Contraction semigroups on $L_\infty(\mathbb{R})$

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Dedicated to the memory of Günter Lumer 1929–2005

Abstract

If $X$ is a non-degenerate vector field on $\mathbb{R}$ and $H = -X^2$ we examine conditions for the closure of $H$ to generate a continuous semigroup on $L_\infty$ which extends to the $L_p$-spaces. We give an example which cannot be extended and an example which extends but for which the real part of the generator on $L_2$ is not lower semibounded.

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1 Introduction

The Lumer–Phillips theorem [LuP] is a cornerstone of the theory of continuous semigroups. The theorem characterizes the generator of a contraction semigroup with the aid of a dissipativity condition. The latter is based on the elementary properties of the operator $-d^2/dx^2$ of double differentiation acting on $C_0(\mathbb{R})$. In this note we analyze contraction semigroups $S$ generated by squares $-X^2$ of vector fields $X = a d/dx$ acting on $C_0(\mathbb{R})$, or $L_\infty(\mathbb{R})$. An integral part of the analysis consists of examining the one-parameter groups $T$ generated by $X$. Throughout we assume $a > 0$. If $a$ is smooth this is the one-dimensional analogue of Hörmander’s condition [Hö].

First, we identify the kernel of $S$ acting on $L_\infty(\mathbb{R})$. Secondly, $T$ is defined as a weak∗ continuous group of contractions on $L_\infty$ and we derive necessary and sufficient conditions for it to extend to a continuous group on the $L_p(\mathbb{R}; \rho dx)$-spaces with $p \in [1, \infty)$, where $\rho: \mathbb{R} \to [0, \infty)$ is a $C^\infty$-function. These conditions also ensure that $S$ extends to a continuous semigroup. Thirdly, we characterize those $S$, or $T$, which extend to a contraction semigroup, or group, on $L_p(\mathbb{R}; \rho dx)$ for some $p \in [1, \infty)$. Fourthly, we give an example of a smooth vector field with a uniformly bounded coefficient for which neither $T$ nor $S$ can be extended to any of the $L_p$-spaces with $p < \infty$. Fifthly, we give an example of a smooth vector field with a uniformly bounded coefficient which is uniformly bounded away from zero for which $T$ and $S$ extend to all the $L_p$-spaces but the real part of the generator of $S$ on $L_2(\mathbb{R}; \rho dx)$ is not lower semibounded. In particular the $L_2$-generator cannot satisfy a Gårding inequality. Since the Gårding inequality is the usual starting point for the analysis of elliptic divergence form operators on $L_2(\mathbb{R}; \rho dx)$, e.g., operators of the form $X^*X$, this example clearly demonstrates that the theory of ‘non-divergent’ form operators such as $-X^2$ on $L_\infty(\mathbb{R})$ is very different. Finally we discuss the volume doubling property for balls (intervals) whose radius (length) is measured by the distance associated with $X$.

2 Preliminaries

Let $a: \mathbb{R} \to [0, \infty)$ be a locally bounded differentiable function and assume the derivative $a'$ is locally bounded. Further assume

$$\int_0^\infty dx a(x)^{-1} = \infty = \int_{-\infty}^0 dx a(x)^{-1}. \quad (1)$$

Equip $\mathbb{R}$ with the measure $\rho dx$ where $\rho: \mathbb{R} \to [0, \infty)$ is a $C^\infty$-function. Consider the vector field $X = a d/dx$ and the corresponding operators $X_{\min}$ and $X_{\max}$ on $L_\infty(\mathbb{R}; \rho dx)$ with domains $D(X_{\min}) = C_c^\infty(\mathbb{R})$ and $D(X_{\max}) = C^1_c(\mathbb{R})$. Set $H_{\min} = -X_{\min}^2$ and $H_{\max} = -X_{\max}^2$. Since we are dealing with operators on $L_\infty$ it is appropriate to deal with the weak∗ topology.

**Proposition 2.1**

I. The operators $X_{\min}$ and $X_{\max}$ are weak∗ closable and $\overline{X}_{\min} = \overline{X}_{\max}$, where the bar denotes the weak∗ closure.

II. The operator $H_{\max}$ is weak∗ closable and its weak∗ closure $\overline{H}_{\max}$ generates a semigroup $S$ which is weak∗ continuous, positive, contractive and holomorphic in the open right half-plane.
III. \( \overline{H}_{\text{max}} = -\overline{X}_{\text{max}}^2 \) and in particular \( \overline{X}_{\text{max}}^2 \) is weak* closed.

IV. If \( a \in C^\infty(\mathbb{R}) \) then \( \overline{H}_{\text{min}} = \overline{H}_{\text{max}} \), where \( \overline{H}_{\text{min}} \) is the weak* closure of \( H_{\text{min}} \).

**Proof** For all \( x_0 \in \mathbb{R} \) the ordinary differential equation \( \dot{x} = a(x) \), with initial data \( x(0) = x_0 \), has a unique maximal solution which we denote by \( t \mapsto e^{tX}x_0 \). Since \( a \) satisfies \( (1) \) this maximal solution is defined for all \( t \in \mathbb{R} \). Moreover, \( e^{sX}e^{tX}x_0 = e^{(s+t)X}x_0 \) and

\[
\int_{x_0}^{e^{tX}x_0} dx \ a(x)^{-1} = t
\]

for all \( s, t \in \mathbb{R} \) and \( x_0 \in \mathbb{R} \). In addition both the maps \( t \mapsto e^{tX}x_0 \) and \( x \mapsto e^{sX}x \) are continuous. In particular for all \( t \in \mathbb{R} \) the map \( T_t: L_\infty \to L_\infty \) defined by \( (T_t\varphi)(y) = \varphi(e^{-tX}y) \) is an isometry and \( T \) is a weak* continuous group on \( L_\infty \). This group is automatically positive and we next show that its generator is the weak* closure of the operator \( X_{\text{min}} \) on \( L_\infty \).

Clearly \( X_{\text{min}} \subseteq X_{\text{max}} \) and by a standard regularization argument it follows that \( \overline{X}_{\text{min}} = \overline{X}_{\text{max}} \). Hence to simplify notation we now set \( X_0 = X_{\text{min}} = \overline{X}_{\text{max}} \).

