Superintegrability and higher order integrals for quantum systems

E G Kalnins\(^1\), J M Kress\(^2\) and W Miller Jr\(^3\)

\(^1\) Department of Mathematics, University of Waikato, Hamilton, New Zealand
\(^2\) School of Mathematics and Statistics, University of New South Wales, Sydney, Australia
\(^3\) School of Mathematics, University of Minnesota, Minneapolis, MN, USA

E-mail: miller@ima.umn.edu

Received 20 February 2010, in final form 10 May 2010
Published 4 June 2010
Online at stacks.iop.org/JPhysA/43/265205

Abstract
We refine a method for finding a canonical form of symmetry operators of arbitrary order for the Schrödinger eigenvalue equation $H \Psi = E \Psi$ on any 2D Riemannian manifold, real or complex, that admits a separation of variables in some orthogonal coordinate system. The flat space equations with potentials $V = \alpha(x + iy)^{k-1}/(x - iy)^{k+1}$ in Cartesian coordinates, and $V = \alpha r^2 + \beta/r^2 \cos^2 k\theta + \gamma/r^2 \sin^2 k\theta$ (the Tremblay, Turbiner and Winternitz system) in polar coordinates, have each been shown to be classically superintegrable for all rational numbers $k$. We apply the canonical operator method to give a constructive proof that each of these systems is also quantum superintegrable for all rational $k$. We develop the classical analog of the quantum canonical form for a symmetry. It is clear that our methods will generalize to other Hamiltonian systems.

PACS numbers: 02.20.Tw, 02.30.Ik, 03.65.Fd
Mathematics Subject Classification: 20C99, 20C35, 22E70

1. Introduction

We consider an $n$-dimensional classical superintegrable system as an integrable Hamiltonian system that not only possesses $n$ mutually Poisson-commuting constants of the motion, but in addition, the Hamiltonian Poisson commutes with $2n - 1$ functions on the phase space that are globally defined and polynomial in the momenta. This notion can be extended to define a quantum superintegrable system to be a quantum Hamiltonian which is one of a set of $n$-independent mutually commuting differential operators that commutes with a set of $2n - 1$ independent differential operators of finite order. We restrict to classical systems of the form $\mathcal{H} = \sum_{i,j=1}^{n} g^{ij} p_i p_j + V$ and corresponding quantum systems $\hat{H} = \Delta_n + \hat{V}$, where $\hat{V}$ is related to $V$ but in general is not equal to it for $n > 2$ due to curvature corrections [1]. (Also there may be no classical or quantum analogs in some cases [2]). These systems,
including the classical Kepler problem and the quantum hydrogen atom, have great historical importance due to their remarkable properties. They are exactly analytically solvable and in multiple ways. They can serve as the foundation for perturbation analysis. There are deep connections with special functions. See [3, 4] for various applications of their theory and usage. Superintegrable systems of first order, i.e. classical systems where the defining symmetries are first order in the momenta and quantum systems where the symmetries are first-order partial differential operators, are directly related to Lie transformation groups and well understood. Superintegrable systems of second order have been well studied and there is now a structure and classification theory [1, 5–7]. The connection between second-order symmetries and separation of variables has been of crucial importance in finding examples and carrying out the classification [8–10]. However for third- and higher-order superintegrable systems much less is known. In particular there have been relatively few examples and there is almost no structure theory, i.e. an understanding of the structure of the Poisson algebra generated by the classical symmetries or the algebra generated by the quantum symmetry operators and their commutators, and no classification theory.

This situation has changed recently with the discovery of many more examples of classical (especially) and quantum superintegrable systems of order higher than 2 [11–16]. Also, the tool of coupling constant metamorphosis (St"ackel transform) has been developed to map superintegrable systems of higher order on one Riemannian space to superintegrable systems of the same order and structure on a different Riemannian space [17–20]. In this paper we will concentrate on the case $n = 2$ where the number of independent symmetries is 3, including the classical Hamiltonian or quantum Schrödinger operator, respectively. In almost all of the families of new examples the second symmetry is of second order and defines an orthogonal separable coordinate system for the classical Hamilton–Jacobi equation or the quantum Schrödinger equation. Only one defining symmetry is of higher order. We are particularly considering the classical example of Tremblay, Turbiner and Winternitz (TTW) [13, 14] where

$$V = \alpha r^2 + \frac{\beta}{r^2 \cos^2 k\theta} + \frac{\gamma}{r^2 \sin^2 k\theta}$$

in polar coordinates. Due to the separation in polar coordinates there are two commuting second-order symmetries. For certain rational values of the parameter $k$ these authors found an additional symmetry (usually of higher order), so that the system was superintegrable both in the classical and quantum sense. They conjectured and provided impressive evidence that these systems were classically and quantum superintegrable for all rational $k$. In [21] it was shown that, in fact all of the classical TTW systems were superintegrable. Quesne [22] used a structure developed by Dunkl to show that the TTW systems for $k$ an odd integer were quantum superintegrable. As a bi-product of the tools developed in this paper we will give a constructive proof that the TTW system is quantum superintegrable for all rational $k$. However, our main contribution is a tool for the verification of classical and quantum superintegrability of higher order that can be applied to a variety of Hamiltonian systems.

In section 1 we review a construction of a canonical form for symmetry operators of all orders of a time-independent Schrödinger equation that admits an orthogonal separation of variables [23, 24]. Then in section 2 we apply this tool to the flat space potential $V = \alpha (x + iy)^{k-1}/(x - iy)^{k+1}$ in Cartesian coordinates (separable in polar coordinates), that has recently been shown to be classically superintegrable for all rational $k$ [21]. We demonstrate that it is also quantum superintegrable for all rational $k$.

In section 3 we give the analogous construction of a canonical form for constants of the motion of all orders for a classical Hamiltonian system. We again treat the example $V = \alpha (x + iy)^{k-1}/(x - iy)^{k+1}$ and give a new proof that it is classically superintegrable for
all rational \( k \). The special interest here is the relation between the classical and quantum construction.

In section 4 we apply the canonical operator method to the TTW case to demonstrate that it is quantum superintegrable for all rational \( k \). In section 5 we discuss our overall strategy and prospects for exploitation and generalization of our methods.

2. The canonical form for a symmetry operator

We consider a Schrödinger equation on a 2D real or complex Riemannian manifold with the Laplace–Beltrami operator \( \Delta \) and potential \( V \)

\[
H \Psi = (\Delta + V) \Psi = E \Psi
\]

that also admits an orthogonal separation of variables. If \( \{u_1, u_2\} \) is the orthogonal separable coordinate system the corresponding Schrödinger operator has the form

\[
H = L_1 = \Delta + V(u_1, u_2) = \frac{1}{f_1(u_1) + f_2(u_2)} \left( \partial^2_{u_1} + \partial^2_{u_2} + v_1(u_1) + v_2(u_2) \right),
\]

and, due to the separability, there is the second-order symmetry operator

\[
L_2 = \frac{f_2(u_2)}{f_1(u_1) + f_2(u_2)} \left( \partial^2_{u_1} + v_1(u_1) \right) - \frac{f_1(u_1)}{f_1(u_1) + f_2(u_2)} \left( \partial^2_{u_2} + v_2(u_2) \right),
\]

i.e. \([L_2, H] = 0\), and the operator identities

\[
f_1(u_1)H + L_2 = \partial^2_{u_1} + v_1(u_1), \quad f_2(u_2)H - L_2 = \partial^2_{u_2} + v_2(u_2).
\]

We look for a partial differential operator \( \tilde{L}(H, L_2, u_1, u_2) \) that satisfies

\[
[H, \tilde{L}] = 0.
\]

We require that the symmetry operator take the standard form

\[
\tilde{L} = \sum_{j,k} (A^{j,k}(u_1, u_2) \partial_{u_1} + B^{j,k}(u_1, u_2) \partial_{u_2} + C^{j,k}(u_1, u_2) \partial_{u_1} u_2 + D^{j,k}(u_1, u_2) \partial_{u_1} u_2^2) H^j L^k.
\]

Note that if the formal operators (4) contained partial derivatives in \( u_1 \) and \( u_2 \) of orders \( \geq 2 \) we could use identities (2), recursively, and rearrange terms to achieve the unique standard form (4).

Using operator identities and writing \( \partial_{u_j} = \partial_j \) we have

\[
[\partial_1, H] = -\frac{f_1'}{f_1 + f_2} H + \frac{v_1'}{f_1 + f_2}, \quad [\partial_2, H] = -\frac{f_2'}{f_1 + f_2} H + \frac{v_2'}{f_1 + f_2},
\]

\[
[\partial_1, L_2] = -\frac{f_1'}{f_1 + f_2} H + \frac{f_2' v_1'}{f_1 + f_2}, \quad [\partial_2, L_2] = \frac{f_1'}{f_1 + f_2} H - \frac{f_2'}{f_1 + f_2},
\]

\[
[H, \partial_2] = \frac{f_2'}{f_1 + f_2} \partial_2 H + \frac{f_1'}{f_1 + f_2} \partial_1 H - \frac{1}{f_1 + f_2} (v_1' \partial_1 + v_2' \partial_2),
\]

\[
[H, F(u_1, u_2)] = \frac{1}{f_1 + f_2} \left( F_{u_1 u_1} + F_{u_2 u_1} + 2 F_{u_1} \partial_1 + 2 F_{u_2} \partial_2 \right).
\]

