On the semistability of instanton sheaves over certain projective varieties

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Abstract

We show that instanton bundles of rank \( r \leq 2n - 1 \), defined as the cohomology of certain linear monads, on an \( n \)-dimensional projective variety with cyclic Picard group are semistable in the sense of Mumford-Takemoto. Furthermore, we show that rank \( r \leq n \) linear bundles with nonzero first Chern class over such varieties are stable. We also show that these bounds are sharp.

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1 Introduction

Let \( X \) be a nonsingular projective variety of dimension \( n \) over an algebraically closed field \( \mathbb{F} \) of characteristic zero, and let \( \mathcal{L} \) denote a very ample invertible sheaf on \( X \); let \( \mathcal{L}^{-1} \) denote its inverse.

Given (finite-dimensional) \( \mathbb{F} \)-vector spaces \( V, W \) and \( U \), a linear monad on
$X$ is a complex of sheaves

$$M_\bullet : 0 \to V \otimes \mathcal{L}^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathcal{L} \to 0 \quad (1)$$

which is exact on the first and last terms, i.e. $\alpha \in \text{Hom}(V, W) \otimes H^0(\mathcal{L})$ is injective while $\beta \in \text{Hom}(W, U) \otimes H^0(\mathcal{L})$ is surjective. The coherent sheaf $E = \ker \beta / \text{Im} \alpha$ is called the cohomology of the monad $M_\bullet$. The set:

$$S = \{ x \in X \mid \alpha(x) \in \text{Hom}(V, W) \text{ is not injective} \}$$

is a subvariety called the *degeneration locus* of the monad $M_\bullet$.

A torsion-free sheaf $E$ on $X$ is said to be a *linear sheaf* on $X$ if it can be represented as the cohomology of a linear monad and it is said to be an *instanton sheaf* on $X$ if in addition it has $c_1(E) = 0$.

Linear monads and instanton sheaves have been extensively studied for the case $X = \mathbb{P}^n$ during the past 30 years, see for instance [6, 8] and the references therein. Buchdahl has studied monads over arbitrary blow-ups of $\mathbb{P}^2$ [2]. In a recent preprint, Costa and Miró-Roig have initiated the study of linear monads and locally-free instanton sheaves over smooth quadric hypersurfaces $Q_n$ within $\mathbb{P}^{n+1}$ ($n \geq 3$) [3]. They have asked whether every such locally free sheaf of rank $n - 1$ is stable (in the sense of Mumford-Takemoto) [3, Question 5.1].

The main goal of this paper is to give a partial answer to their question in a more general context, showing that locally-free instanton sheaves of rank $r \leq 2n - 1$ on an $n$-dimensional smooth projective variety with cyclic Picard group are semistable, while locally-free linear sheaves of rank $r \leq n$ and $c_1 \neq 0$ on such varieties are stable. Furthermore, we also show that the bounds on the rank are sharp by providing examples of rank $2n$ instanton sheaves and rank $n + 1$ linear sheaves on $\mathbb{P}^n$ which are not semistable.

We conclude the paper by studying the semistability of special sheaves on $Q_n$, as introduced by Costa and Miró-Roig. Theorem 16 provides a partial answer to Question 5.2 in [3], showing that every rank $r \leq 2n - 1$ locally-free special sheaf $E$ on $Q_n$ with $c_1 = 0$ is semistable, while every rank $r \leq n$ locally-free special sheaf on $Q_n$ with $c_1 \neq 0$ is stable.
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2 Instanton sheaves on cyclic varieties

Note that if $E$ is the cohomology of a linear monad as in (1), then:

$$\text{rk}(E) = w - v - u \quad \text{and} \quad c_1(E) = (v - u) \cdot \ell$$

where $w = \dim W$, $v = \dim V$, $u = \dim U$ and $\ell = c_1(L)$. Thus any instanton sheaf $E$ can be represented as the cohomology of a monad of the following type:

$$0 \rightarrow (L^{-1})^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus r + 2c} \xrightarrow{\beta} \mathcal{L}^{\oplus c} \rightarrow 0$$

(2)

where $r$ is the rank and $c$ is called the charge of $E$. It also follows that the total Chern class of $E$ is given by, in the case $u = v$:

$$c(E) = \frac{1}{(1 - \ell^2)^c} = (1 + \ell^2 + \ell^4 + \cdots)^c.$$

Remark 1. For $X = \mathbb{P}^n$, instanton sheaves exist for $r \geq n - 1$ and all $c$ [6]. For $X$ being a smooth quadric hypersurface of dimension $n \geq 3$, instanton sheaves exist for $r \geq n - 1$ and all $c$ [3]. It would be very interesting to obtain existence results for a wider class of varieties.

A smooth projective variety $X$ is said to be cyclic if $\text{Pic}(X) = \mathbb{Z}$. Examples of cyclic varieties are projective spaces, grassmannians and complete intersection subvarieties of dimension $n \geq 3$ within $P^N$, $N \geq 4$. We can assume without loss of generality that $L \cong \mathcal{O}_X(l)$ for some $l \geq 1$ and $\omega_X \cong \mathcal{O}_X(\lambda)$ for some integer $\lambda$.

