ANNIHILATOR OF LOCAL COHOMOLOGY MODULES AND STRUCTURE OF RINGS

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ABSTRACT

Let \((R, m)\) be a Noetherian local ring, \(A\) an Artinian \(R\)-module, and \(M\) a finitely generated \(R\)-module. It is clear that \(\text{Ann}_R(R/M/ p M) = p\), for all \(p \in \text{Var}(\text{Ann} R M)\). Therefore, it is natural to consider the following dual property for annihilator of Artinian modules:

\[
\text{Ann}_R(0 : A p) = p, \quad \text{for all } p \in \text{Var}(\text{Ann} R A).
\]

Let \(i \geq 0\) be an integer. Alexander Grothendieck showed that the local cohomology module \(H^i_m(M)\) of \(M\) is Artinian. The property \((*)\) of local cohomology modules is closed related to the structure of the base ring. In this paper, we prove that for each \(p \in \text{Spec}(R)\) such that \(H^i_m(R/ p)\) satisfies the property \((*)\) for all \(i\), then \(R/ p\) is universally catenary and the formal fibre of \(R\) over \(p\) is Cohen-Macaulay.

Keywords: Local cohomology; universally catenary; formal fibre; Artinian module; Cohen-Macaulay ring

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LINH HÓA TỬ CỦA MÔĐUN ĐỐI ĐИỀU ĐỊA PHƯƠNG VÀ CẤU TRÚC VÀNH

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TÓM TẮT

Cho \((R, m)\) là vành Noether địa phương, \(A\) là \(R\)-môđun Artin, và \(M\) là \(R\)-môđun hữu hạn sinh. Ta có \(\text{Ann}_R(R/M/ p M) = p\) với mọi \(p \in \text{Var}(\text{Ann} R M)\). Do đó rất tự nhiên ta xét tính chất sau về linh hóa tử của môđun Artin

\[
\text{Ann}_R(0 : A p) = p, \quad \text{for all } p \in \text{Var}(\text{Ann} R A).
\]

Cho \(i \geq 0\) là số nguyên. Alexander Grothendieck đã chỉ ra rằng môđun đối đồng địa phương \(H^i_m(M)\) là Artin. Tính chất \((*)\) của các môđun đối đồng địa phương liên hệ mật thiết với cấu trúc vành cơ sở. Trong bài báo này, chúng tôi chỉ ra với mỗi \(p \in \text{Spec}(R)\) mà \(H^i_m(R/ p)\) thỏa mãn tính chất \((*)\) với mọi \(i\) thì \(R/ p\) là catenary phổ dụng và các thớ hình thức của \(R\) trên \(p\) là Cohen-Macaulay.

Từ khóa: Đối đồng địa phương; catenary phổ dụng; thớ hình thức; môđun Artin; vành Cohen-Macaulay

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1. Introduction

Throughout this paper, let \((R, m)\) be a Noetherian local ring, \(A\) an Artinian \(R\)-module, and \(M\) a finitely generated \(R\)-module of dimension \(d\). For each ideal \(I\) of \(R\), we denote by \(\text{Var}(I)\) the set of all prime ideals containing \(I\). For a subset \(T\) of \(\text{Spec}(R)\), we denote by \(\min(T)\) the set of all minimal elements of \(T\) under the inclusion.

It is clear that \(\text{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}\), for all \(\mathfrak{p} \in \text{Var}([\text{Ann}_R M])\). Therefore, it is natural to consider the following dual property for annihilator of Artinian modules:

\[
\text{Ann}_R(0 : A \mathfrak{p}) = \mathfrak{p}, \forall \mathfrak{p} \in \text{Var}([\text{Ann}_R A]). (*)
\]

