AdS Dynamics from Conformal Field Theory

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We explore the extent to which a local string theory dynamics in anti-de Sitter space can be determined from its proposed Conformal Field Theory (CFT) description. Free fields in the bulk are constructed from the CFT operators, but difficulties are encountered when one attempts to incorporate interactions. We also discuss general features of black hole dynamics as seen from the CFT perspective. In particular, we argue that the singularity of $AdS_3$ black holes is resolved in the CFT description.

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1. Introduction

It has recently been proposed [1] that string/M theory on $AdS_d \times K$ (where $K$ is a suitable compact space) is equivalent to a conformal field theory (CFT) 'living on the boundary' of the anti-de Sitter space. Some evidence for this conjecture comes from the agreement of the spectrum of supergravity fluctuations with the spectrum of operators in the conformal field theory [2]. In part this follows simply from the isomorphism between the symmetry groups of the two theories, but the correspondence also correctly matches the multiplicities of irreducible representations. The extension of the conjecture to all of string/M theory is based on our expectation that this is the unique quantum completion of supergravity. Further evidence comes from the agreement of certain perturbative interactions [3], and the ability of the CFT to explain the entropy of AdS black holes [4].

Given this AdS/CFT correspondence, it is natural to ask to what extent local spacetime physics on $AdS_d$ can be recovered from the CFT, which is a lower-dimensional field theory. Below we discuss several aspects of this question (a number of details are postponed to a future publication [5]). First we show that one can construct free quantum fields corresponding to all of the modes of string theory in this background. These are operators in the CFT which depend on position in the AdS spacetime, and satisfy the usual causality conditions. One might have worried that since the CFT operators are causal with respect to the boundary causal structure which does not include the AdS radial coordinate, it would be difficult to construct operators which commute whenever they are spacelike separated in AdS. We will see that there is an essentially unique way of avoiding this difficulty in the large $N$ limit in which the string theory becomes free. In this limit, the combination of large $N$ factorization and group theory determines the operator algebra of the CFT to be that of creation and annihilation operators of free string modes on $AdS_d \times K$. From these we can construct local free fields in a unique way. These operators turn out not to be local when interactions are included. It is not yet clear whether our expressions for the spacetime operators can be modified to remain local. Nor have we understood the precise nature of the nonlocality and the extent to which it becomes invisible at low energies. In making these constructions, we work mostly with $AdS_5 \times S^5$, but similar arguments should work for backgrounds of the form $AdS_3 \times S^3 \times M$.

Our construction of local fields provides a clue for the discussion of local motions

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1 Although we adhere to current usage, the CFT does not really live on the boundary; see section 4.1.
in spacetime. In particular, the symmetry group of the CFT generates the action of the corresponding spacetime symmetries by vector fields acting on our local operators. We use the intuition derived from this correspondence to discuss black hole dynamics. We explain how certain objects which fall into an AdS black hole can be described in terms of the CFT. The black hole itself is described by a high energy state which looks approximately thermal. The object falling in is described by a localized excitation in the gauge theory. As it evolves in CFT time, its scale size grows. We show that the time for this excitation to expand to the size of the typical thermal wavelength (and hence become indistinguishable from the background) agrees with the time in AdS for the object to fall from a large radius to the vicinity of the horizon. The natural time evolution in the CFT corresponds to evolution with respect to the external Schwarzschild time of the black hole. So one never sees the object cross the horizon. We argue that observers that cross the horizon are described by evolving the state with respect to an operator different from the CFT Hamiltonian. This prescription is a precise realization of the idea of black hole complementarity. The CFT Hilbert space does not break up into a tensor product of spaces inside and outside of the horizon. Rather there is a single Hilbert space describing both inside and outside. Physics as seen by different observers corresponds to acting on this space with different classes of operators. The operators corresponding to an external observer do not commute with those of an infalling observer. In the case of the three dimensional BTZ black hole, one can use the local symmetries to identify a suitable operator. Since this operator is another conformal generator which acts unitarily on the CFT Hilbert space, the evolution is still nonsingular. We are thus led to the conclusion that in the context of the AdS/CFT correspondence, quantum effects indeed resolve the BTZ singularity! The singularity seen in the classical supergravity description of physics as seen by this observer, must be an artifact of the large $N$ limit.

2. Linearized supergravity fields in the large N limit

2.1. A Little Scaling Argument

We will be working, in this and the following section, with the $AdS_5 \times S^5$ system, in the regime where there is a clear separation between the long distance expansion and the perturbation expansion (as noted in the introduction, similar arguments should work for backgrounds of the form $AdS_3 \times S^3 \times M$). The dual CFT is the $\mathcal{N} = 4$ Super Yang Mills theory (SYM). We will be studying the ’t Hooft limit of the SYM and the $1/N$ expansion.
around it, though we would like the 't Hooft coupling to be large since the radius $R$ of the AdS is given by $\frac{R^4}{\alpha'} = g_{\text{YM}}^2 N \equiv \lambda$. Free Type IIB string theory on $AdS_5 \times S^5$ should then be the leading term in the planar expansion of $\mathcal{N} = 4$, $d = 4$ SYM theory. In order to do perturbative string theory on a space of low curvature we take $\lambda$ large but independent of $N$ and expand amplitudes in inverse powers of $N$. That is, we are in the regime $1 \ll g_{\text{YM}}^2 N \ll N$. The perturbative gauge theory gives a natural classification of properties of operators in the large $N$ limit according to the number of powers of the trace which they involve.

Let $\{\mathcal{O}_i\}$ be a complete basis of single trace operators. Using a standard normalization, their full and connected Green’s functions satisfy the following scaling relations for even $n$:

\[ \langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \rangle \sim N^n \tag{2.1} \]

\[ \langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \rangle_c \sim N^2 \tag{2.2} \]

Note that in CFT the VEV of any nonunit operator vanishes. As a consequence the full and connected three point functions are the same and both scale like $N^2$. Full $2k + 1$ point functions scale like $N^k$ and connected ones like $N^2$.

If we define rescaled operators by $\mathcal{O}_i = \frac{1}{N} \mathcal{O}_i$, then the rescaled operators have unit normalized two point functions and the $1/N$ expansion of their connected Green’s functions looks (combinatorially) like a perturbation expansion around a free field theory, in which the $\mathcal{O}_i$ are independent free fields. Note also that in the operator product expansion $\mathcal{O}_i \mathcal{O}_j = \sum \frac{1}{N} C_{ij}^k \mathcal{O}_k$, we should expect the coefficients $C_{ij}^k$ to be of order one for large $N$.

Multiple trace operators, defined as $\mathcal{O}_{i_1 \ldots i_n} = \frac{1}{N^n} \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n}$ will have two point functions normalized to one. Connected Green’s functions of products of single trace operators with appropriate multiple trace operators will be of order one in the large $N$ limit. In terms of the analogy with perturbation theory described in the previous paragraph, the multiple trace operators behave like composites of the “free fields”. The operator product of two single trace operators contains terms of order one with multiple trace operators.