One computes from \( (2) \) that

\[
\frac{d}{dy} e^{tX}y = \frac{a(e^{tX}y)}{a(y)}
\]

for all \( t \in \mathbb{R} \) and \( y \in \mathbb{R} \). Therefore

\[
\frac{d}{dy} (T_t\varphi)(y) = \varphi'(e^{-tX}y) \cdot \frac{a(e^{tX}y)}{a(y)}
\]

for all \( \varphi \in D(X_{\text{max}}) \), \( y \in \mathbb{R} \) and \( t > 0 \). So \( T_t(D(X_{\text{max}})) \subseteq D(X_{\text{max}}) \) for all \( t > 0 \). Moreover,

\[
t^{-1}(\varphi - T_t\varphi)(y) = -t^{-1} \int_0^t ds \frac{d}{ds} \varphi(e^{-sX}y)
\]

\[
= t^{-1} \int_0^t ds \varphi'(e^{-sX}y) a(e^{-sX}y) = t^{-1} \int_0^t ds (T_sX_{\text{max}}\varphi)(y)
\]

for all \( \varphi \in D(X_{\text{max}}) \), \( t > 0 \) and \( y \in \mathbb{R} \), since \( \varphi' \) is continuous. So \( \lim_{t \to 0} t^{-1}(I - T_t)\varphi = X_{\text{max}}\varphi \) strongly in \( L_\infty \) and \( X_{\text{max}} \) is the restriction of the generator of \( T \). Since \( D(X_{\text{max}}) \) is invariant under \( T \) and weak* dense it follows from Corollary 3.1.7 of \([BrR]\) that \( X_0 = X_{\text{max}} \) is the generator of \( T \).

Next define the semigroup \( S \) by the integral algorithm

\[
S_t = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} ds \ e^{-s^2(4t)^{-1}} T_s.
\]

(3)

Obviously \( S \) is weak* continuous, positive, contractive and holomorphic in the open right half-plane. Let \( H_0 \) denote the weak* closed generator of \( S \). If \( \varphi \in D(X_0^2) \) then

\[
t^{-1} (I - S_t)\varphi = t^{-1} (4\pi t)^{-1/2} \int_{-\infty}^{\infty} ds \ e^{-s^2(4t)^{-1}} (I - T_s)\varphi
\]

\[
= t^{-1} (4\pi t)^{-1/2} \int_{-\infty}^{\infty} ds \ e^{-s^2(4t)^{-1}} \int_0^s du \ (s - u) T_u X_0^2 \varphi
\]

\[
= (4\pi)^{-1/2} \int_{-\infty}^{\infty} ds \ e^{-s^2/4} \int_0^s du \ (s - u) T_{t/2u} X_0^2 \varphi
\]
and it follows in the weak* limit $t \to 0$ that $\varphi \in D(H_0)$. Hence $H_0 \supseteq -X_0^2$. To prove $H_0 = -X_0^2$ it suffices to establish that the range $R(I - X_0^2)$ of $I - X_0^2$ is equal to $L_\infty$. But $X_0$ generates the continuous group $T$. Therefore $R(I \pm X_0) = L_\infty$. Moreover, $I - X_0^2 = (I - X_0)(I + X_0)$. Hence $R(I - X_0^2) = L_\infty$ and $H_0 = -X_0^2$.

Clearly $H_{\max} \subseteq -X_0^2 = H_0$ so $H_{\max}$ is weak* closable. It remains to prove that the weak* closure $\overline{H}_{\max}$ of $H_{\max}$ is equal to $H_0$.

Since $T_tD(X_{\max}) \subseteq D(X_{\max})$ and $X_{\max}T_t\varphi = T_tX_{\max}\varphi$ for all $\varphi \in D(X_{\max})$ one deduces by iteration that $T_tD(X_{\max}^2) \subseteq D(X_{\max}^2)$ and $X_{\max}T_t\varphi = T_tX_{\max}^2\varphi$ for all $\varphi \in D(X_{\max}^2)$. Next it follows from (4), by a Riemann approximation argument, that $S_tD(X_{\max}^2) \subseteq D(X_{\max}^2)$ and $X_{\max}S_t\varphi = S_tX_{\max}^2\varphi$ for all $\varphi \in D(X_{\max}^2)$ and all $t > 0$. Since $S_t$ is continuous it further follows that $S_tD(X_{\max}^2) \subseteq D(X_{\max}^2)$ for all $t > 0$. But $C^1_c(\mathbb{R}) \subseteq D(X_{\max}^2) \subseteq D(\overline{H}_{\max})$ is weak* dense in $L_\infty$ by the assumed differentiability of $a$. Hence by Corollary 3.1.7 of [BrR] it follows that $D(\overline{H}_{\max})$ is a core of $H_0$. Therefore $\overline{H}_{\max} = H_0$.

Finally, if $a \in C^\infty_c(\mathbb{R})$ then $C^\infty_c(\mathbb{R})$ is a core for $X_{\max}^2$. Therefore $\overline{H}_{\min} \supseteq H_{\max}$. Since $H_{\min} \subseteq H_{\max}$ this completes the proof of the proposition. \qed

**Remark 2.2** It follows by definition that $T_tC_0(\mathbb{R}) \subseteq C_0(\mathbb{R})$ for all $t \in \mathbb{R}$ and a simple estimate shows that the restriction of $T$ to $C_0(\mathbb{R})$ is strongly continuous. Therefore $S_tC_0(\mathbb{R}) \subseteq C_0(\mathbb{R})$ for all $t > 0$ and the restriction of $S$ to $C_0(\mathbb{R})$ is also strongly continuous. This is a direct consequence of the algorithm [3]. Thus $T$ is a Feller group and $S$ is a Feller semigroup. Now let $X_{00}$ and $H_{00}$ denote the generators of the restricted group and the restricted semigroup, respectively. Then a slight modification of the foregoing argument allows one to obtain similar characterizations of the generators but in terms of norm closures. For example, $X_{00}$ is the norm closure of $X_{\min}$ which is equal to the norm closure of $X_{\max}$. The discussion of $H_{00}$ can in fact be simplified. Since $X_{00}$ generates a strongly continuous group of isometries the operator $-X_{00}^2$ is dissipative in the sense of Lumer and Phillips [LuP] and it is norm closed by standard estimates (see, for example, [Rob] Lemma III.3.3). But one again has $R(I \pm X_{00}) = L_\infty$. Therefore $R(I - X_{00}^2) = L_\infty$. Then $-X_{00}^2$ generates a strongly continuous contraction semigroup by the Lumer–Phillips theorem and it follows by uniqueness that $H_{00} = -X_{00}^2$.

