The Negativity of the Overlap-Based Topological Charge Density Correlator in Pure–Glue QCD and the Non-Integrable Nature of its Contact Part

I. Horváth\(^1\), A. Alexandru\(^1\), J.B. Zhang\(^2\), Y. Chen\(^3\), S.J. Dong\(^1\), T. Draper\(^1\)
K.F. Liu\(^1\), N. Mathur\(^1\), S. Tamhankar\(^1\) and H.B. Thacker\(^4\)

\(^1\)Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506
\(^2\)CSSM and Department of Physics, University of Adelaide, Adelaide, SA 5005, Australia
\(^3\)Institute of High Energy Physics, Academia Sinica, Beijing 100039, P.R. China
\(^4\)Department of Physics, University of Virginia, Charlottesville, VA 22901

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Abstract

We calculate the lattice two-point function of topological charge density in pure-glue QCD using the discretization of the operator based on the overlap Dirac matrix. Utilizing data at three lattice spacings it is shown that the continuum limit of the correlator complies with the requirement of non-positivity at non-zero distances. For our choice of the overlap operator and the Iwasaki gauge action we find that the size of the positive core is \(\approx 2a\) (with \(a\) being the lattice spacing) sufficiently close to the continuum limit. This result confirms that the overlap-based topological charge density is a valid local operator over realistic backgrounds contributing to the QCD path integral, and is important for the consistency of recent results indicating the existence of a low-dimensional global brane-like topological structure in the QCD vacuum. We also confirm the divergent short-distance behavior of the correlator, and the non-integrable nature of the associated contact part.

1. Introduction. An intriguing property of the topological charge density (TChD) two-point function in Euclidean gauge theory has been pointed out long ago by Seiler and Stamatescu \cite{1}. Specifically, the correlator is non-positive at arbitrary non-zero distance. While not widely known or used, this fact arises straightforwardly as a consequence of reflection positivity and the pseudoscalar nature of the corresponding local field operator. There are (at least) two situations where this seemingly unusual property of the correlator plays a relevant role. The first one involves the discussion of subtleties arising in the derivation of the Witten-Veneziano relation for the \(\eta'\) mass \cite{1, 2}. Indeed, the topological susceptibility is positive by usual definition and yet, it can be equivalently expressed as a space-time integral of the correlator which is non-positive everywhere except at the origin. The expected non-integrable behavior of the (negative) correlator near the origin has to be countered by positive divergent terms (with support at the origin) to yield a finite positive susceptibility. This raises both legitimate conceptual issues about the role of short-distance fluctuations in
the associated physics [1], as well as intriguing questions about how exactly does the cancel-
lation of positive and negative infinities take place in the context of a lattice non-perturbative
definition of the theory. Certain points related to these issues were discussed in the context
of CP(N-1) models in Refs. [3, 4].

The second instance where the negativity of TChD correlator has non-trivial implications
relates to questions about the nature of topological charge fluctuations in the QCD
vacuum [5, 6, 7]. Indeed, if there exists a fundamental structure in typical configurations
contributing to the path integral of the theory with non-trivial ultraviolet behavior (such
as QCD), then the space-time characteristics of such structure should be consistent with
the negativity of the TChD correlator.¹ This requirement means, in particular, that such
fundamental structure cannot be dominated by gauge fields supporting 4-dimensional sign-
coherent regions of TChD [5, 6]. On the other hand, the negativity of the correlator can
be satisfied in an ordered manner if the structure involves interleaved layers of oppositely
charged lower-dimensional regions. The existence of such low-dimensional brane-like struc-
ture in Monte Carlo generated lattice QCD configurations has been demonstrated [6],² and
it was shown that it behaves as an inherently global entity in the sense that its localized
parts are not sufficient to explain the value of topological susceptibility in pure-glue QCD [7].