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2 We assume that the equations of motion of the gauge theory are used to eliminate redundant operators.

3 There are actually many multiple trace operators for any given set of single trace operators, corresponding to all the primary fields in the complete operator product expansion.
2.2. A Little Group Theory

To leading order in the expansion in $g_s$ at fixed $\alpha'$, the conventional description of string theory on a given background suggests that it can be viewed as an infinite number of free quantum field theories propagating on this background. Our first goal is to construct this ‘free string field theory’ from the quantum variables of the CFT. Before doing so, we must remark on the issue of gauge fixing, which we will not address in this paper.

In the standard field theoretic approach to constructing the Hilbert space of quantum gravity, one must choose a gauge, and in most gauges, introduce ghosts and a BRST operator. We restrict attention to a class of gauges which are defined by covariant (e.g. De Donder) conditions on the components of the supergraviton fields, in AdS space. In the present paper we will consider only the Green’s functions of fields which are scalars in AdS, to leading order in perturbation theory. In the above class of gauges, the gauge fixing and ghost Lagrangians will not involve the scalar degrees of freedom. Thus, if we concentrate on scalars, and their leading tree level self-interaction we should be able to ignore the issue of gauge fixing. The details of leading order BRST quantization of the full system should be straightforward but tedious.

The Hilbert space of the free string theory is the Fock space constructed from a collection of unitary irreducible representations of the $AdS_5 \times S^5$ super-isometry group $SU(2,2|4)$. These single particle representations all have positive energy, in terms of the generator of the standard global time translations in AdS space. It was pointed out in [8] that the corresponding Hilbert space for the CFT is obtained by quantization of the gauge theory on $S^3 \times \mathbb{R}$, for it is only here that the conformal group is implemented in a unitary fashion. As shown in [3], the generator $H \equiv K^0 + P^0$ of the conformal group is positive in the class of unitary representations of the conformal group obtained by quantizing a unitary CFT on $S^3 \times \mathbb{R}$; thus we map $H$ to the global time translation generator on the $AdS$ side. We note also that the authors of [2] have argued that in a generic CFT the

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4 in which however, the devil often resides. One interesting problem which we will not address is the relation between the diffeomorphism invariance of supergravity and the gauge invariance of the SYM theory. Since most of the (super) Killing diffeomorphisms of the model are accompanied by gauge transformations in their action on the non gauge invariant part of the SYM Hilbert space, we suspect that a close relation does in fact exist.

5 The quantization of the gauge theory on $\mathbb{R}^{3,1}$ carries a realization of the Lie algebra of conformal transformations, but not a unitary implementation of the group.
global time coordinate $t$ cannot be taken periodic. Thus as in [8] we assume that we are working on the universal cover of AdS space.

A scalar particle in $\text{AdS}_5 \times S^5$ can be labelled by an $S^5$ spherical harmonic which we call $Y$, an “orbital” angular momentum $J$ on $S^3$, and a frequency $\omega > 0$. In terms of these, the mass of the particle is determined by the equation for the Casimir operator of $SU(2, 2|4)$ [10]. The particle states are in one-to-one correspondence with the positive frequency solutions of the scalar wave equation on $\text{AdS}_5 \times S^5$. In (dimensionless) coordinates where the $\text{AdS}_5$ metric is

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{(1 + r^2)} + r^2 d\Omega_3^2,$$

(2.3)

this equation reads

$$\left[ -\frac{1}{(1 + r^2)} \partial_t^2 + \frac{1}{r^3} \partial_r \left[r^3 (1 + r^2) \partial_r \right] + \frac{1}{r^2} L_3^2 + L_5^2 \right] \psi = 0 .$$

(2.4)

Here, $L_3^2$ (respectively, $L_5^2$) is the square of the angular momentum operator on $S^3$ ($S^5$). Of course we must also pick the solution which is normalizable in the usual Klein-Gordon norm on surfaces of constant $t$.

At the risk of being pedantic, we remind the reader of a straightforward consequence of the correspondence between particle states and wavefunctions. The spacetime symmetry group action on supergravity fields is implemented by Killing vector fields $L_a$. The solutions $\psi_{\omega,J,Y}$ of the scalar wave equation satisfy

$$L_a \psi_{\omega,J,Y} = D_{\omega,J,Y}^{\rho,K,Z} \psi_{\rho,K,Z} ,$$

(2.5)

where $D$ is the same matrix which implements the operation of the Hilbert space operators on the states:

$$L_a |\omega, J, Y\rangle = D_{\omega,J,Y}^{\rho,K,Z} |\rho, K, Z\rangle .$$

(2.6)

In the CFT description, states with the transformation properties of single scalar supergraviton states are constructed by acting with conformal primary operators on the conformally invariant vacuum. Note that because of the positivity of $H$, operators which satisfy $[H, O] = \omega O$ with negative $\omega$ must annihilate the vacuum. Thus, if we Fourier expand a local operator

$$O(\Omega_3, t) = \sum_{n=0}^{\infty} [O_{\omega_n}(\Omega_3)e^{-i\omega_n t} + h.c.] ,$$

(2.7)
then $O_\omega$ annihilates the vacuum and the states are created by $O_\omega^\dagger$. Note that a field in a given representation of the conformal group on $S^3 \times \mathbf{R}$ actually contains only a discrete set of frequencies $\omega_n$.

By construction, the transformation properties of the operators $O_\omega^\dagger$ under the Killing symmetries of the background spacetime are the same as those of supergraviton creation operators. We would now like to argue that at leading order in $1/N$ they obey the same algebra. Indeed, we have argued that the single trace operators, $O_i$, have only connected two point functions in the large $N$ limit. Furthermore, the positive frequency components of these fields annihilate the vacuum. As a consequence, we can extract the commutators of the positive and negative frequency components from the norm of the states created from the vacuum by the negative frequency operators. These norms are determined by group theory and are therefore the same for supergravity and the CFT.

It is now an easy generalization of the arguments of Weinberg [11] that, to leading order in $1/N$, there is a unique set of local fields in $AdS_5 \times S^5$ which can be constructed from the single trace conformal fields of the SYM theory. These fields satisfy the fundamental equation

$$\mathcal{L}_a \phi = i[L_a, \phi] , \quad (2.8)$$

which says that the symmetry operators of the CFT act on them like the appropriate Killing vectors of the spacetime.

