THE SOCLE FILTRATIONS OF PRINCIPAL SERIES REPRESENTATIONS OF $SL(3, \mathbb{R})$ AND $Sp(2, \mathbb{R})$

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Abstract. We study the structure of the $(g, K)$-modules of the principal series representations of $SL(3, \mathbb{R})$ and $Sp(2, \mathbb{R})$ induced from minimal parabolic subgroups, in the case when the infinitesimal character is nonsingular. The composition factors of these modules are known by Kazhdan-Lusztig-Vogan conjecture. In this paper, we give complete descriptions of the socle filtrations of these modules.

1. Introduction

Given a representation of a group or an algebra, the determination of its composition series is a natural problem. In [8], Kazhdan and Lusztig conjectured that the composition factors of Verma modules of complex reductive Lie algebras are determined by the value of so-called Kazhdan-Lusztig polynomials at $q = 1$. This conjecture is called a Kazhdan-Lusztig conjecture, and it is proved independently by Beilinson and Bernstein ([2]) and by Brylinski and Kashiwara ([3]).

Let $G$ be a real reductive Lie group, and $K$ a maximal compact subgroup of $G$. The complexified Lie algebra of $G$ is denoted by $g$. By Harish-Chandra’s subquotient theorem, every irreducible $(g, K)$-module, is realized as a subquotient module of some principal series module. Moreover, Langlands ([9]) and Milićić classified the irreducible $(g, K)$-modules in terms of standard modules, namely some kind of generalized principal series modules. As for the composition factor problem for standard modules, Vogan generalized Kazhdan-Lusztig conjecture to the standard modules of $G$ ([19]), which we call the Kazhdan-Lusztig-Vogan (KLV) conjecture. This conjecture was proved by himself ([20]).

As introduced above, the composition factor problem for standard modules is completely solved in 1980’s. On the other hand, the composition series problem, namely the complete determination of the socle filtrations of standard modules, is very difficult. Here, for a $(g, K)$-module $V$ of finite length, the socle is defined as the largest semisimple submodule of $V$. Let $V_1$ be the socle of $V$, and let $V_2$ be the socle of $V/V_1$, and so on. The filtration of $V$ so obtained is called the socle filtration of $V$.

For the groups of real rank one, this problem is completely solved by Collingwood ([2]). As for higher rank groups, there are many researches on the structure of degenerate principal series (for example [7], [10], [11], [12]). But, as far as the authors know, for the principal series representations induced from minimal parabolic subgroups, there are few complete results. The intertwining operators between principal series representations are explored in the cases of $G = SL(3, \mathbb{C})$...
by Tsuchikawa [17], \( G = SL(3, \mathbb{R}) \) by A. I. Fomin [6] and \( G = Sp(2, \mathbb{R}) \) by Muić [13], and they obtained some partial results on the structure of principal series representations. The problem of complete determination of socle filtrations of principal series representations is still open for groups with higher real rank. The main objective of this paper is to describe completely the socle filtrations of principal series modules induced from minimal parabolic subgroups with nonsingular integral infinitesimal characters, in the cases of the groups \( SL(3, \mathbb{R}) \) and \( Sp(2, \mathbb{R}) \). The authors hope that the results of this paper become cornerstones for solving the composition series problem generally.

One application of the results of this paper is the description of the socle filtration of the standard Whittaker \((\mathfrak{g}, K)\)-modules [16] defined by the second author. Since the structures of these modules resemble those of the principal series modules in some parts, the results of this paper play important roles when we determine the socle filtrations of the standard Whittaker \((\mathfrak{g}, K)\)-modules of \( SL(3, \mathbb{R}) \) and \( Sp(2, \mathbb{R}) \). Details of this study on Whittaker modules will be reported elsewhere.

We explain the contents of this paper. In Section 2 we review the Langlands classification of irreducible \((\mathfrak{g}, K)\)-modules after Vogan’s book [18]. In Section 3 we explain the outline of our computation. We frequently use the shift operators of \( K \)-types, so we also explain them in this section. Section 4 consists of the explanation of other tools used in this paper. Section 5 consists of the review of the well-known structure of the finite dimensional representations of \( \mathfrak{sl}(2) \), since the maximal compact subgroups of \( SL(3, \mathbb{R}) \) and \( Sp(2, \mathbb{R}) \) are \( SO(3) \) and \( U(2) \), respectively.

We investigate the structure of principal series modules of \( SL(3, \mathbb{R}) \) in Sections 6–11. In Section 6 we review the classification of irreducible \((\mathfrak{g}, K)\)-modules of \( SL(3, \mathbb{R}) \), and the \( K \)-spectra of these irreducible modules are presented in Section 7. In Section 8 we write down the shift operators of \( K \)-types for \( SL(3, \mathbb{R}) \) explicitly. These operators are used for determination of candidates of irreducible submodules in principal series modules. This is done in Section 9 and 10. After these preparations, we determine the socle filtrations of principal series modules of \( SL(3, \mathbb{R}) \) completely in Section 11. The main result is Theorem 11.1.

We treat the case of \( Sp(2, \mathbb{R}) \) in Sections 12–18. We review the structure of the group \( Sp(2, \mathbb{R}) \) in Section 12 and that of irreducible modules of it in Section 13. \( K \)-spectra of irreducible modules are presented in Section 14. In Section 15 we write down the shift operators of \( K \)-types explicitly. In Section 16 the candidates of irreducible submodules are obtained, and the socle filtrations of the principal series modules in the block \( PSO(4, 1) \) are determined. Those of the principal series modules in the block \( PSO(3, 2) \) are determined in Sections 17 and 18. The main results for the case of \( Sp(2, \mathbb{R}) \) are Theorems 16.16, 17.1 and 18.1.

Before going ahead, we fix some notation. For a Lie group \( L \), its Lie algebra is denoted by the corresponding German letter \( l \) and its complexification by \( l^* \). The complex dual space of \( l \) is denoted by \( l^* \). For a compact Lie group \( L \), the set of equivalence classes of irreducible representations of \( L \) is denoted by \( \hat{L} \). For \( \tau \in \hat{L} \), its representation space is denoted by \( V^L_\tau \). If \( \tau \) is specified by a highest weight \( \lambda \) of it, we also denote it by \( V^L_\lambda \). This notation will be used for irreducible finite dimensional representations of \( \mathfrak{sl}(2) \). Let \( A \) be a Lie group and \( B \) its closed subgroup. For a representation \((\pi, V)\) of \( A \) and an irreducible representation \((\tau, W)\) of \( B \), the \( \tau \)-isotypic subspace of \( V \) is denoted by \( V(\tau) \).
We denote the matrix unit \((\delta_{i,k}\delta_{j,l})_{i,j}\) by \(E_{k,l}\). The diagonal matrix with the entries \(a_1, \ldots, a_k\) is denoted by \(\text{diag}(a_1, \ldots, a_k)\). Once a basis \(\{v_i\}\) of a vector space \(V\) is fixed, we often denote an element of \(V\) by its coordinate with respect to this basis. For example, once a basis \(\{e_i\}\) of a Cartan subalgebra \(t\) of \(K\) is fixed, we write the element \(\alpha = \sum_i a_i e_i\) as \((a_1, \ldots, a_k)\).

Next, we fix the notation of \((g, K)\)-modules. Let \(G\) be a real reductive linear Lie group in the sense of [18]. Choose a Cartan involution \(\theta\) of \(G\), and let \(K\) be the maximal compact subgroup of \(G\) consisting of fixed points of \(\theta\). The corresponding Cartan decomposition of \(g_0\) is denoted by \(g_0 = t_0 + z_0\). For an admissible \((g, K)\)-module \(X\), the \(K\)-spectrum of \(X\), namely the set \(\{V^r_\tau \in \tilde{K} \mid \text{Hom}_K(V^r_\tau, X|_K) \neq 0\}\) of \(K\)-types of \(X\), is denoted by \(\tilde{K}(X)\).

Finally, we fix the notation of principal series modules. Let \((a_m)_{\mathfrak{a}}\) be a maximal commutative subspace of \(z_0\). Define \(A_m := \exp(a_m)\) and \(M_m := Z_K(A)\). Choose a positive system \(\Sigma^+\) of the restricted root system \(\Sigma(g_0, (a_m))\). As usual, half the sum of elements in \(\Sigma^+\) is denoted by \(\rho_m\). Define the nilpotent subgroup \(N_m\) of \(G\) which corresponds to the positive system \(\Sigma^+\). Then, \(P_m := M_m A_m N_m\) is a minimal parabolic subgroup of \(G\). For a representation of \(M\) and a linear character \(\nu \in \mathfrak{a}^*\), define the principal series module by

\[
I(\sigma, \nu) := \text{Ind}_{P_m}^G(\sigma \otimes e^{\rho_m} \otimes 1).\text{-finite}.
\]

As we explained in the introduction, the objective of this paper is to determine the socle filtration of \(I(\sigma, \nu)\) explicitly.

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### 2. Regular characters and the Langlands classification

In this section, we review the Langlands classification of \((g, K)\)-modules by means of regular characters after Vogan’s book [18].

Let \(H\) be a Cartan subgroup of \(G\). An element \(\Lambda\) of \(\mathfrak{h}^*\) defines an infinitesimal character \(\chi_\Lambda\) of \(Z(g)\). If a \((g, K)\)-module \(V\) admits the infinitesimal character \(\chi_\Lambda\), we also say \(V\) admits the infinitesimal character \(\Lambda\).

Suppose that \(H\) is \(\theta\)-stable. Let \(T := H \cap K\) and \(A := H \cap \exp z_0\). Then \(H = TA\). The centralizer of \(A\) in \(G\) is denoted by \(L\), and its Langlands decomposition by \(L = MA\). Then, \(T\) is a compact Cartan subgroup of \(M\).

A regular character ([18] Definition 6.6.1) is an ordered pair \(\gamma = (\Gamma, \pi)\) with \(\Gamma\) an ordinary character of \(H\) and \(\pi \in \mathfrak{h}^*\) satisfying

1. if \(\alpha \in \Delta(m, t)\), then \(\langle \alpha, \pi \rangle\) is real and non-zero.
2. The differential of \(\Gamma\) is

\[
d\Gamma = \pi + \rho_m - 2\rho_{m \cap t},
\]

where \(\rho_m \in \mathfrak{h}^*\) is half the sum of roots in \(\Delta(m, t) := \{\alpha \in \Delta(m, t) \mid \langle \alpha, \pi \rangle > 0\}\), and \(\rho_{m \cap t}\) is half the sum of compact roots in \(\Delta^+(m, t)\).

The set of regular characters of \(H\) is denoted by \(\widehat{H}'\).

Choose a regular character \(\gamma = (\Gamma, \pi) \in \widehat{H}'\). There exists a discrete series module \(\sigma\) of \(M\) such that the highest weight of its minimal \(M \cap K\)-type is \(\Gamma|_\tau\). Put \(\nu := [\pi]|_\tau\). Let \(P = MAN\) a parabolic subgroup of \(G\) with Levi factor \(MA\), and let \(\rho\) be half the sum of roots in \(\Delta(n, a)\). Denote by \(X(\gamma)\) the \((g, K)\)-modules of
Ind$^G_P(σ ⊗ e^{ν+ρ} ⊗ 1)$, and call it the standard modules with parameter $(H, γ)$. Note that this module admits the infinitesimal character $γ$. Since we only treat $(g, K)$-modules with nonsingular infinitesimal characters in this paper, we assume that $γ$ is nonsingular, for simplicity. If we choose $N$ to be negative with respect to $ν$, then $X(γ)$ has a unique irreducible submodule. We call it the Langlands submodule of $X(γ)$ and we denote it by $X(γ)$.

**Theorem 2.1 (Langlands classification).** Let $Λ$ be a nonsingular infinitesimal character. Define $\hat{H}'_Λ$ to be the set of regular characters $γ = (Γ, γ) ∈ \hat{H}'$ which satisfies $χ_γ = χ_Λ$.

1. For an irreducible $(g, K)$-module $π$ with the infinitesimal character $Λ$, there exists a Cartan subgroup $H$ of $G$ and a regular character $γ ∈ \hat{H}'_Λ$ such that $π$ is isomorphic to $X(γ)$.

2. If $X(γ_1) ∼ X(γ_2)$, $γ_i ∈ (H_i)'_Λ$ $(i = 1, 2)$, then $(H_1, γ_1)$ and $(H_2, γ_2)$ are conjugate under the action of $K$.

### 3. Outline of computation

Most of this paper consists of direct computation, so it may be helpful for readers to write the outline of computation here.

Firstly, we divide the set of irreducible modules into blocks ([18, Definition 9.2.1]). Block equivalence of irreducible $(g, K)$-modules is the equivalence relation generated by

$$X \sim Y \iff \text{Ext}^1_{g, K}(X, Y) \neq 0.$$  

The equivalence classes are called blocks. By this definition, we may restrict our consideration within a block.

Secondly, by the translation principle, we may restrict our interest in the case of a special infinitesimal character, especially in the case when the infinitesimal character is trivial.

The most important part of our computation is to seek the candidates for the irreducible submodules of principal series modules in question. This is done by use of shift operators of $K$-types. We explain this part in detail. The KLV-conjecture tells us the composition factors of a principal series module. By the Blattner formula of $K$-spectra of discrete series modules and the $K$-spectra of principal series modules, we can compute the $K$-spectra of all irreducible modules. Suppose that $τ_1$ is a $K$-type of an irreducible $(g, K)$-module $π$, but another irreducible $K$-representation $τ_2$ is not. Suppose moreover that there exists a shift operator of $K$-types $P$ in a principal series module, which sends elements of $τ_1$ to $τ_2$. If $π$ is in the socle of this principal series module, then the kernel of $P$ is non-trivial. In this way, we determine the candidates for irreducible factors in the socle of the principal series module. This method is used for the determination of the embedding of discrete series modules into induced modules in [21]. We apply this method to other irreducible modules.

The shift operators of $K$-types are used also for the determination of the socle filtration. Suppose that a non-zero vector $v_1$ is known to be contained in an irreducible factor $V_1$, and another non-zero vector $v_2$ is known to be contained in an irreducible factor $V_2$ but not in $V_1$. If there exists a shift operator of $K$-types which sends $v_1$ to $v_2$, then the irreducible factor $V_1$ lies in a floor higher than $V_2$. 
Given a vector of the principal series module, it is in general hard to specify the irreducible factor in which this vector is contained, but in some cases it is easy. For example, if the multiplicity of a $K$-type $\tau$ of a principal series module in question is one, we can specify the irreducible factor in which this $K$-type $\tau$ is contained from the information of $K$-spectra of irreducible factors.

We also use some known facts on the structure of $(\mathfrak{g}, K)$-modules. These are summarized in the next section.

We explain the shift operator of $K$-types briefly. Let $(\tau, V^K_\tau)$ be an irreducible representation of $K$, and $(\tau^*, (V^K_\tau)^*)$ be its contragredient representation. There is a natural identification

$$\text{Hom}_K(\tau, I(\sigma, \nu)) \simeq C^\infty(K \backslash G/M_m A_m N_m; \sigma \otimes e^{\nu+\rho_m})$$

$$:= \{ \Phi : G \xrightarrow{C^\infty} (V^K_\tau)^* \otimes V^M_\sigma \mid \Phi(kgman) = a^{-\nu-\rho_m}\tau^*(k) \otimes \sigma(m)^{-1} \Phi(g) \quad k \in K, g \in G, m \in M_m, a \in A_m, n \in N_m \}.$$ 

The correspondence is given by

$$\varphi(v)(g) = \langle \Phi(g), v \rangle, \quad v \in V^K_\tau,$$

where $\varphi \in \text{Hom}_K(\tau, I(\sigma, \nu))$, $\Phi \in C^\infty(K \backslash G/M_m A_m N_m; \sigma \otimes e^{\nu+\rho_m})$ and $\langle , \rangle$ is the paring of $(V^K_\tau)^*$ and $V^K_\tau$.

Let $\{ X_i \}$ be a basis of $\mathfrak{s}$ and let $\{ X^i \}$ be the dual basis of $\mathfrak{s}$ with respect to a non-degenerate invariant bilinear form on $\mathfrak{g}$. The adjoint representation of $K$ on $\mathfrak{s}$ is denoted by $(\text{Ad}_s, \mathfrak{s})$. Define a $K$-equivariant map

$$\nabla : C^\infty(K \backslash G/M_m A_m N_m; \sigma \otimes e^{\nu+\rho_m}) \to C^\infty (K \backslash G/M_m A_m N_m; \sigma \otimes e^{\nu+\rho_m}),$$

by

$$\nabla \Phi(g) := \sum_i L(X_i) \Phi(g) \otimes X^i.$$ 

Here, $L(\ast)$ denotes the left translation.

Let $\Delta(\mathfrak{s}, t)$ be the weight space of the adjoint representation $\text{Ad}_s$ with respect to a Cartan subalgebra $t$ of $\mathfrak{k}$. For the groups $G = \text{SL}(3, \mathbb{R})$ and $Sp(2, \mathbb{R})$, the multiplicities of each weights are all one. If $\tau = \tau_\lambda$, namely if the highest weight of $\tau$ is $\lambda$, then the irreducible decomposition of $\tau_\lambda \otimes \text{Ad}_s$ is given by

$$\tau_\lambda \otimes \text{Ad}_s \simeq \bigoplus_{\alpha \in \Delta(\mathfrak{s}, t)} m(\alpha) \tau_{\lambda + \alpha}, \quad m(\alpha) = 0 \text{ or } 1.$$ 

Let

$$\text{pr}_\alpha : (\tau_\lambda)^* \otimes \text{Ad}_s \to (\tau_{\lambda + \alpha})^*$$

be the natural projection along this decomposition. Then

$$P_\alpha := \text{pr}_\alpha \circ \nabla : \text{Hom}_K(\tau_\lambda, I(\sigma, \nu)) \simeq C^\infty(\tau_\lambda, (K \backslash G/M_m A_m N_m; \sigma \otimes e^{\nu+\rho_m})$$

$$\to C^\infty(\tau_{\lambda + \alpha}; (K \backslash G/M_m A_m N_m; \sigma \otimes e^{\nu+\rho_m}) \simeq \text{Hom}_K(\tau_{\lambda + \alpha}, I(\sigma, \nu))$$

is a $K$-equivariant map obtained from the action of $\mathfrak{g}$ on $I(\sigma, \nu)$. We call these operators the shift operators of $K$-types.
4. Other tools

Our direct method explained in the last section is not sufficient for our purpose. We use several known results on the representations of real reductive groups.

4.1. Dual principal series. It is well known that there is a non-degenerate invariant pairing

$$I(\sigma, \nu) \times I(\sigma^*, -\nu) \to \mathbb{C}, \quad \langle f_1, f_2 \rangle = \int_K f_1(k) f_2(k) \, dk.$$ 

By this, we can compare the socle filtrations of these principal series modules.

4.2. Integral intertwining operators between principal series. We also use the well known integral intertwining operators between principal series modules and their factorizations ([14]). Fix a minimal parabolic subgroup $P_m = M_m A_m N_m$. Let $\Sigma^+$ be the positive system of the root system $\Sigma(g_0, a_0)$ corresponding to $N_m$. Choose $\nu \in a^*$ so that $\text{Re} \nu$ is positive with respect to $\Sigma^+$. Denote by $w \circ$ the longest element of the Weyl group $W(G, A_m)$ with respect to this positive system. Let $w \circ = r_1 r_2 \cdots r_\ell$ be a reduced expression of $w \circ$, and put $w_k := r_k \cdots r_\ell$, $k = 1, 2, \ldots, \ell$. Then, there are series of intertwining operators

$$I(\sigma, \nu) \to I(w_\ell \cdot (\sigma, \nu)) \to I(w_{\ell-1} \cdot (\sigma, \nu)) \to \cdots$$

$$\to I(w_k \cdot (\sigma, \nu)) \to \cdots \to I(w^\circ \cdot (\sigma, \nu)).$$

It is known that the composition of these operators is not zero.

These operators help us in some cases. For example, suppose that the socles of $I(w_k \cdot (\sigma, \nu))$ and $I(w_{k-1} \cdot (\sigma, \nu))$ are known to be identical, and they consist of one irreducible factor. Moreover, suppose that the multiplicity of this irreducible factor in these principal series is one. Then, since there exists a non-trivial intertwining operator between these principal series modules, they are isomorphic.

4.3. Horizontal symmetry. In some case, there is a symmetric structure in the socle filtration of a principal series module.

**Theorem 4.1.** (Vogan, Borel-Wallach, [3 Chapter I, Corollary 7.5]). Let $G$ be a connected real semisimple linear Lie group whose complexification is simply connected. Then there exists an element $\mu$ of $\text{Aut} G$ which satisfies the following properties:

1. $\mu(K) = K$,
2. For any irreducible admissible $(g, K)$-module $(\pi, V)$, the twisted module $(\pi^\mu, V^\mu) := (\pi \circ \mu^{-1}, V)$ is isomorphic to the contragredient $(g, K)$-module $(\pi^*, V^*)$.

In the setting of Theorem 4.1, if $\mu$ stabilizes $A_m$, then $I(\sigma, \nu)^\mu$ is also a principal series module. So if we know the structure of $I(\sigma, \nu)$, then that of $I(\sigma, \nu)^\mu$ is determined. This is a matter of course. But in a special situation, this theorem implies a symmetric structure in $I(\sigma, \nu)$.

In order to state it simply, we need a definition.

**Definition 4.2.** Let $\pi_1$ and $\pi_2$ be admissible $(g, K)$-modules with nonsingular infinitesimal characters. These $(g, K)$-modules are called quasi-isomorphic if they lie in the same coherent family and lie in the same open Weyl chamber. The module $\pi_2$ is quasi-dual to $\pi_1$ if $\pi_2$ is quasi-isomorphic to the contragredient module of $\pi_1$. 
Corollary 4.3 (Horizontal symmetry). In the setting of Theorem 4.1, suppose that $I(\sigma, \nu)^\mu$ is quasi-isomorphic to $I(\sigma, \nu)$. Then an irreducible factor $\pi$ of $I(\sigma, \nu)$ and its quasi-dual module $\pi'$, with the same infinitesimal character as that of $\pi$, appear in $I(\sigma, \nu)$ as a pair $\pi \oplus \pi'$. In other words, they appear in the same floor of the socle filtration of $I(\sigma, \nu)$.

Proof. Put $I(\sigma', \nu') \simeq I(\sigma, \nu)^\mu$. In the situation of this corollary, $\psi_{\nu, \nu'} I(\sigma', \nu') \simeq I(\sigma, \nu)$, where $\psi_{\nu, \nu'}$ is the translation functor. By Theorem 4.1, if $\pi$ is in the $k$-th floor of the socle filtration of $I(\sigma, \nu)$, so is the contragredient module $\pi^*$ of $\pi$ in that of $I(\sigma', \nu')$. By the translation principle, $\pi'$ is in the $k$-th floor of $I(\sigma, \nu)$.

4.4. Parity of length. In [18, Definition 8.1.4], Vogan defined the (integral) length $\ell(\gamma)$ of a regular character $\gamma$. The following theorem is a consequence of KLV-conjecture.

Theorem 4.4. (Vogan, [18, Theorem 9.5.1]). Let $\gamma_1, \gamma_2$ be regular characters of the same nonsingular infinitesimal character. Suppose that they are not conjugate under $K$. Then

$$\text{Ext}^1_{g, K}(\mathcal{X}(\gamma_1), \mathcal{X}(\gamma_2)) \neq 0$$

only if

$$\ell(\gamma_1) - \ell(\gamma_2) \equiv 1 \pmod{2}.$$

This theorem enables us to narrow down the candidates for irreducible factors lying in each floor. We use it in a situation as below.

Corollary 4.5 (Parity condition). Suppose that $V$ is a $(\mathfrak{g}, K)$-module of finite length. If the lengths of the irreducible factors in the $k$-th floor of the socle filtration of $V$ are all even (resp. odd), then those of the factors in $(k+1)$-st floor are all odd (resp. even).

5. Finite dimensional representations of $\mathfrak{sl}_2$

In order to write down the shift operator of $K$-types (Section 3) explicitly, it is needed to write the irreducible decomposition of the tensor product $\tau_\lambda \otimes \text{Ad}_s$. Since the maximal compact subgroups $K$ of $G = SL(3, \mathbb{R})$ and $Sp(2, \mathbb{R})$ are $SO(3)$ and $U(2)$, respectively, we need the irreducible decomposition of the tensor product of representations of $\mathfrak{sl}_2$.

