Factorization in quantum field theory: an exercise in Hopf algebras and local singularities

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I discuss the role of Hochschild cohomology in Quantum Field Theory with particular emphasis on Dyson–Schwinger equations. Talk given at Frontiers in Number Theory, Physics and Geometry; Les Houches, March 2003 [arXiv:hep-th/0306020].

1 Introduction

This paper provides a designated introduction to the Hopf algebra approach to renormalization having a specific goal in mind: to connect this approach in perturbative quantum field theory with non-perturbative aspects, in particular with Dyson–Schwinger equations (DSEs) and with the renormalization group (RG), with particular emphasis given to a proof of renormalizability based on the Hochschild cohomology of the Hopf algebra behind a perturbative expansion.

To achieve this goal we will consider a Hopf algebra of decorated rooted trees. In parallel work, we will treat the Feynman graph algebras of quantum electrodynamics, non-abelian gauge theories and the full Standard Model along similar lines [2].

There are various reasons for starting with decorated rooted trees. One is that Hopf algebra structures of such rooted trees play a prominent role also in the study of polylogarithms [3, 4, 5, 6] and quite generally in the analytic study of functions which appear in high-energy physics [7, 8]. Furthermore, the Hopf algebras of graphs and decorated rooted trees are intimately related. Indeed, resolving overlapping divergences into non-overlapping sectors furnishes a homomorphism from the Feynman graph Hopf algebras to Hopf algebras of decorated rooted trees [9, 10, 11, 12]. Thus the study of decorated rooted trees is by no means a severe restriction of the problem, but allows for
the introduction of simplified models which still capture the crucial features of the renormalization problem in a pedagogical manner.\footnote{Furthermore, the structure of the Dyson–Schwinger equations in gauge theories eliminates overlapping divergences altogether upon use of gauge invariance \cite{13, 14}.}

In particular we are interested to understand how the structure maps of a Hopf algebra allow to illuminate the structure of quantum field theory. We will first review the transition from unrenormalized to renormalized amplitudes \cite{9, 15, 16, 19, 11, 20} and investigate how the Hochschild cohomology of the Hopf algebra of a perturbative expansion directly leads to a renormalization proof.

We then study Dyson–Schwinger equations for rooted trees and show again how the Hochschild cohomology explains the form invariance of these equations under the transition from the unrenormalized to the renormalized equations. For the Hopf algebras apparent in a perturbative expansion, this transition is equivalent to the transition from the action to the bare action, as the study of Dyson–Schwinger equations is equivalent to the study of the corresponding generating functionals \cite{21, 22}.

We then show how the structure of these equations leads to a combinatorial factorization into primitives of the Hopf algebra. While this is easy to achieve for the examples studied here, it is subtly related to the Ward–Takahashi and Slavnov–Taylor identities in the case of abelian and non-abelian quantum gauge field theories. Here is not the space to provide a detailed discussion of factorization in these theories, but at the end of the paper we comment on recent results of \cite{2} concerning the relation between factorization and gauge symmetry. Indeed, the combinatorial factorization establishes a commutative associative product $\vee$ from one-particle irreducible (1PI) graphs to 1PI graphs in the Hopf algebra of 1PI graphs. In general, this product is non-integral \cite{17}:

\begin{equation}
\Gamma_1 \vee \Gamma_2 \neq 0 \not\Rightarrow \Gamma_1 = 0 \text{ or } \Gamma_2 = 0,
\end{equation}

but the failure can be attributed to the one-loop graphs generated from a single closed fermion loop with a suitable number of external background gauge fields coupled. This fermion determinant is the starting point in \cite{2} for an understanding of gauge symmetries based on an investigation of the structure of the ring of graph insertions.

Having a commutative ring at hand of 1PI graphs, or, here, of decorated rooted trees, we can ask how the evaluation of a product of 1PI graphs, or trees, compares with the product of the evaluations. In answering this question, it seems to me, serious progress can be made in our understanding of field theory. Indeed, the integrals which appear in Dyson–Schwinger equations or in the perturbative expansion of field theory are of a distinguished kind: they provide a class of functions which is self-similar under the required integrations. The asymptotics of the integral can be predicted from the asymptotics
of the integrand, as already stressed by previous authors [23]. It is this self-similarity which makes the Dyson–Schwinger equations consistent with the renormalization group. Again, a detailed study has to be given elsewhere but a few comments are scattered in the present paper.

We will now outline this program in some detail, and then first turn to a rich class of toy models to exhibit many of the involved concepts. This serves as a training ground for our ideas. As announced, these toy models are based on a Hopf algebra of decorated rooted trees, with only symbolically specified decorations. We provide toy Feynman rules which suffer from short distance singularities. Each genuine quantum field theory is distinguished from this toy case by the mere fact that the calculation of the decorations is analytically harder than what confronts the reader later on. Any perturbative quantum field theory (pQFT) provides a Hopf algebra structure isomorphic to the models below, for a suitably defined set of decorations, through its skeleton graphs.

Unfortunately, the calculation of higher loop order skeletons is beyond the present analytical skill. Most fascinatingly though, up to six loops, they provide multiple zeta values galore [24], a main subject of our school [7, 8, 6, 4]. At higher loops, they might even provide periods outside this class, an open research question in its own right [25].

Nevertheless, there is still much to be learned about how the underlying skeleton diagrams combine in quantum field theory. Ultimately, we claim that the Hopf- and Lie algebra structures of 1PI graphs are sufficiently strong to reduce quantum field theory to a purely analytical challenge: the explanation of relations between two-particle irreducible (2PI) graphs which will necessitate the considerations of higher Legendre transforms. This is not the purpose of the present paper, but a clear task for the future: while the renormalization problem of 1PI graphs is captured by the algebraic structures of 1PI graphs, the analytic challenge is not: Rosner’s cancellation of transcendentals in the $\beta$ function of quenched QED [26, 14], Cvitanovic’s observation of hints towards non-combinatorial growth of perturbative QED [27] and the observation of (modified) four term relations between graphs [28] all establish relations between 2PI skeleton graphs which are primitives in the Hopf algebra of 1PI graphs. In this sense, the considerations started in this paper aim to emphasize where the true problem of QFT lies: in the understanding of the analytic relations between renormalization primitive graphs. The factorizations into Hopf algebra primitives of the perturbation expansion studied here generalizes the shuffle identity on generalized polylogarithms, which comes, for the latter, from studying the very simple integral representations as iterated integrals. A second source of relations comes from studying the sum representations. The corresponding relations between Feynman diagrams have not yet been found, but the above quoted results are, to my mind, a strong hint towards their existence. Alas, the lack of understanding of these relations is the major conceptual challenge which stops us from understanding QFT in four dimensions. All else is taken care of by the algebraic structures of 1PI graphs.
The Hopf algebra of decorated rooted trees is an adequate training ground for QFT, where the focus is on the understanding of the renormalization problem and the factorization of 1PI graphs into graphs which are primitive with respect to the Hopf algebra coproduct.

