On the Chattering of SARSA with Linear Function Approximation

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Abstract

SARSA, a classical on-policy control algorithm for reinforcement learning, is known to chatter when combined with linear function approximation: SARSA does not diverge but oscillates in a bounded region. However, little is known about how fast SARSA converges to that region and how large the region is. In this paper, we make progress towards solving this open problem by showing the convergence rate of projected SARSA to a bounded region. Importantly, the region is much smaller than the ball used for projection provided that the magnitude of the reward is not too large. Our analysis applies to expected SARSA as well as SARSA(\lambda). Existing works regarding the convergence of linear SARSA to a fixed point all require the Lipschitz constant of SARSA’s policy improvement operator to be sufficiently small; our analysis instead applies to arbitrary Lipschitz constants and thus characterizes the behavior of linear SARSA for a new regime.

Keywords: reinforcement learning, SARSA, approximate policy iteration, linear function approximation, chattering

1. Introduction

SARSA is a classical on-policy control algorithm for reinforcement learning (RL, Sutton and Barto (2018)) dating back to Rummery and Niranjan (1994). The key idea of SARSA is to update the estimate for action values with data generated by following an exploratory and greedy policy (e.g., an \(\epsilon\)-greedy policy) derived from the estimate itself. In this paper, we refer to the operator used for deriving such a policy from the action value estimate as the policy improvement operator.

SARSA is well understood in the tabular setting, where the action value estimates are stored in the form of a look-up table. For example, Singh et al. (2000) confirm the asymptotic convergence of SARSA to the optimal policy provided that the policies from the policy improvement operator satisfy the “greedy in the limit with infinite exploration” condition. Tabular methods, however, are not preferred when the state space is large and generalization is required across states. One possible solution is linear function approximation, which approximates the action values via the inner product of state-action features and a learnable

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The behavior of SARSA with linear function approximation (linear SARSA) is, however, less understood. Gordon (1996); Bertsekas and Tsitsiklis (1996) empirically observe that linear SARSA can chatter: the weight vector does not go to infinity (i.e., it does not diverge) but oscillates in a bounded region. Importantly, this chattering behavior remains even if a decaying learning rate is used. Gordon (2001) further proves that trajectory-based linear SARSA with an $\epsilon$-greedy policy improvement operator converges to a bounded region asymptotically. Unlike standard linear SARSA, where the policy improvement operator is invoked every step to generate a new policy for action selection in the next step, trajectory-based linear SARSA generates a policy at the beginning of each episode and the policy remains fixed during the episode. Intuitively, within an episode, trajectory-based linear SARSA is just linear Temporal Difference (TD, Sutton (1988)) learning for evaluating action values. It, therefore, converges to an approximation of the action value function of the policy generated at the beginning of the episode (Tsitsiklis and Roy, 1996). Since the number of all possible $\epsilon$-greedy policies is finite in a finite Markov Decision Process with a fixed $\epsilon$, trajectory-based linear SARSA oscillates among the (approximate) action value functions of those $\epsilon$-greedy policies, which form a bounded region. Later, Perkins and Precup (2002) prove the asymptotic convergence of fitted linear SARSA (a.k.a. model-free approximate policy iteration) to a fixed point. Similar to trajectory-based SARSA, fitted SARSA alternates between thorough TD learning for policy evaluation under a fixed policy and the application of the policy improvement operator. In other words, it involves bi-level optimization. Then assuming the Lipschitz constant of the policy improvement operator is sufficiently small such that the composition of the policy improvement operator and some other function becomes contractive, convergence of fitted SARSA is obtained thanks to Banach’s fixed point theorem. Despite this progress, the asymptotic behavior of standard linear SARSA, which invokes the policy improvement operator every time step, still remains unclear, as does a potential convergence rate. Understanding the behavior of linear SARSA is one of the four open theoretical questions in RL raised by Sutton (1999).

Several efforts have been made to analyze linear SARSA. Melo et al. (2008) prove the asymptotic convergence of linear SARSA. Zou et al. (2019) further provide a convergence rate of a projected linear SARSA, which uses an additional projection operator in the canonical linear SARSA update. Unlike Gordon (2001), the convergence in Melo et al. (2008); Zou et al. (2019) is to a fixed point instead of a bounded region. Although convergence to a fixed point is preferred, Melo et al. (2008); Zou et al. (2019) require that SARSA’s policy improvement operator is Lipschitz continuous and the Lipschitz constant is sufficiently small. It remains an open problem how linear SARSA behaves when the Lipschitz constant is large.

In this paper, we study projected linear SARSA (Zou et al., 2019). We show that it converges to a bounded region regardless of the magnitude of the Lipschitz constant of the policy improvement operator. Importantly, the bounded region is much smaller than the ball used for projection provided that the magnitude of rewards is not too large, which can be easily fulfilled by scaling down the rewards. We further provide a convergence rate and extend our analysis to expected SARSA (Van Seijen et al., 2009) and SARSA($\lambda$) (Rummery and Niranjan, 1994). The differences between our work and existing works are summarized in Table 1.
Table 1: Comparison with existing works. ✓∗ indicates that the corresponding property is not explicitly documented in the original work. “Per-step policy improvement” means that the policy improvement operator is applied every time step.

2. Background

In this paper, all vectors are column. We use \(\langle x, y \rangle = x^\top y\) to denote the standard inner product in Euclidean spaces. For a positive definite matrix \(D\), we use \(\|x\|_D = \sqrt{x^\top D x}\) to denote the vector norm induced by \(D\). We overload \(\|\cdot\|\) to also denote the induced matrix norm. We write \(\|\cdot\|\) as shorthand for \(\|\cdot\|_I\), where \(I\) is the identity matrix, i.e., \(\|\cdot\|\) denotes the standard \(\ell_2\) norm. When it does not cause confusion, we use vectors and functions interchangeably. For example, if \(f\) is a function \(S \to \mathbb{R}\), we also use \(f\) to denote the vector in \(\mathbb{R}^{|S|}\) whose \(s\)-indexed element is \(f(s)\).

We consider an infinite horizon Markov Decision Process (MDP, Puterman (2014)) with a finite state space \(S\), a finite action space \(A\), a transition kernel \(p : S \times S \times A \to [0, 1]\), a reward function \(r : S \times A \to [-r_{\text{max}}, r_{\text{max}}]\), a discount factor \(\gamma\), and an initial distribution \(p_0 : S \to [0, 1]\). At time step \(t = 0\), an initial state \(S_0\) is sampled from \(p_0(\cdot)\). At time step \(t\), an agent at a state \(S_t\) takes an action \(A_t \sim \pi(\cdot|S_t)\) according to a policy \(\pi : A \times S \to [0, 1]\). The agent then receives a reward \(R_{t+1} = r(S_t, A_t)\) and proceeds to a successor state \(S_{t+1} \sim p(\cdot|S_t, A_t)\). The return at time step \(t\) is defined as

\[
G_t = \sum_{i=0}^{\infty} \gamma^i R_{t+i+1},
\]

which allows us to define the action value function as

\[
q_\pi(s, a) = \mathbb{E}[G_t|S_t = s, A_t = a, \pi, p].
\]

The action value function \(q_\pi\) is closely related to the Bellman operator \(T_\pi\), which is defined as

\[
T_\pi q = r + \gamma P_\pi q,
\]

where \(P_\pi \in \mathbb{R}^{|S| \times |A| \times |S| \times |A|}\) is the state-action pair transition matrix, i.e., \(P_\pi((s, a), (s', a')) = p(a'|s, a)\pi(a'|s').\) In particular, \(q_\pi\) is the only vector \(q \in \mathbb{R}^{|S| \times |A|}\) satisfying

\[
q = T_\pi q.
\]
The goal of control is to find an optimal policy \( \pi^* \) such that \( \forall \pi, s, a, \)
\[
q_{\pi^*}(s, a) \geq q_{\pi}(s, a).
\]
All optimal policies share the same action value function, which is referred to as \( q^* \). One classical approach for finding \( q^* \) is SARSA, which updates an estimate \( q \in \mathbb{R}^{|S \times A|} \) iteratively as
\[
A_{t+1} \sim \pi_q(\cdot | S_{t+1}),
q(S_t, A_t) \leftarrow q(S_t, A_t) + \alpha_t \left( R_{t+1} + \gamma q(S_{t+1}, A_{t+1}) - q(S_t, A_t) \right),
\]
where \( \{\alpha_t\} \) is a sequence of learning rates and \( \pi_q \) denotes that the policy \( \pi \) is parameterized by the action value estimate \( q \). A commonly used \( \pi_q \) is an \( \epsilon \)-greedy policy, i.e.,
\[
\pi_q(a|s) = \begin{cases} \epsilon & \text{if } a = \arg \max \max_{b \in A} q(s, b) \\ (1-\epsilon) \frac{\max_{b \in A} q(s, b)}{\sum_{b} \exp(\frac{q(s, b)}{\iota})} & \text{otherwise} \end{cases},
\]
where \( \epsilon \in [0, 1] \) is a hyperparameter and \( \mathbb{I}_{\text{statement}} \) is the indicator function whose value is 1 if the statement is true and 0 otherwise. Another common example is an \( \epsilon \)-softmax policy, i.e.,
\[
\pi_q(a|s) = \begin{cases} \epsilon & \text{if } a = \arg \max \max_{b \in A} q(s, b) \\ (1-\epsilon) \frac{\exp(q(s, a)/\iota)}{\sum_{b} \exp(q(s, b)/\iota)} & \text{otherwise} \end{cases},
\]
where \( \iota \in (0, \infty) \) is the temperature of the softmax function. This \( \pi_q \) is exactly the policy improvement operator discussed in Section 1: it maps an action value estimate \( q \) to a new policy; it is “improvement” in that it usually has greedification over the action value estimate to some extent. So far we have considered only time-homogeneous policies. One can also consider time-inhomogeneous policies, e.g., a policy \( \pi_{q,t}(a|s) \) that depends on both the action value estimate \( q \) and the time step \( t \). Singh et al. (2000) show that if the time-inhomogeneous policies \( \pi_{q,t} \) satisfy the “greedy in the limit with infinite exploration” (GLIE) condition then the iterates generated by (1) converge to \( q^* \) almost surely.

It is, however, not always practical to use a look-up table for storing our action value estimates, especially when the state space is large or generalization is required across states. One natural solution is linear function approximation, where the action value estimate \( q(s, a) \) is parameterized as \( x(s, a)^\top w \). Here \( x : S \times A \to \mathbb{R}^K \) is the feature function which maps a state-action pair to a \( K \)-dimensional vector and \( w \in \mathbb{R}^K \) is the learnable weight vector. We use \( X \in \mathbb{R}^{|S \times A| \times K} \) to denote the feature matrix, whose \((s, a)\)-indexed row is \( x(s, a)^\top \). We use as shorthand
\[
\pi_w = \pi_{Xw}, \ x_t = x(S_t, A_t), \ x_{\max} = \max_{s,a} \|x(s, a)\|.
\]

SARSA(\( \lambda \)) (Algorithm 1) is a commonly used algorithm for learning \( w \). In Algorithm 1, \( \Gamma : \mathbb{R}^K \to \mathbb{R}^K \) is a projection operator onto a ball of radius \( C_\Gamma \), i.e.,
\[
\Gamma(w) = \begin{cases} w, & \|w\| \leq C_\Gamma, \\ C_\Gamma \frac{w}{\|w\|}, & \|w\| > C_\Gamma. \end{cases}
\]
Algorithm 1: SARSA(λ)

Initialize \( w_0 \) such that \( \|w_0\| \leq C_\Gamma \)
\( S_0 \sim p_0(\cdot), A_0 \sim \pi_{w_0}(\cdot|S_0), e_{-1} \leftarrow 0 \)
\( t \leftarrow 0 \)

while True do
  Execute \( A_t \), get \( R_{t+1}, S_{t+1} \)
  Sample \( A_{t+1} \sim \pi_{w_t}(\cdot|S_{t+1}) \)
  \( e_t \leftarrow \gamma \lambda e_{t-1} + x_t \)
  \( \delta_t \leftarrow R_{t+1} + \gamma x_{t+1}^\top w_t - x_t^\top w_t \)
  \( w_{t+1} \leftarrow \Gamma (w_t + \alpha_t \delta_t e_t) \)
  \( t \leftarrow t + 1 \)
end

The vector \( e_t \) is referred to as the eligibility trace (Sutton, 1988), which is a powerful tool for facilitating credit assignments in RL. For now we consider the setting where \( \lambda = 0 \) and \( C_\Gamma = \infty \), i.e., there is no eligibility trace and \( \Gamma \) is an identity mapping. If the iterates \( \{w_t\} \) generated by SARSA(0) converged to some vector \( w_* \), the expected update at \( w_* \) would have to diminish, i.e.,

\[
\mathbb{E}_{S_t, A_t \sim \pi_{w_*}} \left[ \left( R_{t+1} + \gamma x_{t+1}^\top w_* - x_t^\top w_* \right) x_t \right] = 0, \tag{5}
\]

where for a policy \( \pi \), we use \( d_\pi \in \mathbb{R}^{|S\times A|} \) to denote the stationary state action pair distribution of the chain in \( S \times A \) induced by \( \pi \) (assuming it exists). We can equivalently write (5) in a matrix form as

\[
X^\top D_\pi w_* \left( r + \gamma P_{\pi_{w_*}} X w_* - X w_* \right) = 0, \tag{6}
\]

where for a policy \( \pi \), we use \( D_\pi \in \mathbb{R}^{|S\times A| \times |S\times A|} \) to denote a diagonal matrix whose diagonal entry is \( d_\pi \). Define

\[
A_\pi \doteq X^\top D_\pi (\gamma P_\pi - I) X, \quad b_\pi \doteq X^\top D_\pi r, \tag{7}
\]

we can equivalently write (6) as

\[
A_{\pi_{w_*}} w_* + b_{\pi_{w_*}} = 0.
\]

It is known (see, e.g., Tsitsiklis and Roy (1996)) that \( A_\pi \) is negative definite under mild conditions. We define a projection operator \( \Pi_{D_\pi} \) mapping a vector in \( \mathbb{R}^{|S\times A|} \) to the column space of \( X \) as

\[
\Pi_{D_\pi} q = X \arg\min_w \|Xw - q\|_{D_\pi}^2.
\]

It can be computed that

\[
\Pi_{D_\pi} = X \left( X^\top D_\pi X \right)^{-1} X^\top D_\pi
\]
and it is known (see, e.g., De Farias and Van Roy (2000)) that (6) holds if and only if

\[ \Pi_{D_{\pi w}} T_{\pi w} X w_* = X w_* . \]

In other words, \(X w_*\) is a fixed point of the operator

\[ H(q) = \Pi_{D_{\pi q}} T_{\pi q} q. \]

The operator \(H\) is referred to as the \textit{approximate policy iteration} operator and SARSA(0) is an incremental, stochastic method to find a fixed point of approximate policy iteration. Unfortunately, when \(\pi_{q}(a|s)\) is not continuous in \(q\) (e.g., \(\pi_{q}\) is an \(\epsilon\)-greedy policy, c.f. (2)), \(H\) does not necessarily have a fixed point (De Farias and Van Roy, 2000). Conversely, when \(\pi_{q}(a|s)\) is continuous in \(q\), De Farias and Van Roy (2000) show that \(H\) has at least one.

Perkins and Precup (2002) assume \(\pi_{q}\) is Lipschitz continuous in \(q\) and study a form of fitted SARSA(0), which is a model-free variant of approximate policy iteration. At the \(k\)-th iteration, Perkins and Precup (2002) first invoke TD for learning the action value function of \(\pi_{k}\), which converges to \(w_{k} = A_{\pi_{w_{k}}}^{-1} b_{\pi_{w_{k}}} \) after infinitely many steps. Then the policy for the \((k+1)\)-th iteration is obtained via invoking the policy improvement operator, i.e., \(\pi_{k+1} = \pi_{w_{k}}\). Perkins and Precup (2002) show that

\[ \| \pi_{A_{\pi_{1}}^{-1} b_{\pi_{1}}} - \pi_{A_{\pi_{2}}^{-1} b_{\pi_{2}}} \| \leq O(\mathcal{L}_{\pi}) \| \pi_{1} - \pi_{2} \| , \]

where the policies \(\pi_{1}\) and \(\pi_{2}\) should be interpreted as vectors in \(\mathbb{R}^{S \times A}\) whose \((s, a)\)-indexed element is \(\pi(a|s)\) when computing \(\| \pi_{1} - \pi_{2} \| \) and \(\mathcal{L}_{\pi}\) denotes the Lipschitz constant of the policy improvement operator \(\pi_{w}\), i.e., \(\forall s, a,\)

\[ |\pi_{w_{1}}(a|s) - \pi_{w_{2}}(a|s) | \leq L_{\pi} \| w_{1} - w_{2} \|. \]

Consequently, if \(L_{\pi}\) is sufficiently small, the function \(x \to \pi_{A_{\pi}^{-1} b_{\pi}}\), which maps \(\pi_{k}\) to \(\pi_{k+1}\), becomes a contraction. Banach’s fixed point theorem then confirms the convergence of fitted SARSA(0). From the definition of \(A_{\pi}\) and \(b_{\pi}\) in (7), it is easy to see that (8) can also be expressed as

\[ \| \pi_{A_{\pi_{1}}^{-1} b_{\pi_{1}}} - \pi_{A_{\pi_{2}}^{-1} b_{\pi_{2}}} \| \leq O(\mathcal{L}_{\pi} \| r \|) \| \pi_{1} - \pi_{2} \| . \]

Hence, for any \(L_{\pi}\), if the magnitude of the reward \(\| r \|\) is small enough, the function \(x \to \pi_{A_{\pi}^{-1} b_{\pi}}\) is contractive and fitted SARSA(0) remains convergent. It is also trivial to see that their analysis still holds if we use TD(\(\lambda\)) instead of TD for computing \(w_{k}\) given \(\pi_{k}\). Nevertheless, Perkins and Precup (2002) share the same spirit as Gordon (2001) by holding the policy fixed for sufficiently (possibly infinitely) many steps to wait for the policy evaluation to complete.

When it comes to standard linear SARSA that updates the policy every time step, Melo et al. (2008) consider, for a fixed point \(w_{*}\),

\[
C_{w_*} = \sup_{w} \| A_{\pi w} - A_{\pi w_*} \| + \sup_{w \neq w_*} \frac{\| b_{\pi w} - b_{\pi w_*} \|}{\| w - w_* \|} .
\]
They show that $C_{w*} = O(L_\pi)$ and if $L_\pi$ is small enough such that

$$A_{\pi w*} + C_{w*} I$$

is negative definite, SARSA(0) converge to $w_*$. The convergence of SARSA(0) in Perkins and Precup (2002); Melo et al. (2008) does not require the projection operator (i.e., $C_\Gamma = \infty$) but is only asymptotic, Zou et al. (2019) further provide a convergence rate of SARSA(0) using some $C_\Gamma < \infty$, assuming $L_\pi$ is small enough such that

$$A_{\pi w*} + O(L_\pi (r_{max} + 2x_{max}C_\Gamma)) I$$

is negative definite. It is easy to see that if $L_\pi$ is not small enough, neither (9) nor (10) can be guaranteed to be negative definite no matter how small $\|r\|$ is. This is because the $\sup_w \|A_{\pi w} - A_{\pi w*}\|$ term in $C_{w*}$ and the $2x_{max}C_\Gamma$ term in (10) are independent of $r$. In other words, unlike Perkins and Precup (2002), where the requirement for a sufficiently small $L_\pi$ may be mitigated by scaling down the rewards, a sufficiently small $L_\pi$ is an essential requirement for the analysis of Melo et al. (2008); Zou et al. (2019). The behavior of SARSA(0) with a large $L_\pi$ remains an open problem, and even more so that of SARSA($\lambda$).

To investigate SARSA($\lambda$), we can similarly define the $\lambda$-Bellman operator as

$$T_{\pi,\lambda} q = r_{\pi,\lambda} + P_{\pi,\lambda} q,$$

where

$$r_{\pi,\lambda} = (I - \gamma \lambda P_\pi)^{-1} r,$$

$$P_{\pi,\lambda} = I - (1 - \lambda) \sum_{k=0}^{\infty} \lambda^k (\gamma P_\pi)^{k+1}.$$ 

If SARSA($\lambda$) with $C_\Gamma = \infty$ converged to some $w*$, we would similarly need to have

$$\Pi_{D_{\pi w*}} T_{\pi,\lambda} w* = w*.$$ 

Another natural extension of SARSA(0) is expected SARSA, which uses an expectation for bootstrapping and is summarized as Algorithm 2. We can, of course, define expected SARSA($\lambda$) but we only implicitly cover its analysis in this work as it is a straightforward combination of those of SARSA($\lambda$) and expected SARSA.

3. Stochastic Approximation with Rapidly Changing Markov Chains

To prepare us for the analysis of SARSA(0), we show, in this section, a convergence rate (to a bounded region) of a generic stochastic approximation algorithm. More precisely, we consider the following iterative updates:

$$w_{t+1} = \Gamma (w_t + \alpha_t (F_{\theta_t}(w_t, Y_t) - w_t)).$$

Here $\{w_t \in \mathbb{R}^K\}$ are the iterates generated by the stochastic approximation algorithm, $\{Y_t\}$ is a sequence of random variables evolving in a finite space $\mathcal{Y}$, $\{\theta_t \in \mathbb{R}^L\}$ is a sequence of random variables controlling the transition of $\{Y_t\}$, $F_{\theta}$ is a function from $\mathbb{R}^K \times \mathcal{Y}$ to $\mathbb{R}^K$.
Algorithm 2: Expected SARSA

Initialize $w_0$ such that $\|w_0\| \leq C_\Gamma$

$S_0 \sim p_0(\cdot), A_0 \sim \pi_{w_0}(\cdot|S_0)$

$t \leftarrow 0$

while True do

Execute $A_t$, get $R_{t+1}, S_{t+1}$

Sample $A_{t+1} \sim \pi_{w_t}(\cdot|S_{t+1})$

$\delta_t \leftarrow R_{t+1} + \gamma \sum_{a'} \pi_{w_t}(a'|S_{t+1})^\top w_t - x_t^\top w_t$

$w_{t+1} \leftarrow \Gamma(w_t + \alpha_t \delta_t x_t)$

$t \leftarrow t + 1$
end

parameterized by $\theta$, and $\Gamma$ is the projection operator defined in (4). Importantly, we consider the setting where

$$\forall t, \theta_t \equiv w_t.$$ 

In other words, there is only a single iterate in our setting. To ease presentation, we use $\{w_t\}$ and $\{\theta_t\}$ to denote the same quantity. This emphasizes their different roles as the iterates of interest and as the controller of the transition kernel.

Our analysis is a natural extension of Chen et al. (2021) and Zhang et al. (2021a) but has significant differences to theirs. Chen et al. (2021) consider a time-homogeneous Markov chain (i.e., $\forall t, \theta_t \equiv \theta_0$). Consequently, their results are naturally applicable to policy evaluation problems. Zhang et al. (2021a) consider a time-inhomogeneous Markov chain, where the iterates $\{w_t\}$ are different from the sequence $\{\theta_t\}$. More importantly, Zhang et al. (2021a) assume that $\{\theta_t\}$ changes sufficiently slowly, i.e., there exists another sequence $\{\beta_t\}$ such that

$$\|\theta_{t+1} - \theta_t\| = O(\beta_t)$$

and

$$\lim_{t \to \infty} \frac{\beta_t}{\alpha_t} = 0.$$ 

This is the classical two-timescale setting (see, e.g., Borkar (2009)) and their analysis naturally applies to actor-critic algorithms with $\{w_t\}$ and $\{\theta_t\}$ interpreted as critic and actor respectively. We instead consider the setting where $\forall t, \theta_t = w_t$. In other words, the time-inhomogeneous Markov chain we consider changes rapidly, which is the main challenge of our analysis. As a consequence, we introduce the projection operator $\Gamma$, not required in Chen et al. (2021); Zhang et al. (2021a). The price is that we only show convergence to a bounded region while Chen et al. (2021); Zhang et al. (2021a) show convergence to points. Convergence to a bounded region is, however, sufficient for our purpose of understanding the behavior of SARSA since it matches what practitioners have observed. Furthermore, we believe our analysis might be applicable to other RL algorithms and might also have independent interest beyond RL. We now state our assumptions.
Assumption 3.1 (Time-inhomogeneous Markov chain) There exists a family of parameterized transition matrices $\Lambda_P = \{ P_\theta \in \mathbb{R}^{|Y| \times |Y|} | \theta \in \mathbb{R}^L \}$ such that

$$\Pr(Y_{t+1} = y) = P_{\theta_{t+1}}(Y_t, y).$$

Assumption 3.2 (Uniform ergodicity) Let $\bar{\Lambda}_P$ be the closure of $\Lambda_P$. For any $P \in \bar{\Lambda}_P$, the chain induced by $P$ is ergodic. We use $d_\theta$ to denote the invariant distribution of the chain induced by $P_\theta$.

