HIGH-ORDER SOLITON MATRIX FOR THE THIRD-ORDER FLOW EQUATION OF THE GERDJIKOV-IVANOV HIERARCHY THROUGH THE RIEMANN-HILBERT METHOD

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Abstract. The Gerdjikov-Ivanov (GI) hierarchy is derived via recursion operator, in this paper, we mainly consider the third-order flow GI equation. In the framework of the Riemann-Hilbert method, through a standard dressing procedure, soliton matrices for simple zeros and elementary high-order zeros in the Riemann-Hilbert problem (RHP) for the third-order flow GI equation are constructed. Taking advantage of this result, some properties and asymptotic analysis of single soliton solutions and two-soliton solutions are discussed, and the simple elastic interaction of two-soliton is proved. Compared with soliton solution of the classical second-order flow, we found that the higher-order dispersion term affects the propagation velocity, propagation direction and amplitude of the soliton. Finally, by means of certain limit technique, the high-order soliton solution matrix for the third-order flow GI equation is derived.

1. Introduction

As is well known that completely integrable equations have many important and diverse physical applications such as in water waves, plasma physics, field theory and nonlinear optics [1–3]. Among many integrable systems, the nonlinear Schrödinger (NLS) equation has been recognized as a ubiquitous mathematical model, which governs weakly nonlinear and dispersive wave packets in one-dimensional physical systems. Another integrable system of NLS type, the derivative-type NLS equation

\[ iu_t + u_{xx} - iu^2 u_x + \frac{1}{2} u^4 u_x^2 = 0, \quad (1.1) \]

where \( u^* \) denotes the complex conjugation of \( u \). Eq. (1.1) is first found by Gerdjikov and Ivanov in Ref. [4] also known as the GI equation. It can be regarded as an extension of the NLS when certain higher-order nonlinear effects are taken into account, which is also known as DNLS III. In fact, there are three famous DNLS equations, another two kinds of derivative-type NLS equations are the famous Kaup-Newell (KN) equation [5]

\[ iu_t + u_{xx} + i (u^2 u_x)^x = 0, \quad (1.2) \]

which is a canonical dispersive equation derived from the Magneto-hydrodynamic equations in the presence of the Hall effect and usually called DNLS I, and the Chen-Lee-Liu (C-L-L) equation [6]

\[ iu_t + u_{xx} + iuu_{xx} = 0, \quad (1.3) \]

which appears in optical models of ultrashort pulses and is also referred to as the DNLS II. The unified expression of KN, C-L-L and GI equations was presented in Ref. [11].

In plasma physics, the GI equation (1.1) is a model for Alfvén waves propagating parallel to the ambient magnetic field, where \( u \) being the transverse magnetic field perturbation and \( x \) and \( t \) being space and time coordinates, respectively [9–10]. In recent years, there has been much work on the GI equation, such as its Darboux transformation and Hamiltonian structures [7,8] the algebra-geometric solutions [12], the rogue wave and breather solution [13]. Besides, the long-time asymptotic behavior of solution to the GI equation (1.1) was established in Ref. [19,20]. With the development of research, the importance of the higher-order nonlinear effects in plasma physics and other fields motivates us to consider an integrable model that possesses third dispersion and quintic nonlinearity.

In this work, we mainly consider the soliton solutions and high-order solutions of the third-order flow GI equation

\[ u_t = -\frac{1}{2} u_{xxx} + 3 iuu_x u_x^* - \frac{3}{4} |u|^4 u_x \quad (1.4) \]

with the help of Riemann-Hilbert method. It has been proved in Ref. [11] that Eq. (1.4) is Liouville integrable and have multiple Hamiltonian structures. We all known that the inverse scattering method [22–24] is a powerful method to solve the cauchy problem of nonlinear integrable partial differential equation, it was originally solved by using the Gel’Fand-Levitin-Marchenko (GLM) integral equation, although GLM equation can be used to obtain the solution of the equation, the solution process is very complex. Afterwards Shabat used RHP to reconstruct the inverse scattering method [23]. As a new version of inverse scattering transform method, the Riemann-Hilbert (RH) approach has become the preferred research technique to the researchers in investigating the soliton solutions and the long-time asymptotics of integrable systems in recent years [14–18].

Being an important kind of exact solution of the NLS-type equation, the high-order soliton has wide applications, it can describe a weak bound state of solitons and may appear in the study of train propagation of solitons with nearly equal velocities and amplitudes but having a particular chirp [26], so it is necessary to study the high-order solitons of DNLS equation.
In this article, based on the recursion operator construct the GI hierarchy. In the framework of the RHP, through a standard dressing procedure, soliton matrices for simple zeros and elementary high-order zeros in the RHP for the third-order flow GI equation are constructed, respectively. It is noted that pairs of zeros are simultaneously tackled in the situation, which is different from other NLS-type equation. Based on the determinant solution, some properties and asymptotic analysis of single soliton solution and double soliton solution are studied. Compared with the classical second-order flow GI equation, it is found that the higher-order dispersion term has a great influence on the direction, velocity and amplitude of solitons. In the case of elementary higher-order zeros, the higher-order soliton matrix of the third-order flow GI equation is derived by using the limit process of spectral parameters.

The article is arranged as follows. In Section 2, we derive the GI hierarchy with recursion operator. The RHP based on the Jost solutions to the Lax pair of the third-order flow GI equation and scattering data are constructed in section 3. In Section 4, we discuss solutions to the regular and non-regular RHP by applying Plemelj formula. In Section 5, the N-soliton formula for the third-order flow GI equation is derived by considering the simple zeros in the RHP. In Section 6, the high-order soliton matrix is constructed, which corresponds to the elementary high-order zeros in the RHP. The conclusion and discussion are given in the final section.

2. RECURSION OPERATOR AND THE GI HIERARCHY

In the theory of integrable equations, an important task is to construct new equations which are solvable through the inverse scattering transform method. In this section, we will associate with recursion operator, and construct the GI hierarchy of integrable equations. The GI hierarchy has the following spectral problem:

$$Y_x = MY, \quad M = \begin{pmatrix} -i\lambda^2 - \frac{i}{2}uv & \lambda u \\ \lambda v & i\lambda^2 + \frac{i}{2}uv \end{pmatrix},$$  \hspace{1cm} (2.1)

$$Y_t = NY, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$  \hspace{1cm} (2.2)

where \( \lambda \) is the spectral parameter, \( u = u(x,t) \) and \( v = v(x,t) \) are field variables, \( A, B \) and \( C \) are the quantities depending on field variables and their derivatives and \( \lambda \).