From these results and (2) we obtain

\[
(f_1 + f_2)[H, A \partial_1] = 2 A_{u_1} (f_1 \partial_2 H + \partial_2 L_2 - \partial_1 v_2) + 2 A_{u_2} (f_2 \partial_1 H - \partial_2 L_2 - \partial_1 v_2) + A (f_2 \partial_2 H - f_1 \partial_1 H - v_1' \partial_1 - v_2' \partial_2),
\]

\[
(f_1 + f_2)[H, B \partial_1] = B (f_1 H - v_1') + 2 B_{u_1} (f_1 H + L_2 - v_1) + (B_{u_1 u_1} + B_{u_2 u_1}) \partial_1 + 2 B_{u_2} \partial_1,
\]

\[
(f_1 + f_2)[H, C \partial_1] = C (f_1 H - v_1') + 2 C_{u_1} (f_1 H - L_2 - v_2) + (C_{u_1 u_1} + C_{u_2 u_1}) \partial_2 + 2 C_{u_2} \partial_2,
\]

\[
(f_1 + f_2)[H, D] = D_{u_1 u_1} + D_{u_2 u_2} + 2 D_{u_1} \partial_1 + 2 D_{u_2} \partial_2.
\]
Thus, we have

\[
(f_1(u_1) + f_2(u_2))(H, A(u_1, u_2)\partial_{12} + B(u_1, u_2)\partial_1 + C(u_1, u_2)\partial_2 + D(u_1, u_2))
\]

\[
= (A_{u_1}u_1 + A_{u_2}u_2 + 2B_{u_1}u_1 + 2C_{u_1}u_1)\partial_{12} + (B_{u_1}u_1 + B_{u_2}u_2 - 2A_{u_2}u_2 + 2D_{u_1} - Av'_1)\partial_1
\]

\[
+ (2A_{u_1}f_1 + Af'_{1})\partial_1 H - 2A_{u_2}\partial_1 L_2 + (C_{u_1}u_1 + C_{u_2}u_2 - 2A_{u_1}v_1 + 2D_{u_2} - Av'_2)\partial_2
\]

\[
+ (2A_{u_1}f_1 + Af'_{1})\partial_2 H + 2A_{u_2}v_2 L_2
\]

\[
+ (D_{u_1}u_1 + D_{u_2}u_2 - 2B_{u_1}v_1 - 2C_{u_1}v_2 - Bu'_1) - Cv'_2)
\]

\[
+ (2B_{u_1}f_1 + 2Cu_{u_2}f_2 + Bf'_{1} + C_{u_2}f'_{2})H + (2B_{u_1} - 2Cu_{u_2})L_2.
\]

The symmetry condition (3) is equivalent to the system of equations

\[
\partial_1 A^{j,k} + \partial_2 A^{j,k} + 2\partial_2 B^{j,k} + 2\partial_1 C^{j,k} = 0, \quad (5)
\]

\[
\partial_1 B^{j,k} + \partial_2 B^{j,k} - 2\partial_2 A^{j,k}v_2 + 2\partial_1 D^{j,k} - A^{j,k}v'_2
\]

\[
+ (2\partial A^{j-1,k}f_2 + A^{j-1,k}f'_2) - 2\partial_2 A^{j,k-1} = 0, \quad (6)
\]

\[
\partial_1 C^{j,k} + \partial_2 C^{j,k} - 2\partial_1 A^{j,k}v_1 + 2\partial_2 D^{j,k} - A^{j,k}v'_1
\]

\[
+ (2\partial A^{j-1,k}f_1 + A^{j-1,k}f'_1) + 2\partial_1 A^{j,k-1} = 0, \quad (7)
\]

\[
\partial_1 D^{j,k} + \partial_2 D^{j,k} + 2\partial_1 B^{j,k}v_1 - 2\partial_2 C^{j,k}v_2 - B^{j,k}v'_1 - C^{j,k}v'_2 + (2\partial_1 B^{j-1,k}f_1
\]

\[
+ 2\partial_2 C^{j-1,k}f_2 + B^{j-1,k}f'_1 + C^{j-1,k}f'_2) + (2\partial_1 B^{j,k-1} - 2\partial_2 C^{j,k-1}) = 0. \quad (8)
\]

Note that condition (4) makes sense, at least formally, for infinite-order differential equations. Indeed, one can consider $H, L_2$ as parameters in these equations. Then once $\tilde{L}$ is expanded as a power series in these parameters, the terms are reordered so that the powers of the parameters are on the right-hand side, before they are replaced by explicit differential operators. Alternatively one can consider the operator $\tilde{L}$ as acting on a simultaneous eigenbasis of the commuting operators $H$ and $L_2$, in which case the parameters are the eigenvalues. Of course (4) is defined rigorously for finite-order symmetry operators.

In this view we can write

\[
\tilde{L}(H, L_2, u_1, u_2) = A(u_1, u_2, H, L_2)\partial_{12} + B(u_1, u_2, H, L_2)\partial_1
\]

\[
+ C(u_1, u_2, H, L_2)\partial_2 + D(u_1, u_2, H, L_2), \quad (9)
\]

and consider $\tilde{L}$ as an at most second-order order differential operator in $u_1, u_2$ that is analytic in the parameters $H, L_2$. Then the above system of equations can be written in the more compact form

\[
A_{u_1}u_1 + A_{u_2}u_2 + 2B_{u_1} + 2C_{u_1} = 0, \quad (10)
\]

\[
B_{u_1}u_1 + B_{u_2}u_2 - 2A_{u_2}u_2 + 2D_{u_1} - Av'_1 + (2A_{u_1}f_2 + Af'_{1})H - 2A_{u_2}L_2 = 0, \quad (11)
\]

\[
C_{u_1}u_1 + C_{u_2}u_2 - 2A_{u_1}v_1 + 2D_{u_2} - Av'_1 + (2A_{u_1}f_1 + Af'_{1})H + 2A_{u_2}L_2 = 0, \quad (12)
\]

\[
D_{u_1}u_1 + D_{u_2}u_2 - 2B_{u_1}v_1 + 2C_{u_1}v_2 - Bu'_1 - Cv'_2
\]

\[
+ (2B_{u_1}f_1 + 2Cu_{u_2}f_2 + Bf'_{1} + C_{u_2}f'_{2})H + (2B_{u_1} - 2Cu_{u_2})L_2 = 0. \quad (13)
\]

We can view (10) as an equation for $A, B, C$ and (11) and (12) as the defining equations for $D_{u_1}, D_{u_2}$. Then $\tilde{L}$ is $\tilde{L}$ with the terms in $H$ and $L_2$ interpreted as (4) and considered as partial differential operators.
We note that there is a natural bilinear form determined by the metric, namely the properties of the symmetry operators. To see how these properties relate to our construction.

2.2. Formal self-adjointness properties

In terms of the functions $F, G$ has the canonical form

$$(\Phi, \Psi) = \int \int \Phi(u_1, u_2)\Psi(u_1, u_2)(f_1(u_1) + f_2(u_2)) \, du_1 \, du_2.$$
Here $\Phi, \Psi$ are arbitrary $C^\infty$ functions with compact support. If $K$ is a finite-order partial differential operator on the manifold, we define its formal adjoint $K^*$ by

$$(K \Phi, \Psi) = (\Phi, K^* \Psi)$$

for all $\Phi, \Psi$. Thus $K^*$ is obtained by integration by parts where all boundary terms are assumed to vanish. If $K^* = K$ we say that $K$ is formally self-adjoint. If $K^* = -K$ we say that $K$ is formally skew-adjoint. For specific real Riemannian manifolds we can modify these definitions to obtain the usual definitions of self- and skew-adjointness, but this general definition serves well for our work where we are considering all manifolds as complex and local. Note that both $H$ and $L_2$ are formally self-adjoint. If $\tilde{\mathcal{L}}$ is a symmetry operator, then so is $\tilde{\mathcal{L}}^*$. It follows that the symmetry operator $\tilde{L} = \frac{1}{2} (\mathcal{L} + \mathcal{L}^*)$ is formally self-adjoint and the symmetry operator $A = \frac{1}{2} (\mathcal{L} - \mathcal{L}^*)$ is formally skew-adjoint. If the highest-order nonzero derivative occurring in the expansion of $\tilde{\mathcal{L}}$ is of order $N$, then $S$ will necessarily be a nonzero self-adjoint symmetry operator. If $N$ is odd, then $A$ will necessarily be a nonzero skew-adjoint symmetry operator.

If $L_2$ has the canonical form (4), then $\tilde{\mathcal{L}}^*$ has the canonical form

$$\tilde{\mathcal{L}}^* = \sum_{j,k} H^j L_2^k (A^{j,k}(u_1, u_2) \partial_{u_1 u_2} + B^{j,k}(u_1, u_2) \partial_{u_1} + C^{j,k}(u_1, u_2) \partial_{u_2} + D^{j,k}(u_1, u_2)),$$

where

$$A^* = A, \quad B^* = -B + A_2 + \frac{A f'_1}{f_1 + f_2}, \quad C^* = -C + A_1 + \frac{A f'_1}{f_1 + f_2},$$

$$D^* = D + A_{12} + \frac{A_1 f'_1 + A_2 f'_2}{f_1 + f_2} - B_1 - C_1 = \frac{B f'_1 + C f'_2}{f_1 + f_2}.$$

Note that the terms in (17) are ordered so that the powers of the parameters $H, L_2$ are on the left. Though the expressions for the symmetry operators $S$ and $A$ are cumbersome to write down, for any reasonably low-dimensional $N$ in the constructions to follow, Maple can easily compute $S$ and $A$ from $\tilde{\mathcal{L}}$ in the standard form for partial differential operators.

3. The potential $V = \alpha \frac{(x + iy)^{k-1}}{(x - iy)^{k+1}}$

We consider the flat space Schrödinger operator

$$H = \partial_{xx} + \partial_{yy} + V, \quad V = \alpha \frac{(x + iy)^{k-1}}{(x - iy)^{k+1}}.$$  

(18)

where $x, y$ are Cartesian coordinates. We have shown that the corresponding classical systems are superintegrable for all rational $k$, [21].