Those two assumptions are identical to those of Zhang et al. (2021a). Assumption 3.1 states that the random process $\{Y_t\}$ is a time-inhomogeneous Markov chain. Assumption 3.2 states the ergodicity of the Markov chains. Assumption 3.2 is also used in the analysis of RL algorithms in both on-policy (Marbach and Tsitsiklis, 2001) and off-policy (Zhang et al., 2021a,b) settings. We later show how SARSA($\lambda$) can trivially fulfill Assumption 3.2. The uniform ergodicity in Assumption 3.2 immediately implies uniform mixing.

Lemma 1 (Lemma 1 of Zhang et al. (2021a)) Let Assumption 3.2 hold. Then, there exist constants $C_M > 0$ and $\tau \in (0, 1)$, independent of $\theta$, such that for any $n > 0$,

$$\sup_{y, \theta} \sum_{y'} |P^n_\theta(y, y') - d_\theta(y')| \leq C_M \tau^n.$$ 

As noted in Zhang et al. (2021a), the uniform mixing property in Lemma 1 is usually a direct technical assumption in previous works (e.g., Zou et al. (2019); Wu et al. (2020)).

Assumption 3.3 (Uniform pseudo-contraction) Let

$$\bar{F}_\theta(w) \doteq \sum_{y \in Y} d_\theta(y) F_\theta(w, y),$$

$$f^\alpha_\theta(w) \doteq w + \alpha \left( \bar{F}_\theta(w) - w \right).$$

Then,

(i) For any $\theta$, $\bar{F}_\theta$ has a unique fixed point, i.e., there exists a unique $w^*_\theta$ such that

$$\bar{F}_\theta(w^*_\theta) = w^*_\theta.$$ 

(ii) There exists a constant $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha})$, $f^\alpha_\theta$ is a uniform pseudo-contraction, i.e., there exists a constant $\kappa_\alpha \in (0, 1)$ (depending on $\alpha$), such that for all $\theta,w$,

$$\|f^\alpha_\theta(w) - w^*_\theta\| \leq \kappa_\alpha \|w - w^*_\theta\|.$$ 

Assumption 3.3 is another difference from Chen et al. (2021); Zhang et al. (2021a). Namely, Chen et al. (2021); Zhang et al. (2021a) require $\bar{F}_\theta$ to be a contraction while we only require $f^\alpha_\theta$ to be a pseudo-contraction. It is easy to see that the contraction of $\bar{F}_\theta$ immediately implies the pseudo-contraction of $f^\alpha_\theta$ but not in the opposite direction. In other words, our assumption is weaker.
Assumption 3.4 (Continuity and boundedness) There exist constants $L_F, L'_F, L''_F, U_F, U'_F, U''_F, L_w, U_w, L_P$ such that for any $w, w', y, y', \theta, \theta'$,

(i). $\|F_{\theta}(w, y) - F_{\theta}(w', y)\| \leq L_F \|w - w'\|

(ii). $\|F_{\theta}(w, y) - F_{\theta'}(w, y)\| \leq L'_F \|\theta - \theta'\| (\|w\| + U'_F)

(iii). $\|F_{\theta}(0, y)\| \leq U_F

(iv). $\|F_{\theta}(w) - F_{\theta'}(w)\| \leq L''_F \|\theta - \theta'\| (\|w\| + U''_F)

(v). $\|w^*_\theta - w^*_\theta'\| \leq L_w \|\theta - \theta'\|

(vi). $\sup_\theta \|w^*_\theta\| \leq U_w

(vii). $|P_{\theta}(y, y') - P_{\theta'}(y, y')| \leq L_P \|\theta - \theta'\|

Assumption 3.4 states some regularity conditions for the functions we consider and is identical to that of Zhang et al. (2021a).

Assumption 3.5 (Projection)

(i). $\|w_0\| \leq C_\Gamma, U_w \leq C_\Gamma.

(ii). For any $\theta, w, y$, we have

$P_{\theta} = P_{\Gamma(\theta)}, F_{\theta}(w, y) = F_{\Gamma(\theta)}(w, y), w^*_\theta = w^*_{\Gamma(\theta)}$.

Assumption 3.5 requires that some of the functions we consider are invariant to the projection operator. We will later show that SARSA($\lambda$) trivially satisfies this assumption.

Assumption 3.6 The learning rates $\{\alpha_t\}$ have the form

$\alpha_t \doteq \frac{c_\alpha}{(t_0 + t)^{\epsilon_\alpha}}$,

where $c_\alpha > 0, \epsilon_\alpha \in (0, 1], t_0 > 0$ are constants to be tuned.

Assumption 3.6 is just one of many possible forms of learning rates; we use this particular one to ease presentation. Importantly, the learning rates $\{\alpha_t\}$ here do not verify the Robbins-Monro’s condition (Robbins and Monro, 1951) when $\epsilon_\alpha \leq 0.5$, neither do the learning rates in Wu et al. (2020); Chen et al. (2021).

We now present our analysis. Given the sequences $\{\theta_t\}$ (i.e., $\{w_t\}$) and $\{Y_t\}$ in (11), we define an auxiliary sequence $\{u_t\}$ as

$u_0 \doteq w_0$,  
$u_{t+1} \doteq \Gamma(u_t) + \alpha_t F_{\theta_t}(\Gamma(u_t), Y_t) - \Gamma(u_t))$. (12)

Lemma 2 Let Assumption 3.5 hold. Then for any $t$, $w_t = \Gamma(u_t)$. 

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Proof It follows immediately from induction.

Intuitively, \( \{u_t\} \) is simply the pre-projection version of \( \{w_t\} \). We are interested in \( \{u_t\} \) because it has the following nice property.

**Theorem 3** Let Assumptions 3.1 - 3.6 hold. If \( t_0 \) is sufficiently large, then the iterates \( \{u_t\} \) generated by (12) satisfy

\[
\mathbb{E} \left[ \|u_{t+1} - w^*_{t+1}\|^2 \right] \leq (1 - 2(1 - \kappa_\alpha - \mathcal{O}(\alpha_t^2 \log^2 \alpha_t))) \mathbb{E} \left[ \|\Gamma(u_t) - w^*_{t}\|^2 \right]
+ 2L_w L_\theta \alpha \mathbb{E} \left[ \|\Gamma(u_t) - w^*_{t}\| \right] + \mathcal{O}(\alpha_t^2 \log^2 \alpha_t),
\]

where \( L_\theta \equiv U_F + (L_F + 1)C_\Gamma \).

See Section A.1 for the proof of Theorem 3, where the constants hidden by \( \mathcal{O}(\cdot) \) and how large \( t_0 \) is are also explicitly documented. Theorem 3 gives a recursive form of some error terms. We, however, cannot go further unless we have the domain knowledge of \( \kappa_\alpha \).

**Corollary 4** Let Assumptions 3.1 - 3.6 hold. Assume \( \kappa_\alpha = \sqrt{1 - \eta \alpha} \) for some positive constant \( \eta > 0 \). If \( t_0 \) is sufficiently large, then the iterates \( \{w_t\} \) generated by (11) satisfy

\[
\mathbb{E} \left[ \|w_t - w^*_{w_t}\|^2 \right] = \frac{72L_w^2L_\theta^2}{\eta^2} + \begin{cases} 
\mathcal{O} \left( t^{-\frac{\eta_\alpha}{3\alpha}} \log^2 t \right), & \epsilon_\alpha = 1, \eta_\alpha \in (0, 3) \\
\mathcal{O} \left( \log^3 t \right), & \epsilon_\alpha = 1, \eta_\alpha = 3 \\
\mathcal{O} \left( \log^2 t \right), & \epsilon_\alpha = 1, \eta_\alpha \in (3, \infty) \\
\mathcal{O} \left( \log^2 t \right), & \epsilon_\alpha \in (0, 1)
\end{cases}
\]

See Section A.2 for the proof of Corollary 4, as well as the constants hidden by \( \mathcal{O}(\cdot) \) and how large \( t_0 \) is. Trivially, due to the projection operator, we have the trivial bound

\[
\mathbb{E} \left[ \|w_t - w^*_{w_t}\|^2 \right] \leq 4C_\Gamma^2.
\]

So Corollary 4 is informative only if

\[
\frac{72L_w^2L_\theta^2}{\eta^2} \leq 4C_\Gamma^2.
\]

This is where we need more domain knowledge and the analysis in the next section provides an example.

4. SARSA with Linear Function Approximation

We first analyze SARSA(0) and expected SARSA with the following assumptions.

**Assumption 4.1** (Lipschitz continuity) There exists \( L_\pi > 0 \) such that \( \forall w, w', a, s, \)

\[
\|\pi_w(a|s) - \pi_{w'}(a|s)\| \leq L_\pi \|w - w'\|.
\]
Assumption 4.2 (Uniform ergodicity) Let $\tilde{\Lambda}_\pi$ be the closure of $\{\pi_w \mid w \in \mathbb{R}^{S \times A}\}$. For any $\pi \in \tilde{\Lambda}_\pi$, the chain induced by $\pi$ is ergodic and $\pi(a|s) > 0$.

Assumption 4.3 (Linear independence) The feature matrix $X$ has full column rank.

Assumption 4.1 is also used in Perkins and Precup (2002); Melo et al. (2008); Zou et al. (2019). As noted by Zhang et al. (2021a), Assumption 4.2 is easy to fulfill especially when the chain induced by a uniformly random policy is ergodic, which we believe is a fairly weak assumption. An example policy satisfying those two assumptions is the $\epsilon$-softmax policy in (3) with any $\epsilon \in (0, 1]$, provided that the chain induced by a uniformly random policy is ergodic. Assumption 4.3 is standard in the literature regarding RL with linear function approximation (see, e.g., Tsitsiklis and Roy (1996)).

As discussed in (7), if SARSA(0), as well as expected SARSA, converged to some vector $w^*$, that vector would verify

$$w^* = A_{\pi_w}^{-1} b_{\pi_w}.$$  

This inspires us to define the error function

$$e(w) \doteq \left\| w - \left( X^\top D_{\Gamma(w)} \left( \gamma P_{\Gamma(w)} - I \right) X \right)^{-1} X^\top D_{\Gamma(w)} r \right\|^2$$

to study the behavior of SARSA. Here we have included the projection operator $\Gamma$ in the definition of $e(w)$ because we use a finite $C_\Gamma$ in Algorithms 1 and 2.

Theorem 5 Let Assumptions 3.6 and 4.1 - 4.3 hold. Assume $\|X\| = 1$, $r_{\max} \leq 1$ and $\|r\|$ is not so large such that

$$L_w \doteq O(L_\pi \|r\|) < 1.$$  

Assume $C_\Gamma$ is large enough such that

$$U_w \doteq O(\|r\|) \leq C_\Gamma.$$  

Let $t_0$ be sufficiently large. Then the iterates $\{w_t\}$ generated by Algorithm 1 with $\lambda = 0$ or by Algorithm 2 satisfy

$$\mathbb{E}[\|w_t - w_*\|] = \frac{6\sqrt{2}L_w (1 + 4C_\Gamma)}{\eta(1 - L_w)} + \begin{cases} O\left(t^{-\frac{\eta c_\alpha}{2}} \log t\right), & \epsilon_\alpha = 1, \eta c_\alpha \in (0, 3) \\ O\left(t^{-\frac{1}{2}} \log \frac{1}{t}\right), & \epsilon_\alpha = 1, \eta c_\alpha = 3 \\ O\left(t^{-\frac{1}{2}} \log t\right), & \epsilon_\alpha = 1, \eta c_\alpha \in (3, \infty) \\ O\left(t^{-\frac{\eta c_\alpha}{2}} \log t\right), & \epsilon_\alpha \in (0, 1) \end{cases}$$

where $\eta$ is a positive constant and $w_*$ is the unique vector such that $e(w_*) = 0$.

We prove Theorem 5 mainly by invoking Corollary 4. See Section B.1 for more details. The exact expressions of $L_w, U_w, \eta$ are detailed in the proof, all of which are independent of $C_\Gamma$. Moreover, $\eta$ is also independent of $\|r\|$ and $L_\pi$. The convergence rate in Theorem 5 is
identical to that of Zou et al. (2019). The main difference is that we show the convergence to a ball with any positive $L_\pi$, while Zou et al. (2019) show the convergence to a fixed point with a sufficiently small $L_\pi$ regardless of $\|r\|$. Obviously, due to the use of projection, we have a trivial bound

$$
\mathbb{E}[\|w_t - w_*\|] \leq 2C_\Gamma.
$$

So the results in Theorem 5 is informative only if

$$
\frac{6\sqrt{2}L_w(1 + 4C_\Gamma)}{\eta(1 - L_w)} \leq 2C_\Gamma \iff L_w \leq \left(1 + \frac{3\sqrt{2}(1 + 4C_\Gamma)}{\eta C_\Gamma}\right)^{-1}.
$$

The above inequality holds when $\|r\|$ is not too large and $C_\Gamma$ is sufficiently large.

As discussed in Perkins and Precup (2002); Zou et al. (2019), one problem of requiring $L_\pi$ to be sufficiently small is that the corresponding policies are highly stochastic, making exploitation hard. In Theorem 5, we require $L_w < 1$, which holds when $L_\pi \|r\|$ is small. Importantly, the $L_w$ does not depend on $C_\Gamma$. Consequently, we can choose any large $C_\Gamma$ without affecting the requirement of $L_\pi \|r\|$. By contrast, Zou et al. (2019) require $L_\pi C_\Gamma$ to be sufficiently small (c.f. (10)). Consequently, when $C_\Gamma$ approaches infinity, which is usually preferred, their $L_\pi$ must approach 0, which might be of less interest for control. Though when we scale the rewards down, the resulting policies also tend to become more stochastic, our requirement for $\|r\|$ is independent of $C_\Gamma$. That being said, though our policies are likely to be less stochastic than those of Zou et al. (2019), the problem of lack of exploitation discussed in Perkins and Precup (2002); Zou et al. (2019) is still not fully solved, which we leave for future work.

We further analyze SARSA($\lambda$). Similarly, we define the error function

$$
e_{\lambda}(w) \doteq \left\| w - \left( X^\top D_{\pi_{\Gamma(w)}} \left( P_{\pi_{\Gamma(w)},\lambda} - I \right) X \right)^{-1} X^\top D_{\pi_{\Gamma(w)}} r_{\pi_{\Gamma(w)},\lambda} \right\|^2.
$$

**Theorem 6** Let Assumptions 3.6 and 4.1 - 4.3 hold. Assume $\|X\| = 1$, $r_{\max} \leq 1$ and $\|r\|$ is not so large such that

$$L_w \doteq \mathcal{O}(L_\pi \|r\|) < 1.
$$

Assume $C_\Gamma$ is large enough such that

$$U_w \doteq \mathcal{O}(\|r\|) \leq C_\Gamma.
$$

Let $t_0$ be sufficiently large. Let $\epsilon_0 \in (0, \frac{\epsilon_\alpha}{2})$ be any constant. Then the iterates $\{w_t\}$ generated by Algorithm 1 with any $\lambda \in [0, 1]$ satisfy

$$
\mathbb{E}[\|w_t - w_*\|] = \frac{6\sqrt{2}L_w(1 + 4C_\Gamma)}{\eta(1 - L_w)} + \begin{cases} 
\mathcal{O}\left(t^{-\frac{\epsilon_\alpha}{4\epsilon_0} + \epsilon_0}\right), & \epsilon_\alpha = 1, \eta \epsilon_\alpha \in (0, 3) \\
\mathcal{O}\left(t^{-\frac{1}{2} + \epsilon_0} \log t\right), & \epsilon_\alpha = 1, \eta \epsilon_\alpha = 3 \\
\mathcal{O}\left(t^{-\frac{1}{2} + \epsilon_0}\right), & \epsilon_\alpha = 1, \eta \epsilon_\alpha \in (3, \infty) \ \text{,} \\
\mathcal{O}\left(t^{-\frac{\epsilon_\alpha}{2} + \epsilon_0}\right), & \epsilon_\alpha \in (0, 1)
\end{cases}
$$

where $\eta$ is a positive constant and $w_*$ is the unique vector such that $e_{\lambda}(w) = 0$. 

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Similarly, Theorem 6 is informative only when $\|r\|$ is not too large and $C_\Gamma$ is sufficiently large. Despite this limit, to our knowledge, Theorem 6 is the only theoretical result so far regarding the convergence of SARSA($\lambda$). The $\epsilon_0$ in Theorem 6 is not part of the algorithm and is used solely for presenting the results. The choice of $\epsilon_0$ affects the constants hidden by $O(\cdot)$. A log $t$ term in Theorem 5 is replaced by the $t^{\epsilon_0}$ term in Theorem 6. At this point, we are not sure if this decrease in the convergence rate is a fundamental flaw of using eligibility traces for control problems or merely an artifact of our proof techniques. We leave this as an open problem for future work. Though Theorem 5 can be proved by invoking Corollary 4, Theorem 6 cannot. This is because in Algorithm 1, when $\lambda > 0$, the update to $w_t$ depends on all the history $\{S_i, A_i\}_{i=0, \ldots, t+1}$, which grows to infinity as time progresses. By contrast, when $\lambda = 0$, the update to $w_t$ depend on only $(S_t, A_t, S_{t+1}, A_{t+1})$, which always remains finite. Consequently, parts of the proof of Theorem 6 are significantly different from the counterparts of Theorem 3. See Section B.2 for more details. We do believe that the techniques used here to deal with the eligibility trace can also be used for extending the analysis in Zou et al. (2019) from SARSA(0) to SARSA($\lambda$), provided that $L_\pi$ is sufficiently small. This extension, however, deviates from the main purpose of this work and is left for future work.

5. Related Work

Our results regarding the finite sample analysis of the general stochastic approximation algorithm in Section 3 rely on the pseudo contraction property and follow from Chen et al. (2021); Zhang et al. (2021a). Another family of convergent results regarding stochastic approximation algorithms is usually based on the analysis of the corresponding ordinary differential equations (see, e.g., Benveniste et al. (1990); Kushner and Yin (2003); Borkar (2009)). See Chen et al. (2021) and references therein for more details.

SARSA is an extension of TD for control. The convergence of linear TD, which aims at estimating the value of a fixed policy, is an active research area, see, e.g., Tsitsiklis and Roy (1996); Dalal et al. (2018); Lakshminarayanan and Szepesvári (2018); Bhandari et al. (2018); Srikant and Ying (2019). Analyzing SARSA is more challenging than TD because the policy SARSA considers changes every step.

SARSA is an incremental and stochastic way to implement approximate policy iteration. Other variants of approximate policy iteration include Lagoudakis and Parr (2003); Antos et al. (2008); Farahmand et al. (2010); Lazaric et al. (2012, 2016).

6. Experiments

We use a diagnostic MDP from Gordon (1996) (Figure 1) to illustrate the chattering of linear SARSA. Gordon (1996) tested the $\epsilon$-greedy policy (2), which is not continuous. We further test the $\epsilon$-softmax policy (3), whose Lipschitz constant is inversely proportional to the temperature $\iota$. When $\iota$ approaches 0, the $\epsilon$-softmax policy approaches the $\epsilon$-greedy policy. We run Algorithm 1 in this MDP with $\lambda = 0$ and $C_\Gamma = \infty$, i.e., there is no eligibility trace nor projection. Following Gordon (1996), we set $\epsilon = 0.1, \gamma = 1.0,$ and $\alpha_t = 0.01 \forall t$. As discussed in Gordon (1996), using a smaller discount factor or a decaying learning rate only slows down the chattering but the chattering always occurs. Following Gordon (1996),
Figure 1: A diagnostic MDP from Gordon (1996). The state $s_0$ is the initial state with two actions $a_U$ and $a_L$ available, both of which yield a 0 reward. At $s_1$, only one action $a_1$ is available, which yields a reward -2. At $s_2$, the action $a_2$ yields a reward -1. Both $a_1$ and $a_2$ leads to the terminal state $S_T$.

Figure 2: The action value of $a_U$ during training under different temperatures.

we use the following feature function:

$$x(s_0, a_U) = [1, 0, 0]^\top,$$

$$x(s_0, a_L) = [0, 1, 0]^\top,$$

$$x(s_1, a_1) = x(s_2, a_2) = [0, 0, 1]^\top.$$

In other words, it is essentially state aggregation.

As shown in Figure 2, when the temperature is small (i.e., $\tau = 0.01$), linear SARSA chatters. We further fix $\tau$ to be 0.01 and scale the reward with a multiplier $\alpha_r$. As shown in Figure 3, the chattering behavior disappears with $\alpha_r = 0.1$. This suggests that our results can might be improved such that when the magnitude of the rewards is small enough we can also achieve convergence to a fixed point, instead of a bounded region. We, however, leave this for future work. When we set $\alpha_r = 4.0$, the iterates still only chatter but do not diverge. This might suggest that our requirement for $\|r\|$ might be only sufficient and not necessarily necessary. We, however, leave the development of a necessary condition for future work. All the curves in Figures 2 and 3 are from a single run. Since the policy is stochastic and the initialization of the weight vector is random, we find the peaks and valleys can sometimes average each other out when we average over multiple runs.
Figure 3: The $\alpha_r$-normalized action value of $a_U$ during training with a fixed temperature $\bar{\nu} = 0.01$. The reward of the MDP in Figure 1 is scaled via $\alpha_r$, e.g., the reward for the action $a_1$ is now $-2\alpha_r$.

7. Conclusion

The behavior of linear SARSA is a long-standing open problem in the RL community. This work makes major progress on this open problem via presenting a convergence rate to a bounded region of projected linear SARSA($\lambda$). Previous art Zou et al. (2019) presents the convergence rate of linear SARSA(0) to a fixed point. Zou et al. (2019), however, require the Lipschitz constant of the policy improvement operator to be sufficiently small, no matter how small the magnitude of the rewards is. The desired Lipschitz constant is also inversely proportional to the radius of the ball used for projection. Unfortunately, the radius is usually chosen to be very large, sometimes infinite, in practice. Our analysis instead applies to an arbitrary Lipschitz constant, provided that the magnitude of the rewards is not too large. More importantly, our requirement for the magnitude of the rewards is independent of the radius of the ball used for projection.

Despite the major progress made in this work, there are still many open problems regarding the behavior of linear SARSA. To name a few: how does linear SARSA behave if the policy improvement operator is merely continuous but not Lipschitz continuous? How does linear SARSA behave if both the Lipschitz constant and the magnitude of the rewards are not small? Can we get a convergence rate without using any projection? We hope this work can draw more attention to the convergence of linear SARSA, arguably one of the most fundamental RL algorithms.

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Appendix A. Proofs of Section 3

A.1 Proof of Theorem 3

**Theorem 3** Let Assumptions 3.1 - 3.6 hold. If $t_0$ is sufficiently large, then the iterates $\{u_t\}$ generated by (12) satisfy

$$
\mathbb{E} \left[ \left\| u_{t+1} - w_{\theta_{t+1}}^* \right\|^2 \right] \leq (1 - 2 (1 - \kappa \alpha_t - O (\alpha_t^2 \log^2 \alpha_t))) \mathbb{E} \left[ \left\| \Gamma(u_t) - w_{\theta_t}^* \right\|^2 \right]
$$

$$
+ 2 L w L_\theta \alpha_t \mathbb{E} \left[ \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| \right] + O (\alpha_t^2 \log^2 \alpha_t),
$$

where $L_\theta = U_F + (L_F + 1) C_\Gamma$.

