap W; \mathcal{O}_X)$, where $W$ is a connected open neighborhood of $M$ in $X$, such that $u = b(f)$, that is, $u$ is the boundary value of $f$. If $b(f)$ is analytic on $U$, then $f$ extends holomorphically in a neighborhood of $U$ in $X$. If moreover $f = 0$ on $U$, then $f \equiv 0$ on $\langle M \times \sqrt{-1}\gamma \rangle \cap W$ and thus $u \equiv 0$.

Q.E.D.

2.3 D-modules

Let $(X, \mathcal{O}_X)$ be a complex manifold. One denotes by $\mathcal{D}_X$ the sheaf of rings of finite order holomorphic differential operators on $X$. In the sequel, a $\mathcal{D}_X$-module means a left $\mathcal{D}_X$-module. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Locally on $X$, $\mathcal{M}$ may be represented as the cokernel of a matrix $\cdot P_0$ of differential operators acting on the right:

$$\mathcal{M} \cong \frac{\mathcal{D}_X^{N_0}}{\mathcal{D}_X^{N_1} \cdot P_0}$$

and one shows that $\mathcal{M}$ is locally isomorphic to the cohomology of a bounded complex

$$(2.9) \quad \mathcal{M}^\bullet := 0 \rightarrow \mathcal{D}_X^{N_0} \rightarrow \cdots \rightarrow \mathcal{D}_X^{N_1} \rightarrow \mathcal{D}_X^{N_0} \rightarrow \mathcal{M}^0.$$
Clearly, $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module. It is indeed coherent since $\mathcal{O}_X \cong \mathcal{D}_X/\mathcal{I}$ where $\mathcal{I}$ is the left ideal generated by the vector fields. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, one sets for short

$$\mathcal{I} \text{ol}(\mathcal{M}) := \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Representing (locally) $\mathcal{M}$ by a bounded complex $\mathcal{M}^\bullet$ as above, we get

$$(2.10) \quad \mathcal{I} \text{ol}(\mathcal{M}) \cong 0 \to \mathcal{O}_X^0 \xrightarrow{P_0} \mathcal{O}_X^1 \to \cdots \mathcal{O}_X^{N_f} \to 0,$$

where now $P_0$ operates on the left.

Hence a coherent $\mathcal{D}_X$-module is nothing but a system of linear partial differential equations.

To a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is associated its characteristic variety $\text{char}(\mathcal{M})$, a closed analytic $\mathbb{C}^\times$-conic co-isotropic subset of $T^*X$.

**Theorem 2.10** (see [KS90, Th. 11.3.3]). Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then $\mu\text{supp}(\mathcal{I} \text{ol}(\mathcal{M})) = \text{char}(\mathcal{M})$.

Let $f: Y \to X$ be a morphism of complex manifolds. One can define the inverse image $f^!\mathcal{M}$, an object of $\mathcal{D}^b(\mathcal{D}_Y)$. The Cauchy-Kowalevska theorem has been extended to $\mathcal{D}$-modules in Kashiwara’s thesis of 1970.

**Theorem 2.11** (see [Kas95, Kas03]). Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and assume that $f$ is non characteristic for $\mathcal{M}$, that is, for $\text{char}(\mathcal{M})$. Then

(i) $f^!\mathcal{M}$ is concentrated in degree 0 and is a coherent $\mathcal{D}_Y$-module,

(ii) $\text{char}(f^!\mathcal{M}) = f_!f^{-1}\text{char}(\mathcal{M})$,

(iii) one has a natural isomorphism $f^{-1}\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \cong \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_Y}(f^!\mathcal{M}, \mathcal{O}_Y)$.

**Example 2.12.** Assume $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X \cdot P$ for a differential operator $P$ of order $m$ and $Y$ is a smooth hypersurface, non characteristic for $P$. Let $s = 0$ be a reduced equation of $Y$. Then, $f^!\mathcal{M} \cong \mathcal{D}_Y/(s \cdot \mathcal{D}_Y + \mathcal{D}_X \cdot P)$ and it follows from the Weierstrass division theorem that, locally, $f^!\mathcal{M} \cong \mathcal{D}_Y^m$. In this case, isomorphism (iii) in the above theorem is nothing but the Cauchy-Kowalevska theorem.

