GLOBAL MAXIMIZERS FOR THE SPHERE ADJOINT FOURIER RESTRICTION INEQUALITY

DAMIANO FOSCHI

ABSTRACT. We show that constant functions are global maximizers for the adjoint Fourier restriction inequality for the sphere.

1. INTRODUCTION

Recently, Christ and Shao [1, 2] have proved the existence of maximizers for the adjoint Fourier restriction inequality of Stein and Thomas [5] for the sphere:

\[ \left\| \hat{f} \sigma \right\|_{L^4(\mathbb{R}^3)} \lesssim \left\| f \right\|_{L^2(S^2)}, \]

where \( S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \} \) is the standard unit sphere equipped with its natural surface measure \( \sigma \) induced by the Lebesgue measure on \( \mathbb{R}^3 \). Here the Fourier transform of a function \( f \) supported on the sphere is defined for any \( x \in \mathbb{R}^3 \) by \( \hat{f}(x) = \int_{S^2} e^{-ix \cdot \omega} f(\omega) d\sigma_\omega \).

Let us denote by \( R \) the optimal constant in (1):

\[ R := \sup_{f \in L^2(S^2), \ f \neq 0} \frac{\left\| \hat{f} \sigma \right\|_{L^4(\mathbb{R}^3)}}{\left\| f \right\|_{L^2(S^2)}}. \]

In [1], using concentration compactness methods, they prove that there exist sequences \( \{ f_k \} \) of nonnegative even functions in \( L^2(S^2) \) which converge to some maximizer of the ratio \( \left\| \hat{f} \sigma \right\|_{L^4} / \left\| f \right\|_{L^2} \), but they do not compute the exact value of \( R \). Nevertheless, they show that constant functions are local maximizers and raise the question of whether constants are actually global maximizers. The purpose of this note is to give a positive answer to that question:

**Theorem 1.1.** A nonnegative function \( f \in L^2(S^2) \) is a global maximizers for (1) if and only if it is a non zero constant, and we have

\[ R = \frac{\left\| \hat{1} \sigma \right\|_{L^4(\mathbb{R}^3)}}{\left\| 1 \right\|_{L^2(S^2)}} = 2^{5/4} \pi. \]

When we combine Theorem 1.1 with the results of [2, Theorem 1.2] we obtain that all complex valued global maximizers for (1) are of the form

\[ f(\omega) = ke^{i\theta} e^{i\xi \cdot \omega}, \]

for some \( k > 0, \theta \in \mathbb{R}, \xi \in \mathbb{R}^3 \).

A large part of the analysis carried out in [1] is local in nature and it is based on a comparison between the case of the sphere and that of a paraboloid which approximates the sphere at one point. Here we are able to keep everything global, thanks to an interesting geometric feature of the sphere, which is expressed in Lemma 4.2.
It essentially says: when the sum $\omega_1 + \omega_2 + \omega_3$ of three unit vectors is again a unit vector, then we have

$$|\omega_1 + \omega_2|^2 + |\omega_1 + \omega_3|^2 + |\omega_2 + \omega_3|^2 = 4.$$  

In order to find maximizers for (1), we follow the spirit of the proof of analogous results obtained by the author for the paraboloid and the cone [4]. The main steps are:

- The exponent 4 is an even integer and we can view the $L^4$ norm as a $L^2$ norm of a product, which becomes, thru the Fourier transform, a $L^2$ norm of a convolution. We write the $L^2$ norm of a convolution of measures supported on the sphere as a quadrilinear integral over a submanifold of $(S^2)^4$.

- A careful application of the Cauchy-Schwarz inequality over that submanifold allows us to control the quadrilinear integral by some bilinear integral over $(S^2)^2$.

- Finally, by a spectral decomposition of the bilinear integral using spherical harmonics will show that the optimal bounds for the bilinear integral are obtained when we consider constant data.

We will see that every time an inequality appears, the choice of $f$ constant will correspond to the case of equality.

2. Quadrilinear form associated to the estimate

**Definition 2.1.** Given a complex valued function $f$ defined on $S^2$, its antipodally conjugate $f_\star$ is defined by $f_\star(\omega) := f(-\omega)$.

