A NEW ENGINEERING THEORY DESCRIBING OBLIQUE FREE SURFACE IMPACT BY FLEXIBLE PLATES

A PREPRINT

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January 6, 2022

ABSTRACT

Consideration of slamming loads within the structural design of planning hulls is of critical importance in ensuring adequate structural performance. However, because of the intricacy in the interplay between complex fluid flows and nonlinear structural deformations that accompany slamming, a general engineering theory for slamming has yet to be uncovered, and so design relies on specialized theories. In this paper, we propose one such theory for a design case that has, until now, eluded a proper description. In pursuit of this theory, we employ a specialized implicit, partitioned fluid-structural interaction (FSI) simulation approach, to study the underlying physical mechanisms accompanying the oblique impact of a flexible plate during water entry. In the present work, we first present validation results from flexible plate water entry experiments. Subsequent to validation, we carry out a series of numerical analyses to characterize the impact force and plate out-of-plane deformations. Finally, we use our FSI solver to study the mechanistic evolution of fluid flows and elastic plate deformations that occur during slamming. Based on these observations, we propose a novel, but simple engineering theory for flexible plates obliquely impacting the water free surface (e.g., high speed porpoising water craft reentry).

Keywords Fluid-structure interaction · Volume of Fluid · Oblique slamming impact · Added Mass Approximation · Engineering theory

1 Introduction

In planning watercraft, sailing in head seas at high speeds, the outer hull plating frequently experiences violent impacts with the water free surface. These violent impacts are known as slamming loads. Slamming loads are characterized as instantaneous, acute, and concentrated pressure impulses acting on hull plating and underlying structural components. This type of extreme loading occurs at small spatiotemporal scales and can induce high strain rates and nonlinear behaviors within the structural system – in extreme cases, slamming may result in serious ship accidents, with accompanying loss of life [1]. Indeed, the American Bureau of Shipping, (ABS), in its HSNC 2007 guide for building and classing naval craft, has called out slamming loads as the single most important consideration when
proportioning hull scantlings in high speed watercraft. That said, a proper consideration of slamming loads in the design of planning hulls is of critical importance for safe and satisfactory performance.

1.1 Literature review

Theoretical study of hydrodynamic slamming phenomenology was pioneered by von Karman in his 1929 report focused on the impact of seaplane floats during landing. In that study, von Karman approximated the seaplane float as a rigid wedge-shaped body, and subsequently developed a mathematical framework (based on the conservation of momentum) to estimate the hydrodynamic impact force experienced by the structure of the float [2]. A few years after that seminal work, Herbert Wagner proposed an improvement to von Karman’s theory. He included the so-called splash-up effect in the force calculation, and derived a new mathematical model on the basis of potential flow theory to approximate the impact force [3] acting on the wetted surface of a wedge. Wagner’s model was further extended by many researchers in an effort to explain the various aspects of slamming phenomenology. These studies include, to name a few, three-dimensional impact on various geometries during water entry [4], structural elastic deflections resulting from wave loads [5], the hydroelastic effects on elastic plates and wedges subjected to wetdeck slamming [6-8], etc; many mathematical and analytical models have been developed to better understand the underlying hydrodynamic and hydroelastic impacts centering around slamming [9,10,11,12,13,14].

Accompanying the development of slamming theories, experimental investigations of various slamming contexts have been undertaken in an effort to validate and refine the proposed theories. For example, the importance of hydroelastic effects in wedge drop testing was discussed in [6], water slamming loads and the effects of air entrapment for flat plate slamming was reported in [15], experimental studies of flat plate subjected to high horizontal speed was reported in [16], and the oblique impact of flexible plates was studied in [17,18,19]. Although experimental data concerning slamming impacts have helped in accelerating the development of slamming theories, the existing experimental findings only represent a subset of slamming scenarios that are encountered within design practice. The need for considering many specimen geometries and loading contexts creates a combinatorial “explosion” within the space of possible experiments. The highly complex nature of slamming involves the interaction of complex fluid flows and nonlinear structural response that occurs at a confined spatiotemporal scale in order to properly characterize the underlying phenomenology; this further complicates experimental investigation. As a result of all the foregoing, we pursue a program of investigation involving experimentally validated computer simulations, as a stand-in for large scale experimentation, as we pursue a useful engineering theory suitable for design.

Considerable work has already been undertaken to uncover useful numerical strategies for analyzing slamming impacts: some of these approaches include water entry of a wedge with large deadrise angle using a nonlinear boundary element method [20], numerical simulation of the vertical impact of a two dimensional rigid plate onto water surface via boundary value problem formulation [21,22], and a state-space models using a Newmark \( \beta \) time integration scheme for analyzing a flexible barge [23]. While the previous numerical studies mentioned provide valuable insights into slamming, the fluid and structure domains are solved explicitly within these numerical frameworks (i.e., the slamming problems are treated as one-way coupling FSI problems where the structural bodies are assumed rigid). Since structural deformations have significant effects on the fluid dynamics and vice versa [24], a one-way coupling strategy undermines the importance of the interdependent nature between the structural and fluid responses in the context of water entry and exit problems: a vital element within the current work. As a result, implicit fluid-structure interaction (FSI) analysis is adopted herein; to accurately assess the hydrodynamic and hydroelastic effects accompanying slamming. Exemplars of the FSI formulations for analyzing the hydrodynamics and hydroelastic response in the context of slamming can be found in [25,26,27,28,29,30,31].

1.2 Contribution

The present work describes a state-of-the-art, partitioned, implicit FSI simulation tool built from open source components and experimentally validated for the oblique, hydroelastic plate context that is salient for high speed vessels operating within seaways. We use subsequent high fidelity simulations to
afford unprecedented insight into the physical mechanisms at work within this important engineering context, so that we may propose a useful engineering theory to support design.

1.3 Paper structure

In the present work, we aim to study the slamming phenomenon in unprecedented detail, through an implicit partitioned FSI framework. Specifically, we first validate the impact forces and the out-of-plane deflections of flexible plates using the experimental investigations reported in [19]. Subsequently, we perform a series of complementary simulations for this slamming context, to elucidate the missing details from experimental results. Finally, we furnish a new engineering theory for slamming with engineering insights gleaned from high fidelity FSI simulations.

The outline of this manuscript is as follows: Section 2 and 3 provide the governing equations for the FSI coupled system along with the FSI coupling framework adopted in the present work; Section 4 provides in-depth information regarding the experimental and computational details concerning the flexible plate slamming experiments, as well as FSI validation results; Section 5 presents slamming results from additional FSI simulations germane to the slamming context we study; Section 6 discusses the derivation of the proposed engineering theory; and Section 7 concludes the work with a summary of findings.

2 Mathematical formulations

We now introduce the mathematical formulations that govern the implicitly coupled FSI system employed in the present work. These mathematical models include the governing equations for the fluid and structural domains, along with mathematical formulations for free surface tracking, and imposition of transmission conditions along the FSI interface, as well as the algorithmic approach to implicit coupling within the FSI system.

2.1 Governing equations for incompressible flows

The fluid domain within the present work is modeled as a homogenous, incompressible Newtonian fluid governed by the Navier-Stokes equation; compressibility of the air is assumed negligible in the present work given that no significant air entrapments were observed in oblique slamming impact [15, 32, 33] with angle of attack greater than $5^\circ$. The Navier-Stokes and the continuity equation are expressed in vectorial form in Equation 1

$$\rho^f \left[ \frac{\partial u^f}{\partial t} + (u^f \cdot \nabla) u^f \right] = \rho^f g - \nabla p^f + \mu^f \nabla^2 u^f,$$

$$\nabla \cdot u^f = 0. \tag{1}$$

where $\rho^f$ is the fluid density, $u^f$ represents the fluid velocity, $g$ is the gravitational body force, $p^f$ is the fluid pressure.

2.1.1 Free surface tracking method

Within the present FSI context, it is vitally important to correctly track the free surface: the interface between the air phase and water phase within the fluid domain. In order to efficiently resolve and capture the free surface motion, the volume-of-fluid (VOF) method [34] is used to model the transient fluid dynamic flow of the two immiscible fluids in the present work. The VOF method is a popular interface tracking method in marine application because of its ability to capture the water free surface when undergoing large deformations (e.g., breaking waves) [35]. A visual representation of the VOF methodology is shown in Figure 1. The control volumes within the fluid domain are characterized using so-called $\alpha$ values, which are used to quantify the volume fraction of the water and air phase within the volume cells used to discretize the fluid domain, as well as to track the motion of the free
A representation of the VOF implementation is shown. The cells with blue color represents water and white color represent air. Each cell has an assigned phase fraction value, \( \alpha \), which is used to identify the amount of water within each cell.

\[
\alpha = \begin{cases} 
1 & \text{water,} \\
0 & \text{air,} \\
0 < \alpha < 1 & \text{fluid interface.}
\end{cases}
\] (2)

Within the VOF framework, the air and water phases are each treated as individual continua, and the fluid properties of the air and water are interpolated through a phase fraction variable, \( \alpha \), along the interface, as shown in Equation 3

\[
\rho = \alpha \rho_w + (1 - \alpha) \rho_a, \\
\mu = \alpha \mu_w + (1 - \alpha) \mu_a,
\] (3)

where subscript \( w \) denotes the fluid properties of water and the subscript \( a \) denotes the fluid properties of air. The continuity equation for the phase fraction \( \alpha \) is expressed as

\[
\frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha \vec{u}_f) + \nabla \cdot (\alpha(1 - \alpha) \vec{u}_f) = 0,
\] (4)

where \( \vec{u}_f \) is the relative velocity between water and air (i.e., the artificial compression velocity [36]). It is critical to maintain a sharp interface within the computational domain, in order to maintain appropriate simulation resolution. Since the sharpness in the vicinity of the interface can no longer be guaranteed if the term \( \nabla \cdot (\alpha \vec{u}_f) \) is diffusive, an artificial compressive convective term, \( \nabla \cdot (\alpha(1 - \alpha) \vec{u}_f) \), is added to the continuity equation, in order to enhance interface sharpness.

### 2.2 Governing equations for isotropic linear elastic materials

The structural domain in the present work is described within a Lagrangian frame of reference, wherein the momentum equation appears as

\[
\rho^s \left[ \frac{\partial \vec{d}^s}{\partial t} + (\vec{d}^s \cdot \nabla) \vec{d}^s \right] = \rho^s \vec{g} + \frac{\partial \sigma^s}{\partial x_j},
\] (5)

where \( \rho^s \) is the structure density, \( \vec{d}^s \) represents the structural displacement vector, \( \vec{g} \) is the gravitational body force, \( \sigma^s \) is the Cauchy stress tensor, and \( x_j \) represents the rectangular Cartesian spatial coordinates of the structural response variables, and \( t \) is the temporal variable.
Consider an isotropic linear elastic material model

\[ \sigma_{ij}^s = 2\mu^s \epsilon_{ij}^s + \lambda^s \epsilon_{kk}^s \delta_{ij}, \]

\[ \epsilon_{ij}^s = \frac{1}{2} \left( \frac{\partial d_i^s}{\partial x_j^s} + \frac{\partial d_j^s}{\partial x_i^s} \right), \]

(6)

where \( \epsilon^s \) is the infinitesimal strain tensor, \( \lambda^s \) and \( \mu^s \) are the Lamé constants. \( \mu^s \) is also the shear modulus. The conservation of momentum equation governing within the structural domain may be written as

\[ \rho^s \frac{\partial^2 d_i^s}{\partial t^2} + \rho^s \nabla \cdot (\sigma^s \cdot n^s) = \rho^s \frac{\partial d_i^s}{\partial x_j^s} + \frac{\partial d_j^s}{\partial x_i^s} + \lambda \frac{\partial d_k^s}{\partial x_k^s}. \]

(7)

The advection term of the momentum equation vanishes in a Lagrangian framework; thus the vectorial form of the momentum equation is expressed as

\[ \rho^s \frac{\partial^2 d^s}{\partial t^2} = \rho^s g + \mu^s \nabla d^s + (\lambda + \mu^s)(\nabla \cdot d^s). \]

(8)

2.3 FSI interface transmission conditions

In contrast with an explicit FSI coupling scheme, where equilibrium along the fluid-structure interface is not strictly enforced, the following transmission conditions in Equation 9 are imposed on the fluid-structure interface within the implicit FSI framework adopted herein, to ensure no-slip boundary condition for the fluid flow, as well as force equilibrium along the moving boundary.

\[ u_f = \frac{\partial d^s}{\partial t}, \]

\[ \sigma_f \cdot n_f = \sigma^s \cdot n^s, \]

(9)

where \( n_f \) is the unit outward normal vector w.r.t the fluid domain, and \( n^s \) is the unit outward normal vector w.r.t the structural domain.