A crucial ingredient for both of the above developments is the availability of a new kind of
lattice TChD operator that can be used in the context of a non-perturbative definition of the
theory. Indeed, the recent progress in putting the derivation of Witten-Veneziano formula on
firmer ground [11] is based on the use of topological field associated with Ginsparg-Wilson
fermions [12, 13]. In fact, such a topological field exhibits properties analogous to those in the
continuum and appears to lead to a satisfactory definition of topological susceptibility also in
full QCD [14, 15]. Similarly, the low-dimensional long-range topological structure has been
observed using the operator based on the overlap Dirac matrix, and is not obviously visible
when various naive operators are used [6]. The underlying reason leading to such niceties is
tied to the fact that TChD operators based on chiral fermions appear to have a proper control
of short-distance fluctuations. Indeed, the artificial ultraviolet infinities, present when naive
lattice operators are used, apparently disappear with Ginsparg-Wilson TChD operators and
no power-divergent subtractions to the susceptibility are needed. On the other side of the
coin, the uncontrolled short-distance fluctuations of naive operators mask the presence of
the ordered long-range topological structure in the QCD vacuum which however becomes
apparent when these fluctuations are properly treated [16].

Despite the special significance of TChD operators based on Ginsparg-Wilson fermions,
the negativity of the correlator has not been numerically demonstrated for any particular
choice of the operator.³ The reason why negativity is not obviously satisfied here has to
do with the non-ultralocal nature of Ginsparg-Wilson fermions. Indeed, it was shown [18]
that if $D$ is any (otherwise acceptable) Ginsparg-Wilson operator, then $D_{x,y}$ is non-zero

¹Note that by fundamental structure we mean a structure that contains fluctuations at all scales and is
in principle relevant for all aspects of QCD physics.

²The low-dimensional nature of the fundamental topological field is reflected to some degree also in low-
lying Dirac modes [8, 9] but the precise form of such correspondence is not known. The notion of strictly
low-dimensional structure also emerged recently using indirect projection techniques [10]. Its relation to the
structure in topological field is not clear at this point.

³The early attempt to verify the negativity can be found in Ref. [17].
for arbitrarily large distances $|x - y|$. While not proved rigorously, it is expected that the analogous property holds also in terms of gauge variables in the sense that $D_{x,y}$ receives small but non-zero contributions from gauge paths that extend arbitrarily far away from $x$ and $y$. As a consequence, the TChD operator $q(x) \propto \text{tr} \gamma_5 D_{x,x}$ is non-ultralocal in this sense and cannot be strictly contained in any finite lattice region. This complication makes direct arguments involving reflection positivity inexact at the lattice level even when the underlying lattice action is otherwise reflection positive. Nevertheless, assuming that $q(x)$ is a valid local lattice TChD operator, the consequences of reflection positivity are expected to hold upon taking the continuum limit. In this sense, verifying the negativity of the correlator at arbitrary non-zero physical distances represents a non-trivial check on the locality of the lattice operator and on the consistency of the lattice action defining the theory (in cases where the lattice action is not manifestly reflection positive, such as in full QCD with overlap fermions).

In what follows, we will verify numerically that the TChD operator constructed from overlap Dirac matrix [19] indeed leads to a negative correlator at non-zero physical distances in the continuum limit of pure-glue gauge theory. While our calculation is performed with a particular (but generic) choice of the overlap operator ($\rho = 1.368$) and with a particular choice of ultralocal gauge action (Iwasaki action [20]), we do not expect this conclusion to change for other generic choices. It is of some practical interest to quantify the size of the positive core $r_c^p$ of the correlator at the regularized level. We find that for lattice spacing $a = 0.082$ fm the size is $r_c^p \approx 0.18$ fm. Using data at three different lattice spacings we obtain the continuum-extrapolated shape of the lattice correlator at small lattice distances. From this calculation we conclude that the size of the positive core sufficiently close to the continuum limit is $r_c^p \approx 2a$ in our case (thus shrinking to zero in a corresponding manner). Finally, we verify that the contribution of positive core of the correlator to susceptibility indeed diverges in the continuum limit as expected from general arguments. The nature of this divergence will be discussed quantitatively in an upcoming publication.

Before starting, we wish to emphasize a point of convention. Since our discussion will revolve mostly around lattice objects, we reserve the standard notation (e.g. $x$, $q(x)$, $G(x)$) to represent lattice quantities and/or quantities in lattice units. The corresponding physical counterparts will be distinguished by superscript $p$ (e.g. $x^p$, $q^p(x^p)$, $G^p(x^p)$).

