On Bipartite Graphs Having Minimum Fourth Adjacency Coefficient

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Abstract

Let $G$ be a simple graph with order $n$ and adjacency matrix $A(G)$. The characteristic polynomial of $G$ is defined by $\phi(G; \lambda) = \det(\lambda I - A(G)) = \sum_{i=0}^{n} a_i(G)\lambda^{n-i}$, where $a_i(G)$ is called the $i$-th adjacency coefficient of $G$. Denote by $\mathcal{B}_{n,m}$ the collection of all connected bipartite graphs having $n$ vertices and $m$ edges. A bipartite graph $G$ is referred as 4-Sachs optimal if

$$a_4(G) = \min \{a_4(H) \mid H \in \mathcal{B}_{n,m}\}.$$ 

For any given integer pair $(n, m)$, in this paper we investigate the 4-Sachs optimal bipartite graphs. Firstly, we show that each 4-Sachs optimal bipartite graph is a difference graph. Then we deduce some structural properties on 4-Sachs optimal bipartite graphs. Especially, we determine the unique 4-Sachs optimal bipartite $(n, m)$-graphs for $n \geq 5$ and $n - 1 \leq m \leq 2(n - 2)$. Finally, we provide a method to construct a class of cospectral difference graphs, which disprove a conjecture posed by Andelić et al. (J Czech Math 70:1125–1138, 2020).

Keywords Sachs subgraph • Matching • Characteristic polynomial • Young matrix • Partitions of positive integer

1 Introduction

Throughout the paper graphs are undirected and simple. Denote by $\mathcal{B}_{n,m}$ the collection of all connected bipartite $(n, m)$-graphs, in which each of them contains $n$ vertices and $m$ edges.

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Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A = A(G) = (a_{ij})_{n \times n}$ of $G$ is defined as $a_{ij} = 1$ if and only if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of $G$ is defined by

$$
\phi(G; \lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} a_i(G)\lambda^{n-i},
$$

where $a_i(G)$ is called the $i$-th adjacency coefficient of $G$. More researches concerning the characteristic polynomial of graphs can be found in the literature [1, 6, 8–10, 12, 19, 21, 22] and references therein.

A difference graph is a bipartite graph $G$ whose vertices are partitioned as $X \cup Y$ with $X = \{x_1, x_2, \ldots, x_l\}$ and $Y = \{y_1, y_2, \ldots, y_h\}$ with the properties that

- $(x_1, y_h)$ and $(x_l, y_1)$ are edges in $G$, and
- whenever $(x_k, y_p)$ is an edge in $G$, then so is $(x_i, y_j)$ for any $i \leq k$ and $j \leq p$.

Difference graphs are also called threshold bipartite graphs [20], double nested graphs [2, 5] and chain graphs [24]. Difference graphs have many beautiful structure and possess many important mathematical properties such as being the extreme cases of bipartite graph properties, see e.g., [15, 18, 20, 24].

Two graphs are referred as nonisomorphic cospectral if they are nonisomorphic and share the same spectrum, with respect to their adjacency matrices. Finding or characterizing families of nonisomorphic cospectral graphs is an interesting and challenging problem, which was proposed by Günthard and Primas [14]. In [18], Lazzarin et al. showed that no two nonisomorphic threshold graphs are cospectral. Recently, Andelić et al. conjectured that difference graphs are analogous, that is, there do not exist nonisomorphic cospectral difference graphs; see Conjecture 3.3 in [1].

Denote by $o(G)$ and $c(G)$ respectively the number of components and the number of cycles contained in $G$. The subgraph $H$ of $G$ is called a $p$-Sachs subgraph if the order of $H$ is $p$ and each component of $H$ is either a single edge or a cycle. The following combinatorial interpretation related to the coefficients of a graph in terms of its $i$-Sachs subgraphs is well known; see example [9, Theorem 1.3], that is,

$$
a_i(G) = \sum_{H}(-1)^{o(H)}2^c(H),
$$

where the summation is over all $i$-Sachs subgraphs contained in $G$. For convenience, we below refer $a_i(G)$ as the $i$-Sachs number of $G$.

The problem of characterizing the extremal graphs that maximize or minimize a certain parameter within a certain category of graphs has received much attention [4, 5, 7, 16, 17, 23]. For a given $(n, m)$-graph $G$, from Eq. (1.1), $a_i(G)$ is fixed for each $i$ with $i \in \{0, 1, 2\}$ and $a_i(G)$, $i \geq 3$, is subjected by its structural properties. Especially, $a_3(G)$ is related to the number of its triangles; see [6, Corollary 8.1.3(c)]. Naturally, the problem of characterizing the extremal graphs maximizing or minimizing 4-Sachs number within a certain category of graphs is curious.
In this paper, we focus on studying the bipartite graphs having minimum 4-Sachs number. The bipartite graph $G$ is referred as 4-Sachs optimal if

$$a_4(G) = \min \{a_4(H) \mid H \in \mathcal{B}_{n,m} \}.$$ 

The value $\min \{a_4(H) \mid H \in \mathcal{B}_{n,m} \}$ is called the minimum 4-Sachs number in $\mathcal{B}_{n,m}$.