2.4 Mathematical approach of the FSI coupled system

In the realm of FSI, the computational mesh in the fluid domain needs to track to the interface deformations accompanying the structural deformation, in order to ensure numerical stability. Given that the fluid and structural domains are formulating in different mathematical frameworks (Lagrangian v.s. Eulerian), a unified approach that incorporates the two frames of reference, into a single system context, is needed. The present work adopts the Arbitrary Lagrangian Eulerian (ALE) framework \[37, 38\] to combine the best features of both the Lagrangian and Eulerian descriptions, while allowing for FSI analyses involving large structural deformations. In the ALE formulation, a field variable \( u^m \), associated with the velocity of mesh points is introduced to the convective term in the Navier-Stokes governing equation, in order to capture the mesh motion effects. This additional variable provides flexibility for the fluid computational mesh to move in a Lagrangian fashion, or to be fixed in an Eulerian manner, or to deform in some combination of the two. This moving grid methodology greatly improves the efficiency in FSI simulations by reducing mesh distortions resulting from large deformation in a purely Lagrangian perspective, meanwhile, offering higher resolution in the solution than that afforded by a purely Eulerian viewpoint.

The Navier-Stokes equation formulated in an ALE frame of reference is expressed in Equation 10

\[ \rho^f \left[ \frac{\partial u_f}{\partial t} + ((u_f - u^m) \cdot \nabla) u_f \right] = \rho^f g - \nabla p + \mu^f \nabla^2 u_f, \]

(10)
where \( \mathbf{u}^f \) and \( \mathbf{u}^m \) denote the velocity of the fluid particle and fluid mesh respectively. In the ALE framework above, the Navier-Stokes equation reduces to the Eulerian expression when \( \mathbf{u}^m = 0 \) and Lagrangian if \( \mathbf{u}^m = \mathbf{u}^f \).

In the FSI context, the mesh motion propagates in a way such that the region closest to the FSI interface stays Lagrangian while allowing an Eulerian description to govern within the region away from the deforming FSI interface. In order to avoid spurious field values arising from artificial mass and momentum sources, as a result of the moving grid, the Space Conservation Scheme (SCL) [39] or the Geometric Conservation Law (GCL) [40] needs to be satisfied in the ALE scheme, to preserve a stable solution; especially in nonlinear flows [41]. In other words, the SCL or GCL must be satisfied to ensure that the ALE discretization reproduces exactly a uniform flow, as if it were using a fixed grid. To do so, the velocity of the mesh motion must be computed for all first or second order time accurate methods (i.e., implicit Euler, Crank-Nicholson, etc.) to ensure the condition

\[
\dot{\mathbf{u}}^m = \frac{\mathbf{u}^m_{n+1} - \mathbf{u}^m_n}{\Delta t} \tag{11}
\]

is satisfied [41]. In Equation (11) \( \dot{\mathbf{u}}^m \) denotes the mesh acceleration and the subscript, \( n \), denotes a time instance within the deformation evolution. In the present work, a vertex-based Laplacian motion diffuser [42] is chosen to determine \( \mathbf{u}^m \) (i.e., mesh diffusivity) in Equation (10). Detailed descriptions of the Laplacian motion diffuser is discussed in Section 3.2.

### 3 Partitioned FSI solver

We adopt an implicit, partitioned FSI approach to carry out the FSI analyses in this work. Herein, the fluid and structural systems within the partitioned FSI framework are treated with two open source transient dynamics libraries (OpenFOAM 1.6-ext [43] [44] [45] [46] and CU-BENs [47]). OpenFOAM 1.6-ext is a computational fluid dynamics (CFD) finite volume solver written in C++, while, CU-BENs is a computational structural dynamics (CSD) finite element solver written in C. To ensure the two solvers communicate with each other properly, a coupling library written in Python, denoted as the “Coupler” [48], is used to facilitate information transfer between the two solvers. Figure 2 provides a schematic representation of the implicit partitioned FSI solver coupling framework in the present work. A flow chart summarizing these steps appears as Figure 2.

![Figure 2: Implicit partitioned FSI coupling framework with OpenFOAM as the CFD solver, CU-BENs as the CSD solver, and a Coupler that facilitates all necessary communication between the two solvers to ensure the fluid-structure interface conforms to the transmission conditions.](image)

We will now discuss additional specialized details pertaining to the FSI modeling framework, adopted herein.
3.1 Structural solver

The structural behavior within the present work is modeled using an open source finite element solver, CU-BENs [47]. CU-BENs exploits higher order 3D structural elements (i.e., truss, beam, and Discrete Kirchhoff triangular shell elements) in linear and nonlinear transient dynamic analyses. Details on the functionalities and implementations within CU-BENs can be found in [47]. Within the structural domain, geometric nonlinear transient analyses (as opposed to modal analyses in a few of the related work [28, 31]) were carried out to assess the local deformation on aluminum plates during slamming impact. In addition, Newton-Raphson subiteration was called within each time step to account for the nonlinear behavior within the structural system.

3.1.1 Structural elements

Discrete Kirchhoff triangular (DKT) shell are used as the structural elements in this work [49, 50]. The DKT shell element formulation considers a three node triangular element whose tangent stiffness matrix, $K_T$, comprises the superimposition of the plane stress membrane stiffness, $K_m$, plate bending stiffness, $K_b$, and in-plane rotational stiffness, $K_\theta$

$$K_T = K_m + K_b + K_\theta,$$

which takes the following matrix form

$$K_T = \begin{bmatrix} K_m & 0 & 0 \\ 0 & K_b & 0 \\ 0 & 0 & K_\theta \end{bmatrix}.$$  

(13)

In Equation 12, the plate stress membrane stiffness assumes a constant plane stress, the plate bending stiffness assumes homogeneous isotropic material properties, and the in-plane rotational stiffness assumes the value of $10^{-14}$ times the smallest bending stiffness (a value selected to remove the in-plane rotational singularity from the stiffness matrix in the case where the local element coordinates happen to coincide with the global coordinates). These highly efficient shell elements provide satisfactory performance in the small deflection regime, while also ensuring satisfactory performance in the large deformation regime. Since the present work concerns the deformation of flexible plates subjected to various intensities of slamming impacts, this response regime flexibility in DKT shell elements, is especially favorable when it comes to accurately approximating the plate deformations. Interested readers can refer to [50] for extensive derivations and discussions on the plate bending stiffness, plane stress membrane stiffness, and in-plane rotational stiffness.

3.1.2 Implicit time integration scheme

Numerical stability is often time a concern in FSI analyses. In particular, numerical artifacts arising from coupling two distinct spatial discretization schemes (finite element and finite volume methods) in combination with the presence of added mass effects [51, 52, 53] greatly amplify the spurious high frequency modes within the FSI system; thus, lead to potentially serious numerical instabilities. The generalized-$\alpha$ method appears to be effective in improving numerical stability within the partitioned FSI context, and therefore is used to facilitate time advancement in the structural domain. The generalized-$\alpha$ method was developed by Chung and Hulbert [54] to ameliorate spatially unresolved high frequency responses by introducing maximal numerical dissipation in high frequency structural modes, while, safeguarding solution accuracy in the low frequency structural modes. The generalized-$\alpha$ formulation is expressed as followed,

$$M_\varepsilon \ddot{d}_{n+1-\alpha_m} + (1 - \alpha_f)K_\varepsilon \Delta d = R_{n+1-\alpha_f}^{t+\Delta t},$$

(14)

where

$$\ddot{d}_{n+1-\alpha_m}^{t+\Delta t} = (1 - \alpha_m)\ddot{d}_{n+1}^{t+\Delta t} + \alpha_m \dot{d}_{n}^{t+\Delta t}.$$  

(15)
and

\[
R_{n+1}^{t+\Delta t} = (1 - \alpha_f)R_{n+1}^{t+\Delta t} + \alpha_f R_{n}^{t+\Delta t}.
\]  

(16)

In the equations above, \(M_s\) is the mass matrix, \(K_s\) is the stiffness matrix, \(\ddot{d}\) is the acceleration vector, \(d\) is the displacement vector, and \(R\) is the residual force vector.

Integration constants \(\alpha_f\) and \(\alpha_m\) in Equation 14 to 16 are expressed as

\[
\alpha_f = \frac{\rho_\infty}{\rho_\infty + 1} \quad \text{and} \quad \alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1},
\]

(17)

where \(\rho_\infty\) is the spectral radius – a measure of numerical dissipation in the high frequency limit. \(\rho_\infty = 1\) indicates no numerical dissipation and the algorithm recovers the standard Newmark implicit scheme, while, \(\rho_\infty = 0\) indicates “asymptotic annihilation”: meaning, the high-frequency response is entirely damped out after one time step. In the present work, we set \(\rho_\infty\) to

\[
\rho_\infty = \begin{cases} 
0.4, & \text{if } W_n \geq 0.75 \text{ m/s}, \\
0.2, & \text{otherwise},
\end{cases}
\]

(18)

where \(W_n\) represents the normal slamming impact velocity. The maximum number of iterations for each time step is set to 100.

### 3.2 Fluid solver

The fluid responses in the present work are modeled using an open source finite volume solver, OpenFOAM 1.6-ext [43]. OpenFOAM includes an extensive range of specialized solvers designed for specific continuum mechanics problems involving chemical reactions, turbulence, electromagnetics, multiphase, etc [43, 44, 45, 46]. In particular, the \texttt{interDyMFoam} solver is used as the multiphase fluid dynamics solver adopted in this work.

#### 3.2.1 Mesh motion solver

In contrast with the structural domain, where the structure is discretized into DKT shell elements, the fluid domain within the FSI system is discretized into hexahedral control volumes/cells. Since the DKT shell is a vertex-based element, while the hexahedral cell is a cell-centered volume, this gives rise to complications, if one wishes to transfer field variables between the structural and fluid systems (this constitutes an important aspect of FSI analyses). Due to the need for communication between FEM and FVM meshes through the FSI transmission conditions, the FEM decomposition library [43, 55] within OpenFOAM 1.6-ext is particularly attractive in terms of both communicating with finite element meshes and tracking fluid mesh motions. And therefore the FEM decomposition framework is adopted in the present work to preserve mesh quality and mesh motion boundedness within the FSI solutions.

The mesh motion and diffusivity in the fluid domain is managed and maintained by a Laplacian operator [42, 55, 56]. In particular, a mesh diffusion variable, \(\gamma^f\), is introduced in the Laplace operator, to update the locations of mesh points. In this work, a linear distance-based method is chosen (i.e., \(\gamma^f = \frac{1}{y}\) where \(y\) is the distance between the cell center and the closest moving boundary).

