The Shannon-entropy-based uncertainty relation for $D$-dimensional central potentials

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Abstract
The uncertainty relation based on the Shannon entropies of the probability densities in position and momentum spaces is improved for quantum systems in arbitrary $D$-dimensional spherically symmetric potentials. To find this, we have used the $L^p–L^q$ norm inequality of De Carli and the logarithmic uncertainty relation for the Hankel transform of Omri. Applications to some relevant three-dimensional central potentials are shown.

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1. Introduction

The position–momentum uncertainty principle is likely to be the most prominent difference between classical and quantum physics. The first mathematical realization of this principle was proposed by Heisenberg [1] and Kennard [2] in terms of the standard deviations of the quantum-mechanical probability densities of the particle in position and momentum spaces. However, the so-called Heisenberg uncertainty relation is neither the most appropriate nor the most stringent [3–5]. Indeed, the standard deviation is a measure of separation of the region(s) of concentration of the probability cloud from the centroid (a particular point of the distribution), rather than a measure of the extent to which the distribution is in fact concentrated [4–6]. Information theory [7, 8] provides more appropriate local (Fisher information) and global (Shannon, Rényi and Tsallis entropies) uncertainty measures. The Shannon entropy has been argued to be the best global measure of the spreading of a probability distribution according to some criteria [5, 7, 9]; see also [10, 11] for further details.

The (differential) Shannon entropies of the position and momentum continuous probability distributions, $\rho(\vec{r})$ and $\gamma(\vec{p})$, respectively, are known to fulfill the so-called entropic uncertainty relation (units with $\hbar = 1$ are used)

$$S[\rho] + S[\gamma] \geq D(1 + \ln \pi),$$

(1)
where
\[ S[\rho] = - \langle \ln \rho \rangle = - \int_{\mathbb{R}^D} \rho(\vec{r}) \ln \rho(\vec{r}) \, d^D r \]
(2)
denotes the D-dimensional position Shannon entropy, and \( S[\gamma] \) denotes the corresponding momentum Shannon entropy. This relation was conjectured independently in 1957 by Everett [12] and Hirschman [13] and proved in 1975 by Beckner [14] and Bialynicki-Birula and Mycielski [15]. It provides a strict improvement on the standard Heisenberg relation [5, 15].

The Rényi entropy of the position probability \( \rho(\vec{r}) \) is defined [16] by
\[ R_q[\rho] = \frac{1}{1-q} \ln \int_{\mathbb{R}^D} [\rho(\vec{r})]^q \, d^D r, \quad q > 0, q \neq 1. \]
(3)
Note that this quantity is a generalization of the Shannon entropy given by (2) so that
\[ \lim_{q \to 1} R_q[\rho] = S[\rho]. \]
Due to this property, it is often convenient to consider the general case of the Rényi entropy and use the limit \( q \to 1 \) to obtain the particular results for the Shannon entropy. We shall use this strategy in section 2.

The main goal of this paper is to refine the entropic uncertainty relation (1) for particles moving in a central potential \( V_D(r) \), \( r = |\vec{r}| \). The D-dimensional \( (D \geq 2) \) central-field approximation has been successfully applied in one- and many-body physics from the early days of quantum mechanics up until now, to explain numerous physical properties and phenomena of natural systems (see, e.g., [17–24]). For example, it is known to be the theoretical basis of the periodic table of the chemical elements [25] together with the Pauli exclusion principle. Moreover, the central potentials are very often being used as prototypes for numerous other purposes and systems not only in the three-dimensional world (e.g., oscillator-, Coulomb- and van der Waals-like potentials) but also in non-relativistic and relativistic D-dimensional physics (see, e.g., [21, 22, 24]). Recently, it has also been applied to study the behaviour of the quantum dots and wires as well as to interpret the experiments of dilute bosonic and fermionic systems in magnetic traps at extremely low temperatures [23, 26, 27], which is provoking a fast development of a density-functional theory of independent particles moving in multidimensional central potentials [28, 29].

