Quantum Gauge Equivalence in QED

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We discuss gauge transformations in QED coupled to a charged spinor field, and examine whether we can gauge-transform the entire formulation of the theory from one gauge to another, so that not only the gauge and spinor fields, but also the forms of the operator-valued Hamiltonians are transformed. The discussion includes the covariant gauge, in which the gauge condition and Gauss’s law are not primary constraints on operator-valued quantities; it also includes the Coulomb gauge, and the spatial axial gauge, in which the constraints are imposed on operator-valued fields by applying the Dirac-Bergmann procedure. We show how to transform the covariant, Coulomb and spatial axial gauges to what we call “common form,” in which all particle excitation modes have identical properties. We also show that, once that common form has been reached, QED in different gauges has a common time-evolution operator that defines time-translation for states that represent systems of electrons and photons. By combining gauge transformations with changes of representation from standard to common form, the entire apparatus of a gauge theory can be transformed from one gauge to another.
I. INTRODUCTION

Quantum Electrodynamics (QED) embodies the problems characteristic of a field theory with an infinite number of degrees of freedom, as well as those that attach to a theory with time-independent constraints. Thus, QED manifests the divergent, albeit renormalizable radiative corrections which also characterize other field theories, many of which do not include any gauge fields and are not invariant to any local gauge transformations. QED is also a gauge theory, and must obey a time-independent constraint which imposes relations among fields not only while different particles are in interaction with each other, but also in the so-called “asymptotic limit,” when all particles are far apart and not interacting. Such constraints can conflict with idealized models in which, when two different varieties of particles recede from each other, field interaction effectively cease, and their corresponding asymptotic “in” or “out” fields commute. We note, for example, that the “LSZ” axioms, which were believed to be general enough to accommodate all field theories, are not fully compatible with QED because canonical commutation rules between asymptotic “in” or “out” Dirac and Maxwell fields are incompatible with Gauss’s law.

Because it is difficult to graft the constraints that characterize QED onto the apparatus that generates perturbative $S$-matrix elements, gauge-invariance is not nearly as straightforward in QED as it is in classical electrodynamics. On a superficial level, gauge-invariance in QED is equated to the very well established “test of gauge-invariance,” which requires that $S$-matrix elements vanish when they have external photon lines polarized in the $k^\mu$ direction. This principle is beyond dispute when applied to tree graphs. It also applies to graphs with radiative corrections, although apparent violations can be caused by regularization procedures that are incompatible with it. Furthermore, we can raise the question of whether $S$-matrix elements are identical in different gauges. Here, again, this identity is well established for tree graphs. In graphs with radiative corrections, the renormalization procedure has not been firmly established since the finite parts of divergent are integrals not well defined in non-covariant gauges, such as axial gauges.
In classical electrodynamics, gauge transformations can be implemented simply by adding the gradient \( \partial_\mu \chi \) to \( A_\mu \). In QED, various approaches may be taken. It is possible to consider \( \chi \) to be a \( c \)-number. In that case, the gauge transformation cannot readily shift the formulation of QED from one gauge to another, since different representations of the operator-valued fields and Hamiltonians will generally be required in different gauges. We can also consider \( \chi \) to be an operator-valued field. In that case, \( \partial_\mu \chi \) is not well defined, since the time derivative operator is \( \partial_0 \chi = i[H, \chi] \), and the gauge-dependent Hamiltonian presents an obstacle to the unambiguous definition of time evolution. It would be desirable to be able to demonstrate identical time displacement, in different gauges, of a state vector that represents an observable state—for example, a system of electrons and photons. Such a development might not make any substantial difference in the task of calculating \( S \)-matrix elements or energy levels of quasi-bound states. But it might contribute to our fundamental understanding of the theory.

In this paper, we will address this problem, and review the theoretical apparatus required for its consideration.

II. QED IN COVARIANT GAUGES

The most common and familiar example of a gauge theory is Quantum Electrodynamics (QED), in a manifestly covariant gauge. In a manifestly covariant gauge (or “covariant gauge,” for short), Lorentz transformations do not change the gauge; in other gauges, after a Lorentz boost, a subsequent gauge transformation is required to return to the original gauge. A useful Lagrangian for QED in a manifestly covariant gauge is [5]

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - j_\mu A^\mu - G (\partial_\mu A^\mu) + \frac{1}{2}(1 - \gamma)G^2 + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi
\]

(1)

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( j^\mu = e\bar{\psi}\gamma^\mu \psi \). Here \( G \) is the so-called gauge-fixing field, and \( \gamma \) is a parameter for “tuning” the covariant gauge to various possible alternatives—the Feynman [6], Landau [7], Fried-Yennie [8], or other variants. The Euler-Lagrange equations for the gauge fields derived from this Lagrangian are
\[ \partial_\nu F^{\mu\nu} + j^\mu - \partial^\mu G = 0; \] (2)

it is instructive to rewrite these equations as

\[ \partial_0 E - \nabla \times B + j = -\nabla G \] (3)

and

\[ \nabla \cdot E - j_0 = -\partial_0 G. \] (4)

Further Euler-Lagrange equations are

\[ \partial_\mu A^\mu = (1 - \gamma)G, \] (5)

\[ \partial_\mu \partial^\mu G = 0, \] (6)

and

\[ (i\gamma_\mu D^\mu - m)\psi = 0 \] (7)

where \( D_\mu = \partial_\mu + ieA_\mu \). The Lagrangian, \( \mathcal{L} \), also determines the momenta canonical to the gauge fields, which are given by

\[ \Pi_i = \frac{\delta \mathcal{L}}{\delta (\partial^0 A_i)} = -F_{0i} \] (8)

and

\[ \Pi_0 = \frac{\delta \mathcal{L}}{\delta (\partial^0 A^0)} = -G; \] (9)

where \( \Pi_i \) is a component of \( \mathbf{\Pi} = -\mathbf{E} \), and \( \mathbf{E} \) is the electric field.

The momentum conjugate to \( \psi \) is \( i\psi^\dagger \). We observe that the introduction of the gauge-fixing field \( G \) has provided us with a momentum conjugate to \( A^0 \); this avoids the necessity of imposing primary constraints on operator-valued fields, and allows us to preserve the canonical commutation (or anticommutation) relations among all the participating field operators. On the other hand, as shown in Eqs. (3) and (4), the gauge-fixing field also adds
spurious terms to Maxwell’s equations. In order to guarantee that this theory is really QED, we will ultimately have to prevent the derivatives of $G$ from affecting the validity of Maxwell’s equations. First, however, we proceed with the development of the canonical apparatus, and impose the equal-time commutation (and anticommutation) rules. Because there are no primary operator-valued constraints in this formulation, the following completely canonical rules apply

$$[A_i(x), \Pi_j(y)] = i\delta_{ij}\delta(x - y),$$  \hspace{1cm} (10)

$$[A_0(x), G(y)] = -i\delta(x - y)$$  \hspace{1cm} (11)

and

$$\{\psi^\dagger(x), \psi(y)\} = \delta(x - y).$$ \hspace{1cm} (12)

We evaluate the Hamiltonian for the covariant gauge, $H^{\text{cov}} = \int d\mathbf{x} \ H^{\text{cov}}(\mathbf{x})$, with

$$H^{\text{cov}} = \partial^0 A^i \frac{\delta L}{\delta (\partial^0 A^i)} + \partial^0 A^0 \frac{\delta L}{\delta (\partial^0 A^0)} + \partial^0 \psi \frac{\delta L}{\delta (\partial^0 \psi)} - L;$$  \hspace{1cm} (13)

and we represent $H^{\text{cov}}$ as $H^{\text{cov}} = H_0^{\text{cov}} + H_1^{\text{cov}}$, where

$$H_0^{\text{cov}} = \frac{1}{2} \Pi \cdot \Pi + \frac{1}{4} F_{ij} F^{ij} + G\nabla \cdot A + A^0 \nabla \cdot \Pi - \frac{1}{2} (1 - \gamma) G^2$$

$$+ \psi^\dagger(-i\boldsymbol{\alpha} \cdot \nabla + \beta m) \psi$$  \hspace{1cm} (14)

and

$$H_1^{\text{cov}} = j_0 A^0 - j \cdot A;$$  \hspace{1cm} (15)

we use $\boldsymbol{\alpha} = \gamma^0 \gamma$ and $\beta = \gamma^0$.

In order to obtain a Fock space of particle states, we represent the gauge field in terms of creation and annihilation operators for photons and electrons. There are, however, more

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$^1$Equation (14) assumes implicit integration by parts.
independent degrees of freedom in the gauge fields than we can accommodate with the
two helicity modes available for propagating photons; additional operator-valued excitation
modes are necessary to represent the quantized gauge field. These additional modes must
neither carry energy-momentum, nor have any probability of being observed, so that the
theory’s unitarity is preserved within the space determined by the electrons and the two
helicity modes of observable photons. In order to accommodate these requirements consist-
tently, we make use of “ghost modes,” i.e., excitation modes of the gauge fields which create
zero-norm states. We will designate the annihilation operators for these ghost excitations as
\( a_Q(k) \) and \( a_R(k) \), and the creation operators which are their adjoints in an indefinite metric
space as \( a_Q^*(k) \) and \( a_R^*(k) \), respectively. In order to produce ghost states, these operators
must commute with their respective adjoints. Any state \( a_Q^*(k)|p\rangle \), where \( |p\rangle \) is a normed
Fock state consisting of electrons and observable photons, has the norm
\( \langle p|a_Q(k)a_Q^*(k)|p\rangle \).

