Out-of-Equilibrium Two-Dimensional Yukawa Theory in a Strong Scalar Wave Background

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Abstract—We consider 2D Yukawa theory in a strong scalar wave background. We use operator and functional formalisms. In the latter the Schwinger–Keldysh diagram technique is used to calculate retarded, advanced and Keldysh propagators. We use simplest states in the two formalisms in question, which appear to be different from each other. As a result, the two Keldysh propagators found in different formalisms do not coincide, while the retarded and advanced ones do coincide. We use these propagators to calculate physical quantities such as the fermion stress–energy flux and the scalar current. One needs to know the latter to address the backreaction problem. It happens that while in the functional formalism (for the corresponding simplest state) we find zero fermion flux, in the operator formalism (for the corresponding simplest state) the flux is not zero and is proportional to a Schwarzian derivative. Meanwhile the scalar current is the same in both formalisms if the background field is large and slowly changing.

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1. INTRODUCTION

There is a longstanding backreaction problem in strong background fields (see, e.g., the old textbooks \cite{19, 24, 25} for the introduction). In this respect it would be nice to have a simple enough but nontrivial example of an out-of-equilibrium QFT in a strong background field. A possible option is to consider QFT in strong electric or gravitational fields in 2D rather than in 4D. However, there the electromagnetic and gravitational fields are substantially nondynamical, while to study the backreaction problem it is more appropriate to have dynamical fluctuations over the background field. Then an option is to study strong scalar field background in a 2D QFT.

Namely, we propose to consider the Yukawa theory of fermions interacting with a massless real scalar field in $(1+1)$-dimensional Minkowski spacetime with the action

\begin{equation}
S[\psi, \overline{\psi}, \phi] = \int d^2x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \overline{\psi} i \gamma^\mu \partial_\mu \psi - \lambda \phi \overline{\psi} \psi \right). 
\end{equation}

The signature of the metric is (1, −1); the gamma matrices have the form

\begin{equation}
\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. 
\end{equation}

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In the presence of classical background fields we split \( \psi = \psi_{cl} + \psi_q \) and \( \phi = \phi_{cl} + \phi_q \), where \( \phi_{cl} \) and \( \psi_{cl} \) are solutions of the classical equations of motion and \( \phi_q \) and \( \psi_q \) are quantum fluctuations:

\[
\begin{aligned}
\partial^2 \phi_{cl} + \lambda \psi_{cl} \psi_{cl} &= 0, \\
[i\gamma^\mu \partial_\mu - \lambda \phi_{cl}] \psi_{cl} &= 0.
\end{aligned}
\]

In particular, we consider the following scalar wave solution:

\[
\lambda \phi_{cl}(t, x) = \Phi \left( \frac{t - x}{\sqrt{2}} \right) \quad \text{and} \quad \psi_{cl} = 0.
\]

The other options \( \phi_{cl} = \alpha + \beta t \) and \( \phi_{cl} = \alpha + \beta x \), for constant and real \( \alpha \) and \( \beta \), will be considered in a separate paper.

However, the background scalar fields seem to suffer from a number of disadvantages because they do not share some of the relevant properties of the strong electric and gravitational fields. Indeed, consider a point-like relativistic particle in a scalar field. The simplest classical action in such a situation is as follows:

\[
S = -\int d\tau \left\{ m + \lambda \phi(x(\tau)) \right\},
\]

where \( \tau \) is the proper time, \( \phi(x) \) is the scalar field, \( x(\tau) \) is the world-line of the particle, \( \lambda \) is its "charge" with respect to the scalar field and \( m \) is its mass.

The equation of motion that follows from this action is

\[
[m + \lambda \phi] \ddot{x}_\mu = \lambda \left[ \dot{x}_\nu \partial_\mu \phi - \partial_\nu \phi \dot{x}_\mu \right] \dot{x}_\nu, \quad \mu, \nu = 0, 1,
\]

and \( \dot{x}_\mu \dot{x}^\mu = 1 \). One can show that for the examples of the background scalar fields that have been listed around equation (1.4), the analytically continued equation (1.5) does not have Euclidean world-line instanton solutions. Furthermore, in this paper we will also see that the fermionic effective action in such background fields does not have an imaginary contribution and is analytical on the cut complex plane of the background field. All this means that there is no particle tunneling in the background scalar fields under consideration, unlike the strong electric and gravitational fields. However, as we show in this paper, the situation is not that trivial and in this simple Yukawa theory there are interesting effects related to the particle creation.

To begin with, in this paper we neglect quantum fluctuations of the scalar field \( \phi_q \). (Potentially highly important, as we explain in the concluding section, loop effects due to quantum scalar fluctuations will be considered in a separate paper.) The action in such a case simplifies to

\[
S[\psi, \overline{\psi}] = \int d^2x \left( \overline{\psi}(x, t)i\gamma^\mu \psi(x, t) - \Phi \left( \frac{t - x}{\sqrt{2}} \right) \overline{\psi}(x, t) \psi(x, t) \right).
\]
propagators calculated in the two formalisms are the same, which is appropriate, as is explained in the main body of the paper. However, the Keldysh propagators are different, because they are sensitive to the states of the theory. The difference in the Keldysh propagators leads to the differences in the physical observables. At the end of the paper we discuss a relation between the Keldysh propagators found in different settings.

What sort of observables are we interested in? We calculate the expectation value of the fermionic stress–energy tensor and of the scalar current operator $\psi\psi$. The reason for considering the latter is the following. From the Hamiltonian of the theory (1.1), one obtains the operator equation

$$\partial^2\phi + \lambda\psi\psi = 0,$$

which reproduces one of the classical equations of motion (1.3). To solve this equation iteratively, we take the expectation values of both sides and use the method of successive approximations. At the leading order in the expansion over the quantum fluctuations $\phi_q$ and $\psi_q$, we reproduce the first equation in (1.3) with $\psi_{cl} = 0$. Then we put $\phi = \phi_{cl}$ in the second equation in (1.3) and solve it for the fermionic field. After that we can solve the averaged equation (1.7) for $\langle\phi\rangle$ when $\langle\bar{\psi}\psi\rangle$ is calculated in the background of $\phi = \phi_{cl}$. Thus, as follows from (1.7), the expectation value $\langle\bar{\psi}\psi\rangle$ calculated in the background of $\phi_{cl}$ serves as a response of quantum fluctuations of the fermionic field to the scalar background field.

In this paper we show that for those values of $\phi_{cl}$ that are large and slowly varying, the expectation value of the scalar current $\psi\psi$ is the same in both formalisms that have been briefly described above. As we explain below, there is some sort of universality in the dependence of the scalar current on the background field for those states that lead to Green functions with the proper Hadamard behavior.

However, in the expectation values of the stress–energy tensor we find a disagreement in the two formalisms. In particular, in one formalism we find the zero fermion flux, while in the other it is not zero. As we explain, the disagreement comes from the fact that different states are used in the calculations in the two different formalisms.

The paper is organized as follows. In Section 2 we briefly describe the use of the Schwinger–Keldysh diagram technique in the functional formalism. We derive the Dyson–Schwinger equations for the retarded, advanced and Keldysh propagators and solve them. Then we use the resulting Keldysh propagator to calculate the physical observables.

In Section 3 we use the operator formalism to quantize the Yukawa theory in the strong scalar wave background. We find the exact fermionic basis of modes in this background and then use it to find the retarded, advanced and Keldysh propagators. The retarded and advanced propagators found in this formalism coincide with those found in the functional one. The Keldysh propagators are different. Again we use the new Keldysh propagator to find the physical observables.

In Section 4 we explain where the disagreement between the two Keldysh propagators comes from. We derive an equation relating the two Keldysh propagators, but we cannot solve it at present. In Section 5 we describe the peculiarities of the two Keldysh propagators found in the paper. Section 6 contains conclusions.

