Low Energy Effective Action for Dilatonic Braneworld
–A Formalism for Inflationary Braneworld–

Sugumi Kanno∗ and Jiro Soda†
Department of Physics, Kyoto University, Kyoto 606-8501, Japan
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We derive the low energy effective action for the dilatonic braneworld. In the case of the single-brane model, we find the effective theory is described by the Einstein-scalar theory coupled to the dark radiation. Remarkably, the dark radiation is not conserved in general due to a coupling to the bulk scalar field. The effective action incorporating Kaluza-Klein (KK) corrections is obtained and the role of the AdS/CFT correspondence in the dilatonic braneworld is revealed. In particular, it is shown that CFT matter would not be confined to the braneworld in the presence of the bulk scalar field. The relation between our analysis and the geometrical projection method is also clarified.

In the case of the two-brane model, the effective theory reduces to a scalar-tensor theory with a non-trivial coupling between the radion and the bulk scalar field.

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I. INTRODUCTION

Theoretically, it is usual to assume the existence of extra dimensions. In fact, the superstring theory requires ten spacetime dimensions to keep consistency. Empirically, however, we know our universe is a four-dimensional spacetime. To reconcile these apparently contradicted views, we need a concept of the compactification. Inspired by the recent developments of the superstring theory [1], Randall and Sundrum (RS) have proposed a novel compactification mechanism in the context of the braneworld scenario [2]. Interestingly, in this mechanism, gravity can be localized on the brane due to the warped geometry, which allows even the non-compact extra dimensions. Because of this attractive feature, there are many works on cosmology and black hole in a braneworld [3, 4].

In the superstring theory, scalar fields are ubiquitous. Indeed, a dilaton and moduli exist in the bulk generically, because they arise as modes associated with a closed string. Moreover, when the supersymmetry is spontaneously broken, they may have a non-trivial potential. Hence, it is natural to incorporate a bulk scalar field with non-trivial potential into the RS model, which is often called dilatonic braneworld. In addition to this theoretical motivation, there is a phenomenological interest in this model. Namely, the bulk scalar field can drive the inflation on the brane although the bulk spacetime never inflate [5, 6]. This kind of inflationary scenario certainly deserves further investigations.

In any case, it is nice if a purely 4-dimensional description of the braneworld exists. In the case of the non-dilatonic braneworld, there are two independent approaches to obtaining the 4-dimensional effective theory. One approach is to use a geometrical projection method [7] which is proved to be useful for understanding cosmological perturbations [8]. However, it is not a closed system of equations. In fact, the projected Weyl tensor $E_{\mu\nu} = C_{\mu\nu\rho\sigma}|_{y=0}$ which represents the effect of the bulk geometry can not be determined within this theory. In other words, there exists no action for this projected equations of motion. The other approach is based on the AdS and conformal field theory (CFT) correspondence [9]. There exists an action for this effective theory, however, it is valid only at low energy. This approach also gives rise to an interesting consequence [10]. Because both approaches are complementary and useful, their mutual relations should be understood. This was done by us through the explicit construction of the low energy effective action [11]. It would be beneficial if we could generalize these results to the dilatonic braneworld.

Fortunately, the extension of the geometrical projection method to the dilatonic case has been already known [12] (see also [13]). The 4-dimensional effective Einstein equation reads

$$G_{\mu\nu} = \frac{2\kappa^2}{3} \partial_\mu \varphi \partial_\nu \varphi - \frac{5\kappa^2}{12} g_{\mu\nu} \partial^\alpha \varphi \partial_\alpha \varphi - \Lambda(\varphi) g_{\mu\nu} + \frac{\kappa^4}{6} \sigma(\varphi) T_{\mu\nu} + \kappa^4 \pi_{\mu\nu} - E_{\mu\nu},$$

(1)
where

\[ \Lambda(\varphi) = \frac{\kappa^2}{2} \left[ U(\varphi) + \frac{\kappa^2}{6} \sigma^2(\varphi) - \frac{1}{8} \left( \frac{d\sigma(\varphi)}{d\varphi} \right)^2 \right]. \tag{2} \]

Here, \( \pi_{\mu\nu}, U \) and \( \sigma \) are the quadratic of the energy momentum tensor, the bulk potential and the brane tension, respectively. Curiously, the field \( \varphi \) has non-conventional kinetic terms, hence it can not be derived from an action. This prevent us from interpreting \( \Lambda(\varphi) \) as the potential function. Therefore, the geometrical approach may have disadvantages in applying it to the physical problems, although it has a clear geometrical meaning. This urges us to find the AdS/CFT interpretation of the dilatonic braneworld. Since it is an action approach, the enigma of kinetic terms for the field \( \varphi \) would be resolved, which lead us to more profound understanding of the dilatonic braneworld.

Thus, the purpose of this paper is to construct the low energy effective action and make AdS/CFT interpretation of the dilatonic braneworld. In the course of analysis, we clarify the relation between the geometrical approach and the AdS/CFT approach. Apart from these conceptual developments, we present the action of two-brane system as a concrete result.

The organization of this paper is as follows. In sec.2, we present the action for the dilatonic braneworld and derive basic equations. In sec.3, the low energy approximation scheme is explained briefly. The non-linear low energy effective actions for both the single-brane model and the two-brane model are derived. The relation to the geometric projection method is clarified. In sec.4, the effective action with KK corrections are derived and its implications are discussed. In particular, the role of AdS/CFT correspondence is revealed and the relation to the geometrical projection method is further investigated. The final section is devoted to the conclusion. In the appendix A, useful formula for the calculation are displayed. In the Appendix B, the details of the calculation at second order can be found.

II. MODEL AND BASIC EQUATIONS

Inflation in the braneworld can be driven by a scalar field either on the brane or in the bulk. We derive the effective equations of motion which are useful for both models. In this section, we begin with the single-brane system. Since we know the effective 4-dimensional equations hold irrespective of the existence of other branes [14], the analysis of the single-brane system is sufficient to derive the effective action for the two-brane system as we see in the next section.

We consider a \( Z_2 \) symmetric 5-dimensional spacetime with a brane at the fixed point and a bulk scalar field \( \varphi \) coupled to the brane tension \( \sigma(\varphi) \) but not to the matter \( \mathcal{L}_{\text{matter}} \) on the brane. The corresponding action is given by

\[
S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \, \mathcal{R} - \int d^5x \sqrt{-g} \left[ \frac{1}{2} g^{AB} \partial_A \varphi \partial_B \varphi + U(\varphi) \right] - \int d^4x \sqrt{-h} \, \sigma(\varphi) + \int d^4x \sqrt{-h} \, \mathcal{L}_{\text{matter}}, \tag{3}
\]

where \( \kappa^2, \mathcal{R} \) and \( h_{\mu\nu} \) are the gravitational constant, the scalar curvature in 5-dimensions constructed from the metric \( g_{AB} \) and the induced metric on the brane, respectively. We assume the potential \( U(\varphi) \) for the bulk scalar field takes the form

\[
U(\varphi) = -\frac{6}{\kappa^2 \ell^2} + V(\varphi), \tag{4}
\]

where the first term is regarded as a 5-dimensional cosmological constant and the second term is an arbitrary potential function. The brane tension is also assumed to take the form

\[
\sigma(\varphi) = \sigma_0 + \tilde{\sigma}(\varphi). \tag{5}
\]

The constant part of the brane tension, \( \sigma_0 \) is tuned so that the effective cosmological constant on the brane vanishes. The above setup realizes a flat braneworld after inflation ends and the field \( \varphi \) reaches the minimum of its potential.

We adopt the Gaussian normal coordinate system to describe the geometry of the brane model;

\[
ds^2 = dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu, \tag{6}
\]

where the brane is assumed to be located at \( y = 0 \). Let us decompose the extrinsic curvature into the traceless part \( \Sigma_{\mu\nu} \) and the trace part \( K \) as

\[
K_{\mu\nu} = -\frac{1}{2} g_{\mu\nu:y} = \Sigma_{\mu\nu} + \frac{1}{4} g_{\mu\nu} K. \tag{7}
\]
Then, we can obtain the basic equations off the brane using these variables. First, the Hamiltonian constraint equation leads to
\[
\frac{3}{4}K^2 - \Sigma^\alpha_\beta \Sigma^\beta_\alpha = \left(^{(4)}R - \kappa^2 \nabla^\alpha \varphi \nabla_\alpha \varphi + \kappa^2 (\partial_y \varphi)^2 - 2\kappa^2 U(\varphi)\right),
\]
where \(^{(4)}R\) is the curvature on the brane and \(\nabla_\mu\) denotes the covariant derivative with respect to the metric \(g_{\mu\nu}\). Momentum constraint equation becomes
\[
\nabla_\lambda \Sigma^\lambda_\mu - \frac{3}{4} \nabla_\mu K = -\kappa^2 \partial_y \varphi \partial_\mu \varphi.
\]
Evolution equation in the direction of \(y\) is given by
\[
\Sigma^\mu_{\nu,y} - K \Sigma^\mu_{\nu} = -\left[^{(4)}R^\mu_\nu - \kappa^2 \nabla^\mu_\varphi \nabla^\nu_\varphi\right]_{\text{traceless}}.
\]
Finally, the equation of motion for the scalar field reads
\[
\partial_y^2 \varphi - K \partial_y \varphi + \nabla^\alpha \nabla_\alpha \varphi - U'(\varphi) = 0,
\]
where the prime denotes derivative with respect to the scalar field \(\varphi\).

