Nambu-Dirac Structures on Lie Algebroids

Aïssa WADE
Department of Mathematics, The Pennsylvania State University
University Park, PA 16802.
e-mail: wade@math.psu.edu

Abstract
The theory of Nambu-Poisson structures on manifolds is extended to the context of Lie algebroids in a natural way based on the Vinogradov bracket associated with Lie algebroid cohomology. We show that, under certain assumptions, any Nambu-Poisson structure on a Lie algebroid is decomposable. Also, we introduce the concept of a higher order Dirac structure on a Lie algebroid. This allows to describe both Nambu-Poisson structures on Lie algebroids and Dirac structures on manifolds in the same setting.

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1 Introduction
The purpose of this Letter is to explore an unusual approach to Nambu-Poisson manifolds using the Vinogradov bracket defined below and the theory of Dirac structures. Our first attempt to describe Nambu-Poisson manifolds in terms of Dirac structures can be found in [W]. In the present paper, we work in a context which is general enough to include Nambu-Poisson manifolds and triangular Lie bialgebroids [MX].

Nambu-Poisson structures originate from an idea presented in [Na], where Nambu took some tentative steps toward a generalization of Hamiltonian mechanics. In 1975, Bayen and Flato developed further Nambu’s generalized mechanics (see [BF]). Takhtajan gave an axiomatic formalism for Nambu’s bracket in [T]. Since then, there has been a great deal of interest in the study of Nambu-Poisson structures producing a beautiful series of papers by mathematicians and physicists.

Consider a Lie algebroid $E$ over a smooth manifold $M$. It is well known that the Lie algebroid structure on $E$ induces a differential operator $d_E$ on the space of all
smooth sections of $\Lambda^*E^*$, where $E^*$ is the dual of $E$. The Vinogradov bracket is a derived bracket in the sense of Kosmann-Schwarzbach (see [CV], [K-S]). Precisely, the Vinogradov bracket associated with $E$ is the $\mathbb{R}$-bilinear operation on the space of graded endomorphisms of $\Gamma(\Lambda^*E^*)$ given by $[[a, b]]_E = [[a, d_e], b]$, for any $a, b \in \text{End}(\Gamma(\Lambda^*E^*))$, where $[\cdot, \cdot]$ is the usual commutator on endomorphisms of $\Gamma(\Lambda^*E^*)$. In Theorem 2.1 we show that there exists an injective map $\sigma$ from $\Gamma(E \oplus \Lambda^*E^*)$ to $\text{End}(\Gamma(\Lambda^*E^*))$ and an $\mathbb{R}$-bilinear operation $[[\cdot, \cdot]]_E$ on $\Gamma(E \oplus \Lambda^*E^*)$ such that

(1) for any $e \in \Gamma(E \oplus \Lambda^*E^*)$, $\sigma(e)$ is $C^\infty$-linear;
(2) for any $e \in \Gamma(E \oplus \Lambda^*E^*)$ and for any $f \in C^\infty(M)$, $\sigma(fe) = f\sigma(e)$;
(3) for any $e_1, e_2 \in \Gamma(E \oplus \Lambda^*E^*)$, $\sigma([[e_1, e_2]]_E) = [[\sigma(e_1), \sigma(e_2)]]_E$.

Precisely, $\sigma$ is defined by $\sigma(X, \alpha) = i_X + m(\alpha)$, where $i_X$ is the interior product by $X$ and $m(\alpha)$ is the exterior multiplication by $\alpha$. The bracket $[[\cdot, \cdot]]_E$ is given by:

\[
[[X, \alpha], (Y, \beta)]_E = ([X, Y]_E, \mathcal{L}_X \beta - i_Y d_e \alpha),
\]

for any $(X, \alpha), (Y, \beta) \in \Gamma(E \oplus \Lambda^*E^*)$, where $\mathcal{L}_X = [i_X, d_e]$. In view of this correspondence, one may think of $\Gamma(E \oplus \Lambda^*E^*)$ as locally generated by field operators satisfying the canonical commutation relations. The elements of $\Gamma(E)$ correspond to boson fields and the elements of $\Gamma(E^*)$ to fermion fields.

In the second part of this work, we exploit the bracket $[[\cdot, \cdot]]_E$, and in this way we are led to the definition of a Nambu-Poisson structure on a Lie algebroid $E$. A Nambu-Poisson structure of order $p$ on $E$ is a smooth section $\Pi \in \Gamma(\Lambda^pE)$ such that

\[
[[\Pi \alpha, \Pi \beta]]_E = -\Pi(i_{\Pi \beta} d_e \alpha), \text{ for any } \alpha, \beta \in \Gamma(\Lambda^{p-1}E^*).
\]

This definition includes both triangular Lie bialgebroids (see [MX]) and Nambu-Poisson structures on manifolds, as shown below. We prove that, given an arbitrary Nambu-Poisson structure $\Pi$ of order $p \geq 3$ on a Lie algebroid $E$ with base $M$, $\Pi(d_e f)$ is decomposable for any $f \in C^\infty(M)$. In particular, $\Pi$ is decomposable when $\Gamma(E^*)$ is generated by the elements of the form $d_e f$.

Finally, we study Nambu-Poisson structures on a Lie algebroid $E$ from the point of view of the theory of Dirac structures. Very recently, the concept of a Nambu-Dirac manifold has been introduced and studied in [H]. This extension of the definition of a Dirac manifold is essentially based on the existence of a Leibniz algebroid structure on the vector bundle $\Lambda^{p-1}(T^*M)$ for a Poisson manifold $M$ of order $p$. Note that there is another Leibniz algebroid structure on $\Lambda^{p-1}(T^*M)$ presented in [ILMP]. Here, inspired by the remarkable interplay between skew-symmetric bilinear forms and bivectors arising from linear Dirac structures [C], we use an alternative approach to define a Dirac structure of higher order on a Lie algebroid. We then generalize both strong Nambu-Dirac [H] and Dirac structures on manifolds. We show that, to any
Nambu-Poisson structure $\Pi$ of order $p \geq 2$ on a Lie algebroid $E$, there corresponds a Dirac structure of order $p$. Furthermore, we prove that any Dirac structure of order $p$ on $E$ gives rise to a Nambu-Poisson bracket on the set of its admissible functions. Thus, we recover certain results obtained in [H]. Our main results are Theorem 2.1, Theorem 3.1, and Theorem 4.6.

The paper is organized as follows. In Section 2, we review basic definitions and properties of Leibniz algebras and algebroids, as well as Lie algebroids. Then we define the Vinogradov bracket associated with a Lie algebroid and establish Theorem 2.1. In Section 3, we define Nambu-Poisson structures on Lie algebroids and study their properties. Finally, in Section 4, we study the relationships between Nambu-Poisson structures on Lie algebroids and the theory of Dirac structures.

