CATEGORICAL FRAMEWORKS FOR GENERALIZED FUNCTIONS

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Abstract. We tackle the problem of finding a suitable categorical framework for generalized functions used in mathematical physics for linear and non-linear PDEs. We are looking for a Cartesian closed category which contains both Schwartz distributions and Colombeau generalized functions as natural objects. We study Frölicher spaces, diffeological spaces and functionally generated spaces as frameworks for generalized functions. The latter are similar to Frölicher spaces, but starting from locally defined functionals. Functionally generated spaces strictly lie between Frölicher spaces and diffeological spaces, and they form a complete and cocomplete Cartesian closed category. We deeply study functionally generated spaces (and Frölicher spaces) as a framework for Schwartz distributions, and prove that in the category of diffeological spaces, both the special and the full Colombeau algebras are smooth differential algebras, with a smooth embedding of Schwartz distributions and smooth pointwise evaluations of Colombeau generalized functions.

Contents

1. Introduction: finding a categorical framework for generalized functions 2
1.1. The special and full Colombeau algebras 3
2. Functionally generated diffeologies 5
2.1. Preliminaries on diffeological spaces and Frölicher spaces 5
2.2. Definition and examples of functionally generated diffeologies 10
2.3. Categorical properties of functionally generated spaces 13
2.4. Preservation of limits and (suitable) colimits of manifolds 18
2.5. Categorical frameworks for generalized functions 20
3. Topologies for spaces of generalized functions 21
3.1. Locally convex vector spaces and Cartesian closed categories 21

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4. Spaces of compactly supported functions as functionally generated spaces
4.1. Plots of $\mathcal{D}_K(\Omega)$, $\mathcal{D}(\Omega)$ and Cartesian closedness
4.2. The locally convex topology and the $D$-topology on $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$
5. Spaces for Colombeau generalized functions as diffeological spaces
The space $\mathcal{C}^\infty(\Omega)^I$
The space $\mathcal{E}_{sM}(\Omega)$
The space $\mathcal{A}_q(\Omega)$
The space $U(\Omega)$
The space $\mathcal{E}^e(\Omega)$
The space $\mathcal{E}_{eM}(\Omega)$
The special and full Colombeau algebras
5.1. Colombeau ring of generalized numbers and evaluation of generalized functions
6. Conclusions and open problems
References

1. Introduction: finding a categorical framework for generalized functions

The problem of considering (generalized) derivatives of locally integrable functions arises frequently in Physics, e.g. in idealized models like in shock Mechanics, material points Mechanics, charged particles in Electrodynamics, gravitational waves in General Relativity, etc. (see e.g. [10, 21, 31]). Therefore, the need to perform calculations with discontinuous functions like one deals with smooth functions motivated the introduction of generalized functions (GF) as objects extending, in some sense, the notion of function. As such, generalized functions find deep applications in solutions of singular differential equations ([22, 30]) and are naturally framed in (several) theories of infinite dimensional spaces, from locally convex vector spaces ([24]) and convenient vector spaces ([26]) up to diffeological ([23, 25]) and Frölicher spaces ([11]).

The foundation of a rigorous linear theory of generalized functions has been pioneered by L. Schwartz with a deep use of locally convex vector space theory ([32, 22]), but heuristic multiplications of distributions early appeared e.g. in quantum electrodynamics, elasticity, elastoplasticity, acoustics and other fields ([10, 31]). Despite the impossibility of a straightforward extension of Schwartz linear theory ([33]) to an algebra extending pointwise product of continuous functions, the theory of Colombeau algebras (see e.g. [8, 9, 10, 30, 21, 31]) permits to bypass this impossibility in a very simple way by considering an algebra of generalized functions which extends the pointwise product of smooth functions.
The main aim of the present work is to study different categories as frameworks for generalized functions. In particular, we introduce the category $\mathbf{FDlg}$ of functionally generated spaces. This category has very nice properties and strictly lies between Frölicher and diffeological spaces.

We start by defining the algebras from Colombeau theory that will consider in this work. Henceforth, we will use the notations of [21, 22] for the well-known Schwartz distribution theory.

1.1. The special and full Colombeau algebras.

The special Colombeau algebra. In this section, we fix some basic notations and terminology from Colombeau theory. For details we refer to [21]. We include zero in the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Henceforth, $\Omega$ will always be an open subset of $\mathbb{R}^n$ and we denote by $I$ the interval $(0, 1]$. The (special) Colombeau algebra on $\Omega$ is defined as the quotient $G^s(\Omega) := E_M(\Omega)/N^s(\Omega)$ of moderate nets over negligible nets, where the former is

$$E_M^s(\Omega) := \{(u_\varepsilon) \in C^\infty(\Omega)^I \mid \forall K \subseteq \Omega \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\}$$

and the latter is

$$N^s(\Omega) := \{(u_\varepsilon) \in C^\infty(\Omega)^I \mid \forall K \subseteq \Omega \forall \alpha \in \mathbb{N}^n \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m})\}.$$

Throughout this paper, every asymptotic relation is for $\varepsilon \to 0^+$. Nets in $E_M^s(\Omega)$ are written as $(u_\varepsilon)$, and we use $u = [u_\varepsilon]$ to denote the corresponding equivalence class in $G^s(\Omega)$. For $(u_\varepsilon) \in N^s(\Omega)$ we also write $(u_\varepsilon) \sim 0$. Then $\Omega \mapsto G^s(\Omega)$ is a fine and supple sheaf of differential algebras and there exist sheaf embeddings of the space of Schwartz distributions $D'$ into $G^s$ (cf. [21]). A very simple way to embed $D'$ into $G^s$ is given by the following result ([35]):

**Theorem 1.** There exists a net $(\psi_\varepsilon) \in D(\mathbb{R}^n)^I$ with the properties:

(i) $\forall \varepsilon \in I \forall x \in \text{supp}(\psi_\varepsilon) : |x| < 1$;

(ii) $\int \psi_\varepsilon = 1 \forall \varepsilon \in I$, where the implicit integration is over the whole $\mathbb{R}^n$;

(iii) $\forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} |\partial^\alpha \psi_\varepsilon(x)| = O(\varepsilon^{-N})$;

(iv) $\forall j \in \mathbb{N} \exists \varepsilon_0 \in I \forall \varepsilon \in (0, \varepsilon_0] \forall \alpha \in \mathbb{N}^n : 1 \leq |\alpha| \leq j \Rightarrow \int x^\alpha \psi_\varepsilon(x) dx = 0$;

(v) $\forall \eta \in \mathbb{R}_{>0} \exists \varepsilon_0 \in I \forall \varepsilon \in (0, \varepsilon_0] : \int |\psi_\varepsilon| \leq 1 + \eta$.

In particular, if we set

$$\varepsilon \odot \psi_\varepsilon : x \in \mathbb{R}^n \mapsto \frac{1}{\varepsilon^n} \psi_\varepsilon \left(\frac{x}{\varepsilon}\right) \in \mathbb{R} \quad \forall \varepsilon \in I$$

$$\iota_\Omega(u) := [u \ast (\varepsilon \odot \psi_\varepsilon|_{\Omega})] \quad \forall u \in D'(\Omega)$$

then we have:

(vi) $\iota_\Omega : D'(\Omega) \to G^s(\Omega)$ is a linear embedding;
(vii) $\partial^\alpha (\iota_\Omega(u)) = \iota_\Omega (D^\alpha u)$ for all $u \in \mathcal{D}'(\Omega)$ and all $\alpha \in \mathbb{N}^n$, where $D^\alpha$ and $\partial^\alpha$ are the $\alpha$-partial differential operators on $\mathcal{D}'(\Omega)$ and $\mathcal{G}^s(\Omega)$, respectively;
(viii) $\iota_\Omega(f) = [f]$ for all $f \in C^\infty(\Omega)$.

The ring of constants in $\mathcal{G}^s$ is denoted by $\tilde{\mathbb{R}}$ and is called the ring of Colombeau generalized numbers (CGN). It is an ordered ring with respect to the order defined by $[x_\varepsilon] \leq [y_\varepsilon]$ iff $\exists [z_\varepsilon] \in \tilde{\mathbb{R}}$ such that $(z_\varepsilon) \sim 0$ and $x_\varepsilon \leq y_\varepsilon + z_\varepsilon$ for $\varepsilon$ sufficiently small. Even if this order is not total, we can still define the infimum $[x_\varepsilon] \wedge [y_\varepsilon] := \min(x_\varepsilon, y_\varepsilon)$, and analogously the supremum of two elements. More generally, the space of generalized points in $\Omega$ is $\tilde{\Omega} = \Omega_M/\sim$, where $\Omega_M = \{(x_\varepsilon) \in \Omega^I \mid \exists N \in \mathbb{N} : |x_\varepsilon| = O(\varepsilon^{-N})\}$ is called the set of moderate nets and $(x_\varepsilon) \sim (y_\varepsilon)$ if $|x_\varepsilon - y_\varepsilon| = O(\varepsilon^m)$ for every $m \in \mathbb{N}$. By $\mathcal{N}$ we will denote the set of all negligible nets of real numbers $(x_\varepsilon) \in \mathbb{R}^I$, i.e., such that $(x_\varepsilon) \sim 0$.

The space of compactly supported generalized points $\tilde{\Omega}_c$ is defined by $\Omega_c = \{ (x_\varepsilon) \in \Omega^I : \exists K \in \Omega \exists \varepsilon_0 \forall \varepsilon < \varepsilon_0 : x_\varepsilon \in K \}$ and $\sim$ is the same equivalence relation as in the case of $\tilde{\Omega}$. Any Colombeau generalized function (CGF) $u \in \mathcal{G}^s(\Omega)$ acts on generalized points from $\tilde{\Omega}_c$ by $u(x) := [u_\varepsilon(x_\varepsilon)]$ and is uniquely determined by its point values (in $\tilde{\mathbb{R}}$) on compactly supported generalized points ([21]), but not on standard points. A CGF $[u_\varepsilon]$ is called compactly-bounded (c-bounded) from $\Omega$ into $\Omega'$ if for any $K \in \Omega$ there exists $K' \in \Omega'$ such that $u_\varepsilon(K) \subseteq K'$ for $\varepsilon$ small. This type of CGF is closed with respect to composition. Moreover, if $u \in \mathcal{G}^s(\Omega)$ is c-bounded from $\Omega$ into $\Omega'$ and $v \in \mathcal{G}^s(\Omega')$, then $[v_\varepsilon \circ u_\varepsilon] \in \mathcal{G}^s(\Omega')$. For $x, y \in \mathbb{R}^n$ we will write $x \approx y$ if $x - y$ is infinitesimal, i.e., if $|x - y| \leq r$ for all $r \in \mathbb{R}_{>0}$.

Topological methods in Colombeau theory are usually based on the so-called sharp topology (see e.g. [2] and references therein), which is the topology generated by the balls $B^\varepsilon_p(x) = \{ y \in \mathbb{R}^n \mid |y - x| < \rho \}$, where $| - |$ is the natural extension of the Euclidean norm on $\mathbb{R}^n$, i.e., $|[x_\varepsilon]| := |[x_\varepsilon]| \in \tilde{\mathbb{R}}$, and $\rho \in \mathbb{R}_{>0}$ is positive invertible. Henceforth, we will also use the notation $\tilde{\mathbb{R}}^* := \{ x \in \mathbb{R} \mid x$ is invertible$\}$. Finally, Garetto in [12, 13] extended the above construction to arbitrary locally convex spaces by functorially assigning a space of CGF $\mathcal{G}^*_E$ to any given locally convex space $E$. The seminorms of $E$ can then be used to define pseudovaluations which in turn induce a generalized locally convex topology on the $\mathbb{C}$-module $\mathcal{G}^*_E$, again called sharp topology.

The full Colombeau algebra. Clearly, the embedding $\iota_\Omega$ defined in (1.1) depends on the net of maps $(\psi_\varepsilon) \in \mathcal{D}(\mathbb{R}^n)^I$ whose existence is given by Thm. 1. This shall not be considered only in a negative way: e.g. it is not difficult to choose $(\psi_\varepsilon)$ so that the embedding satisfies the properties that
\[ H(0) = v_R(H)(0) = \left[ \int_{-\infty}^{0} \psi_x \right] = 0 \text{ and } \delta(0) = v_R(\delta)(0) = \left[ \psi_x(0) \right] \text{ is an infinite number of } \mathbb{R} \text{ (here } H \text{ is the Heaviside function and } \delta \text{ is the Dirac delta function). These properties are informally used in several applications.}

The main idea of the full Colombeau algebra is to consider a different set of indices, instead of } I = (0, 1), \text{ so as to obtain an intrinsic embedding.}

**Definition 2.** (i) \( A_0(\Omega) := \{ \varphi \in \mathcal{D}(\Omega) \mid \int \varphi = 1 \} \), \( A_0 := A_0(\mathbb{R}^n) \);

(ii) \( A_q(\Omega) := \{ \varphi \in A_0(\Omega) \mid \forall \alpha \in \mathbb{N}^n : 1 \leq |\alpha| \leq q \Rightarrow \int x^\alpha \varphi(x) \, dx = 0 \} \);

(iii) \( A_q := A_q(\mathbb{R}^n) \);

(iv) \( U(\Omega) := \{ (\varphi, x) \in A_0 \times \Omega \mid \text{supp}(\varphi) \subseteq \Omega - x \} \);

(v) We say that \( R \in \mathcal{E}(\Omega) \) iff \( R : U(\Omega) \rightarrow \mathbb{R} \) and \( \forall \varphi \in A_\Om : R(\varphi, -) \text{ is smooth on } \Omega \cap \{ x \in \mathbb{R}^n \mid \text{supp}(\varphi) \subseteq \Omega - x \} \);

(vi) We say that \( R \in \mathcal{E}_M(\Omega) \) iff \( R \in \mathcal{E}(\Omega) \) and \( \forall K \in \Omega \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} \forall \varphi \in A_N : \sup_{x \in K} |\partial^\alpha R(\varepsilon \odot \varphi, x)| = O(\varepsilon^{-N}) \);

(vii) We say that \( R \in \mathcal{N}(\Omega) \) iff \( R \in \mathcal{E}(\Omega) \) and \( \forall K \in \Omega \forall \alpha \in \mathbb{N}^n \forall m \in \mathbb{N} \exists q \in \mathbb{N} \forall \varphi \in A_q : \sup_{x \in K} |\partial^\alpha R(\varepsilon \odot \varphi, x)| = O(\varepsilon^m) \);

(viii) \( \mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega) \) is called the full Colombeau algebra;

(ix) The above mentioned intrinsic embedding \( \nu_\Om : \mathcal{D}(\Omega) \rightarrow \mathcal{G}(\Omega) \) is defined by \( (\nu_\Om u)(\varphi, x) := \langle u, \varphi(-x) \rangle \). It verifies properties like (vi), (vii) and (viii) in Thm. 1.

For motivations and details, see [21].

2. FUNCTIONALLY GENERATED DIFFELOGIES

2.1. Preliminaries on diffeological spaces and Frölicher spaces. Both diffeological spaces and Frölicher spaces are generalizations of smooth manifolds, introduced by J.M. Souriau and A. Frölicher, respectively, in the 1980’s. The smooth structure (called the diffeology) on a diffeological space is defined by some testing functions from all open subsets of all Euclidean spaces to the given set, subject to a covering condition, a presheaf condition and a sheaf condition (see Def. 3). A possible intuitive description of this structure on a diffeological space \( X \) is that a diffeology is the specification not only of a particular family of smooth functions (like charts on manifolds) but of all the possible smooth maps of the type \( d : U \rightarrow X \) for all open subsets \( U \subseteq \mathbb{R}^n \) and for all \( n \in \mathbb{N} \). We can roughly say that we have to specify what are smooth curves, surfaces, etc. on the space \( X \).

On a Frölicher space \( X \) we consider only \( U = \mathbb{R} \), i.e., the smooth structure on the space is given by a set of smooth curves; moreover, these curves are determined by (and they determine) a given set of functionals, i.e., of smooth functions of the type \( l : X \rightarrow \mathbb{R} \) (see Def. 9). The category \( \mathbf{Fr} \) of all Frölicher spaces is a full subcategory of the category \( \mathbf{Dlg} \) of all diffeological spaces.
In the following subsections, we are going to focus on a family of diffeological spaces called functionally generated (diffeological) spaces, where the diffeological structure is determined by a given family of locally defined smooth functionals. As we will see in the present work, these spaces frequently appear in functional analysis, strictly lie between diffeological spaces and Frölicher spaces, and the category $\mathcal{FDlg}$ of all these spaces behaves nicely – it is complete, cocomplete and Cartesian closed.

To simplify the notation, we write $\mathcal{OR}^\infty$ for the category of open sets in Euclidean spaces and ordinary smooth functions.

