Target Space Duality: The Dilatonic Field

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Abstract

Classical target space duality transformations are studied for the non-linear sigma model with a dilatonic field. Working within the framework of the Hamiltonian formalism we require the duality transformation to be a property only of the target spaces. We obtain a set of restrictions on the geometrical data. The “on-shell duality” integrability conditions are inspected.

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1 Introduction

In a series of previous papers \cite{1,2} a framework was developed for studying classical target space duality between nonlinear sigma models in two dimensional Minkowski space. References to the earlier literature may be found in \cite{1,2}. Here we introduce a dilaton field $\Phi$ coupled to the world sheet curvature scalar $R^{(2)}(\vec{\sigma})$ via the action $S = S_0 + S_B + S_\Phi$ where

$$S_0 + S_B = -\frac{1}{2} \int d^2 \sigma \left\{ \sqrt{-h} h^{\alpha \beta} g_{ij}(x) \partial_\alpha x^i \partial_\beta x^j - \epsilon^{\alpha \beta} \partial_\alpha x^i \partial_\beta x^j B_{ij}(x) \right\},$$  

(1.1)

$$S_\Phi = \int d^2 \sigma \sqrt{-h} \Phi(x) R^{(2)}(\vec{\sigma}).$$  

(1.2)

The target manifolds with their respective geometrical data are denoted by $M(g, B, \Phi)$ and $\tilde{M}(\tilde{g}, \tilde{B}, \tilde{\Phi})$. The Greek indices refer to the world sheet $\Sigma$ with metric $h_{\alpha \beta}$ and coordinates $\vec{\sigma} = (\tau, \sigma)$. In two dimensions it is always possible to find a coordinate transformation that locally puts the metric in conformal form

$$h_{\alpha \beta} = e^{2\mu(\vec{\sigma})} \eta_{\alpha \beta},$$

(1.3)

where $\eta_{\alpha \beta}$ is the flat world sheet metric with the signature $(-, +)$ and $\mu(\vec{\sigma})$ is the conformal factor. Introducing light-cone coordinates on the world sheet by $\sigma^\pm = \tau \pm \sigma$ we see that the curvature scalar is

$$R^{(2)} = 8e^{-2\mu(\vec{\sigma})} \partial^2_{+-} \mu(\vec{\sigma}).$$

(1.4)

A possible multiplicative constant for $S_\Phi$ can be absorbed into the definition of dilaton field $\Phi$. Defining the closed 3-form $H$ by $H = dB$ and the derivatives $x^i_{\pm}$ by pulling back an orthonormal coframe from the target space \footnote[3]{See \cite{2,3} for notation.} gives the classical equations of motion

$$x^i_{++} + \frac{1}{2} H^i_{jk}(x) x^j_{+} x^k_{--} + 2 \Phi'_i(x) \partial^2_{+-} \mu(\vec{\sigma}) = 0$$

(1.5)

where $d\Phi = \Phi'_i \theta^i$ and $x^i_{++}$ is the second covariant derivative. There is an analogous expression for $\tilde{M}$ model. Any transformation of the metric resulting from

$$\mu(\vec{\sigma}) \to \mu(\vec{\sigma}) + \xi(\sigma^+) + \eta(\sigma^-)$$

(1.6)

with arbitrary functions $\xi$ and $\eta$ leaves the form of equations of motion invariant.

We note that the first two terms in the action $S_0 + S_B$ are manifestly independent of the choice of the conformal factor but the term $S_\Phi$ is not \footnote[4]{It is still invariant under simultaneous global scaling of the conformal factor $\mu(\vec{\sigma}) \to a \mu(\vec{\sigma})$ and the dilaton field $\Phi(x) \to \Phi(x)/a$.}

The $\Phi = 0$ models are trivially classically conformally invariant. Therefore, at the classical level it was sufficient to study those models on a flat world sheet because all dependence on the conformal factor was absent. Consequently, the study of classical target space duality in these conformally invariant models reduced to studying models with a flat world sheet. The classical conformal invariance is manifestly broken by the presence of a generic non-zero
dilaton field\footnote{At the quantum level it is consistent to choose the dilaton term to be $O(\hbar)$. The condition for conformal invariance, the tracelessness of the energy-momentum tensor, is obtained by combining contributions (both classical and quantum) from the dilaton term with the quantum corrections from the other terms. This gives a set of restrictions on geometrical data describing a conformally invariant model, see e.g. \cite{4,5}.} This means that the classical behavior of strings propagating on a target space $M$ depends on local metrical properties of the world sheet $\Sigma$. In this article we ask the following question.

