On the Analysis of Optimization with Fixed-Rank Matrices: a Quotient Geometric View

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Abstract

We study a type of Riemannian gradient descent (RGD) algorithm, designed through Riemannian preconditioning, for optimization on $\mathcal{M}^{m \times n}_k$—the set of $m \times n$ real matrices with a fixed rank $k$. Our analysis is based on a quotient geometric view of $\mathcal{M}^{m \times n}_k$: by identifying this set with the quotient manifold of a two-term product space $\mathbb{R}^{m \times k} \times \mathbb{R}^{n \times k}$ of matrices with full column rank via matrix factorization, we find an explicit form for the update rule of the RGD algorithm, which leads to a novel approach to analysing their convergence behavior in rank-constrained optimization. We then deduce some interesting properties that reflect how RGD distinguishes from other matrix factorization algorithms such as those based on the Euclidean geometry. In particular, we show that the RGD algorithm is not only faster than Euclidean gradient descent but also does not rely on balancing techniques to ensure its efficiency while the latter does. We further show that this RGD algorithm is guaranteed to solve matrix sensing and matrix completion problems with linear convergence rate under the restricted positive definiteness property. Numerical experiments on matrix sensing and completion are provided to demonstrate these properties.

Keywords: fixed-rank matrices, Riemannian optimization, quotient manifold, gradient descent methods, nonconvex optimization, matrix recovery

1 Introduction

Optimization with low-rank matrices is a fundamental problem that arises in signal processing, machine learning and computer vision. In principal component analysis, matrix recovery and data clustering for example, the most meaningful information in the data is structured, such that it can be captured by a matrix with an intrinsically low rank [39]. Therefore, low-rank matrix models enjoy the advantage of having a low complexity without compromising the accuracy or representativity. One approach to optimizing low-rank matrix models is by using the matrix nuclear norm [8, 33, 12], which is a convex relaxation of the matrix rank. Another approach is to represent an $m \times n$ matrix $X$ through low-rank matrix factorization such as $X = AB^T$, where $A$ and $B$ are thin factor matrices of size $m \times k$ and $n \times k$ respectively. Although the matrix factorization approach induces a nonconvex optimization problem, it presents several advantages over the previous convex relaxation approach.

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due to much lower cost in memory and computation. Algorithms for low-rank matrix factorization can be regrouped into two main types, alternating minimization [16] [22] [18] and gradient descent algorithms [21] [23] [32]. Recent advances in matrix recovery problems such as compressed sensing and matrix completion [35] [17] [35] [45] shed light on the absence of spurious local minima in these problems under mild conditions, and thus they explain formally the success of (Euclidean) gradient descent algorithms for matrix recovery, despite the nonconvexity of matrix factorization.

Riemannian algorithms, in a similar spirit as Euclidean gradient descent but exploiting non-Euclidean geometries on low-rank matrix space, have shown performances superior to algorithms using the Euclidean geometry. As an important example of low-rank matrix spaces, the set \( \mathcal{M}_k^{m \times n} \) of fixed-rank matrices

\[
\mathcal{M}_k^{m \times n} = \{ X \in \mathbb{R}^{m \times n} : \text{rank}(X) = k \}
\]

is a smooth Riemannian manifold of dimension \((m + n - k)k\) and can be characterized through the factorization of rank-\(k\) matrices (e.g., [11]). Table 1 shows some basic information of different matrix factorization approaches alongside convex relaxation of the matrix rank.

For example, Vandereycken [41] used the rank-\(k\) matrix factorization approaches in the related work, alongside the convex relaxation approach. Wei et al. [42] [36] proposed several variants of the iterative hard thresholding (IHT) algorithm, also based on the embedded Riemannian manifold structure of \(\mathcal{M}_k^{m \times n}\) for matrix recovery problems with the fixed-rank constraint, and provided exact-recovery guarantees of the algorithm for compressed sensing and matrix completion [43] using the restricted isometry property (RIP).

Table 1: Different matrix factorization approaches alongside convex relaxation of the matrix rank.

| Search space | # parameters | Reference |
|--------------|--------------|----------|
| Conv. relax. | \(\mathbb{R}^{m \times n}\) | \(mn\) | [8] [33] [12] |
| \(X = U \Sigma V^T\) | \(\text{St}(m, k) \times S_{++}(k) \times \text{St}(n, k)\) | \((m + n + 1)k\) | [11] [29] |
| \(X = U Y^T\) | \(\text{St}(m, k) \times \mathbb{R}^{n \times k}\) | \((m + n)k\) | [5] |
| \(X = G H^T\) | \(\mathbb{R}_*^{m \times k} \times \mathbb{R}^{n \times k}_* (\mathbb{R}^{m \times k} \times \mathbb{R}^{n \times k})\) | \((m + n)k\) | ours, [27] [28] [42] [36] [37] |

Apart from the embedded manifold view, it is known that \(\mathcal{M}_k^{m \times n}\) [11] can be understood as a quotient space: consider a product space \(\mathcal{M}_k^{m \times n} := \mathbb{R}_*^{m \times k} \times \mathbb{R}^{n \times k}_*\) of matrices with full column rank \(k\), the projection

\[
\pi : \mathbb{R}_*^{m \times k} \times \mathbb{R}^{n \times k}_* \to \mathcal{M}_k^{m \times n} : (G, H) \mapsto G H^T
\]

induces the following equivalence relation \(\sim\):

\[
(G, H) \sim (G', H') \quad \text{if and only if} \quad G H^T = G' H'^T.
\]

Since the equivalent classes of \(\sim\) are the fibers of \(\pi\) and \(\mathcal{M}_k^{m \times n} \subset \mathbb{R}^{m \times n}\) is the image of \(\pi\), the mapping \(\pi\) induces an one-to-one correspondence between \(\mathcal{M}_k^{m \times n}\) and \(\mathbb{R}_*^{m \times k} \times \mathbb{R}^{n \times k}_*/\sim\), hence the quotient structure of \(\mathcal{M}_k^{m \times n}\); see Figure 1. Subsequently, the optimization of a real-valued function \(f\) on \(\mathcal{M}_k^{m \times n}\) can be seen as the following matrix factorization optimization:

\[
\min_{(G, H) \in \mathbb{R}_*^{m \times k} \times \mathbb{R}^{n \times k}_*} \tilde{f}(G, H) := f \circ \pi(G, H).
\]

In view of this quotient structure, Mishra et al. [28] proposed a Riemannian gradient descent (RGD) algorithm using two-term matrix factorization and a preconditioning technique [30] specially adapted to the squares loss function. Tong et al. [37] analyzed the convergence property of
the **preconditioned RGD algorithm** of [28] on the product space $\mathbb{R}_{*}^{m \times k} \times \mathbb{R}_{*}^{n \times k}$ (under the name of ScaledGD) by leveraging a distance on $\mathbb{R}_{*}^{m \times k} \times \mathbb{R}_{*}^{n \times k}$ that is specifically invariant on fixed-rank manifolds. Boumal et al. [5] used the factorization $X = U Y$, where $U \in \mathbb{R}_{*}^{m \times k}$ is considered as a point of the Grassmannian manifold (a quotient space of the Stiefel manifold $\text{St}(m, k)$), and proposed a variable projection method using a Riemannian trust-region algorithm for matrix completion with fixed-rank matrices. We refer to Table 1 and [29, 1] for a thorough overview. More recently, Huang et al. [20] proposed an efficient Riemannian gradient descent algorithm for blind deconvolution based on the quotient manifold structure of the set of rank-1 (complex-valued) matrices with guaranteed convergence using the RIP; Luo et al. [25] proposed a new sketching algorithm on the set of fixed-rank matrices and proved that the algorithm enjoys high-order convergence for low-rank matrix trace regression and phase retrieval under similar conditions.

In this paper, we are interested in how Riemannian algorithms behave in solving problem (2), and how their behavior is related to the landscape of the original problem $\min_{X \in \mathcal{M}^{m \times n}} f(X)$. To answer this question, we start with two noticeable difficulties underlying (2): one difficulty is the nonconvexity of (2), which is inherited from the nonconvex space $\mathcal{M}^{m \times n}$ via the mapping $\pi$. Another difficulty is that $\pi$ being not injective, there is no unique—and in fact an infinite number of—matrix representations $(G, H)$ in $\mathbb{R}_{*}^{m \times k} \times \mathbb{R}_{*}^{n \times k}$ for each $X \in \mathcal{M}^{m \times n}$, and the performances of algorithms such as Euclidean gradient descent generally vary according to the actual locations of the iterates (such as the initial point) inside their respective equivalence classes on $\mathbb{R}_{*}^{m \times k} \times \mathbb{R}_{*}^{n \times k}$. The non-uniqueness of matrix representations also leads to the difficulty that the minima of the problem (2) are degenerate [1]. These difficulties can be visualized in the landscape of even the simplest matrix factorization problem: Figure 2 shows the graph of $\bar{f} : \mathbb{R}_{*} \times \mathbb{R}_{*} \to \mathbb{R} : (x, y) \mapsto \frac{1}{2}(xy - A)^2$, for $A = 1$, along with two different paths (in red and purple) passing through the equivalence classes on $\mathbb{R}_{*} \times \mathbb{R}_{*}$. Concretely, we investigate the properties of the RGD algorithm of [28] through the lens of quotient geometry of $\mathcal{M}^{m \times n}$. We present new results leading to the
confirmation that this algorithm bypasses the above-mentioned difficulties of \(\text{[2]}\). Our results about the RGD algorithm \([28]\) also show substantial differences between algorithms designed with an explicit metric on the quotient manifold \(M_{m,n}^{m\times n}\) and algorithms that are not based on the quotient geometry of \(M_{m,n}^{m\times n}\), such as the Riemannian algorithms using metric projection \([41, 42, 36, 43]\) and algorithms based on the Euclidean geometry \([35]\). More precisely, the main contributions are as follows.

1.1 Contributions

First, we prove that the RGD algorithm of \([28]\) (on \(\mathbb{R}^{m\times k} \times \mathbb{R}^{n\times k}\)) induces a sequence on \(M_{m,n}^{m\times n}\) that admits an explicit update rule, and that the sequence is invariant to changes of its iterate \((G, H) \in \mathbb{R}^{m\times k} \times \mathbb{R}^{n\times k}\) in the equivalence classes. Second, we analyse the convergence behavior of the algorithm under the restricted positive definiteness (RPD) property \([40]\) in solving the following low-rank matrix optimization problems:

**Example 1** (Compressed sensing). Let \(M^*\) be an \(m \times n\) matrix and \(b^* = \Phi(M^*)\) a vector observed through a matrix sensing operator, \(\Phi : \mathbb{R}^{m\times n} \to \mathbb{R}^{d}\). Recover \(M^*\) by minimizing \(f(X) := \frac{1}{2}\|\Phi(X) - b^*\|^2_2\) with a low-rank matrix \(X\).

**Example 2** (Matrix completion). Let \(M^* \in \mathbb{R}^{m\times n}\) be a matrix observed only on an index set \(\Omega \subset [m] \times [n]\). Complete \(M^*\) by minimizing \(f(X) := \frac{1}{2}\|P_\Omega(X - M^*)\|^2_F\) with a low-rank matrix \(X\).

Through the RPD property of the objective function \(f\) around the true hidden matrix \(M^*\), we demonstrate the existence of a region of attraction on \(M_{m,n}^{m\times n}\) in which the landscape of \(f\) in the ambient space \(\mathbb{R}^{m\times n}\) is preserved. Then, we prove results about the local convergence rate of the quotient geometry-based RGD algorithm for the minimization of \(f\) on \(M_{m,n}^{m\times n}\). To the best of our knowledge, this is the first convergence rate analysis of the RGD algorithm of \([28]\) using quotient geometry of the fixed-rank matrix manifold.

We illustrate the main results through numerical experiments: we show that the quotient manifold-based algorithms not only enjoy the benefits of low-rank matrix factorization but also present desirable invariance properties that the Euclidean gradient descent does not possess. In particular, its convergence behavior does not vary with changes in the balancing between the factor matrices, while the performance of Euclidean gradient descent can be easily deteriorated due to unbalanced factor matrices.

In summary: (i) we show formally that the Riemannian gradient descent algorithm under the aforementioned quotient geometric setting enjoys desirable invariance properties. A new result (Lemma\([11]\) about this quotient manifold-based RGD algorithm is given; (ii) for a class of low-rank matrix optimization problems, we provide new results about the geometric properties of the RGD algorithm around critical points (Lemma\([16]\) under the restricted positive definiteness property, about sufficient descent conditions for RGD (Lemma\([20]\) and Corollary\([21]\)), and about the local linear convergence of RGD (Theorem\([23]\)) on \(M_{m,n}^{m\times n}\).

1.2 Preliminaries

Given the integers \(m, n \geq 1\) and \(k \leq \min(m, n)\),

\[
\mathcal{M}_{k}^{m,n} = \mathbb{R}_{+}^{m\times k} \times \mathbb{R}_{+}^{n\times k}
\]  

(3)
denotes the product space of \(m \times k\) and \(n \times k\) real matrices with full column-rank. The set of \(m \times k\) orthonormal real matrices, i.e., the Stiefel manifold, is denoted by \(\text{St}(m, k)\). The set of \(k \times k\) positive definite matrices is denoted by \(S_{++}(k)\).
A point in $\mathcal{M}_k^{m,n}$ is denote by $\bar{x} = (G, H)$, $(G, H)$ or simply $\bar{x}$ indifferently. By default, the symbols $G$ and $H$ signify the $m \times k$ and $n \times k$ matrices of $\bar{x}$ respectively; they also constitute a pair of left and right factor matrices of $X = GH^T$. For any $\bar{x} \in \mathcal{M}_k^{m,n}$, the tangent space to $\mathcal{M}_k^{m,n}$ at $\bar{x}$ is $T_{\bar{x}} \mathcal{M}_k^{m,n} = \mathbb{R}^{m \times k} \times \mathbb{R}^{n \times k}$. A tangent vector $\bar{\xi} \in T_{\bar{x}} \mathcal{M}_k^{m,n}$ is denoted as $\bar{\xi} = (\bar{\xi}^{(1)}, \bar{\xi}^{(2)})$. For any $Y \in \mathbb{R}^{m \times k}$, the Euclidean metric on the tangent space $T_Y \mathbb{R}^{m \times k} \simeq \mathbb{R}^{m \times k}$ is defined and denoted by $\langle V, W \rangle := \text{tr}(V^T W)$, for $V, W \in T_Y \mathbb{R}^{m \times k}$, where $\langle \cdot, \cdot \rangle$ is also called the Frobenius inner product.

The index set $\{1, \ldots, n\}$ is denoted as $[n]$. For any $X \in \mathcal{M}_k^{m \times n}$, the $k$-th singular value of $X$, i.e., the minimal non-zero singular value, is denoted as $\sigma_k^*(X)$. For brevity, we denote the maximal and the minimal non-zero singular values of a given constant matrix $M^*$ by $\sigma_{\text{max}}^*$ and $\sigma_k^*$, respectively. The row vector from the $i$-th row of a matrix $X$ is denoted as $X_{i,:}$ and the 2-norm of the matrix row as $\|X_{i,:}\|_2$. The maximal row norm of a matrix $X$ with $m$ rows is denoted and defined as $\|X\|_{2,\infty} := \max_{1 \leq i \leq m} \|X_{i,:}\|_2$. The spectral norm of a symmetric positive semidefinite matrix $A$ and the operator norm of the linear operator $A : X \mapsto AX$ are denoted by $\|A\|_2$. We denote the adjoint of a linear operator $A$ by $A^*$.

**The RPD property.** Due to the rank constraint of (9), the optimization of $f$ on $\mathcal{M}_k^{m \times n}$ is nonconvex. However, some properties of the quadratic function $f$ are preserved on rank-constrained matrix spaces. The following restricted positive definiteness (RPD) property [40] characterizes the well-conditionedness of $f$ on the set of rank-constrained matrices.

**Definition 3** (RPD property [40, Definition 3.1]). For an integer $r \geq 1$ and a parameter $0 \leq \beta_r < 1$, the operator $A$ of $f$ satisfies a $(\beta_r, r)$-RPD property if

$$\begin{align*}
(1 - \beta_r)\|Z\|^2_F &\leq \langle Z, A(Z) \rangle \leq (1 + \beta_r)\|Z\|^2_F,
\end{align*}
$$

for any $Z \in \mathcal{M}_{\leq r}$. The smallest $\beta_r \geq 0$ for the $(\beta_r, r)$-RPD property to hold is called the RPD constant of $A$ on $\mathcal{M}_{\leq r}$.