Let $\{H, E, F\}$ be the standard $\mathfrak{sl}_2$-triple, namely they satisfy

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

For $\lambda \in \mathbb{Z}_{\geq 0}$, let

$$\{v^\lambda_q | -\lambda \leq q \leq \lambda, \lambda - q \in \mathbb{Z}\}$$

be the basis of the irreducible representation of $\mathfrak{sl}_2$ with the highest weight $2\lambda$, which satisfies

$$Hv^\lambda_q = 2q v^\lambda_q, \quad Ev^\lambda_q = (\lambda - q) v^\lambda_{q+1}, \quad Fv^\lambda_q = (\lambda + q) v^\lambda_{q-1}.$$

Note that the highest weight is not $\lambda$ but $2\lambda$. For later purpose, we need the actions of some elements in $SU(2)$. 
Lemma 5.1. For \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU(2), \)
\[
\tau_{2\lambda}(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) v^\lambda_q = a^{2q} v^\lambda_q, \quad \tau_{2\lambda}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) v^\lambda_q = (-1)^{\lambda-q} v^\lambda_{-q}.
\]

Lemma 5.2 (Irreducible decomposition of \( \tau_{2\lambda} \otimes \tau_{2\mu} \)). For \( \lambda, \mu \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) with \( \lambda \geq \mu \), the irreducible decomposition of the tensor product \( (\tau_{2\lambda}, V^\xi_{2\lambda}) \otimes (\tau_{2\mu}, V^\xi_{2\mu}) \) is given by
\[
v^\lambda_q \otimes v^\mu_r = \sum_{|j| \leq \mu, \mu-j \in \mathbb{Z}} c^{\lambda,\mu}(q, i; j) v^{\lambda^b\mu^b}_{q+j},
\]
where
\[
c^{\lambda,\mu}(q, i; j) = \frac{(\mu + i)!}{(2\mu)!} \prod_{p=0}^{\mu-i} \left( \frac{\mu - i}{p} \right) \prod_{a=0}^{\mu-1} (j + a) (\lambda + q)! \frac{(\lambda - q)!}{(\lambda + q - p)! (\lambda - q + j + p)!}.
\]

6. Irreducible \((g, K)\)-modules of \( SL(3, \mathbb{R}) \)

From this section to section \[\[11\] we set \( G = SL(3, \mathbb{R}) \) and investigate the socle filtrations of the principal series modules for \( G \). In this section, we review the classification of the irreducible \((g, K)\)-modules of \( G \).

\( G = SL(3, \mathbb{R}) \) has two conjugacy classes of Cartan subgroups. One is split and the other is fundamental. Let \( H_s = M_m A_m \) be the split Cartan subgroup, where
\[
A_m := \{ \text{diag}(a_1, a_2, a_3) \in G \mid a_i > 0 \},
M_m := Z_K(A_m) = \{ I, m_{1,1}^{sl}, m_{1,-1}^{sl}, m_{-1,1}^{sl} \}, \quad m_{i,j}^{sl} := \text{diag}(i, j, ij).
\]
The Weyl group \( W(G, H_s) \) is isomorphic to \( W(g, h_s) \cong \mathfrak{S}_3 \), which acts on \( H_s \) by the permutation
\[
(6.1) \quad s \cdot \text{diag}(h_1, h_2, h_3) = \text{diag}(h_{s^{-1}(1)}, h_{s^{-1}(2)}, h_{s^{-1}(3)}),
\]
\[
s \in \mathfrak{S}_3, \quad \text{diag}(h_1, h_2, h_3) \in H_s.
\]

Let \( \hat{M}_m \) be the set of equivalence classes of irreducible representations of \( M_m \). Define \( \sigma_{i,j} \in \hat{M}_m, i, j \in \mathbb{Z}, \) by
\[
\sigma_{i,j}(m_{1,1}^{sl}) = (-1)^i, \quad \sigma_{i,j}(m_{-1,-1}^{sl}) = (-1)^j.
\]
Then \( \hat{M}_m \) consists of four elements \( \sigma_{i,j}, i, j \in \{0, 1\} \). The action of \( W(g, h_s) \cong \mathfrak{S}_3 \) on \( \hat{M}_m \) is given by
\[
(6.2) \quad r_{1,2} \cdot \sigma_{i,j} = \sigma_{j,i}, \quad r_{2,3} \cdot \sigma_{i,j} = \sigma_{i+j,j},
\]
where, \( r_{p,q} \) is the permutation of \( p \) and \( q \).

Let \( f_i \) be the elements of \( \mathfrak{h}_s^* \) defined by
\[
(6.3) \quad f_i(\text{diag}(x_1, x_2, x_3)) = x_i, \quad \text{diag}(x_1, x_2, x_3) \in \mathfrak{h}_s.
\]
Then the root system \( \Delta(g, h_s) \) is \( \{ f_i - f_j \mid 1 \leq i \neq j \leq 3 \} \).

In this paper, we specify regular characters by the numbers used in ATLAS \[\[11\].\]
Let
\[
\Lambda = \sum_{i=1}^{3} \Lambda_i f_i, \quad \Lambda_1 - \Lambda_2, \: \Lambda_2 - \Lambda_3 \in \mathbb{Z}_{>0}
\]
be a nonsingular integral infinitesimal character. Then there are four conjugacy classes of regular characters of \(H_m\) with the infinitesimal character \(\Lambda\). These four correspond to the ATLAS number "3", "4", "5" in the block \(PU(2,1)\) and "0" in the block \(PU(3)\). According to these numbers, we define a regular character \(\gamma_3, \gamma_4, \gamma_5\) and \(\gamma_0'\) by
\[
\begin{align*}
\gamma_3 &= (\sigma_{-1} - \Lambda_2 - \Lambda_3, \Lambda_1, \Lambda_2, \Lambda_3) \sim (\sigma_{-1} - \Lambda_2 - \Lambda_3, \Lambda_2, \Lambda_1, \Lambda_3) \\
\gamma_4 &= (\sigma_{-1} - \Lambda_2 - \Lambda_3, \Lambda_2, \Lambda_3, \Lambda_1) \sim (\sigma_{-1} - \Lambda_2 - \Lambda_3, \Lambda_3, \Lambda_2, \Lambda_1) \\
\gamma_5 &= (\sigma_{-1} - \Lambda_2 - \Lambda_3, \Lambda_3, \Lambda_2, \Lambda_1) \sim (\sigma_{-1} - \Lambda_2 - \Lambda_3, \Lambda_1, \Lambda_2, \Lambda_3) \\
\gamma_0' &= (\sigma_{-1} - \Lambda_3, \Lambda_2, \Lambda_3, \Lambda_1)
\end{align*}
\]
Here, \(\sim\) means the \(K\)-conjugacy, and we denoted regular characters in such a way as \(\gamma = (\sigma, \nu) \in \mathcal{M}_m \times \mathfrak{m}_m\), since a regular character of \(H_m\) is determined by its restriction to \(M_m\) and \(\mathfrak{m}_m\). Such notation will be applied to regular characters of other Cartan subgroups. The lengths of these regular characters are all two.

Next, consider the fundamental Cartan subgroup. Let
\[
H_f = T_f A_f,
\]
\[
T_f := \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \right\}, \quad A_f := \{ \text{diag}(a, a^{-2}) \mid a > 0 \}
\]
be a fundamental Cartan subgroup of \(G\). Define \(L_f\) to be the centralizer of \(A_f\) in \(G\). Then
\[
L_f = \left\{ \begin{pmatrix} A & \ast \\ \ast & (\det A)^{-1} \end{pmatrix} \right\}, \quad A \in GL(2, \mathbb{R}).
\]
Therefore, \(L_f = M_f A_f\), where
\[
M_f = \left\{ \begin{pmatrix} A & \ast \\ \ast & (\det A)^{-1} \end{pmatrix} \right\}, \quad A \in SL^+(2, \mathbb{R}).
\]
The order of the Weyl group \(W(G, H_f)\) is two, and this group is generated by \(\text{diag}(1, -1, -1)\). Since \(H_f\) is connected, there are \#\(W(g, h_f)/W(G, H_f)\) = 3 equivalence classes of irreducible \((g, K)\)-modules with a given integral infinitesimal character, which correspond to the ATLAS numbers "0", "1" and "2" in the block.
and \( A \) for the trivial infinitesimal character \( \Lambda = \rho \).

Here, we denoted the restriction of \( \gamma_i \) to \( T_f \simeq SO(2) \) by its differential. The length of \( \gamma_0 \) is zero and those of \( \gamma_1, \gamma_2 \) are one.

The composition factors of \( X(\gamma_i), \ i = 1, \ldots, 5, 0' \) are known by KLV-conjecture.

**Theorem 6.1.** In the Grothendieck group, \( X(\gamma_i), \ i = 1, \ldots, 5, 0' \), decomposes into irreducible modules as follows:

\[
\begin{align*}
X(\gamma_0) &= \overline{X}(\gamma_0), & X(\gamma_1) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_1), & X(\gamma_2) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_2), \\
X(\gamma_3) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_1) + \overline{X}(\gamma_2) + \overline{X}(\gamma_3), \\
X(\gamma_4) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_1) + \overline{X}(\gamma_4), & X(\gamma_5) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_2) + \overline{X}(\gamma_5), \\
X(\gamma_0') &= \overline{X}(\gamma_0').
\end{align*}
\]

For \( G = SL(3, \mathbb{R}) \), the automorphism \( \mu \) in Section 6.4 is given by the involution \( \mu : G \to G, \quad \mu(g) = t g^{-1} \).

This automorphism stabilizes the Cartan subgroups \( H_m \) and \( H_f \). For two regular characters \( \gamma \) and \( \gamma' \), we write \( \gamma \sim \gamma' \) if \( X(\gamma) \) and \( X(\gamma') \) are in the same coherent family. Then by easy calculation, \( \gamma_0 \circ \mu \sim \gamma_0, \gamma_1 \circ \mu \sim \gamma_2, \gamma_3 \circ \mu \sim \gamma_3 \) and \( \gamma_4 \circ \mu \sim \gamma_5 \). It follows that

**Lemma 6.2.** \( \overline{X}(\gamma_0) \) and \( \overline{X}(\gamma_3) \) are quasi-self dual. \( \overline{X}(\gamma_1) \) is quasi-dual to \( \overline{X}(\gamma_2) \), and \( \overline{X}(\gamma_4) \) to \( \overline{X}(\gamma_5) \).

Put \( \mu' = w \circ \mu (w \circ)^{-1} \), where \( w \circ \) is the longest element of the Weyl group \( W(G, A_m) \). This automorphism stabilizes \( K, M_m, A_m \) and \( N_m \) respectively. For \( \sigma_{i,j} \in M_m \) and \( \nu = (\nu_1, \nu_2, \nu_3) \in a^* \),

\[
\sigma_{i,j} \circ (\mu')^{-1} = \sigma_{i,j+1}, \quad \nu \circ (\mu')^{-1} = (-\nu_3, -\nu_2, -\nu_1) \in a^*.
\]

It follows that

\[
(6.8) \quad I(\sigma_{i,j}, (\nu_1, \nu_2, \nu_3))^\mu \simeq I(\sigma_{i,j+1}, (-\nu_3, -\nu_2, -\nu_1)).
\]

7. **K-types of the standard modules of \( SL(3, \mathbb{R}) \)**

For our calculation, we need the \( K \)-spectra of the irreducible modules \( \overline{X}(\gamma_i), \ i = 1, \ldots, 5 \). In fact, we only need the information of the minimal \( K \)-types of \( \overline{X}(\gamma_i), \ i = 0, 1, 2 \) for general \( \Lambda \) and that of the \( K \)-spectra of \( \overline{X}(\gamma_i), \ i = 0, 1, \ldots, 5 \), for the trivial infinitesimal character \( \Lambda = \rho_m = (1, 0, -1) \).

Before going ahead, we fix the identification \( \mathfrak{k} = \mathfrak{so}(3) \otimes \mathbb{C} \simeq \mathfrak{sl}_2 \).

Let \( A_{ij} = E_{ij} - E_{ji} \) and \( S_{ij} := E_{ij} + E_{ji} \). The commutation relations of them are \([A_{ij}, A_{ik}] = A_{ik}, [A_{ij}, S_{jk}] = S_{ik} \) for \( i \neq j \neq k \neq i \), \([A_{ij}, S_{ij}] = 2(E_{ii} - E_{jj}) \) and \([A_{ij}, E_{jj}] = S_{ij} \).

As a basis of \( \mathfrak{k} \), choose

\[
(7.1) \quad D := \sqrt{-1} A_{21}, \quad Y_+ := \sqrt{-1} A_{32} + A_{31}, \quad Y_- := \sqrt{-1} A_{32} - A_{31}.
\]
Then they satisfy \([2D, Y_+] = 2Y_+, [2D, Y_-] = -2Y_-\) and \([Y_+, Y_-] = 2D\), namely \([2D, Y_+, Y_-]\) is a standard \(\mathfrak{sl}_2\)-triple. Then, the irreducible representation \(V^\lambda_{SO(3)}\), where \(\lambda\) is the highest weight with respect to \(D\), is identified with \(V^\lambda_{\mathfrak{sl}_2}\) as a \(SO(3)\)-module. Under this identification \(\mathfrak{k} \simeq \mathfrak{sl}_2 = \mathfrak{su}(2) \otimes \mathbb{C}\), the surjection map \(SU(2) \rightarrow SO(3)\) sends the elements \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 &\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}\) in \(SU(2)\) to \(m_{-1,1}\) and \(m_{1,-1}\) in \(SO(3)\), respectively. It follows from Lemma 5.1 that the elements \(m_{-1,1}, m_{1,-1} \in M_m \subset SO(3)\) act on \(v_\lambda^m \in V^\lambda_{SO(3)}\) by

\[
(7.2) \quad \tau_\lambda(m_{-1,1})v_\lambda^m = (-1)^{\lambda - q}v_\lambda^{-q} \quad \text{and} \quad \tau_\lambda(m_{1,-1})v_\lambda^m = (-1)^{\lambda}v_\lambda^{-q}.
\]

Firstly, consider the standard modules \(X(\gamma_i), i = 0, 1, 2\). Let \(\pi_{SL^2}^{SO(3)}(\mu)\) be the \((g, K)\)-module of the discrete series representation of \(M_f \simeq SL^2(2, \mathbb{R})\) with the minimal \(K\)-type \(V_{O(2)}^\lambda\). Let

\[
P_f = L_f N_f = M_f A_f N_f
\]

be a parabolic subgroup of \(G\). We choose an appropriate basis of \(\mathfrak{a}_f\), and we identify an element of \(\mathfrak{a}_f\) with a complex number. Then, the standard module \(X(\gamma_i), i = 0, 1, 2\), is the space of \(K\)-finite vectors of

\[
\text{Ind}^G_{P_f}(\pi_{SL^2}^{SL^2}(\Lambda_p - \Lambda_q + 1) \otimes e^{\Lambda_r + 2\Lambda_r + 3} \otimes 1), \quad (p, q, r) = \begin{cases} (1, 3, 2) & (i = 0), \\ (2, 3, 1) & (i = 2). \end{cases}
\]

From this, we know that the \(K \cap M_f = O(2)\)-spectrum of \(\pi_{SL^2}^{SL^2}(\mu)\) is

\[
\{V_{O(2)}^{\mu+2k} \mid k = 0, 1, 2, \ldots\},
\]

and the multiplicities of these are all one. By Frobenius reciprocity, the \(K\)-spectra of \(X(\gamma_i), i = 0, 1, 2\), are

\[
(7.3) \quad \tilde{K}(X(\gamma_i)) = \mathbb{Z}_{\geq \Lambda_p - \Lambda_q + 1}, \quad m_\lambda = \left\lfloor \frac{\lambda - \Lambda_p + \Lambda_q + 1}{2} \right\rfloor,
\]

where, \((p, q) = (1, 3)\) for \(i = 0\), \((1, 2)\) for \(i = 1\) and \((2, 3)\) for \(i = 2\), respectively, and \(m_\lambda\) is the multiplicity of \(V^\lambda_{SO(3)}\) in \(X(\gamma_i)\). Note that the minimal \(K\)-types of \(X(\gamma_0), X(\gamma_1)\) and \(X(\gamma_2)\) are \(V^\lambda_{A_1 - A_3 + 1}, V^\lambda_{A_1 - A_3 + 1}\) and \(V^\lambda_{A_2 - A_3 + 1}\), respectively, and the multiplicities of them are all one.

For the standard modules \(X(\gamma_i), i = 3, 4, 5\), we need the information of \(K\)-types when the infinitesimal character is trivial. In this case,

\[
X(\gamma_i) = \text{Ind}^G_{P_{\gamma_i}}(e^{\rho_{\gamma_i}} \otimes 1), \quad (p, q) = \begin{cases} (0, 0) & (i = 3), \\ (1, 1) & (i = 4), \\ (1, 0) & (i = 5). \end{cases}
\]

Let us calculate the \(K = SO(3)\) spectra of these modules. By [22] and the Frobenius reciprocity, the \(K\)-spectra of \(X(\gamma_i), i = 3, 4, 5\), are as follows:

\[
\hat{K}(X(\gamma_3)) = \mathbb{Z}_{\geq 0}, \quad m_\lambda = \begin{cases} \frac{\lambda}{2} + 1 & (\text{if } \lambda \text{ is even}), \\ \frac{\lambda + 1}{2} & (\text{if } \lambda \text{ is odd}). \end{cases}
\]

\[
\hat{K}(X(\gamma_4)) = \hat{K}(X(\gamma_5)) = \mathbb{Z}_{\geq 1}, \quad m_\lambda = \begin{cases} \frac{\lambda}{2} & (\text{if } \lambda \text{ is even}), \\ \frac{\lambda + 1}{2} & (\text{if } \lambda \text{ is odd}). \end{cases}
\]
From these informations on the $K$-spectra of standard modules $X(\gamma_i)$, $i = 0, 1, \ldots, 5$, and Theorem 6.1 we obtain the following lemma.

**Lemma 7.1.**

1. If the nonsingular infinitesimal character is $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$, $\Lambda_1 - \Lambda_2, \Lambda_2 - \Lambda_3 \in \mathbb{Z}_{\geq 0}$, then the minimal $K$-types of the irreducible modules $X(\gamma_i)$, $i = 0, 1, 2$, are $\Lambda_1 - \Lambda_3 + 1$, $\Lambda_1 - \Lambda_2 + 1$ and $\Lambda_2 - \Lambda_3 + 1$, respectively.
2. If the infinitesimal character $\Lambda$ is trivial, namely $\Lambda = \rho_m = (1, 0, -1)$, then the multiplicities of small $K$-types in the irreducible modules are as follows:

| $K$-type $\lambda$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-------------------|---|---|---|---|---|---|
| $X(\gamma_0)$    | 0 | 0 | 0 | 1 | 1 | 2 |
| $X(\gamma_1)$    | 0 | 0 | 1 | 0 | 0 | 1 |
| $X(\gamma_2)$    | 0 | 0 | 1 | 0 | 1 | 0 |
| $X(\gamma_3)$    | 1 | 0 | 0 | 0 | 0 | 0 |
| $X(\gamma_4)$    | 0 | 1 | 0 | 1 | 0 | 1 |
| $X(\gamma_5)$    | 0 | 1 | 0 | 1 | 0 | 1 |

Here, each row exhibits the multiplicities of $V_{\lambda}^{SO(3)}$ in the irreducible module $X(\gamma_i)$.

Note that, if $\Lambda = \rho_m$, the irreducible module $X(\gamma_3)$ is the trivial representation, which is isomorphic to $V_{0}^{SO(3)}$ as a $K$-module, and the minimal $K$-types of $X(\gamma_i)$, $i = 4, 5$, are $V_{1}^{SO(3)}$ with multiplicity one.

8. Shift operators for $SL(3, \mathbb{R})$

In this section, we write down the shift operators of $K$-types explicitly, in the case $G = SL(3, \mathbb{R})$. Choose $A_m$ as in Section 3. Define

$$N_m = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} \\ x_{12} & 1 & x_{23} \\ 1 & 1 & 1 \end{pmatrix} \mid x_{ij} \in \mathbb{R} \right\}.$$

Then $G = KA_m N_m$ is an Iwasawa decomposition of $G$.

The adjoint representation $(s, \text{Ad}_s)$ of $K$ is irreducible and $s \simeq V_{2}^{SO(3)} = V_{4}^{sl_2}$. By (7.1), this isomorphism is given by

$$v_{\pm 2}^2 = E_{11} - E_{22} \pm \sqrt{-1} S_{12} = E_{11} - E_{22} \pm D \pm 2\sqrt{-1} E_{12},$$

$$v_{\pm 1}^2 = \frac{1}{2}(\mp S_{13} - \sqrt{-1} S_{23}) = -\frac{1}{2} S_{13} \mp E_{13} - \sqrt{-1} E_{23},$$

$$v_0^2 = -\frac{1}{3}(E_{11} + E_{22} - 2E_{33}).$$

Define an invariant bilinear form $\langle , \rangle$ on $g$ by $\langle X, Y \rangle = \text{tr}(XY)$. Then

$$\langle v_2^2, v_2^2 \rangle = 4, \quad \langle v_1^2, v_1^2 \rangle = -1, \quad \langle v_0^2, v_0^2 \rangle = \frac{2}{3}.$$

The operator $\nabla$ defined in Section 3 is given by

$$4\nabla \phi := L(E_{11} - E_{22} - \sqrt{-1} S_{12}) \phi \otimes v_2^2 - 2L(S_{13} - \sqrt{-1} S_{23}) \phi \otimes v_1^2 - 2L(E_{11} + E_{22} - 2E_{33}) \phi \otimes v_0^2 + 2L(S_{13} + \sqrt{-1} S_{23}) \phi \otimes v_{-1}^2 + L(E_{11} - E_{22} + \sqrt{-1} S_{12}) \phi \otimes v_{-2}^2.$$. 
For $(\tau_\lambda, V^\SO_\lambda(3)) \in \hat{K}$ and $\nu = (\nu_1, \nu_2, \nu_3) \in \mathfrak{a}_m$, we define

\[
C^\infty_{(\tau_\lambda)} (K \backslash G/A_m N_m; e^{\nu + \rho_m}) = \{ \phi_\lambda : G \to V^{\tau_\lambda}_\lambda \mid \phi_\lambda(kan) = a^{-\nu - \rho_m}(\tau_\lambda)^*(k) \phi_\lambda(e), \ k \in K, a \in A_m, n \in N_m \}.
\]

Note that we may and do replace $(\tau_\lambda)^*$ by $\tau_\lambda$, since $(\tau_\lambda, V^\SO_\lambda(3))$ is self-dual. For $j = -2, -1, 0, 1, 2$, let $\text{pr}_j$ be the natural projection

\[
V^\SO(3)_\lambda \otimes \mathfrak{g} \simeq \bigoplus_{j=-2}^{2} V^\SO(3)_{\lambda+j} \to V^\SO(3)_{\lambda+j}.
\]

We define the shift operators of $K$-types $P_j$ ($j = -2, -1, 0, 1, 2$) by

\[
P_j = \text{pr}_j \circ \nabla : C^\infty_{\tau_\lambda} (K \backslash G/A_m N_m; e^{\nu + \rho_m}) \to C^\infty_{\tau_\lambda+j} (K \backslash G/A_m N_m; e^{\nu + \rho_m}).
\]

For $\phi \in C^\infty_\tau (K \backslash G/A_m N_m; e^{\nu + \rho_m})$, the action of elements of Lie algebra is as follows;

\[
(L(E_i)\phi)(e) = (\epsilon_i, \nu + \rho_m)\phi(e),
\]

\[
(L(X)\phi)(e) = -\tau_\lambda(X)\phi(e) \quad (X \in \mathfrak{k}),
\]

\[
(L(Y)\phi)(e) = 0 \quad (Y \in \mathfrak{n}).
\]

Put

\[
\phi_\lambda(kan) = a^{-\nu - \rho_m} \tau_\lambda(k)^{-1} \sum_{q=-\lambda}^{\lambda} c(q) v^\lambda_q \in C^\infty_{\tau_\lambda} (K \backslash G/A_m N_m; e^{\nu + \rho_m}), \quad c(q) \in \mathbb{C}.
\]

For notational convenience, we regard $c(q)$ to be zero if $|q| > \lambda$. For $\alpha \in \mathbb{C}$ and non-negative integer $k$, we denote

\[
(\alpha)_{\nu(k)} := \alpha(\alpha - 1) \cdots (\alpha - k + 1) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - k + 1)}.
\]

The shift operators of $K$-types for $G = SL(3, \mathbb{R})$ are as follows.

\[
(P_2 \phi_\lambda)(e) = \sum_{q=-\lambda-2}^{\lambda+2} \left\{ (\nu_1 - \nu_2 + q - 1) c(q - 2) - 2(\nu_1 + \nu_2 - 2\nu_3 + 2\lambda + 3) c(q) + (\nu_1 - \nu_2 - q - 1) c(q + 2) \right\} v^\lambda_q,
\]

(8.1)

\[
(P_1 \phi_\lambda)(e) = \sum_{q=-\lambda-1}^{\lambda+1} \left\{ (\lambda - q + 2)(\nu_1 - \nu_2 + q - 1) c(q - 2) + 2q(\nu_1 + \nu_2 - 2\nu_3 + \lambda + 1) c(q) - (\lambda + q + 2)(\nu_1 - \nu_2 - q - 1) c(q + 2) \right\} v^\lambda_q,
\]

(8.2)
and (8.5), the system of equations

By replacing the parameter $\lambda$

$-\frac{2}{3} (3q^2 - \lambda^2 - \lambda) (\nu_1 + \nu_2 - 2\nu_3) c(q)
+ (\lambda + q + 2)_4 (\nu_1 - \nu_2 - q - 1) c(q + 2) \right\} v_q^\lambda,

(P_{-1}\phi\lambda)(e) = \sum_{q=-\lambda+1}^{\lambda-1} \left\{ (\lambda - q + 2)_4 (\nu_1 - \nu_2 + q - 1) c(q - 2)
- 2q(\lambda - q)(\lambda + q) (\nu_1 + \nu_2 - 2\nu_3 - \lambda) c(q)
- (\lambda + q + 2)_4 (\nu_1 - \nu_2 - q - 1) c(q + 2) \right\} v_q^{\lambda - 1},

(P_{-2}\phi\lambda)(e) = \sum_{q=-\lambda+2}^{\lambda-2} \left\{ (\lambda - q + 2)_4 (\nu_1 - \nu_2 + q - 1) c(q - 2)
- 2(\lambda + q)_4 (\lambda - q)_4 (\nu_1 + \nu_2 - 2\nu_3 - 2\lambda + 1) c(q)
+ (\lambda + q + 2)_4 (\nu_1 - \nu_2 - q - 1) c(q + 2) \right\} v_q^{\lambda - 2}.

