SLOPE ESTIMATES OF ARTIN-SCHREIER CURVES

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Abstract. Let \( f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0 \) be a polynomial of degree \( d \) in \( \mathbb{Q}[x] \). For every prime number \( p \) coprime to \( d \) and \( f(x) \in (\mathbb{Z}_p \cap \mathbb{Q})[x] \), let \( X/\mathbb{F}_p \) be the Artin-Schreier curve defined by the affine equation \( y^p - y = f(x) \mod p \). Let \( \text{NP}_1(X/\mathbb{F}_p) \) be the first slope of the Newton polygon of \( X/\mathbb{F}_p \). We prove that there is a Zariski dense subset \( U \) in the space \( \mathbb{A}^d \) of degree-\( d \) monic polynomials over \( \mathbb{Q} \) such that for all \( f(x) \in U \) the following limit exists and \( \lim_{p \to \infty} \text{NP}_1(X/\mathbb{F}_p) = \frac{1}{2} \). This is a "first slope version" of a conjecture of Wan.

Let \( X/\mathbb{F}_p \) be an Artin-Schreier curve defined by the affine equation \( y^p - y = \tilde{f}(x) \) where \( \tilde{f}(x) = x^d + \tilde{a}_{d-1}x^{d-1} + \ldots + \tilde{a}_0 \). We prove that if \( p > d \geq 2 \) then \( \text{NP}_1(X/\mathbb{F}_p) \geq \frac{\lfloor \frac{d-1}{p} \rfloor}{2} \). If \( p > 2d \geq 4 \), we give a sufficient condition for the equality to hold.

1. Introduction

In this paper a curve is a smooth, projective and geometrically integral algebraic variety of dimension one. Let \( d \) be a natural number. Let \( p \) be a prime coprime to \( d \). Let \( q = p^r \) for some natural number \( r \). Let \( X \) be an Artin-Schreier curve over \( \mathbb{F}_q \) defined by an affine equation \( X : y^p - y = \tilde{f}(x) \), where \( \tilde{f}(x) \in \mathbb{F}_q[x] \) is of degree \( d \). Then \( X \) has genus \( g := \frac{(q-1)(d-1)}{2} \). (See Section 3.)

Write the \( L \) function of \( X \) over \( \mathbb{F}_q \) as

\[
\exp \left( \sum_{n=1}^{\infty} \left( q^n + 1 - \#X(\mathbb{F}_q^n) \right) \frac{T^n}{n} \right) = \frac{1}{P(T)}.
\]

The denominator \( P(T) \) is a polynomial \( 1 + \sum_{t=1}^{2g} b_t T^t \in 1 + T \mathbb{Z}[T] \). Consider the sequence of points \((0,0), (1, \text{ord}_p b_1), (2, \text{ord}_p b_2), \ldots, (2g, \text{ord}_p b_{2g}) \) in \( \mathbb{R}^2 \). (If \( b_t = 0 \), define \( \text{ord}_p b_t = \infty \).) The normalized \( p \)-adic Newton polygon of \( P(T) \) is defined to be the lower convex hull of this set of points. It is called the Newton polygon of \( X/\mathbb{F}_q \), denoted by \( \text{NP}(X/\mathbb{F}_q) \). Let \( \text{NP}_1(X/\mathbb{F}_q) \) denote the first slope of \( \text{NP}(X/\mathbb{F}_q) \), which we call the first slope of \( X/\mathbb{F}_q \).

For any real number \( t \) let \( \lfloor t \rfloor \) denote the least integer greater than or equal to \( t \) and let \( \lceil t \rceil \) be the biggest integer less than or equal to \( t \).

Let \( R \) be a commutative ring with unity. For any \( f(x) \in R[x] \), any natural numbers \( N \) and \( r \), we use \( \lfloor f(x)^N \rfloor_r \) to denote the \( x^r \)-coefficient of \( f(x)^N \).

Theorem 1.1. Fix \( d \geq 2 \). Let \( X/\mathbb{F}_q \) be an Artin-Schreier curve of genus \( g \geq 3 \) whose affine equation is given by \( y^p - y = \tilde{f}(x) \) where \( \tilde{f}(x) \) is monic of degree \( d \).

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(a) If \( p > d \) then \( \text{NP}_1(X/F_q) \geq \left\lceil \frac{p-1}{p-1} \right\rceil \).

(b) If \( p > 2d \) and \( [\tilde{f}(x)^{1/p-1}]_{p-1} \neq 0 \) then \( \text{NP}_1(X/F_q) = \left\lceil \frac{p-1}{p-1} \right\rceil \).

Let \( A^d \) be the set of all polynomials \( f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in \mathbb{Q}[x] \).

**Theorem 1.2.** There exists a Zariski dense subset \( \mathcal{U} \) in \( A^d \) such that for all \( f(x) \in \mathcal{U} \) the Artin-Schreier curve \( X/F_p \) defined by \( y^p - y = f(x) \mod p \) satisfies
\[
\lim_{p \to \infty} \text{NP}_1(X/F_p) = \frac{1}{d}.
\]

If a curve is defined over a perfect field of characteristic \( p \) then its Newton polygon is defined by the “formal types” of the \( p \)-divisible groups associated to the Jacobian of the curve (see \cite{15} or \cite{14}). Theorem 1.1 holds if \( F_q \) is replaced by any perfect field of characteristic \( p \) because its proof remains valid. It is known that these Newton polygons have integral bending points and is symmetric in the sense that any line segment of slope \( \lambda \) of length \( \ell \) occurs in companion with a line segment of slope \( 1 - \lambda \) of the same length.

Artin-Schreier curves are precisely those degree \( p \) abelian covers of the projective line with the point at infinity totally ramified. So their \( p \)-ranks are zero by the Deuring-Shafarevich formula (see \cite{17}). The \( p \)-rank is exactly equal to the length of the slope zero segment of their Newton polygons (see \cite{13}). Thus an Artin-Schreier curve has no zero slope. Suppose \( g = 1 \) or \( 2 \), then an Artin-Schreier curve \( X \) has its first slope equal to \( 1/2 \).

When \( f(x) \) is a monomial then the Frobenius and Verschiebung maps on the first crystalline cohomology of \( X \) have explicit interpretations (see \cite{11} and \cite{13}), which enable one to describe the entire Newton polygon of \( X \) explicitly. (Note that classical literature often refer to this special case as the definition of Artin-Schreier curve.)

This paper is organized as follows: We recall relevant preliminaries in Section 2. Then we develop a method in Section 3 to estimate the first slopes of Artin-Schreier curves. With some technical preparation in Sections 4 and 5, Section 6 proves a lower bound for the first slopes of Artin-Schreier curves, and gives a sufficient condition for the lower bound to be achieved. We prove Theorem 1.1 here. Section 7 detours to prove a similar result by a completely different method. Finally in Section 8 we prove Theorem 1.2. We will explain how this is related to the \( L \) functions of exponential sums and a conjecture of Wan.

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### 2. Sharp slope estimates

This section provides fundamental ingredients for our slope estimates of curves over finite fields. Note that lemmas we need hold valid when the base field is perfect of characteristic \( p \). However, for simplicity we constrain ourselves to finite fields in this paper. Firstly we establish a variation of Katz’s sharp slope estimates in Theorem 2.2. Secondly we recall a method of computing the Verschiebung action on the first de Rham cohomology of a curve by taking power series expansions at
a rational point. This section essentially follows \[10, \] \[17, \] \[4, \] \[5, \] and \[18, \]. Our approach is particularly inspired by Nygaard’s paper \[18, \].

Let \( W \) be the ring of Witt vectors over \( \mathbb{F}_q \), and \( \sigma \) the absolute Frobenius automorphism of \( W \). Throughout this section we assume that \( X/\mathbb{F}_q \) is a curve of genus \( g \) with a rational point. Let \( X/W \) be a smooth and proper lifting of \( X \) to \( W \), together with a lifted rational point \( P \). The Frobenius endomorphism \( F \) (resp., Verschiebung endomorphism \( V \)) are \( \sigma \) (resp., \( \sigma^{-1} \)) linear maps on the first crystalline cohomology \( H^1_{\text{crys}}(X/W) \) of \( X \) with \( FV = VF = p \). It is known that \( H^1_{\text{crys}}(X/W) \) is canonically isomorphic to the first de Rham cohomology \( H^1_{\text{dR}}(X/W) \) of \( X \), and one gets \( F \) and \( V \) actions on \( H^1_{\text{dR}}(X/W) \). Thus the pair \((H^1_{\text{dR}}(X/W), F)\) is a \( \sigma \)-\( F \)-crystal, whereas the pair \((H^1_{\text{dR}}(X/W), V)\) is a \( \sigma^{-1} \)-\( F \)-crystal. The Newton polygon of \( X/\mathbb{F}_q \) is equal to the Newton polygon of the crystals \((H^1_{\text{dR}}(X/W), F)\) and \((H^1_{\text{dR}}(X/W), V)\) as defined in \[10, \] \[15, \].

