CONDENSERS WITH TOUCHING PLATES AND CONSTRAINED MINIMUM RIESZ AND GREEN ENERGY PROBLEMS

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Abstract. We study minimum energy problems relative to the \( \alpha \)-Riesz kernel \( |x-y|^{\alpha-n} \), \( \alpha \in (0,2] \), over signed Radon measures \( \mu \) on \( \mathbb{R}^n \), \( n \geq 3 \), associated with a generalized condenser \( (A_1, A_2) \), where \( A_1 \) is a relatively closed subset of a domain \( D \) and \( A_2 = \mathbb{R}^n \setminus D \). We show that, though \( A_2 \cap \text{Cl}_{\mathbb{R}^n} A_1 \) may have nonzero capacity, this minimum energy problem is uniquely solvable (even in the presence of an external field) if we restrict ourselves to \( \mu \) with \( \mu^+ \leq \xi \), where a constraint \( \xi \) is properly chosen. We establish the sharpness of the sufficient conditions on the solvability thus obtained, provide descriptions of the weighted \( \alpha \)-Riesz potentials of the solutions, single out their characteristic properties, and analyze their supports. The approach developed is mainly based on the establishment of an intimate relationship between the constrained minimum \( \alpha \)-Riesz energy problem over signed measures associated with \( (A_1, A_2) \) and the constrained minimum \( \alpha \)-Green energy problem over positive measures carried by \( A_1 \). The results are illustrated by examples.

1. INTRODUCTION

The purpose of this paper is to study minimum energy problems with external fields (also known in the literature as weighted minimum energy problems) relative to the \( \alpha \)-Riesz kernel \( \kappa_\alpha(x,y) := |x-y|^{\alpha-n} \) of order \( \alpha \in (0,2] \) on \( \mathbb{R}^n \), \( n \geq 3 \), where \( |x-y| \) is the Euclidean distance between \( x,y \in \mathbb{R}^n \) and infimum is taken over classes of (signed) Radon measures \( \mu \) on \( \mathbb{R}^n \) associated with a generalized condenser \( A = (A_1, A_2) \). More precisely, an ordered pair \( A = (A_1, A_2) \) is termed a generalized condenser in \( \mathbb{R}^n \) if \( A_1 \) is a relatively closed subset of a given (connected open) domain \( D \subset \mathbb{R}^n \) and \( A_2 = D^c := \mathbb{R}^n \setminus D \), while \( \mu \) is said to be associated with \( A \) if the positive and negative parts in the Hahn–Jordan decomposition of \( \mu \) are carried by \( A_1 \) and \( A_2 \), respectively.

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Note that, although $A_1 \cap A_2 = \emptyset$, the set $A_2 \cap \text{Cl}_{\mathbb{R}^n} A_1$ may have nonzero (in particular infinite, see Example 9.2 below) $\alpha$-Riesz capacity and may even coincide with the boundary of $D$ relative to $\mathbb{R}^n$. Therefore the classical condenser problem for the generalized condenser $A$, which amounts to the minimum $\alpha$-Riesz energy problem over the class of all $\mu$ associated with $A$ and normalized by $\mu^+(A_1) = \mu^-(A_2) = 1$, can easily be shown to have no solution, see Theorem 4.3. Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x - y|^{-1}$ on $\mathbb{R}^3$, in the case where a minimum energy problem has no solution we say that a short-circuit occurs between the oppositely charged plates of the generalized condenser $A$. It is therefore meaningful to ask what kinds of additional requirements on the objects in question will prevent this blow-up effect, and secure that a solution to the corresponding minimum $\alpha$-Riesz energy problem does exist.

We show that a solution $\lambda_A^\xi$ to the minimum $\alpha$-Riesz energy problem exists (no short-circuit occurs) if we restrict ourselves to $\mu$ with $\mu^+ \leqslant \xi$, where the constraint $\xi$ is properly chosen. More precisely, if $A_2 = D^c$ is not $\alpha$-thin at infinity, then such $\lambda_A^\xi$ exists (even in the presence of an external field) provided that $\xi$ is a positive Radon measure carried by $A_1$ with finite $\alpha$-Riesz energy $E_{\kappa_\alpha}(\xi) := \int \kappa_\alpha(x, y) d(\xi \otimes \xi)(x, y) < \infty$ and with total mass $\xi(A_1) \in (1, \infty)$; see Theorem 6.1. In particular, if the domain $D$ is bounded, then a solution $\lambda_A^\xi$ exists whenever $A = (D, D^c)$, $m_n(D) > 1$ and $\xi = m_n|_D$, where $m_n$ is the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. Theorem 6.1 is sharp in the sense that it no longer holds if the requirement $\xi(A_1) < \infty$ is omitted from its hypotheses, see Theorem 6.2.

We provide descriptions of the weighted $\alpha$-Riesz potentials of the solutions $\lambda_A^\xi$, single out their characteristic properties, and analyze their supports, see Theorems 6.3, 6.4 and 6.5. The results are illustrated by Examples 9.1 and 9.2. The theory of weighted minimum $\alpha$-Riesz energy problems with a (positive) constraint $\xi$ acting only on positive parts of measures associated with $A$, thus developed, remains valid in its full generality for the signed constraint $\xi - \xi^{D^c}$ acting simultaneously on the positive and negative parts of the measures in question, see Section 6.2. (Here $\xi^{D^c}$ is the $\alpha$-Riesz balayage of $\xi$ onto $D^c$.)

The approach developed is mainly based on the establishment of an intimate relationship between, on the one hand, the constrained weighted minimum $\alpha$-Riesz energy problem over (signed) measures associated with $A$ and, on the other hand, the constrained weighted minimum $\alpha$-Green energy problem over positive measures carried by $A_1$ (Theorem 5.2). The proof of Theorem 5.2 uses substantially the required finiteness of $E_{\kappa_\alpha}(\xi)$. Regrettably, a similar assertion in [10], Lemma 4.2, did not require that $E_{\kappa_\alpha}(\xi) < \infty$, being based on a false statement, Lemma 2.4, that the finiteness of the $\alpha$-Green energy $E_g(\mu)$ of a bounded measure $\mu$ on $D$ implies the finiteness of its $\alpha$-Riesz energy (see Example 10.1 below for a counterexample). This caused the incorrectness of some of the formulations.

\footnote{See Section 3 for the notion of $\alpha$-thinness at infinity. The uniqueness of a solution $\lambda_A^\xi$ can be established by standard methods based on the convexity of the class of admissible measures and the pre-Hilbert structure on the linear space of all (signed) Radon measures on $\mathbb{R}^n$ with $E_{\kappa_\alpha}(\mu) < \infty$, see Lemma 4.6.}
and proofs presented in [10]. The present paper rectifies the results on the constrained weighted \( \alpha \)-Riesz and \( \alpha \)-Green energy problems announced in [10].

Regarding the constrained weighted minimum \( \alpha \)-Green energy problem over positive measures carried by \( A_1 \), crucial to the arguments applied in the investigation thereof is the perfectness of the \( \alpha \)-Green kernel \( g \) on a domain \( D \), established recently by the second and the fifth named authors [16], which amounts to the completeness of the cone of all positive Radon measures \( \nu \) on \( D \) with finite \( \alpha \)-Green energy \( E_g(\nu) \) in the topology determined by the energy norm \( \| \nu \|_g := \sqrt{E_g(\nu)} \).

2. Preliminaries

Let \( X \) be a locally compact (Hausdorff) space [4, Chapter I, Section 9, n° 7], to be specified below, and \( \mathcal{M}(X) \) the linear space of all real-valued (signed) Radon measures \( \mu \) on \( X \), equipped with the vague topology, i.e. the topology of pointwise convergence on the class \( C_0(X) \) of all continuous functions on \( X \) with compact support. We refer to [5, 12] for the theory of measures and integration on a locally compact space, to be used throughout the paper; see also [13] for a short survey.

For the purposes of the present study it is enough to assume that \( X \) is metrizable and countable at infinity, where the latter means that \( X \) can be represented as a countable union of compact sets [4, Chapter I, Section 9, n° 9]. Then the vague topology on \( \mathcal{M}(X) \) satisfies the first axiom of countability [11, Remark 2.5], and the vague convergence is entirely determined by convergence of sequences. The vague topology on \( \mathcal{M}(X) \) is Hausdorff, and hence a vague limit of any sequence in \( \mathcal{M}(X) \) is unique (provided that it exists).

Let \( \mu^+ \) and \( \mu^- \) denote the positive and negative parts of a measure \( \mu \in \mathcal{M}(X) \) in the Hahn–Jordan decomposition, \( |\mu| := \mu^+ + \mu^- \) its total variation, and \( S(\mu) = S_X^\mu \) its support. A measure \( \mu \) is said to be bounded if \( |\mu|(X) < \infty \). Given \( \mu \in \mathcal{M}(X) \) and a \( |\mu| \)-measurable function \( u : X \to [-\infty, \infty] \), we shall for brevity write \( \langle u, \mu \rangle := \int u d\mu \).

Let \( \mathcal{M}^+(X) \) stand for the (convex, vaguely closed) cone of all positive \( \mu \in \mathcal{M}(X) \), and let \( \Psi(X) \) consist of all lower semicontinuous (l.s.c.) functions \( \psi : X \to (-\infty, \infty] \), nonnegative unless \( X \) is compact. The following fact is well known, see e.g. [13, Section 1.1].

**Lemma 2.1.** For any \( \psi \in \Psi(X) \) the mapping \( \mu \mapsto \langle \psi, \mu \rangle \) is vaguely l.s.c. on \( \mathcal{M}^+(X) \).

We define a kernel \( \kappa \) on \( X \) as a symmetric positive function from \( \Psi(X \times X) \). Given \( \mu, \nu \in \mathcal{M}(X) \), let \( E_\kappa(\mu, \nu) \) and \( U_\kappa^{\mu} \) denote the mutual energy and the potential relative to

\(^2\)When speaking of a continuous numerical function we understand that the values are finite real numbers.

\(^3\)When introducing notation about numerical quantities we always assume the corresponding object on the right to be well defined (as a finite real number or \( \pm \infty \)).
is universally measurable \[4, \text{Chapter I, Section 3, Proposition 5}\], and hence \[4, \text{Chapter I, Section 9, Proposition 13}\] the locally closed set consists of all the restrictions that \(V\) \(\text{Definition 2}\), this means that for every \(x \in X\) provided that \(U^\mu_\kappa(x)\) or \(U^\mu_-\kappa(x)\) is finite, and then \(U^\mu_\kappa(x) = U^{\mu^+}_\kappa(x) - U^{\mu^-}_\kappa(x)\). In particular, if \(\mu \geq 0\), then \(U^\mu_\kappa\) is defined everywhere on \(X\) and represents a positive l.s.c. function, see Lemma 2.1.

Also note that \(E_\kappa(\mu, \nu), \mu, \nu \in \mathcal{M}(X)\), is well defined and equal to \(E_\kappa(\nu, \mu)\) provided that \(E_\kappa(\mu^+, \nu^-) + E_\kappa(\mu^-, \nu^+)\) or \(E_\kappa(\mu^+, \nu^-) + E_\kappa(\mu^-, \nu^+)\) is finite. For \(\mu = \nu\) the mutual energy \(E_\kappa(\mu, \nu)\) becomes the energy \(E_\kappa(\mu) := E_\kappa(\mu, \mu)\) of \(\mu\). Let \(\mathcal{E}_\kappa(X)\) consist of all \(\mu \in \mathcal{M}(X)\) whose energy \(E_\kappa(\mu)\) is finite, which by definition means that the kernel \(\kappa\) is \((|\mu| \otimes |\mu|)\)-integrable, i.e. \(E_\kappa(|\mu|) < \infty\), and let \(\mathcal{E}_\kappa^+(X) := \mathcal{E}_\kappa(X) \cap \mathcal{M}^+(X)\).

If \(f : X \to [-\infty, \infty]\) is an external field, then the \(f\)-weighted potential \(W^\mu_{\kappa,f}\) and the \(f\)-weighted energy \(G_{\kappa,f}(\mu)\) of \(\mu \in \mathcal{E}_\kappa(X)\) are formally given by

\begin{align}
(2.1) & \quad W^\mu_{\kappa,f} := U^\mu_\kappa + f,
(2.2) & \quad G_{\kappa,f}(\mu) := E_\kappa(\mu) + 2\langle f, \mu \rangle = \langle W^\mu_{\kappa,f} + f, \mu \rangle.
\end{align}

Let \(\mathcal{E}_{\kappa,f}(X)\) consist of all \(\mu \in \mathcal{E}_\kappa(X)\) whose \(f\)-weighted energy \(G_{\kappa,f}(\mu)\) is finite, or equivalently such that \(f\) is \(|\mu|\)-integrable.

