ON THE LOCAL EIGENVALUE SPACINGS FOR CERTAIN ANDERSON-BERNOULLI HAMILTONIANS

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Abstract. The aim of this work is to extend the results from [B2] on local eigenvalue spacings to certain 1D lattice Schrödinger with a Bernoulli potential. We assume the disorder satisfies a certain algebraic condition that enables one to invoke the recent results from [B1] on the regularity of the density of states. In particular we establish Poisson local eigenvalue statistics in those models.

1. Introduction

The aim of this Note is to exploit the results from [B1] on certain Anderson-Bernoulli (A-B) Hamiltonians, in order to extend some of the eigenvalue spacing properties obtained in [B2] for Hamiltonians with Hölder site-distribution to the A-B setting.

As in [B2], all models are 1D. Recall that the A-B Hamiltonian with coupling \( \lambda \) is given by
\[
H = H_\lambda = \Delta + \lambda V
\]  
(1.1)
where \( V = (v_n)_{n \in \mathbb{Z}} \) are IID-variables ranging in \( \{-1, 1\} \), \( \text{Prob}[v_n = -1] = \frac{1}{2} = \text{Prob}[v_n = 1] \). It is believed that for \( \lambda \neq 0 \) sufficiently small, the integrated density of states (IDS) \( N \) of \( H \) is Lipschitz and becomes arbitrary smooth for \( \lambda \to 0 \). A first result in this direction was obtained in [B1], for small \( \lambda \) with certain specific algebraic properties.

Proposition 1. (see [B1]).

Let \( H_\lambda \) be the A-B model considered above and restrict \( |E| < 2 - \delta_0 \) for some fixed \( \delta_0 > 0 \). Given a constant \( C > 0 \) and \( k \in \mathbb{Z}_+ \), there is some \( \lambda_0 = \lambda_0(C, k) > 0 \) such that \( N(E) \) is \( C^k \)-smooth on \( [-2 + \delta_0, 2 - \delta_0] \) provided \( \lambda \) satisfies the following conditions.

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\( |\lambda| < \lambda_0 \)

(1.3) \( \lambda \) is an algebraic number of degree \( d < C \) and minimal polynomial \( P_d(x) \in \mathbb{Z}[X] \) with coefficients bounded by \( (\frac{1}{\lambda})^C \)

(1.4) \( \lambda \) has a conjugate \( \lambda' \) of modulus \( |\lambda'| \geq 1 \)

In what follows, we assume \( \lambda \) satisfies the conditions of Proposition 1 and the energy \( E \) restricted to \([-2 + \delta_0, 2 - \delta_0]\), unless specified differently.

Once we are in the presence of the Hamiltonian with a bounded density of states \( k(E) = \frac{dN}{dE} \), it becomes a natural question to inquire about local eigenvalue statistics for ‘truncated’ models \( H_N \), denoting \( H_N \) the restriction of \( H \) to the interval \([1, N]\) with Dirichlet boundary conditions. This problem was explored in [B2], assuming the site distribution \( v_n \) of \( V \) Hölder regular of some exponent \( \beta > 0 \), and we extended (in 1D) the theorem from [G-K] on Poisson statistics in this setting. Here we consider the A-B situation.

**Proposition 2.** With \( H \) as in Proposition 1, the rescaled eigenvalues of \( H_N \)

\[ \{N(E - E_0)\chi_I(E)\}_{E \in \text{Spec } H_N} \]

where \( I = [E_0, E_0 + \frac{L}{N}] \) and we let first \( N \to \infty \), then \( L \to \infty \), obey Poisson statistics.

This is the analogue of [B2], Proposition 5. Again one could conjecture the above statement to hold under the sole assumption that \( \lambda \) be sufficiently small.

Proposition 2 gives a natural example of a Jacobi Schrödinger operator with bounded density of states where the local eigenvalue spacing distribution differs from that of the potential (cf. S. Jitomirskaya’s talk ‘Eigenvalue statistics for ergodic localization’, Berkeley 11/10/2010).