The rest of the paper is organized as follows. In Sect. 2, we give some preliminary results, showing that each 4-Sachs optimal bipartite graph is a difference graph. Then we establish some formulas on 4-Sachs number of a difference graph in Sect. 3. In Sect. 4, we first deduce some structural properties on 4-Sachs optimal graphs. Then we determine all 4-Sachs optimal $(n, m)$-graphs together with the corresponding minimal 4-Sachs number, for $n \geq 5$ and $n - 1 \leq m \leq 2(n - 2)$. Finally, we provide a method to construct a class of nonisomorphic cospectral difference graphs in Sect. 5, which disproves the conjecture posed by Andelić et al.; see Conjecture 3.3 [1].

## 2 Preliminary

Let $G = (V, E)$ be a graph with $u \in V$. Denote by $N_G(u)$ the neighbors of the vertex $u$. Vertices $u$ and $v$ of $G$ are called duplicate if $N_G(u) = N_G(v)$. Denote by $\text{dis}(u, v)$ the distance between vertices $u$ and $v$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is defined as $(V_1 \cup V_2, E_1 \cup E_2)$. Let $V_1 \subset V$. The subgraph induced by the vertex set $V_1$ is denoted by $G[V_1]$.

The cycle and the path of order $n$ are denoted by $C_n$ and $P_n$, respectively. The complete bipartite graph with bipartition $(X; Y)$ is denoted by $K_{|X|,|Y|}$.

Let $G$ be a difference graph with bipartition $(X; Y)$. From the definition of difference graphs, if $\{u, v\} \in X$ or $\{u, v\} \in Y$, then

$$N_G(u) \subseteq N_G(v) \quad \text{or} \quad N_G(v) \subseteq N_G(u). \quad (2.1)$$

Therefore, we can further suppose that

$$N(X_1) \supset N(X_2) \supset \ldots \supset N(X_k) \quad \text{and} \quad N(Y_1) \supset N(Y_2) \supset \ldots \supset N(Y_p),$$

where $X = \bigcup_{i=1}^k X_i$ and $Y = \bigcup_{i=1}^p Y_i$ such that, for each $i$, both $X_i$ and $Y_i$ are non-empty, and all elements in $X_i$ (resp. $Y_i$) are duplicate. Obviously, $p = k$. We refer the integer $k$ as the character of $G$; see Fig. 1.

Moreover, applying (2.1), we have

$$N_G(x) = \bigcup_{j=1}^{k-i+1} Y_j \quad \text{for} \quad x \in X_i$$

and

$$N_G(y) = \bigcup_{j=1}^{k-i+1} X_j \quad \text{for} \quad y \in Y_i.$$

Thus, for each $i$, both $G[(\bigcup_{j=1}^{k-i+1} X_j) \cup Y_i]$ and $G[(\bigcup_{j=1}^{k-i+1} Y_j) \cup X_i]$ are complete bipartite. Consequently, the vertex set sequence $(X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_p)$
Fig. 1 A difference graph with character $k$

![Image](image.png)

determines the difference graph $G$ and vice versa. Let $|X_i| = x_i$ and $|Y_i| = y_i$ for $i = 1, 2, \ldots, k$. For convenience, we refer $(X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k)$ and $(x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k)$ as the vertex-bipartition and the vertex-eigenvector of the difference graph $G$, respectively. Obviously, the complete bipartite graph $K_{n,m}$ is a difference graph with character 1 and vertex-eigenvector $(n; m)$.

Let $G$ be a difference graph with vertex-bipartition $(X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k)$. Denoted by $G[(\bigcup_{j=1}^{i-1} X_j) \cup Y_{k-i+1}]$ the subgraph induced by the vertex set $V(G) \setminus (\bigcup_{j=1}^{i-1} X_j \cup Y_{k-i+1})$. One can verify that, for each $i$, $G[(\bigcup_{j=1}^{i} X_j) \cup Y_{k-i+1}]$ is a difference graph with vertex-bipartition $(X_{i+1}, \ldots, X_k; Y_1, \ldots, Y_{k-i})$.

In addition, we need to introduce the following lemma, which is an equivalent description of difference graphs.

**Lemma 1** [15, Proposition 2.5 (2)] The connected bipartite graph $G$ is difference if and only if $G$ has no induced subgraph $P_5$.