This is a somewhat shorter version of the text published in arXiv:1909.12805, where one can find all technical details and appendices.

2. FUNCTIONAL FORMALISM AND THE CORRESPONDING SIMPLEST STATE

In nonstationary situations the quantities to consider are the correlation functions (see, e.g., [17, 26, 28])

$$\langle O(t_1, \ldots , t_n) \rangle = \langle \mathrm{st}| U^+ T [O(t_1, \ldots , t_n) U] | \mathrm{st}\rangle$$

(2.1)
rather than amplitudes

\[ A = \frac{\langle \text{out} | T [O(t_1, \ldots, t_n) U] | \text{in} \rangle}{\langle \text{out} | U | \text{in} \rangle}, \tag{2.2} \]

at least because there are no asymptotic states. The amplitudes are more appropriate to calculate in stationary situations (in the proper ground state). Here \(O(t_1, \ldots, t_n)\) is an operator in the theory under consideration in the interaction picture; \(U = T \exp \{ i \int_{-\infty}^{t} dt' H_{\text{int}}(t') \}\) is the evolution operator, where \(H_{\text{int}}(t)\) is the nonlinear part of the full Hamiltonian in the interacting picture; \(|\text{st}\rangle\) is a state out of equilibrium, while \(|\text{in}\rangle\) \((|\text{out}\rangle\) is the true ground state (rotated by a phase) of the normal-ordered free Hamiltonian, if such a state exists.\(^1\)

In equation (2.2) all the expressions are time-ordered and, hence, one can apply the Feynman diagram technique. At the same time, if one converts (2.1) into the functional integral form [26], then there are two copies of the action, which appear in the exponent under the integral. One is coming from \(U\) and the other from \(U^+\). This is how one obtains the so-called Schwinger–Keldysh time contour \(C\), which goes forward from past to future infinity and then back. For convenience we denote fields on the forward branch of the contour \(C\) by \(\psi(t_+, x) \equiv \psi_+(t, x)\) and those on the backward branch by \(\psi(t_-, x) \equiv \psi_-(t, x)\) (see [26]). Then the action in the exponent under the functional integral can be rewritten in terms of these fields as

\[
S[\psi, \overline{\psi}] = \int_{-\infty}^{+\infty} dt \int dx \left[ \overline{\psi}_+(x, t) i \slashed{D} \psi_+(x, t) - \Phi \left( \frac{t - x}{\sqrt{2}} \right) \overline{\psi}_+(x, t) \psi_+(x, t) \right. \\
- \left. \left( \overline{\psi}_-(x, t) i \slashed{D} \psi_-(x, t) - \Phi \left( \frac{t - x}{\sqrt{2}} \right) \overline{\psi}_-(x, t) \psi_-(x, t) \right) \right], \tag{2.3}
\]

where the relative minus sign comes from the reversed direction of the time integration on the backward part of the contour, because \(U^+\) contains the complex conjugate exponent with respect to \(U\).

Then the propagators are defined as follows:

\[
i G(t, x; t', x') \equiv \int D\psi D\overline{\psi} \psi(t, x) \overline{\psi}(t', x') e^{iS[\psi, \overline{\psi}]} = \langle \psi(t, x) \overline{\psi}(t', x') \rangle. \tag{2.4}\]

In terms of \(\psi_+\) and \(\psi_-\) we have the following matrix of propagators:

\[
\tilde{G}(t, x; t', x') = \begin{bmatrix}
-i \langle T \psi_+(t, x) \overline{\psi}_+(t', x') \rangle & i \langle \overline{\psi}_-(t', x') \psi_+(t, x) \rangle \\
-i \langle \overline{\psi}_-(t, x) \overline{\psi}_+(t', x') \rangle & -i \langle T \overline{\psi}_-(t, x) \overline{\psi}_-(t', x') \rangle
\end{bmatrix} \equiv \begin{bmatrix}
G_{++} & G_{<-} \\
G_{>-} & G_{--}
\end{bmatrix}, \tag{2.5}
\]

where\(^2\)

\[
G_{++}(t, x; t', x') = \theta(t - t') G_{>}(t, x; t', x') + \theta(t' - t) G_{<}(t, x; t', x'), \\
G_{--}(t, x; t', x') = \theta(t - t') G_{<}(t, x; t', x') + \theta(t' - t) G_{>}(t, x; t', x'). \tag{2.6}
\]

From this definition it is clear that

\[
G_{++} + G_{--} = G_{>} + G_{<}. \tag{2.7}
\]

\(^1\)Turning on and switching off the interaction term \(H_{\text{int}}\) at past and future infinity is usually assumed in the stationary situations. Note that (2.1) reduces to (2.2) if the quantum average is taken there over the true ground state \(|\text{st}\rangle\rightarrow|\text{in}\rangle\) and if \(|\text{out}U|\text{in}\rangle| = 1\) under the adiabatic turning-on and switching-off of \(H_{\text{int}}\). Meanwhile the direct calculation of loop corrections to (2.2) in a nonstationary situation leads to loop infrared divergences that cannot be canceled out [7, 8, 13].

\(^2\)It is worth mentioning that \(G_{>, G_{<}}\) and \(G_{\pm\pm}\) are \(2 \times 2\) matrices in spinor indices. So, \(G\) itself is a block matrix of \(2 \times 2\) matrices.
The structure of the action (2.3) and of the propagator (2.4) may lead to the conclusion that nondiagonal components of the propagator matrix (2.5) must vanish. In that sense, the functional integral representation of the theory is slightly misleading: it does not contain information about the initial state of the theory, which makes the $\psi_+^\ast$ and $\psi_-$ fields correlated (see, e.g., [26]). And if one works in the functional integral formalism, it seems to be unclear where the information about the initial state of the fermions is hidden. Then it seems that one should always keep in mind the operator formalism, from which the propagators can be derived using the initial density matrix.

It is possible to make the presence of the initial state apparent in the functional formalism by applying the so-called Keldysh rotation [26, 28]. In fact, if we introduce the new pair of fields

\[
\begin{align*}
\Psi_1 &= \frac{1}{\sqrt{2}}(\psi_+^\ast + \psi_-), \\
\Psi_2 &= \frac{1}{\sqrt{2}}(\psi_+^\ast - \psi_-)
\end{align*}
\]

and

\[
\begin{align*}
\overline{\Psi}_1 &= \frac{1}{\sqrt{2}}(\psi_+ - \psi_-), \\
\overline{\Psi}_2 &= \frac{1}{\sqrt{2}}(\psi_+ + \psi_-),
\end{align*}
\]

then the action (2.3) acquires the following form:

\[
S = \int_{-\infty}^{+\infty} dt \, dx \left\{ \overline{\Psi}(x, t) i \theta \Psi(x, t) - \overline{\Psi}(x, t) \hat{\Phi}(t, x) \Psi(x, t) \right\},
\]

where

\[
\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \quad \overline{\Psi} = \begin{bmatrix} \overline{\Psi}_1 \\ \overline{\Psi}_2 \end{bmatrix} \quad \text{and} \quad \hat{\Phi}(t, x) = \begin{bmatrix} \Phi \left( \frac{t-x}{\sqrt{\epsilon}} \right) & 0 \\ 0 & \Phi \left( \frac{t-x-\epsilon}{\sqrt{\epsilon}} \right) \end{bmatrix}.
\]

After such a rotation the propagator matrix transforms into the triangular form:

\[
\hat{G}(t, x; t', x') = \begin{bmatrix} -i \langle \Psi_1(t, x) \overline{\Psi}_1(t', x') \rangle & -i \langle \Psi_1(t, x) \overline{\Psi}_2(t', x') \rangle \\ 0 & -i \langle \Psi_2(t, x) \overline{\Psi}_2(t', x') \rangle \end{bmatrix} = \begin{bmatrix} G^R & G^K \\ 0 & G^A \end{bmatrix},
\]

where

\[
\begin{align*}
G^R(t, x; t', x') &= \theta(t - t')(G^>(t, x; t', x') - G^<(t, x; t', x')), \\
G^A(t, x; t', x') &= \theta(t' - t)(G^<(t, x; t', x') - G^>(t, x; t', x')), \\
G^K(t, x; t', x') &= G^>(t, x; t', x') + G^<(t, x; t', x').
\end{align*}
\]

The tree-level retarded and advanced Green functions (the first two equations in (2.12)) do not depend on the initial state of the theory: they only carry information about causality and spectrum. This is because these propagators are proportional to the anticommutator of $\psi$'s, which is the c-number. At the same time the Keldysh propagator (the last line in (2.12)) does contain information about the initial density matrix. This is precisely the piece of information one needs to correctly define the correlation functions in the functional integral formalism (2.4) (see, e.g., [17, 26]).

