Solutions of $\mathfrak{gl}_{m|n}$ XXX Bethe ansatz equation and rational difference operators

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Abstract

We study solutions of the Bethe ansatz equations of the non-homogeneous periodic XXX model associated to super Yangian $\mathcal{Y}(\mathfrak{gl}_{m|n})$. To a solution we associate a rational difference operator $R$ and a superspace of rational functions $W$. We show that the set of complete factorizations of $R$ is in canonical bijection with the variety of superflags in $W$ and that each generic superflag defines a solution of the Bethe ansatz equation. We also give the analogous statements for the quasi-periodic supersymmetric spin chains.

Keywords: supersymmetric spin chains, Bethe ansatz, difference operators

1. Introduction

The supersymmetric spin chains were introduced back to [Kul] in the 1980s. There is a considerable renewed interest to those models, see [BR1, BR2, KSZ, HLPRS, TZZ]. However, many results available for the even spin chains are still unknown for the supersymmetric case. In this paper we are able to fill in a few gaps.

We use the method of populations of solutions of the Bethe ansatz equations. It was pioneered in [MV2] in the case of the Gaudin model and then extended to the XXX models constructed from the Yangian associated to $\mathfrak{gl}_n$, see [MV1, MV3, MTV2]. We are helped by the recent work on the populations of the supersymmetric Gaudin model [HMVY].

Let us describe our findings in more detail. In this paper we restrict ourselves to tensor products of evaluation polynomial $\mathfrak{gl}_{m|n}$-modules. Moreover, we assume that the evaluation parameters are generic, meaning they are distinct modulo $h\mathbb{Z}$ where $h$ is the shift in the super Yangian relations. We also assume that at least one of the participating $\mathfrak{gl}_{m|n}$-modules is typical.

The crucial observation is the reproduction procedure which given a solution of the Bethe ansatz equation and a simple root of $\mathfrak{gl}_{m|n}$ produces another solution, see theorem 5.1.

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The reproduction procedure along an even root is given in [MV1]. An even component of a solution of the Bethe ansatz equation gives a polynomial solution of a second order difference equation. The reproduction procedure amounts to trading this solution to any other polynomial solution of the difference equation, see (5.1). We call it the bosonic reproduction procedure.

The reproduction procedure along an odd root is different. In fact, an odd component of a solution of the Bethe ansatz equation corresponds to a polynomial which divides some other polynomial, see (5.2). The reproduction procedure changes the divisor to the quotient polynomial with an appropriate shift. We call it the fermionic reproduction procedure. The fermionic reproduction procedure looks similar to a mutation in a cluster algebra.

Then the population is the set of all solutions obtained from one solution by recursive application of the reproduction procedure.

Given a solution of the Bethe ansatz equation, we define a rational difference operator of the form $R = D_0 D_1^{-1}$, where $D_0, D_1$ are linear difference operators of orders $m$ and $n$ with rational coefficients, respectively, see (5.6). The operator $R$ is invariant under reproduction procedures and therefore it is defined for the population, see theorem 5.3. The idea of considering such an operator is found in [HMVY] in the case of the Gaudin model. Such an operator in the case of tensor products of vector representations also appears in [Tsu], in relation to the study of T-systems and analytic Bethe ansatz.

Kernels $V = \ker D_0$, $U = \ker D_1$ are spaces of rational functions of dimensions $m$ and $n$. Under our assumption, that at least one of the representations is typical, we can show $V \cap U = 0$, see lemma 6.1. We consider superspace $W = V \oplus U$, where $W_0 = V$ and $W_1 = U$. Then we show that there are natural bijections between three objects: elements of the population of the solutions of the Bethe ansatz equation, superflags in $W$, and complete factorizations of $R$ into products of linear difference operators and their inverses, see theorem 6.7.

Note that the Bethe ansatz equations depend on the choice of the Borel subalgebra in $\mathfrak{gl}_{m|n}$. The fermionic reproductions change this choice. In general, the Borel subalgebra is determined from the parity of the superflag or, equivalently, from the positions of the inverse linear difference operators in a complete factorization of $R$.

Thus the solutions of the Bethe ansatz equations correspond to superspaces of rational functions. It is natural to expect that all joint eigenvectors of XXX Hamiltonians correspond to such spaces and that there is a natural correspondence between the eigenvectors of the transfer matrix and points of an appropriate Grassmannian. However, the precise formulation of this correspondence is not established in the even case, see [MTV2].

We give a few details in the quasi-periodic case as well, see section 7. In this case we also have concepts of reproduction procedure, the population, and the rational difference operator. Then the elements in the population are in a natural bijection with the permutations of the distinguished flags in the space of functions of the form $f(x) = e^{zr(x)}$, where $r(x) \in \mathbb{C}(x)$ is a rational function and $z \in \mathbb{C}$, see theorem 7.4. A similar picture in the even case is described in [MV3].

The paper is constructed as follows. In section 2 we study rational difference operators and their complete factorizations. We then recall the XXX model associated to $\mathfrak{gl}_{m|n}$ and the corresponding Bethe ansatz equations in section 3. In section 4 we recall the reproduction procedure for $\mathfrak{gl}_2$ and define its analog for $\mathfrak{gl}_{1|1}$. In section 5 we define reproduction procedures for $\mathfrak{gl}_{m|n}$, a population, and a rational difference operator associated to a population. In section 6 we give the bijections between the superflag variety, the set of complete factorizations, and a population. We conclude our paper by generalizing our results to the quasi-periodic XXX model in section 7. The appendix is devoted to the basics of Bethe ansatz in the case of $Y(\mathfrak{gl}_{1|1})$. 
2. Rational difference operators and their factorizations

We study properties of ratios of difference operators, following the treatment of ratios of differential operators in [CDSK]. We also describe the relation between the complete factorizations and the superflag varieties.

2.1. Parity sequences

We use the notation of [HMVY, section 2]. We recall some of them.

Denote by $S_{m+n}$ the set of all sequences $s = (s_1, s_2, \ldots, s_{m+n})$ where $s_i \in \{\pm 1\}$ and 1 occurs exactly $m$ times. Elements of $S_{m+n}$ are called parity sequences. The parity sequence of the form $s_0 = (1, \ldots, 1, -1, \ldots, -1)$ is the standard parity sequence.

A parity sequence $s$ corresponds to a permutation $\sigma_s$ of the permutation group $S_{m+n}$ of $m+n$ elements as follows:

$$\sigma_s(i) = \begin{cases} \#\{j \mid j \leq i, s_j = 1\}, & \text{if } s_i = 1, \\ m + \#\{j \mid j \leq i, s_j = -1\}, & \text{if } s_i = -1. \end{cases}$$

For a parity sequence $s \in S_{m+n}$, we define

$$s_i^+ = \#\{j \mid j > i, s_j = 1\}, \quad s_i^- = \#\{j \mid j < i, s_j = -1\}, \quad i = 1, \ldots, m+n.$$

The permutation $\sigma_s$ is related to $s_i^\pm$ by

$$s_i^+ = \begin{cases} m - \sigma_s(i), & \text{if } s_i = 1, \\ \sigma_s(i) - i, & \text{if } s_i = -1, \end{cases} \quad s_i^- = \begin{cases} i - \sigma_s(i), & \text{if } s_i = 1, \\ \sigma_s(i) - m - 1, & \text{if } s_i = -1. \end{cases}$$

If the parity sequence is dropped from the notation, it means we consider the standard parity sequence.

2.2. Rational difference operators

Fix a non-zero number $h \in \mathbb{C}^\times$. Let $\mathbb{k}$ be the field of complex valued rational functions $\mathbb{k} = \mathbb{C}(x)$, with an automorphism $\tau : \mathbb{k} \to \mathbb{k}$, $(\tau f)(x) : f(x - h)$.

Consider the algebra $\mathbb{k}[\tau]$ of difference operators where the shift operator $\tau$ satisfies

$$\tau \cdot f(x) = f(x - h) \cdot \tau$$

for all $f(x) \in \mathbb{k}$. By definition, an element $D \in \mathbb{k}[\tau]$ has the form

$$D = \sum_{j=0}^{r} a_j \tau^j, \quad a_j \in \mathbb{k}, \quad r \in \mathbb{Z}_{\geq 0}. \tag{2.1}$$

The difference operator $D$ has order $r$, ord $D = r$, if $a_r \neq 0$. One says that $D$ is monic if $a_r = 1$. We call $a_0$ the constant term of $D$.

Let $D \in \mathbb{k}[\tau]$ be a difference operator of order $r$ as in (2.1). We say a difference operator $D$ of order $r$ is completely factorable over $\mathbb{k}$ if there exist $f_i \in \mathbb{k}$, $i = 1, \ldots, r$, such that $D = a_r f_1 \cdots f_r$, where $f_i = \tau - f_i$. We focus on completely factorable difference operators with non-zero constant terms $a_0$. In this case, we consider factorizations of the form $D = a_0 d_1 \cdots d_r$, where $d_i = 1 - f_i \tau$, $f_i \in \mathbb{k}$, $i = 1, \ldots, r$.

Let $\ker D = \{u \in \mathbb{k} \mid Du = 0\}$ be the kernel of $D$. It is clear that if dim (ker $D$) = ord $D$, then $D$ is completely factorable over $\mathbb{k}$.
Let $\mathbb{K}[\tau]$ be the division ring generated by $\mathbb{K}[\tau]$. The division ring $\mathbb{K}[\tau]$ is called the ring of rational difference operators. Elements in $\mathbb{K}[\tau]$ are called rational difference operators.

A fractional factorization of a rational difference operator $\mathcal{R}$ is the equality $\mathcal{R} = D_0 D_1^{-1}$, where $D_0, D_1 \in \mathbb{K}[\tau]$. A fractional factorization $\mathcal{R} = D_0 D_1^{-1}$ is called minimal if $D_1$ is monic and has the minimal possible order.

**Proposition 2.1.** Any rational difference operator $\mathcal{R} \in \mathbb{K}[\tau]$ has the following properties.

(i) There exists a unique minimal fractional factorization of $\mathcal{R}$.

(ii) Let $\mathcal{R} = D_0 D_1^{-1}$ be the minimal fractional factorization. If $\mathcal{R} = D_0 D_1^{-1}$ is a fractional factorization, then there exists $D \in \mathbb{K}[\tau]$ such that $D_0 = D D_1$ and $D_0 D_1^{-1}$. (ii) Let $\mathcal{R} = D_0 D_1^{-1}$ be a fractional factorization such that $\dim(\ker D_0) = \ord D_0$ and $\dim(\ker D_1) = \ord D_1$. Then $\mathcal{R} = D_0 D_1^{-1}$ is the minimal fractional factorization of $\mathcal{R}$ if and only if $\ker D_0 \cap \ker D_1 = 0$.

**Proof.** We have the analogs of [CDSK, proposition 2.1, corollary 2.2, lemma 3.2] for difference operators. Namely, the algebra $\mathbb{K}[\tau]$ is right Euclidean, therefore $\mathbb{K}[\tau]$ satisfies the right Ore condition and every right ideal of $\mathbb{K}[\tau]$ is principal. This statement is proved similarly as [CDSK, proposition 3.4].

We define a rational difference operator $\mathcal{R}$ as an $(m|n)$-rational difference operator if in the minimal fractional factorization $\mathcal{R} = D_0 D_1^{-1}$, $D_0, D_1$ are completely factorized over $\mathbb{K}$, and $\ord(D_0) = m$, $\ord(D_1) = n$, and $D_0, D_1$ have the same non-zero constant term.

Let $\mathcal{R}$ be an $(m|n)$-rational difference operator. Note that $\mathcal{R}$ can also be written in the form $\mathcal{R} = \tilde{D}_1^{-1} \tilde{D}_0$, where $\tilde{D}_1, \tilde{D}_0 \in \mathbb{K}[\tau], \ord(\tilde{D}_0) = m$, and $\ord(\tilde{D}_1) = n$. More generally, let $s \in S_{m|n}$ be a parity sequence. Then we call the form $\mathcal{R} = d_1 \cdots d_{m+n}$, where $d_i = 1 - f_i \tau$, $f_i \in \mathbb{R}$, $i = 1, \ldots, m + n$, a complete factorization with the parity sequence $s$. Let $\mathcal{F}(\mathcal{R})$ be the set of all complete factorizations of $\mathcal{R}$ with parity sequence $s$ and $\mathcal{F}(\mathcal{R}) = \bigcup_{s \in S_{m|n}} \mathcal{F}(\mathcal{R})$ the set of all complete factorizations of $\mathcal{R}$.

Throughout the paper, we use the following useful notation: for any $i \in \mathbb{Z}$ and $f \in \mathbb{K}$, $f[i] := \tau^i(f) = f(x - ih)$.

Define the discrete logarithmic derivative of a function $f(x)$ by $\ln'(f) = f/f[1]$. Consider two $(1|1)$-rational difference operators $\mathcal{R}_1 = (1 - a \tau)(1 - b \tau)^{-1}$ and $\mathcal{R}_2 = (1 - c \tau)^{-1}(1 - d \tau)$, where $a, b, c, d \in \mathbb{K}$, $a \neq b$, and $c \neq d$.

**Lemma 2.2.** We have $\mathcal{R}_1 = \mathcal{R}_2$ if and only if

\[
\begin{align*}
\{ a[1] = d / \ln'(c - d), \\
b[1] = c / \ln'(c - d), \quad \text{or equivalently} \quad \{ c = b[1] \ln'(a - b), \\
d = a[1] \ln'(a - b) \},
\end{align*}
\]

Let $\mathcal{R}$ be an $(m|n)$-rational difference operator with a complete factorization $\mathcal{R} = d_1 \cdots d_{m+n}$, where $d_i = 1 - f_i \tau$. Suppose $s_i \neq s_{i+1}$ and $d_i \neq d_{i+1}$. Using theorem 2.2, one constructs $\tilde{d}_i$ and $\tilde{d}_{i+1}$ such that $d_i^{\tilde{d}_i} d_{i+1}^{\tilde{d}_{i+1}} = d_i^{d_{i+1}} d_{i+1}^{d_i}$. This induces a new complete factorization of $\mathcal{R} = d_1 \cdots d_{i+1}^{\tilde{d}_i} d_{i+1}^{\tilde{d}_{i+1}} \cdots d_{m+n}$ with the new parity sequence $\tilde{s} = s^{[\tilde{d}_i]} = (s_1, \ldots, s_{i+1}, s_i, \ldots, s_{m+n})$. 

C. Huang et al.
Repeating this procedure, we see that there exists a canonical bijection between the sets of complete factorizations with respect to any two parity sequences.

2.3. Complete factorizations and superflag varieties

Let \( W = W_0 \oplus W_1 \) be a vector superspace (i.e. a \( \mathbb{Z}_2 \)-graded vector space) with \( \dim(W_0) = m \) and \( \dim(W_1) = n \). Consider a full flag \( \mathcal{F} \) of \( W \), \( \mathcal{F} = \{ F_1 \subset F_2 \subset \cdots \subset F_{m+n} = W \} \) such that \( \dim(F_i) = i \). A basis \( \{ w_1, \ldots, w_{m+n} \} \) of \( W \) generates the full flag \( \mathcal{F} \) if \( F_i \) is spanned by \( w_1, \ldots, w_i \). A full flag is called a full superflag if it is generated by a homogeneous basis. We denote by \( \mathcal{F}(W) \) the set of all full superflags.

To a homogeneous basis \( \{ w_1, \ldots, w_{m+n} \} \) of \( W \), we associate the unique parity sequence \( s \in S_m \) such that \( s_i = (-1)^{|w_i|} \). We say a full superflag \( \mathcal{F} \) has parity sequence \( s \) if it is generated by a homogeneous basis whose parity sequence is \( s \). We denote by \( \mathcal{F}^s(W) \) the set of all full superflags of parity \( s \).

