Thermal Pions

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I discuss the absorption and dispersion of pions in hot matter. A two-loop calculation in the framework of chiral perturbation theory is presented and its result is compactly written in terms of the two- and three-particle forward $\pi\pi$ scattering amplitudes. At modest temperatures, $T \leq 100$ MeV, the change in the pion mass is small and its dispersion law closely resembles the free space one. At these temperatures, all quantities of interest are given to a good degree of accuracy by the first term in the virial expansion which is linear in the density.

1. INTRODUCTION

The properties of pions in hot hadronic matter are encoded in the pion propagator, in particular in the mass shift due to the influence of the heat bath,

\[(p^0)^2 = \tilde{p}^2 + M^2_\pi + \Pi(p^0, \tilde{p})\]  

(1)

where $M_\pi$ denotes the pion mass, $p^0$ its energy, $\tilde{p}$ its three-momentum and $\Pi$ its self-energy, which is in general a complex quantity. Its real part is related to the dispersion and the group velocity while its imaginary part encodes the information about the pion absorption. Chiral perturbation theory (CHPT) allows to systematically calculate the behaviour of $\Pi$ at low and modest temperatures as will be spelled out below. This calculation is done under the following assumptions. I assume a hadron gas in a state of thermal equilibrium which mostly consists of pions. Therefore, $T$ should not be larger than approximately 100 MeV because at higher temperatures the massive states start to dominate \cite{1}. In particular, there should not be any sizeable baryon contamination in the gas. Therefore, the results which will be presented below are mostly relevant for future experiments and facilities which will have more pions and reach lower temperatures than the present ones. It is instructive to briefly recall the scattering of light on a dilute gas of $N$ molecules in a volume $V$. The index of refraction will in general be complex and to leading order in the density is given by \cite{4}:

\[n + i\kappa \simeq 1 + \frac{2\pi N}{k^2 V} f\]  

(2)

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with $f$ the forward scattering amplitude of photons on a single molecule. Making use of the optical theorem, it follows that $\kappa$ is directly proportional to the total forward scattering cross section. In complete analogy, we will see that the absorptive and dispersive properties of the pions are related to the forward $\pi\pi$ and $\pi\pi\pi$ scattering amplitudes (the latter is related to the effects of second order in the density).

2. EFFECTIVE FIELD THEORY OF QCD AT FINITE TEMPERATURE

At low energies, the QCD Green functions are dominated by the (almost) massless Goldstone bosons related to the spontaneous breakdown of the chiral symmetry, $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$. Furthermore, to a good first approximation the quark masses can be set to zero (which defines the chiral limit of QCD). The consequences of these features can be worked out in a systematic fashion by making use of an effective field theory,

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{eff}}[U, \partial_\mu U, \mathcal{M}]$$

where $U = \exp[i\pi/F_\pi]$ embodies the Goldstone (pion) fields and $\mathcal{M}$ is the quark mass matrix. A fundamental scale of the strong interactions at low energies is set by the pion decay constant $F_\pi$, which is defined via $<0|A_\mu^i|\pi^k> = i\delta^{ik}p_\mu F_\pi$, $F_\pi \simeq 93$ MeV, with $A_\mu^i$ the axial current. One can systematically expand all Green functions around the chiral limit. This is a simultaneous expansion in small momenta, small quark (pion) masses and small temperatures, the corresponding small parameters being $p/\Lambda$, $\mathcal{M}/\Lambda^2$ and $T/\Lambda$. Here, $\Lambda \simeq M_\rho$ is the scale where the non-Goldstone excitations become important. As shown by Weinberg [3], in the effective Lagrangian framework this amounts to an expansion in pion loops. The effective Lagrangian consists of a string of terms with increasing number of derivatives and/or quark mass insertions,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

To lowest order, one calculates tree diagrams using $\mathcal{L}^{(2)}$. This leads to the venerable current algebra results. At next-to-leading order, one has to calculate one-loop diagrams using $\mathcal{L}^{(2)}$ to perturbatively restore unitarity and tree diagrams with exactly one insertion from $\mathcal{L}^{(4)}$. The coefficients of the latter terms are not restricted by symmetry but have to be determined from phenomenology [3].

