The concept of orthogonality in Cartan’s geometry
based on the concept of area

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Abstract

In 1931 Elie Cartan constructed a geometry which was rarely con-
sidered. Cartan proposed a way to define an infinitesimal metric \(ds\)
starting from a variational problem on hypersurfaces in an \(n\)-dimensional
manifold \(M\). This distance depends not only of the point \(m \in M\) but
on the orientation of a hyperplane in the tangent space \(T_m M\). His first
step is a natural definition of the orthogonal direction to such tangent
hyperplane. In this paper we extend it, starting from considerations
from the calculus of variation.

Introduction

Riemann considered the possibility to give to \(ds\), the distance between
two infinitesimally close points, a much more general expression than \(\sqrt{g_{ij}(dx^i, dx^j)}\)
namely to choose any function of \(x\) and \(dx\) which is homogeneous of degree
1 in \(dx\). This more general geometry was later developed by P. Finsler, E.
Cartan, [4], more recently, by Chern, H.Rund and Bryant [2].

In [3] Cartan proposed another generalisation of Riemannian geometry
where the distance between two infinitesimally closed points in \(M\) depends
of the point \(m \in M\) and of the choice of a hyperplane in the tangent space
to the manifold. In the modern language, this amounts to define a metric on
the vector bundle over the Grassmannian bundle of oriented hyperplanes,
\(Gr_{n-1}(\mathcal{M})\) whose fiber at \(m \in E\) is the set of oriented hyperplanes in \(T_m \mathcal{M}\)
(where \(\mathcal{M} \in M\) and \(E\) called “element” by Cartan, denotes an oriented hyperplane in \(T_m \mathcal{M}\)). Moreover Cartan found a way to canonically derive such
a metric from a variational problem on hypersurfaces in \(\mathcal{M}\). He simultane-
ously defined a connection on this bundle. The first step consists in choosing
a natural definition for the orthogonal complement of an element \(E\) and the
metric in the normal direction: The idea is to require that, for any extremal
hypersurface \(\mathcal{H}\) of \(\mathcal{M}\) and any compact subset with a smooth boundary

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Σ ∈ ℋ if we perform a deformation of ∂Σ in the normal direction to Σ and with an arbitrary indensity and consider the family of extremal hypersurfaces spanned by the images of ∂Σ by this deformations, then the area of hypersurfaces is stationary. This uses a formula of De Donder (which is basically an extension to variational problems with several variables of a basic formula in the theory of integral invariants). Let us now present this idea for submanifolds of arbitrary codimension n − p. Such variational problem can be described as follows. Let β be a p-form which, in local coordinates x¹,...,xⁿ, reads β = dx¹ ∧ ... ∧ dxᵖ. Any p-dimensional oriented submanifold 𝑁 such that β|𝑁 > 0 can be locally represented as the graph of a function f = (f¹,...,fⁿ⁻ᵖ) of the variables (x¹,...,xᵖ). We consider functional L of the form

\[ L(f) := \int_N d\sigma \] when

\[ d\sigma = L(x¹,...,xᵖ, f¹,...,fⁿ⁻ᵖ, \nabla f) \beta. \]

Let 𝑁 the critical point of L. To define the orthogonal subspaces to all tangent subspaces to 𝑁 the idea is to consider a 1-parameter family (𝑁ₓ)ₓ of submanifolds which form a foliation of a submanifold 𝑈 of dimension p + 1 in 𝑀 and such that 𝑁₀ = 𝑁. Consider a vector field X on 𝑈 which induces the variation from 𝑁ₓ to 𝑁ₓ+dx and denote

\[ A(x) = L(fₓ). \]

According to Cartan [3] the condition for X to be orthogonal to 𝑁 = 𝑁₀ is that the derivative of A(x) with respect to x at x = 0 is zero. This will allow us to find the orthogonal subspace. We will show that this definition of \( E⊥ \) actually dose not depend on the choice of 𝑁 but uniquelly on \( E ∈ Gr^p_M \).

1 Cartan geometry based on the concept of area

Let 𝑀 be a manifold of n-dimensional then we define the Grassmannian bundle or Grassmannian by

\[ Gr_pM = \{(M,E)|M ∈ 𝑀; E an oriented p-dimensional vector subspace in T_MM\}. \]

And, if β is a p-form which in local coordinates (x¹,...,xⁿ), reads β = dx¹ ∧ ... ∧ dxᵖ where 1 ≤ p ≤ n − 1, then

\[ Gr^β_pM = \{(M,E) ∈ T_MM|β = dx¹ ∧ ... ∧ dxᵖ|E > 0\}. \]

Let \( (q^l)_{1≤l≤p(n−p)} \) be coordinate fonctions on \( Gr^β_pM \) such that \( (x^l, q^l) \) are local coordinates on \( Gr^β_pM \). We denote the projection π by:

\[ π : Gr^β_pM → 𝑀 \]
We consider $\pi^*TM$ the bundle over the Grassmannian whose fiber at $(M, E)$ is $T_mM$, we denote a metric $g$ on $\pi^*TM$ by

$$g_{(M,E)} = g_{ij}(x^k, q^j)dx^i dx^j,$$

We see that the coefficients $g_{ij}$ not only depend on coordinates of $M$, but they also depend on the orientation of the element at $M$.