#### 3.2.2 Pressure-velocity coupling

Due to the mutual linear dependency between the velocity and pressure fields in the incompressible Navier-Stokes system, inter-equation coupling between the velocity and pressure fields are required. The \texttt{PIMPLE} [57, 58, 59] procedure is adopted in the present work for computing the pressure-velocity fields within the fluid domain. The \texttt{PIMPLE} algorithm adopts the procedure of \texttt{PISO} [60, 45] algorithm and solves each time step as a steady-state problem, by introducing under-relaxation to the pressure and velocity fields. The inclusion of under-relaxation amplifies the influence of owner cells –
which increases the dominance of the diagonal in the system – thus leads to better convergence in the numerical solutions. In this work, we define the maximum PIMPLE iterations to be 4.

### 3.2.3 Numerical schemes and iterative solvers

OpenFOAM offers numerous discretization schemes [61]; Table 1 includes a list of the temporal and spatial discretization schemes used to estimate the incompressible Navier-Stokes equation in this work. In particular, a surface normal corrected scheme (a scheme which is used to ensure surface orthogonality) is applied to explicitly correct for non-orthogonality between two adjacent fluid cell faces, as to maintain a second-order accuracy in the solution. Non-orthogonality in the fluid mesh is determined by the angle between the vector that connects the cell centroids of two neighboring control volumes and the vector normal to the shared surface. It is imperative to keep non-orthogonality under $70^\circ$ to maintain a stable solution [61].

| Operators     | Terms                         | Numerical schemes                      |
|---------------|-------------------------------|----------------------------------------|
| Time          | $\frac{d}{dt}$                | Implicit Euler                         |
| Gradient      | $\nabla \mathbf{u}$          | CellLimited Gauss linear 1.0           |
|               | $\nabla \mathbf{p}$          | CellLimited Gauss linear 1.0           |
|               | $\nabla \alpha$              | CellLimited Gauss linear 1.0           |
|               | $\nabla \cdot (\rho \mathbf{u} \cdot \mathbf{u}^f)$ | Gauss linear Upwind                   |
| Divergence    | $\nabla \cdot (\mathbf{u}_{\alpha} \cdot \mathbf{u}^f)$ | Gauss interface Compression            |
|               | $\nabla \cdot (\mathbf{u}_{\alpha} \cdot \mathbf{u}^f)$ | Gauss vanLeer [62]                    |
| Laplacian     | $\nabla^2$                    | Gaussian linear Corrected             |
| Surface normal gradient | $\nabla \perp$               | Corrected                              |
| Interpolation |                                | Linear                                 |

The iterative solvers used in this work for each fluid variable are specified in Table 2. Each iterative solver can be accompanied by a preconditioner and smoother, to improve numeric stability and solution smoothness. For more details on the iterative solvers, one may refer to [63, 64, 65, 66] and the OpenFOAM User Manual [61]. In the present work, we set the convergence tolerances for pressure, velocity, mesh motion to $10^{-7}$; under-relaxation factors of 0.6 and 0.7 are applied to the pressure and velocity fields respectively.

| Operators     | Numerical schemes                      |
|---------------|----------------------------------------|
| p$^f$ correction | Geometric Agglomerated Algebraic MultiGrid (GAMG) with Diagonal Incomplete Cholesky smoother |
| p$^f$ final    | Pre-conditioned conjugate gradient (PCG) with GAMG pre-conditioner and DICGaussSeidel smoother |
| u$^f$          | Gauss Seidel smooth solver             |
| $\alpha$      | Preconditioned bi-conjugate gradient with Diagonal incomplete LU preconditioner |
| FEM mesh motion | Conjugate Gradient with Cholesky preconditioner |

### 3.3 Coupler

In terms of communications between the fluid and structural solvers, we use the Coupler (which was originally developed by the Navel Surface Warfare Center Carderock Division [48] and later modified by the Authors, for use in the present work), to facilitate information passing along the fluid-structure interface. The grid-to-grid mapping and mesh updating is managed using a nonconforming mesh projecting technique [67]. Within the Coupler, inverse distance weighting interpolation (IDW) [68]
is used to convert hydrodynamic pressure, located at the center of each fluid patch (one face of the finite volume), along the interface, into point loads, before they are projected onto the structural mesh. While the fixed-point under-relaxation approach (also known as the Aitken’s relaxation method [30]) was a popular coupling algorithm in partitioned FSI analysis [31, 69] due to its simplicity, it is rather computationally expensive compared to other coupling schemes such as the iterative Quasi-Newton method (IQN-ILS) and Interface-GMRES [70, 71]. As such, the IQN-ILS method is used to ensure solution convergence on the fluid-structure interface within each FSI time step. Detailed descriptions of the algorithm and functionality within the Coupler can be found in [72].

4 Experimental validation

In this section, we discuss validation results for our proposed partitioned FSI modeling framework. In particular, we compare our FSI simulation results against experimental work detailed in [19]. The organization of this section is as follow: 1) we discuss the experimental setup of the oblique free surface impact experiments; 2) we provide details of the FSI computational models used in the present work; and 3) we present mesh convergence and FSI validation results.

4.1 Experiment configuration

The flexible plate oblique impact experiments are conducted at the University of Maryland in an open channel filled with water. The water channel measures 13.41 m long, 2.44 m wide, and 0.96 m deep, as shown in Figure 3. An aluminum plate is mounted to a vertical carriage, which is attached to an electric servo motor. The plate mounting instrument offers flexibility for the plate to be adjusted to the desired angle of attack and deadrise angle (denoted as $\alpha$ and $\beta$ respectively). The flexible plates in this work are oriented at $10^\circ$ angle of attack ($\alpha = 10^\circ$) and zero deadrise ($\beta = 0^\circ$). The horizontal motion of the plate is driven by a horizontal carriage attached to two hydraulic servo motors, located at opposite ends of the tank. Both vertical and horizontal carriages are controlled by a computer-based feedback system that is connected to sensors, so that it precisely drives the position of the horizontal and vertical carriages. This system is used to control the impact velocities with which the plates strike in the water free surface (i.e., the impact velocities).

![Side view and End view of the open channel testing facility at which the flat plate slamming experiments are conducted](image)

Figure 3: The side view and end view of the open channel testing facility at which the flat plate slamming experiments are conducted [19].

The total impact force acting on the plate during the slamming process is recorded by force sensors that are attached at the four corners of the plate, as displayed in Figure 4. The rotation bearings shown in Figure 3 (very stiff rails made of aluminum act as rigid stiffeners along the width of the plate) is modeled as rigid offset pinned condition in our FSI models (in order to properly capture the
subtle kinematic effect). The deflection of the plate is measured by the five displacement gauges attached along the centerline of the plate, each 169 mm apart, as shown in Figure 4b.

![Diagram](image)

(a) Plate mounting system  
(b) Displacement gauge locations

Figure 4: Locations of the force sensors and displacement gauges for tracking the hydrodynamics and hydroelastics responses of the flexible plate slamming experiments [19].

### 4.2 Numerical models

#### 4.2.1 Fluid domain

The fluid domain discretization is generated using a three-dimensional finite element generator, Gmsh [73]. Then, we convert the Gmsh model to a hexahedral mesh for OpenFOAM by issuing the `gmshToFoam` command in OpenFOAM. The fluid model considers a truncated geometry of the open water channel to ease computational demands, while still capturing all salient physics. The water channel is modeled as a rectangular box measuring 3 m long, 1.624 m wide, and 2.12 m tall as presented in Figure 5. The rectangular box is partially filled with water (in red) that is 0.96 m deep. A “placeholder” (i.e., “cut-out” within fluid domain) for the flexible plate structural model is oriented at a 10° angle of attack, and situated at 0.1 m above the water free surface (the white layer in between blue and red).

We assume wall boundary condition (i.e., no slip boundary for velocity and zero gradient for pressure) on all boundaries enclosing the fluid domain, except the top plane and the left plane, looking from the end view, which we assume to be a standard atmosphere and symmetry wall boundary condition, respectively. The surfaces of the placeholder, enclosing the flexible plate model, assume moving wall velocity and fixed flux pressure conditions. The densities of the water and air are taken as $\rho_w = 1000 \text{ kg/m}^3$ and $\rho_a = 1 \text{ kg/m}^3$ respectively. Furthermore, the kinematic viscosities of the water and air are given values of $\nu_w = 1 \times 10^{-6} \text{ m/s}^2$ and $\nu_a = 1.48 \times 10^{-5} \text{ m/s}^2$, respectively.

#### 4.2.2 Structural domain

The structural model of the flexible plate is shown in Figure 6. The plate is 1.079 m long and 0.406 m wide. There are two rectangular beams on two ends of the plate, to represent the high strength aluminum rails shown in Figure 4b. The rectangular beam used to model the rails measures 0.016 m wide and 0.035 m tall. Both the flexible plate and the rigid rails are discretized using DKT shell elements: the latter being used to create the rigid offset pinned condition mentioned previously. As a result of using DKT shells, the thickness of the structure is treated as a shell cross-sectional property; thus, we place the shell elements at the mid-plane of the physical plate model (represented by the red dotted lines) to properly capture the geometry of the validation experiments. A pinned-pinned
boundary condition is applied to the top ends of the rectangular beams. The flexible plate has Young’s modulus, $E_p = 68.9$ GPa, Poisson’s ratio, $\nu_p = 0.35$, and mass density, $\rho_p = 2700 \text{ kg/m}^3$. The rectangular beams have Young’s modulus, $E_b = 689$ GPa, Poisson’s ratio, $\nu_b = 0.35$, and mass density, $\rho_b = 8050 \text{ kg/m}^3$. We assign a high Young’s modulus to the beams to ensure rigidity of the boundary rails. A summary of the engineering properties of the FSI model is provided in Table 3.

Table 3: Engineering properties of the FSI model for flexible plate slamming tests.

| Fluid Properties | Structural Properties |
|------------------|-----------------------|
| Water density, $\rho_w$ | 10$^3$ kg/m$^3$ | Plate density, $\rho_p$ | 2700 kg/m$^3$ |
| Air density, $\rho_a$ | 1 kg/m$^3$ | Beam density, $\rho_b$ | 8050 kg/m$^3$ |
| Water viscosity, $\mu_w$ | 10$^{-6}$ m/s$^2$ | Plate Young’s modulus, $E_p$ | 68.9 GPa |
| Air viscosity, $\mu_a$ | $1.48 \times 10^{-5}$ m/s$^2$ | Beam Young’s modulus, $E_b$ | 689 GPa |
| | | Plate Poisson ratio, $\nu_p$ | 0.35 |
| | | Plate Poisson ratio, $\nu_b$ | 0.35 |

4.3 FSI model validation

In the present work, we perform a total of 24 simulated slamming impact tests on very flexible, moderately deformable, and approximately rigid aluminum plates, each characterized by plate thickness 0.00635 m, 0.00795 m, and 0.0127 m respectively. (thicker plate implies more resistance}
to deformation when subjected to bending.) \[19\] The rigidity of the plate is expressed with the usual flexural rigidity, as

\[
D = \frac{E_s h^3}{12(1 - (\nu_s)^2)},
\]

(19)

where \(h\) is the plate thickness. The flexural rigidity of the three plates considered, in the order of increasing thickness, are 1675 \(Pa\cdot m^3\), 3288 \(Pa\cdot m^3\), and 13403 \(Pa\cdot m^3\) respectively.