In the special case $k = 3$ we have explicitly established quantum superintegrability. Indeed, we obtained the symmetry operators

$$K_1 = (\partial_x - i \partial_y)^3 + \frac{\alpha}{(x - iy)^3} [(3x + iy) \partial_x + (ix - 3y) \partial_y],$$

$$K_2 = (x \partial_x - y \partial_y) (\partial_x - i \partial_y)^3 + \frac{\alpha}{(x - iy)^3} [i(2y^2 - 3ixy - 3x^2) \partial_z^2 - (3iy + x)(iy + 3x) \partial_x \partial_y + i(2x^2 + 3ixy - 3y^2) \partial_z^2 - 2i(3iy + x) \partial_x - 2i(3iy + x) \partial_y - 8i] + i \alpha \frac{(x + iy)^3}{(x - iy)^{k+1}},$$

$$K_3 = (x \partial_x - y \partial_y)^2 + 2i \alpha \frac{(-y^2 + 3x^2)}{(x - iy)^3},$$

6
with the commutation relations

\[ [K_1, K_2] = 3iK_1^2, \quad [K_1, K_3] = 6iK_2 - 9K_1, \]
\[ [K_2, K_3] = 3i(K_1, K_2) + i(27 + 6\alpha)K_1 + 9K_2, \]

and the analog of the constraint

\[ \frac{1}{2}[K_1, K_1, K_3] - 3K_2^2 - \frac{i}{2}[K_1, K_2] + \left( \frac{63}{2} + 3\alpha \right)K_1^2 - 3aH^3 = 0. \]

All of these quantum systems separate in polar coordinates:

\[ u_1 = R, \quad u_2 = \theta, \quad x = e^R \cos \theta, \quad y = e^R \sin \theta. \]

The corresponding symmetry operator is

\[ -L_2 = \partial_\theta^2 + \alpha e^{2i\theta} \]

(For \( k = 3 \) we have \(-L_2 = K_3 + \alpha \).) Furthermore,

\[ H = e^{-2k} \left( \partial_R^2 - L_2 \right), \]

and

\[ f_1 = e^{2R}, \quad f_2 = 0, \quad v_1 = 0, \quad v_2 = \alpha e^{2i\theta}. \]

We assume \( k = p/q \) for positive relatively prime integers \( p,q \). Based on the known expressions for the classical higher-order constants of the motion, derived in \([21]\), we look for an operator constant of the motion \( L, (9) \), where

\[ F = \sum_{a,b} A_{a,b}(\alpha, H, L_2) e^{2iaR + ibk\theta}, \quad G = \sum_{a,b} B_{a,b}(\alpha, H, L_2) e^{2iaR + ibk\theta}. \]

We require that there are only a finite number of nonzero terms in the sums and that the sums are of the form \( a = a_0 + m, b = b_0 + n \) where \( m,n \) run over a subset of the non-negative integers. Thus \( C_{a_0,b_0} \) will be an analog of a lowest weight vector. Substituting all these expressions into equations (14) and (15) and equating coefficients of terms \( e^{2iaR + ibk\theta} \), we obtain the matrix recursion

\[
2(a^2 - k^2b^2) \left( \begin{array}{cc} iakb & a^2 + k^2b^2 - L_2 \\ -a^2 - k^2b^2 + L_2 & 4iakb \end{array} \right) \left( \begin{array}{c} A_{a,b} \\ B_{a,b} \end{array} \right) + \frac{(2a - 1)H}{J(a,b)} \left( \begin{array}{cc} -ikb & 1 - a \\ a & 0 \end{array} \right) \left( \begin{array}{c} A_{a-1,b} \\ B_{a-1,b} \end{array} \right) + \frac{ak(2b - 1)}{J(a,b)} \left( \begin{array}{cc} ia & k(b - 1) \\ -kb & 0 \end{array} \right) \left( \begin{array}{c} A_{a,b-1} \\ B_{a,b-1} \end{array} \right) = 0, \]

or, solving for

\[ C_{a,b} = \left( A_{a,b}, \frac{C_{a-1,b}}{J(a,b)} \right), \]

\[ C_{a,b} + \frac{(2a - 1)H}{J(a,b)} \left( \begin{array}{cc} a(L_2 + 3k^2b^2 - a^2) & 4i(1 - a)akb \\ -ikb(L_2 + k^2b^2) & -(a - 1)(L_2 + a^2 + k^2b^2) \end{array} \right) C_{a-1,b} + \frac{ak(2b - 1)}{J(a,b)} \left( \begin{array}{cc} -kb(L_2 - k^2b^2 + 3a^2) & 4i(b - 1)ak^2b \\ ia(L_2 + a^2) & k(b - 1)(L_2 + a^2 + k^2b^2) \end{array} \right) C_{a,b-1} = 0, \]

where

\[ J(a, b) = 2(a^2 - k^2b^2)((a - kb)^2 - L_2)((a + kb)^2 - L_2). \]
Consider first the case where both \( p, q \) are odd. We see from (21) that we can choose the 2-tuple \( C_{-p/2,q/2} \) arbitrary, i.e. it is not a consequence of a recursion. Thus, we set \( a_0 = -p/2, b_0 = -q/2, \) so that \( a_0^2 - k^2 b_0^2 = 0. \) Further we set \( C_{a,b} = 0 \) unless it can be computed explicitly from \( C_{m,b} \) by a sequence of recursions (22).

Think of the elements \( C_{a,b} \) as laid out on an infinite grid, with rows labeled by \( a \) and columns by \( b. \) The value of \( C_{a_0+m,b_0+n} \) for \( m, \neq 0 \) can be obtained via (22) as the sum of the contributions from all regular paths that lead from gridpoint \( (a_0, b_0) \) to gridpoint \( a_0 + m, b_0 + n. \) A regular path is a connected sequence of vertical segment moves upward: \( (a-1, b) \to (a, b) \) and horizontal segment moves to the right: \( (a, b-1) \to (a, b), \) in arbitrary order. All contributions from gridpoints below or to the left of \( (a_0, b_0) \) are assumed zero. The contribution of a path to the value of \( C_{a_0+m,b_0+n} \) is the ordered product of the contributions of the vertical and horizontal segments that make up the regular path.

We have assumed first that \( p \) and \( q \) are odd, so that \( p/2, q/2 \) are half-integers. Now note that any regular path connecting \( (a_0, b_0) = (-p/2, -q/2) \) to some gridpoint \( (1/2, b) \) necessarily contains a vertical segment \( (-1/2, b) \to (1/2, b) \) and contributes the factor 0, so that the contribution of the path to the sum is 0. Thus, necessarily, \( C_{1/2,b} = 0. \) Similarly, any regular path connecting \( (a_0, b_0) = (-p/2, -q/2) \) to some gridpoint \( (a, 1/2) \) necessarily contains a horizontal segment \( (b, -1/2) \to (b, 1/2) \) and contributes the factor 0, so that the contribution of the path to the sum is 0. Thus, \( C_{a,1/2} = 0. \) We conclude that the only possible nonzero 2-tuples are those in the grid \( (-p/2+m, -q/2+n) \) where \( 0 \leq m < p/2, \) \( 0 \leq n < q/2, \) and these terms are uniquely determined by the choice of \( C_{-p/2,q/2}. \) We get polynomial constants of the motion by taking the terms in \( C_{-p/2,q/2} \) to be suitable finite products of the form

\[
\prod_{a,b} [J(a, b)],
\]

to cancel the denominator terms in the expressions for \( C_{a_0+m,b_0+n} \) which come from recursion (22). Thus, we have constructed a two-parameter family of finite-order constants of the motion. It is a simple exercise to show that the commutators of these symmetries with \( L_2 \) are nonzero, so that the system is operator superintegrable. (Note that this last fact follows also from our original construction of the symmetries \( L. \) We must have \( A \equiv B \equiv C = 0 \) unless \( L \) is functionally independent of \( H \) and \( L_2.)

If \( k = -p/q \) with both \( p, q \) odd, the same construction works with \( (a_0, b_0) = (-p/2, -q/2). \)

The case \( k = 2s \cdot p/q \) with \( s \geq 1 \) and \( p, q \) relatively prime odd integers requires a modified analysis. Now we set \( a_0 = -2s^{-1} p, b_0 = -q/2, \) so that \( a_0^2 - k^2 b_0^2 = 0, \) \( a_0 \) is an integer and, as before, \( b_0 \) is a half-integer. Further we set \( C_{a,b} = 0 \) unless it can be computed explicitly from \( C_{a_0,b_0} \) by a sequence of recursions (22). As before, the value of \( C_{a_0+m,b_0+n} \) for \( m, \neq 0 \) can be obtained via (22) as the sum of the contributions from all regular paths that lead from gridpoint \( (a_0, b_0) \) to gridpoint \( a_0 + m, b_0 + n. \) Now note that any vertical segment connecting a gridpoint \((-1, b)\) to gridpoint \((0, b)\) maps \( C_{-1,b} \) to

\[
C_{0,b} = \left( \begin{array}{c} 0 \\ B_{0,b} \end{array} \right),
\]

i.e. to a 2-vector with upper component 0. Further, if this segment is followed by the horizontal segment connecting \((0, b)\) to \((0, b+1)\) the upper component of the 2-vector will remain 0. Thus, all regular paths that lead from gridpoint \((a_0, b_0)\) to any gridpoint \((0, b)\) on row \( a = 0 \) will produce a 2-vector of the form

\[
C_{0,b} = \left( \begin{array}{c} 0 \\ B_{0,b} \end{array} \right), \tag{23}
\]
Next, note that any vertical segment \((0, \hat{b}) \rightarrow (1, \hat{b})\) will map a special 2-vector \((23)\) to the zero vector. This means that \(C_{a,b} = 0\) for all integers \(a \geq 1\). Just as before, \(\hat{C}_{a,b} = 0\) for all half-integers \(b \geq 1/2\). Thus the only possible nonzero 2-tuples are those in the grid \((-2^{a-1} p + m, -q/2 + n)\) where \(0 \leq m \leq 2^{a-1} p, 0 \leq n < q/2\), and these terms are uniquely determined by the choice of \(J_{p/2, q/2}\). We get polynomial constants of the motion by taking the terms in \(\hat{C}_{p/2, q/2}\) to be suitable finite products of the form

\[
\prod_{a,b} [J(a, b)],
\]

to cancel the denominator terms in the expressions for \(C_{a+m, b+n}\). Thus, we have again constructed a two-parameter family of finite-order constants of the motion and the system is operator superintegrable.