Proposition 2. Let $X$ be a smooth projective cyclic variety of dimension $n$ such that $H^p(\mathcal{O}_X(k)) = 0$ for $1 \leq p \leq n - 1$ and $\forall k$. Let $E$ be the linear sheaf given by the cohomology of the monad:

$$0 \rightarrow \mathcal{O}_X(-l)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(l)^{\oplus c} \rightarrow 0.$$

(3)

Then, we have:
1. for \( n \geq 2 \), \( H^0(E(k)) = H^0(E^*(k)) = 0 \) for all \( k \leq -1 \),

2. for \( n \geq 3 \), \( H^1(E(k)) = 0 \) for all \( k \leq -l - 1 \),

3. for \( n \geq 4 \), \( H^i(E(k)) = 0 \) for all \( k \) and \( 2 \leq i \leq n - 2 \),

4. for \( n \geq 3 \), \( H^{n-1}(E(k)) = 0 \) for all \( k \geq \lambda + l + 1 \),

5. for \( n \geq 2 \), \( H^n(E(k)) = 0 \) for all \( k \geq \lambda + 1 \),

It is not hard to see that complete intersection subvarieties of dimension \( \geq 3 \) within \( P^n \), \( n \geq 4 \) do satisfy the conditions of the theorem. Note also that cyclic Fano varieties (i.e. \( \lambda \leq -1 \)) also satisfy \( H^p(O_X(k)) = 0 \) for \( 1 \leq p \leq n - 1 \).

Indeed, Kodaira Vanishing Theorem tells us that:

\[
H^i(O_X(k)) = 0 \quad \text{for all} \quad i < n \quad \text{and} \quad k \leq -1; \quad \text{and} \\
H^i(O_X(k) \otimes \omega_X) = 0 \quad \text{for all} \quad i > 0 \quad \text{and} \quad k \geq 1. 
\]

By Serre’s duality \( H^i(O_X(k) \otimes \omega_X) \cong H^{n-i}(O_X(-k))^* \). So, we conclude that

\[
H^0(O_X(k)) = 0 \quad \text{for all} \quad k \leq -1, \\
H^i(O_X(k)) = 0 \quad \text{for all} \quad k \quad \text{and} \quad 1 \leq i \leq n - 1, \quad \text{and} \\
H^n(O_X(k)) = 0 \quad \text{for all} \quad k \geq \lambda + 1. 
\]

**Proof.** Assuming that \( E \) is the cohomology of the linear monad (3), let \( K = \ker \beta \); it is a locally-free sheaf of rank \( b - c \) fitting into the sequences:

\[
0 \rightarrow K(k) \rightarrow O_X(k) \overset{\beta}{\rightarrow} O_X(k+l) \overset{\alpha}{\rightarrow} E(k) \rightarrow 0 \quad \text{and} \quad (4) \\
0 \rightarrow O_X(k-l) \overset{\alpha}{\rightarrow} K(k) \rightarrow E(k) \rightarrow 0. \quad (5)
\]

Passing to cohomology, the exact sequence (4) yields, in the appropriate ranges of \( n \):

\[
H^0(K(t)) = 0 \quad \text{for} \quad t \leq -1, \\
H^1(K(t)) = 0 \quad \text{for} \quad t \leq -l - 1.
\]
$H^i(X, K(t)) = 0$ for all $t$ and $2 \leq i \leq n-1$, 

$H^n(K(t)) = 0$ for $t \geq \lambda + 1$.

Passing to cohomology, the exact sequence (5) yields:

$H^0(E(k)) = 0$ for all $k \leq -1$, 

$H^1(E(k)) = 0$ for all $k \leq -l - 1$, 

$H^i(E(k)) = 0$ for all $k$ and $2 \leq i \leq n - 2$, 

$H^{n-1}(E(k)) = 0$ for all $k \geq \lambda + l + 1$, 

$H^n(E(k)) = 0$ for all $k \geq \lambda + 1$, 

as desired.

Dualizing sequences (4) and (5), we obtain:

$0 \to \mathcal{O}_X(-k-l) \otimes c \beta^* \to \mathcal{O}_X(-k) \to K^*(k) \to 0$ and (6)

$0 \to E^*(-k) \to K^*(-k) \otimes \mathcal{O}_X(-k+l) \otimes a \to \mathcal{E}xt^1(E(k), \mathcal{O}_X) \to 0$. (7)

Again, passing to cohomology, (7) forces $H^0(E^*(k)) \subseteq H^0(K^*(k))$ for all $k$, while (6) implies $H^0(K^*(k)) = 0$ for $k \leq -1$. Therefore, $H^0(E^*(k)) = 0$ for all $k \leq -1$. 

Remark 3. It follows from (7) that $\mathcal{E}xt^1(E, \mathcal{O}_X) = \text{coker}\alpha^*$, i.e. the degeneration locus of the monad (2) coincides with the support of $\mathcal{E}xt^1(E, \mathcal{O}_X)$. Furthermore, it also follows from (7) that $\mathcal{E}xt^p(E, \mathcal{O}_X) = 0$ for $p \geq 2$.

Proposition 4. Let $E$ be a linear sheaf on a smooth projective variety $X$ (not necessarily cyclic).