Hence the program which we want to carry out in the following consists of a series of steps which can be set up in any QFT, while in this paper we will utilize the fact that they can be set up in a much wider context. When one considers DSE, one usually obtains them as the quantum equations of motion of some Lagrangian field theory using some generating functional technology in the path integral. Now the DSEs for 1PI Green functions can all be written in the form

\[ \Gamma_\mu = 1 + \sum_{\gamma \in H^{[1]}_\mu} \frac{\alpha |\gamma|}{\text{Sym}(\gamma)} B_+^{\gamma}(X^\gamma_R), \]

where the \( B_+^{\gamma} \) are Hochschild closed one-cocycles of the Hopf algebra of Feynman graphs indexed by Hopf algebra primitives \( \gamma \) with external legs \( \mu \), and \( X^\gamma_R \) is a monomial in superficially divergent Green functions which dress the internal vertices and edges of \( \gamma \). We quote this result from [2] to which we refer the reader for details. It allows to obtain the quantum equations of motion, the DSEs for 1PI Green functions, without any reference to actions, Lagrangians or path integrals, but merely from the representation theory of the Poincaré group for free fields.

Motivated by this fact we will from now on call any equation of the form

\[ X = 1 + \alpha B_+(X^k), \]

with \( B_+ \) a closed Hochschild one-cocycle, a combinatorial Dyson–Schwinger equation.

Thus in this paper we choose as a first Hopf algebra to study the one of decorated rooted trees, without specifying a particular QFT. The decorations play the role of the skeleton diagrams \( \gamma \) above, indexing the set of closed Hochschild one-cocycles and the primitives of the Hopf algebra.

In general, this motivates an approach to quantum field theory which is utterly based on the Hopf and Lie algebra structures of graphs. Let us discuss the steps which we would have to follow in such an approach.

### 1.1 Determination of \( H \)

The first step aims at finding the Hopf algebra suitable for the description of a chosen QFT. For such a QFT consider the set of Feynman graphs corresponding to its perturbative expansion close to its free Gaussian functional integral. Identify the one-particle irreducible (1PI) diagrams. Identify all vertices and propagators in them and define a pre-Lie product on 1PI graphs by using the possibility to replace a local vertex by a vertex correction graph, or, for internal edges, by replacing a free propagator by a self-energy. For any local
QFT this defines a pre-Lie algebra of graph insertions [12]. For a renormalizable theory, the corresponding Lie algebra will be non-trivial for only a finite number of types of 1PI graphs (self-energies, vertex-corrections) corresponding to the superficially divergent graphs, while the superficially convergent ones provide a semi-direct product with a trivial abelian factor [19].

The combinatorial graded pre-Lie algebra so obtained [12] provides not only a Lie-algebra \( L \), but a commutative graded Hopf algebra \( H \) as the dual of its universal enveloping algebra \( U(L) \), which is not cocommutative if \( L \) was non-abelian. Dually one hence obtains a commutative but non-cocommutative Hopf algebra \( H \) which underlies the forest formula of renormalization [9, 10, 15, 19].

1.2 Character of \( H \)

For a so-determined Hopf algebra \( H = H(m, E, \bar{e}, \Delta, S) \), a Hopf algebra with multiplication \( m \), unit \( e \) with unit map \( E : \mathbb{Q} \to H \), \( q \to qe \), with counit \( \bar{e} \), coproduct \( \Delta \) and antipode \( S \), \( S^2 = e \), we immediately have at our disposal the group of characters \( G = G(H) \) which are multiplicative maps from \( H \) to some target ring \( V \). This group contains a distinguished element: the Feynman rules \( \varphi \) are indeed a very special character in \( G \). They will typically suffer from short-distance singularities, and the character \( \varphi \) will correspondingly reflect these singularities. This can happen in various ways depending on the chosen target space \( V \). We will here typically take \( V \) to be the ring of Laurent polynomials in some indeterminate \( z \) with poles of finite orders for each finite Hopf algebra element, and design Feynman rules so as to reproduce all salient features of QFT.

As \( \varphi : H \to V \), with \( V \) a ring, with multiplication \( m_V \), we can introduce the group law
\[
\varphi \ast \psi = m_V \circ (\varphi \otimes \psi) \circ \Delta,
\]
and use it to define a new character
\[
S^\phi_R \ast \varphi \in G,
\]
where \( S^\phi_R \in G \) twists \( \varphi \circ S \) and furnishes the counterterm of \( \varphi(\Gamma) \), \( \forall \Gamma \in H \), while \( S^\phi_R \ast \varphi(\Gamma) \) corresponds to the renormalized contribution of \( \Gamma \) [9, 15, 10, 19]. \( S^\phi_R \) depends on the Feynman rules \( \varphi : H \to V \) and the chosen renormalization scheme \( R : V \to V \). It is given by
\[
S^\phi_R = -R \left[ m_V \circ (S^\phi_R \otimes \varphi) \circ (\text{id}_H \otimes P) \circ \Delta \right],
\]
where \( R \) is supposed to be a Rota-Baxter operator in \( V \), and the projector into the augmentation ideal \( P : H \to H \) is given by \( P = \text{id} - E \circ \bar{e} \).

The \( \bar{R} \) operation of Bogoliubov is then given by
\[
\bar{\phi} := \left[ m_V \circ (S^\phi_R \otimes \varphi) \circ (\text{id}_H \otimes P) \circ \Delta \right],
\]
and
\[ S_R^φ \ast φ \equiv m_V \circ (S_R^φ \otimes φ) \circ Δ = \tilde{φ} + S_R^φ = (id_H - R)(\tilde{φ}) \] (8)
is the renormalized contribution. Note that this second step has been established for all perturbative quantum field theories combining the results of [9, 10, 11, 12, 19]. These papers are rather abstract and will be complemented by explicit formulas for the practitioner of gauge theories in forthcoming work.

1.3 Locality from \( H \)

The third step aims to show that locality of counterterms is utterly determined by the Hochschild cohomology of Hopf algebras. Again, we can dispense of the existence of an underlying Lagrangian and derive this crucial feature from the Hochschild cohomology of \( H \). This cohomology is universally described in [19], see also [18]. What we are considering are spaces \( \mathcal{H}^{(n)} \) of maps from the Hopf algebra into its own \( n \)-fold tensor product,
\[ \mathcal{H}^{(n)} \ni \psi \Leftrightarrow \psi : H \to H^{\otimes n} \] (9)
and an operator
\[ b : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)} \] (10)
which squares to zero: \( b^2 = 0 \). We have for \( \psi \in \mathcal{H}^{(1)} \)
\[ (b\psi)(a) = \psi(a) \otimes e - Δ(\psi(a)) + (id_H \otimes \psi)Δ(a) \] (11)
and in general
\[ (b\psi)(a) = (-1)^{n+1}\psi(a) \otimes e + \sum_{j=1}^{n} (-1)^j Δ(j)(\psi(a)) + (id \otimes \psi)Δ(a), \] (12)
where \( Δ(j) : H^{\otimes n} \to H^{\otimes n+1} \) applies the coproduct in the \( j \)-th slot of \( \psi(a) \in H^{\otimes n} \).