In this section we assume that initially, at $t = -\infty$, when $\Phi(t - x) = 0$, fermions are at the equilibrium and have thermal distribution with temperature $T$. Then, at past infinity the time Fourier transform of the Keldysh propagator is related to the retarded and advanced propagators in the following way\(^3\):

\[
G^K_0(\epsilon, x, x') = F(\epsilon) \left[ G^R_0(\epsilon, x, x') - G^A_0(\epsilon, x, x') \right],
\]

\(^3\)Equation (2.13) constitutes the statement of the so-called fluctuation-dissipation theorem [26, 28]: it implies a rigid relation between the response functions and the correlation functions in equilibrium.
where the index 0 means that we consider the propagators in the absence of the background field, i.e., when \( \Phi = 0 \), and

\[
F(\epsilon) = 1 - 2n_\epsilon = 1 - 2 \frac{1}{e^{\epsilon/T} + 1} = \tanh \frac{\epsilon}{2T}
\]

(2.14)
defines the distribution function; \( G_{0}^{K,R,A}(\epsilon, x, x') \) are time Fourier transforms of the Keldysh, retarded and advanced propagators, respectively.

On the whole, the theory defined with the help of the Keldysh rotation (2.8) is self-consistent; i.e., having established relation (2.13) in the operator formalism, we can forget about this formalism and work only in the functional approach, knowing that all information about the initial density matrix is encoded in (2.13).

2.1. Dyson equation and R–A junction. To find the exact propagator matrix, we treat the \( \overline{\Psi} \hat{\Phi} \Psi \) term in (2.9) as a perturbation and use the causality condition [26]

\[
G^{R/A}(t_{1}) \ldots G^{R/A}(t_{n}, t) = 0,
\]

which follows from the fact that the retarded and advanced propagators are proportional to the \( \theta \)-functions. (Physically this just means that the exact retarded and advanced Green functions, respectively, as their tree-level counterparts.) This way we obtain the Dyson–Schwinger equation for the matrix of exact propagators:

\[
\hat{G}(t, x; t', x') = \hat{G}_{0}(t, x; t', x') + \int d\tau dy \hat{G}_{0}(t, x; \tau, y) \hat{\Phi}(\tau, y) \hat{G}(\tau, y; t', x')
\]

\[
\equiv \hat{G}_{0} + \hat{G}_{0} \circ \hat{\Phi} \circ \hat{G},
\]

(2.15)

where \( \hat{G}_{0} \) is the matrix of propagators for \( \Phi(v) = 0 \) and \( \hat{G} \) is the matrix of exact propagators.

In components this equation can be written as

\[
\begin{cases}
G^{R} = G_{0}^{R} + G_{0}^{R} \circ \Phi \circ G^{R}, \\
G^{A} = G_{0}^{A} + G_{0}^{A} \circ \Phi \circ G^{A}, \\
G^{K} = G_{0}^{K} + G_{0}^{K} \circ \Phi \circ G^{A} + G_{0}^{R} \circ \Phi \circ G^{K}.
\end{cases}
\]

(2.16)

One can see that since the matrix \( \hat{\Phi}(t, x) \) is diagonal, the equations for \( G^{R/A} \) are independent of each other. The explicit form of \( G_{0}^{R} \) can be found below, the \( G_{0}^{A} \) is just conjugate to \( G_{0}^{R} \), and \( G_{0}^{K} \) can be found from (2.13).

Since in the presence of \( \Phi(v) \) there is no time-translation invariance, in the exact propagators one has to perform the Fourier transform in \( t \) and \( t' \) separately:

\[
G(\epsilon, x; \epsilon', x') \equiv \int dt dt' G(t, x; t', x') e^{i\epsilon t - i\epsilon' t'}.
\]

(2.17)

At the same time, when initially (at past infinity) fermions are at thermal equilibrium, the Fourier transform of the Keldysh propagator has the form (2.13) and we can rewrite it as

\[
G_{0}^{K}(\epsilon, x; \epsilon', x') = G_{0}^{R}(\epsilon, x; \epsilon', x') F(\epsilon') - F(\epsilon) G_{0}^{A}(\epsilon, x; \epsilon', x'),
\]

(2.18)

where \( G_{0}^{K,R,A}(\epsilon, x; \epsilon', x') \equiv G_{0}^{K,R,A}(\epsilon, x, x') \delta(\epsilon - \epsilon') \). This is because we have time-translation invariance in the case when \( \Phi(v) = 0 \).

Making the inverse Fourier transform of (2.18), we obtain

\[
G_{0}^{K}(t, x; t', x') = \int d\tau G_{0}^{R}(t, x; \tau, x') f(\tau - t') - \int d\tau f(t - \tau) G_{0}^{A}(\tau, x; t', x')
\]

\[
= G_{0}^{R} \circ f - f \circ G_{0}^{A},
\]

(2.19)
where the Fourier transform of \( F(\epsilon) \) is

\[
  f(\tau) = \int_{-\infty}^{+\infty} \frac{de}{2\pi} e^{-i\epsilon \tau} \tanh \frac{\epsilon}{2T} = -\mathcal{P} \frac{iT}{\sinh(\pi T \tau)}.
\]

(2.20)

With the use of the first two equations in (2.16), the last one there can be rewritten as

\[
  G^K(t, x; t', x') = \int dt' \int dx' \left( f(t - t') - \int dt' \int dx' G^L(\tau, x; t, x') \right)
\]

and

\[
  G^K(t, x; t', x') = \int d\tau \int dx \left( f(\tau - t') - \int d\tau \int dx' G^L(\tau, x; t, x') \right)
\]

(2.21)

i.e., we have expressed the exact Keldysh propagator via the exact retarded and advanced ones. For convenience, let us denote by \( G^K \) the sum of the first two terms, which reproduces the structure of the equilibrium Keldysh propagator (2.19). The third “anomalous” term is the so-called R–A junction [27, 31], which we denote by \( G^K_{\text{an}} \). Note that the “anomalous” term vanishes in the case of constant potential \( \Phi(v) \), while for nontrivial \( \Phi(v) \) the theory is out of thermal equilibrium.