**Proof** We consider a Lyapunov function

$$
M(x) \doteq \frac{1}{2} \|x\|^2.
$$

It is well-known that for any $x, x'$,

$$
M(x') \leq M(x) + \langle \nabla M(x), x' - x \rangle + \frac{1}{2} \|x - x'\|^2.
$$

Using $x' = u_{t+1} - w_{\theta_{t+1}}^*$ and $x = \Gamma(u_t) - w_{\theta_t}^*$ in the above inequality and recalling the update (12)

$$
u_{t+1} = \Gamma(u_t) + \alpha_t (F_{\theta_t}(\Gamma(u_t), Y_t) - \Gamma(u_t))
$$

$$
= f^\alpha_{\theta_t}(\Gamma(u_t)) + \alpha_t (F_{\theta_t}(\Gamma(u_t), Y_t) - \bar{F}_{\theta_t}(\Gamma(u_t)))
$$

yield

$$
\frac{1}{2} \left\| u_{t+1} - w_{\theta_{t+1}}^* \right\|^2 \leq \frac{1}{2} \left\| \Gamma(u_t) - w_{\theta_t}^* \right\|^2 + \langle \Gamma(u_t) - w_{\theta_t}^*, u_{t+1} - \Gamma(u_t) + w_{\theta_t}^* - w_{\theta_{t+1}}^* \rangle
$$

$$
+ \frac{1}{2} \left\| u_{t+1} - \Gamma(u_t) + w_{\theta_t}^* - w_{\theta_{t+1}}^* \right\|^2
$$

$$
= \frac{1}{2} \left\| \Gamma(u_t) - w_{\theta_t}^* \right\|^2
$$

$$
+ \langle \Gamma(u_t) - w_{\theta_t}^*, f^\alpha_{\theta_t}(\Gamma(u_t)) - \Gamma(u_t) \rangle
$$

$$
+ \alpha_t \langle \Gamma(u_t) - w_{\theta_t}^*, F_{\theta_t}(\Gamma(u_t), Y_t) - \bar{F}_{\theta_t}(\Gamma(u_t)) \rangle
$$

$$
+ \alpha_t^2 \left\| F_{\theta_t}(\Gamma(u_t), Y_t) - \Gamma(u_t) \right\|^2
$$

$$
+ \left\| w_{\theta_t}^* - w_{\theta_{t+1}}^* \right\|^2.
$$
Here we do not have $T_3$ because the counterpart in Zhang et al. (2021a) is now 0. To further decompose $T_3$, we define a function $\tau_\alpha$ of $\alpha$ as
\[
\tau_\alpha = \min \{ n \geq 0 \mid C_M \tau^n \leq \alpha \},
\]
where the constants $C_M$ and $\tau$ are given in Lemma 1. In particular, $\tau_\alpha$ denotes the number of steps the chain needs to mix to an accuracy of $\alpha_t$. It is easy to see
\[
\tau_\alpha = O (- \log \alpha), \quad \lim_{\alpha \to 0} \alpha \tau_\alpha = 0.
\]

We now decompose $T_3$ as
\[
T_3 = \langle \Gamma(u_t) - w^*_{\theta_t}, F_{\theta_t}(\Gamma(u_t), Y_t) - \tilde{F}_{\theta_t}(\Gamma(u_t)) \rangle = \left\langle \Gamma(u_t) - w^*_{\theta_t} - \left( \Gamma(u_{t-\tau_\alpha}) - w^*_{\tilde{\theta}_t-\tau_\alpha} \right), F_{\theta_t}(\Gamma(u_t), Y_t) - \tilde{F}_{\theta_t}(\Gamma(u_t)) \right\rangle
\]
\[
+ \left\langle \Gamma(u_{t-\tau_\alpha}) - w^*_{\tilde{\theta}_t-\tau_\alpha}, F_{\theta_t}(\Gamma(u_t), Y_t) - \tilde{F}_{\theta_t}(\Gamma(u_{t-\tau_\alpha})), Y_t \right\rangle
\]
\[
+ \left\langle \Gamma(u_{t-\tau_\alpha}) - w^*_{\tilde{\theta}_t-\tau_\alpha}, F_{\theta_t}(\Gamma(u_{t-\tau_\alpha}), Y_t) \right\rangle.
\]

We further decompose $T_{33}$ as
\[
T_{33} = \left\langle \Gamma(u_{t-\tau_\alpha}) - w^*_{\tilde{\theta}_t-\tau_\alpha}, F_{\theta_t}(\Gamma(u_{t-\tau_\alpha}), Y_t) - \tilde{F}_{\theta_t}(\Gamma(u_{t-\tau_\alpha})) \right\rangle
\]
\[
= \left\langle \Gamma(u_{t-\tau_\alpha}) - w^*_{\tilde{\theta}_t-\tau_\alpha}, F_{\theta_t-\tau_\alpha}(\Gamma(u_{t-\tau_\alpha}), \tilde{Y}_t) - \tilde{F}_{\theta_t-\tau_\alpha}(\Gamma(u_{t-\tau_\alpha})) \right\rangle
\]
\[
+ \left\langle \Gamma(u_{t-\tau_\alpha}) - w^*_{\tilde{\theta}_t-\tau_\alpha}, F_{\tilde{\theta}_t-\tau_\alpha}(\Gamma(u_{t-\tau_\alpha}), Y_t) \right\rangle
\]
\[
+ \left\langle \Gamma(u_{t-\tau_\alpha}) - w^*_{\tilde{\theta}_t-\tau_\alpha}, F_{\tilde{\theta}_t}(\Gamma(u_{t-\tau_\alpha}), Y_t) \right\rangle.
\]

Here $\{\tilde{Y}_t\}$ is an auxiliary chain inspired from Zou et al. (2019). Before time $t - \tau_\alpha - 1$, $\{\tilde{Y}_t\}$ is exactly the same as $\{Y_t\}$. After time $t - \tau_\alpha - 1$, $\tilde{Y}_t$ evolves according to the fixed kernel $P_{\theta_t-\tau_\alpha}$ while $Y_t$ evolves according to the changing kernel $P_{\theta_t-\tau_\alpha}$, $P_{\theta_{t+1}-\tau_\alpha}$, \ldots
\[
\{\tilde{Y}_t\} : \ldots \rightarrow Y_{t-\tau_\alpha - 1} \xrightarrow{P_{\theta_t-\tau_\alpha}} Y_{t-\tau_\alpha} \xrightarrow{P_{\theta_t-\tau_\alpha}} \tilde{Y}_{t-\tau_\alpha + 1} \xrightarrow{P_{\theta_{t-\tau_\alpha}}} \tilde{Y}_{t-\tau_\alpha + 2} \rightarrow \ldots
\]
\[
\{Y_t\} : \ldots \rightarrow Y_{t-\tau_\alpha - 1} \xrightarrow{P_{\theta_t-\tau_\alpha}} Y_{t-\tau_\alpha} \xrightarrow{P_{\theta_t-\tau_\alpha} + 1} Y_{t-\tau_\alpha + 1} \xrightarrow{P_{\theta_{t+1}-\tau_\alpha}} Y_{t-\tau_\alpha + 2} \rightarrow \ldots
\]
We are now ready to present bounds for each of the above terms. To begin, we define some shorthand:

\[ A = 2L_F + 1, \quad B = U_F, \quad C = AU_w + B + A(1 + U_F U_F'). \]

(16)

According to (15), we can select a sufficiently large \( t_0 \) such that

\[ \alpha_t - \tau_{t-1} \leq \frac{1}{4A} \]

holds for all \( t \). This condition is crucial for Lemma 31, which plays an important role in the following bounds.

**Lemma 7** (Bound of \( T_1 \))

\[ T_1 \leq L_w L_\theta \alpha_t \| \Gamma(u_t) - w_{\theta_t}^* \|. \]

The proof of Lemma 7 is provided in Section D.1.

**Lemma 8** (Bound of \( T_2 \))

\[ T_2 \leq -(1 - \kappa \alpha_t) \| \Gamma(u_t) - w_{\theta_t}^* \|^2. \]

The proof of Lemma 8 is provided in Section D.2.

**Lemma 9** (Bound of \( T_{31} \))

\[ T_{31} \leq 8(L_w L_\theta + 1) \alpha_t - \tau_{t-1} \left( A^2 \| \Gamma(u_t) - w_{\theta_t}^* \|^2 + C^2 \right). \]

The proof of Lemma 9 is provided in Section D.3.

**Lemma 10** (Bound of \( T_{32} \))

\[ T_{32} \leq 16 \alpha_t - \tau_{t-1} (1 + L_w L_\theta \alpha_t - \tau_{t-1}) \left( A^2 \| \Gamma(u_t) - w_{\theta_t}^* \|^2 + C^2 \right). \]

The proof of Lemma 10 is provided in Section D.4.

**Lemma 11** (Bound of \( T_{331} \))

\[ \mathbb{E}[T_{331}] \leq 8L_p L_\theta \sum_{j=t-\tau_{t-1}}^{t-1} \alpha_t - \tau_{t-1} \left( 1 + L_w L_\theta \alpha_t - \tau_{t-1} \right) \left( A^2 \mathbb{E} \left[ \| \Gamma(u_t) - w_{\theta_t}^* \|^2 \right] + C^2 \right). \]

The proof of Lemma 11 is provided in Section D.5.

**Lemma 12** (Bound of \( T_{332} \))

\[ \mathbb{E}[T_{332}] \leq \frac{8\|Y\|L_p L_\theta \sum_{j=t-\tau_{t-1}}^{t-1} \alpha_t - \tau_{t-1} \left( 1 + L_w L_\theta \alpha_t - \tau_{t-1} \right)}{A} \left( A^2 \mathbb{E} \left[ \| \Gamma(u_t) - w_{\theta_t}^* \|^2 \right] + C^2 \right). \]

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The proof of Lemma 12 is provided in Section D.6.

**Lemma 13** (Bound of $T_{333}$)

$$T_{333} \leq \frac{4L_F L_\theta \alpha_{t-\tau_\alpha, t-1} (1 + L_w L_\theta \alpha_{t-\tau_\alpha, t-1} - 1)}{A^2} \left( A^2 \| \Gamma(u_t) - w^*_t \|^2 + C^2 \right).$$

The proof of Lemma 13 is provided in Section D.7.

**Lemma 14** (Bound of $T_{334}$)

$$T_{334} \leq \frac{4L_F^2 L_\theta \alpha_{t-\tau_\alpha, t-1} (1 + L_w L_\theta \alpha_{t-\tau_\alpha, t-1} - 1)}{A^2} \left( A^2 \| \Gamma(u_t) - w^*_t \|^2 + C^2 \right).$$

The proof of Lemma 14 is provided in Section D.8.

**Lemma 15** (Bound of $T_5$)

$$T_5 \leq 2 \left( A^2 \| \Gamma(u_t) - w^*_t \|^2 + C^2 \right).$$

The proof of Lemma 15 is provided in Section D.9.

**Lemma 16** (Bound of $T_6$)

$$T_6 = \| w^*_t - w^*_{t+1} \|^2 \leq L_w^2 L_\theta^2 \alpha_t^2.$$  

Lemma 16 follows immediately from Lemma 34.

We now assemble the bounds in Lemmas 7 - 16 back into (13). Define

$$L_{\alpha,t} \equiv \left( \sum_{j=t-\tau_\alpha}^{t} \alpha_{t-\tau_\alpha, j} \right) (1 + L_w L_\theta \max \{1, \alpha_{t-\tau_\alpha, t} \}),$$

$$C_0 \equiv \max \{ A^2, C^2 \} \max \{ 16, L_w^2 L_\theta^2, 8 |Y| L_F L_\theta, 4L_F^2 L_\theta, 4L_F^2 L_\theta \}.$$

Using $A > 1$ and Lemmas 9 - 16, it is easy to see

$$\mathbb{E} [T_3] \leq C_0 L_{\alpha,t} \left( \mathbb{E} \left( \| \Gamma(u_t) - w^*_t \|^2 \right) + 1 \right),$$

$$\alpha_t \mathbb{E} [T_3] \leq C_0 \alpha_t L_{\alpha,t} \left( \mathbb{E} \left( \| \Gamma(u_t) - w^*_t \|^2 \right) + 1 \right),$$

$$\mathbb{E} [T_5] \leq C_0 \left( \mathbb{E} \left( \| \Gamma(u_t) - w^*_t \|^2 \right) + 1 \right),$$

$$\alpha_t^2 \mathbb{E} [T_5] \leq C_0 \alpha_t^2 \left( \mathbb{E} \left( \| \Gamma(u_t) - w^*_t \|^2 \right) + 1 \right),$$

$$\mathbb{E} [T_6] \leq C_0 \alpha_t^2.$$
Then we have
\[
\frac{1}{2}E\left[\left\| u_{t+1} - w^*_{\theta_{t+1}} \right\|^2 \right] \\
\leq \frac{1}{2}E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\|^2 \right] + L_w L_\theta \alpha_t E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\| \right] - (1 - \kappa_\alpha) E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\|^2 \right] \\
+ C_0 \alpha_t L_{\alpha,t} \left( E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\|^2 \right] + 1 \right) + C_0 \alpha_t^2 \left( E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\|^2 \right] + 1 \right) \\
+ C_0 \alpha_t^2,
\]
implicating
\[
E\left[\left\| u_{t+1} - w^*_{\theta_{t+1}} \right\|^2 \right] \leq \left( 1 - 2 \left( 1 - \kappa_\alpha - C_0 \alpha_t L_{\alpha,t} - C_0 \alpha_t^2 \right) \right) E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\|^2 \right] \\
+ 2L_w L_\theta \alpha_t E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\| \right] + 2C_0 \alpha_t L_{\alpha,t} + 4C_0 \alpha_t^2.
\]
Observing that
\[
L_{\alpha,t} = O(\alpha_t \log^2 \alpha_t)
\]
then completes the proof.

A.2 Proof of Corollary 4

**Corollary 4** Let Assumptions 3.1 - 3.6 hold. Assume \( \kappa_\alpha = \sqrt{1 - \eta \alpha} \) for some positive constant \( \eta > 0 \). If \( t_0 \) is sufficiently large, then the iterates \( \{w_t\} \) generated by (11) satisfy
\[
E\left[\left\| w_t - w^*_{w_t} \right\|^2 \right] = \frac{72 L_w^2 L_\theta^2}{\eta^2} + \begin{cases} 
O\left( \frac{\alpha t}{\log^2 \alpha} \right), & \epsilon_\alpha = 1, \eta \alpha \in (0, 3) \\
O\left( \frac{\alpha t}{\log \alpha} \right), & \epsilon_\alpha = 1, \eta \alpha = 3 \\
O\left( \frac{\alpha t}{\log^2 \alpha} \right), & \epsilon_\alpha = 1, \eta \alpha \in (3, \infty) \\
O\left( \frac{\alpha t}{\log \alpha} \right), & \epsilon_\alpha \in (0, 1)
\end{cases}
\]

**Proof** According to Theorem 3, we have
\[
E\left[\left\| u_{t+1} - w^*_{\theta_{t+1}} \right\|^2 \right] \leq \left( 1 - 2 \left( 1 - \kappa_\alpha - C_0 \alpha_t L_{\alpha,t} - C_0 \alpha_t^2 \right) \right) E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\|^2 \right] \\
+ 2L_w L_\theta \alpha_t E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\| \right] + 2C_0 \alpha_t L_{\alpha,t} + 4C_0 \alpha_t^2.
\]
Since \( L_{\alpha,t} = O(\alpha_t \log^2 \alpha_t) \), we conclude that there exists a constant \( C_1 > 0 \) such that
\[
E\left[\left\| u_{t+1} - w^*_{\theta_{t+1}} \right\|^2 \right] \leq \left( 1 - 2 \left( 1 - \sqrt{1 - \eta \alpha} - C_1 \alpha_t^2 \log^2 \alpha_t \right) \right) E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\|^2 \right] \\
+ 2L_w L_\theta \alpha_t E\left[\left\| \Gamma(u_t) - w^*_{\theta_t} \right\| \right] + C_1 \alpha_t^2 \log^2 \alpha_t.
\]
When $t_0$ is sufficiently large, we have $\forall t$,

\[ 1 - 2 \left( 1 - \sqrt{1 - \eta \alpha_t} - C_1 \alpha_t^2 \log^2 \alpha_t \right) > 0. \]

Using

\[ \| \Gamma(u_t) - w_{\theta_t}^* \| = \| \Gamma(u_t) - \Gamma(w_{\theta_t}^*) \| \leq \| u_t - w_{\theta_t}^* \| \]

then yields

\[
\mathbb{E} \left[ \| u_{t+1} - w_{\theta_{t+1}}^* \|^2 \right] \\
\leq \left( 1 - 2 \left( 1 - \sqrt{1 - \eta \alpha_t} - C_1 \alpha_t^2 \log^2 \alpha_t \right) \right) \mathbb{E} \left[ \| u_t - w_{\theta_t}^* \|^2 \right] \\
+ 2L_w L_{\theta \alpha_t} \mathbb{E} \left[ \| u_t - w_{\theta_t}^* \| \right] + C_1 \alpha_t^2 \log^2 \alpha_t
\]

\[
\leq \left( 1 - 2 \left( 1 - \sqrt{1 - \eta \alpha_t} - C_1 \alpha_t^2 \log^2 \alpha_t \right) \right) \mathbb{E} \left[ \| u_t - w_{\theta_t}^* \|^2 \right] \\
+ 2L_w L_{\theta \alpha_t} \sqrt{ \mathbb{E} \left[ \| u_t - w_{\theta_t}^* \|^2 \right] } + C_1 \alpha_t^2 \log^2 \alpha_t \quad \text{(Jensen’s inequality)}.
\]

Since

\[
\lim_{\alpha \to 0} \frac{1 - \sqrt{1 - \eta \alpha}}{\eta \alpha} = 1,
\]

we conclude that when $t_0$ is sufficiently large, $\forall t$,

\[ 1 - \sqrt{1 - \eta \alpha_t} - C_1 \alpha_t^2 \log^2 \alpha_t \geq \frac{\eta}{3} \alpha_t - C_1 \alpha_t^2 \log^2 \alpha_t^2 \geq \frac{\eta}{4} \alpha_t. \]

With

\[ z_t = \sqrt{ \mathbb{E} \left[ \| u_t - w_{\theta_t}^* \|^2 \right] }, \]

we then get

\[ z_{t+1}^2 \leq \left( 1 - \frac{\eta}{2} \alpha_t \right) z_t^2 + 2L_w L_{\theta \alpha_t} z_t + C_1 \alpha_t^2 \log^2 \alpha_t. \]

If

\[ \left( 1 - \frac{\eta}{2} \alpha_t \right) z_t^2 + 2L_w L_{\theta \alpha_t} z_t \leq (1 - \frac{\eta}{3} \alpha_t) z_t^2, \]

\[ \Longleftrightarrow \frac{12L_w L_{\theta \alpha_t}}{\eta} \leq z_t, \quad (17) \]

we have

\[ z_{t+1}^2 \leq (1 - \frac{\eta}{3} \alpha_t) z_t^2 + C_1 \alpha_t^2 \log^2 \alpha_t. \]
If
\[
\frac{12L_w L_\theta}{\eta} \geq z_t,
\]
we have
\[
z_{t+1}^2 \leq \left(1 - \frac{\eta}{2} \alpha_t \right) z_t^2 + 2L_w L_\theta \alpha_t z_t + C_1 \alpha_t^2 \log^2 \alpha_t.
\]
Since for any time \(t\), one of (17) and (18) must hold, we always have
\[
z_{t+1}^2 \leq \left(1 - \frac{\eta}{3} \alpha_t \right) z_t^2 + \frac{24L_w^2 L_\theta^2}{\eta} \alpha_t + C_1 \alpha_t^2 \log^2 \alpha_t.
\]
Telescoping the above inequality from \(t_0\) to \(t\) yields
\[
z_t^2 \leq \prod_{i=t_0}^{t-1} \left(1 - \frac{\eta}{3} \alpha_i \right) z_{t_0}^2 + \frac{24L_w^2 L_\theta^2}{\eta} \sum_{i=t_0}^{t-1} \prod_{j=i+1}^{t-1} \left(1 - \frac{\eta}{3} \alpha_j \right) \alpha_i
\]
\[
+ C_1 \sum_{i=t_0}^{t-1} \prod_{j=i+1}^{t-1} \left(1 - \frac{\eta}{3} \alpha_j \right) \alpha_i^2 \log^2 \alpha_i,
\]
where we adopt the convention that \(\prod_{x=1}^{j} (\cdot) = 1\) if \(i > j\). For \(E_1\), using \(1 + x \leq \exp x\) yields
\[
E_1 \leq \exp \left( -\frac{\eta}{3} \sum_{i=t_0}^{t-1} \alpha_i \right) \leq \exp \left( -\frac{\eta}{3} \int_{x=t_0}^{t} \frac{c_\alpha}{(t_0 + x)^{1-\epsilon_\alpha}} dx \right)
\]
\[
= \begin{cases} 
\left( \frac{2t_0}{t_0 + t} \right)^{\frac{\eta c_\alpha}{3}} \cdot \epsilon_\alpha = 1 \\
\exp \left( \frac{\eta c_\alpha}{3(1-\epsilon_\alpha)} \left( (2t_0)^{1-\epsilon_\alpha} - (t_0 + t)^{1-\epsilon_\alpha} \right) \right), & \epsilon_\alpha \in (0, 1)
\end{cases}
\]
For \(E_2\), define
\[
B_t = \sum_{i=0}^{t} \prod_{j=i+1}^{t} \left(1 - \frac{\eta}{3} \alpha_j \right) \alpha_i.
\]
Then we have
\[
B_t = \alpha_t + \left(1 - \frac{\eta}{3} \alpha_t \right) B_{t-1}.
\]
When \(t_0\) is sufficiently large such that
\[
1 - \frac{\eta}{3} \alpha_t > 0,
\]
it is easy to see

$$B_{t-1} \leq \frac{3}{\eta} \implies B_t \leq \frac{3}{\eta}.$$ 

As $B_0 = \alpha_0$, we have $B_0 < \frac{3}{\eta}$ for sufficiently large $t_0$. We, therefore, conclude by induction that $\forall t$,

$$B_t \leq \frac{3}{\eta}.$$ 

Consequently,

$$E_2 \leq B_{t-1} \leq \frac{3}{\eta}.$$ \hspace{1cm} (21)

For $E_3$, we have

$$E_3 \leq \log^2 \alpha_t \sum_{i=t_0}^{t-1} \prod_{j=i+1}^{t-1} \left(1 - \frac{\eta}{3} \alpha_j \right) \alpha_j^2.$$ \hspace{1cm} (22)

If $\epsilon_\alpha = 1$, we have

$$E_4 \leq \sum_{i=t_0}^{t-1} \exp \left(-\frac{\eta}{3} \int_{t_0}^{t} \frac{c_\alpha}{(t_0 + x)^{\epsilon_\alpha}} dx \right) \frac{c_\alpha^2}{(t_0 + i)^{2\epsilon_\alpha}}$$

$$= \sum_{i=t_0}^{t-1} \left( \frac{t_0 + i + 1}{t + t_0} \right)^{\frac{\eta \epsilon_\alpha}{3}} \frac{c_\alpha^2}{(t_0 + i)^2}$$

$$= \sum_{i=t_0}^{t-1} \left( \frac{t_0 + i + 1}{t + t_0} \right)^{\frac{\eta \epsilon_\alpha}{3}} \frac{c_\alpha^2}{(t_0 + i + 1)^2} \left( \frac{t_0 + i + 1}{t_0 + i} \right)^2$$

$$\leq \frac{4c_\alpha^2}{(t + t_0)^{\frac{\eta \epsilon_\alpha}{3}}} \sum_{i=t_0}^{t-1} \frac{1}{(t_0 + i + 1)^{2 - \frac{\eta \epsilon_\alpha}{3}}}$$

$$\leq \begin{cases} O \left(t^{-\frac{\eta \epsilon_\alpha}{3}}\right), & \eta \epsilon_\alpha \in (0, 3) \\ O \left(\log t\right), & \eta \epsilon_\alpha = 3 \\ O \left(\frac{1}{t}\right), & \eta \epsilon_\alpha \in (3, \infty) \end{cases}$$

If $\epsilon_\alpha \in (0, 1)$, when $t_0$ is sufficiently large, we can use induction (see, e.g., Section A.3.7 of Chen et al. (2021)) to show that

$$E_4 = O \left(\frac{1}{\nu_\alpha}\right).$$
Putting the bounds in (20), (21), and (22) back into (19) yields

\[ z_t^2 = \frac{72L_w^2\lambda^2}{\eta^2} + \begin{cases} O\left(t^{-\frac{\eta c_0}{3}} \log^2 t\right), & \epsilon_\alpha = 1, \eta c_\alpha \in (0, 3) \\ O\left(t^{-\frac{\log t}{3}}\right), & \epsilon_\alpha = 1, \eta c_\alpha = 3 \\ O\left(t^{-\frac{\log t}{2}}\right), & \epsilon_\alpha = 1, \eta c_\alpha \in (3, \infty) \\ O\left(t^{-\frac{\log t}{2\eta c_\alpha}}\right), & \epsilon_\alpha \in (0, 1) \end{cases}. \]

Here we have used the fact that \( E_3 \) always dominates \( E_1 \) for any \( \epsilon_\alpha, c_\alpha \). Using

\[ \mathbb{E} \left[ \| w_t - w^*_w \|^2 \right] = \mathbb{E} \left[ \| \Gamma(u_t) - \Gamma(w^*_w) \|^2 \right] \leq \mathbb{E} \left[ \| u_t - w^*_w \|^2 \right] = z_t^2 \]

then completes the proof. \( \blacksquare \)

Appendix B. Proofs of Section 4

B.1 Proof of Theorem 5

**Theorem 5** Let Assumptions 3.6 and 4.1 - 4.3 hold. Assume \( \| X \| = 1, r_{\max} \leq 1 \) and \( \| r \| \) is not so large such that

\[ L_w \doteq O(L_\pi \| r \|) < 1. \]

Assume \( C_\Gamma \) is large enough such that

\[ U_w \doteq O(\| r \|) \leq C_\Gamma. \]

Let \( t_0 \) be sufficiently large. Then the iterates \( \{ w_t \} \) generated by Algorithm 1 with \( \lambda = 0 \) or by Algorithm 2 satisfy

\[ \mathbb{E} \left[ \| w_t - w_* \| \right] = \frac{6\sqrt{2}L_w (1 + 4C_\Gamma)}{\eta(1 - L_w)} + \begin{cases} O\left(t^{-\frac{\eta c_0}{6}} \log t\right), & \epsilon_\alpha = 1, \eta c_\alpha \in (0, 3) \\ O\left(t^{-\frac{1}{2} \log^2 \frac{3}{2}}\right), & \epsilon_\alpha = 1, \eta c_\alpha = 3 \\ O\left(t^{-\frac{1}{2} \log t}\right), & \epsilon_\alpha = 1, \eta c_\alpha \in (3, \infty) \\ O\left(t^{-\frac{1}{2} \log t}\right), & \epsilon_\alpha \in (0, 1) \end{cases}, \]

where \( \eta \) is a positive constant and \( w_* \) is the unique vector such that \( e(w_*) = 0 \).

