Abstract. In this work we analyze the behavior of Massey products of closed manifolds under the blow-up construction. The results obtained in the article are applied to the problem of constructing closed symplectic non-formal manifolds. The proofs use Thom spaces as an important technical tool. This application of Thom spaces is of conceptual interest.

Introduction

In this paper we suggest a simple general method of constructing non-formal manifolds. In particular, we construct a large family of non-formal symplectic manifolds. Here we detect non-formality via essentiality of rational Massey products (we say that the Massey product is essential if it is non-empty and does not contain the zero element, see 1.1). In this context Thom spaces play the role of a technical tool which allows us to construct essential Massey products in an elegant way, Lemmas 3.4 and 3.5.

In greater detail, we analyze the behavior of Massey products of closed manifolds under a blow-up construction. The knowledge of various homotopic properties of blow-ups is important in many areas and, in particular, in symplectic geometry [Go, M, MS, BT, TO]. In the article we present several results (Theorems 4.9, 4.10 and 5.2) which can be regarded as a qualitative description of what happens to the non-formality under the blow-up. More precisely, the essentiality of Massey products in the ruled submanifold yields essential Massey products in the resulting manifold.

It is clear that the blow-up procedure enables us to construct a large class of manifolds and, on the other hand, many classes of manifolds (complex, Kähler, symplectic, etc) are invariant with respect to blow-up. Because of this, our approach turns out to be useful in constructing non-formal manifolds with certain structures or other prescribed properties.

For example, these results have a nice application to symplectic topology. We suggest a simple way to construct a large family of new examples of closed symplectic non-formal (and, hence, non-Kähler) manifolds, including new simply-connected examples. Note that the problem of constructing symplectic closed non-Kähler manifolds (the Weinstein-Thurston problem) was and still remains of substantial

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interest in symplectic geometry [BT, CFG, FMG, FLS, Go], see [TO] for a de-
tailed survey. It is also worth mentioning that usually the blow-up procedure does
not change the fundamental group of ambient manifold(s), and so our approach is
applicable to manifolds with arbitrary fundamental groups. In particular, we can
produce families of simply-connected non-formal symplectic manifolds. Note that
the construction of closed non-formal symplectic manifolds is regarded as more diffi-
cult if one demands simple connectivity, cf. the Lupton-Oprea problem in [LO,TO].
The first simply-connected examples have very recently appeared in [BT].

As another example, we mention that our method enables us to construct a large
family of algebraic varieties and Kähler manifolds with essential Massey products
in $\mathbb{Z}/p$-cohomology. (Such examples were already known, [E]; we just emphazise
that our method yields a simple construction of a large family.)

It seems that this nice (and a bit surprising) application of Thom spaces is
conceptually interesting in its own right. We want also to mention that our research
was originally initiated by ideas of Gitler [G], who has used Thom spaces in studying
of homology of (complex) blow-ups.

We use the term "F-fibration" for any (Hurewicz) fibration with fiber $F$. Also,
when we write "an F-fibration $F \to E \to B$", it means that $E \to B$ is a fibration
with fiber $F$.

Throughout the paper we fix a commutative ring $R$ with the unit, and $H^*(X)$,
resp $C^*(X)$ denotes the singular cohomology $H^*(X; R)$, resp. singular cochain
complex $C^*(X; R)$ of the space $X$ unless something else is said explicitly. Similarly,$\tilde{H}^*(X)$ denotes the reduced singular cohomology with coefficients in $R$.

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1. Preliminaries on Massey products

Throughout the section we fix a differential graded associative (in the sequel
DGA) algebra $(A, d)$. We denote by $H(A)$ the cohomology ring of $(A, d)$, and we
always assume that $H(A)$ is commutative. Given an element $a \in A$ with $da = 0$, we
denote by $[a]$ the cohomology class of $a$. So, $[a] \in H(A)$.

Given a homogeneous element $a \in A$, we set $\overline{a} = (-1)^{|a|}a$.

1.1. Definition ([K], [Ma]). Given homogeneous elements $a_1, \ldots, a_n \in H(A)$, a
defining system for $a_1, \ldots, a_n$ is a family $\mathcal{X} = \{x_{ij}\}$ of elements of $A$, $1 \leq i < j \leq
n + 1$, with the following properties:

1. $[x_{i,i+1}] = a_i$ for every $i$;
2. $dx_{ij} = \sum_{r=i+1}^{j-1} x_{ir} x_{rj}$ if $i + 1 < j < n + 1$. 

Consider the element $c(\mathcal{X}) := \sum_{r=2}^{n} x_{1r} x_{r,n+1}$. One can prove that it is a cocycle, so we have the class $[c(\mathcal{X})] \in H(A)$. We define

$$\langle \alpha_1, \ldots, \alpha_n \rangle := \{ [c(\mathcal{X})] \mid \mathcal{X} \text{ runs over all defining systems} \} \subset H(A).$$

The family $\langle \alpha_1, \ldots, \alpha_n \rangle$ is called the Massey $n$-tuple product of $\alpha_1, \ldots, \alpha_n$.

The indeterminacy of the Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is the subset

$$\text{Indet}(\alpha_1, \ldots, \alpha_n) := \{ x - y \mid x, y \in (\alpha_1, \ldots, \alpha_n) \}$$

of $H(A)$.

We say that the Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is essential if $\langle \alpha_1, \ldots, \alpha_n \rangle \neq \emptyset$ and $0 \notin \langle \alpha_1, \ldots, \alpha_n \rangle$. The last condition means that there is no defining system $\mathcal{X}$ with $[c(\mathcal{X})] = 0$.

1.2. Remark. Frequently, one says about non-vanishing Massey products instead of essential Massey products. Also, people say that the Massey product is defined, when it is non-empty.

For convenience of references, we fix the following proposition. The proof follows from the definition directly.