By Plancherel’s theorem we have

$$\left\| \hat{\sigma} \right\|_{L^4(R^3)}^2 = \left\| \hat{\sigma} \hat{\sigma} \right\|_{L^2(R^3)} = \left\| \hat{\sigma} \hat{\sigma} \right\|_{L^2(R^3)} = \left\| f_\star f_\star \right\|_{L^2(R^3)} = (2\pi)^{\frac{7}{2}} \left\| f_\star f_\star \right\|_{L^2(R^3)}.$$

When $f$ is constant we can explicitly compute this convolution.

**Lemma 2.2.** For $x \in \mathbb{R}^3$ we have

$$\sigma * \sigma(x) = \int_{S^2} \delta(x - \omega - \nu) \, d\sigma_\omega \, d\sigma_\nu = \frac{2\pi}{|x|} \chi(|x| \leq 2),$$

with norm $\| \sigma * \sigma \|_{L^2(\mathbb{R}^3)} = 2^{5/2} \pi^{3/2}$.

The notation $\delta(\cdot)$ stands for the Dirac’s delta measure concentrated at the origin of $\mathbb{R}^n$.

**Proof.** The surface measure of the sphere can be written as

$$d\sigma_\omega = \delta(1 - |\omega|) \, d\omega = 2 \delta(1 - |\omega|^2) \, d\omega.$$  

The convolution then can be written as

$$\sigma * \sigma(x) = 2 \int_{S^2} \delta(1 - |x - \omega|^2) \, d\omega = 2 \int_{S^2} \delta(2x \cdot \omega - |x|^2) \, d\omega =$$

$$= \frac{2\pi}{|x|} \int_0^\pi \delta(\cos \theta - \frac{|x|}{2}) \sin \theta \, d\theta = \frac{2\pi}{|x|} \int_{-1}^1 \delta \left( c - \frac{|x|}{2} \right) \, dc =$$

$$= \frac{2\pi}{|x|} \chi \left( \frac{|x|}{2} \leq 1 \right).$$
The norm can then be easily computed,
\[
\|\sigma * \sigma\|^2_{L^2(\mathbb{R}^3)} = 4\pi^2 \int_{|x| \leq 2} \frac{dx}{|x|^2} = 4\pi^2 4\pi \int_0^2 dr = 32\pi^3.
\]

For a generic data \(f\), we can write the convolution in (2) as
\[
f \sigma * f_\ast \sigma(x) = \int_{S^2 \times S^2} f(\omega) \overline{f(-\nu)} \delta(x - \omega - \nu) \, d\sigma_\omega \, d\sigma_\nu.
\]
The \(L^2\) norm of the convolution can be written as a quadrilinear integral
\[
\|f \sigma * f_\ast \sigma\|^2_{L^2(\mathbb{R}^3)} = \int_{(S^2)^4} f(\omega_1) \overline{f(-\nu_1)} f(\omega_2) \overline{f(-\nu_2)} \delta(\omega_1 + \nu_1 - \omega_2 - \nu_2) \, d\sigma_{\omega_1} \, d\sigma_{\nu_1} \, d\sigma_{\omega_2} \, d\sigma_{\nu_2} = \int f(\omega_1) \overline{f(-\omega_2)} f(\omega_3) \overline{f(-\omega_4)} \, d\Sigma_\omega = Q(f, f_\ast, f, f_\ast),
\]
where the measure \(\Sigma\) is given by
\[
d\Sigma(\omega_1, \omega_2, \omega_3, \omega_4) := \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) \, d\sigma_{\omega_1} \, d\sigma_{\omega_2} \, d\sigma_{\omega_3} \, d\sigma_{\omega_4},
\]
and \(Q\) is the quadrilinear form defined by
\[
Q(f_1, f_2, f_3, f_4) := \int f_1(\omega_1) f_2(\omega_2) f_3(\omega_3) f_4(\omega_4) \, d\Sigma_\omega.
\]
Observe that \(Q\) is fully symmetric in its arguments.

**Remark 2.3.** The positive measure \(\Sigma\) defined in (4) is supported on the (singular) submanifold \(\Gamma\) of \((S^2)^4\) of (generic) dimension 5 given by
\[
\Gamma := \{ (\omega_1, \omega_2, \omega_3, \omega_4) \in (S^2)^4 : \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0 \}.
\]

One way to visualize and parametrize \(\Gamma\) is to choose freely the unit vectors \(\omega_1\) and \(\omega_2\), then \(\omega_3\) and \(\omega_4\) must be two diametrically opposite points on the circle obtained intersecting the unit sphere centered at 0 with the unit sphere centered at \(-\omega_1 - \omega_2\).