We begin with the Schrödinger equation of the corresponding D-dimensional central force problem:
\[ \left[-\frac{1}{2} \nabla^2_D + V_D(r)\right] \Psi(\vec{r}) = E\Psi(\vec{r}), \]
(4)
with \( \nabla^2_D \) denoting the Laplace operator [21, 30, 31] associated with the position vector in hyperspherical coordinates \( \vec{r} = (r, \theta_1, \ldots, \theta_{D-1}) = (r, \Omega_{D-1}), \Omega_{D-1} \in S^{D-1} \) (surface of the D-dimensional sphere), which is given by
\[ \nabla^2_D \equiv \frac{\partial^2}{\partial r^2} + \frac{D - 1}{r} \frac{\partial}{\partial r} - \frac{\Lambda^2_{D-1}}{r^2}, \]
and the squared hyperangular momentum operator \( \Lambda^2_{D-1} \) is known to fulfil the eigenvalue equation
\[ \Lambda^2_{D-1} \mathcal{Y}_{l,|\mu|}(\Omega_{D-1}) = l(l + D - 2)\mathcal{Y}_{l,|\mu|}(\Omega_{D-1}). \]
The symbol \( \mathcal{Y}_{l,|\mu|}(\Omega_{D-1}) \) denotes the hyperspherical harmonics [30, 32] characterized by the \( D - 1 \) hyperangular quantum numbers \( (l \equiv \mu_1, \mu_2, \ldots, \mu_{D-1} \equiv m) \equiv (l, |\mu|) \), which are integer numbers with values \( l = 0, 1, 2, \ldots \), and \( l \geq \mu_2 \geq \cdots \geq \mu_{D-2} \geq |\mu_{D-1}| \geq 0 \). Note that for \( D = 2 \) we only have one quantum number \( l \in \mathbb{Z} \).
The eigenfunctions $\Psi(\vec{r})$ of this problem may be written as
\[
\Psi(\vec{r}) \equiv R(r)Y_{l,|\mu|}(\Omega_{D-1}) = r^{l+D/2}u(r)Y_{l,|\mu|}(\Omega_{D-1}),
\]
where the reduced radial eigenfunction $u(r)$ is the physical solution of the one-dimensional Schrödinger equation in the radial coordinate $r$:
\[
\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{L(L+1)}{2r^2} + V_D(r)\right]u_{E,l}(r) = Eu_{E,l}(r),
\]
where we have used the notation $L = l + (D - 3)/2$ for the grand hyperangular momentum. Since $L(L + 1) = l(l + D - 2) + (D - 1)(D - 3)/4$, it is worth realizing from equation (6) that a particle moving in a $D$-dimensional potential is subject to two additional forces besides the force coming from the external potential $V_D(r)$: the centrifugal force associated with non-vanishing hyperangular momentum, and a quantum fictitious force associated with the so-called quantum-centrifugal potential $(D - 1)(D - 3)/8r^2$, which has a purely dimensional origin [33]. This potential vanishes for $D = 1$ and 3, being negative for $D = 2$ and positive for $D \geq 4$. Then, the quantum fictitious force, which exists irrespective of the hyperangular momentum and has a quadratic dependence on the dimensionality, has an attractive character when $D = 2$ and is repulsive for $D \geq 4$. Let us also remark that $L = l$ when $D = 3$.

From equation (5), one can find that the allowed quantum-mechanical state $(E, l, \{\mu\})$ has the following probability density:
\[
\rho(\vec{r}) = \frac{|u(r)|^2}{\mu_{D-1}} |Y_{l,|\mu|}(\Omega_{D-1})|^2,
\]
where, according to (4) and (6), we know that $u(r) = u_{E,l}(r)$ and $\rho(\vec{r}) = \rho_{E,l,|\mu|}(\vec{r})$. In addition, the normalization-to-unity of the wavefunction $\Psi(\vec{r}) \equiv \Psi_{E,l,\{\mu\}}(\vec{r})$ yields that
\[
\int_0^\infty |u(r)|^2 dr = 1
\]
for the reduced radial eigenfunction, where we have taken into account the normalization condition of the hyperspherical harmonics:
\[
\int_{g_{D-1}} |Y_{l,|\mu|}(\Omega_{D-1})|^2 d\Omega_{D-1} = 1.
\]