2.2. Solution of the Dyson equation for the retarded and advanced propagators.

Multiplying both sides of the first equation of system (2.16) by \( [G^R_0]^{-1} \), we get

\[
  \left[ [G^R_0]^{-1} - \Phi \left( \frac{t - x}{\sqrt{2}} \right) \right] \circ G^R = 1.
\]

(2.22)

Since \( [G^R_0]^{-1} = i\phi \), the last equation can be written in components as

\[
  \begin{vmatrix}
    -\Phi \left( \frac{t-x}{\sqrt{2}} \right) & i(\partial_t + \partial_x) \\
    i(\partial_t - \partial_x) & -\Phi \left( \frac{t-x}{\sqrt{2}} \right)
  \end{vmatrix}
  \begin{pmatrix}
    G^R_{11}(t, x; t', x') & G^R_{12}(t, x; t', x') \\
    G^R_{21}(t, x; t', x') & G^R_{22}(t, x; t', x')
  \end{pmatrix}
  = \delta(t - t') \delta(x - x') \hat{I},
\]

(2.23)

with the condition that

\[
  G^R(t, x; t', x') = 0 \quad \text{if} \quad t < t'.
\]

Since the potential \( \Phi \) is a function of \( t - x \) only, it is convenient to work in the light-cone coordinates

\[
  u = \frac{t + x}{\sqrt{2}} \quad \text{and} \quad v = \frac{t - x}{\sqrt{2}}.
\]

(2.24)

It can be shown that the solution of (2.23) is as follows:

\[
  \begin{align*}
  G^R_{11}(u, v; u', v') &= -\frac{\Phi(v')}{2} J_0(2\sqrt{(u-u')a(v, v')}) \theta_{uu'} \theta_{vv'}, \\
  G^R_{22}(u, v; u', v') &= -\frac{\Phi(v)}{2} J_0(2\sqrt{(u-u')a(v, v')}) \theta_{uu'} \theta_{vv'}, \\
  G^R_{12}(u, v; u', v') &= -\frac{i}{\sqrt{2}} \delta_{uu'} \theta_{vv'} + \frac{i}{\sqrt{2}} \sqrt{\frac{a(v, v')}{u-u'}} J_1(2\sqrt{(u-u')a(v, v')}) \theta_{uu'} \theta_{vv'}, \\
  G^R_{21}(u, v; u', v') &= -\frac{i}{\sqrt{2}} \theta_{uu'} \delta_{vv'} + \frac{i}{\sqrt{2}} \Phi(v') \frac{\Phi(v)}{2} \sqrt{\frac{u-u'}{a(v, v')}} J_1(2\sqrt{(u-u')a(v, v')}) \theta_{uu'} \theta_{vv'}.
  \end{align*}
\]

(2.25)

Here for simplicity we introduced the notation \( \theta_{xy} \equiv \theta(x - y) \) and \( \delta_{xy} \equiv \delta(x - y) \);

\[
  a(v, v') = \frac{1}{2} \int_{v'}^v dy \Phi^2(y);
\]

(2.26)

and \( J_0(x) \) and \( J_1(x) \) are the Bessel functions.
It is straightforward to check that when $\Phi(v) = 0$, the obtained expression reduces to

$$G^R|_{\Phi=0} = \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}} \theta(v-v') \delta(u-u') \\ -\frac{1}{\sqrt{2}} \delta(v-v') \theta(u-u') & 0 \end{bmatrix} = G^R_0,$$  \hspace{1cm} (2.27)

which is the retarded Green function in the absence of the background scalar field. The advanced Green function can be obtained via the Hermitian conjugation of the retarded one:

$$G^A = [G^R]^\dagger.$$  \hspace{1cm} (2.28)

Hermitian conjugation includes complex conjugation along with the interchange of the arguments $u, v \leftrightarrow u', v'$.

The knowledge of the exact form of $G^{R/A}$ allows one to find the Keldysh propagator from (2.21). This form is bulky and hard to treat in physical terms. That is why we are interested in calculating the expectation values of the scalar current, $\langle \bar{\psi} \psi \rangle$, and of the stress–energy tensor. Then one only needs the trace of the Keldysh propagator over the spinor indices for coincident points.

### 2.3. The expectation value of the scalar current.

The physical observables that we calculate below can be expressed in terms of the Wightman propagator $G^<(x,t; x', t') = i\langle \bar{\psi}(t, x) \psi(t', x') \rangle$, which can be represented as

$$G^<(x, t; x', t') = \frac{1}{2} G^K(x, t; x', t') - \frac{1}{2} \left(G^R(x, t; x', t') - G^A(x, t; x', t')\right).$$  \hspace{1cm} (2.29)

As we will see, the second term on the right-hand side of this expression does not contribute to the quantities that we calculate, at least in the limit that we describe below in this section.

Thus, we need to find the exact propagator $G^K(t, x; t', x')$ when $t' = t$ and $x' = x - \delta$, as $\delta \rightarrow 0$. We split $x$ and $x'$ to single out the divergent part of the correlation function. The divergent part is only in the diagonal components of $G^K$. In principle, using the exact form of $G^K$ at coincident points, one can express the current via integrals of Bessel functions. But to simplify expressions and to reduce them to a physically tractable form, we look for $\langle \bar{\psi} \psi \rangle$ for those values of $\Phi(v)$ which obey the following conditions:

$$\frac{\Phi(v)}{\lambda} \gg 1, \quad \left| \frac{\Phi^{(k)}(v)}{\lambda^2 \Phi(v)} \right| \ll 1 \quad \text{and} \quad \frac{1}{\lambda} \int_v^{v+1} \Phi^2(y) \, dy \gg 1,$$  \hspace{1cm} (2.30)

which essentially means that the scalar field is a large and slowly changing function. Our goal is to find the leading contribution to $\langle \bar{\psi}(t, x - \delta) \psi(t, x) \rangle$ in the limit (2.30).

The result of the calculation is

$$\langle \bar{\psi}(t, x - \delta) \psi(t, x) \rangle \approx \frac{\Phi(v)}{\pi} \ln[\Phi(v)] \delta.$$  \hspace{1cm} (2.31)

We discuss the meaning of this result in the concluding section.

### 2.4. The expectation value of the stress–energy tensor.

The stress–energy operator in the theory under consideration is as follows:

$$T^{\mu\nu} = \frac{i}{4} \left[\bar{\psi} \gamma^\mu \partial^\nu \psi + \bar{\psi} \gamma^\nu \partial^\mu \psi - \partial^\mu \bar{\psi} \gamma^\nu \psi - \partial^\nu \bar{\psi} \gamma^\mu \psi \right] - \eta^{\mu\nu} \mathcal{L}.$$  \hspace{1cm} (2.32)

The component of this tensor of our main interest is $T^{61}$, the flux. We will concentrate on the calculation of this component in the present subsection. The diagonal components of $T^{\mu\nu}$ will be discussed in the next section.
After the point splitting regularization, we can represent the expectation value of the flux via the derivatives of the components of the propagator (2.21):

\[
\langle T^{01}(t, x) \rangle = \frac{i}{4} \left[ \left( \sqrt{2} \partial_v - \sqrt{2} \partial_{v'} \right) \langle \bar{\psi}_1(t', x') \psi_2(t, x) \rangle 
- \left( \sqrt{2} \partial_u - \sqrt{2} \partial_{u'} \right) \langle \bar{\psi}_2(t', x') \psi_1(t, x) \rangle \right] \Big|_{t, x = t', x'},
\]

(2.33)

The calculation is straightforward but tedious, so we show here only the final result. We find that the nondiagonal component of the stress–energy tensor is zero:

\[
\langle T^{01}(t, x) \rangle = 0,
\]

(2.34)

which means that there is no fermion flux in the background field under consideration for the given state in question. This result follows from the fact that in the functional formalism and the corresponding state the left- and right-moving fermions always enter symmetrically. We will see below that for some other states the situation can be different.

3. OPERATOR FORMALISM AND THE CORRESPONDING SIMPLEST STATE

In the previous section we have calculated propagators and expectation values of physical quantities using solutions of the Dyson–Schwinger equation corresponding to an initial (thermal or ground) state at past infinity, which is the simplest one in those settings. Now we want to calculate the same quantities in the operator formalism, also using an initial state that is the simplest one in the new setting. In this way we will encounter subtleties in matching the results for the physical observables in the two formalisms under discussion. We will argue that the discrepancy comes from the fact that in different formalisms we loosely consider different states, which have physically distinct properties. Then, in Section 4 we propose a way to match the states in the two cases.