This norm vanishes when \( a_Q(k) \) and \( a_Q^*(k) \) commute and \( a_Q(k) \) annihilates the Fock space vacuum. However, these ghost operators cannot commute with all other operators, because
then they would become useless in representing the commutation relations of the gauge
fields. A small generalization of the algebra of positive metric Hilbert spaces suffices to
satisfy all these requirements. In this generalized algebra, \( a_Q(k) \) and its adjoint \( a_Q^*(k) \)
commute, as do \( a_R(k) \) and \( a_R^*(k) \). In addition, we impose the commutation rules
\[
[a_Q(k), a_R^*(q)] = [a_R(k), a_Q^*(q)] = \delta_{kq}.
\] (16)

The unit operator in the one-particle ghost sector then is
\[
1_{OPG} = \sum_k [a_Q^*(k)|0\rangle\langle 0|a_R(k) + a_R^*(k)|0\rangle\langle 0|a_Q(k)]
\] (17)

and, in the \( n \)-particle ghost sector, the obvious generalization of Eq. (17) applies. We
represent \( \mathbf{A} \) as \( \mathbf{A} = \mathbf{A}^T + \mathbf{A}^L \), i.e. as a sum of a transverse and a longitudinal field. The
transverse field, \( \mathbf{A}^T \), is represented as a superposition of propagating photons with the two
helicity modes \( \epsilon_i^a(k) \), where \( i \) designates the spatial component and \( a \) refers to one of the
two polarization modes. The components of $A^T$ are given as

$$A_i^T(x) = \sum_{k} \frac{\epsilon_i^n(k)}{\sqrt{2k}} \left[ a_n(k) e^{ik \cdot x} + a_n^\dagger(k) e^{-ik \cdot x} \right]$$  \hspace{1cm} (18)$$

where $a_n^\dagger(k)$ and $a_n(k)$ designate the creation and annihilation operators, respectively, for transversely polarized photons with polarization mode $n$ and momentum $k$. The longitudinal gauge field, $A^L_i$, is represented in terms of ghost excitations as

$$A_i^L(x) = \sum_{k} \frac{k_i}{2k^{3/2}} \left[ a_R(k) e^{ik \cdot x} + a_R^*(k) e^{-ik \cdot x} \right]$$
$$+ \left( 1 - \frac{\gamma}{2} \right) \sum_{k} \frac{k_i}{2k^{3/2}} \left[ a_Q(k) e^{ik \cdot x} + a_Q^*(k) e^{-ik \cdot x} \right],$$  \hspace{1cm} (19)$$

and similarly, the time-like component is represented as

$$A_0(x) = \sum_{k} \frac{1}{2\sqrt{k}} \left[ a_R(k) e^{ik \cdot x} + a_R^*(k) e^{-ik \cdot x} \right]$$
$$- \left( 1 - \frac{\gamma}{2} \right) \sum_{k} \frac{1}{2\sqrt{k}} \left[ a_Q(k) e^{ik \cdot x} + a_Q^*(k) e^{-ik \cdot x} \right].$$  \hspace{1cm} (20)$$

$\Pi$, the momentum conjugate to $A$ as well as the negative of the electric field, is represented as

$$\Pi_i(x) = -i \sum_{k} \epsilon_i^n(k) \sqrt{\frac{k}{2}} \left[ a_n(k) e^{ik \cdot x} - a_n^\dagger(k) e^{-ik \cdot x} \right]$$
$$- i \sum_{k} \frac{k_i}{\sqrt{k}} \left[ a_Q(k) e^{ik \cdot x} - a_Q^*(k) e^{-ik \cdot x} \right];$$  \hspace{1cm} (21)$$

and the gauge-fixing field, $G(x)$, is represented as

$$G(x) = i \sum_{k} \sqrt{k} \left[ a_Q(k) e^{ik \cdot x} - a_Q^*(k) e^{-ik \cdot x} \right].$$  \hspace{1cm} (22)$$

The standard representation of $\psi$ and $\psi^\dagger$ in terms of electron and positron creation and annihilation operators will be implicitly assumed. The choice of these representations of the $A_i$ designates the spatial components of $A^\mu$ which, in relativistic notation, would be represented as contravariant quantities.
gauge fields is determined by a number of requirements. They must implement the equal-time commutation rules given in Eqs. (10) and (11); and they must lead to an implementable Hilbert space. The latter requirement is connected to the gauge-invariance of the theory, and will be discussed later.

When the representations given in Eqs. (18)-(22) are substituted into the Hamiltonians $H_0^{\text{cov}}$ and $H_1^{\text{cov}}$ where $H = \int dx \, \mathcal{H}(x)$, we obtain

$$H_0^{\text{cov}} = \sum_k [a_n^\dagger(k)a_n(k) + a_R^\dagger(k)a_Q(k) + a_Q^\dagger(k)a_R(k) + \gamma a_Q^\dagger(k)a_Q(k)]$$

$$+ \sum_q \omega_q [e_s^\dagger(q)e_s(q) + \bar{e}_s^\dagger(q)\bar{e}_s(q)]$$

(23)

where $\omega_q = \sqrt{q^2 + m^2}$ and $m$ is the electron mass; $e_s(q)$ and $e_s^\dagger(q)$ represent electron annihilation and creation operators, respectively, and the barred symbols designate the corresponding positron operators.

We can construct a Fock space, $\{|h\rangle\}$, based on the perturbative vacuum, $|0\rangle$, which is annihilated by all the annihilation operators, $a_n(k)$, $a_Q(k)$ and $a_R(k)$, as well as, $e_s(k)$ and $\bar{e}_s(k)$. This perturbative Fock space includes all multiparticle states, $|N\rangle$, consisting of observable, propagating particles, i.e., electrons, positrons, and photons, created when $e_s^\dagger(k)$, $\bar{e}_s^\dagger(k)$, and $a_n^\dagger(k)$, respectively, act on $|0\rangle$. All such states, $|N\rangle$, are eigenstates of $H_0^{\text{cov}}$. States in which a single variety of ghost creation operator acts on one of these multiparticle states $|N\rangle$, such as $a_Q^\dagger(k)|N\rangle$ or $a_Q^\dagger(k_1)a_Q^\dagger(k_2)|N\rangle$, have zero norm; they have no probability of being observed, and have vanishing expectation values of energy, momentum, as well as, of all other observables. We will designate the subspace of $\{|h\rangle\}$ that consists of all states $|N\rangle$, and of all states in which a chain of $a_Q^\dagger(k)$ operators (but no $a_R^\dagger(k)$ operators) act on $|N\rangle$, as $\{|n\rangle\}$. States in which both varieties of ghost appear simultaneously, such as $a_Q^\dagger(k_1)a_R^\dagger(k_2)|N\rangle$, also are in the Fock space $\{|h\rangle\}$; but because these states have non-vanishing norms and contain ghosts, they are not interpretable. Their appearance in the course of time evolution signals a catastrophic defect in the theory.

The time evolution operator $\exp(-iH_0^{\text{cov}}t)$—which excludes the effect of the interaction Hamiltonian—has the important property that, if it acts on a state vector $|n_i\rangle$ in $\{|n\rangle\}$,
it can only propagate it within \{\{n\}\}; but it can never translate it into the part of \{\{h\}\} external to \{\{n\}\}. We observe that the only parts of \(H_0^{\text{cov}}\) that could possibly cause a state vector to leave the subspace \{\{n\}\} are those that contain either \(a_R^\ast(k)\) or \(a_R(k)\) operators. The only part of \(H_0^{\text{cov}}\) that has that feature contains the combination of operators \(\Gamma(k) = a_R^\ast(k)a_Q(k) + a_Q^\ast(k)a_R(k)\). When \(a_R(k)\) acts on a state vector \(|n_i\rangle\), it either annihilates the vacuum or it annihilates one of the \(a_Q^\ast(k)\) operators in \(|n_i\rangle\). In the latter case, \(\Gamma\) replaces the annihilated \(a_Q^\ast(k)\) operator with an identical one. When \(a_Q(k)\) acts on a state vector \(|n_i\rangle\), it always annihilates it. It is therefore impossible for \(\Gamma\) to produce a state vector external to \{\{n\}\}, i.e. one in which an \(a_R^\ast(k)\) operator acts on \(|n_i\rangle\). The only effect of \(\Gamma\) is to translate \(|n_i\rangle\) states within \{\{n\}\}. Substitution of Eqs. \((18)-(22)\) into \(H_I^{\text{cov}}\) leads to

\[
H_I^{\text{cov}} = -\sum_k \frac{1}{\sqrt{2k}} \left[ a_n(k, j(-k) \cdot \hat{\epsilon}^n(k) + a_n^\dagger(k) j(k) \cdot \hat{\epsilon}^n(k) \right] \\
- (1 - \frac{\gamma}{2}) \sum_k \frac{1}{2\sqrt{k}} \left[ a_Q(k) \left( j_0(-k) + \frac{k \cdot j(-k)}{|k|} \right) + a_Q^\ast(k) \left( j_0(k) + \frac{k \cdot j(k)}{|k|} \right) \right] \\
+ \sum_k \frac{1}{2\sqrt{k}} \left[ a_R(k) \left( j_0(-k) - \frac{k \cdot j(-k)}{|k|} \right) + a_R^\ast(k) \left( j_0(k) - \frac{k \cdot j(k)}{|k|} \right) \right].
\]

In this expression of \(H_I^{\text{cov}}\), all gauge field excitations appear, including creation and annihilation operators for both varieties of ghosts. This notifies us that \(H_I^{\text{cov}}\) causes transitions from the “safe” subspace \{\{n\}\} into the part of the larger space occupied by uninterpretable state vectors that nevertheless absorb probability amplitude, energy and momentum. In Sec. \[IV\], we will show how implementation of the constraints prevents the catastrophic appearance of these state vectors in the course of time evolution.

### III. THE PERTURBATIVE REGIME

The perturbative theory involves the vertices dictated by the interaction Hamiltonian given in Eq. \((15)\) and the propagators for the interaction-picture operators \(\psi(x), \bar{\psi}(x), \text{ and } A^\mu(x)\). The interaction-picture operators are given by \(P(x) = \exp (iH_0^{\text{cov}}t) P(x) \exp (-iH_0^{\text{cov}}t)\) and, in the case of the gauge fields, they are given by

\[
\]
\[ A_i(x) = \sum_k \frac{\epsilon_k^n(k)}{\sqrt{2k}} \left[ a_n(k)e^{-ik_{\mu}x^\mu} + a_n(k)e^{ik_{\mu}x^\mu} \right] \\
+ \sum_k \frac{k_i}{2k^{3/2}} \left[ a_R(k)e^{-ik_{\mu}x^\mu} + a_R^*(k)e^{ik_{\mu}x^\mu} \right] \\
+ (1 - \frac{\gamma}{2}) \sum_k \frac{k_i}{2k^{3/2}} \left[ a_Q(k)e^{-ik_{\mu}x^\mu} + a_Q^*(k)e^{ik_{\mu}x^\mu} \right] \\
- \sum_k \frac{i\gamma x_0 k_i}{2\sqrt{k}} \left[ a_Q(k)e^{-ik_{\mu}x^\mu} - a_Q^*(k)e^{ik_{\mu}x^\mu} \right] \tag{25} \]

and

\[ A_0(x) = \sum_k \frac{1}{2\sqrt{k}} \left[ a_R(k)e^{-ik_{\mu}x^\mu} + a_R^*(k)e^{ik_{\mu}x^\mu} \right] \\
- (1 - \frac{\gamma}{2}) \sum_k \frac{1}{2\sqrt{k}} \left[ a_Q(k)e^{-ik_{\mu}x^\mu} + a_Q^*(k)e^{ik_{\mu}x^\mu} \right] \\
- \sum_k \frac{i\gamma x_0 \sqrt{k}}{2} \left[ a_Q(k)e^{-ik_{\mu}x^\mu} - a_Q^*(k)e^{ik_{\mu}x^\mu} \right] \tag{26} \]