9. Solutions of the system of equations $P_{-1}\phi\lambda = 0, P_{-2}\phi\lambda = 0$

We will solve the system of equations $P_{-1}\phi\lambda(e) = 0, P_{-2}\phi\lambda(e) = 0$. By (8.4) and (8.5), the system of equations $P_{-1}\phi\lambda(e) = 0, P_{-2}\phi\lambda(e) = 0$ is equivalent to

(9.1) 
$(\lambda - q + 2)_4 (\nu_1 - \nu_2 + q - 1) c(q - 2)
- (\lambda + q)_4 (\nu_1 + \nu_2 - 2\nu_3 - 2\lambda + q + 1) c(q) = 0,
(\lambda + q + 2)_4 (\nu_1 - \nu_2 - q - 1) c(q + 2) = 0

(\nu_1 + \nu_2 + q - 1)$.

By replacing the parameter $q$ of (9.2) by $q - 2$, we get

(9.3) 
$(\lambda - q + 2)_4 (\nu_1 + \nu_2 - 2\nu_3 - 2\lambda + q - 3) c(q - 2)
- (\lambda + q)_4 (\nu_1 - \nu_2 - q + 1) c(q) = 0
(\lambda + q + 2)_4 (\nu_1 - \nu_2 - 2\nu_3 - 2\lambda + q + 1) c(q + 2) = 0

By replacing the parameter $q$ of (9.2) by $q - 2$, we get

(9.3) 
$(\lambda - q + 2)_4 (\nu_1 + \nu_2 - 2\nu_3 - 2\lambda + q - 3) c(q - 2)
- (\lambda + q)_4 (\nu_1 - \nu_2 - q + 1) c(q) = 0
(\lambda + q + 2)_4 (\nu_1 - \nu_2 - 2\nu_3 - 2\lambda + q + 1) c(q + 2) = 0

Suppose $-\lambda + 3 \leq \lambda - 1$, i.e. $\lambda \geq 2$. If

$\begin{vmatrix} \nu_1 + \nu_2 - 2\nu_3 - 2\lambda + q + 1 \\ \nu_1 - \nu_2 - q + 1 \end{vmatrix}
= 4(\nu_1 - \nu_3 + 1)(\nu_2 - 3) \neq 0,$

namely if $\lambda \neq \nu_1, -\nu_3 + 1$ and $\lambda \neq \nu_2 - \nu_3 + 1$, then $c(q) = c(q - 2) = 0$ for $q = -\lambda + 3, -\lambda + 4, \ldots, \lambda - 1$. Therefore, if $\lambda \geq 3$, then $c(q) = 0$ for $-\lambda + 1 \leq q \leq \lambda - 1$, and if $\lambda = 2$, then $c(1) = c(-1) = 0.$
Suppose first that \( \lambda \geq 3 \), \( \lambda \neq \nu_1 - \nu_3 + 1 \) and \( \lambda \neq \nu_2 - \nu_3 + 1 \). In this case,
\[
(\nu_1 - \nu_2 - \lambda + 1) c(-\lambda) = (\nu_1 - \nu_2 - \lambda + 1) c(\lambda) = 0.
\]
So, if \( \lambda \neq \nu_1 - \nu_2 + 1 \), then \( c(\lambda) = c(-\lambda) = 0 \), and if \( \lambda = \nu_1 - \nu_2 + 1 \), then \( c(\lambda) \) and \( c(-\lambda) \) are arbitrary.

Secondly, suppose that \( \lambda = 2 \), \( \lambda \neq \nu_1 - \nu_3 + 1 \) and \( \lambda \neq \nu_2 - \nu_3 + 1 \). In this case, 
\[
c(-1) = c(1) = 0, \quad c(2), c(0), c(-2)
\]
satisfies
\[
(\nu_1 + \nu_2 - 2\nu_3 - 3) c(0) - 6(\nu_1 - \nu_2 - 1) c(-2) = 0,
(\nu_1 + \nu_2 - 2\nu_3 - 3) c(0) - 6(\nu_1 - \nu_2 - 1) c(2) = 0.
\]
Since we are assuming \( 2 = \lambda \neq \nu_2 - \nu_3 + 1 \), if \( \nu_1 - \nu_2 = 1 \), then \( \nu_1 + \nu_2 - 2\nu_3 - 3 = (\nu_1 - \nu_2 - 1) + 2(\nu_2 - \nu_3 - 1) \neq 0 \). Therefore, \( c(0) = 0 \) and \( c(2), c(-2) \) are arbitrary.

Note that the condition \( \nu_1 - \nu_2 = 1 \) is equivalent to \( \lambda = 2 = \nu_1 - \nu_2 + 1 \), and this result is the same as that of the above case for \( \lambda \geq 3 \) and \( \lambda = \nu_1 - \nu_2 + 1 \). If \( \nu_1 - \nu_2 \neq 1 \), namely \( \lambda = 2 \neq \nu_1 - \nu_2 + 1 \), then \( c(2) = c(-2) = \frac{1}{6} \frac{\nu_1 - \nu_2 - 2\nu_3 - 3}{\nu_1 - \nu_2 - 1} c(0) \).

We obtained

**Proposition 9.1.** Suppose \( \lambda \neq \nu_1 - \nu_3 + 1 \) and \( \lambda \neq \nu_2 - \nu_3 + 1 \).

1. If \( \lambda = \nu_1 - \nu_2 + 1 \), then the solution space of \( P_{-1}\phi_\lambda(e) = 0 \) and \( P_{-2}\phi_\lambda(e) = 0 \)
is spanned by \( v_\lambda^i + (-1)^i v_\lambda^{\ast i} \), \( i = 0, 1 \). These vectors are contained in \( V_{\nu_1 - \nu_2}^{SO(3)}(\sigma_{i,\lambda+1}) \), respectively.
2. If \( \lambda \geq 3 \) and \( \lambda \neq \nu_1 - \nu_2 + 1 \), then the solution space of \( P_{-1}\phi_\lambda(e) = 0 \) and \( P_{-2}\phi_\lambda(e) = 0 \) is \{0\}.
3. If \( \lambda = 2 \) and \( \lambda \neq \nu_1 - \nu_2 + 1 \), then the solution space of \( P_{-1}\phi_\lambda(e) = 0 \) and \( P_{-2}\phi_\lambda(e) = 0 \)
is spanned by \( \frac{1}{6} \frac{\nu_1 + \nu_2 - 2\nu_3 - 3}{\nu_1 - \nu_2 - 1} (v_2^2 + v_2^2) + v_0^2 \). This vector is contained in \( V_{\nu_1 - \nu_2}^{SO(3)}(\sigma_{0,0}) \).

9.1. The case \( \lambda = \nu_2 - \nu_3 + 1 \). Next, suppose that \( \lambda = \nu_2 - \nu_3 + 1 \). Then (9.1) is
\[
(\nu_1 - \nu_2 + q - 1) \times \{(\lambda - q + 2)_{\Delta(q)} c(q - 2) - (\lambda + q)_{\Delta(q)} c(q)\} = 0,
(-\lambda + 2 \leq q \leq \lambda - 1),
\]
and (9.3) is
\[
(\nu_1 - \nu_2 - q + 1) \times \{(\lambda - q + 2)_{\Delta(q)} c(q - 2) - (\lambda + q)_{\Delta(q)} c(q)\} = 0,
(-\lambda + 3 \leq q \leq \lambda).
\]
Since \( \nu \) is regular, \( \nu_1 \) and \( \nu_2 \) are different, so if \( \nu_1 - \nu_2 + q - 1 \) is zero, then \( \nu_1 - \nu_2 - q + 1 \) is not zero. It follows that (9.1) and (9.3) are equivalent to
\[
(\nu_1 - \nu_2 - \lambda + 1) \{1 \cdot 2 c(\lambda - 2) - 2\lambda(2\lambda - 1) c(\lambda)\} = 0,
(\lambda - q + 2)_{\Delta(q)} c(q - 2) = (\lambda + q)_{\Delta(q)} c(q), \quad (-\lambda + 3 \leq q \leq \lambda - 1),
(\nu_1 - \nu_2 - \lambda + 1) \{2\lambda(2\lambda - 1) c(-\lambda) - 1 \cdot 2 c(-\lambda + 2)\} = 0.
\]
Therefore, if \( \lambda \neq \nu_1 - \nu_2 + 1 \), then
\[
c(q - 2k) = \frac{(\lambda + q)_{\Delta(q)}}{(\lambda - q + 2k)_{\Delta(2k)}} c(q),
\]
so

\( c(\lambda - 2k) = \frac{(2\lambda - 2k)_0}{(2k)!} c_0, \quad (0 \leq k \leq \lambda), \)

\( c(\lambda - 1 - 2k) = \frac{(2\lambda - 1 - 2k)_1}{(2k)!} c_1, \quad (0 \leq k \leq \lambda - 1) \)

for some constants \( c_0, c_1 \). If \( \lambda = \nu_1 - \nu_2 + 1 \), then \( c(\lambda) \) and \( c(-\lambda) \) are arbitrary and other terms are given by [9.7] and [9.8].

**Proposition 9.2.**

1. If \( \lambda = \nu_2 - \nu_3 + 1 \) and \( \lambda \neq \nu_1 - \nu_2 + 1 \), then the solution space of \( P - 1 \phi_\lambda(e) = 0 \) and \( P - 2 \phi_\lambda(e) = 0 \) is spanned by the two vectors \( \sum_{k=0}^{\lambda-1} \frac{(2\lambda - 1)_k}{(2k+1)!} v^\lambda_{\nu - 2k}, \ i = 0, 1 \). These vectors are contained in \( V^{SO(3)}(\sigma_{i,\lambda}) \), respectively.

2. If \( \lambda = \nu_2 - \nu_3 + 1 \) and \( \lambda = \nu_1 - \nu_2 + 1 \), then the solution space of \( P - 1 \phi_\lambda(e) = 0 \) and \( P - 2 \phi_\lambda(e) = 0 \) is spanned by the four vectors \( v^\lambda_{\nu - 2k}, \ (-1)^i v^\lambda_{\nu - 2k}, \ i = 0, 1 \). These vectors are contained in \( V^{SO(3)}(\sigma_{i,\lambda}) \) and \( V^{SO(3)}(\sigma_{i,\lambda}) \), respectively.

9.2. The case \( \lambda = \nu_1 - \nu_3 + 1 \). Let us consider the case \( \lambda = \nu_1 - \nu_3 + 1 \). Note that \( \lambda \) is not equal to \( \nu_1 - \nu_2 + 1 \) nor to \( \nu_2 - \nu_3 + 1 \) because of the nonsingularity of \( \nu \).

In this case,

\[
\nu_1 + \nu_2 - 2\nu_3 - 2\lambda + q + 1 = -\nu_1 + \nu_2 + q - 1 \quad \text{and} \\
\nu_1 + \nu_2 - 2\nu_3 - 2\lambda - q + 3 = -\nu_1 + \nu_2 - q + 1.
\]

It follows that the equations [9.1] and [9.3] are the same, so the conditions are

\[
\begin{align*}
\frac{(\lambda + q + 2)_1}{(\nu - \nu_2 + 1)_2} (\nu_1 - \nu_2 + q - 1) c(q - 2) \\
+ (\lambda + q)_1 (\nu_1 - \nu_2 + 1) c(q) &= 0, \\
(-\lambda + 2 + \nu_2 - q)_2 &= 0.
\end{align*}
\]

As far as the coefficients in [9.9] are non-zero, \( c(q) \) and \( c(q') \) with \( q \equiv q' \mod 2 \) satisfy

\[
\begin{align*}
c(q) &= \frac{\Gamma(\nu_2 - \nu_1 + 1 + q) \Gamma(\nu_2 - \nu_1 + q')}{\Gamma(\nu_2 - \nu_1 + 1 + q + 1) \Gamma(\nu_2 - \nu_1 + q')} (\lambda + q)! (\lambda - q')! \\
&= \frac{(-1)^{q-q'/2} \Gamma(\nu_2 - \nu_1 + 1 + q) \Gamma(\nu_2 - \nu_1 + 1 - q')}{\Gamma(\nu_2 - \nu_1 + 1 + q + 1) \Gamma(\nu_2 - \nu_1 + 1 - q')} (\lambda + q)! (\lambda - q').
\end{align*}
\]

Some of the coefficients in [9.9] can be zero only if \( q - \nu_1 + \nu_2 \equiv 1 \mod 2 \). It follows that

\[
\sum_{q - \nu_1 + \nu_2 \equiv 0 \mod 2} \frac{\Gamma(\nu_2 - \nu_1 + 1 + q) - \nu^\lambda_1}{\Gamma(\nu_2 - \nu_1 + 1 + q + 1) \Gamma(\nu_2 - \nu_1 + 1 - q')} (\lambda + q)! (\lambda - q')! v^\lambda_q
\]

is a solution of [9.10]. If all the coefficients in [9.10] are non-zero,

\[
\sum_{q - \nu_1 + \nu_2 \equiv 1 \mod 2} \frac{(-1)^{q/2} (\nu_2 - \nu_1 + 1 + q - 1)}{\Gamma(\nu_2 - \nu_1 + 1 + q + 1) \Gamma(\nu_2 - \nu_1 + 1 - q)} (\lambda + q)! (\lambda - q')! v^\lambda_q
\]
Proposition 9.4. If \( \nu_1 - \nu_2 \pm q + k > \nu_3 - \nu_2 + 1 \) and \(-\lambda \leq q \leq \lambda\), this is the case when \( \nu_2 > \nu_3 \) and \( 2\nu_1 \geq \nu_2 + \nu_3 \). Therefore, \( \nu \) must satisfy \( \nu_1 > \nu_2 > \nu_3 \) or \( \nu_2 > \nu_1 > \nu_3 \). Suppose first that \( \nu_1 > \nu_2 > \nu_3 \). In this case, \( 0 < \nu_1 - \nu_2 + 1 \leq \nu_1 - \nu_3 + 1 = \lambda \) and \( c(\pm(\nu_1 - \nu_2 - 1)) = 0 \) by (9.9). By using (9.9) again, we obtain \( c(\nu_1 - \nu_2 - 1 - 2k) = 0 \) for \( 0 \leq k \leq \nu_1 - \nu_2 - 1 \). Therefore (9.10) implies that

\[
\sum_{q \equiv \nu_1 - \nu_2 + 1 \mod 2} \frac{\Gamma\left(\frac{\nu_1 - \nu_2 + 1 + q}{2}\right)}{\Gamma\left(\frac{\nu_2 - \nu_1 + 1 + q}{2}\right)} \left(\lambda + q\right)! \left(\lambda - q\right)! v_q^\lambda
\]

\[
\text{and}
\sum_{q \equiv \nu_1 - \nu_2 + 1 \mod 2} \frac{\Gamma\left(\frac{\nu_1 - \nu_2 + 1 + q}{2}\right)}{\Gamma\left(\frac{\nu_2 - \nu_1 + 1 + q}{2}\right)} \left(\lambda + q\right)! \left(\lambda - q\right)! v_q^\lambda
\]

are solutions of (9.9).

Proposition 9.3. If \( \lambda = \nu_1 - \nu_2 + 1 \) and \( \nu_1 > \nu_2 > \nu_3 \), then the solution space of \( P_{-1} \phi_\lambda(e) = 0 \) and \( P_{-2} \phi_\lambda(e) = 0 \) is

\[
\text{Span}(v_\alpha) := \sum_{q \equiv \nu_1 - \nu_2 + 1 \mod 2} \frac{\Gamma\left(\frac{\nu_1 - \nu_2 + 1 + q}{2}\right)}{\Gamma\left(\frac{\nu_2 - \nu_1 + 1 + q}{2}\right)} \left(\lambda + q\right)! \left(\lambda - q\right)! v_q^\lambda,
\]

\[
v_+ := \sum_{q \equiv \nu_1 - \nu_2 + 1 \mod 2} \frac{\Gamma\left(\frac{\nu_1 - \nu_2 + 1 + q}{2}\right)}{\Gamma\left(\frac{\nu_2 - \nu_1 + 1 + q}{2}\right)} \left(\lambda + q\right)! \left(\lambda - q\right)! v_q^\lambda
\]

\[
v_- := \sum_{q \equiv \nu_1 - \nu_2 + 1 \mod 2} \frac{\Gamma\left(\frac{\nu_1 - \nu_2 + 1 + q}{2}\right)}{\Gamma\left(\frac{\nu_2 - \nu_1 + 1 + q}{2}\right)} \left(\lambda + q\right)! \left(\lambda - q\right)! v_q^\lambda
\]

The vectors \( v_+ + (-1)^i v_- \), \( i = 0, 1 \), are contained in \( V_{SO(3)}(\sigma_{\nu_2 - \nu_3 + i, \nu_1 - \nu_3 + 1 + i}) \) and \( v_0 \) is contained in \( V_{SO(3)}(\sigma_{\nu_1 - \nu_3 + 1, \nu_2 - \nu_3 + 1}) \).

Suppose that \( \nu \) satisfies \( \nu_2 > \nu_1 > \nu_3 \) and \( 2\nu_1 \geq \nu_2 + \nu_3 \). In this case, \( 0 < \nu_2 - \nu_1 + 1 \leq \nu_1 - \nu_3 + 1 = \lambda \) and \( c(\pm(\nu_1 - \nu_2 - 1)) = 0 \) by (9.9). By using (9.9) again, we obtain \( c(\pm(\nu_2 - \nu_1 + 1 + 2k)) = 0 \) for \( k \geq 0 \). Therefore, in this case, (9.13) is a solution if we regard \( 1/\Gamma(-k) = 0 \) for \( k \in \mathbb{Z}_{\geq 0} \).

We have obtained all the solutions of (9.9).

Proposition 9.4. If \( \lambda = \nu_1 - \nu_2 + 1 \), and either \( \nu_2 > \nu_1 > \nu_3 \) or \( \nu_1 > \nu_3 > \nu_2 \), then the solution space of \( P_{-1} \phi_\lambda(e) = 0 \) and \( P_{-2} \phi_\lambda(e) = 0 \) is \( \text{Span}(w_i \mid i = 0, 1) \), where

\[
w_0 = \sum_{q \equiv \nu_1 - \nu_3 + 1 \mod 2} \frac{\Gamma\left(\frac{\nu_1 - \nu_3 + 1 + q}{2}\right)}{\Gamma\left(\frac{\nu_2 - \nu_3 + 1 + q}{2}\right)} \left(\lambda + q\right)! \left(\lambda - q\right)! v_q^\lambda
\]

\[
w_1 = \sum_{q \equiv \nu_1 - \nu_3 + 1 \mod 2} \frac{\left(-1\right)^q}{\Gamma\left(\frac{\nu_1 - \nu_3 + 1 + q}{2}\right)} \Gamma\left(\frac{\nu_2 - \nu_3 + 1 + q}{2}\right) \left(\lambda + q\right)! \left(\lambda - q\right)! v_q^\lambda
\]

These vectors \( w_i \), \( i = 0, 1 \), are contained in \( V_{SO(3)}(\sigma_{\nu_1 - \nu_3 + 1, \nu_2 - \nu_3 + 1 + i}) \), respectively.
10. Candidates for irreducible submodules. $SL(3,\mathbb{R})$ case.

In this section, we seek the candidates for the irreducible submodules of the principal series modules of $SL(3,\mathbb{R})$.

10.1. The irreducible modules $\overline{X}(\gamma_i)$, $i = 0, 1, 2$. The results in the previous section imply the possible principal series modules into which the irreducible modules $\overline{X}(\gamma_i)$, $i = 0, 1, 2$, are embedded.

Let $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ be a dominant nonsingular integral infinitesimal character. Namely, it satisfies $\Lambda_1 - \Lambda_2, \Lambda_2 - \Lambda_3 \in \mathbb{Z}_{\geq 0}$. By Lemma 7.1 (1), we know that the minimal $K$-types of $\overline{X}(\gamma_i)$, $i = 0, 1, 2$, are $\lambda = \Lambda_1 - \Lambda_3 + 1$, $\Lambda_1 - \Lambda_2 + 1$ and $\Lambda_2 - \Lambda_3 + 1$, respectively. Suppose that $\nu = (\nu_1, \nu_2, \nu_3)$ is an element of the orbit $W(G, H_\nu) \cdot \Lambda$. By Propositions 9.1, 9.2, 9.3 and 9.4, we know the principal series modules of which $\overline{X}(\gamma_i)$, $i = 0, 1, 2$, may be an irreducible submodule. Moreover, by 6.5, 6.6 and 6.7, we know which standard module is isomorphic to this principal series, in the Grothendieck group. We write these informations in a table.

Table 1. Possible embeddings of $\overline{X}(\gamma_i)$, $i = 0, 1, 2$

| Irred. Mod. | Propn # | $\nu$ | $\sigma_{i,j}$ | Std Mod. |
|-------------|---------|------|----------------|----------|
| $\overline{X}(\gamma_0)$ | 9.1 (1) | $(\Lambda_1, \Lambda_3, \Lambda_2)$ | $\sigma_{\Lambda_1-\Lambda_2, \Lambda_2-\Lambda_3+1}$ | $k = 5$ |
| | 9.1 (1) | $(\Lambda_1, \Lambda_3, \Lambda_2)$ | $\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3}$ | $k = 4$ |
| | 9.2 (1) | $(\Lambda_2, \Lambda_1, \Lambda_3)$ | $\sigma_{\Lambda_2-\Lambda_3, \Lambda_1-\Lambda_3+1}$ | $k = 4$ |
| | 9.2 (1) | $(\Lambda_2, \Lambda_1, \Lambda_3)$ | $\sigma_{\Lambda_2-\Lambda_3+1, \Lambda_1-\Lambda_3+1}$ | $k = 5$ |
| | 9.3 | $(\Lambda_1, \Lambda_2, \Lambda_3)$ | $\sigma_{\Lambda_1-\Lambda_3+1, \Lambda_2-\Lambda_3}$ | $k = 3$ |
| | 9.3 | $(\Lambda_1, \Lambda_2, \Lambda_3)$ | $\sigma_{\Lambda_1-\Lambda_3+1, \Lambda_2-\Lambda_3+1}$ | $k = 5$ |
| $\overline{X}(\gamma_1)$ | 9.1 (1) | $(\Lambda_1, \Lambda_2, \Lambda_3)$ | $\sigma_{\Lambda_1-\Lambda_2, \Lambda_2-\Lambda_3+1}$ | $k = 3$ |
| | 9.1 (1) | $(\Lambda_1, \Lambda_2, \Lambda_3)$ | $\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3}$ | $k = 4$ |
| | 9.2 (1) | $(\Lambda_3, \Lambda_1, \Lambda_2)$ | $\sigma_{\Lambda_2-\Lambda_3+1, \Lambda_1-\Lambda_2+1}$ | $k = 3$ |
| | 9.2 (1) | $(\Lambda_3, \Lambda_1, \Lambda_2)$ | $\sigma_{\Lambda_2-\Lambda_3+1, \Lambda_1-\Lambda_2+1}$ | $k = 4$ |
| | 9.3 | $(\Lambda_1, \Lambda_3, \Lambda_2)$ | $\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3+1}$ | $k = 3$ |
| | 9.3 | $(\Lambda_1, \Lambda_3, \Lambda_2)$ | $\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3}$ | $k = 4$ |
| $\overline{X}(\gamma_2)$ | 9.1 (1) | $(\Lambda_2, \Lambda_1, \Lambda_3)$ | $\sigma_{\Lambda_2-\Lambda_2+1, \Lambda_1-\Lambda_3}$ | $k = 3$ |
| | 9.1 (1) | $(\Lambda_2, \Lambda_1, \Lambda_3)$ | $\sigma_{\Lambda_2-\Lambda_2+1, \Lambda_1-\Lambda_3}$ | $k = 5$ |
| | 9.2 (1) | $(\Lambda_1, \Lambda_2, \Lambda_3)$ | $\sigma_{\Lambda_1-\Lambda_3+1, \Lambda_2-\Lambda_3+1}$ | $k = 5$ |
| | 9.2 (1) | $(\Lambda_1, \Lambda_2, \Lambda_3)$ | $\sigma_{\Lambda_1-\Lambda_3+1, \Lambda_2-\Lambda_3+1}$ | $k = 5$ |
| | 9.3 | $(\Lambda_2, \Lambda_1, \Lambda_3)$ | $\sigma_{\Lambda_2-\Lambda_2+1, \Lambda_1-\Lambda_3}$ | $k = 3$ |
| | 9.3 | $(\Lambda_2, \Lambda_1, \Lambda_3)$ | $\sigma_{\Lambda_2-\Lambda_2+1, \Lambda_1-\Lambda_3}$ | $k = 5$ |

We see that there exist principal series modules which may have two or more irreducible submodules. For example, $\overline{X}(\gamma_0)$, $\overline{X}(\gamma_1)$ and $\overline{X}(\gamma_2)$ may be submodules of $I(\sigma_{\Lambda_1-\Lambda_3, \Lambda_2-\Lambda_3+1}, (\Lambda_1, \Lambda_2, \Lambda_3))$. Let us explore whether this is true or not. For this purpose, we use the solutions obtained in Propositions 9.1, 9.2, 9.3 and 9.4.

By the translation principle, we may put $\Lambda = \rho_{\mu} = (1,0,-1)$. Firstly, let us consider the case $I(\sigma_{\Lambda_1-\Lambda_3, \Lambda_2-\Lambda_3+1}, (\Lambda_1, \Lambda_2, \Lambda_3)) = I(\sigma_0, (1,0,-1))$. By 6.8, this principal series module has horizontal symmetry (Corollary 4.3). Since $\overline{X}(\gamma_1)$
Proposition 10.1. Suppose that the nonsingular integral infinitesimal character is \( \Lambda = (A_1, A_2, A_3) \), \( A_1 - A_2, A_2 - A_3 \in \mathbb{Z}_{>0} \).

