BLOCKS OF RESTRICTED RATIONAL CHEREDNIK ALGEBRAS FOR
\(G(m, d, n)\)

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ABSTRACT. We study the Dunkl-Opdam subalgebra of the rational Cherednik algebra for wreath products at \(t = 0\), and use this to describe the block decomposition of restricted rational Cherednik algebras for \(G(m, d, n)\).

1. Introduction

1.1. Restricted rational Cherednik algebras have been a topic of recent interest due to connections with cells and families for cyclotomic Hecke algebras, [GM] and [Mar]. This paper is an attempt to understand better the Cherednik side of the picture. In particular, we extend the description of the blocks of the restricted algebras to arbitrary parameters. The case of rational parameters was dealt with in [Gor2] (see [GM, Theorem 2.5]), and the combinatorics reinterpreted in [Mar, Theorem 3.13]. Key to our arguments is the Dunkl-Opdam subalgebra of the rational Cherednik algebra, which we study in the \(t = 0\) situation. This algebra has already been used to great effect in the study of rational Cherednik algebras in the \(t \neq 0\) case, see for example [DG].

1.2. Let us describe our results in a little more detail, for notation see Sections 2 and 3. Let \(W = G(m, 1, n)\) be the wreath product of the symmetric group with the cyclic group of order \(m\). Corresponding to tuples \((t, \kappa, c_1, \ldots c_{m-1}) \in \mathbb{C}^{m+1}\), Etingof and Ginzburg defined in [EG] a flat family of algebras \(H_{t, \kappa}\) called rational Cherednik algebras. When \(t = 0\), these algebras, which are denoted \(H_c\), are finite-dimensional modules over their centres. There exists a particularly interesting finite dimensional quotient algebra \(\overline{H}_c\) of \(H_c\) called the restricted rational Cherednik algebra. The main focus of this paper is to determine the block structure of \(\overline{H}_c\).

1.3. The algebras \(\overline{H}_c\) have many interesting properties. There exist baby Verma modules \(\overline{\Delta}_c(E)\) for each irreducible \(W\)-module \(E\). These are indecomposable modules whose simple heads yield all simple \(\overline{H}_c\)-modules. It is well-known that the irreducible \(W\)-modules are parametrised by \(m\)-multipartitions of \(n\). Given a multipartition \(\lambda = (\lambda^0, \ldots, \lambda^{m-1})\), we denote by \(\overline{\Delta}_c(\lambda)\) the corresponding baby Verma module. Our main result, Theorem [5.5], is that one can use the combinatorics of multipartitions to describe the block decomposition of \(\overline{H}_c\). The numbers \(a_i\) are defined in [5.4] and the polynomials \(\text{Res}_{\lambda^i}(x)\) denote residue of \(\lambda^i\), which is defined in [5.2].

**Theorem.** Let \(\lambda\) and \(\mu\) be \(m\)-multipartitions of \(n\). The baby Verma modules \(\overline{\Delta}_c(\lambda), \overline{\Delta}_c(\mu)\) lie in the same block if and only if \(\sum_{i=0}^{m-1} x^{a_i} \text{Res}_{\lambda^i}(x^{\kappa}) = \sum_{i=0}^{m-1} x^{a_i} \text{Res}_{\mu^i}(x^{\kappa})\).

1.4. It is interesting to note the method of proof. An important role is played by the Dunkl-Opdam subalgebra, which is generated by the elements \(z_i\) from [3.2]. This is a commutative subalgebra of \(H_c\), and a crucial point is that the subalgebra \(z_c\) of symmetric polynomials in the \(z_i\) is central in \(H_c\), Theorem [3.4]. This is proved in Section 3 by direct calculation; a proof using the theory of generalised Jack polynomials was kindly passed on to the author by Stephen Griffeth and we sketch
this argument in [4.5]. It is shown in Theorem 5.5 that the action of $\mathfrak Z_c$ on baby Verma modules determines the blocks for $\mathfrak P_c$. Evaluating the eigenvalues of $\mathfrak Z_c$ on baby Verma modules then yields the block description of Theorem 1.3. We conclude the paper by applying the results from [Bel] to describe the block decomposition of restricted Cherednik algebras for the normal subgroups $G(m,d,n)$, Theorem 5.7.

1.5. Relation to Hecke algebras and further questions. The combinatorics governing the blocks of $\mathfrak P_c$ are conjectured to be related to cells at unequal parameters, [GM], and to Rouquier families for cyclotomic Hecke algebras, [Mar] (where our $a_i$ are written $m_i$). In both these cases the parameters (more precisely the $H_i$s from 3.1) need to be rational in order to be able to interpret the corresponding Hecke algebras. Our results hold for all parameters, and we note here another interpretation of the resulting combinatorics which holds also in non-rational cases. Let us assume that $\kappa \neq 0$, so that without loss of generality we can set $\kappa = -1$, see (1). Let $H^a_n$ be the degenerate cyclotomic Hecke algebra for $S_n$ associated to the parameter $a$ from 1.3 see [Bru]. Then the blocks of $H^a_n$ are precisely determined by the formula in Theorem 1.3, [Bru, Lemma 4.2]. It would be interesting to know whether $\mathfrak P_c$ is more closely related to $H^a_n$, or to the associated parabolic category $\mathcal O$.

It is natural to ask whether analogues of the Dunkl-Opdam subalgebra exist for the exceptional complex reflection groups, and whether the methods presented here could be extended to those cases. These techniques should also be applicable to studying blocks for restricted Cherednik algebras in characteristic $p > 0$ and at $t \neq 0$. We will return to these questions in future work.

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2. Rational Cherednik algebras

2.1. Let $W$ be a complex reflection group and $V$ its reflection representation over $\mathbb C$. Let $\mathcal S \subset W$ denote the set of complex reflections in $W$. For $s \in \mathcal S$ let $H_s \subset V$ denote the reflection hyperplane $\text{Fix}_s(V)$, and let $W_s$ denote the pointwise stabiliser of $H_s$. For every $s \in \mathcal S$ we choose $v_s \in V$ such that $\mathbb C v_s$ is a $W_s$-stable complement to $H_s$, and choose also a linear form $\alpha_s \in V^*$ with kernel $H_s$. Let $\langle , \rangle$ denote the natural pairing of $V^*$ with $V$.

2.2. Rational Cherednik algebras. We introduce formal parameters $t, c_s$ for $s \in \mathcal S$, where we set $c_t = c_s$ if the conjugacy classes of $s$ and $t$ are equal. Let $R$ be the polynomial ring $\mathbb C[t, c_s]_{s \in \mathcal S}$. Let $T_R(V \oplus V^*)$ and $S_R(V \oplus V^*)$ denote the tensor and symmetric algebras, respectively, of $V \oplus V^*$ over $R$.