1. $E$ is locally-free if and only if its degeneration locus is empty;

2. $E$ is reflexive if and only if its degeneration locus is a subvariety of codimension at least 3;

3. $E$ is torsion-free if and only if its degeneration locus is a subvariety of codimension at least 2.
Proof. Let $S$ be the degeneration locus of the linear monad associated to the linear sheaf $E$. From Remark 9 we know that $\mathcal{E}xt^p(E, \mathcal{O}_X) = 0$ for $p \geq 2$ and

$$S = \text{supp } \mathcal{E}xt^1(E, \mathcal{O}_X) = \{x \in X \mid \alpha(x) \text{ is not injective} \}.$$  

The first statement is clear; so it is now enough to argue that $E$ is torsion-free if and only if $S$ has codimension at least 2 and that $E$ is reflexive if and only if $S$ has codimension at least 3.

Recall that the $m$th-singularity set of a coherent sheaf $F$ on $X$ is given by:

$$S_m(F) = \{x \in X \mid dh(F_x) \geq n - m \}$$

where $dh(F_x)$ stands for the homological dimension of $F_x$ as an $\mathcal{O}_x$-module:

$$dh(F_x) = d \iff \begin{cases} \text{Ext}^d_{\mathcal{O}_x}(F_x, \mathcal{O}_x) \neq 0 \\ \text{Ext}^p_{\mathcal{O}_x}(F_x, \mathcal{O}_x) = 0 \ \forall p > d. \end{cases}$$

In the case at hand, we have that $dh(E_x) = 1$ if $x \in S$, and $dh(E_x) = 0$ if $x \notin S$. Therefore $S_0(E) = \cdots = S_{n-2}(E) = \emptyset$, while $S_{n-1}(E) = S$. It follows that [9, Proposition 1.20]:

- if codim $S \geq 2$, then dim $S_m(E) \leq m - 1$ for all $m < n$, hence $E$ is a locally 1st-syzygy sheaf;
- if codim $S \geq 3$, then dim $S_m(E) \leq m - 2$ for all $m < n$, hence $E$ is a locally 2nd-syzygy sheaf.

The desired statements follow from the observation that $E$ is torsion-free if and only if it is a locally 1st-syzygy sheaf, while $E$ is reflexive if and only if it is a locally 2nd-syzygy sheaf [8, p. 148-149].

Remark 5. Note that if $E$ is a locally-free linear sheaf on $X$, which is represented as the cohomology of the linear monad

$$M_\bullet : 0 \to (\mathcal{L}^{-1})^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{L}^{\oplus c} \to 0,$$

its dual $E^*$ is also a linear sheaf, being represented as the cohomology of the dual monad

$$M_\bullet^* : 0 \to (\mathcal{L}^{-1})^{\oplus c} \xrightarrow{\alpha^*} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta^*} \mathcal{L}^{\oplus a} \to 0.$$
In particular, if $E$ is a locally-free instanton sheaf on $X$ then its dual $E^*$ is also an instanton. In general, however, there are non-locally-free instanton sheaves whose duals are not instantons; the simplest example of this situation is a non-locally-free nullcorrelation bundle on $\mathbb{P}^3$.

3 Semistability of instanton sheaves

Fixed an ample invertible sheaf $L$ with $c_1(L) = \ell$ on a projective variety $X$ of dimension $n$, recall that the slope $\mu(E)$ with respect to $L$ of a torsion-free sheaf $E$ on $X$ is defined as follows:

$$\mu(E) := \frac{c_1(E)\ell^{n-1}}{rk(E)}.$$

We say that $E$ is semistable with respect to $L$ if for every coherent sheaf $0 \neq F \hookrightarrow E$ we have $\mu(F) \leq \mu(E)$. Furthermore, if for every coherent sheaf $0 \neq F \hookrightarrow E$ with $0 < rk(F) < rk(E)$ we have $\mu(F) < \mu(E)$ then $E$ is said to be stable. A sheaf $E$ is said to be properly semistable if it is semistable but not stable. It is also important to recall that $E$ is (semi)stable if and only if $E^*$ is (semi)stable if and only if $E \otimes L^k$ is (semi)stable.

The goal of this section is to study the (semi)stability of instanton sheaves.

**Proposition 6.** Every rank 2 torsion-free instanton sheaf on a cyclic variety is semistable.

**Proof.** Let us first consider a rank 2 reflexive sheaf $F$ on $X$ with $c_1(F) = 0$ and $H^0(F(-1)) = 0$; we argue that $F$ is semistable. Indeed, if $F$ is not semistable, then any destabilizing sheaf $L \hookrightarrow F$ with torsion-free quotient $F/L$ must be reflexive (see [8, p. 158]). But every rank 1 reflexive sheaf is locally-free, thus $L = \mathcal{O}_X(d)$ with $d = c_1(L) > 0$ since Pic($X$) = $\mathbb{Z}$. It follows that $H^0(F(-d)) \neq 0$, hence $H^0(F(-1)) \neq 0$ as well.

Now if $E$ is a rank 2 torsion-free sheaf with $c_1(E) = 0$ and $H^0(E^*(-1)) = 0$, then $F = E^*$ is a rank 2 reflexive sheaf with $c_1 F = 0$ and $H^0(F(-1)) = 0$. But we’ve seen that such $F$ is semistable, hence $E$ is also semistable. Together with the first statement in Proposition 2 the desired result follows.
For instanton sheaves of higher rank, we have our first main result:

**Theorem 7.** Let $E$ be a rank $r$ instanton sheaf on a cyclic variety $X$ of dimension $n$. If $E$ is locally-free and $r \leq 2n - 1$, then $E$ is semistable.