Below we will briefly describe some techniques to approximate slopes of these crystals. Let \( L \) be the image of \( H^0(X, \Omega^1_{X/W}) \) in \( H^1_{\text{dR}}(X/W) \), and let \( M \) be a complement of \( L \) such that \( H^1_{\text{dR}}(X/W) = L \oplus M \). The following lemma is prepared for the proof of Theorem \[2.2, \]

Lemma 2.1. Let notation be as above. Then \( L \subset V(L \oplus M) \subset L \oplus pM \). If \( p^m \) divides \( V^nL \) for some \( m > 0 \) and \( a \geq 0 \), then for all \( n > a \) we have

\[
V^nL \subset p^{m-1}L + p^mM.
\]

Proof. Recall an equality due to Mazur and Ogus (see \[16, \] Theorem 3)).

\[
F^{-1}(p(L \oplus M)) \otimes \mathbb{F}_q = L \otimes \mathbb{F}_q.
\]

One easily verifies the following inclusions

\[
L \subset F^{-1}(p(L \oplus M)) + p(L \oplus M) \subset F^{-1}FV(L \oplus M) + VF(L \oplus M) \subset V(L \oplus M).
\]

The rest follows from \[18, \] Lemmas 1.4 and 1.5.

Theorem 2.2. Let \( \lambda \) be a rational number with \( 0 \leq \lambda \leq \frac{1}{2} \). Then \( \text{NP}_1(X/\mathbb{F}_q) \geq \lambda \) if and only if

\[
p^{[n\lambda]} \mid V^{n+g-1}L
\]

for all integer \( n \geq 1 \).

Proof. The main ingredient of the proof is Katz’s sharp slope estimate \[10, \] \[15, \] (1.5), which says that \( \text{NP}_1(X/\mathbb{F}_p) \geq \lambda \) if and only if \( p^{[n\lambda]} \mid V^{n+g} \) for all \( n \geq 1 \).

Suppose \( \text{NP}_1(X/\mathbb{F}_p) \geq \lambda \). Then \( p^{[n\lambda]} \mid V^{n+g} \) for all \( n \geq 1 \). By Lemma 2.1 we have

\[
V^{n+g-1}(L) \subset V^{n+g-1}(\text{Im}V) = V^{n+g}(L \oplus M) \subset p^{[n\lambda]}(L \oplus M).
\]

Conversely, suppose that \( p^{[n\lambda]} \mid V^{n+g-1}L \) for all \( n \geq 1 \). It suffices to show that \( p^{[n\lambda]} \mid V^{n+g} \) for all \( n \geq 0 \). For \( n = 0 \) this statement is trivially true. We proceed by induction on \( n \).

Assume that \( p^{[(n-1)\lambda]} \mid V^{n+g-1} \). By Lemma 2.1 we have \( V(L \oplus M) \subset L \oplus pM \).

So

\[
V^{n+g}(L \oplus M) = V^{n+g-1}V(L \oplus M) \subset V^{n+g-1}(L \oplus pM),
\]

and

\[
p^{[n\lambda]} \mid p^{[(n-1)\lambda]+1} = V^{n+g-1}(pM).
\]
From the hypothesis $p^{[n\lambda]} \mid V^{n+g-1}L$, we have $p^{[n\lambda]} \mid V^{n+g}$. 

**Remark 2.3.** Let $n$ and $m$ be any natural integers. If $p^m - 1$ divides $V^n L$ for some nonnegative integer $a < n$, following Lemma [2.1], the composition of $\frac{V}{p}$ and reduction to $\mathbb{F}_q$ gives a natural endomorphism of $L \otimes \mathbb{F}_q$. This endomorphism of $L \otimes \mathbb{F}_q$ is called a *higher Cartier operator*, denoted by $C(m, n)$. The hypothesis in the theorem above is equivalent to that $C([n\lambda], n+g-1)$ is defined and vanishes for all integer $n \geq 1$. The underneath philosophy of our slope estimates is replacing the traditional Cartier operator by this higher Cartier operator. We will not explore this terminology further in this paper.

Let $\hat{X}/W$ be the formal completion of $X/W$ at the rational point $P$. If $x$ is a local parameter of $P$, then every element of $H^1_{\text{dR}}(\hat{X}/W)$ can be represented as $h(x)\frac{dx}{x}$ for some $h(x) \in xW[[x]]$, and $F$ and $V$ act as follows:

$$F(h(x)\frac{dx}{x}) = ph^\sigma(x^p)\frac{dx}{x}$$

$$V(h(x)\frac{dx}{x}) = h^{\sigma-1}(x^{1/p})\frac{dx}{x} \quad \text{where } x^{m/p} = 0 \text{ if } p \nmid m.$$  

Denote the restriction map $H^1_{\text{dR}}(X/W) \rightarrow H^1_{\text{dR}}(\hat{X}/W)$ by res.

**Lemma 2.4.** The $F$ and $V$ actions on $H^1_{\text{dR}}(X/W)$ and $H^1_{\text{dR}}(\hat{X}/W)$ commute with the restriction map

$$\text{res} : H^1_{\text{dR}}(X/W) \rightarrow H^1_{\text{dR}}(\hat{X}/W).$$

Furthermore,

$$\text{res}^{-1}(pH^1_{\text{dR}}(\hat{X}/W)) = F(H^1_{\text{dR}}(X/W)).$$

**Proof.** The first statement follows from [11, Lemma 5.8.2]. The second is precisely [13, Lemma 2.5].

This lemma will only be used in the proof of Theorem 3.4.

### 3. Slope estimates of Artin-Schreier curves

Assume that $X$ is an Artin-Schreier curve over $\mathbb{F}_q$ defined by an affine equation $y^p - y = \hat{f}(x)$ where $\hat{f}(x) = x^d + \hat{a}_{d-1}x^{d-1} + \ldots + \hat{a}_1x$ and $p \nmid d$. It is easy to observe that every Artin-Schreier curve over $\mathbb{F}_p$ can be written in this form (over some suitable $\mathbb{F}_q$). So $X/\mathbb{F}_q$ has a rational point at the origin.

Let $p$ be coprime to $d$ and $g \geq 3$. Take a lifting $X/W$ defined by $y^p - y = f(x)$ where $f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1x \in W[x]$ with $a_\ell \equiv \hat{a}_\ell \mod p$ for all $\ell$. So $X/W$ has a rational point at the origin with a local parameter $x$. The goal of this section is to prove Theorem 3.4. In particular, we shall prove a highly applicable version in Key-Lemma 3.5.

For any integer $N > 0$ and $0 \leq i \leq p - 2$ let $C_r(i, N)$ be the $x^r$-coefficient of the power series expansion of the function $y'(py^{p-1} - 1)^{p^N-1}$ at the origin $P$:

$$y'(py^{p-1} - 1)^{p^N-1} = \sum_{r=0}^{\infty} C_r(i, N)x^r.$$

We prepare three lemmas before we start to prove Theorem 3.4. We shall reserve the range of $i$ and $j$ to be as in Lemma 3.3.
Lemma 3.1. The curve $X/W$ has genus $(d-1)(p-1)/2$, and for $p-2 \geq i \geq 0$, $j \geq 1$ and $di + pj \leq (p-1)(d-1) - 2 + p$ the differential forms

$$\omega_{i,j} := x^j y^i (py^{p-1} - 1)^{-1} \frac{dx}{x}$$

form a basis for $L$.

Proof. For the special fibre $X/\mathbb{F}_q$, this follows immediately from Proposition VI.4.2 of [21]. Let $QW$ be the field of fractions of $W$. Consider the generic fibre $X/QW$. There are no points $(x, y)$ in $X(\overline{QW})$ with $py^{p-1} - 1 = f'(x) = 0$, since for such $(x, y)$ one can easily show that $y^p - y$ is not integral over $W$, and $f(x)$ is integral over $W$. It follows that the affine part of $X/QW$ is nonsingular. The affine ramification points of the map $X \to \mathbb{P}^1$ defined by $(x, y) \mapsto x$ are those satisfy $py^{p-1} - 1 = 0$. For each such $y$ there are exactly $d$ corresponding values of $x$, since $f'(x) \neq 0$ there. So there are $(p-1)d$ ramification points on the affine part. The function $y^p - y - f(x)$ in $y$ and its first two derivatives have no common zeroes, so all affine ramification points are of index 2. Let $e_\infty$ be the ramification index at $\infty$. By Riemann-Hurwitz, we have $2g - 2 = 2p + (p-1)(d-1) + e_\infty - 1$. It follows that $g \leq (p-1)(d-1)/2$. But the genus of the special fibre $X/\mathbb{F}_q$ is $(p-1)(d-1)/2$, hence the genus of $X/QW$ is $(p-1)(d-1)/2$, and $e_\infty = p$.