The fields have the expansion

$$\phi(t, \Omega_3, \Omega_5, r) = \sum_{\omega > 0, J, Y} [\psi_{\omega, J, Y}(r)e^{-i\omega t}Y_J(\Omega_3)Y_Y(\Omega_5)O_{\omega, J, Y} + h.c.] . \quad (2.9)$$

where the $Y$ index on $O$ denotes the conformal primary operator associated with the $S^5$ spherical harmonic $Y$. Below we will have occasion to use a condensed notation for this formula. If we rewrite this in terms of the local fields $O$ it has the generic appearance

$$\phi(x) = \int_b G(x, b)O(b) \quad (2.10)$$

where $x$ is a point in $AdS_5 \times S^5$ and $b$ a point on its boundary. The Green’s function $G(x, b)$ is implicitly defined by (2.9). It is a solution of the homogeneous Klein-Gordon equation and its properties will be further explored in [3].

The two-point function of $\phi(x)$ is by construction that of the supergravity fields, since the wavefunctions in (2.9) solve the wave equation (determined by group theory), and the
algebra of the modes of the CFT operators $O$ is just that of creation and annihilation operators at this order in $1/N$. In particular, the propagation is causal. The extent to which this can be maintained at higher orders in $1/N$ will be discussed below.

3. The Effect of Interactions

We have thus, modulo the presumably technical questions of gauge fixing, constructed the free local string field theory of this string theory compactification as a set of operators in the CFT which is supposed to encode the exact dynamics of the theory. We now want to generalize these considerations to the interacting theory. Before beginning to calculate, we wish to point out a general property of the CFT which may be crucial for understanding why this theory is different from local field theory.

The algebra of the single trace operators is only approximately that of independent creation and annihilation operators. In the full theory it is highly constrained and not at all free. It is easy to see that the representation space for this algebra should be much smaller than the algebra of independent creation and annihilation operators, due to the operator product relations $O_i O_j = \sum \frac{1}{N} C_{ijk} O_k$. This is, we believe, an indication that the theory has many fewer degrees of freedom than one expects from a field theory. Note that we say field theory rather than String Field Theory. This is because there are operator product relations even between operators in short representations of the superconformal algebra. Thus not even all of the would-be supergraviton creation and annihilation operators are independent in the CFT.

With these philosophical comments dispensed with, we are nearly ready to begin discussing interactions. However we must first see whether our benign neglect of the problem of gauge fixing can affect the discussion. We believe that it will not (within the classes of gauges we have discussed above) if we restrict our attention to three point functions of AdS scalar fields at lowest order in perturbation theory. The only graphs

\footnote{An amusing example with some similar properties is an affine Lie algebra at level $k$, where the large $k$ limit corresponds to our large $N$ limit. At leading order in $1/k$, the currents are abelian and their modes are independent creation and annihilation operators. At next-to-leading order, there are relations among them given by the affine Lie algebra structure constants; this is why $c < \dim G$. Note also that the interaction Lagrangian is not locally expressed in terms of the currents, although here there is a simple way out, by passing to the group field. Below we will argue that there is no such simple fix for the string field theory.}
which contribute to these Green’s functions at this order involve scalar propagators, and the triple scalar coupling in the Lagrangian. Since none of these objects are changed by a change of gauge within the allowed class, we can compute them without a full discussion of gauge fixing.

The formula (2.9) for the scalar fields in lowest order perturbation theory already implies a contribution to the connected three point function in leading order in $1/N$. This comes precisely from the $1/N$ contribution to operator products of single trace operators. Thus, the fields defined by (2.9) have a connected 3 point Wightman function

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle_c = \int G(x_1,b_1)G(x_2,b_2)G(x_3,b_3) \langle O_1(b_1)O_2(b_2)O_3(b_3) \rangle_c .$$

(3.1)

There are two interesting questions to ask of this formula. Does it coincide with the lowest order perturbative formula of supergravity? Does it reproduce the boundary correlation functions of supergravity discussed in [12,13]? Of course, a positive answer to the first of these questions would obviate the need to ask the second.

It is easy to see however that the answer to the first question is no. If we apply the appropriate scalar wave operator to any of the three legs of this Green function, it vanishes. In supergravity, the fields satisfy nonlinear wave equations of the schematic form

$$\Box \phi_1 = \phi_2 \phi_3 ,$$

(3.2)

where we have taken only the relevant trilinear scalar coupling into account. In evaluating the Green’s functions perturbatively, we find a nonvanishing connected three point function only by taking into account this leading nonlinearity in the field equations. Thus, the leading connected three point function satisfies

$$\Box_1 \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \langle \phi_2\phi_3(x_1)\phi_2(x_2)\phi_3(x_3) \rangle ,$$

(3.3)

which is inconsistent with our formula. We will return below to the question of whether it is possible to modify the field to obtain the supergravity formula for the Green functions.

First however, we turn to the second question. Gubser et.al. [12] and Witten [13] have presented a connection between supergravity Green functions and CFT correlators. There is a graphical version of this prescription which resembles the LSZ formula for S-matrix elements in Minkowski space. The CFT correlators are extracted from supergravity by calculating the graphs for Green functions at points in the bulk $AdS_5 \times S^5$ space and then replacing the external lines by a special homogeneous Green function $G_W$ of the
scalar wave operator which implements delta function boundary conditions on the $S^3 \times \mathbb{R}$ boundary.

At large spacelike distance, the Feynman Green function $G_F(x, y)$ for a scalar of mass $m$ behaves like $r^{-d_m}$ where $d_m = 2 + (4 + m^2)^{1/2}$. Here we have used the coordinates (2.3) and go to infinity by taking $r$ large with other coordinates fixed. The function $G_W(x, b)$ is only defined when one of its arguments is on the boundary. We can obtain it from $G_F$ by the formula

$$G_W(x, b) = \lim_{y \to b} G_F(x, y) r_y^{d_m}.$$  

Thus, we can obtain the GKP/W correlators by multiplying the Feynman Green functions by powers of $r$ and taking a limit in which the external points go to the boundary.

$$G_{GKP/W}(b_1 \ldots b_n) = \lim_{y_i \to b_i} \prod r_y^{d_m} G(y_1 \ldots y_n).$$  

There may be some subtleties in this limiting procedure because the interior points of the diagram are integrated over all of AdS space and the measure of integration is concentrated on the boundary. Indeed, in the calculations of [14], great care had to be taken to obtain correct results. We have not studied the question of whether these subtleties invalidate the simple formula (3.5). Note by the way that although we have described the derivation of this formula perturbatively, it would appear to make sense (again modulo questions about gauge variance) as a nonperturbative relation. The formula is also, despite its apparent dependence on a particular coordinate system, coordinate invariant. That is, we can easily replace the explicit coordinate factors in the equation by factors of the geodesic distance between the points $y_i$ and some arbitrarily chosen interior point. Up to an overall constant factor for each external leg (a sort of wave function renormalization), the answer does not depend on the choice of interior point.