**Proof** In this proof, we mainly focuses on SARSA(0). Expected SARSA is covered only implicitly. To start with, define

\[ \mathcal{Y} \doteq \{ (s, a, a') \mid s \in S, a \in A, s' \in S, p(s'|s,a) > 0 \}, \]

\[ Y_t \doteq (S_t, A_t, S_{t+1}, A_{t+1}), \]

\[ y \doteq (s, a, s', a'), \]

\[ \pi_{\theta}(a|s) \doteq \pi_{\Gamma(\theta)}(a|s), \]

\[ P_\theta((s_1, a_1, s'_1, a'_1), (s_2, a_2, s'_2, a'_2)) \doteq \begin{cases} 0 & (s'_1, a'_1) \neq (s_2, a_2) \\ p(s'_2|s_2, a_2)\pi_{\theta}(a'_2|s'_2) & (s'_1, a'_1) = (s_2, a_2). \end{cases} \]

\[ F_\theta(w, y) \doteq \left(r(s, a) + \gamma x(s', a')^T w - x(s, a)^T w\right) x(s, a) + w. \]
Here our \( F_\theta(w, s, a, s') \) is independent of \( \theta \). We choose to keep the subscription \( \theta \) for the easy extension to expected SARSA. That being said, if we were analyzing expected SARSA, we would have defined it as

\[
F_\theta(w, (s, a, s')) = \left( r(s, a) + \gamma \sum_{a'} \pi_\theta(a'|s') x(s', a')^\top w - x(s, a)^\top w \right) x(s, a) + w.
\]

The rest of the analysis of expected SARSA remains similar and is thus omitted.

According to the action selection rule for \( A_{t+1} \) specified in Algorithm 1, we have

\[
\Pr(Y_{t+1} = y) = P_{\theta_{t+1}}(Y_t, y),
\]

Assumption 3.1 is then fulfilled.

Assumption 3.2 is immediately implied by Assumption 4.2. In particular, for any \( \theta \), the invariant distribution of the chain induced by \( P_\theta \) is

\[
d_\pi_\theta(s) \pi_\theta(a|s) p(s'|s, a) \pi_\theta(a'|s').
\]

For Assumption 3.3, it is easy to see

\[
\bar{F}_\theta(w) = X^\top D_{\pi_\theta}(\gamma P_{\pi_\theta} - I) X w + X^\top D_{\pi_\theta} r + w,
\]

Define

\[
w^*_\theta = - \left( X^\top D_{\pi_\theta}(\gamma P_{\pi_\theta} - I) X \right)^{-1} X^\top D_{\pi_\theta} r.
\]

It is then easy to see that \( w^*_\theta \) is the unique fixed point of \( \bar{F}_\theta(w) \). The uniform pseudo-contraction is verified by Lemma 35. In particular, we have

\[
\kappa = \sqrt{1 - \eta \alpha},
\]

\[
\eta = (1 - \gamma) \inf_\theta \lambda_{\min} \left( X^\top D_{\pi_\theta} X \right) > 0,
\]

where \( \lambda_{\min}(\cdot) \) denotes the minimum eigenvalue of a symmetric positive definite matrix.

We now verify Assumption 3.4. To verify Assumption 3.4 (i), we have

\[
\|F_\theta(w, y) - F_\theta(w', y)\| \\
\leq \| \gamma x(s, a)x(s', a')^\top - x(s, a)x(s, a)^\top + I\| \|w - w'\| \\
\leq \frac{(1 + \gamma) x_{max}^2 + 1}{L_F} \|w - w'\|.
\]

Assumption 3.4 (ii) immediately holds since our \( F_\theta(w, y) \) is independent of \( \theta \).

To verify Assumption 3.4 (iii), we have

\[
\|F_\theta(0, y)\| = \|r(s, a) x(s, a)\| \leq \frac{r_{max} x_{max}}{U_F}.
\]
To verify Assumption 3.4 (iv), we have
\[
\| F_\theta(w) - F_{\theta'}(w) \| \\
= \| X^T (D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) - D_{\tilde{\pi}_{\theta'}} (\gamma P_{\tilde{\pi}_{\theta'}} - I)) X w + X^T (D_{\tilde{\pi}_\theta} - D_{\tilde{\pi}_{\theta'}}) r \| \\
\leq \| X \|^2 \| D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) - D_{\tilde{\pi}_{\theta'}} (\gamma P_{\tilde{\pi}_{\theta'}} - I) \| \| w \| + \| X \| \| D_{\tilde{\pi}_\theta} - D_{\tilde{\pi}_{\theta'}} \| \| r \|.
\]

Lemma 32 asserts that $D_{\tilde{\pi}_\theta}$ is Lipschitz continuous in $\theta$. We then conclude, by Lemma 30, that there exist positive constants $L_{DP} > 0, L_D > 0$ such that
\[
\| D_{\pi_\theta} (\gamma P_{\pi_\theta} - I) - D_{\pi_{\theta'}} (\gamma P_{\pi_{\theta'}} - I) \| \leq L_{DP} L_{\pi} \| \theta - \theta' \|, \\
\| D_{\pi_\theta} - D_{\pi_{\theta'}} \|' \leq L_DL_{\pi} \| \theta - \theta' \|.
\]

Importantly, $L_{DP}$ and $L_D$ do not depend on $C_\Gamma$. It is then easy to see that
\[
\| F_\theta(w) - F_{\theta'}(w) \| \leq \left( \| X \|^2 L_{DP} L_{\pi} \| w \| + \| X \| \| r \| L_D L_{\pi} \right) \| \Gamma(\theta) - \Gamma(\theta') \| \\
\leq \left( \| X \|^2 L_{DP} L_{\pi} \| w \| + \| X \| \| r \| L_D L_{\pi} \right) \| \theta - \theta' \|.
\]

To verify Assumption 3.4 (v), we first use Lemma 33 to get
\[
\left\| \left( X^T D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X \right)^{-1} - \left( X^T D_{\tilde{\pi}_{\theta'}} (\gamma P_{\tilde{\pi}_{\theta'}} - I) X \right)^{-1} \right\| \\
\leq \left\| \left( X^T D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X \right)^{-1} \right\| \left\| \left( X^T D_{\tilde{\pi}_{\theta'}} (\gamma P_{\tilde{\pi}_{\theta'}} - I) X \right)^{-1} \right\| \\
\times \left\| \left( X^T D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X - X^T D_{\tilde{\pi}_{\theta'}} (\gamma P_{\tilde{\pi}_{\theta'}} - I) X \right) \right\|.
\]

Thanks to Assumption 4.2, for any $\theta$,
\[
\left( X^T D_{\pi_\theta} (\gamma P_{\pi_\theta} - I) X \right)^{-1}
\]
is well-defined. Since $\Lambda_\pi$ is a compact set, we conclude, by the extreme value theorem, that there exists a constant $U_{inv} > 0$, independent of $C_\Gamma$, such that
\[
\sup_{\theta} \left\| \left( X^T D_{\pi_\theta} (\gamma P_{\pi_\theta} - I) X \right)^{-1} \right\| < U_{inv}.
\]
Recalling that $\tilde{\pi}_\theta(a|s) = \pi_{\Gamma(\theta)}(a|s)$ then yields
\[
\sup_{\theta} \left\| \left( X^T D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X \right)^{-1} \right\| < U_{inv}.
\]

It then follows immediately that
\[
\left\| \left( X^T D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X \right)^{-1} - \left( X^T D_{\tilde{\pi}_{\theta'}} (\gamma P_{\tilde{\pi}_{\theta'}} - I) X \right)^{-1} \right\| \leq U^2_{inv} \| X \|^2 L_{DP} L_{\pi} \| \theta - \theta' \|.
\]
It is also easy to see that
\[
\|X^\top D_{\tilde{\pi}} r\| \leq \|X\| \|r\|,
\]
\[
\|X^\top D_{\tilde{\pi}} r - X^\top D_{\tilde{\pi}'} r\| \leq \|X\| L_D \|L_D\| \|r\|.
\]

Using Lemma 30 again yields
\[
\|w^*_\theta - w^*_{\theta'}\| \leq \left(U_{\text{inv}}^2 \|X\|^2 L_D P \|X\| + U_{\text{inv}} \|X\| L_D \|r\| \right) L_w \|r\| \|\theta - \theta'\|.
\]

To verify Assumption 3.4 (vi), we have
\[
\sup_{\theta} \|w^*_\theta\| \leq U_{\text{inv}} \|X\| \|r\| \cdot
\]
Assumption 3.4 (vii) follows immediately from Assumption 4.1.

We now verify Assumption 3.5. Assumption 3.5 (i) is fulfilled by our selection of $C_\Gamma$. It is easy to see
\[
\tilde{\pi}_\theta(a|s) = \tilde{\pi}_{\Gamma(\theta)}(a|s),
\]
Assumption 3.5 (ii) then follows immediately.

With Assumptions 3.1 - 3.5 satisfied, we conclude by Corollary 4 that the iterates \{w_t\} generated by Algorithm 1 with $\lambda = 0$ satisfy
\[
\mathbb{E}\left[\|w_t - w^*_w\|^2\right] \leq \frac{72 L_w^2 L_\theta^2}{\eta^2} + \begin{cases} \mathcal{O}\left(t^{-\frac{\eta_{\alpha}}{3}} \log^2 t\right), & \epsilon_\alpha = 1, \eta_{\alpha} \in (0, 3) \\
\mathcal{O}\left(\log t\right), & \epsilon_\alpha = 1, \eta_{\alpha} = 3 \\
\mathcal{O}\left(\log^2 t\right), & \epsilon_\alpha = 1, \eta_{\alpha} \in (3, \infty) \\
\mathcal{O}\left(\frac{\log^2 t}{t^{\gamma_{\alpha}}}\right), & \epsilon_\alpha \in (0, 1) \end{cases}
\]
where
\[
L_{\theta} \doteq U_F + (L_F + 1)C_\Gamma = (r_{\text{max}}x_{\text{max}} + ((1 + \gamma)x_{\text{max}}^2 + 2) C_\Gamma) \leq 1 + 4C_\Gamma.
\]
Consequently,
\[
\mathbb{E}\left[\|w_t - w^*_w\|\right] \leq \sqrt{\mathbb{E}\left[\|w_t - w^*_w\|^2\right]} = \frac{6\sqrt{2} L_w L_{\theta}}{\eta} + \begin{cases} \mathcal{O}\left(t^{-\frac{\eta_{\alpha}}{6}} \log t\right), & \epsilon_\alpha = 1, \eta_{\alpha} \in (0, 3) \\
\mathcal{O}\left(t^{-\frac{1}{2}} \log^{\frac{3}{2}} t\right), & \epsilon_\alpha = 1, \eta_{\alpha} = 3 \\
\mathcal{O}\left(t^{-\frac{1}{2}} \log t\right), & \epsilon_\alpha = 1, \eta_{\alpha} \in (3, \infty) \\
\mathcal{O}\left(t^{-\frac{\eta_{\alpha}}{2}} \log t\right), & \epsilon_\alpha \in (0, 1) \end{cases}
\]
If
\[ \|r\| < \frac{1}{U_{inv}^2 \|X\|^2 LDP \|X\| + U_{inv} \|X\| L}, \]
we get
\[ L_w < 1. \]

Since
\[ \mathbb{E} [\|w_t - w_*\|] \]
\[ = \mathbb{E} [\|w_t - w_w^*\|] \]
\[ \leq \mathbb{E} [\|w_t - w_w^*\|] + \mathbb{E} [\|w_w^* - w_w^*\|] \]
\[ \leq \mathbb{E} [\|w_t - w_w^*\|] + L_w \mathbb{E} [\|w_t - w_*\|], \]
we conclude that
\[ \mathbb{E} [\|w_t - w_*\|] \leq \frac{1}{1 - L_w} \mathbb{E} [\|w_t - w_w^*\|] \]
\[ = \frac{6\sqrt{2}L_w (1 + 4C_\Gamma)}{\eta(1 - L_w)} + \begin{cases} 
O \left(t^{-\frac{\eta\alpha}{2}} \log t\right), & \epsilon_\alpha = 1, \eta \alpha \in (0, 3) \\
O \left(t^{-\frac{1}{3}} \log^3 t\right), & \epsilon_\alpha = 1, \eta \alpha = 3 \\
O \left(t^{-\frac{1}{2}} \log t\right), & \epsilon_\alpha = 1, \eta \alpha \in (3, \infty) \\
O \left(t^{-\frac{5}{6}} \log t\right), & \epsilon_\alpha \in (0, 1) 
\end{cases} \]
which completes the proof.

**B.2 Proof of SARSA(\(\lambda\))**

**B.2.1 History-Dependent Stochastic Approximation**

Consider iterates \(\{w_t\}\) generated by
\[ w_{t+1} = \Gamma (w_t + \alpha_t (F(w_t, Y_{0:t}) - w_t)), \]
where \(\{Y_t\}\) is a sequence of random variables controlled by \(\{\theta_t\}\). We use \(Y_{i:j}\) as shorthand for the sequence \((Y_i, Y_{i+1}, \ldots, Y_j)\). Similarly, we define \(y_{i:j} = (y_i, y_{i+1}, \ldots, y_j)\). The function
\[ F : \bigcup_{n=1}^{\infty} (\mathbb{R}^K \times \mathcal{Y}^n) \to \mathbb{R}^K \]
has the form of
\[ F(w, y_{i:j}) = \delta(w, y_j) \sum_{k=0}^{j-i} \xi^k x(y_{j-k}) + w, \]
where \(\xi \in (0, 1)\) is some constant, \(\delta\) is a function \(\mathbb{R}^K \times \mathcal{Y} \to \mathbb{R}\) and \(x\) is a function \(\mathcal{Y} \to \mathbb{R}^K\). We will later realize \(\delta\) as the TD error and \(x\) as the feature mapping in our application for SARSA(\(\lambda\)). Again, we consider the setting \(\theta_t = w_t, \forall t\). Besides Assumptions 3.1, 3.2, and 3.6, we making the following assumptions corresponding to Assumptions 3.3, 3.4, and 3.5.

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Assumption B.1 (Uniform pseudo-contraction) Let
\[ \tilde{F}_\theta^{(n)}(w) \doteq \mathbb{E}_{Y_1 \sim d_\theta(\cdot), Y_{i+1} \sim P_\theta(Y_i, \cdot)} [F(w, Y_1: n)], \]
\[ \tilde{F}_\theta(w) \doteq \lim_{n \to \infty} \tilde{F}_\theta^{(n)}(w), \]
\[ f_\alpha^\theta(w) \doteq w + \alpha (\tilde{F}_\theta(w) - w). \]

Then
(i) For any \( \theta \), \( \tilde{F}_\theta \) has a unique fixed point, i.e., there exists a unique \( w_\theta^* \) such that \( \tilde{F}_\theta(w_\theta^*) = w_\theta^* \).

(ii) There exist a constant \( \bar{\alpha} > 0 \) such that for all \( \alpha \in (0, \bar{\alpha}) \), \( f_\alpha^\theta \) is a uniform pseudo-contraction, i.e., there exists a constant \( \kappa_\alpha \in (0, 1) \) (depending on \( \alpha \)), such that for all \( \theta, w \),
\[ \| f_\alpha^\theta(w) - w_\theta^* \| \leq \kappa_\alpha \| w - w_\theta^* \|. \]

Assumption B.2 (Continuity and boundedness) There exist constants \( L_F, L_F', U_F, U_F'', L_w, U_w, L_P \) such that for any \( w, w', y, y', n, \theta, \theta' \),
(i). \( \| F(w, y_1:n) - F(w', y_1:n) \| \leq L_F \| w - w' \| \)
(ii). \( \| \tilde{F}_\theta(w) - \tilde{F}_\theta(w') \| \leq L_F \| w - w' \| \)
(iii). \( | \delta(w, y) - \delta(w', y) | \leq L_F \| w - w' \| \)
(iv). \( \| F(0, y_1:n) \| \leq U_F \)
(v). \( | \delta(0, y) | \leq U_F \)
(vi). \( \| \tilde{F}_\theta(w) - \tilde{F}_\theta'(w) \| \leq L_F' \| \theta - \theta' \| (\| w \| + U_F'') \)
(vii). \( \| w_\theta^* - w_{\theta'}^* \| \leq L_w \| \theta - \theta' \| \)
(viii). \( \sup_{\theta} \| w_\theta^* \| \leq U_w \)
(ix). \( | P_\theta(y, y') - P_{\theta'}(y, y') | \leq L_P \| \theta - \theta' \| \)

Assumption B.3 (Projection)
(i). \( \| w_0 \| \leq C_\Gamma, C_\Gamma \geq U_w \)
(ii). For any \( \theta, w, y \), we have
\[ P_\theta = P_{\Gamma(\theta)}, w_\theta^* = w_{\Gamma(\theta)}^*. \]

Given the sequences \( \{ \theta_t \} \) and \( \{ Y_t \} \), we define an auxiliary sequence \( \{ u_t \} \) as
\[ u_0 \doteq w_0, \]
\[ u_{t+1} \doteq \Gamma(u_t) + \alpha_t (F(\Gamma(u_t), Y_{0:t}) - \Gamma(u_t)). \]  
Similar to Lemma 2, it is easy to see \( \forall t, w_t = \Gamma(u_t) \). We have
Theorem 17 Let Assumptions 3.1, 3.2, 3.6, B.1, B.2, and B.3 hold. Let $\epsilon \in (0, \epsilon_\alpha)$ be any constant. Let $t_0$ be sufficiently large. Then the iterates $\{u_t\}$ generated by (24) satisfy
\[
\mathbb{E}\left[\left\|u_{t+1} - w_{\theta_{t+1}}^*\right\|^2\right] \leq \left(1 - \frac{2}{1 - \kappa_\alpha - O\left(\frac{1}{(t + t_0)^{2\epsilon - 2\epsilon}}\right)}\right) \mathbb{E}\left[\left\|\Gamma(u_t) - w_{\theta_t}^*\right\|^2\right] + 2L_wL_\theta\alpha_t\mathbb{E}\left[\left\|\Gamma(u_t) - w_{\theta_t}^*\right\|\right] + O\left(\frac{1}{(t + t_0)^{2\epsilon - 2\epsilon}}\right),
\]
where $L_\theta = U_F + (L_F + 1)C_\Gamma$.

Proof The proof mainly replicates the proof of Theorem 3 in Section A.1. However, the analysis of $M_{33}$ in this proof is significantly different from that of $T_{33}$ in Section A.1. For easing presentation, we in this proof focus on the differences and omit the duplicate parts.