**Theorem 1** According to the consistency of space part (2.1) and time part (2.2) of spectral problem, infinite hierarchy of GI system can be obtained by recursive operator:

$$\left( \begin{array}{c} u \\ v \end{array} \right)_t = \left( \begin{array}{c} u_x \\ v_x \end{array} \right), \quad n = 2, 3, \ldots, \hspace{1cm} (2.3)$$

where

$$L_1 = \begin{pmatrix} -1 + iu\partial^{-1}v & iu\partial^{-1}u \\ -iv\partial^{-1}v & -1 - iv\partial^{-1}u \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$L_2 = \begin{pmatrix} -iuw - u\partial^{-1}uv^2 & u\partial^{-1}u^2v \\ v\partial^{-1}uv^2 & iuv - v\partial^{-1}u^2v \end{pmatrix}. \hspace{1cm} (2.4)$$

**Proof.** The corresponding zero-curvature equation or the compatibility condition of (2.1) and (2.2),

$$M_t - N_x + [M, N] = 0, \hspace{1cm} (2.4)$$

which can get

$$\frac{i}{2}(uv)_x + A_x - \lambda uC + \lambda vB = 0, \hspace{1cm} (2.5)$$

$$\lambda u_t - B_x - 2i\lambda^2B - iuB - 2\lambda uA = 0, \hspace{1cm} (2.6)$$

$$\lambda v_t - C_x + 2i\lambda^2C + iuvC + 2\lambda vA = 0. \hspace{1cm} (2.7)$$

From those equations, we can get

$$A = -\frac{i}{2\lambda}(uB_x + uC_x + iuv^2B - iuv^2vC) + A_0, \hspace{1cm} (2.8)$$

where \( \partial^{-1} \) is an antiderivative in \( x \) which can be taken as either \( \partial^{-1} = \int_{-\infty}^{x} dy \) or \( \partial^{-1} = -\int_{x}^{\infty} dy \), and \( A_0 \) are \( x \)-independent.

Using (2.8), Eqs. (2.6) and (2.7) may be rewritten as

$$\lambda \left( \begin{array}{c} u \\ v \end{array} \right)_t + L_1 \left( \begin{array}{c} B_x \\ C_x \end{array} \right) - 2i\lambda^2 \left( \begin{array}{c} B \\ -C \end{array} \right) + L_2 \left( \begin{array}{c} B \\ C \end{array} \right) - 2\lambda A_0 \left( \begin{array}{c} u \\ -v \end{array} \right) = 0, \hspace{1cm} (2.9)$$

where

$$L_1 = \begin{pmatrix} -1 + iu\partial^{-1}v & iu\partial^{-1}u \\ -iv\partial^{-1}v & -1 - iv\partial^{-1}u \end{pmatrix}, \quad L_2 = \begin{pmatrix} -iuw - u\partial^{-1}uv^2 & u\partial^{-1}u^2v \\ v\partial^{-1}uv^2 & iuv - v\partial^{-1}u^2v \end{pmatrix}. \hspace{1cm} (2.10)$$
To obtain the evolution equations, we expand

\[
\begin{pmatrix}
  B \\
  C
\end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix}
  b_j \\
  c_j
\end{pmatrix} (\lambda)^{2j-1},
\]  

(2.10)

Let \( A_0 = -2i\lambda^2 n \). Inserting (2.10) into (2.9) and equating terms of the same power in \( \lambda \), then we get the following equations:

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t + L_1 \begin{pmatrix}
  b_1 \\
  c_1
\end{pmatrix} x + L_2 \begin{pmatrix}
  b_1 \\
  c_1
\end{pmatrix} = 0,
\]

(2.11)

and

\[
\begin{pmatrix}
  b_n \\
  c_n
\end{pmatrix} = 2 \begin{pmatrix}
  u \\
  v
\end{pmatrix},
\]

(2.12)

\[ L_1 \begin{pmatrix}
  b_1 \\
  c_1
\end{pmatrix} x - 2is_3 \begin{pmatrix}
  b_{j-1} \\
  c_{j-1}
\end{pmatrix} + L_2 \begin{pmatrix}
  b_j \\
  c_j
\end{pmatrix} = 0, \quad j = 2...n. \]

Eq. (2.11) and (2.12) are used to iterate and derive the GI hierarchy

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = (-\frac{1}{2})^{n-2}(is_3)^{n-1}(L_1\partial_x + L_2)^{n-1} \begin{pmatrix}
  u_x \\
  v_x
\end{pmatrix}, \quad n = 2, 3, \ldots,
\]

the Theorem can eventually be proved.

Remark: \( \) In the zero curvature equation of GI equation, the derivative of \( M \) principal diagonal to \( t \) is not zero, which leads to GI hierarchy recursive form more complex than KN hierarchy [21] and the AKNS hierarchy [14].

To obtain the evolution equations, we expand

For convenience, we list down the explicit spectral problem of Eq.(1.4) here

\[
\begin{pmatrix}
  \text{Y}_x = M Y, \quad M = -i\lambda^2 \sigma_3 + \lambda Q - \frac{i}{2} Q^2 \sigma_3,
\end{pmatrix}
\]

(2.16)

\[
\begin{pmatrix}
  \text{Y}_t = N Y, \quad N = -2i\lambda^6 \sigma_3 + Z_3 \lambda^5 + Z_4 \lambda^4 + Z_3 \lambda^3 + Z_2 \lambda^2 + Z_2 \lambda + Z_0,
\end{pmatrix}
\]

(2.17)

where

\[
\begin{align*}
  Z_5 &= 2Q, & Z_4 &= -iQ^2 \sigma_3, & Z_3 &= i\sigma_3 Q_x, \\
  Z_2 &= -\frac{1}{4} [Q, Q_x] + \frac{i}{4} Q^4 \sigma_3, & Z_1 &= -\frac{1}{2} Q_{xx} + \frac{i}{2} Q Q_x \sigma_3 - \frac{1}{4} Q^3, & Z_0 &= \frac{i}{4} (Q Q_{xx} + Q_{xx} Q) \sigma_3 - \frac{1}{4} Q_x^2 \sigma_3 + \frac{i}{8} Q^6 \sigma_3.
\end{align*}
\]

(2.18)

It’s easy to see that

\[
Q^\dagger = -Q, \quad \sigma_3 Q \sigma_3 = -Q,
\]

where the superscript ‘\( \dagger \)’ represents the Hermitian of a matrix, and \([A, B]\) denotes AB-BA.
From the above Volterra type integral equations, we can easily prove the existence and uniqueness of the Jost solutions as a fundamental matrix of those linear equations. In our analysis, we mainly consider the zero boundary condition, i.e.

\[ u(x, t_0) \to 0, \quad x \to \pm \infty, \]

which belongs to Schwartz space. Therefore, it is easy to take the form of the solution of Eqs. (2.16) and (2.17) as

\[ Y = J e^{(-i\lambda^2 x - 2i\lambda^3 t)\sigma_3}. \]

The Lax pair of Eqs. (2.16)-(2.17) becomes

\[ J_x + i\lambda^2 [\sigma_3, J] = (\lambda Q - \frac{i}{2} Q^2 \sigma_3) J, \]

\[ J_t + 2i\lambda^6 [\sigma_3, J] = (Z_5 \lambda^5 + Z_4 \lambda^4 + Z_3 \lambda^3 + Z_2 \lambda^2 + Z_1 \lambda + Z_0) J. \]

where \( Q, Z_i (i = 0...5) \) have been given by Eqs. (2.18) and (2.19).