Given a set \(Q \subset X\), let \(\mathcal{M}^+(Q; X)\) consist of all \(\mu \in \mathcal{M}^+(X)\) carried by \(Q\), which means that \(X \setminus Q\) is locally \(\mu\)-negligible, or equivalently that \(Q\) is \(\mu\)-measurable and \(\mu = \mu|_Q\), where \(\mu|_Q = 1_Q \cdot \mu\) is the trace (restriction) of \(\mu\) on \(Q\) \[5, \text{Chapter V, Section 5, n^o 3, Example}\]. (Here \(1_Q\) denotes the indicator function of \(Q\).) If \(Q\) is closed, then \(\mu\) is carried by \(Q\) if and only if it is supported by \(Q\), i.e. \(S(\mu) \subset Q\). It follows from the countability of \(X\) at infinity that the concept of local \(\mu\)-negligibility coincides with that of \(\mu\)-negligibility; and hence \(\mu \in \mathcal{M}^+(Q; X)\) if and only if \(\mu^+(X \setminus Q) = 0\), \(\mu^+(\cdot)\) being the outer measure of a set. Denoting by \(\mu_s(\cdot)\) the inner measure of a set, for any \(\mu \in \mathcal{M}^+(Q; X)\) we thus get

\[\mu^+(Q) = \mu_s(Q) =: \mu(Q)\].

Write \(\mathcal{E}_\kappa^+(Q; X) := \mathcal{E}_\kappa(X) \cap \mathcal{M}^+(Q; X)\), \(\mathcal{M}^+(Q, q; X) := \{\mu \in \mathcal{M}^+(Q; X) : \mu(Q) = q\}\) and \(\mathcal{E}_\kappa^+(Q, q; X) := \mathcal{E}_\kappa(X) \cap \mathcal{M}^+(Q, q; X)\), where \(q \in (0, \infty)\).

Assume for a moment that \(Q\) is locally closed in \(X\). According to \[4, \text{Chapter I, Section 3, Definition 2}\], this means that for every \(x \in Q\) there is a neighborhood \(V\) of \(x\) in \(X\) such that \(V \cap Q\) is a closed subset of the subspace \(Q \subset X\). Being locally closed, the set \(Q\) is universally measurable \[4, \text{Chapter I, Section 3, Proposition 5}\], and hence \(\mathcal{M}^+(Q; X)\) consists of all the restrictions \(\mu|_Q\), \(\mu\) ranging over \(\mathcal{M}^+(X)\). On the other hand, according to \[4, \text{Chapter I, Section 9, Proposition 13}\] the locally closed set \(Q\) itself can be thought of as
a locally compact subspace of \( X \). Thus \( \mathfrak{M}^+(Q; X) \) consists, in fact, of all those \( \nu \in \mathfrak{M}^+(Q) \) for each of which there exists \( \nu \in \mathfrak{M}^+(X) \) with the property

\[
\nu(x) = \int_X \kappa(x, y) \nu(dy) \quad \text{for all } x \in X.
\]

We say that such \( \nu \) extends \( \nu \in \mathfrak{M}^+(Q) \) by 0 off \( Q \) to all of \( X \). A sufficient condition for this to happen is that \( \nu \) be bounded.

In all that follows a kernel \( \kappa \) is assumed to be strictly positive definite, which means that the energy \( E_\kappa(\mu), \mu \in \mathfrak{M}(X) \), is nonnegative whenever defined and it equals 0 only for \( \mu = 0 \). Then \( \mathcal{E}_\kappa(X) \) forms a pre-Hilbert space with the inner product \( E_\kappa(\mu, \mu_1) \) and the energy norm \( \| \mu \|_\kappa := \sqrt{E_\kappa(\mu)} \), see [13]. The (Hausdorff) topology on \( \mathcal{E}_\kappa(X) \) determined by the norm \( \| \cdot \|_\kappa \) is termed strong.

In contrast to [14, 15] where a capacity has been treated as a functional acting on positive numerical functions on \( X \), in the present study we use the (standard) concept of capacity as a set function. Thus the (inner) capacity of a set \( Q \subset X \) relative to the kernel \( \kappa \), denoted \( c_\kappa(Q) \), is defined by

\[
c_\kappa(Q) := \inf_{\mu \in \mathcal{E}_\kappa^+(Q; 1; X)} E_\kappa(\mu)^{-1},
\]

see e.g. [13, 21]. Then \( 0 \leq c_\kappa(Q) \leq \infty \). (As usual, here and in the sequel the infimum over the empty set is taken to be +\( \infty \). We also put \( 1/(+\infty) = 0 \) and \( 1/0 = +\infty \.)

In consequence of the strict positive definiteness of the kernel \( \kappa \),

\[
c_\kappa(K) < \infty \quad \text{for every compact } K \subset X.
\]

Furthermore, by [13, p. 153],

\[
c_\kappa(Q) = \sup c_\kappa(K) \quad (K \subset Q, K \text{ compact}).
\]

An assertion \( \mathcal{U}(x) \) involving a variable point \( x \in X \) is said to hold \( c_\kappa \)-nearly everywhere (\( c_\kappa \)-n.e.) on \( Q \) if \( c_\kappa(N) = 0 \), where \( N \) consists of all \( x \in Q \) for which \( \mathcal{U}(x) \) fails. Throughout the paper we shall often use the fact that \( c_\kappa(N) = 0 \) if and only if \( \mu_+(N) = 0 \) for every \( \mu \in \mathcal{E}_\kappa^+(X) \), see [13, Lemma 2.3.1].

As in [19, p. 134], we call a (signed Radon) measure \( \mu \in \mathfrak{M}(X) \) \( c_\kappa \)-absolutely continuous if \( \mu(K) = 0 \) for every compact set \( K \subset X \) with \( c_\kappa(K) = 0 \). It follows from (2.6) that for such \( \mu \), \( |\mu|_\kappa(Q) = 0 \) for every \( Q \subset X \) with \( c_\kappa(Q) = 0 \). Hence, every \( \mu \in \mathcal{E}_\kappa(X) \) is \( c_\kappa \)-absolutely continuous; but not conversely, see [19, pp. 134–135].

**Definition 2.2.** Following [13], we call a (strictly positive definite) kernel \( \kappa \) perfect if every strong Cauchy sequence in \( \mathcal{E}_\kappa^+(X) \) converges strongly to any of its vague cluster points.\(^4\)

**Remark 2.3.** On \( X = \mathbb{R}^n \), \( n \geq 3 \), the \( \alpha \)-Riesz kernel \( \kappa_\alpha(x, y) = |x - y|^\alpha - n, \alpha \in (0, n) \), is strictly positive definite and moreover perfect [8, 9], and hence so is the Newtonian kernel \( \kappa_2(x, y) = |x - y|^{2-n} \) [7]. Recently it has been shown that if \( X \) is an open set \( D \) in \( \mathbb{R}^n \),

\(^4\)It follows from Theorem 2.1 that for a perfect kernel such a vague cluster point exists and is unique.
n \geq 3$, and $g_D^\alpha$, $\alpha \in (0,2]$, is the $\alpha$-Green kernel on $D$ [19, Chapter IV, Section 5], then $\kappa = g_D^\alpha$ likewise is strictly positive definite and moreover perfect [16, Theorems 4.9, 4.11].

**Theorem 2.4** (see [13]). If a kernel $\kappa$ on a locally compact space $X$ is perfect, then the cone $\mathcal{E}_\kappa^+(X)$ is strongly complete and the strong topology on $\mathcal{E}_\kappa^+(X)$ is finer than the (induced) vague topology on $\mathcal{E}_\kappa^+(X)$.

**Remark 2.5.** In contrast to Theorem 2.4, for a perfect kernel $\kappa$ the whole pre-Hilbert space $\mathcal{E}_\kappa(X)$ is in general strongly incomplete, and this is the case even for the $\alpha$-Riesz kernel of order $\alpha \in (1,n)$ on $\mathbb{R}^n$, $n \geq 3$ (see [7] or [19, Theorem 1.19]). When speaking of a completion of $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$, one needs to consider e.g. tempered distributions of finite Deny-Schwartz energy defined with the aid of the Fourier transform [8]. Recently it has also been shown that if we restrict ourselves to $\nu \in \mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ such that $S_{2\alpha}^\kappa \subset D$, $D$ being a bounded domain in $\mathbb{R}^n$, then the pre-Hilbert space of all those $\nu$ can be isometrically imbedded into its completion, the Sobolev space $\tilde{H}^{-\alpha/2}(D)$, see [18, Corollary 3.3].

**Remark 2.6.** The concept of perfect kernel is a powerful tool in minimum energy problems over classes of positive scalar Radon measures with finite energy. Indeed, if $Q \subset X$ is closed, $c_\kappa(Q) \in (0,\infty)$, and $\kappa$ is perfect, then the minimum energy problem (2.4) has a unique solution $\lambda_Q$, termed the (inner) $\kappa$-capacitary measure on $Q$ [13, Theorem 4.1]. Later the concept of perfectness has been shown to be efficient also in minimum energy problems over classes of vector measures of finite or infinite dimensions associated with a standard condenser, see [22, 25]. The approach developed in [22–25] used substantially the assumption of the boundedness of the kernel on the product of the oppositely charged plates of a condenser, which made it possible to extend Cartan’s proof [7] of the strong completeness of the cone $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ of all positive measures on $\mathbb{R}^n$ with finite Newtonian energy to an arbitrary perfect kernel $\kappa$ on a locally compact space $X$ and suitable classes of signed measures $\mu \in \mathcal{E}_\kappa(X)$; compare with Remark 2.5 above.

### 3. $\alpha$-Riesz balayage and $\alpha$-Green function

In all that follows fix $n \geq 3$, $\alpha \in (0,2]$ and a domain $D \subset \mathbb{R}^n$ with $c_{\kappa_\alpha}(D^c) > 0$, where $D^c := \mathbb{R}^n \setminus D$, and assume that either $\kappa = \kappa_\alpha$ is the $\alpha$-Riesz kernel on $X = \mathbb{R}^n$, or $\kappa = g_D^\alpha$ is the $\alpha$-Green kernel on $X = D$ [19, Chapter IV, Section 5] (or see below). We simply write $\alpha$ instead of $\kappa_\alpha$ if $\kappa_\alpha$ serves as an index, and we use the short form ‘n.e.’ instead of ‘$c_\alpha$-n.e.’ if this will not cause any misunderstanding.

Given $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, write $B(x,r) := \{y \in \mathbb{R}^n : |y - x| < r\}$, $S(x,r) := \{y \in \mathbb{R}^n : |y - x| = r\}$ and $\overline{B}(x,r) := B(x,r) \cup S(x,r)$. Throughout the paper $\partial Q$ denotes the boundary of a set $Q \subset \mathbb{R}^n$ in the topology of $\mathbb{R}^n$.

When speaking of a positive Radon measure $\mu$ on $\mathbb{R}^n$, we always tacitly assume that $U_\alpha^\mu$ is not identically infinite. This implies that

$$
\int_{|y| > 1} \frac{d\mu(y)}{|y|^{n-\alpha}} < \infty,
$$

(3.1)
see [19, Eq. (1.3.10)], and consequently that \( U^\mu_n \) is finite \((c_\alpha^-)\)n.e. on \( \mathbb{R}^n \) [19, Chapter III, Section 1]; these two implications can actually be reversed.

**Definition 3.1.** \( \nu \in \mathcal{M}(D) \) is called *extendible* if there exist \( \nu^+ \) and \( \nu^- \) extending \( \nu^+ \) and \( \nu^- \), respectively, by \( 0 \) off \( D \) to all of \( \mathbb{R}^n \), see [2.3], and if these \( \nu^+ \) and \( \nu^- \) satisfy (3.1). We identify such \( \nu \in \mathcal{M}(D) \) with its extension \( \tilde{\nu} := \nu^+ - \nu^- \), and we therefore write \( \tilde{\nu} = \nu \).

Every bounded measure \( \nu \in \mathcal{M}(D) \) is extendible. The converse holds if \( D \) is bounded, but not in general (e.g. not if \( D^c \) is compact). The set of all extendible measures \( \nu \in \mathcal{M}(D) \) consists of all the restrictions \( \mu|_D \) where \( \mu \) ranges over \( \mathcal{M}(\mathbb{R}^n) \).

The \( \alpha \)-Green kernel \( g = g^\alpha_D \) on \( D \) is defined by
\[
g^\alpha_D(x, y) = U^\varepsilon_D(x) - U^{\varepsilon_D}(x) \quad \text{for all } x, y \in D, \tag{3.2}
\]
where \( \varepsilon_y \) denotes the unit Dirac measure at a point \( y \) and \( \varepsilon^D_y \) its \( \alpha \)-Riesz balayage onto the (closed) set \( D^c \), determined uniquely in the frame of the classical approach by [16, Theorem 3.6]. See also the book by Bliedtner and Hansen [3] where balayage is studied in the setting of balayage spaces.

We shall simply write \( \mu' \) instead of \( \mu^{D^c} \) when speaking of the \( \alpha \)-Riesz balayage of \( \mu \in \mathcal{M}^+(D; \mathbb{R}^n) \) onto \( D^c \). According to [16, Corollaries 3.19, 3.20], for any \( \mu \in \mathcal{M}^+(D; \mathbb{R}^n) \) the balayage \( \mu' \) is \( c_\alpha \)-absolutely continuous, and it is determined uniquely by the relation
\[
\mu' = \int \varepsilon'_y d\mu(y), \tag{3.3}
\]
see [16, Theorem 3.17]. If moreover \( \mu \in \mathcal{E}^+_\alpha(D; \mathbb{R}^n) \), then the balayage \( \mu' \) is in fact the orthogonal projection of \( \mu \) onto the convex cone \( \mathcal{E}^+_\alpha(D^c; \mathbb{R}^n) \), i.e. \( \mu' \in \mathcal{E}^+_\alpha(D^c; \mathbb{R}^n) \) and
\[
||\mu - \theta||_\alpha > ||\mu - \mu'||_\alpha \quad \text{for all } \theta \in \mathcal{E}^+_\alpha(D^c; \mathbb{R}^n), \theta \neq \mu' \tag{3.4}
\]
(see [15, Theorem 4.12] or [16, Theorem 3.1]).

