THE $k$-TACNODE PROCESS

ROBERT BUCKINGHAM AND KARL LIECHTY

Abstract. The tacnode process is a universal behavior arising in nonintersecting particle systems and tiling problems. For Dyson Brownian bridges, the tacnode process describes the grazing collision of two packets of walkers. We consider such a Dyson sea on the unit circle with drift. For any $k \in \mathbb{Z}$, we show that an appropriate double scaling of the drift and return time leads to a generalization of the tacnode process in which $k$ particles are expected to wrap around the circle. We derive winding number probabilities and an expression for the correlation kernel in terms of functions related to the generalized Hastings–McLeod solutions to the inhomogeneous Painlevé-II equation. The method of proof is asymptotic analysis of discrete orthogonal polynomials with a complex weight.

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1. Introduction

Ensembles of nonintersecting particles, growth and tiling processes, random matrix models, and related probabilistic systems exhibit a variety of non-classical universal stochastic processes. These include the sine process describing bulk spacing [46], the Airy process at a soft edge [47], the $k$-Airy process for the initial opening of a new band at a soft edge [7, 6, 1, 10, 8], the Bessel process at a hard edge [49, 29, 11], and the Pearcey process with or without inliers at the merger of two bands.
Here we consider the tacnode process describing the statistics when two band endpoints collide and separate without merging.

The tacnode process first appeared in the paper [3] of Adler, Ferrari, and van Moerbeke in the analysis of continuous-time nonintersecting random walks on $\mathbb{Z}$ with fixed starting and ending points. Their results were expressed in terms of resolvents and Fredholm determinants of the Airy integral operator acting on a semi-infinite interval. Very shortly thereafter, Delvaux, Kuijlaars, and Zhang [28] studied the tacnode process for nonintersecting Brownian bridges on the line with two fixed starting and ending points. They analyzed a certain $4 \times 4$ Riemann–Hilbert problem (see also [30, 40, 44]) associated to the Hastings–McLeod solution to the homogeneous Painlevé-II equation defined below in (1.13) and (1.14). Johansson [39] subsequently computed a resolvent form of the tacnode kernel for nonintersecting Brownian motions, but it was quite different from the one found in [3]. The equivalence between the Airy resolvent formulas appearing in [3] and [39] was shown in [4] by computing a scaling limit of the statistics in the domino tiling of the double Aztec diamond in two different ways. Ferrari and Vető [31] extended the results of Johansson [39] to the non-symmetric case when there are a different number of walkers in the two groups. Then the equivalence of the Riemann–Hilbert form of the kernel [28] and resolvent form of Johansson [39] was shown by Delvaux [25], who was also able to extend the Riemann–Hilbert formulation to the non-symmetric tacnode kernel. Liechty and Wang [45] showed that the even (respectively, odd) parts of the tacnode kernel (with respect to the spatial variables) arise in appropriate scaling limits of nonintersecting Brownian bridges on an interval with reflecting (respectively, absorbing) boundary conditions. Transitions between the tacnode kernel and the Airy and Pearcey kernels were studied by Bertola and Cafasso [11], Girotti [36], and Geudens and Zhang [35].

Liechty and Wang [45] showed the tacnode process occurs in a Dyson sea of nonintersecting Brownian bridges on the unit circle with a single starting and ending point. Suppose $n$ Brownian particles on $\mathbb{T} = \{e^{i\theta} | \theta \in \mathbb{R} \}$ with diffusion parameter (i.e. standard deviation) $n^{-1/2}$ are conditioned to start at angle $\theta = 0$ at time $t = 0$, to return to the starting position at a fixed return time $T$, and to not intersect for $0 < t < T$. In the large-$n$ limit, if $T < \pi^2$ then there is a portion of the unit circle no particles are expected to visit (Figure 1, left panel), whereas if $T > \pi^2$ then there is a time interval in which the bulk of particles is expected to cover the entire circle (Figure 1, right panel). In the critical case $T = \pi^2$, particles are expected to visit the far side of the circle, but only at the halfway time $t = T/2 = \pi^2/2$ (Figure 1, center panel). The behavior when $T \approx \pi^2$ in a neighborhood of $\theta = -\pi$ around the time $t = T/2$ is described by the tacnode process. In the current work, by adding a small amount of drift we introduce a generalization of the tacnode process (the $k$-tacnode process) in which two band endpoints collide and separate, after which $k$ particles are expected to switch from one endpoint to the other (Figure 2, right panel).

![Figure 1](image1.png)

**Figure 1.** Random walk simulations of a Dyson Brownian bridge with 24 walkers in subcritical (left), critical (center), and supercritical (right) cases. The tacnode process describes the statistics of the critical case.
Figure 2. Random walk simulations of a Dyson Brownian bridge with 24 walkers in the critical case $T = \pi^2$ with total winding 0 (left) and $-2$ (right).

More precisely, consider the determinantal process $\text{NIBM}_{0 \to T}^\mu$, introduced in [15]. This process describes $n$ nonintersecting Brownian bridges on the circle with the addition of a (real) drift $\mu$ and is defined as follows. Given a positive integer $n$ and $\tau \in [0,1]$, define the lattice

$$L_{n,\tau} := \left\{ \frac{m + \tau}{n} \mid m \in \mathbb{Z} \right\}.$$ (1.1)

Define $p_{n,j}^{(T,\mu,\tau)}(x)$ to be the monic polynomial of degree $j$ satisfying

$$\sum_{x \in L_{n,\tau}} p_{n,j}^{(T,\mu,\tau)}(x) e^{-\frac{Tn^2}{2}(x^2 - 2i\mu x)} = h_{n,j}^{(T,\mu,\tau)} \delta_{jk},$$ (1.2)

where $\{h_{n,j}^{(T,\mu,\tau)}\}_{j=0}^\infty$ are the normalizing constants. These polynomials are defined to be orthogonal with respect to a non-Hermitian weight, and so their existence is not guaranteed. In this paper we analyze the Riemann–Hilbert problem for these orthogonal polynomials and prove their existence for large enough $n$. Then the existence of the full system of polynomials for small $|\mu|$ follows via a perturbation argument from the $\mu = 0$ case, in which the orthogonality is Hermitian.

Now define the $\tau$-deformed correlation kernel

$$K_{t_i,t_j}(\varphi, \theta) := \tilde{K}_{t_i,t_j}(\varphi, \theta) - W_{[i,j]}^0(\varphi, \theta),$$ (1.3)

where

$$\tilde{K}_{t_i,t_j}(\varphi, \theta) := \frac{n}{2\pi} \sum_{j=0}^{n-1} \frac{1}{h_{n,j}^{(T,\mu,\tau)}} S_{j,T-t}(\varphi) S_j(t(-\theta)), $$ (1.4)

wherein

$$S_{j,a}(\varphi; T, \mu, \tau, n) \equiv S_{j,a}(\varphi) := \frac{1}{n} \sum_{x \in L_{n,\tau}} p_{n,j}^{(T,\mu,\tau)}(x) e^{-an(x^2 - 2i\mu x)/2} e^{i\varphi nx},$$ (1.5)

and

$$W_{[i,j]}^0(\varphi, \theta) := \begin{cases} 0, & t_i \geq t_j, \\ \frac{1}{2\pi} \sum_{s \in L_{n,\tau}} e^{-\frac{(t_j-t_i)s^2}{2}} e^{-in(\theta-\varphi)s}, & t_i < t_j. \end{cases}$$ (1.6)

Then $\text{NIBM}_{0 \to T}^\mu$ is the determinantal process on $\mathbb{T}$ defined by the multi-time extended kernel (1.3) with the parameter $\tau$ set to $\tau = 0$ if $n$ is odd, and $\tau = 1/2$ if $n$ is even. That is, if we fix $m$ times
0 < t_1 \leq t_2 \leq \cdots \leq t_m < T$, then the $m$-point correlation function for the positions of the particles at times $t_1, \ldots, t_m$ is given by

\begin{equation}
R_{0\to T}^{(n)}(\theta_1, t_1; \ldots; \theta_m, t_m; \mu) = \det \left( K_{i_1, t_1}(\theta_i, \theta_j) \right)_{\tau = \epsilon(n)}^{m}_{i,j=1},
\end{equation}

where

\begin{equation}
\epsilon(n) := \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
\frac{1}{2} & \text{if } n \text{ is even}.
\end{cases}
\end{equation}

We note that (1.4) is not a correlation kernel unless $\tau = \epsilon(n)$. The Fourier-type parameter $\tau$ is used for computing winding numbers, as we will see below.

To analyze the $k$-tacnode process, we consider the process NIBM$^\mu_{0\to T}$ as the number of particles $n$ approaches infinity when the total time $T$ is close to the critical time $T_c = \pi^2$, and the drift $\mu$ is of order $O((\log n)/n)$. More specifically, we define the $k$-tacnode regime to be the scaling regime such that

- As $n \to \infty$, $(\pi^2 - T)n^{2/3}$ is bounded.
- For some fixed non-negative integer $k$, $\mu$ satisfies

\begin{equation}
\left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}.
\end{equation}

In order to state our results, we define a parameter $s$ in terms of $n$ and $T$ as

\begin{equation}
s \equiv s(T, n) := - \left( 3\pi n \left( z_1 - i\mu - \int_{i\mu}^{z_1} \rho(w) \, dw \right) \right)^{2/3}
\end{equation}

where

\begin{equation}
\rho(w) := \frac{T}{2\pi} \frac{4}{\sqrt{T}} - (w - i\mu)^2, \quad z_1 := i\mu + \frac{2}{\pi T} \sqrt{T - \pi^2}.
\end{equation}

Here $\rho(w)$ is positive on the interval $(-\frac{2}{\sqrt{T}} + i\mu, \frac{2}{\sqrt{T}} + i\mu)$ and extends analytically into a neighborhood of compact subsets of that interval. One can check that as $T \to \pi^2$,

\begin{equation}
s = \frac{2^{2/3} n^{2/3}}{\pi^2} \left( (\pi^2 - T) + \frac{4}{5\pi^2} (\pi^2 - T)^2 + O((\pi^2 - T)^3) \right),
\end{equation}

so $s$ is bounded as $n \to \infty$ in the $k$-tacnode scaling. We choose to write (1.10) in a form that involves $\mu$ because the density $\rho(w)$ on the interval $(-\frac{2}{\sqrt{T}} + i\mu, \frac{2}{\sqrt{T}} + i\mu)$ naturally appears in our analysis, but it is not hard to see that $s$ does not depend on $\mu$. In fact, $s$ is the same as was defined in [42 Equation (1.43)].

1.1. Results on winding numbers. We first review the known results in the tacnode regime in the zero-drift case $\mu = 0$ [33] and then state our results in the $k$-tacnode regime. Let $u_{\text{HM}}^{(0)}(s)$ be the Hastings–McLeod function, that is, the unique solution to the homogeneous Painlevé-\(\Pi\) equation

\begin{equation}
\frac{d^2}{ds^2} u(s) = 2u(s)^3 + su(s)
\end{equation}

with asymptotics

\begin{equation}
u_{\text{HM}}^{(0)}(s) = \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-\frac{s^3}{2}} \left( 1 + O \left( \frac{1}{s^{3/4}} \right) \right) \text{ as } s \to +\infty.
\end{equation}

One can also write $u_{\text{HM}}^{(0)}(s) \sim \text{Ai}(s)$ as $s \to +\infty$, where $\text{Ai}(s)$ is the Airy function. Let $W_n(T, \mu)$ be the random variable counting the total winding number of particles in the process NIBM$^\mu_{0\to T}$. The first result from the literature describes the distribution of winding numbers in the zero-drift case.
Proposition 1.1. (Winding numbers in the zero-drift tacnode regime [43, Theorem 1.2(b)]). Set \( \mu = 0 \). Suppose \( (\pi^2 - T)n^{2/3} \) remains bounded as \( n \to \infty \), and define \( s \) by (1.10). Then, as \( n \to \infty \),

\[
P(W_n(T, 0) = \omega) = \begin{cases} 
1 - \frac{u_{\text{HM}}^{(0)}}{2(2n)^{1/3}} + O \left( \frac{1}{n^{2/3}} \right), & \omega = 0, \\
1 + \frac{u_{\text{HM}}^{(0)}}{2(2n)^{1/3}} + O \left( \frac{1}{n^{2/3}} \right), & \omega = \pm 1, \\
O \left( \frac{1}{n^{2/3}} \right), & \text{otherwise}.
\end{cases}
\]

(1.15)

Our results for the winding numbers in the \( k \)-tacnode regime are expressed in terms of certain functions \( U_k \) and \( V_k \) that solve a coupled Painlevé-II system and whose logarithmic derivatives are generalized Hastings–McLeod functions, which we now define. For any \( \alpha > - \frac{1}{2} \), the inhomogeneous Painlevé-II equation

\[
u''(s) = 2u(s)^3 + su(s) - \alpha
\]

has a unique solution \([37, 33, 20]\), denoted \( u_{\text{HM}}^{(\alpha)} \) and called the generalized Hastings–McLeod solution, satisfying both

\[
u_{\text{HM}}^{(\alpha)}(s) = \sqrt{-\frac{s}{2}} \left( 1 + O \left( \frac{1}{(-s)^{3/2}} \right) \right) \quad \text{as } s \to -\infty,
\]

and

\[
u_{\text{HM}}^{(\alpha)}(s) = \frac{\alpha}{s} \left( 1 + O \left( \frac{1}{s} \right) \right) \quad \text{as } s \to +\infty \quad (\alpha \neq 0).
\]

(1.16)

(1.17)

For \( \alpha = 0 \) the asymptotic condition (1.14) is used instead of (1.18), giving the standard Hastings–McLeod function \( u_{\text{HM}}^{(0)} \).

Now let \( U_k \) and \( V_k \) be the solutions of the coupled Painlevé-II system

\[
U_k''(s) = 2U_k(s)^2V_k(s) + sU_k(s),
\]

\[
V_k''(s) = 2U_k(s)V_k(s)^2 + sV_k(s),
\]

(1.19)

with asymptotic behavior

\[
U_k(s) = \begin{cases} 
-\frac{i}{s} \sqrt{\pi} e^{-s^{3/2} \left( 1 + O \left( \frac{1}{s^{3/4}} \right) \right)}, & s \to +\infty, \\
\frac{2 \cdot 8^k \sqrt{\pi} s^{(2k+1)/4}}{(-s)^{k(k+1)/2}} \left( 1 + O \left( \frac{1}{s^{3/2}} \right) \right), & s \to -\infty.
\end{cases}
\]

(1.20)

\[
V_k(s) = \begin{cases} 
\frac{8^k \sqrt{\pi} i s^{(2k-1)/4}}{(-s)^{(k-1)/2}} \left( 1 + O \left( \frac{1}{s^{3/4}} \right) \right), & s \to +\infty, \\
\frac{2^{3k-1} \sqrt{2} i}{(-s)^{(2k-1)/2}} \left( 1 + O \left( \frac{1}{s^{3/2}} \right) \right), & s \to -\infty.
\end{cases}
\]

When \( k = 0 \), these functions are simply multiples of the Hastings–McLeod function for \( \alpha = 0 \):

\[
U_0(s) = -iu_{\text{HM}}^{(0)}(s), \quad V_0(s) = iu_{\text{HM}}^{(0)}(s).
\]

(1.21)

For \( k > 0 \), if we define

\[
p_k(s) := \frac{U_k'(s)}{U_k(s)}, \quad q_k(s) := \frac{V_k'(s)}{V_k(s)},
\]

(1.22)

and

\[
P_k(s) := 2^{-1/3}p_k(-2^{-1/3}s), \quad Q_k(x) := 2^{-1/3}q_k(-2^{-1/3}s),
\]

(1.23)
then
\[
P_k(x) \equiv -u_{\text{HM}}^{(k+1/2)}(x), \quad Q_k(x) \equiv u_{\text{HM}}^{(k-1/2)}(x).
\]

Proofs of Equations (1.21) and (1.24) are given in §4.3.4.

**Theorem 1.2** (Winding numbers in the $k$-tacnode regime). Fix a non-negative integer $k$ and choose $\mu$ according to (1.9). Also suppose $(\pi^2 - T)n^{2/3}$ remains bounded as $n \to \infty$, and define $s$ by (1.10). Then

\[
\mathbb{P} (W_n(T, \mu) = \omega) = \begin{cases} 
\frac{F_\nu}{1 + F_\nu + F_\nu} + O \left( \frac{1}{n^{2/3}} \right), & \omega = k - 1, \\
\frac{1}{1 + F_\nu + F_\nu} + O \left( \frac{1}{n^{2/3}} \right), & \omega = k, \\
\frac{F_\nu}{1 + F_\nu + F_\nu} + O \left( \frac{1}{n^{2/3}} \right), & \omega = k + 1, \\
O \left( \frac{1}{n^{2/3}} \right), & \text{otherwise},
\end{cases}
\]

where
\[
F_\nu \equiv F_\nu(s, \mu, n) := \frac{i e^{2n\pi \mu}}{2(2n)^{(2k+1)/3}} U_k(s),
\]
\[
F_\nu \equiv F_\nu(s, \mu, n) := -\frac{i (2n)^{(2k-1)/3}}{2e^{2n\pi \mu}} V_k(s).
\]

Here $F_\nu$ and $F_\nu$ are non-negative real quantities.

Theorem 1.2 is proven in §2.1 (assuming Lemma 2.1, which is proven in §5.3). Note that when $\mu = 0$ (implying $k = 0$), (1.25) reduces to (1.15) using (1.21). When $\mu$ is in the interior of the interval (1.9), both $F_\nu$ and $F_\nu$ are $O(1)$. These terms are smallest when $\mu = \frac{k \log n}{3\pi n}$, i.e., $\mu$ is in the center of the interval, at which point they are $O(n^{-1/3})$. They are largest when $\mu$ is at the endpoints of the interval, at which point they are $O(1)$.

