Minimal subdynamics and minimal flows without characteristic measures

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Abstract

Given a countable group \(G\) and a \(G\)-flow \(X\), a probability measure \(\mu\) on \(X\) is called characteristic if it is Aut\((X, G)\) invariant. Frisch and Tamuz asked about the existence of a minimal \(G\)-flow, for any group \(G\), which does not admit a characteristic measure. We construct for every countable group \(G\) such a minimal flow. Along the way, we are motivated to consider a family of questions we refer to as minimal subdynamics: Given a countable group \(G\) and a collection of infinite subgroups \(\{\Delta_i : i \in I\}\), when is there a faithful \(G\)-flow for which every \(\Delta_i\) acts minimally?

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Given a countable group \(G\) and a faithful \(G\)-flow \(X\), we write Aut\((X, G)\) for the group of homeomorphisms of \(X\) which commute with the \(G\)-action. When \(G\) is abelian, Aut\((X, G)\) contains a natural copy of \(G\) resulting from the \(G\)-action, but in general this need not be the case. Much is unknown about how the properties of \(X\) restrict the complexity of Aut\((X, G)\); for instance, Cyr and Kra [1] conjecture that when \(G = \mathbb{Z}\) and \(X \subseteq 2^\mathbb{Z}\) is a minimal, 0-entropy subshift, then Aut\((X, \mathbb{Z})\) must be amenable. In fact, no counterexample is known even when restricting to any of the three properties ‘minimal’, ‘0-entropy’ or ‘subshift’. In an effort to shed light on this question, Frisch and Tamuz [3] define a probability measure \(\mu\) on \(X\) to be characteristic if it is Aut\((X, G)\)-invariant. They show that 0-entropy subshifts always admit characteristic measures. More recently, Cyr and Kra [2] provide several examples of flows which admit characteristic measures for nontrivial reasons, even in cases where Aut\((X, G)\) is nonamenable. Frisch and Tamuz asked (Question 1.5, [3]) whether there exists, for any countable group \(G\), some minimal \(G\)-flow without a characteristic measure. We give a strong affirmative answer.

Theorem 0.1. For any countably infinite group \(G\), there is a free minimal \(G\)-flow \(X\) so that \(X\) does not admit a characteristic measure. More precisely, there is a free \((G \times F_2)\)-flow \(X\) which is minimal as a \(G\)-flow and with no \(F_2\)-invariant measure.
We remark that the $X$ we construct will not in general be a subshift.

Over the course of proving Theorem 0.1, there are two main difficulties to overcome. The first difficulty is a collection of dynamical problems we refer to as \textit{minimal subdynamics}. The general template of these questions is as follows.

**Question 0.2.** Given a countably infinite group $\Gamma$ and a collection $\{\Delta_i : i \in I\}$ of infinite subgroups of $\Gamma$, when is there a faithful (or essentially free, or free) minimal $\Gamma$-flow for which the action of each $\Delta_i$ is also minimal? Is there a natural space of actions in which such flows are generic?

In [8], the author showed that this was possible in the case $\Gamma = G \times H$ and $\Delta = G$ for any countably infinite groups $G$ and $H$. We manage to strengthen this result considerably.

**Theorem 0.3.** For any countably infinite group $\Gamma$ and any collection $\{\Delta_n : n \in \mathbb{N}\}$ of infinite normal subgroups of $\Gamma$, there is a free $\Gamma$-flow which is minimal as a $\Delta_n$-flow for every $n \in \mathbb{N}$.

In fact, what we show when proving Theorem 0.3 is considerably stronger. Recall that given a countably infinite group $\Gamma$, a subshift $X \subseteq 2^\Gamma$ is \textit{strongly irreducible} if there is some finite symmetric $D \subseteq \Gamma$ so that whenever $S_0, S_1 \subseteq \Gamma$ satisfy $DS_0 \cap S_1 = \emptyset$ (i.e., $S_0$ and $S_1$ are $D$-apart), then for any $x_0, x_1 \in X$, there is $y \in X$ with $y|S_i = x_i|S_i$ for each $i < 2$. Write $S$ for the set of strongly irreducible subshifts, and write $\overline{S}$ for its Vietoris closure. Frisch, Tamuz and Vahidi-Ferdowsi [5] show that in $\overline{S}$, the minimal subshifts form a dense $G_\delta$ subset. In our proof of Theorem 0.3, we show that the shifts in $\overline{S}$ which are $\Delta_n$-minimal for each $n \in \mathbb{N}$ also form a dense $G_\delta$ subset.

This brings us to the second main difficulty in the proof of Theorem 0.1. Using this stronger form of Theorem 0.3, one could easily prove Theorem 0.1 by finding a strongly irreducible $F_2$-subshift which does not admit an invariant measure. This would imply the existence of a strongly irreducible $(G \times F_2)$-subshift without an $F_2$-invariant measure. As not admitting an $F_2$-invariant measure is a Vietoris-open condition, the genericity of $G$-minimal subshifts would then be enough to obtain the desired result. Unfortunately, whether such a strongly irreducible subshift can exist (for any nonamenable group) is an open question. To overcome this, we introduce a flexible weakening of the notion of a strongly irreducible shift.

The paper is organized as follows. Section 1 is a very brief background section on subsets of groups, subshifts and strong irreducibility. Section 2 introduces the notion of a UFO, a useful combinatorial gadget for constructing shifts where subgroups act minimally; Theorem 0.3 answers Question 3.6 from [8]. Section 3 introduces the notion of $B$-irreducibility for any group $H$, where $B \subseteq P_f (H)$ is a right-invariant collection of finite subsets of $H$. When $H = F_2$, we will be interested in the case when $B$ is the collection of finite subsets of $F_2$ which are connected in the standard left Cayley graph. Section 4 gives the proof of Theorem 0.1.

1. Background

Let $\Gamma$ be a countably infinite group. Given $U, S \subseteq \Gamma$ with $U$ finite, then we call $S$ a (one-sided) $U$-\textit{spaced} set if for every $g \neq h \in S$ we have $h \notin Ug$, and we call $S$ a $U$-\textit{syndetic} set if $US = \Gamma$. A maximal $U$-\textit{spaced} set is simply a $U$-spaced set which is maximal under inclusion. We remark that if $S$ is a maximal $U$-spaced set, then $S$ is $(U \cup U^{-1})$-syndetic. We say that sets $S, T \subseteq \Gamma$ are (one-sided) $U$-\textit{apart} if $US \cap T = \emptyset$ and $S \cap UT = \emptyset$. Notice that much of this discussion simplifies when $U$ is symmetric, so we will often assume this. Also, notice that the properties of being $U$-spaced, maximal $U$-spaced, $U$-syndetic and $U$-apart are all right invariant.

