REALIZABILITY OF HYPERGRAPHS AND RAMSEY LINK THEORY

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Abstract. We present short simple proofs of Conway-Gordon-Sachs’ theorem on graphs in 3-dimensional space, as well as van Kampen-Flores’ and Ummel’s theorems on nonrealizability of certain hypergraphs (or simplicial complexes) in 4-dimensional space. The proofs use a reduction to lower dimensions which allows to exhibit relation between these results.

We present a simplified exposition accessible to non-specialists in the area and to students who know basic geometry of 3-dimensional space and who are ready to learn straightforward 4-dimensional generalizations. We use elementary language (e.g. collections of points) which allows to present the main ideas without technicalities (e.g. without using the formal definition of a hypergraph).

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It’s too difficult.’
‘Write simply.’
‘That’s hardest of all.’
I. Murdoch, The Message to the Planet.

1. Introduction

1.1. Impossible constructions and intrinsic linking. ‘Impossible constructions’ like the impossible cube, the Penrose triangle, the blivet etc (see Figure 1 and [Io]) are well-known, mainly due to pictures by M.C. Escher, see also [Br26, CKS+, GSS+]. The pictures do not allow the global spatial interpretation because of collision between local spatial interpretations to each other. In geometry, topology and graph theory there are also famous basic examples of ‘impossible constructions’ (of which local parts are ‘possible’).

Figure 1. The impossible cube, the Penrose triangle, the blivet, an impossible projection

In this paper we exhibit a striking relation of ‘impossible constructions’ in four-dimensional space to ‘intrinsic linking’ results in three-dimensional space. Such a relation was found by M. Skopenkov in [Sk03] and used there to obtain a short proof of the Menger 1929 conjecture and its generalizations, see Remark 1.5 and §1.4. Let us give a beautiful example of ‘intrinsic linking’.

We abbreviate ‘three-dimensional space $\mathbb{R}^3$’ to ‘3-space’. Analogous meaning has ‘4-space’.

By a triangle we mean ‘the interior’ of a triangle (more accurately, the convex hull of three points).

Take two triangles in 3-space no 4 of whose 6 vertices lie in the same plane. The triangles are called linked, if the outline of the first triangle intersects the second triangle exactly at one point. It is not obvious from the definition that the property of being linked is symmetric. For a proof see e.g. [Sk, Symmetry Lemma 4.2].

E.g. the triangles $A_1A_3A_5$ and $A_2A_4A_6$ in Figure 2 are linked. (The distance from the point $A_j$ to the projection plane equals $j$, see Figure 2 left. So the projection in Figure 2 right, is realizable, as opposed to Figure 1 right.)

**Theorem 1.1** (Linear Conway–Gordon–Sachs Theorem; [Sa81, CG83]). If no 4 of 6 points in 3-space lie in the same plane, then there are two linked triangles with vertices at these 6 points.

1A subset of the plane or of $\mathbb{R}^d$ is called convex, if for any two points from this subset the segment joining these two points is in this subset. The convex hull of a subset $X$ of the plane or $\mathbb{R}^d$ is the minimal convex set that contains $X$. 
Moreover, the number of linked unordered pairs of triangles with vertices at these 6 points is odd.

See idea of a short proof in Remark 1.5. Formally, Theorem 1.1 is reduced to Proposition 1.2 below in §2.2. See more results on linking in 3-space in §2.2 and in [Sk] §4.1 ‘Linking of triangles in three-dimensional space’.

1.2. Realizability of hypergraphs. Another example of an ‘impossible construction’ is that one cannot construct 3 houses and 3 wells in the plane and join each house to each well by a path so that paths intersect only at their starting points or endpoints.

Moreover, if no 3 of 5 points in the plane lie in the same line, then the number of intersection points of interiors of segments joining the 5 points is odd.

In this paper we present a natural interesting generalization: beautiful and nontrivial examples of two-dimensional analogues of graphs non-realizable in three- and four-dimensional space.

Remark 1.3 (why this expository paper might be interesting). We present a simplified exposition accessible to non-specialists in the area, see also the second paragraph of §1.6. We state the examples in terms of certain systems of points, see Theorem 1.4 below. So we

\[ K_5 \quad \text{and} \quad K_{3,3} \]

\[ \text{Figure 3. Nonplanar graphs } K_5 \text{ and } K_{3,3} \]

**Proposition 1.2** (see proof in §2.1). From any 5 points in the plane one can choose two disjoint pairs such that the segment joining the first pair intersects the segment joining the second pair.

Moreover, if no 3 of 5 points in the plane lie in the same line, then the number of intersection points of interiors of segments joining the 5 points is odd.

In graph-theoretic terms this means that the complete bipartite graph \( K_{3,3} \) is not planar, see Figure 3 right.

This is a ‘linear’ version of the nonplanarity of the complete graph \( K_5 \) on 5 vertices, see Figure 3 left.

The first sentence of Proposition 1.2 indeed follows by the ‘moreover’ part. This is true because for non-general-position points the first sentence is obvious: if points \( A, B, C \) among given 5 points lie in the same line, \( B \) between \( A \) and \( C \), and \( D \) is any other given point, then segments \( AC \) and \( BD \) intersect. This is also true because we can make a small shift so that no 3 of 5 shifted points lie in the same line, and no intersection points of segments with disjoint vertices are added. Analogous remarks can be made for Theorems 1.1, 1.4 below; such remarks are omitted.
do not use the notions of a hypergraph and its realizability neither for the statements nor for the proofs. (We do mention hypergraphs because the problem of their realizability helps to understand the motivation of the results.) For understanding most of the paper it suffices to know basic geometry of 3-dimensional space and to be ready to learn straightforward 4-dimensional generalizations. We believe that describing simple applications of topological methods in elementary language makes these methods more accessible (although this is called 'detopologization' in [MTW12, §1]).

![Figure 4](image)

**Figure 4.** Left: Realization in $\mathbb{R}^3$ of the complete 3-homogeneous hypergraph on 5 vertices.
Right: Realization in $\mathbb{R}^3$ of the product of the complete graphs on 5 and on 2 vertices.

Such analogues are **3-homogeneous**, or **2-dimensional hypergraphs** defined as collections of 3-element subsets of a finite set $F$. For brevity, we omit ‘3-homogeneous, or ‘2-dimensional’. For instance, a complete hypergraph on $k$ vertices is the collection of all 3-element subsets of a $k$-element set. **Realizability** of a hypergraph in $d$-dimensional Euclidean space $\mathbb{R}^d$ is defined similarly to the realizability of a graph in the plane (one ‘draws’ a triangle for every three-element subset; see Figures 4 and 5). Hypergraphs (and simplicial complexes) play an important role in mathematics. One cannot imagine topology and combinatorics without them. They are also used in computer science and bioinformatics, see, e.g. [PS11].

A ‘small shift’ (or ‘general position’) argument shows that every graph is realizable in $\mathbb{R}^3$. A straightforward generalization shows that every hypergraph is realizable in $\mathbb{R}^5$.

It is easy to see that the complete hypergraph on 6 vertices is non-realizable in $\mathbb{R}^3$ (Proposition 2.4.a). Already in the early history of topology (1920s) mathematicians tried to construct hypergraphs non-realizable in $\mathbb{R}^4$. Egbert van Kampen and A. Flores in 1932-34

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5 In topology such objects are called **pure**, or **dimensionally homogeneous**, 2-dimensional simplicial complexes, but I hope the term hypergraph is more convenient to generic mathematician or computer scientist.