**Definition 2.13.** Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let $L \subset X$ be a real submanifold. One says that the pair $(L, \mathcal{M})$ is elliptic if $\text{char}(\mathcal{M}) \cap T^*_L X \subset T^*_L X$.

If $X$ is a complexification of a real manifold $M$, the pair $(M, \mathcal{M})$ is elliptic if and only if $\mathcal{M}$ is elliptic in the usual sense and Corollary 2.3 gives, for $F = \mathcal{I} \text{ol}(\mathcal{M})$, the isomorphism

$$(2.11) \quad \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \cong \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M).$$

In particular, the hyperfunction solutions of the system $\mathcal{M}$ are real analytic. More generally, we have

**Theorem 2.14** ([Sat70]). Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let $u \in \Gamma(M; \mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M))$. Then $\text{WF}(u) \subset \Gamma_M X \cap \text{char}(\mathcal{M})$.

When $L = Y$ is a complex submanifold of complex codimension $d$, $(Y, \mathcal{M})$ is elliptic if and only if the embedding $Y \hookrightarrow X$ is non-characteristic for $\mathcal{M}$. In this case, Corollary 2.3 gives the isomorphism

$$(2.12) \quad f^{-1}\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \cong \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{R}\Gamma_Y \mathcal{O}_X)[2d].$$
3 Wick rotation for D-modules

3.1 Hyperbolic D-modules

Let $M$ be a real manifold and let $X$ be a complexification of $M$. Recall the embedding $T^*M \hookrightarrow T^*T^*_M X$ of (2.3) and recall that for $S \subset T^*X$, the Whitney cone $C_{T^*_M X}(S)$ is contained in $T^*_M T^*X \simeq T^*T^*_M X$. The next definition is extracted from [KS90]. See [Sch13] for details.

**Definition 3.1.** Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module.

(a) We set

$$\text{hypchar}_M(\mathcal{M}) = T^*M \cap C_{T^*_M X}(\text{char}(\mathcal{M}))$$

and call hypochar$_M(\mathcal{M})$ the hyperbolic characteristic variety of $\mathcal{M}$ along $M$.

(b) A vector $\theta \in T^*M$ such that $\theta \notin \text{hypchar}_M(\mathcal{M})$ is called hyperbolic with respect to $\mathcal{M}$.

(c) A submanifold $N$ of $M$ is called hyperbolic for $\mathcal{M}$ if

$$T^*_N M \cap \text{hypchar}_M(\mathcal{M}) \subset T^*_M M,$$

that is, any nonzero vector of $T^*_N M$ is hyperbolic for $\mathcal{M}$.

(d) For a differential operator $P$, we set $\text{hypchar}(P) = \text{hypchar}_M(\mathcal{D}_X/\mathcal{D}_X \cdot P)$.

**Example 3.2.** Assume we have a local coordinate system $(x + \sqrt{1}y)$ on $X$ with $M = \{y = 0\}$ and let $(x + \sqrt{1}y; \xi + \sqrt{1}\eta)$ be the coordinates on $T^*X$ so that $T^*_M X = \{y = \xi = 0\}$. Let $(x_0; \theta_0) \in T^*M$ with $\theta_0 \neq 0$. Let $P$ be a differential operator with principal symbol $\sigma(P)$. Then $(x_0; \theta_0)$ is hyperbolic for $P$ if and only if

$$\begin{cases} 
\text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ and an open conic} \\
\text{neighborhood } \gamma \text{ of } \theta_0 \in \mathbb{R}^n \text{ such that } \sigma(P)(x; \theta + \sqrt{1}\eta) \neq 0 \text{ for all } \eta \in \mathbb{R}^n, x \in U \text{ and } \theta \in \gamma.
\end{cases}$$

(3.3)

As noticed by M. Kashiwara, it follows from the local Bochner’s tube theorem that Condition (3.3) can be simplified: $(x_0; \theta_0)$ is hyperbolic for $P$ if and only if

$$\begin{cases} 
\text{there exists an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ such that} \\
\sigma(P)(x; \theta_0 + \sqrt{1}\eta) \neq 0 \text{ for all } \eta \in \mathbb{R}^n, \text{ and } x \in U.
\end{cases}$$

(3.4)

Hence, one recovers the classical notion of a (weakly) hyperbolic operator.

