AN ILLUSTRATION OF A SHIODA-INOSE STRUCTURE

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ABSTRACT. We investigate the Shioda-Inose structure of the Jacobian of a smooth complex genus-2 curve $C$ arising from its degree-2 elliptic subcovers and determine the Mordell-Weil groups and lattices in the case of a semistable fibration having exactly four singular fibers.

Key words: Shioda-Inose structure, involution, lattice.

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Let $C$ be a Weierstrass cubic which is one-dimensional integral scheme over $\mathbb{C}$ of arithmetic genus 1 together with a point $p_0$ in its smooth locus. More precisely, it means that $C$ is either an elliptic curve or a rational curve either with a node or a cusp. In other words, these two rational curves are degenerated Kodaira fibers of types $I$ and $II$ (cycle of type $I_1$). $C$ is embedded by linear system $|3p_0|$ as a cubic plane curve in the complex projective plane satisfying the following equation

$$y^2z = x^3 + g_2xz^2 + g_3z^3,$$

where $g_2$ and $g_3$ are specified up to the $\mathbb{C}^*$-action defined by $\mu.(g_2, g_3) = (\mu^4g_2, \mu^6g_3)$. We will refer to such an equation as a Weierstrass model. The type of curves are actually of importance considerable in the advancement to understand the concept of constructing moduli spaces of holomorphic principal $G$-bundles over singular curves $C$ for $G$ a complex reductive algebraic group. These curves appear in elliptic fibrations and make part of fibers called Goreinstein arithmetic genus-one curves. These curves and curves of types $I_N$ are classified essentially in both cases where $f : X \rightarrow S$ is an elliptic fibration with $X$ a fibered elliptic surface and $S$ a smooth curve; and $f : X \rightarrow S$ a morphism of smooth projective varieties where $X$ is an elliptic three fold, having a section. This shows the importance that play the elliptic surfaces in the classification of algebraic surfaces to determine the geometric and arithmetic properties at the levels of groups, lattices and modular curves.

Let $X$ be an algebraic $K3$ surface defined over the field of complex numbers $\mathbb{C}$. Denote by $NS(X)$ the Neron-Severi lattice of $X$ is a sublattice of the unimodular lattice $H^2(X, \mathbb{Z})$ of rank $(3, 19)$. This sublattice is an even lattice of signature $(1, \rho(X) - 1)$, where $\rho$ denotes the rank of the Neron-Severi lattice of $X$. The main property of the Nikulin involution $\varphi$ for a $K3$ surface $X$ is that we can recover a $K3$ surface $\tilde{Y}$ as the minimal resolution of the quotient of $X$ by the subgroup generated by $\varphi$ obtained from the blowing-up $\tilde{X}$ of $X$ at the locus of fixed points of $\varphi$, which induces that $\varphi$ extends to $\varphi$ whose quotient by $<\varphi>$ is smooth. Moreover, if we add a cocycle condition on the transcendental lattices of $K3$ surfaces and the $K3$ surface obtained $Y$ is a Kummer.

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surface, then $X$ admits a Shioda-Inose structure.

On the other hand, we can also consider an automorphism $\Phi : X \to X$ such that there exists a triple $(\varphi_X, S_1, S_2)$, where 
$\varphi_X$ is an elliptic fibration over $\mathbb{P}^1(\mathbb{C})$. 
$S_1, S_2$ are disjoint sections of $\varphi_X$. 
$S_2$ is an element of order 2 in the Mordell-Weil group $MW(\varphi_X, S_1)$.

Then $\Phi$ extends to an involution called Van Geemen-Sarti involution by extending the fibre-wises by an action of group of automorphisms of translations by $S_2$ in the smooth fibers of $\varphi_X$. Its main interest is that when we can construct another elliptic fibration $Y \to \mathbb{P}^1(\mathbb{C})$ which is also a Van Geemen-Sarti involution, then it appears both rational maps from $X \to Y$ and conversely which form a pair of dual two-isogenies between $X$ and $Y$.

Let $X$ be a $N$-polarized algebraic $K3$ surface defined over $\mathbb{C}$, where $N = U \oplus E_7 \oplus E_8$ with $i : N \hookrightarrow NS(X)$ a lattice embedding whose image contains a pseudo-ample class. By Torelli Theorem with a Hodge structure of weight 2 on $T \otimes \mathbb{Q}$, where $T$ is a rank-$5$ lattice $U \oplus U \oplus (-2)$, this determine a bijective map:

$$(X, i) \leftrightarrow (A, \Pi),$$

where $A$ is a principally polarized abelian surface with its polarization $\Pi : A \to J(A)$, where $J(A)$ denotes the Jacobian of $A$. On the one hand, the set of isomorphism classes of $N$-polarized $K3$ surfaces with a canonical extension by the lattice $U \oplus E_8 \oplus E_8$ is identified with the set of isomorphism classes of Humbert surfaces $\mathcal{H}_1$, which identifies with the set of isomorphism classes of complex abelian surfaces in the form:

$$(E_1 \times E_2, \mathcal{O}_{E_1 \times E_2}((E_1 \times \{p_2\} + \{p_1\} \times E_2))).$$

On the other hand, the set of isomorphism classes of $N$-polarized $K3$ surfaces without extension is identified with the open $\mathcal{F}_2 \setminus \mathcal{H}_1$, where $\mathcal{F}_2 = Sp(4, \mathbb{Z})/\mathbb{H}_2$ is the Siegel threefold, which identifies with the set of isomorphism classes of complex abelian surfaces in the form:

$$(J(C), \mathcal{O}_{JC}(\theta)),$$

where $C$ is a smooth genus-$2$ curve. Moreover, the Hodge correspondence provides an isomorphism of analytic spaces between the moduli space $\mathcal{M}_2$ of smooth genus-$2$ curves and $\mathcal{F}_2 \setminus \mathcal{H}_1$. Hence, in this case, we are able to determine the Shioda-Inose structure of the Jacobian of a smooth complex genus-$2$ curve $C$. A number of properties for genus $1$-curves translate in the same way for $N$-polarized $K3$ surfaces or principally abelian varieties obtained as minimal resolutions of a surface in $\mathbb{P}^3$. This is a key to determine the Mordell-Weil groups and lattices of the semistable fibration over $\mathbb{P}^1(\mathbb{C})$ having exactly four singular fibers.