The RPD property is equivalent to the restricted isometry property (RIP) condition in the literature of compressed sensing. This property can be satisfied with overwhelmingly high probability for a large family of random measurement matrices, for example, the normalized Gaussian and Bernoulli matrices [13, 33, 42]. For matrix completion, the RPD property also holds under reasonable assumptions on the hidden true matrix $M^*$ and the sampling distribution [9].

**1.3 Organization**

The rest of this paper is organized as follows. In Section 2 we present algorithms based on a Riemannian preconditioning technique on $\mathcal{M}_k^{m,n}$ and then the quotient manifold structure of $\mathcal{M}_k^{m \times n}$ related to these algorithms. In Sections 3 we propose the main results about the RGD algorithm of Section 2. In Section 4 we present a new convergence analysis of the RGD with performance guarantees for matrix recovery problems on $\mathcal{M}_k^{m \times n}$. Numerical experiments and results are presented in Section 5. We conclude the paper in Section 6.

**2 Algorithms on $\mathcal{M}_k^{m,n}$ and quotient geometry of $\mathcal{M}_k^{m \times n}$**

In this section, we revisit two Riemannian gradient-based algorithms, which are designed for solving (2) through Riemannian preconditioning. The preconditioning technique induces a Riemannian metric on $\mathcal{M}_k^{m,n}$, which turns out to be invariant on the equivalence classes of the matrix product.
mapping \( \pi \) in the total space \( \bar{M}_{k}^{m,n} \). This invariance property is essential to the quotient structure of \( \bar{M}_{k}^{m,n} \) and the properties of the RGD algorithm, as we will present in Section 2.2 and later in Section 3.

### 2.1 Gradient descent on \( \bar{M}_{k}^{m,n} \) by Riemannian preconditioning

First, we describe a non-Euclidean metric on the product space \( \bar{M}_{k}^{m,n} \) used in a Riemannian algorithm for low-rank matrix completion [28].

**Definition 4** ([28]). Given \( \bar{x} := (G,H) \in \bar{M}_{k}^{m,n} \), let \( \bar{g}_{\bar{x}} : T_{\bar{x}}\bar{M}_{k}^{m,n} \times T_{\bar{x}}\bar{M}_{k}^{m,n} \) denote an inner product defined as follows,

\[
\bar{g}_{\bar{x}}(\bar{\xi},\bar{\eta}) = \text{tr}(\bar{\xi}^{(1)T}\bar{\eta}^{(1)}(H^{T}H)) + \text{tr}((\bar{\xi}^{(2)})^{T}\bar{\eta}^{(2)}(G^{T}G)),
\]

for \( \bar{\xi}, \bar{\eta} \in T_{\bar{x}}\bar{M}_{k}^{m,n} \).

It can be shown that \( \bar{g}_{\bar{x}}(\cdot,\cdot) \) is a Riemannian metric since it is symmetric and positive-definite at any \( \bar{x} \in \bar{M}_{k}^{m,n} \) and it is a smooth-varying bilinear form on \( \bar{M}_{k}^{m,n} \). By definition, the Riemannian gradient of a function \( \bar{f} \) at \( \bar{x} \in \bar{M}_{k}^{m,n} \) is the unique vector, denoted as \( \text{grad} \bar{f}(\bar{x}) \in T_{\bar{x}}\bar{M}_{k}^{m,n} \), such that \( \bar{g}_{\bar{x}}(\bar{\xi},\text{grad} \bar{f}(\bar{x})) = D\bar{f}(\bar{x})[\bar{\xi}], \forall \bar{\xi} \in T_{\bar{x}}\bar{M}_{k}^{m,n} \). Therefore, given the metric (5), the Riemannian gradient of \( \bar{f} \) has the following form,

\[
\text{grad} \bar{f}(\bar{x}) = (\partial_{G}\bar{f}(\bar{x})(H^{T}H)^{-1}, \partial_{H}\bar{f}(\bar{x})(G^{T}G)^{-1}),
\]

where \( \partial_{G}\bar{f}(\bar{x}) \) and \( \partial_{H}\bar{f}(\bar{x}) \) are the two partial differentials of \( \bar{f} \).

The metric \( \bar{g} \) defined in (3) was proposed by Mishra et al. [28] for matrix completion (Example 2) using fixed-rank matrices, and can be seen as a metric deduced from the Riemannian preconditioning technique [30] in the context of two-term matrix factorization with a squares-loss function \( \bar{f}(G,H) = \frac{1}{2}\|GH^{T} - M^{*}\|_{F}^{2} \). Indeed, one can see that the Riemannian gradient (6) is in fact the solution to the secant equation in \( \bar{\xi} : \mathcal{H}(\bar{\xi}) = (\partial_{G}\bar{f}(\bar{x}), \partial_{H}\bar{f}(\bar{x})) \), where \( \mathcal{H} : T_{\bar{x}}\bar{M}_{k}^{m,n} \rightarrow T_{\bar{x}}\bar{M}_{k}^{m,n} \) is defined as

\[
\mathcal{H}(\bar{\xi}) := \begin{pmatrix}
\partial_{G}^{2}\bar{f}(G,H) & 0 \\
0 & \partial_{H}^{2}\bar{f}(G,H)
\end{pmatrix}(\bar{\xi}) = (\bar{\xi}^{(1)}H^{T}H, \bar{\xi}^{(2)}G^{T}G).
\]

Note that \( \mathcal{H}(\bar{\xi}) \approx D^{2}\bar{f}(G,H)[\bar{\xi}] \). Therefore, the Riemannian gradient (6) is an approximation of the Newton direction of \( \bar{f} \) at \( (G,H) \). The metric \( \bar{g} \) (5) is also referred to as the preconditioned metric on \( \mathbb{R}^{m \times k}_{+} \times \mathbb{R}^{n \times k}_{+} \).

**Riemannian gradient descent (RGD)** For a matrix factorization problem in the form of (2), we present the RGD as in Algorithm 1. The search direction is set to be the negative Riemannian gradient (6). The operation needed for the gradient descent update step (line 4) on \( \bar{M}_{k}^{m,n} \) is chosen to be the identity map, which is a valid retraction operator on the \( \bar{M}_{k}^{m,n} \). One can find the use of the identity map as a retraction in a total space: \( \bar{M}_{k}^{m,n} \) and \( \mathbb{C}^{n \times k}_{+} \times \mathbb{C}^{m \times n}_{+} \), respectively, in [28] and [20]. The stepsize \( \theta_{t} \) in each iteration is obtained following a backtracking line search procedure with respect to the line search condition (5). In this backtracking procedure, the initial trial stepsize \( \theta^{0}_{t} \) (line 3) is important to the time efficiency of algorithm. We consider the following methods for setting the initial trial stepsize, including notably (i) exact line minimization (e.g., [41]), and (ii) Riemannian Barzilai–Borwein (RBB) stepsize rules [21], which have been shown to be very efficient. The following two RBB rules are considered,

\[
\theta^{BB1}_{t} := \frac{\bar{g}_{\bar{x}_{t-1}}(\bar{z}_{t-1}, \bar{z}_{t-1})}{\bar{g}_{\bar{x}_{t}}(\bar{z}_{t-1}, \bar{y}_{t-1})}, \quad \theta^{BB2}_{t} := \frac{|\bar{g}_{\bar{x}_{t}}(\bar{z}_{t-1}, \bar{y}_{t-1})|}{|\bar{g}_{\bar{x}_{t}}(\bar{z}_{t-1}, \bar{y}_{t-1})|},
\]

where \( \bar{z}_{t-1} = \bar{x}_{t} - \bar{x}_{t-1} \) and \( \bar{y}_{t-1} = \text{grad} \bar{f}(\bar{x}) - \text{grad} \bar{f}(\bar{x}_{t-1}) \).
Algorithm 1: Riemannian Gradient Descent (RGD) using the precomputed metric

**Input:** Initial point $\bar{x}_0 \in \mathcal{M}_k^{m,n}$, parameters $\beta, \sigma \in (0,1)$, $\theta_0 > 0$ and $\epsilon > 0$; $t = 0$.

**Output:** $\bar{x}_t \in \mathcal{M}_k^{m,n}$.

1. **while** $\|\text{grad} f(\bar{x}_t)\| > \epsilon$ **do**
2.  Set $\bar{\eta}_t = -\text{grad} f(\bar{x}_t)$ using (11).
3.  Backtracking line search: find the smallest integer $\ell \geq 0$ such that, for $\theta_t := \theta_0 \beta^\ell$,
   \[
   \bar{f}(\bar{x}_t) - \bar{f}(\bar{x}_t + \theta_t \bar{\eta}_t) \geq \sigma \theta_t g_{\bar{x}_t}(-\text{grad} \bar{f}(\bar{x}_t), \bar{\eta}_t).
   \]
   (8)
4.  Update: $\bar{x}_{t+1} = \bar{x}_t + \theta_t \bar{\eta}_t$; $t \leftarrow t + 1$.
5.  **end while**

### Riemannian Conjugate Gradient Descent (RCG)

Based on the same computational elements as for Algorithm 1, we also consider a Riemannian conjugate gradient (Qprecon RCG) algorithm on the total space $\mathcal{M}_k^{m,n}$. With the same Riemannian gradient definition, the search direction $\bar{\eta}$ of the RCG algorithm at the $t$-th iteration is defined as $\bar{\eta}_t = -\text{grad} \bar{f}(\bar{x}_t) + \beta_t \bar{\eta}_{t-1}$, where $\beta_t$ is computed using a Riemannian version of one of the nonlinear conjugate gradient rules. In the numerical experiments, we choose the Riemannian version of the following modified Hestenes-Stiefel rule [19]:

\[
\tilde{\beta} = \max \{0, \frac{\bar{g}_{\bar{x}_t}(\xi_t, t_{\xi_{t-1}, \bar{\eta}_{t-1}})}{\bar{g}_{\bar{x}_t}(\xi_t, t_{\bar{x}_{t-1}, \bar{\eta}_{t-1}})}\},
\]

where $\tilde{\xi}_t := \text{grad} \bar{f}(\bar{x}_t)$ for $t \geq 0$.

### 2.2 Geometry of $\mathcal{M}_k^{m,n}$ as the Quotient of $\mathcal{M}_k^{m,n} / \text{GL}(k)$

As briefly mentioned in Section 2, the projection $\pi : \mathcal{M}_k^{m,n} \to \mathcal{M}_k^{m,n}$: $(G, H) \mapsto GH^T$ induces the equivalence relation $\sim$ on $\mathcal{M}_k^{m,n}$. The structure of the fibers of $\pi$ can be characterized by the linear group $\text{GL}(k)$: in fact, the operations $(G, H) \mapsto (GF^{-1}, HF^T)$ for $F \in \text{GL}(k)$ correspond to all possible transformations that leave the matrix product $X = GH^T$ unchanged. This means that $X = \pi((G, H))$ is on an one-to-one correspondence with $\{(GF^{-1}, HF^T): F \in \text{GL}(k)\}$, which defines the equivalence class $\{(G', H') \in \mathcal{M}_k^{m,n}: G'H'^T = GH^T\}$. Hence the identification $\mathcal{M}_k^{m,n} \simeq \mathcal{M}_k^{m,n} / \text{GL}(k)$. Moreover, the quotient space $\mathcal{M}_k^{m,n}$ is a quotient submanifold of $\mathcal{M}_k^{m,n}$ [2] §3.4. The product space $\mathcal{M}_k^{m,n}$ is referred to as the total space.

The structure of the tangent space to $\mathcal{M}_k^{m,n}$ can be deduced as follows [2, §3]. Given a matrix $X \in \mathcal{M}_k^{m,n}$ and a tangent vector $\xi \in T_X \mathcal{M}_k^{m,n}$, the mapping $\pi : \mathcal{M}_k^{m,n} \to \mathcal{M}_k^{m,n}$ induces infinitely many representations of $\xi$: for $\bar{x} \in \pi^{-1}(X)$, any element $\bar{\xi} \in T_{\bar{x}} \mathcal{M}_k^{m,n}$ that satisfies $\text{D} \pi(\bar{x})[\bar{\xi}] = \xi$ can be considered as a representation of $\xi$. Indeed, for any smooth function $f : \mathcal{M}_k^{m,n} \to \mathbb{R}$, the function $\tilde{f} := f \circ \pi : \mathcal{M}_k^{m,n} \to \mathbb{R}$, one has the following identification,

\[
\text{D} \tilde{f}(\bar{x})[\bar{\xi}] = \text{D} f(\pi(\bar{x}))[\text{D} \pi(\bar{x})[\bar{\xi}]] = \text{D} f(X)[\xi].
\]

Since $\pi : \mathcal{M}_k^{m,n} \to \mathcal{M}_k^{m,n}$ is surjective, the kernel of $\text{D} \pi(\bar{x}) : T_{\bar{x}} \mathcal{M}_k^{m,n} \to T_X \mathcal{M}_k^{m,n}$ is non-trivial, the matrix representation of $\xi$ in $T_{\bar{x}} \mathcal{M}_k^{m,n}$ is not unique. Nevertheless, one can find a unique representation of $\xi$ in a subspace of $T_{\bar{x}} \mathcal{M}_k^{m,n}$. This is realized by decomposing the tangent space $T_{\bar{x}} \mathcal{M}_k^{m,n} \simeq \mathbb{R}^{m \times k} \times \mathbb{R}^{n \times k}$ into two complementary subspaces, i.e., $T_{\bar{x}} \mathcal{M}_k^{m,n} = \mathcal{V}_{\bar{x}} \oplus \mathcal{H}_{\bar{x}}$, where $\mathcal{V}_{\bar{x}} := T_{\bar{x}}(\pi^{-1}(X))$, called the vertical space, is the tangent space at $\bar{x}$ to the equivalence class $[\bar{x}]$, and $\mathcal{H}_{\bar{x}}$, called the horizontal space, is the complementary of $\mathcal{V}_{\bar{x}}$ in $T_{\bar{x}} \mathcal{M}_k^{m,n}$. One can see that a tangent vector $\xi \in \mathcal{V}_{\bar{x}}$ satisfies $\text{D} \pi(\bar{x})[\bar{\xi}] = 0$.

Consequently, for any $X \in \mathcal{M}_k^{m,n}$ and $\xi \in T_X$, there is a unique representation $\bar{\xi} \in \mathcal{H}_{\bar{x}} \subset T_{\bar{x}} \mathcal{M}_k^{m,n}$ of $\xi$ such that $\text{D} \pi(\bar{x})[\bar{\xi}] = \xi$. The tangent vector $\bar{\xi} \in \mathcal{H}_{\bar{x}}$ is called the horizontal lift of $\xi$. 7
Given the horizontal lifts as the matrix representation of tangent vectors to the quotient manifold $\mathcal{M}_k^{m \times n}$, any metric $\bar{g}$ on the total space that satisfies following invariance property induces a metric on $\mathcal{M}_k^{m \times n}$.

**Definition 5** ([1] §3). For $X \in \mathcal{M}_k^{m \times n}$ and $\bar{x} \in \pi^{-1}(X)$, let $\bar{\xi}, \bar{\eta} \in T_{\bar{x}}\mathcal{M}_k^{m,n}$ denote the horizontal lifts of $\xi$ and $\eta$ respectively, a metric $\bar{g}$ in the total space is said to be invariant along $\pi^{-1}(X)$ if $\bar{g}_\bar{x}(\bar{\xi}, \bar{\eta}) = \bar{g}_\bar{x}(\bar{\xi}_\eta, \bar{\eta}_\beta)$ for any point $\bar{y} \sim \bar{x}$ in $\pi^{-1}(X)$, where $\bar{\xi}_\eta, \bar{\eta}_\beta$ denote the horizontal lifts of $\xi$ and $\eta$ at $\bar{y}$ respectively.

Indeed, given a metric $\bar{g}$ that satisfies the invariance property in Definition 5, the inner product $g_X : T_X\mathcal{M}_k^{m \times n} \times T_X\mathcal{M}_k^{m \times n} \to \mathbb{R}$ such that $g_X(\xi, \eta) = \bar{g}_\bar{x}(\bar{\xi}, \bar{\eta})$ is a metric on the quotient manifold $\mathcal{M}_k^{m \times n}$. The following proposition shows that the metric $\bar{g}$ induces a metric on $\mathcal{M}_k^{m \times n}$.

