Lipschitz Chain Approximation
of Metric Integral Currents

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Abstract
Every integral current in a locally compact metric space $X$ can be approximated by a Lipschitz chain with respect to the normal mass, provided that Lipschitz maps into $X$ can be extended slightly.

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1 Introduction

In [5], after proving their celebrated deformation theorem, Federer and Fleming show that every integral current admits an approximation by Lipschitz chains. More precisely, for each current $T \in \mathbf{I}_{n,c}(\mathbb{R}^N)$ and $\varepsilon > 0$ there is a Lipschitz chain $P \in \mathbf{L}_{n,c}(\mathbb{R}^N)$ such that $\mathbf{N}(T - P) < \varepsilon$, see Theorem 5.8 in [5].

In this paper we prove an analogue of this result for a locally compact metric space $X$ with the property that every Lipschitz map into $X$ can be extended to a neighborhood of its domain. In fact, we need this property to hold only locally and for Lipschitz maps with compact domains. We will work in the context of metric integer rectifiable currents $\mathcal{I}_n(X)$ and metric integral currents $\mathbf{I}_{n,c}(X)$, see [1] and [10]. All relevant concepts and results will be discussed in Section 2.

The abelian group $\mathcal{I}_n(X)$ of integer rectifiable currents is equipped with the mass norm $\mathbf{M}$ and its subgroup $\mathbf{I}_{n,c}(X)$ of integral currents consists of all currents $T \in \mathcal{I}_n(X)$ with compact support and whose boundary $\partial T$ is also integer rectifiable, that is, an element of $\mathcal{I}_{n-1}(X)$. The resulting chain complex $\mathbf{I}_{n,c}(X)$ of integral currents is endowed with a norm $\mathbf{M}$ in each degree and boundary maps $\partial$: $\mathbf{I}_{k,c}(X) \to \mathbf{I}_{k-1,c}(X)$, so that each current $T \in \mathbf{I}_{n,c}(X)$ has mass $\mathbf{M}(T)$ and normal mass $\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$. A current $Z \in \mathbf{I}_{n,c}(X)$ with $\partial Z = 0$ is called a cycle and we denote by $\mathbf{Z}_{n,c}(X)$ the corresponding subgroup of cycles. An element $V \in \mathbf{I}_{n+1,c}(X)$ is called a filling of $S \in \mathbf{Z}_{n,c}(X)$ if $\partial V = S$.

Every singular Lipschitz chain in $X$ with integer coefficients induces a current $P \in \mathbf{L}_{n,c}(X)$, called Lipschitz chain, which is an element of $\mathbf{I}_{n,c}(X)$.

If $X = \mathbb{R}^N$ there is a canonical isomorphism between metric currents in $\mathbf{I}_{n,c}(\mathbb{R}^N)$ and classical integral currents from [5] (see Section 5 in [10]).

We begin with a weaker approximation result for the $\mathbf{M}$-norm. Every compactly supported function $u \in L^1(\mathbb{R}^n)$ induces an integer rectifiable current $\|u\| \in \mathcal{I}_{n,c}(\mathbb{R}^n)$ with $\mathbf{M}(\|u\|) = \|u\|_1$. If $F: \mathbb{R}^n \to X$ is $L$-Lipschitz, then there is a push-forward $F_\#\|u\| \in \mathcal{I}_{n,c}(X)$, and $\mathbf{M}(F_\#\|u\|) \leq L^n \mathbf{M}(\|u\|)$.

Every current $T \in \mathcal{I}_n(X)$ can be written as $T = \sum T_i$, where the sum converges with respect to the mass norm and each $T_i \in \mathcal{I}_{n,c}(X)$ is of the form $T_i = (F_i)_\#\|u_i\|$ for some integer valued $u_i \in L^1(\mathbb{R}^n)$, $F_i: K_i \to F_i(K_i) \subset X$ bi-Lipschitz and $K_i \subset \mathbb{R}^n$ compact containing the support of $u_i$. By a purely measure-theoretic argument it is possible to approximate each $u_i$ in the $L^1$-norm with a finite sum of characteristic functions corresponding to Borel subsets $B_j$ contained in $\text{spt}(u_i)$. In turn, these Borel subsets can be approximated by cubes. This produces an approximation of $\|u_i\|$ by Lipschitz chains with respect to the $\mathbf{M}$-norm in $\mathbb{R}^n$. However, these cubes may leak slightly outside of $B_j$ and in particular outside of $\text{spt}(u_i)$, so that their image in $X$ is not defined. For this reason, we assume that all such maps can be extended to a (small) neighborhood of their domain.
Lemma 1.3. can be deformed into a Lipschitz chain (compare with Lemma 5.7 in [5]).

First prove that if a current $T$, by smartly applying Lemma 1.3 twice, once in dimension $n-1$ and once in dimension $n$, they finally prove the $N$-approximation theorem for $\mathbb{R}^N$. This argument does not translate one to one in our setting, as there is no analogue of the deformation theorem for general metric spaces. The deformation theorem is a powerful result in the classical theory of currents. It provides a way of...
deforming a current into the $n$-skeleton of the cube decomposition of $\mathbb{R}^N$, while keeping uniform bounds on the masses of the currents involved (see Theorem 5.5 in [5], also Theorem 4.2.9 in [4] and Theorem 1, Section 2.6 in [6]).

We solve this issue taking inspiration from De Pauw’s strategy in [3]. We will embed a compact neighborhood $K$ of $\text{spt}(T)$ into $l^\infty(N)$ and exploit the metric approximation property of $l^\infty(N)$ to project it onto a finite dimensional vector subspace, in which we can apply Lemma 1.3.

To be more precise, suppose first that $K$ is contained in an open subset of $X$ having property $L$ and let $\iota: K \to l^\infty(N)$ be an isometric embedding with image $K' := \iota(K)$. By the metric approximation property of $l^\infty(N)$ there is a finite dimensional subspace $V \subset l^\infty(N)$ arbitrarily close to $K'$, and a 1-Lipschitz projection $\pi: K' \to V$. In particular, we can choose $V$ close enough such that the extension of $\iota^{-1}$, say $g$, provided by property $L$ is defined on $\pi(K')$.

Ideally, we could consider the projection $T'' = (\pi \circ \iota)_#T$ of $T$ in $\pi(K')$, apply Federer and Fleming’s N-approximation theorem in $V$ and then map back to $X$ using the extension $g$ of $\iota^{-1}$. The issue is that by doing so we do not have control over the difference between the original current $T$ and $g_#T''$.

This issue can be overcome by using Lemma 1.3 instead of directly applying the approximation result for the $N$-norm, together with the construction of a “homotopy filling” which allows us to control the error produced by $g_#$. (See Section 2.6.)

Similarly to the approximation with respect to the $M$-norm, this argument proves that every current $T \in I_{n,c}(X)$ with support contained in an open subset of $X$ with property $L$ can be approximated by Lipschitz chains with respect to the $N$-norm. As before, assuming that $X$ has local property $L$, we can apply Lemma 1.3 to write an arbitrary current $T \in I_{n,c}(X)$ as $T = T_1 + \cdots + T_k$ where each $T_i \in I_{n,c}(X)$ has support in an open subset of $X$ having property $L$, then using the partial result just described we can prove the following analogue of Theorem 5.8 in [5].

**Theorem 1.4 (N-Approximation).** Let $n \geq 1$, let $X$ be a locally compact metric space with local property $L$, and let $T \in I_{n,c}(X)$. Then for every $\varepsilon > 0$ there is $P \in I_{n,c}(X)$ with $N(T - P) < \varepsilon$ and $\text{spt}(P) \subset \text{spt}(T)_\varepsilon$.

In the case $n = 1$, that is $T \in I_{1,c}(X)$, we can actually reach a stronger conclusion where the approximating chain $P \in I_{1,c}(X)$ not only satisfies $N(T - P) < \varepsilon$ and $\text{spt}(P) \subset \text{spt}(T)_\varepsilon$, but also $\partial P = \partial T$ (see Corollary 4.4).

