We demonstrate that vacuum diagrams in light front field theory are non-zero, contrary to the prevailing opinion. Using the light-front Hamiltonian (time-ordered) perturbation theory, the vacuum amplitudes in self-interacting scalar $\lambda \phi^2$ and $\lambda \phi^4$ models are obtained as $p = 0$ limit of the associated self-energy diagrams, where $p$ is the external momentum. They behave as $C \lambda^2 \mu^2$ in $D=2$, with $\mu$ being the scalar-field mass, or diverge in $D=4$, in agreement with the usual "equal-time" form of field theory, and with the same value of the constant $C$. The simplest case of the vacuum bubble with two internal lines is analyzed in detail. It is shown that, surprisingly, the light-front diagrams are nonvanishing not due to the zero-mode contribution. This is made explicit using the DLCQ method - the discretized (finite-volume) version of the theory, where the light-front zero modes are manifestly absent, but the vacuum amplitudes still converge to their continuum-theory values with the increasing "harmonic resolution" $K$. A brief discussion comparing the status of the scalar-field zero mode in the light-front theory and in the Feynman perturbation theory expressed in light-front variables concludes the paper.

I. INTRODUCTION

Quantum field theory (QFT) formulated in terms of light-front (LF) variables \( \mathbb{H} \) has a few unusual features which are sometimes interpreted as indicating its inconsistency or incompleteness \( \mathbb{R} \). One problem appeared to be paradoxically related to the most celebrated property of the LF quantization, namely vacuum simplicity. As is well known, positivity of the LF momentum $p^+$ together with its conservation implies that the ground state of any dynamical model cannot contain quanta carrying $p^+ \neq 0$. Only a tiny subset of all field modes, namely those carrying $p^+ = 0$ - the LF zero modes - can contribute. In addition, some modes (like the scalar-field zero mode) which appear as dynamical ones in the conventional ("space-like", SL for short) theory become constrained, that is, non-dynamical, in the LF form of the interacting theory (or vanish in the free theory), and hence cannot contribute to vacuum processes directly. A natural question then is if the LF formalism is capable to describe phenomena, related to nontrivial vacuum structure, like spontaneous symmetry breaking. More generally, is the LF form of QFT equivalent to the usual SL one, can it reproduce the known results or even predict something truly new?

The equivalence issue has been realized and studied by Chang and Ma and by T.-M. Yan already in the pioneering LF papers \( \mathbb{R} \), including the vacuum problem at the level of the perturbation theory. Here one should distinguish two methods. In the first method, one starts from the covariant Feynman amplitudes and rewrites the corresponding integrals in terms of LF variables $p^\pm = p^0 \pm p^3$. The delicate step is to perform the integration in $p^-$ variable (using the residue theorem) since $p^+$ is not manifestly covariant and has energy denominators instead of Feynman propagators. The integration over the $p^-$ variable is not present by construction. The same simple kinematical argument (positivity and conservation of $p^+$) suggests that vacuum bubbles are zero in this light-front perturbation theory (LFPT). A vacuum diagram can be non-vanishing only if all internal lines carry $p^+ = 0$, but the rules of LFPT give an ill-defined result in this case \( \mathbb{S} \). Thus, it has not been clear for a long time how the vacuum amplitudes can be correctly evaluated in the LF perturbation theory. As mentioned above, they have been considered as vanishing. One implication was that the cosmological constant problem could be solved in this way \( \mathbb{G} \). If true, this would mean that the conventional SL form and the LF form of the relativistic dynamics are not equivalent, since the vacuum loops diverge (logarithmically or quadratically) in the SL theory in $D=3+1$.  

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Recently, J. Collins drew attention to this contradiction \cite{10}. He used an example with the simplest vacuum diagram (two propagators) to identify the mathematical inconsistency present in the LF computation of the corresponding integrals. He also indicated an alternative approach, based on analyticity, to calculate the LF vacuum amplitudes as the limit of vanishing external momentum of the associated self-energy diagrams.

