Size-dependent free vibrations of electrostatically pre-deformed functionally graded micro-cantilevers

Masoud Tahani1,3, Romesh C. Batra2 and Amir R. Askari1

1Department of Mechanical Engineering, Ferdowsi University of Mashhad, Mashhad, Iran
2Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA
E-mail: mtahani@um.ac.ir

Abstract. Size-dependent free vibration analysis of cantilever-type resonant micro-sensors made of non-homogeneous functionally graded materials with two material phases is the objective of the present paper. For this aim, the modified couple stress theory together with the Euler-Bernoulli beam model is considered, and the size-dependent equation of motion which accounts for the non-linear and distributed electrostatic force is derived using the Hamilton's principle. The natural frequencies of the system are extracted using the Galerkin weighted residual method. It is found that the fundamental frequency of the system is decreased with an increase of applied voltage and becomes zero when the input voltage reaches the pull-in voltage of the system. The findings of the present paper are compared and validated by available results in the literature and excellent agreements between them are observed.

1. Introduction
Micron and sub-micron scale structures are frequently used in different applications such as micro-electro-mechanical systems (MEMS) nowadays. Because of their small size, low power consumption and the reliability of batch fabrications, there are lots of potential applications in engineering [1]. Resonant micro-sensors as one of the largest categories of these systems are extensively used in different applications such as signal filtering and chemical and mass sensing [2, 3]. The building blocks of these sensors, which employ the vibrational characteristics of the mechanical structure as the sensing method, are electrostatically actuated micro-cantilevers and doubly clamped micro-beams [4, 5]. In general, an electrostatically actuated micro-beam is an electrically conductive and elastic thin plate suspended over a stationary rigid electrode [6]. In resonant MEMS sensors, the movable electrode is deflected toward the fixed electrode and vibrates about its static deflection, where the direct current (DC) applied voltage is responsible for the static deflection and the oscillatory motion is occurred due to applying the alternating current (AC) voltage [7]. It is noteworthy that the resonance frequencies of micro-sensors in which the applied AC voltage is much smaller than the DC ones are controlled by the DC voltage [8]. Therefore, free vibration analysis of such structures is quiet useful and has application in determining design parameters of these devices.

3 Address for correspondence: Masoud Tahani, Department of Mechanical Engineering, Ferdowsi University of Mashhad, Mashhad, Iran. E-mail: mtahani@um.ac.ir.
The applied DC voltage has an upper limit in which the non-linear electrostatic attraction
overcomes the elastic restoring force of the beam. In this manner the movable part suddenly collapses
toward the fixed substrate. This unstable behavior is called pull-in instability which is simultaneously
observed by Nathanson et al. [9] and Taylor [10]. Also the upper limit of the DC voltage is called the
pull-in voltage.

Recently, variety of experiments showed that the material mechanical behavior in small scales is
size-dependent [11-14]. Size-dependent behavior is an intrinsic property of certain materials, which
emerges when the characteristic size, e.g. the diameter or the thickness is comparable to the material
length scale parameter. Material length scale parameter for a specific material can be determined using
some typical experiments such as micro-torsion test [11], micro-bend test [12, 13] and micro/nano
indentation test [14-16]. For example, the length scale parameter for single crystals of Al, Ag, Ni,
polycrystalline Cu and ploy-synthetically twinned (PST) lamellar α₂-TiAl and γ-TiAl have been
determined, respectively as: 2762 nm, 6233 nm, 4315 nm, 1120 nm, 74 nm and 49 nm [17].

The classical continuum mechanics cannot predict the size-dependent behavior of materials that
occurs in micron and sub-micron scale structures. In 1960s Toupin [18], Koiter [19] and Mindlin [20,
21] proposed the classical couple stress elasticity theory based on the Cosserat continuum mechanics
[22]. Based on this theory, besides the classical stress tensor, the couple stress components should be
included to describe the manner of media. In comparison to the classical continuum mechanics, the
couple stress theory has two additional parameters (high-order material length scales) other than two
classical Lame's constants in constitutive equations for isotropic elastic materials. Recently, a modified
version of this theory has been elaborated by Yang et al., in which constitutive equations involve only
one additional internal material length scale parameter besides two classical material constants [23].