**Theorem 0.1.** The Mordell-Weil groups of the semistable fibration over $\mathbb{P}^1(\mathbb{C})$ having exactly four singular fibers are:

| $MW(\mathbb{Z}/3)^2$ | $G(I_3)$ | Number of irreducible components of singular fibers |
|------------------------|----------|--------------------------------------------------|
| $\mathbb{Z}/4 \times \mathbb{Z}/2$ | $G(I_4) \times G(I_2)$ | 3, 3, 3, 3 |
| $\mathbb{Z}/5$ | $G(I_5)$ | 4, 4, 2, 2 |
| $\mathbb{Z}/6$ | $G(I_6)$ | 5, 5, 1, 1 |
| $\mathbb{Z}/7$ | $G(I_7)$ | 6, 3, 2, 1 |
| $\mathbb{Z}/8$ | $G(I_8)$ | 8, 2, 1, 1 |
| $\mathbb{Z}/9$ | $G(I_9)$ | 9, 1, 1, 1 |
Note that in the second case, it arises a Van-Geemen Sarti involution.

**Theorem 0.2.** We deduce the Mordell-Weil lattices of Beauville’s semistable fibration.

| MW L | Number of irreducible components of singular fibers |
|------|--------------------------------------------------|
| $(\mathbb{Z}/3)^2/\mathbb{Z}/3$ | 3, 3, 3, 3 |
| $(\mathbb{Z}/4 \times \mathbb{Z}/2)/\mathbb{Z}/2$ | 4, 4, 2, 2 |
| $\{0\}$ | 5, 5, 1, 1 |
| $\mathbb{Z}/6/\mathbb{Z}/3$ | 6, 3, 2, 1 |
| $\mathbb{Z}/4/\mathbb{Z}/2$ | 8, 2, 1, 1 |
| $\{0\}$ | 9, 1, 1, 1 |

The structure of the paper is as follows.
In Sect. 1, we give an explicit description of the double covers of an elliptic curve.
In Sect. 2, we use the Van Geemen-Sarti involution to investigate the Shioda-Inose structure arising from elliptic fibrations.
In Sect. 3, we define the Mordell-Weil group and lattice of an elliptic surface.
In Sect. 4, we illustrate these concepts to determine the Mordell-Weil groups and lattices for a semistable fibration over $\mathbb{P}^1(\mathbb{C})$ having exactly 4 singular fibers whose we extract a Van Geemen-Sarti involution.

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1. **Genus-2 covers of an elliptic curve**

In this section, we first will describe the degree-2 covers of elliptic curves which are curves of genus 2.

**Definition 1.1.** Let $\pi : C \to E$ be a degree-2 map of curves. If $E$ is elliptic, then we say that $C$ is bielliptic and that $E$ is a degree-2 elliptic subcover of $C$.

Legendre and Jacobi observed that any genus-2 bielliptic curve has an equation of the form

$$y^2 = c_0x^6 + c_1x^4 + c_2x^2 + c_3 \quad (c_i \in \mathbb{C}) \quad (1)$$

in appropriate affine coordinates $(x, y)$. It immediately follows that any bielliptic curve $C$ has two elliptic subcovers $\pi_i : C \to E_i$, $E_1 : y^2 = c_0x_1^3 + c_1x_1^2 + c_2x_1 + c_3$, $\pi_1 : (x, y) \mapsto (x_1 = x^2, y)$, and $E_2 : y_2^2 = c_3x_2^3 + c_2x_2^2 + c_1x_2 + c_0$, $\pi_2 : (x, y) \mapsto (x_2 = 1/x^2, y_2 = y/x^3)$. \quad (2)

This description of bielliptic curves, though very simple, depends on an excessive number of parameters. To eliminate unnecessary parameters, we will represent $E_i$ in the form

$$E_i : y_i^2 = x_i(x_i - t_i)(x_i - t_i), \quad (t_i \in \mathbb{C} \setminus \{0, 1\}, t_1 \neq t_2) \quad (3)$$

Note that any pair of elliptic curves $(E_1, E_2)$ admits such a representation even if $E_1 \simeq E_2$.

We will describe the reconstruction of $C$ starting from $(E_1, E_2)$ following [Di]. This procedure will allow us to determine the periods of bielliptic curves $C$ in terms of the periods of their elliptic subcovers $E_1, E_2$. 

3
Let $\varphi_i : E_i \to \mathbb{P}^1$ be the double cover map $(x_i, y_i) \mapsto x_i$ ($i = 1, 2$). Recall that the fibered product $E_1 \times_{\mathbb{P}^1} E_2$ is the set of pairs $(P_1, P_2) \in E_1 \times E_2$ such that $\varphi_1(P_1) = \varphi_2(P_2)$. It can be given by two equations with respect to three affine coordinates $(x, y_1, y_2)$:

$$C := E_1 \times_{\mathbb{P}^1} E_2 : \left\{ \begin{array}{l}
y_1^2 = x(x - 1)(x - t_1) \\
y_2^2 = x(x - 1)(x - t_2)
\end{array} \right.$$  \hspace{1cm} (4)

It is easily verified that $C$ has nodes over the common branch points 0, 1, $\infty$ of $\varphi_i$ and is nonsingular elsewhere. For example, locally at $x = 0$, we can choose $y_i$ as a local parameter on $E_i$, so that $x$ has a zero of order two on $E_i$; equivalently, we can write $x = f_i(y_i)y_i^2$ where $f_i$ is holomorphic and $f_i(0) \neq 0$. Then eliminating $x$, we obtain that $C$ is given locally by a single equation $f_1(y_1)y_1^2 = f_2(y_2)y_2^2$. This is the union of two smooth transversal branches $\sqrt{f_1}(y_1)y_1 = \pm \sqrt{f_2(y_2)y_2}$.

Associated to $C$ is its normalization (or desingularization) $\bar{C}$ obtained by divorcing the two branches at each singular point. Thus $\bar{C}$ has two points over $x = 0$, whilst the only point of $C$ over $x = 0$ is the node, which we will denote by the same symbol 0. We will also denote by 0, +, 0− the two points of $\bar{C}$ over 0. Any of the functions $y_1, y_2$ is a local parameter at 0±. In a similar way, we introduce the points 1, $\infty \in \bar{C}$ and 1±, $\infty \pm \in \bar{C}$.

**Proposition 1.2.** Given a genus-2 bielliptic curve $C$ with its two elliptic subcovers $\pi_i : C \to E_i$, one can choose affine coordinates for $E_i$ in such a way that $E_i$ are given by the equations (5), $C$ is the normalization of the nodal curve $\bar{C} := E_1 \times_{\mathbb{P}^1} E_2$, and $\pi_i = pr_i \circ \nu$, where $\nu : C \to \bar{C}$ denotes the normalization map and $pr_i$ the projection onto the $i$-th factor.

**Proof.** See [Di].

It is curious to know, how the descriptions given by (5) and Proposition 1.2 are related to each other. The answer is given by the following proposition.