In this consideration, the time \( t \) is fixed and is a dummy variable, and thus it will be suppressed in our notation. In the scattering problem, we first introduce matrix Jost solutions \( J(x, \lambda) \) of Eq. (3.2) with the following asymptotics at large distances:

\[ J(x, \lambda) \to I, \quad x \to \pm \infty. \]

It is easy to find that \( J(x, \lambda) \) satisfies the following integral equation

\[ J_-(x, \lambda) = I + \int_{-\infty}^{x} e^{i \lambda^2 \sigma_3 (y-x)} (\lambda Q(y) - \frac{i}{2} Q^2 \sigma_3) J_-(x-y) dy, \]

\[ J_+(x, \lambda) = I - \int_{x}^{\infty} e^{i \lambda^2 \sigma_3 (y-x)} (\lambda Q(y) - \frac{i}{2} Q^2 \sigma_3) J_+(x-y) dy. \]

From the above Volterra type integral equations, we can easily prove the existence and uniqueness of the Jost solutions through standard iteration method. Partitioning \( J_{\pm} \) into columns as \( J = (J^{(1)}, J^{(2)}) \), due to the structure Eq. (3.5) of the potential \( Q \), we have

**Proposition 3.1** The column vectors \( J^{(1)} \) and \( J^{(2)} \) are continuous for \( \lambda \in D_+ \cup R \cup iR \) and analytic for \( \lambda \in D_+ \), while the columns \( J^{(1)}_+ \) and \( J^{(2)}_- \) are continuous for \( \lambda \in D_- \cup R \cup iR \) and analytical for \( \lambda \in D_- \), where

\[ D_+ = \left\{ \lambda \mid \arg \lambda \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \pi\right) \right\}, \quad D_- = \left\{ \lambda \mid \arg \lambda \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \right\}. \]

The distribution area of \( D \) is shown in Fig. 1.

In fact, the \( J_x E \) and \( J_x E \) are the simultaneous solutions for the Lax pair (3.2). Therefore, they have following linear relation by the constant scattering matrix \( S(\lambda) \)

\[ J_- E = J_+ ES(\lambda), \quad \lambda \in R \cup iR, \]

where \( E = e^{i\lambda^2 x \sigma_3} \) and \( S(\lambda) = (s_{ij})_{2x2} \). By using Abel’s formula and \( \text{tr}(Q) = 0 \), we obtain that the determinant of \( J \) is independent of \( x \), then considering the boundary conditions (3.4), we can get

\[ \det J = 1. \]
Thus we can derive \( \det S(\lambda) = 1 \).

**Proposition 3.2** Through the analytic property of \( J_- \), it’s easy to know that \( s_{11} \) allows analytic extension to \( D_+ \), \( s_{22} \) can be analytically extended to \( D_- \).

**Proof.** According to the relation (3.7), we have

\[
S(\lambda) = \lim_{x \to \pm \infty} E^{-1} J_- E = I + \int_{-\infty}^{+\infty} E^{-1}(\lambda Q(y) - \frac{i}{2} Q^2 \sigma_3) J_- E dx, \quad \lambda \in \mathbb{R} \cup i\mathbb{R}.
\]  

(3.8)

So

\[
S(\lambda) = I + \lambda \left( \begin{array}{cc}
\int_{-\infty}^{+\infty} uJ^2 dx & \int_{-\infty}^{+\infty} uJ^2 e^{2i\lambda x} dx \\
-\int_{-\infty}^{+\infty} u^*J^2 dx & -\int_{-\infty}^{+\infty} u^*J^2 e^{2i\lambda x} dx
\end{array} \right).
\]

(3.9)
i.e

\[
s_{11} = 1 + \lambda \int_{-\infty}^{+\infty} uJ^2 dx, \quad s_{22} = 1 - \lambda \int_{-\infty}^{+\infty} u^*J^2 dx.
\]

through the analytic property of \( J_- \), it’s easy to know that \( s_{11} \) allows analytic extension to \( D_+ \), \( s_{22} \) can be analytically extended to \( D_- \).

In order to construct the RHP, introducing the notation

\[
P_+ = (J^1_+, J^2_+) = J_- H_1 + J_+ H_2 = J_+ E \begin{pmatrix} s_{11} & 0 \\ s_{21} & 1 \end{pmatrix} E^{-1},
\]

(3.10)

where \( H_1 = \text{diag}\{1,0\} \) and \( H_2 = \text{diag}\{0,1\} \). Through the previous analysis, we can see that \( P_+ \) is analytic in \( D_- \) and \( \det (P_+) = s_{11} \). To find the boundary condition of \( P_+ \) as \( \lambda \to \infty \), we consider the following asymptotic expansion

\[
P_+ = P_+^{(0)} + \frac{1}{\lambda} P_+^{(1)} + \frac{1}{\lambda^2} P_+^{(2)} + O\left(\frac{1}{\lambda^3}\right).
\]

(3.11)

Substituting (3.11) into (3.2) and equating terms with like powers of \( \lambda \), which lead to

\[
P_+^{(0)} = 0,
\]

(3.12)

without loss of generality, we can set \( P_+^{(0)} = I \). This means

\[
P_+ \to I, \quad \lambda \in D_+ \to \infty.
\]

(3.13)

To obtain the analytic counterpart of \( P_+ \) in \( D_- \), we consider the adjoint scattering equation of (3.2)

\[
\Phi_x = -i\lambda^2 \left[ \sigma_3, \Phi \right] - \lambda \Phi Q + \frac{i}{2} \Phi Q^2 \sigma_3,
\]

(3.14)
it is easy to see that \( J^{-1} \) is the solution of the adjoint Eq. (3.14) and satisfy the boundary condition \( J^{-1} \to I \) as \( x \to \pm \infty \). Taking the similar procedure as above denote matrices \( J^{-1} \) as a collection of rows

\[
J^{-1}_+ = \left( (J^{-1}_+)^{[1]}, (J^{-1}_+)^{[2]} \right)^T, \quad J^{-1}_- = \left( (J^{-1}_-)^{[1]}, (J^{-1}_-)^{[2]} \right)^T,
\]

(3.15)
we can show that the adjoint Jost solutions

\[
P^{-1}_- = H_1 J_-^{-1} + H_2 J_+^{-1} = E \begin{pmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{21} & \hat{s}_{22} \end{pmatrix} E^{-1} J_+^{-1}
\]

(3.16)
analytic for \( D_- \), where

\[
J_-^{-1} = ES^{-1} E^{-1} J_+^{-1}, \quad \hat{S} = \begin{pmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{21} & \hat{s}_{22} \end{pmatrix},
\]

det \( P^{-1}_- = s_{11} \). Through direct calculation, we can get that \( P^{-1}_- \) also satisfies the same boundary condition (3.13), i.e.

\[
P^{-1}_-(x, \lambda) \to I, \quad \lambda \in \mathbb{C}_- \to \infty.
\]

(3.17)
Hence, we have constructed two matrix functions \( P_\pm(x, \lambda) \) which are analytic for \( \lambda \in D_\pm \), respectively. Thus the RHP can be constructed as follow by \( P_+, P_- \)

\[
P_-(x, \lambda)P_+(x, \lambda) = G(x, \lambda) = E \begin{pmatrix} 1 & \hat{s}_{12} \\ \hat{s}_{21} & 1 \end{pmatrix} E^{-1}, \quad \lambda \in \mathbb{R} \cup i\mathbb{R},
\]

(3.18)
with boundary condition

\[
P_\pm \to I, \quad \lambda \to \infty.
\]

(3.19)
At the end of this section, we consider the time evolution of the scattering matrices \( S(\lambda) \) and \( \hat{S}(\lambda) \), since \( J \) satisfies the temporal Eq. (3.3) of the Lax pair and the relation (3.7), then according to the evolution property (3.7) and \( Q \to 0, Z_i(i = 1...5) \to 0 \) as \( |x| \to \infty \), we have

\[
S_t + 2i\lambda^6 [\sigma_3, S] = 0.
\]
And then the time evolution of $\hat{S}(\lambda)$ can be gotten immediately
\[ \hat{S}_t + 2i\lambda^6 [\sigma_3, \hat{S}] = 0. \]

These two equations lead that
\[ s_{11,t} = \hat{s}_{11,t} = 0, \]
\[ s_{12}(t; \lambda) = s_{12}(0; \lambda) \exp \left(-4i\lambda^6 t\right), \quad \hat{s}_{21}(t; \lambda) = \hat{s}_{21}(0; \lambda) \exp \left(4i\lambda^6 t\right). \quad (3.20) \]

4. Solution of the RHP

In this section, we discuss how to solve the matrix RHP [3.18] in the complex $\lambda$ plane. The RHP [3.18] constructed in the above section is regular when $\det(P_+) = s_{11} \neq 0$ and $\det(P_-^*) = \hat{s}_{11} \neq 0$ for all $\lambda$, and is nonregular when $\det(P_+)$ and $\det(P_-)$ can be zero at certain discrete locations of $\lambda$. In fact, a non-regular RHP can be transformed into a regular one, thus we consider the regular case at first.