6 Here is a rigorous definition. A hypergraph $(V, F \subseteq \binom{V}{3})$ is linear realizable in $\mathbb{R}^d$ if there is a set of non-degenerate triangles in $\mathbb{R}^d$ whose vertices correspond to $V$, whose triangles correspond to $F$, and every two triangles either are disjoint, or intersect only at a common vertex, or intersect only by a common side.
proved that the complete hypergraph on 7 vertices is not realizable in $\mathbb{R}^4$ (Theorem 1.4). It is both an early application of combinatorial topology (nowadays called algebraic topology) and one of the first results of topological combinatorics (also an area of ongoing active research).

Before stating Theorem 1.4 observe that ‘typical’ intersection of two segments in the plane is either empty set or a point. Analogously, ‘typical’ intersection of two triangles in 4-space is either empty set or a point. More intuition on 4-space can be developed by reading e.g. [Sk, §4.7 ‘How to work with four-dimensional space?’], see also Remark 1.5 below.

**Theorem 1.4** (Linear Van Kampen-Flores Theorem; [vK32, Fl34]). From any 7 points in 4-space one can choose two disjoint triples such that the two triangles with vertices at the triples intersect.

Moreover, if no 5 of 7 points in 4-space lie in the same 3-dimensional hyperplane, then the number of intersection points of triangles with vertices at these points is odd.

See idea of a short proof in Remark 1.5. Formally, Theorem 1.4 is reduced to Theorem 1.1 in §2.3.

An analogue of Theorem 1.4

- is true for 5 points in the plane or 6 points in 3-space (Propositions 1.2 and 2.4b);
- is false for 4 points in the plane, 5 points in 3-space or 6 points in 4-space (in $\mathbb{R}^n$ take the $n+1$ vertices and an interior point of an $n$-simplex, see Figure 4 left).

**Remark 1.5** (lowering of dimension). A striking idea is that the nonrealizability of hypergraphs in $\mathbb{R}^4$ can be reduced to 3-dimensional results due to John Conway, Cameron Gordon and Horst Sachs (Theorems 1.1 and 2.5). Before reducing the 4-dimensional results to the 3-dimensional results (§2.3), we reduce the 3-dimensional results to certain 2-dimensional results (§2.2), and the 2-dimensional results to certain 1-dimensional result (§2.1). Thus Proposition 1.2 is reduced to Proposition 2.1 below, Theorem 1.1 to Proposition 1.2, and Theorem 1.4 to Theorem 2.1. This pattern is generalized by Theorem 1.6 below. Because of such ‘lowering of dimension’ the reader not familiar with 4-dimensional space need not be scared. See also Historical Remark 2.8.

The non-realizability results may be called ‘Ramsey intersection theory’, just as the Conway–Gordon–Sachs theorem is departure point of Ramsey linking theory. See surveys [RA05, PS05].
1.3. Linking and intersection in higher dimensions. The above relation between intrinsic linking in dimension 3 and non-realizability (i.e. intrinsic intersection) in dimension 4 generalizes to a relation between intrinsic linking and non-realizability in consecutive dimensions. That is, the above results on intrinsic linking and non-realizability turn out to be particular cases of a result in arbitrary dimensions (Theorem 1.6). For simplicity we mention dimensions higher than 4 only in that theorem and state only the ‘quantitative’, ‘moreover’ parts, omitting the ‘existence’ parts.

Take two $k$-dimensional simplices in $(2k-1)$-space of whose $2k+2$ vertices no $2k$ lie in the same $(2k-2)$-dimensional hyperplane. The two simplices are called linked, if the boundary of the first simplex intersects the convex hull of the second simplex exactly at one point.

Theorem 1.6. Take any $n+3$ points in $\mathbb{R}^n$ of which no $n+1$ points lie in the same $(n-1)$-dimensional hyperplane.

For $n$ even mark the intersection points of the interiors of convex hulls of $n/2$-simplices with vertices at these points. Then the number of marked points is odd.

If $n$ is odd, then the number of linked unordered pairs of $(n+1)/2$-simplices with vertices at these points is odd.

This is Proposition [L2] for $n = 2$, is the Linear Conway–Gordon–Sachs Theorem [L4] for $n = 3$, is the linear version of a result by Lovas-Schrijver-Taniyama for odd $n > 3$ [LS98, Corollary 1.1], [Ta00], and is the linear version of the van Kampen-Flores Theorem for $n$ even [vK32, Fl34].

Theorem 1.6 is proved by induction on $n$. The base is $n = 1$ and is trivial. The inductive step is proved in §2 for $n = 2, 3, 4$; the proof for the general case is analogous.

There is also an ‘intersection property’ of odd-dimensional space (Proposition [2.4]b is an analogue of Theorems [1.2, 1.4, 1.6]). It is weaker than the corresponding ‘linking property’ (Theorems [1.1, 1.6]). For ‘unlinking properties’ see Remark [2.9].

1.4. Cartesian product and the Menger conjecture. The (Cartesian) product $F \times F'$ of two figures $F, F'$ in $\mathbb{R}^3$ is the set of points $(x, y, z, x', y', z') \in \mathbb{R}^6$ such that $(x, y, z) \in F$ and $(x', y', z') \in F'$. A combinatorial version of this notion is product of two graphs (not necessarily planar). This product can be considered (although not canonically) as a hypergraph; see Figure 5, left. In Figure 5, middle and right, splitting of quadrilaterals into triangles is not shown.

Karl Menger conjectured in 1929 that the square of a nonplanar graph is not realizable in $\mathbb{R}^4$ [Me29]. This was proved only in 1978 by Brian Ummel [Um78] (Theorem 3.3). A simpler proof was obtained in 2003 by Mikhail Skopenkov [Sk03]. There is a short formula for the minimal number $d$ such that given product of several graphs is realizable in $\mathbb{R}^d$ [Sk03].

The argument of [Sk03] is based on discovery and use of the relation between linking and non-realizability phenomena in dimensions 3 and 4 (illustrated in §3.2 and §3.3).

1.5. Linear, piecewise-linear (PL) and topological versions. We present elementary statements and simple proofs of the linear versions of the above classical results. PL and topological realizations (=embeddings) of hypergraphs are defined and discussed e.g. in [Sk18 §3.2], [Sk §5]. Our proofs are easily generalized to the PL versions [Sk03, Zi13]. The ‘quantitative’ PL versions of Proposition [1.2] and Theorems [1.1, 1.4, 1.6]. For ‘unlinking properties’ see Remark [2.9].

This formula (generalizing the Menger conjecture) was announced in a 1992 preprint of Marek Galecki. However, after an extensive search Robert J. Daverman kindly informed the authors of a corresponding result for manifolds [ARS01] that there is no longer any copy of Galecki’s dissertation (presumably containing a proof) available at the University of Tennessee.
(analogous to their ‘moreover’ parts) imply the PL versions for almost-embeddings (see the PL case of [Sk18, Theorem 1.4.1 and 3.1.6]). The latter imply the topological versions (see explanation in [Sk18, the paragraph after Theorem 1.4.1]).

Proof of the Menger conjecture (see §1.4 in Um78) works for the topological version but is complicated (one computes an obstruction via spectral sequences). Proof in [Sk03] is much simpler but for the topological version uses the Bryant approximation theorem which is not easy. A simpler proof could possibly be obtained by proving ‘quantitative’ PL version of the Menger conjecture (i.e. improvements of Proposition 3.1 and Theorems 3.2, 3.3 analogous to the ‘moreover’ parts of Proposition 1.2 and Theorems 1.1, 1.4, see Problem 3.9).