**Remark 1.1.** If $p = n - 1$ then

$$Gr_{n-1}(M) \sim (T^*M \setminus \{0\})/\mathbb{R}^*.$$

**Definition 1.2.** A *geometry based on the concept of area $(M, F)$* is a differential manifold $M$ equipped with a function $F$ defined over $T^*M$ with values in $\mathbb{R}^+$

$$F : T^*M \rightarrow \mathbb{R}^+,$$

which satisfies the following conditions:

1. $F$ is $C^\infty$ over $T^*M \setminus \{0\} := \bigcup_{m \in M} T^*_mM \setminus \{0\}$.
2. $F$ is homogeneous of degree one in $q^k$

$$F(x^k, \lambda q^k) = \lambda F(x^k, q^k).$$

3. The Hessian matrix defined by

$$(g_{ij}) := \frac{1}{2}(F^2)_{q^i q^j}$$

is positive definite at any point of $Gr_p(M)$.

In other words, $F |_{T^*_mM}$ is a Minkowski norm for all $m \in M$.

2 The concept of orthogonality in the space of Cartan

In the following, since we work locally we shall identity $M$ with $\mathbb{R}^n$ to the coordinate system $(x^i)_i$. 

3
2.1 Lagrangian formulation

Let \( L : Gr^\beta_p(\mathbb{R}^p \times \mathbb{R}^{n-p}) := \{(x^1, ..., x^n, (q^i_1)_{1 \leq i \leq n-p}) \} \) be the Lagrangian function. For any function \( f : \Omega \subset \mathbb{R}^p \rightarrow \mathbb{R}^{n-p} \) of class \( C^\infty \), we denote by \( \Gamma_f \) its graph.

A point \( x \in \Gamma_f \) is defined by \((x^{p+1}, ..., x^n) = (f^1(x^1, ..., x^p), ..., f^{n-p}(x^1, ..., x^p))\) and values of the coordinates \((q^i_1)\) at the tangent space to \( \Gamma_f \) are given by \((\nabla f)(x)\). Let \( \beta = dx^1 \wedge ... \wedge dx^p \) be a \( p \)-form, the action integral \([10] \) is given by

\[
\mathcal{L}(f) = \int_\Omega L(x^1, ..., x^p, f^1, ..., f^{n-p}, \nabla f)\beta = \int_\Omega L(x, f, \nabla f)\beta.
\]

The bundle over the Grassmannian of \( \Gamma_f \) given by

\[
Gr^\beta_p(\Gamma_f) := \{(x, E) ; x \in \Gamma_f, \ E = T_x \Gamma_f \}.
\]

**Definition 2.1.** Let \( \Gamma \) be an oriented \( p \)-dimensional submanifold of \( M \) with boundary \( \Gamma_0 \) which is a critical point of \( L \). A distribution of vector lines \( \mathcal{D} \) in \( T\mathcal{M} \) along \( \Gamma_0 \) is called normal if, for any vector field \( N \) defined along \( \Gamma_0 \) such that \( \forall M \in \Gamma_0, \ N(M) \in \mathcal{D}(M) \), and if \( \partial \Gamma_t := \{ \ell^N(M) | M \in \partial \Gamma, t \in (-\varepsilon, \varepsilon) \} \) and \( \mathcal{A}(t) := L(\Gamma_t) \) then \( \frac{d}{dt}(\mathcal{A}(t))|_{t=0} = 0 \).

**Theorem 2.2.** There exists a vector subbundle \( \pi^*T^\perp \mathcal{M} \) of \( \pi^*T\mathcal{M} \) of rank \( n-p \) whose fiber at \((x, E)\) is denoted by \((\pi^*T^\perp \mathcal{M})_{(x, E)}\) such that for any oriented \( p \)-dimensional critical point \( \Gamma \) of \( L \), a vector field \( N \) along \( \partial \Gamma \) is normal if and only if \( N_x \in (\pi^*T^\perp \mathcal{M})_{(x, T_x \Gamma)} \). In the following we write \((\pi^*T^\perp \mathcal{M})_{(x, T_x \Gamma)} = (T_x \Gamma)^\perp \). Moreover \((T_x \Gamma)^\perp \) is spanned by \((v^1, ..., v^{n-p})\), where

\[
v^1 = \begin{pmatrix}
\frac{\partial L}{\partial q^1_1} \\
\vdots \\
-\frac{\partial L}{\partial q^n_p} \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad v^2 = \begin{pmatrix}
\frac{\partial L}{\partial q^1_2} \\
\vdots \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad ..., \quad v^{n-p} = \begin{pmatrix}
\frac{\partial L}{\partial q^n_p} \\
\vdots \\
0 \\
\vdots \\
0 \\
-\frac{\partial L}{\partial q^n_p} \\
\vdots \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