4.1. Solution to the Regular RHP.

In this subsection, we first consider the regular RHP of [3.18], i.e., in their analytic domain. Rewriting Eq. [3.18] as
\[ (P^+)^{-1}(\lambda) - P_-^*(\lambda) = \hat{G}(\lambda) \left( (P^+)^{-1}(\lambda) \right), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}, \quad (4.1) \]
where
\[ \hat{G} = I - G = -E \begin{pmatrix} 0 & \hat{s}_{12} \\ \hat{s}_{21} & 0 \end{pmatrix} E^{-1}. \quad (4.2) \]

By Plemelj formula, the formal solution of this problem reads as
\[ (P^+)^{-1}(\lambda) = I + \frac{1}{2\pi i} \int_T \frac{\hat{G}(\xi) (P^+)^{-1}(\xi)}{\xi - \lambda} d\xi, \quad \lambda \in D_+, \quad (4.3) \]
and $T = (-\infty, 0] \cup (i\infty, 0] \cup [0, -\infty) \cup [0, \infty)$.

Under the canonical normalization condition [3.19], the solution to this regular RHP is unique. Suppose [3.18] has two sets of solutions $P_{\pm}$ and $\hat{P}_{\pm}$. Then
\[ P_-^*(\lambda) P_+^*(\lambda) = \hat{P}^{-1}_-^*(\lambda) \hat{P}_+^*(\lambda), \]
and thus
\[ \hat{P}_-(\lambda) P_-^*(\lambda) = \hat{P}_+^*(\lambda) P_-^*(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}. \quad (4.4) \]

Since $\hat{P}_-(\lambda) P_-^*(\lambda)$ and $\hat{P}_+^*(\lambda) P_-^*(\lambda)$ are analytic in $D_-$ and $D_+$ respectively, and they are equal to each other on $\mathbb{R} \cup i\mathbb{R}$, they together define a matrix function which is analytic in the whole plane of $\lambda$. Due to the boundary condition [3.19], we have
\[ \hat{P}_-(\lambda) P_-^*(\lambda) = \hat{P}_+^*(\lambda) P_-^*(\lambda) = I, \quad (4.5) \]
for all $\lambda$ by applying the Liouville’s theorem. That is, $\hat{P}_{\pm} = P_{\pm}$, which implies the uniqueness of solution to the above RHP [3.18].

4.2. Solution to the Nonregular RHP.

In the more general case, the RHP [3.18] is not regular, i.e., $\det(P_+^* \lambda) = s_{11}(\lambda)$ and $\det(P_+^* \lambda) = \hat{s}_{11}(\lambda)$ can be zero at certain discrete locations. In order to study a nonregular RHP, we shall consider symmetric property of these zero points. Note that $s_{11}(\lambda)$ and $\hat{s}_{11}(\lambda)$ are time independent, so the roots of $s_{11}(\lambda)$ and $\hat{s}_{11}(\lambda)$ are also time independent.

The Hermitian of the spectral equation [3.2] reads as
\[ (J^t)_x = -i\lambda^2 \left[ \sigma_3, J^t \right] - \lambda J^t Q + \frac{i}{2} J^t Q^2 \sigma_3, \quad (4.6) \]
where $Q^t = -Q$ is used. Thus $J^t(x, \lambda^*)$ satisfies the adjoint scattering Eq. [3.14]. $J^t(x, \lambda^*)$ and $J^{-1}(x, \lambda)$ must be linearly dependent on each other. Recalling the boundary conditions of Jost solutions $J$, we further see that $J^t(x, \lambda^*)$ and $J^{-1}(x, \lambda)$ have the same boundary conditions at $x \to \pm \infty$ and hence they must be the same solutions of the adjoint Eq. [3.14], i.e.
\[ J^t(x, \lambda^*) = J^{-1}(x, \lambda), \quad (4.7) \]
so there is
\[ (P_+)^t(\lambda^*) = P_+^{-1}(\lambda). \quad (4.8) \]

In addition, in view of the scattering relation [3.7] between $J_+$ and $J_-$, it’s easy to know that $S(\lambda)$ also satisfies involution property
\[ S^t(\lambda^*) = S^{-1}(\lambda). \quad (4.9) \]

Besides, from the symmetric property $\sigma_3 Q \sigma_3 = -Q$ and $\sigma_3 Q^2 \sigma_3 = Q$, we conclude that
\[ J(\lambda) = \sigma_3 J(-\lambda) \sigma_3. \quad (4.10) \]
It follows that

\[ P_{\pm}(-\lambda) = \sigma_3 P_{\pm}(\lambda) \sigma_3, \tag{4.11} \]

and

\[ S(-\lambda) = \sigma_3 S(\lambda) \sigma_3. \tag{4.12} \]

From the (4.9) and (4.12), we obtain the relations

\[ s_{11}^+(\lambda) = s_{11}(\lambda), \ s_{21}(\lambda) = s_{21}(\lambda), \ s_{12}(\lambda) = s_{21}(\lambda), \ \lambda \in \mathbb{R} \cup i\mathbb{R}, \tag{4.13} \]

and

\[ s_{11}(\lambda) = s_{11}(-\lambda), \ s_{22}(\lambda) = s_{22}(-\lambda), \ s_{12}(-\lambda) = -s_{12}(\lambda), \ s_{21}(-\lambda) = -s_{21}(\lambda). \tag{4.14} \]

Thus \( s_{11}(\lambda) \) is an even function, and each zero \( \lambda_k \) of \( s_{11} \) is accompanied with zero \(-\lambda_k\). Similarly, \( \check{s}_{11}(\lambda) \) has two zeros \( \pm \check{\lambda}_k \).

Here we first consider the case of simple zeros \( \{ \pm \lambda_k \in D_+, 1 \leq k \leq N \} \) and \( \{ \pm \check{\lambda}_k \in D_-, 1 \leq k \leq N \} \), where \( N \) is the number of these zeros. Due to the involution property \( (4.13) \), the involution relation is obtained as (3.13) and (3.17) is normalization condition \( (3.19) \).

\[ \Gamma(\lambda)|_{\lambda = 0} = 0 \]

and the boundary condition

\[ \Gamma(\lambda)|_{\lambda = \infty} = 0. \tag{4.21} \]

Differentiating both sides of the first equation of (4.16) with respect to \( x \) and \( t \), and recalling the Lax (3.2)-(3.3) we have

\[ P_+(\lambda_k; x) \left( \frac{d|v_k\rangle}{dx} + i\lambda^2 \sigma_3 |v_k\rangle \right) = 0, \quad P_+(\lambda_k; x) \left( \frac{d|v_k\rangle}{dt} + 2i\lambda^6 \sigma_3 |v_k\rangle \right) = 0. \]

It follows that

\[ |v_k\rangle = e^{-i\lambda^2 \sigma_3 x - 2i\lambda^6 \sigma_3 t} |v_{k0}\rangle e^{\int_0^t \alpha_k(y) dy + \int_0^t \beta_k(y) dy} dt, \]

where \( v_{k0} = v_k|_{x = 0} \) and \( \alpha_k(x) \) and \( \beta_k(t) \) are two scalar functions.