For all the Hopf algebras considered here and in future work on QFT, the Hochschild cohomology is rather simple: it is trivial in degree \( n > 1 \), so that the only non-trivial elements in the cohomology are the maps from \( H \to H \) which fulfil the above equation and are non-exact. In QFT these maps are given by maps \( B_+^ω \), indexed by primitive graphs \( γ \), an easy consequence of [19] [18] extensively used in [2].

Locality of counterterms and finiteness of renormalized quantities follow from the Hochschild properties of \( H \): the Feynman graph is in the image of a closed Hochschild one cocycle \( B_+^γ, b B_+^γ = 0 \), i.e.
\[ Δ \circ B_+^γ(X) = B_+^γ(X) \otimes e + (id \otimes B_+^γ) \circ Δ(X), \] (13)
and this equation suffices to prove the above properties by a recursion over the augmentation degree of \( H \). This is a new result: it is the underlying Hochschild
cohomology of the Hopf algebra $H$ of the perturbative expansion which allows to provide renormalization by local counterterms. The general case is studied in [2], but we will study this result in detail for rooted tree algebras below. This result is again valid due to the benign properties of Feynman integrals: we urgently need Weinberg’s asymptotic theorem which ensures that an integrand, overall convergent by powercounting and free of subdivergences, can actually be integrated [23].

1.4 Combinatorial DSEs from Hochschild cohomology

Having understood the mechanism which achieves locality step by step in the perturbative expansion, one can ask for more: how does this mechanism fare in the quantum equations of motion? So we next turn to the Dyson–Schwinger equations.

As mentioned before, they typically are of the form

\[ \Gamma^n = 1 + \sum_{\gamma \in H^{[1]}_{\text{res}(\gamma) = n}} \frac{\alpha |\gamma|}{\text{Sym}(\gamma)} B^\gamma_\text{res}(X^n_R) = 1 + \sum_{\Gamma \in H^{[1]}_{\text{res}(\Gamma) = n}} \frac{\alpha |\Gamma|}{\text{Sym}(\Gamma)} \Gamma^n , \]

where the first sum is over a finite (or countable) set of Hopf algebra primitives $\gamma$,

\[ \Delta(\gamma) = \gamma \otimes e + e \otimes \gamma, \]

indexing the closed Hochschild one-cocycles $B^\gamma$ above, while the second sum is over all one-particle irreducible graphs contributing to the desired Green function, all weighted by their symmetry factors. The equality is non-trivial and needs proof [2]. Here, $\Gamma^n$ is to be regarded as a formal series

\[ \Gamma^n = 1 + \sum_{k \geq 1} c^n_k \alpha^k, c^n_k \in H. \]

Typically, this is all summarized in graphical form as in Fig.(1), which gives the DSE for the unrenormalized Green functions of massless QED as an example (restricting ourselves to the set of superficially divergent Green functions, ie. $n \in R_{\text{QED}} \equiv \{ \rightarrow, \rightarrow \} $). In our terminology this QED system reads for renormalized functions:

\[ \Gamma^n_{\text{R}} = Z \rightarrow + \sum_{\gamma \in H^{[1]}_{\text{res}(\gamma) = n}} \frac{\alpha |\gamma|}{\text{Sym}(\gamma)} B^\gamma \left( [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} / [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} / [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} \right) \]

\[ \Gamma^n_{\text{R}} = Z \rightarrow + \sum_{\gamma \in H^{[1]}_{\text{res}(\gamma) = n}} \frac{\alpha |\gamma|}{\text{Sym}(\gamma)} B^\gamma \left( [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} / [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} / [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} \right) \]

\[ \Gamma^n_{\text{R}} = Z \rightarrow + \sum_{\gamma \in H^{[1]}_{\text{res}(\gamma) = n}} \frac{\alpha |\gamma|}{\text{Sym}(\gamma)} B^\gamma \left( [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} / [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} / [\Gamma^n_{\text{R}} \rightarrow \gamma]_{\rightarrow} \right) \]
where the integers
\[ n_\gamma, \quad n_{\gamma}, \quad n_{\gamma, \text{corr.}} \]
count the numbers of internal vertices, fermion lines and photon lines in \( \gamma \), and the \( B_\gamma^+ \) operator inserts the corresponding Green functions into \( \gamma \), corresponding to the blobs in figure 1. The unrenormalized equations are obtained by omitting the subscript \( R \) at \( \Gamma_{R}^{\pm} \) and setting \( Z_{R} \) to unity. The usual integral equations are obtained by evaluation both sides of the system by the Feynman rules.\(^2\) The form invariance in the transition from the unrenormalized to the renormalized Green functions directly follows from the fact that the equation for the series \( \Gamma_{R}^{\pm} \) is in its non-trivial part in the image of closed Hochschild one-cocycles \( B_\gamma^+ \). It is this fact which ensures that a local Z-factor is sufficient to render the theory finite. The fact that the rhs of a DSE is Hochschild closed ensures the form invariance of the quantum equations of motion in the transition from the unrenormalized to the renormalized Green functions, as indeed in the renormalized system the Hochschild closed one-cocycle acts only on renormalized functions. We will exemplify this for rooted trees below.

1.5 Factorization

Such systems of DSEs can be factorized. The factorization is based on a commutative associative product on one-particle irreducible (1PI) graphs in the Hopf algebra, which maps 1PI graphs to 1PI graphs. We will do this below with considerable ease for the corresponding product on rooted trees. For Feynman graphs, one confronts the problem that the product can be non-integral \(\text{[17]}\).

\(^2\) The system is redundant, as we made no use of the Ward identity. Also, the unrenormalized system is normalized so that the rhs starts with unity, implying an expansion of inverse propagators in the external momentum up to their superficial degree of divergence. This creates their skeleton diagrams \(\text{[16]}\).
A detailed discussion of the relation of this failure to the requirements of identities between Green functions in the case of gauge theories is given in [2], a short discussion appended at the end of this paper.

1.6 Analytic factorization and the RG

In the final step, we pose the question: how relates the evaluation of the product to a product of the evaluations?

That can be carried out in earnest only in the context of a true QFT [2] - there is no exact RG equation available for our toy model of decorated rooted trees. So we will not carry out this step here, but only include in the discussion at the end of the paper an argument why a RG equation is needed for this step.

