The Spectral Dimension of Non-generic Branched Polymer Ensembles

JOÃO D. CORREIA and JOHN F. WHEATER

Department of Physics, University of Oxford
Theoretical Physics,
1 Keble Road,
Oxford OX1 3NP, UK

Abstract. We show that the spectral dimension on non-generic branched polymer models with susceptibility exponent $\gamma > 0$ is given by $d_s = 2/(1 + \gamma)$. For those models with $\gamma < 0$ we find that $d_s = 2$.

PACS: 04.60.Nc, 5.20.-y, 5.60.+w
Keywords: conformal matter, quantum gravity, branched polymer, spectral dimension

1e-mail: j.correia1@physics.ox.ac.uk
2e-mail: j.wheater1@physics.ox.ac.uk
The spectral dimension $d_s$ is an important measure of the dimensionality of the manifold ensembles appearing in quantum gravity. In particular much effort has recently been put into its determination for discretized two-dimensional quantum gravity ensembles. The spectral dimension is defined by a random walk which leaves a fixed vertex at $t = 0$ and at every step is allowed to move from its present position to one of its neighbouring vertices with uniform probability. After $t$ steps the probability that the walk has returned to the initial point is given by

$$P(t) \approx \frac{\text{const}}{t^{d_s/2}}$$

provided that $t \gg 1$ (to negate discretization effects) and that $t \ll N^\Delta$ where $N$ is the number of vertices in the graph and $\Delta$ some exponent (to avoid finite size effects).

The spectral dimension has been investigated numerically for many different values of the central charge $c$ [1] and it has been calculated analytically for the generic branched polymer model [2] and found to be $d_s = \frac{4}{3}$ in excellent agreement with the numerical results at large $c$. We refer the reader to refs [1] and [2] for detailed discussion of the definition of $d_s$ and the ensemble averages involved. In this letter we extend the results of [2] to the non-generic branched polymers.

We start by recalling the essential features of the branched polymer models [3]. A general branched polymer (BP) model has grand canonical partition function $Z(z)$ satisfying

$$Z(z) = z(1 + \sum_{n=1}^\infty \alpha_n Z(z)^n)$$

where the $\alpha_n$ are constants. Iterating this equation shows that $Z(z)$ is the generating function for the set of all rooted trees $B$ made up of links and vertices of all coordination numbers $n + 1$ such that $\alpha_n \neq 0$.

$$Z(z) = \sum_{A \in B} z^{N_A} w_A.$$  

The number of links in $A$ is denoted by $N_A$ and the weight $w_A$ of $A$ is given by

$$w_A = \prod_{v \in A} \alpha_{n(v) - 1}$$

where $v$ runs over all vertices of $A$ and $n(v)$ denotes the coordination number of $v$. We denote the set of all polymers whose first vertex has coordination number $n + 1$ by $B^n$; clearly $B = B^0 \cup B^1 \cup B^2 \cup \ldots$ ($B^0$ contains the polymer consisting of a single link). Any graph $A \in B^n$ has a set of constituents $A_1, \ldots, A_n$ obtained by severing the links connecting the first vertex to the rest of the graph (see fig.1). Note that
Figure 1: The constituents $A_1, \ldots, A_n$ of a branched polymer $A \in B^n$.

For $A \in B^n$

$$ w_A = \alpha_n \prod_{i=1}^{n} w_{A_i} \quad (5) $$

and

$$ N_A = 1 + \sum_{i=1}^{n} N_{A_i} \quad (6) $$

Equation (2) can be rewritten in the form

$$ \frac{1}{z} = F(Z) \quad (7) $$

$$ F(x) = x^{-1}(1 + \sum_{n=1}^{\infty} \alpha_n x^n) \quad (8) $$

For small enough $z$ the solution $Z(z)$ is an analytic function but at some critical value $z = z_{cr}$ it is non-analytic; this is the point at which the graphical expansion (3) diverges. The susceptibility is given by

$$ \chi(z) = \frac{\partial Z}{\partial z} = -\frac{z^{-2}}{F'(Z(z))} = \frac{Z z^{-1}}{1 - z \sum_{n=1}^{\infty} n \alpha_n Z(z)^{n-1}} \quad (9) $$

and so the critical point is where $F'(Z(z_{cr})) = 0$. As $z \to z_{cr}$ the susceptibility has leading non-analytic behaviour given by