Clearly, we have

\[
\mathcal{F}(W) = \bigsqcup_{s \in S_m} \mathcal{F}^s(W), \quad \mathcal{F}^s(W) \cong \mathcal{F}(W_0) \times \mathcal{F}(W_1).
\]

Given a basis \( \{ v_1, \ldots, v_m \} \) of \( W_0 \), a basis \( \{ u_1, \ldots, u_n \} \) of \( W_1 \), and a parity sequence \( s \in S_m \), define a homogeneous basis \( \{ w_1, \ldots, w_{m+n} \} \) of \( W \) by the rule \( w_i = v_i + 1 \) if \( s_i = 1 \) and \( w_i = u_i - 1 \) if \( s_i = -1 \). Conversely, any homogeneous basis of \( W \) gives a basis of \( W_0 \), a basis of \( W_1 \), and a parity sequence \( s \). We say that the basis \( \{ w_1, \ldots, w_{m+n} \} \) is associated to \( \{ v_1, \ldots, v_m \}, \{ u_1, \ldots, u_n \} \), and \( s \).

Define the discrete Wronskian \( \text{Wr} \) (or Casorati determinant) of \( g_1, \ldots, g_r \) by

\[
\text{Wr}^r(g_1, \ldots, g_r) = \det (g_i[i+(i-1)])_{i,j=1}^r = \det (g_i(x \pm (i-1)\hbar))_{i,j=1}^r.
\]

We simply write \( \text{Wr} \) for \( \text{Wr}^r \).

Let \( \mathcal{R} \) be an \( \langle m|n \rangle \)-rational difference operator over \( K \). Let \( \mathcal{R} = D_0 D_1^{-1} \) be a fractional factorization such that \( \text{ord } D_1 = n \) and the constant term of \( D_1 \) is 1. By proposition 2.1, such a fractional factorization of \( \mathcal{R} \) is unique.

Let \( V = W_0 = \ker D_0, U = W_1 = \ker D_1, W = W_0 \oplus W_1 \).

Given a basis \( \{ v_1, \ldots, v_m \} \) of \( V \), a basis \( \{ u_1, \ldots, u_n \} \) of \( U \), and a parity sequence \( s \in S_m \), define \( d_i = 1 - f_i \tau \), where

\[
\begin{align*}
  f_i &= \ln' \frac{\text{Wr}(v_1, v_2, \ldots, v_i+1,u_1,u_2,\ldots,u_n)}{\text{Wr}(v_1, v_2, \ldots, v_i,u_1,u_2,\ldots,u_n)} | \Pi |, & \text{if } s_i = 1, \\
  f_i &= \ln' \frac{\text{Wr}(v_1, v_2, \ldots, v_i,u_1,u_2,\ldots,u_n+1)}{\text{Wr}(v_1, v_2, \ldots, v_i,u_1,u_2,\ldots,u_n)} | \Pi |, & \text{if } s_i = -1.
\end{align*}
\]

Note that if two bases \( \{ v_1, \ldots, v_m \} \), \( \{ \tilde{v}_1, \ldots, \tilde{v}_m \} \) generate the same full flag of \( V \) and two bases \( \{ u_1, \ldots, u_n \} \), \( \{ \tilde{u}_1, \ldots, \tilde{u}_n \} \) generate the same full flag of \( U \), then the coefficients \( f_i \) computed from \( v, u \) and from \( \tilde{v}, \tilde{u} \) are the same.

**Proposition 2.3.** We have a complete factorization of \( \mathcal{R} \) with parity \( s \): \( \mathcal{R} = d_1^s \cdots d_{m+n}^s \).

**Proof.** The statement for the case of \( s = s_0 \) follows from [MV1].

Let \( s \) and \( \tilde{s} \) be two parity sequences which differ only in positions \( i, i+1 \). Explicitly, \( s_j = \tilde{s}_j \) for \( j \neq i, i+1 \) and \( s_i = -s_{i+1} = -\tilde{s}_i = \tilde{s}_{i+1} \). It is clear that \( d_j = \tilde{d}_j \) for \( j \neq i, i+1 \). In
addition, the equality \( d_i^{k} d_{i+1}^{k+1} = d_{i+1}^{k} d_i^{k+1} \) follows from the discrete Wronskian identity, see [MV1, lemma 9.5],

\[
\text{Wr}(\text{Wr}(v_1, v_2, \ldots, v_{k+1}, u_1, u_2, \ldots, u_{k-1})) = \text{Wr}(v_1, v_2, \ldots, v_{k-1}, u_1, u_2, \ldots, u_{k+1}) \]



By proposition 2.3, we have maps \( \varpi : \mathcal{F}(W) \to \mathcal{F}(R) \) and \( \varpi^s : \mathcal{F}(W) \to \mathcal{F}(R) \). The map \( \varpi^s \) is explicitly given as follows. Let \( \mathcal{F} \) be a superflag generated by a basis which is associated to \( \{ v_1, \ldots, v_m \} \), \( \{ u_1, \ldots, u_n \} \), and \( s \). Then

\[
\varpi^s(\mathcal{F}) = (1 - f_1 \tau)^{i_1} (1 - f_2 \tau)^{i_2} \cdots (1 - f_{m+n} \tau)^{i_{m+n}},
\]

where \( f_i \) are given by (2.2). The map \( \varpi \) is the disjoint union of maps \( \varpi^s \) over all distinct parities \( s \).

**Corollary 2.4.** The maps \( \varpi \) and \( \varpi^s \) are bijections.

Thus the set of complete factorizations of \( R \) is canonically identified with the variety of full superflags of \( W \).

### 3. XXX model

In this section we recall the definition of the super Yangian \( Y(\mathfrak{gl}_{m|n}) \) and some facts about the XXX model associated with \( Y(\mathfrak{gl}_{m|n}) \). Our main source is [BR1].

#### 3.1. Super Yangian \( Y(\mathfrak{gl}_{m|n}) \) and transfer matrix

Let \( \mathbb{C}^{m|n} \) be the complex vector superspace with \( \dim(\mathbb{C}^{m|n}) = m \) and \( \dim(\mathbb{C}^{m|n}) = n \). We choose a homogeneous basis \( e_1, \ldots, e_{m+n} \) of \( \mathbb{C}^{m|n} \) such that \( e_i = 0 \) for \( 1 \leq i \leq m \) and \( e_j = 1 \) for \( m + 1 \leq j \leq m + n \). Denote by \( E_{ij} \in \text{End}(\mathbb{C}^{m|n}) \) the linear operator of parity \( |i| + |j| \) such that \( E_{ij}e_k = \delta_{kj}e_i \) for \( 1 \leq i, j, k \leq m + n \).

The **super Yangian** \( Y(\mathfrak{gl}_{m|n}) \) is a unital associative algebra with generators \( L_{ij}^{(k)} \) of parity \( |i| + |j|, i, j = 1, \ldots, m + n, k \in \mathbb{Z}_{>0} \).

Consider the generating series

\[
L_{ij}(x) = \sum_{k=0}^{\infty} L_{ij}^{(k)} x^{-k}, \quad L_{ij}^{(0)} = \delta_{ij},
\]

and combine the series into a linear operator \( L(x) = \sum_{i,j=1}^{m+n} E_{ij} \otimes L_{ij}(x) \in \text{End}(\mathbb{C}^{m|n}) \otimes Y(\mathfrak{gl}_{m|n})[[x^{-1}]] \). The defining relations of \( Y(\mathfrak{gl}_{m|n}) \) are given by

\[
R^{(12)}(x_1 - x_2) L^{(11)}(x_1) L^{(23)}(x_2) = L^{(23)}(x_2) L^{(13)}(x_1) R^{(12)}(x_1 - x_2), \tag{3.1}
\]

where \( R(x) \in \text{End}(\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n}) \) is the super R-matrix defined by

\[
x R(x) = x \text{id} + \hbar \sum_{i,j=1}^{m+n} (-1)^{|i|} E_{ij} \otimes E_{ji}.
\]
Remark 3.1. Note that, for any non-zero $z \in \mathbb{C}^*$, the map $\mathcal{L}_y(x) \mapsto \mathcal{L}_y(x/z)$ induces an isomorphism of the super Yangians $Y(\mathfrak{gl}_{m|n})$ with different non-zero $h$. In particular, we can always rescale $h$ to 1.

The R-matrix $R(x)$ satisfies the graded Yang–Baxter equation,

$$R^{(12)}(x_1 - x_2) R^{(13)}(x_1) R^{(23)}(x_2) = R^{(23)}(x_2) R^{(13)}(x_1) R^{(12)}(x_1 - x_2).$$

The super commutator relations obtained from (3.1) are explicitly given by

$$(x_1 - x_2) [\mathcal{L}_y(x_1), \mathcal{L}_d(x_2)] = (-1)^{|x||y| + |x||d| + |y||d|} h \mathcal{L}_d(x_2) \mathcal{L}_y(x_1) - \mathcal{L}_y(x_1) \mathcal{L}_d(x_2)$$

$$= (-1)^{|x||y| + |x||d| + |y||d|} h \mathcal{L}_d(x_1) \mathcal{L}_y(x_2) - \mathcal{L}_y(x_2) \mathcal{L}_d(x_1)).$$

(3.2)

In particular, one has

$$[\mathcal{L}_{ij}^{(1)}, \mathcal{L}_{kl}(x)] = (-1)^{|k||l| + |i||j| + |i||l|} h (\delta_{ik} \mathcal{L}_{lj}(x) - \delta_{lj} \mathcal{L}_{ik}(x)).$$

(3.3)

The super Yangian $Y(\mathfrak{gl}_{m|n})$ is a Hopf algebra with the coproduct

$$\Delta : \mathcal{L}_y(x) \mapsto \sum_{k=1}^{m+n} (-1)^{|k||l| + |i||j|} \mathcal{L}_{ik}(x) \otimes \mathcal{L}_{kj}(x), \quad i, j = 1, \ldots, m + n.$$

The super Yangian $Y(\mathfrak{gl}_{m|n})$ contains the algebra $U(\mathfrak{gl}_{m|n})$ as a Hopf subalgebra. The embedding is given by the map $e_{ij} \mapsto (-1)^{|j||l|} \mathcal{L}_{ij}^{(1)}/h$ for $1 \leq i, j \leq m + n$, where $e_{ij}$, $1 \leq i, j \leq m + n$, are the standard generators of the Lie superalgebra $\mathfrak{gl}_{m|n}$ with the supercommutator relations

$$[e_{ij}, e_{kl}] = \delta_{ik} e_{lj} - (-1)^{|i||j| + |k||l|} \delta_{lj} e_{ik}.$$

We identify $U(\mathfrak{gl}_{m|n})$ with the image of this map.

The transfer matrix $T(x)$ is defined as the supertrace of $\mathcal{L}(x)$,

$$T(x) = \text{str}(\mathcal{L}(x)) = \sum_{i=1}^{m+n} (-1)^{|i||l|} \mathcal{L}_{ii}(x).$$

It is known that the transfer matrices commute, $[T(x_1), T(x_2)] = 0$. Moreover, the transfer matrix $T(x)$ commutes with the subalgebra $U(\mathfrak{gl}_{m|n})$. Since the transfer matrices commute, the transfer matrix can be considered as a generating function of integrals of motion of an integrable system.

For any complex number $z \in \mathbb{C}$, there is an automorphism

$$\zeta_z : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m|n}), \quad \mathcal{L}_y(x) \to \mathcal{L}_y(x - z),$$

where $(x - z)^{-1}$ is expanded as a power series in $x^{-1}$. The evaluation homomorphism $ev : Y(\mathfrak{gl}_{m|n}) \to U(\mathfrak{gl}_{m|n})$ is defined by the rule:

$$\mathcal{L}_y^{(a)} \mapsto (-1)^{|i||l|} \delta_{i\alpha} h e_{ij},$$

for $a \in \mathbb{Z}_{\geq 0}$.

For any $\mathfrak{gl}_{m|n}$-module $V$ denote by $V(z)$ the $Y(\mathfrak{gl}_{m|n})$-module obtained by pulling back of $V$ through the homomorphism $ev \circ \zeta_z$. The module $V(z)$ is called the evaluation module with the evaluation point $z$.

Let $V$ be a $Y(\mathfrak{gl}_{m|n})$-module. Given a parity sequence $s \in S_{m|n}$, a non-zero vector $v \in V$ is called an $s$-singular vector if

$$\mathcal{L}_y^{(s)}(x)v = \Lambda_i(x)v, \quad L_y^{(s)}(x)v = 0, \quad i > j,$$
where $\Lambda_l(x) \in \mathbb{C}[x^{-1}]$ and $L_{\sigma,a,b}(x) = L_{\sigma(a),\sigma(b)}(x)$.

**Example 3.2.** Let $L_1$ be an irreducible polynomial $gl(n|n)$-module of highest weight $\lambda$ with highest weight vector $\nu_1$. Let $z$ be a complex number. Then the $gl(n|n)$ $s$-singular vector $v^s_\lambda \in L_\lambda(z)$ is a $Y(gl(n|n))$ $s$-singular vector. Moreover, we have

$$L_{\mu}(x)v^s_\lambda = \left(1 + \frac{s_i \lambda_i'(e_{ij})h}{x - z}\right)v^s_\lambda = \frac{x - z + s_i \lambda_i'(e_{ij})h}{x - z}v^s_\lambda, \quad i = 1, 2, \ldots, m + n,$$

where $e_{ij}^s = e_{\sigma(i),\sigma(j)}$.



3.2. Bethe ansatz equation

We fix a parity sequence $s \in S_{m|n}$, a sequence $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ of polynomial $gl(m|n)$ weights, and a sequence $z = (z_1, \ldots, z_p)$ of complex numbers. We call $(\lambda^{(k)})^s$, see [HMVY, section 2.3], the weight at point $z_k$ with respect to $s$. We simply write $\lambda^{(k)}_i$ for $(\lambda^{(k)})^s_i(e_{ij})$.

Let $(\cdot)$ be the non-degenerate invariant bilinear form on $gl(m|n)$ given by $(e_{ij}, e_{kl}) = (-1)^{|i|} \delta_{ik}\delta_{jl}$. Let $h$ be the Cartan subalgebra of $gl(m|n)$ generated by $e_{ii}$, $i = 1, \ldots, m + n$. Let $e_i \in h^*$, $i = 1, \ldots, m + n$, be such that $e_i(e_{ij}) = \delta_{ij}$. Define the $s$-positive simple root $\alpha_i^s = s\alpha_i - s\alpha_{i+1}$, see [HMVY, section 2.2].

Let $I = (l_1, \ldots, l_{m+n-1})$ be a sequence of non-negative integers. Define $l = \sum_{i=1}^{m+n-1} l_i$. Let $t = (t^{(1)}_1, \ldots, t^{(1)}_l, \ldots, t^{(m+n-1)}_1, \ldots, t^{(m+n-1)}_{l_{m+n-1}})$ be a collection of variables. We say that $t^{(j)}_i$ has color $i$. Define the $gl(m|n)$ weight at $\infty$ with respect to $s$, $\lambda$, and $l$ by

$$\lambda^{(r, \infty)} = \sum_{k=1}^{p} (\lambda^{(k)})^s - \sum_{i=1}^{m+n-1} l_i \alpha_i^s.$$

The Bethe ansatz equation (BAE) associated to $s$, $z$, $\lambda$, and $I$, is a system of algebraic equations in variables $t$:

\[
\prod_{k=1}^{p} \frac{t^{(j)}_i - z_k + s_k \lambda^{(k)}_i h}{t^{(j)}_i - z_k + s_k \lambda^{(k)}_i h} \prod_{r=1}^{l} \frac{t^{(j)}_i - t^{(j-1)}_i - s_i h}{t^{(j)}_i - t^{(j-1)}_i - s_i h} \prod_{r=1}^{l_{m+n-1}} \frac{t^{(j)}_i - t^{(j+1)}_i - s_i h}{t^{(j)}_i - t^{(j+1)}_i - s_i h} = 1,
\]

where $i = 1, \ldots, m + n - 1$, $j = 1, \ldots, l_i$. We call the single equation (3.4) the BAE for $t$ related to $t^{(j)}_i$.