If $A$ is a finite set or \textit{alphabet}, then $\Gamma$ acts on $A^\Gamma$ by \textit{right shift}, where given $x \in A^\Gamma$ and $g, h \in \Gamma$, we have $(gx)(h) = x(hg)$. A \textit{subshift} of $A^\Gamma$ is a nonempty, closed, $\Gamma$-invariant subset. Let $\text{Sub}(A^\Gamma)$ denote the space of subshifts of $A^\Gamma$ endowed with the Vietoris topology. This topology can be described as follows. Given $X \subseteq A^\Gamma$ and a finite $U \subseteq \Gamma$, the set of $U$-\textit{patterns} of $X$ is the set $P_U (X) = \{x|_U : x \in X\} \subseteq A^U$. Then the typical basic open neighborhood of $X \in \text{Sub}(A^\Gamma)$ is the set $N_U (X) := \{Y \in \text{Sub}(A^\Gamma) : P_U (Y) = P_U (X)\}$, where $U$ ranges over finite subsets of $\Gamma$. 

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A subshift $X \subseteq A^\Gamma$ is $U$-irreducible if for any $x_0, x_1 \in X$ and any $S_0, S_1 \subseteq \Gamma$ which are $U$-apart, there is $y \in X$ with $y|_{S_i} = x_i|_{S_i}$ for each $i < 2$. If $X$ is $U$-irreducible and $V \supseteq U$ is finite, then $X$ is also $V$-irreducible. We call $X$ strongly irreducible if there is some finite $U \subseteq \Gamma$ with $X$ $U$-irreducible. By enlarging $U$ if needed, we can always assume $U$ is symmetric. Let $S(A^\Gamma) \subseteq \text{Sub}(A^\Gamma)$ denote the set of strongly irreducible subshifts of $A^\Gamma$, and let $\overline{S}(A^\Gamma)$ denote the closure of this set in the Vietoris topology.

More generally, if $2^M$ denotes Cantor space, then $\Gamma$ acts on $(2^M)\Gamma$ by right shift exactly as above. If $k < \omega$, we let $\pi_k : (2^M) \rightarrow (2^k)\Gamma$ denote the restriction to the first $k$ entries. This induces a factor map $\overline{\pi}_k : (2^M)\Gamma \rightarrow (2^k)\Gamma$ given by $\overline{\pi}_k(x)(g) = \pi_k(x(g))$; we also obtain a map $\overline{\pi}_k : \text{Sub}(2^M)\Gamma \rightarrow \text{Sub}(2^k)\Gamma$ (where $2^k$ is viewed as a finite alphabet) given by $\overline{\pi}_k(X) = \overline{\pi}_k[X]$. The Vietoris topology on $\text{Sub}(2^M)\Gamma$ is the coarsest topology making every such $\overline{\pi}_k$ continuous. We call a subflow $X \subseteq (2^M)\Gamma$ strongly irreducible if for every $k < \omega$, the subshift $\overline{\pi}_k(X) \subseteq (2^k)\Gamma$ is strongly irreducible in the ordinary sense. We let $\overline{S}((2^M)\Gamma) \subseteq \text{Sub}(2^M)\Gamma$ denote the set of strongly irreducible subflows of $(2^M)\Gamma$, and we let $\overline{\mathcal{S}}((2^M)\Gamma)$ denote its Vietoris closure.

The idea of considering the closure of the strongly irreducible shifts has its roots in [4]. This is made more explicit in [5], where it is shown that in $\overline{\mathcal{S}}(A^\Gamma)$, the minimal subflows form a dense $G_\delta$ subset. More or less the same argument shows that the same holds in $\overline{S}((2^M)\Gamma)$ (see [6]). Recall that a $\Gamma$-flow $X$ is free if for every $g \in \Gamma \setminus \{1_\Gamma\}$ and every $x \in X$, we have $gx \neq x$. The main reason for considering a Cantor space alphabet is the following result, which need not be true for finite alphabets.

Proposition 1.1. In $\overline{\mathcal{S}}((2^M)\Gamma)$, the free flows form a dense $G_\delta$ subset.

Proof. Fixing $g \in \Gamma$, the set $\{X \in \text{Sub}((2^M)\Gamma) : \forall x \in X \,(gx \neq x)\}$ is open; indeed, if $X_n \rightarrow X$ is a convergent sequence in $\text{Sub}((2^M)\Gamma)$ and $x_n \in X_n$ is a point fixed by $g$, then passing to a subsequence, we may suppose $x_n \rightarrow x \in X$, and we have $gx = x$. Intersecting over all $g \in \Gamma \setminus \{1_\Gamma\}$, we see that freeness is a $G_\delta$ condition.

Thus, it remains to show that freeness is dense in $\overline{\mathcal{S}}((2^M)\Gamma)$. To that end, we fix $g \in \Gamma \setminus \{1_\Gamma\}$ and show that the set of shifts in $\mathcal{S}((2^M)\Gamma)$ where $g$ acts freely is dense. Fix $X \in \mathcal{S}((2^M)\Gamma)$, $k < \omega$ and a finite $U \subseteq \Gamma$; so a typical open set in $\mathcal{S}((2^M)\Gamma)$ has the form $\{X' \in \mathcal{S}((2^M)\Gamma) : P_U(\overline{\pi}_k(X')) = P_U(\overline{\pi}_k(X))\}$. We want to produce $Y \in \text{Sub}((2^M)\Gamma)$ which is strongly irreducible, $g$-free and with $P_U(\overline{\pi}_k(Y)) = P_U(\overline{\pi}_k(X))$. In fact, we will produce such a $Y$ with $\overline{\pi}_k(Y) = \overline{\pi}_k(X)$.

Let $D \subseteq \Gamma$ be a finite symmetric set containing $g$ and $1_\Gamma$. Setting $m = |D|$, consider the subshift $\text{Color}(D, m) \subseteq m^\Gamma$ defined by

$$\text{Color}(D, m) := \{x \in m^\Gamma : \forall i < m \,[x^{-1}(\{i\}) \text{ is } D\text{-spaced}]\}.$$ 

A greedy coloring argument shows that $\text{Color}(D, m)$ is nonempty and $D$-irreducible. Moreover, $g$ acts freely on $\text{Color}(D, m)$. Inject $m$ into $2^{(k, \ldots, \ell-1)}$ for some $\ell > k$ and identify $\text{Color}(D, m)$ as a subflow of $(2^{(k, \ldots, \ell-1)})\Gamma$. Then $Y := \overline{\pi}_k(X) \times \text{Color}(D, m) \subseteq (2^\ell)\Gamma \subseteq (2^M)\Gamma$, where the last inclusion can be formed by adding strings of zeros to the end. Then $Y$ is strongly irreducible, $g$-free and $\overline{\pi}_k(Y) = \overline{\pi}_k(X)$. \qed

2. UFOs and minimal subdynamics

Much of the construction will require us to reason about the product group $G \times F_2$. So for the time being, fix countably infinite groups $\Delta \subseteq \Gamma$. For our purposes, $\Gamma$ will be $G \times F_2$, and $\Delta$ will be $G$, where we identify $G$ with a subgroup of $G \times F_2$ in the obvious way. However, for this subsection, we will reason more generally.