3.1. Modes and canonical commutation relation. To quantize the theory in the operator formalism, we start from the equations of motion for the modes following from the action (1.6):

\[
\begin{bmatrix}
-\Phi(v) \\
\frac{i}{\sqrt{2}} \partial_u \\
\frac{i}{\sqrt{2}} \partial_v \\
-\Phi(v)
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = 0.
\]

(3.1)

Two linearly independent solutions of this equation are as follows:

\[
\chi_p(t, x) = \bar{u}(p)e^{-ipu-ia(v,0)/p} \quad \text{and} \quad \zeta_p(t, x) = \bar{v}(p)e^{ipu+ia(v,0)/p},
\]

(3.2)

where \(v\) and \(u\) are defined in (2.24), \(a(v, v')\) is defined in (2.26) and

\[
\bar{u}(p) = \left. \frac{1}{\Phi(v)} \right|_{2p} \quad \text{and} \quad \bar{v}(p) = \left. \frac{1}{\Phi(v)} \right|_{2p}.
\]

(3.3)

We choose modes such as in (3.2) and (3.3) because in the case when \(\Phi(v) = m\) the corresponding Keldysh propagator is the same as for the standard plane waves. In fact, one can show that if we write the field operator\(^4\) as

\[
\hat{\psi}(u, v) = \int_{-\infty}^{+\infty} \frac{dq}{2\pi \sqrt{2}} \left[ a_q \left[ \frac{1}{\Phi(v)} \right] e^{-iqu-ia(v,0)/q} + a_q^\dagger \left[ \frac{1}{\Phi(v)} \right] e^{iqu+ia(v,0)/q} \right]
\]

(3.4)

with

\[
\{ a_p, a_q^\dagger \} = 2\pi \delta(p - q), \quad \{ b_p, b_q^\dagger \} = 2\pi \delta(p - q)
\]

\(^4\)The form of the field operator under consideration is formal because of the singularity of the modes at \(p = 0\). That is why in the calculations of observables we assume that there is an \(\epsilon\) shift in the exponents: \(u \rightarrow u - i\epsilon\) and \(v \rightarrow v - i\epsilon\).
and then put $\Phi(v) = m$, then the standard theory of massive fermions follows after a trivial Bogoliubov transformation, which does not mix positive and negative modes.

The state that we are going to use in our calculations below is the ground Fock space state $|0\rangle = \hat{a}_p|0\rangle = 0$. We will return to the discussion of other possible states in the next section.

3.2. Retarded and advanced Green functions. As we have explained above, the retarded and advanced Green functions in the Gaussian approximation do not depend on the choice of the initial state. In fact, the retarded Green function is equal to

$$G^R(u, v; u', v') \equiv -i\theta(t - t')\{\psi(u, v), \overline{\psi}(u', v')\},$$

(3.6)

where the anticommutator of $\psi$'s is the c-number. The independence of the state is apparent for the tree-level propagators for the case when $\Phi = 0$. In this subsection we observe the same fact for the exact retarded and advanced propagators in the Gaussian approximation (see, e.g., [3, 26]).

From (3.6) and (3.4) for $t > t'$ we obtain

$$iG^R(u, v; u', v')$$

$$= \left[ \frac{-i\Phi(v')}{2} \int_0^\infty \frac{dp}{p^2} \sin \left( p(u - u') + \frac{a(v,v')}{p} \right) - \frac{1}{\sqrt{2}} \int_0^\infty \frac{dp}{p} \cos \left( p(u - u') + \frac{a(v,v')}{p} \right) \right] \phi(v,v').$$

(3.7)

The integrals here are equal to [23, 29]

$$-\theta(t - t') \frac{i\Phi(v)}{2} \int_0^\infty \frac{dp}{p^2} \sin \left( p(u - u') + \frac{a(v,v')}{p} \right)$$

$$= -\frac{i\Phi(v)}{2} J_0(2\sqrt{(u - u')a(v,v')}) \theta(u - u') \theta(v - v'),$$

(3.8)

and

$$\theta(t - t') \int_0^\infty \frac{dp}{p} \cos \left( p(u - u') + \frac{a(v,v')}{p} \right)$$

$$= -\sqrt{\frac{a(v,v')}{u - u'}} J_1(2\sqrt{(u - u')a(v,v')}) \theta(u - u') \theta(v - v'),$$

(3.9)

where $u - u' \neq 0$. At the same time, when $u - u' = 0$ one has

$$\theta(t - t') \int_0^\infty \frac{dp}{p} \cos \left( \frac{a(v,v')}{p} \right) = \theta(t - t') \int_0^\infty \frac{dp}{p^2} \cos \left( \frac{pa(v,v')}{p} \right)$$

$$= \theta(v - v') \lim_{\epsilon \to 0} \frac{1}{\epsilon} - \frac{1}{2} a(v,v') \theta(v - v').$$

(3.10)

Then it is possible to express the result of the integration as

$$\theta(t - t') \int_0^\infty \frac{dp}{p} \cos \left( p(u - u') + \frac{a(v,v')}{p} \right)$$

$$= \theta(v - v') \delta(u - u') - \sqrt{\frac{a(v,v')}{u - u'}} J_1(2\sqrt{(u - u')a(v,v')}) \theta(u - u') \theta(v - v').$$

(3.11)
Finally, putting all these results together, we find that the retarded Green function has the same form as (3.9):

\[
\frac{\Phi(v)\Phi(v')}{2} \int_0^{+\infty} \frac{dp}{p^2} \cos\left(p(u - u') + \frac{a(v, v')}{p}\right)
= \theta(u - u') \delta(v - v') - \frac{\Phi(v)\Phi(v')}{2} \sqrt{\frac{u - u'}{a(v, v')}} J_1\left(2\sqrt{(u - u')a(v, v')}\right) \theta(u - u') \theta(v - v').
\]

(3.12)

3.3. The expectation value of the scalar current. The Keldysh propagator is equal to

\[
G^K(t', x', t, x) = -i\langle0|\left[\hat{\psi}(t', x'), \hat{\bar{\psi}}(t, x)\right]|0\rangle
= -i \int_0^{+\infty} \frac{dp}{2\pi} \left(\chi_p(t', x')\bar{\chi}_p(t, x) - \zeta_p(t', x')\bar{\zeta}_p(t, x)\right),
\]

(3.13)

where the spinor modes \(\chi_p\) and \(\zeta_p\) are defined in (3.2) and (3.3). In the expectation value under consideration, we use the state \(|0\rangle\), which is annihilated by the operators \(\hat{a}_p\) and \(\hat{b}_p\) from equation (3.4).

Noting that

\[
\int_0^{+\infty} \frac{dp}{2\pi} \cos\left(p(u - u') + \frac{a(v, v')}{p}\right)
= \begin{cases} 
-\frac{1}{2} N_0(2\sqrt{(u - u')a(v, v')}), & (u - u')(v - v') > 0, \\
\frac{1}{\pi} K_0(2\sqrt{|(u - u')a(v, v')}|), & (u - u')(v - v') < 0,
\end{cases}
\]

(3.14)

we can express the trace of the spinor matrix \(G^K\), which we need in order to calculate the scalar current, as

\[
\text{Tr}(G^K(u, v; u', v')) = -i(\Phi(v) + \Phi(v')) \int_0^{+\infty} \frac{dp}{2\pi} \cos\left(p(u - u') + \frac{a(v, v')}{p}\right)
= -i(\Phi(v) + \Phi(v')) \left[-\theta((u - u')(v - v')) \frac{1}{2} N_0(2\sqrt{(u - u')a(v, v')}) \right. \\
+ \theta(-(u - u')(v - v')) \frac{1}{\pi} K_0(2\sqrt{|(u - u')a(v, v')}|) \right],
\]

(3.15)

where \(K_0(x)\) is the Macdonald function and \(N_0(x)\) is the Neumann function.

