Markov Chains, Tensor Products, and Quantum Random Walks

Jeffrey Kuan

May 5, 2014

Abstract

We provide two new constructions of Markov chains which had previously arisen from the representation theory of $U(\infty)$. The first construction uses the combinatorial rule for the Littlewood–Richardson coefficients, which arise from tensor products of irreducible representations of the unitary group. The second arises from a quantum random walk on the von Neumann algebra of $U(n)$, which is then restricted to the center. Additionally, the restriction to a maximal torus can be expressed in terms of weight multiplicities, explaining the presence of tensor products.

1 Introduction

In [3], the authors introduce a family of Markov chains on the Gelfand–Tsetlin set $\mathcal{GT}$. This is the set of infinite sequences $\lambda^{(1)} \prec \lambda^{(2)} \prec \ldots$, where $\lambda^{(k)} = (\lambda^{(k)}_1, \ldots, \lambda^{(k)}_k)$ is a $k$–tuple of nonincreasing integers and $\lambda \prec \mu$ denotes the condition $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \mu_n \geq \lambda_n$. By considering the map

$$\lambda^{(1)} \prec \lambda^{(2)} \prec \ldots \mapsto \{(\lambda^{(k)}_i - i, k)\}_{1 \leq i \leq k < \infty} \subset \mathbb{Z} \times \mathbb{Z}_+,$$

these Markov chains define an interacting particle system on $\mathbb{Z} \times \mathbb{Z}_+$. Drawing lozenges around each particle yields a random tiling of the half plane. Furthermore, the condition $\lambda^{(k)} \prec \lambda^{(k+1)}$ ensures that there is a natural interpretation as a randomly growing stepped surface. This random growth belongs to the $2 + 1$ anisotropic KPZ class of stochastic growth models. This universality class is a variant of the KPZ universality class, which has seen many results in recent years (see [7] for a survey). By considering suitable projections, the Markov chains also reduce to TASEP, PushASEP (introduced in [4]), and the Charlier process from [12].

The Gelfand–Tsetlin set parametrises the branching of irreducible representations of the unitary group. Additionally, the family of Markov chains can be constructed from the representation theory of the infinite–dimensional unitary group $U(\infty)$ [6]. Tools from representation theory have yielded a rich variety of two–dimensional dynamics (e.g. [3, 14]). One of the most general processes arising from representation theory are Macdonald processes [5].
In this paper, we hope to deepen the connections between probability theory and representation theory. To this end, we give two new representation-theoretic constructions for these Markov chains. The first involves the Littlewood–Richardson rule for decomposing tensor products of irreducible representations of \( U(n) \). The second involves a quantum random walk on the von Neumann algebra of \( U(n) \), which is then restricted to the center. These constructions have the advantage of not requiring infinite-dimensional representation theory and are therefore more generalisable to other simple Lie groups.

The structure of the paper is as follows. In section 2 we review the representation theory of \( U(n) \) and introduce the Markov chains from \( \mathbb{T} \). In section 3 we provide a construction the combinatorial description of the Littlewood–Richardson coefficients. In section 4 we provide another construction, this time using a quantum random walk on the von Neumann algebra of \( U(n) \). This will also give a representation theoretic explanation (i.e. using tensor products of representations instead of combinatorics) for the occurence of the Littlewood–Richardson coefficients.

## 2 Markov chains

### 2.1 Background

Before defining the Markov chains, let us review some background on the unitary groups. Let \( U(n) \) denote the compact group of \( n \times n \) unitary matrices. Occasionally, to clean up notation, \( G \) will also refer to \( U(n) \). Let \( T^n \subset U(n) \) be the subgroup of diagonal unitary matrices, which is a maximal torus of \( U(n) \). With respect to this maximal torus, the weight lattice \( P \) of irreducible representations of \( U(n) \) is easy to describe. The Lie algebra of \( T^n \), denoted \( L^n \), consists of imaginary diagonal matrices. The weight lattice \( P \subset (L^n)^* \) is the \( n \)-dimensional lattice generated by the elements \( \epsilon_1, \ldots, \epsilon_n \), where \( \epsilon_j(\text{diag}(u_1, \ldots, u_n)) = u_j/(2\pi i) \). Each \( \lambda_1 \epsilon_1 + \ldots + \lambda_n \epsilon_n \), \( \lambda_j \in \mathbb{Z} \) defines a character of \( T^n \) by sending \((z_1, \ldots, z_n) \) to \( z_1^{\lambda_1} \cdots z_n^{\lambda_n} \). In this way, there is an isomorphism \( e : P \to T^n \). For \( x \in P \) and \( \theta \in T^n \), write \( x(\theta) = e(x)\theta \).

Note that with this notation, \( x(\theta)y(\theta) = (x+y)(\theta) \).

The roots of \( U(n) \) with respect to \( T^n \) are \( \epsilon_i - \epsilon_j, 1 \leq i \neq j \leq n \). The Weyl group is generated by the reflections with respect to the roots. It is isomorphic to the group \( S_n \) acting on \( \{ \epsilon_1, \ldots, \epsilon_n \} \), where the reflection with respect to \( \epsilon_i - \epsilon_j \) is the transposition \( (\epsilon_i \epsilon_j) \). The Weyl chamber is thus \( W := \{ \lambda_1 \epsilon_1 + \ldots + \lambda_n \epsilon_n : \lambda_1 \geq \ldots \geq \lambda_n \in \mathbb{Z} \} \).

Recall that any irreducible representation of any compact, connected, simple Lie group is generated by a highest weight vector, which must lie in the Weyl chamber. Conversely, any weight in the Weyl chamber generates an irreducible representation by successively applying negative roots. Therefore the irreducible unitary representations of \( U(n) \) is parameterised by \( W_n \).

Let \( m_1^{n_1} m_2^{n_2} \ldots \) denote the sequence \((m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots) \). For \( \lambda, \mu \in W_n \), let \( \lambda \prec \mu \) denote the condition \( \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \mu_n \geq \lambda_n \).
For each $\lambda \in \mathcal{W}_n$, let $\pi_\lambda : U(n) \to GL(V_\lambda)$, $\chi_\lambda$ and $\dim \lambda$ denote the corresponding representation, character and dimension. Let $\tilde{\chi}_\lambda$ denote the normalized character $\chi_\lambda/\dim \lambda$. Recall that the conjugacy class of a matrix in $U(n)$ is given by its eigenvalues. Therefore, $\chi^\lambda$ is a function of $\theta = (\theta_1, \ldots, \theta_n)$. Explicitly, $\chi^\lambda$ is just the Schur polynomial $s_\lambda$. Useful formulae are

$$\chi_\lambda(\theta_1, \ldots, \theta_n) = s_\lambda(\theta_1, \ldots, \theta_n) = \frac{\det[\theta^\lambda_{i+j-n}]_{1 \leq i,j \leq n}}{\det[\theta^\lambda_{i-j}]_{1 \leq i,j \leq n}}.$$  (1)

and

$$\chi_\lambda(\theta_1, \ldots, \theta_n) = s_\lambda(\theta_1, \ldots, \theta_n) = \det[h_{\lambda,-i+j}(\theta)],$$  (2)

where $h_k$ is the $k$–th complete homogeneous symmetric polynomial:

$$h_k(\theta) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \theta_{i_1} \cdots \theta_{i_k}.$$  

