Caputo q-Fractional Initial Value Problems and 
a q-Analogue Mittag-Leffler Function

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Abstract

Caputo q-fractional derivatives are introduced and studied. A Caputo-type q-fractional initial value problem is solved and its solution is expressed by means of a new introduced q-Mittag-Leffler function. Some open problems about q-fractional integrals are proposed as well.

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1 Introduction

The concept of fractional calculus is not new. It is believed to have stemmed from a question raised in 1695. However, it has gained considerable popularity and importance during the last three decades or so. This is due to its distinguished applications in numerous diverse fields of science and engineering ([15], [14], [16]). The q-calculus is also not of recent appearance. It was initiated in twenties of the last century. As a survey about this calculus we refer to [8]. Starting from the q-analogue of Cauchy formula [12], Al-Salam started the fitting of the concept of q-fractional calculus. After that he ([11], [10]) and Agarwal R. ([9]) continued on by studying certain q-fractional integrals and derivatives, where they proved the semigroup properties for left and right (Riemann)type fractional integrals but without variable lower limit and variable upper limit, respectively. Recently, the authors in [13] generalized the notion

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of the (left)fractional q-integral and q-derivative by introducing variable lower limit and proved the semigroup properties. However, the case of the (right) q-fractional integral by introducing a variable upper limit is still open. This open problem will be stated clearly in this article.

Very recently and after the appearance of time scale calculus (see for example [6]), some authors started to pay attention and apply the techniques of time scale to discrete fractional calculus ([3],[4],[5],[1]) benefitting from the results announced before in [7]. All of these results are mainly about fractional calculus on the time scales

\[ T_q = \{ q^n : n \in \mathbb{Z} \} \cup \{0\} \]

and \( h\mathbb{Z} \) [2]. However, the study of fractional calculus on time scales combining the previously mentioned time scales is still unknown. Continuing in this direction and being motivated by all above, in this article we define and study Caputo type q-fractional derivatives. This manuscript is organized as follows: Section 2 contains essential definitions and results about fractional q-integrals and q-derivatives, where we present an open problem about the semigroup property. Section 3 is devoted to define and study left and right Caputo q-fractional derivatives. In Section 4, we solve a Caputo q-fractional nonhomogeneous linear dynamic equation, where the solution is expressed by a q-analogue of Mittag-Leffler function.

2 Preliminaries and Essential Results about q-Calculus, Fractional q-Integrals and q-Derivatives

For the theory of q-calculus we refer the reader to the survey [8] and for the basic definitions and results for the q-fractional calculus we refer to [5]. Here we shall summarize some of those basics.

For \(0 < q < 1\), let \( T_q \) be the time scale

\[ T_q = \{ q^n : n \in \mathbb{Z} \} \cup \{0\}. \]

where \( \mathbb{Z} \) is the set of integers. More generally, if \( \alpha \) is a nonnegative real number then we define the time scale

\[ T_q^\alpha = \{ q^{n+\alpha} : n \in \mathbb{Z} \} \cup \{0\}, \]

we write \( T_q^0 = T_q \).

For a function \( f : T_q \to \mathbb{R} \), the nabla q-derivative of \( f \) is given by

\[ \nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in T_q - \{0\} \] (1)

The nabla q-integral of \( f \) is given by

\[ \int_0^t f(s) \nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i) \] (2)
and for $0 \leq a \in T_q$

$$\int_a^tf(s)\nabla_q s = \int_0^tf(s)\nabla_q s - \int_0^af(s)\nabla_q s$$

On the other hand

$$\int_t^\infty f(s)\nabla_q s = \int_t^0 f(s)\nabla_q s - \int_0^\infty f(s)\nabla_q s$$

and for $0 < b < \infty$ in $T_q$

$$\int_t^bf(s)\nabla_q s = \int_t^\infty f(s)\nabla_q s - \int_b^\infty f(s)\nabla_q s$$

By the fundamental theorem in q-calculus we have

$$\nabla_q \int_0^t f(s)\nabla_q s = f(t)$$

and if $f$ is continuous at 0, then

$$\int_0^t \nabla_q f(s)\nabla_q s = f(t) - f(0)$$

Also the following identity will be helpful

$$\nabla_q \int_a^t f(t, s)\nabla_q s = \int_a^t \nabla_q f(t, s)\nabla_q s + f(qt, t)$$

Similarly the following identity will be useful as well

$$\nabla_q \int_t^b f(t, s)\nabla_q s = \int_t^{qt} \nabla_q f(t, s)\nabla_q s - f(t, t)$$

The q-derivative in (7) and (8) is applied with respect to $t$.