From the expression above, we can find that in the limit (2.30) the scalar current is equal to

\[
\lim_{\delta \to 0} \langle 0|\psi(t + \delta^0, x + \delta^j)\psi(t, x)|0\rangle = -\frac{i}{2} \lim_{\delta \to 0} \text{Tr}(G^K(u, v; u + \delta^u, v + \delta^v))
= -\frac{1}{2} \lim_{\delta \to 0} (\Phi(v) + \Phi(v + \delta^v)) \left[-\theta(\delta^v \delta^u) \frac{1}{2} N_0(\Phi(v) \delta) + \theta(-\delta^v \delta^u) \frac{1}{\pi} K_0(\Phi(v) \delta) \right]
\approx \frac{\Phi(v)}{\pi} \ln[\Phi(v) \delta],
\]

(3.16)
where \(\delta \equiv \sqrt{\delta_{\mu}^{\mu} = \sqrt{2\delta_{\mu}^{\mu}}}\). This expression coincides with (2.31), with the scalar current calculated in the functional formalism in the same limit. We explain these observations in the concluding section.

3.4. The expectation value of the stress–energy tensor. In this subsection we discuss the expectation value of the stress–energy operator (2.32) in the operator formalism. Since the modes in (3.4) satisfy equations of motion, we can skip \(\mathcal{L}\) in (2.32). The result of the calculation of the stress–energy tensor in the limit (2.30) is as follows:

\[
\langle T_{\mu\nu}\rangle_{\text{reg}} \approx \frac{\Phi^2(v)}{2\pi} \ln[\Phi(v)\delta] \eta_{\mu\nu} - \frac{1}{48\pi} \{\Phi(v), v\} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]  

(3.17)

where

\[
\overline{\Phi}(v) = \frac{1}{m^2} \int_0^v dy \Phi^2(y)
\]

and \(\{f(z), z\}\) is the Schwarzian derivative:

\[
\{f(z), z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]  

(3.18)

The second term in (3.17) breaks the general covariance. However, this is not surprising since we consider the theory with the position-dependent potential \(\Phi(t - x)\). Note that for the case \(\Phi = m\) the general covariance in (3.17) is restored.

The presence of the first term in (3.17) can be explained as follows. From the definition of the stress–energy operator (2.32), with the help of the equations of motion (3.1), one can deduce that

\[
\langle 0 | T_{\mu}^{\mu} | 0 \rangle = \Phi(v) \langle 0 | \psi(t, x) \psi(t, x) | 0 \rangle.
\]  

(3.19)

Hence, the expectation value of the stress–energy tensor should contain the following term:

\[
\langle 0 | T_{\mu}^{\mu} | 0 \rangle = \frac{1}{2} \Phi(v) \langle 0 | \psi(t, x) \psi(t, x) | 0 \rangle \eta_{\mu\nu} + \ldots = \frac{\Phi^2(v)}{2\pi} \ln[\Phi(v)\delta] \eta_{\mu\nu} + \ldots,
\]  

(3.20)

which is a part of the diagonal component of the expectation value. The same term is present in the diagonal components of the stress–energy tensor found in the functional formalism. As we explain in the concluding section, this term is universal in the limit (2.30).

Slightly more unusual is the presence of the second Schwarzian term in (3.17). In particular, the existence of this term means that there is a nonzero fermion flux \(T^{01}\), which was absent in the functional formalism. So we encounter here a physically distinct state. The situation is somewhat similar to that with the Boulware and Hartle–Hawking states as compared to the Unruh state in the presence of eternal black holes [20].

To understand the origin of the Schwarzian term in (3.17), consider the action of our theory

\[
S = \int du dv \left[ \psi_1^\dagger i \sqrt{2} \partial_u \psi_1 + \psi_2^\dagger i \sqrt{2} \partial_u \psi_2 - \Phi(v)(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) \right]
\]  

(3.21)

and perform the following transformation in it:

\[
\begin{align*}
\psi_1(u, v) &= \psi_1(u', v') = \eta_1(u', v'), \\
\psi_2(u, v) &= \sqrt{\frac{dv'}{dv}} \psi_2(u', v(v')) = \sqrt{\frac{dv'}{dv}} \eta_2(u', v').
\end{align*}
\]  

(3.22)

This is a conformal map, but we perform it in the theory which is not conformally invariant due to the presence of the interaction term.
Under such a transformation the action (3.21) gets converted into
\[
S = \int d\tau' \left[ \eta_1^+ i\sqrt{2} \partial_\tau \eta_1 + \eta_2^+ i\sqrt{2} \partial_\tau \eta_2 - m(\eta_1^+ \eta_2 + \eta_2^+ \eta_1) \right],
\]
which is the theory of free massive fermions in Minkowski space. Furthermore, under such a transformation the field operator \(\hat{\psi}(u, v)\), which is initially defined in (3.4), gets converted into
\[
\hat{\eta}(u', v') = \left[ \eta_1 \eta_2 \right] = \int_0^{+\infty} \frac{dq}{2\pi \sqrt{2}} \left[ \hat{\alpha}_q \left[ \frac{1}{m \sqrt{2q}} \right] e^{-iqu' - im^2v'/2q} + \hat{b}_p^+ \left[ -\frac{1}{m \sqrt{2q}} \right] e^{iqu' + im^2v'/2q} \right]
\]
\[
= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \left[ \hat{a}_u u(p) e^{-iwp + ipx'} + \hat{b}_v v(p) e^{iwp - ipx'} \right].
\]
Thus, under the conformal transformation in question, the theory with the potential \(\Phi\) gets converted into the theory of free massive fermions with the standard plane wave modes. At the same time, one can show that under transformations such as (3.22) the expectation value of the flux operator \(\langle T_{01} \rangle\) gets shifted affinely by a contribution of the Schwarzian derivative type. Meanwhile, the theory (3.23) obviously has zero flux. These observations explain the origin of the Schwarzian contribution to the expectation value of the stress–energy tensor.

4. RELATION BETWEEN PREVIOUSLY FOUND KELDYSH PROPAGATORS

Thus, in Section 2 we have solved the Dyson–Schwinger equation and found the exact retarded and advanced propagators in the functional formalism. Then we have calculated the physical observables following from the exact Keldysh propagator (2.21), which carries the information about the state of the theory.

In Section 3, we worked in the operator formalism and found a complete basis of modes solving the classical equations of motion. But there is an ambiguity in the choice of such a basis. Depending on this choice, there are different “ground” Fock space states in the theory. In fact, instead of (3.2) and (3.3) one could consider the canonically transformed basis of modes:
\[
\tilde{\chi}_p(t, x) = \int \frac{dq}{2\pi} \left[ \alpha_{pq} \chi_q(t, x) + \beta_{pq} \sigma_q(t, x) \right], \quad \tilde{\zeta}_p(t, x) = \int \frac{dq}{2\pi} \left[ \gamma_{pq} \chi_q(t, x) + \eta_{pq} \sigma_q(t, x) \right].
\]
To respect the canonical anticommutation relations for the fermionic fields and for the corresponding creation and annihilation operators, the Bogoliubov coefficients \(\alpha_{pq}, \beta_{pq}, \gamma_{pq}\) and \(\eta_{pq}\) should satisfy the following conditions:
\[
\int \frac{dp}{2\pi} \left( \alpha_{pq} \alpha_{pq}^* + \gamma_{pq} \gamma_{pq}^* \right) = 2\pi \delta(q - q'), \quad \int \frac{dp}{2\pi} \left( \beta_{pq} \beta_{pq}^* + \eta_{pq} \eta_{pq}^* \right) = 2\pi \delta(q - q'),
\]
\[
\int \frac{dp}{2\pi} \left( \alpha_{pq} \beta_{pq}^* + \gamma_{pq} \eta_{pq}^* \right) = 0.
\]
On physical grounds one also should require that
\[
\alpha_{pq} \approx \eta_{pq} \approx \delta(p - q), \quad \beta_{pq} \approx \gamma_{pq} \approx 0
\]
as either \(p\) or \(q\) is taken to infinity. This is necessary for the modes to have the proper UV behavior and accordingly for the propagators to have the proper Hadamard behavior or UV singularity structure.