In the present letter, we extend this argument to the vacuum bubbles of two-dimensional $\phi^3$ and $\phi^4$ models. We show that the self-energy diagrams, when represented in the LF perturbation theory, reduce for external momentum $p = 0$ to nonzero limits equal to the values of vacuum bubbles known from the conventional Feynman diagrams. For example, the value of the vacuum loop of the $\lambda\phi^3$ model is given by the $p = 0$ limit of the two-loop self-energy of the $\lambda\phi^4$ model. Moreover, and this is the main result of our study, the non-vanishing of LF vacuum loops is not due to the LF zero modes: the same results as in the continuum theory are obtained in the DLCQ framework, namely in the finite-volume formulation with (anti)periodic boundary conditions, where the zero mode is manifestly absent. In this approach, the spectrum of field modes is discrete, enumerated by non-negative integers $n, 1 \leq n \leq K - 1$, where $K$ is called the "harmonic resolution". We show that although the Fourier modes with $n = 0$ are not present, the corresponding summations (which replace integrals of the continuum theory) still converge to the same values of the vacuum amplitudes for increasing values of $K$. This adds further evidence to the attitude that the LF form of QFT is fully equivalent to the usual SL one. However, the mechanisms and mathematical appearance may be different, and the LF theory requires fresh and independent thinking.

II. VACUUM AMPLITUDES IN THE CONVENTIONAL AND LIGHT-FRONT THEORY

In the language of perturbation theory, the non-trivial structure of the ground state in relativistic QFT manifests itself by vacuum-polarization amplitudes (diagrams). In the case of the self-interacting $\lambda\phi^3$ and $\lambda\phi^4$ theories, these diagrams are shown in Fig.1 (we shall work in two space-time dimensions). Such a process of creation of two or more particles from the vacuum does not violate the momentum conservation since $p^0$ can acquire both the positive and negative values which add to zero (momentum conservation). Once again, this seems impossible in the LF case where $p^0$ values carried by the two quanta are strictly positive (or vanishing if there exist dynamical LF zero modes in the given model).

Let us remind briefly how the vacuum amplitudes are calculated in the SL form with the example of the $\phi^3$ bubble. The corresponding Feynman rules lead to the double two-dimensional integral expression

$$V_3(\mu) = N\lambda^2 \int \frac{d^2 k_1}{k_1^2 - \mu^2} \int \frac{d^2 k_2}{k_2^2} G(k_1)G(k_2)G(k_1 + k_2), \quad (1)$$

$$G(k) = \frac{i}{k^2 - \mu^2 + i\epsilon}, \quad N = \frac{1}{3!} \frac{1}{i(2\pi)^4}. \quad (2)$$

The coefficient $1/3!$ is the symmetry factor. The above double integral can be evaluated in a few ways: by using the Feynman parameters \cite{11}, by means of representation \cite{12} or via more sophisticated mathematical methods (Mellin-Barnes representation for powers of massive propagators \cite{13}). All of them yield the same result

$$V_3(\mu) = -\frac{i\lambda^2}{\mu^2} \pi^2 NC, \quad C = 2.343908..., \quad (3)$$

where the constant $C$ has a particular representation in each of the computational method. For example, the first method requires to combine the propagators into one denominator by means of the auxiliary integrals in terms of the Feynman parameters $x_i$, then to go over to Euclidean space and calculate the integrals in $k_1$ and $k_2$ variables. The result is the double-integral representation

$$C = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{D(x_1, x_2)}, \quad (4)$$

$$D(x_1, x_2) = x_1(1 - x_1) + x_2(1 - x_2) - x_1x_2.$$  

The first integral can be calculated analytically in terms of $\arctan$ function and square-roots of polynomials, the numerical evaluation of the second integral then yields the above value of $C$.