**Definition 3.** A diffeological space $X = (|X|, \mathcal{D})$ is a set $|X|$ together with a specified family of functions $\mathcal{D} = \cup_{U \in \mathcal{OR}^\infty} \mathcal{D}_U$ with $\mathcal{D}_U \subseteq \text{Set}(U, |X|)$ such that for any $U, V \in \mathcal{OR}^\infty$, the following three axioms hold:

(i) Every constant function $d : U \to |X|$ is in $\mathcal{D}_U$ (Covering condition);
(ii) $d \circ f \in \mathcal{D}_V$ for any $d : U \to |X| \in \mathcal{D}_U$ and any $f \in \mathcal{C}^\infty(V, U)$ (Presheaf condition);
(iii) Let $d \in \text{Set}(U, |X|)$, and let $\{U_i\}_{i \in I}$ be an open covering of $U$. Then $d \in \mathcal{D}_U$ if $d|_{U_i} \in \mathcal{D}_{U_i}$ for each $i \in I$ (Sheaf condition).

For a diffeological space $X = (|X|, \mathcal{D})$, every element in $\mathcal{D}$ is called a plot of $X$. We write $d \in_U X$ to denote that $d \in \mathcal{D}_U$, which will also be called a figure of type $U$ of the space $X$.

**Definition 4.** A morphism (also called smooth map) $f : X \to Y$ between two diffeological spaces $X = (|X|, \mathcal{D}^X)$ and $Y = (|Y|, \mathcal{D}^Y)$ is a function $|f| : |X| \to |Y|$ such that $f \circ d \in \mathcal{D}^Y_Y$ for any $d \in \mathcal{D}^X_U$ and $U \in \mathcal{OR}^\infty$.

If we write $f(d) := f \circ d$, by the covering condition of Def. 3 we have a generalization of the usual evaluation; moreover, $f : X \to Y$ is smooth if and only if for all $U \in \mathcal{OR}^\infty$ and $d \in_U X$, we have $f(d) \in_U Y$, i.e., $f$ take figures of type $U$ on the domain to figures of the same type in the codomain. Moreover, $X = Y$ as diffeological spaces if and only if for all $d$ and $U$, $d \in_U X$ if and only if $d \in_U Y$. These and several other generalization of set-theoretical properties justify the use of the symbol $\in_U$.

All diffeological spaces with smooth maps form a category, which will be denoted by $\mathbf{Dlg}$. Given two diffeological spaces $X$ and $Y$, we write $\mathcal{C}^\infty(X, Y)$ for the set of all smooth maps $X \to Y$.

Here is a list of basic properties of diffeological spaces. We refer readers to the standard textbook [23] for more details.

**Remark 5.** (i) By a smooth manifold, we always assume it is Hausdorff and finite-dimensional. Every smooth manifold $M$ is automatically a diffeological space $M = (M, \mathcal{D})$ with $d \in_U M$ if and only if $d : U \to M$ is smooth in the usual sense. We call this $\mathcal{D}$ the standard diffeology on $M$, and without specification, we always assume a smooth manifold
with this diffeology when viewed as a diffeological space. Moreover, given two smooth manifolds $M$ and $N$, $f : M \rightarrow N$ is smooth if and only if $f : M \rightarrow N$ is smooth in the usual sense. In other words, the category $\text{Man}$ of all smooth manifolds and smooth maps is fully embedded in $\text{Dlg}$. This justifies our notation $\mathcal{C}^\infty(X,Y)$ for the hom-set $\text{Dlg}(X,Y)$. Limits of smooth manifolds that already exist in $\text{Man}$ are preserved by this embedding (see Thm. 27). Generally speaking the same property does not hold for colimits of smooth manifolds that already exist in $\text{Man}$.

(ii) Given a set $X$, the set of all diffeologies on $X$ forms a complete lattice. The smallest diffeology is called the discrete diffeology, which consists of all locally constant functions, and the largest diffeology is called the indiscrete diffeology, which consists of all set functions. Let $A = (X, D_A)$ and $B = (X, D_B)$ be two diffeological spaces with the same underlying set. We simply write $A \subseteq B$ iff $1_X : A \rightarrow B$ is smooth, i.e., iff $D_A \subseteq D_B$.

Therefore, given a family of functions $\{\iota_i : |X_i| \rightarrow Y\}_{i \in I}$ from the underlying sets of the diffeological spaces $X_i$ to a fixed set $Y$, there exists a smallest diffeology on $Y$ making all these maps $\iota_i$ smooth. We call this diffeology the final diffeology associated to $I$. In more detail, $d \in_U Y \text{ iff } \forall u \in U \exists V \text{ neigh. of } u \exists \delta \in V \exists \iota_i \colon \iota_i \circ \delta = d|_V$.

Dually, given a family of functions $\{p_j : X \rightarrow |Y_j|\}_{j \in J}$ from a given set $X$ to the underlying sets of the diffeological spaces $Y_j$, there exists a largest diffeology on $X$ making all these maps $p_j$ smooth. We call this diffeology the initial diffeology associated to $J$. In more detail, $d \in_U X \text{ iff } p_j \circ d \in_U Y_j \forall j \in J$.

In particular, if $Y$ is a quotient set of $|X|$, then the final diffeology on $Y$ associated to the quotient map $|X| \rightarrow Y$ is called the quotient diffeology, and $Y$ with the quotient diffeology is called a quotient diffeological space of $X$. Dually, if $X$ is a subset of $|Y|$, then the initial diffeology on $X$ associated to the inclusion map $X \rightarrow |Y|$ is called the sub-diffeology, and we write $(X \prec Y)$ to denote this new diffeological space. We call $(X \prec Y)$ the diffeological subspace of $Y$. Finally, the initial diffeology associated to the projection maps $p_i : \prod_{i \in I} |X_i| \rightarrow |X_i|$ of an arbitrary product is called the product diffeology, and dually the final diffeology associated to the inclusion maps $|X_j| \rightarrow \prod_{j \in J} |X_j|$ of an arbitrary coproduct is called the coproduct diffeology.

(iii) The category $\text{Dlg}$ is complete and cocomplete. In more detail, let $G : I \rightarrow \text{Dlg}$ be a functor from a small category $I$. Write $|-| : \text{Dlg} \rightarrow \text{Set}$ for the forgetful functor. Then both $\lim G$ and $\colim G$ exist in $\text{Dlg}$ as lifting and co-lifting of limits and colimits in $\text{Set}$. In more detail, $|\lim G| = \lim |G|$ and the diffeology of $\lim G$ is the initial
diffeology associated to the universal cone \( \lim |G| \rightarrow |G(i)| \}_{i \in I} \) in \( \mathsf{Set} \); dually \( |\operatorname{colim} G| = \operatorname{colim} |G| \) and the diffeology of \( \operatorname{colim} G \) is the final diffeology associated to the universal co-cone \( \{|G(i)| \rightarrow \operatorname{colim} |G|\}_{i \in I} \) in \( \mathsf{Set} \).

(iv) The category \( \mathsf{Dlg} \) is Cartesian closed. In more detail, given three diffeological spaces \( X, Y \) and \( Z \), there is a canonical diffeology (called the functional diffeology) on \( C^\infty(X,Y) \) defined by

\[
d \in U C^\infty(X,Y) \iff d^v \in C^\infty(U \times X,Y),
\]

with \( d^v(u,x) = d(u)(x) \) (in the present work, we use the notations of [1]). Without specification, the set \( C^\infty(X,Y) \) is always equipped with the functional diffeology when viewed as a diffeological space. Then Cartesian closedness means that \( f \in C^\infty(X,C^\infty(Y,Z)) \) if and only if \( f^v \in C^\infty(X \times Y,Z) \) (or, equivalently that \( g \in C^\infty(X \times Y,Z) \) if and only if \( g^\wedge \in C^\infty(X,C^\infty(Y,Z)) \), where \( g^\wedge(x)(y) := g(x,y) \)). Therefore, Cartesian closedness permits to equivalently translate an infinite dimensional problem like \( f \in C^\infty(X,C^\infty(Y,Z)) \) into a finite dimensional one \( f^v \in C^\infty(X \times Y,Z) \) and vice versa.

(v) Every diffeological space can be extended with infinitely near points \( X \in \mathsf{Dlg} \mapsto \bullet X \in \mathsf{•Dlg} \), \( X \subseteq \bullet X \), obtaining a non-Archimedean framework similar to Synthetic Differential Geometry (see e.g. [28] and references therein) but compatible with classical logic. The category \( \mathsf{•Dlg} \) of Fermat spaces is defined by generalizing the category of diffeological spaces, but taking suitable smooth functions defined on the extension \( \bullet U \subseteq \mathbb{R}^n \) of open sets \( U \in \mathcal{O}R^\infty \). It is remarkable to note that the so called Fermat functor \( \mathfrak{•}(-) : \mathsf{Dlg} \rightarrow \mathsf{•Dlg} \) has very good preservation properties strictly related to intuitionistic logic. See [14, 15, 16, 17, 20] for more details.

(vi) \( \mathsf{Dlg} \) is a quasi-topos and hence is locally Cartesian closed ([3]).

Every diffeological space has an interesting canonical topology:

**Definition 6.** Let \( X = (|X|,D) \) be a diffeological space. The final topology \( \tau_X \) induced by \( D \) is called the \( D \)-topology.

Without specification, every diffeological space \( X \) is equipped with the \( D \)-topology \( \tau_X \). Elements in \( \tau_X \) are called \( D \)-open subsets.

**Example 7.**

(i) The \( D \)-topology on any smooth manifold is the usual topology.

(ii) The \( D \)-topology on any discrete (indiscrete) diffeological space is the discrete (indiscrete) topology.

**Theorem 8.** [36] \( T_D : \mathsf{Dlg} \rightarrow \mathsf{Top} \) defined by \( T_D(X) = (|X|, \tau_X) \) is a functor\(^1\), which has a right adjoint \( D_T : \mathsf{Top} \rightarrow \mathsf{Dlg} \) defined by \( |D_T(X)| \) :=

\(^1\text{We can recall the symbol } T_D \text{ by saying “topological space from diffeological space”. Analogously we can recall the plenty of symbols for the other functors related to our categories in this paper.} \)
As a consequence, the $D$-topology of a quotient diffeological space of $X$ is same as the quotient topology of $T_D(X)$. However, the $D$-topology of a diffeological subspace of $X$ may be different from the sub-topology of $T_D(X)$.

For more detailed discussion of the $D$-topology of diffeological spaces, see [23, Chapter 2] and [7].

Now, let’s turn to Frölicher spaces. In several spaces of functional analysis (like all those listed in Section 2.5), smooth figures are “generated by smooth functionals”. Therefore, smoothness can also be tested using smooth functionals, similarly as using projections in finite dimensional Euclidean spaces. In Frölicher spaces, we focus our attention also to smooth functions of the type $X \rightarrow \mathbb{R}$.

**Definition 9.** A Frölicher space $(\mathcal{C}, X, \mathcal{F})$ is a set $X$ together with two specified families of functions

$$\mathcal{C} \subseteq \textbf{Set}(\mathbb{R}, X) \text{ and } \mathcal{F} \subseteq \textbf{Set}(X, \mathbb{R})$$

with the following smooth compatibility conditions:

$$c : \mathbb{R} \rightarrow X \in \mathcal{C} \text{ iff } l \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \forall l \in \mathcal{F},$$

and

$$f : X \rightarrow \mathbb{R} \in \mathcal{F} \text{ iff } l \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \forall c \in \mathcal{C}.$$
Frölicher structure on $Y$ the final Frölicher structure associated to $$\{u_i : X_i \to Y\}_{i \in I}.$$ Dually, let $\{p_j : X \to Y_j\}_{j \in J}$ be a family of functions from a fixed set $X$ to the underlying sets of the Frölicher spaces $(C_j, Y_j, \mathcal{F}_j)$. Let $C_X = \{c : \mathbb{R} \to X \mid p_j \circ c \in C_j \forall j\}$ and let $\mathcal{F}_X = \{l : X \to \mathbb{R} \mid l \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \forall c \in C_X\}$. Then $(C_X, X, \mathcal{F}_X)$ is a Frölicher space and all these maps $p_j$ are morphisms between Frölicher spaces. We call this Frölicher structure on $X$ the initial Frölicher structure associated to $\{p_j : X \to Y_j\}_{j \in J}$.

(iii) The category $\mathbf{Fr}$ is complete and cocomplete. In more detail, let $G : \mathcal{I} \to \mathbf{Fr}$ be a functor from a small category $\mathcal{I}$. Write $\lvert - \rvert : \mathbf{Fr} \to \mathbf{Set}$ for the forgetful functor. Then both $\lim G$ and $\colim G$ exist in $\mathbf{Fr}$ as lifting and co-lifting of limits and colimits in $\mathbf{Set}$. In more detail, $\lvert \lim G \rvert = \lim \lvert G \rvert$ and the Frölicher structure of $\lim G$ is the initial Frölicher structure associated to the universal cone $\{\lvert G \rvert \to \lvert G(i) \rvert\}_{i \in \mathcal{I}}$ in $\mathbf{Set}$; dually $\lvert \colim G \rvert = \colim \lvert G \rvert$ and the Frölicher structure of $\colim G$ is the final Frölicher structure associated to the universal co-cone $\{\lvert G(i) \rvert \to \colim \lvert G \rvert\}_{i \in \mathcal{I}}$. In the category of Hausdorff Frölicher spaces, limits and colimits of smooth manifolds that already exist in $\mathbf{Man}$ are preserved by the embedding $\mathbf{Man} \to \mathbf{HFr}$ (see Thm. 31).

(iv) The category $\mathbf{Fr}$ is Cartesian closed. In more detail, given Frölicher spaces $X$ and $Y$, set $C = \{c : \mathbb{R} \to \mathbf{Fr}(X, Y) \mid c^\vee \in \mathbf{Fr}(\mathbb{R} \times X, Y)\}$, and $\mathcal{F} = \{l : \mathbf{Fr}(X, Y) \to \mathbb{R} \mid l \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \forall c \in C\}$. Then one can show that $(C, \mathbf{Fr}(X, Y), \mathcal{F})$ is a Frölicher space. Without specification, $\mathbf{Fr}(X, Y)$ is always equipped with this Frölicher structure when viewed as a Frölicher space.

(v) Given a Frölicher space $(C, Y, \mathcal{F})$, let $\mathcal{D}_U = \{d : U \to Y \mid l \circ d \in C^\infty(U, \mathbb{R}) \forall l \in \mathcal{F}\}$. Then $\mathcal{D}_F(C, Y, \mathcal{F}) := (Y, \mathcal{D} = \cup_{U \in \mathcal{O}} C^\infty(\mathbb{R}, \mathbb{R}) \mathcal{D}_U)$ is a diffeological space. This defines a full embedding $\mathcal{D}_F : \mathbf{Fr} \to \mathbf{Dlg}$. So, there will be no confusion to call smooth maps also the morphisms between Frölicher spaces. Moreover, one can show that $\mathcal{D}_F(\mathbf{Fr}(A, B)) = C^\infty(\mathcal{D}_F(A), \mathcal{D}_F(B))$ as diffeological spaces. This embedding functor has a left adjoint given as follows. For a diffeological space $X = (\lvert X \rvert, \mathcal{D})$, let $\mathcal{F} = C^\infty(X, \mathbb{R})$ and let $C = \{c : \mathbb{R} \to X \mid l \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \forall l \in \mathcal{F}\}$. Then $\mathcal{F}_D(X) := (C, \lvert X \rvert, \mathcal{F})$ is a Frölicher space. Both functors $\mathcal{D}_F$ and $\mathcal{F}_D$ are identities on the morphisms. For more discussion on the relationship between diffeological spaces and Frölicher spaces, see [34, 4].

2.2. Definition and examples of functionally generated diffeologies. Now, let’s introduce a special class of diffeological spaces called functionally generated spaces, which are like Frölicher spaces but with locally defined smooth functionals. The idea is that, in this type of spaces, we can know whether a continuous function $d \in \mathbf{Top}(U, \mathcal{T}_D(X))$ is a figure by testing for
smoothness of the composition with a given family of locally defined smooth functionals $l : (A \prec X) \to \mathbb{R}$.

**Definition 12.** Let $X = (|X|, \mathcal{D})$ be a diffeological space. Let $\mathcal{F} = \{\mathcal{F}_A\}_{A \in \tau_X}$ be a $\tau_X$-family of smooth functions, that is, for each $A \in \tau_X$

\[ \mathcal{F}_A \subseteq \mathcal{C}^\infty(A \prec X, \mathbb{R}). \]

We say that $\mathcal{F}$ generates $\mathcal{D}$ if for any open set $U \in \mathcal{O}\mathbb{R}^\infty$ and any continuous map $d \in \mathbf{Top}(U, T_D(X))$, the condition

\[ \forall A \in \tau_X \forall l \in \mathcal{F}_A : \quad l \circ d|_{d^{-1}(A)} \in \mathcal{C}^\infty(d^{-1}(A), \mathbb{R}) \quad (2.1) \]

implies $d \in_U X$, i.e., $d$ is a plot of $X$. Any map $l \in \mathcal{C}^\infty(A \prec X, \mathbb{R})$ is called a smooth functional of the space $X$. Finally, we say that the diffeological space $X$ is functionally generated if its diffeology can be generated by some family $\mathcal{F}$, and we denote with $\mathbf{FDlg}$ the full subcategory of $\mathbf{Dlg}$ of all functionally generated diffeological spaces.