Definition. The rational Cherednik algebra $H_R$ is the quotient of the smash product $T_R(V \oplus V^*) \rtimes W$ by the relations:

$$[x, x'] = 0, \quad [y, y'] = 0 \quad \text{and} \quad [y, x] = t \langle x, y \rangle - \sum_{s \in \mathcal S} t c_s \frac{\langle \alpha_s, y \rangle < x, v_s \rangle >}{\langle \alpha_s, v_s \rangle s}$$

for $y, y' \in V$ and $x, x' \in V^*$. 2
Let \( t \in \mathbb{C} \) and let \( c : S \to \mathbb{C} \) be a \( W \)-invariant function. We write \( c_s \) for the value \( c(s) \), and identify \( c \) with an element in \( \mathbb{C}^{[S/W]} \). Let \( m_{t,c} \) be the maximal ideal of \( R = \mathbb{C}[\mathbb{C} \times \mathbb{C}^{[S/W]}] \) corresponding to \( (t, c) \). The rational Cherednik algebra specialised at \( (t, c) \) is the algebra \( H_{t,c} := R/m_{t,c} \otimes_R H_R \).

The first important property about \( H_R \) is the following PBW theorem.

**Theorem.** The multiplication map \( S_R(V) \otimes_R RW \otimes_R S_R(V^*) \to H_R \) is an isomorphism of \( R \)-modules.

**Proof.** The analogous property is proved for all \( H_{t,c} \) in [EG] Theorem 1.3. The multiplication map over \( R \) is clearly surjective. If \( x \in S_R(V) \otimes_R RW \otimes_R S_R(V^*) \) lies in the kernel, then there exists a \( (t, c) \) such that the specialisation \( x_{t,c} \) of \( x \) at \( (t, c) \) is nonzero, and furthermore \( x_{t,c} \) lies in the kernel of the multiplication map \( S_C(V) \otimes_C CW \otimes_C S_C(V^*) \to H_{t,c} \), a contradiction. \( \square \)

We can filter \( H_R \) by the putting \( V \) and \( V^* \) in degree 1, and putting \( RW \) in degree 0, and we denote by \( \text{gr}H_R \) the corresponding associated graded algebra. A consequence of the PBW theorem is the following.

**Corollary.** There is an isomorphism of \( R \)-algebras \( \text{gr}H_R \cong S_R(V \oplus V^*) \rtimes W \).

There exists a \( \mathbb{Z} \)-grading on \( H_R \) by putting \( V \) and \( V^* \) in degree 1 and \(-1\), respectively, and putting \( RW \) in degree 0. This is obtained by considering an algebraic \( \mathbb{C}^* \)-action on \( H_R \) with the corresponding weights. In particular, it follows that the filtered pieces of \( H_R \) are also \( \mathbb{Z} \)-graded.

2.3. **Restricted rational Cherednik algebras.** Let \( c = (0, c) \) be a parameter and let \( H_c = H_{(0,c)} \). By [Gor1] Proposition 3.6], there is an algebra monomorphism

\[
A := (S_C V)^W \otimes_C (S_C V^*)^W \hookrightarrow H_c,
\]

whose image lies in the centre of \( H_c \). Let \( A^+ \) denote the ideal in \( A \) generated by polynomials with zero constant term. The quotient algebra \( \overline{H}_c := H_c/\langle A^+ \rangle \) is called the **restricted rational Cherednik algebra**.

By Theorem 2.2 \( H_c \) is a free \( A \)-module, and \( \overline{H}_c \cong (S_C V)^{co W} \otimes_C CW \otimes_C (S_C(V^*))^{co W} \) as vector spaces (here \( (S_C V)^{co W} \) denotes the coinvariant ring). The \( \mathbb{Z} \)-grading on \( H_R \) induces a \( \mathbb{Z} \)-grading on \( H_c \) and also on \( \overline{H}_c \), since \( \langle A^+ \rangle \) is a graded ideal.

For any \( a \in \mathbb{C}^* \), it is easy to show that there is an isomorphism of algebras \( H_c \cong H_{ac} \), which induces isomorphisms

\[
\overline{H}_c \cong \overline{H}_{ac}.
\]

2.4. **Baby Verma modules.** Let \( \text{Irr}CW \) denote the complex irreducible representations of \( W \). Let \( E \in \text{Irr}CW \). We can construct the corresponding baby Verma module as in [Gor1]: it is the induced module \( \overline{H}_c \otimes (S_C V)^{co W} \rtimes_W E \), where the action of an element \( pg \in (S_C V)^{co W} \rtimes_W E \) on \( E \) is given by \( p(0)g \). Denote this module by \( \overline{S_c}(E) \).

**Proposition.** [Gor1] Proposition 4.3] The baby Verma modules \( \overline{S_c}(E) \) are indecomposable \( \overline{H}_c \)-modules. They have simple heads \( L_c(E) \), and the set \( \{L_c(E) : E \in \text{Irr}CW \} \) is a complete set of pairwise nonisomorphic simple modules for \( \overline{H}_c \).
3. Cherednik algebras for $G(m, 1, n)$

3.1. Let $m, n$ be positive integers. Let $C_m$ be the cyclic group of order $m$. We fix a generator $g \in C_m$ and let $\eta \in \mathbb{C}$ be a primitive $m$th root of unity. Let $s_{ij} \in S_n$ denote the transposition which swaps $i$ and $j$. We denote by $s_j$ the simple transposition swapping $j$ and $j+1$. The group $W = G(m, 1, n)$ is the semidirect product $S_n \rtimes (C_m)^n$. We write $g^l_i$ for the element $(1, \ldots, g^l_i, \ldots, 1) \in G(m, 1, n)$ with $g^l_i$ in the $i$th place. Let $V$ be the reflection representation of $G(m, 1, n)$: we fix a basis $\{y_1, \ldots, y_n\}$ of $V$ so that $g^l_i(y_j) =$ \begin{align*}
\eta y_j & \text{ if } i = j \\
y_j & \text{ otherwise}
\end{align*}
and $\sigma(y_j) = y_{\sigma(j)}$, for all $i, j$ and all $\sigma \in S_n$. Let $\{x_1, \ldots, x_n\} \in V^*$ be the dual basis. The conjugacy classes of reflections in $W$ are given by $\{s_{ij}g^l_i : 0 \leq l \leq m-1 \text{ and } i \neq j\}$, and, for each $1 \leq l \leq m-1$, $\{g^l_j : 1 \leq j \leq n\}$.

We fix formal variables $(t; \kappa, c_1, \ldots, c_{m-1})$, where $\kappa$ corresponds to (2) and the $c_l$ correspond to the classes in (3). Let $R = \mathbb{C}[t, \kappa, c_1, \ldots, c_{m-1}]$. In the case $W = G(m, 1, n)$, Definition 2.2 becomes the following.

