FUZZY SEGMENTATIONS OF A STRING

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Abstract. This article discusses a particular case of the data clustering problem, where it is necessary to find groups of adjacent text segments of the appropriate length that match a fuzzy pattern represented as a sequence of fuzzy properties. To solve this problem, a heuristic algorithm for finding a sufficiently large number of solutions is proposed. The key idea of the proposed algorithm is the use of the prefix structure to track the process of mapping text segments to fuzzy properties.

An important special case of the text segmentation problem is the fuzzy string matching problem, when adjacent text segments have unit length and, accordingly, the fuzzy pattern is a sequence of fuzzy properties of text characters. It is proven that the heuristic segmentation algorithm in this case finds all text segments that match the fuzzy pattern.

Finally, we consider the problem of a best segmentation of the entire text based on a fuzzy pattern, which is solved using the dynamic programming method.

Keywords: fuzzy clustering, fuzzy string matching, approximate string matching

1. Introduction

This paper refers to the field of pattern recognition, which “is concerned with the automatic discovery of regularities in data through the use of computer algorithms and with the use of these regularities
to take actions such as classifying the data into different categories” [1]. Specifically, data clustering deals with the grouping of elements into clusters based on similarities defined in one way or another [2]. Due to the difficulties in accurately describing clusters in many applications, modern approaches use fuzzy clusters [3]. The data clustering involves sequence labeling that assigns categorical labels to specific parts of the sequential structure.

We consider the sequence labeling problem for the case when the sequential structure is given in the form of text, i.e., as a sequence of characters in some alphabet. The label category is defined as a pattern represented as a sequence of fuzzy properties of adjacent segments of text. To solve this problem (called the fuzzy local segmentation problem), we propose a heuristic algorithm that finds a significant part of the occurrences of a given pattern in the text. For the efficiency of the proposed algorithm, the concept of a prefix structure is introduced, which is a generalization of the concept of the prefix function array used in the KMP string matching algorithm [4]. A distinctive feature of the prefix structure is the tracking of the process of assigning adjacent text segments to pattern symbols.

In the particular case, when adjacent text segments have unit length, and the pattern, respectively, is a sequence of fuzzy properties of alphabet characters, the fuzzy local segmentation problem is transformed into a fuzzy string matching problem. The latter can be viewed in the context of approximate string matching, variations of which are distance-based string matching [5], string matching using patterns with meta-characters, and, more generally, string matching with patterns represented as regular expression [6, 7]. A detailed review of these works is presented in the monograph [8].

In this paper, we prove that the heuristic algorithm designed to solve the fuzzy local segmentation problem, adapted for the fuzzy string matching problem, finds all occurrences of a fuzzy pattern in the text. This result summarizes the previous research by the authors in the field of string matching. Particularly, in [9] the periodicity in the pattern was used to improve the efficiency of the preprocessing phase in the method of string matching with finite automata and in the KMP algorithm. In [10], a non-deterministic transition system was constructed to describe the possibilities of processing a given text in order to find all occurrences of a fuzzy pattern in it. In [11], an efficient algorithm was proposed for determining all occurrences of a fuzzy pattern in the text, imitating the KMP algorithm with a two-dimensional prefix table. The prefix structure-based solution we propose in this paper improves this result in terms of memory usage.

Another special case of the fuzzy local segmentation problem is achieved when it is necessary to split the entire text into adjacent segments of at least given length in order to best match the fuzzy pattern. We call this problem the fuzzy global segmentation problem. A special case of this problem is the Bellman’s string segmentation problem [12], in which it is required to split the text into adjacent segments so that elements from the same segments would be closely related to each other. (In contrast to this, we assume that segmentation should be done according to a fuzzy pattern in order to match it in a best possible way.) This problem was considered in [13], where an algorithm for finding an optimal solution using the dynamic programming approach was proposed. In this paper, we present this solution in a more general form.

The paper is organized as follows.

Section 2 presents the fuzzy local segmentation problem and a heuristic algorithm for solving it using the prefix structure. Section 3 introduces the fuzzy string matching problem as a special case of the fuzzy local segmentation problem. It is proven that the heuristic algorithm for the fuzzy local
segmentation problem, adapted for this case, finds all occurrences of a fuzzy pattern in the text. Section 4 presents a solution to the global fuzzy segmentation problem. Finally, the conclusion summarizes the obtained results.

2. FUZZY LOCAL SEGMENTATIONS

2.1. Preliminaries

Suppose \((L, \leq, 0, 1)\) is a linearly ordered set with the smallest element 0 and the largest element 1. According to [14], a fuzzy subset \(A\) of the universal set \(U\) is defined by the membership function \(\mu_A : U \rightarrow L\) that associates with each element \(x\) from \(U\) the value \(\mu_A(x)\) from \(L\), called the degree of membership of \(x\) in \(A\). A fuzzy subset \(A\) of \(U\) can be represented by the additive form

\[
A = \sum_{x \in U} x / \mu_A(x).
\]

We say that an element \(x\) certainly belongs to \(A\) if \(\mu_A(x) = 1\), and certainly does not belong to \(A\) if \(\mu_A(x) = 0\). Conversely, if \(0 < \mu_A(x) < 1\), then we say that \(x\) belongs to \(A\) with degree \(\mu_A(x)\).

2.2. Problem definition

Let \(\Sigma\) be an alphabet of characters and \(\Sigma^*\) be the set of all finite length strings in \(\Sigma\).

We define a fuzzy segmentation symbol (or, in short, a segmentation symbol) as a fuzzy subset of \(\Sigma^*\) that allows strings in \(\Sigma\) to be measured by elements from \(L\). Given a segmentation symbol \(\alpha\) and a string \(x \in \Sigma^*\), we say that \(x\) matches \(\alpha\) with degree \(\mu_\alpha(x)\).