In Eqs. (25) and (26), \( k^\mu \) is “on-shell,” i.e., \( k_0 = |k| \). We use \( A_i(x) \) and \( A_0(x) \) in the expression for the propagator,

\[ D^{\mu\nu}(x, y) = \langle 0| T(A^\mu(x)A^\nu(y))|0 \rangle \tag{27} \]

where \( T \) designates time-ordering, and where \( |0\rangle \) is the perturbative vacuum annihilated by all annihilation operators, \( a_n(k), a_Q(k) \) and \( a_R(k) \), as well as, \( e_s(p) \) and \( \bar{e}_s(p) \) for electrons and positrons, respectively. We obtain the expression [3]

\[ D^{ij}(x, y) = \int \frac{dk}{(2\pi)^3} \left\{ \frac{e^{-ik_{\mu}(x-y)^\mu}}{2k} \left[ \delta_{ij} - \frac{\gamma k_i k_j}{|k|^2} \left( 1 + ik(x_0 - y_0) \right) \right] \Theta(x_0 - y_0) \\
+ \frac{e^{ik_{\mu}(x-y)^\mu}}{2k} \left[ \delta_{ij} - \frac{\gamma k_i k_j}{|k|^2} \left( 1 - ik(x_0 - y_0) \right) \right] \Theta(y_0 - x_0) \right\} \right. \tag{28} \]

\[ D^{00}(x, y) = -i \gamma \int \frac{dk}{(2\pi)^3} \left\{ \frac{e^{-ik_{\mu}(x-y)^\mu}}{2k} \left[ 1 - \frac{\gamma}{2} \left( 1 - ik(x_0 - y_0) \right) \right] \Theta(x_0 - y_0) \\
- \frac{e^{ik_{\mu}(x-y)^\mu}}{2k} \left[ 1 - \frac{\gamma}{2} \left( 1 + ik(x_0 - y_0) \right) \right] \Theta(y_0 - x_0) \right\} \tag{29} \]

and

\[ D^{0i}(x, y) = -i \gamma \int \frac{dk}{(2\pi)^3} \frac{k_i(x_0 - y_0)}{4|k|} \left[ e^{-ik_{\mu}(x-y)^\mu} \Theta(x_0 - y_0) - e^{ik_{\mu}(x-y)^\mu} \Theta(y_0 - x_0) \right] \tag{30} \]
which can be represented as

\[
D^{\mu\nu}(x, y) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik_{\lambda}(x-y)\lambda}}{k_{\lambda}k^\lambda + i\epsilon} \left[ g^{\mu\nu} - \frac{\gamma k^\mu k^\nu}{k_{\lambda}k^\lambda + i\epsilon} \right]; \tag{31}
\]

\(\gamma = 0\) corresponds to the Feynman gauge, \(\gamma = 1\) corresponds to the Landau gauge, and \(\gamma = -2\) corresponds to the Fried-Yennie gauge.

The propagator \(D^{\mu\nu}(x, y)\), a corresponding one for the spinor field, and the vertices \(\gamma^\mu\), constitute the Feynman rules for \(S\)-matrix elements in the covariant gauges. The representation of the \(S\)-matrix in terms of Feynman rules is based on an expression for the \(S\)-matrix given by

\[
S_{f,i} = \delta_{fi} - 2\pi i\delta(E_i - E_f)T_{f,i}, \tag{32}
\]

where the subscripts refer to the initial and final states \(|i\rangle\) and \(|f\rangle\) respectively; these states consist of creation operators for electrons, positrons, and transversely polarized photons acting on the perturbative vacuum state \(|0\rangle\). \(T_{f,i}\) can be represented as

\[
T_{f,i} = \sum_{n=1}^{\infty} T_{f,i}^n \tag{33}
\]

with the \(n\)th order transition amplitude, \(T_{f,i}^n\), given by

\[
T_{f,i}^n = \langle f | H^{\text{cov}}_1 \frac{1}{(E_i - H^{\text{cov}}_0)H^{\text{cov}}_1} \frac{1}{(E_i - H^{\text{cov}}_0)H^{\text{cov}}_1} \cdots \frac{1}{(E_i - H^{\text{cov}}_0)H^{\text{cov}}_1} | i \rangle. \tag{34}
\]

Standard procedures transform Eq. (34) into Feynman rules. [10]

The essential point of this brief summary is that the states \(|i\rangle\) and \(|f\rangle\) do not implement Gauss’s law except in the physically uninteresting limit in which all interactions between charged particles and the electromagnetic field have been eliminated. It is easily seen, for example, that in the case of the expectation value of \(\nabla \cdot \mathbf{E}\) in the perturbative one-electron state \(|e_s(p)\rangle\), \(\langle e_s(p)\Pi_{i}(x)|e_s(p)\rangle = 0\); \(\langle e_s(p)|\partial_i\Pi(x) + j_0(x)|e_s(p)\rangle = 0\) would be required by Gauss’s law. In the next section, we will discuss measures that produce a Hilbert space whose states implement Gauss’s law for the complete theory—i.e., one that includes interactions between the electromagnetic field and charged particles.

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IV. IMPLEMENTATION OF CONSTRAINTS

We will define the operator $\mathcal{G}$, that expresses Gauss’s law in the covariant gauge, as

$$\mathcal{G}(x) = \partial_0 \Pi_l(x) + j_0(x)$$  \hspace{1cm} (35)

so that Eq. (4) can be written as $\mathcal{G} = \partial_0 \mathcal{G}$. Substitution of Eq. (21) into Eq. (35) leads to

$$\mathcal{G}(x) = \sum_{k} k^{3/2} [\Omega(k) e^{ik \cdot x} + \Omega^*(k) e^{-ik \cdot x}],$$  \hspace{1cm} (36)

where $\Omega(k)$ is defined as $\Omega(k) = a_Q(k) + j_0(k)/2k^{3/2}$. Similarly, we can express $\partial_\mu A^\mu$ as

$$\partial_\mu A^\mu(x) = i(1 - \gamma) \sum_k \sqrt{k} \left[ \Omega(k) e^{ik \cdot x} - \Omega^*(k) e^{-ik \cdot x} \right].$$  \hspace{1cm} (37)

The pivotal fact about $\Omega(k)$ is that, since $[H, \Omega(k)] = -|k|\Omega(k)$, the corresponding Heisenberg operator, $\Omega(k,t)$, where

$$\Omega(k,t) = \exp \left( iH_{\text{cov}} t \right) \Omega(k) \exp \left( -iH_{\text{cov}} t \right),$$  \hspace{1cm} (38)

has a c-number time-dependence given by $\Omega(k,t) = \Omega(k) e^{-i|k|t}$.

In general, the time-dependence of Heisenberg operators cannot be represented by simple c-number expressions; it can only be expressed by iterative expansions involving their interaction-picture equivalents. $\Omega(k,t)$, however, is an exception in this regard. And it is this property of $\Omega(k)$ that makes it so useful in implementing constraints.

We will now use $\Omega(k)$ to define a subspace $\{|\nu\rangle\}$, of another Fock space, in which all state vectors $|\nu_i\rangle$ obey the condition $\Omega(k)|\nu\rangle = 0$.\hspace{1cm} (39)

We will call $\{|\nu\rangle\}$ the “physical subspace,” because the constraints are implemented in it. For all state vectors $|\nu\rangle$ and $|\nu'\rangle$ in this physical subspace $\{|\nu\rangle\}$, $\langle \nu' | G | \nu \rangle = 0$; similarly $\langle \nu' | \nabla G | \nu \rangle = 0$ and $\langle \nu' | \partial_0 G | \nu \rangle = 0$. These equations demonstrate that, in the subspace $\{|\nu\rangle\}$, both Gauss’s law and the gauge condition hold and Maxwell’s equations are exactly satisfied. Moreover, a state vector initially in the physical subspace will always remain
entirely contained in it, as it develops under time evolution. This follows from the \( c \)- number time-dependence of the Heisenberg operator \( \Omega(\mathbf{k}, t) \). To complete the Fock space in which this physical subspace is embedded, we note that there are unitary transformations, \([12]\) \( U = e^{D_1} \) and \( V = e^D \), for which

\[
U^{-1}\Omega(\mathbf{k})U = V^{-1}\Omega(\mathbf{k})V = a_Q(\mathbf{k}),
\]

where \( D = D_1 + D_2 \) and where

\[
D_1 = \sum_k \frac{1}{2k^{3/2}} [a_R(\mathbf{k})j_0(-\mathbf{k}) - a_R^*(\mathbf{k})j_0(\mathbf{k})]
\]

and

\[
D_2 = \sum_k \frac{\phi(\mathbf{k})}{2k^{3/2}} [a_Q(\mathbf{k})j_0(-\mathbf{k}) - a_Q^*(\mathbf{k})j_0(\mathbf{k})];
\]

the form of \( \phi(\mathbf{k}) \) is arbitrary except that \( \phi(\mathbf{k}) = \phi(-\mathbf{k}) \). Since \( \phi(\mathbf{k}) \) is an arbitrary function, and since with \( \phi(\mathbf{k}) = 0 \), \( V \) reduces to \( U \), we will not differentiate between \( U \) and \( V \) hereafter, but regard \( U \) as a special case of \( V \).

We can use the unitary operator \( V \) to construct the physical subspace \( \{|\nu\rangle\} \). We extract the previously defined subspace \( \{|n\rangle\} \) from the Hilbert space \( \{|h\rangle\} \), and simply set \( |\nu_i\rangle = V|n_i\rangle \) for any operator \( V \). Since \( a_Q(\mathbf{k})|n_i\rangle = 0 \), \( \Omega(\mathbf{k})|\nu_i\rangle = 0 \) follows immediately from the definition of \( V \). As an alternate to explicitly constructing the subspace \( \{|\nu\rangle\} \), it is possible, and often most convenient, to transpose the entire formalism into a unitarily equivalent representation. In this unitarily equivalent representation, we keep \( \{|n\rangle\} \) as the physical subspace which, in the original, untransformed representation, is given by \( \{|\nu\rangle\} \).

We must then also transform all operators, so that for any operator \( P \), \( P \rightarrow V^{-1}PV \). We will designate the transformed operators \( V^{-1}PV \) as \( \tilde{P} \). In the transformed representation, \( a_Q(\mathbf{k})|n_i\rangle = 0 \) is the equation that implements Gauss’s law and the gauge condition for the “complete” theory, i.e., for the theory of the electromagnetic field interacting with the charged electron-positron field.