Even with the smoothness of the IDS at hand, the arguments from [B2] do not carry out immediately to the A-B setting. For instance, the ‘classical’ approach to Minami’s inequality (see [C-G-K]) rests also on regularity of the single site distribution (in addition to a Wegner estimate) which makes it inapplicable in the A-B case. This will require us to develop an alternative argument in order to deal with near resonant eigenvalues.

Roughly speaking, it turns out that for the analysis below, the following ingredients suffice.
(1) The Furstenberg measures $\nu_E$ are absolutely continuous with bounded density

(2) The density of states $k$ is $C^1$

Hence the results from [B2] for Hölder site distribution follow from the present treatment. We believe however that the presentation in [B2] remains of interest since it is considerably simpler than the method from this paper.

As in [B2], the techniques are very much 1D and based on the usual transfer matrix formalism.

Recall that

$$M_n = M_n(E) = \prod_{j=n}^1 \begin{pmatrix} E - v_j & -1 \\ 1 & 0 \end{pmatrix}$$

(1.5)

and that the equation $H\xi = E\xi$ on the positive side is equivalent to

$$M_n \begin{pmatrix} \xi_1 \\ \xi_0 \end{pmatrix} = \begin{pmatrix} \xi_{n+1} \\ \xi_n \end{pmatrix}.$$  

(1.6)

What follows will use extensively ideas and techniques developed in [B1], [B2].

2. Preliminary estimates

Denote $\nu_E$ the Furstenberg measure at energy $E$. This is the unique probability measure on $P_1(\mathbb{R}) \simeq S^1$ which is $\mu = \frac{1}{2}(\delta_{g_+} + \delta_{g_-})$ - stationary, where

$$g_+ = \begin{pmatrix} E + \lambda & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad g_- = \begin{pmatrix} E - \lambda & -1 \\ 1 & 0 \end{pmatrix}.$$  

(2.1)

Thus

$$\nu_E = \sum_g \mu(g)\tau_g^*[\nu_E]$$

(2.2)

where $\tau_g$ denotes the projective action of $g \in SL_2(\mathbb{R})$.

It was proven in [B1] that in the context of Proposition 1, $\nu_E$ is absolutely continuous wrt Haar measure on $S^1$ and moreover $\frac{d\nu_E}{d\theta}$ becomes arbitrarily smooth for $\lambda \to \infty$.

The results of this section are stated for A-B Hamiltonians in general however and rely on general random matrix product theory.
Lemma 1. Let $\xi, \eta$ be unit vectors in $\mathbb{R}^2$. Given $E$ and $\varepsilon > 0$, $N > C(\lambda) \log \frac{1}{\varepsilon}$ we have
\[ \mathbb{E}[|\langle M_N(E)(\xi), \eta \rangle| < \varepsilon \| M_N(E)(\xi) \|] \leq \tau(\varepsilon) \] (2.3)
where $\tau(\varepsilon) = \max_{|I| = \varepsilon} \nu_E(I), I \subset S^1$ an interval.

Proof. Let $\xi = e^{i\theta}, \eta = e^{i\psi}$. If $M_N = g_N \cdots g_1, g_j \in \{g^+, g^-\}$, then
\[ \left| \frac{\langle M_N(\xi), \eta \rangle}{\| M_N(\xi) \|} \right| = |\cos(\tau_{g_N \cdots g_1}(\theta) - \psi)|. \]
Hence the l.h.s. of (2.3) is bounded by
\[ \mathbb{E}_{g_1, \ldots, g_N} \left[ |\tau_{g_N \cdots g_1}(\theta) - \psi'| < \varepsilon \right] (\psi' = \psi^+) \]
\[ = \sum_g \mu^{(N)}(g) 1_{[\psi' - \varepsilon, \psi' + \varepsilon]}(\tau_g(\theta)). \] (2.4)
Let $0 \leq f \leq 1$ be a smooth function on $S^1$ such that $f = 1$ on $[\psi' - \varepsilon, \psi' + \varepsilon]$, supp $f \subset [\psi' - 2\varepsilon, \psi' + 2\varepsilon]$ and $|\partial^\alpha f| \lesssim_\alpha (\frac{\varepsilon}{4})^\alpha$. Then, invoking the large deviation estimate for the $\mu$-random walk, we obtain
\[ (2.4) \leq \sum_g \mu^{(N)}(g) (f \circ \tau_g) \]
\[ \leq \int f d\nu_E + e^{-c(\lambda)N} \| f \|_{C^1} \]
\[ \leq \nu_E([\psi' - 2\varepsilon, \psi' + 2\varepsilon]) + \frac{1}{\varepsilon} e^{-c(\lambda)N} \]
proving the lemma. \qed