Equation (2) is called the first Giambelli formula. The elementary homogeneous symmetric polynomial $e_k$ will also appear:

$$e_k(\theta) = \sum_{1 < i_1 < \cdots < i_k < n} \theta_{i_1} \cdots \theta_{i_k}.$$  

Observe that

$$\det[\theta^\lambda_{i+j-n}]_{1 \leq i,j \leq n} = \begin{cases} h_k, & \text{when } \lambda = k0^{n-1} \\ e_k, & \text{when } \lambda = 1^k0^{n-k} \\ h_k(\theta_1^{-1}, \ldots, \theta_n^{-1}), & \text{when } \lambda = 0^{n-1}(-k) \\ \theta_1^{-1} \cdots \theta_n^{-1}e_{n-k} = e_k(\theta_1^{-1}, \ldots, \theta_n^{-1}), & \text{when } \lambda = 0^{n-k}(-1)^k \\ \end{cases}$$

The third formula follows from the first Giambelli formula. A formula for the dimension is

$$\dim \lambda = \prod_{i<j} \frac{\lambda_i - i - (\lambda_j - j)}{j-i},$$

which extends formally to $\dim : P \to \mathbb{R}$.

Let $L^2(G, dg)^G$ denote the square–integrable class functions on $G$. By the Peter–Weyl theorem, $\{\chi_\lambda\}_{\lambda \in \mathcal{W}_n}$ is an orthonormal basis for $L^2(G, dg)^G$. For any $\kappa \in L^2(G, dg)^G$ and any $\lambda \in \mathcal{W}_n$, let $\hat{\kappa}(\lambda)$ be the Fourier coefficient

$$\hat{\kappa}(\lambda) = \int_G \kappa(g)\overline{\chi_\lambda(g)} dg,$$

so that

$$\kappa(g) = \sum_{\lambda \in \mathcal{W}_n} \hat{\kappa}(\lambda)\chi_\lambda(g).$$  (3)

Formally, this means that $\sum_{\lambda \in \mathcal{W}_n} \chi_\lambda(g)\chi_\lambda(g')$ is the Dirac delta function $\delta_{g^{-1}g'}$.

Restricting the highest weight representation $V_\lambda$ to $T^n$ yields a decomposition into one–dimensional subspaces

$$V_\lambda = \bigoplus_{x \in P} U_x^\oplus n\lambda(x),$$  (4)
where $$U_x = \{ v \in V_\lambda : \theta \cdot v = x(\theta)v \ \text{for all} \ \theta \in \mathbb{T}^n \}$$
and $$n_\lambda(x)$$ are non-negative integers. In terms of characters, this means that

$$\chi_\lambda(\theta) = \sum_{x \in P} n_\lambda(x) x(\theta).$$

For $$\kappa \in L^2(G, \mathbb{C})^G$$, define $$n_\kappa(x)$$ by linear extension, i.e.

$$n_\kappa(x) = \sum_{\lambda \in \mathcal{W}_n} \hat{n}(\lambda)n_\lambda(x).$$

### 2.2 Markov chains

Now review the Markov chains from [6]. Let $$\theta_1, \ldots, \theta_n$$ be fixed nonzero complex numbers, and let $$F$$ be an analytic function in an annulus $$A$$ which contains all the $$\alpha_j^{-1}$$ such that each $$F(\alpha_j^{-1})$$ is nonzero. Given such an $$F$$, define

$$f(m) := \frac{1}{2\pi i} \oint \frac{F(z)}{z^{m+1}} \, dz,$$

where the integral is taken over any positively oriented simple loop in $$A$$. Section 2.3 of [6] defines matrices $$T_n$$ with rows and columns parameterised by $$\mathcal{W}_n$$:

$$T_n(\theta; F)(\lambda, \mu) := \frac{s_\mu(\theta)}{s_\lambda(\theta)} \det[f(\lambda_j + j - \mu_i - i)]_{ij}^n \prod F(\theta_j^{-1}).$$

**Proposition 2.1.** There is the commuting relation $$T_n(\theta; F_1)T_n(\theta; F_2) = T_n(\theta; F_1F_2)$$. For $$\theta = (1, 1, \ldots, 1)$$, $$T_n(\theta; F)$$ is a stochastic matrix.

**Proof.** Proposition 2.10 of [6] gives the commuting relation. Proposition 2.8 of [6] gives the stochastic matrix result.

Let us now describe the functions $$F$$ to be considered. Define the functions

$$F_{\alpha^+, q}(z) = (1 - qz)^{-1}, \quad F_{\alpha^-, q}(z) = (1 - qz^{-1})^{-1}, \quad 1 > q \geq 0$$

$$F_{\beta^+, p}(z) = 1 + pz, \quad F_{\beta^-, p}(z) = 1 + pz^{-1}, \quad 1 \geq p \geq 0.$$

$$F_{\gamma^+, t}(z) = e^{tz}, \quad F_{\gamma^-, t}(z) = e^{tz-1}, \quad t \geq 0.$$

**Lemma 2.2.** For these functions,

$$T_n(\theta; F_{\beta^-, p})(\lambda, \mu) = \frac{p^k}{\prod F(\theta_j^{-1})} \frac{s_\mu(\theta)}{s_\lambda(\theta)}, \quad \text{if each } \mu_j - \lambda_j \in \{0, 1\} \text{ and } \sum (\mu_j - \lambda_j) = k,$$

$$0, \quad \text{otherwise.}$$

$$T_n(\theta; F_{\beta^+, p})(\lambda, \mu) = \frac{p^k}{\prod F(\theta_j^{-1})} \frac{s_\mu(\theta)}{s_\lambda(\theta)}, \quad \text{if each } \mu_j - \lambda_j \in \{-1, 0\} \text{ and } \sum (\mu_j - \lambda_j) = -k,$$

$$0, \quad \text{otherwise.}$$
\[ T_n(\theta; F_{\alpha^\pm})(\lambda, \mu) = \begin{cases} \frac{q^k}{\prod F(\theta_j^{-1})} \frac{s_\mu(\theta)}{s_\lambda(\theta)} & \text{if } \lambda < \mu \text{ and } \sum (\mu_j - \lambda_j) = k, \\ 0, & \text{otherwise.} \end{cases} \]

\[ T_n(\theta; F_{\beta^\pm})(\lambda, \mu) = \begin{cases} \frac{q^k}{\prod F(\theta_j^{-1})} \frac{s_\mu(\theta)}{s_\lambda(\theta)} & \text{if } \mu < \lambda \text{ and } \sum (\mu_j - \lambda_j) = -k, \\ 0, & \text{otherwise.} \end{cases} \]

**Proof.** These are Lemmas 2.12 and 2.13 from [3].

Use the variable \( \xi \) to denote one of the symbols \( \alpha^\pm, \beta^\pm, \gamma^\pm \). For \( \xi = \beta^\pm \) and \( \xi = \gamma^\pm \), the process with transition probabilities \( T_n(1, F_\xi) \) are respectively the Krawtchouk and Charlier processes from [12]. These can be described respectively as the Doob \( h \)-transform (where \( h(\lambda) = \dim \lambda \)) of \( n \) independent Bernoulli walks and \( n \) independent exponential random walks of rate 1.