2 Leibniz Algebras and Vinogradov Bracket

2.1 Definitions

Throughout the text, all vector bundles have finite dimension. We will be using Leibniz algebras and Lie algebroids in many parts of this work, so we briefly recall the definitions.

A left Leibniz bracket on a real vector space $V$ (see [L]) is given by an $\mathbb{R}$-bilinear operation $[\cdot, \cdot]_V : V \times V \to V$ such that

$$[[a, b], c]_V = [a, [b, c]]_V + [b, [a, c]]_V,$$

for any $a, b, c \in V$. Then the pair $(V, [\cdot, \cdot]_V)$ is called a Leibniz algebra. For short, it will be called Leibniz algebra. The bracket $[\cdot, \cdot]_V$ becomes a Lie bracket if, additionally, it is skew-symmetric.

Let $(V_1, [\cdot, \cdot]_1)$ and $(V_2, [\cdot, \cdot]_2)$ be two Leibniz algebras. A morphism $\phi$ from $(V_1, [\cdot, \cdot]_1)$ to $(V_2, [\cdot, \cdot]_2)$ is said to be a morphism of Leibniz algebras if $\phi([a, b]_1) = [\phi(a), \phi(b)]_2$, for any $a, b \in V_1$.

A Leibniz algebra (see [ILMP]) consists of a vector bundle $E \to M$ with a Leibniz bracket $[\cdot, \cdot]_E$ on the space $\Gamma(E)$ of smooth sections of $E$ and a bundle map $\rho : E \to TM$, extended to a map between sections of these bundles such that the following properties are satisfied:

$$[X_1, fX_2]_E = f[X_1, X_2]_E + (\rho(X_1)f)X_2 \quad \text{and} \quad \rho([X_1, X_2]_E) = [\rho(X_1), \rho(X_2)],$$

for any smooth sections $X_1, X_2 \in \Gamma(E)$ and for any smooth function $f$ defined on $M$. Then $\rho$ is called the anchor of the Leibniz algebroid.

A Lie algebroid is a Leibniz algebroid $(E, [\cdot, \cdot]_E, \rho)$ over $M$ such that $[\cdot, \cdot]_E$ is skew-symmetric.
2.2 The Vinogradov bracket of a Lie algebroid

Let $E$ be a Lie algebroid over a smooth $n$-dimensional manifold $M$ with anchor map $\rho$ and let $E^*$ be its dual. Let $\Gamma(\Lambda^*E^*) = \oplus_{k \geq 0}\Gamma(\Lambda^kE^*)$ denote the exterior algebra generated by all smooth sections of $E^*$. Define $d_\xi : \Gamma(\Lambda^{k-1}E^*) \longrightarrow \Gamma(\Lambda^kE^*)$ by:

$$(d_\xi \xi)(e_1, \ldots, e_k) = \sum_i (-1)^{i+1}\rho(e_i)(\xi(e_1, \ldots, \hat{e}_i, \ldots, e_k)) + \sum_{i<j} (-1)^{i+j}\xi([e_i, e_j]_E, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_k),$$

for any $\xi \in \Gamma(\Lambda^{k-1}E^*)$, and for any $e_1, \ldots, e_k \in \Gamma(E)$. It satisfies $d_\xi^2 = 0$. The cohomology of $(\Gamma(\Lambda^*E^*), d_\xi)$ is called the Lie algebroid cohomology of $E$ and denoted by $H^*(E)$. Let $\text{End}^\ell(\Gamma(\Lambda^*E^*))$ denote the space of all graded endomorphisms of degree $\ell$ on $\Gamma(\Lambda^*E^*)$, and $\text{End}(\Gamma(\Lambda^*E^*)) = \oplus_{\ell \geq 0}\text{End}^\ell(\Gamma(\Lambda^*E^*))$.

Then, the Vinogradov bracket associated with the Lie algebroid $E$ is the $\mathbb{R}$-bilinear operation $\llbracket \cdot, \cdot \rrbracket_E$ defined by $\llbracket a, b \rrbracket_E = \llbracket [a, d_\xi], b \rrbracket$, for any $a, b \in \text{End}(\Gamma(\Lambda^*E^*))$, where $\llbracket \cdot, \cdot \rrbracket$ is the usual commutator on endomorphisms of $\Gamma(\Lambda^*E^*)$, i.e.

$$\llbracket a, b \rrbracket = a \circ b - (-1)^{|a|} b \circ a,$$

when $a, b$ are graded endomorphisms with respective degrees $|a|, |b|$.

For instance, the tangent bundle $TM$ of a smooth manifold $M$ defines a Lie algebroid with respect to the usual Lie bracket on vector fields. The Vinogradov bracket associated with $TM$ was used in [CV] as a tool for extending an arbitrary Poisson bracket defined on $C^\infty(M)$ to the space of co-exact forms. As mentioned above, the Vinogradov bracket associated with a Lie algebroid $E$ described here is a particular case of the general construction of derived brackets presented in [KS]. In fact, what Kosmann-Schwarzbach called Vinogradov bracket in [KS] is the skew-symmetrization of $\llbracket \cdot, \cdot \rrbracket_E$. But, we will use the non skew-symmetric version in this work. The main property of $\llbracket \cdot, \cdot \rrbracket_E$ (see [KS]) is that

$$\llbracket a, \llbracket b, c \rrbracket_E \rrbracket_E = \llbracket \llbracket a, b \rrbracket_E, c \rrbracket_E + (-1)^{|a|+1}(-1)^{|b|+1} \llbracket \llbracket b, a \rrbracket_E, c \rrbracket_E,$$

for any graded endomorphisms $a, b,$ and $c$ with respective degrees $|a|, |b|$, and $|c|$.

