On Some Properties of Metals Under Complex Loading

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Abstract: Drucker’s Associated Law of yielding and gradient plasticity cannot always adequately reflect the processes of stress-strain state in deformable continua. Aim of this issue explains these effects via theory of complex loading processes by Ilyushin.

Keywords: Plasticity; Complex loading; Strain; Stress; Process

1. Introduction

The Drucker’s Associated Law of yielding (the phenomenological postulate) and gradient plasticity cannot always adequately reflect the processes of stress-strain state in deformable continua. When studying the processes of complex loading of isotropic strengthening metals, there occur the violations of the Drucker’s postulate. This is due to strain anisotropy, and also to the violation of gradientality principle. In generalized representation by Hodge and Prager, the increments of total strains have the form [1]:

\[ de_{ij} = H_{ijkl} d\sigma_{kl} + 0.5 [P(J_2, J_3) s_{ij} + Q(J_2, J_3) \frac{\partial J_3}{ds_{ij}}] df \]

here the augends is an elastic component of strain increment, the addend is its plastic component, \( J_2 \) and \( J_3 \) are the invariants of stress deviator, \( s_{ij} \) are the components of stress deviator, \( H_{ijkl} \) is a matrix of elastic coefficients of compliance, \( df \) is a loading surface, \( de_{ij} \) and \( d\sigma_{kl} \) - are the increments of strain and stress tensors, respectively.

As seen when moving along yield surface (neutral loading), there should be no increments in plastic components of strains. Attempts to construct a yield surface in strain space give the same results under neutral loading. However, as shown by experiments in [2], this condition is not satisfied under neutral loading both in stress space and in strain space (neutral strain) also other types of complex loading [3,4,5,6,7]. At neutral loading on a surface of loading there is an increase in strains. On the contrary, when loaded at a constant strain rate, the value of equivalent stress drops. Typical process of stress drop is shown in figure 1 for various values of plastic strain development. These experimental data give grounds for assuming that the effect of complex loading on the amount of work under plastic strain can be substantially lower than that at simple loading.

2. Main Part

In processing metals by pressure, it is observed that the work of external forces at various ways of complex strain is of less importance than at simple loading. These effects were established in experimental studies; it was noted that under complex loading of metals, the plastic forming force is reduced compared to simple tension, simple shear and other kinds of simple loading. This can be clearly traced on the processes of loading and strain process with breakpoint. This effect can be explained on the basis of Ilyushin's theory of elastoplastic processes and the so-called "damping memory" principle or the principle of delay of vector and scalar properties of material (confirmed in numerous experiments). In the previous chapters, the properties of metals under loading with breakpoint have been shown.

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doi: 10.18063/ijmp.v1i1.751
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On the basis of this, the following lemma is proposed, which will be useful in what follows. Here, we will specify the positions and hypotheses on which we will base:

we assume that the total strain consists of the sum of elastic and plastic parts, independent of each other;
the work of forces on elastic strains is conservative;
the principle of delay is a clearly manifested property of material.

Lemma: The minimum work of external forces to achieve a given form under plastic strain of the body is achieved only with the implementation of strain processes with breakpoint.

Proof: Based on the generalized Clapeyron theorem, which states that the work of external forces on corresponding displacements is equal to accumulated potential energy of deformable body, we have:

\[ A = \int_{V} \sigma_{ij} d\varepsilon_{ij} dV = \int_{V} \int_{0}^{\varepsilon} \sigma_{ij} d\varepsilon_{ij} dV + \int_{V} \int_{0}^{\varepsilon} s_{ij} d\varepsilon_{ij} dV \]

here \( s \) - is a magnitude of strain path, \( d\varepsilon_{ij} \) - are the components of strain deviation increment, \( \sigma_{ij} \) - is the modulus of spherical stress tensor, and \( d\varepsilon_{0} \) - is the increment in mean tensile strain.

Since the work of forces on elastic strains does not depend on the loading path, the created field of stress vector on strains prior to the onset of flow is conservative. The work of hydrostatic pressure on strains of volume compression is also conservative. This is based on the fact that material is taken as plastically incompressible.

Further we have:

\[ \int_{0}^{\varepsilon} s_{ij} d\varepsilon_{ij} = \int_{0}^{\varepsilon} \sigma d\varepsilon \cos \theta ds = \int_{0}^{\varepsilon} \sigma ds + \int_{0}^{\varepsilon} \sigma \cos \theta ds \]

here \( \theta \) - is an angle of approach of stress vector and the increment of strains.