We allow the following cancellations in the BAE,

\[
\frac{t^{(j)}_i - z_k + s_k \lambda^{(k)}_i h}{t^{(j)}_i - z_k + s_k \lambda^{(k)}_i h} = 1, \quad \text{if } s_k \lambda^{(k)}_i = s_{i+1} \lambda^{(k)}_{i+1}, \quad \frac{t^{(j)}_i - t^{(j)}_i - s_i h}{t^{(j)}_i - t^{(j)}_i - s_i h} = 1, \quad \text{if } s_i = -s_{i+1}.
\]

After these cancellations, we consider only the solutions that do not make the remaining denominators in (3.4) vanish.

In addition, we impose the following condition. Suppose $(\alpha_i^s, \alpha_i^s) = 0$ for some $i$. Consider the BAE for $t$ related to $t^{(j)}_i$ with all $t^{(j)}_i$ fixed, where $a \neq i$ and $1 \leq b \leq l_a$, this equation does not depend on $j$. Let $t^{(j)}_i$ be a solution of this equation with multiplicity $r$. Then we require that the number of $j$ such that $t^{(j)}_i = t^{(j)}_i$ is at most $r$, see lemma 4.3, theorem 5.1.

The group $G_l = G_1 \times \cdots \times G_{l_{m+n-1}}$, acts on $t$ by permuting the variables of the same color.
We do not distinguish between solutions of the BAE in the same $\mathcal{G}_T$-orbit.

**Remark 3.3.** Note that in the quasiclassical limit $h \to 0$, system (3.4) becomes system (4.2) of [MVY], which is the Bethe ansatz equation of Gaudin model associated to $\mathfrak{gl}_m|n$. \hfill \Box

### 3.3. Bethe vector

Let $\lambda = (\lambda(1), \ldots, \lambda(p))$ be a sequence of polynomial $\mathfrak{gl}_m|n$ weights. Let $v^t_{\lambda} = v^t_{\lambda(1)} \otimes \cdots \otimes v^t_{\lambda(p)}$ be an $s$-singular vector in the irreducible $\mathfrak{gl}_m|n$-module $L_\lambda$. Consider the tensor product of evaluation modules $L(\lambda, z) = \bigotimes_{k=1}^p L_\lambda(z_k)$. We also denote by $L(\lambda)$ the corresponding $\mathfrak{gl}_m|n$-module.

Let $I = (l_1, \ldots, l_{m+n-1})$ be a collection of non-negative integers. The weight function is a vector $w^I(t, z)$ in $L(\lambda, z)$ depending on variables $t = (t_1^{(1)}, \ldots, t_1^{(m+n-1)}; \ldots; t_l^{(m+n-1)})$ and parameters $z = (z_1, \ldots, z_p)$. The weight function $w^I(t, z)$ is constructed as follows, see [BR1, section 5.2].

Set $t^a = l_1 + \cdots + l_{a-1}$, $a = 1, \ldots, m+n$. Note that $l = t^m+n$. Consider a series in $l$ variables $t$ with coefficients in $Y(\mathfrak{gl}_m|n)$:

$$
\mathcal{B}^I_l(t) = (\text{str}_{12} \cdots l \otimes \text{id}) \left( \mathcal{L}^{(1)}(t_1) \cdots \mathcal{L}^{(l-1)}(t_l) \mathcal{L}^{(l+1)}(t_{l+1}) \cdots \mathcal{L}^{(m+n-1)}(t_{m+n-1}) \right)
\times \mathcal{G}^{(1 \ldots l)}(t) E_{m+n,m+n-1}^{a_1 a_{m+n-1}} \otimes \cdots \otimes E_{21}^{a_2 a_1} \otimes \mathcal{I}^{1},
$$

where

$$
\mathcal{G}^{(1 \ldots l)}(t) = \prod_{a < b} \prod_{1 \leq j \leq l} \prod_{i \leq a \leq b} \frac{t_{j}^{(b)} - t_{j}^{(a)}}{t_{j}^{(b)} - t_{j}^{(a)} + s_a h} R^{(t^{(a)} - t^{(b)} + s_a h)}(t_j^{(b)} - t_j^{(a)})^{(a)}
$$

and the first product in (3.6) runs over $1 \leq a < b \leq m + n - 1$.

The weight function $w^I(t, z) \in L(\lambda, z)$ is given by

$$
w^I(t, z) = \mathcal{B}^I_l(t) (v^1_1 \otimes \cdots \otimes v^p_p).
$$

**Example 3.4.** Let $m + n = 2$ and $t = (t_1, \ldots, t_l)$, then

$$
w^I(t, z) = (-1)^{[2]} \mathcal{L}^{(1)}_1(t_1) \cdots \mathcal{L}^{(l)}_1(t_l) (v^1_1 \otimes \cdots \otimes v^p_p)
$$

is an example of the weight function. \hfill \Box

The following theorem is known.

**Theorem 3.5 ([BR1]).** Suppose that $\lambda$ is a sequence of polynomial $\mathfrak{gl}_m|n$ weights and $t$ a solution of the BAE associated to $s$, $z$, $\lambda$, and $I$. If the vector $w^I(t, z) \in L(\lambda, z)$ is well-defined and non-zero, then $w^I(t, z) \in L(\lambda, z)$ is an eigenvector of the transfer matrix $T(x)$, $T(x)w^I(t, z) = \mathcal{E}(x)w^I(t, z)$, where the eigenvalue $\mathcal{E}(x)$ is given by

$$
\mathcal{E}(x) = \sum_{a=1}^{m+n} \prod_{k=1}^p \frac{1}{x - z_k + s_a \lambda_a^{(k)}} \frac{1}{x - z_k} \prod_{j=1}^{l-1} \frac{1}{x - t_j^{(a-1)}} \prod_{j=1}^l \frac{1}{x - t_j^{(a)} - s_a h}.
$$

Note that the eigenvalue $\mathcal{E}(x)$ depends on the parameters $t$, $s$, $z$, and $\lambda$. We drop this dependence for our notation.
If \( t \) is a solution of the BAE associated to \( s, z, \lambda, \) and \( l \), then the value of the weight function \( w^s(t, z) \) is called the Bethe vector.

We have the following standard statement regarding to Bethe vectors, see [MTV1, proposition 6.2] and [MVY, theorem 4.3].

**Proposition 3.6.** The Bethe vector \( w^s(t, z) \) is a \( \mathfrak{gl}_{m+n} \) \( s \)-singular vector of weight \( \lambda^{(s, \infty)} \).

**Proof.** Clearly, the Bethe vector \( w^s(t, z) \) is a vector of weight \( \lambda^{(s, \infty)} \). We then show that \( w^s(t, z) \) is \( \mathfrak{gl}_{m+n} \) \( s \)-singular.

We show it for the case of \( m = n = 1 \) with the standard parity \( s_0 \) in section A.2. The general case follows from a similar computation using a combination of nested Bethe ansatz, as in [BR1, section 4], and induction on \( m + n \), see e.g. [MTV1, proposition 6.2]. \( \square \)

### 3.4. Sequences of polynomials

We use the following convenient notation. We say that a sequence \( z = (z_1, \ldots, z_p) \) of complex numbers is \( h \)-generic if \( z_i - z_j \not\in h \mathbb{Z} \) for all \( 1 \leq i < j \leq p \).

Let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)}) \) be a sequence of polynomial \( \mathfrak{gl}_{m+n} \) weights. Let \( z = (z_1, \ldots, z_p) \) be an \( h \)-generic sequence of complex numbers. Fix a parity sequence \( s \in S_{m+n} \).

Define a sequence of polynomials \( T^s = (T^s_1, \ldots, T^s_{m+n}) \) associated to \( s, \lambda \) and \( z \),

\[
T^s_i(x) = \prod_{k=1}^p \prod_{j=1}^i (x - z_k + sjh), \quad i = 1, \ldots, m + n. \tag{3.9}
\]

Note that \( T^s_i(T^s_{i+1})^{-s_i h + 1} \) is a polynomial for all \( i = 1, \ldots, m + n - 1 \).

Let \( l = (l_1, \ldots, l_{m+n-1}) \) be a sequence of non-negative integers.

Let \( t = (t^{(1)}_1, \ldots, t^{(1)}_{l_1}; \ldots; t^{(m+n-1)}_{1}, \ldots, t^{(m+n-1)}_{l_{m+n-1}}) \) be a sequence of complex numbers.

Define a sequence of polynomials \( y = (y_1, \ldots, y_{m+n-1}) \) by

\[
y_i(x) = \prod_{j=1}^{l_i} (x - t^{(i)}_j), \quad i = 1, \ldots, m + n - 1. \tag{3.10}
\]

We say the sequence of polynomials \( y \) represents \( t \). We have \( \text{deg} \ y_i = l_i \).

We also set \( y_0(x) = y_{m+n}(x) = 1 \).

If \( t \) is a solution of the BAE associated to \( s, z, \lambda, \) and \( l \), then the eigenvalue \( E^s(x) \) of the transfer matrix \( T(x) \) acting on the Bethe vector \( w^s(t, z) \), see (3.8), can be written in terms of \( y \) and \( T^s \). Namely, we have

\[
E^s(x) = E^s_y(x) = \sum_{a=1}^{m+n} s_a \frac{T^s_a}{T^s_n[sa]} \frac{y_{a-1}[sa]}{y_{a-1}} \frac{y_a[sa]}{y_a}. \tag{3.11}
\]

We do not consider zero polynomials \( y_i(x) \) and do not distinguish between polynomials \( y_i(x) \) and \( cy_i(x), c \in \mathbb{C}^\times \). Hence, a sequence \( y \) defines a point in \( (\mathbb{P}(\mathbb{C}[x]))^{m+n-1} \), the direct product of \( m + n - 1 \) copies of the projective space associated to the vector space of polynomials.

We say that a sequence of polynomials \( y \) is generic with respect to \( s, \lambda, \) and \( z \) if it satisfies the following conditions:
(i) if $s_0s_{i+1} = 1$, then $y_i$ has only simple roots and $y_i$ has no common roots with the polynomial $y_{i+1}$;
(ii) the polynomial $y_i$ has no common roots with polynomials $y_{i-1}$, $y_{i-1}$, and $y_{i+1}$;
(iii) all roots of $y_i$ are different from the roots of polynomial $T_i(T_{i+1})^{-1}\sim h_{i+1}$.

for $j = 1, \ldots, m + n - 1$.

4. Reproduction procedures for $\mathfrak{gl}_2$ and $\mathfrak{gl}_{1|1}$

In this section, we recall the reproduction procedure for the XXX model associated to $\mathfrak{gl}_2$ from [MV1, section 2] and define its analogue for $\mathfrak{gl}_{1|1}$. We define a rational difference operator associated to a solution of BAE. We also show that the reproduction procedure does not alter the rational difference operator and the corresponding eigenvalues obtained from theorem 3.5.

4.1. Reproduction procedure for $\mathfrak{gl}_2$

Set $m = 2$ and $n = 0$. We have the following identifications $Y(\mathfrak{gl}_{2|0}) \cong Y(\mathfrak{gl}_{0|2}) \cong Y(\mathfrak{gl}_2)$. Let $\Lambda = (\Lambda^{(1)}, \ldots, \Lambda^{(p)}) = ((a_1, b_1), \ldots, (a_p, b_p))$ be a sequence of polynomial $\mathfrak{gl}_2$ weights. We have $a_k, b_k \in \mathbb{Z}$, $a_k \geq b_k \geq 0$, $k = 1, \ldots, p$. Let $z = (z_1, \ldots, z_p)$ be an $h$-generic sequence of complex numbers. We have

$$T_1(x) = \prod_{k=1}^{p} \prod_{j=1}^{a_k} (x - z_k + jh), \quad T_2(x) = \prod_{k=1}^{p} \prod_{j=1}^{b_k} (x - z_k + jh).$$

Let $a = \deg T_1$ and $b = \deg T_2$.

Give a non-negative integer $l$ and variables $t = (t_1, \ldots, t_l)$. The BAE associated to $\Lambda$, $z$, and $l$ is simplified to

$$\prod_{k=1}^{p} \prod_{j=1}^{l_j} \frac{t_j - z_k + a_kh}{t_j - z_k + b_kh} \prod_{i=1, i \neq j}^{l} \frac{t_j - t_i + h}{t_j - t_i} = 1, \quad j = 1, \ldots, l. \quad (4.1)$$

It is known that the BAE (4.1) can be reformulated in terms of discrete Wronskian. Moreover, starting from a generic solution of BAE, one can construct a family of new solutions of the BAE in the following way.

Lemma 4.1 ([MV1]). Let $y$ be a polynomial of degree $l$ which is generic with respect to $\Lambda$ and $z$.

(i) The polynomial $y \in \mathbb{C}[x]$ represents a solution of the BAE (4.1) associated to $\Lambda$, $z$ and $l$, if and only if there exists a polynomial $\tilde{y} \in \mathbb{C}[x]$ such that

$$\text{Wr}^+(y, \tilde{y}) = T_1 T_2^{-1}. \quad (4.2)$$

(ii) If $\tilde{y}$ is generic, then $\tilde{y}$ represents a solution of the BAE associated to $\Lambda$, $z$ and $\tilde{l}$, where $\tilde{l} = \deg \tilde{y}$. \hfill \square

Almost all $\tilde{y}$ are generic with respect to $\Lambda$ and $z$, and therefore by lemma 4.1 represent solutions of the BAE (4.1). Thus, from one solution of the BAE, we obtain a family of new
solutions. Following the terminology of [MV1], we call this construction the \( gl_2 \) reproduction procedure.

Let \( P_y \) be the closure of the set containing \( y \) and all \( \tilde{y} \) as in lemma 4.1 in \( P(C[x]) \). We call \( P_y \) the \( gl_2 \) population originated at \( y \). The population \( P_y \) can be identified with the projective line \( CP^1 \) through the correspondence \( c_1 y + c_2 \tilde{y} \mapsto (c_1 : c_2) \).

The weight at infinity associated to the data \( \lambda \) and \( l \) is given by \( \lambda^{(\infty)} = (a - l, b + l) \). Suppose that the weight \( \lambda^{(\infty)} \) is dominant, namely \( 2l \leq a - b \). If \( \tilde{l} \neq l \), then the weight at infinity associated to \( \tilde{\lambda} \) and \( \tilde{l} \) is

\[
\tilde{\lambda}^{(\infty)} = (a - \tilde{l}, b + \tilde{l}) = (b + l - 1, a - l + 1) = s \cdot \lambda^{(\infty)},
\]

where \( s \in G_2 \) is the non-trivial element in the Weyl group of \( gl_2 \), and the dot denotes the shifted action.

Let \( \tilde{y} = \prod_{i=1}^{l} (x - \tilde{t}_i) \) and \( \tilde{l} = (\tilde{t}_1, \ldots, \tilde{t}_l) \). If \( y \) is generic, then by lemma 4.1, \( \tilde{l} \) is a solution of the BAE (4.1) with \( l \) replaced by \( \tilde{l} \). By proposition 3.6, the value of the weight function \( w(\tilde{l}, z) \) is a singular vector. At the same time, \( \tilde{\lambda}^{(\infty)} \) is not dominant and therefore \( w(\tilde{l}, z) = 0 \) in \( L(\lambda) \). So, in a \( gl_2 \) population only the unique polynomial (the one of the smallest degree) corresponds to an actual eigenvector in \( L(\lambda) \).