From the theory of q-calculus and the theory of time scale more generally, the following product rule is valid

$$\nabla_q (f(t)g(t)) = f(qt)\nabla_q g(t) + \nabla_q f(t)g(t)$$

The q-factorial function for $n \in \mathbb{N}$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1} (t-q^i s)$$

When $\alpha$ is a non positive integer, the q-factorial function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1 - \frac{s}{t} q^i}{1 - \frac{s}{t} q^{i+\alpha}}$$
We summarize some of the properties of q-factorial functions, which can be found mainly in [5], in the following lemma

**Lemma 1.** (i) $(t - s)^{\beta} q^{\gamma} = (t - s)^{\beta} q^{\gamma}$

(ii) $(at - as)^{\beta} = a^{\beta} (t - s)^{\beta}$

(iii) The nabla q-derivative of the q-factorial function with respect to $t$ is

$$\nabla_q (t - s)^{\alpha} = \frac{1 - q^\alpha}{1 - q} (t - s)^{\alpha-1}$$

(iv) The nabla q-derivative of the q-factorial function with respect to $s$ is

$$\nabla_q (t - s)^{\alpha} = -\frac{1 - q^\alpha}{1 - q} (t -qs)^{\alpha-1}$$

where $\alpha, \gamma, \beta \in \mathbb{R}$.

For the q-gamma function, $\Gamma_q(\alpha)$, we refer the reader to [5] and the references therein. We just mention here the identity

$$\Gamma_q(\alpha + 1) = \frac{1 - q^\alpha}{1 - q} \Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad \alpha > 0. \quad (12)$$

The authors in [5] following [9] defines the left fractional q-integral of order $\alpha \neq 0, -1, -2, ...$ by

$$\mathcal{I}_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{\alpha-1} q^{\alpha-1} f(s) \nabla_q s \quad (13)$$

In [9] it was proved that the left q-fractional integral obeys the identity

$$\mathcal{I}_q^{\beta} \mathcal{I}_q^\alpha f(t) = \mathcal{I}_q^{\alpha+\beta} f(t), \quad \alpha, \beta > 0 \quad (14)$$

The left q-fractional integral $\mathcal{I}_q^\alpha$ starting from $0 < a \in T_q$ is to be defined by

$$\mathcal{I}_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)^{\alpha-1} f(s) \nabla_q s \quad (15)$$

It is clear, from the q-analogue of Cauchy’s formula [12], that

$$\nabla_q^n \mathcal{I}_q^\alpha f(t) = f(t) \quad (16)$$

where $n$ is a positive integer and $0 \leq a \in T_q$.

Recently, in Theorem 5 of [13], the authors there have proved that

$$\mathcal{I}_q^{\beta} \mathcal{I}_q^\alpha f(t) = \mathcal{I}_q^{\alpha+\beta} f(t), \quad \alpha, \beta > 0 \quad (17)$$

The right q-fractional integral of order $\alpha$ is defined by [9].
\[
I_q^\alpha f(t) = \frac{q^{-(1/2)\alpha(\alpha-1)}}{\Gamma_q(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(sq^{1-\alpha}) q^{-\alpha s} (18)
\]
and the right q-fractional integral of order \(\alpha\) ending at \(b\) for some \(b \in T_q\) is defined by
\[
b I_q^\alpha f(t) = \frac{q^{-(1/2)\alpha(\alpha-1)}}{\Gamma_q(\alpha)} \int_t^b (s-t)^{\alpha-1} f(sq^{1-\alpha}) q^{-\alpha s} \quad (19)
\]
Note that, while the left q-fractional integral \(q I_a^\alpha\) maps functions defined on \(T_q\) to functions defined on \(T_q\), the right q-fractional integral \(b I_q^\alpha\), \(0 < b \leq \infty\), maps functions defined on \(T_q^{1-\alpha}\) to functions defined on \(T_q\).