Let us now apply this procedure to the time ordered Green’s functions of the fields $\phi$ defined in (2.9). Since our limiting procedure involves the variation of only the spacelike variable $r$, we can apply it to each term in the time ordered product. Thus, we might as well study the limit of Wightman functions with a particular ordering of the operators. If we harmonically expand each field as in (2.9), we see that the harmonically transformed Wightman function is just the ordered product of factors of the form $\psi_{\omega, J, Y}(r) O_{\omega, J, Y}$. Now we exploit the fact that for fixed $Y$ (and fixed values of all other implicit labels on the operator), the large $r$ behavior of all of the $\psi$ functions is the same as that of the Feynman Green function with the same value of the mass. Thus, up to an overall constant, (which
we can absorb in the definition of our LSZ formula) the large $r$ behavior of the Wightman function will be the Wightman function in the CFT of the conformal fields $O_Y(b)$ (we are implicitly assuming that the limit commutes with the integrals which define the harmonic expansion). In view of the remarks at the beginning of this paragraph, the same will be true of the time ordered functions.

To summarize, what we have proved, without recourse to any expansion, is that the Green functions of the field defined by (2.9) reproduce all of the GKP/W correlation functions. This is rather more than we bargained for – a field satisfying the linearized field equations computes the exact nonperturbative “S matrix”.

In fact, this result shows us that the GKP/W correlation functions cannot be thought of as an S-matrix in the sense of an overlap between exact multiparticle eigenstates of the CFT Hamiltonian. If such an interpretation were possible, then the Green functions of $\phi(x)$ with one variable in the bulk and the rest taken to the boundary:

$$G_{GKP/W}(x, b_1, \ldots, b_n) = \lim_{y_i \to b_i} \prod r_{y_i}^{d_m} G(x, y_1 \ldots y_n)$$

would have to be interpreted as the form factor or matrix element:

$$\langle b_1, \ldots, b_k | \phi(x) | b_{k+1} \ldots b_n \rangle.$$  (3.7)

The free field equation for $\phi$ would then imply that this expression could be nonzero only if the two states differed by addition of a single particle of the mass carried by the field. Then, using the formula (3.5) we would conclude that the amplitude $\langle O(b_1) \ldots O(b_k)O(b_{n+1})O(b_{k+1}) \ldots O(b_n) \rangle$ obeyed the same restrictions. This is absurd, because of the complete symmetry of these amplitudes in the boundary points, unless the amplitudes vanish for $n > 2$. But of course we know that this is not the case.

We emphasize that this conclusion is tied to the use of the global time of AdS space as evolution parameter. We have, for example, used global time ordering to define our amplitudes. As mentioned above, the spacetime geometry of this problem suggests that no sensible S-matrix interpretation of the asymptotic limit of the global time evolution is to be expected. Geodesics simply do not separate from each other asymptotically in

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7 The ambiguity in splitting the boundary points into two groups in this expression is another indication that we are not calculating an S-matrix. The spacelike limit we are taking does not naturally separate asymptotic points into future and past.
time. D-brane black hole physics suggests that these amplitudes do have a sensible S-
matrix interpretation in terms of the Minkowski time evolution generated by $P^0$ in the
conformal field theory (as opposed to the global evolution defined by $P^0 + K^0$). Motions
along the flows of the corresponding $AdS$ generator do not correspond to geodesic motion
in $AdS$ space (rather they are like the flows of Rindler time). However they do appear
to correspond to motions of incoming and outgoing particles in the asymptotically flat
geometry of which $AdS_5 \times S^5$ is the near horizon limit.

We now turn to the question of whether we can modify our definition of the field to
order $1/N$ in order to make it coincide with the fields of perturbative supergravity, at least
in some low energy approximation. This is the same as asking whether the field can be
made local. Indeed, textbook arguments [15] imply that local perturbations of free field
theory can always be understood as perturbations of the Hamiltonian by integrals of a
local density. Correspondingly, the connected $n$ point functions of such perturbed local
fields are directly connected to the nonlinear terms in the equations of motion which they
satisfy.

This implies that the field we have constructed in (2.9) is not local at next-to-leading
order in $1/N$. It satisfies a free field equation but has a connected 3 point function at order
$1/N$.

It is of course well known that, to leading order in derivatives, the only consistent
local perturbation of the free supergravity fields is that of interacting supergravity. More
generally, any higher derivative covariant correction to the supergravity action would be an
acceptable perturbation at leading order. Schematically, any local field with a connected
3 point function would have to satisfy an equation of the form

$$\Box \phi = \frac{1}{N} \phi^2.$$  \hspace{1cm} (3.8)

If we call the field of (2.9) $\phi_0$ and write $\phi = \phi_0 + \Delta$, then we can solve for $\Delta$ to leading
order in $1/N$:

$$\Delta = \frac{1}{N} \left( \int G_R(x,y) \phi_0^2(y) + \phi_1 \right)$$  \hspace{1cm} (3.9)

8 Since our field is not local, its time ordered Green’s functions will not be covariant under
symmetries of AdS space which change the definition of the global time. We emphasize that the
physical argument for covariance of these Green’s functions requires us to contemplate perturba-
tions of the system by a local external source. Such perturbations are not allowed in a quantum
theory of gravity because they destroy the covariant conservation law for the stress tensor. This is
perhaps the most primitive reason for believing that the quantum theory of gravity is holographic.
where $G_R$ is the retarded inhomogeneous Green function for the scalar wave operator and $\phi_1$ is a solution of the free wave equation.

If we keep only the first term in $\Delta$ and (correctly to this order) treat $\phi_0$ as a local free field, then we are solving the usual Yang-Feldman equation [16] and we reproduce the expected local supergravity Wightman function. The other terms of order $1/N$ come from the connected part of the $\phi_0$ three point function in CFT, and from insertions of $\phi_1$, with $\phi_0$ treated as a free field. Thus, in order to remain with just the supergravity formula for the three point function we must have a cancellation

$$\langle \phi_0(x_1)\phi_0(x_2)\phi_0(x_3) \rangle_c + \frac{1}{N} \langle \phi_1(x_1)\phi_0(x_2)\phi_0(x_3) \rangle_c + \text{permutations} = 0 \quad (3.10)$$

Here, permutations, refers to the two other possible positions for $\phi_1$ in the Green’s function. Remember that the connected three point function of $\phi_0$ is of order $1/N$ so that these terms are of the same order. Note also that to this order, the expectations of products of $\phi_0$ in the last three terms are to be evaluated as if these were free fields in $AdS$ space.

Because this equation involves integrals over the entire boundary of the singular CFT correlators, making sense of it requires additional information, such as a prescription for analytic continuation. Such issues will be discussed in [3]. At present we are unable to tell whether a field $\phi_1$ satisfying this equation can be constructed. The issue of the existence of local fields including interactions remains to be clarified.