The differential form $\frac{dx}{py^{p-1} - 1} = \frac{dy}{y^{p-1} - 1}$ has no poles at the affine part. The form $dx$ only has affine zeroes (of order 1) at points where the map $(x, y) \mapsto x$ ramifies. At these points, $py^{p-1} - 1 = 0$, so $\frac{dx}{py^{p-1} - 1}$ has no affine zeroes. The degree of a differential form is $2g - 2 = (p-1)(d-1) - 2$, hence $\frac{dx}{py^{p-1} - 1}$ has a zero of order $(p-1)(d-1) - 2$ at $\infty$. The function $x$ has degree $p$ and no poles at the affine part. Hence it has a pole of order $p$ at $\infty$. Similarly, $y$ has a pole of order $d$ at $\infty$. So for $p - 2 \geq i \geq 0$, $j \geq 1$ and $di + pj \leq (p-1)(d-1) - 2 + p$ the form $\omega_{i,j}$ is in $L$. From the assumption that $d$ and $p$ are coprime it follows that for $i$ and $j$ in this range the $\omega_{i,j}$ have zeroes of different order at $\infty$, hence they are independent. From [21], Proposition VI.4.2 (h), it follows that the reduction of these differential forms modulo $p$ form a basis for $H^0(\Omega^1_{X/\mathbb{F}_q})$, hence the $\omega_{i,j}$ form a basis for $L$. □

Lemma 3.2. Let $m$ be a positive integer. If $p \nmid m$ then $x^m(py^{p-1} - 1)^{-1} \frac{dx}{x} \equiv 0 \mod p$ in $H^1_{dR}(\hat{X}/W)$.

Proof. If $p \nmid m$ then $x^m(py^{p-1} - 1)^{-1} \frac{dx}{x} \equiv -d \left( \frac{m}{m} \right) \mod p$, which is zero in $H^1_{dR}(\hat{X}/W)$. □

Lemma 3.3. For all nonnegative integers $a$ and $r$ we have

$$C_r(i, N + a) \equiv C_r(i, N) \mod p^{N+1}.$$

Proof. It is easy to see that $\binom{p^{N}}{\ell} \equiv 0 \mod p^{N+1-\ell}$ if $N + 1 \geq \ell \geq 1$. Thus

$$(1 - py^{p-1})^{p^N} = \sum_{\ell=0}^{p^N} \binom{p^N}{\ell} (-py^{p-1})^\ell \equiv 1 \mod p^{N+1}.$$

Therefore, we have

$$y^i(py^{p-1} - 1)^{p^{N+a}-1} \equiv y^i(py^{p-1} - 1)^{p^{N-1}}(1 - py^{p-1}) \equiv y^i(py^{p-1} - 1)^{p^{N-1}} \mod p^{N+1}. $$
This proves the lemma.

**Theorem 3.4.** Let $\lambda$ be a rational number with $0 \leq \lambda \leq \frac{1}{2}$. Suppose there exists an integer $n_0$ such that

(i) for all $m \geq 1$ and $1 \leq n < n_0$ we have

$$\text{ord}_p(C_{mp^{n+g-1}}(i, n + g - 2)) \geq \lceil n\lambda \rceil;$$

(ii) for all $m \geq 2$ we have

$$\text{ord}_p(C_{mp^n+g-1}(i, n + g - 2)) \geq \lceil n_0\lambda \rceil.$$

Then

$$\left\{ \begin{array}{ll}
p^\lceil n\lambda \rceil | V^{n+g-1}(\omega_{i,j}) & \text{if } n < n_0; \\
p^{\lceil n_0\lambda \rceil - 1} | V^{n_0+g-1}(\omega_{i,j}) & \text{if } n = n_0.
\end{array} \right.$$

Furthermore, we have

$$V^{n_0+g-1}(\omega_{i,j}) = C_{p^{n_0+g-1}}^{p^n+g-2}(i, n_0 + g - 2)(\omega_{0,1}).$$

**Proof.** We will prove the first part by induction. Suppose $1 \leq n \leq n_0$ and

$$p^{\lceil (n-1)\lambda \rceil} | V^{n+g-2}(\omega_{i,j}).$$

Note that this is trivially true if $n = 1$.

Write $h(x) := \text{res}(py^{p-1} - 1)^{-1}$. By [18, Lemma 2.2], we have

$$h(x)p^{n+g-2} = h^n(x^{p^{n+g-2}}) + ph_1^n(x^{p^{n+g-3}}) + \cdots + p^{n+g-2}h_{n+g-2}(x),$$

for some power series $h_1(x), h_2(x), \ldots, h_{g+n-2}(x) \in W[[x]]$. Thus the power series expansion of $\omega_{i,j}$ is

$$\text{res}(\omega_{i,j}) = \text{res} \left( x^j y^i (py^{p-1} - 1)^{-1} \frac{dx}{x} \right)$$

$$= \text{res} \left( x^j y^i (py^{p-1} - 1)p^{n+g-2}h(x) p^{n+g-2} \frac{dx}{x} \right)$$

$$= \sum_{r=0}^{\infty} C_r(i, n + g - 2)x^{r+j}h_r^{n+g-2}(x^{p^{n+g-2}}) \frac{dx}{x}$$

$$+ p^{n+g-2} \sum_{r=0}^{\infty} C_r(i, n + g - 2)x^{r+j}h_{n+g-2}(x) \frac{dx}{x}.$$
Apply the $V^{g+n-2}$ action on the above first differential form. Since the $V$-action commutes with the restriction map (by Lemma 2.4), we have

$$\text{res}(V^{n+g-2} \omega_{ij}) = \sum_{m=1}^{\infty} C_{mp^{n+g-2} - j}^{−(n+g-2)}(i, n + g - 2)x^m h(x) \frac{dx}{x}$$

$$+ p \sum_{m=1}^{\infty} C_{mp^{n+g-3} - j}^{−(n+g-3)}(i, n + g - 2)V \left( x^m h_1(x) \frac{dx}{x} \right)$$

$$+ p^2 \sum_{m=1}^{\infty} C_{mp^{n+g-4} - j}^{−(n+g-4)}(i, n + g - 2)V^2 \left( x^m h_2(x) \frac{dx}{x} \right)$$

$$+ \cdots$$

$$+ p^{[n\lambda]-1} \sum_{m=1}^{\infty} C_{mp^{n+g-1} - [n\lambda] - j}^{−(n+g-1-[n\lambda])}(n + g - 2)$$

$$\cdot V^{[n\lambda]-1} \left( x^m h_{[n\lambda]-1}(x) \frac{dx}{x} \right)$$

$$+ p^{[n\lambda]} \beta,$$

for some $\beta \in H^1_{\text{DR}}(\hat{X}/W)$.

By the hypothesis, $p^{[n\lambda]-1}$ divides $C_{mp^{n+g-2} - j}(i, n + g - 3)$. For all $m \geq 1$, by Lemma 3.3,

$$p^{[n\lambda]-1} \mid C_{mp^{n+g-2} - j}(i, n + g - 2).$$

For $m$ coprime to $p$ it follows from Lemma 2.2 that $p$ divides $x^m h(x) \frac{dx}{x}$. Thus

$$p^{[n\lambda]} \mid C_{mp^{n+g-2} - j}(i, n + g - 2).$$

Otherwise, except possibly when $n = n_0$ and $m = p$, we have

$$p^{[n\lambda]} \mid C_{p^{n+g-1} - j}(i, n + g - 2).$$

Therefore,

$$\sum_{m=1}^{\infty} C_{mp^{n+g-2} - j}^{−(n+g-2)}(i, n + g - 2)x^m h(x) \frac{dx}{x}$$

$$\equiv \sum_{m'=1}^{\infty} C_{mp^{n+g-2} - j}^{−(n+g-2)}(i, n + g - 2)x^{m'} h(x) \frac{dx}{x}$$

$$\equiv \begin{cases} 
0 \mod p^{[n\lambda]} & \text{if } n < n_0 \\
C_{p^{n+g-1} - j}^{−(n+g-2)}(i, n_0 + g - 2)x^{p^\ell} h(x) \frac{dx}{x} \mod p^{[n\lambda]} & \text{if } n = n_0
\end{cases}$$

For all integer $\ell \geq 1$, by the hypothesis of the theorem, we obtain

$$\text{ord}_p(C_{mp^{n+g-\ell} - j}(i, n + g - \ell - 3)) \geq [(n - \ell - 1)\lambda] \geq [n\lambda] - \ell.$$
Hence for such $n$ Lemma 2.4 implies
\[
\frac{V^{n+g-2}(\omega_{i,j})}{p^{|n\lambda|-1}} \in F(H^1_{\text{dr}}(X/W))
\]
so
\[
\frac{V^{n+g-1}(\omega_{i,j})}{p^{|n\lambda|-1}} \in VF(H^1_{\text{dr}}(X/W)) = pH^1_{\text{dr}}(X/W),
\]
which proves the induction hypothesis. If $n = n_0$ then the above implies
\[
\text{res} \left( \frac{V^{n_0+g-2}(\omega_{i,j})}{p^{|n_0\lambda|-1}} \right) - \frac{1}{p^{|n_0\lambda|-1}} C_{p^{n_0+g-1-j}}^{\sigma^{-(n_0+g-2)}}(i, n_0 + g - 2)\omega_{0,p}
\]
lies in $pH^1_{\text{dr}}(\hat{X}/W)$. Lemma 2.4 implies
\[
\frac{V^{n_0+g-2}(\omega_{i,j})}{p^{|n_0\lambda|-1}} - \frac{1}{p^{|n_0\lambda|-1}} C_{p^{n_0+g-1-j}}^{\sigma^{-(n_0+g-2)}}(i, n_0 + g - 2)\omega_{0,p}
\]
lies in $F(H^1_{\text{dr}}(X/W))$. Hence,
\[
\frac{V^{n_0+g-1}(\omega_{i,j})}{p^{|n_0\lambda|-1}} - \frac{1}{p^{|n_0\lambda|-1}} C_{p^{n_0+g-1-j}}^{\sigma^{-(n_0+g-1)}}(i, n_0 + g - 2)\omega_{0,p}
\]
lies in $VFH^1_{\text{dr}}(X/W) = pH^1_{\text{dr}}(X/W)$. Now the theorem follows from $V(\omega_{0,p}) \equiv \omega_{0,1} \text{ mod } p$. \hfill \Box

We summarize everything we need in the key lemma below.