**Proof.** Consider first the case \( p = 2 \) and \( n = 3 \)

The Grassmannian of dimension 5, the Lagrangian \((x, y, z, p, q) \Rightarrow L(x, y, z, p, q) := L(x, y, f(x, y), \nabla f(x, y))\) and the action integral is given by

\[
\mathcal{L}(f) = \int_{\mathbb{R}^2} L(x, y, f(x, y), \nabla f(x, y))\beta
\]
Suppose that this integral is extended to a portion of extremal surface \( \Sigma \) limited by a contour \( C \), deform slightly \( \Sigma \) to a surface \( \Sigma' \) limited by a contour \( C' \). This amounts to change in the preceding integral \( f \) into \( f + \varepsilon g \) where \( g \) has not necessarily a compact support. Then we consider a family \((\Sigma_t)_t\) of surfaces with boundary which forms a foliation of a domain \( U \subset \mathbb{R}^3 \) which coincides in \( t = 0 \) with \( \Sigma \) and in \( t = 1 \) with \( \Sigma' \), depending of a real parameter \( t \in [0,1] \). We suppose that for all \( t \), \((\Sigma_t)_t\) is a critical point of \( \mathcal{L} \) that we will represent by the graph \( \Sigma_t \) of a function \( f_t : \Omega_t \to \mathbb{R} \)

\[
\Sigma_t = \{(x, y, f_t(x, y)) | (x, y) \in \Omega_t \}
\]

Let \( X \) a vector field defined on \( U \) such that, if \( e^{sX} \) is the flow of \( u \), then:

\[
e^{sX}(\Sigma_t) = \Sigma_{t+s}.
\]

Note

\[
\begin{aligned}
f(t, x, y) &= f_t(x, y) \\
f(x, y) &= f(0, x, y) = f_0(x, y) \\
\Phi(t, x, y) &= e^{tX}(x, y, f(x, y))
\end{aligned}
\]

if \( t = 0 \) we have \( \Phi = f = f_0 \) and \( \forall t \in [0,1] \), the function \( (x, y) \to \Phi(t, x, y) \) is a parametrization of \( \Sigma_t \), we denote by \( \Phi = (\phi^1, \phi^2, \phi^3) \) and \( \phi^3(t, x, y) = f(t, \phi^1(t, x, y), \phi^2(t, x, y)) \) so, if we derive with respect to \( t \) then

\[
\frac{\partial \phi^3}{\partial t} = \frac{\partial f}{\partial t}(t, \phi^1, \phi^2) + \frac{\partial f}{\partial x}(t, \phi^1, \phi^2) \frac{\partial \phi^3}{\partial x} + \frac{\partial f}{\partial y}(t, \phi^1, \phi^2) \frac{\partial \phi^3}{\partial y}.
\]

which gives for \( t = 0 \)

\[
X^3(x, y, f) = \frac{\partial f}{\partial t}(0, x, y) + \frac{\partial f}{\partial x}(0, x, y)X^1(x, y, f) + \frac{\partial f}{\partial y}(0, x, y)X^2(x, y, f).
\]

Thus along \( \Sigma = \Sigma_0 \), we have:

\[
\frac{\partial f}{\partial t} = X^3 - X^1 \frac{\partial f}{\partial x} - X^2 \frac{\partial f}{\partial y}.
\]

Let the Lagrangian \((x, y, z, p, q) \to L(x, y, z, p, q)\) and we consider

\[
\mathcal{A}(t) = \int_{\Omega_t} L(x, y, f_t(x, y), \frac{\partial f_t}{\partial x}(x, y), \frac{\partial f_t}{\partial y}(x, y))dxdy
\]

Assuming that \( \Omega_t \) is regular (ie \( \partial \Omega_t \) is a curve \( C^1 \) of plan \( \mathbb{R}^2 \)) then we have

\[
\frac{d\mathcal{A}(t)}{dt} = \int_{\Omega_t} \frac{\partial}{\partial t} L(x, y, f_t, \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y})dxdy + \int_{\partial \Omega_t} L(x, y, f_t, \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y})\langle (X^1, X^2), \nu \rangle d\ell
\]

Where \( \nu \) is a exterior normal of \( \Omega_t \) in \( \mathbb{R}^2 \) and \( \langle (X^1, X^2), \nu \rangle \) is the change in the area of \( \Omega_t \) and \( d\ell \) is a measure of one dimension \( \partial \Omega \), thus
\[
\frac{dA(t)}{dt} \bigg|_{t=0} = \int_{\Omega_0} \frac{\partial}{\partial t} L(x, y, f_t, \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}) \, dx \, dy + \int_{\partial \Omega_0} L(x, y, f_t, \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}) \langle (X^1, X^2), \nu \rangle \, d\ell \]
\[= \int_{\Omega} \frac{\partial L}{\partial z} \frac{\partial f}{\partial t} + \frac{\partial L}{\partial p} \frac{\partial^2 f}{\partial x \partial t} + \frac{\partial L}{\partial q} \frac{\partial^2 f}{\partial y \partial t} + \int_{\partial \Omega} L((X^1, X^2), \nu) \, d\ell \]
\[= \int_{\Omega} \frac{\partial f}{\partial t} \left[ \frac{\partial L}{\partial z} - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial p} \right) - \frac{\partial}{\partial y} \left( \frac{\partial L}{\partial q} \right) \right] + \int_{\partial \Omega} \left\langle \left( \frac{\partial L}{\partial p} \frac{\partial f}{\partial t} + \frac{\partial L}{\partial q} \frac{\partial f}{\partial t} \right), \nu \right\rangle \, d\ell \]
\[= \int_{\Omega} \frac{\partial f}{\partial t} \left[ \frac{\partial L}{\partial z} - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial p} \right) - \frac{\partial}{\partial y} \left( \frac{\partial L}{\partial q} \right) \right] + \int_{\partial \Omega} \left\langle \left( \frac{\partial L}{\partial p} \frac{\partial f}{\partial t} + \frac{\partial L}{\partial q} \frac{\partial f}{\partial t} + L X^1, \frac{\partial L}{\partial q} \frac{\partial f}{\partial t} + L X^2 \right), \nu \right\rangle \, d\ell \]