Based on above analysis, we have the following theorem for the solution to the nonregular RHP with canonical normalization condition \( (3.19) \).

**Theorem 2** The solution to a nonregular RHP \((3.18)\) with simple zeros under the canonical normalized condition \((3.13)\) and \((5.17)\) is

\[ P_+ = \hat{P}_+ \Gamma, \quad P_- = \Gamma^{-1} \hat{P}_-, \tag{4.18} \]

where

\[ \Gamma(\lambda) = \Gamma_N(\lambda)\Gamma_{N-1}(\lambda) \cdots \Gamma_1(\lambda), \quad \Gamma^{-1}(\lambda) = \Gamma_1^{-1}(\lambda)\Gamma_2^{-1}(\lambda) \cdots \Gamma_N(\lambda), \]

\[ \Gamma_k(\lambda) = I + \frac{A_k}{\lambda - \lambda_k} - \frac{\sigma_3 A_k \sigma_3}{\lambda + \lambda_k}, \tag{4.19} \]

\[ \Gamma_k^{-1}(\lambda) = I + \frac{A_k^\dagger}{\lambda - \lambda_k} - \frac{\sigma_3 A_k \sigma_3}{\lambda + \lambda_k}, \quad k = 1, 2, \ldots, N \tag{4.20} \]

\[ A_k = \frac{\lambda_k^2 - \lambda_k^2}{2} \left( \begin{array}{cc} \alpha_k^2 & 0 \\ 0 & \alpha_k \end{array} \right) |w_k\rangle \langle w_k|, \quad \alpha_k^{-1} = \langle w_k| \left( \begin{array}{cc} \lambda_k & 0 \\ 0 & \lambda_k^* \end{array} \right) |w_k\rangle, \tag{4.21} \]

and

\[ \det \Gamma_k(\lambda) = \frac{\lambda^2 - \lambda_k^2}{\lambda^2 - \lambda_k^2} |w_k\rangle = \Gamma_{k-1}(\lambda_k) \cdots \Gamma_1(\lambda_k) |v_k\rangle, \quad |w_k\rangle = |w_k\rangle^\dagger, \tag{4.22} \]

Therefore, \( \Gamma(x, t, \lambda) \) and \( \Gamma^{-1}(x, t, \lambda) \) accumulates all zero of the RHP, and then we obtain the regular RHP

\[ \hat{P}_+^{-1}(\lambda) \hat{P}_+(\lambda) = \Gamma(\lambda) G(\lambda) \Gamma^{-1}(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}, \tag{4.23} \]

and the boundary condition \( \hat{P}_+ \to I \) as \( \lambda \to \infty \), where \( \hat{P}_\pm \) are analytic in \( D_\pm \) respectively.

The proof has been given in Ref. [28], and we will not repeat it here.
5. The inverse problem

The ultimate purpose of inverse scattering is to obtain the potential $u$. Based on (3.11), the potential can be obtained from the asymptotic expansion of Jost solutions $P$ as $\lambda \to +\infty$,

$$Q = i[\sigma_3, P_+^{(1)}],$$

from this formula, we can get the potential

$$u = 2i(P_+^{(1)})_{12}. \quad \text{(5.2)}$$

It is well known that the soliton solutions correspond to the vanishing of scattering coefficients, $G = I, \hat{G} = 0$. Thus, we intend to solve the corresponding RHP (4.23). The product representations $\Gamma(\lambda)$ and $\Gamma^{-1}(\lambda)$ are not convenient to use for later calculations in the inverse scattering transform method, it is necessary to simplify the expression of $\Gamma(\lambda)$ and its inverse, in fact,\n
$$\Gamma(\lambda) = I + \sum_{j=1}^{N} \left[ \frac{B_j}{\lambda - \lambda_j^+} - \frac{\sigma_3 B_j \sigma_3}{\lambda + \lambda_j^+} \right], \quad \text{(5.3)}$$

and

$$\Gamma^{-1}(\lambda) = I + \sum_{j=1}^{N} \left[ \frac{B_j^\dagger}{\lambda - \lambda_j} - \frac{\sigma_3 B_j^\dagger \sigma_3}{\lambda + \lambda_j} \right],$$

with $B_j = |z_j\rangle \langle v_j|$. To determine the form of matrix $B_j$, we consider $\Gamma(\lambda) \Gamma^{-1}(\lambda) = I$. Taking into account the residue condition at $\lambda_j$, we have

$$\text{Res}_{\lambda=\lambda_j} \Gamma(\lambda) \Gamma^{-1}(\lambda) = \Gamma(\lambda_j) B_j^\dagger = 0,$$

and it yields

$$\left[ I + \sum_{k=1}^{N} \left( \frac{|z_k\rangle \langle v_k| - \sigma_3 |z_k\rangle \langle v_k| \sigma_3}{\lambda_j - \lambda_k^+} \right) \right] |v_j\rangle = 0, \quad j = 1, 2, \ldots, N \quad \text{(5.4)}$$

it’s easy to figure out

$$|z_k\rangle_1 = \sum_{j=1}^{N} (M^{-1})_{jk} |v_j\rangle_1, \quad \text{(5.5)}$$

where $|z_k\rangle_l$ denotes the $l$–th element of $|z_k\rangle$, matrix $M$ is defined as

$$(M)_{jk} = \frac{\langle v_k | \sigma_3 | v_j \rangle}{\lambda_j + \lambda_k^+} - \frac{\langle v_k | v_j \rangle}{\lambda_j - \lambda_k^+}. \quad \text{(5.6)}$$

From these equations enable us to have

$$P_+^{(1)} = \sum_{j=1}^{N} (B_j - \sigma_3 B_j \sigma_3),$$

by Eq. (5.1), we can obtain that the potential function $u$ is

$$u = 2i \left[ \sum_{j=1}^{N} (B_j - \sigma_3 B_j \sigma_3)_{12} \right], \quad \text{(5.7)}$$

and substituting above expressions for $|z_k\rangle_l$ and $|v_j\rangle_l$ into Eq. (5.7) gives

$$u = -4i \frac{\det F}{\det M} \quad \text{(5.8)}$$

where $M$ defined as (5.6), and

$$F = \begin{bmatrix}
    M_{11} & \cdots & M_{1N} & |v_1\rangle_1 \\
    \vdots & \ddots & \vdots & \vdots \\
    M_{N1} & \cdots & M_{NN} & |v_N\rangle_1 \\
    \langle v_{11}\rangle_2 & \cdots & \langle v_{1N}\rangle_2 & 0
\end{bmatrix}.$$
HIGH-ORDER SOLITON MATRIX FOR THE THIRD-ORDER FLOW GI EQUATION THROUGH THE RHP

5.1. Single-soliton solution.

To obtain the single-soliton solution, we set $N = 1$ in formula (5.8). The solution is

$$u(x, t) = -2i(\lambda_1^2 - \lambda_1^{*2}) \frac{c_1 e^{\theta_1 - \theta_1^*}}{\lambda_1 e^{-(\theta_1 + \theta_1^*)} + \lambda_1^* e^{\theta_1 + \theta_1^*}}.$$  (5.9)

The velocity for the single soliton is $v_1 = 8\xi_1^2\eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2$, and its behavior occurring along the line

$$x - v_1 t + \frac{1}{4m_1} \ln|c_1| = 0.$$

The amplitudes associated with $|u|^2$ are given by

$$A(q) = \frac{64\xi_1^2\eta_1^2}{2|\lambda_1^2 + \lambda_1^{*2}|^2}.$$  

Besides, it is found that $\alpha(x)$ and $\beta(t)$ are eliminated automatically interior the calculation, so set $\alpha(x) = \beta(t) = 0$ below without loss of generality. More, $\xi_1\eta_1 > 0$ if $\lambda_1 \in D_+$, in the subregion $\xi_1 > \eta_1$ and $\xi_1 < \eta_1$ of $D_+$, the one-soliton is a left traveling wave (see Fig. (2) and Fig. (3)). On the line $\xi_1 = \eta_1$, the one-soliton is a right traveling wave (see Fig. (4)).