**Proposition 1.3.** Under the assumptions and in the notation of Proposition 1.2, apply the following changes of coordinates in the equations of the curves $E_i$:

$$(x_i, y_i) \mapsto (\tilde{x}_i, \tilde{y}_i), \quad \tilde{x}_i = \frac{x_i - t_j}{x_i - t_i}, \quad \tilde{y}_i = \frac{y_i}{(x_i - t_i)^2} \sqrt{\frac{(t_j - t_i)^3}{t_i(1 - t_i)}},$$

where $j = 3 - i$, $i = 1, 2$, so that $\{i, j\} = \{1, 2\}$. Then the equations of $E_i$ acquire the form

$$E_1 : \ \tilde{y}_1^2 = \left(\tilde{x}_1 - \frac{t_2}{t_1}\right) \left(\tilde{x}_1 - \frac{1 - t_2}{1 - t_1}\right) (\tilde{x}_1 - 1),$$

$$E_2 : \ \tilde{y}_2^2 = \left(1 - \frac{t_2}{t_1}\tilde{x}_2\right) \left(1 - \frac{1 - t_2}{1 - t_1}\tilde{x}_2\right) (1 - \tilde{x}_2).$$ \hspace{1cm} (5)

Further, $C$ can be given by the equation

$$\eta^2 = \left(\xi^2 - \frac{t_2}{t_1}\right) \left(\xi^2 - \frac{1 - t_2}{1 - t_1}\right) (\xi^2 - 1),$$

and the maps $\pi_i : C \to E_i$ by $(\xi, \eta) \mapsto (\tilde{x}_i, \tilde{y}_i)$, where

$$(\tilde{x}_1, \tilde{y}_1) = (\xi^2, \eta), \quad (\tilde{x}_2, \tilde{y}_2) = (1/\xi^2, \eta/\xi^3).$$
Proof. We have the following commutative diagram of double cover maps

\[
\begin{array}{ccc}
C & \xrightarrow{f} & E_1 \\
\pi_1 \downarrow & & \downarrow \varphi_1 \\
\mathbb{P}^1 & \xrightarrow{\xi} & \mathbb{P}^1 \\
\varphi_2 \downarrow & & \downarrow \pi_2 \\
E_2 & \xrightarrow{\varphi} & \mathbb{P}^1
\end{array}
\]

in which the branch loci of \( \tilde{\varphi}, \varphi_i, f, \pi_i \) are respectively \( \{t_1, t_2\}, \{0, 1, t_i, \infty\}, \varphi_i^{-1}(\{0, 1, \infty\}), \varphi_i^{-1}(t_j) \) (\( j = 3 - i \)). Thus the \( \mathbb{P}^1 \) in the middle of the diagram can be viewed as the Riemann surface of the function \( \sqrt{\frac{x-t_2}{x-t_1}} \), where \( x \) is the coordinate on the bottom \( \mathbb{P}^1 \). We introduce a coordinate \( \xi \) on the middle \( \mathbb{P}^1 \) in such a way that \( \tilde{\varphi} \) is given by \( \xi \mapsto x, \xi^2 = \frac{x-t_2}{x-t_1} \). Then \( C \) is the double cover of \( \mathbb{P}^1 \) branched in the 6 points \( \tilde{\varphi}^{-1}(\{0, 1, \infty\}) = \{\pm 1, \pm \sqrt{\frac{t_2-t_1}{t_1-t_2}}, \pm \sqrt{\frac{t_1-t_2}{t_2-t_1}}\} \), which implies the equation (\ref{eq:6}) for \( C \). Then we deduce the equations of \( E_i \) in the form (\ref{eq:5}) following the recipe of (\ref{eq:2}), and it is an easy exercise to transform them into (\ref{eq:3}).

From now on, we will stick to a representation of \( C \) in the classical form \( y^2 = F_6(\xi) \), where \( F_6 \) is a degree-6 polynomial. We want that \( E \) is given the Legendre equation \( y^2 = x(x-1)(x-t) \), but \( F_6 \) is not so bulky as in (\ref{eq:6}). Of course, this can be done in many different ways. We will fix for \( C \) and \( f \) the following choices:

\[
f : C = \{y^2 = (t' - \xi^2)(t' - 1 - \xi^2)(t' - t - \xi^2)\} \quad \rightarrow \quad E = \{y^2 = x(x-1)(x-t)\}
\]

(\ref{eq:8})

\[
(\xi, y) \mapsto (x, y) = (t' - \xi^2, y)
\]

(\ref{eq:8})

\[
\text{Lemma 1.4. } \text{For any bielliptic curve } C \text{ with an elliptic subcover } f : C \rightarrow E \text{ of degree } 2, \text{ there exist affine coordinates } \xi, x, y \text{ on } C, E \text{ such that } f, C, E \text{ are given by (8) for some } t, t' \in C \setminus \{0, 1\}, t \neq t'.
\]

Proof. By Proposition(\ref{prop:1}), it suffices to verify that the two elliptic subcovers \( E, E' \) of the curves \( C \) given by (8), as we vary \( t, t' \), run over the whole moduli space of elliptic curves independently from each other. \( E' \) can be determined from (2). It is a double cover of \( \mathbb{P}^1 \) ramified at \( \frac{1}{t'}, \frac{1}{t'-1}, \frac{1}{t'-t}, \infty \). This quadruple can be sent by a homographic transformation to \( 0, 1, t, t' \), hence \( E' \) is given by \( y^2 = x(x-1)(x-t)(x-t') \). If we fix \( t \) and let vary \( t' \), we will obviously obtain all the elliptic curves, which ends the proof.

The only branch points of \( f \) in \( E \) are \( p_{\pm} = (t', \pm y_0) \), where \( y_0 = \sqrt{t'(t' - 1)(t' - t)} \), and thus the ramification points of \( f \) in \( C \) are \( \tilde{p}_{\pm} = (0, \pm y_0) \). In particular, \( f \) is non-ramified at infinity and the preimage of \( \infty \in E \) is a pair of points \( \infty_{\pm} \in C \).

We remark that if \( \varphi : C \rightarrow E \) is a cover of degree \( n \); then there are two points \( P_1, P_2 \) in \( C \) such that their multiplicity \( e_{\varphi}(P_i) = 2, i = 1..2 \) and \( e_{\varphi}(Q) = 1, \forall Q \in C \setminus \{P_1, P_2\} \) such that their images are either different or the same. Otherwise, there is a point \( P \in C \) with \( e_{\varphi}(P) = 3 \) and \( e_{\varphi}(Q) = 1, \forall Q \in C \setminus \{P\} \). Thus \( \varphi \) is branched in one point in two cases and two points in the other case.

Let \( \iota : C \rightarrow C \) be the hyperelliptic involution map of \( C \) whose fixed points are Weierstrass points \( W \) contained in the two-torsions points of \( E \). We can specify the latter according to the degree of the cover. If its degree is odd, then \( \varphi(W) = E[2] \) and for
any $Q \in E[2]$, the cardinal of $\varphi^{-1}(Q) \cap W \cong 1(2)$. If $\deg \varphi = 2k$, then $\varphi(W) \subset E[2]$ and for any $Q \in E[2]$, the cardinal of $\varphi^{-1}(Q) \cap W \cong 0(2)$.

