AUTOMORPHISM GROUPS OF POSITIVE ENTROPY
ON PROJECTIVE THREEFOLDS

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ABSTRACT. We prove two results about the natural representation of a group $G$ of automorphisms of a normal projective threefold $X$ on its second cohomology. We show that if $X$ is minimal, then $G$, modulo a normal subgroup of null entropy, is embedded as a Zariski-dense subset in a semi-simple real linear algebraic group of real rank $\leq 2$. Next, we show that $X$ is a complex torus if the image of $G$ is an almost abelian group of positive rank and the kernel is infinite, unless $X$ is equivariantly non-trivially fibred.

1. Introduction

Let $X$ be a compact Kähler manifold. For an automorphism $g \in \text{Aut}(X)$, its (topological) entropy $h(g) = \log \rho(g)$ is defined as the logarithm of the spectral radius $\rho(g)$ of the pullback action $g^*$ on the total cohomology group of $X$, i.e.,

$$\rho(g) := \max \{ |\lambda| ; \lambda \text{ is an eigenvalue of } g^*| \bigoplus_{i \geq 0} H^i(X, \mathbb{C}) \}.$$ 

By the fundamental result of Gromov and Yomdin, the above definition is equivalent to the original dynamical definition of entropy (cf. [11], [21]).

An element $g \in \text{Aut}(X)$ is of null entropy if its (topological) entropy $h(g)$ equals 0. For a subgroup $G$ of $\text{Aut}(X)$, we define the null subset of $G$ as

$$N(G) := \{ g \in G | g \text{ is of null entropy, i.e., } h(g) = 0 \}$$

which may not be a subgroup. A group $G \leq \text{Aut}(X)$ is of null entropy if every $g \in G$ is of null entropy, i.e., if $G$ equals $N(G)$.

By the classification of surfaces, a complex surface $S$ has some $g \in \text{Aut}(S)$ of positive entropy only if $S$ is bimeromorphic to a rational surface, complex torus, $K3$ surface or Enriques surface (cf. [5]). See [24] for a similar phenomenon in higher dimensions.

Recall that a normal projective variety $X$ is minimal if it has at worst terminal singularities and the canonical divisor $K_X$ is nef (cf. [12] Definition 2.34)). Let $\text{NS}(X)$ be the Neron-Severi group and $\text{NS}_{\mathbb{C}}(X) := \text{NS}(X) \otimes \mathbb{C}$. For a subgroup $G$ of $\text{Aut}(X)$, let $\overline{G} \subseteq \text{GL}(\text{NS}_{\mathbb{C}}(X))$ be the Zariski-closure of the action of $G$ on $\text{NS}_{\mathbb{C}}(X)$ (simply denoted by $G| \text{NS}_{\mathbb{C}}(X)$), and let $R(\overline{G})$ be its solvable radical, both of which are defined over $\mathbb{Q}$ (cf. [14] ChI; 0.11, 0.23]). We have a natural composition of homomorphisms: $\iota : G \to G | \text{NS}_{\mathbb{C}}(X) \to \overline{G}$. Denote by

$$R(G) := \iota^{-1}(\iota(G) \cap R(\overline{G})) \triangleleft G.$$
Theorem 1.1. Let $X$ be a 3-dimensional minimal projective variety and $G \leq \text{Aut}(X)$ a subgroup such that $G|\text{NS}_C(X)$ is not virtually solvable. Then $R(G)|\text{NS}_C(X)$ is virtually unipotent. Replacing $G$ by a suitable finite-index subgroup, $G/R(G)$ is embedded as a Zariski-dense subgroup in $H := \overline{G}/R(\overline{G})$ so that $H(\mathbb{R})$ is a semi-simple real linear algebraic group and is either of real rank 1 (cf. [14, 0.25]) or locally isomorphic to $\text{SL}_3(\mathbb{R})$ or $\text{SL}_3(\mathbb{C})$ (where locally isomorphic means having isomorphic Lie algebras).

The key step of the proof is Theorem 4.2 of which part (1) is a consequence of [7] Theorem 5.1 in which the authors have determined the actions of irreducible lattices in semi-simple real Lie groups of higher rank on threefolds.

For a subgroup $G$ of $\text{Aut}(X)$, the pair $(X, G)$ is non-strongly-primitive, if there are $X'$ bimeromorphic to $X$, a finite-index subgroup $G_1$ of $G$ and a holomorphic map $X' \to Y$ with $0 < \dim Y < \dim X$, such that the induced bimeromorphic action of $G_1$ on $X'$ is biholomorphic and descends to an action on $Y$ with $X' \to Y$ being $G_1$-equivariant. $(X, G)$ is strongly primitive if it is non-strongly-primitive.

Our second main result is Theorem 1.2 (being generalized to higher dimensions in [9]).

Theorem 1.2. Let $X$ be a 3-dimensional normal projective variety with only $\mathbb{Q}$-factorial terminal singularities, and $G \leq \text{Aut}(X)$ a subgroup such that $G_0 := G \cap \text{Aut}_0(X)$ is infinite and the quotient group $G/G_0$ is an almost abelian group of positive rank (cf. Section 2.1 for the terminology). Suppose that the pair $(X, G)$ is strongly primitive. Then $X$ is a complex 3-torus and $G_0$ is Zariski-dense in $\text{Aut}_0(X)$.

We remark that the almost abelian condition (as defined in [21] on $G/G_0$ is used to show that the extremal rays on $X$ are $G$-periodic (cf. [25] Theorem 2.13, or the Appendix]).

Corollary 1.3. Let $X$ be a 3-dimensional normal projective variety with only $\mathbb{Q}$-factorial rational singularities, and $G \leq \text{Aut}(X)$ a subgroup of null entropy such that $G|\text{NS}_C(X)$ is almost abelian of positive rank. Assume that $(X, G)$ is strongly primitive. Then, $\text{Aut}_0(X) = \{\text{id}_X\}$ and $h^1(X, \mathcal{O}_X) = 0$.

We do not have any example satisfying all the hypotheses of Corollary 1.3.

Let $\tau$ be a primitive cubic or quartic root of 1, $E := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ and $X := E^n/\langle\text{diag}[\tau, \ldots, \tau]\rangle$ (cf. [17] Thm. (0.3]), [20] Ex. 1.7]). Take some $g$ in $\text{SL}_n(\mathbb{Z})$ such that $g$ acts on $X$ as an automorphism of infinite order and null entropy. When $n = 3$, the group $G := \langle g \rangle$ satisfies the hypotheses of Corollary 1.3 except for the strong primitivity which seems hard to verify. The action of $\langle g \rangle$ on $E^n$ is not strongly primitive (cf. Proof of Corollary 3.8).

Remark 1.4. (1) In Theorem 1.1 suppose that $c_1(X) \neq 0$. Then the Iitaka fibration $X \to Y$ is $G$-equivariant, and also non-trivial by the abundance theorem or the classification of surfaces. Replacing $G$ by a subgroup of finite index, we may assume that the induced action of $G$ on $Y$ is trivial (cf. [20] Theorem 14.10]), so $G$ acts faithfully on a general fibre $S$ and the group $G|S$ is not of null entropy (since the same holds for $G|X$; see Theorem 2.2). Since $K_S = K_X|S \sim_0 0$, our $S$ is a complex 2-torus, $K3$ or Enriques surface. So there is a homomorphism $G|X = G|S \to \text{SO}(1, \rho(S) - 1) \leq \text{SL}(\text{NS}_S(S))$ ($G$ being replaced by a subgroup of index $\leq 2$) with kernel virtually contained in $\text{Aut}_0(S)$ and the Picard number $\rho(S) \leq 20$. 


(2) In Theorem 1.2, the strong primitivity assumption on \((X, G)\) is necessary by considering \(X = S \times T\) and \(G = \langle g \rangle \times \text{Aut}_0(T)\), where \(g\) is of positive entropy on a K3 surface \(S\) and \(T\) a homogeneous curve (\(\mathbb{P}^1\) or elliptic).

(3) The projectivity of \(X\) in Theorem 1.1 is used in applying the characterization of a quotient of an abelian threefold (cf. [18]). The projectivity of \(X\) in Theorem 1.2 is used in running the minimal model program (only for uniruled varieties).

The following Theorem 1.5 is a direct consequence of [23, Theorem 1.1]; see also the discussion in [6, §6]. It extends the classical Tits alternative [19, Theorem 1].

A compact Kähler manifold \(X\) is ruled if it is bimeromorphic to a manifold with a \(\mathbb{P}^1\)-fibration. By a result of Matsumura, \(X\) is ruled if \(\text{Aut}_0(X)\) is not a compact torus (cf. [10, Proposition 5.10]). When \(X\) is a compact complex Kähler manifold (or a normal projective variety), set \(L := H^2(X, \mathbb{Z})/\text{(torsion)}\) (resp.
\(L := \text{NS}(X)/\text{(torsion)}\)), \(L_{\mathbb{R}} := L \otimes \mathbb{R}\), and \(L_{\mathbb{C}} := L \otimes \mathbb{C}\).

**Theorem 1.5.** Let \(X\) be a compact Kähler (resp. projective) manifold of dimension \(n\) and \(G \leq \text{Aut}(X)\) a subgroup. Then one of the following properties holds.