Compared with the classical second-order flow GI equation (1.1), for $\lambda$ belongs to $D_+$, soliton solutions of GI equation have three different traveling wave directions: left traveling wave, right traveling wave and stationary wave. By choosing the same parameters as Ref. [32], We find that for the soliton solutions of the third-order flow GI equation (1.4), there are only two kinds of traveling wave solutions, left traveling wave and right traveling wave. And the wave propagation velocity also changed, the classical GI equation velocity is: $4(\xi_1^2 - \eta_1^2)$, and the velocity of high-order GI equation is: $8\xi_1^2\eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2$. Besides, the amplitude of the soliton solution of the higher-order GI equation is also affected. Compared with the classical GI, the amplitude of the third-order flow GI equation becomes higher. That is to say, the introduction of third-order dispersion and fifth-order nonlinearity will affect the velocity, direction and amplitude of solution.
5.2. Two-soliton solutions.

When $N = 2$, the two-soliton solutions of the third-order flow GI equation can be written out explicitly as follows

$$u(x, t) = \frac{a_1 e^{\Theta_1 - \Theta_2} + a_2 e^{\Theta_1' + \Theta_2'} + a_3 e^{\Theta_1 - \Theta_2'} + a_4 e^{\Theta_1' + \Theta_2}}{b_1 e^{-\Theta_1 - \Theta_2} + b_2 e^{\Theta_1 + \Theta_2} + b_3 e^{\Theta_1 - \Theta_2} + b_4 e^{-\Theta_1' - \Theta_2'} + b_5 e^{\Theta_1 - \Theta_2} + b_6 e^{-\Theta_1 + \Theta_2}}. \tag{5.10}$$

where

$$\Theta_1 = \Theta_1 + \Theta_1', \quad \Theta_1' = \Theta_1 - \Theta_2', \quad \Theta_2 = \Theta_2 + \Theta_2', \quad \Theta_2' = \Theta_2 - \Theta_2'.$$

$$a_1 = c_1 \lambda_2 (\lambda_1 - \lambda_1')(\lambda_2^2 - \lambda_1^2);$$

$$a_2 = c_1 |c_2|^2 \lambda_2 (\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_2^2)(\lambda_2^2 - \lambda_2^2);$$

$$a_3 = c_2 \lambda_1 (\lambda_2^2 - \lambda_2^2)(\lambda_2^2 - \lambda_2^2)(\lambda_2^2 - \lambda_2^2);$$

$$a_4 = c_1 |c_2|^2 \lambda_2 (\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_2^2)(\lambda_2^2 - \lambda_2^2);$$

$$b_1 = 2c_1^2 |c|^2 \lambda_2 (\lambda_1^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2);$$

$$b_2 = 2c_1^2 |c|^2 \lambda_1 (\lambda_1^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2);$$

$$b_3 = -2c_1^2 |c_2|^2 \lambda_1 (\lambda_2^2 - \lambda_2^2)(\lambda_2^2 - \lambda_2^2);$$

$$b_4 = -2c_1^2 |c_2|^2 \lambda_1 (\lambda_2^2 - \lambda_2^2)(\lambda_2^2 - \lambda_2^2);$$

$$b_5 = 2c_1 |c_1|^2 \lambda_1 (\lambda_1^2 - \lambda_1^2)(\lambda_2^2 - \lambda_2^2);$$

$$b_6 = 2c_1 |c|^2 \lambda_1 (\lambda_1^2 - \lambda_1^2)(\lambda_2^2 - \lambda_2^2).$$

We show the typical solution behaviors in Fig. 5 with $\lambda_1 = 1 + 0.3i, c_1 = 1, \lambda_2 = 1 + 0.5i, c_2 = 1$.

We see from Fig. (5) (a) that as $t \to -\infty$, the solution consists of two single-solitons which are far apart and moving toward each other. When they collide, they interact strongly. But when $t \to \infty$, these solitons reemerge out of interactions without any change of shape and velocity, and there is no energy radiation emitted to the far field. Thus the interaction of these solitons is elastic. Indeed, after the interaction, each soliton acquires a position shift and a phase shift. The position of each soliton is always shifted forward, as if the soliton accelerates during interactions.

To show this fact, we analyze the asymptotic states of the solution (5.8) as $t \to \pm \infty$. Without loss of generality, let us assume that $\xi_i, \eta_i > 0$ and $v_1 < v_2$. This means that at $t \to -\infty$, soliton-1 is on the right side of soliton-2 and moves slower. In the moving frame with velocity $v_i = 8\xi_i^2 \eta_i^2 - 6(\xi_i^2 - \eta_i^2)^2$, note that $z_1 = 2m_1(x - v_1 t), z_2 = 2m_2(x - v_2 t)$, it yields

$$m_2 z_1 - m_1 z_2 = 2m_1 m_2 (v_2 - v_1) t.$$

When $t \to -\infty$, $|z_1| < \infty, z_2 \to \pm \infty$. In this case, simple calculations show that the asymptotic state of the solution (5.10) is

$$u(x, t) \to -2i(\lambda_1^2 - \lambda_1^2) e^{\Theta_1 - \Theta_1'} \frac{c_1^{-} e^{\Theta_1 - \Theta_1'} c_2^{-}}{\lambda_1 e^{\Theta_1 + \Theta_1'} + \lambda_1^* |c_1|^2 e^{\Theta_1 + \Theta_1'}}, \quad t \to -\infty,$$

where $c_1^{-} = c_1(\lambda_1^2 - \lambda_1^2)$. Comparing this expression with (5.8), we see that this asymptotic solution is a single-soliton solution with peak amplitude $\frac{64 \lambda_1^2 |c_1|^2}{(\lambda_1 + \lambda_1^*)(\lambda_1 + \lambda_1^*)}$ and velocity $8\xi_1^2 \eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2$.