If the codomain of a continuous map $f : X \to Y$ is functionally generated, then we can also test the smoothness of $f$ by smooth functionals of $Y$:

**Theorem 13.** Let $f : |X| \to |Y|$ be a map with $X \in \mathbf{Dlg}$ and $Y \in \mathbf{FDlg}$. Assume that the diffeology of $Y$ is generated by the family $\{\mathcal{F}_A\}_{A \in \tau_Y}$. Then the following are equivalent

(i) $f \in \mathcal{C}^\infty(X, Y)$

(ii) $f \in \mathbf{Top}(T_D(X), T_D(Y))$ and

\[ \forall A \in \tau_Y \forall l \in \mathcal{F}_A : \quad l \circ f|_{f^{-1}(A)} \in \mathcal{C}^\infty(f^{-1}(A) \prec X, \mathbb{R}). \quad (2.2) \]

**Proof.** Since the implication (i) $\Rightarrow$ (ii) is clear, we only prove the opposite one. For any $d \in_U X$, since $d \in \mathbf{Top}(U, T_D(X))$, $f \circ d \in \mathbf{Top}(U, T_D(Y))$. Then for any $A \in \tau_Y$ and any $l \in \mathcal{F}_A$, by (2.2) we get $l \circ f|_{f^{-1}(A)} \in \mathcal{C}^\infty(f^{-1}(A) \prec X, \mathbb{R})$ and hence $l \circ f|_{f^{-1}(A)} \circ d|_{d^{-1}(f^{-1}(A))} = l \circ (f \circ d)|_{f \circ d^{-1}(A)} \in \mathcal{C}^\infty((f \circ d)^{-1}(A), \mathbb{R})$. Since the diffeology of $Y$ is generated by $\{\mathcal{F}_A\}_{A \in \tau_Y}$, the conclusion $f \circ d \in_U Y$ follows.

Here is a list of basic properties and examples of functionally generated spaces:

**Remark 14.**

(i) The notion of functionally generated space is of local nature, i.e., we can equivalently say that $\mathcal{F}$ generates $\mathcal{D}$ if for any $U \in \mathcal{O}\mathbb{R}^\infty$ and any $d \in \mathbf{Set}(U, |X|)$, the condition

\[ \forall u \in U \forall A \in \tau_X \forall l \in \mathcal{F}_A : \quad d(u) \in A \Rightarrow \exists V \text{ neigh. of } u : \quad d(V) \subseteq A, \quad l \circ d|_V \in \mathcal{C}^\infty(V, \mathbb{R}) \]

implies $d \in_U X$. 

\[ \]
(ii) We can also equivalently ask that \( \mathcal{F}_A \subseteq \text{Set}(A, \mathbb{R}) \) and for all continuous \( d \in \text{Top}(U, T_D(X)) \) and any open set \( U \in \mathcal{O}\mathbb{R}^\infty \) we have \( d \in_U X \) if and only if (2.1) holds. Therefore, smooth functionals determine completely the figures (plots) of the underlying diffeological space, i.e., if \( D_1, D_2 \) are diffeologies on \( |X| \) and \( \mathcal{F} \) generates both \( D_1 \) and \( D_2 \), then \( D_1 = D_2 \).

(iii) Let \( \mathcal{F} \) generate \( D \). Define \( \mathcal{M}^X_A := \mathcal{C}^\infty(A \times X, \mathbb{R}) \) for any \( A \in \tau_X \). Then \( \mathcal{M}^X \) also generates \( D \). Of course, \( \mathcal{M}^X \) is the maximum family of smooth functionals which can be used to test whether a continuous map \( d \in \text{Top}(U, T_D(X)) \) is a figure or not, and the interesting problem is to find a smaller family \( \mathcal{F} \subseteq \mathcal{M}^X \) generating the same set of plots of \( X \).

(iv) The diffeology generated by a Frölicher space \((\mathcal{C}, X, \mathcal{F})\) is functionally generated by globally defined smooth functionals. That is, it suffices to consider \( \tilde{\mathcal{F}} \) defined by \( l \in \tilde{\mathcal{F}}_A \) if and only if \( A = X \) and \( l \in \mathcal{F} \). Therefore, the functor \( F_D : \text{Fr} \rightarrow \text{Dlg} \) has values in \( \text{FDlg} \). In particular, every smooth manifold and every discrete diffeological space is functionally generated. However, there are functionally generated spaces which do not come from Frölicher spaces; see Ex. 25.

In a functionally generated space, besides the usual \( D \)-topology \( \tau_X \), we can consider the initial topology \( \tau_F \) with respect to all smooth functionals \( \bigcup_{A \in \tau_X} \mathcal{F}_A \) (which is analogous to the weak topology, see e.g. [24]). In particular, the topology \( \tau_{\mathcal{M}^X} \) is called the \textit{functional topology} on \( X \). In general, \( \tau_F \) is coarser than the \( D \)-topology, see Ex. 25, but in every functionally generated space the functional topology and the \( D \)-topology coincide, as stated in the following theorem.

**Theorem 15.** Let \( \mathcal{F} \) generate the diffeology of the space \( X \in \text{Dlg} \). Then \( \tau_F \subseteq \tau_X \). If \( \mathcal{F}_A \neq \emptyset \) for all \( A \in \tau_X \), then \( \tau_F = \tau_X \). In particular, \( \tau_{\mathcal{M}^X} = \tau_X \).

**Proof.** The topology \( \tau_F \) has the set
\[
\{ l^{-1}(V) \mid l \in \mathcal{F}_A, V \in \tau_\mathbb{R}, A \in \tau_X \}
\]
as a subbase. For any \( l^{-1}(V) \) in this subbase and any plot \( d \in_U X \) the set
\[
d^{-1}(l^{-1}(V)) = (l \circ d|_{d^{-1}(A)})^{-1}(V)
\]
is open in \( U \) since \( l \circ d|_{d^{-1}(A)} \) is smooth by Def. 12. Hence \( \tau_F \subseteq \tau_X \). Vice versa, if \( A \in \tau_X \) and \( l \in \mathcal{F}_A \neq \emptyset \) is any functional, then \( A = l^{-1}(\mathbb{R}) \in \tau_F \). So \( \tau_X \subseteq \tau_F \) if \( \mathcal{F}_A \neq \emptyset \) for all \( A \in \tau_X \). \( \square \)

Here are some examples of diffeological spaces which are not functionally generated.

**Example 16.**

(i) Let \((X, D)\) be the irrational torus \( \mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z}) \), for some \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), with the quotient diffeology \( D \); see [23]. Then \((X, D)\) is not functionally generated. Indeed, since \( \mathbb{Z} + \theta \mathbb{Z} \) is dense in \( \mathbb{R} \), the \( D \)-topology \( \tau_X \) is
indiscrete, that is, \( \tau_X = \{ \emptyset, X \} \). Hence, every smooth map \( X \to \mathbb{R} \) is constant. Therefore, for any function \( d : U \to X \), for any \( l \in \mathcal{M}_X \), the composition \( l \circ d \) is constant, hence smooth. Therefore, there does not exist a family \( \mathcal{F} \) that generates \( \mathcal{D} \).

(ii) For any \( n \geq 2 \), let \( \mathbb{R}^n_w = (\mathbb{R}^n, \mathcal{D}^w) \) be \( \mathbb{R}^n \) with the wire diffeology \( \mathcal{D}^w \); see [23]. Then the \( D \)-topology of \( \mathbb{R}^n_w \) is the usual Euclidean topology, and by Boman’s theorem [5], \( \mathcal{C}^\infty(A \ltimes \mathbb{R}^n_w, \mathbb{R}) = \mathcal{C}^\infty(A \ltimes \mathbb{R}^n, \mathbb{R}) = \mathcal{M}^w_{\mathbb{R}} \) so that if \( \mathbb{R}^n_w \) is functionally generated, we would have \( 1_{\mathbb{R}^n} \in_{\mathbb{R}^n} \mathbb{R}^n_w \), which is false for the wire diffeology. Therefore, \( \mathbb{R}^n_w \) is not functionally generated.

2.3. Categorical properties of functionally generated spaces. In this subsection, we are going to prove some nice categorical properties for the category \( \text{FDlg} \) of all functionally generated spaces with smooth maps, that is, \( \text{FDlg} \) is complete, cocomplete and Cartesian closed.

Although the family \( \mathcal{F} \) that generates a diffeology is a \( \tau_X \)-family of smooth functions, in practice, we usually only need a \( \mathcal{B} \)-family with \( \mathcal{B} \subseteq \tau_X \). In other words, \( \mathcal{F}_A \) can be any subset (in particular, the empty set) of \( \mathcal{C}^\infty(A \ltimes X, \mathbb{R}) \) if \( A \in \tau_X \setminus \mathcal{B} \). We have already met such examples in (iv) of Rem. 14. Here is another big class of examples:

**Theorem 17.** Let \( \{ p_i : X \to |X_i| \}_{i \in I} \) be a family of functions from a given set \( X \) to the underlying sets of the diffeological spaces \( X_i = (|X_i|, \mathcal{D}^i) \). Assume that each \( \mathcal{D}^i \) is generated by \( \mathcal{F}^i \), and let \( \mathcal{D} \) be the initial diffeology on \( X \) associated to this family (i.e., \( d \in U \mathcal{D} \) iff \( p_i \circ d \in_U \mathcal{D}^i \forall i \)). For all \( A \in \tau_{(X, \mathcal{D})} \) set \( l \in \mathcal{F}_A \) iff \( l \in \mathcal{C}^\infty(A \ltimes X, \mathbb{R}) \) and

\[
\exists i \in I \exists B \in \tau_{X_i} \exists \lambda \in \mathcal{F}_B^i : A = p_i^{-1}(B) , l = \lambda \circ p_i|_A.
\]

Then the diffeology \( \mathcal{D} \) is generated by \( \mathcal{F} \).

The proof follows directly from Def. 12. Note that \( \mathcal{F}_A = \emptyset \) if \( A \) is not of the form \( A = p_i^{-1}(B) \) for some \( i \in I \) and \( B \in \tau_{X_i} \), so that we are essentially considering only smooth functionals defined on \( D \)-open subsets in \( B = \{ p_i^{-1}(B) | B \in \tau_{X_i}, i \in I \} \subseteq \tau_{(X, \mathcal{D})} \). In this sense, \( \mathcal{F} \) is also the smallest family of smooth functionals generating \( (X, \mathcal{D}) \) and containing all the smooth functionals of the form \( \lambda \circ p_i|_{p_i^{-1}(B)} \).

In particular, every subset of a functionally generated space with the sub-diffeology is again functionally generated. Analogously, every product of functionally generated spaces with the product diffeology is functionally generated, so that

**Corollary 18.** The category \( \text{FDlg} \) is complete.

Similarly, one can show that every coproduct of functionally generated spaces with the coproduct diffeology is functionally generated.

Let \( f, g : X \to Y \) be smooth maps between functionally generated spaces. In general, the coequalizer in \( \text{Dlg} \) may not be functionally generated. For example, let \( X = \mathbb{R} \) be equipped with the discrete diffeology,
and let $Y = \mathbb{R}$ be equipped with the standard diffeology. Let $\theta$ be some irrational number. Fix a representative in $\mathbb{R}$ for each element in $\mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z})$, i.e., a function $\rho : \mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z}) \to \mathbb{R}$ such that $\rho(c) \in c$ for all $c \in \mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z})$.

Let $f : X \to Y$ be the identity function and let $g : X \to Y$ be the function defined by $g(r) := \rho(c)$ for all $r \in c \in \mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z})$, i.e., sending every point in the subset $\rho(c) + \mathbb{Z} + \theta \mathbb{Z}$ to the fixed representative $\rho(c) \in \mathbb{R}$. It is clear that both $f$ and $g$ are smooth because we equip $X$ with locally constant figures, and the coequalizer in $\text{Dlg}$ is the irrational torus because the equivalence relation of $\mathbb{R}/(\mathbb{Z} + \theta \mathbb{Z})$ is the smallest one where $f(r) = r$ is equivalent to $g(r) = \rho(c)$ for $r \in c$. We already know from (i) of Ex. 16 that the irrational torus is not functionally generated. However, we will show below that the category $\text{FDlg}$ is cocomplete, and the coequalizer of the above diagram in $\text{FDlg}$ is the underlying set of the irrational torus with the indiscrete diffeology.

Now, we want to see how to define a new functionally generated diffeology starting from a diffeological space and a $\tau_X$-family of smooth functionals.

**Definition 19.** Let $X = (\|X\|, \sigma) \in \text{Top}$ be a topological space, and let $\mathcal{F} = \{\mathcal{F}_B\}_{B \subseteq \sigma}$ be a $\mathcal{B}$-family of functions, i.e., $\mathcal{F}_B \subseteq \text{Set}(B, \mathbb{R})$.

For $U \in \mathcal{O}^{\mathbb{R}}$, write $d \in \mathcal{D}_U \mathcal{F}_U$ (or $\mathcal{D}^X \mathcal{F}_U$; if we need to show the dependence from $X$) if and only if $d \in \text{Top}(U, X)$ and
\[
\forall B \in \mathcal{B} \forall l \in \mathcal{F}_B : l \circ d|_{d^{-1}(B)} \in \mathcal{C}^\infty(d^{-1}(B), \mathbb{R}).
\]

We set $\mathcal{D} \mathcal{F} := \bigcup_{U \in \mathcal{O}^{\mathbb{R}}} \mathcal{D}_U \mathcal{F}_U$ and call $\hat{X}_\mathcal{F} := (\|X\|, \mathcal{D} \mathcal{F})$ the diffeological space generated by $X$ and $\mathcal{F}$. If $Y \in \text{Dlg}$ is a diffeological space, we will always apply the above construction with respect to the $D$-topology, i.e., with $X = T_D(Y)$ and considering only smooth functions: $\mathcal{F}_B \subseteq \mathcal{C}^\infty(B \prec X, \mathbb{R})$ for all $B \in \mathcal{B}$.

One can show directly from the definitions that

**Remark 20.** In the hypotheses of the previous Def. 19, the following properties hold:

(i) We can trivially extend the $\mathcal{B}$-family $\mathcal{F}$ to the whole $\sigma$-family by setting $\mathcal{F}_A := \emptyset$ if $A \notin \mathcal{B}$. We will always assume to have extended $\mathcal{F}$ in this way;

(ii) $\mathcal{D} \mathcal{F}$ is a diffeology on $\|X\|$;

(iii) For all $A \in \sigma$ and $l \in \mathcal{F}_A$, we have $l \in \mathcal{C}^\infty(A \prec \hat{X}_\mathcal{F}, \mathbb{R})$;

(iv) The diffeology $\mathcal{D} \mathcal{F}$ of $\hat{X}_\mathcal{F}$ is functionally generated by $\mathcal{F}$.

Moreover, if $X = (\|X\|, D) \in \text{Dlg}$ is a diffeological space, then

(v) $\mathcal{D}_U \subseteq \mathcal{D}_U \mathcal{F}_U$. Hence, $\mathcal{C}^\infty(Y \prec X, \mathbb{R}) \supseteq \mathcal{C}^\infty(Y \prec \hat{X}_{M_X}, \mathbb{R}) \forall Y \subseteq \|X\|$. Together with the above Property (iii), we have $\mathcal{C}^\infty(A \prec X, \mathbb{R}) = \mathcal{C}^\infty(A \prec \hat{X}_{M_X}, \mathbb{R}) \forall A \in \tau_X$;

(vi) $\mathcal{F}$ generates $D$ if and only if $\mathcal{D}_U = \mathcal{D}_U \mathcal{F}_U$;

(vii) the $D$-topology on $\hat{X}_\mathcal{F}$ coincides with the $D$-topology $\tau_X$ on $X$.
(viii) If $F$ generates $D$, then $X = \hat{X}_F$. So $\hat{X}_{\hat{F}} = \hat{X}_F$ for any $\tau_X$-family $F$.
And if $X$ is functionally generated, then $X = \hat{X}_{M^X}$;
(ix) $D M^X$ is the smallest functionally generated diffeology on $|X|$ containing $D$.

In particular, if we take $F$ to be the empty $\tau_X$-family, that is, $F_A = \emptyset$ for all $A \in \tau_X$, then $DF_U = \text{Top}(U, T_D(X))$.