Lemma 2. Assume the Lyapounov exponent Hölder regular of exponent $\alpha > 0$. Then
\[ \max_{|E - E_0| < \kappa} \log \| M_N(E) \| < L(E_0)N + c\kappa^\alpha N \] (2.5)
outside a set $\Omega$ of measure at most $e^{-\kappa' N}$.

Proof. Recall the large deviation theorem for the Lyapounov exponent
\[ \mathbb{E}\left[ \left| \frac{1}{N} \log \| M_N(E) \| - L(E) \right| > \sigma \right] \lesssim e^{-\sigma' N}. \] (2.6)
Set $\kappa_1 = C\kappa^\alpha$. It follows from (2.6) that for $|E - E_0| < \kappa$
\[ \mathbb{E}[\log \| M_N(E) \| > L(E_0)N + \kappa_1 N] \lesssim e^{-\kappa' N} \] (2.7)
since $|L(E) - L(E_0)| \lesssim \kappa^\alpha$.

More generally, given indices $N > \ell_1 > \ell_2 > \cdots > \ell_r > 1$ ($r = O(1)$), we have for $|E - E_0| < \kappa$ that
\[
\mathbb{E}[\log \|M_{N-\ell_1}^{(v_{\ell_1+1})}(E)\| + \log \|M_{\ell_1-\ell_2-1}^{(v_{\ell_1-1},v_{\ell_2+1})}(E)\| + \cdots + \log \|M_{\ell_r-1}^{(v_{\ell_1},\cdots,v_{\ell_r})}(E)\|]
\geq L(E_0)N + \kappa_1 N \lesssim e^{-\kappa'N}.
\] (2.8)

Set $\theta = e^{-\frac{1}{2}\kappa'}$ and let $E$ be a finite subset of $|E - E_0| < \kappa$, $|E| < \frac{1}{\theta}$ such that $\max_{|E-E_0|<\kappa} \operatorname{dist}(E,E) < \theta$. Take $r = r(\kappa)$ and $\Omega \subset \{1, -1\}^N$ such that (2.8) holds for $V \not\in \Omega, E \in \mathcal{E}$ and all $N > \ell_1 > \cdots > \ell_r > 1$,
\[
|\Omega| \lesssim N^{r}, |\mathcal{E}| e^{-\kappa'N} < e^{-\frac{1}{4}\kappa'N}.
\] (2.9)

Take then $E \in [E_0 - \kappa, E_0 + \kappa]$ and $E_1 \in \mathcal{E}, |E - E_1| < \theta$. Using a truncated Taylor expansion, we get
\[
M_N(E) = M_N(E_1) + \left( \sum_{1 \leq \ell \leq N} M_{N-\ell}(E_1) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) M_{\ell-1}(E_1) \right) (E - E_1) + \cdots
\]
\[
+ \frac{1}{r!} \left[ \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_r \leq N} M_{N-\ell_1}(E_1) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) M_{\ell_1-\ell_2-1}(E_1) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \cdots M_{\ell_r-1}(E_1) \right] (E - E_1)^r
\]
\[+ O(C^N \theta^{r+1}).
\] (2.10)

Taking $r \sim \frac{1}{\kappa'}$, we ensure the remainder term $< e^{-N}$, while for $V \not\in \Omega$
\[
\|(1.8)\| < e^{(L(E_0)+\kappa_1)N} + \theta e^{(L(E_0)+\kappa_1)N} + \cdots + \theta^r e^{(L(E_0)+\kappa_1)N}
\]
Lemma 2 follows.