Under active loading process, a minimum of the functional of work is achieved when the approach angle is \( \pm \pi/2 \). This is possible only if the stress vector \( \sigma \) is normal to the Frenet basis vector \( p_{1} \) (tangent to strain path), i.e. at path breakpoint. 

Maintaining the angle of approach at that value is not possible over the whole process of strain, according to the principle of delay. This means that when loading up to the exhaustion of delay trace of vector properties of material, it is...
necessary to produce a bend point of strain path. Such breakpoint in strain path, generally speaking, realize multi-link strain processes. In [8], the problem of optimal control of strain path under complex loading of metals is stated and solved. Solution of the optimization problem is obtained by using dynamic programming method. Strain process of a homogeneous cylindrical sample is considered. It is shown that the Ilyushin theory of elastic-plastic processes describes the effect of reducing the work of plastic forming of metals under complex loading.

This effect finds application in the optimization of technological processes of metal working with pressure [9,10]. The disadvantage is that the problem of optimal control is not solved with a fixed right end, i.e. the given point in strain space is not reached.

In manufacturing steel hollow cylinder by the method of reverse extrusion (pressing-out), the application of zigzag strain paths leads to a significant decrease in extrusion force.

Conclusion: To attach a certain shape to material in metal processing with pressure, by punching, etc., the possibility of optimal control of loading process is important. Thus, to achieve the goal, it is necessary to put as a term the optimal number of links of the path. With this term, it is possible to determine at what stage to produce a breakpoint. Here, in addition to classical physical parameters of material, one should also know its delay trace.

Note: When implementing loading processes with breakpoint (stress space), a situation arises when the scalar product of the stress vector $\sigma$ and the vector of strain increments $\frac{\Delta \text{e}}{\lambda}$ is a minimum. However the difference in the breakpoint in strain space and in stress space is that in addition to the change in vector properties of material, the stress "leaps" occur in the former case, which gives the minimum of work precisely under the strain processes with breakpoint.

The possibility of using the effect of reducing the external force in development of new technologies for plastic processing of metals and controlling the strain paths of these processes is considered in [8]. Strain processes of a specified length in the paper with minimization of the energy expended are studied also. Thus, consideration of complex loading and identification of the effects arising in this case, besides revealing the reserves of bearing capacity of materials, allows to minimize the work of external forces (to reduce energy dissipation) in technological processes.

For further discussion, we need to accept the following statements: the Ilyushin’s isotropy postulate, the delay principle, universality of loading diagram $\sigma = \sigma(s)$.

As previously noted, the universality of the function $\vartheta$ for material at the same angles at breakpoint was observed experimentally. Based on this, empirical formulas (1-3) for the approximations of function $\vartheta$ were proposed. With proposed approximations, graphs have been constructed showing the changes in the approach angle $\vartheta$ (Figure 2) and $\cos \vartheta$ (Figure 3), depending on the angle at breakpoint $\vartheta$. The delay trace of vector material properties is assumed as equal to $\lambda = 5\epsilon_i$.

$$
\vartheta = \frac{\theta}{\exp(k(1+\frac{1}{\lambda})\Delta s)}, \text{ where } k = \ln 16/(1+\lambda)
$$

$$
\vartheta = \frac{l}{\Delta s - \lambda b}, \text{ where } H = 1, \quad b = \frac{\ln(\theta/16+1)}{\ln(\theta/16+1)-\ln(\theta+1)}
$$

$$
\vartheta = \text{Carcctg}\left(\frac{r}{\lambda}\Delta s\right), \text{ here } \quad C = \frac{2\theta}{\pi}, \quad r = \text{ctg}\left(\frac{\pi}{32}\right)
$$

As seen from the graphs, it is possible to reduce the amount of work of plastic forming. The approximation (1) better reflects the vector properties of material before the exhaustion of conditional delay trace, but decreases after passing this boundary more rapidly than in experiment. The approximations (2) and (3), on the contrary, decrease faster than experimental curves, but after exhaustion of trace of delay they approach the experimental curves.
**Figure 2.** The angle of approach at different angles at breakpoint

- a) $\theta = \pi/6$, b) $\theta = \pi/4$, c) $\theta = \pi/3$, d) $\theta = \pi/2$

$* - 1, \square - 2, \Delta - 3$
The work of forming is expressed as: \[ \int_0^s \overline{\sigma} d\overline{E} = \int_0^s \sigma \cos \theta ds \]. As seen from the graphs above, with a certain strain path the work under complex loading will be less than under simple loading. Here it is necessary to impose certain limitations on the geometry of strain process. Consider these conditions in multi-link strain processes.