2. SHIODA-INOSE STRUCTURE

**Definition 2.1.** A $K3$ surface $X$ admits a Shioda-Inose structure if there exists a Nikulin involution $\iota : X \to X$ such for any holomorphic $(2,0)$-form $\omega$, then $\iota^*(\omega) = \omega$ and there is a rational quotient map $\varphi : X \to Y$ such that $Y$ is a Kummer surface with in addition a cocycle condition $T_X(2) \cong T_Y$ which expresses the isomorphism of transcendental lattices induced by the map $\varphi_*$. Moreover, this induces both degree-2 rational maps $\varphi_1 : X \to Y$ and $\varphi_2 : Z \to Y$, where $Y$ is the Kummer surface associated to the 2-dimensional complex torus $Z$.

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_1} & Z \\
& \searrow & \nearrow \\
& Y & \xleftarrow{\varphi_2}
\end{array}
\]

(9)

We give a criterion to determine if a $K3$ surface $X$ admits a Shioda-Inose structure.

**Theorem 2.2.** An algebraic $K3$ surface $X$ admits a Shioda-Inose structure if there exists a lattice primitive embedding $T_X \hookrightarrow U^{\oplus 3}$.

**Proof.** See Theorem 6.3 of [Mor].

We come back our bielliptic curve $C$ arising from its elliptic subcovers $E$ and $E'$, and investigate the Shioda-Inose structure.

**Theorem 2.3.** In an abstract way, we can construct a bielliptic curve $C$ from its elliptic subcovers $E$ and $E'$ and conversely from the following commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & JC \\
\downarrow{f \times f'} & & \downarrow{f^* + f'^*} \\
E \times E' & & \\
\end{array}
\]

in which $f^* + f'^*$ is an isogeny of degree 2 and $f \times f'(C)$ is the graph of a $(2,2)$ correspondence between $E, E'$. From this, we have

\[
\begin{array}{ccc}
SI(J(C)) & \xrightarrow{2:1} & Kum(J(C)) \\
\downarrow{2:1} & & \downarrow{2:1} \\
Kum(J(E) \times J(E'))
\end{array}
\]

where $SI(JC)$ denotes the Jacobian of $C$ endowed with a Shioda-Inose structure corresponding to a $N$-polarized $K3$ surface without extension.

**Proof.** We can first assume that $C$ is a generic smooth genus-2 curve (i.e. such that its Jacobian surface has $p(JC) = 1$) and prove in this case that the morphism $Kum(J(E) \times J(E')) \to Kum(J(C))$ is an isomorphism. $JC$ is a principally polarized abelian variety and if $H$ is the principal polarization, then we have $H^2 = 2$ and $T = T_{JC} = U^{\oplus 2} \oplus (-2)$. Note that $T$ embeds in $\Lambda$, then there exists a $K3$ surface $X$ such that we have a Hodge isometry $T \cong T_X$. We construct an embedding lattice from $T_X \hookrightarrow U^{\oplus 3}$ in the following way, we send the first two copies of $U \subset T_X$ to the corresponding ones of $U^{\oplus 3}$ and the remaining element $-2$ to $e_1^3 - e_2^3$, where $(e_i^j)_{j=1...3,i=1,2}$ is a basis of $U^{\oplus 3}$. Then
by the previous theorem, \( X \) admits a Shioda-Inose structure and we have the following diagram,
\[
\begin{array}{ccc}
X & \to & Z = E \times E' \\
\varphi_1 & & \varphi_2 \\
Y & \downarrow & \\
\end{array}
\]
with Hodge isometries \( T_X \simeq T_{JC} \) and \( T_Z \simeq T_{JC} \). Then these are Fourier-Mukai partners and \( \text{Kum}(JC) \simeq \text{Kum}(Z) \) after Main Theorem of [H-L-O-Y]. We at present consider the framework general of a smooth genus-2 curves. As \( JC \) and \( E \times E' \) are only two-isogeneous, this only implies an isomorphism of \( T_{JC} \) and \( T_{E \times E'} \) over \( \mathbb{Q} \) not over \( \mathbb{Z} \); and we do not recover the isomorphism in the generic case. The Hodge correspondence gives an isomorphism between the set of isomorphism classes of \( \mathbb{N} \)-polarized K3 surfaces without extension and this one of principally polarized abelian varieties \((JC, \mathcal{O}(\Theta))\). This induces an isomorphism between the open region \( \mathcal{F}_2 \setminus \{\mathcal{H}_1\} \) and the moduli space of the genus-2 curves. Then we rely on the first parts of the Sect. 4 of the paper of Clingher-Doran [Cl-D1] to get the diagram given above. \( \square \)

For consequently, by lemma 4.1 of [Cl-D1], the Shioda-Inose structure of the Jacobian of a smooth genus-2 curve is:

**Proposition 2.4.** The Shioda-Inose structure of the Jacobian of a smooth complex genus-2 curve is a K3-surface \( X(k, m, n) \) with transcendental Hodge isometric lattice \( T(k, m, n) = U(k) \oplus U(m) \oplus \langle -2n \rangle \). Moreover by Hodge correspondence, the K3-surface \( X(k, m, n) \) is represented by a principally polarized abelian surface of type \((1, n)\).

3. MORDELL-WEIL GROUP AND LATTICE OF AN ELLIPTIC SURFACE

We want to determine the Mordell-Weil group and lattice of an elliptic surface in a particular case. So, we need to put in place some materials to realize this. We first need to know what happens at the level of the fibers after a change base on the new constructed elliptic surface and to solve its singularities.

We start with an elliptic surface \( S \) over a smooth curve \( C \). The change base of \( S \) requires a smooth projective curve \( B \) together with an onto morphism \( \pi : B \to C \) such that the following diagram commutes
\[
\begin{array}{ccc}
S' & \overset{\tilde{f}}{\to} & B \\
\rho \downarrow & & \pi \\
S & \to & C \\
\end{array}
\]
where \( S' = S \times_C B \) is the constructed elliptic surface. If \( S \to C \) the elliptic surface has only smooth fibers then those translate into \( S' \) by the same smooth fibers whose the number copies is the degree of \( \pi \). In the case where it only appears in the elliptic surface \( S \) of multiplicative fibers \( I_n \) over unramified points of \( \pi \), then those translate into \( S' \) in \( I_{nd} \), where \( d \) is the index of ramification of \( \pi \). In the case where it only appears of additive fibers, we make reference to the Tate’s table (See [Sc-Sh]).