1.3. Proposition. For every morphism $f : (A,d) \to (A',d')$ of DGA algebras and every classes $\alpha_1, \ldots, \alpha_n \in H(A)$ we have

$$f_* (\alpha_1, \ldots, \alpha_n) \subset (f_* \alpha_1, \ldots, f_* \alpha_n).$$

In particular, if $\langle \alpha_1, \ldots, \alpha_n \rangle \neq \emptyset$ then $f_* (\alpha_1, \ldots, f_* \alpha_n) \neq \emptyset$, and if, in addition, $\langle f_* \alpha_1, \ldots, f_* \alpha_n \rangle$ is essential then $\langle \alpha_1, \ldots, \alpha_n \rangle$ is. □

1.4. Proposition. Given a DGA algebra $A$, take an element $\xi \in H(A)$ such that $\xi$ is represented by a central element $a \in A$, i.e., $ab = ba$ for every $b \in A$. Then for every $\alpha_1, \ldots, \alpha_n \in H(A)$ and every $k$ we have

$$\xi \langle \alpha_1, \ldots, \alpha_n \rangle \subset \langle \alpha_1, \ldots, \alpha_{k-1}, \xi \alpha_k, \alpha_{k+1}, \ldots, \alpha_n \rangle$$

Proof. Consider a defining system $\{x_{ij}\}$ for $\alpha_1, \ldots, \alpha_n$. We set

$$y_{ij} = \begin{cases} ax_{ij} & \text{if } i \leq k \text{ and } j \geq k + 1, \\ x_{ij} & \text{otherwise} \end{cases}$$

It is easy to see (using the equality $ba = ab$ for every $b \in A$) that $\{y_{ij}\}$ is a defining system for $\alpha_1, \ldots, \alpha_{k-1}, \xi \alpha_k, \alpha_{k+1}, \ldots, \alpha_n$. Furthermore,

$$\sum_{r=1}^{n} y_{1r} y_{r,n+1} = b \sum_{r=1}^{n} x_{1r} x_{r,n+1},$$

and so

$$\left[ \sum_{r=1}^{n} y_{1r} y_{r,n+1} \right] = \xi \left[ \sum_{r=1}^{n} x_{1r} x_{r,n+1} \right].$$
Thus, \[ \xi\langle \alpha_1, \ldots, \alpha_n \rangle \subset \langle \alpha_1, \ldots, \alpha_{k-1}, \xi \alpha_k, \alpha_{k+1}, \ldots, \alpha_n \rangle. \]

Now let us consider the special case of Massey triple products \( \langle \alpha, \beta, \gamma \rangle \). The above definition leads to the following description. Let \( a, b, c \in A \) be such that \( \alpha = [a], \beta = [b] \) and \( \gamma = [c] \). Suppose that \( \alpha \beta = 0 = \beta \gamma \) and consider \( x, y \in A \) such that \( dx = ab \) and \( dy = bc \). Then \( \langle \alpha, \beta, \gamma \rangle \) consists of all classes of the form \([ay + xc]\). Furthermore, the indeterminacy of \( \langle \alpha, \beta, \gamma \rangle \) is the set of elements of the form \( \alpha u + v \gamma \) where \( u, v \in H(A) \) are arbitrary elements with \( |\alpha u| = |v \gamma| = |\alpha| + |\beta| + |\gamma| - 1 \).

For convenience, we formulate the properties of the Massey triple products as the following proposition.

1.5. Proposition. \( \langle \alpha, \beta, \gamma \rangle \neq \emptyset \) if and only if \( \alpha \beta = 0 = \beta \gamma \). If \( \langle \alpha, \beta, \gamma \rangle \neq \emptyset \) then \( \text{Indet}(\alpha, \beta, \gamma) \) is the ideal \( (\alpha, \gamma) \), and \( \langle \alpha, \beta, \gamma \rangle \) is a coset in \( H(A) \) with respect to \( (\alpha, \gamma) \). So, \( \langle \alpha, \beta, \gamma \rangle \) is essential if and only if there exists \( u \in \langle \alpha, \beta, \gamma \rangle \) with \( u /\in (\alpha, \gamma) \). □

Now, let \( X \) be a topological space. If we wish to consider Massey products in \( H^*(X) \), we need a DGA algebra whose cohomology equals \( H^*(X) \). (It can be the singular cochain complex, or the Sullivan model, etc, see below.) Consider a functor

\[ \Phi : \mathcal{T} \to \mathcal{D} \]

where \( \mathcal{T} \) is a subcategory of the category of topological spaces and \( \mathcal{D} \) is the category of DGA \( R \)-algebras. Furthermore, we require that \( H \circ \Phi \) coincides with the (singular) cohomology functor. Here \( H \) means the cohomology functor for DGA algebras.

Now, given \( \alpha_1, \ldots, \alpha_n \in H^*(X) \), we can consider the Massey products

\[ \langle \alpha_1, \ldots, \alpha_n \rangle_{\Phi} \subset H(\Phi(X)) = H^*(X) \]

with respect to \( \Phi \) (i.e. the defining system for \( \alpha_1, \ldots, \alpha_n \) lies in \( \Phi(X) \)). A priori, these Massey products depend on \( \Phi \). In future we omit the subscript \( \Phi \) and write \( \langle \alpha_1, \ldots, \alpha_n \rangle \) instead of \( \langle \alpha_1, \ldots, \alpha_n \rangle_{\Phi} \), because we always say explicitly what is \( \Phi \), (i.e. which concrete \( \Phi \) do we use).