$$ \chi(z) \simeq (z_{cr} - z)^{-\gamma} \quad (10) $$

where $\gamma$ is the susceptibility exponent. (Throughout this letter we use the symbol $\simeq$ to denote the leading singularity as $z \to z_{cr}$.) The nature of the critical point and the value of $\gamma$ depend upon how $F'$ vanishes. The multi-critical BPs are obtained
by supposing that $F^{(n)}(Z(z_{cr})) = 0$, $n = 0, \ldots, k$ but that $F^{(k+1)}(Z(z_{cr})) \neq 0$ so, close to $z_{cr}$,

$$\chi(z) \simeq \frac{z^{-2}}{|Z(z) - Z(z_{cr})|^k}. \quad (11)$$

However

$$Z(z) \simeq Z(z_{cr}) - \text{const.}(z_{cr} - z)^{1-\gamma} \quad (12)$$

so comparing singularities we obtain

$$\gamma = k(1-\gamma)$$

or

$$\gamma = 1 - \frac{1}{k+1}. \quad (13)$$

The generic case, requiring no special tuning of the $\alpha_n$ (except that at least one $\alpha_n$ ($n \geq 2$) must be non-zero), is $k = 1$, $\gamma = \frac{1}{2}$; higher values of $k$, which require at least $k$ of the $\alpha_n$ ($n \geq 2$) to be non-zero and of varying sign to enforce the vanishing of higher derivatives, give the multi-critical BPs (MCBP from now on).

The MCBPs are not the only ones with $\gamma \neq \frac{1}{2}$. By allowing an infinite number of the $\alpha_n$ to be non-zero we can arrange that $F'$ vanishes non-analytically [4, 5]. Consider

$$F(Z) = \frac{1}{Z} + \mu + \frac{Z}{A^2} + \lambda \left(1 - \frac{Z}{A}\right)^{\beta+1} \quad (14)$$

where $A$, $\lambda$, $\mu$ and $\beta$ are positive constants. Then

$$F'(Z) = -\frac{1}{Z^2} + \frac{1}{A^2} - \frac{\lambda(\beta+1)}{A} \left(1 - \frac{Z}{A}\right)^{\beta} \quad (15)$$

As $z$ increases from zero so does $Z(z)$. The first zero of $F'$ is at $Z = A$ when $z_{cr} = (2/A + \mu)^{-1}$; provided that $1 > \beta > 0$ the non-analytic term in (14) dominates as $z \rightarrow z_{cr}$. Inserting this dominant behaviour into (11), using (12), and comparing singularities, we obtain

$$\gamma = \frac{\beta}{\beta+1}. \quad (16)$$

Thus we get models with continuously varying $\gamma$ in the range 0 to $\frac{1}{2}$ as discussed in [5]. However the restriction on the range of $\beta$ is easily removed; by tuning the coefficients $\alpha_n$ so that the analytic part of $F'$ in (13) vanishes as $(Z - A)^2$ the non-analytic behaviour dominates for $2 > \beta > 0$ so now we can get continuously varying $\gamma$s up to $\frac{2}{3}$. If the coefficients are tuned so that the analytic part of $F'$ in (13) vanishes as $(Z - A)^m$ then $\beta$ can be as large as $m$. In this way we see that by combining the multi-critical strategy and the non-analytic form of $F$ used in [4, 5] we can obtain all $\gamma$ values in the range 0 to 1. If $\beta$ is an integer these models just
reproduce the MCBP; when $\beta$ is not an integer we will call them the ‘continuous critical branched polymers’, or CCBP.

The CCBPs actually continue to negative gamma. Suppose that $F'$ is finite at $z_{cr}$ but that $F''$ diverges; then by (14) the susceptibility $\chi$ is finite at $z_{cr}$. However

$$\frac{\partial \chi}{\partial z} = \frac{2z^{-3}}{F'(Z(z))} + \frac{z^{-2}F''(Z(z))\chi}{F'(Z(z)^2)} \tag{17}$$

so the derivative of $\chi$ diverges at the critical point and $\gamma$ is negative. This is easily arranged by (for example) eliminating the linear term in (14) so that

$$F(Z) = \frac{1}{Z} + \mu + \lambda \left(1 - \frac{Z}{\Lambda}\right)^{\beta+1} \tag{18}$$

with $1 > \beta > 0$ and comparison of the behaviour as $z \to z_{cr}$ then gives

$$\gamma = \frac{-\beta}{2 - \beta}. \tag{19}$$

In our calculation of the spectral dimension we will need to know the behaviour of

