POISSON SIGMA MODELS
ON SURFACES WITH BOUNDARY:
CLASSICAL AND QUANTUM ASPECTS

Memoria presentada por

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Chapter 1

Introduction
The understanding of Yang-Mills and instanton equations led to major mathematical advances in topology and geometry which proved to be connected to the quantum theory. It is the study of such relations what has been called *topological quantum field theory*.

The first examples of topological field theories appear in the works of Schwarz ([60]) and Witten ([73], [72]). Their different constructions include all known topological field theories up to now and one usually talks about topological field theories of the Schwarz or Witten type. The importance of Witten’s approach has been enormous. In [76] he gave a field theory description of Donaldson’s work on four-manifolds. In [78] he gave a three-dimensional interpretation of the polynomial invariant of knots previously found by Jones ([43]). Different examples of topological field theories were provided also by Witten in [77], [75] in relation to the construction of topological invariants of complex manifolds.

All these models share common features which allow to give a definition of a topological field theory. Roughly speaking, they do not depend on a metric tensor. More precisely, a topological (quantum) field theory consists of:

(i) A set of (Grassmann graded) fields \( \varphi \) defined on a Riemannian manifold \( (\Sigma, g) \).

(ii) An odd nilpotent operator \( Q \).

(iii) An energy momentum tensor which is \( Q \)-exact. That is, there exist some functionals \( F_{\mu\nu} \) of the fields and the metric tensor such that

\[
T_{\mu\nu} = \{ Q, F_{\mu\nu}(\varphi, g) \}.
\]

Property (ii) allows to define a cohomology on the Hilbert space so that physical states are \( Q \)-cohomology classes, whereas (iii) ensures that expectation values of \( Q \)-closed (and metric independent) observables do not depend on the metric tensor, i.e. they are topological invariants.

As mentioned above, all known topological field theories are classified as being of either Witten or Schwarz type ([13]). Witten type theories are those in which the complete quantum action is \( Q \)-exact. On the other hand, theories of the Schwarz type are those in which one starts from a classical action \( S_c(\varphi) \) which does not depend on the metric and the quantization procedure adds terms which are \( Q \)-exact, so that the full quantum action is of the form

\[
S_q(\varphi, g) = S_c(\varphi) + \{ Q, V(\varphi, g) \}.
\]

This dissertation is devoted to a thorough study of a two-dimensional topological field theory of Schwarz type known as *Poisson sigma model* (PSM). It was introduced in [59] as a generalization of the mathematical structure underlying a number of two-dimensional gauge theories such as 2D Yang-Mills, gauged WZW and models of 2D gravity. The mathematical object common to these theories is a Poisson structure \( \Pi \) on the target.

The fields of the PSM are given by a bundle map \((X, \eta)\) from the tangent bundle of a surface \( \Sigma \) to the cotangent bundle of a Poisson manifold \((M, \Pi)\).
Here $X$ stands for the base map and $\eta$ for the fiber map. Apart from its original motivation, the interest of the PSM resides in the fact that it naturally encodes the Poisson geometry of the target and allows to unravel it by means of techniques from Classical and Quantum Field Theory. This became manifest for the first time in 1999 with the work of Cattaneo and Felder [19], who used the PSM defined on the disk to give a field theoretical interpretation of Kontsevich’s formula for deformation quantization in terms of Feynman diagrams (see also [21]).

The same authors showed in [20] the connection between the structure of the phase space of the PSM and the symplectic groupoid (when the latter exists) integrating the target Poisson manifold. Their ideas inspired to a large extent the crucial works [27],[26] of Crainic and Fernandes on the integrability of Lie algebroids.

In these works the boundary conditions considered for the PSM are such that $X$ is free at $\partial \Sigma$ and $\eta$ vanishes on vectors tangent to $\partial \Sigma$.

The study of more general boundary conditions for the PSM was started in [22]. The question is what submanifolds (branes) $N \subset M$ are admissible in the sense that the field $X$ can be consistently restricted to $N$ at the boundary. In [22] it was shown that perturbative branes, i.e. branes admitting a perturbative quantization around $\Pi = 0$, coincide with coisotropic submanifolds of $M$.

In this dissertation we perform a general study of the PSM defined on a surface $\Sigma$ with boundary at both the classical and quantum level and show that much more general branes are allowed.

In Chapter 2 we give a self-contained introduction to Poisson geometry with emphasis in the reduction of Poisson manifolds. We introduce Dirac structures, beautiful objects living on $TM \oplus T^* M$ and generalizing simultaneously the notions of Poisson and presymplectic manifolds. Then, we present some original results dealing with the reduction of Dirac structures. The chapter ends with the statement of the problem of deformation quantization of Poisson manifolds and Kontsevich’s solution.

In Chapter 3 we present the PSM and perform the general study on a surface with boundary. We identify the classically admissible branes and show that the (pre-)symplectic structure on the phase space is related to the Poisson bracket induced on the brane. In this chapter we also recall the results of Cattaneo and Felder regarding the symplectic groupoid integrating the Poisson manifold $M$ by means of the PSM on the strip with free boundary conditions. Then, we see how everything is modified for more general branes.

On the light of the results of Chapter 3, it is tempting to conjecture that the perturbative quantization of the PSM with a general brane is related to the deformation quantization of the brane with the induced Poisson bracket. In Chapter 4 we show that this actually holds, although the suitable approach is quite different from that used in [19],[22]. We shall see
that non-coisotropic branes are non-perturbative in a sense and require a redefinition of the perturbative series.

Chapters 5 and 6 are devoted to an interesting particular case of the PSM, in which the target is a Lie group and the Poisson structure is compatible with the product on the group, i.e. a Poisson-Lie group. In Chapter 5 we briefly introduce the results on Lie bialgebras and Poisson-Lie groups that will be needed in Chapter 6. Poisson-Lie groups come in dual pairs $G, G^*$. In Chapter 6 we solve the models with target $G$ and $G^*$ for any simple, connected and simply connected Poisson-Lie group $G$. Poisson-Lie structures on simple Lie groups are either factorizable or triangular. In the first case the PSM over $G^*$ is known to be locally equivalent to $G/G$ theory. We show that in the second case it is equivalent to BF-theory. We end Chapter 6 by finding a family of branes which preserve the duality between the models.

Chapter 7 deals with the relation between extended supersymmetry in two-dimensional first order sigma models and generalized complex geometry from a phase space point of view. The particular case in which the metric in the target vanishes coincides with the PSM or more generally, if a WZ term is included, with the so-called twisted or WZ-Poisson sigma model. We generalize the definitions of Chapter 2 concerning the geometry of $TM \oplus T^*M$ to the twisted case. Then, we work out the conditions for extended supersymmetry in the WZ-Poisson sigma model and give a simple geometrical interpretation.
Chapter 2

Poisson geometry
In any course on analytical mechanics the Poisson bracket of two functions on phase space, in coordinates \((q^i, p_i), i = 1, \ldots, n\), is introduced:

\[
\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).
\]

In the study of some systems, in particular systems with constraints, more general brackets and then a generalization of the notion of Poisson bracket, are needed. The general theory of Poisson manifolds, i.e. manifolds whose set of smooth functions is equipped with a Poisson bracket, has been developed since the 1970s. The importance of Poisson geometry from the point of view of Physics resides not only on a geometrical formulation of classical mechanics, but also on the issue of quantization. It is a well-known fact that the starting point in the (canonical) quantization of a classical theory is precisely the Poisson bracket.

For the purposes of this dissertation some knowledge on Poisson geometry is obviously needed. This chapter provides the necessary geometrical background to carry out the general study of the PSM on surfaces with boundary in Chapters 3 and 4. We review some basic facts on Poisson geometry and we develop the theory of reduction of Poisson manifolds. Then we introduce more advanced topics related to Dirac structures, which generalize and contain as particular cases symplectic and Poisson structures. Finally, we state the problem of deformation quantization of Poisson manifolds and present the solution given by Kontsevich.

### 2.1 Poisson algebras and Poisson manifolds

Let \(\mathcal{A}\) be an associative commutative algebra with unit over the real or complex numbers equipped with a bilinear bracket \(\{\cdot, \cdot\} : \mathcal{A} \to \mathcal{A}\). We say that \(\mathcal{A}\) is a Poisson algebra if

(i) \(\{f, g\} = -\{g, f\}\) \hspace{1cm} (antisymmetry)

(ii) \(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0\) \hspace{1cm} (Jacobi)

(iii) \(\{f, gh\} = g\{f, h\} + \{f, g\}h\) \hspace{1cm} (Leibniz)

\(\forall f, g, h \in \mathcal{A}\).

We shall be mainly interested in the case in which \(\mathcal{A}\) is the algebra of smooth functions on an \(m\)-dimensional manifold \(M\), which is then called a Poisson manifold. A Poisson algebra structure on \(C^\infty(M)\) determines in a unique way a bivector field \(\Pi \in \Gamma(\wedge^2 TM)\) such that the Poisson bracket of two functions is given by:

\[
\{f, g\}(p) = \iota(\Pi_p)(df \wedge dg)_p, \ p \in M.
\]
Taking local coordinates \((x^1, \ldots, x^m)\) on \(M\) the components of \(\Pi\) are \(\Pi^{ij}(x) = \{x^i, x^j\}\). The Jacobi identity for the Poisson bracket reads in terms of \(\Pi^{ij}\):

\[
\Pi^{ji} \partial_i \Pi^{kl} + \Pi^{ki} \partial_i \Pi^{lj} + \Pi^{li} \partial_i \Pi^{jk} = 0 \tag{2.2}
\]

where summation over repeated indices is understood and \(\partial_i := \frac{\partial}{\partial x^i}\). A bivector field satisfying (2.2) is called a Poisson tensor or a Poisson structure on \(M\). It is obvious that a Poisson tensor defines a Poisson bracket on \(C^\infty(M)\) through formula (2.1).

Some definitions are in order at this point:

**Definition 2.1.** Let \(V(M) = \{V_p(M), p \in M\}\) be a set of linear subspaces of the tangent space \(T_pM\) at each point of \(M\). \(V(M)\) is called a general differentiable distribution if for every \(p \in M\), there exist vector fields \(X_1, \ldots, X_k \in V(M)\) such that \(V_p(M) = \text{span}\{X_1(p), \ldots, X_k(p)\}\). If \(\dim(V_p(M))\) is constant, \(V(M)\) is a differentiable distribution in the usual sense.

**Definition 2.2.** A function \(f : M \to \mathbb{R}\) is upper semicontinuous at \(p_0\) if for every \(\varepsilon > 0\) there exists a neighborhood \(U\) of \(p_0\) such that

\[
f(p) < f(p_0) + \varepsilon, \quad \forall p \in U.
\]

**Definition 2.3.** A function \(f : M \to \mathbb{R}\) is lower semicontinuous at \(p_0\) if for every \(\varepsilon > 0\) there exists a neighborhood \(U\) of \(p_0\) such that

\[
f(p) > f(p_0) - \varepsilon, \quad \forall p \in U.
\]

A function \(f\) is continuous if and only if it is upper and lower semicontinuous.

**Remark 2.1.** The dimension of a general differentiable distribution is a lower semicontinuous function on \(M\).

Now define \(\Pi^\sharp : T^*M \to TM\) by

\[
\beta(\Pi^\sharp(\alpha)) = \iota(\Pi)(\alpha \wedge \beta), \quad \alpha, \beta \in T^*M. \tag{2.3}
\]

For any \(f \in C^\infty(M)\) we call \(X_f := \Pi^\sharp df\) the Hamiltonian vector field of \(f\). Clearly,

\[
\{f, g\} = X_f(g), \quad \forall f, g \in C^\infty(M).
\]

By virtue of Jacobi identity the image of \(\Pi^\sharp\),

\[
\text{Im}(\Pi^\sharp) := \bigcup_{p \in M} \text{Im}(\Pi^\sharp_p)
\]

is a completely integrable general differential distribution and \(M\) admits a generalized foliation. \(M\) is foliated into leaves which may have varying
dimensions. The Poisson structure can be consistently restricted to a leaf and this restriction defines a non-degenerate Poisson structure on it. That is why we shall also refer to the leaves as symplectic leaves and to the foliation as the symplectic foliation of \( M \). This result comes from a generalization of the classical Frobenius theorem for regular distributions.

Next we give some examples of Poisson manifolds and Poisson algebras.

**Example 2.1.** \( \Pi = 0 \) is the trivial Poisson structure.

**Example 2.2.** Take \( M = g^* \), where \( g \) is a Lie algebra. Linear functions on \( g^* \) can be viewed as elements of \( g \). The Poisson bracket of two such functions is given by the Lie bracket on \( g \):

\[
\{ f, g \} = [f, g], \quad f, g \in g
\]

and is extended to \( C^\infty(\mathfrak{g}^*) \) by the Leibniz rule. This defines the so-called Kostant-Kirillov Poisson structure.

The symplectic leaves in this case correspond to the orbits under the coadjoint representation of any connected Lie group \( G \) with Lie algebra \( g \) and have, in general, varying dimensions; in particular, the origin is always a symplectic leaf since the Poisson structure vanishes.

**Example 2.3.** Take \( M = \mathbb{R}^3 \). Then,

\[
\Pi^{12}(x) = -(x^3)^2 + \frac{1}{4}, \quad \Pi^{23}(x) = x^1, \quad \Pi^{31}(x) = x^2
\]

defines a Poisson structure invariant under rotations around the \( x^3 \) axis. This example will appear in Chapter 3 in the context of two-dimensional \( R^2 \)-gravity.

**Example 2.4.** A presymplectic structure \( \omega \) on \( M \) is a closed 2-form on \( M \). We can define a Poisson algebra \( \mathcal{A}_\omega \) consisting of functions that possess a Hamiltonian vector field, i.e. those functions \( f \in C^\infty(M) \) for which the equation

\[
\iota(X_f)\omega = -df
\]

has a solution \( X_f \in \mathfrak{X}(M) \). Given \( f, g \in \mathcal{A}_\omega \) with Hamiltonian vector fields \( X_f, X_g \) respectively, \( fg \) has Hamiltonian vector field \( fX_g + gX_f \) and then \( \mathcal{A}_\omega \) is a subalgebra of \( C^\infty(M) \). The Poisson bracket is defined by

\[
\{ f, g \} = \omega(X_f, X_g).
\]

Note that in general the Hamiltonian vector field \( X_f \) for \( f \in \mathcal{A}_\omega \) is not uniquely defined but the ambiguities are in the kernel of \( \omega \) and then it leads to a well defined Poisson bracket.
Due to the closedness of $\omega$, $\{f,g\} \in A_\omega$, $\forall f,g \in A_\omega$ and the Jacobi identity is satisfied, so that $A_\omega$ is actually a Poisson algebra. It is worth mentioning that in this case the center of $A_\omega$ (Casimir functions) is the set of constant functions on $M$.

If $\omega^\flat: TM \to T^*M$ is invertible at every point of $M$, $\omega$ is called symplectic. A presymplectic structure $\omega$ defines a Poisson structure on $M$ (equivalently, $A_\omega = C^\infty(M)$) if and only if it is symplectic.

We end this section by giving some definitions.

**Definition 2.4.** Given two Poisson algebras $(A_1, \{\cdot, \cdot\}_1)$, and $(A_2, \{\cdot, \cdot\}_2)$ and a homomorphism of abelian associative algebras, $\Psi: A_1 \to A_2$, we say that $\Psi$ is a Poisson homomorphism if

$$\Psi(\{f,g\}_1) = \{\Psi(f), \Psi(g)\}_2, \forall f, g \in A_1.$$  

Analogously, we say that $\Psi$ is an anti-Poisson homomorphism if

$$\Psi(\{f,g\}_1) = -\{\Psi(f), \Psi(g)\}_2, \forall f, g \in A_1.$$  

**Definition 2.5.** Let $(M_1, \Pi_1)$ and $(M_2, \Pi_2)$ be two Poisson manifolds. A smooth map $F: M_1 \to M_2$ is called a Poisson map if

$$\{f,g\}_2 \circ F = \{f \circ F, g \circ F\}_1, \forall f, g \in C^\infty(M_2).$$

and an anti-Poisson map if

$$\{f,g\}_2 \circ F = -\{f \circ F, g \circ F\}_1, \forall f, g \in C^\infty(M_2).$$

That is, $F^*: C^\infty(M_2) \to C^\infty(M_1)$ is a (anti-)Poisson homomorphism.

**Definition 2.6.** Let $N \subset M$ be a submanifold of $M$. $N$ is called a Poisson submanifold if the inclusion $i: N \hookrightarrow M$ is a Poisson map.

## 2.2 Reduction of Poisson manifolds

Let $N$ be a closed submanifold of $(M, \Pi)$. Can we define in a natural way a Poisson structure on $N$? The answer is negative, in general. What we can always achieve is to endow a certain subset of $C^\infty(N)$ with a Poisson algebra structure. The canonical procedure below follows in spirit reference [44], although we present some additional, new results.

We adopt the notation $A = C^\infty(M)$ and take the ideal (with respect to the pointwise product of functions in $A$. We will use the term Poisson ideal when we refer to an ideal with respect to the Poisson bracket),

$$I = \{f \in A \mid f(p) = 0, \forall p \in N\}. $$
Define \( \mathcal{F} \subset \mathcal{A} \) as the set of \textit{first-class functions}, also called the \textit{normalizer} of \( \mathcal{I} \),

\[
\mathcal{F} = \{ f \in \mathcal{A} \mid \{ f, \mathcal{I} \} \subset \mathcal{I} \}.
\]

Note that due to the Jacobi identity and the Leibniz rule \( \mathcal{F} \) is a Poisson subalgebra of \( \mathcal{A} \) and \( \mathcal{F} \cap \mathcal{I} \) is a Poisson ideal of \( \mathcal{F} \). Then, we have canonically defined a Poisson bracket in the quotient \( \mathcal{F}/(\mathcal{F} \cap \mathcal{I}) \). However, this is not what we want, as our problem was to find a Poisson bracket on \( C^\infty(N) \cong \mathcal{A}/\mathcal{I} \) (or, at least, in a subset of it). To that end we define an injective map

\[
\phi : \mathcal{F}/(\mathcal{F} \cap \mathcal{I}) \longrightarrow \mathcal{A}/\mathcal{I}
\]

\[
f + \mathcal{F} \cap \mathcal{I} \longmapsto f + \mathcal{I}
\]

which is a homomorphism of abelian associative algebras with unit and then induces a Poisson algebra structure \( \{\cdot, \cdot\}_N \) in the image, that will be denoted by \( \mathcal{C}(\Pi, M, N) \), i.e.: \( \{f_1 + \mathcal{I}, f_2 + \mathcal{I}\}_N = \{f_1, f_2\} + \mathcal{I} \), \( f_1, f_2 \in \mathcal{F} \).

\[\text{(2.7)}\]

\textit{Remark 2.2.} Poisson reduction is a generalization of the symplectic reduction in the following sense: If the original Poisson structure is non-degenerate, it induces a symplectic structure \( \omega \) in \( M \). Then, we may canonically define on \( N \) the closed two-form \( i^*\omega \), where \( i : N \hookrightarrow M \) is the inclusion map. As described before, this presymplectic two-form on \( N \) defines a Poisson algebra for a certain subset of \( C^\infty(N) \cong \mathcal{A}/\mathcal{I} \). The Poisson algebra obtained this way coincides with the one defined above.

\textit{Remark 2.3.} The elements of \( \mathcal{F} \cap \mathcal{I} \) are, in the language of physicists, the \textit{generators of gauge transformations} or, in Dirac’s terminology, the \textit{first-class constraints}. They are constraints whose Hamiltonian vector fields are tangent to the submanifold \( N \).

The problem is that in general \( \phi \) is not onto and \( N \) cannot be made a Poisson manifold. The goal now is to use the geometric data of the original Poisson structure to interpret the algebraic obstructions.

Let \( T^\circ N \) (or \( \text{Ann}(TN) \)) be the conormal bundle of \( N \) (or annihilator of \( TN \)), i.e. the subbundle of the pull-back \( i^*(T^* M) \) consisting of covectors that kill all vectors in \( TN \). One has the following

\textit{Theorem 2.1.} Assume that:

a) \( \dim(\Pi_p^0(T_pN^0) + T_pN) = \text{const.}, \forall p \in N \), and

b) \( \Pi_p^0(T_pN^0) \cap T_pN = \{0\}, \forall p \in N \).

Then, the map \( \phi \) of \( (2.6) \) is an isomorphism of associative commutative algebras with unit.
2.2. Reduction of Poisson manifolds

Proof: Condition b) implies that

\[ T_p^*M = \text{Ann}(\Pi_p^*(T_p N^0) \cap T_p N) = T_p N^0 + \Pi_p^{-1}(T_p N) \]

and, then, \(\Pi_p^*(T_p^*M) \subseteq \Pi_p^*(T_p N^0) + T_p N, \quad \forall p \in N.\)

Now define a smooth bundle map:

\[ \Upsilon : TN \oplus T^0 N \to i^* TM \]

that maps \((v_p, \alpha_p) \in T_p N \oplus T_p N^0\) to \(v_p + \Pi_p^* \alpha_p.\) Due to condition a) the map is of constant rank and then every smooth section of its image has a smooth preimage.

Take \(f \in \mathcal{A}.\) As shown before \(\Pi_p^*(df) \in \Pi_p^*(T_p N^0) + T_p N\) for any \(p \in N.\) Then, the restriction to \(N\) of \(\Pi_p^* df\) is a smooth section of the image of \(\Upsilon.\)

Let \((v, \alpha)\) be a smooth section of \(TN \oplus T^0 N\) with \(\Upsilon(v, \alpha)_p = \Pi_p^*(df)_p\) for \(p \in N.\) Now for any section \(\alpha\) of \(T^0 N\) there exists a function \(g \in \mathcal{I}\) such that \(\alpha_p = (dg)_p\) for any \(p \in N.\)

Hence, one has that \(\tilde{f} = f - g \in \mathcal{F}\) and \(\phi(\tilde{f} + \mathcal{F} \cap \mathcal{I}) = f + \mathcal{I}.\) \(\square\)

Condition a) of Theorem 2.2 is not necessary as shown by the following Example 2.5. Take \(M = sl(2)^*.\) In coordinates \((x_1, x_2, x_3)\) the linear Poisson bracket is given by \(\{x_i, x_j\} = \epsilon^{ijk}x_k.\) Now define \(N\) by the constraints: \(x_1 = 0, x_2 = 0.\) Clearly,

\[ \dim(\Pi_p^*(T_p N^0) + T_p N) = \begin{cases} 3 & \text{for } p \neq 0 \\ 1 & \text{for } p = 0 \end{cases} \]

and for any \(f \in C^\infty(M)\) we may define \(\tilde{f} = f - x_1 \partial_1 f - x_2 \partial_2 f \in \mathcal{F}\) such that \(\phi(\tilde{f} + \mathcal{F} \cap \mathcal{I}) = f + \mathcal{I},\) i.e. \(\phi\) is onto. The Poisson structure induced in this case is, of course, zero.

Condition b), however, is necessary:

**Theorem 2.2.** If the map \(\phi\) of (2.7) is onto then \(\Pi_p^*(T_p N^0) \cap T_p N = \{0\}.\)

Proof: Assume that \(\exists v_p \neq 0, v_p \in \Pi_p^*(T_p N^0) \cap T_p N.\) It is enough to take a function \(f \in \mathcal{A}\) such that its derivative at \(p\) in the direction of \(v_p\) does not vanish. Then \(f + \mathcal{I}\) is not in the image of \(\phi.\) \(\square\)

This result tells us that when \(\Pi_p^*(T_p N^0) \cap T_p N \neq \{0\}\) one cannot endow \(N\) with a Poisson structure. The only functions on \(N\) that have got a well-defined Poisson bracket (i.e. the physical observables) are those in the image
of $\phi$. On the other hand, it is easy to see that all functions in the image of $\phi$ belong to the subalgebra of gauge invariant functions

$$A_{\text{inv}} = \{ f \in A | \{ f, F \cap I \} \subset I \}.$$ 

One may wonder when the physical observables are precisely the gauge invariant functions. A sufficient condition is given by the following

**Theorem 2.3.** If $\dim(\Pi_p^\sharp(T_pN^0) + T_pN) = \text{const.}, \forall p \in N$, then $\phi(F/F \cap I) = A_{\text{inv}}/I$.

Before proving the theorem we shall establish a Lemma that will be useful in the following.

**Lemma 2.1.** The following two statements are equivalent:

a) $\dim(\Pi_p^\sharp(T_pN^0) + T_pN) = \text{const.}, \forall p \in N$

b) $\Pi_p^{\sharp-1}(T_pN) \cap T_pN^0 = \{(dg)_p | g \in F \cap I \}, \forall p \in N$.

**Proof:**

a) $\Rightarrow$ b): Assume that $\dim(\Pi_p^\sharp(T_pN^0) + T_pN)$ is constant on $N$. Then, $\text{Ann}_p(\Pi_p^\sharp(T_pN^0) + T N) = \Pi_p^{\sharp-1}(T_pN) \cap T_pN^0$ is also of constant dimension and $\Pi_p^{\sharp-1}(T N) \cap T_pN^0$ is a subbundle of $T N^0$ whose fiber at every point of the base is spanned by a set of sections. For every section $\alpha$ of this subbundle there exists $g \in I$ such that $\alpha_p = (dg)_p$. But since $(dg)_p \in \Pi_p^{\sharp-1}(T_pN)$, it follows that $g \in F \cap I$.

The other inclusion is trivial as differential of first-class constraints are in $T N^0$ (because they are constraints) and their Hamiltonian vector fields transform constraints into constraints (because they are first-class) so their restrictions to $N$ are in $T N$.

b) $\Rightarrow$ a): Assuming b) one has that $\dim(\Pi_p^{\sharp-1}(T_pN) \cap T_pN^0)$ is a lower semicontinuous function on $N$ because the fiber of $\Pi_p^{\sharp-1}(T_pN) \cap T_pN^0$ at every point is spanned by local sections (see [71]). For the same reason, $\dim(\Pi_p^\sharp(T_pN^0) + T_pN)$ is also lower semicontinuous. But from the relation

$$\Pi_p^\sharp(T_pN^0) + T_pN = \text{Ann}_p(\Pi_p^{\sharp-1}(T N) \cap T N^0)$$

we infer that $\dim(\Pi_p^\sharp(T_pN^0) + T_pN)$ is upper semicontinuous, so it is continuous and, being integer valued it is indeed constant. $\square$

**Proof of Theorem 2.3.** First note that $f \in A_{\text{inv}}$ implies that $\Pi_p^\sharp(df)_p \in \text{Ann}_p(\{dg | g \in F \cap I \})$. But from the previous Lemma we have that the latter is equal to $\Pi_p^\sharp(T_pN^0) + T_pN$. 


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Then, \( \forall f \in A_{inv} \) one has \( \Pi_p^\sharp (df)_p \in \Pi_p^\sharp (T_p N^0) + T_p N \). And from here on the proof is like that of Theorem 2.2. \( \square \)

At first sight we might expect a result analogous to Theorem 2.2 for the case with gauge transformations in the constrained submanifold, namely that a necessary condition for \( \phi \) mapping onto the space of gauge invariant functions on \( N \) is that the space of Hamiltonian vector fields of first-class constraints at every point coincides with \( T_p N \cap \Pi_p^\sharp (T_p N^0) \). This is not true, however, as shown by the following example in which the spaces above differ in some points whereas the image of map \( \phi \) of (2.6) is \( A_{inv}/I \).

**Example 2.6.** Take \( M = \mathbb{R}^6 = \{(q_1, q_2, q_3, p_1, p_2, p_3)\} \) with the standard Poisson bracket \( \{p_i, q_j\} = \delta_{ij} \). Now consider the constraints

\[
g_i := p_i - q_i q_{\sigma(i)} \quad i = 1, 2, 3
\]

with \( \sigma \) the cyclic permutation of \( \{1, 2, 3\} \) s. t. \( \sigma(1) = 2 \). In this case

\[
\dim(\Pi_p^\sharp (T_x N^0) \cap T_x N) = \begin{cases} 
1 & \text{for } x \neq 0 \\
3 & \text{for } x = 0
\end{cases}
\]

while the gauge transformations are restrictions to \( N \) of Hamiltonian vector fields of \( fg \) with \( f \in C^\infty(M) \) and \( g = q_2 g_1 + q_3 g_2 + q_1 g_3 \). It implies that at \( x = 0 \) the gauge transformations vanish and, hence, they do not fill \( \Pi_p^\sharp (T_p N^0) \cap T_p N \).

We will show that the image of the map \( \phi \) of (2.6) is \( A_{inv}/I \). In every class of \( A_{inv}/I \) we may take the only representative independent of the \( p_i \)'s. Gauge invariant functions \( f(q_1, q_2, q_3) \) are then characterized by:

\[
(q_2 \partial_{q_1} + q_3 \partial_{q_2} + q_1 \partial_{q_3}) f = 0,
\]

and for any of them we may define

\[
\tilde{f} = f + \sum_i a_i g_i
\]

with \( a_i \) smooth, given by

\[
\begin{align*}
a_1(q_1, q_2, q_3) &= \frac{1}{q_1} [\partial_{q_2} f(q_1, q_2, q_3) - \partial_{q_2} f(0, q_2, q_3)], \\
a_2(q_1, q_2, q_3) &= \frac{1}{q_1} [\partial_{q_1} f(q_1, q_2, q_3) - \partial_{q_1} f(0, q_2, q_3)], \\
a_3(q_1, q_2, q_3) &= \frac{1}{q_2} \partial_{q_2} f(0, q_2, q_3) - \frac{1}{q_3} \partial_{q_1} f(0, q_2, q_3).
\end{align*}
\]

Now \( \tilde{f} \) is first class and \( \phi(\tilde{f} + \mathcal{F} \cap I) = f + I \). This shows that in this case the image of \( \phi \) fills \( A_{inv}/I \).
For later purposes it will be interesting to introduce coordinates adapted to the submanifold and to the Poisson structure. This is not possible for an arbitrary submanifold \(N\), but we must require some regularity conditions on it. The sufficient condition of Theorem \(2.3\)

\[
\dim(\Pi^\sharp_p(T_pN^0) + T_pN) = k + \dim(N), \quad \forall p \in N.
\]

will be called the strong regularity condition (here \(k\) is a non-negative constant). If it is satisfied, we can choose in a neighborhood \(U \subset M\) of every \(p \in N\) adapted local coordinates on \(M\), \((x^a, x^\mu, x^A)\), with \(a = 1, \ldots, \dim(N)\), \(\mu = \dim(N) + 1, \ldots, \dim(M) - k\) and \(A = \dim(M) - k + 1, \ldots, \dim(M)\), verifying:

1. \(N \cap U\) is defined by \(x^\mu = x^A = 0\).
2. \(\{x^\mu, x^\nu\}|_{N \cap U} = \{x^\mu, x^A\}|_{N \cap U} = 0\), i.e. \(x^\mu\) are first-class constraints.
3. \(\det(\{x^A, x^B\}(p)) \neq 0\), \(\forall p \in N \cap U\), i.e. \(x^A\) are second-class constraints.

It is clear that in these adapted coordinates the Poisson structure satisfies:

\[
\Pi^\mu\nu|_{N \cap U} = 0, \quad \Pi^A\mu|_{N \cap U} = 0, \quad \det(\Pi^{AB})|_{N \cap U} \neq 0.
\]

Recall that by Lemma \(2.1\) the strong regularity condition is equivalent to

\[
\Pi^\sharp_p(T_pN^0) \cap T_pN^0 = \{(df)_p | f \in \mathcal{F} \cap \mathcal{I}\}, \quad \forall p \in N.
\]

**Definition 2.7.** \(N \subset M\) is called coisotropic if \(\Pi^\sharp_p(T_pN^0) \subset T_pN\), \(\forall p \in N\).

For a coisotropic submanifold \(N\) the strong regularity condition \((2.8)\) is obviously satisfied and every constraint is first-class. In addition, \((2.7)\) is the original bracket on \(M\) restricted to the gauge-invariant functions on \(N\). The other extreme case is given by the absence of first-class constraints:

**Definition 2.8.** We say that \(N\) is a second-class submanifold if

\[
\Pi_p^{\sharp-1}(T_pN) \cap T_pN^0 = \{0\}, \quad \forall p \in N.
\]

Equivalently, \(N\) is a second-class submanifold if every constraint defining it is second-class. In the adapted coordinates defined above there are no Greek indices and the strong regularity condition is fulfilled. In this case the matrix of the Poisson brackets of the constraints \(\Pi^{AB} = \{x^A, x^B\}\) is invertible on \(N\). Defining on \(N\) the matrix \(\omega_{AB}\) by \(\omega_{AB}\Pi^{BC} = \delta_C^A\) the Poisson bracket \((2.7)\) can be written locally:

\[
\{f + \mathcal{I}, f' + \mathcal{I}\}_N = \{f, f'\} - \{f, x^A\} \omega_{AB} \{x^B, f'\} + \mathcal{I}
\]
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which is the usual definition of the Dirac bracket restricted to $N$. In this case every function on $M$ is trivially gauge-invariant (since $\mathcal{F}\cap\mathcal{I} = 0$), the image of $\phi$ is $C^\infty(N)$ and we get a Poisson structure on $N$. In adapted coordinates the components of the canonical Poisson tensor on $N$ corresponding to (2.10) are given by:

$$\Pi^a_b = \Pi^{ab} - \Pi^{aA}\omega_{AB}\Pi^{Bb} \quad (2.11)$$

where the subscript $D$ stands for Dirac.

When first-class constraints are present one can still use formula (2.11). Given a choice of adapted coordinates $(x^a, x^\mu, x^A)$ the expression

$$\Pi^{pq}_D = \Pi^{pq} - \Pi^{pA}\omega_{AB}\Pi^{Bq} \quad (2.12)$$

with the indices $p, q = 1, \ldots, \dim(M) - k$ running over $a$ and $\mu$ values, defines a Poisson bracket on the submanifold $N'$ on which the second-class constraints vanish (if we assume $\det(\Pi^{AB}) \neq 0$ on $N'$). The submanifold $N'$ is not uniquely defined, as it depends on the concrete choice of the set of second-class constraints. $N$ is now a coisotropic submanifold of $N'$ and the Poisson algebra induced by $\Pi_D$ on the gauge invariant functions on $N$ is indeed canonical, independent of the choice of $N'$, and equals the one given by (2.7).