2 Locality and Hochschild cohomology

The first result we want to exhibit in some detail is the close connection between the Hochschild cohomology of a Hopf algebra and the possibility to obtain local counterterms.

We will first study the familiar Hopf algebra of non-planar decorated rooted trees. We will invent toy Feynman rules for it such that we have a non-trivial renormalization problem. Then we will show how the structure maps of the Hopf algebra precisely allow to construct local counterterms and finite renormalized amplitudes thanks to the fact that each non-trivial Hopf algebra element is in the image of a closed Hochschild one-cocycle.

2.1 The Hopf algebra of decorated rooted trees

To study the connection between renormalization and Hochschild cohomology in a most comprehensive manner we thus introduce the Hopf algebra of decorated rooted trees. Let Dec be a (countable) set of decorations, and \( H = H(\text{Dec}) \) be the Hopf algebra of decorated rooted trees (non-planar). For any such tree \( T \) we let \( T^{[0]} \) be the set of its vertices and \( T^{[1]} \) be the set of its edges. To each vertex \( v \in T^{[0]} \) there is assigned a decoration \( \text{dec}(v) \in \text{Dec} \).

For \( T_1, T_2 \) in \( H \), we let their disjoint union be the product, we write \( e \) for the unit element in \( H \), and define the counit by

\[
\bar{e}(e) = 1, \quad \bar{e}(X) = 0 \text{ else.} \tag{17}
\]

We write \( P : H \to H, P = \text{id}_H - E \circ \bar{e} \) for the projection into the augmentation ideal of \( H \).

The coproduct is given by

\[
\Delta[e] = e \otimes e, \quad \Delta[T_1 \ldots T_k] = \Delta[T_1] \ldots \Delta[T_k], \tag{18}
\]
\[ \Delta[T] = T \otimes e + e \otimes T + \sum_{\text{adm cut}} P_C(T) \otimes R_C(T), \quad (19) \]

as in [19]. The introduction of decorations does not require any changes, apart from the fact that the operators \( B_+ \) are now indexed by the decorations. We have, as \( T = B_+^c(X) \), for some \( X \in H \) and \( c \in \text{Dec} \),

\[ \Delta[T] = T \otimes e + [\text{id} \otimes B_+^c] \Delta[X]. \quad (20) \]

The antipode is given by \( S(e) = e \) and

\[ S[B_+^c(X)] = -B_+^c(X) - m \circ [S \circ P \otimes B_+^c] \circ \Delta[X]. \quad (21) \]

A distinguished role is played by the primitive elements \( \bullet e, \forall c \in \text{Dec} \), with

\[ \Delta(\bullet e) = \bullet e \otimes e + e \otimes \bullet e. \quad (22) \]

Let now \( G \) be the group of characters of \( H \), \( \phi \in G \Leftrightarrow \phi : H \to V \), \( \phi(T_1 T_2) = \phi(T_1) \phi(T_2) \), with \( V \) a suitable ring. Feynman rules provide such characters for the Hopf algebras of QFT, and we will now provide a character for the Hopf algebra of rooted trees which mimicks the renormalization problem faithfully.

2.2 The toy Feynman rule

We choose \( V \) to be the ring of Laurent series with poles of finite order. To understand the mechanism of renormalization in an analytically simple case we define toy Feynman rules using dimensional regularization,

\[ \phi(B_+^c[X]) \left\{ \frac{q^2}{\mu^2}; z \right\} = \left[ \frac{\mu^2}{q^2} \right]^{x - \frac{D}{2}} \int f_c(|y|) \phi(X) \left\{ \frac{q^2}{\mu^2}; z \right\} \frac{y^2}{y^2 + q^2} [y^{2z} - y^{2z - (2 - D)/2}] d^D y, \quad (23) \]

for some functions \( f_c(y) \) which turn to a constant for \( |y| \to \infty \). In the following, we assume that \( f_c \) is simply a constant, in which case the above Feynman rules are elementary to compute. In the above, \( \phi(T) \) is a function of a dimensionless variable \( q^2/\mu^2 \) and the regularization parameter \( z = (2 - D)/2 \).

Furthermore

\[ \phi[X_1 X_2] = \phi(X_1) \phi(X_2) \quad (24) \]

as part of the definition and therefore \( \phi(e) = 1 \). Hence, indeed, \( \phi \in G \). We also provided for each decoration \( c \) an integer degree \( |c| \geq 1 \), which resembles the loop number of skeleton diagrams.

The sole purpose of this choice of \( \phi \in G \) for the Feynman rules is to provide a simple character which suffers from short-distance singularities in quite the same way as genuine Feynman diagrams do, without confronting the reader with overly hard analytic challenges at this moment. Note that, \( \forall c \in \text{Dec} \),
\[
\phi(\bullet c) = f_c \left[ \frac{q^2}{\mu^2} \right] \frac{\Gamma(1 + |c|z)}{|c|z} \pi^{D/2}, \tag{25}
\]

exhibiting the obvious pole at \( D = 2 \).

Using
\[
\int d^Dy \left[ \frac{[y^2]}{y^2 + q^2} \right]^{z-u} \frac{\Gamma(-u + D/2) \Gamma(1 + u - D/2)}{\Gamma(D/2)}, \tag{26}
\]
evaluations of decorated rooted trees are indeed elementary. The reader can convince himself that the degree of the highest order pole of \( \phi(T) \) equals the augmentation degree \( \text{aug}(T) \) of \( T \), which, for a single tree, is the number of vertices, see (35) below.

Having defined the character \( \phi \), we note that, for \( T = B_+^c(X) \)
\[
\phi \circ S[B_+^c(X)] = -\phi(T) - m \circ [\phi \circ S \otimes \phi \circ B_+^c] \circ \Delta[X]. \tag{27}
\]

We then twist \( \phi \circ S \in G \) to \( S_R \in G \) by
\[
S_R(T) := -R[\phi(T)] + m \circ (S_R^\phi \otimes \phi \circ B_+^c) \circ \Delta[X], \tag{28}
\]
where the renormalization scheme \( R : V \to V \) is a Rota–Baxter map and hence fulfills \( R[ab] + R[a]R[b] = R[R(a)b] + R[aR(b)] \), which suffices [15] to guarantee that \( S_R \in G \), as it guarantees that \( S_R^\phi \circ m_H = m_V \circ (S_R^\phi \otimes S_R^\phi) \).

Set
\[
G \ni \phi_R(T) \equiv S_R^\phi \ast \phi(T) \equiv m \circ (S_R^\phi \otimes \phi) \circ \Delta[T]. \tag{30}
\]
Furthermore, assume that \( R \) is chosen such that
\[
\lim_{z \to 0} (\phi(X) - R[\phi(X)]) \quad \text{exists} \quad \forall X \in H. \tag{31}
\]

2.3 Renormalizability and Hochschild Cohomology

We now can prove renormalization for the Hopf algebra \( H \) and the toy Feynman rules \( \phi \) in a manner which allows for a straightforward generalization to QFT.

