A Small Observation on Co-categories

Peter LeFanu Lumsdaine
plumsdai@andrew.cmu.edu

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Abstract

Various concerns suggest looking for internal co-categories in categories with strong logical structure. It turns out that in any coherent category $E$, all co-categories are co-equivalence relations.

Definition 1 Let $E$ be any category. An (internal) co-category $Q$ in $E$ is an internal category in $E^{op}$, i.e. objects and morphisms in $E$

$$Q^0 \xrightarrow{l} Q^1 \xrightarrow{q} Q^1 +_{Q^0} Q^1$$

such that the following diagrams commute:

Definition 2 A co-category $Q$ is a co-preorder if the maps $l, r$ are jointly epimorphic.

A co-category $Q$ is a co-groupoid if there is a map $s : Q^1 \rightarrow Q^1$ satisfying the duals of the usual identities for the inverse map of a groupoid.

A co-groupoid $Q$ is a co-equivalence relation if it is a co-preorder.

Remark 1 In a co-preorder, the co-composition $q$ is uniquely determined by the maps $l, r, i$; likewise, in a co-groupoid, the co-inverse map $s$ is determined by the rest of the structure.
Together with the obvious maps, these give categories and full inclusions

\[
\text{CoEqRel}(\mathcal{E}) \longrightarrow \text{CoPreOrd}(\mathcal{E}) \longrightarrow \text{CoCat}(\mathcal{E}).
\]

**Example** If \(\mathcal{E}\) has all (or enough) pushouts and \(m : S \to A\) is any monomorphism, then the co-kernel pair of \(m\) is a co-equivalence relation

\[
A \leftarrow \nu_1 \downarrow_{\nu_1, \nu_3} \leftarrow A + S A \rightarrow \nu_2 \downarrow_{\nu_2} \rightarrow A + S A.
\]

This gives the object part of a functor \(\text{Mono}(\mathcal{E}) \to \text{CoEqRel}(\mathcal{E})\), which (almost by definition) is one half of an equivalence whenever \(\mathcal{E}\) is co-exact. (Here \(\text{Mono}(\mathcal{E})\) denotes the full subcategory of \(\mathcal{E}\) on monomorphisms.)

**Example** A paradigmatic example is the interval \(I\) in \(\text{Top}\), where \(I^0\) is a singleton, \(I^1\) is the unit interval, \(l\) and \(r\) are the endpoints, \(I^1 + I^1\) is two copies of the interval joined end to end, and \(q\) is the obvious “stretching” map. Unfortunately, this is also of course not an actual co-category — the axioms hold only up to homotopy. However, it provides a very useful mental picture for the arguments below; and if we delete the interior of the interval, we obtain a genuine co-category. See also the examples below for more versions of the interval.

**Definition 3** A coherent category is a category with all finite limits, and images and unions that are stable under pullback.

[4] A1.3–4] gives various basic results on coherent categories, which we will use here without comment.

**Definition 4** Coherent logic is the fragment of first-order logic built up from atomic formulæ using finite con-/disjunction and existential quantification.

Coherent logic is discussed in [4] D1.1–2; the essential point is that coherent logic can be interpreted soundly in coherent categories, and so may be used as an internal language for working in them.

**Proposition 1** In a coherent category \(\mathcal{E}\), every co-category \(Q\) is a co-equivalence relation.
Proof First, we show that any co-category \( Q \) is a co-preorder.

Arguing in the internal logic: given \( x \) in \( Q^1 \), consider \( q(x) \), in \( Q^1 +_Q 0^1 \). Either there is some \( y \) in \( Q^1 \) with \( q(x) = \nu_1(y) \), or else some \( y \) with \( q(x) = \nu_2(y) \). In the first case, we then have \( x = [li, 1]q(x) = li(y) \); in the second, \( x = ri(y) \). Thus any \( x \) in \( Q^1 \) is in the image of either \( l \) or \( r \), i.e. \( l \) and \( r \) are jointly covering, hence epi. (Indeed, in the first case \( x = li(y) = l(li(y)) = li(x) \), and in the second, \( x = ri(x) \).

Restating this diagrammatically: \( Q^1 +_Q 0^1 \) is the union of the subobjects \( \nu_j : Q^1 \rightarrow Q^1 +_Q 0^1 \), so \( Q^1 \) is the union of the subobjects \( m_j = q^*(\nu_j) \):

\[
\begin{array}{ccc}
P_j & \xrightarrow{q_j} & Q^1 \\
\downarrow m_j & & \downarrow \nu_j \\
Q^1 & \xrightarrow{q} & Q^1 +_Q 0^1
\end{array}
\]

In particular, the pair \( m_1, m_2 \) are jointly covering. But by the co-unit identities, \( liq_1 = [li, 1]q_1 = [li, 1]q_m = m_1 \), and similarly \( riq_2 = m_2 \). Thus \( liq_1, riq_2 \) are jointly covering, and hence so are \( l, r \).