One can also extend the Poisson tensor to a tubular neighborhood of $N'$ by taking $\Pi^A_p = \Pi^B_p = 0$. If one considers the tubular neighborhood equipped with the Dirac bracket, $N$ is coisotropic in it and $N'$ is a Poisson submanifold.

Later on in this dissertation, it will be important to consider the following weak regularity condition:

$$\dim\{ (df)_p | f \in \mathcal{F}\cap\mathcal{I} \} = \dim(M) - \dim(N) - k, \quad \forall p \in N \quad (2.13)$$

for some non-negative constant $k$.

That the strong regularity condition (2.8) implies the weak one with the same value for the constant $k$ is clear from (2.9). The weak regularity condition is equivalent to the existence of local coordinates on a tubular neighborhood of every patch of $N$ with a maximal (and constant) number of coordinates which are first-class constraints. In other words, (2.13) holds if and only if there exist local coordinates satisfying (i), (ii) as above and

(iii)' $\det(\{x^A, x^B\}(p)) \neq 0$ for $p$ in an open dense subset of $N \cap \mathcal{U}$.

However, the weak regularity condition is not enough to guarantee that $\phi(\mathcal{F}/\mathcal{F}\cap\mathcal{I}) = \mathcal{A}_{inv}/\mathcal{I}$.

In general, if $\dim(\Pi_p^T(T_pN^0 + T_pN))$ is not constant on $N$ we cannot define a Poisson bracket on the set of gauge invariant functions. The only thing we can assert is that we have a Poisson algebra on the subset of $C^\infty(N)$ given by the image of $\phi$. However, an efficient description of the functions in the image (the space of observables) is not available in the general case.
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2.2.1 Poisson-Dirac submanifolds

In this section we would like to make contact between our previous results and terminology in absence of gauge transformations and those appearing in two papers by Crainic and Fernandes [26] and Vaisman [70].

In [26], the interpretation of a Poisson structure in terms of Dirac structures (see Section 2.3) is adopted to carry out the reduction procedure. This geometrical approach is slightly more general than our algebraic methods as we shall see below. A submanifold \( N \subset M \) satisfying

\[
\Pi^\sharp_p(T_pN^0) \cap T_pN = \{0\}, \quad \forall p \in N
\]

is called pointwise Poisson-Dirac in [26]. Recall that this is condition b) of Theorem 2.2.

In this case, for any \( p \in N \) there exists a unique map \( \hat{\Pi}^\sharp_p \) that makes the following diagram commutative. If the maps \( \hat{\Pi}^\sharp_p : T^*_pN \to T_pN \) define a smooth bundle map, the latter gives a Poisson structure on \( N \), which is then called a Poisson-Dirac submanifold. With these definitions, Poisson-Dirac submanifolds are the most general submanifolds which inherit in a canonical way a Poisson structure from \((M, \Pi)\). It is clear that \( \phi \) onto implies that \( N \) is a Poisson-Dirac submanifold. The following is an example in which \( \phi \) is not onto while \( N \) is still a Poisson-Dirac submanifold, being possible to endow it with a Poisson structure.

**Example 2.7.** Consider \( M = \mathbb{R}^4 = \{(q_1, q_2, p_1, p_2)\} \) with Poisson structure \( \{p_i, q_j\} = \delta_{ij}q_i \exp(-1/q_i^2) \) smoothly extended to \( q_i = 0 \) and \( N \) defined by the constraints \( g_1 = p_1 - q_2^2/2, \quad g_2 = p_2 + q_1^2/2 \). We can take \( \sigma_i := q_i \) as coordinates on \( N \).

\( \Pi^\sharp_p(T_xN^0) \cap T_xN = \{0\} \) on \( N \) but \( \phi \) is not onto. For instance, take \( f_i = q_i \in C^\infty(M) \). If we try to find a first-class function in the class \( f_1 + \mathcal{I} \) (its pre-image by \( \phi \)) we obtain for \( q_i \neq 0 \)

\[
\tilde{f}_1 := f_1 - \frac{q_1 \exp(-1/q_1^2)}{q_1^2 \exp(-1/q_1^2) + 2 \exp(-1/q_2^2)}
\]
which fails to extend continuously to \( q_i = 0 \). Then, \( f_1 \) does not belong to the image of \( \phi \). However, the Hamiltonian vector field associated to this singular \( \tilde{f}_1 \) is smooth and we can define a Poisson structure on \( N \):

\[
\Pi_{12}^N(\sigma_1, \sigma_2) = \{ \tilde{f}_1, f_2 \}(\sigma_1, \sigma_2, \frac{1}{2} \sigma_2^2, -\frac{1}{2} \sigma_1^2) = \frac{\sigma_1 \sigma_2}{\sigma_1^2 \exp(1/\sigma_2^2) + \sigma_2^2 \exp(1/\sigma_1^2)}.
\]

If \( \phi \) is onto and, in addition, \( \text{dim}(\Pi_p^N(T_p^*N^0) + T_p^*N) \) is constant on \( N \) (i.e. the situation of Theorem 2.2), we have what is called in [26] a constant rank Poisson-Dirac submanifold.

The surjectivity of \( \phi \) is equivalent to the existence of an algebraically compatible normal bundle in the language of [70]. Following the latter paper, define a normalization of \( N \) by a normal bundle \( \nu N \) as a splitting \( TM|_N = TN \oplus \nu N \). For every \( p \in N \) there exists a neighborhood \( U \) where we can choose adapted coordinates \((g^A, z^a)\) such that, locally, \( g^A|_{N \cap U} = 0 \) and \( z^a \) are coordinates on \( N \cap U \). Vaisman calls \( \nu N \) algebraically \( \Pi \)-compatible if, in these coordinates, \( \Pi^{Aa}|_N = 0 \). Then, we have the following

**Theorem 2.4.** \( \phi \) is onto iff there exists an algebraically \( \Pi \)-compatible normal bundle.

**Proof:**

Only local properties in a neighborhood of each point of \( N \) matter for this proof.

\( \Rightarrow \) Let \((g^A, z^a)\) be local coordinates such that \( N \) is locally defined by \( g^A = 0 \) and \( z^a \) are coordinates on \( N \). Take the pre-image by \( \phi \) of the coordinate functions \( z^a \) and denote them by \( y^a \). \((g^A, y^a)\) are local coordinates such that \( \Pi^{Aa}|_N = 0 \).

\( \Leftarrow \) For any \( f(g^A, y^a) \in C^\infty(U) \), \( \tilde{f}(g^A, y^a) = f(0, y^a) \in \mathcal{F} \) and \( \tilde{f} - f \in \mathcal{I} \). Then, \( \phi \) is onto. \( \square \)

### 2.3 Dirac structures

As stressed before the structure underlying the reduction of Poisson or symplectic manifolds is a Poisson algebra. Such an algebraic structure can be encoded in geometric terms through the concept of Dirac structure, which generalizes the Poisson and presymplectic geometries by embedding them in the context of the geometry of \( TM \oplus T^*M \). Dirac structures were introduced in the remarkable paper by T. Courant [25]. Therein, they are related to the Marsden and Weinstein [56] reduction and to the Dirac bracket [30] on a submanifold of a Poisson manifold. This simple but powerful construction allows to deal with mechanical situations in which we have both gauge symmetries and Casimir functions.
Dirac structures (more precisely their twisted counterparts) will play an essential role in Chapter 7 when working out the conditions for extended supersymmetry in the WZ-PSM. In addition, the fact that Dirac structures can be seen as a generalization of Poisson structures suggests generalizations of the PSM (see Dirac sigma models in Section 3.1.1) and of the deformation quantization scheme \cite{63}. Below we describe Dirac structures and give some original results on their relation to Poisson algebras and on reduction issues.

It will be useful for later purposes to introduce the notion of Lie algebroid, of which a Dirac structure is a particular case. A Lie algebroid over a smooth manifold $M$ is a vector bundle $E$ with the following additional structure:

(i) a Lie bracket $[\cdot, \cdot]$ on sections of $E$.

(ii) A bundle map $\rho : E \to TM$, called the anchor, inducing a Lie algebra homomorphism on sections.

(iii) $[\alpha, f\beta] = f[\alpha, \beta] + (\rho(\alpha)f)\beta$, where $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma(E)$.

Let us provide some elementary examples of Lie algebroids:

Example 2.8. $M$ a point and $\rho$ trivial. Then, $E$ is a Lie algebra.

Example 2.9. $M$ arbitrary and $\rho = 0$. Then, $E$ is a bundle of Lie algebras.

Example 2.10. $E = TM$ endowed with the ordinary bracket and $\rho = \text{id}_{TM}$.

Example 2.11. Let $(M, \Pi)$ be a Poisson manifold. $E = T^*M$ is a Lie algebroid with anchor $\rho = \Pi^\sharp$ and bracket defined on exact elements by $[df, dg] = d\{f, g\}$ and extended by the Leibniz rule to all 1-forms.

Dirac structures are special types of Lie algebroids contained in the vector bundle $TM \oplus T^*M$. We shall use the notation $\rho_1 : TM \oplus T^*M \to TM$, $\rho_2 : TM \oplus T^*M \to T^*M$ for the canonical projections.

The Courant bracket on sections of $TM \oplus T^*M$ is defined by

$$[(X, \xi), (X', \xi')] = ([X, Y], \iota(X)d\xi' - \iota(X')d\xi + \frac{1}{2}d(\iota(X)\xi' - \iota(X')\xi)).$$

It is bilinear and antisymmetric but not a Lie bracket, since it does not satisfy the Jacobi identity in general.

Consider the natural symmetric bilinear form on $TM \oplus T^*M$,

$$\langle (X, \xi), (X', \xi') \rangle = \iota(X)\xi' + \iota(X')\xi.$$ \hfill (2.15)

Following \cite{25}, we shall define a Dirac subbundle as a subbundle $D \subset TM \oplus T^*M$ maximally isotropic with respect to (2.15). Maximal isotropy

\footnote{In Chapter [7] we shall define the twisted version of the Courant bracket by a closed 3-form and the twisted analogues of the rest of objects introduced in this section.}
implies that $D^\perp = D$, where $D^\perp$ stands for the orthogonal of $D$. In particular, $\dim(D) = \dim(M)$. Dirac subbundles have been recently considered (see [14]) in connection to the singular reduction of implicit Hamiltonian systems.

A Dirac structure is a Dirac subbundle $D$ such that its sections close under the Courant bracket. In this case, as shown in [25], the restriction to $D$ of the Courant bracket fulfills the Jacobi identity and $D$ with anchor $\rho_1|_D$ is a Lie algebroid.

Example 2.12. For a 2-form $\omega$, the graph $L_\omega$ of $\omega^b : TM \to T^*M$ is a Dirac structure in $TM \oplus T^*M$ if and only if $d\omega = 0$. This type of Dirac structure is characterized by the fact that $\rho_1(L_\omega) = TM$ at every point of $M$.

Example 2.13. Let $\Pi$ be a bivector field on $M$. The graph $L_{\Pi^\sharp} : T^*M \to TM$ is a Dirac structure if and only if $\Pi$ is a Poisson structure. In this case $\rho_2$ maps $L_{\Pi^\sharp}$ onto $T^*M$.

The definition of a symmetry of a Dirac structure is a straightforward generalization of the definitions in the presymplectic and Poisson cases:

Definition 2.9. A vector field $Y$ is a symmetry of the Dirac structure $D \subset TM \oplus T^*M$ if $(L_YX, L_Y\xi) \in \Gamma(D)$ for every $(X, \xi) \in \Gamma(D)$.

2.3.1 Dirac structures and Poisson algebras

Let $D$ be a Dirac structure on $M$. We say that $f \in C^\infty(M)$ is an admissible function if there exists a vector field $X_f$ such that $(X_f, df)$ is a section of $D$. It is not difficult to see that the admissible functions form an algebra with the pointwise product. In addition, given two admissible functions $f$ and $g$,

$$\{f, g\} = X_f(g)$$

defines a Poisson bracket. Thus the set of admissible functions is a Poisson algebra. The converse, i.e. whether given a Poisson algebra we may define a Dirac structure $D$ associated to it, is a more subtle issue. The situation is as follows.

Let $A \subset C^\infty(M)$ be a Poisson algebra and let us define its associated Dirac structure $D$. For any point $p \in M$, $(X_p, \xi_p) \in D_p$ if and only if

a) There exists $f \in A$ such that $\xi_p = (df)_p$.

b) $\iota(X_p)(dg)_p = \{f, g\}(p)$ for every $g \in A$.

We are tacitly assuming that $\forall f \in A$ there exists a vector field satisfying condition b). Equivalently, we assume that $\rho_2(D_p) = \{(df)_p, f \in A\}$. This is mandatory if we want to define a meaningful Dirac structure associated to the Poisson algebra $A$.

Notice that $\rho_2(D_p) = \{(df)_p | f \in A\}$ if and only if $(dg)_p = 0, g \in A \Rightarrow \{f, g\}(p) = 0, \forall f \in A$. This requirement is not met for an arbitrary Poisson algebra of functions, as shown by the next example.
Example 2.14. Take $M = \mathbb{R}^2$ with coordinates $(x, y)$ and $\mathcal{A}$ consisting of functions of the form $f(x^2, y^2)$. We endow $\mathcal{A}$ with the Poisson bracket $\{x^2, y^2\} = 1$. The function $x^2 \in \mathcal{A}$ has vanishing differential at the origin and non-zero Poisson bracket with $y^2 \in \mathcal{A}$. Consequently, the vector field $X$ of condition b) above does not exist for every $f \in \mathcal{A}$.

However, it is not difficult to show that for Poisson algebras of functions obtained by the reduction procedure of Section 2.2, the aforementioned condition $\rho_2(D_p) = \{(df)_p|f \in \mathcal{A}\}$ actually holds. To see this, consider the Poisson algebra $\mathcal{C}(\Pi, M, N)$ with Poisson bracket given by (2.7). Take $g + \mathcal{I} \in \mathcal{C}(\Pi, M, N)$ with $g \in \mathcal{F}$ such that $(dg)_p|TN = 0$ at a point $p \in N$. For any $f + \mathcal{I} \in \mathcal{C}(\Pi, M, N)$ with $f \in \mathcal{F}$, formula (2.7) obviously implies that $\{f, g\}_N = 0$. Then, conditions a), b) above define a Dirac structure associated to $\mathcal{C}(\Pi, M, N)$.

From now on we assume that $\rho_2(D_p) = \{(df)_p|f \in \mathcal{A}\}$.

It is clear that $D_p$ is a linear subspace. Given $(X_p, \xi_p), (X'_p, \xi'_p) \in D_p$ corresponding to $f$ and $f'$, we have that $(aX_p+a'X'_p, a\xi_p+a'\xi'_p) \in D_p, \forall a, a' \in \mathbb{R}$ and corresponds to $af + a'f' \in \mathcal{A}$.

$D_p$ is isotropic with respect to (2.15) as a consequence of the antisymmetry of the Poisson bracket i.e. given $(X_p, \xi_p), (X'_p, \xi'_p) \in D_p$ associated to functions $f$ and $f'$ we have

$$\langle (X_p, \xi_p), (X'_p, \xi'_p) \rangle = \iota(X_p)(df')_p + \iota(X'_p)(df)_p$$

$$= \{f, f'\}(p) + \{f', f\}(p) = 0.$$

We can show that $D_p = (D_p)\perp$ by a dimensional argument. Since

$$\rho_2(D_p) = \{(df)_p|f \in \mathcal{A}\}$$

and

$$\text{Ker}\rho_2|D_p = \{X_p| \iota(X_p)(df)_p = 0, \forall f \in \mathcal{A}\}$$

it is clear that we have the relation

$$(\text{Ker}\rho_2|D_p)^0 = (\text{Im}\rho_2|D_p)$$

and then

$$\dim(D_p) = \dim(\text{Im}\rho_2|D_p) + \dim(\text{Ker}\rho_2|D_p) = \dim(M).$$

This proves that $D_p$ is maximally isotropic.

One can also show that sections of $TM \oplus T^*M$ with values in $D_p$ as defined before close under the Courant bracket. Firstly, consider sections of the form $(X_f, df)$ with $f \in \mathcal{A}$; their bracket reads
2.3. Dirac structures

\[ [(X_f, df), (X_{f'}, df')] = ([X_f, X_{f'}], \frac{1}{2} d(X(f') - X'(f)) \]
\[ = ([X_f, X_{f'}], df, f') \]

and due to the Jacobi identity for the Poisson bracket,

\[ [X_f, X_{f'}](g) = \{f, f'\}, \forall g \in A \]

so that exact sections close under the Courant bracket. Using now that for two orthogonal sections of \( TM \oplus T^* M \) and \( h \in C^\infty(M) \) we have

\[ [(hX, h\xi), (X', \xi')] = h[(X, \xi), (X', \xi')] - X'(h)(X, \xi) \]

the closedness relation can be extended to arbitrary sections. Hence, if the fibers \( D_p \) define a subbundle \( D \) of \( TM \oplus T^* M \), then \( D \) is a Dirac structure. Although the fibers have constant dimension, in general they do not define a subbundle. This can be shown in the following simple example.

**Example 2.15.** Consider \( M = \mathbb{R}^3 \) and the algebra \( A \) of smooth functions in \( M \) invariant under the flow of the vector field:

\[ x_2 \partial_{x_1} + x_3 \partial_{x_2} + x_1 \partial_{x_3}. \]

This is a Poisson algebra if we endow it with the zero Poisson bracket.

Now for \((x_1, x_2, x_3) \neq 0\) we have

\[ D_{(x_1, x_2, x_3)} = \text{span}\{(x_2 \partial_{x_1} + x_3 \partial_{x_2} + x_1 \partial_{x_3}, 0), (0, \xi_1), (0, \xi_2)\} \]

where \( \xi_1, \xi_2 \) are two independent covectors in \( T^*_{(x_1, x_2, x_3)} M \) that kill the vector \( x_2 \partial_{x_1} + x_3 \partial_{x_2} + x_1 \partial_{x_3} \). However, as the differential of any function in \( A \) vanishes in \((0, 0, 0)\) we have

\[ D_{(0,0,0)} = T_{(0,0,0)} M, \]

which shows that the bunch of fibers is not a subbundle of \( TM \oplus T^* M \).

Of course, when the Poisson algebra is induced by a Dirac structure the latter is recovered form the former as sketched above.

### 2.3.2 Induced Dirac structures

In this paragraph we shall introduce a method to obtain a Dirac structure from another one. The method is very simple but it has important applications like the generalization of the Dirac bracket, reduction of Dirac structures to submanifolds or the Marsden and Ratiu reduction of Poisson manifolds [55].
Take a subbundle $S \subset TM \oplus T^*M$ isotropic with respect to \eqref{2.15}, i.e. $S \subset S^\perp$. It is easy to show that given a Dirac subbundle $D$ we may obtain another one $D^S = (D \cap S^\perp) + S$ provided that it is a subbundle. We must show that it is maximally isotropic, but this is immediate:

$$(D^S)^\perp = (D^\perp + S) \cap S^\perp = (D \cap S^\perp) + S = D^S,$$

where in the last line we have used that $D$ is maximally isotropic and $S$ is a subset of $S^\perp$. In a sense $D^S_S$ is the Dirac subbundle closest to $D$ among those containing $S$, as stated in the following

**Theorem 2.5.** Let $D, S$ and $D^S$ be as above and let $D'$ be a Dirac subbundle such that $S \subset D'$. Then, $D' \cap D \subset D^S \cap D$. In addition, $D' \cap D = D^S \cap D$ if and only if $D' = D^S$.

**Proof:**

From the isotropy of $D'$ and given that $S \subset D'$ we deduce that $D' \subset S^\perp$. Hence,

$$D' \cap D \subset S^\perp \cap D = D^S \cap D.$$

If the equality $D' \cap D = D^S \cap D$ holds, then $D' \supset D' \cap D = S^\perp \cap D$. Since $S \subset D'$, we deduce that $D^S = (D \cap S^\perp) + S \subset D'$. But $D^S$ and $D'$ have the same dimension and hence they are equal. \(\square\)

In general even when $D^S$ is a maximally isotropic smooth subbundle its sections do not close under the Courant bracket. We discuss now some particularly interesting examples and applications of this construction.

**Dirac bracket (or Dirac Dirac structure):** Consider an integrable distribution $\Phi \subset TM$ and take $S = \Phi^0 \subset T^*M$. Then, for any Dirac structure $D$ on $M$, $D^S$ (if smooth) is a Dirac structure such that $\rho_1(D^S)$ is everywhere tangent to the foliation. That is,

$$(D^S)_p = \{(X_p, \xi_p + \nu_p)|(X_p, \xi_p) \in D_p, X_p \in \Phi_p, \nu_p \in S_p\}.$$

Let us see that sections of $D^S$ close under the Courant bracket:

$$[(X, \xi + \nu), (X', \xi' + \nu')] =$$

$$= ([X, X'] + \iota(X)d\xi - \iota(X')d\xi + \frac{1}{2}d(\iota(X)\xi' - \iota(X')\xi)) +$$

$$+ (0, \iota(X)d\nu' - \iota(X')d\nu). \quad \quad \quad (2.16)$$

$(X, \xi), (X', \xi')$ are sections of the Dirac structure and then so it is the first line of the right hand side of \eqref{2.16}. $X, X'$ are vector fields in the integrable distribution $\Phi$ and then $[X, X']$ is also a vector field of $\Phi$. Finally, we have
to show that the third line of (2.16) is a section of $\Phi^0$. Contracting with a vector field $Y$ in $\Phi$ we obtain:

$$\iota(Y)(\iota(X)d\nu') = X(\iota(Y)\nu') - Y(\iota(X)\nu') - \iota([X,Y])\nu' = 0$$

where we have used that $\nu'$ is a section of $S = \Phi^0$ and $X, Y$ are sections of $\Phi$, which is an integrable distribution. Therefore, we have proven that $D^S$ is a Dirac structure on $M$ (assuming it is a subbundle).

$D^S$ can be considered as a generalization of the Dirac bracket extended to the whole manifold for the case of Dirac structures. Recall that the Dirac structure $D^S$ is the graph of a bivector field (which is a Poisson structure due to involutivity) if and only if $\rho_2(D^S) = T^*M$ at every point of $M$. This is equivalent to

$$D^S + TM = TM \oplus T^*M. \quad (2.17)$$

Taking the orthogonal of (2.17) we get the more familiar (and equivalent) condition

$$(D + \Phi^0) \cap \Phi = \{0\}. \quad (2.18)$$

If this holds, $D^S$ corresponds to a Poisson bracket which should be called Dirac bracket. Indeed, it coincides with the standard Dirac bracket if $D$ itself comes from a Poisson structure $\Pi$. In this case (2.18) can be rewritten as $\Pi^2(\Phi^0) \cap \Phi = \{0\}$. Notice that $D^S_N$ corresponds to a Poisson structure if and only if (2.18) is satisfied on $N$, i.e. if

$$(D|_N + TN^0) \cap TN = \{0\}. \quad (2.19)$$

If $D$ is the graph of a bivector field $\Pi$, (2.19) is equivalent to requiring $\Pi^2(TN^0) \cap TN = \{0\}$ at every point of $N$. That is, $N \subset M$ is a pointwise Poisson-Dirac submanifold (recall (2.14)). The smoothness of the induced bivector field is ensured by our regularity assumptions on $D^S$ so that $N$ is a Poisson-Dirac submanifold and is equipped with a Poisson bracket.
2.3. Dirac structures

Projection along an integrable distribution: Now, let $S \subset TM$ be an integrable distribution. Assuming that both $D \cap S^\perp$ and $D^S$ are subbundles, any section of $D^S$ can be written as $(X + Y, \xi)$ where $(X, \xi)$ is a section of $D \cap (TM \oplus S^0)$ and $Y$ is a section of $S$.

$D^S$ is not Courant involutive in general. This is not strange since one would expect to be able to define a Dirac structure only on $M/S$, the space of leaves of the foliation defined by $S$ (assuming $M/S$ is a manifold). Objects on $M$ which descend suitably to $M/S$ will be called \textit{projectable along} $S$. Functions on $M/S$, $C^\infty(M)_{pr}$, can be viewed as the set

$$C^\infty(M)_{pr} = \{ f \in C^\infty(M) \mid X(f) = 0, \forall X \in S \}.$$ 

Vector fields on $M/S$ are then defined as derivations on $C^\infty(M)_{pr}$, i.e.

$$\mathfrak{X}(M)_{pr} = \{ X \in \mathfrak{X}(M) \mid X(C^\infty(M)_{pr}) \subset C^\infty(M)_{pr} \}.$$ 

Notice that $X$ belongs to $\mathfrak{X}(M)_{pr}$ if and only if $Z(X(f)) = 0$, $\forall f \in C^\infty(M)_{pr}$, $\forall Z \in \Gamma(S)$. Or equivalently, $[Z, X](f) = (L_Z X)(f) = 0$ using that $Z(f) = 0$. Hence, we obtain the more useful characterization

$$\mathfrak{X}(M)_{pr} = \{ X \in \mathfrak{X}(M) \mid L_Z X \in \Gamma(S), \forall Z \in \Gamma(S) \}.$$ 

Analogously, 1-forms on $M/S$, $\Omega^1(M)_{pr}$, are linear maps from $\mathfrak{X}(M)_{pr}$ to $C^\infty(M)_{pr}$. An analogous argument to that followed for vector fields yields:

$$\Omega^1(M)_{pr} = \{ \xi \in \Gamma(S^0) \mid L_Z \xi = 0, \forall Z \in \Gamma(S) \}.$$ 

We are now ready to prove that sections $(X + Y, \xi)$ of $D^S$ which are projectable along $S$, i.e.

a) $\xi$ is a section of $S^0$,

b) $\forall Z \in \Gamma(S), \ (L_Z (X + Y), L_Z \xi) = (Z', 0)$, with $Z' \in \Gamma(S)$

are Courant involutive. Notice that the integrability of $S$ implies that $L_Z X = [Z, X]$ in condition b) must be a section of $S$. Hence, if $(X + Y, \xi)$ is a projectable section, $X$ itself must be a projectable vector field.

Take two such sections $(X + Y, \xi)$ and $(X', Y', \xi')$. First notice that

$$[X + Y, X' + Y'] - [X, X'] \in \Gamma(S)$$

because $[X, Y'], [Y, X']$ and $[Y, Y']$ are sections of $S$ due to condition b) and the integrability of $S$. Therefore,

$$[X + Y, X' + Y'] = [X, X'] + W, \ W \in \Gamma(S). \quad (2.20)$$

We must prove that $L_Z [X + Y, X' + Y'] = [Z, [X + Y, X' + Y']]$ is a section of $S$ for any $Z \in \Gamma(S)$. Using $(2.20)$, this amounts to prove that $[Z, [X, X']]$ belongs to $\Gamma(S)$. By the Jacobi identity,

$$[Z, [X, X']] = [[[Z, X], X'] - [[Z, X'], X] \quad (2.21)$$
which is a section of \( S \) because \( X \) and \( X' \) are projectable vector fields.

For the cotangent part of the Courant bracket of projectable sections we have

\[
\begin{align*}
\iota(X + Y)\xi' &= \iota(X)\xi' + \iota(Y)\xi' - d\iota(X)\xi' = \iota(X)d\xi' \\
\iota(X + Y)d\xi' &= \iota(X)d\xi' + \mathcal{L}_Y\xi' - d\iota(Y)\xi' = \iota(X)d\xi'
\end{align*}
\tag{2.22}
\]

where we have used conditions a) and b) above. We want to show that the right-hand side of both equations in (2.22) are projectable sections, i.e. both are in \( \Gamma(S^0) \) and have vanishing Lie derivative along any section \( Z \in \Gamma(S) \).

This is not difficult:

\[
\begin{align*}
\iota(Z)(\iota(X)d\xi') &= -\iota(X)(\mathcal{L}_Z\xi' - d(\iota(Z)\xi')) = 0, \\
\iota(Z)d(\iota(X)d\xi') &= \mathcal{L}_Z(\iota(X)\xi') = \iota(\mathcal{L}_Z X)\xi' + \iota(X)\mathcal{L}_Z\xi' = 0, \\
\mathcal{L}_Z(\iota(X)d\xi') &= \iota(\mathcal{L}_Z X)d\xi' + \iota(X)\mathcal{L}_Zd\xi' = \\
&= \mathcal{L}_Zd(\iota(Z)\xi') + \iota(X)d(\mathcal{L}_Z \xi') - \iota(X)d(\iota(Z)\xi') = 0, \\
\mathcal{L}_Z(d(\iota(X)\xi')) &= d(\iota(Z)d(\iota(X)\xi')) = 0,
\end{align*}
\]

where \( Z' = \mathcal{L}_Z X = [Z, X] \in \Gamma(S) \).

If for any point \( p \in M \) and any \( (X_p, \xi_p) \in (D^S)_p \), there exists a (local) projectable section of \( D^S \) such that it coincides with \( (X_p, \xi_p) \) at \( p \), then \( S \) is a symmetry of \( D^S \) in the sense of Definition 2.9. To see this, denote by \( \{\mathcal{X}_i\}_{i=1}^m, m = \dim(M) \), a basis of \( \Gamma(D^S) \) formed by projectable sections. Any local section of \( D^S \) is written as

\[
\mathcal{X} = \sum_{i=1}^m f^i \mathcal{X}_i
\]

for some smooth functions \( f^i \). Taking the Lie derivative with respect to \( Z \in \Gamma(S) \),

\[
\mathcal{L}_Z \mathcal{X} = \sum_{i=1}^m (\mathcal{L}_Z f^i) \mathcal{X}_i + \sum_{i=1}^m f^i (\mathcal{L}_Z \mathcal{X}_i)
\]

where the last term belongs to \( \Gamma(S) \) due to the projectability of the \( \mathcal{X}_i \)s.

Hence, for any \( \mathcal{X} \in \Gamma(D^S) \) we have shown that \( \mathcal{L}_Z \mathcal{X} \in \Gamma(D^S) \), \( \forall Z \in \Gamma(S) \) and \( S \) is a symmetry of \( D^S \).

In this situation we can project the Dirac structure onto \( M/S \). Given the projection \( \tau : M \to M/S \) the image of

\[
\tau_* \oplus (\tau^*)^{-1} : D^S \to T(M/S) \oplus T^*(M/S)
\]

correctly defines a Dirac structure on \( M/S \).
Generalized Marsden-Ratiu reduction: The third example is a combination of the two previous ones and is inspired in the work of Marsden and Ratiu \cite{55}. It consists in reducing a Dirac structure $D$ to a submanifold $N$ along a symmetry $S$ of $D^S$. $S$ is an integrable distribution as above and we view $N$ as a leaf of another integrable distribution $\Phi \subset TM$.

As we have learnt $D^S$ induces a Dirac structure in $M/S$. The latter can be reduced to $N/(S \cap TN)$ and is the generalization to any Dirac structure of the procedure of \cite{55} for Poisson manifolds.

One may wonder when the reduced Dirac structure, which we denote by $\hat{D}$, is actually a Poisson structure on the orbit space $N/(S \cap TN)$. $\hat{D}$ is Poisson if and only if

$$\hat{D} \cap T(N/(S \cap TN)) = \{0\}.$$ 

Equivalently, in terms of the original Dirac structure $D$, $\hat{D}$ is Poisson if and only if for any $p \in N$ we have that

$$(X_p, \xi_p) \in D_p \text{ s.t. } \xi_p \in (S^0)_p \cap (TN^0)_p \Rightarrow X_p \in S.$$ 

This can be reformulated as

$$TM \oplus (S^0 \cap TN^0) \subset D \cap (S \oplus T^*M) + TM$$ 

and taking the orthogonal with respect to the symmetric pairing

$$TM \cap (D + S^0) \subset S + TN,$$  \hspace{1cm} (2.23)

which can be phrased by saying that the (generalized) Poisson vector fields of $S^0$ are sections of $S + TN$. This is precisely the same condition that Marsden and Ratiu found for their reduction to yield a Poisson structure in the reduced space.

We have then shown that Marsden and Ratiu reduction fits perfectly in the framework of Dirac structures in the sense that the reduction of an invariant Dirac structure always produces a Dirac structure (assuming the existence of the different subbundles) in the reduced space. In the particular case in which (2.23) holds the final Dirac structure is actually Poisson.

2.4 Deformation quantization of Poisson manifolds

We end this chapter by introducing the problem of deformation quantization of Poisson manifolds, which will be a central subject in this dissertation due to its relation with the Poisson sigma model.

The program of deformation quantization and the notion of star product was established by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in the 1970s \cite{10}. The idea is to quantize the algebra of classical observables
on a Poisson manifold $M$ by deforming the commutative pointwise product
of functions by a non-commutative associative product, the star product,
with the Planck’s constant controlling the non-commutativity.

More precisely, the problem of deformation quantization of a Poisson
manifold $(M, \Pi)$ is to find an $\mathbb{R}[[\hbar]]$-bilinear product $\star$ (a star product) on
$C^\infty(M)[[\hbar]]$

\[
C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]] \\
(f, g) \mapsto f \star g
\]

such that $\forall f_1, f_2, f_3 \in C^\infty(M)$:

(i) $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$

(ii) $f_1 \star f_2 = f_1 f_2 + \{f_1, f_2\} \hbar + O(\hbar^2)$

i.e. an associative deformation of the pointwise product on $C^\infty(M)$ which
at first order in $\hbar$ coincides with the Poisson bracket.

The existence of star products for any symplectic manifold was shown
first by DeWilde and Lecomte ([29]). A more geometrical proof was given
later by Omori, Maeda and Yoshioka ([57]). Finally, Fedosov provided a
purely geometrical construction ([33],[34]).

The case in which $M$ is a general Poisson manifold turned out to be much
harder and was solved by Kontsevich in 1997 ([49]) as a consequence of his
formality theorem. He proved that every Poisson manifold admits a star
product and gave an explicit formula for $M = \mathbb{R}^n$ (see [23] for globalization
aspects) which can be expressed in terms of diagrams.