1. \(G|L_C \geq \mathbb{Z} \ast \mathbb{Z}\) (the non-abelian free group of rank two), and hence \(G \geq \mathbb{Z} \ast \mathbb{Z}\).
2. \(G|L_C\) is virtually solvable and \(G \geq K \cap L(\text{Aut}_0(X)) \geq \mathbb{Z} \ast \mathbb{Z}\) where \(L(\text{Aut}_0(X))\) is the linear part of \(\text{Aut}_0(X)\) (cf. [10, Definition 3.1, p. 157]) and \(K = \text{Ker}(G \to \text{GL}(L_C))\), so \(X\) is ruled (cf. [10, Proposition 5.10]).
3. There is a finite-index solvable subgroup \(G_1\) of \(G\) such that the null subset \(N(G_1)\) of \(G_1\) is a normal subgroup of \(G_1\) and \(G_1/N(G_1) \cong \mathbb{Z}^\oplus r\) for some \(r \leq n - 1\).

In particular, either \(G \geq \mathbb{Z} \ast \mathbb{Z}\) or \(G\) is virtually solvable. In Cases (2) and (3) above, \(G|L_C\) is finitely generated.

### 2. Entropy and Algebraic Group Action

In this section, we shall recall some definitions and technical results needed in the proofs and establish some easy consequences or already known facts.

#### 2.1. Terminology and notation are as in [12]. Below are some more conventions.

Let \(X\) be a compact complex Kähler manifold (resp. a normal projective variety). As in the introduction, set \(L := H^2(X, \mathbb{Z})/\text{(torsion)}\) (resp. \(L := \text{NS}(X)/\text{(torsion)}\)), \(L_{\mathbb{R}} := L \otimes \mathbb{R}\), and \(L_{\mathbb{C}} := L \otimes \mathbb{C}\). Let \(\overline{P(X)}\) be the closure of the Kähler cone (resp. the nef cone \(\text{Nef}(X)\), i.e., the closure of the ample cone) of \(X\). Elements in \(\overline{P(X)}\) are called nef.

For \(g \in \text{Aut}(X)\), let

\[
d_1(g) := \max\{ |\lambda| ; \lambda \text{ is an eigenvalue of } g^* | H^{1,1}(X) \}
\]

be the first dynamical degree of \(g\) (cf. [8, §2.2]). By the generalization of the Perron-Frobenius theorem (cf. [3]) applied to \(\overline{P(X)}\), for every \(g \in \text{Aut}(X)\), there is a non-zero nef class \(L_g\) (not unique) such that

\[
g^* L_g = d_1(g) L_g.
\]

We remark that \(g\) is of null entropy if and only if so is \(g^{-1}\); if this is the case, then for every non-trivial \((1, 1)\)-class \(M\) with \(g^* M = \lambda M\), we have \(|\lambda| = 1\).

\(G|Y\) denotes a naturally (from the context) induced action of \(G\) on \(Y\). A subvariety \(Z \subset X\) is \(G\)-periodic if \(Z\) is stabilized by a finite-index subgroup of \(G\). For a complex torus \(X\) (as a variety), we have \(\text{Aut}_{\text{variety}}(X) = T \times \text{Aut}_{\text{group}}(X)\).
where \( T = \text{Aut}_0(X) (\cong X) \) consists of all the translations of \( X \) and \( \text{Aut}_{\text{group}}(X) \) is the group of bijective homomorphisms of \( X \) (as a torus).

A group \( G \) is virtually unipotent (resp. virtually abelian, or virtually abelian of rank \( r \)) if a finite-index subgroup \( G_1 \) of \( G \) is unipotent (resp. abelian, or isomorphic to \( \mathbb{Z}^{\oplus r} \)). A group \( G \) is virtually solvable (resp. almost abelian, or almost abelian of finite rank \( r \); cf. definition after [16 Thm 1.2]) if it has a finite-index subgroup \( G_1 \) and an exact sequence

\[
1 \to H \to G_1 \to Q \to 1
\]
such that \( H \) is finite and \( Q \) is solvable (resp. abelian, or isomorphic to \( \mathbb{Z}^{\oplus r} \)). By Lemma 2.2, \( G \) is almost abelian of finite rank \( r \) if and only if \( G \) is virtually abelian of rank \( r \). Replacing \( G_1 \) by a finite-index subgroup, we may assume that the conjugation action of \( G_1 \) (and hence of \( H \)) on \( H \) is trivial. So in the above definition of virtually solvable group, we may also assume \( H = 1 \), so that our definition here coincides with the usual definition.

Theorem 2.2 below follows from Oguiso [16 Lemma 2.5] and Tits [19 Theorem 1].

**Theorem 2.2.** Let \( X \) be a compact Kähler (or projective) manifold of dimension \( n \) and \( G \) a subgroup of \( \text{Aut}(X) \). Then we have:

1. Suppose that \( G \) is of null entropy. Then \( G \mid L_C \) is virtually unipotent and hence virtually solvable (cf. Section 2.1 for notation). Moreover, \( G \mid L_C \) is finitely generated.
2. Suppose that \( G \mid L_C \geq \mathbb{Z} \ltimes \mathbb{Z} \). Then \( G \) contains an element of positive entropy.

Let \( X \) be a compact Kähler (resp. projective) manifold of dimension \( n \). A sequence \( 0 \neq L_1 \cdots L_k \in H^{k,k}(X) (1 \leq k < n) \) is quasi-nef if it is inductively obtained in the following way: first \( L_1 \in \overline{P}(X) \); once \( L_1 \cdots L_{j-1} \in H^{j-1,j-1}(X) \) is defined, we define

\[
L_1 \cdots L_j = \lim_{t \to \infty} L_{1} \cdots L_{j-1} \cdot M_t
\]

for some \( M_t \in \overline{P}(X) \) (cf. [23 §2.2]). We remark that for \( j \geq 2 \), the \( L_j \) which appears in the construction of the \((j, j)\)-class \( L_1 \cdots L_j \) may not belong to \( \overline{P}(X) \). A group \( G \leq \text{Aut}(X) \) is polarized by the quasi-nef sequence \( L_1 \cdots L_k (1 \leq k < n) \) if

\[
g^*(L_1 \cdots L_k) = \chi_1(g) \cdots \chi_k(g)(L_1 \cdots L_k)
\]

for some characters \( \chi_j : G \to (\mathbb{R}_{>0}, \times) \).

Theorem 2.3 gives criteria of virtual solvability, with (3) proved in [23 Theorem 1.2].

**Theorem 2.3.** Let \( X \) be a compact Kähler (resp. projective) manifold of dimension \( n \) and \( G \) a subgroup of \( \text{Aut}(X) \). Then we have (cf. Section 2.1 for notation of \( L_C \)):

1. Suppose that \( G \mid L_C \) is virtually solvable and its Zariski-closure in \( \text{GL}(L_C) \) is connected. Then \( G \) is polarized by a quasi-nef sequence \( L_1 \cdots L_k (1 \leq k < n) \).
2. Conversely, suppose that \( G \) is polarized by a quasi-nef sequence \( L_1 \cdots L_k (1 \leq k < n) \). Then \( G \mid L_C \) is virtually solvable.
3. \( G \mid L_C \) is virtually solvable if and only if there exists a finite-index subgroup \( G_1 \) of \( G \) such that \( N(G_1) \triangleleft G_1 \) and \( G_1 / N(G_1) \cong \mathbb{Z}^{\oplus r} \) for some \( r \leq n - 1 \).

We need the following lemmas for the proof of the theorems.
Lemma 2.4. Let $G$ be a group, and $H < G$ a finite normal subgroup. Then we have:

(1) Suppose that for some $r \geq 1$ and $g_i \in G$ we have:
\[ G/H = \langle \bar{g}_1 \rangle \times \cdots \times \langle \bar{g}_r \rangle \cong \mathbb{Z}^r. \]
Then there is an integer $s > 0$ such that the subgroup $G_1 := \langle g_1^s, \ldots, g_r^s \rangle$ satisfies
\[ G_1 = \langle g_1^s \rangle \times \cdots \times \langle g_r^s \rangle \cong \mathbb{Z}^r \]
and is of finite-index in $G$. Further, the quotient map $\gamma : G \to G/H$ restricts to an isomorphism $\gamma | G_1 : G_1 \to \gamma(G_1)$ onto a finite-index subgroup of $G/H$.

(2) A group is almost abelian of finite rank $r$ if and only if it is virtually abelian of rank $r$.

Proof. (2) follows from (1). For (1), we only need to find $s > 0$ such that $g_i^s$ commutes with $g_j^s$ for all $i, j$. Since $G/H$ is abelian, the commutator subgroup $[G, G] \leq H$. Thus the commutators $[g_1^t, g_2^t]$ for $t > 0$ all belong to $H$. The finiteness of $H$ implies that $[g_1^{t_1}, g_2^{t_1}] = [g_1^{t_2}, g_2^{t_2}]$ for some $t_2 > t_1$, which implies that $g_1^{s_{12}}$ commutes with $g_2$, where $s_{12} := t_2 - t_1$. Similarly, we can find an integer $s_{1j} > 0$ such that $g_1^{s_{1j}}$ commutes with $g_j$. Set $s_i := s_{1j} \times \cdots \times s_{1r}$. Then $g_i^{s_i}$ commutes with every $g_j$. Similarly, for each $i$, we can find an integer $s_i > 0$ such that $g_i^{s_i}$ commutes with $g_j$ for all $j$. Now $s := s_1 \times \cdots \times s_r$ will do the job. This proves the lemma.

Lemma 2.5. Let $X$ be a compact complex Kähler manifold (resp. normal projective variety), and $G$ a subgroup of $\text{Aut}(X)$. Then, replacing $G$ by a suitable finite-index subgroup, the following are true (cf. Section 2.1 for notation $L_C$).