It is easy to extend these arguments to the cases \(k = -2^s p/q\) and \(k = \pm p/2^s q\) where \(p, q\) are relatively prime odd integers. Thus the system \((18)\) is operator superintegrable for all rational \(k\). It is also tedious but straight-forward to verify from the results of subsection 2.1 that for any of our symmetry operators \(\hat{L}\) for this system the commutator \([L_2, \hat{L}]\) also belongs to the algebra of solutions that we have constructed.

**Example.** We consider the case \(k = 3\) where explicit operators proving superintegrability are known to be given by \((19)\). The third-order operator \(K_1\) can be cast into canonical form to give

\[
A = 12i e^{-3(R+i\theta)}, \quad B = (8 + e^{2R}H + 4L_2) e^{-3(R+i\theta)},
\]

\[
C = -(8i + 3i e^{2R}H + 4iL_2) e^{-3(R+i\theta)}, \quad D = -(4e^{2R}H + 12L_2) e^{-3(R+i\theta)}.
\]

It is straightforward to check that equations \((10), (11), (12)\) and \((13)\) are satisfied, as well as the recurrence relations \((21)\). Here, \(K_1\) is a skew symmetry operator. We can also find other solutions of the recurrence relations and construct symmetry operators, though they will generally be of higher order. For example, a solution is

\[
A = \frac{9i}{4} H e^{-R-3i\theta}, \quad B = \frac{3}{4} (2H e^{2R} + 4L_2^2 + 20L_2 + 16 - HL_2 e^{2R}) e^{-3(R+i\theta)},
\]

\[
C = \frac{3i}{4} (6H e^{2R} - 4L_2^2 - 20L_2 - 16 + 3HL_2 e^{2R}) e^{-3(R+i\theta)}, \quad D = -\frac{3}{4} H(8 + 5L_2) e^{-R+3i\theta}.
\]

The corresponding operator is

\[
\hat{L} = A\hat{a}_{R_0} + B\hat{a}_R + C\hat{a}_b + D = \frac{9}{8} K_1 K_3 - \frac{3}{8} K_3 K_1 + \frac{3}{4}(\alpha + \frac{13}{2}) K_1,
\]

clearly a symmetry. Since \(K_3\) is a second-order self-adjoint operator we see that

\[
\hat{L}^* = -\frac{3}{8} K_3 K_1 + \frac{3}{2} K_1 K_3 = \frac{3}{4}(\alpha + \frac{13}{2}) K_1.
\]

Thus \(S = \frac{11}{8} (K_1, K_3)\), which is fourth-order self-adjoint (and proportional to the commutator of \(L_2\) with \(K_1\)), and \(A = \frac{9}{8} (K_1 K_3 + K_3 K_1) + \frac{3}{4}(\alpha + \frac{13}{2}) K_1\), which is fifth-order skew-adjoint. Neither one of these symmetries is of minimal order, but both are functionally independent of the generators \(H, K_3\), so they each verify superintegrability.

**4. The classical analog**

Here we first describe the classical analog of our infinite-order symmetry operator construction and then apply it to the same example as in the previous section. We construct constants of the motion of all orders for the Hamiltonian system

\[
\mathcal{H} = \sum_{j,k=1}^2 g^{jk} p_j p_k + V = E
\]

\((24)\)
We require that the constant of the motion take the standard form
\[ \mathcal{H} = \mathcal{L}_1 = \frac{1}{f_1(u_1) + f_2(u_2)} \left( p_{u_1}^2 + p_{u_2}^2 + v_1(u_1) + v_2(u_2) \right), \]
and, due to the separability, there is the second-order constant of the motion
\[ \mathcal{L}_2 = \frac{f_2(u_2)}{f_1(u_1) + f_2(u_2)} \left( p_{u_1}^2 + v_1(u_1) \right) - \frac{f_1(u_1)}{f_1(u_1) + f_2(u_2)} \left( p_{u_2}^2 + v_2(u_2) \right), \]
i.e. \([\mathcal{L}_2, \mathcal{H}] = 0\), where \([\cdot, \cdot]\) is the usual Poisson bracket, and we have phase space identities
\[ f_1(u_1) \mathcal{H} + \mathcal{L}_2 = p_{u_1}^2 + v_1(u_1), \quad f_2(u_2) \mathcal{H} - \mathcal{L}_2 = p_{u_2}^2 + v_2(u_2). \tag{25} \]

We look for a constant of the motion \(\tilde{\mathcal{H}}(\mathcal{H}, \mathcal{L}_2, u_1, u_2)\), i.e. a function on the phase space that satisfies
\[ \{\mathcal{H}, \tilde{\mathcal{H}}\} = 0. \tag{26} \]

We require that the constant of the motion take the standard form
\[ \tilde{\mathcal{L}} = \sum_{j,k} \left( A^{j,k}(u_1, u_2) p_{u_1} p_{u_2} + B^{j,k}(u_1, u_2) p_{u_1} + C^{j,k}(u_1, u_2) p_{u_2} + D^{j,k}(u_1, u_2) \right) \mathcal{H}^j \mathcal{L}_2^k. \tag{27} \]
Note that if the formal symmetries (27) contained polynomial terms in \(p_{u_1}\) or \(p_{u_2}\), of orders \(\geq 2\) we could use identities (25), recursively, and rearrange terms to achieve the unique standard form (27).

We find that the symmetry condition (26) is equivalent to the system of equations
\[
\begin{align*}
\partial_2 B^{j,k} + \partial_1 C^{j,k} & = 0, \tag{28} \\
-2 \partial_2 A^{j,k} v_2 + 2 \partial_1 D^{j,k} - A^{j,k} v'_2 + 2 \partial_2 A^{j-1,k} f_2 + A^{j-1,k} f'_2 - 2 \partial_2 A^{j-1,k-1} & = 0, \tag{29} \\
-2 \partial_1 A^{j,k} v_1 + 2 \partial_2 D^{j,k} - A^{j,k} v'_1 + 2 \partial_1 A^{j-1,k} f_1 + A^{j-1,k} f'_1 + 2 \partial_1 A^{j-1,k-1} & = 0, \tag{30} \\
-2 \partial_1 B^{j,k} v_1 - 2 \partial_2 C^{j,k} v_2 - B^{j,k} v'_1 - C^{j,k} v'_2 + 2 \partial_1 B^{j-1,k} f_1 & + 2 \partial_2 C^{j-1,k} f_2 + B^{j-1,k} f'_1 + C^{j-1,k} f'_2 + 2 \partial_1 B^{j-1,k-1} - 2 \partial_2 C^{j-1,k-1} = 0. \tag{31}
\end{align*}
\]

Note that condition (27) makes sense, at least formally, for infinite-order constants of the motion, and one can consider \(\mathcal{H}, \mathcal{L}_2\) as parameters in these equations.

In this view we can write
\[ \tilde{\mathcal{L}}(\mathcal{H}, \mathcal{L}_2, u_1, u_2) = A(u_1, u_2, \mathcal{H}, \mathcal{L}_2) p_1 p_2 + B(u_1, u_2, \mathcal{H}, \mathcal{L}_2) p_1 + C(u_1, u_2, \mathcal{H}, \mathcal{L}_2) p_2 + D(u_1, u_2, \mathcal{H}, \mathcal{L}_2), \tag{32} \]
and consider \(\tilde{\mathcal{L}}\) as an at most second-order constant of the motion that is analytic in the parameters \(\mathcal{H}, \mathcal{L}_2\). Then the above system of equations can be written in the more compact form
\[
\begin{align*}
B_{u_2} + C_{u_1} & = 0, \tag{33} \\
-2 A_{u_2} v_2 + 2 D_{u_1} - A v'_2 + (2 A_{u_2} f_2 + A f'_2) H - 2 A_{u_1} L_2 & = 0, \tag{34} \\
-2 A_{u_1} v_1 + 2 D_{u_2} - A v'_1 + (2 A_{u_1} f_1 + A f'_1) H + 2 A_{u_2} L_2 & = 0, \tag{35} \\
-2 B_{u_1} v_1 - 2 C_{u_2} v_2 - B v'_1 - C v'_2 & + (2 B_{u_1} f_1 + 2 C_{u_2} f_2 + B f'_1 + C f'_2) H + (2 B_{u_1} - 2 C_{u_2}) L_2 = 0. \tag{36}
\end{align*}
\]
We can view (33) as an equation for $B, C$ and (34) and (35) as the defining equations for $D_1, D_2$.