A diagram $\Gamma$ of order $n$ consists of $n$ numbered vertices of first type
$\{1, \ldots, n\}$ and two vertices of second type labeled by $\{L, R\}$ (standing for
Left and Right). Two ordered oriented edges start at each of the vertices
of the first type and end at any vertex (first or second type), but an edge
cannot start and end at the same vertex. There are no edges starting at
vertices of the second type. The endpoints of the edges starting at vertex $i$
are denoted $v_1(i), v_2(i)$.

One then associates to every diagram $\Gamma$ a bidifferential operator $B_\Gamma$ as
follows:

- Place a function $f$ at $L$ and a function $g$ at $R$.
- Associate to each vertex $v$ of the first type the Poisson tensor $\Pi^{j_1j_2}$.
- Associate a partial derivative $\partial_{j_1}$ ($\partial_{j_2}$) to the first (second) edge starting
  from $v$ and contract it with the first (second) index of $\Pi^{j_1j_2}$. The partial
  derivative acts on the function or Poisson tensor placed at the endpoint of
  the edge.
2.4. Deformation quantization of Poisson manifolds

The Kontsevich’s formula for the star product of \( f, g \in C^\infty(M) \) reads:

\[
f \star g = fg + \sum_{n \geq 1} \left( \frac{i\hbar}{2} \right)^n \frac{1}{n!} \sum_{\gamma \text{ order } n} w_\gamma B_\gamma(f, g).
\]  

(2.24)

Consider the upper half plane \( H_+ \) endowed with the Poincaré metric

\[
ds^2 = \frac{dx^2 + dy^2}{y^2}
\]

and define the configuration space

\[
H_n := \{(u_1, \ldots, u_n) \in H^n | u_k \in H, u_k \neq u'_k \text{ for } k \neq k' \}.
\]

Now take a complex coordinate \( z = x + iy \) and let \( \phi(z, w) \) be the angle (measured counterclockwise) between the geodesic passing through \( z \) and \( i\infty \) and the geodesic passing through \( z \) and \( w \),

\[
\phi(z, w) = \frac{1}{2i} \log \left( \frac{(z-w)(z-w)}{(z-w)(z-w)} \right).
\]  

(2.25)

Define \( d\phi(z, w) := \partial_z \phi(z, w)dz + \partial_w \phi(z, w)dw \). The weight \( w_\gamma \) is given by:

\[
w_\gamma = \frac{1}{(2\pi)^2n} \int_{H_n} \bigwedge_{i=1}^n d\phi(u_i, u_{v_1(i)}) \wedge d\phi(u_i, u_{v_2(i)})
\]

setting \( u_L = 0, u_R = 1 \).

This combinatorial, diagrammatic construction of the Kontsevich’s star product resembles Feynman expansions in perturbative Quantum Field Theory. It was Cattaneo and Felder who identified the theory from which Kontsevich’s formula can be recovered and it turned out to be the Poisson sigma model, which we introduce in Chapter 3.

\footnote{What Kontsevich calls \( \hbar \) is here \( i\hbar/2 \). This convention will be convenient in connection with the perturbative quantization of the Poisson sigma model of Chapter 4.}
Chapter 3

Classical Poisson sigma model and branes
In this chapter we introduce, at the classical level, the field theory which is the main object of study of this dissertation: the Poisson sigma model. In Section 3.1 we discuss its original motivation, some interesting theories which can be seen as particular cases of the Poisson sigma model and a number of generalizations which have been recently proposed.

In Section 3.2 a thorough study of the classical Poisson sigma model on a surface with boundary is carried out, identifying the admissible branes.

In Section 3.3 it is shown how the phase space of the model is related to the Poisson geometry of the brane.

3.1 Introduction to the PSM

The Poisson sigma model (PSM) is a two-dimensional topological sigma model defined on a surface $\Sigma$ and with a finite dimensional Poisson manifold $(M, \Pi)$ as target. The fields are given by a bundle map $(X, \eta) : T\Sigma \to T^*M$, where $X : \Sigma \to M$ is the base map and $\eta$ is a 1-form on $\Sigma$ with values in $\Gamma(X^*T^*M)$. The action functional has the form

$$S(X, \eta) = \int_\Sigma \langle \eta, \wedge dX \rangle + \frac{1}{2} \langle \Pi \circ X, \eta \wedge \eta \rangle,$$  \hspace{1cm} (3.1)

where $\langle \cdot, \cdot \rangle$ denotes the pairing between vectors and covectors of $M$.

In this section we shall not be concerned with the problem of defining consistent boundary conditions for the action (3.1), since it will be the central subject in the subsequent ones. Hence, in the remainder of this section we assume that $\Sigma$ is closed, i.e. $\partial \Sigma = \emptyset$.

If $X^i$ are local coordinates on $M$, $\sigma^\kappa$, $\kappa = 1, 2$ local coordinates on $\Sigma$, $\Pi^{ij}$ the components of the Poisson structure in these coordinates and $\eta^i = \eta^i_\kappa d\sigma^\kappa$, the action reads

$$S(X, \eta) = \int_\Sigma \eta^i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X)\eta^i \wedge \eta^j,$$  \hspace{1cm} (3.2)

The equations of motion in the bulk are:

$$dX^i + \Pi^{ij}(X)\eta_j = 0$$  \hspace{1cm} (3.3a)

$$d\eta^i + \frac{1}{2} \partial_l \Pi^{lok}(X)\eta_k \wedge \eta_l = 0.$$  \hspace{1cm} (3.3b)

The necessity for the bivector field $\Pi$ to be a Poisson structure is a consequence of the consistency of the equations of motion (3.3). Differentiating (3.3a) we obtain:

$$\partial_l \Pi^{ij}(X)dX^k \wedge \eta_j + \Pi^{ij}(X)d\eta^j = 0$$

which, using again (3.3a), yields

$$\Pi^{ij}(X)d\eta^j = \Pi^{kl} \partial_k \Pi^{ij}(X)\eta^l \wedge \eta^j.$$  \hspace{1cm} (3.4)
3.1. Introduction to the PSM

Contracting (3.3b) with $\Pi$ and applying (3.4) gives

$$
\frac{1}{2} \left( \Pi^{li} \partial_i \Pi^{jk} + \Pi^{ji} \partial_i \Pi^{kl} + \Pi^{ki} \partial_i \Pi^{lj} \right) \eta_j \wedge \eta_k = 0 \quad (3.5)
$$

which holds for arbitrary $\eta$ only if $\Pi$ satisfies the Jacobi identity (2.2).

The infinitesimal transformations

$$
\delta_\epsilon X^i = \Pi^{ij}(X) \epsilon_j \quad (3.6a)
$$

$$
\delta_\epsilon \eta_i = -d\epsilon_i - \partial_i \Pi^{jk}(X) \eta_j \epsilon_k \quad (3.6b)
$$

where $\epsilon(\sigma) = \epsilon_i(\sigma) dX^i$ is a section of $X^* T^* M$, change the action (3.2) by a boundary term

$$
\delta_\epsilon S = - \int_{\Sigma} d(dX^i \epsilon_i) \quad (3.7)
$$

which vanishes if $\Sigma$ is closed. Up to trivial gauge transformations (that is, symmetry transformations proportional to the equations of motion) the expression (3.6) gives a complete set of local symmetries for the PSM. In particular, the obvious invariance of (3.2) under diffeomorphisms of $\Sigma$ should be recovered by a suitable choice of the gauge parameter. If $\sigma^\mu \mapsto \sigma^\mu + \xi^\mu(\sigma)$ is the infinitesimal form of the diffeomorphism, we must take $\epsilon_i = -\xi^\mu \eta_{i\mu}$ so that

$$
\delta_\epsilon X^i = \mathcal{L}_\xi X^i - \iota(\xi)(dX^i + \Pi^{ij}(X) \epsilon_j) \quad (3.8a)
$$

$$
\delta_\epsilon \eta_i = \mathcal{L}_\xi \eta_i - \iota(\xi)(d\eta_i + \frac{1}{2} \partial_i \Pi^{jk}(X) \eta_j \wedge \eta_k) \quad (3.8b)
$$

which on-shell gives the expected transformations of the fields.

It is worth pointing out that the form of the transformations (3.6) is not invariant under a change of coordinates on $M$. However, it is straightforward to check that they are well-defined when evaluated on solutions of the equations of motion. Let $X^\mu(X^j)$ be a change of coordinates on $M$ and denote by

$$
\epsilon'_i = \frac{\partial X^j}{\partial X'^i} \epsilon_j, \quad \eta'_i = \frac{\partial X^j}{\partial X'^i} \eta_j, \quad \Pi'^{ij} = \frac{\partial X^\mu}{\partial X'^i} \frac{\partial X^\nu}{\partial X'^j} \Pi^{k\ell}
$$

the objects in the new coordinates $X'^i$. Using (3.6), we can work out the transformation of the fields in these coordinates:

$$
\delta_\epsilon X'^i = \Pi'^{ij}(X) \epsilon'_j
$$

$$
\delta_\epsilon \eta'_i = -d\epsilon'_i - \partial'_i \Pi'^{jk}(X) \eta'_j \epsilon'_k - \frac{\partial X^j}{\partial X'^i} \frac{\partial^2 X'^r}{\partial X'^j \partial X'^l}(dX'^l + \Pi'^{lk} \eta'_k) \epsilon'_r
$$

which, on-shell, are of the form (3.6).
3.1. Introduction to the PSM

Also notice that
\begin{equation}
[\delta_{\epsilon}, \delta_{\epsilon'}]X^i = \delta_{[\epsilon, \epsilon']}^i X^i \quad (3.9a)
\end{equation}
\begin{equation}
[\delta_{\epsilon}, \delta_{\epsilon'}]\eta_i = \delta_{[\epsilon, \epsilon']} \eta_i - \epsilon_i \epsilon' \partial_i \partial_j \Pi^{ij}(dX^j + \Pi^{ij}(X)\eta_j) \quad (3.9b)
\end{equation}
where $[\epsilon, \epsilon']^i := -\partial_i \Pi^{ij}(X)\epsilon_j\epsilon'_j$. The term in parenthesis in (3.9b) is the equation of motion (3.3a). Hence, the commutator of two transformations of type (3.6) is a transformation of the same type only on-shell and the gauge transformations (3.6) form an open algebra. As we shall see in Chapter 4 this makes the quantization of the model quite involved, since the standard BRST techniques do not work and the more sophisticated Batalin-Vilkovisky procedure is needed.

The PSM was introduced in [59], [42] as a generalization of the underlying mathematical structure of two-dimensional gravity models and Yang-Mills theories. In relation to the latter, it is worth noticing that given a volume form $\varepsilon$ on $\Sigma$ and a Casimir function $C$ on $M$ we can add a non-topological term
\begin{equation}
S_C(X) = \lambda \int_{\Sigma} \varepsilon C(X(\sigma)), \quad \lambda \in \mathbb{R} \quad (3.10)
\end{equation}
to the action (3.2) without spoiling the gauge symmetry (3.6). The equation of motion (3.3a) does not change whereas (3.3b) now reads:
\begin{equation}
d\eta_i + \frac{1}{2} \partial_i \Pi^{jk}(X) \eta_j \wedge \eta_k + \lambda \partial_i C(X) = 0.
\end{equation}

2D Yang-Mills theories are related in this framework to linear Poisson structures (see Example 2.2). Let $M$ be a linear space and $\Pi^{ij}(X) = f^{ij}_k X^k$. The Jacobi identity implies that $M$ is the dual of a Lie algebra $\mathfrak{g}$ of which $f^{ij}_k$ are the structure constants. After integrating by parts, the topological action (3.2) takes the form
\begin{equation}
S_{\text{lin}} = \int_{\Sigma} X^i F_i \quad (3.11)
\end{equation}
with $F_i := d\eta_i + \frac{1}{2} f^{jk}_i \eta_j \wedge \eta_k$. The field $\eta \in \Omega(\Sigma) \otimes \mathfrak{g}$ can be viewed as a 1-form connection and $F$ as the curvature two-form. The action (3.11) actually corresponds to BF-theory (11). A first order formulation of Yang-Mills theory is obtained by adding to $S_{\text{lin}}$ a term (3.10) with $C(X) = \sum_i X^i X^i$, the quadratic Casimir of $\mathfrak{g}$. Namely,
\begin{equation}
S_{\text{YM}}^{fo} = \int_{\Sigma} X^i \left( d\eta_i + \frac{1}{2} f^{jk}_i \eta_j \wedge \eta_k \right) + \lambda \varepsilon \sum_i X^i X^i
\end{equation}
which can be seen to be equivalent to the ordinary second order formulation of Yang-Mills theory
\begin{equation}
S_{\text{YM}} = -\frac{1}{4\lambda} \int_{\Sigma} \text{tr}(F \wedge *F) \quad (3.12)
\end{equation}
by using the field equations to get rid of $X$.

It is remarkable that most two-dimensional models of pure gravity can be formulated as a PSM (see [66] for a thorough discussion). As an example we work out the case of $R^2$-gravity, whose action is given by

$$S_{R^2} = \frac{1}{4} \int_{\Sigma} \sqrt{\det(g)} \left( \frac{1}{4} R^2 + 1 \right) d^2 \sigma$$

(3.13)

where $g$ is a (Riemannian) metric on $\Sigma$ and $R$ is the scalar curvature of the torsionless connection compatible with $g$. The action (3.13) can be written as a PSM with $\dim(M) = 3$ via Einstein-Cartan variables. Let $e^a$, $a = 1, 2$ denote the zweibein and $\omega$ the connection 1-form. The torsion two-form in these variables reads

$$De^a := de^a + \varepsilon^a_{\ b} \omega^b.$$ Thus one can see that (3.13) is equivalent to the first-order action

$$S_{fo}^{R^2} = \int_{\Sigma} X^a De^a + X^3 d\omega + \left( \frac{1}{4} - (X^3)^2 \right) \varepsilon.$$ 

(3.14)

With the identification $(\eta_1, \eta_2, \eta_3) \equiv (e^1, e^2, \omega)$ and after an integration by parts, the action (3.14) is a PSM with Poisson structure given by (2.5).

Interestingly, the PSM can also be viewed as a limit of the action of the bosonic string in a generic background of massless fields ([9]). The latter reads (setting the dilaton field to zero for simplicity)

$$S_{str} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2 \sigma \sqrt{\det(h)} \left[ g_{ij}(X)h^{\mu\nu} + \epsilon^{\mu\nu} B_{ij}(X) \right] \partial_\mu X^i \partial_\nu X^j$$

(3.15)

where $h$ is the metric on $\Sigma$, $g$ the metric on the target space $M$ and $B$ the two-form on $M$. Let us write a first order action equivalent to (3.15). To this end, introduce a one-form $\eta$ on $\Sigma$ with values in $X^* (T^* M)$ and take

$$S_{str}^{fo} = \int_{\Sigma} \eta_i \wedge dX^i + \pi\alpha' G^{ij}(X) \eta_i \wedge * \eta_j + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j$$

(3.16)

where

$$(g + B)^{-1} = G + \frac{1}{2\pi\alpha'} \Pi.$$

The Seiberg-Witten limit ([61]) consists in taking the limit $\alpha' \to 0$ keeping $G$ and $\Pi$ fixed. This implies

$$g \sim (2\pi\alpha')^2 \Pi^{-1} G (\Pi^t)^{-1},$$

$\varepsilon = e^1 \wedge e^2$ is the volume form of the metric $g$, which in Einstein-Cartan variables is just $g = e^1 \wedge e^1 + e^2 \wedge e^2$. 


3.1. Introduction to the PSM

$$B \sim 2\pi \alpha' \Pi^{-1}.$$

One can show (see [9]) that the conditions for the theory to describe a
conformal sigma model impose that in the Seiberg-Witten limit $dB = 0$ and
then $d\Pi^{-1} = 0$. Hence, in this limit we obtain

$$S_{\text{string}}^f = \int_\Sigma \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij} \eta_i \wedge \eta_j$$

which is a PSM with (symplectic) Poisson structure $\Pi$.

3.1.1 Generalizations of the PSM

Several generalizations of the action (3.2) and the equations of motion (3.3)
have been proposed by Strobl et al.. In this section we briefly discuss some
of them.

- **WZ-Poisson sigma model.**

  In [48] a modification of the action (3.2) by adding a WZ term

  $$S_{\text{WZ}} = \int_\Sigma d^{-1}H$$

  was considered. Here $H$ is a closed 3-form on $M$. The consistency of the
  model leads to the interesting notion of twisted Poisson manifolds ([44]),
in which the Jacobi identity is violated by a term proportional to $H$. We
  shall consider the WZ-Poisson sigma model and twisted Poisson manifolds
  in Chapter 7.

- **Lie algebroid sigma models.**

  The idea behind this generalization of the PSM (due to Strobl) is the ob-
  servation that the equations of motion of the PSM (3.3) can be interpreted
  as a morphism of Lie algebroids (in the sense described in [15]) from $T\Sigma$
to $T^*M$. Hence, the aim is to generalize the PSM so that the new theo-
  ries, associated to general Lie algebroids, yield equations of motion which
  have an analogous geometrical interpretation. In this framework, Strobl
  has found new gravity models ([68]) and generalizations of Yang-Mills theo-
  ries ([67]). Let us sketch what the mathematical concepts underlying these
  constructions are.

  Let us take a Lie algebroid $E$ of dimension $n$ over an $m$-dimensional
  manifold $M$. If $\{X^i\}_{i=1}^m$ are local coordinates on $M$ and $\{b_I\}_{I=1}^n$
denotes a local basis of $E$, the bracket and anchor give rise to structure functions
  $c_{IJ}^K(X)$ and $\rho^I_I(X)$,

  $$[b_I, b_J] = c_{IJ}^K b_K, \quad \rho(b_I) = \rho^I_I \partial_I.$$
The compatibility conditions in the definition of a Lie algebroid (see Section 2.3) are translated into differential equations for the structure functions:

\[ c^S_{IJK} + c^{ij}_{IJ,K} + \text{cycl.}(IJK) = 0 \]  
\[ c^K_{IJ} \rho^i_I - \rho^j_J \rho^i_J + \rho^j_J \rho^i_J = 0. \]

where a comma denotes a partial derivative.

For the particular case of a Poisson manifold \( M, E = T^*M, b_I \sim dX^i, \rho^j_J \sim \Pi^j_i \) and \( c^K_{IJ} \sim \partial_k \Pi^{ij} \). It is straightforward to check that in this case (3.16b) gives the Jacobi identity and (3.16a) its derivative, so that it is a consequence of the former.

Although we shall not use it, let us mention that differential calculus on \( T^*M \) can be generalized to any Lie algebroid \( E \). Take the local basis of \( E^* \), \( \{ b^I_i \}_{i=1}^n \), dual to \( \{ b_I^i \}_{i=1}^n \). The differential \( d_E : \Omega^k_E(M) \rightarrow \Omega^{k+1}_E(M) \), with \( \Omega^k_E(M) := \Gamma(\wedge^k E^*) \), \( \Omega^0_E(M) := C^\infty(M) \), is defined by

\[ d_E f = f, i \rho^i_I b^I \]
\[ d_E b^I = -\frac{1}{2} c^K_{JK} b^J \wedge b^K \]

and extended by a graded Leibniz rule. One can show that, in particular, \( d^2_E = 0 \).

The generalization of the equations of motion of the PSM is:

\[ dX^i - \rho^i_I \eta^I = 0 \]
\[ d\eta^J + \frac{1}{2} c^I_{JK} \eta^J \wedge \eta^K = 0 \]

and the generalization of the gauge symmetries (3.6) (which also have a geometrical meaning in terms of homotopies) is:

\[ \delta_x X^i = \rho^i_I \epsilon^I \]
\[ \delta_x \eta^J = d\epsilon^J + c^I_{JK} \eta^J \wedge \eta^K \quad \text{(on-shell)}. \]

In [68] an interpretation of these equations in terms of gravity theories was discussed. The paper [67] is devoted, as said above, to a generalization of Yang-Mills theories in the context of Lie algebroids. Actions whose equations of motion are (3.17) can be constructed for arbitrary \( d = \text{dim}(\Sigma) \), but only through the introduction of Lagrange multipliers in the general case. Namely,

\[ S = \int_{\Sigma} B_i \wedge (dX^i - \rho^i_I \eta^I) + B_I \wedge (d\eta^I + \frac{1}{2} c^K_{JK} \eta^J \wedge \eta^K) \]
where $B_i, B_I$ are respectively $(d-2)$- and $(d-1)$- forms. Only for very special choices of $E$ and $\dim(\Sigma)$ one can construct topological actions yielding Lie algebroid morphisms up to homotopies. For $\dim(\Sigma) = 2$ and $E = T^*M$ one has the PSM, whereas for $\dim(\Sigma) = 3$ and $E$ a quadratic Lie algebra we get the Chern-Simons theory.

- **Dirac sigma models.**

The Dirac sigma model, introduced in [50], is a generalization of the $G/G$ WZW model and the WZ-Poisson sigma model at the same time. The fields of the model are given by a bundle map from the tangent bundle of a surface $\Sigma$ to a Dirac structure $D$. In order to ensure that the equations of motion produce Lie algebroid morphisms from $T\Sigma$ to $D$, metric tensors both on $\Sigma$ and $M$ are needed, acting as a sort of regulators.

These models are mathematically appealing, and in principle might be related to the quantization of Dirac structures in the same way as the PSM is related to the quantization of Poisson manifolds. However, at the present time the applications of the Dirac sigma models are still to be investigated.

### 3.2 Classically admissible branes

In this section we carry out the classical study of the PSM defined on a surface with boundary and search for the boundary conditions (BC) which make the theory consistent.

In order to preserve the topological character of the theory one must choose the BC independent of the point of the boundary, as far as we move along one of its connected components. For the sake of clarity we shall restrict ourselves in this section to one connected component. In the next section we shall discuss the relation between the BC in the possible different connected components of the boundary.

In surfaces with boundary a new term appears in the variation of the action (3.2) under a change of $X$ when performing the integration by parts:

$$
\delta_X S = -\int_{\partial \Sigma} \delta X^i \eta_i + \int_{\Sigma} \delta X^i (d \eta_i + \frac{1}{2} \partial_j \Pi^{jk}(X) \eta_j \wedge \eta_k)
$$

The BC must be such that the boundary term vanishes. Let us take the field

$$
X|_{\partial \Sigma} : \partial \Sigma \to N
$$

(3.18)

for an arbitrary (for the moment) closed submanifold $N$ of $M$ (brane, in a more stringy language). Cattaneo and Felder concluded in [22] that maximally symmetric branes are given by the coisotropic submanifolds of $M$. In the sequel we show that much more general branes are admissible, however.
Denote by $\eta_t = \eta \circ dX^i$ the contraction of $\eta$ with vector fields tangent to $\partial \Sigma$. The BC (3.13) implies that $\delta X \in T_X N$ at every point of the boundary, and consequently $\eta_t$ must belong to $T_X N^0$ (the fiber over $X$ of the conormal bundle of $N$).

On the other hand, by continuity, the equations of motion in the bulk must be satisfied also at the boundary. In particular,

$$\partial_t X = \Pi^A \eta_t$$

where by $\partial_t$ we denote the derivative in the direction of a vector on $\Sigma$ tangent to the boundary. As $\partial_t X$ belongs to $T_X N$ it follows that $\eta_t \in \Pi^A_{X} (T_X N^0)$.

Both conditions for $\eta_t$ imply that

$$\eta_t (u) \in \Pi^A_{X(u)} (T_{X(u)} N) \cap T_{X(u)} N^0, \text{ for any } u \in \partial \Sigma$$

(3.19)

which is the boundary condition we shall take for $\eta_t$. We should check now that the BC are consistent with the gauge transformations (3.6).

In order to cancel the boundary term (3.7) $\epsilon|_{\partial \Sigma}$ must be a smooth section of $TN^0$ and if (3.6) is to preserve the boundary condition of $X$, $\epsilon|_{\partial \Sigma}$ must belong to $\Pi^{A}_{N}$. Hence,

$$\epsilon (u) = \Pi^A_{X(u)} (T_{X(u)} N) \cap T_{X(u)} N^0, \text{ for any } u \in \partial \Sigma.$$  

(3.20)

Next, we shall show that the the gauge transformations (3.6) with (3.20) also preserve (3.19) provided that the brane satisfies some (mild) regularity conditions. Let us restrict ourselves to weakly regular branes, i.e. branes verifying the weak regularity condition (2.13). This allows to choose adapted coordinates $(X^a, X^\mu, X^A)$ satisfying the properties (i), (ii), (iii)' of Section 2.2. Recall that in these coordinates the Poisson tensor satisfies:

$$\Pi^{\mu \nu}|_N = 0, \quad \Pi^{\mu A}|_N = 0$$

$$\det(\Pi^{AB})|_N \neq 0 \quad \text{in an open dense set.}$$

The boundary condition (3.19) translates in these coordinates into $\eta_t = \eta \circ dX^\mu$. Hence, we must show that $\delta \eta_t = \delta \eta_A = 0$. Recalling (3.20) we also may write $\epsilon|_N = \epsilon_{\mu} dX^{\mu}$ and therefore,

$$\delta \eta_t = \partial_a \Pi^{\mu \nu}|_N \eta_{\mu A} \epsilon_{\nu}|_N$$

which vanishes because $\Pi^{\mu \nu}|_N = 0 \Rightarrow \partial_a \Pi^{\mu \nu}|_N = 0$.

Showing that

$$\delta \eta_A = \partial_A \Pi^{\mu \nu}|_N \eta_{\mu A} \epsilon_{\nu}|_N$$

also vanishes on $N$ is more tricky, but it does, as a consequence of the Jacobi identity:
3.3 Hamiltonian study of the PSM

We proceed now to the Hamiltonian study of the model with the BC of the previous section (in each connected component of the boundary) when $\Sigma = \mathbb{R} \times [0, 1]$ (open string). The fields in the Hamiltonian formalism are a smooth map $X : [0, 1] \to M$ and a 1-form $\eta$ on $[0, 1]$ with values in the pull-back $X^*T^*M$; in coordinates, $\eta = \eta_i dX^i d\sigma$.

The space of maps $(X, \eta)$ can be given the structure of a Banach manifold by requiring $X$ to be differentiable and $\eta$ to be continuous. This infinite dimensional manifold is equipped with a canonical symplectic structure $\omega$. The action of $\omega$ on two vector fields (denoted for shortness $\delta, \delta'$) reads

$$\omega(\delta, \delta') = \int_0^1 (\delta X^i \delta' \eta_i - \delta' X^i \delta \eta_i) d\sigma. \quad (3.21)$$

The phase space $P(M; N_0, N_1)$ of the theory is defined by the constraints:

$$\partial X^i + \Pi^{ij}(X) \eta_j = 0, \ i = 1, \ldots, \dim(M). \quad (3.22)$$
and the BC: \(X(0) \in N_0\) and \(X(1) \in N_1\) for two closed submanifolds \(N_u \subset M, u = 0, 1\). We use the notation \(\partial \equiv \partial_\sigma\).

This geometry, with a boundary consisting of two connected components, raises the question of the relation between the BC at both ends. Note that due to (3.22) the field \(X\) varies within a symplectic leaf of \(M\). This implies that in order to have solutions the symplectic leaf must have non-empty intersection both with \(N_0\) and \(N_1\). In other words, only points of \(N_0\) and \(N_1\) that belong to the same symplectic leaf lead to points of \(\mathcal{P}(M; N_0, N_1)\).

In the following we shall assume that this condition is met for every point of \(N_0\) and \(N_1\) and correspondingly for the tangent spaces. That is, if we denote by \(J_0, J_1\) the maps

\[
J_0 : \mathcal{P}(M; N_0, N_1) \longrightarrow N_0 \\
(X, \eta) \longmapsto X(0)
\]

\[
J_1 : \mathcal{P}(M; N_0, N_1) \longrightarrow N_1 \\
(X, \eta) \longmapsto X(1)
\]

we are assuming that both maps are surjective submersions.

Vector fields tangent to the phase space satisfy the linearization of (3.22), i.e. \(\delta \eta_j\) and \(\delta X^i\) are such that

\[
\partial \delta X^i = \partial_j \Pi^{ki}(X) \eta_k \delta X^j + \Pi^{ij} \delta \eta_j
\]  

(3.23)

with \(\delta X(u) \in T_{X(u)}N_u, u = 0, 1\).

The solution to the differential equation (3.23) is

\[
\delta X^i(\sigma) = R^i_j(\sigma, 0) \delta X^j(0) - \int_0^\sigma R^i_j(\sigma, \sigma') \Pi^{jk}(X(\sigma')) \delta \eta_k(\sigma') d\sigma' 
\]  

(3.24)

where \(R\) is given by the path-ordered integral

\[
R(\sigma, \sigma') = P \exp \left[ \int_{\sigma'}^\sigma A(z) dz \right], \quad A^i_j(z) = (\partial_j \Pi^{ki})(X(z)) \eta_k(z)
\]

and the boundary conditions are such that \(\forall \xi_0 \in T_{X(0)}N_0^0, \forall \xi_1 \in T_{X(1)}N_1^0\),

\[
\xi_0 \partial \delta X^i(0) = 0, \\
\xi_1 \left( R^i_j(1, 0) \delta X^j(0) - \int_0^1 R^i_j(1, \sigma') \Pi^{jk}(X(\sigma')) \delta \eta_k(\sigma') d\sigma' \right) = 0. \quad (3.25)
\]

The canonical symplectic 2-form restricted to \(\mathcal{P}(M; N_0, N_1), \omega_P\), is only presymplectic. It is instructive to give a detailed calculation of its kernel. Assume \((\delta X(\sigma), \delta \eta(\sigma)) \in \text{Ker}(\omega_P)\). Then, \(\omega(\delta, \delta') = 0\) for any \((\delta X(\sigma), \delta' \eta(\sigma))\)
tangent to $\mathcal{P}(M; N_0, N_1)$. Using (3.24) and with a change in the order of integration we get:

$$
0 = -\delta' X^j(0) \int_0^1 R_j^i(\sigma, 0) \delta \eta_i(\sigma) d\sigma + 
+ \int_0^1 \left[ \delta X^i(\sigma) + \int_0^1 R^i_j(\sigma', \sigma) \Pi^{ji}(X(\sigma)) \delta \eta_k(\sigma') d\sigma' \right] \delta' \eta_i(\sigma) d\sigma.
$$

From (3.25) we deduce that there exist covectors $\xi_0 \in T_{X(0)}M$ and $\xi_1 \in T_{X(1)}M$ such that

$$
- \int_0^1 R^i_j(\sigma, 0) \delta \eta_i(\sigma) d\sigma = \xi_{0j} + \xi_{1i} R^i_j(1, 0),
$$

$$
\delta X^i(\sigma) + \int_0^1 R^i_j(\sigma', \sigma) \Pi^{ji}(X(\sigma)) \delta \eta_k(\sigma') d\sigma' = -\xi_{1k} R^k_j(1, \sigma) \Pi^{ji}(X(\sigma)).
$$

Now, defining

$$
\epsilon_j(\sigma) = \int \sigma^1 R^i_j(\sigma', \sigma) \delta \eta_i(\sigma') d\sigma' + \xi_{1i} R^i_j(1, \sigma)
$$

we can write

$$
\delta X^i(\sigma) = \Pi^{ij}(X) \epsilon_j(\sigma).
$$

Finally, differentiating (3.26) we find

$$
\delta \eta_i(\sigma) = -\partial \epsilon_i(\sigma) - \partial \Pi^{jk}(X(\sigma)) \eta_j(\sigma) \epsilon_k(\sigma).
$$

Hence, the kernel is given by:

$$
\delta \epsilon_i X^i = \Pi^{ij}(X) \epsilon_j
$$

$$
\delta \epsilon_i \eta_i = -\partial \epsilon_i - \partial \Pi^{jk}(X) \eta_j \epsilon_k
$$

(3.27)

where $\epsilon$ is subject to the BC

$$
\epsilon(u) \in \Pi^{-1}_{X(u)}(T_{X(u)} N_u) \cap (T_{X(u)} N_u)^0, \text{ for } u = 0, 1
$$

because (3.27) must be tangent to $\mathcal{P}(M; N_0, N_1)$.

As discussed in Example 2.4, the presymplectic structure induces a Poisson algebra $\mathcal{A}_\omega$ on the phase space $\mathcal{P}(M; N_0, N_1)$. On the other hand, the canonical reduction of $\Pi$ explained in Section 2.2 defines Poisson algebras in $N_0$ and $N_1$. We turn now to study the relation between them. First, we wish to figure out when a function on the phase space of the form $F(X, \eta) = f(X(0))$, $f \in C^\infty(M)$ belongs to $\mathcal{A}_\omega$, i.e. when it has a Hamiltonian vector field $\delta_F$. Solving the corresponding equation

$$
\int_0^1 (\delta_F X^i \delta' \eta_i - \delta' X^i \delta_F \eta_i) d\sigma = \partial_i f(X(0)) \delta X^i(0)
$$
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along the same lines as above one can show that the general solution is of the form (3.27) with

\[ \epsilon(0) - df_{X(0)} \in T_{X(0)}N_0^0, \quad \epsilon(0) \in \Pi^{x-1}_{X(0)}(T_{X(0)}N_0) \] (3.28)

and

\[ \epsilon(1) \in \Pi^{x-1}_{X(1)}(T_{X(1)}N_1) \cap T_{X(1)}N_1^0. \]

As we have already learnt (see proof of Theorem 2.3), equation (3.28) can be solved in \( \epsilon(0) \) if and only if \( F \) is a gauge invariant function (i.e. it is invariant under (3.27)). This is equivalent to saying that \( f + I_0 \) belongs to the Poisson algebra \( C(\Pi, M, N_0) \). (Here \( I_0 \) is the ideal of functions that vanish on \( N_0 \)).