One can associate a distance with the vector field $X$ by the definition

$$d(x ; y) = \sup \{ |\psi(x) - \psi(y)| ; \psi \in C^\infty_c(\mathbb{R}) , \|X\psi\|_\infty \leq 1 \} .$$

Clearly one has

$$|\psi(x) - \psi(y)| = \left| \int_x^y dz \psi'(z) \right| \leq \left| \int_x^y dz \, a(z)^{-1} \right|$$

for all $\psi \in C^\infty_c(\mathbb{R})$ with $\|X_{\min}\psi\|_\infty \leq 1$. So

$$d(x ; y) \leq \left| \int_x^y dz \, a(z)^{-1} \right| .$$

But by regularizing $a^{-1}$ on a compact interval one deduces that the inequality is in fact an equality, i.e.,

$$d(x ; y) = \left| \int_x^y dz \, a(z)^{-1} \right|$$
for all $x, y \in \mathbb{R}$. Note that by setting $x = e^{-sX}y$ and using (2) one finds
\[
d(e^{-sX}y; y) = \left| \int_y^{e^{-sX}y} dz \, a(z)^{-1} \right| = |s|.
\]
(5)

Therefore the distance is invariant under the flow in the sense that
\[
d(e^{-tX}x; e^{-tX}y) = d(x; y)
\]
for all $x, y \in \mathbb{R}$ and all $t \geq 0$. This follows by setting $x = e^{-sX}y$ and
\[
d(e^{-tX}x; e^{-tX}y) = d(e^{-sX}e^{-tX}y; e^{-tX}y) = |s| = d(e^{-sX}y; y) = d(x; y),
\]
where we have used (5).

Now one can calculate the kernel of the semigroup $S$.

**Proposition 2.3** The kernel $K$ of the semigroup $S$ on $L_\infty(\mathbb{R})$ is given by
\[
K_t(x; y) = (4\pi t)^{-1/2} \left( a(y) \rho(y) \right)^{-1} e^{-d(x;y)^2/(4t)}^{-1}
\]
(6)

for all $x, y \in \mathbb{R}$ and $t > 0$. Moreover, $K_t$ is continuous and $\int dy \, \rho(y) \, K_t(x; y) = 1$ for all $x \in \mathbb{R}$.

**Proof** First by (3) one has
\[
(S_t \varphi)(x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} ds \, e^{-s^2/(4t)}^{-1} \varphi(e^{-sX}x)
\]
for all $\varphi \in C^\infty_c(\mathbb{R})$, $t > 0$ and $x \in \mathbb{R}$. Therefore by a change of variables $y = e^{-sX}x$ one deduces that
\[
(S_t \varphi)(x) = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} dy \, a(y)^{-1} e^{-d(x;y)^2/(4t)}^{-1} \varphi(y)
\]
since $|s| = d(x; y)$ by (5). The representation (6) follows immediately.

Clearly $K_t$ is continuous and $H_{\max} \mathbb{1} = 0$. So $S_t \mathbb{1} = 1$ in $L_\infty$-sense. Therefore $\int dy \, \rho(y) \, K_t(x; y) = 1$ for all $t > 0$ and almost every $x \in \mathbb{R}$. Moreover, the map $x \mapsto \int dy \, \rho(y) \, K_t(x; y)$ is continuous. Hence $\int dy \, \rho(y) \, K_t(x; y) = 1$ for all $t > 0$ and $x \in \mathbb{R}$. $\square$

### 3 Extension properties

Although $T$ is defined as a group of isometries and $S$ as a contraction semigroup on $L_\infty$ they do not automatically extend to the $L_p$-spaces. This requires extra boundedness conditions on the coefficient function $a$ and the density function $\rho$. The following proposition gives necessary and sufficient conditions for $T$ to extend to a continuous group and sufficient conditions for $S$ to extend to a continuous semigroup.

**Proposition 3.1** Let $T$ be the group of isometries of $L_\infty(\mathbb{R}; \rho \, dx)$ defined by $(T_t \varphi)(y) = \varphi(e^{-tX}y)$. The following conditions are equivalent for all $C \geq 1$ and $\omega \geq 0$. 

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Therefore

I. There is a \( \omega \) for all \( p \in [1, \infty) \) such that \( T \) extends to a (strongly) continuous group on \( L_p(\mathbb{R}; \rho \, dx) \) satisfying the bounds \( \|T_t\|_{p \to p} \leq C^{1/p} e^{\omega |t|/p} \) for all \( t \in \mathbb{R} \).

II. For all \( p \in [1, \infty) \) the group \( T \) extends to a (strongly) continuous group on \( L_p(\mathbb{R}; \rho \, dx) \) satisfying the bounds \( \|T_t\|_{p \to p} \leq C^{1/p} e^{\omega |t|/p} \) for all \( t \in \mathbb{R} \).

III. \( a(y) \rho(y) \leq C e^{\omega d(x;y)} a(x) \rho(x) \) for all \( x, y \in \mathbb{R} \).

Moreover, if these conditions are satisfied then the semigroup \( S \) extends to a (strongly) continuous semigroup on all the \( L_p \)-spaces, \( p \in [1, \infty] \), satisfying the bounds

\[
\|S_t\|_{p \to p} \leq \left( (2C)^{1/p} e^{\omega^2 t/p} \right) \wedge \left( 2C^{1/p} e^{\omega^2 t/p^2} \right)
\]

if \( \omega > 0 \) and \( \|S_t\|_{p \to p} \leq C^{1/p} \) if \( \omega = 0 \), for all \( t > 0 \).

