Second order intuitionistic propositional logic of the real line is decidable*

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Abstract

It is known that the set of tautologies of second order intuitionistic propositional logic, IPC2, is undecidable. Here, we prove that the sets of formulas of IPC2 which are true in the algebra of open subsets of reals or rationals are decidable.

1 Basic definitions

We investigate the second order intuitionistic propositional logic, denoted as IPC2 (for a detailed treatment of this logic we refer to the book [SU]). The set of formulas of this logic is the same as in the classical case. We have standard propositional connectives, universal and existential quantifiers and the set of propositions. Firstly, we present the set of axioms and rules in the Gentzen style. Then, we define various semantics for IPC2 and describe their status with respect to completeness and decidability of the tautology problem.

Below, $\Gamma$ is a multiset of formulas of IPC2, $\psi$, $\varphi$ and $\rho$ are formulas of IPC2 and $p$ is a proposition. The letters I and E in names of rules stand for the “introduction” and “elimination”, respectively.

1. Axioms:

   $\Gamma, \psi \vdash \psi.$

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2. Rules for conjunction:

\[ \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \land \psi} \] (\&I),

\[ \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi} \] (\&L),

\[ \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \psi} \] (\&R).

3. Rules for disjunction:

\[ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \lor \psi} \] (\lor I1),

\[ \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \lor \psi} \] (\lor I2),

\[ \frac{\Gamma, \varphi \vdash \rho \quad \Gamma, \psi \vdash \rho}{\Gamma \vdash \varphi \lor \psi} \] (\lor E).

4. Rules for implication:

\[ \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi} \] (\Rightarrow I),

\[ \frac{\Gamma \vdash \varphi \Rightarrow \psi}{\Gamma \vdash \varphi} \] (\Rightarrow E).

5. The rule ex falso quodlibet:

\[ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \] (\bot E).

6. Rules for quantifiers:

\[ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall p \varphi} \] (\forall I),

\[ \frac{\Gamma \vdash \forall p \varphi}{\Gamma \vdash \varphi[p := \psi]} \] (\forall E),

\[ \frac{\Gamma \vdash \varphi[p := \psi]}{\Gamma \vdash \exists p \varphi} \] (\exists I),

\[ \frac{\Gamma \vdash \exists p \varphi \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi} \] (\exists E).

In rules (\forall I) and (\exists E) we have a restriction that the variable \( p \) should not occur as a free variable of \( \Gamma \) or \( \psi \).

We denote the above calculus with IPC2 as well. Later, we consider sets of tautologies of IPC2 for various kinds of algebraic semantics. However, these sets will contain the set of theorems of the above calculus with one exception of IPC2\(^-\) (defined later). By IPC we denote intuitionistic propositional logic (without quantification) and the corresponding calculus defined by removing from the above one the rules for quantifiers.

In the above formulation we have no special rules and even no symbol for negation. This is so, because we do not treat negation as a primitive but rather we define \( \neg \varphi \) as \( \varphi \Rightarrow \bot \). We define also \( \varphi \leftrightarrow \psi \) as \((\varphi \Rightarrow \psi) \land (\psi \Rightarrow \varphi)\).
It is known that in the given calculus one can define from $\forall$ and $\Rightarrow$ all other connectives and quantifiers. So, the following formulas are provable,

$$\bot \iff \forall p \, p$$
$$\varphi \lor \psi \iff \forall p ((\varphi \Rightarrow p) \Rightarrow ((\psi \Rightarrow p) \Rightarrow p)),$$
$$\varphi \land \psi \iff \forall p ((\varphi \Rightarrow (\psi \Rightarrow p)) \Rightarrow p),$$
$$\exists q \varphi (q) \iff \forall p (\forall q (\varphi (q) \Rightarrow p) \Rightarrow p).$$

In the proof of our main theorem we use this fact implicitly by restricting our translation to formulas with $\forall$ and $\Rightarrow$, only. Let us mention that we do need $\forall$ quantifier to define other connectives, see e.g. [SU] or [SU10].