Let $u$ and $v$ be two vertices of the graph $G$. Define

$$N_{G}(u, v) = \{x \in V(G) \setminus \{u, v\} : xu \in E(G), xv \in E(G)\}$$

and

$$N_{G}(u, \bar{v}) = \{x \in V(G) \setminus \{u, v\} : xu \in E(G), xv \notin E(G)\}.$$  

Denote by $G_{u \rightarrow v}$ the graph formed by deleting all edges between $u$ and $N_{G}(u, v)$ and adding all edges from $v$ to $N_{G}(u, \bar{v})$. This operation is called the compression of $G$ from $u$ to $v$; see Definition 2.4 in [17]. It is clear that $G_{u \rightarrow v}$ has the same number of edges as $G$.

A $k$-matching in a given graph $G$ is a subset with $k$ edges such that every vertex of $V(G)$ is incident with at most one edge in it. The $k$-matching number, denoted by $\mathbf{m}_k(G)$, is defined as the number of $k$-matchings contained in $G$. Due to Keough and Radcliffe [17], a result comparing the number of $k$-matchings between $G$ and $G_{u \rightarrow v}$ was given as follows.
Lemma 2  [17, Lemma 3.2] For all graphs $G$ and all $u,v \in V(G)$

$$m_k(G) \geq m_k(G_{u \rightarrow v}).$$

Applying the method parallel to the proof of Lemma 2, we can obtain a more strengthened result on counting the number of $k$-matchings, $k \geq 2$, of a graph. Since the proof is similar to that of Lemma 2, we omit the detail.

Lemma 3  Let $G$ be a graph and $u,v \in V(G)$. Then for any $k (k \geq 2)$

$$m_k(G) \geq m_k(G_{u \rightarrow v})$$

with equality holding if and only if either $N_G(u,v) = \emptyset$ or $N_G(u,v) = \emptyset$.

Applying Lemma 3, we have

Theorem 4  Let $G$ be a graph with $u,v \in V(G)$. If $\text{dis}(u,v) = 2$, then

$$a_4(G) \geq a_4(G_{u \rightarrow v})$$

with equality holding if either $N_G(u,v) = \emptyset$ or $N_G(u,v) = \emptyset$.

Proof  Let $H := G_{u \rightarrow v}$. Denote by $Q(G)$ the set of all quadrangles of $G$ and set $q(G) = |Q(G)|$. From (1.1), $a_4(G) = m_2(G) - 2q(G)$. Then, applying Lemma 3, it is sufficient to prove that

$$q(H) \geq q(G).$$

(2.2)

To prove (2.2), we construct an injection from $Q(G) \setminus Q(H)$ to $Q(H) \setminus Q(G)$ that preserves the number of quadrangles. Firstly, we define a replacement function $r : E(G) \rightarrow E(H)$ by

$$r(e) = \begin{cases} 
va, & \text{if } e = ua \text{ with } a \in N_G(u); \\
ub, & \text{if } e = vb \text{ with } b \in N_G(u,v); \\
e, & \text{otherwise.}
\end{cases}$$

Given $e \in E(G)$, we claim that $r(e)$ is an edge in $H$. If $y \in N_G(u)$, then $r(uy) = vy \in E(H)$; if $y \in N_G(u,v)$, then $r(vy) = uy \in E(H)$ and $r(e) = e \in E(H)$ if $e \in E(G) \setminus (E_1 \cup E_2)$, where $E_1 = \{ux \mid x \in N_G(u)\}$ and $E_2 = \{vx \mid x \in N_G(u,v)\}$.

Now we define an injection $\phi : Q(G) \setminus Q(H) \rightarrow Q(H) \setminus Q(G)$ by

$$\phi(C) = \{r(e) \mid e \in C, C \in Q(G) \setminus Q(H)\},$$

where $C$ is an arbitrary 4-cycle of $Q(G) \setminus Q(H)$. Then $C$ must contain an edge $uv$ with $w \in N_G(u,v)$ and another edge $ux$ with $x \in N_G(u)$, regardless $x \in N_G(u,v)$ or $x \in N_G(u,v)$, that is, $C = uwxy$ with $y \in N_G(w,x)$. By the definition of $r(e)$, $r(uw) = vw$, $r(ux) = vx$ and $r(e) = e$ if $e \notin \{uw, uy\}$. Then $\phi(C) = vwxyv$ and thus $\phi(C) \in Q(H) \setminus Q(G)$.

It remains to show that $\phi$ has a left inverse. Consider $r' : E(H) \rightarrow E(G)$ defined by
\[
r'(e) = \begin{cases} 
  ua, & \text{if } e = va \text{ with } a \in N_G(u); \\
  vb, & \text{if } e = ub \text{ with } b \in N_G(u,v); \\
  e, & \text{otherwise.}
\end{cases}
\]

Define \( \phi' : Q(H) \setminus Q(G) \rightarrow Q(G) \setminus Q(H) \) by \( \phi'(C) = \{ r(e) : e \in C \} \). It is straightforward to check that \( \phi'(\phi(C)) = C \). Thus \( \phi \) has a left inverse and so \( \phi \) is injective. Consequently, the result follows.