Since smooth quadric hypersurfaces are cyclic, the above statement provides in particular a partial answer to the questions raised in [3] Questions 5.1 and 5.2.

The proof of Theorem 7 is based on a very useful to decide whether a locally-free sheaf on cyclic variety is (semi)stable.

Recall that for any rank $r$ locally-free sheaf $E$ on a cyclic variety $X$, there is a uniquely determined integer $k_E$ such that $-r + 1 \leq c_1(E(k_E)) \leq 0$. We set $E_{\text{norm}} := E(k_E)$ and we call $E$ normalized if $E = E_{\text{norm}}$. We then have the following criterion.

**Proposition 8.** Let $E$ be a rank $r$ locally-free sheaf on a cyclic variety $X$. If $H^0((\wedge^q E)_{\text{norm}}) = 0$ for $1 \leq q \leq r - 1$, then $E$ is stable. If $H^0((\wedge^q E)_{\text{norm}}(-1)) = 0$ for $1 \leq q \leq r - 1$, then $E$ is semistable.

**Proof.** [5]; Lemma 2.6.

**Proof of Theorem 7.** We argue that every instanton sheaf on an $n$-dimensional cyclic variety $X$ satisfying the conditions of the theorem fulfills Hoppe’s criterion (see Proposition 8).

Indeed, let $E$ be a rank $r$ locally-free instanton sheaf on $X$. Assume that $E$ can be represented as the cohomology of the linear monad as in [3].

Considering the long exact sequence of symmetric powers associated to the sheaf sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_X^{\oplus r+2c} \rightarrow \mathcal{O}_X(l)^{\oplus c} \rightarrow 0$$

twisted by $\mathcal{O}_X(k)$, we have:

$$0 \rightarrow \wedge^q K(k) \rightarrow \wedge^q (\mathcal{O}_X^{\oplus r+2c})(k) \rightarrow \cdots$$

$$\cdots \rightarrow \mathcal{O}_X^{\oplus r+2c}(k) \otimes S^{q-1}(\mathcal{O}_X(l)^{\oplus c}) \rightarrow S^q(\mathcal{O}_X(l)^{\oplus c}) \rightarrow 0$$
Cutting into short exact sequences, passing to cohomology and using the fact that \( H^p(\mathcal{O}_X(k)) = 0 \) for \( p \leq n - 1 \) and \( k \leq -1 \) (Kodaira vanishing theorem), we conclude that
\[
H^p(\wedge^q K(k)) = 0 \quad \text{for} \quad 1 \leq q \leq r + c = \text{rk}(K) \quad , \quad p \leq n - 1 \quad \text{and} \quad k \leq -pl - 1.
\]

Now consider the long exact sequence of exterior powers associated to the sheaf sequence
\[
0 \to \mathcal{O}_X(-l)^{\oplus c} \to K \to E \to 0
\]
and twisted by \( \mathcal{O}_X(-1) \):
\[
0 \to \mathcal{O}_X(-ql - 1)^{\oplus (c+q+1)} \to K((-q+1)l - 1)^{\oplus (c+q-2)} \to \cdots
\]
\[
\cdots \to \wedge^{q-1} K(-1 - l)^{\oplus c} \to \wedge^q K(-1) \to \wedge^q E(-1) \to 0. 
\] (8)
Cutting into short exact sequences and passing to cohomology, we obtain that
\[
H^0(\wedge^p E(-1)) = 0 \quad \text{for} \quad 1 \leq p \leq n - 1 .
\] (9)
If \( \text{rk}(E) \leq n \), this proves that \( E \) is semistable by Proposition 8.

If \( \text{rk}(E) = n + 1 \), we have, since \( c_1(E) = 0 \) and \( E \) is normal:
\[
H^0(\wedge^n E(-1)) \simeq H^0(E^*(E(-1)) = 0 ,
\] (10)
thus \( E \) is also semistable.

Assume \( \text{rk}(E) > n + 1 \). The dual \( E^* \) is also a locally-free instanton sheaf on \( X \), so
\[
H^0(\wedge^q E^*(-1)) = 0 \quad \text{for} \quad 1 \leq q \leq n - 1 .
\] (11)
But \( \wedge^p(E) \simeq \wedge^{r-p}(E^*) \), since \( \det(E) = \mathcal{O}_X \); it follows that:
\[
H^0(\wedge^p E(-1)) = H^0(\wedge^{r-p}(E^*)(-1)) = 0 \quad \text{for} \quad 1 \leq r - p \leq n - 1
\]
\[
\implies r - n + 1 \leq p \leq r - 1 .
\] (12)
Together, (11) and (12) imply that if \( E \) is a rank \( r \leq 2n - 1 \) locally-free instanton sheaf, then:
\[
H^0(\wedge^p E(-1)) = 0 \quad \text{for} \quad 1 \leq p \leq 2n - 2
\]
hence $E$ is semistable by Proposition 8.