Let $i_X$ denote the interior product by $X \in \Gamma(E)$ and $m(\alpha)$ the multiplication by $\alpha \in \Gamma(\Lambda^*E^*)$, i.e. $m(\alpha)(\eta) = \alpha \wedge \eta, \forall \eta \in \Gamma(\Lambda^*E^*)$. We associate the $C^\infty(M)$-linear map $\sigma : \Gamma(E \oplus \Lambda^*E^*) \rightarrow \text{End}(\Gamma(\Lambda^*E^*))$ given by $\sigma(X, \alpha) = i_X + m(\alpha)$, for any $(X, \alpha) \in \Gamma(E \oplus \Lambda^*E^*)$. With these notations, we have:

**Theorem 2.1** Let $(E, \llbracket \cdot, \cdot \rrbracket_E, \rho)$ be a Lie algebroid over $M$ and let $\llbracket \cdot, \cdot \rrbracket_E$ be its corresponding Vinogradov bracket. Then, there is a $\mathbb{R}$-bilinear operation $\llbracket \cdot, \cdot \rrbracket_E$ on
\( \Gamma(E \oplus \Lambda^*E^*) \) such that \( \sigma : \Gamma(E \oplus \Lambda^*E^*) \rightarrow (\text{End}(\Gamma(\Lambda^*E^*)), \llbracket \cdot, \cdot \rrbracket_E) \) is an injective morphism of Leibniz algebras. Precisely, \( \llbracket \cdot, \cdot \rrbracket_E \) is defined by:

\[
\llbracket (X, \alpha), (Y, \beta) \rrbracket_E = ([X,Y]_E, \mathcal{L}^E_X \beta - i_Y d_e \alpha),
\]

for any \((X, \alpha), (Y, \beta) \in \Gamma(E \oplus \Lambda^*E^*)\), where \( \mathcal{L}^E_X = [i_X, d_e] \).

To prove this theorem, we need the following lemma.

**Lemma 2.2** Let \( X \in \Gamma(E) \), \( \alpha, \beta \in \Gamma(\Lambda^*A^*) \). Then,

(i) \( [i_X, m(\alpha)] = m(i_X \alpha) \);

(ii) \( [\mathcal{L}^E_X, m(\alpha)] = m(\mathcal{L}^E_X \alpha) \);

(iii) \( [d_e, m(\alpha)] = m(d_e \alpha) \);

(iv) \( [m(\alpha), m(\beta)] = 0 \);

(v) \( [\mathcal{L}^E_X, \mathcal{L}^E_Y] = \mathcal{L}^E_{(X,Y)} \).

The proof of this lemma is straightforward.

**Proof of Theorem 2.1.** For any \( X, Y \in \Gamma(E) \), we have \( \llbracket i_X, i_Y \rrbracket_E = [\mathcal{L}^E_X, i_Y] = [X,Y]_E \).

Applying Property (ii) of Lemma 2.2 we get \( \llbracket i_X, m(\beta) \rrbracket_E = [\mathcal{L}^E_X, m(\beta)] = m(\mathcal{L}^E_X \beta) \), for any \( X \in \Gamma(E) \), \( \beta \in \Gamma(\Lambda^*E^*) \). Using (iii), (i) and the symmetry property of the commutator, one gets:

\[
\llbracket m(\alpha), i_Y \rrbracket_E = \llbracket m(\alpha), d_e \rrbracket_E, \ i_Y \rrbracket = - \llbracket i_Y, m(d_e \alpha) \rrbracket = -m(i_Y d_e \alpha),
\]

for any \( Y \in \Gamma(E) \), \( \alpha \in \Gamma(\Lambda^*A^*) \). Finally, Property (iv) implies \( \llbracket m(\alpha), m(\beta) \rrbracket_E = 0 \), for any \( \alpha, \beta \in \Gamma(\Lambda^*A^*) \). Thus combining the above terms, we obtain

\[
\llbracket i_X + m(\alpha), i_Y + m(\beta) \rrbracket_E = i_{[X,Y]}_E + m(\mathcal{L}^E_X \beta - i_Y d_e \alpha),
\]

for any \( X, Y \in \Gamma(E) \), \( \alpha, \beta \in \Gamma(\Lambda^*E^*) \). Equivalently,

\[
\llbracket \sigma(X, \alpha), \sigma(Y, \beta) \rrbracket_E = \sigma([X,Y]_E, \mathcal{L}^E_X \beta - i_Y d_e \alpha) = \sigma(\llbracket (X, \alpha), (Y, \beta) \rrbracket_E).
\]

Since \( \sigma \) is injective and \( \llbracket \cdot, \cdot \rrbracket_E \) is a Leibniz bracket, it follows that \( \llbracket \cdot, \cdot \rrbracket_E \) is a Leibniz bracket. Hence, \( \sigma \) is an injective morphism of Leibniz algebras. This completes the proof of Theorem 2.1.
Example 1. Let $E = TM$ be the tangent bundle of a smooth manifold $M$. Then, applying Theorem 2.1 we get an operation $[[\cdot, \cdot]]$ on $\Gamma(TM \oplus \Lambda^*T^*M)$ which is “isomorphic” to the Vinogradov bracket via the map $\sigma$. Namely, $[[\cdot, \cdot]]$ is given by

$$[[\langle X, \alpha \rangle, \langle Y, \beta \rangle]] = ([X, Y], L_\alpha \beta - i_Y d\alpha),$$

where $[X, Y]$ is the Lie bracket of the vector fields $X$ and $Y$, $L_\alpha$ is the Lie derivative with respect to the vector field $X$ and $d$ is the de Rham differential. The restriction of $[[\cdot, \cdot]]$ to $\Gamma(TM \oplus T^*M)$ coincides with the Courant bracket (see [C]).

Remark. Theorem 2.1 pushes further an observation made by Kosmann-Schwarzbach in an unpublished paper, where it is shown that the Courant bracket is obtained by using $\sigma|_{\Gamma(TM \oplus T^*M)}$.

3 Nambu-Poisson Structures

3.1 Nambu-Poisson Structures on Manifolds

Let $M$ be a smooth $n$-dimensional manifold. A Nambu-Poisson structure on $M$ of order $p$ (with $2 \leq p \leq n$) is given by a $p$-vector field $\Pi$ which satisfies the fundamental identity:

$$\{f_1, \ldots, f_{p-1}, \{g_1, \ldots, g_p\}_\Pi\} = \sum_{k=1}^{n} \{g_1, \ldots, g_{k-1}, \{f_1, \ldots, f_{p-1}, g_k\}_\Pi, g_{k+1}, \ldots, g_p\}_\Pi$$

for any $f_1, \ldots, f_{p-1}, g_1, \ldots, g_p \in C^\infty(M)$, where $\{ \}$ is defined by

$$\{f_1, \ldots, f_p\}_\Pi = \Pi(df_1, \ldots, df_p) \quad \forall f_1, \ldots, f_p \in C^\infty(M).$$

A manifold equipped with such a structure is called a Nambu-Poisson manifold. Nambu-Poisson structures of order 2 on $M$ are Poisson structures. It is easy to see that the fundamental identity is equivalent to $[X_{f_1, \ldots, f_{p-1}}, \Pi]_{SN} = 0$, for any $f_1, \ldots, f_{p-1} \in C^\infty(M)$, where $[\cdot, \cdot]_{SN}$ is the Schouten-Nijenhuis on multi-vector fields and $X_{f_1, \ldots, f_{p-1}}$ is the Hamiltonian vector field given by $\langle X_{f_1, \ldots, f_{p-1}}, dg \rangle = \{f_1, \ldots, f_{p-1}, g\}_\Pi$.