Is it possible to have a classical duality transformations between strings propagating on target spaces $M$ and $\tilde{M}$ such that the duality transformation is only a property of the target spaces and it is independent of the metrical geometry of the world sheet $\Sigma$, i.e., independent of the conformal factor $\mu$?

N.B. The sigma models are not necessarily conformally invariant.

The world sheet stress-energy tensor $T^{\alpha\beta}$ is defined variationally by

\[
\delta S = \frac{1}{2} \int d^2\sigma \sqrt{-h} T^{\alpha\beta} \delta h_{\alpha\beta} .
\]  

(1.7)

The contribution from $S_0$ is

\[
T^{\alpha\beta}_{(0)} = 4e^{-4\mu(\sigma)} \begin{pmatrix} x_- x_- & 0 \\ 0 & x_+ x_+ \end{pmatrix} ,
\]

(1.8)

the term $S_B$ does not contribute to the stress-energy tensor although it contribute to the equations of motion. Using the equations of motion in the calculation of $\nabla_\alpha T^{\alpha\beta}$ you find terms such as $H_{ijk} x_i^+ x_j^- x_k^+$ that vanish due to the antisymmetry of $H_{ijk}$.

To calculate the contribution from $S_\Phi$ we have to integrate by parts twice and to observe that the Einstein tensor vanishes because the Hilbert-Einstein action is a topological invariant in 2 dimensions. After some algebra we arrive at the result

\[
T^{\alpha\beta}_{(\Phi)} = 2(h^{\gamma\alpha}h^{\delta\beta} \Phi_{\gamma\delta} - h^{\alpha\beta}h^{\gamma\delta} \Phi_{\gamma\delta}) ,
\]

(1.9)

where $\Phi_\gamma := x^i_\gamma \partial_\gamma \Phi(x)$ and $\Phi_{\gamma\delta}$ is the covariant derivative on the world sheet. The only non-vanishing connection coefficients for the metric (1.3) are $\Gamma^+_{++} = 2\partial_+ \mu(\tilde{\sigma})$ and $\Gamma^-_{--} = 2\partial_- \mu(\tilde{\sigma})$.

The explicit expression is

\[
T^{\alpha\beta}_{(\Phi)} = 8e^{-4\mu} \left( 2 \partial_- \mu \frac{\Phi'_{ij} x^-_i x^-_j - \Phi'_{ij} x^-_i x^-_j - \Phi'_{ii} x^-_i x^-_i}{\Phi'_{ij} x^-_i x^-_j + \Phi'_{ii} x^-_i x^-_i} \right) (1.10)
\]

Here, the derivatives $\Phi'_i$ are defined as components of the 1-form $d\Phi$ with respect to the orthonormal coframe of $\tilde{M}$ and $\Phi'_{ij}$ is a covariant derivative on the target space. For the model $\tilde{M}(\tilde{g}, \tilde{B}, \tilde{\Phi})$ these derivatives will be denoted as $\tilde{\Phi}'_i$ and $\tilde{\Phi}'_{ij}$ respectively. Note that the stress-energy tensor does not vanish in the limiting case of a flat world sheet $\mu(\tilde{\sigma}) \to 0$, although in this limit both the action and the classical equations of motion are the same as for the flat world sheet case. In the limit $\mu(\tilde{\sigma}) \to 0$, $T_{(\Phi)}^{\alpha\beta}$ has the property that its divergence vanishes identically, i.e. $\nabla_\beta T_{(\Phi)}^{\alpha\beta} = 0$ for any $x^i(\sigma^\mu)$ and not just for the solution of equation of motion.
In the case of \([1, 2]\) where the action was \(S_0 + S_R\), the transformation equation for the “on-shell” duality could have been written down by inspecting the stress-energy tensor. Here there are two difficulties. The stress-energy tensor contains the terms \(\Phi'_i(x)\) and \(\Phi'_{ij}(x)\) which have dependence on \(x\)’s and the duality transformation involves derivatives of \(x\)’s, therefore integrability issues arise. The other difficulty is just the mentioned possibility that two equivalent expressions may differ by a contribution whose divergence vanishes identically and this contribution has to be clearly identified.