2. Reflection Positivity and Locality. We will discuss the 2-point function of TChD

$$G^p(x^p) \equiv \langle q^p(x^p) q^p(0) \rangle$$

in Euclidean gauge theory. To show that $G^p(x^p) \leq 0$ for $|x^p| > 0$ is straightforward. Indeed, let us put the origin of the new coordinate system at the midpoint between 0 and $x^p$ and consider a reflection $\theta$ with respect to the axis connecting these points, so that $x^p = \theta(0)$. Due to the pseudoscalar nature of $q^p(x^p)$ we then have $q^p(x^p) = -\Theta q^p(0)$, if $|x^p| > 0$. Here $\Theta$ is the (antilinear) reflection operator. Consequently

$$\langle q^p(x^p) q^p(0) \rangle = -\langle \Theta q^p(0) q^p(0) \rangle \leq 0$$

where the last inequality follows from reflection positivity of the theory. The point $x^p = 0$ is singular in this regard and the correlator is obviously positive at the origin.
Figure 1: For ultralocal $q(x)$ with range $r_u$ one can directly prove the negativity of $G(r,a)$ (in reflection-positive lattice theory) for $|x| > 2r_u$ (left). For $|x| < 2r_u$ (right) the operator extends beyond the reflection “plane” and the positive core involving non-zero lattice distances may develop in the 2-point function.

The situation on the lattice is more complicated with complications coming from two sources. (1) The lattice theory defined via particular action $S$ may not be strictly reflection-positive, i.e. it cannot be proved (via neither site nor link reflection) that $\langle \Theta F F \rangle \geq 0$ for any operator function $F$ depending on field variables at arbitrary positive lattice times. (2) The lattice operator $q(x)$ can extend over several lattice spacings, and thus for sufficiently small (lattice) $x$ one cannot make the direct argument even if $S$ is strictly reflection positive. Indeed, if $q(x)$ is ultralocal with lattice radius $r_u$ then for $|x| < 2r_u$ the above reflection positivity reasoning breaks down since the operator will extend beyond the reflection “plane”. This is schematically illustrated in Fig. 1. Nevertheless, the consequences of reflection positivity will be recovered in the continuum limit since the physical size $2r_u a$ of the (possibly) violating region will go to zero. The situation is more involved if operator $q(x)$ is non-ultralocal (such as an operator based on Ginsparg-Wilson kernel). In this case the operator always extends beyond the reflection plane and the exact lattice arguments based on reflection positivity cannot be used even if $S$ is reflection positive. However, if a non-ultralocal operator is exponentially local with the associated finite lattice range $r_{\text{exp}}$, then one expects the behavior similar to that of an ultralocal operator with comparable range. Specifically, $G(x)$ could contain a positive core with radius $r_c \approx r_{\text{exp}}$. In fact, one could use the measured size of the positive core as a very rough estimate of the lattice range of such an operator (assuming that the lattice theory is otherwise reflection positive). If the lattice operator is not exponentially local, the negativity of the correlator could be violated at finite physical distance in the continuum limit even if the underlying lattice theory is strictly reflection positive. If that happens, the corresponding operator should be viewed as non-local and discarded.

In later sections we will focus on demonstrating the negativity of $G(x)$ in the continuum limit of pure-glue QCD defined by the Iwasaki gauge action and using the TChD operator based on the overlap Dirac matrix. We will thus be dealing with a lattice theory without strict reflection positivity, but with an ultralocal action (with the extent of just two lattice
spacings) for which there is little doubt that the consequences of reflection positivity will hold accurately even before taking the continuum limit. Consequently, the demonstration of negativity for $G(x)$ will represent mainly a check on the locality properties of the overlap-based $q(x)$. In case of very smooth (“admissible”) gauge fields the locality of $q(x)$ follows from arguments given in Ref. [21] for locality of the overlap Dirac operator. However, the explicit check of locality for the overlap-based TChD operator over realistic lattice QCD ensembles has not been done and remains a very relevant issue. Moreover, the question of effective range of the operator (and the size of the positive core of the correlator) is of practical interest.