It was proved by L"ob (see [L76]) that the above calculus has an undecidable provability problem. Even the $\forall$–free fragment of this logic is undecidable (see [SU10]). Before L"ob’s article, Gabbay in [G74] considered IPC2 extended by a scheme called the axiom of constant domains (CD),

$$\forall p (\varphi \lor \psi (p)) \Rightarrow (\varphi \lor \forall p \psi (p)),$$

where $p$ is not free in $\varphi$. In the context of first order intuitionistic logic this scheme was introduced by Grzegorczyk in [G64] and it is also called Grzegorczyk scheme (one should not confuse this with Grzegorczyk axiom in modal logic). Gabbay showed undecidability of (IPC2 + CD), see [G74], but his proof was later corrected by Sobolev in [S77]. Gabbay claimed that his result generalizes to the case without CD. According to Gabbay, the generalization could be obtained by the finite axiomatizability of CD over IPC2. Nevertheless, it seems that there is no obvious method to define such an axiomatization.

Sobolev also considered logics without full comprehension axioms which correspond in our setting to rules $\forall E$ and $\exists I$. In the restricted versions of both rules we demand that the formula $\psi$ is atomic. Let us call this logic IPC2$^-$. Sobolev showed then that any logic between IPC2$^-$ and (IPC2+CD) is undecidable.

Now, we will discuss various semantics for IPC2. There are two kinds of popular semantics for intuitionistic logics, one constructed using Kripke models and the other one based on Heyting algebras. In the Kripkean approach semantics is given by Kripke frames $(C, \leq, \{D_c : c \in C\})$, where each $D_c$ is a subset of $\mathcal{P}(C)$ and all $X \in D_c$ are upward closed with respect to $\leq$. 3
Moreover, for each \( c \leq c' \in C \) we have \( D_c \subseteq D_{c'} \). The set \( C \) is the set of possible worlds and, for each \( c \in C \), the set \( D_c \) is the range of second order quantification in the world \( c \). Then, the value of a given formula \( \varphi \) as the set of possible worlds at which \( \varphi \) is true may be given by a usual inductive definition. In order to satisfy unrestricted versions of rules \( \forall E \) and \( \exists I \) one needs to require that for any formula \( \varphi \), any \( c \in C \) and for any valuation \( v \) of propositions into \( D_c \), if \( X_{\varphi,v} \) is the set of worlds greater or equal \( c \) at which \( \varphi \) is satisfied with \( v \), then \( X_{\varphi,v} \in D_c \), for each \( c \in C \). This class of models gives a sound and complete semantics for IPC2. If we drop the last condition concerning rules \( \forall E \) and \( \exists I \) then we get a sound and complete semantics for IPC2$^{-}$. The logics with CD are complete w.r.t. “constant domains semantics” where all \( D_c \)'s are equal, see [G74]. Then, the range for quantifiers is the same in all possible worlds. Let us stress that \( D_c \)'s may not contain all upward closed subsets of \( C \). Adding the condition that \( D_c \)'s contain all upward closed subsets of \( C \) gives us the, so called, \textit{principal} semantics. The set of tautologies of principal Kripkean semantics is recursively isomorphic to the classical second order logic as proved by Kremer, see [K97a].

Now, we turn to algebraic semantics. In the classical case the sound and complete semantics for second order propositional logic is given by boolean algebras. In the intuitionistic case the sound and complete semantics for IPC is given by Heyting algebras.

A Heyting algebra \((H, \leq, \cap, \cup, \to, 0, 1)\) is a distributive lattice with top and bottom elements augmented with the pseudo-complement operation \( \to \) which interprets implication. It is required that the following is well defined for \( a, b \in H \),

\[
    a \to b = \max\{c \in H : c \cap a \leq b\}.
\]

In the case of IPC, Heyting algebras give the sound and complete semantics, e.g., by constructing a Heyting algebra from a Kripke model where all upward closed sets from the Kripke model form the universe of the algebra.

In the algebraic semantics we have two ways in which we can interpret quantification. In the, so called, principal semantics quantifiers range over all elements of a given algebra and their meaning is given by infinite joins and meets, which have to exist. In the non-principal semantics quantifiers range over a distinguished subset of an algebra. In the second order case the relation between Kripkean and algebraic semantics is not as straightforward as in the quantifier free case. An obvious way to translate a Kripke frame \((C, \leq, \ldots)\) into a special kind of Heyting algebra, a topology, would be to
define a topology of all upward closed subsets of $C$. However, not all upward closed subsets of a Kripke model are within its domain of quantification. Moreover, for different possible worlds $c \in C$ we may have different domains $D_c$. Only if we consider principal Kripkean semantics then for a given frame $(C, \leq, \ldots)$ we may define a topology of all upward closed subsets of $C$ (see, e.g., Exercise 2.8 in [SU]). For such topology the satisfaction relation is preserved from the one of the Kripke frame.