There are three important examples of cochain functors \( \Phi \). First, given a space \( X \), consider the singular cochain complex \( C^*(X) \) of \( X \). We equip \( C^*(X) \) with the standard (Alexander–Whitney) associative cup product pairing. Then \( (C^*(X), \delta) \) turns into a DGA algebra. Now, since \( H^*(X) = H(C^*(X), \delta) \), we use the previous construction and define the Massey product

\[ \langle \alpha_1, \ldots, \alpha_n \rangle \subset H^*(X), \quad \alpha_i \in H^*(X). \]

Sullivan minimal models give us the second example. In greater detail, for every space \( X \) there is a natural commutative DGA algebra \( (\mathcal{M}_X, d) \) over the field of rational numbers \( \mathbb{Q} \) which is a homotopy invariant of \( X \). Furthermore, if \( X \) is a simply-connected (or more generally, nilpotent) CW-space of finite type then \( (\mathcal{M}_X, d) \) completely determines the rational homotopy type of \( X \), see [DGMS], [L].
A space \( X \) is called \textit{formal} if there exists a DGA-morphism
\[
\rho : (\mathcal{M}_X, d) \rightarrow (H^*(X; \mathbb{Q}), 0)
\]
inducing isomorphism on the cohomology level. Formality is an important homotopy property, since the rational homotopy type of any nilpotent formal space can be reconstructed by some “formal” procedure from its cohomology algebra. Kähler manifolds are formal [DGMS]. Examples of formal and non-formal manifolds occurring in various geometric situations can be found in [TO].

Since the cochain functor \( \Phi(X) = (H^*(X), 0) \) yields inessential Massey products in \( H^*(X) \), we conclude that, for every formal space \( X \), all the Massey products in \( H^*(X; \mathbb{Q}) \) with respect to the Sullivan model are inessential, cf. [DGMS, TO]. In other words, a space \( X \) is not formal if we can find an essential Massey product (with respect to the Sullivan model) in \( H^*(X; \mathbb{Q}) \).

The third example is the de Rham algebra \((\text{DR}(X), d)\) of differential forms on a smooth manifold \( X \). We have \( H(\text{DR}(X), d) = H^*(X; \mathbb{R}) \), and we can define and compute the corresponding Massey products in \( H^*(X; \mathbb{R}) \).

One can prove that the Sullivan model (tensored by \( \mathbb{R} \)) yields the same Massey product as the de Rham complex does, see [L, Theorem III.7]. One can also prove that the singular cochain complex with rational coefficients yields the same Massey product as the Sullivan model does, but the proof is more complicated: the crucial ingredients are [Ma, Theorem 1.5] and [BG, Prop. 3.3]. However, we do not need this fact and do not discuss it here.

Since the DGA algebra of singular cochains is not commutative, Proposition 1.4 does not hold for the Massey product which comes from singular cochain complex. However, there is the following analog of 1.4.

\textbf{1.7. Proposition.} Let the functor \( \Phi \) from (1.6) be the singular cochain functor. Let \( \alpha_1, \ldots, \alpha_n \in H^*(X) \) be such that \( \langle \alpha_1, \ldots, \alpha_n \rangle_\Phi \neq \emptyset \). Then for every \( \xi \in H^*(X) \) and every \( k \) we have
\[
\xi \langle \alpha_1, \ldots, \alpha_n \rangle \subseteq \pm \langle \xi \alpha_1, \ldots, \alpha_n \rangle,
\]
\[
\xi \langle \alpha_1, \ldots, \alpha_n \rangle \subseteq \pm \langle \alpha_1, \ldots, \xi \alpha_n \rangle.
\]
\[
\langle \alpha_1, \ldots, \alpha_{k-1}, \xi \alpha_k, \alpha_{k+1}, \ldots, \alpha_n \rangle \cap \pm \langle \alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \xi \alpha_{k+1}, \ldots, \alpha_n \rangle \neq \emptyset.
\]
Furthermore, for Massey triple products we have:
\[
\langle \alpha, \beta, \xi \gamma \rangle \subseteq \pm \langle \alpha, \xi \beta, \gamma \rangle
\]
whenever \( \langle \alpha, \beta, \gamma \rangle \neq \emptyset \).

Here \( \pm A := \{ a | a \in A \text{ or } (-a) \in A \} \).

\textbf{Proof.} The first two inclusions and the inequality
\[
\langle \alpha_1, \ldots, \alpha_{k-1}, \xi \alpha_k, \alpha_{k+1}, \ldots, \alpha_n \rangle \cap \pm \langle \alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \xi \alpha_{k+1}, \ldots, \alpha_n \rangle \neq \emptyset
\]
follow from [K, Theorem 6]. Now, for Massey triple products we have
\[
\text{Indet} \langle \alpha, \beta, \xi \gamma \rangle = \langle \alpha, \xi \gamma \rangle \subseteq (\alpha, \gamma) = \text{Indet} \langle \alpha, \xi \beta, \gamma \rangle
\]
and the desired inclusion follows from 1.5 since \( \langle \alpha, \xi \beta, \gamma \rangle \cap \pm \langle \alpha, \beta, \xi \gamma \rangle \neq \emptyset. \) \( \square \)
2. $\mathbb{C}P^k$-fibrations

2.1. Proposition. Let

$$\mathbb{C}P^k \overset{j}{\rightarrow} E \overset{p}{\twoheadrightarrow} B$$

be a $\mathbb{C}P^k$-fibration over a path connected base, and let $\xi \in H^2(E)$ be an element such that $j^*(\xi) \in H^2(\mathbb{C}P^k)$ generates the $R$-module $H^2(\mathbb{C}P^k) = R$. Then the following holds:

(i) every element $a \in H^*(E)$ can be represented as

$$a = \sum_{i=0}^k \xi^i p^*(a_i), \quad a_i \in H^*(B)$$

where the elements $a_i \in H^*(B)$ are uniquely determined by $a$;

(ii) let $x, a_1, \ldots, a_m \in H^*(B)$ be such that $\xi^n p^*x \in (p^*a_1, \ldots, p^*a_m)$ for some $n \leq k$. Then $x \in (a_1, \ldots, a_m)$;

(iii) let the functor $\Phi$ from (1.6) either take the values in the subcategory of commutative DGA algebras or is the singular cochain complex functor. Let $\alpha, \beta, \gamma \in H^*(B)$ be such that the Massey triple product $\langle \alpha, \beta, \gamma \rangle$ is essential. Then the Massey triple product

$$\langle \xi^l p^*\alpha, \xi^m p^*\beta, \xi^n p^*\gamma \rangle$$

is essential whenever $l, m, n$ are non-negative integer numbers with $l + m + n \leq k$.