3. Lattice Data. We will work with lattice TChD given by [12]

$$q(x) = \frac{1}{2\rho} \text{tr} \gamma_5 D_{x,x} \equiv -\text{tr} \gamma_5 (1 - \frac{1}{2\rho} D_{x,x})$$

(3)

where $D$ is the overlap Dirac operator [19] based on the Wilson-Dirac kernel with mass $-\rho$. For the numerical results presented here we use the value $\rho = 1.368 (\kappa = 0.19)$. Details of the numerical implementation for overlap matrix–vector operation needed to evaluate $q(x)$ can be found in Ref. [22]. The 2-point correlation function was calculated over the ensembles of Iwasaki action at three different lattice spacings with details specified in Table 1. The scale has been determined from string tension and the physical size $L_p = 1.32$ fm is the same for all ensembles. The density $q(x)$ was evaluated at every point of the lattice and, consequently, all the correlators computed include contributions from all possible pairs of points (“all-to-all” correlators). For ensemble $\mathcal{E}_1$ a single point source has been used to evaluate the density individually for each point. For ensembles $\mathcal{E}_2$ ($\mathcal{E}_3$) the superposition of 2 (8) maximally separated point sources was used to evaluate the density at 2 (8) points simultaneously, thus speeding the calculation up accordingly. In case of $\mathcal{E}_3$ this leads to a typical relative error of calculation around $10^{-5}$, and this error is better than $2 \times 10^{-3}$ for all but 2% of the least intense (as measured by $|q(x)|$) points. Such deviations have a negligible effect on the correlator. The precision is even better for $\mathcal{E}_1$ and $\mathcal{E}_2$.

Computed lattice correlation functions for ensembles $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are shown in Fig. 2 as a function of $r \equiv |x|$. The bottom part of the figure displays the detail of the behavior for small values of $G(r)$. One can clearly see that for all three ensembles the correlator exhibits a positive core followed by a negative behavior at larger lattice distances. For the analysis that will follow we wish to highlight a particular observation indicated by our raw lattice data.

\footnote{Note that discussion in Ref. [21] focuses on locality properties of $D$ in terms of fermionic degrees of freedom. What is relevant here is the locality in gauge variables.}

| ensemble | $a$ [fm] | $V$ | $V^p$ [fm$^4$] | configs |
|----------|---------|-----|---------------|---------|
| $\mathcal{E}_1$ | 0.165 | $8^4$ | 3.0 | 50 |
| $\mathcal{E}_2$ | 0.110 | $12^4$ | 3.0 | 50 |
| $\mathcal{E}_3$ | 0.082 | $16^4$ | 3.0 | 25 |

Table 1: Ensembles of Iwasaki gauge configurations for overlap TChD calculation.
Figure 2: Lattice 2-point functions of TChD for ensembles $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$. Details of the behavior for small values of $G(r, a)$ are shown on the bottom plot.
Observation 1: The range (width) of the positive core in lattice units decreases as the continuum limit is approached.

There are other interesting properties exhibited by the data that we will discuss in a forthcoming publication. Here we wish to generalize the above observation into the corresponding precise statement which can be verified by further simulations. We suggest that the conjecture below is valid at least for the set of standard ultralocal gauge actions such as Wilson, Iwasaki, Lüscher-Weisz, and DBW2 actions\(^5\) and probably much more generally. Also, we expect it to be valid for a generic value of \(\rho\) (i.e. \(0 < \rho < 2\)) in the definition of the overlap matrix and the corresponding TChD operator. In arguments that follow we implicitly assume that a sufficiently large physical volume (e.g. larger than 1 fm\(^4\)) is kept fixed as the lattice spacing is changed toward the continuum limit.

Conjecture 1: There exists a finite lattice spacing \(a_0\) satisfying the following requirements.\(^6\)

(i) For all \(a \leq a_0\) there is a finite lattice distance \(r_c(a)\) ("size" of the positive core) such that \(G(r, a) \geq 0\) for \(r \leq r_c(a)\) and \(G(r, a) < 0\) for \(r > r_c(a)\). (ii) The function \(r_c(a)\) is non-increasing with decreasing \(a\) for \(a \leq a_0\).