**Key-Lemma 3.5.** Let $\lambda$ be a rational number with $0 \leq \lambda \leq \frac{1}{2}$.

i) If for all $i, j$ within the range of Lemma 2.4 and for all $m \geq 1, n \geq 1$ we have
\[
\text{ord}_p(C_{mp^{n+s-1-j}}(i, n + g - 2)) \geq [n\lambda]
\]
then
\[
NP_1(X/\mathbb{F}_q) \geq \lambda.
\]

ii) Let $i, j$ be within the range.

a) Let $n_0 \geq 1$. Suppose that for all $m \geq 1$ and $1 \leq n < n_0$ we have
\[
\text{ord}_p(C_{mp^{n+s-1-j}}(i, n + g - 2)) \geq [n\lambda];
\]

b) suppose that for all $m \geq 2$ we have
\[
\text{ord}_p(C_{mp^{n_0+s-1-j}}(i, n_0 + g - 2)) \geq [n_0\lambda];
\]

c) suppose
\[
\text{ord}_p(C_{p^{n_0+s-1-j}}(i, n_0 + g - 2)) < [n_0\lambda];
\]

Then
\[
NP_1(X/\mathbb{F}_q) < \lambda.
\]

**Proof.** i) The hypotheses in Theorem 3.4 are satisfied for all positive integers $n_0$ and for all possible $i$ and $j$. Thus our statement follows from Theorem 2.3.

ii) If $NP_1(X/\mathbb{F}_q) \geq \lambda$ then $p^{|n_0\lambda|} | V^{n_0+g-1}(\omega_{i,j})$ for all $i, j$ in the range of Lemma 3.1 by Theorem 2.2. This implies that for the particular $i, j$ satisfying the hypothesis of Theorem 3.4 we have
\[
\text{ord}_p(C_{p^{n_0+s-1-j}}(i, n_0 + g - 2)) \geq [n_0\lambda].
\]
This proves the Key-Lemma. \hfill \Box
4. \( p \)-adic behavior of coefficients of power series

In this section we study the \( p \)-adic behavior of coefficients of two power series.

To make this paper as self-contained as possible, we recall Lagrange inversion formula from mathematical analysis [4 IX, § 189]. Let \( z \) and \( y \) be two functions such that \( y = z\mu(y) \) for some function \( \mu(y) \) which can be developed into a power series in \( y \). Then the power series expansion of any function \( h(y) \) in \( z \) is

\[
(8) \quad h(y) = \sum_{k_1=1}^{\infty} \frac{1}{k_1!} \left( \left( \mu(y)^{k_1}h'(y) \right)^{(k_1-1)} \bigg|_{y=0} \right) z^{k_1},
\]

where the upper corner \( (k_1-1) \) denotes the \((k_1 - 1)\)-th derivative and \( h'(y) \) denotes the first derivative of \( h(y) \) in terms of \( y \).

**Lemma 4.1.** Let \( a > 0 \) and let \( y \in W[[z]] \) be a power series that satisfies \( y^a - y = z \) and \( y(0) = 0 \). Then

\[
y^a = \sum_{k_1=1}^{\infty} D_{k_1}(a)z^{k_1},
\]

where \( D_{k_1}(a) = 0 \) if \( k_1 \not\equiv a \mod p−1 \); otherwise,

\[
D_{k_1}(a) = (-1)^{a+k_1-a} a \left( \frac{k_1 + \frac{k_1-a}{p-1} - 1}{k_1! \left( \frac{k_1-a}{p-1} ! \right)} \right).
\]

**Proof.** Note that \( y = z(y^{p-1} - 1)^{-1} \). Apply (8) to this equation, we get

\[
(9) \quad y^a = \sum_{k_1=1}^{\infty} \frac{a}{k_1!} \left( \left( (y^{p-1} - 1)^{-k_1}y^{a-1} \right)^{(k_1-1)} \bigg|_{y=0} \right) z^{k_1}.
\]

We have

\[
\left. (y^{p-1} - 1)^{-k_1}y^{a-1} \right|_{y=0}^{(k_1-1)} = \left( \sum_{\ell=0}^{\infty} (-1)^{(p-1)\ell+k_1} \left( \frac{-k_1}{\ell} \right) y^{(p-1)\ell+a-1} \right)^{(k_1-1)} \bigg|_{y=0}.
\]

Clearly, this is 0 if \( k_1 \not\equiv a \mod p−1 \); otherwise, it is equal to

\[
(-1)^a(k_1 - 1)! \left( \frac{-k_1}{p-1} \right).
\]

Plugging this into (9) yields the desired value for \( D_{k_1}(a) \).

For any natural numbers \( k_1 \) and \( a \), we will keep the notation \( D_{k_1}(a) \) as defined in Lemma 4.1. We also define \( D_{k_1}(a) = 1 \) if \( a = k_1 = 0 \) and \( D_{k_1}(a) = 0 \) if only one of \( k_1 \) and \( a \) is 0. For any integer \( k \geq 0 \) denote by \( s_p(k) \) the sum of all digits in the “base \( p \)” expansion of \( k \).

**Lemma 4.2.** If \( a > 0 \) and \( k_1 \equiv a \mod p−1 \), write \( a = i + \ell(p−1) \) with integers \( \ell \) and \( 1 \leq i \leq p−1 \), then

\[
\begin{align*}
\ord_p(D_{k_1}(a)) &= \frac{s_p(k_1) - i}{p-1} & \text{if } \ell = 0; \\
\ord_p(D_{k_1}(a)) &\geq \frac{s_p(k_1) - i}{p-1} - (\ell - 1) & \text{if } \ell \geq 1.
\end{align*}
\]
Proof. Let $k_1 \equiv a \mod p - 1$. Using the well-known identity $(p - 1)\text{ord}_p(k!) = k - s_p(k)$ for all natural number $k$, one gets that

$$\text{ord}_p(D_{k_1}(a)) = \text{ord}_p(a) + \frac{1}{p - 1} \left( s_p(k_1) + s_p \left( \frac{k_1 - a}{p - 1} \right) + s_p \left( a - 1 + \frac{k_1 - a}{p - 1} \right) \right).$$

If $\ell = 0$ then

$$s_p \left( a - 1 + \frac{k_1 - a}{p - 1} \right) = i - 1 + s_p \left( \frac{k_1 - a}{p - 1} \right).$$

If $\ell = 1$ then

$$s_p \left( a - 1 + \frac{k_1 - a}{p - 1} \right) \leq (p - 1)\text{ord}_p(a) + i - 1 + s_p \left( \frac{k_1 - a}{p - 1} \right).$$

If $\ell > 1$ then

$$s_p \left( a - 1 + \frac{k_1 - a}{p - 1} \right) \leq i - 1 + s_p(\ell(p - 1)) + s_p \left( \frac{k_1 - a}{p - 1} \right) \leq i - 1 + (\ell - 1)(p - 1) + s_p \left( \frac{k_1 - a}{p - 1} \right).$$

Substitute these back in (10), we obtain the desired (in)equalities.