$$\Omega_l = \frac{d^{l+1}}{dz^{l+1}} \sum_{n=1}^{\infty} a_n Z(z)^n = \frac{d^{l+1}}{dz^{l+1}} ZF(Z) \tag{20}$$

close to the critical point. Using the properties of $F$ discussed above we find that for $\gamma > 0$

$$\Omega_l \simeq (Z(z) - Z(z_{cr}))^{\beta-l} + \frac{1}{z_{cr}} \delta_{l,0}$$

$$= (z_{cr} - z)^{\gamma-l(1-\gamma)} + \frac{1}{z_{cr}} \delta_{l,0} \tag{21}$$

unless $\beta$ is an integer (ie MCBP) in which case for $l \geq \beta$ we find that $\Omega_l$ is finite at $z_{cr}$. If $\gamma < 0$ then $F'$ is finite and it follows from (14) that

$$1 - z_{cr} \Omega_0(z_{cr}) = C \tag{22}$$

where $C$ is finite.

Our calculation of the spectral dimension follows the same method as in [2] but generalized to take account of the presence of vertices of varying order. The return probability generating function $P_A(y)$ on a given polymer $A \in \mathcal{B}^n$ is related to that on its $n$ constituents $A_1, \ldots, A_n$ (see fig.1). It is convenient to define for any polymer $A \in \mathcal{B}$

$$P_A(y) = \frac{1}{1 - y \frac{1}{h_A(y)}} \tag{23}$$
and then we find for $A \in B$ \[ h_A(y) = \frac{1 + \sum_{i=1}^{n} h_{A_i}(y)}{1 + (1 - y) \sum_{i=1}^{n} h_{A_i}(y)}. \] (24)

Note that at $y = 1$ the solution is

$$h_A(1) = N_A$$ (25)

for all polymers $A \in B$. The spectral dimension is found by considering the quantity

$$\tilde{Q}(z, y) = -\frac{d}{dy} (1 - y) \sum_{A \in B} z^{N_A} w_A P_A(y) = \frac{d}{dy} \sum_{A \in B} z^{N_A} w_A h_A(y).$$ (26)

At $y = 1$ the $n$th derivative with respect to $y$ of $\tilde{Q}(z, y)$ behaves as

$$\tilde{Q}_n(z) = \frac{\partial^n}{\partial y^n} \tilde{Q}(z, y) \bigg|_{y=1} \simeq (z_{ct} - z)^{\beta - n\Delta}$$ (27)

where the exponents $\beta$ and $\Delta$ are related to the spectral dimension by

$$\beta = 1 - \gamma + \Delta (d_s/2 - 1).$$ (28)

The behaviour (27) can be established by considering the quantities

$$H^{(n_1, n_2, \ldots, n_p)} = \sum_{A \in B} z^{N_A} w_A \prod_{i=1}^{p} \frac{d^{n_i}}{dy^{n_i}} h_A(y) \bigg|_{y=1}$$ (29)

which have leading singular behaviour as $z \to z_{ct}$ given by \[ H^{(n_1, n_2, \ldots, n_p)} \simeq (z_{ct} - z)^{a - \beta - c \sum_{i=1}^{p} n_i} \] (30)

where we will determine the constants $a, b, c$.

First we will consider the case when $\gamma > 0$. Let us compute

$$H^{(1)} = \sum_{A \in B} z^{N_A} w_A h_A'(1) = \sum_{n} \sum_{A \in B^n} z^{N_A} w_A h_A'(1).$$ (31)

Now we use (24) to relate $h_A'$ to the corresponding quantity for the constituents of $A$ and \[ H^{(1)} = \sum_{n} \alpha_n \sum_{A \in B^n} \sum_{A_1 \cup A_2 \cup \ldots \cup A_n = A} z^{1+N_{A_1}+\ldots+N_{A_n}} w_{A_1} \ldots w_{A_n} \left\{ \sum_{i=1}^{n} h_{A_i}'(1) + \sum_{i=1}^{n} h_{A_i}(1) \left( 1 + \sum_{j=1}^{n} h_{A_j}(1) \right) \right\}. \] (32)
Note that although $A \in \mathcal{B}^n$ in the above expression there is no such restriction on its constituent polymers, $A_i \ldots A_n$ which are drawn from the entire ensemble $\mathcal{B}$. The terms on the r.h.s. of (32) which involve $h'_{A_m}(1)$, of which there are $n$, simply reproduce $H^{(1)}$ multiplied by powers of the partition function; by using (25) we see that the remaining pieces simply produce derivatives of the partition function so we may rearrange (32) to obtain

$$H^{(1)} \left( 1 - z \sum_{n=1}^{\infty} n \alpha_n \mathcal{Z}(z)^{n-1} \right) = z^2 \frac{\partial^2}{\partial z^2} \mathcal{Z}. \quad (33)$$