We consider a Lyapunov function
\[
M(x) = \frac{1}{2}\|x\|^2.
\]
It is well-known that for any $x, x'$,
\[
M(x') \leq M(x) + \langle \nabla M(x), x' - x \rangle + \frac{1}{2}\|x - x'\|^2.
\]
Using $x' = u_{t+1} - w_{\theta_{t+1}}^*$ and $x = \Gamma(u_t) - w_{\theta_t}^*$ in the above inequality and
\[
u_{t+1} = \Gamma(u_t) = \alpha_t(F(\Gamma(u_t), Y_{0:t}) - \Gamma(u_t)) = f_{\theta_t}^\alpha(\Gamma(u_t)) + \alpha_t\left(F(\Gamma(u_t), Y_{0:t}) - \tilde{F}_{\theta_t}(\Gamma(u_t))\right)
\]
yields
\[
\frac{1}{2}\left\|u_{t+1} - w_{\theta_{t+1}}^*\right\|^2 \\
\leq \frac{1}{2}\left\|\Gamma(u_t) - w_{\theta_t}^*\right\|^2 + \left\langle \Gamma(u_t) - w_{\theta_t}^*, u_{t+1} - \Gamma(u_t) + w_{\theta_t}^* - w_{\theta_{t+1}}^*\right\rangle \\
+ \frac{1}{2}\left\|u_{t+1} - \Gamma(u_t) + w_{\theta_t}^* - w_{\theta_{t+1}}^*\right\|^2 \\
= \frac{1}{2}\left\|\Gamma(u_t) - w_{\theta_t}^*\right\|^2 \\
+ \left\langle \Gamma(u_t) - w_{\theta_t}^*, f_{\theta_t}^\alpha(\Gamma(u_t)) - \Gamma(u_t)\right\rangle \\
+ \left\langle \Gamma(u_t) - w_{\theta_t}^*, f_{\theta_t}^\alpha(\Gamma(u_t)) - \Gamma(u_t)\right\rangle \\
+ \alpha_t\left(\Gamma(u_t) - w_{\theta_t}^*, F(\Gamma(u_t), Y_{0:t}) - \tilde{F}_{\theta_t}(\Gamma(u_t))\right) \\
+ \alpha_t^2\left\|F_{\theta_t}(\Gamma(u_t), Y_{0:t}) - \Gamma(u_t)\right\|^2 \\
+ \left\|w_{\theta_t}^* - w_{\theta_{t+1}}^*\right\|^2.
\]
To further decompose $M_3$, we define

$$\tau_{\alpha_t} \doteq (t_0 + t)^e$$

for some small $\epsilon \in (0, \epsilon_{\alpha_t})$. It is easy to see

$$\lim_{t \to \infty} \alpha_t \tau_{\alpha_t} = \lim_{t \to \infty} \frac{c_{\alpha_t}}{(t + t_0)^{\epsilon_{\alpha} - \epsilon}} = 0.$$ 

It is also easy to verify that if $t_0$ is sufficiently large, then $\forall t$,

$$C_M T^{\frac{\tau_{\alpha_t}}{2}} \leq \alpha_t.$$ 

Here we recall that $C_M$ and $\tau$ are given in Lemma 1. In the proof of Theorem 3, $\tau_{\alpha_t}$ is selected to be of the order $O(\log t)$ (c.f. (14)). Here we use a much larger one of the order $O(t^e)$. The reason will be clear by the end of this proof. We now decompose $M_3$ as

$$M_3 = \left\langle \Gamma(u_t) - w_{t_0}(\cdot), F(\Gamma(u_t), Y_{0:t}) - \tilde{F}_{\theta t}(\Gamma(u_t)) \right\rangle = \left\langle \Gamma(u_t) - w_{t_0} - \left(\Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}\right), F(\Gamma(u_t), Y_{0:t}) - \tilde{F}_{\theta t}(\Gamma(u_t)) \right\rangle_M$$

$$+ \left\langle \Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}, F(\Gamma(u_t), Y_{0:t}) - F(\Gamma(u_{t-\tau_{t}}), Y_{0:t}) + \tilde{F}_{\theta t}(\Gamma(u_{t-\tau_{t}})) - \tilde{F}_{\theta t}(\Gamma(u_t)) \right\rangle_M$$

$$+ \left\langle \Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}, F(\Gamma(u_{t-\tau_{t}}), Y_{0:t}) - \tilde{F}_{\theta t}(\Gamma(u_{t-\tau_{t}})) \right\rangle_M.$$ 

We further decompose $M_{33}$ as

$$M_{33} = \left\langle \Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}, F(\Gamma(u_{t-\tau_{t}}), Y_{0:t}) - \tilde{F}_{\theta t}(\Gamma(u_{t-\tau_{t}})) \right\rangle$$

$$= \left\langle \Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}, F(\Gamma(u_{t-\tau_{t}}), Y_{0:t}) - F(\Gamma(u_{t-\tau_{t}}), Y_{t-\tau_{t}:t}) \right\rangle_M$$

$$+ \left\langle \Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}, F(\Gamma(u_{t-\tau_{t}}), Y_{t-\tau_{t}:t}) - F(\Gamma(u_{t-\tau_{t}}), \tilde{Y}_{t-\tau_{t}:t}) \right\rangle_M$$

$$+ \left\langle \Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}, F(\Gamma(u_{t-\tau_{t}}), \tilde{Y}_{t-\tau_{t}:t}) - F(\Gamma(u_{t-\tau_{t}}), \tilde{F}_{\theta_{t-\tau_{t}}}(\Gamma(u_{t-\tau_{t}}))) \right\rangle_M$$

$$+ \left\langle \Gamma(u_{t-\tau_{t}}) - w_{\theta_{t-\tau_{t}}}, \tilde{F}_{\theta_{t-\tau_{t}}}(\Gamma(u_{t-\tau_{t}})) - \tilde{F}_{\theta_{t-\tau_{t}}}(\Gamma(u_{t-\tau_{t}})) \right\rangle_M.$$

$$M_{33} = 32.$$
Here \( \{\tilde{Y}_t\} \) is an auxiliary chain inspired from Zou et al. (2019). Before time \( t - \tau_{\alpha_t} - 1 \), \( \{\tilde{Y}_t\} \) is exactly the same as \( \{Y_t\} \). After time \( t - \tau_{\alpha_t} - 1 \), \( \tilde{Y}_t \) evolves according to the fixed kernel \( P_{t_{\tau_{\alpha_t}}} \) while \( Y_t \) evolves according to the changing kernel \( P_{t_{\tau_{\alpha_t}+1}}, \ldots \)

\[
\{\tilde{Y}_t\} : \ldots \rightarrow Y_{t-\tau_{\alpha_t}-1} \xrightarrow{P_{t_{\tau_{\alpha_t}}}} Y_{t-\tau_{\alpha_t}} \xrightarrow{P_{t_{\tau_{\alpha_t}+1}}} \tilde{Y}_{t-\tau_{\alpha_t}+1} \xrightarrow{P_{t_{\tau_{\alpha_t}}}} \tilde{Y}_{t-\tau_{\alpha_t}+2} \rightarrow \ldots
\]

\[
\{Y_t\} : \ldots \rightarrow Y_{t-\tau_{\alpha_t}-1} \xrightarrow{P_{t_{\tau_{\alpha_t}}}} Y_{t-\tau_{\alpha_t}} \xrightarrow{P_{t_{\tau_{\alpha_t}+1}}} Y_{t-\tau_{\alpha_t}+1} \xrightarrow{P_{t_{\tau_{\alpha_t}}}} Y_{t-\tau_{\alpha_t}+2} \rightarrow \ldots
\]

Define

\[
A = 2L_F + 1, \quad B = U_F, \quad C = AU_w + B + A(1 + U_F^w).
\]

Let \( t_0 \) be sufficiently large such that

\[
\alpha_{t-\tau_{\alpha_t}, t-1} \leq \frac{1}{4A}.
\]

This ensures that Lemma 31 also holds in the context of (23). We now bound the error terms one by one.

**Lemma 18** (Bound of \( M_1 \))

\[
M_1 \leq L_w L_0 \alpha_t \|\Gamma(u_t) - w^*_{\theta_t}\|.
\]

The proof of Lemma 18 is identical to that of Lemma 7 in Section D.1 and is thus omitted.

**Lemma 19** (Bound of \( M_2 \))

\[
M_2 \leq -(1 - \kappa_{\alpha_t}) \|\Gamma(u_t) - w^*_{\theta_t}\|^2.
\]

The proof of Lemma 19 is identical to that of Lemma 8 in Section D.2 is thus omitted.

**Lemma 20** (Bound of \( M_{31} \))

\[
M_{31} \leq 8(L_w L_0 + 1) \alpha_{t-\tau_{\alpha_t}, t-1} \left( A^2 \|\Gamma(u_t) - w^*_{\theta_t}\|^2 + C^2 \right).
\]

**Proof** The proof of Lemma 20 is mostly identical to that of Lemma 9 in Section D.3 up to replacing \( Y_t \) with \( Y_{0:t} \). The only difference is the following term.

\[
\|F(\Gamma(u_t), Y_{0:t}) - \tilde{F}_{\theta_t}(\Gamma(u_t))\| \\
\leq \|F(\Gamma(u_t), Y_{0:t})\| + \|\tilde{F}_{\theta_t}(\Gamma(u_t)) - \tilde{F}_{\theta_t}(w^*_{\theta_t})\| + \|w^*_{\theta_t}\| \quad \text{(Assumption B.1(i))}
\leq U_F + L_F \|\Gamma(u_t)\| + \|\tilde{F}_{\theta_t}(\Gamma(u_t)) - \tilde{F}_{\theta_t}(w^*_{\theta_t})\| + \|w^*_{\theta_t}\| \quad \text{(Similar to Lemma 36)}
\leq U_F + L_F \|\Gamma(u_t)\| + L_F \|\Gamma(u_t) - w^*_{\theta_t}\| + \|w^*_{\theta_t}\|
\leq U_F + L_F \|\Gamma(u_t) - w^*_{\theta_t}\| + L_F \|w^*_{\theta_t}\| + L_F \|\Gamma(u_t) - w^*_{\theta_t}\| + \|w^*_{\theta_t}\|
\leq A \|\Gamma(u_t) - w^*_{\theta_t}\| + A \|w^*_{\theta_t}\| + B.
\]

\[\blacksquare\]
Lemma 21 \textit{(Bound of $M_{32}$)}

$$M_{32} \leq 16\alpha_t \tau_{\alpha_t,t-1}(1 + L_w L_\theta \alpha_t \tau_{\alpha_t,t-1}) \left( A^2 \| \Gamma(u_t) - w^*_{\theta_t} \|^2 + C^2 \right).$$

\textbf{Proof} The proof of Lemma 21 is mostly identical to that of Lemma 10 in Section D.4 up to replacing $Y_t$ with $Y_{0:t}$. The only difference is the following term.

$$\| F(\Gamma(u_t), Y_{0:t}) - F(\Gamma(u_{t-\tau_{\alpha_t}}), Y_{0:t}) \|
\leq \| F(\Gamma(u_t), Y_{0:t}) - F(\Gamma(u_{t-\tau_{\alpha_t}}), Y_{0:t}) \| + \| F(\Gamma(u_{t-\tau_{\alpha_t}})) - F(\Gamma(u_t)) \|
\leq 2 L_F \| \Gamma(u_{t-\tau_{\alpha_t}}) - \Gamma(u_t) \|.$$


Lemma 22 \textit{(Bound of $M_{331}$)}

$$M_{331} \leq \frac{8 \max \xi^{\tau_{\alpha_t} + 1}(1 + L_w L_\theta \alpha_t \tau_{\alpha_t,t-1})}{(1 - \xi) A} \left( A^2 \| \Gamma(u_t) - w^*_{\theta_t} \|^2 + C^2 \right).$$

\textbf{Proof} $M_{331} \leq \left\| \Gamma(u_{t-\tau_{\alpha_t}}) - w^*_{\theta_{t-\tau_{\alpha_t}}} \right\| \| F(\Gamma(u_{t-\tau_{\alpha_t}}), Y_{0:t}) - F(\Gamma(u_{t-\tau_{\alpha_t}}), Y_{t-\tau_{\alpha_t},t}) \|.$

For the first term, similar to (33), we have

$$\left\| \Gamma(u_{t-\tau_{\alpha_t}}) - w^*_{\theta_{t-\tau_{\alpha_t}}} \right\| \leq \frac{2(1 + L_w L_\theta \alpha_t \tau_{\alpha_t,t-1})}{A} \left( A \| \Gamma(u_t) - w^*_{\theta_t} \| + C \right).$$

For the second term, we have

$$\| F(\Gamma(u_{t-\tau_{\alpha_t}}), Y_{0:t}) - F(\Gamma(u_{t-\tau_{\alpha_t}}), Y_{t-\tau_{\alpha_t},t}) \|
\leq \frac{x_{\max} \xi^{\tau_{\alpha_t} + 1}}{1 - \xi} \left\| \delta(\Gamma(u_{t-\tau_{\alpha_t}}), Y_t) \right\|
\leq \frac{x_{\max} \xi^{\tau_{\alpha_t} + 1}}{1 - \xi} (U_F + L_F \| \Gamma(u_{t-\tau_{\alpha_t}}) \|) \quad \text{(Similar to Lemma 36)}
\leq \frac{x_{\max} \xi^{\tau_{\alpha_t} + 1}}{1 - \xi} (U_F + L_F \| \Gamma(u_t) \| + L_F \| \Gamma(u_t) - \Gamma(u_{t-\tau_{\alpha_t}}) \|)
\leq \frac{x_{\max} \xi^{\tau_{\alpha_t} + 1}}{1 - \xi} \left( U_F + L_F \| \Gamma(u_t) \| + L_F \left( \| \Gamma(u_t) \| + \frac{B}{A} \right) \right) \quad \text{(Lemma 31)}
\leq \frac{x_{\max} \xi^{\tau_{\alpha_t} + 1}}{1 - \xi} (U_F + 2 L_F \| \Gamma(u_t) \| + B)
\leq \frac{x_{\max} \xi^{\tau_{\alpha_t} + 1}}{1 - \xi} (U_F + 2 A \| \Gamma(u_t) - w^*_{\theta_t} \| + 2 A \| w^*_{\theta_t} \| + B)
\leq \frac{2 x_{\max} \xi^{\tau_{\alpha_t} + 1}}{1 - \xi} \left( A \| \Gamma(u_t) - w^*_{\theta_t} \| + C \right). \quad (25)
Putting the two bounds together completes the proof. ■

**Lemma 23** *(Bound of $M_{332}$)*

$$
\mathbb{E}[M_{332}] \leq 8|\gamma|LP L_\theta \frac{1}{A} \sum_{j=t-\tau_\alpha}^{t-1} \alpha_{t-\tau_\alpha, j} (1 + L_w L_\theta \alpha_{t-\tau_\alpha, t-1}) \left( A^2 \mathbb{E} \left[ \left\| \Gamma(u_t) - w_{\theta_t}^* \right\|^2 \right] + C^2 \right).
$$

**Proof**

$$
\mathbb{E}[M_{332}] \\
\leq \mathbb{E} \left[ \left\| \Gamma(u_{t-\tau_\alpha}) - w_{\theta_{t-\tau_\alpha}}^* \right\| \right] \mathbb{E} \left[ F(\Gamma(u_{t-\tau_\alpha}), Y_{t-\tau_\alpha}; t) - F(\Gamma(u_{t-\tau_\alpha}), \bar{Y}_{t-\tau_\alpha}; t) \right] \left\| \frac{\theta_{t-\tau_\alpha}}{w_{t-\tau_\alpha}} \right\|.
$$

(Similar to (34))

$$
\leq \mathbb{E} \left[ \frac{2(1 + L_w L_\theta \alpha_{t-\tau_\alpha, t-1})}{A} \left( A \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| + C \right) \right. \\
\times 2|\gamma|LP L_\theta \sum_{j=t-\tau_\alpha}^{t-1} \alpha_{t-\tau_\alpha, j} \left( A \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| + C \right) \left. \right]
$$

(Using (33) and Lemma 38)

$$
\leq \frac{8|\gamma|LP L_\theta \sum_{j=t-\tau_\alpha}^{t-1} \alpha_{t-\tau_\alpha, j} (1 + L_w L_\theta \alpha_{t-\tau_\alpha, t-1})}{A} \left( A^2 \mathbb{E} \left[ \left\| \Gamma(u_t) - w_{\theta_t}^* \right\|^2 \right] + C^2 \right),
$$

which completes the proof. ■

**Lemma 24** *(Bound of $M_{333}$)*

$$
\mathbb{E}[M_{333}] \leq \frac{24\xi_{max} \max \left\{ \alpha_t, \frac{\xi \tau_\alpha}{2} \right\} (1 + L_w L_\theta \alpha_{t-\tau_\alpha, t-1})}{(1 - \xi)A} \left( A^2 \mathbb{E} \left[ \left\| \Gamma(u_t) - w_{\theta_t}^* \right\|^2 \right] + C^2 \right).
$$

**Proof** Define an auxiliary chain $\{\bar{Y}_i\}_{i=t-\tau_\alpha, \ldots, t}$ as

$$
\bar{Y}_{t-\tau_\alpha} \sim d_{\theta_{t-\tau_\alpha}}(\cdot), \bar{Y}_{i+1} \sim P_{\theta_{t-\tau_\alpha}}(\bar{Y}_i, \cdot).
$$

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Then we have

\[
E[M_{333}] \\
\leq E \left[ \| \Gamma(u_{t-\tau_{t\alpha}}) - w^*_\theta_{t-\tau_{t\alpha}} \| \right] E \left[ F(\Gamma(u_{t-\tau_{t\alpha}}), \tilde{Y}_{t-\tau_{t\alpha}} : t) - F(\Gamma(u_{t-\tau_{t\alpha}}), Y_{t-\tau_{t\alpha}} : t) \mid \theta_{t-\tau_{t\alpha}} \right] \\
= E \left[ \| \Gamma(u_{t-\tau_{t\alpha}}) - w^*_\theta_{t-\tau_{t\alpha}} \| \right] E \left[ F(\Gamma(u_{t-\tau_{t\alpha}}), \tilde{Y}_{t-\tau_{t\alpha}} : t) - F(\Gamma(u_{t-\tau_{t\alpha}}), Y_{t-\tau_{t\alpha}} : t) \mid \theta_{t-\tau_{t\alpha}} \right] \\
\leq E \left[ \frac{2(1 + LwL\alpha_{t-\tau_{t\alpha}} - t - 1)}{A} (A\|\Gamma(u_t) - w^*_\theta_t\| + C) \right. \\
\left. \times \frac{6x_{\text{max}}}{1 - \xi} \max \left\{ \alpha_t, \xi^{\frac{\tau_{t\alpha}}{2}} \right\} (A\|\Gamma(u_t) - w^*_\theta_t\| + C) \right] \quad \text{(Using (33) and Lemma 39),}
\]

which completes the proof.

\[\blacksquare\]

**Lemma 25** \textit{(Bound of }M_{334}\text{)}

\[
M_{334} \leq \frac{8|S|x_{\text{max}}\xi^{\tau_{t\alpha}} + 2(1 + LwL\alpha_{t-\tau_{t\alpha}}, t - 1)}{(1 - \xi)A} \left( A^2\|\Gamma(u_t) - w^*_\theta_t\|^2 + C^2 \right)
\]

**Proof** In this proof, we use \(X \in \mathbb{R}^{|Y| \times K}\) to denote a matrix whose \(y\)-th row is \(x(y)^\top\). We use \(D_\theta \in \mathbb{R}^{|Y| \times |Y|}\) to denote the diagonal matrix whose diagonal entry is \(d_\theta\). For a matrix \(X\), we use \(X_{y,:}\) to denote its \(y\)-th row, which is understood as a column vector. We use
\( \delta(w) \in \mathbb{R}^{|Y|} \) to denote a vector whose \( y \)-th element is \( \delta(w, y) \).

\[
\ddot{F}(w) - \ddot{F}^{(n)}(w) = \mathbb{E}_{Y_1 \sim d_\theta(\cdot), Y_{i+1} \sim P_\theta(Y_i, \cdot)} [F(w, Y_{1:n})] = \mathbb{E} [F(w, Y_{1:n}) | Y_n] = \mathbb{E} \left[ \delta(w, Y_n) \mathbb{E} \left[ \sum_{k=0}^{n} \xi^k x(Y_{n-k}) | Y_n \right] \right] + w
\]

\[
= \sum_{y_n} \Pr(Y_n = y_n) \delta(w, y_n) \sum_{k=0}^{n} \sum_{y_{n-k}} \Pr(Y_{n-k} = y_{n-k} | Y_n = y_n) \xi^k x(y_{n-k}) + w
\]

\[
= \sum_{y_n} \Pr(Y_n = y_n) \delta(w, y_n) \sum_{k=0}^{n} \frac{\Pr(Y_{n-k} = y_{n-k}, Y_n = y_n)}{\Pr(Y_n = y_n)} \xi^k x(y_{n-k}) + w
\]

\[
= \sum_{y_n} \Pr(Y_n = y_n) \delta(w, y_n) \sum_{k=0}^{n} d_\theta(y_{n-k}) P_\theta^k (y_{n-k}, y_n) d_\theta(y_n) - \xi^k x(y_{n-k}) + w
\]

\[
= \sum_{y_n} \Pr(Y_n = y_n) \delta(w, y_n) \xi^k \left( D_\theta^{-1}(P_\theta)^k D_\theta X \right)_{y_n} + w
\]

\[
= \sum_{y_n} \Pr(Y_n = y_n) \delta(w, y_n) \left( D_\theta^{-1} \sum_{k=0}^{n} (\xi P_\theta)^k D_\theta X \right)_{y_n} + w
\]

\[
= \left( X^\top D_\theta \sum_{k=0}^{n} (\xi P_\theta)^k D_\theta^{-1} \right) D_\theta \delta(w) + w
\]

Consequently, we have

\[
\left\| \ddot{F}(w) - \ddot{F}^{(n)}(w) \right\| \leq \left\| X^\top \right\| \left\| \sum_{k=n+1}^{\infty} (\xi P_\theta)^k \delta(w) \right\| \leq \left\| X^\top \right\| \left\| \sum_{k=n+1}^{\infty} (\xi P_\theta)^k \delta(w) \right\| \leq \sqrt{|S|} x_{\max} \left\| \sum_{k=n+1}^{\infty} (\xi P_\theta)^k \delta(w) \right\| \leq \sqrt{|S|} x_{\max} \frac{\xi^{n+1}}{1 - \xi} \max_y |\delta(w, y)|.
\]
We, therefore, conclude that

\[ M_{334} = \langle (u_{t-\tau_\alpha}) - w_{\theta_{t-\tau_\alpha}}^*, \tilde{F}_{\theta_{t-\tau_\alpha}}^{(\tau_\alpha+1)} (\Gamma(u_{t-\tau_\alpha})) - \tilde{F}_{\theta_{t-\tau_\alpha}} (\Gamma(u_{t-\tau_\alpha})) \rangle \]

\[ \leq \| (u_{t-\tau_\alpha}) - w_{\theta_{t-\tau_\alpha}}^* \| \| \tilde{F}_{\theta_{t-\tau_\alpha}}^{(\tau_\alpha+1)} (\Gamma(u_{t-\tau_\alpha})) - \tilde{F}_{\theta_{t-\tau_\alpha}} (\Gamma(u_{t-\tau_\alpha})) \| \]

\[ \leq \frac{2(1 + L_w L_\theta \alpha_{t-\tau_\alpha}, t-1)}{A} (A\| \Gamma(u_t) - w_{\theta_t}^* \| + C) \]

\times \left| S \right| x_{\max} \frac{\xi^{\tau_\alpha+2}}{1 - \xi} \max_y \| \delta(\Gamma(u_t), y) \| \quad \text{(Using (33))} \]

\[ \leq \frac{4(1 + L_w L_\theta \alpha_{t-\tau_\alpha}, t-1)}{A} (A\| \Gamma(u_t) - w_{\theta_t}^* \| + C)^2 \left| S \right| x_{\max} \frac{\xi^{\tau_\alpha+2}}{1 - \xi} \quad \text{(Using (25))} , \]

which completes the proof.

\[ \text{Lemma 26 (Bound of } M_{335} \text{)} \]

\[ M_{335} \leq \frac{4L_w^2 L_\theta \alpha_{t-\tau_\alpha}, t-1}{A^2} \left(1 + L_w L_\theta \alpha_{t-\tau_\alpha}, t-1\right) \left(A^2\| \Gamma(u_t) - w_{\theta_t}^* \|^2 + C^2\right) . \]

The proof of Lemma 26 is identical to that of Lemma 14 in Section D.8 and is thus omitted.

\[ \text{Lemma 27 (Bound of } M_5 \text{)} \]

\[ M_5 \leq 2 \left(A^2\| \Gamma(u_t) - w_{\theta_t}^* \|^2 + C^2\right) . \]

The proof of Lemma 27 is identical to that of Lemma 15 in Section D.9 up to replacing \( Y_t \) with \( Y_{0:t} \) and is thus omitted.

\[ \text{Lemma 28 (Bound of } M_6 \text{)} \]

\[ M_6 = \left\| w_{\theta_t}^* - w_{\theta_{t+1}}^* \right\|^2 \leq L_w^2 L_\theta^2 \alpha_t^2 . \]

The proof of Lemma 28 follows immediately from Lemma 34.