Proof First assume Condition \( \text{III} \) is satisfied. Then for all \( \varphi \in L_p \) one has

\[
\|T_t \varphi\|_p^p = \int_{\mathbb{R}} dy \, \rho(y) |\varphi(e^{-tX} y)|^p
\]

Secondly, by a change of variables \( x = e^{-tX} y \) one finds

\[
\|T_t \varphi\|_p^p = \int_{\mathbb{R}} dx \, \frac{a(e^{tX} x)}{a(x)} \rho(e^{tX} x) |\varphi(x)|^p = \int_{\mathbb{R}} dx \, \rho(x) \left( \frac{a(e^{tX} x)}{a(x)} \rho(e^{tX} x) \right) |\varphi(x)|^p
\]

Therefore

\[
\sup_{x \in \mathbb{R}} \left( \frac{a(e^{tX} x)}{a(x)} \rho(e^{tX} x) \right)^{1/p} = \|T_t\|_{p \to p} \leq C^{1/p} e^{\omega |t|/p}
\]

for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \). Hence

\[
a(e^{tX} x) \rho(e^{tX} x) \leq C e^{\omega t} a(x) \rho(x)
\]

for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \). Setting \( y = e^{tX} x \) and noting that \( d(x; y) = |t| \) one deduces that Condition \( \text{III} \) is satisfied. Conversely, the same calculation shows that if Condition \( \text{III} \) is satisfied then

\[
\|T_t \varphi\|_p \leq C^{1/p} e^{\omega |t|/p} \|\varphi\|_p
\]

for all \( p \in [1, \infty) \), \( \varphi \in L_p \) and \( t \in \mathbb{R} \). In addition if \( \varphi \in C_c^\infty \) then one calculates that

\[
\varphi - T_t \varphi = \int_0^t ds \, T_s X_{\min} \varphi
\]

Hence using \( \text{III} \) and the density of \( C_c^\infty \) in \( L_p \) one concludes that \( T_t \) extends to a continuous semigroup on \( L_p \) satisfying the bounds \( \text{III} \), i.e., Condition \( \text{II} \) is valid. The implication \( \text{II} \to \text{III} \) is trivial.

If the conditions are satisfied then \( S \) extends to the \( L_p \)-spaces by \( \text{III} \). The estimates on the norms of \( S_t \) are established in two steps. First, if \( \omega > 0 \) then it follows from \( \text{III} \) and the estimates on \( \|T_s\|_{1 \to 1} \) that

\[
\|S_t\|_{1 \to 1} \leq 2C e^{\omega^2 t}
\]

for all \( t > 0 \). Since \( S \) is contractive on \( L_\infty \) one deduces from interpolation that

\[
\|S_t\|_{p \to p} \leq (2C)^{1/p} e^{\omega^2 t/p}
\]
for all \( p \in (1, \infty) \) and \( t > 0 \). Alternatively, one can reverse the reasoning and use the interpolated bounds \( \| T_s \|_{p \rightarrow p} \leq C^{1/p} e^{\omega|s|/p} \) together with (3) to calculate that
\[
\| S_t \|_{p \rightarrow p} \leq 2 C^{1/p} e^{\omega^2 t/p^2}
\]
for all \( p \in [1, \infty] \) and \( t > 0 \).

If \( \omega = 0 \) similar arguments apply and both lead to the bounds \( \| S_t \|_{p \rightarrow p} \leq C^{1/p} \). \( \square \)

The situation described by the proposition simplifies if \( C = 1 \). Then Condition (III) together with (3) implies that
\[
\pm (a\rho)'(y) a(y) = \lim_{t \downarrow 0} t^{-1} \left( (a\rho)(e^{\pm tX} y) - (a\rho)(y) \right)
\leq \limsup_{t \downarrow 0} t^{-1} (e^{\omega t} - 1)(a\rho)(y) = \omega (a\rho)(y)
\]
for all \( y \in \mathbb{R} \). Thus \( \| \rho^{-1}(a\rho)' \|_{\infty} \leq \omega \). Conversely, if \( \| \rho^{-1}(a\rho)' \|_{\infty} \leq \omega \) then
\[
\rho(e^{tX} y)^{-1} \frac{d}{dt} \left( e^{-\omega t} (a\rho)(e^{\pm tX} y) \right) \leq 0
\]
for all \( t \geq 0 \). Hence Condition (III) is satisfied with \( C = 1 \). But the condition \( \| \rho^{-1}(a\rho)' \|_{\infty} \leq \omega \) can be expressed in terms of the vector field. Therefore one has the following corollary.

**Corollary 3.2** The following conditions are equivalent for all \( \omega \geq 0 \).

I. There is a \( p \in [1, \infty) \) such that \( T \) extends to a continuous group on \( L^p(\mathbb{R}; \rho dx) \) satisfying the bounds \( \| T_t \|_{p \rightarrow p} \leq e^{\omega|t|/p} \) for all \( t \in \mathbb{R} \).

II. For all \( p \in [1, \infty) \) the group \( T \) extends to a continuous group on \( L^p(\mathbb{R}; \rho dx) \) satisfying the bounds \( \| T_t \|_{p \rightarrow p} \leq e^{\omega|t|/p} \) for all \( t \in \mathbb{R} \).

III. \( \| \rho^{-1}(a\rho)' \|_{\infty} \leq \omega \).

IV. \( |(\psi, (X + X^*)\varphi)| \leq \omega \| \psi \|_q \| \varphi \|_p \) for all \( \varphi, \psi \in C^\infty_c(\mathbb{R}) \) and for one pair (for all pairs) of dual exponents \( p, q \in [1, \infty) \).

Moreover, if these conditions are satisfied then the semigroup \( S \) extends to a continuous semigroup on all the \( L^p \)-spaces, \( p \in [1, \infty) \), satisfying the bounds
\[
\| S_t \|_{p \rightarrow p} \leq e^{\omega^2 t/p^2}
\]
for all \( t > 0 \). In addition \( H_{\text{max}} \) satisfies a Gårding inequality. Precisely,
\[
\text{Re}(\varphi, H_{\text{max}} \varphi) \geq (1 - \varepsilon)\| X \varphi \|^2_2 - (4\varepsilon)^{-1}\| X + X^* \|_{2 \rightarrow 2}^2 \| \varphi \|^2_2
\]
for all \( \varphi \in C^\infty_c(\mathbb{R}) \) and \( \varepsilon > 0 \).

**Proof** The equivalence of the first three conditions and the existence of the extension of the semigroup \( S \) follow from Proposition 2.11 and the above discussion. Conditions (III) and (IV) are equivalent because
\[
(\psi, X \varphi) + (X \psi, \varphi) = \int_\mathbb{R} dx (a\rho)(x) \left( \psi(x) \varphi'(x) + \psi'(x) \varphi(x) \right)
= \int_\mathbb{R} dx \rho(x) \left( \rho(x)^{-1}(a\rho)'(x) \right) \psi(x) \varphi(x)
\]
for all $\varphi, \psi \in C_c^\infty(\mathbb{R})$. It remains to prove the Gårding inequality.