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2 Preliminaries

Let $X = (X,d)$ be a metric space. We write $B_x(r) := \{y \in X : d(x,y) \leq r\}$ for the closed ball of radius $r \geq 0$ and center $x \in X$.

Given a subset $A \subset X$ and $\varepsilon > 0$ we denote by $A_\varepsilon := \{x \in X : d(A,x) < \varepsilon\}$ the open $\varepsilon$-neighborhood of $A$ in $X$, where $d(A,x)$ is the infimum over all $d(a,x)$ with $a \in A$.

A map $f: X \rightarrow Y$ into another metric space $Y = (Y,d)$ is $L$-Lipschitz, for some constant $L \geq 0$, if $d(f(x),f(x')) \leq Ld(x,x')$ for all $x, x' \in X$. The Lipschitz constant $\text{Lip}(f)$ of $f$ is the infimum over all such $L$. A map $f$ is Lipschitz if it is $L$-Lipschitz for some $L$, it is locally Lipschitz if every point in $X$ has a neighborhood on which $f$ is Lipschitz. We denote by $\text{Lip}_{loc}(X)$ and $\text{Lip}_c(X)$ the spaces of functions $X \rightarrow \mathbb{R}$ which are locally Lipschitz or Lipschitz with compact support, respectively. A map $f$ is a bi-Lipschitz embedding if it is injective and both $f$ and $f^{-1}$ are Lipschitz.

2.1 Metric Currents

Metric currents of finite mass were introduced by Ambrosio and Kirchheim in [1]. Here we will work with a variant of this theory for locally compact metric spaces, as described by Lang in [10]. In this section we provide some background on this theory and refer the reader to [10] for more details. We will assume throughout that the underlying metric space $X$ is locally compact.

For every integer $n \geq 0$ let $\mathcal{D}^n(X) := \text{Lip}_c(X) \times [\text{Lip}_{loc}(X)]^n$, that is, the set of $(n+1)$-tuples $(f,\pi_1,\ldots,\pi_n)$ of real valued functions on $X$ such that $f$ is Lipschitz with compact support and $\pi_1,\ldots,\pi_n$ are locally Lipschitz. We will use $(f,\pi)$ as a shorthand for $(f,\pi_1,\ldots,\pi_n)$. The idea is that $(f,\pi_1,\ldots,\pi_n) \in \mathcal{D}^n(X)$ represents the compactly supported differential $n$-form $fd\pi_1 \wedge \ldots \wedge d\pi_n$ if $X$ is (an open subset of) $\mathbb{R}^N$ and the functions $f,\pi_1,\ldots,\pi_n$ are smooth; and roughly speaking, a current (with some additional properties defined below) is a map $\mathcal{D}^n(X) \rightarrow \mathbb{R}$ representing integration on a submanifold of $\mathbb{R}^N$.

**Definition 2.1.** An $n$-dimensional current $T$ on $X$ is a function $T: \mathcal{D}^n(X) \rightarrow \mathbb{R}$ satisfying the following three properties:

1. (multilinearity) $T$ is $(n+1)$-linear;
2. (continuity) if $f^k \rightarrow f$, $\pi_i^k \rightarrow \pi_i$ pointwise on $X$, $\sup_k \text{Lip}(f^k|_K) < \infty$, $\sup_k \text{Lip}(\pi_i^k|_K) < \infty$ for every compact set $K \subset X$ (for each $i = 1,\ldots,n$) and $\bigcup_k \text{spt}(f^k) \subset K$ for some compact set $K \subset X$, then
   \[T(f^k,\pi_1^k,\ldots,\pi_n^k) \rightarrow T(f,\pi_1,\ldots,\pi_n);\]
3. (locality) $T(f,\pi_1,\ldots,\pi_n) = 0$ whenever one of the functions $\pi_1,\ldots,\pi_n$ is constant on a neighborhood of $\text{spt}(f)$.
The vector space of all $n$-dimensional currents on $X$ is denoted by $\mathcal{D}_n(X)$. Every function $u \in L^1_{loc}(\mathbb{R}^n)$ induces a current $\|u\| \in \mathcal{D}_n(\mathbb{R}^n)$ defined by

$$\|u\| (f, \pi_1, \ldots, \pi_n) := \int u f \det \left( \frac{\partial \pi_i}{\partial x^j} \right)_{i,j=1}^n \, dx = \int u f \det(D\pi) \, dx$$

for all $(f, \pi_1, \ldots, \pi_n) \in \mathcal{D}^n(\mathbb{R}^n)$, where the partial derivatives in the Jacobian $D\pi$ of $\pi = (\pi_1, \ldots, \pi_n)$ exist almost everywhere according to Rademacher’s theorem (see Proposition 2.6 in [10]). This corresponds to the integration of the differential form $fd\pi_1 \wedge \ldots \wedge \pi_n$ over $\mathbb{R}^n$, multiplied by $u$. If $W \subset \mathbb{R}^n$ is a Borel set and $\chi_W$ is its characteristic function, we set $\|W\| := \|\chi_W\|.

2.2 Support, Push-forward, and Boundary

Let $T \in \mathcal{D}_n(X)$ be an $n$-dimensional current. The support $\text{spt}(T)$ of $T$ is the smallest closed subset of $X$ such that the value $T(f, \pi_1, \ldots, \pi_n)$ depends only on the restrictions of $f, \pi_1, \ldots, \pi_n$ to it.

For a proper Lipschitz map $F$: $\text{spt}(T) \rightarrow Y$ into another locally compact metric space $Y$, the push-forward $F_\#T \in \mathcal{D}_n(Y)$ is defined by

$$F_\#(f, \pi_1, \ldots, \pi_n) := T(f \circ F, \pi_1 \circ F, \ldots, \pi_n \circ F)$$

for all $(f, \pi) \in \mathcal{D}(Y)$. It holds that $\text{spt}(F_\#T) \subset F(\text{spt}(T))$.

For $n \geq 1$, the boundary $\partial T \in \mathcal{D}_{n-1}(X)$ of $T$ is defined by

$$(\partial T)(f, \pi_1, \ldots, \pi_{n-1}) = T(\sigma, f, \pi_1, \ldots, \pi_{n-1})$$

for $(f, \pi_1, \ldots, \pi_{n-1}) \in \mathcal{D}^{n-1}(X)$, where $\sigma$ is any compactly supported Lipschitz function, which is identically 1 on $\text{spt}(f) \cap \text{spt}(T)$. It holds that $\partial \circ \partial = 0$, $\text{spt}(\partial T) \subset \text{spt}(T)$, and $F_\#(\partial T) = \partial(F_\#T)$ for $F$ as above. (For more details, see Section 3 in [10].)

2.3 Mass

Let $T \in \mathcal{D}_n(X)$ be an $n$-dimensional current. For an open set $U \subset X$, the mass $\|T\|(U) \in [0, \infty]$ of $T$ in $U$ is defined as the supremum of $\sum_{i=1}^k T(f^i, \pi_1^i, \ldots, \pi_n^i)$ over all finite families $(f^i, \pi_1^i, \ldots, \pi_n^i)_{i=1}^k \subset \mathcal{D}_n(X)$ such that the restrictions of $\pi_1^i, \ldots, \pi_n^i$ to $\text{spt}(f^i)$ are 1-Lipschitz for all $i$, $\bigcup_{i=1}^k \text{spt}(f^i) \subset U$ and $\sum_{i=1}^k |f^i| \leq 1$.

This defines a regular Borel measure $\|T\|$ on $X$. The total mass $\|T\|(X)$ of $T$ is denoted $M(T)$ and is called the mass of $T$. If $S \in \mathcal{D}_n(X)$ is another current, then $\|T + S\| \leq \|T\| + \|S\|$, and in particular

$$M(T + S) \leq M(T) + M(S).$$

For every Borel set $B \subset X$, it is possible to define the restriction $T|_{\mathbb{R}^n} B \in \mathcal{D}_n(X)$ of $T$ to $B$; the measure $\|T|_{\mathbb{R}^n} B\|$ coincides with the restriction $\|T\|_{\mathbb{R}^n} B$ of the measure $\|T\|$. 