Before we discuss the implications of the above conjecture for the negativity of the TChD 2-point function in the continuum limit, we wish to emphasize the (perhaps) unintuitive nature of Conjecture 1. Indeed, the standard expectation is that the typical gauge fields become "smoother" in terms of lattice distances as the continuum limit is approached. However, one obviously needs to be careful about the interpretation of this expectation since according to Conjecture 1 the typical lattice distance over which \(q(x)\) changes sign actually shrinks with decreasing lattice spacing. This effect has already been noted in Ref. [6] where it manifested itself via the fact that the size of maximal connected regions built from sign-coherent 4-d hypercubes decreases even in lattice units as the continuum limit is approached. This trend is presumably associated with increasingly more definite formation of low-dimensional sign-coherent structure in the vacuum closer to the continuum limit. Related to this is another trend exhibited by our data shown in Fig. 2. In particular, there is a definite lattice distance \(r_d(a)\) (clearly identifiable for \(\mathcal{E}_2\) and \(\mathcal{E}_3\)) for which the maximal negative value of the correlator in lattice units \(G^{\text{min}}(a) \equiv \min_r G(r, a) < 0\) is achieved. The function \(r_d(a)\) is non-increasing (similarly to \(r_c(a)\)) with decreasing lattice spacing. Moreover, for the window of lattice spacings studied here, the magnitude \(-G^{\text{min}}(a)\) of maximal anticorrelation increases with decreasing lattice spacing. At the same time \(G^{\text{max}}(a) \equiv \max_r G(r, a) = G(0, a) > 0\) decreases. In other words, while the typical value of \(q(0)^2\) decreases closer to the continuum limit, the typical magnitude of maximal anticorrelation \(q(0)q(r_d)\) grows in this range of lattice spacings.\(^7\) We again associate this unusual behavior with the presence of a low-dimensional sign-coherent structure in typical configurations [6], and the fact that this structure becomes more sharply defined closer to the continuum limit.

\(^5\)Our earlier results with Wilson gauge action [17] support this conclusion.

\(^6\)One would normally say in mathematics that \(r_c(a)\) is a non-decreasing function in the vicinity of \(a = 0\). However, in lattice gauge theory one usually thinks of changing \(a\) from finite values to zero rather than vice-versa.

\(^7\)We should emphasize that we do not predict the increasing trend for \(-G^{\text{min}}(a)\) to continue arbitrarily close to the continuum limit.
4. The Negativity of the Correlator. Conjecture 1 has immediate consequences relevant for the negativity of the overlap-based TChD correlator in the continuum limit.

**Corollary 1:** The size of the positive core in physical units $r_c^p(a) \equiv ar_c(a)$ vanishes in the continuum limit.

Indeed, according to Conjecture 1 we have

$$r_c^p(a) = ar_c(a) \leq ar_c(a_0) \rightarrow 0 \text{ for } a \rightarrow 0$$

We thus conclude that the TChD 2-point function obtained as a continuum limit of the lattice correlator using an overlap-based TChD operator is negative at arbitrary non-zero distances as required by reflection positivity arguments in the continuum.

To see pictorially how the size of the positive core in physical units decreases as the lattice spacing is lowered, we show the computed lattice correlation functions against physical distance in Fig. 3 (top) with detail of the negative behavior shown on the right. The shrinking of the positive core toward zero physical range can be most clearly seen by separating the lattice-spacing dependence of the shape of the correlator from that of its magnitude. In other words, we write

$$G(r, a) \equiv G(0, a) G^N(r, a)$$

where $G^N(r, a)$ (the “shape”) is normalized to unity at the origin. The behavior of $G^N(r^p, a)$ is shown in Fig. 3 (middle) with detail on the right. We note that the errorbars on these correlators were determined by applying the jacknife procedure directly to $G^N(r^p, a)$ and thus are zero at the origin.

We now wish to use our lattice data to extract more detailed information on how the size of the positive core behaves close to the continuum limit. Conjecture 1 straightforwardly implies the following statement.