![Parametrization of the manifold Γ](image-url)
3. Symmetrization

It is evident that \( |Q(f_1, f_2, f_3, f_4)| \leq Q(|f_1|, |f_2|, |f_3|, |f_4|) \), with equality when the functions \( f_k \) are nonnegative. Hence, we can reduce to consider nonnegative functions only. We may say more.

**Definition 3.1.** Given a complex valued function \( f \) defined on \( S^2 \) we define its **nonnegative antipodally symmetric rearrangement** \( f_\sharp \) by

\[
 f_\sharp(\omega) := \sqrt{\frac{|f(\omega)|^2 + |f(-\omega)|^2}{2}}, \quad \omega \in S^2.
\]

The function \( f_\sharp \) is also uniquely determined by the conditions

\[
 f_\sharp(\omega) = f_\sharp(-\omega) \geq 0, \quad f_\sharp(\omega)^2 + f_\sharp(-\omega)^2 = |f(\omega)|^2 + |f(-\omega)|^2.
\]

Moreover, we have \( \|f_\sharp\|_{L^2(S^2)} = \|f\|_{L^2(S^2)} \).

**Proposition 3.2.** We always have the pointwise estimate

\[
 |f\sigma * f_\sharp\sigma(x)| \leq f_\sharp\sigma * f_\sharp\sigma(x), \quad \forall x \in \mathbb{R}^3.
\]

By (2) and (3) the proposition immediately implies:

**Corollary 3.3 ([1]).** We always have that

\[
 Q(f, f_\sharp, f_\sharp, f_\sharp) \leq Q(f_\sharp, f_\sharp, f_\sharp, f_\sharp) \quad \text{and} \quad \|\hat{f}\sigma\|_{L^4(\mathbb{R}^3)} \leq \|\hat{f_\sharp}\sigma\|_{L^4(\mathbb{R}^3)}.
\]

We also have equality when \( f \) is a nonnegative constant function, since in that case \( f = f_\sharp = f_\sharp \). Corollary 3.3 was proved in [1], our proof here is much shorter and simpler.

**Proof of Proposition 3.2.** We may assume that \( f \) is nonnegative. By the symmetry of the convolution,

\[
 2f\sigma * f_\sharp\sigma(x) = f\sigma * f_\sharp\sigma(x) + f_\sharp\sigma * f\sigma(x) = 
  \int_{(S^2)^2} \left( f(\omega)f(-\nu) + f(-\omega)f(\nu) \right) \delta(x - \omega - \nu) \, d\sigma_\omega \, d\sigma_\nu.
\]

Now we use Cauchy-Schwarz inequality in its simplest form:

\[
 AC + BD \leq \sqrt{A^2 + B^2} \sqrt{C^2 + D^2},
\]

applied with \( A = f(\omega), B = f(-\omega), C = f(-\nu), D = f(\nu) \). We obtain

\[
 f(\omega)f(-\nu) + f(-\omega)f(\nu) \leq 2f_\sharp(\omega)f_\sharp(\nu).
\]

We plug this into (7) and obtain (6). \(\square\)

**Remark 3.4.** When \( A, B, C, D \geq 0 \), we have equality in (8) if and only if \( AD = BC \). Suppose now that the equality \( Q(f, f_\sharp, f_\sharp, f_\sharp) = Q(f_\sharp, f_\sharp, f_\sharp, f_\sharp) \) holds for some nonnegative function \( f \). It follows from the proof of Proposition 3.2 that

\[
 f(\omega)f(\nu) = f(-\omega)f(-\nu),
\]

for almost every \( (\omega, \nu) \in (S^2)^2 \). If we integrate this identity with respect to \( \nu \in S^2 \) we obtain that \( f(\omega) = f(-\omega) \) for almost every \( \omega \in S^2 \), which means that \( f = f_\sharp \) is antipodally symmetric.

From now on, we may assume that \( f = f_\sharp \) is a nonnegative antipodally symmetric function.
4. Reduction to a quadratic form estimate

Our goal now is to bound $Q(f, f, f, f)$ in terms of the $L^2$ norm of $f$. We may try to use Cauchy-Schwartz inequality with respect to the measure $\Sigma$.