**Definition.** The rational Cherednik algebra for $G(m, 1, n)$, $H_R = H_R(G(m, 1, n))$, is the quotient of the smash product $T_R(V \oplus V^*) \rtimes W$ by the relations:

\[
[x_i, x_j] = 0, \quad [y_i, y_j] = 0, \quad [y_i, x_i] = \frac{t - \kappa}{m-1} \sum_{l=0}^{m-1} s_{ij}g^{-l}_i g^l_j - \sum_{l=1}^{m-1} c_l (1 - \eta^{-l}) g^l_i, \quad [y_i, x_j] = \kappa \sum_{l=0}^{m-1} \eta^{-l} s_{ij}g^{-l}_i g^l_j.
\]

As in 2.3, if we are given a tuple $c = (0; \kappa, c_1, \ldots, c_{m-1}) \in \mathbb{C}^{m+1}$, then we can specialise $H_R$ to the algebra $H_c$. We will also use the formal parameters $(t, h, H_0, \ldots, H_{m-1})$ from [Gor2]. Thus, $H_0 + \cdots + H_{m-1} = 0$, and the $h, H_j$ are related to the parameters above via

\[
h = -\kappa \quad \text{and} \quad -c_l (1 - \eta^{-l}) = \sum_{j=0}^{m-1} \eta^{-lj} H_j.
\]

We will denote by $(0, h, H_0, \ldots, H_{m-1})$ the image of the $(t, h, H_j)$ in $R/m_{(0,c)}$ - these then become another set of parameters for $H_c$. 

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3.2. The Dunkl-Opdam operators. For all $1 \leq i \leq n$, we define elements in $H_R$:

$$z_i = y_i x_i - \frac{1}{2} \mathcal{L} + \sum_{l=0}^{m-1} \sum_{1 \leq j \leq i} s_{ij} g_i^l g_j^{-l} - \sum_{l=1}^{m-1} \varphi_l g_l^i. \quad (5)$$

$$= x_i y_i + \frac{1}{2} \mathcal{L} - \sum_{l=0}^{m-1} \sum_{i < j} s_{ij} g_j^l g_i^{-l} - \sum_{l=1}^{m-1} \varphi_l g_l^i. \quad (6)$$

**Remark.** The elements $z_i$ have already appeared in [DO] and [Gri1] for the rational Cherednik algebra specialised at $t = 1$, where their action on Verma modules is studied. Our definition differs from these because we have added the terms $-\frac{1}{2} \mathcal{L}$ and $-\sum_{l=1}^{m-1} \varphi_l g_l^i$ in (5). This is important for Theorem 3.3, but using this definition changes little in the $t = 1$ applications.

We define

$$\xi_{ij} = \sum_{l=0}^{m-1} g_l^j g_i^{-l}.$$

**Lemma.** The following relations hold in $H_R$:

(a) $[z_i, z_j] = 0$;

(b) $z_i g_k = g_k z_i$;

(c) $z_i s_i = s_i z_i + 1 - \mathcal{L} \xi_{i+1}$

(d) $z_i s_j = s_j z_i$ for $i \neq j, j + 1$.

**Proof.** By [Gri1] Propositions 4.2 and 4.3, these hold for the elements $\tilde{z}_i = z_i + \sum_{l=1}^{m-1} \eta_l g_l^i + \frac{1}{2} \mathcal{L}$. It is straightforward to check that the relations (a)-(d) above can be deduced from this. \hfill \Box

By Theorem 2.2, the algebra $t_R$ generated over $R$ by the $z_i$ and $W$ is isomorphic to a graded Hecke algebra for $G(m, 1, n)$ as introduced in [RS, § 5]. In particular, the subalgebra generated by the $z_i$s is a polynomial algebra in $n$ variables. It follows from [Gri1] Lemma 5.1 that the $R$-subalgebra $Z_R$ of symmetric polynomials in the $z_i$ is central in $t_R$, and in particular commutes with $W$.

3.3. An involution on $H_R$. We define an automorphism $\psi$ of $H_R$ via:

$$\mathcal{L} \mapsto -\mathcal{L}, \quad \mathcal{E} \mapsto -\mathcal{E}, \quad \mathcal{O} \mapsto \eta^l \mathcal{O}^{-l},$$

$$s_i \mapsto s_{n-i}, \quad g_i \mapsto g_{n-i+1}^{-1}, \quad x_i \mapsto y_{n-i+1}, \quad y_i \mapsto x_{n-i+1}.$$

It is an easy check using the relations in (3.1) that $\psi$ is indeed a $C$-algebra homomorphism, and clearly $\psi^2 = \text{Id}$. Furthermore, by the equality of (5) and (6) it follows that $\psi(z_i) = z_{n-i+1}$ for all $i$.

3.4. A central theorem. We now state one of our main theorems, the proof of which will occupy Section 4.

**Theorem.** Let $\mathfrak{S}_r = \sum_{1 \leq i_1 < j_2 < \ldots < j_r \leq n} \mathcal{Z}_{i_1 j_1} \ldots \mathcal{Z}_{i_r j_r}$ denote the $r$th symmetric polynomial in the $z_i$s. Then $[x_1, \mathfrak{S}_r]$ lies in $tH_R$. In particular, the element $\mathfrak{S}_r$ specialised at $c = (0, c)$ is central in $H_c$.

Let us explain how the second claim follows from the first. We assume that $\mathfrak{S}_r$ is specialised at $c$. We know that $\mathfrak{S}_r$ commutes with $W$, so it remains to show that $[\mathfrak{S}_r, x_i] = [\mathfrak{S}_r, y_i] = 0$ for all $i$. From the first part of the theorem we know that $[\mathfrak{S}_r, x_1] = 0$. Thus $s_{1i} [\mathfrak{S}_r, x_1] s_{1i} = [s_{1i} \mathfrak{S}_r s_{1i}, x_i] = [\mathfrak{S}_r, x_i] = 0$. Now we apply the involution from (3.1) to obtain $[\mathfrak{S}_r, y_{n-i+1}] = \psi([\mathfrak{S}_r, x_i]) = 0$ for all $i$. 

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Remark. In the theory of rational Cherednik algebras, an important role is played by the Euler element

$$e_u := \sum_{i=1}^{n} x_i y_i + \frac{n}{2} t - n \sum_{i \neq j} s_{ij} g_i^{-1} g_j - \sum_{i=1}^{n} c_i g_i^l.$$ 

An easy check shows that the sum $\sum_{i=1}^{n} z_i$ equals the $e_u$ in $H_R$. It is known that specialising $e_u$ at $c$ produces a central element, see [GGOR, Section 3.1 (4)]. This verifies the theorem in the special case $r = 1$.

4. Proof of Theorem 3.4

4.1. Identities in the graded Hecke algebra. We begin our proof of Theorem 3.4 by listing some relations in $H_R$. We define for all $i \neq j$,

$$\gamma_{ij} = -\kappa m - 1 \sum_{l=0}^{s_{ij} - 1} s_{ij} g_i - l g_j.$$ 

To simplify notation, let $\gamma_j = \gamma_{1j}$ for all $j > 1$. The next relations all follow directly (by sometimes fairly lengthy computations) from the Definition 2.2 and Lemma 3.2:

$$\gamma_{ij} = \gamma_{ji}, \quad (7)$$

$$\sigma \gamma_{ij} \sigma^{-1} = \gamma_{\sigma(i) \sigma(j)} \text{ for all } \sigma \in S_n; \quad (8)$$

$$\gamma_u z_v = \begin{cases} 
  z_u \gamma_u + \sum_{1 < t \leq u} \gamma_u \gamma_t & \text{if } v = 1, \\
  z_v \gamma_u & \text{if } u < v, \\
  (z_1 - \sum_{1 < t \leq u} \gamma_t) \gamma_u & \text{if } u = v, \\
  z_v \gamma_u + \gamma_v \gamma_u - \gamma_u \gamma_v & \text{if } u > v \neq 1.
\end{cases} \quad (9)$$