We at present make refer to T.Shioda [Sh-1] and [Sh-2] to define in a first time Mordell-Weil groups and in a second time the Mordell-Weil lattices of an elliptic surface. Let \( K = \mathbb{C}(C) \) be the function field on a smooth complex projective curve \( C \). Let \( E \) be an elliptic
curve defined over $K$ with a $K$-rational point. Let $E(K)$ be the group of $K$-rational points of $E$, with origin $O$. The Kodaira-Néron model of $E/K$ is an elliptic surface $f : S \to C$, where $S$ is a smooth complex projective surface and $f$ has no exceptional $(-1)$ curves. So, the group $E(K)$ of $K$-rational points of $E$ called the Mordell-Weil group of $E$ can be indentified with the group of sections of $f$. Each $P \in E(K)$ determines a section of $f$ is interpreted as a divisor on $S$, which is a curve denoted $\bar{P}$.

Assume thought that $f$ has at least one singular fiber. Then $E(K)$ is a finitely generated (Mordell-Weil theorem) and the Néron-Severi group $\text{NS}(S)$ of $S$ is none than $\text{Pic}(S)$, which becomes an integral lattice of finite rank with respect to the intersection pairing $D.D'$. Denote us $T$ its sublattice generated by the zero section $(O)$, by a general fiber $F$ and the irreducible components of fibers. Then $T$ can be written as follows:

$$T = (\mathbb{Z}(O) \oplus \mathbb{Z}F) \oplus (\oplus_{v \in N} T_v),$$

where $N = \{v \in C \mid F_v \text{ is reducible}\}$, and $T_v$ is generated by the irreducible components of $F_v$ not meeting the zero section $(O)$.

The map $P \mapsto P \mod T$ induces a group isomorphism:

$$E(K) \simeq \text{NS}(S)/T,$$

and then a unique homomorphism:

$$\varphi : E(K) \to \text{NS}(S) \otimes \mathbb{Q}$$

such that

$$\varphi(P) = \bar{P} \mod T \otimes \mathbb{Q}, \quad \text{im}(\varphi) \perp T.$$

Now, we want to consider the torsion sections and how those work in the case of an elliptic surface with singularities. Assume that $P \in E(K)$ has torsion, then its image $\bar{P}$ lies in the primitive closure $T'$ of $T$ defined as follows

$$T' = (T \otimes \mathbb{Q}) \cap \text{NS}(S).$$

Denote $t_p$ the automorphism of elliptic surfaces without forgetting their elliptic structure. Then the group $<t_p>$ is finite of order $m$ that acts on the generic fiber by fixed point free and the quotient of $S$ by $<t_p>$ gives an elliptic curve $E'$ together with an isometry from $E' \to E$, whose Kodaira-Néron model is an elliptic surface obtained as the minimal resolution of the quotient $S/<t_p>$. In the case where the characteristic of the field $p$ does not divide $m$, then we can study the type of Kodaira fibers appear in the elliptic surface since the operation is separable in the sense that the $j$-form of an elliptic surface is defined as the morphism $j : C \to \mathbb{P}^1(\mathbb{C})$ that is separable. We treat the various cases separately. In the multiplicative fibers, if we assume that $m$ is a prime number to simplify the case, and $P$ meets the zero component of an $I_n$ fiber, then each component of $I_n$ is fixed by $<t_p>$. Hence, there are $n$ fixed points at the intersection of components. Each of them attains an $A_{m-1}$ singularity in the quotient whose desingularization is $I_{m^2}$. If $P$ does not meet the zero section, then $m | n$ by corollary 3.1 $t_p$ rotates the singular fiber (i.e cycle of rational curves) by an angle of $\frac{2\pi}{m}$ and the resulting fiber in $S'$ has type $I_{m^2}$. In the case of additive fibers, $P$ has to meet the non-trivial singular fiber.

**Corollary 3.1.** Restricted to the torsion subgroup of $E(K)$, the group homomorphism $\varphi$ is one-to-one:

$$\varphi : E(K)_{\text{tors}} \to \prod_{v \in R} G(F_v).$$

**Proof.** See 11.9 of [Sc-Sh].
The case among which we are interested is the case of an elliptic rational extremal elliptic surface.

**Definition 3.2.** An elliptic surface $S$ with section is said extremal if $\rho(S)$ is maximal and $E(K)$ is finite.

**Remark 3.3.** $\rho(S)$ maximal means that the bounds given by Igusa and Lefschetz are attained. In other words, $\rho(S) = h^{1,1}(S)$ (resp. $= b_2(S)$).

An extremal rational elliptic surface is featured by the discriminant of its trivial lattice as follows

$$\text{disc}(T(S)) = -(\#E(K))^2.$$ 

We use the following corollary to deduce the number of singular fibers in such an elliptic surface.

**Corollary 3.4.** Let $S$ be an elliptic surface with section. Denote the generic fiber by $E$. Then

$$\rho(S) = \text{rk}(T(S)) + \text{rk}(E(K)) = 2 + \sum_{v \in \mathbb{R}} (m_v - 1) + \text{rk}(E(K)).$$

**Proof.** See section 6 of [Sc-Sh]. □

Hence, it will appear at least four singular fibers in this elliptic surface whose classification is:

1. 4 multiplicative singular fibers.
2. 2 multiplicative and 1 additive singular fibers.
3. 2 additive singular fibers.

(We denote in the case where there is a wild ramification, we have at the same time of additive and multiplicative singular fibers.) One classifies the extremal rational elliptic surfaces with section in terms of configuration of singular fibers which we write as tuples

$$[[n_1, n_2, ...]], \text{ with entries } 1, 2, ..., 0^*, 1^*, ..., II, III, ..., II^*,$$

representing the fiber types. This configuration is a method of determining the Mordell-Weil group. One can use quotients by translation by torsion sections to limit the possible orders. Denote that is no longer true in the case of K3 elliptic surfaces.

In our case, we are only interested in the multiplicative fibers. Then, the existence of the Mordell Weil groups is ensured by corollary 3.1.

We want to compute the euler number $e(S)$ of an elliptic surface $S$. It is known that in the case of fiber $F_v$, we get the following results.

$$e(F_v) = \begin{cases} 
0, & \text{if } F_v \text{ is smooth;} \\
m_v, & \text{if } F_v \text{ is multiplicative;} \\
m_v + 1, & \text{if } F_v \text{ is additive.} 
\end{cases} \quad (12)$$

**Theorem 3.5.** For an elliptic surface $S$ over $C$, we have

$$e(S) = \sum_{v \in \mathbb{C}} (e(F_v) + \delta_v),$$

where $\delta_v$ is the wild ramification defined as $\delta_v = v(\Delta) - 1$-numbers of components of an additive fiber.