Fix two integers $N > 0$ and $0 \leq i \leq p - 2$. Let $y \in W[[z]]$ still be the power series satisfying $y^p - y = z$. Define coefficients $E_{k_1}(i, N)$ by

$$y^i (py^{p-1} - 1)^{p^N-1} = \sum_{k_1=0}^{\infty} E_{k_1}(i, N) z^{k_1}.$$

For any integer $r \geq 0$ let $\mathbf{K}_r$ denote the set of transposes $\mathbf{k} = \{k_1, \ldots, k_d\}$ of $d$-tuple integers with $k_1 \geq k_2 \geq \ldots \geq k_d \geq 0$ and $\sum_{\ell=1}^{d} k_\ell = r$. We define

$$s_p(\mathbf{k}) := s_p(k_1 - k_2) + \ldots + s_p(k_d - 1 - k_d) + s_p(k_d).$$

Note that from the definition of the coefficients $C_r(i, N)$ in (3) we find

$$\sum_{r=0}^{\infty} C_r(i, N)x^r = \sum_{k_1=0}^{\infty} E_{k_1}(i, N)f(x)^{k_1}.$$

Expanding the powers of $f(x)$ yields

$$(11) \quad C_r(i, N) = \sum_{\mathbf{k} \in \mathbf{K}_r} E_{k_1}(i, N) \prod_{\ell=1}^{d-1} \binom{k_\ell}{k_{\ell+1}} a_{\ell-1}^{k_{\ell-1}-k_{\ell+1}}.$$

Lemma 4.3. Let $\mathbf{k} = \{k_1, \ldots, k_d\} \in \mathbf{K}_r$. If $k_1 \not\equiv i \mod p - 1$ then $E_{k_1}(i, N) = 0$. If $k_1 \equiv i \mod p - 1$ then

$$\text{ord}_p(E_{k_1}(i, N)) = \frac{s_p(k_1) - i}{p - 1},$$

$$\text{ord}_p \left( E_{k_1}(i, N) \prod_{\ell=1}^{d-1} \binom{k_\ell}{k_{\ell+1}} \right) = \frac{s_p(\mathbf{k}) - i}{p - 1}.$$
Proof. Take the identity
\[
y^i(p^yp^{-1} - 1)p^{N-1} = \sum_{\ell=0}^{p-1} (-1)^{p^{N-1-\ell}} \left( \frac{p^N - 1}{\ell} \right) p^\ell y^{i+\ell(p-1)}.
\]
Substitute the power series expansion of $y^{i+\ell(p-1)}$ in (12); we get
\[
E_{k_1}(i, N) = \sum_{\ell=0}^{p-1} (-1)^{p^{N-1-\ell}} \left( \frac{p^N - 1}{\ell} \right) D_{k_1}(i + \ell(p-1))p^\ell.
\]
If $k_1 \not\equiv i \mod p-1$ then $D_{k_1}(i + \ell(p-1)) = 0$ by Lemma 1.1; hence $E_{k_1}(i, N) = 0$. This proves the first part of the lemma.
If $k_1 = i = 0$ then $E_{k_1}(i, N) = (-1)^{p^{-1}}$ and $s_p(k_1) - i = 0$. If $i = 0$, $k_1 > 0$ and $i \equiv k_1 \mod p-1$ then, by Lemma 4.2 the term with minimal valuation in (13) occurs at $\ell = 1$. We have
\[
\text{ord}_p(E_{k_1}(i, N)) = 1 + \text{ord}_p(D_{k_1}(p-1)) = 1 + \frac{s_p(k_1) - (p-1)}{p-1}.
\]
If $i > 0$ and $k_1 \equiv i \mod p-1$ then the term with minimal valuation in (13) occurs at $\ell = 0$. We have
\[
\text{ord}_p(E_{k_1}(i, N)) = \text{ord}_p(D_{k_1}(i)) = \frac{s_p(k_1) - i}{p-1}.
\]
This implies the second assertion.
By $\text{ord}_p(k!) = \frac{k-s_p(k)}{p-1}$ we have that $\text{ord}_p(\prod_{\ell=1}^{d-1} \left( \frac{k_{\ell+1}}{k_{\ell+1}} \right)) = \frac{s_p(k) - s_p(k_1)}{p-1}$. Thus
\[
\text{ord}_p(\prod_{\ell=1}^{d-1} \left( \frac{k_{\ell+1}}{k_{\ell+1}} \right)) = \frac{s_p(k) - s_p(k_1)}{p-1}.
\]
So the third assertion follows from this equality and the second assertion. \qed

5. $p$-Adic Behavior of $C_r(i, N)$

To apply Theorem 1.4 one needs to have in hands an efficient formula of the $p$-adic valuations of the coefficients in (3). This formula is in Lemma 5.3, which is prepared for Section 5.3.
Let $k = (k_1, \ldots, k_d) \in K_r$. For $1 \leq \ell \leq d$, let $k_\ell = \sum_{v \geq 0} k_{\ell,v}p^v$ be the “base $p^v$” expansion of $k_{\ell}$, we introduce a dot representation
\[
\hat{k}_{\ell} := [\ldots, \hat{k}_{\ell,2}, \hat{k}_{\ell,1}, \hat{k}_{\ell,0}]
\]
in the following way: for $\ell = d$, let $\hat{k}_{d,v} = k_{d,v}$ for all $v \geq 0$; for $1 \leq \ell < d$, it is defined inductively by
\[
\hat{k}_{\ell-1,v} := \hat{k}_{\ell,v} + p^v\text{-coefficient in the “base $p^v$” expansion of $(k_{\ell-1} - k_\ell)$},
\]
for all $v \geq 0$. It can be verified that $k_\ell = \sum_{v \geq 0} \hat{k}_{\ell,v}p^v$ for $1 \leq \ell \leq d$. Since $k_{\ell} \geq k_{\ell+1}$ we have $\hat{k}_{\ell-1,v} \geq \hat{k}_{\ell,v}$ for all $v$. It is not hard to observe
\[
s_p(k) = \sum_{v \geq 0} \left( \sum_{\ell=1}^{d-1} (\hat{k}_{\ell,v} - \hat{k}_{\ell+1,v}) + \hat{k}_{d,v} \right) = \sum_{v \geq 0} \hat{k}_{1,v}.
\]
For any natural number $a$, define a subset of $K_r$ as follows
\[
K_r^a := \{ k \in K_r \mid \hat{k}_{\ell,v} = 0 \text{ for } v \geq a, 1 \leq \ell \leq d \}.
\]
More explicitly $K^a_r$ consists of all $k \in K_r$ with $\hat{k}_t = [\ldots, 0, \hat{k}_{t,a-1}, \ldots, \hat{k}_{t,1}, \hat{k}_{t,0}]$ for all $1 \leq t \leq d$. Then we have an obvious filtration $K^a_1 \subseteq \ldots \subseteq K^n_1 \subseteq K^a_r \subseteq \ldots \subseteq K_r$.

**Lemma 5.1.** Let $p > d$. Let $1 \leq j \leq p - 1$, let $a, m, n \geq 1$ and $r = mp^a - j$. If $k \in K^a_r$, then

$$s_p(k) \geq \left\lfloor \frac{(m-1)p}{d} \right\rfloor + (a - 1) \left\lfloor \frac{p-1}{d} \right\rfloor + \left\lfloor \frac{p-j}{d} \right\rfloor.$$  

If $p > 2d$, $m = 1$ and the equality holds, then

$$\hat{k}_1 = [\ldots, 0, \left\lfloor \frac{p-1}{d} \right\rfloor, \ldots, \left\lfloor \frac{p-1}{d} \right\rfloor, \left\lfloor \frac{p-j}{d} \right\rfloor].$$

**Proof.** We will prove this lemma by induction on $a$. Let $k \in K^a_r$. Suppose $a = 1$. Note that all real numbers $c_1$ and $c_2$ satisfy $\lfloor c_1 + c_2 \rfloor \geq \lfloor c_1 \rfloor + \lfloor c_2 \rfloor$. So we have

$$s_p(k) = \hat{k}_{1,0} \geq \left\lfloor \frac{r}{d} \right\rfloor \geq \left\lfloor \frac{(m-1)p}{d} \right\rfloor + \left\lfloor \frac{p-j}{d} \right\rfloor.$$

Suppose $m = 1$ and the equality holds. It reads $\hat{k}_{1,0} = \left\lfloor \frac{p-1}{d} \right\rfloor$.

Now suppose $a \geq 2$. Let $k \in K^a_r$. Let $k' := \sum_{v=0}^{a-2} k_{v,p^v}$ for all $1 \leq t \leq d$. One can find a natural number $m'$ such that $m'p^a - j = \sum_{t=1}^{d} k'_t$. Then $k' := \langle k'_1, \ldots, k'_d \rangle \in K^{a-1}_{m'p^{a-1} - j}$. We have $(m-1)p + p-1 = (m'-1) + \sum_{t=1}^{d} \hat{k}_{t,a-1} \leq (m' - 1) + dk_{1,a-1}$. So

$$\hat{k}_{1,a-1} \geq \left\lfloor \frac{(m-1)p - (m'-1) + p-1}{d} \right\rfloor$$

$$\geq \left\lfloor \frac{(m-1)p}{d} \right\rfloor - \left\lfloor \frac{m'-1}{d} \right\rfloor + \left\lfloor \frac{p-1}{d} \right\rfloor.$$  

On the other hand, by induction hypothesis on $k' \in K^{a-1}_{m'p^{a-1} - j}$, one has

$$\sum_{v=0}^{a-2} \hat{k}_{1,v} \geq \left\lfloor \frac{(m'-1)p}{d} \right\rfloor + (a-2) \left\lfloor \frac{p-1}{d} \right\rfloor + \left\lfloor \frac{p-j}{d} \right\rfloor.$$  

Combining (14) and (15), one gets

$$s_p(k) = \sum_{v=0}^{a-1} \hat{k}_{1,v} \geq \left\lfloor \frac{(m-1)p}{d} \right\rfloor + (a-1) \left\lfloor \frac{p-1}{d} \right\rfloor + \left\lfloor \frac{p-j}{d} \right\rfloor + A,$$

where $A := \left\lfloor \frac{(m'-1)p}{d} \right\rfloor - \left\lfloor \frac{m'-1}{d} \right\rfloor$. Using $p > d$ one easily observes that $A \geq 0$, and the first part of the lemma follows from (16).