**Theorem 1.** i) \( \lim_{z \to 0} \phi_R(T) \left\{ \frac{q^2}{\mu^2}; z \right\} \) exists and is a polynomial in \( \log \frac{q^2}{\mu^2} \) ("finiteness")

ii) \( \lim_{z \to 0} \frac{\partial}{\partial \log \frac{q^2}{\mu^2}} \phi_R(T) \left\{ \frac{q^2}{\mu^2}; z \right\} \) exists ("local counterterms").

To prove this theorem, we use that \( B_+^c \) is a Hochschild closed one-cocycle \( \forall c \in \text{Dec} \).

**Proof.** For us, Hochschild closedness just states [19] that
\[ \Delta \circ B^c_+ (X) = B^c_+ (X) \otimes e + (\text{id} \otimes B_+^c) \Delta (X) \Leftrightarrow b \cdot B^c_+ = 0 . \] (32)

We want to prove the theorem in a way which goes through unmodified in the context of genuine field theories. That essentially demands that we only use Hopf algebra properties which are true regardless of the chosen character representing the Feynman rules. To this end we introduce the augmentation degree. Let \( P \) be the projection into the augmentation ideal, as before.

Define, \( \forall k \geq 2 \),
\[
\mathcal{P}^k : H \to H \otimes \ldots \otimes H \quad \text{by} \quad [P \otimes \ldots \otimes P] \circ \Delta^{k-1}, \quad \mathcal{P}^1 := P, \quad \mathcal{P}^0 := \text{id} .
\] (34)

For every element \( X \) in \( H \), there exists a largest integer \( k \) such that \( \mathcal{P}^k(X) \neq 0 \), \( \mathcal{P}^{k+1}(X) = 0 \). We set
\[
\text{aug} [X] = k .
\] (35)

(This degree is called bidegree in [29].) We prove the theorem by induction over this augmentation degree. It suffices to prove it for trees \( T \in H_L \).

Start of the induction: \( \text{aug}(T) = 1 \).

Then, \( T = \bullet c \) for some \( c \) in \( \text{Dec} \). Indeed
\[
\mathcal{P}^1(\bullet c) \neq 0, \quad \mathcal{P}^2(T) = (P \otimes P) [\bullet c \otimes e + e \otimes \bullet c] = 0 .
\] (36)

\[ S^\phi_R(\bullet c) = -R[\phi(\bullet c)] , \] (37)

and
\[ S^\phi_R \ast \phi(\bullet c) = \phi(\bullet c) - R[\phi(\bullet c)] , \] (38)

which is finite by assumption [34]. Furthermore, \( \lim_{z \to 0} \partial / \partial \log(q^2 / \mu^2) \phi(\bullet c) \) exists \( \forall c \), so we obtain a start of the induction.

Induction: Now, assume that \( \forall T \) up to \( \text{aug}(T) = k \), we have that
\[
\lim_{z \to 0} \frac{\partial}{\partial \log \frac{q^2}{\mu^2}} \tilde{\phi}(T)
\] (39)

exists, and \( S^\phi_R \ast \phi(T) \) is a finite polynomial in \( \log \frac{q^2}{\mu^2} \) at \( z = 0 \). We want to prove the corresponding properties for \( T \) with \( \text{aug}(T) = k + 1 \).

So, consider \( T \) with \( \text{aug}(T) = k + 1 \). Necessarily (each \( T \) is in the image of some \( B^c_+ ) \), \( T = B^c_+(X) \) for some \( c \) in \( \text{Dec} \) and \( X \in H \). Then, from
\[
\Delta \circ B^c_+(X) = B^c_+(X) \otimes e + (\text{id} \otimes B^c_+) \Delta [X] , \] (40)

indeed the very fact that \( B^c_+ \) is Hochschild closed, we get
\[
S^\phi_R(\bullet c[X]) \left\{ \frac{q^2}{\mu^2}; z \right\} = -R \left[ \int \frac{(y^2 - (\frac{q}{\mu})^2) d^D y}{[\mu^2 - \frac{q^2}{\mu^2}]} \frac{f_c}{y^2 + q^2} \phi(X) \left\{ \frac{y^2}{\mu^2}; z \right\} \right]
\]
\[
+ \sum \int \frac{(y^2 - (\frac{q}{\mu})^2) d^D y}{[\mu^2 - \frac{q^2}{\mu^2}]} \frac{f_c}{y^2 + q^2} S^\phi_R(X') \phi(X'') \left\{ \frac{y^2}{\mu^2}; z \right\} , \] (41)
where we abbreviated $\Delta[X] = \sum X' \otimes X''$, and the above can be written, using the definition (28) of $S^\phi_R$, as

$$S^\phi_R \left( B_+^c [X] \right) \left\{ \frac{q^2}{\mu^2} ; z \right\} = -R \left[ \int \frac{(y^2)^{-\left(\frac{\mu^2}{y^2} - 1\right)} y d^D y}{\left(y^2 + q^2\right)} \right] S^\phi_R * \phi(X) \left\{ \frac{y^2}{\mu^2} ; z \right\} \quad (42)$$

This is the crucial step: the counterterm is obtained by replacing the subdivergences in $\phi(B_+^c (X))$ by their renormalized evaluation $S^\phi_R * \phi(X)$, thanks to the fact that $bB_+^c = 0$.

Now use that $\text{aug} [X] = k$, and that $X$ is a product $X \equiv \prod_i \tilde{T}_i$ say, so that

$$S^\phi_R * \phi(X) = \prod_i S^\phi_R * \phi(\tilde{T}_i) \quad (43)$$

We can apply the assumption of the induction to $S^\phi_R * \phi(X)$. Hence there exists an integer $r_X$ such that

$$S^\phi_R * \phi(X) \left\{ \frac{y^2}{\mu^2} ; z \right\} = \sum_{j=0}^{r_X} c_j(z) \left[ \log \left( \frac{y^2}{\mu^2} \right) \right]^j \quad (44)$$

for some coefficient functions $c_j(z)$ which are regular at $z = 0$.

A simple derivative with respect to $\log \frac{y^2}{\mu^2}$ shows that $\tilde{\phi}_R(T)$ has a limit when $z \to 0$ which proves locality of $S^\phi_R$. Here, we use that our integrands belong to the class of functions analyzed in [23]. The needed results for $S^\phi_R * \phi(T)$ follow similarly.

We encourage the reader to go through these steps for a rooted tree with augmentation degree three say.

3 DSEs and factorization

We start by considering combinatorial DSEs. Those we define to be equations which define formal series over Hopf algebra elements. As before, we consider a Hopf algebra of decorated rooted trees, with the corresponding investigation of DSEs in the Hopf algebra of graphs to be given in [2].