Define

\[ L_{\alpha,t} = \max \left\{ \xi^{\frac{\tau_\alpha}{2}}, \sum_{j=t-\tau_\alpha}^t \alpha_{t-\tau_\alpha,j} \left(1 + L_w L_\theta \max \{1, \alpha_{t-\tau_\alpha,t}\}\right) \right\} , \]

\[ C_0 = \max \{ A^2, C^2 \} \max \left\{ 16, L_w L_\theta, 24 x_{\max} |S|, L_w^2 L_\theta^2, 8 |Y| L_P L_\theta, 4L_w L_\theta, 4L''_w L_\theta \right\} . \]
Similar to the proof of Theorem 3 in Section A.1, we can get
\[
\mathbb{E}\left[\|u_{t+1} - w_{\hat{b}_{t+1}}^*\|^2\right] \leq (1 - 2(1 - \kappa_{\alpha_t} - C_0\alpha_t L_{\alpha,t} - C_0\alpha_t^2)) \mathbb{E}\left[\|\Gamma(u_t) - w_{\hat{b}_t}^*\|^2\right] + 2L_w L_{\theta t} \mathbb{E}\left[\|\Gamma(u_t) - w_{\hat{b}_t}^*\|\right] + 2C_0\alpha_t L_{\alpha,t} + 4C_0\alpha_t^2.
\]
Observing that
\[
L_{\alpha,t} = O\left(\frac{1}{(t + t_0)^{\epsilon_{\alpha} - 2\epsilon}}\right)
\]
then completes the proof.

It now becomes clear why we set \(\tau_{\alpha_t}\) to be of the order of \(O\left(\tau_{\alpha_t}\right)\). If \(\tau_{\alpha_t}\) was of the order \(O\left(\log t\right)\) just like the proof of Theorem 3, then \(L_{\alpha,t}\) would have been of the order \(O\left(\xi \ln t\right)\), which decays much slower.

**Corollary 29** Let Assumptions 3.1, 3.2, 3.6, B.1, B.2, and B.3 hold. Assume \(\kappa_{\alpha} = \sqrt{1 - \eta_{\alpha}}\) for some positive constant \(\eta > 0\). Let \(\epsilon \in (0, \frac{\eta_{\alpha}}{2})\) be any constant. Let \(t_0\) be sufficiently large. Then the iterates \(\{w_t\}\) generated by (23) satisfy
\[
\mathbb{E}\left[\|w_t - w_{w_t}^*\|^2\right] = \frac{72L_w^2 L_{\theta t}^2}{\eta^2} + \begin{cases}
O\left(\frac{1}{t^{\epsilon_{\alpha} - 2\epsilon}}\right), & \epsilon_{\alpha} = 1, \eta_{\alpha} \in (0, 3) \\
O\left(\frac{\log t}{t^{2\epsilon}}\right), & \epsilon_{\alpha} = 1, \eta_{\alpha} = 3 \\
O\left(\frac{1}{t^{2\epsilon}}\right), & \epsilon_{\alpha} = 1, \eta_{\alpha} \in (3, \infty) \end{cases}.
\]

**Proof** The proof is mostly identical to that of Corollary 4 in Section A.2. We omit the duplicate parts to avoid verbatim repetition. Since
\[
\lim_{t \to \infty} \frac{1 - \sqrt{1 - t}}{\alpha_t L_{\alpha,t}} = \infty,
\]
we can similarly get
\[
z_t^2 \leq \prod_{i=t_0}^{t-1} \left(1 - \frac{\eta}{3} \alpha_i\right) z_{t_0}^2 + \frac{24L_w^2 L_{\theta t}^2}{\eta} \sum_{i=t_0}^{t-1} \prod_{j=i+1}^{t-1} \left(1 - \frac{\eta}{3} \alpha_j\right) \alpha_i^{\epsilon} + C_1 \sum_{i=t_0}^{t-1} \prod_{j=i+1}^{t-1} \left(1 - \frac{\eta}{3} \alpha_j\right) \alpha_i^{\epsilon} (i + t_0)^{2\epsilon},
\]
where \(C_1\) is some positive constant and
\[
z_t \doteq \sqrt{\mathbb{E}\left[\|u_t - w_{\hat{b}_t}^*\|^2\right]}.
\]
The terms $F_1$ and $F_2$ are identical to $E_1$ and $E_2$ in Section A.2. For $F_3$, we have

$$F_3 \leq (t + t_0)^2 e^{t-1} \prod_{i=t_0}^{t-1} \left( 1 - \frac{\eta}{3} \alpha_j \right) \alpha_i^2.$$

The term $F_4$ is identical to $E_4$ in Section A.2, so does the rest of the proof.

**B.2.2 Proof of Theorem 6**

**Theorem 6** Let Assumptions 3.6 and 4.1 - 4.3 hold. Assume $\|X\| = 1, r_{\max} \leq 1$ and $\|r\|$ is not so large such that

$$L_w = O(L_{\pi} \|r\|) < 1.$$

Assume $C_\Gamma$ is large enough such that

$$U_w = O(\|r\|) \leq C_\Gamma.$$

Let $t_0$ be sufficiently large. Let $\epsilon_0 \in (0, \frac{\eta}{2})$ be any constant. Then the iterates $\{w_t\}$ generated by Algorithm 1 with any $\lambda \in [0, 1]$ satisfy

$$E[\|w_t - w_*\|] = \frac{6 \sqrt{2} L_w (1 + 4 C_\Gamma)}{\eta (1 - L_w)} + \begin{cases} O \left( t^{-\frac{\eta c_\alpha}{6} + \epsilon_0} \right), & \epsilon_\alpha = 1, \eta c_\alpha \in (0, 3) \\ O \left( t^{-\frac{\eta c_\alpha}{2} + \epsilon_0} \right) \log t, & \epsilon_\alpha = 1, \eta c_\alpha = 3 \\ O \left( t^{-\frac{\eta c_\alpha}{4} + \epsilon_0} \right), & \epsilon_\alpha = 1, \eta c_\alpha \in (3, \infty) \\ O \left( t^{-\frac{\eta c_\alpha}{2} + \epsilon_0} \right), & \epsilon_\alpha \in (0, 1) \end{cases}$$

where $\eta$ is a positive constant and $w_*$ is the unique vector such that $e_{\lambda}(w) = 0$.

**Proof** We first rewrite Algorithm 1 in the form of (23). To this end, define

$$Y \doteq \{(s, a, s', a') \mid s \in S, a \in A, s' \in S, p(s' | s, a) > 0\},$$

$$Y_i \doteq (S_i, A_i, S_{i+1}, A_{i+1}),$$

$$y \doteq (s, a, s', a'),$$

$$\pi_{\theta}(a|s) \doteq \pi_{\Gamma(\theta)}(a|s),$$

$$P_{\theta}((s_1, a_1, s'_1, a'_1), (s_2, a_2, s'_2, a'_2)) \doteq \begin{cases} 0 & (s'_1, a'_1) \neq (s_2, a_2) \\ p(s'_2 | s_2, a_2) \pi_{\theta}(a'_2 | s'_2) & (s'_1, a'_1) = (s_2, a_2) \end{cases},$$

$$\delta(w, y) \doteq r(s, a) + \gamma x(s', a')^\top w - x(s, a)^\top w,$$

$$x(y) \doteq x(s, a),$$

$$F(w, y_{i:j}) \doteq \delta(w, y_j) \sum_{k=0}^{j-i} \xi^k x(y_{j-k}) + w.$$
The update of \( \{w_t\} \) in Algorithm 1 can then be expressed as

\[
w_{t+1} = \Gamma (w_t + \alpha_t (F(w_t, Y_{0:t}) - w_t)).
\]

We now proceed by verifying Assumptions 3.1\,3.2, B.1, B.2, and B.3 and thus invoking Corollary 29. The verification of Assumptions 3.1\,3.2, B.2, and B.3 are similar to that of the proof of Theorem 5 in Section B.1 and are thus omitted. We focus on Assumption B.1 in this proof. With our definition of \( \delta(w, y) \) above, it can be computed, similarly to (26), that

\[
\bar{F}_\theta^n (w) = X^\top D_{\tilde{\pi}_\theta} \sum_{k=0}^{n} (\gamma \lambda P_{\tilde{\pi}_\theta})^k (r + \gamma P_{\tilde{\pi}_\theta} Xw - Xw) + w.
\]

It is then easy to see that

\[
\bar{F}_\theta (w) = X^\top D_{\tilde{\pi}_\theta} (I - \gamma \lambda P_{\tilde{\pi}_\theta})^{-1} (r + \gamma P_{\tilde{\pi}_\theta} Xw - Xw) + w,
\]

\[
f_\theta^\alpha (w) = w + \alpha X^\top D_{\tilde{\pi}_\theta} (I - \gamma \lambda P_{\tilde{\pi}_\theta})^{-1} (r + \gamma P_{\tilde{\pi}_\theta} Xw - Xw).
\]

Using

\[
(I - \gamma \lambda P_{\tilde{\pi}_\theta})^{-1} (I - \gamma P_{\tilde{\pi}_\theta}) = (I - \gamma \lambda P_{\tilde{\pi}_\theta})^{-1} (I - \gamma \lambda P_{\tilde{\pi}_\theta} + \gamma \lambda P_{\tilde{\pi}_\theta} - \gamma P_{\tilde{\pi}_\theta}) = I - P_{\tilde{\pi}_\theta, \lambda}
\]

then yields

\[
f_\theta^\alpha (w) = w + \alpha X^\top D_{\tilde{\pi}_\theta} \left( X^\top D_{\tilde{\pi}_\theta} X \right)^{-1} X^\top D_{\tilde{\pi}_\theta} X (r_{\tilde{\pi}_\theta, \lambda} + P_{\tilde{\pi}_\theta, \lambda} Xw - Xw)
\]

\[
= w + \alpha X^\top D_{\tilde{\pi}_\theta} \left( \Pi_{D_{\tilde{\pi}_\theta}} T_{\tilde{\pi}_\theta, \lambda} Xw - Xw \right).
\]

The rest of the verification of Assumption 3.3 is identical to Lemma 35 up to replacing \( T_{\tilde{\pi}_\theta} \) with \( T_{\tilde{\pi}_\theta, \lambda} \) and using \( \Pi_{D_{\tilde{\pi}_\theta}} T_{\tilde{\pi}_\theta, \lambda} \) is a \( \frac{\gamma (1-\lambda)}{1-\gamma \lambda} \) contraction w.r.t. \( \|\cdot\|_{D_{\tilde{\pi}_\theta}} \) (see, e.g., Lemma 6.6 of Bertsekas and Tsitsiklis (1996)). In particular, we have

\[
\kappa_{\alpha} = \sqrt{1 - \frac{1 - \gamma}{1 - \gamma \lambda} \inf_{\lambda \in [0, 1]} \lambda_{\min} \left( X^\top D_{\pi_{\theta}} X \right) \alpha}.
\]

Appendix C. Technical Lemmas

**Lemma 30** Let \( f_1(x), f_2(x) \) be two Lipschitz continuous functions with Lipschitz constants \( L_1, L_2 \). Assume \( \|f_1(x)\| \leq U_1, \|f_2(x)\| \leq U_2 \), then \( L_1 U_2 + L_2 U_1 \) is a Lipschitz constant of \( f(x) = f_1(x) f_2(x) \).
\textbf{Lemma 31} Given positive integers \(t_1 < t_2\) satisfying
\[ \alpha_{t_1, t_2-1} \leq \frac{1}{4A}, \]
we have, for any \(t \in [t_1, t_2]\),
\[
\begin{align*}
\| \Gamma(u_t) - \Gamma(u_{t_1}) \| &\leq 2\alpha_{t_1, t_2-1}(A\|\Gamma(u_{t_1})\| + B), \\
\| \Gamma(u_t) - \Gamma(u_{t_2}) \| &\leq 4\alpha_{t_1, t_2-1}(A\|\Gamma(u_{t_2})\| + B), \\
\| \Gamma(u_t) - \Gamma(u_{t_1}) \| &\leq \min \{\|\Gamma(u_{t_1})\|, \|\Gamma(u_{t_2})\|\} + \frac{B}{A}. 
\end{align*}
\]

\textbf{Proof} Notice that
\[
\begin{align*}
\| \Gamma(u_{t+1}) \| - \| \Gamma(u_t) \| &\leq \| \Gamma(u_{t+1}) - \Gamma(u_t) \| \\
&\leq \| u_{t+1} - \Gamma(u_t) \| \\
&= \alpha_t \| F_{\theta_t}(\Gamma(u_t), Y_t) - \Gamma(u_t) \| \\
&\leq \alpha_t (\| F_{\theta_t}(\Gamma(u_t), Y_t) \| + \| \Gamma(u_t) \|) \\
&\leq \alpha_t (U_F + (L_F + 1)\| \Gamma(u_t) \|) \quad \text{(Lemma 36)} \\
&\leq \alpha_t (A\| \Gamma(u_t) \| + B) \quad \text{(Using (16))} 
\end{align*}
\]

The rest of the proof follows from the proof of Lemma A.2 of Chen et al. (2021) up to changes of notations. We include it for completeness. Rearranging terms of the above inequality yields
\[
\| \Gamma(u_{t+1}) \| + \frac{B}{A} \leq (1 + \alpha_t A) \left( \| \Gamma(u_t) \| + \frac{B}{A} \right),
\]

implying that for any \(t \in (t_1, t_2]\),
\[
\| \Gamma(u_t) \| + \frac{B}{A} \leq \prod_{j=t_1}^{t-1} (1 + A\alpha_j) \left( \| \Gamma(u_{t_1}) \| + \frac{B}{A} \right).
\]

Notice that for any \(x \in [0, \frac{1}{2}]\), \(1 + x \leq \exp(x) \leq 1 + 2x\) always hold. Hence
\[
\alpha_{t_1, t_2-1} \leq \frac{1}{4A}
\]
implies
\[ \prod_{j=t_1}^{t-1} (1 + A\alpha_j) \leq \exp(A\alpha_{t_1,t-1}) \leq 1 + 2A\alpha_{t_1,t-1}. \]

Consequently, for any \( t \in (t_1, t_2) \), we have
\[ \|\Gamma(u_t)\| + \frac{B}{A} \leq (1 + 2A\alpha_{t_1,t-1}) \left( \|\Gamma(u_{t_1})\| + \frac{B}{A} \right) \]
\[ \implies \|\Gamma(u_t)\| \leq (1 + 2A\alpha_{t_1,t-1}) \|\Gamma(u_{t_1})\| + 2B\alpha_{t_1,t-1}, \]
which together with (30) yields that for any \( t \in (t_1, t_2 - 1) \)
\[ \|\Gamma(u_{t+1}) - \Gamma(u_t)\| \leq \alpha_t (A\|\Gamma(u_t)\| + B) \]
\[ \leq \alpha_t (A (1 + 2A\alpha_{t_1,t-1}) \|\Gamma(u_{t_1})\| + 2AB\alpha_{t_1,t-1} + B) \]
\[ \leq 2\alpha_t (A\|\Gamma(u_{t_1})\| + B) \quad \text{(Using } \alpha_{t_1,t-1} \leq \frac{1}{4A} \text{)} . \]

Consequently, for any \( t \in (t_1, t_2) \), we have
\[ \|\Gamma(u_t) - \Gamma(u_{t_1})\| \leq \sum_{j=t_1}^{t-1} \|\Gamma(w_{j+1}) - \Gamma(w_j)\| \leq \sum_{j=t_1}^{t-1} 2\alpha_j (A\|\Gamma(u_t)\| + B) \]
\[ = 2\alpha_{t_1,t-1} (A\|\Gamma(u_{t_1})\| + B) \leq 2\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_1})\| + B), \]
which completes the proof of (27). For (28), we have from the above inequality
\[ \|\Gamma(u_{t_2}) - \Gamma(u_{t_1})\| \leq 2\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_1})\| + B) \]
\[ \leq 2\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_1}) - \Gamma(u_{t_2})\| + A\|\Gamma(u_{t_2})\| + B) \]
\[ \leq \frac{1}{2}\|\Gamma(u_{t_1}) - \Gamma(u_{t_2})\| + 2\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_2})\| + B), \]
implying
\[ \|\Gamma(u_{t_2}) - \Gamma(u_{t_1})\| \leq 4\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_2})\| + B). \]

Consequently, for any \( t \in [t_1, t_2] \),
\[ \|\Gamma(u_t) - \Gamma(u_{t_1})\| \leq 2\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_1})\| + B) \]
\[ \leq 2\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_1}) - \Gamma(u_{t_2})\| + A\|\Gamma(u_{t_2})\| + B) \]
\[ \leq 2\alpha_{t_1, t_2-1} (A4\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_2})\| + B) + A\|\Gamma(u_{t_2})\| + B) \]
\[ \leq 4\alpha_{t_1,t_2-1} (A\|\Gamma(u_{t_2})\| + B) \quad \text{(Using } \alpha_{t_1,t_2-1} \leq \frac{1}{4A} \text{)} , \]
which completes the proof of (28). (27) implies
\[ \|\Gamma(u_t) - \Gamma(u_{t_1})\| \leq \|\Gamma(u_{t_1})\| + \frac{B}{A}, \]
(28) implies
\[ \| \Gamma(u_t) - \Gamma(u_{t_1}) \| \leq \| \Gamma(u_{t_2}) \| + \frac{B}{A}, \]
then (29) follows immediately, which completes the proof. \[ \square \]

**Lemma 32** Let Assumptions 4.1 and 4.2 hold. Then there exists a constant \( L'_\pi \) such that
\[ \forall \theta, \theta', a, s, \]
\[ |d_{\pi_\theta}(s, a) - d_{\pi_{\theta'}}(s, a)| \leq L'_\pi \| \theta - \theta' \|. \]

**Proof** See, e.g., Lemma 9 of Zhang et al. (2021b). \[ \square \]

**Lemma 33** For any \( \| \cdot \| \), we have
\[ \| X^{-1} - Y^{-1} \| \leq \| X^{-1} \| \| X \| \| Y \| \| Y^{-1} \|. \]

**Proof**
\[ \| X^{-1} - Y^{-1} \| = \| X^{-1} Y Y^{-1} - X Y Y^{-1} \| \leq \| X^{-1} \| \| X \| \| Y \| \| Y^{-1} \|. \]
\[ \square \]

**Lemma 34** Recall that
\[ L_\theta = U_F + (L_F + 1)C_\Gamma, \]
then for any \( j > i, y, y', w, \)
\[ \| w_{\theta_j} - w_{\theta_i} \| \leq L_\omega L_\theta \alpha_{i,j-1}, \]
\[ |P_{\theta_j}(y, y') - P_{\theta_i}(y, y')| \leq L_F L_\theta \alpha_{i,j-1}, \]
\[ \| F_{\theta_j}(w, y) - F_{\theta_i}(w, y) \| \leq L'_F L_\theta \alpha_{i,j-1} (\| w \| + U'_F), \]
\[ \| F_{\theta_j}(w) - F_{\theta_i}(w) \| \leq L'_F L_\theta \alpha_{i,j-1} (\| w \| + U'_F). \]
Proof

\[
\left\| w_{\theta_j}^* - w_{\theta_i}^* \right\| \leq \sum_{k=i}^{j-1} \left\| w_{\theta_{k+1}}^* - w_{\Gamma(\theta_k)}^* \right\| \quad \text{(Assumption 3.5)}
\]

\[
\leq \sum_{k=i}^{j-1} L_w \left\| \theta_{k+1} - \Gamma(\theta_k) \right\| 
\]

\[
= \sum_{k=i}^{j-1} L_w \left\| w_{k+1} - \Gamma(w_k) \right\|
\]

\[
= \sum_{k=i}^{j-1} L_w \left\| \Gamma(u_{k+1}) - \Gamma(u_k) \right\| \quad \text{(Lemma 2)}
\]

\[
\leq \sum_{k=i}^{j-1} L_w \left\| u_{k+1} - \Gamma(u_k) \right\|
\]

\[
= \sum_{k=i}^{j-1} L_w \alpha_k \left\| F_{\theta_k}(\Gamma(u_k), Y_k) - \Gamma(u_k) \right\|
\]

\[
\leq \sum_{k=i}^{j-1} L_w \alpha_k (U_F + L_F \|\Gamma(u_k)\| + \|\Gamma(u_k)\|) \quad \text{(Lemma 36)}
\]

\[
\leq \sum_{k=i}^{j-1} L_w \alpha_k (U_F + L_F C_\Gamma + C_\Gamma)
\]

\[
= L_w L_\theta \alpha_{i,j-1}.
\]

Similarly we can get

\[
\left| F_{\theta_j}(y, y') - F_{\theta_i}(y, y') \right| \leq L_P L_\theta \alpha_{i,j-1}.
\]

Moreover,

\[
\left\| F_{\theta_j}(w, y) - F_{\theta_i}(w, y) \right\| \leq \sum_{k=i}^{j-1} \left\| F_{\theta_{k+1}}(w, y) - F_{\theta_k}(w, y) \right\|
\]

\[
\leq \sum_{k=i}^{j-1} \left\| F_{\theta_{k+1}}(w, y) - F_{\Gamma(\theta_k)}(w, y) \right\| \quad \text{(Assumption 3.5)}
\]

\[
\leq \sum_{k=i}^{j-1} L_F' \left\| \theta_{k+1} - \Gamma(\theta_k) \right\| \left( \|w\| + U_F' \right)
\]

\[
\leq L_F' L_\theta \alpha_{i,j-1} \left( \|w\| + U_F' \right).
\]
Since $P_\theta = P_{\Gamma(\theta)}$, it is easy to see $d_\theta(y) = d_{\Gamma(\theta)}(y)$. Consequently, $\tilde{F}_\theta(w) = \tilde{F}_{\Gamma(\theta)}(w)$. We can then similarly get

$$\| \tilde{F}_{\theta_i}(w) - \tilde{F}_{\theta_j}(w) \| \leq L_{\tilde{F}}^\prime \|w\| + U_{\tilde{F}}^\prime,$$

which completes the proof. ■

**Lemma 35 (Lemma 5.4 of De Farias and Van Roy (2000))** There exists an $\tilde{\alpha}$ such that for all $\alpha \in (0, \tilde{\alpha})$ and all $\theta$,

$$\| f^\alpha_\theta(w) - w^*_\theta \| \leq \kappa_\alpha \|w - w^*_\theta\|,$$

where

$$\kappa_\alpha \doteq \sqrt{1 - (1 - \gamma) \inf_\theta \lambda_{\text{min}}(X^\top D_{\tilde{\pi}_\theta} X) \alpha < 1}.$$

Here $\lambda_{\text{min}}(\cdot)$ denotes the minimum eigenvalue of a symmetric positive definite matrix.