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If $\mathbf{M}(T) < \infty$ and $F$: spt($T$) $\rightarrow Y$ is a proper $L$-Lipschitz map into a locally compact metric $Y$, and $B' \subset Y$ is a Borel set, then $(F_\#T) \setminus B' = F_\#(T \setminus F^{-1}(B'))$ and $\|T\#\|(B') \leq L^n\|T\|(F^{-1}(B'))$, in particular,

$$\mathbf{M}(F_\#T) \leq L^n \mathbf{M}(T).$$

The normal mass of $T$ in $U \subset X$ is defined as $\|T\|(U) + \|\partial T\|(U)$ and the normal mass of $T$ as $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T)$. We say that $T$ is normal if $\mathbf{N}(T) < \infty$.

The real vector space of all $T \in \mathcal{D}_n(X)$ with finite mass is denoted by $\mathbf{M}_n(X)$, and the subspace consisting of currents with compact support is denoted by $\mathbf{M}_{n,c}(X)$; they are both normed vector spaces when endowed with the mass norm $\mathbf{M}$. Similarly, $\mathbf{N}_n(X)$ and $\mathbf{N}_{n,c}(X)$ denote the vector spaces of normal currents and normal currents with compact support, respectively.

If $u \in L^1(\mathbb{R}^n)$, then $\mathbf{M}([u]) = \|u\|_{L^1}$, in particular, if $K \subset \mathbb{R}^n$ is a Borel set, then $\mathbf{M}([K]) = \mathcal{L}^n(K)$ and $\partial [K]$ has finite mass $\mathbf{M}(\partial [K])$ whenever $\chi_K$ has finite variation $V(\chi_K)$. (For more details, see Sections 2, 4 and 7 of [10].)

### 2.4 Integral Currents

A subset $E \subset X$ is countably $n$-rectifiable if there are countably many Lipschitz maps $F_i: A_i \rightarrow X$, $A_i \subset \mathbb{R}^n$, such that $E \subset \bigcup_i F_i(A_i)$. The set $E \subset X$ is countably $\mathcal{H}^n$-rectifiable if there is a countably $n$-rectifiable set $E' \subset X$ such that $\mathcal{H}^n(E \setminus E') = 0$, where $\mathcal{H}^n$ is the $n$-dimensional Hausdorff measure on $X$.

A current $T \in \mathbf{M}_n(X)$ is called integer rectifiable if $\|T\|$ is concentrated on some $\mathcal{H}^n$-rectifiable Borel set $E \subset X$ and the following integer multiplicity condition holds: for every Borel set $B \subset X$ with compact closure and for every Lipschitz map $\pi_i: X \rightarrow \mathbb{R}^n$ the push-forward $\pi_\#(T \setminus B) \in \mathcal{D}_n(\mathbb{R}^n)$ is of the form $[u]$, for some integer valued $u = u_{B,\pi} \in L^1(\mathbb{R}^n)$. The abelian group of integer rectifiable $n$-currents in $X$ is denoted by $\mathcal{I}_n(X)$; it is closed under push-forwards and restrictions to Borel sets. We write $\mathcal{I}_{n,c}(X)$ for the subgroup of integer rectifiable currents with compact support.

A current $T \in \mathcal{I}_{n,c}(X)$ is an integral current with compact support, or simply an integral current, if whenever $n \geq 1$, its boundary $\partial T$ is integer rectifiable as well. We denote the corresponding abelian groups by $\mathcal{I}_{n,c}(X)$, and observe that they form a chain complex. It is worth mentioning that by Theorem 8.7 (boundary rectifiability) in [10], $T \in \mathcal{I}_{n,c}(X)$ is integral if $\mathbf{M}(\partial T) < \infty$ (that is, $T$ is normal), equivalently, $\mathcal{I}_{n,c}(X) = \mathcal{I}_{n,c}(X) \cap \mathbf{N}_{n,c}(X)$.

If $K \subset \mathbb{R}^n$ is a bounded Borel set, then $[K]$ is an element of $\mathcal{I}_{n,c}(\mathbb{R}^n)$, and it is in $\mathcal{I}_{n,c}(\mathbb{R}^n)$ whenever $\chi_K$ has finite variation $V(\chi_K)$.

An integral current $T$ is a cycle whenever $\partial T = 0$ and we denote by $\mathcal{Z}_{n,c}(X) \subset \mathcal{I}_{n,c}(X)$ the subgroup of integral cycles. An element of $\mathcal{I}_{0,c}(X)$ is an integer linear combination of currents of the form $[x]$, where $x \in X$ and $[x](f) = f(x)$ for all compactly supported Lipschitz functions $f \in \mathcal{D}(X)$. In this case $\mathcal{Z}_{n,c}(X) \subset \mathcal{I}_{0,c}(X)$ denotes the subgroup of integer linear combinations whose coefficients sum to zero. Note that $\partial: \mathcal{I}_{n,c}(X) \rightarrow \mathcal{I}_{n-1,c}(X)$ for
all $n \geq 1$, and if $F: X \to Y$ is a proper Lipschitz map into a locally compact metric space $Y$, then the push-forward $F_{\#}$ maps $I_{n,c}(X)$ to $I_{n,c}(Y)$ and $Z_{n,c}(X)$ to $Z_{n,c}(Y)$. Given $Z \in Z_{n,c}(X)$ we call a current $V \in I_{n+1,c}(X)$ a filling of $Z$ if $\partial V = Z$. (For more details, see Section 8 of [10].)

In general the restriction $T|_B$ of an integral current $T$ to an arbitrary Borel subset $B \subset X$ is not integral but only integer rectifiable. However, as a special case of a more general construction, one can show that for $T \in I_{n,c}(X)$ and $x \in X$, the restriction $T|_B(x)$ is in $I_{n,c}(X)$ for almost every $r \geq 0$ (see Section 6 and Theorem 8.5 in [10], and Section 2.6 in [8]).

2.5 Polyhedral and Lipschitz Chains

An $n$-dimensional polyhedron $K$ in $\mathbb{R}^n$, such as a (hyper-)cube or an $n$-simplex, is the convex hull of finitely many (non coplanar) points in $\mathbb{R}^n$. As remarked before $\|K\|$ is in $I_{n,c}(\mathbb{R}^n)$ and one can show that $M(\partial \|K\|) = V(\partial K) = \mathcal{H}^{n-1}(\partial K) < \infty$, so that in fact $\|K\|$ is in $I_{n,c}(\mathbb{R}^n)$.

A polyhedral $n$-chain in $\mathbb{R}^n$ is a finite sum of the form $P = \sum_{i=1}^t a_i [D_i]$, where $a_i \in \mathbb{Z}$, and $D_i \subset \mathbb{R}^n$ are $n$-dimensional polyhedra.

A Lipschitz $n$-chain in $X$ is a finite sum of the form

$$L = \sum_{i=1}^t a_i (\varphi_i)_{\#}[D_i],$$

where $a_i \in \mathbb{Z}$, $D_i \subset \mathbb{R}^n$ are $n$-dimensional polyhedra and $\varphi_i: D_i \to X$ are Lipschitz maps.

We denote by $P_{n,c}(\mathbb{R}^n)$ and $L_{n,c}(X)$ the abelian groups of polyhedral $n$-chains in $\mathbb{R}^n$ and Lipschitz $n$-chains in $X$. Rearranging the polyhedra $D_i$ in the definition, one can show that every Lipschitz $n$-chain $L \in L_{n,c}(X)$ can be written as the push-forward $L = \varphi_{\#}P$ of some polyhedral $n$-chain $P \in P_{n,c}(\mathbb{R}^n)$.

In $\mathbb{R}^n$ every polyhedral $n$-chain is a Lipschitz chain and in general every Lipschitz chain is an integral current, that is, $P_{n,c}(\mathbb{R}^n) \subset L_{n,c}(\mathbb{R}^n)$ and $L_{n,c}(X) \subset I_{n,c}(X)$. It is possible to define polyhedral $m$-chains in $\mathbb{R}^n$ also for $m < n$ but for our purposes it suffices to consider them as Lipschitz $m$-chains in $\mathbb{R}^n$.