Given two such functions \( F_1 \) and \( F_2 \) associated to the classes \( f_1 + I_0 \) and \( f_2 + I_0 \in C(\Pi, M, N_0) \) and with gauge parameter \( \epsilon_1 \) and \( \epsilon_2 \) respectively, one immediately computes the Poisson bracket \( \{F_1, F_2\}_P = \omega(\delta F_1, \delta F_2) \) to give

\[ \{F_1, F_2\}_P = \Pi^i j \epsilon_{1i}(0) \epsilon_{2j}(0) \]

This coincides with the restriction to \( N_0 \) of \( \{f_1 + I_0, f_2 + I_0\}_N \) and defines a Poisson homomorphism between \( C(\Pi, M, N_0) \) and the Poisson algebra of \( P(M, N_0, N_1) \). This homomorphism is \( J_0^* \), the pull-back defined by \( J_0 \), and the latter turns out to be a Poisson map. In an analogous way we can show that \( J_1 \) is an anti-Poisson map and besides

\[ \{f_0 \circ J_0, f_1 \circ J_1\} = 0 \quad \text{for any } f_u \in C(\Pi, M, N_u), \quad u = 0, 1. \]

The previous considerations can be summarized in the following diagram

\[ \xymatrix{ C(\Pi, M, N_0) \ar[r]^{J_0^*} & \mathcal{A}_\omega \ar[r]^{J_1^*} & C(\Pi, M, N_1) } \]

in which \( J_0^* \) is a Poisson homomorphism, \( J_1^* \) antihomomorphism and the image of each map is the commutant (with respect to the Poisson bracket) of the other. In particular it implies that the reduced phase space is finite-dimensional. This can be considered as a generalization of the symplectic dual pair to the context of Poisson algebras.

#### 3.3.1 Symplectic groupoid structure of the phase space

As we already know, a Poisson bracket on a manifold \( M \) defines a Lie bracket on the space of 1-forms on \( M \). This bracket is determined by:

(i) \[ [df, dg] = d\{f, g\}, \quad f, g \in C^\infty(M) \]

(ii) \[ [\alpha, f\beta] = f[\alpha, \beta] + \Pi^1 \beta(f)\alpha, \quad \alpha, \beta \in \Omega^1(M), \quad f \in C^\infty(M) \]
which makes $T^*M$ into a Lie algebroid over $M$ with anchor $\rho = \Pi^\sharp$.

As mentioned in the introductory chapter, Cattaneo and Felder proved in [20] that the reduced phase space of the Poisson sigma model defined on $\Sigma = [0,1] \times \mathbb{R}$ and with $X$ free at the boundary can be viewed (in the integrable case) as the symplectic groupoid for the Poisson manifold $M$. In this section we recall the basic concepts and constructions and study what happens when a second-class brane is introduced.

The global version of Lie algebroids are Lie groupoids ([18]). A Lie groupoid over a manifold $M$ is a manifold $G$ together with the following structure maps (all in the smooth category) satisfying a set of axioms. An injection $j : M \hookrightarrow G$, two surjections $l, r : G \to M$ and a product on pairs $g, h \in G$ defined only if $r(g) = l(h)$. Denote $G_{x,y} = l^{-1}(x) \cap r^{-1}(y)$. These maps must verify:

(i) $l \circ j = r \circ j = \text{id}_M$.
(ii) If $g \in G_{x,y}$ and $h \in G_{y,z}$, then $gh \in G_{x,z}$.
(iii) $j(x)g = gj(y) = g$, if $g \in G_{x,y}$.
(iv) For every $g \in G_{x,y}$ there exists $g^{-1} \in G_{y,x}$ such that $gg^{-1} = j(x)$.
(v) The product on $G$ is associative whenever it is defined.

Some examples of Lie groupoids are:

**Example 3.1.** $M$ a point and $G$ a Lie group. $j$ maps to the identity of the group and $l, r$ are trivial.

**Example 3.2.** $G$ a vector bundle over $M$, $l = r$ the projection, $j$ the zero section, the multiplication given by $(p, v_1)(p, v_2) = (p, v_1 + v_2)$ and the inverse $(p, v)^{-1} = (p, -v)$.

**Example 3.3.** $G = M \times M$. Then, $l, r$ are projections onto the left and right components respectively and $j$ the diagonal map. The multiplication is defined by $(p_1, p_2)(p_2, p_3) = (p_1, p_3)$ and the inverse by $(p_1, p_2)^{-1} = (p_2, p_1)$.

Assume that $G$ is equipped with a symplectic structure $\omega$. $G$ is called the symplectic groupoid of the Poisson manifold $M$ if the following axioms are also satisfied:

(vi) $j(M)$ is a Lagrangian (i.e. maximally isotropic) submanifold.
(vii) $l$ is a Poisson map and $r$ is anti-Poisson.
(viii) Define $G_0 = \{(g, h) \in G | r(g) = l(h)\}$ and let $\tau_1, \tau_2$ the projections onto the first and second factor. If $\mu : G_0 \to G$ denotes the product on $G_0$, then $\mu^* \omega = \tau_1^* \omega + \tau_2^* \omega$.
(ix) The inverse map is anti-Poisson.

The basic example of a symplectic Lie groupoid is:
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Example 3.4. $M = g^*$ the dual of a Lie algebra with the Kostant-Kirillov Poisson structure. Define $j : g^* \to T^* G$ as the inclusion for any Lie group integrating the Lie algebra $g$. The maps $l, r : T^* G \to g^*$ are naturally defined by left (right) translations to the cotangent space at the identity of $G$. Take two elements $(g_1, \xi_1)$ and $(g_2, \xi_2)$ such that $r(g_1, \xi_1) = l(g_2, \xi_2)$. The multiplication law is given by $(g, \xi) = (g_1, \xi_1)(g_1, \xi_1)$ with $g = g_1g_2$ and $\xi = (dR_\epsilon(g^*)^{-1}\xi_1 = (dL_g(h^*)^{-1}\xi_2$.

The question of integrability for general Lie algebroids was solved by Crainic and Fernandes in [27], much inspired by the work of Cattaneo and Felder [20] on the particular case of Poisson manifolds. They proved that the reduced phase space of the PSM, $\mathcal{G} := P(M; M, M)/\sim$ (where the quotient is taken with respect to the gauge transformations (3.27) with $\epsilon$ vanishing at the boundary) is the symplectic groupoid of the Poisson manifold $M$ (in the integrable case, or equivalently, when $\mathcal{G}$ is a manifold).

The algebraic groupoid structure is defined by composition of paths. One can show (20) that in each equivalence class $[(X, \eta)]$ in $\mathcal{G}$ there exists a representative with $\eta(0) = \eta(1) = 0$. Then, the composition law $[(X, \eta)] = [(X_1, \eta_1)][(X_2, \eta_2)]$ is given by choosing such representatives and

$$X(\sigma) = \begin{cases} X_1(2\sigma), & 0 \leq \sigma \leq \frac{1}{2} \\ X_2(2\sigma - 1), & \frac{1}{2} \leq \sigma \leq 1 \end{cases}$$

$$\eta(\sigma) = \begin{cases} 2\eta_1(2\sigma), & 0 \leq \sigma \leq \frac{1}{2} \\ 2\eta_2(2\sigma - 1), & \frac{1}{2} \leq \sigma \leq 1 \end{cases}$$

as long as $X_1(1) = X_2(0)$.

In this case the remaining objects entering the definition of a symplectic groupoid are as follows. The map $j$ sends a point $x \in M$ to the class of the constant solution $X(\sigma) = x, \eta(\sigma) = 0$ and the maps $r, l$ give the values of $X$ at the endpoints. Namely, $l(X, \eta) = X(0), r(X, \eta) = X(1)$. The symplectic structure is provided by (3.21), which is obviously symplectic on $\mathcal{G}$. We refer the reader to [20] for the explicit proof that $\mathcal{G}$ satisfies (i)-(ix).

It would be interesting to investigate what happens when we set a second-class brane at both endpoints of the string. Hence, let us take $N$ a second-class brane. $\mathcal{P}(M; N, N)$ is given by paths

$$\partial X^i + \Pi^i_j(X)\eta_j = 0, \quad i = 1, \ldots, \dim(M).$$

(3.29)

satisfying $X^A(0) = X^A(1)$ in adapted coordinates. The characteristic foliation of $\omega$ restricted to $\mathcal{P}(M; N, N)$ is of the form (3.27) with $\epsilon(0) = \epsilon(1) = 0$. We want to study the structure of the reduced phase space $\mathcal{P}(M; N, N)/\sim$. 
Locally, one can choose $\epsilon_A(\sigma)$ so that $X^A(\sigma) = 0, \forall \sigma \in [0, 1]$. This is easily seen by writing
\[
\delta_\epsilon X^A = \Pi^{An} \epsilon_n + \Pi^{AB} \epsilon_B
\tag{3.30}
\]
and recalling that $\Pi^{AB}$ is invertible in a neighborhood of $N$.

The characteristic foliation is tangent to $\mathcal{P}(M; N, N)$ and a representative satisfying $X^A(\sigma) = 0$ still must obey (3.29), which implies
\[
\eta_A = -\omega_{AB} \Pi^{Ba} \eta_a
\tag{3.31}
\]
which gives, for lower-case indices:
\[
\partial X^a = -\Pi^{ab} \eta_b
\tag{3.32}
\]
where $\Pi_D$ is the Dirac bracket (2.11).

The remaining freedom in the choice of representative is given by gauge transformations which preserve $X^A(\sigma) = 0$. This corresponds to those $\epsilon$ verifying $\epsilon_A = -\omega_{AB} \Pi^{Ba} \epsilon_a$ and leaves the freedom for lower-case indices:
\[
\delta_\epsilon X^a = \Pi^{ab} \epsilon_b
\tag{3.33}
\]

Analogously, one can check that the remaining freedom for $\eta_a$ is
\[
\delta_\epsilon \eta_a = -\partial \epsilon_a - \partial_n \Pi^{bc} \eta_b \epsilon_c
\tag{3.34}
\]

Hence, from (3.32), (3.33), (3.34) we deduce that $\mathcal{P}(M; N, N)/\sim$ can be viewed (if smooth), up to topological obstructions, as the symplectic groupoid integrating the brane equipped with the Dirac bracket. Not only it might happen that $\Pi^{AB}$ be degenerate for paths far enough from the brane, but the obstructions might appear in a different, more essential way. Notice that if the intersection of $N$ with a symplectic leaf is disconnected, there exist paths with can not be taken to lie on $N$ by means of gauge transformations. Below we give an example of a Poisson manifold and a second-class brane where this obstacle shows up.

**Example 3.5.** Consider $M = \mathbb{R}^3$ with coordinates $(x, y, z)$ and Poisson structure defined by
\[
\{x, z\} = x, \quad \{y, z\} = y, \quad \{x, y\} = 0
\]

The symplectic leaves of this Poisson structure are the level sets of the Casimir function $\theta = \arctan(y/x)$. Take $N$ a closed curve (without self-intersections) winding twice around the $z$ axis. It is clear that $N$ can be chosen so that it is second-class and has disconnected intersection with some symplectic leaves.
Chapter 4

Branes in the quantum Poisson sigma model
In [19] Cattaneo and Felder gave a field theoretical interpretation of Kontsevich’s formula ([19]) for the deformation quantization of a Poisson manifold. They showed that Kontsevich’s formula can be obtained from Feynman expansion of certain Green’s functions of the PSM when $\Sigma$ is the unit disk $D$ and the base map $X : \Sigma \to M$ has free boundary conditions.

We have proven in Section 3.2 that classically the field $X$ can be consistently restricted at the boundary $\partial \Sigma$ to an almost arbitrary submanifold $N$. Then, we have shown in Section 3.3 that the symplectic structure on the reduced phase space of the model is related to the Poisson bracket canonically induced on $(a$ subset of) $C^\infty(N)$.

On the light of these results it is natural to conjecture that the perturbative quantization of the model with general $N$ be related to the deformation quantization of the induced Poisson bracket on (certain functions on) $N$. We shall work out in detail the case in which $N$ can be defined by a set of second-class constraints which is, in some sense, opposite to the coisotropic one. The quantization of the coisotropic case ([22]) presents some intricacies due to the fact that gauge transformations do not vanish at the boundary (see Section 4.2.5). If $N$ is defined by second-class constraints (second-class brane) they do vanish and one would expect to have a clean quantization recovering Kontsevich’s formula, this time not for $\Pi$ but for the Dirac bracket on $N$. We show that this expected result holds and that it emerges in quite a different way from the coisotropic case. Finally, we shall give the quantization of the PSM with a general brane defined by a mixture of first and second class constraints.

### 4.1 Quantization of gauge theories

The content of this section is now standard material in modern Quantum Field Theory. However, the Batalin-Vilkovisky scheme, needed for the quantization of gauge theories with open algebras like the PSM, is much less known than its simplified version for closed algebras (the standard BRST method). That is why we find it useful to give a brief introduction to this topic. The exposition is based on the excellent reference ([28]).

#### 4.1.1 Gauge algebras

Let $\varphi^i$ be the fields of our theory and $S[\varphi]$ the action functional. With the notation

$$y_i(\varphi) = \frac{\delta S}{\delta \varphi^i}$$

the stationary points of $S$ are given by the field equations $y_i = 0, \forall i$. 
4.1. Quantization of gauge theories

A symmetry transformation of the action $S$ is given by a set of independent operators $R^i_\alpha(\varphi)$ verifying

$$y_i(\varphi)R^i_\alpha(\varphi)\epsilon^\alpha = 0 \quad (4.1)$$

where we adopt the DeWitt notation, understanding an integration in space-time whenever there is a summation over $i$.

If (4.1) holds only for constant $\epsilon^\alpha$ we have a global symmetry. If it holds for space-time dependent $\epsilon^\alpha$ we talk about a gauge symmetry and the summation over $\alpha$ includes an integration in space-time. The problems when quantizing a theory with gauge symmetries manifest in several ways. For instance, the quadratic part of the action is not invertible and the perturbative expansion is not well-defined. From a path integral point of view, the source of the problems is that when naively integrating over the fields, many physically equivalent configurations are added in a redundant way and one cannot make sense of the resulting expressions. Hence, some prescriptions for the quantization of gauge theories must be developed.

We shall assume that our set of gauge generators $R^i_\alpha(\varphi)$ is complete in the following sense:

$$y_i(\varphi)f^i(\varphi) = 0 \Rightarrow f^i(\varphi) = R^i_\alpha(\varphi)\epsilon^\alpha + y_j(\varphi)M^{ij}(\varphi) \quad (4.2)$$

for some $M^{ij}(\varphi)$ such that $M^{ij}(\varphi) = (-1)^{\epsilon_i\epsilon_j+1}M^{ji}(\varphi)$, where $\epsilon_i$ is the Grassmann parity of $\varphi^i$.

Denote by $\epsilon_\alpha$ the Grassmann parity of $\epsilon^\alpha$. Due to (4.2) the most general form of the graded commutator of two gauge transformations is given by

$$\frac{\epsilon^-}{\delta \varphi^j} R^i_\alpha R^j_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta}\frac{\delta R^i_\alpha}{\delta \varphi^j} R^j_\beta = 2(-1)^{\epsilon_\alpha\epsilon_\gamma} R^i_\alpha T^\gamma_{\alpha\beta} - 4(-1)^{\epsilon_i+\epsilon_\alpha}y_j E^j_{\alpha\beta} \quad (4.3)$$

for some $T^\gamma_{\alpha\beta}(\varphi)$ and $E^j_{\alpha\beta}(\varphi)$. The signs and numerical factors have been chosen for later convenience. If $E^j_{\alpha\beta}(\varphi) = 0$ one calls the gauge algebra closed. If $E^j_{\alpha\beta}(\varphi) \neq 0$ the gauge algebra is called open.

4.1.2 BRST quantization

The idea of the BRST method ([11],[69]) is to enlarge the space of fields so that the resulting theory has no gauge symmetries and the integration over field configurations is well-defined. At the end of the day, one should be able to eliminate the states associated to the fields which were not present in the original theory. Let us formulate these ideas in a precise way.

Besides the $\mathbb{Z}_2$ gradation corresponding to the Grassmann parity we introduce a $\mathbb{Z}$ gradation called ghost number. The action must have even Grassmann parity and ghost number zero.
For every gauge generator we introduce a *ghost field* \( c^\alpha \), an *antighost field* \( b_\alpha \) and a Lagrange multiplier \( \lambda_\alpha \). The Grassmann parities are \( \varepsilon_{c^\alpha} = \varepsilon_{b_\alpha} = \varepsilon_\alpha + 1 \), \( \varepsilon_{\lambda_\alpha} = \varepsilon_\alpha \). Finally, we assign ghost numbers \( \text{gh}(\varphi^i) = \text{gh}(\lambda_\alpha) = 0 \), \( \text{gh}(c^\alpha) = 1 \), \( \text{gh}(b_\alpha) = -1 \).

The essential object in the BRST formalism is an odd (right) derivation of ghost number one \( \delta \). It is defined by its action on the basic fields:

\[
\begin{align*}
\delta \varphi^i &= R^i_\alpha(\varphi)c^\alpha \\
\delta c^\alpha &= T^\alpha_{\beta\gamma}(\varphi)c^\beta c^\gamma \\
\delta b_\alpha &= \lambda_\alpha \\
\delta \lambda_\alpha &= 0
\end{align*}
\]

and extendend to any functional of the fields through the Leibniz rule, \( \delta(F_1 F_2) = F_1 \delta F_2 + (-1)^{\varepsilon_{F_2}} \delta F_1 F_2 \). It is straightforward to check that \( \delta^2 = 0 \) for closed gauge algebras.

Now, given a set of admissible gauge-fixing conditions \( F^\alpha(\varphi) \) we define the gauge-fixing fermion \( \Psi = b_\alpha F^\alpha \). The gauge fixed action is defined as

\[
S_{gf} = S + \delta \Psi.
\]

\( S_{gf} \) has no gauge symmetries and can be used for path integral quantization of the theory. Notice that the original local symmetry has been replaced by the global BRST symmetry given by \( \delta \). Recall, however, that this has been done at the expense of introducing spurious fields (which would translate into spurious states in a Hilbert space formulation) and we should give some prescription to get rid of them after quantizing. The key point is the nilpotency of \( \delta \), which allows to define a cohomology on the space of fields. A gauge invariant observable translates into a non-trivial cohomology class of \( \delta \) at ghost number zero.

Notice that this construction fails when dealing with open algebras because \( \delta \) is not nilpotent anymore. Instead, \( \delta^2 \varphi^i \) is proportional to the field equations. The appropriate procedure in this case is given by the Batalin-Vilkovisky quantization method. In the next section we discuss in detail how the quantization of gauge theories with *closed* algebras fits into the BV scheme. Then, we shall be ready to give the suitable prescriptions for open algebras, which is our main objective.

### 4.1.3 BV formalism

Assume we have a gauge theory with *closed* gauge algebra given by a classical action \( S \). Let us denote by \( \varphi^A \) the original fields \( \varphi^i \), the ghosts \( c^\alpha \), the antighosts \( b_\alpha \) and the Lagrange multipliers \( \lambda_\alpha \). Their BRST transformation rules \( \delta \) are summarized as

\[
\delta \varphi^A = R^A(\varphi^B).
\]

\(^1\)More precisely, by \( \delta := \hat{\delta} \).
In the BV formalism ([4],[6],[5],[7],[8]) we double the number of fields by introducing an antifield $\varphi^+_A$ for each field $\varphi^A$ such that:

(i) $\varphi^+_A$ has Grassmann parity opposite to that of $\varphi^A$

(ii) $gh(\varphi^+_A) = -1 - gh(\varphi^A)$.

We define the BV action by

$$S_{BV}^0[\varphi,\varphi^+] = S[\varphi^i] + \varphi^+_A R^A(\varphi).$$

Finally, given a gauge-fixing fermion $\Psi$ the partition function is taken to be

$$Z = \int e^{i S_{BV}^0[\varphi,\varphi^+] - \delta (\varphi^+_A - \overleftarrow{\delta} \Psi \delta \varphi^A)} D\varphi D\varphi^+.$$ (4.6)

It is easy to show that after integrating over the antifields one is left with the partition function defined by the action $S_{BV}^0$, hence recovering the BRST quantization formalism.

The essential object in the BV formalism, which allows its generalization for the quantization of theories with open gauge algebras, is the *Batalin-Vilkovisky antibracket* for functionals of the fields and antifields:

$$(F,G) = \left< \frac{\overleftarrow{\delta} F}{\delta \varphi^A} \frac{\overrightarrow{\delta} G}{\delta \varphi^+_A} - \frac{\overleftarrow{\delta} F}{\delta \varphi^+_A} \frac{\overrightarrow{\delta} G}{\delta \varphi^A} \right>.$$

The construction we have given for closed gauge algebras ensures that the classical master equation

$$(S_{BV}^0, S_{BV}^0) = 0$$

is satisfied. It might be necessary to add quantum corrections to $S_{BV}^0$ in order to keep the BRST symmetry at the quantum level. It is possible to show that this is equivalent to impose on $S_{BV} = S_{BV}^0 + O(\hbar)$ that it be a solution of the quantum master equation

$$(S_{BV}, S_{BV}) - 2i\hbar \Delta S_{BV} = 0$$ (4.7)

where

$$\Delta F = (-1)^{\epsilon_A+1} \frac{\overleftarrow{\delta}}{\delta \varphi^+_A} \frac{\overleftrightarrow{\delta}}{\delta \varphi^A} F.$$

For open algebras the prescription is as follows. Take the BRST transformations ([4],[3]). The starting point is

$$S_{BV}^0 = S + \varphi^+_A \delta \varphi^A.$$
4.2 Perturbative quantization of the PSM on the disk

Take $\Sigma$ the unit disk $D = \{ \sigma \in \mathbb{R}^2, |\sigma| \leq 1 \}$. Cattaneo and Felder showed in [19] that the perturbative expansion of certain Green's functions of the PSM defined on $D$ with $N = M$ (free BC) yields the Kontsevich $\star$-product corresponding to the Poisson manifold $M$, namely

$$
\langle f(X(p_1))g(X(p_2))\delta(X(p_3) - x) \rangle = f \star g(x)
$$

(4.8)

where $p_1, p_2$ and $p_3$ are three cyclically ordered points at the boundary of $D$, $x \in M$, and the expectation value is calculated using (4.6). In a later work [22] the same authors studied the quantization with a general coisotropic brane $N$, which turned out to be related with the deformation quantization of the submanifold $N \hookrightarrow M$.

In both papers [19], [22] the Green’s functions (4.8) are worked out in the same fashion: first, one takes the Lorentz gauge $d \ast \eta = 0$, where the Hodge operator acting on 1-forms requires the introduction of a complex structure on $D$. The Feynman expansion in powers of $\hbar$ is then performed around the constant classical solution where $X(\sigma) = x \in N$ and the rest of the fields vanish. In this case expanding in powers of $\hbar$ amounts to expanding in powers of $\Pi$ or, equivalently, to expanding around zero Poisson structure.

The next section is a review of the seminal work [19]. Then, we shall try to work out (4.8) following the steps enumerated in the last paragraph when $N$ is non-coisotropic. We shall see that when second-class constraints are present the propagator does not exist, but a natural redefinition of the unperturbed (or quadratic) part of the action will yield a well-defined perturbative expansion, showing that the non-coisotropic branes also make sense at the quantum level. However, this leads to a messy expression whose interpretation is far from clear. In Section 4.2.3 we shall see that a change of the gauge fixing is illuminating in order to unravel the relation between the quantization of the PSM with a non-coisotropic brane and the Kontsevich formula. In Section 4.2.4 we deal with the quantization of a general strongly regular brane. Section 4.2.5 is a summary of the results of [22] on the quantization of coisotropic branes.

4.2.1 Quantization with free boundary conditions. The Kontsevich formula

As we already know, the gauge algebra of the PSM is open (see (3.9)) and we must use the BV techniques learnt in Section 4.1.3 to formulate the quantum
theory. The BRST transformations (4.4) in the case of the PSM model read

\[ \delta X^i = \Pi^{ij}(X) \beta_j \]
\[ \delta \eta_i = -d \beta_i - \partial_i \Pi^{jk}(X) \eta_j \beta_k \]
\[ \delta \beta_i = \frac{1}{2} \partial_i \Pi^{jk}(X) \beta_j \beta_k \]
\[ \delta \gamma^i = \lambda^i \]
\[ \delta \lambda^i = 0 \]

where \( \beta_i \) are the ghost fields, \( \gamma^i \) the antighost fields and \( \lambda^i \) the Lagrange multipliers. Applying the machinery of Section 4.1.3 one gets the BV action for the PSM (19)

\[ S_{BV} = \int_D \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j + X^i_+ \Pi^{ij}(X) \beta_j - \]
\[ \eta^i \wedge (d \beta_i + \partial_i \Pi^{kl}(X) \eta_k \beta_l) - \frac{1}{2} \beta^i \partial_i \Pi^{jk}(X) \beta_j \beta_k - \]
\[ \frac{1}{4} \eta^i \wedge \eta^{i+} \partial_i \partial_j \Pi^{kl}(X) \beta_k \beta_l - \lambda^i \gamma^i_+ \]  

(4.10)

Let us take the Lorentz gauge \( d* \eta = 0 \). The gauge fixing fermion is then

\[ \Psi = \int_D \gamma^i d* \eta_i \]

and the gauge fixed action with the antifields integrated out becomes

\[ S_{gf} = \int_D \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - \star d \gamma^i \wedge (d \beta_i + \partial_i \Pi^{kl}(X) \eta_k \beta_l) - \]
\[ \star d \gamma^i \wedge \star d \gamma^j \wedge \partial_i \partial_j \Pi^{kl}(X) \beta_k \beta_l - \lambda^i \star d \eta_i. \]  

(4.11)

Now write \( X^i(\sigma) = x^i + \xi^i(\sigma) \) and choose

\[ S_0 = \int_D \eta_i \wedge d \xi^i - \star d \gamma^i \wedge d \beta_i - \star d \eta_i \lambda^i \]  

(4.12)

as the quadratic part which defines the propagators. This is the perturbative expansion defined in [19] in order to make contact with Kontsevich’s formula. Therein the field \( X \) is taken to be free at the boundary, i.e. the brane is the whole manifold \( M \) and consequently the contraction of \( \eta \) with vectors tangent to \( \partial D \) must vanish. The fields \( \beta_i, \gamma_i \) and \( \lambda^i \) must also vanish at the boundary.

\[ ^{2}\text{The path integral quantization of the PSM in the particular case of 2D gravity was first carried out in [51].} \]
4.2. Perturbative quantization of the PSM on the disk

It is convenient to map conformally the disk onto the upper complex half plane $H_+$ (recall the conformal invariance of $S_{gf}$) and use a complex coordinate $z \in H_+$. The boundary of the disk is mapped onto the (compactified) real line.

The equations defining the propagators are obtained as Schwinger-Dyson equations of the form

$$\frac{\delta}{\delta \varphi^i(w, \bar{w})} \int \varphi^i(z, \bar{z}) e^{iS_0} D\varphi = 0 \quad (4.13)$$

where $\varphi^i$ stands for any field entering the gauge-fixed action $S_{gf}$. For example, taking $\varphi^i(z, \bar{z}) = \beta_i(z, \bar{z})$, $\varphi^j(w, \bar{w}) = \beta_j(w, \bar{w})$, we get

$$\partial_w \partial_{\bar{w}} \langle \gamma^j(w, \bar{w}) \beta_i(z, \bar{z}) \rangle_0 = \frac{ih}{4} \delta^j_i \delta^2(w - z).$$

Hence, the propagators for the $\beta_i$ and $\gamma^j$ fields are given by the Green’s function of the Laplacian with Dirichlet boundary conditions. Namely,

$$\langle \gamma^j(w, \bar{w}) \beta_j(z, \bar{z}) \rangle_0 = \frac{ih}{2\pi} \delta^j_i \log \left| \frac{w - z}{w - \bar{z}} \right|$$

with $\partial_{\bar{z}} \frac{1}{z} = \pi \delta^2(z)$.

The other non vanishing components of the propagator are conveniently expressed in terms of the complex fields $\zeta^i = \xi^i + i\lambda^i$ and $\bar{\zeta}^i = \xi^i - i\lambda^i$. The relevant Schwinger-Dyson equations (4.13) yield

$$\partial_z \langle \bar{\zeta}(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 - \partial_{\bar{z}} \langle \zeta(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = -i\hbar \delta^2(w - z)$$

$$\partial_z \langle \zeta(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 - \partial_{\bar{z}} \langle \bar{\zeta}(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = 0$$

$$\partial_{\bar{w}} \langle \zeta(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = -\frac{\hbar}{2} \delta^2(w - z)$$

$$\partial_{\bar{w}} \langle \bar{\zeta}(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = \frac{\hbar}{2} \delta^2(w - z)$$

$$\partial_z \langle \zeta(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 + \partial_{\bar{z}} \langle \bar{\zeta}(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = 0$$

$$\partial_z \langle \bar{\zeta}(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 + \partial_{\bar{z}} \langle \zeta(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = 0$$

$$\partial_{\bar{w}} \langle \bar{\zeta}(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = 0$$

$$\partial_{\bar{w}} \langle \zeta(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = 0.$$

It is not difficult to show that the general solution (i.e. before imposing the boundary conditions) is:

$$\langle \zeta^j(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = \frac{\hbar}{2\pi} \delta^j_i \left( -\frac{dz}{w - z} + f_j(w, z) dz + g_j(w, \bar{z}) d\bar{z} \right)$$
4.2. Perturbative quantization of the PSM on the disk

\[ \langle \bar{\zeta}^i(w, \bar{w}) \eta_j(z, \bar{z}) \rangle_0 = \frac{\hbar}{2\pi} \delta^i_j \left( \frac{d\bar{z}}{\bar{w} - \bar{z}} + \bar{f}_j(\bar{w}, \bar{z}) d\bar{z} + \bar{g}_j(\bar{w}, z) dz \right) \]  

(4.14)

where no sum in \( j \) is assumed and \( f_j, \bar{f}_j, g_j, \bar{g}_j \) are holomorphic in their arguments with domains given by \( w, z \in H_+ \). The boundary conditions imply \( f_i = \bar{f}_i = 0 \) and

\[ g_i(w, \bar{z}) = \frac{1}{w - \bar{z}}, \quad \bar{g}_i(\bar{w}, z) = -\frac{1}{\bar{w} - w}. \]

Define \( d_z := dz \partial / \partial z + d\bar{z} \partial / \partial \bar{z} \). We can rewrite the propagators in terms of the Green’s function of the Laplacian

\[ \psi(z, w) = \log \left| \frac{z - w}{\bar{z} - \bar{w}} \right| \]

and the Kontsevich’s angle function (recall (2.25))

\[ \phi(z, w) = \frac{1}{2i} \log \left( \frac{z - w}{z - \bar{w}} \right) \left( \frac{\bar{z} - \bar{w}}{\bar{z} - w} \right). \]

This will be convenient to identify the different ingredients of the Kontsevich formula in the perturbative expansion.

Thus we have

\[ \langle \gamma^k(w) \beta_j(z) \rangle_0 = \frac{i\hbar}{2\pi} \delta^k_j \psi(z, w) \]

\[ \langle \xi^k(w) \eta_j(z) \rangle_0 = \frac{i\hbar}{2\pi} \delta^k_j d_z \phi(z, w) \]

\[ \langle \lambda^k(w) \eta_j(z) \rangle_0 = -\frac{i\hbar}{2\pi} \delta^k_j d_z \psi(z, w) \]

Observing that \( *d \psi(z, w) = dw \phi(z, w) \), so that

\[ \langle *d \gamma^k(w) \beta_j(z) \rangle_0 = \delta^k_j i\hbar dw \phi(z, w) \]

it follows that the propagators can be combined into a superpropagator

\[ \langle \xi^k(w) \eta_j(z) \rangle_0 + \langle *d \gamma^k(w) \beta_j(z) \rangle_0 = \frac{i\hbar}{2\pi} \delta^k_j D \phi(z, w) \]

where \( D := d_z + dw \). In terms of superfields \( \tilde{\eta}_j(z, \theta) = \beta_j(z) + \theta^\mu \eta_{j\mu}(z) \) and \( \tilde{\xi}^k(w, \rho) = \xi^k(w) + \rho^\mu \eta_{j\mu}^k(w) \) with \( \eta^{+j} = *d \gamma^j \), the superpropagator reads

\[ \langle \tilde{\xi}^k(w, \rho) \tilde{\eta}_j(z, \theta) \rangle_0 = \frac{i\hbar}{2\pi} \delta^k_j D \phi(z, w) \]

\(^3\text{When no confusion is possible we shall simplify the notation related to the variables on which the complex functions depend. Hence, we shall write } \beta_j(z) \text{ instead of } \beta_j(z, \bar{z}) \text{, and so on.}\)
4.2. Perturbative quantization of the PSM on the disk

with \( D := \theta^\mu \frac{\partial}{\partial \xi^\mu} + \rho^m u \frac{\partial}{\partial \omega^m} \).

The perturbative series is defined by taking \( S_{gf} = S_0 + S_1 \) and expanding:

\[
\langle O \rangle = \int e^{i S_{gf}} O = \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!} \int e^{i S_0} (S_1)^n O.
\]

The Feynman expansion is obtained by expanding \( S_1 \) and the observable \( O \) in powers of \( \tilde{\xi}, \tilde{\eta} \), giving the vertices

\[
S_1 = \frac{1}{2} \int_D \int d^2 \theta \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{j_1} \ldots \partial_{j_k} \Pi^{lr}(x) \tilde{\xi}^{j_1} \ldots \tilde{\xi}^{j_k} \tilde{\eta}^l \tilde{\eta}^r.
\]

Take three cyclically ordered points at \( \partial D \), 0, 1 and \( \infty \). The perturbative expansion of the expectation value of

\[
O = f(\tilde{X}(0))g(\tilde{X}(1))\delta(X(\infty) - x)
\]

reproduces Kontsevich formula for \( \Pi \) except for the presence of tadpoles, i.e. diagrams containing at least an edge which starts and ends at the same vertex. The divergence of the Poisson tensor, \( \partial_i \Pi^{ij} \), and the ill-defined expression \( d\phi(z, z) \) enter into the tadpole diagrams. However, it is easy to regularize these terms by a point-splitting procedure defining

\[
d_z \phi(z, z) = \kappa(z; \tau) := \lim_{\epsilon \to 0} d_z \phi(z, z + \epsilon \tau(z))
\]

where \( \tau(z) \) is a vector field which does not vanish in the interior of the disk. The limit exists but depends on \( \tau(z) \). Writing \( \tau(z) = r(z)e^{i\alpha(z)} \) in polar coordinates, then \( \kappa(z; \tau) = da(z) \). Choosing \( \alpha \) constant we can get rid of the tadpole diagrams\(^4\) and the Kontsevich formula is obtained.