However, an indication that modifications along these lines are on the wrong track comes from the expression for the $SU(2,2|4)$ generators that one would deduce from the free fields (2.3). Each of the infinite tower of free fields $\phi_\alpha$ ($\alpha$ labels all the quantum numbers besides those of $AdS_5 \times S^5$) has its own generator of $SU(2,2|4)$; the full generator that implements the isometries on the supergravity fields (or even the string field of closed string field theory) has the form

$$G_{\alpha}^{SUGRA} = \sum_\alpha \phi_\alpha^\dagger [L_a, \phi_\alpha]. \quad (3.11)$$

This expression, which contains terms of arbitrarily high order in the SYM fields, has essentially no relation to the proper generators in terms of the CFT (super)stress tensor $T_{CFT}$ and superconformal Killing fields $v_a$

$$G_a^{CFT} = \int_b v_a \cdot T_{CFT} \quad (3.12)$$
which are bilinear in the SYM field strengths. Small $1/N$ corrections to the relation between the $\phi_\alpha$ and the SYM fields will not repair this disparity; the SYM theory has a fundamentally different structure, with far fewer degrees of freedom, from which the supergravity fields are built as highly composite objects.

In addition, the fact that the free fields we have constructed compute the exact interacting correlation functions (once the correct CFT operator algebra is used) in such a simple way, suggests to us that the operator solutions of the interacting supergravity field equations in the CFT Hilbert space are needlessly complicated objects.

Nevertheless, an important reason to attempt a construction of local fields is the following. To the extent that our framework precisely reproduces supergravity it will be exactly local, and such a construction must exist. On the other hand, previous work on the black hole information paradox strongly suggests that the full theory must contain some breakdown of locality. Therefore, the attempt to construct local field theory should break down at some level, and it would be extremely interesting to know where this happens.

The limit in which locality is recovered is likely to be highly context-dependent. There cannot be a universal limit in which high-frequency excitations are averaged out. For instance, an examination of well-separated objects in M-theory would lead one to conclude that membrane and fivebrane excitations should be integrated out below the Planck scale. Yet it is precisely these objects (when bound together in combination with supergravitons) that are responsible for generating the extremely small gap – which can be arbitrarily smaller than the Planck scale – observed in near-extremal black holes in four and five dimensions. The recovery of approximately local physics in the interacting theory is likely to be a rather delicate issue.

4. Aspects of black hole dynamics

While a detailed understanding of the interacting theory requires further investigation, qualitative aspects of black hole dynamics can already be understood. This section is divided into five parts. We begin by setting the stage for our discussion of black holes: We consider first the relation between scale in the CFT and radial position in $AdS$, and then some general features of the spectrum of string theory on $AdS_d \times S^p$. The next two parts discuss objects falling toward a black hole, including wrapped fundamental strings and waves on a three brane. Finally, we explain how one might describe an observer falling into a black hole in terms of the CFT.
4.1. IR-UV duality

There is an important correspondence between small/large radial position in AdS and IR/UV phenomena in the CFT [1,17]. One can gain insight into this by considering the frequently asked question: “Where are the branes?” According to the map (2.10), the superconformally invariant vacuum of the CFT maps to the vacuum state in AdS, suggesting that the branes are everywhere. A qualitative way to see that the branes fill the entire AdS spacetime is to consider the operator \( \text{tr}[X^2] \) in the example of \( AdS_5 \times S^5 \). This operator measures the mean square radial position of the branes. In the strong coupling limit, it has a large anomalous dimension \( \Delta \sim (g_s N)^{1/4} \sim R/\ell_s \). Fluctuations in the radial position are diagnosed by the correlator

\[
\langle \text{tr}[X^2(z)] \text{tr}[X^2(0)] \rangle \sim |z|^{-2\Delta}.
\]  

(4.1)

Thus, locality in the 3+1 dimensions of the gauge theory is in direct conflict with locality in the radial position. As we consider shorter and shorter distance scales in the gauge theory, fluctuations in the radial position become arbitrarily large. Conversely, fluctuations on the longest length scales probe the ‘center’ of the AdS space. We see qualitatively why a scale in the CFT corresponds to a radial location \( r \) in the bulk. We also see why \( \text{tr}[X^2] \) acquires a large anomalous dimension – it is the radial coordinate on \( AdS_5 \). Operator insertions on the branes are pointlike, hence extreme UV in character; this is why they are conventionally regarded as acting on the boundary of AdS. On the other hand, a given field configuration, such as an instanton, has a scale, putting it at the corresponding radial position [20].

As a consequence, the CFT does not really ‘live’ on the boundary of the AdS spacetime; rather, it fills the bulk. One can regard it as a representation of M-theory dynamics, much as worldsheet CFT is a representation of perturbative string dynamics. Indeed, the perturbative string also fills spacetime due to its quantum fluctuations; pointlike UV perturbations – the vertex operators – represent perturbations at the asymptotic boundary of spacetime, yet one would not say that the string worldsheet resides at the (conformal) boundary of Minkowski space.

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9 Such a correspondence was foreshadowed by the observed IR/UV duality of brane probes of background geometry [18], and indeed by similar well-known phenomena in perturbative string theory (c.f. [19]).

10 Once we have chosen a given generator \( P^0 + K^0 \) as the CFT Hamiltonian, the ‘center’ of AdS can be defined as the point where the corresponding Killing vector has minimal norm.
4.2. The supergravity spectrum

We now consider some general features of the spectrum of string theory on $AdS_d \times S^p$. At sufficiently high energy, the typical state in the gauge theory describes an $AdS_d$ Schwarzschild black hole (which is constant on $S^p$), with horizon size $r_+ > R$ and positive specific heat [13]. For lower energies, such that $\ell_s < r_+ < R$, one has a phase of ordinary $(d + p)$-dimensional Schwarzschild black holes [21]; the black hole localizes on $S^p$ due to the Gregory-Laflamme instability [22]. This latter phase is stable microcanonically, as is the ‘Hagedorn’ phase that appears below the (correspondence point) energy $E_{\text{corr}}$ where $r_+ \sim \ell_s$ [23]. These two phases would be missed in an analysis of the canonical ensemble where, because of the negative specific heat, once the energy reaches the string scale the external heat bath pumps energy into the system until it reaches the threshold to form an AdS Schwarzschild black hole. At very low energies, one expects a gas of supergravity particles in AdS to prevail.