**Proof** The proof is due to De Farias and Van Roy (2000); we rewrite it in our notation for completeness. We first recall

$$f^\alpha_\theta(w) = w + \alpha \left( X^\top D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X w + X^\top D_{\tilde{\pi}_\theta} r \right),$$

$$w^*_\theta = - \left( X^\top D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X \right)^{-1} X^\top D_{\tilde{\pi}_\theta} r.$$

Define

$$g_\theta(w) = X^\top D_{\tilde{\pi}_\theta} (\gamma P_{\tilde{\pi}_\theta} - I) X w + X^\top D_{\tilde{\pi}_\theta} r$$

$$= X^\top D_{\tilde{\pi}_\theta} X \left( X^\top D_{\tilde{\pi}_\theta} X \right)^{-1} X^\top D_{\tilde{\pi}_\theta} (\tilde{\pi}_\theta X w - X w)$$

$$= X^\top D_{\tilde{\pi}_\theta} \Pi_{D_{\tilde{\pi}_\theta}} \tilde{\pi}_\theta X w - X^\top D_{\tilde{\pi}_\theta} X w$$

$$= X^\top D_{\tilde{\pi}_\theta} \left( \Pi_{D_{\tilde{\pi}_\theta}} \tilde{\pi}_\theta X w - X w \right).$$

By the contraction property (see, e.g., Tsitsiklis and Roy (1996)),

$$\left\| \Pi_{D_{\tilde{\pi}_\theta}} \tilde{\pi}_\theta X w - X w^*_\theta \right\|_{D_{\tilde{\pi}_\theta}} \leq \gamma \|X w - X w^*_\theta\|_{D_{\tilde{\pi}_\theta}}.$$

Consequently,

$$\left( w - w^*_\theta \right)^\top g_\theta(s)$$

$$= (X w - X w^*_\theta)^\top D_{\tilde{\pi}_\theta} \left( \Pi_{D_{\tilde{\pi}_\theta}} \tilde{\pi}_\theta X w - X w \right)$$

$$= (X w - X w^*_\theta)^\top D_{\tilde{\pi}_\theta} \left( \Pi_{D_{\tilde{\pi}_\theta}} \tilde{\pi}_\theta X w - X w^*_\theta + X w^*_\theta - X w \right)$$

$$\leq \|X w - X w^*_\theta\|_{D_{\tilde{\pi}_\theta}} \left\| \Pi_{D_{\tilde{\pi}_\theta}} \tilde{\pi}_\theta X w - X w^*_\theta \right\|_{D_{\tilde{\pi}_\theta}} - \|X w - X w^*_\theta\|_{D_{\tilde{\pi}_\theta}}^2$$
(Cauchy-Schwarz inequality)

\[ \leq (\gamma - 1) \| Xw - Xw_\theta^* \|_{D_\theta}^2 \]

\[ = (\gamma - 1)(w - w_\theta^*)^\top \left( X^\top D_\pi X \right) (w - w_\theta^*). \]

Since \( X^\top D_\pi X \) is symmetric and positive define, eigenvalues are continuous in the elements of the matrix, \( \Lambda_\pi \) is compact, we conclude, by the extreme value theorem, that

\[ C_1 \doteq \inf_\theta \lambda_{\min} \left( X^\top D_\pi X \right) > 0. \]

Consequently, for any \( y \) and \( \theta \),

\[ y^\top X^\top D_\pi X y \geq C_1 \| y \|^2, \]

implying

\[ y^\top X^\top D_\pi X y \geq C_1 \| y \|^2. \]

It follows immediately that

\[ (w - w_\theta^*)^\top g_\theta(w) \leq -(1 - \gamma)C_1 \| w - w_\theta^* \|^2. \tag{31} \]

Moreover, let \( x_i \) be the \( i \)-th column of \( X \), we have

\[ \| g_\theta(w) \|^2 = \sum_{i=1}^K \left( x_i^\top D_\pi \left( \Pi_{D_\theta} T_{D_\theta} Xw - Xw \right) \right)^2 \]

\[ \leq \sum_{i=1}^K \| x_i \|^2_{D_\theta} \left\| \Pi_{D_\theta} T_{D_\theta} Xw - Xw \right\|_{D_\theta}^2 \]

(Cauchy-Schwarz inequality)

\[ \leq \sum_{i=1}^K \| x_i \|^2_{D_\theta} \left( \left\| \Pi_{D_\theta} T_{D_\theta} Xw - Xw_\theta^* \right\|_{D_\theta} + \| Xw_\theta^* - Xw \|_{D_\theta} \right)^2 \]

\[ \leq (1 + \gamma)^2 \sum_{i=1}^K \| x_i \|^2_{D_\theta} \left( \left\| Xw_\theta^* - Xw \right\|^2_{D_\theta} \right) \]

\[ = (1 + \gamma)^2 \left( \sum_{i=1}^K \| x_i \|^2_{D_\theta} \right) \left\| X^\top D_\pi X \left\| w - w_\theta^* \right\| \right. \]

According to the extreme value theorem,

\[ C_2 \doteq \sup_\theta \left( \sum_{i=1}^K \| x_i \|^2_{D_\theta} \right) \left\| X^\top D_\pi X \right\| < \infty. \]
Consequently, we have
\[ \|g_\theta(w)\|^2 \leq (1 + \gamma)^2 C_2\|w - w^*_\theta\|^2. \] (32)

Combining (31) and (32) yields
\[
\|f_\theta^\alpha(w) - w^*_\theta\|^2 = \|w + \alpha g_\theta(w) - w^*_\theta\|^2
= \|w - w^*_\theta\|^2 + 2\alpha (w - w^*_\theta)^\top g_\theta(w) + \alpha^2\|g_\theta(w)\|^2
\leq (1 - 2\alpha(1 - \gamma)C_1 + (1 + \gamma)^2 \alpha^2 C_2) \|w - w^*_\theta\|^2.
\]

Consequently, if
\[
\alpha < \bar{\alpha} = \frac{(1 - \gamma)C_1}{(1 + \gamma)^2 C_2},
\]
we have
\[
1 - 2\alpha(1 - \gamma)C_1 + (1 + \gamma)^2 \alpha^2 C_2 \leq 1 - (1 - \gamma)C_1 \alpha.
\]
Defining
\[
\kappa_\alpha = \sqrt{1 - (1 - \gamma)C_1 \alpha}
\]
then completes the proof. Importantly, both $C_1$ and $C_2$ here are independent of $C_T$. ■

Appendix D. Proof of Auxiliary Lemmas

D.1 Proof of Lemma 7

Lemma 7 (Bound of $T_1$)
\[
T_1 \leq L_w L_\theta \alpha_t \|\Gamma(u_t) - w^*_{\theta_t}\|.
\]

Proof
\[
T_1 = \langle \Gamma(u_t) - w^*_{\theta_t}, w^*_{\theta_t} - w^*_{\theta_{t+1}} \rangle
\leq \|\Gamma(u_t) - w^*_{\theta_t}\| \|w^*_{\theta_t} - w^*_{\theta_{t+1}}\|
\leq \|\Gamma(u_t) - w^*_{\theta_t}\| L_w L_\theta \alpha_t \quad \text{(Lemma 34)}
\]

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D.2 Proof of Lemma 8

Lemma 8 (Bound of $T_2$)

\[ T_2 \leq -(1 - \kappa_{\alpha_t}) \| \Gamma(u_t) - w_{\theta_t}^* \|^2. \]

Proof

\[ T_2 = \langle \Gamma(u_t) - w_{\theta_t}^*, f_{\theta_t}^\alpha (\Gamma(u_t)) - \Gamma(u_t) \rangle = \langle \Gamma(u_t) - w_{\theta_t}^*, f_{\theta_t}^\alpha (\Gamma(u_t)) - w_{\theta_t}^* \rangle - \langle \Gamma(u_t) - w_{\theta_t}^*, \Gamma(u_t) - w_{\theta_t}^* \rangle \]
\[ \leq \| \Gamma(u_t) - w_{\theta_t}^* \| \| f_{\theta_t}^\alpha (\Gamma(u_t)) - w_{\theta_t}^* \| - \| \Gamma(u_t) - w_{\theta_t}^* \|^2 \]
\[ \leq \| \Gamma(u_t) - w_{\theta_t}^* \| \kappa_{\alpha_t} \| \Gamma(u_t) - w_{\theta_t}^* \| - \| \Gamma(u_t) - w_{\theta_t}^* \|^2 \quad (\text{Assumption 3.3}) \]
\[ = -(1 - \kappa_{\alpha_t}) \| \Gamma(u_t) - w_{\theta_t}^* \|^2. \]

\[ \square \]

D.3 Proof of Lemma 9

Lemma 9 (Bound of $T_{31}$)

\[ T_{31} \leq 8(L_w L_\theta + 1)\alpha_{t - \tau_{\alpha_t}, t - 1} \left( A^2 \| \Gamma(u_t) - w_{\theta_t}^* \|^2 + C^2 \right). \]

Proof

\[ T_{31} = \langle \Gamma(u_t) - w_{\theta_t}^*, \left( \Gamma(u_{t-\tau_{\alpha_t}}) - w_{\theta_{t-\tau_{\alpha_t}}}^* \right), F_{\theta_t}(\Gamma(u_t), Y_t) - \tilde{F}_{\theta_t}(\Gamma(u_t)) \rangle \]
\[ \leq \| \Gamma(u_t) - w_{\theta_t}^* \| \left( \| \Gamma(u_{t-\tau_{\alpha_t}}) - w_{\theta_{t-\tau_{\alpha_t}}}^* \| \right) \| F_{\theta_t}(\Gamma(u_t), Y_t) - \tilde{F}_{\theta_t}(\Gamma(u_t)) \|. \]

For the first term, we have

\[ \| \Gamma(u_t) - w_{\theta_t}^* - \left( \Gamma(u_{t-\tau_{\alpha_t}}) - w_{\theta_{t-\tau_{\alpha_t}}}^* \right) \| \]
\[ \leq \| \Gamma(u_t) - \Gamma(u_{t-\tau_{\alpha_t}}) \| + \| w_{\theta_t}^* - w_{\theta_{t-\tau_{\alpha_t}}}^* \| \]
\[ \leq \| \Gamma(u_t) - \Gamma(u_{t-\tau_{\alpha_t}}) \| + L_w L_\theta \alpha_{t-\tau_{\alpha_t}, t - 1} \quad (\text{Lemma 34}) \]
\[ \leq 4 \alpha_{t - \tau_{\alpha_t}, t - 1} (A \| \Gamma(u_t) \| + B) + L_w L_\theta \alpha_{t-\tau_{\alpha_t}, t - 1} \quad (\text{Lemma 31}) \]
\[ \leq 4 \alpha_{t - \tau_{\alpha_t}, t - 1} (A \| \Gamma(u_t) - w_{\theta_t}^* \| + A \| w_{\theta_t}^* \| + B) + L_w L_\theta \alpha_{t-\tau_{\alpha_t}, t - 1} \]
\[ \leq 4 \alpha_{t - \tau_{\alpha_t}, t - 1} (L_w L_\theta + 1) (A \| \Gamma(u_t) - w_{\theta_t}^* \| + A \| w_{\theta_t}^* \| + B + 1). \]
For the second term, we have

\[
\begin{align*}
&\|F_{\theta_t}(\Gamma(u_t), Y_t) - \bar{F}_{\theta_t}(\Gamma(u_t))\| \\
\leq &\|F_{\theta_t}(\Gamma(u_t), Y_t)\| + \|\bar{F}_{\theta_t}(\Gamma(u_t)) - \bar{F}_{\theta_t}(w_{\theta_t}^*)\| + \|w_{\theta_t}^*\| \\
\leq &U_F + L_F\|\Gamma(u_t)\| + \|\bar{F}_{\theta_t}(\Gamma(u_t)) - \bar{F}_{\theta_t}(w_{\theta_t}^*)\| + \|w_{\theta_t}^*\| \\
= &U_F + L_F\|\Gamma(u_t)\| + \left|\sum_y d_{\theta_t}(y) \left(F_{\theta_t}(\Gamma(u_t), y) - \bar{F}_{\theta_t}(w_{\theta_t}^*, y)\right)\right| + \|w_{\theta_t}^*\| \\
\leq &U_F + L_F\|\Gamma(u_t)\| + \sum_y d_{\theta_t}(y) \left(F_{\theta_t}(\Gamma(u_t), y) - \bar{F}_{\theta_t}(w_{\theta_t}^*, y)\right) \\
\leq &U_F + L_F\|\Gamma(u_t)\| + \sum_y d_{\theta_t}(y) \left(F_{\theta_t}(\Gamma(u_t), y) - \bar{F}_{\theta_t}(w_{\theta_t}^*, y)\right) \\
\leq &A\|\Gamma(u_t) - w_{\theta_t}^*\| + A\|w_{\theta_t}^*\| + B.
\end{align*}
\]

Combining the two inequalities together yields

\[
\begin{align*}
&\langle \Gamma(u_t) - w_{\theta_t}^* - (\Gamma(u_{t-\tau_{\alpha_t}}) - w_{\theta_t}^*), F_{\theta_t}(\Gamma(u_t), Y_t) - \bar{F}_{\theta_t}(\Gamma(u_t)) \rangle \\
\leq &4(L_w L_\theta + 1)\alpha_{t-\tau_{\alpha_t}, t-1}(A\|\Gamma(u_t) - w_{\theta_t}^*\| + C)^2 \\
\leq &8(L_w L_\theta + 1)\alpha_{t-\tau_{\alpha_t}, t-1}(A^2\|\Gamma(u_t) - w_{\theta_t}^*\|^2 + C^2),
\end{align*}
\]

which completes the proof.

\section*{D.4 Proof of Lemma 10}

\textbf{Lemma 10 (Bound of $T_{32}$)}

\[T_{32} \leq 16\alpha_{t-\tau_{\alpha_t}, t-1}(1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t}, t-1}) \left(A^2\|\Gamma(u_t) - w_{\theta_t}^*\|^2 + C^2\right).
\]

Proof

\[
T_{32} = \langle \Gamma(u_{t-\tau_{\alpha_t}}) - w_{\theta_t-\tau_{\alpha_t}}^*, F_{\theta_t}(\Gamma(u_t), Y_t) - \bar{F}_{\theta_t}(\Gamma(u_{t-\tau_{\alpha_t}}), Y_t) + \bar{F}_{\theta_t}(\Gamma(u_{t-\tau_{\alpha_t}})) - \bar{F}_{\theta_t}(\Gamma(u_t)) \rangle
\]

\[
\leq \|\Gamma(u_{t-\tau_{\alpha_t}}) - w_{\theta_t-\tau_{\alpha_t}}^*\| \|F_{\theta_t}(\Gamma(u_t), Y_t) - \bar{F}_{\theta_t}(\Gamma(u_{t-\tau_{\alpha_t}}), Y_t) + \bar{F}_{\theta_t}(\Gamma(u_{t-\tau_{\alpha_t}})) - \bar{F}_{\theta_t}(\Gamma(u_t))\|,
\]
For the second term, we have

\[
\left\| \Gamma(u_{t \rightarrow \tau_{\alpha}}) - w_{\theta_{t \rightarrow \tau_{\alpha}}}^* \right\| \\
\leq \left\| \Gamma(u_{t \rightarrow \tau_{\alpha}}) - w_{\theta_{t \rightarrow \tau_{\alpha}}}^* \right\| + \left\| w_{\theta_t} - w_{\theta_t}^* \right\| \\
\leq \left\| \Gamma(u_{t \rightarrow \tau_{\alpha}}) - w_{\theta_t}^* \right\| + \left( L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \quad \text{(Lemma 34)} \\
\leq \left\| \Gamma(u_{t \rightarrow \tau_{\alpha}}) - \Gamma(u_t) \right\| + \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| + \left( L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \\
\leq \left\| \Gamma(u_t) \right\| + \frac{B}{A} + \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| + \left( L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \quad \text{(Lemma 31)} \\
\leq \left( 1 + L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \left( \left\| w_{\theta_t}^* \right\| + \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| + \frac{B}{A} \right)
\leq 2\left( 1 + L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \left( \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| + 1 \right)
\leq 2\left( 1 + L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \frac{1}{A} \left( \left\| \Gamma(u_t) - w_{\theta_t}^* \right\| + C \right)
\]

For the second term,

\[
\left\| F_{\theta_t} \Gamma(u_t), Y_t \right\| - F_{\theta_t} \Gamma(u_{t \rightarrow \tau_{\alpha}}, Y_t) + \bar{F}_{\theta_t} \Gamma(u_{t \rightarrow \tau_{\alpha}}, \Gamma(u_t)) - \bar{F}_{\theta_t} \Gamma(u_t)), \right\|
\leq \left\| F_{\theta_t} \Gamma(u_t), Y_t \right\| - F_{\theta_t} \Gamma(u_{t \rightarrow \tau_{\alpha}}, Y_t) + \left\| \Gamma(u_{t \rightarrow \tau_{\alpha}}) \right\| + \left\| \Gamma(u_t) \right\|
\leq L_F \left\| \Gamma(u_t) - \Gamma(u_{t \rightarrow \tau_{\alpha}}) \right\| + \left\| \sum_y d_{\theta_t}(y) \left( F_{\theta_t} \Gamma(u_{t \rightarrow \tau_{\alpha}}, y) - \bar{F}_{\theta_t} \Gamma(u_t), y) \right\|
\leq 2L_F \left\| \Gamma(u_t) - \Gamma(u_{t \rightarrow \tau_{\alpha}}) \right\|
\leq A \left\| \Gamma(u_t) - \Gamma(u_{t \rightarrow \tau_{\alpha}}) \right\|
\leq 4A \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \left( \left\| \Gamma(u_t) \right\| + B \right) \quad \text{(Lemma 31)}
\leq 4A \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \left( \left\| \Gamma(u_t) \right\| - w_{\theta_t}^* \right) + A \left\| w_{\theta_t}^* \right\| + B).
\]

Combining the two inequalities together yields

\[
T_{32} \leq 8 \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \left( 1 + L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \left( \left\| \Gamma(u_t) \right\| - w_{\theta_t}^* \right) + C^2\right)
\leq 16 \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \left( 1 + L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \left( A^2 \left\| \Gamma(u_t) \right\|^2 + C^2 \right),
\]

which completes the proof.

\textbf{D.5 Proof of Lemma 11}

\textbf{Lemma 11 (Bound of }T_{331}\text{)}

\[
\mathbb{E} [T_{331}] \leq \frac{8 \alpha_t \left( 1 + L_w L_{\theta_t} \alpha_t \Gamma(u_{t \rightarrow \tau_{\alpha}}, t-1) \right) \left( A^2 \mathbb{E} \left[ \left\| \Gamma(u_t) \right\|^2 \right] + C^2 \right) .
\]
Proof

\[ \mathbb{E}[T_{331}] \]

\[ = \mathbb{E} \left[ \langle \Gamma(u_{t-\tau_{\alpha_{t}}}) - w_{\theta_{t-\tau_{\alpha_{t}}}}^* \rangle, F_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}}), \tilde{Y}_t) - \tilde{F}_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}})) \rangle \right] \]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ \langle \Gamma(u_{t-\tau_{\alpha_{t}}}) - w_{\theta_{t-\tau_{\alpha_{t}}}}^* \rangle, F_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}}), \tilde{Y}_t) - \tilde{F}_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}})) \rangle \mid \theta_{t-\tau_{\alpha_{t}}} \right] \right] \]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ \langle \Gamma(u_{t-\tau_{\alpha_{t}}}) - w_{\theta_{t-\tau_{\alpha_{t}}}}^* \rangle, F_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}}), \tilde{Y}_t) - \tilde{F}_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}})) \rangle \mid \theta_{t-\tau_{\alpha_{t}}} \right] \mid \theta_{t-\tau_{\alpha_{t}}} \right] \]

\[ \leq \mathbb{E} \left[ \langle \Gamma(u_{t-\tau_{\alpha_{t}}}) - w_{\theta_{t-\tau_{\alpha_{t}}}}^* \rangle \right] \mathbb{E} \left[ \left\| F_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}}), \tilde{Y}_t) - \tilde{F}_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}})) \right\| \bigg| \theta_{t-\tau_{\alpha_{t}}} \right] \]

We now bound the inner expectation.

\[ \left\| \mathbb{E} \left[ F_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}}), \tilde{Y}_t) - \tilde{F}_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}})) \right] \right\| \]

\[ = \left\| \sum_y \mathbb{P}(\tilde{Y}_t = y \mid \theta_{t-\tau_{\alpha_{t}}}, u_{t-\tau_{\alpha_{t}}}) - d_{\theta_{t-\tau_{\alpha_{t}}}}(y) \right\| \]

\[ \leq \max_y \left\| F_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}}), y) \right\| \sum_y \mathbb{P}(\tilde{Y}_t = y \mid \theta_{t-\tau_{\alpha_{t}}}, u_{t-\tau_{\alpha_{t}}}) - d_{\theta_{t-\tau_{\alpha_{t}}}}(y) \]

\[ \leq \max_y \left\| F_{\theta_{t-\tau_{\alpha_{t}}}}(\Gamma(u_{t-\tau_{\alpha_{t}}}), y) \right\| \mathbb{P}(\tilde{Y}_t = y \mid \theta_{t-\tau_{\alpha_{t}}}, u_{t-\tau_{\alpha_{t}}}) \] (Definition of \( \tau_{\alpha_{t}} \))

\[ \leq \alpha_t \left( U_F + L_F \left\| \Gamma(u_{t-\tau_{\alpha_{t}}}) \right\| \right) \] (Lemma 36)

\[ \leq \alpha_t \left( U_F + L_F \left\| \Gamma(u_{t-\tau_{\alpha_{t}}}) - \Gamma(u_t) \right\| + L_F \left\| \Gamma(u_t) \right\| \right) \]

\[ \leq \alpha_t \left( B + A \left( \left\| \Gamma(u_t) \right\| + \frac{B}{A} \right) + A \left\| \Gamma(u_t) \right\| \right) \] (Lemma 31)

\[ \leq \alpha_t (2B + (A + 1) \left\| \Gamma(u_t) \right\|) \]

\[ \leq 2\alpha_t \left( B + A \left\| \Gamma(u_t) \right\| \right) \]

\[ \leq 2\alpha_t \left( B + A \left\| \Gamma(u_t) - w_{\theta_{t}}^* \right\| \right) \]

\[ \leq 2\alpha_t \left( A \left\| \Gamma(u_t) - w_{\theta_{t}}^* \right\| + C \right) \]

Using the above inequality and (33) yields

\[ \mathbb{E}[T_{331}] \]

\[ \leq \mathbb{E} \left[ \frac{4\alpha_t (1 + L_u L_{\theta_{t-\tau_{\alpha_{t}},t-1}})}{A} \left( A \left\| \Gamma(u_t) - w_{\theta_{t}}^* \right\| + C \right) \right] \]

\[ \leq \mathbb{E} \left[ \frac{8\alpha_t (1 + L_u L_{\theta_{t-\tau_{\alpha_{t}},t-1}})}{A} \left( A^2 \left\| \Gamma(u_t) - w_{\theta_{t}}^* \right\|^2 + C^2 \right) \right] , \]

which completes the proof. \( \blacksquare \)
D.6 Proof of Lemma 12

Lemma 12 (Bound of $T_{332}$)

$$E[T_{332}] \leq \frac{8|\mathcal{Y}|L_p L_\theta \sum_{j=t-\tau_{\alpha_t}}^{t-1} \alpha_{t-\tau_{\alpha_t},j} (1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A} \left( A^2 E \left[ \|\Gamma(u_t) - w^*_\theta \|^2 \right] + C^2 \right).$$

Proof

$$E[T_{332}]$$

$$= E \left[ \left( \Gamma(u_{t-\tau_{\alpha_t}}) - w^*_\theta_{t-\tau_{\alpha_t}}, F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), Y_t) - F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), \tilde{Y}_t) \right) \right]$$

$$\leq E \left[ \left\| \Gamma(u_{t-\tau_{\alpha_t}}) - w^*_\theta_{t-\tau_{\alpha_t}} \right\| E \left[ F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), Y_t) - F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), \tilde{Y}_t) \right] \right].$$

(Similar to (34))

$$\leq E \left[ \frac{2(1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A} \left( A \|\Gamma(u_t) - w^*_\theta \| + C \right) \right.$$

$$\times 2|\mathcal{Y}|L_p L_\theta \frac{t-1}{\sum_{j=t-\tau_{\alpha_t}}^{t-1} \alpha_{t-\tau_{\alpha_t},j} (A \|\Gamma(u_t) - w^*_\theta \| + C) \right]$$

(Using (33) and Lemma 37)

$$\leq \frac{8|\mathcal{Y}|L_p L_\theta \sum_{j=t-\tau_{\alpha_t}}^{t-1} \alpha_{t-\tau_{\alpha_t},j} (1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A} \left( A^2 E \left[ \|\Gamma(u_t) - w^*_\theta \|^2 \right] + C^2 \right),$$

which completes the proof.