In this unitarily equivalent representation, the theory gives rise to the same equations of motion, and implements the same constraints as in the original representation. The
choice between the transformed and the untransformed representations is entirely a matter of convenience. The transformed Hamiltonian, \( \tilde{H}^{\text{cov}} \), is given by

\[
\tilde{H}^{\text{cov}} = H_C + H_Q
\]

(43)

where \( H_C \) and \( H_Q \) are given by

\[
H_C = \sum_k ka_n^\dagger(k)a_n(k) + \sum_q \omega_q \left[ e_s^\dagger(q)e_s(q) + \bar{e}_s^\dagger(q)\bar{e}_s(q) \right]
+ \sum_k j_0(k)j_0(-k) \frac{2k^2}{2k^2} - \sum_k \frac{\epsilon_l^m(k)}{\sqrt{2k}} \left[ a_n(k)j_l(-k) + a_n^\dagger(k)j_l(k) \right]
\]

(44)

and

\[
H_Q = \sum_k k [a_R^*(k)a_Q(k) + a_Q^*(k)a_R(k) + \gamma a_Q^*(k)a_Q(k)]
- (1 + \frac{\gamma}{2}) \sum_k \frac{1}{2\sqrt{k}} [a_Q(k)j_0(-k) + a_Q^*(k)j_0(k)]
- \sum_k \frac{\phi(k)}{2\sqrt{k}} [a_Q(k)j_0(-k) + a_Q^*(k)j_0(k)]
- (1 - \frac{\gamma}{2}) \sum_k \frac{k_i}{2k^3/2} [a_Q(k)j_i(-k) + a_Q^*(k)j_i(k)]
+ \sum_k \frac{\phi(k)k_i}{2k^3/2} [a_Q(k)j_i(-k) + a_Q^*(k)j_i(k)],
\]

(45)

respectively. We can similarly transform the operator-valued fields in this theory, leading to

\[
\tilde{A}_0(x) = A_0(x) + (1 - \frac{\gamma}{2}) \sum_k \frac{j_0(k)e^{ik \cdot x}}{2k^2} - \sum_k \frac{\phi(k)j_0(k)e^{ik \cdot x}}{2k^2},
\]

(46)

\[
\tilde{G}(x) = G(x),
\]

(47)

\[
\tilde{A}_l(x) = A_l(x),
\]

(48)

\[
\tilde{\Pi}_l(x) = \Pi_l(x) + i \sum_k \frac{k_je_0(k)e^{ik \cdot x}}{k^2},
\]

(49)

and

\[
\tilde{\psi}(x) = \psi(x)e^{D(x)},
\]

(50)
with

\[
D(x) = -ie \int dy \frac{1}{4\pi|x-y|} \left[ \nabla \cdot A(y) - \frac{1}{2}(1 - \frac{\gamma}{2})G(y) \right] \\
\quad + \frac{ie}{2} \int dy \ G(y) \frac{1}{\nabla^2} \phi(x-y)
\]  

(51)

where

\[
\phi(y) = \frac{1}{(2\pi)^3} \int d\kappa \ \phi(\kappa)e^{i\kappa \cdot y}. 
\]  

(52)

The currents therefore remain untransformed, with \( \tilde{j}_0(x) = j_0(x) \) and \( \tilde{j}(x) = j(x) \)

We make the following observations about \( \tilde{H}^{\text{cov}} \). \( H_C \) is precisely the Hamiltonian for QED in the Coulomb gauge. Except for a single piece proportional to \( \Gamma(k) = a_R^*(k)a_Q(k) + a_Q^*(k)a_R(k) \), \( H_Q \) consists entirely of Hermitian combinations of \( a_Q(k) \) and \( a_Q^*(k) \) operators, in various combinations with each other and with the current densities \( j_0(\pm k) \) and \( j(\pm k) \). Every component of \( H_Q \) is proportional to \( a_Q(k) \) or \( a_Q^*(k) \); and, most importantly, except for \( \Gamma(k) \), there are no \( a_R(k) \) or \( a_R^*(k) \) operators in \( H_Q \). Previously we showed that, within the subspace \( \{|n\rangle\} \), \( \Gamma(k) \) can propagate \( a_Q^*(k) \) operators, but cannot cause any state vectors in \( \{|n\rangle\} \) to make transitions out of \( \{|n\rangle\} \). \( H_Q \) therefore cannot affect the observable consequences of the time evolution imposed by the evolution operator \( \exp\left(-i\tilde{H}^{\text{cov}}t\right) \).

Figure 1 is useful in illustrating the role that \( H_Q \) can have in the time evolution of state vectors. The Hilbert space \( \{|h\rangle\} \) contains the subspace \( \{|n\rangle\} \), and \( \{|n\rangle\} \) contains the set of states \( |N_i\rangle \), which constitute a quotient space consisting of electrons, positrons and transversely polarized photons. The subspace \( \{|n\rangle\} \) includes all the \( |N_i\rangle \) states, as well as all other states in which chains of \( a_Q^*(k) \) operators act on \( |N_i\rangle \) states. \( \{|h\rangle\} \) contains all of \( \{|n\rangle\} \), as well as all other states in which chains of \( a_R^*(k) \) and \( a_Q^*(k) \) operators act on \( \{|n\rangle\} \) states. When the time evolution operator in the transformed representation, \( \exp\left(-i\tilde{H}^{\text{cov}}t\right) \), acts on a state \( |N\rangle \), it time-translates it so that it moves on the sheet representing the quotient space and also spreads along the fiber that represents a set of states in \( \{|n\rangle\} \). The time-translated state extends along the fiber from a point on the sheet that represents the quotient space;
but it remains entirely within \( \{|n\rangle\} \). The states that are along the fiber, but no longer part of the quotient space, all are zero-norm states. They never affect the trajectory of states \( \exp\left(-i\tilde{H}^{\text{cov}}t\right)|N\rangle \) within the quotient space because they have no further interactions with the states that form the projection of \( \exp\left(-i\tilde{H}^{\text{cov}}t\right)|N\rangle \) onto the quotient space. Because all the states in \( \{|n\rangle\} \) that are not part of the quotient space have zero norm, the theory is manifestly unitary within the quotient space alone. And since the interaction Hamiltonian in \( \tilde{H}^{\text{cov}} \) is entirely devoid of \( a_{k^*}(k) \) operators, \( H_Q \) cannot be involved in the generation of internal loops, and therefore also cannot change any of the radiative corrections. In fact, if \( H_Q \) were entirely eliminated from \( \tilde{H}^{\text{cov}} \), there would be no change in the trajectory of the point that represents the projection of the state \( \exp\left(-i\tilde{H}^{\text{cov}}t\right)|N\rangle \) onto the quotient space. All amplitudes \( \langle f | \exp\left(-i\tilde{H}^{\text{cov}}t\right)|N\rangle \), where \( |f\rangle \) is one of the \( |N_i\rangle \) states, are entirely identical to the corresponding \( \langle f | \exp\left(-iH_Ct\right)|N\rangle \). Equation (49) demonstrates that in the transformed representation, \( \langle n_b| (\nabla \cdot \tilde{E} - \tilde{j}_0)|n_a\rangle = 0 \). Gauss’s law is implemented for the complete theory, which includes interactions between the electromagnetic field and charged particles, when constraints are imposed.

The condition \( a_Q(k)|n_i\rangle = 0 \) and the subspace \( \{|n\rangle\} \) sometimes are also used in the formulation of the theory in the untransformed representation. The perturbative formulation summarized in Sec. III is an illustration of that practice. But use of the subspace \( \{|n\rangle\} \) with the untransformed Hamiltonian \( H^{\text{cov}} \), does not guarantee the validity of Maxwell’s equations at all times. In that case, the constraint \( a_Q(k)|n_i\rangle = 0 \) implements Gauss’s law only in the interaction-free limit, in which all interactions between the electromagnetic field and the charged particles have been eliminated. It is therefore natural to ask why the perturbative theory outline in Sec. III, and the Feynman rules to which it gives rise, can be used to evaluate \( S \)-matrix elements. To answer that question we compare the transition amplitude that leads to the Feynman rules for covariant gauges,

\[
T_{f,i} = \langle f | H_1^{\text{cov}} + H_1^{\text{cov}}(E_i - H^{\text{cov}} + i\epsilon)^{-1}H_1^{\text{cov}}|i\rangle, \tag{53}
\]

with the transition amplitude, \( \tilde{T}_{f,i} \), that results when Gauss’s law is implemented. We
can express $\tilde{T}_{f,i}$ in one of two forms. In one form, we let $\tilde{H}^{\text{cov}} = H_0^{\text{cov}} + \hat{H}_1^{\text{cov}}$ so that $(\tilde{H}^{\text{cov}} - E_i)|i\rangle = \hat{H}_1^{\text{cov}}|i\rangle$, but $\hat{H}_1^{\text{cov}} \neq V^{-1}H_1^{\text{cov}}V$. We then find that

$$\tilde{T}_{f,i} = \langle f|\hat{H}_1^{\text{cov}} + \hat{H}_1^{\text{cov}}(E_i - \tilde{H}^{\text{cov}} + i\epsilon)^{-1}\hat{H}_1^{\text{cov}}|i\rangle.$$  \hspace{1cm} (54)

Alternatively, we can express $H^{\text{cov}}$ in the form $H^{\text{cov}} = H_0 + H_1$, such that the states in the subspace $\{|\nu\rangle\}$ are eigenstates of $H_0$, i.e., $(H_0 - E_i)|\nu_i\rangle = 0$. Then

$$\tilde{T}_{f,i} = \langle f|H_1^{\text{cov}} H_1^{\text{cov}}(|E_i - H^{\text{cov}} + i\epsilon)^{-1}H_1^{\text{cov}}|\nu_i\rangle.$$  \hspace{1cm} (55)

We will show that, because the sets of states $|\nu\rangle$ and $|n\rangle$ are unitarily equivalent, $T_{f,i}$ may safely be substituted for the correct $\tilde{T}_{f,i}$ in the $S$-matrix, although $T_{f,i}$ is not in general identical to $\tilde{T}_{f,i}$. The argument proceeds as follows: \cite{12, 14} We rewrite $T_{f,i}$ in the form

$$T_{f,i} = \langle f|H_1^{\text{cov}}|\Psi_i\rangle,$$  \hspace{1cm} (56)

where $|\Psi_i\rangle$ is the scattering state with outgoing boundary conditions,

$$|\Psi_i\rangle = \left(1 + (E_i - H^{\text{cov}} + i\epsilon)^{-1}H_1^{\text{cov}}\right)|i\rangle.$$  \hspace{1cm} (57)

Similarly, we express $\tilde{T}_{f,i}$ in the form

$$\tilde{T}_{f,i} = \langle f|\hat{H}_1^{\text{cov}}|\hat{\Psi}_i\rangle$$  \hspace{1cm} (58)

where $|\hat{\Psi}_i\rangle$ is the scattering state

$$|\hat{\Psi}_i\rangle = \left(1 + (E_i - \tilde{H}^{\text{cov}} + i\epsilon)^{-1}\hat{H}_1^{\text{cov}}\right)|i\rangle$$  \hspace{1cm} (59)

that corresponds to the formulation in which Gauss’s law has been implemented.