3.2 Nambu-Poisson structures on Lie algebroids

It is known that, given a Lie algebroid $(E, [\cdot, \cdot]_E, \rho)$ over $M$, the algebra $\Gamma(\Lambda^*E)$ endowed with the exterior product and the generalized Schouten bracket determines a Gerstenhaber algebra. Consider a smooth section $\Pi \in \Gamma(\Lambda^pE)$. We say that $\Pi$ is a Nambu-Poisson structure of order $p$ on $E$ if

$$[\Pi \alpha, \Pi \beta]_E = -\Pi(\Pi \beta d_\Pi \alpha), \text{ for any } \alpha, \beta \in \Gamma(\Lambda^{p-1}E^*).$$
In particular, this property implies that \([\Pi \alpha, \Pi] = 0\) when \(d_E \alpha = 0\). This definition agrees with that of a Nambu-Poisson structure on a manifold, which is obtained when \(E = TM\).

It is natural to ask if a Nambu-Poisson structure of order \(p > 2\) on a Lie algebroid \(E\) is decomposable. It is well known that such a phenomenon occurs for Nambu-Poisson structures of order \(p > 2\) on a smooth manifold \(M\). It turns out that any Nambu-Poisson structure of order \(p > 2\) on a Lie algebroid \(E\) is decomposable if \(\Gamma(E^*)\) is generated by the sections \(d_E f\), for \(f \in C^\infty(M)\). Precisely, we have the following result:

**Theorem 3.1** Let \(\Pi\) be a Nambu-Poisson structure of order \(p \geq 3\) on a Lie algebroid \(E\). For any \(f \in C^\infty(M)\), \(\Pi(d_E f)\) is decomposable. In particular, if \(\Gamma(E^*)\) is generated by elements of the form \(d_E f\) then \(\Pi\) is decomposable.

**Proof:** Assume that \(\Pi\) is a Nambu-Poisson structure of order \(p \geq 3\) on \(E\). Take \(f \in C^\infty(M)\), \(\eta \in \Gamma(\Lambda^{p-1}E^*)\) and set \(\alpha = fd_E f \wedge \eta\). On the one hand, for any \(\beta \in \Gamma(\Lambda^{p-2}E^*)\) we have

\[
[\Pi \alpha, \Pi]_E \beta = -\Pi(i_{\Pi \beta} d_E \alpha) = f \Pi \left( i_{\Pi \beta} (d_E f \wedge d_E \eta) \right).
\]

On the other hand,

\[
[\Pi \alpha, \Pi]_E \beta = [f \Pi(d_E f \wedge \eta), \Pi]_E \beta \\
= -\left( \pm \Pi(d_E f) \wedge \Pi(d_E f \wedge \eta) + f [\Pi(d_E f \wedge \eta), \Pi]_E \right) \beta \\
= -\left( \pm \Pi(d_E f) \wedge \Pi(d_E f \wedge \eta) \right) \beta + f \Pi \left( i_{\Pi \beta} (d_E f \wedge d_E \eta) \right).
\]

Therefore,

\[
\Pi(d_E f) \wedge \Pi(d_E f \wedge \eta) = 0, \text{ for any } f \in C^\infty(M), \eta \in \Gamma(\Lambda^{p-2}E^*).
\]

Hence, \(\Pi(d_E f)\) is decomposable (see \([MVM]\)). There follows the result.

**Proposition 3.2** Consider a Lie algebroid \(E\) with base \(M\). Then \(\Pi\) is a Nambu-Poisson structure of order 2 on \(E\) if and only if \(\Pi \Pi = 0\).

This proposition shows that the definition of Nambu-Poisson structure on \(E\) agrees with that of a Poisson structure on \(E\) when \(p = 2\). To prove it, we need the following lemma which has been shown in \([K-SM]\) and \([LX]\).

**Lemma 3.3** For any \(\eta_1\) and \(\eta_2 \in \Gamma(E^*)\), we have

\[
\Pi \left( \mathcal{L}_{\eta_1} \eta_2 - \mathcal{L}_{\eta_2} \eta_1 - d_E (\Pi(\eta_1, \eta_2)) \right) = \frac{1}{2} \Pi \Pi (\eta_1 \wedge \eta_2).
\]
Proposition 3.4. Therefore, we get the following result:

\[ \lfloor \lfloor \Pi \eta_1, \Pi \eta_2 \rfloor_E = [\Pi \eta_1, \Pi] \eta_2 + \Pi([\mathcal{L}_E^\Pi \eta_2]). \]

Add \( \Pi(i_{\eta_1} d_\xi \eta_2) \) in both sides of this last equation. We get

\[ [\Pi \eta_1, \Pi \eta_2]_E - \Pi([\mathcal{L}_E^\Pi \eta_2] + \Pi(i_{\eta_1} d_\xi \eta_2) = [\Pi \eta_1, \Pi] \eta_2 + \Pi(i_{\eta_2} d_\xi \eta_1). \]

Now using Lemma 3.3, we get \( \frac{1}{2}[\Pi, \Pi]_E (\eta_1 \wedge \eta_2) = -\left( [\Pi \eta_1, \Pi] \eta_2 + \Pi(i_{\eta_2} d_\xi \eta_1) \right) \). Hence, \( [\Pi, \Pi]_E = 0 \iff [\Pi \eta_1, \Pi] \eta_2 + \Pi(i_{\eta_2} d_\xi \eta_1) = 0 \), for any \( \eta_1 \) and \( \eta_2 \in \Gamma(E^*) \).

Proof of Proposition 3.2: Given a Lie algebroid \( E \) with base \( M \) gives rise to a natural Nambu-Poisson bracket of order \( p \) on the algebra \( C^\infty(M) \) of functions on \( M \). Define the bracket \( \{ \ldots \} \) by

\[ \{ f_1, \ldots, f_p \}_E = \Pi(d_\xi(f_1) \wedge \ldots \wedge d_\xi(f_p)), \]

for \( f_1, \ldots, f_p \in C^\infty(M) \). Obviously, this bracket is skew-symmetric and \( \{ f_1, \ldots, f_{p-1}, \ldots \}_E \) is a derivation with respect to the pointwise product of functions. We denote by \( X_{f_1, \ldots, f_{p-1}} \) the corresponding vector field. Then, we get

\[ \mathcal{L}_{X_{f_1, \ldots, f_{p-1}}} g = \{ f_1, \ldots, f_{p-1}, g \}_E. \]

Furthermore, since \( [\Pi \eta_1, \Pi]_E = 0 \), it follows

\[ \{ f_1, \ldots, f_{p-1}, \{ g_1, \ldots, g_p \}_E = \mathcal{L}_{X_{f_1, \ldots, f_{p-1}}} \{ g_1, \ldots, g_p \}_E \]

\[ = \sum_{i=1}^p \{ g_1, \ldots, g_{i-1}, \{ f_1, \ldots, f_{p-1}, g_i \}_E, \ldots, g_p \} \]

Therefore, we get the following result:

Proposition 3.4. Given a Lie algebroid \( (E, [\cdot, \cdot]_E, \rho) \) with base \( M \), there is a natural Nambu-Poisson bracket on \( C^\infty(M) \) arising from any Nambu-Poisson structure on \( E \).