If now \( \nu \in \mathcal{M}(D) \) is an extendible (signed Radon) measure, then \( \nu' := \nu^{D^c} := (\nu^+)' - (\nu^-)' \) is said to be a balayage of \( \nu \) onto \( D^c \). It follows from [19, Chapter III, Section 1, n° 1, Remark] that the balayage \( \nu' \) is determined uniquely by (3.2) with \( \nu \) in place of \( \mu \) among the \( c_\alpha \)-absolutely continuous signed measures on \( \mathbb{R}^n \) supported by \( D^c \).

\footnote{In the literature the integral representation \( \nu' \) seems to have been more or less taken for granted, though it has been pointed out in [5, Chapter V, Section 3, n° 1] that it requires that the family \( \langle \varepsilon'_y \rangle_{y \in D} \) be \( \mu \)-adequate in the sense of [5, Chapter V, Section 3, Definition 1]; see also counterexamples (without \( \mu \)-adequacy) in Exercises 1 and 2 at the end of that section. A proof of this adequacy has therefore been given in [16, Lemma 3.16].}
The following definition goes back to Brelot [6, Theorem VII.13]. A closed set \( F \subset \mathbb{R}^n \) is said to be \( \alpha \)-thin at infinity if either \( F \) is compact, or the inverse of \( F \) relative to \( S(0,1) \) has \( x = 0 \) as an \( \alpha \)-irregular boundary point (cf. [19, Theorem 5.10]).

**Theorem 3.2** (see [16, Theorem 3.22]). The set \( D^c \) is not \( \alpha \)-thin at infinity if and only if for every bounded measure \( \mu \in \mathcal{M}^+(D) \) we have \( \mu'(\mathbb{R}^n) = \mu(\mathbb{R}^n) \).

As noted in Remark 2.3, the \( \alpha \)-Riesz kernel \( \kappa_{\alpha} \) on \( \mathbb{R}^n \) as well as the \( \alpha \)-Green kernel \( g_{D^c}^\alpha \) on \( D \) is strictly positive definite and moreover perfect. Furthermore, the kernel \( \kappa_{\alpha} \) (with \( \alpha \in (0,2) \)) satisfies the complete maximum principle in the form stated in [19, Theorems 1.27, 1.29]. Regarding a similar result for the kernel \( g \), the following assertion holds.

**Theorem 3.3** (see [16, Theorem 4.6]). Let \( \mu \in \mathcal{E}^+_g(D) \), let \( \nu \in \mathcal{M}^+(D) \) be extendible, and let \( v \) be a positive \( \alpha \)-superharmonic function on \( \mathbb{R}^n \) (see [19, Chapter I, Section 5, n° 20]). If moreover \( U_\mu^\nu \leq U_\nu^\nu + v \) \( \mu \)-a.e. on \( D \), then the same inequality holds on all of \( D \).

The following three lemmas establish relations between potentials and energies relative to the kernels \( \kappa_{\alpha} \) and \( g_{D^c}^\alpha \).

**Lemma 3.4.** For any extendible measure \( \mu \in \mathcal{M}(D) \) the \( \alpha \)-Green potential \( U_\mu^\nu \) is finite (\( c_{\alpha}^{-}\)n.e. on \( D \)) and given by

\[
U_\mu^\nu = U_\alpha^\mu - \mu' \quad \text{n.e. on } D.
\]

**Proof.** It is seen from Definition 3.1 that \( U_\alpha^\mu \) is finite n.e. on \( \mathbb{R}^n \), and hence so is \( U_\alpha^\mu \). Applying (3.3) to \( \mu^\pm \), we get by [5, Chapter V, Section 3, Theorem 1]

\[
U_\mu^\nu = \int [U_\alpha^\mu - U_\alpha^\nu] \, d\mu(y) = U_\alpha^\mu - U_\alpha^\nu
\]

n.e. on \( D \), as was to be proved. \( \Box \)

**Lemma 3.5.** If \( \mu \in \mathcal{M}(D) \) is extendible and its extension belongs to \( \mathcal{E}_\alpha(\mathbb{R}^n) \), then

\[
\begin{align*}
\mu &\in \mathcal{E}_g(D), \\
\mu - \mu' &\in \mathcal{E}_\alpha(\mathbb{R}^n), \\
\|\mu\|_g^2 &\leq \|\mu - \mu'\|_\alpha^2 = \|\mu\|_\alpha^2 - \|\mu'\|_\alpha^2.
\end{align*}
\]

**Proof.** In view of the definition of a signed measure of finite energy (see Section 2), we obtain (3.6) from the inequality\( ^6 \)

\[
\begin{align*}
\nu_\alpha^\mu(x,y) < \kappa_\alpha(x,y) \quad &\text{for all } x, y \in D,
\end{align*}
\]

\( ^6 \)In general, \( \nu^D(\mathbb{R}^n) \leq \nu(\mathbb{R}^n) \) for every \( \nu \in \mathcal{M}^+(\mathbb{R}^n) \) [16, Theorem 3.11].

\( ^7 \)If \( Q \) is a given subset of \( D \), then any assertion involving a variable point holds n.e. on \( Q \) if and only if it holds \( c_{\alpha}^{-}\)-n.e. on \( Q \), see [10, Lemma 2.6].

\( ^8 \)The strict inequality in (3.9) is caused by our convention that \( c_{\alpha}(D^c) > 0 \).
while (3.7) from [16, Corollary 3.7] or [16, Theorems 3.1, 3.6]. According to Lemma 3.4 and footnote 7, \( U^\mu_g \) is finite \( c_\nu \)-n.e. on \( D \) and given by (3.5), while by (3.6) the same holds \(|\mu|\)-a.e. on \( D \); see [13, Lemma 2.3.1]. Integrating (3.5) with respect to \( \mu^\pm \), we therefore obtain by subtraction

\[
\infty > E_g(\mu) = E_\alpha(\mu - \mu', \mu).
\]

As \( U^{\mu'-\mu}_\alpha = 0 \) n.e. on \( D^c \) by (3.2), while \( \mu' \) is \( c_\alpha \)-absolutely continuous, we also have

\[
E_\alpha(\mu - \mu', \mu') = 0,
\]

which results in the former equality in (3.8) when combined with (3.10). In view of (3.7), (3.11) takes the form \( \|\mu\|_\alpha = E_\alpha(\mu, \mu') \), and the former equality in (3.8) therefore implies the latter.

Lemma 3.6. Assume that \( \mu \in \mathcal{M}(D) \) has compact support \( S^\mu_D \). Then \( \mu \in \mathcal{E}_g(D) \) if and only if its extension belongs to \( \mathcal{E}_\alpha(\mathbb{R}^n) \).

Proof. According to Lemma 3.5, it is enough to establish the necessity part of the lemma. We may clearly assume that \( \mu \) is positive. Since \( U^{\mu'}_\alpha \) is continuous on \( D \), and hence bounded on the compact set \( S^\mu_D \), we have

\[
E_\alpha(\mu, \mu') < \infty.
\]

On the other hand, \( E_g(\mu) \) is finite by assumption, and hence likewise as in the preceding proof relation (3.10) holds. Combining (3.10) with (3.12) yields \( \mu \in \mathcal{E}_\alpha(\mathbb{R}^n) \).

Remark 3.7. The proof of Lemma 3.5 uses substantially the requirement \( \mu \in \mathcal{E}_\alpha(\mathbb{R}^n) \). Being founded on the weaker assumption \( \mu \in \mathcal{E}_g(D) \), a similar assertion in [10] there was incorrect, as will be shown by Example 10.1 below. The revision of [10] provided in present paper is based significantly on the current version of Lemma 3.5 as well as on the perfectness of the kernel \( g^{\alpha}_D \), discovered recently in [16].

4. Minimum \( \alpha \)-Riesz energy problems for generalized condensers

4.1. A generalized condenser. Under the (permanent) assumptions stated at the beginning of Section 3, fix a (not necessarily proper) subset \( A_1 \) of \( D \) which is relatively closed in \( D \). The pair \( A = (A_1, A_2) \), where \( A_2 := D^c \), is said to form a generalized condenser in \( \mathbb{R}^n \), and \( A_1 \) and \( A_2 \) are termed its positive and negative plates. To avoid triviality, we shall always require that \( c_\alpha(A_i) > 0 \), and hence

\[
c_\alpha(A_i) > 0 \quad \text{for} \quad i = 1, 2.
\]

The generalized condenser \( A = (A_1, A_2) \) is said to be standard if \( A_1 \) is closed in \( \mathbb{R}^n \).

9If the measure in question is positive, then Lemma 3.6 can be generalized to any bounded \( \mu \in \mathcal{M}(D) \) such that the Euclidean distance between \( S^\mu_D \) and \( \partial D \) is \( > 0 \), see [17, Lemma 3.4].

10The notion of generalized condenser thus defined differs from that introduced in our recent work [11]; cf. Remark 6.7 below.
Example 4.1. Let \( A_1 = B(0, r) = D, \ r \in (0, \infty) \). Then \( A = (A_1, A_2) \) is a generalized condenser in \( \mathbb{R}^n \), which certainly is not standard. See Example 9.1 for constraints under which the constrained minimum \( \alpha \)-Riesz energy problem (Problem 4.4) for such \( A \) admits a solution (has no short-circuit) despite the fact that \( A_2 \cap \text{Cl}_{\mathbb{R}^n} A_1 = S(0, r) \).

Unless explicitly stated otherwise, in all that follows \( A = (A_1, A_2) \) is assumed to be a generalized condenser in \( \mathbb{R}^n \). We emphasize that, though \( A_1 \cap A_2 = \emptyset \), the set \( A_2 \cap \text{Cl}_{\mathbb{R}^n} A_1 = S(0, r) \) may have nonzero \( \alpha \)-Riesz capacity and may even coincide with the whole \( \partial D \).

Let \( \mathfrak{M}(A; \mathbb{R}^n) \) consist of all (signed Radon) measures on \( \mathbb{R}^n \) whose positive and negative parts in the Hahn–Jordan decomposition are carried by \( A_1 \) and \( A_2 \), respectively, and let \( \mathcal{E}_\alpha(A; \mathbb{R}^n) := \mathfrak{M}(A; \mathbb{R}^n) \cap \mathcal{E}_\alpha(\mathbb{R}^n) \). For any vector \( a = (a_1, a_2) \) with \( a_1, a_2 > 0 \) write \( E_\alpha(A, a; \mathbb{R}^n) := \{ \mu \in \mathcal{E}_\alpha(A; \mathbb{R}^n) : \mu_+(A_1) = a_1, \mu_-(A_2) = a_2 \} \).

This class is nonempty, which is clear from (4.1) by [13, Lemma 2.3.1], and it therefore makes sense to consider the problem on the existence of \( \lambda A \in E_\alpha(A, a; \mathbb{R}^n) \) with

\[
\| \lambda A \|_\alpha^2 = w_\alpha(A, a) := \inf_{\mu \in \mathcal{E}_\alpha(A, a; \mathbb{R}^n)} \| \mu \|_\alpha^2.
\]

This problem will be referred to as the condenser problem. By the (strict) positive definiteness of the kernel \( \kappa_\alpha \),

\[
w_\alpha(A, a) \geq 0.
\]

Remark 4.2. Assume for a moment that \( A \) is a standard condenser in \( \mathbb{R}^n \). If moreover it possesses the separation property

\[
\inf_{(x,y) \in A_1 \times A_2} |x - y| > 0,
\]

then the assumption

\[
c_\alpha(A_i) < \infty \quad \text{for } i = 1, 2
\]

is sufficient for problem (4.2) to be (uniquely) solvable for every normalizing vector \( a \). See e.g. [24] where this result has actually been established even for infinite dimensional vector measures in the presence of a vector-valued external field and for an arbitrary perfect kernel on a locally compact space. However, if (4.4) fails to hold, then in general there exists a vector \( a' \) such that the corresponding extremal value \( w_\alpha(A, a') \) is not an actual minimum, see [24] [14]. Therefore it was interesting to give a description of the set of all vectors \( a \) for which the condenser problem nevertheless is solvable. Such a characterization has been established in [25]. On the other hand, if the separation condition (1.4) is omitted, then the approach developed in [23] [25] breaks down and (4.4) does not guarantee anymore the existence of a solution to problem (4.2). This has been illustrated by [11] Théorème 4.6 pertaining to the Newtonian kernel.

\[\text{In the case of the } \alpha \text{-Riesz kernels of order } 1 < \alpha \leq 2 \text{ on } \mathbb{R}^3 \text{ some of the (theoretical) results on the solvability or unsolvability of the condenser problem, mentioned in [24], have been illustrated in [13, 20] by means of numerical experiments.} \]
The following theorem shows that for a generalized condenser $A$ the condenser problem in general has no solution. Denote $1 := (1, 1)$.

**Theorem 4.3.** If $A_2$ is not $\alpha$-thin at infinity and $c_g(A_1) = \infty$, then

$$w_\alpha(A, 1) = [c_g(A_1)]^{-1} = 0.$$  

Hence, $w_\alpha(A, 1)$ cannot be an actual minimum because $0 \notin E_\alpha(A, 1; \mathbb{R}^n)$.