**Proof.** See Proposition 5.16 of [Co-D]. □
We at present define the Mordell-Weil lattice of an elliptic surface. For \( P, P' \in E(K) \), let
\[
< P, P' > = \varphi(P) \varphi(P').
\]
Then with this height pairing the group \( E(K)/E(K)_{tor} \) becomes a positive definite lattice, called the Mordell-Weil lattice of \( E/K \) or of \( f : S \to C \). These are not integral lattices in general.

We can rewrite down the pairing defined above as follows:
\[
< P, P' > = \chi + P \cdot O + P' \cdot O - P \cdot P' - \sum_{v \in N} \text{contrv}(P, P').
\]
Here \( \chi \) is the euler characteristic of the surface \( S \), \( P \cdot Q = \bar{P} \cdot \bar{Q} \), and the term in the sum expresses that \( P \) and \( P' \) pass through non-identity components of \( F_v \).

The subgroup \( E(K)^0 \) of \( E(K) \) consisting of those sections meeting the identity of every fiber is a torsion-free subgroup of \( E(K) \) of index finite, and becomes a positive definite even lattice, called the narrow Mordell-Weil lattice of \( E/K \) or of \( f : S \to C \). This lattice is isomorphic via the map \( \varphi \) to the opposite of the orthogonal complement \( T^\perp \) of \( T \) in the Néron-Severi lattice \( NS(S) \).

Note that
\[
< P, P' > = \chi + P \cdot O + P' \cdot O - P \cdot P'
\]
if \( P \) or \( P' \) are in \( E(K)^0 \),
\[
< P, P > = 2\chi + 2P \cdot O \geq 2\chi,
\]
for \( P \in E(K)^0, P \neq O \).

4. Semistable fibration - Beauville's case

We are going to illustrate these various concepts in a particular case where the fibration is semistable said of Beauville.

**Definition 4.1.** Let \( S \) be a surface and \( C \) a smooth projective curve, \( f : S \to C \) is said a semistable fibration if \( S \) is a smooth surface and the fibers are connected rational curves of genus \( g \geq 1 \) having at worst ordinary double points. Moreover, the fibers do not contain the exceptional curves.

From (5), we can construct the following diagram of semistable fibrations.

**Proposition 4.2.**

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_1} & X' \\
\downarrow f & & \downarrow \bar{\varphi} \\
\bar{X} & \xrightarrow{\varphi_1} & \bar{X}' \\
\downarrow \pi_2 & & \downarrow \varphi_2 \\
X'' & \xrightarrow{\bar{\varphi}} & \bar{X} \\
\end{array}
\]

where \( \bar{\varphi} \) is a semistable fibration having 4 singular fibers of genus 1, \( \varphi_1, \varphi_2 \) are semistable fibrations consisting of 6 singular fibers of genus 3, \( f \) from a 4-fold \( X \) is a double cover of \( \bar{X} \) algebraic surface and the \( \pi_i, i = 1..2 \) are double covers of semistable fibrations \( X' \) and \( X'' \) are both algebraic surfaces. The composition \( f \circ \bar{\varphi} \) is a semistable fibration with 8 singular fibers of genus 5.
Proof. The proof is based on the arguments given by Beauville in [Be-1]. So, for the reader, we remind this proof. We start with a smooth curve $C$, endowed with a morphism $\varphi : C \to \mathbb{P}^1(\mathbb{C})$ of degree $n$ and a homography $u$ of $\mathbb{P}^1(\mathbb{C})$ such that the following conditions hold:

(i) the set $R$ of the ramification points are simple;
(ii) $R$ is stable under $u$ and has no fixed points of $u$. So, there exists a double cover $\pi : X' \to C \times \mathbb{P}^1(\mathbb{C})$ ramified along $\Gamma_{\varphi} \cup \Gamma_{u\circ \varphi}$ which are linearly equivalent divisors since $PGL(2, \mathbb{C})$ is a rational variety. Let $g = pr_2 \circ u$ be the composition morphism. Let $t \in \mathbb{P}^1(\mathbb{C})$ be a point, the fiber $g^{-1}(t)$ is also a double cover ramified along the divisor $\varphi^{-1}(t) + \varphi^{-1}(u^{-1})(t)$. For either $t \neq 0$ or a fixed point of $u$, the divisor is not reduced and multiplicity 2. By blowing-up the double points of $X'$, we get a semistable fibration $X \to \mathbb{P}^1(\mathbb{C})$, that admits $Card(R) + 2g(C)$ singular fibers whose genus is $n - 1 + 2g(C)$. On the one hand, we can construct a morphism of degree $n$ even having four simple ramification points $\varphi : E \to \mathbb{P}^1(\mathbb{C})$, $g(E) = 1$, obtained as the composition of a double cover ramified into these 4 points with an étale morphism of degree $\frac{2}{7}$. Hence, in our case, for $n=2$, $g^{-1}(t)$ is a singular genus-2 curve and get a semistable fibration having 6 singular fibers whose genus is 3. We proceed in the same way of constructing another algebraic surface $X''$ and deduce the 4-fold $X$. On the other hand, as $C$ is a smooth genus-2 curve and $f : C \to \mathbb{P}^1(\mathbb{C})$ is branched in 6 points, so composition $f \circ \tilde{\varphi}$ is a semistable fibration with 8 singular fibers of genus 5.

Note if $n$ is odd, we construct a morphism $\varphi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree $n$ having four simple ramification points. Then, we get a semistable fibration having six singular fibers whose genus is $n - 1$.

We remark that for $n = 3$, a semistable fibration at 6 singular fibers of the genus-2 curves.

We want to determine the Mordell-Weil groups and lattices of the semistable fibration having four singular fibers over $\mathbb{P}^1(\mathbb{C})$.

We first set up a few notation. Let $\Gamma$ be the group of index finite in $SL_2(\mathbb{Z})$ satisfying the relation $(SS)$: $\Gamma$ whose trace is different from the values $\{-2, -1, 0, 1\}$. Note that $\Gamma$ acts freely over the upper-half plane $\mathbb{H}$; the semi-direct product of $\Gamma$ by $\mathbb{Z}^2$ acts freely and properly on $\mathbb{H} \times \mathbb{C}$ by the formula:

$$(\gamma, p, q).(\tau, z) = (\gamma \tau, (c \tau + d)^{-1}(z + p + q)),$$

for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

$$(p, q) \in \mathbb{Z}^2, \tau \in \mathbb{H}, z \in \mathbb{C}.$$

We denote $X^0_1$ the quotient surface, and $B^0_1$ the curve $\mathbb{H}/\Gamma$. By (2), the smooth elliptic fibration extends in a only one way in a semistable fibration $X_1 \to B_1$: this is the modular family associated to $\Gamma$.