It is clear, from the q-analogue of Cauchy’s formula \([12]\), that
\[
\nabla_q^n b I_q^\alpha f(t) = (-1)^n f(t) \quad (20)
\]
In \([10]\) it was proved that the right q-fractional integral obeys the identity
\[
b I_q^\beta I_q^\alpha f(t) = I_q^{\alpha+\beta} f(t), \quad \alpha, \beta > 0 \quad (21)
\]
Taking into account the domain and the range of the right q-fractional integral, as mentioned above, we note that the formula \((21)\) is valid under the condition that \(f\) must be at least defined on \(T_q, T_q^{1-\beta}, T_q^{1-\alpha}\) and \(T_q^{1-(\alpha+\beta)}\).

A particular case of the identity \((21)\) is
\[
I_q^{\alpha-\alpha} I_q^\alpha f(t) = I_q^\alpha f(t), \quad \alpha > 0. \quad (22)
\]

Lemma 2. For \(\alpha, \beta > 0\) and a function \(f\) fitting suitable domains, we have
\[
\int_b^\infty (t-x)^{\beta-1} b I_q^\alpha f(tq^{1-\beta}) q^{-\alpha s} (1-\beta) = 0 \quad (23)
\]
Proof. From \((3)\) we can write
\[
\int_b^\infty (t-x)^{\beta-1} b I_q^\alpha f(tq^{1-\beta}) q^{-\alpha s} = \sum_{i=0}^\infty (1-q) b \sum_{i=0}^\infty q^{-i} (bq^{-i} - x)^{\beta-1} b I_q^\alpha f(q^{1-\beta} bq^{-i}) \quad (24)
\]
From the fact that \((t-r)^{\beta-1} = 0\), when \(t < r\) we conclude that \(b I_q^\alpha f(q^{1-\beta} bq^{-i}) = 0\) and hence the result follows.

Problem 1: Can we use Lemma \([2]\) and following similar ideas to that in \([13]\) to prove that
\[
b I_q^\beta b I_q^\alpha f(t) = b I_q^{\alpha+\beta} f(t), \quad \alpha, \beta > 0, \quad 0 < b \in T_q \quad (25)
\]
Alternatively, can we define the q-analogue of the Q-operator and prove that
\[ Q_q I_q^\alpha f(t) = b I_q^\alpha Q f(t) \]
Then apply the Q-operator to the identity
\[ q I_a^\beta q I_a^\alpha g(t) = q I_a^\alpha + \beta g(t), \quad \alpha, \beta > 0 \]
with \( g(t) = Q f(t) \) to obtain (25). Recall that in the continuous case \( Q f(t) = f(a + b - t) \).

In connection to Problem 1, the following open problem is also raised

**Problem 2:** Is it possible to obtain a by-part formula for q-fractional derivatives when the lower limit \( a \) and the upper limit \( b \) both exist. That is on the interval \([a, b]_q\). As for the \((0, \infty)\) case there is a formula was early obtained by Agarwal in [9].

As for the left and right (Riemann) q-fractional derivatives of order \( \alpha > 0 \), as traditionally done in fractional calculus, they are defined respectively by

\[ q \nabla^n_a f(t) \triangleq \nabla^n_q q I_a^{-\alpha} f(t) \quad \text{and} \quad b \nabla^n_q f(t) \triangleq (-1)^n \nabla^n q I_q^{1 - (n - \alpha)} f(t) \]
where \( n = [\alpha] + 1 \) and \( a, b \in T_q \cup \{\infty\} \) with \( 0 \leq a < b \leq \infty \). We usually remove the endpoints in the notation when \( a = 0 \) or \( b = \infty \). Here, we point that the operator \( q \nabla^n_a \) maps functions defined on \( T_q \) to functions defined on \( T_q \), while the operator \( b \nabla^n_q \) maps functions defined on \( T_q^{1 - (n - \alpha)} \) to functions defined on \( T_q \). Also, particularly, one has to note that

\[ q \nabla^n_a f(t) = \nabla^n_q f(t) \quad \text{and} \quad b \nabla^n_q f(t) = (-1)^n \nabla^n_q f(t) \]
where \( \nabla^n_q \) always denotes the \( n \)-th q-derivative (i.e. the q-derivative applied \( n \) times).

### 3 Caputo q-Fractional Derivative

In this section, before defining Caputo-type q-fractional derivatives and relating them to Riemann ones, we first state and prove some essential preparatory lemmas.