**Proof.** Consider an exhaustion of $A_1$ by an increasing sequence $\{K_j\}_{j \in \mathbb{N}}$ of compact sets. By (2.6),

$$c_g(K_j) \uparrow c_g(A_1) = \infty \quad \text{as } j \to \infty,$$

and there is therefore no loss of generality in assuming that every $c_g(K_j)$ is $> 0$. Furthermore, since the $\alpha$-Green kernel $g$ is strictly positive definite and moreover perfect (Remark 2.3), we see from (2.5) that $c_g(K_j) < \infty$ and hence, by Remark 2.6 there exists a (unique) $g$-capacitary measure $\lambda_j$ on $K_j$, i.e. $\lambda_j \in \mathcal{E}_g(K_j, 1; D)$ with

$$\|\lambda_j\|_g^2 = 1/c_g(K_j) < \infty.$$

According to Lemma 3.6, $E_\alpha(\lambda_j)$ is finite along with $E_g(\lambda_j)$ and hence, by Lemma 3.5

$$\|\lambda_j\|_g^2 = \|\lambda_j - \lambda_j'\|_\alpha^2.$$

As $A_2$ is not $\alpha$-thin at infinity, we see from Theorem 3.2 that $\lambda_j - \lambda_j' \in \mathcal{E}_\alpha(A, 1; \mathbb{R}^n)$, which together with the two preceding displays yields

$$1/c_g(K_j) = \|\lambda_j\|_g^2 = \|\lambda_j - \lambda_j'\|_\alpha^2 \geq w_\alpha(A, 1) \geq 0.$$

Letting here $j \to \infty$, we obtain the theorem from (4.5). \qed

Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x - y|^{-1}$ on $\mathbb{R}^3$, we say that under the hypotheses of Theorem 4.3 a short-circuit occurs between the oppositely charged plates of the generalized condenser $A$. It is therefore meaningful to ask what kinds of additional requirements on the objects in question will prevent this blow-up effect, and secure that a solution to the corresponding minimum $\alpha$-Riesz energy problem does exist. To this end we have succeeded in working out a substantive theory by imposing a suitable upper constraint on the measures under consideration, thereby rectifying the results on the constrained $\alpha$-Riesz energy problem announced in [10], cf. Remark 3.7 above.

### 4.2. A constrained $f$-weighted minimum $\alpha$-Riesz energy problem for a generalized condenser.

In the rest of the paper we shall always require that $A_2$ is not $\alpha$-thin at infinity and that $a = 1$. When speaking of an external field $f$, see Section 2 we shall tacitly assume that either of the following Case I or Case II holds:

1. $f \in \Psi(\mathbb{R}^n)$ and moreover

$$f = 0 \text{ n.e. on } A_2;$$

(4.6)
II. $f = U_{\alpha}^{\xi-\zeta'}$, where $\zeta$ is a signed extendible Radon measure on $D$ with $E_\alpha(\zeta) < \infty$.

Note that relation (4.6) holds also in Case II, see (3.2). Since a set with $c_\alpha(\cdot) = 0$ carries no measure with finite $\alpha$-Riesz energy [13, Lemma 2.3.1], we thus see that in either Case I or Case II no external field acts on the measures from $\mathcal{E}_\alpha^+(A_2; \mathbb{R}^n)$. The $f$-weighted $\alpha$-Riesz energy $G_{\alpha,f}(\mu)$, cf. (2.2), of $\mu \in \mathcal{E}_\alpha(A; \mathbb{R}^n)$ can therefore be defined as

$$G_{\alpha,f}(\mu) := \|\mu\|_{\alpha}^2 + 2\langle f, \mu \rangle = \|\mu\|_{\alpha}^2 + 2\langle f, \mu^+ \rangle.$$  

If Case II takes place, then for every $\mu \in \mathcal{E}_\alpha(A; \mathbb{R}^n)$ we moreover get

$$\infty > G_{\alpha,f}(\mu) = \|\mu\|_{\alpha}^2 + 2E_\alpha(\zeta - \zeta', \mu)$$

$$= \|\mu + \zeta - \zeta'\|_{\alpha}^2 - \|\zeta - \zeta'\|_{\alpha}^2 \geq -\|\zeta - \zeta'\|_{\alpha}^2 > -\infty.$$  

Thus in either Case I or Case II,

$$G_{\alpha,f}(\mu) \geq -M > -\infty \text{ for all } \mu \in \mathcal{E}_\alpha(A; \mathbb{R}^n).$$

Indeed, in Case I this is obvious by (4.7), while in Case II it follows from (4.8).

By a constraint for measures from $\mathcal{E}_\alpha^+(A_1, 1; \mathbb{R}^n)$ we mean any $\xi$ such that

$$\xi \in \mathcal{E}_\alpha^+(A_1; \mathbb{R}^n) \text{ and } \xi(A_1) > 1.$$  

Let $\mathcal{C}(A_1; \mathbb{R}^n)$ consist of all such constraints. Given $\xi \in \mathcal{C}(A_1; \mathbb{R}^n)$, write

$$\mathcal{E}_\alpha^\xi(A, 1; \mathbb{R}^n) := \{ \mu \in \mathcal{E}_\alpha(A, 1; \mathbb{R}^n) : \mu^+ \leq \xi \},$$

where $\mu^+ \leq \xi$ means that $\xi - \mu^+ \geq 0$. Note that we do not impose any constraint on the negative parts of measures $\mu \in \mathcal{E}_\alpha(A, 1; \mathbb{R}^n)$. If

$$\mathcal{E}_\alpha^\xi(A, 1; \mathbb{R}^n) = \mathcal{E}_\alpha^\xi(A, 1; \mathbb{R}^n) \cap \mathcal{E}_\alpha,f(\mathbb{R}^n) \neq \emptyset$$

(see Section 4 for the definition of the class $\mathcal{E}_\alpha,f(\mathbb{R}^n)$), or equivalently by [12]

$$G_{\alpha,f}(A, 1; \mathbb{R}^n) := \inf_{\mu \in \mathcal{E}_\alpha^\xi(A, 1; \mathbb{R}^n)} G_{\alpha,f}(\mu) < \infty,$$

then the following constrained $f$-weighted minimum $\alpha$-Riesz energy problem makes sense.

**Problem 4.4.** Does there exist $\lambda_\alpha^{\xi} \in \mathcal{E}_\alpha^\xi(A, 1; \mathbb{R}^n)$ with

$$G_{\alpha,f}(\lambda_\alpha^{\xi}) = G_{\alpha,f}(A, 1; \mathbb{R}^n)?$$

Conditions which guarantee (4.11) are provided by the following Lemma 4.5. Write

$$A_1^0 := \{ x \in A_1 : |f(x)| < \infty \}.$$  

**Lemma 4.5.** Relation (4.11) holds if either Case II takes place, or (in the presence of Case I) if

$$\xi(A_1^0) > 1.$$  

---

12If (4.11) is fulfilled, then $G_{\alpha,f}(A, 1; \mathbb{R}^n)$ is actually finite, see (3.2).
Proof. Assume first that (4.14) holds; then there exists by (4.13) a compact set \(K \subset A_0\) such that \(|f| \leqslant M < \infty\) on \(K\) and \(\xi(K) > 1\). Define \(\mu = \mu^+ - \mu^-\), where \(\mu^+ := \xi|_K / \xi(K)\) while \(\mu^-\) is any measure from \(E^+_\alpha(A_2, 1; \mathbb{R}^n)\) (such \(\mu^-\) exists because \(c_\alpha(A_2) > 0\)). Noting that \(\xi|_K \in E^+_\alpha(K; \mathbb{R}^n)\) by (4.10), we get \(\mu \in E^\xi_\alpha,f(A, 1; \mathbb{R}^n)\), or equivalently (4.11). To complete the proof of the lemma, it is left to note that (4.13) holds automatically whenever Case II takes place, since then \(U^\xi_{\alpha-\xi'}\) is finite n.e. on \(\mathbb{R}^n\), hence \(\xi\)-a.e. by (4.10). \(\square\)

Lemma 4.6. A solution \(\lambda^\xi_A\) to Problem 4.4 is unique (whenever it exists).

Proof. This can be established by standard methods based on the convexity of the class \(E^\xi_{\alpha,f}(A, 1; \mathbb{R}^n)\) and the pre-Hilbert structure on the space \(E_\alpha(\mathbb{R}^n)\). Indeed, if \(\lambda\) and \(\tilde{\lambda}\) are two solutions to Problem 4.4 then

\[
4G^\xi_{\alpha,f}(A, 1; \mathbb{R}^n) = 4G_{\alpha,f}\left(\frac{\lambda + \tilde{\lambda}}{2}\right) = \|\lambda + \tilde{\lambda}\|^2_\alpha + 4\langle f, \lambda + \tilde{\lambda} \rangle.
\]

On the other hand, applying the parallelogram identity in \(E_\alpha(\mathbb{R}^n)\) to \(\lambda\) and \(\tilde{\lambda}\) and then adding and subtracting \(4\langle f, \lambda + \tilde{\lambda} \rangle\) we get

\[
\|\lambda - \tilde{\lambda}\|^2_\alpha = -\|\lambda + \tilde{\lambda}\|^2_\alpha - 4\langle f, \lambda + \tilde{\lambda} \rangle + 2G_{\alpha,f}(\lambda) + 2G_{\alpha,f}(\tilde{\lambda}).
\]

When combined with the preceding relation, this yields

\[
0 \leqslant \|\lambda - \tilde{\lambda}\|^2_\alpha \leqslant -4G^\xi_{\alpha,f}(A, 1; \mathbb{R}^n) + 2G_{\alpha,f}(\lambda) + 2G_{\alpha,f}(\tilde{\lambda}) = 0.
\]

Since \(\|\cdot\|_\alpha\) is a norm, the lemma follows. \(\square\)

5. Relations between minimum \(\alpha\)-Riesz and \(\alpha\)-Green energy problems

We are keeping the (permanent) assumptions on \(A\), \(f\) and \(\xi\) stated in Sections 4.1 and 4.2. Since \(M^+(A_1; \mathbb{R}^n) \subset M^+(A_1; D)\), the constraint \(\xi\) can be thought of as an extendible measure from \(M^+(A_1; D)\) such that its extension has finite \(\alpha\)-Riesz energy (and total mass \(\xi(A_1) > 1\)). Define

\[
E_\xi^\xi(A_1, 1; D) := \{\mu \in E_\alpha^+(A_1, 1; D) : \mu \leqslant \xi\},
\]

and let \(E_\xi^\xi_f(A_1, 1; D)\) consist of all \(\mu \in E_\alpha(A_1, 1; D)\) such that

\[
G_{\alpha,f}(\mu) := G_{\alpha,f|_D}(\mu) = \|\mu\|^2_g + 2\langle f|_D, \mu \rangle
\]

is finite, cf. (2.2). We have used here the fact that \(\nu^*(D^c) = 0\) for every \(\nu \in M^+(D; \mathbb{R}^n)\), see Section 2. If the class \(E_\xi^\xi_f(A_1, 1; D)\) is nonempty, or equivalently if

\[
G_{\alpha,f}(\mu) = \inf_{\mu \in E_\xi^\xi_f(A_1, 1; D)} G_{\alpha,f}(\mu) < \infty,
\]

then the following constrained \(f\)-weighted minimum \(\alpha\)-Green energy problem makes sense.
Problem 5.1. Does there exist \( \lambda^\xi_{A_1} \in \mathcal{E}^\xi_{g,f}(A_1, 1; D) \) with

\[
G_{g,f}(\lambda^\xi_{A_1}) = G_{g,f}^\xi(A_1, 1; D) ?
\]

Based on the convexity of the class \( \mathcal{E}^\xi_{g,f}(A_1, 1; D) \) and the pre-Hilbert structure on the space \( \mathcal{E}_g(D) \), likewise as in the proof of Lemma 4.6 we see that a solution \( \lambda^\xi_{A_1} \) to Problem 5.1 is unique whenever it exists (see [10, Lemma 4.1]).

Theorem 5.2. Under the stated assumptions,

\[
G^\xi_{\alpha,f}(A, 1; \mathbb{R}^n) = G^\xi_{g,f}(A_1, 1; D),
\]

Assume moreover that either of the (equivalent) assumptions (4.11) or (5.2) is fulfilled. Then the solution to Problem 4.4 exists if and only if so does that to Problem 5.1 and in the affirmative case they are related to each other by the formula

\[
\lambda^\xi_A = \lambda^\xi_{A_1} - (\lambda^\xi_{A_1})'.
\]

Proof. We begin by establishing the inequality

\[
G^\xi_{g,f}(A_1, 1; D) \geq G^\xi_{\alpha,f}(A, 1; \mathbb{R}^n).
\]

Assuming \( G^\xi_{g,f}(A_1, 1; D) < \infty \), choose \( \nu \in \mathcal{E}^\xi_{g,f}(A_1, 1; D) \). Being bounded, this \( \nu \) is extendible. Furthermore, its extension has finite \( \alpha \)-Riesz energy, for so does the extension of the constraint \( \xi \) by [4,10]. Applying (3.8) and (4.7) we get

\[
G_{g,f}(\nu) = \|\nu - \nu'\|^2 + 2\langle f | D, \nu \rangle.
\]

As \( A_2 \) is not \( \alpha \)-thin at infinity, we see from Theorem 3.2 that \( \theta := \nu - \nu' \in \mathcal{E}^\xi_{\alpha}(A, 1; \mathbb{R}^n) \). Furthermore, by (4.7),

\[
\langle f, \theta \rangle = \langle f | D, \nu \rangle < \infty.
\]

Thus \( \theta \in \mathcal{E}^\xi_{\alpha,f}(A, 1; \mathbb{R}^n) \) and \( G_{\alpha,f}(\theta) = G_{g,f}(\nu) \), the latter relation being valid according to the two preceding displays. This yields

\[
G_{g,f}(\nu) = G_{\alpha,f}(\theta) \geq G^\xi_{\alpha,f}(A, 1; \mathbb{R}^n),
\]

which establishes (5.6) by letting here \( \nu \) range over \( \mathcal{E}^\xi_{g,f}(A_1, 1; D) \).