3. A Wegner estimate

**Proposition 3.** Assume the Furstenberg measures of $H$ have bounded density. Then
\[
\mathbb{E}[\text{Spec } H_N \cap I \neq \emptyset] < CN |I| + Ce^{-cN}
\] (3.1)
if $I \subset R$ is an interval.
Proof. What follows is an adaptation of the argument used in [B2]. Let 
\( I = [E_0 - \delta, E_0 + \delta] \) and assume \( H_N \) has an eigenvalue \( E \in I \) with eigenvector \( \xi = (\xi_j)_{1 \leq j \leq N} \). Then

\[
M_N(E) \begin{pmatrix} \xi_1 \\ 0 \\ \vdots \\ \xi_N \end{pmatrix} = \begin{pmatrix} 0 \\ \xi_N \end{pmatrix}.
\]

Assume \( |\xi_1| \geq |\xi_N| \) (otherwise, we replace \( M_N \) by \( M_N^{-1} \)). It follows that

\[
\| M_N(E)e_1 \| \leq 1.
\]

On the other hand, from the large deviation theorem

\[
\log \| M_N(E_0)e_1 \| > L(E_0)N - \kappa N
\]

for \( V \notin \Omega \), where

\[
|\Omega| < e^{-\kappa'N}.
\]

Here \( \kappa > 0 \) is an appropriate constant.

In view of Lemma 2, we may moreover assume that for \( |E - E_0| < \delta \)

\[
\max_n \| M_{N-n}^{(v_N, \ldots, v_{n+1})}(E) \| \| M_{n-1}^{(v_n-1, \ldots, v_n)}(E) \| < e^{L(E_0)N+\kappa N}
\]

if \( V \notin \Omega \).

Denote

\[
B = e^{L(E_0)N-2\kappa N}.
\]

Then, for \( V \notin \Omega \),

\[
\kappa N \leq \left| \log(\| M_N(E)e_1 \| + B) - \log(\| M_N(E_0)e_1 \| + B) \right| \\
\leq \int_{-\delta}^{\delta} \frac{d}{dt} \log(\| M_N(E_0 + t)e_1 \| + B) \, dt.
\]

The integrand in (3.6) is clearly bounded by

\[
\sum_{j=1,2} \sum_{n=1}^{N} \frac{|\langle M_{N-n}^{(v_N, \ldots, v_{n+1})}(E_0 + t)e_1, e_j \rangle| \langle M_{n-1}^{(v_n-1, \ldots, v_n)}(E_0 + t)e_1, e_1 \rangle}{\| M_{N-n}^{(v_N, \ldots, v_{n+1})}(E_0 + t)e_1 \| + B}
\]

and we estimate the \( n \)-norm by

\[
\| (M_{N-n}^{(v_N, \ldots, v_{n+1})}(E_0 + t)) e_j \| \cdot \| M_{n-1}^{(v_n-1, \ldots, v_n)}(E_0 + t)e_1 \| \]

\[
\left| \langle (M_{N-n}^{(v_N, \ldots, v_{n+1})}(E_0 + t))^* e_j, M_{n-1}^{(v_n-1, \ldots, v_n)}(E_0 + t)e_1 \rangle \right|.
\]
We distinguish two cases. If \( n \geq \frac{N}{2} \), set
\[
\eta = \frac{(M_{N-n}^{(v_{N},...,v_{n+1})}(E_0 + t))^* e_j}{\| (M_{N-n}^{(v_{N},...,v_{n+1})}(E_0 + t))^* e_j \|}
\]
which is independent from \( v_1, \ldots, v_n \). From Lemma 1, we get the distributional inequality
\[
\mathbb{E}_{v_1,\ldots,v_{n-1}}[\| (M_{n-1}^{(v_{n-1},v_1)}(E_0 + t)e_1, \eta) \| < \varepsilon \| M_{n-1}^{(v_{n-1},v_1)}(E_0 + t)e_1 \|] \leq C\varepsilon
\]
(3.9) since by assumption \( \tau(E) \leq C\varepsilon \). If \( n < \frac{N}{2} \), set
\[
\eta = \frac{M_{n-1}^{(v_{n-1},v_1)}(E_0 + t)e_1}{\| M_{n-1}^{(v_{n-1},v_1)}(E_0 + t)e_1 \|}
\]
and argue similarly, considering \( (M_{N-n}^{(v_N,\ldots,v_{n-1})}(E_0 + t))^* \).