Proof. (i) This is the Leray–Hirsch Theorem (see e.g. [S, Theorem 15.47]).

(ii) We have

$$\xi^n p^*x = \sum_{r=1}^m p^*a_r \sum_{j=0}^k \xi^j p^*u_{rj} = \sum_{j=0}^k \xi^j \sum_{r} p^*(a_r u_{rj}).$$

Now, by (i), we conclude that $x = \sum_r a_r u_{rn}$, i.e. $x \in (a_1, \ldots, a_n)$.

(iii) First, notice that

$$\langle \xi^l p^*\alpha, \xi^m p^*\beta, \xi^n p^*\gamma \rangle \neq \emptyset$$

by 1.5 and 1.3. Take $u \in \langle \alpha, \beta, \gamma \rangle$ with $u \notin \langle \alpha, \gamma \rangle$. Then, by (ii), $\xi^{l+m+n} p^*u \notin (p^*\alpha, p^*\gamma)$, and hence

$$\xi^{l+m+n} p^*u \notin (\xi^l p^*\alpha, \xi^n p^*\gamma) \subset (p^*\alpha, p^*\gamma).$$

So, because of 1.5, it suffices to prove that

$$\xi^{l+m+n} p^*u \in \pm (\xi^l p^*\alpha, \xi^m p^*\beta, \xi^n p^*\gamma).$$

Now, if $\Phi(E)$ is a commutative DGA algebra then, by 1.4.

$$\xi^{l+m+n} p^*u \in \xi^{l+m+n}(p^*\alpha, p^*\beta, p^*\gamma) \subset (\xi^l p^*\alpha, \xi^m p^*\beta, \xi^n p^*\gamma)$$

If $\Phi(X) = (C^*(X), \delta)$ then, by 1.7,

$$\xi^{l+m+n} p^*u \in \xi^{l+m+n}(p^*\alpha, p^*\beta, p^*\gamma) \subset \pm (\xi^l p^*\alpha, p^*\beta, \xi^m p^*\gamma) \subset \pm (\xi^l p^*\alpha, \xi^m p^*\beta, \xi^n p^*\gamma). \quad \Box$$
3. Thom spaces and Massey products

In this section the functor $\Phi$ is an arbitrary functor as in (1.6).

We need some preliminaries on Thom spaces of normal bundles. Standard references are [B], [R]. Let $M, X$ be two closed smooth manifolds, and let $i : M \to X$ be a smooth embedding. Let $\nu$ be the normal bundle of $i : M \subset X$, $\dim \nu = d$, and let $T\nu$ be the Thom space of $\nu$. We assume that $\nu$ is orientable, choose an orientation of $\nu$ and denote by $U \in \tilde{H}^d(T\nu)$ the Thom class of $\nu$.

Let $N$ be a closed tubular neighborhood of $i(M)$ in $X$. Let $V$ be the interior of $N$, and set $\partial N = N \setminus V$. The Thom space $T\nu$ can be identified with $X/X \setminus V = N/\partial N$.

We denote by

\[
\varphi : \tilde{H}^i(T\nu) \otimes H^j(M) \to \tilde{H}^{i+j}(T\nu)
\]

of the form

$H^i(N, \partial N) \otimes H^j(N) \to H^{i+j}(N, \partial N)$

and the pairing

\[
\psi : \tilde{H}^i(T\nu) \otimes H^j(X) \to \tilde{H}^{i+j}(T\nu)
\]

of the form

$H^i(X, X \setminus V) \otimes H^j(X) \to H^{i+j}(X, X \setminus V)$.

It is well known and easy to see that the diagram

\[
\begin{array}{ccc}
\tilde{H}^i(T\nu) \otimes H^j(M) & \xrightarrow{\varphi} & \tilde{H}^{i+j}(T\nu) \\
1 \otimes \iota & \downarrow & \downarrow >
\end{array}
\]

commutes; here $\Delta = \Delta_X$ is induced by the diagonal $X \to X \times X$ and $\varepsilon$ is the canonical inclusion $\tilde{H}^j(T\nu) \subset H^j(T\nu)$.
As usual, for the sake of simplicity we denote each of the products $\varphi(a \otimes b)$, $\psi(a \otimes b)$ and $\Delta(a \otimes b)$ by $ab$. Also, below we will use the same letter for an element $x \in \tilde{H}^*(T\nu)$ and its image $\varepsilon(x)$ under the canonical inclusion $\tilde{H}^*(T\nu) \subset H^*(T\nu)$. For example, we can consider the Thom class $U \in H^d(T\nu)$.

Finally, we recall that the Euler class $\chi = \chi(\nu)$ of $\nu$ is defined as $\chi := z^*(U) \in H^d(M)$ where $z : M \to T\nu$ is the zero section of the Thom space. Furthermore,

$$(3.3) \quad z^*(Ua) = \chi a \quad \text{for every } a \in H^*(M),$$

see e.g. [R, Prop. V.1.27].

3.4. Lemma. If $\langle \alpha, \beta, \gamma \rangle \neq \emptyset$ for some $\alpha, \beta, \gamma \in H^*(M)$ then $\langle \chi\alpha, \chi\beta, \chi\gamma \rangle \neq \emptyset$. If, in addition, $\langle \chi\alpha, \chi\beta, \chi\gamma \rangle$ is essential then there are $u, v, w \in H^*(X)$ such that the Massey product $\langle u, v, w \rangle$ is essential.