In the slamming test, the plate is subjected to a constant impact velocity in the direction normal to the plate surface (the impact velocity deviates by a maximum of 1.35% in the experiment as a result of the closed-loop control system setup). The normal impact velocity can be be divided into two components, namely, the forward and downward velocities. The various slamming velocity conditions are shown in Figure 7 (Laminar model is used in all FSI simulations in the present work). Therein, \(U\) and \(W\) denote horizontal and downward impact velocities, respectively. \(W_n\) denotes impact velocities orthogonal to the plate surface, expressed as \(W_n = U\sin(\alpha) + W\cos(\alpha)\), where \(\alpha\) is the angle of attack (in this work, \(\alpha = 10^\circ\)). In each test, we compute the displacements at the plate center and the total slamming impact force normal to the plate surface. Subsequently, we compare our FSI model results against experimental data.

![Figure 7: The U/W ratios and normal impact velocities of the slamming tests from [19] that are considered in the present work.](image)

All numerical simulations are performed on the Centennial high performance computing system, located at the U.S. Army Research Laboratory (ARL) DoD Supercomputing Resource Center (DSRC). Each compute node contains 40 Intel Xeon E5-2698v4 Broadwell cores, with a 2.2GHz clock speed and 128GB RAM per node. All FSI simulations are run in serial on one core (the large plate deformation in some slamming cases leads to inconsistent fluid patch area between adjacent cells among processors – this causes simulation failures when the FSI analyses are run in parallel).

4.3.1 Mesh convergence tests

In order to arrive at the most suitable FSI model for validation, we perform mesh convergence tests to identify a suitable mesh sizes in both the structural and fluid domains. We begin with a structural dynamics mesh convergence study. A uniformly distributed load, \(P_n = 5000 N/m^2\), is applied perpendicularly to the surface of the moderately deformable plate (plate thickness 0.00795 m) for
0.1 sec with $\Delta t = 0.0005$ sec. The structural model is discretized into 64, 128, and 256 DKT shell elements for the mesh refinement study. The normal displacements at the plate center is obtained using the standard Newmark implicit scheme. Based on the results shown in Figure 8, we observe that 64 DKT shell elements slightly underpredicts the displacements, while displacement results with 128 and 256 DKT shell elements are virtually identical. Since the 256 DKT shell model does not provide much improvement in displacement predictions, compared to the 128 DKT shell model, we use 128 DKT shell model for the FSI validation tests.

![Figure 8: Normal displacements at plate center of the flexible plate model using 64, 128, and 256 DKT shell elements for time $t = [0, 0.1]$ sec.](image)

After we obtain a satisfactory structural model, we focus on the fluid domain, and the OpenFOAM mesh discretization that offers the best FSI results in both the plate displacement and total normal impact force. We focus on refining the mesh in the region bounded by the dotted black line in Figure 9 (the physical behavior within the refinement box is most relevant to the hydrodynamics and hydroelastics histories of the slamming test.) Furthermore, the mesh size away from the refinement box is adjusted correspondingly, based on appropriate aspect ratios (i.e., the mesh is proportionally coarsened w.r.t it’s distance from the plate). The fluid mesh within the refinement box is discretized into 348, 3120, and 21218 fluid cells respectively, for the mesh refinement study.

![Figure 9: A snapshot of the FSI simulation. (a) The region enclosed by the dotted white line is subjected to grid refinement to identify a suitable mesh size for the FSI slamming analysis. The dimension of each fluid cell within the refinement box is: $dx = 0.018$ m, $dy = 0.068$ m, and $dz = 0.015$ m. (b) The fluid mesh resolution of the medium mesh in the region near the plate.](image)

We perform the mesh refinement study on the fluid model using a slamming condition of $U/W = 8.33$ and $W_n = 1.386$ m/s. The plate is assumed rigid in this case. The total normal impact force acting on
the plate, for the three different refinements, are shown in Figure 10. In Figure 10, $dz/Z \in [-0.2, 0]$ suggests the plate is in air – the plate is reaching toward the water free surface as $dz/Z$ increases. Positive $dz$ represents the total vertical distance the plate has traveled in water, and $Z$ represents the total vertical distance from the lower to the upper edge of the plate. In other words, $dz/Z \in [0, 1]$ represents the fraction of the plate submerged within the water. Since the free surface in the present work is modeled by linearly interpolating the fluid properties between the air and water using a VOF formulation, the non-zero fluid density of the free surface introduces additional force to the plate as it approaches the free surface. As such, non-zero forces are observed for zero submergence in the range closes to $dz/Z = 0$ in Figure 10. However, as the mesh is refined, the artificial forces at the free surface are reduced.

It is observed that all three meshes produce somewhat similar force histories. Figure 10b shows the pressure profile along the plate for all three meshes – the finest mesh shows excellent performance in resolving the sharp pressure peak. The CPU times for the three meshes, with increasing level of refinement, are 1071 sec, 9543 sec, and 61702 sec, respectively. Given that the medium mesh performs nearly as well as the fine mesh but requires much less computational time (which is a limitation in the present work) and that the total impact force is more relevant to the present work, the fluid mesh with 3120 control volumes within the refinement box is chosen for carrying out all FSI analyses.

Figure 10: (a) Total normal impact forces of the fluid model with 348, 3120, 21218 control volumes within the refinement box; (b) the pressure profile on all three plates with varying mesh densities.

4.3.2 Validation results

In the following, we will present three sets of validation results. First, we present validation results for oblique slamming scenarios with $W_n = 1.386$ m/s and $U/W = 8.33, 6.28, 5.50$, and 4.50; higher $U/W$ ratio indicates higher forward velocity ($U$) but lower downward velocity ($W$). The $U$ and $W$ of the slamming tests in this validation set are listed in Table 4. Displacement and normal force responses for plate thickness 0.00635 m, 0.00795 m, and 0.0127 m are shown in Figure 11. In Figure 11a, c, and e, we observe that thinner plates have higher deflections at the plate center. This observation agrees with our intuition. (as the plate gets thinner, it has lower flexural rigidity, and therefore it admits larger deformation.) Figure 11a and 11e imply that plate deformation is independent of $U/W$ ratios. This means for plates with thickness 0.00795 m, or larger (i.e., flexural rigidity 3288 Pa · m$^3$ or higher), forward impact velocity has negligible influence on plate deformations, as long as the normal impact velocity is fixed. However, the observation mentioned does not apply to the case with plate thickness of 0.00635 m. For such highly flexible plates, higher $U/W$ leads to larger plate deformation, even if the plates are subjected to the same normal impact.
Table 4: Forward and downward velocities of slamming tests with \( W_n = 1.386 \) m/s. \( F_r \) denotes the Froude number expressed as \( F_r = \frac{W_n}{\sqrt{gL}} \), where \( g \) is gravity and \( L \) is the length of the plate.

| \( W_n \) (m/s) | \( U/W \) | \( U \) (m/s) | \( W \) (m/s) | \( F_r \) |
|-----------------|----------|--------------|-------------|---------|
| 1.386           | 8.33     | 4.75         | 0.57        | 0.426   |
| 1.386           | 6.28     | 4.19         | 0.668       | 0.426   |
| 1.386           | 5.50     | 3.93         | 0.715       | 0.426   |
| 1.386           | 4.50     | 3.53         | 0.785       | 0.426   |

In Figure 11b, d, and f, it is observed that the normal impact force histories resulting from the slamming impacts are almost identical between plate thickness 0.00795 m and 0.0127 m; these two plates experience linear force loading within the interval \( dz/Z = [0, 0.8] \) and forces unloading within \( dz/Z = [0, 0.85] \), and the impact force levels off afterward. Investigations into the inconsistency in structural behaviors with regard to plate rigidity are discussed in Section 5, wherein we identify the plate thickness at which the \( U/W \) ratio becomes important to plate deformations and normal impact force behaviors.

The \( L_2 \) relative error norms in impact forces and plate deformations of the first set of FSI validation tests are presented in Table 5; these norms are defined as

\[
L_2 \text{ relative error norm} = \sqrt{\frac{\sum_{i=0}^{N} (S_{exp,i} - S_{fsi,i})^2}{\sum_{i=0}^{N} S_{exp,i}^2}} \quad (20)
\]

where \( S_{exp,i} \) represents the \( i^{th} \) index of the experiment solution vector, \( S_{fsi,i} \) represents the \( i^{th} \) index of the FSI solution vector, and \( N \) represents the total number of sampled points. The \( L_2 \) error norms in impact forces among the three plates fall within the interval \([0, 0.201]\), with a median of 0.164 and a mean of 0.165. Likewise, the \( L_2 \) error norms in displacements among the three plates occur within the interval \([0, 0.222]\), with a median of 0.152 and a mean of 0.149. The \( L_2 \) error norms are consistent in this validation set and are only a small fraction of the typical impact forces and plate deflections; thus indicating favorable agreements between the models and experiments.

Table 5: \( L_2 \) relative error norms in slamming impact forces and plate deformations from slamming tests with \( W_n = 1.386 \) m/s.

| Plate thickness (m) | \( L_2 \) error, impact force | \( L_2 \) error, displacement |
|---------------------|-----------------------------|-----------------------------|
|                     | 0.00635 | 0.00795 | 0.0127 | 0.00635 | 0.00795 | 0.0127 |
| U/W 8.33            | 0.201   | 0.170   | 0.161  | 0.132   | 0.149   | 0.222  |
| U/W 6.28            | 0.178   | 0.149   | 0.159  | 0.155   | 0.159   | 0.128  |
| U/W 5.50            | 0.171   | 0.146   | 0.160  | 0.158   | 0.146   | 0.104  |
| U/W 4.50            | 0.167   | 0.150   | 0.156  | 0.162   | 0.160   | 0.113  |

In the second set of FSI validation tests, we present validation results of slamming cases with \( U/W = 8.33 \) and \( W_n = 1.386, 1.313, 1.167, \) and 0.875 m/s; higher \( W_n \) indicates higher \( U \) and \( W \). The \( U \) and \( W \) of the slamming tests are listed in Table 6. In Figure 12a, c, and e, we observe that plate deflection increases with increasing \( W_n \) and decreasing plate thickness. This observation is reasonable and agrees with our intuition. In Figure 12b, d, and f, once again, we observe nearly identical force histories in 0.00795 m and 0.0127 m plates, wherein the plates experience bilinear force loading-unloading behaviors. In Figure 12b, we observe that the 0.00635 m plate experiences nonlinear force loading in \( W_n = 1.386 \) m/s and \( W_n = 1.313 \) m/s, while in \( W_n = 1.167 \) m/s and \( W_n = 0.875 \) m/s, the 0.00635 m plate follows a similar bilinear force pattern as the two stiffer plates. These diverging structural behaviors in Figure 12 suggest that the plate’s oblique slamming response is related to its rigidity and the intensity of impact.
Figure 11: Plate deflection and normal impact force histories of the highly flexible (with plate thickness 0.00635 m), moderately deformable (with plate thickness 0.00795 m), and nearly rigid (with plate thickness 0.0127 m) plates subjected to slamming impacts with $W_n = 1.386$ m/s.

Table 6: Forward and downward velocities of slamming tests with $U/W = 8.33$.

| $W_n$ (m/s) | $U/W$ | $U$ (m/s) | $W$ (m/s) | $F_c$ |
|------------|-------|-----------|-----------|-------|
| 1.386      | 8.33  | 4.75      | 0.57      | 0.426 |
| 1.313      | 8.33  | 4.50      | 0.54      | 0.404 |
| 1.167      | 8.33  | 4.00      | 0.48      | 0.359 |
| 0.875      | 8.33  | 3.00      | 0.36      | 0.269 |

The $L_2$ relative error norms in impact forces and plate deformations of the second set of FSI validation tests are presented in Table 7. The $L_2$ error norms in impact forces among the three plates fall within the interval $[0.161, 0.226]$, with a median of 0.180 and a mean of 0.187. The $L_2$ error norms in displacements among the three plates occur within the interval $[0.104, 0.379]$, with a median of 0.1885 and a mean of 0.205. The second validation set yields higher $L_2$ error norms, compared to the first FSI validation set. In Table 7 we observe that, in general, the $L_2$ error norm increases with
Figure 12: Plate deflection and normal impact force histories of the highly flexible (with plate thickness 0.00635 m), moderately deformable (with plate thickness 0.00795 m), and nearly rigid (with plate thickness 0.0127 m) plates subjected to slamming impacts with $U/W = 8.33$.