3.1 The general structure of DSEs

In analogy to the situation in QFT, our toy DSE considered here is of the form
\[ X = 1 + \sum_{c \in S \subseteq \text{Dec}} \alpha^{|c|-1} B^c_\alpha [X^{|c|}], \]  

where \( \forall c \in \text{Dec}, \ |c| \) is an integer chosen \( \geq 2 \), and the above is a series in \( \alpha \) with coefficients in \( H \equiv H(S) \). Note that every non-trivial term on the rhs is in the image of a closed Hochschild one-cocycle \( B^c_+ \). The above becomes a series,

\[ X = 1 + \sum_{k=2}^{\infty} c_k \alpha^{k-1}, \ c_k \in H, \]

such that \( c_k \) is a weighted sum of all decorated trees with weight \( k \). Here, the weight \(|T|\) of a rooted tree \( T \) is defined as the sum of the weights of its decorations:

\[ |T| := \sum_{v \in T^{[0]}} |\text{dec}(v)|. \]

This is typical for a Dyson–Schwinger equation, emphasizing the dual role of the Hochschild one-cocycles \( B^c_\alpha \): their Hochschild closedness guarantees locality of counterterms, and they define quantum equations of motion at the same time. In the above, \( \alpha \) plays the role corresponding to a coupling constant and provides a suitable grading of trees by their weight.

Let us now assign to a given unordered set \( I \subset \text{Dec} \) of decorations the linear combination of rooted trees

\[ T(I) := \sum_{T \in H} a_{I \in \{T \in T^{[0]}\text{dec}(v)\}} \frac{\alpha^{|T|-1} c_T}{\text{sym}(T)} T. \]

Here, the symmetry factor of a tree \( T \) is the rank of its automorphism group, for example

\[ \text{sym}(\bullet_a \bullet_b) = 2, \ \text{sym}(\bullet_a \bullet_b) = 1. \]

To define \( c_T \), let for each vertex \( v \) in a rooted tree \( f_v \) be the number of outgoing edges as in \( \text{I5} \). Then

\[ c_T := \prod_{v \in T^{[0]}} \frac{|\text{dec}(v)|!}{(|\text{dec}(v)| - f_v)!}. \]

If a tree \( T \) appears in such a sum, we write \( T \in T(I) \). It is then easy to see that for such a linear combination \( T(I) \) of rooted trees we can recover \( I \) from \( \mathcal{P}^{\text{aug}(T)}(T) \). For two sets \( I_1, I_2 \) we then define

\[ T(I_1) \lor T(I_2) := T(I_1 \cup I_2). \]
Theorem 2. For the DSE above, we have

i) \( X = 1 + \sum_{T \in H(S)} a^{|T|-1} \frac{\text{sym}(T)}{\text{sym}(T)} T \),

ii) \( \Delta(c_k) = \sum_{i=0}^k \text{Pol}_i \otimes c_{k-i} \), where Pol\(_i\) is a degree \( i \) polynomial in the \( c_j \).

Thus, these coefficients \( c_j \) form a closed subcoalgebra.

iii) \( X = \prod_{T \in S} \frac{1}{1 - a_T} \). The solution factorizes in terms of geometric series with respect to the product \( \vee \).

This theorem is a special case of a result in [2], to which we have to refer the reader for a proof. The factorization in the third assertion is a triviality thanks to the definition of \( T \). It only becomes interesting in the QFT case where the pre-Lie product of graphs is degenerate [17].

3.2 Example

To have a concrete example at hand, we focus on the equation:

\[
X = 1 + \alpha B_a^a (X^2) + \alpha^2 B_b^b (X^3),
\]

where we have chosen \(|a| = 2\) and \(|b| = 3\). For the first few terms the expansions of \( X \) reads

\[
c_1 = \bullet^a, \\
c_2 = \bullet^a b + 2 \bullet^a a,
\]

\[
c_3 = 2 \bullet^a b + 3 \bullet^a a + 4 \bullet^a a + a \bullet^a a,
\]

\[
c_4 = 3 \bullet^a b + 4 \bullet^a a + 6 \bullet^a a + 6 \bullet^a a + 2 b \bullet^a a + 3 \bullet^a a + 3 \bullet^a a + 8 \bullet^a a
\]

As rooted trees, we have non-planar decorated rooted trees, with vertex fertility bounded by three in this example. In general, in the Hopf algebra of decorated rooted trees, the trees with vertex fertility \( \leq k \), always form a sub Hopf algebra.

Let us calculate the coproducts of \( c_i \), \( i = 1, \ldots, 4 \) say, to check the second assertion of the theorem. We confirm

\[
\Delta(c_1) = c_1 \otimes e + e \otimes c_1,
\]

\[
\Delta(c_2) = c_2 \otimes e + e \otimes c_2 + 2c_1 \otimes c_1,
\]

\[
\Delta(c_3) = c_3 \otimes e + e \otimes c_3 + 3c_1 \otimes c_2 + \left[ 2c_2 + c_1c_1 \right] \otimes c_1,
\]

\[
\Delta(c_4) = c_4 \otimes e + e \otimes c_4 + 4c_1 \otimes c_3 + \left[ 3c_2 + 3c_1c_1 \right] \otimes c_2
\]

\[+ \left[ 2c_3 + 2c_1c_2 \right] \otimes c_1.\]
3.3 Analytic Factorization

The crucial question now is what has the evaluation of all the terms in \( X \) as given by (45),

\[
\phi(X) \left\{ \frac{q^2}{\mu^2}; \alpha; z \right\} = 1 + \sum_{T \in H(S)} \frac{c_T \alpha^{|T|-1}}{\text{sym}(T)} \phi(T) \left\{ \frac{q^2}{\mu^2}; z \right\},
\]

(61)
to do with

\[
\prod_{c \in S} \frac{1}{1 - \alpha^{|c|-1} \phi(c) \left\{ \frac{q^2}{\mu^2}; z \right\}}? \tag{62}
\]

If the evaluation of a tree would decompose into the evaluation of its decorations, we could expect a factorization of the form

\[
\phi(T(I)) = N_I \prod_{c \in I} \phi(c), \tag{63}
\]

where \( N_I \) is the integer \( \sum_{T \in T(I)} c_T \). It is easy to see that the highest order pole terms at each order of \( \alpha \) in the unrenormalized DSE are in accordance with such a factorization [31], but that we do not get a factorization for the non-leading terms.