Now suppose $p > 2d$, $m = 1$, the equality holds in (16) and $A = 0$. This can only happen if $m' = 1$. It follows by induction that $\hat{k}_{1,0} = \left\lfloor \frac{p-2}{d} \right\rfloor$ and $\hat{k}_{1,v} = \left\lfloor \frac{p-1}{d} \right\rfloor$ for $1 \leq v < a - 1$. From the equality in (16) it follows that $\hat{k}_{1,a-1} = \left\lfloor \frac{p-1}{d} \right\rfloor$.  

**Lemma 5.2.** Let $a$ be a natural number and $p > d$. For a polynomial $f(x) \in W[x]$ of degree $d$ we have

$$[f(x)]^\sum_{v=0}^{a-1} \left[ \frac{p-1}{d} \right] p^v \equiv \prod_{v=0}^{a-1} \left[ (f(x)) \left( \frac{p^v}{d} \right) \right] p^v \mod p.$$
Proof. Write
\[ f(x)\sum_{v=0}^{a-1} \lfloor \frac{p-1}{d} \rfloor \varphi^v = \prod_{v=0}^{a-1} f(x) \lfloor \frac{p-1}{d} \rfloor \varphi^v \equiv \prod_{v=0}^{a-1} f\varphi^v (x\varphi^v) \mod p. \]

Now we write
\[ x^{p^a-1} = \prod_{v=0}^{a-1} x^{p^v(p-1)}. \]

Consider contributions of each factor of the product of (17) in the coefficient of (18).

Each \( v \)-th factor of (17) contributes to the coefficients of \( x^{p^m} \) for some \( m \), where \( 1 \leq m \leq d \lfloor \frac{p-1}{d} \rfloor < 2p - 1 \). When \( v = 0 \) then it has to contribute to the coefficient of \( x^{p-1} \). Inductively for each \( v = 1, \ldots, a - 1 \) the \( v \)-th factor contributes precisely the coefficient to \( x^{p^v(p-1)} \). It is easy to see that
\[ [f\varphi^v (x\varphi^v) \lfloor \frac{p-1}{d} \rfloor \varphi^v] \equiv [f(x) \lfloor \frac{p-1}{d} \rfloor \varphi^v] \mod p. \]

Thus our assertion follows. \( \square \)

Lemma 5.3. Let \( p > d \). Let \( a, m, N \) be natural numbers. Let \( i, j \) be as in Lemma 5.1. Then
\[ \text{ord}_p (C_{mp^a-j} (i, N)) \geq \left\lfloor \frac{(a-1) \lfloor \frac{p-1}{d} \rfloor + \lfloor \frac{p-1}{d} \rfloor - i}{p-1} \right\rfloor. \]

Moreover, for \( p > 2d \) we have
\[ \text{ord}_p (C_{mp^a-1} (i, N)) = \frac{a \lfloor \frac{p-1}{d} \rfloor - i}{p-1} \]
if and only if
\[ \left\{ \begin{array}{l}
  m = 1;
  \frac{a \lfloor \frac{p-1}{d} \rfloor}{p-1} \equiv i \mod p-1;
  f(x) \lfloor \frac{p-1}{d} \rfloor \neq 0 \mod p.
\end{array} \right. \]

Proof. Let \( \mathbf{k} = \langle k_1, \ldots, k_d \rangle \in \mathbf{K}_{mp^a-j} \). Let \( \mathbf{k}' = \langle k'_1, \ldots, k'_d \rangle \) where \( k'_i = \sum_{v=0}^{a-1} k_{i,v} p^v \), then \( \mathbf{k}' \in \mathbf{K}_r [a] \). Let \( r' := \sum_{i=1}^d k'_i \), write \( r' = m' p^a - j \) for some \( m' \).

From Lemma 5.1 it follows that
\[ s_p (\mathbf{k}) = \sum_{v=0}^{a-1} k_{1,v} \geq \sum_{v=0}^{a-1} k_{1,v} = s_p (\mathbf{k}') \geq (a-1) \left\lfloor \frac{p-1}{d} \right\rfloor + \left\lfloor \frac{p-j}{d} \right\rfloor. \]

Then by (13) and Lemma 5.3 one easily verifies that (19) holds.

Assume (20) holds. Then there is a \( \mathbf{k} \) such that the equality in (21) holds for \( j = 1 \), which implies that \( m = 1 \), \( k_{1,v} = 0 \) for \( v \geq a \), \( k_{1,v} = \lfloor \frac{p-1}{d} \rfloor \) for \( 0 \leq v \leq a - 1 \) by Lemma 5.3. Thus \( k_1 = \sum_{v=0}^{a-1} \lfloor \frac{p-1}{d} \rfloor p^v \). So \( s_p (k_1) = a \lfloor \frac{p-1}{d} \rfloor \equiv i \mod p-1 \). Those \( \mathbf{k} \in \mathbf{K}_{mp^a-1} \) which contribute terms in the sum (11) with minimal valuation necessarily have \( k_1 \equiv i \mod p-1 \). By the identity
\[ C_{p^a-1} (i, N) = \sum_{k_1=0}^{\infty} E_{k_1}(i, N) \cdot [f(x) k_1]_{p^a-1}, \]

we have by Lemma 4.3

$$\text{ord}_p(C_{p^n-1}(i, N)) \geq \text{ord}_p(E_{k_1}(i, N)) + \text{ord}_p([f(x)^{k_1}]_{p^n})$$

$$= \frac{s_p(k_1) - \lambda}{p - 1} + \text{ord}_p([f(x)^{k_1}]_{p^n})$$

This is equal to $a_{\left\lceil \frac{n-1}{p-1} \right\rceil - i}$ if and only if $[f(x)^{k_1}]_{p^n} \neq 0 \bmod p$. By Lemma 5.2 this is equivalent to $[f(x)^{k_1}]_{p^n-1} \neq 0 \bmod p$.

Conversely, the conditions imply that the contribution of $k \in K_{p^n-1}$ with $k_1 = \sum_{n=0}^{a-1} [\frac{p-1}{d}] p^n \text{ to } \text{ord}_p(C_{p^n-1}(i, N))$ in (11) has valuation $a_{\left\lceil \frac{a-1}{p-1} \right\rceil - i}$. Contribution from other $k \in K_{p^n-1}$ has higher valuation by the above arguments. Thus $\text{ord}_p(C_{p^n-1}(i, N)) = a_{\left\lceil \frac{n-1}{p-1} \right\rceil - i}$. This finishes the proof of this lemma.

6. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. It suffices to prove the theorem for the case that $\tilde{f}(x)$ has constant coefficient $\tilde{a}_0 = 0$. On the one hand, $[\tilde{f}(x)^{\left\lceil \frac{n-1}{p-1} \right\rceil}]_{p-1}$ is independent of $\tilde{a}_0$; on the other hand, the curves $g^p - y = \tilde{f}(x)$ and $g^p - y = f(x) + \tilde{a}_0$ are isomorphic over $\mathbb{F}_p$ for any $\tilde{a}_0$, and hence have the same Newton polygon. With the assumption $\tilde{a}_0 = 0$ we can use the results of Section 3.

a) Set $\lambda_0 := \left\lfloor \frac{p-1}{d} \right\rfloor$. By the hypothesis on $d$, $p$ and $g$ it is elementary to check that for $i$ and $j$ in the range of Lemma 3.3 we have $(g - 2) \left[ \frac{p-1}{d} \right] + \left[ \frac{p-1}{d} \right] \geq i$ thus $\left( n + g - 2 \right) \lambda_0 + \left\lfloor \frac{n+1}{d} - i \right\rfloor \geq \left\lfloor n \lambda_0 \right\rfloor$ for all $n \geq 1$. By Lemma 5.2 we have

$$\text{ord}_p(C_{mp^n+s-1-j}(i, n + g - 2)) \geq \left( n + g - 2 \right) \lambda_0 + \left\lfloor \frac{n+1}{d} - i \right\rfloor \geq \left\lfloor n \lambda_0 \right\rfloor.$$ 

Thus $\text{NP}_1(X/\mathbb{F}_q) \geq \lambda_0$ by Lemma 3.3.

b) Choose a value of $i$ in the range of Lemma 3.3 for $j = 1$ such that the following congruence has a solution for $a$,

$$a \left[ \frac{p-1}{d} \right] \equiv i \bmod p-1.$$ 

For any integer $n > 1$ define

$$\lambda_n := \frac{(n + g - 2) \left[ \frac{p-1}{d} \right] - i}{(n-1)(p-1)}.$$ 

