\[ \mathcal{N} = 1 \text{ and } \mathcal{N} = 2 \text{ Super Yang-Mills theories from wrapped branes} \]

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Abstract

We consider supergravity solutions of D5 branes wrapped on supersymmetric 2-cycles and use them to discuss relevant features of four-dimensional \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) super Yang-Mills theories with gauge group \( SU(N) \). In particular in the \( \mathcal{N} = 1 \) case, using a gravitational dual of the gaugino condensate, we obtain the complete NSVZ \( \beta \)-function. We also find non-perturbative corrections associated to fractional instantons with charge \( 2/N \). These non-perturbative effects modify the running of the coupling constant which remains finite even at small scales in a way that resembles to the soft confinement scenario of QCD.

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1 Introduction

Recently, it has become more and more evident that a lot of relevant information about supersymmetric Yang-Mills (SYM) theories can be obtained by studying their dual supergravity backgrounds produced by stacks of D branes. This gauge/gravity correspondence has been thoroughly investigated in the maximally supersymmetric and conformal case, where a precise duality can be established between the $\mathcal{N} = 4$ SYM theory in four dimensions with gauge group $SU(N)$ and the type IIB supergravity in $AdS_5 \times S_5$ which is the near horizon geometry of $N$ D3 branes in flat space [1]. Prompted by the remarkable success of this AdS/CFT duality, a lot of activity has been devoted to extend the gauge/gravity correspondence also to less supersymmetric and/or non-conformal theories [2].

One way to reduce the amount of supersymmetry is to place a stack of D3 branes at the apex of an orbifold [3] or of a conifold [4]. Depending on the details of the background, the number of preserved supercharges can be reduced to eight or four, and thus the SYM theories that correspond to these configurations will possess

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1See the introduction of Ref. [2] for a more detailed discussion.
\( \mathcal{N} = 2 \) or \( \mathcal{N} = 1 \) supersymmetry in four dimensions. Furthermore, both for the orbifold and the conifold there is a very natural way to break conformal invariance, namely by means of fractional D3 branes \([5]\). Therefore, in this way one can realize interesting non-conformal SYM theories in four dimensions with \( \mathcal{N} = 2 \) or \( \mathcal{N} = 1 \) supersymmetry (for recent reviews on this approach see, for example, Refs. \([6, 7]\)).

Another possibility to reduce the number of preserved supercharges is to consider D branes whose world-volume is partially wrapped on a supersymmetric cycle inside a K3 manifold or a Calabi-Yau space. The unwrapped part of the brane world-volume remains flat and supports a gauge theory. In order to preserve at least some supersymmetry, the normal bundle to the wrapped D branes has to be partially twisted \([8]\), and as a consequence of this twist, some world-volume fields become massive and decouple. This procedure has been first used in Ref. \([9]\) to study the pure \( \mathcal{N} = 1 \) SYM theory in four dimensions, and later it has been generalized to many other cases with different space-time dimensions and different amounts of supersymmetry \([10, 11, 12, 13, 14]\).

In this paper we will use the gauge/gravity correspondence to study SYM theories with \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) supersymmetry in four dimensions. In particular we reconsider in some detail the supergravity solutions corresponding to \( N \) D5-branes wrapped on a 2-sphere that have been first found in Refs. \([9]\) and \([10, 11]\) for the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) case respectively. Even if these two solutions have been discussed already in several papers, here we review again their derivation in order to set up the notation, to show their similarities and differences, and also to be self-contained. In doing this, we also provide a derivation of the Maldacena-Núñez (MN) solution of Ref. \([11]\) using the first order formalism \([A]\). We then consider the (bosonic) massless open-string modes that propagate on the flat part of the D5 world-volume and study their effective action at energies where the higher string modes, as well as the Kaluza-Klein excitations around the 2-cycles, decouple. This resulting theory is simply a four-dimensional SYM theory with gauge group \( SU(N) \). In both the \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) case, it turns out that the coupling constant \( g_{YM} \) and the vacuum angle \( \theta_{YM} \) of this SYM theory can be expressed in terms of the ten-dimensional supergravity solution representing \( N \) wrapped D5 branes in a very explicit and simple form \([B]\). In particular, if we denote by \( \rho \) the radial coordinate on which the classical supergravity fields depend, we find that

\[
\frac{1}{g_{YM}^2} = F(\rho) \tag{1.1}
\]
where the function $F(\rho)$ is an explicitly computable function (see eqs. (3.8) and (4.7) below).

The other crucial ingredient for the gauge/gravity correspondence is the relation between the radial parameter of the supergravity solution and the energy scales of the gauge theory. For non-conformal theories in general it may be ambiguous and difficult to establish such a relation \[22\]; however, for the two cases under consideration we manage to find a supergravity realization of a protected operator of the gauge theory, so that a definite radius/energy relation can be obtained\[5\]. The protected operators we will consider are the complex scalar field $\Psi$ of the vector multiplet in the $\mathcal{N} = 2$ theory, and the gaugino condensate $\langle \lambda^2 \rangle$ in the $\mathcal{N} = 1$ theory. In general, the energy/radius relation that we obtain in this way takes the form

$$\mu \Lambda = G(\rho)$$

where $\mu$ is the subtraction point at which the theory is defined, $\Lambda$ is the scale dynamically generated by quantum corrections and $G(\rho)$ is an explicit function which depends on the model (see eqs. (3.13) and (4.16) below).

Once the functions $F(\rho)$ and $G(\rho)$ are determined, one can exploit them to find explicit results for the gauge theory. For example, by eliminating $\rho$ between (1.1) and (1.2), one can obtain the running coupling constant and hence the $\beta$-function. In the $\mathcal{N} = 2$ case, these manipulations can be performed in an analytic way and lead to the exact perturbative $\beta$-function. In the $\mathcal{N} = 1$ case, instead, these manipulations can only be performed using asymptotic expansions but nevertheless lead to interesting results. In particular, we will show that the complete NSVZ $\beta$-function \[23\] of the $\mathcal{N} = 1$ theory can be obtained from the MN solution, once the appropriate energy/radius relation is enforced. It is remarkable to see that the entire perturbative $\beta$-function, and not only the 1-loop approximation, is encoded in a classical supergravity solution! Actually, the MN solution also shows the presence of non-perturbative effects in the form of fractional instantons with charge $2/N$ which smooth out the running of the gauge coupling constant that remains finite even at small energy. This smooth behavior as well as the absence of the Landau pole indicate that the theory softly flows to the confining phase, in analogy to the soft-confinement scenario of QCD \[24\]. Furthermore, the MN solution also accounts for the chiral anomaly, the chiral symmetry breaking and the correct action for the gauge instantons which we manage to realize explicitly as wrapped D strings.

The analysis of the IR regime of a gauge theory and the study of its non-perturbative features by means of a supergravity solution are possible only if the

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\[5\] Again, this situation is similar to the one of fractional D3 branes in orbifold models (see, in particular, Ref. \[21\]).
latter is non singular. This is precisely the case of the MN solution which is regular even at small distances due the presence of a field that, in Ref. [14], has been identified with the gravitational dual of the gaugino condensate. In this respect, the MN solution is very different from the \( \mathcal{N} = 2 \) solution of Ref. [10, 11] which exhibits a naked singularity of repulson type and an enhançon locus [25] that prevent from obtaining information on the IR regime of the dual \( \mathcal{N} = 2 \) SYM theory and in particular on its instanton corrections [6]. On the contrary, the MN solution is similar to the Klebanov-Strassler solution (KS) [26] which describes a system of regular and fractional D3 branes on a deformed conifold. The KS solution is free of singularities and can be successfully used to describe many interesting features of a four dimensional \( \mathcal{N} = 1 \) gauge theory with chiral matter that, through a series of duality cascades, eventually flows to the pure \( \mathcal{N} = 1 \) SYM theory in the deep IR to which our analysis can be extended [27]. A qualitative study of the general implications of the MN and KS solutions for \( \mathcal{N} = 1 \) gauge theories can be found in Ref. [28].

The paper is organized as follows. In Section 2 we review the \( \mathcal{N} = 2 \) supergravity solution of Ref. [10, 11] and the \( \mathcal{N} = 1 \) solution Ref. [9] in a first order formalism. In Section 3 we discuss the pure \( \mathcal{N} = 2 \) SYM theory using the dual supergravity solution of Section 2.1. In particular, we derive the running of the coupling constant and the chiral anomaly, but differently from Ref. [10] we do not use the probe technique. In Section 4 we study the pure \( \mathcal{N} = 1 \) SYM theory and, by exploiting the supergravity solution of Section 2.2, we discuss the chiral anomaly, the gaugino condensate, the running coupling constant, the NSVZ \( \beta \)-function and the instanton action. Finally, in Section 5 we present our conclusions, while in the Appendix we give some technical details about the parameterizations used to derive the supergravity solutions of Section 2.

## 2 The Supergravity Solutions

In this section we review the type IIB supergravity solutions that correspond to \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) SYM theories in four dimensions with gauge group \( SU(N) \) and no matter.