The relations (7)-(9) yields the following identities (we assume that $u, v \neq 1$):

$$\gamma_u (z_v + \gamma_v) = \begin{cases} 
  (z_v + \gamma_{uv}) \gamma_u & \text{if } 1 \neq u < v, \\
  (z_1 - \sum_{1 < t < u} \gamma_t) \gamma_u & \text{if } u = v, \\
  (z_u + \gamma_u) \gamma_u & \text{if } u > v;
\end{cases} \quad (10)$$

$$\gamma_u (z_v + \gamma_v) = \begin{cases} 
  \gamma_u (z_v + \gamma_{uv}) & \text{if } 1 \neq u < v, \\
  \gamma_u (z_1 - \sum_{1 < t < u} \gamma_t) & \text{if } u = v, \\
  \gamma_u (z_v + \gamma_v) & \text{if } u > v.
\end{cases} \quad (11)$$

Let $1 < k \leq u$. An important consequence of (11) for us will be:

$$\gamma_u (z_v + \gamma_v) = \gamma_u (z_v - \sum_{1 < t < u} \gamma_t) + (z_v + \gamma_v) \gamma_u (z_v + \gamma_v) = \gamma_u z_v - \gamma_v \sum_{1 < t < u} \gamma_t + \sum_{k < t < u} \gamma_t \gamma_v. \quad (12)$$
We now record commutation relations for the $z_i$ in $H_R$. Let $1 \leq i, j \leq n$ be integers. Then

$$[x_i, z_j] = \begin{cases} 
-\lambda x_i - \sum_{1 \leq k < i} x_i \gamma_{ki} - \sum_{i < k \leq n} \gamma_{ik} x_i & \text{if } i = j, \\
\gamma_{ij} x_i & \text{if } i < j, \\
x_i \gamma_{ij} & \text{if } i > j;
\end{cases} \quad (13)$$

and

$$[y_i, z_j] = \begin{cases} 
\lambda y_i + \sum_{1 \leq k < i} \gamma_{ki} y_i + \sum_{i < k \leq n} y_i \gamma_{ik} & \text{if } i = j, \\
-\gamma_{ij} y_i & \text{if } i < j, \\
-\gamma_{ij} y_i & \text{if } i > j.
\end{cases} \quad (14)$$

4.2. We define elements in the Dunkl-Opdam subalgebra which will play a significant role in our proof of Theorem 3.4.

**Definition.** Given integers $1 < j_1 < \cdots < j_r \leq n$, let

$$P_{j_1, \ldots, j_r} := \sum z_{k_1} \cdots z_{k_s} \gamma_{l_1} \cdots \gamma_{l_t},$$

where the sum is taken over all $1 < k_1 < \cdots < k_s \leq n$, $1 < l_1 < \cdots < l_t \leq n$ such that $t \geq 1$ and \{k_1, \ldots, k_s\} \cup \{l_1, \ldots, l_t\} = \{j_1, \ldots, j_r\}$. We also define

$$\tilde{P}_{j_1, \ldots, j_r} := \sum z_{k_1} \cdots z_{k_s} \gamma_{l_1} \cdots \gamma_{l_t},$$

where the sum is taken as above, except that we allow $t = 0$.

It is clear from the definitions that

$$\tilde{P}_{j_1, \ldots, j_r} = P_{j_1, \ldots, j_r} + z_{j_1} \cdots z_{j_r}. \quad (15)$$

Since $\gamma_{j_t} z_{j_s} = z_{j_s} \gamma_{j_t}$ if $t < s$ by (9), it follows that

$$\tilde{P}_{j_1, \ldots, j_r} = (z_{j_1} + \gamma_{j_1}) \cdots (z_{j_r} + \gamma_{j_r}). \quad (16)$$

Now combining (15) and (16) we obtain:

$$P_{j_1, \ldots, j_r} = (z_{j_1} + \gamma_{j_1}) P_{j_2, \ldots, j_r} + \gamma_{j_1} z_{j_2} \cdots z_{j_r}. \quad (17)$$

**Lemma.** Let $1 < j_1 < \cdots < j_r \leq n$. In the algebra $H_R$ we have $[x_1, z_{j_1} z_{j_2} \cdots z_{j_r}] = P_{j_1, \ldots, j_r} x_1$ or equivalently, $x_1 z_{j_1} z_{j_2} \cdots z_{j_r} = P_{j_1, \ldots, j_r} x_1$.

**Proof.** The proof is by induction on $r$. The case $r = 1$ follows from (13). Now by (13), induction and (17),

$$[x_1, z_{j_1} z_{j_2} \cdots z_{j_r}] = [x_1, z_{j_1}] z_{j_2} \cdots z_{j_r} + z_{j_1} [x_1, z_{j_2} \cdots z_{j_r}] = \gamma_{j_1} x_1 z_{j_2} \cdots z_{j_r} + z_{j_1} P_{j_2, \ldots, j_r} x_1 = \gamma_{j_1} [x_1, z_{j_2} \cdots z_{j_r}] + \gamma_{j_1} z_{j_2} \cdots z_{j_r} x_1 + z_{j_1} P_{j_2, \ldots, j_r} x_1 = \gamma_{j_1} P_{j_2, \ldots, j_r} x_1 + \gamma_{j_1} z_{j_2} \cdots z_{j_r} x_1 + z_{j_1} P_{j_2, \ldots, j_r} x_1 = P_{j_1, \ldots, j_r} x_1.$$ 

The equivalence of the equations is a consequence of (15). \qed
4.3. **Key calculation.** We require one further technical result.

**Proposition.** Keep the notation from 4.2. Let \(1 \leq k < j_1\). Then

\[
\sum_{k < j_1 < \cdots < j_r \leq n} (z_1 - \sum_{k < t \leq n} \gamma_t) \tilde{P}_{j_1, \ldots, j_r} = \sum_{k < j_1 < \cdots < j_r \leq n} z_1 z_{j_1} \cdots z_{j_r} - \sum_{k < j_1 < \cdots < j_{r+1} \leq n} P_{j_1, \ldots, j_{r+1}} + \sum_{k < j_1 < \cdots < j_r \leq n} \sum_{1 < t \leq k} \gamma_t P_{j_1, \ldots, j_r}.
\]

**Proof.** We prove this by induction on \(r\). We begin with the case \(r = 1\). Let \(1 \leq k < j < n\). Then by (10),

\[
(z_1 - \sum_{k < t \leq n} \gamma_t)(z_j + \gamma_j) = z_1 z_j + z_1 \gamma_j - \sum_{k < t < j} \gamma_t(z_j + \gamma_j) - z_1 \gamma_j + \sum_{1 < t < j} \gamma_t \gamma_j - \sum_{j < t \leq n} (z_j + \gamma_j) \gamma_t
\]

Thus,

\[
\sum_{k < j \leq n} (z_1 - \sum_{k < t \leq n} \gamma_t)(z_j + \gamma_j)
= \sum_{k < j \leq n} z_1 z_j - \sum_{k < j \leq n} \left( \sum_{k < t < j} \gamma_t z_j + \sum_{j < t \leq n} (z_j + \gamma_j) \gamma_t \right) + \sum_{k < j \leq n} \sum_{1 < t \leq k} \gamma_t \gamma_j
= \sum_{k < j \leq n} z_1 z_j - \sum_{k < j_1 < j_2 \leq n} P_{j_1, j_2} + \sum_{k < j \leq n} \sum_{1 < t \leq k} \gamma_t P_j,
\]

where (19) follows from (17).