**Proposition 6.** For any matrix $X \in \mathcal{M}_k^{m \times n}$, the preconditioned metric $g(5)$ satisfies the invariance property as in Definition 5, that is, for two horizontal lifts $\bar{x} \in \mathcal{M}_k^{m \times n}$ and $\bar{\xi}, \bar{\eta} \in T_{\bar{x}}\mathcal{M}_k^{m,n}$, $\bar{g}_\bar{x}(\bar{\xi}, \bar{\eta}) = \bar{g}_\bar{x}(\bar{\xi}_\eta, \bar{\eta}_\beta)$, for any $\bar{y} \sim \bar{x}$, where $\bar{\xi}, \bar{\eta} \in T_{\bar{x}}\mathcal{M}_k^{m,n}$ are the horizontal lifts at $\bar{x}$ such that $D\bar{\pi}(\bar{x})[\bar{\xi}] = D\pi(\bar{x})[\bar{\xi}]$ and $D\bar{\pi}(\bar{x})[\bar{\eta}] = D\pi(\bar{x})[\bar{\eta}]$.

### 3 Main results about RGD on $\mathcal{M}_k^{m \times n}$

In this section, we investigate how the matrix factorization-based RGD algorithm (Algorithm 1) designed for solving the reformulated problem (2)—behaves in relation with the landscape of the original problem

$$\min_{X \in \mathcal{M}_k^{m \times n}} f(X).$$

(9)

The matrix factorization approach (2) converts the rank constraint of (9) on $\mathbb{R}^{m \times n}$ into an unconstrained problem on $\mathbb{R}_n^{m \times k} \times \mathbb{R}_n^{k \times p}$, which allows for efficient algorithms with a significant reduction in both memory (from $O(mn)$ to $O((m+n)k)$) and computation time costs.

However, some difficulties persist with the reformulation (2). Note that the lifted objective function $\bar{f}$ is invariant along the equivalence classes in $\mathcal{M}_k^{m,n}$, i.e., $f(\bar{x}) = f(\bar{x}')$, for any $\bar{x}' \sim \bar{x}$. This poses issues to algorithms for (2). For example, due to the invariance of $\bar{f}$ in equivalence classes, any (isolated) local minimum of $f$ on $\mathcal{M}_k^{m \times n}$ corresponds to a whole equivalence class in $\mathcal{M}_k^{m,n}$, which contains an infinite number of nondegenerate minima. This makes the landscape of $\bar{f}$ on $\mathcal{M}_k^{m,n}$ different and actually more complicated than (9), even under the smallest dimensions (see Figure 2).

Interestingly, we will deduce concrete results in Section 3.1–3.2 to show that the RGD algorithm (Algorithm 1) bypasses the above difficulties, using the invariance property (Proposition 6) of the preconditioned metric.

### 3.1 Riemannian gradient on the quotient manifold

Under the preconditioned metric $\bar{g}$ (5), the gradient vector field of $\bar{f}$ induces a Riemannian gradient vector field of $f$ in the tangent bundle of $\mathcal{M}_k^{m \times n}$. Indeed, from Proposition 6 the metric $\bar{g}$ (5) is invariant along the equivalence classes and therefore induces a metric $g$ in $\mathcal{M}_k^{m \times n}$ such that, for any $X \in \mathcal{M}_k^{m \times n}$ and $\xi, \eta \in T_X\mathcal{M}_k^{m,n}$,

$$g_X(\eta, \xi) = g_{\bar{x}}(\bar{\eta}, \bar{\xi}),$$

(10)

where $\bar{x}$ is an element in $\pi^{-1}(X)$ and $\bar{\eta}$ and $\bar{\xi}$ are the horizontal lifts (at $\bar{x}$) of $\eta$ and $\xi$ respectively. It follows that, for any $\bar{x} \in \pi^{-1}(X)$, $\bar{\eta} := \text{grad} f(\bar{x})$ and any $\bar{\xi} \in T_{\bar{x}}\mathcal{M}_k^{m,n}$:

$$g_{\bar{\pi}(\bar{x})}(\bar{\xi}, \text{grad} f(\bar{\pi}(\bar{x}))) = g_{\bar{x}}(\bar{\eta}, \bar{\xi}) = D\bar{f}(\bar{x})[\bar{\xi}] = D(f \circ \pi)(\bar{x})[\bar{\xi}] = Df(\pi(\bar{x}))[D\pi(\bar{\xi})].$$
Hence the horizontal component of $\tilde{\eta} = \text{grad} \tilde{f}(\tilde{x})$ is the horizontal lift of $\text{grad} f(X)$ at $\tilde{x} \in \pi^{-1}(X)$. Note that $\text{grad} \tilde{f}(\tilde{x})$ belongs to the horizontal space $\mathcal{H}_X$: since $\tilde{f}$ is invariant along the equivalence classes, it is constant on $\pi^{-1}(X)$, which entails that for any $\tilde{\xi} \in V_\tilde{x} = T_{\tilde{x}}(\pi^{-1}(X)), D\tilde{f}(\tilde{x})[\tilde{\xi}] = \tilde{g}_\tilde{x}(\tilde{\xi}, \text{grad} \tilde{f}(\tilde{x})) = 0$. Therefore, $\text{grad} \tilde{f}(\tilde{x})$ is the horizontal lift of $\text{grad} f(X)$ at $\tilde{x}$.

As a consequence, all horizontal lifts $\text{grad} \tilde{f}(\tilde{x})$ for different $\tilde{x} \in \pi^{-1}(X)$ are equivalent. This means the induced Riemannian gradient $\text{grad} f(X)$ is invariant to the location of $\tilde{x}$ in the equivalence class $\pi^{-1}(X)$; which can be reflected by an explicit form of $\text{grad} f(X)$ independent of $\tilde{x}$ as follows.

**Proposition 7.** Given a matrix $X \in \mathcal{M}_k^{m \times n}$, let $X = U \Sigma V^T$ denote its SVD. Let $g$ be the metric on $\mathcal{M}_k^{m \times n}$ induced by the preconditioned metric [5], the Riemannian gradient of $f$ on the quotient manifold $(\mathcal{M}_k^{m \times n}, g)$ satisfies

$$\text{grad} f(X) = \mathcal{G}_X(\nabla f(X)),$$

where $\nabla f(X)$ is the Euclidean gradient of $f$ in $\mathbb{R}^{m \times n}$ and $\mathcal{G}_X : \mathbb{R}^{m \times n} \to T_X \mathcal{M}_k^{m \times n}$ is a linear operator such that, for any $Z \in \mathbb{R}^{m \times n}$,

$$\mathcal{G}_X(Z) = P_U Z + Z P_V,$$

where $P_U := U U^T$ and $P_V := V V^T$ are the orthogonal projections onto the column and row subspaces of $X$ respectively.

**Remark 8.** Proposition [7] is a result in the thesis [14] §5.2.3. In a more recent work [26, Remark 9], the same result is given as an example of the geometric connections of embedded and quotient geometries in Riemannian fixed-rank matrix optimization.

**Remark 9.** In the notation of (12), $P_U$ and $P_V$ are the matrices of the orthogonal projections. We also denote by abuse the action of these projection operators as $P_U(Z) = P_U Z$ and $P_V(Z) = Z P_V$ respectively. The action of $\mathcal{G}_X$ (12) can be rewritten as $\mathcal{G}_X(Z) = P_U Z (I - P_V) + (I - P_U) Z P_V + 2P_U Z P_V$. This implies that the operator $\mathcal{G}_X$ is related to the orthogonal projection $P_{T_X \mathcal{M}_k^{m \times n}}$ (see [11, (2.5)]) by

$$\mathcal{G}_X(Z) = P_{T_X \mathcal{M}_k^{m \times n}}(Z) + P_U Z P_V, \quad \text{for all } Z \in \mathbb{R}^{m \times n}. \quad (13)$$

The next lemma relates the operator $\mathcal{G}_X$ to the orthogonal projection operator $P_{T_X \mathcal{M}_k^{m \times n}}$.

**Lemma 10.** Let $X$ be a matrix in $\mathcal{M}_k^{m \times n}$ and let $X := U \Sigma V^T$ denotes its rank-$k$ SVD. The linear operator $\mathcal{G}_X : \mathbb{R}^{m \times n} \to T_X \mathcal{M}_k^{m \times n}$ defined in (12) satisfies the following properties.

(i): $\mathcal{G}_X$ is symmetric and positive semidefinite. In particular, $\mathcal{G}_X(Z) = 2Z$ if $Z \in T_X \mathcal{M}_k^{m \times n}$, and $\mathcal{G}_X(Z) = 0$ if $Z \in (T_X \mathcal{M}_k^{m \times n})^\perp$.

(ii): Let $X,Y$ be two matrices in $\mathcal{M}_k^{m \times n}$, and let $(\mathcal{G}_X - 2I)(Z) := \mathcal{G}_X(Z) - 2Z$ denote the evaluation of the operator $(\mathcal{G}_X - 2I)$ at $Z \in \mathbb{R}^{m \times n}$. It holds that

$$(\mathcal{G}_X - 2I)(X - Y) = 2(I - P_{T_X \mathcal{M}_k^{m \times n}})(Y), \quad (14)$$

where $P_{T_X \mathcal{M}_k^{m \times n}}$ is the orthogonal projector onto the tangent space to $\mathcal{M}_k^{m \times n}$ at $X$.

### 3.2 Induced sequence in the quotient manifold

Now, we are ready to deduce the image of the sequence $\{\tilde{x}_t\}_{t \geq 0} \subset \mathcal{M}_k^{m,n}$ by RGD (Algorithm 11) under the projection $\pi$. We refer to the image $\{\pi(\tilde{x}_t)\}_{t \geq 0}$ as the *induced* sequence, which is denoted by $\{X_t\}_{t \geq 0} \subset \mathcal{M}_k^{m \times n}$ with $X_t = \pi(\tilde{x}_t) = G_{\tilde{x}_t} H_{\tilde{x}_t}^T$. 


Lemma 11. Let $(\bar{x}_t)_{t\geq 0} \in \mathcal{M}_{k,n}^{m,n}$ denote the sequence generated by Algorithm 4 and let $\{X_t\}_{t\geq 0}$ be the induced sequence in $\mathcal{M}_{k,n}^{m,n}$. Then the iterates satisfy

$$X_{t+1} = X_t - \theta_t \nabla f(X_t) + \theta_t^2 \nabla f(X_t) X_t \nabla f(X_t),$$

where $X_t^\dagger \in \mathbb{R}^{n \times m}$ denotes the Moore–Penrose pseudoinverse of $X_t$.

Note that the characterization in Proposition 7 is not a computational component of the RGD algorithm (Algorithm 1) but is an update rule on $\mathcal{M}_{k,n}^{m,n}$ induced by the algorithm. The quotient manifold-based algorithms in the framework of Section 2.1 (such as Algorithm 1) produce iterates in the total space $\mathcal{M}_{k,n}^{m,n}$.

The invariance property in Proposition 7 and Lemma 11 enables us to qualify these algorithms as Riemannian descent algorithms on the quotient space $\mathcal{M}_{k,n}^{m,n}$. The explicit form in Lemma 11 allows for a local convergence analysis on $\mathcal{M}_{k,n}^{m,n}$ that is substantially different and simpler than those based on the specific matrix factorization of $X \in \mathbb{R}^{m \times n}$, as we present next.

4 Convergence behavior in low-rank matrix recovery

In this section, we consider the problem (9) with a quadratic objective function $f$ as follows:

$$f(X) := \frac{1}{2} \langle X, A(X) \rangle - \langle B^*, X \rangle,$$

where $A : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ is a linear operator that is symmetric and positive semidefinite, $B^* = A(M^*)$ is a data matrix containing observations of an unknown matrix $M^* \in \mathbb{R}^{m \times n}$, and $\langle \cdot, \cdot \rangle$ is the Frobenius inner product of two matrices.

The function $f$ (16) generalizes the objective function of many matrix recovery problems [15, 9, 16, 35, 17, 20]. For compressed sensing (Example 1), the objective function

$$f(X) = \frac{1}{2} (\|\Phi(X) - b^*\|_2^2 - \|b^*\|_2^2)$$

(17)
can be written in the form of (16), with $A = \Phi^* \Phi$ and $B^* = \Phi^* b^*$. For matrix completion (Example 2), the recovery of the hidden matrix $M^*$ can be realized by the minimizing the following quadratic function

$$f(X) = \frac{1}{2p} (\|P_\Omega (X - M^*)\|_F^2 - \|P_\Omega (M^*)\|_F^2),$$

(18)
where $\Omega$ is the index set of the known entries, $p = |\Omega|/mn$, and $P_\Omega : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ is the projection operator that retains only the entries on $\Omega$ and projects all other entries to zero. In this case, $A = \frac{1}{p} P_\Omega$ and $B^* = \frac{1}{p} P_\Omega (M^*)$.

4.1 Critical points

A matrix $X^* \in \mathcal{M}_{\leq k}$ is a critical point for the minimization of $f$ (16) on $\mathcal{M}_{\leq k}$ if the Euclidean gradient $\nabla f(X^*)$ belongs to the polar cone of the tangent cone of $\mathcal{M}_{\leq k}$ at $X^*$. As characterized in [34] and [40, Proposition 2.4], such a critical point $X^*$ can be a critical point of $f$ on $\mathcal{M}_{k,n}^{m,n}$, or a solution to the (global) optimality condition $A(X) - B^* = 0$.

In general, the operator $A$ of (16) admits a nontrivial kernel and there are thus more than one point in $\mathcal{M}_{\leq k}$ that satisfy either of the above two characterizations. In particular, the hidden matrix $M^*$ is a critical point and also the global optimum of (9), as long as the rank of $M^*$ is equal
to $k$ in (9). This scenario is often referred to as the noiseless case in the low-rank matrix recovery problems.

Recent advances in low-rank matrix recovery have results about guarantees of recovering $M^*$ via (9). It is shown in [9] Theorem 3.5 and corollaries], under the RPD property with a bounded RPD constant $\beta_{3k} \leq C_p < 1$ where $C_p$ is a constant depending on the Riemannian Hessian of $f$ (16) at $M^*$, that: (i) the optimality equation $A(X) = B^*$ admits $M^*$ as the unique solution (ii) $M^*$ is the unique global minimum of $f$ and (iii) all other critical points $X' \neq M^*$ of $f$ are strict saddle points. This implies, in particular, that one can use Riemannian descent algorithms on $M_k^{m \times n}$ to find $M^*$, provided that the RPD constant $\beta$ satisfies the given upper-bound condition.

Instead of studying the uniqueness of $M^*$ (as a local minimum of $f$ on $M_k^{m \times n}$), we focus on the properties of $f$ near $M^*$ without requiring $M^*$ to be the unique local minimum. This motivates us to make a refined local convergence analysis with respect to the hidden matrix $M^*$, as we will present in Section 4.2. The RPD property (Definition 3) with respect to $M^* \in M_k^{m \times n}$ is used but without a specific bound on the RPD constant $0 \leq \beta < 1$.

**Proposition 12.** Suppose $A$ satisfies the $(\beta, 2k)$-RPD property. Then for any $M^* \in M_k^{m \times n}$

$$
(1 - \beta)\|X - M^*\|_F^2 \leq \langle X - M^*, A(X - M^*) \rangle \leq (1 + \beta)\|X - M^*\|_F^2
$$

holds for $X \in M_k^{m \times n}$.

**Proof.** This is because $(X - M^*) \in M_{\leq 2k}$ for any $X$ and $M^*$ in $M_k^{m \times n}$.

### 4.2 Convergence analysis

Through Lemma [11], the study of convergence properties of the RGD algorithm (Algorithm 1) on the product space $M_k^{m \times n}$ can be conducted instead on $M_k^{m \times n}$ directly. Here, $M_k^{m \times n}$ is considered as an embedded manifold of $\mathbb{R}^{m \times n}$ now that all objects in the quotient manifold representation have explicit expressions through Section 3.1 Section 3.2. Proofs to the results in this section are given in Appendix C.

In the noiseless case, i.e., the hidden matrix $M^*$ has a low rank $k$, we investigate the Riemannian Hessian of $f$ (16) at $M^*$.

**Lemma 13.** Let $f$ be a quadratic function (16) such that the hidden matrix $M^*$ has a low rank $k$ and $A$ satisfies the $(\beta, 2k)$-RPD property. Then the Riemannian Hessian of $f$ at $M^*$ is positive definite:

$$
\lambda_{\min}(f) := \min_{W \in T_{M^*}M_k^{m \times n}} \left( \frac{\langle \Hess f(M^*)[W], W \rangle}{\|W\|_F^2} \right) \geq 1 - \beta > 0.
$$

Consequently, the following immediate result about Algorithm I can be deduced, provided that the algorithm converges.

**Theorem 14.** Let $f$ be a quadratic function (16) such that the hidden matrix $M^*$ has a low rank $k$ and $A$ satisfies the $(\beta, 2k)$-RPD property. Then, if the sequence $\{X_t\}_{t \geq 0}$ induced by Algorithm 7 converges to $M^*$, the local convergence rate of $\{X_t\}_{t \geq 0}$ is linear.