Definition 2.1. Let $\Delta \subseteq \Gamma$ be countably infinite groups. A finite subset $U \subseteq \Gamma$ is called a $(\Gamma, \Delta)$-UFO if for any maximal $U$-spaced set $S \subseteq \Gamma$, we have that $S$ meets every right coset of $\Delta$ in $\Gamma$.

We say that the inclusion of groups $\Delta \subseteq \Gamma$ admits UFOs if for every finite $U \subseteq \Gamma$, there is a finite $V \subseteq \Gamma$ with $V \supseteq U$ which is a $(\Gamma, \Delta)$-UFO.

As a word of caution, we note that the property of being a $(\Gamma, \Delta)$-UFO is not upwards closed.
The terminology comes from considering the case of a product group, that is, $\Gamma = \mathbb{Z} \times \mathbb{Z}$ and $\Delta = \mathbb{Z} \times \{0\}$. Figure 1 depicts a typical UFO subset of $\mathbb{Z} \times \mathbb{Z}$.

**Proposition 2.2.** Let $\Delta$ be a subgroup of $\Gamma$. If $|\bigcap_{u \in U} u\Delta u^{-1}|$ is infinite for every finite set $U \subseteq \Gamma$, then $\Delta \subseteq \Gamma$ admits UFOs. In particular, if $\Delta$ contains an infinite subgroup that is normal in $\Gamma$, then $\Delta \subseteq \Gamma$ admits UFOs.

**Proof.** We prove the contrapositive. So assume that $\Delta \not\subseteq \Gamma$ does not admit UFOs. Let $U \subseteq \Gamma$ be a finite symmetric set such that no finite $V \subseteq \Gamma$ containing $U$ is a $(\Gamma, \Delta)$-UFO. Let $D \subseteq \Delta$ be finite, symmetric and contain the identity. It will suffice to show that $C = \bigcap_{u \in U} uDu^{-1}$ satisfies $|C| \leq |U|$.

Set $V = U \cup D^2$. Since $V$ is not a $(\Gamma, \Delta)$-UFO, there is a maximal $V$-spaced set $S \subseteq \Gamma$ and $g \in \Gamma$ with $S \cap \Delta g = \emptyset$. Since $S$ is $V$-spaced and $u^{-1}C^2u \subseteq D^2 \subseteq V$, the set $C_u = (uS) \cap (Cg)$ is $C^2$-spaced for every $u \in U$. Of course, any $C^2$-spaced subset of $Cg$ is empty or a singleton, so $|C_u| \leq 1$ for each $u \in U$. On the other hand, since $S$ is maximal we have $VS = \Gamma$, and since $S \cap \Delta g = \emptyset$ we must have $Cg \subseteq US$. Therefore, $|C| = |Cg| = \sum_{u \in U} |C_u| \leq |U|$. □

In the spaces $\overline{S}(k^\Gamma)$ and $\overline{S}((2^\mathbb{N})^\Gamma)$, the minimal flows form a dense $G_\delta$. However, when $\Delta \subseteq \Gamma$ is a subgroup, we can ask about the properties of members of $\overline{S}(k^\Gamma)$ and $\overline{S}((2^\mathbb{N})^\Gamma)$ viewed as $\Delta$-flows.

**Definition 2.3.** Given a subshift $X \subseteq k^\Gamma$ and a finite $E \subseteq \Gamma$, we say that $X$ is $(\Delta, E)$-minimal if for every $x \in X$ and every $p \in P_E(X)$, there is $g \in \Delta$ with $(gx)|_E = p$. Given a subflow $X \subseteq (2^\mathbb{N})^\Gamma$ and $n \in \mathbb{N}$, we say that $X$ is $(\Delta, E, n)$-minimal if $\overline{p_n}(X) \subseteq (2^n)^\Gamma$ is $(\Delta, E)$-minimal. When $\Delta = \Gamma$, we simply say that $X$ is $E$-minimal or $(E, n)$-minimal.

The set of $(\Delta, E)$-minimal flows is open in $\text{Sub}(k^\Gamma)$, and $X \subseteq k^\Gamma$ is minimal as a $\Delta$-flow iff it is $(\Delta, E)$-minimal for every finite $E \subseteq \Gamma$. Similarly, the set of $(\Delta, E, n)$-minimal flows is open in $\text{Sub}((2^\mathbb{N})^\Gamma)$, and $X \subseteq (2^\mathbb{N})^\Gamma$ is minimal as a $\Delta$-flow iff it is $(\Delta, E, n)$ minimal for every finite $E \subseteq \Gamma$ and every $n \in \mathbb{N}$.

In the proof of Proposition 2.4, it will be helpful to extend conventions about the shift action to subsets of $\Gamma$. If $U \subseteq \Gamma$, $g \in G$ and $p \in k^U$, we write $g \cdot p \in k^{Ug^{-1}}$ for the function where given $h \in Ug^{-1}$, we have $(g \cdot p)(h) = p(hg)$.

**Proposition 2.4.** Suppose $\Delta \subseteq \Gamma$ are countably infinite groups and that $\Delta \subseteq \Gamma$ admits UFOs. Then $\{X \in \overline{S}(k^\Gamma) : X \text{ is minimal as a } \Delta \text{-flow}\}$ is a dense $G_\delta$ subset. Similarly, $\{X \in \overline{S}((2^\mathbb{N})^\Gamma) : X \text{ is minimal as a } \Delta \text{-flow}\}$ is a dense $G_\delta$ subset.

**Proof.** We give the arguments for $k^\Gamma$, as those for $(2^\mathbb{N})^\Gamma$ are very similar.

It suffices to show for a given finite $E \subseteq \Gamma$ that the collection of $(\Delta, E)$-minimal flows is dense in $\overline{S}(k^\Gamma)$. By enlarging $E$ if needed, we can assume that $E$ is symmetric.