We can simplify this system, and easily compare it to the operator system, by noting that there are two functions $F(u_1, u_2, \mathcal{H}, L_2), G(u_1, u_2, \mathcal{H}, L_2)$ such that (33) is satisfied by

\[ A = F, \quad B = -\partial_1 G, \quad C = \partial_2 G. \]

Then the integrability condition for (34) and (35) is (with the shorthand $\partial_j F = F_j, \partial_j \ell F = F_j \ell$, etc, for $F$ and $G$)

\[
2F_{22}(v_2 - f_2 \mathcal{H} + L_2) + 3F_2(v_2' - f_2' \mathcal{H}) + F(v_2'' - f_2'' \mathcal{H}) = 2F_{11}(v_1 - f_1 \mathcal{H} - L_2) + 3F_1(v_1' - f_1' \mathcal{H}) + F(v_1'' - f_1'' \mathcal{H}),
\]

and equation (36) becomes

\[
2G_{11}(v_1 - f_1 \mathcal{H} - L_2) + G_1(v_1' - f_1' \mathcal{H}) = 2G_{22}(v_2 - f_2 \mathcal{H} + L_2) + G_2(v_2' - f_2' \mathcal{H}).
\]

Now we use this classical construction to study the flat space Hamiltonian system

\[
\mathcal{H} = p_1^2 + p_2^2 + V, \quad V = \alpha \frac{(x + iy)^k}{(x - iy)^{k+1}},
\]

where $x, y$ are Cartesian coordinates. We have already shown that this system is superintegrable for all rational $k$ [21].

All of these classical systems separate in polar coordinates:

\[
u_1 = R, \quad u_2 = \theta, \quad x = e^R \cos \theta, \quad y = e^R \sin \theta,
\]

with corresponding constants of the motion

\[-L_2 = p_\theta^2 + \alpha e^{2k\theta}.
\]

Furthermore,

\[\mathcal{H} = e^{-2R} (p_R^2 - L_2),\]

and

\[f_1 = e^{2R}, \quad f_2 = 0, \quad v_1 = 0, \quad v_2 = \alpha e^{2k\theta}.
\]

We assume $k = p/q$ for relatively prime integers $p, q$. Based on the known expressions for the classical higher-order constants of the motion, derived in [21], we look for a standard form constant of the motion $\mathcal{L}$, (32), where

\[
F = \sum_{a,b} A_{a,b}(\alpha, \mathcal{H}, L_2) e^{2(aR+ib\theta)}, \quad G = \sum_{a,b} B_{a,b}(\alpha, \mathcal{H}, L_2) e^{2(aR+ib\theta)}. \]

We require that there are only a finite number of nonzero terms in the sums and that the sums are of the form $a = a_0 + m, b = b_0 + n$ where $m, n$ run over a subset of the non-negative integers.

Substituting all these expressions into equations (37) and (38) and equating coefficients of terms $e^{2(aR+ib\theta)}$, we obtain the matrix recursion

\[
2\mathcal{L}_2(a^2 - k^2b^2) \left( \begin{array}{cc} A_{a,b} & B_{a,b} \\ B_{a,b} & A_{a,b} \end{array} \right) + (2a - 1)H \left( \begin{array}{cc} 0 & 0 \\ 0 & a - 1 \end{array} \right) \left( \begin{array}{cc} A_{a-1,b} & B_{a-1,b} \\ B_{a-1,b} & A_{a-1,b} \end{array} \right) \\
- \alpha k^2(2b - 1) \left( \begin{array}{cc} 0 & 0 \\ 0 & b - 1 \end{array} \right) \left( \begin{array}{cc} A_{a,b-1} & B_{a,b-1} \\ B_{a,b-1} & A_{a,b-1} \end{array} \right) = 0.
\]
Although this system of equations is much simpler than the corresponding operator equations (21) and (22), it shares essential features with them so that the details of the proof of superintegrability are essentially unchanged. As before we set

\[ C_{a,b} = \left( \frac{A_{a,b}}{B_{a,b}} \right). \]

Consider first the case where \( p, q \) are both odd and positive. We see from (41) that we can choose the 2-tuple \( C_{-p/2,q/2} \) arbitrarily. Thus we set \( a_0 = -p/2, b_0 = -q/2 \), so that \( a_0^2 - k^2 b_0^2 = 0 \) and we set \( C_{a,b} = 0 \) unless it can be computed explicitly from \( C_{a_0,b_0} \) by a sequence of recursions (41).

The value of \( C_{a_0+m,b_0+n} \) for \( m, n \neq 0 \) can be obtained via (41) as the sum of the contributions from all regular paths that lead from gridpoint \((a_0, b_0)\) to gridpoint \(a_0 + m, b_0 + n\). Since \( p/2, q/2 \) are half-integers, any regular path connecting \((a_0, b_0)\) to some gridpoint \((1/2, b)\) necessarily contains a vertical segment \((-1/2, b) \rightarrow (1/2, b)\) and contributes the factor \(0\), so that the contribution of the path to the sum is 0. Thus, necessarily, \( C_{1/2, b} = 0 \). Similarly, any regular path connecting \((a_0, b_0)\) to some gridpoint \((a, 1/2)\) necessarily contains a horizontal segment \((b, -1/2) \rightarrow (b, 1/2)\) and contributes the factor \(0\), so that the contribution of the path to the sum is 0. Thus, \( C_{a,1/2} = 0 \). We conclude that the only possible nonzero 2-tuples are those in the grid \((-p/2 + m, -q/2 + n)\) where \( 0 \leq m < p/2, 0 \leq n < q/2 \), and these terms are uniquely determined by the choice of \( C_{-p/2,q/2} \). We get polynomial constants of the motion by taking the terms in \( C_{-p/2,q/2} \) to be suitable powers of \( \mathcal{L}_2 \) to cancel the denominator terms in the the expressions for \( C_{a_0+m,b_0+n} \) which come from recursion (41). Thus, we have constructed a two-parameter family of finite-order constants of the motion. It is easy to show that the Poisson brackets of these symmetries with \( \mathcal{L}_2 \) are nonzero, so that the system is classically superintegrable. There is a special simplification here in that the recursion (41) decouples into separate equations for \( A_{a,b} \) and for \( B_{a,b} \).

If \( k = -p/q \) with \( p, q \) both odd, the same construction works with \((a_0, b_0) = (-p/2, -q/2)\).

For the case \( k = 2^sp/q \) with \( s \geq 1 \) and \( p, q \) relatively prime odd positive integers, we set \( a_0 = -2^{s-1}p, b_0 = -q/2 \), so that \( a_0^2 - k^2 b_0^2 = 0 \), \( a_0 \) is an integer and, as before, \( b_0 \) is a half-integer. Again we set \( C_{a,b} = 0 \) unless it can be computed explicitly from \( C_{a_0,b_0} \) by a sequence of recursions (41). As before, the value of \( C_{a_0+m,b_0+n} \) for \( m, n \neq 0 \) can be obtained via (41) as the sum of the contributions from all regular paths that lead from gridpoint \((a_0, b_0)\) to gridpoint \(a_0 + m, b_0 + n\). However, any vertical segment connecting a gridpoint \((-1, b)\) to gridpoint \((0, b)\) maps \( C_{-1,b} \) to

\[ C_{0,b} = \left( \frac{0}{B_{0,b}} \right), \]

i.e. to a 2-vector with upper component 0. If this segment is followed by the horizontal segment connecting \((0, b)\) to \((0, b+1)\) the upper component of the 2-vector will remain 0, so all regular paths that lead from gridpoint \((a_0, b_0)\) to any gridpoint \((0, b)\) on row \( a = 0 \) will produce a 2-vector of the form

\[ C_{0,b} = \left( \frac{0}{B_{0,b}} \right). \] (42)

Note that any vertical segment \((0, b) \rightarrow (1, b)\) will map a special 2-vector (42) to the zero vector. Thus \( C_{a,b} = 0 \) for all integers \( a \geq 1 \), and as before, \( C_{a,b} = 0 \) for all half-integers \( b \geq 1/2 \). Thus the only possible nonzero 2-tuples are those in the grid \((-2^{s-1}p+m, -q/2+n)\)
where \(0 \leq m \leq 2^{s-1} p, 0 \leq n < q/2\), and these terms are uniquely determined by the choice of \(C_{-p/2,-q/2}\). We get polynomial constants of the motion by taking the terms in \(C_{-p/2,-q/2}\) to be the suitable powers of \(L_2\) to cancel the denominator terms in the the expressions for \(C_{m+n,0}\). We have again constructed a two-parameter family of finite-order constants of the motion and the system is classically superintegrable. There is a significant simplification here, due to the decoupling of (41) into separate equations for \(A_{a,b}\) and for \(B_{a,b}\). If we choose \(B_{0,0} = 0\) then all vectors \(C_{0,b} = 0\), so that row 0 can be removed from the grid.

Again it is easy to extend these arguments to the cases \(p = -2^i p, q = \pm p/2^j q\) where \(p, q\) are relatively prime odd integers. Thus we have a new proof that the system (39) is classically superintegrable for all rational \(k\), and have clarified the relation between the classical and operator symmetries for this system.

**Example.** For system (39) with \(k = 3\) the recurrence relations can be solved easily to give

\[
F = L_2 e^{-3R-3\theta} + \frac{1}{2} \mathcal{H} e^{-R-3\theta}, \quad G = L_2 e^{-3R-3\theta} + \frac{3}{4} \mathcal{H} e^{-R-3\theta},
\]

so

\[
A = F, \quad B = 3L_2 e^{-3R-3\theta} + \frac{3}{4} \mathcal{H} e^{-R-3\theta},
\]

\[
C = -3iL_2 e^{-3R-3\theta} - \frac{9}{4} \mathcal{H} e^{-R-3\theta}, \quad D = \frac{i}{4} L_2 (4L_2 + 3e^{2R} \mathcal{H}) e^{-3R-3\theta}.
\]

This gives us the canonical forms for the third- and fourth-order constants of the motion \(K_1, K_2\) as given in [21]. Indeed, \(A_{pR} p_0 + D = \frac{3}{4} K_1, B_{pR} + C p_0 = \frac{1}{4} K_2\).