As we have the singular source at the brane position, we must consider the junction conditions. Assuming a \(Z_2\) symmetry of spacetime, we obtain the junction conditions for the metric and the scalar field
\[
\left[\Sigma^\mu_\nu - \frac{3}{4} \delta^\mu_\nu K\right]_{y=0} = -\frac{\kappa^2}{2} \sigma(\varphi) \delta^\mu_\nu + \frac{\kappa^2}{2} T^\mu_\nu,
\]
\[
\left[\partial_y \varphi\right]_{y=0} = \frac{1}{2} \sigma'(\varphi),
\]
where \(T^\mu_\nu\) is the energy-momentum tensor for the matter fields on the brane.

### III. LOW ENERGY EFFECTIVE ACTION

In order to derive the effective action, we take the following strategy. First, we solve the bulk equations and obtain the bulk fields as the functional of the induced metric and the scalar field on the brane. After that, we will impose the junction conditions on the solutions which can be regarded as conditions on the induced fields. Thus, we obtain the effective equations of motion from which one can read off the effective 4-dimensional action for the dilatonic braneworld. Needless to say, most interesting phenomena occur at low energy in the sense that the additional energy due to the bulk scalar field is small, \(\kappa^2 \ell^2 V(\varphi) \ll 1\), and the curvature on the brane \(R\) is also small, \(R \ell^2 \ll 1\). It should be stressed that the low energy does not necessarily implies weak gravity on the brane. Under these circumstances, we can use a gradient expansion scheme to solve the bulk equations of motion. Then, following the above procedure, we can derive the effective low energy action with corrections coming from Kaluza-Klein modes. Let us explain this more concretely (see [11] for detailed calculations and discussions).

At zeroth order, we take the brane tension \(\sigma(\varphi)\) to be constant \(\sigma_0\) and ignore matters on the brane. Then, from the junction condition (12), we have
\[
\left[\Sigma^\mu_\nu - \frac{3}{4} \delta^\mu_\nu K\right]^{(0)}_{y=0} = -\frac{\kappa^2}{2} \sigma_0 \delta^\mu_\nu.
\]
As the right hand side of (14) contains no traceless part, we get
\[
\Sigma^\mu_\nu^{(0)} = 0.
\]
We also take the potential for the bulk scalar field $U(\varphi)$ to be $-6/(\kappa^2 \ell^2)$. We discard the terms with 4-dimensional derivatives since one can neglect the long wavelength variation in the direction of $x^\mu$ at low energies. Thus, the equations to be solved are given by

$$\frac{3}{4}K^2 = \kappa^2 (\partial_y \varphi)^2 + \frac{12}{\ell^2},$$  \hspace{1cm} (16)

$$\partial_y^2 \varphi - K \partial_y \varphi = 0.$$  \hspace{1cm} (17)

The junction condition at this order

$$\left[ \partial_y \varphi \right]_{y=0} = 0$$  \hspace{1cm} (18)

tells us that the solution of Eq. (17) must be

$$\varphi = \eta (x^\mu),$$  \hspace{1cm} (19)

where $\eta(x^\mu)$ is an arbitrary constant of integration. Now, the solution of Eq. (17) yields

$$K = \frac{4}{\ell}.$$  \hspace{1cm} (20)

Other Eqs. (9) and (10) are trivially satisfied at zeroth order. Using the definition $K_{\mu \nu} = -g_{\mu \nu, y}/2$, we have the lowest order metric

$$g_{\mu \nu}(y, x^\mu) = b^2(y) h_{\mu \nu}(x^\mu), \quad b(y) \equiv e^{-y/\ell},$$  \hspace{1cm} (21)

where the induced metric on the brane, $h_{\mu \nu} \equiv g_{\mu \nu}(y = 0, x^\mu)$, arises as a constant of integration. The junction condition for the induced metric merely implies well known relation $\kappa^2 \sigma_0 = 6/\ell$ and that for the scalar field is trivially satisfied. At this leading order analysis, we can not determine the constants of integration $h_{\mu \nu}(x^\mu)$ and $\eta(x^\mu)$ which are constant as far as the short length scale $\ell$ variations are concerned, but are allowed to vary over the long wavelength scale. These constants should be constrained by the next order analysis.

Now, we take into account the effect of both the bulk scalar field and the matter on the brane perturbatively. Our iteration scheme is to write the metric $g_{\mu \nu}$ and the scalar field $\varphi$ as a sum of local tensors built out of the induced metric and the induced scalar field on the brane, in the order of expansion parameters, that is, $O(\ell^2)^n$ and $O(\kappa^2 \ell^2 V(\varphi))^n$, $n = 0, 1, 2, \cdots$ Then, we expand the metric and the scalar field as

$$g_{\mu \nu}(y, x^\mu) = b^2(y) \left[ h_{\mu \nu}(x^\mu) + \varphi_{(1)} + \varphi_{(2)} + \cdots \right],$$

$$\varphi(y, x^\mu) = \eta(x^\mu) + \varphi_{(1)} + \varphi_{(2)} + \cdots.$$  \hspace{1cm} (22)

Here, we put the boundary conditions

$$g_{\mu \nu}(y = 0, x^\mu) = 0, \quad i = 1, 2, 3, \cdots \hspace{1cm} (23)$$

$$\varphi(y = 0, x^\mu) = 0, \hspace{1cm} (24)$$

so that we can interpret $h_{\mu \nu}$ and $\eta$ as induced quantities. Extrinsic curvatures can be also expanded as

$$K = \frac{4}{\ell} + \varphi_{(1)} + \varphi_{(2)} + \cdots,$$  \hspace{1cm} (25)

$$\Sigma_{\mu \nu} = \varphi_{(1)} + \varphi_{(2)} + \cdots.$$  \hspace{1cm} (26)

Below, we discuss the single-brane model and the two-brane model, separately.
A. Single-brane Model

The 5-dimensional geometry is deformed by the additional bulk scalar field and matter field on the brane. We show that junction conditions at first order lead to the effective equations on the brane and determine this deformation.

At the first order, the Hamiltonian constraint equation (8) becomes

$$K = \ell \left[ R - \kappa^2 \nabla^\alpha \varphi \nabla_\alpha \varphi \right]^{(1)} - \frac{\ell}{3} \kappa^2 V(\eta).$$

Using the formula such as $R^{(4)} g_{\mu\nu} = R^{(h)}_{\mu\nu}/b^2$, we obtain the solution

$$K = \frac{\ell}{4b^2} \left[ R(h) - \kappa^2 \eta^{\alpha} \eta_{\alpha} \right] - \frac{\ell}{3} \kappa^2 V(\eta),$$

where $R(h)$ is the scalar curvature of $h_{\mu\nu}$ and $\nabla$ denotes the covariant derivative with respect to $h_{\mu\nu}$. The evolution equation (10) at this order reads

$$\frac{\Sigma^{\mu\nu}}{\ell} - \frac{4}{\ell} \frac{\Sigma}{\ell} = - \left[ \frac{4}{\ell} R^{(h)}_{\mu\nu} - \kappa^2 \eta^{\alpha} \eta_{\alpha} \right]^{(1)}_{\text{traceless}}.$$

Substituting the results at zeroth order solutions into Eq. (29), we obtain

$$\frac{(1)}{\ell} \Sigma^{\mu\nu} = \frac{\ell}{2b^2} \left[ R^{(h)}_{\mu\nu} - \kappa^2 \eta^{\alpha} \eta_{\alpha} \right]_{\text{traceless}} + \frac{\chi^{\mu\nu}}{b^4},$$

where $R^{(h)}_{\mu\nu}$ denotes the Ricci tensor of $h_{\mu\nu}$ and $\chi^{\mu\nu}$ is a constant of integration which satisfies the constraint $\chi^{\mu\mu} = 0$. Hereafter, we omit the argument of the curvature for simplicity. Integrating the scalar field equation (11) at first order

$$\frac{\partial^2 y}{\partial \varphi} - \frac{4}{\ell} \frac{\partial y}{\partial \varphi} = - \left[ \nabla^\alpha \nabla_\alpha \varphi \right]^{(1)}.$$

we have

$$\frac{\partial y}{\partial \varphi} = \frac{\ell}{2b^2} \Box \eta - \frac{\ell}{4} V'(\eta) + \frac{C}{b^4},$$

where $C$ is also a constant of integration. At first order in this iteration scheme, we get two kinds of constants of integration, $\chi^{\mu\nu}$ and $C$.