**Theorem 21.** The inclusion functor $\text{DFG} : \text{FDlg} \longrightarrow \text{Dlg}$ is a right adjoint of the functor $\text{FGD} : X \in \text{Dlg} \mapsto \hat{X}_{M^X} \in \text{FDlg}$ (both functors act as identity on arrows). Therefore, for all $X \in \text{Dlg}$ and $Y \in \text{FDlg}$, we have

$$C_\infty(X, Y) = C_\infty(\hat{X}_{M^X}, Y).$$

We call $\text{FGD}(X) = (|X|, D M^X)$ the functional extension of $X$.

**Proof.** It follows by applying Def. 12, Def. 19 and Rem. 20. □

**Corollary 22.** Let $G : I \longrightarrow \text{FDlg}$ be a functor from a small category $I$. Then

$$\text{FGD} \left( \text{colim}_{i \in I} D \text{FGD}(G_i) \right) \simeq \text{colim}_{i \in I} G_i.$$  

Therefore, the category $\text{FDlg}$ is cocomplete.

**Proof.** Since $\text{FGD}$ is a left adjoint, it preserves colimits

$$\text{FGD} \left( \text{colim}_{i \in I} D \text{FGD}(G_i) \right) \simeq \text{colim}_{i \in I} \text{FGD}(D \text{FGD}(G_i)) = \text{colim}_{i \in I} \text{FGD}(G_i).$$

But $G_i \in \text{FDlg}$ is functionally generated, so $\text{FGD}(G_i) = G_i$ from (viii) of Rem. 20. □

**Corollary 23.** Let $X \in \text{Dlg}$ and let $S \in \tau_X$ be a $D$-open subset. Then $\text{FGD}(S \prec X) = (S \prec \text{FGD}(X))$.

**Proof.** By [7, Lem. 3.17], $\tau_{S \prec X} = \{ A \cap S \mid A \in \tau_X \}$ since $S$ is open. By (vii) of Rem. 20 we have $\tau_{\text{FGD}(X)} = \tau_X$ and hence $T_D(\text{FGD}(S \prec X)) = T_D(S \prec X) = T_D(S \prec \text{FGD}(X))$. The smoothness of the identity set map $\text{FGD}(S \prec X) \rightarrow (S \prec \text{FGD}(X))$ follows from Thm. 21, and the smoothness of the inverse set map essentially follows from (v) of Rem. 20. □

Since the coequalizer in $\text{FDlg}$ in general is different from the coequalizer in $\text{Dlg}$, the forgetful functor $\text{DFG} : \text{FDlg} \longrightarrow \text{Dlg}$ has no right adjoint.

Here is another interesting example that colimit in $\text{FDlg}$ is different from the corresponding colimit in $\text{Dlg}$:

**Example 24.** Let $X$ be the pushout of

$$\begin{array}{ccc}
\mathbb{R} & \overrightarrow{0} & \mathbb{R}^0 & \overrightarrow{0} & \mathbb{R} \\
\end{array}$$

(2.3)
Then we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}^0 & \rightarrow & \mathbb{R} \\
\downarrow & & \downarrow i \\
\mathbb{R} & \rightarrow & \mathbb{R}^2 \\
\end{array}
\]

in \(\mathbf{Dlg}\) with \(i(x) = (x,0)\) and \(j(y) = (0,y)\). This induces a smooth injective map \(X \to \mathbb{R}^2\). Write \(Y \in \mathbf{Dlg}\) for the image of this map with the sub-diffeology of \(\mathbb{R}^2\). One can show that

(i) the induced smooth map \(X \to Y\) is not a diffeomorphism;
(ii) the \(D\)-topology on both \(X\) and \(Y\) coincide with the sub-topology of \(\mathbb{R}^2\);
(iii) for any open subset \(A\) of \(\mathbb{R}^2\), \(C^\infty(A \cap X \prec X, \mathbb{R}) = C^\infty(A \cap Y \prec Y, \mathbb{R})\), which implies that \(X\) is not functionally generated;
(iv) \(Y\) is Frölicher because \(\mathbf{Fr}\) is closed with respect to subobjects, so \(Y \in \mathbf{FDlg}\).

Hence, by Cor. 22, the pushout of (2.3) in \(\mathbf{FDlg}\) is \(\mathbf{FGD}(X) \simeq Y \not\simeq X\).

Now we show that the embedding \(\mathbf{FGF} : \mathbf{Fr} \to \mathbf{FDlg}\) is not essentially surjective, that is, there are functionally generated spaces which are not from Frölicher spaces:

**Example 25.** Let \(Y = (-\infty, 0) \cup (0, \infty)\), and let \(X\) be the pushout of

\[
\begin{array}{ccc}
\mathbb{R} & \rightarrow & Y \\
\downarrow & & \downarrow \\
\mathbb{R} & \rightarrow & \mathbb{R} \\
\end{array}
\]

in \(\mathbf{Dlg}\). Then \(C^\infty(X, \mathbb{R}) \simeq C^\infty(\mathbb{R}, \mathbb{R})\). Since no element in \(C^\infty(X, \mathbb{R})\) can detect the double points at origin, there is no Frölicher space such that its image under the embedding \(\mathbf{DF} : \mathbf{Fr} \to \mathbf{Dlg}\) is \(X\). But since the two colimit maps \(\mathbb{R} \to X\) are injective and open, \(X\) is functionally generated. In other words, for any \(U \in \mathcal{O}^\infty\), \(C^\infty(X, \mathbb{R})\) can not detect whether an arbitrary function \(U \to X\) is smooth, but it can detect whether a continuous function \(U \to X\) is smooth. Moreover, the initial topology on \(X\) with respect to \(C^\infty(X, \mathbb{R})\) is strictly coarser than the \(D\)-topology.

**Theorem 26.** The category \(\mathbf{FDlg}\) is Cartesian closed.

**Proof.** Since \(\mathbf{Dlg}\) is Cartesian closed and the product in \(\mathbf{FDlg}\) is the same as the product in \(\mathbf{Dlg}\), it suffices to show that if \(X\) is a diffeological space and \(Y\) is a functionally generated space, then the functional diffeology of the space \(C^\infty(X, Y)\) is functionally generated.

We split the proof of the claim into three steps.

**Step 1:** We prove that if \(C^\infty(\mathbb{R}^n, Y)\) is functionally generated for all \(n \in \mathbb{N}\), then \(C^\infty(X, Y)\) is functionally generated.

To prove that \(C^\infty(X, Y)\) is functionally generated, by (vi) of Rem. 20, for any \(d \in_u \mathbf{FGD}(C^\infty(X, Y))\) we need to show that \(d \in_u C^\infty(X, Y)\), i.e., that
$d^\vee : U \times X \to Y$ is smooth. This is equivalent to show that for any plot $p : \mathbb{R}^n \to X$, the composition

\[
U \times \mathbb{R}^n \xrightarrow{1_U \times p} U \times X \xrightarrow{d^\vee} Y
\]

is a plot of $Y$. This is again equivalent to show that the composition

\[
U \xrightarrow{d} C^\infty(X,Y) \xrightarrow{p^*} C^\infty(\mathbb{R}^n, Y)
\]

is smooth. By assumption $C^\infty(\mathbb{R}^n, Y)$ is functionally generated, and the map $p^*$ is smooth, so the adjunction $FG_D \dashv D_{FG}$ (Thm. 21) implies that $p^* : FG_D(C^\infty(X,Y)) \to C^\infty(\mathbb{R}^n, Y)$ is smooth. But $d \in_U FG_D(C^\infty(X,Y))$. So $p^* \circ d : U \to C^\infty(\mathbb{R}^n, Y)$ is smooth, which prove our first claim.

**Step 2:** We prove below that if $d : U \to C^\infty(\mathbb{R}^n, Y)$ is a continuous map, then the induced function $d^\vee : U \times \mathbb{R}^n \to Y$ is continuous.

Let $A$ be a $D$-open subset of $Y$, and let $(u, x) \in (d^\vee)^{-1}(A)$. Since $d(u) \in C^\infty(\mathbb{R}^n, Y)$, $(d(u))^{-1}(A)$ is an open neighborhood of $x \in \mathbb{R}^n$. Take a relatively compact open neighborhood $V$ of $x \in \mathbb{R}^n$ such that its closure $\bar{V} \subseteq (d(u))^{-1}(A)$. Write $\tilde{A} = \{ f \in C^\infty(\mathbb{R}^n, Y) \mid f(V) \subseteq A \}$. Since the $D$-topology on $C^\infty(\mathbb{R}^n, Y)$ contains the compact-open topology ([7, Prop. 4.2]), $\tilde{A}$ is $D$-open in $C^\infty(\mathbb{R}^n, Y)$. Hence, $W := d^{-1}(\tilde{A})$ is an open neighborhood of $u \in U$. Therefore, $W \times V$ is an open neighborhood of $(u, x) \in (d^\vee)^{-1}(A)$, which implies that the map $d^\vee$ is continuous.

**Step 3:** We prove below that $C^\infty(\mathbb{R}^n, Y)$ is functionally generated.

Let $d \in_U FG_D(C^\infty(\mathbb{R}^n, Y))$. We need to show that the induced function $d^\vee : U \times \mathbb{R}^n \to Y$ is smooth. From Step 2, we know that $d^\vee$ is continuous. Since $Y$ is functionally generated, it is enough to show that for any $D$-open subset $A$ of $Y$ and any $l \in C^\infty(A \prec Y, \mathbb{R})$, the composition

\[
(d^\vee)^{-1}(A) \xrightarrow{(d^\vee)^{-1}(A)} A \xrightarrow{l} \mathbb{R}
\]

is smooth. For any $(u, x) \in (d^\vee)^{-1}(A)$, use the notations $\tilde{A}$, $V$ and $W$ defined in Step 2. Since smoothness is a local condition, it is enough to show that the composition

\[
W \times V \xrightarrow{(d^\vee)^{-1}(A)} A \xrightarrow{l} \mathbb{R}
\]

is smooth. Equivalently, we need to show that the composition

\[
W \xrightarrow{d|_W} (\tilde{A} \prec C^\infty(\mathbb{R}^n, Y)) \xrightarrow{\text{Res}} C^\infty(V, A) \xrightarrow{l_*} C^\infty(V, \mathbb{R})
\]

is smooth, where Res is the restriction map. It is easy to see that the map $\text{Res} : (\tilde{A} \prec C^\infty(\mathbb{R}^n, Y)) \to C^\infty(V, A)$ is smooth, so $l_* \circ \text{Res} : (\tilde{A} \prec C^\infty(\mathbb{R}^n, Y)) \to C^\infty(V, \mathbb{R})$ is smooth. Since $d \in_U FG_D(C^\infty(\mathbb{R}^n, Y))$, we get $d|_W \in_W (\tilde{A} \prec FG_D(C^\infty(\mathbb{R}^n, Y)))$. But both $V$ and $\mathbb{R}$ are Frölicher
spaces, so \( C^\infty(V, \mathbb{R}) \) is functionally generated, and the adjunction \( \text{FG}_\text{D} \vdash \text{D}_\text{FG} \) (Thm. 21) implies that the map \( l_\ast \circ \text{Res} : \text{FG}_\text{D}(\tilde{A} \prec C^\infty(\mathbb{R}^n, Y)) \longrightarrow C^\infty(V, \mathbb{R}) \) is smooth. By Cor. 23 we have \( \text{FG}_\text{D}(\tilde{A} \prec C^\infty(\mathbb{R}^n, Y)) = (\tilde{A} \prec \text{FG}_\text{D}(C^\infty(\mathbb{R}^n, Y))) \), so the conclusion follows. \( \square \)

2.4. Preservation of limits and (suitable) colimits of manifolds. In this subsection, we are going to discuss the question that if a limit (or colimit) exists in \textit{Man}, the category of smooth manifolds and smooth maps, then is it the same as the corresponding limit (or colimit) in \textit{FDlg}? The statements of the main results and the idea of proofs mainly come from [29].

\textbf{Theorem 27.} [29] \textit{Let} \( F : \mathcal{I} \longrightarrow \text{Man} \textit{be a functor. Assume that} \lim F \textit{exists in} \textit{Man}. Write} \( \text{FG}_\text{M} : \text{Man} \longrightarrow \text{FDlg} \textit{for the embedding functor. Then} \text{FG}_\text{M}(\lim F) \simeq \lim(\text{FG}_\text{M} \circ F). \)

\textit{Proof.} By the universal property of limit in \textit{FDlg}, there is a canonical smooth map \( \eta : \text{FG}_\text{M}(\lim F) \longrightarrow \lim(\text{FG}_\text{M} \circ F). \)

First we prove that \( |\eta| \) is surjective. Note that any \( x \in |\lim(\text{FG}_\text{M} \circ F)| \) corresponds to a smooth map \( x : \mathbb{R}^0 \longrightarrow \lim(\text{FG}_\text{M} \circ F). \) So we have a cone \( x \longrightarrow F. \) Since \( \mathbb{R}^0 \) is a smooth manifold and \( \lim F \) exists in \textit{Man}, by the universal property of limit in \textit{Man} and \textit{FDlg}, there exists \( y : \mathbb{R}^0 \longrightarrow \lim F \) such that \( x = \eta \circ \text{FG}_\text{M}(y) \), which implies that \( |\eta| \) is surjective.

Next we prove that \( |\eta| \) is injective. If \( a, a' \in |\text{FG}_\text{M}(\lim F)| \) such that \( |\eta|(a) = |\eta|(a') \), then the two cones \( a \longrightarrow F \) and \( a' \longrightarrow F \) have the same image in the target. By the universal property of limit in \textit{Man}, \( a = a' \).

Finally, we prove that \( \eta^{-1} \) is smooth. Let \( d \in \lim(\text{FG}_\text{M} \circ F). \) Since the functor \( \text{FG}_\text{M} \) is fully faithful, we get a cone \( U \longrightarrow F. \) Note that \( \text{FG}_\text{M}(U) = U. \) By the universal property of limit in \textit{Man} and \textit{FDlg}, we get a smooth map \( f : U \longrightarrow \lim F \) such that \( \eta^{-1} \circ d = \text{FG}_\text{M}(f). \) Hence, \( \eta^{-1} \) is smooth.

Therefore, \( \text{FG}_\text{M}(\lim F) \simeq \lim(\text{FG}_\text{M} \circ F). \) \( \square \)

\textbf{Remark 28.} Note that the category \( \mathcal{O}R^\infty \) with the usual open coverings is a site. [3] showed that \( \mathcal{O}R^\infty \) is a concrete site, and the category \( \text{Dlg} \) is equivalent to the category \( \mathcal{C}\text{Sh}((\mathcal{O}R^\infty)) \) of concrete sheaves over \( \mathcal{O}R^\infty \).

We write \( \mathcal{C}\text{Pre}(\mathcal{O}R^\infty) \), \( \mathcal{P}\text{re}(\mathcal{O}R^\infty) \) and \( \mathcal{S}\text{h}(\mathcal{O}R^\infty) \) for the category of concrete presheaves over \( \mathcal{O}R^\infty \), the category of presheaves over \( \mathcal{O}R^\infty \) and the category of sheaves over \( \mathcal{O}R^\infty \), respectively. There are embedding functors

\[ \text{Man} \longrightarrow \text{Fr} \longrightarrow \text{FDlg} \longrightarrow \text{Dlg} \longrightarrow \mathcal{C}\text{Pre}(\mathcal{O}R^\infty) \longrightarrow \mathcal{P}\text{re}(\mathcal{O}R^\infty) \]

and

\[ \text{Man} \longrightarrow \text{Dlg} \longrightarrow \mathcal{S}\text{h}(\mathcal{O}R^\infty). \]

Moreover, if a limit exists in \textit{Man}, then the corresponding limits in all the other categories listed above are isomorphic to that limit in the corresponding categories.

In general, if a colimit in \textit{Man} exists, when viewed as a functionally generated space it may be different from the corresponding colimit in \textit{FDlg}.
Example 29. Recall that in Ex. 25, we showed that the underlying set of
the pushout $X$ of

$$\begin{array}{ccc}
\mathbb{R} & \rightarrow & \mathbb{R} \setminus \{0\} \\
\downarrow & & \downarrow \\
\mathbb{R} & \leftarrow & \mathbb{R}
\end{array}$$

in $\text{FDlg}$ has double points at origin. Moreover, the $D$-topology $\tau_X$ is not
Hausdorff. One can also show by continuity that the pushout of this diagram
in $\text{Man}$ exists, and it is $\mathbb{R}$. Therefore, $X \not\simeq \mathbb{R}$ in $\text{FDlg}$.

Theorem 30. [29] Let $G : J \rightarrow \text{Man}$ be a functor such that $\text{colim } G$ exists
in $\text{Man}$. Then the canonical smooth map

$$\eta : \text{colim}(\text{FM} \circ G) \rightarrow \text{FM}(\text{colim } G)$$

induces a surjective map

$$|\eta| : |\text{colim}(\text{FM} \circ G)| \rightarrow |\text{FM}(\text{colim } G)|.$$

Proof. The canonical smooth map $\eta : \text{colim}(\text{FM} \circ G) \rightarrow \text{FM}(\text{colim } G)$
comes from the universal property of colimits in $\text{Man}$ and $\text{FDlg}$.

