Epistemic Uncertainty from an Averaged Hamilton–Jacobi Formalism

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Abstract
In recent years, the non-relativistic quantum dynamics derived from three assumptions: (i) probability current conservation, (ii) average energy conservation, and (iii) an epistemic momentum uncertainty (Budiyono and Rohrlich in Nat Commun 8:1306, 2017). Here we show that, these assumptions can be derived from a natural extension of classical statistical mechanics.

Keywords Bohmian mechanics · Exact uncertainty · Quantum potential · Many Interacting Worlds

... we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines [1].

P. A. M. Dirac (1951)

1 Introduction

The difference between classical and quantum mechanics can be intuitively understood via uncertainty relations. In fact, the uncertainty relations lie at the core of quantum mechanics and can quantitatively determine other fundamental quantum features, such as quantum interference and quantum non-locality [2–6]. Remarkably, the non-relativistic Schrödinger equation recently derived from three assumptions: (i) probability current conservation, (ii) average energy conservation, and (iii) an exact form of uncertainty relation [7–9]. This begs a question: Are there profound

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and natural principles determining this exact form [10–12]? Here we show that, all these three assumptions can be derived from a natural extension of classical statistical mechanics; an averaged version of classical Hamilton–Jacobi formalism.

This make a new answer to an old—but still interesting [7, 13–18]—question: Can we reconstruct quantum mechanics as a clear modification of classical statistical mechanics? Moreover, this axiomatic quantization method leads naturally to a reasonable formulation of mixed classical-quantum dynamics; i.e. Hall’s theory [19, 20].

2 Ontology and Notation

The proposed formalism can be applied to some different interpretations of quantum theory. Nonetheless, for clarity, here we only use of “Many Interacting World” (MIW) interpretation [14–17, 21, 22] (or very similar proposal which called “Real Ensemble” interpretation [23, 24]). In this interpretation, the wave-function dose not exist; instead there is a real ensemble of classical “worlds”, and all quantum effects merely arise from a universal interaction between these worlds. This interpretation leads to a reasonable solution for measurement problem [16, 17]. Furthermore, it is supported by PBR theorem, which claims that the quantum state must be corresponds to something real in nature [23, 25].

Let us also fix our notation. Consider an ensemble of classical-like n-particle systems, worlds, which state of any of them is determined by a point in phase-space, \((x, p) = (x_1, ..., x_n, p_1, ..., p_n)\). The locale average of any classical “observable”, \(O(x, p)\), which is represented by \(O(x)\), is defined as

\[
\bar{O}(x) \equiv \frac{1}{\rho(x)} \int f(p, x)O(x, p) d^{3n}p,
\]

where \(f(x, p)\) is phase-space distribution and \(\rho(x) \equiv \int f(x, p) dp\) is configuration-space distribution of worlds-ensemble. Moreover, the local variance, \(\bar{O}^2 - \bar{O}^2\), is represented by \((\delta O)^2\), and similarly, associated global average and variance are defined as \(\langle O \rangle \equiv N^{-1} \int \rho(x)\bar{O}(x)d^{3n}x\) and \((\Delta O)^2 \equiv N^{-2} \int \rho(x)(\delta O)^2d^{3n}x\), respectively. Note that, these global ensemble averages are comparable with expectation values of associated operator, \(\langle \hat{O} \rangle_w\), in the standard Hilbert-space formalism.

3 Assumptions and Results

In fact we only use MIW ontology without considering any special form for interworld interaction priorly. Instead, we assume that the time evolution of worlds ensemble, associated to a pure quantum state, is describable by an average version of classical Hamilton–Jacobi formalism: Explicitly, we assume that, there is a smooth
function, \( S(x, t) \), which is related to local average momentum as \( \bar{p}_i = \partial_i S \), and satisfy flowing axioms.