If $\varepsilon > 0$ then

$$
\text{Re}(\varphi, H_{\max}\varphi) = -\text{Re}(X^*\varphi, X\varphi) = \|X\varphi\|_2^2 - \text{Re}((X^* + X)\varphi, X\varphi) \\
\geq \|X\varphi\|_2^2 - \|(X^* + X)\varphi\|_2\|X\varphi\|_2 \\
\geq (1 - \varepsilon)\|X\varphi\|_2^2 - (4\varepsilon)^{-1}\|X + X^*\|_{2 \to 2}\|\varphi\|_2^2
$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. $\square$

The corollary, applied with $\omega = 0$, gives the following criteria for $T$ or $S$ to extend to a contraction group or semigroup on the $L_p$-spaces.

**Proposition 3.3** The following are equivalent.

I. There is a $p \in [1, \infty)$ such that $T$ extends to a continuous contraction group on $L_p(\mathbb{R}; \rho \, dx)$.

II. For all $p \in [1, \infty)$ the group $T$ extends to a continuous contraction group on $L_p(\mathbb{R}; \rho \, dx)$.

III. There is a $p \in [1, \infty)$ such that $S$ extends to a continuous contraction group on $L_p(\mathbb{R}; \rho \, dx)$.

IV. For all $p \in [1, \infty)$ the semigroup $S$ extends to a continuous contraction group on $L_p(\mathbb{R}; \rho \, dx)$.

V. The function $a\rho$ is constant.

**Proof** The implications $\text{V} \Rightarrow \text{I} \Rightarrow \text{II} \Rightarrow \text{IV}$ follow from Corollary 3.2 and the implication $\text{V} \Rightarrow \text{III}$ is trivial.

The proof of the implication $\text{III} \Rightarrow \text{V}$ relies on the reasoning of Lumer and Phillips.

If Condition III is valid for some $p \in [1, 2]$ then it follows by interpolation with the contraction semigroup on $L_\infty$ that Condition III is valid for all $p > 2$. Hence it suffices to show that if $p \in \langle 2, \infty \rangle$ and $S$ extends to a continuous contraction group on $L_p(\mathbb{R}; \rho \, dx)$ then the function $a\rho$ is constant, i.e., Condition V is valid. Fix $p \in \langle 2, \infty \rangle$ and assume $S$ extends to a continuous contraction group on $L_p(\mathbb{R}; \rho \, dx)$. Then it follows from the Lumer–Phillips theorem, [LuP] Theorem 3.1, that the generator $H$ of the semigroup $S$ on $L_p(\mathbb{R}; \rho \, dx)$ is dissipative. So if $[\cdot, \cdot]$ is a semi-inner product on $L_p(\mathbb{R}; \rho \, dx)$ then $\text{Re}[H\varphi, \varphi] \geq 0$ for all $\varphi \in D(H)$. If $\varphi \in C_c^2(\mathbb{R})$ is real valued then $\varphi \in D(H_{\max})$ and $H_{\max}\varphi \in L_p(\mathbb{R}; \rho \, dx)$. So $\varphi \in D(H)$ and $H_{\max}\varphi = H\varphi$. Moreover,

$$
\int d(a \rho \varphi^{p-1}) a(d \varphi) = \int \rho \varphi^{p-1} H_{\max}\varphi = \int \rho \varphi^{p-1} H\varphi = \|\varphi\|_p^{p-2}[H\varphi, \varphi] \geq 0
$$

where $d = d/dx$. Hence

$$
\int d(a \rho \varphi^{p-1}) a(d \varphi) \geq 0
$$

for all real valued $\varphi \in W^{1,\infty}_c(\mathbb{R})$ by approximation.

Next fix $\tau \in C_c^\infty(\mathbb{R})$ such that $0 \leq \tau \leq 1$, $\tau(0) = 1$ and $\tau$ is decreasing on $[0, \infty)$. For all $n \in \mathbb{N}$ define $\varphi_n \in W^{1,\infty}_c(\mathbb{R})$ by

$$
\varphi_n = (a\rho)^{-1/p}(\tau \circ \Phi_n)
$$

for all $\varphi, \psi \in C_c^\infty(\mathbb{R})$. It remains to prove the Gårding inequality.

If $\varepsilon > 0$ then

$$
\text{Re}(\varphi, H_{\max}\varphi) = -\text{Re}(X^*\varphi, X\varphi) = \|X\varphi\|_2^2 - \text{Re}((X^* + X)\varphi, X\varphi) \\
\geq \|X\varphi\|_2^2 - \|(X^* + X)\varphi\|_2\|X\varphi\|_2 \\
\geq (1 - \varepsilon)\|X\varphi\|_2^2 - (4\varepsilon)^{-1}\|X + X^*\|_{2 \to 2}\|\varphi\|_2^2
$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. $\square$
Then
\[ \Phi_n(x) = n^{-1} d(0 ; x)^2 = n^{-1} \left( \int_0^x a^{-1} \right)^2. \]

Then
\[
\varphi'_n(x) = -p^{-1} (a \rho')(x)^{-1-p^{-1}} (a \rho)'(x) \tau(\Phi_n(x))
\]
\[ + 2n^{-1} (a \rho)(x)^{-1/p} \tau'(\Phi_n(x)) \left( \int_0^x a^{-1} \right) a(x)^{-1}. \]

and
\[
(a \rho \varphi'_n)(x) = -p^{-1} (a \rho)(x)^{-1/p} (a \rho)'(x) \tau(\Phi_n(x))
\]
\[ + 2n^{-1} \rho(x) (a \rho)(x)^{-1/p} \tau'(\Phi_n(x)) \left( \int_0^x a^{-1} \right). \]

Similarly, \((a \rho \varphi_n^{p-1})(x) = (a \rho)(x)^{1/p} \tau(\Phi_n(x))^{p-1}\) and
\[
(a \rho \varphi_n)'(x) = p^{-1} (a \rho)(x)^{-1+p^{-1}} (a \rho)'(x) \tau(\Phi_n(x))^{-1+p^{-1}}
\]
\[ + 2n^{-1} (p - 1) \rho(x) (a \rho)(x)^{-1+p^{-1}} \tau(\Phi_n(x))^{p-2} \tau'(\Phi_n(x)) \left( \int_0^x a^{-1} \right). \]