**Notation 3.3.** As usual, we shall write $\mathcal{R}\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{E}_M)$ instead of $\mathcal{R}\text{Hom}_{\pi^{-1}\mathcal{D}_X}(\mathcal{M}, \mathcal{E}_M)$ and similarly with other sheaves on cotangent bundles.

3.2 Main tool

Consider as above a real manifold $M$ and a complexification $X$ of $M$, a closed submanifold $N$ of $M$, and $Y$ a complexification of $N$ in $X$. Denote as above by $f: Y \hookrightarrow X$ the embedding. Consider also another closed real submanifold $L \subset X$ such that $L \cap M = N$ and the intersection is clean. Denote by $g: L \hookrightarrow X$ the embedding and consider the Diagram 2.4.
Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and consider the hypotheses:

(3.5) the pair $(L, \mathcal{M})$ is elliptic,
(3.6) the submanifold $N$ is hyperbolic for $\mathcal{M}$ on $M$,
(3.7) $Y$ is non characteristic for $\mathcal{M}$.

Set $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Then hypothesis (a) of Theorem 2.4 is translated as hypothesis (3.5) and hypothesis (b) is translated as hypothesis (3.6).

We shall constantly use the next result.

**Lemma 3.4** (see [JS16, Lem. 3.5]). Hypothesis (3.6) implies hypothesis (3.7).

**Theorem 3.5.** Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module. Assume (3.5) and (3.6). Then one has the natural isomorphism

$$Rg_{Nd!}g_{N!}^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M) \sim \mu_N(\omega_{L/N} \otimes g^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)).$$

**Proof.** Apply Theorem 2.4 together with Lemma 2.7 to the sheaf $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. We get:

$$Rg_{Nd!}(\omega_{N/M} \otimes g_{N!}^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_M(\mathcal{O}_X))) \simeq \mu_N(\omega_{L/N} \otimes g^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)).$$

Equivalently, we have

$$Rg_{Nd!}g_{N!}^{-1}(\omega_{X/M} \otimes R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_M(\mathcal{O}_X))) \simeq \mu_N(\omega_{L/N} \otimes g^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)).$$

Finally $\omega_{X/M} \otimes \mu_M(\mathcal{O}_X) \simeq \mathcal{C}_M$. Q.E.D.

**Example 1: Cauchy problem for microfunctions**

Let $M, X, L, N$ and $f$ be as above and assume that $L = Y$, hence $f = g$.

**Corollary 3.6.** Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module. Assume (3.6). Then one has the natural isomorphism

$$f_{Nd!}f_{N!}^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(f^D\mathcal{M}, \mathcal{C}_N).$$

**Proof.** Applying Theorem 2.11, we get $f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(f^D\mathcal{M}, \mathcal{O}_Y)$. (Recall that (3.6) implies (3.7).) Moreover, $\omega_{Y/N} \otimes \mu_N(\mathcal{O}_Y) \simeq \mathcal{C}_N$. Finally, since $f_{Nd}$ is finite on char($\mathcal{M}$), we may replace $Rf_{Nd!}$ with $f_{Nd!}$. Q.E.D.

### 3.3 Boundary values

Let $M$ be a real $n$-dimensional manifold, $N$ a closed submanifold of codimension $d$, $X$ a complexification of $M$ and $Y$ a complexification of $N$ in $X$. We denote by $f: Y \hookrightarrow X$ the embedding.