4.2.2 Perturbation expansion for non-coisotropic branes

Now, our aim is to show that the perturbative expansion defined in the previous section makes sense only for coisotropic branes (recall that \( N = M \) is coisotropic) and breaks down when second-class constraints are present.

In other words, we want to show that if the brane \( N \) is not coisotropic the propagator defined by (4.12) cannot fulfill the appropriate boundary conditions.

In the adapted coordinates of Section 2.1 the index \( i \) splits into \( a, \mu, A \) where \( \xi^a \) are coordinates along the brane (free at the boundary) and \( \xi^\mu \) and \( \xi^A \) are respectively first-class and second-class coordinates transverse to the brane and must vanish at the boundary. For the rest of the fields we have the following boundary conditions,

\[^4\]Equivalently, one can add a counterterm proportional to \( \partial_i \Pi^{ij} \) an renormalize the tadpoles to zero.
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- Dirichlet: $\lambda^a, \lambda^A, \eta_{at}, (\eta_\mu)_t, \eta_{At}, \beta_a, \beta_A, \gamma^a$ and $\gamma^A$.
- Neumann: $\beta_\mu, \lambda^H$ and $\gamma^H$.

The propagators for the $\beta$ and $\gamma$ fields are given by the Green’s function of the Laplacian with appropriate boundary conditions:

\[
\langle \gamma^a(w, \bar{w}) \beta_b(z, \bar{z}) \rangle_0 = \frac{i\hbar}{2\pi} \delta^a_b \log \frac{|w - z|}{|w - \bar{z}|},
\]

\[
\langle \gamma^\mu(w, \bar{w}) \beta_\nu(z, \bar{z}) \rangle_0 = \frac{i\hbar}{2\pi} \delta^\mu_\nu \log(|w - z|)|w - \bar{z}|,
\]

\[
\langle \gamma^A(w, \bar{w}) \beta_B(z, \bar{z}) \rangle_0 = \frac{i\hbar}{2\pi} \delta^A_B \log \frac{|w - z|}{|w - \bar{z}|}.
\]

(4.15)

The problem comes when we try to impose the boundary conditions to the remaining components of the propagator (4.14). It is straightforward to prove that the boundary conditions imply $f_a = f_\mu = \bar{f}_a = \bar{f}_\mu = 0$ and

\[
g_a(w, \bar{z}) = -g_\mu(w, \bar{z}) = \frac{1}{w - \bar{z}}, \quad \bar{g}_a(\bar{w}, z) = -\bar{g}_\mu(\bar{w}, z) = \frac{-1}{\bar{w} - z}.
\]

However, if we try to fulfill the boundary conditions for the components corresponding to the second-class constraints ($A, B, \ldots$ indices) we find a contradiction. Let us show this in detail.

We shall omit the second-class subscripts for simplicity. The BC for $\eta_t$ implies

\[
\langle \zeta(w, \bar{w})(\eta_z(z, \bar{z}) + \eta_{\bar{z}}(z, \bar{z})) \rangle_{|z=\bar{z}} = 0
\]

\[
\langle \zeta(w, \bar{w})(\eta_z(z, \bar{z}) + \eta_{\bar{z}}(z, \bar{z})) \rangle_{|z=\bar{z}} = 0
\]

which translates into

\[
g(w, \bar{z}) = \frac{1}{w - \bar{z}} - h(w, \bar{z}), \text{ s.t. } h(w, u) = f(w, u), u \in \mathbb{R},
\]

\[
\bar{g}(\bar{w}, z) = -\frac{1}{\bar{w} - z} - \bar{h}(\bar{w}, z), \text{ s.t. } \bar{h}(w, u) = \bar{f}(w, u), u \in \mathbb{R}.
\]

(4.16)

Dirichlet BC on second-class components of the field $X$ impose

\[
\langle (\zeta(w, \bar{w}) + \bar{\zeta}(w, \bar{w}))\eta_z(z, \bar{z}) \rangle_{|w=\bar{w}} = 0
\]

\[
\langle (\zeta(w, \bar{w}) + \bar{\zeta}(w, \bar{w}))\eta_{\bar{z}}(z, \bar{z}) \rangle_{|w=\bar{w}} = 0
\]

so that

\[
\bar{g}(\bar{w}, z) = \frac{1}{\bar{w} - z} - \bar{h}'(\bar{w}, z), \text{ s.t. } \bar{h}'(u, z) = \bar{f}(u, z), u \in \mathbb{R},
\]

\[
g(w, \bar{z}) = -\frac{1}{w - \bar{z}} - \bar{h}'(w, \bar{z}), \text{ s.t. } \bar{h}'(u, \bar{z}) = \bar{f}(u, \bar{z}), u \in \mathbb{R}.
\]

(4.17)
4.2. Perturbative quantization of the PSM on the disk

From (4.16) and (4.17) we deduce that \( f \) and \( \bar{f} \) extend to entire functions in \( z \) and \( w \) and consequently they are constant. In addition, also from (4.16) and (4.17) we derive the relation

\[
\bar{f}(w, \bar{z}) - f(w, \bar{z}) = -\frac{2}{w - \bar{z}}
\]

which is obviously impossible if \( f \) and \( \bar{f} \) are entire. Hence, the propagator does not exist.

At this point one might be tempted to conclude that only coisotropic branes make sense at the quantum level. But this is a too sloppy conclusion since we have only shown that the perturbative expansion defined by the choice of (4.12) as the unperturbed part ceases to exist when second-class constraints appear. The question is whether there is a different definition of the perturbative expansion leading to a well-defined result in this case. The situation is reminiscent to the contradiction found by Dirac in imposing the second-class constraints on the states of the physical Hilbert space; he proposed to circumvent this difficulty with the help of the Dirac bracket [30].

From now on we shall restrict to branes which satisfy the strong regularity condition (2.8) and, in order to make the presentation simpler, we shall assume that there are no first-class constraints, i.e. we restrict to second-class branes.

The strategy to solve the problem is the well-known technique of using our opponent’s strength against him. The origin of the non-existence of the propagator for the second-class coordinates is that \( \det(\Pi^{AB}) \neq 0 \) implies that if \( X^A = 0 \) at the boundary then \( \eta_A \) must also vanish. And the propagator cannot satisfy these two conditions simultaneously. But precisely due to the fact that \( \det(\Pi^{AB}) \neq 0 \) and given that the \( \eta_A \) fields appear at most quadratically in the gauge fixed action (4.11) we can perform the Gaussian integration over them in order to get an effective action \( S^{\text{eff}} \). This action can be used to compute the correlation functions of observables that do not involve \( \eta_A \) fields as it is our case.

Once the integration has been performed there is a splitting of \( S^{\text{eff}} \) which defines a consistent perturbative expansion. Take \( S^{\text{eff}} = S_0^{\text{eff}} + S_{\text{pert}}^{\text{eff}} \) with

\[
S_0^{\text{eff}} = \int_D \eta_a \wedge d\xi^a - d*\eta_a \lambda^a + \omega_{AB}(x) d\xi^A \wedge *d\lambda^B - *d\gamma^i \wedge d\beta_i.
\]

The \( \beta, \gamma \) propagators are as before (see eq. (4.15)), as well as those for \( \zeta^a \) and \( \eta_a \). In addition, \( S_0^{\text{eff}} \) yields well-defined propagators for the other fields, the only non-zero components being

\[
\langle \lambda^A(w, \bar{w}) \xi^B(z, \bar{z}) \rangle_0^{\text{eff}} = \frac{i\hbar}{2\pi} \Pi^{AB}(x) \log \frac{|w - z|}{|w - \bar{z}|}.
\]
4.2. Perturbative quantization of the PSM on the disk

Now one can expand $S_{\text{eff}}^{\text{pert}}$ into vertices and define a perturbative expansion for Green’s functions of the form (4.8). However, from the resulting perturbative series it seems very hard to find out whether the formula (4.8) defines an associative product. A simpler derivation providing a positive answer is given in the next section.

4.2.3 Second-class branes and the Kontsevich formula for the Dirac bracket

Let us take advantage of our opponent’s strength in a more profound sense. Using that $\Pi^{AB}$ is invertible at every point of $N$, and consequently in a tubular neighborhood of $N$, we can show that the gauge fixing

\[ d^*\eta_a = 0, \quad \xi^A = 0 \]  

(4.18)
is reachable, at least locally: $d^*\eta_a = 0$ can be obtained by choosing suitably $\epsilon_a$ in (3.65). Now, write (3.6a) for upper-case Latin indices

\[ \delta_i \xi^A = \Pi^{Aa}(x + \xi)\epsilon_a + \Pi^{AB}(x + \xi)\epsilon_B. \]

Since $\Pi^{AB}$ is invertible one can solve for $\epsilon_B$ and get $\xi^A = 0$.

We want to stress that the analog of (4.18) is not an admissible gauge-fixing in the coisotropic case. For second-class branes both the Lorentz gauge and (4.18) are admissible but, as we shall see, the latter makes the perturbative quantization transparent and is the appropriate approach to the problem.

Let us go back to the BV action (4.10), set the indices of the antighosts $\gamma$ and Lagrange multipliers $\lambda$ upstairs or downstairs as demanded by (4.18) and take

\[ \Psi = \int_D \gamma^a d^*\eta_a + \int_D \gamma_A X^A \]

where $\gamma_A$ are anticommuting 2-form fields on $\Sigma$.

On the submanifold $\varphi^+_i = \frac{\delta \Psi}{\delta \varphi^i}$ we have

\[ \beta^{+a} = \beta^{+A} = 0 \]
\[ \eta^{+a} = *d\gamma^a, \eta^{+A} = 0 \]
\[ X^+_a = 0, \quad X^+_A = -\gamma_A \]
\[ \gamma^+_a = d^*\eta_a, \quad \gamma^{+A} = X^A. \]

And the gauge fixed action with the antifields integrated out reads now

\[ \tilde{S}_{gf} = \int_D \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - *d\gamma^a \wedge (d\beta_a + \partial_a \Pi^{kl}(X) \eta_k \beta_l) - \frac{1}{4} *d\gamma^a \wedge *d\gamma^b \partial_a \partial_b \Pi^{kl}(X) \beta_k \beta_l - \lambda^a d^*\eta_a - \gamma_A \Pi^{Ai}(X) \beta_i - \lambda A X^A. \]
Recall that we are interested in calculating the expectation value of functionals depending only on $X$. Hence, integration over $\lambda_A$ sets $X^A = 0$ and we can write:

$$
\tilde{S}^\prime_{gf} = \int_D \eta_a \wedge dX^a + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - *d\gamma^a \wedge (d\beta_a + \partial_a \Pi^{kl}(X) \eta_k \beta_l) - \frac{1}{4} *d\gamma^a \wedge *d\gamma^b \partial_a \partial_b \Pi^{kl}(X) \beta_k \beta_l - \lambda^a d*\eta_a - *\gamma_A \Pi^{Ai}(X) \beta_i
$$

where $\Pi$ is evaluated on $X^A = 0$.

Now, integrating over $\gamma_A$ forces

$$\Pi^{Ai} \beta_i = 0 \Leftrightarrow \beta_A = -\omega_{AB} \Pi^{Ba} \beta_a$$

(4.19)

which is a crucial relation which can be used to get rid of the components of the fields with upper-case indices and get an effective action depending only on the lower-case components. Notice that writing

$$*d\gamma^a \wedge \partial_a \Pi^{kl}(X) \eta_k \beta_l = *d\gamma^a \wedge \partial_a (\Pi^{bc}(X) \eta_b \beta_c + \Pi^{Ai}(X) \eta_A \beta_i + \Pi^{bA}(X) \eta_b \beta_A)$$

and applying (4.19) to the second and third terms in parentheses we obtain the Dirac Poisson structure (2.11) in a beautiful way:

$$*d\gamma^a \wedge \partial_a \Pi^{bc} \eta_b \beta_c = *d\gamma^a \wedge \partial_a \Pi^{bc} \eta_b \beta_c.$$

Doing the same for the term quadratic in $d\gamma$ in $S_{BV}^\prime$ we get:

$$\tilde{S}_{gf} = \int_D \eta_a \wedge dX^a + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - *d\gamma^a \wedge (d\beta_a + \partial_a \Pi^{bc}(X) \eta_b \beta_c) - \frac{1}{4} *d\gamma^a \wedge *d\gamma^b \partial_a \partial_b \Pi^{cd}(X) \beta_c \beta_d - \lambda^a d*\eta_a - i\hbar \log \det(\Pi^{AB}(X))$$

where the last term in the action comes from the Jacobian corresponding to the delta distribution

$$\delta(\Pi^{Ai} \beta_i) = \delta(\beta_A + \omega_{AB} \Pi^{Ba} \beta_a) \det(\Pi^{BC}(X)).$$

The final step is to integrate out the $\eta_A$ fields. The integral is Gaussian (due again to the non-degeneracy of $\Pi^{AB}$) and the determinant coming from it cancels the contribution from the $\delta$ function. Finally,

$$\tilde{S}_{gf} = \int_D \eta_a \wedge dX^a + \frac{1}{2} \Pi^{ab}(X) \eta_a \wedge \eta_b - *d\gamma^a \wedge (d\beta_a + \partial_a \Pi^{cd}(X) \eta_c \beta_d) - \frac{1}{4} *d\gamma^a \wedge *d\gamma^b \partial_a \partial_b \Pi^{cd}(X) \beta_c \beta_d - \lambda^a d*\eta_a$$

which is Cattaneo and Felder’s gauge-fixed BV action for a PSM defined on $D$, with target $(\mathcal{N}, \Pi_D)$ (recall that we set $X^A = 0$) and boundary conditions such that $X^a$ is free and $\eta_a$ vanishes on vectors tangent to $\partial D$. In
other words, we have ended up with the situation studied in [19]. Invoking the results therein we can deduce our announced relation, namely that the perturbative expansion of

\[ \langle f(X(0))g(X(1))\delta(X(\infty) - x) \rangle, \ f, g \in C^\infty(M) \]

yields the Kontsevich’s formula for \( \Pi_D \) applied to the restrictions to \( N \) of \( f \) and \( g \).

In the derivation of this result the second-class boundary conditions seem to play no role, as they are not used to compute any propagator. Notice, however, that the gauge fixing \([4,13]\) makes sense only if the fields \( \xi^A \) vanish at the boundary before fixing the gauge. As stressed above the fact that \( \det(\Pi^{AB}) \neq 0 \) is also essential. This ties inextricably the present result and the use of second-class branes together.

We would like to stress the interesting cancellation of the determinant coming from the integration of the \( \gamma_A \) and \( \beta_A \) fields with that coming from the integration of the \( \eta_A \) fields. It would be worth finding out whether there is some underlying symmetry behind it.

### 4.2.4 Quantization with a general brane

Once the quantization of the PSM with a second-class brane has been understood, it is straightforward to describe the procedure for the quantization of the model with an arbitrary strongly regular brane defined by both first and second-class constraints. The appropriate gauge fixing fermion in the general case is

\[ \Psi = \int_D \gamma^a d\eta_a + \gamma^\mu d\eta_\mu + \gamma_A X^A. \] (4.20)

Then, we integrate out the second-class components of the fields exactly as above and we are left with

\[ \tilde{\mathcal{S}}_{gf}^{ef} = \int_D \eta_p \wedge dX^p + \frac{1}{2} \Pi_D^{pq}(X) \eta_p \wedge \eta_q - *d\gamma^p \wedge (d\beta_p + \partial_p \Pi_D^{qr}(X) \eta_q \beta_r) \]

\[ - \frac{1}{4} *d\gamma^p \wedge *d\gamma^q \partial_p \partial_q \Pi_D^{rs}(X) \beta_r \beta_s - \lambda^p d*\eta_p \]

where the indices now run over \( a \) and \( \mu \) values and \( \Pi_D \) is the Dirac bracket of \([2,12]\). This is Cattaneo and Felder’s gauge-fixed BV action for the PSM with target given by local coordinates \((X^a, X^\mu)\), Poisson structure \( \Pi_D \) and a coisotropic brane defined by \( X^\mu = 0 \). At this point we can apply the results of \([22]\).

An interesting question is how the choice of a set of second-class constraints affects the final result. In our case the choice of second class constraints was made through the gauge fixing fermion, i.e. a different choice
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amounts to a change of gauge-fixing. Since the expectation values of gauge-invariant observables do not depend on this particular choice, we conclude that the final result is independent of the choice of second-class constraints.

The whole derivation parallels that of Dirac’s quantization of constrained systems ([30]): one gets rid of the second-class constraints by defining the appropriate Dirac bracket that can be quantized with the first class constraints imposed on the states. It is nice that this result is obtained in a quantum-field theoretical context by tuning the boundary conditions of the fields.

Remark: The need for strong regularity in the quantum case can be seen, for example, from the fact that we need \( \det(\Pi^{AB}) \neq 0 \) at every point of a tubular neighborhood of \( N \) in order to perform the Gaussian integration over the \( \eta_A \) fields.

4.2.5 Comments on the coisotropic case

The classical and quantum study of the PSM with coisotropic boundary conditions was performed in detail by Cattaneo and Felder in [19]. As we have already mentioned, the results on the perturbative quantization in this case have some subtleties which we would like to discuss in a more detailed way.

We proved in Section 4.2.2 that the perturbative expansion valid in the free case (\( N = M \)), i.e. around \( \Pi = 0 \), is also valid for any coisotropic brane \( N \). This expansion was used in [19] and its relation with deformation quantization was investigated. Concretely, it was found that the perturbative expansion on the disk \( D \) of

\[
 f \star g(x) := \int_{X(p_3) = x} e^{\frac{i}{\hbar}S} f(X(p_1))g(X(p_2))
\]

with \( p_1, p_2, p_3 \) cyclically ordered at the boundary of \( D \) and \( x \in N \), leads to a modification of the Kontsevich’s formula (2.24). It can be described as follows:

\[
 f \star g = fg + \sum_{n \geq 1} \frac{\varepsilon^{n} \hbar}{n!} \sum_{\pi \in G_{n,2}} w_{\pi} B_{\pi}(f, g), \quad f, g \in C^{\infty}(N) \quad (4.21)
\]

where \( \varepsilon = i\hbar/2 \). A graph in \( G_{n,2} \) has \( n \) vertices of the first type and 2 vertices of the second type. The edges are oriented and they also come in two types, called in [19] straight and wavy. Exactly as described for the Kontsevich’s formula (2.24), two edges emerge from each vertex of the first type and no vertex emerges from a vertex of the second type. An edge cannot start and end at the same vertex.

Take local adapted coordinates on \( M \), \( (x^a, x^\mu), a = 1, \ldots, \dim(N), \mu = \dim(N) + 1, \ldots, \dim(M) \). Recall that functions on \( N \) which inherit a Poisson
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bracket from \( \Pi \) are those invariant under the flow of the vector fields

\[
Z^\mu = \Pi^\mu i, \quad \mu = \dim(N) + 1, \ldots, \dim(M)
\]
on \( N \), i.e. gauge-invariant functions.

The bidifferential operator \( B_\gamma \) associated to the graph \( \gamma \) is constructed as the one entering (2.24). The only difference is that if an edge is straight, the sum is performed only over \( a \) indices whereas if the edge is wavy, it is performed only over \( \mu \) indices.

The weight \( w_\gamma, \gamma \in G_{n,2} \) is

\[
w_\gamma = \frac{1}{(2\pi)^2n} \int_{H^n} \prod_{\text{edges } e} d\phi_e.
\]

Let \( e \) be an edge going from \( k \) to \( k' \). \( d\phi_e \) is \( d\phi(z_k, z_{k'}) \) if \( e \) is straight and \( d\phi(z_{k'}, z_k) \) if it is wavy, where \( \phi(z, w) \) is the Kontsevich’s angle function (2.25).

Of course, the product defined by (4.21) coincides with Kontsevich’s formula (2.24) when \( N = M \). However, for general coisotropic \( N \) the formula (4.21) does not define an associative product. The description of the obstructions requires some new objects which we proceed to introduce.

Let \( Y, Y' \) be two vector fields on \( M \) and define a differential operator \( A(Y) \) on \( C^\infty(N)[[\varepsilon]] \) and a function \( F(Y, Y') \in C^\infty(N)[[\varepsilon]] \) by

\[
A(Y)f = Yf + \sum_{n \geq 1} \frac{\varepsilon^n}{n!} \sum_{\gamma \in G_{n+1,1}} w_\gamma B_\gamma(Y)f, \quad (4.22a)
\]

\[
F(Y, Y') = \sum_{n \geq 0} \frac{\varepsilon^n}{n!} \sum_{\gamma \in G_{n+2,0}} w_\gamma B_\gamma(Y, Y'). \quad (4.22b)
\]

The bidifferential operators \( B_\gamma \) are defined as for \( G_{n,2} \) with the following exceptions: graphs in \( G_{n+1,1} \) have one additional vertex of the first type corresponding to \( Y \) and only one vertex of the second type. Exactly one edge starts from the vertex of the first type associated to \( Y \). Graphs in \( G_{n+2,0} \) have two new vertices of the first type corresponding to \( Y \) and \( Y' \) and there are no vertices of the second type.

The claim in [19] is that if \( F(Z^\mu, Z^{\nu}) = 0, \quad \dim(N) + 1 \leq \mu, \nu \leq \dim(M) \) then, formula (4.21) defines an associative product on

\[
C^\infty_{inv}(N) := \{ f \in C^\infty(N)[[\varepsilon]] \mid A(Z^\mu)f = 0, \dim(N) + 1 \leq \mu \leq \dim(M) \}
\]

which in general is not a deformation of the set of gauge-invariant functions on \( N \) as discussed in [19].
4.3 The cyclicity of the star product

In this section we would like to draw the attention to an interesting application of the field theoretical interpretation of Kontsevich’s formula given by Cattaneo and Felder in [19]. Let us set the issue of branes aside and take $M = \mathbb{R}^n$. As we already know, Kontsevich’s formula for the Poisson structure $\Pi$ can be obtained as the perturbative expansion of the following Green’s functions of the PSM defined on the disk with free boundary conditions for $X$:

$$f \ast g(x) = \langle f(X(p_1))g(X(p_2))\delta(X(p_3) - x) \rangle$$

with $p_1, p_2, p_3$ three cyclically ordered points at the boundary of the disk. More generally, one can consider the expectation value

$$\langle f(X(p_1))g(X(p_2))h(X(p_3)) \rangle$$

which seems invariant under cyclic permutations of $f, g, h$ because the action of the PSM is invariant under orientation preserving diffeomorphisms of the disk. Hence, one might be tempted to conclude that the Kontsevich’s star product satisfies

$$\int_{\mathbb{R}^n} (f \ast g)h \, d^n x = \int_{\mathbb{R}^n} (g \ast h)f \, d^n x.$$  \hspace{1cm} (4.23)

However, as proved in [33], this claim is not generally true. The correct statement is given by the following

**Theorem 4.1.** Take a constant volume form $\Omega$ and a Poisson tensor $\Pi$ such that $\text{div}_\Omega \Pi = 0$. Then, the Kontsevich’s star product satisfies \[ \text{(4.27)} \].

The pitfall of the heuristic argument above resides in the fact that the regularization of tadpole diagrams in the perturbative expansion of the PSM involves a counterterm proportional to $\partial_i \Pi^{ij}$, which breaks topological invariance. However, if the Poisson tensor is divergence-free the heuristic argument applies.

We would like to point out that the cyclicity property \[ \text{(4.23)} \] implies, taking $h = 1$,

$$\int_{\mathbb{R}^n} f \ast g \, d^n x = \int_{\mathbb{R}^n} fg \, d^n x $$ \hspace{1cm} (4.24)

and the star product is called closed. It is interesting to notice that the implication also works in the opposite direction. Assume that \[ \text{(4.24)} \] holds. Take $g = g' \ast h$,

$$\int_{\mathbb{R}^n} f \ast (g' \ast h) \, d^n x = \int_{\mathbb{R}^n} f(g' \ast h) \, d^n x.$$
Now, using associativity

\[ \int_{\mathbb{R}^n} (f \ast g') \ast h \, d^n x = \int_{\mathbb{R}^n} f(g' \ast h) \, d^n x. \]

Finally, applying the closedness property to the left-hand side of the last equality gives the desired result,

\[ \int_{\mathbb{R}^n} (f \ast g')h \, d^n x = \int_{\mathbb{R}^n} (g' \ast h)f \, d^n x. \]
Chapter 5

Lie bialgebras and Poisson-Lie groups
Poisson-Lie groups are Lie groups endowed with a Poisson structure satisfying a compatibility condition with the group multiplication. The tangent space at the identity $e$ of the group, $\mathfrak{g}$, is a Lie algebra. In addition, the Poisson structure induces in a natural way a Lie algebra structure on the cotangent space at $e$, $\mathfrak{g}^*$, which is the dual space of $\mathfrak{g}$. The compatibility condition between the group multiplication and the Poisson structure yields a compatibility condition between both Lie brackets, defining a Lie bialgebra structure.

The theory of Lie bialgebras and Poisson-Lie groups was developed mainly by Drinfeld and Semenov-Tian-Shansky. These objects, which play a fundamental role in the theory of classical integrable systems, are classical limits of the so-called quantum groups, a notion which emerged from the quantum inverse scattering method, due to Faddeev and collaborators. The classical counterpart plays a fundamental role in the theory of classical integrable systems.

In this chapter we introduce the basic facts on Lie bialgebras and Poisson-Lie groups which will be needed in the study of the PSM when the target is a group manifold.

5.1 Poisson-Lie groups and Lie bialgebras

Consider a Lie group $G$ equipped with a Poisson structure $\{\cdot , \cdot \}_G$. It is natural to demand the Poisson structure to be compatible with the group multiplication. To this end, endow $G \times G$ with the product Poisson structure

$$\{ f_1, f_2 \}_G \cdot G (g_1, g_2) = \{ f_1(\cdot, g_2), f_2(\cdot, g_2) \}_G (g_1) + \{ f_1(g_1, \cdot), f_2(g_1, \cdot) \}_G (g_2)$$

for $f_1, f_2 \in C^\infty(G \times G)$. $G$ is called a Poisson-Lie group if the multiplication $\mu : G \times G \to G$ is a Poisson map, i.e. if

$$\{ f, h \}_G (g_1 g_2) = \{ f(g_2), h(g_2) \}_G (g_1) + \{ f(g_1), h(g_1) \}_G (g_2)$$

(5.1)

for $f, h \in C^\infty(G)$. It is evident from (5.1) that a Poisson-Lie structure always vanishes at the unit $e$ of $G$. Therefore, the linearization of the Poisson structure at $e$ provides a Lie algebra structure on $\mathfrak{g}^* = T^*_e(G)$ by the formula

$$[df(e), dh(e)]_{\mathfrak{g}^*} = d\{ f, h \}_G (e), \ f, h \in C^\infty(G).$$

The infinitesimal form of equation (5.1) yields a compatibility condition between the Lie brackets of $\mathfrak{g}$ and $\mathfrak{g}^*$. Taking $X, Y \in \mathfrak{g}$, $\xi, \zeta \in \mathfrak{g}^*$ and denoting by $\langle \cdot , \cdot \rangle$ the natural pairing between elements of a vector space and its dual, it reads

\[\text{ad}^* \xi \text{denotes the coadjoint representation of a Lie algebra on its dual vector space. Hence, } \xi \in \mathfrak{g}^* \mapsto \text{ad}^* \xi \text{ is the coadjoint representation of } (\mathfrak{g}^* , [\cdot, \cdot]_{\mathfrak{g}^*}) \text{ on } \mathfrak{g}^*.\]
\[ \langle [\xi, \zeta]_\mathfrak{g}^*, [X, Y] \rangle + \langle \text{ad}^*_X \zeta, \text{ad}^*_Y X \rangle - \langle \text{ad}^*_Y \xi, \text{ad}^*_X X \rangle - \langle \text{ad}^*_X \zeta, \text{ad}^*_Y Y \rangle + \langle \text{ad}^*_X \xi, \text{ad}^*_Y Y \rangle = 0 \quad (5.2) \]

This is the condition defining a Lie bialgebra structure for \( \mathfrak{g} \) (or, by symmetry, for \( \mathfrak{g}^* \)).

Now take \( \mathfrak{g} \oplus \mathfrak{g}^* \) equipped with the non-degenerate symmetric bilinear form

\[ (X + \xi | Y + \zeta) = \langle \xi, X \rangle + \langle \zeta, Y \rangle, \quad X, Y \in \mathfrak{g}, \ \xi, \zeta \in \mathfrak{g}^*. \quad (5.3) \]

There exists a unique Lie algebra structure on \( \mathfrak{g} \oplus \mathfrak{g}^* \) such that \( \mathfrak{g} \) and \( \mathfrak{g}^* \) are Lie subalgebras and that (5.3) is invariant:

\[ [X + \xi, Y + \zeta] = [X, Y] + [\xi, \zeta]_\mathfrak{g}^* - \text{ad}^*_X \zeta + \text{ad}^*_Y \xi + \text{ad}^*_Y \zeta - \text{ad}^*_X Y. \quad (5.4) \]

The vector space \( \mathfrak{g} \oplus \mathfrak{g}^* \) with the Lie bracket (5.4) is called the double of \( \mathfrak{g} \) and is denoted by \( \mathfrak{g} \triangleright \triangleleft \mathfrak{g}^* \) or \( \mathfrak{d} \).

If \( G \) is connected and simply connected, (5.2) is enough to integrate \( [\cdot, \cdot]_\mathfrak{g}^* \) to a Poisson structure on \( G \) that makes it Poisson-Lie and the Poisson structure is unique. Hence, there is a one-to-one correspondence between Poisson-Lie structures on \( G \) and Lie bialgebra structures on \( \mathfrak{g} \). The symmetry between \( \mathfrak{g} \) and \( \mathfrak{g}^* \) in (5.2) implies that one also has a Poisson-Lie group \( G^* \) with Lie algebra \( \langle \mathfrak{g}^*, [\cdot, \cdot]_\mathfrak{g}^* \rangle \) and a Poisson structure \( \{\cdot, \cdot\}_G^* \) whose linearization at \( e \) gives the Lie bracket of \( \mathfrak{g} \). \( G^* \) is the dual Poisson-Lie group of \( G \). The connected and simply connected Lie group with Lie algebra \( \mathfrak{g} \triangleright \triangleleft \mathfrak{g}^* \) is known as the double group of \( G \) and denoted by \( \mathfrak{D} \).

\( G \) and \( G^* \) are subgroups of \( \mathfrak{D} \) and there exists a neighborhood \( \mathfrak{D}_0 \) of the identity of \( \mathfrak{D} \) such that every element \( \nu \in \mathfrak{D}_0 \) can be factorized as \( \nu = u g = \tilde{g} \tilde{u}, \ g, \tilde{g} \in G, \ u, \tilde{u} \in G^* \) and both factorizations are unique (notice that \( G_0 := G \cap G^* \subset \mathfrak{D} \) is a discrete subgroup). These factorizations define a local left action of \( G^* \) on \( G \) and a local right action of \( G^* \) on \( G \) by

\[ u g = \tilde{g}, \quad u^g = \tilde{u}. \]

Starting with the element \( gu \in \mathfrak{D} \) we can define in an analogous way a left action of \( G \) on \( G^* \) and a right action of \( G^* \) on \( G \). These are known as dressing transformations or dressing actions. The symplectic leaves of \( G \) (resp. \( G^* \)) are the connected components of the orbits of the right or left dressing action of \( G^* \) (resp. \( G \)).

There is a natural Poisson structure on \( \mathfrak{D} \) which will be important for us since it will show up in the analysis of the reduced phase space of the Poisson-Lie sigma models. Its main symplectic leaf is \( \mathfrak{D}_0 = GG^* \cap G^*G \).
5.1. Poisson-Lie groups and Lie bialgebras

We write its inverse in $D_0$, which is a symplectic form defined at a point $ug = \tilde{g}\tilde{u} \in D_0$ as

$$\Omega(ug) = \langle d\tilde{g}\tilde{g}^{-1} \wedge d\tilde{u} \tilde{u}^{-1} \rangle + \langle g^{-1}dg \wedge \tilde{u}^{-1}d\tilde{u} \rangle$$

(5.5)

where $\langle \cdot, \cdot \rangle$ acts on the values of the Maurer-Cartan one-forms. $D$ endowed with the Poisson structure yielding $\Omega$ is known as the Heisenberg double (31), (2).

A Poisson-Lie subgroup $H \subset G$ is a subgroup which is Poisson-Lie and such that the inclusion $i : H \hookrightarrow G$ is a Poisson map. In particular $H$ is a coisotropic submanifold of $G$. Let us call $h \subset g$ the Lie algebra of $H$ and $h^0 \subset g^*$ its annihilator. $H$ is a Poisson-Lie subgroup if and only if $h^0$ is an ideal of $g^*$, i.e. $[\xi, \zeta]_{g^*} \in h^0, \forall \xi \in g^*, \forall \zeta \in h^0$. This property permits to restrict the bialgebra structure to $h$, which is then called a Lie subbialgebra of $g$. The Poisson-Lie group $H$ associated to $h$ is a subgroup of $G$. However, in general there is no natural way to realize the dual Poisson-Lie group $H^*$ as a subgroup of $G^*$.

5.1.1 Poisson-Lie structures on simple Lie groups

Let us take $G$ a complex, simple, connected and simply connected Lie group and give the above construction explicitly. The (essentially unique) non-degenerate, invariant, bilinear form $\text{tr}(\ )$ on $g$ establishes an isomorphism between $g$ and $g^*$. The Poisson structure $\Pi$ contracted with the right-invariant forms on $G$, $\theta_R(X) = \text{tr}(dgg^{-1}X), X \in g$, will be denoted by $\pi$,

$$\pi_g(X,Y) = \iota(\Pi_g)\theta_R(X) \wedge \theta_R(Y).$$

(5.6)

For a general Poisson-Lie structure on $G$ (54),

$$\pi_g^r(X,Y) = \frac{1}{2} \text{tr}(XrY - X\text{Ad}_g^{-1}r \text{Ad}_gY)$$

(5.7)

where $r : g \rightarrow g$ is an antisymmetric endomorphism such that

$$r[rX,Y] + r[X,rY] - [rX,rY] = \alpha[X,Y], \alpha \in \mathbb{C}$$

(5.8)

which is sometimes called modified Yang-Baxter identity. Such an operator is what we shall understand by an r-matrix.