Assume that $|\eta|$ is not surjective. Say $y \in |\text{FM}(\text{colim } G)|$ is not in the
image. Then $A := \text{colim } G \setminus \{y\}$ is a smooth manifold, and $G(j) \rightarrow \text{colim } G$
factors through $A \hookrightarrow \text{colim } G$ for each $j \in J$ since $|\text{colim}(\text{FM} \circ G)| =
\text{colim} |\text{FM} \circ G|$. Hence, by the universal property of colimit in $\text{Man}$, the
identity map $\text{colim } G \rightarrow \text{colim } G$ must factor through $A \hookrightarrow \text{colim } G$, which
is impossible. Therefore, $|\eta|$ is surjective. \qed

Theorem 31. [29] Let $G : J \rightarrow \text{Man}$ be a functor such that $\text{colim } G$
exists in $\text{Man}$. If $\text{colim}(\text{FM} \circ G)$ is a Frölicher space and its $D$-topology
is Hausdorff, then $\text{colim}(\text{FM} \circ G) \simeq \text{FM}(\text{colim } G)$.

Proof. Let $X$ be a diffeological space. It is direct to show that

1. If $C^\infty(X, \mathbb{R})$ separates points, that is,

   $$\forall x \neq x' \in X, \exists l \in C^\infty(X, \mathbb{R}) : l(x) \neq l(x'),$$

   then the $D$-topology $\tau_X$ is Hausdorff.

2. If the $D$-topology $\tau_X$ is Hausdorff, then any plot $\mathbb{R} \rightarrow X$ with finite
   image must be constant.

3. If any plot $\mathbb{R} \rightarrow X$ with finite image must be constant and $X$ is
   Frölicher, then $C^\infty(X, \mathbb{R})$ separates points.

By Thm. 30, the canonical smooth map

$$\eta : \text{colim}(\text{FM} \circ G) \rightarrow \text{FM}(\text{colim } G)$$

induces a surjective map

$$|\eta| : |\text{colim}(\text{FM} \circ G)| \rightarrow |\text{FM}(\text{colim } G)|.$$

For any $l \in C^\infty(\text{colim}(\text{FM} \circ G), \mathbb{R})$, by the universal property of colimits
in $\text{Man}$ and $\text{FDlg}$, there exists a unique smooth map $f : \text{colim } G \rightarrow \mathbb{R}$
such that $\text{FM}(f) \circ \eta = l$. 
Since \( \text{colim}(FG_M \circ G) \) is Frölicher and its \( D \)-topology is Hausdorff, from the above we know that \( C^\infty(\text{colim}(FG_M \circ G), \mathbb{R}) \) separates points. Then the equality \( FG_M(f) \circ \eta = l \) implies that \( |\eta| \) is injective.

For any \( d \in U \text{colim}(G) \), \( l \circ \eta^{-1} \circ d = FG_M(f) \circ d \) is smooth for any \( l \in C^\infty(\text{colim}(FG_M \circ G), \mathbb{R}) \). Since \( \text{colim}(FG_M \circ G) \) is Frölicher, \( \eta^{-1} \circ d \in U \text{colim}(FG_M \circ G) \). Hence, \( \eta^{-1} \) is smooth.

Therefore, \( \text{colim}(FG_M \circ G) \simeq FG_M(\text{colim} G) \). \( \square \)

Here is an immediate application:

**Example 32.** Recall from Ex. 24 that the pushout of

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{0} & \mathbb{R}^0 & \xrightarrow{0} & \mathbb{R}
\end{array}
\]

in \( \text{FDlg} \) is the union of the two axes in \( \mathbb{R}^2 \) with the sub-diffeology, which is a Frölicher space with Hausdorff \( D \)-topology, but clearly not a smooth manifold. By Thm. 31, the pushout of the above diagram does not exist in \( \text{Man} \).

2.5. **Categorical frameworks for generalized functions.** We start by asking the following question: What is a “good” category to frame spaces like \( \mathcal{D}(\Omega), \mathcal{D}'(\Omega), \mathcal{A}_0(\Omega), U(\Omega), G^*(\Omega) \) and \( G^c(\Omega) \)?

The following list of remarks permits to restrict the range of choices:

**Remark 33.**

(i) Schwartz distribution theory is classically framed using locally convex topological vector spaces (LCTVS), so it is natural to search for a category which contains the category \( \text{LCS} \) of LCTVS and continuous linear maps as a subcategory.

(ii) The space \( \mathcal{A}_0(\Omega) \) is an affine space and is usually identified with its underlying vector space (see e.g. [21]). However, it seems that the necessity of this identification is only due to the choice of a category like \( \text{LCS} \), which is not closed with respect to arbitrary subspaces. It would be better to choose a complete category.

(iii) \( U(\Omega) \) and \( \mathcal{A}_0(\Omega) \) can be viewed as manifolds modelled in convenient vector spaces (CVS, [11, 26]). However, the category of this type of manifolds is not Cartesian closed ([26]), whereas Cartesian closedness is a basic choice preferred by several mathematicians working with infinite dimensional spaces (see e.g. [16] and references therein).

(iv) The candidate category shall contain the category \( \text{Con}^\infty \) of convenient vector spaces and generic smooth maps ([11, 26]) as a (full) subcategory because the differential calculus of these spaces is used in the study of Colombeau algebras ([21]). Note that we have embeddings \( \text{Con} \subseteq \text{LCS} \subseteq \text{Con}^\infty \), where \( \text{Con} \) is the category of CVS and continuous linear maps.

(v) The candidate category must be closed with respect to arbitrary quotient spaces, so as to contain the quotient algebras \( G^*(\Omega) \) and \( G^c(\Omega) \).
Of course, a better choice would be to consider a cocomplete category. Since, generally speaking, CVS are not closed with respect to quotient spaces (see [26, page 22]), the candidate category cannot be $\text{Con}^\infty$.

(vi) The candidate category must also contain nonlinear maps like the product of GF, e.g.

$$(u, v) \in G^s(\Omega) \times G^s(\Omega) \mapsto u \cdot v \in G^s(\Omega)$$

or, more generally, any nonlinear smooth operation $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ which can be extended to an operation of our algebras of GF, e.g.

$$(u_1, \ldots, u_n) \in (G^s(\Omega))^n \mapsto [f(u_1(-), \ldots, u_n(-))] \in G^s(\Omega).$$

Another feature of the candidate category we are looking for is to contain as arrows the maps between infinite dimensional spaces like convolutions, derivatives and integrals of GF.

In the literature, there are only two categories satisfying all these requirements: the category $\text{Dlg}$ of diffeological spaces, and the category $\text{Fr}$ of Frölicher spaces. In the present work, we will also introduce the category $\text{FDlg}$ of functionally generated spaces as another framework for GF, trying to take the best ideas and properties of both $\text{Dlg}$ and $\text{Fr}$. We will see that all the spaces $D_K(\Omega), D(\Omega), D'(\Omega), A_q(\Omega), U(\Omega), G^s(\Omega)$ and $G^e(\Omega)$ are objects of these categories, and in this paper, we in particular study them as functionally generated spaces.

3. Topologies for spaces of generalized functions

[26, page 2] declared that “locally convex topology is not appropriate for non-linear questions in infinite dimensions”. Indirectly, this is also confirmed by the fact that topology plays a less important role in categories like $\text{Dlg}$ or $\text{Fr}$. The main aim of this section is to highlight some relationship between Cartesian closedness and locally convex topology.

3.1. Locally convex vector spaces and Cartesian closed categories.

The problems that arise in relating locally convex topology and Cartesian closedness can be expressed as follows:

**Theorem 34.** Let $F' \in \text{LCS}$, and let $(\mathcal{T}, U)$ be a Cartesian closed concrete category over $\text{Top}$, with exponential objects given by the hom-functor $\mathcal{T}(-, -)$ and the forgetful functor $U : \mathcal{T} \to \text{Top}$ acting as identity on arrows. Assume that $R, \bar{F} \in \mathcal{T}$, $U(R) = \mathbb{R}$ and $U(\bar{F}) = F$. Set $F' := \text{LCS}(F, \mathbb{R})$ for the continuous dual of $F$ and assume that

$$F' \subseteq |U(\mathcal{T}(\bar{F}, R))|,$$

(3.1)

$$|U(\bar{F} \times \mathcal{T}(\bar{F}, R))| = |U(\bar{F}) \times U(\mathcal{T}(\bar{F}, R))|$$

and the topology of the space $U(\bar{F} \times \mathcal{T}(\bar{F}, R))$ is coarser than the product topology of $U(\bar{F}) \times U(\mathcal{T}(\bar{F}, R))$. Finally assume that for all $g \in F'$ the map
(\lambda \in \mathbb{R} \mapsto \lambda \cdot g \in F') is continuous with respect to the topology induced on $F'$ by (3.1). Then the locally convex topology on the space $F$ is normable.

**Proof.** The idea for the proof is only a reformulation of the corresponding result in [26, page 2]. Since $\mathcal{T}$ is Cartesian closed, every evaluation

$$ev_{XY}(x, f) := f(x) \quad \forall x \in X \forall f \in \mathcal{T}(X, Y)$$

is an arrow of $\mathcal{T}$ (this is a general result in every Cartesian closed category, see e.g. [1]). Thus, $U(ev_{XY}) = ev_{XY}$ is a continuous function. In particular, $ev_{\bar{F}R} : U(\bar{F} \times \mathcal{T}(\bar{F}, R)) \rightarrow U(R) = \mathbb{R}$ is continuous. By assumption, also $ev_{\bar{F}R} : F \times U(\mathcal{T}(\bar{F}, R)) \rightarrow \mathbb{R}$ is continuous. Therefore, also its restriction to the subspace $F' = \text{LCS}(F, \mathbb{R}) \subseteq [U(\mathcal{T}(\bar{F}, R))]$ is (jointly) continuous:

$$\varepsilon := ev_{\bar{F}R}|_{F \times F'} : F \times F' \rightarrow \mathbb{R}.$$ 

Hence, we can find neighborhoods $U \subseteq F$ and $V \subseteq F'$ of zero such that $\varepsilon(U \times V) \subseteq [-1, 1]$, that is

$$U \subseteq \{ u \in F | \forall f \in V : |f(u)| \leq 1 \}.$$ 

But then, because the map $(\lambda, g) \mapsto \lambda \cdot g \in F'$ is continuous, taking a generic functional $g \in F'$, we can always find $\lambda \in \mathbb{R} \neq 0$ such that $\lambda g \in V$, and hence $|g(u)| \leq 1/\lambda$ for every $u \in U$. Any continuous functional is thus bounded on $U$, so the neighborhood $U$ itself is bounded (see e.g. [24]). Since the topology of any locally convex vector space with a bounded neighborhood of zero is normable (see e.g. [24]), we get the conclusion. □

If, in this theorem, we take $F = C^\infty(\mathbb{R}, \mathbb{R})$ or $F = \mathcal{D}(\Omega)$ or any other non-normable LCTVS, there are two possibilities to make the space $F$ an object in a Cartesian closed category:

(a) $\mathcal{F}$ belongs to a Cartesian closed category $\mathcal{T}$, but $\mathcal{T}$ is not a concrete category over $\text{Top}$. This is the solution used in CVS theory which are embedded in the Cartesian closed category $\text{Con}^\infty$. Note that $\text{LCS}$ is not a full subcategory of $\text{Con}^\infty$ since not every arrow of $\text{Con}^\infty$ is continuous. A typical example of a $\text{Con}^\infty$-smooth but not continuous map (with respect to the given locally convex topology instead of the $D$-topology) is the evaluation

$$ev : (x, g) \in F \times F' \mapsto g(x) \in \mathbb{R}.$$ 

(b) For the solution adapted by using diffeological spaces, we can take $\mathcal{T} = \text{Dlg}$. But then several assumptions of Thm. 34 fail: e.g. in general $\tau_{X \times Y} \supseteq \tau_X \times \tau_Y$, but not the opposite as required; moreover, the $D$-topology on $\mathcal{D}(\Omega)$ is not normable since it is finer than the usual locally convex topology, which is also not normable. On the contrary, there is no problem regarding the continuity of the product by scalar, as stated in the following:
Theorem 35. Let $F$ be any one of the spaces $C^\infty(\Omega, \mathbb{R})$ or $D(\Omega)$, and let $\tau_F$ be the $D$-topology on $F$. Let

$$F'_s := \{ l \in C^\infty(F, \mathbb{R}) \mid l \text{ is linear} \} \times C^\infty(F, \mathbb{R})$$

be the smooth dual of $F$, and let $\tau_{F'_s}$ be the $D$-topology on $F'_s$. Then, with respect to pointwise operations, both spaces $(F, \tau_F)$ and $(F'_s, \tau_{F'_s})$ are topological vector spaces.

Proof. We proceed for the case $F = C^\infty(\Omega, \mathbb{R})$ since the other one is very similar. For simplicity set $Y^X := C^\infty(X, Y)$, and

$$\langle -, - \rangle : (u, v) \in F \times F \mapsto (r \in \Omega \mapsto (u(r), v(r)) \in \mathbb{R}^2) \in (\mathbb{R}^2)^\Omega$$

$$\gamma_1 : (u, v) \in (\mathbb{R}^2)^\Omega \times \mathbb{R}^2 \mapsto v \circ u \in F$$

$$\gamma_2 : (u, v) \in \mathbb{R}^R \times F \mapsto u \circ v \in F$$

$$s_R : (r, s) \in \mathbb{R}^2 \mapsto r \in \mathbb{R}$$

$$p_R : (r, s) \in \mathbb{R}^2 \mapsto r \cdot s \in \mathbb{R}$$

$$(-,-) : (u, v) \in F^R \times F^R \mapsto ((\lambda, f) \in \mathbb{R} \times F \mapsto (u(\lambda, f), v(\lambda, f)) \in F \times F) \in (F \times F)^R \times F^R.$$ 

It is easy to prove that the pointwise sum and pointwise product by scalars are given by $(-) + (-) = \gamma_1(-, s_R) \circ (-, -)$ and $(-) \cdot (-) = \gamma_2 \circ (p_R \circ q_1, q_2)$, where $q_1 : \mathbb{R} \times F \rightarrow \mathbb{R}$ and $q_2 : \mathbb{R} \times F \rightarrow F$ are the projections. Therefore, both sum and product in $F$ are composition or pairing of smooth functions, and hence they are smooth and continuous in the $D$-topology. Analogously, we can proceed with the smooth dual $F'_s$ by considering the properties of the operator $(- \prec -)$.

4. Spaces of compactly supported functions as functionally generated spaces

It is very easy to see that the spaces $\mathcal{D}(\Omega) = \{ f \in C^\infty(\Omega, \mathbb{R}) \mid \text{supp}(f) \subseteq \Omega \}$ and $\mathcal{D}_K(\Omega) = \{ f \in \mathcal{D}(\Omega) \mid \text{supp}(f) \subseteq K \}$ with $K \subseteq \Omega$ are functionally generated spaces. Recall that $\mathcal{D}_K(\Omega)$ is a LCTVS whose topology is induced by the family of norms (Fréchet structure)

$$\| \varphi \|_{K, m} := \max_{|\alpha| \leq m} \max_{x \in K} \| \partial^\alpha \varphi(x) \| \quad \forall \varphi \in \mathcal{D}_K(\Omega) \forall m \in \mathbb{N}.$$ (4.1)

Also the space $\mathcal{D}(\Omega)$ is a LCTVS obtained as the inductive limit (colimit) of $\mathcal{D}_K(\Omega)$ for $K \subseteq \Omega$, i.e.:

$A$ is open in $\mathcal{D}(\Omega) \iff \forall K \subseteq \Omega : i_K^{-1}(A) = A \cap \mathcal{D}_K(\Omega)$ is open in $\mathcal{D}_K(\Omega)$,

(4.2)
where $i_K : D_K(\Omega) \hookrightarrow D(\Omega)$. Recall that on the space $D'(\Omega)$ of distributions (i.e., linear maps $l : |D(\Omega)| \rightarrow \mathbb{R}$ which are continuous with respect to the locally convex topology) there is a topology called weak* topology, i.e., the coarsest topology such that each evaluation $ev_\varphi : u \in D'(\Omega) \mapsto \langle u, \varphi \rangle \in \mathbb{R}$ is continuous. With respect to this topology, $D'(\Omega)$ is a LCTVS.