The eigenvalues corresponding to the solutions \( y \) and \( \tilde{y} \), see (3.11), are given by

\[
\mathcal{E}(x) = \frac{T_1 y[1]}{T[1y]} + \frac{T_2 y[-1]}{T[2y]} \quad \text{and} \quad \tilde{\mathcal{E}}(x) = \frac{T_1 \tilde{y}[1]}{T[1\tilde{y}]} + \frac{T_2 \tilde{y}[-1]}{T[2\tilde{y}]}.
\]

**Lemma 4.2.** The eigenvalues \( \mathcal{E}(x) \) and \( \tilde{\mathcal{E}}(x) \) are the same.

**Proof.** Note that

\[
\tilde{\mathcal{E}}(x) - \mathcal{E}(x) = \frac{\text{Wr}^+(y, \tilde{y})[1]}{y\tilde{y}} \frac{T_1}{T[1y]} - \frac{\text{Wr}^+(y, \tilde{y})}{y\tilde{y}} \frac{T_2}{T[2y]}.
\]

By (4.2), we have

\[
\frac{\text{Wr}^+(y, \tilde{y})[1]}{\text{Wr}^+(y, \tilde{y})[1]} = \frac{T_1 T_2[1]}{T[1T][2]}
\]

Therefore the lemma follows.

This fact can be reformulated in the following form.

Define a difference operator

\[
\mathcal{D}(y) = \left( 1 - \frac{T_1 y[1]}{T[1y]} \right) \left( 1 - \frac{T_2 y[-1]}{T[2y]} \right).
\]

The operator \( \mathcal{D}(y) \) does not depend on a choice of polynomial \( y \) in a population, \( \mathcal{D}(y) = \mathcal{D}(\tilde{y}) \).

**4.2. Reproduction procedure for \( gl_{1|1} \)**

Set \( m = n = 1 \). We have \( S = \{(1, -1), (-1, 1)\} \). Let \( s \) and \( \tilde{s} \) be two different parity sequences in \( S \). Let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)}) \) be a sequence of polynomial \( gl_{1|1} \) weights. For each \( k = 1, \ldots, p \), let us write \( (\lambda^{(k)})^{(1)} = (a_k, b_k) \), where \( a_k, b_k \in \mathbb{Z}_{\geq 0} \) and if \( a_k = 0 \) then
\( b_k = 0 \). Note that \( \lambda^{(k)} \) is atypical if and only if \( a_k + b_k = 0 \). Let \( z = (z_1, \ldots, z_p) \) be an \( h \)-generic sequence of complex numbers.

Let

\[
\tilde{a}_k = \begin{cases} 
  b_k + 1 & \text{if } a_k + b_k \neq 0, \\
  0 & \text{if } a_k + b_k = 0,
\end{cases} \quad \tilde{b}_k = \begin{cases} 
  a_k - 1 & \text{if } a_k + b_k \neq 0, \\
  0 & \text{if } a_k + b_k = 0.
\end{cases}
\]

Equation (3.9) becomes

\[
T_1^* = \prod_{k=1}^{p} \prod_{j=1}^{p} (x - z_k + s_{1j} h), \quad T_2^* = \prod_{k=1}^{p} \prod_{j=1}^{p} (x - z_k + s_{2j} h),
\]

\[
T_1^\prime = \prod_{k=1}^{p} \prod_{j=1}^{p} (x - \tilde{z}_k + s_{1j} \tilde{h}), \quad T_2^\prime = \prod_{k=1}^{p} \prod_{j=1}^{p} (x - \tilde{z}_k + s_{2j} \tilde{h}).
\]

Let \( a = \deg T_1^* \), \( b = \deg T_2^* \). Similarly, let \( \tilde{a} = \deg T_1^\prime \), \( \tilde{b} = \deg T_2^\prime \). Suppose the number of typical weights in \( \lambda \) is \( r \), then \( \tilde{a} = b + r \) and \( \tilde{b} = a - r \).

Let \( t \) be a non-negative integer. Let \( t = (t_1, \ldots, t_l) \) be a collection of variables. The Bethe ansatz equation associated to \( s, \lambda, z \), and \( l \), is given as follows,

\[
\prod_{k=1}^{p} \prod_{j=1}^{p} (1 - \frac{t_j - z_k + s_{1k} h}{t_j - z_k + s_{2k} h}) = 1, \quad j = 1, \ldots, l. \tag{4.3}
\]

The Bethe ansatz equation (4.3) can be rewritten in the form

\[
\varphi^s(t_j) - \psi^s(t_j) = 0,
\]

where

\[
\varphi^s = \prod_{k=1 \atop a_k+b_k \neq 0} (x - z_k + s_{1k} h), \quad \psi^s = \prod_{k=1 \atop a_k+b_k \neq 0} (x - z_k + s_{2k} h).
\]

Note that \( \varphi^s = \psi^{s[-1]} \) and \( \psi^s = \varphi^{s[-1]} \). Thus, in the case of \( \mathfrak{gl}_{1|1} \), the BAEs (4.3) associated to \( s \) and \( \tilde{s} \) coincide up to a shift.

We call a sequence of polynomial \( \mathfrak{gl}_{1|1} \) weights \( \lambda \) typical if at least one of the weights \( \lambda^{(k)} \) is typical. Note that \( \lambda \) is typical if and only if \( a + b \neq 0 \). In other words, \( \lambda \) is typical if and only if \( T_1^* T_2^* \neq 1 \).

The BAE (4.3) is reformulated as follows, see [GLM, equation (A.12)].

**Lemma 4.3.** \( \lambda \) be a polynomial of degree \( l \). Let \( \lambda \) be typical.

(i) The polynomial \( y \) represents a solution of the BAE (4.3) associated to \( s, z, \lambda, l \), if and only if there exists a polynomial \( \tilde{y} \), such that

\[
y \cdot \tilde{y}[-s_1] = \varphi^s - \psi^s. \tag{4.4}
\]

(ii) The polynomial \( \tilde{y} \) represents a solution of the BAE (4.3) associated to \( \tilde{s}, z, \lambda, l \), where \( l = \deg \tilde{y} = r - 1 - l \).

For each solution \( y \), we can construct exactly one solution \( \tilde{y} \). We call this construction the \( \mathfrak{gl}_{1|1} \) reproduction procedure.

The set \( P_s \) consisting of \( y \) and \( \tilde{y} \) is called the \( \mathfrak{gl}_{1|1} \) population originated at \( y \).
The weight at infinity associated to $s, \lambda$, and $l$ is $\lambda_{\mathfrak{g}}^{(\infty)} = (a - l, b + l)$. The weight at infinity associated to $\tilde{s}, \lambda$, and $\tilde{l}$ is $\lambda_{\tilde{\mathfrak{g}}}^{(\infty)} = (\tilde{a} - \tilde{l}, \tilde{b} + \tilde{l}) = (b + l + 1, a - l - 1)$. Thus we have $\lambda^{(\infty)} = \tilde{\lambda}^{(\infty)} + \alpha^s$. In particular, in contrast to the case of $\mathfrak{gl}_2$, both $y$ and $\tilde{y}$ correspond to actual eigenvectors of the transfer matrix.

If $\lambda$ is not typical, then all participating representations are one-dimensional, where the situation is trivial. In particular, we have $y(x) = 1$. We do not discuss this case.

4.3. Motivation for $\mathfrak{gl}_{1|1}$ reproduction procedure

Suppose $y$ and $\tilde{y}$ are in the same $\mathfrak{gl}_{1|1}$ population as in section 4.2. Parallel to the $\mathfrak{gl}_2$ reproduction procedure, we show that the eigenvalues of transfer matrix corresponding to the Bethe vectors obtained from polynomials $y$ and $\tilde{y}$ coincide.

Let $y = \prod_{i=1}^p (x - t_i)$, $\tilde{y} = \prod_{i=1}^p (x - \tilde{t}_i)$. Let $t = (t_1, \ldots, t_p)$, $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_p)$. By theorem 3.5 and (3.11), we have $T(x)w^t(t, z) = E(x)w^t(t, z)$ and $T(x)w^\tilde{t}(\tilde{t}, z) = \tilde{E}(x)w^\tilde{t}(\tilde{t}, z)$, where

$$E(x) = s_1 \frac{T_1 y[s_1]}{T_1[s_1]} + s_2 \frac{T_2 y[-s_2]}{T_2[s_2]}, \quad \tilde{E}(x) = \tilde{s}_1 \frac{T_1 y[\tilde{s}_1]}{T_1[\tilde{s}_1]} + \tilde{s}_2 \frac{T_2 y[-\tilde{s}_2]}{T_2[\tilde{s}_2]}.$$

**Lemma 4.4.** The eigenvalues $E(x)$ and $\tilde{E}(x)$ of transfer matrix are the same.

**Proof.** By (4.4), we have

$$E(x) = s_1 \frac{y[s_1]}{y} (\varphi^s - \psi^s) \prod_{a_i, b_i > 0}^p (x - z_i)^{-1} = s_1 y[s_1] y[-s_1] \prod_{a_i, b_i > 0}^p (x - z_i)^{-1},$$

and

$$\tilde{E}(x) = \tilde{s}_1 \frac{\tilde{y}[-\tilde{s}_1]}{\tilde{y}} (\varphi^s[s_1] - \psi^s[s_1]) \prod_{a_i, b_i > 0}^p (x - z_i)^{-1} = s_1 y[s_1] y[-s_1] \prod_{a_i, b_i > 0}^p (x - z_i)^{-1}.$$ 

Therefore the lemma follows. 

Define a rational difference operator:

$$R^s(y) = \left(1 - \frac{T_1 y[s_1]}{T_1[s_1]} \right)^s \left(1 - \frac{T_2 y[-s_2]}{T_2[s_2]} \right)^s.$$ 

It is clear that $R^s(y) = 1$ if $\lambda$ is not typical.

We have the following lemma.

**Lemma 4.5.** If $\lambda$ is typical, then $R^s(y)$ is a $(1|1)$-rational difference operator. Moreover, this $(1|1)$-rational difference operator is independent of a choice of a polynomial in a population, $R^s(y) = R^\tilde{s}(\tilde{y})$.

**Proof.** The lemma is proved by a direct computation using lemma 2.2 and (4.4). 


5. Reproduction procedure for $\mathfrak{g}l_{m|n}$

We define the reproduction procedure and the populations in the general case.

5.1. Reproduction procedure

Let $s \in S_{m|n}$ be a parity sequence. Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ be a sequence of polynomial $\mathfrak{g}l_{m|n}$ weights. Let $z = (z_1, \ldots, z_p)$ be an $h$-generic sequence of complex numbers. Let $T^s$ be a sequence of polynomials associated to $s, \lambda$, and $z$, see (3.9). If $s_i \neq s_{i+1}$, we also set
\[
\psi^s_i = \prod_{\lambda_i^{(a)}+\lambda_i^{(a)} = 0} (x - z_k + s_i \lambda_i^{(a)} h), \quad \psi^s_i = \prod_{\lambda_i^{(a)}+\lambda_i^{(a)} = 0} (x - z_k + s_{i+1} \lambda_i^{(a)} h).
\]

Let $l = (l_1, \ldots, l_{m+n-1})$ be a sequence of non-negative integers.

For $i \in \{1, \ldots, m+n-1\}$, set $s[i] = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{m+n})$. Set $y_0 = y_{m+n} = 1$.

For $g_1, g_2 \in \mathbb{K}$, we also use the notation
\[
W^s_i(g_1, g_2) = g_1 g_2 [-s_i] - g_2 g_1 [-s_i].
\]

We now reformulate the BAE (3.4) which allows us to construct a family of new solutions.

**Theorem 5.1.** Let $y = (y_1, \ldots, y_{m+n-1})$ be a sequence of polynomials generic with respect to $s, \lambda$, and $z$, such that $\deg y_k = l_k, k = 1, \ldots, m+n-1$.

(i) The sequence $y$ represents a solution of the BAE (3.4) associated to $s, \lambda$, and $z$ if and only if for each $i = 1, \ldots, m+n-1$, there exists a polynomial $\tilde{y}_i$, such that
\[
W^s_i(y_i, \tilde{y}_i) = T^s_i (T^s_{i+1})^{-1} y_{i-1}[-s_i] y_{i+1}, \quad \text{if} \quad s_i = s_{i+1}, \quad \text{(5.1)}
\]

\[
y_i \tilde{y}_i[-s_i] = \psi^s_i y_{i-1}[-s_i] y_{i+1} - \psi^s_{i+1} y_{i-1} y_{i+1}[-s_i], \quad \text{if} \quad s_i \neq s_{i+1}. \quad \text{(5.2)}
\]

(ii) Let $i \in \{1, \ldots, m+n-1\}$ be such that $\tilde{y}_i \neq 0$. If $y^{[i]} = (y_1, \ldots, \tilde{y}_i, \ldots, y_{m+n-1})$ is generic with respect to $s^{[i]}, \lambda$, and $z$, then $y^{[i]}$ represents a solution of the BAE associated to $s^{[i]}, \lambda$, $z$, and $l^{[i]}$, where $l^{[i]} = (l_1, \ldots, \hat{l}_i, \ldots, l_{m+n-1})$. $\tilde{l}_i = \deg \tilde{y}_i$.

**Proof.** Part (i) follows from lemmas 4.1 and 4.3.

Now we consider Part (ii). Let $y_r = \prod^{J_r}_{j=1} (x - t_j^{(r)})$ and $\tilde{y}_r = \prod^{\tilde{J}_r}_{j=1} (x - \tilde{t}_j^{(r)})$, $r = 1, \ldots, m+n-1$. Let $t = (t^{(r)})_{r=1, \ldots, m+n-1}$ and $\tilde{t} = (\tilde{t}^{(r)})_{r=1, \ldots, m+n-1}$. Clearly, the BAEs for $\tilde{t}$ and $t$ related to $t^{(r)}$ with $|r - \tilde{r}| > 1$ are the same. On the other hand, the BAE for $\tilde{t}$ related to $\tilde{t}^{(r)}$ holds by lemmas 4.1 and 4.3.

We only need to establish the BAE for $\tilde{t}$ related to $\tilde{t}^{(r-1)}$ and $\tilde{t}^{(r+1)}$. We have two main cases depending on the sign of $s_i s_{i+1}$.

Suppose $s_i = s_{i+1}$. Dividing (5.1) by $y_i [-s_i] \tilde{y}_i [-s_i]$ and evaluating at $x = \tilde{t}_j^{(r-1)} - s_i h$ and $x = \tilde{t}_j^{(r+1)}$, we obtain
Thus, the BAE for \( \tilde{t} \) related to \( t_j^{(i-1)} \) follows from the BAE for \( t \) related to \( t_j^{(i+1)} \).

If \( s_i = -s_{i+1} \), then the argument depends on \( s_{i-1}, s_{i+2} \). Here we only treat the case of \( s_{i-1} = -s_i \). All other cases are similar, we omit further details.