The coefficient of $H^{(1)}$ is $1 - z \Omega_0$ so by (10) and (21) we obtain

$$H^{(1)} \simeq (z_{cr} - z)^{-1-2\gamma} \quad (34)$$

and also, since $H^{(1)}$ is the second derivative of $\tilde{Q}_0(z)$ with respect to $z$, that

$$\tilde{Q}_0(z) \simeq (z_{cr} - z)^{1-2\gamma} \quad (35)$$

which gives $\beta = 1 - 2\gamma$.

Next we compute

$$H^{(1,1)} = \sum_{A \in \mathcal{B}} z^{N_A} w_A h'_A(1) h'_A(1)$$

$$= \sum_{n} \alpha_n \sum_{A \in \mathcal{B}^n} \sum_{A_1 \cup A_2 \cup \ldots \cup A_n = A} z^{1+N_{A_1} + \ldots + N_{A_n}} w_{A_1} \ldots w_{A_n} \left\{ \sum_{i=1}^{n} h'_{A_i}(1) \right\}^2$$

$$+ 2z^2 \frac{\partial^2}{\partial z^2} \sum_{n} \alpha_n \sum_{A \in \mathcal{B}^n} \sum_{A_1 \cup A_2 \cup \ldots \cup A_n = A} z^{1+N_{A_1} + \ldots + N_{A_n}} w_{A_1} \ldots w_{A_n} \left\{ \sum_{i=1}^{n} h'_{A_i}(1) \right\}$$

$$+ \left( z^2 \frac{\partial^2}{\partial z^2} \right)^2 \mathcal{Z}. \quad (36)$$

Again the first sum contains pieces which reproduce $H^{(1,1)}$; the remaining terms on the r.h.s. of (36) can be expressed in terms of $H^{(1)}$ and $\mathcal{Z}$. We obtain

$$H^{(1,1)} \left( 1 - z \sum_{n=1}^{\infty} n \alpha_n \mathcal{Z}(z)^{n-1} \right) = z \sum_{n=2}^{\infty} n(n-1) \alpha_n \mathcal{Z}^{n-2} (H^{(1)})^2$$

$$+ 2z^2 \frac{\partial^2}{\partial z^2} z \sum_{n=1}^{\infty} n \alpha_n \mathcal{Z}(z)^{n-1} H^{(1)}$$

$$+ \left( z^2 \frac{\partial^2}{\partial z^2} \right)^2 \mathcal{Z}. \quad (37)$$

Noting that the coefficient of the $(H^{(1)})^2$ term is simply $z \Omega_1$ and that the coefficient of the $H^{(1)}$ term is $z \Omega_0$ which tends to 1 as $z \rightarrow z_{cr}$ we see that the first two terms on
the r.h.s. of (37) both vary as \((z_{cr} - z)^{-3-2\gamma}\) whilst the last term is sub-leading since it varies as \((z_{cr} - z)^{-3-\gamma}\). The coefficient of \(H^{(1,1)}\) varies, as before, like \((z_{cr} - z)^{\gamma}\) so we find

\[
H^{(1,1)} \simeq (z_{cr} - z)^{-3-3\gamma}.
\] (38)

We can iterate this process; \(H^{(1,1,1)}\) is given by

\[
H^{(1,1,1)} (1 - z\Omega_0) = z\Omega_2 \left( H^{(1)} \right)^3 + z\Omega_1 H^{(1,1)} H^{(1)} + 3z^2 \frac{\partial^2}{\partial z^2} z\Omega_0 H^{(1,1)}
\]

\[+ \text{ (less singular terms).} \] (39)

The presence of the coefficients \(\Omega_{1,2}\) in the first two terms on the r.h.s. changes their degree of divergence to \((z_{cr} - z)^{-5-3\gamma}\) which is the same as that of the third term and hence we conclude that

\[
H^{(1,1,1)} \simeq (z_{cr} - z)^{-5-4\gamma}.
\] (40)