(1) \( \mathbf{X}(\gamma_0) \) can be a submodule of \( I(\sigma, \nu) \) only if one of the following conditions holds:
   (a) \( I(\sigma, \nu) \approx X(\gamma_3) \) and \( \nu = (A_1, A_2, A_3) \). In this case, \( \mathbf{X}(\gamma_1) \oplus \mathbf{X}(\gamma_2) \) lies in the higher floor than \( \mathbf{X}(\gamma_0) \).
   (b) \( I(\sigma, \nu) \approx X(\gamma_4) \) and \( \nu = (A_1, A_2, A_3) \). In this case, \( \mathbf{X}(\gamma_1) \) lies in the higher floor than \( \mathbf{X}(\gamma_0) \).
   (c) \( I(\sigma, \nu) \approx X(\gamma_5) \) and \( \nu = (A_1, A_2, A_3) \). In this case, \( \mathbf{X}(\gamma_2) \) lies in the higher floor than \( \mathbf{X}(\gamma_0) \).
   (d) \( I(\sigma, \nu) \approx X(\gamma_4) \) and \( \nu = (A_2, A_1, A_3) \).
   (e) \( I(\sigma, \nu) \approx X(\gamma_5) \) and \( \nu = (A_2, A_1, A_3) \). In this case, \( \mathbf{X}(\gamma_2) \) lies in the higher floor than \( \mathbf{X}(\gamma_0) \).
   (f) \( I(\sigma, \nu) \approx X(\gamma_4) \) and \( \nu = (A_1, A_3, A_2) \). In this case, \( \mathbf{X}(\gamma_1) \) lies in the higher floor than \( \mathbf{X}(\gamma_0) \).
   (g) \( I(\sigma, \nu) \approx X(\gamma_5) \) and \( \nu = (A_1, A_3, A_2) \).

(2) \( \mathbf{X}(\gamma_1) \) can be a submodule of \( I(\sigma, \nu) \) only if one of the following conditions holds:
   (a) \( I(\sigma, \nu) \approx X(\gamma_3) \) and \( \nu = (A_4, A_1, A_2) \).
   (b) \( I(\sigma, \nu) \approx X(\gamma_4) \) and \( \nu = (A_4, A_1, A_2) \).
   (c) \( I(\sigma, \nu) \approx X(\gamma_3) \) and \( \nu = (A_1, A_3, A_2) \).

(3) \( \mathbf{X}(\gamma_2) \) can be a submodule of \( I(\sigma, \nu) \) only if one of the following conditions holds:
   (a) \( I(\sigma, \nu) \approx X(\gamma_3) \) and \( \nu = (A_2, A_1, A_3) \).
   (b) \( I(\sigma, \nu) \approx X(\gamma_3) \) and \( \nu = (A_2, A_1, A_3) \).
   (c) \( I(\sigma, \nu) \approx X(\gamma_3) \) and \( \nu = (A_2, A_3, A_1) \).

10.2. The irreducible module \( \mathbf{X}(\gamma_3) \). If \( \Lambda \) is trivial, then \( \mathbf{X}(\gamma_3) \) is the trivial \((g, K)\)-module and then its \( K \)-type is \( V_0^{SO(3)} \) alone. So an embedding corresponds to a non-trivial solution of \((P_1\phi_0)(e) = 0 \) and \((P_2\phi_0)(e) = 0 \).
For \( \lambda = 0 \), the equation \( (P_1 \phi_0)(e) = 0 \) is trivial. By \( \text{(5.1)} \), \( (P_2 \phi_0)(e) = 0 \) is

\[
\begin{cases}
(\nu_1 - \nu_2 + 1) c(0) = 0 \\
(\nu_1 + \nu_2 - 2\nu_3 + 3) c(0) = 0
\end{cases}
\]

This system of equations has non-trivial solution if and only if \( \nu_2 = \nu_1 + 1 \) and \( \nu_3 = \nu_1 + 2 \). Since we are considering the case of trivial infinitesimal character, the possible \( \nu \) is only \((-1, 0, 1)\). By Casselman’s subrepresentation theorem, every irreducible admissible \( (g, K) \)-module is an submodule of at least one principal series module. So the unique solution obtained above really corresponds to an embedding. By translation principle, we obtained the following proposition.

**Proposition 10.2.** In the case when the infinitesimal character is \( \Lambda = (\Lambda_1, \Lambda_2, \Lambda_3), \Lambda_1 - \Lambda_2, \Lambda_2 - \Lambda_3 \in \mathbb{Z}_{> 0} \), the irreducible finite dimensional module \( \overline{X}(\gamma_3) \) is embedded into only one principal series module \( I(\sigma_{\Lambda_1 - \Lambda_2, \Lambda_1 - \Lambda_2 + 1}, (\Lambda_3, \Lambda_2, \Lambda_1)) \).

10.3. The irreducible modules \( \overline{X}(\gamma_4) \) and \( \overline{X}(\gamma_5) \). Suppose that \( \Lambda \) is trivial. Then Lemma \( \text{(7.1)} \) says that both \( \overline{X}(\gamma_4) \) and \( \overline{X}(\gamma_5) \) contain the \( K \)-type \( \lambda = 1 \), with multiplicity one and do not contain \( \lambda = 2 \). It follows that if \( \text{Ind}_{P_m}^{G_m} (\sigma_{i,j} \otimes e^{-\nu - \rho_m}) \), \( \nu \in W(G, H) \cdot \rho_m \), contains \( \overline{X}(\gamma_k) \), \( k = 4 \) or \( 5 \), as a submodule, then the equation \( (P_1 \phi_1)(e) = 0 \) has a non-zero solution. For \( \phi_1(e) = \sum_{q=-1}^1 c(q) v_q^1 \), this equation is equivalent to

\[
\begin{align*}
(10.1) & \quad (\nu_1 - \nu_2 + 1) c(0) = 0, \\
(10.2) & \quad (\nu_1 - \nu_2) c(-1) + (\nu_1 + \nu_2 - 2\nu_3 + 2) c(1) = 0, \\
(10.3) & \quad (\nu_1 + \nu_2 - 2\nu_3 + 2) c(-1) + (\nu_1 - \nu_2) c(1) = 0,
\end{align*}
\]

by \( \text{(8.2)} \).

The equation \( \text{(10.1)} \) has a non-zero solution if and only if \( \nu_2 = \nu_1 + 1 \), i.e. \( \nu = (0, 1, -1) \) or \((-1, 0, 1)\), and the solution space is \( \mathbb{C} v_1^1 \). Since \( v_0^1 \in V_{1}^{SO(3)}(\sigma_{1,1}) \), this vector corresponds to the composition factor \( \overline{X}(\gamma_4) \subset I(\sigma_{1,1}, (0, 1, -1)) \approx X(\gamma_4) \) and \( \overline{X}(\gamma_5) \subset I(\sigma_{1,1}, (-1, 0, 1)) \approx X(\gamma_5) \). Here, we used \( \text{(6.6)} \) and \( \text{(6.7)} \).

The system of equations \( \text{(10.2)} \) and \( \text{(10.3)} \) has a non-zero solution if and only if

\[
(\nu_1 + \nu_2 - 2\nu_3 + 2)^2 - (\nu_1 - \nu_2)^2 = 4(\nu_1 - \nu_3 + 1)(\nu_2 - \nu_3 + 1) = 0.
\]

If \( \nu_3 = \nu_1 + 1 \), i.e. \( \nu = (-1, 1, 0) \) or \((0, -1, 1)\), then the solution space is \( \mathbb{C}(v_1^1 + v_1^{-1}) \). If \( \nu_3 = \nu_2 + 1 \), i.e. \( \nu = (-1, 0, 1) \) or \((1, -1, 0)\), then the solution space is \( \mathbb{C}(v_1^1 - v_1^{-1}) \). Since \( v_1^1 + v_1^{-1} \in V_{1}^{SO(3)}(\sigma_{0,1}) \) and \( v_1^1 - v_1^{-1} \in V_{1}^{SO(3)}(\sigma_{1,0}) \), the informations \( \text{(6.6)} \) and \( \text{(6.7)} \) imply that these vectors correspond to (the possibilities of the) composition factors \( \overline{X}(\gamma_4) \subset I(\sigma_{0,1}, (0, -1, 1)), I(\sigma_{1,0}, (-1, 0, 1)) \approx X(\gamma_4) \) and \( \overline{X}(\gamma_5) \subset I(\sigma_{0,1}, (-1, 1, 0)), I(\sigma_{1,0}, (1, -1, 0)) \approx X(\gamma_5) \). We have obtained the following results.

**Proposition 10.3.** Suppose that the nonsingular integral infinitesimal character is \( \Lambda = (\Lambda_1, \Lambda_2, \Lambda_3), \Lambda_1 - \Lambda_2, \Lambda_2 - \Lambda_3 \in \mathbb{Z}_{> 0} \).

1. \( \overline{X}(\gamma_4) \) can be a submodule of \( I(\sigma, \nu) \) only if \( I(\sigma, \nu) \approx X(\gamma_4) \) and \( \nu = (\Lambda_2, \Lambda_1, \Lambda_3), (\Lambda_2, \Lambda_3, \Lambda_1) \) or \( (\Lambda_3, \Lambda_2, \Lambda_1) \).

2. \( \overline{X}(\gamma_5) \) can be a submodule of \( I(\sigma, \nu) \) only if \( I(\sigma, \nu) \approx X(\gamma_5) \) and \( \nu = (\Lambda_3, \Lambda_2, \Lambda_1), (\Lambda_3, \Lambda_1, \Lambda_2) \) or \( (\Lambda_1, \Lambda_3, \Lambda_2) \).
11. Determination of the socle filtration. \( SL(3, \mathbb{R}) \) case.

From Propositions 10.1, 10.2 and 10.3 we can determine the socle filtrations of the principal series modules.

For the \( SL(3, \mathbb{R}) \) case, the dual module of \( I(\sigma_{i,j}, \nu) \) is \( I(\sigma_{i,j}, -\nu) \) and the \( \mu' \)-twisted module (see (6.3)) of the latter is isomorphic to \( I(\sigma_{i,j}, w^0, \nu) \), where \( w^0 \) is the longest element of \( W(G, A_m) \). It follows that the socle filtrations of

\[
I(\sigma_{i,j}, (\nu_1, \nu_2, \nu_3)) \quad \text{and} \quad I(\sigma_{i,j}, (\nu_3, \nu_2, \nu_1))
\]

are upside down.

We make a list of the candidates for the irreducible submodules of \( I(\sigma, \nu) \). In Table 2, the first column means that the principal series in consideration is isomorphic to \( X(\gamma_i) \) in the Grothendieck group. The second and fourth columns are the "\( \nu \)" part of \( I(\sigma, \nu) \), and the third and fifth columns are the candidates for irreducible submodules of \( I(\sigma, \nu) \).

Table 2. Candidates for irreducible submodules

| \( X(\gamma_3) \) | \( \nu \) | Irred.Sub. | \( \nu \) | Irred.Sub. |
|------------------|--------|------------|--------|------------|
| \( (A_1, A_2, A_3) \) | \( \nu_2 \) | \( X(\gamma_0) \) | \( (A_3, A_2, A_1) \) | \( X(\gamma_0) \) |
| \( (A_2, A_1, A_3) \) | \( \nu_2 \) | \( X(\gamma_0) \) | \( (A_1, A_3, A_2) \) | \( X(\gamma_0) \) |
| \( (A_3, A_2, A_1) \) | \( \nu_2 \) | \( X(\gamma_0) \) | \( (A_1, A_3, A_2) \) | \( X(\gamma_0) \) |
| \( X(\gamma_4) \) | \( \nu_2 \) | \( X(\gamma_0) \) | \( (A_3, A_2, A_1) \) | \( X(\gamma_0) \) |
| \( (A_2, A_3, A_1) \) | \( \nu_2 \) | \( X(\gamma_0) \) | \( (A_1, A_3, A_2) \) | \( X(\gamma_0) \) |
| \( X(\gamma_5) \) | \( \nu_2 \) | \( X(\gamma_0) \) | \( (A_3, A_2, A_1) \) | \( X(\gamma_0) \) |
| \( (A_2, A_3, A_1) \) | \( \nu_2 \) | \( X(\gamma_0) \) | \( (A_1, A_3, A_2) \) | \( X(\gamma_0) \) |

We know the composition factors of \( X(\gamma_3) \), \( X(\gamma_4) \) and \( X(\gamma_5) \) (Theorem 6.1). We also know that the lengths (Section 4.3) of regular characters are

\[
\ell(\gamma_0) = 0, \quad \ell(\gamma_1) = \ell(\gamma_2) = 1, \quad \ell(\gamma_3) = \ell(\gamma_4) = \ell(\gamma_5) = 2.
\]

From Table 2, the parity condition (Corollary 4.5) and (11.1), all the socle filtrations of principal series modules are completely determined. For example, consider the principal series \( I(\sigma_{\Lambda_1-\Lambda_3, \Lambda_2-\Lambda_3+1}, (A_1, A_2, A_3)) \approx X(\gamma_3) \). The irreducible factors of this module are \( X(\gamma_1), i = 0, 1, 2, 3 \), and the multiplicities are all one. The second row of Table 2 says that the candidates for the irreducible submodules of this module is \( X(\gamma_0) \) only, so it is really the unique submodule. It also tells us that \( X(\gamma_3) \) is the unique quotient module of it. The lengths of the other irreducible factors \( X(\gamma_1) \) and \( X(\gamma_2) \) are odd, but those of \( X(\gamma_0) \) and \( X(\gamma_3) \) are even. Then by the parity condition, the socle filtration of \( I(\sigma_{\Lambda_1-\Lambda_3, \Lambda_2-\Lambda_3+1}, (A_1, A_2, A_3)) \) is

\[
X(\gamma_3) \quad X(\gamma_1) \oplus X(\gamma_2) \quad X(\gamma_0)
\]

(11.2)

This diagram means that the socle of \( I := I(\sigma_{\Lambda_1-\Lambda_3, \Lambda_2-\Lambda_3+1}, (A_1, A_2, A_3)) \) is \( X(\gamma_0) \); the socle of \( I/X(\gamma_0) \) is \( X(\gamma_1) \oplus X(\gamma_2) \); and the socle of \( (I/X(\gamma_0))/X(\gamma_1) \oplus \)
\( \mathcal{X}(\gamma_2)/\mathcal{X}(\gamma_0) \) is \( \mathcal{X}(\gamma_3) \). Hereafter, for notational convenience, we abbreviate (11.2) as

\[
(11.3) \quad \frac{\mathcal{X}}{\mathcal{Y}} \oplus \frac{\mathcal{Y}}{\mathcal{Z}}
\]

In the same way, we obtain the main results for the \( SL(3,\mathbb{R}) \) case.

**Theorem 11.1.** Let \( \Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \) be a nonsingular dominant integral infinitesimal character of \( SL(3,\mathbb{R}) \). The socle filtrations of the principal series modules \( I(\sigma, \nu) \) with \( \nu \in W(G, A_m) \cdot \Lambda \) are as follows:

1. The cases \( I(\sigma, \nu) \approx X(\gamma_3) \).

| \( I(\sigma_{\Lambda_1-\Lambda_3, \Lambda_2-\Lambda_3+1}(A_1, \Lambda_2, \Lambda_3)) \) | \( I(\sigma_{\Lambda_1-\Lambda_3, \Lambda_2-\Lambda_3+1}(A_3, \Lambda_2, \Lambda_1)) \) |
| --- | --- |
| \( \frac{\mathcal{J}}{\mathcal{T} \oplus \mathcal{Z}} \) | \( \frac{\mathcal{U}}{\mathcal{T} \oplus \mathcal{Z}} \) |

2. The cases \( I(\sigma, \nu) \approx X(\gamma_4) \).

| \( I(\sigma_{\Lambda_1, \Lambda_2-\Lambda_3}(A_1, \Lambda_2, \Lambda_3)) \), \( I(\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3}(A_1, \Lambda_1, \Lambda_3)) \) | \( I(\sigma_{\Lambda_1, \Lambda_2-\Lambda_3}(A_3, \Lambda_2, \Lambda_1)) \), \( I(\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3}(A_1, \Lambda_1, \Lambda_3)) \) |
| --- | --- |
| \( \frac{\mathcal{J}}{\mathcal{T} \oplus \mathcal{Z}} \) | \( \frac{\mathcal{U}}{\mathcal{T} \oplus \mathcal{Z}} \) |

3. The cases \( I(\sigma, \nu) \approx X(\gamma_5) \).

| \( I(\sigma_{\Lambda_1-\Lambda_3+1, \Lambda_2-\Lambda_3+1}(A_1, \Lambda_2, \Lambda_3)) \), \( I(\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3+1}(A_2, \Lambda_1, \Lambda_3)) \) | \( I(\sigma_{\Lambda_1-\Lambda_3+1, \Lambda_2-\Lambda_3+1}(A_3, \Lambda_2, \Lambda_1)) \), \( I(\sigma_{\Lambda_1-\Lambda_2+1, \Lambda_2-\Lambda_3+1}(A_1, \Lambda_1, \Lambda_3)) \) |
| --- | --- |
| \( \frac{\mathcal{J}}{\mathcal{T} \oplus \mathcal{Z}} \) | \( \frac{\mathcal{U}}{\mathcal{T} \oplus \mathcal{Z}} \) |

12. **The group** \( Sp(2,\mathbb{R}) \)

From this section to Section 18, we consider the group

\[ G = Sp(2,\mathbb{R}) = \left\{ g \in GL(4,\mathbb{R}) \mid g \begin{pmatrix} O & -I \\ I & O \end{pmatrix} g = \begin{pmatrix} O & -I \\ I & O \end{pmatrix} \right\} \]

and investigate the socle filtrations of the principal series modules of \( G \).
Let

\[
K := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \text{Sp}(2, \mathbb{R}) \right\} \simeq U(2), \quad \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B
\]

be a maximal compact subgroup of $G$.

Let $g_0 = \mathfrak{t}_0 \oplus \mathfrak{z}_0$ be the Cartan decomposition of $g_0$ with respect to the Cartan involution $\theta(X) = -^t X$. Choose a maximal commutative subspace $(a_m)_0 = \{ \text{diag}(a_1, a_2, -a_1, -a_2) \in G \mid a_i \in \mathbb{R} \}$ of $\mathfrak{z}_0$ and define

\[
A_m = \{ \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \in G \mid a_i > 0 \} \quad \text{and} \quad N_m = \left\{ \begin{pmatrix} 1 & x_{12} \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_{13} & x_{14} \\ 1 & x_{14} & x_{24} \end{pmatrix} \mid x_{ij} \in \mathbb{R} \right\}.
\]

Then $G = KA_m N_m$ is an Iwasawa decomposition of $G$.

Define a basis $H_1, H_2$ of $(a_m)_0$ by

\[
H_i := E_{ii} - E_{2+i,2+i}, \quad i = 1, 2.
\]

Let $f_1, f_2$, be the dual basis of $(a_m)_0^*$ defined by $f_i(H_j) = \delta_{i,j}$. The root system $\Sigma = \Sigma(g_0, (a_m)_0)$ is $\{ \pm 2f_1, \pm 2f_2, \pm 2f_1, \pm 2f_2 \}$. We choose $\Sigma^+ = \{ f_1 \pm f_2, 2f_1, 2f_2 \}$ as a positive system of it. Define root vectors for positive roots by

\[
X_{f_1 - f_2} := E_{12} - E_{43}, \quad X_{f_1 + f_2} := E_{14} + E_{23}, \quad X_{2f_1} := E_{13}, \quad X_{2f_2} := E_{24},
\]

and for negative roots $-\alpha \ (\alpha \in \Sigma^+)$, define $X_{-\alpha} = {}^t X_\alpha$.

As a basis of $\mathfrak{t}$, we choose

\[
D_i := \sqrt{-1}(X_{-2f_i} - X_{2f_i}), \quad i = 1, 2,
\]

\[
Y_{\pm} := \frac{1}{2}\{ (X_{-f_1 + f_2} - X_{f_1 - f_2}) + \sqrt{-1}(X_{f_1 - f_2} - X_{f_1 + f_2}) \}.
\]

Under the identification \((12.1)\), these vectors correspond to $E_{ii}, E_{12}, E_{21}$ of $\mathfrak{gl}(2, \mathbb{C})$, respectively.

Let $T := \{ \exp(x_1 D_1 + x_2 D_2) \mid x_1, x_2 \in \mathbb{R} \}$ be a Cartan subgroup of $K$ (also of $G$). Define a basis $\{ e_1, e_2 \}$ of $\mathfrak{t}^*$ by $e_i(D_j) = \delta_{i,j}$.

Now, we fix a basis of an irreducible representation of $K = U(2)$. Let \(\{v^\lambda_q\}\) be the basis of irreducible representation of $SU(2)$ with the highest weight $2\lambda$, which is defined in Section 5. Let $1_\mu$ be the basis of the one dimensional representation of the center of $U(2)$ defined by

\[
\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot 1_\mu = a^\mu, \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in U(2).
\]

Let $(\tau_{(\lambda_1, \lambda_2)}, V_{\langle \lambda_1, \lambda_2 \rangle})$ be the irreducible representation of $K = U(2)$ with the highest weight $\lambda = (\lambda_1, \lambda_2)$. We define a basis $\{ v_{q}^{(\lambda_1, \lambda_2)} \mid \lambda_1 \leq \lambda \}$ of it by

\[
v_{q}^{(\lambda_1, \lambda_2)} := v_{q - (\lambda_1 + \lambda_2)/2}^{(\lambda_1 - \lambda_2)/2} \otimes 1_{\lambda_1 + \lambda_2}.
\]

The action of elements in $\mathfrak{t}$ is given by

\[
D_1 v_{q}^{(\lambda_1, \lambda_2)} = q v_{q}^{(\lambda_1, \lambda_2)}, \quad D_2 v_{q}^{(\lambda_1, \lambda_2)} = (\lambda_1 + \lambda_2 - q) v_{q}^{(\lambda_1, \lambda_2)},
\]

\[
Y_{+} v_{q}^{(\lambda_1, \lambda_2)} = (\lambda_1 - q) v_{q+1}^{(\lambda_1, \lambda_2)}, \quad Y_{-} v_{q}^{(\lambda_1, \lambda_2)} = (q - \lambda_2) v_{q-1}^{(\lambda_1, \lambda_2)}.
\]
13. Irreducible \((\mathfrak{g}, K)\)-modules of \(Sp(2, \mathbb{R})\)

There are four conjugacy classes of Cartan subgroups in \(G = Sp(2, \mathbb{R})\). One of them is the split Cartan subgroup \(H_s = M_s \cdot A_m\), where

\[ M_s := Z_K(A_m) = \{ I, m_{1,1}^{sp}, m_{1,-1}^{sp}, m_{-1,1}^{sp} \}, \quad m_{i,j}^{sp} := \text{diag}(i, j, i, j). \]

Let \(\widehat{M}_m\) be the set of equivalence classes of irreducible representations of \(M_m\).

Define \(\sigma_{i,j} \in \widehat{M}_m, i, j \in \mathbb{Z}\), by

\[ \sigma_{i,j}(m_{i,j}^{sp}) = (-1)^i, \quad \sigma_{i,j}(m_{1,-1}^{sp}) = (-1)^j. \]

Then \(\widehat{M}_m\) consists of four elements \(\sigma_{i,j}, i, j \in \{0, 1\}\). The action of \(W(G, H_s) \cong W(B_2)\) on \(\widehat{M}_m\) is given by

\[
(13.1) \quad r_{12} \cdot \sigma_{i,j} = \sigma_{j,i}, \quad \epsilon_1 \cdot \sigma_{i,j} = \sigma_{i,j} = \sigma_{i,j},
\]

where \(r_{12}\) is the permutation of 1 and 2, and \(\epsilon_i, i = 1, 2\), is the reflection with respect to the root \(2f_i\).

Let

\[
(13.2) \quad \Lambda = \Lambda_1 f_1 + \Lambda_2 f_2, \quad \Lambda_1, \Lambda_2 \in \mathbb{Z}, \quad \Lambda_1 > \Lambda_2 > 0
\]

be a nonsingular integral infinitesimal character. There are four conjugacy classes of regular characters of \(H_s\) with a given infinitesimal character, which correspond to the ATLAS numbers "10", "11" in the block \(PSO(3, 2)\), "4" in the block \(PSO(4, 1)\) and "0" in the block \(PSO(5)\). According to these numbers, we define regular characters \(\gamma_1, \gamma_11, \gamma_{4'}, \gamma_{0''}\) of \(H_s\) by

\[
(13.3) \quad \gamma_{10} = (\sigma_{\Lambda_1, \Lambda_1+1}, (\pm \Lambda_1, \pm \Lambda_2)) \sim (\sigma_{\Lambda_2+1, \Lambda_1}, (\pm \Lambda_2, \pm \Lambda_1)),
\]

\[
(13.4) \quad \gamma_{11} = (\sigma_{\Lambda_2, \Lambda_1+1}, (\pm \Lambda_2, \pm \Lambda_1)) \sim (\sigma_{\Lambda_1+1, \Lambda_2}, (\pm \Lambda_1, \pm \Lambda_2)),
\]

\[
(13.5) \quad \gamma_{4'} = (\sigma_{\Lambda_1+1, \Lambda_2+1}, (\pm \Lambda_1, \pm \Lambda_2)) \sim (\sigma_{\Lambda_2+1, \Lambda_1+1}, (\pm \Lambda_2, \pm \Lambda_1)),
\]

\[
(13.6) \quad \gamma_{0''} = (\sigma_{\Lambda_1, \Lambda_2}, (\pm \Lambda_1, \pm \Lambda_2)) \sim (\sigma_{\Lambda_2, \Lambda_1}, (\pm \Lambda_2, \pm \Lambda_1)).
\]

Here, the notation is the same as in the case of \(SL(3, \mathbb{R})\). Note that the lengths of these regular characters are all three.