A segmentation symbol \(\alpha\) is said to be regular if

- The \(\mu_\alpha(\epsilon)\) value can be computed in \(O(1)\) time, and
- For any \(c \in \Sigma\), the values \(\mu_\alpha(xc)\) and \(\mu_\alpha(cx)\) can be obtained from the value \(\mu_\alpha(x)\) in constant time with appropriate tracking of the calculation of the value \(\mu_\alpha(x)\).

It follows from this definition that for any string \(x\) and for any regular segmentation symbol \(\alpha\), the value \(\mu_\alpha(x)\) can be computed in \(O(|x|)\) time. From now on, we assume that the segmentation symbols are regular, unless otherwise stated.

Define the text as a sequence \(T[1..n]\) of characters from \(\Sigma\), where \(n\) is the length of the text. The problem of fuzzy segmentation of a text that we are considering is based on the concept of a fuzzy segmentation pattern (or, in short, a segmentation pattern), which is defined as an array \(P[1..m]\) of segmentation symbols.

We define the problem of finding a segmentation pattern in the text using 2 parameters, the first of which is a restriction on the length of the text segment, and the second is a restriction on the degree of matching. More precisely, let us define the first parameter as a pair \(\lambda = (\lambda_1, \lambda_2)\), where the numbers \(\lambda_1\) and \(\lambda_2\) determine the minimum and maximum lengths of the string to be found, and the second parameter as \(\mu \in L\) that determines the minimum degree of matching. For \(x \in \Sigma^*\) and a segmentation symbol \(\alpha\), we say that \(x\) matches \(\alpha\) and write \(x \sim \alpha\) if \(\lambda_1 \leq |x| \leq \lambda_2\) and \(\mu_\alpha(x) \geq \mu\).
Given the text \( T[1..n] \) the segmentation pattern \( P[1..m] \) and the restrictions \((\lambda, \mu)\), we define a valid \((P, \lambda, \mu)\) - segmentation of \( T \) as a sequence

\[
s_1 = [low_1, high_1], \ldots, s_m = [low_m, high_m], high_k + 1 = low_{k+1}, 1 \leq k \leq m - 1,
\]

of adjacent segments of \( T \) that meet the \((\lambda, \mu)\) - restrictions, that is \( s_k \sim P[k] \) for all \( k, 1 \leq k \leq m \).

We define the \((P, \lambda, \mu)\) - fuzzy local segmentation problem (or, in short, the \((P, \lambda, \mu)\) - segmentation problem) as the problem of finding all valid \((P, \lambda, \mu)\) - segmentations of \( T \).

Example 2.1. Let \( L \) be the segment \([0, 1]\) of ordered reals and \( \Sigma = \{0, 1\} \) be a two-element alphabet. Consider the segmentation symbols \( \alpha_0 \) and \( \alpha_1 \) such that for all \( x \in \Sigma^* \), \( \mu_{\alpha_0}(x) \) and \( \mu_{\alpha_1}(x) \) are the relative numbers of 0’s and 1’s in \( x \), respectively (it is easy to check that the specified symbols are regular).

Suppose that \( T = 101100011 \), \( P = \alpha_1 \alpha_0 \alpha_1 \), and the segmentation parameters are \( \lambda = (2, 3), \mu = 2/3 \). Then, there are the following valid \((P, \lambda, \mu)\) - segmentations of \( T \):

\[
T = (101)(100)(011), T = 1(011)(000)(11), T = 1(011)(00)(011).
\]

\( \Box \)

2.3. Brute force solution

A brute force solution to the \((P, \lambda, \mu)\) - segmentation problem can be obtained by considering all increasing sequences \( J = j_1, \ldots, j_{m+1} \) of the text positions and checking if the segmentation

\[
s_1 = (j_1, j_2 - 1), s_2 = (j_2, j_3 - 1), \ldots, s_m = (j_m, j_{m+1} - 1)
\]

generated by \( J \) is a valid \((P, \lambda, \mu)\) - segmentation.

This method of finding the valid segmentations is inefficient since the number of sequences to be considered is \( \binom{n}{m+1} \), which is exponential in \( n \) if \( m \in \Theta(n) \). Based on the KMP string matching algorithm, we propose a heuristic method for constructing a sufficiently large number of segmentations, although not necessarily all of them.

2.4. Segment capture heuristic (SC-Heuristic)

The proposed heuristic is based on the KMP string matching algorithm. Note that in the KMP algorithm, moving forward in the text is carried out by one position. On the contrary, the proposed algorithm moves forward through the text by the length of the shortest segment that satisfies the \((\lambda, \mu)\) - restriction for the next segmentation symbol. The found segment is captured in subsequent matches with symbols in the pattern.

We use the following functions to move through the text:

- \( j = lookAhead(T, i, \alpha) \). Using the global parameters \( \lambda \) and \( \mu \), this function returns the rightmost position of the shortest segment that starts at position \( i \) of \( T \), and \((\lambda, \mu)\) - matches the segmentation symbol \( \alpha \) (assume that this function returns \(-1\) if no such segment exists). It follows from the regularity of \( \alpha \) that the value \( lookAhead(T, i, \alpha) \) can be calculated in time \( O(n) \).
• **increment**\((i, j)\). This function returns \(i + 1\) if \(j = -1\), and \(j + 1\) otherwise.