**Notation 3.7.** We set

$$\tilde{\mathcal{B}}_N = R\Gamma_N(\mathcal{O}_X) \otimes \text{or}_N [n] \simeq H^n_{\mathcal{O}_X}(\mathcal{O}_X) \otimes \text{or}_N.$$
We shall not confuse the sheaf $\tilde{B}_N$ with the sheaf $B_N$ of hyperfunctions on $N$. We have an isomorphism

$$\tilde{B}_N \cong \Gamma_N B_M \otimes \text{o}_N M \cong \Gamma_N B_M \otimes \omega_{M/N} [-d].$$

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Applying the functor $R\Gamma_N (\cdot) \otimes \text{o}_N [n - d]$ to the isomorphism (iii) in Theorem 2.11 together with isomorphism (2.12) one recovers a well known result:

**Lemma 3.8.** Assume (3.7). One has a natural isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \tilde{B}_N) [d] \cong R\mathcal{H}om_{\mathcal{D}_N} (f^D \mathcal{M}, \mathcal{B}_N).$$

Applying the functor $D'_X$ to the morphism $C_M \rightarrow C_N$, we get the morphism $D'_X (C_N) \rightarrow D'_X (C_M)$, that is, the morphism $\text{o}_N [d + n] \rightarrow \text{o}_M [n]$. Applying the functor $R\mathcal{H}om (\cdot, \mathcal{O}_X)$ we get the “restriction” morphism

$$\rho_{MN} : B_M \rightarrow \tilde{B}_N [d] \cong \Gamma_N B_M \otimes \omega_{M/N}. \tag{3.8}$$

For a closed cone $\lambda \subset T^*_M X$, we set for short

$$\mathcal{B}_{M, \lambda} := \pi_M \ast \Gamma_X C_M. \tag{3.9}$$

For an open cone $\gamma \subset N_M$, we set for short:

$$\Gamma_{\gamma} \mathcal{B}_{NM} := \tau_N \ast \Gamma_{\gamma} (\nu_N (\mathcal{B}_M)). \tag{3.10}$$

(In the sequel, we shall use this notation for another real manifold $L$ instead of $M$.)

Hence, for a closed convex proper cone $\delta \subset T^*_M X$ with $\delta \supset N$, setting $\gamma = \text{Int}(\delta^a)$, we have by (2.2):

$$\pi_N \ast \Gamma_{\delta} (\mu_N \mathcal{B}_M) \otimes \omega_{M/N} \cong \Gamma_{\gamma} \mathcal{B}_{NM}. \tag{3.11}$$

One can use (3.11) and the morphism $\pi_N \ast \Gamma_{\delta} (\mu_N \mathcal{B}_M) \rightarrow \pi_N \ast \mu_N \mathcal{B}_M \cong \Gamma_N \mathcal{B}_M$ to obtain the morphism

$$b_{\gamma, N} : \Gamma_{\gamma} \mathcal{B}_{NM} \rightarrow \Gamma_N \mathcal{B}_M \otimes \omega_{M/N}. \tag{3.12}$$

One can also construct (3.12) directly as follows. Let $U$ be an open subset of $M$ such that $\mathcal{U} \supset N$, $U$ is locally cohomologically trivial (see [KS90, Exe. III.4]). Then the morphism $C_U \rightarrow C_N$ gives by duality the morphism $\text{o}_N [d + n] \rightarrow \text{o}_U [n]$ and one gets the morphism $\Gamma_U \mathcal{B}_M \rightarrow \Gamma_N \mathcal{B}_M \otimes \omega_{M/N}$ by applying $R\mathcal{H}om (\cdot, \mathcal{O}_X)$ similarly as for $\rho_{MN}$. Taking the inductive limit with respect to the family of open sets $U$ such that $C_M (X \setminus U) \cap \gamma = \emptyset$ (see [KS90, Th. 4.2.3]), we recover the morphism (3.12).

In particular, for a coherent $\mathcal{D}_X$-module $\mathcal{M}$ we get the morphisms

$$\rho_{MN} : R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \mathcal{B}_{M, \lambda}) \rightarrow R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \tilde{B}_N) [d],$$

$$b_{\gamma, N} : R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \Gamma \mathcal{B}_{NM}) \rightarrow R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \tilde{B}_N) [d].$$
3.4 Wick rotation

Let $M$, $X$, $Y$, $N$, $L$, $f$ and $g$ be as above. Now, we also assume that $L$ is a real manifold of the same dimension than $M$ and $X$ is a complexification of $L$. We still consider diagram (2.4).