For the induction step, we first apply (18) to obtain

\[
\sum_{k < j_1 < \cdots < j_r \leq n} (z_1 - \sum_{k < t \leq n} \gamma_t) \tilde{P}_{j_1, \ldots, j_r}
= \sum_{k < j_1 < \cdots < j_r \leq n} (z_1 - \sum_{k < t \leq n} \gamma_t)(z_{j_1} + \gamma_{j_1}) \tilde{P}_{j_2, \ldots, j_r}
= \sum_{k < j_1 < \cdots < j_r \leq n} \left( (z_{j_1} + \gamma_{j_1})(z_1 - \sum_{j_1 < t \leq n} \gamma_t) - \sum_{k < j_1} \gamma_t z_{j_1} + \sum_{1 < t \leq k} \gamma_t \gamma_{j_1} \right) \tilde{P}_{j_2, \ldots, j_r}.
\]

By induction, this equals

\[
\sum_{k < j_1 < \cdots < j_r \leq n} (-\gamma_{j_1} z_1 - \sum_{k < t < j_1} \gamma_t z_{j_1} + \sum_{1 < t \leq k} \gamma_t \gamma_{j_1}) \tilde{P}_{j_2, \ldots, j_r}
+ \sum_{k < j_1 < \cdots < j_r \leq n} (z_{j_1} + \gamma_{j_1})(z_1 z_{j_2} \cdots z_{j_r} - \sum_{j_r < j_{r+1} \leq n} P_{j_2, \ldots, j_{r+1}} + \sum_{1 < t \leq j_1} \gamma_t P_{j_2, \ldots, j_r}).
\]
By applying (15) and rearranging terms we obtain

\[ \sum_{k<j_1<\ldots<j_r\leq n} (-\gamma_{j_1} z_1 - \sum_{k<t<j_1} \gamma_t z_{j_1} + \sum_{1<t\leq k} \gamma_t \gamma_{j_1} + (z_{j_1} + \gamma_{j_1})z_{j_2} \ldots z_{j_r}) \\
+ \sum_{k<j_1<\ldots<j_r\leq n} (\gamma_{j_1} z_1 + \sum_{1<t\leq k} \gamma_t z_{j_1})P_{j_2,\ldots,j_r}, \quad (20) \]

where line (20) follows from (12). We now cancel terms to obtain

\[ \sum_{k<j_1<\ldots<j_r\leq n} (- \sum_{k<t<j_1} \gamma_t \gamma_{j_1} + \sum_{1<t\leq k} \gamma_t \gamma_{j_1} + (z_{j_1} + \gamma_{j_1})P_{j_2,\ldots,j_r+1}) \\
+ \sum_{k<j_1<\ldots<j_r\leq n} (\sum_{1<t\leq k} \gamma_t \gamma_{j_1})P_{j_2,\ldots,j_r}. \]

By rearranging terms and rewriting the summation indices, we obtain

\[ \sum_{k<j_1<\ldots<j_r\leq n} z_1 z_{j_1} \ldots z_{j_r} - \sum_{k<j_1<\ldots<j_r\leq n} (\gamma_{j_1} z_{j_2} \ldots z_{j_r+1} + (z_{j_1} + \gamma_{j_1})P_{j_2,\ldots,j_r+1}) \\
+ \sum_{k<j_1<\ldots<j_r\leq n} \gamma_t (\gamma_{j_1} z_{j_2} \ldots z_{j_r} + (z_{j_1} + \gamma_{j_1})P_{j_2,\ldots,j_r})). \]

Now by (17), this equals

\[ \sum_{k<j_1<\ldots<j_r\leq n} z_1 z_{j_1} \ldots z_{j_r} - \sum_{k<j_1<\ldots<j_r\leq n} P_{j_1,\ldots,j_r+1} + \sum_{k<j_1<\ldots<j_r\leq n} \sum_{1<t\leq k} \gamma_t P_{j_1,\ldots,j_r}, \]

as required.

4.4. Proof of Theorem 3.4. Recall that we want to show that \([x_1, \mathcal{S}_r] \in \mathcal{L}H_R\) for any \(r\).

We calculate

\[ [x_1, \mathcal{S}_r] = \bigg[ x_1, \sum_{1\leq j_1<\ldots<j_r\leq n} z_{j_1} \ldots z_{j_r} \bigg] \]

\[ = \bigg[ x_1, \sum_{1=t\leq k} \sum_{1<j_1<\ldots<j_r\leq n} z_{j_1} \ldots z_{j_r} \bigg] + \bigg[ x_1, \sum_{1<j_1<\ldots<j_r\leq n} z_{j_1} \ldots z_{j_r} \bigg] \]

\[ = \sum_{1<j_1<\ldots<j_r\leq n} \bigg( (x_1, z_{j_1})z_{j_2} \ldots z_{j_r} + z_1 (x_1, z_{j_2} \ldots z_{j_r}) \bigg) + \sum_{1<j_1<\ldots<j_r\leq n} P_{j_1,\ldots,j_r} x_1 \quad (21) \]

\[ = \sum_{1<j_1<\ldots<j_r\leq n} [x_1, z_{j_1}]z_{j_2} \ldots z_{j_r} + \sum_{1<j_1<\ldots<j_r\leq n} z_1 P_{j_2,\ldots,j_r} x_1 + \sum_{1<j_1<\ldots<j_r\leq n} P_{j_1,\ldots,j_r} x_1. \quad (22) \]

The lines (21) and (22) follow from Lemma 4.2.

By (13) and Lemma 4.2,

\[ \sum_{1<j_2<\ldots<j_r\leq n} [x_1, z_{j_1}]z_{j_2} \ldots z_{j_r} = \sum_{1<j_2<\ldots<j_r\leq n} x_1 z_{j_2} \ldots z_{j_r} - \sum_{1<j_2<\ldots<j_r\leq n} \gamma_t x_1 z_{j_2} \ldots z_{j_r} \]

\[ = \sum_{1<j_2<\ldots<j_r\leq n} x_1 z_{j_2} \ldots z_{j_r} - \sum_{1<j_2<\ldots<j_r\leq n} \gamma_t P_{j_2,\ldots,j_r} x_1. \]
Plugging this back into (22), it suffices to show that
\[
\sum_{1 < j_2 < \cdots < j_r \leq n} \sum_{1 \leq t \leq n} -\gamma_t \tilde{p}_{j_2, \ldots, j_r} + \sum_{1 < j_2 < \cdots < j_r \leq n} z_1 p_{j_2, \ldots, j_r} + \sum_{1 < j_1 < j_2 < \cdots < j_r \leq n} p_{j_1, \ldots, j_r} = 0. \quad (23)
\]
Equivalently, we want to show that
\[
\sum_{1 < j_2 < \cdots < j_r \leq n} (z_1 - \sum_{1 \leq t \leq n} \gamma_t) \tilde{p}_{j_2, \ldots, j_r} - \sum_{1 < j_2 < \cdots < j_r \leq n} z_1 z_{j_2} \cdots z_{j_r} + \sum_{1 < j_1 < j_2 < \cdots < j_r \leq n} p_{j_1, \ldots, j_r} = 0. \quad (24)
\]
This is the statement of Proposition 14.3 (in the case \(k = 1\)).