Given an array \(z = z[1..k]\) and an index \(s, 1 \leq s \leq k\), we denote by \(z^*[1..s]\) the \(s\)-length postfix of \(z\) for which

\[
z^*[j] = z[k-s+j]\text{ for all }j, 1 \leq j \leq s.
\]

For a given segmentation pattern \(P = P[1..m]\), an array \(x = x[1..q]\) \((q \leq m)\) of strings in the alphabet \(\Sigma\) such that \(\lambda_1 \leq |x[i]| \leq \lambda_2, 1 \leq i \leq q\), and the minimum matching degree \(\mu\), we define the \(x\)-border of \(P\) as a subarray \(P[1..k](k < m)\) such that

\[
x^k[i] \sim P[i] \text{ for all } i, 1 \leq i \leq k
\]

(that is, the last \(k\) components of \(x\) match the first \(k\) symbols of \(P\) with a degree of at least \(\mu\)). We denote \(LB_P(x)\) the longest \(x\)-border of \(P\).

Let us define the \(x\)-prefix function for \(P\) as a mapping

\[
\pi : \{1, \ldots, q\} \rightarrow \{0, \ldots, q-1\}
\]

such that for all \(i, 1 \leq i \leq q,\)

\[
\pi(i) = \text{size}(LB_P(x[1..i])).
\]

Additionally, suppose that \(\pi(0) = 0\).

**Example 2.2.** In addition to the segmentation symbols \(\alpha_0\) and \(\alpha_1\) defined in Example 2.1, let us introduce the segmentation symbols \(\alpha_2\) and \(\alpha_3\) so that for all \(x \in \Sigma^*\), \(\mu_{\alpha_2}(x)\) and \(\mu_{\alpha_3}(x)\) are the maximum number of consecutive 0’s and 1’s divided by \(|x|\), respectively. The regularity of \(\alpha_2\) and \(\alpha_3\) follows from the fact that the values of \(\mu_{\alpha_2}(xc)\) (resp., \(\mu_{\alpha_2}(cx)\)) and \(\mu_{\alpha_3}(xc)\) (resp., \(\mu_{\alpha_3}(cx)\)) can be obtained from \(\mu_{\alpha_2}(x)\) and \(\mu_{\alpha_3}(x)\) by keeping track of the number of consecutive 0’s and 1’s along with the number of last (resp., first) 0’s and 1’s in \(x\).

Suppose \(m = q = 6, P = \alpha_3\alpha_1\alpha_0\alpha_2\alpha_1\alpha_3, x = <010, 110, 101, 001, 011, 11>, \mu = 2/3\). Then, the \(x\)-prefix function for \(P\) will be defined as follows:

\[
\pi(1) = 0, \pi(2) = 1, \pi(3) = 2, \pi(4) = 3, \pi(5) = 1, \pi(6) = 2.
\]

For a given segmentation pattern \(P[1..m]\), let us define the **\(P\)-based prefix structure** as a triplet \(\Pi = <q, x, \pi>\), where

- \(q\) is the length of the structure, \(0 \leq q \leq m,\)
- \(x = x[1..q]\) is a \(q\)-length array of strings in the alphabet \(\Sigma\) such that

\[
\lambda_1 \leq |x[i]| \leq \lambda_2, 1 \leq i \leq q,
\]

- \(\pi = \pi[1..q]\) is a \(q\)-length array of values of the \(x\)-prefix function for \(P\).
Let us define two basic operations over the prefix structure \( \Pi = \langle q, x, \pi \rangle \):

- **reduce(\(\Pi\)).** This operation converts the \(P\)-based prefix structure \(\Pi' = \langle q', x', \pi' \rangle\) such that \(q' = k, x' = x^k, \pi' = \pi[1..k]\), where \(k = \pi[q]\). As a result of performing this operation, the length of the non-empty prefix structure is decreased by at least 1.

- **extend(\(\Pi, y\)), \(y \in \Sigma^*, \lambda_1 \leq |y| \leq \lambda_2\).** Assuming that \(q < m\), this operation converts the \(P\)-based prefix structure \(\Pi\) to the \(P\)-based prefix structure \(\Pi' = \langle q', x', \pi' \rangle\) such that
  
  - \(q' = q + 1,\)
  - \(x'[i] = x[i]\) for \(1 \leq i \leq q, x'[q + 1] = y,\)
  - \(\pi'[i] = \pi[i]\) for \(1 \leq i \leq q, \pi'[q'] = k + 1,\)

  where \(k\) is the first element of the sequence \(\pi[q], \pi^2[q] = \pi[\pi[q]], \pi^3[q] = \pi[\pi^2[q]], \ldots\) for which \(\mu_{P[k+1]}(y) \geq \mu\). Additionally, suppose that \(k = -1\) if there is no such element.

As a result of performing this operation the length of the prefix structure is increased by 1.

Assuming the arrays \(x\) and \(\pi\) are organized as **multi-queue** and **multi-stack** respectively, with operations

- **\(\Box\ push(element)\):** inserts element into multi-queue/multi-stack,
- **\(\Box\ mulipop(quantity)\):** removes quantity elements from multi-queue/multi-stack,
consider the following implementations of these operations:

---

**Algorithm 1: reduce** // Reduces a prefix structure

**Input:** A prefix structure $\Pi = \langle q, x, \pi \rangle$

**Output:** The $\Pi$ is reduced

1. $k = q - \pi[q]$
2. $q = \pi[q]$
3. $x.multipop(k)$; //leaves the last $\pi[q]$ elements of $x$
4. $\pi.multipop(k)$; //leaves the first $\pi[q]$ elements of $\pi$