Next, consider the fundamental Cartan subgroup \(H_f = T \simeq U(1)^{\times 2}\). There are four conjugacy classes of regular characters of \(H_f\) with a given nonsingular integral infinitesimal character, which correspond to the ATLAS numbers "0", "1", "2" and "3" in the block \(PSO(3, 2)\). According to these numbers, we define regular characters \(\gamma_j, j = 0, 1, 2, 3\), of \(H_f\) by

\[
(13.7) \quad \gamma_0 = (\Lambda_1 + 1, -\Lambda_2), \quad \gamma_1 = (\Lambda_2, -\Lambda_1 - 1),
\]

\[
\gamma_2 = (\Lambda_1 + 1, \Lambda_2 + 2), \quad \gamma_3 = (\Lambda_2 - 2, -\Lambda_1 - 1).
\]

Here we denoted \(\gamma_j \in \widehat{H_f}\) by its differential. The standard module \(X(\gamma_2)\) and \(X(\gamma_3)\) are the holomorphic and anti-holomorphic discrete series, and \(X(\gamma_0)\) and \(X(\gamma_1)\) are large discrete series modules.

There are two other Cartan subgroups of \(Sp(2, \mathbb{R})\). One is \(H_{long} = T_{long} A_{long}\), which is the Cayley transform of the compact Cartan subgroup through a long non-compact imaginary root. The groups \(T_{long}\) and \(A_{long}\) are isomorphic to \(U(1) \times \{\pm 1\}\) and \(\mathbb{R}_{>0}\), respectively, so \(H_{long}\) has two connected components. The Levi subgroup \(L_{long} := Z_G(A_{long})\) is isomorphic to \(SL(2, \mathbb{R}) \times \{\pm 1\} \times \mathbb{R}_{>0}\). The Weyl group \(W(G, H_{long})\) is isomorphic to \(S_2\), which is the reflection on \(a_{long}\). Therefore, when a
nonsingular integral infinitesimal character is fixed, there are $8/2 \times 2 = 8$ conjugacy classes of regular characters attached to $H_{\text{long}}$. They correspond to the ATLAS numbers "5", "6", "7", "8" in the block $PSO(3, 2)$ and "0", "1", "2", "3" in the block $PSO(4, 1)$. According to these numbers, we define regular characters $\gamma_j$, $j = 5, 6, 7, 8, 0', 1', 2', 3'$ by

\[
\begin{align*}
\gamma_5 &= (\Lambda_1 + 1, \text{sgn}^{A_2}, \Lambda_2), & \gamma_6 &= (-\Lambda_1 - 1, \text{sgn}^{A_2}, \Lambda_2), \\
\gamma_7 &= (\Lambda_2 + 1, \text{sgn}^{A_1}, \Lambda_1), & \gamma_8 &= (-\Lambda_2 - 1, \text{sgn}^{A_1}, \Lambda_1), \\
\gamma_{0'} &= (\Lambda_1 + 1, \text{sgn}^{A_2+1}, \Lambda_2), & \gamma_{1'} &= (-\Lambda_1 - 1, \text{sgn}^{A_2+1}, \Lambda_2), \\
\gamma_{2'} &= (\Lambda_2 + 1, \text{sgn}^{A_1+1}, \Lambda_1), & \gamma_{3'} &= (-\Lambda_2 - 1, \text{sgn}^{A_1+1}, \Lambda_1).
\end{align*}
\]

Here, the expression $\gamma_j = (\varepsilon_j(\Lambda_{a(j)} + 1), \text{sgn}^{b(j)}, \Lambda_{e(j)})$ means that the restriction of $\gamma_j$ to $T_{\text{long}} \simeq U(1) \times \{\pm 1\}$ is $(\varepsilon_j(\Lambda_{a(j)} + 1), \text{sgn}^{b(j)}) \in U(1) \times \{\pm 1\}$ and its restriction to $a_{\text{long}} = \Lambda_{e(j)}$. The lengths of these regular characters are $\ell(\gamma_5) = \ell(\gamma_6) = \ell(\gamma_7) = \ell(\gamma_8) = 1$ and $\ell(\gamma_{0'}) = \ell(\gamma_{1'}) = \ell(\gamma_{2'}) = \ell(\gamma_{3'}) = 2$.

The other intermediate Cartan subgroup is $H_{\text{short}} = T_{\text{short}}A_{\text{short}}$, which is the Cayley transform of the compact Cartan subgroup through a short non-compact imaginary root. The groups $T_{\text{short}}$ and $A_{\text{short}}$ are isomorphic to $SO(2)$ and $\mathbb{R}_{>0}$, respectively, so $H_{\text{short}}$ is connected. The Levi subgroup $L_{\text{short}} := Z_G(A_{\text{short}})$ is isomorphic to $GL(2, \mathbb{R}) \simeq SL^\pm(2, \mathbb{R}) \times \mathbb{R}_{>0}$. The Weyl group $W(G, H_{\text{short}})$ is isomorphic to $S_2 \times S_2$, generated by the reflections on $t_{\text{short}}$ and $a_{\text{short}}$. Therefore, when a nonsingular integral infinitesimal character is fixed, there are $8/4 = 2$ conjugacy classes of regular characters. They correspond to the ATLAS numbers "4" and "9" in the block $PSO(3, 2)$. According to these numbers, we define regular characters $\gamma_4$ and $\gamma_9$ of $H_{\text{short}}$ by

$\gamma_4 = (\Lambda_1 + \Lambda_2 + 1, \Lambda_2 - \Lambda_1), \quad \gamma_9 = (\Lambda_1 - \Lambda_2 + 1, \Lambda_1 + \Lambda_2) \in \widehat{SO(2)} \times a_{\text{short}}$.

The length of $\gamma_4$ is one and that of $\gamma_9$ is two.

14. $K$-TYPES OF THE IRREDUCIBLE MODULES OF $Sp(2, \mathbb{R})$

In this section, we calculate the $K$-spectra of the irreducible modules of $Sp(2, \mathbb{R})$. For the determination of the socle filtration of principal series modules, we need them in the case when the infinitesimal character $\Lambda$ is trivial and the highest weight $\lambda = (\lambda_1, \lambda_2)$ of the irreducible representation $V^U(2)$ of $K \simeq U(2)$ satisfies $|\lambda_1|, |\lambda_2| \leq 3$. The $K$-spectra of irreducible modules can be computed by using the KLV-conjecture and the Blattner formula.

**Theorem 14.1.** In the Grothendieck group, the standard modules $X(\gamma_i)$ decomposes into irreducible modules as follows:

1. (Block $PSO(3, 2)$)

\[
\begin{align*}
X(\gamma_i) &= \overline{X}(\gamma_i), & i &= 0, 1, 2, 3, \\
X(\gamma_4) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_1) + \overline{X}(\gamma_4), \\
X(\gamma_5) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_2) + \overline{X}(\gamma_5), & X(\gamma_6) &= \overline{X}(\gamma_1) + \overline{X}(\gamma_3) + \overline{X}(\gamma_6), \\
X(\gamma_7) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_4) + \overline{X}(\gamma_5) + \overline{X}(\gamma_7), \\
X(\gamma_8) &= \overline{X}(\gamma_1) + \overline{X}(\gamma_4) + \overline{X}(\gamma_6) + \overline{X}(\gamma_8), \\
X(\gamma_9) &= \overline{X}(\gamma_0) + \overline{X}(\gamma_1) + \overline{X}(\gamma_4) + \overline{X}(\gamma_5) + \overline{X}(\gamma_6) + \overline{X}(\gamma_9),
\end{align*}
\]
\[ X(\gamma_{10}) = \overline{X}(\gamma_0) + \overline{X}(\gamma_1) + 2 \times \overline{X}(\gamma_4) + \overline{X}(\gamma_5) + \overline{X}(\gamma_6) \]
\[ + \overline{X}(\gamma_7) + \overline{X}(\gamma_8) + \overline{X}(\gamma_9) + \overline{X}(\gamma_{10}), \]
\[ X(\gamma_{11}) = \overline{X}(\gamma_0) + \overline{X}(\gamma_1) + \overline{X}(\gamma_2) + \overline{X}(\gamma_3) \]
\[ + \overline{X}(\gamma_4) + \overline{X}(\gamma_5) + \overline{X}(\gamma_6) + \overline{X}(\gamma_9) + \overline{X}(\gamma_{11}). \]

(2) (Block PSO(4,1))
\[ X(\gamma_i) = \overline{X}(\gamma_i), \quad i = 0', 1', \]
\[ X(\gamma_{2}) = \overline{X}(\gamma_{0'}) + \overline{X}(\gamma_{2'}), \quad X(\gamma_{3'}) = \overline{X}(\gamma_{1'}) + \overline{X}(\gamma_{3'}), \]
\[ X(\gamma_{4}) = \overline{X}(\gamma_{0'}) + \overline{X}(\gamma_{1'}) + \overline{X}(\gamma_{2'}) + \overline{X}(\gamma_{3'}). \]

(3) (Block PSO(5))
\[ X(\gamma_{0'}) = \overline{X}(\gamma_{0'}). \]

Let us calculate the \( K \)-spectra of the irreducible modules. The multiplicities of \( K \)-types in a discrete series module \( X(\gamma_j), j = 0, 1, 2, 3 \), are given by the Blattner formula \((15)\)
\[ m(\lambda) = \sum_{w \in W(K,T)} e(w) \mathcal{Q}(w(\lambda + \rho_c) - \tau - \rho_n). \]

Secondly, consider the standard modules \( X(\gamma_d) \) and \( X(\gamma_9) \). These are the generalized principal series modules
\[ X(\gamma_i) = \text{Ind}^G_{P_{\text{short}}} (\pi_{DS}^{SL^\pm(2,\mathbb{R})}(\Lambda_1 + \epsilon_1\Lambda_2 + 1) \otimes e^{\Lambda_1 - \epsilon_1\Lambda_2 + 1}), \quad \epsilon_i = \begin{cases} 1 & (i = 4) \\ -1 & (i = 9) \end{cases}, \]
where \( P_{\text{short}} = M_{\text{short}}A_{\text{short}}N_{\text{short}} \) is a parabolic subgroup of \( G \) with the Levi factor \( L_{\text{short}} = A_{\text{short}}, \pi_{DS}^{SL^\pm(2,\mathbb{R})}(\alpha) \) is the discrete series module of \( SL^\pm(2,\mathbb{R}) \) with the minimal \( K \cap L_{\text{short}} \cong O(2) \)-type, and \( e^\beta \) is the one dimensional representation of \( A_{\text{short}} \cong \mathbb{R}_{>0} \), defined by \( a \mapsto e^{\beta \log a} \). Therefore, their \( K \)-spectra are
\[ \mathcal{K}(X(\gamma_i)) = \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{U}(2) \mid \lambda_1 - \lambda_2 \in \Lambda_1 + \epsilon_1\Lambda_2 + 1 + 2\mathbb{Z}_{\geq 0} \}, \]
with the multiplicities
\[ m_{\lambda} = \frac{1}{2} (\lambda_1 - \lambda_2 - \Lambda_1 - \epsilon_1\Lambda_2 + 1). \]

Thirdly, consider the standard modules \( X(\gamma_j), i = 5, 6, 7, 8, 0', 1', 2', 3' \). These are generalized principal series modules induced from a parabolic subgroup \( P_{\text{long}} = M_{\text{long}}A_{\text{long}}N_{\text{long}} \) of \( G \), where \( M_{\text{long}} \cong SL(2,\mathbb{R}) \times \{ \pm 1 \} \) and \( A_{\text{long}} \cong \mathbb{R}_{>0} \). In Section \( 13 \) we wrote the regular characters \( \gamma_j, j = 5, 6, 7, 8, 0', 1', 2', 3' \), in a form like
\[ \gamma_j = (\epsilon_j(\Lambda_{a(j)} + 1), \text{sgn}^{b(j)}(\Lambda_{c(j)}), \quad \epsilon_i = +1 \text{ or } -1. \]
Under this notation,
\[ X(\gamma_j) = \text{Ind}^G_{P_{\text{long}}} (\pi_{DS}^{SL(2,\mathbb{R})}(\epsilon_j(\Lambda_{a(j)} + 1)) \otimes \text{sgn}^{b(j)}(\Lambda_{c(j)} + e^{\Lambda_{c(j)} + 1}). \]
Therefore, the \( K \)-spectra \( \mathcal{K}(X(\gamma_j)) \) are
\[ \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{U}(2) \mid \exists l \in \mathbb{Z}_{\geq 0} \text{ s.t. } \lambda_1 \geq \epsilon_i(\Lambda_{a(j)} + 1 + 2l) \geq \lambda_2, \]
and \( \lambda_1 + \lambda_2 \equiv \Lambda_{a(j)} + b(j) + 1 \text{ (mod 2)} \).
and the multiplicities are

\[ m_\lambda = \# \{ l \in \mathbb{Z}_{\geq 0} \mid \lambda_1 \geq \epsilon_j (\Lambda_{a(j)} + 1 + 2l) \geq \lambda_2 \}. \]

Finally, consider the standard modules \( X(\gamma_j), j = 10, 11, 4', 0'' \). These are the principal series modules induced from a minimal parabolic subgroup of \( G \). In Section 13 we wrote the regular characters \( \gamma_j, j = 10, 11, 4', 0'' \), in a form like

\[ \gamma_j = (\sigma_{a,b}, \nu), \quad \sigma_{a,b} \in \widehat{M}_m, \quad \nu \in W(G, A_m) \cdot \Lambda. \]

Under this notation,

\[ X(\gamma_j) \cong \text{Ind}_{\mathcal{E}^m}^G (\sigma_{a,b} \otimes e^{\nu + \rho_m}). \]

Under the identification \( \{12, 1\} \), the elements \( m_1, m_2 \in M_m, \epsilon_1, \epsilon_2 \in \{ \pm 1 \} \), correspond to \( \text{diag}(\epsilon_1, \epsilon_2) \in U(2) \). Therefore, Lemma 5.1 implies the following lemma.

**Lemma 14.2.**

1. The action of \( M_m \) on the bases of \( V^{U(2)}_{(\lambda_1, \lambda_2)} \) is given by

\[ \tau_\lambda (m_1, m_2) v_q^{(\lambda_1, \lambda_2)} = \epsilon_1 \epsilon_2 v_q^{(\lambda_1 + \lambda_2)} \epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_2 v_q^{(\lambda_1 + \lambda_2)} \epsilon_1 \epsilon_2. \]

2. If \( \lambda_1 + \lambda_2 \) is even, then the restriction of \( V^{U(2)}_{(\lambda_1, \lambda_2)} \) to \( M_m \) decomposes into \( \sigma_{0,0} \) and \( \sigma_{1,1} \)-isotypic subspaces:

\[ V^{U(2)}_{(\lambda_1, \lambda_2)} = V^{U(2)}_{(\sigma_{0,0})} \oplus V^{U(2)}_{(\sigma_{1,1})}, \]

where

\[ V^{U(2)}_{(\sigma_{0,0})} = \text{Span}(v_q^{(\lambda_1, \lambda_2)} \mid q \text{ is even}), \quad V^{U(2)}_{(\sigma_{1,1})} = \text{Span}(v_q^{(\lambda_1, \lambda_2)} \mid q \text{ is odd}). \]

3. If \( \lambda_1 + \lambda_2 \) is odd, then the restriction of \( V^{U(2)}_{(\lambda_1, \lambda_2)} \) to \( M_m \) decomposes into \( \sigma_{0,1} \) and \( \sigma_{0,1} \)-isotypic subspaces:

\[ V^{U(2)}_{(\lambda_1, \lambda_2)} = V^{U(2)}_{(\sigma_{0,1})} \oplus V^{U(2)}_{(\sigma_{0,1})}, \]

where

\[ V^{U(2)}_{(\sigma_{0,1})} = \text{Span}(v_q^{(\lambda_1, \lambda_2)} \mid q \text{ is even}), \quad V^{U(2)}_{(\sigma_{0,1})} = \text{Span}(v_q^{(\lambda_1, \lambda_2)} \mid q \text{ is odd}). \]

By this lemma, the \( K \)-spectra \( \hat{K}(X(\gamma_i)) = \hat{K}(\text{Ind}_{\mathcal{E}^m}^G (\sigma_{a,b} \otimes e^{\nu + \rho_m})) \) are

\[ \{ \lambda = (\lambda_1, \lambda_2) \mid |\lambda_1 - \lambda_2| \leq 1 + 2 |\lambda_1 - \lambda_2| \geq 0 \} \quad \text{if } a \equiv b \equiv 0 \pmod{2}, \]

\[ \{ \lambda = (\lambda_1, \lambda_2), |\lambda_1 - \lambda_2| \leq 1 + 2 |\lambda_1 - \lambda_2| \geq 0 \} \quad \text{if } a \equiv b \equiv 1 \pmod{2}, \]

and the multiplicities are

\[ m_\lambda = \begin{cases} \frac{1}{2}(|\lambda_1 - \lambda_2| + 1) & \text{if } \lambda_1, \lambda_2 \text{ are even}, \\ \frac{1}{2}(|\lambda_1 - \lambda_2|) & \text{if } \lambda_1, \lambda_2 \text{ are odd}, \end{cases} \quad \text{if } a \equiv b \equiv 0 \pmod{2}, \]

\[ m_\lambda = \begin{cases} \frac{1}{2}(|\lambda_1 - \lambda_2|) & \text{if } \lambda_1, \lambda_2 \text{ are even}, \\ \frac{1}{2}(|\lambda_1 - \lambda_2| + 1) & \text{if } \lambda_1, \lambda_2 \text{ are odd}, \end{cases} \quad \text{if } a \equiv b \equiv 1 \pmod{2}, \]

\[ m_\lambda = \frac{1}{2}(|\lambda_1 - \lambda_2| + 1), \quad \text{if } a \equiv b \equiv 1 \pmod{2}. \]

From these data and Theorem 14.1 we obtain the \( K \)-spectra of all irreducible modules.

**Proposition 14.3.** Suppose that the infinitesimal character is trivial, namely \( \Lambda = (\Lambda_1, \Lambda_2) = (2, 1) \). The multiplicities of the \( K \)-types \( (\tau_\lambda, V^{U(2)}_\lambda) \) with \( \lambda = (\lambda_1, \lambda_2), |\lambda_1|, |\lambda_2| \leq 3 \), in the irreducible modules are as follows:
(1) Block $PSO(3,2)$

| Irred.Mod. | $\lambda$        | Irred.Mod. | $\lambda$        |
|-----------|------------------|-----------|------------------|
| $X(\gamma_0)$ | $(3, -1), (3, -3)$ | $X(\gamma_1)$ | $(1, -3), (3, -3)$ |
| $X(\gamma_2)$ | $(3, 3)$         | $X(\gamma_3)$ | $(-3, -3)$        |
| $X(\gamma_4)$ | $(2, -2)$         | $X(\gamma_5)$ | $(3, 1)$          |
| $X(\gamma_6)$ | $(-1, -3)$        | $X(\gamma_7)$ | $(2, 2), (2, 0)$  |
| $X(\gamma_8)$ | $(-2, -2), (0, -2)$ | $X(\gamma_9)$ | $(2, 0), (1, -1), (0, -2)$ |
| $X(\gamma_{10})$ | $(0, 0)$         | $X(\gamma_{11})$ | $+1, (1, 3), (1, -3)$ |

The multiplicities of these $K$-types are all one.

(2) Block $PSO(4,1)$

| Irred.Mod. | $\lambda$        | Irred.Mod. | $\lambda$        |
|-----------|------------------|-----------|------------------|
| $X(\gamma')$ | $(3, \pm2), (3, 0)$ | $X(\gamma'')$ | $(\pm2, -3), (0, -3)$ |
| $X(\gamma''')$ | $(2, \pm1), (2, -3)$ | $X(\gamma''''')$ | $(\pm1, -2), (3, -2)$ |
| $X(\gamma''''''')$ | $(1, 0), (0, -1), (3, 0), (2, -1)$ | $X(\gamma''''''''')$ | $(1, -2), (0, -3), (3, -2), (2, -3)$ |

The multiplicities of these $K$-types are all one.

15. Shift operators for $Sp(2, \mathbb{R})$

In this section, we write down the shift operators of $K$ types explicitly, in the case of $G = Sp(2, \mathbb{R})$. The adjoint representation of $\mathfrak{k}$ on $\mathfrak{s}$ decomposes into two irreducible representations $s_{\pm}$, whose highest weights are $(2, 0)$ and $(0, -2)$, respectively. Define

$$X_{\pm 2e_i} := H_i \pm \sqrt{-1}(X_{2f_i} + X_{-2f_i}) = H_i \pm D_i \pm 2\sqrt{-1}X_{2f_i}, \quad i = 1, 2,$$

$$X_{\pm (e_1 + e_2)} := \pm(X_{f_1 - f_2} + X_{-f_1 + f_2}) + \sqrt{-1}(X_{f_1 + f_2} + X_{-f_1 - f_2}) = 2(Y_7 \pm X_{f_1 - f_2} + \sqrt{-1}X_{f_1 + f_2}).$$

Then, $X_{\pm 2e_i}, X_{\pm 2e_2}, X_{\pm (e_1 + e_2)}$ is a basis of $s_{\pm}$, and we may identify

$$(15.1) \quad v_2^{(2, 0)} = X_{2e_i}, \quad v_1^{(2, 0)} = \frac{1}{2}X_{e_1 + e_2}, \quad v_0^{(2, 0)} = X_{2e_2},$$

$$(15.2) \quad v_0^{(0, -2)} = X_{-2e_2}, \quad v_{-1}^{(0, -2)} = \frac{1}{2}X_{-e_1 - e_2}, \quad v_{-2}^{(0, -2)} = X_{-2e_i}.$$

Define an invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by $\langle X, Y \rangle = \text{tr}(XY)$. Then

$$\langle X_{2e_i}, X_{-2e_i} \rangle = 4, \quad \langle X_{2e_2}, X_{-2e_2} \rangle = -8, \quad \langle X_{e_1 + e_2}, X_{-e_1 - e_2} \rangle = 4.$$

The operator $\nabla$ defined in Section 3 is

$$(15.3) \quad 4\nabla \phi = \sum_{i=1}^{2} \sum_{\epsilon \in \{-1, 1\}} L(H_i - \epsilon D_i - 2\sqrt{-1}\epsilon X_{2f_i})\phi \otimes X_{2e_i}$$

$$- \sum_{\epsilon \in \{-1, 1\}} L(Y_7 - \epsilon X_{f_1 - f_2} + \sqrt{-1}X_{f_1 + f_2})\phi \otimes X_{(e_1 + e_2)}.$$
The contragredient representation of $(\tau_{(\lambda_1, \lambda_2)}, V^{U(2)}_{(\lambda_1, \lambda_2)})$ is $(\tau_{(-\lambda_2 + \lambda_1)}, V^{U(2)}_{(-\lambda_2, -\lambda_1)})$.