It is easy to see that

$$\hat{H}_1^{\text{cov}} = H_1^{\text{cov}}V + (1 - V)\hat{H}_1^{\text{cov}} - H_0^{\text{cov}}(1 - V)$$  \hspace{1cm} (60)

and

$$H_1^{\text{cov}} = V\hat{H}_1^{\text{cov}} + H_1^{\text{cov}}(1 - V) - (1 - V)H_0^{\text{cov}},$$  \hspace{1cm} (61)

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and that, therefore,

$$|\hat{\Psi}_i\rangle = V^{-1}|\Psi_i\rangle - i\epsilon(E_i - \hat{H}_i^{\text{cov}} + i\epsilon)^{-1}(V^{-1} - 1)|i\rangle.$$  

(62)

Substitution of these relations into Eqs. (56) and (58) leads to

$$T_{f,i} = \tilde{T}_{f,i} + (E_f - E_i)T_{f,i}^{(\alpha)} + i\epsilon T_{f,i}^{(\beta)}$$

(63)

where

$$T_{f,i}^{(\alpha)} = \langle f|(1-V)|\hat{\Psi}_i\rangle$$

(64)

and

$$T_{f,i}^{(\beta)} = \langle f| \left[ \frac{1}{E_i - H_i^{\text{cov}} + i\epsilon} (1-V) - \frac{1}{E_i - H_i^{\text{cov}} + i\epsilon} \hat{H}_i^{\text{cov}} \right] |i\rangle.$$  

(65)

Equation (63) establishes that, although the transition amplitudes $T_{f,i}$ and $\tilde{T}_{f,i}$ can differ, they are the same “on-shell”, when $E_i = E_f$, provided that $T_{f,i}^{(\beta)}$ remains bounded as $\epsilon \to 0$. But in the case of QED, we find that because of singularities in $T_{f,i}^{(\beta)}$, $i\epsilon T_{f,i}^{(\beta)}$ generates a series of additional $S$-matrix elements in which $D_1$ vertices connect with $\hat{H}_i^{\text{cov}}$ or $H_i^{\text{cov}}$ vertices. Since neither $H_i^{\text{cov}}$ nor $\hat{H}_i^{\text{cov}}$ commute with $H_0^{\text{cov}}$, the propagator $(E_i - H_0^{\text{cov}} + i\epsilon)^{-1}$ cannot bypass $H_i^{\text{cov}}$ and/or $\hat{H}_i^{\text{cov}}$ to act on $|i\rangle$ and/or $|f\rangle$ directly and reduce Eq. (63) to a trivial identity. If we set $\phi(k) = 0$, we find that $D = D_1$, and since two $D_1$ vertices cannot connect, the states $e^{D_1}|n\rangle$ are normalized states. We further observe that the factor $(i\epsilon)^{-1}$ can arise only in self-energy insertions to external electron lines. There they appear as new, spurious self-energy insertions, in which propagators connect a $D_1$ with either an $\hat{H}_i^{\text{cov}}$ or an $H_i^{\text{cov}}$ vertex in a self-energy part that is entirely disconnected from the rest of the $S$-matrix element. Multiple $(i\epsilon)^{-n}$ contributions, with $n > 1$, cannot arise when the theory has been mass-renormalized, so that the energy continua of $\hat{H}_i^{\text{cov}}$ and $H_i^{\text{cov}}$ coincide. The effect of substituting $T_{f,i}$ for $\tilde{T}_{f,i}$ therefore only produces extra, spurious, self-energy insertions to external lines, and these are absorbed in the renormalization constants. The physical predictions are not affected by this substitution. These extra contributions from
$i\epsilon T_{F_\delta}^{(\beta)}$ to the renormalization constants have long been known and have been designated “gauge-dependent parts” of the renormalization constants.\[15\] In fact, when the constraints are implemented, the renormalization constants are identical in the different forms of the covariant gauge, as well as, in the Coulomb gauge. The $i\epsilon T_{F_\delta}^{(\beta)}$ contributions arise because the constraints are not implemented in the perturbative theory in covariant gauges, \[12\] and they are responsible for the fact that the renormalized, rather than the unrenormalized $S$-matrix elements are gauge-independent. \[16\]

V. QED IN THE COULOMB AND THE COVARIANT GAUGES

The preceding discussion provides us with a basis for understanding the relation between QED in the covariant gauge and in the Coulomb gauge, as well as among the different forms of the covariant gauge. Gauge transformations can be implemented by the unitary operator $\mathcal{V} = e^\tau$, where $\tau = i \int dx \mathcal{G}(x)\chi(x)$, in which $\chi$ is a $c$-number function. Under this gauge transformation the gauge field and the charged particle field transform as

$$A_\mu \rightarrow A_\mu - \partial_\mu \chi$$

(66)

and

$$\psi \rightarrow \psi e^{i\epsilon \chi}.$$  \hspace{1cm} (67)

The unitary transform $\mathcal{V}$ can be used to study the behavior of various operator-valued quantities under a $c$-number gauge transformation within a particular gauge formulation. But it is more complicated to understand the relation between two different gauge formulations of QED. For example, it is not immediately obvious that $H^{\text{cov}}$, the Hamiltonian for the covariant gauge, and the Coulomb gauge Hamiltonian, $H_C$, describe identical interactions among charged particles and photons. It is known that $S$-matrix elements with external photons polarized in the $k_\mu$ direction vanish, unless the regularization procedure used to control divergences introduces non-vanishing contributions. These $S$-matrix elements describe the
creation of “forbidden” $R$-type photon ghosts, and the condition that they vanish is often ascribed to “gauge-invariance of the $S$-matrix.” It is also known that in tree approxima-
tion Feynman rules give identical results for $S$-matrix elements in different gauges. When radiative corrections are included, the renormalization program in non-covariant gauges is still highly problematical, and the finite parts of divergent $S$-matrix elements are still not well defined. The gauge-invariance of $S$-matrix elements and Feynman rules have been widely studied. But the comparison of canonical formulations of QED in different gauges has not been the subject of systematic investigation.

Equations (43)–(45) demonstrate that QED in the Coulomb and covariant gauges are identical theories, in the sense that when $\tilde{H}^{\text{cov}}$, the unitary transform of $H^{\text{cov}}$, is projected onto the quotient space composed of state vectors $|N_i\rangle$, that projection is identical to the Coulomb gauge Hamiltonian $H_C$. There are two differences between $H^{\text{cov}}$ and $H_C$. The chief difference is that the particle creation (and annihilation) operators that appear in the two Hamiltonians refer to different excitations, though they are commonly represented as though they were identical; and, when these creation operators act on the vacuum, they represent different states in the covariant and Coulomb gauges. In the covariant gauge formulation that leads to the Hamiltonian $H^{\text{cov}}$, $e_s^\dagger(q)|0\rangle$ represents a pure “Dirac” electron totally devoid of all electric or magnetic fields. In the Coulomb gauge formulation however, $e_s^\dagger(q)|0\rangle$ represents an electron accompanied by the longitudinal electric field $E(x) = -\nabla \int dy \, j_0(y)/4\pi|x - y|$. The unitary transformation $P \rightarrow \tilde{P} = V^{-1}PV$ shifts to a representation in which $e_s^\dagger(q)|0\rangle$ represents an electron accompanied by its Coulomb field, in the covariant gauge as well. $\tilde{H}^{\text{cov}}$ and $H_C$ therefore are expressed in terms of the same particle excitation operators.

The remaining difference between $\tilde{H}^{\text{cov}}$ and $H_C$ is entirely in the “ghost” part of the spectrum, i.e., along the fiber within the subspace $\{|n\rangle\}$ but not on the quotient space of states $|N_i\rangle$. The difference between $\tilde{H}^{\text{cov}}$ and $H_C$ affects the equations of motion of the gauge field; but it has no effect on the time evolution of state vectors in the quotient space, which alone contains state vectors that describe physically observable configurations of particles.
That there is a difference in the renormalization program for QED in different gauges is entirely consistent with the fact that, in the perturbative theory, Gauss’s law is not implemented. The differences in the renormalization constants ensue from that fact. In non-covariant gauges, these renormalization constants are frame-dependent, and this contributes to the difficulties that beset the renormalization program in these gauges. But QED in the covariant and the Coulomb gauges, as quantum field theories, are entirely equivalent when Gauss’s law is implemented in both, as shown in the preceding section. Later, we will extend this result to other gauges.

VI. QED IN THE SPATIAL AXIAL GAUGE

A gauge which presents an interesting contrast to the covariant gauge is the spatial axial, or the $A_3 = 0$ gauge. [17][19] The photon propagator for this gauge has been known for some time, but its canonical quantization has not been discussed extensively. Since $A_0$ is not involved in the gauge condition, the gauge fixing term cannot simultaneously serve the purpose of imposing $A_3 = 0$ and providing $A_0$ with a canonically conjugate momentum. Primary constraints on operator-valued fields therefore are inevitable, unless the gauge is approached as a limit of a more general axial gauge [18]. The Lagrangian for this model is

$$L = -\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}F_{0i}F_{0i} + j_i A_i - j_0 A_0 - A_3 G + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi$$  \hspace{1cm} (68)

where $F_{ij} = \partial_j A_i - \partial_i A_j$ and $F_{0i} = \partial_0 A_i + \partial_i A_0$. The Euler-Lagrange equations that follow from this Lagrangian are

$$\partial_0 F_{0i} - \partial_j F_{ij} - j_i + \delta_{i3} G = 0,$$  \hspace{1cm} (69)

$$\partial_i F_{0i} + j_0 = 0$$  \hspace{1cm} (70)

\footnote{In this gauge, in which we use non-relativistic notation for the gauge field, $j_i$ and $A_i$ refer to the contravariant quantities and $\partial_i$ to the covariant quantity.}
and

\[ A_3 = 0. \]  \hfill (71)

The momenta conjugate to the fields are given by \( \Pi_i = \delta \mathcal{L} / \delta (\partial_0 A_i) = F_{0i} \), \( \Pi_0 = \delta \mathcal{L} / \delta (\partial_0 A_0) = 0 \) and \( \Pi_G = \delta \mathcal{L} / \delta (\partial_0 G) = 0 \). The charged particle field is treated precisely as in the covariant gauge. \( \Pi_0 \approx 0 \) and \( \Pi_G \approx 0 \) are primary constraints. We will use the sign \( \approx \) to designate weak equalities that hold only by virtue of constraints. Commutation rules must be modified to be consistent with primary constraints, as well as with further derived constraints. We use the Dirac-Bergmann method to carry out this program. \[20,21\] We first form the Hamiltonian density

\[ \mathcal{H}_{\text{spat}} = \partial_0 A_i \frac{\delta \mathcal{L}}{\delta (\partial_0 A_i)} + \partial_0 \psi \frac{\delta \mathcal{L}}{\delta (\partial_0 \psi)} - \mathcal{L} + \Pi_0 U_0 + \Pi_G U_G, \]  \hfill (72)

which becomes

\[ \mathcal{H}_{\text{spat}} = \frac{1}{2} \Pi_i \Pi_i + \frac{1}{4} F_{ij} F_{ij} + A_0 (\nabla \cdot \Pi + j_0) + A_3 G - j \cdot A \]
\[ + \psi^\dagger (-i \alpha \cdot \nabla + \beta m) \psi + \Pi_0 U_0 + \Pi_G U_G \]  \hfill (73)

where \( U_0 \) and \( U_G \) are undetermined \( c \)-number fields. Secondary constraints are obtained by assuming canonical (Poisson) commutation rules for \( \Pi_0 \) and \( \Pi_G \), and by setting \( i[H_{\text{spat}}, \Pi_0] \approx 0 \) and \( i[H_{\text{spat}}, \Pi_G] \approx 0 \). The resulting secondary constraints are

\[ \nabla \cdot \Pi + j_0 \approx 0, \]  \hfill (74)

with \( \Pi = -E \), and

\[ A_3 \approx 0, \]  \hfill (75)

respectively. Further, tertiary constraints, obtained from \( i[H_{\text{spat}}, \nabla \cdot \Pi + j_0] \approx 0 \) and \( i[H_{\text{spat}}, A_3] \approx 0 \), are

\[ \partial_3 G \approx 0 \]  \hfill (76)

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and
\[ \partial_3 A_0 - \Pi_3 \approx 0, \quad (77) \]
respectively. Since \( i[H^{\text{spat}}, \partial_3 G] \) and \( i[H^{\text{spat}}, \Pi_3 - \partial_3 A_0] \) involve the \( c \)-number fields \( U_0 \) and \( U_G \), they do not lead to any further constraints.