The starting point is a stack of \( N \) D5 branes with two longitudinal directions wrapped on a supersymmetric 2-cycle. The difference between \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) arises from the way in which the 2-cycle is embedded in the ambient space, a Calabi-Yau threefold for \( \mathcal{N} = 1 \) and a K3 manifold for \( \mathcal{N} = 2 \). The four unwrapped

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\[6\] The same problems occur in the fractional D3 brane solutions [17, 18, 19, 2].
longitudinal directions of the D5 branes span a flat world-volume where the SYM theory lives. In order to preserve the proper amount of supersymmetry the normal bundle to the five-branes has to be topologically twisted, this is achieved by embedding the $U(1)$ spin-connection of the 2-cycle into $SO(4)$ which is the R-symmetry group of the D5 branes. As remarked in [29, 8], an efficient way to derive the geometry corresponding to this set-up is to consider the $SO(4)$ gauged supergravity in seven dimensions, find domain-walls wrapped on a 2-sphere, and then lift them up in ten dimensions.

The fields of the $SO(4)$ gauged supergravity in $d = 7$ are the metric $G_{\mu \nu}$, the gauge fields $A_{\mu}^{ij} = -A_{\mu}^{ji}$ with $i, j = 1, \ldots, 4$ being vector indices of $SO(4)$, a symmetric matrix $T_{ij}$ of scalars and a 2-form potential $A_{\mu}^{(2)}$. The Lagrangian of this supergravity theory is

$$L_7 = \sqrt{-\det G} \left\{ \mathcal{R}(G) - \frac{5}{16} \partial_\mu y \partial^\mu y - \frac{1}{4} \tilde{T}_{ij}^{-1} \mathcal{D}_\mu \tilde{T}_{jk} \tilde{T}_{kl}^{-1} \mathcal{D}^{\mu \nu} \tilde{T}_{\nu} \right. \\
- \frac{1}{8} \epsilon^{-y/2} \tilde{T}_{ik}^{-1} \tilde{T}_{j\ell}^{-1} F_{\mu \nu}^{ij} F^{\mu \nu}_{k\ell} - \frac{1}{12} \epsilon^{-y} H_{\mu \nu \rho} H^{\mu \nu \rho} - V \right\}$$

(2.1)

where

$$e^y = \det T , \quad \tilde{T}_{ij} = e^{-y/4} T_{ij} ,$$

(2.2)

and

$$\mathcal{D}_\mu \tilde{T}_{ij} = \partial_\mu \tilde{T}_{ij} + \lambda \left( A_{\mu k} T_{kj} + A_{\mu}^{jk} \tilde{T}_{ik} \right) ,$$

(2.3)

$$F_{\mu \nu}^{ij} = \partial_\mu A_{\nu}^{ij} - \partial_\nu A_{\mu}^{ij} + \lambda \left( A_{\mu k} A_{\nu}^{kj} - A_{\nu k} A_{\mu}^{kj} \right) ,$$

(2.4)

$$V = \frac{\lambda^2}{2} e^{y/2} \left( 2\tilde{T}_{ij} \tilde{T}_{ij} - (\tilde{T}_{ii})^2 \right) .$$

(2.5)

Finally, $H_{\mu \nu \rho}$ are the components of the following 3-form

$$H^{(3)} = dA^{(2)} + \frac{1}{8} \epsilon_{ijkl} \left( F_{ij}^{kl} A^{k\ell} + \lambda A_{ij}^{k} A^{km} A^{n\ell} \right) .$$

(2.6)

In these equations $\lambda$ denotes the $SO(4)$ gauge coupling constant which has dimensions of a (length)$^{-1}$.

As mentioned above, we are interested in domain-walls of this supergravity theory that are wrapped on a 2-sphere. Thus we look for metrics of the form

$$ds_7^2 = e^{2f(r)} \left( dx_{1,3}^2 + dr^2 \right) + \frac{1}{\lambda^2} e^{2g(r)} d\Omega_2^2$$

(2.7)

where $dx_{1,3}^2$ is the Minkowski metric on $\mathbb{R}_{1,3}$, $r$ is the transverse coordinate to the domain-wall, and $d\Omega_2^2 = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2$ (with $0 \leq \tilde{\theta} \leq \pi$ and $0 \leq \tilde{\varphi} \leq 2\pi$) is the
metric of a unit 2-sphere. To fully determine the domain-wall configuration we must specify also the profile of the other supergravity fields. However, to do this we have to distinguish between the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases which require different types of Ansatz. In the next subsection we review the $\mathcal{N} = 2$ case \cite{10, 11}, while the $\mathcal{N} = 1$ case \cite{9} will be considered in the last subsection.

2.1 The $\mathcal{N} = 2$ Solution

In order to obtain a solution corresponding to a gauge theory with eight supercharges in four dimensions, one must first truncate the $SO(4)$ gauge group of the $d = 7$ supergravity to its diagonal $U(1)$ subgroup, i.e. to the abelian part of the diagonal $SU(2)_D \subset SO(4)$. This truncation can be achieved, for example, by setting

$$A_{\mu}^{12} \equiv - A_{\mu}$$

(2.8)

with all other $A_{\mu}^{ij}$'s put to zero. The consistency of this abelian reduction requires that the unimodular matrix $\tilde{T}_{ij}$ takes the form \cite{31}

$$\tilde{T}_{ij} = \text{diag}(e^x, e^x, e^{-x}, e^{-x}) .$$

(2.9)

With these positions it is easy to realize that the Lagrangian (2.1) becomes

$$L_{\gamma} = \sqrt{- \det G} \left\{ R(G) - \frac{5}{16} \partial_{\mu} y \partial^{\mu} y - \partial_{\mu} x \partial^{\mu} x - \frac{1}{4} e^{-2x-y/2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} e^{-y} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4 \lambda^2 e^{y/2} \right\}$$

(2.10)

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ and $H_{\mu\nu\rho} = \partial_{[\mu} A_{\nu\rho]}^{(2)}$.

We now look for solutions of the field equations following from the Lagrangian (2.10) in which the metric has the form (2.7) and the fields $x$ and $y$ are functions only of the transverse coordinate $r$. To implement the topological twist that preserves eight supercharges, we must identify the $U(1)$ gauge field with the spin-connection on the 2-sphere and thus write

$$A = - \frac{1}{\lambda} \cos \tilde{\theta} d\tilde{\varphi}$$

(2.11)

where the coupling constant $\lambda$ has been introduced for dimensional reasons. Furthermore, we can consistently set $A^{(2)} = 0$.

The task is then reduced to find the four functions $f(r), g(r), x(r)$ and $y(r)$ that solve the field equations. This has been done explicitly in Ref. \cite{10} \footnote{Notice that the factor of $\lambda^{-2}$ in (2.7) is necessary for dimensional reasons.} where it

\footnote{See also Ref. \cite{11}.}
has been shown that it is consistent to impose the relation \( y = -4f \) and that, after defining
\[
    h = g - f, \quad k = \frac{3}{2} f + g, \tag{2.12}
\]
it is possible to derive the field equations for \( h(r) \), \( k(r) \) and \( x(r) \) from the auxiliary Lagrangian
\[
    \mathcal{L} = e^{2k} \left[ 4\dot{k}^2 - 2\dot{h}^2 - \dot{x}^2 - \mathcal{V} \right] \tag{2.13}
\]
where dots denote derivative with respect to \( r \) and
\[
    \mathcal{V} = -4\lambda^2 - 2\lambda^2 e^{-2h} + \frac{\lambda^2}{2} e^{-4h-2x} \tag{2.14}
\]
together with the additional condition that the Hamiltonian \( \mathcal{H} \) associated to \( \mathcal{L} \) vanishes. This condition, which reads
\[
    \mathcal{H} = \frac{1}{4} e^{-2k} \left( \frac{1}{4} \dot{p}_k^2 - \frac{1}{2} \dot{p}_h^2 - \dot{p}_x^2 \right) + e^{2k} \mathcal{V} = 0, \tag{2.15}
\]
is a signal of the fact that the second-order field equations can be derived from a system of first-order (or BPS) equations, as one can show with a Hamilton-Jacobi approach \[82\]. In fact, by introducing the principal function \( \mathcal{F}(k, h, x) \) such that \( p_k = \frac{\partial \mathcal{F}}{\partial k} \), \( p_h = \frac{\partial \mathcal{F}}{\partial h} \), \( p_x = \frac{\partial \mathcal{F}}{\partial x} \) and assuming that \( \mathcal{F}(k, h, x) = e^{2k} \mathcal{W}(h, x) \), it is easy to see that (2.13) becomes
\[
    \frac{1}{8} \left( \frac{\partial \mathcal{W}}{\partial h} \right)^2 + \frac{1}{4} \left( \frac{\partial \mathcal{W}}{\partial x} \right)^2 - \frac{1}{4} \mathcal{W}^2 = \mathcal{V}, \tag{2.16}
\]
which is solved by
\[
    \mathcal{W} = -4\lambda \cosh x - \lambda e^{-2h-x}. \tag{2.17}
\]
The remaining Hamilton equations yield the following first-order system
\[
    \dot{k} = \frac{1}{4} \mathcal{W}, \quad \dot{h} = -\frac{1}{4} \frac{\partial \mathcal{W}}{\partial h}, \quad \dot{x} = -\frac{1}{2} \frac{\partial \mathcal{W}}{\partial x}, \tag{2.18}
\]
which, as shown in Ref. \[10\], can be conveniently solved in terms of the variable \( z \equiv e^{2h} \). In fact, one finds
\[
    e^{-2x} = 1 - \frac{1 + ce^{-2z}}{2z}, \quad e^{2k+x} = ze^{2z} \tag{2.19}
\]
where \( c \) is an arbitrary integration constant. Thus, choosing \( z \) as new radial coordinate, the functions appearing in the domain-wall solution are
\[
    f(z) = \frac{2}{5} z - \frac{1}{5} x(z), \quad g(z) = f(z) + \frac{1}{2} \log(z), \quad y(z) = -4f(z) \tag{2.20}
\]
with \( x(z) \) given in (2.19).