**Lemma 3.** For any \( \alpha > 0 \), the following equality holds:

\[ q I_a^\alpha \nabla_q f(t) = \nabla_q q I_a^\alpha f(t) - \frac{(t - a)^{\alpha - 1}}{\Gamma_q(\alpha)} f(a) \]

**Proof.** From (9) and (iv) of Lemma 1, we obtain the following q-integration by parts:

\[ \nabla_q ((t - s)^{\alpha - 1} f(s)) = (t - qs)^{\alpha - 1} \nabla_q f(s) - \frac{1 - q^{\alpha - 1}}{1 - q} (t - qs)^{\alpha - 2} f(s) \]
Applying (30) leads to

\[ qI_a^\alpha \nabla_q f(t) = \frac{(t-s)^{\alpha-1}}{\Gamma_q(\alpha)} f(s) \Big|^t_a + \frac{1-q^{\alpha-1}}{1-q} \int_a^t (t-qs)^{\alpha-2} f(s) \nabla_q s \]  \hspace{1cm} (31)

or

\[ qI_a^\alpha \nabla_q f(t) = -\frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)} f(a) + \frac{1-q^{\alpha-1}}{1-q} \int_a^t (t-qs)^{\alpha-2} f(s) \nabla_q s \]

On the other hand, and by the help of (iii) of Lemma 1, (7) and the identity (12), we find that

\[ \nabla_q qI_a^\alpha f(t) = \frac{1-q^{\alpha-1}}{1-q} \int_a^t (t-qs)^{\alpha-2} f(s) \nabla_q s, \]  \hspace{1cm} (32)

which completes the proof.

**Theorem 4.** For any real \( \alpha > 0 \) and any positive integer \( p \) such that \( \alpha - p + 1 \) is not negative integer or 0, in particular \( \alpha > p - 1 \), the following equality holds:

\[ qI_a^\alpha \nabla_q^p f(t) = \nabla_q^p qI_a^\alpha f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha + k - p + 1)} \nabla_q^k f(a) \]  \hspace{1cm} (33)

**Proof.** The proof can be achieved by following inductively on \( p \) and making use of Lemma 3, (iii) of Lemma 1 and (12).

Now we obtain an analogue to Lemma 3 for the right \( q \)-integrals.

**Lemma 5.** For any \( \alpha > 0 \), the following equality holds:

\[ q^{-1}bI_q^\alpha b \nabla_q f(t) = b \nabla_q bI_q^\alpha f(t) - \frac{r(\alpha)}{\Gamma_q(\alpha)} (b-qt)^{\alpha-1} f(q^{1-\alpha} q^{-1}b) \]  \hspace{1cm} (34)

where

\[ r(\alpha) = q^{(-1/2)\alpha(\alpha-1)} \]  \hspace{1cm} (35)

and

\[ b \nabla_q f(t) = -\nabla_q f(t) \]

**Proof.** First, by the help of (iii) of Lemma 1 and (11), the following q-calculus by-parts version is valid:

\[ (s-t)^{\alpha-1} \nabla_q f(sq^{1-\alpha}) q^{1-\alpha} = \]
\[ \nabla_q ((s-t)_q^{\alpha-1} f(sq^{1-\alpha})) - \frac{1-q^{\alpha-1}}{1-q} (s-t)^{\alpha-2} f(sq^{2-\alpha}) \] (36)

where the q-derivative is applied with respect to s. Using (36) we obtain

\[ q^{-1}b I^\alpha_q b \nabla_q f(t) = \]

\[ \frac{q^{\alpha-1} r(\alpha)}{\Gamma_q(\alpha)} \left( \frac{1-q^{\alpha-1}}{1-q} \int_t^{q^{-1}b} (s-t)^{\alpha-2} f(q^{2-\alpha}s) \nabla_q s - (s-t)^{\alpha-1} f(q^{1-\alpha}s) \right) \] (37)

\[ = \frac{q^{\alpha-1} r(\alpha)}{\Gamma_q(\alpha)} \left( \frac{1-q^{\alpha-1}}{1-q} \int_t^{q^{-1}b} (s-t)^{\alpha-2} f(q^{2-\alpha}s) \nabla_q s - (q^{-1}b-t)^{\alpha-1} f(q^{1-\alpha}(q^{-1}b)) \right) \] (38)

On the other hand (8) and (iv) of Lemma 1 imply

\[ b \nabla_q b I^\alpha_q f(t) = \frac{q^{\alpha-1} r(\alpha)}{\Gamma_q(\alpha)} \left( \frac{1-q^{\alpha-1}}{1-q} \int_t^{q^{-1}b} (s-t)^{\alpha-2} f(q^{2-\alpha}s) \nabla_q s \right) \] (39)

Taking into account (38) and (39), identity (34) will follow and the proof is complete.