1.6. **Comparison with other expositions.** The (linear, PL and topological) van Kampen-Flores theorem has an alternative simple proof using the van Kampen number, see e.g. [Sk18, §1.4], [Sk] §1.4, §5]. That proof and the proof sketched in this paper, are presumably the simplest known proofs (‘proofs from the Book’). Proofs of the Menger conjecture (see §1.4) using an analogue of the van Kampen number or the Borsuk-Ulam theorem are not known.

Usually the van Kampen-Flores theorem is proved using the Borsuk-Ulam theorem [Pr07, §10.3], [Ma03, §5]. As opposed to this paper (and to the alternative simple proof using the van Kampen number), this requires some knowledge of algebraic topology. And this knowledge does not make things simpler: no known proof of the Borsuk-Ulam theorem (see [Ma03] and the references therein) is easier than direct proof of the van Kampen-Flores theorem (presented here or in [Sk18, §1.4], [Sk] §1.4, §5]). E.g. the Borsuk-Ulam theorem is usually proved using the degree analogously to the direct proof of the van Kampen-Flores theorem using the van Kampen number.

Short algebraic proofs of the linear versions of the van Kampen-Flores and the Conway–Gordon–Sachs in the spirit of the ‘standard’ proof of the Radon theorem are given in [BM15]. However, those proofs do not generalize to PL (or topological) versions.

1.7. **Further generalizations.** The results discussed in this survey are in the basis of ongoing research.

An important area is study of realizability of (higher-dimensional) hypergraphs, including applications of algebraic topology to algorithmic problems. For recent surveys see [Sk08, §4, §5], [MTW11, §1], [Sk18, §3.2]. For a recent application of the relation between intrinsic linking and non-realizability in computer science see [Pa15, Sk18o].

Realizations (=embeddings) are maps without self-intersections. For topological combinatorics and discrete geometry it is interesting to study of maps whose self-intersections are ‘not too complicated’. This is similar to study of smooth maps where one needs to study maps whose singularities are ‘not too complicated’, i.e. to develop singularity theory. An important particular case is studying maps without triple intersections and, more generally, maps without r-tuple intersections, see e.g. survey [Sk18, §3.3]. For relation of this subject to the topological Tverberg conjecture see survey [Sk16] and references therein.

For analogous problem on embedding dynamical systems see [LT14] and references therein.

2. **Proofs and further results**

By $k$ **points in** $\mathbb{R}^d$ (in this paper mostly $d \leq 4$) we mean a $k$-element subset of $\mathbb{R}^d$; so these $k$ points are assumed to be pairwise distinct.

2.1. **Intersection in the plane.** Proposition 1.2 is easily proved by analyzing the convex hull of the points. In order to illustrate the ‘lowering of dimension’ argument in the simplest
situations, let us present another proof of Proposition 1.2 based on reduction to the following obvious 1-dimensional result.

Take 4 points on a line, 2 red and 2 blue. The red and the blue pairs of points are called linked if they alternate: red-blue-red-blue or blue-red-blue-red. The following result is obvious:

**Proposition 2.1.** Every 4 points in a line can be colored in 2 red and 2 blue so that the red pair is linked with the blue pair.

Moreover, the number of linked unordered pairs of pairs with vertices at these 4 points is odd.

**Proof of the first sentence in Proposition 1.2.** We may assume that $O$ is the unique point among given ones whose first coordinate $a$ is maximal. Consider a line $x = b$, where $b$ is slightly smaller than $a$. Denote by $A, B, C, D$ the remaining points.

**Figure 6.** Left: to the proof of Proposition 1.2. Right: to Proposition 2.3.b.

If for some two points $X, Y \in \{A, B, C, D\}$ the point $X$ belongs to the segment $OY$, then we are done. Otherwise we can assume that the points $A, B, C, D$ are seen from $O$ in this order, see Figure 6. Then by the following Lemma 2.2 the outlines of the triangles $OAC$ and $OBD$ have an intersection point different from $O$. Hence some two sides of the triangles have disjoint vertices and intersect. □

**Lemma 2.2** (See figure 6, left). Assume that two triangles $\Delta, \Delta'$ in the plane have a common vertex $O$, and no 3 of their vertices lie in the same line. Then the outlines $\partial \Delta, \partial \Delta'$ of the triangles intersect at an even number of points if and only if the intersection $\partial \Delta \cap \Delta'$ contains only one segment with vertex $O$.

This lemma is trivial. It is explicitly stated in order to illustrate higher-dimensional generalizations (Lemmas 2.6 and 3.8).

The ‘moreover’ part of Proposition 1.2 follows by a simple additional counting analogous to the proof of the Linear Conway-Gordon-Sachs Theorem 1.1 in §2.2 and using the ‘moreover’ part of Proposition 2.1.

The following propositions are proved analogously to Proposition 1.2. They are used for some 3-dimensional results (Proposition 3.1 and Theorems 3.2, 2.5) in §2.2 and §3.2.

**Proposition 2.3.** (a) (See figure 3, right, and Theorem 2.7.) Two triples of points are given in the plane. Then there exist two intersecting segments without common vertices and such that each segment joins the points from distinct triples.

(b) (See figure 4, right) Suppose that there are 4 red and 2 blue points $B_1, B_2$ in the plane. Suppose further that any two segments joining points of different colors either are disjoint or intersect at their common vertex. Then there are 2 red points $R_1, R_2$ such that the quadrilateral $R_1B_1R_2B_2$ does not have self-intersections and the remaining 2 red points
lie on different sides of the quadrilateral. (i.e. a general position polygonal line joining the remaining 2 red points intersects the outline of the quadrilateral at an odd number of points.)

See more results in [Sk18, §1.1].

2.2. Linking and intersection in three-dimensional space. First we illustrate the ‘lowering of the dimension’ idea (see Remark [1,5]) of proof of the Linear Conway–Gordon–Sachs Theorem [1,1] by proving its weaker versions.

**Proposition 2.4.** (a) From any 6 points in 3-space one can choose 5 points $O, A, B, C, D$ such that the triangles $OAB$ and $OCD$ have a common point other than $O$.

(b) From any 6 points in 3-space one can choose disjoint pair and triple such that the segment joining points of the pair intersects the triangle spanned by the triple.

![Figure 7](image)

**Figure 7.** To the proofs of Proposition 2.4a and Theorem 1.1. A plane in $\mathbb{R}^3$ intersects the segments $OA_1, \ldots, OA_5$ by points $A'_1, \ldots, A'_5$.

**Proof of (a).** Without loss of generality we may assume that there is a unique ‘highest’ point $O$ among the given ones. Consider a ‘horizontal’ plane slightly below the point $O$. Consider the intersection of this plane with the union of triangles $OAB$ for all pairs $A, B$ of given points. Now the assertion follows by Proposition 1.2. □

Part (b) follows from (a). Part (b) is an improvement of (a) and is a spatial analogue of Proposition 1.2 (without the ‘moreover’ part).

Figure 4, left, shows that the analogue of (a) for 5 points is false.