We now up-lift this solution to ten dimensions in order to exhibit it as a D5 brane configuration of Type IIB supergravity. This up-lift can be explicitly performed using the formulas of Ref. [31] which lead to a metric, a dilaton and a 2-form of magnetic type in ten dimensions. In this respect, however, we would like to point out that in Ref. [31] only the up-lift to NS5 brane configurations is considered. Indeed, the magnetic 2-form that is produced in ten dimensions has to be identified with the NS-NS antisymmetric tensor \( B_{\mu\nu} \) in view of the sign that is chosen for the exponential coupling with the dilaton in the Einstein frame. However, it is straightforward to adjust the up-lift formulas of Ref. [31] to the other sign and thus produce a magnetic 2-form which can be identified with a R-R potential \( C^{(2)} \) in ten dimensions. In this way, the seven-dimensional solution can be directly up-lifted to a D5 brane configuration of type IIB, which is what one needs to discuss the correspondence with a dual SYM theory in four dimensions.

With this in mind, we parameterize the 3-sphere that leads from seven to ten dimensions with the angles \( \theta, \phi_1 \) and \( \phi_2 \) (with \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( 0 \leq \phi_{1,2} \leq 2\pi \)) as discussed in Appendix A (see eq. (A.2)), and use the (modified) up-lift formulas of Ref. [31] to find a ten-dimensional (string frame) metric given by

\[
ds_{10}^2 = e^\Phi \left[ dx_{1,3}^2 + \frac{z}{\lambda^2} \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} \, d\tilde{\phi}^2 \right) + \frac{1}{\lambda^2} e^{2x} \, dz^2 \right. \\
+ \left. \frac{1}{\lambda^2} \left( d\theta^2 + \frac{e^{-x}}{\Omega} \cos^2 \theta \left( d\phi_1 + \cos \tilde{\theta} \, d\tilde{\phi} \right)^2 + \frac{e^x}{\Omega} \sin^2 \theta \, d\phi_2^2 \right) \right] ,
\]

a dilaton given by

\[
e^{2\Phi} = e^{2z} \left[ 1 - \sin^2 \theta \, \frac{1 + e^{-2z}}{2z} \right] ,
\]

and a magnetic R-R 2-form given by

\[
C^{(2)} = \frac{1}{\lambda^2} \phi_2 \, d \left[ \frac{\sin^2 \theta}{\Omega \, e^x} \left( d\phi_1 + \cos \tilde{\theta} \, d\tilde{\phi} \right) \right]
\]

where

\[
\Omega = e^x \cos^2 \theta + e^{-x} \sin^2 \theta .
\]

We remark that this D5 brane solution agrees with the one of Ref. [10] which is obtained by performing a S-duality transformation on a NS5 brane configuration. However, the metric (2.21) is written in a way in which the role of the different coordinates and factors is not immediately clear. Thus, in analogy with what has been done in Ref. [13], we perform the following change of variables

\[
\rho = \sin \theta \, e^z , \quad \sigma = \sqrt{z} \, \cos \theta \, e^{z-x}
\]
and rewrite eq. (2.21) as follows

$$ds_{10}^2 = H^{-1/2} \left[ dx_{1,3}^2 + \frac{z}{\lambda^2} \left( d\bar{\theta}^2 + \sin^2 \bar{\theta} \, d\bar{\varphi}^2 \right) \right] + \frac{H^{1/2}}{\lambda^2} \left( d\rho^2 + \rho^2 d\phi_2^2 \right)$$

$$+ \frac{H^{1/2}}{\lambda^2 z} \left[ d\sigma^2 + \sigma^2 \left( d\phi_1 + \cos \bar{\theta} \, d\bar{\varphi} \right)^2 \right]$$

(2.26)

where $H \equiv e^{-2\Phi}$. In this form the structure of the metric is much clearer. In fact, one can distinguish the transverse space into a plane parameterized by $\rho$ and $\phi_2$ (see the last two terms in the first line of (2.26)), and a non-trivial twisted plane pertaining to the K3 manifold (see the second line of (2.26)). Thus one can say that the two (dimensionless) coordinates $\rho$ and $\sigma$ defined in (2.25) represent two radial directions, respectively in the flat transverse space and in the curved space transverse to the brane but non-trivially fibered on the 2-cycle along which the brane is wrapped. In the next section we will show that the flat radial coordinate $\rho$ is directly related to the energy scale of the dual four dimensional gauge theory, and that shifts in the angle $\phi_2$ are directly related to chiral transformations.

We conclude this subsection by observing that the R-R charge of the configuration (2.21)-(2.23) is given by

$$Q_5 = \frac{1}{2\kappa_{10}^2} \int_{S_3} dC^{(2)} = \frac{2\pi^2}{\kappa_{10}^2} \lambda$$

(2.27)

where $\kappa_{10} = 8\pi^{7/2} g_s \alpha'^2$ is the gravitational coupling constant of the Type IIB string theory and $S_3$ is the transverse 3-sphere at infinity. Since this charge must be an integer multiple of the elementary D5 charge

$$\tau_5 = (2\pi)^{-5} g_s^{-1} \alpha'^{-3},$$

(2.28)

we deduce that the seven-dimensional coupling constant $\lambda$ can be written in terms of string theory parameters as follows

$$\frac{1}{\lambda^2} = N g_s \alpha'$$

(2.29)

where $N$ is the number of wrapped D5 branes.

### 2.2 The $\mathcal{N} = 1$ Solution

We now discuss the MN supergravity solution of Ref. [9] which corresponds to a gauge theory with four supercharges in four dimensions. As before, we start from the $SO(4)$ gauged supergravity Lagrangian (2.1), but this time we gauge one of the
two $SU(2)$ factors inside $SO(4)$ (say, for example, $SU(2)_L$). This is achieved by imposing the anti-selfduality constraint

$$A^{ij}_\mu = -\frac{1}{2} \epsilon^{ijkl} A^k_\mu$$

which explicitly reads

$$A^{34}_\mu = -A^{12}_\mu \equiv A^3_\mu \ , \quad A^{24}_\mu = -A^{31}_\mu \equiv A^2_\mu \ , \quad A^{14}_\mu = -A^{23}_\mu \equiv A^1_\mu \ ,$$

(2.31)

The vectors $A^a_\mu$ (with $a = 1, 2, 3$) are the gauge fields of $SU(2)_L$ whose field strengths are $F^a = dA^a + \lambda \varepsilon^{abc} A^b \wedge A^c$ as one can see by inserting (2.30) and (2.31) into (2.4). The consistency of this $SU(2)_L$ reduction requires that the unimodular matrix $\tilde{T}_{ij}$ be simply [31]

$$\tilde{T}_{ij} = \delta_{ij} \ .$$

(2.32)

With these positions the Lagrangian (2.1) becomes

$$L_7 = \sqrt{-\det G} \left\{ R(G) - \frac{5}{16} \partial_\mu y \partial^\mu y - \frac{1}{2} \frac{1}{e^{y/2}} F^a_\mu F^{a \mu} - \frac{1}{12} e^{-y} H_{\mu \nu \rho} H^{\mu \nu \rho} + 4 \lambda^2 e^{y/2} \right\}$$

(2.33)

where $H_{\mu \nu \rho}$ are the components of the 3-form

$$H^{(3)} = dA^{(2)} + F^a \wedge A^a - \frac{\lambda}{12} \epsilon^{abc} A^a \wedge A^b \wedge A^c \ .$$

(2.34)

Like in the $N = 2$ case, we look again for domain-wall solutions where the metric is of the form (2.7) and the scalar $y$ is function only of the radial coordinate $r$. To implement the topological twist that preserves four supercharges we should identify the spin-connection on the 2-sphere with the gauge field of a $U(1) \subset SU(2)_L$. If we simply do this, the corresponding supergravity solution turns out to be singular and unphysical. However, as discovered in Ref. [3], the singularity can be smoothed out by considering a more general Ansatz in which all gauge fields of $SU(2)_L$ are switched-on, namely by taking [33]

$$A^1 = -\frac{1}{2\lambda} a(r) \, d\tilde{\theta} \ , \quad A^2 = \frac{1}{2\lambda} a(r) \sin \tilde{\theta} \, d\tilde{\varphi} \ , \quad A^3 = -\frac{1}{2\lambda} \cos \tilde{\theta} \, d\tilde{\varphi} \ .$$

(2.35)

With this Ansatz it is easy to realize that the only non-vanishing components of the field strengths are

$$F^1_{r\tilde{\theta}} = -\frac{1}{2\lambda} \dot{a} \ , \quad F^2_{r\tilde{\varphi}} = \frac{1}{2\lambda} \sin \tilde{\theta} \dot{a} \ , \quad F^3_{\tilde{\theta}\tilde{\varphi}} = \frac{1}{2\lambda} \sin \tilde{\theta} \left(1 - a^2\right) \ .$$

(2.36)

Moreover one finds that $F^a \wedge A^a = A^1 \wedge A^2 \wedge A^3 = 0$ so that the 3-form (2.34) simply becomes $H^{(3)} = dA^{(2)}$. Therefore, it is consistent to set $A^{(2)} = 0$ and look
for a classical configuration that depends only on the functions \( f(r) \), \( g(r) \), \( y(r) \) and \( a(r) \).