## 2 A Toy Model

As a guideline consider a classical mechanics time-dependent Hamiltonian system. By prescribing a generating function \(F(q, \tilde{q}, t)\) one obtains both the canonical transformation and the relation between the hamiltonians by

\[
\dot{p} = \frac{\partial F(q, \tilde{q}, t)}{\partial \tilde{q}}, \quad -p = \frac{\partial F(q, \tilde{q}, t)}{\partial q}, \quad \tilde{H} - H = -\frac{\partial F(q, \tilde{q}, t)}{\partial t}. \tag{2.1}
\]

Here we consider the inverse problem. Both hamiltonians are given and we want to determine the conditions that have to be satisfied in order to establish a canonical transformation. We assume the Hamiltonians are of the form

\[
H = \frac{1}{2} (p - A(q, t))^2 + V(q, t), \quad \tilde{H} = \frac{1}{2} (\tilde{p} - \tilde{A}(\tilde{q}, t))^2 + \tilde{V}(\tilde{q}, t). \tag{2.2}
\]

Consider a generating function of the form

\[
F(q, \tilde{q}, t) = q\tilde{q} + f(t)(\tilde{W}(\tilde{q}) - W(q)) \tag{2.3}
\]

with \(f(t), \tilde{W}(\tilde{q}), W(q)\) to be determined. Using (2.1) we see that

\[
q = \tilde{p} - f\tilde{W}', \quad \tilde{q} = -(p - fW') \tag{2.4}
\]

and

\[
\frac{1}{2} (\tilde{p} - \tilde{A})^2 + \tilde{V} - \left\{\frac{1}{2} (p - A)^2 + V\right\} = -\dot{f} (\tilde{W} - W). \tag{2.5}
\]

We rewrite (2.5) using (2.4) and group together terms according to their \(q\) and \(\tilde{q}\) dependence:

\[
0 = \frac{1}{2} q^2 - V - \frac{1}{2} (A - fW') - \dot{f} W
- \left\{\frac{1}{2} \tilde{q}^2 - \tilde{V} - \frac{1}{2} (\tilde{A} - f\tilde{W}') - \dot{f} \tilde{W}\right\}
- q(\tilde{A} - f\tilde{W}') - \tilde{q}(A - fW'). \tag{2.6}
\]

To eliminate the mixed \(q\) and \(\tilde{q}\) dependence we require that the summands in the last line of the equation above satisfy

\[
A - fW' = h(t)q, \quad \tilde{A} - f\tilde{W}' = -h(t)\tilde{q}. \tag{2.7}
\]
The remaining part of (2.6) gives immediately the conditions

\[ V(q, t) = \frac{1}{2} (1 - h(t))^2 q^2 - \dot{f} W, \]
\[ \tilde{V}(\tilde{q}, t) = \frac{1}{2} (1 - h(t))^2 \tilde{q}^2 - \dot{f} \tilde{W}. \] (2.8)

To make this example more similar to the sigma model case consider the special case where

\[ A(q, t) = f(t) B(q), \quad A(\tilde{q}, t) = f(t) \tilde{B}(\tilde{q}). \] (2.9)

From (2.7) we see that

\[ B - W' = q, \quad \tilde{B} - \tilde{W}' = \tilde{q}, \quad h(t) = f(t). \] (2.10)

Integrating we obtain the generating function for this transformation

\[ F(q, \tilde{q}, t) = q \tilde{q} + h(t) \left( -\frac{1}{2} (\tilde{q}^2 - q^2) + \int_0^q d\tilde{q}' \tilde{B}(\tilde{q}') - \int_0^q dq' B(q') \right). \] (2.11)

3 Target Space Duality

In the field theory case we have to consider hamiltonian densities of the form

\[ \mathcal{H} = \frac{1}{2} g^{ik}(\pi_i - B_{ij} x^j_\sigma)(\pi_k - B_{kl} x^l_\sigma) + g_{ik} \frac{1}{2} x^i_\sigma x^k_\sigma + 2\eta^{\alpha\beta}(\partial_\alpha \mu) \Phi(x) \] (3.1)

and the analogous expression for the \( \tilde{M} \) model. The explicit time dependence enters via the conformal factor. By analogy to (2.1) the imposed requirement is that the Hamiltonians of both models differ only by a time derivative of the generating functional.