Consider the hypothesis

$$\text{(3.13)}$$

in a neighborhood of $N$, char$(\mathcal{M}) \cap T^*_N X$ is contained in the union of two closed cones $\lambda$ and $\lambda'$ such that $\lambda \cap \lambda' = M \times_X T^*_X X$.

(Here, $M \times_X T^*_X X$ stands for the zero-section of $T^*_M X$.)

**Lemma 3.9.** Assume (3.13). Then we have the natural isomorphism

$$\text{(3.14)}$$

$$g_{\mathcal{N}^*}^{-1}\Gamma \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^*, \mathcal{C}_M) \cong \Gamma \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^*, \mathcal{C}_M).$$

**Proof.** (i) Set for short $F = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^*, \mathcal{C}_M)$, $j = g_{\mathcal{N}^*}$, $A = \lambda$, $B = j^{-1} A$. With these new notations, we have to prove the morphism

$$\text{(3.15)}$$

$$j^{-1}\Gamma A F \cong \Gamma B j^{-1} F$$

is an isomorphism.

(ii) The morphism (3.15) is an isomorphism outside of the zero-section of $T^*_M X$ since then supp$(F) = A \cup C$ with $A$ and $C$ closed and $A \cap C = \emptyset$, by the hypothesis (3.13).

(iii) Consider the diagram in which $s_N$ and $s_M$ denote the embeddings of the zero-sections:

$$\text{(3.16)}$$

$$\begin{array}{ccc}
N \times_M T^*_M X & \xrightarrow{j} & T^*_M X \\
\pi_N \downarrow & & \downarrow \pi_M \\
N & \xrightarrow{j} & M.
\end{array}$$

Since $R\pi_{N*} \simeq s_N^{-1}$, when applied to conic sheaves, it remains to show that (3.15) is an isomorphism after applying the functor $R\pi_{N*}$.

(iv) Consider the morphism of Sato’s distinguishing triangles:

$$\begin{array}{cccccc}
R\pi_{N!*} j^{-1}\Gamma A F & \xrightarrow{\mu} & R\pi_{N!*} j^{-1}\Gamma A F & \xrightarrow{\pi N!*\Gamma A F} & +1 \\
\downarrow & & \downarrow & & \downarrow & \\
R\pi_{N!*} \Gamma B j^{-1} F & \xrightarrow{v} & R\pi_{N!*} \Gamma B j^{-1} F & \xrightarrow{\pi N!*\Gamma B j^{-1} F} & +1
\end{array}$$

It follows from (ii) that the vertical arrow $w$ on the right is an isomorphism. We are thus reduced to prove the isomorphism

$$\text{(3.17)}$$

$$R\pi_{N!*} j^{-1}\Gamma A F \cong R\pi_{N!*} \Gamma B j^{-1} F.$$ 

(v) Using the fact that $A \supset M$ and $B \supset N$ and that Diagram (3.16) with the arrows going down is Cartesian, we get

$$R\pi_{N!*} j^{-1}\Gamma A F \simeq j^{-1} R\pi_{M!*} \Gamma A F \simeq j^{-1} s_M^{-1} R\Gamma A F \simeq j^{-1} s_M^{-1} F \simeq j^{-1} \Gamma B j^{-1} F \simeq s_M^{-1} \Gamma B j^{-1} F \simeq R\pi_{N!*} \Gamma B j^{-1} F.$$ 

Q.E.D.
Consider
\[(3.18) \quad \gamma \subset T_N L \text{ an open convex cone such that } \gamma \text{ contains the zero-section } N\]
and recall notations (3.9) and (3.10).

**Theorem 3.10 (Wick isomorphism Theorem).** Let \(\mathcal{M}\) be a coherent left \(\mathcal{D}_X\)-module and let \(\gamma\) be as in (3.18). Assume (3.5), (3.6), (3.13) and also

\[(3.19) \quad g_{N_\pi}^{-1}(\lambda) = g_{Nd}^{-1}(\gamma^{\circ a}).\]

Then one has the commutative diagram in which the horizontal arrow is an isomorphism:

\[(3.20) \quad \begin{array}{ccc}
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{M,\lambda})|_N & \sim & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\gamma \mathcal{B}_{NL}) \\
\rho_{MN} & & b_{\gamma N} \\
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_N)[d] & & \\
\end{array}\]