There is a chain isomorphism $I_{n,c}(\mathbb{R}^n) \to I_{n,c}^{LF}(\mathbb{R}^n)$ between (metric) integral currents in $\mathbb{R}^n$ and "classical" Federer-Fleming integral currents of [5] which is bi-Lipschitz with respect to the $\mathcal{M}$-norm with constants depending only on the dimensions, and which restricts to an isomorphism between the respective subchains of polyhedral and Lipschitz chains (see Theorem 5.5 in [10]). In particular we can apply Lemma 1.3 to metric integral currents in $I_{n,c}(\mathbb{R}^n)$.

Finally, note that all 0-dimensional integral currents are by definition Lipschitz chains, that is, $I_{0,c}(X) = L_{0,c}(X)$, and therefore an approximation theorem for the $\mathcal{N}$-norm is not necessary in dimension 0.
2.6 Homotopies

We recall a useful technique to produce fillings of cycles. Let $X, Y$ be locally compact metric spaces and $H : [0, 1] \times X \rightarrow Y$ a Lipschitz homotopy between the Lipschitz maps $f, g : X \rightarrow Y$. Given $T \in \mathbb{Z}_{n,c}(X)$, $n \geq 1$, one can construct the product current $[0, 1] \times T \in \mathcal{I}_{n+1,c}([0, 1] \times X)$ and take the push-forward with respect to $H$. This produces the current $H_#([0, 1] \times T) \in \mathcal{I}_{n+1,c}(Y)$ with support in $H([0, 1] \times \text{spt}(T)) \subset Y$ and boundary

$$\partial H_#([0, 1] \times T) = g_#T - f_#T,$$

where $f_#T$ and $g_#T$ are in $\mathcal{I}_{n,c}(Y)$.

If $H(t, \cdot) \big|_{\text{spt}(T)} : \text{spt}(T) \rightarrow Y$ is $L$-Lipschitz for all $t \in [0, 1]$ and $H(\cdot, x) : [0, 1] \rightarrow Y$ is a geodesic of length at most $D$ for all $x \in \text{spt}(T)$, then

$$M(H_#([0, 1] \times T)) \leq (n + 1) L^n D M(T).$$

For more details see Theorem 2.9 in [13] and Section 2.7 of [8].

Lipschitz chains are closed under products and push-forwards so that $[0, 1] \times P$ is in $\mathcal{L}_{n+1,c}([0, 1] \times X)$ and $H_#([0, 1] \times P)$ is in $\mathcal{L}_{n+1,c}(Y)$ whenever $P$ is an element of $\mathcal{L}_{n,c}(X)$.

We describe a special case of this construction. Let $Y$ denote a normed vector space and $K \subset Y$ a compact subset. Let $\varphi, \psi : K \rightarrow Y$ be $L$-Lipschitz maps with $|\psi(x) - \varphi(x)| \leq D$ for all $x \in K$, and consider the affine homotopy $H : [0, 1] \times K \rightarrow Y$ from $\varphi$ to $\psi$, that is, $H(t, x) := t\psi(x) + (1 - t)\varphi(x)$. Then, if $P$ is an element of $\mathcal{L}_{n,c}(K)$ with $\partial P = 0$, the push-forward $H_#([0, 1] \times P)$ is in $\mathcal{L}_{n+1,c}(Y)$ has support contained in $H([0, 1] \times K)$ and satisfies

$$\partial H_#([0, 1] \times P) = \psi_#P - \varphi_#P,$$

$$M(H_#([0, 1] \times P)) \leq (n + 1) L^n D M(P).$$

We call $H_#([0, 1] \times P)$ the affine (homotopy) filling of $\psi_#P - \varphi_#P$.

2.7 Finite Dimensional Projections

As mentioned in the introduction, in order to exploit the deformation theorem we project integral currents defined on a metric space $X$ into a finite dimensional vector space. This is done in two steps.

First, every metric space $X$ embeds isometrically into the Banach space $l^\infty(X)$ of bounded maps on $X$ via the map $x \mapsto d(x, \cdot) - d(x_0, \cdot)$, for any base point $x_0 \in X$. If $X$ is compact, or more generally separable, and $(x_i)_{i \in \mathbb{N}} \subset X$ is a countable dense subset, then $x \mapsto (d(x_i, x))_{i \in \mathbb{N}}$ is an isometric embedding into $l^\infty(\mathbb{N})$, and the second term $d(x_i, x_0)$ is not necessary if $X$ is bounded. This allows us to embed a compact neighborhood of the support of $T \in \mathcal{I}_{n,c}(X)$ into $l^\infty(\mathbb{N})$.

Then, we find a finite dimensional subspace of $l^\infty(\mathbb{N})$ which is “close enough” to the image of the embedding. Recall that a Banach space $V$ has the bounded
**approximation property** if there exists \( \lambda \geq 1 \) such that the following holds. For every compact subset \( K \subset V \) and \( \varepsilon > 0 \) there is a finite dimensional vector subspace \( V' \leq V \) and a \( \lambda \)-Lipschitz map \( \pi: K \rightarrow V' \) satisfying \( |\pi(x) - x| \leq \varepsilon \) for all \( x \in K \). We say that \( V \) has the metric approximation property in the case \( \lambda = 1 \). Conveniently, \( l^\infty(\mathbb{N}) \) has this property.

**Proposition 2.2.** \( l^\infty(\mathbb{N}) \) has the metric approximation property.

For a detailed proof of this fact we refer to Theorem A.6 in [3]. In the next section we discuss the property needed to go back from the finite dimensional subspace of \( l^\infty(\mathbb{N}) \) to \( X \).

### 3 Lipschitz Extensions

We compare the Lipschitz extension property mentioned in the introduction with other similar properties found in the literature. In this section we do not assume \( X \) to be locally compact, unless otherwise specified. We recall the definition.

**Definition 3.1.** A metric space \( X \) has property \( L \) if the following holds. For every metric space \( Y \), every compact subset \( K \subset Y \), and every \( 1 \)-Lipschitz map \( g: K \rightarrow X \), there exist \( \varepsilon = \varepsilon(g) > 0 \) and \( L = L(g) \geq 1 \) such that \( g \) admits an \( L \)-Lipschitz extension \( \pi: K_{\varepsilon} \rightarrow X \). We say that \( X \) has **local property \( L \)** if every point in \( X \) has a neighborhood with property \( L \).

**Lemma 3.2.** Let \( X \) be a locally compact metric space with property \( L \). Then \( X \) is semi-locally quasi-convex, that is, for every point \( o \in X \) there are constants \( r = r(o) > 0 \) and \( L = L(o) \geq 1 \) such that any two points \( x, y \in B_o(r) \) are joined by a curve of length \( \leq Ld(x,y) \) contained in \( B_o(2Lr) \).

Suppose that \( X \) is a locally compact metric space with local property \( L \). Then each point has an open neighborhood \( U \) which is locally compact and has property \( L \) and hence this lemma implies that \( X \) is semi-locally quasi-convex.

**Proof.** Let \( o \in X \) and take \( \delta > 0 \) small enough such that \( K := B_o(\delta) \) is compact. Consider the isometric embedding \( \iota: K \rightarrow l^\infty(\mathbb{N}) \) with image \( K' := \iota(K) \). By assumption there exist \( \varepsilon > 0, L \geq 1 \) and an \( L \)-Lipschitz extension \( g: K'_{\varepsilon} \rightarrow X \) of \( \iota^{-1}: K' \rightarrow X \).

Let \( r := \min\{\frac{\varepsilon}{2}, \delta\} \) and consider the possibly smaller ball \( B_o(r) \). For \( x, y \in B_o(r) \) let \( \gamma: [0, 1] \rightarrow l^\infty(\mathbb{N}) \) be the straight segment \( \gamma(t) := \iota(x)+t(\iota(y)-\iota(x)) \) from \( \iota(x) \) to \( \iota(y) \) of length \( |\iota(x)-\iota(y)| = d(x,y) \leq 2r \). The image of \( \gamma \) is within distance at most \( r \) from \( \{\iota(x), \iota(y)\} \subset K' \) and hence contained in \( K'_{\varepsilon} \). Thus \( g \circ \gamma: [0, 1] \rightarrow X \) is a curve from \( x \) to \( y \) of length at most \( Ld(x,y) \) and contained in \( B_o(r + Lr) \subset B_o(2Lr) \).