Lately, Philip Kramer in a personal communication expressed his strong confidence that the complexity of the set of tautologies for principal algebraic semantics is as hard as in the case of principal Kripkean semantics. Kramer made his statement in his article [K97b] (p. 296) claiming that a nontrivial extension of methods from [K97a] would be needed. In a non-principal case a recent article by Kramer, [K13], establishes its completeness w.r.t. IPC2.

A special case of Heyting algebras is given by topologies. Let us describe a satisfiability relation for IPC2 and topological principal semantics. Let $T = (T, \mathcal{O}(T))$ be an arbitrary topology, where $\mathcal{O}(T)$ is the set of open subsets of $T$, and let $v : \text{PROP} \to \mathcal{O}(T)$ be a valuation from the set of propositions. Then, we may define a value of a given formula of IPC2 in $T$ under $v$, denoted as $\llbracket \varphi \rrbracket_v^T$, by recursion on the complexity of the formula:

1. $\llbracket \bot \rrbracket_v^T = 0$,
2. $\llbracket p \rrbracket_v^T = v(p)$,
3. $\llbracket \psi \land \gamma \rrbracket_v^T = \llbracket \psi \rrbracket_v^T \cap \llbracket \gamma \rrbracket_v^T$,
4. $\llbracket \psi \lor \gamma \rrbracket_v^T = \llbracket \psi \rrbracket_v^T \cup \llbracket \gamma \rrbracket_v^T$,
5. $\llbracket \psi \Rightarrow \gamma \rrbracket_v^T = \text{int}((T \setminus \llbracket \psi \rrbracket_v^T) \cup \llbracket \gamma \rrbracket_v^T),$
6. $\llbracket \exists p \psi \rrbracket_v^T = \bigcup_{a \in \mathcal{O}(T)} \llbracket \psi \rrbracket_v^{T(p \mapsto a)}$,
7. $\llbracket \forall p \psi \rrbracket_v^T = \text{int}(\bigcap_{a \in \mathcal{O}(T)} \llbracket \psi \rrbracket_v^{T(p \mapsto a)}).

We say that $\varphi$ is true in $T$ under $v$ if $\llbracket \varphi \rrbracket_v^T = T$.

It is known that the topologies of open subsets of $\mathbb{R}$ or $\mathbb{Q}$ form a sound and complete semantics for IPC. It can be easily shown that it is not the case for IPC2. Indeed, for each two $r, r' \in \mathbb{R}$ there is a homeomorphism of $\mathbb{R}$ into itself mapping $r$ to $r'$. It follows that if we have a sentence $\psi$ of IPC2 then either $\llbracket \psi \rrbracket_v^\mathbb{R} = \mathbb{R}$ or $\llbracket \psi \rrbracket_v^\mathbb{Q} = \emptyset$ (note that the value of $\llbracket \psi \rrbracket_v^\mathbb{R}$ does not depend on
Therefore, for each sentence $\psi$ of IPC2, $\psi \lor \neg \psi$ is true in $\mathbb{R}$. Of course, some sentences of that form are not provable in IPC2. One can check that if we take an arbitrary quantifier free formula $\varphi(p_1, \ldots, p_n)$ which is a classical tautology and is not an intuitionistic tautology then for $\psi := \forall p_1 \ldots \forall p_n \varphi$, the formula $\psi \lor \neg \psi$ is not provable in IPC2. The very same argument works also for $\mathbb{Q}$.

On the other hand the sentence $\neg \forall p(p \lor \neg p)$ is true in $\mathbb{R}$ (and in $\mathbb{Q}$) though it is not valid intuitionistically and moreover it is a classical contrtautology. It follows that the IPC2 theory of $\mathbb{R}$ or $\mathbb{Q}$ is not a subset of classical tautologies.