(1) There is a normal subgroup $U < G$ such that $U \mid L_C$ is unipotent and $G/U$ is embedded as a Zariski-dense subgroup in a reductive complex linear algebraic group.

(2) There is a normal subgroup $R < G$ such that $R \mid L_C$ is solvable and $G/R$ is embedded as a Zariski-dense subgroup in a semi-simple complex linear algebraic group $H$.

Proof. Let $\overline{G}$ be the Zariski-closure of $G \mid L_C \subseteq GL(L_C)$, and $\iota$ the composite: $G \to G \mid L_C \to \overline{G}$. Replacing $G$ by the intersection of $G$ and the $\iota$-inverse of the identity connected component of $\overline{G}$, we may assume that $\overline{G}$ is connected. Let $U(\overline{G})$ (resp. $R(\overline{G})$) be the unipotent radical (resp. the radical) of $\overline{G}$. Let $U \leq G$ (resp. $R \leq G$) be the $\iota$-inverse of $U(\overline{G})$ (resp. $R(\overline{G})$). Then the embeddings $G/U \to \overline{G}/U(\overline{G})$ and $G/R \to \overline{G}/R(\overline{G})$, and $U$ and $R$ here meet the requirements of the lemma.

Lemma 2.6. We use the notation $L_C$ of Section 2.1. A group $G \leq \text{Aut}(X)$ has finite restriction $G \mid L_C$ if and only if the index $|G : G \cap \text{Aut}_0(X)|$ is finite.

Proof. Consider the exact sequence
\[ 1 \to K \to G \to G \mid L_C \to 1. \]
For an ample divisor or Kähler class $\omega$ of $X$, our $K$ is a subgroup of $\text{Aut}_\omega(X) := \{ g \in \text{Aut}(X) \mid g^* \omega = \omega \}$, where the latter contains $\text{Aut}_0(X)$ as a group of finite-index (cf. [13, Proposition 2.2]). Now the last group below is a finite group
\[ K/(K \cap \text{Aut}_0(X)) \cong (K \text{ Aut}_0(X))/\text{Aut}_0(X) \leq \text{Aut}_\omega(X)/\text{Aut}_0(X). \]
The lemma follows since the connected group $\text{Aut}_0(X)$ acts trivially on the lattice $L$ (and hence on $L_C$) so that $G \cap \text{Aut}_0(X) = K \cap \text{Aut}_0(X)$. \hfill $\square$

A more precise version of Lemma 2.7 below was proved in [6, 6.1] for finitely generated groups.

**Lemma 2.7.** Let $G$ be a group of automorphisms of a compact Kähler manifold $X$. Consider an exact sequence of groups:

$$1 \to N \to G \to Q \to 1.$$ 

Suppose $N$ is contained in the union of finitely many connected components of $\text{Aut}(X)$. Suppose both $N$ and $Q$ are virtually solvable. Then $G$ is also virtually solvable.

**Proof.** Let $\overline{N} \subseteq \text{Aut}(X)$ be the Zariski-closure of $N$. Replacing $G$ by a suitable finite-index subgroup, we may assume that $Q$ is solvable. Since $\overline{N} \cap G \triangleleft G$, we have $(\overline{N})_0 \cap G \triangleleft G$ for the identity connected component $(\overline{N})_0$ of $\overline{N}$. Hence $M := (\overline{N})_0 \cap N \triangleleft G$. Now

$$N/M \cong (N(\overline{N})_0)/(\overline{N})_0 \leq \overline{N}/(\overline{N})_0,$$

where the latter is a finite group. We have an exact sequence

$$1 \to N/M \to G/M \to G/N = Q \to 1.$$ 

Replacing $G/M$ by a finite-index subgroup we may assume that the conjugate action of $G/M$ (and hence of $N/M$) on the finite group $N/M$ is trivial. Thus $N/M$ is abelian and hence $G/M$ is solvable. Since $N$ is virtually solvable, so is $\overline{N}$. Hence $(\overline{N})_0$ is solvable. Thus $M$ is solvable. Therefore, $G$ is solvable. \hfill $\square$

**Lemma 2.8.** Let $X$ be a compact complex Kähler manifold (resp. a normal projective variety), and $G \leq \text{Aut}(X)$ a subgroup. Assume the following two conditions:

1. $H \triangleleft G$, and $H$ has a finite-index subgroup $H_1$ such that the null set $N(H_1)$ is a (normal) subgroup of $H_1$ and $H_1/N(H_1) = \langle h \rangle \cong \mathbb{Z}$ for some $h \in H_1$.

2. Suppose that there is a common nef eigenvector $L_1$ of $H_1$, and further that $h^*L_1 = d_1(h)L_1$, i.e., $L_1$ equals some $L_h$ up to scalar. Suppose also that for every $s \neq 0$ and every nef $M$ so that $(h^s)^*M = \lambda M$ with $\lambda \neq 1$, we have $M$ parallel to either one of $L_{h^\pm 1}$ (which are two fixed nef eigenvectors).

Then the stabilizer subgroup $\text{Stab}_{L_h}(G) := \{g \in G \mid g^*L_h \text{ is parallel to } L_h \}$ has index $\leq 2$ in $G$.

**Proof.** We begin with:

**Claim 2.9.** For every $g \in G$, the class $g^*L_h$ is parallel to one of $L_{h^\pm 1}$. 

We prove the claim. Take any element $g$ of $G$. Since $H \triangleleft G$, we have $ghg^{-1} \in H \setminus N(H)$. Hence $gh^a g^{-1}$ is in $H_1$ with $a = |H : H_1|$, so it equals $h^n$ for some $b \neq 0$ and $n \in N(H_1)$. Since the element $n$ of $N(H_1)$ fixes $L_1 = L_h$ (cf. Section 2.1),

$$(gh^a g^{-1})^*L_h = d_1(h^b)L_h, \quad (h^a)^*(g^*L_h) = d_1(h)^b(g^*L_h).$$

Now condition (2) applied to $h^a$ and $\lambda := d_1(h)^b \neq 1$ implies the claim.

Return to the proof of Lemma 2.8. Suppose there is some $g_1 \in G \setminus \text{Stab}_G(L_h)$. Take any $g \in G \setminus \text{Stab}_G(L_h)$. By Claim 2.9, we have (with equalities all up to scalars)
\[ g^*L_h = L_h^{-1} = g_1^*L_h, \quad (gg_1^{-1})^*L_h = (g_1^{-1})^*g^*L_h = L_h. \] Hence \( gg_1^{-1} \in \text{Stab}_G(L_h) \) and \( g = (gg_1^{-1})g_1 \in \text{Stab}_G(L_h)g_1. \) So \( G = \text{Stab}_G(L_h) \cup \text{Stab}_G(L_h)g_1. \) The lemma follows.

2.10. **Proof of Theorem 2.2.** Assertion (2) follows from assertion (1), so we only need to prove Theorem 2.2(1).

We follow the proof of Oguiso [16, Prop 2.2]. Since \( G \) is of null entropy, the subset
\[ U := \{ g \in G ; g \mid L_C \text{ is unipotent} \} \]
is a normal subgroup of \( G. \) If \( G \mid L_C \) is not virtually solvable, then by the classical Tits alternative theorem [19, Theorem 1], there are \( g_i \in G \) such that \( \langle g_1, g_2 \rangle \mid L_C = \langle g_1 \rangle \mid L_C \ast \langle g_2 \rangle \mid L_C = \mathbb{Z} \ast \mathbb{Z}. \) As observed in [16], \( g_i \in U \) for some \( s \geq 1, \) and hence \( \mathbb{Z} \ast \mathbb{Z} = \langle g_1^s, g_2^s \rangle \mid L_C \leq U \mid L_C \) which is unipotent (and hence solvable). This is absurd.

Thus, \( G \mid L_C \) is virtually solvable. Replacing \( G \) by a suitable finite-index subgroup, we may assume that \( G \mid L_C \) is solvable and its closure \( \overline{G} \) in \( \text{GL}(L_C) \) is connected (and solvable). Write \( \overline{G} = U \times T, \) where \( U \) is the unipotent radical and \( T \) a maximal torus in \( \overline{G} \). As observed in [16], the image of \( G \) via the quotient map \( G \to T \) is a torsion group in \( \text{GL}(L_C) \) with bounded exponent and hence a finite group by Burnside’s theorem. Thus the index \( |G : U| < \infty. \)

To finish the proof of assertion (1), we may assume that \( G = U, \) and it suffices to show that \( G \mid L \) is generated by \( \ell(\ell - 1)/2 \) elements where \( \ell = \text{rank} L. \) Regarding \( \overline{G} \) as a subgroup of upper triangular matrices, there is a standard normal series
\[ 1 < U_1 < U_2 < \cdots < U_{\ell(\ell - 1)/2} = \overline{G} \]
such that the factor groups are all 1-dimensional. Restricting the series to \( G \mid L, \) we get a normal series of discrete groups whose factor groups are cyclic groups. Thus \( G \mid L \) is generated by \( \ell(\ell - 1)/2 \) elements. This proves Theorem 2.2.

2.11. **Proof of Theorem 2.3.** (1) was proved in [23, Theorem 1.2]. For (2), suppose that \( G \) is polarized by a quasi-nef sequence \( L_1 \cdots L_k \) \( (1 \leq k < n) \) so that \( g^*(L_1 \cdots L_k) = \chi_1(g) \cdots \chi_k(g)L_1 \cdots L_k. \) As in the proof of [23, Theorem 1.2], the homomorphism
\[ \varphi : G \to (\mathbb{R}, +), \quad g \mapsto (\log \chi_1(g), \ldots, \log \chi_n(g)) \]
has \( \text{Ker}(G) = N(G) \), and \( \varphi(G) = \mathbb{Z}^\oplus r \) a lattice in \( \mathbb{R}^{n-1}. \) By Theorem 2.2, \( N(G) \mid L_C \) is virtually solvable; so is \( G \mid L_C, \) since \( G/N(G) \) is abelian and by Lemma 2.7.