**Axiom 1**

- **Average Hamilton–Jacobi equation:**
  \[
  \int \rho \left( \frac{\partial S}{\partial t} + \bar{H} \right) d^3n_x = 0. \tag{1}
  \]

**Axiom 2**

- **Least average action:**
  \[
  \delta \int \rho \left( \frac{\partial S}{\partial t} + \bar{H} \right) d^3n_x dt = 0. \tag{2}
  \]

where, \( \bar{H}(x) \), is local average of classical Hamiltonian, \( \bar{H} = \sum_i \frac{p_i^2}{2m} + V(x) \), which explicitly reads \( \bar{H} = \sum_i \frac{1}{2m} \left( \nabla_i S \right)^2 + \left( \bar{p}_i \right)^2 / 2m + V(x) \). It is worth noting that, the time evolution of an classical ensemble, which is describable by classical Hamilton–Jacobi formalism, satisfy these axioms (see Fn 2). The main result of this work is that above axioms determine the time evolution of \( \rho \) and \( S \) up to some free parameters, which is include classical and quantum dynamics as special cases: see following theorem.

**Theorem** The axioms (1, 2), beside the assumption that the local momentum variance is a local function of probability density as \( (\delta p_i)^2 = \mu_i(\rho, \partial \rho) \), leads to

\[
(\delta p_i)^2 = \lambda_i^2 \left( \frac{\nabla_i \rho}{\rho} \right)^2 \tag{3}
\]

and following evolution equations, uniquely:

---

1 Perhaps, the meaning of “average” is more cleared, if we rewrite this axiom as \( \int f(x, p) \left( \partial_S + H(x, p) \right) dxdp = 0 \)

2 This variational principle is a direct generalization of the variational formulation of classical H–J formalism, i.e. \( \delta \int \rho \partial_S + H(x, VS) d^3n_x dt = 0 \), which leads to classical H–J and continuity equations [26]. Similar generalized actions, also used in some other methods for derivation of Schroedinger equation [8, 9, 27–29]. Moreover, one can use of an “average Hamilton’s equations” instead of this variational principle; i.e. \( \partial \rho = \delta(H) / \delta S \) and \( \partial_S = -\delta(H) / \delta \rho \), where \( \rho \) and \( S \) are considered as canonical conjugate variables [19, 20]. In addition, it is easy to see that, the equation-of-motions which finally we find from these axioms, (4) and (5), have clear physical meanings in our interpretation: probability and average classical energy conservation, \( d\langle H \rangle / dt = 0 \), which together ensure conservation of the total classical energy of ensemble.
\[
\frac{\partial \rho}{\partial t} = - \sum_i \nabla_i \left( \rho \frac{\nabla S}{m_i} \right) \tag{4}
\]

\[
\frac{\partial S}{\partial t} = - \sum_i \frac{(\nabla_i S)^2}{2m_i} + \frac{\lambda_i^2}{2m_i} \frac{\nabla S}{\sqrt{\rho}} - U(x) \quad \tag{5}
\]

where \(\lambda_i^2\)'s are non-negative constants.

**Proof** For simplicity, here we prove the theorem for one-particle case, however it is completely similar to the N-particle case. The axiom-2 leads to following evolution equations:

\[
\frac{\partial \rho}{\partial t} = -\nabla \left( \rho \frac{\nabla S}{m} \right) \quad \tag{6}
\]

\[
\frac{\partial S}{\partial t} = -\frac{(\nabla S)^2}{2m} - U(x) - Q_0, \quad \tag{7}
\]

in which \(Q_0 \equiv \left[ \frac{\partial}{\partial \rho} - \frac{\partial}{\partial i} \frac{\rho}{\partial \rho} \right](\rho \mu)\). The Eq. (7) beside the axiom-1 leads to

\[
\int \rho \hat{H} d^3x = \int \rho \left[ \frac{(\nabla S)^2}{2m} + U(x) + Q_0 \right] d^3x, \quad \tag{8}
\]

which, using the axiom-2, leads to

\[
\frac{\partial S}{\partial t} = -\frac{(\nabla S)^2}{2m} - U(x) + Q_1. \quad \tag{9}
\]

where \(Q_1 \equiv \left[ \frac{\partial}{\partial \nu} \frac{\rho}{\partial \nu} - \frac{\partial}{\partial k} \frac{\rho}{\partial \nu} + \frac{\partial}{\partial \rho} \right](\rho Q_0)\). The Eq. (9) is consistent with Eq. (7), if and only if

\[
Q_1 = Q_0. \quad \tag{10}
\]