Despite the above facts, the topologies of the real and rational lines are natural semantics for intuitionistic logics. Firstly, it is natural to ask about second order propositional theory of these models which are kind of standard models for IPC. Secondly, one can see IPC2 over $\mathbb{R}$ or $\mathbb{Q}$ as a language capable of expressing some properties of these topologies. Thus, we may ask about the decidability of topological theories of $\mathbb{R}$ or $\mathbb{Q}$ expressible in IPC2.

We show here, that IPC2 tautologies of principal semantics of reals or rationals are easier than in the general case, namely decidable. The method used in the proof is an interpretation of these theories into the monadic theory of infinite binary tree, proved to be decidable by Rabin’s result (for details on this subject we refer to [GWT02]).

Let $T^\omega = \{0, 1\}^*$ be the set of finite binary sequences. The infinite binary tree is a structure $\mathcal{T}^\omega = (T^\omega, s_0, s_1, \leq)$, where $s_0(u) = u0$ and $s_1(u) = u1$ and $u \leq v$ when $u$ is an initial segment of $v$. A path in $\mathcal{T}^\omega$ is an infinite set $P \subseteq T^\omega$ such that $P$ is closed on initial segments and is linearly ordered by $\leq$. The empty sequence is denoted by $\varepsilon$.

The monadic second order logic is an extension of first order logic by second order quantifiers ranging over subsets of a given universe. Rabin’s theorem states that the monadic second order theory of $\mathcal{T}^\omega$ is decidable. We will denote this theory by S2S. It should be noted that the complexity of S2S is non-elementary. It became a standard method to show decidability of various problems by reducing them to S2S. In the next section, we exhibit a reduction for theories of IPC2 of reals and rationals.
decidability on $\mathbb{R}$ and $\mathbb{Q}$

2.1 Interpretation in S2S

We give interpretations of IPC2 theories of $\mathcal{O}(\mathbb{R})$ and $\mathcal{O}(\mathbb{Q})$ in S2S. A similar though a bit simpler interpretation was used in [Z04] showing decidability of IPC2 (and S4 with propositional quantification) on trees of height and arity $\leq \omega$ (in the principal semantics).

**Theorem 1** The IPC2 theories of the open subsets of reals and the open subsets of rational numbers are interpretable in S2S.

**Proof.** Let us recall that we write $s_0(x)$ and $s_1(x)$ to denote respectively the left and the right successors, and $x \leq y$ to denote that $x$ is on the path from the root of the tree to $y$.

Firstly, we give an interpretation of the IPC2 theory of reals. Instead of thinking about $\mathbb{R}$ we take an open interval $(0,1)$ which has the same topological properties. In particular any topological operation is taken in $(0,1)$, e.g., the closure of $(0,1/3)$ is $(0,1/3]$.

Each real number $r \in (0,1)$ may be seen as its binary representation $0.a_0a_1a_2\ldots$, where $a_i \in \{0,1\}$ and $r = \sum_{i=0}^{\infty} a_i 2^{-i-1}$. Such representations can be interpreted as infinite paths in $T^\omega$. A binary sequence $0.a_0a_1a_2\ldots$ is therefore a path $\{\varepsilon, s_{a_0}(\varepsilon), s_{a_1}s_{a_0}(\varepsilon), \ldots\}$. In what follows we will use both representations: infinite $\{0,1\}$-sequences $a_0a_1a_2\ldots$ (without the leading “0.”) and paths in $T^\omega$. In order to have the unique representation of each real we exclude sequences which have only finitely many zeros.

We can define on such infinite paths in $T^\omega$ the topology inherited from $(0,1)$. What we need is to assure that formulas of IPC2 can be effectively translated into formulas of S2S such that they will be equivalent modulo the translation of open sets of both topologies.

An infinite binary path represents a real from $(0,1)$ if and only if it is not of the form $0^\omega$ or $u1^\omega$ for some $u \in \{0,1\}^*$. The set of such paths is obviously definable in S2S. A formula $\text{Path}(X)$ stating that $X$ is an infinite path is just

$$
(X(x) \land X(y) \Rightarrow (x \leq y \lor y \leq x)) \land \\
\forall x(X(x) \Rightarrow (X(s_0(x)) \lor X(s_1(x)))) \land X(\varepsilon).
$$
Then a formula $U(X)$ defining the set of paths which represent some real can be written as:

$$
\text{Path}(X) \land \forall x (X(x) \Rightarrow \exists z \geq x \ X(s_0(z))) \land \exists x \ X(s_1(x)).
$$

The first conjunct of the above formula states that $X$ is an infinite path, the second one states that there are infinitely many 0’s in $X$ and the third conjunct states that there is at least one 1 in $X$. For a set $X$ such that $\text{Path}(X)$, let $r(X)$ be a real represented by $X$.

