Lower bound for ground state energy of BEC in a rotating optical lattice

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We use the frustrated XY model approximation of BEC in a rotating optical lattice and formulate the problem of the ground state in terms of eigenvectors and eigenvalues of frustrated adjacency matrix (coupling matrix). By using this formulation, we show that there is a lower bound for ground state energy in terms of maximum eigenvalue of this matrix.

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I. INTRODUCTION

Effect of the rotation of optical lattice on physical properties of BEC is an interesting problem. This effect can be observed as vortex in lattice and in this way is related to many problems in condensed matter physics, e.g. Josephson junctions arrays.

It can be shown that with appropriate approximations, the problem of the ground state of BEC in a rotating optical lattice can be formulated as frustrated XY model. In this formulation, rotation plays the role of frustration parameter.

Here, we focus on ground state of BEC in a rotating optical lattice in this approximation and by use of properties of frustrated XY model, we find a lower bound for ground state energy for different optical lattices.

II. BEC IN ROTATING OPTICAL LATTICE: FRUSTRATED XY MODEL

It can be shown that in appropriate approximation, Hamiltonian for BEC in a rotating optical lattice can be reduce to frustrated XY Hamiltonian,

\[ H = - \sum_{<i,j>} J_{ij} \cos(\theta_i - \theta_j + A_{ij}) \] (2.1)

Suppose that \( n \) is number of atoms in \( i \)th site and \( \theta_i \) is the phase of localized wave function on \( i \)th site. \( J_{ij} \) is proportional to \( \sqrt{n _i n_j} \) and \( A_{ij} \) is line integral of \( A = \Omega z \times r \). If we define frustration parameter \( f \) as \( f = 2\Omega m /h \), where \( m \) is mass of condensate and \( h \) is plank constant, then we can define \( 2\pi B_{ij} f = A_{ij} \) for nonzero \( f \) and

\[ H = - \sum_{<i,j>} J_{ij} \cos(\theta_i - \theta_j + 2\pi B_{ij} f) \] (2.2)

Hamiltonian can be written as

\[ H = - \sum_{<i,j>} J_{ij} \exp(i(\theta_i - \theta_j + 2\pi B_{ij} f)) \] (2.3)

where \( Re\{x\} \) is real part. By introducing frustrated adjacency matrix, \( S_{ij} = J_{ij} \exp(i2\pi B_{ij} f) \), we have,

\[ H = -N/2Re\{\xi^* S \xi \} \] (2.4)

where \( N \) is number of sites and \( \xi_i = \exp(-i\theta_i)/\sqrt{N} \). The factor 2 enters because we consider both \( (i,j) \) and \( (j,i) \) terms in this case.

Now we focus on formulating of the problem in terms of eigenvalues and eigenvectors of \( S \). Suppose that \( \eta^k \) is eigenvector correspond to eigenvalue \( \lambda^k \) of matrix \( S \), i.e. \( S \eta^k = \lambda^k \eta^k \). Also we suppose that set of eigenvectors of \( S \) are complete and orthogonal. Then we can write following expansion for \( \xi \),

\[ \xi = \sum_k c^k \eta^k \] (2.5)

Using this expansion in Hamiltonian

\[ H = -N/2Re\{\sum_k c^k c^* \lambda_k \} \] (2.6)

But there is some restriction on \( c \) coming from properties of \( \xi \). Because of \( \xi \xi^* = 1 \), we have \( \sum_k c^k c^* = 1 \) and therefore \( |c^k| \leq 1 \) for any \( k \). Also from definition of \( \xi_i \) we have,

\[ \xi_i \xi_i^* = \frac{1}{N} \] (2.7)

which leads to,

\[ \sum_{m,k} c^m c^* \eta_i^m \eta_i^* = \frac{1}{N} \] (2.8)

The problem of minimization of Hamiltonian is equivalent to minimization of Hamiltonian with respect to complex numbers \( c_i \)'s and with constrains.

III. LOWER BOUND FOR GROUND STATE ENERGY

We focus on minimization problem defined by equations and restrictions.

We consider simple situation, a plaquette with unit length and uniform \( J_{ij} \), which can be shown that frustrated XY model for it can be solved exactly. In this case, \( S \) is given as, \( S_{12} = 1, S_{23} = 1, S_{34} = \exp(-2\pi i f), S_{41} = 1, \) and \( S_{ji} = S_{ij}^* \) (which is property of \( S \)). Numerical results shows that in this case, \( E/N = -\lambda_{max}/2 \).
From the general Hamiltonian \(2.6\) and the constrains on \(c^i\)'s \(|c^i| \leq 1\), we can see that,

\[
\frac{H}{N} \geq -\frac{\lambda_{\text{max}}}{2} \quad (3.1)
\]

Therefore the minimum of Hamiltonian also satisfies this inequality. It means that

\[
\frac{E}{N} \geq -\frac{\lambda_{\text{max}}}{2} \quad (3.2)
\]

IV. CONCLUSION

We have shown that ground state energy of a BEC in an rotating optical lattice in frustrated XY approximation, has a lower bound which determined by coupling matrix of system. Formulation of the problem in terms of properties of coupling matrix, maybe can be used for more analytical results especially relation between XY models with different coupling matrix.

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