The so-called canonical diffeology on these spaces is a particular case of the following:

**Definition 36.** Let $V$ be a topological vector space. The canonical diffeology $D(V) = \bigcup_{U \in O_{\mathbb{R}\infty}} D_U(V)$, is given by the sets $D_U(V)$ of all maps $d : U \rightarrow V$ which are smooth when tested by continuous linear functionals, i.e.,

$$\forall l : V \rightarrow \mathbb{R} \text{ continuous linear: } l \circ d \in C^\infty(U, \mathbb{R}).$$

Therefore (see Def. 19 and Rem. 20), $(V, D(V)) \in \mathcal{FDlg}$ and, since the functionals are globally defined, this is also a Frölicher space, i.e.,

$$D_F(F_D(V, D(V))) = (V, D(V)).$$

We will continue to denote our spaces by $D(\Omega)$ and $D_K(\Omega)$ even when we think of them as diffeological spaces with the canonical diffeology. When we want to underscore that we are considering them only as LCTVS with the topology given by (4.2) and (4.1), we will use the notations $D_L^C(\Omega)$ and $D_K^L(\Omega)$.

### 4.1. Plots of $D_K(\Omega)$, $D(\Omega)$ and Cartesian closedness

It is also interesting to reformulate the property of being a plot $d \in_u D(\Omega)$, or $d \in_u D_K(\Omega)$, using Cartesian closedness. This permits to compare better the canonical diffeology on these spaces as LCTVS with the diffeology induced on them as subspaces of $C^\infty(\Omega, \mathbb{R})$. We will denote with $D^s(\Omega)$ this diffeological space, so that $d \in_u D^s(\Omega)$ if and only if $d : U \rightarrow |D(\Omega)|$ and $i \circ d \in_u C^\infty(\Omega, \mathbb{R})$, where $i : D(\Omega) \hookrightarrow C^\infty(\Omega, \mathbb{R})$ is the inclusion. Recall that $i \circ d \in_u C^\infty(\Omega, \mathbb{R})$ holds if and only if $(i \circ d)^\vee \in C^\infty(U \times \Omega, \mathbb{R})$. Analogously, we define $D_K^s(\Omega)$. In performing this comparison, we will use Lem. 2.1, Lem. 2.2 and Thm. 2.3 of [25] which are cited here for reader’s convenience. In this subsection, without confusion we use the same notation for morphisms in different categories when the functions for the underlying sets are the same.

**Lemma 37** (2.1 of [25]). If $U \in O_{\mathbb{R}^\infty}$ and $f \in C^\infty(U, D(\Omega))$, then $f : U \rightarrow D^L_C(\Omega)$ is continuous.

To state the other cited results of [25], we need the following:

**Definition 38.** Let $U \in O_{\mathbb{R}^\infty}$ and let $f : U \times \Omega \rightarrow \mathbb{R}$ be a map. We say that $f$ is of uniformly bounded support (with respect to $U$) if

$$\exists K \Subset \Omega \forall u \in U : \text{supp}(f(u, -)) \subseteq K.$$
We say that
\[ f \] is locally of uniformly bounded support
if
\[ \forall u \in U \ \exists V \text{ open neigh. of } u \in U : f|_{V \times \Omega} \text{ is of uniformly bounded support.} \]
Finally we say that
\[ f \] is pointwise of bounded support
if
\[ \forall u \in U \ \exists K \subseteq \Omega : \text{supp}(f(u, -)) \subseteq K. \]

Using this definition, we can state

**Lemma 39** (2.2 of [25]). Let \( U \in \mathcal{O}^{\mathbb{R}\infty} \) and assume that \( f \in C^\infty(U \times \Omega, \mathbb{R}) \) is pointwise of bounded support. Then the following are equivalent:

(i) \( f \) is locally of uniformly bounded support;
(ii) \( f^\wedge : U \to D_L^L(\Omega) \) is continuous.

**Theorem 40** (2.3 of [25]). Let \( U \in \mathcal{O}^{\mathbb{R}\infty} \). Then the following are equivalent:

(i) \( f \in C^\infty(U, D(\Omega)) \);
(ii) \( f^\vee \in C^\infty(U \times \Omega, \mathbb{R}) \) and \( f^\wedge \) is locally of uniformly bounded support.

In other words, Thm 40 says that \( d \in U \ D(\Omega) \) if and only if \( d \in U \ D^s(\Omega) \) and \( d^\wedge \) is locally of uniformly bounded support, and hence we have \( D(\Omega) \subseteq D^s(\Omega) \).

From these results we can also solve the same problem for the spaces \( D_K(\Omega) \) and \( D^s_K(\Omega) \). The following lemma is analogous to Lem. 37 for \( D_K(\Omega) \).

**Lemma 41.** If \( f \in C^\infty(U, D_K(\Omega)) \) with \( U \in \mathcal{O}^{\mathbb{R}\infty} \) and \( K \subseteq \Omega \), then \( f : U \to D_L^L(\Omega) \) is continuous.

**Proof.** Since the inclusion map \( i_K : D_L^L(\Omega) \to D_L^L(\Omega) \) is continuous linear, by post-composition it also takes continuous linear functionals \( l : D_L^L(\Omega) \to \mathbb{R} \) into continuous linear functionals \( l \circ i_K : D_L^L(\Omega) \to \mathbb{R} \). From Thm. 13 it follows that \( i_K \in C^\infty(D_K(\Omega), D(\Omega)) \) and hence \( i_K \circ f \in C^\infty(U, D(\Omega)) \). Therefore, Lem. 37 implies that \( i_K \circ f \) is continuous and hence the conclusion since the topology on \( D_L^L(\Omega) \) coincides with the initial topology induced by \( i_K \).

The following lemma is analogous to Lem. 39 for \( D_K(\Omega) \).

**Lemma 42.** Let \( U \in \mathcal{O}^{\mathbb{R}\infty} \) and let \( f \in C^\infty(U \times \Omega, \mathbb{R}) \). If there exists \( K \subseteq \Omega \) such that
\[ \forall u \in U : \text{supp}(f(u, -)) \subseteq K \] (4.3) then \( f^\wedge : U \to D_L^L(\Omega) \) is continuous.

**Proof.** Clearly \( f \) is also locally of uniformly bounded support. Apply Lem. 39, we know that \( i_K \circ f^\wedge : U \to D_L^L(\Omega) \) is continuous. Since \( D_L^L(\Omega) \) has the initial topology from \( i_K : D_L^L(\Omega) \to D_L^L(\Omega), f^\wedge : U \to D_L^L(\Omega) \) is continuous.  \( \square \)
Finally, the following theorem is analogous to Thm. 40 for $\mathcal{D}_K(\Omega)$.

**Theorem 43.** Let $U \in \mathcal{O}_{\mathbb{R}^\infty}$ and let $K \subseteq \Omega$. Then the following are equivalent:

(i) $f \in C^\infty(U, \mathcal{D}_K(\Omega))$;
(ii) $f^\vee \in C^\infty(U \times \Omega, \mathbb{R})$ and $\text{supp}(f^\vee(u,-)) \subseteq K$ for all $u \in U$.

**Proof.** (i) $\Rightarrow$ (ii). We already proved in Lem. 41 that the inclusion map $i_K \in C^\infty(\mathcal{D}_K(\Omega), \mathcal{D}(\Omega))$, so $i_K \circ f \in C^\infty(U, \mathcal{D}(\Omega))$. By Thm. 40 we have $(i_K \circ f)^\vee = f^\vee \in C^\infty(U \times \Omega, \mathbb{R})$. The second part of the conclusion follows from the codomain $\mathcal{D}_K(\Omega)$ of $f$ in (i).

(ii) $\Rightarrow$ (i). Assumption (ii) implies that $f$ is locally of uniformly bounded support. From Thm. 40 we thus obtain that $f \in C^\infty(U, \mathcal{D}(\Omega))$. But our assumption implies that $f(U) \subseteq |\mathcal{D}_K(\Omega)|$. So the conclusion follows from the following Lem. 44.

**Lemma 44.** If $K \subseteq \Omega$, then $|\mathcal{D}_K(\Omega)| \prec \mathcal{D}(\Omega) = \mathcal{D}_K(\Omega)$.

**Proof.** We have to prove that figures of both spaces are equal.

(1) $|\mathcal{D}_K(\Omega)| \prec \mathcal{D}(\Omega)$: This follows directly from the fact that the inclusion map $i_K \in C^\infty(\mathcal{D}_K(\Omega), \mathcal{D}(\Omega))$.

(2) $|\mathcal{D}_K(\Omega)| \prec \mathcal{D}_K(\Omega)$: Assume that $d : U \rightarrow \mathcal{D}_K(\Omega)$ is a map such that $i_K \circ d \in_U \mathcal{D}(\Omega)$, i.e., $\lambda \circ i_K \circ d \in C^\infty(U, \mathbb{R})$ for all $\lambda \in \mathcal{D}'(\Omega)$. We need to prove that $l \circ d \in C^\infty(U, \mathbb{R})$ for all continuous linear maps $l : \mathcal{D}^LC_K(\Omega) \rightarrow \mathbb{R}$. So the problem is to extend any such given $l$ to some $\lambda \in \mathcal{D}'(\Omega)$. To this end, we can repeat the usual proof of the local form of distributions as derivatives of continuous functions to obtain the following: $\square$

**Theorem 45.** For any continuous linear map $l : \mathcal{D}^LC_K(\Omega) \rightarrow \mathbb{R}$ there exist $g \in C^0(\Omega, \mathbb{R})$ and $\alpha \in \mathbb{N}^n$ such that

$$l(\varphi) = \langle \partial^\alpha g, \varphi \rangle \quad \forall \varphi \in \mathcal{D}_K(\Omega).$$

Therefore, the continuous functional $\langle \partial^\alpha g, \cdot \rangle : \mathcal{D}^LC_K(\Omega) \rightarrow \mathbb{R}$ extends the functional $l$.

The conclusion follows by applying this theorem. $\square$

**Corollary 46.** If $K \subseteq \Omega$, then $\mathcal{D}_K(\Omega) = \mathcal{D}^*(\Omega)$.

**Fact.** Indeed Thm. 43 says that $d \in_U \mathcal{D}_K(\Omega)$ if and only if $d \in_U \mathcal{D}^*(\Omega)$.

4.2. The locally convex topology and the $D$-topology on $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$. In this section we present some results about functionals on the spaces $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$ which are continuous with respect to the locally convex topology and the $D$-topology. The first result follows at once from Lem. 37 and Lem. 41.

**Corollary 47.** On the spaces $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$, the $D$-topology is finer than the locally convex topology.
It remains an open problem whether the \( D \)-topology is strictly finer than the locally convex topology or not. We first study the behaviour of maps of the form \( \lambda : D(\Omega) \to D(\Omega') \), where henceforth we always assume that \( \Omega' \subseteq \mathbb{R}^d \) is open.

**Theorem 48.**

(i) \( D(\Omega) \) is a CVS.

(ii) If \( T \in C^\infty(D(\Omega), \mathbb{R}) \) is linear, then \( T : D^{LC}(\Omega) \to \mathbb{R} \) is continuous.

The same results hold for \( D_K(\Omega) \).

**Proof.** See [25, page 5, 6, 9] or [26, Lem. 6.2, page 67]. \( \square \)

The following lemma is a trivial consequence of Thm. 13, but we prefer to state it for completeness.

**Lemma 49.** If \( \lambda : D^{LC}(\Omega) \to D^{LC}(\Omega') \) is continuous linear, then \( \lambda \in C^\infty(D(\Omega), D(\Omega')) \).

In the following results, we show that if a linear map \( |D(\Omega)| \to \mathbb{R} \) is continuous with respect to the \( D \)-topology, then it is a distribution:

**Theorem 50.** If \( l : T_D(D(\Omega)) \to \mathbb{R} \) is continuous linear, then \( l \in C^\infty(D(\Omega), \mathbb{R}) \cap D'(\Omega) \).

The schema to prove this theorem is the following: we need to prove that \( l \circ d \in C^\infty(U, \mathbb{R}) \) whenever \( d \in V \) \( D(\Omega) \), i.e., by Thm. 40, if \( d' \in C^\infty(U \times \Omega, \mathbb{R}) \) and \( d' \) is locally of uniformly bounded support. We are going to prove that:

(i) For any \( u \in U \) the limit

\[
\lim_{h \to 0} \frac{d(u + he_i) - d(u)}{h}
\]

exists in \( T_D(D(\Omega)) \), where

\[
e_i = (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^n \supseteq U.
\]

In fact, this limit is \( \left( \frac{d d'}{d e_i} \right)^U \), which is again a figure of type \( U \) of \( D(\Omega) \).

(ii) Since \( l : T_D(D(\Omega)) \to \mathbb{R} \) is continuous linear, we can apply (i) and commute \( l \) with the limit and the incremental ratio to prove that \( \frac{d}{d e_i}(l \circ d) \) exists and is of the form \( l \circ p \) with \( p \in U D(\Omega) \). The conclusion then follows by induction.

Before proving (i), it is indispensable to have the following:

**Lemma 51.** Let \( V \) be an open set in \( \mathbb{R}^n \). Then the spaces \( T_D(C^\infty(V, \mathbb{R})) \) and \( T_D(D(V)) \) are Hausdorff.

**Proof.** Note that for any \( v \in V \), the evaluation maps \( l_v : h \in C^\infty(V, \mathbb{R}) \mapsto h(v) \in \mathbb{R} \) and \( \tilde{l}_v : h \in D(V) \mapsto h(v) \in \mathbb{R} \) are smooth, and hence the maps \( T_D(l_v) : T_D(C^\infty(V, \mathbb{R})) \to \mathbb{R} \) and \( T_D(\tilde{l}_v) : T_D(D(V)) \to \mathbb{R} \) are both continuous. Therefore, the functional topology on \( C^\infty(V, \mathbb{R}) \) and \( D(V) \) are Hausdorff. The conclusion then follows by Thm. 15. \( \square \)
We now prove Thm. 50:

Proof. To prove the existence of the limit in (i), we first fix \( d \in \mathcal{D}(\Omega) \), \( u \in U \), \( e_i = (0, \ldots, i-1, 1, 0, \ldots, 0) \in \mathbb{R}^n \supseteq U \) and \( r \in \mathbb{R}_{>0} \) such that \( B_r(u) \subseteq U \). Then there exist an open neighbourhood \( V \) of \( u \) in \( U \) and \( a \in \mathbb{R}_{>0} \) such that \( v + he_i \in B_r(u) \) for all \( v \in V \) and \( h \in (-a, a) \). Set \( H := (-a, a) \), and for any \( h \in H \), define

\[
\delta(h) := \left((v, x) \in V \times \Omega \mapsto \int_0^1 \frac{\partial d}{\partial e_i}(v + she_i, x) \, ds \in \mathbb{R} \right).
\]

Clearly \( \delta(0) = \frac{\partial d}{\partial e_i}|_{V \times \Omega} \). Thm. 40 implies \( d^\vee \in C^\infty(U \times \Omega, \mathbb{R}) \), so \( \delta^\vee \in C^\infty(H \times V \times \Omega, \mathbb{R}) \) and hence \( \delta \in \mathcal{C}_u^\infty(V \times \Omega, \mathbb{R}) =: \mathbb{R}^{V \times \Omega} \). Also note that for any non-zero \( h \in H \) and for any \( (v, x) \in V \times \Omega \), by the fundamental theorem of calculus, we have

\[
\delta^\vee(h, v, x) = \frac{d^\vee(v + he_i, x) - d^\vee(v, x)}{h}.
\] (4.5)

We prove below that \( \lim_{h \to 0} \delta(h) = \frac{\partial d}{\partial e_i}|_{V \times \Omega} \) in the space \( \mathbb{R}^{V \times \Omega} \) which has the underlying set

\[
\left[R^{V \times \Omega}\right] := \{ \varphi \in R^{V \times \Omega} \mid \varphi^\vee \in \mathcal{D}(\Omega) \},
\]

and figures defined by \( p \in \mathbb{R}^{V \times \Omega} \) iff \( p^\vee : W \times V \times \Omega \to \mathbb{R} \) is smooth and locally of uniformly bounded support with respect to \( W \times V \).