**Remark 1** Let \( G \) be a graph with \( u, v \in V(G) \). Then, by the same method, the result \( a_4(G) \geq a_4(G_{u\leftrightarrow v}) \) is also true if \( \text{dis}(u,v) > 2 \). The restriction ensures that the resultant graph \( G_{u\leftrightarrow v} \) is connected.

Combining with Lemma 1 and Theorem 4, we have

**Theorem 5** Each 4-Sachs optimal bipartite graph is a difference graph.

**Proof** Let \( G \) be a 4-Sachs optimal bipartite graph. Assume that \( G \) contains an induced subgraph \( P_5 \), then there exist vertices \( u \) and \( v \) such that \( u \) and \( v \) lie in the same partition and satisfy

\[
N_G(u) \not\supseteq N_G(v) \quad \text{and} \quad N_G(v) \not\supseteq N_G(u).
\]

Applying Theorem 4, \( a_4(G) > a_4(G_{u\leftrightarrow v}) \), which is a contradiction to the optimality of the given graph \( G \). Then \( G \) is \( P_5 \)-free and thus \( G \) is a difference graph by Lemma 1.

**3 Formulas on 4-Sachs Number of Difference Graphs**

From Theorem 5, each 4-Sachs optimal bipartite graph is difference. Then we are focus on difference graphs. In [1], a formula on the \( p \)-Sachs number of a difference graph is given in terms of the theory of tridiagonal matrix; see Theorem 3.1 [1]. In the following, we will establish a combinatorial version formula on the \( p \)-Sachs number of a difference graph.

Let \( G \) be a graph, \( C \) an even cycle of \( G \) with length \( 2l \) and \( H \) a \( 2r \)-Sachs subgraph of \( G \). Suppose that \( r \geq l \). We say the cycle \( C \) is embedded in \( H \) if \( C \cap H \) forms a \( 2l \)-Sachs subgraph and \( C \cup H \) forms a \( 2r \)-Sachs subgraph; see [13]. Applying the formula (1.1), we have

**Lemma 6** Let \( G \) be a bipartite graph and \( C_4 \) a given 4-cycle of \( G \). Denote by \( \mathcal{H}(C_4, 2r) \) the set of all \( 2r \)-Sachs subgraphs, of \( G \), embedding the cycle \( C_4 \). Then

\[
\sum_{H \in \mathcal{H}(C_4, 2r)} (-1)^{o(H)} 2^{e(H)} = 0,
\]

where the summation is over all \( 2r \)-Sachs subgraphs of \( \mathcal{H}(C_4, 2r) \).
Obviously, \( r \geq 2 \). Let \( C_4 = x_1y_1x_2y_2x_1 \). Since \( G \) is bipartite, 
\[ G[\{x_1, y_1, x_2, y_2\}] = C_4 \]. If \( r = 2 \), then \( H((C_4, 2r)) \) contain exactly three elements: two disjoint 2-matchings of \( C_4 \), named as \( M_1 = \{x_1y_1, x_2y_2\} \) and \( M_2 = \{y_1x_2, y_2x_1\} \), and \( C_4 \) itself. Thus

\[
\sum_{H \in H(C_4, 2r)} (-1)^{\sigma(H)} 2^{\sigma(C_4)} = (-1)^{\sigma(C_4)} 2^{\sigma(C_4)} + (-1)^{\sigma(M_1)} 2^{\sigma(M_1)} + (-1)^{\sigma(M_2)} 2^{\sigma(M_2)} = 0.
\]

If \( r > 2 \), then each \( H \in H(C_4, 2r) \) contains \( M_1 \) or \( M_2 \) or \( C_4 \) as a subgraph. Let \( H = H_1 \cup H_2 \), where \( H_1 \) is the \( 2r - 4 \)-Sachs subgraph of \( G \setminus C_4 \) and \( H_2 \in \{M_1, M_2, C_4\} \). Thus

\[
\sum_{H \in H(C_4, 2r)} (-1)^{\sigma(H)} 2^{\sigma(H)} = \sum_{H_i \in G \setminus C_4} (-1)^{\sigma(H_i)} 2^{\sigma(C_4)} \left( (-1)^{\sigma(M_1)} 2^{\sigma(M_1)} + (-1)^{\sigma(M_2)} 2^{\sigma(M_2)} \right)
\]

\[= 0. \]

Consequently, the result follows.

As a consequence of Lemma 6, a formula on 4-Sachs number of difference graphs can be deduced as follows.