The modified couple stress theory (MCST) has been successfully utilized to predict the mechanical
behavior of micro-structures in recent years. Here some of these works are reviewed. Park and Gao
[24] showed that the bending rigidity predicted by the MCST is larger than that calculated by the
classical theory (CT) and the difference between the deflections predicted by these two models is
significant when the beam thickness is small. Kong et al. [25] investigated the size effect on natural
frequencies of the Euler-Bernoulli micro-beam. Ke et al. [26] studied the thermal effect on the free
vibration and buckling of micro-beams using the MCST and Timoshenko beam theory through the
differential quadrature method (DQM). Rahaeifard et al. [27] investigated the size effect on the
deflection and static pull-in voltage using the MCST. They could remove the gap between the
experimental observations and the results of CT for the static pull-in voltage and calculated the silicon
length scale parameter as \( l = 0.592 \mu \text{m} \). Kong [28] introduced an analytic approximate solution to
static pull-in problem and calculated pull-in voltage and pull-in displacement based on the MCST
using the Rayleigh–Ritz method. He found that pull-in voltage predicted by the MCST is 3.1 times
greater than that predicted by the CT when the micro-beam thickness is equal to material length scale
parameter. Furthermore, the normalized pull-in displacement is size-independent and equals to 0.448
and 0.398 for cantilever and clamped-clamped micro-beams, respectively.

To achieve all material and economical requirements, functionally graded materials (FGMs) have
been frequently employed and analyzed in the past years [29-41]. Recently these types of materials are
extensively utilized in MEMS structures [42-45]. Therefore, many researchers are motivated to
develop size-dependent mechanical models for functionally graded structures at micron and sub-
micron scales using the MCST. Reddy [46] developed size-dependent non-linear Euler-Bernoulli and
Timoshenko beam theories for micro-beams made of FGM with two material phases based on the
MCST. He also presented some analytic solutions as well as finite element models for investigating
free vibrations, bending, buckling and post-buckling responses in such systems. Ke and Wang [47]
investigated dynamic buckling instability of functionally graded (FG) micro-beams utilizing the
MCST and Timoshenko beam model. Ke et al. [48] also investigated non-linear free vibration of FG
Timoshenko micro-beam using the MCST and solved the resulting equations through iterative DQM.
Furthermore, Akgöz and Civalek [49] investigated free vibrations of tapered axially FG micro-beams

2
based on the Euler-Bernoulli beam model and the MCST. They extracted the natural frequencies of vibrations using the Rayleigh–Ritz method.

Although many researchers have dealt with the mechanical behavior of micro-beams, the research effort devoted to free vibration analysis of electrostatically pre-deformed micro-beams are very limited. The objective of the present work is to establish a size-dependent electromechanical model for free vibration analysis of FG micro-cantilevers pre-formed by an electric field on the basis of MCST.

The rest of the paper is organized as follows. In sections two and three we present a brief review of the MCST and describe the present size-dependent electro-mechanical model for FG micro-beams, respectively. In section four we provide details of the solution procedure. In section five we compare and validate our findings with the available results in the literature. Also the rest of the section is devoted to a parametric study which indicates the significant effects of couple stress components on the natural frequencies of the system. The main concluding remarks of the present study are summarized in section six.

2. The modified couple stress theory

According to the MCST presented by Yang et al. [23] in 2002, both strain tensor (conjugated with the stress tensor) and curvature tensor (conjugated with the couple stress tensor) are included in the strain energy density. Based on this theory, the strain energy $U$ in a deformed isotropic linear elastic material occupying region $\Pi$ is given by

$$U = \frac{1}{2} \int_\Pi \left( \bar{\sigma} : \bar{\varepsilon} + m : \bar{\chi} \right) \, d\Pi$$

where $\bar{\sigma}$, $\bar{\varepsilon}$, $m$ and $\bar{\chi}$ are the Cauchy stress, strain, deviatoric part of couple stress and symmetric curvature tensors, respectively. These tensors for cases with small slopes and deflections can be written as

$$\bar{\sigma} = \lambda \text{tr}(\bar{\varepsilon}) \, I + 2\mu \bar{\varepsilon}$$