Proof. Clearly, $\chi\alpha\beta = 0 = \chi\beta\chi\gamma$, and so $\langle \chi\alpha, \chi\beta, \chi\gamma \rangle \neq \emptyset$. Set $u := c^*(U\alpha)$, $v := c^*(U\beta)$, $w := c^*(U\gamma)$ and prove that $uv = 0 = vw$. Consider the diagram (where $H$ denotes $H^*$)

$$
\begin{array}{ccc}
\tilde{H}(T\nu) \otimes H(M) \otimes \tilde{H}(T\nu) \otimes H(M) & \xrightarrow{\varphi \otimes \varphi} & \tilde{H}(T\nu) \otimes H(T\nu) \\
\downarrow T & & \downarrow \Delta \\
\tilde{H}(T\nu) \otimes \tilde{H}(T\nu) \otimes H(M) \otimes H(M) & \xrightarrow{\Delta \otimes \Delta} & \tilde{H}(T\nu) \otimes H(T\nu) \\
\end{array}
$$

where $T(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d$. This diagram commutes up to sign, and so

$$(U\alpha)(U\beta) = \Delta((U\alpha) \otimes (U\beta)) = \Delta \circ (\varphi \otimes \varphi)(U \otimes \alpha \otimes U \otimes \beta)
= \pm \varphi(\Delta \otimes \Delta)(U \otimes U \otimes \alpha \otimes \beta) = \pm \varphi((UU) \otimes (\alpha\beta)) = 0$$

since $\alpha\beta = 0$. So, $uv = c^*((U\alpha)(U\beta)) = 0$.

Similarly, $vw = 0$, and so $\langle u, v, w \rangle \neq \emptyset$. Furthermore, the map

$$M \overset{i}{\to} X \overset{\nu}{\to} T\nu$$

coinsides with the zero section $z : M \to T\nu$. Now,

$$0 \notin \langle \chi\alpha, \chi\beta, \chi\gamma \rangle = \langle z^*(U\alpha), z^*(U\beta), z^*(U\gamma) \rangle = \langle i^*u, i^*v, i^*w \rangle,$$

and the result follows from 1.3. □

3.5. Lemma. Let $\alpha, \beta \in H^*(M)$ and $w \in H^*(X)$ be such that $\langle \alpha, \beta, i^*w \rangle \neq \emptyset$. Then $\langle \chi\alpha, \chi\beta, i^*w \rangle \neq \emptyset$. If, in addition, $\langle \chi\alpha, \chi\beta, i^*w \rangle$ is essential then there are $u, v \in H^*(X)$ such that the Massey product $\langle u, v, w \rangle$ is essential.

Proof. Clearly, the Massey product $\langle \chi\alpha, \chi\beta, i^*w \rangle$ is not empty. As in the proof of 3.4, we set $u := c^*(U\alpha)$, $v := c^*(U\beta)$. We have proved in 3.4 that $uv = 0$. Now we prove that $vw = 0$, i.e. that $\langle u, v, w \rangle \neq \emptyset$. Consider the commutative diagram

$$
\begin{array}{ccc}
\tilde{H}^*(T\nu) \otimes H^*(M) \otimes H^*(M) & \xrightarrow{1 \otimes \Delta} & \tilde{H}^*(T\nu) \otimes H^*(M) \\
\varphi \otimes 1 & \downarrow & \varphi \\
\tilde{H}^*(T\nu) \otimes H^*(M) & \xrightarrow{\varphi} & \tilde{H}^*(T\nu) \\
\end{array}
$$
Notice that
\begin{equation}
\varphi(U \beta \otimes i^* w) = \varphi \circ (\varphi \otimes 1)(U \otimes \beta \otimes i^* w) = \varphi(1 \otimes \Delta)(U \otimes \beta \otimes i^* w)
= \varphi(U \otimes \beta i^* w) = 0.
\end{equation}

Furthermore, in the diagram (3.2) we have
\begin{equation*}
vw = \Delta(v \otimes w) = c^* \psi((U \beta) \otimes w) = c^* \varphi(U \beta \otimes i^* w) = 0,
\end{equation*}
the last equality follows from (3.6). Now the proof can be completed just as the proof of 3.4. \(\square\)

3.7. Remark. Certainly, in 3.5 we can consider the Massey product \(\langle i^* w, \alpha, \beta \rangle\), resp. \(\langle \alpha, i^* w, \beta, \rangle\), and get the essential Massey product \(\langle w, u, v \rangle\), resp. \(\langle u, w, v, \rangle\). We leave it to the reader to formulate and prove the corresponding parallel results.

4. Applications to blow-up

In this section the functor \(\Phi\) is assumed to be as in 2.1(iii).

4.1. Recollection. Let \(\zeta\) be a \((k+1)\)-dimensional complex vector bundle over a space \(Y\). We assume that \(\zeta\) is equipped with a Hermitian metric and let \(\text{Prin}(\zeta) = \{P \to Y\}\) be the corresponding principal \(U(k+1)\)-bundle. Recall that the projectivization of \(\zeta\) is a locally trivial \(\mathbb{C}P^k\)-bundle
\[\widetilde{Y} \to Y\]
where \(\widetilde{Y} := P \times_{U(k+1)} \mathbb{C}P^k\) and the \(U(k+1)\)-action on \(\mathbb{C}P^k\) is induced by the canonical \(U(k+1)\)-action on \(\mathbb{C}^{k+1}\).

Now we define a canonical complex line bundle \(\lambda_\zeta\) over \(\widetilde{Y}\) as follows. The total space \(L\) of the canonical line bundle \(\eta_k\) over \(\mathbb{C}P^k\) has the form
\[L = \{(z, l) \in \mathbb{C}^{k+1} \times \mathbb{C}P^k \mid z \in l\}.
\]
In particular, the projection \(\pi : L \to \mathbb{C}P^k\), \((z, l) \mapsto l\) is an \(U(k+1)\)-equivariant map. We define \(\lambda_\zeta\) to be the induced map
\[1 \times \pi : P \times_{U(k+1)} L \to P \times_{U(k+1)} \mathbb{C}P^k = \widetilde{Y}.
\]
Notice that if \(Y\) is the one-point space then \(\widetilde{Y} = \mathbb{C}P^k\) and the canonical bundle \(\lambda\) coincides with \(\eta_k\).