**5. The quantum TTW system**

Now we apply our constructions to the quantum TTW system [13, 14]. Here,

\[
u_1 = R, \quad u_2 = \theta, \quad f_1 = e^{2R}, \quad f_2 = 0, \quad v_1 = \alpha e^{4R},
\]

\[
u_2 = \frac{\beta}{\cos^2(k\theta)} + \frac{\gamma}{\sin^2(k\theta)} = \frac{2(\gamma + \beta)}{\sin^2(2k\theta)} + \frac{2(\gamma - \beta) \cos(2k\theta)}{\sin^2(2k\theta)}.
\]

Based on the results of [21] for the classical case, we postulate expansions of \(F, G\) in finite series

\[
F = \sum_{a,b,c} A_{a,b,c} e^{i a R} e^{b R} e^{c R} (R, \theta), \quad G = \sum_{a,b,c} B_{a,b,c} e^{i a R} e^{b R} e^{c R} (R, \theta),
\]

\[
E_{a,b,0} = e^{2a R} \sin^b (2k\theta), \quad E_{a,b,1} = e^{2a R} \sin^b (2k\theta) \cos(2k\theta).
\]

The sum is taken over terms of the form \(a = \alpha + m, b = \beta + n, \text{ and } c = 0, 1, \text{ where } m, n\) are integers. The point \((a_0, b_0)\) could in principle be any point in \(\mathbb{Z}^2\); however, for reasons discussed below, we will take \(a_0\) to be a positive integer and \(b_0\) to be a negative integer.

Taking coefficients with respect to the basis (44) in each of equation (14) and (15) gives recurrence relations for these coefficients. For example, the coefficient of \(e^{2a R} \sin^b (2k\theta) \cos(2k\theta)\) in equation (14) gives the equation

\[
8k^2 (b - 1)(L_2 - 2k^2 (b - 1) + \gamma + \beta) A_{a,b,1} + 32ak^3 b (b^2 - 1) B_{a,b+1,0} + 8k^2 (b - 1)(b^2k^2 + 2b k^2 + 2 \beta + 2 \gamma) A_{a,b+2,1} + 16k^2 (b^2 - 1)(\gamma - \beta) A_{a,b+2,0}
\]

\[
- 8k^2 (2b - 1)(\gamma - \beta) A_{a,b,0} + 8(a^2 - k^2 (b - 1)^2)(L_2 - a^2)
\]

\[
- k^2 (b - 1)^2) A_{a,b-1,0} + 4Ha (2a - 1) A_{a-1,b-2,1} - 8aa (a - 1) A_{a-2,b-2,1}
\]

\[
+ 32ak (b - 1)(a^2 - k^2 (b - 1)^2) B_{a,b-1,0} = 0.
\]
Figure 1. The template. Points contributing to the recurrence relation are marked with large dot (•). The large dot in the upper-left corner corresponds to the position \((a, b)\).

The shifts in the indices of \(A\) and \(B\) are integers and so we can view this as an equation on a two-dimensional lattice with integer spacings. While the shifts in the indices are of integer size, we have not required that the indices themselves be integers, although they may be integers in particular examples.

Taking the coefficient of \(e^{2aR} \sin^b(2k\theta)\) in equation (14) and the coefficients of \(e^{2aR} \sin^{b-1}(2k\theta) \cos(2k\theta)\) and \(e^{2aR} \sin^{b-1}(2k\theta)\) in equation (15) gives a further three recurrence relations. At a general point in the lattice there are four coefficients, and these four equations will be shown to be independent. The equations are linear and homogeneous and so there must be some points where the independence of the equations breaks down and allows at least one coefficient to be arbitrarily chosen.

The different powers of \(\sin(2k\theta)\) used in obtaining these equations have been chosen as a matter of convenience after many experiments conducted using the computer algebra package Maple.

All four recurrence relations are of a similar complexity, but rather than writing them out separately, we will combine them into a matrix recurrence relation by defining

\[
C_{a,b} = \begin{pmatrix} A_{a,b,0} \\ B_{a,b-1,0} \\ A_{a,b-2,1} \\ B_{a,b-1,1} \end{pmatrix}.
\]

We can now write the four recurrence relations in matrix form as

\[
0 = M_{a,b}C_{a,b} + M_{a,b-2}C_{a,b-2} + M_{a,b-4}C_{a,b-4} + M_{a,b-6}C_{a,b-6} + M_{a-1,b}C_{a-1,b} \\
+ M_{a-1,b+2}C_{a-1,b+2} + M_{a-2,b}C_{a-2,b} + M_{a-2,b+2}C_{a-2,b+2}.
\]  

\[ (45) \]

where each \(M_{i,j}\) is a \(4 \times 4\) matrix given below. It is useful to visualize the the set of points in the lattice which enter into this recurrence for a given choice of \((a, b)\). These are represented in figure 1 in which the upper-left corner is the point \((a, b)\). From this it is clear that when \(M_{a,b}\) is nonsingular, the value of \(C_{a,b}\) can be uniquely determined from the eight points to its right and below. In that case:

\[
C_{a,b} = -M_{a,b}^{-1}\left( M_{a,b-2}C_{a,b-2} + M_{a,b-4}C_{a,b-4} + M_{a,b-6}C_{a,b-6} + M_{a-1,b}C_{a-1,b} \\
+ M_{a-1,b+2}C_{a-1,b+2} + M_{a-2,b}C_{a-2,b} + M_{a-2,b+2}C_{a-2,b+2} \right).
\]  

\[ (46) \]

This allows us to construct an iterative procedure that calculates the values of \(C_{a,b}\) at points in the lattice using only other points where the values of \(C_{i,j}\) are already known. Since the point \((a, b)\) corresponds to the top-left corner of the collection of points in figure 1, this process will calculate the coefficients in a sequence that moves from right to left and bottom to top. Note that the matrices corresponding to the right-hand ‘corners’ of the set of points in
finite sums and hence we seek solutions that are confined to a finite rectangle in the lattice.

We are interested in finding a solution to the recurrence relation that gives $F$ and $G$ as finite sums and hence we seek solutions that are confined to a finite rectangle in the lattice. Since the equations for the $A_{i,j,k}$ and $B_{i,j,k}$ are linear and homogeneous, they always admit the trivial solution and so we need to demonstrate that a nonzero solution can be found. Our approach is as follows.

For a finite solution, there must be a lowest nonzero row and in that row, a rightmost nonzero element. Label this rightmost point in the bottom row as $(a_0, b_0)$. Since all elements to the right and below this point are zero, $M_{a_0,b_0}$ must be singular, otherwise we could use (46) to show that $C_{a_0,b_0}$ must vanish, contradicting its definition.

Since
\[
det(M_{a,b}) = -4096(a^2 - k^2 b^2)^2(a^2 - k^2(b - 1)^2)(L_2 - (a + k(b - 1)))^2 \\
\times (L_2 - (a - k(b - 1)))^2(L_2 - (a +kb))^2(L_2 - (a - kb))^2
\]
we must choose our starting point so that either $a^2 = k^2b^2$ or $a^2 = k^2(b - 1)^2$, that is, if $k = p/q$ with $p$ and $q$ a pair of relatively prime positive integers, we can choose $(a_0, b_0)$ to be one of $(np, nq)$, $(-np, nq)$, $(np, nq + 1)$ or $(np, -nq + 1)$ for any real number $n$. At all of these points, the rank of $M_{a_0,b_0}$ is 2 and hence at these points we can choose $4 - \text{rank}(M_{a,b}) = 2$ components of $C_{a,b}$ to be the arbitrary parameters.

In order to have a finite solution, we must eventually reach a point in the lattice beyond which all entries vanish or can be chosen to vanish. Examining the matrices defining the recurrence, we see that it may be possible to achieve this on a left-hand boundary due to the many terms with factors such as $(a - b - 1)$, $(b + 1)$, $b + 3$ and on an upper boundary because of factors of $a$ and $a - 1$. For this reason, we will now take $a_0$ to be $-p$ and $b_0$ to be $q$ or $q + 1$ and examine how the cutoffs on the left and top occur.

As the recurrence relations (45) and (46) only involve shifts of multiples of two units in the second index, it is easy to verify that all entries in columns that are an odd number of steps away from column $b_0$ must vanish or can be chosen to vanish. Furthermore, we have two candidates for $b_0$, $q$ and $q + 1$. We will choose $b_0$ to be whichever of these is odd. We can then assume that even numbered columns have only vanishing entries and can traverse the lattice in steps of two to the left starting from column $b_0$.

We now work our way across the row $a_0$ starting from the column $b_0$ taking steps of two units to the left. At the first point, $(a, b) = (a_0, b_0)$, $M_{a,b}$ has rank 2 and so the components of $C_{a,b}$ depend linearly on two arbitrary parameters. At other points in the bottom row, $M_{a,b}$ is nonsingular (unless we reach $b = -b_0$) and so we can solve for $C_{a_0,b}$. At the points when $b = 1$, $-1$, $-3$, this takes a special form. Note that in the bottom row, all lower points have vanishing $C_{i,j}$ and so we need to only consider contributions from the points $(a, b + 2)$, $(a, b + 4)$ and $(a, b + 6)$. We will initially assume that $q \neq 1, 2, 3, 4, 5, 6$ so that $M_{-p-1}$, $M_{-p-1}$ and $M_{-p-5}$ are all nonsingular. This is not essential, but the argument is simpler in this case.