For an ideally plastic materials we get:

\[ \int_0^s \overline{\sigma} d\overline{E} = \sigma_i [n] \] - under simple loading;

\[ \int_0^s \overline{\sigma} d\overline{E} = \sigma_i \int_0^s \cos \theta ds \] - under complex loading.

Provided that all links of the multi-link path are assumed to be equal to \( \Delta \) and the changes in the approach angles at each link are identical (i.e., the approach angle varies by one law), then

\[ \sigma_i \int_0^s \cos \theta ds = \sigma_i n \int_0^\Delta \cos \theta ds \].

For lineally strengthening material:

\[ \int_0^s \overline{\sigma} d\overline{E} = \sigma_i [n] + E_i [n]^2 / 2 \] - under simple loading;

\[ \int_0^s \overline{\sigma} d\overline{E} = \sigma_i \int_0^s \cos \theta ds + E_i \int_0^s s \cos \theta ds \] under complex loading.

Let us show the possibility of finding a multi-link strain path at which the work of stresses on the increments of strains will be less than under simple loading and the condition under which this can be achieved. For simplicity, consider the processes with identical angles at breakpoint on a path in the form of a zigzag (\( \beta_i = \pm x \)). In this case we have for the magnitudes of strain vector:

\[ [\overline{\epsilon}] = n\Delta \sqrt{\frac{1 + \cos(x - \alpha_i)}{2}} \] at an even number of links.
\[ |\mathcal{E}| = \Delta \sqrt{\frac{n^2 + 1}{2} + \frac{n^2 - 1}{2} \cos(x - \alpha_i)} \] at an odd number of links

Without loss of generality of the processes under consideration, we can take \( \alpha_i = 0 \). Then for a multi-link zigzag path with equal angles at breakpoint \( x \), the condition that the work of complex loading under plastic strain is less than that of simple loading has the form:

at ideal plasticity

\[
\sigma, n \int_0^\Delta \cos \vartheta ds \leq \sigma, n \Delta \sqrt{\frac{1 + \cos x}{2}} \quad \text{or} \quad \sigma, n \int_0^\Delta \cos \vartheta ds \leq \sigma, n \Delta \sqrt{\frac{1 + n^2}{2} + \cos x \frac{n^2 - 1}{2}}
\]

Simplifying, we get:

\[
\int_0^\Delta \cos \vartheta ds \leq n \Delta \sqrt{\frac{1 + \cos x}{2}} \tag{4}
\]

For a linearly strengthening material, the conditions have the form:

\[
\begin{align*}
\int_0^\Delta \cos \vartheta ds & \leq n \Delta \sqrt{\frac{1 + \cos x}{2}} \\
\int_0^\Delta s \cos \vartheta ds & \leq \frac{n^2}{2} \frac{1 + \cos x}{2}
\end{align*}
\tag{5}
\]

Note: Active loading processes are considered, i.e. \( |\beta| \leq \pi/2 \).

In Figure 4 graphs are constructed for functions \( f(s) = \int_0^\Delta \cos \vartheta ds \) and \( F(s) = \Delta \sqrt{\frac{1 + \cos x}{2}} \) for different approximations for the approach angle \( \vartheta \) (1-3). As seen from the graphs, there is an area in which the work of external forces at complex loading area is less than that at simple loading.

![Graphs](image)

a) \( \vartheta = \pi/3 \), b) \( \vartheta = \pi/2 \)

**Figure 4.** Areas of CL work decrease for different types of approximation

To determine the critical link length for which conditions (4) and (5) are satisfied, we expand
\( \cos \vartheta = 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} - \frac{\vartheta^6}{6!} + \ldots \) into the series at \( \vartheta = \theta \exp(-\alpha \xi) \)

Leaving the first 3 terms, we get:

\[
\cos \vartheta = 1 - \frac{\vartheta^2 \exp(-2\alpha \xi)}{2!} + \frac{\vartheta^4 \exp(-4\alpha \xi)}{4!} \geq \cos \vartheta
\]

Calculate the integral

\[
\int_0^\Delta \cos \vartheta ds = \Delta + \frac{\vartheta^2 \exp(-2\alpha \xi)}{2\alpha} - \frac{\vartheta^4 \exp(-4\alpha \xi)}{4\alpha} - \frac{\vartheta^2}{2!} \frac{1}{2\alpha} + \frac{\vartheta^4}{4!} \frac{1}{4\alpha}
\]