When $t \to \infty$, $|z_1| < \infty, z_2 \to -\infty$. In this case, the asymptotic state of the solution (5.10) is

$$u(x, t) \to -2i(\lambda_1^2 - \lambda_1^2) e^{\Theta_1 - \Theta_1'} \frac{c_1^{+} e^{\Theta_1 - \Theta_1'} c_2^{+}}{\lambda_1 e^{\Theta_1 + \Theta_1'} + \lambda_1^* |c_1|^2 e^{\Theta_1 + \Theta_1'}}, \quad t \to \infty,$$
where \( c^+_1 = c_1^+ \frac{(\lambda^2_1 - \lambda^2_2)}{(\lambda^2_1 - \lambda^2_2)} \). This is also a single-soliton solution with peak amplitude \( \frac{64\xi_1^2\eta_1^2}{2(\lambda^2_1 + \lambda^2_1 + \lambda^2_2)} \) and velocity \( 8\xi_1^2\eta_1^2 - 6(\xi_1^2 - \eta_1^2)^2 \). This indicates that this soliton does not change its shape and velocity after collision. Its position and phase have shifted, however, as Fig. 6(b) has shown. The position shift is

\[
\Delta x_{01} = -\frac{1}{8\xi_1\eta_1} (\ln |c^+_1| - \ln |c^-_1|) = -\frac{1}{4\xi_1\eta_1} \ln \left| \frac{\lambda^2_1 - \lambda^2_2}{\lambda^2_1 - \lambda^2_2} \right|, 
\]

and the phase shift is

\[
\Delta \sigma_{01} = \arg (c^+_1) - \arg (c^-_1) = -2 \arg \left( \frac{\lambda^2_1 - \lambda^2_2}{\lambda^2_1 - \lambda^2_2} \right). 
\]

Notice that \( \Delta x_{01} < 0 \) since \( \lambda_k \in D_+ \), and thus the (slower) soliton-1 acquires a negative position shift.

Following similar calculations, we find that soliton-2 in the moving frame with velocity \( 8\xi_2^2\eta_2^2 - 6(\xi_2^2 - \eta_2^2)^2 \), as \( t \to \pm \infty \), the asymptotic solutions are both single soliton with the same peak amplitude \( 8\xi_2^2\eta_2 \), and the soliton constants \( c^+_2 \) before and after collision are related as

\[
c^+_2 = c_2^+ \frac{(\lambda^2_1 - \lambda^2_2)}{(\lambda^2_1 - \lambda^2_2)}.
\]

Thus, after collision, this second soliton acquires a position shift

\[
\Delta x_{02} = -\frac{1}{8\xi_2\eta_2} (\ln |c^+_2| - \ln |c^-_2|) = -\frac{1}{4\xi_2\eta_2} \ln \left| \frac{\lambda^2_1 - \lambda^2_2}{\lambda^2_1 - \lambda^2_2} \right|, 
\]

and a phase shift

\[
\Delta \sigma_{02} = \arg (c^+_2) - \arg (c^-_2) = 2 \arg \left( \frac{\lambda^2_1 - \lambda^2_2}{\lambda^2_1 - \lambda^2_2} \right). 
\]

Notice that \( \Delta x_{02} > 0 \), indicating that the (faster) soliton-2 acquires a positive position shift. In addition,

\[
\frac{\Delta x_{02}}{\Delta x_{01}} = \frac{\xi_1\eta_1}{\xi_2\eta_2},
\]

thus the amount of each soliton’s position shift is inversely proportional to its amplitude.

### 6. Soliton matrices for high-order zeros

In this section, we will consider the high-order zeros in RHP of the third-order flow GI equation. We assume \( \det P_+(\lambda) \) have high-order zeros \( \{\pm \lambda_i\}_{i=1}^N \), from the symmetries (4.2) and (4.14), we know that \( \{\pm \lambda^*_i\}_{i=1}^N \) are high-order zeros of \( \det P^{-1}(\lambda) \). So \( P_+(\lambda) \) and \( P^{-1}(\lambda) \) can be expanded as:

\[
\det P_+(\lambda) = s_{11}(\lambda) = (\lambda^2 - \lambda^2_1)^{n_1} (\lambda^2 - \lambda^2_2)^{n_2} \cdots (\lambda^2 - \lambda^2_N)^{n_N} s_0(\lambda),
\]

\[
\det P^{-1}(\lambda) = s_{11}(\lambda) = (\lambda^2 - \lambda^2_1)^{n_1} (\lambda^2 - \lambda^2_2)^{n_2} \cdots (\lambda^2 - \lambda^2_N)^{n_N} s_0(\lambda),
\]

where \( s_0(\lambda) \neq 0 \) for all \( \lambda \in D_+ \), and \( s_0(\lambda) \neq 0 \) for all \( \lambda \in D_- \).

First of all, let functions \( P_+(\lambda) \) and \( P^{-1}(\lambda) \) from above RHP have only one pair of zero of order \( n_1 \), i.e. \( \{\lambda_1, -\lambda_1\} \) and \( \{\lambda^*_1, -\lambda^*_1\} \). Hence, one needs to construct the dressing matrix \( \Gamma(\lambda) \) whose determinant is \( \frac{(\lambda^2 - \lambda^2_1)^{n_1}}{(\lambda^2 - \lambda^2_1)} \). For multiple zeros, its kernel vector will no longer be one. The geometric multiplicity of \( \pm \lambda_1(\pm \lambda^*_1) \) is defined as the number of the null vectors in the kernel of \( \det P_+(\lambda) \). It can be easily shown that the order of a zero is always greater or equal to its geometric multiplicity. It is also obvious that the geometric multiplicity of a zero is less than the matrix dimension.

Below we derive the soliton matrix \( \Gamma(\lambda) \) and its inverse for an elementary high-order zero. The results are presented in the following lemma.

**Lemma 1** Consider a pair of elementary high-order zeros of order \( n: \{\lambda_1, -\lambda_1\} \) in \( D_+ \) and \( \{\lambda^*_1, -\lambda^*_1\} \) in \( D_- \). Then the corresponding soliton matrix \( \Gamma(\lambda) \) and its inverse can be cast in the following form

\[
\Gamma^{-1}(\lambda) = I + ([p_1], \ldots, [p_n]) D(\lambda) \left( \begin{array}{c} \langle q_1 \rangle \\ \vdots \\ \langle q_n \rangle \end{array} \right), 
\]

\[
\Gamma(\lambda) = I + ([\bar{q}_1], \ldots, [\bar{q}_1]) \overline{D}(\lambda) \left( \begin{array}{c} \langle \bar{p}_1 \rangle \\ \vdots \\ \langle \bar{p}_1 \rangle \end{array} \right), 
\]

where the matrices \( D(\lambda) \) and \( \overline{D}(\lambda) \) are defined as

\[
D(\lambda) = \begin{pmatrix} \mathcal{K}^+(\lambda - \lambda_1) & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathcal{K}^+(\lambda + \lambda_1) \end{pmatrix}, \quad \overline{D}(\lambda) = \begin{pmatrix} \mathcal{K}^-(\lambda - \lambda^*_1) & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathcal{K}^-(\lambda + \lambda^*_1) \end{pmatrix},
\]
$\mathcal{K}^+(s), \mathcal{K}^-(s)$ are upper-triangular and lower-triangular Toeplitz matrices defined as:

$$
\mathcal{K}^+(s) = \begin{pmatrix}
 s^{-1} & s^{-2} & \cdots & s^{-n} \\
 0 & s^{-2} & \cdots & \vdots \\
 \vdots & \ddots & \ddots & \vdots \\
 0 & \cdots & s^{-1} & s^{-2} \\
 \end{pmatrix},
\mathcal{K}^-(s) = \begin{pmatrix}
 s^{-1} & 0 & \cdots & 0 \\
 s^{-2} & s^{-1} & \cdots & \vdots \\
 \vdots & \ddots & \ddots & \vdots \\
 s^{-n} & \cdots & s^{-2} & s^{-1} \\
 \end{pmatrix},
$$

and vectors $|p_j], [\tilde{p}_j], [q_j], [\tilde{q}_j], [\tilde{q}_j]$ ($j = 1, \ldots, n$) are independent of $\lambda$.