On the other hand, for any \( \mu \in \mathcal{E}^\xi_{\alpha,f}(A, 1; \mathbb{R}^n) \) we have \( \mu^+ \in \mathcal{E}^\xi_{\alpha}(\mathbb{R}^n) \) by the definition of a signed measure of finite energy, and hence \( \mu^+ \in \mathcal{E}^\xi_{g,f}(A_1, 1; D) \) by (3.6) and (4.7). Because of (5.4), (5.8) and (4.7),

\[
G_{\alpha,f}(\mu) = \|\mu\|^2 + 2\langle f, \mu^+ \rangle = \|\mu^+ - \mu^-\|^2 + 2\langle f, \mu^+ \rangle \\
\geq \|\mu^+ - (\mu^+)^'\|^2 + 2\langle f, \mu^+ \rangle = \|\mu^+\|^2_\alpha + 2\langle f, \mu^+ \rangle \\
= G_{g,f}(\mu^+) \geq G^\xi_{g,f}(A_1, 1; D).
\]

As \( \mu \in \mathcal{E}^\xi_{\alpha,f}(A, 1; \mathbb{R}^n) \) has been chosen arbitrarily, this together with (5.6) proves (5.4).
Let now $\lambda_{A_1}^\xi \in \mathcal{E}_{g,f}^\xi (A_1; 1; D)$ satisfy (5.3). In the same manner as in the first paragraph of the present proof we see that $\bar{\mu} := \lambda_{A_1}^\xi - (\lambda_{A_1}^\xi)' \in \mathcal{E}_{\alpha,f}^\xi (A; 1; \mathbb{R}^n)$. Substituting $\bar{\mu}$ in place of $\theta$ into (5.7) and then combining the relation thus obtained with (5.4), we see that in fact $G_{\alpha,f}(\bar{\mu}) = G_{\alpha,f}^\xi (A; 1; \mathbb{R}^n)$. Hence there exists the (unique) solution $\lambda_{A}^\xi := \bar{\mu}$ to Problem 4.4 and it is related to $\lambda_{A_1}^\xi$ by means of formula (5.5).

To complete the proof, assume next that $\lambda_{A}^\xi = \lambda^+ - \lambda^- \in \mathcal{E}_{\alpha,f}^\xi (A; 1; \mathbb{R}^n)$ satisfies (4.12). Similarly as in the second paragraph of the present proof, we have $\lambda^+ \in \mathcal{E}_{g,f}^\xi (A_1; 1; D)$. Furthermore, by (5.4) and (5.8), the latter with $\lambda_{A}^\xi$ in place of $\mu$,

$G_{g,f}^\xi (A_1, 1; D) = G_{\alpha,f}(\lambda_{A}^\xi) \geq \|\lambda^+ - (\lambda^+)'\|_\alpha^2 + 2\langle f, \lambda^+ \rangle = \|\lambda^+\|_\alpha^2 + 2\langle f, \lambda^+ \rangle = G_{g,f}(\lambda^+) \geq G_{g,f}^\xi (A_1, 1; D)$.

Hence, all the inequalities in the last display are, in fact, equalities. This shows that $\lambda_{A_1}^\xi := \lambda^+$ solves Problem 5.1 and also, on account of (5.4), that $\lambda^- = (\lambda^+)' = (\lambda_{A_1}^\xi)'$. □

When investigating Problem 5.1 we shall need the following assertion, see [10, Lemma 4.3].

**Lemma 5.3.** Assume that (5.2) holds. Then $\lambda \in \mathcal{E}_{g,f}^\xi (A_1, 1; D)$ solves Problem 5.1 if and only if

$\langle W_{g,f}^\lambda, \nu - \lambda \rangle \geq 0 \text{ for all } \nu \in \mathcal{E}_{g,f}^\xi (A_1, 1; D)$,

where it is denoted $W_{g,f}^\lambda := W_{g,f}^\lambda |_D := U_g^\lambda + f |_D$, cf. (2.1).

6. **Main results**

We keep all the (permanent) assumptions on $A$, $f$ and $\xi$ imposed in Sections 4.1 and 4.2.

6.1. **Formulations of the main results.** In the following Theorem 6.1 we require that relation (4.11) holds; see Lemma 4.3 providing sufficient conditions for this to occur.

**Theorem 6.1.** Suppose moreover that the constraint $\xi \in \mathcal{C}(A_1; \mathbb{R}^n)$ is bounded, i.e.

(6.1) $\xi(A_1) < \infty$.

Then in either Case I or Case II Problem 4.4 is (uniquely) solvable.

Theorem 6.1 is sharp in the sense that it does not remain valid if requirement (6.1) is omitted from its hypotheses (see the following Theorem 6.2).

**Theorem 6.2.** Condition (6.1) is actually necessary (and sufficient) for the solvability of Problem 4.4. More precisely, suppose that $c_\alpha(A_1) = \infty$ and that Case II holds with $\zeta \geq 0$. Then there exists a constraint $\xi \in \mathcal{C}(A_1; \mathbb{R}^n)$ with $\xi(A_1) = \infty$ such that $G_{\alpha,f}^\xi (A; 1; \mathbb{R}^n) = 0$, and hence $G_{\alpha,f}^\xi (A; 1; \mathbb{R}^n)$ cannot be an actual minimum.
The following three assertions establish descriptions of the $f$-weighted $\alpha$-Riesz potential $W^{\lambda \xi}_{\alpha,f}$, cf. (2.1), of the solution $\lambda ^{\xi}_{\dot{A}}$ to Problem 4.4 (whenever it exists) and single out its characteristic properties. An analysis of the support of $\lambda ^{\xi}_{\dot{A}}$ is also provided.

**Theorem 6.3.** Let assumption (4.14) hold and let $f$ be lower bounded on $A_1$. Fix an arbitrary $\lambda \in \mathcal{E}_{\alpha,f}(A,1;\mathbb{R}^n)$; such $\lambda$ exists by Lemma 4.5. Then in either Case I or Case II the following two assertions are equivalent:

(i) $\lambda$ is a solution to Problem 4.4.

(ii) There exists a number $c \in \mathbb{R}$ possessing the properties

\begin{align}
W^{\lambda}_{\alpha,f} &\geq c \quad (\xi - \lambda^+)^{-a.e.} , \\
W^{\lambda}_{\alpha,f} &\leq c \quad \lambda^+^{-a.e.} ,
\end{align}

and in addition it holds true that

\begin{equation}
W^{\lambda}_{\alpha,f} = 0 \quad \text{n.e. on } A_2 .
\end{equation}

If moreover Case II holds, then relation (6.4) can be rewritten equivalently in the following apparently stronger form:

\begin{equation}
W^{\lambda}_{\alpha,f} = 0 \quad \text{on } A_2 \setminus I_{\alpha,A_2},
\end{equation}

where $I_{\alpha,A_2}$ denotes the set of all $\alpha$-irregular (boundary) points of $A_2$.

Let $\dot{A}_2$ denote the $\kappa_\alpha$-reduced kernel of $A_2$ [19, p. 164], namely the set of all $x \in A_2$ such that $c_\alpha(B(x,r) \cap A_2) > 0$ for every $r > 0$.

In the following Theorems 6.4 and 6.5 we suppose that there exists the solution $\lambda ^{\xi}_{\dot{A}} = \lambda^+ - \lambda^-$ to Problem 4.4. For the sake of simplicity of formulation, in Theorem 6.4 we also assume that in the case $\alpha = 2$ the domain $D$ is simply connected.

**Theorem 6.4.** It holds that

\begin{equation}
S^{\lambda^-}_{\mathbb{R}^n} = \begin{cases}
\dot{A}_2 & \text{if } \alpha < 2 , \\
\partial D & \text{if } \alpha = 2 .
\end{cases}
\end{equation}

**Theorem 6.5.** Let $f = 0$. Then

\begin{equation}
W^{\lambda \dot{A}}_{\alpha,f} = U^{\lambda \dot{A}}_{\alpha} = \begin{cases}
U^{\lambda^+} & \text{n.e. on } D , \\
0 & \text{on } D^c \setminus I_{\alpha,D^c}.
\end{cases}
\end{equation}

\[\text{In Case I the assumption of the lower boundedness of } f \text{ on } A_1 \text{ is automatically fulfilled. Furthermore, in Case I relation (6.5) is equivalent to the following apparently stronger assertion: } W^{\lambda}_{\alpha,f} \leq c \text{ on } S^{\lambda^+}_D .\]
Furthermore, assertion (ii) of Theorem 6.3 holds, and relations (6.2) and (6.3) now take respectively the following (equivalent) form:

\[(6.8)\]
\[U_\alpha^{\lambda_A} = c (\xi - \lambda^+) - \text{a.e.},\]
\[(6.9)\]
\[U_\alpha^{\lambda_A} \leq c \text{ on } \mathbb{R}^n,\]

where \(0 < c < \infty\). In addition, in the present case \(f = 0\) relations (6.8) and (6.9) together with \(U_\alpha^{\lambda_A} = 0\) n.e. on \(D^c\) determine uniquely the solution \(\lambda_\alpha^\xi\) to Problem 4.4 within the class \(E_{\alpha,f}^{\xi}(A,1;\mathbb{R}^n)\) of admissible measures. If moreover \(U_\alpha^{\xi}\) is (finitely) continuous on \(D\), then also

\[(6.10)\]
\[U_\alpha^{\lambda_A} = c \text{ on } S_D^{\xi - \lambda^+},\]
\[(6.11)\]
\[c_g^D(S_D^{\xi - \lambda^+}) < \infty.\]

Omitting now the requirement of the continuity of \(U_\alpha^{\xi}\), assume next that \(\alpha < 2\) and \(m_n(D^c) > 0\), where \(m_n\) is the \(n\)-dimensional Lebesgue measure. Then

\[(6.12)\]
\[S_D^{\lambda^+} = S_D^\xi,\]
\[(6.13)\]
\[U_\alpha^{\lambda_A} < c \text{ on } D \setminus S_D^\xi \left( = D \setminus S_D^{\lambda^+}\right).\]

The proofs of Theorems 6.1–6.5 are presented in Section 7.

6.2. An extension of the theory. Parallel with a constraint \(\xi \in \mathcal{C}(A_1;\mathbb{R}^n)\) given by relation (4.10) and acting only on (positive) measures from \(E_{\alpha}^{+}(A_1,1;\mathbb{R}^n)\), consider also the measure \(\sigma = \sigma^+ - \sigma^- \in \mathfrak{M}(A;\mathbb{R}^n)\) defined as follows:

\[(6.14)\]
\[\sigma^+ = \xi, \text{ while } \sigma^- \geq \xi'.\]

Since \(\sigma^-(\mathbb{R}^n) \geq \xi'(\mathbb{R}^n) = \xi(\mathbb{R}^n) > 1\), where the equality is obtained from Theorem 3.2, this \(\sigma\) can be thought of as a signed constraint acting on (signed) measures from \(\mathcal{E}_{\alpha}(A;1;\mathbb{R}^n)\). Let \(\mathcal{E}_{\alpha}^{\sigma}(A,1;\mathbb{R}^n)\) consist of all \(\mu \in \mathcal{E}_{\alpha}(A,1;\mathbb{R}^n)\) such that \(\mu^\pm \leq \sigma^\pm\), and let

\[(6.15)\]
\[G_{\alpha,f}^{\sigma}(A,1;\mathbb{R}^n) := \inf_{\mu \in \mathcal{E}_{\alpha}^{\sigma}(A,1;\mathbb{R}^n)} G_{\alpha,f}(\mu),\]

where \(\mathcal{E}_{\alpha,f}(A,1;\mathbb{R}^n) := \mathcal{E}_{\alpha,f}^{\xi}(A,1;\mathbb{R}^n) \cap \mathcal{E}_{\alpha,f}(\mathbb{R}^n)\).