Then, according to equations (2) and (7), the Shannon entropy of the $D$-dimensional density $\rho(\vec{r})$ is given by
\[
S[\rho] = S[\omega] + (D - 1)\langle \ln r \rangle + S(Y_{l,|\mu|}),
\]
where $S[\omega]$ denotes the Shannon entropy of the one-dimensional probability density $\omega(r) = |u(r)|^2$:
\[
S[\omega] = -\int_0^\infty \omega(r) \ln \omega(r) \, dr.
\]
Similarly, $S(Y_{l,|\mu|})$ gives the Shannon entropy of the hyperangular probability density $|Y_{l,|\mu|}|^2$:
\[
S(Y_{l,|\mu|}) = -\int_{g_{D-1}} |Y_{l,|\mu|}|^2 \ln |Y_{l,|\mu|}|^2 \, d\Omega_{D-1}
\]
with the volume element
\[
d\Omega_{D-1} = \prod_{j=1}^{D-1} (\sin \theta_j)^{\alpha_j} d\theta_j, \quad \alpha_j = D - j - 1.
\]
Moreover, the logarithmic expectation value $\langle \ln r \rangle$ is defined as
\[
\langle \ln r \rangle = \int_0^\infty \rho(\vec{r}) \ln r \, dr.
\]
A parallel analysis in momentum space yields the following expression:

$$S[\gamma] = S[\tilde{\omega}] + (D - 1)(\ln p) + S[\mathcal{Y}_{l,\mu}^\gamma],$$

(10)

for the momentum Shannon entropy of a particle in the $D$-dimensional central potential $V_D(r)$, where $S[\tilde{\omega}]$ and $(\ln p)$ have analogous expressions to $S[\omega]$ and $(\ln r)$ but with respect to the momentum density

$$\gamma(\vec{p}) = |\tilde{\Psi}(\vec{p})|^2 = \frac{[\tilde{u}(\vec{p})]^2}{2p^{D-1}}|\mathcal{Y}_{l,\mu}^\gamma(\Omega_{D-1})|^2,$$

where $\tilde{\Psi}(\vec{p})$ is the Fourier transform of $\Psi(\vec{r})$ (keep in mind that the hyperspherical harmonics remains invariant under this transformation). The reduced radial momentum eigenfunction $\tilde{u}(p)$ is related to $u(r)$ by means of the Hankel transform

$$\tilde{u}(p) = (-i)^l \int_0^\infty \sqrt{rp}J_{l+D/2-1}(rp)u(r) \, dr,$$

(11)

with $J_l(x)$ symbol being the first-kind Bessel function of order $\nu$. Let us call $\tilde{\omega}(p) = |\tilde{u}(p)|^2$ the momentum-reduced radial probability density. The overall phase $(-i)^l$ will play no role in our investigations.

Summing equations (8) and (10), we obtain

$$S[\rho] + S[\gamma] = S[\omega] + S[\tilde{\omega}] + (D - 1)(\ln r) + (\ln p) + 2S[\mathcal{Y}_{l,\mu}],$$

(12)

which will be the basic starting point to obtain our goal. Let us emphasize that the hyperangular Shannon entropy $S[\mathcal{Y}_{l,\mu}]$ is under control since the hyperspherical harmonics are well-known mathematical objects so that they do not depend on the external potential $V_D(r)$. On the other hand, the remaining terms $S[\omega] + S[\tilde{\omega}]$ and $(\ln r) + (\ln p)$ do depend on $V_D(r)$ so that we will try to bound them from below in terms of the hyperquantum number $l$ that characterizes the state. This will be done in sections 2 and 3 by use of the $L^p - L^q$ norm inequality of De Carli [34] and the logarithmic uncertainty relation of Omri [35], respectively. In section 4, the new entropic uncertainty relation is given and discussed. Let us advance that the encountered lower bound to the Shannon-entropy sum only depends on the hyperangular quantum numbers in an analytical form. Then, in section 5, the new entropic relation is examined for two relevant three-dimensional central potentials: the Coulomb and the harmonic oscillator potentials. Finally, some conclusions and open problems are given in section 6.