Inserting this Ansatz into the supergravity field equations that follow from (2.33), we find that it is possible to set \( y = -4f \) and derive the remaining equations from the auxiliary Lagrangian

\[
\mathcal{L} = e^{2k} \left[ 4\dot{k}^2 - 2\dot{h}^2 - \frac{\dot{a}^2}{2} e^{-2h} - \mathcal{V} \right]
\]

(2.37)

where \( h \) and \( k \) are defined as in (2.12) and

\[
\mathcal{V} = -4\lambda^2 - 2\lambda^2 e^{-2h} + \frac{\lambda^2 (1 - a^2)^2}{4} e^{-4h}
\]

(2.38)

provided we require that the Hamiltonian \( \mathcal{H} \) associated to \( \mathcal{L} \) vanishes. This condition explicitly reads

\[
\mathcal{H} = \frac{1}{2} e^{-2k} \left( \frac{1}{8} p_k^2 - \frac{1}{4} p_h^2 - e^{2h} p_a^2 \right) + e^{2k} \mathcal{V} = 0 \,.
\]

(2.39)

We now proceed as in the \( \mathcal{N} = 2 \) case with the Hamilton-Jacobi method and introduce the principal function \( \mathcal{F}(k, h, a) \) such that \( p_k = \frac{\partial \mathcal{F}}{\partial k} \), \( p_h = \frac{\partial \mathcal{F}}{\partial h} \), \( p_a = \frac{\partial \mathcal{F}}{\partial a} \). Then, if we assume that \( \mathcal{F}(k, h, a) = e^{2k} \mathcal{W}(h, a) \), eq. (2.39) becomes

\[
\frac{1}{8} \left( \frac{\partial \mathcal{W}}{\partial h} \right)^2 + \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial a} \right)^2 e^{2h} - \frac{1}{4} \mathcal{W}^2 = \mathcal{V} \,.
\]

(2.40)

which is solved by (see also Ref. [16])

\[
\mathcal{W} = \lambda e^{-2h} \sqrt{(1 + 4e^{2h})^2 + 2(-1 + 4e^{2h})a^2 + a^4} \,.
\]

(2.41)

Having the explicit form of \( \mathcal{W} \), the problem is reduced to a system of first-order (BPS) equations given by

\[
\dot{k} = \frac{1}{4} \mathcal{W} \,, \quad \dot{h} = -\frac{1}{4} \frac{\partial \mathcal{W}}{\partial h} \,, \quad \dot{a} = -e^{2h} \frac{\partial \mathcal{W}}{\partial a}
\]

(2.42)

which can be explicitly solved. Indeed one finds

\[
e^{2h} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4} \,,
\]

(2.43)

\[
e^{2k} = \frac{e^h \sinh 2\rho}{2} \,,
\]

(2.44)

\[
a = \frac{2\rho}{\sinh 2\rho}
\]

(2.45)
where \( \rho \equiv \lambda r \). This solution has been first found in Ref. [33], although in a different context. The functions originally appearing in the domain-wall solution are then

\[
f(\rho) = \frac{1}{5} \log \left( \frac{\sinh 2\rho}{2 e^{h}} \right), \quad g(\rho) = f(\rho) + h(\rho), \quad y(\rho) = -4f(\rho) \quad (2.46)
\]

with \( h(\rho) \) given in (2.43). It is interesting to observe that \( a \sim \rho e^{-2\rho} \) for \( \rho \to \infty \); thus only the \( A^3 \) component of the \( SU(2)_L \) gauge field effectively survives in the large \( \rho \) region and the gauge group reduces to \( U(1) \).

We now up-lift this solution to ten dimensions to exhibit it as a D5 brane configuration of Type IIB supergravity. To this aim we could choose the same parameterization of the 3-sphere that we used in the previous subsection when we up-lifted the \( \mathcal{N} = 2 \) solution. However, for the applications to the dual gauge theory that we will discuss in the following, it is more convenient to choose a different parameterization and describe the 3-sphere with the Euler angles \( \theta' \), \( \phi \) and \( \psi \) (with \( 0 \leq \theta' \leq \pi \), \( 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi \leq 4\pi \) as described in (A.4), and use the corresponding left-invariant 1-forms \( \sigma^a \) given in (A.5). Then, using the (modified) up-lift formulas of Ref. [31], we find a ten-dimensional (string frame) metric

\[
ds_{10}^2 = e^\Phi \left[ dx_{1,3}^2 + \frac{e^{2h}}{\lambda^2} \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right) \right] + \frac{e^\Phi}{\lambda^2} \left[ d\rho^2 + \sum_{a=1}^{3} (\sigma^a - \lambda A^a)^2 \right] \quad (2.47)
\]

a dilaton

\[
e^{2\Phi} = \frac{\sinh 2\rho}{2 e^{h}}, \quad (2.48)
\]

and a magnetic R-R 2-form

\[
C^{(2)} = \frac{1}{4\lambda^2} \left[ \psi \left( \sin \theta' d\theta' \wedge d\phi - \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\phi} \right) - \cos \theta' \cos \tilde{\theta} d\phi \wedge d\tilde{\phi} \right] + \frac{a}{2\lambda^2} \left[ d\tilde{\theta} \wedge \sigma^1 - \sin \tilde{\theta} d\tilde{\phi} \wedge \sigma^2 \right] \quad (2.49)
\]

with field strength

\[
F^{(3)} = \frac{2}{\lambda^2} \left( \sigma^1 - \lambda A^1 \right) \wedge \left( \sigma^2 - \lambda A^2 \right) \wedge \left( \sigma^3 - \lambda A^3 \right) - \frac{1}{\lambda} \sum_{a=1}^{3} F^a \wedge \sigma^a. \quad (2.50)
\]

By computing the R-R charge \( Q_5 \) of this configuration (see eq.(2.27)) and imposing its quantization in units of the elementary D5 charge \( \tau_5 \), we find again that

\[
\frac{1}{\lambda^2} = N g_s \quad (2.51)
\]

where \( N \) is the number of wrapped five-branes. The solution (2.47)-(2.50) entirely agrees with the one of Ref. [3].

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We conclude this section by observing that there are many common features and similarities between the ten-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 1$ solutions as we have presented them here, despite the fact that we have used different angular variables in the two cases. After all, this is not too surprising since both solutions originate from the same Ansatz (2.7) in seven dimensions; but nevertheless it is noteworthy to see that in the first-order formalism the two solutions are very similar and that a flow between them could be in principle established in terms of the “superpotentials” $\mathcal{W}$ defined in eqs. (2.17) and (2.41). Of course there are also many important differences between the two solutions. In particular we would like to emphasize that in the $\mathcal{N} = 2$ case one can distinguish a plane in the transverse space where the wrapped D5 branes can move, while in the $\mathcal{N} = 1$ case no such plane exists. Moreover the $\mathcal{N} = 2$ solution is singular at short distance while the $\mathcal{N} = 1$ solution is regular. These facts have important consequences for the dual SYM theories as we shall discuss in the next sections.

3 The $\mathcal{N} = 2$ Super Yang-Mills Theory

The supergravity solution presented in subsection 2.1 describes the geometry produced by $N$ D5 branes wrapped on a supersymmetric 2-cycle in such a way that one quarter of the 32 supercharges of Type IIB are preserved. The four-dimensional part of the D5 world-volume that is not involved in the wrapping process remains flat and is the Minkowski space-time $\mathbb{R}_{1,3}$ where a supersymmetric gauge theory with 8 supercharges is defined. To specify this theory one must determine which massless open-string modes can propagate in $\mathbb{R}_{1,3}$. By applying for example the methods of Ref. [8], one can find that these modes fill a $\mathcal{N} = 2$ vector multiplet in the adjoint representation of $SU(N)$, and thus the low-energy four-dimensional gauge theory is a pure $\mathcal{N} = 2$ SYM theory with gauge group $SU(N)$.