**Theorem 7** Let \((X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k)(k \geq 1)\) be the vertex-bipartition of the difference graph \( G \). Then

\[
a_4(G) = \sum_{i=1}^{k-1} a_2(G[X_i; \cup_{j=1}^{k-1} Y_j]) a_2(G[X_i; \cup_{j=1}^{k-1} Y_j]). \tag{3.1}
\]

**Proof** Recall that \( a_2(G) \) equals the opposite number of the edges contained in \( G \) by Eq. (1.1). By the discussion above, \( E(G) \) can be partitioned as

\[ \bigcup_{i=1}^{k} E(G[X_i; \cup_{j=1}^{k-1} Y_j]). \]

Applying Lemma 6, to compute \( a_4(G) \), it is sufficient to count the number of all 2-matchings in which no two edges are contained in any quadrangle. Let \( M \) be such an 2-matching. If \( e \in G[X_i; \cup_{j=1}^{k-1} Y_j] \) \( (i = 1, 2, \ldots, k - 1) \), then the another edges of \( M \) must be contained in \( G[X_i; \cup_{j=1}^{k-1} Y_j] \). Conversely, each pair of edges \((e_1, e_2)\) with \( e_1 \in G[X_i; \cup_{j=1}^{k-1} Y_j] \) and \( e_2 \in G[X_i; \cup_{j=1}^{k-1} Y_j] \) forms a 2-matching which is not embedded in any quadrangle. Consequently, the result follows.

To simplify the formula (3.1), we introduce the Young diagram, which can be used to represent difference graphs more intuitively; see [17].
**Definition 1** Let \((X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k)\) be the vertex-bipartition of the difference graph \(G\). The Young diagram, or Young matrix, \(Y(G) = (y_{ij})\) is defined by: First, we set the rows of \(Y\) correspond to the vertices \(x_1^1, \ldots, x_1^{X_1}, \ldots, x_k^1, \ldots, x_k^{X_k}\) and the columns of \(Y\) correspond to the vertices \(y_1^1, \ldots, y_1^{Y_1}, \ldots, y_k^1, \ldots, y_k^{Y_k}\), respectively. Then we define \(y_{ij} = 1\) if and only if the vertices corresponding to the row \(i\) and the column \(j\) are adjacent, and \(y_{ij} = 0\) otherwise.

**Theorem 8** Let \(G\) be a difference graph with Young matrix \(Y(G)\), defined as above. Denote by \(r_1, r_2, \ldots, r_h\) the row sum of \(Y\), respectively. Then

\[
a_4(G) = \sum_{i=1}^{h-1} \sum_{j=i+1}^{h} (r_i - r_{i+1})i r_j. \tag{3.2}
\]

**Proof** Suppose that \((X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k)\) is the vertex-bipartition of \(G\). Then \(h := \sum_{i=1}^{k} |X_i|\) and \(r_1 \geq r_2 \geq \ldots \geq r_h\) obviously. If \(r_i - r_{i+1} > 0\) for some \(i\), then there exists an integer \(t\) such that \(\sum_{s=1}^{t} |X_s| = i\), thus \((c_i - c_{i+1})i\) is equal to the number of edges contained in \(G[\bigcup_{j=1}^{t} X_j; Y_{k-t+1}]\) and \(\sum_{j=i+1}^{h} r_j\) is equal to the number of edges contained in the difference graph \(G[\bigcup_{j=1}^{t} X_j; Y_{k-t+1}]\). Then applying Theorem 7, the result follows (Fig. 2).

**Example 1** Let \(G\) be a difference graph with vertex-eigenvector \((1, 2, 2, 1; 2, 2, 3, 1)\). The corresponding Young matrix \(Y\) and graph are respectively given as follows:

![Fig. 2 A difference graph with vertex-eigenvector (1, 2, 2, 1; 2, 2, 3, 1)]
Then the character of $G$ is 4 and the row sum sequence of $Y$ is $(r_1, r_2, r_3, r_4, r_5, r_6) = (8, 7, 7, 4, 4, 2)$. Applying the formula (3.2), we have $$a_4(G) = (8 - 7) \times 1 + 12 + (7 - 4) \times 3 + 10 + (4 - 2) \times 5 \times 2 = 134.$$ 

From Theorem 8, the problem of determining the minimum 4-Sachs number among all difference graphs in $B_{n,m}$ is equivalent to the following optimization problem.

$$\min \sum_{i=1}^{h-1} \sum_{j=i+1}^{h} (r_i - r_{i+1})i r_j,$$

s.t.

$$\begin{cases}
    r_1 \geq r_2 \geq \ldots \geq r_h > 0; \\
    r_1 + h = n; \\
    h \geq \left\lfloor \frac{n-1}{2} \right\rfloor; \\
    \sum_{i=1}^{h} r_i = m,
\end{cases}$$

where $r_1, r_2, \ldots, r_h$ denotes the row sum sequence with respect to its Young diagram. The optimization problem above is related to the partition of a positive integer, which was first studied by Leibniz; see [3, 11].

### 4 4-Sachs Optimal Bipartite Graphs

Although the problem on computing the minimum 4-Sachs number among all difference graphs in $B_{n,m}$ is equivalent to the optimization problem above, it is difficult to solve the optimization problem. In this section, we will deduce some structural properties on 4-Sachs optimal bipartite graphs. Especially, we determine the unique 4-Sachs optimal bipartite graphs in $B_{n,m}$ with $n \geq 5$ and $n - 1 \leq m \leq 2(n - 2)$.