Or we know that \( \Sigma = \Sigma_0 \) is a critical point of \( \int_{\Omega} L \), then the Euler Lagrange equations are verified, thus

\[
\frac{dA(t)}{dt} \bigg|_{t=0} = \int_{\partial \Omega} \left\langle \left( \frac{\partial L}{\partial p} \frac{\partial f}{\partial t} + L X^1, \frac{\partial L}{\partial q} \frac{\partial f}{\partial t} + L X^2 \right), \nu \right\rangle \, d\ell.
\]

We now assume that \( X|_{\partial \Gamma} \) has the form \( \psi N_0 \in D \) where \( \psi \in C^\infty(\partial \Gamma) \) with values in \( \mathbb{R} \) and where \( N_0 \) is a fixed non-vanishing tangent defined along \( \partial \Gamma \), to be determined we seek a condition to \( N_0 \) such that for any regular function \( \psi \), \( \frac{dA(t)}{dt} \bigg|_{t=0} = 0 \). We can choose a function \( f_t \) depends of \( \psi \) such that \( \frac{\partial f_t}{\partial t} \bigg|_{t=0} = \psi \frac{\partial f_t}{\partial t} \bigg|_{t=0} \). We replace this in the previous integral, thus

\[
\frac{dA(t)}{dt} \bigg|_{t=0} = \int_{\partial \Omega} \left\langle \left( \frac{\partial L}{\partial p} \frac{\partial f}{\partial t} + \psi L N_0^1, \frac{\partial L}{\partial q} \frac{\partial f}{\partial t} + \psi L N_0^2 \right), \nu \right\rangle \, d\ell.
\]

The condition for \( \frac{dA(t)}{dt} \bigg|_{t=0} = 0 \) for all \( \psi \) regular function on \( \partial \Gamma \) and \( \nu \) exterior normal of \( \Gamma \) is that \( \left\langle \left( \frac{\partial L}{\partial p} \frac{\partial f}{\partial t} + \psi L N_0^1, \frac{\partial L}{\partial q} \frac{\partial f}{\partial t} + \psi L N_0^2 \right), \nu \right\rangle = 0 \). If we denote by \( \lambda = \frac{\partial f}{\partial t} \bigg|_{t=0} \), then this enthusiasm that

\[
\begin{align*}
N_0^1 &= \lambda \frac{\partial L}{\partial p} \\
N_0^2 &= \lambda \frac{\partial L}{\partial q}
\end{align*}
\]
Form (1), we have
\[ N^3_0 = -\lambda L + \lambda \frac{\partial L}{\partial p} \frac{\partial f}{\partial x} + \lambda \frac{\partial L}{\partial q} \frac{\partial f}{\partial y}. \]

Thus, so that \( \frac{dA(t)}{dt} |_{t=0} = 0 \) it suffices that
\[ X = \left( \frac{\partial L}{\partial p}, \frac{\partial L}{\partial q}, \frac{\partial L}{\partial p} + q \frac{\partial L}{\partial q} - L \right). \]

**Lets now** \( n > 0 \) and \( p = n - 1 \)

The grassmanian bundle is of dimension \( 2n - 1 \). It is the same as previous, thus the orthogonal of \( \pi^*TM \) of rank \( n - 1 \) is spanned by
\[ X = \left( \frac{\partial L}{\partial p_1}, ..., \frac{\partial L}{\partial p_{n-1}}, \sum_{i=1}^{n-1} p_i \frac{\partial L}{\partial p^i} - L \right), \tag{2} \]

where \( p^i = \frac{\partial f}{\partial x_i} \) for \( i = 1, ..., n - 1 \), and \( L \) be the Lagrangian on \( Gr_{n-1}(\Sigma) \).

**Now the case** \( n > 3 \) and \( p < n \)

For \( 1 \leq p \leq n - 1 \) let \( \Omega \) be a regular open set of \( \mathbb{R}^p \) and \( f = (f^1, ..., f^{n-p}) : \Omega \to \mathbb{R}^{n-p} \), we denote its graph by
\[ \mathcal{N} := \{(x, f(x)) \mid x \in \Omega \}. \]

Let \( \beta = dx^1 \wedge ... \wedge dx^p \) be a \( p \)-form, and \( L \) be the Lagrangian on \( Gr_p^\beta(\mathcal{N}) \), thus the action integral is given by
\[ \mathcal{L}(f) := \int_{\mathcal{N}} L(x^1, ..., x^p, f^1, ..., f^{n-p}, \nabla f) \beta. \]