Hence we proved that
\[
\mathbb{E}_{v_1,\ldots,v_N}[ (3.8) > \lambda ] < C\lambda^{-1} \text{ for } \lambda < c \log N. \tag{3.10}
\]
On the other hand, the \( n \)-term in (3.7) is also bounded by
\[
\frac{1}{B} \| M_{N-n}^{(v_N,\ldots,v_{n+1})}(E_0 + t) \| \| M_{n-1}^{(v_{n-1},v_1)}(E_0 + t) \| < e^{3\kappa N} \tag{3.11}
\]
since \( V \not\in \Omega \), by (3.5). Therefore, taking \( \kappa \) in (3.11) appropriately, according to (3.10), it follows that
\[
\mathbb{E}[(3.7) 1_{\Omega^c}] \lesssim N^2. \tag{3.12}
\]
Consequently, recalling (3.6), we obtain from (3.12) and Tchebychev’s inequality
\[
\mathbb{E}[ Spec H_N \cap I \neq \phi] \leq |\Omega| + C \frac{\delta}{\kappa N} N^2 < e^{-c'N} + \frac{C}{\kappa} \delta N
\]
proving (3.1).

Let \( H \) be as in Proposition 1 on the sequel.

The energy range is restricted to \([-2 + \delta_0, 2 - \delta_0]\) according to Proposition 1.

Using Proposition 3 and Anderson localization, one deduces then the analogue of Proposition 3 in \( \text{[B2]} \). We leave the details to the reader (see \( \text{[B2]} \)).
Proposition 4. Assuming \( \log \frac{1}{\delta} < c(\lambda)N \), we have for \( I = [E_0 - \delta, E_0 + \delta] \)
that
\[
\mathbb{E}[\text{Tr} X_I(H_N)] = N k(E_0)|I| + O\left(N \delta^2 + \delta \log^2 \left(N + \frac{1}{\delta}\right)\right).
\] (3.13)

4. Near resonances

In what follows, we develop an alternative to Minami’s argument that is applicable in the A-B context (recall that \( H = H_\lambda \) with \( \lambda \) satisfying the conditions from Proposition 1).

Lemma 3. Let \( I = E_0 - \delta, E_0 + \delta \) be as above. Let \( N \in \mathbb{Z}_+ \).

The probability for existence of a pair of orthogonal unit vectors \( \xi, \xi' \in \mathbb{R}^n \) satisfying
\[
\| (H_N - E_0)\xi \|_2 < \delta, \| (H_N - E_0)\xi' \|_2 < \delta \quad (4.1)
\]
\[
\max_{j < \sqrt{N}} (|\xi_j|, |\xi_{N-j}|, |\xi'_j|, |\xi'_{N-j}|) < \frac{1}{N^{10}} \quad (4.2)
\]
is at most
\[
CN^7 \delta^2 + e^{-c\sqrt{N}}. \quad (4.3)
\]

Proof. We take \( \sqrt{N} < \nu < N - \sqrt{N} \) such that \( |\xi_{\nu}| \gtrsim \frac{1}{\sqrt{N}} \). Since \( \xi \perp \xi' \),
\[
\| \xi_{\nu} \xi'_{\nu} - \xi'_{\nu} \xi_{\nu} \|_2 \gtrsim \frac{1}{\sqrt{N}} \quad (4.4)
\]
Again by (4.2). \( \sqrt{N} < \nu_1 < N - \sqrt{N} \). Set further for \( 1 \leq j \leq N \)
\[
(\xi_j, \xi'_j) = (\xi_j^2 + (\xi'_j)^2)^{\frac{1}{2}} e^{i\theta_j}. \quad (4.5)
\]
Hence (4.4) certainly implies that
\[
|\sin(\theta_{\nu} - \theta_{\nu_1})| > \frac{1}{N}. \quad (4.6)
\]
Assume \( \nu < \nu_1 \) (the other alternative is similar). We distinguish two cases.