**Theorem 6.6.** With these assumptions and notations, we have

\[(6.16)\]
\[G_{\alpha,f}^{\sigma}(A,1;\mathbb{R}^n) = G_{\alpha,f}^{\xi}(A,1;\mathbb{R}^n).\]

If these (equal) extremal values are finite, then Problem 4.4 (with the positive constraint \(\xi\)) is solvable if and only so is problem (6.13) (with the signed constraint \(\sigma\)), and in the affirmative case their solutions coincide.
**Proof.** Indeed, \( G_{\alpha,f}^{\xi}(A,1;\mathbb{R}^n) \geq G_{\alpha,f}^\xi(A,1;\mathbb{R}^n) \) follows directly from the relation
\[
\mathcal{E}_{\alpha,f}^{\xi}(A,1;\mathbb{R}^n) \subset \mathcal{E}_{\alpha,f}^\xi(A,1;\mathbb{R}^n).
\]
To prove the converse inequality, assume \( G_{\alpha,f}^\xi(A,1;\mathbb{R}^n) < \infty \) and fix \( \nu \in \mathcal{E}_{\alpha,f}^\xi(A,1;\mathbb{R}^n) \).
Define \( \mu := \nu^+ - (\nu^+)\prime \). It is obvious that \( \mu \in \mathcal{E}_{\alpha}(A;\mathbb{R}^n) \), while Theorem 3.2 shows that
\( (\nu^+)\prime(A_2) = \nu^+(A_1) = 1 \). Furthermore, \( (\nu^+)\prime \leq \xi' \leq \sigma^+ \) by the linearity of balayage and (6.14), and so altogether \( \mu \in \mathcal{E}_{\alpha}(A,1;\mathbb{R}^n) \). According to (5.5) and (6.17), we thus have
\[
G_{\alpha,f}(\nu) = \|\nu\|^2_\alpha + 2\langle f,\nu^+ \rangle \geq \|\nu^+ - (\nu^+)\prime\|^2_\alpha + 2\langle f,\nu^+ \rangle
\geq \|\mu\|^2_\alpha + 2\langle f,\mu^+ \rangle = G_{\alpha,f}(\mu) \geq G_{\alpha,f}^\xi(A,1;\mathbb{R}^n),
\]
which establishes (6.16) by letting here \( \nu \) range over \( \mathcal{E}_{\alpha,f}(A,1;\mathbb{R}^n) \).

Assume now that (4.11) holds. If there is a solution \( \lambda_\mathcal{A}^\xi \) to problem (6.15), then this \( \lambda_\mathcal{A}^\xi \)
also solves Problem 4.4 which is clear from (6.16) and (6.17). Conversely, if \( \lambda_\mathcal{A}^\xi = \lambda^+ - \lambda^- \)
solves Problem 4.4 then by (5.5) it holds that \( \lambda^- = (\lambda^+)\prime \), and in the same manner as in the preceding paragraph we get \( \lambda_\mathcal{A}^\xi \in \mathcal{E}_{\alpha,f}(A,1;\mathbb{R}^n) \). Hence, \( \lambda_\mathcal{A}^\xi \) also solves problem (6.15) because \( G_{\alpha,f}(\lambda_\mathcal{A}^\xi) = G_{\alpha,f}^\xi(A,1;\mathbb{R}^n) = G_{\alpha,f}^\xi(A,1;\mathbb{R}^n) \) by (6.16).

Thus the theory of weighted minimum \( \alpha \)-Riesz energy problems with a (positive) constraint \( \xi \in \mathcal{C}(A_1;\mathbb{R}^n) \) acting only on positive parts of measures from \( \mathcal{E}_{\alpha}(A,1;\mathbb{R}^n) \), developed in Section 6.1, remains valid in its full generality for the signed constraint \( \sigma \), defined by (6.14) and acting simultaneously on positive and negative parts of \( \mu \in \mathcal{E}_{\alpha}(A,1;\mathbb{R}^n) \).

**Remark 6.7.** Assume for a moment that a generalized condenser is a finite collection \( K = (K_i)_{i \in I} \) of compact sets \( K_i \subset \mathbb{R}^n, i \in I \), with the sign \( s_i = \pm 1 \) prescribed such that
\[
c_\alpha(K_i \cap K_j) = 0 \quad \text{whenever} \quad s_is_j = -1.
\]
Problem 4.4 formulated for \( K \) in place of \( A \), has been analyzed in our recent work [11] for the \( \alpha \)-Riesz kernel of any order \( \alpha \in (0,n) \), any normalizing vector \( a = (a_i)_{i \in I} \), a vector-valued external field \( f = (f_i)_{i \in I} \), and a vector constraint \( (\xi_i)_{i \in I} \) such that \( U_\alpha^{\xi_i} \) is (finitely) continuous on \( K_i \); see e.g. Theorem 6.1 therein. (Compare with [2] where a similar problem with \( I = \{1,2\} \) and \( f = 0 \) was treated for the logarithmic kernel on the plane.) However, the approach developed in [11] was based substantially on the requirement (6.18), and can not be adapted to the present case where \( A_2 \cap \text{Cl}_{\mathbb{R}^n} A_1 \) may have nonzero \( \alpha \)-Riesz capacity.

7. **Proofs of the Assertions Formulated in Section 6.1**

Observe that if Case II takes place, then
\[
(7.1) \quad \zeta \in \mathcal{E}_\theta(D),
\]
\[
(7.2) \quad f = U_\alpha^{\zeta-\zeta'} = U_\alpha^\zeta \quad \text{c.g.-n.e. on } D.
\]
Indeed, (7.1) is obvious by (3.6), and (7.2) holds by Lemma 3.4 and footnote 7. By (7.1) and (7.2) we get in Case II for every $\nu \in E^+_g(A_1; D)$

$$G_{g,f}(\nu) = \|\nu\|_g^2 + 2E_g(\zeta, \nu) = \|\nu + \zeta\|_g^2 - \|\zeta\|_g^2.$$

7.1. **Proof of Theorem 6.1.** By Theorem 5.2, Theorem 6.1 will be proved once we have established the following assertion.

**Theorem 7.1.** Under the assumptions of Theorem 6.1, Problem 5.1 is solvable.

**Proof.** Under the assumptions of Theorem 6.1 Problem 5.1 makes sense since, by (5.4), (5.2) holds. (The value $G_{g,f}(A_1, 1; D)$ is then actually finite, which is clear from (5.4) and footnote 12.) In view of (5.2), there is a sequence $\{\mu_k\} \subset E^{+}_{g,f}(A_1, 1; D)$ such that

$$\lim_{k \to \infty} G_{g,f}(\mu_k) = G_{g,f}(A_1, 1; D).$$

Since $E_{g,f}(A_1, 1; D)$ is a convex cone and $E_g(D)$ is a pre-Hilbert space with the inner product $E_g(\nu, \nu_1)$ and the energy norm $\|\nu\|_g = \sqrt{E_g(\nu)}$, arguments similar to those in the proof of Lemma 4.6 can be applied to the set $\{\mu_k : k \in \mathbb{N}\}$. This gives

$$0 \leq \|\mu_k - \mu_\ell\|_g^2 \leq -4G_{g,f}(A_1, 1; D) + 2G_{g,f}(\mu_k) + 2G_{g,f}(\mu_\ell).$$

Letting here $k, \ell \to \infty$ and then combining the relation thus obtained with (7.4), we see in view of the finiteness of $G_{g,f}(A_1, 1; D)$ that $\{\mu_k\} \subset E^{+}_{g,f}(A_1, 1; D)$ forms a strong Cauchy sequence in the metric space $E^{+}_g(D)$. In particular, this implies

$$\sup_{k \in \mathbb{N}} \|\mu_k\|_g < \infty.$$

Since $A_1$ is (relatively) closed in $D$ and the cone $\mathcal{M}^+(D)$ is vaguely closed in $\mathcal{M}(D)$, so is the cone $\mathcal{M}^C(A_1; D) := \{\nu \in \mathcal{M}^+(A_1; D) : \nu \leq \xi\}$. Furthermore, the set $\mathcal{M}^C(A_1, 1; D) := \mathcal{M}^C(A_1; D) \cap \mathcal{M}^+(A_1, 1; D)$ is vaguely bounded, and hence it is vaguely relatively compact according to [5, Chapter III, Section 2, Proposition 9]. Thus, there exists a vague cluster point $\mu$ of the sequence $\{\mu_k\} \subset \mathcal{M}^C(A_1, 1; D)$ chosen above, and this $\mu$ belongs to $\mathcal{M}^C(A_1; D)$. Passing to a subsequence and changing notations, we can certainly assume that

$$\mu_k \to \mu \text{ vaguely in } \mathcal{M}^+(D) \text{ as } k \to \infty.$$

We assert that this $\mu$ is a solution to Problem 5.1.

Applying Lemma 2.1 to $1D \in \Psi(D)$, we obtain from (7.6)

$$\mu(A_1) = \mu(D) \leq \lim_{k \to \infty} \mu_k(D) = 1.$$

We proceed by showing that equality prevails in the inequality here, and so altogether

$$\mu \in \mathcal{M}^C(A_1, 1; D).$$
Consider an exhaustion of $A_1$ by an increasing sequence $\{K_j\}_{j \in \mathbb{N}}$ of compact sets. Since $1_{K_j}$ is upper semicontinuous on $D$ (and of course bounded), we get from (7.6) and Lemma 2.1 with $X = D$ and $\psi = -1_{K_j}$

$$1 \geq \mu(A_1) = \lim_{j \to \infty} \mu(K_j) \geq \lim_{j \to \infty} \limsup_{k \to \infty} \mu_k(K_j) = 1 - \lim_{j \to \infty} \liminf_{k \to \infty} \mu_k(A_1 \setminus K_j).$$

Relation (7.7) will therefore follow if we show that

$$(7.8) \quad \lim_{j \to \infty} \liminf_{k \to \infty} \mu_k(A_1 \setminus K_j) = 0.$$ 

Since by (6.1) $\infty > \xi(A_1) = \lim_{j \to \infty} \xi(K_j)$, we have

$$\lim_{j \to \infty} \xi(A_1 \setminus K_j) = 0.$$ 

When combined with

$$\mu_k(A_1 \setminus K_j) \leq \xi(A_1 \setminus K_j) \quad \text{for any } k, j \in \mathbb{N},$$

this implies (7.8) and consequently (7.7).

Another consequence of (7.6) is that $\mu_k \otimes \mu_k \rightarrow \mu \otimes \mu$ vaguely in $\mathcal{M}^+(D \times D)$ [5] Chapter III, Section 5, Exercise 5]. Applying Lemma 2.1 to $X = D \times D$ and $\psi = g$, we thus get

$$E_g(\mu) \leq \liminf_{k \to \infty} \|\mu_k\|^2_g < \infty,$$

where the latter inequality is valid by (7.5). Hence, $\mu \in \mathcal{E}_g^+(D)$. Combined with (7.7), this yields $\mu \in \mathcal{E}_g^+(A_1; D)$. As $G_{g,f}(\mu) > -\infty$, the assertion that $\mu$ solves Problem 5.1 will therefore be established once we have shown that

$$(7.9) \quad G_{g,f}(\mu) \leq \lim_{k \to \infty} G_{g,f}(\mu_k).$$

Since the kernel $g$ is perfect [16] Theorem 4.11], the sequence $\{\mu_k\}_{k \in \mathbb{N}}$, being strong Cauchy in $\mathcal{E}_g^+(D)$ and vaguely convergent to $\mu$, converges to the same limit strongly in $\mathcal{E}_g^+(D)$, i.e.

$$(7.10) \quad \lim_{k \to \infty} \|\mu_k - \mu\|_g = 0.$$ 

Also note that the mapping $\nu \mapsto G_{g,f}(\nu)$ is vaguely l.s.c., resp. strongly continuous, on $\mathcal{E}_g^+(D) \cap \mathcal{M}^+(A_1; D)$ if Case I, resp. Case II, holds. In fact, since $\|\nu\|_g$ is vaguely l.s.c. on $\mathcal{E}_g^+(D)$, the former assertion follows from Lemma 2.1. As for the latter assertion, it is obvious by (7.3). This observation enables us to obtain (7.9) from (7.6) and (7.10). \qed
7.2. Proof of Theorem 6.2. In view of Theorem 6.1 it is enough to establish the necessity part of the theorem. Assume that the requirements of the latter part of the theorem are fulfilled. Since Case II with $\zeta \geq 0$ takes place, we get from (7.1) and (7.2)

\begin{equation}
G_{g,f}(\nu) = \|\nu\|^2 + 2E_g(\zeta, \nu) \in [0, \infty) \quad \text{for all} \quad \nu \in \mathcal{E}_g^+(A_1; D).
\end{equation}

Consider numbers $r_j > 0$, $j \in \mathbb{N}$, such that $r_j \uparrow \infty$ as $j \to \infty$, and write $B_{r_j} := B(0, r_j)$, $A_{1,r_j} := A_1 \cap B_{r_j}$. As $c_{\alpha}(B_{r_j}) < \infty$ and $c_{\alpha}(A_1) = \infty$, it follows from the subadditivity of $c_{\alpha}(\cdot)$ on universally measurable sets \[^{13}\text{Lemma 2.3.5}\] that $c_{\alpha}(A_1 \setminus B_{r_j}) = \infty$. For every $j \in \mathbb{N}$ there is therefore $\xi_j \in \mathcal{E}_\alpha^+(A_1 \setminus B_{r_j}, 1; \mathbb{R}^n)$ of compact support $S_D^{\xi_j}$ such that

\begin{equation}
\|\xi_j\|_\alpha \leq j^{-2}.
\end{equation}

Clearly, the $r_j$ can be chosen successively so that $A_{1,r_j} \cup S_D^{\xi_j} \subset A_{1,r_{j+1}}$. Any compact set $K \subset \mathbb{R}^n$ is contained in a ball $B_{r_{j_0}}$ with $j_0$ large enough, and hence $K$ has points in common with only finitely many $S_D^{\xi_j}$. Therefore $\xi$ defined by the relation

\begin{equation}
\xi(\varphi) := \sum_{j \in \mathbb{N}} \xi_j(\varphi) \quad \text{for any} \quad \varphi \in C_0(\mathbb{R}^n)
\end{equation}

is a positive Radon measure on $\mathbb{R}^n$ carried by $A_1$. Furthermore, $\xi(A_1) = \infty$ and $\xi \in \mathcal{E}_\alpha^+(\mathbb{R}^n)$. To prove the latter, note that $\eta_k := \xi_1 + \cdots + \xi_k \in \mathcal{E}_\alpha^+(\mathbb{R}^n)$ in view of (7.12) and the triangle inequality in $\mathcal{E}_\alpha(\mathbb{R}^n)$. Also observe that $\eta_k \to \xi$ vaguely in $\mathcal{M}(\mathbb{R}^n)$ because for any $\varphi \in C_0(\mathbb{R}^n)$ there is $k_0$ such that $\xi(\varphi) = \eta_k(\varphi)$ for all $k \geq k_0$. As $\|\eta_k\|_\alpha \leq M := \sum_{j \in \mathbb{N}} j^{-2} < \infty$ for all $k \in \mathbb{N}$, Lemma 2.4 with $X = \mathbb{R}^n \times \mathbb{R}^n$ and $\psi = \kappa_\alpha$ yields $\|\xi\|_\alpha \leq M$. It has thus been shown that $\xi$ given by (7.13) is an element of $\mathcal{C}(A_1; \mathbb{R}^n)$ with $\xi(A_1) = \infty$.