Proposition 3.5. Given a Lie algebroid \( (E, [\cdot, \cdot]_E, \rho) \) over \( M \) and a Nambu-Poisson structure \( \Pi \) on \( E \), the triplet \( (\Lambda^{p-1}E^*, [\cdot, \cdot]_s, \rho \circ \Pi) \) determines a Leibniz algebroid, where \( [\cdot, \cdot]_s \) is defined by

\[ [\alpha, \beta]_s = \mathcal{L}^\Pi_{\alpha \beta} - d_\xi \alpha \beta, \text{ for any } \alpha, \beta \in \Gamma(\Lambda^{p-1}E^*) \]

Proof: By simple computations, one gets

\[ [\alpha, f \beta]_s = f [\alpha, \beta]_s + (\rho(\Pi \alpha) f) \beta \quad \text{and} \quad \rho(\Pi [\alpha, \beta]_s) = \rho(\Pi \alpha), \rho(\Pi \beta)), \]

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for any $\alpha, \beta \in \Gamma(\Lambda^{p-1}E^*)$. Moreover, we have:
\[
\begin{align*}
\lfloor \lfloor \lfloor \alpha, \beta \rfloor, \gamma \rfloor \rfloor \rfloor \ = \ L_{\Pi \alpha}(\lfloor \lfloor \beta, \gamma \rfloor \rfloor \lfloor \lfloor \alpha, \gamma \rfloor \rfloor \rfloor \lfloor \lfloor \beta, \alpha \rfloor \rfloor \rfloor \ = \ L_{\Pi \alpha}(L_{\Pi \beta} \gamma) - L_{\Pi \alpha}(i_{\Pi \gamma}d\beta) \\
- L_{\Pi \beta}(i_{\Pi \gamma}d\alpha) + i_{\Pi \gamma}(L_{\Pi \beta}d\alpha).
\end{align*}
\]

By similar computations we obtain
\[
\begin{align*}
\lfloor \lfloor \lfloor \lfloor \alpha, \beta \rfloor \rfloor, \gamma \rfloor \rfloor \rfloor \ = \ L_{\Pi \alpha}(\lfloor \lfloor \beta, \gamma \rfloor \rfloor \lfloor \lfloor \alpha, \gamma \rfloor \rfloor \rfloor \rfloor \ = \ L_{\Pi \alpha}(L_{\Pi \beta} \gamma) - L_{\Pi \alpha}(i_{\Pi \gamma}d\beta) \\
- L_{\Pi \beta}(i_{\Pi \gamma}d\alpha) + i_{\Pi \gamma}(L_{\Pi \beta}d\alpha).
\end{align*}
\]

Thus, $\lfloor \lfloor \lfloor \lfloor \alpha, \beta \rfloor \rfloor, \gamma \rfloor \rfloor \rfloor \ = \ L_{\Pi \alpha}(\lfloor \lfloor \beta, \gamma \rfloor \rfloor \lfloor \lfloor \alpha, \gamma \rfloor \rfloor \rfloor \rfloor \rfloor \ = \ L_{\Pi \alpha}(L_{\Pi \beta} \gamma) - L_{\Pi \alpha}(i_{\Pi \gamma}d\beta) \\
- L_{\Pi \beta}(i_{\Pi \gamma}d\alpha) + i_{\Pi \gamma}(L_{\Pi \beta}d\alpha)$. The proposition follows immediately.

\section{Higher order Dirac structures}

\subsection{Linear Dirac structures}

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ and $V^*$ its dual. Consider the canonical symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V \oplus V^*$ given by
\[
\langle (v_1, \omega_1), (v_2, \omega_2) \rangle = \frac{1}{2}(\omega_1(v_2) + \omega_2(v_1)).
\]

A linear Dirac structure on $V$ (see [C]) is a subspace $L$ of $V \oplus V^*$ which is maximally isotropic with respect to $\langle \cdot, \cdot \rangle$. There are two equivalent characterizations of linear Dirac structures given by the following propositions.

**Proposition 4.1** [C] There is a bijective correspondence between linear Dirac structures on $V$ and the pairs $(P, \Omega)$ consisting of a subspace $P$ of $V$ and a skew-symmetric bilinear form $\Omega : P \times P \to \mathbb{R}$

**Proof:** Let $\pi : V \oplus V^* \to V$ and $\pi^* : V \oplus V^* \to V^*$ be the canonical projections. If $L \subset V \oplus V^*$ is a linear Dirac structure on $V$, then there exists a well defined skew-symmetric bilinear map $\Omega_L : \pi(L) \times \pi(L) \to \pi(L)$ given by $\Omega_L(v_1, v_2) = \omega_1(v_2)$, for any $(v_1, \omega_1), (v_2, \omega_2) \in L$.

Conversely, let $P$ be a subspace of $V$ equipped with a skew-symmetric bilinear form $\Omega : P \to P^*$. Define $L = \{ (v, \omega) \in V \oplus V^* \mid v \in P \text{ and } \Omega(v) = \omega_P \}$. $L$ is isotropic with respect to $\langle \cdot, \cdot \rangle$ since $\Omega$ is skew-symmetric. Moreover,
\[
\dim L = \dim P + \dim \ker \pi_P = \dim P + \dim \text{Ann}(P) = n,
\]
where Ann($P$) denotes the annihilator of the subspace $P$. Therefore, $L$ is a linear Dirac structure on $V$.

Similarly, one gets the proposition:

**Proposition 4.2** There is a bijective correspondence between linear Dirac structures on $V$ and pairs $(S, \Pi)$ consisting of a subspace $S$ of $V^*$ and a 2-vector $\Pi \in \Lambda^2 S^*$.

### 4.2 Higher order Dirac structures on vector spaces

All vector spaces considered in this section have finite dimension.

**Definition.** Let $V$ and $W$ be two vector spaces. A linear quasi Dirac structure of order $p$ on $V$ relative to $W$ consists of a subspace $P$ of $V$ together with a skew-symmetric $p$-form $\Omega : P \times \ldots \times P \rightarrow W$.