The family \( (\mathcal{N}_t)_t \) of submanifolds of board form a foliation in a submanifold \( U \subset \mathbb{R}^{p+1} \), we suppose that for all \( t \), \( \mathcal{N}_t \) is a critical point of \( \mathcal{L} \). Let \( X \) be a vector field defined on \( U \) such that, if \( e^{\alpha X} \) is the flow of \( u \), then \( e^{\alpha X}(\mathcal{N}_t) = \mathcal{N}_{t+s} \), denote
\[
\left\{ \begin{array}{l}
 f(t, x^1, ..., x^p) = f_t(x^1, ..., x^p) \iff f^i(t, x^1, ..., x^p) = (f^i)_t(x^1, ..., x^p) \forall t = 1, ..., n-p, \\
 f(x^1, ..., x^p) = f(0, x^1, ..., x^p) = f_0(x^1, ..., x^p) \iff \forall t = 1, ..., n-p \text{ we have} \\
 f^i(x^1, ..., x^p) = f^i(0, x^1, ..., x^p) = (f^i)_0(x^1, ..., x^p), \\
 \Phi(t, x^1, ..., x^p) = e^{tX}(x^1, ..., x^p, f^1, ..., f^{n-p}), 
\end{array} \right.
\]

The function \( \Phi \) is the parametrization of \( \mathcal{N}_t \), we denote:
\[
\left\{ \begin{array}{l}
 \Phi = (\varphi^1, ..., \varphi^p, \varphi^{p+1}, ..., \varphi^n), \\
 \varphi^{p+i}(t, x^1, ..., x^p) = f^i(t, \varphi^1, ..., \varphi^p) \text{ for } i = 1, ..., n-p,
\end{array} \right.
\]
Thus, \( \forall i = 1, \ldots, n - p \):

\[
\frac{\partial \varphi^{p+i}}{\partial t} = \frac{\partial f^i}{\partial t}(t, \varphi^1, \ldots, \varphi^p) + \sum_{j=1}^p \frac{\partial f^i}{\partial x^j}(t, \varphi^1, \ldots, \varphi^p) \frac{\partial \varphi^j}{\partial t}.
\]

For \( t = 0, \forall i = 1, \ldots, n - p \), thus

\[
X^{p+i}(x, f) = \frac{\partial f^i}{\partial t}(0, x^1, \ldots, x^p) + \sum_{j=1}^p \frac{\partial f^i}{\partial x^j}(0, x^1, \ldots, x^p) X^j(x, f),
\]

this given along \( N = N_0 \) and \( \forall i = 1, \ldots, n - p \)

\[
\frac{\partial f_i}{\partial t} = X^{p+i} - \sum_{j=1}^p \frac{\partial f_i}{\partial x^j} X^j. \tag{3}
\]

We have

\[
A(t) = L(f_t) = \int_{N_t} L(x^1, \ldots, x^p, (f^1)_t, \ldots, (f^{n-p})_t, \nabla f_t) \beta,
\]

thus

\[
\frac{dA(t)}{dt} = \int_{N_t} \frac{\partial}{\partial t} L(x^1, \ldots, x^p, (f^1)_t, \ldots, (f^{n-p})_t, \nabla f_t) \beta
\]

\[
+ \int_{\partial N_t} L(x^1, \ldots, x^p, (f^1)_t, \ldots, (f^{n-p})_t, \nabla f_t) \langle (X^1, \ldots, X^p), \nu \rangle d\ell,
\]

where \( \nu \) is the exterior normal to \( N_t \) in \( \mathbb{R}^n \), \( \langle (X^1, \ldots, X^p), \nu \rangle \) represents the change in volume of \( N_t \) and \( d\ell \) is a measure of one dimension \( \partial N_t \). Thus for \( t = 0 \) we have

\[
\frac{dA(t)}{dt} \bigg|_{t=0} = \int_{N_0} \frac{\partial}{\partial t} L(x^1, \ldots, x^p, (f^1)_t, \ldots, (f^{n-p})_t, \nabla f_t) \beta + \int_{\partial N_0} L((X^1, \ldots, X^p), \nu) d\ell.
\]

We calculate \( \frac{dA(t)}{dt} \bigg|_{t=0} \)

\[
\int_N \frac{\partial}{\partial t} L(x, f_t, \nabla f_t) \beta = \int_N \left( \sum_{i=1}^{n-p} \frac{\partial L}{\partial x^i} \frac{\partial f^i}{\partial t} + \sum_{1 \leq j \leq p}^{n-p} \sum_{1 \leq i \leq n-p} \frac{\partial F}{\partial p_j} \frac{\partial^2 f^i}{\partial x^i \partial x^j \partial t} \right) \beta
\]

\[
= \sum_{i=1}^{n-p} \int_N \left( \frac{\partial L}{\partial x^i} \frac{\partial f^i}{\partial t} + \sum_{j=1}^p \frac{\partial F}{\partial p_j} \frac{\partial^2 f^i}{\partial x^j \partial t} \right) \beta
\]

\[
= \sum_{i=1}^{n-p} \int_N \left( \frac{\partial L}{\partial x^i} \frac{\partial f^i}{\partial t} + \sum_{j=1}^p \frac{\partial}{\partial x^j} \left( \frac{\partial L}{\partial q_j} \frac{\partial f^i}{\partial t} \right) - \frac{\partial f^i}{\partial t} \sum_{j=1}^p \frac{\partial}{\partial x^j} \left( \frac{\partial L}{\partial q_j} \right) \right) \beta
\]