**Proof of Theorem 1.1.** We may assume that $O$ is the unique point among given ones whose first coordinate $a$ is maximal. Consider a plane $x = b$, where $b$ is slightly smaller than $a$. Denote by $A'_1, \ldots, A'_5$ the intersection points of this plane and segments joining $O$ to other given points. See Figure 7

In 3-space a segment $p$ is below a segment $q$ (looking from point $O$), if there exists a half-line $OX$ with the endpoint $O$ that intersects the segment $p$ at a point $P := p \cap OX$, the segment $q$ at a point $Q := q \cap OX$, $P \neq Q$, so that $Q$ is in the segment $OP$. So in the plane $x = b$ we can draw a figure analogous to Figure 2, right. Since no 4 of the given points $O, A_1, \ldots, A_5$ lie in the same plane, the number of those sides of the triangle $A_3A_4A_5$ that are higher than $A_1A_2$ equals to the number of intersection points of the outline of the triangle $A_3A_4A_5$ with the triangle $OA_1A_2$. Also, a segment cannot intersect a triangle by more than 2 points. All this implies that the triangles $OA_1A_2$ and $A_3A_4A_5$ are linked if and only if $A_1A_2$ is below an odd number of sides of the triangle $A_3A_4A_5$.

For the existence of linked triangles it suffices to prove that if no 3 of 5 points in the plane lie in the same line and the intersection points (different from vertices) of segments joining these points are marked so as to show that one segment ‘passes below the other’, then
there is a segment that is below exactly one side of its ‘complementary’ triangle. This can be proved by considering all possible cases. Instead of giving details, let us present a counting argument that gives the ‘moreover’ part.

The following numbers have the same parity:

- the number of linked unordered pairs of triangles formed by given points;
- the number of segments \( A_iA_j \) that are below an odd number of sides of their ‘complementary’ triangles \( A_kA_lA_m \), \( \{i,j,k,l,m\} = \{1,2,3,4,5\} \);
- the number of ordered pairs \( (A_iA_j, A_kA_l) \) of segments of which the first is below the second;
- the number of intersection points of segments whose vertices are \( A_1', \ldots, A_5' \).

By Proposition 1.2 the latter number is odd. \( \square \)

The following version of Theorem 1.1 is analogously reduced to Proposition 2.3.b \([Zi13]\) and is used for some 4-dimensional result (Theorem 3.3) in \( \S 3.4 \).

Take two space quadrilaterals (i.e. closed quadrangular polygonal lines) \( ABCD \) and \( A'B'C'D' \) in 3-space no 4 whose 8 vertices lie in the same plane. The quadrilaterals are called \textit{linked modulo 2} if the number of intersection points of the polygonal line \( ABCD \) with the union of the triangles \( A'B'C' \) and \( A'D'C' \) is odd. (As opposed to triangles, there are space quadrilaterals \textit{linked} but not linked modulo 2 \([Wl]\).) Proposition 3.7 illustrates this notion of linking.

**Theorem 2.5** (Linear Sachs Theorem; \([Sa81]\)). Suppose that there are 8 general position points in 3-space, 4 red and 4 blue. Then there are two linked space quadrilaterals with vertices at these points consisting of segments joining points of different colors.

2.3. \textbf{Linking and intersection in four-dimensional space.} This and the following two subsections are independent of each other (except that \( \S 3.4 \) uses the statement of Lemma 3.8), so they can be read in any order.

**Proof of the first sentence in the Linear Van Kampen-Flores Theorem 1.4.** We may assume that no 5 of the given 7 points \( O, A_1, \ldots, A_6 \) lie in the same 3-dimensional hyperplane (see the sentence after Proposition 1.2). We may also assume that \( O \) is the unique point among them whose first coordinate \( a \) is maximal. Consider a 3-dimensional hyperplane \( x = b \), where \( b \) is slightly smaller than \( a \).

![Figure 8](image)

**Figure 8.** To the proof of Theorem 1.4. A hyperplane in \( \mathbb{R}^4 \) (shown as a plane in \( \mathbb{R}^3 \)) intersects the segments \( OA_1, \ldots, OA_6 \) at 6 points \( A_1', \ldots, A_6' \) which are vertices of two linked triangles.

Take the 6 intersection points \( A_1', \ldots, A_6' \) of the hyperplane with the segments \( OA_1, \ldots, OA_6 \); see Figure 8. Clearly, no 4 of the obtained 6 points lie in the same plane. Hence by the Linear Conway–Gordon–Sachs Theorem 1.1 there are two linked triangles with vertices at
these points. Without loss of generality, the vertices of the first triangle belong to the segments joining $O$ to $A_2, A_3, A_4$, and the vertices of the second triangle belong to the segments joining $O$ to $A_1, A_5, A_6$. The above triangles are the intersections with the hyperplane of the tetrahedra $OA_2A_3A_4$ and $OA_1A_5A_6$.

Since the triangles are linked, the outline of $A'_2A'_3A'_4$ intersects the triangle $A'_1A'_5A'_6$ at exactly one point. Hence either triangles $A_2A_3A_4$ and $A_1A_5A_6$ intersect (then we are done) or the surface of $OA_2A_3A_4$ intersects the convex hull of $OA_1A_5A_6$ at exactly one segment. In the second case by the following Lemma 2.6 the surfaces of the tetrahedra have an intersection point distinct from $O$. Since no 5 of the given 7 points lie in the same 3-dimensional hyperplane, any two triangles spanned by the 7 points and having one common vertex intersect only at the vertex. Hence some two faces of the tetrahedra $OA_2A_3A_4$ and $OA_1A_5A_6$ have disjoint vertices and intersect. □

**Lemma 2.6.** Assume that two tetrahedra $\Delta, \Delta'$ in 4-space have a common vertex $O$, and no 5 of their 7 vertices lie in the same 3-dimensional hyperplane. Then the surfaces $\partial \Delta, \partial \Delta'$ of the tetrahedra intersect at an even number of points if and only if the intersection $\partial \Delta \cap \Delta'$ contains only one segment with vertex $O$.

This lemma (and Lemma 3.8 below) is not as obvious as its low-dimensional analogues (Lemma 2.2 and analogous result for a triangle and a tetrahedron in 3-space) because the surface of a tetrahedron in 4-space does not split 4-space. Lemma 2.6 is reduced to Lemma 2.2 by proving that the intersection plane of 3-dimensional hyperplanes spanned by the tetrahedra intersects each tetrahedron by a triangle.

The condition on $\partial \Delta \cap \Delta'$ of Lemma 2.6 is equivalent to the following: a small 3-dimensional sphere containing $O$ in its interior intersects $\Delta$ and $\Delta'$ by two triangles which are linked in the sphere. Cf. Lemma 3.8.

The ‘moreover’ part of Theorem 1.4 follows by a simple additional counting (analogous to the proof of Theorem 1.1 in §2.2) using the ‘moreover’ part of Theorem 1.1.

The following result can perhaps be deduced analogously to Theorem 1.4 from some 3-dimensional linking result and some 4-dimensional parity lemma.

**Theorem 2.7** (cf. Proposition 2.3a; [Fl34]). Three triples of points in 4-space are given. Then there exist two intersecting triangles without common vertices such that the vertices of each triangle belong to distinct triples.

**Remark 2.8** (historical). Of course general ‘lowering of dimension’ or ‘the link construction’ ideas are simple and well-known. Proofs of the Radon theorem on convex hulls based on this idea are given in [Pe72, Ko]. For a recent application in computer science see [DF94, proof of 2.3.i]. Also well-known is relation between linking and intersection in consecutive dimensions (e.g. the linking number of two disjoint closed polygonal lines in 3-dimensional sphere $\partial D^4$ equals to the intersection number of two general position 2-dimensional disks in 4-dimensional ball $D^4$ spanning the two polygonal lines). An elaboration of this idea to a relation between intrinsic linking and non-realizability in consecutive dimensions is non-trivial (cf. the difference between Proposition 2.4a and Theorem 1.1). Proofs that discover and use that relation seem to have not been published

- before [RST, RST'], Alexander Shapovalov’s 2003 solution of an olympic problem, [RSS+, Zi13] for reduction of intrinsic linking to non-realizability in lower dimension (the Conway–Gordon–Sachs theorem),

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8See e.g. [Sk16, §1] for the statement of the Radon theorem. See [Sk16, §4] for relations between the Radon, the van Kampen-Flores and the Conway–Gordon–Sachs theorems.
• before [Sk03, Example 2, Lemmas 2 and 1’, RSS+] for reduction of non-realizability to intrinsic linking in lower dimension (the van Kampen-Flores theorem and the Menger conjecture, see below).