**Corollary 2:** The function $r_c(a)$ has a well-defined non-diverging continuum limit, i.e.

$$0 \leq \lim_{a \to 0} r_c(a) \equiv r_c(0) < \infty$$

Moreover, there exists a non-zero lattice spacing $a_c < a_0$ such that $r_c(a) = r_c(0)$ for $a \leq a_c$.

Indeed, since $r_c(a)$ is bounded from below and non-increasing as $a \to 0$, the limit is guaranteed to exist. Moreover, $r_c(a)$ is a discrete-valued function with possible values such that $r_c^2 \in \{0, 1, 2, \ldots\}$ (non-negative integers). This means that there is only a finite number of possible values smaller than $r_c(a_0)$ thus implying the second part of the statement. Our goal is to determine $r_c(0)$. To do that, it is actually practical to attempt a more general calculation. In particular, we will determine the shape of the correlator (i.e. $G^N(r, a)$) at short lattice distances and arbitrarily close to the continuum limit. More precisely, we will assume that the point-wise continuum limit $\lim_{a \to 0} G^N(r, a) \equiv G^N(r)$ exists, and that $G^N(r, a)$ can be power-expanded around it for arbitrary fixed lattice $r$, i.e.

$$G^N(r, a) = G^N(r) + \sum_{k=1}^{\infty} a^k G^N_k(r)$$

Here $G^N_k(r)$ are finite functions (with values depending on the choice of units for $a$). The convergence is expected to be non-uniform across the domain of $r$. The correlators $G^N(r, a)$...
Figure 3: (Top) Lattice 2-point functions of TChD from Fig. 2 plotted against physical distance. (Middle) 2-point function normalized at the origin versus physical distance. The detail is on the right. (Bottom) Short (lattice) distance behavior of $G^N(r, a)$ together with the continuum extrapolation is shown on the left. The individual extrapolations for $r^2 = 1, \ldots, 6$ (in this order from top down) are shown on the right.
for small lattice distances are shown in Fig. 3 (bottom, left). The lattice spacing dependence of $G^N(r, a)$ for $r^2 = 1, \ldots, 6$ is shown on the right, indicating that the linear terms in the expansion (5) are dominant for this range of lattice spacings/lattice distances. We thus perform a linear extrapolation to the continuum limit to estimate $G^N(r)$. The result of the extrapolation is shown together with normalized correlators. One can simply read off the plot that $r_c(0) = \sqrt{3}$. We need to point out here that while the statistical significance of this result is very good (the errorbars are barely visible), there will be a small systematic effect present due to the fact that in the continuum extrapolation we neglect higher orders in lattice spacing. Indeed, a very small positive curvature can be seen under close inspection of Fig. 3 (bottom, right). This will lead to a small shift of the extrapolated values around $r_c(0)$ in the upward direction. A simple estimate from the curvature gives a correction such that $G^N(r = 2) \approx 0$. We thus conclude that $r_c(0) = \sqrt{3}$ or $2$ and that sufficiently close to the continuum limit the size of the positive core is $r_p^c \approx 2a$.

It should be emphasized that contrary to the content of Conjecture 1, we do not expect the specific value of $r_c(0)$ to be strictly universal with respect to the set of “standard” gauge actions or the family of TChD operators labeled by mass parameter $\rho$. For example, changing $\rho$ can probably have some effect on $r_c(0)$ since the precise lattice localization range of the corresponding TChD operator can change [21].

5. Divergent Contact Part. Finally, we wish to illustrate how the expected non-integrable nature of the contact part in TChD 2-point function manifests itself in the regularized correlator as the continuum limit is approached. Let us first recall the standard expectation that $g^p(x^p)$ is a dimension 4 operator (i.e. has no anomalous dimension) and thus $G^p(x^p)$ should behave near the origin as $\sim -|x^p|^{-8}$ up to possible logarithms. Such a singularity is clearly non-integrable and thus, in order to obtain a finite positive space-time integral (susceptibility), the appropriate non-integrable counterterms must be present with support at the origin [1]. In particular, the contact part is expected to have the form [1, 11]

\[ c_1 \delta(x^p) + c_2 \Delta \delta(x^p) + c_3 \Delta^2 \delta(x^p) \]

with $\Delta$ denoting the Laplacian and $c_1$, $c_2$ and $c_3$ being free parameters.\(^8\)