Then by \(\square\) it follows that
\[ 0 \leq \int \rho^{-1} d(a \rho \varphi_n^{p-1}) \rho (d \varphi_n)
\]
\[ = \int dx \left( -p^{-2} \rho(x)^{-1} (a \rho)(x)^{-1}(a \rho)'(x)^2 \left( \tau(\Phi_n(x)) \right)^2
\]
\[ - 2n^{-1} (1 - 2p^{-1}) (a \rho)(x)^{-1}(a \rho)'(x) \tau(\Phi_n(x))^{p-1} \tau'(\Phi_n(x)) \left( \int_0^x a^{-1} \right)
\]
\[ + 4n^{-2} (p - 1) \rho(x) (a \rho)(x)^{-1} \tau(\Phi_n(x))^{p-1} \left( \tau'(\Phi_n(x)) \right)^2 d(0 ; x)^2 \right). \]

Using the estimate \(a b \leq \varepsilon a^2 + (4 \varepsilon)^{-1} b^2\) for the second term, setting \(\varepsilon = (2p(p - 2))^{-1}\) and rearranging one finds
\[
(2p^2)^{-1} \int \rho^{-1} (a \rho)^{-1} ((a \rho)')^2 (\tau \circ \Phi_n)^2
\]
\[ \leq n^{-1} \int \rho (a \rho)^{-1} \left( 4(p - 1) \tau(\Phi_n)^{p-2} + 2(p - 2)^2 (\tau(\Phi_n)^{2p-2}) (\tau' \circ \Phi_n)^{2} \Phi_n \right) \]
\[ \text{for all } n \in \mathbb{N}. \] There are \(b, c > 0\) such that
\[ y \left( 4(p - 1) \tau(y)^{p-2} + 2(p - 2)^2 \tau(y)^{2p-2} \right) (\tau'(y))^2 \leq c e^{-\left(4b\right)^{-1}y}
\]
for all \(y \in [0, \infty)\). Then
\[ \left( (a \rho)^{-1} \left( 4(p - 1) (\tau \circ \Phi_n)^{p-2} + 2(p - 2)^2 (\tau \circ \Phi_n)^{2p-2} \right) (\tau' \circ \Phi_n)^{2} \Phi_n \right)(x)
\]
\[ \leq c (a \rho)(x)^{-1} e^{-d(0;x)^2(4bn)^{-1}}
\]
\[ = c (4 \pi b n)^{1/2} K_{bn}(0 ; x) \]

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uniformly for all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \). Using Proposition 2.3 one deduces that
\[
\int \rho (a \rho)^{-1} (4(p-1)(\tau \circ \Phi_n)^{p-2} + 2(p-2)^2(\tau \circ \Phi_n)^{2p-2}) (\tau' \circ \Phi_n)^2 \Phi_n \leq c (4 \pi b n)^{1/2}
\]
for all \( n \in \mathbb{N} \). Finally and the monotone convergence theorem establishes that
\[
(2p^2)^{-1} \int \rho^{-1} (a \rho)^{-1} ((a \rho)'^2 = \lim_{n \to \infty} (2p^2)^{-1} \int \rho^{-1} (a \rho)^{-1} ((a \rho)'^2 (\tau \circ \Phi_n)^2 \leq \lim_{n \to \infty} n^{-1} c (4 \pi b n)^{1/2} = 0
\]
Therefore \((a \rho)' = 0\) as required. \(\square\)

In the unweighted case, i.e., \( \rho = 1 \), the proposition establishes that \( S \) extends to a contraction semigroups on one of the \( L_p \)-spaces with \( p < \infty \) only in the case that \( X \) is proportional to \( d/dx \).

### 4 Examples

Next we give two examples of rather unexpected properties although there is nothing inherently pathological about the weight \( \rho \) or the coefficient \( a \). In fact in both examples \( \rho = 1 \) and the coefficient \( a \) of the vector field is strictly positive, smooth and uniformly bounded. The first example gives a continuous group \( T \) and semigroup \( S \) which do not extend from \( L_\infty \) to the other \( L_p \) spaces. The principal reason for this singular behaviour is the fact that \( \inf a = 0 \), i.e., there is a mild degeneracy at infinity.

**Example 4.1** Let \( \rho = 1 \). For all \( n \in \mathbb{N}_0 \) define \( h_n = n!^{-1} \). Define \( y_n \in \mathbb{R} \) for all \( n \in \mathbb{N}_0 \) by \( y_0 = 0 \) and inductively
\[
y_{n+1} = y_n + 4^{-1}(h_n + h_{n+1}) + 2^{-1}
\]
for all \( n \in \mathbb{N} \). Define \( \tilde{a}: \mathbb{R} \to (0, \infty) \) by
\[
\tilde{a}(x) = \begin{cases} 
  h_n & \text{if } x \in [y_n - 4^{-1}h_n, y_n + 4^{-1}h_n) \quad (n \in \mathbb{N}_0), \\
  1 & \text{if } x \in [y_n + 4^{-1}h_n, y_n + 4^{-1}h_n + 2^{-1}) \quad (n \in \mathbb{N}_0), \\
  1 & \text{if } x \in (-\infty, 0].
\end{cases}
\]
Then \( \tilde{a}(y_n) = h_n \) and \( \int_{y_n}^{y_{n+1}} dx \tilde{a}(x)^{-1} = 1 \) for all \( n \in \mathbb{N} \). Next we regularize \( \tilde{a}^{-1} \). For all \( n \in \mathbb{N}_0 \) let \( \chi_n \in C_0^\infty(\mathbb{R}) \) be such that \( \chi_n \geq 0, \int \chi_n = 1, \supp \chi_n \subseteq [-8^{-1}h_n, 8^{-1}h_n] \) and \( \chi_n(-x) = \chi_n(x) \) for all \( x \in \mathbb{R} \). Define \( a \in C_0^\infty(\mathbb{R}) \) by
\[
a(x)^{-1} = \begin{cases} 
  (\chi_0 \ast \tilde{a}^{-1})(x) & \text{if } x \leq 0, \\
  (\chi_n \ast \tilde{a}^{-1})(x) & \text{if } n \in \mathbb{N}_0 \text{ and } x \in [y_n - 4^{-1}h_n - 4^{-1}, y_n + 4^{-1}h_n + 4^{-1})
\end{cases}
\]
Then \( a(y) = h_n \) for all \( y \in [y_n - 8^{-1}h_n, y_n + 8^{-1}h_n] \) and \( \int_{y_n}^{y_{n+1}} dx a(x)^{-1} = 1 \) for all \( n \in \mathbb{N} \). Hence \( d(y_n, y_{n+1}) = 1 \) for all \( n \in \mathbb{N} \). But \( a(y_n) = (n + 1) a(y_{n+1}) \) for all \( n \in \mathbb{N} \). Therefore Condition [III] of Proposition 3.1 is not valid. In particular the group \( T \) does not extend to
any of the other \( L_p \) spaces. Next we show that the semigroup \( S \) also does not extend to another \( L_p \) space.