**Remark 2.9** (unlinking properties). (2) There are 5 general position points in the plane such that every segment joining 2 of these points intersects the outline of the triangle formed by the remaining points at an even number of points.

This means that every pair of points is unlinked with the triangle formed by the remaining points. We do not spell out analogous interpretations of properties (3), (4-2) and (4-3) below.

(2’) For every 5 general position points in the plane the number of those segments joining 2 of these points that intersect the outline of the triangle formed by the remaining points exactly at one point, is even.

Proofs of (2,2’) are easy and are left to the reader.

In 3-space instead of unlinking properties (2,2’) there are a linking property (Theorem 1.1) and the following unlinking properties.

(3) There are 6 general position points in 3-space such that every segment joining 2 of these points intersects the surface of the tetrahedron formed by the remaining points at an even number of points.

(3’) For every 6 general position points in 3-space the number of those segments joining 2 of these points that intersect the surface of the tetrahedron formed by the remaining points exactly at one point, is even.

For (3) we can take points on a helix, see Figure 2. For (3’) we can use the symmetry of linking [Sk03, Symmetry Lemma 4.2] to prove that this number is twice the number from the ‘moreover’ part of Theorem 1.1.

The odd-dimensional analogue of the ‘moreover’ part of Proposition 1.2 fails by (3’). So under transition from dimension 2 to dimension 3 the property of the existence of intersection is preserved, while the parity of the number of intersections change.

It would be interesting to prove the following conjectures and their higher-dimensional analogues. (I am grateful to M. Tancer for sending me proof of the PL version of (4-3).)

(4-3) There are 7 general position points in 4-space such that every triangle formed by 3 of these points intersects the surface of the tetrahedron formed by the remaining points at an even number of points.

(4’-3) For every 7 general position points in 4-space the number of those triangles spanned by 3 of these points that intersect exactly at one point the surface of the tetrahedron formed by the remaining points, is even.

(4-2) There are 7 general position points in 4-space such that every segment joining 2 of these points intersects the surface of the 4-simplex formed by the remaining points at an even number of points.

(4’-2) For every 7 general position points in 4-space the number of those segment joining 2 of these points that intersect exactly at one point the surface of the 4-simplex formed by the remaining points, is even.

3. **Realizability of products and the Menger conjecture**

3.1. **Realizability of products.** For motivations see §1.4. Suppose that we have mn points $A_{jp}$, where $j \in [m] := \{1, 2, \ldots, m\}$ and $p \in [n]$, in 3- or 4-space. For two-element subsets $\{j, k\} \subset [m]$, $j < k$, and $\{p, q\} \subset [n]$, $p < q$, denote by $jk \times pq$ the collection, or the union, of two triangles $A_{jp}A_{kq}A_{jq}$ and $A_{jp}A_{kq}A_{kp}$ having a common side (see Figure 5 left). This union could be, but need not be, a plane quadrilateral. An $(m, n)$-**product** is a collection
of $2mn$ triangles from

$$jk \times pq, \quad \text{where} \quad 1 \leq j < k \leq m, \quad 1 \leq p < q \leq n.$$  

The union of triangles of $(m, n)$-product is a polyhedral and possibly self-intersecting
- square, if $m = n = 2$ (Fig. 5 left);
- lateral surface of a cylinder, if $m = 3$ and $n = 2$ (Fig. 5 middle);
- torus, if $m = n = 3$ (Fig. 5 right).

A typical example is the Cartesian product of $m$ points in the plane and $n$ points in the line (or in the plane).

**Proposition 3.1.** Any $(4, 4)$-product in 3-space has two triangles which have disjoint vertices but intersect.

Proposition 3.1 is reduced to Proposition 2.3.a in §3.2.

In terms of hypergraphs or complexes Proposition 3.1 implies that $K_4 \times K_4$ is not linearly realizable in 3-space. We do not spell out analogous corollaries of the following two theorems.

**Theorem 3.2** (Product; [Sk03]). Any $(5, 3)$-product in 3-space has two triangles which have disjoint vertices but intersect.

The Product Theorem 3.2 is reduced to Proposition 2.3.b in §3.2.

**Theorem 3.3** (Square; [Um78, Sk03]). Any $(5, 5)$-product in 4-space has two triangles which have disjoint vertices but intersect.

The Square Theorem 3.3 is reduced to the Linear Sachs Theorem 2.5 in §3.4.

**Example 3.4.** The analogues of Theorems 3.2 and 3.3 are false for
(a) $(2, n)$-products in 3-space for every $n$ (for $n \leq 4$ this is obvious; for $n = 5$ see Figure 4 right: the vertices of the parallelograms are the required 10 points; for $n \geq 6$ the construction is analogous, see [RSS] Theorem 1.5);
(b) $(3, n)$-products in 3-space for every $n \leq 4$ (for $n \leq 3$ this is obvious, see Figure 5 right; for $n = 4$ the construction is analogous, see [3.2]);
(c) $(4, n)$-products in 4-space for every $n$ (see [3.4]).

**3.2. Realizability of products in three-dimensional space.**

**Proof of Example 3.4.a.** Let $(0, 0, 0), V, A_{11}, \ldots, A_{1n}$ be points in $\mathbb{R}^3$ of which no 4 lie in the same plane. For every $p \in [n]$ denote $A_{2p} := V + A_{1p}$. If $V$ is close enough to $(0, 0, 0)$, then the points $A_{jp}, j \in \{1, 2\}, p \in [n]$, are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect.

Indeed, $12 \times pq$ is a parallelogram for every $p \neq q$. Since no 4 of the points $(0, 0, 0), V, A_{11}, \ldots, A_{1n}$ lie in the same plane, for any distinct $p, q, r, s$ the segments $A_{1p}A_{1q}$ and $A_{1r}A_{1s}$ are disjoint. Since $V$ is close enough to $(0, 0, 0)$, the same holds for 1 replaced by 2. Then any two (convex hulls of) parallelograms $12 \times pq$ and $12 \times rs$ that have no common side are disjoint. Therefore the points $A_{jp}$ are as required.

**Proof of Example 3.4.b.** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation through $\frac{2\pi}{3}$ w.r.t. $x$-axis. Let

$$(A_{11}, A_{12}, A_{13}, A_{14}) = ((1, 0, 1), (-1, 0, 1), (0, 0, 2), (0, 0, 3)).$$

9Proof of Proposition 3.1 shows that even $K_{3,1} \times K_{3,1}$ is not linearly realizable in 3-space. Analogous improvements of the following two theorems are false.
Let $A_{2p} = f(A_{1p})$ and $A_{3p} = f(f(A_{1p}))$ for every $p \in [4]$. Cf. Figure 5 right. Then the points $A_{jp}$, $j \in [3]$, $p \in [4]$, are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect.