The generalized Hastings–McLeod functions $u_{\text{HM}}^{(\alpha)}(s)$ have previously appeared in at least three different random matrix and nonintersecting particle problems. Claeys, Kuijlaars, and Vanlessen [20] showed these functions arise in the double-scaling limit of the eigenvalue correlation kernel near the origin for $n \times n$ unitary random matrix ensembles of the form

\[
Z_{n,N}^{-1} \left| \det M \right|^{2\alpha} e^{-N \text{Tr} V(M)} dM, \quad \alpha > -\frac{1}{2}.
\]

Here $V(x)$ is the potential and $Z_{n,N}$ is the partition function. They study an associated $2 \times 2$ Riemann–Hilbert problem with jumps on four rays and a pole of order $\alpha$ at the origin. For $\alpha = 0$, this Riemann–Hilbert problem agrees with Riemann–Hilbert Problem 4.4 below with $k = 0$, a fact we will use to identify the functions $U_k$ and $V_k$. A different random matrix setting in which $u_{\text{HM}}^{(\alpha)}(s)$ appears is the chiral two-matrix model for $n \times (n + \alpha)$ rectangular matrices $\Phi_1$ and $\Phi_2$ with distribution

\[
Z_n^{-1} \exp(-n \text{Tr} (V(\Phi_1^* \Phi_1) + W(\Phi_2^* \Phi_2) - \tau (\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1))) d\Phi_1 d\Phi_2, \quad \tau \in \mathbb{R}.
\]

Here $V$ and $W$ are polynomial potentials, $Z_n$ is the partition function, and $\tau$ is a coupling constant. In the case where $V$ is linear and $W$ is quadratic, Delvaux, Geudens, and Zhang [27] analyzed the problem using a $4 \times 4$ Riemann–Hilbert problem with jumps on ten rays and a pole of order $\alpha$ at the origin whose solution can be expressed in terms of $u_{\text{HM}}^{(\alpha)}(s)$. This same Riemann–Hilbert problem was previously derived by Delvaux [26] in the study of the so-called hard-edge tacnode. Here $n$ nonintersecting squared Bessel paths start and end at the same point, chosen so the paths just osculate against the hard edge. The squared Bessel process is a $\mathbb{R}_+^0$-valued stochastic process with
a transition probability density defined in terms of Bessel functions. It depends on a parameter \(\alpha > -1\), which is the parameter of the Bessel function (in the special case \(\alpha = \frac{d}{2} - 1\) for a positive integer \(d\), the squared Bessel process behaves like the square of the distance to the origin of a \(d\)-dimensional Brownian motion). When a group of nonintersecting squared Bessel paths is tuned to approach and then separate from the hard edge at \(0\), the hard-edge tacnode process appears and is given in terms of the generalized Hastings–McLeod functions \(u_{\text{HM}}^{(\alpha + \frac{1}{2})}(s)\). For certain Bessel parameters, the hard-edge tacnode kernel was related to the even and odd parts of the (standard) tacnode kernel by Liechty and Wang [15]. Note that all of these processes are different from the \(k\)-tacnode process we study here, as they involve the generalized Hastings–McLeod functions directly as opposed to the tau functions \(U_k(s)\) and \(V_k(s)\). To the best of our knowledge, this is the first time the tau functions for the generalized Hastings–McLeod functions have appeared in the literature. Interestingly, it is the tau functions for the rational solutions to the inhomogeneous Painlevé-II equation that arise in the analysis of a librational-rotational transition for the semiclassical sine-Gordon equation [16].

1.2. Results on the correlation kernel. Next, we define the zero-drift tacnode kernel and describe results from the literature on the local convergence of the kernel \([1.4]\) with \(\mu = 0\) before giving our results for nonzero drift. Let \(\Psi(\zeta; s)\) be the \(2 \times 2\) matrix-valued function satisfying the differential equation

\[
\frac{d}{d\zeta} \Psi(\zeta; s) = \begin{bmatrix} -4i\zeta^2 - i(s + 2u_{\text{HM}}^{(0)}(s))^2 & 4\zeta u_{\text{HM}}^{(0)}(s) + 2iu_{\text{HM}}^{(0)}(s)'(s) \\ 4\zeta u_{\text{HM}}^{(0)}(s) - 2iu_{\text{HM}}^{(0)}(s)'(s) & 4i\zeta^2 + i(s + 2u_{\text{HM}}^{(0)}(s))^2 \end{bmatrix} \Psi(\zeta; s)
\]

and the asymptotic condition

\[
\Psi(\zeta; s)e^{i(\frac{1}{2}\zeta^3 + s\zeta)} = \mathbb{I} + \mathcal{O}(\zeta^{-1}), \quad \zeta \to \pm \infty
\]

(here the Pauli matrix \(\sigma_3\) is defined in \([1.41]\)). Equation \((1.27)\) is part of the Flaschka–Newell Lax pair for the homogeneous Painlevé-II equation (specialized to the Hastings–McLeod solution). Then define

\[
f_0(u; s) := \begin{cases} -[\Psi(u; s)]_{12}, & \Im u > 0, \\ [\Psi(u; s)]_{11}, & \Im u < 0, \end{cases} \quad g_0(u; s) := \begin{cases} -[\Psi(u; s)]_{22}, & \Im u > 0, \\ [\Psi(u; s)]_{21}, & \Im u < 0, \end{cases}
\]

and

\[
\phi_{\tau_i, \tau_j}(\xi, \eta) := \begin{cases} 0, & \tau_i \geq \tau_j, \\ \frac{1}{\sqrt{2\pi(\tau_j - \tau_i)}} e^{-(\xi - \eta)^2/(2(\tau_j - \tau_i))}, & \tau_i < \tau_j. \end{cases}
\]

Also define \(\Sigma_T\) to be the oriented contour consisting of two unbounded components, the first composed of three straight segments from \(e^{i\pi/6} \cdot \infty \to \sqrt{3} + i \to -\sqrt{3} + i \to e^{5\pi i/6} \cdot \infty\), and the second composed of three straight segments from \(e^{-5\pi i/6} \cdot \infty \to -\sqrt{3} - i \to \sqrt{3} - i \to e^{-i\pi /6} \cdot \infty\). Then set

\[
\widetilde{K}_{\tau_i, \tau_j}^{(0)}(\xi, \eta; s) := \frac{1}{2\pi} \int_{\Sigma_T} du \int_{\Sigma_T} dv \, e^{\tau_i u^2/2 - \tau_j v^2/2 - i(\eta - \eta_0)} f_0(u; s)g_0(v; s) - g_0(u; s)f_0(v; s) \frac{2\pi i(u - v)}{2\pi i(u - v)}
\]

and define the (zero-drift) tacnode kernel as

\[
K_{\tau_i, \tau_j}^{(0)}(\xi, \eta; s) := \widetilde{K}_{\tau_i, \tau_j}^{(0)}(\xi, \eta; s) - \phi_{\tau_i, \tau_j}(\xi, \eta).
\]

The following result from the literature states that the tacnode kernel is the limiting behavior of the kernel \([1.4]\) in the tacnode regime.
Proposition 1.3. (The correlation kernel in the zero-drift tacnode regime [13, Theorem 1.3(b)]).

Set \( \mu = 0 \). Suppose \((\pi^2 - T)n^{2/3}\) remains bounded as \( n \to \infty \), and define \( s \) by \((1.10)\). Scale

\[
    t_i = \frac{T}{2} + \frac{2 - 10/3 \pi^2}{n^{1/3}} s, \quad t_j = \frac{T}{2} + \frac{2 - 10/3 \pi^2}{n^{1/3}} \tau, \quad \varphi = -\pi - \frac{2 - 5/3 \pi}{n^{2/3}} s, \quad \theta = -\pi - \frac{2 - 5/3 \pi}{n^{2/3}} \eta.
\]

Then

\[
    \lim_{n \to \infty} K_{t_i, t_j}(\varphi, \theta) \left| \frac{dy}{d\eta} \right| = K^{(0)}_{\tau_i, \tau_j}(\xi, \eta; s).
\]

We now present our results for the correlation kernel in the \( k \)-tacnode regime. Let \( \tilde{L}_k(\zeta; s) \) be the \( 2 \times 2 \) matrix-valued function satisfying the differential equation

\[
    \frac{\partial}{\partial \zeta} \tilde{L}_k(\zeta; s) = \begin{bmatrix} -4i\zeta^2 - i(s + 2U_k(s)V_k(s)) & 4iU_k(s) - 2U_k(s) \\ -4i\zeta V_k(s) - 2V_k(s) & 4i\zeta^2 + i(s + 2U_k(s)V_k(s)) \end{bmatrix} \tilde{L}_k(\zeta; s),
\]

where \( U_k(s) \) and \( V_k(s) \) are defined in \((1.19)\) and \((1.20)\), with the boundary condition

\[
    \tilde{L}_k(\zeta; s)\zeta^{-k\sigma_3} e^{i(\xi^2 + s + k)\sigma_3} = \mathbb{I} + O\left(\frac{1}{\zeta}\right) \quad \text{as} \quad \zeta \to \pm \infty;
\]

compare to \((1.27)\) and \((1.28)\). Then define the functions

\[
    f_k(u; s) := \begin{cases} \tilde{L}_k(u; s)_{12}, & \Im u > 0, \\ \tilde{L}_k(u; s)_{11}, & \Im u < 0, \end{cases} \quad g_k(u; s) := \begin{cases} \tilde{L}_k(u; s)_{22}, & \Im u > 0, \\ \tilde{L}_k(u; s)_{21}, & \Im u < 0. \end{cases}
\]

Set

\[
    \tilde{K}^{(k)}_{\tau_i, \tau_j}(\xi, \eta; s) := \frac{1}{2\pi} \int_{\Sigma_T} \int_{\Sigma_T} du \, dv \, e^{i(\xi u - \zeta v + \eta v)} f_k(u; s) g_k(v; s) - g_k(u; s) f_k(v; s) \frac{1}{2\pi i(u - v)}
\]

and define the \( k \)-tacnode kernel to be

\[
    K^{(k)}_{\tau_i, \tau_j}(\xi, \eta; s) := \tilde{K}^{(k)}_{\tau_i, \tau_j}(\xi, \eta; s) - \phi_{\tau_i, \tau_j}(\xi, \eta),
\]

where \( \phi_{\tau_i, \tau_j}(\xi, \eta) \) is defined in \((1.30)\).

Theorem 1.4 (The correlation kernel in the \( k \)-tacnode regime). Fix a non-negative integer \( k \) and let \( \mu = \frac{k\log n}{3n} \). Also suppose \((\pi^2 - T)n^{2/3}\) remains bounded as \( n \to \infty \), and define \( s \) by \((1.10)\). Scale \( t_i, t_j, \varphi, \) and \( \theta \) as in \((1.33)\). Then

\[
    \lim_{n \to \infty} K_{t_i, t_j}(\varphi, \theta) \left| \frac{dy}{d\eta} \right| = K^{(k)}_{\tau_i, \tau_j}(\xi, \eta; s).
\]

Theorem 1.4 is proven in \([2, 2]\) (assuming Lemmas \([2, 2]\) and \([2, 3]\) which are proven in \([5, 3]\)). Note that in Theorem 1.4 we have chosen \( \mu \) to be exactly at the midpoint of the interval \((1.9)\). In fact the theorem holds with slower convergence for any \( \mu \) in the interior of this interval, provided it does not approach the endpoints. For ease of the analysis we choose the \( \mu \) that gives the best convergence.

1.3. Outline and notation. We begin in \( \S 2 \) with the proofs of Theorems 1.2 and 1.4, assuming three lemmas describing the asymptotic behavior of the discrete Gaussian orthogonal polynomials with a complex weight. The remainder of the paper is dedicated to proving these lemmas. In \( \S 3 \) we recall the meromorphic Riemann–Hilbert problem associated to the discrete orthogonal polynomials derived in [13] and carry out a series of transformations in order to arrive at a controllable problem. The resulting (sectionally analytic) Riemann–Hilbert problem has jumps that are close to the identity matrix everywhere except in a neighborhood of a certain line segment (the band). Discarding the jumps except on the band leads to the outer model problem, which is constructed in \( \S 4 \) in a nonstandard way due to a pole of order \( k \) at the point \( i\mu \) on the band. In \( \S 4 \) we also construct
parametrices near the point $i\mu$ and the band endpoints. The parametrix near $i\mu$ is constructed in terms of the Painlevé-II tau functions $U_k$ and $V_k$, and for certain values of $\mu$ contributes to the leading-order behavior of the solution to the full Riemann–Hilbert problem. This contribution is captured through the construction of a parametrix for the error that is carried out in §5. In this section we also complete the error analysis and prove the three lemmas on discrete orthogonal polynomials.

Notation. With the exception of 
\begin{equation}
(1.41) \quad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad 0 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{equation}
matrices are denoted by bold capital letters. We denote the $(jk)$th entry of a matrix $M$ by $[M]_{jk}$.

In reference to a smooth, oriented contour $\Sigma$, for $z \in \Sigma$ we denote by $f^+(z)$ (respectively, $f^-(z)$) the non-tangential limit of $f(\zeta)$ as $\zeta$ approaches $\Sigma$ from the left (respectively, the right).

2. Proofs of the main theorems assuming results on orthogonal polynomials

In this section we prove Theorems 1.2 and 1.4 using asymptotic results for the orthogonal polynomials (1.2). These asymptotic results are stated here and are proved using the Riemann–Hilbert method in the remaining sections of the paper.

2.1. Proof of Theorem 1.2 (winding numbers). Define the Hankel determinant
\begin{equation}
(2.1) \quad H_n(T, \mu, \tau) := \det \left( \frac{1}{n} \sum_{x \in L_{n,\tau}} x^{j+m-2} e^{-T_n(x^2 - 2i\mu x)} dx \right)_{j,m=1}^n
\end{equation}
(recall $L_{n,\tau}$ from (1.1)). Then the total winding number $W_n(T, \mu)$ has the following probability mass function (see [15, Equation (1.16)] and [43, Equation (185)]):
\begin{equation}
(2.2) \quad P(W_n(T, \mu) = \omega) = e^{2\pi i \omega \epsilon(n)} \int_0^1 \frac{H_n(T, \mu, \tau)}{H_n(T, \mu, \epsilon(n))} e^{-2\pi i \omega \tau} d\tau,
\end{equation}
where $\epsilon(n)$ was defined in (1.8). We exploit the natural connection between Hankel determinants for discrete weights and discrete orthogonal polynomials. Define $c_{n,m,j}^{(T,\mu,\tau)}$ as the coefficient of the term $x^j$ in $p_{n,m}^{(T,\mu,\tau)}(x)$:
\begin{equation}
(2.3) \quad p_{n,m}^{(T,\mu,\tau)}(x) = x^m + \sum_{j=0}^{m-1} c_{n,m,j}^{(T,\mu,\tau)} x^j.
\end{equation}
It was shown in [15, Proposition 4.2] that
\begin{equation}
(2.4) \quad \log \left( \frac{H_n(T, \mu, \tau)}{H_n(T, \mu, \epsilon(n))} \right) = \int_0^\tau \left( i n T \mu + T c_{n,n-1}^{(T,\mu,v)} \right) dv.
\end{equation}
Combining (2.2) and (2.4) gives
\begin{equation}
(2.5) \quad P(W_n(T, \mu) = \omega) = e^{2\pi i \omega \epsilon(n)} \int_0^1 \exp \left( \int_0^\tau \left( i n T \mu + T c_{n,n-1}^{(T,\mu,v)} \right) dv \right) e^{-2\pi i \omega \tau} d\tau.
\end{equation}
Define
\begin{equation}
(2.6) \quad R_V \equiv R_V(s, \mu, n) := \frac{(-1)^n (2n)^{(2k-1)/3}}{\pi e^{2n \pi \mu}} V_k(s), \quad R_U \equiv R_U(s, \mu, n) := \frac{(-1)^n e^{2n \pi \mu}}{\pi (2n)^{(2k+1)/3}} U_k(s).
\end{equation}
The following lemma gives the expansion of $c_{n,n-1}^{(T,\mu,\tau)}$ we need.
Lemma 2.1. Fix a non-negative integer $k$ and choose $\mu$ by (1.9). Also suppose $(\pi^2 - T)n^{2/3}$ remains bounded as $n \to \infty$, and define $s$ by (1.10). Then

\[ c_{n,n-1}^{(T,\mu,\tau)} = \frac{2ik}{\pi} + \frac{RVe^{-2i\tau\pi}}{1 - \frac{\pi}{2i}RVe^{-2i\tau\pi}} + \frac{RUe^{2i\tau\pi}}{1 + \frac{\pi}{2i}RUe^{2i\tau\pi}} - in\mu + O\left(\frac{1}{n^{2/3}}\right), \]

where the error term is uniformly bounded in $\tau$.

Lemma 2.1 is proven in §5.3. Assuming this, we are now ready to prove our main result on winding numbers.