We prove the BAE for \( \tilde{t} \) related to \( t_j^{(i-1)} \), which has the form

\[
\prod_{a=1}^{l} \left( \frac{t_j^{(i-1)} - t_a^{(i)}}{t_j^{(i-1)} - t_a^{(i)} - s_i h} \right) = \prod_{a=1}^{l} \left( \frac{t_j^{(i-1)} - t_a^{(i)}}{t_j^{(i-1)} - t_a^{(i)} + s_i h} \right).
\]

Substituting \( x = t_j^{(i-1)} - s_i h \) and \( x = t_j^{(i-1)} \) to (5.2) and dividing, we get

\[
\frac{\tilde{y}(t_j^{(i-1)})}{y(t_j^{(i-1)} + s_i h)} = -\frac{\psi\phi}{\psi\phi} = \frac{\psi\phi}{\psi\phi} = -1. \quad (5.3)
\]

Changing \( i \) in (5.2) to \( i-1 \) (recall \( s_{i-1} = -s_i \)) and substituting \( x = t_j^{(i-1)} \), we have

\[
\frac{\psi\phi}{\psi\phi} = \frac{\psi\phi}{\psi\phi} = 1. \quad (5.5)
\]

Equation (5.3) follows from (5.4), (5.5), and the equality

\[
\frac{\psi\phi}{\psi\phi} = \frac{\psi\phi}{\psi\phi}.
\]

**Remark 5.2.** Suppose \( s_i \neq s_{i+1} \). It is not hard to see that if \( \psi\phi_{[s_{i-1}],s_{y}}{[s_{i+1}]y} \) and \( \psi\phi_{[s_{i-1}],s_{y}}{[s_{i+1}]y} \) in (5.2) have common roots, then \( \psi\phi \) is not generic with respect to \( s\psi\phi, \lambda \), and \( z \).

If \( s_i = s_{i+1} \), then starting from a solution of the BAE we construct a family of new solutions represented by sequences \( y\psi\phi \). Here we use (5.1) and the parity sequence remains unchanged. We call this construction the **bosonic reproduction procedure in ith direction**.

If \( \psi\phi_{[s_{i-1}],s_{y}}{[s_{i+1}]y} \neq \psi\phi_{[s_{i-1}],s_{y}}{[s_{i+1}]y} \), then starting from a solution of the BAE we construct a single new solution represented by \( y\psi\phi \). We use (5.2) and the parity sequence changes from \( s \) to \( s\psi\phi \). We call this construction the **fermionic reproduction procedure in ith direction**.

From the very definition of the fermionic reproduction procedure, \( (y\psi\phi)\psi\phi = y \).

If \( y\psi\phi \) is generic with respect to \( s\psi\phi, \lambda \), then by theorem 5.1 we can apply the reproduction procedure again.
Let
\[ P_{(y,s)} \subseteq (\mathbb{P}(\mathbb{C}[x]))^{m+n-1} \times S_{m|n} \]
be the closure of the set of all pairs \((y,s)\) obtained from the initial pair \((y,s)\) by repeatedly applying all possible reproductions. We call \(P_{(y,s)}\) the \(g_{m|n}\) population of solutions of the BAE associated to \(s, z, \lambda\), originated at \(y\). By definition, \(P_{(y,s)}\) is a disjoint union over parity sequences,
\[ P_{(y,s)} = \bigcup_{i \in S_n} p_{(y,s)}^i, \quad p_{(y,s)}^i = P_{(y,s)} \cap \left((\mathbb{P}(\mathbb{C}[x]))^{m+n-1} \times \{s\}\right). \]

5.2. Rational difference operator associated to population
We define a rational difference operator which does not change under the reproduction procedure.

Let \(s \in S_{m|n}\) be a parity sequence. Let \(z = (z_1, \ldots, z_p)\) be an \(h\)-generic sequence of complex numbers. Let \(\lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})\) be a sequence of polynomial \(g_{m|n}\) weights. The sequence \(T^s = (T^s_1, \ldots, T^s_{m+n})\) is given by (3.9).

Let \(y = (y_1, \ldots, y_{m+n-1})\) be a sequence of polynomials. Recall our convention that \(y_0 = y_{m+n} = 1\). Define a rational difference operator \(R^s(y)\) over \(\mathbb{K} = \mathbb{C}(x)\),
\[ R^s(y) = \prod_{1 \leq i \leq m+n} \left(1 - \frac{T^s_{y_{i-1}}[-s_i]y_i[s]_i}{T^s_{[s_i]}y_{i-1}y_i} \right)^{s_i}. \] (5.6)

The following theorem is the main result of this section.

**Theorem 5.3.** Let \(P\) be a \(g_{m|n}\) population. Then the rational difference operator \(R^s(y)\) does not depend on the choice of \(y\) in \(P\).

**Proof.** We want to show
\[
\left(1 - \frac{T^s_{y_{i-1}}[-s_i]y_i[s_i]}{T^s_{[s_i]}y_{i-1}y_i}\right)^{s_i} \left(1 - \frac{T^s_{y_{i+1}}[-s_{i+1}]y_{i+1}[s_{i+1}]}{T^s_{[s_{i+1}]}y_{i+1}y_{i+1}}\right)^{s_{i+1}} = \left(1 - \frac{T^s_{y_{i-1}}[-s_{i+1}]y_{i+1}[s_{i+1}]}{T^s_{[s_{i+1}]}y_{i+1}y_{i+1}}\right)^{s_{i+1}} \left(1 - \frac{T^s_{y_{i+1}}[-s_i]y_i[s_i]}{T^s_{[s_i]}y_{i+1}y_{i}}\right)^{s_i}.
\]

We have four cases, \((s_i, s_{i+1}) = (\pm 1, \pm 1)\). The cases of \(s_i = s_{i+1}\) are proved similarly to lemma 4.2. The case of \(s_i = -s_{i+1} = 1\) is similar to lemma 4.5. Namely, we want to show
\[
\left(1 - \frac{T^{s_i}_{y_{i-1}}[1]y_{i+1}[1]}{T^{s_i}_{[1]}y_{i-1}y_{i+1}}\right)^{-1} \left(1 - \frac{T^{s_i}_{y_{i+1}}[1]y_{i}[1]}{T^{s_i}_{[1]}y_{i}y_{i+1}}\right)^{-1} = \left(1 - \frac{T^{s_i}_{y_{i-1}}[1]y_{i+1}[1]}{T^{s_i}_{[1]}y_{i-1}y_{i+1}}\right) \left(1 - \frac{T^{s_i}_{y_{i+1}}[1]y_{i}[1]}{T^{s_i}_{[1]}y_{i}y_{i+1}}\right).
\]

This equation is proved by a direct computation using lemma 2.2 and (5.2). We only note that the following identities
\[
\frac{T_i^{[p]}(h)}{T_i^{[p]}(1)} \frac{T_i^{[p]}(1)}{T_i^{[p]}(1)} = \frac{T_i^{[p]}(1)}{T_i^{[p]}(1)} \frac{T_i^{[p]}(1)}{T_i^{[p]}(1)} = \prod_{k=1}^{P} \frac{x - z_k - h}{x - z_k}
\]

are used.

The case of \( s_i = -s_{i+1} = -1 \) is similar. \( \square \)

We denote the rational difference operator corresponding to a population \( P \) by \( \mathcal{R}_P \).

**Remark 5.4.** Taking the quasiclassical limit \( h \to 0 \), a solution \( t_h \) of BAE (3.4) tends to a solution of BAE for the Gaudin model associated to \( \mathfrak{gl}_{m|n} \) represented by a tuple \( \mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_{m+n-1}) \), see remark 3.3. Note that \( \tau = e^{-h\lambda} \), we have

\[
1 - \frac{T_i^{[p]}(h)}{T_i^{[p]}(1)} \frac{T_i^{[p]}(1)}{T_i^{[p]}(1)} \tau = h \left( \partial_x - s_i \left( \ln \frac{\mathcal{P}_i^{[p]}(\mathcal{Y}_{i+1})}{\mathcal{P}_i^{[p]}(\mathcal{Y}_i)} \right) \right) + O(h^2),
\]

where \( \mathcal{P}_i^{[p]} = \prod_{k=1}^{P} (x - z_k)^{s_i^{[p]}} \), \( \mathcal{Y}_0 = \mathcal{Y}_{m+n} = 1 \). Ignoring the terms in \( O(h^2) \) for each factor, one gets from \( \mathcal{R}_P^i(y) \) the rational pseudo-differential operator \( \mathcal{R}_P^i(\mathcal{Y}) \) defined in [HMVY, equation (6.5)]. \( \square \)

The transfer matrix \( \mathcal{T}(x) \) (associated to the vector representation) can be included in a natural commutative algebra \( \mathcal{B} \) generated by transfer matrices associated to other finite dimensional representations of \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \), see [KSZ, TZZ]. We expect that similar to the even case, the rational difference operator \( \mathcal{R}_P^i(y) \) encodes eigenvalues of algebra \( \mathcal{B} \) acting on the Bethe vector corresponding to \( y \), see [Tsu]. Then, theorem 5.3 would assert that formulas for eigenvalues of \( \mathcal{B} \) acting on \( L(\lambda, z) \) do not depend on a choice of \( y \) in the population.

Similar to lemmas 4.2 and 4.4, we show that formula for eigenvalue (3.8) or (3.11) does not change under \( \mathfrak{gl}_{m|n} \) reproduction procedure.

**Lemma 5.5.** Let \( y = (y_1, \ldots, y_{m+n-1}) \) be a sequence of polynomials such that there exists a polynomial \( \tilde{y}_i \) satisfying (5.1) if \( s_i = s_{i+1} \) or (5.2) if \( s_i = -s_{i+1} \). Then \( \mathcal{E}_y(x) = \mathcal{E}_{\tilde{y}_i}(x) \), where \( \mathcal{y}^{[p]} = (y_1, \ldots, y_{m+n-1}) \).

**Proof.** The proof is similar to proofs of lemmas 4.2 and 4.4. \( \square \)

**5.3. Example of a \( \mathfrak{gl}_{2|1} \) population**

In this section, we give an example of a population for the case of \( \mathfrak{gl}_{2|1} \).

Set \( m = 2, n = 1 \), and \( p = 3 \). There are three parity sequences in \( S_{2|1} \), namely, \( s_0 = (1, 1, -1) \), \( s_1 = (1, -1, 1) \), and \( s_2 = (-1, 1, 1) \).

Let \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \), where \( \lambda^{(i)} = (1, 1, 0) \), for \( i = 1, 2, 3 \), in standard parity sequence \( s_0 \). Let \( I = (0, 0) \) and \( y = (y_1, y_2) = (1, 1) \). We also set \( h = 1 \).

Let \( z = (0, \sqrt{2}, -\sqrt{2}) \). Our choice of \( z \) is such that \( z_i - z_j \notin h\mathbb{Z} \) for \( i \neq j \). We have \( T = T^{10} = \sqrt{3}^2 + x - 1, x^3 + 3x^2 + x - 1 \). We consider the population \( P_{(1,1)} \) of solutions of the BAE associated to \( s_0, z, \lambda \), originated at \( y \).

(i) Applying bosonic reproduction procedure in the first direction to \( y \), we have \( s_0^{[1]} = s_0 \), \( T^{10} = T \), and \( y^{[1]} = (y_1^{[1]}, y_2^{[1]}) = (x - c, 1) \), where \( c \in \mathbb{C}^{p^4} \). Note that \( y^{[1]} = (1, 1) = y \).
(ii) We then apply fermionic reproduction procedure in the second direction to \( y_1^{[1]} \). We have \((s_0)^{[2]} = s_1 \) and \( T^{s_1} = (x^3 + 3x^2 + x - 1, x^3 - 3x + x + 1, 1) \). We have
\[
(y_1^{[1]})^{[2]} = (x - c, 4x^3 - (6 + 3c)x^2 + 3cx + c + 1).
\]

(iii) Finally, apply fermionic reproduction procedure in the first direction to \((y_1^{[1]})^{[2]} \). We have \((s_1)^{[1]} = s_2 \) and \( T^{s_2} = ((x - 1)(x - 2)(x^2 - 2x - 1)(x^2 - 4x + 2), 1, 1) \). We have
\[
((y_1^{[1]})^{[2]})^{[1]} = (6(x - 1)^4 - 9(x - 1)^2 + 1, 4x^3 - (6 + 3c)x^2 + 3cx + c + 1).
\]

It is easy to check that all further reproduction procedures cannot create a new pair of polynomials. Therefore the \( \mathfrak{gl}_{2|1} \) population \( P_{(1,1)} \) is the union of three \( \mathbb{C}P^1 \), \( P_{(1,1)}^1 = \{ (x - c, 1) \mid c \in \mathbb{C}P^1 \} \), \( P_{(1,1)}^2 = \{ (x - c, 4x^3 - (6 + 3c)x^2 + 3cx + c + 1) \mid c \in \mathbb{C}P^1 \} \), and \( P_{(1,1)}^3 = \{ (6(x - 1)^4 - 9(x - 1)^2 + 1, 4x^3 - (6 + 3c)x^2 + 3cx + c + 1) \mid c \in \mathbb{C}P^1 \} \).

6. Populations and superflag varieties

In this section, we show that \( \mathfrak{gl}_{m|n} \) populations associated to typical \( \lambda \) are isomorphic to the variety of the full superflags.

6.1. Discrete exponents and dominants

Following [HMVY], we introduce the following partial ordering on the set of partitions with \( r \) parts. Let \( a = (a_1 \leq a_2 \leq \cdots \leq a_r) \) and \( b = (b_1 \leq b_2 \leq \cdots \leq b_r) \), \( a_i, b_i \in \mathbb{Z}_{\geq 0} \), be two partitions with \( r \) parts. If \( b_i \geq a_i \) for all \( i = 1, \ldots, r \), we say that \( b \) dominates \( a \).

For a partition \( a \) with \( r \) parts, we call the smallest partition with \( r \) distinct parts that dominates \( a \) the dominant of \( a \) and denote it by \( a_\lambda = (a_1 < a_2 < \cdots < a_r) \). Namely, the partition \( a_\lambda \) is such that \( a_\lambda \) dominates \( a \) and if a partition \( a' \) with \( r \) distinct parts dominates \( a \) then \( a' \) dominates \( a_\lambda \). The partition \( a_\lambda \) is unique.

We identify a set of non-negative integers with a partition by rearranging their elements into weakly increasing order.

This definition is motivated by the relation of exponents for a sum of spaces of functions to exponents of the summands. We describe this phenomenon for the discrete exponents of spaces of functions.

Let \( V \) be an \( r \)-dimensional space of functions. Let \( z \in \mathbb{C} \) be such that all functions in \( V \) are well-defined at \( z - h\mathbb{C} \). Then there exists a partition with \( r \) distinct parts \( c = (c_1 < \cdots < c_r) \) and a basis of \( \{ v_1, \ldots, v_r \} \) of \( V \) such that for \( i = 1, \ldots, r \), we have \( v_j(z - jh) = 0 \) for \( j = 1, \ldots, c_i \) and \( v_1(z - (c_i + 1)h) \neq 0 \). This sequence of integers is defined uniquely and will be called the sequence of discrete exponents of \( V \) at \( z \). We denote the set \( c \) by \( E_c(V) \).

Let \( V_1, \ldots, V_l \) be spaces of functions such that the sum \( V = \sum_{i=1}^l V_i \) is a direct sum. Let \( a_z = \bigwedge_{i=1}^l E_{c_i}(V_i) \), then \( E_c(V) \) dominates \( a_z \). Moreover, for generic spaces of functions \( V_\nu \), we have the equality \( E_c(V) = a_z \).

6.2. Space of rational functions associated to a solution of BAE

Let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)}) \) be a sequence of polynomial \( \mathfrak{gl}_{m|n} \) weights. Let \( z = (z_1, \ldots, z_p) \) be an \( h \)-generic sequence of complex numbers.
Let \( y = (y_1, \ldots, y_{m+n-1}) \) represent a solution of the BAE associated to \( \lambda, z \), and the standard parity sequence \( s_0 \). Suppose further that \( y \) is generic with respect to \( \lambda, z, s_0 \). Recall the rational difference operator \( \mathcal{R}^m(y) = D_0(y)D_1^{-1}(y) \) associated to the population \( P_{(x,n)} \) generated by \( y \), see (5.6). Let \( V_T = \ker D_0(y) \) and \( U_T = \ker D_1(y) \).