If \(R\) is complete with respect to \(m\)-adic topology, it follows by Matlis duality that the property (*) is satisfied for all Artinian \(R\)-modules. However, there are Artinian modules which do not satisfy this property. For example, by [1, Example 4.4], the Artinian \(R\)-module \(H^i_m(R)\) does not satisfy the property (*), where \(R\) is the Noetherian local domain of dimension 2 constructed by M. Ferrand and D. Raynaud [2] (see also [3, App. Ex. 2] [Ex. 2]) such that its \(m\)-adic completion \(\hat{R}\) has an associated prime \(q\) of dimension 1. In [4], N. T. Cuong, L. T. Nhan and N. T. Dung showed that the top local cohomology module \(H^i_m(M)\) satisfies property (*) if and only if the ring \(R/\text{Ann}_R(M/U_M(0))\) is catenary, where \(U_M(0)\) is the largest submodule of \(M\) of dimension less than \(d\). The property (*) of local cohomology modules is closed related to the structure of the ring. In [5], L. T. Nhan and the author proved that if \(H^i_m(M)\) satisfies the property (*) for all \(i\), then \(R/\mathfrak{p}\) is unmixed for all \(\mathfrak{p} \in \text{Ass}M\) and the ring \(R/\text{Ann}_R M\) is universally catenary. The following conjecture was given by N. T. Cuong in his seminar.

**Conjecture 1.1.** The following statements are equivalent:

(i) \(H^i_m(R)\) satisfies the property (*) for all \(i\);

(ii) \(R\) is universally catenary and all its formal fibers are Cohen-Macaulay.

L. T. Nhan and T. D. M. Chau proved in [6] that \(H^i_m(M)\) satisfies the property (*) for all \(i\), for all finitely generated \(R\)-module \(M\) if and only if \(R\) is universally catenary and all its formal fibers are Cohen-Macaulay. The following result is the main result of this paper. We hope that we can use this to give a positive answer for the above conjecture.

**Theorem 1.2.** Assume \(\mathfrak{p} \in \text{Spec}(R)\) such that \(H^i_m(R/\mathfrak{p})\) satisfies the property (*) for all \(i\). Then \(R/\mathfrak{p}\) is universally catenary and the formal fibre of \(R\) over \(\mathfrak{p}\) is Cohen-Macaulay.

2. Proof of the main results

The theory of secondary representation was introduced by I. G. Macdonald (see [7]) which is in some sense dual to that of primary decomposition for Noetherian modules. Note that every Artinian \(R\)-module \(A\) has a minimal secondary representation \(A = A_1 + \ldots + A_n\), where \(A_i\) is \(p_i\)-secondary. The set \(\{p_1, \ldots, p_n\}\) is independent of the choice of the minimal secondary representation of \(A\). This set is called the set of attached prime ideals of \(A\), and denoted by \(\text{Att}_R A\). Note also that \(A\) has a natural structure as an \(\hat{R}\)-module. With this structure, a subset of \(A\) is an \(R\)-submodule if and only if it is an \(\hat{R}\)-submodule of \(A\). Therefore, \(A\) is an Artinian \(\hat{R}\)-module.

**Lemma 2.1.** (i) The set of all minimal elements of \(\text{Att}_R A\) is exactly the set of all minimal elements of \(\text{Var}(\text{Ann}_R A)\).

(ii) \(\text{Att}_R A = \{\hat{p} \cap R : \hat{p} \in \text{Att}_\hat{R} A\}\).
R. N. Roberts introduced the concept of Krull dimension for Artinian modules (see [8]). D. Kirby changed the terminology of Roberts and referred to Noetherian dimension to avoid confusion with Krull dimension defined for finitely generated modules (see [9]). The Noetherian dimension of $A$ is denoted by $\text{N-dim}_R(A)$. In this paper, we use the terminology of Kirby (see [9]).

Lemma 2.2 ([1]). (i) $\text{N-dim}_R(A) \leq \dim(R/\text{Ann}_R A)$, and the equality holds if $A$ satisfies the property (*).