For instance, let $V = W = \mathbb{R}^n$. Consider a skew-symmetric bilinear operation $B$ on $\mathbb{R}^n$. Then $(\mathbb{R}^n, B)$ is a linear quasi Dirac structure of order 2 on $\mathbb{R}^n$ relative to $\mathbb{R}^n$.

**Definition.** A linear Dirac structure of order $p$ on a real vector space $V$ relative to $\mathbb{R}$ is a linear quasi Dirac structure $(P, \Omega)$ of order $p$ on $V$ such that for any $Z_1, \ldots, Z_{p-1} \in \Lambda^{p-1} P$, there exists $v \in P$ satisfying

$$(\text{H}) \quad (iZ_1 \Omega) \wedge \ldots \wedge (iZ_{p-1} \Omega) = i_v \Omega.$$ 

Obviously, (H) is always satisfied when $p = 2$. In other words, any linear quasi Dirac structure $(P, \Omega)$ of order 2 is a linear Dirac structure of order 2.

For a better understanding of the condition (H) for $p > 2$, let us consider the subspace $P^*$ given by $E_\Omega = \text{Span} \{ i_{v_1} \ldots i_{v_{p-1}} \Omega \mid v_i \in P \}$. It is known that $\dim(E_\Omega) \geq p$ and $\dim(E_\Omega) = p$ if and only if $\Omega$ is decomposable. Let $\{ \xi_1, \ldots, \xi_r \}$ be a basis for $E_\Omega$. Then,

$$\Omega = \sum_{1 \leq i_1 < \ldots < i_{p-1} \leq r} \xi_{i_1} \wedge \ldots \wedge \xi_{i_{p-1}} \wedge \xi_{i_1 \ldots i_{p-1}},$$

where $\xi_{i_1 \ldots i_{p-1}} \in E_\Omega$. The condition (H) means that none of the 1-forms $\xi_{i_1 \ldots i_{p-1}}$ is zero. We deduce that (H) is always true when $\Omega$ is decomposable.

Now, consider a linear Dirac structure $(P, \Omega)$ of order $p$ on a real vector space $V$ relative to $\mathbb{R}$. On the one hand, $\Omega$ can be viewed as a map from $P$ to $\Lambda^{p-1} P^*$. In this case, we associate the subspace $L$ of $V \times \Lambda^{p-1} V^*$:

$$L = \{ (v, \omega) \mid v \in P \text{ and } i_v \Omega = \omega_{\Lambda^{p-1} P} \}.$$ 

On the other hand, regarding $\Omega$ as a map from $\Lambda^{p-1} P$ to $P^*$, the natural subspace $L^\perp \subset \Lambda^{p-1} V \times V^*$ to be defined is:

$$L^\perp = \{ (Z, \eta) \mid Z \in \Lambda^{p-1} P \text{ and } i_Z \Omega = \eta_{\Lambda^p} \}.$$
For any \((v, \omega) \in L\) and for any \((Z, \eta) \in L^\perp\), we have
\[
\eta_p(v) + (-1)^p \omega_{\Lambda^{p-1}p}(Z) = 0,
\]
since \(\Omega(Z)(v) = (-1)^{p-1}\Omega(v)(Z)\).

**Notation.** Let \(\pi^*(L^\perp)\) denote the projection of \(L^\perp\) onto \(V^*\) and \(\pi^*(L)\) the projection of \(L\) onto \(\Lambda^{p-1}V^*\). Then we have the following lemmas.

**Lemma 4.3**
\[
\text{Ann}(\pi^*(L^\perp)) = L \cap V.
\]
**Proof:** In fact, \(v \in \text{Ann}(\pi^*(L^\perp))\) if and only if \(\eta(v) = 0, \forall (Z, \eta) \in L^\perp\). This is equivalent to say that \(\Omega(Z)(v) = (-1)^{p-1}\Omega(v)(Z) = 0, \forall Z \in \Lambda^{p-1}P\). But, \(\Omega(v) = 0\) means that \(v \in L \cap V\). There follows the lemma.

**Lemma 4.4**
\[
\Lambda^{p-1}\left(\pi^*(L^\perp)\right) \subset \pi^*(L).
\]
**Proof:** For any \((Z_1, \eta_1) \ldots (Z_{p-1}, \eta_{p-1}) \in L^\perp\), we have
\[
\eta_1 \wedge \ldots \wedge \eta_{p-1} |_{\Lambda^{p-1}P} = (i_{Z_1}, \Omega) \wedge \ldots \wedge (i_{Z_{p-1}}, \Omega).
\]
We deduce from the definition of a linear Dirac structure of order \(p\) the existence of an element \(v \in P\) such that \(i_i \Omega = \eta_1 \wedge \ldots \wedge \eta_{p-1} |_{\Lambda^{p-1}P}\). Thus \((v, \eta_1 \wedge \ldots \wedge \eta_{p-1})\) is in \(L\). There follows the lemma.

Now, we set \(S = \pi^*(L^\perp)\) and identify \(S^*\) with \(V/L \cap V\). Consider the \(p\)-vector \(\Pi \in \Lambda^pS^*\) defined by \(\Pi(\eta) = Z |_{\Lambda^{p-1}S}\), for any \((Z, \eta) \in L^\perp\). \(\Pi\) is well defined. Indeed, if \((Z_1, \eta)\) and \((Z_2, \eta)\) are in \(L^\perp\), then \((Z_1 - Z_2, 0) \in L^\perp\). Let \(\eta_1, \ldots, \eta_{p-1}\) be in \(S\). There exists \(v\) in \(P\) such that \((v, \eta_1 \wedge \ldots \wedge \eta_{p-1}) \in L\). Thus, \((Z_1 - Z_2)(\eta_1 \wedge \ldots \wedge \eta_{p-1}) = 0\). This shows that \(Z_1 = Z_2\) on \(\Lambda^{p-1}S\). Since \(\Pi\) is skew-symmetric, we get
\[
Z_1(\eta_2) + Z_2(\eta_1) |_{\Lambda^{p-1}S} = 0, \forall (Z_1, \eta_1), (Z_2, \eta_2) \in L^\perp.
\]
If \((v, \eta_1 \wedge \ldots \wedge \eta_{p-1}) \in L\), with \(\eta_i \in S\), then \(\Pi(\eta_1 \wedge \ldots \wedge \eta_{p-1}) = v |_S\). This follows from the fact that, for any \((Z_p, \eta_p) \in L^\perp\), we have
\[
\Pi(\eta_1 \wedge \ldots \wedge \eta_{p-1})(\eta_p) = (-1)^{p-1}(\Pi(\eta_p))(\eta_1 \wedge \ldots \wedge \eta_{p-1})
\]
\[
= (-1)^{p-1}Z_p(\eta_1 \wedge \ldots \wedge \eta_{p-1}) = v(\eta_p).
\]
Furthermore, \( \Pi \) has the following property: for any \( \omega_1, \ldots, \omega_{p-1} \in \Lambda^{p-1}S \), there exists \( \eta \in S \) such that

\[
(\mathcal{H}') \quad \Pi(\omega_1) \wedge \ldots \wedge \Pi(\omega_{p-1}) = \Pi(\eta).
\]