We illustrate with two examples. For $AdS_5 \times S^5$, the hierarchy of scales is ($\ell_{\text{pl}}$ here denotes the 10d Planck scale)

$$E_{\text{AdS-Schw}} \sim R^7 \ell_{\text{pl}}^{-8} \sim N^2 R^{-1},$$

$$E_{\text{corr}} \sim \ell_s^{-1} \ell_{\text{pl}}^{-8} \sim N^2 R^{-1} (g_s N)^{-7/4},$$

$$E_{\text{Hag}} \sim \ell_s^{-1} (g_s N)^{9/4} \sim R^{-1} (g_s N)^{5/2},$$

such that the entropy is $S(E) \sim N^2 (R E/N^2)^{3/4}$ for $E > E_{\text{AdS-Schw}}$, and one has five-dimensional AdS Schwarzschild black holes; $S(E) \sim N^2 (R E/N^2)^{8/7}$ for $E_{\text{corr}} < E < E_{\text{AdS-Schw}}$, and one has ten-dimensional Schwarzschild black holes; $S(E) \sim (\ell_s E) \sim R E (g_s N)^{-1/4}$ for $E_{\text{Hag}} < E < E_{\text{corr}}$, where fundamental strings dominate the entropy; and $S(E) \sim (R E)^{9/10}$ for $E < E_{\text{Hag}}$, where the entropy is dominated by a gas of supergravity particles in $AdS_5 \times S^5$. We have assumed strong coupling $g_s N > 1$, otherwise the supergravity gas and the 10d black hole phase disappear.

Similarly, in $AdS_3 \times S^3 \times M$, there is a corresponding set of scales (here $\ell_{\text{pl}}^{(3)}$ denotes the 3d Planck scale, $R$ is the $AdS_3$ radius, $g_s$ is the 6d string coupling, and $k = Q_1 Q_5$)

$$E_{\text{BTZ}} \sim (\ell_{\text{pl}}^{(3)})^{-1} \sim k R^{-1},$$

$$E_{\text{corr}} \sim k \ell_s^{-1} (g_s^2 k)^{-3/4} \sim k R^{-1} (g_s^2 k)^{-1/2},$$

$$E_{\text{Hag}} \sim \ell_s^{-1} (g_s^2 k)^{5/4} \sim R^{-1} (g_s^2 k)^{3/2}. \quad (4.3)$$

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11 We thank J. Maldacena for helpful discussions which improved our understanding of this spectrum.
For completeness, we note that the Kaluza-Klein modes and string winding modes on $M$ have typical scales $R^{-1/2}(g_s Q_5)$ and $R^{-1/2}(g_s Q_1)$, respectively.

One might ask what regime of parameters is described by the $S^k(M)$ symmetric orbifold CFT which is a candidate for the dual CFT. Roughly, the $n^{th}$ twisted sector has oscillators with $O(1/n)$ moding; the energy threshold to reach this sector is $RE \sim O(n)$. Thus the total density of states is approximately

$$\rho(E) \sim \min (k, RE) \sum_{n=0}^{k} \rho_n \exp \left[ \beta_0 \sqrt{n(RE - n)} \right]$$

for some constants $\beta_0, \rho_n$. For a given $E < k/2R$, the probable value of $n$ is $O(RE/2)$, thus $\rho(E) \sim \exp[\beta_0 RE/2]$ is a Hagedorn spectrum. We conclude that the symmetric orbifold describes a regime where $\ell_s > R (g_s^2 k < 1)$, since there is no regime where the system looks like a supergravity gas in $AdS_3 \times S^3$, and there is no 6d black hole phase. To attain the supergravity limit requires an understanding of the CFT away from the orbifold point.

4.3. Falling toward a black hole

We are now ready to consider some simple examples of objects falling toward a black hole and explain how to describe them in terms of the gauge theory. The basic idea is that objects initially far from the black hole are described by localized excitations of the gauge theory. The evolution toward the black hole is represented by an expansion of the size of the excitation. This is a dynamical effect, and not e.g. just a change in the UV cut-off. For definiteness, we consider the case of four dimensional SYM which describes string theory in $AdS_5 \times S^5$.

$AdS_5$ can be written in the form

$$ds^2 = \frac{r^2}{R^2} \left[ -dt^2 + dx^2 + dy^2 + dz^2 \right] + \frac{R^2 dr^2}{r^2}$$

where $\partial/\partial t$ becomes null at the horizon $r = 0$. Let us compactify $z$ with period $L$. Any excitation of (4.5) will evolve toward $r = 0$. For example, consider a fundamental string wound once around $z$. If we start the string at rest at large $r$, it will collapse toward $r = 0$ moving close to the speed of light. String theory on this background is described by the SYM on $S^1 \times R^{2,1}$. Time evolution in the gauge theory corresponds

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12 We understand that similar ideas are being explored in [24].

13 This produces a conical singularity on the horizon, but this difficulty will be removed shortly.
to evolution in $t$ in (4.5). The string at $r = \infty$ corresponds to a Wilson loop (in the fundamental representation) wrapping around the circle in the gauge theory \[21\]. Acting on the vacuum, this Wilson loop creates an infinitesimally thin flux tube. This state has infinite energy, but so does the string wound around the infinitely long circle at $r = \infty$. A better starting point is to introduce an UV cut-off $\delta$ and smear the flux tube out over a thickness $\delta$. By the UV/IR connection (c.f. [1,17]), this corresponds to starting the string at $r = R^2/\delta$. The flux tube is not a stationary state in the SYM theory. Its width will expand in time, eventually filling all space as $t \to \infty$. This is the analog of the string falling toward $r = 0$. Even though it takes only a finite proper time to reach $r = 0$, the coordinate $t$ diverges. This interpretation is consistent with the above arguments that the radial direction corresponds to a length scale in SYM.

Suppose we now start with the near extremal black three-brane. The near horizon geometry is given by

$$ds^2 = \frac{r^2}{R^2} \left[ - \left( 1 - \frac{r_0^4}{r^4} \right) dt^2 + dx^2 + dy^2 + dz^2 \right] + \left( 1 - \frac{r_0^4}{r^4} \right)^{-1} \frac{R^2 dr^2}{r^2}$$ \hspace{1cm} (4.6)$$

It has a Hawking temperature $T = r_0 / (\pi R^2)$ and an energy density $\mu = 3\pi^2 N^2 T^4 / 8$. In the gauge theory, the geometry (4.6) represents a typical state with energy density $\mu$. Since the number of quanta is large\[14\], such a state is closely approximated by a thermal state with temperature $T$. Let us again compactify $z$, wrap a string around it and let it fall in. In the gauge theory, the wrapped string again corresponds to a thin flux tube wrapped around $z$. The flux will expand as before, but now it only expands until it reaches the typical wavelength $1/T$ of the thermal state. At that point, it thermalizes. We claim that this corresponds to the wrapped string approaching the horizon. To see this, note that the infalling string will approximately follow a null geodesic. So the time it takes to fall from a large radius to $r$ of order $r_0$ is $t_0 = \int R^2 dr / r^2 = R^2 / r_0$. If we assume the flux also expands at close to the speed of light, after this time it will have a width of order $t_0$ which is indeed of order the thermal wavelength $t_0 = R^2 / r_0 \sim 1/T$. This agreement is quite robust: It is independent of the details of the initial starting point, and would hold in other dimensions as well.