Consider a nonempty open $Q \subseteq \overline{S}(k^\Gamma)$. By shrinking $Q$ and/or enlarging $E$ if needed, we can assume that for some $X \in S(k^\Gamma)$, we have $Q = N_E(X) \cap \overline{S}(k^\Gamma)$. We will build a $(\Delta, E)$-minimal shift $Y$ with $Y \in N_E(X) \cap S(k^\Gamma)$. Fix a finite symmetric $D \subseteq \Gamma$ so that $X$ is $D$-irreducible. Then fix a finite $U \subseteq \Gamma$ which is large enough to contain an $EDE$-spaced set $Q \subseteq U \cap \Delta$ of cardinality $|P_E(X)|$, and enlarging $U$ if needed, choose such a $Q$ with $EQ \subseteq U$. Fix a bijection $Q \to P_E(X)$ by writing $P_E(X) = \{p_g : g \in Q\}$. Because $X$ is $D$-irreducible, we can find $\alpha \in P_U(X)$ so that $(gq)|_E = p_g$ for every $g \in Q$. By Proposition 2.2, fix a finite $V \subseteq \Gamma$ with $V \supseteq UDU$ which is a
(Γ, Δ)-UFO. We now form the shift
\[ Y = \{ y \in X : \exists \text{ a max. } V\text{-spaced set } T \text{ such that } \forall g \in T \ (g \cdot y)|_U = \alpha \}. \]

Because \( V = UDU \) and \( X \) is \( D \)-irreducible, we have that \( Y \neq \emptyset \). In particular, for any maximal \( V \)-spaced set \( T \subseteq \Gamma \), we can find \( y \in Y \) so that \( (g \cdot y)|_U = \alpha \) for every \( g \in T \). We also note that \( Y \in N_E(X) \) by our construction of \( \alpha \).

To see that \( Y \) is \((\Delta, E)\)-minimal, fix \( y \in Y \) and \( p \in P_E(Y) \). Suppose this is witnessed by the maximal \( V \)-spaced set \( T \subseteq \Gamma \). Because \( V \) is a \( (\Gamma, \Delta) \)-UFO, find \( h \in \Delta \cap T \). So \((hy)|_U = \alpha \). Now, suppose \( g \in Q \) is such that \( p = p_g \). We have \((ghy)|_E = (g \cdot ((hy)|_U)|_E = p_g \).

To see that \( Y \in S(k^\Gamma) \), we will show that \( Y \) is \( DUVUD \)-irreducible. Suppose \( y_0, y_1 \in Y \) and \( S_0, S_1 \subseteq \Gamma \) are \( DUVUD \)-apart. For each \( i < 2 \), fix \( T_i \subseteq \Gamma \) a maximal \( V \)-spaced set which witnesses that \( y_i \) is in \( Y \). Set \( B_i = \{ g \in T_i : DUG \cap S_i \neq \emptyset \} \). Notice that \( B_i \subseteq UDS_i \). It follows that \( B_0 \cup B_1 \) is \( V \)-spaced, so extend to a maximal \( V \)-spaced set \( B \). It also follows that \( S_i \cup UB_i \subseteq U^2 DS_i \). Since \( V \subseteq UDU \) and by the definition of \( B_i \), the collection of sets \( \{ S_i \cup UB_i : i < 2 \} \cup \{ Ug : g \in B \setminus (B_0 \cup B_1) \} \) is pairwise \( D \)- apart. By the \( D \)-irreducibility of \( X \), we can find \( y \in X \) with \( y|_{S_i \cup UB_i} = y_i|_{S_i \cup UB_i} \) for each \( i < 2 \) and with \((g \cdot y)|_U = \alpha \) for each \( g \in B \setminus (B_0 \cup B_1) \). Since \( B_i \subseteq T_i \), we actually have \((g \cdot y)|_U = \alpha \) for each \( g \in B \). So \( y \in Y \) and \( y|_{S_i} = y_i|_{S_i} \) as desired.

**Proof of Theorem 0.3.** By Proposition 2.4, the generic member of \( \overline{S}((\mathbb{Z}^N)^\Gamma) \) is minimal as a \( \Delta_n \)-flow for each \( n \in \mathbb{N} \), and by Proposition 1.1, the generic member of \( \overline{S}((\mathbb{N}^N)^\Gamma) \) is free.

In contrast to Theorem 0.1, the next example shows that Question 0.2 is nontrivial to answer in full generality.

**Theorem 2.5.** Let \( G = \sum_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z}) \), and let \( X \) be a \( G \) flow with infinite underlying space. Then there exists an infinite subgroup \( H \) such that \( X \) is not minimal as an \( H \) flow.

**Proof.** We may assume that \( X \) is a minimal \( G \)-flow, as otherwise we may take \( H = G \). We construct a sequence \( X \supseteq K_0 \supseteq K_1 \supseteq \cdots \) of proper, nonempty, closed subsets of \( X \) and a sequence of group elements \( \{ g_n : n \in \mathbb{N} \} \) so that by setting \( K = \bigcap_{n} K_n \) and \( H = \langle g_n : n \in \mathbb{N} \rangle \), then \( K \) will be a minimal \( H \)-flow. Start by fixing a closed, proper subset \( K_0 \subseteq X \) with nonempty interior. Suppose \( K_n \) has been created and is \( \langle g_0, \ldots, g_{n-1} \rangle \)-invariant. As \( X \) is a minimal \( G \)-flow, the set \( S_n := \{ g \in G : \text{Int}(gK_n \cap K_n) \neq \emptyset \} \) is infinite. Pick any \( g_n \in S_n \setminus \{ 1_G \} \), and set \( K_{n+1} = g_n K_n \cap K_n \). As \( g_n^2 = 1_G \), we see that \( K_{n+1} \) is \( g_n \)-invariant, and as \( G \) is abelian, we see that \( K_{n+1} \) is also \( g_i \)-invariant for each \( i < n \). It follows that \( K \) will be \( H \)-invariant as desired.

Before moving on, we give a conditional proof of Theorem 0.1, which works as long as some nonamenable group admits a strongly irreducible shift without an invariant measure. It is the inspiration for our overall construction.

**Proposition 2.6.** Let \( G \) and \( H \) be countably infinite groups, and suppose that for some \( k < \omega \) and some strongly irreducible flow \( Y \subseteq k^H \) that \( Y \) does not admit an \( H \)-invariant measure. Then there is a minimal \( G \)-flow which does not admit a characteristic measure.

**Proof.** Viewing \( Z = k^G \times Y \) as a subshift of \( k^{G \times H} \), then \( Z \) is strongly irreducible and does not admit an \( H \)-invariant probability measure. The property of not possessing an \( H \)-invariant measure is an open condition in \( \text{Sub}(k^{G \times H}) \); indeed, if \( X_n \rightarrow X \) is a convergent sequence in \( \text{Sub}(k^{G \times H}) \) and \( \mu_n \) is an \( H \)-invariant probability measure supported on \( X_n \), then by passing to a subsequence, we may suppose that the \( \mu_n \) weak∗-converge to some \( H \)-invariant probability measure \( \mu \) supported on \( X \). By Proposition 2.4, we can therefore find \( X \subseteq k^{G \times H} \) which is minimal as a \( G \)-flow and which does not admit an \( H \)-invariant measure. As \( H \) acts by \( G \)-flow automorphisms on \( X \), we see that \( X \) does not admit a characteristic measure. \( \square \)
3. Variants of strong irreducibility

In this section, we investigate a weakening of strong irreducibility that one can define given any right-invariant collection $\mathcal{B}$ of finite subsets of a given countable group. For our overall construction, we will consider $F_2$ and $G \times F_2$, but we give the definitions for any countably infinite group $\Gamma$. Write $\mathcal{P}_f(\Gamma)$ for the collection of finite subsets of $\Gamma$.