Let $\text{pr}_{(i,j)}$, $(i, j) = \pm (2, 0), \pm (1, 1)$ or $\pm (0, 2)$, be the natural projection

$$(\tau_{(\lambda_1, \lambda_2)})^* \otimes 5 \rightarrow \tau_{(-\lambda_2 - j, -\lambda_1 - i)} \simeq (\tau_{(\lambda_1 + i, \lambda_2 + j)})^*.$$ 

We define the shift operators of $K$-types $P_{(i,j)}$ by

$$P_{(i,j)} = 4 \times \text{pr}_{(i,j)} \circ \nabla :$$

$$C^\infty(\tau_{(\lambda_1, \lambda_2)}), (K \backslash G/A_m N_m; e^{\rho}) \rightarrow C^\infty(\tau_{(\lambda_1 + i, \lambda_2 + j)}), (K \backslash G/A_m N_m; e^{\rho}).$$

An element $\phi_\lambda(g)$ in $C^\infty(\tau_{(\lambda_1, \lambda_2)}), (K \backslash G/A_m N_m; e^{\rho})$ is determined by its value at $g = e$ by Iwasawa decomposition. So we write it as

$$\phi_\lambda(e) = \sum_{q = -\lambda_1}^{-\lambda_2} c(q) v_q^{(-\lambda_2, -\lambda_1)}.$$ 

For notational convenience, we put $c(q) = 0$ if $q > -\lambda_2$ or $q < -\lambda_1$. By Lemma 5.2 and [12.2], the irreducible decompositions of $\tau_{(-\lambda_2 - \lambda_1)} \otimes \tau_{(2, 0)}$ and $\tau_{(-\lambda_2, -\lambda_1)} \otimes \tau_{(0, -2)}$ are given by

$$v_q^{(-\lambda_2, -\lambda_1)} \otimes v_i^{(2,0)} = \sum_{j=0}^{2} d^+(q, i; j) v_{q+i}^{(-\lambda_2 + j, -\lambda_1 + 2 - j)},$$

$$v_q^{(-\lambda_2, -\lambda_1)} \otimes v_i^{(0,-2)} = \sum_{j=-2}^{0} d^-(q, i; j) v_{q+i}^{(-\lambda_2 + j, -\lambda_1 - 2 - j)},$$

where

$$d^+(q, i; 2) = 1 \quad (0 \leq i \leq 2), \quad d^+(q, i; 0) = 1 \quad (-2 \leq i \leq 0),$$

$$d^+(q, 2; 1) = d^- (q, 0; -1) = - (\lambda_2 + q),$$

$$d^+(q, 2; 0) = d^- (q, 0; -2) = (\lambda_2 + q + 1)_{\downarrow (2)},$$

$$d^+(q, 1; 1) = d^- (q, -1; -1) = - \frac{1}{2} (\lambda_1 - \lambda_2 + 2q),$$

$$d^+(q, 1; 0) = d^- (q, -1; -2) = (\lambda_1 + q)(\lambda_2 + q),$$

$$d^+(q, 0; 1) = d^- (q, -2; -1) = (\lambda_1 + q),$$

$$d^+(q, 0; 0) = d^- (q, -2; -2) = (\lambda_1 + q)_{\downarrow (2)}.$$ 

Therefore, by (15.1), (15.2) and (15.3), we obtain the explicit form of the shift operators $P_{(i,j)}$ of $K$-types:

$$P_{(2,0)} \phi_\lambda(e) = \sum_{q = -\lambda_1}^{-\lambda_2} \left\{ (\nu_1 + 2\lambda_1 + 4 + q) c(q + 2) \right. + (\nu_2 + \lambda_1 + \lambda_2 + 1 + q) c(q) \} v_q^{(-\lambda_2, -\lambda_1-2)},$$

$$P_{(1,1)} \phi_\lambda(e) = - \sum_{q = -\lambda_1}^{-\lambda_2-1} \left\{ (\lambda_1 + q + 2)(\nu_1 + \lambda_1 + \lambda_2 + q + 2) c(q + 2) \right. + (\lambda_2 + q)(\nu_2 + \lambda_1 + \lambda_2 + q + 1) c(q) \} v_q^{(-\lambda_2-1, -\lambda_1-1)},$$

(15.4)
\[ P_{(0,2)} \phi_\lambda(e) = \sum_{q=-\lambda_1}^{\lambda_2 - 2} \{(\lambda_1 + q + 2)_4(\nu_1 + 2\lambda_2 + q + 2) c(q + 2)
+ (\lambda_2 + q + 1)_4(\nu_2 + \lambda_1 + \lambda_2 + q + 1) c(q)\} v_q^{(-\lambda_2 - 2, -\lambda_1)}, \]

(15.6)

\[ P_{(0,-2)} \phi_\lambda(e) = \sum_{q=-\lambda_1}^{\lambda_2 + 2} \{(\nu_1 - 2\lambda_2 - q + 4) c(q - 2)
+ (\nu_2 - \lambda_1 - \lambda_2 - q + 1) c(q)\} v_q^{(-\lambda_2 + 2, -\lambda_1)}, \]

(15.7)

\[ P_{(-1,-1)} \phi_\lambda(e) = - \sum_{q=-\lambda_1+1}^{\lambda_2 + 1} \{(\lambda_2 + q - 2)(\nu_1 - \lambda_1 - \lambda_2 - q + 2) c(q - 2)
+ (\lambda_1 + q)(\nu_2 - \lambda_1 - \lambda_2 - q + 1) c(q)\} v_q^{(-\lambda_2 + 1, -\lambda_1 + 1)}, \]

(15.8)

\[ P_{(-2,0)} \phi_\lambda(e) = \sum_{q=-\lambda_1 + 2}^{\lambda_2} \{(\lambda_2 + q - 1)_4(\nu_1 - 2\lambda_1 - q + 2) c(q - 2)
+ (\lambda_1 + q)_4(\nu_2 - \lambda_1 - \lambda_2 - q + 1) c(q)\} v_q^{(-\lambda_2, -\lambda_1 + 2)}. \]

(15.9)

16. Candidates for irreducible submodules. \(Sp(2, \mathbb{R})\) case.

In this section, we seek candidates for irreducible submodules of principal series modules. By the translation principle, we will restrict our calculation to the case when the infinitesimal character \( \Lambda = (\Lambda_1, \Lambda_2) \) is trivial, namely \( \Lambda = \rho_m = (2, 1) \).

Firstly, consider the block \( PSO(3, 2) \). The regular characters of the split Cartan subgroup which are contained in this block are \( \gamma_{10} \) and \( \gamma_{11} \). If the infinitesimal character is trivial, they are

\[ \gamma_{10} = (\sigma_{0,0}, (2,1)), \quad \gamma_{11} = (\sigma_{1,1}, (2,1)), \]

so we consider the socle filtrations of the principal series modules

\[ I(\sigma_{0,0}, w \cdot (2,1)) \quad \text{and} \quad I(\sigma_{1,1}, w \cdot (2,1)), \quad w \in W(G, H_s) \cong W(B_2). \]

For \( G = Sp(2, \mathbb{R}) \), the automorphism \( \mu \) in section 10 is given by the involution

\[ \mu : G \rightarrow G, \quad \mu(g) = x_{\mu} g (x_{\mu})^{-1}, \quad x_{\mu} := \text{diag}(1, 1, -1, -1). \]

This automorphism acts on \( H_f \) as the inverse map. Therefore \( \gamma_0 \circ \mu \) and \( \gamma_2 \circ \mu \) are conjugate to \( \gamma_1 \) and \( \gamma_3 \), respectively. Next, \( \mu \) acts on \( A_{long} \) trivially and on \( T_{long} \) as the inverse map, up to inner automorphisms. Therefore, \( \gamma_5 \circ \mu = \gamma_6, \gamma_7 \circ \mu = \gamma_8, \gamma_9 \circ \mu = \gamma_1, \) and \( \gamma_2 \circ \mu = \gamma_3 \). Finally, \( \mu \) acts on \( H_s \) and \( H_{short} \) trivially, up to inner automorphisms. Therefore, \( \overline{X}(\gamma_i), \ i = 4, 9, 10, 11, 14', 0'', \) are self-dual. Moreover, since \( \mu \) stabilizes \( N_m \),

\[ I(\sigma, \nu) \mu \cong I(\sigma, \nu). \]

By these and the horizontal symmetry, we have obtained the following:

**Lemma 16.1.**

1. \( \overline{X}(\gamma_i), \ i = 4, 9, 10, 11, 14', 0'' \) are self-dual.
2. If \( (i,j) = (0,1), (2,3), (5,6), (7,8), (0',1') \) or \( (2',3') \), then \( \overline{X}(\gamma_i) \) and \( \overline{X}(\gamma_j) \) are dual.
Lemma 16.2. The irreducible modules $\mathcal{X}(\gamma_2)$ and $\mathcal{X}(\gamma_3)$. If the infinitesimal character is trivial, then the minimal $K$-type of the holomorphic discrete series $\mathcal{X}(\gamma_2)$ is $\lambda = (\Lambda_1 + 1, \Lambda_2 + 2) = (3, 3)$, and $\lambda = (3, 1)$ is not a $K$-type of it. Therefore, if $\mathcal{X}(\gamma_2)$ is a submodule of $I(\sigma_{1,1} \otimes e^{-\nu-\rho})$, then there exists a non-zero function

$$\phi_2^{(3,3)}(\text{kan}) = a^{-\nu-\rho} r(3^{-1}, -3) (k^{-1}) c(-3) v_{-3}^{(-3, -3)}$$

which satisfies $P_{(0,-2)} \phi_2^{(3,3)} = 0$. By (15.7), this is equivalent to

$$(P_{(0,-2)} \phi_2^{(3,3)})(e) = c(-3) \left\{ (\nu_1 - 1) v_{-1}^{(-1,-3)} + (\nu_2 - 2) v_{-3}^{(-1,-3)} \right\} = 0.$$ 

Therefore, the principal series of which $\mathcal{X}(\gamma_2)$ can be a submodule is $I(\sigma_{1,1}, (1, 2))$. Since there is no other possibility, Casselman’s subrepresentation theorem implies that $\mathcal{X}(\gamma_2)$ is really a submodule of $I(\sigma_{1,1}, (1, 2))$.

By Lemma 16.1 we have obtained the next lemma.

**Lemma 16.2.** If $\Lambda$ is trivial, $\mathcal{X}(\gamma_2)$ and $\mathcal{X}(\gamma_3)$ are submodules of $I(\sigma_{1,1}, (1, 2)) = I(\sigma_{1,1}, (\Lambda_2, \Lambda_1))$, and there is no other principal series representation of which these irreducible modules $\mathcal{X}(\gamma_2), \mathcal{X}(\gamma_3)$ are submodules.

16.2. The irreducible modules $\mathcal{X}(\gamma_0)$ and $\mathcal{X}(\gamma_1)$. In the same way as in the previous subsection, we determine the principal series representations of which $\mathcal{X}(\gamma_0)$ is a submodule.

If the infinitesimal character is trivial, then the minimal $K$-type of the large discrete series $\mathcal{X}(\gamma_0)$ is $\lambda = (\Lambda_1 + 1, -\Lambda_2) = (3, -1)$, and $\lambda = (3, 1), (1, -1), (2, -2)$ are not $K$-types of it. Therefore, if $\mathcal{X}(\gamma_0)$ is a submodule of $I(\sigma_{0,0}, (\nu_1, \nu_2)) \oplus I(\sigma_{1,1}, (\nu_1, \nu_2))$, then there exists a non-zero function

$$\phi_0^{(0,-1)}(\text{kan}) = a^{-\nu-\rho} r_{(1,-3)} (k^{-1}) \sum_{q=-3}^{1} c(q) v_{q}^{(1,-3)}$$

which satisfies

$$P_{(0,2)} \phi_0^{(0,-1)} = 0 \quad \text{and} \quad P_{(-2,0)} \phi_0^{(0,-1)} = 0 \quad \text{and} \quad P_{(-1,-1)} \phi_0^{(0,-1)} = 0.$$ 

By (15.6), (15.9) and (15.8), this system of equations is equivalent to

\begin{align*}
(16.1) & \left\{ \begin{array}{l}
6(\nu_1 - 2) c(0) + 6(\nu_2 + 1) c(-2) = 0, \\
6(\nu_1 - 4) c(-2) + 6(\nu_2 - 1) c(0) = 0, \\
-3(\nu_1 - 2) c(0) = 0, \\
-3\nu_1 c(-2) + 3(\nu_2 - 1) c(0) = 0, \\
(\nu_2 + 1) c(-2) = 0;
\end{array} \right. \\
(16.2) & \left\{ \begin{array}{l}
12(\nu_1 - 1) c(1) + 2(\nu_2 + 2) c(-1) = 0, \\
2(\nu_1 - 3) c(-1) + 12(\nu_2 - 2) c(-3) = 0, \\
2(\nu_2 - 5) c(-1) + 12(\nu_2 - 2) c(1) = 0, \\
12(\nu_1 - 3) c(-3) + 2\nu_2 c(-1) = 0, \\
-2(\nu_1 - 1) c(-1) + 4(\nu_2 - 2) c(1) = 0, \\
-4(\nu_1 + 1) c(-3) + 2\nu_2 c(-1) = 0.
\end{array} \right. 
\end{align*}
Lemma 16.3.  (1) The system of equations (16.1) has a non-zero solution if and only if $\nu = (2,1) = (\Lambda_1, \Lambda_2)$ or $(2,-1) = (\Lambda_1, -\Lambda_2)$. In these cases, the solutions are

$$
\phi^0_{(3,-1)}(e) = \alpha_1 v_{0}^{(1,-3)}, \quad \phi^0_{(3,-1)}(e) = \alpha_2 (v_{0}^{(1,-3)} - v_{-2}^{(1,-3)}),
$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$, respectively.

(2) The system of equations (16.2) has a non-zero solution if and only if $\nu = (2,1) = (\Lambda_1, \Lambda_2)$, $(2,-1) = (\Lambda_1, -\Lambda_2)$ or $(1,2) = (\Lambda_2, \Lambda_1)$. In these cases, the solutions are

$$
\phi^0_{(1,-3)}(e) = \alpha \times \begin{cases} 
3v_1^{(1,-3)} - 6v_{-1}^{(1,-3)} - v_{-3}^{(1,-3)}, & \\
v_1^{(1,-3)} - 6v_{-1}^{(1,-3)} + v_{-3}^{(1,-3)}, & \alpha \in \mathbb{C},
\end{cases}
$$

respectively.

By Lemmas 14.2, 16.1 and 16.3, we have obtained the next corollary.

Corollary 16.4. If the infinitesimal character $\Lambda$ is trivial, then $\mathcal{X}(\gamma_0) \oplus \mathcal{X}(\gamma_1)$ is a submodule of $I(\sigma, \nu)$ only if

$$(\sigma, \nu) = (\sigma_0, 0), (\sigma_0, (\Lambda_1, \Lambda_2)), (\sigma_1, (\Lambda_1, \Lambda_2)), (\sigma_1, (\Lambda_1, -\Lambda_2)) \text{ or } (\sigma_1, (\Lambda_2, \Lambda_1)).$$

16.3. The irreducible modules $\mathcal{X}(\gamma_5)$ and $\mathcal{X}(\gamma_6)$. If the infinitesimal character is trivial, the representation $\mathcal{X}(\gamma_5)$ has a $K$-type $\lambda = (3,1)$ with multiplicity one, and $\lambda = (3,3), (3,-1), (2,0), (1,1)$ are not $K$-types of it. Therefore, if $\mathcal{X}(\gamma_5)$ is a submodule of $I(\sigma, \nu)$, then there exists a non-zero vector

$$
\phi^5_{(3,1)}(k\alpha) = a^{-\nu - \rho} r_{(-1,-3)}(k^{-1}) \sum_{q=-3}^{-1} c(q) v_{-3}^{(-1,-3)}
$$

which satisfies

$$
P_{(0,2)} \phi^5_{(3,1)} = 0, \quad P_{(0,-2)} \phi^5_{(3,1)} = 0, \quad P_{(-1,-1)} \phi^5_{3,1} = 0, \quad \text{and} \quad P_{(1,-2)} \phi^5_{3,1} = 0.
$$

By (16.6), (16.7), (16.8) and (16.9), this system of equations is equivalent to

$$(16.3) \quad \begin{cases} 
(\nu_1 + 2) c(-2) = 0, & (\nu_2 - 1) c(-2) = 0, \\
(\nu_1 - 2) c(-2) = 0, & (\nu_2 - 1) c(-2) = 0, \\
2(\nu_1 + 1) c(-1) + 2(\nu_2 + 2) c(-3) = 0, & \nu_1 + 1) c(-1) = 0,
\end{cases}
$$

and

$$(16.4) \quad \begin{cases} 
(\nu_1 + 3) c(-3) + (\nu_2 - 2) c(-1) = 0, \\
\nu_2 c(-3) = 0, \\
-2(\nu_1 - 1) c(-3) + 2(\nu_2 - 2) c(-1) = 0, \\
2(\nu_1 - 3) c(-3) + 2(\nu_2 - 2) c(-1) = 0.
\end{cases}
$$

Lemma 16.5.  (1) The solution of the system of equations (16.3) is \{0\}.

(2) The system of equations (16.3) has a non-trivial solution if and only if $(\nu_1, \nu_2) = (-1,2) = (-\Lambda_2, \Lambda_1)$. In this case, the solution is

$$
\phi^5_{(-1,-3)}(e) = \alpha v_{-1}^{(-1,-3)}, \quad \alpha \in \mathbb{C}.
$$
(3) If the infinitesimal character is trivial, then the module \( \overline{X}(\gamma_5) \oplus \overline{X}(\gamma_6) \) is a submodule of \( I(\sigma, \nu) \) if and only if \( (\sigma, \nu) = (\sigma_{1,1}, (-\Lambda_2, \Lambda_1)) \).

Proof. (3) By Lemma \[16.1\] \( \overline{X}(\gamma_5) \) and \( \overline{X}(\gamma_6) \) appear as a pair. Therefore, we may consider only the embedding of \( \overline{X}(\gamma_5) \). The "only if" part follows from (1) and (2).

By the subrepresentation theorem, \( \overline{X}(\gamma_5) \) is a submodule of at least one principal series representation. As a result of (1), (2), there is only one possibility, so it is the unique embedding. This completes the proof of (3). \( \square \)

16.4. The irreducible module \( \overline{X}(\gamma_4) \). If the infinitesimal character is trivial, the irreducible representation \( \overline{X}(\gamma_4) \) has a \( K \)-type \( \lambda = (2, -2) \) with multiplicity one, and \( \lambda = (3, -1), (2, 0), (1, -3), (0, -2) \) are not a \( K \)-type of it. Therefore, if \( \overline{X}(\gamma_4) \) is a submodule of \( I(\sigma, \nu) \), then there exists a non-zero function

\[
\phi_{(2, -2)}^4(\kappa \alpha) = a^{-\nu - \rho} r_{(2, -2)}(k^{-1}) \sum_{q=-2}^{2} c(q) v_{q}^{(2, -2)}
\]

which satisfies

\[
P_{(1,1)}\phi_{(2, -2)}^4 = 0, \quad P_{(0,2)}\phi_{(2, -2)}^4 = 0, \quad P_{(-1, -1)}\phi_{(2, -2)}^4 = 0, \quad P_{(-2, 0)}\phi_{(2, -2)}^4 = 0.
\]

By \[15.5\], \[15.6\], \[15.8\] and \[15.9\], this system of equations is equivalent to

\[
\begin{cases}
-4(\nu_1 + 2) c(2) + 2(\nu_2 + 1) c(0) = 0, \\
-2\nu_1 c(0) + 4(\nu_2 - 1) c(-2) = 0, \\
12(\nu_1 - 2) c(2) + 2(\nu_2 + 1) c(0) = 0, \\
2(\nu_1 - 4) c(0) + 12(\nu_2 - 1) c(-2) = 0, \\
2\nu_1 c(0) - 4(\nu_2 - 1) c(2) = 0, \\
4(\nu_1 + 2) c(-2) - 2(\nu_2 + 1) c(0) = 0, \\
2(\nu_1 - 4) c(0) + 12(\nu_2 - 1) c(2) = 0, \\
12(\nu_1 - 2) c(-2) + 2(\nu_2 + 1) c(0) = 0, \\
(\nu_2 + 2) c(1) = 0, \\
-3(\nu_2 + 1) c(1) + 3\nu_2 c(-1) = 0, \\
-(\nu_1 - 1) c(-1) = 0, \\
6(\nu_1 - 3) c(1) + 6\nu_2 c(-1) = 0, \\
(\nu_1 - 1) c(1) = 0, \\
3(\nu_1 + 1) c(-1) - 3\nu_2 c(1) = 0, \\
-\nu_2 c(1) = 0, \\
6(\nu_1 - 3) c(-1) + 6\nu_2 c(1) = 0.
\end{cases}
\]

Lemma 16.6. (1) The system of equations \[16.5\] has a non-zero solution if and only if \( (\nu_1, \nu_2) = (1, 2) = (\Lambda_2, \Lambda_1) \) or \( (1, -2) = (\Lambda_2, -\Lambda_1) \). In these cases, the solutions are

\[
\phi_{(2, -2)}^4(e) = \alpha \times \begin{cases}
\nu_2^{(2, -2)} + 2\nu_0^{(2, -2)} + v_{-2}^{(2, -2)} \\
\nu_2^{(2, -2)} - 6\nu_0^{(2, -2)} + v_{-2}^{(2, -2)}
\end{cases}, \quad \alpha \in \mathbb{C},
\]

respectively.
(2) The system of equations (16.6) has a non-zero solution if and only if \((\nu_1, \nu_2) = (1, -2) = (\Lambda_2, -\Lambda_1)\). In this case, the solution is
\[
\phi_{(2, -2)}(e) = \alpha (v_1^{(2, -2)} - v_{-1}^{(2, -2)}), \quad \alpha \in \mathbb{C}.
\]

(3) If the infinitesimal character is trivial, then \(X(\gamma_4)\) can be a submodule of \(I(\sigma, \nu)\) only if
\[
(\sigma, \nu) = (\sigma_{0, 0}, (\Lambda_2, \Lambda_1)), (\sigma_{0, 0}, (-\Lambda_2, -\Lambda_1)) \text{ or } (\sigma_{1, 1}, (\Lambda_2, -\Lambda_1)).
\]

16.5. The irreducible modules \(X(\gamma_7)\) and \(X(\gamma_8)\). If the infinitesimal character is trivial, then \(\lambda = (2, 0)\) is a \(K\)-type of the irreducible representation \(X(\gamma_7)\) with multiplicity one, and \(\lambda = (3, 1), (2, -2), (1, -1), (0, 0)\) are not. Therefore, if \(X(\gamma_7)\) is a submodule of \(I(\sigma, \nu)\), then there exists a non-zero function
\[
\phi_7^{(2, 0)}(\text{kan}) = a^{\nu-\rho} \tau_{(0, -2)}(k^{-1}) \sum_{q=-2}^{0} c(q) v_q^{(0, -2)}
\]
which satisfies
\[
P_{(1, 1)} \phi_7^{(2, 0)} = 0, \quad P_{(0, -2)} \phi_7^{(2, 0)} = 0, \quad P_{(-1, -1)} \phi_7^{(2, 0)} = 0, \quad P_{(-2, 0)} \phi_7^{(2, 0)} = 0.
\]

Since \(X(\gamma_7)\) is not a composition factor of \(I(\sigma_{1, 1}, \nu) \approx X(\gamma_{11})\) (Theorem 14.1), we may assume that \(c(q) = 0\) if \(q\) is odd. Then, by (15.5), (15.7), (15.8) and (15.9), this system of equations is equivalent to
\[
\begin{align*}
-2(\nu_1 + 2) c(0) + 2(\nu_2 + 1) c(-2) &= 0, \\
(\nu_1 + 2) c(0) &= 0, \\
(\nu_2 + 1) c(-2) &= 0, \\
2\nu_1 c(-2) - 2(\nu_2 - 1) c(0) &= 0, \\
2(\nu_1 - 2) c(-2) + 2(\nu_2 - 1) c(0) &= 0.
\end{align*}
\]

Lemma 16.7. (1) There exists a non-trivial solution of (16.7) if and only if \((\nu_1, \nu_2) = (-2, 1) = (-\Lambda_1, \Lambda_2)\). In this case, the solution is
\[
\phi_7^{(0, -2)}(e) = \alpha v_0^{(0, -2)}, \quad \alpha \in \mathbb{C}.
\]

(2) If the infinitesimal character is trivial, the the representation \(X(\gamma_7) \oplus X(\gamma_8)\) is a submodule of \(I(\sigma, \nu)\) if and only if \((\sigma, \nu) = (\sigma_{0, 0}, (-\Lambda_1, \Lambda_2))\).

Proof. The proof of (2) is the same as that of Lemma 16.5.

16.6. The irreducible module \(X(\gamma_9)\). If the infinitesimal character is trivial, then \(\lambda = (0, -2)\) is a \(K\)-type of the irreducible representation \(X(\gamma_9)\) with multiplicity one, but \(\lambda = (0, 0), (-1, -3), (-2, -2)\) are not. Therefore, if \(X(\gamma_9)\) is a subrepresentation of \(I(\sigma, \nu)\), then there exists a non-zero function
\[
\phi_9^{(0, -2)}(\text{kan}) = a^{\nu-\rho} \tau_{(2, 0)}(k^{-1}) \sum_{q=0}^{2} c(q) v_q^{(2, 0)}
\]
which satisfies
\[
P_{(0, 2)} \phi_9^{(0, -2)} = 0, \quad P_{(-1, -1)} \phi_9^{(0, -2)} = 0, \quad P_{(-2, 0)} \phi_9^{(0, -2)} = 0.
\]
By (15.6), (15.8) and (15.9), this system of equations is equivalent to
\[
\begin{align*}
2(v_1 - 2)c(2) + 2(v_2 - 1)c(0) &= 0, \\
2(v_1 + 2)c(0) - 2(v_2 + 1)c(2) &= 0, \\
2v_1 c(0) + 2(v_2 + 1)c(2) &= 0, \\
(v_1 + 1)c(1) &= 0, \\
-(v_2 + 2)c(1) &= 0.
\end{align*}
\]

(16.8)

(16.9)

Lemma 16.8. (1) The system of equations (16.8) has a non-zero solution if and only if \((\nu_1, \nu_2) = (2, -1) = (\Lambda_1, -\Lambda_2), (-1, 2) = (-\Lambda_2, \Lambda_1)\) or \((-1, -2) = (-\Lambda_2, -\Lambda_1)\). In these cases, the solutions are
\[
\phi^0_{(0,-2)}(e) = \alpha_1 v_2^{(2,0)}, \quad \alpha_2(v_2^{(2,0)} + 3v_0^{(2,0)}) \quad \text{and} \quad \alpha_3(v_2^{(2,0)} - v_0^{(2,0)}),
\]
\[
\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \text{ respectively.}
\]

(2) The system of equations (16.9) has a non-zero solution if and only if \((\nu_1, \nu_2) = (-1, -2) = (-\Lambda_2, -\Lambda_1)\). In this case, the solution is
\[
\phi^0_{(0,-2)}(e) = \alpha_4 v_1^{(2,0)}, \quad \alpha_4 \in \mathbb{C}.
\]

(3) If the infinitesimal character \(\Lambda\) is trivial, then the irreducible representation \(\overline{X}(\gamma_0)\) can be a submodule of \(I(\sigma, \nu)\) only if
\[
(\sigma, \nu) = (\sigma_{0,0}, (\Lambda_1, -\Lambda_2)), (\sigma_{0,0}, (\Lambda_2, \Lambda_1)),
\]
\[
(\sigma_{0,0}, (-\Lambda_2, -\Lambda_1)) \text{ or } (\sigma_{1,1}, (\Lambda_2, -\Lambda_1)).
\]

16.7. The irreducible module \(\overline{X}(\gamma_{11})\). If the infinitesimal character is trivial, the irreducible representation \(\overline{X}(\gamma_{11})\) has the \(K\)-types \(\lambda = (-1, -1)\) and \((-1, -3)\) with multiplicity one, and \(\lambda = (-3, -3), (0, -2)\) are not \(K\)-types of it. Therefore, if \(\overline{X}(\gamma_{11})\) is a submodule of \(I(\sigma_{1,1}, \nu)\), then there exists a non-zero function
\[
\phi^{11}_{(-1, -1)}(kan) = a^{-\nu - \rho_{\gamma_{11}}} (k^{-1}) c v_1^{(1,1)}
\]
which satisfies
\[
P_{(-2,0)} \circ P_{(0,-2)} \phi^{11}_{(-1,-1)} = 0, \quad P_{(1,1)} \circ P_{(0,-2)} \phi^{11}_{(-1,-1)} = 0.
\]

By (15.7), (15.9) and (15.5), this system of equations is equivalent to
\[
\begin{align*}
(2(v_1 + 1)(v_2 + 2) + 2(v_2 + 2)(v_1 + 3))c &= 4(v_1 + 2)(v_2 + 2)c = 0, \\
(-2(v_1 - 1)(v_1 + 3) + 2(v_2 - 2)(v_2 + 2))c &= 2(-v_1 + 1)^2 + v_2^2)c = 0.
\end{align*}
\]

Since we are assuming the infinitesimal character \(\Lambda\) is \(\rho_m = (2, 1)\), the parameter \(\nu = (\nu_1, \nu_2)\) is conjugate to \(\rho_m\) under the action of Weyl group \(W(B_2)\). It follows that this system of equations has a non-zero solution if and only if \((\nu_1, \nu_2) = (-2, 1), (-2, -1)\) or \((1, -2)\).