**Lemma 4.1.** Let $B(F, G)$ be the bilinear form given by

$$B(F, G) = \int_\Gamma F(\omega_1, \omega_2) G(\omega_3, \omega_4) \, d\Sigma_{\omega},$$

for functions $F$ and $G$ defined on $\mathbb{S}^2 \times \mathbb{S}^2$. Then

$$|B(F, G)|^2 \leq B(|F|^2, 1) B(|G|^2, 1),$$

with equality if and only if there exist two constants $\lambda, \mu$ and a measurable function $h(x)$ defined on $|x| \leq 2$ such that

$$F(\omega, \nu) = \lambda h(\omega + \nu), \quad G(\omega, \nu) = \mu h(-\omega - \nu), \quad \text{for almost every } \omega, \nu \in \mathbb{S}^2.$$

**Proof.** Apply Cauchy-Schwartz with respect to the measure $d\Sigma$. We have equality when $F(\cdot, \cdot) \otimes 1 \otimes 1$ and $1 \otimes 1 \otimes G(\cdot, \cdot)$ are linearly dependent on the support of $\Sigma$. If $F$ and $G$ are not identically zero, that happens when there are non zero constants $\lambda, \mu$ such that

$$\frac{F(\omega_1, \omega_2)}{\lambda} = \frac{G(\omega_3, \omega_4)}{\mu} = h(x),$$

for almost every $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Gamma$, with $x = \omega_1 + \omega_2 = -\omega_3 - \omega_4$. \hfill $\Box$

In our case $Q(f, f, f, g) = B(f \otimes f, g \otimes g)$. Lemma 4.1 and Lemma 2.2 imply that

$$Q(f, f, f, f) \leq Q(f^2, f^2, 1, 1) =$$

$$= \iint_{(\mathbb{S}^2)^2} f(\omega_1)^2 f(\omega_2)^2 \left( \iint_{(\mathbb{S}^2)^2} \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) \, d\sigma_{\omega_3} \, d\sigma_{\omega_4} \right) \, d\sigma_{\omega_1} \, d\sigma_{\omega_2} =$$

$$= \iint_{(\mathbb{S}^2)^2} f(\omega_1)^2 f(\omega_2)^2 \frac{2\pi}{|\omega_1 + \omega_2|} \, d\sigma_{\omega_1} \, d\sigma_{\omega_2},$$

but unfortunately the last integral is too singular for our purposes.

The next lemma contains the geometric information about the symmetries of the support of the measure $\Sigma$ which allows us to neutralize the singularity of the previous integral.

**Lemma 4.2.** Let $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{S}^2$ be such that $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$. Then

$$|\omega_1 + \omega_2| |\omega_3 + \omega_4| + |\omega_1 + \omega_3| |\omega_2 + \omega_4| + |\omega_1 + \omega_4| |\omega_2 + \omega_3| = 4.$$

**Proof.** Let $X := \omega_1 \cdot \omega_2 + \omega_1 \cdot \omega_3 + \omega_2 \cdot \omega_3$. We have $\omega_1 + \omega_2 + \omega_3 = -\omega_4 \in \mathbb{S}^2$. This implies that

$$1 = |\omega_4|^2 = |\omega_1 + \omega_2 + \omega_3|^2 = 3 + 2X.$$

Hence $X = -1$. Then

$$|\omega_1 + \omega_2|^2 + |\omega_1 + \omega_3|^2 + |\omega_2 + \omega_3|^2 = 6 + 2X = 4.$$

To conclude the proof it is enough to observe that $|\omega_j + \omega_k| = |\omega_m + \omega_n|$ whenever $(j, k, m, n)$ is any permutation of $(1, 2, 3, 4)$. \hfill $\Box$

We combine the result of lemma 4.2 with the symmetry properties of $Q$ and obtain

$$Q(f, f, f, f) = \frac{3}{4} \iint_{(\mathbb{S}^2)^2} f(\omega_1) f(\omega_2) |\omega_1 + \omega_2| f(\omega_3) f(\omega_4) |\omega_3 + \omega_4| \, d\Sigma_{\omega} = \frac{3}{4} B(F, F),$$

as desired.
where \( F(\omega, \nu) := f(\omega)f(\nu)|\omega + \nu| \). We apply the Cauchy-Schwarz inequality of Lemma 4.1, use again Lemma 2.2 and obtain
\[
(10) \quad B(F, F) \leq B(F^2, 1) = 2\pi \int_{(\mathbb{S}^2)^2} f(\omega_1)^2 f(\omega_2)^2 |\omega_1 + \omega_2| \, d\sigma_{\omega_1} d\sigma_{\omega_2}.
\]

**Remark 4.3.** We have equality in (10) if and only if \( f(\omega)f(\nu) = h(\omega + \nu) \) for almost every \((\omega, \nu) \in (\mathbb{S}^2)^2\) and for some measurable function \( h(x) \) defined on \(|x| \leq 2\); this happens for example when \( f \) is a constant function.