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\[
\begin{align*}
&= \sum_{i=1}^{n-p} \int_N \frac{\partial f^i}{\partial t} \left[ \frac{\partial L}{\partial x^i} - \sum_{j=1}^p \frac{\partial}{\partial x^j} \left( \frac{\partial L}{\partial q^j} \right) \right] + \sum_{j=1}^{n-p} \int_{\partial N} \left\langle \left( \frac{\partial L}{\partial q^j_1} \frac{\partial f^j}{\partial t}, \ldots, \frac{\partial L}{\partial q^j_{n-p}} \frac{\partial f^j}{\partial t} \right), \nu \right\rangle d\ell
\end{align*}
\]

We have \( N = N_0 \) is a critical point of \( L \), thus the Euler Lagrange equations are verified
\[
\frac{\partial L}{\partial x^i} - \sum_{j=1}^p \frac{\partial}{\partial x^j} \left( \frac{\partial L}{\partial q^j} \right) = 0
\]
this gives
\[
\frac{dA(t)}{dt} \bigg|_{t=0} = \sum_{i=1}^{n-p} \int_{\partial \Omega} \left\langle \left( \frac{\partial L}{\partial q^i_1} \frac{\partial f^i}{\partial t} + LX^1, \ldots, \frac{\partial L}{\partial q^i_{n-p}} \frac{\partial f^i}{\partial t} + LX^p \right), \nu \right\rangle d\ell
\]

Using the previous definition we can also consider a regular function \( \psi : \partial N \rightarrow \mathbb{R} \) such that \( f_{t,\psi} \) such that \( \frac{\partial f_{t,\psi}}{\partial t} = \psi \frac{\partial f}{\partial t} \). By the same as the previous, so that \( \frac{dA(t)}{dt} \bigg|_{t=0} = 0 \), it suffices that for all \( j = 1, \ldots, p \) we have
\[
\sum_{i=1}^{n-p} \frac{\partial L}{\partial q^j_i} \frac{\partial f^j_i}{\partial t} + LX^j = 0
\]

if we denote \( \lambda_i = \psi \frac{\partial f^i}{\partial t} \) and \( \nabla f := \left( \frac{\partial f^i}{\partial q^j} \right)_{1 \leq i \leq n-p, 1 \leq j \leq p} \) then
\[
X^j = \sum_{i=1}^{n-p} \lambda_i \frac{\partial L}{\partial q^j_i}
\]
for all \( j = 1, \ldots, p \), from (3), for \( i = 1, \ldots, n-p \) thus
\[
X^{p+i} = -\lambda_i L + \sum_{j=1}^p \lambda_i q^j_i \frac{\partial L}{\partial q^j_i}
\]
wich gives \( X = (X^1, \ldots, X^n) \) with
\[
\begin{align*}
X^1 &= \lambda_1 \frac{\partial L}{\partial q^1_1} + \lambda_2 \frac{\partial L}{\partial q^1_2} + \ldots + \lambda_{n-p} \frac{\partial L}{\partial q^1_{n-p}} \\
&\vdots \\
X^p &= \lambda_1 \frac{\partial L}{\partial q^p_1} + \lambda_2 \frac{\partial L}{\partial q^p_2} + \ldots + \lambda_{n-p} \frac{\partial L}{\partial q^p_{n-p}} \\
X^{p+1} &= \lambda_1 \left( -L + q^1_1 \frac{\partial L}{\partial q^1_1} + \ldots + q^1_p \frac{\partial L}{\partial q^1_p} \right) \\
&\vdots \\
X^n &= \lambda_{n-p} \left( -L + q^p_1 \frac{\partial L}{\partial q^p_1} + \ldots + q^p_{n-p} \frac{\partial L}{\partial q^p_{n-p}} \right)
\end{align*}
\]
Example 2.3. In the case $n = 3$ and $p = 2$, the functional area is $L(x, y, z, p, q) = \sqrt{1 + p^2 + q^2}$ then we find

$$X = \left( \frac{p}{\sqrt{1 + p^2 + q^2}}, \frac{q}{\sqrt{1 + p^2 + q^2}}, \frac{p^2}{\sqrt{1 + p^2 + q^2}}, \frac{q^2}{\sqrt{1 + p^2 + q^2}} \right) - \sqrt{1 + p^2 + q^2}$$

which have the same direction as the normal $n$ to the hypersurface $\Sigma$ spanned by $(1, 0, p)$ and $(1, 0, q)$ in the classical euclidean sens.