Indeed, $j k \times p q$ is a parallelogram for every $j \neq k, p \neq q$. Since every two segments joining points $A_{1p}$ either are disjoint or intersect at a common vertex, any two of such parallelograms that have no common side are disjoint. Therefore the points $A_{jp}$ are as required. \hfill \square

Proof of Proposition 3.1. (The proof is analogous to Proposition 2.4.) Take a small tetrahedron containing $A_{11}$ in its interior. For every $j = 2, 3, 4$ color in red the intersection point of the surface $S$ of the tetrahedron with the segment $A_{11} A_{j1}$, see Figure 9, left. For every $k = 2, 3, 4$ color in blue the intersection point of $S$ with the segment $A_{11} A_{1k}$. (The intersection of $S$ with the union of the triangles of the $(4, 4)$-product is the image of a piecewise linear map of the graph $K_{3,3}$ to $S$.) Then by an analogue of Proposition 2.3a (cf. Sk18 Remark 1.5.1.d]) there are $2 \leq j < k \leq 4$ and $2 \leq p < q \leq 4$ such that the triangles $A_{11} A_{1p} A_{j1}$ and $A_{11} A_{1q} A_{k1}$ have a common point other than $A_{11}$. Hence without loss of generality the segment $A_{1p} A_{j1}$ intersects the triangle $A_{11} A_{1q} A_{k1}$. So the triangles $A_{jp} A_{1p} A_{j1}$ and $A_{11} A_{1q} A_{k1}$ have disjoint vertices but intersect. \hfill \square

![Figure 9](image) Figure 9. To the proofs of Proposition 3.1 (left) and the Product Theorem 3.2 (right)

Given 9 points $A_{jk}$, $j, k \in \{u, v, w\}$, in 3- or 4-space denote by $T_{uvw}$ the body of the corresponding $(3, 3)$-product, i.e. the union of products $j k \times p q$ (defined at the beginning of §3.1) taken for every 2-element subsets $\{j, k\}, \{p, q\} \subset \{u, v, w\}$. See Figure 5 right. (As opposed to the figure, $T_{uvw}$ can have self-intersections.) We abbreviate ‘the body of a $(3, 3)$-product’ to ‘a $(3, 3)$-product’.

Proof of the Product Theorem 3.2. Take a small tetrahedron containing $A_{11}$ in its interior. For every $j = 2, 3, 4, 5$ color in red the intersection point of the surface $S$ of the tetrahedron with the segment $A_{11} A_{j1}$, see Figure 9 right. For every $k = 2, 3$ color in blue the intersection point of $S$ with the segment $A_{11} A_{1k}$. (The intersection of $S$ with the union of the triangles of the $(5, 3)$-product is the image of a piecewise linear map of the graph $K_{4,2}$ to $S$.)

Denote the blue points by $B_1, B_2$. The intersection of a triangle $A_{11} A_{j1} A_{1p}$, with $S$ is called an arc. Analogously to the last sentences from the proof of Proposition 3.1 either some two triangles of the $(5, 3)$-product have disjoint vertices and intersect, or any two arcs joining points of different colors can only intersect at their common vertex. In the second case

\footnote{It is here that we use a specific triangulation of $K_4 \times K_4$. Thus the point $A_{11}$ is not interchangeable with other $A_{1p}$. So we have to consider a tetrahedron instead of a (hyper)plane as in Theorems 1.1, 1.4 and 3.3. Analogous remark applies for the proof of the Product Theorem 3.2 below.}
by an analogue of Proposition 2.3b there are 2 red points $R_1, R_2$ such that the polygonal line $R_1B_1R_2B_2$ formed by arcs does not have self-intersections and the remaining two red points $R_3, R_4$ lie in $S$ on different sides of the polygonal line. Without loss of generality, $R_1, B_1, R_2, B_2$ belong to the segments joining $A_{11}$ to $A_{21}$, $A_{12}$, $A_{31}$, $A_{13}$, respectively, and $R_3, R_4$ belong to the segments joining $A_{11}$ to $A_{41}$, $A_{51}$, respectively. Then the points $R_3$ and $R_4$ are intersection points of $S$ and the outline of the triangle $A_{11}A_{41}A_{51}$. The intersection of $S \cap A_{11}A_{41}A_{51}$ is a polygonal line joining $R_3$ and $R_4$. The polygonal line $R_1B_1R_2B_2$ is the intersection of $S$ and the (3, 3)-subproduct $T_{123}$. Since $R_3, R_4$ lie in $S$ on different sides of the polygonal line, $(S \cap A_{11}A_{41}A_{51}) \cap (S \cap T_{123}) \neq \emptyset$. Thus $A_{11}A_{41}A_{51} \cap T_{123} \neq \emptyset$. Hence one of the two triangles $A_{11}A_{41}A_{51}$, $A_{45}A_{41}A_{51}$ and some triangle from $T_{123}$ have disjoint vertices but intersect. 

\[ \Box \]

### 3.3. Parity Lemmas

For the proof of the Square Theorem 3.3 we need Lemma 3.8 whose simpler analogues were already used above (see Lemmas 2.2, 2.6 and an argument on a triangle and a (3, 3)-product in 3-space from the proof of the Product Theorem 3.2).

Proof of Lemma 3.8 allows to exhibit a basic idea of homology theory (i.e. Poincaré Lemma on the homology of Euclidean space) in an elementary language accessible to non-specialists. See a similar alternative proof in [Zu] and more on parity lemmas in [Sk18 §1.3], [Sk §4].

In order to illustrate the idea in the simplest situation, we start with a planar version of a 3-dimensional ‘general position’ parity lemma (Lemma 3.6) required for Lemma 3.8.

Some points in the plane are in general position, if no three of them lie in the same line and no three segments joining them have a common interior point.

**Lemma 3.5** (Parity; [Sk18, Parity Lemma 1.3.7]). Any two closed polygonal lines in the plane whose vertices are in general position intersect at an even number of points.

We need a generalization of the following evident fact: if no 4 of the vertices of a polygonal line and of a tetrahedron in 3-space lie in the same plane, then the polygonal line and the surface of the tetrahedron intersect at an even number of points.

Some points in 3-space are in general position, if no 4 of them lie in the same plane, and for every pair, triple and triple of the points the common points of their convex hulls is the same as the convex hull of the set of their common points. (In particular, if the pair, triple and triple are pairwise disjoint, then their convex hulls do not have a common point.) E.g. in general position are

- the set of 6 points in Figure 2 (Consider a regular hexagon in a horizontal plane. Point $A_j$ lies exactly above the vertices of the hexagon at the height $j = 1, 2, \ldots 6$.)
- the set of points with Cartesian coordinates $(t; t^2; t^3)$, where $t \in (0, 1)$ (‘moment curve’).

A 2-cycle is a collection of (different) triangles such that every segment is the side of an even number (possibly, zero) of triangles from the collection. The vertices of a 2-cycle are the vertices of its triangles. The body of a 2-cycle is the union of its triangles.

An example of a 2-cycle is the surface of a tetrahedron (possibly, degenerate). Also, the (3, 3)-product $T_{uvw}$ defined in §3.2 is the body of a 2-cycle.

**Lemma 3.6** (Parity). If the vertices of a polygonal line and a 2-cycle in 3-space are in general position, then the polygonal line intersects the body of the 2-cycle at an even number of points.

**Sketch of the proof.** The lemma follows by its particular case when the closed polygonal line is a triangle (analogously to [Sk18 §1.3, proof of the Parity Lemma 1.3.7]). This particular case is reduced to (the case when one polygonal line is a triangle of) the Parity Lemma 3.5.
by proving that the intersection of the 2-cycle and the plane containing the triangle is the union of closed polygonal lines.