Proof of Theorem 1.2. Combining (2.4), Lemma 2.1, and (1.12) (and using the uniform boundedness of the error term to integrate with respect to $\tau$) shows

\[ \log \left( \frac{H_n(T,\mu,\tau)}{H_n(T,\mu,\epsilon(n))} \right) = \int_{\epsilon(n)}^{\tau} \left( \frac{1}{2i} + \frac{\pi^2 RV}{2i} e^{-2i\pi\tau iv} + \frac{\pi^2 RU}{2i} e^{2i\pi\tau iv} dv + O\left(\frac{1}{n^{2/3}}\right) \right) \]

Taking the exponent of both sides gives

\[ \frac{H_n(T,\mu,\tau)}{H_n(T,\mu,\epsilon(n))} = \frac{(-1)^{-(n+1)k}}{1 + \frac{(-1)^{n+1}R(U - RU)}{2i}} \left( 1 + \frac{\pi}{2i} RU e^{2i\pi\tau iv} - \frac{\pi}{2i} RV e^{-2i\pi\tau iv} \right) e^{2ik\pi\tau} + O\left(\frac{1}{n^{2/3}}\right). \]

Here we have used $e^{2\pi\tau iv(n)} = (-1)^{n+1}$ from (1.8) and $RU RV = O(n^{-2/3})$ from (2.6). The last step is to use (2.2) to determine the winding numbers.

\[ \mathbb{P}(W_n(T,\mu) = \omega) = \frac{(-1)^{(n+1)(\omega-k)}}{1 + \frac{(-1)^{n+1}R(U - RU)}{2i}} \int_0^1 \left( 1 + \frac{\pi}{2i} RU e^{2i\pi\tau iv} - \frac{\pi}{2i} RV e^{-2i\pi\tau iv} \right) e^{2i\pi\tau(k-\omega)} d\tau + O\left(\frac{1}{n^{2/3}}\right) \]

\[ \begin{cases} \frac{\pi}{2i} (-1)^n RV & \omega = k - 1, \\ 1 + \frac{\pi}{2i} (-1)^n RU + \frac{\pi}{2i} (-1)^n RV + O\left(\frac{1}{n^{2/3}}\right), & \omega = k, \\ 1 + \frac{\pi}{2i} (-1)^n RU + \frac{\pi}{2i} (-1)^n RV + O\left(\frac{1}{n^{2/3}}\right), & \omega = k + 1, \\ O\left(\frac{1}{n^{2/3}}\right), & \text{otherwise.} \end{cases} \]

Along with the definitions of $F_U$ and $F_V$ in (1.26), this completes the proof of Theorem 1.2. \qed

2.2. Proof of Theorem 1.4 (the correlation kernel). We will compute the asymptotics of the kernel (1.3) in the scalings (1.12) and (1.33). Our proof closely follows that of [43, Section 5.3]. We begin with two results on discrete orthogonal polynomials that are proven in [55, 53]. In order to state these two lemmas we define the so-called $g$-function and Lagrange multiplier $\ell$:

\[ g(z) \equiv g(z; T, \mu) := g_0(z - i\mu; T), \quad \ell \equiv \ell(T, \mu) := \ell_0(T) - \frac{T}{2} \mu^2, \]
where
\[ \text{(2.12)} \quad g_0(z; T) := \frac{T}{4} z \left( z - \sqrt{z^2 - \frac{4}{T}} \right) - \log \left( z - \sqrt{z^2 - \frac{4}{T}} \right) - \frac{1}{2} + \log \frac{2}{T}, \quad \ell_0(T) := -1 - \log T, \]

with the logarithm and square root indicating principal branches. More details concerning the function \( g \) are given in [3].

The following lemma is the extension of [3, Proposition 3.7] to include the parameter \( \mu \).

**Lemma 2.2.** Fix a non-negative integer \( k \) and let \( \mu = \frac{k \log n}{3n} \). Also suppose \((\pi^2 - T)n^{2/3}\) remains bounded as \( n \to \infty \), and define \( s \) by (1.10). Then the discrete Gaussian orthogonal polynomials (1.2) satisfy the estimates (2.13)
\[ p^{(T,\mu,\tau)}_{n,n}(z) = e^{ng(z)}[M_k^{(\text{out})}(z)]_{11}(1 + \mathcal{O}(n^{-1/3})), \quad \frac{p^{(T,\mu,\tau)}_{n,n-1}(z)}{h^{(T,\mu,\tau)}_{n,n-1}} = e^{n(g(z) - \ell)}[M_k^{(\text{out})}(z)]_{21}(1 + \mathcal{O}(n^{-1/3})) \]
as \( n \to \infty \) in the domain \( \{ z \mid \Re z > \epsilon \} \) for any fixed \( \epsilon > 0 \). Here the function \( g(z) \) is defined in (2.11) and the matrix function \( M_k^{(\text{out})}(z) \) is defined in (4.7). The errors are uniform in \( z \).

The following lemma extends [3, Proposition 3.8] to include the parameter \( \mu \).

**Lemma 2.3.** Fix a non-negative integer \( k \) and a real number \( \delta > 0 \). Set \( \mu = \frac{k \log n}{3n} \). Also suppose \((\pi^2 - T)n^{2/3}\) remains bounded as \( n \to \infty \), and define \( s \) by (1.10). Then there exists \( \epsilon > 0 \) such that for all \( z, w \in \{ z \in \mathbb{C} \mid |z| < e^{-\epsilon} \} \) the following asymptotic formula holds:
\[ \text{(2.14)} \quad e^{-\frac{nT}{2} (z^2 - 2iuz + w^2 - 2iww)} \frac{p^{(T,\mu,\tau)}_{n,n}(z)}{p^{(T,\mu,\tau)}_{n,n-1}(w)} \frac{p^{(T,\mu,\tau)}_{n,n-1}(z)}{p^{(T,\mu,\tau)}_{n,n-1}(w)} = \frac{1}{2\pi i (z - w)} \left[ \frac{e^{i\pi(nz-\tau)}}{e^{i\pi(nz-\tau)}} \right] \left[ \frac{e^{i\pi(nw-\tau)}}{e^{i\pi(nw-\tau)}} \right] \left( 1 + \mathcal{O}(n^{-1/3}) + \mathcal{O}(n^{1/3-2\delta}) + \mathcal{O}(n^{-\delta}) \right), \]

where \( d = 2^{-5/3} \pi \). Also, the following estimates hold uniformly in \( \{ z \in \mathbb{C} \mid |z| < \epsilon \} \):
\[ \text{(2.15)} \quad p^{(T,\mu,\tau)}_{n,n}(z) = \mathcal{O} \left( n^{\frac{2k}{3}} e^{ng(z) + \ell n^{1/3}} \right), \quad \frac{p^{(T,\mu,\tau)}_{n,n-1}(z)}{h^{(T,\mu,\tau)}_{n,n-1}} = \mathcal{O} \left( n^{\frac{2k}{3}} e^{n(g(z) - \ell) + \ell n^{1/3}} \right) \]
for some constant \( r \in \mathbb{R} \).

Assuming these two results, we can now prove Theorem 1.4.

**Proof of Theorem 1.4.** From [1.3], we need to understand \( \tilde{K}_{\ell_1,\ell_2}(\varphi, \theta) \) and \( \hat{W}^{\circ}_{[i,j]}(\varphi, \theta) \). The second function is exactly the one in [3, Equation (133)], and in this scaling it was shown in [3, Equation (270)] that
\[ \text{(2.16)} \quad \lim_{n \to \infty} \frac{\pi}{\nu^{2/3} n^{2/3}} \hat{W}^{\circ}_{[i,j]}(\varphi, \theta) = \phi_{\ell_1, \ell_2}(\xi, \eta). \]

We now compute \( \tilde{K}_{\ell_1,\ell_2}(\varphi, \theta) \). Set
\[ \text{(2.17)} \quad d := 2^{-5/3} \pi \]
and define \( \Sigma \) to be the contour that in the upper half-plane connects \( \infty \cdot e^{i0} \) to \( \sqrt{3} + i \) to \( (\sqrt{3} + i) d^{-1} n^{-1/3} \) to \( (-\sqrt{3} + i) d^{-1} n^{-1/3} \) to \( -\sqrt{3} + i \) to \( \infty \cdot e^{i\pi} \) (oriented right-to-left), and in the lower
half-plane is the reflection of the contour in the upper half-plane through the origin (oriented left-to-right). Our starting point for asymptotics is the formula

\[(2.18) \quad \tilde{K}_{t_i,t_j}(\varphi, \theta) = \frac{n}{2\pi} \int_{\Sigma} dz \int_{\Sigma + (i/2d)n^{-2/3}} dw J(z, w),\]

where

\[(2.19) \quad J(z, w) := \left( e^{-\frac{\pi}{4} \left( (z^2 - 2i\mu z + w^2 - 2i\mu w) \right)} \left( p^{(T,\mu,\tau)}_{h,n,n-1}(z) p^{(T,\mu,\tau)}_{h,n,n-1}(w) - p^{(T,\mu,\tau)}_{h,n,n-1}(z) p^{(T,\mu,\tau)}_{h,n,n-1}(w) \right) \right) \times e^{\frac{\pi}{4} \left( (w^2 - 2i\mu w) \right)} e^{-in \frac{1}{4} d(\xi - \eta w) - e^{\pi i(nz - \tau)} e^{\pi i(nw - \tau)}} (e^{2\pi i(nz - \tau)} - 1)(e^{2\pi i(nw - \tau)} - 1).\]

Equations (2.18) and (2.19) are the same as [43, Equations (258) and (259)] when \(\mu = 0\). We denote by \(\Sigma_{\text{local}}, \Sigma_{\text{upper}}\), and \(\Sigma_{\text{lower}}\) the contours

\[(2.20) \quad \Sigma_{\text{local}} := \Sigma \cap \{ z \in \mathbb{C} \mid |z| < n^{-\frac{1}{3}} \}, \quad \Sigma_{\text{upper}} := \Sigma_{\text{local}} \cap \mathbb{C}^+, \quad \Sigma_{\text{lower}} := \Sigma_{\text{local}} \cap \mathbb{C}^- .\]

We claim that for large \(n\) the integral (2.18) becomes localized on \(\Sigma_{\text{local}}\). To justify this claim, we apply Lemma 2.2 and Equation (2.15) to the function \(J(z, w)\) to obtain the estimate that for all \(z \in \Sigma \setminus \Sigma_{\text{local}}\) and \(w \in \{ \Sigma + (i/2d)n^{-2/3} \} \setminus \{ \Sigma_{\text{local}} + (i/2d)n^{-2/3} \},\)

\[(2.21) \quad |J(z, w)| = \mathcal{O}(e^{n\frac{1}{3}(\xi - \eta w) - e^{\pi i(nz - \tau)} e^{\pi i(nw - \tau)}} \times \mathcal{O}(n^{\frac{4}{3}} e^{2\pi n^{1/3}}) e^{\frac{\pi}{4} d(\tau_i(z^2 - 2i\mu z) - \tau_j(w^2 - 2i\mu w))} e^{-in \frac{1}{4} d(\xi - \eta w)} \mathcal{O}(n^{\frac{2}{3}}),\]

where \(\tilde{g}\) is defined as

\[(2.22) \quad \tilde{g}(z) := \begin{cases} g(z) & \text{if } \Im z > \mu, \\ g(z) + i\pi - 2\pi iz & \text{if } \Im z < \mu. \end{cases}\]

The \(\mathcal{O}(n^{2/3})\) contribution comes from \((z - w)^{-1}\). By direct calculation, we find that for \(z \in \Sigma \setminus \Sigma_{\text{local}}, \Re(\tilde{g}(z) - T(z^2 - 2i\mu z)/4 + \pi iz)\) decreases as \(z\) moves away from 0. Hence, by a standard steepest-descent argument we have that

\[(2.23) \quad \tilde{K}_{t_i,t_j}(\theta, \varphi) = \frac{n}{2\pi} \int_{\Sigma} dz \int_{\Sigma + \frac{i}{2} d^{-1} n^{-2/3}} dw J(z, w) = \frac{n}{2\pi} \int_{\Sigma_{\text{local}}} dz \int_{\Sigma_{\text{local}} + \frac{i}{2} d^{-1} n^{-2/3}} dw J(z, w) + \mathcal{O}(e^{-cn^{1/3}}),\]

where \(c\) is a positive constant.

Inserting the estimates (2.15) into the integral (2.23) and making the change of variables \(u = dn^{1/3}z\) and \(v = dn^{1/3}w\), we obtain that

\[(2.24) \quad \frac{n}{2\pi} \int_{\Sigma_{\text{local}}} dz \int_{\Sigma_{\text{local}} + \frac{i}{2} d^{-1} n^{-2/3}} dw J(z, w) = \frac{n^2}{4\pi^2 d} \int_{\Sigma_{\tau}^+} du \int_{\Sigma_{\tau}^+ + \frac{i}{2} d^{-1} n^{-2/3}} dv \frac{e^{\frac{1}{4} (\tau_i u^2 - \tau_j v^2 - i(\xi u - \eta v))}}{1 - e^{2\pi i(nz - \tau)}} \left( 1 + \mathcal{O}(n^{-\frac{1}{6}}) \right),\]

where \(\Sigma_{\tau}^+ := \Sigma_{\tau} \cap \{ z : |z| < n^{1/12} \}\) and \(\Sigma_{\tau}\) is defined following (1.30). Noting that the factors \(1/(1 - e^{2\pi i(nz - \tau)})\) and \(1/(1 - e^{2\pi i(nw - \tau)})\) are either \(\mathcal{O}(e^{-2n^{2/3}/d})\) or \((1 + \mathcal{O}(e^{-2n^{2/3}/d}))\) depending on \(z\) and/or
where \( \phi \) (Discrete Gaussian orthogonal polynomial problem) Riemann–Hilbert Problem 3.1

the following Riemann–Hilbert problem \([15]\).

\[
\int_{\Sigma_T + \frac{i}{2}} dw J(z, w)
\]

where \( f_k(u; s) \) and \( g_k(u; s) \) are defined in \((1.37)\). Combining \((2.25)\) and \((2.16)\) we obtain

\[
\lim_{n \to \infty} \frac{\pi}{25/3 n^{2/3}} K_{t_i, t_j}(\varphi, \theta) = \tilde{K}^{(k)}_{t_i, t_j}(\xi, \eta; s) - \phi_{t_i, t_j}(\xi, \eta),
\]

where \( \phi_{t_i, t_j}(\xi, \eta) \) and \( \tilde{K}^{(k)}_{t_i, t_j}(\xi, \eta; s) \) are as defined in \((1.30)\) and \((1.38)\), respectively, and the contour \( \Sigma_T + \frac{i}{2} \) is easily deformed to \( \Sigma_T \) after taking the limit. \( \square \)

3. Setup of the Riemann–Hilbert problem

We obtain our asymptotic results on the discrete Gaussian orthogonal polynomials by analyzing the following Riemann–Hilbert problem \([15]\).

Riemann–Hilbert Problem 3.1 (Discrete Gaussian orthogonal polynomial problem). Fix \( n \in \{1, 2, 3, \ldots \} \) and \( \tau \in [0, 1] \) and find a \( 2 \times 2 \) matrix-valued function \( P_n(z) \) with the following properties:

**Analyticity:** \( P_n(z) \) is a meromorphic function of \( z \) and is analytic for \( z \in \mathbb{C} \setminus L_{n, \tau} \).

**Normalization:** There exists a function \( r(x) > 0 \) on \( L_{n, \tau} \) such that

\[
\lim_{x \to \infty} r(x) = 0
\]

and such that, as \( z \to \infty \), \( P_n(z) \) admits the asymptotic expansion

\[
P_n(z) = \left( I + \frac{P_{n,1}}{z} + \frac{P_{n,2}}{z^2} + O \left( \frac{1}{z^3} \right) \right) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix}, \quad z \in \mathbb{C} \setminus \bigcup_{x \in L_{n, \tau}} D(x, r(x))
\]

where \( P_{n,1} \) and \( P_{n,2} \) are independent of \( z \), and \( D(x, r(x)) \) denotes a disk of radius \( r(x) > 0 \) centered at \( x \).

**Residues at poles:** At each node \( x \in L_{n, \tau} \), the elements \( [P_n(z)]_{11} \) and \( [P_n(z)]_{21} \) of the matrix \( P_n(z) \) are analytic functions of \( z \), and the elements \( [P_n(z)]_{12} \) and \( [P_n(z)]_{22} \) have a simple pole with the residues

\[
\text{Res}_{z=x} [P_n(z)]_{j2} = \frac{1}{n} e^{-\frac{\pi}{2} i (z^2 - 2 i \mu x)} [P_n(x)]_{j1}, \quad j = 1, 2.
\]

Define the weighted discrete Cauchy transform \( C \) as

\[
C f(z) := \frac{1}{n} \sum_{x \in L_{n, \tau}} f(x) e^{-\frac{\pi}{2} i (x^2 - 2 i \mu x)} (z - x).
\]

Then the unique solution to Riemann–Hilbert Problem 3.1 (see \([34, 14]\)) is

\[
P_n(z) := \begin{bmatrix} (T_{\mu, \tau})^{(n)}_{n, n} (z) & C_{n, n} \left( \frac{T_{\mu, \tau}}{n, n} \right) (z) \\ (h_{n, n-1})^{(n, n-1)}_{n, n} (z) & (h_{n, n-1})^{(n, n-1)} \left( \frac{T_{\mu, \tau}}{n, n-1} \right) (z) \end{bmatrix}.
\]

In particular, the coefficient \( c_{n, n, n-1}^{(T, \mu, \tau)} \) can be calculated via

\[
c_{n, n, n-1}^{(T, \mu, \tau)} = [P_{n,1}]_{11}.
\]
For subcritical drift values, this Riemann–Hilbert problem was transformed in [15] via consecutive changes of variables $P_n \to R_n \to T_n \to S_n$ to a controlled problem with either constant or near-identity jumps. As we will see below, using exactly the same changes of variables in the $k$-tacnode regime leads to a Riemann–Hilbert problem where the jumps are controlled except in a neighborhood of the origin. The jumps near the origin will be controlled by an additional change of variables $S_n \to S_n^{\text{crit}}$ and the use of a local parametrix. We begin by combining the interpolation of poles, introduction of the $g$-function, and opening of lenses into one change of variables $P_n \to S_n$.

Define

\[
D_{n \pm}(z) := \begin{bmatrix} 1 & -\frac{\pi}{2\pi} e^{-nT(z^2 - 2\mu z)/2} e^{\pm i\pi(nz - \tau)} \\ 0 & 1 \end{bmatrix}
\]

and $A := \begin{bmatrix} 1 & 0 \\ 0 & -2\pi i \end{bmatrix}$.

Recall the $g$-function $g$ and Lagrange multiplier $\ell$ defined in (2.11). Also define the complex numbers $a$ and $b$ as

\[
a \equiv a(T, \mu) := -\frac{2}{\sqrt{T}} + i\mu, \quad b \equiv b(T, \mu) := \frac{2}{\sqrt{T}} + i\mu,
\]

and the potential function

\[
V(z) \equiv V(z; T, \mu) := \frac{Tz^2}{2} - iT\mu z.
\]

We will denote the horizontal line segment between $a$ and $b$ by $(a, b)$, and the half-infinite horizontal ray $\{x + i\mu \mid x \leq 2/\sqrt{T}\}$ by $(-\infty + i\mu, b]$. Here we note that the function $g(z)$ is the one appearing in the asymptotic analysis of Hermite polynomials [22] up to rescaling and a shift, and we record some of its analytic properties.

- $g(z)$ is also given by the integral formula

\[
g(z) = \int_a^b \log(z - w)\rho(w) \, dw,
\]

where the principal branch of the logarithm is taken, and $\rho(w)$ is the semicircle density on the interval $(a, b)$ as defined in (1.11).

- $g(z)$ is analytic for $z \in \mathbb{C} \setminus (-\infty + i\mu, b]$.