Example 2.4. If we take $n = 4$ and $p = 2$ then $f : \mathbb{R}^2 \to \mathbb{R}^2$, $\Sigma_t$ are a domains with boundary of dimension 2 of $\mathbb{R}^4$, we define the functional area by $L(x, y, f_1, f_2, q_1, q_2, q_1^2, q_2^2) := \sqrt{(q_1^2)^2 + (q_2^2)^2 + (q_1 q_2 q_1 q_2)^2 + (q_1^2 q_2^2 - q_2^2 q_1^2)^2}$, an easy calculation gives us that the normal subspace to $\mathcal{N}_x$ is $\mathbf{V} = (v^1, v^2)$ with

$$v^1 = \frac{\sqrt{(q_1^2)^2 + (q_2^2)^2 + (q_1 q_2 q_1 q_2)^2} - (q_1^2)^2 - (q_2^2)^2}{\sqrt{(q_1^2)^2 + (q_2^2)^2 + (q_1 q_2 q_1 q_2)^2}}, \quad v^2 = \frac{\sqrt{(q_1^2)^2 + (q_2^2)^2 + (q_1 q_2 q_1 q_2)^2} - (q_1^2)^2 - (q_2^2)^2}{\sqrt{(q_1^2)^2 + (q_2^2)^2 + (q_1 q_2 q_1 q_2)^2}}$$

then

$$\mathbf{V} = \frac{1}{L} \left( \begin{array}{c} q_1^2 - q_2^2 (q_1 q_2 q_1 q_2) \\ q_2^2 + q_1 (q_1 q_2 - q_2 q_1) \\ -(q_2^2)^2 - (q_1^2)^2 \\ 0 \end{array} \right) \left( \begin{array}{c} q_1^2 - q_2^2 (q_1 q_2 q_1 q_2) \\ q_2^2 + q_1 (q_1 q_2 - q_2 q_1) \\ -(q_2^2)^2 - (q_1^2)^2 \\ 0 \end{array} \right)$$
Remark 2.5. The tangent space to $\Sigma_M$ in neighborhood of a point $M$ is spanned by

$$T\Sigma_M = \langle \begin{pmatrix} 1 \\ 0 \\ q_1^1 \\ q_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ q_2^1 \\ q_2^2 \end{pmatrix} \rangle$$

Then, in the Euclidean case, the subspace orthogonal to $\Sigma_M$ is spanned by

$$(T\Sigma_M)^\perp = \langle \begin{pmatrix} q_1^1 \\ q_2^1 \\ -1 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} q_1^2 \\ q_2^2 \\ 0 \\ -1 \end{pmatrix} \rangle$$

which coincides with our result when

$$\begin{cases} (q_1^1)^2 + (q_2^1)^2 = (q_2^2)^2 + (q_1^2)^2 = 1 \\ q_1^1 q_2^2 - q_1^2 q_2^1 = 0 \end{cases}$$

In other form

$$\begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ \cos \theta & \sin \theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix} = \begin{pmatrix} -\cos \theta & \sin \theta \\ -\cos \theta & \sin \theta \end{pmatrix}$$

So we can see the difference between our space and an euclidean space.

3 Determination of the normal unit vector to a hypersurface

Theorem 3.1. In the space of $n$ dimension the determinant of Gram calculates the volume $\mathcal{V}$ of a parallelepiped formed by $n$ vectors $\xi_1, ..., \xi_n$ with $\xi_i = \xi_i^j e_j$ for $i, j = 1, ..., n$ if the espace is equiped by a metric $g$, then

$$\mathcal{V} = \sqrt{G(\xi_1, ..., \xi_n)}.$$ 

where

$$G(\xi_1, ..., \xi_n) := \begin{vmatrix} g(\xi_1, \xi_1) & \cdots & g(\xi_1, \xi_n) \\ \vdots & \ddots & \vdots \\ g(\xi_n, \xi_1) & \cdots & g(\xi_n, \xi_n) \end{vmatrix}$$

Proof. The proof of this theorem is made by recurrence on $n$.

If $n = 1$ obvious case, assuming the property is true for any family of $n$ vectors, and prove to $n + 1$. The volume $v$ of the parallelepiped $n + 1$ dimension is by definition the volume of the basis $F$, space generated by the first $n$ vectors of volume equal $\sqrt{G(\xi_1, ..., \xi_n)}$, by assumption of recurrence, multiplied by the height of $\xi_{n+1}$. Then $v = \sqrt{G(\xi_1, ..., \xi_{n+1})}$ due to the third point of the following consequence.
Consequence 3.2. 1. The volume constructed on \( n \) vectors of the same origin is equal to the determinant formed by the components of these vectors, multiplied by the volume constructed by \( n \) unit vectors.

\[
\mathcal{V}(\xi_1, \ldots, \xi_n) = \mathcal{V} \left( \frac{\xi_1}{||\xi_1||}, \ldots, \frac{\xi_n}{||\xi_n||} \right) \mid \xi_1, \ldots, \xi_n \mid
\]

2. In the case where the vectors \( \xi_i \) are unit we have \( g(\xi_i, \xi_j) = g_{ij} \) then the volume formed by \( n \) unit vectors is:

\[
G(\xi_1, \ldots, \xi_n) = \sqrt{g}
\]

3. Let \( v \) the orthogonal vector to parallelepiped formed by \( n \) vectors \( \xi_1, \ldots, \xi_n \) then we have:

\[
G(v, \xi_1, \ldots, \xi_n) = \sqrt{g(v, v)} G(\xi_1, \ldots, \xi_n)
\]

Proof. 2 and 3 are obvious, we proved only 1.