---

**Algorithm 2: extend** // Extends a prefix structure

**Input:**
- A prefix structure $\Pi = \langle q, x, \pi \rangle$, where $q < m$
- A $y \in \Sigma^*$ such that $\lambda_1 \leq |y| \leq \lambda_2$, $\mu_{P[q+1]}(y) \geq \mu$
  - $(\lambda = (\lambda_1, \lambda_2)$ and $\mu$ are global parameters)

**Output:** The $\Pi$ is extended by $y$

1. $x.push(y)$; //adds $y$ to the multi-queue $x$
2. $k = (q > 0) ? \pi[q] : 0$
3. while $k > 0$ and $\mu_{P[k+1]}(y) < \mu$ do
   4. $k = \pi[k]$
5. end
6. $\pi.push((\mu_{P[k+1]}(y) \geq \mu) ? k + 1 : 0)$; //adds new element to the multi-stack $\pi$
7. $q = q + 1$;

---

The prefix structure based implementation of the $SC$ – Heuristic algorithm using the *reduce* and *extend* operations is shown in Figure 1.
Algorithm 3: $SC – Heuristic$  // Prints a set of valid segmentations of $T$

Input:
Text $T = T[1..n]$, pattern $P = P[1..m]$, global parameters $\lambda$ and $\mu$ 

Output: A set of valid $(P, \lambda, \mu)$ - segmentations of $T$

1. $\Pi = \langle 0, \text{empty, empty} \rangle$; // current prefix structure
2. $i = 1$; // current position in the text
3. while $(k = \text{length} (\Pi)) > 0$ and $\text{lookAhead}(T, i, P[k+1]) == -1$ do
4.    reduce ($\Pi$);
5.   end
6. if $(j = \text{lookAhead}(T, i, P[k+1])) \neq -1$ then
7.   extend ($\Pi, T[i..j]$);
8. end
9. if length ($\Pi$) == $m$ //entire pattern matched then
10.   Print ($\Pi.x$);
11.   reduce ($\Pi$);
12. end
13. increment ($i, j$);
14. end

Figure 1. Prefix structure based implementation of the $SC – Heuristic$ algorithm.

Example 2.3. Suppose the segmentation symbols $\alpha_0, \alpha_1, \alpha_2$ and $\alpha_3$ are defined as in Example 2.2

$T = 010110010100110011$, $P = \alpha_0 \alpha_1 \alpha_2 \alpha_3$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\mu = 2/3$.

Consider the following description of the processing of these data by the $SC – Heuristic$ algorithm:

$\diamond$ $\alpha_0 \alpha_1 \alpha_2$ matches $T[1..8]$, generating the segmentation $T[1..8] = (010)(11)(100)$.

$\diamond$ Then, we have a mismatch for $\alpha_3$ in position $i = 9$.

$\diamond$ Since $\pi[3] = 1$, we fix the matching $\alpha_0$ with the segment $T[6..8] = (100)$ and continue processing for $T[9..20]$ and $P[2..4]$.

$\diamond$ $\alpha_1 \alpha_2 \alpha_3$ matches $T[9..15]$, generating the first valid segmentation

$T[6..8] = (100) \sim \alpha_0, T[9..11] = (101) \sim \alpha_1, T[12..13] = (00) \sim \alpha_2, T[14..15] = (11) \sim \alpha_3$.

$\diamond$ Since $\pi[4] = 2$, we fix the matching $\alpha_0 \alpha_1$ with $T[12..15] = (00)(11)$ and continue processing for $T[16..20]$ and $P[3..4]$.

$\diamond$ $\alpha_2 \alpha_3$ matches $T[16..20]$, generating the second valid segmentation

$T[12..13] = (00) \sim \alpha_0, T[14..15] = (11) \sim \alpha_1$,
T[16..18] = (100) \sim \alpha_2, T[19..20] = (11) \sim \alpha_3.

\diamond As a result, the SC – Heuristic algorithm generates the following two valid \((P, \lambda, \mu)\) - segmentations of \(T\):

\[
T = 01011(100 \sim \alpha_0)(101 \sim \alpha_1)(00 \sim \alpha_2)(11 \sim \alpha_3)11011.
\]

\[
T = 01011100101(00 \sim \alpha_0)(11 \sim \alpha_1)(100 \sim \alpha_2)(11 \sim \alpha_3).
\]

\diamond Note that the following \((P, \lambda, \mu)\) - valid segmentation of \(T\) is not found by the SC – Heuristic algorithm:

\[
T = 010111(00 \sim \alpha_0)(101 \sim \alpha_1)(00 \sim \alpha_2)(11 \sim \alpha_3)11011.
\]

(See illustration in Figure 2)

Figure 2. Illustration of the execution of the SC – Heuristic algorithm.

□

ANALYSIS

Let us use the potential method to estimate the complexity of the SC – Heuristic algorithm. Define the potential before each execution of the body of the external while loop to be \(\text{length}(\Pi) = \Pi.q\), which is initially 0 and never becomes negative. Considering that the text segments included in \(\Pi\) can be identified by pairs of indices, suppose that the elements of multi-queue \(x\) are pairs of index values.

Ignoring for now the lookAhead(…) function call in line 5, we can argue that the while loop in lines 5-7 has \(O(1)\) amortized complexity, since the actual cost of an iteration is compensated by a decrease in potential.

The extend(…) operation in line 9 has \(O(m\lambda_2)\) amortized complexity. Indeed, its actual cost is \(O(m\lambda_2)\) due to \(O(m)\) iterations of the while loop in lines 3-5 of the extend(…) procedure, at each of which we calculate the matching degree in \(O(\lambda_2)\) time. At the same time, this operation increases the potential by 1, which gives \(O(m\lambda_2)\) for both actual and amortized costs.