4.5. **An alternative proof of Theorem 3.4** We sketch now a proof of Theorem 3.4 using the theory of Jack polynomials. We use freely the notation from [Gri1]. Let \(W = G(m, 1, n)\) and let \(H_R = H_R(W)\). Let \(\lambda = ((n), \emptyset, \ldots, \emptyset)\) be the \(m\)-multipartition corresponding to the trivial representation of \(W\), see 5.3 below. Then \(H_R\) has the standard polynomial representation \(\Delta_R(\lambda) = S_R(V^*)\). It is a faithful representation of \(H_R\). Indeed, if we consider \(\Delta_K(\lambda) := K \otimes_R H_R\), which contains \(H_R\) as a subring, then we obtain a representation \(K \otimes_R \Delta(\lambda)\) of \(H_K\), and this has a non-degenerate contravariant form. Therefore \(\Delta_K(\lambda)\) and also \(\Delta_R(\lambda)\) are faithful.

There is a basis \(f_\mu\) of \(\Delta_K(\lambda)\) consisting of non-symmetric Jack polynomials, where \(\mu \in \mathbb{Z}^n\); see [Gri1] Section 6. The Jack polynomials do not have poles at \(t = 0\). So in calculations involving \(f_\mu\) it makes sense to specialize \(t = 0\) (this is not necessarily true for the polynomials \(f_{\mu, T}\) in other standard modules, [Gri2]).

In the notation of the proof of Lemma 3.2 we have
\[
z_i = \tilde{z}_i - \frac{1}{2} t - \sum_{l=1}^{m-1} \eta^{-l} \xi_i^l g_i^l = \tilde{z}_i - \frac{1}{2} t - \sum_{0 \leq l \leq m-1} d_{l-1} \pi_{i, l},
\]
where the idempotents \(\pi_{i, l}\) are given by
\[
\pi_{i, l} = \frac{1}{m} \sum_{0 \leq j \leq m-1} \eta^{-j l} g_i^j,
\]
and
\[
d_{l-1} = \sum_{k=1}^{m-1} \eta^{k(l-1)} \xi_k.
\]
Then by [Gri1] Theorem 6.1,
\[
\pi_{i, l} f_\mu = \begin{cases} f_\mu & \text{if } l = -\mu_i \text{ and} \\ 0 & \text{else.} \end{cases}
\]

It follows from the above and [Gri1] Theorem 6.1 that \(z_i\) acts on \(f_\mu\) as
\[
z_i f_\mu = \left( t(\mu_i + \frac{1}{2}) - d_0 - m(w_\mu(i) - 1)k \right) f_\mu. \quad (25)
\]
Working modulo \(t\), for any \(\mu, \nu \in \mathbb{Z}^n \geq 0\) we have an equality of multisets:
\[
\{t(\mu_i + \frac{1}{2}) - d_0 - m(w_\mu(i) - 1)k\}_{1 \leq i \leq n} = \{t(\nu_i + \frac{1}{2}) - d_0 - m(w_\nu(i) - 1)k\}_{1 \leq i \leq n} \mod t.
\]
Therefore for any symmetric polynomial \( f = f(z_1, \ldots, z_n) \) in the operators \( z_1, \ldots, z_n \) the eigenvalues by which \( f \) acts on \( f_\mu \) and \( f_\nu \) are equal modulo \( t \). Applying this with \( \nu = \phi, \mu \) and \( \nu = \phi^{-1}, \mu \) where \( \phi \) is defined in equation (5.6) of [Gri1], it follows that
\[
[\Phi, f].f_\mu = 0 \mod t \quad \text{and} \quad [\Psi, f].f_\mu = 0 \mod t \quad \text{for all } \mu \in \mathbb{Z}_n^+.
\]
Each monomial \( x^\mu \) is an linear combination of \( f_\mu \), so we have also
\[
[\Phi, f].x^\mu = 0 \mod t \quad \text{and} \quad [\Psi, f].x^\mu = 0 \mod t \quad \text{for all } \mu \in \mathbb{Z}_n^+.
\]
Since the polynomial representation is faithful the relations \([\Phi, f] = [\Psi, f] = 0 \mod t\) hold in \( H_R \). But \( uf = fw \) for all \( w \in W \) by [Gri1] Lemma 5.1, so since \( H_R \) is generated by \( \Phi, \Psi, \) and \( W \) it follows that \( f \) becomes central upon setting \( t = 0 \).

Remark. More generally, \( z_i \) acts on the basis \( \{f_{\mu,T}\} \) of the standard module \( \Delta_K(\lambda') \) constructed in [Gri2] by
\[
z_i.f_{\mu,T} = (t(\mu_i + 1) - d_\beta(T^{-1}w_\mu(i))) - mct(T^{-1}w_\mu(i))_x f_{\mu,T}.
\]
Setting \( t = 0 \) gives the eigenvalues by which symmetric polynomials in the \( z_i \)'s will act on the baby Verma module for \( \lambda' \), even though \( f_{\mu,T} \) may have a pole at \( t = 0 \). This is written out in 5.4.

5. Blocks of \( \overline{H}_c \) for \( G(m, d, n) \)

5.1. Jucys-Murphy elements for wreath products. Let \( W = G(m, 1, n) \). We recall a generalisation of the beautiful construction of representations of the symmetric group described in [OV], which is explained in [Pus]. Let \( GZ_n \) be the subalgebra of \( CW \) generated by elements \( g_j \) for \( 1 \leq j \leq n \) and \( 0 \leq l \leq m - 1 \), and
\[
u_i := \sum_{l=0}^{m-1} \sum_{1 \leq j < i} s_{ij} g_i^{-1} g_j^l
\]
for \( 1 \leq i \leq n \). The algebra \( GZ_n \) is a maximal commutative subalgebra of \( CW \) which acts diagonally on the irreducible representations of \( W \).