We need to represent not only real numbers but also subsets of $(0, 1)$. For a subset $S \subseteq T^\omega$, by $R(S)$ we denote a set of reals such that their corresponding paths are contained in $S$,

$$
R(S) = \{r(X) : U(X) \land X \subseteq S\}.
$$

We will represent open subsets of $(0, 1)$ by their closed complements. For a set $S \subseteq T^\omega$, $R(S)$ is closed in $(0, 1)$ if the following formula, $\text{Closed}(S)$, holds:

$$
\forall X \subseteq S\{[\text{Path}(X) \land \exists y (X(s_0(y)) \land \forall z \geq s_0(y) \neg X(s_0(z)))] \Rightarrow \exists Y \subseteq S \exists y [\text{Path}(Y) \land Y(y) \land X(s_0(y)) \land \forall z \geq s_0(y) (X(z) \Rightarrow X(s_1(z))) \land Y(s_1(y)) \land \forall z \geq s_1(y) (Y(z) \Rightarrow Y(s_0(z)))].
$$

The formula above states that if a path $X$ of the form $u01^\omega$ is a subset of $S$ then there is a path $Y \subseteq S$ of the form $u1^\omega$. The condition is necessary because we do not allow paths of the form $u1^\omega$ to represent reals (and satisfy the predicate $U(X)$). Thus, if we have such a path $X \subseteq S$, then we require that $S$ contains also a path $Y$ such that $r(X) = r(Y)$ and $U(Y)$. Otherwise, it could happen that for some sequence of sets $\{X_i \subseteq S : i \in \omega \land U(X_i)\}$ such that $\lim_{i \to \infty} r(X_i) = r(X)$ (in fact all $r(X_i)$ may be less than $r(X)$) there is no $Y \subseteq S$ such that $U(Y)$ and $r(Y) = r(X)$.

We use two facts about sets satisfying formula $\text{Closed}(S)$.

**Claim 2**

1. For any $C$ closed in $(0, 1)$ there exists $S \subseteq T^\omega$ such that $\text{Closed}(S)$ and $R(S) = C$.

2. For any $S \subseteq T^\omega$ such that $\text{Closed}(S)$, $R(S)$ is closed in $(0, 1)$. 

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Proof. To show 1 it is enough to take $S = \bigcup \{ X \subseteq T^{\omega} : U(X) \land r(X) \in C \}$. Then, $\text{Closed}(S)$ holds. Indeed, let a path $u01^\omega$ be a subset of $S$, then there is a sequence of reals $r_i \in C$, for $i \in \omega$ such that $r_i = r(u01^i v_i)$, for infinite binary words $v_i$ where all paths $u01^i v_i$ are subsets of $C$. The sequence $r_i$ converges to a real $r(u01^\omega)$ and since $C$ is closed $r(u01^\omega) \in C$ and so $u01^\omega$ is a subset of $S$.

Obviously, $C \subseteq R(S)$. To prove the converse let us assume that $r = r(X) \in R(S)$ for some $X \subseteq R(S)$ such that $U(X)$. Let us assume towards a contradiction that $r \not\in C$. We have two cases to consider. The first one when $X$ is of the form $u01^\omega$ and the second, complementary case. We consider the latter. Then, for each $i \in \omega$ there is $r_i \in C$ and $X_i \subseteq S$ such that $U(X_i)$, $r_i = r(X_i)$ and $X$ has a common initial segment with $X_i$ of length $i$. This is so because any element of $S$ belongs to a path representing a real from $C$. Now, $r = \lim_{i \to \infty} r_i$ and, since $C$ is closed, $r \in C$, a contradiction. As for the case of $X = u01^\omega$ we repeat the same reasoning either with $X$ or with a path $u01^\omega$. In both cases we get the same contradiction $r(X) = r(u01^\omega) \in C$.