Finally, if statement in lines 11-14 has \(O(m)\) amortized complexity due to \(O(m)\) actual cost of the Print(…) operation and \(O(1)\) amortized cost of the reduce(…) procedure.
Thus, we get the $O(m\lambda_2)$ amortized complexity for the body of the outermost while loop. Since it is executed $O(n/\lambda_1)$ times, we have $O(mn\lambda_2/\lambda_1)$ total complexity ignoring $\text{lookAhead}(...)$ function calls.

It follows from the regularity of the segmentation symbols that a single call to $\text{lookAhead}(...)$ function takes $O(\lambda_2)$ time. The total number of calls is $O(n/\lambda_1)$ since each call to this function in line 5 is accompanied by a decrease in potential. Obviously, the total number of calls to the same function in line 8 is again $O(n/\lambda_1)$. Thus, we get the total time $O(n\lambda_2/\lambda_1)$ for all calls to the $\text{lookAhead}(...)$ function.

Summarizing the above, we conclude that the $\text{SC} - \text{Heuristic}$ algorithm has time complexity $O(mn\lambda_2/\lambda_1) + O(n\lambda_2/\lambda_1) = O(mn\lambda_2/\lambda_1)$.

The algorithm uses $O(m)$ extra memory required to represent the prefix structure $\Pi$.

It is important to figure out what fraction of valid segmentations the $\text{SC} - \text{Heuristic}$ algorithm is guaranteed to recognize. The following statement sheds light on this question.

**Proposition 2.4.** There is an extreme case of behavior of the $\text{SC} - \text{Heuristic}$ algorithm when it finds only $1$ of $\lambda_2$ valid $(P, \lambda, \mu)$ - segmentations of $T$.

**Proof:**
Let us define the segmentation symbols $\alpha_1, \ldots, \alpha_m$ and the minimum matching degree $\mu$ so that the string $x \in \Sigma^*$ satisfies the property $\alpha_i$ iff $|x| = \lambda_2$ and only one position of $x$ contains the value $i$.

Suppose that $T = (0^{\lambda_2-1})(0^{\lambda_2-1}2)(0^{\lambda_2-1}m)...(0^{\lambda_2-1}0^{\lambda_2-1})P = \alpha_1...\alpha_m$.

In this case, the algorithm produces a single valid segmentation of $T$ (starting at position 1), while there are $\lambda_2$ valid segmentations starting at positions $1, \ldots, \lambda_2$, respectively.

3. FUZZY STRING MATCHING

3.1. Problem definition
Let us consider a specific case of the fuzzy segmentation problem when $\lambda_1 = \lambda_2 = 1$. In this case, we rename the fuzzy segmentation symbol to fuzzy symbol that can be defined as a fuzzy subset of $\Sigma$. Similarly, we rename the fuzzy segmentation pattern to fuzzy pattern that is defined as a sequence $P[1..m]$ of fuzzy symbols of length $m$. Finally, the $(P, \lambda, \mu)$ - fuzzy segmentation problem we rename to the $(P, \mu)$ - fuzzy string matching problem and formulate it as follows.

Given text $T[1..n]$, fuzzy pattern $P[1..m]$ and threshold $\mu$, find all positions $s, 1 \leq s \leq n-m+1$, (hereinafter $(P, \mu)$ - match positions) in $T$ such that

$$\mu_{P[k]}(T[s + k - 1]) \geq \mu,$$

for all $k, 1 \leq k \leq m$.

**Example 3.1.** Let us choose $\Sigma = \{1, 2, 3, 4, 5\}, L = \{0, 0.25, 0.5, 0.75, 1\}$ and define the fuzzy symbols $S(\text{small}), M(\text{medium})$ and $L(\text{large})$ as follows:

$$S = 1/1 + 2/0.75 + 3/0.5 + 4/0.25 + 5/0,$$
Then for \( P = SMSL, T = 13231425 \) and \( \mu = 0.75 \) there are two \((P, \mu)\) - match positions in \( T \), which are \( s = 3 \) and \( s = 5 \).

\[ M = 1/0 + 2/0.75 + 3/1 + 4/0.75 + 5/0, \]

\[ L = 1/0 + 2/0.25 + 3/0.5 + 4/0.75 + 5/1. \]

The fuzzy string matching problem is a direct generalization of the classical string matching problem. It was investigated in [10], where a non-deterministic transition system was constructed to describe the possibilities of processing a given text to find all occurrences of a fuzzy pattern in it, and an efficient algorithm was proposed to determine a certain part of the occurrences.

3.2. Solution to the fuzzy string matching problem

In [11], an \( O(mn) \)-time algorithm was proposed to solve the fuzzy string matching problem using a two-dimensional prefix table, which is a generalization of the one-dimensional prefix array used in the KMP algorithm. In this section, we propose a new \( O(mn) \)-time algorithm to the same problem, which is the result of customization of the \( SC - Heuristic \) algorithm. Unlike the algorithm from [11], which has \( O(mn) \)-space complexity, the proposed algorithm is more efficient in terms of memory usage and has \( O(m) \)-space complexity.

Let us note a number of simplifications in the \( SC - Heuristic \) algorithm and related data structures, which result in a solution to the fuzzy string matching problem.

- In the prefix structure \( \Pi = \langle q, x, \pi \rangle \), \( x \) becomes an array of characters from \( \Sigma \) instead of an array of strings in \( \Sigma \),

- The condition \( \text{lookAhead}(T, i, P[k+1]) = -1 \) in line 5 becomes \( \mu_{P[k+1]}(T[i]) < \mu \),

- The if statement in lines 8-10 becomes

\[
\text{if } \mu_{P[k+1]}(T[i]) \geq \mu \text{ then extend}(\Pi, T[i])
\]

end

- \( \text{Print}(\Pi.x) \) procedure in line 12 can be simplifies as \( \text{Print}(i - m + 1) \),

- The increment \((i, j)\) statement in line 15 becomes \( i = i + 1 \).