Note that the sequence \( (y_1, \ldots, y_{m-1}) \) represents a solution of the BAE associated to the Lie algebra \( gl_m \). It follows from [MV1] that one can generate a \( gl_m \) population starting from \( (y_1, \ldots, y_{m-1}) \) using bosonic reproduction procedures. Moreover, the corresponding difference operator to this population is given by \( y_m \cdot D_0(y) \cdot (y_m)^{-1} \). Therefore, by [MV1, proposition 4.7], the space \( y_m \cdot V_T \) is a \( m \)-dimensional space of polynomials. Similarly, since \( (y_m, \ldots, y_{m+n-1}) \) represents a solution of the BAE associated to the Lie algebra \( gl_m \), the space \( T_{m+1}[-1]y_m \cdot U_T \) is an \( n \)-dimensional space of polynomials. In particular, \( V_T \) and \( U_T \) are spaces of rational functions.

In the remainder of section 6, we impose the condition that \( y_m(z_i + kh) \neq 0 \) for \( i = 1, \ldots, p \) and \( k \in \mathbb{Z} \).

Since \( z \) is \( h \)-generic and \( y_m(z_i + kh) \neq 0 \) for \( 1 \leq i \leq p \) and \( k \in \mathbb{Z} \), it follows from [MTV2, corollary 7.5] that the sequence of discrete exponents \( E_{\lambda, y_m} \cdot V_T \) is given by

\[
(\lambda_m^{(i)} < \lambda_{m-1}^{(i)} + 1 < \cdots < \lambda_{m-k+1}^{(i)} + k - 1 < \cdots < \lambda_1^{(i)} + m - 1).
\]

Therefore the sequence of discrete exponents \( E_{\lambda, y_m} \cdot V_T \) is given by

\[
(\lambda_m^{(i)} + \lambda_{m-1}^{(i)} < \lambda_{m-2}^{(i)} + \lambda_{m-1}^{(i)} + 1 < \cdots < \lambda_{m-k+1}^{(i)} + \lambda_{m-1}^{(i)} + k - 1 < \cdots < \lambda_1^{(i)} + \lambda_{m-1}^{(i)} + m - 1).
\]

Similarly, the sequence of discrete exponents \( E_{\lambda, y_m} \cdot U_T \) is given by

\[
(0 < \lambda_{m+1}^{(i)} - \lambda_{m+2}^{(i)} + 1 < \cdots < \lambda_{m+n}^{(i)} - \lambda_{m+n+k}^{(i)} + k - 1 < \cdots < \lambda_{m+n}^{(i)} - \lambda_{m+n}^{(i)} + n - 1).
\]

**Lemma 6.1.** If \( \lambda \) is typical, then \( V_T \cap U_T = 0 \).

**Proof.** Since \( \lambda \) is typical, there exists some \( i_0 \in \{1, \ldots, p\} \) such that \( \lambda_m^{(i_0)} \geq n \). Therefore the largest discrete exponent of \( T_{m+1}[-1]y_m \cdot U_T \) at \( z_{i_0} + \lambda_m^{(i_0)}h \) is strictly less than the smallest discrete exponent of \( T_{m+1}[-1]y_m \cdot V_T \) at \( z_{i_0} + \lambda_m^{(i_0)}h \), namely,

\[
\lambda_{m+1}^{(i_0)} - \lambda_{m+n}^{(i_0)} + n - 1 < n + \lambda_{m+1}^{(i_0)} \leq \lambda_m^{(i_0)} + \lambda_m^{(i_0)}.
\]

Therefore, by the definition of discrete exponents, we have \( (T_{m+1}[-1]y_m \cdot U_T) \cap (T_{m+1}[-1]y_m \cdot V_T) = 0 \), which completes the proof. \( \square \)

Therefore, by proposition 2.1, the operator \( \mathcal{R}^m(y) \) is an \( (m|n) \)-rational difference operator.

**Remark 6.2.** If \( \lambda \) is not typical, then the intersection \( V_T \cap U_T \) may be non-trivial. For example, consider the tensor product of the vector representations, namely \( L(\lambda) = (\mathbb{C}^{m|n}) \otimes p \), and the sequence of polynomials \( y = (1, \ldots, 1) \). Then we have \( T_i(x) = (x - z_i + h) \cdots (x - z_p + h) \) and \( T_i(x) = 1 \) for \( i = 2, \ldots, m + n \). Therefore for the rational difference operator \( \mathcal{R}^m(y) = D_0(y)D_1^{-1}(y) \), we have

\[
D_0(y) = \left(1 - \frac{(x - z_i + h) \cdots (x - z_p + h)}{(x - z_1) \cdots (x - z_p)} \tau\right)(1 - \tau)^{m-1}, \quad D_1(y) = (1 - \tau)^n.
\]
Fix \( a \in \{0, 1, \ldots, m\} \) and \( b \in \{0, 1, \ldots, n\} \). For each \( 1 \leq i \leq p \), set

\[
A_i = (\lambda^{(i)}_m + \lambda^{(i)}_{m+1} < \lambda^{(i)}_{m-1} + \lambda^{(i)}_{m+1} + 1 < \cdots < \lambda^{(i)}_{m-a+1} + \lambda^{(i)}_{m+1} + a - 1),
\]

\[
B_i = (0 < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+2} + 1 < \cdots < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+b} + b - 1).
\]

**Lemma 6.3.** If \( b \leq \lambda^{(i)}_m \), then the dominant of \( A_i \cup B_i \) is given by

\( (0 < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+2} + 1 < \cdots < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+b} + b - 1 < \lambda^{(i)}_m + \lambda^{(i)}_{m+1} < \cdots < \lambda^{(i)}_{m-a+1} + \lambda^{(i)}_{m+1} + a - 1) \).

If \( \lambda^{(i)}_{m-j+1} < b \leq \lambda^{(i)}_{m-j} \) for some \( 1 \leq j \leq a - 1 \), then the dominant of \( A_i \cup B_i \) is given by

\( (0 < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+2} + 1 < \cdots < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+b} + b - 1 < \lambda^{(i)}_{m+1} + b < \lambda^{(i)}_{m+1} + b + 1 < \cdots < \lambda^{(i)}_{m-a+1} + \lambda^{(i)}_{m+1} + a - 1) \).

If \( \lambda^{(i)}_{m+a-1} < b \), then the dominant of \( A_i \cup B_i \) is given by

\( (0 < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+2} + 1 < \cdots < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+b} + b - 1 < \lambda^{(i)}_m + b + 1 < \cdots < \lambda^{(i)}_{m+1} + b + a - 1) \).

**Proof.** If \( b \leq \lambda^{(i)}_m \), the statement is clear. If \( \lambda^{(i)}_{m-j+1} < b \leq \lambda^{(i)}_{m-j} \) for some \( 1 \leq j \leq a - 1 \). Let \( \lambda^{(i)}_m = \ell \). Since \( \lambda^{(i)}_m \) is a polynomial \( \mathfrak{gl}_{m+n} \) weight, we have \( \lambda^{(i)}_{m+\ell+k} = 0 \) for \( k = 1, \ldots, b - \ell \). In particular, the last \( b - \ell \) numbers in \( B_i \) are consecutive integers from \( \lambda^{(i)}_{m+1} + \ell \) to \( \lambda^{(i)}_{m+1} + b - 1 \). Adding \( \lambda^{(i)}_m + \lambda^{(i)}_{m+1} \) into \( B_i \), the dominant of the new set is obtained by changing \( \lambda^{(i)}_m + \lambda^{(i)}_{m+1} \) to \( \lambda^{(i)}_{m+1} + b \). We add the numbers of \( A_i \) one by one (from left to right) into \( B_i \). Inductively, adding \( \lambda^{(i)}_{m+1} + \lambda^{(i)}_{m-k+1} + k - 1 \), if \( \lambda^{(i)}_{m-k+1} < b \), then the dominant is obtained by changing \( \lambda^{(i)}_{m+1} + \lambda^{(i)}_{m-k+1} + k - 1 \) to \( \lambda^{(i)}_{m+1} + b + k - 1 \). Therefore the lemma follows. \( \square \)

### 6.3. Polynomials \( \pi_{a,b} \)

Let \( s \in S_{m|n} \) be a parity sequence. Let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)}) \) be a sequence of polynomial \( \mathfrak{gl}_{m+n} \) weights. Let \( z = (z_1, \ldots, z_p) \) be an \( h \)-generic sequence of complex numbers. Let \( T^s \) be a sequence of polynomials associated to \( s, \lambda \), and \( z \), see (3.9). We set \( T_i = T^s_i \) the polynomials corresponding to the standard parity \( s_0 \).

Define polynomials \( \pi^\lambda_{a,b} \) by

\[
\pi^\lambda_{a,b}(x) = \prod_{k=1}^{p} \prod_{i=1}^{a} \prod_{j=1}^{\min(b, \lambda^{(i)}_{m+k})} (x - z_k + (i + j - a - b - 1)h).
\]

We often abbreviate \( \pi^\lambda_{a,b} \) to \( \pi_{a,b} \).

The polynomials \( T^s_i \) can be expressed in terms of \( T_i \) and \( \pi_{a,b} \). Recall that we have

\[
s^+_{i} = \begin{cases} 
m - \sigma_i(i), & \text{if } s_i = 1, \\
\sigma_i(i) - i, & \text{if } s_i = -1, \end{cases}
\]

\[
s^-_{i} = \begin{cases} 
i - \sigma_i(i), & \text{if } s_i = 1, \\
\sigma_i(i) - m - 1, & \text{if } s_i = -1. \end{cases}
\]

21
Theorem 6.4. We have

\[ T_i^s = T_{\sigma(i)}[s_i^-] \frac{\pi_{k^+}^s x_i^-}{\pi_{k^+}^s x_i^+ + 1}, \text{ if } s_i = 1; \quad T_i^s = T_{\sigma(i)}[s_i^+] \frac{\pi_{k^+}^s x_i^+ + 1}{\pi_{k^+}^s x_i^+ + 1}, \text{ if } s_i = -1. \]

Proof. It is not hard to see that

\[ \lambda_j^{(k_x)} = \begin{cases} \lambda^{(k)}_{\sigma(i)} - \min \{ s_i^-, \lambda^{(k)}_{\sigma(i)} \}, & \text{if } s_i = 1, \\ \lambda^{(k)}_{\sigma(i)} + \# \{ j \mid \lambda^{(k)}_{m-j+1} > s_i^-, j = 1, \ldots, s_i^+ \}, & \text{if } s_i = -1. \end{cases} \]

The theorem follows from a direct computation. \[ \square \]

Note that polynomials \( \pi_{a,b} \) are discrete versions of \( \pi_{a,b} \) in [HMVY, equation (7.1)], even though our definition here is more explicit. In particular, theorem 6.4 is the counterpart of [HMVY, theorem 7.2].

The polynomial \( \pi_{a,b} \) is related to the dominants of \( A_i \sqcup B_i \) for all \( 1 \leq i \leq p \). Write the dominant \( A_i \sqcup B_i \) of \( A_i \sqcup B_i \) as

\[ 0 = c^{(i)}_{a+b} < c^{(i)}_{a+b-1} + 1 < \cdots < c^{(i)}_{a+b-j} + j < \cdots < c^{(i)}_1 + a + b - 1, \]

where \( c^{(i)}_j \) are computed explicitly from lemma 6.3. Let \( z_i = z_j + \lambda^{(i)}_{m+1} h \) and set

\[ \mathcal{F}(x) = \prod_{k=1}^p \prod_{j=0}^{c^{(i)}_j} (x - z_k + jh). \] (6.1)

Proposition 6.5. We have

\[ \pi_{a,b} \prod_{j=1}^a \mathcal{F}(j) = \prod_{i=1}^a (T_m-a+b)[b+i]T_{m+1}[i-1]. \]

Proof. The lemma is obtained from lemma 6.3 by a direct computation. \[ \square \]

6.4. Generating map

Recall the notation from the beginning of section 6.2, where \( V_y = \ker D_0(y) \) and \( U_y = \ker D_1(y) \).

For \( a \in \{0, 1, \ldots, m\}, b \in \{0, 1, \ldots, n\}, \nu_1, \ldots, \nu_a \in V_y, u_1, \ldots, u_b \in U_y \), we define the function

\[ y_{a,b} = \text{Wr}(\nu_1, \ldots, \nu_a, u_1, \ldots, u_b)[1] \pi_{a,b} y_{m}[a+b] \frac{T_{m+1}[a+b-1] \cdots T_{m+b[a]}[b+1]}{T_m[a+b] \cdots T_{m-a+1}[b+1]} \]

We impose the technical condition that \( y_{m} \) has only simple roots and is relatively prime to \( y_{m}[k] \) for all non-zero integers \( k \).

Proposition 6.6. The function \( y_{a,b} \) is a polynomial.

Proof. This proposition is proved in section 6.5. \[ \square \]
In the following, we assume that $\lambda$ is typical. Set $W_f = V_f \oplus U_f$. Given a parity sequence $s$ and a full superflag $F \in \mathcal{F}(W_f)$ generated by a homogeneous basis $\{w_1, \ldots, w_{m+n}\}$, we define polynomials $y_i(F)$, $i = 1, \ldots, m+n-1$, by the formula

$$y_i(F) = \begin{cases} y_i^{s_i} & \text{if } s_i = 1, \\ y_i^{s_i+1} & \text{if } s_i = -1, \end{cases}$$

where we choose $\{v_1, \ldots, v_m\}$ and $\{u_1, \ldots, u_n\}$ such that the basis $\{w_1, \ldots, w_{m+n}\}$ is associated to $\{v_1, \ldots, v_m\}, \{u_1, \ldots, u_n\}$, and $s$, see section 2.3.

Define the generating map by

$$\beta^s : \mathcal{F}(W_f) \to (P(\mathbb{C}[x]))^{m+n-1}, \quad F \mapsto y(F) = (y_1(F), \ldots, y_{m+n-1}(F)).$$

The following theorem is our main result of this section.

**Theorem 6.7.** For any superflag $F \in \mathcal{F}(W_f)$, we have $\beta^s(F) \in \mathcal{P}_s^{(Y_{s_0})}$. Moreover, the generating map $\beta^s : \mathcal{F}(W_f) \to \mathcal{P}_s^{(Y_{s_0})}$ is a bijection and the complete factorization $\omega^s(\mathcal{F})$ of $\mathcal{R}^s(\mathcal{F})$ given by (5.6) coincides with $\mathcal{R}^s(\beta^s(F))$ given by (5.6).

**Proof.** Note that the even case of this theorem is proved in [MV1, theorem 4.16]. Due to theorem 6.4 and proposition 6.6, the proof is parallel to that of [HMYV, theorem 7.9].

This theorem does not rely on the technical condition imposed above proposition 6.6, see remark 6.10.

### 6.6. Proof of proposition 6.6

We prepare several lemmas which will be used in the proof.

**Lemma 6.8.** For any $v \in V_f, u \in U_f$, the function $T_{m+1}y_{m}[1]Wt(v, u)$ is a polynomial. In particular, if $v \in V_f, u \in U_f$ are not regular at $z$, then there exists a $c \in \mathbb{C}$ such that $(u + cv)(z - h) = 0$.

**Proof.** The case of $g_{\{1\}}$ is clear. Now we assume that either $m \geq 2$ or $n \geq 2$.

If the fermionic reproduction in the $m$th direction is not applicable, then we can slightly change $y_{m-1}$ or $y_{m+1}$ using bosonic reproduction procedure such that the fermionic reproduction in the $m$th direction can be applied to the new tuple of polynomials $\tilde{y}$. Therefore we can assume that the fermionic reproduction in the $m$th direction is applicable to $y$ at the beginning.

It follows from (2.2) and theorem 5.1 that

$$T_{m+1}y_{m}[1]Wt(v, u) = T_{m+1}y_{m+1}[1].$$

Here $\tilde{y}$ depends on $u$ and $v$.