$$\bar{\varepsilon} = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right]$$

$$m = 2l^2 \mu \bar{\chi}$$

$$\bar{\chi} = \frac{1}{2} \left[ \nabla \theta + (\nabla \theta)^T \right]$$

where $\nabla = e_x \frac{\partial}{\partial x} + e_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z}$, $u$ is the displacement vector, $\lambda$ and $\mu$ are Lame's constants ($\mu$ is also known as shear modulus), $l$ is a material length scale parameter, $I$ is the identity tensor and $\theta$ is the rotation vector expressed as

$$\theta = \frac{1}{2} \text{curl} \, u$$

Obviously, only one length scale parameter $l$ involved in addition to two Lame's constants in the constitutive equation of the MCST. The material length scale parameter $l$ is mathematically defined as the square of the ratio of the curvature modulus to the shear modulus. Also, it is physically treated as a material property characterizing the effect of couple stress components [50]. It is noted that both $\bar{\sigma}$ and $m$, as respectively defined in equations (2) and (4) are symmetric due to the symmetry of $\bar{\varepsilon}$ and $\bar{\chi}$ given in equations (3) and (5), respectively.
3. The MCST formulation for FG Euler-Bernoulli micro-beam pre-deformed by an electric field

Figure 1 shows a schematic of clamped-free electrically actuated micro-beam under the action of electrostatic force. The thickness, length, width and density of micro-beam are $h$, $L$, $b$ and $\rho$, respectively. The initial gap between the non-actuated beam and the stationary electrode is $d$. Also, $x$, $y$ and $z$ are the coordinates along the length, width and thickness, respectively, $w$ is deflection of the beam and $t$ is time.

![Figure 1. Schematic of an electrostatically pre-deformed micro-cantilever.](image)

The electrostatic excitation by polarized DC voltage $V$ without the effect of fringing field per unit length of the beam can be expressed as [4]:

$$ F_{es} = -\frac{\varepsilon b V^2}{2(d - w)^2} $$

where $\varepsilon$ is the dielectric constant of the medium. It is noted that the fringing field does not have a sizable effect especially for the case of wide micro-beams [51].

According to the basic hypothesis of the Euler-Bernoulli beam model, the displacement field $(\vec{u}, \vec{w})$ of an arbitrary point on the micro-beam can be expressed as [52]:

$$ \vec{u} = -z \frac{\partial w}{\partial x}, \quad \vec{w} = w(x,t) $$

where $w$ is the transverse displacement of a point on the mid-plane of micro-beam (i.e. $z = 0$). For the displacement field introduced in equation (8), the strain components can be written as [52]:

$$ \varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \varepsilon_z = \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0 $$

Substitution of equation (8) into equation (6) and the subsequent result into equation (5) yields

$$ \chi_{xy} = -\frac{1}{2} \frac{\partial^2 w}{\partial x^2}, \quad \chi_x = \chi_y = \chi_z = \chi_{xz} = \chi_{yz} = 0 $$

For slender beam, the Poisson's effect is negligible [46], so by substituting equations (9) and (10) into equations (2) and (4), respectively, the non-zero components of the Cauchy stress and high-order couple stress can be determined as:

$$ \sigma_x = -E(z) z \frac{\partial^2 w}{\partial x^2} $$
\[ m_{xy} = -\mu(z)l^2(z)\frac{\partial^2 w}{\partial x^2} \]  

(12)

where \( E(z) \), \( \mu(z) \) and \( l(z) \) are the Young modulus, the shear modulus and the material length scale parameter of the micro-beam and vary continuously through the beam thickness according to the power-law [46, 53, 54]. We assume that the beam is made of a two-constituent FGM through the thickness. Hence, the Young modulus, the shear modulus, the material length scale parameter and the density of the beam take the form

\[
E(z) = E_1 + \left(\frac{2z+h}{2h}\right)^n (E_2 - E_1) \\
\mu(z) = \frac{E(z)}{2(1+\nu)} = \frac{1}{2(1+\nu)} \left[ E_1 + \left(\frac{2z+h}{2h}\right)^n (E_2 - E_1) \right] \\
l(z) = l_1 + \left(\frac{2z+h}{2h}\right)^n (l_2 - l_1) \\
\rho(z) = \rho_1 + \left(\frac{2z+h}{2h}\right)^n (\rho_2 - \rho_1)
\]