The simplified \( SC - Heuristic \) algorithm renamed to \( Fuzzy - String - Matching \) is presented in Figure 3.
Algorithm 4: Fuzzy - String - Matching // Prints all (P, µ) - match positions in T

Input:
Text T = T[1..n], pattern P = P[1..m] and global parameter µ

Output: All (P, µ) - match positions in T

1  Π = < 0, empty, empty >; //current prefix structure
2  for i = 1 to n do
3    while (k = length(Π)) > 0 and (µP[k+1](T[i]) < µ) do
4      reduce(Π);
5    end
6    if (µP[k+1](T[i]) ≥ µ) then
7      extend(Π, T[i]);
8    end
9  if length(Π) == m //entire pattern matched then
10    Print(i - m + 1);
11    reduce(Π);
12  end
13 end

Figure 3. The Fuzzy - String - Matching algorithm.

CORRECTNESS

For the q-length prefix structure Π, we denote by Π* the sequence Π0, Π1, ..., Πq of prefix structures such that Π0 = Π, and for all i, 1 ≤ i ≤ q, Πi = reduce(Πi−1).

The following lemma is a generalization of the prefix-function iteration lemma of the KMP algorithm [4].

Lemma 3.2. (Prefix-structure iteration lemma)
Let Π = < q, x, π > be a P-based prefix structure of length q. Then P[1..k] is the x-border of P for k < q iff there exists i, 1 ≤ i ≤ q, such that Πi = < k, xk, π[1..k] >.

Proof:
We first prove that if Πi = < k, xk, π[1..k] >, 1 ≤ i ≤ q, then P[1..k] is an xk - border of P.

If i = 1, then this follows from Π1 = reduce(Π0) and the definition of the reduce operation.

If i > 1, then Πi = reduce(Πi−1), Πi−1 = < s, xs, π[1..s] >. Assuming by induction that P[1..s] is an xs - border of P, we get that P[1..π[s]] is an xπ[s] - border of P.

On the other hand, suppose, on the contrary, that j is the largest integer such that j < q, P[1..j] is the x-border of P, but no prefix structure in Π* has length j. Let j' denote the smallest integer in the sequence q, π[q], π2[q], ..., πq[q] that is greater than j. But in this case Πj'+1 = reduce(Πj') must have length j, which contradicts our assumption. □

Theorem 3.3. The Fuzzy - String - Matching algorithm finds all (P, µ) - match positions in T.
Remark 3.4. This statement is consistent with the Proposition 2.4 for $\lambda = 1$.

Proof:
For each $i, 1 \leq i \leq n$, denote by $\nu(i) \geq 0$ the length $k$ of the maximum proper prefix of $P$ such that $T[i−k+1..i]$ matches $P[1..k]$ and which we have in position $i$ during the processing of the algorithm. Accordingly, we call the position $p$ the starting point for the algorithm if $p = i − \nu(i) + 1$ for some position $i$.

Suppose there is a match position $s$ not found by the algorithm. It follows from this assumption that there should be two starting points $p_1$ and $p_2$, sequentially processed by the algorithm, such that $p_1 < s < p_2$. Suppose that $q_1 \geq p_2 − 1$ is the endpoint for $p_1$, that is $p_1 = q_1 − \nu(q_1) + 1$. We have that $T[p_2..q_1]$ matches $P[1..q_1 − p_2 + 1]$, which means that $P[1..q_1 − p_2 + 1]$ is a $T[p_1..q_1]$ - border of $P[1..m]$. On the other hand, $T[s..q_1]$ matches $P[1..q_1 − s + 1]$, since $s$ is a match position. Thus, we get that there is a $T[p_1..q_1]$ - border of $P[1..m]$ greater than $P[1..q_1 − p_2 + 1]$ not found by the algorithm.

According to the Lemma 3.2 we get a contradiction.

Example 3.5. Consider how the Fuzzy – String – Matching algorithm processes the text $T$ using the pattern $P$ and the minimum matching degree $\mu$, where

$T[1..8] = 13231425, P[1..4] = SMSL, \mu = 0.75.$

○ $T[1..3] = 132$ matches $P[1..3] = SMS$.

○ Then, we have a mismatch for $L$ in position 4.

○ Since $\pi[3] = 1$, we continue processing for $T[4..8] = 31425$ and $P[2..4] = MSL$.

○ $T[4..6] = 314$ matches $P[2..4] = MSL$, generating the first match position 3.

○ Since $\pi[4] = 2$, we continue processing for $T[7..8] = 25$ and $P[3..4] = SL$.

○ $T[7..8] = 25$ matches $P[3..4] = SL$, generating the second match position 5.

(See illustration in Figure 4.)

□

ANALYSIS

As follows from the analysis of the SC – Heuristic algorithm for $\lambda_1 = \lambda_2 = 1$, the Fuzzy – String – Matching algorithm has time complexity $O(mn)$ and space complexity $O(m)$. □

4. FUZZY GLOBAL SEGMENTATIONS

4.1. Problem definition

In this problem, we remove the $\lambda_2$ restriction on the maximum length of substrings and consider the problem of splitting the entire string into $m$ substrings with the minimum length $\lambda_1$ for optimal
matching to the segmentation pattern \( P[1..m] \). For a more accurate assessment of the quality of segmentation, we add an accumulation operation to the lattice \( L \), and instead of the minimum degree of matching of text segments with pattern symbols, use the value of the accumulation operation applied to text segments. In [14], a dynamic programming algorithm was proposed to find a best solution for a particular case of this problem. Below we formulate it in a more general form and present an algorithm for constructing a best solution.