In view of the fact that $a_4(G) \geq 0$ holds for any bipartite graph $G$, then the following result is obvious.

**Theorem 9** Let positive integers $t$, $n$ and $m$ satisfy $m = t(n - t)$. Then the complete bipartite graph $K_{t,n-t}$ is the unique bipartite optimal graph in $B_{n,m}$. 

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Since each difference graph with character 1 is complete bipartite, we are focus on studying the difference graphs with character at least 2. Motivated by the Young diagrams, we define another matrix, named as the characteristic matrix, as follows.

**Definition 2** Let \( (x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k) \) be the vertex-eigenvector of the difference graph \( G \). The characteristic matrix \( T(G) = (t_{ij})_{k \times k} \) is defined as follows: 

\[
T = \begin{pmatrix}
    x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{k-1} & x_1 y_k \\
    x_2 y_1 & x_2 y_2 & \cdots & x_2 y_{k-1} & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    x_{k-1} y_1 & x_{k-1} y_2 & \cdots & 0 & 0 \\
    x_k y_1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

Let \( A = (a_{ij})_{n \times n} \) be a matrix and \( S, T \) be two sub-index sets of \( \{1, 2, \ldots, n\} \). Set \( \{1, 2, \ldots, n\} =: \langle n \rangle \) and \( S := \langle n \rangle \setminus S \). Denote by \( A(S; T) \) the submatrix of \( A \) by deleting the rows indicated by \( \overline{S} \) and the columns indicated by \( \overline{T} \). The column matrix \( A(\langle n \rangle; \{i\}) \) will be written as \( A(\cdot; i) \) for simplication. In addition, we use \( s(A) \) to denote the sum of all entries in \( A \). Then we have

**Theorem 10** Let \( G \) be a difference graph with vertex-eigenvector \( (x_1, \ldots, x_k; y_1, \ldots, y_k) \). Suppose that the characteristic matrix of \( G \) is \( T = (t_{ij})_{k \times k} \). Then

\[
a_4(G) = \sum_{i=1}^{k-1} s(T(\cdot; k - i + 1))s(T(\langle i \rangle; \langle k - i \rangle)). \tag{4.1}
\]

**Proof** Applying Theorem 7, we only need to count the number of those 2-matchings in which each pair of them is not embedded in any quadrangle. Let \( M \) be such a matching. If \( e \in M \) is contained in \( G[\bigcup_{j=1}^i X_j; Y_{k-i+1}] \), then the another edge of \( M \) must be contained in \( G[\bigcup_{j=1}^i X_j; Y_{k-i+1}] \). Note that the number of edges contained in \( G[\bigcup_{j=1}^i X_j; Y_{k-i+1}] \) is \( s(T(\cdot; k - i + 1)) \) and the number of edges contained in \( G[\bigcup_{j=1}^i X_j; Y_{k-i+1}] \) is \( s(T(\langle i \rangle; \langle k - i \rangle)) \). Thus the result follows. \( \square \)

**Example 2** Let \( G \) be the difference graph described as Example 1. Then its characteristic matrix \( T \) is

\[
T = \begin{pmatrix}
    2 & 2 & 3 & 1 \\
    4 & 4 & 6 & 0 \\
    4 & 4 & 0 & 0 \\
    2 & 0 & 0 & 0
\end{pmatrix}.
\]

Applying the formula (4.1), we have
Applying the formula (4.1), we can deduce some properties on the vertex-eigenvector of a 4-Sachs optimal bipartite graph as follows.

**Theorem 11** Let \( G \) be a 4-Sachs optimal bipartite graph in \( \mathcal{B}_{n,m} \). Let also \((x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k) (k \geq 2)\) be its vertex-eigenvector. Suppose that \( \sum_{i=1}^{k} x_i \geq \sum_{j=1}^{k} y_j \). Then

\[
x_1 > y_1.
\]

**Proof** Assume to the contrary that \( x_1 \leq y_1 \), say \( y_1 = x_1 + y_1^* \) with \( y_1^* > 0 \). Let \( G_1 \) be the difference graph with vertex-eigenvector \((x_1 + x_2, x_3, \ldots, x_k, y_k; x_1, y_1^*, y_2, \ldots, y_{k-1})\). Then

\[
T_1 := T(G) = \begin{pmatrix}
x_1 (x_1 + y_1^*) & x_1 y_2 & \cdots & x_1 y_{k-1} & x_1 y_k \\
x_2 (x_1 + y_1^*) & x_2 y_2 & \cdots & x_2 y_{k-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{k-1} (x_1 + y_1^*) & x_{k-1} y_2 & \cdots & 0 & 0 \\
x_k (x_1 + y_1^*) & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

and

\[
T_2 := T(G_1) = \begin{pmatrix}
(x_1 + x_2) x_1 & (x_1 + x_2) y_1^* & (x_1 + x_2) y_2 & \cdots & (x_1 + x_2) y_{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{k-1} x_1 & x_{k-1} y_1^* & x_{k-1} y_2 & \cdots & 0 \\
x_k x_1 & x_k y_1^* & 0 & \cdots & 0 \\
x_1 y_k & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