In fact, the rest of the vector parameters in (6.1) can be derived by calculating the poles of each order in the identity

$$
\Gamma(\lambda)\Gamma^{-1}(\lambda) = I \text{ at } \lambda = \lambda_1 \text{ and } \lambda = -\lambda_1,
$$

where

$$
\Gamma(\lambda) = \begin{pmatrix}
 \Gamma(\lambda) & 0 & \cdots & 0 \\
 \frac{\partial}{\partial \lambda} \Gamma(\lambda) & \Gamma(\lambda) & \cdots & 0 \\
 \vdots & \ddots & \ddots & \vdots \\
 \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \Gamma(k) & \cdots & \frac{\partial}{\partial \lambda} \Gamma(\lambda) & \Gamma(\lambda) \\
 \end{pmatrix}.
$$

Hence, in terms of the independent vector parameters, results (6.1) can be formulated in a more compact form as in Ref. [27], and here we just avoid these overlapped parts. Using this method, the process of solving soliton solution is very complex. In the following, we derive dressing matrix of higher-order poles via the method of unipolar point limit. The specific results are given by the following theorem.

**Theorem 2** In the case of one pair of elementary high-order zero, the dressing matrix for the third-order flow GI equation can be represented as:

$$
\Gamma = \Gamma_1^{-1} \cdots \Gamma_1^{-1}, \quad \Gamma^{-1} = \Gamma_1^{-1} \cdots \Gamma_1^{-1},
$$

where

$$
\Gamma_1^{[j]} = I + A_1^{[j]} - \frac{\sigma_3 A_1^{[j]} \sigma_3}{\lambda - \lambda_1}, \quad \Gamma^{-1}_1^{[j]} = I + A_1^{[j]} - \frac{\sigma_3 A_1^{[j]} \sigma_3}{\lambda + \lambda_1},
$$

$$
A_1^{[j]} = \frac{\lambda_1^2 - \lambda^2}{2} \begin{pmatrix}
 \alpha_1^{[j]} & 0 \\
 0 & \alpha_1^{[j]} \\
 \end{pmatrix} |v_1^{[j]}(v_1^{[j]}), (\alpha_1^{[j]})^{-1} = \begin{pmatrix} v_1^{[j]} & 0 \\
 0 & \lambda_1 \\
 \end{pmatrix} |v_1^{[j]}|,
$$

and

$$
|v_1^{[j]}| = \lim_{\delta \to 0} \frac{\Gamma_1^{[j]}|v_1|}{\delta} |v_1^{[j]}(v_1^{[j]}), (\alpha_1^{[j]})^{-1} = \begin{pmatrix} v_1^{[j]} & 0 \\
 0 & \lambda_1 \\
 \end{pmatrix} |v_1^{[j]}|,
$$

$$
\langle v_1^{[j]}| = \lim_{\delta \to 0} \langle v_1| |v_1^{[j]}(v_1^{[j]}), (\alpha_1^{[j]})^{-1} = \begin{pmatrix} v_1^{[j]} & 0 \\
 0 & \lambda_1 \\
 \end{pmatrix} |v_1^{[j]}|.
$$

Then by techniques similar to those used above, we can get

$$
u = 2i \left( \sum_{j=0}^{n-1} [B_1^{[j]} - \sigma_3 B_1^{[j]} \sigma_3]_{12} \right).
$$

As before, the above formulas also could be rewritten with the determinant form

$$
u = -4i \frac{\det \tilde{F}}{\det \tilde{M}}, \quad (6.2)
$$

where

$$
\tilde{F} = \begin{pmatrix}
 \tilde{M}_{11} & \tilde{M}_{12} & \cdots & \tilde{M}_{1n} & |v_1^{[0]}| \\
 \tilde{M}_{21} & \tilde{M}_{22} & \cdots & \tilde{M}_{2n} & |v_1^{[1]}| \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 \tilde{M}_{n1} & \tilde{M}_{n2} & \cdots & \tilde{M}_{nn} & |v_1^{[n-1]}| \\
 |v_1^{[0]}| & |v_1^{[1]}| & \cdots & |v_1^{[n-1]}| & 0 \\
 \end{pmatrix},
$$

and

$$
\tilde{M}_{kl} = \frac{1}{(k-1)! (l-1)!} \frac{\partial^{k+l-2}}{\partial \lambda^{k-1} \partial \lambda^*^{l-1}} \langle v_1 | v_1 \rangle_{\lambda = \lambda_1, \lambda^* = \lambda_1}.
$$
Where \(|v_1|^{(j)}, \langle v_1 \rangle^{(j)}\) can be written as follows
\[
|v_1|^{(j)} = \frac{1}{(j)!} \frac{\partial^j}{\partial (\lambda_j)^j}\! |v_1\rangle |_{\lambda_j=\lambda_1}, \quad \langle v_1 \rangle^{(j)} = \frac{1}{(j)!} \frac{\partial^j}{\partial (\lambda_j)^j}\! \langle v_1 \rangle |_{\lambda_j=\lambda_1^*}.
\]
Hence, formula (6.2) leads to the elementary high-order zeros solution formula. When \(N = 1\), it corresponds to a single soliton solution, and when \(N \geq 2\), it corresponds to higher-order soliton. Notice that the general expression of the high-order soliton solution of Eq. (6.2) is very complicated and is not given explicitly. However, with the aid of computer softwares such as Maple and Matlab, one can easily get the corresponding double-pole solution for different parameters by using Eq. (6.2). Explicitly, taking \(N = 2\), in (6.2) by choosing appropriate parameters considering the simplest higher-order 1-soliton solution case, which is plotted in Fig. 6.

7. Conclusion and discussion

In summary, the GI hierarchy is derived by using recursive operator. The recursive operator here contains two operators, which is more complex than the form of AKNS hierarchy and KN hierarchy. The main reason is that the derivative of the main diagonal of \(M\) to \(t\) is not 0, but a function related to the potential functions \(u\) and \(v\), which leads to the complex expression of \(A\). Then the inverse scattering method has been applied to the third-order flow GI equation and by considering the associated RHP, we successfully give a simple representation for the N-soliton in the determinant form. Owing to the symmetry properties of Jost solution and scattering data, the corresponding zeros in the RHP for higher-order GI equation appear in pairs, which is the same as the \(3 \times 3\) Sasa-Satsuma equation [29]. Later, taking single-soliton solution and two-soliton solutions as examples, the long-time behavior of the solution is studied. Compared with the classical GI direction of the second-order flow, it is found that the motion direction and wave height of the soliton solution are affected by the third-order dispersion and the fifth-order nonlinearity. These analysis results have important reference value for the study of GI hierarchy or other nonlinear integrable dynamic systems of higher order flow equations, and provide a theoretical basis for possible experimental research and application. Finally, the corresponding higher-order soliton solution matrix is derived by analyzing the limiting behavior of spectral parameters.

In recent years, there are many achievements in the study of the classical second-order flow GI equation with non-zero boundary conditions [33–35]. In this paper, we only consider the simple zeros and a pair of elementary higher-order zeros of the third-order flow GI equation with vanishing boundary conditions. Whether the behavior of soliton solutions with non-zero boundary and more multiplicity will have more abundant forms and long time behavior can be studied in the future.

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