In a similar way, one can show that, given any pair \((S, \Pi)\) of a subspace \( S \) of \( V^* \) and a \( p \)-vector on \( S \) that satisfies \((\mathcal{H}')\), there corresponds a linear Dirac structure \((P, \Omega)\) of order \( p \) on \( V \). There follows the result:

**Proposition 4.5** There is a one-to-one correspondence between linear Dirac structures of order \( p \) on a real vector space \( V \) relative to \( \mathbb{R} \) (with \( 2 \leq p \leq \dim V \)) and the pairs \((S, \Pi)\) consisting of a subspace \( S \subset V^* \) and a \( p \)-vector \( \Pi \in \Lambda^p S^* \) satisfying the property \((\mathcal{H}')\).

### 4.3 Higher order Dirac structures on Lie algebroids

**Definition.** Let \((E, [\cdot, \cdot]_E, \rho)\) be a Lie algebroid over \( M \). A Dirac structure of order \( p \) on \( E \) is a sub-bundle \( L \) of \( E \oplus \Lambda^p E^* \) which determines a linear Dirac structure of order \( p \) relative to \( \mathbb{R} \) at each point and whose sections are closed under the Leibniz bracket \([\cdot, \cdot]_E\) defined as in (1.1). The fact that \( L \) determines a linear Dirac structure of order \( p \) means that there exist a sub-bundle \( P \) of \( E \) and a skew-symmetric bilinear form \( \Omega_L \) on \( P \) satisfying \((\mathcal{H}')\) and such that the fibre of \( L \) at a point \( x \in M \) is

\[
L_x = \\left\{ (X, \alpha)_x \mid X \in P_x \text{ and } (\Omega_L(X))_x = \alpha |_{\Lambda^{|x|(p-1)}P_x} \right\}.
\]

Observe that if the sections of a sub-bundle \( L \) of \( E \oplus \Lambda^p E^* \) are closed under \([\cdot, \cdot]_E\) then \([L, [\cdot, \cdot]_E, \rho_L] \) determines a Leibniz algebroid, where \( \rho_L(X, \alpha) = \rho(X) \) at every point.

**Example 2.** A Dirac structure of order 2 on \( T M \) is just a Dirac structure on \( M \) in the sense of Courant (see [C]).

**Example 3.** Let \( \Pi \) be a Nambu-Poisson structure of order \( p \geq 2 \) on a Lie algebroid \( E \). With the notation \( X_{f_1 \ldots f_{p-1}} = \Pi(d_{x_1} f_1 \wedge \ldots \wedge d_{x_p} f_{p-1}) \), consider

\[
L = \text{Span } \{ (X_{f_1 \ldots f_{p-1}}, d_{x_1} f_1 \wedge \ldots \wedge d_{x_p} f_{p-1}) \mid f_i \in C^\infty(M) \}.
\]

For \( f_1, \ldots, f_{p-1}, g_1, \ldots, g_{p-1} \in C^\infty(M) \), we have

\[
[[X_{f_1 \ldots f_{p-1}}, d_{x_1} g_1 \wedge \ldots \wedge d_{x_p} g_{p-1}], (X_{g_1 \ldots g_{p-1}}, d_{x_1} f_1 \wedge \ldots \wedge d_{x_p} f_{p-1})]]_E = \\
\sum_{i=1}^{p-1} d_{x_i} g_1 \wedge \ldots \wedge d_{x_i} (f_1, \ldots, f_{p-1}, g_i)_{\Pi} \wedge \ldots \wedge d_{x_p} g_{p-1}.
\]

Moreover for any \((X, \alpha)\) and \((Y, \beta)\) \( \in \Gamma(L) \), \( f, g \in C^\infty(M) \),
|f(X, α), g(Y, β)|_E = (fg[X, Y]_E + f⟨d_ε g, X⟩Y − g⟨d_ε f, Y⟩X, γ),

where

\[ \gamma = fg(\mathcal{L}_X β - iy_ε d_ε α) + f⟨d_ε g, X⟩β - g⟨d_ε f, Y⟩α + gd_ε f ∧ (i_ε β + iy_ε α). \]

But, one deduces from the fundamental identity that

\[ \Pi(d_ε f ∧ (i_ε β + iy_ε α)) = 0, \]

for any \( X = X_{f_1} \ldots f_{p−1}, \ \alpha = d_ε f_1 \ldots d_ε f_{p−1}, \ Y = X_{g_1} \ldots g_{p−1}, \ \text{and} \ \beta = d_ε g_1 \ldots d_ε g_{p−1}. \) Therefore, the sections of \( L \) are closed under the bracket defined as in (1.1). So, using Proposition 4.5 one sees that \( L \) determines a Nambu-Poisson structure of order \( p \) on \( E \).

**Example 4.** Let \( M \) be an \( n \)-dimensional manifold. Recall from [H] that an *almost Nambu-Dirac structure* of order \( p \) on \( M \) is a sub-bundle \( L \subset TM × \Lambda^{p−1} T^* M \) such that \( \omega_1(X_2) + \omega_2(X_1)|_{\Lambda^{p−1}\pi(L)} = 0 \), for any \( (X_1, \omega_1), (X_2, \omega_2) \in L \) and \( \Lambda^{p−1}\pi(L) = \pi(\bar{L}), \) where \( \bar{L} = \{(Z, \eta) \in \Lambda^{p−1} TM × T^∗ M \mid \omega(Z) + (−1)^p \eta(X) = 0, \ \forall X, \omega \in L \} \), here \( \pi(L) \) and \( \pi(\bar{L}) \) denote the projections from \( L \) and \( \bar{L} \) onto their first components, respectively. If the sections of \( L \) are closed under the bracket defined as in (1.1), then \( L \) is called a Nambu-Dirac structure of order \( p \) on \( M \). It is called a strong Nambu-Dirac structure of order \( p \) on \( M \) if additionally \( \Lambda^{p−1}(\pi^*(\bar{L})) \subset \pi^*(L) \), where \( \pi^*(L) \) and \( \pi^*(\bar{L}) \) denote the projections from \( L \) and \( \bar{L} \) onto their second components, respectively. It has been shown in [H] that an almost Nambu-Dirac structure of order \( p \) on \( M \) induces a skew-symmetric \( p \)-form \( \Omega_L \) on \( \pi(L) \). Any strong Nambu-Dirac structure of order \( p \) on \( M \) is a Nambu-Dirac structure of order \( p \) on the Lie algebroid \( E = TM \) as defined above.