One can find that both \( T_1 \) and \( T_2 \) are square matrices with order \( k \). Applying Theorem 10, we have
\[ a_4(G) - a_4(G_1) = \sum_{i=1}^{k-1} s(T_1(\cdot; k-i+1)) s(T_1(\{i\}; \{k-i\})) - \sum_{i=1}^{k-1} s(T_2(\cdot; k-i+1)) s(T_2(\{i\}; \{k-i\})) \]

\[ = x_1^2 y_k \left( \sum_{i=2}^{k} x_i - \sum_{j=2}^{k-1} y_j - y_1^* \right) \]

\[ = x_1^2 y_k \left( \sum_{i=1}^{k} x_i - \sum_{j=1}^{k-1} y_j \right) \]

\[ > 0, \]

which yields a contradiction to the hypothesis that \( G \) is 4-Sachs optimal. Consequently, the result follows.

Applying all preliminary results above, we can determine the unique 4-Sachs optimal bipartite graphs together with its corresponding minimal 4-Sachs number in \( B_{n,m} \) with \( n \geq 5 \) and \( n - 1 < m < 2(n - 2) \) as follows.

**Theorem 12** Let \( n \geq 5 \) and \( n - 1 < m < 2(n - 2) \). Then the difference graph with vertex eigenvector \((1, 1; m-n+2, 2n-4-m)\) is the unique 4-Sachs optimal bipartite graph in \( B_{n,m} \).

**Proof** Let \( G \) be a 4-Sachs optimal bipartite graph in \( B_{n,m} \). By Theorem 5, \( G \) is difference. Suppose that the vertex-eigenvector of \( G \) is \((x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k)\). Since \( n - 1 < m < 2(n - 2) \), \( k \geq 2 \). By Theorem 9, \( y_1 = 1 \). Then

\[ a_4(G) \geq x_k \left( m - \sum_{i=1}^{k} x_i \right) \]

with equality if and only if \( k = 2 \). Moreover, \( \sum_{i=1}^{k-1} x_i \leq \frac{m-x_k}{2} \) with equality holding if and only if \( \sum_{i=1}^{k} y_i = 2 \), then \( m - \sum_{i=1}^{k} x_i \geq \frac{m-x_k}{2} \) with equality holding if and only if \( \sum_{i=1}^{k} y_i = 2 \). Consequently,

\[ a_4(G) \geq \frac{x_k(m-x_k)}{2} = (2n-4-m)(m-n+1) \quad (4.2) \]

with equality if and only if \( k = 2 \) and \( y_2 = 1 \). Thus the vertex eigenvector of \( G \) is \((1, 1; m-n-2, 2n-4-m)\), whose character is 2. Consequently, the result follows.

**5 Nonisomorphic Cospectral Difference Graphs**

In this section, we provides a method to construct a class of nonisomorphic cospectral difference graphs.
Firstly, we give a combinatorial interpretation on the coefficients of the characteristic polynomial of a difference graph. Let
\[ b_{2i}(G) = (-1)^i a_{2i}(G). \]

As a consequence of Lemma 6, a recurrence formula for \( b_{2i}(G) \), \( i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \), of difference graphs can be given as follows.

**Theorem 13** Let \((X_1, X_2, \ldots, X_k; Y_1, Y_2, \ldots, Y_k) (k \geq 1)\) be the vertex bipartition of the difference graph \( G \). Then
\[
\begin{equation}
\begin{split}
b_{2}(G) &= \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} |X_i||Y_j|, \\
b_{2l}(G) &= \sum_{i=1}^{k} b_{2}(G[\bigcup_{j=1}^{i} X_j; Y_i])b_{2l-2}(G[\bigcup_{j=1}^{i} X_j; Y_i])
\end{split}
\end{equation}
\]

for \( l = 2, \ldots, k \) and \( b_p(G) = 0 \) for \( p > 2k \).

**Proof** Applying Lemma 6, to compute \( b_{2l}(G) \) \( (l \geq 2) \), it is sufficient to count the number of all \( l \)-matchings in which no two edges are contained in any cycle. Let \( H \) be an \( l \)-matchings in which no two edges are contained in an arbitrary cycle. If \( e \in G[\bigcup_{j=1}^{i} X_j; Y_i] \) \( (i = 1, 2, \ldots, k - 1) \), then all other edges of \( H \) must contained in \( G[\bigcup_{j=1}^{i} X_j; Y_i] \). Conversely, each pair of edges \((e_1, e_2)\) with \( e_1 \in G[\bigcup_{j=1}^{i} X_j; Y_i] \) and \( e_2 \in G[\bigcup_{j=1}^{i} X_j; Y_i] \) forms a 2-matching which does not embedded in any cycle. Consequently, the result follows by Theorem 7. \( \square \)

Let \( M \) be an \( m \times n \) matrix. Denote by \( r_i(M) (i = 1, 2, \ldots, m) \) the \( i \)-th row vector of \( M \) and by \( c_i(M) (i = 1, 2, \ldots, n) \) the \( i \)-th column vector of \( M \).