¿From the definition [23] for our toy model Feynman rule \( \phi \) we can write the DSE for the unrenormalized toy Green function \( \phi(X) \) as

\[
\phi(X) \left\{ \frac{q^2}{\mu^2}; \alpha; z \right\} = 1 + \sum_{c \in S} \alpha^{|c|-1} \int d^D y \frac{[y^2] z^{(\frac{|c|}{2} - 1)} f_c}{y^2 + q^2} \phi(X) \left\{ \frac{y^2}{\mu^2}; \alpha; z \right\}. \tag{64}
\]

As the \( B^c_e \) in (45) are Hochschild closed, the corresponding renormalized DSE is indeed of the same form

\[
\phi_R(X) \left\{ \frac{q^2}{\mu^2}; \alpha; z \right\} = Z_X + \sum_{c \in S} \alpha^{|c|-1} \int d^D y \frac{[y^2] z^{(\frac{|c|}{2} - 1)} f_c}{y^2 + q^2} \phi_R(X) \left\{ \frac{y^2}{\mu^2}; \alpha; z \right\}, \tag{65}
\]

where \( Z_X = S_R^\phi(X) \).

Now assume we would have some "RG-type" information about the asymptotic behaviour of \( \phi_R(X) \), for example

\[
\phi_R(X) \left\{ \frac{q^2}{\mu^2}; \alpha \right\} = F(X)(\alpha) \left[ \frac{q^2}{\mu^2} \right]^{-\gamma(\alpha)}, \tag{66}
\]

consistent with the renormalized DSE. Then, our toy model would regulate itself, as
\[ \phi_R(X) \left\{ \frac{q^2}{\mu^2}; \alpha \right\} = [\mu^2]^{\gamma(\alpha)} \sum_{c \in S} \alpha^{[c]-1} \times \int d^2 y \frac{|y^2|^{[c]-1} \gamma(\alpha) f_c}{y^2 + q^2} \left[ \frac{y^2}{\mu^2}; \alpha \right]^{[c]} \] \tag{67}

\[ = [\mu^2]^{\gamma(\alpha)} \sum_{c \in S} \alpha^{[c]-1} [F(X)(\alpha)]^{[c]} \int d^2 y \frac{|y^2|^{\gamma(\alpha) f_c}}{y^2 + q^2}. \tag{68} \]

with no need for a regulator, as long as we assume that \( \gamma(\alpha) \) serves that purpose, possibly by means of analytic continuation.

Then, we would be in much better shape: the "toy anomalous dimension" \( \gamma(\alpha) \) could be defined from the study of scaling in the complex Lie algebra \( \mathcal{L} \) underlying the dual of \( H(S) \) \[30\], while \( F(X)(\alpha) \) could be recursively determined at \( q^2 = \mu^2 \) from Feynman rules which imply factorization for a tree \( T = B_+(U) \) as

\[ \phi_R(B_+^c(U)) \left\{ \frac{q^2}{\mu^2}; \alpha \right\} = \phi_R(\bullet c) \left\{ \frac{q^2}{\mu^2}; \alpha \right\} \phi(U) \left\{ 1; \alpha \right\}, \tag{69} \]

by (68).

Alas, we do not have a renormalization group at our disposal here. But in QFT we do. While it might not tell us that we have scaling \[32\], it will indeed give us information about the asymptotic behaviour, which combines with the present analysis of DSEs in a profitable manner: what is needed is information how the asymptotic behaviour of the integrand which corresponds to \( B_+^c \) under the Feynman rules relates to the asymptotic behaviour of the integral. This is just what field theory provides. We will in \[2\] then indeed set out to combine the DSEs and the RG so as to achieve a factorization in terms of Hopf algebra primitives, using the Hochschild closedness of suitable \( B_+^c \) operators, the RG, as well as a dedicated choice of Hopf algebra primitives so as to isolate all short-distance singularities in Green functions which depend only on a single scale. As it will turn out, this makes the Riemann–Hilbert approach of \[11, 30\] much more powerful.

### 3.4 Remarks

Let us understand how the above theorem fares in the context of QFT. Consider all 1PI graphs together with their canonical Hopf- and Lie algebra structures of 1PI graphs. The set of primitive graphs is then well-defined. We use it to form a set of equations

\[ \Gamma_{\pm} = 1 + \sum_{\gamma \in A_{[1]}^\pm} \frac{g^{[\gamma]-1}}{\text{Sym}(\gamma)} B_+^\gamma(X_R). \tag{70} \]
These equations define 1PI Green functions, in a normalization such that its tree level value is unity, recursively, via insertion of such Green functions (combined in a monomial $X^\gamma_R$) into prime graphs $\gamma$, graphs which are themselves free of subgraphs which are superficially divergent. They define formal series in graphs such that the evaluation by the Feynman rules delivers the usual quantum equations of motion, the DSEs. This gives us an independent way to find such equations of motion: the above equation can be described as a canonical problem in Hochschild cohomology, without any reference to the underlying physics. Investigating these equations from that viewpoint has many interesting consequences \[2\] which generalize the toy analysis in this talk:

1. The $\Gamma^{\mu}$ are determined as the sum over all 1PI graphs with the right weights so as to determine the 1PI Green functions of the theory:

$$
\Gamma^{\mu} = 1 + \sum_{\Gamma \in \mathcal{H}_{\text{res}}(\gamma)} \frac{g^{\Gamma}}{\text{Sym}(\Gamma)} \Gamma,
$$

where the latter sum is over all 1PI graphs $\Gamma$ with external legs (“residue”).

2. The maps $B^{\gamma_i}_+$ are suitably defined so that they are Hochschild closed for a sub Hopf algebra of saturated sums of graphs $\Sigma_\Gamma = \sum_i \gamma_i \ast X_i$ which contain all maximal forests:

$$
\sum_i \Delta B^{\gamma_i}_+(X_i) = \sum_i B^{\gamma_i}_+(X_i) \otimes e + \sum_i (\text{id} \otimes B^{\gamma_i}_+) \Delta(X_i).
$$

3. This delivers a general proof of locality of counterterms and finiteness of renormalized Green functions by induction over the augmentation degree precisely as above:

$$
\sum_i \Delta B^{\gamma_i}_+(X_i) = \sum_i B^{\gamma_i}_+(X_i) \otimes e + \sum_i (\text{id} \otimes B^{\gamma_i}_+) \Delta(X_i) \Leftrightarrow \sum_i S^{\phi}_R(B^{\gamma_i}_+(X_i)) = (\text{id} - R) \sum_i \int D(\gamma) (S^{\phi}_R \ast \phi(X_i)),
$$

so that the $R$-bar operation and the counterterm are obtained by replacing the divergent subgraphs by their renormalized contribution.