Since \( d^\vee \) is locally of uniformly bounded support (Thm. 40), we may assume that \( V \) and \( H \) are sufficiently small so that \( \delta^\vee : H \times V \times \Omega \to \mathbb{R} \) is of uniformly bounded support with respect to \( H \times V \). Thus

\[
\delta \in \mathcal{C}_u^\infty(V \times \Omega).
\] (4.6)

To prove the above mentioned limit equality, let \( A \) be a \( D \)-open subset of \( \mathbb{R}^{V \times \Omega} \) such that \( \frac{\partial d}{\partial e_i}|_{V \times \Omega} \in A \). From (4.6) we know that \( \delta^{-1}(A) =: B \) is open in \( H \). Moreover, \( \delta(0) = \frac{\partial d}{\partial e_i}|_{V \times \Omega} \in A \) so \( 0 \in B \). This proves that

\[
\lim_{h \to 0} \delta(h) = \frac{\partial d}{\partial e_i}|_{V \times \Omega} \text{ in } \mathbb{R}^{V \times \Omega}.
\]

Now, we apply this limit to the adjoint map

\[
(-)^\vee : \varphi \in \left[R^{V \times \Omega}\right] \mapsto \varphi^\vee \in \left|\mathcal{D}(\Omega)^V\right|,
\] (4.7)

where the domain is the diffeological space \( \mathbb{R}^{V \times \Omega} \), and the codomain is the space \( \mathcal{D}(\Omega)^V \) with \( \left|\mathcal{D}(\Omega)^V\right| = |\mathcal{C}^\infty(V, \mathcal{D}(\Omega))| \) the underlying set and figures defined by \( q \in \mathbb{R}^{V \times \Omega} \) iff \( q^\vee : \hat{W} \times V \times \Omega \to \mathbb{R} \) is smooth and locally of uniformly bounded support. We claim that the adjoint map (4.7) is also smooth with respect to these diffeological structures on its domain and codomain. In fact, if \( p \in \mathbb{R}^{V \times \Omega} \), then \( ((-)^\vee \circ p)^\vee = p^\vee \) which is locally of uniformly bounded support by the definition of the diffeology on
\(\mathbb{R}^{V \times (\Omega)}\). Therefore \((-)^{\wedge} : \mathbb{R}^{V \times (\Omega)} \rightarrow \mathcal{D}(\Omega) \uparrow V\) is smooth and hence it is also \(D\)-continuous:

\[
\left. \frac{\partial d^V}{\partial e_i} \right|_{V \times \Omega} = \left( \lim_{h \to 0} \delta(h) \right)^{\wedge} = \lim_{h \to 0} \delta(h)^{\wedge} \quad \text{in } \mathcal{D}(\Omega) \uparrow V.
\]

Now consider the evaluation at \(v \in V \subseteq U\):

\[
ev_v : \varphi \in |\mathcal{D}(\Omega) \uparrow V| = |\mathcal{D}(\Omega)^V| \mapsto \varphi(v) \in |\mathcal{D}(\Omega)|.
\]

We claim that \(ev_v : \mathcal{D}(\Omega) \uparrow V \rightarrow \mathcal{D}(\Omega)\) is smooth. In fact, for any \(q \in \mathcal{D}(\Omega) \uparrow V\), \(\mathcal{D}(\Omega) \uparrow V\), i.e.,

\[
(q^\vee)^{\wedge} : \hat{W} \times V \times \Omega \rightarrow \mathbb{R} \text{ is smooth and locally of uniformly bounded support. \quad (4.8)}
\]

We need to prove that \((ev_v \circ q)^\vee : \hat{W} \times \Omega \rightarrow \mathbb{R}\) is also smooth and locally of uniformly bounded support. Take \(w \in \hat{W}\), and from (4.8) we have open neighbourhoods \(C\) of \(w\) and \(D\) of \(v\) so that \((q^\vee)^\wedge |_{C \times D \times \Omega}\) is of uniformly bounded support. We may assume that \(\text{supp} \left[(q^\vee)^\wedge (w', v', -)\right] \subseteq K \in \Omega\) for all \((w', v') \in C \times D\). But \((q^\vee)^\wedge (w', v', -) = q(w')(v') = ev_v(q(w'))\). Therefore, for all \(w' \in C\) we have \(\text{supp}[(ev_v \circ q)^\wedge (w', -)] = \text{supp}[q(w')(v)] \subseteq K\). By Cartesian closedness, \(ev_v \circ q\) is smooth and hence it is a figure of \(\mathcal{D}(\Omega)\).

This proves that \(ev_v : \mathcal{D}(\Omega) \uparrow V \rightarrow \mathcal{D}(\Omega)\) is smooth and hence it is also \(D\)-continuous. So we have:

\[
\frac{\partial d^V}{\partial e_i}(v, -) = ev_v \left[ \left. \left( \frac{\partial d^V}{\partial e_i} \right|_{V \times \Omega} \right)^{\wedge} \right] = ev_v \left[ \lim_{h \to 0} \delta(h)^{\wedge} \right]
\]

\[
= \lim_{h \to 0} \delta(h)^{\wedge}(v) = \lim_{h \to 0} \frac{d^V(v + he_i, -) - d^V(v, -)}{h}
\]

\[
= \lim_{h \to 0} \frac{d(v + he_i) - d(v)}{h} \quad \forall v \in V.
\]

Therefore, this limit exists in \(\mathcal{D}(\Omega)\). By assumption, \(l : |\mathcal{D}(\Omega)| \rightarrow \mathbb{R}\) is \(D\)-continuous and linear, so

\[
l \left( \frac{\partial d^V}{\partial e_i}(v, -) \right) = l \left[ \lim_{h \to 0} \frac{d(v + he_i) - d(v)}{h} \right]
\]

\[
= \lim_{h \to 0} \frac{l(d(v + he_i)) - l(d(v))}{h}
\]

\[
= \frac{\partial (l \circ d)}{\partial e_i}(v).
\]

This proves that the first partial derivatives of \(l \circ d\) exist and are continuous because both \(l\) and \(\frac{\partial d^V}{\partial e_i}\) are \(D\)-continuous. We can now apply the same procedure to the figure

\[
\left( \frac{\partial d^V}{\partial e_i} \right)^{\wedge} \in_v \mathcal{D}(\Omega)
\]
obtaining that also the second partial derivatives of \( l \circ d \) exist and are continuous. By applying inductively this process, we get the conclusion \( l \circ d \in C^\infty(U, \mathbb{R}) \). Finally, from Thm. 48 we have \( l \in \mathcal{D}'(\Omega) \).

We also have the following

**Corollary 52.** Let \( l : |\mathcal{D}(\Omega)| \to \mathbb{R} \) be a linear map. Then the following are equivalent:

(i) \( l \) is continuous in the locally convex topology, i.e., it is a distribution.
(ii) \( l \) is continuous in the \( D \)-topology on \( \mathcal{D}(\Omega) \).
(iii) \( l \in C^\infty(\mathcal{D}(\Omega), \mathbb{R}) \)

Proof. (i) \( \Rightarrow \) (ii): From Cor. 47; (ii) \( \Rightarrow \) (iii): From Thm. 50; (iii) \( \Rightarrow \) (i): From Thm. 48.

From the proof of Thm. 50 we have

**Corollary 53.** Let \( U \) be an open set in \( \mathbb{R}^n \) and let \( d \in U \mathcal{D}(\Omega) \). Then \( d \) is smooth in the usual sense, i.e., for all \( \alpha \in \mathbb{N}^n \) the partial derivative \( \partial^\alpha d : U \to |\mathcal{D}(\Omega)| \) exists as the limit of a suitable incremental ratio in the topological vector space \( |\mathcal{D}(\Omega)| \) with the \( D \)-topology. Moreover, \( \partial^\alpha d \in U \mathcal{D}(\Omega) \).

By applying this result to a curve \( d \in U \mathcal{D}(\Omega) \), and knowing that the \( D \)-topology is finer than the usual locally convex topology, we get an independent proof that \( \mathcal{D}(\Omega) \) is a CVS.

We close this section with the following result, which underscores the difference between \( \mathcal{D}(\Omega) \) and its counterpart \( \mathcal{D}^s(\Omega) \); in its statement, if \( F \in \mathcal{Dlg} \) is also a vector space, then we set

\[ F'_s := \{ l \in C^\infty(D^s(\Omega), \mathbb{R}) \mid l \text{ is linear} \} \]

for its smooth dual space (this notation has been used for the special cases in Thm. 35).

**Corollary 54.**

(i) \( |\mathcal{D}'(\Omega)| = |\mathcal{D}(\Omega)'_s| \) and \( \mathcal{D}'(\Omega) \supseteq \mathcal{D}(\Omega)'_s \).

(ii) \( |\mathcal{D}'(\Omega)| \supseteq |\mathcal{D}^s(\Omega)'_s| = \{ l \in C^\infty(D^s(\Omega), \mathbb{R}) \mid l \text{ is linear} \} \).

Proof. (i): We first prove that the underlying sets are equal, i.e., \( |\mathcal{D}'(\Omega)| = |\mathcal{D}(\Omega)'_s| \). In fact, this follows from the equivalence (i) \( \iff \) (iii) of Cor. 52. Now, if \( d \in U \mathcal{D}(\Omega)'_s \), then \( d' : U \times \mathcal{D}(\Omega) \to \mathbb{R} \) is smooth. The space \( \mathcal{D}'(\Omega) \) is functionally generated by all linear functionals \( l : |\mathcal{D}'(\Omega)| \to \mathbb{R} \) which are continuous with respect to the weak* topology. Since each one of these functionals is of the form \( l = ev_\varphi \) for some \( \varphi \in \mathcal{D}(\Omega) \), we only need to consider \( (ev_\varphi \circ d)(u) = ev_\varphi[d(u)] = d(u)(\varphi) = d'(u, \varphi) \) for each \( u \in U \). Therefore, \( l \circ d = ev_\varphi \circ d = d'(\cdot, \varphi) \) is smooth, which implies that \( d \in U \mathcal{D}'(\Omega) \).

(ii): As a consequence of Thm. 40, we know that \( \mathcal{D}(\Omega) \subseteq \mathcal{D}^s(\Omega) \). Therefore, if \( l \in C^\infty(\mathcal{D}^s(\Omega), \mathbb{R}) \) is linear, then we also have \( l \in C^\infty(\mathcal{D}(\Omega), \mathbb{R}) \). Now \( l \in |\mathcal{D}'(\Omega)| \) follows from Cor. 52.
5. Spaces for Colombeau Generalized Functions as Diffeological Spaces

It is natural to view all the spaces used to define CGF as diffeological spaces. We will start with $C^\infty(\Omega)^I$, $\mathcal{E}^s_M(\Omega)$, $\mathcal{A}_q(\Omega)$, $U(\Omega)$, $\mathcal{E}^\nu(\Omega)$ and $\mathcal{E}^s_M(\Omega)$, with the aim to prove that also the quotient spaces $\mathcal{G}^s(\Omega)$, $\mathcal{G}^\nu(\Omega)$ are smooth differential algebras.

The space $C^\infty(\Omega)^I$. Elements $(u_\varepsilon)$ of $C^\infty(\Omega)^I$ are arbitrary nets, indexed in $\varepsilon \in I$, of smooth functions on $\Omega$. There are studies of Colombeau-like algebras with smooth or continuous $\varepsilon$-dependence (see [6, 19] and references therein). In [18] it has been proved that a very large class of equations have no solution if we request continuous dependence with respect to $\varepsilon \in I$. For this reason, it is natural to think $I$ as a space with the discrete diffeology (see (ii) of Rem. 5), i.e., where only locally constant maps $d : U \rightarrow I$ are figures $d \in_U I$. With this structure, the space $I$ is functionally generated by $\text{Set}(I, \mathbb{R})$. If we think of $C^\infty(\Omega)$ as the space $C^\infty(\Omega, \mathbb{R}) \in \text{Dlg}$, then by Cartesian closedness (Thm. 26) we have $u \in |C^\infty(\Omega)^I|$, i.e., $u \in C^\infty(I, C^\infty(\Omega, \mathbb{R}))$, iff $u \in \text{Set}(I, C^\infty(\Omega))$. The space $C^\infty(\Omega)^I$ with this diffeological structure will be denoted by $C^\infty(\Omega, \mathbb{R})^I$. Figures $d \in_U C^\infty(\Omega, \mathbb{R})^I$ are maps $d : U \rightarrow \text{Set}(I, C^\infty(\Omega))$ such that $(d^\nu)^\vee (-, \varepsilon, -) \in C^\infty(U \times \Omega, \mathbb{R})$ for all $\varepsilon \in I$.

The space $\mathcal{E}^s_M(\Omega)$. The natural diffeology on

$$\mathcal{E}^s_M(\Omega) = \{ (u_\varepsilon) \in C^\infty(\Omega)^I \mid \forall K \subseteq \Omega \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \}$$

is the sub-diffeology of $C^\infty(\Omega, \mathbb{R})^I$:

$$\mathcal{E}^s_M(\Omega) := (\mathcal{E}^s_M(\Omega) \triangleleft C^\infty(\Omega, \mathbb{R})^I) .$$

Its figures $d \in_U \mathcal{E}^s_M(\Omega)$ are maps $d : U \rightarrow \mathcal{E}^s_M(\Omega)$ such that $(d^\nu)^\vee (-, \varepsilon, -) \in C^\infty(U \times \Omega, \mathbb{R})$ for all $\varepsilon \in I$.

The space $\mathcal{A}_q(\Omega)$. The set $\mathcal{A}_0(\Omega) = \{ \varphi \in |\mathcal{D}(\Omega)| \mid \int \varphi = 1 \}$ has a natural diffeology, the sub-diffeology of $\mathcal{D}(\Omega)$. So

$$\mathcal{A}_0(\Omega) := (\mathcal{A}_0(\Omega) \triangleleft \mathcal{D}(\Omega)) \in \text{Dlg} .$$

Analogously, the set

$$\mathcal{A}_q(\Omega) = \left\{ \varphi \in \mathcal{A}_0(\Omega) \mid \forall \alpha \in \mathbb{N}^n : 1 \leq |\alpha| \leq q \Rightarrow \int x^\alpha \varphi(x) \, dx = 0 \right\}$$

has a natural diffeology, the sub-diffeology of $\mathcal{A}_0(\Omega)$ So

$$\mathcal{A}_q(\Omega) := (\mathcal{A}_q(\Omega) \triangleleft \mathcal{A}_0(\Omega)) = (\mathcal{A}_q(\Omega) \triangleleft \mathcal{D}(\Omega)) \in \text{Dlg} ,$$

where we used the property $(S \triangleleft (T \triangleleft X)) = (S \triangleleft X)$ if $S \subseteq T \subseteq |X|$ and $X \in \text{Dlg}$. Therefore, figures $d \in_U \mathcal{A}_q(\Omega)$ are maps $d : U \rightarrow \mathcal{A}_q(\Omega)$ such
that \( d^\vee \in C^\infty(U \times \Omega, \mathbb{R}) \) and \( d^\vee \) is locally of uniformly bounded support (Thm. 40).

Note that \( \int_\Omega : \mathcal{D}(\Omega) \to \mathbb{R} \) is a linear and diffeologically smooth map. Therefore, \( A_0(\Omega) \) is an affine space which is closed in the locally convex topology of \( \mathcal{D}(\Omega) \). An isomorphism with the corresponding vector space \( A_{00}(\Omega) := \ker \left( \int_\Omega \right) \) is given by \( \varphi \in A_0(\Omega) \mapsto \varphi - \varphi_0 \in A_{00}(\Omega) \), where \( \varphi_0 \in A_0(\Omega) \) is any fixed element. This isomorphism is clearly diffeologically smooth. This solves the problem stated in (ii) of Rem. 33.

The space \( U(\Omega) \). In Def. 2 of the full Colombeau algebra, the set \( U(\Omega) \) serves as domain of the representatives \( R : U(\Omega) \to \mathbb{R} \) of CGF in \( \mathcal{G}^0(\Omega) \). These representatives are requested to be smooth in the \( \Omega \) slot (note that \( U(\Omega) \subseteq A_0 \times \Omega \) but with no particular regularity in the \( A_0 \) slot (which serves as an index set for the full Colombeau algebra, analogous to the interval \( I \) as an index set for the special one)). This means that we shall consider the discrete diffeology on \( A_0 \) and the standard diffeology on \( \Omega \). If we identify the set \( A_0 \) with the corresponding diffeological space with the discrete diffeology, then

\[
U(\Omega) := \left( \{ (\varphi, x) \in A_0 \times \Omega \mid \text{supp}(\varphi) \subseteq \Omega - x \} \right) < A_0 \times \Omega)
\]

\[
= (U(\Omega) < |\mathcal{D}(\mathbb{R}^n)| \times \mathbb{R}^n) \in \text{Dlg},
\]

where we used the property \((A < D) \times (O < R) = (A \times O < D \times R)\) which holds in \( \text{Dlg} \). Therefore, figures \( d \in V U(\Omega) \) are maps \( d : V \to U(\Omega) \) such that the two projections verify \( d_1 \in \text{Set}(V, |\mathcal{D}(\mathbb{R}^n)|) \) and \( d_2 \in C^\infty(V, \Omega) \).