Case 1. There is some \( \nu < j_1 < \nu_1 \) such that
\[
|\sin(\theta_{\nu} - \theta_{j_1})| > \frac{1}{10N} \text{ and } |\sin(\theta_{\nu_1} - \theta_{j_1})| > \frac{1}{10N}. \quad (4.7)
\]

Define the vector
\[
\eta = \frac{\xi_{j_1} \xi' - \xi'_{j_1} \xi}{(\xi_{j_1}^2 + (\xi'_{j_1})^2)^{\frac{1}{2}}}. \]
Obviously $\|\eta\|_2 = 1$ and $\eta_{j_1} = 0$. Also, from (4.1) it easily follows that

$$
\|(H_N - E_0)\eta\|_2 < 2\delta. \quad (4.8)
$$

From (4.7)

$$
|\eta_{\nu}| = \left(\xi_{\nu}^2 + (\xi_{\nu}')^2\right)^{\frac{1}{2}}|\sin(\theta_{j_1} - \theta_{\nu})| \gtrsim \frac{1}{N^{3/2}} \quad (4.9)
$$

$$
|\eta_{\nu_{1}}| \gtrsim \frac{1}{N^2}. \quad (4.10)
$$

Next, we introduce the vectors

$$
\eta^{(1)} = \eta|_{[1, j_1]} \quad \text{and} \quad \eta^{(2)} = \eta|_{[j_1 + 1, N]}
$$

as well as the restrictions

$$
H^{(1)} = H|_{[1, j_1]} \quad \text{and} \quad H^{(2)} = H|_{[j_1 + 1, N]} \quad (4.11)
$$

with Dirichlet boundary conditions. By (4.9), (4.10), $\|\eta^{(1)}\|_2, \|\eta^{(2)}\|_2 \gtrsim \frac{1}{N^2}$ while by (4.8) and $\eta_{j_1} = 0$, it follows that $\|(H^{(1)} - E_0)\eta^{(1)}\|_2 < 2\delta, \|\eta^{(2)}_j\|_2 < 2\delta$. Hence

$$
\text{dist}(E, \text{Spec } H^{(1)}) < N^2\delta, \text{dist}(E, \text{Spec } H^{(2)}) < N^2\delta. \quad (4.12)
$$

Note that $H^{(1)}, H^{(2)}$ are independent as functions of $V = (v_n)_{1 \leq n \leq N}$ and by construction, $\sqrt{N} \leq j_1 \leq N - \sqrt{N}$. Involving Proposition 3, it follows that the probability for the joint event (4.12) is at most

$$
c[j_1 N^2 \delta + e^{c j_1}][(N - j_1)N^2\delta + e^{-c(N-j_1)}] < CN^6 \delta^2 + e^{-c\sqrt{N}}. \quad (4.13)
$$

**Case 2.** For all $\nu \leq j \leq \nu_1$, either $|\sin(\theta_{\nu} - \theta_j)| \leq \frac{1}{10N}$ or $|\sin(\theta_{\nu_1} - \theta_j)| \leq \frac{1}{10N}$. Take then the smallest $\nu < j_1 \leq \nu_1$ for which $|\sin(\theta_{\nu_1} - \theta_{j_1})| \leq \frac{1}{10N}$. Hence $|\sin(\theta_{\nu} - \theta_{j_1-1})| \leq \frac{1}{10N}$. Denote

$$
\eta^{(1)} = \frac{\xi_{j_1}\xi' - \xi_{j_1}'\xi}{(\xi_{j_1}^2 + (\xi_{j_1}')^2)^{\frac{1}{2}}} \bigg|_{1 \leq j \leq j_1 - 1}
$$