This process may be continued to longer strings \((1,1,1,\ldots,1)\) and to strings involving higher derivatives of \(h_\Lambda(y)\) (obtained by successive differentiation of (24)). Following [2] we can set up a proof by induction [3] that

\[
H^{(n_1,n_2,\ldots,n_p)} \simeq (z_{cr} - z)^{1-\gamma-p-(1+\gamma)\sum_{i=1}^p n_i}
\] (41)

from which it follows, by inserting this result in (27) (see [4] for details), that \(\Delta = 1 + \gamma\). Substituting this and \(\beta = 1 - 2\gamma\) into (28) we find that

\[
d_s = \frac{2}{1+\gamma}.
\] (42)

Note that the spectral dimension also satisfies the scaling relation \(d_s\Delta = 2\). As was discussed in [7] this scaling relation is expected to be true for graphs whose Laplacian has no level-dependent degeneracy in its eigenvalue spectrum; clearly the BP ensembles, most of whose graphs have no symmetries, fall into this category.

The case of \(\gamma < 0\) is simpler. The factor \(1 - z\Omega_0\) does not vanish as \(z \to z_{cr}\) so we now find that

\[
H^{(1)} \simeq (z_{cr} - z)^{-1-\gamma}
\] (43)

and therefore that

\[
\tilde{Q}_0(z) \simeq (z_{cr} - z)^{1-\gamma}
\] (44)

which implies \(\beta = 1 - \gamma\). Using (28) we see immediately that \(d_s = 2\). Of course it is necessary to check that \(\Delta \neq 0\) before reaching this conclusion; in fact it is straightforward to show that \(\Delta = 1\) [3] and so the scaling relation \(d_s\Delta = 2\) is still satisfied.
It is interesting to compare these results with a scaling relation recently found by Ambjørn et al \[8\]. They showed that

$$\frac{2}{d_H} = \Delta \left(1 - \frac{d_s}{2}\right)$$  \(\text{(45)}\)

where \(d_H\) is the *extrinsic* Hausdorff dimension. For the \(k\)th multicritical model \(d_H\) has been calculated \[3\] and is known to be given by

$$d_H = \frac{2(k + 1)}{k}.$$  \(\text{(46)}\)

Assuming that \(d_s \Delta = 2\) we can then use (45) to determine that \[8\]

$$d_s = \frac{2}{2 - \frac{1}{k+1}} = \frac{2}{1 + \gamma}$$  \(\text{(47)}\)

in agreement with our calculations. The value of \(d_H\) has not been calculated explicitly for the CCBPs but we can now use the scaling relation (45) and our results to show that \(\gamma d_H = 2\) if \(\gamma > 0\). When \(\gamma < 0\) we have \(d_s = 2\) so we expect \(d_H = \infty\). It is amusing that the two-dimensional quantum gravity models at \(c < 1\) also have \(d_H = \infty\) and hence \(d_s = 2\) \[8\].

We acknowledge valuable conversations with Jan Ambjørn who told us of the scaling relation result (45) prior to writing the paper \[8\]. J.C. acknowledges a grant from Praxis XXI.

**References**

[1] J.Ambjørn, J.Jurkiewicz and Y.Watabiki, Nucl. Phys. B454 (1995) 313.

[2] T. Jonsson and J.F. Wheater, “The spectral dimension of the branched polymer phase of two dimensional quantum gravity”, Oxford University Preprint OUTP-97-33P, [hep-lat/9710024].

[3] J.Ambjørn, B.Durhuus and T.Jonsson, Phys. Lett. B244 (1990) 403.

[4] J.Ambjørn, B. Durhuus, J. Fröhlich and P.Orland, Nucl. Phys. B270 [FS16] 457.

[5] P. Bialas and Z. Burda, Phys. Lett. B 384 (1996) 75.

[6] J.D. Correia, D.Phil Thesis, Oxford University 1998.
[7] T. Jonsson and J.F. Wheater, “The spectral dimension on branched polymer ensembles”, to appear in the proceedings of NATO workshop *New Developments in Quantum Field Theory*, eds. P.H. Damgaard and J. Jurkiewicz, published by Plenum Publishing Corp., New York.

[8] J. Ambjørn, D. Boulatov, J.L. Nielsen, J. Rolf and Y. Watabiki, “The spectral dimension of 2d quantum gravity”, Niels Bohr Institute preprint NBI-HE-97-62.