There is a general construction for building multivariate Markov chains out of \( \{ T_n : n = 1, 2, 3, \ldots \} \). This construction requires an intertwining relation between the transition probabilities (see section 2 of [3]). It is still an open problem to find a representation-theoretic interpretation of the commutation relation.

Let \( P_n(\theta; F)(\mu) = T_n(\theta; F)(0, \mu) \). From Lemma 2.2 we see that for \( F = F_{\alpha^\pm}, F_{\beta^\pm} \), these are geometric random variables weighted by the dimension of the representation. The construction then proceeds as follows. First, for \( F = F_{\alpha^\pm} \) or \( F_{\beta^\pm} \), construct \( T_n(F) \) out of \( P_n(F) \) (Lemmas 3.2 and 4.3 below). Second, we show that there is a commuting relation that is analogous to the one in Proposition 2.1 (Proposition 3.3 and Lemma 4.4 below). Finally, use a continuity argument (Lemmas 3.5 and 4.5 below) to give \( F_{\gamma^\pm} \).

### 3 Littlewood–Richardson Coefficients

Let us briefly recall the Littlewood–Richardson coefficients. For any two partitions \( \lambda, \tau \) such that \( \lambda_j \leq \tau_j \) for each \( j \), the **skew diagram** of \( \tau \setminus \lambda \) is the set-theoretic difference of the Young diagrams of \( \lambda \) and \( \tau \). A **skew Tableau** of shape \( \tau \setminus \lambda \) and weight \( \mu \) is obtained by filling in the skew diagram of \( \tau \setminus \lambda \) with positive integers such that the integer \( k \) appears \( \mu_k \) times. A skew Tableau is **semistandard** if it the entries weakly increase along each row and strictly increase down each column. A **Littlewood–Richardson tableau** is a semistandard skew Tableau with the additional property that in the sequence obtained by concatenating the reversed rows, every initial part of the sequence contains any number \( k \) at least as often as it contains \( k + 1 \). See figure 1 for an example.

In the special case \( \mu = k0^{n-1} \), the Littlewood–Richardson rule is known as **Pieri’s formula**. In this case, the semistandard skew Tableau can only be filled with 1’s, so the condition on the concatenated sequence

5
is automatically satisfied. The only requirement is that the skew diagram of \( \tau \setminus \lambda \) does not contain two boxes in the same column. In other words,

\[
\text{when } \mu = k0^{n-1}, \quad c_{\lambda \mu}^\tau = \begin{cases} 
1, & \text{if } \lambda \prec \tau \text{ and } \sum (\tau_j - \lambda_j) = k, \\
0, & \text{else.} 
\end{cases} \tag{6}
\]

We also need the special case \( \mu = 1k0^{n-k} \). The integers appearing in the semistandard skew tableau are \( \{1, 2, \ldots, k\} \), so the condition on the concatenated sequence can only hold if the skew diagram of \( \tau \setminus \lambda \) does not contain two boxes in the same row. In other words,

\[
\text{when } \mu = 1k0^{n-k}, \quad c_{\lambda \mu}^\tau = \begin{cases} 
1, & \text{if } \tau_j - \lambda_j \in \{0, 1\} \text{ and } \sum (\tau_j - \lambda_j) = k, \\
0, & \text{else.} 
\end{cases} \tag{7}
\]

The Littlewood-Richardson coefficients are related to representation theory by the decomposition

\[
V_\lambda \otimes V_\mu = \bigoplus_{\tau \in GT_n} c_{\lambda \mu}^\tau V_\tau.
\]

Since the character of \( V_\lambda \) is the Schur polynomial \( s_\lambda \) it is equivalent to say

\[
s_\lambda s_\mu = \sum_{\tau \in \mathcal{W}_n} c_{\lambda \mu}^\tau s_\tau.
\]

Also define the coefficients \( c_{\lambda \sigma \nu}^\tau \) by

\[
s_\lambda s_\sigma s_\nu = \sum_{\tau \in \mathcal{W}_n} c_{\lambda \sigma \nu}^\tau s_\tau.
\]

It follows immediately that

\[
\sum_{\mu \in \mathcal{W}_n} c_{\lambda \mu}^\tau c_{\sigma \nu}^\mu = c_{\lambda \sigma \nu}^\tau.
\]

In [8], the author considers a discrete–time particle system which arises from Pieri’s formula for the orthogonal groups. In [13], it is proven that this discrete–time particle system also arises from representations of \( O(\infty) \). Thus, the next theorem is a generalization in the unitary group case.
Theorem 3.1. For $F = F_{\xi}$,

$$
\sum_{\mu \in W_n} \mathbb{P}_n(\theta, F)(\mu) c^\tau_{\lambda_{\mu}} \frac{s_{\mu}(\theta)}{s_{\lambda}(\theta)s_{\mu}(\theta)} = T_n(\theta, F)(\lambda, \tau). \quad (8)
$$

Proof. Let $\mathcal{C}$ be the set of all functions $F : A \to \mathbb{C}$ such that $\mathcal{R}$ holds.

Lemma 3.2. The functions $F = F_{\alpha z}, F_{\beta z}$ are in $\mathcal{C}$.

Proof. Start with $1 + pz$. By lemma 2.2

$$
\mathbb{P}_n(\theta; F)(\mu) = \begin{cases} 
    s_{\mu}(\theta) \frac{p^k}{\Pi F(\alpha_j^{-1})}, & \mu = 1^k0^{n-k}, \\
    0, & \text{else}
\end{cases} \quad (9)
$$

Thus it suffices to consider at the the value of $c^\tau_{\mu}$ when $\mu = 1^k0^{n-k}$. By using Pieri’s formula (6) and another application of lemma 2.2, $F_{\beta z} \in \mathcal{C}$.

Now let $\tilde{F}(m) = f(-m)$,

$$
\mathbb{P}_n(\theta; F_{\beta z})(\mu) = \begin{cases} 
    s_{\mu}(\theta) \frac{p^k}{\Pi F(\alpha_j^{-1})}, & \mu = 0^{n-k}(-1)^k, \\
    0, & \text{else}
\end{cases}
$$

Since $c^\tau_{\lambda_{\mu}} = c^\tau_{\mu + 1, \lambda}$, then for $\mu = 0^{n-k}(-1)^k$,

$$
c^\tau_{\lambda_{\mu}} = \begin{cases} 
    1, & \text{if each } \tau_i - \lambda_i \in \{-1, 0\} \text{ and } \sum(\tau_j - \lambda_j) = -k, \\
    0, & \text{else}
\end{cases}
$$

Therefore, by lemma 2.2 $F_{\beta z} \in \mathcal{C}$.

Now consider the function $F(z) = (1 - qz)^{-1}$ By lemma 2.2

$$
\mathbb{P}_n(\theta; F)(\mu) = \begin{cases} 
    s_{\mu}(\theta) \frac{q^k}{\Pi F(\alpha_j^{-1})}, & \mu = k0^{n-1}, \\
    0, & \text{else}
\end{cases}
$$

By (7) and lemma 2.2 $F_{\alpha z} \in \mathcal{C}$.