$$
\eta^{(2)} = \frac{\xi_{j_1-1}\xi' - \xi_{j_1-1}'\xi}{(\xi_{j_1-1}^2 + (\xi_{j_1-1}')^2)^{\frac{1}{2}}} \bigg|_{j_1 \leq j \leq N}
$$
and \(H^{(1)} = H_{[1,j_1]}, H^{(2)} = H_{[j_1,N]}\). Since \(\eta^{(1)}_{j_1} = 0, \| (H^{(1)} - E_0) \eta^{(1)} \|_2 < 2\delta\).

Also \(\| \eta^{(1)} \|_2 \geq |\eta_{\nu}| \gtrsim \frac{1}{\sqrt{N}} |\sin(\theta_\nu - \theta_{j_1})| \gtrsim \frac{1}{\sqrt{N}} (\frac{1}{10N} - \frac{1}{10N}) > \frac{1}{10N}\), implying \(\text{dist}(E, \text{Spec} H^{(1)}) < N^2\delta\). Similarly \(\text{dist}(E, \text{Spec} H^{(2)}) < N^2\delta\) and we conclude as in Case 1.

Summing (4.13) over \(j\), Lemma 3 follows. \(\square\)

We may now establish an analogue of Proposition 4 in [B2] for Anderson-Bernoulli Hamiltonians as considered above.

**Proposition 5.** Let \(I = [E_0 - \delta, E_0 + \delta] \subset [-2 + \delta_0], [2 + \delta_0]\) and \(\log \frac{1}{\delta} < c\sqrt{N}\).

Then

\[
\mathbb{E}[H_N \text{ has at least two eigenvalues in } I] \leq CN^2\delta^2 + C\delta \log \left(N + \frac{1}{\delta}\right). \tag{4.14}
\]

**Proof.** Proceeding as in [B2], set \(M = C \log^2 (N + \frac{1}{\delta})\) for an appropriate constant \(C\). From the theory of Anderson localization, the eigenvectors \(\xi_\alpha\) of \(H_N \{\xi_\alpha\} = 1\), satisfy

\[
|\xi_\alpha(j)| < e^{-c|j - j_\alpha|} \text{ for } |j - j_\alpha| > \frac{M}{10} \tag{4.15}
\]

with probability at least \(1 - e^{-cM}\), with \(j_\alpha\) the center of localization of \(\xi_\alpha\). We may therefore introduce a collection \((\Lambda_s)_{1 \leq s \leq s_1}, s_1 \sim \frac{N}{M}\), of size \(M\) subinterval of \([1, N]\) such that for each \(\alpha\), there is some \(1 \leq s \leq s_1\) satisfying \(j_\alpha \in \Lambda_s\) and \(\| \xi_\alpha \| \{1, N\} \setminus \Lambda_s \|_2 < e^{-cM}\) \(\tag{4.16}\)

\[
\| (H_{\Lambda_s} - E_\alpha) \xi_\alpha \|_2 < e^{-cM}, \tag{4.17}
\]

where \(\xi_{\alpha,s} = \xi_\alpha \mid_{\Lambda_s}\). For \(1 < s < s_1\), we may moreover insure that

\[
|\xi_\alpha(j)| < e^{-cM} \text{ if } \text{dist}(j, \partial \Lambda_s) < \frac{M}{10} \tag{4.18}
\]

By (4.16), (4.17), \(\text{dist}(E_\alpha, \text{Spec } H_{\Lambda_s}) < e^{-cM}\) and hence \(\text{Spec } H_{\Lambda_s} \cap \tilde{I} \neq \phi, \tilde{I} = [E_0 - 2\delta, E_0 + 2\delta]\), if \(E_\alpha \in I\). According to Proposition 3, by our choice of \(M\)

\[
\mathbb{E}[\text{Spec } H_{\Lambda_s} \cap \tilde{I} \neq \phi] < CM\delta + c e^{-cM} < CM\delta. \tag{4.19}
\]