We now show how the supergravity solution (2.22), (2.23) and (2.26) can be used to extract information on the corresponding gauge theory. First of all, we observe that in the transverse space to the D5 branes one can select a two-dimensional sub-space in which the branes can freely move. This is the locus $\sigma = 0$ and thus is the plane parameterized by $\rho$ and $\phi_2$. To find this result we can use the “probe technique” [34, 3] and study the dynamics of a wrapped D5 brane in the background geometry (2.22)-(2.26). The (string frame) world-volume action for a probe D5 brane is

$$S = -\tau_5 \int d^6 \xi \ e^{-\Phi} \sqrt{-\det (G + 2\pi \alpha' F)} + \tau_5 \int \left( \sum_n C^{(n)} \wedge e^{2\pi \alpha' F} \right)_{6\text{-form}}$$

(3.1)
where $\tau_5$ is defined in (2.28), $F$ is a world-volume gauge field and all bulk fields are understood to be the pull-backs onto the brane world-volume which is parameterized by $\xi = \{x^0, x^1, x^2, x^3, \tilde{\theta}, \tilde{\phi}\}$. By expanding the action (3.1) in powers of $\alpha'$ and substituting in it the solution (2.22), (2.23) and (2.26), we find that there is a static potential between the probe and the source given by

$$- \tau_5 \int d^6\xi \frac{z}{\lambda^2 H} \sin \tilde{\theta} \left[ \left( 1 + \frac{\lambda^2 H \sigma^2}{z^2 \tan^2 \tilde{\theta}} \right)^{1/2} - 1 \right].$$

(3.2)

In order to satisfy the “zero-force” condition and hence describe a BPS supersymmetric configuration, this potential term should vanish. Thus, we should require that the probe brane be placed at $\sigma = 0$. This means that in order to preserve supersymmetry the branes cannot move arbitrarily in their transverse space but only in the locus $\sigma = 0$, i.e. in the $(\rho, \phi_2)$-plane.

From now on, we abandon the probe approach but still we set $\sigma = 0$. Then, the relevant part of the transverse D5 brane metric (2.26) can be simply written as

$$ds^2_{\text{transv}} = H^{1/2} dZ d\overline{Z}$$

(3.3)

where we have defined

$$Z = \frac{1}{\lambda} \rho e^{i\phi_2},$$

(3.4)

and the R-R 2-form (2.23) simply reduces to

$$C^{(2)} = -\frac{1}{\lambda^2} \phi_2 \sin \tilde{\theta} d\overline{\theta} \wedge d\overline{\phi}.$$  

(3.5)

After these preliminaries we now study the dynamics of the $SU(N)$ gauge fields that propagate on the wrapped D5 branes. As discussed in Refs. [17, 22] for the analogue case of the fractional branes, the low-energy action for the bosonic degrees of freedom of the non-abelian gauge theory can be inferred from the abelian world-volume action (3.1) with the following procedure: take the limit $\alpha' \to 0$ keeping fixed the combination

$$\Psi = (2\pi \alpha')^{-1} Z$$

(3.6)

that has to play the role of the complex scalar of the $\mathcal{N} = 2$ vector multiplet, and then promote $F$ and $\Psi$ to $SU(N)$ fields by giving them an adjoint index $A$ and replacing all derivatives with the covariant ones. If we normalize the $SU(N)$ generators in such a way that $\text{tr}(T_A T_B) = (1/2) \delta^{AB}$ for the fundamental representation, then the above procedure leads to

$$S_{YM} = -\frac{1}{g_{YM}^2} \int d^4x \left\{ \frac{1}{4} F_A^{\alpha\beta} F_A^{\alpha\beta} + \frac{1}{2} D_{\alpha} \overline{\Psi}^A D^\alpha \Psi^A \right\} + \frac{\theta_{YM}}{32\pi^2} \int d^4x F_A^{\alpha\beta} \tilde{F}_A^{\alpha\beta}$$

(3.7)

9In flat space or in orbifold backgrounds where the classical D brane solutions can be explicitly obtained using boundary states (see e.g. Ref. [33]), this procedure can be made more rigorous by computing string scattering amplitudes as done for example in Ref. [34].
where
\[
\frac{1}{g_{\text{YM}}^2} = \frac{\tau_5 (2\pi)^2 \alpha'^2}{2} \int_0^{2\pi} d\tilde{\phi} \int_0^\pi d\tilde{\theta} \ e^{-\Phi} H \sqrt{-\det G} = \frac{N}{4\pi^2} \log \rho , \quad (3.8)
\]
\[
\theta_{\text{YM}} = \tau_5 (2\pi)^2 \alpha'^2 \int_0^{2\pi} d\tilde{\phi} \int_0^\pi d\tilde{\theta} C^{(2)}_{\tilde{\phi} \tilde{\theta}} = -2N \phi \quad . \quad (3.9)
\]

In other words, the inverse square of the YM coupling constant is proportional to the volume of the 2-sphere along which the D5 branes are wrapped computed in the ten-dimensional metric, whereas the YM \( \theta \)-angle is proportional to the flux of the R-R 2-form across this 2-sphere. Notice that in deriving these results we have used eqs.\((2.28)\) and \((2.29)\), the explicit form of the metric \((2.26)\), of the dilaton \((2.22)\) and of the R-R 2-form \((3.5)\), as well as the fact that eq.\((2.25)\) implies that \( z = \log \rho \) for \( \sigma = 0 \). In the following two sub-sections we will use the results \((3.8)\) and \((3.9)\) to derive the \( \beta \)-function and the chiral anomaly of the \( \mathcal{N} = 2 \) SYM theory.

### 3.1 Running Coupling Constant and \( \beta \)-function

To obtain further information on the gauge theory we have to find the precise relationship between the gravitational coordinates and the scales of the gauge theory which, in some sense, is the essence of the gauge/gravity correspondence. In the present case, it is not difficult to find this relationship. In fact, let us denote by \( \mu \) the (arbitrary) mass-scale at which the theory is defined and by \( \Lambda \) the (fixed) dynamically generated scale which is the analogue of \( \Lambda_{\text{QCD}} \) for quantum chromodynamics. Under a scale transformation with parameter \( s \) the mass \( \mu \) transforms according to
\[
\mu \to s \mu , \quad (3.10)
\]
but the physical mass \( \Lambda \) remains fixed. This is the signal of the lack of conformal invariance in this theory. On the other hand, it is known that the complex scalar \( \Psi \) of the \( \mathcal{N} = 2 \) vector multiplet has a (protected) mass-dimension 1, that is
\[
\Psi \to s \Psi , \quad (3.11)
\]
If we now use eqs.\((3.6)\) and \((3.4)\), we easily conclude that the scale transformation \((3.11)\) implies
\[
\rho \to s \rho , \quad (3.12)
\]
\[i.e.\] in the dual gravitational description the scale transformation of the gauge theory is realized as a dilatation of the 2-plane in the transverse space of the wrapped D5 branes. Combining the fact that \( \rho \) is dimensionless with the transformations \((3.10)\) and \((3.12)\), we are lead the following identification
\[
\rho = \frac{\mu}{\Lambda} \quad . \quad (3.13)
\]
This is a sort of “holographic” relation, at least in an extended sense, because it establishes a correspondence between a quantity of the gravity theory (the coordinate \( \rho \)) and a quantity of the gauge theory (the scale \( \mu \)) even if it does not follow from a standard holographic bulk/boundary relation. The fact that \( \rho \) and \( \mu \) are directly proportional to each other can also be established by looking at the energy of a string that stretches up to a distance \( \rho \) in the transverse plane. The energy \( E \) of such a string is proportional to its world-volume per unit time \([22, 7]\), namely

\[
E \sim \int_0^\rho \sqrt{-G_{00}G_{\rho\rho}} \, d\rho \sim \rho
\]

where we have used the metric \([2.24]\). This energy is the natural scale to use in order to regulate the theory and hence it is natural to take \( E \sim \mu \), which in turn leads to \((3.13)\). Notice that this relation implies that the UV (IR) regime of the gauge theory corresponds to the large (small) distance region in the dual gravitational description.

If we insert \((3.13)\) into \((3.8)\) we can obtain the expression of the running coupling constant at the scale \( \mu \), namely

\[
\frac{1}{g_{YM}^2(\mu)} = \frac{N}{4\pi^2} \log \frac{\mu}{\Lambda},
\]

from which we can derive the \( \beta \)-function

\[
\beta(g_{YM}) = -\frac{N}{8\pi^2} g_{YM}^3.
\]

This is the correct field-theory result, numerical factors included. We would like to stress that this \( \beta \)-function has been obtained without using the probe approach which instead is appropriate to describe the theory in the Coulomb branch (for a recent review, see Ref. \([6]\) \[^{10}\]).