5.2. Combinatorics. To describe the eigenvalues of \( GZ_n \) on irreducible representations we need to introduce some combinatorics, we follow the conventions of [Mar] Section 3. We denote by \( P(m, n) \) the set of \( m \)-multipartitions of \( n \), \( P(m, n) := \{ (\lambda^0, \ldots, \lambda^{m-1}) : \sum_{i=0}^{m-1} |\lambda^i| = n \} \). We will often identify a multipartition with its \( m \)-tuple of Young diagrams. Given an \( m \)-multipartition \( \lambda = (\lambda^0, \ldots, \lambda^{m-1}) \) we define a standard tableau on \( \lambda \) to be a numbering of the Young diagram of \( \lambda \) with \( \{1, \ldots n\} \) such that the numbers in the Young diagram corresponding to each \( \lambda^i \) are row increasing and column increasing. One can think of a tableau as a map \( T : \{1, \ldots, n\} \to \text{Boxes of } \lambda \). For example, this is a standard tableau on the multipartition \((3, 3), (2, 1, 1)):
\[
\begin{pmatrix}
1 & 3 & 5 \\
4 & 9 & 10
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 3 \\
6 & 7
\end{pmatrix}.
\]

Let \( b \) be a box in \( \lambda \). We denote by \( \beta(b) \) the \( i \) such that \( b \) lies in \( \lambda^i \). The content of a box \( b \) in \( \lambda \) is the number \( ct(b) := j - i \), where \( \beta(b) = k \) for some \( k \) and \( b \) lies in column \( j \) and row \( i \) of \( \lambda^k \). The residue of a partition \( \lambda_i \) is the polynomial \( \text{Res}_{\lambda_i}(x) := \sum_{b \in \lambda_i} x^{ct(b)} \).
5.3. The irreducible representations of $G(m,1,n)$. The irreducible representations of $W$ are labelled by the set $\mathcal{P}(m,n)$. To each $\lambda \in \mathcal{P}(m,n)$ we denote by the $V_\lambda$ the corresponding representation. By [Pus, Theorem 9], there exists a basis $\{v_T : T$ is a standard tableau on $\lambda\}$ of $V_\lambda$ such that the $v_T$ are eigenvectors for $GZ_n$. Let us describe the eigenvalues. Let $T$ be a standard tableau on $\lambda$ and let $\beta_T(i) = \beta(T(i))$. Then $u_i$ and $g_j^i$ act on $v_T$ by scalars $m.ct(T(i))$ and $\eta^{\beta_T(i)}$, respectively.

5.4. Characters of $3_c$. Let $c = (0, \kappa, c_1, \ldots, c_{m-1}) \in \mathbb{C}^{m+1}$. Recall the notation of [2.3] and [2.4]. We calculate the character of $3_c := R/m_{(0,c)} \otimes_R 3_R$ on a baby Verma module $\overline{\Delta}_c(\lambda)$. Let us first note that $3_c$ acts on $\overline{\Delta}_c(\lambda)$ by a scalar. This is because $3_c$ acts on the subspace $1 \otimes V_\lambda \subset \overline{\Delta}_c(\lambda)$, and since $3_c$ is central (and in particular commutes with $W$) its acts on $1 \otimes V_\lambda$, and therefore also on $\overline{\Delta}_c(\lambda)$, via a scalar.

Let $T$ be a standard tableau on $\lambda$. Since $y_i$ annihilates the subspace $1 \otimes V_\lambda$, $z_i \in 3_c$ acts on $1 \otimes V_\lambda$ via the element

$$-\kappa \sum_{l=0}^{m-1} \sum_{i<j} s_{i\beta} g_i^{-l} g_j^l - \sum_{l=1}^{m-1} c_l g_j^l.$$ 

This equals

$$w_0(-\kappa u_{n-i+1} - \sum_{l=1}^{m-1} c_l g_j^l)) w_0,$$

where $w_0 \in S_n$ denotes the longest element. Thus the set of $\{w_0 \cdot (1 \otimes v_T)\}$ forms a basis of eigenvectors for the action of the $z_i$. By [5.3] $z_{n-i+1}$ acts on $w_0 \cdot (1 \otimes v_T)$ via the eigenvalue

$$-\kappa m.ct(T(i)) - \sum_{l=1}^{m-1} c_l \eta^{\beta_T(i)}.$$ (29)

Recall the parameters $H_j$ from [3.1] and the formula relating the $c_i$ and $H_j$, (4). Let us fix $i$ and set $\beta = \beta_T(i)$, then

$$-\sum_{l=1}^{m-1} c_l \eta^{\beta_l} = \sum_{l=1}^{m-1} \sum_{j=0}^{m-1} \eta^{\beta_l} \frac{\eta^{-lj}}{(1-\eta^{-l})} H_j = \sum_{j=1}^{m-1} \sum_{l=1}^{m-1} \eta^{\beta_l} \frac{(\eta^{-lj} - 1)}{(1-\eta^{-l})} H_j$$

$$= -\sum_{j=1}^{m-1} \left(\sum_{l=1}^{m-1} \eta^{\beta_l} (1 + \eta^{-l} + \cdots + \eta^{-(l-j)})\right) H_j$$

$$= \sum_{j=1}^{\beta} j H_j + \sum_{j=\beta+1}^{m-1} (j-m) H_j$$

$$= \sum_{j=1}^{m-1} (j-m) H_j + \sum_{j=1}^{\beta} m H_j.$$ (30)

Let $C = \sum_{j=1}^{m-1} (j-m) H_j$, and let $a = (0, H_1, H_1 + H_2, \ldots, H_1 + \cdots + H_{m-1})$.

Since $3_c$ is generated by symmetric polynomials in the $z_i$, the character of $3_c$ on $\overline{\Delta}_c(\lambda)$ is given by the unordered $n$-tuple of the eigenvalues from (29). It is convenient to replace the term
generators for By Weyl’s Theorem, [Wey], a generating set is given by the power sums
\[ Z_{\kappa} \] of \(-q, r\) where
\[ \text{any lifts } \tilde{f}, \text{ and only if the characters of } \tilde{f} \text{ as comparing characters for the latter algebra. This was calculated in the previous section.} \]

This is precisely the condition obtained in [Mar, Theorem 3.13] for rational \(H_i\).

In the case \(\kappa = 0\), we get that \(\overline{\Delta}(\lambda), \overline{\Delta}(\mu)\) lie in the same block if and only if \(\sum_{i=0}^{m-1} x^{|\lambda^i|} = \sum_{i=0}^{m-1} x^{|\mu^i|} = \sum_{i=0}^{m-1} x^{a_i} |\lambda^i|\).

5.5. Blocks for wreath products.

**Theorem.** Let \(c = (0, \kappa, c_1, \ldots, c_{m-1}) \in \mathbb{C}^{m+1}\) be a parameter as in 5.4. Let \(\lambda, \mu \in \mathcal{P}(m, n)\). Let \(a = (0, H_1, H_1 + H_2, \ldots, H_1 + \cdots + H_{m-1})\). Then the baby Verma modules \(\overline{\Delta}(\lambda), \overline{\Delta}(\mu)\) lie in the same block if and only if \(\sum_{i=0}^{m-1} x^{|\lambda^i|} = \sum_{i=0}^{m-1} x^{|\mu^i|}\).

**Proof.** Recall that there is a grading on \(H_c\) where the \(y_i\) are in degree 1, the \(x_i\) in degree \(-1\), and \(W\) in degree 0. Let \(Z_c\) denote the centre of \(H_c\), a graded subalgebra of \(H_c\). Let \(f_1, \ldots, f_k\) be a set of homogeneous algebra generators for \(S[V \oplus V^\ast]^W\). Since \(\text{gr}Z_c = S[V \oplus V^\ast]^W\) is a domain, any lifts \(\tilde{f}_i\) of the \(f_i\) form a set of algebra generators for \(Z_c\). Furthermore, since the filtered pieces of \(Z_c\) are graded, we can choose the \(\tilde{f}_i\) to be homogeneous of the same degree as the \(f_i\).