Assume that the binary monotonic accumulation operation \( \otimes \) is defined on the set of measures \( L \), so that \( (L, \leq, 0, 1, \otimes) \) is a commutative monoid with respect to \( \otimes \), with neutral element 1 and zero element 0. That is, for all \( a, b, c \in L \),

\[
a \otimes 0 = 0, \\
a \otimes 1 = a, \\
a \leq b \Rightarrow a \otimes c \leq b \otimes c.
\]

For a text \( T = T[1..n] \), an integer \( m \) and a restriction \( \lambda \) such that \( m\lambda \leq n \), define the \((m, \lambda)\)-decomposition of \( T \) as any sequence

\[
t_1 = T[j_0 + 1..j_1], t_1 = T[j_1 + 1..j_2], \ldots, t_m = T[j_m - 1 + 1..j_m]],
\]

\[
j_{k+1} - j_k \geq \lambda(0 \leq k \leq m-1), j_0 = 0, j_m = n,
\]

of \( m \) adjacent segments of \( T \) of at least \( \lambda \) length that cover \( T \).

Given text \( T \), segmentation pattern \( P = P[1..m] \) and restriction \( \lambda \), we define the \((P, \lambda)\)-fuzzy global segmentation problem (or, in short, the \((P, \lambda)\)-global segmentation problem) as an \((m, \lambda)\)-decomposition \( t_1, \ldots, t_m \) of \( T \) that maximizes the value

\[
\mu_P(t_1, \ldots, t_m) = \bigotimes_{i=1}^{m} \mu_{P[i]}(t_i).
\]

Let us denote

\[
\sigma_{P, \lambda}(T) = \max\{\mu_P(t_1, \ldots, t_m) | \text{for all } (m, \lambda)\text{-decompositions of } T \text{ into substrings } t_1, \ldots, t_m\}.
\]
Example 4.1. Suppose that the segmentation symbols $\alpha_0$ and $\alpha_1$, as previously, are defined as the relative number of 0’s and 1’s, respectively, $T = 101110001101$, $P = \alpha_1 \alpha_0 \alpha_1$, $\lambda = 2$. Suppose also that $L = [0, 1]$ is the segment of real numbers and the operation $\otimes$ is the multiplication.

In this case, the only solution to the global segmentation problem is the decomposition $T = t_1 t_2 t_3$, where

$$t_1 = 10111, t_2 = 000, t_3 = 1101$$

with $\mu_P(t_1, t_2, t_3) = (4/5) \cdot (3/3) \cdot (3/4) = 3/5$.

Note that when $\lambda = 1$, then there is another solution $T = t'_1 t'_2 t'_3$, where

$$t'_1 = 1, t'_2 = 0, t'_3 = 1110001101.$$

□

4.2. Solution to the global segmentation problem

Theorem 4.2. (Optimal substructure of the global segmentation problem)

Suppose $n \geq m\lambda$. Then

1. $m = 1 \Rightarrow [\sigma_{P,\lambda}(T) = \mu_{P[1]}(T[1..n])]$.

2. $2 \leq m \leq n \Rightarrow [\sigma_{P,\lambda}(T) = \max\{\sigma_{P[1..m-1],\lambda}(T[1..k-1]) \otimes \mu_{P[i]}(T[k..n])\}$

   for all $k, (m-1)\lambda + 1 \leq k \leq n - \lambda + 1$.

Proof:
The first statement is obvious. The second statement follows from the fact that for a solution $t = t_1, ..., t_m$ to the $(P, \lambda)$-global segmentation problem for $T$, we have that

- $P[m]$ must match a substring $T[k..n]$ for some value $k$ such that $n-k+1 \geq \lambda$ (to have the last segment of at least $\lambda$ length), and $k-1 \geq (m-1)\lambda$ (to have solution to the $(P[1..m-1], \lambda)$-global segmentation problem for $T[1..k-1]$).

- The segmentation $t_1, ..., t_{m-1}$ should be a solution to the $(P[1..m-1], \lambda)$-global segmentation problem for $T[1..k-1]$, since otherwise, the solution $t$ can be improved.

□

Recursive computation of $\sigma_P(T)$ based on Theorem 4.2 will be inefficient due to overlapping subproblems. To avoid this, let us use the dynamic-programming approach in the bottom-up version.

For $1 \leq i \leq m, i\lambda \leq j \leq n$, denote $s[i, j] = \sigma_{P[1..i],\lambda}(T[1..j]) \in L$. The optimal substructure of the global segmentation problem dictates the following recurrent equation for calculating $s[i, j]$:

$$s[i, j] = \begin{cases} 
\mu_{P[1]}(T[1..j]), & \text{if } i = 1, 1 \leq j \leq n \\
\max_{\{k|(i-1)\lambda+1 \leq k \leq j-\lambda+1\}} (s[i-1, k-1] \otimes \mu_{P[i]}(T[k..j])), & \text{if } 2 \leq i \leq m, i\lambda \leq j \leq n \\
\text{undefined}, & \text{otherwise}
\end{cases} \quad (4.2.1)$$
The optimal cost of the global segmentation is obviously $s[m, n]$. To construct an optimal segmentation as well, let us maintain the value $b[i, j], 2 \leq i \leq m, i\lambda \leq j \leq n$, equal to the index $k$ maximizing the value $s[i - 1, k - 1] \otimes \mu_{P[i]}(T[k..j])$ in formula (4.2.1).

The memoization and construction phases of the proposed algorithm for solving the $(P, \lambda)$-global optimization problem are provided in Figures 5 and 6.