At this point, since \(|\omega_1 + \omega_2| \leq 2\), we can immediately deduce the estimate
\[
(11) \quad B(F^2, 1) \leq 4\pi \|f\|^4_{L^2}.
\]
and hence prove the inequality (1), but the constant is not the optimal one and we will have strict inequality also for \( f \) constant.

5. Spectral decomposition of the quadratic form

We consider now the quadratic functional
\[
(12) \quad H(g) := \int_{(\mathbb{S}^2)^2} g(\omega)g(\nu) |\omega - \nu| \, d\sigma_\omega d\sigma_\nu,
\]
which is well defined, real valued and continuous on \( L^1(\mathbb{S}^2) \). It is easy to verify that
\[
|H(g_1) - H(g_2)| \leq 2 \left( \|g_1\|_{L^1(\mathbb{S}^2)} + \|g_2\|_{L^1(\mathbb{S}^2)} \right) \|g_1 - g_2\|_{L^1(\mathbb{S}^2)}.
\]

We want to show that the value of \( H(g) \) does not decrease when we replace \( g \) with a constant function with the same mean value.

**Theorem 5.1.** Let \( g \in L^1(\mathbb{S}^2) \). Let \( \mu = \frac{1}{4\pi} \int_{\mathbb{S}^2} g(\omega) \, d\sigma_\omega \) be the mean value of \( g \) on the sphere. Then \( H(g) \leq H(\mu 1) = |\mu|^2 H(1) \). Moreover, equality holds if and only if \( g \) is constant.

By the continuity of \( H \) on \( L^1(\mathbb{S}^2) \), it is enough to prove the theorem for functions in a dense subset of \( L^1(\mathbb{S}^2) \), for example in the Hilbert space \( L^2(\mathbb{S}^2) \). When \( g \in L^2(\mathbb{S}^2) \), we consider the decomposition of \( g \) as a sum of its spherical harmonics components. A spherical harmonic \( Y_k \) of degree \( k \) is an eigenfunction of \( \Delta_{\mathbb{S}^2} \) corresponding to the eigenvalue \(-k(k+1)\),
\[
\Delta_{\mathbb{S}^2} Y_k = -k(k+1)Y_k,
\]
where \( \Delta_{\mathbb{S}^2} \) stands for the Laplace-Beltrami operator on the sphere acting on scalar functions. Any function in \( L^2(\mathbb{S}^2) \) can be expanded as a sum of orthogonal spherical harmonics (see for example [6, chapter IV]).

Spherical harmonics are related to Legendre polynomials. The latter can be defined in terms of a generating function: when \(|r| < 1\) and \(|t| \leq 1\), if we write the power series expansion
\[
(1 - 2rt + r^2)^{-\frac{1}{2}} = \sum_{k \geq 0} P_k(t)r^k,
\]
then, for any integer \( k \geq 0 \), the coefficient \( P_k(t) \) is the Legendre polynomial of degree \( k \). These polynomials form a complete orthogonal system in \( L^2([-1, 1]) \) and we have
\[
\int_{-1}^{1} P_k(t)^2 \, dt = \frac{2}{2k+1}.
\]
We are going to need the following facts about spherical harmonics and Legendre polynomials.
Lemma 5.2 (Funk-Hecke formula). Let \( \phi \) be a continuous function on \([-1, 1]\) and \( Y_k \) be a spherical harmonics of degree \( k \). Then for any \( \omega \in L^2(S^2) \) we have
\[
\int_{S^2} \phi(\omega \cdot \nu) Y_k(\nu) \, d\sigma_{\nu} = 2\pi \lambda_k Y_k(\omega),
\]
where
\[
\lambda_k = \int_{-1}^1 \phi(t) P_k(t) \, dt,
\]
and \( P_k \) is the Legendre polynomial of degree \( k \).

A proof of Lemma 5.2 and its generalization to higher dimensions can be found in [3, p. 247].