First consider $b = 1$. The form of the matrices giving contributions from points to the right are the first two matrices in table 1. It is clear from these that the third component of $C_{a_0,1}$ must vanish. Next, consider $b = -1$. Only nonzero matrix elements occur in column 3 of $M_{a_0,-1}$ and so
\[
C_{a_0,-1} = M_{a_0,-1}^{-1} M_{a_0,1} C_{a_0,1} + M_{a_0,-1}^{-1} M_{a_0,3} C_{a_0,3} + M_{a_0,-1}^{-1} M_{a_0,5} C_{a_0,5} = 0.
\]
A similar calculation shows that $C_{a_0,-3} = C_{a_0,-5} = 0$ and hence $C_{a_0,j} = 0$ for all $j \leq -1$. 


cases, that is, the third component of \( \frac{b}{b} \) = −

J. Phys. A: Math. Theor. 43 (2010) 265205

E G Kahnis et al

Table 1. Matrices giving contributions near the left boundary. A ‘∗’ represents a nonzero entry.

| \( b = 1 \): | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
|---|---|---|---|---|---|---|---|---|
| \( M_{a,b}^{-1} \) | \( M_{a,b}^{-1} \) | \( M_{a,b}^{-1} \) | \( M_{a,b}^{-1} \) | \( M_{a,b}^{-1} \) |
| \( b = -1 \): | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b = -3 \): | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( b = -5 \): | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

Next we repeat the process for the row above, that is, row \( a_0 + 1 \) starting from the right-hand end, and then again for row \( a_0 + 2 \) and so on. The argument showing that all \( C_{i,j} \) vanish for \( j \leq -1 \) is essentially the same as for row \( a_0 \) except there are a few extra terms to consider since the elements in the two rows below are no longer all zero.

To see how the cutoff occurs at the top, start at the right-hand end of the 0 row, that is in position \( (0, b_0) \). All elements to the right are zero and so the only contributions to \( C_{0,b_0} \) come from below, that is, from \( (-1, b_0) \) and \( (-2, b_0) \). It is clear from the corresponding matrices in table 2 that the first and third components of \( C_{0,b_0} \) are zero. Stepping across the row in step of two to the left, it is easy to check that this is maintained for all elements of this row.

Next consider row 1. From the form of the matrices given for \( a = 1 \) in table 2 it is clear that only the first and third components of \( C_{0,j} \) can contribute to any \( C_{1,j} \). However, since these components have already been shown to vanish we conclude that \( C_{1,j} = 0 \) for all \( j \).

The last step is to consider row 2. As for row 1, when \( a = 2 \) the form of the matrices given for \( a = 2 \) in table 2 clearly shows that only the first and third components of \( C_{0,j} \) can contribute to any \( C_{2,j} \). As these components have already been shown to vanish we conclude that \( C_{2,j} = 0 \) for all \( j \).

Since we have two completely zero rows, it is now clear that \( C_{i,j} = 0 \) for all \( i \geq 1 \).

The above argument needs modification to see that the left-hand cutoff can be achieved when treating the bottom row for \( q = 1, 2, 3, 4, 5 \) or 6 as \( M_{a,b} \) will be singular in one of the columns \( b = -1, -3 \) or −5. However, it is a simple matter to use the original matrix recurrence relations (45) to check that the same conclusions can be reached in each of these cases, that is, the third component of \( C_{i,1} \) vanishes and each entry of \( C_{a_0,j} \) for \( j \leq 1 \) is either required to vanish or can be chosen to vanish.
Example 1. It is well known that the TTW system is quantum superintegrable in the case $k = 2$, $(p = 2, q = 1)$, [13, 21]. The generating operators, expressed in Cartesian coordinates, are

\[
H = \partial_x^2 + \partial_y^2 + \alpha (x^2 + y^2) + \beta \frac{(x^2 + y^2)}{(x^2 - y^2)^2} + \gamma \frac{(x^2 + y^2)}{4x^2y^2} - L_2 = (x \partial_y - y \partial_x)^2 + 4\beta \frac{x^2y^2}{(x^2 - y^2)^2} + \gamma \frac{(x^4 + y^4)}{4x^2y^2} + \beta + \frac{\gamma}{2}.
\]

\[
\bar{L} = (\partial_t^2 - \partial_y^2)^2 + \left(2\alpha x^2 + 2\beta \frac{(x^2 + y^2)}{(x^2 - y^2)^2} - \gamma \frac{(x^2 - y^2)}{2x^2y^2}\right) \partial_t^2 + \left(-4\alpha xy + \frac{8\beta xy}{(x^2 - y^2)^2}\right) \partial_x \partial_y + \left(2\alpha y^2 + 2\beta \frac{(x^2 + y^2)}{(x^2 - y^2)^2} + \gamma \frac{(x^2 - y^2)}{2x^2y^2}\right) \partial_y^2 + \left(2\alpha x - \frac{\gamma}{x^2}\right) \partial_x + \left(2\alpha y - \frac{\gamma}{y^2}\right) \partial_y + \alpha^2 (x^2 - y^2)^2 + \beta^2 \frac{x^2}{(x^2 - y^2)^2} + \frac{\gamma^2 (x^2 - y^2)^2}{16x^4y^2} + 8\alpha \beta \frac{x^2y^2}{(x^2 - y^2)^2} + \frac{\beta \gamma}{2x^2y^2} + 3\gamma \left(\frac{1}{2x^4} + \frac{1}{2y^4}\right).
\]

By expressing the fourth-order self-adjoint symmetry operator $\bar{L}$ in polar coordinates and converting to canonical form we can read off the functions $A, B, C, D$ and then determine
$F$, $G$ and the nonzero expansion coefficients. The results are

$$C_{-2,1} = \begin{pmatrix} -44 - 4L_2 \\ 4(\gamma - \beta) \\ 0 \\ -28 - 8L_2 \end{pmatrix}, \quad C_{-1,1} = \begin{pmatrix} -2H \\ 0 \\ 0 \\ -4H \end{pmatrix}, \quad C_{0,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ 

It is easy to check that these terms satisfy all our recurrence relations.

**Example 2.** The nonzero vectors in the solution to the recurrence for the $k = 1/3$ quantum TTW system are $C_{-1,1}$, $C_{-1,3}$, $C_{0,3}$ and $C_{0,1}$. The solution to the recurrence depends linearly on two parameters that can be taken to be $A_{-2,1,0}$ and $B_{-2,0,1}$. To obtain solutions for $A$, $B$, $C$ and $D$ that are polynomial in $H$ and $L_2$, we must choose the free parameters so as to cancel any denominators. With the choice

$$B_{-2,0,1} = 36H(18L_2 + 13) \quad \text{and} \quad A_{-2,1,0} = 8(81L_2^2 + 765L_2 + 274),$$

we find that $\hat{L}$ is a sixth-order symmetry and the expressions for $A$, $B$, $C$ and $D$ are given below written in terms of the $u_1 = R$ and $u_2 = \theta$ coordinates:

$$A = 8(81L_2^2 + 765L_2 + 274)e^{-2R}\sin^3(\frac{2\theta}{3}) - 144(\gamma - \beta)(9L_2 + 26)e^{-2R}\sin(\frac{2\theta}{3})\cos(\frac{2\theta}{3}) + 6(6(23 + 9L_2)(\gamma + \beta) - 81(\gamma - \beta)^2 - 81L_2^2 - 765L_2 - 274)\sin(\frac{2\theta}{3})e^{-2R}$$

$$B = 40(8 + 81L_2^2 + 135L_2)e^{-2R}\sin^2(\frac{2\theta}{3})\cos(\frac{2\theta}{3}) + 48(101 + 144L_2)(\gamma - \beta)e^{-2R}\sin^2(\frac{2\theta}{3})\cos(\frac{2\theta}{3}) + 12(315L_2 + 229)(\gamma + \beta)$$

$$- 2754(\gamma - \beta)^2 - 80 - 810L_2^2 - 1350L_2)e^{-2R}\cos(\frac{2\theta}{3}) + 12(162(\gamma - \beta)(\gamma + \beta) - (450L_2 + 319)(\gamma - \beta))e^{-2R}$$

$$C = -40(8 + 81L_2^2 + 135L_2)e^{-2R}\sin^3(\frac{2\theta}{3})\cos(\frac{2\theta}{3}) + 144(27L_2 + 8)(\gamma - \beta)\cos(\frac{2\theta}{3})\sin(\frac{2\theta}{3})e^{-2R} + 6(6(-27L_2 - 5)(\gamma + \beta) - 81(\gamma - \beta)^2 - 405L_2^2 + 40 + 675L_2)e^{-2R}\sin(\frac{2\theta}{3})$$

$$- 72H(18L_2 + 13)\sin^3(\frac{2\theta}{3}) + 1296H(\gamma - \beta)\sin(\frac{2\theta}{3})\cos(\frac{2\theta}{3}) + 54H(-6(\gamma + \beta) + 18L_2 + 13)\sin(\frac{2\theta}{3})$$

$$D = -12(81L_2^2 + 423L_2 + 40)(\gamma + \beta) - 162(9L_2 + 8)(\gamma - \beta)^2\cos(\frac{2\theta}{3})e^{-2R} + 12(9L_2 + 40)(9L_2 + 1)(\gamma - \beta)\cos(\frac{2\theta}{3})e^{-2R} + 2(81L_2^2 + 765L_2 + 274) \times L_2\cos(2\theta) e^{-2R} (36(-27L_2 - 23)(\gamma - \beta)(\gamma + \beta) + 162(\gamma - \beta)^3$$

$$+ 6(81L_2^2 + 441L_2 + 40)(\gamma - \beta))e^{-2R} + 81(-2(3L_2 + 7)(\gamma + \beta) + 9(\gamma - \beta)^2)H\cos(\frac{2\theta}{3}) + 486(L_2 + 2)H(\gamma - \beta)\cos(\frac{2\theta}{3})$$

$$+ (81L_2^2 + 441L_2 + 40)H\cos(2\theta).$$

These give the operator $\hat{L}$ (9). In order to construct the symmetry operator in the standard form (4), the $H$ and $L_2$, which have been treated as parameters throughout the calculation, must be moved to the right. For example, after expanding $C$, the coefficient of $HL_2$ is

$$-1296\sin^3\left(\frac{2\theta}{3}\right) + 972\sin\left(\frac{2\theta}{3}\right),$$

and so this contributes the term

$$\left(-1296\sin^3\left(\frac{2\theta}{3}\right) + 972\sin\left(\frac{2\theta}{3}\right)\right)\frac{\partial}{\partial \theta}HL_2$$
to the differential operator $\tilde{L}$, in which $H$ and $L_2$ are now treated as differential operators. We have used Maple to verify that the this operator does in fact commute with the $k = 1/3$ quantum TTW Hamiltonian. In this case $S = \frac{1}{2}(\tilde{L} + \tilde{L}^*)$ is sixth-order self-adjoint.