- $g(z)$ takes limiting values from above or below the ray $(-\infty + i\mu, b]$, which we denote $g_+(z)$ and $g_-(z)$, respectively. These functions satisfy the variational condition

\[
g_+(z) + g_-(z) - V(z) - \ell \begin{cases} = 0, & z \in \{\Re z = \mu\} \cap \{|\Re z| \leq 2/\sqrt{T}\}, \\
< 0, & z \in \{\Re z = \mu\} \cap \{|\Re z| > 2/\sqrt{T}\}.
\end{cases}
\]

- As $z \to \infty$, $g(z)$ has the expansion

\[
g(z) = \log(z) - \frac{i\mu}{z} + O\left(\frac{1}{z^2}\right).
\]

Now fix small positive constants $\epsilon$ and $\delta$ and define the following regions:

\[
\Omega_+ := \left\{ z : |\Re z| < \frac{2}{\sqrt{T}} \text{ and } \mu < \Im z < \mu + \epsilon \right\}, \quad \Omega_- := \left\{ z : |\Re z| < \frac{2}{\sqrt{T}} \text{ and } -\delta < \Im z < \mu \right\},
\]

\[
\Omega^\circ_+ := \left\{ z : |\Re z| > \frac{2}{\sqrt{T}} \text{ and } \mu < \Im z < \mu + \epsilon \right\}, \quad \Omega^\circ_- := \left\{ z : |\Re z| > \frac{2}{\sqrt{T}} \text{ and } -\delta < \Im z < \mu \right\}.
\]

See Figure 3. The boundary between $\Omega_+$ and $\Omega_-$ is the band $(a, b)$. The jump matrices will decay to the identity as $n \to \infty$ except on this interval. Also define the function...
Figure 3. The jump contours for $S_n$ along with their orientations and the regions $\Omega_{\pm}$ and $\Omega_{\pm}$. The real axis is the dotted horizontal line.

(3.14) $G(z) \equiv G(z; T, \mu) := \begin{cases} 2g(z; T, \mu) - V(z; T, \mu) - \ell, & z \in \Omega_+, \\ -2g(z; T, \mu) + V(z; T, \mu) + \ell, & z \in \Omega_- \end{cases}$

From the equilibrium condition (3.11), we see that $G(z)$ is analytic in $\Omega_+ \cup \Omega_-$, and on the band $(a, b)$ it is given by the formula

(3.15) $G(z) = g_+(z) - g_-(z), \quad z \in (a, b)$.

Furthermore, combining (3.10) and (3.15) gives the formula

(3.16) $G(z) = 2\pi i \int_{z}^{b} \rho(w) \, dw, \quad z \in (a, b)$,

which naturally extends analytically into $\Omega_+ \cup \Omega_-$. We are now ready to define the matrix $S_n(z)$. Set

(3.17) $S_n(z) := \begin{cases} A e^{-n\ell\sigma_3/2} P_n(z) D_+^u(z) e^{-n(g(z) - \ell/2)\sigma_3} A^{-1} \begin{bmatrix} 1 & 0 \\ -e^{-nG(z)} & 1 \end{bmatrix}, & z \in \Omega_+, \\ A e^{-n\ell\sigma_3/2} P_n(z) D_-^u(z) e^{-n(g(z) - \ell/2)\sigma_3} A^{-1} \begin{bmatrix} 1 & 0 \\ e^{nG(z)} & 1 \end{bmatrix}, & z \in \Omega_-, \\ A e^{-n\ell\sigma_3/2} P_n(z) D_+^u(z) e^{-n(g(z) - \ell/2)\sigma_3} A^{-1}, & z \in \Omega_+, \\ A e^{-n\ell\sigma_3/2} P_n(z) D_-^u(z) e^{-n(g(z) - \ell/2)\sigma_3} A^{-1}, & z \in \Omega_-, \\ A e^{-n\ell\sigma_3/2} P_n(z) e^{-n(g(z) - \ell/2)\sigma_3} A^{-1}, & \text{otherwise.} \end{cases}$

Here the matrices $D_\pm^u(z)$ and $A$ are for the interpolation of poles, the diagonal matrices involving $g(z)$ and $\ell$ are how the $g$-function is introduced, and the lower-triangular matrices involving $G(z)$ are for the opening of lenses. As $z \to \infty$, $S_n(z)$ satisfies

(3.18) $S_n(z) = I + O(z^{-1})$. 
Also, $S_n(z)$ satisfies the jump conditions $S_{n+}(z) = S_{n-}(z) V(S)(z)$, where the orientation is given in Figure 3 and the jumps are given by

$$V(S)(z) := \begin{cases} 
1 & z \in (\mathbb{R} + i(\mu + \epsilon)) \setminus (a + i\epsilon, b + i\epsilon), \\
0 & z \in (a + i\epsilon, b + i\epsilon), \\
1 & z \in (a, a + i\epsilon) \cup (b, b + i\epsilon), \\
0 & z \in (\mathbb{R} + i\mu) \setminus (a, b), \\
0 & z \in (a, b), \\
1 & z \in (a, a - i\delta) \cup (b, b - i\delta), \\
0 & z \in (\mathbb{R} - i\delta) \setminus (a - i\delta, b - i\delta). 
\end{cases}$$

(3.19)

The jump conditions and the boundary condition (3.18) determine $S_n$ uniquely (see, for example, [21, Theorem 7.18]).

In the $k$-tacnode scaling, the convergence of the jump matrices to the identity matrix fails in a neighborhood of the origin and we need to make a local transformation in a small neighborhood of $z = i\mu$. To that end, first define the regions $\Omega^\Delta_\pm$ as

$$\Omega^\Delta_+ = \{ z = x + iy : -y + \mu < x < y - \mu, \mu < y < \mu + \epsilon \},$$

$$\Omega^\Delta_- = \{ z = x + iy : -y - \mu < x < y + \mu, -\delta < y < \mu \}$$

(3.20)

(see Figure 4). Now define the matrix $S_{n}^{\text{crit}}(z)$ as

$$S_{n}^{\text{crit}}(z) := \begin{cases} 
S_n(z) \left[ \begin{array}{cc} 1 & \pm e^{\pm nG(z)} e^{\pm 2\pi i(nz - \tau)} \\
0 & 1 \end{array} \right], & z \in \Omega^\Delta_+, \\
S_n(z), & \text{otherwise},
\end{cases}$$

(3.21)

where $S_n(z)$ is defined in (3.17). The matrix function $S_{n}^{\text{crit}}(z)$ satisfies a Riemann–Hilbert problem similar to $S_n(z)$, but with additional jumps in the boundaries of $\Omega^\Delta_\pm$, which we denote $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$, and orient as shown in Figure 4.

It is straightforward to check that the jump matrices for $S_{n}^{\text{crit}}(z)$ are the same as those for $S_n(z)$, see (3.19), except on the contours $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$. The jump conditions are

$$S_{n+}^{\text{crit}}(z) = S_{n-}^{\text{crit}}(z) V^{\text{crit}}(z),$$

(3.22)
where the orientations are given in Figure 4 and

\[
V(3.23)\]

\[
\begin{align*}
V^{\text{crit}}(z) &= \begin{cases} 
(1 - e^{2\pi i(nz - \tau)})^{-1} & z \in \gamma_1, \\
-e^{-nG(z)} & e^{-2\pi i(nz - \tau)} \\
0 & 1 - e^{2\pi i(nz - \tau)} \\
1 & e^{2\pi i(nz - \tau)} \\
0 & e^{-nG(z)} \\
1 & e^{nG(z)} \\
(1 - e^{-2\pi i(nz - \tau)})^{-1} & z \in \gamma_4, \\
0 & e^{-2\pi i(nz - \tau)} \\
1 & 1 - e^{-2\pi i(nz - \tau)} \\
\end{cases}
\end{align*}
\]

\[
V^{(S)}(z),
\]

otherwise.

We now show that, as \( n \to \infty \), the jump matrices for \( S_n^{\text{crit}}(z) \) decay exponentially to the identity matrix except in a neighborhood of the band \((a, b)\). Let \( D_a, D_b, \) and \( D_{i\mu} \) be small fixed circular neighborhoods centered at \( a, b, \) and \( i\mu, \) respectively, small enough so their closures do not intersect \( \mathbb{R} + i(\mu + \epsilon), \mathbb{R} - i\delta, \) or each other.

**Lemma 3.2.** Fix a non-negative integer \( k \), choose \( \mu \) according to \((1.9)\), and suppose \((\pi^2 - T)n^{2/3}\) remains bounded as \( n \to \infty \). Then there exists a constant \( c > 0 \) such that

\[
V^{\text{crit}}(z) = I + \mathcal{O}(e^{-cn}) \quad \text{as} \quad n \to \infty \quad \text{for} \quad z \in \Sigma^{\text{crit}} \cap (D_a \cup D_b \cup D_{i\mu})^C
\]

(where \( C \) denotes the complement).

**Proof.** Throughout the proof we assume \( z \notin D_a \cup D_b \cup D_{i\mu} \) in order to avoid rewriting this condition. From \((3.23)\), it is enough to prove the following:

- \( \Re G(z) > 0 \) for \( z \in (a + i\epsilon, b + i\epsilon) \cup (a, a + i\epsilon) \cup (b, b + i\epsilon), \)
- \( \Re G(z) < 0 \) for \( z \in (a - i(\mu + \delta), b - i(\mu + \delta)) \cup (a, a - i(\mu + \delta)) \cup (b, b - i(\mu + \delta)), \)
- \( \Re(G(z) + 2\pi iz) < 0 \) for \( z \in \gamma_2 \cup [\mathbb{R} + i(\mu + \epsilon) \backslash \gamma_1], \)
- \( \Re(G(z) + 2\pi iz) > 0 \) for \( z \in \gamma_3 \cup [\mathbb{R} - i\delta \backslash \gamma_4], \)
- \( \Re(g_+(z) + g_-(z) - V(z) - \ell) < 0 \) for \( z \in (\mathbb{R} + i\mu) \backslash (a, b). \)

Each of these conditions follow from the properties \((3.10) - (3.11)\), and was shown in the tacnode scaling regime with \( \mu = 0 \) in \([12]\). (In fact, the arguments used are the same as in the case when \( T \in (0, \pi^2) \); the only significant change if \( T \approx \pi^2 \) occurs near \( z = i\mu). \) All of the quantities involved \( (G(z; T\mu), g(z; T\mu), V(z; T\mu), \ell(T, \mu), a(T, \mu), b(T, \mu)) \) are continuous as functions of \( \mu. \) Therefore, if \( \mu \) is given by \((1.9)\), then these conditions must hold as long as \( n \) is sufficiently large. \( \square \)
4. Initial construction of parametrices

We build an approximation, or model, of \( S_n^{\text{crit}}(z) \) in four pieces. Inside the disks \( \mathbb{D}_{i\mu}, \mathbb{D}_a, \) and \( \mathbb{D}_b \), we construct the inner model solutions or parametrices \( M_k^{(i\mu)}(z) \), \( M_k^{(a)}(z) \), and \( M_k^{(b)}(z) \) to satisfy exactly the same jumps as \( S_n^{\text{crit}}(z) \). These constructions each involve a local conformal change of variables. When necessary we assume without further comment the jump contours for \( S_n^{\text{crit}}(z) \) have been locally deformed in order to map exactly onto the parametrix jump contours. Outside of the three disks \( S_n^{\text{crit}}(z) \) is approximated by the outer model solution \( M_k^{(\text{out})}(z) \). It is necessary to closely match the inner model solutions to the outer model solution on the disk boundaries. It turns out that if we use the same outer model problem used in \([43]\) in the tacnode case, the inner and outer model solutions do not match well on \( \partial \mathbb{D}_{i\mu} \). This difficulty can be circumvented by requiring the outer model problem to have a pole of a specified order at \( i\mu \). This technique has previously been used to analyze the emergence of a spectral cut in unitarily invariant random matrix ensembles \([12]\), a smooth-to-oscillatory transition for the semiclassical Korteweg-de Vries equation \([19]\), a librational-to-rotational transition for the spectral cut in unitarily invariant random matrix ensembles \([12]\), and at the edge of the pole region for rational solutions for the Painlevé-II equation \([17]\). With this motivation, we therefore pose the outer model problem as follows.

Riemann–Hilbert Problem 4.1 (Outer model problem near criticality). Fix \( k \in \mathbb{Z} \) and determine a \( 2 \times 2 \) matrix-valued function \( M_k^{(\text{out})}(z) \) satisfying:

Analyticity: \( M_k^{(\text{out})}(z) \) is analytic in \( z \) off \([a,b]\) and is Hölder continuous up to \((a,b)\) except in \( \mathbb{D}_{i\mu} \) with at most quarter-root singularities at \( a \) and \( b \). Furthermore, the function

\[
\widetilde{M}_k^{(\text{out})}(z) := \begin{cases} 
M_k^{(\text{out})}(z)(z-i\mu)^{k\sigma_3}, & z \in \mathbb{D}_{i\mu} \cap \{ \Im(z) > \mu \}, \\
M_k^{(\text{out})}(z)(z-i\mu)^{-k\sigma_3}, & z \in \mathbb{D}_{i\mu} \cap \{ \Im(z) < \mu \}
\end{cases}
\]

is analytic in its domain of definition.

Normalization:

\[
\lim_{z \to \infty} M_k^{(\text{out})}(z) = \mathbb{I}.
\]

Jump condition: Orienting \([a,b]\) left-to-right, the solution satisfies

\[
M^{(\text{out})}_{k+}(z) = M^{(\text{out})}_{k-}(z) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad z \in [a,b].
\]

To solve the outer model problem we begin by defining \( R(z) \) to be the function satisfying

\[
R(z)^2 = (z-a)(z-b)
\]

with branch cut \([a,b]\) and asymptotics \( R(z) = z + \mathcal{O}(1) \) as \( z \to \infty \). Next, define the function

\[
d(z) := \frac{R(z) + \frac{2i}{\sqrt{T}}}{z-i\mu}.
\]

The following lemma records some properties of \( d(z) \) that are easily checked directly.

Lemma 4.2. (a) \( d(z) \) is analytic off \([a,b]\).

(b) \( d_+(z)d_-(z) = -1 \) for \( z \in (a,b) \), where \((a,b)\) is oriented left-to-right.

(c) \( d(z) = \frac{4i}{\sqrt{T}} \frac{1}{z-i\mu} + \mathcal{O}(z-i\mu) \) for \( z \in \mathbb{D}_{i\mu} \cap \{ \Im z > 0 \} \).
(d) \( d(z) = \frac{\sqrt{T}}{4i}(z - i\mu) + \mathcal{O}((z - i\mu)^3) \) for \( z \in \mathbb{D}_{i\mu} \cap \{\Im z < 0\} \).

(e) \( d(z) = -i + \mathcal{O}(\sqrt{a - z}) \) for \( z \in \mathbb{D}_a \).

(f) \( d(z) = i + \mathcal{O}(\sqrt{z - b}) \) for \( z \in \mathbb{D}_b \).

(g) \( d(z) = 1 + \frac{2i}{\sqrt{T}z} + \mathcal{O}\left(\frac{1}{z^2}\right) \) as \( z \to \infty \).

Also define

\[
\gamma(z) := \left(\frac{z - a}{z - b}\right)^{1/4}
\]

with branch cut \([a, b]\) and \( \lim_{z \to \infty} \gamma(z) = 1 \). Now Riemann–Hilbert Problem 4.1 is solved by

\[
M_k^{(\text{out})}(z) := e^{-ik\pi\sigma_3/2} \left[\begin{array}{cc}
\frac{\gamma(z) + \gamma(z)^{-1}}{2} & \frac{\gamma(z) - \gamma(z)^{-1}}{2i} \\
\frac{\gamma(z) - \gamma(z)^{-1}}{2i} & \frac{\gamma(z) + \gamma(z)^{-1}}{2}
\end{array}\right] e^{ik\pi\sigma_3/2} d(z)^{k\sigma_3}.
\]

4.2. The inner model problem near \( z = i\mu \). We now construct the function \( M_k^{(i\mu)} \) that satisfies the same jumps as \( S^{\text{crit}}(z) \) for \( z \in \mathbb{D}_{i\mu} \) and approximately matches \( M_k^{(\text{out})}(z) \) for \( z \in \partial\mathbb{D}_{i\mu} \). It is convenient to work in a local variable \( \zeta(z) \) in which the jump conditions take a particularly nice form. For \( z \in \mathbb{D}_{i\mu} \), the jump exponent has the expansion

\[
n(G(z) + 2\pi iz) - 2\pi i\tau
\]

\[
= (ni\pi - 2\pi\mu n - 2\pi i\tau) + 2in(\pi - \sqrt{T})(z - i\mu) + \frac{inT^{3/2}}{12}(z - i\mu)^3 + \mathcal{O}((z - i\mu)^4).
\]

Note that at criticality (i.e. \( T = \pi^2 \) and \( \mu = 0 \)) the coefficient of \( z - i\mu \) vanishes while that of \( (z - i\mu)^3 \) does not. It is therefore reasonable to expect that the exponent can be modeled by a cubic polynomial. Indeed, following Chester, Friedman, and Ursell [18] (see also [16, 42]), there is, for \( z \) sufficiently close to \( i\mu \) and \( T \) and \( \mu \) sufficiently close to criticality, an invertible conformal mapping \( \zeta(z) = \zeta(z; \mu, T) \) as well as analytic functions \( s(\mu, T) \) and \( \theta(\mu) \) such that \( \zeta(i\mu) = 0 \) and

\[
n(G(z) + 2\pi iz) - 2\pi i\tau = 2i\left(\frac{4}{3}\zeta(z)^3 + s(\zeta(z)) - \theta\right).
\]

If necessary, we shrink the size of \( \mathbb{D}_{i\mu} \) to ensure these conditions hold for all \( z \in \mathbb{D}_{i\mu} \). By plugging the expansion [4.8] into [4.9] and matching constant terms we find that

\[
\theta(\mu) = -\frac{n\pi}{2} - in\pi\mu + \pi\tau.
\]

The change of variables [4.9] is nearly identical to the one presented in [42, Section 4.9], up to a shift by \( i\mu \) in the definition of \( G(z) \). The parameters \( s \) and \( \theta \) are defined in terms of the stationary points of the left-hand-side of [4.9], and the analysis presented in [42, Section 4.9] applies to [4.9] as well. The result is the formula [1.10] for \( s \), compare [42, Equation (1.43)]. For a similar application of the Chester–Friedman–Ursell change of variables with more details given, see also [16, Section 4.3].