One has to note that the above formula (34) holds under the request that f must be at least defined on \( T_q \) and \( T_q^{1-\alpha} \).

**Definition 6.** Let \( \alpha > 0 \). If \( \alpha \notin \mathbb{N} \), then the \( \alpha \)-order Caputo left q-fractional and right q-fractional derivatives of a function \( f \) are, respectively, defined by

\[ qC^\alpha_a f(t) \triangleq qI_q^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-qs)^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s \] (40)

and

\[ bC^\alpha_q f(t) \triangleq bI_q^{(n-\alpha)} \nabla_q^n f(t) = \frac{q^{(-1/2)\alpha(\alpha-1)}}{\Gamma_q(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} b \nabla_q^n f(sq^{1-\alpha}) \nabla_q s \] (41)

where \( n = \lfloor \alpha \rfloor + 1 \).

If \( \alpha \in \mathbb{N} \), then \( qC^\alpha_a f(t) \triangleq \nabla_q^n f(t) \) and \( bC^\alpha_q f(t) \triangleq \nabla_q^n = (-1)^n \nabla_q^n \)

Also, it is clear that \( qC^\alpha_a \) maps functions defined on \( T_q \) to functions defined on \( T_q \), and that \( bC^\alpha_q \) maps functions defined on \( T_q^{1-\alpha} \) to functions defined on \( T_q \)
If, in Lemma 3 and Lemma 5 we replace $\alpha$ by $1 - \alpha$. Then, we can relate the left and right Riemann and Caputo q-fractional derivatives. Namely, we state

**Theorem 7.** For any $0 < \alpha < 1$, we have

$$qC^\alpha_a f(t) = q\nabla^\alpha_a f(t) - \frac{(t-a)^{-\alpha}}{\Gamma_q(1-\alpha)} f(a)$$

(42)

and

$$q^{-1}bC^\alpha_a f(t) = b\nabla^\alpha_a f(t) - \frac{r(1-\alpha)}{\Gamma_q(1-\alpha)} (b-qt)^{-\alpha} f(q^\alpha q^{-1}b)$$

(43)

4 A Caputo q-fractional Initial Value Problem and q-Mittag-Leffler Function

The following identity which is useful to transform Caputo q-fractional difference equations into q-fractional integrals, will be our key in this section.

**Proposition 8.** Assume $\alpha > 0$ and $f$ is defined in suitable domains. Then

$$qI^\alpha_a qC^\alpha_a f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma_q(k+1)} \nabla^k_q f(a)$$

(44)

and if $0 < \alpha \leq 1$ then

$$qI^\alpha_a qC^\alpha_a f(t) = f(t) - f(a)$$

(45)

The proof followed by definition of Caputo q-fractional derivatives, Lemma 3 and Theorem 3.

The following identity [13] is essential to solve linear q-fractional equations

$$qI^\alpha_a (x-a) = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)^{\mu+\alpha}$$

(46)

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.

**Example 9.** Let $0 < \alpha \leq 1$ and consider the left Caputo q-fractional difference equation

$$qC^\alpha_a y(t) = \lambda y(t) + f(t), \quad y(a) = a_0, \quad t \in T_q.$$  

(47)

if we apply $qI^\alpha_a$ on the equation (47) then by the help of (46) we see that

$$y(t) = a_0 + \lambda qI^\alpha_a y(t) + qI^\alpha_a f(t).$$
To obtain an explicit clear solution, we apply the method of successive approximation. Set \( y_0(t) = a_0 \) and

\[
y_m(t) = a_0 + \lambda \quad q I^\alpha_a y_{m-1}(t) + q I^\alpha_a f(t), \quad m = 1, 2, 3, \ldots.
\]

For \( m = 1 \), we have by the power formula (47)

\[
y_1(t) = a_0[1 + \frac{\lambda(t-a)^{\alpha}}{\Gamma_q(\alpha + 1)}] + q I^\alpha_a f(t).
\]