To show 2 let $r_i \in R(S)$ be a sequence of reals converging to some $r \in (0,1)$. Let $P_i \subseteq S$ be such that $U(P_i)$ and $r_i = r(P_i)$ and let $P \subseteq T^{\omega}$ be such that $U(P)$ and $r = r(U)$. If $P_i$ are of the form $u01^{n_i} v_i$, for some strictly increasing sequence $n_i$, then $P$ is a path $u01^\omega$ and, by $\text{Closed}(S)$, $P \subseteq S$. It follows that $r \in R(S)$. Otherwise, $P$ is a path with infinitely many 1’s and $r = \sum_{i \in \omega} 2^{-n_i}$, for some strictly increasing sequence $n_i$. Now, if $|r - r_i| < 2^{-n_i-2}$ then $P$ and $P_i$ have a common initial segment of length $n_i$. We obtain that $P \subseteq \bigcup_{i \in \omega} P_i \subseteq S$ and, therefore, $r \in R(S)$. □

The above claim shows that sets of the form $\text{Closed}(S)$ are a good representation of closed subsets of $(0,1)$. We can write an S2S formula $\text{clBelong}(X, S)$ expressing that a real $r(X)$ belongs to a closed set $R(S)$. It has the form

$$U(X) \land \text{Closed}(S) \land X \subseteq S.$$ 

Similarly, we can express that a closed set $R(S)$ is included in a set $R(T)$ with

$$\forall X(\text{clBelong}(X, S) \Rightarrow \text{clBelong}(X, T)).$$

Let us state a useful lemma about definability in S2S.

**Lemma 3** For each S2S formula $\varphi(X)$ with $X$ a free second order variable and possibly with some first and second order parameters there exists a formula $\min_{\varphi}(X)$ such that
• if there exists a unique minimal closed set $C \subseteq (0, 1)$ such that $\varphi(X)$ is true for any $X$ with $C = R(X)$, then $\min \varphi(X)$ is true only about sets $X$ satisfying $C = R(X)$,

• $\min \varphi(X)$ if false for any set $X$, otherwise.

**Proof.** We write a formula $\min \varphi(X)$ as

$$
\varphi(X) \land \text{Closed}(X) \land \\
\forall Y((\text{Closed}(Y) \land \varphi(Y)) \Rightarrow \forall Z((U(Z) \land \text{clBelong}(Z, X)) \Rightarrow \text{clBelong}(Z, Y))).
$$

□

Now, we define an inductive translation of an IPC2 formula $\varphi(p_1, \ldots, p_n)$ into an S2S formula $\varphi^*(T, T_1, \ldots, T_n)$. We represent open sets by their closed complements. We require the following property: for all open subsets $R, R_1, \ldots, R_n$ of $(0, 1)$ and all $X, X_1, \ldots, X_n \subseteq T^\omega$ such that $\text{Closed}(X), R = (0, 1) \setminus R(X)$ and $\text{Closed}(X_i), R_i = (0, 1) \setminus R(X_i)$, for $i \leq n$, we have the equivalence,

$$
[\varphi]^{(0,1)}_{\{p_0 \mapsto R_1\}} = R \text{ if and only if } \\
(\{0, 1\}^*, s_0, s_1, \leq) \models \varphi^*[X, X_1, \ldots, X_n].
$$

If $\varphi = \bot$, then $\varphi^* = \forall x \; T(x)$ (note that if $X = T^\omega$ then $R(X) = (0, 1)$ and we want the complement of $X$ to be the empty set). If $\varphi = p_i$, then $\varphi^* = \forall Y(U(Y) \Rightarrow (\text{clBelong}(Y, T) \Leftrightarrow \text{clBelong}(Y, T_i)))$.

For $\varphi = (\psi_1 \Rightarrow \psi_2)$, we have

$$
[\varphi]^{(0,1)}_{v} = \text{int} \left( ((0, 1) \setminus [\psi_1]^{v,(0,1)}_{v}) \cup [\psi_2]^{v,(0,1)}_{v} \right) \\
= \max\{O \subseteq (0, 1) \colon O \text{ is open } \land O \subseteq ((0, 1) \setminus [\psi_1]^{v,(0,1)}_{v}) \cup [\psi_2]^{v,(0,1)}_{v} \} \\
= (0, 1) \setminus \min\{C \subseteq (0, 1) \colon C \text{ is closed } \land \\
((\psi_1^{v,(0,1)} \cap (0, 1) \setminus [\psi_2]^{v,(0,1)}_{v})) \subseteq C \}.
$$