\[ \tilde{H} - H = \int d\sigma (\tilde{H} - \mathcal{H}) = -\frac{\partial F}{\partial \tau} \] (3.2)

The general form of the generating functional is taken to be of the form

\[ F[x, \tilde{x}] = \int \alpha + \int (\partial_\sigma \mu Y + \partial_\tau \mu Z) d\sigma, \] (3.3)

where \( Y(x, \tilde{x}) \) and \( Z(x, \tilde{x}) \) are functions on \( M \times \tilde{M} \) and \( \alpha(x, \tilde{x}) \) is a 1-form on \( M \times \tilde{M} \) written as

\[ \alpha = \alpha_i(x, \tilde{x}) dx^i + \tilde{\alpha}_i(x, \tilde{x}) d\tilde{x}^i. \] (3.4)

The associated canonical transformation is

\[ \pi_i = m_{ji} \frac{d\tilde{x}^j}{d\sigma} + l_{ij} \frac{dx^j}{d\sigma} - \partial_\sigma \mu \frac{\partial Y}{\partial x^i} - \partial_\tau \mu \frac{\partial Z}{\partial x^i}, \] (3.5)
\[ \tilde{\pi}_i = m_{ij} \frac{dx^j}{d\sigma} + \tilde{l}_{ij} \frac{d\tilde{x}^j}{d\sigma} + \partial_\sigma \mu \frac{\partial Y}{\partial \tilde{x}^i} + \partial_\tau \mu \frac{\partial Z}{\partial \tilde{x}^i}. \] (3.6)
where \( m_{ij}, l_{ij} \) and \( \tilde{l}_{ij} \) are given by

\[
d\alpha = -\frac{1}{2} l_{ij}(x, \bar{x}) \, dx^i \wedge dx^j + \frac{1}{2} \tilde{l}_{ij}(x, \bar{x}) \, d\bar{x}^i \wedge d\bar{x}^j + m_{ij}(x, \bar{x}) \, d\bar{x}^i \wedge dx^j. \tag{3.7}
\]

Here it would be desirable to maintain a symmetric formulation between tilded and untilded quantities. Therefore we introduce the following definitions:

\[
n \equiv l - B, \quad \tilde{n} \equiv \bar{l} - \bar{B}, \quad \tilde{m}_{ij} \equiv m_{ji}, \tag{3.8}
\]

\[
dY = Y^i_\tau \theta^\tau - \tilde{Y}^i_\tau \tilde{\theta}^\tau, \quad dZ = Z^i_\tau \theta^\tau - \tilde{Z}^i_\tau \tilde{\theta}^\tau, \tag{3.9}
\]

where \((\theta, \tilde{\theta})\) is an orthonormal coframe of \( M \times \tilde{M} \). We use \( \| \| \) and \( \langle , \rangle \) to denote the norms and inner products on the target spaces, we also suppress target space indices \( i, j \ldots \) hereafter. Using the form of the canonical transformations we see that the integrand of (3.2) is

\[
\tilde{\mathcal{H}} - \mathcal{H} = \frac{1}{2} \| m x_\sigma + \tilde{n} \tilde{x}_\sigma - \mu_\sigma Y'' - \mu_\tau Z'' \|^2 + \frac{1}{2} \| \tilde{x}_\sigma \|^2 + \mu_\sigma Y' - \mu_\tau Z' \|^2 + \frac{1}{2} \| x_\sigma \|^2 + 2(-\mu_{\tau\tau} + \mu_{\sigma\sigma})(\tilde{\Phi} - \Phi). \tag{3.12}
\]

The next step is to group together terms with different \( x \) and \( \tilde{x} \) behavior:

\[
\tilde{\mathcal{H}} - \mathcal{H} = \frac{1}{2} \langle x_\sigma, (m^i m - n^i \tilde{n} - I) \, x_\sigma \rangle - \frac{1}{2} \langle \tilde{x}_\sigma, (\tilde{m}^i \tilde{m} - \tilde{n}^i \tilde{n} - I) \, \tilde{x}_\sigma \rangle + \langle \tilde{x}_\sigma, (\tilde{n}^i m - \tilde{m}^i n) \, x_\sigma \rangle + \mu_\sigma[-\langle x_\sigma, m^i \, Z'' - n^i \, Z' \rangle + \langle \tilde{x}_\sigma, \tilde{m}^i Z' - \tilde{n}^i \, Z'' \rangle] + \mu_\sigma[-\langle x_\sigma, m^i \, Y'' - n^i \, Y' \rangle + \langle \tilde{x}_\sigma, \tilde{m}^i Y' - \tilde{n}^i \, Y'' \rangle] + \frac{1}{2} \| \mu_\sigma Y'' + \mu_\tau Z'' \|^2 - \frac{1}{2} \| \mu_\sigma Y' + \mu_\tau Z' \|^2 - 2(-\mu_{\tau\tau} + \mu_{\sigma\sigma})(\tilde{\Phi} - \Phi). \tag{3.13}
\]

On the level of Hamiltonian densities the condition (3.2) is expressed as

\[
\tilde{\mathcal{H}} - \mathcal{H} = -(\mu_\sigma Y + \mu_{\tau\tau} Z) + \frac{\partial}{\partial \sigma} \mathcal{H}. \tag{3.14}
\]

It will be convenient to have this condition rewritten as

\[
\tilde{\mathcal{H}} - \mathcal{H} = \mu_\tau (Y^i_\tau x^i_\sigma - Y''^i_\tau \tilde{x}^i_\sigma) + \mu_\sigma (Z^i_\tau x^i_\sigma - Z''^i_\tau \tilde{x}^i_\sigma) + (-\mu_{\tau\tau} + \mu_{\sigma\sigma}) Z + \frac{d}{d\sigma} \left(h - \mu_\tau Y - \mu_\sigma Z \right). \tag{3.14}
\]
Looking at \(x_\sigma x_\sigma, \tilde{x}_\sigma \tilde{x}_\sigma, \bar{x}_\sigma \bar{x}_\sigma\) terms in (3.13) we recover the relations
\[
\begin{align*}
\tilde{m}^t \tilde{m} &= I + \tilde{n}^t \tilde{n} = I - \tilde{n}^2 = m^t, \\
m^t m &= I + n^t n = I - n^2, \\
-m m &= \tilde{n} m,
\end{align*}
\] (3.15)
which are the same as the ones calculated in [1]. Incorporating these in the remaining terms of (3.13) and using (3.14) we obtain
\[
\frac{d}{d \sigma} \left( h - \mu_\tau Y - \mu_\sigma Z \right) = \mu_\tau \left[ -\langle x_\sigma, m^t Z'' + n Z' + Y'' \rangle + \langle \bar{x}_\sigma, \tilde{m}^t \tilde{Z}' + \tilde{n} \tilde{Z}'' + Y'' \rangle \right] \\
+ \mu_\sigma \left[ -\langle x_\sigma, m^t Y'' + n Y' + Z' \rangle + \langle \bar{x}_\sigma, \tilde{m}^t \tilde{Y}' + \tilde{n} \tilde{Y}'' + Z'' \rangle \right] \\
+ \frac{1}{2} \| \mu_\sigma Y'' + \mu_\tau Z'' \|^2 - \frac{1}{2} \| \mu_\sigma Y' + \mu_\tau Z' \|^2 \\
+ (\mu_\tau - \mu_\sigma) [2(\tilde{\Phi} - \Phi) + Z].
\] (3.18)
Now we require that our construction be independent of the conformal factor \(\mu\). The terms linear in \(\mu_\tau\) and \(\mu_\sigma\) should vanish which gives us the equations:
\[
\begin{align*}
\begin{pmatrix} Y' \\ Y'' \end{pmatrix} &= -\begin{pmatrix} n & m^t \\ \tilde{m}^t & \tilde{n} \end{pmatrix} \begin{pmatrix} Z' \\ Z'' \end{pmatrix}, \\
\begin{pmatrix} Z' \\ Z'' \end{pmatrix} &= -\begin{pmatrix} n & m^t \\ \tilde{m}^t & \tilde{n} \end{pmatrix} \begin{pmatrix} Y' \\ Y'' \end{pmatrix}.
\end{align*}
\] (3.19)
(3.20)
The above matrix equations are equivalent because
\[
\begin{pmatrix} n & m^t \\ \tilde{m}^t & \tilde{n} \end{pmatrix}^2 = 1.
\] (3.21)
Next we concentrate on terms quadratic in first derivatives of the conformal factor. They give us respectively the following equations:
\[
\begin{align*}
\mu_\tau^2 : & \iff \| Z'' \|^2 - \| Z' \|^2 = 0, \\
\mu_\sigma^2 : & \iff \| Y'' \|^2 - \| Y' \|^2 = 0, \\
\mu_\tau \mu_\sigma : & \iff \langle Y'', Z'' \rangle - \langle Y', Z' \rangle = 0.
\end{align*}
\] (3.22)
(3.23)
(3.24)
From the term linear in second-order derivatives we obtain that
\[
Z = 2(\tilde{\Phi} - \Phi).
\] (3.25)
This gives us immediately a condition
\[
\| \Phi' \|^2 = \| \tilde{\Phi}' \|^2.
\] (3.26)
The L.H.S. of (3.26) is only a function of \(x\), the R.H.S. only of \(\bar{x}\) which means that it is in fact a restriction saying that the 1-forms \(d \Phi\) and \(d \tilde{\Phi}\) have the same norm in their respective metrics.
Now we are ready to write down the duality equation
\[ (\tilde{x}_\sigma + 2\mu_\sigma \tilde{\Phi}^\nu) = \left( \begin{array}{cc} -(m_t)^{-1}n & (m_t)^{-1} \\ n - \tilde{n}(m_t)^{-1}n & \tilde{n}(m_t)^{-1} \end{array} \right) \left( \begin{array}{c} x_\sigma + 2\mu_\sigma \Phi' \\ x_\tau + 2\mu_\tau \Phi' \end{array} \right), \] (3.27)
\[ = \left( \begin{array}{cc} -(m_t)^{-1}n & (m_t)^{-1} \\ (m_t)^{-1} & \tilde{n}(m_t)^{-1} \end{array} \right) \left( \begin{array}{c} x_\sigma + 2\mu_\sigma \Phi' \\ x_\tau + 2\mu_\tau \Phi' \end{array} \right). \] (3.28)