### 2. Entropic uncertainty relations for the reduced radial wavefunctions

In this section, we will find the Rényi-entropy-based and the Shannon-entropy-based uncertainty relations of the reduced radial wavefunctions $u(r)$ and $\tilde{u}(p)$ in position and momentum spaces, respectively.

Taking into account that $\tilde{u}(p)$ is the Hankel transform (11) of $u(r)$, we can directly apply theorem 1.2 of De Carli [34] to write

$$\left(\int_0^\infty |\tilde{u}(p)|^q \, dp\right)^{1/q} \leq C(q, q'; v)\left(\int_0^\infty |u(r)|^{q'} \, dr\right)^{1/q'},$$

(13)

for $1 < q' \leq 2$, and $1/q' + 1/q = 1$, with the constant

$$C(q, q'; v) = \frac{A(q'; v)}{A(q; v)}, \quad A(q; v) = 2^{\frac{v}{2}} \frac{\sqrt{\pi \Gamma\left(\frac{v}{2}\right)}}{\Gamma\left(\frac{v + 1}{2}\right)},$$

where $v = l + D/2 - 1$. 

Let us now rewrite this inequality for the reduced radial probability densities \( \omega(r) \) and \( \tilde{\omega}(p) \). For this purpose, it is convenient to use the parameters \((\alpha, \beta)\) related to \((q, q')\) by \( q = 2\alpha \) and \( q' = 2\beta \), so that \( 1/\alpha + 1/\beta = 2 \). Then, equation (13) transforms into the following inequality:
\[
\left( \int_0^\infty [\tilde{\omega}(p)]^\alpha \, dp \right)^{\frac{1}{\alpha}} \leq C \left( \int_0^\infty [\omega(r)]^\beta \, dr \right)^{\frac{1}{\beta}},
\]
where \( C = C(2\alpha, 2\beta; \nu) \). Then, the Neperian logarithm of this inequality multiplied by the negative factor \( 1/(1 - \alpha) \) (since \( q \geq 2 \iff \alpha \geq 1 \)) yields that
\[
\frac{1}{1 - \alpha} \ln \left( \int_0^\infty [\tilde{\omega}(p)]^\alpha \, dp \right) \geq \frac{2\alpha \ln C}{1 - \alpha} - \frac{1}{1 - \beta} \ln \left( \int_0^\infty [\omega(r)]^\beta \, dr \right),
\]
where we have used the fact that \( \alpha/(1 - \alpha) = -\beta/(1 - \beta) \).

Then, recalling that the \( q \)-th order Rényi entropy of a probability density \( f(x) \), with \( 0 < x < \infty \), is given according to equation (3) by
\[
R_q[f] = -\frac{1}{1 - q} \ln \int_0^\infty [f(x)]^q \, dx,
\]
the last inequality gives rise to the following Rényi-entropy-based uncertainty relation for the position- and momentum-reduced radial probability densities:
\[
R_\rho[\omega] + R_\beta[\tilde{\omega}] \geq \frac{2\alpha \ln[A(2\alpha; \nu)]}{\alpha - 1} + \frac{2\beta \ln[A(2\beta; \nu)]}{\beta - 1}.
\]
Finally, let us highlight that since \( \beta = \alpha/(2\alpha - 1) \) and making the limit \( \alpha \to 1 \), this inequality yields the Shannon-entropy-based uncertainty relation for the reduced radial probability densities \( \omega(r) \) and \( \tilde{\omega}(p) \):
\[
S[\omega] + S[\tilde{\omega}] \geq C_\nu
\]
with
\[
C_\nu = 2l + D + 2 \ln \left[ \frac{\Gamma(l + \nu)}{\Gamma(l + D)} \right] = (2l + D - 1)\psi \left( l + \frac{D}{2} \right),
\]
where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) denotes the well-known digamma function. To obtain (15) from equation (14), we have taken into account that the Shannon entropy of a probability density \( f(x) \), \( S[f] = -\int_0^\infty f(x) \ln f(x) \, dx \), (see [7]) is the limiting case \( \alpha \to 1 \) of the Rényi entropy \( R_\alpha[f] \).