When the string becomes very close to the horizon, the time $t$ diverges logarithmically. It is tempting to interpret this as reflecting the approach to thermal equilibrium in

\[14\] This is true for any $\mu$, since the volume in the $(x,y)$ directions is arbitrarily large.
the gauge theory. Note that the ordinary Hamiltonian evolution in the gauge theory does
not describe the string crossing the horizon. This is not surprising, since time evolution in
the gauge theory corresponds to evolution in the asymptotic time $t$ in the spacetime. As
we discuss below, to see evolution across the horizon, one has to evolve the state using an
operator which is different from the usual Hamiltonian.

We have described objects falling into an existing black hole in the gauge theory. How does one describe the formation of the black hole itself? A generic initial state of high energy $E > N^2/R$ in $AdS_5$ will collapse to form a Schwarzschild-AdS black hole, which then radiates. Since the negative cosmological constant acts like a confining box, the black hole eventually comes into thermal equilibrium with the gas outside. This is not an exotic process in the gauge theory on $S^3 \times \mathbb{R}$. Rather, it is simply the statement that most high energy states will evolve into a state which is closely approximated by the thermal state. This is also true for states in the gauge theory with energy sufficiently less than $N^2/R$, but they correspond to a gas of particles in AdS without a black hole.

While this is the generic behavior, it is worth emphasizing that special states can
behave differently. For example, consider a low energy supergravity excitation in AdS
which is boosted in one direction until it has a large energy with respect to the global
time translation. Its evolution will be approximately a geodesic which oscillates from one
side of AdS to the other. The point is that the choice of a time coordinate generated by
a globally timelike generator of $SO(4,2)$ is nonunique. A state which is static with respect
to a given Hamiltonian will be an oscillating coherent state with respect to another. In
the SYM, this oscillating state is described by evolving the original SYM state with the
conformal generator corresponding to the new global time in AdS. The resulting state will
no longer be static. Its evolution will localize it near one point of the three-sphere, then
cause it to expand and relocalize near the antipodal point on the sphere, and then repeat.
So changing the spatial scale is not always associated with a renormalization group flow:
Sometimes the motion is reversible.

4.4. Waves on a three-brane

Another type of object which can be introduced into the IIB theory is a three-brane. One can think of the $AdS_5$ background as produced by a large number $N$ of three-branes, leading to the representation as large $N$ gauge theory on $\mathbb{R}^{3,1}$. As discussed in [1], a state containing a three-brane parallel to these at radius $r$ (in the coordinates (4.5)) is described by a gauge theory vacuum with a non-zero scalar vev, having a single non-zero
eigenvalue (call it $\vec{X}_{11}$) with $|\langle \vec{X}_{11} \rangle| = r$. This breaks the gauge group to $U(N-1) \times U(1)$ and excitations in the $U(1)$ can be interpreted as waves on the isolated brane.

Since we know the $r$ coordinate for this brane, this identification gives us another way to produce localized excitations in AdS. However, to describe excitations crossing a horizon, it would be more interesting to have a description of the three-brane in the global coordinates $(\text{2.3})$, in other words as a state in gauge theory on $S^3 \times R$.

The appropriate state is easy to find, given the fact that $R^{3,1}$ can be conformally embedded into $S^3 \times R$. We simply take the classical solution $\vec{X}_{11} = \vec{x}$ on $R^{3,1}$ and apply the standard transformation law for the conformally coupled scalar $X$. This leads to the configuration

$$\langle X_{11} \rangle = \frac{\vec{x}}{\cos t + \cos \chi}$$

(where the metric on $S^3$ is $d\Omega_3^2 = d\chi^2 + \sin^2 \chi \ d\Omega_2^2$, and $t$ is the global time). This is a classical solution preserving 16 of the 32 superconformal symmetries. As such, it can be expected to be an exact solution of the gauge theory. This solution should describe the same three-brane, but in the global coordinate system.

This solution again breaks the gauge symmetry to $U(N-1) \times U(1)$ and excitations in the $U(1)$ are again expected to be excitations of the isolated brane. To the extent that these excitations are confined to the brane, they naturally propagate in Minkowski time, but one can see how such propagation looks in global time.

More importantly, we can act on this solution with $SO(4,2)$ to obtain new three-brane solutions. In particular, translation of the global time variable produces the solutions

$$\langle X_{11} \rangle = \frac{\vec{x}}{\cos(t - t_0) + \cos \chi}$$

which extend out of the region covered by the coordinates $(\text{4.5})$. (A stationary three-brane in the coordinates $(\text{4.5})$ will asymptote to the boundary of this region at large $r$. Since the boundary is at $\cos t = 1/\sqrt{1 + r^2}$, any shift of $t$ will cause the brane to cross the boundary.)

This is an example of a brane solution which crosses the horizon of an extreme black hole. Is it possible to construct a brane which crosses the event horizon of a nonextreme black hole background?

4.5. *Falling into BTZ black holes*

We have seen in sec. 4.3 that when an AdS black hole is represented as a state in the dual CFT, the natural Hamiltonian evolution corresponds to evolving with respect to the
external Schwarzschild time of the black hole. On the other hand, since the global time in AdS is not unique, it is natural to consider different evolutions in the spacetime which correspond to different operators in the CFT playing the role of the Hamiltonian. Combining these ideas, one is led to consider alternative evolutions in the black hole case. In particular, one might ask whether there is an operator which approximates the experience of an infalling observer. We will argue below that the answer is yes. The fact that there is a single Hilbert space, with different (noncommuting) operators describing external and infalling observers, is a concrete realization of the ideas of black hole complementarity [6].

An ideal context to illustrate this is the BTZ black hole [7], whose spacetime geometry is locally $AdS_3$. Some of the local isometries have orbits which cross the horizon, so the corresponding conformal generator in the CFT is a natural candidate for evolving the state for these infalling observers. To be specific, we start with the description of $AdS_3$ as the hyperboloid

$$-T_1^2 - T_2^2 + X_1^2 + X_2^2 = -R^2$$

(4.9) in $\mathbb{R}^{2,2}$. A convenient way to parameterize this surface is to introduce coordinates associated with two commuting symmetries. If we let $t, \varphi$ parameterize rotations in the $T_1, T_2$ and $X_1, X_2$ planes respectively, then the metric takes the standard form

$$ds^2 = -\left(\frac{r^2}{R^2} + 1\right) dt^2 + \left(\frac{r^2}{R^2} + 1\right)^{-1} dr^2 + r^2 d\varphi^2$$

(4.10)

If, instead, we let $t, \varphi$ parameterize boosts in the $T_1, X_1$ and $T_2, X_2$ planes respectively, the metric takes the form

$$ds^2 = -\left(\frac{r^2}{R^2} - 1\right) dt^2 + \left(\frac{r^2}{R^2} - 1\right)^{-1} dr^2 + r^2 d\varphi^2$$

(4.11)

The BTZ black hole is obtained by periodically identifying $\varphi$ in (4.11), and the total energy depends on the choice of period. Alternatively, one can rescale $r$ so that the period of $\varphi$ is always $2\pi$ and the metric becomes:

$$ds^2 = -\left(\frac{r^2}{R^2} - m\right) dt^2 + \left(\frac{r^2}{R^2} - m\right)^{-1} dr^2 + r^2 d\varphi^2$$

(4.12)

The total energy of this solution relative to the $AdS_3$ ground state (4.10) is related to the parameter $m$ by $E \sim E_{BTZ} + mk/R$, where $k = Q_1 Q_5$ and $E_{BTZ} = k/R$ as in (4.3).