Finally let $\tilde{F}(z) = (1 - qz)^{-1}$. Then

$$
\mathbb{P}_n(\theta; F)(\mu) = \begin{cases} 
    s_{\mu}(\theta) \frac{q^k}{\Pi F(\alpha_j^{-1})}, & \mu = 0^{n-1}(-k), \\
    0, & \text{else}
\end{cases}
$$

Using the identity

$$
s_{\lambda}(\theta^{-1}) = (\theta_1 \cdots \theta_n)^{-\lambda_1}s_{(\lambda_1-\lambda_n, \ldots, \lambda_1-\lambda_2, 0)}(\theta),
$$

we get for $\mu = 0^{n-1}(-k)$

$$
s_{(\lambda_1-\lambda_n, \ldots, \lambda_1-\lambda_2, 0)}(\theta)s_{0^{n-1}}(\theta) = (\theta_1 \cdots \theta_n)^{\lambda_1}s_{\lambda}(\theta^{-1}) \cdot s_{\mu}(\theta^{-1})
$$

$$
= (\theta_1 \cdots \theta_n)^{\lambda_1} \sum_{\tau} c^\tau_{\lambda_{\mu}} s_{\tau}(\theta^{-1})
$$

$$
= \sum_{\tau} c^\tau_{\lambda_{\mu}} \tilde{s}_{(\tau_1-\tau_n, \ldots, \tau_1-\tau_2, 0)}(\theta)
$$

$$
= \sum_{\tau} c^\tau_{\lambda_{\mu}} s_{(\tau_1-\tau_n, \ldots, \tau_1-\tau_2)}(\theta),
$$
so
\[ c_{\lambda \mu} = \begin{cases} 
1, & \text{if } (\lambda_1 - \lambda_n, \ldots, \lambda_1 - \lambda_2, 0) \prec (\lambda_1 - \tau_n, \ldots, \lambda_1 - \tau_1) \text{ and } \sum (\tau_j + \lambda_j) = k, \\
0, & \text{else.} 
\end{cases} \]

Equivalently,
\[ c_{\lambda \mu} = \begin{cases} 
1, & \text{if } \tau \prec \lambda \text{ and } \sum (\tau_j - \lambda_j) = -k \\
0, & \text{else.} 
\end{cases} \]

Therefore, by lemma 2.2, \( F_\alpha \in \mathcal{C} \).

Proposition 3.3. If \( F_1, F_2 \in \mathcal{C} \), then \( F_1 F_2 \in \mathcal{C} \).

Proof. We start with a lemma.

Lemma 3.4. Suppose \( \Pr_X, \Pr_Y, \Pr_Z \) are complex valued measures on the countable sets \( X, Y, Z \). Suppose \( \Pr_f \) is a complex valued measure on \( Y^X \) with total weight 1, (one should think of \( f : X \to Y \) as a random map). Suppose \( h : X \to Z \) and \( g : Y \to Z \) are deterministic maps such that \( h = g \circ f \) almost surely. If \( h_* \Pr_X = \Pr_Z \) and \( f_* \Pr_X = \Pr_Y \) (in the sense that \( \Pr_Y(B) = \sum_{x \in X} \Pr_X(x) \Pr_f(f(x) \in B) \)), then \( g_* \Pr_Y = \Pr_Z \).

Proof. Since \( h = g \circ f \), then for a fixed \( x \in X \) and \( E \subset Z \),
\[ \Pr_f(f(x) \in g^{-1}E) = \begin{cases} 
1, & x \in h^{-1}E, \\
0, & x \notin h^{-1}E. 
\end{cases} \]

Thus
\[ \Pr_Z(E) = \Pr_X(h^{-1}E) = \sum_{x \in X} \Pr_X(x) \Pr_f(f(x) \in g^{-1}E) = \Pr_Y(g^{-1}E), \]
i.e. \( g_* \Pr_Y = \Pr_Z \).

Fix \( \lambda \) and \( \tau \). To apply the lemma, use the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{W}^2 & \xrightarrow{f} & \mathcal{W} \\
\downarrow & & \downarrow \quad g \\
\mathcal{W}^1 & \rightarrow & \{0\}
\end{array}
\]

defined by
\[
\begin{array}{ccc}
(\sigma, \gamma, \nu) & \xrightarrow{f} & \mu \\
\downarrow & & \downarrow \\
(\gamma, \nu) & \rightarrow & \gamma 
\end{array}
\]
Every map is projection except for $f$, which is defined by

$$Pr_f(f(\sigma, \gamma, \nu) = \mu) = \begin{cases} \frac{c^\mu_{\sigma \nu} c^\gamma_{\lambda \mu}}{c^\gamma_{\lambda \sigma \nu}}, & \text{if } c^\gamma_{\lambda \sigma \nu} \neq 0, \\ 0, & \text{if } c^\gamma_{\lambda \sigma \nu} = 0 \end{cases}$$

Let $h : W^3 \to \{0\}$ be the composition along the bottom row of the diagram. Define the measures $Pr_X$ on $W^3_N$, $Pr_Y$ on $W_N$ and $Pr_Z$ on $\{0\}$ by

$$Pr_X(\sigma, \gamma, \nu) = P_n(\theta; F_1)(\sigma)c^\gamma_{\lambda \sigma \nu} \frac{s_{\gamma}(\theta)}{s_{\lambda}(\theta)s_{\sigma}(\theta)} P_n(\theta; F_2)(\nu)c^\nu_{\sigma \mu} \frac{s_{\nu}(\theta)}{s_{\sigma}(\theta)s_{\mu}(\theta)}$$

$$Pr_Y(\mu) = P_n(\theta; F_1) c^\gamma_{\lambda \sigma \nu} \frac{s_{\gamma}(\theta)}{s_{\lambda}(\theta)s_{\sigma}(\theta)} P_n(\theta; F_2)(\nu) c^\nu_{\sigma \mu} \frac{s_{\nu}(\theta)}{s_{\sigma}(\theta)s_{\mu}(\theta)}$$

$$Pr_Z(\{0\}) = T_n(\theta; F_1 F_2)(\lambda, \tau).$$

In this formulation, the proposition states that $g_* Pr_X = Pr_Z$ for all $\lambda \in W_N$. Thus the proposition follows if the conditions of the lemma hold.

First, it is immediate from the definitions that the weights of $f$ sum to 1 and that $h = g \circ f$ almost surely.

Second, we need to check that $h_* Pr_X = Pr_Z$. Since $F_1 \in C$, the first projection sends $Pr_X$ to

$$T_n(\theta; F_1)(\lambda, \gamma) P_n(\theta; F_2)(\nu) c^\nu_{\sigma \mu} \frac{s_{\nu}(\theta)}{s_{\sigma}(\theta)s_{\mu}(\theta)}$$

Since $F_2 \in C$, the second projection sends this to

$$T_n(\theta; F_1)(\lambda, \gamma) T_n(\theta; F_2)(\gamma, \tau).$$

Since $T_n(\theta; F_1) T_n(\theta; F_2) = T_n(\theta; F_1 F_2)$, the third projection sends this to $T_n(\theta; F_1 F_2)(\lambda, \tau) = Pr_Z$, as needed.