**Algorithm 5: GS – Memoization**// Constructs the auxiliary matrices $s$ and $b$

**Input:**
Text $T = T[1..n]$, pattern $P = P[1..m]$ and global parameter $\lambda$

**Output:**
The $L$-value matrix $s[1..m, 1..n]$ and the integer matrix $b[2..m, 1..n]$

1. for $j = \lambda$ to $n$ do
2.     $s[1, j] = \mu_{P[1]}(T[1..j])$;
3. end

4. for $i = 2$ to $m$ do
5.     for $j = i\lambda$ to $n$ do
6.         $s[i, j] = 0$;
7.         for $(k = j - \lambda + 1$ downto $(i - 1)\lambda + 1$ do
8.             $r = s[i - 1, k - 1] \otimes \mu_{P[i]}(T[k..j])$;
9.             if $r > s[i, j]$ then
10.                $s[i, j] = r, b[i, j] = k$;
11.            end
12.        end
13.    end
14. end
15. return $s$ and $b$;

Figure 5. Construction of the $L$-value matrix $s$ and the integer matrix $b$. 
Algorithm 6: \textit{GS−Print} // Prints a solution to the \((P[1..i], \lambda)\) - glob. segm. problem for \(T[1..j]\)

\begin{itemize}
  \item \textbf{Input:}
    \begin{itemize}
      \item Global parameter \(\lambda\)
      \item Matrix \(b\) and indices \(i\) and \(j\) such that \(i\lambda \leq j \leq n\)
    \end{itemize}
  \item \textbf{Output:}
    \begin{itemize}
      \item The pairs of indices determining a best solution to the \((P[1..i], \lambda)\) - global segmentation problem for \(T[1..j]\)
    \end{itemize}
\end{itemize}

\begin{enumerate}
  \item \textbf{if} \(i == 1\) \textbf{then}
  \item \quad \textbf{Print}(1, j);
  \item \textbf{end}
  \item \textbf{else}
  \item \quad \textbf{GS−Print}(b, i − 1, b[i, j] − 1);
  \item \quad \textbf{Print}(b[i, j], j);
  \item \textbf{end}
\end{enumerate}

Figure 6. Extraction a solution based on the matrix \(b\).

Remark 4.3. The initial call to the \textit{GS−Print} procedure is \textit{GS−Print}(\(b, m, n\)).

Example 4.4. With the initial data taken from the Example 4.1, the \textit{GS−Memoization} procedure creates the matrices \(s\) and \(b\) shown in Figure. [7]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\(T\): & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
\hline
\(\alpha_1\) & - & 1/2 & 2/3 & 3/4 & 4/5 & 2/3 & 4/7 & 1/2 & 5/9 & 3/5 & 6/11 & 7/12 \\
\hline
\(\alpha_0\) & - & - & - & 0 & 0 & 3/8 & 4/5 & 4/5 & 3/5 & 12/25 & 8/15 & 16/35 \\
\hline
\(\alpha_1\) & - & - & - & - & - & 0 & 0 & 0 & 2/5 & 4/5 & 8/15 & 3/5 \\
\hline
\end{tabular}
\caption{Matrix \(s\)}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
2 & - & - & - & 0 & 0 & 5 & 6 & 6 & 6 & 6 & 6 & 6 \\
3 & - & - & - & - & 0 & 0 & 0 & 8 & 9 & 9 & 9 & 9 \\
\hline
\end{tabular}
\caption{Matrix \(b\)}
\end{table}

The procedure \textit{GS−Print} processing the matrix \(b\), prints the pair \((b[3, 12], 12) = (9, 12)\) last; the pair \((b[2, 8], 8) = (6, 8)\) in penultimate; the pair \((1, 5)\) first. These pairs correspond to the following...
optimal \((m = 3, \lambda = 2)\) decomposition of \(T\):

\[
t_1 = T[1..5] = 10111, t_2 = T[6..8] = 000, t_3 = T[9..12] = 1101,
\]

with \(\mu_P(t_1, t_2, t_3) = \sigma_P(T) = s[3, 12] = 3/5. \)

\[\square\]

5. CONCLUSION

The paper considers the problem of text segmentation according to a fuzzy pattern. This problem is being investigated in the following two aspects:

- As a fuzzy segmentation problem aimed at finding text segmentations matching the pattern with given lower and upper limits on the length of the segmentation units, and

- As a fuzzy decomposition problem aimed at decomposing the entire text into adjacent segments in order to best match the pattern, with a given lower limit on the length of the decomposition units.

For the fuzzy segmentation problem, a heuristic algorithm is proposed for finding a sufficiently large number of occurrences of a pattern in the text. In the special case, when it is required that segments have a unit length, this problem is transformed into the fuzzy string matching problem, when the occurrence of a pattern in the text means that there is a segment in the text having a length equal to the length of the pattern, with one-to-one correspondence between segment characters and pattern symbols. It is proven that the heuristic segmentation algorithm adapted for this particular case finds all occurrences of the pattern in the text.

For the fuzzy decomposition problem, an algorithm for finding a best solution is developed using the dynamic programming approach.

All proposed algorithms are implemented and verified on test cases.

If \(n\) and \(m\) are the text and the pattern lengths, respectively, \(\lambda_1\) and \(\lambda_2\) are limits on segment length, then the proposed algorithms have the following time and space complexities:

- Fuzzy segmentation problem: \(O(mn\lambda_2/\lambda_1), O(m)\),
- Fuzzy string matching problem: \(O(mn), O(m)\),
- Fuzzy decomposition problem: \(O(mn^2), O(mn)\).
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