Lemma 5.3. For any integer \( k \geq 1 \) we have
\[
(2k + 1)P_k(t) = \frac{d}{dt} \left( (k + 1)P_{k+1}(t) - kP_{k-1}(t) \right).
\]

Proof. Differentiate (13) with respect to \( r \),
\[
(t - r) \left( 1 - 2rt + r^2 \right)^{-\frac{3}{2}} = \sum_{k \geq 0} kP_k(t) r^{k-1}.
\]

Multiply on both sides by \( 1 - 2rt + r^2 \),
\[
(t - r) \sum_{k \geq 0} P_k(t) r^k = (1 - 2rt + r^2) \sum_{k \geq 0} kP_k(t) r^{k-1}.
\]

From this identity, equate the coefficients which multiply the same power \( r^k \), for any \( k \geq 1 \), and obtain Bonnet’s recursion formula
\[
(2k + 1)tP_k(t) = (k + 1)P_{k+1}(t) + kP_{k-1}(t).
\]

Differentiate with respect to \( t \),
\[
(2k + 1)P_k(t) = (k + 1)P'_{k+1}(t) - (2k + 1)tP'_{k}(t) + kP'_{k-1}(t).
\]

Now, differentiate (13) with respect to \( t \),
\[
\left( 1 - 2rt - r^2 \right)^{-\frac{3}{2}} = \sum_{k \geq 1} P'_k(t) r^{k-1}.
\]

Again, multiply on both sides by \( 1 - 2rt + r^2 \), and obtain
\[
\sum_{k \geq 0} P_k(t) r^k = (1 - 2rt + r^2) \sum_{k \geq 1} P'_k(t) r^{k-1}.
\]

From this identity, equate the coefficients which multiply the same power \( r^k \), for any \( k \geq 1 \), and obtain another recurrence formula,
\[
P_k(t) = P'_{k+1}(t) - 2tP'_k(t) + P'_{k-1}(t).
\]

To end the proof, multiply (16) by 2 and subtract (17) multiplied by \( 2k + 1 \) to get (15).

We also need to know the sign of the coefficients (14) for the function \( \phi(t) = \sqrt{2 - 2t} \).

Lemma 5.4. The integrals \( \Lambda_k := \int_{-1}^1 \sqrt{2 - 2t} P_k(t) \, dt \) are negative numbers for all \( k \geq 1 \).
**Proof.** Let \( k \geq 1 \). We use Lemma 5.3 and integration by parts,
\[
(2k + 1) \Lambda_k = \int_{-1}^{1} \sqrt{2 - 2t} \left( P'_{k+1}(t) - P'_{k-1}(t) \right) dt = A_{k+1} - A_{k-1},
\]
where
\[
A_k := \int_{-1}^{1} \frac{P_k(t)}{\sqrt{2 - 2t}} dt = \lim_{r \to 1} \int_{-1}^{1} \frac{P_k(t)}{\sqrt{1 - 2rt + t^2}} dt.
\]
The convergence of the limit follows from Lebesgue’s dominated convergence theorem, since we can use the inequality \( 1 - 2rt + t^2 \geq 2r(1 - t) \) to bound the denominator. From the generating function identity (13) and the orthogonality properties of Legendre polynomials we deduce that
\[
A_k = \lim_{r \to 1} t^k \int_{-1}^{1} P_k(t)^2 dt = \frac{2}{2k + 1}.
\]
This shows that the coefficients \( A_k \) form a decreasing sequence, and by (18) it follows that \( \Lambda_k \) is negative for any \( k \geq 1 \).

**Proof of Theorem 5.1.** When \( g \) is a function in \( L^2(\mathbb{S}^2) \) we decompose it into the sum \( g = \sum_{k \geq 0} Y_k \), where \( Y_k \) is a spherical harmonic of degree \( k \). In particular, the spherical harmonic component of \( f \) of degree 0 is given by the constant function \( \mu 1 \), where \( \mu \) is the mean value of \( f \) on \( \mathbb{S}^2 \). We have
\[
H(g) = \sum_{j,k \geq 0} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} Y_j(\omega) Y_k(\nu) |\omega - \nu| d\sigma_\nu d\sigma_\omega.
\]
By the Funk-Hecke formula of Lemma 5.2 we have that
\[
\int_{\mathbb{S}^2} |\omega - \nu| Y_k(\nu) d\sigma_\nu = \int_{\mathbb{S}^2} \sqrt{2(1 - \omega \cdot \nu)} Y_k(\nu) d\sigma_\nu = 2\pi \Lambda_k Y_k(\omega),
\]
where \( \Lambda_k \) are the coefficients computed in Lemma 5.4. By the orthogonality properties of spherical harmonics we deduce that
\[
H(g) = 2\pi \sum_{k \geq 0} A_k \|Y_k\|^2_{L^2(\mathbb{S}^2)} \leq 2\pi \Lambda_0 \|Y_0\|^2_{L^2(\mathbb{S}^2)} = H(\mu 1),
\]
since we know by Lemma 5.4 that \( \Lambda_k < 0 \) when \( k \geq 1 \). Here we have equality if and only if \( Y_k \equiv 0 \) for all \( k \geq 1 \), which means that \( f = Y_0 \) is a constant function.