We now explore stronger properties.
Definition 3.3. A metric space $X$ is an absolute 1-Lipschitz retract if whenever $\iota: X \to Y$ is an isometric embedding into a metric space $Y$, there is a 1-Lipschitz retraction $\phi: Y \to \iota(X)$. A metric space $X$ is injective if for every metric space $B$ and every subset $A \subset B$, every 1-Lipschitz map $f: A \to X$ admits a 1-Lipschitz extension $\tilde{f}: B \to X$.

Basic examples of injective metric spaces are $\mathbb{R}$ and the Banach space $l^\infty(J)$ of bounded functions on a set $J$. Exploiting the isometric embedding of $X$ into the injective space $l^\infty(X)$, one can prove the following equivalence (see for example Proposition 2.2 in [9]).

Proposition 3.4. A metric space $X$ is injective if and only if it is an absolute 1-Lipschitz retract.

Since every separable metric space, in particular every compact metric space, embeds isometrically into $l^\infty(\mathbb{N})$, the same argument used in this proof can be applied to show that property $L$ is equivalent to the following apparently weaker properties.

(1) For every compact subset $K \subset l^\infty(\mathbb{N})$ and every 1-Lipschitz map $g: K \to X$, there exist $\varepsilon = \varepsilon(g) > 0$ and $L = L(g) \geq 1$ such that $g$ admits an $L$-Lipschitz extension $\tilde{g}: K_{\varepsilon} \to X$.

(2) For every Banach space $V$, every compact subset $K \subset V$, and 1-Lipschitz map $g: K \to X$, there exist $\varepsilon = \varepsilon(g) > 0$ and $L = L(g) \geq 1$ such that $g$ admits an $L$-Lipschitz extension $\tilde{g}: K_{\varepsilon} \to X$.

All injective metric spaces, equivalently all absolute 1-Lipschitz retracts, have property $L$. In fact all metric spaces for which every compact subset is contained in an injective subspace, have property $L$. The converse is not necessarily true as the Lipschitz extension of property $L$ possibly has a worse Lipschitz constant and is defined only on a neighborhood of its original domain.

We say that a metric space is an absolute Lipschitz retract (ALR) if the retraction $\phi: Y \to \iota(X)$ in the definition above is allowed to be Lipschitz (instead of 1-Lipschitz). Also in this case $X$ has property $L$ if it is an ALR (or all its compact subsets are contained in one).

Lang and Schlichenmaier [11] provide an instance in which $X$ is an ALR and so has property $L$ (see Corollary 1.8 in [11]).

Theorem 3.5. Suppose that $X$ is a metric space with finite Nagata dimension $\dim_{\mathcal{N}}(X) \leq n < \infty$. Then $X$ is an ALR if and only if $X$ is complete and Lipschitz $n$-connected.

Recall that a family $\mathcal{B}$ of subsets of $X$ is $D$-bounded, for some $D \geq 0$, if the diameter of all subsets in $\mathcal{B}$ is uniformly bounded by $D$. For $s > 0$, its $s$-multiplicity is the infimum over all integers $n \geq 0$ such that each subset of $X$ with diameter $\leq s$ meets at most $n$ members of $\mathcal{B}$. The Nagata dimension $\dim_{\mathcal{N}}(X)$ of $X$ is the infimum over all integers $n$ with the following property:
there exists a constant \( c > 0 \) such that for all \( s > 0 \), \( X \) has a \( cs \)-bounded covering with \( s \)-multiplicity at most \( n + 1 \). For example, every doubling metric space has finite Nagata dimension. (See Section 2 of [11].)

For an integer \( m \geq 0 \) denote by \( S^m \) and \( B^{m+1} \) the unit sphere and closed ball in \( \mathbb{R}^{m+1} \), endowed with the induced metric. A metric space \( X \) is \textit{Lipschitz \( n \)-connected} for some integer \( n \geq 0 \) if there is a constant \( \gamma \) such that for every \( m \in \{0, 1, \ldots, n\} \), every \( \lambda \)-Lipschitz map \( f: S^m \to X \), \( \lambda > 0 \), admits a \( \gamma \lambda \)-Lipschitz extension \( \overline{f}: B^{m+1} \to X \).

We now consider Lipschitz extensions which are possibly not defined on the whole ambient space of their domain.

**Definition 3.6.** A metric space \( X \) is an \textit{absolute Lipschitz neighborhood retract} (ALNR) if whenever \( \iota: X \to Y \) is an isometric embedding into a metric space \( Y \), there exist a neighborhood \( W \) of \( \iota(X) \) in \( Y \) and a Lipschitz retraction \( \pi: W \to \iota(X) \).

If the neighborhood \( W \) is of the form \( \iota(X) \varepsilon \) for some \( \varepsilon > 0 \) we say that \( X \) is an \textit{absolute Lipschitz uniform neighborhood retract} (ALUNR).

Using the injectivity of \( l^\infty(X) \) one can show that if \( X \) is an ALNR, then there exists \( \varepsilon_0 \) such that the neighborhood \( W \) is of the form \( W = \iota(X)\varepsilon_0 \) for every metric space \( Y \) and every isometric embedding isometric embedding \( \iota: X \to Y \). Similarly to Proposition 3.4 the following equivalence holds.

**Proposition 3.7.** A metric space \( X \) is an ALNR if and only if for every metric space \( B \) and every subset \( A \subset B \), every Lipschitz map \( g: A \to X \) admits a Lipschitz extension \( \overline{g}: U \to X \) to a neighborhood \( U \) of \( A \) in \( B \).

An analogous statement holds for ALUNR instead of ALNR. Also in this case \( X \) has property \( L \) if it is an ALNR or even if every compact set is contained in an ALNR subset of \( X \).

In general, the opposite implication is not true because property \( L \) extends only Lipschitz maps whose domains are compact sets. Also, to prove that a subset \( X' \subset X \) is an ALNR one needs to consider Lipschitz maps into \( X' \), but the images of their extensions provided by property \( L \) might land outside \( X' \).

Another argument is that if \( X' \subset X \) is an ALNR, then by definition there exists an open neighborhood \( W \subset X \) of \( X' \) and a Lipschitz retraction \( \phi: W \to X' \). This is a property that spaces with property \( L \) generally do not have.

From the point of view of maps within \( X \), property \( L \) does not seem to be restrictive in the sense that if \( X \) has property \( L \), then for each compact subset \( K \subset X \) the inclusion extends to a Lipschitz map \( W \to X \) for some neighborhood \( W \) of \( K \) in \( X \). This map might as well be the identity \( X \to X \), so all metric spaces satisfy this property.

In the context of continuous extensions of continuous maps, ANRs are defined similarly to ALNRs where the maps involved are continuous and not necessarily Lipschitz. If every \( x \in X \) has an ANR neighborhood, then \( X \) is an ANR (see for instance Theorem 3.2 in [7]). A local to global result of this type does
not hold for ALRs as Example 3.9 below shows, and we now indicate why it is not expected for property L since generally it is not possible to glue Lipschitz maps.

Let $K$ be a compact subset of a metric space $Y$ and let $h: K \to X$ be a 1-Lipschitz map with image $h(K) \subset U \cup V$, where both $U$ and $V$ are open subsets of $X$ with property $L$. Suppose that we can write $K = K_1 \cup K_2$ where $K_1$ and $K_2$ are overlapping compact subsets with $h(K_1) \subset U$ and $h(K_2) \subset V$. By assumption we can find Lipschitz extensions $\tilde{f}: (K_1)_{\varepsilon} \to X$, and $\tilde{g}: (K_2)_{\varepsilon} \to X$ of $f := h|_{K_1}$ and $g := h|_{K_2}$, respectively. The issues here are that the extensions might differ on $(K)_{\varepsilon} \setminus (K_1 \cap K_2)$, and that $\tilde{f}, \tilde{g}$ need not coincide with $h$ on $(K_1)_{\varepsilon} \cap K$ and on $(K_2)_{\varepsilon} \cap K$, respectively. The following example shows that gluing two Lipschitz functions agreeing on some subset of their respective domain in general does not yield a Lipschitz function (compare with Section 2.5 of [1]).