#### 4.2.1 Local convergence analysis

The convergence result of Theorem 14 requires the additional assumption that the algorithm converges to $M^*, a priori$, because the sole RPD property (with an unspecified parameter $\beta$) does not rule out the existence of other critical points of $f$ on $M_k^{m \times n}$ different than $M^*$. Also, it is
Remark 17. The value of the radius \( \delta_0 = (1 - \beta) \frac{\sigma_k^2}{L_f} \) in Lemma 16 is exactly twice the radius within which inequality (23) of Lemma 15 holds. Since the larger value \((1 - \beta) \frac{\sigma_k^2}{L_f}\) will appear more often in the next few technical lemmas, we denoted it with the symbol \(\delta_0\).
Sufficient descent conditions. Now we study conditions for RGD to ensure sufficient descents in a compact subset of the neighborhood $B^*(\delta)$ (21), parametrized by a radius $\delta > 0$ and a limit singular value $\bar{\sigma} > 0$ as follows:

$$C^*_{\delta, \bar{\sigma}} := B^*(\delta) \cap \{ X : \sigma_k(X) \geq \bar{\sigma} \}. \tag{25}$$

For the expression of the RGD update (15) (Lemma 11), we use the following shortened notations whenever needed:

$$X_+ = X - \theta g_X + \theta^2 \Gamma_X.$$ 

Here, the subscript index $t$ and $t+1$ are omitted, $g_X := \text{grad } f(X)$ is short for the Riemannian gradient, and $\Gamma_X := \nabla f(X) X^\dagger \nabla f(X)$ denotes the higher-than-first order term.

We apply the inequality of Lemma 15 to two consecutive RGD iterates $X := X_t$ and $X_+ := X_{t+1}$ given by (15), which entails $f(X_+) \leq f(X) - \theta \| g_X \|^2_F + \theta^2 (g_X, \Gamma_X) + \frac{L_g \theta^2}{2} \| g_X - \theta \Gamma_X \|^2_F$. In other words, the residuals $R_t := f(X_t) - f(M^*)$ satisfy

$$R_{t+1} - R_t = f(X_+) - f(X) \leq D(\theta) := -\theta \| g_X \|^2_F + \theta^2 (g_X, \Gamma_X) + \frac{L_g \theta^2}{2} \| g_X - \theta \Gamma_X \|^2_F, \tag{26}$$

where $D(\theta)$ can be rewritten as

$$D(\theta) = -\frac{\theta}{2} \left( (2 - L_g \theta) \| g_X \|^2_F - 2(1 - L_g \theta) (g_X, \theta \Gamma_X) - L_g \theta \| \theta \Gamma_X \|^2_F \right).$$

On the other hand, through Lemma 15 (ii), we have for all $X \in M_k^{m \times n}$ such that $\| X - M^* \|^2_F \leq \frac{1}{2} (1 - \beta) \theta^2 F$:

$$R_t = f(X) - f(M^*) \leq (g_X, X - M^*) - \frac{\mu}{2} \| X - M^* \|^2_F$$

The inner product on the right-hand side, $(g_X, X - M^*)$, satisfies the following trigonometric equation:

$$(g_X, X - M^*) = \frac{1}{2\theta} (\| X - M^* \|^2_F - \| X_+ - M^* \|^2_F) + \frac{\theta}{2} (\| g_X - \theta \Gamma_X \|^2_F + 2 (\Gamma_X, X - M^*)).$$

Therefore,

$$R_t \leq \frac{1 - \theta \mu}{2\theta} \| X - M^* \|^2_F - \frac{1}{2\theta} \| X_+ - M^* \|^2_F + \frac{\theta}{2} \left( \| g_X - \theta \Gamma_X \|^2_F + 2 (\Gamma_X, X - M^*) \right), \tag{27}$$

where the sum of the last two terms is denoted as $\tilde{D}(\theta)$. In the notation of both $D(\theta)$ and $\tilde{D}(\theta)$, ‘$X$’ is omitted for brevity.

We proceed by checking the following two descent conditions in terms of $D(\theta)$, $\tilde{D}(\theta)$ in (26)–(27):

$$- D(\theta) > 0 \quad \text{(descent condition)} \tag{28}$$

$$- \rho D(\theta) - \tilde{D}(\theta) \geq 0 \quad \text{('strong' descent condition)} \tag{29}$$

for $\rho > 1$ and $0 < \theta < \frac{2}{L_g}$. Notice that these two conditions, combined with (26)–(27), entail the following inequality:

$$\rho R_{t+1} - (\rho - 1) R_t \leq \frac{1 - \theta \mu}{2\theta} \| X - M^* \|^2_F - \frac{1}{2\theta} \| X_+ - M^* \|^2_F, \tag{30}$$
which will be used to deduce the convergence rate of \( \{X_t\}_t \). The roles of \( \rho > 1 \) and \( \sigma \) are similar to those in the analysis of [44] for geodesically convex optimization, but the difference is that the manifold \( M_{k,n} \) in this work is an open space and \( \sigma \) will be applied in a compact subset of \( M_{k,n} \). The next couple of lemmas are about descent properties of RGD in such a compact subset.

The following lemmas are used for checking the two decent conditions \([28]-[29]\). Based on Lemma [16], the next proposition gives properties of \( \nabla f(X) \) and \( \Gamma X \) in \( C_{\delta,\rho} \) \([25]\).

**Proposition 18.** For \( \delta_0 = (1-\beta)\frac{\rho^2}{L_f} \) and \( \bar{\sigma} > 0 \), there exist \( 0 < \bar{C}_1 \leq 1 \) and \( 0 < \bar{C}_2 \leq 1 \) such that for any \( X \in C_{\delta_0,\bar{\sigma}} \) \([25]\):

\[
\| \nabla f(X) \|_F \geq 2\bar{C}_1\| \nabla f(X) \|_F , \quad \text{and when } f \text{ satisfies the } (\beta,2k)\text{-RPD property,}
\]

\[
\| \Gamma X \|_F \leq \frac{L_f}{2\bar{C}_1\bar{\sigma}}\| \nabla f(X) \|_F \| X - M^* \|_F .
\]

**Remark 19.** The inequality \([31]\) in the lemma above can be seen as the Lojasiewicz inequality (with exponent \( \frac{1}{2} \)) in a restricted neighborhood of \( M^* \).

**Lemma 20.** For \( \delta_0 = (1-\beta)\frac{\rho^2}{L_f} \) and \( \bar{\sigma} > 0 \), there exists \( \rho \equiv (1+L_g) \) such that for any \( X \in C_{\delta_0,\bar{\sigma}} \) \([25]\) different than \( M^* \), the RGD update \([15]\) from \( X \) satisfies the descent conditions \([28]-[29]\), provided that the stepsize \( 0 < \theta \leq (1 - \frac{1}{\rho})(\frac{1}{L_g})_\theta \), and that

\[
\| \Gamma X \|_F \leq C_{\rho}L_g\| \nabla f(X) \|_F
\]

where \( C_{\rho} = \min(1, \frac{\rho}{\sqrt{(\rho-1)(\rho^2-1)}}) \).

**Corollary 21.** For \( \delta_0 = (1-\beta)\frac{\rho^2}{L_f} \) and \( 0 < \sigma \leq \sigma^*_k \), there exist \( \rho \equiv (1 + L_g) \), \( \bar{C}_3 \geq 1 \) and \( \delta_0 = \min(\frac{1}{2}\delta_0, \bar{C}_3\delta_0) \) such that for any \( X \in C_{\delta_0,\bar{\sigma}} \) \([25]\) different than \( M^* \), the RGD update \([15]\) from \( X \) satisfies the descent conditions \([28]-[29]\), provided that the stepsize \( 0 < \theta \leq (1 - \frac{1}{\rho})(\frac{1}{L_g})_\theta \).

The proofs are given in Appendix [C,1].

**Remark 22.** In Lemma [20] Corollary [21], the parameter \( \rho \) is defined as \( \rho = 1 + \frac{L_g}{2L_c} \), and the parameter \( \bar{C}_3 \) in Corollary [21] is \( \bar{C}_3 = \frac{L_g\sigma^2\bar{C}_1C_\rho}{1-\beta} \), where \( L_g \) is given in Lemma [15] and \( \bar{C}_1, \bar{C}_2 \) are given in Proposition [18]. The value of \( C_{\rho} \) evolves with \( \rho \) continuously as follows: \( C_{\rho} = 1 \) if \( 1 < \rho \leq \rho^* \approx 2.24 \) and \( C_{\rho} = \frac{\rho}{\sqrt{(\rho-1)(\rho^2-1)}} \) otherwise \( (\rho > \rho^*) \). When \( L_g \approx \sqrt{\beta} < \sqrt{2} \) and \( \bar{C}_2 \approx 0.5 \), for example, we have \( \rho \approx 2.41 \) and \( C_{\rho} \approx 0.93 \). The value of \( \bar{C}_3 \) under Corollary [21] varies around 1 since \( \frac{2\bar{C}_1C_\rho}{1-\beta} \approx 1 \) and \( \frac{L_g\sigma^2}{\bar{\sigma}} \approx 1 \).

**Local linear convergence.** Given the lemmas above, we have the following theorem.

**Theorem 23.** Let \( f \) be a quadratic function \([16]\) such that the hidden matrix \( M^* \) has a low rank \( k \) and that \( A \) satisfies the \((\beta,2k)\)-RPD property. Then, there exist \( \rho \equiv 1 + L_g \) and \( \bar{C} \equiv 1 \) such that given an initial point \( \bar{x}_0 \in M_{k,n} \) satisfying \( \| \pi(\bar{x}_0) - M^* \|_F \leq \min(\frac{1}{2}\delta_0, \bar{C}\delta_0) \), where \( \delta_0 = (1-\beta)\frac{\rho^2}{L_f} \), and an initial stepsize \( \theta_0 = (1 - \frac{1}{\rho})(\frac{1}{L_g})_\theta \). Algorithm [7] converges to \( M^* \) linearly, and the sequence \( \{x_t\}_{t \geq 0} \) induced by Algorithm [7] satisfies

\[
f(x_t) - f(M^*) \leq (1 - \epsilon)^tC_{f,k}\| x_0 - M^* \|_F^2 \quad \text{for } t > 0.
\]

Here \( \epsilon = \min(\frac{1}{\rho}, \theta_0\mu_f) \in (0,1) \) and \( C_{f,k} > 0 \) are constants depending only on \( (\beta,L_g,\mu_f) \).
Remark 24. The radius $\delta_0 = (1 - \beta) \frac{\sigma_1}{\sqrt{t_f}}$ in the initial condition of Theorem [23] and its lemmas is proven in the related work to be attainable by methods such as the spectral initialization (e.g., [23]); see, e.g., [35] Claim V.2 and [37] Lemma 15.

Discussion. In [37], the convergence properties of a gradient descent algorithm called ScaledGD, using the same preconditioned gradient $\Theta^2$, is analyzed in the context of low-rank matrix estimation. The local convergence result of Algorithm 1 given in Theorem 23 is similar to [37, Theorem 11]. More precisely, the initial closeness condition (0.1$\sigma_k(M^*) \frac{\sqrt{p}}{\sqrt{t_f}}$ in [37]), the linear rate ($(1 - \theta_0 \mu)$ in [37]) and the range of stepsize in both results are the same in the order of magnitudes (taking their difference in the search space and distance function into account). In particular, the convergence rate of Algorithm 1 and ScaledGD [37] do not depend on the condition number $\kappa$ of $M^*$, while Euclidean gradient descent algorithms [32,4] have linear convergence rates that are slower by a factor of $\kappa$.

This work differs with [37] in that our analysis is based on connections between embedded and quotient geometries in the optimization of $f$ over $\mathcal{M}_k^{m \times n}$ (Section 3). The analysis of [37], on the other hand, is conducted on the product space $\mathbb{R}^{m \times r} \times \mathbb{R}^{n \times r}$; a special distance that is invariant on equivalence classes (i.e., $\{(GQ, HQ^{-1}) : Q \in \text{GL}(r)\}$) is used. For the difference in abstractions, our proof techniques can be applied (e.g., descent condition in Lemma 20) or extended (e.g., geometric properties in Proposition 18) to other Riemannian metrics on $\mathcal{M}_k^{m \times n}$ while the techniques in [37] are more tailored for the gradient $\Theta$.

4.3 Results adapted for low-rank matrix completion

The matrix completion problem (Example 2) on $\mathcal{M}_k^{m \times n}$ with an objective function (18) is:

$$\min_{X \in \mathcal{M}_k^{m \times n}} f(X) = \frac{1}{2p} (\|P_{\Omega}(X - M^*)\|_F^2 - \|P_{\Omega}(M^*)\|_F^2).$$

It is well understood that the subsampling operator $P_{\Omega}$ can be badly conditioned with unbalanced sampling pattern or on matrices that are highly coherent [9] §1.1.1. In such case, $P_{\Omega}$ does not satisfy the RPD property. Nevertheless, it is also shown in [9] that the RPD property can be satisfied by $P_{\Omega}$ under the following assumptions: (i) the distribution of the observed entries follow either an uniform or a Bernoulli model: e.g., $(i, j) \in \Omega$ with probability $p = \frac{\Omega}{mn}$, where $p$ is a given sampling rate, and (ii) the unknown matrix $M^*$ is $\mu$-incoherent, as defined below.

Definition 25 (Incoherence [9]). $Z = U\Sigma V^T \in \mathbb{R}^{m \times n}$ is $\mu$-incoherent if $\|U_{i,:}\|_2 \leq \sqrt{\frac{\mu k}{m}}$ and $\|V_{j,:}\|_2 \leq \sqrt{\frac{\mu k}{n}}$ for all $(i, j) \in [m] \times [n]$.

The incoherence constant $\mu$ is a bound that measures the maximal row norms of $U$ and $V$. Since the SVD factor matrix $U$ (respectively $V$) is orthonormal, such that $\|U\|_F^2 = \sum_{i=1}^n \|U_{i,:}\|_2 \equiv k$, the incoherence constant $\mu$ is small if the variations of the entries in $U$ and $V$ are moderate; on the contrary, $\mu$ is large if $U$ and $V$ has columns with spikes. It can be shown that the smallest possible incoherence constant is $\mu = 1$.

The RPD inequality As mentioned in Section 4, the subsampling operator $P_{\Omega}$ generally does not satisfy the RPD property (Definition 3). However, under certain conditions on the subsampling pattern, the inequality (19) in Proposition 12 can be satisfied in some neighborhood of the hidden
matrix $M^*$, provided that $M^*$ is sufficiently incoherent. Consider the following set adapted from [35, §3]:

$$B_{\delta_0}^* (C_B, \mu) = B^* (\delta_0') \cap \left\{ X \in \mathcal{M}_{\leq k} : \max (\sqrt{m} \|X\|_{2,\infty}, \sqrt{n} |X^T|_{2,\infty}) \leq C_B \sqrt{\mu} k \right\},$$

(34)

where $B^* (\delta_0')$ is defined by (21) for $\delta_0' := \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}, \kappa := \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}, C_d > 0$ are constants, and $C_B > 0$ is a parameter to be specified.

The following proposition, adapted from a key result in [35], validates the inequality (19) on a neighborhood of $M^*$ in the form of (34).

**Proposition 26.** Assume that a rank-$k$ matrix $M^* \in \mathbb{R}^{m \times n}$ is $\mu$-incoherent (with $\mu > 0$). Suppose the condition number of $M^*$ is $\kappa$ and $\alpha = m/n \geq 1$. Then there exist numerical constants $C_0 > 0$ and $C^* > 1$ such that: if the indices in $\text{B}_{\delta_0}^* (C_B, \mu)$ are uniformly generated with size $|\Omega| \geq C_0 \alpha \kappa \kappa^2 \max (\mu \log (n), \sqrt{\alpha} \kappa^6 \mu^2 \kappa^4),$

then with probability at least $1 - 2n^{-4}$,

$$\frac{1}{\sqrt{2}} \left\| X - M^* \right\|^2_F \leq \frac{1}{p} \left\| P_\Omega (X - M^*) \right\|^2_F \leq 2 \left\| X - M^* \right\|^2_F,$$

(36)

for any $X \in \mathcal{B}_{\delta_0}^* (C_B, \mu)$ defined in (34) for $C_B := \sigma_{\text{max}} \sqrt{2C^* k}$.

**Remark 27.** The inequality (36) of Proposition 26 is an instance of the RPD inequality (19), since $\frac{1}{p} \left\| P_\Omega (X - M^*) \right\|^2_F = (X - M^*, A (X - M^*))$ for $A = \frac{1}{p} P_\Omega$. Therefore, Proposition 26 provides a condition on the index set $\Omega$ under which the RPD inequality (19) holds, hence, provides a way to applying the main result (Lemma 16, Theorem 23) to the matrix completion problem.