The case for a generic \( g \in L^1(\mathbb{S}^2) \) follows by a density argument and by the continuity of \( H \) on \( L^1(\mathbb{S}^2) \).

6. **Constants are (the only real valued) maximizers**

We are now ready to put together all the steps we need in order to prove estimate (1) with its best constant. From (2), (3) and Corollary 3.3 we have
\[
\|f \sigma \|^4_{L^4(\mathbb{R}^3)} = (2\pi)^4 \|f \sigma * f \sigma\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3 Q(f, f, f, f),
\]
where \( Q \) was defined in (5). By Remark 3.4, when \( f \) is a nonnegative function we have equality here if and only if \( f = f_2 \) is antipodally symmetric.

From (9), (10) and the symmetry of \( f_2 \), we get
\[
(2\pi)^3 Q(f_2, f_2, f_2, f_2) \leq \frac{3}{4} (2\pi)^4 \int_{\mathbb{S}^2} f_2(\omega)^2 f_2(\nu)^2 |\omega + \nu| d\sigma_\omega d\sigma_\nu =
\]
\[
= 12\pi^4 \int_{\mathbb{S}^2} f_2(\omega)^2 f_2(\nu)^2 |\omega - \nu| d\sigma_\omega d\sigma_\nu = 12\pi^4 H(f_2^2),
\]
where \( H \) was defined in (12). As observed in Remark 4.3, we have equality here when \( f \) is constant.
The mean value of $f_\sharp^2$ on $S^2$ is

$$\mu := \frac{1}{4\pi} \int_{S^2} f_\sharp^2(\omega) \, d\sigma_\omega = \frac{1}{4\pi} \|f\|^2_{L^2(S^2)}.$$  

By Theorem 5.1 we have that

$$12\pi^4 H(f_\sharp^2) \leq 12\pi^4 \mu^2 H(1) = \frac{3}{4} \pi^2 H(1) \|f\|_{L^2(S^2)}^4.$$  

Here equality holds if and only if $f_\sharp$ is constant. The value of $H(1)$ is simple to compute,

$$H(1) = \int_{(S^2)^2} |\omega - \nu| \, d\sigma_\nu \, d\sigma_\omega = \int_{(S^2)^2} \sqrt{2(1 - \omega \cdot \nu)} \, d\sigma_\nu \, d\sigma_\omega =$$

$$= 4\pi \cdot 2\pi \cdot \sqrt{2} \int_{-1}^1 \sqrt{1 - t} \, dt = \frac{32}{3} \pi^2.$$  

The chain of inequalities collected in this section gives us $\|\hat{\sigma}\|_{L^4(R^3)}^4 \leq 8\pi^4 \|f\|_{L^2(S^2)}^4$, with equality if and only if $f = f_\sharp$ is constant. This proves Theorem 1.1.

ACKNOWLEDGMENTS

The author is grateful to Nicola Visciglia for suggesting to look at [1] and work on this problem, and for his helpful comments on the first draft, and to Rupert Frank for a remark which allowed to considerably simplify the proof of Theorem 5.1.

REFERENCES

[1] Michael Christ and Shuanglin Shao. Existence of extremals for a Fourier restriction inequality. *Anal. PDE*, 5(2):261–312, 2012.
[2] Michael Christ and Shuanglin Shao. On the extremizers of an adjoint Fourier restriction inequality. *Adv. Math.*, 230(3):957–977, 2012.
[3] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. II*. Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original.
[4] Damiano Foschi. Maximizers for the Strichartz inequality. *J. Eur. Math. Soc. (JEMS)*, 9(4):739–774, 2007.
[5] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
[6] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI FERRARA, ITALY

E-mail address: damiano.foschi@unife.it