In a light cone basis these equations become
\[ \tilde{x}_\pm + 2\mu_\pm \tilde{\Phi}^\nu = \pm T_\pm (x_\pm + 2\mu_\pm \Phi'), \] (3.29)
where \( T_\pm \) are orthogonal matrices given by
\[ T_\pm = (m_t)^{-1}(I \mp n). \] (3.30)

Specifying to the case \( n = 0 \) the above become
\[ \tilde{x}_\pm + 2\mu_\pm \tilde{\Phi}^\nu = \pm T (x_\pm + 2\mu_\pm \Phi'), \] (3.31)
with a single orthogonal matrix \( T \).

A final curiosity is that \( h \) according to eq. (3.18) is the Hodge dual of the respective term in the generating function (3.3).

### 4 Integrability Conditions

Here we study the integrability conditions for the classical duality equations. It is instructive to study momentarily a more general duality equation than (3.31). Dimensional considerations impose the form
\[ \tilde{x}_i^\pm + 2\mu_\pm \tilde{u}^i = \pm x_i^\pm \pm 2\mu_\pm u^i. \] (4.1)

These equations are interpreted as equations on a bundle of orthonormal coframes as in references \[3, 6\]. The vector valued functions \( u^i \) and \( \tilde{u}^i \) are functions on the same bundle. We denote by \( ' \) and \( '' \) we denote the derivatives with respect to \( x \) and \( \tilde{x} \). Taking the derivative of (4.1) we have
\[ \tilde{x}_i^\pm \mp x_i^\mp + \tilde{\omega}^i_j \mp \tilde{x}_j^\pm = 2\mu_\pm \left[ (\tilde{-u}^{j\mp} \pm u^{j\pm}) x_i^\pm + (\tilde{-u}^{m\mp} \pm u^{m\pm}) x_i^\mp + \omega^i_j \mp \tilde{u}^i \mp \omega^j_i \pm u^j \right] + 2\partial^2_\pm \mu \left[ \mp \tilde{u}^i \pm u^i \right]. \] (4.2)