At finite temperatures, only a few modifications occur. First, if one works in a real-time approach, one has to choose a proper path for the time integration in the action. This path is called $\mathcal{C}$ and extends from $-\infty$ to $+\infty$ above the real axis, returns below the real axis and goes down to $-\infty - i\beta$ parallel to the imaginary axis [4]. Here, $\beta = 1/T$. Second, the pion propagator is modified according to well-known rules and finally, the conventional time-ordering at $T = 0$ is substituted by path-ordering along $\mathcal{C}$ for $T \neq 0$, e.g.

$$<0|T(A_\mu A_\nu)|0> \rightarrow <0|T_\mathcal{C}(A_\mu A_\nu)|0>$$

More details on CHPT and its application to finite temperature physics can be found e.g. in the review [3] or in Gerber and Leutwyler [4].
3. TWO-LOOP CALCULATION OF THE PION SELF-ENERGY

3.1. One-loop calculation and virial expansion

It is instructive to recall how one calculates the pion properties in CHPT at \( T = 0 \). One particularly useful choice of the interpolating pion field is the axial current due to the PCAC relation. The correlator of two axial currents reads

\[
i \int dx \, e^{ipx} \langle 0|T(A_i^i(x)A_i^k(0))|0 \rangle = \delta^{ik} \left\{ \frac{p_\mu p_\nu F_\pi^2}{M_\pi^2 - p^2} + g_{\mu\nu} F_\pi^2 + \ldots \right\}
\]

(6)

This can straightforwardly be extended to the finite temperature case. Another method is the virial expansion to which I will come back later. The calculation of the pion self-energy to one loop order has been done by Goity and Leutwyler \[7\], Shuryak \[8\] and Schenk \[9\]. One splits the pion self-energy as \( \Pi = \Pi^0 + \Pi^T \); where \( \Pi^T \) vanishes at \( T = 0 \) and describes the modification due to the heat bath. Instead of Re \( \Pi \) and Im \( \Pi \), one conventionally writes

\[
p^0 = \omega(p) - \frac{i}{2} \gamma(p)
\]

(7)

with \( \omega(p) \) the frequency of the pionic waves and \( M_\pi(T) = \omega(p = 0) \) the effective mass. The damping coefficient \( \gamma(p) \) is the inverse time within which the intensity of the wave is attenuated by a factor \( 1/e \). In the one-loop approximation, one simply finds

\[
\omega(p) = \sqrt{p^2 + M_\pi^2(T)}, \quad \gamma(p) = 0
\]

(8)

This is a quite common phenomenon in CHPT calculations - to get to the imaginary part of a certain quantity with the same accuracy as the real part, one has to work harder. It is simply related to the fact that to lowest order one calculates tree diagrams, which are real. The temperature-dependent pion mass \( M_\pi(T) \) has e.g. been given by Gasser and Leutwyler \[10\] in the framework of CHPT. Notice that there is no absorption to one loop, the only effect of the interaction with the heat bath is that the pion mass and decay constant become \( T \)-dependent. There is another way of presenting the one-loop result. If one inspects the corresponding Feynman diagrams, one sees that one effectively deals with the \( \pi \pi \) scattering amplitude in forward direction, \( T_{\pi\pi}(s) \) with \( \sqrt{s} \) the cm energy of the collision. To this order, the pion self-energy is linear in the density and one can write

\[
\Pi^T(\omega_p, \vec{p}) = -\int \frac{d^3q}{(2\pi)^32\omega_q} n_B(\omega_q) T_{\pi\pi}(s)
\]

(9)

with \( \omega_k = \sqrt{k^2 + M_\pi^2} \) and \( n_B^{-1} = (\exp[\beta\omega] - 1) \) is the canonical Boltzmann factor. This is also the first term in the virial expansion. For a general proof, see \[11\]. This last formula is very powerful. If one uses empirical input for \( T_{\pi\pi}(s) \), this form allows one to go beyond the strict one loop CHPT result. In particular, the important effect of the \( \rho \) resonance can thus be implemented fully, whereas in the one-loop representation of \( T_{\pi\pi}(s) \) one only sees the tail of the \( \rho \) (for more details see e.g. \[9\]). One result of employing this procedure is that the pion mass depends only very weakly on the temperature \[9\].
3.2. Two-loop calculation