Given the brane tension $\tilde{\sigma}(\eta)$ and the matter fields $T_{\mu\nu}$ on the brane, the junction condition (12) becomes

$$\left[ \frac{(1)}{\ell} \Sigma^{\mu\nu} - \frac{3}{4} \delta^{\mu\nu} K \right]^{(1)}_{y=0} = - \frac{\kappa^2}{2} \delta^{\mu}_{\nu} \tilde{\sigma} + \frac{\kappa^2}{2} T^{\mu}_{\nu}.$$

Substituting the solutions (28) and (30) into the junction condition (33), we obtain the effective equation

$$G_{\mu\nu} = \frac{\kappa^2}{\ell} T_{\mu\nu} + \kappa^2 \left[ \eta^{\mu}_{\nu} \eta_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \eta^{\alpha}_{\alpha} - \delta^{\mu\nu} V_{\text{eff}} \right] - \frac{2}{\ell} \chi^{\mu\nu},$$

where we have defined an effective potential

$$V_{\text{eff}} = \frac{1}{\ell} \tilde{\sigma} + \frac{1}{2} V.$$

We note that $\chi^{\mu\nu}$ corresponds to “dark radiation” in the case of homogeneous cosmology, so we also refer to $\chi^{\mu\nu}$ as “dark radiation”. This is the standard 4-dimensional Einstein-scalar equations with dark radiation. Strictly speaking, the induced scalar field $\eta$ should be normalized so as to have standard dimension $\eta^{(4)} = \eta/\sqrt{\ell}$. As it does not cause any confusion, however, we leave it in the original form in this paper.
At this order, the junction condition \(^{(1)} \partial_y \varphi\) yields
\[
\left[ \partial_y \varphi \right]_{y=0} = \frac{1}{2} \varphi'.
\] (36)
After substituting the solution \(^{(2)}\) into the junction condition \(^{(36)}\) and using the effective potential \(^{(35)}\), we obtain the Klein-Gordon equation
\[
\Box \eta - V'_\text{eff} = -\frac{2}{\ell} C,
\] (37)
where we refer to \(C\) as “dark source” originated from the bulk scalar field. The momentum constraint \(^{(9)}\) gives constraint on dark radiation and dark source
\[
\chi'^{\mu\nu} = -\kappa^2 C \eta_{\mu\nu}.
\] (38)
Unlike the non-dilatonic theory, the dark radiation is not conserved due to the existence of the bulk scalar field. This result is consistent with the previous analysis \(^{17}\). The Bianchi identity applied to Eq. \(^{(34)}\) gives the conservation of the energy-momentum tensor for the matter fields, \(T_{\mu\nu}|_{\mu} = 0\) provided the relation \(^{(38)}\).

Now, the action can be read off from the Einstein equation,
\[
S = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[ R - \kappa^2 |^{\alpha\alpha} \eta_{\alpha} - 2\kappa^2 V_{\text{eff}} \right] + \int d^4x \sqrt{-h} L_{\text{matter}} + S_{\chi},
\] (39)
where \(S_{\chi}\) is defined by
\[
\frac{1}{\sqrt{-h}} \delta S_{\chi} = \frac{1}{\kappa^2} \chi_{\mu\nu}
\] (40)
and
\[
\frac{1}{\sqrt{-h}} \delta S_{\chi} = 2 C.
\] (41)
These tell us that the dark radiation couples with the scalar field through \(C\). The relation \(^{(38)}\) guarantees the diffeomorphism invariance of \(S_{\chi}\) (see Appendix A).

To determine \(\chi^{\mu\nu}\) and \(C\), we need the boundary condition at the AdS horizon. One natural choice is to impose the regularity at the AdS horizon. In the non-dilatonic case, the dark radiation \(\chi^{\mu\nu}\) corresponds to the black hole singularity in the bulk. In the dilatonic case, Eq. \(^{(38)}\) implies \(C\) induces \(\chi^{\mu\nu}\). Hence, to keep the regularity at the AdS horizon, we must impose \(\chi^{\mu\nu} = C = 0\) at this order. Thus, \(S_{\chi} = 0\). When \(S_{\chi} = 0\), the inflation occurs on the brane under the slow-roll condition, \(V'' \ll H^2\). Moreover, it is obvious the standard primordial curvature fluctuations with the almost scale invariant spectrum is generated. Therefore, at least at this order, the braneworld inflation leads to the prediction consistent with WMAP data \(^{16}\).

Using the solutions \(^{(28)}\) and \(^{(30)}\), first order metric is given by
\[
\left. g_{\mu\nu} \right|_{y=0}^{(1)} = -\frac{\ell^2}{2} \left( \frac{1}{b^2} - 1 \right) \left[ R_{\mu\nu} - \frac{1}{6} h_{\mu\nu} R - \kappa^2 \left( \eta_{\mu\nu} \eta_{\alpha} - \frac{1}{6} h_{\mu\nu} \eta_{\alpha} \eta_{\alpha} \right) \right]
\] (42)
where we imposed the boundary condition \(^{(28)}\). This tells us how we construct the bulk geometry using the 4-dimensional data.

Let us now compare our results with the geometrical approach by Maeda and Wands where the projected Weyl tensor \(E_{\mu\nu} = C_{\mu\nu\rho\sigma}|_{y=0}\) and \(\Phi_2 = \partial_y \varphi|_{y=0}\) gives us information about the bulk. At first order, their equations take the form
\[
G_{\mu\nu} = \frac{2\kappa^2}{3} \eta_{\mu\nu} - \frac{5\kappa^2}{12} h_{\mu\nu} \eta_{\alpha} \eta_{\alpha} - \Lambda(\eta) h_{\mu\nu} + \frac{\kappa^2}{\ell} T_{\mu\nu} - \left. E_{\mu\nu} \right|_{y=0},
\] (43)
\[
\Box \eta = V' + \frac{\kappa^2}{3} \sigma_0 \sigma' - \frac{\kappa^2}{12} \sigma'^2 T - \left. \Phi_2 \right|_{y=0},
\] (44)
where $\Lambda(\eta) = k^2 V/2 + k^4 \sigma_0 \delta/6$. As we have solved the bulk equations, we can write $E_{\mu \nu}$ or $\Phi_2$ explicitly in terms of 4-dimensional quantities. At first order, we obtain

\[
E^{\mu \nu}_{(1)} = \frac{2}{\ell} \chi^{\mu \nu} - \frac{\kappa^2}{3} \left( \eta^{\mu}_{\nu|\nu} - \frac{1}{4} \delta_{\nu}^{\mu} \eta^{\alpha}_{|\alpha} \right).
\]

In cosmology, this Weyl tensor leads to $\frac{2}{\ell} \chi_{0}^{0} + \frac{\kappa^2}{4} \hat{\eta}^2$ which coincides with the one used by Langlois and Sasaki. Substituting Eq. (45) into Eq. (43), we see the geometrical approach agrees with our result (44). Remarkably, the energy-momentum tensor for the field $\eta$ is transformed to the standard form thanks to $E_{\mu \nu}$. We also obtain

\[
\Phi_2 = \Box \eta + \frac{4}{\ell} C.
\]

Substituting Eq. (46) into Eq. (41), the geometrical approach agrees with Eq. (47). It explains the factor $1/2$ in the effective potential (45).

**B. Two-brane Model**

Now we can apply our results in the previous section to two-brane system and write down effective equations on each brane.