Then inequality (3.4) can be written in the form:

\[
\Delta + \frac{1}{\alpha} \left[ \cos \frac{\theta}{\sqrt{2}} - \cos \frac{\theta \exp(-\alpha \xi)}{\sqrt{2}} \right] \leq \Delta \sqrt{\frac{1 + \cos \theta}{2}}
\]

or

\[
1 + \frac{1}{\alpha \Delta} \left[ \cos \frac{\theta}{\sqrt{2}} - \cos \frac{\theta \exp(-\alpha \xi)}{\sqrt{2}} \right] \leq \sqrt{\frac{1 + \cos \theta}{2}} \tag{6}
\]

Below are the graphs for different angles at breakpoint \( \theta \), here \( f = 1 + \frac{1}{\alpha \Delta} \left[ \cos \frac{\theta}{\sqrt{2}} - \cos \frac{\theta \exp(-\alpha \xi)}{\sqrt{2}} \right] \).

\( \alpha = \ln 16/\lambda \).

From the second condition (for strengthening material) we have:

\[
F = 1 + \frac{2}{(\alpha \Delta)^2} \left[ \cos \frac{\theta}{2} - \cos \frac{\theta \exp(-\alpha \xi)}{2} \right] + \frac{2}{\alpha \Delta} \left[ 1 - \cos \frac{\theta \exp(-\alpha \xi)}{\sqrt{2}} \right] \leq \frac{1 + \cos \theta}{2} \tag{7}
\]

It is clearly seen from the graphs (Figure 5) at which values of delay trace of vector properties of material \( \lambda \) the condition (6) or (7) cannot be satisfied.
a) $\theta = \pi / 3$, b) $\theta = \pi / 2$

Figure 5. Boundaries for realization of conditions (6) and (7)

Figure 6 shows the limit of permissible length of the link, depending on delay trace of material at different angles at breakpoint.

Graphically presented conditions are sufficient, but not necessary. Condition (4) is completely feasible even in the "worst-case". It suffices to relate the angle at breakpoint to the length of each link of strain path. Thus, the possibility is shown that, in spite of the increase in strain path under complex loading in comparison with simple one, it is possible to reduce the work of plastic strain.

Figure 6. Allowable lengths of links of strain path

Next, we determine for which angle at breakpoint the minimum of the functional $\int_{0}^{\pi} \sigma \cos \theta ds$ is attained.

As follows from the previous lemma, to minimize the work of forming, it is necessary to consider the processes with breakpoint. With this choice, the number of links $n$, the lengths of the links $\Delta s^{(l)}$ and the angles at breakpoint $\beta^{(l)}$ are unknown in the general case. In real processes, it is required to reach the given point in strain or stress spaces.

Consider the strain path (Figure 7), in which all links of strain process and the stress vector are coplanar. This strain state corresponds to a wide class of plane problems of the theory of plasticity. Without loss of generality, according to the postulate of isotropy, we can consider processes in the plane $\mathcal{E}_{1} \sim \mathcal{E}_{2}$ (strain vectors). In this case, the problem of determining the direction cosines of stress vector is possible on the basis of the law of approach. We introduce the following quantities: $\lambda = (l=1,n)$ - are the slope angles of strain path to the axis $O \mathcal{E}_{1}$, $\gamma = (l=2,n)$ - are the angles at breakpoint of strain path, $\beta = (l=2,n)$ - are the slope angles of stress vector at breakpoint to the previous link (strain zone), $\alpha = (l=1,n)$ - are the slope angles of stress vector at breakpoint to the subsequent link (strain section). The positive direction of the angles is determined if the turn from the previous link to the next one goes clockwise.

We introduce the following quantities: $\alpha^{(l)} = (l=1,n)$ - are the slope angles of the strain path to the axis $OE_{1}$, $\beta^{(l)} = (l=1,n)$ - are the angles at breakpoint of strain path, $\gamma^{(l)} = (l=2,n)$ - are the slope angles of stress vector at breakpoint to the previous link (strain zone), $\theta^{(l)} = (l=2,n)$ - are the slope angles of stress vector at breakpoint to the subsequent link (strain zone). The positive direction of the angles is determined if the turn from the previous link to the next one goes clockwise.
As consequence: when force is applied to isotropic material, there is always an active process of complex loading at which the work of external forces will be less than under simple loading upon reaching the same strained state.

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