We define the constraint functionals \( \xi_1 = \Pi_0, \xi_2 = \nabla \cdot \Pi + j_0, \xi_3 = \partial_3 G, \xi_4 = \Pi_G, \xi_5 = A_3, \) and \( \xi_6 = \partial_3 A_0 - \Pi_3 \). And we establish the matrix of “Poisson” commutators, in which each field is assumed to have canonical commutation rules with its adjoint momentum, even though that commutator may not be consistent with the constraints. We let \( \mathcal{M}_{i,j}(x,y) = [\xi_i(x), \xi_j(y)] \), and obtain the matrix
\[
\mathcal{M}(x,y) = 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & i \frac{\partial}{\partial x_3} \\
0 & 0 & 0 & 0 & -i \frac{\partial}{\partial x_3} & 0 \\
0 & 0 & 0 & i \frac{\partial}{\partial x_3} & 0 & 0 \\
0 & 0 & -i \frac{\partial}{\partial x_3} & 0 & 0 & 0 \\
0 & i \frac{\partial}{\partial x_3} & 0 & 0 & 0 & i \\
i \frac{\partial}{\partial x_3} & 0 & 0 & 0 & i & 0
\end{pmatrix} \delta(x-y). \quad (78)
\]

The six \( \xi_i \) constitute a second class system of constraints, so that the matrix \( \mathcal{M}(x,y) \) has an inverse, which we define as \( \mathcal{Y}(x,y) \); then \( \int dz \mathcal{M}_{i,n}(x,z) \mathcal{Y}_{n,j}(z,y) = \delta_{ij} \delta(x-y) \). We find that
\[
\mathcal{Y}(x,y) = 
\begin{pmatrix}
0 & -i \left( \frac{\partial}{\partial x_3} \right)^{-1} & 0 & 0 & 0 & -i \\
i \left( \frac{\partial}{\partial x_3} \right)^{-1} & 0 & 0 & i & 0 \\
0 & 0 & i & 0 & 0 \\
0 & 0 & -i & 0 & 0 \\
0 & i & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0
\end{pmatrix} \left( \frac{\partial}{\partial x_3} \right)^{-1} \delta(x-y). \quad (79)
\]

The constrained “Dirac” commutator, \( [\chi_a(x), \chi_b(y)]^D \), for two fields \( \chi_a(x) \) and \( \chi_b(y) \), is given by

23
\[ [\chi_a(x), \chi_b(y)]^D = [\chi_a(x), \chi_b(y)] - \int dz dz' [\chi_a(x), \xi_i(z)] Y_{ij}(z, z')[\xi_j(z'), \chi_b(y)]. \]  

(80)

The Dirac commutators for QED in the \( A_3 = 0 \) gauge are

\[ [A_0(x), A_i(y)]^D = i \left[ \left( \frac{\partial}{\partial x_3} \right)^{-2} \frac{\partial}{\partial x_i} \right] \delta(x - y) \quad \text{for } i = 1, 2; \quad (81) \]

\[ [A_0(x), \psi(y)]^D = e \left[ \left( \frac{\partial}{\partial x_3} \right)^{-2} \psi(y) \right] \delta(x - y); \quad (82) \]

\[ [A_i(x), \Pi_j(y)]^D = i \left[ \delta_{ij} - \delta_{j3} \left( \frac{\partial}{\partial x_3} \right)^{-1} \frac{\partial}{\partial x_i} \right] \delta(x - y); \quad (83) \]

and

\[ [\Pi_3(x), \psi(y)]^D = e \left[ \left( \frac{\partial}{\partial x_3} \right)^{-1} \psi(y) \right] \delta(x - y). \quad (84) \]

The Dirac commutators \([A_3(x), \varphi(y)]^D = 0\) for any \( \varphi(y) \); in fact, \([\xi_i(x), \varphi(y)]^D = 0\) for any \( \varphi(y) \) and any of the six \( \xi_i(x) \).

These Dirac commutators demonstrate that relationships exist among the constrained quantities, which can be exploited to simplify the Hamiltonian by reducing the number of independent quantities when constraints are imposed. In the case of the \( A_3 = 0 \) gauge, these relations are

\[ A_0(x) \approx - \left( \frac{\partial}{\partial x_3} \right)^{-2} \left[ \frac{\partial}{\partial x_n} \Pi_n(x) + j_0(x) \right], \quad (85) \]

where the summation extends over \( n = 1 \) and 2 only, and

\[ A_0(x) \approx \left( \frac{\partial}{\partial x_3} \right)^{-1} \Pi_3(x). \quad (86) \]

These same equalities also follow simply, on the classical level, from setting \( A_3 = 0 \) in Maxwell’s equations. When these relations are substituted in Eq. (73), we obtain

\[ H^\text{spat} \approx \frac{1}{2} \Pi_i \Pi_i + \frac{1}{4} F_{ij} F_{ij} - j_i A_i + \psi^\dagger (-i \alpha \cdot \nabla + \beta m) \psi \\
- \frac{1}{2} \left[ \left( \frac{\partial}{\partial x_i} \Pi_i(x) + j_0(x) \right) \left( \frac{\partial}{\partial x_3} \right)^{-2} \left( \frac{\partial}{\partial x_j} \Pi_j(x) + j_0(x) \right) \right] \\
+ \frac{1}{2} \left( \frac{\partial}{\partial x_3} A_i(x) \right) \left( \frac{\partial}{\partial x_3} A_i(x) \right), \quad (87) \]
where the summation now extends only over \( i, j = 1 \) and \( 2 \), and where an implicit integration by parts has been included. When we interpret the time derivative \( \partial_0 = i[H_{\text{spat}}, \cdot] \), we find that the Hamiltonian \( H_{\text{spat}} = \int d\mathbf{x} \mathcal{H}_{\text{spat}}(\mathbf{x}) \) reproduces Maxwell’s equations with the constraints imposed. Thus we obtain

\[
\partial_0 \Pi_i - \partial_j F_{ij} - \partial_3 \partial_3 A_i - j_i = 0
\]

and

\[
\partial_0 \Pi_3 - \partial_3 \partial_3 A_i - j_3 = 0
\]

for \( i = 1, 2 \).

We note that the Hamiltonian \( H_{\text{spat}} \) is not rotationally invariant, and that it bears very little resemblance to the Hamiltonians \( H_{\text{cov}} \) or \( H_C \), the covariant gauge and Coulomb gauge Hamiltonians respectively. It is therefore relevant to ask to what extent \( H_{\text{spat}} \), the Hamiltonian for QED in the \( A_3 = 0 \) gauge, describes the same physics as \( H_{\text{cov}} \) or \( H_C \).

To investigate this question, we expand the gauge field and the canonical momenta in terms of photon creation and annihilation operators. We must choose a representation that is consistent with the constraint \( A_3 = 0 \), as well as with the canonical expression for the magnetic field,

\[
\mathbf{B}(\mathbf{x}) = \sum_{\mathbf{k}} \frac{i\mathbf{k} \times \mathbf{e}^n(\mathbf{k})}{\sqrt{2k}} \left[ a_n(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} - a_n^\dagger(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \right].
\]

Since \( \mathbf{B}(\mathbf{x}) \) is gauge invariant, it should have the identical form in terms of photon creation and annihilation operators as in every other gauge. A representation of \( A_i(\mathbf{x}) \) that satisfies these conditions is

\[
A_i(\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2k}} \left[ \epsilon^i_n(\mathbf{k}) - \frac{k_i}{k_3} \epsilon^3_n(\mathbf{k}) \right] \left[ a_n(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + a_n^\dagger(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \right].
\]

The representation for the canonically conjugate momentum is again constrained by the gauge-invariance of the electric field and by Gauss’s law. We therefore represent \( \Pi_i \), for \( i = 1 \) and \( 2 \), as in all other gauges, by
\[ \Pi_i(x) = -i \sum_k \epsilon_i^n(k) \sqrt{\frac{k}{2}} \left[ a_n(k)e^{ik \cdot x} - a_n^\dagger(k)e^{-ik \cdot x} \right]. \quad (92) \]

For \( i = 3 \), we use Eqs. (85) and (86) and obtain
\[ \Pi_3(x) = -i \sum_k \epsilon_3^n(k) \sqrt{\frac{k}{2}} \left[ a_n(k)e^{ik \cdot x} - a_n^\dagger(k)e^{-ik \cdot x} \right] - \partial_3^{-1} j_0(x). \quad (93) \]

These representations are manifestly consistent with Gauss’s law. When we use Eqs. (91) and (92) to evaluate the commutator \([A_i(x), \Pi_j(y)]\), the result reproduces the Dirac commutator given in Eq. (83), without requiring any ghost components for either \( A_i(x) \) or \( \Pi_j(y) \). When we substitute Eqs. (91) and (92) into (87), we obtain
\[ H^{\text{spat}} = H_0^{\text{spat}} + H_1^{\text{spat}} \quad (94) \]

where
\[ H_0^{\text{spat}} = \sum_n k a_n^\dagger(k)a_n(k) + \int dx \psi^\dagger(x)(\beta m - i\alpha \cdot \nabla)\psi(x), \quad (95) \]
and the summation over \( n \) extends over the two photon helicity modes of the photon; and
\[ H_1^{\text{spat}} = -\int dx \left[ j_i(x) A_i(x) + \partial_i \Pi_i(x) \left( \frac{\partial}{\partial x_3} \right)^{-2} j_0(x) + \frac{1}{2} j_0(x) \left( \frac{\partial}{\partial x_3} \right)^{-2} j_0(x) \right], \quad (96) \]
where the summation extends over \( i = 1 \) and 2 only.

We observe that \( H_0^{\text{spat}} \) represents the kinetic energies of the non-interacting photons and electrons correctly; but the interactions represented by \( H_1^{\text{spat}} \) are still frame-dependent, lack rotational invariance, and are not manifestly equivalent to the covariant and Coulomb gauges. We can demonstrate that these features stem from the fact that, in this \( A_3 = 0 \) gauge formulation, the state \( e_s^\dagger(q)|0\rangle \) represents an electron with a physically unrealistic, severely asymmetric electric field. For example, we note that the expectation value of the electric field for the charged particle state \( e_s^\dagger(q)|0\rangle \) is
\[ - \langle e_s(p)|\Pi_i(x)|e_s(p)\rangle = 0 \quad (97) \]
for \( i = 1, 2 \) and

26
\begin{align}
- \langle e_s(p) | \Pi_3(x) | e_s(p) \rangle &= -\langle e_s(p) | \left( \frac{\partial}{\partial x_3} \right)^{-1} j_0(x) | e_s(p) \rangle. 
\end{align}

Although formally Gauss’s law is preserved by these equations, the spatial asymmetry of the electric field indicates that the Hilbert space \{ | n \rangle \} is not appropriate for this representation of the charged fermion field. We therefore need to construct a Hilbert space that satisfies the requirement of spatial symmetry as well as the implementation of constraints. We can construct such a Hilbert space by transforming the states in the subspace \{ | n \rangle \} to establish the states

\begin{align}
| \phi_i \rangle &= e^{-\Delta} | n_i \rangle
\end{align}

with \( \Delta \) given by

\begin{align}
\Delta &= \sum_k \frac{\epsilon_3^n(k)}{k_3 \sqrt{2k}} \left[ a_n(k) j_0(-k) - a_n^\dagger(k) j_0(k) \right].
\end{align}

The states \( | \phi_i \rangle \) incorporate spatial asymmetries that just compensate for those in \( H_{\text{spat}} \).