Finally, we need to check that $f_* Pr_X = Pr_Y$. By definition,

$$f_* Pr_X(\mu) = \sum_{\sigma, \nu, \gamma \in W_n} P_n(\theta; F_1)(\sigma) c^\gamma_{\lambda \sigma \nu} \frac{s_{\gamma}(\theta)}{s_{\lambda}(\theta)s_{\sigma}(\theta)} P_n(\theta; F_2)(\nu) c^\nu_{\sigma \mu} \frac{s_{\nu}(\theta)}{s_{\sigma}(\theta)s_{\mu}(\theta)} c^\mu_{\sigma \nu} c^\gamma_{\lambda \mu}$$

By summing over $\gamma$,

$$f_* Pr_X(\mu) = \sum_{\sigma, \nu \in W_n} P_n(\theta; F_1)(\sigma) c^\nu_{\lambda \sigma \mu} \frac{s_{\nu}(\theta)}{s_{\lambda}(\theta)s_{\sigma}(\theta)s_{\mu}(\theta)} P_n(\theta; F_2)(\nu) c^\mu_{\sigma \nu} c^\gamma_{\lambda \mu}$$

Now look at $Pr_Y(\mu)$. Since $F_2 \in C$,

$$P_n(\theta; F_1 F_2)(\mu) = \sum_{\sigma \in W_n} P_n(\theta; F_1)(\sigma) T_n(\theta; F_2)(\sigma, \mu)$$

$$= \sum_{\sigma \in W_n} P_n(\theta; F_1)(\sigma) \sum_{\nu \in W_n} P_n(\theta; F_2)(\nu) c^\mu_{\nu \sigma} \frac{s_{\mu}(\theta)}{s_{\nu}(\theta)s_{\sigma}(\theta)}.$$

Thus,

$$Pr_Y(\mu) = \sum_{\sigma, \nu \in W_n} P_n(\theta; F_1)(\sigma) P_n(\theta; F_2)(\nu) c^\nu_{\lambda \sigma \mu} c^\gamma_{\lambda \mu} \frac{s_{\gamma}(\theta)}{s_{\lambda}(\theta)s_{\sigma}(\theta)s_{\mu}(\theta)}$$

so $Pr_Y = f_* Pr_X$, as needed. This proves the proposition.
Lemma 3.5. If \( \{ f_k \} \) is a sequence of functions in \( C \) which converges to \( f \) uniformly in \( A \), then \( F \in C \).

Proof. It is immediate that \( \{ f_k \} \) converges to \( f \) uniformly. Since the determinant is a continuous function of its entries, \( T_n(\theta; F_k)(\lambda, \tau) \) converges to \( T_n(\theta; F)(\lambda, \tau) \). Since sum in the left–hand side of \( E_\tau \) only has finitely many terms, convergence must hold as well.

Finally, since \( e^x = \lim_{k \to \infty} (1 + x/k)^k = \lim_{k \to \infty} (1 - x/k)^{-k} \), Lemma 3.2, Proposition 3.3 and Lemma 3.5 prove Theorem 3.1.

4 Quantum Random Walk

Let us introduce some notation, which will follow [1] closely.

Let \( G \) be a compact topological group, let \( dq \) denote its Haar measure (normalized to have total weight 1), and let \( H = L^2(G, dq) \) be the Hilbert space of square–integrable functions. Let \( \alpha \) denote the representation of \( G \) on \( H \) by left translation. In other words, for \( f \in B(H) \) a unitary operator on \( H \), the map \( \alpha : G \to B(H) \) is defined by \( [\alpha(g)](f)(x) = f(xg) \). The von Neumann algebra of \( G \), denoted \( vN(G) \), is the closure (under the strong operator topology) of the \(*\)-subalgebra of \( B(H) \) generated by \( \alpha(G) \).

Let \( \kappa \) be a continuous, positive type function on \( G \) which sends the identity to 1. This defines a state \( \varphi \) on \( vN(G) \) by \( \varphi(\alpha(g)) = \kappa(g) \), and also defines a completely positive map on \( vN(G) \) by \( Q(\alpha(g)) = \kappa(g) \alpha(g) \).

Since \( vN(G) \) is a unital \( C^* \)-algebra, we can define the infinite tensor product \( vN(G)^{\otimes \infty} \), which is also a \( C^* \)-algebra. Let \( \varphi^{\otimes \infty} \) be the state on \( vN(G)^{\otimes \infty} \) defined by \( \varphi^{\otimes \infty}(x_1 \otimes x_2 \otimes \cdots) = \varphi(x_1) \varphi(x_2) \cdots \). The Gelfand–Naimark–Segal construction produces a von Neumann algebra \( W \). For nonnegative integers \( n \), define \( j_n : vN(G) \to W \) by \( j_0(\alpha(g)) = \text{Id}_W \), and \( j_n(\alpha(g)) = \alpha(g)^{\otimes n} \otimes \text{Id} \otimes \text{Id} \otimes \cdots \). The \( j_n \) form what is called a “non-commutative Markov process”. There is a projection map \( E_n \) from \( W \) to \( W_n \), the von Neumann subalgebra generated by the images of \( j_0, \ldots, j_n \). For \( n \leq m \), there is the Markov property \( E_n \circ j_m = j_n \circ Q^{m-n} \). One could think of these objects with the following analogy:

| Classical | Quantum | State Space | \( (\Omega, \mathcal{F}) \) | \( (\Omega, \mathcal{F}_n) \) | \( X_n : \Omega \to S \) | \( E(\cdot|\mathcal{F}_n) \) | \( E_n(\cdot) \) | \( \varphi^{\otimes \infty} \) |
|-----------|---------|-------------|-----------------|-----------------|----------------|----------------|----------------|----------------|
| \( vN(G) \) | \( vN(G) \) | \( W \) | \( W_n \) | \( j_n \) | \( E_n \) | \( \varphi \) |

4.1 Restriction to Center

Let \( Z(vN(G)) \) be the center of \( vN(G) \). The Peter–Weyl theorem gives an isomorphism \( \chi : Z(vN(G)) \to L^\infty(\hat{G}) \), where \( \hat{G} \) is the set of equivalence classes of irreducible representations of \( G \). If \( \kappa \) is constant on conjugacy classes, then \( Q \) sends \( Z(vN(G)) \) to itself. Because it is completely positive, the map \( \chi \circ Q \circ \chi^{-1} \) defines a transition matrix for a (classical) Markov chain with state space \( \hat{G} \). By a slight abuse of notation, let \( Q_n(\kappa)(x, y) \) denote the transition probabilities.

10
Now let \( G = U(n) \). Define \( \kappa_F : U(n) \to \mathbb{C} \) to be the class function defined by
\[
\kappa_F(\theta) = \prod_{j=1}^{n} \frac{F(\theta_j)}{F(1)}.
\]

Here, \( \theta = (\theta_1, \ldots, \theta_n) \) are the eigenvalues of the unitary matrix on which \( \kappa_F \) is applied. If \( F = F_\xi \), write \( \kappa_\xi = \kappa_{F_\xi} \).