We have \( g(\xi_i, \xi_j) = \delta_{ij} \) if we suppose \( \xi_i = (\xi^1_i, \ldots, \xi^n_i) \) then:

\[
G(\xi_1, \ldots, \xi_n) = \begin{vmatrix}
\xi^1_1 & \cdots & \xi^n_1 \\
\vdots & \ddots & \vdots \\
\xi^1_n & \cdots & \xi^n_n
\end{vmatrix}
\begin{vmatrix}
g_{11} & \cdots & g_{1n} \\
\vdots & \ddots & \vdots \\
g_{n1} & \cdots & g_{nn}
\end{vmatrix}
\begin{vmatrix}
\xi^1_1 & \cdots & \xi^n_1 \\
\vdots & \ddots & \vdots \\
\xi^1_n & \cdots & \xi^n_n
\end{vmatrix}
\]

So using 2, we can find 1.

Definition 3.3. If we denote \((e^1_1, \ldots, e^*_n)\) the dual basis in the espace \( E \) of \( n \) dimension, let \( n \) vectors \( \xi_1, \xi_2, \ldots, \xi_n \) if we considered that \( 1 \leq i_1 < \ldots < i_p \leq n \) we define by \( e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}(\xi_1, \xi_2, \ldots, \xi_p) \) the determinant of the matrix of order \( p \) formed by \( i_k^{th} \) component of vectors \( \xi_j \) where \( j, k = 1, \ldots, n \) and \( i_k \in \{1, \ldots, n\} \):

\[
e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}(\xi_1, \xi_2, \ldots, \xi_p) = \begin{vmatrix}
\xi^1_1 & \cdots & \xi^1_{i_p} \\
\vdots & \ddots & \vdots \\
\xi^n_1 & \cdots & \xi^n_{i_p}
\end{vmatrix}
\]

Theorem 3.4. The length \( \ell \) of the normal vector \( v \) to the hypersurface \( \Sigma \) is

\[
\sqrt{g}.
\]

Proof. Note that the subspace generated by \( n - 1 \) vectors \( p_i = (p^1_i, \ldots, p^n_i) \) for \( i = 1, \ldots, n - 1 \) is the tangent space of the hypersurface \( \Sigma \) and his volume is \( d\sigma \) the volume of the parallelepiped of \( n \) dimension spanned by \( \Sigma \) and \( v \) is \( V = \ell d\sigma \), to simplify the calculation, we introduce the variables \( \xi_1, \ldots, \xi_n \) as \( \xi_n = -\frac{\xi_n}{p_{n-1}} = \ldots = -\frac{\xi_n-1}{p_{n-1}} \) and we denote the function \( F \) as \( \xi_n L(x^1, \ldots, x^n; \frac{\xi_1}{\xi_n}, \ldots, \frac{\xi_n}{\xi_n}) = \)
\(F(x^1, \ldots, x^n; \xi_1, \ldots, \xi_n)\) and like \(F\) homogeneous and of degree 1 \(\xi_1, \ldots, \xi_n\) then we write (3):

\[
v = \left(\frac{\partial F}{\partial \xi_1}, \ldots, \frac{\partial F}{\partial \xi_n}\right)
\]

(4)

secondly according to one of the previous consequence then:

\[
V = \sqrt{g} \begin{vmatrix}
\frac{\partial F}{\partial \xi_1} & \ldots & \frac{\partial F}{\partial \xi_n} \\
\xi_1 & \ldots & \xi_n \\
\vdots & \ddots & \vdots \\
\xi_{n-1} & \ldots & \xi_{n-1}
\end{vmatrix}
\]

If we denote \((e^*_1, \ldots, e^*_n)\) the dual basis then the element of surface \(d\sigma = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial F}{\partial \xi_i} e^*_i \wedge \ldots \wedge e^*_i \wedge e^*_i \ldots e^*_n\), now it remains to calculate:

\[
d\sigma(\xi_1, \ldots, \xi_{n-1}) = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial F}{\partial \xi_i} e^*_i \wedge \ldots \wedge e^*_i \wedge e^*_i \ldots e^*_n(\xi_1, \ldots, \xi_{n-1})
\]

which giving \(\ell = \sqrt{g}\).

**Consequence 3.5.** The components of \(\nu\) on dual basis are:

\[
\sqrt{g} \left(\frac{\xi_1}{F}, \ldots, \frac{\xi_n}{F}\right).
\]

**Proof.** Denote respectively \(\ell^i\) and \(\ell_i\) the components of \(\nu\) in the basis and in the dual basis, then use (4) we have \(\ell^i = \frac{1}{\sqrt{g}} \frac{\partial F}{\partial \xi_i}\) and like \(\nu\) is normal then \(\ell^i \ell_{i} = 1\), or \(F\) is homogeneous of degree one in \(\xi_i\) then

\[
\frac{1}{\sqrt{g}} \frac{\partial F}{\partial \xi_i} = \frac{1}{\sqrt{g}} F \Rightarrow \frac{1}{\sqrt{g}} \frac{\partial F}{\partial \xi_i} \sqrt{g} \frac{\xi_i}{F} = 1
\]

which gives the normal component of unit vector in the dual basis. \(\square\)
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