For (3), the “if part” follows from Theorem 2.2 and Lemma 2.7. The “only if” part is by [23, Theorem 1.2, Remark 1.3]. This proves Theorem 2.3.

2.12. **Proof of Theorem 1.5.** We may assume that assertion (1) is not satisfied. Replacing \( G \) by a suitable finite-index subgroup and by [19, Thm 1], we may assume that \( G \mid L_C \) is solvable and its closure \( \overline{G} \) in \( \text{GL}(L_C) \) is connected (and solvable). Let \( K = \text{Ker}(G \to G \mid L_C) \) be as in Lemma 2.6.

Suppose that \( K \) is virtually solvable. Then so is \( G \) by Lemma 2.7. Thus Theorem 1.5(3) occurs, by [23, Theorem 1.2, Remark 1.3].
Suppose that $K$ is not virtually solvable. Consider the exact sequence

$$1 \to L_A \to \text{Aut}_0(X) \to T \to 1,$$

where $L_A$ is the linear part of $\text{Aut}_0(X)$ and $T$ a compact complex torus (cf. Theorem 3.12]). This induces the exact sequence (with $Q$ abelian):

$$1 \to K \cap L_A \to K \cap \text{Aut}_0(X) \to Q \to 1.$$

We may assume that Theorem 1.5(2) does not occur. So $K \cap L_A$ is virtually solvable by the Tits alternative. Thus so is $K \cap \text{Aut}_0(X)$ by the exact sequence above and Lemma 2.7. Now $K/(K \cap \text{Aut}_0(X))$ is a finite group by Lemma 2.6. Hence $K$ is also virtually solvable, contradicting our extra assumption.

For the final assertion, replacing $G$ by a suitable finite-index subgroup, we may assume that $G/L$ is solvable and has connected Zariski-closure in $\text{GL}(L_C)$. Then $G/N(G) \cong \mathbb{Z}^{\oplus r}$ by [23, Theorem 1.2]. This and Theorem 2.2 for $N(G)$ imply the assertion.

3. Strong primitivity for threefolds

We prove Theorem 1.2. Replacing $G_0$ by the identity connected component of its Zariski-closure in $\text{Aut}_0(X)$ we may further assume that $G_0 = G \cap \text{Aut}_0(X)$ is connected, positive-dimensional and closed in $\text{Aut}_0(X)$. Because $G$ acts naturally on the quotient of $X$ by $G_0$ and because of our assumption, we may assume that one orbit of $G_0$ is a Zariski-dense open subset of $X$, i.e., $X$ is almost homogeneous (cf. [23, Lemma 2.14]).

Claim 3.1. Suppose the irregularity $q(X) = h^1(X, \mathcal{O}_X) > 0$. Then Theorem 1.2 is true.

We prove Claim 3.1. By the proof of [23, Lemma 2.13], the Albanese map $a: X \to A := \text{Alb}(X)$ is surjective, birational and necessarily $\text{Aut}(X)$-equivariant. Our $G_0$ induces an action on $A$ and we denote it by $G_0|A$. Since $G_0|A$ also has a Zariski-dense open orbit in $A$, we have $G_0|A = \text{Aut}_0(X)$ (as $A$). Let $B \subset A$ be the locus over which $a$ is not an isomorphism. Note that $B$ and $a^{-1}(B)$ are $G_0$-stable. Since $G_0|A = \text{Aut}_0(X)$, we have $B = \emptyset$. Claim 3.1 is proved.

We continue the proof of Theorem 1.2. By Claim 3.1 we may assume that $q(X) = 0$. Thus $G_0 \leq \text{Aut}_0(X)$ is a linear algebraic group and has a Zariski-dense open orbit in $X$. In particular, $X$ is ruled and unirational, because linear algebraic groups are rational varieties by a classical result of Chevalley.

In the rest of the proof, we shall derive a contradiction. Let $U \subseteq X$ be the open dense $G_0$-orbit and $F := X \setminus U$. Then $F$ consists of finitely many prime divisors and some subvarieties of codimension $\geq 2$. Since $G_0 \leq G$, we may assume that both $U$ and all irreducible components of $F$ are $G$-stable, after replacing $G$ by a suitable finite-index subgroup. $X$ has only finitely many $G_0$-periodic prime divisors, all of which are contained in $F$ and are $G$-stable.

By the minimal model program (MMP) in dimension three (cf. [12, §3.31, §3.46]), the end product of a uniruled variety (like our $X$ here) is an extremal Fano contraction $f: X_m \to Y$ with a general fibre $X_{m,y}$, i.e., by definition, the restriction $-K_{X_m}|X_{m,y}$ of the canonical divisor $-K_{X_m}$ is ample and the Picard numbers satisfy $\rho(X_m) = 1 + \rho(Y)$. 
Claim 3.2. (1) Every $G_0$-periodic subvariety of $X$ is actually $G_0$-stable.
(2) There are a composite $X = X_0 \rightarrow X_1 \cdots \rightarrow X_m$ of birational extremal contractions and an extremal Fano contraction $X_m \rightarrow Y$ with $\dim Y < \dim X$. The induced birational action of $G_0$ on each $X_i$ is biregular. $G_0|X_m$ descends to an action on $Y$ so that $X_m \rightarrow Y$ is $G_0$-equivariant.
(3) In (2), for every finite-index subgroup $G_1$ of $G$, there is at least one $i \in \{1, \ldots, m\}$ such that the induced action of $G_1$ on $X_i$ is not biregular.
(4) In (2), let $s \leq m$ be the largest integer such that $X_i \rightarrow X_{i+1}$ is divisorial for every $i \in \{0, 1, \ldots, s - 1\}$. Then, replacing $G$ by a suitable finite-index subgroup, the induced birational action of $G$ on each $X_i (i < s)$ is biregular and hence each map $X_{i-1} \rightarrow X_i$ is $G$-equivariant. In particular, $s < m$.

Remark 3.3. By the choice of $s$ in (4), $X_s \rightarrow X_{s+1}$ is a flip with a flipping contraction $X_s \rightarrow Y_s$ and with $X_{s+1} = \text{Proj}_{Y_s} (\bigoplus_{m \geq 0} O_{Y_s} (mK_{Y_s}))$ (cf. [12] Cor. 6.4 or Thm. 3.52).

We now prove Claim 3.2. (1) is true because $G_0$ is a connected group. For the first part of (2), see [12] §3.31, §3.46 when $\dim X = 3$ and [2] Corollary 1.3.2 when $\dim X$ is arbitrary. The second part of (2) is true because $G_0$ acts trivially on $H^1(X, \mathbb{Z})$, and also on $\text{NS}_C (X)$ and the extremal rays of $\overline{\text{NE}} (X)$ (cf. [25] Lemmas 2.12 and 3.6).

For (4), suppose that $X = X_0 \rightarrow X_1$ is a divisorial contraction of an extremal ray $R := \mathbb{R}_{>0}[f]$ with an exceptional divisor $D_0$. Since $G_0$ acts trivially on the extremal rays of $\overline{\text{NE}} (X)$, this $D_0$ is $G_0$-stable. So $D_0$ is contained in $F$ and is $G$-stable.

Since the natural map $G/G_0 \rightarrow \text{Aut} (H^2(X, \mathbb{Z}))$ has finite kernel (cf. [13] Proposition 2.2) and by the assumption, $G/G_0$ is almost abelian of finite rank $r > 0$. By Lemma 2.4 and replacing $G$ by a finite-index subgroup, there is some $H_0 \triangleleft G$ such that $H_0$ contains $G_0$ as a subgroup of finite index and $G/H_0 = \langle g_1 \rangle \oplus \cdots \oplus \langle g_r \rangle \cong \mathbb{Z}^\oplus r$ for some $g_i \in G$.

In [25] Lemma 3.7, it is proved that a positive power of $g_i$ preserves the extremal ray $R$ and hence descends to a biregular automorphism of $X_1$. Thus $X \rightarrow X_1$ is $G$-equivariant after $G$ is replaced by a finite-index subgroup. Indeed, we may assume that $g_i (R) = R$ so that $\{g(R) \mid g \in G\}$ consists of no more than $|H_0 : G_0|$ extremal rays so that a finite-index subgroup of $G$ fixes $R$. This and the second sentence of the next paragraph are the places where we need $G/G_0$ to be almost abelian.

For (3), suppose to the contrary that $G$ (replaced by a finite-index subgroup) acts biregularly on all $X_i$. Then, as in the proof of (4) above, by [25] Theorem 2.13, or the Appendix], we may assume that $G$ (replaced by its finite-index subgroup) fixes the extremal ray giving rise to the extremal Fano contraction $X_m \rightarrow Y$, and hence $X_m \rightarrow Y$ is $G$-equivariant. By the strong primitivity assumption, we have $\dim Y = 0$, so the Picard number $\rho (X_m) = 1$ and $-K_{X_m}$ is ample. Since $G$ fixes the ample class of $-K_{X_m}$, it is a finite extension of $G_0$ (cf. [13] Proposition 2.2). This contradicts the assumption. Claim 3.2 is proved.

Claim 3.4. It is impossible that $\text{NS}_C (X_i)$ with $0 \leq i \leq m$ is spanned by $-K_{X_i}$ and $G_0$-periodic divisors, or that $\text{NS}_C (Y)$ is spanned by $G_0$-periodic divisors.