(13a, b, c, d)

where the parameters with 1 and 2 sub-scripts are referred to the properties of the material used in the bottom and top surfaces of the beam, respectively. Also \( n \) is the power-law index and \( \nu \) is the Poisson’s ratio of the beam. It is also assumed that the Poisson’s ratio is a constant for the FGM [46]. It should be noted that by setting \( n = 0 \), the present formulation will be simplified to that of homogeneous micro-beams. Upon substitution of equations (9)-(12) into equation (1), the following result is obtained

\[
U = \frac{1}{2} \left[ (EI)_{eq} + (\mu Al^2)_{eq} \right] \int_0^L \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx 
\]

(14)

where \((EI)_{eq}\) and \((\mu Al^2)_{eq}\) are

\[
(EI)_{eq} = \int_A z^2 E(z) dA \\
(\mu Al^2)_{eq} = \frac{1}{2(1+\nu)} \int_A l^2(z) E(z) dA
\]

(15a, b)

in which \( A \) is referred to the cross-sectional area of the micro-beam. The virtual work done by the electrostatic loading is

\[
\delta W_{es} = \int_0^L F_{es} \delta w dx = \int_0^L \frac{ebv^2}{2(d-w)^2} \delta w dx
\]

(16)

The kinetic energy of the slender micro-beam with symmetric cross-section can be expressed as:

\[
KE = \frac{1}{2} (\rho A)_{eq} \int_0^L \left( \frac{\partial w}{\partial t} \right)^2 dx
\]

(17)
where \((\rho A)_{\text{eq}}\) is

\[
(\rho A)_{\text{eq}} = \int_A \rho \, dA
\]  \hspace{1cm} (18)

The Hamilton principle for an elastic body states [55]

\[
\int_0^t (\delta K - \delta U + \delta W_{\text{es}}) \, dt = 0
\]  \hspace{1cm} (19)

By substitution of equations (14), (16) and (17) into equation (19), the equation of motion is obtained as:

\[
(\rho A)_{\text{eq}} \frac{\partial^2 w}{\partial t^2} + \left[ (EI)_{\text{eq}} + (\mu A l^2)_{\text{eq}} \right] \frac{\partial^4 w}{\partial x^4} = \frac{\varepsilon b V^2}{2(d - w)^2} \]  \hspace{1cm} (20)

The corresponding boundary conditions for clamped-free micro-beam are also determined as:

\[
w(0, t) = 0, \quad \frac{\partial w}{\partial x}_{|x=0,t} = 0 \]  \hspace{1cm} (21a)

\[
\frac{\partial^2 w}{\partial x^2}_{|x=L,t} = 0, \quad \frac{\partial^3 w}{\partial x^3}_{|x=L,t} = 0 \]  \hspace{1cm} (21b)

It is to be noted that for free vibration analysis, no initial conditions are required. Also, for convenience, the following dimensionless variables are introduced

\[
\hat{w} = \frac{w}{d}, \quad \hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{t}{T}
\]  \hspace{1cm} (22)

where

\[
T = \sqrt{\frac{(\rho A)_{\text{eq}} L^4}{(EI)_{\text{eq}}}} \]  \hspace{1cm} (23)

Upon substitution of the dimensionless quantities given in equation (22) into equation (20) and dropping the hats, the following result will be obtained

\[
\hat{w} + (1 + \alpha) w''' = \frac{\beta}{(1 - w)^3}
\]  \hspace{1cm} (24)

where dot and prime signs denote derivatives with respect to \(t\) and \(x\), respectively. The dimensionless parameters of the system are also introduced as:

\[
\alpha = \left(\frac{\mu A l^2}{(EI)_{\text{eq}}}\right)_{\text{eq}}, \quad \beta = \frac{\varepsilon b V^2 L^4}{2d^3 (EI)_{\text{eq}}}
\]  \hspace{1cm} (25)