### 3. Logarithmic uncertainty relation for central potentials

In this section, we will show that the position and momentum probability densities of a particle moving in a spherically symmetric potential \( V_D(r) \) satisfy the following uncertainty relation:
\[
\langle \ln r \rangle + \langle \ln p \rangle \geq \psi \left( \frac{2l + D}{4} \right) + \ln 2, \quad l = 0, 1, 2, \ldots
\]
(16)

This inequality improves for central potentials the Beckner logarithmic uncertainty relation [36] of general validity, in which the lower bound on the logarithmic sum is \( \psi(D/4) + \ln 2 \).

To prove expression (16), we begin with the logarithmic uncertainty relation of Omri [35]:
\[
\int_0^\infty |f(r)|^2 \ln r \, dr \, d\lambda_\alpha(r) + \int_0^\infty |\tilde{f}(p)|^2 \ln p \, d\lambda_\mu(p) \geq \left[ \psi \left( \frac{\mu + 1}{2} \right) + \ln 2 \right] N_\mu,
\]

(17)
where \( f \in L^2(0, \infty) \), the measure \( d\lambda_\mu(r) \), \( r \in [0, \infty) \), is given by
\[
d\lambda_\mu(r) = \frac{r^{2\mu+1}}{2^{\mu} \Gamma(\mu + 1)} \, dr,
\]
and the Hankel transform \( \tilde{f}(p) \) of order \( \mu \) is defined by
\[
\tilde{f}(p) = \int_0^\infty f(r) j_\mu(rp) \, d\lambda_\mu(r), \quad \mu \geq -\frac{1}{2},
\]
where \( j_\mu(z) \) is the normalized spherical Bessel function of the first kind and is given by
\[
j_\mu(z) = \frac{2^\mu \Gamma(\mu + 1)}{\pi^\mu} J_\mu(z).
\]
Moreover, the normalization constant \( N_\mu \) is defined as
\[
N_\mu = \int_0^\infty |f(r)|^2 \, d\lambda_\mu(r).
\]
If we take the following function:
\[
f(r) = r^{l-D/2} u(r),
\]
together with the value \( \mu = l + D/2 - 1 \), we find that
\[
\begin{align*}
\int_0^\infty |f(r)|^2 \ln r \, d\lambda_\mu(r) &= N_\mu \int_0^\infty \omega(r) \ln r \, dr = N_\mu \langle \ln r \rangle, \quad (18) \\
\int_0^\infty |\tilde{f}(p)|^2 \ln p \, d\lambda_\mu(p) &= N_\mu \int_0^\infty \tilde{\omega}(p) \ln p \, dp = N_\mu \langle \ln p \rangle, \quad (19)
\end{align*}
\]
and the explicit value of the normalization constant is given by
\[
N_\mu = \frac{1}{2^{\mu} \Gamma(\mu + 1)}. \quad (20)
\]
With expressions (18)–(20), the inequality (17) boils down to the required relation (16).

4. Shannon-entropy-based uncertainty relation for central potentials

In this section, we will propose and discuss the Shannon-entropy-based uncertainty relation for general \( D \)-dimensional central potentials, which provides a lower bound to the sum \( S[\rho] + S[\gamma] \) in terms of \( D \) and the hyperquantum numbers of the corresponding state.