**Example 3.8.** Let

$$C := \{(t,0) : t \in [0,1]\} \subset \mathbb{R}^2,$$

$$D := \{(t,t^2) : t \in [0,1]\} \subset \mathbb{R}^2,$$

and define $f: C \cup D \to \mathbb{R}$ as $f(t,0) := 0$, $f(t,t^2) := t$. Then $f|_C$ and $f|_D$ coincide in $(0,0)$ and are both 1-Lipschitz, but $f$ itself is not. Indeed, if it were $L$-Lipschitz for some $L \geq 1$, then

$$t = d(f(t,t^2), f(t,0)) \leq Ld((t,t^2), (t,0)) = Lt^2,$$

a contradiction.

We conclude with an example illustrating some of the properties listed above (see Example 4.2 in [12]).

**Example 3.9.** Consider the unit sphere $S^1$ equipped with the induced metric $d$ and the inner metric $d'$. They are bi-Lipschitz homeomorphic to each other, they are not simply-connected and $(S^1, d)$ is not geodesic. According to Lemma 4.3 in [12], an absolute 1-Lipschitz uniform neighborhood retract is geodesic and simply-connected, hence in this case both spaces do not have this property.

Embedding $(S^1, d)$ into $\mathbb{R}^2$ isometrically we see that both spaces are not ALRs, but the radial projection from a neighborhood of $(S^1, d)$ is Lipschitz, thus showing that both spaces are ALUNRs and so have property $L$.

Every point in $(S^1, d')$ has a neighborhood isometric to $(-\frac{\pi}{2}, \frac{\pi}{2})$, which is an absolute 1-Lipschitz retract, but the whole space is not even an ALR.
4 N-Approximation

We begin this section with the decomposition lemma already mentioned and used in the introduction to prove Proposition 1.2 (M-Approximation).

Lemma 4.1. Let \( n \geq 1 \), and let \( X \) be locally compact metric space with local property \( L \). Every \( T \in I_{n,c}(X) \) admits a decomposition \( T = T_1 + \cdots + T_k \) with \( T_1 \in I_{n,c}(X) \) such that each \( \text{spt}(T_i) \) is contained in \( \text{spt}(T) \) and has a neighborhood with property \( L \). Suppose in addition that \( T \in I_{n,c}(X) \), then each \( T_i \in I_{n,c}(X) \) as well.

Proof. Suppose that \( T \in I_{n,c}(X) \); the argument for \( T \in I_{n,c}(X) \) is simpler but the one presented here applies as well. By assumption there exist finitely many points \( x_1, \ldots, x_k \in \text{spt}(T) \) and radii \( r_1, \ldots, r_k > 0 \) such that \( \text{spt}(T) \subset \bigcup_{i=1}^{k} B_{x_i}(\frac{r_i}{2}) \) and each \( B_{x_i}(r_i) \) has a neighborhood with property \( L \).

Take \( s_i \in (\frac{r_i}{2}, r_1) \) such that \( T_1 := T \cup B_{x_i}(s_i) \in I_{n,c}(X) \), then \( \text{spt}(T_1) \subset \text{spt}(T) \). Then \( T_1 = T \cup (X \setminus B_{x_i}(s_i)) \in I_{n,c}(X) \) has support in \( \text{spt}(T) \) and covered by \( \bigcup_{i=2}^{k} B_{x_i}(\frac{r_i}{2}) \). Then proceed analogously for \( r_2, \ldots, r_k \).

We now prove a version of the N-Approximation Theorem for an integral current whose boundary is already a Lipschitz chain. We will then upgrade it to Theorem 1.4 (N-Approximation) by first showing that property \( L \) is enough to fill arbitrarily small cycles in \( X \) (in an appropriate sense).

Proposition 4.2. Let \( n \geq 1 \), let \( X \) be a locally compact metric space, and let \( U \subset X \) be an open subset with property \( L \). Let \( T \in I_{n,c}(X) \) with \( \partial T \in I_{n-1,c}(X) \) and \( \text{spt}(T) \subset U \). Then for every \( \varepsilon > 0 \) there is \( R \in I_{n,c}(X) \) with \( M(T-R) < \varepsilon \), \( \partial T = \partial R \) and \( \text{spt}(R) \subset \text{spt}(T)_\varepsilon \), in particular \( N(T-R) < \varepsilon \).

Since 0-dimensional cycles in \( X \) are by definition Lipschitz chains, that is, \( Z_{0,c}(X) \subset I_{0,c}(X) \), it follows that any \( T \in I_{1,c}(X) \) automatically satisfies the assumptions of this proposition.

Proof. (See also Figure 1 for a schematic diagram of the proof.)

Let \( K \) denote the closed \( \frac{\varepsilon}{2} \)-neighborhood of \( \text{spt}(T) \) in \( X \), without loss of generality we might assume that \( K \) is compact and that \( \text{spt}(T)_\varepsilon \subset U \). Let \( \iota: K \to l^\infty(\mathbb{N}) \) be an isometric embedding with compact image \( K' := \iota(K) \).

By property \( L \) there exist \( \varepsilon_0 > 0 \), \( L \geq 1 \) and an \( L \)-Lipschitz extension

\[ g: K_{\varepsilon_0} \to X \]

of \( \iota^{-1} = g|_{K'} \) to the open \( \varepsilon_0 \)-neighborhood of \( K' \) in \( l^\infty(\mathbb{N}) \). According to Proposition 1.2 (M-Approximation) we find \( P \in I_{n,c}(X) \) with \( \text{spt}(P) \subset \text{spt}(T)_{\varepsilon/2} \subset K \) and \( M(T-P) < \varepsilon/2^n < \varepsilon \).

By the metric approximation property of \( l^\infty(\mathbb{N}) \), Proposition 2.2 there is a finite dimensional subspace \( V \subset l^\infty(\mathbb{N}) \) and a 1-Lipschitz projection \( \pi: l^\infty(\mathbb{N}) \to V \), such that \( |x - \pi(x)| \leq \frac{\delta}{2} \) for all \( x \in K' \), where

\[ \delta := \min \left\{ \frac{\varepsilon_0}{2}, \frac{\varepsilon}{3nL^n M(\partial T - \partial P)}, \frac{\varepsilon}{4L} \right\}. \]
in particular, \( K' \subseteq \pi(K') \subseteq K_{\varepsilon/2} \cap K'_{\varepsilon/(4L)}. \)

Now, consider

\[
T' \triangleq \iota_g T \in \mathbf{I}_{n,c}(l^\infty(N)), \quad T'' \triangleq \pi_g T' \in \mathbf{I}_{n,c}(V),
\]

\[
P' \triangleq \iota_g P \in \mathbf{I}_{n,c}(l^\infty(N)), \quad P'' \triangleq \pi_g P' \in \mathbf{I}_{n,c}(V).
\]

Note that \( \partial T' \in \mathbf{L}_{n-1,c}(l^\infty(N)), \partial T'' \in \mathbf{L}_{n-1,c}(V), \mathbf{M}(T'' - P'') \leq \mathbf{M}(T'' - P') = \mathbf{M}(T - P) \), the supports of \( T', P' \) are contained in \( K' \) and the supports of \( T'', P'' \) are contained in \( K'' \).