For \( m = 2 \), we also see that

\[
y_2(t) = a_0 + \lambda a_0 \quad q I^\alpha_a [1 + \frac{(t-a)^\alpha}{\Gamma_q(\alpha + 1)}] + q I^\alpha_a f(t) + \lambda q I^{2\alpha}_a f(t)
\]

\[
= a_0[1 + \frac{\lambda(t-a)^\alpha}{\Gamma_q(\alpha + 1)}] + \frac{\lambda^2(t-a)^{2\alpha}}{\Gamma_q(2\alpha + 1)} + q I^\alpha_a f(t) + \lambda q I^{2\alpha}_a f(t)
\]

If we proceed inductively and let \( m \to \infty \) we obtain the solution

\[
y(t) = a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k(t-a)^{k\alpha}}{\Gamma_q(k\alpha + 1)}] + \int_t^\infty \left[ \sum_{k=1}^{\infty} \frac{\lambda^k}{\Gamma_q(ak)} (t-qs)^{(ak-1)} \right] f(s) \Delta_q s
\]

\[
= a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k(t-a)^{k\alpha}}{\Gamma_q(k\alpha + 1)}] + \int_t^\infty \left[ \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma_q(ak + \alpha)} (t-qs)^{(ak+\alpha-1)} \right] f(s) \Delta_q s
\]

\[
= a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k(t-a)^{k\alpha}}{\Gamma_q(k\alpha + 1)}] + \int_t^\infty (t-qs)^{(\alpha-1)} \left[ \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma_q(ak + \alpha)} (t-qs)^{(ak)} \right] f(s) \Delta_q s
\]

If we set \( \alpha = 1, \lambda = 1, a = 0 \) and \( f(t) = 0 \) we come to a \( q \)-exponential formula \( e_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma_q(k+1)} \) on the time scale \( T_q \), where \( \Gamma_q(k+1) = [k]_q! = [1]_q[2]_q[3]_q \ldots [k]_q \) with \( [r]_q = \frac{1-q^r}{1-q} \). It is known that \( e_q(t) = E_q((1-q)t) \), where \( E_q(t) \) is a special case of the basic hypergeometric series, given by

\[
E_q(t) = 1_{\phi_0(0; q, t)} = \Pi_{n=0}^{\infty} (1 - q^n t)^{-1} = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n},
\]

where \( (q)_n = (1-q)(1-q^2)\ldots(1-q^n) \) is the \( q \)-Pochhammer symbol.

If we compare with the classical case, then the above example suggests the following \( q \)-analogue of Mittag-Leffler function

**Definition 10.** For \( z, z_0 \in \mathbb{C} \) and \( \Re(\alpha) > 0 \), the \( q \)-Mittag-Leffler function is defined by

\[
q E_{\alpha, \beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \frac{\lambda^k (z-z_0)^{\alpha k}}{\Gamma_q(\alpha k + \beta)}.
\]

When \( \beta = 1 \) we simply use \( q E_{\alpha}(\lambda, z - z_0) := q E_{\alpha, 1}(\lambda, z - z_0) \).
According to Definition 10 above, the solution of the q-Caputo-fractional equation in Example 9 is expressed by

\[ y(t) = a_0 q E_{\alpha}(\lambda, t - a) + \int_{a}^{t} (t - qs)^{\alpha-1} q E_{\alpha,\alpha}(\lambda, t - q^\alpha s)f(s)\nabla q s. \]

**Remark 11.**

1) Note that the above proposed definition of the q-analogue of Mittag-Leffler function agrees with time scale definition of exponential functions. As it depends on the three parameters other than \( \alpha \) and \( \beta \).

2) The power term of the q-Mittag-Leffler function contains \( \alpha \) (the term \((z - z_0)^{\alpha k}\)). We include this \( \alpha \) in order to express the solution of q-Caputo initial value problem explicitly by means of the q-Mittag-Leffler function. This is due to that in general it is not true for the q-factorial function to satisfy the power formula \((z - z_0)^{\alpha k} = [(z - z_0)^{\alpha}]^k\). But for example the latter power formula is true when \( z_0 = 0 \). Therefore, for the case \( z_0 = 0 \), we may drop \( \alpha \) from the power so that the q-Mittag-Leffler function will tend to the classical one when \( q \to 1 \).

3) Once Problem 1 raised in section 2 is solved an analogue result to Proposition 8 can be obtained for right Caputo q-fractional derivatives.

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