Let \( p \in [1, \infty) \), \( t > 0 \) and let \( q \) be the dual exponent of \( p \). For all \( n \in \mathbb{N} \) set \( I_n = [y_n - 8^{-1}h_n, y_n + 8^{-1}h_n] \). Let \( n \in \mathbb{N} \). Set \( \varphi = 1_{I_{n+1}} \) and \( \psi = 1_{I_n} \). Then \( \| \varphi \|_p = |I_{n+1}|^{1/p} \) and \( \| \psi \|_q = |I_n|^{1/q} \). Moreover,

\[
(\psi, S_t \varphi) = (4\pi t)^{-1/2} \int_{I_n} dx \int_{I_{n+1}} dy a(y)^{-1} e^{-d(x,y)^2(4t)^{-1}} \geq (4\pi t)^{-1/2} \int_{I_n} dx \int_{I_{n+1}} dy a(y)^{-1} e^{-3d(x,y)^2t^{-1}} = (4\pi t)^{-1/2} |I_n| |I_{n+1}| h_{n+1}^{-1} e^{-3d(x,y)^2t^{-1}}.
\]

So

\[
\| S_t \|_{p \to p} \geq (4\pi t)^{-1/2} |I_n|^{1/p} |I_{n+1}|^{1/q} h_{n+1}^{-1} e^{-3d(x,y)^2t^{-1}} = (64\pi t)^{-1/2} (n + 1)^{1/p}.
\]

Hence the operator \( S_t \) on \( L_\infty \) does not extend to a continuous operator on \( L_p \) for any \( p \in [1, \infty) \) or \( t > 0 \). \( \square \)

In the next example the coefficient \( a \) of \( X \) is uniformly bounded above and below by a positive constant but \( \sup a' = \infty \). The semigroup \( S \) extends to a continuous semigroup on all the \( L_p \)-spaces but the real part of the generator of \( S \) on \( L_2 \) is not lower semibounded. This contrasts with the case of continuous self-adjoint semigroups where boundedness of the semigroup immediately implies lower semiboundedness of the generator.

**Example 4.2** First, let \( \rho = 1 \) and let \( \chi \in C^\infty(\mathbb{R}) \) be such that \( 0 \leq \chi \leq 3 \), \( \chi' \geq 0 \), \( \chi(x) = 0 \) if \( x \leq 0 \), \( \chi(x) = 3 \) if \( x \geq 3 \) and \( \chi(x) = x \) if \( 1 \leq x \leq 2 \). Define \( a: \mathbb{R} \to [1, 4] \) by

\[ a(x) = 1 + \sum_{n=1}^\infty \left( \chi(n(x - 16n)) - \chi(n(x - (16n + 8))) \right). \]

Thus \( a = 1 \) on an infinite sequence of intervals of length almost equal to 8 spaced at distance 8 one from the other. On the intermediate intervals \( a \) increases smoothly to the value 4 and then decreases in a similar fashion to the value 1. The rate of increase and decrease, however, becomes larger with the distance of the interval from the origin. Nevertheless \( a \in C^\infty(\mathbb{R}) \) and the bounds of Proposition 3.1.1 are valid with \( C = 4 \) and \( \omega = 0 \). In particular \( S_t \) extends to the \( L_p \)-spaces and \( \| S_t \|_{p \to p} \leq 4^{1/p} \).

Secondly, let \( n \in \mathbb{N} \) with \( n \geq 4 \). Let \( \psi \in C^\infty(\mathbb{R}) \) be such that \( \psi(x) = 3 \) for all \( x \leq 16n + 8 \), \( 0 \leq \psi' \leq n^{1/2} \), \( \psi'(x) = 0 \) for all \( x \geq 16n + 8 + 4n^{-1} \) and \( \psi'(x) = n^{1/2} \) for all \( x \in [16n + 8 + n^{-1}, 16n + 8 + 2n^{-1}] \). Then \( 3 \leq \psi(16n + 8 + 4n^{-1}) \leq 5 \). Now define \( \varphi \in C_c^\infty(\mathbb{R}) \) by

\[
\varphi(x) = \begin{cases} 
\chi(x - (16n + 4)) & \text{if } x \leq 16n + 8 \\
\psi(x) & \text{if } x \in [16n + 8, 16n + 8 + 4n^{-1}] \\
3^{-1}\psi(16n + 8 + 4n^{-1})\left(3 - \chi(x - (16n + 8 + 4n^{-1})\right) & \text{if } x \geq 16n + 8 + 4n^{-1}
\end{cases}
\]
Then \( \| \varphi \|_2 \leq 5 \cdot (12)^{1/2} = (300)^{1/2} \) and
\[
\| \varphi' \|_2 \leq 2 \| \chi' \|_{\infty} + n^{1/2}(4n^{-1})^{1/2} + 3^{-1}\psi(16n + 8 + 4n^{-1}) \| \chi' \|_{\infty} \leq 2 + 4 \| \chi' \|_{\infty} .
\]
But \( a' a \varphi \varphi' \leq 0 \) and
\[
-(a' \varphi, X\varphi) \geq \int_{16n+8+2n^{-1}} 16n+8+2n^{-1} (-a' a \varphi \varphi') \geq \int_{16n+8+2n^{-1}} n \cdot 2 \cdot 3 \cdot n^{1/2} = 6n^{1/2}
\]
by the previous estimates. Therefore
\[
\text{Re}(\varphi, H_{\min}\varphi) = \| X\varphi \|_2^2 + \text{Re}(a' \varphi, X\varphi) \leq \| a \|_{\infty}^2 (2 + 4 \| \chi' \|_{\infty})^2 - 8n^{1/2} \leq -300^{-1} \left( 6n^{1/2} - 16(2 + 4 \| \chi' \|_{\infty})^2 \right) \| \varphi \|_2^2 .
\]
Consequently, \( \text{Re} H_{\min} \) is not lower semibounded. This is despite the uniform boundedness of \( S \) on \( L_2 \).