On the other hand, we have:

**Proposition 9.** Let $H = h^0(L)$. For $r > (H - 2)c$, there are no stable rank $r$ instanton sheaves of charge $c$ on $X$.

In particular, for $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$, it follows that every locally-free instanton sheaf on $\mathbb{P}^n$ of charge 1 and rank $r$ with $n \leq r \leq 2n - 1$ must be properly semistable; for $X = Q_n$ and $L = \mathcal{O}_{Q_n}(1)$, every locally-free instanton sheaf on $Q_n$ of charge 1 and rank $r$ with $n + 1 \leq r \leq 2n - 1$ must be properly semistable.

**Proof.** For the second part, note that if $E$ is a stable torsion-free sheaf with $c_1(E) = 0$, then $H^0(E) = 0$. Indeed, if $H^0(E) \neq 0$, then there is a map $\mathcal{O}_X \to E$, which contradicts stability.

It follows from the sequences (4) and (5) for $k = 0$ that:

$$H^0(E) \simeq H^0(K) \simeq \ker \{ H^0(\beta) : H^0(\mathcal{O}_X^{\oplus r-2c}) \to H^0(L^{\oplus c}) \}.$$  

If $r > (H - 2)c$, then the map $H^0(\beta)$ cannot be injective, $H^0(E) \neq 0$, and $E$ cannot be stable.  

Now dropping the $c_1(E) = 0$ condition, we obtain:

**Theorem 10.** Let $E$ be a rank $r \leq n$ linear sheaf on a cyclic variety $X$ of dimension $n$. If $E$ is locally-free and $c_1(E) \neq 0$, then $E$ is stable.

**Proof.** Since $E$ is a linear sheaf, it is represented as the cohomology of a linear monad

$$0 \to \mathcal{O}_X(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0,$$

so that $c_1(E) = (a - c)\ell$.

Assuming $a - c > 0$, we have $\mu(\wedge^q E) = q(a - c)/r > 0$, hence $(\wedge^q E)_{\text{norm}} = (\wedge^q E)(t)$ for some $t \leq -1$.

On the other hand, arguing as in the proof of Theorem 7, we get

$$H^0((\wedge^q E)(-1)) = 0 \text{ for all } q \leq n - 1.$$  

(13)
Therefore, if $E$ is a rank $r \leq n$ locally-free sheaf represented as the cohomology of a linear monad and $c_1(E) > 0$, then:

$$H^0((\wedge^p E)_{\text{norm}}) = 0 \text{ for } 1 \leq p \leq r - 1.$$  

Hence $E$ is stable by Proposition 8.

For the second statement, note that if $E$ is a locally-free linear sheaf with $c_1(E) < 0$, then $E^*$ is a locally-free linear sheaf with $c_1(E^*) > 0$. By the argument above, $E^*$ is stable; hence $E$ is stable whenever $c_1(E) \neq 0$, as desired.

We will end this section with examples which illustrate that the upper bounds in the rank given in Theorems 7 and 10 are sharp. To establish them, we first need to provide the following useful cohomological characterization of linear sheaves on projective spaces.

**Proposition 11.** Let $F$ be a torsion-free sheaf on $\mathbb{P}^n$. $F$ is the cohomology of a linear monad

$$0 \to \mathcal{O}_\mathbb{P}^n(-1)^{\oplus a} \overset{\alpha}{\to} \mathcal{O}_\mathbb{P}^n(\alpha) \overset{\beta}{\to} \mathcal{O}_\mathbb{P}^n(1)^{\oplus c} \to 0$$

if and only if the following cohomological conditions hold:

- for $n \geq 2$, $H^0(F(-1)) = 0$ and $H^n(F(-n)) = 0$;
- for $n \geq 3$, $H^1(F(k)) = 0$ for $k \leq -2$ and $H^{n-1}(F(k)) = 0$ for $k \geq -n + 1$;
- for $n \geq 4$, $H^p(F(k)) = 0$ for $2 \leq p \leq n - 2$ and all $k$.

**Proof.** The fact that linear sheaves satisfy the cohomological conditions above is a consequence of Proposition 2.

For the converse statement, first note that $H^0(F(-1)) = 0$ implies that $H^0(F(k)) = 0$ for $k \leq -1$, while $H^n(F(-n)) = 0$ implies that $H^n(F(k)) = 0$ for $k \geq -n$. Moreover, we claim that $(g = 0, \ldots, n$ and $p = 0, -1, \ldots, -n)$:

$$H^q(F(-1) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \text{ for } q \neq 1 \text{ and for } q = 1, p \leq -3 . \quad (14)$$
Now the key ingredient is the Beilinson spectral sequence \[8\]: for any coherent sheaf \(G\) on \(\mathbb{P}^n\), there exists a spectral sequence \(\{E^p_{r}\}\) whose \(E^1\)-term is given by \((q = 0, \ldots, n\) and \(p = 0, -1, \ldots, -n)\):

\[E^p_{1,q} = H^q(G \otimes \Omega^{-p}_{\mathbb{P}^n}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)\]

which converges to

\[E^\infty = \begin{cases} G, & \text{if } p + q = 0 \\ 0, & \text{otherwise} \end{cases} \]

Applying the Beilinson spectral sequence to \(G = F(-1)\), it then follows that it degenerates at the \(E^2\)-term, so that the monad

\[0 \to H^1(F(-1) \otimes \Omega^2_{\mathbb{P}^n}(2)) \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \to \cdots \]

has \(F(-1)\) as its cohomology. Tensoring \((15)\) by \(\mathcal{O}_{\mathbb{P}^n}(1)\), we conclude that \(F\) is the cohomology of a linear monad, as desired.