The entropy sum given in equation (12) is bounded from below as
\[
S[\rho] + S[\gamma] \geq B_{l,\{\mu\}}, \quad (21)
\]
where
\[
B_{l,\{\mu\}} = 2l + D + 2 \ln \left[ \frac{\Gamma\left(\frac{l + D}{2}\right)}{\frac{\Gamma\left(\frac{l + D}{2} - 1\right)}{2}} \right] - (2l + D - 1) \psi\left(\frac{l + D}{2}\right)
\]
\[
+ (D - 1) \left( \psi\left(\frac{2l + D}{4}\right) + \ln 2 \right) + 2S(Y_{l,\{\mu\}}), \quad (22)
\]
where \( S(Y_{l,\{\mu\}}) \) is given by equation (9), and inequalities (15) and (16) have been taken into account to bound the terms \( S[\omega] + S[\tilde{\omega}] \) and \( \langle \ln r \rangle + \langle \ln p \rangle \), respectively. Note that this bound depends on the hyperangular quantum numbers \( (l, \{\mu\}) \) and the dimensionality \( D \), but not on the principal (energetic) quantum number \( n \) because the analytical form of the central potential \( V_D(r) \) was not specified.
For completeness, the central bound (22) and the general bound given by (1) are represented in figure 1, for $D = 3$, as a function of the quantum numbers $(l, m)$. The horizontal line represents the general bound (equal to $3(1 + \ln \pi)$ in this case), while the points represent the values of the central bound for different quantum numbers $(l, m)$. The comparison of these bounds shows that the new lower bound is bigger (hence, better) than the general one for all values $(l, m)$ except for $l = m = 0$. For dimensions other than 3, it might occur that the bound does not get improved for more than one set of values $(l, \mu_j)$; this is, e.g., the case of the values $(l, \mu_2, \mu_3) = (0, 0, 0), (1, 0, 0)$ and $(1, 1, 0)$, when $D = 4$. The reason might be due either to the separation made in equation (12) between the sum of the Shannon entropies of the reduced densities and the logarithmic uncertainty sum, or to the fact that, when $l = 0$, the logarithmic uncertainty relation (16) does not represent any improvement with respect to the general inequality by Beckner [36], i.e. not sharp enough in this case.

Finally, let us point out that in the asymptotic case $l \to \infty$, the Shannon entropy of the hyperspherical harmonics $\mathcal{Y}_{l, \{\mu\}}$ have the asymptotic behaviour [37]

$$S(\mathcal{Y}_{l, \{\mu\}}) = O(1)$$

for finite values of $\mu_j$, and

$$S(\mathcal{Y}_{l, \{\mu\}}) = -\frac{1}{2}(K - 1) \ln l + O(1),$$

having the parameters $\mu_j$ ($j = 2, \ldots, D - 1$), the values $l - a_j$, with $a_j \in \mathbb{N}$ fixed, for $j = 2, \ldots, K$, being finite otherwise (i.e. for $j = K + 1, \ldots, D - 1$). Then, the asymptotic behaviour of the new bound (22) in these two cases is given by

$$B_{l, \{\mu\}} = (D - 1) \ln l + O(1)$$

and

$$B_{l, \{\mu\}} = (D - K) \ln l + O(1),$$

respectively.

Thus, the larger the value of $l$, the bigger the new central bound, and the larger the improvement with respect to the general bound (1).
5. Applications to hydrogenic and isotropic oscillator potentials

In this section, we will discuss the Shannon-entropy-based uncertainty relation \( (21) \) for the two main prototypes of central systems: the hydrogenic and isotropic oscillator systems.

5.1. Hydrogenic systems

In this case, \( V_D(r) = -\frac{1}{r} \) (atomic number = 1). The densities in the position and momentum spaces are [38]

\[
\rho_{n,l,\{\mu\}}(\mathbf{r}) = N_{n,l} e^{-\frac{2r}{\eta}} \left( \frac{2r}{\eta} \right)^{2l} \left[ \frac{(2l+D-2)}{(n-l-1)!} \right]^{2} \left| Y_{l,\{\mu\}}(\Omega) \right|^2,
\]

and

\[
\gamma_{n,l,\{\mu\}}(\mathbf{p}) = \tilde{N}_{n,l} \frac{(\eta p)^{2l}}{(1 + \eta^2 p^2)^{2l+D+1}} \left[ C_{n-l-1}^{(l+\frac{D-1}{2})} \left( 1 - \frac{\eta^2 p^2}{1 + \eta^2 p^2} \right) \right]^{2} \left| Y_{l,\{\mu\}}(\Omega) \right|^2,
\]

where

\[
N_{n,l} = \left( \frac{2}{\eta} \right)^{2l} \frac{(n-l-1)!}{2n(n+l+D-3)!},
\]

\[
\tilde{N}_{n,l} = \frac{(n-l-1)!}{2\pi(n+l+D-3)!} \eta^{D+1} 4^{2l+D+1} \left( l + \frac{D-1}{2} \right) \eta^{D+1},
\]

and \( L_k^{(\alpha)}(\cdot) \) and \( C_k^{(\alpha)}(\cdot) \) are the Laguerre and Gegenbauer polynomials of degree \( k \) and parameter \( \alpha \), respectively. The principal quantum number takes the values \( n = 0, 1, 2, \ldots \), and \( l = 0, 1, \ldots, n-1 \).