-decreasing $W_n$. Interestingly, we observe in Figure 12 that the discrepancies between the FSI and experimental results are somewhat consistent among all four slamming tests. Because lower $W_n$ yields lower impact forces and plate deformations, any modeling errors become more significant, as a percentage of response, in these cases.

In the last set of FSI validation tests, we present validation results for slamming tests with $U/W = 0$ and $W_n = 0.875$, 0.584, 0.438, and 0.292 m/s. These cases are subjected to $W$ only; higher $W_n$ indicates higher $W$. The $U$ and $W$ of the slamming tests are listed in Table 8. Displacement and normal force responses for plate thickness 0.00635 m, 0.00795 m, and 0.0127 m are shown in Figure 13. Similarly, we observe that the higher impact velocity and smaller plate thickness lead to higher plate deformation. In Figure 13b, d, and f, we observe bilinear force behaviors in $W_n = 0.875$ m/s and $W_n = 0.584$ m/s, while the forces are monotonically increasing in $W_n = 0.438$ m/s and $W_n = 0.292$ m/s.
Table 7: $L_2$ relative error norms in slamming impact forces and plate deformations from slamming tests with $U/W = 8.33$.

| Plate thickness (m) | $L_2$ error, impact force | $L_2$ error, displacement |
|--------------------|----------------------------|----------------------------|
| $W_n$ 1.386        | 0.00635 0.00795 0.0127     | 0.00635 0.00795 0.0127     |
| $W_n$ 1.313        | 0.177 0.182 0.171          | 0.182 0.168 0.205          |
| $W_n$ 1.167        | 0.166 0.194 0.178          | 0.228 0.124 0.195          |
| $W_n$ 0.875        | 0.201 0.226 0.219          | 0.379 0.104 0.377          |

Table 8: Forward and downward velocities of slamming tests with $U/W = 0$.

| $W_n$ (m/s) | U/W | U (m/s) | W (m/s) | $F_p$ |
|-------------|-----|---------|---------|-------|
| 0.875       | 0   | 0.889   | 0.269   |       |
| 0.584       | 0   | 0.593   | 0.179   |       |
| 0.438       | 0   | 0.445   | 0.135   |       |
| 0.292       | 0   | 0.296   | 0.090   |       |

The $L_2$ relative error norms in impact forces and plate deformations of this last set of FSI validation tests are presented in Table 9. The $L_2$ error norms in impact forces among the three plates fall within the interval $[0.189, 0.348]$, with a median of 0.292 and a mean of 0.277. The $L_2$ error norms in displacements among the three plates occur within the interval $[0.091, 0.670]$, with a median of 0.270 and a mean of 0.313. Similar to the second set of FSI validation tests, it is observed that lower $W_n$ leads to higher $L_2$ errors.

Table 9: $L_2$ relative error norms in slamming impact forces and plate deformations from slamming tests with $U/W = 0$.

| Plate thickness (m) | $L_2$ error, impact force | $L_2$ error, displacement |
|--------------------|----------------------------|----------------------------|
| $W_n$ 0.875        | 0.189 0.215 0.208          | 0.331 0.091 0.136          |
| $W_n$ 0.584        | 0.252 0.274 0.253          | 0.511 0.102 0.221          |
| $W_n$ 0.438        | 0.310 0.309 0.319          | 0.621 0.162 0.318          |
| $W_n$ 0.292        | 0.318 0.329 0.348          | 0.670 0.151 0.446          |

Figure 14 and 15 provide snapshots of the slamming events for $W_n = 0.875$ m/s and 0.292 m/s within the interval $dz/Z = [0.7, 1]$, as well as the corresponding pressure distributions on the plate; to better understand the discrepancy in force behaviors in Figure 13. As the plate is submerged into the water, it experiences a highly concentrated pressure at the spray root (i.e., the intersection point between the water free surface and the plate); the concentrated pressure is represented by red within the pressure distribution contour in Figure 14e and f. As shown in Figure 14f, this highly concentrated pressure leaves the plate at $dz/Z = 0.8$, as the spray root passes the leading edge of the plate. Once the high pressure leaves the plate, the fluid pressure on the plate is substantially reduced; thus leading to force unloading in the case $W_n = 0.875$ m/s.

Contrasting to the $W_n = 0.875$ m/s case, the $W_n = 0.292$ m/s case displays a different fluid behavior, as shown in Figure 15. Since the downward velocity in $W_n = 0.292$ m/s is much lower, the impact resulting from the plate entering the water free surface does not generate enough of the momentum for the water to splash away from the trailing edge. Rather, the water from trailing edge splashes over onto the plate as shown in Figure 15a to d. The water splash over induces a second pressure source onto the plate; this time counteracting the slamming pressure. As a result, force unloading is offset by the second high pressure impact in the $W_n = 0.292$ m/s case.

The pressure histories of the $W_n = 0.875$ m/s and $W_n = 0.292$ m/s cases on the 0.00795 m plate are presented in Figure 16. The pressure histories are extracted at five different locations along the
Figure 13: Plate deflection and normal impact force histories of the highly flexible (with plate thickness 0.00635 m), moderately deformable (with plate thickness 0.00795 m), and nearly rigid (with plate thickness 0.0127 m) plates subjected to slamming impacts with $U/W = 0$.

plate as shown in Figure 16a (the black dots signal the locations of the pressure probes). In Figure 16b, we see that, due to the water splashing from the trailing edge, the pressure in all five locations of the plate is monotonically increasing. This leads to the monotonically increasing force behavior in Figure 13 in $W/n = 0.292$ m/s. While in Figure 16c, pressure unloading is observed where the pressure drops after the spray root passes the locations of the pressure probes. This reflects the bilinear impact force behavior shown in Figure 13 in $W/n = 0.875$ m/s.

In this section, we have presented three sets of validation tests. The first testing set examines the influence of $U/W$ ratio on structural behaviors in cases with $W/n = 1.386$ m/s; the second testing set investigates the effects of $F_r$ on cases with $U/W = 8.33$; and the third testing set studies the structural behaviors of the plate without influences of forward velocity, $U$. In all these validation tests, our proposed FSI solver successfully captures the global force and plate deformation features in these cases. Furthermore, the FSI solver offers satisfactory agreements with experimental displacement data, especially for the 0.00795 m and 0.0127 m plates (the relative errors in the present work for
Figure 14: Snapshots of the water free surface evolution and the corresponding pressure distribution over the plate from $dz/Z = [0.7, 1.0]$ of the slamming test with $W_n = 0.875$ m/s.

Figure 15: Snapshots of the water free surface evolution and the corresponding pressure distribution over the plate from $dz/Z = [0.7, 1.0]$ of the slamming test with $W_n = 0.292$ m/s.

Figure 16: The pressure histories of slamming impacts with $W_n = 0.292$ m/s and $W_n = 0.875$ m/s. Figure (a) displays the locations at which the pressures are measured in both cases.

forces are comparable to [74] and our displacement errors are substantially lower). Validation results of all slamming tests listed in Figure 7 can be found in [A].
5  Additional FSI investigations into oblique slamming impact

In this section, we use our validated FSI solver to carry out a series of complementary FSI slamming analyses, in an effort to elucidate some missing physical insight from the flexible plate oblique impact experiments. Specifically, we apply our validated FSI simulation method to the study of global trends that arise within the engineering context of oblique slamming of flexible plates.

5.1 The effect of horizontal velocity

In the previous section, we presented and discussed the structural behaviors of the plates under slamming cases with fixed \( W_n, U/W, \) and \( U \). In this section, we discuss FSI results with fixed \( W \), to offer a more comprehensive picture of how velocities influence the normal impact force and plate deformation. The FSI analyses in this section focus on the 0.00795 m plate thickness case. We perform a total of 12 FSI slamming simulations; the slamming tests are divided into three groups with \( W = 1.5 \) m/s, \( W = 1 \) m/s, and \( W = 0.5 \) m/s, respectively. The \( U \) and \( W \) of the slamming tests are listed in Table 10. The plate deflection and impact force histories are shown in Figure 17. As expected, by keeping \( W \) fixed, higher \( U \) results in higher \( W_n \) and therefore leads to higher plate deflection and impact force. In addition, we observe in Figure 17a-c that the plate reaches its peak deformation faster (indicated by smaller \( dz/Z \) value) and with lower \( W \). Similarly, the impact force also reaches its peak sooner, with lower \( W \). However, since the plate deformation is affected by the pressure distribution on the plate, rather than the normal impact force (the sum of pressure on the plate), the time of peak plate deformation, and peak force, do not necessarily coincide.

|          | Group 1 |       | Group 2 |       | Group 3 |       |
|----------|---------|-------|---------|-------|---------|-------|
| \( W_n \) (m/s) | 2.172   | 1.998 | 1.824   | 1.650 | 1.679   | 1.506 |
| \( U/W \)        | 2.67    | 2.00  | 1.33    | 0.67  | 4.00    | 3.00  |
| \( U \) (m/s)    | 4.00    | 3.00  | 2.00    | 1.00  | 4.00    | 3.00  |
| \( W \) (m/s)    | 1.50    | 1.50  | 1.50    | 1.50  | 1.00    | 1.00  |
| \( F_r \)        | 0.668   | 0.614 | 0.561   | 0.506 | 0.516   | 0.463 |

Figure 17: Plate deflection and normal impact force histories of the 0.00795 m plate subjected to slamming impacts with \( W = 1.5, 1, \) and 0.5 m/s.

In naval application, peak deflections and impact forces are of most interest, as these engineering responses directly connect to the adequacy in design. The peak deflections and normal impact forces, w.r.t \( W_n \), from the 12 FSI slamming simulations are presented in Figure 18 to better understand the
relations between peak deflection/normal impact force and $W_n$. In Figure 18, we observe positive relation between maximum plate deflections/normal impact forces and $W_n$; the maximum plate deflection increases in an approximately quadratic manner, with increasing $W_n$, while the maximum impact force appears to increase linearly with increasing $W_n$.

5.2 The effect of U/W ratios and plate flexural rigidity

Motivated by the validation results from the first set of FSI validation tests, the influence of U/W ratios on plates in relation to their rigidities is now examined. We prescribe slamming conditions $W_n = 1.386 \text{ m/s}$ and $U/W = 8.33, 6.28, 5.50,$ and $4.50$ on aluminum plates with thickness 0.00675 m, 0.00715 m, and 0.00755 m, respectively. Although the cases of 0.00795 m and 0.0127 m thick plates appear insensitive to $U/W$ ratio, we observed that higher $U/W$ ratio yields higher deformations on the 0.00675 m thick plate. The plate deflection and normal impact force of the 0.00675 m, 0.00715 m, and 0.00755 m thick plates are presented in Figure 19. In the figures, we observed that the deformation discrepancy between $U/W$ increases with decreasing plate thickness. While the deviation between impact force seems insignificant in each of the plates, it appears that the thicker plate precipitates a steeper force unloading.