To ensure that the iterates of RGD remain in the domain $\mathcal{B}_{\delta_0}^* (\epsilon, \mu)$, we adapt the normalization operator in [35] from the product space of factor matrices to an $m \times n$ matrix space (e.g., $\mathcal{M}_k^{m,n}$): given $X \in \mathbb{R}^{m \times n}$, $\mathcal{P}_B (X) = D_B^{(1)} X D_B^{(2)}$, where $D_B^{(1)} := D^{(1)}, D_B^{(2)} := D^{(2)}$ are diagonal square matrices with compatible sizes such that

$$D_{ii}^{(1)} = \min (1, \frac{B}{\sqrt{m} \| X_{i,:} \|_2}) \quad \text{and} \quad D_{jj}^{(2)} = \min (1, \frac{B}{\sqrt{n} \| X_{:,j} \|_2}),$$

(37)

for a parameter $B > 0$. By abuse of notation, $\mathcal{P}_B$ acts on a pair of factor matrices $\tilde{x} = (G_x, H_x) \in \mathcal{M}_k^{m,n}$ by row-wise normalizations $\mathcal{P}_B (\tilde{x}) = (D_B^{(1)} G_x, D_B^{(2)} H_x)$. Note that the operator $\mathcal{P}_B$ depends on the input (\( \tilde{x} \) and/or $X$), which is omitted in the notation.

We consider the following variation of RGD using the normalization operator $\mathcal{P}_B$: in line 4 of Algorithm 1 (given $\tilde{x}_t \in \mathcal{M}_k^{m,n}$, gradient direction $\tilde{\eta}_t = -\nabla \ell (\tilde{x}_t)$, and stepsize $\theta_t$), update by generating $\tilde{x}_{t+1} := \mathcal{P}_B (\tilde{x}_t - \theta_t \tilde{\eta}_t)$. The induced update rule on $\mathcal{M}_k^{m,n}$ consists of the following steps:

$$\tilde{X} := X - \theta (\nabla \ell (X) - \theta \Gamma_X)$$

(38)

$$X_+ := \mathcal{P}_B (\tilde{X}) = D_B^{(1)} \tilde{X} D_B^{(2)},$$

(39)

where $X := \pi (\tilde{x}_t)$, and $\nabla \ell (X)$ and $\Gamma_X$ are defined in Lemma 11.

Given Proposition 26, we show that this normalized RGD rule enjoys some properties that maintain the validity of the convergence results on the $m \times n$ matrix space $\mathcal{M}_k^{m,n}$ Section 4.2.
Proposition 28. Suppose that $X$ satisfies (i) $\|X - M^*\|_F \leq \epsilon_0 \sigma_k(M^*)$, (ii) $\|X - M^*\|_{2,\infty} \leq \epsilon_0 \|M^*\|_{2,\infty}$, and (iii) $\sigma_k^{-1}(X) \sigma_k(M^*) \leq \bar{r}_k$ for constants $\epsilon_0 > 0$ and $\bar{r}_k \leq 1$. Then, for $B \geq (1 + \epsilon) \sqrt{\|k\sigma^*_\max\}$ and $\epsilon_\theta := (1 + \bar{r}_k) \theta \epsilon_0 + \bar{r}_k \theta^2 \epsilon_0^2 \leq \theta \epsilon_0 (1 + \epsilon \epsilon_0)$, the normalized RGD update $P_B(\bar{X})$ given $X$ and stepsize $\theta$ satisfies:

\[
\max(\sqrt{m} \|P_B(\bar{X})\|_2, \sqrt{n} \|P_B(\bar{X})\|_2) \leq B \quad \text{(same incoherence bound),}
\]

\[
\|P_B(\bar{X}) - M^*\|_F \leq \|\bar{X} - M^*\|_F \quad \text{(non-expansiveness),}
\]

as long as $\max(\sqrt{m} \|X\|_2, \sqrt{n} \|X^T\|_2) \leq B$.

The following remark is about Theorem 23 to the incoherence-restricted region $B^*_\delta_0(C_B, \mu)$.

Corollary 29. Under the same assumptions of Proposition 28, there exist constants $C_0$, $C^*$, $C_B$ such that for a sample size $m$, and an initial point $X_0 \in B^*_\delta_0(C_B, \mu)$, the normalized RGD update $P_B(\bar{X})$ ensures a sequence $\{X_t\}_{t \geq 0}$ that converges to $M^*$ with a linear rate.

This result for matrix completion, similar to the previous work, is based on a lemma (Proposition 28) that ensures the RPD property. The difference, however, is that the convergence result in this subsection is built upon the generalized convergence result in Section 4.2.

5 Numerical experiments

In this section, we conduct experiments to test the main algorithms (Section 2.1) on compressed sensing and matrix completion, which correspond to (9) with the objective function (17) and (18) respectively. Below is a list of all tested algorithms.

The main algorithms: the RGD (Algorithm 1) is labeled as Qprecon RGD. The term ‘Qprecon’ signifies the preconditioned metric (5) used in Algorithm 1 that induces the quotient metric on $\mathcal{M}_k^{m \times n}$. Similarly, the RCG algorithm (Section 2.1) is labeled as Qprecon RCG. The trial stepsize (Algorithm 1 line 3) is computed using (i) exact line minimization (linemin), (ii) the Armijo linesearch method (Armijo), and (iii) the Riemannian Barzilai–Borwein (RBB) stepsize (7). The RBB stepsize rule is also tested directly without backtracking linesearch; the underlying rule is labeled by ‘(RBB nols)’.

The Euclidean algorithms: Euclidean gradient descent (Euclidean GD) and nonlinear conjugate gradient (Euclidean CG) refer to the GD and CG algorithms on the product space $\mathcal{M}_k^{m \times n}$ using the Euclidean gradients. The stepsize selection rules are the same as the main algorithms in Section 2.1. The Euclidean algorithms are implemented in the same package as the main algorithms.

Existing algorithms on $\mathcal{M}_k^{m \times n}$: LRGeomCG [41], NIHT and CGIHT [42], and ASD and Scaled-ASD [36] are tested.

The computation environment of all algorithms above is MATLAB. Given that $(m + n)k \ll |\Omega| = mnn$ in low-rank problems, the dominant cost in all algorithms is the computation of the $m \times n$ residual matrix $S = P_\Omega(X - M^*)$, which costs $O(|\Omega|/k)$. The computation of $S$ in all algorithms is implemented in MEX functions.

All numerical experiments were performed on a workstation with 8-core Intel Core 549 i7-4790 CPUs and 32GB of memory running Ubuntu 16.04 and MATLAB R2015. The source code is available at https://gitlab.com/shuyudong.x11/ropt_mk/. Implementations of the existing algorithms are also publicly available.
5.1 Compressed sensing

The sensing operator $\Phi : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ in (17) is represented by a matrix $\hat{\Phi} : \mathbb{R}^{p \times mn}$ in which each row is an i.i.d. Gaussian vector: $[\hat{\Phi}]_{ij} \sim \mathcal{N}(0, 1)$ for $1 \leq i \leq p$ and $1 \leq j \leq mn$. The sensing operator $\Phi$ is defined as $\Phi(X) = \hat{\Phi}(\text{vec}(X))$. In the following test, the dimensions of the problem are $m = 100$, $n = 100$, and $p/mn = 0.25$. We generate a low-rank matrix $M^\star = G^\star H^\star^T$, where $G^\star$ and $H^\star$ are thin Gaussian matrices of size $m \times r$ and $n \times r$ respectively, with $r = 5$. The observed vector from the sensing operator is $b^\star = \Phi(M^\star)$ $\in \mathbb{R}^p$. We test the algorithms using the initial point $X_0 := \Phi^\ast(b^\star)$. The performances of Qprecon RGD (Armijo and RBB) and Euclidean GD (Armijo) are shown in Figure 3.

![Figure 3: Compressed sensing results.](image)

We observe from Figure 3 that Qprecon RGD, with both the Armijo and RBB rules, are faster than Euclidean GD by orders of magnitude. From the iteration history in Figure 3 we also observe that the decay of the residual $\|X_t - M^\star\|_F$ is linear, which agrees with the result given in Theorem 23.

In particular, the RBB stepsize rule (7) is tested both with and without backtracking linesearch under the same experimental setting. Figure 4 shows the iteration history of the gradient norms and stepsizes of these two variants (RBB and RBB nols). In Figure 4: $SS = \bar{g}_{\bar{x}_t}(\bar{z}_t - 1, \bar{z}_{t - 1})$, $SY = \bar{g}_{\bar{x}_t}(\bar{y}_t - 1, \bar{y}_{t - 1})$, and $YY = \bar{g}_{\bar{x}_t}(\bar{y}_t - 1, \bar{y}_{t - 1})$ are the values appearing in the RBB formula (7).

![Figure 4: Gradient and stepsize information of Qprecon RGD (RBB).](image)

We observe from Figure 4 that the decay of the gradients’ norm (‘Gradnorm’) is linear, and the
RBB stepizes (‘SY/YY’, in purple) are rather stable, fluctuating around $O(1)$ with small variation. The RBB stepizes (both with and without linesearch) that enable the convergence agree with the characterization in Theorem 23 (stating that there exists $\theta^* > 0$ such that the algorithm converges with a linear rate, for all stepizes $\theta_i < \theta^*$).

5.2 Matrix completion

In the following experiments, the partially observed matrix $M^* \in \mathbb{R}^{m \times n}$ is generated as follows: for a rank parameter $k \ll \min(m, n)$, $M^* = A^*B^*$, where $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{n \times k}$ are composed of entries drawn from the Gaussian distribution $\mathcal{N}(0, 1)$. The index set of the observed entries (training data) consists of indices sampled from the Bernoulli distribution $\mathcal{B}(p)$: for $(i, j) \in [m] \times [n]$,

$$(i, j) \in \Omega, \text{ with probability } p.$$  

Optimization paths on $\mathcal{M}_k^{m \times n}$ Lemma 11 shows that Qprecon RGD (Algorithm 1), by producing a sequence $\{\bar{x}_t\}_{t \geq 0} \subset \mathcal{M}_k^{m \times n}$, induces a sequence $\{X_t\}_{t \geq 0}$ on $\mathcal{M}_k^{m \times n}$ along the negative Riemannian gradients in $T\mathcal{M}_k^{m \times n}$. This means that the sequence does not depend on the locations of the iterates $\{\bar{x}_t\}_{t \geq 0}$ in the vertical space $\mathcal{V}_x$. To demonstrate this property, we observe next the iteration histories and paths of the induced sequences of Qprecon RGD and Euclidean GD on $\mathcal{M}_k^{m \times n}$.

The matrix completion tests are conducted on noiseless observations of $M^*$ on $\Omega$ with a sampling rate $p = 0.8$. Both Qprecon RGD and Euclidean GD are initialized with the same initial point in the following two settings: (i) the balanced setting: each algorithm is initialized with $x_0 = (G_0, H_0)$ using the spectral initialization method such that $\|G_0\| = \|H_0\|$, and (ii) the unbalanced setting: the initial point is defined as $x_0' = (\lambda G_0, H_0/\lambda)$ for $\lambda = 5$, where $G_0$ and $H_0$ are the same as in the balanced setting. The comparative results are given in Figure 5.

From Figure 5, we observe that the two sequences of Qprecon RGD overlap, which shows indeed that the path of the sequence generated by Qprecon RGD does not vary with the change in the initial point. We also observe that these overlapping sequences converge linearly, with much faster speed than Euclidean GD with the unbalanced initial point. In fact, one can see from the figure that the convergence of Euclidean GD is significantly slowed down with the unbalanced initial point $x_0'$ compared to the case with $x_0$.

Comparisons with existing algorithms We compare the performances of Qprecon RGD and Qprecon RCG with the other listed algorithms [11, 32, 33]. More precisely, LRGBmCG [11], NIHT and CGIHT [42] are based on the same manifold structure of $\mathcal{M}_k^{m \times n}$ and the same retraction operator. These algorithms produce iterates on the product space $\text{St}(m, k) \times S_{++}(k) \times \text{St}(n, k)$ using gradients defined with the Euclidean metric. A retraction is needed to ensure the fixed-rank constraint, for which the projection-like retraction [3] (also called the IHT for “iterative hard thresholding”) is used. ASD and ScaledASD [36] use alternating steepest descent on $\mathbb{R}^{m \times k} \times \mathbb{R}^{n \times k}$.

In Figure 6, we show the performances of the algorithms in recovering low-rank matrices from the same matrix model as the previous experiment. From Figure 6(a-d), we observe that our algorithms (Qprecon RGD and Qprecon RCG) perform similarly as ScaledASD and they are all either faster than or comparable to the rest of the algorithms under all three different rank values. We also observe that algorithms using Riemannian geometry on $\mathcal{M}_k^{m \times n}$—ours, scaledASD, LRGBmCG, and NIHT/CGIHT—outperform those (Euclidean GD/CG, ASD) based on Euclidean geometry, with only one exception, where NIHT attained stationarity a bit later than Euclidean CG when $r^* = 30$. In particular, when $r^* = 10$ and all algorithms attain stationarity, both our algorithms,
Figure 5: Iteration histories of the algorithms with a balanced and an unbalanced initial point. The size of $M^*$ is $100 \times 200$, with rank $r^* = 3$. The sampling rate $p = 0.8$. (a)–(b): test RMSEs by time. (c)–(d): paths of the matrix entries $([X_1]_{1,1}, [X_1]_{2,1})$ of the iterates $\{\pi(\bar{x}_t)\}$. 
Figure 6: Performances of algorithms on matrix completion. The matrix size is $800 \times 900$ with sampling rate $p = 0.6$ for all tests. The rank parameter $k = r^*$ in each setting. (a-d): iteration history under rank values $r^* \in \{10, 20\}$. (e): average time per-iteration with ranks varying in $\{5, 10, \ldots, 30\}$. 
scaled ASD and LRGeomCG achieve a speedup of around 10 times over Euclidean GD; and when \( r^* \in \{20, 30\} \), Euclidean GD and ASD exceeded the maximal time budget before attaining the target stationarity precision. From Figure 6(e), we observe that Qprecon RGD (RBB), ASD and Scaled ASD are the three most efficient algorithms in terms of per-iteration cost, with all different rank values varying in \( \{5, 10, \ldots, 30\} \). In particular, these three algorithms have better scalability in the rank \( k \) than the rest of the algorithms.

6 Conclusion

We investigated methods for the matrix completion problem with a fixed-rank constraint and focused on a Riemannian gradient descent algorithm in the framework of optimization on the quotient manifold of fixed-rank matrices. We showed that the Riemannian gradient descent algorithm under the aforementioned quotient geometric setting not only enjoys the advantage of matrix factorization but can also be analyzed in a more convenient way than existing matrix factorization methods. We developed novel results for analyzing the quotient manifold-based algorithm and proved that this algorithm solves the fixed-rank matrix completion problem with a linear convergence rate. Moreover, the convergence property of the algorithm has desirable invariance properties in contrast to Euclidean gradient descent algorithms. Because of the efficient iteration efficiency and its light per-iteration cost, the time efficiency of this algorithm is also shown to be much faster than the Euclidean gradient descent algorithms and is faster than many other Riemannian algorithms on the set of fixed-rank matrices. Through the convergence analysis, we also provided a novel understanding of the graph-based regularization in the theoretical framework of matrix completion.

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A Proofs in Section 2

Proof of Proposition 6. For any \( \bar{x} := (G, H) \in \bar{M}_k^{m,n} \) and \( \bar{x}' \sim \bar{x} \), there exists an invertible matrix \( F \in GL(k) \), such that \( \bar{x}' = (GF^T, HF^{-1}) \). Since the tangent vector \( \tilde{\xi}' \) must satisfy \( D\pi(\bar{x}')[\tilde{\xi}'] = D\pi(\bar{x})[	ilde{\xi}] \), and note that \( D\pi(\bar{x})[\tilde{\xi}] = G(\tilde{\xi}^{(2)})^T + \tilde{\xi}^{(1)}H^T \) for \( \bar{x} = (G, H) \), \( \tilde{\xi}' \) and \( F \) must satisfy

\[
G(\tilde{\xi}^{(2)})^T + \tilde{\xi}^{(1)}H^T = GF^T(\tilde{\xi'}^{(2)})^T + \tilde{\xi'}^{(1)}(HF^{-1})^T,
\]
for any $\tilde{\xi} \in T_\bar{x} \mathcal{M}^{m,n}_k$, which yields
\[
\tilde{\xi}^{(1)} = \xi^{(1)} F^T \text{ and } \tilde{\xi}^{(2)} = \xi^{(2)} F^{-1}.
\]
The same relation holds for $\bar{\eta}, \bar{\eta}'$ and $F$. Applying these equalities into the expression of $\bar{g}_x' (\tilde{\xi}, \bar{\eta})$, where $\bar{g}$ is the preconditioned metric \([53]\), we recover immediately the expression of $\bar{g}_x (\xi, \eta)$.