**Definition 3** Let \( m, n \) and \( k \) be positive integer with \( k \geq 2 \) and let \( M, X \) and \( Y \) be matrices with order \( m \times n, km \times n \) and \( m \times kn \), respectively. We refer \( X \) as a \( k \)-row expansion from \( M \) if \( r_i(M) = r_{ki-k+1}(X) = \ldots = r_{ki}(X) \) for \( i = 1, 2, \ldots, m \), and refer \( Y \) as a \( k \)-column expansion from \( M \) if \( c_i(M) = c_{kj-k+1}(Y) = \ldots = c_{kj}(Y) \) for \( j = 1, 2, \ldots, n \).

**Example 3** Let
\[
Y(G_1) = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
Y(G_2) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then matrices \( Y_1 \) and \( Y_2 \) are respectively a 2-column expansion and a 2-row expansion from \( M \).
Theorem 14 Let $Y_i$ be Young matrices of the difference graphs $G_i$ for $i = 1, 2, 3$. Suppose that $Y_1$ is an asymmetrical $n \times n$ matrix, and $Y_2$ and $Y_3$ are, respectively, a $t$-column expansion from $Y_1$ and a $t$-row expansion from $Y_1$ with $t \geq 2$. Then $G_2$ and $G_3$ are nonisomorphic cospectral graphs.

Proof Obviously, both $G_2$ and $G_3$ are difference and have the same number of vertices and edges.

Let the bipartition of $G_i$ be $(U_i; W_i)$ for $i = 1, 2, 3$. Suppose further that the vertex corresponding to the $p$-th row vector of $Y_i$ is labelled as $u^i_p$ and the vertex corresponding to the $q$-th column vector of $Y_i$ is labelled as $w^i_q$. Denote by $x^i_p$ and $y^i_p$ the degree of vertices $u^i_p$ and $w^i_q$, respectively. Then the degree sequence of each graph $G_i$ is the union of two subsequence $(x^i_1, x^i_2, \ldots, x^i_{|U_i|})$ and $(y^i_1, y^i_2, \ldots, y^i_{|W_i|})$.

Recall that, for each $i$, $Y_i$ is the Young matrix of the difference graph $G_i$, then both $(x^i_1, x^i_2, \ldots, x^i_{|U_i|})$ and $(y^i_1, y^i_2, \ldots, y^i_{|W_i|})$ are descending order. By the relations of $Y_1$, $Y_2$ and $Y_3$, the cardinal number of the sets $U_i$ and $W_i$ for $i = 1, 2, 3$ satisfy

$$|W_2| = |U_3| = t \mid U_1 | = t \mid W_1 | = t \mid U_2 | = t |W_3| = m,$$

which compels that

$$(x^2_1, x^2_2, \ldots, x^2_n) = (y^3_1, y^3_2, \ldots, y^3_n)$$

if $G_2$ and $G_3$ are isomorphic. However, note that $Y_1$ is asymmetrical, then

$$(x^1_1, x^1_2, \ldots, x^1_n) \neq (y^1_1, y^1_2, \ldots, y^1_n)$$

and thus

$$(x^2_1, x^2_2, \ldots, x^2_n) \neq (y^3_1, y^3_2, \ldots, y^3_n)$$

as $(x^2_1, x^2_2, \ldots, x^2_n) = (x^1_1, x^1_2, \ldots, x^1_n)$ and $(y^3_1, y^3_2, \ldots, y^3_n) = (y^1_1, y^1_2, \ldots, y^1_n)$. Consequently, $G_2$ and $G_3$ are nonisomorphic.

Suppose now that the vertex-eigenvector of $G_1$ is $(x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k)$. Since $Y_2$ is a $k$-column expansion from $Y_1$, the vertex-eigenvector of $G_2$ is $(x_1, x_2, \ldots, x_k; ty_1, ty_2, \ldots, ty_k)$. Similarly, the vertex-eigenvector of $G_3$ is $(tx_1, tx_2, \ldots, tx_k; y_1, y_2, \ldots, y_k)$. Then one can verify that $G_2$ and $G_3$ have the same characteristic matrix. Thus $G_2$ and $G_3$ are cospectral. Consequently, the result follows. 

Theorem 14 provides us a way to construct infinite pair cospectral difference graphs, which refutes Conjecture 3.3 [1].

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