4. The terms of a given order in a 1PI Green functions form a closed Hopf subalgebra:

$$
\Gamma^{\mu} = 1 + \sum_k \frac{\mu_k}{k} g^k \Rightarrow \Delta(\frac{\mu_k}{k}) = \sum_{j=0}^k \text{Pol}_{\frac{\mu_k}{j}} \ast \frac{\mu_k}{k-j},
$$

where the $\text{Pol}_{\frac{\mu_k}{j}}$ are monomials in the $\frac{\mu_k}{j}$ of degree $j$, where $\mu_k \in \mathcal{R}$. Thus, the space of polynomials in the $\frac{\mu_k}{j}$ is a closed Hopf sub(co)algebra of $H$. 
This is a subtle surprise: to get this result, it is necessary and sufficient to impose relations between Hopf algebra elements:

\[ \forall \gamma_1, \gamma_2 \in \psi^1, \ X_{\mathcal{R}}^{\gamma_1} = X_{\mathcal{R}}^{\gamma_2}. \] (75)

These relations turn out to be good old friends, reflecting the quantum gauge symmetries of the theory: they describe the kernel of the characters \( \phi, S_{\mathcal{R}}^\phi, S_{\mathcal{R}}^\phi \ast \phi \), and translate to the Slavnov–Taylor identities

\[ \frac{Z^{ass}}{Z} = \frac{Z}{Z^{ass}} = \frac{Z}{Z^{as^2}} = \frac{Z^{as^2}}{Z}, \] (76)

where \( Z^{as} = S_{\mathcal{R}}^\phi(\Gamma^{as}) \).

5. The effective action, as a sum over all 1PI Green functions, factorizes uniquely into prime graphs with respect to a commutative associativ e product on 1PI graphs \( \vee \):

\[ S_{\text{eff}} = \sum_m I_m = \prod_{\gamma \in H_L[1]} \frac{1}{1 - g^{|\gamma| - 1} I(\gamma)}. \] (77)

Integrality of this product again relates back to relations between graphs which correspond to Ward identities.

With these remarks, we close and invite the reader to participate in the still exciting endeavour to understand the structure of renormalizable quantum field theories in four dimensions.

Acknowledgments

It is a pleasure to thank participants and organizers of our school for a wonderful (and everywhere dense) atmosphere. thanks to K. Ebrahimi-Fard for proofreading the ms. This work was supported in parts by NSF grant DMS-0205977 at the Center for Mathematical Physics at Boston University.

References

1. Kreimer, D.: New mathematical structures in renormalizable quantum field theories. Annals Phys. 303 (2003) 179 [Erratumibid. 305 (2003) 79]
2. Kreimer, D.: Renormalization, Hochschild cohomology and quantum equations of motion. In preparation.
3. Gangl, H: Multiple polylogarithms, rooted trees and algebraic cycles. Talk at this school.
4. Cartier, P: Motivic aspects of polylogarithms. Lectures at this school.
5. Goncharov, A.: Galois symmetries of fundamental groupoids and noncommutative geometry. IHES/M/02/56. www.ihes.fr.
6. Zagier, D.: Polylogarithms. Lectures at this school.
7. Bern, Z.: Perturbative calculations in gauge and gravity theories. Talk at this school.
8. Weinzierl, S.: Algebraic algorithms in perturbative calculations. Talk at this school [arXiv:hep-th/0305260].
9. Kreimer, D.: On the Hopf algebra structure of perturbative quantum field theories. Adv. Theor. Math. Phys. 2 (1998) 303 [arXiv:q-alg/9707029].
10. Kreimer, D.: On overlapping divergences. Commun. Math. Phys. 204 (1999) 669 [arXiv:hep-th/9810022].
11. Connes, A., Kreimer, D.: Renormalization in quantum field theory and the Riemann-Hilbert problem. I: The Hopf algebra structure of graphs and the main theorem. Commun. Math. Phys. 210 (2000) 249. [arXiv:hep-th/9912092].
12. Connes, A., Kreimer, D.: Insertion and elimination: The doubly infinite Lie algebra of Feynman graphs. Annales Henri Poincare 3 (2002) 411 [arXiv:hep-th/0201157].
13. Johnson, K., R. Willey, R., and Baker, M.: Vacuum Polarization In Quantum Electrodynamics. Phys. Rev. 163 (1967) 1699.
14. Broadhurst, D.J., Delbourgo, R., Kreimer, D.: Unknotting the polarized vacuum of quenched QED. Phys. Lett. B 366 (1996) 421 [arXiv:hep-ph/9509296].
15. Kreimer, D.: Chen’s iterated integral represents the operator product expansion. Adv. Theor. Math. Phys. 3 (1999) 627 [arXiv:hep-th/9901099].
16. Broadhurst, D.J., Kreimer, D.: Renormalization automated by Hopf algebra. J. Symb. Comput. 27 (1999) 581 [arXiv:hep-th/9810087].
17. Kreimer, D.: Unique factorization in perturbative QFT. Nucl. Phys. Proc. Suppl. 116 (2003) 392 [arXiv:hep-ph/0211188].
18. Foissy, L.: Les algèbres des Hopf des arbres enracinés décorées. [arXiv:math.QA/0105212].
19. Connes, A., Kreimer, D.: Hopf algebras, renormalization and noncommutative geometry. Commun. Math. Phys. 199 (1998) 203 [arXiv:hep-th/9808042].
20. Broadhurst, D.J., Kreimer, D.: Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality. Nucl. Phys. B 600 (2001) 403 [arXiv:hep-th/0012146].
21. Rivers, R.J.: Path integrals methods in quantum field theory. CUP, Cambridge (1987).
22. Cvitanovic, P.: Field Theory. RX-1012 (NORDITA) [http://www.cns.gatech.edu/FieldTheory/].
23. Weinberg, S.: High-Energy Behavior In Quantum Field Theory. Phys. Rev. 118 (1960) 838.
24. Kreimer, D.: Knots And Feynman Diagrams. CUP, Cambridge (2000).
25. Belkale P., Brosnan, P.: Matroids, Motives and a conjecture of Kontsevich. Duke Math. J. 116 (2002) 147 [arXiv:math.AG/00012198].
26. Rosner, J.L.: Sixth Order Contribution to Z3 in Finite Quantum Electrodynamics. Phys. Rev. Lett. 17 (1966) 1190.
27. Cvitanovic, P.: Asymptotic Estimates And Gauge Invariance. Nucl. Phys. B 127 (1977) 176.
28. Broadhurst, D.J., Kreimer, D.: Feynman diagrams as a weight system: Four-loop test of a four-term relation. Phys. Lett. B 426 (1998) 339 [arXiv:hep-th/9612011].
29. Broadhurst, D.J., Kreimer, D.: Towards cohomology of renormalization: Bi-grading the combinatorial Hopf algebra of rooted trees. Commun. Math. Phys. 215 (2000) 217 [arXiv:hep-th/0001202].

30. Connes, A., Kreimer, D.: Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group. Commun. Math. Phys. 216 (2001) 215 [arXiv:hep-th/0003188].

31. Kreimer, D., Delbourgo, R.: Using the Hopf algebra structure of QFT in calculations. Phys. Rev. D 60 (1999) 105025 [arXiv:hep-th/9903249].

32. Coleman, S.: Aspects of Symmetry, Lect. 3: Dilatations. Cambridge University Press, Cambridge (1985).