Note that $\lambda_n$ is monotonically decreasing as a function in $n$, and it converges to $\lambda_0$ as $n$ approaches $\infty$. Suppose $\text{NP}_1(X/\mathbb{F}_q) > \lambda_0$, then there exists a positive integer $n_0$ large enough such that $\text{NP}_1(X/\mathbb{F}_q) > \lambda_{n_0}$. Choose such an $n_0$, and such that $a = n_0 + g - 1$ is a solution to the congruence above and such that $\left( g^{-1} \left[ \frac{p-1}{d} \right] - i \right) \frac{p-1}{(n-1)(n_0-1)} \leq 1$. For all $1 \leq n < n_0$ we have

$$\lambda_{n_0} \leq \lambda_{n+1} = \frac{(n + g - 1) \left[ \frac{p-1}{d} \right] - i}{n(p-1)}.$$
Thus, for all $m \geq 1$ and $1 \leq n < n_0$ we have by Lemma 5.3 that
\[
\text{ord}_p(C_{mp^n+s-i-1}(i, n + g - 2)) \geq \left\lceil \frac{(n + g - 1)\left\lceil \frac{p-1}{d} \right\rceil - i}{p-1} \right\rceil \geq [n\lambda_{n_0}].
\]

On the other hand, since
\[
0 < n_0\lambda_{n_0} - \frac{(n_0 + g - 1)\left\lceil \frac{p-1}{d} \right\rceil - i}{p-1} = \frac{(g - 1)\left\lceil \frac{p-1}{d} \right\rceil - i}{(p-1)(n_0 - 1)} \leq 1,
\]
by our assumption we have
\[
[n_0\lambda_{n_0}] = \frac{(n_0 + g - 1)\left\lceil \frac{p-1}{d} \right\rceil - i}{p-1} + 1.
\]
Hence, for all $m \geq 2$ one has by Lemma 5.3 that
\[
\text{ord}_p(C_{mp^n+s-i-1}(i, n_0 + g - 2)) \geq \frac{(n_0 + g - 1)\left\lceil \frac{p-1}{d} \right\rceil - i}{p-1} + 1 = [n_0\lambda_{n_0}] - 1,
\]
where the equality holds if and only if $\left\lceil f(x)^\frac{p-1}{d} \right\rceil_{p-1} \neq 0 \mod p$. In this case, we have $\text{NP}_1(X/\mathbb{F}_q) < \lambda_{n_0}$ (by Lemma 5.3), which contradicts our assumption that $\text{NP}_1(X/\mathbb{F}_q) > \lambda_{n_0}$. Therefore, we have $\text{NP}_1(X/\mathbb{F}_q) = \lambda_0$. \(\square\)

7. Ax’s version of Warning theorem and its application to slope estimates over finite fields

This section is independent of the rest of the paper. Main goal is to give a lower bound for $\text{NP}_1(X/\mathbb{F}_q)$ without the assumption $p > d$. Proposition 7.2 is due to Daqing Wan. To begin, we present a simple lemma due to the fact that we were not able to locate a suitable reference.

**Lemma 7.1.** Let $X$ be any curve over $\mathbb{F}_q$ where $q = p^r$. Let $\lambda$ be a rational number with $0 \leq \lambda \leq 1/2$. The following two statements are equivalent

(a) $p^{\lfloor \nu n \lambda \rfloor} \mid (\#X(\mathbb{F}_{q^n}) - 1)$ for all $n \geq 1$;

(b) $\text{NP}_1(X/\mathbb{F}_q) \geq \lambda$.

**Proof.** The denominator of the $L$ function of $X$ of (genus $g$) is $P(T) = \prod_{i=1}^{2g} (1-\pi_i T)$ where $\pi_i$’s are eigenvalues of the Frobenius endomorphism of $X$ relative to $\mathbb{F}_q$. We consider the $\pi_i$ as elements of $\mathbb{Q}_p$. Extend the valuation $\text{ord}_p(.)$ to $\mathbb{Q}_p$.

Let $\text{NP}_1(X/\mathbb{F}_q) \geq \lambda$. Then $\text{ord}_p(\pi_i^n) \geq \nu n \lambda$ for all $n \geq 1$ (see [12, Lemma 4, Chapter IV]). But
\[
\#X(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{i=1}^{2g} \pi_i^n,
\]
so $\text{ord}_p(q^n + 1 - \#X(\mathbb{F}_{q^n})) \geq \nu n \lambda$. Thus $p^{\lfloor \nu n \lambda \rfloor} \mid (q^n + 1 - \#X(\mathbb{F}_{q^n}))$. 

Conversely, suppose \( p^{[\nu \lambda]} \mid (q^n + 1 - \#X(F_{q^n})) \) for every \( n \geq 1 \). One easily derives from (2) that
\[
\exp \left( \sum_{n=1}^{\infty} (q^n + 1 - \#X(F_{q^n})) \frac{T^n}{n} \right) = \frac{1}{\prod_{i=1}^{2g}(1 - \pi_i T)}.
\]
Taking natural logarithm and then derivative at both sides, we get
\[
\sum_{n=1}^{\infty} (q^n + 1 - \#X(F_{q^n})) T^{n-1} = \prod_{i=1}^{2g} \frac{\pi_i}{1 - \pi_i T}.
\]
Then the left hand side of this power series converges \( p \)-adically for all \( T \) with \( \ord_p(T) \geq -\nu \lambda \). Comparing to the right hand side series we have \( \ord_p(\pi_i) \geq \nu \lambda \) for all \( i \). Therefore, \( \NP(X/F_q) \geq \lambda \). \( \square \)

**Proposition 7.2.** Let \( X \) be an Artin-Schreier curve over \( F_p \) given by an equation \( y^p - y = f(x) \) where \( f(x) = x^d + \tilde{a}_d - 1 x^{d-1} + \ldots + \tilde{a}_1 x \) and \( p \not| d \). Then
\[
\NP(X/F_q) \geq \frac{1}{\max_{\tilde{a}_k \neq 0} s_p(k)} \geq \frac{1}{d}.
\]

**Proof.** Let \( \tilde{f}(x) \) be a polynomial over \( F_q \) with \( q = p^\nu \) for some \( \nu \in \mathbb{N} \). For any \( n \in \mathbb{N} \), write \( r = \nu n \). Let \( \{\alpha_k\}_{k=1,\ldots,r} \) be a basis for the degree \( r \) extension \( F_{p^r}/F_p \). For any \( x \in F_{p^r} \), write \( x = \sum_{i=1}^{l} x_i \alpha_i \) for some \( x_i \in F_p \). For any \( k \in \mathbb{N} \), take its \( p \)-adic expansion \( k = \sum_{s=1}^{l} k_s p^s \) with \( 0 \leq k_s \leq p-1 \) and some \( l \in \mathbb{N} \). Then
\[
x^k = \sum_{i=1}^{r} x_i \alpha_i \cdot \sum_{s=1}^{l} k_s p^s = (\sum_{i=1}^{r} x_i \alpha_i) \cdot \sum_{s=1}^{l} k_s p^s = (\sum_{i=1}^{r} x_i \alpha_i) \cdot \sum_{s=1}^{l} k_s p^s.
\]
From this, one observes that \( \tilde{a}_k x^k \) can be considered as a polynomial in \( x_1, \ldots, x_r \) over \( F_{p^r} \) of total degree \( s_p(k) = k_0 + k_1 + \ldots + k_l \). Write \( \Tr \) for \( \Tr_{F_{p^r}/F_p} \), then \( \Tr(\tilde{f}(x)) = \sum_{k=1}^{d} \Tr(\tilde{a}_k x^k) \) is a polynomial in \( x_1, \ldots, x_r \) over \( F_p \) of total degree \( D := \max_{\tilde{a}_k \neq 0} s_p(k) \). Then we observe that
\[
\#X(F_{q^n}) - 1 = p \cdot \#\{ x \in F_{p^n} \mid \Tr(\tilde{f}(x)) = 0 \}.
\]
On the other hand, Ax’s theorem (3) indicates
\[
p^{[\frac{\nu \lambda}{p}]-1} \mid \#\{ x \in F_{p^n} \mid \Tr(\tilde{f}(x)) = 0 \}.
\]
Thus
\[
p^{[\frac{\nu \lambda}{p}]} \mid (\#X(F_{q^n}) - 1).
\]
Applying Lemma (7.3), we have \( \NP(X/F_q) \geq \frac{1}{D} \). The second inequality is elementary. \( \square \)

**Remark 7.3.** Note that \( \frac{[\frac{\nu \lambda}{p}]}{p-1} \geq \frac{1}{d} \) and the equality holds if \( p \equiv 1 \mod d \). Thus for \( p > d \) Theorem (1.1) a) is stronger than Proposition (7.2).
Remark 7.4. The supersingularity of curves over $\mathbb{F}_p$ of the form
\begin{equation}
y^p - y = \sum_{\ell \geq 0} \tilde{a}_{p^{\ell+1}} x^{p^{\ell} + 1},
\end{equation}
as in \cite{24, 23} follows from Proposition 7.2. We conjecture that if $g = (p-1)p^h$ for some $h \geq 1$ then $X$ is supersingular if and only if $X$ has an equation as in (22).