Figure 19: Plate deflection and normal impact force histories of flexible plates with thickness 0.00675 m, 0.00715 m, and 0.00755 m subjected to slamming impacts with $W_n = 1.386 \text{ m/s}$.
The plate deformation as a function of flexural rigidity for all six plates (three from FSI validation test and three from current modeling) is presented in Figure 20. The flexural rigidity of each plate is computed following Equation 19 and used in the plots of Figure 20 where an inverse relationship between plate deflection and flexural rigidity is observed, as expected. The relationship can be divided into two regions, demarcated by a flexural rigidity of $3200 \text{ Pa} \cdot \text{m}^3$. Within Region A (i.e., flexural rigidity $\leq 3200 \text{ Pa} \cdot \text{m}^3$), plate deformation appears to decrease quadratically with increasing flexural rigidity. On the other hand, within Region B (i.e., flexural rigidity $> 3200 \text{ Pa} \cdot \text{m}^3$), plate deformation appears to decrease approximately linearly with increasing flexural rigidity. Moreover, the influence of $U/W$ is negligible in Region B, while visible differences in deflection are observed in Region A, between various $U/W$ (the influence of $U/W$ is more prominent with smaller flexural rigidity). Note that the threshold $3200 \text{ Pa} \cdot \text{m}^3$ is purely phenomenological, based on the flexible plate geometry in the present work. More extensive studies are needed to identify a general threshold for differentiating the linear and nonlinear region in the deformation vs flexural rigidity relation.

![Figure 20: Comparison of plate deflection among various $U/W$ as a function of flexural rigidity.](image)

To summarize this section, a series of complementary FSI slamming simulations are performed to better understand the effects of downward impact velocity, $W$, and plate flexural rigidity on the structural behaviors of the flexible plates (i.e., we are using our experimentally validated simulations to study global trends in oblique slamming behavior). Through the analyses, we have learned 1) both the plate’s peak displacement and peak impact force are positively related to $W$, with peak displacement increasing quadratically and peak impact force increases linearly with increasing $W$; and 2) the plate’s peak displacement is negatively related to flexural rigidity. The plate deflection decreases quadratically with increasing flexural rigidity up to $3200 \text{ Pa} \cdot \text{m}^3$, afterwards, the plate deflection reduces nearly linearly as a function of flexural rigidity.

6 Engineering theory for flexible plates under oblique slamming impacts

In this section, we leverage our insights from Section 5, obtained using our validated simulation approach, to pursue useful design-equation-type mathematical models. Previous works on added mass approximation have been reported in [75, 76, 77]. However, the reported works are limited to rigid, symmetric bodies (e.g., circular cylinder, elliptic paraboloid, and horizontal plate). Additionally, many of the relevant theories in the literature are purely theoretical, without the benefit of experimental observation. In the current work, we aim to use our experimental validated software to offer
unprecedented mechanistic insight as we develop a simple engineering theory for flexible plate oblique slamming. We begin from a mathematical model based on Newton’s second law, to characterize the normal impact force resulting from a flexible plate entering a quiescent water free surface at constant downward velocity. As we move the plate into the water, the water surrounding the plate is disturbed by the motion of the plate; this creates an unsteady flow. The unsteady flow generates an additional inertial force, known as the *added mass force*, that contributes to the plate deformation. To approximate the total normal impact force acting on the plate, we begin from Newton’s second law, written as

\[ F_n = \frac{d}{dt} \left[(m_p + m_a)W_t^2\right], \tag{21} \]

where \( F_n \) is the normal impact force, \( m_p \) is the mass of the plate, \( m_a \) is the added mass from the unsteady flow, and \( W_t^2 \) is the impact velocity orthogonal to plate surface at time \( t \). Surface tension, buoyancy, and the steady state drag force are assumed negligible in the present work.

In the case of constant impact velocity, the added mass is varying throughout the slamming impact event, while the mass of the plate is fixed, Equation 21 becomes

\[ F_n = \frac{dm_a}{dt} W_t^2. \tag{22} \]

Equation 22 suggests the normal impact force of flexible plate slamming at a constant impact velocity is solely dependent on the change of added mass in time, as well as the impact velocity. However, to the Authors’ knowledge, a general analytical solution for determining the exact added mass has yet to be established – indeed the added mass will take a suitable form, depending on the geometry of the structure at play, along with other factors such as the engineering properties of the structure, the viscosity of the fluid, etc. That said, it is also very difficult to measure the so called added mass force systematically in experimental settings. In this work, we aim to ascertain the added mass influence by studying the pressure contours within the water phase that accompanies the slamming events within the context of the FSI validation testing set in Subsection 4.3.2 (*i.e.*, the testing cases subjected to vertical impact only). From these “empirical” results, obtained through pressure contour investigations within our simulations, we uncover the evolutions of the added mass: this allows us to generalize the shape of the added mass for our context. Thereafter, we derive a simple, yet accurate mathematical formulation, to approximate the normal impact forces in slamming under constant vertical velocity, that may be suitable for use in design.

### 6.1 Added mass surface modeling

Theoretically, the added mass shape can be uncovered either through the velocity or the dynamic pressure contours (since dynamic pressure is defined as \( p_d = \frac{\rho u^2}{2} \)). However, during post-processing, the color gradient of the velocity contour falls within a very small range, which makes it nearly impossible to identify the added mass boundary beneath the plate from this measure. Consequently, dynamic pressure contours are preferred in the present work, for their wider range of color gradient, and therefore are used to identify the flow motion and the size of the added mass.

The adopted image processing procedure supporting the added mass identification is illustrated in Figure 21. At each \( dz/Z \), we obtain four slices of pressure contours along the longitudinal direction of the plate. The width of the plate is bounded by \( y = [0, 0.406] \), and so the four pressure slices are made at \( y = 0.05, 0.15, 0.25, \) and \( 0.35 \), respectively. The top left figure in Figure 21 shows the computational model with four pressure contour slices. For each pressure contour, we first convert the pressure contour into gray scale (top right). Then, we apply image thresholding to isolate the region bounded by the cyan layer in the original pressure contour (bottom right). Lastly, we create a boundary to enclose the region of interest (bottom center). Bottom left in Figure 21 shows a point cloud of the added mass boundaries extracted from the four pressure contours. The choice of the cyan color as an indicator of the added mass boundary is based on numerical experimentations – the cyan color provides a more accurate added mass prediction. The described procedure is repeated.
for \( dz/Z = 0, 0.125, 0.25, ..., 1 \), for each slamming test considered. Note, the pressure contours are scaled such that the pressure values correspond to the same color gradient across all \( dz/Z \) values.

![Added mass acquisition](image)

Figure 21: The image processing procedure adopted in the present work to extract the point cloud which surrounds the added mass.

After we obtain an added mass point cloud, we adopt a stable and robust quadratic surface fitting method, the nonlinear Koopman method \([78]\), to recover the shape of the added mass. The nonlinear Koopman method described in \([3]\) is formulated specifically to fit our data to an ellipsoid, which suits our needs given the shape of the added mass point cloud observed within the totality of simulations undertaken in this work.

### 6.2 Added mass surface fitting results

We now present the added masses obtained for the test cases in the last FSI validation set (slamming tests with impact velocities \( W_n = 0.292, 0.438, 0.584, \) and \( 0.875 \) m/s) using the added mass acquisition procedure detailed in Section 6.1 and \([3]\). We focus on recovering the normal impact force of the moderately deformable plate with plate thickness 0.00795 m. The approximated added mass boundaries for each of the slamming cases are presented in Figure 22 to 24. As shown, added mass among the four slamming cases, at each \( dz/Z \), share similar shape. In all four cases, the added mass started off with the shape of a quarter of a long and narrow ellipsoid. As we submerge the plate in the water, both the major and minor radii of the ellipsoid are proportionally increasing in a way such that the semi-major and semi-minor radii of the ellipsoid are roughly in a 1 : 1 ratio.

The added mass can be obtained by multiplying the volume bounded by the approximated added mass surface, by the water density, shown in Figure 26. All four cases have similar added mass, except slamming case \( W_n = 0.292 \) m/s. The \( W_n = 0.292 \) m/s exhibits higher added mass due to water splashing over from the trailing edge as discussed in Section 4.3.2. We adopt the added mass obtained using these ellipsoidal fits to approximate the normal impact force, \( F_n \), using Equation 22.

The empirical results are compared against experimental and FSI simulation results, presented in Figure 27. Using the ellipsoidal fits for the added mass with Equation 22, we observe good agreement between validation experiments, high fidelity simulations, and Equation 22 (the latter using ellipsoidal added mass fits).
Figure 22: Added mass surface approximation via ellipsoid surface fitting method (see B) for the slamming cases with $W_n = 0.292$ m/s.

Figure 23: Added mass surface approximation via ellipsoid surface fitting method (see B) for the slamming cases with $W_n = 0.438$ m/s.

Figure 24: Added mass surface approximation via ellipsoid surface fitting method (see B) for the slamming cases with $W_n = 0.584$ m/s.
6.3 Proposed added mass generalization

In the sequel, we aim to develop a generalized mathematical formulation to characterize the impact force arises from slamming. The force behavior up to force unloading is of interest, as this corresponds to the maximal loading for design. Referring back to the summary of the slamming analyses for the case of $W_n = 0.875$ m/s in Figure 14, we observe that the force unloading is directly related to the location of the spray root. Specifically, the spray root generates a highly concentrated pressure that positively relates to the slamming impact force. As this spray root leaves the plate, the high pressure is released, and as a result, the impact force is reduced. Given this information, the lengths of the spray root (i.e., the effective wetted length of the plate) are measured within our simulations, to determine the instant when the spray root departs the plate. In Figure 28, $L_w$ denotes the wetted length, and $L$ denotes the plate’s overall length. The effective wetted length is approximated by linear fitting to the measured data. The data show that the spray root exits the plate (when $L_w/L$ equals to one) at $1.293dz/Z$. In other words, the effective $L_w$ is approximately 1.293 times longer than the nominal $L_w$. The wetted length coefficient is likely to be higher in cases with smaller angle of attack, $\alpha$, as higher pressure is generated between the plate and the water free surface.
Figure 27: Normal impact force comparison for slamming cases with $W_n = 0.292$, 0.438, 0.584, and 0.875 m/s.

Here, we propose to approximate the added mass volume as a quarter ellipsoid with dimension $L_e/2 \times L_e/2 \times w$, where $L_e$ is the effective $L_w$, and $w$ is the width of the plate, as shown in Figure 29. The added mass of the proposed quarter ellipsoid is formulated as

$$m_a = \frac{\rho_w \pi w}{3} \left( \frac{L_e}{2} \right)^2,$$  \hspace{1cm} (23)
where $L_e = 1.293L_n$ and $L_n = \frac{W_t}{\sin(\alpha)}$. Substitute in $L_e$, Equation 23 is re-formulated as

$$m_a = \frac{\rho_w \pi w}{3} \frac{(1.293W_t)^2}{2\sin(\alpha)}. \quad (24)$$

Integrating Equation 24 w.r.t. time, we obtain an expression for the change of added mass in time as,

$$\frac{dm_a}{dt} = 1.293^2 \frac{\rho_w \pi w}{6} \frac{W^2 \sin^2(\alpha)}{\sin^2(\alpha)} t. \quad (25)$$

Substituting Equation 25 into Equation 22, the generalized form of the normal impact force is arrived at:

$$F_n = \frac{1.293^2 \rho_w \pi w}{6} \frac{W^3 \cos(\alpha)}{\sin^2(\alpha)} t \text{ for } dz/Z \leq \frac{1}{1.293}. \quad (26)$$

This mathematical form is a compact expression suitable for design. It leverages deep phenomenological insights, gleaned from our validated FSI simulations, within the classical context of Newton’s second law of motion, to furnish a useful engineering theory.