**Definition 3.1.** Fix a right-invariant subset $\mathcal{B} \subseteq \mathcal{P}_f(\Gamma)$. Given $k \in \mathbb{N}$, we say that a subshift $X \subseteq k^\Gamma$ is $\mathcal{B}$-irreducible if there is a finite $D \subseteq \Gamma$ so that for any $m < \omega$, any $B_0, \ldots, B_{m-1} \in \mathcal{B}$, and any $x_0, \ldots, x_{m-1} \in X$, if the sets $\{B_0, \ldots, B_{m-1}\}$ are pairwise $D$-apart, then there is $y \in X$ with $y|_{B_i} = x_i|_{B_i}$ for each $i < m$. We call $D$ the witness to $\mathcal{B}$-irreducibility. If we have $D$ in mind, we can say that $X$ is $\mathcal{B}$-$D$-irreducible.

We say that a subflow $X \subseteq (2^\mathbb{N})^\Gamma$ is $\mathcal{B}$-irreducible if for each $k \in \mathbb{N}$, the subshift $\pi_k(X) \subseteq (2^k)^\Gamma$ is $\mathcal{B}$-irreducible.

We write $S_{\mathcal{B}}(k^\Gamma)$ or $S_{\mathcal{B}}((2^\mathbb{N})^\Gamma)$ for the set of $\mathcal{B}$-irreducible subflows of $k^\Gamma$ or $(2^\mathbb{N})^\Gamma$, respectively, and we write $\overline{S}_{\mathcal{B}}(k^\Gamma)$ or $\overline{S}_{\mathcal{B}}((2^\mathbb{N})^\Gamma)$ for the Vietoris closures.

**Remark.**

1. If $\mathcal{B}$ is closed under unions, it is enough to consider $m = 2$. However, this will often not be the case.
2. By compactness, if $X \subseteq k^\Gamma$ is $\mathcal{B}$-$D$-irreducible, $\{B_n : n < \omega\} \subseteq \mathcal{B}$ is pairwise $D$-apart, and $\{x_n : n < \omega\} \subseteq X$, then there is $y \in X$ with $y|_{B_i} = x_i|_{B_i}$.
3. If $\mathcal{B} \subseteq \mathcal{B}'$, then $S_{\mathcal{B}'}(k^\Gamma) \subseteq S_{\mathcal{B}}(k^\Gamma)$ and $S_{\mathcal{B}'}((2^\mathbb{N})^\Gamma) \subseteq S_{\mathcal{B}}((2^\mathbb{N})^\Gamma)$.

When $\mathcal{B}$ is the collection of all finite subsets of $H$, then we recover the notion of a strongly irreducible shift. Again, we consider Cantor space alphabets to obtain freeness.

**Proposition 3.2.** For any right-invariant collection $\mathcal{B} \subseteq \mathcal{P}_f(\Gamma)$, the generic member of $\overline{S}_{\mathcal{B}}((2^\mathbb{N})^\Gamma)$ is free.

**Proof.** Analyzing the proof of Proposition 1.1, we see that the only properties that we need of the collections $S_{\mathcal{B}}(k^\Gamma)$ and $S_{\mathcal{B}}((2^\mathbb{N})^\Gamma)$ for the proof to generalize are that they are closed under products and contain the flows $\text{Color}(D, m)$. If $k, \ell \in \mathbb{N}$ an $X \subseteq k^\Gamma$ and $Y \subseteq \ell^\Gamma$ are $\mathcal{B}$-$D$-irreducible and $\mathcal{B}$-$E$-irreducible for some finite $D, E \subseteq \Gamma$, then $X \times Y \subseteq (k \times \ell)^\Gamma$ will be $\mathcal{B}$-$(D \cup E)$-irreducible. And as $\text{Color}(D, m)$ is strongly irreducible, it is $\mathcal{B}$-irreducible. \hfill $\Box$

Now, we consider the group $F_2$. We consider the left Cayley graph of $F_2$ with respect to the standard generating set $A := \{a, b, a^{-1}, b^{-1}\}$. We let $d : F_2 \times F_2 \to \omega$ denote the graph metric. Write $D_n = \{s \in F_2 : d(s, 1_{F_2}) \leq n\}$.

**Definition 3.3.** Given $n$ with $1 \leq n < \omega$, we set

$$B_n = \{D \in \mathcal{P}_f(F_2) : \text{connected components of } D \text{ are pairwise } D_n\text{-apart}\}.$$  

Write $B_\omega$ for the collection of finite, connected subsets of $F_2$.

**Proposition 3.4.** Suppose $X \subseteq k^{F_2}$ is $B_\omega$-irreducible. Then there is some $n < \omega$ for which $X$ is $B_n$-irreducible.
Proof. Suppose $X$ is $B_\omega$-$D_n$-irreducible. We claim $X$ is $B_n$-$D_n$-irreducible. Suppose $m < \omega$, $B_0, \ldots, B_{m-1} \in B_n$ are pairwise $D_n$-apart, and $x_0, \ldots, x_{m-1} \in X$. For each $i < m$, we suppose $B_i$ has $n_i$-many connected components, and we write $\{C_{i,j} : j < n_i\}$ for these components. Then the collection of connected sets $\bigcup_{i < m} \{C_{i,j} : j < n_i\}$ is pairwise $D_n$-apart. As $X$ is $B_\omega$-$D_n$-irreducible, we can find $y \in X$ so that for each $i < m$ and $j < n_i$, we have $y|_{C_{i,j}} = x_i|_{C_{i,j}}$. Hence, $y|_{B_i} = x_i|_{B_i}$, showing that $X$ is $B_n$-$D_n$-irreducible.

We now construct a $B_\omega$-irreducible subshift with no $F_2$-invariant measure. We consider the alphabet $A^2$ and write $\pi_0, \pi_1 : A^2 \to A$ for the projections. We set

$$X_{padox} = \{x \in (A^2)^2 : \forall g \in F_2 \forall i, j < 2 (i, g) \neq (j, h) \Rightarrow \pi_i(x(g)) \cdot g \neq \pi_j(x(h)) \cdot h\}.$$ 

More informally, the flow $X_{padox}$ is the space of ‘2-to-1 paradoxical decompositions’ of $F_2$ using $A$. We remark that here, our decomposition need not be a partition of $F_2$; we just ask for disjoint $S_0, S_1 \subseteq F_2$ such that for every $g \in G$ and $i < 2$, we have $Ag \cap S_i \neq \emptyset$. This is in some sense the prototypical example of an $F_2$-shift with no $F_2$-invariant measure.