Indeed, note that $\text{NS}_C (X_m)$ is spanned by $-K_{X_m}$ (which is ample over $Y$) and the pullback of $\text{NS}_C (Y)$, and $\text{NS}_C (X)$ is spanned by the pullback of $\text{NS}_C (X_i)$ and (necessarily $G_0$-stable) exceptional divisors of $X \rightarrow X_i$. Thus we only need to rule
out the possibility that $\text{NS}_C(X)$ is spanned by $-K_X$, and $G_0$-stable divisors $D_i$ all of which are necessarily contained in $F$ and hence $G$-stable.

Write an ample divisor $M$ on $X$ as a combination of $-K_X$ and $D_i$'s. Then $G \leq \text{Aut}_M(X)$, so $|G/G_0| < \infty$ as in the proof of Lemma 2.6 contradicting the assumption.

**Claim 3.5.** $X_m$ and hence $Y$ contain a $G_0$-fixed point (here we use the fact that $\dim X = 3$).

Indeed, note that a smooth threefold has no flip and that a flip preserves the singularity type of a threefold. By Claim 3.2 for some $m - 1 \geq t \geq s$, $X_t \rightarrow X_{t+1}$ is a flip and $X_{t+1} \rightarrow \cdots \rightarrow X_m$ is the composite of extremal divisorial contractions. So the non-empty finite set $\text{Sing} X_{t+1}$ (cf. [12, Corollary 5.18]) and its image on $X_m$ are fixed by $G_0$.

**Claim 3.6.** It is impossible that $\dim Y \leq 1$.

Indeed, if $\dim Y = 0$, then $\text{NS}_C(X_m)$ is of rank one and spanned by $-K_{X_m}$ (which is ample over $Y$). This contradicts Claim 3.4. If $\dim Y = 1$, then the rank two space $\text{NS}_C(X_m)$ is spanned by $-K_{X_m}$ (which is ample over $Y$) and the fibre over a $G_0$-fixed point $y_0$ (cf. Claim 3.5). This contradicts Claim 3.4.

We continue the proof of Theorem 1.2. Take an extremal ray on $X_s$ generated by a rational curve $\ell$ and let $X_s \rightarrow X_{s+1}$ be the flip (cf. Claim 3.2 for $s$). Note that $G_0$ stabilizes all irreducible components $E_i$ of the exceptional locus of the flipping contraction $X_s \rightarrow Y_s$ and $G$ (replaced by a finite-index subgroup) stabilizes all irreducible components $D_{ij}$ of the Zariski closure of $\bigcup_{g \in G} gE_i$, because $G_0 \triangleleft G$. These $D_{ij}$ are unions of ‘small’ $G_0$-orbits and hence are contained in the image of the algebraic subset $F \subset X$.

If $\dim D_{ij} = \dim E_i = 1$, then $G$ preserves the extremal ray $\mathbb{R}_{\geq 0}[\ell] \subseteq \text{NE}(X_s)$ and we can descend $G$ to a biregular action on $X_{s+1}$ (cf. [25, Lemma 3.6]). Now apply MMP on $X_{s+1}$ and continue the process.

Assume that $\dim D_{ij} = 2 > \dim E_i = 1$. If $G_0$ acts trivially on some $g_0E_i$ in the set $\{gE_i \mid g \in G\}$, then $G_0 = gG_0g^{-1}$ acts trivially on $g_0E_i$, i.e., on all $g'E_i$ ($g' \in G$). Hence $G_0 \mid D_{ij} = \text{id}$. This contradicts Claim 3.7 below.

Suppose that $G_0$ acts non-trivially on some $g_0E_i$ and hence on all $gE_i$ ($g \in G$). Then these extremal curves $gE_i$ are fibres of the quotient map $D_{ij} \rightarrow D_{ij}/G_0 =: B$ over a curve $B$, hence homologous to each other. So they give rise to one and the same class in the extremal ray $\mathbb{R}_{\geq 0}[\ell] \subseteq \text{NE}(X_s)$. Thus $G$ preserves this extremal ray and we can descend $G$ to a biregular action on $X_{s+1}$ (cf. [25, Lemma 3.6]). Now apply MMP on $X_{s+1}$ and continue the process. Therefore, we can continue the $G$-equivariant MMP and reach an extremal Fano fibration $X_m \rightarrow Y$ which is a contradiction (cf. Claim 3.2).

To complete the proof of Theorem 1.2 we still need to prove:

**Claim 3.7.** It is impossible that $\dim Y = 2$, $\dim D_{ij} = 2$ and $G_0 \mid D_{ij} = \text{id}$.

We now prove Claim 3.7. $X_m \rightarrow Y$ is known as an extremal conic fibration. We may take a $G_0$-equivariant blowup $\tilde{X} \rightarrow X_s$ to resolve indeterminacy of the composite $\pi_s : X_s \rightarrow X_m \rightarrow Y$ so that the induced map $\tilde{\pi} : \tilde{X} \rightarrow Y$ is holomorphic and $G_0$-equivariant. By [15, Theorem 4.8], there exist blowups $\sigma_x : X' \rightarrow \tilde{X}$ and $\sigma_y : Y' \rightarrow Y$ with $X'$ and $Y'$ smooth, and extremal conic fibration $\pi' : X' \rightarrow Y'$.
such that \( \hat{\pi} \circ \sigma_x = \sigma_y \circ \pi' \). We may also assume that the four maps above are \( G_0 \)-equivariant by taking extra blowups so that they are equivariant (noting that \( G_0 \) stabilizes extremal rays).

We will reach a contradiction to Claim 3.4. To do so, we consider both \( Y \) and \( Y' \).

Indeed, if \( K_{Y'}^2 \leq 7 \), then \( \text{NS}_C(Y') \) (and hence \( \text{NS}_C(Y) \)) are spanned by \( G_0 \)-stable curves (i.e., the negative curves on \( Y' \)). This contradicts Claim 3.4. Therefore, we may assume that \( K_{Y'}^2 = 9 \) or \( 8 \), and \( Y' = \mathbb{P}^2 \) or a Hirzebruch surface \( F_d \) of degree \( d \geq 0 \).

If \( Y' = \mathbb{P}^2 \) or \( Y' = \mathbb{P}^1 \times \mathbb{P}^1 \), then \( Y' \) has no negative curve to contract, so \( Y' = Y \).

If \( Y = F_d \), then \( G_0 \) stabilizes a fibre passing through a fixed point of \( y_0 \) of \( G_0|Y \) (cf. Claim 3.5), and the zero-section through \( y_0 \) (resp. the unique \((-d\)-curve) when \( d = 0 \) (resp. \( d \geq 1 \)). This contradicts Claim 3.4.

Therefore, we may assume that either \( Y = Y' = \mathbb{P}^2 \), or \( F_d = Y' \rightarrow Y \) (with \( d \geq 1 \)) is the contraction of the unique \((-d\)-curve). Thus the Picard number \( \rho(X_m) = 1 + \rho(Y) = 2 \).

Let \( D_{ij}' \subseteq X' \) be the proper transform of \( D_{ij} \subseteq X_s \). Then \( G_0 \) acts trivially on \( D_{ij}' \) because so does \( G_0 \) on \( D_{ij} \). Since every fibre of \( \pi' : X' \rightarrow Y' \) is 1-dimensional, the image \( C_{ij} \subseteq Y' \) of \( D_{ij}' \) is the whole \( Y' \) or a curve, and \( G_0|C_{ij} = \text{id} \). Since \( G_0 \) is abelian and equals \( \mathbb{Z}_2 \), we may assume that \( C_{ij} \) is a curve in \( Y' \). If \( Y' = \mathbb{P}^2 \) (resp. \( Y' = F_d \rightarrow Y \) is the contraction of the \((-d\)-curve), then \( G_0|Y \) stabilizes \( C_{ij} \) (resp. the image of \( C_{ij} \) or every generating line). This contradicts Claim 3.4. Claim 3.7 is proved.

**Corollary 3.8.** Let \( X \) be a 3-dimensional normal projective variety and \( G \leq \text{Aut}(X) \) a subgroup of null entropy such that \( G_0 := G \cap \text{Aut}_0(X) \) is infinite and the quotient \( G/G_0 \) is an almost abelian group of positive rank. Then \((X,G)\) is not strongly primitive.