Each $\xi_j$ belongs to $\mathcal{E}_\alpha^+(A_1, 1; D)$ and moreover, by (3.9) and (7.12),

\begin{equation}
\|\xi_j\|_g \leq \|\xi_j\|_\alpha \leq j^{-2}.
\end{equation}

Since Case II takes place, $\xi_j \in \mathcal{E}_{g,f}^+(A_1, 1; D)$ for all $j \in \mathbb{N}$ by (7.11). By the Cauchy–Schwarz (Bunyakowski) inequality in the pre-Hilbert space $\mathcal{E}_g(D)$,

\[0 \leq G_{g,f}^\xi(A_1, 1; D) \leq \lim_{j \to \infty} \left[ \|\xi_j\|^2_g + 2E_g(\zeta, \xi_j) \right] \leq 2\|\xi\|_g \lim_{j \to \infty} \|\xi_j\|_g = 0,
\]

where the first and the second inequalities hold by (7.11), while the third inequality and the equality are valid by (7.14). Hence, $G_{g,f}^\xi(A_1, 1; D) = 0$. As seen from (7.11), such infimum can be attained only at zero measure, which is impossible because $0 \notin \mathcal{E}_{g,f}^\xi(A_1, 1; D)$. Combined with Theorem 6.2, this establishes the claimed assertion.

7.3. Proof of Theorem 6.3. Fix $\lambda = \lambda^+ - \lambda^- \in \mathcal{E}_{\alpha,f}^\xi(A, 1; \mathbb{R}^n)$, and note that since $f = 0$ n.e. on $A_2$, relation (6.4) can alternatively be rewritten as $U_\lambda^f = U_\lambda^{\lambda^+ - \lambda^-} = 0$ n.e. on $A_2$, which in view of the $c_{\alpha}$-absolute continuity of $\lambda$ and (3.2) is equivalent to the equality

\begin{equation}
\lambda^- = (\lambda^+)'.
\end{equation}
Taking Theorem 5.2 into account, we thus see that, while proving the equivalence of assertions (i) and (ii) of Theorem 6.3, there is no loss of generality in assuming that the given measure $\lambda$ satisfies (7.15). By (3.5) we therefore get

$$U_\alpha^\lambda = U_\alpha^\lambda - (\lambda')' = \begin{cases} U_\alpha^\lambda & \text{a.e. on } D, \\ 0 & \text{a.e. on } A_2, \end{cases}$$

and hence

$$W_{\alpha,f}^\lambda = \begin{cases} W_{g,f}^\lambda & \text{a.e. on } D, \\ 0 & \text{a.e. on } A_2, \end{cases}$$

where $W_{g,f}^\lambda := W_{g,f}|_D = U_g^\lambda + f|_D$, cf. (2.11). If moreover Case II holds, then

$$W_{\alpha,f}^\lambda = U_\alpha^\lambda + \zeta - U_\alpha^\lambda (\lambda' + \zeta)' \text{ n.e. on } \mathbb{R}^n.$$  

According to [16, Corollary 3.14], the function on the right (hence that on the left) in this relation takes the value 0 at every $\alpha$-regular point of $A_2$, which establishes (6.5).

Combined with Theorem 5.2, what has been shown just above yields that Theorem 6.3 will be proved once we have established the following theorem.

**Theorem 7.2.** Under the hypotheses of Theorem 6.3 the following two assertions are equivalent for any $\mu \in \mathcal{E}_{g,f}(A_1, 1; D)$:

1. $\mu$ is a solution to Problem 5.1.
2. There exists a number $c \in \mathbb{R}$ possessing the properties

$$W_{g,f}^\mu \geq c \text{ (} \xi - \mu \text{)-a.e.,}$$

$$W_{g,f}^\mu \leq c \text{ } \mu \text{-a.e.}$$

**Proof.** Throughout the proof we shall use permanently the fact that both $\xi$ and $\mu$ have finite $\alpha$-Riesz energy, are hence they are $c_\alpha$-absolutely continuous.

Suppose first that assertion (i') holds. Inequality (7.16) is valid for $c = L$, where

$$L := \sup \{ q \in \mathbb{R} : W_{g,f}^\mu \geq q \text{ (} \xi - \mu \text{)-a.e.} \}.$$  

In turn, (7.16) with $c = L$ implies that $L < \infty$ because $W_{g,f}^\mu < \infty$ holds n.e. on $A_1^0$ and hence $(\xi - \mu)$-a.e. on $A_1^0$, while $(\xi - \mu)(A_1^0) > 0$ by (4.14). Also note that $L > -\infty$, for $W_{g,f}^\mu$ is lower bounded on $A_1$ by assumption.

We next proceed by establishing (7.17) with $c = L$. To this end write for any $w \in \mathbb{R}$

$$A_1^+(w) := \{ x \in A_1 : W_{g,f}^\mu (x) > w \} \quad \text{and} \quad A_1^-(w) := \{ x \in A_1 : W_{g,f}^\mu (x) < w \}.$$  

Assume on the contrary that (7.17) with $c = L$ fails, i.e. $\mu(A_1^+(L)) > 0$. Since $W_{g,f}^\mu$ is $\mu$-measurable, one can choose $w' \in (L, \infty)$ so that $\mu(A_1^+(w')) > 0$. At the same time, as
Proof of Theorem 6.4. For any \( x \in D \) let \( K_x \) be the inverse of \( \text{Cl}_{\mathbb{R}^n} A_2 \) relative to \( S(x, 1), \mathbb{R}^n \) being the one-point compactification of \( \mathbb{R}^n \). Since \( K_x \) is compact, there is the (unique) \( \kappa_x \)-equilibrium measure \( \gamma_x \in E^+(K_x; \mathbb{R}^n) \) on \( K_x \) with the properties \( \| \gamma_x \|^2 = \gamma_x(K_x) = c_0(K_x) \),

\[ \gamma_x = \gamma_\alpha = 1 \text{ n.e. on } K_x, \]

and \( U_\gamma = 1 \) on \( \mathbb{R}^n \). Note that \( \gamma_x \neq 0 \), for \( c_0(K_x) > 0 \) in consequence of \( c_0(A_2) > 0 \), see \cite{19} Chapter IV, Section 5, n° 19]. We assert that under the stated requirements

\[ S^{\gamma_x}_{\mathbb{R}^n} = \begin{cases} K_x & \text{if } \alpha < 2, \\ \partial_{\mathbb{R}^n} K_x & \text{if } \alpha = 2. \end{cases} \]

The latter equality in (7.20) follows from \cite{19} Chapter II, Section 3, n° 13]. To establish the former equality\footnote{We have brought here this proof, since we did not find a reference for this possibly known assertion.} we first note that \( S^{\gamma_x}_{\mathbb{R}^n} \subseteq K_x \) by the \( c_0 \)-absolute continuity of \( \gamma_x \). As for the converse inclusion, assume on the contrary that there is \( x_0 \in K_x \) such that \( x_0 \notin S^{\gamma_x}_{\mathbb{R}^n} \).

Choose \( r > 0 \) with the property \( B(x_0, r) \cap S^{\gamma_x}_{\mathbb{R}^n} = \emptyset \). But \( c_0(B(x_0, r) \cap K_x) > 0 \), hence there is \( y \in B(x_0, r) \) such that \( U_\gamma^{\gamma_x}(y) = 1 \). The function \( U_\gamma^{\gamma_x} \) is \( \alpha \)-harmonic on \( B(x_0, r) \) \cite{19} Chapter I, Section 5, n° 20], continuous on \( B(x_0, r) \), and takes at \( y \in B(x_0, r) \) its maximum value 1. Applying \cite{19} Theorem 1.28] we obtain \( U_\gamma^{\gamma_x} = 1 \) m.e. on \( \mathbb{R}^n \), hence everywhere on \( (K_x)^c \) by the continuity of \( U_\gamma^{\gamma_x} \) on \( (S^{\gamma_x}_{\mathbb{R}^n})^c \), and altogether n.e.
This means that \( \gamma_x \) serves as the \( \alpha\)-Riesz equilibrium measure on the whole of \( \mathbb{R}^n \), which is impossible.

Based on (5.5) and the integral representation (3.3), we arrive at (6.6) with the aid of the fact that for every \( x \in D \), \( \varepsilon_x' \) is the Kelvin transform of the \( \kappa_\alpha \)-equilibrium measure \( \gamma_x \), see [16, Section 3.3].

7.5. Proof of Theorem 6.5. Since \( \lambda^- = (\lambda^+)' \) by (5.5) and \( f = 0 \) by assumption, the function

\[
W_{\alpha,f} = U_{\alpha,f}^\lambda = U_\alpha^{\lambda^+} - U_\alpha^{(\lambda^+)^'}
\]

is well defined and finite n.e. on \( \mathbb{R}^n \). In particular, it is well defined on all of \( D \) and it equals there the strictly positive function \( U_g^{\lambda^+} \), see Lemma 3.4. This together with (6.5) proves (6.7). Combining (6.7) with (6.3) shows that under the stated assumptions the number \( c \) from Theorem 6.3 is \( > 0 \), while (6.2) now takes the (equivalent) form

\[
(7.21) \quad U_{\alpha,f}^\lambda \geq c > 0 \quad (\xi - \lambda^+)-a.e.
\]

Having rewritten (6.3) as

\[
U_\alpha^{\lambda^+} \leq U_\alpha^{\lambda^-} + c \quad \lambda^+-a.e.,
\]

we infer from [19] Theorems 1.27, 1.29, 1.30 that the same inequality holds on all of \( \mathbb{R}^n \), which amounts to (6.9). In turn, (6.9) yields (6.8) when combined with (7.21). It follows directly from Theorem 6.3 that relations (6.8) and (6.9) together with \( U_{\alpha,f}^\lambda = 0 \) n.e. on \( D^c \) determine uniquely the solution \( \lambda^\xi_A \) to Problem 4.4 within the class \( E_{\xi,f}(A,1;\mathbb{R}^n) \) of admissible measures.

Assume now that \( U_{\alpha,f}^\lambda \) is continuous on \( D \). Then so is \( U_{\alpha,f}^{\lambda^+} \). Indeed, since \( U_{\alpha,f}^{\lambda^+} \) is l.s.c. and \( U_{\alpha,f}^{\lambda^+} = U_{\alpha,f}^\xi - U_{\alpha,f}^{\xi-\lambda^+} \) with \( U_{\alpha,f}^\xi \) continuous on \( D \) and \( U_{\alpha,f}^{\xi-\lambda^+} \) l.s.c., it follows that \( U_{\alpha,f}^{\lambda^+} \) is also upper semicontinuous, and hence continuous. Therefore, by the continuity of \( U_{\alpha,f}^{\lambda^+} \) on \( D \), (6.8) implies (6.10). Thus, by (6.7) and (6.10),

\[
U_{\alpha,f}^{\lambda^+} = c \quad \text{on} \quad S_{D}^{\xi-\lambda^+},
\]

which implies (6.11) in view of [13] Lemma 3.2.2 with \( \kappa = g \).

Omitting now the requirement of the continuity of \( U_{\alpha,f}^\xi \), assume next that \( \alpha < 2 \) and \( m_n(D^c) > 0 \). If on the contrary (6.12) fails, then there is \( x_0 \in S_{D}^{\xi} \) such that \( x_0 \notin S_{D}^{\lambda^+} \). Thus one can choose \( r > 0 \) so that

\[
(7.22) \quad \overline{B}(x_0,r) \subset D \quad \text{and} \quad \overline{B}(x_0,r) \cap S_{D}^{\lambda^+} = \emptyset.
\]

Then \( (\xi - \lambda^+)(\overline{B}(x_0,r)) > 0 \), and hence by (6.8) there exists \( y \in \overline{B}(x_0,r) \) with the property \( U_{\alpha,f}^{\lambda^+}(y) = c \), or equivalently

\[
(7.23) \quad U_{\alpha,f}^{\lambda^+}(y) = U_{\alpha,f}^{\lambda^-}(y) + c.
\]
As $U^+_\alpha$ is $\alpha$-harmonic on $B(x_0, r)$ and continuous on $\overline{B}(x_0, r)$, while $U^-\alpha + c$ is $\alpha$-superharmonic on $\mathbb{R}^n$, we obtain from (6.9) and (7.23) with the aid of [19, Theorem 1.28]

(7.24)  
$$U^+\alpha = U^-\alpha + c \text{ m.a.e. on } \mathbb{R}^n.$$  

This implies $c = 0$, for $U^+\alpha = U^-\alpha + c$ holds n.e. on $D^c$, and hence $m_a$-e. on $D^c$. A contradiction.