By the use of equations of motion (1.5) we may eliminate second derivatives on the L.H.S of (4.2) which now reads as
\[ -2\partial^2_\pm \mu (\Phi^{m\mp} \mp \Phi^{m\pm}) + \frac{1}{2} \left( \mp \tilde{H}^{j\pm}_i \mp \tilde{x}_j^\pm + \mp H^{j\pm}_i \mp x_j^\pm \right) - \omega^i_j \pm \tilde{x}_j^\pm \mp \omega^j_i \pm x_j^\pm = (4.3) \]
Now the strategy is to use the duality equation (4.11) in order to replace selectively $\tilde{x}^i_\mu$ with $x^i_\mu$. The L.H.S is thus

$$-2\partial^2_{\pm\mu}(-\tilde{\Phi}^m_\mu + \Phi^m_\mu) + \frac{1}{2} (\pm \tilde{H}^i_{jk} + H^i_{jk}) x^j_\pm x^k_\pm + (\omega^i_{j\mp} - \omega^i_{j\mp}) x^j_\pm - 4\mu\pm\mu\mp \tilde{H}^i_{jk} \tilde{u}^j u^k \quad (4.4)$$

$$+ \mu\pm \tilde{H}^i_{jk} (-\tilde{u}^j \pm u^j) x^k_\pm + \mu\mp \tilde{H}^i_{jk} (-\tilde{u}^j \pm u^j) x^k_\pm - 2\mu\pm \omega^i_{jk} (-\tilde{u}^j \pm u^j). \quad (4.5)$$

Here, we have to identify as before [3, 6] the orthogonal groups in both coframes bundles by

$$\tilde{\omega}_{ij} + \frac{1}{2} H_{ijk} \tilde{\omega}^k = \omega_{ij} + \frac{1}{2} \tilde{H}_{ijk} \omega^k. \quad (4.6)$$

Using (4.6), then again (4.1) and collecting the terms we obtain

$$0 = 2\partial^2_{\pm\mu}(-\tilde{\Phi}^m_\mu + \tilde{u}^i \mp u^i) + 4\mu\pm\mu\mp(\tilde{u}^n_{i\pm} \mp u^n_{i\pm})(\tilde{u}^i \pm u^i)$$

$$+ \mu\pm\left( \tilde{H}^i_{jk} \tilde{u}^k + H^i_{jk} u^k - 2(\tilde{u}^n_{i\pm} \mp u^n_{i\pm} \mp \tilde{u}^m_{i\mp} \pm u^n_{i\pm} \mp \tilde{u}^m_{i\pm} \pm u^n_{i\pm}) \right) x^j_\pm$$

$$+ \mu\mp\left( \mp H^i_{jk} (-\tilde{u}^i \pm u^i) + \tilde{H}^i_{jk} (-\tilde{u}^i \pm u^i) \right) x^j_\pm. \quad (4.7)$$

From $x^k_\pm$ and $x^j_\pm$ terms we learn respectively:

$$H^i_{jk} u^k + \tilde{H}^i_{jk} \tilde{u}^k = 0, \quad (4.8)$$

$$H^i_{jk} \tilde{u}^k + \tilde{H}^i_{jk} u^k = 0, \quad (4.9)$$

$$\tilde{u}^n_{i\pm} \mp u^n_{i\pm} = 0, \quad (4.10)$$

$$\tilde{u}^m_{i\mp} \pm u^n_{i\pm} = 0. \quad (4.11)$$

From the first line of (4.7) and requiring $\mu$ independence leads to

$$u^i = \Phi^i, \quad \tilde{u}^i = \tilde{\Phi}^i. \quad (4.12)$$

and

$$\tilde{u}^m_{i\mp} \tilde{u}^j - u^n_{i\pm} u^j = 0, \quad (4.13)$$

$$u^m_{i\pm} \tilde{u}^j - \tilde{u}^m_{i\mp} u^j = 0.$$  

Here (4.12) tells that the integrable duality equation is (3.31), the one obtained already by studying the Hamiltonian formalism. In this case $u^i$ are only functions of $x$’s and $\tilde{u}^i$ of $\tilde{x}$’s, therefore mixed derivatives $u^m_{i\pm}$ and $\tilde{u}^m_{i\mp}$ vanish. Substituting (4.10) and (4.11) into (4.13) and using (4.12) we obtain

$$\Phi^i_{ij} \Phi^j = \tilde{\Phi}^m_{ij} \tilde{\Phi}^m_{ij} \quad (4.14)$$

which corresponds to (3.20), being expressed in a differential way.