### 3.2 Chiral Anomaly

We now discuss the chiral anomaly from the dual gravitational point of view. It is well known that the \( \mathcal{N} = 2 \) SYM theory possesses at the classical level an abelian \( U(1)_R \) symmetry which becomes anomalous at the quantum level. For definiteness, we fix the \( R \)-charge of the complex scalar \( \Psi \) to be 2, so that the gauginos have the conventional \( R \)-charge 1. This means that under a chiral transformation with parameter \( \varepsilon \) we have

\[
\Psi \rightarrow e^{2i\varepsilon} \Psi,
\]

\[^{10}\]The \( \beta \)-function for the \( \mathcal{N} = 2 \) SYM theory in the Coulomb branch has been first derived from the wrapped D5 brane solution in Ref. \([10]\) \[^{16}\] up to some numerical factors.
and that the chiral anomaly corresponds to the following transformation of the \( \theta \)-angle

\[
\theta_{\text{YM}} \rightarrow \theta_{\text{YM}} - 4N \varepsilon .
\]  

(3.18)

From eqs. (3.6) and (3.4) it is natural to interpret the transformation (3.17) as due to a shift in the angle \( \phi_2 \), namely

\[
\phi_2 \rightarrow \phi_2 + 2 \varepsilon ,
\]  

(3.19)

which is also consistent with the anomaly transformation (3.18) in view of eq. (3.9).

Thus, the chiral \( U(1)_R \) transformations of the gauge theory are realized in the dual gravitational description as phase rotations in the transverse plane where the branes can move with a multiplicity factor (2 in our case) that depends on the charge assignments. As one can easily see from the explicit expressions (2.22), (2.23) and (2.26), the transformation (3.19) is an isometry of the metric but it is not a symmetry of the full solution since the R-R 2-form \( C^{(2)} \) explicitly depends on the angle \( \phi_2 \) and is not invariant under (3.19). This fact is the gravitational counterpart of the occurrence of a chiral anomaly in the dual gauge theory [37, 21]. However, not all different values of the R-R 2-form \( C^{(2)} \) are physically different: in fact what really matters is the value of the flux across the 2-sphere

\[
\frac{1}{4\pi^2 \alpha' g_s} \int_{S_2} C^{(2)} = - \frac{N \phi_2}{\pi}
\]

(3.20)

which is allowed to change by integer values. Thus, the following shifts

\[
\phi_2 \rightarrow \phi_2 + \frac{\pi}{N} k
\]

(3.21)

(with \( k \) integer) are true symmetries of the supergravity solution. On the gauge theory side, these shifts correspond to the non-anomalous \( \mathbb{Z}_{4N} \) rotations with parameter \( \varepsilon = (\pi/2N)k \), which change the \( \theta \)-angle by integer multiples of \( 2\pi \), as one can easily see from (3.18).

4 The \( \mathcal{N} = 1 \) Super Yang-Mills Theory

The supergravity solution presented in subsection 2.2 describes the geometry produced by \( N \) D5 branes wrapped on a supersymmetric 2-cycle in such a way that one eighth of the 32 supercharges of Type IIB are preserved. The four-dimensional unwrapped part of the D5 world-volume is the Minkowski space-time where a supersymmetric gauge theory with 4 supercharges is defined. By using the methods
of Ref. [8], one can see that the massless open string modes that propagate in this Minkowski space fill a \( \mathcal{N} = 1 \) vector multiplet in the adjoint representation of \( SU(N) \), and thus the low-energy four-dimensional gauge theory is a pure \( \mathcal{N} = 1 \) SYM theory with gauge group \( SU(N) \).

We now show how to use the supergravity solution (2.47)-(2.49) to extract information on the corresponding gauge theory. To this aim, we consider again the world-volume action (3.1) of a wrapped five-brane with a gauge field strength \( F \) and then extract from it the quadratic terms in \( F \). In contrast to the \( \mathcal{N} = 2 \) case discussed in the previous section, this time there is no direction in which the branes can move, since the entire transverse space is curved as a consequence of the topological twist of the normal bundle that has been performed. Thus, no scalars survive and the non-abelian bosonic action that is obtained with the procedure discussed in Section 3, is simply

\[
S_{YM} = -\frac{1}{4g_{YM}^2} \int d^4x \, F^A_{\alpha \beta} F^{\alpha \beta}_A + \frac{\theta_{YM}}{32\pi^2} \int d^4x \, F^A_{\alpha \beta} \tilde{F}^{\alpha \beta}_A
\]

(4.1)

where

\[
\frac{1}{g_{YM}^2} = \frac{\tau_5 (2\pi)^2 \alpha'^2}{2} \int_0^{2\pi} d\tilde{\phi} \int_0^\pi d\tilde{\theta} \, e^{-3\Phi} \sqrt{-\det G} ,
\]

(4.2)

\[
\theta_{YM} = \frac{\tau_5 (2\pi)^4 \alpha'^2}{2} \int_0^{2\pi} d\tilde{\phi} \int_0^\pi d\tilde{\theta} \, C^{(2)}_{\tilde{\phi} \tilde{\theta}} .
\]

(4.3)

Exactly as in the \( \mathcal{N} = 2 \) case (see eqs.(3.8) and (3.9)) the inverse square of the YM coupling constant is proportional to the volume of the 2-sphere on which the D5 branes are wrapped and the YM vacuum angle is proportional to the flux of the R-R 2-form across this 2-sphere. However, we would like to stress that our definition (4.2) of the YM coupling constant is different from the one used in Ref. [9] and several other papers that followed afterwards, because we compute the volume of the 2-sphere using the ten-dimensional metric of the wrapped D5 branes, and not the metric (2.7) of the domain-wall solution of the seven-dimensional gauged supergravity. Our definition is the natural one from a string-theory point of view; on the other hand, no one doubts that the YM \( \theta \)-angle is related to the flux of the R-R form in ten-dimensions, and thus it appears more acceptable that also the other parameter of the SYM action be related to quantities of the ten-dimensional solution as in (4.2). The difference between our definition of the YM constant and the one of Ref. [9] will have interesting consequences for the gauge theory, especially in the IR regime.

If we insert the explicit form of the supergravity solution (2.47)-(2.49) in eq.(4.2), after simple calculations we obtain

\[
\frac{1}{g_{YM}^2} = \frac{N}{32\pi^2} Y(\rho) \int_0^\pi d\tilde{\theta} \, \sin \tilde{\theta} \left[ 1 + \frac{\cot^2 \tilde{\theta}}{Y(\rho)} \right]^{1/2}
\]

(4.4)
with
\[ Y(\rho) = 4 e^{2h(\rho)} + a(\rho)^2 = 4\rho \coth 2\rho - 1 \quad (4.5) \]
where in the last step we have used eqs. (2.43) and (2.45). It is interesting to observe that the right hand side of eq. (4.4) can be written in terms of the complete elliptic integral of the second kind
\[ E(x) \equiv \int_0^{\pi/2} d\phi \sqrt{1 - x^2 \sin^2 \phi} \quad ; \quad (4.6) \]
in fact, we find
\[ \frac{1}{g_{YM}^2} = \frac{NY(\rho)}{16\pi^2} E\left(\sqrt{\frac{Y(\rho) - 1}{Y(\rho)}}\right) \quad . \quad (4.7) \]

Using the properties of the elliptic integral, it is easy to see that
\[ \frac{1}{g_{YM}^2} \simeq \frac{N\rho}{4\pi^2} \quad \text{for} \ \rho \to \infty \quad , \quad (4.8) \]
\[ \frac{1}{g_{YM}^2} \simeq \frac{N}{32\pi} \quad \text{for} \ \rho \to 0 \quad . \quad (4.9) \]

The large-\( \rho \) behavior (4.8) is the same as in Ref. [9], but the small-\( \rho \) behavior (4.9) is very different, since we get a finite coupling at \( \rho = 0 \), in contrast to the divergent one of Ref. [9].

Finally, inserting in eq. (4.3) the classical profile of the R-R 2-form (2.49), we find
\[ \theta_{YM} = -\frac{N}{2} \psi \quad . \quad (4.10) \]

In the following we will exploit these results to discuss some relevant features of the pure \( \mathcal{N} = 1 \) SYM theory from the gravitational point of view.

### 4.1 Chiral Anomaly

As is well-known, the \( \mathcal{N} = 1 \) SYM theory possesses a classical abelian \( U(1)_R \) symmetry which becomes anomalous at the quantum level. If we assign \( R \)-charge 1 to the gaugino \( \lambda(x) \) so that under a chiral transformation with parameter \( \varepsilon \)
\[ \lambda(x) \to e^{i\varepsilon} \lambda(x) \quad , \quad (4.11) \]
then the presence of the anomaly implies that
\[ \theta_{YM} \to \theta_{YM} - 2N\varepsilon \quad . \quad (4.12) \]

From eq. (4.10) it is clear that the chiral transformations of the SYM theory must be realized in the gravitational description as shifts in the angle \( \psi \). The
only question is which is the multiplicity factor. Now we argue that the correct transformation law of $\psi$ is

$$\psi \rightarrow \psi + 4 \varepsilon,$$  \hfill (4.13)

which indeed allows to obtain the anomaly rule (4.12) directly from (4.10). The factor of 4 in (4.13) has a natural explanation: in fact it is $\psi/2$ that is the appropriate angular variable with period $2\pi$ which must transform with multiplicity 2, just like the angle $\phi_2$ in the $\mathcal{N} = 2$ case (see eq. (3.19)). This argument can be made more rigorous by observing that there is a simple relation between the Euler angles that we used to parameterize the 3-sphere in the up-lifting process of the $\mathcal{N} = 1$ solution and the angles that instead we used in the up-lift of the $\mathcal{N} = 2$ solution. In particular, as is discussed in the Appendix, $\psi = \phi_1 + \phi_2$ (see (A.8)), and since under a chiral transformation of parameter $\varepsilon$ both $\phi_1$ and $\phi_2$ shift by $2\varepsilon$, then eq. (4.13) immediately follows.