(ii) $\text{N-dim}_R(H^i_m(M)) \leq i$, for all $i$.

The following property of attached primes of the local cohomology under localization is known as Weak general Shifted Localization Principle (see [10]).

Lemma 2.3. We have $\text{Att}_{R_p}(H^{i-\dim R/p}_p(M_p))$ is the subset of $\{ q R_p : q \in \min \text{Att}_R(H^i_m(M) ; q \subseteq p) \}$, for all $p \in \text{Spec}(R)$.

For an integer $i \geq 0$, following M. Brodmann and R. Y. Sharp (see [11]), the $i$-th pseudo support of $M$, denoted by $\text{Psupp}_R^i(M)$, is defined by the set

$$\{ p \in \text{Spec}(R) \mid H^{i-\dim R/p}_p(M_p) \neq 0 \}.$$

Note that the role of $\text{Psupp}_R^i(M)$ for the Artinian $R$-module $A = H^0_m(M)$ is in some sense similar to that of $\text{Supp} L$ for a finitely generated $R$-module $L$, cf. [11], [5]. Although, we always have $\text{Supp} L = \text{Var}(\text{Ann}_R L)$, but the analogous equality $\text{Psupp}_R^i(M) = \text{Var}(\text{Ann}_R H^i_m(M))$ is not valid in general. The following lemma gives a necessary and sufficient conditions for the above equality.

Lemma 2.4 ([5]). Let $i \geq 0$ be an integer. Then the following statements are equivalent:

(i) $H^i_m(M)$ satisfies the property (*).

(ii) $\text{Var}(\text{Ann}_R H^i_m(M)) = \text{Psupp}_R^i(M)$.

In particular, if $H^i_m(M)$ satisfies the property (*) then

$$\min \text{Att}_R(H^i_m(M)) = \min \text{Psupp}_R^i(M).$$

In 2010, N. T. Cuong, L. T. Nhan and N. T. K. Nga (see [12]) used pseudo support to describe the non-Cohen-Macaulay locus of $M$. Recall that $M$ is equidimensional if $\dim(R/p) = d$, for all $p \in \text{min}(\text{Ass}(R))$.

Lemma 2.5 ([12]). Suppose that $M$ is equidimensional and the ring $R/\text{Ann}_R M$ is catenary. Then $\text{Psupp}_R^i(M)$ is closed for $i = 0, 1, d$ and $\text{nCM}(M) = \bigcup_{i=0}^{d-1} \text{Psupp}_R^i(M)$, where $\text{nCM}(M)$ is the Non Cohen-Macaulay locus of $M$.

Following M. Nagata ([3]), we say that $M$ is unmixed if $\dim(\widehat{R}/\widehat{p}) = d$ for all prime ideals $\widehat{p} \in \text{Ass}(\widehat{M})$, and $M$ is quasi unmixed if $\widehat{M}$ is equidimensional. The next lemma show that the property (*) for the local cohomology modules $H^i_m(M)$ of levels $i < d$ is closed related to the universal catenaryicity and unmixedness of certain local rings.

Lemma 2.6 ([5]). Assume that $H^i_m(M)$ satisfies the property (*) for all $i < d$. Then $R/p$ is unmixed for all $p \in \text{Ass}(R)$ and the ring $R/\text{Ann}_R M$ is universally catenary.

Proof of Theorem 1.2. It follows from the Lemma 2.6 that $R/p = R/\text{Ann}_R(R/p)$ is universally catenary.