If we consider the self-energy diagram instead of the vacuum bubble ($G(k_1 + k_2)$ replaced by $G(p - k_1 - k_2)$ in Eq.\,(1)), the analogous calculation yields

$$\Sigma_4(p^2) = N \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{\lambda^2}{A(x_1, x_2)p^2 - D(x_1, x_2)\mu^2},$$

$$A(x_1, x_2) = x_1x_2(1 - x_1 - x_2). \quad (5)$$

Our goal now is to demonstrate that the result \,(3) can be obtained also in the LF perturbation theory, contrary to the general belief. The LFPT \cite{2, 3, 11, 13} is not manifestly covariant (no $T$-product is introduced), since the integrals over the LF time are explicitly performed in the iterative solution of the equation for the $S$-matrix in the interaction representation. As the result, LFPT acquires the $x^+$-ordered structure automatically, with the usual Feynman propagators replaced by the LF energy denominators. The loss of manifest covariance is of no harm and the advantage is that one does not need to perform extra integration over the LF energy. However, when the LFPT rules are applied to vacuum diagrams, the result is ill-defined \cite{8}, as the delta function requires all three momenta be equal to zero:

$$\tilde{V} \sim \int_0^\infty \frac{dk_1^+}{k_1^+} \int_0^\infty \frac{dk_2^+}{k_2^+} \int_0^\infty \frac{dk_3^+}{k_3^+} \delta(k_1^+ + k_2^+ + k_3^+) \frac{1}{k_1^+ + k_2^+ + k_3^+}. \quad (6)$$
It is actually not very difficult to resolve this inconsistency. One simply has to start with the graph with nonvanishing external momentum and to use the LFPT rules to represent this amplitude. The expected analyticity in \( p \) then permits one to consider its value at \( p = 0 \) - the vacuum loop emerges simply as the limit of the associated self-energy graph for vanishing external momentum. In this way, the expression \( \Sigma \) is first replaced by the LF self-energy amplitude

\[
\Sigma_4(p^2) = \tilde{N} \lambda^2 \int_0^{p^+} \frac{dk^+}{k^+} \int_0^{p^+ - k^+} \frac{dl^+}{l^+(p^+ - k^+ - l^+)} \times \frac{1}{p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{l^+} - \frac{\mu^2}{p^+ - k^+ - l^+} + i\epsilon}, \quad \tilde{N} = \frac{1}{3!} (4\pi)^2. \tag{7}
\]

Introducing the dimensionless variables \( x = \frac{k^+}{p^+}, \ y = \frac{l^+}{p^+}, \Sigma_4(p) \) becomes

\[
\Sigma_4(p^2) = \int_0^1 dx \int_0^{1-x} dy \frac{\tilde{N} \lambda^2}{y(1-x-y) - \frac{\mu^2}{x} - \frac{\mu^2}{y} - \frac{\mu^2}{1-x-y}}. \tag{8}
\]

Now we can set \( p = 0 \). The integral over the variable \( y \) can be performed explicitly, yielding

\[
F(x) = \frac{1}{\mu^2} \frac{4x}{\sqrt{3x^2 - 2x - 1}} \arctan \frac{1 - x}{\sqrt{3x^2 - 2x - 1}}, \quad x = \frac{1}{2}, \tag{9}
\]

The numerical computation of the second integral gives

\[
\Sigma_4(0) \equiv V_3(\mu) = -\frac{\lambda^2}{\mu^2} \tilde{N} C, \quad C = 2.343908, \ldots, \tag{10}
\]

in the complete agreement with the space-like result \( \Sigma_3 \). The overall situation is in fact very simple. Multiplying out the terms in the denominator of Eq.\( \Sigma_4 \) for \( p = 0 \), we find that the corresponding double integral is precisely equal the representation of the constant \( C \) in Eq.\( \Sigma_3 \). The LF and SL schemes match at this point. The only difference is that in the SL theory one can start directly with the vacuum diagram while in the LF case the latter emerges as the the limit of the associated self-energy diagram for vanishing external momentum. The \( p = 0 \) value can be taken after changing the LF momenta to relative variables which makes the integrand covariant, that is depending symmetrically on both \( p^+ \) and \( p^- \). Obviously, \( \Sigma_4(p^2) \) is also the same in the both schemes, the significant difference being that the LF scheme needs for that just two steps while the conventional Feynman procedure is by an order of magnitude longer.