We note that

\[
\zeta(z) = \mathcal{O}(n^{1/3}) \text{ for } z \in \mathbb{D}_{i\mu}
\]

as well as the facts that

\[
\frac{\zeta(z)}{z - i\mu} = \mathcal{O}(1) \text{ and } \frac{z - i\mu}{\zeta(z)} = \mathcal{O}(1) \text{ for } z \in \mathbb{D}_{i\mu}.
\]
More precisely, inserting $\zeta(z) = \zeta'(i\mu)(z - i\mu) + O((z - i\mu)^2)$ into \eqref{eq:4.8} and \eqref{eq:4.9}, and then using \eqref{eq:1.12}, shows
\begin{equation}
\zeta'(i\mu) = \frac{n(\pi - \sqrt{T})}{s} = \frac{\pi n^{1/3}}{2^{5/3}} + O\left(\frac{1}{n^{2/3}}\right).
\end{equation}

We are now ready to pose the inner model problem.

**Riemann–Hilbert Problem 4.3** (The inner model problem in $\mathbb{D}_{i\mu}$ near criticality). Fix $s \in \mathbb{R}$ and $k \in \mathbb{Z}$. Determine a $2 \times 2$ matrix-valued function $M_k^{(i\mu)}(\zeta(z))$ satisfying:

**Analyticity**: $M_k^{(i\mu)}(\zeta)$ is analytic for $\zeta \in \mathbb{D}_{i\mu}$ off the six rays $\arg(\zeta) \in \{0, \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}, \pi\}$. In each sector the solution can be analytically continued into a larger sector, and is Hölder continuous up to the boundary in a neighborhood of $\zeta = 0$.

**Normalization**:
\begin{equation}
M_k^{(i\mu)}(\zeta(z)) = (I + o(1))M_k^{\text{out}}(z) \quad \text{as} \quad n \to \infty \quad \text{for} \quad z \in \partial \mathbb{D}_{i\mu}.
\end{equation}

**Jump condition**: The solution satisfies $M_k^{(i\mu)}(\zeta) = M_k^{(i\mu)}(\zeta)V^{(i\mu)}(\zeta)$, with jumps as shown in Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{The jump contours $\Sigma^{(i\mu)}$ and jump matrices $V^{(i\mu)}(\zeta)$.}
\end{figure}

We now perform a series of changes of variables
\[ M_k^{(i\mu)} \to Z_k^{(1)} \to Z_k^{(2)} \to Z_k \]

for the Riemann–Hilbert problem for $M_k^{(i\mu)}$ into a Riemann–Hilbert problem associated with the Painlevé-II equation. Note that
\begin{equation}
E_k(z) := \begin{cases}
M_k^{\text{out}}(z) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left( \frac{n^{1/3}}{\zeta(z)} \right)^{k\sigma_3}, & z \in \mathbb{D}_{i\mu} \cap \{ \Im z > \mu \}, \\
M_k^{\text{out}}(z) \left( \frac{n^{1/3}}{\zeta(z)} \right)^{k\sigma_3}, & z \in \mathbb{D}_{i\mu} \cap \{ \Im z < \mu \}
\end{cases}
\end{equation}
is analytic and invertible for $z \in \mathbb{D}_{i\mu}$. Given the required normalization for $M_k^{(i\mu)}(\zeta(z))$, we have
\begin{equation}
M_k^{(i\mu)}(\zeta(z)) = \begin{cases}
(\mathbb{I} + o(1))E_k(z) \left( \frac{\zeta(z)}{n^{1/3}} \right)^{k\sigma_3} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & z \in \partial \mathbb{D}_{i\mu} \cap \{ \Im z > \mu \}, \\
(\mathbb{I} + o(1))E_k(z) \left( \frac{\zeta(z)}{n^{1/3}} \right)^{k\sigma_3}, & z \in \partial \mathbb{D}_{i\mu} \cap \{ \Im z < \mu \}.
\end{cases}
\end{equation}
We now pull out this analytic factor from \( M_k^{(i\mu)} \):

\[
Z_k^{(1)}(\zeta(z)) := E_k(z)^{-1} M_k^{(i\mu)}(\zeta(z)) \quad \text{for } z \in \mathbb{D}_{i\mu}.
\]

Now \( Z_k^{(1)} \) has the same jumps as \( M_k^{(i\mu)} \), but the normalization changes to

\[
Z_k^{(1)}(\zeta(z)) = \begin{cases} 
(\mathbb{I} + o(1)) \left( \frac{\zeta(z)}{n^{1/3}} \right)^{k\sigma_3} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & z \in \partial \mathbb{D}_{i\mu} \cap \{ \Im z > \mu \}, \\
(\mathbb{I} + o(1)) \left( \frac{\zeta(z)}{n^{1/3}} \right)^{k\sigma_3}, & z \in \partial \mathbb{D}_{i\mu} \cap \{ \Im z < \mu \}.
\end{cases}
\]

The next transformation removes the jump on the real axis and switches the triangularity of the jump matrices in the upper half-plane.

\[
Z_k^{(2)}(\zeta) := \begin{cases} 
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Z_k^{(1)}(\zeta) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \Im \zeta > 0, \\
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} Z_k^{(1)}(\zeta) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & \Im \zeta < 0.
\end{cases}
\]

The jumps for \( Z_k^{(2)}(\zeta) \) are shown in Figure 6. This also has the effect of simplifying the normalization:

\[
\begin{bmatrix} e^{2i(\frac{4}{3}\zeta^3 + s\zeta - \theta)} \\ 1 \end{bmatrix} 0 \\
1 \begin{bmatrix} e^{-2i(\frac{4}{3}\zeta^3 + s\zeta - \theta)} \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ e^{-2i(\frac{4}{3}\zeta^3 + s\zeta - \theta)} \end{bmatrix} 0 \\
0 \begin{bmatrix} 1 \\ -e^{-2i(\frac{4}{3}\zeta^3 + s\zeta - \theta)} \end{bmatrix}
\]

**Figure 6.** The jump contours and matrices for \( Z_k^{(2)}(\zeta) \).

Here \( o(1) \) refers to growth in \( n \), but because \( \zeta(z) \) grows like \( n^{1/3} \), we could just as well think of it as referring to growth in \( \zeta \). This is advantageous as we want to pose a problem in the \( \zeta \) plane with no reference to \( z \) or \( n \). The final transformation removes the dependence on \( n \), \( \mu \), and \( \tau \) from the Riemann–Hilbert problem:

\[
Z_k(\zeta) := n^{k\sigma_3/3} e^{-i\theta \sigma_3} Z_k^{(2)}(\zeta) e^{i\theta \sigma_3}.
\]

For future reference, we note that combining (4.17), (4.19), and (4.21) gives

\[
M_k^{(i\mu)}(\zeta(z)) = \begin{cases} 
E_k(z) \sigma_3 e^{i\theta \sigma_3} n^{-k\sigma_3/3} Z_k(\zeta(z)) e^{-i\theta \sigma_3} \sigma_1, & z \in \mathbb{D}_{i\mu} \cap \{ \Im z > \mu \}, \\
E_k(z) \sigma_3 e^{i\theta \sigma_3} n^{-k\sigma_3/3} Z_k(\zeta(z)) e^{-i\theta \sigma_3} \sigma_3, & z \in \mathbb{D}_{i\mu} \cap \{ \Im z < \mu \}.
\end{cases}
\]

We state the Riemann–Hilbert problem for \( Z_k(\zeta) \) and relate its solution to the generalized Hastings–McLeod functions in the following subsection.
4.3. The Jimbo–Miwa RHP for generalized Hastings–McLeod functions. The function $Z_k(\zeta)$ defined in [4.21] is the unique solution to the following Riemann–Hilbert problem.

**Riemann–Hilbert Problem 4.4** (Jimbo–Miwa problem for generalized Hastings–McLeod functions). Fix $s \in \mathbb{R}$ and $k \in \mathbb{Z}$ and determine a $2 \times 2$ matrix-valued function $Z_k(\zeta; s)$ satisfying:

**Analyticity:** $Z_k(\zeta; s)$ is analytic in $\zeta$ off the four rays $\arg(\zeta) \in \{\pm \frac{\pi}{6}, \pm \frac{5\pi}{6}\}$. In each sector the solution can be analytically continued into a larger sector, and is Hölder continuous up to the boundary in a neighborhood of $\zeta = 0$.

**Normalization:** Uniformly with respect to $\arg(\zeta)$ in each sector of analyticity,

\begin{equation}
\lim_{\zeta \to \infty} Z_k(\zeta; s)\zeta^{-k\sigma_3} = I.
\end{equation}

**Jump condition:** Orienting the four jump rays towards infinity, the solution satisfies

$Z_{k+}(\zeta; s) = Z_{k-}(\zeta; s)V(\zeta; s)$, as shown in Figure 7.

\[
\begin{bmatrix}
1 & 0 \\
-e^{2i(\frac{4}{3}\zeta + s)} & 1
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 \\
e^{2i(\frac{4}{3}\zeta + s)} & 1
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 \\
e^{-2i(\frac{4}{3}\zeta + s)} & 1
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 \\
e^{-2i(\frac{4}{3}\zeta + s)} & 1
\end{bmatrix}
\end{equation}

**Figure 7.** The jump contours $\Sigma(Z)$ and jump matrices $V(Z)(\zeta; s)$.

This is a special case of the Riemann–Hilbert problem posed in Fokas et al., Chapter 3, §3.2 [33]. Specifically, $Y(\lambda) := Z_\nu(\lambda; x)\lambda^{-\nu\sigma_3}$ satisfies that Riemann–Hilbert problem with the specific choice of Stokes data $s_1 = s_4 = 1$, $s_2 = s_5 = 0$, and $s_3 = s_6 = -1$. Fokas et al. show that the solution to this problem exists and is unique. They furthermore (see [33 Chapter 4, §2.5]) show that the function $L_k(\zeta; s)$ defined as

\begin{equation}
L_k(\zeta; s) := Z_k(\zeta; s)e^{-i(\frac{4}{3}\zeta + s)\sigma_3}
\end{equation}

satisfies the Jimbo–Miwa (or Jimbo–Miwa–Garnier) Lax pair [38] (4.25)

\[
\frac{\partial L_k}{\partial s} = \left(-i\sigma_3\zeta + i \begin{bmatrix} 0 & U_k(s) \\ -V_k(s) & 0 \end{bmatrix} \right) L_k,
\]

\[
\frac{\partial L_k}{\partial \zeta} = \left(-4i\sigma_3\zeta^2 + 4i \begin{bmatrix} 0 & U_k(s) \\ -V_k(s) & 0 \end{bmatrix} \right) \zeta + \left[\begin{bmatrix} -2iU_k(s)V_k(s) - is & -2U_k'(s) \\ 2iU_k(s)V_k'(s) + is & -2V_k'(s) \end{bmatrix} \right] L_k,
\]

and that this overdetermined system is equivalent to the coupled Painlevé-II system (1.19). They also show that the logarithmic derivatives of $U_k(s)$ and $V_k(s)$ satisfy (uncoupled) Painlevé-II equations, a calculation we carry out below. This implies that $U_k(s)$ and $V_k(s)$ are tau functions for certain Painlevé-II transcendents. The remainder of the section is devoted to identifying these transcendents as certain generalized Hastings–McLeod functions. We begin by expressing certain terms in the large-$\zeta$ expansion of $Z_k(\zeta)$ in terms of $U_k(s)$ and $V_k(s)$. With the help of this expansion, we then identify the Painlevé-II equations satisfied by the logarithmic derivatives of $U_k(s)$.
and \( V_k(s) \). Next, we obtain the Bäcklund transformations relating \( U_{k+1}(s) \) and \( V_{k+1}(s) \) to \( U_k(s) \). Combining the Bäcklund transformations with the fact that \( U_0(s) \) and \( V_0(s) \) can be identified as Hastings–McLeod functions from the Riemann–Hilbert problem they satisfy allows us to identify the logarithmic derivatives of \( U_k(s) \) and \( V_k(s) \).

4.3.1. Expansion of \( Z_k \). We need the large-\( \zeta \) expansion of \( Z_k(\zeta; s) \):

\[
Z_k(\zeta; s) \zeta^{-k \sigma_3} = I + \frac{A_k(s)}{\zeta} + \frac{B_k(s)}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right).
\]

To compute this expansion, we begin by assuming \( L_k(\zeta; s) \) has the expansion

\[
L_k(\zeta) = \left( I + \frac{Y_k^{(1)}}{\zeta} + \frac{Y_k^{(2)}}{\zeta^2} + \frac{Y_k^{(3)}}{\zeta^3} + O\left(\frac{1}{\zeta^4}\right) \right) \times \exp\left( D_k^{(3)} \zeta^3 + D_k^{(2)} \zeta^2 + D_k^{(1)} \zeta + O(\log \zeta) + O\left(\frac{1}{\zeta^3}\right) \right),
\]

where \( Y_k^{(j)}, j = 1, 2, 3 \) are independent of \( \zeta \) and off-diagonal, while \( D_k^{(j)}, j = -3, 2, 1, 0, -1 \) are independent of \( \zeta \) and diagonal. We insert this ansatz into the \( \zeta \)-derivative equation in the Lax pair (4.25) and group diagonal and off-diagonal terms in each power of \( \zeta \).

\[
\begin{align*}
O(\zeta^2) \text{ diagonal:} & \quad 3D_k^{(3)} = -4i \sigma_3, \\
O(\zeta) \text{ diagonal:} & \quad D_k^{(2)} = 0, \\
O(\zeta) \text{ off-diagonal:} & \quad 3Y_k^{(1)} D_k^{(3)} = -4i \sigma_3 Y_k^{(1)} + 4i \left[ \begin{array}{cc} 0 & U_k \\ -V_k & 0 \end{array} \right], \\
O(1) \text{ diagonal:} & \quad D_k^{(1)} = 4i \left[ \begin{array}{cc} 0 & U_k \\ -V_k & 0 \end{array} \right] Y_k^{(1)} + \left[ \begin{array}{cc} 0 & 2U_k' \\ -2U_k & 0 \end{array} \right] Y_k^{(0)}, \\
O(1) \text{ off-diagonal:} & \quad 2Y_k^{(1)} D_k^{(2)} + 3Y_k^{(2)} D_k^{(3)} = -4i \sigma_3 Y_k^{(2)} + \left[ \begin{array}{cc} 0 & 0 \\ -2V_k' & 0 \end{array} \right] Y_k^{(1)}, \\
O(\zeta^{-1}) \text{ diagonal:} & \quad D_k^{(0)} = 4i \left[ \begin{array}{cc} 0 & U_k \\ -V_k & 0 \end{array} \right] Y_k^{(2)} + \left[ \begin{array}{cc} 0 & 0 \\ -2V_k' & 0 \end{array} \right] Y_k^{(1)}, \\
O(\zeta^{-1}) \text{ off-diagonal:} & \quad Y_k^{(1)} D_k^{(1)} + 2Y_k^{(2)} D_k^{(2)} + 3Y_k^{(3)} D_k^{(3)} \\
& \quad = -4i \sigma_3 Y_k^{(3)} - (2iU_k V_k + is) \sigma_3 Y_k^{(1)}, \\
O(\zeta^{-2}) \text{ diagonal:} & \quad -D_k^{(-1)} = 4i \left[ \begin{array}{cc} 0 & U_k \\ 0 & -V_k \end{array} \right] Y_k^{(3)} + \left[ \begin{array}{cc} 0 & 0 \\ -2V_k' & 0 \end{array} \right] Y_k^{(2)}. 
\end{align*}
\]

Solving these equations sequentially yields

\[
\begin{align*}
D_k^{(3)} & = -\frac{4}{3} i \sigma_3, & D_k^{(2)} & = 0, & D_k^{(1)} & = -i s \sigma_3, \\
D_k^{(0)} & = (U_k V_k' - V_k U_k') \sigma_3, & D_k^{(-1)} & = \frac{i}{2} (U_k^2 V_k^2 + s U_k V_k + U_k' V_k') \sigma_3, \\
Y_k^{(1)} & = \left[ \begin{array}{cc} 0 & \frac{1}{2} U_k \\ \frac{1}{2} V_k & 0 \end{array} \right], & Y_k^{(2)} & = \left[ \begin{array}{cc} 0 & \frac{i}{4} U_k' \\ -i \frac{1}{4} V_k' & 0 \end{array} \right], & Y_k^{(3)} & = -\frac{1}{8} \left[ \begin{array}{cc} 0 & 0 \\ U_k^2 V_k^2 + s V_k & 0 \end{array} \right]. 
\end{align*}
\]

Combining (4.24) and (4.26) gives

\[
L_k(\zeta; s) e^{i(\frac{1}{2} \zeta^3 + s \zeta) \sigma_3} \zeta^{-k \sigma_3} = I + \frac{A_k(s)}{\zeta} + \frac{B_k(s)}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right).
\]
Using (4.27), we see
\begin{equation}
(4.31) \quad A_k = Y_k^{(1)} + D_k^{(-1)}, \quad B_k = Y_k^{(1)} D_k^{(-1)} + Y_k^{(2)} + \frac{1}{2} \left( D_k^{(-1)} \right)^2 + D_k^{(-2)}.
\end{equation}
Applying (4.29),
\begin{equation}
(4.32) \quad A_k = \frac{1}{2} \begin{bmatrix} i(U_k^2 V_k' + s U_k V_k' + U_k V_k) & \frac{U_k}{V_k} \\ -i(U_k^2 V_k' + s U_k V_k + U_k V_k') & U_k - U_k(U_k^2 V_k' + s U_k V_k + U_k V_k') \end{bmatrix},
\end{equation}
\begin{equation}
(4.33) \quad B_k = \frac{1}{4} \begin{bmatrix} -V_k' + V_k(U_k^2 V_k' + s U_k V_k + U_k V_k') & 0 \\ 0 & U_k' - U_k(U_k^2 V_k' + s U_k V_k + U_k V_k') \end{bmatrix} + \text{diagonal}.
\end{equation}
We will not need the diagonal terms in $B_k$.