\section{Proofs in Section 3}

\begin{proof}[Proof of Proposition 7] Let $\text{grad} \bar{f} (\bar{x}) = (\bar{\eta}^{(1)}, \bar{\eta}^{(2)})$ denote the two components of the gradient \([6]\) of $\bar{f}$, where $\bar{x} := (G, H) \in \pi^{-1} (X)$ given $X = U \Sigma V^T \in \mathcal{M}^{m \times n}_k$. We have
\[
\bar{\eta}^{(1)} = \partial_G \bar{f} (\bar{x}) \left( H^T H \right)^{-1} \text{ and } \bar{\eta}^{(2)} = \partial_H \bar{f} (\bar{x}) \left( G^T G \right)^{-1},
\]
where $\partial_G \bar{f} (\bar{x}) = \nabla f (X) H$ and $\partial_H \bar{f} (\bar{x}) = \nabla f (X)^T G$. Since $\text{grad} \bar{f} (\bar{x})$ is the horizontal lift of $\text{grad} f (X)$, by definition, $\text{grad} f (X) = D\pi (\bar{x}) [\text{grad} \bar{f} (\bar{x})]$. Therefore, we have
\[
\text{grad} f (X) = D\pi (\bar{x}) [\text{grad} \bar{f} (\bar{x})] = G (\bar{\eta}^{(2)})^T + \bar{\eta}^{(1)} H^T = G (G^T G)^{-1} G^T \nabla f (X) + \nabla f (X) H (H^T H)^{-1} H = P_U \nabla f (X) + \nabla f (X) P_V,
\]
where $P_U$ and $P_V$ are the matrices defined in the statement.
\end{proof}

\begin{proof}[Proof of Lemma 11] To simplify the notations, we omit the subscript $t$ of all terms related to the $t$-th iterate $(G_t, \bar{\eta}_t, \text{stepsize } \theta_t)$ and denote $X_{t+1}$ by $X_+$ in the following equations. Let $\bar{\eta} := \text{grad} \bar{f} (\bar{x})$ denote the Riemannian gradient of $\bar{f}$ on the current iterate $\bar{x} := (G, H)$. From the update step in Algorithm 1, we have
\[
X^+ := \pi (\bar{x} - \theta \text{grad} \bar{f} (\bar{x})) = \pi (\bar{x}) - \theta \left( G (\bar{\eta}^{(2)})^T + \bar{\eta}^{(1)} H^T \right) + \theta^2 \bar{\eta}^{(1)} (\bar{\eta}^{(2)})^T
\]
\[
= X - \theta \text{grad} f (X) + \theta^2 \partial_G \bar{f} (\bar{x}) \left( H^T H \right)^{-1} (G^T G)^{-1} \left( \partial_H \bar{f} (\bar{x}) \right)^T
\]
\[
= X - \theta \text{grad} f (X) + \theta^2 \nabla f (X) H \left( H^T H \right)^{-1} \left( G^T G \right)^{-1} G^T \nabla f (X) \phi_1
\]
\[
= X - \theta \text{grad} f (X) + \theta^2 \nabla f (X) X^T \nabla f (X),
\]

which proves \([15]\). The equation \([43]\) is obtained by identifying $(G (\bar{\eta}^{(2)})^T + \bar{\eta}^{(1)} H^T)$ with $D\pi [(G, H)] (\bar{\eta}) := D\pi [(G, H)] (\text{grad} \bar{f} (\bar{x})) = \text{grad} f (X)$.

The equation \([43]\) is obtained as follows. Let $X := U \Sigma V^T$ denote the SVD of $X \in \mathcal{M}^{m \times n}_k$, where where $U \in \text{St}(m, k)$ and $V \in \text{St}(n, k)$ and $\Sigma \in \mathbb{R}^{k \times k}$. Then we use the fact that there exists $F \in \text{GL} (k)$, for any $(G, H) \in \pi^{-1} (X)$, such that $G = U \Sigma G F^T$ and $H = V \Sigma H F^{-1}$, where $\Sigma_G$ and $\Sigma_H$ are $k \times k$ diagonal matrices such that $\Sigma_G \Sigma_H = \Sigma$. Therefore, $\phi_1$ in the last term of the right-hand side of \([44]\) reads:
\[
\phi_1 = \nabla f (X) V \Sigma_H F^{-1} (F \Sigma_H^2 F^T) (F^{-T} \Sigma_G^{-2} F^{-1}) F \Sigma_G U^T \nabla f (X)
\]
\[
= \nabla f (X) V \Sigma_H (\Sigma_H^2 \Sigma_G^{-2}) \Sigma_G U^T \nabla f (X)
\]
\[
= \nabla f (X) V \Sigma_H^{-1} \Sigma_G^{-1} U^T \nabla f (X)
\]
\[
= \nabla f (X) X^T \nabla f (X).
\]
Proof of Lemma 10. (i): From the definition (12), \( \mathcal{G}_X(Z) = P_U(Z) + P_V(Z) \), we deduce that \( \mathcal{G}_X \) is a symmetric operator, since the orthogonal projections \( P_U(\cdot) \) and \( P_V(\cdot) \) are symmetric operators. We prove the remaining claims as follows. From (13) in Remark 9, we deduce that \( \mathcal{G}_X(Z) = P_{T_XM_k^{m \times n}}(Z) + UU^T ZV^T = Z \) if \( Z \in T_XM_k^{m \times n} \). If \( Z' \in (T_XM_k^{m \times n})^\perp \), then there exists \( Y \in \mathbb{R}^{m \times n} \) such that \( Z' = (I - P_{T_XM_k^{m \times n}})(Y) \), and consequently

\[
\mathcal{G}_X(Z') = P_{T_YM_k^{m \times n}}((I - P_{T_XM_k^{m \times n}})(Y)) + UU^T((I - P_{T_XM_k^{m \times n}})(Y))V^T = 0.
\]

Therefore, for any \( Z \in \mathbb{R}^{m \times n} \), \( \mathcal{G}_X(Z) = \mathcal{G}_X(P_{T_XM_k^{m \times n}}(Z)) = 2P_{T_XM_k^{m \times n}}(Z) \), which entails that

\[
0 \leq 2\|P_{T_XM_k^{m \times n}}(Z)\|_F^2 = \langle \mathcal{G}_X(Z), Z \rangle \leq 2\|Z\|_F^2.
\]

(ii): The matrix \( Z := Y - X \) can be decomposed as \( Z = \tilde{Z} + \Delta Z \), where \( \tilde{Z} := P_{T_XM_k^{m \times n}}(Z) \) and \( \Delta Z := (I - P_{T_XM_k^{m \times n}})(Z) = (I - P_{T_XM_k^{m \times n}})(Y) \), where the last equality holds since \( (I - \tilde{P}_{T_XM_k^{m \times n}})(X) = 0 \). Therefore, we have

\[
\begin{align*}
(\mathcal{G}_X - 2I)(X - Y) &= -(\mathcal{G}_X - 2I)(\tilde{Z} + \Delta Z) \\
&= (2I - \mathcal{G}_X)(\Delta Z) = 2\Delta Z := 2(I - P_{T_XM_k^{m \times n}})(Y),
\end{align*}
\]

where the equalities in (46) are obtained by using the fact that \( \tilde{Z} \in T_XM_k^{m \times n} \) and \( \Delta Z \in (T_XM_k^{m \times n})^\perp \) and the properties \( \mathcal{G}_X|_{T_XM_k^{m \times n}} = 2I \) and \( \mathcal{G}_X|_{(T_XM_k^{m \times n})^\perp} = 0 \), proven in (i). \( \square \)

C Proofs in Section 4

C.1 Proofs of lemmas in Section 4.2

Proof of Lemma 13. The Riemannian Hessian of \( f \) at \( Y \in M_k^{m \times n} \), by definition [2, Proposition 5.5.4], satisfies

\[
\langle \text{Hess} f(Y)[Z], Z \rangle = \frac{d^2}{dt^2}(f(\text{Exp}_Y(tZ)))\bigg|_{t=0},
\]

where \( \text{Exp}_Y : T_YM_k^{m \times n} \to M_k^{m \times n} \) is the exponential map at \( Y \). We prove the spectral lower bound of \( \text{Hess} f(M^*) \) as follows. First, the exponential map at \( Y := M^* \) has the following expression (e.g., [3, 11, Proposition A1], [10, Appendix A]),

\[
\text{Exp}_{M^*}(X) = M^* + Z + \Delta_{M^*}(Z),
\]

where \( \Delta_{M^*}(Z) := (I - P_{U^*})ZM^*\dagger Z(I - P_{V^*}) + O(\|Z\|_F^3) \) satisfies \( \|\Delta_{M^*}(Z)\|_F \lesssim \|Z\|_F^2 \). From (47)–(48), it follows that the quadratic function \( f \) (16) satisfies, for any \( Z \in T_{M^*}M_k^{m \times n} \),

\[
\langle \text{Hess} f(M^*)[Z], Z \rangle = \langle A(Z), Z \rangle.
\]

Next, we bound (49) using Proposition 12. Consider \( \tilde{f} : X \mapsto f(X) - f(M^*) \) where \( f \) is defined in (16), we have

\[
\begin{align*}
\tilde{f}(X) &= \frac{1}{2} \langle A(X), X \rangle - \langle B^*, X \rangle - f(M^*) \\
&= \frac{1}{2} \langle A(X), X \rangle - \langle A(M^*), X \rangle - \langle A^*(M^*), M^* \rangle \\
&= \frac{1}{2} \langle A(X - M^*), X - M^* \rangle \geq \frac{1 - \beta}{2\lambda} \|X - M^*\|_F^2,
\end{align*}
\]

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where the equality \((50)\) holds due to the fact that \(B^* = A(M^*)\) by definition \((16)\), the equality in \((51)\) holds since \(A\) is a symmetric operator, and the last inequality is a direct result of Proposition \((12)\). It follows that, for any \(Z \in T_{M^*} \mathcal{M}^{m \times n}_k\) and \(X := \text{Exp}_{M^*}(Z) \in \mathcal{M}^{m \times n}_k\),

\[
\tilde{f}(\text{Exp}_{M^*}(Z)) \geq \frac{1 - \beta}{2}\|\text{Exp}_{M^*}(Z) - M^*\|^2_F,
\]

which can be rewritten via the expression \((48)\) of \(\text{Exp}_{M^*}(Z)\) as follows,

\[
\tilde{f}(\text{Exp}_{M^*}(Z)) = \langle A(Z), Z \rangle + \varphi_f(Z) \geq (1 - \beta)\|Z\|^2_F + \varphi_F(Z),
\]

where \(\varphi_f(Z)\) and \(\varphi_F(Z)\) are the sums of third- and higher-order terms of \(Z\) on the two sides of \((52)\), i.e., \(|\varphi_f(Z)| \leq C_1\|Z\|^3_F\) and \(|\varphi_F(Z)| \leq C_2\|Z\|^3_F\), for constants \(C_1 > 0\) and \(C_2 > 0\).

Combining \((53)\) and \((49)\), we have, for any \(Z \in T_{M^*} \mathcal{M}^{m \times n}_k\),

\[
\lambda_f(Z) := \frac{(\text{Hess } f(M^*))[Z, Z]}{\|Z\|^2_F} = \frac{\langle A(Z), Z \rangle}{\|Z\|^2_F} \geq (1 - \beta) - \frac{|\varphi_f(Z)| + |\varphi_F(Z)|}{\|Z\|^2_F}
\]

\[
\geq (1 - \beta) - (C_1 + C_2)\|Z\|^3_F,
\]

which entails \((20)\):

\[
\lambda_f(Z) \geq \sup_{Z \in T_{M^*} \mathcal{M}^{m \times n}_k} (1 - \beta) - (C_1 + C_2)\|Z\|^3_F = 1 - \beta,
\]

where the supremum of the right-hand side is obtained with \(W \in T_{M^*} \mathcal{M}^{m \times n}_k\) in a certain direction, such that \(\|W\|^3_F \to 0\). \(\square\)

**Proof of Theorem \((14)\)**. Under the \((\beta, 2k)\)-RPD property, the result of Lemma \((13)\) holds, i.e., the Riemannian Hessian of \(f\) at \(M^*\) is positive definite; see \((20)\). Therefore, according to \([2, \text{Theorem } 4.5.6]\), if \(\{X_t\}_{t \geq 0}\) converges to \(M^*\), the convergence rate of \(\{X_t\}_{t \geq 0}\) is linear. \(\square\)

**Proof of Lemma \((15)\)**

(i) The inequality \((22)\) holds since \(f\) \((16)\) has Lipschitz-continuous gradient in \(\mathbb{R}^{m \times n}\), which is true since it is twice-differentiable and is composed of quadratic terms. The constant \(L_f = \sqrt{1 + \beta}\) when \(A\) is a projection (e.g., \((13)\)). The existence of constant \(L_g > 0\) is deduced from \([6, \text{Lemma } 2.7 \text{ and Theorem } 2.8]\).

(ii) Since \(f\) is a quadratic function, it is strongly convex on \(\mathbb{R}^{m \times n}\): there exists \(\mu_f > 0\) such that for any \(X, Y \in \mathbb{R}^{m \times n}\),

\[
f(Y) - f(X) \geq \langle \nabla f(X), Y - X \rangle + \frac{\mu_f}{2}\|Y - X\|^2_F.
\]

By setting \(Y := M^*\), the above inequality can be re-written as

\[
f(X) - f(M^*) \leq \langle \nabla f(X), X - M^* \rangle - \frac{\mu_f}{2}\|X - M^*\|^2_F.
\]

Next, we prove an inequality similar to the above result but with the term \(\nabla f(X)\) replaced by the Riemannian gradient \(\text{grad } f(X)\). The right hand-side (RHS) of \((53)\) can be written as

\[
\text{RHS} = \langle \text{grad } f(X), X - M^* \rangle - \frac{\mu_f}{2}\|X - M^*\|^2_F - \langle \text{grad } f(X) - \nabla f(X), X - M^* \rangle
\]

\[
= \langle \text{grad } f(X), X - M^* \rangle - \frac{\mu_f}{2}\|X - M^*\|^2_F - \langle (G_X - I)(\nabla f(X)), X - M^* \rangle,
\]

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where the third term above with $G_X(\cdot)$ is a result of (111) in Proposition 7.