**Proof.** Taking a \( G \)-equivariant resolution, we may assume that \( X \) is smooth. With our assumption and the proof of Theorem 1.2, we may assume that \( X \) is a complex torus, and \( G_0 \) is connected, is closed and has a Zariski-dense open orbit in \( X \). Thus \( G_0 = \text{Aut}_0(X) \). By Lemma 2.3 and replacing \( G \) by a suitable finite-index subgroup, we may assume that \( G/G_0 \) is abelian and equals \( \langle g_1, \ldots, g_r \rangle \) for \( g_i \in G \), where the order \( o(g_i) = \infty \); moreover, \( g_i \) has a unipotent representation matrix on \( H^0(X, \Omega_X^1) \) using Kronecker’s theorem as in [24, Lemma 2.14]. Write \( g_i = T_{t_i} \circ h_i \) where \( T_{t_i} \) is the translation by \( t_i \) and \( h_i \) is a group automorphism. As in [24, Lemma 2.15], the identity connected component \( B \) of the fixed locus \( X^{h_1} \) (pointwise) has dimension equal to that of \( \ker(h_1^* - \text{id}) \subset H^0(X, \Omega_X^1) \) and is hence between 1 and \( \dim X - 1 \). Note that \( h_1 h_j = h_j h_1 \) holds modulo \( \text{Aut}_0(X) \) and hence holds in \( \text{Aut}(X) \) since both sides fix the origin. Thus \( h_j(B) \) is contained in \( X^{h_1} \) and hence equals \( B \) since it contains the origin. Now \( g_j(x + B) = g_j(x) + B \). So \( g_j \) permutes cosets of the quotient torus \( X/B \); the same is true for elements of \( \text{Aut}_0(X) \). Thus, the quotient map \( X \rightarrow X/B \) is \( G \)-equivariant. This proves Corollary 3.8. \( \square \)

3.9. **Proof of Corollary 1.3.** Taking a \( G \)-equivariant resolution, we may assume that \( X \) is smooth. If \( q(X) > 0 \), then, by Claim 3.1 we may assume that \( X \) is a complex torus so that \( \text{Aut}_0(X) \neq \{\text{id}_X\} \). Thus, we may always assume that \( \text{Aut}_0(X) \neq \{\text{id}_X\} \).
Replacing $G$ by $G$. $\text{Aut}_0(X)$ we may assume that $G \geq G_0 := \text{Aut}_0(X)$. According to Lemma 2.6 we have $G|\text{NS}_C(X) = G/K$, where $|K/G_0| < \infty$. Thus $G/G_0$ is also almost abelian of positive rank by assumption. Now Corollary 1.3 follows from Corollary 3.8.

4. Minimal threefolds

Below sufficient conditions for being a quotient of a torus are given.

**Theorem 4.1.** Let $X$ be a 3-dimensional minimal projective torus. Assume that one of the following two properties is satisfied:

1. The first Chern class $c_1(X) = 0$. The second Chern class $c_2(X)$ (as a linear form on $\text{NS}_C(X)$ as in [18, p. 265]) has zero intersection with a nef and big $\mathbb{R}$-divisor.

2. There is a subgroup $G \leq \text{Aut}(X)$ such that the null set $N(G)$ is a subgroup of $G$ and $G/N(G) \cong \mathbb{Z}^\oplus 2$.

Set $B := \text{Aut}(X)$. Then there is a $B$-equivariant birational surjective morphism $X \to X'$ such that $X' \cong T/F$ for a finite group $F$ acting freely outside a finite set of an abelian variety $T$ of dimension three. Further, the action of $B$ on $X'$ lifts to an action of a group $\overline{B}$ on $T$ such that $\overline{B}/F \cong B$.

**Proof.** Assume condition (1) in Theorem 4.1. Let

$$D := \text{Nef}(X) \cap c_2(X)^\perp = \{ M \in \text{Nef}(X) \mid M.c_2(X) = 0 \}$$

be a closed subcone of the nef cone $\text{Nef}(X)$ of $X$. Let

$$C := \overline{\text{NE}}(X) \cap D^\perp = \{ [\ell] \in \overline{\text{NE}}(X) \mid \ell.D_0 = 0 \text{ for all } D_0 \in D \}$$

be a closed subcone of the closed cone $\overline{\text{NE}}(X)$ of effective curves on $X$. Then $c_2(X) \in C$ by definition and using Miyaoka’s pseudo-effectivity of $c_2$ for any minimal variety $X$ of dimension $n$: $c_2(X) \cdot (H_1 \cdots H_{n-2}) \geq 0$ for all nef divisors $H_i$ on $X$ (cf. [18, Theorem 4.1, Proposition 1.1]).

By assumption, $D$ contains a nef and big $\mathbb{R}$-divisor. Let $A$ be an interior element of $D$. As in [2, Theorem 3.9.1], there is a birational contraction

$$\sigma : X \to X'$$

such that a curve $\ell \subset X$ is contracted to a point if and only if the class $[\ell]$ is contained in $C$, and such that $A = \sigma^*A'$ for some ample $\mathbb{R}$-divisor $A'$.

By the projection formula and since $A$ is contained in $D$, we have $A'.c_2(X') = \sigma^*A'.c_2(X) = A.c_2(X) = 0$. For any ample $\mathbb{R}$-divisor $P$ on $X'$, a small perturbation $A'_\varepsilon := A' - \varepsilon P$ of the ample divisor $A'$ is still ample because the ample cone of $X'$ is open. By Miyaoka’s pseudo-effectivity of $c_2$ for minimal variety, we have

$$0 \leq \varepsilon P.c_2(X') \leq (A'_\varepsilon + \varepsilon P).c_2(X') = A'.c_2(X') = 0.$$

So $P.c_2(X') = 0$. Since $\text{NS}_C(X')$ is spanned by ample divisors, we then obtain $c_2(X') = 0$ as a linear form on $\text{NS}_C(X')$.

Thus, $c_1(X)$ and $c_2(X)$ vanish, and by [18, Corollary, p. 266], we have $X' = T/F$ where $F$ is a finite group acting on the abelian variety $T$ freely outside a finite set. Since $D$ and hence $C$ are stable under the action of $B := \text{Aut}(X)$, the contraction $\sigma : X \to X'$ is $B$-equivariant. By [1, §3, especially Proposition 3] applied to étale-in-codimension-one covers, replacing $T$ by the finite cover corresponding to the
maximal lattice in $\pi_1(X' \setminus \text{Sing } X')$, we can lift the action of $B$ on $X'$ to an action of a group $\tilde{B}$ on $T$ such that $\tilde{B}/\text{Gal}(T/X') \cong B$. This proves Theorem 4.1 under condition (1).

Next, assume condition (2) in Theorem 4.1. The maximality of the rank of $G/N(G)$ and [23] Lemma 2.11 imply the Kodaira dimension $\kappa(X) = 0$. The abundance theorem for minimal threefolds implies $K_X \sim_{\mathbb{Q}} 0$ (cf. [12] 3.13]). Replacing $G$ by a finite-index subgroup, we may assume that $G \mid \text{NS}_C(X)$ is solvable and has connected Zariski-closure in $\text{GL}(\text{NS}_C(X))$ (cf. Theorem 2.2 or 2.3). By [26] Claim 2.5(1)], $c_2(X)$ is perpendicular to a nef and big $\mathbb{R}$-divisor. We are reduced to condition (1). This proves Theorem 4.1. \hfill \square

The next is the key step towards Theorem 1.1.

We recall the notation in the Introduction: For a subgroup $G$ of $\text{Aut}(X)$, let $\overline{G} \subseteq \text{GL}(\text{NS}_C(X))$ be the Zariski-closure of $G \mid \text{NS}_C(X)$ and $R(\overline{G})$ its solvable radical, both of which are defined over $\mathbb{Q}$. We have a natural composition of homomorphisms:

$$\iota : G \to G \mid \text{NS}_C(X) \to \overline{G}.$$  

**Theorem 4.2.** Let $X$ be a 3-dimensional minimal projective variety and $G \leq \text{Aut}(X)$ a subgroup such that $G \mid \text{NS}_C(X)$ is not virtually solvable. Then we have:

1. Suppose that $R(G) := \iota^{-1}(\iota(G) \cap R(\overline{G}))$ is of null entropy. Then $R(G) \mid \text{NS}_C(X)$ is virtually unipotent and hence of null entropy. Replacing $G$ by a suitable finite-index subgroup, $G/R(G)$ is embedded as a Zariski-dense subgroup in $H := \overline{G}/R(\overline{G})$ so that $H(\mathbb{R})$ is a semi-simple real linear algebraic group and is either of real rank 1 (cf. [14] 0.25]) or locally isomorphic to $\text{SL}_3(\mathbb{R})$ or $\text{SL}_3(\mathbb{C})$.

2. Suppose that $R(G)$ is not of null entropy. Set $B := \text{Aut}(X)$. Then there is a $B$-(and hence $G$-)equivariant birational surjective morphism $X \to X'$ such that $X' \cong T/F$ for a finite group $F$ acting freely outside a finite set of an abelian variety $T$ of dimension three. Further, the action of $B$ on $X'$ lifts to an action of a group $\tilde{B}$ on $T$ such that $\tilde{B}/F \cong B$.

We now prove Theorem 4.2. Note that $\iota(G)$ is contained in $\overline{G}(\mathbb{Q})$ and is Zariski-dense in $\overline{G}$. As in Lemma 2.5, replacing $G$ by a suitable finite-index subgroup, we may assume $\overline{G}$ is connected. Set $R := R(G)$. The $\iota : G \to \overline{G}$ above induces an injective homomorphism:

$$\gamma : G/R \to H := \overline{G}/R(\overline{G}).$$

Indeed, $\gamma$ is defined over $\mathbb{Q}$ (cf. [14] 0.11]). Of course, $H$ is semi-simple. $R \mid \text{NS}_C(X)$ is solvable, being embedded in the solvable group $R(\overline{G})$.

**Lemma 4.3.** Up to finite index, $H(\mathbb{R})$ is either semi-simple and of real rank one, or locally isomorphic to $\text{SL}_3(\mathbb{R})$ or $\text{SL}_3(\mathbb{C})$.