Set $S$ to be the image of $R \setminus p$ in $\widehat{R}$. We have

$$R_p / p R_p \otimes_R \widehat{R} \cong S^{-1}(\widehat{R}/p \widehat{R}).$$

We need to prove $(S^{-1}(\widehat{R}/p \widehat{R}))_{S^{-1}q}$ is Cohen-Macaulay for all $q \in \text{Spec}(\widehat{R})$ such
that \((\hat{q} \cap R) \cap S = \emptyset\). Assume that the statement is not true. Since
\[
(S^{-1}(\hat{R}/p \hat{R}))_{S^{-1}q} \cong (\hat{R}/p \hat{R})_{\hat{q}}
\]
as \(\hat{R}_{\hat{q}}\)-module, there exists \(\hat{q} \in \text{Spec}(\hat{R})\), \(\hat{q} \cap S = \emptyset\) such that \((\hat{R}/p \hat{R})_{\hat{q}}\) is not Cohen-Macaulay. Then there exists \(\hat{p} \in \text{Spec}(\hat{R})\), \(\hat{q} \supseteq \hat{p}\), \((\hat{p} \cap R) \cap S = \emptyset\) and \(\hat{p} \in \text{Min } \text{nCM}(\hat{R}/p \hat{R})\). Hence,
\[
\text{nCM}(\hat{R}/p \hat{R})_{\hat{p}} = \left\{ \hat{p} \hat{R}_{\hat{p}} \right\}.
\]
We have \(R/p\hat{R}\) is unmixed by Lemma 2.6. So \(\hat{R}/p \hat{R}\) is equidimensional. Hence \((\hat{R}/p \hat{R})_{\hat{p}}\) is equidimensional. On the other hand, since \((\hat{R}/p \hat{R})_{\hat{p}}\) is the image of a Cohen-Macaulay ring, \((\hat{R}/p \hat{R})_{\hat{p}}\) is generalized Cohen-Macaulay.

Set \(s = \dim \hat{R}/p \hat{R} = \text{ht} (p/p \hat{R})\). By Lemma 2.5, we have
\[
\text{nCM}(\hat{R}/p \hat{R})_{\hat{p}} = \bigcup_{i=0}^{s-1} \text{Psupp}_{\hat{R}}^{i}(\hat{R}/p \hat{R})_{\hat{p}}.
\]
Therefore, there exists \(i < s\) such that \(H^{i}_{P \hat{R}}(\hat{R}/p \hat{R})_{\hat{p}} \neq 0\). On the other hand,
\[
\ell(H^{i}_{P \hat{R}}(\hat{R}/p \hat{R})_{\hat{p}}) < \infty.
\]
Then
\[
\text{Att}_{\hat{R}}(H^{i}_{P \hat{R}}(\hat{R}/p \hat{R})_{\hat{p}}) = \left\{ p \hat{R}_{\hat{p}} \right\}.
\]
It is followed by Weak general Shifted Localization Principle (Lemma 2.3) that \(\hat{p} \in \text{Att}_{\hat{R}}(H^{i+\dim \hat{R}/p \hat{R}}_{P \hat{R}}(\hat{R}/p \hat{R}))\). Set \(j = i + \dim \hat{R}/p \hat{R}\). We have
\[
j < \text{ht} \hat{p}/p \hat{R} + \dim \hat{R}/p \hat{R} \leq \dim \hat{R}/p \hat{R} = \dim R/p.
\]
Hence, \(p \in \text{Att}_{R}(H^{j}_{m}(R/p))\) by Lemma 2.1. By Lemma 2.2
\[
\text{N-dim } H^{j}_{m}(R/p) \leq j < \dim R/p \leq R/\text{Ann}_{R} H^{j}_{m}(R/p).
\]
This implies that \(H^{j}_{d}(R/p)\) does not satisfy the property (*). It is in contradiction to the hypothesis. Therefore, all its formal fibers over \(p\) are Cohen-Macaulay.

\[\square\]

3. Conclusion

The paper gives a relation between the property (*) of local cohomology module and structure of base ring. In detail, we prove that for each \(p \in \text{Spec}(R)\) such that \(H^{i}_{m}(R/p)\) satisfies the property (*) for all \(i\), then \(R/p\) is universally catenary and the formal fibre of \(R\) over \(p\) is Cohen-Macaulay.

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