The construction \(\lambda_\zeta\) is natural in the following sense. Let \(\zeta\) be a complex vector bundle over \(Y\), and let \(f : Z \to Y\) be an arbitrary map. Then the obvious map \(I : f^*(\text{Prin}\zeta) \to \text{Prin}\zeta\) yields a commutative diagram
\[\begin{array}{ccc}
\widetilde{Z} & \xrightarrow{f} & \widetilde{Y} \\
\downarrow & & \downarrow \\
Z & \xrightarrow{f} & Y
\end{array}
\]
where \(\widetilde{Z} \to Z\) is the projectivization of \(f^*\zeta\) and \(\widetilde{f}\) is induced by \(I\).
4.2. Proposition. \( \tilde{j}^* \lambda_\zeta \) is naturally isomorphic to \( \lambda_{j^*\zeta} \). \( \square \)

4.3. Corollary. If \( CP^k \xrightarrow{j} \tilde{Y} \to Y \) is the projectivization of a complex vector bundle \( \zeta \), then \( j^* \lambda_\zeta \) is isomorphic to the canonical line bundle \( \eta_k \) over \( CP^k \). \( \square \)

4.4. Definition. Let \( M \) and \( X \) be two closed connected smooth manifolds, and let \( i : M \to X \) be a smooth embedding of codimension \( 2k + 2 \). A blow-up along \( i \) is a commutative diagram of smooth manifolds and maps

\[
\begin{array}{ccc}
CP^k & \xrightarrow{j} & \tilde{M} \\
\downarrow & & \downarrow \tilde{i} \\
M & \xrightarrow{i} & X
\end{array}
\]

such that the following holds:

(i) \( CP^k \xrightarrow{j} \tilde{M} \xrightarrow{p} M \) is a locally trivial bundle (with fiber \( CP^k \)) which is a projectivization of a complex \((k + 1)\)-dimensional vector bundle \( \zeta \). In particular, \( \tilde{M} \) is a closed connected manifold;

(ii) \( \tilde{X} \) is a closed connected manifold, \( \tilde{i} : \tilde{M} \to \tilde{X} \) is a smooth embedding of codimension 2, and the line bundle \( \lambda_\zeta \) is isomorphic (as a real vector bundle) to the normal bundle \( \nu \) of \( \tilde{i} \);

(iii) there is a closed tubular neighborhood \( N \) of \( i(M) \) such that \( \tilde{N} := q^{-1}(N) \) is a tubular neighborhood of \( \tilde{i}(\tilde{M}) \) and

\[
q|_{\tilde{X} \setminus \text{Int } N} : \tilde{X} \setminus \text{Int } N \to X \setminus \text{Int } N
\]

is a diffeomorphism.

By 4.4(ii), the normal bundle \( \nu \) of \( \tilde{i} \) is isomorphic to \( \lambda_\zeta \), and hence \( \nu \) is orientable. Take an orientation of \( \nu \) and consider the Euler class \( \chi(\nu) \in H^2(\tilde{M}) \).

4.6. Proposition. The class \( j^* \chi(\nu) \) generates the \( R \)-module \( H^2(CP^2) = R \).

Proof. By 4.4(ii) and 4.3, \( j^*\nu \) is isomorphic to \( \eta_k \), and hence \( j^*\chi(\nu) = \chi(\eta_k) \).

4.7. Proposition. Consider a blow-up diagram (4.5). If \( k \geq 1 \) then \( q_* : \pi_1(\tilde{X}) \to \pi_1(X) \) is an isomorphism.

Proof. Let \( V \) denote the interior of \( N \) and set \( \partial N = N \setminus V \). Similarly, \( \tilde{V} := q^{-1}(V) \) and \( \partial \tilde{N} := \tilde{N} \setminus \tilde{V} \). Since \( k \geq 1 \), the inclusion \( X \setminus V \subset X \) induces an isomorphism of fundamental groups. So, we must prove that the inclusion \( \tilde{X} \setminus \tilde{V} \subset \tilde{X} \) induces an isomorphism of fundamental groups. But this follows from the van Kampen Theorem, since the inclusion \( \partial \tilde{N} \subset \tilde{N} \) induces an epimorphism \( \pi_1(\partial \tilde{N}) \to \pi_1(\tilde{N}) \). The last claim holds, in turn, because \( \partial \tilde{N} \to \tilde{M} \) is a locally trivial bundle with connected fiber. \( \square \)

4.8. Remark. Usually the term “blow-up” is reserved for a canonical procedure which, in particular, leads to a diagram like (4.5), cf. [G], [M], [MS]. In Definition 4.4 we have just axiomatized certain useful (for us) properties.
4.9. **Theorem.** Consider a blow-up diagram (4.5). Suppose that \( k \geq 3 \) and that \( M \) possesses an essential Massey triple product. Then \( \tilde{X} \) possesses an essential Massey triple product.

**Proof.** Let \( \langle \alpha, \beta, \gamma \rangle \) be an essential Massey product in \( M \), and let \( \chi \) be the Euler class of the normal bundle of \( \tilde{i} \). Then, by 4.6 and 2.1(iii), the Massey triple product

\[
\langle \chi p^* \alpha, \chi p^* \beta, \chi p^* \gamma \rangle \subset H^*(\tilde{M})
\]

is essential. Thus, by 3.4, \( \tilde{X} \) possesses an essential Massey triple product. \( \square \)

Sometimes it is useful to replace the assumption \( k \geq 3 \) in 4.9 by a weaker assumption \( k \geq 2 \). We do it as follows.

4.10. **Theorem.** Consider a blow-up diagram as in (4.5). Suppose that \( k \geq 2 \) and that there are elements \( \alpha, \beta \in H^*(M) \) and \( w \in H^*(X) \) such that at least one of the Massey triple product \( \langle \alpha, \beta, i^*w \rangle, \langle \alpha, i^*w, \beta, \rangle, \langle i^*w, \alpha, \beta, \rangle \) is essential. Then \( \tilde{X} \) possesses an essential triple Massey product.