We adopt the normal impact force formulation in Equation 26 and approximate the normal impact forces for slamming cases with $W_n = 0.292, 0.438, 0.584, \text{ and } 0.875 \text{ m/s}$. The theoretical results are compared against FSI simulation and pressure contour inspection results. As shown in Figure 30, Equation 26 exhibits excellent agreements in comparison to FSI simulations and pressure contour inspections, but with a greatly reduced effort in comparison with full scale experiments, or even high fidelity FSI simulations (which consume up to 82 core hours in the representative case).

7 Conclusion

In this work, we have developed a novel engineering theory for the practically important slamming context of a flexible flat plate obliquely impacting the water free surface. To develop this theory, we leverage detailed phenomenological insights derived from experimentally validated FSI simulations. To confirm the veracity of the proposed implicit, partitioned FSI framework, we first validated our FSI solver against experimental data reported in [19] (the FSI validation results have displayed excellent agreement with experiments). Subsequently, we applied our experimentally validated FSI framework to carry out a program of complementary FSI simulations in an effort to gain comprehensive engineering insight into the context of oblique slamming impact. Finally, we leveraged these insights to obtain a compact and simple mathematical model, derived from Newton’s second law of motion,
for flexible plates obliquely impacting the water free surface. Further slamming analyses are needed in order to finalize the parameters obtained in the current study. Nonetheless, the novel approach of deriving a closed form solution for slamming in the present work is potentially transformative and lays down a foundation towards a generalized slamming theory. This new engineering theory requires much less effort in comparison to experiments and FSI simulations, yet it displays very promising predictive power. Therefore, it presents considerable benefit and potential for use in the design of planning hulls.

8 Acknowledgement

This work is supported by the Office of Naval Research (ONR), under the grant N00014-19-1-2034. The authors thank Jonathan Stergiou, Ronald Miller, Hua Shan, and Dory Lummer from Naval Surface Warfare Center Carderock Division, who had provided guidance and suggestions in the development of the FSI software. The authors would also like to acknowledge An Wang and Professor James Duncan from the University of Maryland, for providing the experimental data reported within this paper.
A Validation Results

Here, we expand on the FSI validation results from all slamming test cases listed in Figure 7.

Figure 31: Plate deflection and normal impact force histories of the highly flexible, moderately deformable, and nearly rigid plates subjected to slamming impacts with $W_n = 1.386$ m/s.
Figure 32: Plate deflection and normal impact force histories of the highly flexible, moderately deformable, and nearly rigid plates subjected to slamming impacts with $W_n = 1.313$ m/s.
Figure 33: Plate deflection and normal impact force histories of the highly flexible, moderately deformable, and nearly rigid plates subjected to slamming impacts with $W_n = 1.167$ m/s.
Figure 34: Plate deflection and normal impact force histories of the highly flexible, moderately deformable, and nearly rigid plates subjected to slamming impacts with $W_n = 0.875$ m/s.
Figure 35: Plate deflection and normal impact force histories of the highly flexible, moderately deformable, and nearly rigid plates subjected to slamming impacts with $W_n = 0.584$ m/s.
Figure 36: Plate deflection and normal impact force histories of the highly flexible, moderately deformable, and nearly rigid plates subjected to slamming impacts with $U/W = 0$. 
Here, we provide detail exposition on the ellipsoidal procedure. Our departure point is the equation of a second degree quadric surface within three dimensional Euclidean space

\[ a_1 x^2 + a_2 y^2 + a_3 z^2 + 2a_4 yz + 2a_5 xz + 2a_6 xy + 2a_7 x + 2a_8 y + 2a_9 z + a_{10} = 0, \]  

(27)

where \( x, y, \) and \( z \) are the point cloud data expressed in Cartesian coordinates, and \( a_1, a_2, \ldots, a_{10} \) are the least square fitted coefficient parameters.

Let the invariants of Equation 27 under rotation and translation be

\[ I = a_1 + a_2 + a_3, \]  

(28)

\[ J = a_1 a_2 + a_2 a_3 + a_1 a_3 - a_4^2 - a_5^2 - a_6^2, \]  

(29)

and

\[ K = \begin{bmatrix} a_1 & a_6 & a_5 \\ a_6 & a_2 & a_4 \\ a_5 & a_4 & a_3 \end{bmatrix}. \]  

(30)

The fitting shape of the Equation 27 represents an ellipsoid when the constraint \( kJ - I^2 > 0 \), where \( k \) is a positive real number, is satisfied. Moreover, when \( k = 4 \), Equation 27 is guaranteed to be an ellipsoid [78]. The procedure of ellipsoid surface fitting is as follows. Let \( p_i(x_i, y_i, z_i) \) \( i = 1 \ldots n \) be the set of points in the added mass point cloud. For each point \( p_i(x_i, y_i, z_i) \), let Equation 27 be reformulated in the following matrix form

\[ X_i^T a = 0, \]  

(31)

where

\[ X_i = \begin{bmatrix} x_i^2 \\ y_i^2 \\ z_i^2 \\ 2y_i z_i \\ 2x_i z_i \\ 2x_i y_i \\ 2x_i \\ 2y_i \\ 2z_i \end{bmatrix}^T, \]  

(32)

and

\[ a = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{bmatrix}^T. \]  

(33)

To ensure Equation 31 corresponds to an ellipsoid, the following condition is imposed

\[ \min \| Da \|^2 \quad \text{s.t.} \quad kJ - I^2 = 1. \]  

(34)

\( D \) is a design matrix in \( \mathbb{R}^{10 \times n} \) defined as \( D = [X_1 \quad X_2 \quad \ldots \quad X_n] \). Finally, we solve the following system of equations

\[ D D^T v = \lambda C v, \]  

\[ v^T C v = 1, \]  

(35)
using the Lagrange multiplier method, to minimize the condition in Equation 34. Herein,

\[
C = \begin{bmatrix}
-1 & \frac{k}{2} - 1 & \frac{k}{2} - 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{k}{2} - 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & \frac{k}{2} - 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -k & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -k & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k \\
\end{bmatrix}.
\] (36)

The system \(DD^T v = \lambda C v\) recovers the general eigenvector associated with the unique positive eigenvalues when \(k > 3\) [78]. In other words, Equation 35 has a unique solution when \(k > 3\). To arrive at a solution for Equation 35, we consider the following matrix decomposition

\[
DD^T = \begin{bmatrix}
S_{11} & S_{12} \\
S_{12}^T & S_{22}
\end{bmatrix}
\]
and \(v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\).

(37)

In the equation above, \(S_{11} \in \mathbb{R}^{6 \times 6}, S_{12} \in \mathbb{R}^{6 \times 4}, S_{22} \in \mathbb{R}^{4 \times 4}, v_1 \in \mathbb{R}^{6 \times 1}\), and \(v_2 \in \mathbb{R}^{4 \times 1}\). Equation 35 becomes

\[
(S_{11} - \lambda C_1) v_1 + S_{12} v_2 = 0,
\]
\[
S_{12}^T v_1 + S_{22} v_2 = 0.
\] (38)

Note that the matrix \(C\) has the eigenvalues \([k - 3, -\frac{k}{2}, -\frac{k}{2}, -k, -k, 0, 0, 0, 0]\) and that the element \(c_{ij}\) in matrix \(C\) is zero when the corresponding \(i\) and \(j\) are greater than 6.

Assume the given data set is not coplanar and therefore \(S_{22}\) is nonsingular, Equation 38 is subsequently rewritten as

\[
a_2 = -S_{22}^{-1} S_{12}^T v_1,
\]
\[
(S_{11} - S_{12} S_{22}^{-1} S_{12}^T) v_1 = \lambda C_1 v_1.
\] (39)

In cases when \((S_{11} - S_{12} S_{22}^{-1} S_{12}^T)\) is positive definite, the general solution to Equation 35 is \(u = [u_1^T, u_2^T]^T\) where \(u_1\) is the eigenvector associated with the general system in Equation 39 and \(u_2 = -S_{22}^{-1} S_{12}^T u_1\). On the other hand, in cases when \((S_{11} - S_{12} S_{22}^{-1} S_{12}^T)\) is singular, \(u_1\) is replaced by the eigenvector associated with the largest eigenvalue of the system.

While the above surface fitting method guarantees an ellipsoid when \(k = 4\), this condition only represents a subset of an ellipsoid. To ensure an ellipsoid is recovered, the following iterative procedure is carried out.

**Algorithm 1: Iterative ellipsoid surface fitting**

**Result:** \(k\)

Initialize \(k\) to a sufficiently large number;

Solve Equation 35

if fitting is an ellipsoid then

STOP;

else

Run a binary search to identify a \(k\) that produces an ellipsoid;

Perform bisection algorithm to identify a minimum \(k\) that produces an ellipsoid;

end
References

[1] G. Kapsenberg. Slamming of ships: where are we now? Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 369(1947):2892–2919, 2011.

[2] T. Von Karman. The impact on seaplane floats during landing. Technical report, National Advisory Committee for Aeronautics, 1929.

[3] Heribert Wagner. Über stoß- und gleitvorgänge an der oberfläche von flüssigkeiten. ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik, 12(4):193–215, 1932.

[4] S. Howison, J. Ockendon, and S. Wilson. Incompressible water-entry problems at small deadrise angles. Journal of Fluid Mechanics, 222:215–230, 1991.

[5] J. Newman. Wave effects on deformable bodies. Applied Ocean Research, 16(1):47 – 59, 1994.

[6] Odd Faltinsen, Jan Kvalsvold, and Jan Aarsnes. Wave impact on a horizontal elastic plate. Journal of Marine Science and Technology, 2(2):87–100, Jun 1997.

[7] O. Faltinsen. The effect of hydroelasticity on ship slamming. Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 355(1724):575–591, 1997.

[8] Odd Faltinsen. Water entry of a wedge by hydroelastic orthotropic plate theory. 43:180–193, 1999.

[9] Z. Dobrovol’skaya. On some problems of similarity flow of fluid with a free surface. Journal of Fluid Mechanics, 36(4):805–829, 1969.

[10] N. de Divitiis and L. de Socio. Impact of floats on water. Journal of Fluid Mechanics, 471:365–379, 2002.

[11] Alexander Korobkin. Analytical models of water impact. European Journal of Applied Mathematics, 15(6):821–838, 2004.

[12] J. Oliver. Second-order wagner theory for two-dimensional water-entry problems at small deadrise angles. Journal of Fluid Mechanics, 572:59–85, 2007.

[13] Reza Taghipour, Tristan Perez, and Torgeir Moan. Hybrid frequency–time domain models for dynamic response analysis of marine structures. Ocean Engineering, 35(7):685 – 705, 2008.

[14] Odd Faltinsen and Yuriy Semenov. Nonlinear problem of flat-plate entry into an incompressible liquid. Journal of Fluid Mechanics, 611:151–173, 2008.

[15] F. Huera, D. Jeon, and M. Gharib. Experimental investigation of water slamming loads on panels. Ocean Engineering, 38(11):1347 – 1355, 2011.

[16] A. Iafrati, S. Grizzi, M.H. Siemann, and L. Benítez Montañés. High-speed ditching of a flat plate: Experimental data and uncertainty assessment. Journal of Fluids and Structures, 55:501–525, 2015.

[17] A. Wang, S. Wang, E. Balaras, D. Conroy, T. O’Shea, and J. Duncan. Spray formation during the impact of a flat plate on a water surface. Monterey, California, 2016. Proceedings of the 31st Symposium on Naval Hydrodynamics.

[18] A. Wang, H. Kim, K. Wong, M. Yu, K. Kiger, and J. Duncan. Spray formation and structural deformation during the oblique impact of a flexible plate on a quiescent water surface. Journal of Ship Research, 63(3), 2019.