Schenk [12] has extended the above to two loops. This calculation is much more tedious - there are not only more diagrams, but one also has to account for three-particle scattering as inspection of some Feynman diagrams reveals. In particular, one finds that

\[ \Pi^T = \Pi^{(1)} + \Pi^{(2)} \] (10)

which means that one has contributions which are linear and quadratic in the Boltzmann factor (density). The diagrams which contribute to \( \Pi^T \) involve insertions from \( L^{(2)}_{\text{eff}} \) and \( L^{(4)}_{\text{eff}} \), the contribution from \( L^{(6)}_{\text{eff}} \) is temperature-independent. Therefore, the final result for \( \Pi^{(1)} \) is entirely fixed in terms of \( M_\pi, F_\pi \) and four low-energy constants determined already in [4]. The term of second order in the density involves an integration over three pion momenta and a function which is of fourth order in \( M_\pi \) and the pion momenta. The result is correct up to and including order \( p^6 \) in CHPT. The explicit formulae are given in [12]. Instead of presenting the rather lengthy expressions, let me rather discuss a more compact form involving S-matrix elements of two- and three-pion scattering. One expects that at two-loop order the thermal distribution will depend on the effective mass, i.e. on \( n_B(\omega^T_p) \) with \( (\omega^T_p)^2 = \vec{p}^2 + M_\pi^2(T) \). In addition, there are the diagrams related to three-particle scattering. Now the forward limit of the \( 3 \to 3 \) scattering amplitude does not exist due to diagrams where one pion is exchanged between clusters and is close to its mass shell. One thus defines a proper amplitude \( \hat{T}_{33} \) [12]

\[ \hat{T}_{33} = T_{33} + \sum_q T_{22} \frac{1}{q^2 - M_\pi^2} T_{22} \] (11)

which has a well-defined forward limit (\( T_{22} \) is the two-particle scattering amplitude). For a more detailed discussion see [12]. Consequently, one can express the result of the two-loop calculation as

\[ \Pi^T(\omega_p, \vec{p}) = - \int \frac{d^3q}{(2\pi)^32\omega_q} n_B(\omega^T_q) T_{\pi\pi}(s) \]

\[ - \frac{1}{2} \int \frac{d^3q}{(2\pi)^32\omega_q} \frac{d^3k}{(2\pi)^32\omega_k} n_B(\omega_q) n_B(\omega_k) \hat{T}^R_{\pi\pi\pi}(p, q, k) \] (12)

with \( \hat{T}^R_{\pi\pi\pi} \) the proper retarded \( 3 \to 3 \) forward scattering amplitude. Notice again that in the first term on the r.h.s. of \( \Pi^T \) the effective pion mass appears, i.e. the thermal distribution of the pions actually depends not longer on the bare mass as in the one-loop case. The second term is the new one, it is believed to be the most general term quadratic in the density (which is, however, not strictly proven). Notice that one could also use \( T \)-dependent energies in the \( n_B^2 \) term, but this goes beyond the accuracy of the two loop calculation. Let me stress again that on can not further simplify this result and express e.g. in terms of a \( T \)-dependent two-particle scattering amplitude only. For details, see [12]. The damping rate \( \gamma(p) \) follows using unitarity form eq.(12). If one neglects Bose correlations in the initial and final states, it can compactly be written as

\[ \gamma(p) = \omega_p^{-1} \int \frac{d^3q}{(2\pi)^32\omega_q} n_B(\omega_q) \sqrt{s(s - 4M_\pi^2)\sigma_{\pi\pi}(s)} \] (13)
Indeed, the r.h.s. of this formula represents the collision rate for pions with momentum $p$ moving through a pionic target whose momenta are distributed according to the Bose factor $n_B(\omega)$. The mean damping rate is defined by

$$ < \gamma(p) > = \int d^3p n_B(\omega_p) \gamma(p) / \int d^3p n_B(\omega_p) $$  \hspace{1cm} (14)

### 3.3. Choice of the $\pi\pi$ scattering amplitude

Before presenting results, a few remarks on the $\pi\pi$ scattering amplitude are in order. One can rewrite eq.(9) in terms of a thermal weight function, $W(p, s)$. An elementary consideration leads one to conclude that for temperatures $T < 200$ MeV the exponential behaviour of $W(p, s)$ suppresses the scattering contributions with cm energies $\sqrt{s} \lesssim 1$ GeV. In this energy range, only the S and P partial waves are non-negligible, so that one can write the forward $\pi\pi$ scattering amplitude as

$$ T_{\pi\pi}(s) = \frac{32\pi}{3} \left( T^0_0 + 9T^1_1 + 5T^2_0 \right) $$  \hspace{1cm} (15)

where the upper index denotes the isospin. As it is well-known, while the isospin zero S-wave and the P-wave are attractive, the smaller exotic S-wave is repulsive and thus there are cancellations in $T_{\pi\pi}(s)$. As is discussed in detail in ref.[13], the one-loop CHPT results are reliable up to energies of $\sqrt{s} \simeq 500$ MeV. Beyond this energy, the corrections become too large and also, one misses completely the peak due to the $\rho$ in $T^1_1$. Finally, unitarity is violated above 700 MeV. Therefore, to make full use of the virial expansion, one can either use a semi-phenomenological parametrization of the phase shifts imposing CHPT constraints at low energies [12] or use an extended effective Lagrangian including also the low-lying resonance fields $R, L_{\text{eff}}[U, R]$. Such an approach has been shown to be successful in the description of $\pi\pi$ and $\pi K$ scattering data [14]. In the following section, I will therefore present results based on the exact two-loop CHPT result (which is correct to order $p^6$) and also making use of the virial expansion and a parametrization of the $\pi\pi$ scattering amplitude which extends to energies of about 1 GeV.