**Proposition 3.7.** Let $ABCD$ and $A'B'C'D'$ be two closed quadrangular polygonal lines in 3-space no 4 of whose 8 vertices lie in the same plane.

(a) The polygonal lines are linked if and only if an odd number among the following pairs of triangles are linked pairs:

$$(ABC, A'B'C'), \ (ABC, A'D'C'), \ (ADC, A'B'C'), \ (ADC, A'D'C').$$

(b) Assume that $\Delta_1, \ldots, \Delta_k$ are triangles in 3-space such that $\Delta_1, \ldots, \Delta_k, ABC, ADC$ is a 2-cycle and the union of their vertices is in general position. (Such a collection of triangles is called a coboundary of $ABCD$.) Assume that $\Delta'_1, \ldots, \Delta'_k$ is an analogous collection of triangles for $A'B'C'D'$. The polygonal lines are linked if and only if an odd number among the $kk'$ pairs $(\Delta_j, \Delta'_j)$ of triangles are linked pairs.

**Proof.** Part (a) is a particular case of (b) for $k = k' = 2$, $\Delta_1 = ABC$, $\Delta_2 = ADC$, $\Delta'_1 = A'B'C'$, $\Delta'_2 = A'D'C'$.

Denote by $\partial \Delta$ the outline of a triangle or a quadrilateral $\Delta$. For a finite set $S$ denote by $|S|$ the number of elements in $S$. By $\equiv 2$ denote congruence modulo 2. Part (b) follows because

$$|ABCD \cap (A'B'C' \cup A'D'C')| \equiv 2 \sum_{j=1}^{k'} |ABCD \cap \Delta'_j| \equiv 2 \sum_{j=1}^{k} |(\partial \Delta_j) \cap \Delta'_j|.$$  

Here the first congruence follows by the Parity Lemma 3.6.

**Lemma 3.8.** Assume that two $(3,3)$-products $T_{123}$ and $T_{145}$ in 4-space intersect at a unique point $A_{11}$, which is their common vertex, no 5 of whose vertices lie in the same 3-dimensional hyperplane, and the triangles of $(3,3)$-products having disjoint vertices are disjoint. Consider the intersection of the union of triangle of $T_{123}$ containing $A_{11}$ and the union of (the convex hulls of) tetrahedra $A_{11}A_{14}A_{41}A_{15}$ and $A_{11}A_{14}A_{41}A_{51}$. Then this intersection contains an even number of segments with vertex $A_{11}$.

**Proof.** The conclusion of the lemma is equivalent to the following: a small 3-dimensional sphere containing $O$ in its interior intersects $T_{123}$ and $T_{145}$ by two quadrangular polygonal lines which are linked in the sphere.

Denote by $\Delta_1, \ldots, \Delta_9$ ($\Delta'_1, \ldots, \Delta'_9$) those triangles of $T$ (of $T'$) that do not contain $O$. Let $OX = \text{conv}\{O \cup X\}$ be the cone over $X$ with the center $O$. Then $(T \cap T') - \{O\} = \emptyset$ consists of an even number of points. Hence there is an even number of pairs $(j, j') \in [9]^2$ such that the surfaces of tetrahedra $O\Delta_j$ and $O\Delta'_j$ intersect at an odd number of points. By (a spherical analogue of) Lemma 2.6 the latter number has the same parity as the number of pairs $(j, j') \in [9]^2$ such that the triangles $\pi \cap O\Delta_j$ and $\pi \cap O\Delta'_j$ are linked. So the lemma follows by (a spherical analogue of) Proposition 3.7.b.

**3.4. Realizability of products in four-dimensional space.**

**Sketch of the proof of a weaker version of Example 3.4.c: $(3,5)$-product in 4-space.** Take a 3-dimensional hyperplane in $\mathbb{R}^4$ (shown in Figure 10 left, as a plane in 3-space). In this hyperplane take 10 vertices $A_{jp}$, where $j \in [5]$, $p \in \{1, 2\}$, shown in Figure 11 right. Take a vector $v$ not parallel to the hyperplane. Set $A_{j1} := A_{j1} + v$. (In Figure 10 left, we see the lateral surface of the prismoid $A_{41}A_{42}A_{43}A_{51}A_{52}A_{53}$.) Then the points $A_{jp}$, $j \in [5]$, $p \in [3]$, ...
are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect. □

**Figure 10.** Left: to realization in \( \mathbb{R}^4 \) of the *product* of the complete graphs on 5 and on 3 vertices. Right: to realization in \( \mathbb{R}^4 \) of the *product* of the complete graphs on 5 and on 4 vertices.

**Sketch of the proof of Example 3.4.c.** See Figure 10 right. Take points \( A_{jp} \in \mathbb{R}^3 \subset \mathbb{R}^4 \), \( j \in \{1, 2\} \), \( p \in [n] \) from the proof of Example 3.4.a. Then \( A_{1p}A_{1q} = A_{2p}A_{2q} \) for every \( p \neq q \). Take vectors \( v_3, v_4 \in \mathbb{R}^4 \) not parallel to the hyperplane \( \mathbb{R}^3 \subset \mathbb{R}^4 \). Denote \( A_{jp} := A_{1p} + v_j \), \( j \in \{3, 4\} \). We can take \( v_3, v_4 \) so that \( A_{14} \) is an interior point of the triangle \( A_{11}A_{12}A_{13} \). Then the points \( A_{jp}, j \in [4], p \in [n] \), are as required: there are no two triangles with vertices at these points which have disjoint vertices but intersect. □

**Figure 11.** To the proof of the Square Theorem 3.3.

**Proof of the Square Theorem 3.3.** We may assume that no 5 of the given 25 points \( A_{jp} \) lie in the same 3-dimensional hyperplane (see the sentence after Proposition 1.2). We may also assume that \( A_{11} \) is the unique point among them whose first coordinate \( a \) is maximal. Consider a 3-dimensional hyperplane \( x = b \), where \( b \) is slightly smaller than \( a \).

For every \( j = 2, 3, 4, 5 \) color in red the intersection point of the hyperplane with the segment \( A_{11}A_{1j} \); see Figure 11. For every \( p = 2, 3, 4, 5 \) color in blue the intersection point of the hyperplane with the segment \( A_{11}A_{1p} \). Clearly, no 5 of the 8 colored points in the hyperplane lie in the same plane. Hence by the Linear Sachs Theorem 2.5 there are two linked closed quadrangular polygonal lines whose vertices are the colored points and whose edges have endpoints of different colors. Without loss of generality, the vertices of the first polygonal line belong to the segments joining \( A_{11} \) to \( A_{12}, A_{21}, A_{13}, A_{31} \), and the vertices of the second polygonal line belong to the segments joining \( A_{11} \) to \( A_{14}, A_{41}, A_{15}, A_{51} \). Then the polygonal lines are the intersections with the hyperplane of the \((3, 3)\)-products \( T_{123} \) and \( T_{145} \). By Lemma 3.8 \( T_{123} \) and \( T_{145} \) have an intersection point distinct from \( A_{11} \). Hence analogously
to the last two sentences in the proof of Theorem 1.4 some two triangles of $T_{123}$ and $T_{145}$ have disjoint vertices but intersect. □

**Problem 3.9.** Find a subset

$$M \subset \left\{ (X, Y), (X', Y') : X, Y, X', Y' \in \binom{[5]}{2}, X \cap X' = \emptyset \text{ or } Y \cap Y' = \emptyset \right\}$$

such that for any 25 general position points $A_{jp}, j, p \in [5]$, in 4-space there is an odd number of pairs $\{(X, Y), (X', Y')\} \in M$ for which the intersection $(X \times Y) \cap (X' \times Y')$ consists of an odd number of points.