It is possible to show that $\text{Ad}_{g_0}r = r\text{Ad}_{g_0}, g_0 \in G_0$.

The matrix $r$ allows to define a second Lie bracket on $g$,

$$[X,Y]_r = \frac{1}{2}([X,rY] + [rX,Y])$$

(5.9)

2In the literature what we call $r$ is often denoted by $R$, keeping $r$ for elements of $g \otimes g$. 
which is nothing but the linearization of (5.7) at the unit of $G$. Denoting by $\mathfrak{g}_r$ the vector space $\mathfrak{g}$ equipped with the Lie bracket $\cdot, \cdot_r$, we have that $\mathfrak{g}_r$ is isomorphic to $(\mathfrak{g}^*, [\cdot, \cdot]_r)$.

In fact, every Lie bialgebra structure on a simple Lie algebra is given by an $r$-matrix as defined above. The pair $(\mathfrak{g}, r)$ is called a factorizable (resp. non-factorizable or triangular) Lie bialgebra if $\alpha \neq 0$ (resp. $\alpha = 0$). The corresponding Poisson-Lie groups will be called either factorizable or triangular accordingly.

Using $\text{tr}(\ )$ it is easy to show that $\mathfrak{g} \rhd \mathfrak{g}^* \cong (\mathfrak{g} \oplus \mathfrak{g}, [\cdot, \cdot]_d)$ as Lie algebras, where
\[
\{[X,Y], (X', Y')\}_d =
\{(X, X') + r[X',Y] + [Y, X'] + [Y', X], [X, Y'] + [Y, X'] + [Y, Y']\}_d.
\]

We would like to point out that $r$-matrices are used not only to construct Poisson-Lie structures but also to construct more general Poisson structures on Lie groups. Following Semenov-Tian-Shansky (62), if $r$ and $r'$ are antisymmetric and satisfy (5.8) with the same value of $\alpha$, we can define the Poisson structure (contracted with the right-invariant forms),
\[
\pi_{r,r'}(X, Y) = \frac{1}{2} \text{tr}(XrY + XAd_g r' Ad_g^{-1} Y).
\] (5.10)

$G$ equipped with (5.10) is denoted by $G_{r,r'}$. Given three $r$-matrices $r$, $r'$ and $r''$ with the same value of $\alpha$, we can define the Poisson structure (contracted with the right-invariant forms),

5.2 The double of a Lie bialgebra

It turns out that the double $\mathfrak{d}$ has a very different aspect for $\alpha \neq 0$ and $\alpha = 0$. In this section we rederive an approach (appeared already in 65) that allows us to understand the cases $\alpha \neq 0$ and $\alpha = 0$ in a unified way.

Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}[[\varepsilon]]$ of polynomials on a variable $\varepsilon$ with coefficients in $\mathfrak{g}$ (always a simple complex Lie algebra from now on) where
\[
\sum_{m=0}^{M} \sum_{n=0}^{N} X_m \varepsilon^m Y_n \varepsilon^n = \sum_{m=0}^{M} \sum_{n=0}^{N} \varepsilon^{m+n} [X_m, X'_n].
\]

$\mathfrak{g}_\alpha = \langle \varepsilon^2 - \alpha \rangle \mathfrak{g}$ is an ideal of $\mathfrak{g}$, hence $\mathfrak{g}/\mathfrak{g}_\alpha$ inherits in a canonical way a Lie algebra structure from $\mathfrak{g}$. In practice, it is more useful to think of $\mathfrak{g}/\mathfrak{g}_\alpha$ as the set $\{X + Y \varepsilon | X, Y \in \mathfrak{g}, \varepsilon^2 = \alpha\}$. Then, the Lie bracket of two elements of $\mathfrak{g}/\mathfrak{g}_\alpha$ can be written as
\[
[X + Y \varepsilon, X' + Y' \varepsilon] = [X, X'] + \alpha [Y, Y'] + ([X, Y'] + [Y, X'] \varepsilon].
\]
5.2. The double of a Lie bialgebra

There exists an isomorphism of Lie algebras between \((g \oplus g, [\cdot, \cdot]_d)\) and \(\mathfrak{g}/\mathfrak{G}_\alpha\) given by \((X, Y) \mapsto X + \frac{1}{2}rY + \frac{1}{2}Y\varepsilon\) and, consequently, \(\mathfrak{d} \cong \mathfrak{g}/\mathfrak{G}_\alpha\). Furthermore,

\[ g \cong \{X \mid X \in g \}, \quad g_r \cong \{rX + X\varepsilon \mid X \in g \} \subset \mathfrak{g}/\mathfrak{G}_\alpha. \]

5.2.1 Factorizable Lie bialgebras

It is clear that if the Lie bialgebra is factorizable (i.e. \(\alpha \neq 0\)), the subalgebras \(\{(1 + \frac{1}{\sqrt{\alpha}})X \mid X \in g\}\), \(\{(1 - \frac{1}{\sqrt{\alpha}})X \mid X \in g\}\) commute with one another and both are isomorphic to \(g\). In fact, \(\mathfrak{g}/\mathfrak{G}_\alpha \cong g \oplus g\) with the natural Lie bracket:

\[[[X_1, X_2], [Y_1, Y_2]] = ([X_1, Y_1], [X_2, Y_2])\]

with the isomorphism given by \(X + Y\varepsilon \mapsto (X + Y, X - Y)\). Hence, the double of a factorizable Lie bialgebra is isomorphic to \(g \oplus g\) and \(D = G \times G\).

As deduced from \(r\pm = \frac{1}{2}(r \pm \sqrt{\alpha})\) are Lie algebra morphisms from \(g_r\) to \(g\), i.e.

\[ r_\pm [X, Y]_r = [r_\pm X, r_\pm Y] \quad (5.11) \]

and we have the following embeddings of \(g\) and \(g_r\) in \(g \oplus g\):

\[ g_d = \{(X, X) \mid X \in g\}, \quad g_r = \{(r_+ X, r_- X) \mid X \in g\}. \]

We shall use the same notation \(g_r\) for \((g, [\cdot, \cdot]_r)\) and for its embedding in \(g \oplus g\). This should not lead to any confusion.

The map \(X \mapsto (r_+ X, r_- X)\) is non-degenerate as long as \(\alpha \neq 0\). Thus we can recover \(X\) through the formula \(X = \alpha^{-\frac{1}{2}}(r_+ X - r_- X)\). We shall often use the notation \(X_\pm = r_\pm X\).

Notice that \(g_\pm := r_\pm g\) are Lie subalgebras of \(g\) and denote by \(G_\pm\) the subgroups of \(G \times G\) integrating \(g_\pm\). We have the following embeddings of \(G\) and \(G^*\):

\[ G_d = \{(g, g) \in D \mid g \in G\}, \quad G_r = \{(g_+, g_-) \in D \mid g_+ \in G_+, g_- \in G_-\}. \]

The dressing transformations are given by the solutions of the factorization problem

\[ (h_+, h_-)(g, g) = (\tilde{g}, \tilde{g})(\tilde{h}_+, \tilde{h}_-). \]

Consider the map

\[ \nu : \quad G_r \quad \longrightarrow \quad G \]

\[ (g_+, g_-) \quad \longmapsto \quad g_- g_+^{-1}. \quad (5.12) \]

It is a submersion so that \(\nu(G_r)\) is an open connected subset of \(G_r\) containing the unit. Clearly, \(\nu(g_+, g_-) = \nu(\tilde{g}_+, \tilde{g}_-)\) if and only if \(\tilde{g}_\pm = g_\pm g_0\) with \(g_0 \in G_0\).
5.2. The double of a Lie bialgebra

We can write now explicitly the Poisson-Lie structure on $G_r$ dual to \((5.7)\). After contraction with the right-invariant forms on the group $G_r$, we have \(\vartheta^r_R(X) = \text{tr}[\{dg_+g_+^{-1} - dg_-g_-^{-1}\}X]\) for $X \in \mathfrak{g}$, it takes the form

$$\pi_{(g_+,g_-)}(X,Y) = \text{tr}\left(X(\text{Ad}_{g_+} - \text{Ad}_{g_-})(r_-\text{Ad}_{g_+}^{-1} - r_+\text{Ad}_{g_-}^{-1})Y\right) \quad (5.13)$$

which verifies, in particular, that its linearization at the unit of $G_r$ gives the Lie bracket of $\mathfrak{g}$. The symplectic leaves of \((5.13)\) are the connected components of the preimages by the map $\nu$ \((5.12)\) of the conjugacy classes in $G$ (see [62]).

Given an $r$-matrix in $G$ it is possible to define an $r$-matrix in $D = G \times G$ (which we shall denote by $R$) in a natural way: $R := P_d - P_r$ where $P_d$ and $P_r$ are the projectors on $\mathfrak{g}$ and $\mathfrak{g}_r$ respectively, parallel to the complementary subalgebra. Hence, $R^+_r = P_d$, $R^- = -P_r$.

Using the explicit description of the double group we can write the symplectic structure \((5.5)\) at a point $(h_+, h_-) = (g\tilde{h}_+, \tilde{g}h_-) \in D_0$ as

$$\Omega((h_+, h_-)) = \text{tr}(d\tilde{g}g^{-1} \wedge (dh_+h_+^{-1} - dh_-h_-^{-1})) + \text{tr}(g^{-1}dg \wedge (\tilde{h}_+d\tilde{h}_+ - \tilde{h}_-d\tilde{h}_-)) \quad (5.14)$$

which corresponds to $D_{R,R}$ with the notation introduced in Section \([5.1.1]\).

The $r$-matrix allows to write the closed 3-form $\chi = \frac{1}{3}\text{tr}[(g^{-1}dg)^\wedge 3]$ on $G$ as the differential of a 2-form on $\nu(G_r) \subset G$. Let us show this in detail because it will be important in Chapter \([\text{6}]\) in relation with the $G/G$ WZW model. The pullback of $\chi$ by $\nu$ has the form

$$\nu^*\chi = \frac{1}{3}\text{tr}\{(g^{-1}dg_- - g_+^{-1}dg_+)^\wedge 3\}$$

$$= d\text{tr}[g^{-1}dg_- \wedge g_+^{-1}dg_+] + \frac{1}{3}\text{tr}[(g^{-1}dg_-)^\wedge 3] - \frac{1}{3}\text{tr}[(g_+^{-1}dg_+)^\wedge 3]. \quad (5.15)$$

Now define $\gamma = g_+^{-1}dg_+ - g_-^{-1}dg_- \in \Omega^1(G_r) \otimes \mathfrak{g}$. One has $r_+\gamma = g_+^{-1}dg_+$ and

$$\text{tr}[(g_+^{-1}dg_+)^\wedge 3] = \text{tr}[(r_+\gamma)^\wedge 3]$$

$$= \frac{1}{2}\text{tr}[r_+(\gamma \wedge r\gamma + r\gamma \wedge \gamma) \wedge r_+\gamma]$$

$$= -\frac{1}{2}\text{tr}[(\gamma \wedge r\gamma + r\gamma \wedge \gamma) \wedge r_-r_+\gamma] \quad (5.16)$$

where we have used \([5.11]\) and the antisymmetry of $r$ with respect to the bilinear form $\text{tr}(\cdot)$. The same result is obtained for $\text{tr}[(g_-^{-1}dg_-)^\wedge 3]$, so that
the last two terms in (5.15) cancel each other. Hence, \( \chi = d\rho \) on \( \nu(G_r) \) where \( \rho \) is a 2-form on \( \nu(G_r) \) such that

\[
\nu^* \rho = \text{tr}[g_-^{-1}dg_+ \wedge g_+^{-1}dg_+].
\]

(5.17)

A straightforward computation leads to an equivalent expression for the symplectic form \( \Omega \) on the leaf \( D_0 \) in the Heisenberg double:

\[
\Omega(hg_+, hg_-) = \text{tr}[(g^{-1}dg + dgg^{-1} + g^{-1}h^{-1}dhg)h^{-1}dh] + \rho(g) - \rho(hgh^{-1}).
\]

(5.18)

The right hand side may be viewed as the 2-form on the space of pairs \((h, g)\) such that both \(g = g_-g_+^{-1}\) and \(hgh^{-1} = \tilde{g}_-\tilde{g}_+^{-1}\) are in \(\nu(G_r)\). We shall denote it then as \(\Omega(h, g)\).

\textbf{Example 5.1.} Let \(g\) be a simple Lie algebra over \(\mathbb{C}\) and \(\Delta\) its set of roots. The decomposition in root spaces reads

\[
g = t \oplus \left( \bigoplus_{\beta \in \Delta} g_\beta \right)
\]

(5.19)

where \(t\) is a Cartan subalgebra of \(g\) and

\[
g_\beta = \{CX_\beta \mid [T, X_\beta] = \beta(T)X_\beta, \forall T \in t\}.
\]

Given a splitting into positive and negative roots, \(\Delta = \Delta_+ \cup \Delta_-\), any element \(X \in g\) can be written in a unique way as \(X = X^{(+)} + X^{(-)} + T\), where \(X^{(\pm)} \in \text{span}(X_\beta, \beta \in \Delta_{\pm})\) and \(T \in t\).

The \textit{standard r-matrix} is defined by \(r = r_+ + r_-\) with

\[
r_+X = X^{(+)} + \frac{1}{2}T
\]

\[
r_-X = -X^{(-)} - \frac{1}{2}T
\]

(5.20)

which is a factorizable \(r\)-matrix with \(\alpha = 1\).

Take as a particular case \(g = \mathfrak{sl}(n, \mathbb{C})\) with the standard \(r\)-matrix. Then, \(\mathfrak{sl}(n, \mathbb{C})_r \subset \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})\) consists of pairs \((X_+, X_-)\) where \(X_+\) (resp. \(X_-\)) is upper (resp. lower) triangular and \(\text{diag}(X_+) = -\text{diag}(X_-)\).

At the group level, \(SL(n, \mathbb{C})_r \subset SL(n, \mathbb{C}) \times SL(n, \mathbb{C})\) is the set of pairs \((g_+, g_-)\) such that \(g_+\) (resp. \(g_-\)) is upper (resp. lower) triangular and \(\text{diag}(g_+) = \text{diag}(g_-)^{-1}\).

\subsection*{5.2.2 Triangular Lie bialgebras}

In this subsection we describe the double of a triangular Lie bialgebra and the double of the associated Poisson-Lie groups. We shall use these results for writing explicitly the Poisson structure dual to (5.7).
5.2. The double of a Lie bialgebra

If \( \alpha = 0, \) \( r \) degenerate to \( \frac{1}{2} r, \) the map \( X \mapsto (\frac{1}{2} r X, \frac{1}{2} r X) \) is not invertible and \( \mathfrak{g}/\mathfrak{g}_0 \) is no longer isomorphic to \( \mathfrak{g} \oplus \mathfrak{g} \). Indeed, \( \mathfrak{g}/\mathfrak{g}_0 = \{ X + Y \varepsilon \mid \varepsilon^2 = 0 \} \) is not semisimple, for the elements \( X \varepsilon \) form an abelian ideal as seen from the Lie bracket

\[
[X + Y \varepsilon, X' + Y' \varepsilon] = [X, X'] + ([X, Y'] + [Y, X']) \varepsilon. \tag{5.21}
\]

This is the Lie algebra of the tangent bundle of \( G, TG \cong G \times \mathfrak{g} \), with the natural group structure given by the semidirect product

\[
(g, X)(g', X') = (gg', \text{Ad}_gX' + X). \tag{5.22}
\]

Hence, the double of a triangular Lie bialgebra is isomorphic to the tangent bundle of \( G \) with the product given by (5.22).

We can represent the elements of the double as

\[
\mathbf{D} = \left\{ \begin{pmatrix} e & 0 \\ X & e \end{pmatrix} g \mid g \in G, \ X \in \mathfrak{g} \right\}
\]

where the product is now the formal product of matrices, resulting the semidirect product mentioned above. Its Lie algebra with this notation is

\[
\mathfrak{d} = \left\{ \begin{pmatrix} X & 0 \\ Y & X \end{pmatrix} \mid X, Y \in \mathfrak{g} \right\}
\]

and the Lie bracket (5.21) is given by the formal commutator of matrices. The embeddings of \( \mathfrak{g} \) and \( \mathfrak{g}_r \) in \( \mathfrak{d} \) are given by

\[
\mathfrak{g}_d = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \mid X \in \mathfrak{g} \right\}, \quad \mathfrak{g}_r = \left\{ \begin{pmatrix} rX & 0 \\ X & rX \end{pmatrix} \mid X \in \mathfrak{g} \right\}.
\]

Both subalgebras exponentiate to subgroups \( G_d, G_r \subset \mathbf{D} \). Clearly,

\[
G_d = \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \in G \right\}
\]

whereas for \( G_r \) the description is less explicit. It is the subgroup of \( \mathbf{D} \) generated by elements of the form:

\[
\begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} e^{rX} \quad \text{with} \quad Y = \int_0^1 \text{Ad}_{e^{rX}}X ds, \ X \in \mathfrak{g}. \tag{5.23}
\]

We will denote a general element of \( G_r \) by

\[
\check{Y} = \begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} h_Y. \tag{5.24}
\]
where, in the general case, $Y$ belongs to a dense subset of $\mathfrak{g}$ and determines $h_Y$ up to multiplication by an element of $G_0$. This means that

$$\left( \begin{array}{cc} e & 0 \\ Y & e \end{array} \right) h_Y \in G_r, \text{ then} \left( \begin{array}{cc} e & 0 \\ Y & e \end{array} \right) h'_Y \in G_r \Leftrightarrow h'_Y = h_Y g_0, \ g_0 \in G_0.$$ 

As a consequence, the notation $\bar{Y}$ has a small ambiguity, but we shall use it for brevity wherever it does not lead to confusion.

The right-invariant forms on $G_r$ are

$$\theta^r_r(Y) = \text{tr} \left( (dX + [X,dh_X h_X^{-1}]) Y \right) = \text{tr} \left( d \left( \text{Ad}^{-1}_{h_X} X \right) \text{Ad}^{-1}_{h_X} Y \right)$$

for $Y \in \mathfrak{g}$ and $\left( \begin{array}{cc} e & 0 \\ X & e \end{array} \right) h_X \in G_r$. Whereas the left invariant forms read

$$\theta^r_L(Y) = \text{tr} \left( Y \text{Ad}^{-1}_{h_X} dX \right).$$

The concrete realization of $G^*$ described above allows us to write the Poisson structure dual to (5.7) in explicit terms. It can be checked after a straightforward (although lengthy) calculation that the dual Poisson-Lie structure contracted with the right-invariant forms is

$$\pi^r_r(Y,Z) = \text{tr} \left( Y[X,Z] - [X,Y] \text{Ad}_{h_X} r \text{Ad}^{-1}_{h_X} [X,Z] \right). \ (5.25)$$

In the triangular case, the symplectic structure of the Heisenberg double (5.5) at a point $\bar{X}g = \tilde{g} \tilde{X} \in D_0$ can be written as

$$\Omega(\bar{X}g) = \text{tr} \left( d\tilde{g}^{-1}\tilde{g}^{-1} \right) \left( dX + [X,dh_X h_X^{-1}] \right) + g^{-1} dg \wedge \text{Ad}^{-1}_{h_X} d\tilde{X} \). \ (5.26)$$

Let us finish this chapter with two examples of triangular Lie bialgebras:

**Example 5.2.** $r = 0$ is an $r$-matrix with $\alpha = 0$. It endows $G$ with trivial Poisson bracket $\{\cdot,\cdot\}_G = 0$ and $\mathfrak{g}^*$ with trivial Lie bracket $[\cdot,\cdot]_{\mathfrak{g}^*} = 0$. The dual Poisson Lie group $G^*$ is $\mathfrak{g}^*$ viewed as an abelian group and equipped with the Kostant-Kirillov Poisson bracket defined in Example 2.4.

**Example 5.3.** Take $\mathfrak{g}$ a complex simple Lie algebra. If $\tau_t : \mathfrak{g} \to t$ is the projector onto the Cartan subalgebra $t$ with respect to the decomposition (5.19) and $\mathcal{O} : t \to t$ is an antisymmetric endomorphism of $t$ with respect to $\text{tr} (\cdot)$, then $r = \mathcal{O}\tau_t$ is an $r$-matrix with $\alpha = 0$.

It is worth studying in detail the structure of $G_r$ for this $r$. As we know, $G_r$ is generated by elements of the form

$$\left( \begin{array}{cc} e & 0 \\ Y & e \end{array} \right) e^{rX}, \ Y = \int_0^1 \text{Ad}_{e^{rX}} X ds, \ X \in \mathfrak{g}. \ (5.27)$$
5.2. The double of a Lie bialgebra

The elements \(\{rX \mid X \in \mathfrak{g}\}\) span a subalgebra of \(\mathfrak{t}\) (therefore abelian) and its exponentiation will be a subgroup of the Cartan subgroup of \(G\), so we concentrate on \(Y\).

Take

\[
X = T + \sum_{\beta \in \Delta} a_\beta X_\beta, \ T \in \mathfrak{t}.
\]

Observing that

\[
\text{Ad}_{e^{srX}} = e^{\text{ad}_{srX}}
\]

we straightforwardly obtain

\[
\text{Ad}_{e^{srX}} X = T + \sum_{\beta \in \Delta} e^{s\beta(rT)} a_\beta X_\beta
\]

and, therefore,

\[
Y = T + \sum_{\beta \in \Delta} \frac{1}{\beta(rT)} \left( e^{\beta(rT)} - 1 \right) a_\beta X_\beta
\]

where, if some \(\beta(rT) = 0\), the limit \(\beta(rT) \to 0\) must be understood in the last expression.

The product of \(n\) elements of the form (5.27)

\[
\begin{pmatrix} e & 0 \\ Y_1 & e \end{pmatrix} e^{rX_1} \cdots \begin{pmatrix} e & 0 \\ Y_n & e \end{pmatrix} e^{rX_n} := \begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} T_Y
\]

can be computed explicitly:

\[
T_Y = e^{rT_1 + \cdots + rT_n}
\]

\[
Y = T_1 + \cdots + T_n + \sum_{\beta \in \Delta} \left[ \frac{1}{\beta(rT_1)} \left( e^{\beta(rT_1)} - e^{\beta(rT_2) + \cdots + rT_n) \right) a_{\beta,1} + \right.
\]

\[
+ \frac{1}{\beta(rT_2)} \left( e^{\beta(rT_2) + \cdots + rT_n) - e^{\beta(rT_3) + \cdots + rT_n) \right) a_{\beta,2} + \right.
\]

\[
+ \cdots + \frac{1}{\beta(rT_n)} \left( e^{\beta(rT_n)} - 1 \right) a_{\beta,n} \right] X_\beta
\]

where we have used the notation

\[
X_i = T_i + \sum_{\beta \in \Delta} a_{\beta,i} X_\beta, \ T_i \in \mathfrak{t}.
\]

It is clear that, in this case, \(G_0 = e\) and that \(Y\) fills \(\mathfrak{g}\). As a consequence, the dressing actions are globally defined, \(D = GG_r = G_rG\) and the factorizations are unique.
Chapter 6

The Poisson sigma model over Lie groups
In this chapter we shall be interested in the Poisson sigma model when the target is a Lie group. In this case it is natural to demand the compatibility between its Poisson and group structures and this leads to the concept of Poisson-Lie group introduced in Chapter 5. Examples of Poisson-Lie sigma models have been studied in \[32,3\] in connection to \(G/G\) Wess-Zumino-Witten theories. Here we pursue a systematic study of the matter.

One of the simplest examples of Poisson-Lie sigma model is the linear one: \(M\) is a vector space (abelian group) and its Poisson structure is linear. This model is related to \(BF\) and Yang-Mills theories and can be considered as dual of the trivial Poisson-Lie sigma model in an (in general) non-abelian group with vanishing Poisson bracket. Our models can be regarded as the simplest generalizations of the linear ones with which they share some properties that will be stressed in the sequel.

Another aspect that will deserve our attention is that of duality, i.e. we shall try to relate the two dual models, which is not obvious at the Lagrangian level. However, in the Hamiltonian approach, in the open geometry, the duality becomes evident: it consists in the exchange of bulk and boundary degrees of freedom. It would be interesting to relate this fact with the non abelian T-duality discussed in \[1,45,46,47\], but this point is not clear to us at the present time.

When the target manifold is a Lie group the action of the Poisson sigma model can be recast in terms of a set of fields adapted to the group structure. \(T^*G\) can be identified, by right translations, with \(G \times g^*\) and, using \(\text{tr}(\ )\), with \(G \times g\). Then, in (3.1) we can take \(A \in \Omega^1(\Sigma) \otimes g\) and \(g : \Sigma \to G\) as fields (equivalently, \(\eta \equiv \text{tr}(dg^{-1} \wedge A)\)) and use the Poisson structure contracted with the right-invariant forms on \(G\) (5.6). Denoting by \(\pi^g_\#: g \to g\) the endomorphism induced by \(\pi_g\) using \(\text{tr}(\ )\) we can write the action of the Poisson sigma model as

\[
S(g, A) = \int_\Sigma \text{tr}(dg^{-1} \wedge A) - \frac{1}{2} \text{tr}(A \wedge \pi^g_\#A). \tag{6.1}
\]

In particular, for the Poisson-Lie structure (5.7) we have

\[
S_{PL}(g, A) = \int_\Sigma \text{tr}(dg^{-1} \wedge A) - \frac{1}{4} \text{tr}(A \wedge (r - \text{Ad}_g r \text{Ad}_g^{-1}) A) \tag{6.2}
\]

which is the action of what we shall call Poisson-Lie sigma model with target \(G\).

The equations of motion are

\[
dg^{-1} + \frac{1}{2} (r - \text{Ad}_g r \text{Ad}_g^{-1}) A = 0 \tag{6.3a}
\]

\[
d\bar{A} + [\bar{A}, \bar{A}]_r = 0, \quad \bar{A} := \text{Ad}_g^{-1} A \tag{6.3b}
\]
from which a zero curvature equation can be also derived for $A$,

$$dA + [A, A]_r = 0. \quad (6.4)$$

The infinitesimal gauge symmetry of the action, for $\beta : \Sigma \to \mathfrak{g}$ is

$$\delta_{\beta} g g^{-1} \equiv \frac{1}{2} (\text{Ad}_g r \text{Ad}_{g^{-1}} - r) \beta \quad (6.5a)$$

$$\delta_{\beta} A = d\beta + [A, \beta]_r - \frac{1}{2} [d g g^{-1} + \frac{1}{2} (r - \text{Ad}_g r \text{Ad}_{g^{-1}}) A, \beta] \quad (6.5b)$$

which corresponds to the right dressing vector fields of [54] translated to the origin by right multiplication in $G$. Its integration (local as in general the vector field is not complete) gives rise to the dressing transformation of $g$. On-shell, $[\delta_{\beta_1}, \delta_{\beta_2}] = \delta_{[\beta_1, \beta_2]_r}$, so that the ‘structure functions’ of the commutation relations are field-independent, unlike in the general case (see equation (3.9)). Thus we can talk properly about a gauge group. In fact, the gauge group of the Poisson sigma model over $G$ is its dual $G^*$. 

Up to here we have not needed to distinguish between $\alpha = 0$ and $\alpha \neq 0$. In order to study further (6.2) and its dual model we need to make use of the embeddings of $G$ and $G^*$ in the double $\mathbf{D}$. As we have learnt, $\mathbf{D}$ is very different in the factorizable and triangular cases and we must analyse them separately.

### 6.1 Factorizable Poisson-Lie sigma models

Without loss of generality we take $\alpha = 1$ in equation (5.8). Since the equation of motion for $A$ is independent of $g$, we first solve the equations (6.3b) and (6.4). This is in contrast with the general case of [16]. Locally,

$$A = h_+ dh_+^{-1} - h_- dh_-^{-1}$$

$$\tilde{A} = \tilde{h}_+ d\tilde{h}_+^{-1} - \tilde{h}_- d\tilde{h}_-^{-1}.$$ 

We need to work a bit more to get explicit solutions for $g$. The equation of motion (6.3a) can be equivalently written as

$$dgg^{-1} + (r_+ - \text{Ad}_g r_+ \text{Ad}_{g^{-1}}) A = 0$$

and recalling that $\tilde{A} = \text{Ad}_{g^{-1}} A$,

$$dgg^{-1} + r_+ A - \text{Ad}_g r_+ \tilde{A} = 0.$$

The operator $r_+$ projects on to the + component of the elements of $\mathfrak{g}$ and yields

$$dgg^{-1} + h_+ dh_+^{-1} - Ad_g \tilde{h}_+ d\tilde{h}_+^{-1} = 0$$
which is equivalent to
\[ \tilde{h}_+^{-1}g^{-1}h_+d(h_+^{-1}g\tilde{h}_+) = 0 \Rightarrow h_+^{-1}g\tilde{h}_+ = \hat{g}, \quad \hat{g} = g(\sigma_0) \in G. \]

The same procedure can be carried out by using \( r_- \),
\[ h_-^{-1}g\tilde{h}_- = \hat{g}. \]

The solutions of the equations of motion can then be expressed by the single relation in \( D \):
\[ (h_+(\sigma)\hat{g}, h_-(\sigma)\hat{g}) = (g(\sigma)\tilde{h}_+(\sigma), g(\sigma)\tilde{h}_-(\sigma)) \tag{6.7} \]
which in particular implies that \( (h_+(\sigma)\hat{g}, h_-(\sigma)\hat{g}) \in D_0 \), and \( g \) take values in the connected component of orbits of \( \hat{g} \) by dressing transformations. These orbits are the symplectic leaves of the Poisson-Lie group \( G \).

We go on to study the reduced phase space of the model in the open geometry, \( \Sigma = \mathbb{R} \times [0,1] \). For the moment we take \( g \) free at the boundary. Equivalently, we take a brane which is the whole target manifold \( G \). The \( A \) field must then vanish on vectors tangent to \( \partial \Sigma \), so that \( h_\pm, \tilde{h}_\pm \) are constant along the connected components of the boundary. Writing \( \sigma = (t,x) \), we have \( h_\pm(t,0) = h_{0\pm}, \tilde{h}_\pm(t,0) = \tilde{h}_{0\pm}, h_\pm(t,1) = h_{1\pm}, \tilde{h}_\pm(t,1) = \tilde{h}_{1\pm} \).

Denote \( I = [0,1] \). The canonical symplectic form on the space of continuous maps \((g,A) : TI \rightarrow G \times \mathfrak{g} \) with continuously differentiable base map is:
\[ \omega = \int_0^1 \text{tr}(\delta gg^{-1} \wedge \delta gg^{-1} A - \delta gg^{-1} \wedge \delta A) dx. \]

When restricted to the solutions of the equations of motion the symplectic form \( \omega \) becomes degenerate, its kernel given by the gauge transformations \( [6.5] \) which vanish at \( x = 0,1 \). By definition, the reduced phase space \( \mathcal{P}(G;G,G) \) is the (possibly singular) quotient of the space of solutions by the kernel of \( \omega \).

If we parametrize the solutions in terms of \( h_\pm(\sigma) \) and \( \hat{g} \) we obtain
\[ \omega = \frac{1}{2} \int_0^1 \partial_x \Omega((h_+(\sigma)\hat{g}, h_-(\sigma)\hat{g})) dx. \]
That is, \( \omega \) depends only on the values of the fields at the boundary (i.e. the degrees of freedom of the theory are all at the boundary, as expected from the topological nature of the model) and is expressed in terms of the symplectic structure on the Heisenberg double \( [5.14] \). Namely,
\[ \omega = \frac{1}{2} [\Omega((h_{1+}\hat{g}, h_{1-}\hat{g})) - \Omega((h_{0+}\hat{g}, h_{0-}\hat{g}))]. \]

Or if we take \( \sigma_0 = (t_0,0) \), i.e. \( h_{0\pm} = \tilde{h}_{0\pm} = e \)
\[ \omega = \frac{1}{2} \Omega((h_{1+}\hat{g}, h_{1-}\hat{g})) \]
The reduced phase space $\mathcal{P}(G; G, G)$ is the set of pairs $([(h_+, h_-),  \hat{g}])$ with $[(h_+, h_-)]$ a homotopy class of maps from $[0, 1]$ to $G_r$ which are the identity at 0, are fixed at 1 and such that $(h_+(x)  \hat{g}, h_-(x)  \hat{g}) \in D_0, \forall x \in [0, 1]$. The symplectic form on $\mathcal{P}(G; G, G)$ can be viewed as the pull-back of $\Omega$ by the map $([(h_+, h_-),  \hat{g}) \mapsto (h_1+ \hat{g}, h_1- \hat{g})].$

### 6.1.1 The dual factorizable model and $G/G$ theory

Using the Poisson structure given in (5.13) the action of the dual model reads

$$S^*_{PL}(g_+, g_-, A) = \int_{\Sigma} \text{tr}[(dg_+ g_+^{-1} - dg_- g_-^{-1}) \wedge A + \frac{1}{2} A \wedge (\text{Ad}_{g_+} - \text{Ad}_{g_-})(r_+ \text{Ad}_{g+}^{-1} - r_- \text{Ad}_{g-}^{-1})A].$$

(6.8)

The equations of motion of the model can be written

$$g_\pm^{-1}dg_\pm + r_\pm(\text{Ad}_{g_+}^{-1} - \text{Ad}_{g_-}^{-1})A = 0$$

$$dA + [A, A] = 0.$$  

(6.9)

The gauge transformations, for $\beta : \Sigma \to g$, are:

$$g_\pm^{-1}\delta_\beta g_\pm = r_\pm(\text{Ad}_{g_-}^{-1} - \text{Ad}_{g_+}^{-1})\beta$$

$$\delta_\beta A = d\beta + [A, \beta] + \frac{1}{2}(r_+ \text{Ad}_{g_+} + r_- \text{Ad}_{g_-})[g_+^{-1}dg_+ - g_-^{-1}dg_- + (\text{Ad}_{g_+}^{-1} - \text{Ad}_{g_-}^{-1})A, \tilde{\beta}]$$

where $\tilde{\beta} := (r_- \text{Ad}_{g_+}^{-1} - r_+ \text{Ad}_{g_-}^{-1})\beta$. The gauge transformations close on-shell. Namely, $[\delta_{\beta_1}, \delta_{\beta_2}] = \delta_{[\beta_1, \beta_2]}$ which corresponds now to the gauge group $G$.