### 4. RESULTS AND DISCUSSION

Here, I will only present the most prominent features of the extensive study presented in ref.[12]. These can be summarized as follows:

First, let us consider the temperature dependence of the effective mass $M_\pi(T)$. If one uses the full CHPT two loop result, one finds that for $T = 100(150)$ MeV the mass is lowered by 2.5 (14) percent. Also, the effects of second order in the density are tiny, they amount to a shift of 2.5 percent at 150 MeV. These $n^2_B$ corrections stem almost entirely from three-body collisions, the effects of the mass shift in the Bose factor are negligible. The full two loop CHPT result is also not very different from the first term in the virial expansion making use of a better description of the forward $\pi\pi$ scattering amplitude. Notice that the current algebra result for the $\pi\pi$ forward scattering amplitude leads to an increasing pion mass because the momentum-dependent part cancels in eq.(15) and the remainder is constant and negative.
For the quasi-particle energy $\omega(p)$ one finds that the two-loop CHPT result and the analysis based on the phenomenological description of the $\pi\pi$ scattering amplitude agree within 20 per cent at temperatures and momenta below 150 MeV. At higher momenta, the influence of the $\rho$ becomes more and more pronounced and the straightforward CHPT result is not reliable any more. Most important, however, is the fact that temperature effects on the dispersion are small. The dispersion law of pions in a medium closely resembles the one in free space. At $T = 150$ MeV, the scattering with the gas modifies the frequency by less than 20 per cent. This means that in the low $T$, long-wavelength limit one does not find any indication for a substantial change in the dispersion law as suggested by Shuryak [15]. Again, the effects of order $n_B^2$ are small, i.e. the first term in the virial expansion dominates.

Let us now consider the mean damping rate. The effects of Bose correlations are of the order of a few per cent, indicating again that the main contribution to the damping rate stems form the terms of first order in the density. It is important to notice that since the damping rate is proportional to the imaginary parts of the partial waves (which are positive due to unitarity), no cancellations occur between the S- and P-waves. As a consequence, using the imaginary parts of the one loop CHPT prediction for the $\pi\pi$ scattering amplitude leads to a result which agrees within 30 per cent with the virial expansion at $T < 100$ MeV. Using the full one-loop CHPT result [13], i.e. the real and the imaginary parts of the partial waves, the mean damping grows much too strongly as $T$ increases. This is due to the fact of the unitarity violation in the $I = 0$ S-wave above 600 MeV cm energy (notice that at $T = 100$ MeV, both collision partners have energies of about 300 MeV). For temperatures between 100 and 200 MeV, the leading term in the virial expansion predicts a decrease of the mean damping rate by a factor of two. For these temperatures, however, one has to account for the massive states like the $K$, $\rho$, $\eta$, . . . These increase the damping rate. For temperatures above 100 MeV, one expects that $< \gamma(p) > \sim T^5/12F_\pi^4$ [10].

To summarize, I have shown that the propagation properties of pions in hot matter are determined by the pole position of various Green functions, like e.g. $< 0 | T_C(AA) | 0 >$. The pion energy is in general complex, $p^0 = \omega(p) - i\gamma(p)/2$. At low temperatures, the virial expansion of $\omega(p)$ and $\gamma(p)$ is appropriate. To first order in the density, the frequency $\omega(p)$ and the damping rate $\gamma(p)$ are determined by the forward $\pi\pi$ scattering amplitude, $T_{\pi\pi}(s)$. To second order in the density, one has additional contributions related to the proper three-particle scattering amplitude $\hat{T}_{\pi\pi\pi}$ and the effective pion mass enters the Boltzmann factor in the first term of the virial expansion. However, the effects of order $n_B^2$ are small. The main contributions to $\omega(p)$ and $\gamma(p)$ stem from the first term in the virial expansion. Furthermore, the temperature effects on the energy and mass are small. The dispersion law of pions at finite temperatures closely resembles the one in the vacuum.

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