### III. THE SIMPLEST DIAGRAM WITH TWO INTERNAL LINES

The above result can be confirmed in a different way. Consider for simplicity the one-loop self energy diagram in the \( \lambda \phi^3 \) theory (see Fig. 1 (a) with two external lines attached). The corresponding Feynman amplitude is

\[
-i\Sigma_3(p^2) = \frac{1}{2} \frac{(-i\lambda)^2}{(2\pi)^3} \int d^2k \ G(k) G(p - k), \tag{11}
\]

The vacuum bubble \( (p = 0, \lambda = 1) \) rewritten in terms of the LF variables takes the form

\[
V_2(\mu) = \frac{i}{16\pi^2} \int_{-\infty}^{+\infty} dk^- \int_{-\infty}^{+\infty} dk^+ \frac{1}{(k^+ - k^- + \mu^2 + i\epsilon)^2}. \tag{12}
\]

To correctly evaluate the integral over \( k^- \), one has to impose a cutoff \( \Lambda \), leading to \( (c = -i/16\pi^2) \)

\[
V_2(\mu) = \frac{c}{\mu^2} \int_{-\infty}^{+\infty} dk^+ \left[ \frac{1}{k^+ + i\epsilon} - \frac{1}{k^- + i\epsilon} \right]. \tag{13}
\]

This result coincides with those obtained in the usual SL calculation \( \Sigma_3 \) and with the direct LF computation \( \Sigma_4(\mu) \). The same diagram, being the simplest one, sheds light upon the mechanism at work in the genuine LF case. The LFPT formula for the self-energy \( \Sigma_3(\mu) \) is

\[
\Sigma_3(p) = \frac{\lambda^2}{8\pi} \int_0^{p^+} \frac{dk^+}{k^+ (p^+ - k^+)} \frac{1}{p^- - \frac{\mu^2}{k^+} + \frac{1}{\eta - k^+ + i\epsilon}}. \tag{14}
\]

Going over to the variable \( x = \frac{k^+}{p^+} \), one transforms the denominator to the form \( x(1-x)p^2 - \mu^2 + i\epsilon \) and for \( p = 0 \) one indeed easily reproduces \( \Sigma_3(\mu) \). Alternatively, we can work directly with the form \( \Sigma_3(\mu) \). Taking \( p^+ = p^- = \eta \) for simplicity, we have \( (\lambda = 1 \) henceforth)

\[
\Sigma_3(\mu) = \frac{1}{8\pi} \int_0^\eta \frac{dk^+}{k^+ (\eta - k^+)} \frac{1}{\eta - \frac{\mu^2}{k^+} + i\epsilon}. \tag{15}
\]

The integral can be evaluated exactly with the result

\[
\Sigma_3(\eta) = -\frac{1}{4\pi} (G(\eta) - G(0)), \quad G(k) = \frac{\arctan \left( \frac{2k - \eta}{\sqrt{4\mu^2 - k^2}} \right)}{\eta \sqrt{4\mu^2 - k^2}}. \tag{16}
\]

The expansion for infinitesimal \( \eta \) gives

\[
\Sigma_3(\eta) = -\frac{1}{8\pi \mu^2} \left[ 1 + \frac{\eta^2}{4\mu^2} + \mathcal{O}(\eta^4) \right]. \tag{17}
\]
One can see that the correct result is recovered for $\eta = 0$. The technical reason is simple: the integrand in (18) is $\frac{1}{\eta^2} \frac{1}{\eta} \left( k^+ \frac{1}{\eta} - k^+ - \mu^2 \right)^{-1}$. For very small $\eta$ the expression in the brackets has practically a constant value very close to $\left( -\mu^2 \right)$ at the interval $(0, \eta)$, while the diverging $\eta^{-1}$ factor is canceled by the length $\eta$ of the integration domain. However, setting $\eta = 0$ from very beginning as in the formula (18) yields the wrong (ill-defined) result.