As in the case of the covariant gauge, we can choose to apply the unitary transform to the operators instead of to the states, and generate the transformed operators \( P \rightarrow \bar{P} = e^{\Delta} Pe^{-\Delta} \). The expressions for the transformed gauge fields are

\begin{align}
\bar{A}_i(x) &= \sum_k \frac{1}{\sqrt{2k}} \left[ \epsilon_3^n(k) - \frac{k_i}{k_3} \epsilon_3^n(k) \right] \left[ a_n(k) e^{ik \cdot x} + a_n^\dagger(k) e^{-ik \cdot x} \right], \\
\bar{A}_0(x) &= -\sum_k \frac{\epsilon_3^n(k)}{k_3} \sqrt{\frac{k}{2}} \left[ a_n(k) e^{ik \cdot x} - a_n^\dagger(k) e^{-ik \cdot x} \right] - \frac{1}{\sqrt{2}} j_0(x)
\end{align}

where \( i = 1, 2 \). The correspondingly transformed electric field obtained from \( \Pi = -E \) and from

\begin{align}
\bar{\Pi}_i(x) &= \Pi_i(x) - \partial_i \nabla^{-2} j_0(x)
\end{align}

for \( i = 1, 2 \), and

\begin{align}
\Pi_3(x) &= -i \sum_k \frac{\epsilon_3^n(k)}{2} \left[ a_n(k) e^{ik \cdot x} - a_n^\dagger(k) e^{-ik \cdot x} \right] - \frac{\partial_3}{\sqrt{2}} j_0(x)
\end{align}
shows that for the transformed representation, spatial symmetry is restored and Gauss's law is implemented in the subspace \( \{|n\rangle\} \). The transformed Hamiltonian \( \bar{H}^{\text{spat}} \) takes the form

\[
\bar{H}^{\text{spat}} = H_0^{\text{spat}} - \sum_{k,i} \frac{\epsilon_i^n(k)}{\sqrt{2k}} \left[ a_n(k) j_i(-k) + a_n^\dagger(k) j_i(k) \right] + \int dxdy \frac{j_0(x)j_0(y)}{8\pi|x-y|}
\]

(105)

where the \( \sum_i \) here extends from \( i = 1 \) to \( 3 \). \( \bar{H}^{\text{spat}} \) is rotationally symmetric and manifestly identical to \( H_C \), the Hamiltonian for the Coulomb gauge.

The propagator for the perturbative theory is evaluated from the vacuum state \( |0\rangle \) that is part of the subspace \( \{|n\rangle\} \) and from the interaction-picture gauge field operators,

\[
A_i(x) = \sum_k \frac{1}{\sqrt{2k}} \left( \epsilon_i^n(k) - \frac{k_i}{k_3} \epsilon_3^n(k) \right) \left[ a_n(k)e^{i(k\cdot x - |k|t)} + a_n^\dagger(k)e^{-i(k\cdot x - |k|t)} \right]
\]

(106)

and

\[
A_0(x) \approx -\left( \frac{\partial}{\partial x_3} \right)^2 \frac{\partial}{\partial x_i} \Pi_i(x)
\]

(107)

where the summation extends over \( i = 1 \) and \( 2 \). The resulting propagators are

\[
D_{ij}(x,y) = \langle 0|T(A_i(x)A_j(y))|0\rangle,
\]

given by

\[
D_{ij}(x,y) = \sum_k \frac{1}{2k} \left( \delta_{ij} + \frac{k_i k_j}{k_3^2} \right) e^{i[k\cdot (x-y) - k|x_0 - y_0|]},
\]

(108)

for \( i = 1 \) or \( 2 \), and \( D_{ij}(x,y) = 0 \) when either \( i \) or \( j \) is in the \( z \)-direction. The propagator

\[
D_{i0}(x,y) = \langle 0|T(A_i(x)A_0(y))|0\rangle
\]

for \( i = 1 \) or \( 2 \) is

\[
D_{i0}(x,y) = \sum_k \frac{k_i}{2k_3^2} e^{i(k\cdot (x-y))} \left[ e^{-i(k\cdot (x_0 - y_0))}\Theta(x_0 - y_0) - e^{i(k\cdot (x_0 - y_0))}\Theta(y_0 - x_0) \right];
\]

(109)

similarly, \( D_{00}(x,y) \) is given by

\[
D_{00}(x,y) = \sum_k \frac{k}{2k_3^2} \left( 1 - \frac{k_3^2}{k^2} \right) e^{i[k\cdot (x-y) - k|x_0 - y_0|]}.
\]

(110)

It is straightforward to show that these expressions for the propagator can be obtained from

\[
D_{\mu\nu}(x,y) = \frac{1}{(2\pi)^4} \int d^4k \; D_{\mu\nu}(k)e^{-ik\lambda(x-y)}
\]

(111)

and from the axial gauge propagator,
\[ D_{\mu\nu}(k) = \frac{-i}{k_\lambda k_\lambda} + i\epsilon \left( g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k_\lambda n_\lambda} + \frac{n_\lambda n_\lambda k_\mu k_\nu}{(k_\lambda n_\lambda)^2} \right) \],

(112)

where \( n_\mu = \delta_\mu^3 \).

In the spatial axial gauge, the perturbative Feynman rules are obtained by combining the Hamiltonian \( H^{\text{spat}} \) with the states \( |n_i\rangle \), instead of the states \( |\phi_i\rangle \) needed for equivalence with the rotationally invariant theory in the Coulomb gauge. However, in this case as in the covariant gauges, the argument presented in Sec. \( \text{IV} \) maintains the validity of the \( S \)-matrix when the \( |n_i\rangle \) are substituted for the corresponding \( |\phi_i\rangle \), except in so far as the renormalization of QED in the spatial axial gauge is affected by this substitution. It is also worth noting that, since

\[ \langle 0|\Delta^2|0 \rangle = \frac{1}{(2\pi)^3} \int d\mathbf{k} \frac{\epsilon_3^n(\mathbf{k})\epsilon_3^n(\mathbf{k})}{2k_3^2} j_0(\mathbf{k})j_0(-\mathbf{k}), \]

(113)

\( \langle \phi_i|\phi_i \rangle \) will not, in general, be finite; \( |\phi_i\rangle \) therefore cannot be assumed to be a normalizable state.

VII. QUANTUM GAUGE TRANSFORMATIONS

The mathematical apparatus we have developed in the preceding sections of this work now enables us to implement gauge transformations by adding \( A_\mu \) in one gauge to the four-dimensional gradient of an operator-valued field, in order to arrive at \( A'_\mu \), the gauge field in another gauge. The new gauge, \( A'_\mu \), may differ from its usual version only by being embedded in the Hilbert space of the original gauge \( A_\mu \), which may, for example, be the space \( \{|n\rangle\} \) instead of the quotient space of states \( |N_i\rangle \). Thus \( A'_\mu \) may have a spurious component whose operator content will be limited to \( a_Q(\mathbf{k}) \) and \( a_Q^*(\mathbf{k}) \) excitation operators. We are able to carry out such operator gauge transformations, because the Hamiltonians, as well as all operator-valued fields in the gauges being considered, have been brought into what we will call “common form.” In this common form, when excitation operators such as \( e_{s^\dagger}(\mathbf{p}) \) and \( e_s(\mathbf{p}) \) act on the perturbative vacuum or on the states built on it, they create and annihilate particle modes with identical properties in all gauges. In common form, the Hamiltonians for
the Coulomb, covariant and spatial axial gauges are $H_C$, $\tilde{H}^{cov}$, and $\tilde{H}^{spat}$, respectively (with $\tilde{H}^{spat} = H_C$); and all three of these Hamiltonians generate the identical time displacement for a state vector $|N_i\rangle$ within the previously defined quotient space.

We define the operator-valued field $\chi^{cov} \to C$ as

$$\chi^{cov} \to C = i \sum_k \frac{1}{2k^{3/2}} \left[ a_R(k)e^{i\mathbf{k} \cdot \mathbf{x}} - a_R^\ast(k)e^{-i\mathbf{k} \cdot \mathbf{x}} \right]$$

$$+ i(1 - \frac{\gamma}{2}) \sum_k \frac{1}{2k^{3/2}} \left[ a_Q(k)e^{i\mathbf{k} \cdot \mathbf{x}} - a_Q^\ast(k)e^{-i\mathbf{k} \cdot \mathbf{x}} \right]$$

and note that its gradient is given by

$$\partial_i \chi^{cov} \to C = -\sum_k \frac{k_i}{2k^{3/2}} \left[ a_R(k)e^{i\mathbf{k} \cdot \mathbf{x}} + a_R^\ast(k)e^{-i\mathbf{k} \cdot \mathbf{x}} \right]$$

$$- (1 - \frac{\gamma}{2}) \sum_k \frac{k_i}{2k^{3/2}} \left[ a_Q(k)e^{i\mathbf{k} \cdot \mathbf{x}} + a_Q^\ast(k)e^{-i\mathbf{k} \cdot \mathbf{x}} \right]$$

and $\partial_0 \chi^{cov} \to C = i[\tilde{H}^{cov}, \chi^{cov} \to C]$, so that

$$\partial_0 \chi^{cov} \to C = -(1 + \frac{\gamma}{2}) \sum_k \frac{1}{2\sqrt{k}} [a_R(k)e^{i\mathbf{k} \cdot \mathbf{x}} + a_R^\ast(k)e^{-i\mathbf{k} \cdot \mathbf{x}}]$$

$$+ \sum_k \frac{\phi(k)}{2\sqrt{k}} j_0(k)e^{i\mathbf{k} \cdot \mathbf{x}} - \sum_k \frac{\phi(k)}{2\sqrt{k}} j_0(k)e^{-i\mathbf{k} \cdot \mathbf{x}}$$

$$+ (1 + \frac{\gamma}{2}) \sum_k \frac{1}{2\sqrt{k}} \left[ a_Q(k)e^{i\mathbf{k} \cdot \mathbf{x}} + a_Q^\ast(k)e^{-i\mathbf{k} \cdot \mathbf{x}} \right].$$

It follows from these expressions for $\partial_\mu \chi^{cov} \to C$, that for $\tilde{A}_\mu^{cov}(x)$, given in Eqs. (18), (19), (20), (46) and (48),

$$\tilde{A}_\mu^{cov}(x) - \partial_\mu \chi^{cov} \to C = A_\mu^C(x) + A_\mu^T(x).$$