The Shannon entropies of these two densities have not yet been analytically calculated despite much effort (see [32] for a recent review). So, we have evaluated \( S[\rho] \) and \( S[\gamma] \) numerically to obtain the entropy sum \( S[\rho] + S[\gamma] \). The ratio

\[
\Xi_{n,l,m} = \frac{S[\rho] + S[\gamma]}{B_{l,m}} \tag{23}
\]

between the entropy sum and the central bound \( (22) \), for \( D = 3 \), is represented in figure 2 for several states. Naturally, in all the cases, \( \Xi_{n,l,m} \geq 1 \). Moreover, the ratio is lower for larger values of \( l \) (\( n \) fixed), as suggested by the analysis of section 4.

Figure 2. Ratio \( (23) \) (●) for several states \( (n, l, m) \) of the hydrogen atom.
5.2. Isotropic oscillator systems

In this case, $V_D(r) = \frac{1}{2}\lambda r^2$, $\lambda > 0$. The densities in the position and momentum spaces are

$$\rho_{n,l,m}(\vec{r}) = \frac{2n!\lambda^{l+\frac{D}{2}}}{\Gamma(n + l + \frac{D}{2})} r^{2l} e^{-\lambda r^2} \left[ L_n^{(l+\frac{D}{2}-1)}(\lambda r^2) \right]^2 |Y_l,m(\Omega_D)|^2,$$

and

$$\gamma_{n,l,m}(\vec{p}) = \frac{1}{\lambda^D} \rho_{n,l,m}\left(\frac{\vec{p}}{\lambda}\right),$$

where $n, l = 0, 1, 2, \ldots$.

Analogously to the previous subsection, the Shannon entropies, $S[\rho]$ and $S[\gamma]$, for these two densities are evaluated numerically to obtain the entropy sum $S[\rho] + S[\gamma]$. The ratio

$$\Phi_{n,l,m} = \frac{S[\rho] + S[\gamma]}{B_{l,m}} \quad (24)$$

between the entropy sum and the central bound (22), for $D = 3$ and $\lambda = 1$, is represented in figure 3 for several states. Like in the hydrogenic systems, in all the cases, $\Phi_{n,l,m} \geq 1$. Moreover, the ratio is lower for larger values of $l$ ($n$ fixed), as suggested by the analysis of section 4. It is worth remarking that for this system the entropy sum of the ground state $(n, l, m) = (0, 0, 0)$ is equal to $3(1 + \ln(\pi))$ (i.e. exactly the bound (1) for $D = 3$). However, as we can see in figure 3, the ratio of this value with respect to $B_{0,0}$ is $\Phi_{0,0,0} \simeq 1.15$, which is clearly greater than unity obtained with the general bound (1), as the ground state $(0, 0, 0)$ of the isotropic oscillator saturates this latter bound. Thus, again we observe that the new bound improves the general bound (1) for all states $(l, m)$ except when $l = 0$.

6. Conclusions and open problems

In this work, the Shannon-entropy-based uncertainty relation (1) of the $D$-dimensional quantum systems has been improved for spherically symmetric potentials. We have obtained that the resulting lower bound does not depend on the specific form of the potential since it only depends on the hyperangular quantum numbers $(l, \{\mu\})$. Moreover, we have observed that this bound is indeed a strict improvement of the general bound (1) for all values of $(l, m)$ except...
when \( l = m = 0 \). The latter is because the logarithmic uncertainty relation (16) does not represent for \( l = 0 \) any improvement with respect to the general Beckner inequality (which is not sharp enough for \( s \) states). It would be nice to overcome this limitation in the future. Moreover, we have studied and discussed the new ‘central’ bound for various states and for some relevant spherically symmetric potentials of Coulomb and oscillator types.

Finally, let us point out an important open problem closely related to that resolved here, namely, to improve for central potentials the general Rényi-entropy-based uncertainty relation of Bialynicki-Birula, and Zozor and Vignat [39, 40].

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