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6The subsequent equation may be not therefore explicitly “tilded-untilded” symmetric, the final result however has to be as neither $M$ nor $\tilde{M}$ is formally distinguished.
In the light-cone coordinates the trace of energy-momentum tensor is proportional to $T^{+-}$:

$$\text{Tr} [T^{\alpha\beta}] = e^{2\mu} T^{+-} = 8 e^{-2\mu} \left( \Phi'_{ij} x^i_+ x^j_+ + \Phi'_{i} x^i_- \right),$$  \hspace{1cm} (4.15)$$

$$= 8 e^{-2\mu} \left( \Phi'_{ij} x^i_+ x^j_+ - \frac{1}{2} \Phi_i H^i_{jk} x^j_+ x^k_+ - 2 \Phi'_i \Phi'_i \partial^2_{+-} \mu(\bar{\sigma}) \right).$$  \hspace{1cm} (4.16)$$

In the last line the equation of motion was used. Inspecting equations (4.8) and (4.9) we may establish their connection to the relationship between $\text{Tr} [T^{\alpha\beta}]$ and $\text{Tr} [\tilde{T}^{\alpha\beta}]$. Using the antisymmetry of $H^i_{jk}$ together with (4.12) we write (4.8) and (4.9) as

$$\Phi'_{ij} H^i_{jk} + \tilde{\Phi}'_{ij} \tilde{H}^i_{jk} = 0,$$  \hspace{1cm} (4.17)$$

$$\tilde{\Phi}'_{ij} H^i_{jk} + \Phi'_{ij} \tilde{H}^i_{jk} = 0.$$  \hspace{1cm} (4.18)$$

Now we contract the equation of motion (1.5) with $\Phi'_{ij}$ and obtain

$$\Phi'_{ij} x^i_+ x^j_+ = -\frac{1}{2} \Phi'_i H^i_{jk} x^j_+ x^k_+ - 2 \Phi'_i \Phi''_i \partial^2_{+-} \mu(\bar{\sigma})$$

$$= \frac{1}{2} \tilde{\Phi}''_i \tilde{H}^i_{jk} x^j_+ x^k_+ - 2 \tilde{\Phi}''_i \tilde{\Phi}'_i \partial^2_{+-} \mu(\bar{\sigma}).$$  \hspace{1cm} (4.19)$$

In the last line we took advantage of (4.17) and (3.26). Using the duality equation (4.1) we eliminate $x^i_+$’s in favor of their tilded counterparts. Having in mind the antisymmetry of $\tilde{H}^i_{jk}$ and (4.18) it is clear that the terms $\tilde{H}^i_{jk} \Phi^{ij} \Phi^{jk}$, $\tilde{H}^i_{jk} \tilde{\Phi}^{ij} \tilde{\Phi}^{jk}$, $\tilde{H}^i_{jk} \tilde{\Phi}''_j \tilde{\Phi}''_k$ vanish. Hence we obtain

$$\Phi'_{ij} x^i_+ x^j_+ = -\frac{1}{2} \tilde{\Phi}''_i \tilde{H}^i_{jk} x^j_+ x^k_+ - 2 \tilde{\Phi}''_i \tilde{\Phi}'_i \partial^2_{+-} \mu(\bar{\sigma}) = \tilde{\Phi}''_i \tilde{x}^i_+,$$  \hspace{1cm} (4.20)$$

which for the case $\Phi'_{ij} = 0$ is a statement that $\text{Tr} [T^{\alpha\beta}] = \text{Tr} [\tilde{T}^{\alpha\beta}]$. We understand that equations (4.8) and (4.9) are a condition which guarantees that form of the trace of the energy-momentum tensor is “preserved on-shell” by the duality transformation.

### 5 Conclusions

In order to set the duality equations we have to impose the constraints allowing only the coupling of a curvature scalar to the dilaton fields whose differentials have the same norm in their respective metric. The special case is a linear dilaton field. Here we might have expected to obtain a strong restriction, we required the “conformal covariance” as a local symmetry of Hamiltonian formalism and subsequently of the integrability conditions.

At the one-loop quantum level the condition that the model has to satisfy in order to be conformally invariant involves the second derivatives of a dilaton field, e.g., [4, 5]. It raises therefore a question whether it is possible to establish at a quantum level a more general form of duality transformation which leads to preserving of the form of the beta function and what would be the role of classical duality within such a construction.

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7By imposing the vanishing of beta function.
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