Equation (10) is a strong constrain on the local momentum variance. Moreover, since the local momentum variance, \((\delta p)^2\), is a scaler under rotation transformation, it must be a function of the \(\rho\) and \((\nabla \rho)^2\); i.e. \((\delta p)^2 = \mu(\rho, \eta)\), where \(\eta = (\nabla \rho)^2\). Therefore, the Eq. (10), by tedious but straightforward calculations, leads to

\[
\mu(\rho, \eta) = a \frac{\eta}{\rho^2} + \frac{b}{\rho} + c, \quad \tag{11}
\]

where \(a\), \(b\) and \(c\) are arbitrary constants. It is easy to see that the Eq. (8) leads to \(b = 0\), and also the constant \(a\) must be a non-negative real number, because the \((\delta p)^2\) has this property. Hence, we get \(Q_0 = Q_1 = \frac{\lambda^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} + c\), where \(\lambda^2 \equiv 2ma\). Finally note that, in the Eq. (7), the constant \(c\) can be absorbed in external potential \(U(x)\); In
fact, it must be equal to zero, if we assume that the momentum of particles ensemble associated to a plane wave, $e^{i\mathbf{p}_0 \cdot \mathbf{x}/\hbar}$, be equal to $\mathbf{p}_0$.

\[ \square \]

### 4 Discussion

Above theorem means the average H–J formalism, axioms (1, 2), determine the time evolution of $p$ and $S$ up to some unknown constants. Firstly, let us consider the case that all $\lambda_i$s are same; $\lambda_i = \lambda$. In this case, if we consider $\lambda = 0$, the Eq. (5) reduce to classical H–J equation, which, beside Eq. (4), describe dynamics of a classical (non-interacting) ensemble. On the other hand, if we consider $\lambda = \hbar$, the Eqs. (4) and (5) describe dynamics of a quantum system: although we do not consider any wave function fundamentally, one can define an effective wave function associated to worlds ensemble as $\psi = \sqrt{p} e^{iS/\hbar}$, which satisfies Schrödinger equation,

$$
\mathbf{i} \hbar \frac{\partial \psi}{\partial t} = \sum_i \frac{\hbar^2}{2m_i} \nabla_i^2 \psi + U(x)\psi.
$$

Moreover, the Eq. (3) ensures the uncertainty relation, $\Delta x_i \Delta p_i \geq \hbar$ [7, 30].

Another interesting case is that considering $\lambda_i = \hbar$ for one part of the system and $\lambda_i = 0$ for the rest; This choice leads to configuration ensemble theory for mixed classical-quantum systems [19, 20]. Finally note that, one can use other values, $0 < \lambda < \hbar$, to making (at least a phenomenological model for) a smooth description of quantum-classical transition [31–38]. Nonetheless, in the rest of the discussion, we consider only pure quantum systems, i.e. $\lambda_i = \hbar$ for all components of the system.

It worth noting that, the Eq. (3) can be derived from an “exact” form of uncertainty relation, $\delta x_i \delta p_i = \hbar$ [8, 9], and then using of the Eq. (3), the Eqs. (4) and (5) can be derived from probability conservation and average classical energy conservation assumptions, respectively [7, 39]. The advantage of our derivation is that, we priory do not assume a cornerstone of quantum theory, i.e uncertainty relation, however it is derived from our “average Hamilton–Jacobi” formalism. Moreover, the meaning of Eq. (3) in our interpretation is different with exact uncertainty approach [8, 9], Nelson’s stochastic mechanics [38–41], and other similar formalism [18, 42, 43]: we do not consider any random motion or inherent fluctuations, and the Eq. (3) merely represents the statistical momentum width of worlds ensemble [13].

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