4.3.2. Differential equations for the logarithmic derivatives. We note that
\begin{equation}
(4.34) \quad \lambda_k := \mathcal{U}_k(s) V_k'(s) - V_k(s) U_k'(s)
\end{equation}
is an $s$-independent quantity. To see this, simply multiply the second equation in (1.19) by $U_k(s)$, multiply the first equation by $V_k(s)$, and subtract:
\begin{equation}
(4.35) \quad \frac{d}{ds} (\mathcal{U}_k(s) V_k'(s) - V_k(s) U_k'(s)) = 0.
\end{equation}
To see exactly what $\lambda_k$ is in terms of $k$, we recall from (4.29) that the same combination $\mathcal{U}_k V_k' - V_k U_k'$ appears in $D_k^{(0)}$:
\begin{equation}
(4.36) \quad D_k^{(0)} = \lambda_k \sigma_3.
\end{equation}
From the expansion (4.27) for $L_k(\zeta)$ and the expressions for the coefficients (4.29), we see
\begin{equation}
(4.37) \quad L_k(\zeta) e^{\frac{4}{3} \zeta^3 + i s \zeta} e^{-\lambda_k \sigma_3} = \mathbb{1} + \mathcal{O}\left( \frac{1}{\zeta} \right).
\end{equation}
Using the definition of $L_k(\zeta)$ in (4.24), this gives
\begin{equation}
(4.38) \quad Z_k(\zeta) e^{-\lambda_k \sigma_3} = \mathbb{1} + \mathcal{O}\left( \frac{1}{\zeta} \right).
\end{equation}
Comparing with the expansion (4.26) for $Z_k(\zeta)$, we obtain
\begin{equation}
(4.39) \quad \lambda_k \equiv k.
\end{equation}
Next, by using the first equation in (1.19) to express $V_k(s)$ in terms of $U_k(s)$, a direct calculation shows
\begin{equation}
(4.40) \quad \frac{1}{2} \frac{d^2}{ds^2} \left( \frac{U_k'(s)}{U_k(s)} \right) = \left( \frac{U_k'(s)}{U_k(s)} \right)^3 - \frac{U_k'(s)}{U_k(s)} + \frac{1}{2} + U_k(s) V_k'(s) - V_k(s) U_k'(s).
\end{equation}
Similarly, using the second equation in (1.19) to express $U_k(s)$ in terms of $V_k(s)$,
\begin{equation}
(4.41) \quad \frac{1}{2} \frac{d^2}{ds^2} \left( \frac{V_k'(s)}{V_k(s)} \right) = \left( \frac{V_k'(s)}{V_k(s)} \right)^3 - \frac{V_k'(s)}{V_k(s)} + \frac{1}{2} - (U_k(s) V_k'(s) - V_k(s) U_k'(s)).
\end{equation}
Thus, from (4.33) and (4.39), the logarithmic derivatives $p_k(s) := U_k'(s)/U_k(s)$ and $q_k(s) := V_k'(s)/V_k(s)$ (cf. (1.22)) satisfy the uncoupled inhomogeneous Painlevé-II equations
\begin{equation}
(4.42) \quad \frac{1}{2} p_k''(s) = p_k(s)^3 - s p_k(s) + \frac{1}{2} + k, \quad \frac{1}{2} q_k''(s) = q_k(s)^3 - s q_k(s) + \frac{1}{2} - k.
\end{equation}
By scaling $P_k(x) := 2^{-1/3}p_k(-2^{-1/3}x)$ and $Q_k(x) := 2^{-1/3}q_k(-2^{-1/3}x)$ (cf. (1.23)), we can bring these into the standard Painlevé-II form matching (1.16):

$$P''_k(x) = 2P'_k(x)^3 + xP_k(x) + \frac{1}{2} + k, \quad Q''_k(x) = 2Q'_k(x)^3 + xQ_k(x) + \frac{1}{2} - k.$$  

### 4.3.3. Schlesinger and Bäcklund transformations

Fix $k$ and assume the functions $Z_k(\zeta; s)$, $U_k(s)$, and $V_k(s)$ are known. Then it is possible to obtain $Z_{k\pm 1}(\zeta; s)$, $U_{k\pm 1}(s)$, and $V_{k\pm 1}(s)$. The maps $Z_k(\zeta; s) \to Z_{k\pm 1}(\zeta; s)$ are called Schlesinger transformations, while the maps $\{U_k(s), V_k(s)\} \to \{U_{k\pm 1}(s), V_{k\pm 1}(s)\}$ are called Bäcklund transformations. We begin with the ansatz

$$Z_{k+1}(\zeta; s) = (\Omega_k(s) + \mathfrak{R}_k(s))Z_k(\zeta; s)$$

and determine the matrices $\Omega_k(s)$ and $\mathfrak{R}_k(s)$. Inserting (4.26) into (4.44) gives

$$\Omega_k \zeta + \mathfrak{R}_k \begin{pmatrix} 1 + \frac{A_k}{\zeta} + \frac{B_k}{\zeta^2} + \mathcal{O}\left(\frac{1}{\zeta^3}\right) \end{pmatrix} \zeta^{-\sigma_3} = \mathbb{I} + \mathcal{O}\left(\frac{1}{\zeta}\right).$$

Grouping terms in each power of $\zeta$ gives

$$\mathcal{O}(\zeta^2) : \quad \Omega_k \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 0,$$

$$\mathcal{O}(\zeta) : \quad (\Omega_k A_k + \mathfrak{R}_k) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = 0,$$

$$\mathcal{O}(1) : \quad \Omega_k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + (\Omega_k B_k + \mathfrak{R}_k A_k) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} = \mathbb{I}.$$  

The $\mathcal{O}(\zeta^2)$ equation gives $[\Omega_k]_{12} = [\Omega_k]_{21} = 0$, while the first column of the $\mathcal{O}(1)$ equation gives $[\Omega_k]_{11} = 1$ and $[\Omega_k]_{21} = 0$. Using this in the $\mathcal{O}(\zeta)$ equation shows $[\mathfrak{R}_k]_{12} = -[A_k]_{12}$ and $[\mathfrak{R}_k]_{22} = 0$. Then the second column of the $\mathcal{O}(1)$ equations shows $[\mathfrak{R}_k]_{21} = [A_k]_{12}^2$ and $[\mathfrak{R}_k]_{11} = [A_k]_{22} - [A_k]_{12}^2 [B_k]_{12}^2$. Along with the expressions for $A_k$ and $B_k$ in (4.32), we have

$$\Omega_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{R}_k = \begin{bmatrix} -\frac{i}{2}U_k U_k^{-1} & -\frac{1}{2}U_k \\ 2U_k^{-1} & 0 \end{bmatrix}.$$  

From the Schlesinger transformation we can now obtain the Bäcklund transformations. Using the expansion (4.26) in the Schlesinger transformation (4.44) gives the equation

$$\Omega_k \zeta + \mathfrak{R}_k \begin{pmatrix} 1 + \frac{A_k}{\zeta} + \frac{B_k}{\zeta^2} + \mathcal{O}\left(\frac{1}{\zeta^3}\right) \end{pmatrix} \zeta^{-\sigma_3} = \mathbb{I} + \mathcal{O}\left(\frac{1}{\zeta}\right).$$

Reading off the (21)-entry of the $\mathcal{O}(\zeta^{-1})$ term gives $[\mathfrak{R}_k]_{21} = [A_k]_{12}^2$, or $2U_k^{-1} = \frac{1}{2}V_{k+1}$. Then we can take the second equation in (1.19) with $k \to k + 1$, i.e. $V_{k+1}'' = 2U_k V_{k+1} + s V_{k+1}$, solve for $U_{k+1}$ in terms of $V_{k+1}$, and use $V_{k+1} = 4U_k^{-1}$. We therefore obtain the Bäcklund transformation

$$U_{k+1} = \frac{1}{4} \frac{(U_k')^2}{U_k} - \frac{1}{8}U_k'' - \frac{s}{8}U_k, \quad V_{k+1} = \frac{4}{U_k}.$$  

### 4.3.4. Identification of $U_k$ and $V_k$

Our next objective is to identify the associated Painlevé functions $U_k(s)$ and $V_k(s)$. The uncoupled Painlevé-II equation (1.16) is the compatibility condition for the well-studied Flaschka–Newell Lax pair (32) (see also, for example, [24 33 42 20])

$$\frac{\partial \Psi}{\partial s} = \begin{bmatrix} -i\zeta & u(s) \\ u(s) & i\zeta \end{bmatrix} \Psi,$$

$$\frac{\partial \Psi}{\partial \zeta} = \begin{bmatrix} -4i\zeta^2 - is - 2iu(s)^2 & 4\zeta u(s) + 2iu'(s) + \alpha/\zeta \\ 4\zeta u(s) - 2iu'(s) + \alpha/\zeta & 4i\zeta^2 + is + 2iu(s) \end{bmatrix} \Psi.$$
The Riemann–Hilbert problem for this Lax pair has jumps on six semi-infinite rays and a pole at the origin of order $\alpha$. However, at $\alpha = 0$ this Riemann–Hilbert problem reduces to the Riemann–Hilbert problem for the Jimbo–Miwa Lax pair with $k = 0$. We will not need the full Flaschka–Newell Riemann–Hilbert problem, but simply note that Riemann–Hilbert Problem 4.4 with $k = 0$ is a special case. Indeed, it is well known [24, 33, 20, 42] that the Painlevé function associated to Riemann–Hilbert Problem 4.4 with $k = 0$ is the Hastings–McLeod function $u_{HM}^{(0)}(s)$ satisfying (1.13) with asymptotics (1.14). Now, matching the Jimbo–Miwa Lax pair (4.25) with the Flaschka–Newell Lax pair (4.50) (with $\alpha = 0$) we arrive at (1.21), i.e. $u_0(s) = -iu_{HM}^{(0)}(s)$ and $V_0(s) = iu_{HM}^{(0)}(s)$.

For $k > 0$ we solve the problem iteratively by Schlesinger and Bäcklund transformations, relating the functions $U_k(s)$ and $V_k(s)$ to the generalized Hastings–McLeod functions. First, we use the uniformly convergent expansions (1.17) and (1.14), the relationship (1.21), and the Bäcklund transformations (4.49) to find the asymptotic behavior of $U_1(s)$ and $V_1(s)$.

$$U_1(s) = \begin{cases} \frac{-i}{16\sqrt{\pi}s^{3/4}}e^{-\frac{x}{3}s^{3/2}} \left( 1 + O \left( \frac{1}{s^{3/4}} \right) \right), & s \to +\infty, \\ \frac{i}{8\sqrt{2}} \left( 1 + O \left( \frac{1}{(-s)^{3/2}} \right) \right), & s \to -\infty, \end{cases}$$

(4.51)

$$V_1(s) = \begin{cases} 8i\sqrt{\pi}s^{1/4}e^{\frac{x}{3}s^{3/2}} \left( 1 + O \left( \frac{1}{s^{3/4}} \right) \right), & s \to +\infty, \\ 4i\sqrt{\frac{2}{s}} \left( 1 + O \left( \frac{1}{(-s)^{3/2}} \right) \right), & s \to -\infty. \end{cases}$$

A straightforward inductive argument applying the Bäcklund transformations (4.49) repeatedly to (4.51) shows the asymptotics (1.20) for $U_k$ and $V_k$. Then (1.22) and (1.23) give

$$P_k(x) = \begin{cases} \frac{-x}{2k + \frac{1}{2}} \left( 1 + O \left( \frac{1}{(-x)^{3/2}} \right) \right), & x \to -\infty, \\ \frac{k + \frac{1}{2}}{x} \left( 1 + O \left( \frac{1}{(-x)^{3/2}} \right) \right), & x \to +\infty, \end{cases}$$

(4.52)

$$Q_k(x) = \begin{cases} \frac{-x}{2k - \frac{1}{2}} \left( 1 + O \left( \frac{1}{(-x)^{3/2}} \right) \right), & x \to -\infty, \\ \frac{k - \frac{1}{2}}{x} \left( 1 + O \left( \frac{1}{(-x)^{3/2}} \right) \right), & x \to +\infty. \end{cases}$$

From (4.43) and (4.52), $(-P_k(x))$ satisfies the inhomogeneous Painlevé–II equation (1.16) and both boundary conditions (1.17)–(1.18) with $\alpha = k + \frac{1}{2}$. Similarly, $Q_k(x)$ satisfies (1.16) and (1.17)–(1.18) with $\alpha = k - \frac{1}{2}$. Therefore, we can identify $P_k(x)$ and $Q_k(x)$ in terms of generalized Hastings–McLeod solutions as $P_k(x) \equiv -u_{HM}^{(k + \frac{1}{2})}(x)$ and $Q_k(x) \equiv v_{HM}^{(k - \frac{1}{2})}(x)$ (cf. (1.24)).

4.4. **Airy parametrices.** It remains to build the inner parametrices $M_k^{(a)}(z)$ and $M_k^{(b)}(z)$ in the open disks $D_a$ and $D_b$. This construction is standard (see, for instance, [22, 14, 9]) and is done in terms of Airy functions. The important fact for us is that the Airy parametrices can be matched to the outer parametrix with an $O(n^{-1})$ error (as opposed to the parametrix at $i\mu$, where the error is $O(n^{-1/3})$). This means the explicit form of the parametrix only comes into play when computing quantities of $O(n^{-1})$. Since we limit ourselves to computing $O(n^{-1/3})$ terms, it is sufficient for our purposes to note the existence and uniqueness of functions satisfying the following problem.

**Riemann–Hilbert Problem 4.5** (The Airy parametrix Riemann–Hilbert problems). Fix a positive integer $k$. For either $* = a$ or $* = b$, determine a $2 \times 2$ matrix-valued function $M_k^{(*)}(z)$ satisfying:
Analyticity: \( M_k^{(s)}(z) \) is analytic in \( \mathbb{D}_s \setminus \Sigma^{\text{crit}} \). In each wedge the solution can be analytically continued into a larger wedge, and is Hölder continuous up to the boundary in a neighborhood of \( z = \ast \).

Normalization:

\[
(4.53) \quad M_k(z) = M_k^{(\text{out})}(z)(I + \mathcal{O}(n^{-1})), \quad z \in \partial \mathbb{D}_s.
\]

Jump condition: For \( z \in \mathbb{D}_s \cap \Sigma^{\text{crit}} \), \( M_k^{(s)}(z) \) satisfies

\[
M_k^{(s)}(z) = M_{k-}^{(s)}(z)V^{\text{crit}}(z),
\]

5. Error analysis

5.1. The error problem. We can now define

\[
(5.1) \quad M^{(k)}(z) := \begin{cases} 
M_k^{(\text{out})}(z), & z \in \mathbb{C} \setminus \{\mathbb{D}_{i\mu} \cup \mathbb{D}_a \cup \mathbb{D}_b\}, \\
M_k^{(\mu)}(\zeta(z)), & z \in \mathbb{D}_{i\mu}, \\
M_k^{(a)}(z), & z \in \mathbb{D}_a, \\
M_k^{(b)}(z), & z \in \mathbb{D}_b.
\end{cases}
\]

Then the error function is

\[
(5.2) \quad X_n^{(k)}(z) := S_n^{\text{crit}}(z)M^{(k)}(z)^{-1},
\]

and it satisfies the jump condition

\[
(5.3) \quad X_n^{(k)}(z) = X_{n-}^{(k)}(z)V^{(X)}(z)
\]

on the jump contours \( \Sigma^{(X)} \) shown in Figure 8. From Lemma 3.2 and (4.53), we have

\[
(5.4) \quad V^{(X)}(z) = I + \mathcal{O}(n^{-1}), \quad z \in \Sigma^{(X)} \setminus \partial \mathbb{D}_{i\mu}.
\]

The jump on \( \partial \mathbb{D}_{i\mu} \) will be analyzed in (5.1). The error function has the expansion

\[
(5.5) \quad X_n^{(k)}(z) = I + X_n^{(k)}1 + \mathcal{O} \left( \frac{1}{z^2} \right), \quad z \to \infty,
\]

where \( X_n^{(k)}1 \) is independent of \( z \).

5.2. Parametrix for the error. The jump matrix \( V^{(X)}(z) \) is well controlled (i.e. close to the identity asymptotically as \( n \to \infty \)) for \( z \in \Sigma^{(X)} \setminus \partial \mathbb{D}_{i\mu} \). However, we will now see that, for certain \( \mu \), the jump on \( \partial \mathbb{D}_{i\mu} \) is not asymptotically small. This will necessitate approximating \( X_n^{(k)}(z) \) by its own parametrix \( Y_n^{(k)}(z) \). We begin by calculating the jump on this circle. From (5.3) and (5.2), and using the fact that \( S_n^{\text{crit}}(z) \) has no jump on \( \partial \mathbb{D}_{i\mu} \),

\[
(5.6) \quad V^{(X)}(z) \big|_{\partial \mathbb{D}_{i\mu}} = M_k^{(i\mu)}(\zeta(z))M_k^{(\text{out})}(z)^{-1}.
\]
Now, using (4.15) and (4.22), and then (4.26),
\[
\begin{align*}
V^{(X)}(z)\bigg|_{\partial D_{i\mu}} = E_k(z)\sigma_3 e^{i\theta_3 n^{-k\sigma_3/3}} Z_k(\zeta(z)) \zeta(z)^{-k\sigma_3 n^{k\sigma_3/3}} e^{-i\theta_3 \sigma_3 E_k(z)} - 1
\end{align*}
\]
(5.7)
\[
E_k(z)\sigma_3 e^{i\theta_3 n^{-k\sigma_3/3}} \left( I + \frac{A_k(s)}{\zeta(z)} + O\left( \frac{1}{\zeta(z)^2} \right) \right) n^{k\sigma_3/3} e^{-i\theta_3 \sigma_3 E_k(z)} - 1.
\]
Using the fact that \( E_k(z) \) is bounded as \( n \to \infty \) and the fact that \( \zeta = O(n^{1/3}) \) (see (4.11)) shows
\[
V^{(X)}(z)\bigg|_{\partial D_{i\mu}} = I + \begin{bmatrix} 0 & O(e^{2n\mu n^{-2k+1/3}}) \\ O(e^{-2n\mu n^{-2k+1/3}}) & 0 \end{bmatrix} + O\left( \frac{1}{n^{1/3}} \right).
\]
(5.8)
To make these jumps bounded as \( n \to \infty \) we require \( k \) to be chosen so
\[
\frac{1}{3\pi} \left( k - 1 \right) \log \frac{n}{n} < \mu < \frac{1}{3\pi} \left( k + 1 \right) \log \frac{n}{n}.
\]
(5.9)
Note that if \( \mu = \frac{k\log n}{3\pi n} \), i.e., it is in the center of the interval (5.9), then the jump (5.8) is \( O(n^{-1/3}) \).
The error increases as \( \mu \) moves away from the center, and at the endpoints \( \mu = \frac{1}{3\pi} \left( k \pm 1 \right) \log \frac{n}{n} \), the error function \( X_{\mu}^{(k)} \) no longer satisfies a small-norm Riemann–Hilbert problem. To circumvent this problem and to give a uniform error for \( \mu \) throughout the interval (5.9) we build a \textit{parametrix for the error}.