The dimensionless boundary conditions also take the form

\[
w(0, t) = \hat{w}'(0, t) = \hat{w}''(1, t) = \hat{w}'''(1, t) = 0 \]  \hspace{1cm} (26)

To obtain the free vibration equation of micro-beams pre-deformed by an electric field, the deflection is divided into two counterparts as:
where \( w_s \) denotes the static deflection of the micro-beam occurred due to the application of the electrical field and \( \Delta w \) describes the oscillating part of the deflection. Upon substitution of equation (27) into equation (24), utilizing the static equilibrium equation (i.e. equation (24) without the inertia terms) and retain linear terms in \( \Delta w \), one can obtain

\[
(1 + \alpha) \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - \frac{2\beta}{(1 - w_s)^3} \Delta w = 0
\]  

Equation (28) governs free vibration of the pre-deformed electrostatically actuated micro-beams. Assuming \( \Delta w = \varphi_n(x) \exp(i\omega_n t) \), the following eigenvalue problem will be obtained

\[
(1 + \alpha) \frac{\partial^4 \varphi_n}{\partial x^4} - \left( \frac{2\beta}{(1 - w_s)^3} + \omega_n^2 \right) \varphi_n = 0
\]  

where \( i = \sqrt{-1} \). Also \( \omega_n \) and \( \varphi_n \) denote the \( n \)th natural frequency and its associated mode-shape of the electrostatically pre-deformed micro-beam, respectively.

4. Solution procedure

4.1. Obtaining the static deflection

To find the natural frequencies of the present electrostatically pre-deformed micro-cantilever, the static deflection of the micro-beam under the applied electrostatic force will be required. The equation that governs the static equilibrium of the beam can be obtained by setting all time derivatives in equation (24) to zero. Doing so, one would get

\[
(1 + \alpha) w_s''' = \frac{\beta}{(1 - w_s)^2}
\]  

Due to the high non-linearity involved in equation (30), a closed-form solution for this equation cannot be found. Hence, an approximate solution will be developed here through the Galerkin weighted residual method. Based on this procedure, the nano/micro-beam deflection can be expressed as a linear combination of a complete set of linearly independent basis functions [55]. It is noted that these functions must satisfy all kinematic boundary conditions [55]. Therefore, linear and undamped mode shapes of the undeformed nano/micro-beam can be used as these basis functions. It is proved that using only the first mode for static, dynamic, vibrational and pull-in analysis of electrically actuated nano/micro-beams maybe very accurate [56-59]. Hence, the deflection of the beam based on one-mode solution can be expressed as:

\[
w_s(x) = \overline{w}\varphi_1(x)
\]  

where \( \overline{w} \) is an unknown parameter determined through the Galerkin procedure and \( \varphi_1(x) \) is the first linear and undamped mode shape of the undeformed nano/micro-beam determined as [60]:

\[
\varphi_1(x) = \cosh(\gamma_1 x) - \cos(\gamma_1 x) - \sigma_1 \left[ \sinh(\gamma_1 x) - \sin(\gamma_1 x) \right]
\]  

where the parameters \( \gamma_1 \) and \( \sigma_1 \) for clamped-free nano/micro-beam are given by [60]:

\[
\gamma_1 = 1.8751, \quad \sigma_1 = 0.7341
\]
It should be noted that the mode-shapes of the present size-dependent FG micro-cantilever are as same as those of homogenous cantilevers introduced in equation (32) [61]. Because their governing boundary value differential equation of free vibrations, as well as their corresponding boundary conditions are the same. According to the Galerkin procedure, we multiply equation (30) by $\varphi_1(x)$, substitute equation (31) into the resulting equation, integrate the outcome from $x = 0$ to 1 and obtain

$$K\bar{w} - \beta \int_0^1 \left\{ \varphi_1(x) \left[ 1 - \bar{w}\varphi_1(x) \right]^{-2} \right\} dx = 0$$

where

$$K = (1 + \alpha) \int_0^1 \varphi_1(x)'''' dx$$

The deflection of the nano/micro-beam under the electrical load can be determined by solving the algebraic equation (34). In this paper, this equation is solved using the method presented by Tahani and Askari [58]. According to the iterative nature of this procedure [58], the non-linear equation (34) is rewritten in the following form

$$K\bar{w} + \mathcal{N}(\bar{w}) = 0$$

where

$$\mathcal{N}(\bar{w}) = -\beta \int_0^1 \left\{ \varphi_1(x) \left[ 1 - \bar{w}\varphi_1(x) \right]^{-2} \right\} dx$$