Similar arguments enable us to establish (6.13). Indeed, if (6.13) fails at some $x_1 \in D \setminus S^+_D$, then relation (7.23) would be valid with $x_1$ in place of $y$, see (6.9); and moreover one could choose $r > 0$ so that (7.22) would be fulfilled with $x_1$ in place of $x_0$. Therefore, using the $\alpha$-harmonicity of $U^+\alpha$ on $B(x_1, r)$ as well as the $\alpha$-superharmonicity of $U^-\alpha + c$ on $\mathbb{R}^n$, we would arrive again at (7.24), and hence at the equality $c = 0$. The contradiction thus obtained completes the proof of the theorem.

8. Duality relation between non-weighted constrained and weighted unconstrained minimum $\alpha$-Green energy problems

As above, fix a (not necessarily proper) subset $A_1$ of $D$ which is relatively closed in $D$ and fix a constraint $\xi \in \mathcal{C}(A_1; \mathbb{R}^n)$, see (4.10), with $1 < \xi(A_1) < \infty$; such $\xi$ exists because of the (permanent) assumption $c_\alpha(A_1) > 0$. According to Theorem 7.1, the non-weighted ($f = 0$) constrained minimum $\alpha$-Green energy problem over the class $\mathcal{E}_{\alpha}^+_g(A_1, 1; D)$ is (uniquely) solvable, i.e. there exists $\lambda = \lambda_\xi^\alpha \in \mathcal{E}_{\alpha}^+_g(A_1, 1; D)$ with

(8.1) $$\|\lambda\|^2_g = \min_{\nu \in \mathcal{E}^+_g(A_1, 1; D)} \|\nu\|^2_g.$$  

Write $q := [\xi(A_1) - 1]^{-1}$ and

$$\theta := q(\xi - \lambda), \quad f_0 := -qU_x.$$  

**Theorem 8.1.** Assume moreover that $U_x^\xi$ is (finitely) continuous on $D$. Then the measure $\theta$ is a (unique) solution to the $f_0$-weighted unconstrained minimum $\alpha$-Green energy problem over $\mathcal{E}^+_g(A_1, 1; D)$, i.e. $\theta \in \mathcal{E}^+_g(A_1, 1; D)$ and

(8.2) $$G_{g, f_0}(\theta) = \inf_{\nu \in \mathcal{E}^+_g(A_1, 1; D)} G_{g, f_0}(\nu).$$  

Moreover, there exists $\eta \in (0, \infty)$ such that

(8.3) $$W^{\theta}_{g, f_0} = -\eta \text{ on } S^0_D,$$

(8.4) $$W^{\theta}_{g, f_0} \geq -\eta \text{ on } D,$$

and these two relations determine uniquely a solution to problem (8.2) among the measures of the class $\mathcal{E}^+_g(A_1, 1; D)$.  

Proof. Under the stated assumptions, relations (7.16) and (7.17) for the solution \( \lambda \) to the (non-weighted constrained) problem (8.1) take the (equivalent) form

\[
U^\lambda_g \geq c \ (\xi - \lambda)\text{-a.e.,} \tag{8.5}
\]
\[
U^\lambda_g \leq c \ \lambda\text{-a.e.} \tag{8.6}
\]

Thus \( c > 0 \), see (8.6). Applying Theorem 3.3 with \( v = c \), from (8.6) we therefore obtain

\[
U^\lambda_g \leq c \text{ on } D.
\]

Combined with (8.5), this gives \( U^\lambda_g = c (\xi - \lambda)\text{-a.e.,} \) and hence \( U^\lambda_g = c \text{ on } D(\xi - \lambda) \).

With the chosen notations the two preceding displays can alternatively be rewritten as (8.3) and (8.4) with \( \eta := qc \). In turn, (8.3) and (8.4) imply that \( \theta, f_0 \) and \( -\eta \) satisfy relations (7.9) and (7.10) in [24], which according to [24, Theorem 7.3] establishes (8.2). \( \square \)

9. Examples

The purpose of the examples below is to illustrate the assertions from Section 6.1. Observe that both in Example 9.1 and Example 9.2 the set \( A_2 = D^c \) is not \( \alpha \)-thin at infinity.

Example 9.1. Let \( n \geq 3 \), \( 0 < \alpha < 2 \), \( A_1 = D = B(0, r) \), where \( r \in (0, \infty) \), and let \( A_2 = D^c \), \( f = 0 \). Define \( \xi := q \lambda_r \), where \( q > 1 \) and \( \lambda_r \) is the \( \kappa_\alpha \)-capacitary measure on \( \overline{B}(0, r) \), see Remark 2.6. As follows from [19, Chapter II, Section 3, n° 13], \( \xi \in \mathcal{E}_\alpha^+(A_1, q; \mathbb{R}^n) \), \( S^\xi_D = D \) and \( U_\alpha^\xi \) is continuous on \( \mathbb{R}^n \). Since \( f = 0 \), Problem 4.4 reduces to the problem of minimizing \( E_\alpha(\mu) \) over the class of all (signed Radon) measures \( \mu \in \mathcal{E}_\alpha(A_1; \mathbb{R}^n) \) with \( \mu^+ \leq \xi \), which by Theorem 5.2 is equivalent to the problem of minimizing \( E_g(\nu) \) where \( \nu \) ranges over \( \mathcal{E}_\alpha^+(A_1, 1; D) \). According to Theorems 5.2, 6.1 and Lemma 4.6 these two constrained minimum energy problems are uniquely solvable (no short-circuit occurs) and their solutions, denoted respectively by \( \lambda^{\xi}_A = \lambda^+ - \lambda^- \) and \( \lambda^{\xi}_{A_1} \), are related to each other as in (5.5). Furthermore, by (6.6), (6.11) and (6.12) we obtain

\[
S_{D^c}^{\lambda^+} = S_D^{\lambda^{\xi}_A} = S_D^\xi = D, \quad S_{\mathbb{R}^n}^{\lambda^-} = D^c, \tag{9.1}
\]

and finally by (6.5) and (6.10) we have

\[
U_\alpha^\lambda = \begin{cases} 
    c & \text{on } S_D^{\xi - \lambda^+}, \\
    0 & \text{on } D^c, 
\end{cases} \tag{9.2}
\]
where $c > 0$, while by (6.9)

\begin{equation}
U_\alpha^\lambda \leq c \quad \text{on} \quad D \setminus S_D^{\xi-\lambda^+}.
\end{equation}

Moreover, according to Theorem 6.3 relations (9.2) and (9.3) determine uniquely the solution $\lambda_\mathbf{A}^\xi$ among the class of admissible measures.

**Example 9.2.** Let $n = 3$, $\alpha = 2$, $f = 0$ and let $D := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0\}$. Define $A_1$ as the union of $K_k$ over $k \in \mathbb{N}$, where

$$K_k := \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = \frac{1}{k}, x_2^2 + x_3^2 \leq k^2 \}, \quad k \in \mathbb{N}.$$  

Let $\lambda_k$ be the $\kappa_2$-capacitary measure on $K_k$, see Remark 2.6, hence $\lambda_k(K_k) = 1$ and $||\lambda_k||^2 = \pi^2/(2k)$ by [19, Chapter II, Section 3, n° 14]. Define

$$\xi := \sum_{k \in \mathbb{N}} \frac{\lambda_k}{k^2}.$$  

In the same manner as in the proof of Theorem 6.2 one can see that $\xi$ is a bounded positive Radon measure carried by $A_1$ with $E_2(\xi) < \infty$. Therefore it follows from Theorem 6.1 that Problem 4.4 for the constraint $\xi$ and the generalized condenser $\mathbf{A} = (A_1, D^c)$ has a solution $\lambda_\mathbf{A}^\xi$ (no short-circuit occurs), although $D^c \cap \mathrm{Cl}_{\mathbb{R}^3} A_1 = \partial D = \{x_1 = 0\}$ and hence

$$c_2(D^c \cap \mathrm{Cl}_{\mathbb{R}^3} A_1) = \infty.$$  

Furthermore, since each $U_2^{\lambda_k}$, $k \in \mathbb{N}$, is continuous on $\mathbb{R}^n$ and bounded from above by $\pi^2/(2k)$, the potential $U_2^\xi$ is continuous on $\mathbb{R}^n$ by uniform convergence of the sequence $\sum_{k \in \mathbb{N}} k^{-2} U_2^{\lambda_k}$. Hence, (9.1), (9.2) and (9.3) also hold in the present case with $\alpha = 2$, again with $c > 0$, and relations (9.2) and (9.3) determine uniquely the solution $\lambda_\mathbf{A}^\xi$ within the class of admissible measures. Also note that $S_{\mathbb{R}^n}^\lambda = \partial D$ according to (6.6).

**10. Appendix**

The following example shows that even for positive bounded (hence extendible) measures on an open ball in $\mathbb{R}^3$ the finiteness of the $\alpha$-Green energy does not necessarily imply the finiteness of the $\alpha$-Riesz energy, contrary to what was stated in [10, Lemma 2.4].

**Example 10.1.** Let $\alpha = 2$. For technical simplicity we first construct the analogous example with the ball replaced by the half-space $D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0\}$ (next we apply a Kelvin transformation). The boundary $\partial D$ (replacing the sphere) is then the plane $\{x_1 = 0\}$. For $r > 0$ let $\mu_r$ denote the $\kappa_2$-capacitary measure on the closed 2-dimensional disc $K_r \subset \partial D$ of radius $r$ centered at $(0,0,0)$, see Remark 2.6. Such $\mu_r$ exists since $0 < c_2(K_r) < \infty$ (in fact $c_2(K_r) = 2r/\pi^2$, see [19, Chapter II, Section 3, n° 14]). The Newtonian energy $E_2(\mu_r)$ equals $E_2(\mu_1)/r$, where $0 < E_2(\mu_1) = 1/c_2(K_1) < \infty$. For real numbers $z_1$ and $z_2$ and a measure $\nu \in \mathfrak{M}^+(\partial D; \mathbb{R}^3)$ denote by $\nu^{z_1,z_2}$ the translation of $\nu$ in
Consider decreasing sequences \( \{c_k\}_{k \in \mathbb{N}} \) which is possible in view of (10.1). Now define the functional \( K \) disks \( K \in E \) belong to \( \varepsilon, \eta \) in (10.2) in view of the triangle inequality in \( E \). Furthermore, the partial sums (10.3) \( \|g\| = 1 \) from Lemma 2.1 with \( x = D \times D \) and \( \psi = g \).
On the other hand, being bounded, $\mu$ is extendible to a positive Radon measure on $\mathbb{R}^3$ and

$$E_2(\mu) \geq \sum_{k \in \mathbb{N}} E_2(c_k \mu_{r_k}) = \sum_{k \in \mathbb{N}} c_k^2 E_2(\mu_{r_k}) = \sum_{k \in \mathbb{N}} c_k^2 r_k^{-1} E_2(\mu_1) = \infty,$$

where the last equality follows from the latter equality in (10.2). This verifies Example 10.1 for a half-space.

For treating the ball, apply the inversion relative to the sphere with center $(2,0,0)$ and radius 2. It maps the above half-space $D$ on the ball $D^*$ centered at $(1,0,0)$ and with radius 1. The above measure $\mu$ has bounded Newtonian potential $U_2^\mu$ at the point $(2,0,0)$ because $\mu$ is bounded and supported by the closed strip $\{0 \leq x_1 \leq 1\}$ not containing $(2,0,0)$. Therefore, the Kelvin transform $\mu^*$ of $\mu$ is a bounded measure, see [19, Eq. (4.5.3)], and can be written in the form

$$\mu^* = \sum_{k \in \mathbb{N}} c_k (\mu_{r_k}^{\varepsilon_k})^*,$$

the Kelvin transformation of positive measures being clearly countably additive. Since $\kappa_2$-energy is preserved by Kelvin transformation, so is $g_2^D$-energy of the measure $\mu_{r_k}^{\varepsilon_k} \in \mathcal{E}_+(D)$, as seen by combining [19 Eqs. (4.5.2), (4.5.4)] and (3.5) above. Denoting by $g^*$ the Green kernel for the above ball $D^*$ we therefore obtain by (10.3)

$$\|\mu^*\|_{g^*} \leq \sum_{k \in \mathbb{N}} c_k \|\mu_{r_k}^{\varepsilon_k}\|^*_{g^*} = \sum_{k \in \mathbb{N}} c_k \|\mu_{r_k}^{\varepsilon_k}\|_g \leq 1.$$

And clearly $E_2(\mu^*) = E_2(\mu) = \infty$. This verifies Example 10.1 also for a ball.

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