Our system is described by the action

\[
S = \frac{1}{2\kappa^2} \int d^5 x \sqrt{-g} \left( \mathcal{R} + \frac{12}{\ell^2} \right) - \sum_{i=\oplus,\ominus} \hat{\sigma}(\varphi) \int d^4 x \sqrt{-g^{\text{brane}}} - \sum_{i=\oplus,\ominus} \int d^4 x \sqrt{-g^{\text{brane}}} \mathcal{L}_{\text{matter}},
\]

where $g^{\text{brane}}_{\mu \nu}$ and $\hat{\sigma}$ are the induced metric and the brane tension on the $i$-brane, respectively. We consider an $S_1/Z_2$ orbifold spacetime with the two branes as the fixed points. In the first Randall-Sundrum (RS1) model, the two flat 3-branes are embedded in $\text{AdS}_5$ and the brane tensions given by $\hat{\sigma} = 6/(\kappa^2 \ell)$ and $\hat{\sigma} = -6/(\kappa^2 \ell)$.

The point is that the equations of motion on each brane take the same form if we use the induced metric on each brane (see also (14) for other approaches). The effective Einstein equations on each positive ($\oplus$) and negative ($\ominus$) tension brane at low-energies yield

\[
G^{\mu \nu}(h) = \kappa^2 \left( \eta^{\mu}_{\nu|\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \eta^{\alpha}_{|\alpha} - \frac{1}{2} \delta^{\mu}_{\nu} V \right) - \frac{2}{\ell} \chi^{\mu \nu} + \frac{\kappa^2}{\ell} \left( \bar{T}^{\mu \nu} - \delta^{\mu}_{\nu} \bar{\sigma} \right),
\]

\[
G^{\mu \nu}(f) = \kappa^2 \left( \eta^{\mu}_{\nu|\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \eta^{\alpha}_{|\alpha} - \frac{1}{2} \delta^{\mu}_{\nu} V \right) - \frac{2}{\ell} \chi^{\mu \nu} + \frac{\kappa^2}{\ell} \left( \bar{T}^{\mu \nu} - \delta^{\mu}_{\nu} \bar{\sigma} \right)
\]

where $f_{\mu \nu}$ is the induced metric on the negative tension brane and $; \sigma$ denotes the covariant derivative with respect to $f_{\mu \nu}$. When we set the position of the positive tension brane at $y = 0$, that of the negative tension brane $\tilde{y}$ in general depends on $x^\mu$, i.e. $\tilde{y} = \tilde{y}(x^\mu)$. Hence, the warp factor at the negative tension brane $\Omega(x^\mu) \equiv b(\tilde{y}(x))$ also depends on $x^\mu$. Because the metric always comes into equations with derivatives, the zeroth order relation is enough in this first order discussion. Hence, the metric on the positive tension brane is related to the metric on the negative tension brane as $f_{\mu \nu} = \Omega^2 h_{\mu \nu}$. Although Eqs. (48) and (49) are non-local individually, with undetermined $\chi^{\mu \nu}$, one can combine both equations to reduce them to local equations for each brane. Eq. (49) can be rewritten using the induced metric on the positive tension brane as

\[
\frac{1}{\Omega^2} \left[ G^{\mu \nu}(h) - 2 (\log \Omega)^{|\mu}_{|\nu} + 2 \delta^{\mu}_{\nu} (\log \Omega)^{|\alpha}_{|\alpha} + 2 (\log \Omega)^{|\mu}_{|\nu} (\log \Omega)^{|\mu}_{|\nu} + \delta^{\mu}_{\nu} (\log \Omega)^{|\alpha}_{|\alpha} \right] = \kappa^2 \left( \eta^{\mu}_{\nu|\nu} - \frac{1}{2} \delta^{\mu}_{\nu} \eta^{\alpha}_{|\alpha} \right) - \kappa^2 \left( \bar{T}^{\mu \nu} - \delta^{\mu}_{\nu} \bar{\sigma} \right).
\]
We can therefore easily eliminate $\chi^\mu_\nu$ from Eqs. (48) and (50), since $\chi^\mu_\nu$ appears only algebraically. Eliminating $\chi^\mu_\nu$ from both Eqs. (48) and (50), we obtain

$$G^\mu_\nu = \frac{\kappa^2}{\ell^2} T^\mu_\nu + \frac{\kappa^2 (1 - \Psi)^2}{\ell^2} \delta^\mu_\nu,$$

$$+ \frac{1}{\Psi} \left[ \Psi^{\mu_\nu} - \delta^\mu_\nu \Psi^{\alpha_\mu} \right] + \frac{3}{2} \frac{1}{1 - \Psi} \left( \Psi^{\mu_\nu} \Psi^{\alpha_\mu} - \frac{1}{2} \delta^\mu_\nu \Psi^{\alpha_\mu} \Psi^{\alpha_\mu} \right)$$

$$+ \kappa^2 \left( \eta^{\mu_\nu} - \frac{1}{2} \delta^\mu_\nu \eta^{\alpha_\mu} \eta^{\alpha_\mu} - \delta^\mu_\nu V_{\text{eff}} \right), \quad (51)$$

where we defined a new field $\Psi = 1 - \Omega^2$ which we refer to by the name “radion”, and the effective potential takes the form

$$V_{\text{eff}} = \frac{1}{\ell} \left[ \frac{1}{\Psi} \sigma' + \frac{(1 - \Psi)^2 \sigma'}{\Psi} \right] + \frac{2 - \Psi}{2} V. \quad (52)$$

The bulk scalar field induces the energy-momentum tensor of the conventional 4-dimensional scalar field with the effective potential which depends on the radion.

We can also determine the dark radiation $\chi^\mu_\nu$ by eliminating $G^\mu_\nu(h)$ from Eqs. (48) and (50),

$$\frac{2}{\ell} \chi^\mu_\nu = - \frac{1}{\Psi} \left[ \Psi^{\mu_\nu} - \delta^\mu_\nu \Psi^{\alpha_\mu} \right] + \frac{3}{2} \frac{1}{1 - \Psi} \left( \Psi^{\mu_\nu} \Psi^{\alpha_\mu} - \frac{1}{2} \delta^\mu_\nu \Psi^{\alpha_\mu} \Psi^{\alpha_\mu} \right)$$

$$+ \frac{\kappa^2}{2} \frac{1}{1 - \Psi} \left[ \eta^{\mu_\nu} \eta^{\alpha_\mu} - \delta^\mu_\nu \eta^{\alpha_\mu} \right] \right] + \frac{2}{\ell} \left( 1 - \Psi \right) \delta^\mu_\nu \Psi^{\alpha_\nu} \right) \right]. \quad (53)$$

Due to the property $\chi^\mu_\mu = 0$, we have

$$\Box \Psi = \frac{\kappa^2}{3\ell} \left[ \frac{1}{\Psi} \left( \frac{\circ}{\circ} + \frac{3}{2} \frac{1}{1 - \Psi} \right) + \frac{1}{2(1 - \Psi)} \Psi^{\alpha_\mu} \Psi^{\mu_\nu}$$

$$- \frac{2\kappa^2}{3} \Psi \left( 1 - \Psi \right) V - \frac{4\kappa^2}{3\ell} \left( 1 - \Psi \right) \left[ \frac{\circ}{\circ} + (1 - \Psi) \frac{\circ}{\circ} \right]. \quad (54)$$

Note that Eqs. 51 and 53 are derived from a scalar-tensor type theory coupled to the additional scalar field.

Similarly, the equations for the scalar field on branes become

$$\Box_h \eta - \frac{V'}{2} + \frac{2}{\ell} C = \frac{1}{\ell} \frac{\circ}{\circ}, \quad (55)$$

$$\Box_f \eta - \frac{V'}{2} + \frac{2}{\ell} C = - \frac{1}{\ell} \frac{\circ}{\circ}, \quad (56)$$

where the subscripts refer to the induced metric on each brane. Notice that the scalar field takes the same value for both branes at this order. Eq. 55 can be written using the induced metric on the positive tension brane as

$$\frac{1}{\Omega^2} \Box_h \eta + \frac{1}{\Omega^2} \left( \log \Omega^2 \right)^{\mu_\nu} \eta^{\mu_\nu} - \frac{V'}{2} + \frac{2}{\ell} C = - \frac{1}{\ell} \frac{\circ}{\circ}. \quad (57)$$

Eliminating the dark source $C$ from these Eqs. 55 and 56, we find the equation for the scalar field takes the form

$$\Box_h \eta - V_{\text{eff}}' = - \frac{\Psi^{\mu_\nu}}{\Psi} \eta^{\mu_\nu}. \quad (58)$$

Notice that the radion acts as a source for $\eta$. And we can also get the dark source as

$$\frac{2}{\ell} C = - \frac{V'}{2} (1 - \Psi) + \frac{\Psi^{\mu_\nu}}{\Psi} \eta^{\mu_\nu} - \frac{1}{\ell \Psi} \left[ \frac{\circ}{\circ} + (1 - \Psi) \frac{\circ}{\circ} \right]. \quad (59)$$

Interestingly, $\chi^\mu_\nu$ and $C$ vanishes in the single brane limit, $\Psi \to 1$ as can be seen from 53 and 59.
Now the effective action for the positive tension brane which gives Eqs. (61), (64) and (68) can be read off as

$$S = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[ \Psi R - \frac{3}{2(1 - \Psi)} \Psi^{\alpha\beta} \Psi_{\alpha\beta} - \kappa^2 \Psi \left( \eta^\alpha \eta_{\alpha} + 2V_{\text{eff}} \right) \right]$$

$$+ \int d^4x \sqrt{-h} \tilde{L} + \int d^4x \sqrt{-h} \left( 1 - \Psi \right)^2 \tilde{L}, \quad (60)$$

where the last two terms represent actions for the matter on each brane. Thus, we found the radion field couples with the induced metric and the induced scalar field on the brane non-trivially. Surprisingly, at this order, the nonlocality of $\chi_{\mu\nu}$ and $C$ are eliminated by the radion. As this is a close system, we can analyze a primordial spectrum to predict the cosmic background fluctuation spectrum [20].