Initially, we have $v(y) = T_{m}y_{m-1}[1]/y_{m}$ and $u(y) = y_{m+1}[1]/(T_{m+1}y_{m+1}[1]y_{m})$. Generic $u$ and $v$ can be obtained from $y$ using only bosonic reproduction procedures. Moreover, the polynomial $y_{m}$ never changes. Note that, by theorem 5.1, $\tilde{y}_{m}$ is a polynomial for generic $u$ and $v$. Therefore the first part of the lemma follows.

Recall that $y_{m}$ has only simple zeros and $v$ is relatively prime to $y_{m}[1]$. In addition, none of zeros of $y_{m}$ belongs to the sets $z_k + h\mathbb{Z}$, $k = 1, \ldots, p$. If $v \in V_f, u \in U_f$ are not regular at $z$, then $z$ is a root of $y_{m}$. Moreover, $v$ and $u$ have simple pole at $x = z$. The second statement follows directly from the first statement.
Suppose $V$ is an $r$-dimensional space of polynomials with the sequence of discrete exponents at $z$ given by $c_r < c_{r-1} + 1 < \cdots < c_{r-i} + i < \cdots < c_1 + r - 1$. Let $\mathcal{F}_y(x) = (x - z + h) \cdots (x - z + ch), i = 1, \ldots, r$.

The following lemma is well-known, see e.g. [MTV3, theorem 3.3].

**Lemma 6.9.** Let $f_1, \ldots, f_s \in V$, then $\text{Wr}(f_1, \ldots, f_s)$ is divisible by $\prod_{j=1}^r \mathcal{F}_{r-1-j}[i-j]$. □

**Proof of proposition 6.6.** Clearly, we only need to consider the case when $v_1, \ldots, v_a, u_1, \ldots, u_b$ are linearly independent. The rational function $y_{a,b}$ can only have poles at $z_i + h\mathbb{Z}$, $1 \leq i \leq p$, and at zeros of the product of polynomials $\prod_{j=1}^{a+b} y_m[j]$. Therefore, it follows from theorem 5.1 that $\text{Wr}(v_1, \ldots, v_a, u_1, \ldots, u_b)$ is divisible by $\prod_{j=1}^{a+b} \mathcal{F}_{r-1-j}[j-1]$, where $\mathcal{F}_y$ are defined in (6.1). It follows from proposition 6.5 that the function $y_{a,b}$ is regular at $z_i + h\mathbb{Z}$, $1 \leq i \leq p$.

Write $y_m = \prod_{j=1}^r (x - z_j' + h)$, then by assumption $z_i' - z_j' \notin h\mathbb{Z}$ for $1 \leq i < j \leq r$. It follows from [MTV2, corollary 7.5] that $E_{c_i} (\text{span}(\tilde{v}_1, \ldots, \tilde{v}_a))$ dominates the partition $(0 < 2 < 3 < \cdots < a)$ with $a$ parts and $E_{c_i} (\text{span}(\tilde{u}_1, \ldots, \tilde{u}_b))$ dominates the partition $(0 < 2 < 3 < \cdots < b)$ with $b$ parts. Therefore, it follows from lemma 6.8 that $E_{c_i} (W_{a,b})$ dominates the partition $(0 < 2 < 3 < \cdots < a + b)$ with $a + b$ parts. Hence, by lemma 6.9, $\text{Wr}(\tilde{v}_1, \ldots, \tilde{v}_a, \tilde{u}_1, \ldots, \tilde{u}_b)$ is divisible by $\prod_{j=2}^{a+b} y_m[j-2]$. In particular, $\text{Wr}(v_1, \ldots, v_a, u_1, \ldots, u_b) y_m[a + b - 1]$ is regular at zeros of the product of polynomials $\prod_{j=1}^{a+b} y_m[j-1]$. □

**Remark 6.10.** If $\lambda$ is typical, the proof of proposition 6.6 can be simplified as follows. Since $\lambda$ is typical, generically the reproduction procedure is applicable for all parity sequences and all directions. Therefore, it follows from theorem 5.1 that $y_{a,b}$ is a polynomial for generic $v_1, \ldots, v_a, u_1, \ldots, u_b$. Hence $y_{a,b}$ is a polynomial for all $v_1, \ldots, v_a, u_1, \ldots, u_b$. □

7. Quasi-periodic case

In this section, we generalize our results to the quasi-periodic case.

7.1. Twisted transfer matrix and Bethe ansatz

We follow the notation in section 3.2.

Let $\kappa = (\kappa_1, \ldots, \kappa_{m+n})$ be a sequence of complex numbers such that $e^{i\kappa_i} \neq e^{i\kappa_j}$ for $1 \leq i < j \leq m + n$. Let $Q_\kappa$ be the diagonal matrix $\text{diag}(e^{i\kappa_1}, \ldots, e^{i\kappa_{m+n}})$. Define the twisted transfer matrix $T_\kappa(x)$ by

$$T_\kappa(x) = \text{str}(Q_\kappa \mathcal{L}(x)) = \sum_{i=1}^{m+n} (-1)^i e^{i\kappa_i} \mathcal{L}_i(x).$$

It is known that the twisted transfer matrices commute, $[T_\kappa(x_1), T_\kappa(x_2)] = 0$. Moreover, $T_\kappa(x)$ commutes with the subalgebra $U(h)$.

The Bethe ansatz equation associated to $s, z, \lambda, \kappa$, and $l$ is a system of algebraic equations in variables $t$:  

24
where \( i = 1, \ldots, m + n - 1, \ j = 1, \ldots, l_i \).

After making cancellations as in (3.5), we require that the solutions do not make the remaining denominators in (7.1) vanish.

We also impose the same condition, as see section 3.2, for variables which correspond to a simple odd root of the same color. Suppose \( (\alpha_i^*, \alpha_i^*) = 0 \) for some \( i \). Consider the BAE for \( t \) related to \( t_j^{(a)} \) with all \( t_j^{(a)} \) fixed, where \( a \neq i \) and \( 1 \leq b \leq l_a \), this equation does not depend on \( j \). Let \( t_j^{(a)} \) be a solution of this equation with multiplicity \( r \). Then we require that the number of \( j \) such that \( t_j^{(a)} = t_j^{(b)} \) is at most \( \tilde{r}_{si} \).

Suppose that \( \lambda \) is a sequence of polynomial \( \text{gl}_{m+n} \) weights and \( t \) a solution of the BAE (7.1) associated to \( s, z, \lambda, \kappa, \) and \( l_i \) to (7.1) and part (ii) are similar to that of theorem 5.1.

Proof. (ii) If \( \kappa(x) = \kappa(1) \) is well-defined and non-zero, then \( w^s(t, z) \in L(\lambda, z) \) which is an eigenvector of twisted transfer matrix, \( T_\kappa(x)w^s(t, z) = E_\kappa(x)w^s(t, z) \), where the eigenvalue \( E_\kappa(x) \) is given by

\[
E_\kappa(x) = \sum \left\{ \sum_{a=1}^{m+n} \frac{s_a e^{bs_a}}{T^n_a \sigma_a \frac{y_{a-1}}{y_a}} \cdot \sum_{a=1}^{m+n} \frac{s_a e^{b}}{T^n_a \sigma_a \frac{y_{a-1}}{y_a}} \right\} .
\]

Let \( y = (y_1, \ldots, y_{m+n-1}) \) be a sequence of polynomials representing the solution \( t \), then

\[
E_\kappa(x) = E_{(y, x)}(x) = \sum_{a=1}^{m+n} s_a e^{bs_a} \frac{T^n_a \sigma_a \frac{y_{a-1}}{y_a}}{y_{a-1}} .
\]

7.2. Reproduction procedure and rational difference operators

Recall the notation given at the beginning of section 5.1. Set \( \kappa^{[i]} = (\kappa_1, \ldots, \kappa_i, \kappa_{i+1}, \ldots, \kappa_n) \).

Theorem 7.1. Let \( y = (y_1, \ldots, y_{m+n-1}) \) be a sequence of polynomials generic with respect to \( s, \lambda, z, \) and \( \kappa \) such that \( \deg y_k = l_k, k = 1, \ldots, m + n - 1 \).

(i) The sequence \( y \) represents a solution of the BAE (7.1) associated to \( s, z, \lambda, \kappa, \) and \( l_i \), if and only if for each \( i = 1, \ldots, m + n - 1 \), there exists a unique polynomial \( \tilde{y}_i \), such that

\[
\text{Wr}^s \left( y_i e^{(\kappa_i - \kappa_i + 1)^2} \right) = e^{(\kappa_i - \kappa_i + 1)^2} \left( T_\kappa^{[i]} \right)^{-1} y_{i-1} \left[ -s_i \right] y_{i+1} , \text{if } s_i = s_{i+1} ,
\]

\[
y_i \tilde{y}_i \left[ -s_i \right] = e^{b s_i} e^{b \kappa_i s_i} y_i \left[ -s_i \right] y_{i+1} e^{b s_i + b \kappa_i s_i} y_{i+1} \left[ -s_i \right] , \text{if } s_i \neq s_{i+1} .
\]

(ii) If \( y^{[i]} = (y_1, \ldots, \tilde{y}_i, \ldots, y_{m+n-1}) \) is generic with respect to \( s^{[i]}, \lambda, \) and \( z, \) then \( y^{[i]} \) represents a solution of the BAE (7.1) associated to \( s^{[i]}, \lambda, \kappa^{[i]}, \) and \( l^{[i]} \), where \( l^{[i]} = (l_1, \ldots, l_i, \ldots, l_{m+n-1}) \).

Proof. For part (i), the case of (7.2) is proved in [MV3, theorem 7.4]. The proofs of (7.3) in part (i) and part (ii) are similar to that of theorem 5.1.

\[\square\]
Thanks to theorem 7.1, we define similarly the twisted bosonic and fermionic reproduction procedures in \(i\)th direction, the twisted \(\mathfrak{gl}_{m+n}\) population \(P(\mathbf{y}, \kappa)\) of solutions of the BAE associated to \(s, z, \lambda\), originated at \((\mathbf{y}, \kappa)\). Here the reproduction procedure in \(i\)th direction sends \((\mathbf{y}, \kappa)\) to \((\mathbf{y}^{[i]}, \kappa^{[i]})\). Note that for both twisted bosonic and fermionic reproduction procedures, the sequence \(\kappa\) is changed to \(\kappa^{[i]}\).

Define a rational difference operator \(\mathcal{R}^i(\mathbf{y}, \kappa)\) over \(\mathbb{K} = \mathbb{C}(x)\),

\[
\mathcal{R}^i(\mathbf{y}, \kappa) = \prod_{1 \leq i \leq m+n} \left( 1 - e^{bc_i} \frac{T_i^m y_{i-1}^{[-si]y_i^{[s_i]}}}{T_i^n [s_i]y_{i-1} y_i^l} \right)^{s_i}.
\]  

(7.4)

**Theorem 7.2.** Let \(P\) be a twisted \(\mathfrak{gl}_{m+n}\) population. Then the rational difference operator \(\mathcal{R}^i(\mathbf{y}, \kappa)\) does not depend on a choice of \((\mathbf{y}, \kappa)\) in \(P\).

**Proof.** The proof is similar to that of theorem 5.3. \(\square\)

**Proposition 7.3.** Let \(\mathbf{y} = (y_1, \ldots, y_{m+n-1})\) be a sequence of polynomials such that there exists a sequence of polynomials \(\mathbf{y}^{(i]} = (y_1, \ldots, y_i, \ldots, y_{m+n-1})\) satisfying (7.2) if \(si = s_{i+1}\) or (7.3) if \(si = -s_{i+1}\). Then \(E_{(y, \kappa)}(x) = E_{y^{(i]}, \kappa^{[i]}}(x)\).

**Proof.** The proof is similar to proofs of lemmas 4.2 and 4.4. \(\square\)

Let \(\sigma_i\) be the permutation \((i, i+1)\) in the symmetric group \(\mathfrak{S}_{m+n}\). There is a natural action of \(\mathfrak{S}_{m+n}\) on the set of sequences of \(m+n\) complex numbers. Namely, for a sequence \(\kappa\), we have \(\sigma_i \kappa = \kappa^{[i]}\).

**Theorem 7.4.** The map \(P(\mathbf{y}, \kappa) \to \mathfrak{S}_{m+n}\mathbb{K}\) given by \((\mathbf{y}, \kappa) \mapsto \kappa\) is a bijection between the twisted population \(P(\mathbf{y}, \kappa)\) and the orbit of \(\kappa\) under the action of symmetric group \(\mathfrak{S}_{m+n}\). In particular, it gives a bijection between the twisted population \(P(\mathbf{y}, \kappa)\) and the symmetric group \(\mathfrak{S}_{m+n}\).

**Proof.** The proof is similar to that of [MV3, corollary 4.12]. \(\square\)

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**Appendix. The Bethe ansatz for \(Y(\mathfrak{gl}_{1|1})\)**

In this section, we give the basics of Bethe ansatz for the \(\mathfrak{gl}_{1|1}\) XXX model (supersymmetric spin chains associated to \(\mathfrak{gl}_{1|1}\)). We follow the notation of section 4.2. We also set \(h = 1\).

**A.1. Super Yangian \(Y(\mathfrak{gl}_{1|1})\) and its representations**

Recall that for \(Y(\mathfrak{gl}_{1|1})\) we have

\[
[E_{\ell}(x_1), E_{\mu}(x_2)] = 0, \quad E_{\ell}(x_1) E_{\mu}(x_2) = \frac{x_1 - x_2 - (-1)^{|\ell|}}{x_2 - x_1 - (-1)^{|\ell|}} E_{\mu}(x_2) E_{\ell}(x_1),
\]
\[ L_{ik}(x_1) L_{ij}(x_2) = \frac{x_1 - x_2 - (-1)^{|i|} x_1 - x_2}{x_1 - x_2} L_{ji}(x_2) L_{ik}(x_1) + \frac{(-1)^{|i|} x_1 - x_2}{x_1 - x_2} L_{ij}(x_1) L_{ik}(x_2), \]

where \( i \neq j \) and \( i, j, k \in \{1, 2\} \).

In what follows we work with the standard parity sequence \( s_0 \).

The description of finite dimensional irreducible representations of \( Y(\mathfrak{gl}_{1|1}) \) is well-known.

Let \( \lambda = (\lambda_1, \lambda_2) \) be a \( \mathfrak{gl}_{1|1} \) weight, we say that \( \lambda \) is non-degenerate if \( \lambda_1 + \lambda_2 \neq 0 \).

Clearly, \( L_{\lambda} \) is two-dimensional if \( \lambda \) is non-degenerate and one-dimensional otherwise. Let \( \Lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \) be a sequence of non-degenerate \( \mathfrak{gl}_{1|1} \) weights, \( z \) a sequence of complex numbers. Let \( \lambda^{(k)} = (a_k, b_k), a_k, b_k \in \mathbb{C} \),

\[
\begin{align*}
    a &= \sum_{k=1}^{p} a_k, \\
    b &= \sum_{k=1}^{p} b_k, \\
    \varphi(x) &= \prod_{k=1}^{p} (x - z_k + a_k), \\
    \psi(x) &= \prod_{k=1}^{p} (x - z_k - b_k).
\end{align*}
\]

**Theorem A.1 ([Zha]).** Every finite dimensional irreducible representation of \( Y(\mathfrak{gl}_{1|1}) \) is a tensor product of evaluation \( Y(\mathfrak{gl}_{1|1}) \)-modules up to twisting by a one-dimensional \( Y(\mathfrak{gl}_{1|1}) \)-module. Moreover, \( L(\lambda, z) \) is irreducible if and only if \( \varphi(x) \) and \( \psi(x) \) are relatively prime.