The solutions of the equations of motion can be obtained along the same lines as before. Locally,

$$A = h dh^{-1}$$

and $(g_+(\sigma), g_-(\sigma))$ is obtained as the solution of

$$(g_+(\sigma)h(\sigma), g_-(\sigma) \hat{h}(\sigma)) = (h(\sigma)  \hat{g}_+, h(\sigma)  \hat{g}_-)$$

which means that $(g_+, g_-)$ is the dressing-transformed of $(\hat{g}_+, \hat{g}_-)$ by $h$. At this point it is evident the symmetry between both dual models under the exchange of the roles of $G$ and $G_r$.

In the open geometry, $\Sigma = \mathbb{R} \times [0, 1]$, with free boundary conditions, $h$ is constant along connected components of the boundary and one may take
6.1. Factorizable Poisson-Lie sigma models

\[ h(t, 0) = \tilde{h}(t, 0) = e, \; h(t, 1) = h_1, \; \tilde{h}(t, 1) = \tilde{h}_1. \]

The symplectic form, which we denote by \( \omega^* \), can be written

\[ \omega^* = \frac{1}{2} \Omega((h_1 \hat{g}_+, h_1 \hat{g}_-)). \]

The duality between \( \tilde{\mathcal{P}}(G; G, G) \) and \( \tilde{\mathcal{P}}(G_r; G_r, G_r) \) was pointed out in [17]. The symplectic forms of the two models coincide upon the exchange of \( h_1 \) with \( \hat{g}^{-1} \) and \( (\hat{g}_+, \hat{g}_-) \) with \( (h^{-1}_{1+}, h^{-1}_{1-}) \). Hence, one can talk about a bulk-boundary duality between the Poisson-Lie sigma models for \( G \) and \( G^* \) since the exchange of degrees of freedom maps variables associated to the bulk of one model to variables associated to the boundary of the other one.

At this point we would like to recall the equivalence between the Poisson-Lie sigma model with target \( G^* \) and the \( G/G \) theory. Such equivalence was first discovered in [3] in the Hamiltonian formalism and was given a covariant description in [32], which we present below.

The \( G/G \) theory is the gauged version of the WZW model [74] with the group \( G \) as target. The fields of this model on a two-dimensional oriented surface \( \Sigma \) equipped with a conformal or a pseudo-conformal structure are \( g : \Sigma \to G \) and a gauge field \( A \), a \( g \)-valued 1-form on \( \Sigma \). The action functional of the model has the form

\[
S_{WZW}(g, A) = \frac{1}{4\pi} \int_{\Sigma} \text{tr}[(g^{-1} \partial_t g)(g^{-1} \partial_r g)] dx^l \wedge dx^r + S_{WZ}(g) \\
+ \frac{1}{2\pi} \int_{\Sigma} \text{tr} [\partial_t gg^{-1} A_r - A_l gg^{-1} \partial_r g] dx^l \wedge dx^r \\
+ \frac{1}{2\pi} \int_{\Sigma} \text{tr} [A_l A_r - gA_l g^{-1} A_r] dx^l \wedge dx^r \tag{6.10}
\]

where \( x^l = z, \; x^r = \bar{z} \) are the complex variables in the Euclidean signature and the light-cone ones \( x^l = x + t, \; x^r = x - t \) in the Minkowski signature. The derivatives \( \partial_t = \frac{\partial}{\partial x^l}, \; \partial_r = \frac{\partial}{\partial x^r} \) and \( A = A_l dx^l + A_r dx^r \). The Wess-Zumino term \( S_{WZ}(g) \) is often written as \( \frac{1}{4\pi} \int_{\Sigma} g^* d^{-1} \chi \) where \( d^{-1} \chi \) stands for a 2-form whose differential is equal to the 3-form \( \chi \) on \( G \). Since such 2-forms do not exist globally, some choices are needed. On closed surfaces, one may replace \( \int_{\Sigma} g^* d^{-1} \chi \) by \( \int_B \tilde{g}^* \chi \) where \( B \) is an oriented 3-manifold with \( \partial B = \Sigma \) and \( \tilde{g} : B \to G \) extends \( g \). This gives a well defined amplitude \( e^{ikS_{WZW}} \) for integer \( k \). When \( \Sigma \) has a boundary, the definition of the amplitude is more problematic and, in general, it makes sense only as an element of a product of line bundles over the loop group \( LG \) [38]. To extract a numerical value of such an amplitude one has to use sections of the line bundle that are not globally defined.
What we shall do here is to restrict the values of the field $g$ of the gauged WZW model to the subset $\nu(G_r) \subset G$ and to define

$$S_{WZW}(g) = \frac{1}{4\pi} \int_{\Sigma} g^* \rho$$  \hspace{1cm} (6.11)

where $\rho$ is given by (5.17). In the geometric language, this corresponds to a particular choice of a section in the (trivial) restriction of the line bundle over $LG$ to the loop space $L\nu(G_r)$. We shall call the resulting field theory the restricted $G/G$ coset model.

The model defined above has simple transformation properties under the gauge transformations of fields

$$h_g = hgh^{-1} \quad h_A = hAh^{-1} + hdh^{-1}$$

for $h : \Sigma \to G$ such that $h_g$ takes also values in $\nu(G_r)$ (this is always accomplished for $h$ sufficiently close to unity). The action transforms according to

$$S_{WZW}(h_g, h_A) = S_{WZW}(g, A) + \int_{\Sigma} \Omega(h, g)$$

where $\Omega(h, g)$ is the closed 2-form given by (5.18). In particular, it follows that the action is invariant under infinitesimal gauge transformations that vanish on the boundary.

The equations of motion of the model $\delta S_{WZW} = 0$ for field variations vanishing at the boundary are

$$D_l(g^{-1}D_rg)dx^l \wedge dx^r + F(A) = 0$$ \hspace{1cm} (6.12a)

$$g^{-1}D_rg = 0, \quad gD_lg^{-1} = 0$$ \hspace{1cm} (6.12b)

where $D$ stands for the covariant derivative $D_{l,r} = \partial_{l,r} + [A_{l,r}, \cdot]$ and $F(A) = dA + A \wedge A$ is the field strength of $A$.

Although (6.12a) is a second order differential equation we can write a system of first order equations equivalent to (6.12), by simply taking

$$F(A) = 0, \quad g^{-1}Dg = 0.$$ \hspace{1cm} (6.13)

Since we have first order equations of motion it should be possible to get the system from a first order Lagrangian. As mentioned above, the Poisson-Lie sigma model with $G_r$ target and fields $(g_+, g_-)$ and $A$ is closely related to the gauged WZW model with the target $\nu(G_r)$ and fields $g = g_-g_+^{-1}$ and $A$. The action of the latter is given by equation (6.10) with the Wess-Zumino term defined by (6.11). More explicitly,
\[ S_{\text{WZW}}(g^-, g^+_1, A) = \frac{1}{4\pi} \int_{\Sigma} \text{tr} \left[ (g^+_1 \partial_l g_+) (g^-_1 \partial_r g_-) + (g^+_1 \partial_l g_+) (g^-_1 \partial_r g_-) \right. \\
\quad \left. -2(g^+_1 \partial_l g_+) (g^-_1 \partial_r g_-) \right] dx^l \wedge dx^r \\
- \frac{1}{2\pi} \int_{\Sigma} \text{tr} [A, g^+_1 \partial_l g + (g \partial g^-_1) A_r \\
\quad + gA, g^-_1 A_r - A_r A] dx^l \wedge dx^r. \]

The equations of motion (6.13) read in terms of \((g_+, g_-)\) as

\[ E_{l, \pm} \equiv g^+_1 \partial_l g_+ + r_\pm (\text{Ad}_{g^+_1} - \text{Ad}_{g^-_1}) A_l = 0, \]
\[ E_{r, \pm} \equiv g^-_1 \partial_r g_- + r_\pm (\text{Ad}_{g^-_1} - \text{Ad}_{g^+_1}) A_r = 0. \]

Defining \( E = E_l dx^l + E_r dx^r, \) \( E_{l,r} = E_{l,r,+} - E_{l,r,-} \) and after a straightforward calculation one obtains the identity:

\[ S_{\text{WZW}}(g^-, g^+_1, A) = -\frac{1}{8\pi} \int_{\Sigma} \text{tr} (E \wedge r E) + \frac{1}{4\pi} \int_{\Sigma} \text{tr} (E_l E_r) dx^l \wedge dx^r - \frac{1}{2\pi} S_{\text{PL}}(g^+, g^-, A) \]

where \( S_{\text{PL}} \) is the action (6.8).

Note that the difference between \( S_{\text{WZW}} \) and \( S_{\text{PL}} \) is quadratic in the equations of motion for \( S_{\text{WZW}} \). This implies that neither the classical solutions nor the phase space structure of both models (based on first functional derivatives of the action evaluated on-shell) differ, as long as the boundary conditions coincide.

### 6.2 Triangular Poisson-Lie sigma models

We now go back to equations (6.9) and assume \( r \) is triangular \((\alpha = 0)\). We start noting that whereas \( A \) is a pure gauge of the group \( G_r, \frac{1}{2} r A \) is a pure gauge of the group \( G, \) i.e.

\[ d\left( \frac{1}{2} r A \right) + \left[ \frac{1}{2} r A, \frac{1}{2} r A \right] = 0. \]

Now take

\[ \tilde{X}(\sigma) = \begin{pmatrix} e & 0 \\ X(\sigma) & e \end{pmatrix} h_X(\sigma) \in G_r. \]

Then, locally,

\[ \tilde{X}^{-1} d\tilde{X} = \begin{pmatrix} h_X^{-1} dh_X & 0 \\ h_X^{-1} dX h_X & h_X^{-1} dh_X \end{pmatrix} = \begin{pmatrix} r A & 0 \\ A & r A \end{pmatrix}. \]
\( \tilde{A} \) is also a pure gauge of the group \( G_r \), so \( \tilde{A} = h^{-1}_X d\tilde{X}h_{\tilde{X}} \) and the equation of motion for \( g \) reads,

\[
dgg^{-1} + \frac{1}{2} (h^{-1}_X dh_X - \text{Ad}_g h^{-1}_X dh_{\tilde{X}}) = 0
\]

which implies

\[
g = h^{-1}_X \hat{g} h_{\tilde{X}}, \quad \hat{g} \in G.
\]

\( \tilde{A} = \text{Ad}_g^{-1} A \Rightarrow \tilde{X} = \text{Ad}_\hat{g}^{-1} X \) and we can write, locally, the solution as an equation in the double:

\[
\begin{pmatrix}
e \\
X 
\end{pmatrix} h_X \begin{pmatrix}
e \\
0 
\end{pmatrix} g = \begin{pmatrix}
e \\
0 
\end{pmatrix} \hat{g} \begin{pmatrix}
e \\
\tilde{X} 
\end{pmatrix} h_{\tilde{X}}.
\]

(6.14)

The analysis of the reduced phase space when \( \Sigma = \mathbb{R} \times [0,1] \) and \( g|_{\partial \Sigma} \) is free works as in the factorizable case. The field \( A \) must vanish on vectors tangent to \( \partial \Sigma \) and hence \( \tilde{X} \) is constant along each connected component of the boundary. By using the explicit solution (6.14) we can identify \( \bar{P}(G; G, G) \).

Notice that we can always choose \( \tilde{X}(t,0) = \tilde{X}(t,0) = e \). With this choice and defining \( X_1 := X(t,1), \tilde{X}_1 := \tilde{X}(t,1), g_1 := g(t,1) \), a straightforward calculation yields

\[
\omega = \frac{1}{2} \text{tr} \left( \delta X_1 + [X_1, \delta h X_1 h^{-1}_X] \right) \wedge \delta \hat{g}^{-1} \delta \hat{g}^{-1} + \text{Ad}_{h_{\tilde{X}_1}^{-1}} \delta \hat{X}_1 \wedge g_{1}^{-1} \delta g_{1}
\]

\[
= \frac{1}{2} \Omega(\tilde{X}_1 \hat{g}).
\]

The reduced phase space \( \bar{P}(G; G, G) \) turns out to be the set of pairs \( ([\tilde{X}], \hat{g}) \) with \( \tilde{X} \) a homotopy class of maps from \([0,1] \) to \( G_r \) which are the identity at \( x = 0 \) and have fixed value at \( x = 1 \).

### 6.2.1 The dual triangular model and BF-theory

Take (5.25) and write the action of the Poisson sigma model with target \( G_r \):

\[
S_{PL}^{\ast}(\tilde{X}, A) = \int_{\Sigma} \text{tr} \left( (dX \wedge A + dX \wedge \text{Ad}_{g_X} r \text{Ad}_{g_X}^{-1} [X, A] + \frac{1}{2} A \wedge [X, A] - \frac{1}{2} [X, A] \wedge \text{Ad}_{g_X} r \text{Ad}_{g_X}^{-1} [X, A] \right)
\]

(6.15)

with fields \( A \in \Omega^1(\Sigma) \otimes \mathfrak{g}, \tilde{X} : \Sigma \rightarrow G_r \).

Note that the action is actually determined by \( X \) and \( A \), since it is invariant under \( g_X \mapsto g_X g_0, \ g_0 \in G_0 \).

Varying the action with respect to \( A \) we get the equation of motion for \( X \),

\[
dX + [A, X] = 0.
\]

(6.16)
Taking variations with respect to $\bar{X}$ and after a rather cumbersome calculation we obtain the equation of motion for $A$,

$$\text{d}A + [A, A] = 0. \quad (6.17)$$

The infinitesimal gauge symmetry for $\beta : \Sigma \to \mathfrak{g}$ is:

$$\delta_\beta X = [X, \beta] \quad (6.18a)$$

$$\delta_\beta A = \text{d}\beta + [A, \beta] - r[\text{d}X + [A, X], \beta + \text{Ad}_{g_X}^{-1}[X, \beta]] \quad (6.18b)$$

which this time corresponds to the vector fields of the infinitesimal form of the right dressing action of $G$ on $G^\ast$. On-shell, $[\delta_{\beta_1}, \delta_{\beta_2}] = \delta_{[\beta_1, \beta_2]}$.

The solutions of the equations of motion are, locally,

$$A = h^{-1}\text{d}h \quad X = \text{Ad}_h^{-1}\hat{X}. \quad (6.19)$$

They may also be rewritten as an equation in $D$:

$$\begin{pmatrix} e & 0 \\ \hat{X} & e \end{pmatrix} g_X \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \tilde{h} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} h \begin{pmatrix} e & 0 \\ X & e \end{pmatrix} g_X$$

which can be obtained from (6.14) taking $X \to \hat{X}$, $g \to \tilde{h}$, $Y \to Y$. Now, $\hat{X} = h^{-1}\hat{X}$.

Now consider $\Sigma = \mathbb{R} \times [0, 1]$ and $\bar{X}|_{\partial\Sigma}$ free. $h$ must be constant along each connected component of the boundary. By choosing $h(t, 0) = \tilde{h}(t, 0) = e$, defining $h_1 := h(t, 1)$, $\tilde{h}_1 := \tilde{h}(t, 1)$, $X_1 := X(t, 1)$ and plugging in (6.19), we get

$$\omega^* = \frac{1}{2} \text{tr} \left( \delta \hat{X} + [\hat{X}, \delta g_X \hat{g}_X^{-1}] \right) \wedge \delta \tilde{h}_1 \hat{h}_1^{-1} + \text{Ad}_{g_X}^{-1}[X_1 \wedge \tilde{h}_1^{-1} \delta \hat{h}] = $$

$$= \frac{1}{2} \Omega(h_1 \hat{X}). \quad (6.20)$$

The reduced phase space $\bar{P}(G_r; G_r, G_r)$ is the set of pairs $([h], \hat{X})$ with $[h]$ a homotopy class of maps from $[0, 1]$ to $G$ which are the identity at $x = 0$ and have fixed value at $x = 1$. Notice the duality between $\bar{P}(G; G, G)$ and $\bar{P}(G_r; G_r, G_r)$ under the interchange $\hat{g} \leftrightarrow h_1$, $X_1 \leftrightarrow \hat{X}$. This is the triangular version of the bulk-boundary duality found in [17] and recalled in Section 5.1 for the factorizable case.

Now, consider as target of the Poisson sigma model the dual of a simple complex Lie algebra $\mathfrak{g}^\ast$ with the Kostant-Kirillov Poisson bracket. As mentioned in Example 5.2, this is the dual Poisson-Lie group of the simply
connected Lie group $G$ whose Lie algebra is $\mathfrak{g}$ endowed with the zero Poisson structure. The action in this particular case is:

$$S_{\text{BF}}(X, A) = \int_{\Sigma} \text{tr} \left( dX \wedge A - \frac{1}{2} [X, A] \wedge A \right) \tag{6.21}$$

with $X : \Sigma \to \mathfrak{g}$ and $A \in \Omega^1(\Sigma) \otimes \mathfrak{g}$. This is the action of BF-theory \([41]\) up to a boundary term.

The equations of motion are

$$\begin{align*}
    dX + [A, X] & = 0 \\
    dA + [A, A] & = 0.
\end{align*} \tag{6.22}$$

For $\epsilon \in C^\infty(M) \otimes \mathfrak{g}$ the gauge transformation

$$\begin{align*}
    \delta_\epsilon X & = [X, \epsilon] \\
    \delta_\epsilon A & = d\epsilon + [A, \epsilon]
\end{align*}$$

induces the change of the action \((6.21)\) by a boundary term

$$\delta_\epsilon S_{\text{BF}} = - \int_{\Sigma} d\text{tr}(dX\epsilon). \tag{6.24}$$

Note that in this case the gauge transformations close even off-shell

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_{[\epsilon, \epsilon']}$$

and induce the Lie algebra structure of $C^\infty(\Sigma) \otimes \mathfrak{g}$ in the space of parameters.

The equations of motion \((6.16), (6.17)\) are the same as \((6.22)\). We would like to understand this fact at the level of the action. A direct computation shows that the following equality holds:

$$S_{\text{PL}}^{\mathcal{r}}(\tilde{X}, A) = S_{\text{BF}}(X, A) - \frac{1}{2} (dX + [A, X]) \text{Ad}_{\mathcal{g}X} r \text{Ad}_{\mathcal{g}X}^{-1} (dX + [A, X]).$$

Hence, both actions differ by terms quadratic in the equations of motion. This means that the Poisson-Lie sigma model with target $G_{\mathcal{r}}$ is equivalent, for any triangular $\mathcal{r}$-matrix, to the Poisson-Lie sigma model over $G_{\mathcal{r}=0}$, i.e. BF-theory. This is the triangular version of the connection encountered in the factorizable case \((32)\), where every Poisson-Lie sigma model with target $G_{\mathcal{r}}$ for any factorizable $\mathcal{r}$-matrix is (locally) equivalent to the $G/G_{\mathcal{WZW}}$ model with target $G$. 
6.3 More general Poisson sigma models over Lie groups

In this section we solve the model with Poisson structure (5.10) following the lines of Section 6.1. As remarked in Section 5.1.1 this Poisson structure does not make $G$ into a Poisson-Lie group. In the resolution of the model we shall introduce a generalization of the dressing transformations and of the Heisenberg double and we shall be able to identify the symplectic leaves for the Poisson structure of $G$. For concreteness, we shall consider $G$ factorizable, although an analogous procedure might be carried out in the triangular case.

The action for the model is

$$S(g, A) = \int_{\Sigma} \text{tr}(dgg^{-1} \wedge A) - \frac{1}{4} \text{tr}(A \wedge (r + Adgr'Ad^{-1}g)A)$$

where $r$ and $r'$ are two solutions of the modified Yang-Baxter equation (5.8) with $\alpha = 1$. The equations of motion are

$$dgg^{-1} + \frac{1}{2}(r + Adgr'Ad^{-1}g)A = 0 \quad (6.25a)$$
$$d\tilde{A} - [\tilde{A}, \tilde{A}]_{r'} = 0, \quad \tilde{A} := Adg^{-1}A. \quad (6.25b)$$

From the previous equations, or performing in the action variations of the fields that keep $\tilde{A}$ unchanged, we obtain

$$dA + [A, A]_r = 0.$$ 

Take $\beta : \Sigma \to \mathfrak{g}$. The gauge symmetry in its infinitesimal form reads:

$$\delta \beta gg^{-1} = -\frac{1}{2}(r + Adgr'Ad^{-1}g)\beta$$
$$\delta \beta A = d\beta + [A, \beta]_r - \frac{1}{2}[dgg^{-1} + \frac{1}{2}(r + Adgr'Ad^{-1}g)A, \beta].$$

The transformation for $g$ corresponds to the right dressing vector field that comes from the contraction of the Poisson structure with the right-invariant forms. On-shell, $[\delta \beta_1, \delta \beta_2] = \delta [\beta_1, \beta_2]_r$. Hence, the symmetry group is the one corresponding to the matrix $r$, i.e. $G_r$. The reason for the preferred role of $r$ against $r'$ in the gauge symmetry is simply the choice of right-invariant one forms to express the Poisson structure. Had we chosen left-invariant forms (i.e. changing the variable $A$ by $\tilde{A}$ in the action) the symmetry algebra would have been that of $r'$.

We can consider $G_r$ or $G_{r'}$ as different subgroups of the same double group $D$. We shall denote by $(h_+, h_-)$ (resp. $(h_{r'}, h_{r'})$) the elements of $G_r$ (resp. $G_{r'}$).
Using the procedure of Section 6.1, we can easily solve the model. As before, we first write locally the solutions for $A$ and $\tilde{A}$,

$$
A = h_+dh_+^{-1} - h_- dh_-^{-1} \\
\tilde{A} = \tilde{h}_- dh_+^{-1} - \tilde{h}_+ dh_-^{-1},
$$

with $h_{\pm}(\sigma_0) = \tilde{h}_{\pm}(\sigma_0) = e$.

The equation of motion (6.25a) can be transformed into

$$
dgg^{-1} + r_\pm A + \text{Ad}_g r'_\mp \tilde{A} = 0
$$

and inserting the solutions for $A$, $\tilde{A}$

$$
dgg^{-1} + h_\pm dh_\pm^{-1} - \text{Ad}_g h'_\mp \tilde{h}'_\mp = 0.
$$

Or equivalently,

$$
h_{\pm}^{-1}g\tilde{h}_{\mp} = h_{\pm}^{-1}g\tilde{h}'_{\mp} = \hat{g}
$$

with $\hat{g} = g(\sigma_0) \in G$.

We can write the general solution as an equation in $D$:

$$
(g(\sigma)\tilde{h}_{\mp}(\sigma), g(\sigma)\tilde{h}'_{\pm}(\sigma)) = (h_{\pm}(\sigma)\hat{g}, h_{\pm}(\sigma)\hat{g}). \quad (6.26)
$$

If we now define $D'_0 := G_rG_d \cap G_dG_{-r'}$, we see that for solutions of the equations of motion, $(h_{\pm}(\sigma)\hat{g}, h_{\pm}(\sigma)\hat{g}) \in D'_0$. The symplectic leaves of $G_{r,r'}$ are connected components of the orbits of the generalized dressing transformation of $\hat{g}$ by $(h_+, h_-) \in G_r$, that comes from solving equation (6.26) in $g(\sigma)$.

In order to describe the presymplectic structure on the space of solutions we need to introduce a new Poisson bracket on $D$. Recall that the Heisenberg double was defined in Section 5.1.1 as $D_{R,R}$ with $R = P_d - P_\tau$. We can generalize this construction by introducing another $r$-matrix $r'$ that gives rise to $R' = P_d - P_{-r'}$. The Poisson structure in the double we are interested in is $D_{R,R'}$. It is non-degenerate around the unit and its main symplectic leaf is denoted by $D'_0$. If we parametrize the points in $D'_0$ by $(\eta_+, \eta_-) = (\tilde{\xi}_\mp, \tilde{\eta}_\mp, \tilde{\eta}'_\pm)$, the symplectic structure in $D'_0$ obtained by inverting the Poisson bracket is

$$
\Omega'(\eta_+, \eta_-) = \text{tr}[d\tilde{\xi}_\mp^{-1}\wedge(d\eta_+\eta'^{-1}_+ - d\eta_-\eta'^{-1}_- - \xi^{-1}d\xi \wedge(\eta'^{-1}_+ d\eta'_+ - \eta'^{-1}_- d\eta'_-)].
$$

Note again that although for a given point of $D'_0$ factors $\xi, \eta_\pm, \tilde{\xi}, \tilde{\eta}_\pm$ in general are not uniquely determined (different choices differ by elements of the discrete groups $G_0$ or $G'_0$) the form $\Omega'$ is not affected by the ambiguity and is indeed well defined in $D'_0$.

The presymplectic structure of the $r,r'$ Poisson sigma model, $\omega'$, in the open geometry, $\Sigma = \mathbb{R} \times [0, 1]$, can then be written in terms of $\Omega'$. It reads

$$
\omega' = \Omega'((h_1 + \hat{g}, h_{1-}\hat{g}) - \Omega'((h_0 + \hat{g}, h_{0-}\hat{g})
$$
with \( h_{0\pm} = h_\pm(0) \) \( h_{1\pm} = h_\pm(1) \). And if we take \( \sigma_0 = (0, t_0) \), i.e. \( h_{0\pm} = e \),
\[
\omega' = \Omega'((h_{1+}\hat{g}, h_{1-}\hat{g})).
\]

The discussion of the gauge transformations and the reduced phase space goes parallel to the previous models. Points of the reduced phase space are pairs \( \{([h_+, h_-]), \hat{g}\} \) with \([h_+, h_-]\) the homotopy class of maps \([0, 1] \to G_r\) with fixed boundary values and such that \((h_+(x)\hat{g}, h_-(x)\hat{g}) \in D'_0\). The symplectic form on the reduced phase space can be obtained as the pullback of \( \Omega' \) by the map \( \{([h_+, h_-]), \hat{g}\} \mapsto (h_{1+}\hat{g}, h_{1-}\hat{g}) \in D'_0\).

### 6.4 Branes in Poisson-Lie sigma models

In the previous section we have seen that, if \( \Sigma = \mathbb{R} \times [0, 1] \), the moduli spaces of solutions of the Poisson-Lie sigma models over \( G \) and \( G_r \cong G^* \) coincide when \( g \) is free at the boundary (i.e. when the brane is the whole target group). We would like to find out whether such duality holds for more general boundary conditions. That is, we address the problem of finding pairs of branes \( N \subset G \) and \( N^* \subset G^* \) such that \( \mathcal{P}(G; N, N) \cong \mathcal{P}(G^*; N^*, N^*) \).

Let us restrict \( g|_{\partial\Sigma} \) to a closed submanifold (brane) \( N \subset M \). It is natural to ask the brane \( N \) to respect the Poisson-Lie structure of \( G \) given by the \( r \)-matrix. To this end we consider a simple subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) such that it is \( r \)-invariant, i.e. \( r|_{\mathfrak{h}} \subset \mathfrak{h} \). The restriction of \( r \) to \( \mathfrak{h} \), \( r|_{\mathfrak{h}} \) is an \( r \)-matrix in \( \mathfrak{h} \).

Since \( \mathfrak{h} \) is simple, its Killing form coincides (up to a constant factor) with the restriction to \( \mathfrak{h} \) of the Killing form in \( \mathfrak{g} \). Let \( H \subset G \) be the subgroup with Lie algebra \( \mathfrak{h} \). Then, for \( g \in H, X, Y \in \mathfrak{h} \),
\[
\pi^r_g (X, Y) = \frac{1}{2} \text{tr}((Xr|_\mathfrak{h})Y - X\text{Ad}_g r|_\mathfrak{h}\text{Ad}_g^{-1}Y) \tag{6.27}
\]
defines a Poisson-Lie structure on \( H \). The nice point is that we can realize \( H^* \), the dual Poisson-Lie group of \( H \), as a subgroup of \( G_r \). \( H^* \) is simply identified with the subgroup \( H^* \subset G_r \) corresponding to the Lie subalgebra \( \{[\cdot, \cdot], r\} \) of \( \{[\cdot, \cdot], r\} \) of \( \mathfrak{g} \). We claim that the Poisson-Lie sigma model with target \( G \) and brane \( H \) is dual to the Poisson-Lie sigma model with target \( G_r \) and brane \( H_r \). That is to say, there is a bulk-boundary duality between \( \mathcal{P}(G; H, H) \) and \( \mathcal{P}(G_r; H_r, H_r) \).

Before describing the duality we shall make some general considerations about the properties of these branes. The results of Section 3.2 on the boundary conditions of the fields are written in terms of the field \( \eta \in \Gamma(T^*\Sigma \otimes X^*T^*M) \). Let us rewrite them in terms of the field \( A \) appearing in the action \( \mathcal{L} \).

When taking variations of \( \mathcal{L} \) with respect to \( g \), a boundary term \(-\int_\Sigma d\text{tr}(dg g^{-1} \wedge A) \) appears. Its cancellation requires \( A_{\perp} \in \mathfrak{h}^\perp \) (\( \perp \) means orthogonal with respect to \( \text{tr} \)). On the other hand, the continuity of \( \mathcal{L} \)
at the boundary imposes $\pi^\sharp A_\tau \in \mathfrak{h}$. Consequently, the boundary condition for $A_\tau$ is
\begin{equation}
A_\tau(\sigma) \in \mathfrak{h}^\perp \cap \pi^\sharp_{g(\sigma)} \mathfrak{h}, \quad \sigma \in \partial \Sigma 
\end{equation}
and the gauge transformation parameter $\beta$ at the boundary is restricted by the same condition.

The condition (2.14) for pointwise Poisson-Dirac branes, applied to our present situation reads
\begin{equation}
\mathfrak{h} \cap \pi^\sharp g \mathfrak{h}^\perp = 0, \quad \forall g \in H.
\end{equation}
In particular, if $H$ is Poisson-Dirac the gauge transformations do not act on $g$ at the boundary.

We have that $H$ is coisotropic if
\begin{equation}
\pi^\sharp_g \mathfrak{h}^\perp \subseteq \mathfrak{h}, \quad \forall g \in H.
\end{equation}

We show now that an $r$-invariant, simple, subgroup $H$ is a Poisson-Dirac submanifold of $G$, and its dual $H_r$ is also Poisson-Dirac in $G_r$. Denote by $\mathfrak{h}^\perp \subset \mathfrak{g}$ the subspace orthogonal to $\mathfrak{h}$ with respect to $\text{tr}(\ )$. Firstly, the $r$-invariance of $\mathfrak{h}$ implies that $\pi^\sharp_r \mathfrak{h} \subseteq \mathfrak{h}, \forall g \in H$. Using that $\pi^\sharp_r$ is antisymmetric we obtain that $\pi^\sharp_r \mathfrak{h}^\perp \subseteq \mathfrak{h}^\perp, \forall g \in H$. Finally, recalling that $\mathfrak{h}$ simple $\Rightarrow \mathfrak{h} \cap \mathfrak{h}^\perp = 0$, one immediately deduces that $H$ is Poisson-Dirac.

Observing that $\mathfrak{h}_r$ (i.e. $\mathfrak{h}$ equipped with the Lie bracket $[\cdot, \cdot]_{\mathfrak{r}\mathfrak{h}}$) is the same as $\mathfrak{h}$ as a vector subspace of $\mathfrak{g}$ and reasoning as above one shows that $H_r$ is a Poisson-Dirac submanifold in $G_r$.

$H$ and $H_r$ inherit a (smooth) Poisson structure (the Dirac bracket) from $G$ and $G_r$, respectively. They coincide with the Poisson structures defined by $r|_{\mathfrak{h}}$ on $H$ and $H_r$ (formula (6.27) for $H$, and analogously for $H_r$), making them into a pair of dual Poisson-Lie groups. Notice, however, that this induced Poisson structure does not make $H$ (resp. $H_r$) into a Poisson submanifold of $G$ (resp. $G_r$), so that in general it is not a Poisson-Lie subgroup, it is so if and only if it is coisotropic.

We now address the issue of the duality of the models with a pair of branes $H$ and $H_r$ as above. The general picture is as follows. In the case of the model over $G$ with brane $H$ the space of solutions, once reduced by the gauge transformations in the bulk, can be identified with the universal covering of $G_rH_d \cap H_dG_r \subset \mathbf{D}_0$. The symplectic form $\Omega$ (see (3.5)) in $\mathbf{D}_0$ (corresponding to free boundary conditions) becomes degenerate in $G_rH_d \cap H_dG_r$. This reflects the existence of gauge transformations at the boundary. Since $\mathfrak{h}$ is $r$-invariant, there is a natural choice of gauge fixing for these transformations: $H_rH_d \cap H_dH_r$. The pullback of $\Omega$ to $H_rH_d \cap H_dH_r$ is nondegenerate and the infinitesimal gauge transformations span a complementary subspace to $T_p(H_rH_d \cap H_dH_r)$ in $T_p(G_rH_d \cap H_dG_r)$ for every $p \in H_rH_d \cap H_dH_r$. 


The dual model over $G_r$ with brane $H_r$ behaves in an analogous way. The space of solutions of the equations of motion can be identified with the set $G_d H_r \cap H_r G_d$, but there are still gauge transformations acting on this space. The gauge fixing is given again by considering the restriction to $H_r H_d \cap H_d H_r$, which makes this model equivalent by bulk-boundary duality to the previous one.

The considerations of the previous paragraphs did not care about the existence of singular points or whether the gauge fixing is local or global. These subtleties may depend on the concrete model. Let us work out an example in which all properties of regularity and global gauge fixing are met.