This problem is a particular case of the following generalized Menger problem: Complexes $K, L$ have non-trivial van Kampen obstructions to embeddability in $\mathbb{R}^m$ and in $\mathbb{R}^n$, respectively (see definition e.g. in [Fo04], [Sk18, §1.5] [SR, §5]). Does the cartesian product $K \times L$ of $K$ and $L$ have non-trivial van Kampen obstruction to embeddability in $\mathbb{R}^{m+n}$?

**References**

[AM5+] S. Avvakumov, I. Mabillard, A. Skopenkov and U. Wagner. Eliminating Higher-Multiplicity Intersections, III. Codimension 2, arxiv:1511.03501.

[ARS01] P. Akhmetiev, D. Repovš and A. Skopenkov, Embedding products of low-dimensional manifolds in $\mathbb{R}^m$, Topol. Appl. 2001. 113. P. 7-12.

[BE82] V.G. Boltyansky and V.A. Efremovich. Intuitive Combinatorial Topology. Springer.

[BM15] I. Bogdanov and A. Matushkin. Algebraic proofs of linear versions of the Conway–Gordon–Sachs theorem and the van Kampen–Flores theorem, arXiv:1508.03185.

[Br26] P. Bruegel. The Magpie on the Gallows, 1526, https://en.wikipedia.org/wiki/The_Magpie_on_the_Gallows.

[CG83] J. H. Conway and C. M. A. Gordon, Knots and links in spatial graphs, J. Graph Theory 7 (1983), 445–453.

[Clo94] New ways of weaving baskets, presented by G. Chelnokov, Yu. Kudryashov, A. Skopenkov and A. Sossinsky, http://www.turgor.ru/lktg/2004/index.htm.

[DE94] T.K. Dey and H. Edelsbrunner. Counting triangle crossings and halving planes, Discrete Comput. Geom, 12 (1994), 281–289.

[Fl34] A. Flores, Über n-dimensionale Komplexe die im $E^{2n+1}$ absolut selbstverschlungen sind, Ergeb. Math. Koll. 6 (1934) 4–7.

[Fl04] R. Fokkink. A forgotten mathematician, Eur. Math. Soc. Newsletter 52 (2004) 9–14.

[GSS+] Projections of skew lines, presented by A. Gaifullin, A. Shapovalov, A. Skopenkov and M. Skopenkov, http://www.turgor.ru/lktg/2001/index.php.

[Ko] E. Kolpakov. A proof of Radon’s Theorem via lowering of dimension, Mat. Prosvesch enie, submitted.

[LT14] E. Lindenstrauss and M. Tsukamoto, Mean dimension and an embedding problem: an example, Israel J. Math. 199 (2014).

[Ma03] J. Matoušek. Using the Borsuk-Ulam theorem: Lectures on topological methods in combinatorics and geometry. Springer Verlag, 2008.

[Me29] K. Menger. Über plättbare Dreiergraphen und Potenzen nicht plättbarer Graphen, Ergebnisse Math. Kolloq., 2 (1929) 30–31.

[MTW11] J. Matoušek, M. Tancer, U. Wagner. Hardness of embedding simplicial complexes in $\mathbb{R}^d$, J. Eur. Math. Soc. 13:2 (2011), 259–295. arXiv:0807.0336.

[MTW12] J. Matoušek, M. Tancer, U. Wagner. A geometric proof of the colored Tverberg theorem, Discr. and Comp. Geometry, 47:2 (2012), 245–265. arXiv:1008.5275.

[Pa15] S. Parsa. On links of vertices in simplicial d-complexes embeddable in the euclidean 2d-space, Discrete Comput. Geom. 59:3 (2018), 663–679. arXiv:1512.05164v4.

[Pe72] B. B. Peterson. The Geometry of Radon’s Theorem, Amer. Math. Monthly 79 (1972), 949-963.

[Pr07] V. V. Prasolov. Elements of homology theory. 2007, GSM 74, AMS, Providence, RI.

[PS05] V. V. Prasolov and M.B. Skopenkov. Ramsey link theory, Mat, Prosveschenie, 9 (2005), 108–115.
[PS11] Y. Ponty and C. Saule. A combinatorial framework for designing (pseudoknotted) RNA algorithms, Proc. of the 11th Intern. Workshop on Algorithms in Bioinformatics, WABI'11, 250–269.

[RA05] * J. L. Ramírez Alfonsín. Knots and links in spatial graphs: a survey. Discrete Math., 302 (2005), 225–242.

[RSS'] D. Repovš, A. B. Skopenkov and E. V Ščepin. On embeddability of $X \times I$ into Euclidean space, Houston J. Math. 1995. 21. P. 199-204.

[RSS+] * A. Rukhovich, A. Skopenkov, M. Skopenkov, A. Zimin, Realizability of hypergraphs, http://www.turgor.ru/lktg/2013/1/index.htm

[RST] N. Robertson, P. Seymour, and R. Thomas, A survey of linkless embeddings, Graph Structure Theory (Seattle, WA, 1991), Contemp. Math. 147, (1993) 125–136.

[RST'] N. Robertson, P. Seymour, and R. Thomas, Linkless embeddings of graphs in 3-space, Bull. of the AMS, 21 (1993) 84–89.

[Sa81] H. Sachs. On spatial representation of finite graphs, in: Finite and infinite sets, Colloq. Math. Soc. Janos Bolyai, North Holland, Amsterdam (37) 1981.

[Sk03] M. Skopenkov. Embedding products of graphs into Euclidean spaces, Fund. Math. 2003. 179. P. 191-198.

[Sk08] * A. Skopenkov. Embedding and knotting of manifolds in Euclidean spaces, London Math. Soc. Lect. Notes, 347 (2008) 248–342; arXiv:math/0604045.

[Sk16] * A. Skopenkov, A user’s guide to the topological Tverberg Conjecture, Russian Math. Surveys, 73:2 (2018), 323–353. Earlier version: arXiv:1605.05141v4. §4 available as A. Skopenkov, On van Kampen-Flores, Conway-Gordon-Sachs and Radon theorems, arXiv:1704.00300.

[Sk18] * A. Skopenkov. Invariants of graph drawings in the plane, arXiv:1805.10237.

[Sk] * A. Skopenkov. Algebraic Topology From Algorithmic Viewpoint, draft of a book, mostly in Russian, http://www.mccme.ru/circles/oim/algor.pdf

[Sk18o] * A. Skopenkov. A short exposition of S. Parsa’s theorem on intrinsic linking and non-realizability.

[Ta00] K. Taniyama, Higher dimensional links in a simplicial complex embedded in a sphere, Pacific Jour. of Math. 194:2 (2000), 465-467.

[Um78] B. Ummel. The product of nonplanar complexes does not imbed in 4-space, Trans. Amer. Math. Soc., 242 (1978) 319–328.

[vK32] E. R. van Kampen, Komplexe in euklidischen Räumen, Abh. Math. Sem. Hamburg, 9 (1932) 72–78; Berichtigung dazu, 152–153.

[WI] https://en.wikipedia.org/wiki/Whitehead_link

[Zi13] A. Zimin. Alternative proofs of the Conway-Gordon-Sachs Theorems, arXiv:1311.2882.

[Zu] J. Zang. A non-general-position Parity Lemma, http://www.turgor.ru/lktg/2013/1/parity.pdf

Books, surveys and expository papers in this list are marked by the stars.