Here is some useful information about \( Q_n(\kappa) \):

**Proposition 4.1.** 1. For any \( \kappa \in L^2(G, dg)^G \),
\[
Q_n(\kappa)(\lambda, \mu) = \frac{\dim \mu}{\dim \lambda} \int_{U(n)} \chi_\lambda(g) \overline{\chi_\mu(g)} \kappa(g) dg. \tag{10}
\]

2. The map \( Q_n : BC(G, \mathbb{C})^G \to Mat(\mathcal{W}_n \times \mathcal{W}_n) \) from complex-valued bounded continuous class functions on \( G \) to matrices with rows and columns indexed by \( \mathcal{W}_n \) is a morphism of \(*\)–algebras.

**Proof.** 1. This is Theorem 3.2 from [2]. Although the result there is only stated for certain \( \kappa \), by following the proof one sees that it holds more generally.

2. The fact that \( Q_n \) preserves linearity and \(*\) follows immediately from (10). By applying (3) to (10), it is immediate that multiplication is also preserved. Another way to see this is to use the quantum random walk: let \( Q_1, Q_2, \) and \( Q_{12} \) be the maps \( vN(G) \to vN(G) \) defined by sending \( \alpha(g) \to \kappa_1(g) \alpha(g) \), \( \kappa_2(g) \alpha(g) \) and \( \kappa_2(g) \kappa_1(g) \alpha(g) \) respectively. By construction, \( Q_n(\kappa_1), Q_n(\kappa_2), \) and \( Q_n(\kappa_1 \kappa_2) \) are the respective restrictions to \( Z(\mathcal{W}_n(G)) \). Since \( Q_1 \circ Q_2 = Q_{12} \), the result follows.

Now specialize to \( \kappa_\xi \).

**Theorem 4.2.** For any symbol \( \xi \), \( Q_n(\kappa_\xi) = T_n(1, F_\xi) \).

**Proof.** Start with:

**Lemma 4.3.** Theorem 4.2 holds for \( \xi = \alpha^\pm, \beta^\pm \).

**Proof.** By Weyl’s integration formula and (1), equation (10) implies
\[
Q_n(\kappa)(\lambda, \mu) = \frac{\dim \mu}{\dim \lambda} \frac{1}{n!} \int_{\mathbb{T}^n} \det[\theta_1^{\lambda_1+n-1}] \det[\mu_1^{\mu_1+n-1}] \kappa(\theta_1, \ldots, \theta_n) d\theta_1 \cdots d\theta_n.
\]

Changing to complex analytic notation and using that the Haar measure \( d\theta \) on \( \mathbb{T} \) is \( dz/2\pi iz \) implies
\[
Q_n(\kappa)(\lambda, \mu) = \frac{\dim \mu}{\dim \lambda} \frac{1}{n!} \left( \frac{1}{2\pi i} \right)^n \int_{\mathbb{T}^n} \det[z_1^{\lambda_1+n-1}] \det[z_1^{\mu_1+n-1}] \kappa(z_1, \ldots, z_n) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}. \tag{11}
\]

Note that
\[
\kappa_{\alpha^\pm}(z) = \prod_{j=1}^{n} \frac{1 - q z_j}{(1 - q)^{-1}} = \prod_{j=1}^{n} \frac{1 + q z_j + (q z_j)^2 + \cdots}{(1 - q)^{-1}} = \sum_{k=0}^{\infty} \frac{q^k h_k(z)}{(1 - q)^{-n}}. \tag{12}
\]
some \sigma, \tau be so that by (12)

\sum_{\lambda \leq \mu} \lambda, \mu \rightarrow \dim \lambda 1^n = \dim \lambda n! (1 - q)^{-n} \sum_{\sigma, \tau} \text{Con} \sigma, \tau.

Proceed with three steps:

I If \mu_i < \lambda_i for some 1 \leq i \leq n, then \text{Con} \sigma, \tau = 0 for all \sigma, \tau \in S_n.

II If \mu_i > \lambda_{i-1} for some 1 < i \leq n, then Q_n(\sigma, \lambda, \mu) = 0.

III If \lambda < \mu, then \frac{1}{n!} \sum_{\sigma, \tau} \text{Con} \sigma, \tau = 1.

For I, fix an i such that \mu_i < \lambda_i and suppose \text{Con} \sigma, \tau is nonzero for some \sigma, \tau \in S_n. If there is a j < i which satisfies \tau^{-1}(\sigma^{-1}(j)) \leq \lambda_i, then \lambda_{j-1} \leq \mu_{\tau^{-1}(\sigma(j))} - \tau^{-1}(\sigma(j)) \leq \mu_i - i, implying that \mu_i > \lambda_j \geq \lambda_i, which contradicts \mu_i < \lambda_i. Therefore \tau^{-1} sends the set \{1, 2, \ldots, i-1\} to itself, so \tau^{-1}(\sigma^{-1}(i)) \geq i. Thus \lambda_{i-1} \leq \mu_{\tau^{-1}(\sigma^{-1}(i))} - \tau^{-1}(\sigma^{-1}(i)) \leq \mu_i - i, so \lambda_i \leq \mu_i. Again, this is a contradiction. Therefore, the only possibility is that all \text{Con} \sigma, \tau are zero.

For II, suppose that \mu_i > \lambda_{i-1}. I claim that for some j \leq i, there is no k such that \mu_j - j < \lambda_j - k \leq \mu_{j-1} - (j - 1). This is simply because there are i - 1 intervals (\mu_j - j, \mu_{j-1} - (j - 1)], but only i - 2 numbers \lambda_k - k that can fit into these intervals, so at least one interval must be empty. The claim implies that the inequality \lambda_j - k \leq \mu_j - j holds if and only if the inequality \lambda_k - k \leq \mu_{j-1} - (j - 1) holds. Therefore \text{Con} \sigma, \tau + \text{Con} \sigma, \tau (j - j - 1) = 0, so the sum \sum_{\sigma, \tau} \text{Con} \sigma, \tau is zero.
For III, suppose that $\text{Con}(\sigma, \tau) \neq 0$. Then, using that $\lambda < \mu$, a strong induction argument on $j$ implies that $\tau^{-1} \sigma(j) = j$ for all $j$. In other words, $\text{Con}(\sigma, \tau) \neq 0$ implies that $\sigma = \tau$. Since the converse is immediate, the sum $\sum_{\sigma, \tau} \text{Con}(\sigma, \tau)$ simplifies to $\sum_{\sigma \in S_n} \text{Con}(\sigma, \sigma)$, which equals $|S_n| = n!$.

Together, I, II and III imply that

$$Q_n(\kappa_{\alpha^+})(\lambda, \mu) = \frac{\sum_{i=1}^{n} \mu_i - \lambda_i \dim \mu}{(1 - q)^{-n}} \dim \lambda_{\lambda < \mu},$$

which is just $T_n(1, F_{\alpha^+})$.