The transformation (4.13) is not a symmetry of the supergravity solution (2.47)-(2.49) and is not even an isometry of the metric. This situation is thus very different from the $\mathcal{N} = 2$ case discussed in the previous section. However, there is a region where the transformation (4.13) is an isometry, namely the large-$\rho$ region where the function $a(\rho)$ becomes exponentially small and can be neglected. In fact, if we remove $a$, then all explicit $\psi$ dependence disappears from the metric (2.47) but still remains in the R-R 2-form (2.49), which therefore is not invariant under (4.13). However, as we discussed in Section 3.2, the relevant quantity that should be considered is the flux of $C^{(2)}$ across the 2-sphere, i.e.

$$\frac{1}{4\pi^2\alpha'g_s} \int_{S^2} C^{(2)} \bigg|_{a=0} = - \frac{N \psi}{4\pi}$$  \hfill (4.14)

which is allowed to change by integer values. Thus the transformations (4.13) with $\varepsilon = (\pi/N)k$ and $k$ integer are true symmetries of the supergravity background in the large-$\rho$ region. These are precisely the non-anomalous $\mathbb{Z}_{2N}$ transformations of the gauge theory that correspond to shifts of the $\theta$-angle by integer multiples of $2\pi$. What we have described here is therefore the gravitational counterpart of the well-known breaking of $U(1)_R$ down to $\mathbb{Z}_{2N}$.

However, in the true supergravity solution $a(\rho)$ is not vanishing and thus even the non-anomalous $\mathbb{Z}_{2N}$ transformations are not symmetries of the background. In fact, both in the metric and in the R-R 2-form, $a(\rho)$ appears in front of terms that explicitly involve $\cos \psi$ and $\sin \psi$, and these are clearly invariant only under shifts of $2\pi$. Thus, the only non-anomalous symmetries of the supergravity background are given by (4.13) with $\varepsilon = k\pi$ and $k$ integer. This phenomenon is the gravitational counterpart of the spontaneous breaking of the chiral symmetry from $\mathbb{Z}_{2N} \to \mathbb{Z}_2$. 

20
4.2 Gaugino Condensate

In the previous subsection we have seen that the presence of a non-vanishing \( a(\rho) \) in the supergravity solution is responsible for the spontaneous chiral symmetry breaking to \( \mathbb{Z}_2 \), which, on the gauge theory side, is accompanied by the presence of a non-vanishing value of the gaugino condensate \( \langle \lambda^2 \rangle \equiv \langle 0 \big| \left( \frac{1}{16\pi^2} \right) \lambda^2 | 0 \rangle \). Thus, it appears very natural to conjecture that the gravitational dual of this condensate is precisely the function \( a(\rho) \) that is present in the supergravity solution (2.47)-(2.49). As a matter of fact, this idea has been made more precise in Ref. [14] where a direct relation between \( \langle \lambda^2 \rangle \) and \( a(\rho) \) has been established by adapting the techniques of the AdS/CFT correspondence to the wrapped branes.

The gaugino condensate \( \langle \lambda^2 \rangle \) belongs to a class of gauge invariant operators which do not acquire any anomalous dimensions. This happens because the gaugino condensate is the lowest component of the so-called anomaly multiplet whose scale dimensions are protected by virtue of the fact that its top component is the trace of the energy-momentum tensor which is a conserved current. Thus, since the engineering dimension of \( \langle \lambda^2 \rangle \) is 3, we have

\[
\langle \lambda^2 \rangle = c \Lambda^3
\]  

(4.15)

where \( \Lambda \) is the (exact) dynamical scale of the \( \mathcal{N} = 1 \) SYM theory and \( c \) a computable constant. Several explicit calculations lead to \( c = 1 \) [38].

In view of the findings of Ref. [14] and of our previous discussion, we now propose to identify the function \( a(\rho) \) given in (2.45) with the gaugino condensate \( \langle \lambda^2 \rangle \) measured in units of the (arbitrary) mass scale \( \mu \) that is introduced to regulate the theory. Thus, we write

\[
a(\rho) = \frac{\Lambda^3}{\mu^3}.
\]  

(4.16)

This crucial equation allows us to establish a precise relation between the supergravity radial coordinate \( \rho \) and the scales of the gauge theory, and is the strict analogue of the “holographic” relation (3.13) of the \( \mathcal{N} = 2 \) model. Notice that since \( a(\rho) \to 0 \) for \( \rho \to \infty \), once again the large-\( \rho \) region corresponds to the UV regime of the gauge theory, and vice-versa the small-\( \rho \) region corresponds to the IR regime. In the following we will exploit the relation (4.16) to compute the perturbative \( \beta \)-function of the pure \( \mathcal{N} = 1 \) SYM theory and get some insights on non-perturbative corrections.

4.3 Running Coupling Constant and \( \beta \)-function

The gauge coupling constant has been defined in (4.4) as a function of the radial coordinate \( \rho \) which, in turn, is related to the scales of the gauge theory as prescribed
Thus, by combining these two equations and eliminating the \( \rho \) dependence, we could in principle obtain an exact expression for the running coupling constant of the \( \mathcal{N} = 1 \) SYM theory. Unfortunately, it appears very difficult to manipulate eqs. (4.4) and (4.16) in an analytic way and exhibit the running coupling constant in a closed form. However, despite this fact, many interesting results can still be derived.

First of all, if we use in (4.16) the explicit expression of \( a(\rho) \) given in (2.45), we easily obtain

\[
\frac{\partial \rho}{\partial \log(\mu/\Lambda)} = \frac{3}{2} \left[ \frac{1}{1 - (2\rho)^{-1} + 2 e^{-4\rho} (1 - e^{-4\rho})^{-1}} \right].
\] (4.17)

The right hand side has a nice interpretation in the large-\( \rho \) region, i.e. in the UV. In fact, for \( \rho \to \infty \) the fraction in square brackets receives two types of contributions: one from terms that are negative powers of \( \rho \), and the other from terms that involve also powers of \( e^{-4\rho} \). If we recall that in the UV region \( \rho \) is directly proportional to the inverse square of the gauge coupling constant as shown in (4.8), it is very easy to realize that the first type of terms corresponds to perturbative loop contributions while the second describes non-perturbative instanton-like effects. Actually, we can be more precise and write explicit formulas. Let us first concentrate on the power-like part and neglect all terms that vanish exponentially for \( \rho \to \infty \). Then, if we use the leading UV asymptotic behavior (4.8) and thus trade \( \rho \) for \( 4\pi^2/(N g_{YM}^2) \), on the one hand we have

\[
\frac{\partial \rho}{\partial \log(\mu/\Lambda)} = \frac{-8\pi^2}{Ng_{YM}^3} \beta(g_{YM})
\] (4.18)

where \( \beta(g_{YM}) \) is the \( \beta \)-function, and on the other hand from (4.17) we have

\[
\frac{\partial \rho}{\partial \log(\mu/\Lambda)} = \frac{3}{2} \left[ 1 - \frac{Ng_{YM}^2}{8\pi^2} \right]^{-1}.
\] (4.19)

Combining these two equations, we obtain

\[
\beta(g_{YM}) = -\frac{3Ng_{YM}^3}{16\pi^2} \left[ 1 - \frac{Ng_{YM}^2}{8\pi^2} \right]^{-1}.
\] (4.20)

This is the complete perturbative NSVZ \( \beta \)-function of the pure \( \mathcal{N} = 1 \) SYM theory with gauge group \( SU(N) \) in the Pauli-Villars regularization scheme [23]. It is remarkable to see that a classical supergravity solution representing wrapped D5 branes is able to completely reproduce it!

It is interesting to observe that the 1-loop approximation corresponds to keep in the supergravity solution only the leading terms in the \( \rho \to \infty \) expansion. In
fact, if in (4.17) we approximate the square bracket to 1, we get \( \rho \sim \frac{3}{2} \log(\mu/\Lambda) \) which is indeed the correct relation between the radial coordinate and the scale \( \Lambda \) at 1-loop [14]. If instead we keep also the \( \rho^{-1} \) term in (4.17), we obtain the complete perturbative result (4.19) and the NSVZ \( \beta \)-function. However, the full supergravity solution contains more than this, since there are also terms that vanish exponentially for large \( \rho \) and correspond to non-perturbative effects. Keeping these terms in (4.17) amounts to replace the expression inside the square brackets of (4.19) and (4.20) with

\[
1 - \frac{N g_{YM}^2}{8\pi^2} + \frac{2 \exp\left(-\frac{16\pi^2}{Ng_{YM}^2}\right)}{1 - \exp\left(-\frac{16\pi^2}{Ng_{YM}^2}\right)}.
\]

(4.21)

From this formula we see that the non-perturbative corrections have the form of instantons with fractional charge \( 2k/N \) where \( k \) is a positive integer. The fractional instantons have recently been shown in Ref. [39] to play a crucial role in the pure \( \mathcal{N} = 1 \) SYM theory since they are the elementary field configurations that directly contribute to the gaugino condensate. It would be very interesting to check with a field theory analysis whether they also contribute to the \( \beta \)-function as our supergravity analysis suggests.