Let \( H : [0, 1] \times K' \to l^\infty(N) \) denote the affine homotopy between id_{K'} and \( \pi_{K''} \), and let \( W \triangleq H_g ([0, 1] \times (\partial T' - \partial P')) \in \mathbf{L}_{n,c}(l^\infty(N)) \) be the affine filling of \( (\partial T' - \partial P') \) as defined in Section 2.6. Note that \( H(t, \cdot) : K' \to l^\infty(N) \) is 1-Lipschitz for all \( t \in [0, 1] \) and \( H(\cdot, x) : [0, 1] \to l^\infty(N) \) has length at most \( \delta/2 \) for all \( x \in K' \). Therefore the support \( \text{spt}(W) \) of \( W \) is contained in \( K'_{\delta} \subseteq K'_{\varepsilon/2} \cap K'_{\varepsilon/(4L)} \) and its mass is bounded by

\[
\mathbf{M}(W) \leq \frac{\delta}{2} \mathbf{M}(\partial T' - \partial P') \leq \frac{\epsilon}{6L^n}.
\]

As \( \partial(T'' - P'') = \partial T'' - \partial P'' \in \mathbf{L}_{n-1,c}(V) \), by Lemma 1.3 we find \( S \in \mathbf{I}_{n+1,c}(V) \) satisfying

\[
\mathbf{N}(S) \leq \eta := \min \left\{ \frac{\varepsilon_0}{2}, \frac{\epsilon}{6L^n} \right\} < \frac{\epsilon}{4L},
\]

\[
\text{spt}(S) \subseteq \text{spt}(T'' - P'')_\eta \subseteq K''_\eta \subseteq K''_{\eta + \delta} \subseteq K_{\varepsilon_0} \cap K'_{\varepsilon/(2L)},
\]

\[
T'' - P'' - \partial S \in \mathbf{L}_{n,c}(V),
\]

(in fact, \( \text{spt}(S) \) is contained in the open \( \eta \)-neighborhood of \( \text{spt}(T'' - P'') \) in \( V \)).

Finally, note that \( T'' - P'' - \partial S \) and \( W \) are both Lipschitz \( n \)-chains with supports in \( K'_{\varepsilon_0} \cap K'_{\varepsilon/(2L)} \), so that \( g_\#(T'' - P'' - \partial S - W) \in \mathbf{L}_{n,c}(X) \) is well defined, has support in \( g(K'_{\varepsilon/(2L)}) \subseteq K_{\varepsilon/2} \), mass

\[
\mathbf{M}(g_\#(T'' - P'' - \partial S - W)) \leq L^n \left( \mathbf{M}(T'' - P'') + \mathbf{M}(\partial S) + \mathbf{M}(W) \right) \leq \frac{\epsilon}{2}
\]

and boundary

\[
\partial(g_\#(T'' - P'' - \partial S - W)) = g_\#(\partial T'' - \partial P'' - (\partial T'' - \partial P'' - \partial T' + \partial P'))
\]

\[
= g_\#(\partial T' - \partial P')
\]

\[
= \partial T - \partial P,
\]

where in the last equality we have used that \( g_{|K'} = \iota^{-1} \). Overall, the Lipschitz \( n \)-chain

\[
R := P + g_\#(T'' - P'' - \partial S - W) \in \mathbf{L}_{n,c}(X)
\]

satisfies \( \mathbf{M}(T - R) \leq \mathbf{M}(T - P) + \mathbf{M}(R - P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \partial R = \partial P + \partial T - \partial P = \partial T \) and \( \text{spt}(R) \subseteq \text{spt}(P) \cup K_{\varepsilon/2} \subseteq \text{spt}(T)_\varepsilon. \)
Notably, this proposition implies a version of Theorem 1.4 (N-Approximation) for cycles.

**Corollary 4.3.** Let $n \geq 1$, let $X$ be a locally compact metric space, and let $U \subset X$ be an open subset with property $L$. Then for every $Z \in \mathbf{Z}_{n,c}(X)$ with $\text{spt}(Z) \subset U$, and every $\varepsilon > 0$, there is $R \in \mathbf{L}_{n,c}(X)$ with $\partial R = 0$, $N(Z - R) < \varepsilon$, and $\text{spt}(R) \subset \text{spt}(Z)_{\varepsilon}$.

If we assume that $X$ has local property $L$ we are not able to prove statements like Proposition 4.2 and Corollary 4.3 for integral currents in $X$ whose boundary is a Lipschitz chain but without restrictions on their supports. This is because the decomposition $T = T_1 + \cdots + T_k$ of Lemma 4.1 does not preserve some properties of the boundary. More specifically, even if $\partial T \in \mathbf{L}_{n-1,c}(X)$ or $\partial T = 0$, it might happen that $\partial T_i \not\in \mathbf{L}_{n-1,c}(X)$ or $\partial T_i \neq 0$ for some $i$.

Nonetheless, if $n = 1$ then it does holds that $\partial T_i \in \mathbf{Z}_{0,c}(X) \subset \mathbf{L}_{0,c}(X)$ for all $i$ and hence we can apply Proposition 4.2 to each component to obtain $R_1, \ldots, R_k \in \mathbf{L}_{1,c}(X)$ with $M(T_i - R_i) < \varepsilon/k$, $\partial R_i = \partial T_i$, and $\text{spt}(R_i) \subset \text{spt}(T_i)_{\varepsilon}$ for all $i$. This implies a stronger version of Theorem 1.4 for $n = 1$.

**Corollary 4.4.** Let $X$ be a locally compact metric space with local property $L$. Then for every $T \in \mathbf{I}_{1,c}(X)$ and $\varepsilon > 0$ there exists $R \in \mathbf{L}_{1,c}(X)$ with $\partial P = \partial T$, $N(T - P) < \varepsilon$ and $\text{spt}(P) \subset \text{spt}(T)_{\varepsilon}$.

In particular, $\partial P = 0$ whenever $T \in \mathbf{Z}_{1,c}(X)$.

As in the proof of Proposition 4.2 given an open subset $U$ of $X$ with property $L$ and a compact subset $K \subset U$, we can consider the isometric embedding $\iota: K \rightarrow l^\infty(\mathbb{N})$ and the Lipschitz extension $g: K'_\varepsilon \rightarrow X$ of $\iota^{-1} = g|_K$. If $Z \in \mathbf{Z}_{n,c}(X)$ has support in $K$ we can fill $\iota g Z$ in $l^\infty(\mathbb{N})$ and if $M(Z)$ is small enough we expect to be able to push the filling back into $X$ using $g$. The next proposition establishes this result precisely.

**Proposition 4.5.** Let $n \geq 1$, let $X$ be a locally compact metric space, and let $U \subset X$ be an open subset with property $L$. Then for every compact subset $K \subset U$ and $\varepsilon > 0$ there exists $M > 0$ such that every $Z \in \mathbf{Z}_{n,c}(X)$ with $\text{spt}(Z) \subset K$ and $M(Z) < M$ possesses a filling $S \in \mathbf{I}_{n+1,c}(X)$ with $\text{spt}(S) \subset \text{spt}(Z)_{\varepsilon}$ and $M(S) < \varepsilon$.

The proof is similar to that of Proposition 4.2 in the sense that we are going to exploit known properties of a finite dimensional subspace $V$ of $l^\infty(\mathbb{N})$, namely that $V$ admits a Euclidean isoperimetric inequality for $\mathbf{Z}_{n,c}(V)$, and the existence of solutions to the Plateau problem. Therefore every $Z \in \mathbf{Z}_{n,c}(V)$ admits a filling $S \in \mathbf{I}_{n+1,c}(V)$ with $M(S) \leq C M(Z)^{(n+1)/n}$ and support $\text{spt}(S)$ within distance at most $(n + 1) C M(Z)^{1/n}$ from $\text{spt}(Z)$, where $C$ is a constant depending only on $n$. This result was shown for classical integral currents in [5] and holds more generally for metric currents in the sense of Ambrosio-Kirchheim [1] and Lang [10] (see Theorem 1.2 and Theorem 1.6 in [13], as well as Section 2.7 and Section 2.8 in [8]).
Proof. Let \( \iota: K \rightarrow l^\infty(\mathbb{N}) \) be an isometric embedding with compact image \( K' := \iota(K) \). By property \( L \) there exist \( \varepsilon_0 > 0 \), \( L \geq 1 \) and an \( L \)-Lipschitz extension \( g: K'_\varepsilon \rightarrow X \) of \( \iota^{-1} \), where we might assume that \( \varepsilon \leq 1 \) and \( \varepsilon_0 < \varepsilon/L \). We might assume that \( \varepsilon_0 < \varepsilon/L \) so that \( g(K'_\varepsilon) \subset K_\varepsilon \). By the metric approximation property of \( l^\infty(\mathbb{N}) \), there exist a finite dimensional subspace \( V \subset l^\infty(\mathbb{N}) \) and a \( 1 \)-Lipschitz map \( \pi: l^\infty(\mathbb{N}) \rightarrow V \) such that \( |x - \pi(x)| \leq \varepsilon_0/4 \) for all \( x \in K' \). In particular, let \( C \geq 1 \) be the constant from above, set

\[
M := \min\left\{ \left( \frac{\varepsilon_0}{2(n+1)C} \right)^n, \frac{\varepsilon}{(C + \frac{n+1}{4} \varepsilon_0) L^{n+1}} \right\}
\]

and let \( Z \in \mathbb{Z}_{n,c}(X) \) with support (Z) \( \subset K \) and \( M(Z) < M \).