It remains to prove that $\alpha_1 \geq 0$ under the $\delta_0$-closeness condition: the term $\alpha_1$ satisfies the following property, given that $\nabla f(X) = A(X - M^*)$,

$$\alpha_1 = \langle (G_X - I)(A(X - M^*)), X - M^* \rangle = \langle A(X - M^*), (G_X - I)(X - M^*) \rangle$$

$$= \langle A(X - M^*), (G_X - 2I)(X - M^*) \rangle + \langle A(X - M^*), X - M^* \rangle$$

$$= 2 \langle A(X - M^*), (I - P_{T_XM_k^{m \times n}})(M^*) \rangle + \langle A(X - M^*), X - M^* \rangle,$$

where the last equality is due to the property of the operator $G_X$ (Lemma 10) that $(G_X - 2I)(X - M^*) = 2(I - P_{T_XM_k^{m \times n}})(M^*)$. We proceed by evoking the following lemma about the Frobenius norm of $(I - P_{T_XM_k^{m \times n}})(M^*)$:

**Lemma 30 ([42] Lemma 4.1).** Let $X$ and $Y$ be two matrices in $M_k^{m \times n}$. Then it holds that

$$\|(I - P_{T_XM_k^{m \times n}})(Y)\|_F \leq \frac{1}{\sigma_k(Y)}\|Y - X\|_F^2. \tag{55}$$

Consequently, $\alpha_1$ is lower-bounded as follows:

$$\alpha_1 = 2 \langle A(X - M^*), (I - P_{T_XM_k^{m \times n}})(M^*) \rangle + \langle A(X - M^*), X - M^* \rangle$$

$$\geq -2\|A(X - M^*)\|_F\|(I - P_{T_XM_k^{m \times n}})(M^*)\|_F + (1 - \beta)\|X - M^*\|_F^2$$

$$\geq -\frac{2L_f}{\sigma_k}\|X - M^*\|_F^2 + (1 - \beta)\|X - M^*\|_F^2,$$

where the $(1 - \beta)$ term in the first inequality is due to the RPD property, and the first term in the last inequality is a result of (a) Lipschitz-smoothness of $f$ (Lemma 15 (i)), i.e.,

$$\|A(X - M^*)\|_F = \|\nabla f(X) - \nabla f(M^*)\|_F \leq L_f\|X - M^*\|_F;$$

and (b) the upper-bound (55). Therefore, we have

$$\alpha_1 \geq \left(1 - \beta - \frac{2L_f}{\sigma_k}\|X - M^*\|_F\right)\|X - M^*\|_F^2 \geq 0$$

for any $X$ satisfying $\|X - M^*\|_F \leq \frac{1}{L_f}(1 - \beta)^{-\frac{1}{2}}\|X - M^*\|_F^2$.

\[\square\]

**Proof of Lemma 16.** We prove the inequality (24) as follows. First, through Proposition 7 the Riemannian gradient of $f$ (16) is

$$\nabla f(X) = G_X(\nabla f(X)) = G_X(A(X - M^*), \tag{56}$$

for all $X \in M_k^{m \times n}$. Therefore, we have

$$\langle \nabla f(X), X - M^* \rangle = \langle G_X(A(X - M^*)), X - M^* \rangle$$

$$= \langle A(X - M^*), G_X(X - M^*) \rangle$$

$$= 2 \langle A(X - M^*), X - M^* \rangle + \langle A(X - M^*), (G_X - 2I)(X - M^*) \rangle$$

$$= 2 \langle A(X - M^*), X - M^* \rangle + 2 \langle A(X - M^*), (I - P_{T_XM_k^{m \times n}})(M^*) \rangle$$

$$\geq 2 \langle A(X - M^*), X - M^* \rangle - 2\|A(X - M^*)\|_F\|(I - P_{T_XM_k^{m \times n}})(M^*)\|_F, \tag{59}$$

\[26\]
where (57) holds since \(G_X\) is a symmetric operator, (58) is obtained by noticing that \((G_X - 2I)(X - M^*) = 2(I - P_{TX, M_k^{m \times n}})(M^*)\); see (14) in Lemma 10. The terms \(a_1\) and \(a_2\) in (59) have the following bounds,

\[
a_1 := \langle A(X - M^*), X - M^* \rangle \geq (1 - \beta)\|X - M^*\|_F^2,
\]

\[
a_2 := \|A(X - M^*)\|_F \leq L_f \|X - M^*\|_F,
\]

where (60) is the result of Proposition 12 and (61) holds according to Lemma 15 (recall that \(A(X - M^*) = \nabla f(X) - \nabla f(M^*)\)).

From (55) of Lemma 30, the term \(a_3\) in (59) has the following bound,

\[
a_3 := \|(I - P_{TX, M_k^{m \times n}})(M^*)\|_F \leq \frac{1}{\sigma_k^*} \|X - M^*\|_F^2,
\]

where \(\sigma_k^* := \sigma_k(M^*)\). Applying (60)–(62) to (59), we have

\[
\langle \text{grad} f(X), X - M^* \rangle \geq 2\left(1 - \beta - \frac{L_f}{\sigma_k^*} \|X - M^*\|_F\right)\|X - M^*\|_F^2.
\]

Note that the coefficient in the right-hand side of (63) is strictly positive if \(X \in M_k^{m \times n}\) satisfies \(\|X - M^*\|_F \leq \delta\), with \(\delta\) satisfying \(0 \leq \delta < (1 - \beta)^{\frac{1}{2L_f}}\).

**Proof of Proposition 18.** First, note that \(\langle \nabla f(X), \text{grad} f(X) \rangle = \frac{1}{2}\|\text{grad} f(X)\|_F^2\). Indeed, from Lemma x, we have \(G_X \circ G_X(Z) = 2G_X(Z)\). Consequently,

\[
\langle g_X, 2\nabla f(X) \rangle = \langle g_X, (2I - G_X + G_X)(\nabla f(X)) \rangle = \|g_X\|_F^2 + \langle g_X, (2I - G_X)\nabla f(X) \rangle = \|g_X\|_F^2 + \langle (2I - G_X) \circ G_X(\nabla f(X)), \nabla f(X) \rangle = \|g_X\|_F^2,
\]

where the last line holds because \((2I - G_X) \circ G_X(Z) = 0\) for any \(Z\).

(i) Consider a decomposition into two orthogonal components \(\text{grad} f(X) := P_{TX, M_k^{m \times n}}(\nabla f(X)) + P_{\perp}(\nabla f(X))\). Given that \(G_X(Z) = 2Z\) if \(Z \in TX, M_k^{m \times n}\) and \(G_X(Z) = 0\) if \(Z \in (TX, M_k^{m \times n})_\perp\), we have

\[
\frac{1}{2}\|g_X\|_F^2 = \langle \nabla f(X), g_X \rangle = \langle \nabla f(X), G_X(\nabla f(X)) \rangle = \langle \nabla f(X), P_{TX, M_k^{m \times n}}(\nabla f(X)) \rangle = 2\|P_{TX, M_k^{m \times n}}(\nabla f(X))\|_F^2.
\]

The set \(C^*_\delta, \theta = B^*(\delta) \cap \{X : \sigma_k(X) \geq \theta\}\) is a compact subset of \(B^*(\delta)\). Moreover, through Lemma 10 all \(X \in C^*_\delta, \theta\) different than \(M^*\) are non-stationary, i.e., \(\|g_X\|_F \neq 0\). Therefore, there exists a ratio \(0 < C_1 \leq 1\) such that for all \(X \in C^*_\delta, \theta\),

\[
\|g_X\|_F = 2\|P_{TX, M_k^{m \times n}}(\nabla f(X))\|_F \geq 2C_1\|\nabla f(X)\|_F.
\]

(ii) When \(A\) satisfies the RPD property, there exists \(\lambda_2 \in (0, 1)\) such that

\[
\|\nabla f(X)\|_F^2 = \|A(X - M^*)\|_F^2 = \langle X - M^*, A^2(X - M^*) \rangle \geq \lambda_2 \|X - M^*\|_F^2
\]

since the linear operator \(A^2\) is symmetric and positive definite. This entails (31) with \(C_2 = C_1\sqrt{\lambda_2}\). For (32): the term \(\Gamma_X = \nabla f(X)^T \nabla f(X)\) satisfies

\[
\|
\|\Gamma_X\|_F \leq \frac{1}{\sigma_k(X)}\|\nabla f(X)\|_F^2 \leq \frac{L_f}{\sigma_k(X)}\|\nabla f(X)\|_F\|X - M^*\|_F \leq \frac{L_f}{2C_1\sigma_k(X)}\|g_X\|_F\|X - M^*\|_F,
\]

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where the first inequality is obtained since the operator norm of $X^*$ is $\frac{1}{\sigma_k(X)}$, and the second and third inequalities are obtained from the $L_f$-Lipschitz continuity of $\nabla f$ and \((64)\) respectively. Then \((62)\) follows given that $0 < \sigma \leq \sigma_k(X)$.

**Proof of Lemma 20.** Throughout the proof, we use the short notation $g_X := \text{grad } f(X)$ for the Riemannian gradient \((11)\).

(i) To verify \((28)\): we observe that for a stepsize $0 < \theta \leq \frac{1}{L_g}$

$$-D(\theta) = \frac{\theta}{2} \left( (2 - L_g \theta) \|g_X\|^2_F - 2(1 - L_g \theta) \langle g_X, \theta \Gamma_X \rangle - L_g \theta \|\theta \Gamma_X\|^2_F \right)$$

$$= \frac{\theta}{2} \left( 2\|g_X\|^2_F - (L_g \|g_X\|^2_F + 2 \langle g_X, \Gamma_X \rangle) \theta - (L_g \|\Gamma_X\|^2_F - 2L_g \langle g_X, \Gamma_X \rangle) \theta^2 \right)$$

$$\geq \frac{\theta}{2} \left( 2\|g_X\|^2_F - (L_g \|g_X\|^2_F + 2 \langle g_X, \Gamma_X \rangle) \theta - (\|\Gamma_X\|^2_F - 2L_g \langle g_X, \Gamma_X \rangle) \theta^2 \right)$$

\(= \|g_X\|_F \|\Gamma_X\|_{F^*} P_1(\theta)\) \hfill (65)

where the inequality holds due to $-L_g \theta \|\theta \Gamma_X\|^2_F \geq -\|\theta \Gamma_X\|^2_F$ given that $0 < L_g \theta \leq 1$. In \((65)\), $P_1(\theta)$ is a quadratic polynomial with the following expression:

$$P_1(\theta) = \frac{1}{\|g_X\|_F \|\Gamma_X\|_F} \left( 2\|g_X\|^2_F - (L_g \|g_X\|^2_F + 2 \langle g_X, \Gamma_X \rangle) \theta - (\|\Gamma_X\|^2_F - 2L_g \langle g_X, \Gamma_X \rangle) \theta^2 \right)$$

$$= \frac{2}{\gamma_X} - \left( \frac{L_g}{\gamma_X} + 2 \cos(\alpha_X) \right) \theta - (\gamma_X - 2L_g \cos(\alpha_X)) \theta^2,$$ \hfill (66)

where $\gamma_X := \frac{\|\Gamma_X\|_F}{\|g_X\|_F} > 0$ and $\cos(\alpha_X) := \frac{\langle g_X, \Gamma_X \rangle}{\|g_X\|_F \|\Gamma_X\|_F}$. From \((65)\), we have $-D(\theta) \geq 0$ if $P_1(\theta) \geq 0$.

Now, we show that $P_1(\theta) \geq 0$ for all $0 < \theta \leq \bar{\theta} := \frac{1}{L_g}$ if $P_1(\bar{\theta}) \geq 0$. First, notice that $P_1(0) = \frac{2}{\gamma_X} > 0$. Then, it suffices to have $P_1(\bar{\theta}) \geq 0$. This is true for all types of quadratic polynomials with $P_1(0) > 0$; in the only unobvious case when this quadratic polynomial is decreasing then increasing on $[0, +\infty)$, it suffices to verify that $\bar{\theta}$ is smaller than the point $\theta^*$ where $P_1$ attains the minimum, i.e., it suffices to verify that

$$\bar{\theta} = \frac{1}{L_g} \leq \theta^* = \frac{1}{2} \frac{L_g}{\gamma_X} + 2 \cos(\alpha_X).$$

The required inequality above is equivalent to $\frac{L_g}{\gamma_X} + 2 \gamma_X \geq 2 \cos(\alpha_X)$, which is always true because $\frac{L_g}{\gamma_X} + 2 \gamma_X \geq 2 \sqrt{\frac{L_g}{\gamma_X} \cdot 2 \gamma_X L_g} = 2\sqrt{2}$ while the right hand-side is bounded by 2.

Note that the condition $P_1(\bar{\theta}) \geq 0$ reads

$$P_1(\bar{\theta}) = \frac{1}{L_g} \left( \frac{2L_g}{\gamma_X} - \left( \frac{L_g}{\gamma_X} + 2 \cos(\alpha_X) \right) - (\frac{\gamma_X}{L_g} - 2 \cos(\alpha_X)) \right) = \frac{1}{L_g} \left( \frac{L_g}{\gamma_X} - \frac{\gamma_X}{L_g} \right) \geq 0$$

where $L_g > 0$ and $\gamma_X > 0$, which holds if and only if $X$ satisfies

$$\gamma_X \leq L_g, \text{ i.e., } \|\Gamma_X\|_F \leq L_g \|g_X\|_F,$$ \hfill (67)

hence the sufficient condition for \((28)\) with any stepsize $0 < \theta \leq \frac{1}{L_g}$.

(ii) To prove \((29)\): We deduce an additional sufficient condition under the condition \((67)\) obtained in (i).
By multiplying both sides of the inequality above with $\rho \geq 1$ and adding $-\bar{D}(\theta)$, we have

$$-\rho D(\theta) - \bar{D}(\theta) \geq \frac{\theta}{2} \left((2\rho - 1)\|g_X\|_F^2 - 2\langle \Gamma_X, X - M^* \rangle - (\rho L_g\|g_X\|_F^2 + 2(\rho - 1)\langle g_X, \Gamma_X \rangle)\theta - ((\rho + 1)\|\Gamma_X\|_F^2 - 2\rho L_g\langle g_X, \Gamma_X \rangle)\theta^2 \right) =: \frac{\theta}{2}\|g_X\|_F\|\Gamma_X\|_F P_\rho(\theta),$$

(68)

where the $P_\rho(\theta)$ can be expressed in $\gamma_X$ and $\cos(\alpha_X)$ as follows:

$$P_\rho(\theta) = \frac{2\rho - 1}{\gamma_X} - 2\bar{I}_X - \left(\frac{\rho L_g}{\gamma_X} + 2(\rho - 1)\cos(\alpha_X)\right)\theta - ((\rho + 1)\gamma_X - 2\rho L_g\cos(\alpha_X))\theta^2,$$

(69)

where $\bar{I}_X := \frac{\langle \Gamma_X, X - M^* \rangle}{\|g_X\|_F\|\Gamma_X\|_F}$. For (69) to be positive, first let the zero-th order term be strictly positive, similar to $P_1(\theta)$, which requires $P_\rho(0) = 2\rho - 1 - 2\bar{I}_X \gamma_X > 0$. Note that for any $X$ satisfying (67), i.e., $X \in \{X \in M_k^{m \times n} : \|\Gamma_X\|_F \leq L_g\|\text{grad}(f(X))\|_F\}$, we have

$$\gamma_X P_\rho(0) + 1 = 2\rho - 2\bar{I}_X \gamma_X = 2\rho - \frac{2\|\Gamma_X\|_F\|X - M^*\|_F}{\|g_X\|_F^2} \geq \frac{2\rho - 2L_g\|X - M^*\|_F}{\|g_X\|_F} \geq \frac{2\rho - L_g}{C_2},$$

(70)

where (70) is obtained from Proposition 18 for a constant $0 < \bar{C}_2 \leq 1$. Therefore, we will have $P_\rho(0) \geq \frac{1}{\gamma_X}$ (same order of magnitude as $P_1(0)$) if $2\rho - \frac{L_g}{C_2} - 1 \geq 1$, i.e., if

$$\rho \geq 1 + \frac{L_g}{2C_2} \times O(1 + L_g).$$

(71)

Subsequently, given $\rho$ as in (71), we use the same technique for (i) to show that $P_\rho(\theta) \geq 0$ for all $0 < \theta \leq \bar{\theta}_\rho := (1 - \frac{1}{\rho})\frac{1}{L_g}$, if $P_\rho(\bar{\theta}_\rho) \geq 0$. Indeed, this can be verified for all types of quadratic polynomials with $P_\rho(0) > 0$; in the only unobvious case when $P_\rho(\theta)$ is decreasing then increasing on $[0, \infty)$, it holds that for any $\rho > 1$, $\bar{\theta}_\rho \leq \theta_\rho^*$, where $\theta_\rho^*$ is the minimizer of $P_\rho$. Hence, it always suffices to require $P_\rho(\bar{\theta}_\rho) \geq 0$. Note that we have

$$P_\rho(\bar{\theta}_\rho) = \frac{1}{L_g} \left((\rho - 2\bar{I}_X \gamma_X) \frac{L_g}{\gamma_X} - (\rho - 1)(\rho^2 - 1) \frac{\gamma_X}{L_g}\right) \geq \frac{1}{L_g} \left(\frac{L_g}{\gamma_X} - (\rho - 1)(\rho^2 - 1) \frac{\gamma_X}{L_g}\right),$$

which means $P_\rho(\bar{\theta}_\rho) \geq 0$ if $\frac{L_g}{\gamma_X} \left(\frac{L_g}{\gamma_X} - (\rho - 1)(\rho^2 - 1) \gamma_X \right) \geq 0$, hence the sufficient condition

$$\gamma_X \leq \frac{L_g}{\sqrt{(\rho - 1)(\rho^2 - 1)}}$$

(72)

where $\rho$ is given in (71). By combining (67) and (72), we have the sufficient condition (33) on $X$. \hfill \Box

**Proof of Corollary 21.** For any $X \in C_{\delta_0, \bar{\sigma}}$, the inequality (32) in Proposition 18 entails that

$$\frac{\|\Gamma_X\|_F}{\|g_X\|_F} \leq \frac{L_f}{\bar{C}_1 \bar{\sigma}} \|X - M^*\|_F,$$

and therefore the condition (33) can be satisfied if $\frac{L_f}{\bar{C}_1 \bar{\sigma}} \|X - M^*\|_F \leq \frac{2\bar{C}_1 C_\rho L_g}{L_f} \bar{\sigma} := \bar{C}_3 \delta_0$

where $\bar{C}_3 = \frac{L_g \sigma' 2\bar{C}_1 C_\rho}{\sigma\beta(1 - \beta)}$. \hfill \Box
Proof of Theorem 23. We start by proving that \( \{X_t\}_{t \geq 0} \) converges to \( M^* \) by checking initial conditions including the \( \delta \)-closeness conditions of Corollary 21.