At the first step, the non-linear term in equation (34) (i.e., $\mathcal{N}(\bar{w})$) is calculated when $\bar{w}_0 = 0$ as:

$$\mathcal{N}_0 = -\beta \int_0^1 \left\{ \varphi_1(x) \left[ 1 - \bar{w}_0\varphi_1(x) \right]^{-2} \right\} dx = -\beta \int_0^1 \varphi_1(x) dx$$

Substituting $\mathcal{N}_0$ from equation (38) into equation (36), yields the first estimation of the unknown parameter $\bar{w}$ (i.e. $\bar{w}_1$) as:

$$\bar{w}_1 = -\frac{\mathcal{N}_0}{K}$$

The next estimation of $\mathcal{N}(w)$ (i.e., $\mathcal{N}_1$) will be obtained by substituting $\bar{w}_1$ from equation (39) into equation (37) as:

$$\mathcal{N}_1 = -\beta \int_0^1 \left\{ \varphi_1(x) \left[ 1 - \bar{w}_1\varphi_1(x) \right]^{-2} \right\} dx$$

Substituting $\mathcal{N}_1$ from equation (40) into equation (36), leads to the second estimation of the unknown parameter $\bar{w}$ (i.e. $\bar{w}_2$). This iterative procedure is continued till the convergence achieves or pull-in happens. The convergence criterion is defined as:

$$\left\| (\bar{w}_i - \bar{w}_{i-1}) \bar{w}_i^{-1} \right\| \leq 10^{-6}$$

and the pull-in will happen if

$$\bar{w}_i \geq 1$$
It is noted that the integral \( \int_0^1 \varphi_1 (1 - \bar{w}, \varphi_1)^{-2} \, dx \) should be calculated numerically and repeated at each step. After finding \( \bar{w} \) with sufficient accuracy, the static deflection of the beam can be obtained through equation (31).

### 4.2. Extracting the natural frequencies

After finding the static deflection corresponding to the applied voltage \( V \), using the Galerkin procedure, the associated \( i \)th dimensionless natural frequency of the pre-deformed micro-cantilever can be easily obtained from equation (29) as [62]:

\[
\omega_i(V) = \left( 1 + \alpha \right) \int_0^1 \varphi_i \varphi_i^{''''} \, dx - 2 \beta \int_0^1 \frac{\varphi_i^2}{(1 - w_x)} \, dx \right)^{1/2} \quad (43)
\]

The \( i \)th dimensional natural frequency of the system (Hz) can also be determined as:

\[
f_i(V) = \frac{1}{2\pi f} \left( 1 + \alpha \right) \int_0^1 \varphi_i \varphi_i^{''''} \, dx - 2 \beta \int_0^1 \frac{\varphi_i^2}{(1 - w_x)} \, dx \right)^{1/2} \quad (44)
\]

where the \( i \)th mode-shape of the micro-cantilever takes the form [60]

\[
\varphi_i(x) = \cosh(\gamma_i x) - \cos(\gamma_i x) - \sigma_i \left[ \sinh(\gamma_i x) - \sin(\gamma_i x) \right] \quad (45)
\]

in which the values of \( \gamma_i \) and \( \sigma_i \) for the first four mode-shapes are given in table 1 [60]. It is to be noted that the present micro-beam's mode-shapes are normalized such that \( \int_0^1 \varphi_i^2 \, dx = 1 \). Also the \( i \)th dimensional natural frequency of the undeformed micro-cantilever (Hz) can also be extracted using equation (44) when \( V = 0 \) as:

\[
f_i(0) = \frac{\gamma_i^2}{2\pi f} \left\{ \left[ (EI)_{eq} + \left( \mu A l^2 \right)_{eq} \right] \left( \rho A l^3 \right)^{-1} \right\}^{1/2} \quad (46)
\]

which is a more comprehensive version of the exact expression derived by Asghari et al. [61] for the natural frequencies of undeformed micro-cantilevers. It is to be mentioned that, Asghari et al. [61] did not account for the variation of the length scale parameter of the FGM through its thickness. However, the present expression (i.e. equation (46)) accounts for the through thickness variation of the length scale parameter as one of the micro-beam properties. It is also noteworthy that this can be considered as verification for present analysis.