In the ideal situation, all of the above statements should be verified using the measured lattice correlator $G(r, a)$ as the sole input. Here we restrict ourselves to illustrating that the non-integrable positive core indeed emerges in the lattice definition of the theory with overlap-based TChD operator. We will thus simply assume that the finite continuum limit

\[ G^p(r^p) = \lim_{a \to 0} G^p(r^p, a) = \lim_{a \to 0} \frac{G(r^p/a, a)}{a^8} \]

exists for arbitrary $r^p > 0$. To monitor what happens at $r^p = 0$ (i.e. in the lattice positive core which shrinks to $r^p = 0$ in physical units) as the continuum limit is approached, we plot $G^p(r^p, a)$ for our ensembles in Fig. 4. One can see that the emergence of the divergent positive core is very drastic as the continuum limit is approached. The precise nature of this divergence will be discussed quantitatively in a forthcoming publication. To see that

\(^8\)The fact that these parameters do not seem to be fixed by general considerations while topological susceptibility depends on $c_1$ is one of the intriguing aspects of this subject.
the singularity is indeed non-integrable we compute the contribution of the positive core to susceptibility, namely

$$\chi^{+p}(a) = a^{-4} \sum_{r \leq r_c(0)} G(r, a) N(r)$$

where the factor $N(r)$ represents the multiplicity of points $x$ on the hypercubic lattice such that $|x| = r$. Using both $r_c(0) = \sqrt{3}$ and $r_c(0) = 2$ (see discussion in the previous section) we plot $\chi^{+p}(a)$ for our ensembles in Fig. 5. The data clearly exhibits the convex increasing behavior of $\chi^{+p}(a)$ when approaching the continuum limit in agreement with the expected non-integrable nature of the singularity in $G^p(r^p)$. In order to obtain finite positive susceptibility, the diverging contribution of the positive core to the integral of $G^p(x^p)$ has to be canceled by the opposite divergence in the negative part. The emergence of divergent negative dip as the continuum limit is approached is clearly visible in Fig. 4 (right).

6. Conclusions. We have performed the first calculation of the lattice 2-point function of TChD in pure-glue QCD using the topological field operator based on the overlap Dirac matrix. In this initial study, we have focused on basic properties of the correlator expected on general grounds. Thus, our main motivation was to check whether an overlap-based TChD operator offers a valid definition of the topological field in the continuum limit. Our main conclusions are the following.

(i) The correlator exhibits a positive core over finite lattice distance $r_c(a)$ and is negative for larger lattice distances. The function $r_c(a)$ is non-increasing as the continuum limit is approached, implying that the size of the positive core in physical units is zero in the continuum limit. Consequently, the overlap-based TChD operator complies with the requirement imposed by reflection positivity. This result also indicates, in an indirect manner, that the overlap-based TChD operator is local for ensembles used in realistic lattice simulations.

9High statistics numerical evidence confirming that the susceptibility defined via overlap-based TChD operator is indeed finite can be found in Refs. [23, 24]. Our data is consistent with this finding.

10Very recently a manuscript dealing with related issues has been released [25].
Figure 5: The contribution of the positive core to the topological susceptibility using $r_c(0) = \sqrt{3}$ and $r_c(0) = 2$. Data indicates the diverging behavior in the continuum limit.

(ii) Data presented in this article indicate that the value of the correlator at the origin diverges in physical units as does the value at the maximal negative “dip” of the correlator. Moreover, the contribution of the positive core to susceptibility also exhibits divergent behavior thus indicating the presence of a positive non-integrable contact part. This is in agreement with conclusions obtained by formal considerations [1, 11].

Let us finally remark that we find it quite intriguing to explicitly see how the two infinities (the negative one coming from strong power-law behavior and the positive one due to the contact part) manifest themselves in the regularized version of the theory. Since the contact part has strict support at the origin in the continuum, it is natural to expect that both infinities should coexist in the lattice correlator, and emerge simultaneously (via lattice point splitting) as the continuum limit is approached. The diverging positive core and the diverging negative dip of Fig. 4 illustrate how this happens in the overlap-based TChD correlator.

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