**Proof.** Let $S$ be a Levi subgroup of $\overline{G}$ such that $\overline{G} = R(\overline{G})S$. Our $S$ can be chosen to be defined over $\mathbb{Q}$ and the induced composite homomorphism $S \to \overline{G} \to H$ is a $\mathbb{Q}$-isogeny (cf. [4] Proof of Proposition 11.23]). Now the argument in [7] Theorem 5.1, Proposition 5.2] for the action $S(\mathbb{R}) \mid \text{NS}_R(X)$ ($\mathbb{Q}$-isogeny to $H(\mathbb{R})$) implies that either $H(\mathbb{R})$ is of real rank $\leq 1$, or $H(\mathbb{R})$ is of real rank $\geq 2$ and is locally isomorphic to $\text{SL}_3(\mathbb{R})$ or $\text{SL}_3(\mathbb{C})$. In fact, our $S(\mathbb{R})$ and indeed even the larger group $\overline{G}(\mathbb{R})$ already act on $\text{NS}_R(X)$ as the extension of the geometrically induced action.
of the Zariski-dense subgroup $G \mid NS_{\mathbb{R}}(X)$ of $\overline{G}(\mathbb{R})$, so we do not need Margulis’ condition on $G \mid NS_{\mathbb{R}}(X)$ there, for the extension of the action. To be precise, one main purpose of the extra assumption in [7] on the rank of a lattice (acting on $X$) of a semi-simple real Lie group is to extend the action of the lattice on cohomology groups of $X$ to an action of the real Lie group.

If $H$ has real rank $r_{\mathbb{R}} H = 0$, then $H(\mathbb{R})$ is compact. The image of $G$ in $H(\mathbb{R})$ is contained in an arithmetic subgroup of $H$, hence is discrete and finite. This image is Zariski-dense in $H(\mathbb{R})$. Thus $H(\mathbb{R})$ is a finite group and hence $G$ is a finite extension of $R$, so $G \mid NS_{\mathbb{C}}(X)$ is virtually solvable, which contradicts the assumption. Thus, $r_{\mathbb{R}} H$ is at least one and the lemma is proved. □

We return to the proof of Theorem 4.2. Suppose that $R$ is of null entropy. As mentioned in the proof of Theorem 2.2 (cf. [16] Proposition 2.2) the set

$$U(R) := \{g \in R; g^* \mid \text{NS}_{\mathbb{C}}(X) \text{ is unipotent}\}$$

is a finite-index subgroup of $R$. So Theorem 1.2(1) occurs by the lemma above.

Hence we may assume that $R$ is not of null entropy. We will deduce Theorem 4.2(2). Take $R_1 \leq R$ a finite-index subgroup such that $R_1 \mid \text{NS}_{\mathbb{C}}(X)$ has connected Zariski-closure in $\text{GL}(\text{NS}_{\mathbb{C}}(X))$ (and is solvable). Thus $R_1/N(R_1) \cong \mathbb{Z}^{\oplus r}$ for some $1 \leq r \leq \dim X - 1 = 2$ by [23] Theorem 1.2. If $r = 2$, then Theorem 4.2(2) holds by Theorem [7].

Thus we may assume $r = 1$, i.e., $\langle h \rangle = R_1/N(R_1) \cong \mathbb{Z}$ with $h \in R_1$ of positive entropy.

Lemma 4.4. $K_X \sim_\mathbb{Q} 0$.

Proof. Suppose to the contrary that the lemma is false. By the 3-dimensional minimal model program and abundance theorem (cf. [12] 3.13), the Kodaira dimension $\kappa(X)$ is positive, and $|mK_X|$ (for some $m > 0$) defines a holomorphic map $\psi : X \rightarrow Y$ with connected fibres and $\dim Y = \kappa(X)$. The induced action of $G$ on $Y$ is trivial if $G$ is replaced by a suitable finite-index subgroup (cf. [20] Theorem 14.10)). Hence $G$ acts faithfully on a general fibre $S$ of $\psi$. Under the identification $G \cong G \mid S$, we have $N(G \mid S) = N(G) \mid S$ (cf. [24] 2.1(11) Remark]). Since $G \neq N(G)$, our $G \mid S$ is not of null entropy. Hence $\dim S \geq 2$. Also $\dim S = \dim X - \dim Y \leq \dim X - 1 = 2$. Thus $\dim S = 2$.

In the notation above, $(R_1 \mid S)/N(R_1 \mid S) \cong \mathbb{Z}$. Hence the restrictions of $R_1$ and $R$ on $\text{NS}_{\mathbb{C}}(S)$ are virtually solvable (cf. Theorem 2.2 or 2.3). Lemma 2.8 is applicable to $R \mid S < G \mid S$ since the conditions in Lemma 2.8(2) (for surfaces) follow from the condition in Lemma 2.8(1). Indeed, by the cone theorem of Lie-Kolchin type [23] Theorem 1.6], $R_1$ (replaced by a finite-index subgroup) has a common non-zero nef eigenvector $L_1$; thus the class $h^*L_1$ is parallel to $L_1$. After switching $h$ with $h^{-1}$ if necessary, [22] Lemma 2.12] implies that $h^*L_1 = d_1(h)L_1$, and also the second condition in Lemma 2.8(2). So, by Lemma 2.8 replacing $G$ by its subgroup of index $\leq 2$, $L_h$ gives rise to a character $\chi : G \mid S \rightarrow (\mathbb{R}_{>0}, \times)$ and that (for surfaces) $\text{Ker}(\chi) = N(G \mid S)$. So the null set $N(G \mid S)$ is a subgroup and $G/N(G) = (G \mid S)/N(G \mid S) \cong \text{Im} \chi$, an abelian group. Hence $G \mid L_{\mathbb{C}}$ is virtually solvable (cf. Theorem 2.2 or 2.3), contradicting the assumption. □

In notation of Subsection 2.4 there exist two non-zero nef divisors $L_h$, $L_{h^{-1}}$ (which will be fixed) such that $(h^{\pm 1})^*L_{h^{\pm 1}} = d_1(h^{\pm 1})L_{h^{\pm 1}}$ with $d_1(h^{\pm 1}) > 1$. 

1634

FREDERIC CAMPANA, FEI WANG, AND DE-QI ZHANG
Claim 4.5. Suppose there are a nef $\mathbb{R}$-divisor $M$, a real number $\lambda \neq 1$ and an integer $s \neq 0$ such that $(h^s)^*M = \lambda M$ and $M$ is not parallel to $L_{h^{\pm 1}}$. Then Theorem 4.2(2) holds.

Proof. Note that $(h^s)^*L_h = d_1(h^s)L_h$, $(h^s)^*L_{h^{-1}} = d_1(h^{-1})^{-s}L_{h^{-1}}$, and $(h^s)^*M = \lambda^s M$. Rewriting $h^s$ as $h$, we may assume $s = 1$. We have $M.c_2(X) = h^*(M.c_2(X)) = h^*M.h^*c_2(X) = \lambda M.c_2(X)$. Hence $M.c_2(X) = 0$ for $\lambda \neq 1$. Similarly, $L_{h^{\pm 1}}.c_2(X) = 0$. By the assumption, $M.L_{h^{\pm 1}} \neq 0$ (cf. [8 Corollary 3.2]). Thus, since $M.L_h, L_{h^{-1}}$ are nef eigenvectors of $h^*$ corresponding to eigenvalues $\lambda, d_1(h), 1/d_1(h^{-1})$, and since $d_1(h) \neq 1/d_1(h^{-1})$, [8 Lemma 4.4] implies that the product of these three nef divisors is nonzero and hence the sum of these three is a nef and big divisor, perpendicular to $c_2(X)$. So Theorem 4.2(2) holds true, by Theorem 4.1(1) and Lemma 4.3.

We return to the proof of Theorem 4.2. As proved above, the closed cone $\text{Nef}(X) \cap c_2(X)^\perp = \{M \in \text{Nef}(X) | M.c_2(X) = 0\}$ contains $L_{h^{\pm 1}}$. Since $R_1 | \text{NS}_C(X)$ is solvable, the cone theorem of Lie-Kolchin type (cf. e.g. [23, Theorem 2.6]) implies that the above closed cone contains a non-zero common nef divisor $L_1$ (with $L_1.c_2(X) = 0$) of $R_1$, after $R_1$ is replaced by a finite-index subgroup. Write $g^*L_1 = \chi(g)L_1$ and consider the homomorphism

$$\varphi : R_1 \rightarrow (\mathbb{R}, +), \ g \mapsto \log \chi(g).$$

Clearly, $N(R_1) \leq \text{Ker}(\varphi)$ (cf. Subsection 2.1). If $g \in \text{Ker}(\varphi) \setminus N(R_1)$, then the product of the three nef eigenvectors $L_1, L_{g^{\pm 1}}$ (corresponding to different eigenvalues $1, d_1(g) \neq 1/d_1(g^{-1})$ of $g^*$) is nonzero by [8 Lemma 4.4] and hence the sum of these three vectors is a nef and big $\mathbb{R}$-divisor class perpendicular to $c_2(X)$. Thus, by Lemma 4.4, we can apply Theorem 4.1(1) to conclude Theorem 4.2(2).

Therefore, we may assume that $\text{Ker}(\varphi) = N(R_1)$. In particular, $\chi(h) \neq 1$, where $h^*L_1 = \chi(h)L_1$. Possibly switching $h$ with $h^{-1}$, we may assume that $\chi(h) > 1$. By Claim 4.5 we may assume that $L_1 = L_h$, which is a common eigenvector of $R_1$. Thus condition (1) of Lemma 2.8 is satisfied while condition (2) can be assumed in view of Claim 4.5. So, by Lemma 2.8 replacing $G$ by its subgroup of index $\leq 2$, we may assume that $G$ fixes $L_h$ up to scalars. Write $g^*L_h = \chi(g)L_h$. Let

$$\psi : G \rightarrow (\mathbb{R}, +), \ g \mapsto \log \chi(g)$$

so that $G/\text{Ker}(\psi)$ is mapped to an abelian subgroup of $[\mathbb{R}, +]$. If $\text{Ker}(\psi) \neq N(G)$, then as in the case of $\text{Ker}(\varphi) \neq N(G)$ above, we take $g \in \text{Ker}(\psi) \setminus N(G)$, so $c_2(X)$ is perpendicular to the nef and big divisor $L_1 + L_g + L_{g^{-1}}$, and hence Theorem 4.2(2) occurs.