The claim \((14)\) follows from repeated use of the exact sequence

\[H^q(F(k))^{\oplus m} \to H^q(F(k + 1) \otimes \Omega^{-p-1}_{\mathbb{P}^n}(-p - 1)) \to \cdots \]

\[H^{q+1}(F(k) \otimes \Omega^{-p}_{\mathbb{P}^n}(-p)) \to H^{q+1}(F(k))^{\oplus m} \]

associated with Euler sequence for \(p\)-forms on \(\mathbb{P}^n\) twisted by \(F(k)\):

\[0 \to F(k) \otimes \Omega^{-p}_{\mathbb{P}^n}(-p) \to F(k)^{\oplus m} \to F(k) \otimes \Omega^{-p-1}_{\mathbb{P}^n}(-p) \to 0 \]

where \(q = 0, \ldots, n, p = 0, -1, \ldots, -n\) and \(m = \left(\begin{array}{c} n + 1 \\ -p \end{array}\right)\).

We are finally ready to construct rank \(2n\) locally-free instanton sheaves on \(\mathbb{P}^n\) which are not semistable; in other words the bound \(r \leq 2n - 1\) in the second part of Theorem \(7\) is sharp.

**Example 12.** Let \(X = \mathbb{P}^n, n \geq 4\). By Fløystad’s theorem \(\text{[H]}\), there is a linear monad:

\[0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n + 3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \to 0 \]

whose cohomology \(F\) is a locally-free sheaf of rank \(n\) on \(\mathbb{P}^n\) and \(c_1(F) = 1\).
Dualizing we get a linear monad:

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^n(-1)}^{\oplus n+3} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0
\]

whose cohomology is \( F^* \), hence it is a locally-free linear sheaf of rank \( n \) on \( \mathbb{P}^n \) and \( c_1(F^*) = -1 \).

Take an extension \( E \) of \( F^* \) by \( F \):

\[
0 \to F \to E \to F^* \to 0.
\]

Using the cohomological criterion given in Proposition 11, it is easy to see that the extension of linear sheaves is also a linear sheaf. Moreover, \( c_1(E) = 0 \), i.e. \( E \) is a rank \( 2n \) locally-free instanton sheaf of charge 3 which is not semistable.

Such extensions are classified by \( \text{Ext}^1(F^*, F) = H^1(F \otimes F) \). We claim that there are non-trivial extensions of \( F^* \) by \( F \). Indeed, we consider the exact sequences

\[
0 \to K = \ker(\beta) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+3} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0 \tag{19}
\]

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2} \to K \to F \to 0 \tag{20}
\]

associated to the linear monad (18). We apply the exact covariant functor \( \cdot \otimes F \) to the exact sequences (19) and (20) and we obtain the exact sequences

\[
0 \to K \otimes F \to F^{\oplus n+3} \to F(1) \to 0
\]

\[
0 \to F(-1)^{\oplus 2} \to K \otimes F \to F \otimes F \to 0
\]

Using Proposition 2, we obtain \( H^i(K \otimes F) = H^i(F \otimes F) = 0 \) for all \( i \geq 3 \). Hence, \( \chi(F \otimes F) = h^0((F \otimes F)) - h^1((F \otimes F)) + h^2((F \otimes F)) \). On the other hand,

\[
\chi(F \otimes F) = \chi(K \otimes F) - 2\chi(F(-1)) = (n + 3)\chi(F) - \chi(F(1)) - 2\chi(F(-1)) = 8 - \frac{n^2}{2} - \frac{n}{2} \leq 0 , \text{ if } n \geq 4
\]

Thus if \( n \geq 4 \), we must have \( h^1((F \otimes F)) > 0 \), hence there are non-trivial extensions of \( F^* \) by \( F \).

For \( X = \mathbb{P}^n \), \( 2 \leq n \leq 3 \), arguing as above, we can construct a rank \( 2n \) locally-free instanton which is not semistable as a non-trivial extension \( E \) of
$F^*$ by $F$, where $F$ is a linear sheaf represented as the cohomology of the linear monad

$$0 \to \mathcal{O}_\mathbb{P}^n(-1)^{\oplus 4} \xrightarrow{\alpha} \mathcal{O}_\mathbb{P}^{n+7} \xrightarrow{\beta} \mathcal{O}_\mathbb{P}^n(1)^{\oplus 3} \to 0.$$  

To conclude this section, we show that the upper bound in the rank given in Theorem 10 is also sharp:

**Example 13.** Let $X = \mathbb{P}^n$, $n \geq 2$. By Floystad’s theorem [4], there is a linear monad:

$$0 \to \mathcal{O}_\mathbb{P}^n(-1)^{\oplus 4} \xrightarrow{\alpha} \mathcal{O}_\mathbb{P}^{n+9} \xrightarrow{\beta} \mathcal{O}_\mathbb{P}^n(1)^{\oplus 5} \to 0 \quad (21)$$

whose cohomology $G$ is a locally-free sheaf of rank $n$ on $\mathbb{P}^n$ and $c_1(G) = -1$.