Lemma 3.5. $X_{padox}$ has no $F_2$-invariant measure.

Proof. For $u \in A^2$ set $Y_u = \{x \in X_{padox} : x(1_G) = u\}$. Notice that if $y \in Y_u, i < 2$ and $x = \pi_i(u)y$, then $x(\pi_i(u)^{-1}) = y(1_G) = u$. Consequently, if $u, v \in A^2, x \in \pi_i(u)Y_u \cap \pi_j(v)Y_v$ then, since $x \in X_{padox}$ and

$$\pi_i(x(\pi_i(u)^{-1}))\pi_i(u)^{-1} = 1_G = \pi_j(x(\pi_j(v)^{-1}))\pi_j(v)^{-1},$$

we must have that $(i, \pi_i(u)) = (j, \pi_j(v))$, and hence also

$$\pi_{1-i}(u) = \pi_{1-i}(x(\pi_i(u)^{-1})) = \pi_{1-j}(x(\pi_j(v)^{-1})) = \pi_{1-j}(v).$$

Therefore, $\pi_i(u)Y_u \cap \pi_j(v)Y_v = \emptyset$ whenever $(i, u) \neq (j, v)$.

If $\mu$ were an invariant Borel probability measure on $X_{padox}$, then we would have

$$2\mu(X_{padox}) = 2 \sum_{u \in A^2} \mu(Y_u) = \sum_{i < 2} \sum_{u \in A^2} \mu(\pi_i(u)Y_u) \leq \mu(X)$$

which is a contradiction.

When proving that $X_{padox}$ is $B_\omega$-irreducible, note that $D_1 = A \cup \{1_F\}$.

Proposition 3.6. $X_{padox}$ is $B_\omega$-$D_4$-irreducible.

Proof. The proof will use a 2-to-1 instance of Hall’s matching criterion [7] which we briefly describe. Fix a bipartite graph $G = (V, E)$ with partition $V = V_0 \cup V_1$. Given $S \subseteq V_0$, write $N_G(S) = \{v \in V_1 : \exists u \in S(u, v) \in E\}$. Then the matching condition we need states that if for every finite $S \subseteq V_0$, we have $|N_G(S)| \geq 2S$, then there is $E' \subseteq E$ so that in the graph $G' := (V, E')$, $d_G(u) = 2$ for every $u \in V_0$.

Let $B_0, \ldots, B_{k-1} \in B_\omega$ be pairwise $D_4$-apart. Let $x_0, \ldots, x_{k-1} \in X_{padox}$. To construct $y \in X_{padox}$ with $y|_{B_i} = x_i|_{B_i}$ for each $i < k$, we need to verify a 2-to-1 Hall’s matching criterion on every finite subset of $F_2 \setminus \bigcup_{i < k} B_i$. Call $s \in F_2$ matched if for some $i < k$, some $g \in B_i$ and some $j < 2$, we have $s = \pi_j(x_i(g)) \cdot g$. So we need for every finite $E \subseteq \mathcal{P}_f(F_2 \setminus \bigcup_{i < k} B_i)$ that $AE$ contains at least $2|E|$-many unmatched elements. Towards a contradiction, let $E \subseteq \mathcal{P}_f(F_2 \setminus \bigcup_{i < k} B_i)$ be a minimal failure of the Hall condition.

In the left Cayley graph of $F_2$, given a reduced word $w$ in alphabet $A = \{a, b, a^{-1}, b^{-1}\}$, write $N_w$ for the set of reduced words which end with $w$. Now, find $t \in E$ (let us assume the leftmost character of $t$ is $a$) so that all of $E \cap N_{at}, E \cap N_{bt}$ and $E \cap N_{b^{-1}t}$ are empty. If any two of $at, bt$ and $b^{-1}t$ is an unmatched point in $AE$, then $E \setminus \{t\}$ is a smaller failure of Hall’s criterion. So there must be
some $i < k$, some $g \in B_i$ and some $j < 2$, we have $\pi_j(x_i(g)) \cdot g \in \{at, bt, b^{-1}t\}$. Let us suppose $\pi_j(x_i(g)) \cdot g = at$. Note that since $g \notin E$, we must have $g \in \{bat, a^2t, b^{-1}at\}$. But then since $B_i$ is connected, we have $D_1B_i \cap \{bt, b^{-1}t\} = \emptyset$, and since the other $B_q$ are at least distance 5 from $B_i$, we have $D_1B_q \cap \{bt, b^{-1}t\} = \emptyset$ for every $q \in k \setminus \{i\}$. In particular, $bt$ and $b^{-1}t$ are unmatched points in $AE$, a contradiction. \qed

We remark that $X_{p_dox}$ is not $D_n$-irreducible for any $n \in \mathbb{N}$. See Figure 2.

### 4. The construction

Our goal for the rest of the paper is to use $X_{p_dox}$ to build a subflow of $(2^{|I|})^{G \times F_2}$ which is free, $G$-minimal and with no $F_2$-invariant measure. In what follows, given an $F_2$-coset $\{g\} \times F_2$, we endow this coset with the left Cayley graph for $F_2$ using the generating set $A$ exactly as above. We extend the definition of $B_n$ to refer to finite subsets of any given $F_2$-coset.

**Definition 4.1.** Given $n$ with $1 \leq n \leq \omega$, we set

$$B_n^* = \{D \in \mathcal{P}_f (G \times F_2) : \text{ for each } F_2 \text{- coset } C, D \cap C \in B_n\}.$$  

Given $y \in k^{G \times F_2}$ and $g \in G$, we define $y_g \in k^{F_2}$ where given $s \in F_2$, we set $y_g(s) = y(g, s)$. If $X \subseteq k^{F_2}$ is $B_n^*$-irreducible, then the subshift $X^G \subseteq k^{G \times F_2}$ is in $S_{B_n^*}$, where we view $X^G$ as the set $\{y \in k^{G \times F_2} : \forall g \in G (y_g \in X)\}$. In particular, $(X_{p_dox})^G$ is $B_n^*$-irreducible. By encoding $(X_{p_dox})^G$ as a subshift of $(2^{|I|})^{G \times F_2}$ for some $m \in \mathbb{N}$ and considering $\tilde{\pi}_m^{-1}((X_{p_dox})^G) \subseteq (2^{|I|})^{G \times F_2}$, we see that there is a $B_n^*$-irreducible subflow of $(2^{|I|})^{G \times F_2}$ for which the $F_2$-action doesn’t fix a measure. It follows that such subflows constitute a nonempty open subset of $\Phi := \bigcup_n S_{B_n^*}((2^{|I|})^{G \times F_2})$. Combining the next result with Proposition 3.2, we will complete the proof of Theorem 0.1.