Clearly, the \( Y(\mathfrak{gl}_{1|1}) \)-module \( L(\lambda, z) \) is irreducible if and only if \( z_i - z_j - a_i - b_j \neq 0 \) for all \( i \neq j \). Moreover, it satisfies the binary property. Namely, \( L(\lambda, z) \) is irreducible if and only if \( L_{\lambda^{(0)}}(z_1) \otimes L_{\lambda^{(0)}}(z_2) \) is irreducible for all \( 1 \leq i < j \leq p \). Furthermore, every finite dimensional irreducible representation of \( Y(\mathfrak{gl}_{1|1}) \) has dimension \( 2^r \) for some non-negative integer \( r \).

Let \( v_1^{(k)} \) be the highest weight vector of \( L_{\lambda^{(0)}} \) with respect to the standard root system, and \( v_2^{(k)} = e_2 w_1^{(k)} \). Then \( v_1^{(k)}, v_2^{(k)} \) is a basis of \( L_{\lambda^{(0)}} \). We use the shorthand notation \([0]\) for \( v_1^{(1)} \otimes \cdots \otimes v_1^{(p)} \).

Let \( E_{ij}, i, j = 1, 2 \), be the linear operator in \( \text{End}(L_{\lambda^{(0)}}) \) of parity \( |i| + |j| \) such that \( E_{ij} w^{(k)} = \delta_{ij} w^{(k)} \) for \( r = 1, 2 \).

The R-matrix \( R(x) \in \text{End}(L_{\lambda^{(0)}}) \otimes \text{End}(L_{\lambda^{(0)}}) \) is given by

\[
R(x) = E_{11} \otimes E_{11} - \frac{b_1 + a_1 + x}{a_1 + b_1 - x} E_{22} \otimes E_{22} + \frac{b_1 - b_1 - x}{a_1 + b_1 - x} E_{11} \otimes E_{22} + \frac{a_1 - a_1 - x}{a_1 + b_1 - x} E_{22} \otimes E_{11} - \frac{a_1 + b_1 + x}{a_1 + b_1 - x} E_{11} \otimes E_{22} + \frac{a_1 + b_1 + x}{a_1 + b_1 - x} E_{22} \otimes E_{11}.
\]

Clearly, \( L_{\lambda^{(0)}}(z) \otimes L_{\lambda^{(0)}}(z) \) is irreducible if and only if \( R(z_i - z_j) \) is well-defined and invertible.

Define an anti-automorphism \( \iota : Y(\mathfrak{gl}_{1|1}) \to Y(\mathfrak{gl}_{1|1}) \) by the rule, \( \iota(\mathcal{L}_i(x)) = (-1)^{|i|+|i|} \mathcal{L}_i(x), i, j = 1 \). One has \( \iota(X_1 X_2) = (-1)^{|X_1||X_2|} \iota(X_2) \iota(X_1) \) for \( X_1, X_2 \in Y(\mathfrak{gl}_{1|1}) \).

Recall that \( T(x) = \mathcal{L}_{11}(x) - \mathcal{L}_{22}(x) \), therefore \( \iota(T(x)) = T(x) \).

The Shapovalov form \( B_{\lambda^{(0)}} \) on \( L_{\lambda^{(0)}} \) is a bilinear form such that

\[
B_{\lambda^{(0)}}(e_j w_1, w_2) = (-1)^{|[i]|+|j|} w_1 B_{\lambda^{(0)}}(w_1, (-1)^{|[i]|+|j|} e_j w_2),
\]

for all \( i, j \) and \( w_1, w_2 \in L_{\lambda^{(0)}} \), and \( B_{\lambda^{(0)}}(v_1^{(0)}, v_1^{(0)}) = 1 \). Explicitly, it is given by

\[
B_{\lambda^{(0)}}(v_1^{(0)}, v_1^{(0)}) = 1, \quad B_{\lambda^{(0)}}(v_1^{(0)}, v_2^{(0)}) = B_{\lambda^{(0)}}(v_2^{(0)}, v_1^{(0)}) = 0, \quad B_{\lambda^{(0)}}(v_2^{(0)}, v_2^{(0)}) = -(a_1 + b_1).
\]
The Shapovalov forms \( B_{\lambda_0} \) on \( L_{\lambda_0} \) induce a bilinear form \( B_\lambda = \bigotimes_{i=1}^p B_{\lambda_i} \) (following the usual sign convention) on \( L(\lambda) \).

Let \( R_{\lambda z} \in \text{End}(L(\lambda)) \) be the product of R-matrices,
\[
R_{\lambda z} = \prod_{1 \leq i \leq p} \prod_{i < j \leq p} R^{(i,j)}(z_i - z_j).
\]

Define a bilinear form \( B_{\lambda z} \) on \( L(\lambda, z) \) by
\[
B_{\lambda z}(w_1, w_2) = B_\lambda(w_1, R_{\lambda z}w_2),
\]
for all \( w_1, w_2 \in L(\lambda, z) \).

One shows that, see \([MTV1, \text{section 7}]\),
\[
B_{\lambda z}(\{0\}, \{0\}) = 1, \quad B_{\lambda z}(Xw_1, w_2) = (-1)^{|X||w_1|} B_{\lambda z}(w_1, \iota(X)w_2),
\]
for all \( X \in \mathfrak{g} \mathfrak{l}_{1|1} \), \( w_1, w_2 \in L(\lambda, z) \). In addition, if \( L(\lambda, z) \) is irreducible, then \( B_{\lambda z} \) is non-degenerate.

### A.2. Bethe ansatz for \( \mathfrak{g} \mathfrak{l}_{1|1} \) XXX model

In this section, we study the spectrum of the transfer matrix \( T(x) = \mathcal{L}_{11}(x) - \mathcal{L}_{22}(x) \).

Let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)}) \) be a sequence of non-degenerate \( \mathfrak{g} \mathfrak{l}_{1|1} \) weights. Recall from section 4.2 that if \( y = (x - t_1) \cdots (x - t_l) \) is a divisor of \( \varphi(x) - \psi(x) \), then \( t = (t_1, \ldots, t_l) \) is a solution of the BAE associated to \( s_0, \lambda, z \), and \( l \).

It is convenient to renormalize the Bethe vector \( w(t, z) \) associated to \( t \), see (3.7),:
\[
\tilde{w}(t, z) = c_0 w(t, z), \quad c_0 = \prod_{i=1}^l \prod_{k=1}^p (t_i - z_k).
\]
The factor \( c_0 \) clears up the denominators and the Bethe vector \( \tilde{w}(t, z) \) is well-defined for all \( z, t \).

The following theorem is well known, see e.g. \([BR1]\).

**Theorem A.2.** If the Bethe vector \( \tilde{w}(t, z) \) is non-zero, then \( \tilde{w}(t, z) \) is an eigenvector of the transfer matrix \( T(x) \) with the corresponding eigenvalue
\[
\mathcal{E}(x) = \frac{y[1]}{y} (\varphi - \psi) \prod_{k=1}^p (x - z_k)^{-1}.
\]

**Proof.** For \( j = 1, 2 \), one has the following relation,
\[
\mathcal{L}_0(x) \mathcal{L}_{12}(t_1) \cdots \mathcal{L}_{12}(t_l) = \xi(x; t) \mathcal{L}_{12}(t_1) \cdots \mathcal{L}_{12}(t_l) \mathcal{L}_0(x)
\]
\[
+ \sum_{i=1}^l \xi(x; t) \mathcal{L}_{12}(x) \mathcal{L}_{12}(t_1) \cdots \widehat{\mathcal{L}_{12}(t_i)} \cdots \mathcal{L}_{12}(t_l) \mathcal{L}_0(t_i).
\]

Here the symbol \( \widehat{\mathcal{L}_{12}(t_i)} \) means the factor \( \mathcal{L}_{12}(t_i) \) is skipped and the functions \( \xi(x; t) \) and \( \xi_i(x; t) \) are given by
\[
\xi(x; t) = \prod_{1 \leq r < j} \frac{x - t_r - 1}{x - t_r} = \frac{y[1]}{y}, \quad \xi_i(x; t) = (-1)^{i-1} \frac{1}{x - t_i} \prod_{1 \leq r < i} \frac{t_i - t_r + 1}{t_i - t_r} \prod_{i < r \leq l} \frac{t_i - t_r - 1}{t_i - t_r}.
\]
We have
\[ T(x)|0\rangle = (\varphi - \psi) \prod_{k=1}^{p}(x-z_k)^{-1}|0\rangle. \]

Since \( t \) is a solution of the BAE, we have \( c_0T(t_i)|0\rangle = 0 \) for \( i = 1, \ldots, l \). Therefore it follows from (A.2) that
\[ T(x)\hat{w}(z,t) = \frac{\gamma[1]}{\gamma} (\varphi - \psi) \prod_{k=1}^{p}(x-z_k)^{-1}\hat{w}(z,t). \]

Recall that the transfer matrix \( T(x) \) commutes with the subalgebra \( U(\mathfrak{gl}_1) \) of \( \mathcal{Y}(\mathfrak{gl}_1) \).

**Proposition A.3.** The Bethe vector \( \hat{w}(t,z) \) is \( \mathfrak{gl}_1 \) singular.

**Proof.** By (3.3), one has the following relation,
\[ [\hat{L}_1^{(1)} , \hat{L}_{12}(t_1) \cdots \hat{L}_{12}(t_l) ] = \sum_{i=1}^{l} \nu_i(t)\hat{L}_{12}(t_1) \cdots \hat{L}_{12}(t_i) T(t_i). \]

The functions \( \nu_i(t) \) are given by
\[ \nu_i(t) = (-1)^i \prod_{1 \leq r < i} \frac{t_i-t_r+1}{t_i-t_r} \prod_{i < r \leq l} \frac{t_i-t_r-1}{t_i-t_r}. \]

Note that \( \hat{L}_1^{(1)}|0\rangle = 0 \) and \( c_0 T(t_i)|0\rangle = 0 \) for \( i = 1, \ldots, l \), therefore the statement follows.

**Proposition A.4.** Suppose \( \varphi \neq \psi \). Let \( t \) and \( \tilde{t} \) be two different solutions of Bethe ansatz equation associated to \( s_0 \), \( \lambda \), \( z \), then the Bethe vectors \( \hat{w}(t,z) \) and \( \hat{w}(\tilde{t},z) \) are orthogonal with respect to the form \( B_{\lambda z} \).

**Proof.** Let \( y \) and \( \tilde{y} \) represent \( t \) and \( \tilde{t} \) respectively. Note that we have
\[ B_{\lambda z}(T(x)\hat{w}(t,z),\hat{w}(\tilde{t},z)) = B_{\lambda z}(\hat{w}(t,z),T(x)\hat{w}(\tilde{t},z)). \]

It follows from theorem A.2 that
\[ \left( \frac{\gamma[1]}{\gamma} - \frac{\tilde{y}[1]}{\tilde{y}} \right) (\varphi - \psi) \prod_{k=1}^{p}(x-z_k)^{-1}B_{\lambda z}(\hat{w}(t,z),\hat{w}(\tilde{t},z)) = 0. \]

Since \( y \) and \( \tilde{y} \) are linearly independent and \( \varphi \neq \psi \), the statement follows.

The following theorem is a particular case of [HLPRS, theorem 4.1] which asserts that the square of the norm of the Bethe vector is essentially given by the Jacobian of the BAE.

**Theorem A.5 ([HLPRS]).** The square of the norm of the Bethe vector \( \hat{w}(t,z) \) is given by
\[ B_{\lambda z}(\hat{w}(t,z),\hat{w}(t,z)) = (-1)^{(l-1)/2} \prod_{1 \leq r < l} \left( \frac{t_i-t_j-1}{t_i-t_j} \right)^2 \prod_{i=1}^{l} \prod_{k=1}^{p} (t_i-z_k + a_k(t_i-z_k-b_k)) \prod_{i=1}^{l} \sum_{k=1}^{p} \frac{a_k + b_k}{(t_i-z_k + a_k)(t_i-z_k-b_k)}. \]

□
Theorem A.6. Suppose \( a + b \neq 0 \). For generic \( z \), the Bethe ansatz is complete. In other words, there are exactly \( 2^{p-1} \) solutions \( t_i, i = 1, \ldots, 2^{p-1} \), to the BAE associated to \( s_0, \lambda, z \), and \( l \) such that the corresponding Bethe vectors \( \tilde{w}(t_i, z), i = 1, \ldots, 2^{p-1} \), form a basis of \( L(\lambda, z) \)\(_{\text{sing}}\).

Proof. Since \( a + b \neq 0 \), we have \( \deg(\varphi - \psi) = p - 1 \). It is not difficult to see that \( \dim L(\lambda) \)\(_{\text{sing}}\) = \( 2^{p-1} \) and for generic \( z \) there are exactly \( 2^{p-1} \) distinct monic divisors of the polynomial \( \varphi - \psi \). Each monic divisor of \( \varphi - \psi \) corresponds to a solution \( t_i, i = 1, \ldots, 2^{p-1} \), of BAE associated to \( s_0, \lambda, z \), with possibly different \( l \). Due to proposition A.3 and theorem A.5, the Bethe vectors \( \tilde{w}(t_i, z) \) are singular and non-zero. Moreover, it follows from proposition A.4 that \( \tilde{w}(t_i, z), i = 1, \ldots, 2^{p-1} \), are linearly independent and hence form a basis of \( L(\lambda, z) \)\(_{\text{sing}}\).

Let \( \lambda^{(k)} = (1, 0) \) and \( z_k = 0 \) for all \( k = 1, \ldots, p \). This case is the homogeneous super XXX model. We obtain the completeness of homogeneous super XXX model.

Let \( \theta \) be a primitive \( p \)th root of unity. Set \( \vartheta_i = 1/((\theta^i - 1)), i = 1, \ldots, p - 1 \).

Corollary A.7. The Bethe ansatz is complete for the super homogeneous XXX model. Explicitly, the Bethe vectors form a basis of \((\mathbb{C}^{11(0)})^{\otimes p}\)\(_{\text{sing}}\) and the transfer matrix \( T(x) \) acts on \((\mathbb{C}^{11(0)})^{\otimes p}\)\(_{\text{sing}}\) diagonally with simple spectrum. Moreover, the spectrum of \( T(x) \) acting on \((\mathbb{C}^{11(0)})^{\otimes p}\)\(_{\text{sing}}\) is given by

\[
\left\{ \frac{(x - \vartheta_{i_1} - 1) \cdots (x - \vartheta_{i_l} - 1)}{(x - \vartheta_{i_1}) \cdots (x - \vartheta_{i_l})}, \frac{(x + 1)^p - x^p}{x^p}, 1 \leq i_1 < i_2 < \cdots < i_l \leq p - 1, l = 0, \ldots, p - 1 \right\}.
\]

Proof. Note that \( \varphi(x) = (x + 1)^p \) and \( \psi(x) = x^p \). Clearly, we have \( \varphi - \psi = p(x - \vartheta_1) \cdots (x - \vartheta_{p-1}) \). It is easy to see that \( \vartheta_i - \vartheta_j \neq 0, 1 \) for \( i \neq j \) and \( \vartheta_i \notin \mathbb{Z} \). Therefore we have exactly \( 2^{p-1} \) distinct monic divisors \( (x - \vartheta_{i_1}) \cdots (x - \vartheta_{i_l}), 1 \leq i_1 < i_2 < \cdots < i_l \leq p - 1, l = 0, \ldots, p - 1 \), of the polynomial \( \varphi - \psi \) and hence \( 2^{p-1} \) different solutions \( t_i, i = 1, \ldots, 2^{p-1} \), of BAE. Therefore, as in theorem A.6, the Bethe vectors \( \tilde{w}(t_i, z), i = 1, \ldots, 2^{p-1} \), form a basis of \((\mathbb{C}^{11(0)})^{\otimes p}\)\(_{\text{sing}}\).

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