We begin by finding a set of homogeneous generators for \(S[V \oplus V^\ast]^W\). This amounts to finding generators for
\[
(\mathbb{C}[x_i, y_j : 1 \leq i, j \leq n]^{C_m})^{S_n} = \mathbb{C}[x_i^m, x_j^m, y_k^m : 1 \leq i, j, k \leq n]^{S_n}.
\]

By Weyl’s Theorem, [Wey], a generating set is given by the power sums
\[
P_{q, r, s} := \sum_{1 \leq i \leq n} (x_i^m)^q (x_i^m)^r (y_i^m)^s,
\]
where \(q, r\) and \(s\) run through all values such that \(1 \leq q + r + s \leq n\). The degree of \(P_{q, r, s}\) equals \(-mq + ms\), so the only degree zero generators in this set are the \(P_{q, r, q}\). A lift of \(P_{q, r, q}\) is given by \(\tilde{P}_{q, r, q} := \sum_{1 \leq i \leq n} x_i^{mq + r} \in \mathcal{Z}_c\). Let \(\tilde{P}_{q, r, s}\) denote homogeneous lifts of the remaining generators.

By Müller’s Theorem, [BG, Theorem III.9.2], the blocks of \(\overline{H}_c\) are determined by the characters of \(Z_c\) on baby Verma modules. For any generator \(\tilde{P}_{q, r, s}\) of degree not equal to zero, we know that \(\tilde{P}_{q, r, s}|_{\overline{\Delta}_c}\) is a nilpotent operator, since the action of \(\tilde{P}_{q, r, s}\) factors through the finite-dimensional \(Z_c\)-graded algebra \(\overline{H}_c\). Thus \(\overline{\Delta}_c(\lambda), \overline{\Delta}_c(\mu)\) lie in the same block if and only if the characters of \(\tilde{P}_{q, r, q}\) on these modules are equal. Since \(\mathbb{C}[\tilde{P}_{q, r, q}] = \mathcal{Z}_c\), this is the same as comparing characters for the latter algebra. This was calculated in the previous section. \(\square\)

**Remark.** If \(\kappa \neq 0\), then we can assume without loss of generality that \(\kappa = -1\), see [11]. Thus \(\overline{\Delta}_c(\lambda), \overline{\Delta}_c(\mu)\) lie in the same block if and only if
\[
\sum_{i=0}^{m-1} x^a \text{Res}_{\lambda^i}(x) = \sum_{i=0}^{m-1} x^a \text{Res}_{\mu^i}(x).
\]

This is precisely the condition obtained in [Mar, Theorem 3.13] for rational \(H_i\).
5.6. **Irreducible modules for** \(G(m, d, n)\). Let \(d\) be a positive integer which divides \(m\), and let \(p = \frac{m}{d}\). We assume throughout that \(n > 2\), or \(n = 2\) and \(d\) is odd - this avoids degeneration of conjugacy classes below. We describe a parametrisation of irreducible modules for the normal subgroup \(G(m, d, n)\) of \(G(m, 1, n)\). See [Bel, Section 5] for a detailed discussion of the group \(G(m, d, n)\). Let \(\delta\) be a generator of the cyclic group \(C_d\). We define an action of \(C_d\) on \(\mathcal{P}(m, n)\) by
\[
\delta \cdot (\lambda^0, \ldots, \lambda^{m-1}) = (\lambda^{m-p}, \lambda^{m+1-p}, \ldots, \lambda^{m-2}, \lambda^{m-1}, \lambda^0, \lambda^1, \ldots, \lambda^{m-p-1}).
\]
Let \(C_\lambda\) be the stabiliser of \(\lambda\) under this action, and let \(\{\lambda\}\) denote the orbit of \(\lambda\). The irreducible representations of \(G(m, d, n)\) are parameterized by distinct pairs \(\{(\lambda), \epsilon\}\), where \(\lambda \in \mathcal{P}(m, n)\) and \(\epsilon \in C_\lambda\).

The conjugacy classes of reflections in \(G(m, d, n)\) are given by:
\[
\{s_{ij}g_i^{-l}g_j^l : 0 \leq l \leq m-1\text{ and } i \neq j\},
\]
and, for each \(1 \leq l \leq p - 1\),
\[
\{g_j^{dl} : 1 \leq j \leq n\}.
\]
In particular, a parameter for the rational Cherednik algebra of \(G(m, d, n)\) is given by a tuple \(c' = (0, c_d, c_{2d}, \ldots, c_{dp-1}) \in \mathbb{C}^{p+1}\). If we set \(c = (0, \kappa, c_1, \ldots, c_{m-1})\), where \(c_l = 0\) if \(d \nmid l\), then there is an algebra monomorphism
\[
H_{c'}(G(m, d, n)) \hookrightarrow H_{c'}(G(m, 1, n)).
\]

5.7. **Blocks for** \(G(m, d, n)\). Let us write \(\lambda = (\lambda^0, \ldots, \lambda_{d-1})\) where \(\lambda^i = (\lambda^{ip}, \ldots, \lambda^{(i+1)p-1})\). A multipartition \(\lambda\) is called \(d\)-stuttering if \(\lambda^i = \lambda^j\) for all \(0 \leq i, j \leq d-1\). The following theorem is [Bel, Corollary 6.10], whose proof remains valid for arbitrary \(c'\) once Theorem 5.5 is known.

**Theorem.** Let \(c'\) and \(c\) be as in 5.6. Let \(\{(\lambda), \epsilon\}, \{(\mu), \eta\}\) be pairs labelling irreducible \(G(m, d, n)\)-modules. Then:

- if \(\{(\lambda)\} \neq \{(\mu)\}\), then \(\overline{\Sigma}_{c'}(\{(\lambda), \epsilon\})\) and \(\overline{\Sigma}_{c'}(\{(\mu), \eta\})\) lie in the same block if and only if
  \[
  \sum_{i=0}^{m-1} x^{a_i} \text{Res}_{\lambda^i}(x^{-\kappa}) = \sum_{i=0}^{m-1} x^{a_i} \text{Res}_{\mu^i}(x^{-\kappa});
  \]
- if \(\lambda\) is a \(d\)-stuttering multipartition and
  \[
  \sum_{i=0}^{m-1} x^{a_i} \text{Res}_{\lambda^i}(x^{-\kappa}) \neq \sum_{i=0}^{m-1} x^{a_i} \text{Res}_{\mu^i}(x^{-\kappa})
  \]
  for all \(\lambda \neq \mu \in \mathcal{P}(m, n)\), then \(\overline{\Sigma}_{c'}(\{(\lambda), \epsilon\})\) and \(\overline{\Sigma}_{c'}(\{(\lambda), \eta\})\) are in the same block if and only if \(\epsilon = \eta\);
- otherwise, \(\overline{\Sigma}_{c'}(\{(\lambda), \epsilon\})\) and \(\overline{\Sigma}_{c'}(\{(\lambda), \eta\})\) are in the same block.

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