Define
\[
Q_+(z) \equiv Q_+(z; s, \mu, \tau, n) = \frac{U_k(s)}{2\zeta(z)} e^{2i\theta} \frac{n^{2k/3}}{n^{2k/3}}, \quad Q_-(z) \equiv Q_-(z; s, \mu, \tau, n) = \frac{V_k(s)}{2\zeta(z)} e^{2i\theta}
\]
for \( z \in \partial D_{i\mu} \). We record that, at worst,
\[
Q_+(z) = \begin{cases} O(n^{-1/3}), & \frac{1}{3\pi} \left( k - 1 \right) \log \frac{n}{n} \leq \mu \leq \frac{1}{3\pi} \left( k + 1 \right) \log \frac{n}{n}, \\ O(1), & \frac{1}{3\pi} \left( k - 1 \right) \log \frac{n}{n} < \mu \leq \frac{1}{3\pi} \left( k + 1 \right) \log \frac{n}{n}, \end{cases}
\]
(5.11)
\[
Q_-(z) = \begin{cases} O(n^{-1/3}), & \frac{1}{3\pi} \left( k - 1 \right) \log \frac{n}{n} \leq \mu \leq \frac{1}{3\pi} \left( k + 1 \right) \log \frac{n}{n}, \\ O(1), & \frac{1}{3\pi} \left( k - 1 \right) \log \frac{n}{n} < \mu \leq \frac{1}{3\pi} \left( k + 1 \right) \log \frac{n}{n}. \end{cases}
\]
We can now rewrite (5.7) as
\[
V^{(X)}(z)\bigg|_{\partial D_{i\mu}} = I - E_k(z) \begin{bmatrix} 0 & Q_+(z) \\ Q_-(z) & 0 \end{bmatrix} E_k(z)^{-1} + O\left( \frac{1}{n^{1/3}} \right).
\]
(5.12)
Here we have used that \( k \) is given so (5.9) is satisfied in order to bound the terms proportional to \( \zeta^{-2} \). If we discard all \( O(n^{-1/3}) \) terms we are led to the following model problem.

Riemann–Hilbert Problem 5.1 (The parametrix for the error). \textit{Determine the} \( 2 \times 2 \) \textit{matrix-valued function} \( Y_n^{(k)}(z) \) \textit{satisfying:}

\textbf{Analyticity:} \( Y_n^{(k)}(z) \) is analytic for \( z \in \mathbb{C}\setminus\partial D_{i\mu} \) with Hölder-continuous boundary values on \( \partial D_{i\mu} \).

\textbf{Normalization:}
\[
Y_n^{(k)}(z) = I + O\left( \frac{1}{n} \right) \text{ as } z \to \infty.
\]
(5.13)
**Jump condition:** Orienting the jump contour negatively, the solution satisfies

\[(5.14) \quad Y^{(k)}_{n^+}(z) = Y^{(k)}_{n^-}(z) V(Y)(z) \equiv Y^{(k)}_{n^-}(z) \left( I - E_k(z)Q(z)E_k(z)^{-1} \right), \quad z \in \partial \mathbb{D}_{i\mu}, \]

where

\[(5.15) \quad Q(z) := \begin{cases} 
0 & 0 \\
Q_-(z) & 0 \\
0 & Q_+(z) \\
0 & 0 
\end{cases}, \quad \left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}.
\]

Before we solve this Riemann–Hilbert problem explicitly, we define the ratio

\[(5.16) \quad Z^{(k)}_{n^-}(z) := X^{(k)}_{n^-}(z) Y^{(k)}_{n^-}(z)^{-1}
\]

and note that the jump \(V(Z)(z)\) for this function is uniformly close to the identity for large \(n\) on its jump contour \(\Sigma(Z) = \Sigma(X)\). Indeed, the jumps for \(z \notin \partial \mathbb{D}_{i\mu}\) are controlled since the jumps for \(X^{(k)}_{n^-}(z)\) are (and \(Y^{(k)}_{n^-}(z)\) is continuous there), while at the same time (5.12) and (5.14) give

\[(5.17) \quad V(Z)(z) \bigg|_{\partial \mathbb{D}_{i\mu}} = Z^{(k)}_{n^-}(z)^{-1} Z^{(k)}_{n^+}(z)
\]

\[= Y^{(k)}_{n^-}(z) X^{(k)}_{n^-}(z)^{-1} X^{(k)}_{n^+}(z) Y^{(k)}_{n^+}(z)^{-1}
\]

\[= Y^{(k)}_{n^-}(z) V(X)(z) V(Y)(z)^{-1} Y^{(k)}_{n^-}(z)^{-1}
\]

\[= I - Y^{(k)}_{n^-}(z) E_k(z) \hat{Q}(z) E_k(z)^{-1} Y^{(k)}_{n^-}(z)^{-1} + O\left(\frac{1}{n}\right),
\]

wherein

\[(5.18) \quad \hat{Q}(z) := \begin{cases} 
0 & Q_+(z) \\
0 & 0 \\
0 & 0 \\
Q_-(z) & 0 
\end{cases}, \quad \left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}.
\]

To invert \(V(Y)(z)\) we used the fact that \(Q(z)\) is nilpotent. Equation (5.11) shows us that \(\hat{Q}(z) = O(n^{-1/3})\) (assuming the correct choice of \(k\), and so

\[(5.19) \quad V(Z)(z) = O(n^{-1/3}), \quad z \in \partial \mathbb{D}_{i\mu}.
\]

Away from \(\partial \mathbb{D}_{i\mu}\), (5.4) gives the stronger bound

\[(5.20) \quad V(Z)(z) = O(n^{-1}), \quad z \in \Sigma(Z) \setminus \partial \mathbb{D}_{i\mu},
\]

with the largest contribution coming from \(\partial \mathbb{D}_a\) and \(\partial \mathbb{D}_b\).

**5.2.1. Solution of Riemann–Hilbert Problem** \(\text{5.1 for the error parametrix} Y^{(k)}_{n}(z)\). Guided by the calculations in [L7, §3.5.2], we now solve Riemann–Hilbert Problem 5.1 exactly. The meromorphic continuation of \(Y^{(k)}_{n}(z)\) from the exterior to the interior of \(\mathbb{D}_{i\mu}\) is

\[(5.21) \quad \tilde{Y}^{(k)}_{n}(z) := \begin{cases} 
Y^{(k)}_{n}(z), \quad z \in \mathbb{C} \setminus \mathbb{D}_{i\mu}, \\
Y^{(k)}_{n}(z) \left( I - E_k(z)Q(z)E_k(z)^{-1} \right), \quad z \in \mathbb{D}_{i\mu}.
\end{cases}
\]

Equivalently, we can use the nilpotency of \(Q\) to write

\[(5.22) \quad Y^{(k)}_{n}(z) := \begin{cases} 
\tilde{Y}^{(k)}_{n}(z), \quad z \in \mathbb{C} \setminus \mathbb{D}_{i\mu}, \\
\tilde{Y}^{(k)}_{n}(z) \left( I + E_k(z)Q(z)E_k(z)^{-1} \right), \quad z \in \mathbb{D}_{i\mu}.
\end{cases}
\]
This function tends to the identity matrix as $z \to \infty$ (since $Y_n^{(k)}(z)$ does), and is analytic for all $z$ except for a simple pole at $z = i\mu$. Therefore, $\tilde{Y}_n^{(k)}(z)$ necessarily has the form

$$\tilde{Y}_n^{(k)}(z) = I + \frac{1}{z - i\mu}B,$$

where $B \equiv B(s, \mu, \tau, n)$ is independent of $z$. It only remains to determine $B$, which can be done using the fact that $Y_n^{(k)}(z)$ is analytic at $z = i\mu$. This analyticity implies

$$Y_n^{(k)}(z) = \left( I + \frac{1}{z - i\mu}B \right) \left( I + E_k(z)Q(z)E_k(z)^{-1} \right) = O(1)$$

for $z \to i\mu$ via (5.21) and (5.23). We expand the constituent functions around $z = i\mu$, keeping in mind that $Q(z)$ has a simple pole at $z = i\mu$:

$$E_k(z) = F + (z - i\mu)G + O((z - i\mu)^2), \quad Q(z) = \frac{1}{z - i\mu}R + S + O((z - i\mu)^2).$$

Here the matrices $F$, $G$, $R$, and $S$ are independent of $z$. Plugging (5.25) into (5.24) and isolating the Laurent terms gives

$$O\left( \frac{1}{(z - i\mu)^2} \right) : \text{BFR}^{-1} = 0,$$

$$O\left( \frac{1}{z - i\mu} \right) : \text{B} + \text{FR}^{-1} - \text{BFR}^{-1}\text{GF}^{-1} + \text{BFS}^{-1} + \text{BGR}^{-1} = 0$$

The invertibility of $F$ follows from the invertibility of $F_k^{(\text{out})}(z)$ at $z = i\mu$. The first equation is equivalent to $\text{BFR} = 0$. This also implies that $\text{BFS} = 0$ since $S$ is a (nonzero) constant multiple of $R$. Using these facts in the second equation gives the simplified system

$$\text{BFR} = 0, \quad \text{BF + FR + BGR} = 0.$$

Looking at the first equation, we recall that $R$ is either strictly upper-triangular (in which case the first column of $BF$ is zero) or strictly lower-triangular (in which case the second column of $BF$ is zero). This can be used along with the second equation and the fact that $R$ is strictly triangular to solve for $BF$. After multiplying on the right by $F^{-1}$, we find

$$B = \begin{cases} -\text{FRF}^{-1} \\ 1 + [F^{-1}G]_{12}[R]_{21}, \quad \left( k - \frac{1}{2} \right) \log n \leq k \frac{\log n}{3\pi n}, \\
-\text{FRF}^{-1} \\ 1 + [F^{-1}G]_{21}[R]_{12}, \quad k \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}. \end{cases}$$

Only the $(11)$-entry will be necessary to compute the winding number probabilities:

$$|B|_{11} = \begin{cases} -[F]_{12}[F]_{21}[R]_{21}, \quad \left( k - \frac{1}{2} \right) \log n \leq k \frac{\log n}{3\pi n}, \\
[F]_{11}[F]_{21}[R]_{21}, \quad k \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}. \end{cases}$$

We note immediately from (5.25), (5.15), (5.10), and (4.13) that

$$R = \begin{cases} \frac{2^{3/2}V_k(s)}{\pi} \cdot \frac{n^{(2k-1)/3}}{e^{2i\theta}} + O\left( \frac{1}{n} \right), \quad \left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq k \frac{\log n}{3\pi n}, \\
0 \quad \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}. \end{cases}$$
Later we will also need the expansion of \( \hat{Q} \), so we note

\[
(5.31) \quad \hat{Q}(z) = \frac{1}{z-i\mu} \hat{R} + O(1),
\]

where

\[
(5.32) \quad \hat{R} := \begin{cases}
0 & \frac{2^{2/3} U_k(s)}{\pi} \cdot \frac{e^{2i\theta}}{n^{(2k+1)/3}} + O\left(\frac{1}{n}\right), \quad \left(k-\frac{1}{2}\right) \frac{\log n}{3\pi n} < \mu \leq \left(k+\frac{1}{2}\right) \frac{\log n}{3\pi n}, \\
0 & \frac{2^{2/3} V_k(s)}{\pi} \cdot \frac{e^{2i\theta}}{n^{(2k-1)/3}} + O\left(\frac{1}{n}\right), \quad k \frac{\log n}{3\pi n} < \mu \leq \left(k+\frac{1}{2}\right) \frac{\log n}{3\pi n}.
\end{cases}
\]

Next we compute \([F]_{12}[F]_{22}\) and \([F]_{11}[F]_{21}\). From (4.15) and (4.7),

\[
(5.33) \quad E_k(z) = \begin{bmatrix}
\gamma(z) + \gamma(z)^{-1} & \gamma(z) - \gamma(z)^{-1} \\
2 & 2i e^{ik\pi} \\
2i e^{ik\pi} & \gamma(z) + \gamma(z)^{-1} \\
0 & 0
\end{bmatrix} \begin{bmatrix}
0 & -\left(\frac{\bar{d}(z)\zeta(z)}{\pi n/3}\right)^k \\
\left(\frac{\pi n}{3}\right)^k & 0
\end{bmatrix},
\]

where \(\gamma(z)\) and \(\bar{d}(z)\) are defined for \(z \in \mathbb{D}_\mu\) to be the analytic continuations of \(\gamma(z)\) and \(d(z)\) from the upper half-plane. From Lemma 4.2 (c),

\[
(5.34) \quad \bar{d}(z)\zeta(z) = \frac{4i\zeta'(i\mu)}{\sqrt{T}} + O(z - i\mu).
\]

Then (4.13) and (1.12) show

\[
(5.35) \quad \frac{4\zeta'(i\mu)}{\sqrt{T}} = (2n)^{1/3} + O\left(\frac{1}{n^{1/3}}\right).
\]

Combining the previous three equations along with \(F \equiv E_k(i\mu)\) and \(\gamma(i\mu)^2 = -i\) gives

\[
(5.36) \quad [F]_{12}[F]_{22} = \frac{\bar{d}(z)\zeta(z)}{\pi n/3} = -2^{(2k-3)/3} + O\left(\frac{1}{n^{2/3}}\right),
\]

\[
(5.37) \quad [F]_{11}[F]_{21} = \left\{ \begin{array}{ll}
-2^{2k/3} + O\left(\frac{1}{n^{2/3}}\right) & \text{if } (k - \frac{1}{2}) \frac{\log n}{3\pi n} < \mu \leq (k + \frac{1}{2}) \frac{\log n}{3\pi n}, \\
\end{array} \right.
\]

Recall the definitions of \(R_y\) and \(R_u\) in (2.6). Then, from (5.30) and (5.36),

\[
(5.37) \quad [F]_{12}[F]_{22}[R]_{21} = -R_y e^{-2i\pi x} + O\left(\frac{1}{n^{2/3}}\right), \quad \left(k - \frac{1}{2}\right) \frac{\log n}{3\pi n} < \mu \leq \left(k+\frac{1}{2}\right) \frac{\log n}{3\pi n},
\]

\[
(5.37) \quad [F]_{11}[F]_{21}[R]_{12} = R_u e^{2i\pi x} + O\left(\frac{1}{n^{2/3}}\right), \quad k \frac{\log n}{3\pi n} < \mu \leq \left(k+\frac{1}{2}\right) \frac{\log n}{3\pi n}.
\]
To find $[F^{-1}G]_{12}$ and $[F^{-1}G]_{21}$, we differentiate (5.33) and use (5.34)–(5.35) to write

$$G \equiv \frac{dE_k(z)}{dz} \bigg|_{z=i\mu}$$

(5.38)

$$\frac{\gamma'(i\mu)}{\gamma(i\mu)} = \frac{\sqrt{T}}{4} = \frac{\pi}{4} + O \left( \frac{1}{n^{2/3}} \right).$$

(5.39)

Using both (5.35) and (5.39) then gives

$$[F^{-1}G]_{12} = i(-1)^k \left( \frac{4i\zeta'(i\mu)}{n^{1/3}\sqrt{T}} \right)^{2k} \frac{\gamma'(i\mu)}{\gamma(i\mu)} = -i2^{(2k-6)/3} \pi + O \left( \frac{1}{n^{2/3}} \right),$$

(5.40)

$$[F^{-1}G]_{21} = -i(-1)^k \left( \frac{4i\zeta'(i\mu)}{n^{1/3}\sqrt{T}} \right)^{-2k} \frac{\gamma'(i\mu)}{\gamma(i\mu)} = -i2^{(-2k-6)/3} \pi + O \left( \frac{1}{n^{2/3}} \right).$$

Therefore

$$[F^{-1}G]_{12}[R]_{21} = -\frac{\pi}{2i} R_y e^{-2\pi i r} + O \left( \frac{1}{n^{2/3}} \right), \quad \left( k - \frac{1}{2} \right) \log n \leq \mu \leq k \left( \log n \frac{3\pi}{\pi n} \right),$$

(5.41)

$$[F^{-1}G]_{21}[R]_{12} = \frac{\pi}{2i} R_{\ell} e^{2\pi i r} + O \left( \frac{1}{n^{2/3}} \right), \quad k \left( \log n \frac{3\pi}{\pi n} \right) \leq \mu \leq \left( k + \frac{1}{2} \right) \log n \frac{3\pi}{\pi n}.$$ 

We now use (5.29), (5.37), and (5.41) to find

$$[B]_{11} = \begin{cases} R_y e^{-2\pi i r} + O \left( \frac{1}{n^{2/3}} \right), & \left( k - \frac{1}{2} \right) \log n \leq \mu \leq k \left( \log n \frac{3\pi}{\pi n} \right), \\ R_{\ell} e^{2\pi i r} + O \left( \frac{1}{n^{2/3}} \right), & k \left( \log n \frac{3\pi}{\pi n} \right) \leq \mu \leq \left( k + \frac{1}{2} \right) \log n \frac{3\pi}{\pi n}. \end{cases}$$

5.3. **Proofs of Lemmas 2.1, 2.2 and 2.3** We are now ready to prove the three results on discrete orthogonal polynomials needed to establish Theorems 1.2 and 1.4.