Now, we check that any co-preorder is a co-equivalence relation. (We give only the diagrammatic version. Exercise: restate this in the internal logic!) We want to define \( s : Q^1 \rightarrow Q^1 \) with \( sl = r, sr = l \). Since \( l, r \) are monos with union \( Q^1 \), the pullback square

\[
\begin{array}{ccc}
\pi_1 & \xrightarrow{\pi_2} & Q^0 \\
\downarrow & & \downarrow r \\
Q^0 & \xrightarrow{l} & Q^1
\end{array}
\]

is also a pushout, so to construct \( s \) as above, it is enough to show that \( r\pi_1 = l\pi_2 \). But \( \pi_1 = il\pi_1 = ir\pi_2 = \pi_2 \), so \( r\pi_1 = r\pi_2 = l\pi_1 = l\pi_2 \), and we are done. \( \square \)

Corollary 2 If \( \mathcal{E} \) is a coherent category with co-kernel pairs of monos, then \( \text{CoCat}(\mathcal{E}) \simeq \text{Mono}(\mathcal{E}) \). (In particular, this holds if \( \mathcal{E} \) is a pretopos [4, A1.4.8].)

Proof A coherent category is certainly co-effective, so if it has co-kernel pairs, it is co-exact. \( \square \)

Corollary 3 For any topos \( \mathcal{E} \), \( \text{CoCat}(\mathcal{E}) \simeq (\mathcal{E}/\Omega)_{colax} \).

(A colax map \( (X, \varphi) \rightarrow (Y, \psi) \) is a map \( f : X \rightarrow Y \) such that \( \varphi \leq \Omega \psi f \).)

In particular, inspecting this equivalence, we see that in this case there is a universal internal co-category in \( \mathcal{E} \), from which every co-category in \( \mathcal{E} \) may be obtained uniquely by pullback: it is the co-kernel pair of \( \top : 1 \rightarrow \Omega \).

Example The condition that unions are preserved by pullback is crucial: \( \text{AbGp} \), for instance, is regular, and has unions, but there is a non-co-preorder co-category corresponding to the interval pictured above, given by the objects

\[
Q^0 = \langle v_0 \rangle \quad Q^1 = \langle v_0, e_1, v_1 \rangle \quad Q^1 +_Q Q^1 = \langle v_0, e_1, v_1, e_2, v_2 \rangle
\]
(with the natural maps making this a pushout), and maps given by the matrices

\[
l = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad i = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \quad q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

This example may be given more structure; it is, for instance, the total space of an natural co-category in \(\text{Ch}(\text{AbGp})\). Since all the underlying groups are free and of finite rank, dualising by transposing matrices also gives corresponding categories in \(\text{AbGp}\) and \(\text{Ch}(\text{AbGp})\).

However, any category in a Mal’cev category is a groupoid (this has been observed by various authors, e.g. in [3]), so any co-category in a co-Mal’cev category (e.g. in an Abelian category, or a topos [2]) is a co-groupoid.

**Example** An example of a non-co-groupoid co-category is the interval \(I\) in \(\text{Cat}\), with \(I^0 = (\cdot)\), \(I^1 = (\cdot \rightarrow \cdot)\); seen as a co-simplicial object, this is just the usual inclusion functor \(\Delta \hookrightarrow \text{PreOrd} \hookrightarrow \text{Cat}\).

Indeed, the functor \(\text{Cat} \rightarrow \text{SSet} \rightarrow \text{SAbGp} \rightarrow \text{Ch}(\text{AbGp}) \rightarrow \text{Ch}(\text{AbGp})\) “take nerve; take free abelian groups; normalise to a complex; quotient out by subcomplex generated in degrees \(\geq 2\)” sends \(I\) to the co-category in \(\text{Ch}(\text{AbGp})\) of the previous example.

Co-categories arise as candidate “interval objects” when using 2-categories to model intensional type theory [1]. There, one seeks them in categories with some sort of “weakened” logical structure; the present result confirms the suspicion that examples in classical “strict” logical categories are necessarily fairly trivial.

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**References**

[1] Steve Awodey and Michael A. Warren, *Homotopy-theoretic models of identity types*, Math. Proc. Cambridge Philos. Soc. 146 (2009), no.1, pp.45–55, [arXiv:0709.0248 [math.LO]]

[2] Dominique Bourn, *Mal’cev Categories and Fibration of Pointed Objects*, Applied Categorical Structures, Vol.4 (1996), pp.307–327

[3] A. Carboni, G.M. Kelly, M.C. Pedicchio *Some remarks on Mal’tsev and Goursat categories*, Applied Categorical Structures, Vol.1 (1993), pp.385–421

[4] Peter Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, Oxford University Press (2002)