Take $G = SL(n, \mathbb{C})$ with the Poisson-Lie structure \( (5.7) \) given by the standard $r$-matrix \( (5.20) \) and $H = \{ (A_{00} 0 I) \in SL(n, \mathbb{C}), s.t. A \in SL(k, \mathbb{C}) \}$ \( (6.29) \) for a given $k < n$. The dual group $H_r \subset G_r \subset G \times G$ is easily described:

$$ H_r = \left\{ (g_+, g_-) \in G_r, \text{ s.t. } g_{\pm} = \begin{pmatrix} A_{\pm} & 0 \\ 0 & I \end{pmatrix}, A_{\pm} \in SL(k, \mathbb{C}) \right\}. \quad (6.30) $$

In this case,

$$ h_r^\perp = \left\{ \begin{pmatrix} \lambda I & B \\ C & X \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{C}) \right\}. \quad (6.31) $$

An easy calculation shows that $\pi_r'^- h_r^\perp = 0$, i.e. $H \subset G$ is Poisson-Dirac and coisotropic. In particular, the inclusion map $i : H \rightarrow G$ is a Poisson map and $H$ is a Poisson-Lie subgroup of $G$.

The situation is different for the dual model with target $G_r$ and brane $H_r$. Recall that $h_r$ is the same as $h$ as a vector space, and hence also their orthogonal complements. In this case $h_r^\perp \cap \pi_r'^- h_r^\perp \neq h_r^\perp$ and $H_r \subset G_r$ is not coisotropic. In fact, in generic points

$$ h_r^\perp \cap \pi_r'^- h_r^\perp = \left\{ \begin{pmatrix} \lambda I & 0 \\ 0 & X \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{C}) \right\}. \quad (6.32) $$

so that the brane $H_r$ is classically admissible.

The solutions of the equations of motion for $g$ in the model with target $G$ are given by

$$ g(\sigma) = (h_+(\sigma), h_-(\sigma)) \hat{g} \quad (6.33) $$

where $g(t, 0), g(t, 1) \in H$. One can always take $(h_+(t, 1), h_-(t, 1)) = (e, e)$ and $(h_+(t, 1), h_-(t, 1)) \in H_r$. One can fix the gauge freedom for $A_t$ at the boundary imposing $A_t = 0$. Then, $h_{\pm}$ are constant at every connected component of the boundary of $\Sigma$. Therefore, the reduced phase space $P(G; H, H)$ covers the set of pairs $((h_+, h_-), \hat{g})$ where $\hat{g} \in H$ and $(h_+, h_-) \in H_r$ and $(h_+ \hat{g}, h_- \hat{g}) \in H_r H_d \cap H_d H_r$. 
6.4. Branes in Poisson-Lie sigma models

For the Poisson-Lie sigma model with target $G_r$ the solutions of the equations of motion for $(g_+, g_-)$ are

$$(g_+ (\sigma), g_- (\sigma)) = h(\sigma) (\hat{g}_+ , \hat{g}_-)$$

with $(g_+ (t, 0), g_- (t, 0)), (g_+ (t, 1), g_- (t, 1)) \in H_r$.

An analogous argument shows that $\mathcal{P}(G_r; H_r, H_r)$ is (a covering of) the set of pairs $(h, (\hat{g}_+ , \hat{g}_-))$ where $(\hat{g}_+ , \hat{g}_-) \in H_r$, $h \in H$ and $(h\hat{g}_+, h\hat{g}_-)$ belongs to $H_d H_r \cap H_r H_d$.

The duality then exchanges degrees of freedom at the boundary with degrees of freedom in the bulk, exactly in the same way as it does for the free boundary conditions.

Notice that the duality described so far exists only for very special branes given by $r$-invariant subalgebras. If one considers more general situations the result is not that clean and one has, in the dual model, non-local boundary conditions that relate the fields at both connected components of $\partial \Sigma$. A more comprehensive treatment of this case will be done elsewhere.
Chapter 7

Supersymmetric WZ-Poisson sigma model
Generalized complex structures \([40, 39]\) on a manifold \(M\) are objects defined on \(TM \oplus T^*M\) which unify complex and symplectic geometry and interpolate between them. Recently, there has been much activity related to field theory realizations of generalized complex structures. Recall that extended supersymmetry in two-dimensional sigma models is related to complex geometry \([37]\) in the target. On the other hand, in the first-order formulation the fields of the models take values in \(TM \oplus T^*M\). This led to investigate the conditions for the existence of extended supersymmetry in first-order actions. The task was started in \([52]\) and systematically carried out in \([53]\), where it was shown that extended supersymmetry is closely related to generalized complex geometry in the target.

Later, in \([79]\), it was shown that the Hamiltonian formalism allows to rederive and extend the results of \([53]\) in a transparent fashion. In particular, it is proved in a model-independent way that if the generators of a second supersymmetry (denoted by \(Q_2(\epsilon)\)) are to satisfy the supersymmetry algebra, the target must be a (twisted, if a WZ term is present) generalized complex manifold. Choosing a concrete model (i.e. a concrete Hamiltonian) and imposing that \(Q_2(\epsilon)\) be a symmetry should yield some compatibility conditions relating the geometrical objects defining the model and the generalized complex structure \(J\). In the general case \([53]\) these conditions include a number of algebraic as well as differential equations.

Following the approach of \([79]\) we work out these compatibility conditions for the case of vanishing metric in the first-order sigma model, including a WZ term. When the closed 3-form \(H\) defining this twisting term vanishes, it is the supersymmetric version of the PSM. For \(H \neq 0\) it is the supersymmetric version of the WZ-Poisson sigma model \([48]\), whose target is equipped with a twisted Poisson structure \(\Pi\). We show that the notion of contravariant connections on twisted Poisson manifolds is the key for unraveling the differential compatibility conditions between \(\Pi\) and \(J\). We prove that, remarkably, the differential compatibility conditions are implied by the algebraic ones, for which we also give a simple geometrical interpretation.

### 7.1 Twisted Poisson manifolds

The effect of closed 3-form fields on Poisson manifolds has recently received attention in String Theory (see \([58, 24]\)). The presence of such closed 3-form leads to a modification of the Jacobi identity and to the notion of twisted Poisson manifolds, introduced in \([48]\). Following \([64]\), in this section we introduce these concepts and show that twisted Poisson structures fit naturally in the framework of Courant algebroids and Dirac structures.

Let \(M\) be an \(m\)-dimensional manifold. A *twisted Poisson structure* or *\(H\)-Poisson structure* on \(M\) is defined by a bivector field \(\Pi\) and a closed
3-form $H$ which in local coordinates $(x^1, \ldots, x^m)$ satisfy a modified Jacobi identity:

$$\Pi^{i\ell} \partial_\ell \Pi^{jk} + \Pi^{j\ell} \partial_\ell \Pi^{ki} + \Pi^{k\ell} \partial_\ell \Pi^{ij} = \Pi^{i\ell} \Pi^{jn} \Pi^{kr} H_{lnr}. \quad (7.1)$$

In the untwisted case, $H = 0$, $\Pi$ is a Poisson structure and (7.1) reduces to the Jacobi identity.

In general, the bracket of two functions $f, g \in C^\infty(M)$,

$$\{f, g\}(p) = \iota(\Pi_p)(df \wedge dg)_p, \quad p \in M$$

is not a Lie bracket anymore. Instead, we have

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} - \iota(X_f)\iota(X_g)\iota(X_h) H$$

where, for any function $f \in C^\infty(M)$, $X_f = \Pi^\sharp df$.

Recall from Section 2.3 that Poisson structures are particular examples of the much more general and powerful notion of Dirac structures. Nicely, the Courant bracket on $TM \oplus T^*M$ can be twisted by a closed 3-form in such a way that the twisted version of Dirac structures can be consistently defined and twisted Poisson structures fit in this picture.

The $H$-twisted Courant bracket on $TM \oplus T^*M$ is defined by

$$[(X_1, \xi_1), (X_2, \xi_2)]_H =$$

$$([X_1, X_2], \iota(X_1)d\xi_2 - \iota(X_2)d\xi_1 + \frac{1}{2}d(\iota(X_1)\xi_2 - \iota(X_2)\xi_1) + \iota(X_1)\iota(X_2)H)$$

for $(X_1, \xi_1), (X_2, \xi_2) \in \Gamma(TM \oplus T^*M)$.

An $H$-Dirac structure in $TM \oplus T^*M$ is defined analogously to the untwisted case of Section 2.3. It is a maximally isotropic subbundle of $TM \oplus T^*M$ with respect to the symmetric pairing

$$\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \xi_1(X_2) + \xi_2(X_1) \quad (7.2)$$

and whose sections are closed under the $H$-twisted Courant bracket.

As expected, the graph of a bivector field $\Pi$,

$$L_\Pi = \{(\Pi^\sharp \xi, \xi) \in TM \oplus T^*M | \xi \in T^*M\} \quad (7.3)$$

is an $H$-Dirac structure if and only if $\Pi$ is an $H$-Poisson structure. The identification of $L_\Pi$ with $T^*M$ by projection to the second component makes $T^*M$ into a Lie algebroid with anchor $\Pi^\sharp$ and Lie bracket defined on exact sections by

$$[df, dg] = df \{f, g\} + \iota(X_f)\iota(X_g)H.$$
7.1. Twisted Poisson manifolds

The action of two-forms on $TM \oplus T^*M$ will play a very important role in twisted Poisson geometry and twisted generalized complex geometry. Let $b$ be a 2-form on $M$ and define its action on $TM \oplus T^*M$ as

$$e^b(X, \xi) := (X, \xi + \iota(X)b) \quad (7.4)$$

which is called a $b$-transform. The interesting point is that this action is a morphism of Courant algebroids. Concretely,

$$e^b([X_1, \xi_1], [X_2, \xi_2])_H = [e^b(X_1, \xi_1), e^b(X_2, \xi_2)]_{H+d\Pi}.$$

Under a $b$-transform (7.4), the $H$-Dirac structure $L_\Pi$ is transformed into $e^b(L_\Pi)$, which is an $(H+d\Pi)$-Dirac structure. However, the subbundle $e^b(L_\Pi)$ is the graph of a bivector field if and only if $1 + b\Pi$ is invertible. In this case, $e^b(L_\Pi) = L_{\Pi b}$, where

$$\Pi b := \Pi(1 + b\Pi)^{-1}$$

is an $(H + d\Pi)$-Poisson structure. In particular, if $H = dB$, a $b$-transform with $b = -B$ untwists the twisted Poisson structure, so that $\Pi b$ is an ordinary Poisson structure (provided that $1 + b\Pi$ is invertible, of course).

If $b$ is closed the transformation (7.4) gives an orthogonal automorphism of the twisted Courant bracket. The most general orthogonal automorphism of the twisted Courant bracket (39) is a semidirect product of a diffeomorphism of $M$ and a $b$-transform with $b \in \Omega^2_{\text{closed}}(M)$.

### 7.1.1 Contravariant connections

Assume we want to endow the twisted Poisson manifold $(M, \Pi, H)$ with a covariant connection. It is natural to demand this connection to be compatible with $\Pi$, so that the covariant derivative of $\Pi$ vanishes. Already in the untwisted case one can prove (71) that a compatible covariant connection exists if and only if the rank of the Poisson tensor is constant. Much of the interest of Poisson and twisted Poisson structures resides in the fact that the rank of $\Pi$ can be non-constant, differing in an essential way from symplectic and twisted symplectic manifolds. Therefore, the notion of covariant derivative is not appropriate for twisted Poisson manifolds. The relevant concepts in Poisson manifolds are those of contravariant derivatives introduced by Vaisman (71) and contravariant connections developed by Fernandes (36). Next, we extend their definitions and some results to the case of twisted Poisson manifolds.

A contravariant derivative on a vector bundle $E$ over the twisted Poisson manifold $(M, \Pi, H)$ is an operator $\nabla$ such that for each $\alpha \in \Omega^1(M)$, $\nabla_\alpha$ maps sections of $E$ to sections of $E$ satisfying:
7.2. Twisted generalized complex structures

(i) \( \nabla_{\alpha_1 + \alpha_2} \psi = \nabla_{\alpha_1} \psi + \nabla_{\alpha_2} \psi \)
(ii) \( \nabla_{\alpha}(\psi_1 + \psi_2) = \nabla_{\alpha} \psi_1 + \nabla_{\alpha} \psi_2 \)
(iii) \( \nabla_{f\alpha} \psi = f\nabla_{\alpha} \psi + \Pi^\beta \alpha(f) \psi \).

A contravariant derivative defines a contravariant connection in an analogous way to the covariant case. We shall be interested in defining the contravariant derivative of tensor fields on \( M \). To this end it is enough to have a contravariant derivative on \( E = T^*M \). Take local coordinates \( x^i \) on \( M \) and define the Christoffel symbols \( \Gamma_{ij}^k \) by

\[
\nabla_{dx^i} dx^j = \Gamma_{ij}^k dx^k.
\]

The contravariant derivative of a tensor field \( K \) of type \( (p,q) \) is given by

\[
\nabla^n K_{i_1...i_p j_1...j_q} = \Pi^{il} \partial_l K_{i_1...i_p j_1...j_q} - \sum_{\mu=1}^p \Gamma_{ijl}^{\mu} K_{i_1...l...i_p j_1...j_q} + \sum_{\nu=1}^q \Gamma_{i_1...j_1...l...j_q}^{\nu} K_{i_1...i_p j_1...j_q}.
\]

A tensor field \( K \) is called parallel if \( \nabla K = 0 \). The relevant result for us is given by the following

**Theorem:** Let \((M, \Pi, H)\) be a twisted Poisson manifold. Then, there exists a contravariant connection such that \( \Pi \) is parallel.

**Proof:** Let \( \{U_n\} \) be an open cover of \( M \). Take local coordinates \( x^i \) on \( U_n \) and define

\[
\Gamma_{(n)ij}^k = \partial_k \Pi^{ij} - \frac{1}{2} \Pi^{il} \Pi^{jm} H_{klm}.
\]

On \( U_n \), \( \Pi \) is parallel for the contravariant connection \( \nabla_{(n)} \) with symbols \( \Gamma_{(n)} \). If \( \sum_n f_n = 1 \) is a partition of unity subordinated to the cover \( \{U_n\} \), \( \nabla = \sum_n f_n \nabla_{(n)} \) gives a contravariant connection on \( M \) such that \( \nabla \Pi = 0 \). \( \square \)

7.2 Twisted generalized complex structures

If \( x^i \) are local coordinates on \( M \) and take \( (\partial_i, dx^i) \) as a basis in the fibers of \( TM \oplus T^*M \), the bilinear form \( (7.2) \) reads

\[
\mathcal{I} = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}.
\]

An almost generalized complex structure on the manifold \( M \) is a linear map \( \mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M \) such that \( \mathcal{J}^t \mathcal{J} = \mathcal{I} \) and \( \mathcal{J}^2 = -1_{2m} \).
For such $J$ define $J_{\pm} = \frac{1}{2}(1_{2m} \pm iJ)$. The almost generalized complex structure $J$ is an $H$-twisted generalized complex structure if
\[
J_{\pm}[J_{\pm}(X_1 + \xi_1), J_{\pm}(X_2 + \xi_2)]_{H} = 0, \quad (7.6)
\]
$\forall X_1 + \xi_1, X_2 + \xi_2 \in \Gamma(TM \oplus T^*M)$. It follows that the $b$-transform of an $H$-twisted generalized complex structure $J$, \[ J_b := e^b J e^{-b} \]
is an $(H + db)$-twisted generalized complex structure.

In the coordinate basis $(\partial_i, dx^i)$ we can write,
\[
J = \begin{pmatrix} J & P \\ L & K \end{pmatrix}
\]
with $J : TM \to TM, P : T^*M \to TM, L : TM \to T^*M, K : T^*M \to T^*M$.

The condition $J^T I J = I$ becomes

\[
J^i_j + K^i_j = 0, \quad P^{ij} = -P^{ji}, \quad L_{ij} = -L_{ji}
\]
whereas $J^2 = -1_{2m}$ translates into

\[
J^i_k J^k_j + P^{ik} L_{kj} = -\delta^i_j \\
J^i_k P^{kj} + P^{ik} K^j_k = 0 \\
K^k_i K^j_k + L_{ik} P^{kj} = -\delta^i_j \\
K^k_i L_{kj} + L_{ik} J^k_j = 0.
\]

The integrability condition (7.6) is equivalent to the following differential equations:

\[
J^i_[jJ^i_k]_j + J^i_j J^j_[l,k] + P^{ij}(L_{[lk,j]} + J^a_i H_{k]nj}) = 0 \\
P^{[ij} P^{lk]} = 0 \\
J^i_j k P^{kl} + P^{kl} j - J^i_k_j = J^j_j k P^{ik} + P^{ik} j J^k_j - P^{i} j k J^k_j - P^{ln} P^{ik} H_{nkj} = 0 \\
J^i_j L_{[lk,r]} + J^i_r L_{jk,l} + J^i_{lj} L_{jk,l} + J^i_{lk} L_{jr,l} + J^i_{lk} L_{tr,j} + +H_{kjr} - J^i_{[k J^a_i H_{j]ln} = 0
\]
where a comma denotes a partial derivative and the brackets stand for antisymmetrization. Notice that $P^{ij}$ is always a Poisson structure.
7.3 WZ-Poisson sigma model

In this section we introduce the twisted Poisson sigma model or WZ-Poisson sigma model (WZ-PSM) in the Lagrangian formalism and see how twisted Poisson geometry arises.

Let \((M, \Pi, H)\) be a twisted Poisson manifold. We assume for simplicity that \(H = dB\) (which is always locally true). The action of the WZ-PSM is

\[
S = \int_{\Sigma} \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j - \frac{1}{2} B_{ij} dX^i \wedge dX^j \tag{7.8}
\]

which is obtained from the action of the PSM \(\text{(3.2)}\) by adding the WZ term \(-\frac{1}{2} \int_{\Sigma} \eta \wedge d\eta\). The equations of motion in the bulk are:

\[
dX^i + \Pi^{ij} \eta_j = 0 \tag{7.9a}
\]
\[
d\eta_i + \frac{1}{2} \partial_i \Pi^{jk} \eta_j \wedge \eta_k - \frac{1}{2} H_{ijk} dX^i \wedge dX^j = 0 \tag{7.9b}
\]

with \(H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}\). Notice that \((7.9a)\) does not get any contribution from \(B\) whereas the additional term in \((7.9b)\) only depends on \(H\). In fact, if \(\partial \Sigma = \emptyset\), the model depends on \(H\) and not on \(B\). Analogously to the case of the ordinary PSM, the consistency of the equations of motion \((7.9)\) require that \(\Pi\) and \(H\) satisfy \((7.1)\), so that \(\Pi\) is an \(H\)-Poisson structure.

The action \((7.8)\) is invariant (up to boundary terms) under the local transformations:

\[
\delta \epsilon_i X^i = \Pi^{ij}(X) \epsilon_j \\
\delta \epsilon_i \eta_i = -d\epsilon_i - \partial_i \Pi^{jk}(X) \eta_j \epsilon_k + H_{ijk} \Pi^{lm} \eta_l \eta_m \epsilon_k
\]

with \(\epsilon(\sigma) = \epsilon_i(\sigma) dX^i\) the gauge parameter.

The treatment of twisted Poisson structures in terms of Dirac structures given in Section \(7.1\) shows that \(b\)-transforms act on \(\Pi\) connecting it with other twisted Poisson structures. In particular, if \(H = dB\) and \(1 - B\Pi\) is invertible, the transformed object \(\bar{\Pi} := \Pi - B\Pi(1 - B\Pi)^{-1}\) is an ordinary Poisson structure. It is then natural to ask whether, in this case, there exists a field redefinition transforming \((7.8)\) into an untwisted PSM with Poisson structure \(\bar{\Pi}\). On-shell this is actually possible, as we proceed to show.

Take \(\tilde{\eta} = (1 - B\Pi) \eta\). The action \((7.8)\) written as a functional of \(X\) and \(\tilde{\eta}\) reads:

\[
S = \int_{\Sigma} \tilde{\eta}_i \wedge dX^i + \frac{1}{2} \tilde{\Pi}^{ij}(X) \tilde{\eta}_i \wedge \tilde{\eta}_j - \frac{1}{2} B_{ij} (dX^i + \tilde{\Pi}^{ik} \tilde{\eta}_k) \wedge (dX^j + \tilde{\Pi}^{jl} \tilde{\eta}_l)
\]

which differs from the PSM with Poisson structure \(\bar{\Pi}\) by terms quadratic in the equations of motion and therefore it is classically equivalent to it.

Consequently, if \(H = dB\) and \(1 - B\Pi\) is invertible the WZ-PSM is equivalent to an ordinary PSM. However, if \(H\) is not exact or \(1 - B\Pi\) is
7.4. Supersymmetric WZ-Poisson sigma model

Let $S^{1,1}$ be the supercircle with coordinates $\bar{\sigma} = (\sigma, \theta)$. In local coordinates the cotangent bundle of the superloop space, $T^*LM$, is given by scalar superfields $\Phi^i(\sigma, \theta)$ and spinorial superfields $\Psi_i(\sigma, \theta)$. In components,

$$\Phi^i(\sigma) = X^i(\sigma) + \theta \lambda^i(\sigma), \quad \Psi_i(\sigma) = \rho_i(\sigma) + \theta \eta_i(\sigma)$$

where $X^i$ and $\eta_i$ are bosonic fields.

Assume that $M$ is equipped with a closed 3-form $H$. Then, the following 2-form on $T^*LM$ is symplectic:

$$\omega = \int_{S^{1,1}} d\sigma d\theta \left( \delta \Phi^i \wedge \delta \Psi_j - \frac{1}{2} H_{ijk} D \Phi^i \delta \Phi^j \wedge \delta \Phi^k \right) \quad (7.11)$$

where $\delta$ stands for the de Rham differential and $D = \partial_\theta - \theta \partial_\sigma$. Since $\omega$ is closed and non-degenerate, it defines a Poisson bracket on functions of the superfields which we shall denote by $\{\cdot, \cdot\}$.

The basic Poisson brackets read:

$$\begin{align*}
\{ \Phi^i(\bar{\sigma}), \Phi^j(\bar{\sigma'}) \} &= 0 \\
\{ \Phi^i(\bar{\sigma}), \Psi_j(\bar{\sigma'}) \} &= \delta^i_j \delta(\bar{\sigma} - \bar{\sigma'}) \\
\{ \Psi_i(\bar{\sigma}), \Psi_j(\bar{\sigma'}) \} &= H_{ijk} D \Phi^k \delta(\bar{\sigma} - \bar{\sigma'})
\end{align*}$$

with $\delta(\bar{\sigma} - \bar{\sigma'})$ the superspace delta distribution. Notice that

$$\Psi_i \mapsto \Psi_i - B_{ij} D \Phi^j \quad (7.12)$$

is a canonical transformation for closed $B$.

The Hamiltonian formulation of the $N=1$ supersymmetric WZ-PSM is as follows. The phase space of the theory, denoted by $\mathcal{P}$, is the set of points of $T^*LM$ satisfying the constraints:

$$D \Phi^i(\sigma, \theta) + \Pi^{ij}(\Phi) \Psi_j(\sigma, \theta) = 0, \quad i = 1, \ldots, m.$$ 

which are a consequence of the singular nature of the Lagrangian of the WZ-PSM ([2]). The Hamiltonian of the WZ-PSM can be written:

$$\mathcal{H} = \int_{S^{1,1}} F_i(\sigma, \theta) \left( D \Phi^i(\sigma, \theta) + \Pi^{ij}(\Phi) \Psi_j(\sigma, \theta) \right) d\sigma d\theta$$

where the fields $F_i$ act as Lagrange multipliers. As we already know, the consistency of the model requires $\Pi$ to be an $H$-Poisson structure. In the
Hamiltonian formalism this is obtained as the condition for the dynamics to preserve the submanifold $\mathcal{P}$, i.e. $\{D\Phi^i + \Pi^i\Psi_j, \mathcal{H}\}|_\mathcal{P} = 0$.

By construction, the WZ-PSM is invariant under the supersymmetry transformation generated by:

$$Q_1(\epsilon) = \int_{S^1} d\sigma \epsilon \left( \eta^i \lambda^i - \rho_i \partial X^i - \frac{1}{3} H_{ijk} \lambda^i \lambda^j \lambda^k \right)$$

where $\epsilon$ is a constant anticommuting parameter and $\partial X^i := \frac{\partial X^i}{\partial \sigma}$.

We address now the issue of extended supersymmetry in the WZ-PSM. The most general ansatz for a second supersymmetry transformation is given by (7.12)

$$Q_2(\epsilon) = -\frac{1}{2} \int_{S^1,1} d\sigma d\theta \epsilon (2D\Phi^i \Psi_j J^j_i + D\Phi^i D\Phi^j L_{ij} + \Psi_i \Psi_j P^{ij}). \quad (7.13)$$

As shown in [79] in a model-independent way, the generators $Q_2(\epsilon)$ satisfy the supersymmetry algebra if and only if

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & -J^i \end{pmatrix}$$

is an $H$-twisted generalized complex structure. In this context, the canonical transformation (7.12) is identified with a $b$-transform for closed $B$, i.e. an automorphism of the Courant algebroid. If $dB \neq 0$, $\Psi_i \mapsto \Psi_i - B_{ij} D\Phi^j$ is not a canonical transformation and it changes the twisting term in (7.11).

In particular, if $H = dB$ it untwists the symplectic structure.

It remains to find out when $Q_2(\epsilon)$ generates a symmetry transformation of the WZ-PSM. That is, when

$$\{Q_2(\epsilon), \mathcal{H}\}|_\mathcal{P} = 0. \quad (7.14)$$

At this stage one expects that (7.14) holds only if some compatibility conditions relating $\mathcal{J}$ and $\Pi$ are satisfied. These conditions were worked out in the Lagrangian formalism for a general (untwisted) first-order sigma model in [53]. It turns out that in the general case some algebraic as well as differential conditions must be imposed. Our aim is to prove that in the WZ-PSM the differential conditions are automatically implied by the algebraic ones. We shall see that the contravariant connections introduced in Section 7.1.1 are extremely helpful in the derivation of this result.

A direct calculation shows that (7.14) holds if and only if the following two conditions are met:

Algebraic condition:

$$P^{ij} + J^i_k \Pi^{kj} + \Pi^{ik} J^j_k - \Pi^{ik} L_{kj} \Pi^{ij} = 0. \quad (7.15)$$
Differential condition:
\[
\frac{1}{2} \left[ \Pi^i_l \partial_l P^{jk} - \partial_i \Pi^{ij} P^{lk} - \partial_i \Pi^{ik} P^{jl} + (\Pi^i_l \partial_l J^j_n - \partial_i \Pi^{ij} J^j_n + \partial_i \Pi^{il} J^j_l) \Pi^{nk} + \Pi^{jn}(\Pi^i_l \partial_l J^k_n + \partial_i \Pi^{il} J^k_l) - \Pi^{jn}(\Pi^i_l \partial_l L_{nr} + \partial_n \Pi^{il} L_{lr} + \partial_r \Pi^{il} L_{nl}) \Pi^{rk} \right] + (P^r_j + J^r_l \Pi^{lj}) \Pi^{in} \Pi^{lk} H_{lnr} = 0. \quad (7.16)
\]

The untwisted version of the differential condition (7.16) was deduced in [53] (see equation (6.35) therein) for a general untwisted sigma model. In the case of non-vanishing metric it was interpreted as a condition of constancy with respect to a covariant derivative compatible with the metric tensor. The WZ-PSM is the limit of vanishing metric of the twisted first-order sigma model and there is no such covariant derivative at hand. We have learnt in Section 7.1.1 that the natural objects which allow to compare tangent spaces at different points of a twisted Poisson manifold are contravariant connections. In fact, after some manipulations we can rewrite condition (7.16) in terms of the local symbols (7.5):
\[
\nabla^i P^{jk} + (\nabla^i J^j_n) \Pi^{nk} + \Pi^{jn}(\nabla^i J^k_n) - \Pi^{jn}(\nabla^i L_{nr}) \Pi^{rk} + (P^r_j + J^r_l \Pi^{lj}) \Pi^{in} \Pi^{lk} H_{lnr} = 0.
\]

Using that \( \nabla \Pi = 0 \) we can go one step further and write:
\[
\nabla^i \left( P^{jk} + J^j_n \Pi^{nk} + \Pi^{jn} J^k_n - \Pi^{jn} L_{nr} \Pi^{rk} \right) + (P^r_j + J^r_l \Pi^{lj}) \Pi^{in} \Pi^{lk} H_{lnr} = 0
\]
which is obviously satisfied if the algebraic condition (7.15) holds. Hence, we obtain the remarkable result that the WZ-PSM has a second supersymmetry (7.13) if and only if \( J \) is a generalized complex structure and the algebraic condition (7.15) is satisfied. This gives an enormous simplification with respect to the general sigma model, in which the differential condition analogous to (7.16) is not necessarily implied by the algebraic ones.

Now, we would like to give a geometrical interpretation of (7.15), which imposes a compatibility condition between the \( H \)-Poisson structure \( \Pi \) and the Poisson structure \( P \). Define two endomorphisms of \( TM \oplus T^*M \) by
\[
\tau_1 := \begin{pmatrix} 0 & \Pi \\ 0 & 1_m \end{pmatrix}, \quad \tau_2 := \begin{pmatrix} 1_m & -\Pi \\ 0 & 0 \end{pmatrix}.
\]

Notice that \( \tau_2^2 = \tau_1 \), \( i = 1, 2 \). \( \tau_1 \) projects onto the Dirac structure \( L_\Pi \) (see definition (7.3)) along \( TM \). \( \tau_2 \) projects onto \( TM \) along \( L_\Pi \). In particular, \( \text{Im}(\tau_1) = \text{Ker}(\tau_2) = L_\Pi \), which will be the important point for us.

---

1 We omit the subscript \((n)\) referring to an open set of a cover of \( M \).
2 The algebraic condition (7.15) was already deduced in [12], but a geometrical interpretation was lacking.
In matrix notation the condition (7.15) can be expressed as

\[
\tau_2 \begin{pmatrix} J & P \\ L & -J^t \end{pmatrix} \tau_1 = 0 \tag{7.17}
\]

which says that \( J \) can be restricted to an endomorphism of \( L_\Pi \). That is, the WZ-PSM supports extended supersymmetry if and only if

\[ J(L_\Pi) \subset L_\Pi. \]

Note that the canonical transformation (7.12) can be viewed as a \( b \)-transform with \( b = -B \in \Omega^2_{\text{closed}}(M) \) acting on \( J \) and \( \Pi \). Since a canonical transformation does not modify the Poisson brackets, one would expect that (7.15) hold for the transformed objects

\[
J_B = \begin{pmatrix} J_B & P_B \\ L_B & -J^t_B \end{pmatrix}, \quad \Pi_B = \Pi(1 + B\Pi)^{-1}.
\]

This is not evident from (7.15). However, the result follows easily by using that the expression (7.17) is manifestly invariant under a \( b \)-transform. But the argument is purely algebraic so that the result holds even if \( B \) is not closed.

The fact that in the WZ-PSM there is only one algebraic compatibility condition between \( J \) and \( \Pi \), given by (7.15), should make easier the search of backgrounds admitting extended supersymmetry. Notice that when trying to solve (7.15) one can use that it is invariant under a \( b \)-transform. Therefore, if \( H = dB \) and \( 1 + B\Pi \) is invertible one can look for solutions of (7.15) in terms of the untwisted objects and twist back at the end of the day.
Conclusions
We end this dissertation by collecting and summarizing our main results:

- We have presented a purely algebraic approach to the reduction of Poisson manifolds. We describe the reduced Poisson algebra and give sufficient conditions for the submanifold to inherit a Poisson structure.

- We give a natural way of producing new Dirac subbundles from a given one. The case in which the original Dirac subbundle is actually a Dirac structure is especially interesting. We apply our procedure to the reduction and projection of Dirac structures, generalizing previous results in the literature. The situation in which the original or the induced Dirac structure are Poisson is also investigated.

- A detailed study of the classical PSM defined on a surface with boundary has been performed. We have identified very general sufficient conditions (*weak regularity*) under which a submanifold is an admissible brane for the model.

- The Hamiltonian analysis of the PSM on the strip with two arbitrary branes has been carried out. We have described how the phase space encodes the Poisson geometry of the branes:
  - We showed that the presymplectic structure on the phase space is related to the Poisson bracket induced on each brane through a dual pair structure.
  - When the brane is second-class, we have proven that there exists a connection between the reduced phase space and the symplectic groupoid integrating the brane.

- At the quantum level, we have shown that the PSM on the disk with a brane satisfying the *strong regularity condition* (a slightly stronger version of the previous one) can be perturbatively quantized. When second-class constraints are present, the standard Feynman expansion valid for coisotropic branes breaks down and a redefinition of the perturbative series is needed. In the particular case of a second-class brane we prove that the perturbative quantization of the model is related to the Kontsevich formula for the Dirac Poisson structure on the brane.

- We have tackled the study of the PSM when the target is a Poisson-Lie group. We gave a thorough analysis for any Poisson-Lie structure associated to a simple Lie group $G$ (its dual group being denoted by $G^*$). Such structures are divided in two classes: factorizable and triangular.
  - We provide explicit realizations of any factorizable or triangular structure and solve the models over $G$ and $G^*$. We also solve some
related PSMs whose targets are Lie groups but not Poisson-Lie groups.

- In the Hamiltonian formalism, in the open geometry with free boundary conditions, we discovered a duality transformation relating the reduced phase space of the models corresponding to a pair of dual groups $G$ and $G^*$.

- Then, we identified a broad family of pairs of branes which preserve the duality.

- We have shown that, if the Poisson-Lie structure is triangular, the model over $G^*$ is equivalent to BF-theory.

- We have worked out the conditions for extended supersymmetry in the twisted Poisson sigma model. They reduce to an algebraic equation with a beautiful geometrical interpretation relating the twisted Poisson structure and the twisted generalized complex structure on the target.

- In the derivation of this result the notion of contravariant connection is very useful. We have proven that any twisted Poisson manifold has a compatible contravariant connection, extending the existing theorems on (standard) Poisson manifolds.
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