IV. KALUZA-KLEIN CORRECTIONS

In the previous section, we showed that the leading order results in our iteration scheme yield the usual Einstein-scalar gravity in the case of the single-brane system. In the non-dilatonic braneworld, it is now well known that Kaluza-Klein (KK) modes generate a correction to the Newtonian force. Therefore, we expect that the next order calculation in our iteration scheme gives a similar correction to the usual Einstein-scalar gravity. In this section, therefore, we consider corrections coming from KK modes. We derive solutions at this order in Appendix B.

Substituting solutions (B2) and (B4) into the junction condition, we obtain the effective Einstein equation

$$G^\mu_\nu = \frac{\kappa^2}{\ell} T^\mu_\nu + \kappa^2 \left[ \eta^{\mu|\nu}_{\beta} - \frac{1}{2} \delta^{\mu}_{\nu} \eta^{\alpha|\beta}_{\alpha} - \delta^{\mu}_{\nu} V_{\text{eff}} \right]$$

$$+ \frac{\ell^2}{2} S^\mu_\nu - \frac{\ell^2}{4} \kappa^2 \left( V'' - \frac{\kappa^2}{3} V \right) U^\mu_\nu - \frac{\ell^2}{12} \kappa^2 Q^\mu_\nu + \frac{\kappa^2}{\ell} \tau^\mu_\nu, \quad (61)$$

where the effective potential at this order is defined by

$$V_{\text{eff}} = \frac{1}{\ell} \tilde{\sigma} + \frac{1}{2} V + \frac{\ell^2 \kappa^2}{48} V^2 - \frac{\ell^2}{64} V^{-2}. \quad (62)$$

Here $S^\mu_\nu$, $U^\mu_\nu$ and $Q^\mu_\nu$ are the quantities for which one can write the Lagrangian explicitly in terms of $R^\mu_\nu$ and $\eta$. We present the explicit expressions for these quantities in Appendix A. On the other hand, we cannot write the local Lagrangian for $\tau^\mu_\nu$ given by

$$\tau^\mu_\nu = -\frac{\ell^3}{12\kappa^2} \left[ -R R^\mu_\nu + \kappa^2 R^\mu_\nu \eta^{\alpha|\beta}_{\alpha} + \kappa^2 R \eta^{\mu|\nu}_{\beta} \right.$$

$$\left. + \frac{1}{2} \delta^{\mu}_{\nu} R^\alpha_\beta \eta^{\beta}_{\alpha} - 2 \kappa^2 R \eta^{\alpha|\beta}_{\alpha} + 3 \kappa^2 R \eta^{\alpha}_{\beta} \eta^{\beta}_{\alpha} - \frac{3}{2} \kappa^2 (\Box \eta)^2 - \kappa^4 (\eta^{\alpha}_{\beta} \eta^{\beta}_{\alpha})^2 \right]$$

$$+ \frac{\ell^3}{4\kappa^2 \ell^2} \left[ 2 \kappa^2 J^\mu_\nu - \frac{2}{3} \kappa^2 \mathcal{L}^\mu_\nu - \kappa^2 P^\mu_\nu - \frac{5}{6} \kappa^4 M^\mu_\nu \right] - \frac{2}{\kappa^2} \frac{\ell^2}{(2\kappa^2)} \chi^\mu_\nu. \quad (63)$$

Here $J^\mu_\nu$, $\mathcal{L}^\mu_\nu$, $P^\mu_\nu$ and $M^\mu_\nu$ are also defined in Appendix A. Interestingly, the trace of $\tau^\mu_\nu$ becomes the following form

$$\tau = \frac{\ell^3}{4\kappa^2} \left[ R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 - 2 \kappa^2 R \eta^{\alpha}_{\beta} \eta_{\alpha} + \frac{2}{3} \kappa^2 R \eta^{\alpha}_{\alpha} \eta^{\alpha}_{\beta} + \kappa^2 (\Box \eta)^2 + \frac{2}{3} \kappa^4 (\eta^{\alpha}_{\beta} \eta^{\beta}_{\alpha})^2 \right]. \quad (64)$$

This is the more familiar form of trace-anomaly [21]. It is here AdS/CFT correspondence comes into the braneworld. Namely, $\tau_{\mu\nu}$ can be interpreted as the energy-momentum tensor of the CFT matter. Note that there is an ambiguity to separate the action into CFT part and other part. However, the same problem exists even in the non-dilatonic case [11]. Despite of this defect, it is generally believed that AdS/CFT interpretation is useful at least qualitatively. Our finding must be also beneficial for understanding of the non-linear physics in the braneworld.

Using the Eq. (60) in Appendix B, the junction condition for the scalar field leads to

$$\Box \eta - V'_{\text{eff}} + \frac{\ell^2}{24} V' \left( R + \kappa^2 \eta^{|\alpha}_{\alpha} \eta_{\alpha} \right) - \frac{\ell^2}{4} \left( V'' - \frac{\kappa^2}{3} V \right) \Box \eta - \frac{\ell^2}{8} V'' \eta^{|\alpha}_{\alpha} \eta_{\alpha} = J. \quad (65)$$
Here we have defined the source $J$ with KK corrections as
\[
J = \frac{\ell^2}{4} \left( \Box^2 \eta + 2R^{\alpha\beta} \eta_{\alpha\beta} - \frac{1}{3} R \Box \eta + \frac{1}{3} R \eta_{\alpha} - \frac{8}{3} \kappa^2 \eta^{\alpha} \eta^{\beta} \eta_{\alpha\beta} - \frac{5}{3} \kappa^2 \eta^{\alpha} \eta_{\alpha} \Box \eta \right) - \frac{2}{\ell} \frac{(2)}{C} .
\] (66)

Using the Bianchi identities (or momentum constraint), we obtain the constraint on $\tau_{\mu\nu}$ and the source $J$
\[
\tau_{\mu\nu} = -\ell J \eta_{\mu} .
\] (67)

This tells us that CFT matter cannot be confined to the brane in contrast to the non-dilatonic braneworld.

Since the Lagrangian for each parts of the effective Einstein equations is known (see Appendix A), we can now deduce the effective action
\[
S = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[ \left( 1 + \frac{\ell^2}{12} \kappa^2 V \right) R - \kappa^2 \left( 1 + \frac{\ell^2}{12} \kappa^2 V - \frac{\ell^2}{4} V'' \right) \eta^{\alpha} \eta_{\alpha} - 2\kappa^2 V_{\text{eff}} \right. \]
\[
- \frac{\ell^2}{4} \left( R^\alpha_{\mu\nu} R^\alpha_{\nu\mu} - \frac{1}{3} R^2 \right) \left. \right] + \int d^4x \sqrt{-\hbar} \mathcal{L}_{\text{matter}} + S_{\text{CFT}} ,
\] (68)

where the last term comes from the energy-momentum tensor of CFT matter $\tau_{\mu\nu}$. Although it is difficult to obtain the explicit form of the non-local Lagrangian for $\tau_{\mu\nu}$, we know variations with respect to $h_{\mu\nu}$ and $\eta$. By definition, we have
\[
-2 \frac{\delta S_{\text{CFT}}}{\sqrt{-\hbar} \delta h_{\mu\nu}} = \tau_{\mu\nu} .
\] (69)

The variation with respect to $\eta$ yields the equation of motion for the scalar field. Comparing with the result from junction condition for the scalar field, we get the following relation
\[
\frac{1}{\sqrt{-\hbar}} \frac{\delta S_{\text{CFT}}}{\delta \eta} = -\ell J .
\] (70)

This means the CFT matter couples with the scalar field. Note that Eq. (67) guarantees the diffeomorphism invariance of $S_{\text{CFT}}$. Here, we do not intend to insist that we have determined the constant of integration which is nonlocal in general, rather we have transformed this unknown quantity to some known nonlocal CFT matter. Of course, which kind of CFT matter should be chosen is remained to be solved. Admittedly, this is the defect of our approach. We just hope the proper consideration of the boundary condition or more fundamental development of the superstring theory change the situation.