| Table 1. The first four mode-shape parameters for clamped-free micro-beams [60]. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Mode-shape parameters | \( i = 1 \) | \( i = 2 \) | \( i = 3 \) | \( i = 4 \) |
| \( \gamma_i \) | 1.8751 | 4.6941 | 7.8548 | 10.9955 |
| \( \sigma_i \) | 0.7341 | 1.0185 | 0.9992 | 1.0000 |

### 5. Results and discussion

#### 5.1. Comparison and validation

As discussed at the end of the previous section, the present solution procedure leads to the exact solution for size-dependent natural frequencies of undeformed FG micro-cantilevers. Since, according to the best of authors’ knowledge, there are no published results in the open literature for size-
dependent natural frequencies of FG electrostatically pre-deformed micro-beams, we neglect the effects of size and through thickness variation of the material properties (i.e. set \( n = l_1 = l_2 = 0 \)) and compare our findings with those obtained by De and Aluru [63] in figure 2 for an 110-direction silicon micro-cantilever with properties listed in table 2. It is to be noted that the 110-direction silicon will be abbreviated to silicon hereinafter.

Table 2. Geometric and material properties of present silicon micro-cantilever.

| \( L (\mu m) \) | \( b (\mu m) \) | \( h (\mu m) \) | \( E (GPa) \) | \( \nu \) | \( \rho (kg/m^3) \) |
|-----------------|-----------------|-----------------|---------------|-----|-----------------|
| 80              | 10              | 0.5             | 169           | 0.33 | 2331            |

Figure 2. Variation of the fundamental natural frequency (kHz) versus the applied voltage (V) for micro-cantilevers with properties presented in table 2.

It is seen from figure 2 that the present findings agree well with those obtained by De and Aluru [63] except for near pull-in cases. It should be noted that De and Aluru [63] solved the coupled non-linear equations of motion developed based on the Lagrangian description of both mechanical and electrical domains using the relaxation and Newton schemes. However, the present paper employs the improved version of the simple and accurate electromechanical model which was introduced by Batra et al. [4] for homogenous materials. It should be mentioned here that the accuracy of this model in comparison to three-dimensional (3-D) finite element (FE) simulations was proven by Batra et al. [4].

According to the results of figure 2, by increasing the applied voltage, the fundamental natural frequency of the system decreases and suddenly drops to zero when the input voltage reaches the pull-in voltage of the system. To validate the accuracy of present results for near pull-in cases, the calculated pull-in voltages for silicon micro-cantilevers with material properties presented in table 2 and geometric properties listed in table 3 are compared with experimental observations reported by Osterberg [64] in figure 3. It is to be noted that the value of the length scale parameter is also set to \( l = 0.592 \mu m \) for preparing the results of figure 3 [27]. Based on the results of this figure and those of figure 2, one can conclude that the present procedure can provide reliable and high-accurate results for both far from and near pull-in cases. However, as it is seen from figure 2, the results reported by De and Aluru [63] are not accurate for cases under input voltage close to their pull-in voltage.

Table 3. Geometric properties of silicon micro-cantilevers tested by Osterberg [64].

| The range for length (\( \mu m \)) | \( b (\mu m) \) | \( h (\mu m) \) | \( d (\mu m) \) |
|----------------------------------|-----------------|-----------------|---------------|
| 75-250                           | 50              | 2.94            | 1.05          |
5.2. The effect of size on FG micro-cantilevers

This section investigates the effect of size on natural frequencies of micro-cantilevers made of FGM with two distinct material phases. It is assumed that the beam is made of FG poly-SiAg layer structure where silver (Ag) and silicon (Si) are used as material phase 1 and material phase 2, respectively. The material properties of both silicon and silver are listed in table 4. Also, it is assumed that geometric properties of the micro-cantilever are as same as those presented in table 2.