$A_\mu^C(x)$ is the form that $A_\mu(x)$ takes in the Coulomb gauge. $A_i^C(x) = A_i^T(x)$, the transverse part of $A_i(x)$; $A_0^C(x)$ is given by

$$A_0^C(x) = \int d\mathbf{y} \frac{j_0(y)}{4\pi|x - y|}.$$
\[ A_0^C(x) = \sum_k \frac{1}{\sqrt{k}} \left[ a_Q(k)e^{ik \cdot x} + a_Q^*(k)e^{-ik \cdot x} \right] \] (119)

In the Hilbert space appropriate for the common form representation, matrix elements of \( A_0^C(x) \) always vanish, so that its presence does not interfere with the gauge condition or the expression for Gauss's law in the Coulomb gauge. When the spinor field is gauge-transformed, we transform \( \tilde{\psi} \), given in Eq. (50) and represented as

\[ \tilde{\psi}(x) = \exp \left\{ e \sum_k \frac{1}{2k^{3/2}} \left[ a_R(k)e^{ik \cdot x} - a_R^*(k)e^{-ik \cdot x} \right] \right\} \psi(x) \] (120)

Under a gauge transformation, \( \tilde{\psi} \) transforms as

\[ \tilde{\psi} \rightarrow \tilde{\psi}' = \tilde{\psi} \exp \left[ ie^{\chi_{\text{cov}} \rightarrow C} \right] . \] (121)

The gauge-transformed spinor wave function, \( \tilde{\psi}' \), is most conveniently expressed in the form

\[ \tilde{\psi}' = \psi + \Upsilon, \] where

\[ \Upsilon = \left\{ \exp \left[ e \sum_k \frac{\phi(k) - 1 + \gamma/2}{2k^{3/2}} \left( a_Q(k)e^{ik \cdot x} - a_Q^*(k)e^{-ik \cdot x} \right) \right] - 1 \right\} \psi(x) \] (122)

and has vanishing matrix elements in the subspace \(|n\rangle\). \( \psi \) is the expression for the spinor field in the Coulomb gauge.

We can gauge-transform from the covariant to the spatial axial gauge, by using \( \chi_{\text{cov}} \rightarrow \text{spat} \) given by

\[ \chi_{\text{cov}} \rightarrow \text{spat} = i \sum_k \frac{1}{2k^{3/2}} \left[ a_R(k)e^{ik \cdot x} - a_R^*(k)e^{-ik \cdot x} \right] \]

\[ + i(1 - \frac{\gamma}{2}) \sum_k \frac{1}{2k^{3/2}} \left[ a_Q(k)e^{ik \cdot x} - a_Q^*(k)e^{-ik \cdot x} \right] \]

\[ + i \sum_k \frac{\epsilon^3_n(k)}{k_3 \sqrt{2k}} \left[ a_n(k)e^{ik \cdot x} - a_n^\dagger(k)e^{-ik \cdot x} \right] , \] (123)

with

\[ \partial_i \chi_{\text{cov}} \rightarrow \text{spat} = - \sum_k \frac{k_i}{2k^{3/2}} \left[ a_R(k)e^{ik \cdot x} + a_R^*(k)e^{-ik \cdot x} \right] \]

\[ - (1 - \frac{\gamma}{2}) \sum_k \frac{k_i}{2k^{3/2}} \left[ a_Q(k)e^{ik \cdot x} + a_Q^*(k)e^{-ik \cdot x} \right] \]

\[ - \sum_k \frac{k_i \epsilon^3_n(k)}{k_3 \sqrt{2k}} \left[ a_n(k)e^{ik \cdot x} + a_n^\dagger(k)e^{-ik \cdot x} \right] \] (124)
and
\[
\partial_0 \chi^{\text{cov} \rightarrow \text{spat}} = -(1 - \frac{\gamma}{2}) \sum_k \frac{1}{2k^2} j_0(k) e^{ik \cdot x} - \sum_k \frac{\phi(k)}{2k^2} j_0(k) e^{ik \cdot x} \\
+ \sum_k \frac{1}{2\sqrt{k}} [a_R(k) e^{ik \cdot x} + a_R^*(k) e^{-ik \cdot x}] \\
+ (1 - \frac{\gamma}{2}) \sum_k \frac{1}{2\sqrt{k}} [a_Q(k) e^{ik \cdot x} + a_Q^*(k) e^{-ik \cdot x}] \\
+ \sum_k \frac{\epsilon_3 n(k)}{k_3} \sqrt{\frac{k}{2}} [a_n(k) e^{ik \cdot x} + a_n^*(k) e^{-ik \cdot x}] .
\] (125)

We observe that
\[
\tilde{A}_\mu^{\text{cov}}(x) - \partial_\mu \chi^{\text{cov} \rightarrow \text{spat}} = \tilde{A}_\mu^{\text{spat}}(x) + \tilde{A}_\mu^{\text{spat}}(x),
\] (126)

where \( \tilde{A}_\mu^{\text{spat}}(x) \) is given in Eqs. (101) and (102), and \( \tilde{A}_\mu^{\text{spat}}(x) \) has vanishing matrix elements in \( \{|n\rangle\} \).

Finally, the field \( \chi^{\text{spat} \rightarrow \text{C}} \), given by
\[
\chi^{\text{spat} \rightarrow \text{C}} = i \sum_k \frac{\epsilon_3 n(k)}{k_3 \sqrt{2k}} [a_n(k) e^{ik \cdot x} - a_n^*(k) e^{-ik \cdot x}],
\] (127)
has a gradient consisting of
\[
\partial_i \chi^{\text{spat} \rightarrow \text{C}} = -\sum_k \frac{k_3 \epsilon_3 n(k)}{k_3 \sqrt{2k}} [a_n(k) e^{ik \cdot x} + a_n^*(k) e^{-ik \cdot x}]
\] (128)
and \( \partial_0 \chi^{\text{spat} \rightarrow \text{C}} = i[H^{\text{C}}, \chi^{\text{spat} \rightarrow \text{C}}] \) given by
\[
\partial_0 \chi^{\text{spat} \rightarrow \text{C}} = \sum_k \frac{\epsilon_3 n(k)}{k_3} \sqrt{\frac{k}{2}} [a_n(k) e^{ik \cdot x} + a_n^*(k) e^{-ik \cdot x}].
\] (129)

With \( \chi^{\text{spat} \rightarrow \text{C}} \) we obtain
\[
\tilde{A}_\mu^{\text{spat}}(x) + \partial_\mu \chi^{\text{spat} \rightarrow \text{C}} = A_\mu^{\text{C}}(x).
\] (130)

These gauge transformations allow us to connect QED in different gauges. Not only the gauge fields, but the apparatus of the entire theory can be transformed in this fashion. We can, for example, start with QED in the covariant gauge, and express the Hamiltonian in common form either by substituting the expressions for \( \tilde{A}_\mu^{\text{cov}}(x) \) in Eqs. (18) and (19), or by
transforming the Hamiltonian given in Eqs. (23) and (24), so that \( H^{\text{cov}} \rightarrow \tilde{H}^{\text{cov}} = V^{-1} H^{\text{cov}} V \).

We can then transform \( \tilde{H}^{\text{cov}} \) “backwards” from the common form, treating it now as the spatial axial gauge Hamiltonian. We transform \( \tilde{H}^{\text{cov}} \rightarrow e^{-\Delta} \tilde{H}^{\text{cov}} e^{\Delta} \), and since \( \tilde{H}^{\text{cov}} = H_C + H_Q \), and \( H_Q \) commutes with \( \Delta \), \( \tilde{H}^{\text{cov}} \rightarrow e^{-\Delta} H_C e^{\Delta} + H_Q \). \( e^{-\Delta} H_C e^{\Delta} \) is the rotationally asymmetric \( H^{\text{spat}} \). \( H_Q \) may either be amputated and discarded, or retained without affecting the dynamics of state vectors in the quotient space. To maintain consistency with the common form versions of QED in all these gauges, \( H^{\text{cov}} \) and \( H^{\text{spat}} \) must operate on quite different Hilbert spaces. \( H^{\text{cov}} \) must operate on the space \( \{ |\nu\rangle \} \) defined in Sec. IV, while the states on which \( H^{\text{spat}} \) operates are the non-normalizable \( |\phi_i\rangle \) defined in Eq. (99).

ACKNOWLEDGMENTS

This research was supported by the Department of Energy under Grant No. DE-FG02-92ER40716.00.
REFERENCES

[1] H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 1, 205 (1955).

[2] K. Haller, Phys. Rev. D 18, 3045 (1978).

[3] See, for example, J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).

[4] For a review of this work, see G. Leibbrandt, Rev. Mod. Phys. 59, 1067 (1987).

[5] B. Lautrup, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 35, Vol. 11 (1967), sec. 8.2.

[6] R. P. Feynman, Phys. Rev. 76, 769 (1949).

[7] L. D. Landau and I. M. Khalatnikov, Sov. Phys. JETP 2, 69 (1956).

[8] H. M. Fried and D. R. Yennie, Phys. Rev. 112, 1391 (1958).

[9] S. P. Tomczak and K. Haller, Nuovo Cimento 8B, 1 (1975).

[10] See for example, K. Gottfried and V. F. Weisskopf, Concepts in Particle Physics Vol. II (Oxford University Press, 1986), sec. IIB; M. Kaku, Quantum Field Theory (Oxford University Press, 1993), chap. 5.

[11] S. N. Gupta, Proc. Phys. Soc. London 63, 681 (1950); K. Bleuler, Helv. Phys. Acta 23, 567 (1950).

[12] K. Haller and L. F. Landovitz, Phys. Rev. D 2, 1998 (1970).

[13] K. Haller, Acta Phys. Austr. 42, 163 (1975).

[14] K. Haller and R. B. Sohn, Phys. Rev. A 20, 1541 (1979).

[15] K. Johnson and B. Zumino, Phys. Rev. Letters 3, 351 (1959).

[16] I. Bialynicki-Birula, Phys. Rev. D 2, 2877 (1970).
[17] W. Kummer, Acta Phys. Austr. 14, 149 (1961).

[18] K. Haller, Phys. Lett. B 251, 575 (1990).

[19] R. L. Arnowitt and S. I. Fickler, Phys. Rev. 127, 184 (1962).

[20] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).

[21] P. G. Bergmann and I. Goldberg, Phys. Rev. 98, 531 (1955).
Figure 1. The Hilbert space for QED in the covariant gauge. The sheet in the interior of the diagram represents the quotient space of states $|N_i\rangle$ consisting of electrons, positrons and transversely polarized photons. The ellipse surrounding the sheet represents the Hilbert space $\{|n\rangle\}$, in which the states $|N_i\rangle$ are augmented with zero norm states in which chains of $a_Q^*(k)$ operators act on $|N_i\rangle$ states. The rectangle surrounding the ellipse represents the space $\{|h\rangle\}$, in which the space $\{|n\rangle\}$ is augmented with further states in which chains of $a_R^*(k)$ and $a_Q^*(k)$ operators act on $|N_i\rangle$ states. The vertical line rising from the sheet represents a fiber of $|n_i\rangle$ states consisting of a single $|N_a\rangle$ and the set of all possible zero norm $|n_i\rangle$ states in which chains of $a_Q^*(k)$ operators act on $|N_a\rangle$. 