**Proof of Lemma 2.1** From (2.5) and (3.6), we merely need $[P_{n,1}]_{11}$ in order to compute the winding probabilities. Reversing the changes of variables used in the nonlinear steepest-descent analysis,

$$P_n(z) = \begin{bmatrix} 1 & 0 \\ 0 & -2\pi i \end{bmatrix}^{-1} e^{n\sigma_3/2} Z_n(z) Y_n(z) M^{(k)}(z) e^{n(g(z) - \ell/2)\sigma_3} \begin{bmatrix} 1 & 0 \\ 0 & -2\pi i \end{bmatrix}$$

(5.43)
for $|z|$ sufficiently large. The matrices $M^{(k)}(z)$, $Y^{(k)}_n(z)$, and $Z^{(k)}_n(z)$ can be expanded as
\begin{equation}
M^{(k)}(z) = I + \frac{M_1^{(k)}}{z} + O\left(\frac{1}{z^2}\right), \quad Y^{(k)}_n(z) = I + \frac{B}{z} + O\left(\frac{1}{z^2}\right), \quad Z^{(k)}_n(z) = I + \frac{Z^{(k)}_{n,1}}{z} + O\left(\frac{1}{z^2}\right),
\end{equation}
where $M_1^{(k)}$, $B$, and $Z^{(k)}_{n,1}$ are independent of $z$ and the expansion for $Y^{(k)}_n(z)$ follows from (5.22) and (5.23). Thus, using (3.12) to expand $g(z)$,
\begin{equation}
\int + P_{n,1} e\left(\frac{1}{z^2}\right)\begin{bmatrix} 1 & 0 \\ 0 & -2\pi i \end{bmatrix}^{-1} e^{n\ell\sigma_3/2} \left(I + \frac{M_1^{(k)} + B + Z^{(k)}_{n,1}}{z} + O\left(\frac{1}{z^2}\right)\right)
\end{equation}
\begin{equation}
\times \exp \left(-n \left(\frac{i\mu}{z} + O\left(\frac{1}{z^2}\right)\right)\right) e^{-n\ell\sigma_3/2} \begin{bmatrix} 1 & 0 \\ 0 & -2\pi i \end{bmatrix}.
\end{equation}
Thus
\begin{equation}
[P_{n,1}]_{11} = [M_1^{(k)}]_{11} + [B]_{11} + [Z^{(k)}_{n,1}]_{11} - i\mu.
\end{equation}
For $[B]_{11}$ refer to (5.42).

We continue by calculating $[M_1^{(k)}]_{11}$. For $|z|$ large, we have that $M^{(k)}(z)$ is given by $M^{(out)}_k(z)$ (see (5.1)). By using first (4.7) and then (4.6) and Lemma 4.2 (g),
\begin{equation}
[M^{(out)}_k(z)]_{11} = \frac{\gamma(z) + \gamma(z)^{-1}}{2} d(z)^k = \left(1 + O\left(\frac{1}{z^2}\right)\right) \left(1 + \frac{2ik}{\sqrt{T}z} + O\left(\frac{1}{z^2}\right)\right).
\end{equation}
Therefore, using (1.12) in the second equation,
\begin{equation}
[M^{(k)}_1]_{11} = \frac{2ik}{\sqrt{T}} = \frac{2ik}{\pi} + O\left(\frac{1}{n^{2/3}}\right).
\end{equation}

We now calculate $[Z^{(k)}_{n,1}]_{11}$. From (5.19) and (5.20), the small-norm theory of Riemann–Hilbert problems (see, for example, [23] and [42, §5]) shows that $Z^{(k)}_n(z)$ can be computed via a convergent Neumann series, with
\begin{equation}
Z^{(k)}_{n,1} = -\frac{1}{2\pi i} \int_{\Sigma(Z)} \left(V(Z)(u) - I\right) du + O\left(\frac{1}{n^{2/3}}\right) = -\frac{1}{2\pi i} \int_{\partial\mathbb{D}_{i\mu}} \left(V(Z)(u) - I\right) du + O\left(\frac{1}{n^{2/3}}\right).
\end{equation}
The last integral can be computed by Cauchy’s integral formula using (5.17), (5.25), and (5.31) (recall $\partial\mathbb{D}_{i\mu}$ is oriented in the negative direction):
\begin{equation}
Z^{(k)}_{n,1} = -Y^{(k)}_n(i\mu) \tilde{F} \tilde{R} F^{-1} Y^{(k)}_n(i\mu)^{-1} + O\left(\frac{1}{n^{2/3}}\right).
\end{equation}
We now show $Y^{(k)}_n(i\mu)$ and its inverse can be neglected without increasing the error. From the formula (5.24) for $Y^{(k)}_n(z)$ and the formulas (5.15) and (5.28) for $Q(z)$ and $B$ (along with the boundedness of $B_k(z)$ in $n$),
\begin{equation}
Y^{(k)}_n(i\mu) = \begin{cases}
I + O\left(\frac{n^{(2k-1)/3}}{e^{2n\pi\mu}}\right), & k - \frac{1}{2} \leq \mu \leq k - \frac{1}{2} \frac{\log n}{3\pi n}, \\
I + O\left(\frac{n^{(2k+1)/3}}{e^{2n\pi\mu}}\right), & k - \frac{1}{2} \frac{\log n}{3\pi n} < \mu \leq \left(k + \frac{1}{2}\right) \frac{\log n}{3\pi n}.
\end{cases}
\end{equation}
The same bounds hold for $Y_n^{(k)}(i\mu)^{-1}$. On the other hand,

\[
\hat{R} = \begin{cases} 
O \left( \frac{e^{2n\pi\mu}}{n^{(2k+1)/3}} \right), & \left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq k \frac{\log n}{3\pi n}, \\
O \left( \frac{n^{(2k-1)/3}}{e^{2n\pi\mu}} \right), & k \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}.
\end{cases}
\]

Plugging these relations into (5.50) and expanding shows all but the identity terms from $Y_n^{(k)}(i\mu)$ and $Y_n^{(k)}(i\mu)^{-1}$ can be absorbed into the error term, leaving

\[
Z_{n,1}^{(k)} = -F\hat{R}F^{-1} + O \left( \frac{1}{n^{2/3}} \right).
\]

We note that $\det F = \det E_k(i\mu) = 1$ since $\det E_k(z) \equiv 1$. Thus

\[
\begin{align*}
Z_{n,1}^{(k)} &= \begin{bmatrix} [F]_{11} [F]_{21} \hat{R} & + O \left( \frac{1}{n^{2/3}} \right), & \left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq k \frac{\log n}{3\pi n}, \\
-[F]_{12} [F]_{22} \hat{R} & + O \left( \frac{1}{n^{2/3}} \right), & k \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}.
\end{align*}
\]

From (5.32), (5.36), and (2.6),

\[
\begin{align*}
Z_{n,1}^{(k)} &= \begin{bmatrix} R_{\ell} e^{2i\pi \tau} + O \left( \frac{1}{n^{2/3}} \right), & \left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq k \frac{\log n}{3\pi n}, \\
R_{\ell} e^{-2i\pi \tau} + O \left( \frac{1}{n^{2/3}} \right), & k \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n}.
\end{bmatrix}
\end{align*}
\]

Comparing this to (5.42), we see that it is advantageous to instead write

\[
\begin{align*}
Z_{n,1}^{(k)} &= \begin{bmatrix} \frac{R_{\ell} e^{2i\pi \tau}}{1 + \frac{k}{\pi} R_{\ell} e^{2i\pi \tau}} + O \left( \frac{1}{n^{2/3}} \right), & \left( k - \frac{1}{2} \right) \frac{\log n}{3\pi n} < \mu \leq k \frac{\log n}{3\pi n}, \\
\frac{R_{\ell} e^{-2i\pi \tau}}{1 - \frac{k}{\pi} R_{\ell} e^{-2i\pi \tau}} + O \left( \frac{1}{n^{2/3}} \right), & k \frac{\log n}{3\pi n} < \mu \leq \left( k + \frac{1}{2} \right) \frac{\log n}{3\pi n},
\end{bmatrix}
\end{align*}
\]

which is legal by (5.52). Now, combining (5.46), (5.42), (5.48), and (5.56), $[P_{n,1}]_{11}$ can be written in the unified form

\[
P_{n,1} = \frac{2i}{\pi} + \frac{R_{\ell} e^{-2i\pi \tau}}{1 - \frac{k}{\pi} R_{\ell} e^{-2i\pi \tau}} + \frac{R_{\ell} e^{2i\pi \tau}}{1 + \frac{k}{\pi} R_{\ell} e^{2i\pi \tau}} - i\mu + O \left( \frac{1}{n^{2/3}} \right).
\]

Since $[P_{n,1}]_{11} = c_{n,n,n-1}^{(T,\mu,\tau)}$, this completes the proof of Lemma 2.1.

**Proof of Lemma 2.2.** In the domain $\{z \mid |\Im z| > \max(\delta, \mu + \epsilon)\}$ the transformations of the Riemann–Hilbert problem give that

\[
P_n(z) = \begin{bmatrix} 1 & 0 \\
0 & -2\pi i \end{bmatrix}^{-1} e^{\pi \sigma_3/2} X_n^{(k)}(z) M_k^{(\text{out})}(z)e^{\pi(g(z) - \ell/2)\sigma_3} \begin{bmatrix} 1 & 0 \\
0 & -2\pi i \end{bmatrix}.
\]

With the choice $\mu = \frac{k\log n}{3\pi n}$, the matrix $X_n^{(k)}(z)$ is uniformly close to the identity matrix:

\[
X_n^{(k)}(z) = I + O(n^{-1/3}).
\]

Thus, expanding (5.58) and recalling that the entries of $P_n(z)$ are given in (3.5), we immediately obtain the results of Lemma 2.2. \qed
Proof of Lemma 2.3. We first prove the rough estimate (2.15). From (3.17), (5.1), and (5.2) we have

\[ P_n(z) = e^{\frac{nf}{2} \sigma_3} A^{-1} X_n^{(k)}(z) M_k^{(i\mu)}(z) H(z) \left[ \begin{array}{cc} 1 & 0 \\ \pm e^{\mp nG(z)} & 1 \end{array} \right] A e^{n(g(z)-\ell/2)\sigma_3} D_{\pm}^{n}(z)^{-1} \quad \text{for } \pm(3z-\mu) > 0, \]

where the matrix \( H(z) \) is defined piecewise as

\[ H(z) := \begin{cases} 
1 & \mp e^{\pm nG(z)} e^{\pm 2\pi i(nz-\tau)} \\
0 & 1 \\
1, & \text{otherwise.}
\end{cases} \quad z \in \Omega_{\pm}^\Delta, \]

Note that both \( M_k^{(i\mu)}(z) \) and \( X_n^{(k)}(z) \) are uniformly bounded in \( n \). Furthermore, from the formula (3.16), we find that for \( x \) close to zero and small \( y > 0 \),

\[ \Re G(x + i\mu \pm iy) = \pm y T \sqrt{\frac{4}{T} - x^2} + O(y^2). \]

This implies that \( \epsilon \) may be chosen small enough so that \( e^{\mp nG(z)} = O(1) \) for all \( z \) such that \( |z-\mu| < \epsilon \) and \( \pm(3z-\mu) > 0 \). Thus, we have

\[ \left[ \begin{array}{cc} 1 & 0 \\ \pm e^{\mp nG(z)} & 1 \end{array} \right] = O(1). \]

Also, using (5.62) in the scalings \( T = \pi^2 (1 + O(n^{-2/3})) \) and \( \mu = \frac{k \log n}{3\pi n} \), we find that

\[ e^{\pm nG(z)} e^{\pm 2\pi i(nz-\tau)} = O(n^{2k} e^{r n^{1/3}}), \]

where \( r := \epsilon \sup_{n} |\sqrt{T} - \pi| n^{2/3} \) is an explicit constant. This estimate is very rough and comes from the fact that the density \( T \pi \sqrt{\frac{4}{T} - x^2} \) may be slightly bigger than one in the \( k \)-tacnode scaling, so the sum \( G(z) + 2\pi i z \) could be positive. If \( T < \pi^2 \) then \( e^{\pm nG(z)} e^{\pm 2\pi i(nz-\tau)} \) is in fact small throughout a neighborhood of \( i\mu \), but the rough estimate (5.64) is good enough for our purposes. Thus, we have

\[ H(z) = O(n^{2k} e^{r n^{1/3}}). \]

Plugging (5.63) and (5.65) into (5.60) we find

\[ P_n(z) = e^{\frac{nf}{2} \sigma_3} O(n^{2k} e^{r n^{1/3}}) e^{n(g(z)-\ell/2)\sigma_3} D_{\pm}^{n}(z)^{-1} \quad \text{for } \pm(3z-\mu) > 0. \]

Since the quantities estimated in (2.15) are given by the entries of the first column of \( P_n(z) \), expanding (5.66) immediately gives (2.15).

Now let us give a more precise formula for \( P_n(z) \) in a neighborhood of \( z = i\mu \). Using (4.22) we can write (5.60) as

\[ P_n(z) = e^{\frac{nf}{2} \sigma_3} A^{-1} X_n^{(k)}(z) E_k(z) e^{i\hat{\sigma}_3 n - k \sigma_3/3} Z_k(\zeta(z)) e^{-i\hat{\sigma}_3 \sigma_2 \zeta_1} H(z) \times \left[ \begin{array}{cc} 1 & 0 \\ \pm e^{\mp nG(z)} & 1 \end{array} \right] \quad \text{for } \pm(3z-\mu) > 0. \]

From (4.24) and (4.9) we find that

\[ Z_k(\zeta(z)) e^{-i\hat{\sigma}_3} = L_k(\zeta(z)) e^{\frac{nG(z)}{2} \sigma_3} e^{i\pi(nz-\tau)\sigma_3}. \]
If we denote by \( \tilde{L}_k(\zeta(z)) \) the analytic continuation of \( L_k(\zeta(z)) \) from the sectors that include the real line, then the jump conditions for \( L_k(\zeta(z)) \) imply that

\[
\begin{align*}
\mathbf{L}_k(\zeta(z)) = & \begin{cases} 
\begin{bmatrix}
1 & 0 \\
1 & 1 
\end{bmatrix}, & z \in \Omega^{\Delta}, \\
\begin{bmatrix}
1 & 1 \\
0 & 1 
\end{bmatrix}, & \text{otherwise}, 
\end{cases}
\end{align*}
\]

(5.69)

Combining the three preceding equations, we can write the more uniform expression

\[
P_n(z) = e^{\frac{n}{2}g_3} A^{-1} X_n^{(k)}(z) E_k(z) e^{i\theta \sigma_3 n - k \sigma_3/3} \tilde{L}_k(\zeta(z)) e^{i\pi(\nu - \tau)\sigma_3} \begin{bmatrix} 1 & \frac{1+i}{2} \\ 1 & \frac{1-i}{2} \end{bmatrix} \times e^{\frac{n}{2}G_3} \nu A e^{n(g(z) - \ell/2)\sigma_3} \mathbf{D}_{\pm}(z)^{-1} \quad \text{for } \pm (3z - \mu) > 0.
\]

Note that for \( \mu = \frac{k \log n}{3\pi n} \), the error \( X_n^{(k)}(z) \) is uniformly \( \mathbb{I} + \mathcal{O}(n^{-1/3}) \). Also using (4.10), we see that

\[
e^{i\theta \sigma_3 n - k \sigma_3/3} = e^{-\frac{i\pi}{2} \sigma_3} e^{i\pi \sigma_3}.
\]

From [3.5] and the fact that \( \det P_n(\mu) \equiv 1 \), we have the formula

\[
e^{-\frac{n}{2}(z^2 - 2i\nu z + w^2 - 2i\nu w) \mathcal{P}^{(T,\mu,\tau)}_{n,n} \mathcal{P}^{(T,\mu,\tau)}_{n,n-1}(w) - \mathcal{P}^{(T,\mu,\tau)}_{n,n-1}(z) \mathcal{P}^{(T,\mu,\tau)}_{n,n}(w)} = \frac{e^{-n(V(z) + V(w))/2}}{z - w} \begin{bmatrix} 1 & 1 \end{bmatrix} P_n(z)^{-1} P_n(w) \begin{bmatrix} 0 & 1 \end{bmatrix},
\]

where \( V(z) \) is defined in (3.9). Plugging (5.70) into this formula, we find that

\[
\frac{e^{-n(V(z) + V(w))/2}}{z - w} \begin{bmatrix} 0 & 1 \end{bmatrix} P_n(z)^{-1} P_n(w) \begin{bmatrix} 1 & 0 \end{bmatrix} = \frac{1}{2\pi i(z - w)} \begin{bmatrix} -e^{-i\pi(\nu - \tau)} & e^{i\pi(\nu - \tau)} \end{bmatrix} \times \tilde{L}_k(\zeta(z))^{-1} e^{-i\pi \sigma_3} e^{\frac{i\pi}{2} \sigma_3} E_k(z)^{-1} X_n^{(k)}(z)^{-1} X_n^{(k)}(w) E_k(w) e^{-\frac{i\pi}{2} \sigma_3} e^{i\pi \sigma_3} \tilde{L}_k(\zeta(w)) \begin{bmatrix} e^{i\pi(\nu - \tau)} \\ e^{i\pi(\nu - \tau)} \end{bmatrix},
\]

which is obtained by direct calculation and the relation (3.14) between \( g(z) \), \( G(z) \), and \( V(z) \). Recall that \( X_n^{(k)}(z) = \mathbb{I} + \mathcal{O}(n^{-1/3}) \). Since \( E_n(z) \) is analytic at \( z = i\mu \), we find that \( E_k(z)^{-1} E_k(w) = \mathbb{I} + \mathcal{O}(n^{-\delta}) \) for \( |z - i\mu| < \epsilon n^{-\delta} \) and \( |w - i\mu| < \epsilon n^{-\delta} \). Finally, for \( |z - i\mu| < \epsilon n^{-\delta} \),

\[
\zeta(z) = dn^{1/3} z(1 + \mathcal{O}(n^{1/3 - 2\delta})),
\]

which implies

\[
\tilde{L}_k(\zeta(z)) = \tilde{L}_k(dn^{1/3} z) \left( 1 + \mathcal{O}(n^{1/3 - 2\delta}) \right).
\]

The equation (5.73) reduces to (2.14), and the proof of Lemma 2.3 is complete. \( \square \)

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