We consider the following subgroups of $SL_2(\mathbb{Z})$:

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv c \equiv 0, a \equiv 1(mod.n) \right\},$$

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})c \equiv 0, a \equiv 1(mod.n) \right\},$$

for $n$ fixed. $\Gamma(n)$ acts freely over the modular surface $H/\Gamma(n)$, and $\Gamma_0(n)$ acts freely properly over $H/\Gamma_0(n)$.
\[ \Gamma_0(n) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \mid b \equiv c \equiv 0(\text{mod.} n) \right\}, \quad (17) \]

Consider elsewhere a pencil of cubics in \( \mathbb{P}^2 \), such that the only one singularities of the pencil are ordinary double points. By blowing-up of the nine base points of the pencil, we have an elliptic semistable fibration over \( \mathbb{P}^1 \), said deduced family of the pencil of cubics.

We state a theorem from Beauville (See [Be-2]).

**Theorem 4.3.** Let \( f : X \to \mathbb{P}^1(\mathbb{C}) \) be an elliptic semistable fibration having exactly four singular fibers. Then \( f \) is isomorphic to the family of the modular curves associated to the group \( \Gamma \) of index finite in \( SL_2(\mathbb{Z}) \) satisfying the relation \((SS)\); which identifies with the family of the cubics induced by a pencil of cubics:

| \( \Gamma \) | Pencil Equations | Number of irreducible components of singular fibers |
|------------|------------------|-----------------------------------------------|
| \( \Gamma(3) \) | \( X^3 + Y^3 + Z^3 + tXYZ = 0 \) | 3, 3, 3, 3 |
| \( \Gamma_0(4) \cap \Gamma(2) \) | \( X(X^2 + Z^2 + 2ZY) + tZ(X^2 - Y^2) = 0 \) | 4, 4, 2, 2 |
| \( \Gamma_0(5) \) | \( X(X - Z)(Y - Z) + tZY(X - Y) = 0 \) | 5, 5, 1, 1 |
| \( \Gamma_0(6) \) | \( (X + Y)(Y + Z)(Z + X) + tXYZ = 0 \) | 6, 3, 2, 1 |
| \( \Gamma_0(8) \) | \( (X + Y)(XY - Z^2) + tXYZ = 0 \) | 8, 2, 1, 1 |
| \( \Gamma_0(9) \cap \Gamma_0(3) \) | \( X^2Y + Y^2Z + Z^2X + tXYZ = 0 \) | 9, 1, 1, 1 |

We adapt this to the case of Beauville’s semistable fibration, where \( K = \mathbb{C}(\mathbb{P}^1) \), the field of rational functions over the complex projective line, and \( T = (\mathbb{Z}(O) \oplus \mathbb{Z}F) \oplus (\sum_{i=1}^{12} \oplus T_i) \). Note that Beauville’s semistable fibration belongs to the family of extremal rational elliptic surfaces. Then the Kodaira-Néron model of this elliptic surface will define a structure of an algebraic group scheme over all the fibers. Hence, the group scheme of the identity component is \( \mathbb{G}_m \) if \( F_v \) is multiplicative and \( \mathbb{G}_a \) if \( F_v \) is additive. The quotient by this group scheme is a finite abelian group will be denoted \( G(F_v) \) depending on the fiber.

**Theorem 4.4.** The Mordell-Weil groups of the semistable fibration over \( \mathbb{P}^1(\mathbb{C}) \) having exactly four singular fibers are:

| \( MW \) | \( G(F_v) \) | Number of irreducible components of singular fibers |
|----------|----------------|-----------------------------------------------|
| (\( \mathbb{Z}/3 \))^2 | \( G(I_3)^2 \) | 3, 3, 3, 3 |
| \( \mathbb{Z}/4 \times \mathbb{Z}/2 \) | \( G(I_4) \times G(I_2) \) | 4, 4, 2, 2 |
| \( \mathbb{Z}/5 \) | \( G(I_5) \) | 5, 5, 1, 1 |
| \( \mathbb{Z}/6 \) | \( G(I_6) \) | 6, 3, 2, 1 |
| \( \mathbb{Z}/4 \) | \( G(I_4) \) | 8, 2, 1, 1 |
| \( \mathbb{Z}/3 \) | \( G(I_3) \) | 9, 1, 1, 1 |

**Proof.** Let \( j \in K \subset \{0, 12^3\} \). Then the following elliptic curve in generalized Weierstrass form has \( j \)-invariant:

\[ E : y^2 + xy = x^3 - \frac{36}{j - 12^3}x - \frac{1}{j - 12^3}. \quad (18) \]

Denote that there are both another elliptic curves admitting extra automorphisms defined by:

\[ j = 0 : y^2 + y = x^3, \quad j = 12^3 : y^2 = x^3 + x, \quad \text{12} \]
in characteristic of $K$ distinct from 2 and 3.

We use the normal form for an elliptic surface with $j$-given. The generic fiber is defined over $K(t)$. An integral model is obtained from (18) by a simple scaling of $x$ and $y$ of the type $x \mapsto u^2x$ and $y \mapsto u^3y$.

$$S : y^2 + (t - 12^3)xy = x^3 - 36(t - 12^3)x - (t - 12^3)^5.$$

We want to determine the singular fibers of $S$. We first compute the discriminant $\Delta = t^2(t-12^3)^9$. As the characteristic of $K$ is distinct from 2 and 3, We carry out of translations in $x$ and $y$ to yield the Weierstrass form

$$y^2 = x^3 - \frac{1}{48}t(t-12^3)x + \frac{1}{864}t(t-12^3)^5.$$ Then, we deduce that the fibers at 0 and $12^3$ are additive fibers which are of type $II$ (resp. of type $III^*$) arising from the Tate’s table. Otherwise, the two additive fibers collapse so that the valuation of the discriminant $\Delta$ at 0 becomes 11. Tate’algorithm only terminates at fiber of type $II^*$. As concerns the point at the infinity, we carry out the following change of coordinates

$$t \mapsto \frac{1}{s}, \quad x \mapsto \frac{x}{s^2}, \quad y \mapsto \frac{y}{s^3}.$$ Hence the discriminant $\Delta = s(1-12^3s)^9$, since the vanishing order of $\Delta$ at $s = 0$ is 1, $S$ acquires a nodal rational curve.

After a suitable base change, the additive fibers are replaced by semistable fibers in the case of semistable fibration of Beauville that are of multiplicative fibers of type $I_n$.

In our case, $j : C \to \mathbb{P}^1(C)$, is a morphism of degree $e(S) = 12$. According to the Tate’s table, this guarantees that in the pull-back of the normal surface via $j$, the additive fibers $II$ and $III^*$ are replaced by 4 fibers (resp. 6) fibers of type $I_6^*$. Then these can be eliminated by quadrating twists to recover the semistable fibration $S \to C$. Hence, $S$ acquires of multiplicative fibers $I_n$.

We remark that in the case where $K$ is an algebraically closed field, we do not need to use quadratic twist to ensure the uniqueness of elliptic curves up to isomorphisms.