Now $G^*$ is the cohomology of the dual monad

$$0 \to \mathcal{O}_\mathbb{P}^n(-1)^{\oplus 5} \xrightarrow{\beta^*} \mathcal{O}_\mathbb{P}^{n+9} \xrightarrow{\alpha^*} \mathcal{O}_\mathbb{P}^n(1)^{\oplus 4} \to 0.$$  

It follows that:

$$H^1(G^*) = H^1(\ker \alpha^*) = \text{coker}\{H^0\alpha^* : H^0(\mathcal{O}_\mathbb{P}^{n+9}) \to H^0(\mathcal{O}_\mathbb{P}^n(1)^{\oplus 4})\}.$$  

Since $n \geq 2$ forces $4n + 4 > n + 9$, the generic map $\alpha$ will have $\text{coker}(H^0\alpha^*) \neq 0$. In other words, there exists a rank $n$ locally-free linear sheaf $G$ on $\mathbb{P}^n$ with $c_1(G) = -1$ and $H^1(G^*) \neq 0$.

Take an extension $E$ of such a linear sheaf $G$ by $\mathcal{O}_\mathbb{P}^n$:

$$0 \to \mathcal{O}_\mathbb{P}^n \to E \to G \to 0. \quad (22)$$

Using the cohomological criterion given in Proposition 11, it is easy to see that $E$ is a rank $n + 1$ locally-free linear sheaf with $c_1(E) = c_1(G) = -1$. It is not stable, since $H^0(E) \neq 0$.

Note also that there are nontrivial extensions of $G$ by $\mathcal{O}_\mathbb{P}^n$ since $H^1(G^*) \neq 0$. Furthermore, the dual $E^*$ is an example of a rank $n + 1$ locally-free linear sheaf with $c_1(E) > 0$ which is not stable.

We do not know how to establish the semistability of torsion-free instanton sheaves of rank higher than 3. However, for each $n \geq 2$, it is easy to show, using
the same technique as in the examples above, that there are unstable torsion-free instanton sheaves of rank $n + 1$ in $\mathbb{P}^n$, see [3, Example 3]. The natural, sharp conjecture would be that every torsion-free instanton sheaf of rank $r \leq n$ on a cyclic variety $X$ of dimension $n$ is semistable; this statement is true for $n = 2$, see Proposition 6 above.

For reflexive linear sheaves, one can construct rank $n + 2$ reflexive instanton sheaves which are not semistable; in this case, one can expect that every reflexive instanton sheaf of rank $r \leq n + 1$ on a cyclic variety $X$ of dimension $n$ is semistable.

4 Special sheaves on smooth quadric hypersurfaces

Now we restrict ourselves to the set-up in [3], and we assume that $Q_n$ is a smooth quadric hypersurface within $\mathbb{P}^{n+1}$, $n \geq 3$; such varieties are cyclic.

Recall that a special sheaf $E$ on $Q_n$ is defined [3, Definition 3.4] as either the cohomology of a linear monad

\[(M1) \quad 0 \to \mathcal{O}_{Q_n}(-1)^{\oplus a} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0,\]

or the cohomology of a monad of the following type

\[(M2.1) \quad 0 \to \Sigma(-1)^{\oplus a} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0, \quad \text{if } n \text{ is odd,}\]

\[(M2.2) \quad 0 \to \Sigma_1(-1)^{\oplus a_1} \oplus \Sigma_2(-1)^{\oplus a_2} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0, \quad \text{if } n \text{ is even,}\]

where $\Sigma$ is the Spinor bundle for $n$ odd, and $\Sigma_1, \Sigma_2$ are the Spinor bundles for $n$ even.

Clearly, instanton sheaves on $Q_n$ are special sheaves of the first kind with zero degree.

**Proposition 14.** Let $E$ be a special sheaf on $Q_n$, $n \geq 3$. Then one of the following conditions holds:

1. $E$ is the cohomology of a linear monad, and
\[ H^0(E(k)) = H^0(E^*(k)) = 0 \text{ for all } k \leq -1, \]
\[ H^1(E(k)) = 0 \text{ for all } k \leq -2, \]
\[ H^i(E(k)) = 0 \text{ for all } k \text{ and } 2 \leq i \leq n - 2, \]
\[ H^{n-1}(E(k)) = 0 \text{ for all } k \geq -n + 2, \]
\[ H^n(E(k)) = 0 \text{ for all } k \geq -n + 1, \]

and if \( E \) is locally-free:

\[ H^n(E^*(k)) = 0 \text{ for all } k \geq -n + 1; \text{ or} \]

2. \( E \) is the cohomology of a monad of type (M2.1) and (M2.2), and

\[ H^0(E(k)) = H^0(E^*(k)) = 0 \text{ for all } k \leq -1, \]
\[ H^1(E(k)) = 0 \text{ for all } k \leq -2, \]
\[ H^i(E(k)) = 0 \text{ for all } k \text{ and } 2 \leq i \leq n - 2, \]
\[ H^{n-1}(E(k)) = 0 \text{ for all } k \geq -n + 1, \]
\[ H^n(E(k)) = 0 \text{ for all } k \geq -n + 1, \]

and if \( E \) is locally-free:

\[ H^n(E^*(k)) = 0 \text{ for all } k \geq -n + 1 \]

Proof. (1) It is analogous to the proof of Proposition \( \square \).