---

5 After all, the appearance of the Schwarzian term is not so surprising, because the flux operator \(T_{01}\) in 2D generates the Virasoro algebra irrespective of whether the theory is conformally invariant or not. We would like to thank G. Jorjadze for pointing out this fact to us.
Thus, there is no unique way to chose the basis of modes, and all possibilities in (4.1) are in principle allowed and may lead to different physical situations. This fact is apparent when there is no preferable basis of special functions, which is obviously the case in the present situation.

For a given choice of modes one can define a new Fock space “ground” state:

\[ \tilde{a}_p |\alpha, \beta, \gamma, \eta \rangle = \tilde{b}_p |\alpha, \beta, \gamma, \eta \rangle = 0, \]  

(4.4)

where \( \tilde{a}_p \) and \( \tilde{b}_p \) are canonically transformed annihilation operators. Then one can calculate the corresponding matrix of propagators. The retarded and advanced propagators will be the same due to (4.1)-(4.3). However, one will obtain different Keldysh propagators.

The new Keldysh propagator calculated with the use of an excited state on top of (4.4) has the following form:

\[
\begin{align*}
\bar{G}^K(t, x; t', x') &\equiv -i \langle \Omega | \bar{\psi}(t, x), \bar{\psi}(t', x') | \Omega \rangle \\
&= -i \int dp \int dq \left[ \chi_p(t, x) \chi_q(t', x') (2\pi \delta(p - q) - 2n'_{pq}) + \xi_p(t, x) \xi_q(t', x') (2\pi \delta(p - q)) \right] \\
&= G^K(t, x; t', x') + 2i \int dp \int dq \left[ n_{qp} \chi_p(t, x) \chi_q(t', x') - n_{pq} \chi_p(t, x) \chi_q(t', x') \right] \\
&\quad - \kappa_{pq} \chi_p(t, x) \xi_q(t', x') - \kappa_{pq}^\dagger \xi_q(t, x) \chi_p(t', x') \\
&\equiv \int d\tau G^R(t, x; \tau, x') f(\tau - t') - \int d\tau f(t - \tau) G^A(\tau, x; t', x') \\
&\quad + \int dy \int d\tau_1 d\tau_2 G^R(t, x; \tau_1, y) G^A(\tau_2, y; t', x') [\Phi(\tau_2, y) - \Phi(\tau_1, y)] f(\tau_1 - \tau_2). 
\end{align*}
\]

(4.5)

where \( G^K(t, x; t', x') \) is given by (3.13), \( \bar{\psi} \) is a field operator rewritten in terms of \( \bar{\chi} \) and \( \bar{\xi} \) and

\[ n'_p \equiv \langle \Omega | \tilde{a}_p a_q | \Omega \rangle, \quad n'_{pq} \equiv \langle \Omega | \tilde{b}_p b_q | \Omega \rangle. \]

Also

\[ n_{qp} \equiv \int \frac{dk_1}{2\pi} \gamma_{k_1 p} \gamma_{k_1 q}^* + \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \left[ n_{k_2 k_1} \alpha_{k_1 p} \alpha_{k_2 q}^* - \bar{n}^\prime_{k_2 k_1} \gamma_{k_2 q} \gamma_{k_1 p} \right]. \]

(4.7)

\[ \bar{n}_{pq} \equiv \int \frac{dk_1}{2\pi} \bar{\beta}_{k_1 p} \bar{\beta}_{k_1 q}^* + \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \left[ \bar{n}^\prime_{k_2 k_1} \eta_{k_2 q} \eta_{k_1 p}^* - \bar{n}^\prime_{k_2 k_1} \bar{\beta}_{k_2 q} \bar{\beta}_{k_1 p} \right]. \]

(4.8)

and

\[ \kappa_{pq} \equiv \int \frac{dk_1}{2\pi} \alpha_{k_1 p} \beta_{k_1 q}^* + \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \left[ \bar{n}^\prime_{k_2 k_1} \gamma_{k_1 p} \eta_{k_2 q}^* - \bar{n}^\prime_{k_2 k_1} \beta_{k_2 p} \alpha_{k_1 q} \right]. \]

(4.9)

On the other hand, we know the exact form of the Keldysh propagator for a given distribution function \( f \). Then one should expect that for a state \( |\Omega\rangle \) and for some choice of \( \alpha, \beta, \gamma \) and \( \eta \) the following relation holds:

\[
\begin{align*}
G^K(t, x; t', x') &\equiv 2i \int dp \int dq \left[ n_{qp} \chi_p(t, x) \chi_q(t', x') - n_{pq} \chi_p(t, x) \chi_q(t', x') \right] \\
&\quad - \kappa_{pq} \chi_p(t, x) \chi_q(t', x') - \kappa_{pq}^\dagger \chi_q(t, x) \chi_p(t', x') \\
&\equiv \int d\tau G^R(t, x; \tau, x') f(\tau - t') - \int d\tau f(t - \tau) G^A(\tau, x; t', x') \\
&\quad + \int dy \int d\tau_1 d\tau_2 G^R(t, x; \tau_1, y) G^A(\tau_2, y; t', x') [\Phi(\tau_2, y) - \Phi(\tau_1, y)] f(\tau_1 - \tau_2). 
\end{align*}
\]

(4.10)

If we solve this equation for \( n \)’s and \( \kappa \)’s, then this will establish a connection between the states in the operator and functional formalisms.
5. THE BEHAVIOR OF THE PREVIOUSLY FOUND KELDYSH PROPAGATORS AT PAST INFINITY

At past infinity the exact Keldysh propagator $G^K$ found in the functional formalism has the form

$$G^K(v, u; v', u') \approx \left[ \frac{0}{\mathcal{P} \sinh[\sqrt{2\pi T}(v' - v)]} \right] \frac{\mathcal{T}}{0} \mathcal{P} \left[ \frac{1}{\sinh[\sqrt{2\pi T}(v' - u)]} \right],$$

which follows directly from the Dyson–Schwinger equation if the background field is switched off, $\Phi(v) \to 0$ as $v \to -\infty$, as we have assumed.

Let us see what happens at past infinity with the Keldysh propagator calculated in the operator formalism (3.13). It is straightforward to see that its diagonal components behave as

$$\begin{cases}
G^K_{11}(v, u; v', u') \approx \Phi(v) \ln \sqrt{(u - u')a(v, v')} \approx \Phi \ln(\Phi) \approx 0, & \text{when } \Phi(v) \to 0.
\end{cases}$$

One of the nondiagonal components is

$$G^K_{12}(v', u'; v, u) = -\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dp \sin \left( p(u - u') + \frac{a(v, v')}{p} \right)$$

$$= -\frac{1}{\sqrt{2\pi}} \lim_{\delta^v, \delta^u \to 0} \int_0^{+\infty} dp \exp \left\{ -[\delta^u - i(u - u')]p - \frac{\delta^v - ia(v, v')}{p} \right\}$$

$$= -\frac{1}{\sqrt{2\pi}} \lim_{\delta^u \to 0} \int_0^{+\infty} dp \exp \left\{ -[\delta^u - i(u - u')]p - \frac{\delta^v - ia(v, v')}{p} \right\}$$

while the other is

$$G^K_{21}(v', u'; v, u) = -\frac{1}{\sqrt{2\pi}} \frac{\Phi(v)\Phi(v')}{2} \int_0^{+\infty} dp \sin \left( p(u - u') + \frac{a(v, v')}{p} \right)$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{\Phi(v)\Phi(v')}{2} \int_0^{+\infty} dp \exp \left\{ -[\delta^u - i(u - u')]p - \frac{\delta^v - ia(v, v')}{p} \right\} \left( p = \frac{1}{q} \right)$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{\Phi(v)\Phi(v')}{2} \lim_{\delta^v \to 0} \int_0^{+\infty} dq \exp \left\{ -\frac{\delta^u - i(u - u')}{q} - \frac{\delta^v - ia(v, v')}{q} \right\}$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{\Phi(v)\Phi(v')}{2} \lim_{\delta^v \to 0} \int_0^{+\infty} dq \exp \left\{ -\frac{\delta^u - i(u - u')}{q} - \frac{\delta^v - ia(v, v')}{q} \right\}$$

$$= -\frac{1}{\sqrt{2\pi}} \frac{\Phi(v)\Phi(v')}{2} \lim_{\delta^v \to 0} \int_0^{+\infty} dq \exp \left\{ -\frac{\delta^u - i(u - u')}{q} - \frac{\delta^v - ia(v, v')}{q} \right\}.$$
but is due to the fact that the modes in equation (3.2) contain integrals over \( v \), which is some sort of memory encoded into their basis. Let us examine the situation in greater detail.