**Proposition 4.2.** With $\Phi$ as above, the $G$-minimal flows are dense $G_\delta$ in $\Phi$.

**Proof.** We show the result for $\Phi_k := \bigcup_n S_{B_n^*}(k^{G \times F_2})$ to simplify notation; the proof in full generality is almost identical.

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*Figure 2. A pair of outgoing edges, drawn in solid red, is chosen at each of $v_{00}, v_{01}, v_{10}$ and $v_{11}$. Edges which must consequently be oriented in a particular direction are indicated with dashed red arrows. Most importantly, $v_\varnothing$ is forced to direct an edge to $u_\varnothing$. By considering the generalization of this picture for any length of binary string, we see that $X_{p_dox}$ cannot be $D_n$-irreducible for any $n \in \mathbb{N}$.***
We only need to show density. To that end, fix a finite symmetric \( E \subseteq G \times F_2 \) which is connected in each \( F_2 \)-coset. It is enough to show that the \((G, E)\)-minimal subshifts are dense in \( \Phi_k \). Fix some nonempty open \( O \subseteq \Phi_k \). By enlarging \( E \) and/or shrinking \( O \), we may assume that for some \( n < \omega \) and 

\[ X \in S_{B_n^*}(k^{G \times F_2}) \text{ so that } O = \{ X' \in \Phi_k : P_E(X') = P_E(X) \}. \]

We will build a \((G, E)\)-minimal subshift \( Y \subseteq k^{G \times F_2} \) so that \( P_E(Y) = P_E(X) \) and so that for some \( N < \omega \), we have \( Y \in S_{B_n^*}(k^{G \times F_2}) \).

Recall that \( D_n \subseteq F_2 \) denotes the ball of radius \( n \). Fix a finite, symmetric \( D \subseteq G \times F_2 \) so that 

\[ \{1_G\} \times D_{2n} \subseteq D \text{ and } X \in B_{n}^*-D\text{-irreducible.} \]

Find a finite symmetric \( U_0 \subseteq G \) with \( 1_G \subseteq U_0 \) and \( r < \omega \) so that upon setting \( U = U_0 \times D_r \subseteq G \times F_2 \), then \( U \) is large enough to contain an \( EDE \)-spaced set \( Q \subseteq G \) with \( EQ \subseteq U \). As \( X \) is \( B_{n}^*-D\text{-irreducible, there is a pattern } \alpha \in P_U(X) \) so that 

\[ \{(ga)_E : g \in Q\} = P_E(X). \]

Let \( V \supseteq UD^2U \) be a \((G \times F_2, G)\)-UFO. We remark that for most of the remainder of the proof, it would be enough to have \( V \supseteq UDU \); we only use the stronger assumption \( V \supseteq UD^2U \) in the proof of the final claim. Consider the following subshift:

\[ Y = \{ y \in X : \exists \text{ a max. } V\text{-spaced set } T \text{ so that } \forall g \in T (gy)|_{V} = \alpha \}. \]

The proof that \( Y \) is nonempty and \((G, E)\)-minimal is exactly the same as the analogous proof from Proposition 2.4. Note that by construction, we have \( P_E(Y) = P_E(X) \).

We now show that \( Y \in S_{B_n^*}(k^{G \times F_2}) \) for \( N = 4r + 3n \). Set \( W = DUVUD \). We show that \( Y \) is \( B_{N}^*\text{-W}\text{-irreducible.} \)

Suppose \( m < \omega \), \( y_0, \ldots, y_{m-1} \in Y \) and \( S_0, \ldots, S_{m-1} \in B_{n}^* \) are pairwise \( \mathcal{V}\text{-apart.} \)

Suppose for each \( i < m \) that \( T_i \subseteq G \times F_2 \) is a maximal \( \mathcal{V}\text{-spaced set which witness that } v_i \in Y \). Set 

\[ B_i = \{ g \in T_i : \text{ DU}g \cap S_i \neq \emptyset \}. \]

Then \( \bigcup_{i < m} B_i \) is \( \mathcal{V}\text{-spaced, so enlarge to a maximal } \mathcal{V}\text{-spaced set } B \subseteq G \times F_2. \)

For each \( i < m \), we enlarge \( S_i \cup UB_i \) to \( J_i \in B_{n}^* \) as follows. Suppose \( C \subseteq G \times F_2 \) is an \( F_2\)-coset. Each set of the form \( C \cap Ug \) is connected. Since \( S_i \in B_{n}^* \), it follows that given \( g \in B_i \), there is at most one connected component \( \Theta_{C,g} \) of \( S_i \cap C \) with \( Ug \cap \Theta_{C,g} = \emptyset \), but \( Ug \cap D_n \Theta_{C,g} \neq \emptyset \). We add the line segment in \( C \) connecting \( \Theta_{C,g} \) and \( Ug \). Upon doing this for each \( g \in B_i \) and each \( F_2\)-coset \( C \), this completes the construction of \( J_i \). Observe that \( J_i \subseteq D_{n-1} S_i \cap UB_i \).

**Claim.** Let \( C \) be an \( F_2\)-coset, and suppose \( Y_0 \) is a connected component of \( S_i \cap C \). Let \( Y \) be the connected component of \( J_i \cap C \) with \( Y_0 \subseteq Y \). Then \( Y \subseteq D_{2r+n}Y_0 \). In particular, if \( Y_0 \neq \emptyset \) are two connected components of \( S_i \cap C \), then \( Y_0 \) and \( Z_0 \) do not belong to the same component of \( J_i \cap C \).

**Proof.** Let \( L = \{ x_j : j < \omega \} \subseteq C \) be a ray with \( x_0 \in Y_0 \) and \( x_j \notin Y_0 \) for any \( j \geq 1 \). Then \( \{ j < \omega : x_j \in J_i \cap C \} \) is some finite initial segment of \( \omega \). We want to argue that for some \( j \leq 2r + n + 1 \), we have \( x_j \notin J_i \cap C \). First, we argue that if \( x_n \in J_i \cap C \), then \( x_n \notin UB_i \). Otherwise, we must have \( x_n \in D_{n-1} S_i \). But since \( x_n \notin D_{n-1} Y_0 \), there must be another component \( Y_1 \) of \( S_i \cap C \) with \( x_n \in D_n Y_1 \). But this implies that \( Y_0 \) and \( Y_1 \) are not \( D_{2n-1}\text{-apart, a contradiction since } 2n - 1 \leq 4r - 3n = N \).

Fix \( g \in B_i \) with \( x_n \in Ug \). Let \( q < \omega \) be least with \( q > n \) and \( x_q \notin U_g \). We must have \( q \leq 2r + n + 1 \).