(2) If \( n \) is odd we consider the exact sequences

\[ 0 \to \ker(\delta) \to \mathcal{O}_{\mathbb{Q}_n}^{\oplus b} \xrightarrow{\delta} \mathcal{O}_{\mathbb{Q}_n}(1)^{\oplus c} \to 0, \]
\[ 0 \to \Sigma(-1)^{\oplus a} \to \ker(\delta) \to E \to 0 \]

and if \( n \) is even we consider the exact sequences

\[ 0 \to \ker(\psi) \to \mathcal{O}_{\mathbb{Q}_n}^{\oplus b} \xrightarrow{\psi} \mathcal{O}_{\mathbb{Q}_n}(1)^{\oplus c} \to 0, \]
\[ 0 \to \Sigma_1(-1)^{\oplus a_1} \oplus \Sigma_1(-2)^{\oplus a_2} \to \ker(\psi) \to E \to 0 \]

and we argue as in the proof of Proposition \( \square \) taking into account that

\[ H^0(\Sigma(k)) = H^0(\Sigma_1(k)) = H^0(\Sigma_2(k)) = 0 \text{ for all } k \leq -1, \]
\[ H^i(\Sigma(k)) = H^i(\Sigma_1(k)) = H^i(\Sigma_2(k)) = 0 \text{ for all } k \text{ and } 1 \leq i \leq n - 1, \text{ and} \]
\[ H^n(\Sigma(k)) = H^n(\Sigma_1(k)) = H^n(\Sigma_2(k)) = 0 \text{ for all } k \geq n. \]
Proposition 15. Every rank 2 torsion-free special sheaf $E$ on $Q_n$ with $c_1(E) = 0$ is semistable.

Proof. Since every torsion-free special sheaf $E$ on $Q_n$ satisfies $H^0(E(k)) = H^0(E^*(k)) = 0$, simply use the argument in the proof of Proposition [6].

Finally, for higher rank locally-free special sheaves on $Q_n$, we have:

Theorem 16. Let $E$ be a rank $r$ locally-free special sheaf on $Q_n$.

• If $r \leq 2n - 1$ and $c_1(E) = 0$, then $E$ is semistable;
• if $r \leq n$ and $c_1(E) \neq 0$, then $E$ is stable.

It is interesting to note that, by [3, Proposition 4.7], there are no rank $r \leq n - 1$ linear sheaves $E$ on $Q_n$ with $c_1(E) < 0$ or rank $r \leq n - 2$ linear sheaves $E$ on $Q_n$ with $c_1(E) = 0$.

Proof. For locally-free special sheaves which are represented as cohomologies of the monad $(M1)$, the statement follows from Theorem [6] and [10] and for locally-free special sheaves which are represented as cohomologies of the monad $(M2.1)$ and $(M2.2)$ an analogous argument works.

Note that using the Fløystad type existence theorem for linear sheaves on $Q_n$ established in [4, Proposition 4.7], one can easily produce examples of rank $2n$ locally-free instanton sheaves on $Q_n$ as well as rank $n + 1$ locally-free linear sheaves on $Q_n$ which are not semistable, following the ideas in Examples [12] and [13].

However, we do not know whether the bounds in the rank are sharp for locally-free sheaves on $Q_n$ which are the cohomology of monads of type $(M2.1)$ and $(M2.2)$. For instance, is there an unstable rank $2n$ locally-free sheaf on $Q_n$ which can be represented as the cohomology of a non-linear special monad?
5 Conclusion

In this paper we have studied the semistability of torsion-free sheaves on nonsingular projective varieties with cyclic Picard group that arise as cohomologies of a particular type of monad. Many interesting questions regarding linear monads and instanton sheaves remain unanswered.

First of all, one would like to have a generalizations of Fløystad’s (resp. Costa and Miró-Roig’s) existence result \[4\] (resp. \[3\]) and of Proposition \[11\] establishing the existence of instanton sheaves over varieties other than \(\mathbb{P}^n\) (resp. \(\mathbb{Q}_n\)) and their intrinsic cohomological characterization.

The semistability of instanton sheaves of low rank indicate the existence of a well-behaved moduli space of instanton sheaves on cyclic varieties. One approach to study the moduli space of instanton sheaves would be the construction of the moduli space of linear monads, using methods from geometric invariant theory. This task is probably deeply linked with the theory of representation of quivers, since a monad can be regarded as the representation of a quiver, the one whose underlying graph is the Dynkyn diagram for \(A_3\), into the category of sheaves, see \[7\].

This also brings up the question of a reasonable stability condition for monads, meaning compatible with geometric invariant theory, and how does it compare with the slope stability of its cohomology sheaf. Notice that a monad can also be regarded as an element in the derived category \(D^b(X)\) of bounded complexes of coherent sheaves on \(X\); the concept of stability on triangulated categories has been recently introduced by Bridgeland in \[1\], but it is still unclear what does it have to do with moduli spaces. We hope that the study of the moduli space of instanton sheaves will shed some light on this topic.

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