8. A conjecture of Wan

We first introduce a conjecture of Daqing Wan, then link it to Artin-Schreier curves. We prove Theorem 8.3, which clearly indicates Theorem 1.2.

For every integer $\ell \geq 1$ let
\[ S_\ell(f) := \sum_{x \in \mathbb{F}_{p^{\ell}}} \zeta_p^{\text{Tr}_{p^{\ell}/p}(f(x))}. \]

The $L$ function of $f(x) \mod p$ is defined by
\[ L(f \mod p; T) = \exp \left( \sum_{\ell=1}^{\infty} S_\ell(f) \frac{T^\ell}{\ell} \right). \]

It is a theorem of Dwork-Bombieri-Grothendieck that $L(f \mod p; T) = 1 + b'_1 T + \ldots + b'_{d-1} T^{d-1} \in \mathbb{Z}[\zeta_p][T]$ for some $p$-th root of unity $\zeta_p$ in $\mathbb{C}$. Define the Newton polygon of $f \mod p$ as the lower convex hull of the points $(k, \text{ord}_p(b'_k))$ in $\mathbb{R}^2$ for $0 \leq k \leq d - 1$. We denote it by $\text{NP}(f \mod p)$. It is the $p$-adic Newton polygon of the polynomial $L(f \mod p; T)$. Define the Hodge polygon $\text{HP}(f)$ as the convex hull in $\mathbb{R}^2$ of the points $(k, \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{d})$ for $0 \leq k \leq d - 1$. It is proved by Bombieri \cite{11} that the Newton polygon is always lying above or equal to the Hodge polygon. See also \cite{20} and \cite{2} for generalizations.

Remark 8.1. Some literature call $f \mod p$ ordinary if these two polygons coincide (see \cite{10}).

Conjecture 8.2 (Wan). There is a Zariski dense subset $U$ in $k^d$ such that for all $f(x) \in U$ we have the following limit exists and
\[ \lim_{p \to \infty} \text{NP}(f \mod p) = \text{HP}(f). \]

This conjecture was proposed by Wan in the Berkeley number theory seminar in the fall of 2000, a general form of which will appear in Section 2.5 \cite{24}. The cases $\deg(f) = 3$ and $4$ are proved in \cite{20} and \cite{3}, respectively. It is also known for all prime $p \equiv 1 \mod d$, in which case the Newton polygon is always equal to the Hodge polygon (see \cite{1}).

It is not hard to verify
\[ L(X/\mathbb{F}_p; T) = \frac{1}{N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(L(f \mod p; T))}, \]
where $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\cdot)$ represents the norm map. If we “normalize” the Newton polygon $\text{NP}(X/\mathbb{F}_p)$ by shrinking by a factor of $p^{-1}$ horizontally and vertically, then we obtain the Newton polygon $\text{NP}(f \mod p)$. Obviously these two Newton polygons have the same “shape” thus they have the same slope. Therefore, Theorem 1.2 confirms a first slope version of Conjecture 8.2.

\[ \text{NP}(X/\mathbb{F}_p) \]
Theorem 8.3. Let $d \geq 2$. Let $U$ be the set of all monic polynomials $f(x) \in \mathbb{A}^d$ such that $f(x)$ has
\[ f(x)[\frac{1}{p^{d-1}}]_{p-1} \equiv 0 \mod p \]
for finitely many primes $p$. It is Zariski dense in $\mathbb{A}^d$. For every $f(x) \in U$ we have
\[ \lim_{p \to \infty} NP_1(X/\mathbb{F}_p) = \frac{1}{d}. \]
Proof. We fix a natural number $d \geq 2$. Define $F(x) = x^d + A_{d-1}x^{d-1} + \ldots + A_0$ as an element of the polynomial ring $\mathbb{Q}[A_0, \ldots, A_{d-1}, x]$ in $d + 1$ variables.

Let $k$ be any integer with $0 \leq k \leq d - 1$ and $\gcd(k - 1, d) = 1$. For every prime $p > d$ and $p \equiv 1 - k \mod d$, we write $p = Nd - k + 1$ for some integer $N$. So $N = \lceil \frac{k-1}{d} \rceil$. The following lemmas are suggested by Bjorn Poonen.

Lemma 8.4. We have $[F(x)^N]_{p-1} \in \mathbb{Q}[A_0, A_1, \ldots, A_{d-1}]$, and it can be written as a polynomial in $N, A_0, \ldots, A_{d-1}$ with rational coefficients. Let $f_k(A_0, \ldots, A_{d-1})$ denote the evaluation of this polynomial at $N = \frac{k-1}{d}$. Then $f_k(A_0, \ldots, A_{d-1})$ is not the zero polynomial in $\mathbb{Q}[A_0, \ldots, A_{d-1}]$. For any $f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0$ in $\mathbb{A}^d$ we have
\[ f(x)[\frac{1}{p^{d-1}}]_{p-1} = f_k(a_0, \ldots, a_{d-1}) \mod p. \]
Proof. Define an auxiliary polynomial $h(T) = A_{d-1} + A_{d-2}T + \ldots + A_0T^{d-1}$. Note that $[F(x)^N]_{p-1}$ is equal to the $T^k$-coefficient of
\[ (1 + Th(T))^N = \sum_{\ell=0}^{k} \binom{N}{\ell} (Th(T))^\ell. \]
For fixed $\ell$ the binomial coefficient $\binom{N}{\ell}$ is a polynomial in $N$ with rational coefficients. The first assertion follows. Consider $[F(x)^N]_{p-1}$ as a polynomial in $A_0, \ldots, A_{d-1}$, its $A_k^d$-coefficient is equal to $\binom{N}{k}$ by inspecting (23). Note that $\binom{k-1}{d} \neq 0$, so $f_k$ is not the zero polynomial in $\mathbb{Q}[A_0, \ldots, A_{d-1}]$. Since $p = Nd - k + 1$, we have $\frac{k-1}{d} \equiv k-1 \mod p$; hence
\[ f(x)[\frac{1}{p^{d-1}}]_{p-1} = f_k(a_0, \ldots, a_{d-1}) \mod p. \]
This finishes the proof of Lemma 8.4. □

Lemma 8.5. Let $f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0$ be in $\mathbb{A}^d$. The following statements are equivalent
1) $[f(x)[\frac{1}{p^{d-1}}]_{p-1} \equiv 0 \mod p$ for infinitely many primes $p$;
2) there is a $k$ with $0 \leq k \leq d-1$ and $\gcd(k-1, d) = 1$, such that $[f(x)[\frac{1}{p^{d-1}}]_{p-1} \equiv 0 \mod p$ for infinitely many primes $p \equiv 1 - k \mod d$;
3) there is a $k$ with $0 \leq k \leq d-1$ and $\gcd(k-1, d) = 1$, such that $f_k(a_0, \ldots, a_{d-1}) \equiv 0 \mod p$ for infinitely many prime $p \equiv 1 - k \mod d$;
4) there is a $k$ with $0 \leq k \leq d-1$ and $\gcd(k-1, d) = 1$, such that $f_k(a_0, \ldots, a_{d-1}) = 0$.
Proof. Parts 1) and 2) are clearly equivalent. Parts 2) and 3) are equivalent by Lemma 8.4. Parts 3) and 4) are equivalent because $f_k(a_0, \ldots, a_{d-1}) \in \mathbb{Q}$ has to vanish if it vanishes modulo $p$ for infinitely many primes $p$; conversely, since $\gcd(k-1, d) = 1$, there are infinitely many prime $p \equiv 1 - k \mod d$ by Dirichlet. This concludes Lemma 8.5. □
The complement $U^c$ of $U$ in $\mathbb{A}^d$ is the set of all $f(x)$ in $\mathbb{A}^d$ such that

$$[f(x)]_{p^{-1}} = 0 \mod p$$

for infinitely many prime $p$. Write $G(A_0, \ldots, A_{d-1}) := \prod_k f_k(A_0, \ldots, A_{d-1})$. By Lemma 8.5, $U^c$ is equal to the set of all $f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0$ in $\mathbb{A}^d$ such that $(a_0, \ldots, a_{d-1})$ is a zero of a polynomial $G(A_0, \ldots, A_{d-1})$ in $\mathbb{Q}[A_0, \ldots, A_{d-1}]$. But $G(A_0, \ldots, A_{d-1})$ is not the zero polynomial by Lemma 8.4, so $U^c$ is Zariski closed. Therefore, $U$ is Zariski open and hence dense in $\mathbb{A}^d$. This concludes the first part of the theorem.

Now let $f(x) \in U$. Then there exists a natural number $M$ such that for all $p \geq M$ we have

$$[f(x)]_{p^{-1}} \neq 0 \mod p,$$

and hence $\text{NP}_1(X/\mathbb{F}_p) = \frac{[f(x)]_{p^{-1}}}{p^{-1}}$ by Theorem 2.4. Therefore, for every $f(x) \in U$ we have

$$\lim_{p \to \infty} \text{NP}_1(X/\mathbb{F}_p) = \frac{1}{d}.$$ 

This finishes the proof of Theorem 8.3. 

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