First, we show that \( \{X_t\}_{t \geq 0} \) by Algorithm 1 is included in a compact subset of \( \mathcal{M}_k^{m \times n} \), in view of the set \( C_{\delta, \sigma}^* \) in Lemma 20 and Corollary 21. It suffices to ensure that the sequence \( \{X_t\} \) by Algorithm 1 does not get arbitrarily close to any point in \( \mathcal{M}_{\leq(k-1)} \). Note that (i) for any \( Y \in \mathcal{M}_{\leq(k-1)} \), \( \|Y - M^*\|_F \geq \sigma_k^* \), and (ii) \( \{X_t\}_{t \geq 0} \) is monotonically decreasing in \( f \) values, hence it suffices for \( X_0 \) to satisfy
\[
f(X_0) - f(M^*) \leq \min_{Y \in \mathcal{M}_{\leq(k-1)}} \{f(Y) - f(M^*)\},
\]
which, under the \((\beta, 2k)\)-RPD property, can be satisfied if \((1+\beta)\|X_0 - M^*\|_F^2 \leq (1-\beta) \min_{Y \in \mathcal{M}_{\leq(k-1)}} \|Y - M^*\|_F^2 \), i.e., if \( \|X_0 - M^*\|_F \leq \delta_1 := \sqrt{\frac{1-\beta}{1+\beta}} \sigma_k^* \). Therefore, under this \( \delta_1 \)-closeness condition, \( \{X_t\}_{t \geq 0} \) is closed, i.e., there exists \( \sigma_k(X_t) \geq \sigma \) for all \( t \geq 0 \). Consequently \( \{f(X_t)\}_{t \geq 0} \) is closed given that \( f \) is continuous. Then it follows that \( \{f(X_t)\}_{t \geq 0} \) converges, since \( \{f(X_t)\}_{t \geq 0} \) is monotonically decreasing by Algorithm 1.

Furthermore, through the RPD property, \( \{f(X_t)\}_{t \geq 0} \) being decreasing and convergent entails that (i) \( \{X_t\}_{t \geq 0} \) converges and (ii) the sequence \( \{\max_{s \geq t} \|X_s - M^*\|_F\}_{t \geq 0} \) is decreasing. Consequently, when \( \|X_0 - M^*\|_F \leq \min(\delta_1, \delta_0) \) for \( \delta_0 \) given in Corollary 21 the limit point \( X^* := \lim_{t \geq 0} X_t \) is included in \( C_{\delta_0, \sigma}^* \) (25). Finally, we show that the limit point \( X^* \) is \( M^* \). Suppose that \( X^* \in C_{\delta_0, \sigma}^* \) is not \( M^* \), then according to Corollary 21 there exists a stepsize \( 0 < \theta \leq (1 - \frac{1}{\rho}) \frac{1}{L_g} \) such that Algorithm 1 admits a strict descent from \( X^* \) in the \( f \) value, hence a contradiction. Therefore, in conclusion, \( \{X_t\}_{t \geq 0} \) converges to \( M^* \), given that \( \|X_0 - M^*\|_F \leq \delta_1 := \min(\delta_0, \delta_1) \), where \( \delta_0 \neq \delta_0 \) and \( \delta_1 = \delta_0 \).

Moreover, since \( X_0 \) satisfy the proximity conditions required in Corollary 21 \( (X_0, X_1) \) satisfies (30), which, combined with the RPD property, entails that \( \|X_1 - M^*\|_F \leq (1 + \theta_0 \Delta_{f,k}) \|X_0 - M^*\|_F \), where \( -1 < \Delta_{f,k} < 2\rho - 1 \) is a constant depending on \( (\beta, L_g, \mu) \). Hence, \( \|X_1 - M^*\|_F \leq \delta_0 \) holds if \( X_0 \) satisfies \( \|X_0 - M^*\|_F \leq \frac{1}{1 + \Delta_{f,k}} \delta_0 \). In conclusion, there exists a constant \( C \approx 1 \) such that, when \( \|X_0 - M^*\|_F \leq \min(\delta_0, C \delta_0) \), the two descent conditions (28)-(29) hold for all iterates, i.e.,
\[
R_{s+1} \leq \left( 1 - \frac{1}{\rho} \right) R_s + \frac{1}{\rho} \left( 1 - \theta_0 \mu f \frac{1}{2\theta_0} \|X_s - M^*\|_F^2 \right) \forall s \geq 0,
\]
where \( \theta_0 = (1 - \frac{1}{\rho}) \frac{1}{L_g} \) is in the range of valid stepsizes given by Corollary 21. Finally, for a strictly positive constant \( \epsilon := \min(\frac{1}{\rho}, \mu \theta_0) < 1 \), we obtain the following inequality by summing both sides of (73) times \((1 - \epsilon)^{-s}\) for \( s \in \{0, 1, \ldots, t\} \),
\[
(1 - \epsilon)^{-t} R_{t+1} \leq \left( 1 - \frac{1}{\rho} \right) R_0 + \frac{1}{\rho} \left( \frac{1 - \mu \epsilon \theta_0}{2\theta_0} \|X_0 - M^*\|_F^2 \right),
\]
where \((1 - \frac{1}{\rho}) \leq (1 - \epsilon)\). Note that \( R_0 = \langle A(X_0 - M^*), X_0 - M^* \rangle \leq (1 + \beta) \|X_0 - M^*\|_F^2 \), hence we have \( f(X_{t+1}) - f(M^*) \leq (1 - \epsilon)^{t+1} C_{f,k} \|X_0 - M^*\|_F^2 \) for a constant \( C_{f,k} > 0 \) depending only on \( (\beta, L_g, \mu_f) \).

**C.2 Proofs for Section 4.3**

**Lemma 31.** Suppose that \( X, M^* \in \mathcal{M}_k^{m \times n} \) are sufficiently incoherent:
\[
\max(\sqrt{m}\|Y\|_{2,\infty}, \sqrt{n}\|Y^T\|_{2,\infty}) \leq B \quad \text{for } Y \in \{X, M^*\},
\]
and that $X$ satisfies $\|X - M^*\|_F \leq \epsilon_0 \sigma_k(M^*)$, and $\|X - M^*\|_{2,\infty} \leq \epsilon_0 \|M^*\|_{2,\infty}$ for $\epsilon_0 > 0$. Then the RGD update $\tilde{X} := X - \theta (\nabla f(X) - \theta \Gamma_X)$ satisfies

$$\max(\sqrt{m}\|\tilde{X}\|_{2,\infty}, \sqrt{n}\|\tilde{X}^T\|_{2,\infty}) \leq (1 + \epsilon_0)B$$

where the parameter $\epsilon_\theta := (1 + \bar{r}_k)\epsilon_0 + \bar{r}_k \theta^2 \epsilon_0^2$ with $\bar{r}_k \geq \sigma_k^{-1}(X)\sigma_k^* > 0$.

**Remark 32.** The upper bound in this lemma will be an over-estimation if the RGD update with stepsize $\theta$ ensures an effective decrease in the $(2, \infty)$-norm. However, we are content with this bound in order not to add complication (or interfere with) to the contraction results in the Frobenius norm in Section [4.2].

**Proof.** First, the gradient term is an $m \times n$ matrix sufficiently incoherent: $\nabla f(X) = G_X(\nabla f(X)) = P_U P_\Omega(Z) + P_\Omega(Z)P_V$ where $Z := X - M^*$ and $(U, V)$ are the unitary matrices in the $k$-SVD $X := U \Sigma V^T$. Hence for every $i$ we have:

$$|(P_U P_\Omega(Z))_{i,:}| \leq \|(U\Sigma)^{-1}U^T P_\Omega(Z)\|_{op} \leq \|X_{i,:}\|_2 \|X^TP_\Omega(Z)\|_{op},$$

(74)

where the equality in (74) is due to $\|(U\Sigma)_{i,:}\|_2 = \|X_{i,:}\|_2$ and $\|(\Sigma^{-1}U^T P_\Omega(Z)y)\|_2 = \|V(y)\|_2$ for any $y \in \mathbb{R}^n$ (since $V^T V = I_{k \times k}$), and inequality (75) is obtained after $\|P_\Omega(Z)\|_{op} \leq \|P_\Omega(Z)\|_F \leq \|Z\|_F$. On the other hand, it holds that

$$|(P_\Omega(Z)P_V)_{i,:}| \leq \|(P_\Omega(Z))_{i,:}\|_2 \leq \|Z_{i,:}\|_2,$$

(76)

where the first inequality is because $P_V$ is an orthogonal projection onto a strict subspace of $\mathbb{R}^n$ (with dimension $k < n$). By combining (75) and (76), along with the two $\epsilon_0$-proximity bounds of $X$ (in terms of $Z = X - M^*$), we have $\|(\nabla f(X))_{i,:}\|_2 \leq (1 + \bar{r}_k)\epsilon_0 \frac{B}{\sqrt{m}}$.

Second, by applying the same techniques, we have

$$|(\Gamma_X)_{i,:}| \leq \|(P_\Omega(Z))_{i,:}\|_2 \|X^TP_\Omega(Z)\|_{op} \leq \|Z_{i,:}\|_2 \|X^TP_\Omega(Z)\|_{op} \leq \|Z_{i,:}\|_2 (\sigma_k^{-1}(X)\|Z\|_F).$$

(77)

Given the $\epsilon_0$-proximity bounds, (77) entails that $|(\Gamma_X)_{i,:}| \leq \|X - M^*\|_{2,\infty} (\sigma_k^{-1}(X)\|X - M^*\|_F) \leq \bar{r}_k \epsilon_0 \frac{B}{\sqrt{m}}$.

Finally, we have $\sqrt{m}\|\tilde{X}_{i,:}\|_2 \leq \left(1 + \theta(1 + \bar{r}_k)\epsilon_0 + \theta^2 \bar{r}_k \epsilon_0^2\right)B$ for all $i$. Hence the parameter $\epsilon_\theta = \theta(1 + \bar{r}_k)\epsilon_0 + \theta^2 \bar{r}_k \epsilon_0^2$. The same conclusion applies to $\sqrt{n}\|\tilde{X}^T\|_{2,\infty}$ by the transpose operation and exchange of roles of $U$ and $V$, noticing that $P_\Omega$ (and $\mathcal{A}$ in general) is a symmetric operator. □

**Proof of Proposition 26.** We show that the assumptions of this proposition can be converted into the assumption: of [35] Theorem 3.1, Claim 3.1, Lemma 3.1], based on the fact that they all describe an incoherence neighborhood of $M^*$; the only difference is in the matrix representations: in this proposition, we use the $m \times n$ matrix representation, while [35] Claim 3.1] uses the matrix factorization representation $(G, H)$ such that $GHT = X \in \mathbb{R}^{m \times n}$, and the incoherence neighborhood is $K_1 \cap K_2 \cap K_3$ (see [35, eq.(30)]), where $K_3 := \{(G, H)\|GHT - M^*\|_F \leq \delta\}$, and $K_1$ and $K_2$ are sets of $(G, H)$ with bounded max row-norms and Frobenius norms respectively.

For any $X \in B_{\epsilon_0}(C_B, \mu)$, let $(G, H) \in \mathbb{R}^{m \times k} \times \mathbb{R}^{n \times k}$ be a pair of matrices satisfying $GHT = X$ and $\|G\|_F = \|H\|_F$. Then, it follows that (i) $(G, H) \in K_1$, because $G$ and $H$ are balanced and share the same set of left and right singular vectors with $X$, respectively, hence they are
incoherent: \(|G|_{2,\infty} \leq \sqrt{2C^* \sigma_{\max}^* k \frac{\mu k}{m}}\) and \(|H|_{2,\infty} \leq \sqrt{2C^* \sigma_{\max}^* k \frac{\mu k}{n}}\). (ii) \((G, H) \in K_2\) (with an uniform bound “\(\beta_T\)” on Frobenius norms) holds with the given choice \(|G|_F = |H|_F\), provided that the bound \(\beta_T\) fits \(\max_{X \in B(\delta)} |X|_F\); and (iii) \((G, H) \in K_\delta\), simply because \(GH^T = X\) and \(|X - M^*|_F \leq \delta\). Hence the assumptions of [35] Lemma 3.1 hold, therefore the conclusion.  

**Proof of Proposition 28.** By construction, \(\mathcal{P}_B(\tilde{X})\) has the same incoherence bound \(\|\|\) as a result of the normalization operator \(\mathcal{P}_B\) at \(\tilde{X}\). We show the non-expansiveness \(\|\|\) as follows.

The following property [37, 45] is used:

**Property 33.** For vectors \(u, u^* \in \mathbb{R}^n\) and \(\lambda \geq \frac{|u^*|^2}{|u|^2}\), it holds that \(|(1 + \lambda)u - u^*|_2 \leq |u - u^*|_2\).

In view of Property 33, it holds that, for \(D_{i(i)} = \min(1, \lambda_i)\) and \(\lambda_i : = \frac{\mu}{\sqrt{m} |X_i|_2}\),

\[
\|\mathcal{P}_B(\tilde{X}) - M^*\|_{i(i)}_2 = \|D_{i(i)}(\tilde{X}D(2))_{i(i)} - M_{i(i)}^*\|_2 \leq \|D_{i(i)}(\tilde{X}D(2) - M^*)_{i(i)}\|_2
\]

(78)

as long as \(\lambda_i = \frac{B}{\sqrt{m} |X_i|_2} \geq \frac{\|M_{i(i)}^*\|_2}{\|X(2)i(i)\|_2}\), which can be satisfied when \(B \geq (1 + \epsilon_0)\sigma_{\max}^* \sqrt{\mu k}\). More precisely, we have

1. \(|M_{i(i)}^*|_2 \leq \|U_{i(i)}^* \Sigma^*\|_2 \leq \sigma_{\max}^* \|U^*\|_{2,\infty} \leq \sigma_{\max}^* \sqrt{\frac{\mu k}{m}}\).

2. The ratio \(\frac{\|\tilde{X}_{i(i)}\|_2}{\|X(2)i(i)\|_2} \leq 1 + \epsilon_0\). This is because: for every \(j\), \((\tilde{X}D(2))_{ij} = D_{jj}^{(2)} \tilde{X}_{ij}\) where \(D_{jj}^{(2)} = \min(1, \frac{B}{\sqrt{m} |X_{i(j)}|_2})\) is lower-bounded since \(|\tilde{X}_{i(j)}|_2\) is also small enough. Indeed, through Lemma 31 we have \(\sqrt{n} \|\tilde{X}_{i(j)}\|_2 \leq (1 + \epsilon_0)B\) for \(\epsilon_0\) given in the statement. Hence for all \(j\), \(\frac{1}{1 + \epsilon_0} \leq D_{jj}^{(2)} \leq 1, \) and \(\frac{1}{1 + \epsilon_0} |\tilde{X}_{ij}| \leq |(\tilde{X}D(2))_{ij}| \leq |\tilde{X}_{ij}|\).

As a consequence of (78) for all \(i\), we have \(|\mathcal{P}_B(\tilde{X}) - M^*|_F \leq \|\tilde{X}D(2) - M^*|_F\).

Finally, using the same argument, we have further that \(|\tilde{X}D(2) - M^*|_F \leq \|\tilde{X} - M^*|_F\). This is because: for every \(j\), it holds that \(|(\tilde{X}D(2) - M^*)_{i(j)}|_2 \leq \|D_{jj}^{(2)} \tilde{X}_{i(j)} - M_{i(j)}^*\|_2 \leq |\tilde{X}_{i(j)} - M_{i(j)}^*|_2\) for \(D_{jj}^{(2)} = \min(1, \frac{B}{\sqrt{m} |X_{i(j)}|_2})\), if \(\lambda_j : = \frac{B}{\sqrt{m} |X_{i(j)}|_2} \geq \frac{\|M_{i(j)}^*\|_2}{\|X_{i(j)}\|_2}\), i.e., if \(\frac{B}{\sqrt{m}} \geq \|M_{i(j)}^*\|_2,\) which is true since \(B \geq \sqrt{n} \|M^*\|_{2,\infty} \geq \sqrt{n} \|M_{i(j)}^*\|_2\) (for the same reason as point 1. above).  

**Proof of Corollary 29.** The linear convergence to \(M^*\) is a result of Theorem 23 given Proposition 26 and Proposition 28. Proposition 28 confirms that the normalized RGD rule preserves the incoherence bound of \(X_t\) and is also non-expansive, which ensure that \(\{X_t\}_{t \geq 0} \subset B_{\delta_0}^* (C_B, \mu)\) [34]. Hence the RPD property [35] holds for all \(X_t\).

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