The space \( \mathcal{E}^e(\Omega) \). The space \( \mathcal{E}^e(\Omega) \) (see Def. 2) inherits its diffeological structure from \( C^\infty(U(\Omega), \mathbb{R}) \in \text{Dlg} \):

\[
\mathcal{E}^e(\Omega) := (\mathcal{E}^e(\Omega) < C^\infty(U(\Omega), \mathbb{R}))
\]

Figures \( d \in V \mathcal{E}^e(\Omega) \) are maps \( d : V \to \mathcal{E}^e(\Omega) \) such that \( d^\vee \in C^\infty(V \times U(\Omega), \mathbb{R}) \). We give an equivalent characterization of \( \mathcal{E}^e(\Omega) \) as follows: For \( \varphi \in A_0 \), set

\[
\Omega_\varphi := \Omega \cap \{ x \in \mathbb{R}^n \mid \text{supp}(\varphi) \subseteq \Omega - x \}.
\]

As a convention, when \( \Omega_\varphi = \emptyset \), we think of \( C^\infty(\Omega_\varphi, \mathbb{R}) \) as a set with a single element. Since \( R \in \mathcal{E}^e(\Omega) \) iff \( R^\wedge : A_0 \to \bigcup_{\varphi \in A_0} C^\infty(\Omega_\varphi, \mathbb{R}) \) and \( R(\varphi, -) \in C^\infty(\Omega_\varphi, \mathbb{R}) \) for all \( \varphi \in A_0 \), \( R^\wedge \in \prod_{\varphi \in A_0} C^\infty(\Omega_\varphi, \mathbb{R}) \). By Cartesian closedness of \( \text{Dlg} \):

\[
\mathcal{E}^e(\Omega) \simeq \prod_{\varphi \in A_0} C^\infty(\Omega_\varphi, \mathbb{R}).
\]

Therefore, up to smooth isomorphism, figures of \( \mathcal{E}^e(\Omega) \) can be described as maps \( d : V \to \prod_{\varphi \in A_0} C^\infty(\Omega_\varphi, \mathbb{R}) \) such that \( d(-)(\varphi)^\vee \in C^\infty(V \times \Omega_\varphi, \mathbb{R}) \) for all \( \varphi \in A_0 \).
The space $\mathcal{E}_M^e(\Omega)$. The natural diffeology on the space of moderate functions $\mathcal{E}_M^e(\Omega)$ is the sub-diffeology of $\mathcal{E}^e(\Omega)$. Hence,

$$\mathcal{E}_M^e(\Omega) := (\mathcal{E}_M^e(\Omega) \prec \mathcal{E}^e(\Omega)) \in \text{Dlg}.$$ 

Figures $d \in_v \mathcal{E}_M^e(\Omega)$ are maps $d : V \to \mathcal{E}_M^e(\Omega)$ such that $d(-)(\varphi, -) \in \mathcal{C}^\infty(V \times \Omega, \varphi) \in \mathbb{R}$ for all $\varphi \in \mathcal{A}_0$.

The special and full Colombeau algebras. Since the category $\text{Dlg}$ of diffeological spaces is cocomplete, both quotient algebras $\mathcal{G}^s(\Omega)$ and $\mathcal{G}^e(\Omega)$ can be viewed as objects of $\text{Dlg}$:

$$\mathcal{G}^s(\Omega) := \mathcal{E}_M^s(\Omega)/\mathcal{N}^s(\Omega)$$
$$\mathcal{G}^e(\Omega) := \mathcal{E}_M^e(\Omega)/\mathcal{N}^e(\Omega).$$

Figures of these spaces can be described using the notion of quotient diffeology. E.g. $d \in_v \mathcal{G}^s(\Omega)$ iff $d : U \to \mathcal{G}^s(\Omega)$ and for any $u \in U$ we can find an open neighbourhood $V$ of $u$ in $U$ and a map $\delta : V \to \mathcal{C}^\infty(\Omega)$ such that

(i) $\delta(v)$ is moderate for all $v \in V$
(ii) $(\delta^V)^{-}(\varepsilon, -) \in \mathcal{C}^\infty(V \times \Omega, \varepsilon) \in \mathbb{R}$ for all $\varepsilon \in I$
(iii) $d|_V = \pi \circ \delta$, where $\pi : (u_{\varepsilon}) \in \mathcal{E}_M^s(\Omega) \mapsto [u_{\varepsilon}] \in \mathcal{G}^s(\Omega)$ is the projection on the quotient.

Analogously, we can describe figures of the full Colombeau algebra.

We can now state the following natural result:

**Theorem 55.** Both for the special and the full Colombeau algebras $\mathcal{G}(\Omega) \in \{\mathcal{G}^s(\Omega), \mathcal{G}^e(\Omega)\}$, the sum, product and derivation maps

$$+ : \mathcal{G}(\Omega) \times \mathcal{G}(\Omega) \to \mathcal{G}(\Omega)$$
$$\cdot : \mathcal{G}(\Omega) \times \mathcal{G}(\Omega) \to \mathcal{G}(\Omega)$$
$$\partial^\alpha : \mathcal{G}(\Omega) \to \mathcal{G}(\Omega) \quad \forall \alpha \in \mathbb{N}^n$$

are smooth. Therefore, with respect to the $\mathcal{D}$-topology, $\mathcal{G}(\Omega)$ is a topological algebra with continuous derivations.

Moreover, if $(\psi_{\varepsilon}) \in \mathcal{D}(\Omega)^I$ is a net verifying properties (i), (ii), (iii), (iv), (v) of Thm. 1, and let $\iota_{\Omega}$ be defined as in (1.1) and let $\sigma_{\Omega}(f) := [f] \in \mathcal{G}(\Omega)$ for all $f \in \mathcal{C}^\infty(\Omega)$, then the embeddings

$$\iota_{\Omega} : |\mathcal{D}'(\Omega)| \to \mathcal{G}(\Omega)$$
$$\sigma_{\Omega} : \mathcal{C}^\infty(\Omega) \to \mathcal{G}(\Omega)$$

are smooth maps if we equip $|\mathcal{D}'(\Omega)|$ with the sub-diffeology of $\mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R})$.

**Proof.** We prove that the maps are smooth for the case $\mathcal{G}(\Omega) = \mathcal{G}^s(\Omega)$, since the proof is similar for the case $\mathcal{G}(\Omega) = \mathcal{G}^e(\Omega)$.

Concerning the smoothness of the sum map, let $d \in_v \mathcal{G}^s(\Omega) \times \mathcal{G}^s(\Omega)$, i.e., $p_i \circ d \in_v \mathcal{G}^s(\Omega)$, where $p_i : \mathcal{G}^s(\Omega) \times \mathcal{G}^s(\Omega) \to \mathcal{G}^s(\Omega), i = 1, 2$, are the projections. Hence, by the definition of the quotient diffeology on $\mathcal{G}^s(\Omega)$,
for any \( u \in U \) we can write \( (p_i \circ d) |_{V_i} = \pi \circ \delta_i \), where \( V_i \in \tau_U, u \in V_1 \cap V_2, \delta_i \in \mathcal{E}^s_M(\Omega) \). Thus, we can write the composition
\[
(\circ d) |_{V_1 \cap V_2} : v \mapsto \pi [\delta_1(v)] + \pi [\delta_2(v)] = \pi [\delta_1(v) + \delta_2(v)].
\]
Since \( \delta_1 + \delta_2 \in \mathcal{E}^s_M(\Omega) \), the conclusion follows from the definition of the quotient diffeology.

Analogously, we can prove that the product map is smooth.

Concerning the smoothness of the partial derivative \( \partial^\alpha \), if \( d \in \mathcal{G}^s(\Omega) \), then for any \( u \in U \) we can write \( d |_V = \pi \circ \delta \), where \( u \in V \in \tau_U \) and \( \delta \in \mathcal{E}^s_M(\Omega) \). Therefore, we have
\[
(\partial^\alpha \circ d) |_V : v \mapsto \partial^\alpha (d(v)) = \partial^\alpha (\pi(\delta(v))) = \pi [\partial^\alpha \delta(v)].
\]
But it is not difficult to show that \( \partial^\alpha \in \mathcal{C}^\infty(\mathcal{E}^s_M(\Omega), \mathcal{E}^s_M(\Omega)) \). Hence, \( \partial^\alpha \delta \in \mathcal{E}^s_M(\Omega) \), and the conclusion follows.

Concerning the smoothness of the embeddings, we only need to prove that \( \iota_\Omega \) is smooth, since the smoothness of \( \sigma_\Omega \) follows directly from the definition of figures of a quotient diffeology. Let \( d \in \mathcal{G}^s(\Omega) \), \( \mathcal{G}^s(\Omega) \subset \mathcal{C}^\infty(\mathcal{D}(\Omega), \mathbb{R}) \), i.e., \( d' \in \mathcal{C}^\infty(U \times \mathcal{D}(\Omega), \mathbb{R}) \). We can compute that
\[
(\iota_\Omega \circ d)(u) = [d(u) \ast (\epsilon \circ \psi_\epsilon |_\Omega)] = \left[ x \in \Omega \mapsto \langle d(u), \frac{1}{\epsilon^n} \psi_\epsilon \left( \frac{x - ?}{\epsilon} \right) \rangle \right].
\]
For any fixed \( \epsilon \in I \), we show below that the map
\[
\delta_\epsilon : (u, x) \in U \times \Omega \mapsto \langle d(u), \frac{1}{\epsilon^n} \psi_\epsilon \left( \frac{x - ?}{\epsilon} \right) \rangle = d' \left[ u, \frac{1}{\epsilon^n} \psi_\epsilon \left( \frac{x - ?}{\epsilon} \right) \right] \in \mathbb{R}
\]
is smooth (for the moderateness property, see [21, 35]). Define the maps
\[
S : \epsilon, \varphi \in I \times \mathcal{D}(\mathbb{R}^n) \mapsto \epsilon \circ \varphi \in \mathcal{D}(\mathbb{R}^n) \quad (5.1)
\]
\[
\tilde{T} : (x, \varphi) \in \mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n) \mapsto \varphi(x - ?) \in \mathcal{D}(\mathbb{R}^n). \quad (5.2)
\]
Then
\[
\delta_\epsilon (u, x) = d' \left[ u, S \left( \epsilon, \tilde{T}(x, \psi_\epsilon) \right) \right] \quad \forall (u, x) \in U \times \Omega.
\]
Therefore, \( \delta_\epsilon \) is smooth once we prove that both maps \( S \) and \( \tilde{T} \) are diffeologically smooth. This is done in the following lemma:

**Lemma 56.** The maps defined in (5.1), (5.2) and
\[
T : (x, \varphi) \in \mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n) \mapsto \varphi(\cdot - x) \in \mathcal{D}(\mathbb{R}^n)
\]
are diffeologically smooth, i.e., \( S \in \mathcal{C}^\infty(I \times \mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)), \tilde{T} \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)), T \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)).
\]

**Proof.** We only proceed for \( S \), since the other two cases are similar. If \( \epsilon \in I \) and \( p \in U \mathcal{D}(\mathbb{R}^n) \), then \( p' \in \mathcal{C}^\infty(U \times \mathbb{R}^n, \mathbb{R}) \) and \( p' \) is locally of uniformly bounded support with respect to \( U \) (Thm. 40). But \( [S(\epsilon, \cdot) \circ p]'(u, x) = \frac{1}{\epsilon^n} p' \left( u, \frac{x}{\epsilon} \right) \) for all \( (u, x) \in U \times \Omega \), and this shows that \( [S(\epsilon, \cdot) \circ p]' \in \mathcal{C}^\infty(U \times \mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)) \).
\(C^\infty(U \times \mathbb{R}^n, \mathbb{R})\) and it is locally of uniformly bounded support with respect to \(U\).

Since \(I\) has the discrete diffeology, all these \(\delta_i\)'s together induce a smooth map \(\delta : U \rightarrow \mathcal{E}^s_M(\Omega)\) such that \(\pi \circ \delta = \iota_\Omega \circ d\). By the definition of the quotient diffeology on \(\mathcal{G}^s(\Omega)\), the embedding \(\iota_\Omega : |\mathcal{D}'(\Omega)| \rightarrow \mathcal{G}^s(\Omega)\) is smooth.

5.1. Colombeau ring of generalized numbers and evaluation of generalized functions. In this subsection we consider only the case of the special Colombeau algebra \(\mathcal{G}^s(\Omega)\) since it is mostly studied in the literature. The case of the full algebra can be treated analogously.

One of the main features of Colombeau theory is the possibility to define a point evaluation of every CGF. Hence, it is natural to ask whether this evaluation map

\[
\text{ev} : (u, x) \in \mathcal{G}^s(\Omega) \times \tilde{\Omega}_c \mapsto u(x) \in \mathbb{R}^{\tilde{\Omega}c} \tag{5.3}
\]

is a smooth map or not (see Section 1.1 for the definitions of \(\tilde{\Omega}_c\) and \(\mathbb{R}^{\tilde{\Omega}c}\)).

The diffeology we consider on \(\tilde{\Omega}_c\) and \(\mathbb{R}^{\tilde{\Omega}c}\) are the natural ones:

**Definition 57.** All the following are diffeological spaces:

- (i) \(R_M := (\mathbb{R}M \times C^\infty(I, \mathbb{R}))\)
- (ii) \(\tilde{\mathbb{R}} := R_M/\sim\)
- (iii) \(\Omega_M := (\Omega_M \times C^\infty(I, \Omega))\)
- (iv) \(\tilde{\Omega} := \Omega_M/\sim\)
- (v) \(\tilde{\Omega}_c := (\tilde{\Omega}/\sim)\)

Note, e.g., that \(d \in \mathcal{U}_V \Omega_M\) iff \(d : U \rightarrow \Omega_M\) and \(d'(\cdot, \varepsilon) \in C^\infty(U, \Omega)\) for all \(\varepsilon \in I\).

**Theorem 58.** The evaluation map (5.3) is smooth.

**Proof.** Let \(a \in \mathcal{U}_V \mathcal{G}^s(\Omega)\) and let \(b \in \tilde{\Omega}_c\). We need to prove that \(\text{ev} \circ \langle a, b \rangle \in \tilde{\mathbb{R}}\). For any fixed \(u \in \mathcal{U}_V\), by definition of the quotient diffeologies, we can write \(a|_V = \pi_1 \circ \alpha\) and \(i \circ b|_V = \pi_2 \circ \beta\), where \(u \in V \in \mathcal{U}_\mathbf{V}, \alpha \in V \mathcal{E}^s_M(\Omega), \beta \in V \Omega_M, i : \tilde{\Omega}_c \rightarrow \tilde{\Omega}\) is the inclusion, and \(\pi_1 : \mathcal{E}^s_M(\Omega) \rightarrow \mathcal{G}^s(\Omega), \pi_2 : \Omega_M \rightarrow \tilde{\Omega}\) are the projections. Hence, for any \(v \in V\), we have \((\text{ev} \circ \langle a, b \rangle)(v) = \text{ev}(a(v), b(v)) = \text{ev}(\pi_1(a(v)), \pi_2(\beta(v))) = \text{ev}((\langle \varepsilon \rangle^V (v, \varepsilon, -)), (\varepsilon, \beta(v, \varepsilon)))\). Note that for any \(\varepsilon \in I\), \((\langle \varepsilon \rangle^V (-, \varepsilon, -)) \in C^\infty(V \times \Omega, \mathbb{R})\) and \(\beta^V(-, \varepsilon) \in C^\infty(V, \Omega)\), so the restriction \((\text{ev} \circ \langle a, b \rangle)|_V\) can be written as an ordinary smooth function defined on \(V\) composed with the projection \(\pi : \mathbb{R}M \rightarrow \mathbb{R}\). Therefore, \(\text{ev} \circ \langle a, b \rangle \in \tilde{\mathbb{R}}\). \(\square\)
6. Conclusions and open problems

We explore why the categories Fr of Frölicher spaces, Dlg of diffeological spaces and FDlg of functionally generated (diffeological) spaces work as good frameworks both for the classical spaces of functional analysis and for the Colombeau algebras. On the one hand, there seem to be few differences between FDlg and Fr: we can say that the former seems better than the latter because in FDlg we don’t have the problem of extending to the whole space locally defined functionals; but in the latter, it is easier to work directly with globally defined functionals when the D-topology of the space is unknown. On the other hand, the usual counter-examples about locally Cartesian closedness of Fr do not seem to work in FDlg. Moreover, if compared to Dlg, functionally generated spaces seem to be closer to spaces used in functional analysis, where testing smoothness using functionals is customary. On the other hand, Thm. 55 and Thm. 58, show that Dlg can be considered a promising categorical framework for Colombeau algebras. Some open problems underscored by the present work are the following:

• A clear and useful example of functionally generated space which is not Frölicher and where locally defined functionals cannot be extended to the whole space is missing.
• The problem to show that Dlg gives also a sufficiently simple infinite dimensional calculus for the diffeomorphism invariant Colombeau algebra (see [21]) remains open. In particular, we note that the differentiable uniform boundedness principle ([21, Thm. 2.2.7]) is used in [21] only to prove the analogy of Lem. 56, whereas the other results of [21, Section 2.2.1] seem repeatable in Dlg without the need to know the calculus on convenient vector spaces.
• The relationship between the locally convex topology and the D-topology on $\mathcal{D}(\Omega)$ is only partially solved (see Cor. 47).
• The relationship between the space of Schwartz distributions $\mathcal{D}'(\Omega)$ and the smooth dual $\mathcal{D}(\Omega)'$ is only partially solved (see Cor. 54).
• The preservation of colimits from the category of smooth manifolds to FDlg is only partially solved.

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