15 We restrict our attention to nonrotating black holes; there is a simple generalization to the rotating case involving periodically identifying a linear combination of $t$ and $\varphi$. 

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Hawking temperature is \( T = \sqrt{m}/(2\pi R) \). Note that the form of the metric (4.12) also includes \( AdS_3 \) (without identifications) by setting \( m = -1 \).

We have seen that a rotation in the \( T_1, T_2 \) plane corresponds to the global time translation in \( AdS_3 \) (4.10), but not in the BTZ black hole (4.12) (with \( m > 0 \)). Nevertheless, in the latter case, it is still a local isometry which is simply not invariant under translations of \( \varphi \) by \( 2\pi \). When \( m > 0 \), the radial coordinate in (4.12) is related to the embedding variables by \( r^2 = m(T_2^2 - X_2^2) \). So if we start with \( r^2 > 0 \) and \( T_1 = 0 \) (which corresponds to \( t = 0 \) in (4.12)) and rotate \( T_1, T_2 \) keeping \( X_1, X_2 \) fixed, then \( r \) decreases to zero after a finite rotation. In other words, the orbits of the local symmetry \( T_1 \partial/\partial T_2 - T_2 \partial/\partial T_1 \) are timelike curves which fall into the black hole and hit the singularity in finite proper time.

How is this described in the CFT? The usual Hamiltonian, \( L_0 + \bar{L}_0 \), generating time translations on \( S^1 \times \mathbb{R} \), always corresponds to translations of \( t \) in (4.12). This is true for either sign of \( m \). Similarly, \( L_0 - \bar{L}_0 \) always corresponds to translations of \( \varphi \). Thus even though the local isometries of the spacetime, \( SO(2,2) \), agree with the conformal symmetries of the 1 + 1 dimensional CFT, the relation between the generators depends on the energy. For \( m < 0 \) (\( E < E_{BTZ} \)), the spacetime generator \( T_1 \partial/\partial T_2 - T_2 \partial/\partial T_1 \) is represented in the CFT by \( L_0 + \bar{L}_0 \), but for \( m > 0 \) (\( E > E_{BTZ} \)), it is not. Since \( L_0 + \bar{L}_0 \) and \( L_0 - \bar{L}_0 \) now correspond to boosts in the \( T_1, X_1 \) and \( T_2, X_2 \) planes, the algebra implies that

\[
T_1 \frac{\partial}{\partial T_2} - T_2 \frac{\partial}{\partial T_1} = \frac{1}{2} (L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}) .
\]

(4.13)

If we adjust the constants in \( L_0, \bar{L}_0 \) so that they vanish for the NS ground state, the BTZ black hole is represented on the CFT side by the ensemble of states with energy \( L_0 = \bar{L}_0 > k/4 \). In this regime, the canonical and microcanonical ensembles are equivalent. In \( AdS_5 \times S^5 \), the transition from a gas of particles in AdS to a black hole in AdS is associated with a ‘deconfinement’ transition in the gauge theory \[13\]. In \( AdS_3 \), the analogous order parameter involves the permutation symmetry of the symmetric product \( S^k(M) \) \[27\]; singlets dominate the gas phase, while large representations dominate the BTZ phase. At finite \( k \), this crossover is not singular, and the CFT ‘sums’ over all topologies in supergravity that are asymptotically \( AdS_3 \times S^3 \times M \) (including 6d Schwarzschild black holes, as discussed above). At large \( k \), and well into the BTZ regime \( L_0, \bar{L}_0 \gg k/4 \), the classical BTZ spacetime is the only relevant geometry. As we have said, the time conjugate to \( L_0 + \bar{L}_0 \) is to be related to the static external time of the black hole solution. In direct analogy with the \( AdS_5 \times S^5 \) case, a probe dropped into the black hole will be seen to be
thermalized in the CFT as it evolves according to the asymptotic time variable. However, here the bulk spacetime is locally $AdS_3$, so we can also evolve the CFT state using (4.13), corresponding to an infalling frame of reference. The black hole state in the CFT is not an eigenstate of this operator, corresponding to the fact that the black hole geometry is not static in the infall frame. The fact that the operator (4.13) is not invariant under $2\pi$ shifts of $\varphi$ in the BTZ spacetime is not a problem, since all of the generators of the conformal symmetry act on all of the states in the CFT – even those which are not invariant. To summarize, the evolution operators for infalling and static observers simply correspond to different (noncommuting) generators of $SO(2,2)$, whose action on the CFT states is canonical.

Neither evolution is singular in the finite $k$ CFT. The static evolution is just that – the black hole states are eigenstates of $L_0 + \bar{L}_0$. The infall evolution is also nonsingular, since any $SO(2,2)$ generator acts unitarily on the CFT Hilbert space. This shows that, given the AdS/CFT duality, quantum effects indeed resolve the BTZ black hole singularity! The infall evolution generator (4.13) should develop a singularity in the classical limit $k \to \infty$, $E > k/4$, in the regime $\ell_s \ll R$ where one can trust supergravity; this possibility is currently under investigation. Such a singularity would simply reflect an impropriety of the limit (as in geometric optics near a focal point) rather than of the theory itself. The ‘long string’ model that pertains to the orbifold locus in the CFT moduli space does not appear to exhibit such a singularity for finite infall evolution parameter even in this limit, perhaps because this CFT describes $AdS_3$ with $\ell_s > R$, so that the geometry is stringy. Details will appear in [5].

In higher dimensions, we believe that a similar story should hold: Observers that fall into a black hole are described in the CFT by evolving the state by an operator (or family of operators) which are different from the usual Hamiltonian. To determine this operator and find the behavior of a black hole state near the curvature singularity is an outstanding problem. In this regard, it is interesting to note the following. The above observation that the relation between the spacetime and CFT symmetry generators depends on the energy of the state, seems to be a special feature of $2+1$ dimensions. In higher dimensions, the Schwarzschild-AdS black hole does not have constant curvature. But in the asymptotically AdS region, the time translation symmetry outside the black hole is the same as a global time translation in AdS, i.e., a compact generator of $SO(n,2)$. This can be seen from the fact that the boost symmetries do not commute with all the rotation symmetries $SO(n)$. 

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present outside the black hole.

A different approach to some of the questions addressed in this paper has recently been given in [28].

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