D.7 Proof of Lemma 13

Lemma 13 (Bound of $T_{333}$)

$$T_{333} \leq \frac{4L'_F L_\theta \alpha_{t-\tau_{\alpha_t},t-1} (1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A^2} \left( A^2 \|\Gamma(u_t) - w^*_\theta \|^2 + C^2 \right).$$

Proof

$$T_{333} = E \left[ \left( \Gamma(u_{t-\tau_{\alpha_t}}) - w^*_\theta_{t-\tau_{\alpha_t}}, F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), Y_t) - F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), \tilde{Y}_t) \right) \right]$$

$$\leq \left\| \Gamma(u_{t-\tau_{\alpha_t}}) - w^*_\theta_{t-\tau_{\alpha_t}} \right\| \left\| F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), Y_t) - F_{\theta_t} (\Gamma(u_{t-\tau_{\alpha_t}}), \tilde{Y}_t) \right\|$$

$$\leq \frac{2(1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A} \left( A \|\Gamma(u_t) - w^*_\theta \| + C \right)$$

$$\times L'_F L_\theta \alpha_{t-\tau_{\alpha_t},t-1} \left( \|\Gamma(u_{t-\tau_{\alpha_t}})\| + U'_F \right)$$

(Using (33) and Lemma 34)
Since
\[ \left\| \Gamma(u_t - \tau_{\alpha_t}) \right\| \]
\[ \leq \left\| \Gamma(u_t - \tau_{\alpha_t}) - \Gamma(u_t) \right\| + \left\| \Gamma(u_t) \right\| \]
\[ \leq 2\left\| \Gamma(u_t) \right\| + \frac{B}{A} \quad \text{(Lemma 31)} \]
\[ \leq 2\left\| \Gamma(u_t) \right\| - w^*_\theta_t \| + 2\| w^*_\theta_t \| + \frac{B}{A}, \]
we have
\[
T_{333} \leq \frac{8L'_F L_\theta \alpha_{t-\tau_{\alpha_t},t-1} (1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A^2} \left( A^2 \left\| \Gamma(u_t) - w^*_\theta_t \right\|^2 + C^2 \right),
\]
which completes the proof. \[ \Box \]

D.8 Proof of Lemma 14

Lemma 14 (Bound of $T_{334}$)

\[ T_{334} \leq \frac{4L'_F L_\theta \alpha_{t-\tau_{\alpha_t},t-1} (1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A^2} \left( A^2 \left\| \Gamma(u_t) - w^*_\theta_t \right\|^2 + C^2 \right). \]

Proof
\[
T_{334} = \left\langle \Gamma(u_t - \tau_{\alpha_t}) - w^*_\theta_t - \tau_{\alpha_t}, \bar{F}_{\theta_t - \tau_{\alpha_t}} (\Gamma(u_t - \tau_{\alpha_t})) - \bar{F}_{\theta_t} (\Gamma(u_t - \tau_{\alpha_t})) \right\rangle
\]
\[ \leq \left\| \Gamma(u_t - \tau_{\alpha_t}) - w^*_\theta_t - \tau_{\alpha_t} \right\| \left\| \bar{F}_{\theta_t} (\Gamma(u_t - \tau_{\alpha_t})) - \bar{F}_{\theta_t - \tau_{\alpha_t}} (\Gamma(u_t - \tau_{\alpha_t})) \right\| \]
\[ \leq \frac{2(1 + L_w L_\theta \alpha_{t-\tau_{\alpha_t},t-1})}{A} \left( A \left\| \Gamma(u_t) - w^*_\theta_t \right\| + C \right)
\]
\[ \times L'_F L_\theta \alpha_{t-\tau_{\alpha_t},t-1} \left( \left\| \Gamma(u_t - \tau_{\alpha_t}) \right\| + U'_F \right) \quad \text{(Using (33) and Lemma 34)} . \]

Using (36) completes the proof. \[ \Box \]

D.9 Proof of Lemma 15

Lemma 15 (Bound of $T_5$)

\[ T_5 \leq 2 \left( A^2 \left\| \Gamma(u_t) - w^*_\theta_t \right\|^2 + C^2 \right). \]
Proof

\[ T_5 = \| F_{\theta_t}(\Gamma(u_t), Y_t) - \Gamma(u_t) \|^2 \]
\[ \leq (\| F_{\theta_t}(\Gamma(u_t), Y_t) \| + \| \Gamma(u_t) \|)^2 \]
\[ \leq (U_F + (L_F + 1)\| \Gamma(u_t) \|)^2 \] (Lemma 36)
\[ \leq (B + A\| \Gamma(u_t) \|)^2 \]
\[ \leq (B + A\| \Gamma(u_t) - w^*_\theta \| + A\| w^*_\theta \|)^2 \]
\[ \leq 2 \left( A^2\| \Gamma(u_t) - w^*_\theta \|^2 + C^2 \right) \]

Lemma 36 For any \( \theta, w, y \),

\[ \| F_{\theta}(w, y) \| \leq U_F + L_F\| w \| \]

Proof Assumption 3.4 implies that

\[ \| F_{\theta}(w, y) \| - \| F_{\theta}(0, y) \| \leq \| F_{\theta}(0, y) - F_{\theta}(w, y) \| \]
\[ \leq L_F\| w - 0 \|, \]

which completes the proof.

Lemma 37

\[ \left\| \mathbb{E} \left[ F_{\theta_l-\tau_{\alpha_l}}(\Gamma(u_{l-\tau_{\alpha_l}}, Y_{l-\tau_{\alpha_l}}), \bar{Y}_l) \mid \theta_{l-\tau_{\alpha_l}}, \bar{Y}_{l-\tau_{\alpha_l}} \right] \right\| \]
\[ \leq 2|\mathcal{Y}|L_PL_\theta \sum_{j=1-\tau_{\alpha_l}}^{l-1} \alpha_{l-\tau_{\alpha_l},j} (A\| \Gamma(u_t) - w^*_\theta \| + C) \]

Proof In this proof, all \( \text{Pr} \) and \( \mathbb{E} \) are implicitly conditioned on \( u_{l-\tau_{\alpha_l}}, \theta_{l-\tau_{\alpha_l}}, Y_{l-\tau_{\alpha_l}} \). We use \( \Theta_t \) to denote the set of all possible \( \theta_t \) given \( u_{t-\tau_{\alpha_t}}, \theta_{t-\tau_{\alpha_t}}, Y_{t-\tau_{\alpha_t}} \). Obviously, \( \Theta_t \) is a finite set. We have

\[ \text{Pr}(Y_l = y') = \sum_{y} \sum_{z \in \Theta_t} \text{Pr}(Y_l = y', Y_{l-1} = y, \theta_t = z) \]
\[ = \sum_{y} \sum_{z \in \Theta_t} \text{Pr}(Y_l = y' \mid Y_{l-1} = y, \theta_t = z) \text{Pr}(Y_{l-1} = y, \theta_t = z) \]
\[ = \sum_{y} \sum_{z \in \Theta_t} P_z(y, y') \text{Pr}(Y_{l-1} = y) \text{Pr}(\theta_t = z \mid Y_{l-1} = y) \]
\[
\begin{align*}
\Pr(\tilde{Y}_t = y') &= \sum_y \Pr(\tilde{Y}_{t-1} = y) P_{\theta_{t-\tau_1}}(y, y') \\
&= \sum_y \Pr(\tilde{Y}_{t-1} = y) P_{\theta_{t-\tau_1}}(y, y') \sum_{z \in \Theta_t} \Pr(\theta_t = z | Y_{t-1} = y) \\
&= \sum_y \sum_{z \in \Theta_t} \Pr(\tilde{Y}_{t-1} = y) P_{\theta_{t-\tau_1}}(y, y') \Pr(\theta_t = z | Y_{t-1} = y)
\end{align*}
\]

Consequently,
\[
\sum_{y'} \left| \Pr(Y_t = y') - \Pr(\tilde{Y}_t = y') \right| \\
\leq \sum_{y, y'} \sum_{z \in \Theta_t} \left| \Pr(Y_{t-1} = y) P_{\theta_{t-\tau_1}}(y, y') - \Pr(\tilde{Y}_{t-1} = y) P_{\theta_{t-\tau_1}}(y, y') \Pr(\theta_t = z | Y_{t-1} = y) \right|
\]

Since for any \( z \in \Theta_t \),
\[
\left| \Pr(Y_{t-1} = y) P_{\theta_{t-\tau_1}}(y, y') - \Pr(\tilde{Y}_{t-1} = y) P_{\theta_{t-\tau_1}}(y, y') \Pr(\theta_t = z | Y_{t-1} = y) \right| \\
\leq \left| \Pr(Y_{t-1} = y) - \Pr(\tilde{Y}_{t-1} = y) \right| \left| P_{\theta_{t-\tau_1}}(y, y') \right| + L_P L_\theta L_\alpha_{t-\tau_1, t-1} \Pr(\tilde{Y}_{t-1} = y) \quad \text{(Lemma 34)},
\]

we have
\[
\sum_{y'} \left| \Pr(Y_t = y') - \Pr(\tilde{Y}_t = y') \right| \\
\leq \sum_{y} \left| \Pr(Y_{t-1} = y) - \Pr(\tilde{Y}_{t-1} = y) \right| + |Y| L_P L_\theta L_\alpha_{t-\tau_1, t-1}.
\]

Applying the above inequality recursively yields
\[
\sum_{y'} \left| \Pr(Y_t = y') - \Pr(\tilde{Y}_t = y') \right| \leq |Y| L_P L_\theta \sum_{j=t-\tau_1}^{t-1} \alpha_{t-\tau_1, j}.
\]
Consequently,

\[
\left\| \mathbb{E} \left[ F_{\theta_{t-\tau_{t}}} (\Gamma(u_{t-\tau_{t}}), Y_t) - F_{\theta_{t-\tau_{t}}} (\Gamma(u_{t-\tau_{t}}), \tilde{Y}_t) \right] \right\| \\
= \sum_y \left( \Pr(Y_t = y) - \Pr(\tilde{Y}_t = y) \right) \cdot \left\| F_{\theta_{t-\tau_{t}}} (\Gamma(u_{t-\tau_{t}}), y) \right\| \\
\leq \max_y \left\| F_{\theta_{t-\tau_{t}}} (\Gamma(u_{t-\tau_{t}}), y) \right\| \cdot |\mathcal{Y}| \cdot L_P \cdot \sum_{j=t-\tau_{t}}^{t-1} \alpha_{t-\tau_{t}, j} \\
\leq 2 |\mathcal{Y}| \cdot L_P \cdot \sum_{j=t-\tau_{t}}^{t-1} \alpha_{t-\tau_{t}, j} \cdot (A \cdot |\Gamma(u_t)| - w_{\theta_t}^* + C) \quad \text{(Using (35))},
\]

which completes the proof.

\[\blacksquare\]

**Lemma 38**

\[
\left\| \mathbb{E} \left[ F(\Gamma(u_{t-\tau_{t}}), Y_{t-\tau_{t}}; t) - F(\Gamma(u_{t-\tau_{t}}), \tilde{Y}_{t-\tau_{t}}; t) \right] \right\| \\
\leq 2 |\mathcal{Y}| \cdot L_P \cdot \sum_{j=t-\tau_{t}}^{t-1} \alpha_{t-\tau_{t}, j} \cdot (A \cdot |\Gamma(u_t)| - w_{\theta_t}^* + C)
\]

**Proof** In this proof, all \(\Pr\) and \(\mathbb{E}\) are implicitly conditioned on \(u_{t-\tau_{t}}, \theta_{t-\tau_{t}}, y_{0:t-\tau_{t}}\). We use \(\Theta_t\) to denote the set of all possible \(\theta_t\) given \(u_{t-\tau_{t}}, \theta_{t-\tau_{t}}, y_{0:t-\tau_{t}}\). Obviously, \(\Theta_t\) is a finite set. For any \(n\), we have

\[
\Pr(Y_{t-n:t} = y_{0:n}) = \sum_{z \in \Theta_t} \Pr(Y_t = y_n, Y_{t-n-1} = y_{0:n-1}, \theta_t = z) \\
= \sum_{z \in \Theta_t} \Pr(Y_t = y_n | Y_{t-n-1} = y_{0:n-1}, \theta_t = z) \cdot \Pr(Y_{t-n-1} = y_{0:n-1}, \theta_t = z) \\
= \sum_{z \in \Theta_t} P_z(y_{n-1}, y_n) \cdot Pr(Y_{t-n-1} = y_{0:n-1}) \cdot \Pr(\theta_t = z | Y_{t-n-1} = y_{0:n-1})
\]

\[
\Pr(\tilde{Y}_{t-n:t} = y_{0:n}) = \Pr(\tilde{Y}_{t-n-1} = y_{0:n-1}) \cdot \mathbb{P}_{\theta_{t-\tau_{t}}} (y_{n-1}, y_n) \\
= \Pr(\tilde{Y}_{t-n-1} = y_{0:n-1}) \cdot \mathbb{P}_{\theta_{t-\tau_{t}}} (y_{n-1}, y_n) \sum_{z \in \Theta_t} \Pr(\theta_t = z | Y_{t-n-1} = y_{0:n-1}).
\]
Consequently,
\[
\sum_{y_{0:n}} \left| \Pr(Y_{t-n:t} = y_{0:n}) - \Pr(\tilde{Y}_{t-n:t} = y_{0:n}) \right|
\leq \sum_{y_{0:n}} \sum_{z \in \Theta_t} \left| \Pr(Y_{t-n:t-1} = y_{0:n-1}) P_z(y_{n-1}, y_n) - \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) P_{\theta_t-\tau_{\alpha_t}}(y_{n-1}, y_n) \right| 
\times \Pr(\theta_t = z \mid Y_{t-n:t-1} = y_{0:n-1}).
\]
Since for any \( z \in \Theta_t, \)
\[
\left| \Pr(Y_{t-n:t-1} = y_{0:n-1}) P_z(y_{n-1}, y_n) - \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) P_{\theta_t-\tau_{\alpha_t}}(y_{n-1}, y_n) \right|
\leq \left| \Pr(Y_{t-n:t-1} = y_{0:n-1}) P_z(y_{n-1}, y_n) - \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) P_z(y_{n-1}, y_n) \right|
+ \left| \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) P_z(y_{n-1}, y_n) - \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) P_{\theta_t-\tau_{\alpha_t}}(y_{n-1}, y_n) \right|
\leq \left| \Pr(Y_{t-n:t-1} = y_{0:n-1}) - \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) \right| P_z(y_{n-1}, y_n)
+ L_P L_\theta \alpha_{t-\tau_{\alpha_t},t-1} \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) \quad \text{(Lemma 34),}
\]
we have
\[
\sum_{y_{0:n}} \left| \Pr(Y_{t-n:t} = y_{0:n}) - \Pr(\tilde{Y}_{t-n:t} = y_{0:n}) \right|
\leq \sum_{y_{0:n-1}} \left| \Pr(Y_{t-n:t-1} = y_{0:n-1}) - \Pr(\tilde{Y}_{t-n:t-1} = y_{0:n-1}) \right| + |\mathcal{Y}| L_P L_\theta \sum_{j=t-\tau_{\alpha_t}}^{t-1} \alpha_{t-\tau_{\alpha_t},j}.
\]
Letting \( n = \tau_{\alpha_t} \) and repeating the above procedure yield
\[
\sum_{y_{0:\tau_{\alpha_t}}} \left| \Pr(Y_{t-\tau_{\alpha_t}:t} = y_{0:\tau_{\alpha_t}}) - \Pr(\tilde{Y}_{t-\tau_{\alpha_t}:t} = y_{0:\tau_{\alpha_t}}) \right| \leq |\mathcal{Y}| L_P L_\theta \sum_{j=t-\tau_{\alpha_t}}^{t-1} \alpha_{t-\tau_{\alpha_t},j}.
\]
Consequently,
\[
\| \mathbb{E} \left[ F(\Gamma(u_{t-\tau_{\alpha_t}}), Y_{t-\tau_{\alpha_t}:t}) - F(\Gamma(\hat{u}_{t-\tau_{\alpha_t}}), \tilde{Y}_{t-\tau_{\alpha_t}:t}) \right] \|
\leq \sum_{y_{0:\tau_{\alpha_t}}} \left( \Pr(Y_{t-\tau_{\alpha_t}:t} = y_{0:\tau_{\alpha_t}}) - \Pr(\tilde{Y}_{t-\tau_{\alpha_t}:t} = y_{0:\tau_{\alpha_t}}) \right) \|F(\Gamma(u_{t-\tau_{\alpha_t}}), y_{0:\tau_{\alpha_t}})\|
\leq \max_{y_{0:\tau_{\alpha_t}}} \|F(\Gamma(u_{t-\alpha_t}), y_{0:\tau_{\alpha_t}})\| |\mathcal{Y}| L_P L_\theta \sum_{j=t-\tau_{\alpha_t}}^{t-1} \alpha_{t-\tau_{\alpha_t},j}
\leq 2|\mathcal{Y}| L_P L_\theta \sum_{j=t-\tau_{\alpha_t}}^{t-1} \alpha_{t-\tau_{\alpha_t},j} (A\|\Gamma(u_t) - \hat{w}_t\| + C) \quad \text{(Similar to (35)),}
\]
which completes the proof.
Lemma 39

\[ \left\| \mathbb{E} \left[ F(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t_{\tau_{\alpha_{t}}}}, t)) - F(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t_{\tau_{\alpha_{t}}}}, t)) \right] \right\| \leq \frac{6x_{\max}}{1 - \xi} \max \left\{ \alpha_t, \xi^{\left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor} \right\}(A\|\Gamma(u_t) - \hat{w}_t^*\| + C) \]

**Proof** All the \( \mathbb{E} \) and \( \text{Pr} \) in this proof are implicitly conditioned on \( \theta_{t_{\tau_{\alpha_{t}}}}, u_{t_{\tau_{\alpha_{t}}}}, Y_{0:t_{\tau_{\alpha_{t}}}} \). We omit them for easing the presentation. According to the definition of \( F \), we have

\[ F(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t_{\tau_{\alpha_{t}}}}, t)) - F(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t_{\tau_{\alpha_{t}}}}, t)) = \delta(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t}) \sum_{k=0}^{\tau_{t_{\alpha_{t}}}} \xi^k x(\tilde{Y}_{t-k}) - \delta(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t}) \sum_{k=0}^{\tau_{t_{\alpha_{t}}}} \xi^k x(\tilde{Y}_{t-k}) \right) .

For \( k \geq \left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor \), we have

\[ \| \delta(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t}) x(\tilde{Y}_{t-k}) - \delta(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t}) x(\tilde{Y}_{t-k}) \right) \| \leq 2x_{\max} \max_y \| \delta(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, y)) \| \leq 4x_{\max} (A\|\Gamma(u_t) - \hat{w}_t^*\| + C) \quad \text{(Using (25))} \]

implying

\[ \sum_{k=\left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor}^{\tau_{t_{\alpha_{t}}}} \| \xi^k \delta(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t}) x(\tilde{Y}_{t-k}) - \delta(\Gamma(u_{t_{\tau_{\alpha_{t}}}}, \tilde{Y}_{t}) x(\tilde{Y}_{t-k}) \right) \| \leq \frac{4x_{\max} \xi^\left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor}{1 - \xi} (A\|\Gamma(u_t) - \hat{w}_t^*\| + C) . \tag{37} \]

For \( k < \left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor \), we first recall that we set \( t_0 \) to be sufficiently large such that \( \forall t \),

\[ C_M \tau^\left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor \leq \alpha_t . \]

Then

\[ \sup_{y, \theta} \sum_{y'} \left| P_{\theta}^{\left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor}(y, y') - d_\theta(y') \right| \leq C_M \tau^\left\lfloor \frac{\tau_{t_{\alpha_{t}}}}{2} \right\rfloor \leq \alpha_t . \]

Further, since

\[ \|P_\theta\|_\infty = 1, \]

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we have for any $i \geq 0$,
\[
\left\| P_{\theta}^{\left\lfloor \frac{\tau_{\alpha t}}{2} \right\rfloor + i} - 1d_{\theta} \right\|_{\infty} = \left\| P_{\theta}^{\left\lfloor \frac{\tau_{\alpha t}}{2} \right\rfloor} - 1d_{\theta} P_{\theta} \right\|_{\infty} \\
\leq \left\| P_{\theta}^{\left\lfloor \frac{\tau_{\alpha t}}{2} \right\rfloor} - 1d_{\theta} \right\|_{\infty} \left\| P_{\theta}^{i} \right\| = \left\| P_{\theta}^{\left\lfloor \frac{\tau_{\alpha t}}{2} \right\rfloor} - 1d_{\theta} \right\|_{\infty}.
\]
Consequently,
\[
\sup_{y,\theta} \sum_{y'} \left| P_{\theta}^{\left\lfloor \frac{\tau_{\alpha t}}{2} \right\rfloor + i}(y, y') - d_{\theta}(y') \right| \leq \alpha_t.
\]
When $k < \lfloor \frac{\tau_{\alpha t}}{2} \rfloor$, we immediately have
\[
t - k > t - \lfloor \frac{\tau_{\alpha t}}{2} \rfloor = t - \tau_{\alpha t} + \tau_{\alpha t} - \lfloor \frac{\tau_{\alpha t}}{2} \rfloor \geq t - \tau_{\alpha t} + \lfloor \frac{\tau_{\alpha t}}{2} \rfloor,
\]
i.e., at time $t - k$, the chain $\{\tilde{Y}_t\}$ must have evolved according to $P_{\theta_{t-\tau_{\alpha t}}}$ for at least $\lfloor \frac{\tau_{\alpha t}}{2} \rfloor$ steps, which implies that
\[
\sum_y \left| \Pr(\tilde{Y}_{t-k} = y) - d_{\theta_{t-\tau_{\alpha t}}}(y) \right| \leq \alpha_t.
\]
We, therefore, have
\[
\| E \left[ \xi_k \left( \delta \left( (u_{t-\tau_{\alpha t}}), \tilde{Y}_t \right) x(\tilde{Y}_{t-k}) - \delta \left( (u_{t-\tau_{\alpha t}}), \tilde{Y}_t \right) x(\tilde{Y}_{t-k}) \right) \right] \|
= \xi_k \left\| \sum_{y,y'} \left( \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y) - \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y') \right) \delta \left( (u_{t-\tau_{\alpha t}}), y' \right) x(y) \right\|
\leq \xi_k \sum_{y,y'} \left| \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y) - \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y) \right| \left\| \delta \left( (u_{t-\tau_{\alpha t}}), y' \right) x(y) \right\|
\leq \xi_k x_{\text{max}} \max_y \left| \delta \left( (u_{t-\tau_{\alpha t}}), y \right) \right| \left\| \sum_{y,y'} \left| \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y) - \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y) \right| \right|
\leq \xi_k 2x_{\text{max}} (A\|\Gamma(u_t) - w_{t_0}^*\| + C) \sum_{y,y'} \left| \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y) - \Pr(\tilde{Y}_t = y', \tilde{Y}_{t-k} = y) \right|
\leq \alpha_t \xi_k 2x_{\text{max}} (A\|\Gamma(u_t) - w_{t_0}^*\| + C) .
\]
Consequently,

\[
\sum_{k=0}^{\lceil \tau a \rceil - 1} \| \mathbb{E} \left[ \xi^k \left( \delta \left( \Gamma(u_{t-\tau a}), \tilde{Y}_t \right) x(\tilde{Y}_{t-k}) - \delta \left( \Gamma(u_{t-\tau a}), \bar{Y}_t \right) x(\bar{Y}_{t-k}) \right) \right] \|
\]

\[
= \sum_{k=0}^{\lceil \tau a \rceil - 1} \alpha_t \xi^k 2x_{max} (A\|\Gamma(u_t) - w_{0_t}\| + C) = \frac{2x_{max} \alpha_t}{1 - \xi} (A\|\Gamma(u_t) - w_{0_t}\| + C) \tag{38}
\]

Combining (37) and (38) yields

\[
\sum_{k=0}^{\lceil \tau a \rceil - 1} \| \mathbb{E} \left[ F(\Gamma(u_{t-\tau a}), \tilde{Y}_{t-\tau a}, t) - F(\Gamma(u_{t-\tau a}), \bar{Y}_{t-\tau a}, t) \right] \|
\]

\[
= \sum_{k=0}^{\lceil \tau a \rceil - 1} \| \mathbb{E} \left[ \xi^k \left( \delta \left( \Gamma(u_{t-\tau a}), \tilde{Y}_t \right) x(\tilde{Y}_{t-k}) - \delta \left( \Gamma(u_{t-\tau a}), \bar{Y}_t \right) x(\bar{Y}_{t-k}) \right) \right] \|
\]

\[
\leq \sum_{k=0}^{\lceil \tau a \rceil - 1} \max \left\{ \alpha_t, \xi^k \right\} \frac{2x_{max}}{1 - \xi} \max \{ \alpha_t, \xi^k \} (A\|\Gamma(u_t) - w_{0_t}\| + C),
\]

which completes the proof.

\[\Box\]

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