**Proof.** (i) As a particular case of Theorem 3.5 and using the fact that \(g^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_L)\), we get the isomorphism

\[Rg_{Nd}g_{N_\pi}^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_N \mathcal{B}_L) \otimes \omega_{L/N}.\]

(ii) Set for short \(F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)\). Using Lemma 3.9 and the fact that \(g_{Nd}\) is proper on \(\text{supp } F\), we have the isomorphism

\[Rg_{Nd}g_{N_\pi}^{-1}R\Gamma_\lambda F \simeq Rg_{Nd}R\Gamma g_{N_\pi}^{-1}(\lambda)g_{N_\pi}^{-1}F \simeq R\Gamma^{\circ a}Rg_{Nd}g_{N_\pi}^{-1}F.\]

Therefore, we have proved the isomorphism

\[(3.21) \quad Rg_{Nd}g_{N_\pi}^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\lambda \mathcal{B}_M) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\gamma^{\circ a} \mu_N \mathcal{B}_L) \otimes \omega_{L/N}.\]

(iii) Let us apply the functor \(R\pi_{N*}\) to (3.21). Since \(g_{Nd}\) is proper on \(\text{supp } F\), setting \(G = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\lambda \mathcal{B}_M)\), we have (see Diagram 2.6)

\[R\pi_{N*}Rg_{Nd}g_{N_\pi}^{-1}G \simeq R\pi_{N}g_{N_\pi}^{-1}G \simeq (R\pi_{M*}G)|_N.\]

Hence, we have proved the isomorphism

\[R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{M,\lambda})|_N \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \pi_{N*} \Gamma_\gamma^{\circ a} \mu_N \mathcal{B}_L) \otimes \omega_{L/N}\]

and the result follows from (3.11). \(\text{Q.E.D.}\)

### 3.5 The classical Wick rotation

Let us treat the classical Wick rotation. Hence, we assume that \(M = N \times \mathbb{R}\) and \(L = N \times \sqrt{-1}\mathbb{R}\). As usual, \(Y\) is a complexification of \(N\) and \(X = Y \times \mathbb{C}\). We denote by \(t + is\) the holomorphic coordinate on \(\mathbb{C}\), by \((t + is; \tau + i\sigma)\) the symplectic coordinates on \(T^* \mathbb{C}\) and by \((x; \eta)\) a point of \(T_N^* Y\). We identify \(N\) and \(N \times \{0\} \subset X\).
Let $P$ is a differential operator of order $m$, elliptic on $L$ and (weakly) hyperbolic on $M$ in the $\pm dt$ codirections. A typical example is the wave operator on a globally hyperbolic spacetime $N \times \mathbb{R}_t$. Set

$$L^+ = N \times \{t + is; t = 0, s > 0\}, \quad \lambda = T^*_N Y \times \{(t + is; \tau + i\sigma); s = 0, \tau = 0, \sigma \leq 0\}.$$

The map $g_{Nd} : N \times_M T^*_M X \to T^*_N L$ is given by

$$(x, 0; i\eta, i\sigma) \mapsto (x; \sigma).$$

We shall apply the preceding result with $\gamma = L^+$. In that case, $\gamma^{\alpha} = \lambda$ and (3.19) is satisfied.

Let $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X \cdot P$. In the sequel we write for short $\mathcal{B}^P_M$ instead of $\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \mathcal{B}_M)$ and similarly with other sheaves. Note that $\mathcal{E}xt^1_{\mathcal{D}_X} (\mathcal{M}, \mathcal{B}_N) \simeq \mathcal{B}_N / P \cdot \mathcal{B}_N$.

As a particular case of Theorem 3.10, we get:

**Corollary 3.11.** We have a commutative diagram in which the horizontal arrow is an isomorphism:

$$\begin{array}{ccc}
\mathcal{B}^P_M |_N & \sim & \mathcal{B}^P_{L^+} |_N \\
\mathcal{B}_N / P \cdot \mathcal{B}_N & \hookrightarrow & \mathcal{B}_N^m
\end{array}$$

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