It is remarkable that the same result for the LF vacuum bubble is obtained in the discretized (finite-volume) treatment with (anti)periodic boundary conditions (BC). In both cases, the field mode carrying $k^+ = 0$ is manifestly absent. The field expansion at $x^+ = 0$ is

$$\phi(0, x^-) = \frac{1}{\sqrt{2L}} \sum_{n}^{\infty} \frac{1}{\sqrt{k_n^+}} \left[ a_n e^{-ik_n^+ x^-} + a_n^\dagger e^{ik_n^+ x^-} \right], \quad (18)$$

where $k_n^+ = 2\pi n/L$ and $L$ is the box length. The index $n$ runs over half-integers for antiperiodic BC (no ZM by construction) and over integers for periodic BC, with $n = 0$ excluded (the field equation in the ZM sector $\mu^2 \phi_0 = 0$ requires the field mode $\phi_0 \equiv \phi(k^+ = 0)$ to vanish for $\mu \neq 0$). The DLCQ analog of the $\Sigma_3(p)$ amplitude is

$$\Sigma_3(p) = \mathcal{N} \sum_{k^+}^p \frac{1}{k^+ (p^+ - k^+)} \left[ \frac{p^+ - \mu^2}{k^+ - \mu^2} - \frac{\mu^2}{p^+ - k^+} \right], \quad (19)$$

$$p^+ = \frac{2\pi}{L} K, \quad k^+ \equiv k_n^+ = \frac{2\pi}{L} n, \quad n = 1, 2, \ldots, K - 1. \quad (20)$$

$\mathcal{N}$ is the normalization constant. For $p = 0$, choosing periodic BC, $\Sigma_3(0) = V_2(\mu)$ becomes

$$V_2(\mu) = -\frac{1}{8\pi} \mu^2 S,$$

$$S = \sum_{n=1}^{K-1} \frac{1}{n(K-n)} \left[ \frac{1}{n} + \frac{1}{K-n} \right] = -\frac{K-1}{K}. \quad (21)$$

$K$ plays the role of $\eta^{-1}$ here (cf. discussion below Eq. (17)). For $K \to \infty$, $S$ obviously converges to the continuum value 1. The same result is obtained for the antiperiodic BC [17]. In Table I, the smooth approach of the self-energy value to the vacuum-loop value as $p \to 0$ is shown. The same pattern is valid also for the LF vacuum loops with three and four internal lines: the discrete representation of the corresponding $\Sigma_4(0)$ and $\Sigma_5(0)$ amplitudes involves two (three) summations with the result converging to the continuum result [17]. Generalization to an arbitrary number of the internal lines is straightforward and may be of interest to the LF solution of the sine-Gordon model [5, 10].

It is not difficult to generalize these results to the case of vacuum loops with internal lines corresponding to two fields with different masses like for example in the Yukawa model. Analogous calculations of the vacuum loops in $D = 3 + 1$ scalar self-interacting models also agree with their SL analogs. In this case the corresponding integral diverges logarithmically in both schemes.

Finally, we make a remark concerning one conceptual issue. It has been a long lasting concern that LF fields (scalar and fermion) have a singularity at $k^+ = 0$ that indicates certain pathology. It was suggested within an axiomatic approach that this singularity of the scalar field has to be dealt with by means of a specific class of test functions, vanishing at $k^+ = 0$ [21]. On the other hand, some physical effects were attributed to the zero mode, in particular within the Feynman perturbation theory with LF variables [21, 23]. Our results indicate a natural solution to this contradiction: when working within the genuine LF form (no reference to the Feynman perturbation theory), zero modes of the free scalar and fermion fields do not exist and there is no singularity issue related to $k^+ = 0$ mode. The physical results are correctly obtained without this type of the LF zero mode.
IV. CONCLUSIONS

We have demonstrated that the vacuum amplitudes are non-zero in LF perturbation theory and match the known values. The zero modes do not play any role in this phenomenon, as was shown within the discretized (finite-volume) form of the theory where the zero modes are manifestly absent.

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