[19] A. Wang, H. Kim, Z. Wong, M. Yu, K. Kiger, and J. Duncan. The impact of a flexible plate on a quiescent water surface. Osaka, Japan, June 2020. 33rd Symposium on Naval Hydrodynamics.

[20] R. Zhao and O. Faltinsen. Water entry of two-dimensional bodies. Journal of Fluid Mechanics, 246:593–612, 1993.

[21] Alessandro Iafrati and Alexander Korobkin. Initial stage of flat plate impact onto liquid free surface. Physics of Fluids, 16(7):2214–2227, 2004.
[22] Alessandro Iafrati and Alexander Korobkin. Hydrodynamic loads during early stage of flat plate impact onto water surface. *Physics of Fluids*, 20(8):082104, 2008.

[23] Reza Taghipour, Tristan Perez, and Torgeir Moan. Time domain hydroelastic analysis of a flexible marine structure using state-space models. (42703):519–527, 2007.

[24] A. Korobkin and T. Khabakhpasheva. Regular wave impact onto an elastic plate. *Journal of Engineering Mathematics*, 55(1):127–150, Aug 2006.

[25] C. Lu, Y. He, and G. Wu. Coupled analysis of nonlinear interaction between fluid and structure during impact. *Journal of Fluids and Structures*, 14(1):127 – 146, 2000.

[26] E. Walhorn, A. Kölke, B. Hübner, and D. Dinkler. Fluid–structure coupling within a monolithic model involving free surface flows. *Computers & Structures*, 83(25):2100 – 2111, 2005.

[27] A. Korobkin, T. Khabakhpasheva, and G. Wu. Coupled hydrodynamic and structural analysis of compressible jet impact onto elastic panels. *Journal of Fluids and Structures*, 24(7):1021 – 1041, 2008.

[28] Kwang Paik, Pablo Carrica, Donghee Lee, and Kevin Maki. Strongly coupled fluid–structure interaction method for structural loads on surface ships. *Ocean Engineering*, 36(17):1346 – 1357, 2009.

[29] P. Tallec and J. Mouro. Fluid structure interaction with large structural displacements. *Computer Methods in Applied Mechanics and Engineering*, 190(24):3039 – 3067, 2001.

[30] Wolfgang Wall, Steffen Genkinger, and Ekkehard Ramm. A strong coupling partitioned approach for fluid–structure interaction with free surfaces. *Computers & Fluids*, 36(1):169 – 183, 2007.

[31] Dominic Piro and Kevin Maki. Hydroelastic analysis of bodies that enter and exit water. *Journal of Fluids and Structures*, 37:134 – 150, 2013.

[32] Peter D. Hicks and Richard Purvis. Air cushioning and bubble entrapment in three-dimensional droplet impacts. *Physics of Fluids*, 649:135–163, 2010.

[33] T. I. Khabakhpasheva and A. A. Korobkin. Oblique elastic plate impact on thin liquid layer. *Physics of Fluids*, 32(6):062101, 2020.

[34] C. Hirt and B. Nichols. Volume of fluid (vof) method for the dynamics of free boundaries. *Journal of Computational Physics*, 39(1):201 – 225, 1981.

[35] J. Pedersen, B. Larsen, H. Bredmore, and H. Jasak. A new volume-of-fluid method in open-foam. In *MARINE 2017 Computational Methods in Marine Engineering VII*, pages 266–278. International Center for Numerical Methods in Engineering.

[36] Edin Berberović, Nils van Hinsberg, Suad Jakirlić, Ilia Roisman, and Cameron Tropea. Drop impact onto a liquid layer of finite thickness: Dynamics of the cavity evolution. *Phys. Rev. E*, 79:036306, Mar 2009.

[37] Jean Donea, Antonio Huerta, J.. Ponthot, and A. Rodríguez. *Arbitrary Lagrangian–Eulerian methods*, chapter 14. American Cancer Society, 2004.

[38] C. Hirt, A. Amsden, and J. Cook. An arbitrary lagrangian-eulerian computing method for all flow speeds. *Journal of Computational Physics*, 14(3):227 – 253, 1974.

[39] I. Demirdžić and M. Perić. Space conservation law in finite volume calculations of fluid flow. *International Journal for Numerical Methods in Fluids*, 8(9):1037–1050, 1988.

[40] P. Thomas and C. Lombard. Geometric conservation law and its application to flow computations on moving grids. *AIAA Journal*, 17(10):1030–1037, 1979.

[41] Charbel Farhat, Philippe Geuzaine, and Céline Grandmont. The discrete geometric conservation law and the nonlinear stability of ale schemes for the solution of flow problems on moving grids. *Journal of Computational Physics*, 174(2):669 – 694, 2001.

[42] Hrvoje Jasak and Zeljko Tuković. Automatic mesh motion for the unstructured finite volume method, 2004.
[43] IUC. *OpenFOAM: A C++ library for complex physics simulations*, Dubrovnik, Croatia, September 2007. International Workshop on Coupled Methods in Numerical Dynamics.

[44] H. Weller and G. Tabor. A tensorial approach to computational continuum mechanics using object-oriented techniques. *Computers in Physics*, 12(6):620, 1998.

[45] Hrvoje Jasak. *Error analysis and estimation for the finite volume method with applications to fluid flows*. PhD thesis, Imperial College of Science, Technology and Medicine, 1996.

[46] Bernhard Gschaider, Hakan Nilsson, Henrik Rusche, Hrvoje Jasak, Martin Beaudoin, and Vanja Skuric. OpenFOAM-1.6-ext.

[47] Wensi Wu, Justyna Kosianka, Heather Reed, Christopher Stull, and Christopher Earls. CU-BENs: A structural modeling finite element library. *SoftwareX*, 11:100485, 2020.

[48] Ronald Miller, Sung Kim, and Sung Won Lee. A computational framework for fluid–structure interaction on flexible propellers involving large deformation. Pasadena, California, September 2010. 28th Symposium on Naval Hydrodynamics.

[49] Klaus Bathe and Lee Ho. A simple and effective element for analysis of general shell structures. *Computers & Structures*, 13(5):673–681, 1981.

[50] Christopher Earls. CUBENs_theory_manual, September 2016.

[51] P. Causin, J. Gerbeau, and F. Nobile. Added-mass effect in the design of partitioned algorithms for fluid–structure problems. *Computer Methods in Applied Mechanics and Engineering*, 194(42):4506 – 4527, 2005.

[52] Christiane Förster, Wolfgang A. Wall, and Ekkehard Ramm. The artificial added mass effect in sequential staggered fluid–structure interaction algorithms. TU Delft, The Netherlands, 2006. European Conference on Computational Fluid Dynamics.

[53] Christiane Förster, Wolfgang A. Wall, and Ekkehard Ramm. Artificial added mass instabilities in sequential staggered coupling of nonlinear structures and incompressible viscous flows. *Computer Methods in Applied Mechanics and Engineering*, 196(7):1278 – 1293, 2007.

[54] J. Chung and G. Hulbert. A time integration algorithm for structural dynamics with improved numerical dissipation: the generalized-α method. *Journal of applied mechanics*, 60(2):371–375, 1993.

[55] Hrvoje Jasak. *Dynamic mesh handling in OpenFOAM*.

[56] Christophe Kassiotis. Which strategy to move the mesh in the computational fluid dynamic code openfoam. 2008.

[57] A. Passalacqua and R. Fox. Implementation of an iterative solution procedure for multi-fluid gas–particle flow models on unstructured grids. *Powder Technology*, 213(1):174 – 187, 2011.

[58] Charbel Habchi, Serge Russeil, Daniel Bougeard, Jean Harion, Thierry Lemenand, Akram Ghanem, Dominique Valle, and Hassan Peerhossaini. Partitioned solver for strongly coupled fluid–structure interaction. *Computers & Fluids*, 71:306 – 319, 2013.

[59] E. Robertson, V. Choudhury, S. Bhushan, and D. Walters. Validation of openfoam numerical methods and turbulence models for incompressible bluff body flows. *Computers & Fluids*, 123:122 – 145, 2015.

[60] R. Issa. Solution of the implicitly discretised fluid flow equations by operator-splitting. *Journal of Computational Physics*, 62(1):40 – 65, 1986.

[61] *The open source CFD toolbox – user guide version 7*, 2019.

[62] Bram van Leer. Towards the ultimate conservative difference scheme. ii. monotonicity and conservation combined in a second-order scheme. *Journal of Computational Physics*, 14(4):361 – 370, 1974.

[63] Owe Axelsson. *Iterative Solution Methods*. Cambridge University Press, 1994.

[64] Joel Ferziger and Milovan Peric. *Computational methods for fluid dynamics*. Springer-Verlag Berlin Heidelberg, 2002.
[65] Hrvoje Jasak, Aleksandar Jemcov, and Joseph Maruszewski. Preconditioned linear solvers for large eddy simulation. 2007.

[66] F. Moukalled, L. Mangani, and M. Darwish. *The finite volume method in computational fluid dynamics*. Springer International Publishing, 2016.

[67] C. Farhat, M. Lesoinne, and P. Tallec. Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces: Momentum and energy conservation, optimal discretization and application to aeroelasticity. *Computer Methods in Applied Mechanics and Engineering*, 157(1):95 – 114, 1998.

[68] Donald Sheard. A two-dimensional interpolation function for irregularly-spaced data. In *Proceedings of the 1968 23rd ACM National Conference*, ACM ’68, pages 517–524, New York, NY, USA, 1968. Association for Computing Machinery.

[69] R.L. Campbell and E.G. Paterson. Fluid–structure interaction analysis of flexible turbomachinery. *Journal of Fluids and Structures*, 27(8):1376–1391, 2011.

[70] Joris Degroote, Klaus Bathe, and Jan Vierendeels. Performance of a new partitioned procedure versus a monolithic procedure in fluid–structure interaction. *Computers & Structures*, 87(11):793 – 801, 2009.

[71] Joris Degroote, Robby Haelterman, Sebastiaan Annerel, Peter Bruggeman, and Jan Vierendeels. Performance of partitioned procedures in fluid–structure interaction. *Computers & Structures*, 88(7):446 – 457, 2010.

[72] Wensi Wu, Christophe Bonneville, and Christopher Earls. A principled approach to design using high fidelity fluid-structure interaction simulations. *Finite Elements in Analysis and Design*, 194:103562, 2021.

[73] Christophe Geuzaine and Jean Remacle. Gmsh: A 3-d finite element mesh generator with built-in pre- and post-processing facilities. *International Journal for Numerical Methods in Engineering*, 79(11):1309–1331, 2009.

[74] Riccardo Pellegrini, Matteo Diez, Zhaoyuan Wang, Frederic Stern, An Wang, Z. Wong, Miao Yu, Kenneth T. Kiger, and James H. Duncan. High-fidelity fsi simulations and v&v of vertical and oblique flexible plate slamming. 33rd ONR Symposium on Naval Hydrodynamics, 2020.

[75] Odd M. Faltinsen. *Hydrodynamics of High-Speed Marine Vehicles*. Cambridge University Press, 2005.

[76] Y.-M Scolan and A.A Korobkin. Energy distribution from vertical impact of a three-dimensional solid body onto the flat free surface of an ideal fluid. *Journal of Fluids and Structures*, 17(2):275–286, 2003.

[77] A.A. Korobkin, T.I. Khabakhpasheva, and Kevin J. Maki. Hydrodynamic forces in water exit problems. *Journal of Fluids and Structures*, 69:16–33, 2017.

[78] Qingde Li and J. Griffiths. Least squares ellipsoid specific fitting. In *Geometric Modeling and Processing, 2004. Proceedings*, pages 335–340, April 2004.