Let us see the relation of our results to the geometric approach at this order as well. If we rewrite Eqs. (61) and (65) using the Weyl tensor $E_{\mu\nu}$ and $\Phi_2$ whose explicit forms at second order can be found in Eqs. (B11) and (B12) in Appendix B and reduce them using Eqs. (44) and (87) at first order, finally we obtain
\[
G^\mu_{\nu} = \frac{2\kappa^2}{3} \eta^\mu \eta^\nu - \frac{5\kappa^2}{12} \delta^\mu_{\nu} \eta^\alpha \eta_{\alpha} - \delta^\mu_{\nu} \Lambda(\eta) + \frac{\kappa^4}{6} \eta^\mu \eta^\nu + \frac{\kappa^4}{12} \delta^\mu \eta^\nu + \frac{\kappa^4}{12} \delta_\mu \eta^\nu - \frac{(1)}{4} \mathbf{E}_{\mu\nu} - \frac{(2)}{2} \mathbf{E}_{\mu\nu} ,
\] (71)
\[
\Box \eta = V' + \frac{\kappa^2}{3} \sigma^\alpha \sigma^\alpha - \frac{\kappa^2}{3} \delta^\alpha \delta^\beta T - \frac{(1)}{2} \Phi_2 - \frac{(2)}{2} \Phi_2 ,
\] (72)
where
\[
\Lambda(\eta) = \frac{\kappa^2}{\ell} \delta + \frac{\kappa^2}{2} V + \frac{\kappa^4}{12} \delta^2 - \frac{\kappa^2}{16} \delta^2 ,
\] (73)
\[
\pi_{\mu\nu} = -\frac{1}{4} T_{\lambda\nu} T_{\lambda\nu} + \frac{1}{12} T T_{\mu\nu} + \frac{1}{8} g_{\mu\nu} \left( T^\alpha_{\mu\nu} T^\alpha_{\nu\mu} - \frac{1}{3} T^2 \right) .
\] (74)

Here $\tau_{\mu\nu}$ is absorbed in Weyl tensor. These are nothing but equations obtained by Maeda and Wands (12).

At this point, we give a rather general result as a byproduct of our analysis. Indeed, our result suggests the following generalization of arguments advocated by Langlois and Sasaki (18). Let us rewrite equations obtained by Maeda and Wands as
\[
G_{\mu\nu} = \kappa^2 \partial_\mu \varphi \partial_\nu \varphi - \frac{\kappa^2}{2} g_{\mu\nu} \partial^\alpha \varphi \partial_\alpha \varphi - \Lambda(\varphi) g_{\mu\nu} + X_{\mu\nu} ,
\] (75)
\[
\Box \varphi = \frac{1}{\kappa^2} \Lambda'(\varphi) + \tilde{J} ,
\] (76)
where, following Langlois and Sasaki, we have defined

\[ X_{\mu\nu} = \kappa^4 \left( \frac{\sigma}{6} T_{\mu\nu} + \kappa^4 \pi_{\mu\nu} - E_{\mu\nu} - \frac{\kappa^2}{3} \left( \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{4} g_{\mu\nu} \partial_\alpha \varphi \partial_\alpha \varphi \right) \right), \tag{77} \]

\[ \tilde{J} = \frac{1}{8} \sigma' \sigma'' - \frac{\kappa^2}{24} \sigma' T + \frac{1}{2} \Box \varphi - \Phi_2 \]. \tag{78} \]

Here, we have transformed the non-conventional kinetic term into the standard form. Thus, \( \Lambda \) can be interpreted as the potential functional for the scalar field \( \varphi \). Bianchi identity implies

\[ \nabla_\nu X^\nu_{\mu} = -\kappa^2 \tilde{J} \nabla_\mu \varphi. \tag{79} \]

In the absence of the matter \( T_{\mu\nu} \) on the brane, we have \( X^{\mu}_{\mu} = 0 \), namely, we can interpret \( X^\mu_{\nu} \) as the energy momentum tensor for the dark radiation. In this case, Eq. (79) describes the leak of the dark radiation due to the bulk scalar field. It should be stressed that no approximation is employed in the above argument.

V. CONCLUSION

We derived the non-linear low energy effective action for the dilatonic braneworld. We considered the bulk scalar field with a nontrivial potential and the brane tension coupled to the bulk scalar field. Provided that the energy density of the matter on the brane is less than the brane tension and the potential energy in the bulk is less than the bulk vacuum energy, we formulated the systematic low energy expansion scheme to solve the bulk equations and derived the effective 4-dimensional action for both the single-brane model and the two-brane model through the junction conditions.

In the case of the single-brane model, the effective theory is described by the Einstein-scalar theory with the dark radiation. Remarkably, the dark radiation is not conserved in general due to the coupling to the bulk scalar. We also clarified the role of \( E_{\mu\nu} \) in the dilatonic braneworld, namely, enigmatic kinetic terms in the formula of the geometric equations reduce to that of the conventional scalar field thanks to \( E_{\mu\nu} \). A factor 1/2 in the effective potential \( \Phi_2 \) is also naturally explained by \( \Phi_2 \). Admittedly, the dark radiation and the dark source can not be determined without imposing the boundary conditions at AdS horizon. If we choose the regularity condition at the horizon, these terms must vanish. This is the reason why the conventional Einstein theory can be recovered at the low energy.

In the case of the two-brane model, junction conditions at both branes give the boundary conditions. Then, the constant of integration is determined completely. As a result, the effective theory reduces to the scalar-tensor theory with the non-trivial coupling between the radion and the bulk scalar field. It turns out that \( \chi_{\mu\nu} \) and \( C \) becomes zero when two branes get separated infinitely. However, we should be careful to treat this limit, because next order corrections might diverge in this limit as is suggested by analysis of the linearized gravity.

As for the single-brane model, we also constructed the effective action with Kaluza-Klein corrections and revealed the role of the AdS/CFT correspondence. In particular, it was shown that CFT matter would not be confined to the braneworld in the presence of the bulk scalar field. The relation between our analysis and the geometrical projection method was also clarified at this order. In particular, we have shown that the quadratic part of the energy-momentum tensor \( \pi_{\mu\nu} \) can be reproduced from our effective action. Hence, by analyzing our effective action, we can investigate both high-energy effects and effects of \( E_{\mu\nu} \). It should be noted that we did not determine the constant of integration which is nonlocal in general. Instead we transformed the issue to another form, namely, choosing the appropriate CFT matter suitable for the physical situation.

Moreover, we generalized the conjecture of Langlois and Sasaki which is explicitly advocated only in the homogeneous model. The key observation is that the enigmatic kinetic part of the scalar field can be transformed to the standard form by subtracting the traceless combination of \( \partial_\mu \varphi \partial_\nu \varphi \) which is absorbed into the energy-momentum tensor of dark radiation.

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APPENDIX A: USEFUL FORMULA

In this Appendix, we give possible local functionals in terms of $R_{\mu\nu}$ and $\eta$ up to second order in our low-energy expansion.

First of all, we consider the following variation

$$\delta \int d^4 x \sqrt{-h} \ L(h_{\mu\nu}, \eta) = \int d^4 x \sqrt{-h} \left[ \delta h_{\mu\nu} X_{\mu\nu} + \delta \eta \frac{\delta L}{\delta \eta} \right]. \quad (A1)$$

Provided the invariance under the diffeomorphism $\delta h_{\mu\nu} = -\xi_{\mu|\nu} - \xi_{\nu|\mu}$ and $\delta \eta = -\xi^\mu \eta_{\mu}$, we get the identity

$$X_{\mu\nu|\nu} + \frac{1}{2} \frac{\delta L}{\delta \eta} \eta_{\mu} = 0 . \quad (A2)$$

Conversely, this identity guarantees the diffeomorphism invariance of the system. This fact play an important role in the discussion related to the AdS/CFT correspondence.

Besides the above theoretical importance, the above identity reduces the necessary calculations considerably. Hence, we apply the above identity to the following tensors.