Lemma 16.9. If the infinitesimal character \(\Lambda\) is trivial, then the irreducible representation \(\overline{X}(\gamma_{11})\) can be a submodule of \(I(\sigma, \nu)\) only if
\[
(\sigma, \nu) = (\sigma_{1,1}, (-\Lambda_1, \Lambda_2)), (\sigma_{1,1}, (-\Lambda_1, -\Lambda_2)) \text{ or } (\sigma_{1,1}, (\Lambda_2, -\Lambda_1)).
\]

In these cases, the corresponding vectors \(P_{(0,-2)} \phi^{11}_{(-1,-1)}(e)\) are given by
\[
P_{(0,-2)} \phi^{11}_{(-1,-1)}(e) = \alpha_1 v_3^{(3,1)} + 3v_1^{(3,1)}, \quad \alpha_2(v_3^{(3,1)} + v_1^{(3,1)}), \quad \text{and} \quad \alpha_3 v_3^{(3,1)},
\]
\[
\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \text{ respectively.}
16.8. The irreducible module $\overline{X}(\gamma_{10})$. If the infinitesimal character is trivial, then the irreducible representation $\overline{X}(\gamma_{10})$ is the trivial representation of $G$. Its $K$-type is $\lambda = (0, 0)$ alone. If it is a submodule of $I(\sigma, \nu)$, then there exists a non-zero function

$$\phi_{(0,0)}^{(10)}(kan) = a^{-\nu - \rho} r_{0}(0,0)(k^{-1})c v_{0}^{(0,0)}$$

which satisfies $P_{(2,0)}\phi_{(0,0)}^{(10)} = 0$ and $P_{(0,-2)}\phi_{(0,0)}^{(10)} = 0$. By (15.4) and (15.7), this system of equations is equivalent to

$$(\nu_1 + 2) c = (\nu_2 + 1) c = 0.$$  

**Lemma 16.10.** If the infinitesimal character $\Lambda$ is trivial, then the irreducible module $\overline{X}(\gamma_{10})$ is a submodule of $I(\sigma, \nu)$ if and only if $(\sigma, \nu) = (\sigma_{0,1}, (\pm \Lambda_1, -\Lambda_2))$.

16.9. The irreducible modules $\overline{X}(\gamma_{0'})$ and $\overline{X}(\gamma_{1'})$. Since $\overline{X}(\gamma_{0'})$ and $\overline{X}(\gamma_{1'})$ are dual, they appear in a pair, by the horizontal symmetry.

We know that $\lambda = (3, 2)$ is a $K$-type of $\overline{X}(\gamma_{0'})$ but $\lambda = (2, 1)$ is not. So if $\overline{X}(\gamma_{0'})$ is a submodule of $I(\sigma, (\nu_1, \nu_2))$, then there exists a non-zero vector

$$\phi_{(3,2)}^{(0')}(kan) = a^{-\nu - \rho} r_{-2,-3}(k^{-1}) \sum_{q=-3}^{-2} c(q) v_{q}^{(-2,-3)}$$

which satisfies

$$P_{(-1,-1)}\phi_{(3,2)}^{(0')}(e) = 0.$$  

By (15.8), this system of equations is equivalent to

$$(16.10) \quad (\nu_1 - 2) c(-3) = 0, \quad -(\nu_2 - 2) c(-2) = 0.$$  

**Lemma 16.11.** Suppose that the infinitesimal character is trivial.

1. The system of equations (16.10) has a non-zero solution if and only if $(\nu_1, \nu_2) = (2, \pm 1) = (\Lambda_1, \pm \Lambda_2)$ or $(\pm 1, 2) = (\pm \Lambda_2, \Lambda_1)$. In these cases, the corresponding vectors $\phi_{(3,2)}^{(0')}(e)$ are constant multiple of $v_{-3}^{(-2,-3)}$ and $v_{-2}^{(-2,-3)}$, respectively.

2. The module $\overline{X}(\gamma_{0'}) \oplus \overline{X}(\gamma_{1'})$ can be a submodule of $I(\sigma, \nu)$ only if $(\sigma, \nu) = (\sigma_{0,0}, (\Lambda_1, \pm \Lambda_2))$ or $(\sigma_{0,1}, (\pm \Lambda_2, \Lambda_1))$.

16.10. The irreducible modules $\overline{X}(\gamma_{2'})$ and $\overline{X}(\gamma_{3'})$. We know that $\lambda = (2, 1)$ is a $K$-type of $\overline{X}(\gamma_{2'})$ but $\lambda = (3, 2)$ and $(1, 0)$ are not. So if $\overline{X}(\gamma_{2'})$ is a submodule of $I(\sigma, (\nu_1, \nu_2))$, then there exists a non-zero vector

$$\phi_{(2,1)}^{(0')}(kan) = a^{-\nu - \rho} r_{-1,-2}(k^{-1}) \sum_{q=-2}^{1} c(q) v_{q}^{(-1,-2)}$$

which satisfies

$$P_{(1,1)}\phi_{(2,1)}^{(0')}(e) = 0, \quad P_{(-1,-1)}\phi_{(2,1)}^{(0')}(e) = 0.$$  

By (15.9) and (15.10), this system of equations is equivalent to

$$(16.11) \quad (\nu_2 + 2) c(-2) = (\nu_1 - 1)c(-2) = 0,$$

$$(16.12) \quad -(\nu_1 + 2) c(-1) = -(\nu_2 - 1)c(-1) = 0.$$  

**Lemma 16.12.** Suppose that the infinitesimal character is trivial.
(1) The system of equations (16.14) has a non-zero solution if and only if 
\((\nu_1, \nu_2) = (1, -2) = (A_2, -A_1)\). In this case, the corresponding solution is a constant multiple of \(v^{(-1,-2)}_{-1}\).

(2) The system of equations (16.12) has a non-zero solution if and only if 
\((\nu_1, \nu_2) = (-2, 1) = (-A_1, A_2)\). In this case, the corresponding solution is a constant multiple of \(v^{(-1,-2)}_{-1}\).

(3) The module \(\mathfrak{X}(\gamma_2') \oplus \mathfrak{X}(\gamma_3')\) can be a submodule of \(I(\sigma, \nu)\) only if \((\sigma, \nu) = (\sigma_{0,1}, (A_2, -A_1))\) or \((\sigma_{1,0}, (-A_1, A_2))\).

16.11. The irreducible module \(\mathfrak{X}(\gamma_4')\). The irreducible module \(\mathfrak{X}(\gamma_4')\) has a 
\(K\)-type \(\lambda = (1, 0)\) with multiplicity one, and \(\lambda = (2, 1)\) is not a \(K\)-type of it. Therefore, if \(\mathfrak{X}(\gamma_4')\) is a submodule of \(I(\sigma, \nu)\), then there are non-zero vectors

\[
\phi_{(1,0)}^\nu(k\alpha n) = a^{-\nu-\rho} \tau_{(0,-1)}(k^{-1}) \sum_{q=-1}^{0} c(q) v^{(0,-1)}_q
\]

which satisfies

(16.13) \(P_{(1,1)} \phi_{(1,0)}^\nu(e) = (\nu_2 + 1) c(-1) v^{(-1,-2)}_{-1} - (\nu_1 + 1) c(0) v^{(-1,-2)}_{0} = 0\).

Lemma 16.13. Suppose that the infinitesimal character is trivial.

(1) The system of equations (16.13) has a non-zero solution if and only if 
\((\nu_1, \nu_2) = (-2, 1) = (-A_1, A_2)\) or \((1, 2) = (A_2, -A_1)\). In these cases, the corresponding vectors \(\phi_{(1,0)}^\nu(e)\) are constant multiples of \(v_{-1}^{(0,-1)}\) and \(v_{0}^{(0,-1)}\), respectively.

(2) \(\mathfrak{X}(\gamma_4')\) can be a submodule of \(I(\sigma, \nu)\) only if \((\sigma, \nu) = (\sigma_{0,1}, (\pm A_1, -A_2))\) or \((\sigma_{0,1}, (-A_2, \pm A_1))\).

16.12. Summary of the candidates for submodules of principal series. We summarize the results obtained in this section.

Proposition 16.14. Suppose that the infinitesimal character \(\Lambda\) is trivial.

(1) (Block \(PSO(3,2)\)) The candidates for irreducible submodules of the principal series modules \(I(\sigma_i, \nu), i = 0, 1, \nu \in W(B_2) \cdot \Lambda\) are as follows:

| \(\sigma\) | \(\nu\) | Irred.Mod \(\mathfrak{X}(\gamma_j)\) | \(\nu\) | Irred.Mod \(\mathfrak{X}(\gamma_j)\) |
|---|---|---|---|---|
| \(\sigma_{0,0}\) | \((A_1, A_2)\) | 
\((A_1, -A_2)\) | \(j = 0, 1\) | 
\((-A_1, -A_2)\) | \(j = 10\) |
| \((A_2, A_1)\) | \(j = 4\) | 
\((-A_2, -A_1)\) | \(j = 9\) |
| \((A_2, -A_1)\) | \(j = 4\) | 
\((-A_2, A_1)\) | \(j = 9\) |
| \(\sigma_{1,1}\) | \((A_1, A_2)\) | 
\((A_1, -A_2)\) | \(j = 0, 1\) | 
\((-A_1, -A_2)\) | \(j = 11\) |
| \((A_2, A_1)\) | \(j = 0, 1\) | 
\((-A_2, -A_1)\) | \(j = 11\) |
| \((A_2, -A_1)\) | \(j = 4, 11\) | 
\((-A_2, A_1)\) | \(j = 5, 6\) |

(2) The candidates in (1) are actually submodules at least except for the following two cases:

(a) \((\sigma, \nu) = (\sigma_{0,0}, (A_1, -A_2)), \) irreducible factor \(\mathfrak{X}(\gamma_9)\).

(b) \((\sigma, \nu) = (\sigma_{1,1}, (A_2, -A_1)), \) irreducible factors \(\mathfrak{X}(\gamma_4)\) and \(\mathfrak{X}(\gamma_{11})\).

(3) (Block \(PSO(4,1)\)) The candidates for submodules of the principal series representations \(I(\sigma, \nu), (\sigma, \nu) \in W(B_2) \cdot (\sigma_{1,0}, \Lambda)\) are as follows:
\begin{align*}
\sigma & \quad \nu \quad \text{Irred.Mod } \chi(\gamma_j) \quad \nu \quad \text{Irred.Mod } \chi(\gamma_j) \\
\sigma_{1.0} & \quad (\Lambda_1, \Lambda_2) \quad j = 0', 1' \quad (-\Lambda_1, -\Lambda_2) \quad j = 4' \\
& \quad (\Lambda_1, -\Lambda_2) \quad j = 0', 1', 4' \quad (-\Lambda_1, \Lambda_2) \quad j = 2', 3' \\
\sigma_{0.1} & \quad (\Lambda_2, \Lambda_1) \quad j = 0' \quad (-\Lambda_2, \Lambda_1) \quad j = 4' \\
& \quad (\Lambda_2, -\Lambda_1) \quad j = 2', 3' \quad (-\Lambda_2, \Lambda_1) \quad j = 0', 1', 4'
\end{align*}

**Proof.** (1) and (3) are summaries of the results obtained in this section.

(2) Firstly, for a fixed principal series, if there is only one candidate for its submodule, then it is really the unique submodule. From the table in (1), we see that results analogous to (1) holds for every discrete series module and every principal series. Then it is actually a submodule because of the Casselman's subrepresentation theorem. It follows that \( \chi(\gamma_{10}) \) in \( I(\sigma_{0.0}, (-\Lambda_1, -\Lambda_2)); \chi(\gamma_9) \) in \( I(\sigma_{1.0}, (-\Lambda_2, -\Lambda_1)), i = 0, 1 \), and \( I(\sigma_{0.0}, (-\Lambda_2, \Lambda_1)); \chi(\gamma_4) \) in \( I(\sigma_{0.0}, (\Lambda_2, \pm\Lambda_1)); \) and \( \chi(\gamma_{11}) \) in \( I(\sigma_{1.1}, (-\Lambda_1, \pm\Lambda_2)) \) are submodules.

Secondly, if an irreducible module can be a submodule of only one principal series, then it is actually a submodule because of the Casselman's subrepresentation theorem. It follows that \( \chi(\gamma), \chi(\gamma_8) \) in \( I(\sigma_{0.0}, (-\Lambda_1, \Lambda_2)); \) and \( \chi(\gamma_5), \chi(\gamma_6) \) in \( I(\sigma_{1.1}, (-\Lambda_2, \Lambda_1)) \) are submodules.

Thirdly, by a theorem of Yamashita (21), embeddings of discrete series modules into principal series modules are characterized by the system of equations \( P_{\Phi} \phi_\lambda = 0 \), if its infinitesimal character is far from the walls. Here \( \alpha \) runs through the set of positive noncompact roots and \( \lambda \) is the minimal \( K \)-type of this discrete series. Though we calculated only the case of trivial infinitesimal character, it is easy to see that results analogous to (1) holds for every discrete series module and every infinitesimal character. It follows that the \( \chi(\gamma_0), \chi(\gamma_2), \chi(\gamma_3) \) in the list in (1) are submodules.

Afterwords, we will show that all irreducible factors in (2) are submodules.

In order to determine the socle filtration of principal series representations, we use the tools explained in Section 4.

First of all, for \( G = Sp(2, \mathbb{R}) \), the dual module \( I(\sigma, \nu)^* \) is isomorphic to \( I(\sigma, -\nu) \).

Secondly, the integral intertwining operators for \( G = Sp(2, \mathbb{R}) \) are

\begin{align}
(16.14) & \quad I(\sigma_{i,j}, (\Lambda_1, \Lambda_2)) \to I(\sigma_{j,i}, (\Lambda_2, \Lambda_1)) \to I(\sigma_{j,i}, (\Lambda_2, -\Lambda_1)) \\
& \quad \to I(\sigma_{i,j}, (-\Lambda_1, \Lambda_2)) \to I(\sigma_{i,j}, (-\Lambda_1, -\Lambda_2)) \quad \text{and}

(16.15) & \quad I(\sigma_{i,j}, (\Lambda_1, \Lambda_2)) \to I(\sigma_{i,j}, (\Lambda_1, -\Lambda_2)) \to I(\sigma_{j,i}, (-\Lambda_2, \Lambda_1)) \\
& \quad \to I(\sigma_{i,j}, (-\Lambda_2, -\Lambda_1)) \to I(\sigma_{i,j}, (-\Lambda_2, -\Lambda_2)).
\end{align}

By using these intertwining operators, we can deduce some information about the socle filtrations.

**Lemma 16.15.** Suppose that the infinitesimal character is trivial.

1. The irreducible module \( \chi(\gamma_9) \) lies in the socle of \( I(\sigma_{0.0}, (\Lambda_1, -\Lambda_2)) \).
2. The socle of \( I(\sigma_{1.1}, (\Lambda_2, -\Lambda_1)) \) is \( \chi(\gamma_4) \) or \( \chi(\gamma_4) \oplus \chi(\gamma_{11}) \).
3. There exist isomorphisms \( I(\sigma_{0.0}, (\pm\Lambda_2, \Lambda_1)) \simeq I(\sigma_{0.0}, (\pm\Lambda_2, -\Lambda_1)) \) and \( I(\sigma_{1.1}, (\pm\Lambda_1, \Lambda_2)) \simeq I(\sigma_{1.1}, (\pm\Lambda_1, -\Lambda_2)) \).

**Proof.** (1) Suppose \( \chi(\gamma_9) \) is not in the socle of \( I(\sigma_{0.0}, (\Lambda_1, -\Lambda_2)) \). Then by Proposition 16.14 (1), the socles of \( I(\sigma_{0.0}, (\Lambda_1, \Lambda_2)) \) and \( I(\sigma_{0.0}, (\Lambda_1, -\Lambda_2)) \) are both \( \chi(\gamma_9) \oplus \chi(\gamma_1) \), and the multiplicities of these irreducible factors in the principal series modules are both one. Since there is a non-zero intertwining operators between these
two principal series modules, they must be isomorphic. On the other hand, the socles of their dual modules $I(\sigma_{0,0}, (-\Lambda_1, -\Lambda_2))$ and $I(\sigma_{0,0}, (-\Lambda_1, \Lambda_2))$ are different; the socle of the former is $\mathcal{X}(\gamma_{10})$ but that of the latter is $\mathcal{X}(\gamma_7) \oplus \mathcal{X}(\gamma_8)$. This is a contradiction, so $\mathcal{X}(\gamma_{11})$ lies in the socle of $I(\sigma_{0,0}, (\Lambda_1, -\Lambda_2))$.

(2) is proved just in the same way as in (1). There is a non-zero intertwining operator $I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1)) \rightarrow I(\sigma_{1,1}, (-\Lambda_1, \Lambda_2))$. Compare the submodules of these principal series modules and those of their dual modules. Then we know that the socles of them must be different, so the socle of $I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1))$ does not consist of $\mathcal{X}(\gamma_{11})$ alone.

(3) There are non-zero intertwining operators between these pairs of principal series modules. For each pair, Proposition 16.14 says that the two principal series modules have the same socles. Moreover, the multiplicities of $\mathcal{X}(\gamma_{11})$, $j = 0, 1, 4, 11$, in these principal series modules are all one. It follows that these intertwining operators are isomorphisms.

Next, consider the parity condition (Corollary 4.5). The lengths of irreducible modules of $Sp(2, \mathbb{R})$ are as follows:

| Length | Irred. Mod. $\mathcal{X}(\gamma_j)$ |
|--------|----------------------------------|
| 0      | $j = 0, 1, 2, 3$                 |
| 1      | $j = 4, 5, 6, 0', 1'$           |
| 2      | $j = 7, 8, 9, 2', 3'$           |
| 3      | $j = 10, 11, 4', 0'$            |

By Proposition 16.14 the lengths of the irreducible factors in the socle of each principal series module have the same parity. So we can apply Corollary 4.5 to these cases. Especially, the socle filtrations of the principal series modules $I(\sigma, \nu)$ in the block $PSO(4, 1)$ are completely determined by the parity condition.

For notational convenience, we use the notation $\mathcal{I}_{11.3}$.

**Theorem 16.16.** Let $\Lambda = (\Lambda_1, \Lambda_2)$ be a dominant nonsingular integral infinitesimal character of $G = Sp(2, \mathbb{R})$. Then the socle filtrations of the principal series modules $I(w \cdot (\sigma_{1,1}, \Lambda_2 + 1), (\Lambda_1, \Lambda_2))$, $w \in W(B_2)$, are:

| $I(\sigma_{1,1}, \Lambda_2 + 1, (\Lambda_1, \Lambda_2))$ | $I(\sigma_{1,1}, \Lambda_2 + 1, (-\Lambda_1, -\Lambda_2))$, $I(\sigma_{1,1}, \Lambda_2 + 1, (-\Lambda_1, \Lambda_2)), I(\sigma_{1,1}, \Lambda_2 + 1, (\Lambda_1, -\Lambda_2))$ |
|-----------------------------------------------|-----------------------------------------------|
| $\mathcal{T} \oplus \mathcal{V}$               | $\mathcal{V} \oplus \mathcal{I}'$           |
| $\mathcal{V} \oplus \mathcal{I}'$             | $\mathcal{T}$                                |

| $I(\sigma_{1,1}, \Lambda_2 + 1, (\Lambda_1, -\Lambda_2))$, $I(\sigma_{1,1}, \Lambda_2 + 1, (-\Lambda_1, \Lambda_2))$, $I(\sigma_{1,1}, \Lambda_2 + 1, (\Lambda_1, -\Lambda_2))$ |
|-----------------------------------------------|-----------------------------------------------|
| $\mathcal{V} \oplus \mathcal{I}'$           | $\mathcal{V} \oplus \mathcal{I}'$           |
| $\mathcal{V} \oplus \mathcal{I}'$           | $\mathcal{V} \oplus \mathcal{I}'$           |

**Proof.** We will show this theorem in the case when the infinitesimal character $\Lambda$ is trivial, namely $\Lambda = (\Lambda_1, \Lambda_2) = (2, 1)$. If this is done, then the cases of general infinitesimal characters follow from the translation principle.

Firstly, we will show that all the candidates in Proposition 16.14 (3) are actually submodules. By the horizontal symmetry, $\mathcal{V} \oplus \mathcal{I}'$ and $\mathcal{T} \oplus \mathcal{V}$ appear in the same places of the socle filtrations of principal series modules. Since every non-zero module has its socle (this is trivial!), the candidates in Proposition 16.14 (3) are actually submodules at least $\sigma, \nu \neq (\sigma_{1,0}, (\Lambda_1, -\Lambda_2))$ and $\sigma, \nu \neq (\sigma_{0,1}, (-\Lambda_2, -\Lambda_1))$. 

For these remaining cases, the socles of their dual modules are \( \mathcal{F} \oplus \mathcal{F} \). We know that the lengths of \( X(\gamma_j) \) are odd if \( j = 0', 1', 4' \) and even if \( j = 2', 3' \). We also know from Theorem 14.1 that the irreducible factors of the principal series modules \( I(w \cdot (\sigma_{0,0}, (2, 1))) \) are \( X(\gamma_j) \), \( j = 0', 1', 2', 3', 4' \), whose multiplicities are all one. It follows from the parity condition (Corollary 4.5) that the socles of both \( I(\sigma_{0,0}, (\Lambda_1, -\Lambda_2)) \) and \( I(\sigma_{0,1}, (-\Lambda_2, -\Lambda_1)) \) are \( \mathcal{F} \oplus \mathcal{T} \oplus \mathcal{T} \).

Finally, for each principal series module, there is only one possibility of the socle filtration which satisfies Proposition 16.14 the parity condition (Corollary 4.5) and the dual principal series property (Section 4.1). The possible filtration is the one indicated in the theorem.

\[ \square \]

17. Socle filtrations of \( I(w \cdot (\sigma_{\Lambda_1, \Lambda_2+1}, \Lambda)) \), \( w \in W(B_2) \)

In this section, we determine the socle filtrations of the principal series modules which are isomorphic to \( X(\gamma_{10}) \) in the Grothendieck group.

Firstly we summarize the results.

**Theorem 17.1.** Let \( \Lambda = (\Lambda_1, \Lambda_2) \) be a nonsingular dominant integral infinitesimal character of \( Sp(2, \mathbb{R}) \). The socle filtrations of \( I(w \cdot (\sigma_{\Lambda_1, \Lambda_2+1}, \Lambda)) \), \( w \in W(B_2) \), are as follows:

|   | \( I(\sigma_{\Lambda_1, \Lambda_2+1}, (\Lambda_1, \Lambda_2)) \) | \( I(\sigma_{\Lambda_1, \Lambda_2+1}, (-\Lambda_1, -\Lambda_2)) \) |
|---|---|---|
| (1) | \( 0 \oplus \mathcal{T} \) | \( 0 \oplus \mathcal{T} \) |
|   | \( \mathcal{T} \oplus \mathcal{S} \oplus \mathcal{G} \) | \( \mathcal{T} \oplus \mathcal{S} \oplus \mathcal{G} \) |
| | \( \mathcal{T} \oplus \mathcal{I} \oplus \mathcal{F} \oplus \mathcal{G} \) | \( \mathcal{T} \oplus \mathcal{I} \oplus \mathcal{F} \oplus \mathcal{G} \) |

Let us prove this theorem.

By Theorem 14.1 the composition factors of this principal series module are \( \mathcal{F} \oplus \mathcal{T}, 2 \times \mathcal{T}, \mathcal{F} \oplus \mathcal{S}, \mathcal{F}, \mathcal{T}, \mathcal{I}, \mathcal{G} \).