![Figure 3](image)

Figure 3. Comparison between present results and experimental observations reported by Osterberg [64] for pull-in voltages of silicon micro-cantilevers with geometric properties presented in table 3.

| Table 4. Material properties of silver and silicon. |
|---------------------------------|---|---|---|---|
| Material | $E$ (GPa) | $\nu$ | $\rho$ (kg/m$^3$) | $l$ (μm) |
| 1. Silver (Ag) | 83 | 0.37 | 10490 | 6.233 [17] |
| 2. Silicon (Si) | 169 | 0.33 | 2331 | 0.592 [27] |

Table 5 represents the first two natural frequencies of the undeformed micro-cantilevers made of both homogenous and non-homogenous FG materials obtained by both MCST and CT. According to the results of this table, by increasing the values of power-law index (i.e. $n$) the values of the natural frequencies lead to those of silver; because the volume fraction of silver increases with an increase of the power-law index value (i.e. $n$). It is noteworthy that the volume fraction of each material phase takes the form [46]

$$VF_{Ag}(\varepsilon) = 1 - \left(\frac{2\varepsilon + h}{2h}\right)^n, \quad VF_{Si}(\varepsilon) = \left(\frac{2\varepsilon + h}{2h}\right)^n$$ (47)

Based on equation (47), the properties of the FGM may lead to those of silicon and silver, when the power-law index leads to zero and infinity, respectively.

| Table 5. The first two natural frequencies (kHz) for undeformed micro-cantilevers made of both homogenous and non-homogenous FG materials. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Material | $f_1$ (MCST) | $f_1$ (CT) | $f_2$ (MCST) | $f_2$ (CT) |
| Silicon (Si) | 288.8889 | 107.4354 | 1810.4502 | 673.2917 |
| FGM, $n = 0.1$ | 501.6917 | 181.0473 | 3144.0736 | 561.1063 |
| FGM, $n = 1$ | 822.7274 | 55.9488 | 5155.9866 | 350.6280 |
| FGM, $n = 10$ | 922.7817 | 40.3917 | 5783.0212 | 253.1326 |
| Silver (Ag) | 926.7893 | 35.4993 | 5808.1364 | 222.4721 |
As one can observe from table 5, neglecting the effect of size may lead to much more in-accurate results for silver than silicon, since the material length scale parameter of silver is much more than that of silicon. Because, based on equation (46), the effect of size may be more considerable for cases with...
greater values of material length scale parameter. Furthermore, according to the results of table 5, accounting for the effect of couple stress components increases the bending rigidity of the micro-beam that results in increasing the natural frequencies of the system.

Variation of the fundamental natural frequency of present FG poly-SiAg, as well as silicon and silver micro-cantilevers versus the applied voltage is depicted in figure 4 for systems with different initial gaps. According to the results of this figure, increasing the initial gap of the system increases its pull-in voltage. Furthermore, increasing the power-law index increases the bending rigidity of the system which results in the increase of its pull-in voltage and natural frequencies. Because by increasing the value of power-law index, the properties of the FGM are more and more similar to those of silver that is stiffer than silicon in such dimensions. It should be noted that, although the Young modulus of silver is smaller than that of silicon, it is stiffer than silicon. Because its larger material length scale parameter plays a crucial role in the dimensions of present micro-cantilever. This is the concept of size-dependent behavior at micron and sub-micron scales which must be taken into account for designing MEMS.

6. Concluding remarks

Extracting the size-dependent natural frequencies of FG micro-cantilevers pre-deformed by an electric field was the objective of the present paper. To this end, a size-dependent Euler-Bernoulli beam model was employed and the equation of motion, which accounts for the through thickness variation of material properties and distributed non-linear electrostatic attraction, was derived using the Hamilton's principle. The eigenvalue equation which governs on the free vibrations of electrostatically pre-deformed micro-cantilevers was also obtained and solved using the Galerkin weighted residual method. The extracted natural frequencies, as well as the pull-in voltage of the system were compared and validated by available results in the literature and excellent agreements between them were observed. The results showed that the fundamental natural frequency of the system decreases with an increase of applied voltage and suddenly drops to zero when the applied voltage reaches the pull-in voltage of the system.

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