Now move on to $\xi = \beta^+$. Define the contribution from $\sigma$ and $\tau$ to be

$$\text{Con}'(\sigma, \tau) = \begin{cases} \text{sgn}(\sigma) \text{sgn}(\tau), & \text{if each } \lambda_j - j - (\mu_{\tau - 1}(j) - \tau \sigma^{-1}(j)) \in \{0, -1\} \\ 0, & \text{else} \end{cases}$$

so that

$$Q_n(\kappa_{\beta^+})(\lambda, \mu) = \frac{\dim \mu}{\dim \lambda} \frac{\sum_{\sigma, \tau} \text{Con}'(\sigma, \tau)}{n!} (1 + p)^n \sum_{\sigma, \tau} \text{Con}'(\sigma, \tau).$$

Again we prove three steps:

1. If $\mu_i < \lambda_i$ for some $1 \leq i \leq n$, then $\text{Con}'(\sigma, \tau) = 0$ for all $\sigma, \tau \in S_n$.
2. If $\mu_i - \lambda_i \notin \{0, 1\}$ for some $1 < i \leq n$, then $Q_n(\kappa)(\lambda, \mu) = 0$.
3. If all $\mu_i - \lambda_i$ are 0 or 1, then $\frac{1}{n!} \sum_{\sigma, \tau} \text{Con}'(\sigma, \tau) = 1$.

For 1, notice that $\text{Con}(\sigma, \tau) = 0$ implies that $\text{Con}'(\sigma, \tau) = 0$, and I above shows that $\text{Con}(\sigma, \tau)$ is always 0.

For 2, we already know that $Q_n(\kappa)(\lambda, \mu) = 0$ if some $\mu_i < \lambda_i$, so we can assume that all $\mu_i \geq \lambda_i$. Now fix some $j$ such that $\mu_j - \lambda_j \geq 2$, and suppose $\text{Con}'(\sigma, \tau) \neq 0$. Then $\tau \sigma^{-1}(j) \leq j$ would imply $\mu_{\tau \sigma^{-1}(j)} - \tau \sigma^{-1}(j) \geq \mu_j - j$, which implies that $\lambda_j - j - (\mu_{\tau \sigma^{-1}(j)} - \tau \sigma^{-1}(j)) \leq \lambda_j - \mu_j \leq -2$, which contradicts $\text{Con}'(\sigma, \tau) \neq 0$. So $\tau \sigma^{-1}(j) > j$. Thus there must be some $i > j$ such that $\tau \sigma^{-1}(i) \leq j$ (or else $\tau \sigma^{-1}$ would map $\{j, \ldots, n\}$ to $\{j + 1, \ldots, n\}$). This implies that $\lambda_i - i < \lambda_j - j - \mu_j - j \leq \mu_{\tau \sigma^{-1}(i)} - \tau \sigma^{-1}(i) < -2$, which again contradicts $\text{Con}'(\sigma, \tau) \neq 0$. Thus, $\text{Con}(\sigma, \tau)$ must always be zero.

For 3, suppose that $\text{Con}(\sigma, \tau) \neq 0$ with $\sigma \neq \tau$, and let $j$ be the smallest integer such that $\sigma(j) \neq \tau(j)$. Then $\tau \sigma^{-1}(j) > j$ and there is some $i > j$ such that $\tau \sigma^{-1}(i) = j$. This implies that $\lambda_i - i < \lambda_j - j \leq \mu_{\tau \sigma^{-1}(j)} - \tau \sigma^{-1}(j) < \mu_j - j = \mu_{\tau \sigma^{-1}(i)} - \tau \sigma^{-1}(i)$, which implies that $\lambda_i - i - (\mu_{\tau \sigma^{-1}(i)} - \tau \sigma^{-1}(i)) \leq -2$, which is a contradiction. Therefore $\text{Con}'(\sigma, \tau) = 1$ exactly when $\tau = \sigma$, and the result follows.

For the $\alpha^-$ case, it is almost identical to the $\alpha^+$ case.

Now move on to the $\beta^-$ case. Since

$$e_k(z^{-1}) = z_1^{-1} \ldots z_n^{-1} e_{n-k}(z),$$

the contribution is now

$$\text{Con}''(\sigma, \tau) = \begin{cases} \text{sgn}(\sigma) \text{sgn}(\tau), & \text{if each } \lambda_j - j - (\mu_{\tau \sigma^{-1}(j)} - \tau \sigma^{-1}(j)) \in \{0, 1\} \\ 0, & \text{else} \end{cases}$$
and
\[ Q_n(\kappa_{\beta^-})(\lambda, \mu) = \frac{\dim \mu}{\dim \lambda} \frac{1}{\dim \mu!} \sum_{\sigma \tau} \text{Con}''(\sigma, \tau). \]

From here, the proof is essentially identical to the $$\beta^+$$ case, except with negative signs inserted and inequalities reversed. \(\square\)

Lemma 4.4. If Theorem 4.2 holds for two functions $$F_1$$ and $$F_2$$, then it holds for $$F_1 F_2$$.

Proof. This follows from Proposition 4.1. \(\square\)

Lemma 4.5. If Theorem 4.2 holds for a sequence of functions $$F_k$$ which converge uniformly to a function $$F$$ on $$A$$, then the theorem also holds for $$F$$.

Proof. With $$f_k$$ defined as in (5), it is immediate that $$f_k$$ converges uniformly. Since the determinant is a continuous function of its entries, $$T_n(1; F_k)(\lambda, \mu)$$ converges to $$T_n(1; F)(\lambda, \mu)$$. By (11), $$Q_n(\kappa F_k)(\lambda, \mu)$$ converges to $$Q_n(\kappa F)(\lambda, \mu)$$ as well. \(\square\)

Finally, since $$e^x = \lim_{k \to \infty} (1 + x/k)^k = \lim_{k \to \infty} (1 - x/k)^{-k}$$, Lemmas 4.3, 4.4 and 4.5 finish the proof of Theorem 4.2. \(\square\)

Let us also prove a statement similar to Theorem 3.1.

Proposition 4.6. For $$\kappa \in L^2(G, \mathbb{C})^G$$,
\[ \sum_{\mu \in \mathcal{W}_n} Q_n(\kappa)(0, \mu) c_{\mu} \frac{\dim \tau}{\dim \lambda \dim \mu} = Q_n(\kappa)(\lambda, \tau). \]

Proof. By linearity, it suffices to prove the result when $$\kappa = \chi_\beta$$. By (10),
\[ \sum_{\mu \in \mathcal{W}_n} Q_n(\chi_\beta)(0, \mu) c_{\mu} \frac{\dim \tau}{\dim \lambda \dim \mu} = \sum_{\mu \in \mathcal{W}_n} \frac{\dim \tau}{\dim \lambda} c_{\mu} \int_{U(n)} \chi_{\mu}(g) \chi_\beta(g) dg = \frac{\dim \tau}{\dim \lambda} c_{\chi_\beta}. \]

On the other side,
\[ Q_n(\chi_\beta)(\lambda, \tau) = \frac{\dim \tau}{\dim \lambda} \int_{U(n)} \chi_\lambda(g) \overline{\chi_\tau(g)} \chi_\beta(g) dg = \frac{\dim \tau}{\dim \lambda} \int_{U(n)} \overline{\chi_\tau(g)} \sum_{\mu \in \mathcal{W}_n} c_{\mu}^{\lambda} \chi_\mu(g) dg = \frac{\dim \tau}{\dim \lambda} c_{\chi_\beta}. \]
\(\square\)
4.2 Restriction to Maximal Torus

The purpose of this subsection is to demonstrate that there is a natural representation theoretic reason for the occurrence of tensor products in the transition probabilities. To see this, we will consider the restriction of the quantum random walk to the von Neumann algebra of the maximal torus. This is a natural restriction to consider: in [12], it is shown that the Krawtchouk and Charlier processes and Doob h–transforms of Bernoulli and exponential random walks; while in [2], it is shown that representations whose highest weight is miniscule, the restriction of the quantum random walk to the center is the Doob h–transform of the restriction to the maximal torus.