**Proof.** We consider the case when the Massey product \( \langle \alpha, \beta, i^*w \rangle \subset H^*(M) \) is essential, all the other cases can be considered similarly. Let \( \chi \) be the Euler class of the normal bundle of \( \tilde{i} \). Then, by 4.6 and 2.1(iii), \( \langle \chi p^* \alpha, \chi p^* \beta, \tilde{i}^*q^*w \rangle \neq \emptyset \), and the Massey product

\[
\langle \chi p^* \alpha, \chi p^* \beta, \tilde{i}^*q^*w \rangle \subset H^*(\tilde{M})
\]

is essential. Thus, by 3.5, \( \tilde{X} \) possesses an essential Massey product of the form \( \langle u, v, q^*w \rangle \). \( \square \)

5. **APPLICATION TO SYMPLECTIC MANIFOLDS**

In this section the functor \( \Phi \) is assumed to be the Sullivan minimal model.

5.1. **Theorem** ([M]). Let \( M \) and \( X \) be two closed symplectic manifolds, and let \( i : M \to X \) be a symplectic embedding. Then there exists a blow-up diagram where the map \( \tilde{i} \) is a symplectic embedding. In particular, \( \tilde{X} \) is a symplectic manifold. \( \square \)

We define a **symplectic blow-up** to be a blow-up with \( i \) and \( \tilde{i} \) symplectic.

Actually, McDuff [M] suggested a canonical construction of symplectic blow-up. In particular, the bundle \( \zeta \) in 4.4(i) turns out to be the normal bundle of the embedding \( i \). A detailed exposition of this construction can be found in [MS], [TO].

This theorem enables us to use the above Theorems 4.9, 4.10 in order to construct non-formal symplectic manifolds. To start with, we must have at least one symplectic manifold with an essential Massey triple product.

5.2. **Example** (Kodaira–Thurston). Let \( H \) be the Heisenberg group, i.e. the group of the \( 3 \times 3 \) matrices of the form

\[
\alpha = \begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]
with \(a, b, c \in \mathbb{R}\), and let \(\Gamma\) be the subgroup of \(H\) with integer entries. We set
\[
K := (H/\Gamma) \times S^1.
\]

So, the Kodaira–Thurston manifold \(K\) is a 4-dimensional nilmanifold. Its Sullivan minimal model has the form
\[
(\Lambda(x_1, x_2, x_3, x_4), d) \quad \text{with} \quad dx_1 = dx_2 = dx_4 = 0, \quad dx_3 = x_1x_2,
\]
where \(\deg x_i = 1\) for \(i = 1, 2, 3, 4\) and the generators \(x_1, x_2, x_3\) come from the Heisenberg manifold. We set \(\alpha = [x_1], \beta = [x_2], \gamma = x_4\) and \(u = [x_1x_4 + x_2x_3]\).

Then
\[
H^1(K; \mathbb{Q}) = \mathbb{Q}^3 = \{\alpha, \beta, \gamma\},
\]
\[
H^2(K; \mathbb{Q}) = \mathbb{Q}^4 = \{\alpha\gamma, \beta\gamma, u, [x_1x_3]\},
\]
\[
H^3(K; \mathbb{Q}) = \mathbb{Q}^3 = \{\alpha u, \gamma u, [x_1x_3x_4]\}
\]
where \(\{\ldots\}\) denotes “the \(\mathbb{Q}\)-vector space with the basis ...”. Finally, \(K\) possesses a symplectic form \(\omega\), and the class \(u \otimes 1 \in H^*(K; \mathbb{Q}) \otimes \mathbb{R} = H^*(K; \mathbb{R})\) coincides with the de Rham cohomology class of \(\omega\). See [TO] for details.

It is easy to see that the Massey product \(\langle \alpha, \alpha, \beta \rangle\) is essential, but we need more.

**5.3. Proposition.** The Massey products \(\langle \alpha, \alpha, \beta \rangle\) and \(\langle \beta, \beta, u \rangle\) are essential.

**Proof.** We prove the essentiality of the second Massey product, because the proof for the first one is simpler. Clearly \(\beta \beta = 0\). Furthermore,
\[
\beta u = [x_2x_1x_4 + x_2x_2x_3] = [x_2x_1x_4] = [-d(x_3x_4)] = 0,
\]
and hence \(\langle \beta, \beta, u \rangle \neq \emptyset\). Furthermore,
\[
x_2(x_1x_4 + x_2x_3) = -d(x_3x_4),
\]
and hence \([-x_1x_3x_4] \in \langle \beta, \beta, u \rangle\). But \([x_1x_3x_4] \notin \langle \beta, u \rangle\) because
\[
\langle \beta, u \rangle \cap H^3(K, \mathbb{Q}) = \{\alpha u, \gamma u\}.
\]
Thus, \(\langle \beta, \beta, u \rangle\) is essential. \(\square\)

**5.4. Theorem.** Consider a symplectic embedding \(i : K \to X\) and any symplectic blow-up
\[
\begin{array}{ccc}
\mathbb{CP}^k & \xrightarrow{j} & \tilde{K} \\
\downarrow & & \downarrow i \\
K & \xrightarrow{i} & X
\end{array}
\]
along i. If \( k \geq 2 \) then \( \bar{X} \) possesses an essential rational Massey triple product. In particular, \( \bar{X} \) is not a formal space.

**Proof.** For \( k \geq 3 \) this follows from 4.9, because \( K \) possesses an essential Massey triple product. If \( k = 2 \), consider the symplectic form \( \omega_X \) on \( X \). Then \( i^* \omega_X = \omega_K \) where \( \omega_K \) is the symplectic form on \( K \). Hence,

\[
    u \in \text{Im}\{i^*: H^2(X; \mathbb{Q}) \rightarrow H^2(K; \mathbb{Q})\}.
\]

So, according to 5.3, \( \langle \beta, \beta, i^*w \rangle \) is essential for some \( w \in H^2(X; \mathbb{Q}) \). Now, the result follows from 4.10. \( \square \)

In particular, if the initial ambient space \( X \) is simply-connected, then the space \( \bar{X} \) in 5.4 gives us an example of non-formal simply-connected symplectic manifold.