We determine the Mordell-Weil group for the case $[1, 1, 1, 9]$ of a Beauville’ semistable fibration. To carry out this, we transform the cubic equation after a suitable base change into a Weierstrass form $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2$, with $P = (0, 0)$ is not 2-torsion. Since $-2P = (a_2, 0)$ is 3-torsion if only if $a_2 = 0$. As we work in an extremal rational elliptic surface, so, after a suitable base change, we can recover both minimal rational elliptic surfaces defined as follows:

$$S : y^2 + xy + ty = x^3, \quad \text{and} \quad S' : y^2 + ty = x^3.$$ The trivial lattice has rank 10 as $e(S) = e(S') = 12$, and $\rho(S) = 10$. We deduce from corollary (3.4) that the Mordell-Weil rank is zero. Since $T$ has discriminant $-9$, there can only be 3-torsion in either case.

$$E(K) = E'(K) = \{O, P, -P\} \simeq \mathbb{Z}/3.$$ We can also get this result from the height. $E(K)$ is torsion free with generator $P$ meeting the component $\Theta_3$ of the fiber $I_0$ whose $h(P) = 2 - 3.5/9 = 1/3$, so that $P \in 1/3\mathbb{Z}$, and we find the Mordell-Weil group $\mathbb{Z}/3$ again. The another cases of determining the Mordell-Weil groups treat in the similar way. \qed
Note that in the second case, it arises a Van-Geemen Sarti involution.

We at present want to determine the Mordell-Weil lattices of Beauville’s semistable fibration. Hence, we first establish the list of the torsion parts of the various Mordell-Weil groups in this case.

**Corollary 4.5.** The subgroups $E(K)^0$ of the Mordell-Weil of the semistable fibration over $\mathbb{P}^1(\mathbb{C})$ having exactly four singular fibers are:

| $E(K)^0$ | $G(F_v)$ | Number of irreducible components of singular fibers |
|---|---|---|
| $\mathbb{Z}/3$ | $G(I_3)$ | 3, 3, 3 |
| $\mathbb{Z}/2$ | $G(I_4) \times G(I_2)$ | 4, 4, 2, 2 |
| $\mathbb{Z}/5$ | $G(I_5)$ | 5, 5, 1, 1 |
| $\mathbb{Z}/2 \times \mathbb{Z}/3$ | $G(I_6)$ | 6, 3, 2, 1 |
| $\mathbb{Z}/2$ | $G(I_4)$ | 8, 2, 1, 1 |
| $\mathbb{Z}/3$ | $G(I_3)$ | 9, 1, 1, 1 |

**Proof.** The proof is based on the same arguments than the proof of determining the Mordell-Weil groups.

**Theorem 4.6.** Then, we deduce the Mordell-Weil lattices of Beauville’s semistable fibration.

| $\text{MWL}$ | Number of irreducible components of singular fibers |
|---|---|
| $(\mathbb{Z}/3)^2/\mathbb{Z}/3$ | 3, 3, 3 |
| $(\mathbb{Z}/4 \times \mathbb{Z}/2)/\mathbb{Z}/2$ | 4, 4, 2, 2 |
| $\{0\}$ | 5, 5, 1, 1 |
| $\mathbb{Z}/6/\mathbb{Z}/3$ | 6, 3, 2, 1 |
| $\mathbb{Z}/4/\mathbb{Z}/2$ | 8, 2, 1, 1 |
| $\{0\}$ | 9, 1, 1, 1 |

The invariants of Mordell-Weil lattices can be expressed in terms of the geometric data of the surface $S$. Let $M = E(K)^0$ be the narrow Mordell-Weil lattice of $f : S \to C$. Then

$$\text{rk}(M) = \rho(S) - 2 - \sum_{v \in N} (m_v - 1),$$

lies in $\{0, 1\}$ for the Beauville’s case.

$$\det(M) = \nu^2 \det(NS(S)/\det(T)), \nu = [E(K) : E(K)^0],$$

lies in $\{8, 9, 18, 25, 108, 162\}$ for the Beauville’s case.

$$\mu(M) = 2\chi + 2 \min\{P.O : P \in M, P \neq O\} \geq 2\chi,$$

where $\rho(S)$ is the Picard number of $S$ and $m_v$ is the number of irreducible components of the singular fibers $F_v$, and $\mu(M)$ is the square of the minimal norm of $M$.

Note that there exits another approach of determining the Mordell-Weil lattices of Beauville’s semistable fibration. We just give a sketch of this approach. Consider the Lamé connections which are defined as the irreducible rank 2-connections on $\mathbb{P}^1(\mathbb{C})$ having four regular singular points at 0, 1, $t$, $\infty$ with exponents $-1/2, 1/2$ (see [LvdPU]) for more details.

Let $f : E \to \mathbb{P}^1$ be the double cover, its generic fiber is a two dimensional irreducible differential module $M$ over $C(z)$, the field of rational functions on $\mathbb{P}^1(\mathbb{C})$. Then $M$ is imprimitive and $\text{Sym}^2 M$ has a one differential module such that the corresponding quadratic extension $L$ has the form $L = C(z)(w)$. 

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We want to investigate the differential Galois group of a connection to get some information on the Mordell-Weil lattices.

**Definition 4.7.** Let \((K, \mathcal{O}) \subset (L, \mathcal{O})\) be an extension of differential fields with field of constants \(\mathbb{C}\). The differential Galois group \(\text{DGal}(L/K)\) is the group consisting of all the \(\mathcal{O}\)-automorphisms \(\sigma\) of \(L\) such that \(\sigma(f') = (\sigma(f))'\) for all \(f \in L\).

If \(L\) is finitely generated as a \(K\)-algebra, say, by \(p\) elements, then \(\text{DGal}(L/K)\) can be embedded onto \(GL(p, \mathbb{C})\), and it is an algebraic group if considered as a subgroup of \(GL(p, \mathbb{C})\) in this embedding.

We apply this definition to \(K = C(z)\), the derivation \(\cdot'\) being the differentiation with respect to some nonconstant function \(z \in K\). Given a connection \(\nabla_E\) on \(E\) over \(\mathbb{P}^1(\mathbb{C})\), we can consider a fundamental matrix \(\Phi\) of its solutions, and set \(L\) to be the field generated by all the matrix elements of \(\Phi\). The group \(\text{DGal}(\nabla_E)\) is defined to be \(\text{DGal}(L/K)\). See \(\text{[vdP]}\) for more details. Note also that the \(\text{DGal}(\nabla_E)\) of order \(m\) if and only if the \(\text{DGal}(f^*(\nabla_E))\) is cyclic and order \(2m\) if and only if there exists a point \((z_0, w_0) \in E\) of order \(2m\). This description enables us to compute the Mordell-Weil lattices of Beauville’semistable fibration.

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