6. Discussion

The key to our method for proof of superintegrability is the canonical form for symmetry operators of all orders. It enables us to replace the computation of the commutator of $H$ with operators of arbitrary high order by verification of equations (14) and (15). In these equations $H$ and $L_2$ can be treated as parameters until the very last step when the canonical form is reinterpreted as an operator. Since (14) and (15) are linear and homogeneous in $F$ and $G$ the solutions of these equations form a vector space. There are, of course, many solutions but most are not polynomials in $H, L_2$. To prove superintegrability we have to find a nontrivial solution $F(u_1, u_2, H, L_2), G(u_1, u_2, H, L_2)$ that has polynomial dependence on $H, L_2$. If there is one such solution, there will be an infinite number of others, since any polynomial function of a finite symmetry is a finite symmetry, as is the commutator of $L_2$ with a finite symmetry. To prove superintegrability we need find only one such solution. We choose the simplest ansatz that leads to success. The method we employ leads to an algebra of symmetry operators, virtually any one of which proves superintegrability. It will not necessarily lead to the symmetry operator of lowest order. It is a nontrivial issue to select out of this algebra the operator of lowest possible order.

Our strategy is to postulate a set of basis functions and to expand $F$ and $G$ in terms of it. The basis has to be chosen so that (14) and (15) reduce to a set of recurrence relations between the coefficients of the basis functions. We will succeed if we can find some nonzero solutions of these recurrences such that only a finite number of the coefficients are nonzero. The coefficients will then be rational functions of $H, L_2$, but arbitrary up to a scale factor $K(H, L)$. We choose $K$ such that all coefficients become polynomials in $H, L_2$, and then we are done! We have used two different methods to solve the recurrences in our two examples. The template method for the more complicated TTW problem is more general and its step-by-step evaluation of the expansion coefficients probably makes it the preferred tool to treat additional examples.

How can one determine an appropriate set of basis functions? In the examples appearing in this paper we used the known expressions for the corresponding classical superintegrable systems as computed in [21] to determine the basis for the classical constants of the motion, and then used the same basis for the quantum system. This worked although the classical expansion coefficients differed from the quantum coefficients, as would be expected. We note that the canonical operator construction permits easy generation of explicit expressions for the defining operators in a large number of examples. Once the basic rectangle of nonzero solutions is determined it is easy to compute dozens of explicit examples via Maple and simple Gaussian elimination. The generation of explicit examples is easy; the proofs that the method works for all orders are more challenging.

It is clear that the methods of this paper will apply to many Hamiltonian systems, but each system will have its own peculiarities. Also the canonical form for symmetry operators can clearly be extended to higher dimensions in the cases where the separable coordinates are of the subgroup type as treated in [16]. Of particular interest is the relation between the classical constants of the motion and the quantum symmetries. We intend to pursue these lines of inquiry.
7. Appendix. The $M$ matrices for the quantum TTW system

$$M_{a,b} = \begin{pmatrix}
8k^2(2b - 1)(b - 1)(\beta - \gamma) & 32ak(b - 1)(a^2 - k^2(b - 1)^2) \\
8(a^2 - k^2b_1^2)(L_2 - a^2 - b_k^2) & 0 \\
8ak(b - 1)(\beta - \gamma) & 0 \\
8ak(2b - 1)(\beta - \gamma) & -8(a^2 - k^2(b - 1)^2)(L_2 - a^2 - k^2(b - 1)^2)
\end{pmatrix}
$$

$$M_{a,b+2} = \begin{pmatrix}
8k^2(2b - 1)(b - 1)(\beta - \gamma) & 32ak^2(b^2 - 1) \\
8k^2(2b + 1)(\beta - \gamma) & 0 \\
8ak(b + 1)(\beta - \gamma) & 0 \\
8akk(b^2 - 1)(\beta - \gamma)
\end{pmatrix}
$$

$$M_{a,b+4} = \begin{pmatrix}
8k^2(2b + 1)(\beta - \gamma) & 0 \\
8k^2(2b + 1)(\beta - \gamma) & 0 \\
8ak(b + 1)(\beta - \gamma) & 0 \\
8akk(b^2 + 1)(\beta - \gamma)
\end{pmatrix}
$$

$$M_{a,b+6} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
$$

$$M_{a-1,b} = \begin{pmatrix}
4Ha(2a - 1) & 0 \\
-4Hk(2a - 1) & 0 \\
-4Ha(2a - 1)(a - 1) & 4Hk(2a - 1)(a - 1)
\end{pmatrix}
$$

$$M_{a-1,b+2} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
$$

$$M_{a-2,b} = \begin{pmatrix}
-8ak(b - 1)(a - 2) & 0 \\
8ak(a - 1)(a - 2) & 0 \\
8ak(b - 1)(a - 2) & 0
\end{pmatrix}
$$

$$M_{a-2,b+2} = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
$$
References

[1] Kalnins E G, Kress J M and Miller W Jr 2006 Second-order superintegrable systems in conformally flat spaces. V: 2D and 3D quantum systems J. Math. Phys. 47 093501

[2] Gravel S 2004 Hamiltonians separable in Cartesian coordinates and third-order integrals of motion J. Math. Phys. 45 1003–19

[3] Tempesta P, Winternitz P, Miller W and Pogosyan G (ed) 2005 Superintegrability in Classical and Quantum Systems vol 37 (Providence, RI: American Mathematical Society)

[4] Eastwood M and Miller W Jr (ed) 2008 Symmetries and Overdetermined Systems of Partial Differential Equations (IMA Volumes in Mathematics and its Applications) vol 144 (New York: Springer)

[5] Kalnins E G, Kress J M, Miller W Jr and Pogosyan G S 2001 Completeness of superintegrability in two-dimensional constant curvature spaces J. Phys. A: Math. Gen. 34 4705

[6] Daskaloyannis C and Ypsilantis K 2006 Unified treatment and classification of superintegrable systems with integrals quadratic in momenta on a two-dimensional manifold J. Math. Phys. 47 042904

[7] Kalnins E G, Kress J M and Miller W Jr 2007 Nondegenerate 3D complex Euclidean superintegrable systems and algebraic varieties J. Math. Phys. 48 113518

[8] Frit J, Smorodinski Y A, Uhlir A M and Winternitz P 1967 Symmetry groups in classical and quantum mechanics Sov. J. Nucl. Phys. 4 444

[9] Makarov A A, Smorodinsky Ya A, Valiev Kh and Winternitz P A 1967 Systematic search for nonrelativistic systems with dynamical symmetries Nuovo Cimento A 52 1061

[10] Kalnins E G, Kress J M and Miller W Jr 2005 Second-order superintegrable systems in conformally flat spaces: II. The classical 2D Stäckel transform J. Math. Phys. 46 053510

[11] Evans N W and Verrier P E 2008 Superintegrability of the caged anisotropic oscillator J. Math. Phys. 49 092902 (arXiv:0808.2146)

[12] Verrier P E and Evans N W 2008 A new superintegrable Hamiltonian J. Math. Phys. 49 022902 (arXiv:0712.3677)

[13] Tremblay F, Turbiner V A and Winternitz P 2009 An infinite family of solvable and integrable quantum systems on a plane J. Phys. A: Math. Theor. 42 242001

[14] Tremblay F, Turbiner A and Winternitz P 2010 Periodic orbits for an infinite family of classical superintegrable systems J. Phys. A: Math. Theor. 43 015202

[15] Kalnins E G, Miller W Jr and Post S 2010 Coupling constant metamorphosis and Nth-order symmetries in classical and quantum mechanics J. Phys. A: Math. Theor. 43 035202

[16] Kalnins E G, Kress J M and Miller W Jr 2010 Families of classical subgroup separable superintegrable systems. J. Phys. A: Math. Theor. 43 092001 (arXiv:0912.3158v1 [math-ph])

[17] Hietarinta J, Grammaticos B, Dorizzi B and Ramani A 1987 Direct methods for the search of the second invariant Phys. Rep. 147 87–154

[18] Kalnins E G, Miller W Jr and Pogosyan G S 2010 Superintegrability and higher order constants for classical and quantum systems Phys. Atomic Nuclei (arXiv:0912.2278v1 [math-ph]) submitted

[19] Quesne C 2010 Superintegrability of the Tremblay–Turbiner–Winternitz quantum Hamiltonians on a plane for odd k J. Phys. A: Math. Theor. 43 082001

[20] Hietarinta J 1987 Direct methods for the search of the second invariant Phys. Rep. 147 87–154

[21] Kalnins E G, Miller W Jr and Pogosyan G S 2010 Superintegrability and higher order constants for classical and quantum systems Proc. Inst. Math. NAS of Ukraine 50 Part 1 184–95

[22] Quesne C 2010 Superintegrability of the Tremblay–Turbiner–Winternitz quantum Hamiltonians on a plane for odd k J. Phys. A: Math. Theor. 43 082001

[23] Kalnins E G, Miller W Jr and Pogosyan G S 2004 Infinite order symmetries for two-dimensional separable Schrödinger equations Proc. Inst. Math. NAS of Ukraine 50 Part 1 184–95

[24] Kalnins E G, Kress J M, Miller W Jr and Pogosyan G S 2005 Infinite order symmetries for quantum separable systems Phys. Atomic Nuclei 68 1756–63

[25] Eisenhart L P 1934 Separable systems of Stäckel Ann. Math. 35 284–305