Here, \( \mathcal{F} \oplus \mathcal{T} \) means that these two factors appear in a pair (cf. Corollary 4.3) and \( 2 \times \mathcal{T} \) means that the multiplicity of \( \mathcal{T} \) is two (and those of others are one). The lengths of irreducible modules are indicated above Theorem 16.16.

Hereafter, we calculate the socle filtration of principal series modules in the case when the infinitesimal character \( \Lambda \) is trivial, namely \( \Lambda = (\Lambda_1, \Lambda_2) = (2, 1) \). In this case, \( \sigma_{\Lambda_1, \Lambda_2+1} \) and \( \sigma_{\Lambda_2+1, \Lambda_1} \) are both \( \sigma_{0,0} \).

17.1. **Proof of Theorem 17.1 (1).** By Proposition 16.14 (1), the socle of the module \( I(\sigma_{0,0}, (\Lambda_1, \Lambda_2)) \) is \( \mathcal{F} \oplus \mathcal{T} \) and that of \( I(\sigma_{0,0}, (-\Lambda_1, -\Lambda_2)) \) is \( \mathcal{T} \). The remaining factors are \( \mathcal{T} \times \mathcal{S}, \mathcal{S} \oplus \mathcal{G} \), whose lengths are odd, and \( \mathcal{T} \oplus \mathcal{S}, \mathcal{F} \), whose lengths are even. Since the lengths of \( \mathcal{F}, \mathcal{T} \) are even and \( \mathcal{T} \) odd, Corollary 4.5 says that the second floor of \( I(\sigma_{0,0}, (\Lambda_1, \Lambda_2)) \) consists of irreducible factors of odd length, the third floor
of even length, and so on, and the top floor is $\mathfrak{F}$ whose length is odd. We shall show that

\[(17.1) \quad \mathfrak{T} \oplus \mathfrak{S} \text{ and } \mathfrak{E} \text{ lie above the factors } \mathfrak{F} \text{ and } \mathfrak{F} \oplus \mathfrak{E}.\]

This forces that there is only one possibility of the socle filtration of $I(\sigma_{0,0}, (\Lambda_1, \Lambda_2))$, which is nothing but the one stated in this proposition. The socle filtration of $I(\sigma_{0,0}, (-\Lambda_1, -\Lambda_2))$ is obtained from it by the duality of principal series modules.

Let us show (17.1) by direct computation. Since $M_m$ acts by $\sigma_{0,0}$, the vectors $\phi^j(e)$ which correspond to the $\lambda$-isotypic subspace of the irreducible factors $\mathfrak{T}$ are

\[\phi^5_{(3,1)}(e) = v^{(-1,-3)}_2, \quad \phi^7_{(2,2)}(e) = v^{(-2,-2)}_2,\]
\[\phi^8_{(-2,-2)}(e) = v^{(2,2)}_2, \quad \phi^9_{(1,-1)}(e) = v^{(1,-1)}_0,\]

respectively (cf. Proposition 14.3). Since

\[P_{(1,1)} P_{(0,-2)} \phi^9_{(2,2)}(e) = -8v^{(-1,-3)}_2 = -8\phi^5_{(3,1)}(e) \quad \text{and} \]
\[P_{(1,1)} P_{(1,1)} \phi^9_{(1,-1)}(e) = -16v^{(-1,3)}_2 = -16\phi^3_{(1,1)}(e),\]

there exist $\sigma$-actions which send non-zero elements of $\mathfrak{T}$ and $\mathfrak{E}$ to non-zero elements of $\mathfrak{F}$. Therefore, in the socle filtration of $I(\sigma_{0,0}, (2, 1))$, the irreducible factors $\mathfrak{T}$ and $\mathfrak{E}$ lie above the irreducible factor $\mathfrak{F}$. By the horizontal symmetry (Corollary 14.3), $\mathfrak{T} \oplus \mathfrak{S}$ and $\mathfrak{E}$ lie above the factor $\mathfrak{F} \oplus \mathfrak{S}$.

Next, consider the factors $2 \times \mathfrak{F}$. Since the three vectors

\[P_{(0,-2)} P_{(1,1)} \phi^9_{(1,-1)}(e) = 4(2v^{(2,-2)}_2 - 3v^{(2,-2)}_0 - v^{(2,-2)}_2),\]
\[P_{(2,0)} P_{(-1,-1)} \phi^9_{(1,-1)}(e) = 4(v^{(2,-2)}_2 - 3v^{(2,-2)}_0 - 2v^{(2,-2)}_2),\]
\[P_{(-1,-1)} P_{(2,0)} \phi^9_{(1,-1)}(e) = 12(3v^{(2,-2)}_0 - v^{(2,-2)}_2)\]

span the three dimensional space $V^{U(2)}_{(2,-2)}(\sigma_{0,0})$, $\mathfrak{E}$ lies above $2 \times \mathfrak{F}$. Moreover, since the two vectors

\[P_{(0,-2)} P_{(0,-2)} \phi^7_{(2,2)}(e) = 8v^{(2,-2)}_2 \quad \text{and} \quad P_{(2,0)} P_{(2,0)} \phi^8_{(-2,-2)}(e) = 8v^{(2,-2)}_2\]

span two dimensional subspace of $V^{U(2)}_{(2,-2)}(\sigma_{0,0})$, at least two factors in $\mathfrak{E}$, $2 \times \mathfrak{F}$ lie below $\mathfrak{T} \oplus \mathfrak{S}$. But we know that $\mathfrak{E}$ lies above $2 \times \mathfrak{F}$. It follows that the two $\mathfrak{T}$s lie below $\mathfrak{T} \oplus \mathfrak{S}$, and (17.1) is proved.

17.2. Proof of Theorem 17.1 (2). By Proposition 16.14 and Lemma 16.15 (1), the socle of $I(\sigma_{0,0}(\Lambda_1, -\Lambda_2))$ is a direct sum $\mathfrak{U} \oplus \mathfrak{T} \oplus \mathfrak{E}$ of irreducible modules of even length, and that of $I(\sigma_{0,0}(\Lambda_1, \Lambda_2))$ is a direct sum $\mathfrak{T} \oplus \mathfrak{S}$ of even length. The remaining irreducible factors are $2 \times \mathfrak{T}$, $\mathfrak{S} \oplus \mathfrak{S}$ and $\mathfrak{T} \oplus \mathfrak{S}$, all of whose lengths are odd. It follows from Corollary 16.15 that there is only one possibility of the socle filtration of $I(\sigma_{0,0}(\Lambda_1, -\Lambda_2))$, which is the one stated in this proposition.

17.3. Proof of Theorem 17.1 (3). Since $I(\sigma_{0,0}, (\pm \Lambda_2, \Lambda_1))$ is isomorphic to $I(\sigma_{0,0}, (\pm \Lambda_2, -\Lambda_1))$ by Lemma 16.15 (3), it suffices to determine the socle filtrations of $I(\sigma_{0,0}, (\pm \Lambda_2, \Lambda_1))$. By Proposition 16.14, the socle of $I(\sigma_{0,0}(\Lambda_2, \Lambda_1))$ is $\mathfrak{F}$, whose length is odd, and that of $I(\sigma_{0,0}(\Lambda_2, -\Lambda_1))$ is $\mathfrak{F}$, whose length is even. The remaining composition factors are $\mathfrak{U} \oplus \mathfrak{T}$, $\mathfrak{F} \oplus \mathfrak{S}$, whose lengths are even, and $1 \times \mathfrak{T}$,
\( \mathfrak{g} \oplus \mathfrak{s} \) and \( \mathfrak{t} \), whose lengths are odd. We shall show that, in the socle filtration of \( I(\sigma_{0,0}, (\Lambda_2, \Lambda_1)) \),

\[(17.2) \quad 4, 5 \oplus 6 \text{ lie above } \overline{0} \oplus \overline{1}, \overline{7} \oplus \overline{8}, \text{ and } \overline{10} \text{ lies above } \overline{7} \oplus \overline{8}.\]

By Proposition 11.3, the vector \( \phi^5_{(3,1)}(e) = v^{(-1,-3)} \) corresponds to \( \overline{5} \), and \( V_{\rho(2)}^{(3,-1)}(\sigma_{0,0})^* \simeq V_{SU(2)}^{(1,-3)}(\sigma_{0,0}) \) is a sum of one dimensional subspaces which corresponds to the factor \( \overline{9} \) and one dimensional subspace which corresponds to the factor \( \overline{7} \). Since \( \overline{9} \) is the unique quotient of \( I(\sigma_{0,0}, (\Lambda_2, \Lambda_1)) \), the vector

\[
P_{(0,-2)} \phi^5_{(3,1)}(e) = 3v_0^{(1,-3)} + v_2^{(1,-3)}
\]

corresponds to \( \overline{9} \). Since this vector is non-zero, there is a \( \mathfrak{g} \)-action which sends a vector in \( \overline{5} \) to a non-zero vector in \( \overline{9} \). It follows that \( \overline{5} \) lies above \( \overline{9} \) in the socle filtration of \( I(\sigma_{0,0}, (\Lambda_2, \Lambda_1)) \). By the horizontal symmetry, \( \overline{5} \oplus \overline{6} \) lies above \( \overline{1} \oplus \overline{7} \). Just in the same way,

\[
P_{(-1,-1)} \phi^5_{(3,1)}(e) = v_0^{(0,-2)} + v_2^{(0,-2)} \neq 0
\]

implies that \( \overline{5} \oplus \overline{6} \) lies above \( \overline{7} \oplus \overline{8} \), and

\[
P_{(2,0)} \phi_{(0,0)}^5 (e) = P_{(2,0)} v_0^{(0,0)} = 3(v_0^{(0,-2)} + v_2^{(0,-2)}) \neq 0
\]

implies that \( \overline{10} \) lies above \( \overline{7} \oplus \overline{8} \).

Next, we will check that the factor \( \mathfrak{t} \), which is not in the socle, lies above \( \overline{0} \).

As mentioned above, we know that \( V_{\rho(2)}^{(1,-3)}(\sigma_{0,0}) \) consists of two 1-dimensional subspaces, one of which corresponds to the factor \( \overline{9} \) and the other to \( \overline{7} \). On the other hand, \( V_{\rho(2)}^{(2,-2)}(\sigma_{0,0}) \) consists of three 1-dimensional subspaces one of which corresponds to \( \overline{9} \) and the other two to the two factors isomorphic to \( \overline{7} \). The image of the map

\[
P_{(1,1)} : V_{\rho(2)}^{(2,-2)}(\sigma_{0,0})^* \simeq V_{\rho(2)}^{(2,-2)}(\sigma_{0,0}) \rightarrow V_{\rho(2)}^{(1,-3)}(\sigma_{0,0}) \simeq V_{\rho(2)}^{(3,-1)}(\sigma_{0,0})^*,
\]

where

\[
\sum_{j=1}^{1} \alpha_{2j} v_{2j}^{(2,-2)} \mapsto -6(2\alpha_2 - \alpha_0) v_0^{(1,-3)} - 2(\alpha_0 - 2\alpha_2) v_2^{(1,-3)}
\]

is two dimensional. Since \( \overline{9} \) is the irreducible quotient module, there is a \( \mathfrak{g} \)-action sending a vector in a factor isomorphic to \( \mathfrak{t} \) to a non-zero vector in \( \overline{9} \). It follows that one of the factors isomorphic to \( \mathfrak{t} \) lies above \( \overline{0} \), and, by the horizontal symmetry, it lies above \( \overline{0} \oplus \overline{1} \).

The proof of the fact that \( \mathfrak{t} \) lies above \( \overline{7} \oplus \overline{8} \) is almost the same. Indeed, \( V_{\rho(2)}^{(2,-2)}(\sigma_{0,0})^* \simeq V_{\rho(2)}^{(2,-2)}(\sigma_{0,0}) \) consists of two 1-dimensional subspaces, one of which corresponds to \( \overline{9} \) and the other to \( \overline{7} \). The image of

\[
P_{(0,2)} : V_{\rho(2)}^{(2,-2)}(\sigma_{0,0})^* \simeq V_{\rho(2)}^{(2,-2)}(\sigma_{0,0}) \rightarrow V_{\rho(2)}^{(0,-2)}(\sigma_{0,0}) \simeq V_{\rho(2)}^{(0,0)}(\sigma_{0,0})^*,
\]

where

\[
\sum_{j=1}^{1} \alpha_{2j} v_{2j}^{(2,-2)} \mapsto 6(-2\alpha_2 + \alpha_0) v_0^{(0,-2)} - 6(\alpha_0 - 2\alpha_2) v_2^{(0,-2)}
\]

is two dimensional. Therefore, there exists a \( \mathfrak{g} \)-action which sends a vector in a factor isomorphic to \( \mathfrak{t} \) to a non-zero vector in \( \overline{7} \). This implies that one of the \( \overline{7} \) lies above \( \overline{7} \oplus \overline{8} \).
There are two possibilities of the socle filtration of \( I(\sigma_{0,0}, (\Lambda_2, \Lambda_1)) \) which satisfy both (17.2) and Corollary 14.3, namely

\[
\begin{array}{c}
\mathcal{S} & 4+5+6+10 \\
\mathcal{T} & 6+10 \\
\mathcal{U} & 10 \\
\mathcal{X} & 10 \\
\mathcal{Y} & 10 \\
\end{array}
\]

or

\[
\begin{array}{c}
\mathcal{S} & 4+5+6 \\
\mathcal{T} & 6+10 \\
\mathcal{U} & 10 \\
\mathcal{X} & 10 \\
\mathcal{Y} & 10 \\
\end{array}
\]

But the latter is impossible. Indeed, since the \( K \)-spectrum of \( 10 \) is not adjacent to those of \( \mathcal{U} \) and \( \mathcal{T} \), \( \text{Ext}^1_{K}(\mathcal{T}, \mathcal{U} \oplus \mathcal{T}) = 0 \). This completes the proof.

### 18. Socle filtration of \( I(w \cdot (\sigma_{A_1+1,A_2}, \Lambda)) \), \( w \in W(B_2) \).

Finally, let us determine the socle filtrations of the principal series modules which are isomorphic to \( X(\gamma_1) \) in the Grothendieck group. As in the previous section, we first summarize the result.

**Theorem 18.1.** Let \( \Lambda = (\Lambda_1, \Lambda_2) \) be a nonsingular dominant integral infinitesimal character of \( Sp(2, \mathbb{R}) \). The socle filtrations of \( I(w \cdot (\sigma_{A_1+1,A_2}, \Lambda)) \), \( w \in W(B_2) \), are as follows:

| (1) | \( I(\sigma_{A_1+1,A_2}, (\Lambda_1, \pm \Lambda_2)) \) | \( I(\sigma_{A_1+1,A_2}, (-\Lambda_1, \mp \Lambda_2)) \) |
|-----|---------------------------------|---------------------------------|
|     | \( 10 \)                         | \( 10 \)                         |
|     | \( 4 \oplus 5 + 6 + 10 \)       | \( 4 \oplus 5 + 6 \)             |
|     | \( 6 + 10 \)                     | \( 6 + 10 \)                     |
|     | \( 10 \)                         | \( 10 \)                         |

| (2) | \( I(\sigma_{A_2,A_1+1}, (\Lambda_2, \Lambda_1)) \) | \( I(\sigma_{A_2,A_1+1}, (-\Lambda_2, -\Lambda_1)) \) |
|-----|---------------------------------|---------------------------------|
|     | \( 9 \)                         | \( 9 \)                         |
|     | \( 4 \oplus 5 + 6 + 10 \)       | \( 4 \oplus 5 + 6 + 10 \)       |
|     | \( 6 + 10 \)                     | \( 6 + 10 \)                     |
|     | \( 10 \)                         | \( 10 \)                         |

| (3) | \( I(\sigma_{A_2,A_1+1}, (\Lambda_2, -\Lambda_1)) \) | \( I(\sigma_{A_2,A_1+1}, (-\Lambda_2, \Lambda_1)) \) |
|-----|---------------------------------|---------------------------------|
|     | \( 4 \oplus 5 + 6 \)            | \( 4 \oplus 5 + 6 \)            |
|     | \( 4 \oplus 5 + 6 + 10 \)       | \( 4 \oplus 5 + 6 + 10 \)       |
|     | \( 6 + 10 \)                     | \( 6 + 10 \)                     |
|     | \( 10 \)                         | \( 10 \)                         |

Let us prove this theorem.

By Theorem [14.1] the composition factors of this principal series module are \( \mathcal{U} \oplus \mathcal{T} \), \( \mathcal{Z} \oplus \mathcal{U} \), \( \mathcal{Y} \oplus \mathcal{X} \), \( \mathcal{U} \oplus \mathcal{T} \).

As in the last section, we consider the case when the infinitesimal character \( \Lambda \) is trivial. In this case, \( \sigma_{A_1+1,A_2} \) and \( \sigma_{A_2,A_1+1} \) are both \( \sigma_{1,1} \). Note that the highest weight \( \lambda = (\lambda_1, \lambda_2) \) of each \( K \)-type of the principal series module \( I(\sigma_{1,1}, \nu) \) satisfies \( \lambda_1 + \lambda_2 \in 2\mathbb{Z} \), and the \( \sigma_{1,1} \)-isotypic subspace \( V^{(2)}_{\Lambda}(\sigma_{1,1}) \) is \( \text{Span}(\phi_{(1,1)} \mid q \text{ is odd}) \).

**18.1. Proof of Theorem 18.1 (1).** By Lemma [16.13] (3), \( I(\sigma_{1,1}, (\pm \Lambda_1, \Lambda_2)) \) and \( I(\sigma_{1,1}, (\pm \Lambda_1, -\Lambda_2)) \) are isomorphic, so it suffices to show the case \( I(\sigma_{1,1}, (\pm \Lambda_1, \Lambda_2)) \). By Proposition [16.14] the socle of \( I(\sigma_{1,1}, (\Lambda_1, \Lambda_2)) \) is \( \mathcal{U} \oplus \mathcal{T} \) and that of its dual module \( I(\sigma_{1,1}, (-\Lambda_1, -\Lambda_2)) \) is \( \mathcal{T} \). The remaining composition factors are \( \mathcal{Z} \oplus \mathcal{U} \) and \( \mathcal{U} \), whose lengths are even, and \( \mathcal{Y} \) and \( \mathcal{Y} \oplus \mathcal{X} \), whose lengths are odd.

We first show that, in the socle filtration of \( I(\sigma_{1,1}, (\Lambda_1, \Lambda_2)) \), \( \mathcal{Z} \oplus \mathcal{U} \) lies above \( \mathcal{T} \) and \( \mathcal{Y} \oplus \mathcal{X} \) lies above \( \mathcal{U} \) and \( \mathcal{Y} \).
The space $V_{(3,3)}^{U(2)}(\sigma_{1,1})^*$ isomorphic to $\mathbb{F}$ and $V_{(3,3)}^{U(2)}(\sigma_{1,1})^*$ isomorphic to $V_{(-1,-3)}^{U(2)}(\sigma_{1,1})$ consists of two 1-dimensional subspaces one of which corresponds to $\mathbb{F}$ and the other to $\mathbb{F}$. Since $\mathbb{F}$ is the unique irreducible quotient of $I(\sigma_{1,1}, (\Lambda_1, \Lambda_2))$, the image of

$$P_{(0,-2)}: V_{(-3,-3)}^{U(2)}(\sigma_{1,1}) \rightarrow V_{(-1,-3)}^{U(2)}(\sigma_{1,1}), \hspace{1cm} v_{-3} \mapsto v_{-1}^{(-1,-3)} - v_{-3}^{(-1,-3)} \neq 0$$

corresponds to the factor $\mathbb{F}$. It follows that $\mathbb{F}$ lies above $\mathbb{F}$, so $\mathbb{F} \oplus \mathbb{F}$ lies above $\mathbb{F} \oplus \mathbb{F}$.

For $\mathbb{F}$, we discuss analogously. The space $V_{(2,0)}^{U(2)}(\sigma_{1,1})^* \simeq V_{(0,-2)}^{U(2)}(\sigma_{1,1}) = \mathbb{C}v_{(0,-2)}$ corresponds to $\mathbb{F}$. Since the image of

$$P_{(1,1)}: V_{(0,-2)}^{U(2)}(\sigma_{1,1}) \rightarrow V_{(-1,-3)}^{U(2)}(\sigma_{1,1}), \hspace{1cm} v_{-1} \mapsto 3(v_{-1}^{(-1,-3)} - v_{-3}^{(-1,-3)})$$

is non-zero, the factor $\mathbb{F}$ lies above $\mathbb{F}$, so $\mathbb{F} \oplus \mathbb{F}$ lies above $\mathbb{F} \oplus \mathbb{F}$.

The one-dimensional space $V_{(0,-2)}^{U(2)}(\sigma_{1,1})^* \simeq V_{(2,0)}^{U(2)}(\sigma_{1,1}) = \mathbb{C}v_{(2,0)}$ is also contained in $\mathbb{F}$. The two-dimensional space $V_{(2,-2)}^{U(2)}(\sigma_{1,1})$ consists of two 1-dimensional subspaces, one of which corresponds to $\mathbb{F}$ and the other to $\mathbb{F}$. Since

$$P_{(0,-2)}v_{-1}^{(0,-2)} = 5v_{-1}^{(2,-2)} + v_{-1}^{(2,-2)} \quad \text{and} \quad P_{(2,0)}v_{1}^{(2,0)} = v_{1}^{(2,-2)} + 5v_{-1}^{(2,-2)}$$

span the space $V_{(2,-2)}^{U(2)}(\sigma_{1,1})$, there is a $g$-action which sends a vector in $\mathbb{F}$ to a non-zero vector in $\mathbb{F}$. It follows that $\mathbb{F}$ lies above $\mathbb{F}$.

In order to complete the proof of Theorem 18.1 (1), we shall show that the extension group $\text{Ext}^1_{g,K}(\mathbb{F}, \mathbb{F} \oplus \mathbb{F})$ vanishes.

By the Blattner formula (14.1), the $K$-spectra of $\mathbb{F}$ and $\mathbb{F}$ are contained in $\{ \lambda = (\Lambda_1, \Lambda_2) \mid \Lambda_2 \geq \Lambda_2 + 2 \}$ and $\{ \lambda = (\Lambda_1, \Lambda_2) \mid \Lambda_1 \leq -\Lambda_2 - 2 \}$, respectively. On the other hand, by using the Blattner formula again, we see that, if $\Lambda_2 \geq -\Lambda_2$ (resp. $\Lambda_1 \leq -\Lambda_2$), then the multiplicity of $(\gamma_{1,2}, V_{(1,2)}^{U(2)})$ in $\mathbb{F}$ (resp. $\mathbb{F}$) is $\frac{1}{2}(\Lambda_1 - \Lambda_2 + 1)$, which is the same as the multiplicity in $X(\pi_{4})$ (cf. (14.1)). Since $X(\pi_{4}) = -X(\gamma_{4}) + X(\gamma_{1}) + X(\gamma_{4})$, the $K$-spectrum of $X(\pi_{4})$ is contained in $\{ \lambda = (\Lambda_1, \Lambda_2) \mid \Lambda_1 \geq \Lambda_2 + 1, \Lambda_2 \leq -\Lambda_2 - 1 \}$. It follows that the $K$-spectrum of $\mathbb{F}$ is not adjacent to those of $\mathbb{F}$ and $\mathbb{F}$, so $\text{Ext}^1_{g,K}(\mathbb{F}, \mathbb{F} \oplus \mathbb{F}) = 0$.

Finally, by a discussion analogous to the last part of the proof of Theorem 17.1 (3), we see that there is only one possibility of the socle filtration of $I(\sigma_{1,1}, (\Lambda_1, \Lambda_2))$, which is nothing but the one stated in this proposition.

18.2. **Proof of Theorem 18.1 (2)**. By Proposition 16.14 again, the socle of $I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1))$ is $\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$, and that of $I(\sigma_{1,1}, (-\Lambda_2, -\Lambda_1))$ is $\mathbb{F}$. The lengths of these modules are all even. The remaining composition factors are $\mathbb{F}$, $\mathbb{F} \oplus \mathbb{F}$ and $\mathbb{F}$, whose lengths are all odd. Then by Corollary 14.3 the socle filtration is uniquely determined as in the statement of this proposition.

18.3. **Proof of Theorem 18.1 (3)**. By Proposition 16.14 and Lemma 16.15 (2), the socle of $I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1))$ is $\mathbb{F}$ or $\mathbb{F} \oplus \mathbb{F}$, and that of $I(\sigma_{1,1}, (-\Lambda_2, -\Lambda_1))$ is $\mathbb{F} \oplus \mathbb{F}$.

We first show that the socle of $I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1))$ is $\mathbb{F} \oplus \mathbb{F}$. Assume that it is not, namely suppose that the socle consists of $\mathbb{F}$ alone. Then, the socle filtration of $I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1))$ (Theorem 18.1 (2)) implies that the factor $\mathbb{F}$ in $I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1))$ is contained in the kernel of the intertwining operator

$$I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1)) \rightarrow I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1)),$$
But it is well known that the composition
\[ I(\sigma_{1,1}, (\Lambda_1, \Lambda_2)) \rightarrow I(\sigma_{1,1}, (-\Lambda_1, -\Lambda_2)) \]
of the sequence of intertwining operators (16.14) is a non-zero map and its image is \( \mathcal{T} \). This is a contradiction, so the sole of \( I(\sigma_{1,1}, (\Lambda_2, -\Lambda_1)) \) is \( \mathcal{T} \oplus \mathcal{T} \).

The remaining factors are \( \mathcal{T} \oplus \mathcal{T} \), \( \mathcal{T} \oplus \mathcal{F} \) and \( \mathcal{F} \), whose lengths are all even. As before, the possibility of socle filtration of this principal series is unique and it is the one stated in this proposition.

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