If we assume that \( \Phi(v) = 0 \) when \( v < v_0 \) for some \( v_0 \) and \( v, v' \) in \( a(v, v') \) are both smaller than \( v_0 \), then equation (5.5) reduces to (5.1) for \( T = 0 \); i.e., in this case the background field does not affect the propagator at past infinity. However, consider, for example, a class of background fields that behave as
\[
\Phi(v) \approx C e^{\alpha v} \quad \text{when} \quad v \to -\infty.
\] (5.6)

Then, because
\[
a(v, v') = \frac{1}{2} \int_{v'}^{v} dy \Phi^2(y) = \frac{C^2}{4\alpha} (e^{2\alpha v} - e^{2\alpha v'}) = \frac{C^2}{2\alpha} e^{\alpha(v + v')} \sinh[\alpha(v - v')],
\]
the \( G_{21}^K \) component of (5.5) behaves as
\[
-\frac{1}{\sqrt{2\pi}} \frac{\Phi(v)\Phi(v')}{2} \frac{1}{P} a(v, v') \approx \frac{P T_0}{\sinh[\sqrt{2\pi} T_0 (v' - v)]},
\] (5.7)
where \( T_0 \equiv \alpha/(\sqrt{2\pi}) \). As a result, the Keldysh propagator under consideration behaves as
\[
G^K(v, u; v', u') \approx \begin{pmatrix}
0 & \frac{1}{\sqrt{2\pi}} P T_0 \frac{1}{u - u'}
\frac{1}{\sinh[\sqrt{2\pi} T_0 (v' - v)]} & 0
\end{pmatrix}
\] when \( t \to -\infty \).

So, if we assume that the background field is turned on exponentially at past infinity, then it means that initially the right-handed fermions are at the ground state, while the left-handed fermions are at the thermal equilibrium with some effective temperature \( T_0 \) set up by the way one turns on the background field. Moreover, with adiabatic turning-on of the background field \( \Phi \) the integral \( a(v, v') \) can be nontrivial, which results in a nontrivial \( G^K \) at past infinity due to the specifics of the process of turning on the background field. This observation is also consistent with the expression for the stress–energy tensor. One can check that the Schwarzian derivative in that limit is proportional to \( \sim \alpha^2 \sim T_0^2 \), as it should be for chiral massless fermions at finite temperature. Thus, in the operator formalism (as opposed to the functional one) and the corresponding state, there is no symmetry between the left- and right-moving fermions.

### 6. CONCLUSIONS

We have found that the exact retarded and advanced propagators are the same in the functional and operator formalisms. This is essentially just a consistency check for the validity of our calculations.

Then we have found that physical observables calculated with the use of these two formalisms have different values, which is related to the fact that there are physically distinct states in the calculations in these two approaches. However, there are certain quantities that have the same values independently of the approach. Namely, the form of the scalar current is the same in both situations for large and slowly changing values of the background field \( \Phi \). The subleading terms in the current are, of course, state-dependent.

Furthermore, the form of the scalar current can be traced to the Feynman in–out effective action calculated in the approximation (2.30). In fact, if one took the T-ordered Feynman functional integral,
\[
e^{iS_{\text{eff}}[\Phi]} = \int D\psi D\bar{\psi} e^{iS[\psi, \bar{\psi}; \Phi]},
\]

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in the limit (2.30) the result would be \( S_{\text{eff}}[\Phi] \approx \int d^2 x V_{\text{eff}}[\Phi] \). It is a textbook calculation to see that \( V_{\text{eff}} \) is exactly such that

\[
\langle \tilde{\psi}(t, x - \delta) \psi(t, x) \rangle \approx \frac{\partial V_{\text{eff}}(\Phi)}{\partial \Phi} \approx -\frac{\Phi(v)}{\pi} \ln \left[ \Phi(v) \delta \right].
\]

There are several points which are worth stressing here. First, these observations mean that for the properly chosen modes and/or propagators (i.e., such that they have proper UV Hadamard behavior), the form of the current in the limit (2.30) is universal and state-independent. This seems to be natural, because in this limit the field is very strong and, hence, is not sensitive, at the leading order, to the properties of the low-lying ground state.

Second, the Feynman effective action does not have any imaginary contribution, contrary to what happens, for example, in the strong electric field. Moreover, unlike the case of the strong electric field, the effective action in this case is an analytic function on the complex cut \( \Phi \)-plane, which means that there is no tunneling of fermions in the background scalar field. This seems to signal that there is no particle creation in the situation we are discussing here.

However, as we observe in Subsection 3.4, the situation is not that simple, because there we encounter a nontrivial fermion flux in the operator formalism. The flux is given by

\[
\langle T_{01}^{\text{reg}} \rangle = -\frac{1}{48\pi} \{ \varpi(v), v \}, \quad \text{where} \quad \varpi(v) = \frac{1}{m^2} \int dy \Phi^2(y),
\]

with \( \{ f(z), z \} \) the Schwarzian derivative (3.18). This fact definitely indicates fermion creation. At the same time, in the functional formalism we find that the flux is zero. The situation is similar to that on the black hole background over the Boulware and Hartle–Hawking states as compared to the Unruh state [20].

Furthermore, the QFT in the background field \( \Phi \) as given by equation (3.1) is similar to the QFT in the presence of the nonideal mirror [4, 18]. Such a mirror is transparent for high energy modes unlike the ideal one [21, 22], which reflects waves of any energy. Moreover, similar to the case under consideration, the Schwarzian type of the stress–energy flux naturally appears in the case of moving mirrors [21, 22].

Third, we have seen that \( \Phi(t - x) \) solves the classical equation of motion \( \partial^2 \phi = 0 \). However, on the quantum level, zero point fluctuations create an effective potential and the expectation value of equation (1.7) reduces to

\[
\partial^2 \phi \approx -\lambda^2 \frac{\phi}{\pi} \ln[\phi \delta]
\]

for large and slowly changing values of \( \phi \).

Now \( \Phi(t - x) \) is no more a solution of the effective equations of motion. The correct solution should describe the rolling of the field \( \phi \) down to the minimum of the effective potential. Actually, to describe the rolling, one probably needs a slightly more expanded form of the effective action. The latter depends on the choice of the quantum ground state.

During this rolling there will probably be a quantum loop amplification of the tree-level particle flux due to the change of the anomalous quantum averages and of the level population for fermions and scalars. However, this can be seen only in the loops when the field \( \phi \) is made dynamical. This is at least what we have already seen in the de Sitter space background [1, 2, 6, 9, 11, 12, 15], in the black hole collapse background [10], in the strong electric field backgrounds [5, 14] and in the case of moving mirrors [4, 16] (see also the case of nonlinear quantum mechanics with time-dependent frequency [30]).
OUT-OF-EQUILIBRIUM TWO-DIMENSIONAL YUKAWA THEORY

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