Note that if \(\Lambda_s \cap \Lambda_{s'} = \phi\), then \(H_{\Lambda_s}, H_{\Lambda_{s'}}\) are independent.
Hence, by construction and (4.19),

$$\mathbb{E}[\text{there are } \alpha, \alpha' \text{ s.t. } E_\alpha, E_{\alpha'} \in I \text{ and } |j_\alpha - j_{\alpha'}| > 4M] \leq C \sum_{s,s'} (M\delta)^2 < CN^2\delta^2.$$  \hspace{1cm} (4.20)

with the $s, s'$-sum performed over pairs such that $\Lambda_s \cap \Lambda_{s'} = \emptyset$.

It remains to consider the case $|j_\alpha - j_{\alpha'}| \leq 4M$. If $\text{dist}(j_\alpha, \{1, N\}) < 2M$ then $\text{Spec} H_{[1,4M]} \cap \tilde{I} \neq \emptyset$ or $\text{Spec} H_{[N-4M,N]} \cap \tilde{I} \neq \emptyset$.

Again by Proposition 3, the probability for this event is less than

$$CM\delta < C\log(N + \frac{1}{\delta})\delta.$$  \hspace{1cm} (4.21)

Next assume moreover $\text{dist}(\{j_\alpha, j_{\alpha'}\}, \{1, N\}) \geq 2M$. Then

$$j_\alpha \in \Lambda_t, j_{\alpha'} \in \Lambda_{t'} \text{ where } 1 < t, t' < s_1 \text{ and } |t - t'| < 10.$$

Introduce an interval $\Lambda$ obtained as union of at most 10 consecutive $\Lambda_s$ intervals, such that $\Lambda_t, \Lambda_{t'} \subset \Lambda$. By (4.16), (4.18), setting $\tilde{\xi}_\alpha = \xi_\alpha|_\Lambda, \tilde{\xi}_{\alpha'} = \xi_{\alpha'}|_\Lambda$, we get

$$\| (H_\Lambda - E_\alpha) \tilde{\xi}_\alpha \|_2 < e^{-cM}, \| (H_\Lambda - E_{\alpha'}) \tilde{\xi}_{\alpha'} \|_2 < e^{-cM}$$

so that for $E_\alpha, E_{\alpha'} \in I$

$$\| (H_\Lambda - E_0) \tilde{\xi}_\alpha \|_2 < 2\delta, \| (H_\Lambda - E_0) \tilde{\xi}_{\alpha'} \|_2 < 2\delta.$$  \hspace{1cm} (4.22)

Also, by (4.18), $\max_{\text{dist}(j, \partial \Lambda)} \frac{M}{10} (|\xi_\alpha(j)|, |\xi_{\alpha'}(j)|) < e^{-cM} < \frac{1}{|\Lambda|^{d_0}}$. Hence, Lemma 3 applies to $H_\Lambda$. According to (4.3), the probability that $H_\Lambda$ satisfies the above property is at most (again by our choice of $M$)

$$CM^7\delta^2 + e^{-c\sqrt{M}} < CM^7\delta^2.$$  \hspace{1cm} (4.23)

Summing over the different boxes $\Lambda$ introduced above gives then

$$CNM^7\delta^2 < CN \left( \log(N + \frac{1}{\delta}) \right)^7 \delta^2.$$  \hspace{1cm} (4.24)

Adding the contributions (4.20), (4.21), (4.24) and noting that the last is majorized by the first two, inequality (4.14) follows.  \hspace{1cm} \square
5. LOCAL EIGENVALUE STATISTICS

Following the same argument as in Proposition 5 of [B2], Proposition 4 and Proposition 5 above permit to establish Poisson statistics for the local eigenvalue spacings. Thus we obtain Proposition 2 stated in the Introduction.

The proof is completely analogous to that of Proposition 5 in [B2], except that instead of choosing $M = K \log N$, $M_1 = K_1 \log N$, we take say $M = (\log N)^4$, $M_1 = (\log N)^3$.

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