The relevant local functional with fourth order derivatives which consists of $R_{\mu\nu}$ is given by

$$\delta \int d^4 x \sqrt{-h} \frac{1}{2} \left[ R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 \right] = \int d^4 x \sqrt{-h} \ S_{\mu\nu} \delta h_{\mu\nu} , \quad (A3)$$

where

$$S_{\mu\nu} = R_{\mu\alpha} R^{\alpha\nu} - \frac{1}{3} R R_{\mu\nu} - \frac{1}{4} h_{\mu\nu} \left( R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 \right) - \frac{1}{2} \left( R_{\mu|\alpha} + R_{\nu|\alpha} - \frac{2}{3} R_{\mu\nu} - \Box R_{\mu\nu} + \frac{1}{6} h_{\mu\nu} \Box R \right) . \quad (A4)$$

The local functionals with fourth order derivatives which consist of $R_{\mu\nu}$ and $\eta$ are

$$\delta \int d^4 x \sqrt{-h} \left( R \eta^{\alpha|\alpha} \right) = \int d^4 x \sqrt{-h} \ L_{\mu\nu} \delta h_{\mu\nu} , \quad (A5)$$

$$\delta \int d^4 x \sqrt{-h} \left( R^{\alpha\beta} \eta_{\alpha|\beta} \right) = \int d^4 x \sqrt{-h} \ J_{\mu\nu} \delta h_{\mu\nu} , \quad (A6)$$

where

$$L_{\mu\nu} = -\frac{1}{2} h_{\mu\nu} R \eta^{\alpha|\alpha} \eta_{\alpha|\beta} + R_{\mu/\nu} \eta^{\alpha|\alpha} \eta_{\alpha/\beta} + \eta_{\mu/\nu} \eta^{\alpha|\alpha} + h_{\mu\nu} \Box \eta^{\alpha|\alpha} + R \eta_{\mu/\nu} \eta^{\alpha|\alpha} , \quad (A7)$$

$$J_{\mu\nu} = -\frac{1}{2} h_{\mu\nu} R^{\alpha\beta} \eta^{\alpha|\alpha} \eta_{\alpha|\beta} + R_{\mu/\nu} \eta^{\alpha|\alpha} \eta_{\alpha/\beta} + R_{\alpha/\beta} \eta^{\alpha|\alpha} \eta_{\mu/\nu}$$

$$- \frac{1}{2} \left( \eta_{\mu/\nu} \eta^{\alpha|\alpha} \right) \eta_{\alpha/\beta} + \frac{1}{2} \Box \left( \eta_{\mu/\nu} \eta^{\alpha|\alpha} \right) \eta_{\alpha/\beta} + \frac{1}{2} h_{\mu\nu} \eta^{\alpha|\beta} \eta_{\alpha|\beta} . \quad (A8)$$

The local functionals with fourth order derivatives which consist of $\eta$ are

$$\delta \int d^4 x \sqrt{-h} \ (\Box \eta)^2 = \int d^4 x \sqrt{-h} \ P_{\mu\nu} \delta h_{\mu\nu} , \quad (A9)$$

$$\delta \int d^4 x \sqrt{-h} \ (\eta^{\alpha|\alpha})^2 = \int d^4 x \sqrt{-h} \ M_{\mu\nu} \delta h_{\mu\nu} , \quad (A10)$$

where

$$M_{\mu\nu} = -\frac{1}{2} h_{\mu\nu} (\eta^{\alpha|\alpha})^2 + 2 \eta_{\mu/\nu} \eta^{\alpha|\alpha} \eta_{\alpha} + \frac{1}{2} \Box \eta^{\alpha|\alpha} \eta_{\alpha} , \quad (A11)$$

$$P_{\mu\nu} = \frac{1}{2} h_{\mu\nu} (\Box \eta)^2 - (\Box \eta)_{\mu/\nu} \eta_{\alpha} - (\Box \eta)_{\nu/\mu} \eta_{\alpha} + h_{\mu\nu} (\Box \eta)^2 \eta_{\alpha} . \quad (A12)$$
The local functionals with second order derivatives are

$$\delta \int d^4x \sqrt{-h} \eta^\alpha \eta_\alpha = \int d^4x \sqrt{-h} \mathcal{U}_{\mu\nu} \delta h^{\mu\nu},$$  \hspace{1cm} (A13)$$

$$\delta \int d^4x \sqrt{-h} R V(\eta) = \int d^4x \sqrt{-h} \mathcal{Q}_{\mu\nu} \delta h^{\mu\nu},$$ \hspace{1cm} (A14)

where

$$\mathcal{U}_{\mu\nu} = -\frac{1}{2} h_{\mu\nu} \eta^\alpha \eta_\alpha + \eta_\mu \eta_\nu,$$  \hspace{1cm} (A15)$$

$$\mathcal{Q}_{\mu\nu} = V \left( R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R \right) - \left( V'' \eta_\mu \eta_\nu + V' \eta_\mu \eta_\nu \right) + h_{\mu\nu} \left( V'' \eta^\alpha \eta_\alpha + V' \Box \eta \right).$$ \hspace{1cm} (A16)

These formula can be used to read off action from the Einstein equations.

Using Eq. (A2), the above quantities satisfy the following identities

$$S_{\mu\nu} |^\nu = 0,$$  \hspace{1cm} (A17)$$

$$L_{\mu\nu} |^\nu = R^\alpha \eta_\alpha \eta_\mu + R(\Box \eta) \eta_\mu,$$ \hspace{1cm} (A18)$$

$$J_{\mu\nu} |^\nu = \frac{1}{2} R^\alpha \eta_\alpha \eta_\mu + R^{\alpha\beta} \eta_\alpha \eta_\beta \eta_\mu,$$ \hspace{1cm} (A19)$$

$$P_{\mu\nu} |^\nu = -\Box \eta \eta_\mu,$$ \hspace{1cm} (A20)$$

$$M_{\mu\nu} |^\nu = 4 \eta_\alpha \eta_\beta \eta^\alpha \eta^\beta \eta_\mu + 2 \eta^\alpha \eta_\alpha (\Box \eta) \eta_\mu,$$ \hspace{1cm} (A21)$$

$$\mathcal{U}_{\mu\nu} |^\nu = (\Box \eta) \eta_\mu,$$ \hspace{1cm} (A22)$$

$$\mathcal{Q}_{\mu\nu} |^\nu = -\frac{1}{2} R V' \eta_\mu.$$ \hspace{1cm} (A23)

These identities are useful to check the momentum constraint and the Bianchi identity.

It is also useful to know the traces of these quantities

$$S = 0,$$ \hspace{1cm} (A24)$$

$$L = 3 \Box (\eta^\alpha \eta_\alpha),$$ \hspace{1cm} (A25)$$

$$J = \frac{1}{2} \Box (\eta^\alpha \eta_\alpha) + (\eta^\alpha \eta^\beta)_{\alpha \beta},$$ \hspace{1cm} (A26)$$

$$P = 2(\Box \eta)^2 + 2(\Box \eta)^\alpha \eta_\alpha,$$ \hspace{1cm} (A27)$$

$$M = 0,$$ \hspace{1cm} (A28)$$

$$\mathcal{U} = -\eta^\alpha \eta_\alpha,$$ \hspace{1cm} (A29)$$

$$\mathcal{Q} = -VR + 3V'' \eta^\alpha \eta_\alpha + 3V' \Box \eta.$$ \hspace{1cm} (A30)

**APPENDIX B: SECOND ORDER CALCULATIONS**

In this Appendix we show the solutions of basic equations (8)-(11) at second order in our iteration scheme. We take $\chi_{\mu\nu} = C = 0$, which is consistent with the results in the single-brane limit, $\Psi \rightarrow 1$, for the 2-brane system.