We claim that \( x_q \notin J_i \cap C \). Towards a contradiction, suppose \( x_q \in J_i \cap C \). We cannot have \( x_q \in UB_i \), so we must have \( x_q \in D_{n-1} S_i \). But now there must be some component \( Y_1 \) of \( S_i \cap C \) with \( x_q \in D_n Y_1 \). But then \( D_{2r+2n} Y_0 \cap Y_1 \neq \emptyset \), a contradiction as \( Y_0 \) and \( Y_1 \) are \( D_n\text{-apart.} \) This concludes the proof that \( Y \subseteq D_{2r+n}Y_0 \).

Now, suppose \( Y_0 \neq \emptyset \) are two connected components of \( S_i \cap C \). Then \( Y_0 \) and \( Z_0 \) are \( N\text{-apart.} \) In particular, \( Z_0 \notin D_{2r+n} Y_0 \), so they do not belong to the same connected component of \( J_i \cap C \) as \( Y_0 \).

**Claim.** \( J_i \in B_{n}^* \).

**Proof.** Fix an \( F_2\)-coset \( C \) and two connected components \( Y \neq Z \) of \( J_i \cap C \). By the previous claim, each of \( Y \) and \( Z \) can only contain at most one nonempty component of \( S_i \cap C \). The claim will be proven after considering three cases.

1. First, suppose each of \( Y \) and \( Z \) contain a nonempty component of \( S_i \cap C \), say \( Y_0 \subseteq Y \) and \( Z_0 \subseteq Z \).

Then since \( Y_0 \) and \( Z_0 \) are \( D_{4r+3n}\text{-apart, the previous claim implies that } Y \) and \( Z \) are \( D_{n}\text{-apart.} \)

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2. Now, suppose $Y$ contains a nonempty component $Y_0$ of $S_i \cap C$ and that $Z$ does not. Then for some $g \in B_i$, we have $Z = Ug \cap C$. Towards a contradiction, suppose $D_n Y \cap Ug \neq \emptyset$. Let $L = \{x_j : j \leq M\}$ be the line segment connecting $Y$ and $Ug$ with $L \cap Y = \{x_0\}$ and $L \cap Ug = \{x_M\}$. We must have $M \leq n$. We cannot have $x_0 \in UB_i$, so we must have $x_0 \in D_{n-1}S_i$. This implies that $x_0 \in D_{n-1}Y_0$. We cannot have $x_0 \in Y_0$, as otherwise, we would have connected $Y_0$ and $Ug \cap C$ when constructing $J_i$. It follows that for some $h \in B_i$, we have that $x_0$ is on the line segment $L' = \{x'_j : j \leq M'\}$ connecting $Y_0$ and $Uh \cap C$, and we have $M' \leq n$. But this implies that $Ug \cap D_{2n}Uh \neq \emptyset$, a contradiction since $V \supseteq UDU$ and $D \supseteq D_{2n}$.

3. If neither $Y$ nor $Z$ contain a component of $S_i \cap C$, then there are $g \neq h \in B_i$ with $Y = Uh \cap C$ and $Z = Ug \cap C$. It follows that $Y$ and $Z$ are $D_n$-apart.

**Claim.** Suppose $i \neq j < m$. Then $J_i$ and $J_j$ are $D$-apart.

**Proof.** We have that $J_i \subseteq D_{n-1}S_i \cup UB_i$, and likewise for $j$. As $UB_i \subseteq U^2DS_i$ and as $D \supseteq D_{2n}$, we have $J_i \subseteq U^2DS_i$, and likewise for $j$. As $S_i$ and $S_j$ are $W$-apart and as $V \supseteq UDU$, we see that $J_i$ and $J_j$ are $D$-apart.

**Claim.** Suppose $g \in B \setminus \bigcup_{i \leq n} B_i$. Then $Ug$ and $J_i$ are $D$-apart for any $i < m$.

**Proof.** As $g \notin B_i$, we have $Ug$ and $S_i$ are $D$-apart. Also, for any $h \in B$ with $g \neq h$, we have that $Ug$ and $Uh$ are $D$-apart. Now, suppose $DUg \cap J_i \neq \emptyset$. If $x \in DUg \cap J_i$, then on the coset $C = F_2\chi$, $x$ must belong on the line between a component of $S_i \cap C$ and $Uh$ for some $h \in B_i$. Furthermore, we have $x \in D_{n-1}Uh$. But since $D_{2n} \subseteq D$, this contradicts that $Ug$ and $Uh$ are $D^2$-apart (using the full assumption $V \supseteq UDU$).

We can now finish the proof of Proposition 4.2. The collection $\{J_i : i < m\} \cup \{Ug : g \in B \setminus \bigcup_{i < m} B_i\}$ is a pairwise $D$-apart collection of members of $B_n^*$. As $X$ is $B_n^*$-irreducible, we can find $y \in X$ with $y|J_i = y_i|J_i$ for each $i < m$ and with $(gy)|_U = \alpha$ for each $g \in B \setminus \bigcup_{i < m} B_i$. As $J_i \supseteq UB_i$ and since $B_i \subseteq T_i$, we actually have $(gy)|_U = \alpha$ for each $g \in B$. As $B$ is a maximal $V$-spaced set, it follows that $y \in Y$ and $y|S_i = y_i|S_i$ as desired.

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**References**

[1] V. Cyr and B. Kra, ‘The automorphism group of a minimal shift of stretched exponential growth’, *J. Mod. Dyn.* **10** (2016), 483–495.

[2] V. Cyr and B. Kra, ‘Characteristic measures for language stable subshifts’, *Monatshefte fur Mathematik* **201**(3) (2023), 659–701.

[3] J. Frisch and O. Tamuz, ‘Characteristic measures of symbolic dynamical systems’, *Ergodic Theory Dyn. Sys.* **42**(5) (2022), 1655–1661.

[4] J. Frisch and O. Tamuz, ‘Symbolic dynamics on amenable groups: the entropy of generic shifts’, *Ergodic Theory Dyn. Sys.* **37**(4) (2017), 1187–1210.

[5] J. Frisch, O. Tamuz, and P. Vahidi Ferdowsi, ‘Strong amenability and the infinite conjugacy class property’, *Inventiones Mathematicae* **218** (2019), 833–851.

[6] E. Glasner, T. Tsankov, B. Weiss, and A. Zucker, ‘Bernoulli disjointness’, *Duke Mathematical Journal* **170**(4) (2021), 615–651.

[7] P. Hall, ‘On representatives of subsets’, *J. London Math. Soc.* **10** (1) (1935), 26–30.

[8] A. Zucker, ‘Minimal flows with arbitrary centralizer’, *Ergodic Theory Dyn. Sys.* **42**(1) (2022), 310–320.