Thus we may assume that $N(G) = \text{Ker}(\psi)$, which is hence a subgroup of $G$. Since $G/N(G) \cong \text{Im} \psi$ is abelian, $G | \text{NS}_C(X)$ is virtually solvable by Theorem 2.2 or 2.3, contradicting the assumption. The proof of Theorem 4.2 is completed.

4.6. Proof of Theorem 1.1. We may assume that Theorem 4.2(2) occurs and use the notation there. Let $\tilde{G}$ be the lifting to $T$ of $G | X'$ with $\tilde{G}/F = G | X'$. As sets (and set of left cosets), we have equalities $N(G)/F = N(G | X') = N(G) | X'$; so $N(G) \leq G$ if and only if $N(\tilde{G}) \leq \tilde{G}$, and if this is the case $\tilde{G}/N(\tilde{G}) \cong G/N(G)$ (cf. [21 Lemma 2.6]). Thus, by Theorem 2.3(3), as on $X$, neither $G | \text{NS}_C(X')$ nor $\tilde{G} | \text{NS}_C(T)$ is virtually solvable. By the same reasoning, the lifting to $T$ of $R(G) | X'$ has virtually solvable action on $\text{NS}_C(T)$, is normal in $\tilde{G}$ and is not of null
entropy. $R(\tilde{G})$ contains this lifting up to finite index, so it is not of null entropy. Hence we may assume that $X = T$, a complex torus.

Let $\tilde{G} \leq GL(H^0(X, \Omega_X^1)^\vee) = GL_3(\mathbb{C})$ be the Zariski-closure of the action $G \cdot H^0(X, \Omega_X^1)^\vee$. Since every $g \in G$ acts on $H^1(X, \mathbb{Z})$ invertibly, its matrix representation has determinant $\pm 1$; note that also $H^1(X, \mathbb{C}) = H^0(X, \Omega_X^1) \oplus H^0(X, \Omega_X^1)^\vee$.

Hence we may assume that $\tilde{G}$ is contained in $SL_3(\mathbb{C})$ and is connected, after $G$ is replaced by a finite-index subgroup.

Since $H^*(X, \mathbb{C}) := \bigoplus_{i \geq 0} H^i(X, \mathbb{C})$ is generated by wedge products of $H^0(X, \Omega_X^1)$ and its conjugate, the null set $N(G)$ is equal to $\{ g \in G \mid g \cdot H^0(X, \Omega_X^1) $ is of null entropy}\}.

Let $R := G \cap R(\tilde{G}) \triangleleft G$, $U := G \cap U(\tilde{G}) \triangleleft G$.

Then $R(\tilde{G}) \mid H^*(X, \mathbb{C})$ and hence $R \mid H^*(X, \mathbb{C})$ are solvable. By Theorem 2.3, $R$ has a finite-index subgroup $R_1$ such that

$$Z^{\oplus r} \cong R_1/N(R_1) \leq R/N(R).$$

Also $|N(R) : U| < \infty$ (cf. Theorem 2.2). Thus $R/U$ contains a copy of $Z^{\oplus 2}$ as a subgroup of finite index (cf. Lemma 2.4). Consider the natural embedding

$$G/U \to J := \hat{G}/U(\hat{G})$$

into the reductive group $J$ of real rank $\leq \text{rk}_\mathbb{R} SL_3(\mathbb{C}) = 2$.

If $r \geq 2$, then $\text{rk}_\mathbb{R} J = 2$. The Zariski-closure of $R/U \subset J$ contains a copy of $Z^{\oplus 2}$ and hence a maximal torus of $J$, and is normal in $J$, so this closure equals $J$. Thus $J$ (like $R/U$) and hence the actions of $G/U$ and $G$ on $H^*(X, \mathbb{C})$ are all solvable, contradicting the assumption.

Consider the case $r = 0$, i.e., $R \subseteq N(G)$. This contradicts the extra assumption that $R(G)$ is not of null entropy and the assertion (*): $R(G)$ equals $R$ up to finite index. Indeed, as in the proof of Lemma 2.6, $G \mid \text{NS}_\mathbb{C}(X) = G/K$ with $|K : G \cap \text{Aut}_0(X)| < \infty$, while $G \mid H^0(X, \Omega_X^1)^\vee = G/(G \cap \text{Aut}_0(X))$. Hence $G \mid \text{NS}_\mathbb{C}(X)$ equals $G \mid H^0(X, \Omega_X^1)^\vee$ modulo a finite group. Now the assertion (*) follows from the definitions of $R(G)$ and $R$.

Finally, assume $r = 1$, i.e., $R_1/N(R_1) \cong \mathbb{Z}$. For any $g_1 \in G$, the group $G_1 := \langle g_1, R \rangle$ (replaced by its finite-index subgroup) has $G_1 \mid H^*(X, \mathbb{C})$ solvable and $G_1/N(G_1) \cong Z^{\oplus s}$ (cf. Theorem 2.3). We claim that $s \geq 2$ for some $g_1$. If the claim is false, then for any $g_1 \in G$, we have $s \leq 1$ and hence $g_1^n = h^n$ for some $a \geq 1$, $n \in N(G)$, where $\langle h \rangle = R_1/N(R_1) \leq G_1/N(G_1)$. Thus $g_1$ (mod $R$) has a positive power acting as a unipotent element on $H^0(X, \Omega_X^1)^\vee$ because the same is true for $n \in N(G)$. So the subgroup $G/R$ of an arithmetic subgroup of $\hat{G}/R(\hat{G})$ (defined over $\mathbb{Q}$; cf. [14, ChI; 0.11]) has a unipotent group $U(G/R)$ as its subgroup of finite index, by Burnside’s theorem as in [16, Proposition 2.2] or Theorem 2.2. Thus its Zariski-closure $\hat{G}/R(\hat{G})$ is both unipotent and semi-simple and hence trivial. So $G \mid H^*(X, \mathbb{C})$ is solvable, contradicting the assumption.

Thus the claim is true and hence some $G_1 := \langle g_1, R \rangle$ has $G_1/N(G_1) \cong Z^{\oplus s}$ for some $s \geq 2$. So by [26, Paragraph before §2.8], $U(G_1)$ and hence $N(G_1)$ and $N(R)$ act as finite groups on $H^0(X, \Omega_X^1)$ and also on $H^*(X, \mathbb{C})$ (cf. Proof of Theorem 2.2).

Thus, when restricted on $H^0(X, \Omega_X^1)^\vee$, our $R$ (containing a finite-index subgroup $R_1$ with $R_1/N(R_1) \cong \mathbb{Z}$) is virtually infinite cyclic and normalized by $G$, so it is contained in the centre of $G \mid H^0(X, \Omega_X^1)^\vee$ and of $\hat{G}$, by replacing $G$ by a finite-index subgroup and considering the conjugate action on the derived series of $R$. 

Now we follow the referee’s suggestion. Take an element \( h \in R \setminus N(R) \). If \( h \mid H^0(X, \Omega^1_X)^\vee \in \text{SL}_3(\mathbb{C}) \) has three distinct eigenvectors, then \( h \) and all elements of \( G \) are simultaneously diagonalizable and hence \( G \mid H^*(X, \mathbb{C}) \) is abelian, contradicting the assumption.

Therefore, relative to a suitable basis \( B \) of \( H^0(X, \Omega^1_X)^\vee \), our \( h \mid H^0(X, \Omega^1_X)^\vee \) is in one of the Jordan canonical forms

\[
\text{block diag}[\alpha^{-2}, J_2(\alpha)], \quad \text{diag}[\alpha^{-2}, \alpha, \alpha],
\]

and the matrix representation \( g \mid H^0(X, \Omega^1_X)^\vee = (a_{ij}) \) of every \( g \in G \) is especially upper triangular. Consider the projection

\[
\tau : G \rightarrow C^*, \quad g \mapsto a_{11}.
\]

If \( \text{Ker} \tau \subseteq N(G) \), then the actions of \( \text{Ker} \tau \) and hence of \( (\text{Ker} \tau) R \) on \( H^0(X, \Omega^1_X)^\vee \) are virtually solvable (cf. Theorem 2.2), so is that of \( \text{Ker} \tau \) \( R \) is a quotient of the abelian group \( \text{Im} \tau \). This contradicts the assumption.

Thus, we can take \( g_1 \in \text{Ker} \tau \setminus N(G) \). Then \( g_1 \mid H^0(X, \Omega^1_X)^\vee \) has 3 eigenvalues \( 1, \lambda \pm 1 \) (with \( |\lambda| \neq 1 \)); it has a unique (up to scalar) eigenvector \( w \in H_1(X, \mathbb{Z}) = (\text{the lattice } \Lambda \text{ of the torus } X = \mathbb{C}^3/\Lambda) \) corresponding to the eigenvalue \( 1 \in \mathbb{Q} \) and is proportional to the column vector \( (1, 0, 0)^t \) (in basis \( B \)). Now \( h \) or \( h^{-1} \) takes \( w \) to \( \alpha^{-2}w \) with \( |\alpha^{-2}| < 1 \). This contradicts the fact that \( h(\Lambda) = \Lambda \), which is discrete in \( \mathbb{C}^3 = \mathbb{R}^6 \). Theorem 1.1 is proved.

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