### 4.4 The Nambu-Poisson Algebra of Admissible Functions

In this section, we give a definition of admissible functions for a Dirac structure of order \( p \) on a Lie algebroid. Also, we show that, given any arbitrary Dirac structure of order \( p \) on a Lie algebroid, there exists a Nambu-Poisson bracket defined on the set of admissible functions.

Let \( (E, [\cdot, \cdot]_E, ρ) \) be a Lie algebroid over \( M \). Consider a sub-bundle \( L \) of \( E ⊕ \Lambda^{p−2} E^* \). Assume that \( L \) determines a Dirac structure of order \( p \) on \( E \), that is, \( Γ(L) \) is closed under the bracket \([\cdot, \cdot]_E \) defined as in (1.1) and there exist a sub-bundle \( P \) of \( E \) and a skew-symmetric 2-form \( Ω_L \) on \( P \) satisfying (\( \mathcal{H} \)) such that the fibre of \( L \) at a point \( x \in M \) is \( L_x = \{(X, α)_x \mid X_x \in P_x \text{ and } (Ω_L(X))_x = α_{X}|_{\Lambda^{p−2}_x} \} \). A function \( f \in C^α(M) \) is said to be \( L \)-admissible if \( d_ε f|_{L∩E} = 0 \).
Now, we denote by $L^\perp$ the sub-bundle of $\Lambda^{p-1} E \oplus E^*$ whose fibre at a point $x \in M$ is given by:

$$L^\perp_x = \left\{ (Z, \eta)_x \mid (\eta(X) + (-1)^p \alpha(Z))_x = 0, \forall (X, \alpha)_x \in L_x \right\}.$$

In view of the above discussion, $f$ is $L$-admissible if and only if there exists a $(p-1)$-vector field $X_f$ such that $e_f = (X_f, d_x f) \in \Gamma(L^\perp)$. Remark that for a given $L$-admissible function $f$, $e_f$ is unique up to sections of $L^\perp \cap \Lambda^{p-2} E$. Let $f_1, \ldots, f_p \in C^\infty(M)$ be $L$-admissible functions. Define

$$\{f_1, \ldots, f_p\} = X_{f_1}(d_{x} f_2 \wedge \ldots \wedge d_{x} f_p) = -X_{f_2}(d_{x} f_1 \wedge \ldots \wedge d_{x} f_p)$$

This skew-symmetric bracket is well defined and is local. The product $gh$ of two $L$-admissible functions $g$ and $h$ is $L$-admissible since $(gX_h + hX_g, d_x (gh))$ becomes a section of $L^\perp$ when $e_g = (X_g, d_x g)$ and $e_h = (X_h, d_x h)$ are sections of $\pi(L^\perp)$. We have,

$$\{f_1, \ldots, f_p-1, gh\} = g\{f_1, \ldots, f_p, h\} + h\{f_1, \ldots, f_p, g\}.$$

Furthermore, we have the following result:

**Theorem 4.6** Let $L$ be a Dirac structure of order $p$ on a Lie algebroid $E$. Then the set of $L$-admissible functions has an induced Nambu-Poisson bracket.

**Proof:** We only have to prove that the fundamental identity holds. But, this is an consequence of the fact that the sections of $L$ are closed under the bracket $[\cdot, \cdot, \cdot]_E$. Indeed, if $f_1, \ldots, f_{p-1}, g_1, \ldots, g_{p-1}$ are $L$-admissible then $(X_{f_1}, \ldots, f_{p-1}, d_x f_1 \wedge \ldots \wedge d_x f_{p-1})$ and $(X_{g_1}, \ldots, g_{p-1}, d_x g_1 \wedge \ldots \wedge d_x g_{p-1})$ are sections of $L$, where

$$X_{f_1, \ldots, f_{p-1}} = X_{f_1}(d_{x} f_2 \wedge \ldots \wedge d_{x} f_{p-1}).$$

Hence, we have

$$\left( [X_{f_1, \ldots, f_{p-1}}, X_{g_1, \ldots, g_{p-1}}]_E, \mathcal{L}^E_{X_{f_1, \ldots, f_{p-1}}} (d_x g_1 \wedge \ldots \wedge d_x g_{p-1}) \right) \in \Gamma(L).$$

Since

$$\mathcal{L}^E_{X_{f_1, \ldots, f_{p-1}}} (d_x g_1 \wedge \ldots \wedge d_x g_{p-1}) = \sum_{i=1}^{p-1} d_x g_1 \wedge \ldots \wedge d_x \{ f_1, \ldots, f_{p-1}, g_i \} \wedge \ldots \wedge d_x g_{p-1}.$$

We deduce from Equation (1.2) that for any arbitrary $L$-admissible function $h$, we have

$$\left\langle d_x h, [X_{f_1, \ldots, f_{p-1}}, X_{g_1, \ldots, g_{p-1}}]_E \right\rangle = \sum_{i=1}^{p-1} \{ g_1, \ldots, \{ f_1, \ldots, f_{p-1}, g_i \}, \ldots, g_{p-1}, h \}.$$
There follows immediately the fundamental identity.

If a sub-bundle $L \subset E \times \Lambda^{p-1}E^{*}$ determines a Dirac structure of order $p$ on $E$, then $\rho(L \cap E)$ is an integrable and singular distribution. Let $\mathcal{F}$ denote the singular foliation associated with $\rho(L \cap E)$, which is called characteristic foliation of $L$. Any $L$-admissible function is constant along the leaves of $\mathcal{F}$. Assume that the characteristic foliation is simple, that is, $\mathcal{F}$ is a regular foliation such that $M/\mathcal{F}$ is a smooth manifold and $\text{pr} : M \to M/\mathcal{F}$ is a submersion. In this case, $L$ is said to be reducible. The set of all $L$-admissible functions can be identified with $C^{\infty}(M/\mathcal{F})$. Thus, $M/\mathcal{F}$ has a Nambu-Poisson bracket. It would be interesting to get a result analogous to Theorem 3.3 of [LWX] by showing that, given an integrable distribution whose foliation $\mathcal{F}$ is simple, any Nambu-Poisson structure of order $p$ on $M/\mathcal{F}$ gives rise to a reducible Dirac structure of order $p$ on $E$.

We end this paper by noting that the problem of how to place Nambu-Jacobi manifolds in the setting of Dirac structures is left open. We hope that we will return to this question in the future.

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