Next, since \( S \) is uniformly bounded on each of the \( L_p \)-spaces, the spectrum \( \sigma(H) \) of the generator \( H \) of the semigroup on \( L_p \) is contained in the right half-plane. But \( a(x) \in [1,4] \) for all \( x \in \mathbb{R} \). Therefore \( 4^{-1}|x - y| \leq d(x; y) \leq |x - y| \) and Proposition 2.3 implies that
\[
K_t(x; y) \leq (4\pi t)^{-1/2} e^{-|x-y|^2/(64t)} t > 0 .
\]
for all \( x, y \in \mathbb{R} \) and \( t > 0 \). Hence it follows from Kun or LiV that \( \sigma(H) \) is independent of \( p \in [1, \infty] \). On the other hand \( \text{Re} H_{\min} \) is not lower semibounded on \( L_2 \) and the above estimates establish that \( (-\infty, 0] \subset \Theta(H) \), the \( L_2 \)-numerical range of \( H \). Therefore \( \Theta(H) \neq \sigma(H) \) on \( L_2 \).

In fact this example illustrates the extreme situation that the spectrum of \( H \) is contained in the right half plane but the numerical range is the whole complex plane. This follows since one can establish that the numerical range \( \Theta(H) = C \) by a small modification of the foregoing estimates applied to the function \( \tilde{\varphi} \in C_0^\infty(\mathbb{R}) \) defined by
\[
\tilde{\varphi}(x) = e^{i\lambda x} \tau(x) + \varphi(x) ,
\]
where \( \lambda \in \mathbb{R} \) and \( \tau \in C_0^\infty((-1,4)) \) is fixed such that \( 0 \leq \tau \leq 1 \) and \( \tau|_{[0,3]} = 1 \). One also uses the observation that the numerical range is convex.

Finally note that the semigroup \( S \) has a bounded holomorphic extension to the open right half-plane on each of the \( L_p \)-spaces, \( p \in [1, \infty] \). This follows from the explicit form of the kernel given in Propositions 2.3. Therefore the operator \( H \) is of type \( S_{0^+} \). Nevertheless, since \( \Theta(H) = C \) the operator \( H \) is not sectorial.

5 Volume doubling

Let \( V(x ; r) \) denote the measure of the ball of radius \( r \) centred at \( x \), i.e., the set \( \{ y : d(x ; y) < r \} = (e^{-rX}x, e^{rX}x) \). Then \( V \) is defined, as usual, to have the volume doubling property if there is a \( c > 0 \) such that
\[
V(x ; 2r) \leq c V(x ; r)
\]
for all \( r > 0 \). This property can be immediately related to the conditions of Proposition 3.3 which are necessary and sufficient for the continuous extension of \( T \) to the \( L_p \)-spaces.
Proposition 5.1

I. If the equivalent conditions of Proposition 3.1 are satisfied then

\[ V(x; 2r) \leq 2C^2 e^{3\omega} V(x; r) \quad (10) \]

for all \( x \in \mathbb{R} \) and \( r \in (0, 1] \) where \( C \) and \( \omega \) are the parameters of Proposition 3.1. Moreover if \( \omega = 0 \) then (10) is valid for all \( x \in \mathbb{R} \) and \( r > 0 \).

II. If there exist \( c > 0 \) and a function \( v: (0, \infty) \rightarrow \mathbb{R} \) such that

\[ c^{-1} v(r) \leq V(x; r) \leq cv(r) \]

for all \( x \in \mathbb{R} \) and \( r \in (0, 1] \) then Condition III of Proposition 3.1 is satisfied with \( \omega = 0 \).

Proof

It follows by definition that

\[ V(x; r) = \int_{-rXx}^{e^{rXx}} dy \rho(y) . \]

But

\[ \frac{d}{dr} V(x; r) = (a\rho)(e^{rXx}) + (a\rho)(e^{-rXx}) . \]

Hence

\[ V(x; r) = \int_0^r ds \left( (a\rho)(e^{sXx}) + (a\rho)(e^{-sXx}) \right) = \int_{-r}^r ds (a\rho)(e^{sXx}) . \]

Therefore if Condition III of Proposition 3.1 is satisfied one estimates that

\[ 2C^{-1} r e^{-\omega r} (a\rho)(x) \leq V(x; r) \leq 2C r e^{\omega r} (a\rho)(x) \]

for all \( x \in \mathbb{R} \) and \( r > 0 \). These bounds imply (11) for all \( x \in \mathbb{R} \) and \( r \in (0, 1] \) or, if \( \omega = 0 \), for all \( r > 0 \).

If, however, the assumptions of the second statement are valid then

\[ c^{-1} v(r) \leq V(x; r) = \int_0^r ds (a\rho)(e^{sXx}) + (a\rho)(e^{-sXx}) \leq r \max_{y \in [e^{-Xx}, e^{Xx}]} (a\rho)(y) \]

for all \( x \in \mathbb{R} \) and \( r \in (0, 1] \). Similarly

\[ cv(r) \geq r \min_{y \in [e^{-Xx}, e^{Xx}]} (a\rho)(y) . \]

Hence there exists a \( c_1 > 0 \) such that \( c_1^{-1} r \leq v(r) \leq c_1 r \) for all \( r \in (0, 1] \). But then

\[ 2(a\rho)(x) = \lim_{r \downarrow 0} r^{-1} \int_0^r ds (a\rho)(e^{sXx}) + (a\rho)(e^{-sXx}) = \lim_{r \downarrow 0} r^{-1} V(x; r) \leq \limsup_{r \downarrow 0} r^{-1} c v(r) \leq cc_1 \]

for all \( x \in \mathbb{R} \). Similarly \( 2(a\rho)(x) \geq (cc_1)^{-1} \). Hence \( 2cc_1^{-1} \leq a\rho \leq 2^{-1} cc_1 \) and Condition III of Proposition 3.1 is satisfied with \( \omega = 0 \). \( \square \)
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