Let $T$ be the subalgebra of $vN(G)$ generated by $\{\alpha(\theta) : \theta \in \mathbb{T}^n\}$. Since every element of $G$ is conjugate to exactly one element of $T^n$, we can decompose the Haar measure on $G$ as a measure on $T^n \times T^n \setminus G$. Thus $L^2(G, dg) \cong L^2(T^n, d\theta) \otimes L^2(T^n \setminus G)$, where $d\theta$ is Haar measure on $T^n$. With this isomorphism, $\alpha(\theta)$ acts as the identity element on $L^2(T^n \setminus G)$.

Therefore $T$ is isomorphic to the group von Neumann algebra of $T^n$. Since the character group of $T^n$ is isomorphic to $P$, there is an isomorphism of $\ast$–algebras $\zeta : T \rightarrow L^\infty(P)$ such that $\zeta(\alpha(\theta))$ sends $x \in P$ to $e(x)(\theta)$. Since $Q$ sends $T$ to itself, the map $\zeta \circ Q \circ \zeta^{-1}$ defines a classical Markov chain with state space $P$. Identity $P$ with $Z^n$ naturally, and write $P_n(x, y)$ for the transition matrix of this Markov chain.

**Proposition 4.7.** 1. For any $\kappa \in L^2(G, \mathbb{C})^G$,

\[
P_n(\kappa)(x, y) = n_\kappa(y - x)
\]

Furthermore, for any $\sigma$ in the Weyl group, $P_n(\kappa)(x, y) = P_n(\kappa)(\sigma x, \sigma y)$.

2. The map $P_n : BC(G, \mathbb{C})^G \rightarrow Mat(P \times P)$ is a morphism of $\ast$–algebras.

**Proof.** 1. By Proposition 3.1 in [11],

\[
P_n(\kappa)(x, y) = \int_{\mathbb{T}^n} e(x)(\theta)e(y)(\theta)\kappa(\theta)d\theta,
\]

which implies that

\[
P_n(\kappa)(x, y) = \int_{\mathbb{T}^n} e(y - x)(\theta)\sum_{\lambda \in W_n} \hat{\kappa}(\lambda)\chi_\lambda(\theta)d\theta
\]

\[= \int_{\mathbb{T}^n} e(y - x)(\theta)\sum_{\lambda \in W_n} \hat{\kappa}(\lambda)\sum_{z \in P} n_\lambda(z) \cdot e(z)(\theta)d\theta
\]

\[= \int_{\mathbb{T}^n} e(y - x)(\theta) \cdot e(y - x)(\theta)\sum_{\lambda \in W_n} \hat{\kappa}(\lambda)n_\lambda(y - x)d\theta
\]

\[= \sum_{\lambda \in W_n} \hat{\kappa}(\lambda)n_\lambda(y - x) = n_\kappa(y - x).
\]

Furthermore, since the weight multiplicities are invariant under the action of the Weyl group, it follows that the transition probabilities are invariant under the Weyl group.
2. The fact that $P_\kappa$ is linear and preserves $^*$ follows from (16). Since $\sum_{z \in P} e(z)(\theta)e(z)(\theta')$ is the Dirac delta $\delta_{\theta\theta'-1}$, it follows that from (16) multiplication is also preserved. This can also be seen from the construction of the quantum random walk, as in the proof of Proposition 4.1.2.

There is also a proof which illuminates the occurrence of tensor products. To show that $P_\kappa$ preserves multiplication, by (15) it suffices to show that the map $n : BC(G, \mathbb{C})^G \rightarrow B(P, \mathbb{C})$ defined by $n(\kappa) = n_\kappa$ from bounded, continuous complex–valued class functions on $G$ to bounded complex–valued functions on $P$ is a morphism of $^*$–algebras, where the multiplication in $B(P, \mathbb{C})$ is usual convolution. By definition, $n$ is linear, so it suffices to show that

$$n_{\chi_\lambda \chi_\mu} = n_{\chi_\lambda} \ast n_{\chi_\mu}.$$

Letting $W(\pi)$ denote the multiset of weight multiplicities (i.e. the number of times that $x \in P$ appears in $W(\pi)$ is $n_{\chi_\pi}(x)$, which is the multiplicity of the weight $x$ in the representation $V_\pi$), this is equivalent to

$$W(\pi_1 \otimes \pi_2) = W(\pi_1) + W(\pi_2),$$

where $A + B$ denotes the usual addition of multisets, $A + B = \{a + b : a \in A, b \in B\}$. However, by [4], this follows immediately.

References

[1] Biane P. Quantum random walk on the dual of $SU(n)$. Probab. Th. Rel. Fields 89, 117–129 (1991).
[2] Biane P. Minuscule weights and random walks on lattices. Quantum Probability and Related Topics Vol. VII, 51–65 (1992).
[3] Borodin, A.; Ferrari, P.L. Anisotropic growth of random surfaces in $d+1$ dimensions. J. Stat. Mech. (2009) P02009. arXiv:0804.3035v1
[4] Borodin, A.; Ferrari, P.L. Large time asymptotics of growth models on space-like paths I: PushASEP. Elec. J. Prob. Volume 13, Number 50 (2008), 1380–1418. arXiv:0707.2813
[5] Borodin, A; Corwin, I. Macdonald processes arXiv:1111.4408
[6] Borodin, A.; Kuan, J. Asymptotics of Plancherel measures for the infinite-dimensional unitary group. Adv. Math. 219 (2008), 894–931. arXiv:0712.1848v1
[7] Corwn, I. The Kardar-Parisi-Zhang equation and universality class. arXiv:1106.1596
[8] Defosseux, M. An interacting particle model and a Pieri-type formula for the orthogonal group. J. Theor. Probab, Feb 2012. arXiv:1012.0117v1
[9] Defosseux, M. Interacting particle models and the Pieri-type formulas : the symplectic case with non equal weights. http://arxiv.org/abs/1104.4457
[10] Dixmier, J. Les $C^*$-algèbres et leurs representations. Paris: Gauthier-Villars 1964
[11] Fulton, W.; Harris, J. *Representation theory: a first course*. Graduate Texts in Mathematics, Vol. 129. Springer, New York, 1991.

[12] König, W; O’Connell, N; Roch, S; *Non–colliding random walks, tandem queues and discrete orthogonal polynomial ensembles*, Elec. J. Prob, Volume 7, Number 1 (2002), 1–24.

[13] Kuan, J. *Discrete-time particle system with a wall and representations of O(infinity)*. [arXiv:1203.1660v1](arXiv:1203.1660v1)

[14] Warren, J; Windridge, P. *Some Examples of Dynamics for Gelfand–Tsetlin Patterns*. Elec. J. Prob, Volume 14, Number 59 (2009), 1745–1769. [arXiv:0812.0022](arXiv:0812.0022)