Consider

\[
Z' := \iota_\#Z \in \mathbb{Z}_{n,c}(l^\infty(\mathbb{N})), \quad Z'' := \pi_\#Z' \in \mathbb{Z}_{n,c}(V),
\]

which have supports in \( K' \) and \( K'' \), respectively, and satisfy \( M(Z'') \leq M(Z') = M(Z) < M \). Let \( H: [0, 1] \times \text{spt}(Z') \rightarrow l^\infty(\mathbb{N}) \) denote the affine homotopy between \( \text{id}_{\text{spt}(Z')} \) and \( \pi|_{\text{spt}(Z')} \), and let \( Q := H_\#([0, 1] \times Z') \in \mathbb{I}_{n+1,c}(l^\infty(\mathbb{N})) \) be the affine filling of \( Z'' - Z' \), as defined in Section 2.6. Note that \( H(t, \cdot): \text{spt}(Z') \rightarrow l^\infty(\mathbb{N}) \) is 1-Lipschitz for all \( t \in [0, 1] \) and \( H(\cdot, x): [0, 1] \rightarrow l^\infty(\mathbb{N}) \) has length at most \( \varepsilon_0/4 \) for all \( x \in \text{spt}(Z') \). Thus the support \( \text{spt}(Q) \) of \( Q \) is contained in \( \text{spt}(Z')_{\varepsilon_0} \subset K'_\varepsilon \) and its mass is bounded by

\[
M(Q) \leq (n + 1) \frac{\varepsilon_0}{4} M(Z') < \frac{n+1}{4} \varepsilon_0 M.
\]

As noted above \( Z'' \) possesses a filling \( S'' \in \mathbb{I}_{n+1,c}(V) \) with mass

\[
M(S'') \leq C M(Z'')^{n+1} < C M^{\frac{n+1}{n}} \leq C M
\]

and support within distance at most \( (n+1) C M(Z'')^{1/n} < \varepsilon_0/2 \) from \( \text{spt}(Z'') \subset \pi(\text{spt}(Z')) \), in particular it is contained in \( \pi(\text{spt}(Z'))_{\varepsilon_0/2} \subset \text{spt}(Z')_{\varepsilon_0} \subset K'_\varepsilon \).

Finally, \( S'' \) and \( Q \) have support in \( \text{spt}(Z')_{\varepsilon_0} \subset K'_\varepsilon \) so that \( S := g_\#(S'' - Q) \in \mathbb{I}_{n+1,c}(X) \) is well defined, has support in \( g(\text{spt}(Z')_{\varepsilon_0}) \subset \text{spt}(Z)_{\varepsilon_0} \), has boundary \( \partial S = g_\#(\partial S'' - \partial Q) = g_\#(Z'' - Z'' + Z') = Z \) and its mass is bounded by

\[
M(S) \leq L^{n+1}(M(S'') + M(Q)) < L^{n+1}\left(C + \frac{n+1}{4} \varepsilon_0\right) M \leq \varepsilon.
\]

We are now in a position to upgrade Proposition 4.2 to any current \( T \in \mathbb{I}_{n,c}(X) \).

**Proposition 4.6.** Let \( n \geq 1 \), let \( X \) be a locally compact metric space, and let \( U \subset X \) be an open subset with property \( L \). Then for every \( T \in \mathbb{I}_{n,c}(X) \) with \( \text{spt}(T) \subset U \), and every \( \varepsilon > 0 \), there is \( P \in \mathbb{I}_{n,c}(X) \) with \( N(T - P) < \varepsilon \) and \( \text{spt}(P) \subset \text{spt}(T)_\varepsilon \).
A stronger conclusion holds already for the 1-dimensional case from Proposition 4.2 and the comment thereafter, so that in the proof we can assume that \( n \geq 2 \) and apply Proposition 4.5 in dimension \( n - 1 \geq 1 \).

**Proof.** Suppose \( n \geq 2 \). Let \( K \) denote the closed \( \frac{\varepsilon}{2} \)-neighborhood of \( \text{spt}(T) \) in \( X \), without loss of generality we might assume that \( K \) is compact and that \( \text{spt}(T) \subset U \). Let \( M > 0 \) be the constant of Proposition 4.5 for \( K \) and \( \varepsilon/4 \); up to decreasing it we might assume that \( M \leq \varepsilon/4 \).

Consider \( T' := \partial T \in Z_{n-1,c}(X) \) and note that \( \text{spt}(T') \subset \text{spt}(T) \subset U \). By Proposition 4.2 we can find \( P' \in L_{n-1,c}(X) \) with \( \partial P' = \partial T' (= 0) \), \( M(T' - P') < M \leq \varepsilon/4 \) and \( \text{spt}(P') \subset \text{spt}(T') \subset K \).

According to Proposition 4.5 and the choice of \( M \), there exists a filling \( S \in I_{n,c}(X) \) of \( T' - P' \) with \( M(S) < \varepsilon/4 \) and \( \text{spt}(S) \subset \text{spt}(T') \subset U \).

Note that \( T - S \in I_{n,c}(X) \) has support contained in \( \text{spt}(T) \subset U \) and boundary \( \partial(T - S) = T' - (T' - P') = P' \in L_{n-1,c}(X) \) so applying Proposition 4.2 a second time we find \( P \in L_{n,c}(X) \) with \( M(T - S - P) < \varepsilon/2 \), \( \partial P = \partial(T - S) = P' \), and \( \text{spt}(P) \subset \text{spt}(T - S) \subset \text{spt}(T) \).

Therefore \( P \) satisfies:

\[
M(T - P) \leq M(T - S - P) + M(S) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \leq \varepsilon,
\]

\[
M(\partial T - \partial P) = M(T' - P') < M < \varepsilon.
\]

The proof of Theorem 1.4 (N-Approximation) now follows by combining Lemma 4.1 and Proposition 4.6.

**Proof of Theorem 1.4** Let \( X \) be a locally compact metric space with local property \( L \), \( T \in I_{n,c}(X) \) and \( \varepsilon > 0 \). By Lemma 4.1 we can write \( T = T_1 + \cdots + T_k \) with each \( T_i \in I_{n,c}(X) \) having support contained in both \( \text{spt}(T) \) and in an open subset of \( X \) having property \( L \). By Proposition 4.6 there exist \( P_i \in L_{n,c}(X) \) with \( N(T_i - P_i) < \varepsilon/k \) and \( \text{spt}(P_i) \subset \text{spt}(T_i) \subset \text{spt}(T) \), so that \( P := P_1 + \cdots + P_k \in L_{n,c}(X) \) is the desired Lipschitz approximation of \( T \) with \( N(T - P) < \varepsilon \) and \( \text{spt}(P) \subset \text{spt}(T) \).

\[\square\]
Figure 1: In this figure we illustrate schematically some of the currents used in the proof of Proposition 4.2. By construction $T' - P'$ is a current in $K' \subset l^\infty(N)$ and has small mass $M(T' - P')$; $T'' - P'' - \partial S$ is a current in a finite dimensional space $V$ and $T'' - P'' - \partial S$ is in $L_{n,c}(V)$. $W$ is the affine homotopy filling of $\partial(T'' - P'') - \partial(T' - P')$ and overall $P' + (T'' - P'' - \partial S - W)$ is a Lipschitz chain with boundary $\partial T'$ and the error term $M(T'' - P'' - \partial S - W)$ is small.
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