The Hamiltonian constraint (8) at second order takes the form

$$K^{(2)} = -\frac{\ell}{8} K^{(1)} + \frac{\ell}{6} \Sigma^\alpha \Sigma^\beta \phi \phi_\alpha \phi_\beta + \frac{\ell}{6} \left[ (4) R - \kappa^2 \nabla^\alpha \phi \nabla_\alpha \phi \right]^{(2)} + \frac{\ell}{6} \kappa^2 \left( \partial_y^\alpha \varphi \right)^2 - \frac{\ell \kappa^2}{3} V'^{(1)}} \varphi.$$ \hspace{1cm} (B1)
Substituting the solutions (28) – (32) into above equation (B1), we get

\[
\frac{(2)}{K} = \frac{\ell^3}{b^4} \left[ -\frac{1}{72} R^2 + \frac{1}{24} R R^\beta R^\alpha + \frac{\kappa^2}{36} R \eta^{\alpha \eta} - \frac{\kappa^2}{12} R R^\alpha \eta_{\alpha} + \frac{\kappa^2}{24} (\Box \eta)^2 + \frac{\kappa^4}{36} (\eta^{\eta \eta}_{\eta})^2 \right] \\
+ \frac{\ell^3}{b^2} \kappa^2 \left[ \frac{1}{72} R V - \frac{\kappa^2}{72} V \eta^{\alpha \eta} - \frac{1}{8} V^\alpha \Box \eta \right] \\
+ \frac{\ell^3}{12} \left( \frac{b^2}{b^2} \mu^2 \right) \left[ R R^\beta R^\gamma + \frac{\mu^2}{6} R^2 + \kappa^2 (\Box \eta)^2 - 2 \kappa^2 R \eta^\alpha \eta_{\beta} + \frac{\kappa^2}{3} R \eta^{\eta \eta} - \frac{5}{6} \kappa^4 (\eta^{\eta \eta}_{\eta})^2 \right] \\
- \frac{\ell^3}{72} V^2 + \frac{\ell^3}{96} \kappa^2 V^2 + \frac{\ell^3}{12} V^2 \Box \eta \\
+ \frac{\ell^3}{12} \kappa^2 \log b \left[ V^\alpha \Box \eta - \frac{\kappa^2}{3} V \eta^\alpha \eta_{\alpha} + \frac{\kappa^4}{3} \kappa^2 V^2 \log b \right].
\] (B2)

The evolution equation (10) becomes

\[
\frac{(2)}{\Sigma_{\mu \nu}} - \frac{4}{\ell} \Sigma_{\mu \nu} = \frac{(1)}{K} \Sigma_{\mu \nu} - \left[ \frac{(4)}{R \eta^{\mu \nu} - \kappa^2 \nabla^\mu \phi \nabla^\nu \phi} \right] \text{traceless}. \tag{B3}
\]

We can integrate Eq. (B3) easily as

\[
\frac{(2)}{\Sigma_{\mu \nu}} = -\frac{\ell^3}{12} \log b \left[ R^\mu \eta_{\nu} - \frac{1}{4} \delta^\mu_{\nu} R - \kappa^2 \eta^\mu \eta_{\nu} + \frac{\kappa^2}{4} \delta^\mu_{\nu} \eta^\alpha \eta_{\alpha} \right] \\
+ \frac{\ell^3}{12} \kappa^2 V \left[ R^\mu \eta_{\nu} - \frac{1}{4} \delta^\mu_{\nu} R - \kappa^2 \eta^\mu \eta_{\nu} + \frac{\kappa^2}{4} \delta^\mu_{\nu} \eta^\alpha \eta_{\alpha} \right] \\
- \frac{\ell^3}{4} \left( \log b - \frac{2}{b^2} \right) \left[ -2 S^\mu_{\nu} + 2 \kappa^2 J^\mu_{\nu} - \frac{2}{3} \kappa^2 \beta^\mu_{\nu} - \kappa^2 \beta^\mu_{\nu} - \frac{5}{6} \kappa^4 \mathcal{M}^\mu_{\nu} \right] \\
- \frac{1}{3} R R^\mu_{\nu} + \frac{\kappa^2}{3} R^\mu_{\nu} \eta^\alpha \eta_{\alpha} + \frac{\kappa^2}{3} R^\mu_{\nu} \eta^\mu \eta_{\nu} \text{traceless} \\
+ \frac{\ell^3}{12} \kappa^2 \left( \log b - \frac{2}{b^2} \right) \left[ -3 V^\mu \eta^\mu \eta_{\nu} - \kappa^2 V^\mu \eta^\mu \eta_{\nu} + 2 V R^\mu_{\nu} \right] \text{traceless} + \frac{(2)}{\chi_{\mu \nu}}. \tag{B4}
\]

Finally, the equation for the scalar field (11) gives

\[
\partial_y (2) \phi - \frac{4}{\ell} \partial_y (2) \phi = \frac{(1)}{K} \partial_y (1) \phi - [\nabla^\alpha \nabla_\alpha \phi]^{(2)} + V^{(1)} \phi. \tag{B5}
\]

The result of integration becomes

\[
\partial_y (2) \phi = \frac{(2)}{C} - \frac{\ell^3}{12} \log b \left[ \left( R - \kappa^2 \eta^\alpha \eta_{\alpha} \right) \Box \eta - 3 \Box \eta - 6 R \eta^\alpha \eta_{\alpha} + R \Box \eta - R^\alpha \eta_{\alpha} + 8 \kappa^2 \eta^\alpha \eta_{\alpha} \eta^\beta \eta_{\alpha} + 5 \kappa^2 \eta^\alpha \eta_{\alpha} \Box \eta \right] \\
+ \frac{\ell^3}{48} b^2 \left[ \left( R - \kappa^2 \eta^\alpha \eta_{\alpha} \right) V^' + 4 \kappa^2 V^' \Box \eta - 3 \kappa^2 (V^')^2 - \frac{2}{3} \kappa^2 V^' \eta^\alpha \eta_{\alpha} - 3 \kappa^2 (V^')^2 + \frac{2}{3} \kappa^2 V^' \Box \eta - 6 \kappa^2 V^' \Box \eta \right] \\
+ \left( -6 \Box \eta - 12 \kappa^2 \eta_{\alpha \beta} + 2 R \Box \eta - 2 R^\alpha \eta_{\alpha} + 16 \kappa^2 \eta^\alpha \eta_{\alpha} \eta^\beta \eta_{\alpha} + 10 \kappa^2 \eta^\alpha \eta_{\alpha} \Box \eta \right] \\
+ \frac{\ell^3}{48} \kappa^2 V V' + \frac{\ell^3}{16} V \Box \eta + \frac{\ell^3}{64} V V' + \frac{\ell^3}{8} \log b \left[ (V^')^2 - \frac{2}{3} \kappa^2 V^' \eta^\alpha \eta_{\alpha} + (V^')^2 + \frac{2}{3} \kappa^2 V^' \Box \eta \right] \\
- \frac{\ell^3}{16} \log b \left( V'^2 \right). \tag{B6}
\]

Similar to results at first order, we get two kinds of constant of integration (2) \chi_{\mu \nu} and (2) C. Momentum constraint (9) at this order

\[
\Sigma_{\mu \nu}^{(2)} - \frac{3}{4} K_{\mu \nu} + \Gamma_{\alpha \beta}^{(1)} \Sigma_{\mu \nu}^{(1)} - \Gamma_{\mu \nu}^{(1)} \Sigma_{\mu \nu}^{(2)} = -\kappa^2 \left( \partial_y (2) \phi \right) \eta_{\mu} - \kappa^2 \left( \partial_y (1) \phi \right) \phi_{\mu}. \tag{B7}
\]
also gives the constraint on these constants of integration
\[
E_{\mu\nu}^{(2)} = \left( \frac{\ell^2}{4} + \frac{\ell^3}{24} \left[ -2R_{\mu
u}^{\alpha\beta} + \frac{2\kappa^2}{3} R_{\mu\nu}^{\alpha\beta} \eta_{\alpha\beta} + \frac{5}{6} \kappa^4 M_{\mu\nu} \right]
\right)
\]
and the correspondent quantity for the scalar field is defined by
\[
\Phi_2 = \left[ \frac{\partial^2 \varphi}{\partial y^2} \right]_{y=0} .
\]
Using these formulas, we get the above quantities at second order
\[
E_{\mu\nu}^{(2)} = \frac{\ell^2}{4} \left[ -2S_{\mu\nu}^{\alpha\beta} + \frac{2\kappa^2}{3} S_{\mu\nu}^{\alpha\beta} \eta_{\alpha\beta} - \frac{5}{6} \kappa^4 M_{\mu\nu} \right]
\]
and
\[
\Phi_2 = \frac{\ell^2}{4} \left[ \left( R + 3\kappa^2 \eta_{\alpha\beta} \right) \Box \eta - 2\Box^2 \eta - 4R_{\mu\nu}^{\alpha\beta} \eta_{\alpha\beta} - \frac{2\kappa^2}{3} R_{\mu\nu}^{\alpha\beta} \eta_{\alpha\beta} + \frac{16}{3} \kappa^2 \eta_{\alpha\beta} \eta_{\alpha\beta} \right]
\]
Using these results Eqs. 11 and 12, we can discuss the relation between AdS/CFT approach and the geometrical approach.

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