We conclude by observing that the presence of these non-perturbative effects modifies the running of the coupling constant which remains finite even at low energy. In fact, from (4.16) we see that the IR limit \( \mu \to \Lambda \) corresponds to the limit \( \rho \to 0 \) in which the coupling constant smoothly tends to a finite value (see eq. (4.9)). This smooth behaviour and the absence of a Landau pole qualitatively resembles the soft confinement scenario of QCD.

4.4 Instantons

It well-known that a system of \( N \) D3 branes and \( k \) D(-1) branes in flat space describes the \( k \)-instanton sector of the maximally supersymmetric \( SU(N) \) Yang-Mills theory in four dimensions [40]. This idea has been successfully generalized to other less supersymmetric configurations by considering, for example, systems of D3/D(-1) fractional branes in orbifold backgrounds. Thus, it appears very natural to think that the same happens also in the context of wrapped branes. The idea is then to consider a system of \( N \) D5 branes and \( k \) D1 branes wrapped on the same supersymmetric 2-cycle and see whether the \( k \) wrapped D-strings branes account for the \( k \) instanton contributions. This analysis has been performed in Ref. [28] where, however, negative conclusions have been reached. We now show that this is not the case and that, also in the wrapped brane case, the instantons are correctly represented by D branes with four dimensions less than the branes which support the gauge degrees of freedom.
The world-volume action of a Euclidean D1 brane in the string frame is given by

\[ S = \tau_1 \int d^2 \xi \ e^{-\Phi} \sqrt{\det G} - i \tau_1 \int C^{(2)} \]  

(4.22)

where

\[ \tau_1 = \frac{1}{2\pi g_s \alpha'} \]  

(4.23)

and the bulk fields are understood to be the pull-backs onto the brane world-volume which is parameterized \( \xi = \{ \tilde{\theta}, \tilde{\varphi} \} \). If the D1 brane is in the background geometry produced by \( N \) wrapped D5 branes, we must use in the above action the metric, the dilaton and the R-R 2-form of the solutions reviewed in Section 2. In particular in the \( \mathcal{N} = 1 \) case, by using the explicit form of the metric given in (2.47) we get

\[ \sqrt{\det G} = \frac{\psi}{4\lambda^2} Y(\rho) \sin \tilde{\theta} \sqrt{1 + \frac{\cot^2 \tilde{\theta}}{Y(\rho)}} \]  

(4.24)

where \( Y(\rho) \) is defined in (4.5). Inserting this result into the action (4.22), recalling from (2.49) that the relevant component of the R-R 2-form is

\[ C^{(2)}_{\tilde{\theta} \tilde{\varphi}} = - \frac{1}{4\lambda^2} \psi \sin \tilde{\theta} \]  

(4.25)

and performing the integrals over the world-volume coordinates, we find

\[ S = \tau_1 \frac{\pi}{\lambda^2} Y(\rho) E \left( \sqrt{\frac{Y(\rho) - 1}{Y(\rho)}} \right) + i \tau_1 \frac{\pi \psi}{\lambda^2} \]  

(4.26)

Finally, if we express \( \tau_1 \) and \( \lambda \) in terms of the string parameters as in (4.23) and (2.51), and use the definitions of the YM coupling constant and the \( \theta \)-angle given in (4.7) and (4.10), it is easy to see that

\[ S = \frac{8\pi^2}{g_{\text{YM}}^2} - i \theta_{\text{YM}} \]  

(4.27)

which is the correct form of the instanton action! Clearly, to obtain the \( k \)-instanton action we should consider \( k \) wrapped D1 branes in the background of \( N \) wrapped D5 branes, which amounts simply to multiply the result (4.27) by \( k \).

Therefore, we conclude that the gauge theory instantons of the \( \mathcal{N} = 1 \) SYM theory are indeed represented by D1 branes that wrap the same supersymmetric 2-cycle of the D5 branes, as one should expect from general considerations \(^{11}\). We also notice that the above calculations can be performed in the \( \mathcal{N} = 2 \) SYM theory using the supergravity solution of Section 2.1 which leads to similar results.

\(^{11}\)The discrepancies with Ref. [28] originate from a different choice of normalization for the \( SU(N) \) generators as well as from a different definition of the YM coupling constant that for us is proportional to the volume of the 2-cycle computed with the ten-dimensional metric and not with the seven-dimensional domain-wall metric as in Ref. [28].
5 Conclusions

We have shown that many interesting features of the pure $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM theories in four dimensions are quantitatively encoded in the supergravity solutions that describe D5 branes wrapped on supersymmetric 2-cycles. These features comprise the running of the gauge coupling constant, the $\beta$-function, the chiral anomaly and instantons. In the $\mathcal{N} = 1$ case we have also discussed the gravitational dual of the gaugino condensate and exploited it to obtain the complete NSVZ $\beta$-function. Moreover we have found the occurrence of non-perturbative corrections in the form of fractional instantons with topological charge $2/N$ that smooth out the running of the coupling constant which never diverges. The precise meaning of the fractional instanton corrections and their relevance for the IR physics deserve, however, further study from a field theory point of view. Finally, we observe that in the $\mathcal{N} = 2$ case the supergravity analysis that we have presented allows to obtain only the perturbative part of the $\beta$-function and no instanton corrections. In this respect it is worth pointing out that, differently from the $\mathcal{N} = 1$ case, the $\mathcal{N} = 2$ supergravity solution exhibits a naked singularity at small distances where it becomes unphysical. It would very interesting to see whether it is possible to resolve this singularity and eventually find the known non-perturbative instanton corrections of the $\mathcal{N} = 2$ SYM theory from a resolved supergravity solution. Another interesting development would be to explore if the MN solution and the results we have found here can be used to quantitatively investigate other features of the $\mathcal{N} = 1$ SYM theory and if they can be extended also to theories in which supersymmetry is completely broken.

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A Parameterizations of the 3-sphere

There are several ways to describe a 3-sphere $S^3$. Here we briefly describe the two parameterizations we have used in Section 2 to up-lift the domain wall-solution from seven to ten dimensions.
The defining equation for a unit 3-sphere is

\[ \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 1 \]  
(A.1)

One explicit parameterization of the \( \mu_i \)'s is the following

\[ \mu_1 = \cos \theta \cos \phi_1 \, , \, \mu_2 = \cos \theta \sin \phi_1 \, , \]
\[ \mu_3 = \sin \theta \cos \phi_2 \, , \, \mu_4 = \sin \theta \sin \phi_2 \]  
(A.2)

with \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( 0 \leq \phi_{1,2} \leq 2\pi \). In this parameterization, the metric of the 3-sphere is

\[ ds^2 = \sum_{i=1}^{4} d\mu_i^2 = d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2 \]  
(A.3)

These are the coordinates that we used in Section 2.1 in order to up-lift to ten dimensions the domain-wall solution in the \( N = 2 \) case.

A second convenient parameterization of the \( \mu_i \)'s is in terms of the Euler angles, namely

\[ \mu_3 = \cos \frac{\theta'}{2} \cos \frac{\psi + \phi}{2} \, , \, \mu_4 = \cos \frac{\theta'}{2} \sin \frac{\psi + \phi}{2} \, , \]
\[ \mu_2 = \sin \frac{\theta'}{2} \cos \frac{\psi - \phi}{2} \, , \, \mu_1 = \sin \frac{\theta'}{2} \sin \frac{\psi - \phi}{2} \]  
(A.4)

with \( 0 \leq \theta' \leq \pi \), \( 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi \leq 4\pi \). The corresponding left-invariant 1-forms of \( S^3 \) are

\[ \sigma^1 = \frac{1}{2} \left[ \cos \psi d\theta' + \sin \theta' \sin \psi d\phi \right] \, , \, \sigma^2 = -\frac{1}{2} \left[ \sin \psi d\theta' - \sin \theta' \cos \psi d\phi \right] \, , \]
\[ \sigma^3 = \frac{1}{2} \left[ d\psi + \cos \theta' d\phi \right] \]  
(A.5)

which close the \( SU(2) \) differential algebra

\[ d\sigma^a = -\epsilon^{abc} \sigma^b \wedge \sigma^c \]  
(A.6)

In this parameterization the metric of the unit 3-sphere is

\[ ds^2 = \sum_{i=1}^{4} d\mu_i^2 = \sum_{a=1}^{3} (\sigma^a)^2 = \frac{1}{4} \left[ (d\psi + \cos \theta' d\phi)^2 + \sin^2 \theta' d\phi^2 + d\theta'^2 \right] \]  
(A.7)

These are the coordinates that we used in Section 2.2 in order to up-lift to ten dimensions the domain-wall solution in the \( N = 1 \) case.

By comparing (A.2) and (A.4), it is immediate to see that the two sets of angles are related as follows

\[ \theta' = 2\theta \, , \, \phi = \phi_1 - \phi_2 \, , \, \psi = \phi_1 + \phi_2 \]  
(A.8)
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