By properties of the topology the above maximum and minimum exist. We need to write a formula $\psi^*(T, T_1, \ldots, T_n)$ such that with parameters $X_1, \ldots, X_n$ substituted for $T_1, \ldots, T_n$, respectively, it will be true only about the unique $T$ with

$$
R(T) = C_0 = \min\{C \subseteq (0, 1) \colon C \text{ is closed } \land ((\psi_1^{v,(0,1)} \cap (0, 1) \setminus [\psi_2]^{v,(0,1)}_{v})) \subseteq C \}.
$$

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Let $\varphi(T, T_1, \ldots, T_n)$ be a formula

$$\text{Closed}(T) \land \psi_1^*(T_1^*, T_1, \ldots, T_n) \land \psi_2^*(T_2^*, T_1, \ldots, T_n) \land$$

$$\forall X ((U(X) \land \neg \text{clBelong}(X, T_1^*) \land \text{clBelong}(X, T_2^*)) \Rightarrow \text{clBelong}(X, T)).$$

The formula above expresses the definitional property of $C_0$ in S2S and the topology of $T^{\omega}$ inherited from $(0, 1)$. Now, as $\varphi^*(T, T_1, \ldots, T_n)$ we take the formula $\min \hat{\varphi}(T)$ from Lemma 3 where the minimum is taken over $T$. The formula $\varphi^*(T, \ldots)$ is true only about the set $C_0$ what proves the inductive thesis for $\varphi$.

If $\varphi = \forall p_n \psi(p_n)$ then

$$[\varphi]_{(0,1)} = \text{int} \left( \bigcap_{O \text{ is open}} [\psi]_{v(p \mapsto O)}^{(0,1)} \right)$$

$$= \max \{ S \subseteq (0, 1) : S \text{ is open and for all open } O \subseteq (0, 1), S \subseteq [\psi]_{v(p \mapsto O)}^{(0,1)} \}$$

$$= (0, 1) \setminus \min \{ C \subseteq (0, 1) : C \text{ is closed and for all open } O \subseteq (0, 1), (0, 1) \setminus [\psi]_{v(p \mapsto O)}^{(0,1)} \subseteq C \}.$$ 

Since any topology is a complete Heyting algebras, the above sets are well defined. The last expression can be translated to an S2S formula. Let $\hat{\varphi}(T)$ be the following formula

$$\text{Closed}(T) \land$$

$$\forall W \forall T_n [\text{Closed}(W) \land \text{Closed}(T_n) \land \text{psi}(W, T_1, \ldots, T_n)] \Rightarrow$$

$$\forall Y ((U(Y) \land \text{clBelong}(Y, W)) \Rightarrow \text{clBelong}(Y, T))).$$

Now, using Lemma 3 we can write $\varphi^*(T)$ as $\min_{\hat{\varphi}(T)}(T, T_1, \ldots, T_{n-1})$.

The above translation gives us decidability of IPC2 on $(0, 1)$ since for any IPC2 sentence $\varphi$,

$$\varphi \text{ is true in } (0, 1) \text{ if and only if}$$

$$\forall T \forall X ((\text{Closed}(T) \land \varphi^*(T) \land U(X)) \Rightarrow \neg \text{clBelong}(X, T)) \text{ is true in } T^{\omega}.$$ 

A similar procedure gives also decidability of the IPC2 theory of open subsets of rationals. One needs to use the fact that the topology of dyadic rationals from $(0, 1)$ is isomorphic to the topology of $\mathbb{Q}$. Then, the set of paths which correspond to these rationals is easily definable as paths of the form $u10^u$, for some $u \in \{0, 1\}^*$. Now, let $U_Q(X)$ be a formula which defines
these paths. In order to obtain an interpretation of the IPC2 theory of open subsets of rationals one should restrict the universe of the given above interpretation to infinite paths satisfying $U_Q$. Syntactically, one should replace each occurrence of $U(X)$ with $U_Q(X)$. □

The above reduction gives a non elementary upper bound on the complexity of IPC2 on reals or rationals. We conjecture that the complexity of these theories is in fact elementary.

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