It is well known that, for every \( n \geq 5 \), there exists a symplectic embedding \( K \hookrightarrow \mathbb{C}P^n, [\text{Gr}, T] \). Consider a symplectic blow-up along such an embedding, and let \( \bar{\mathbb{C}P}^n \) denote the corresponding space (in the top right corner of the diagram (4.5)). Then 5.2 yields the following corollary.

**5.5. Corollary** (Babenko–Taimanov [BT]). Every space \( \bar{\mathbb{C}P}^n, n \geq 5 \) possesses an essential Massey triple product, and hence it is not a formal space. \( \square \)

**5.6. Remarks.** 1. Generalizing 5.2, consider the so-called Iwasawa manifolds. Namely, we set

\[
    I(p, q) = (H(1, p) \times H(1, q))/\Gamma
\]

where \( H(1, p) \) consists of all matrices of the form

\[
    \alpha = \begin{pmatrix}
    I_p & A & C \\
    0 & 1 & b \\
    0 & 0 & 1
    \end{pmatrix}
\]

It is proved in [CFG] that \( I(p, q) \) always possesses an essential rational Massey triple product. So, we can use these manifolds for construction of families of non-formal symplectic manifolds. In particular, if both \( p \) and \( q \) are even then \( I(p, q) \) is symplectic, and so in this way we can also get a large family of non-formal (and so non-Kähler) symplectic manifolds.

2. Certainly, the blow-up construction can be iterated, i.e. starting from \( M \subset X \) and getting \( \bar{X} \), we can embed \( \bar{X} \) into another manifold and construct a new blow-up, and so on. So, here we really have a lot of possibilities to construct manifolds with essential Massey products and, in particular, non-formal manifolds.

6. **Massey products in algebraic and Kähler manifolds**

In this section \( \Phi^*(X) = (C^*(X; \mathbb{Z}/p), \delta) \) where \( p \) is an odd prime number.

According to [DMGS], every Kähler manifold \( M \) is formal, and so every Massey product in \( H^*(M; \mathbb{Q}) \) is inessential. However, there are complex projective algebraic varieties (and in particular Kähler manifolds) \( S \) with essential Massey products in
$H^*(S;\mathbb{Z}/p)$. Indeed, Kraines [K] proved that, for every prime $p > 2$ and every $x \in H^{2n+1}(X;\mathbb{Z}/p)$, we have

$$\beta P^n(x) \in -(x,x,\ldots,x) \subset H^{2np+2}(X;\mathbb{Z}/p)$$

\[ \text{where } \beta : H^i(X;\mathbb{Z}/p) \to H^{i+1}(X;\mathbb{Z}/p) \text{ is the Bockstein homomorphism and} \]

$$P^n : H^i(X;\mathbb{Z}/p) \to H^{i+2n(p-1)}(X;\mathbb{Z}/p)$$

\[ \text{is the reduced Steenrod power. For instance, if } H_1(X;Z) = \mathbb{Z}/3 \text{ then } H^1(X;\mathbb{Z}/3) = \mathbb{Z}/3 \text{ and } \beta(x) \neq 0 \text{ for every } x \in H^1(X;\mathbb{Z}/3), x \neq 0. \text{ So, the Massey product } \langle x,x,x \rangle \text{ has zero indeterminacy, and } \langle x,x,x \rangle \neq 0 \text{ provided } x \neq 0. \]

So, if $H_1(X;Z) = \mathbb{Z}/3$, then $H^*(X;\mathbb{Z}/3)$ possesses an essential Massey triple product. Certainly, one can find many examples of algebraic varieties and Kähler manifolds with such homology groups, see e.g. [ABCKT]. Now, since the classes of Kähler manifolds and algebraic varieties are invariant under the blow-up construction, we can use Theorems 4.9 and 4.10 and construct a large class of Kähler manifolds and algebraic varieties with essential Massey triple $\mathbb{Z}/3$-products, including simply-connected objects as well.

Perhaps, the following observation looks interesting. If we have a Kähler manifold or algebraic variety $X_0$ with $u_0 \in H^1(X_0;\mathbb{Z}/3)$ such that $0 \notin \langle u_0,u_0,u_0 \rangle$, then we can perform a blow-up along $X_0$ (in the corresponding category) and get a resulting object $X_1$, and if $\dim X_1 - \dim X_0 \geq 6$ then there is an element $u_1 \in H^3(X_1;\mathbb{Z}/3)$ with $0 \notin \langle u_1,u_1,u_1 \rangle$. In particular, $\beta P^1(u_1) \neq 0$. Similarly, performing an obvious induction, we can construct $X_n$ (algebraic or Kähler) and an element $u_n \in H^{2n+1}(X_n;\mathbb{Z}/3)$ with $\beta P^n(u) \neq 0$.

Can one generalize these results for modulo $p$ cohomology? One can construct a Kähler manifold with the fundamental group $\mathbb{Z}/p$, but we cannot go ahead because we are not able to prove an analog of 2.1(iii) for $p$-tuple Massey products. However, there is the following analog of the above results.

For every $u \in H^*(X;\mathbb{Z}/p)$, Kraines [K] defined a “small” Massey product

$$\langle u \rangle^k \subset \langle u,u,\ldots,u \rangle$$

\[ \text{by considering special defining systems where } x_{ij} = x_{kl} \text{ whenever } j - i = l - k. \]

Moreover, he proved that

$$-\beta P^i(u) = \langle u \rangle^p \text{ for every } u \in H^{2i+1}(X;\mathbb{Z}/p).$$

It is remarkable that $\langle u \rangle^p$ has zero indeterminacy. Now we can prove an analog of 2.1(iii) for $\langle u \rangle^p$ provided $k \geq p$, i.e. to prove that $\langle xu \rangle^p \neq 0$ provided $\langle u \rangle^p \neq 0$. (One can prove it directly, or can use [K, Corollary 7].) Thus, we can proceed (by blowing up) and construct a large family of Kähler manifolds with $\langle u \rangle^p \neq 0$. 


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