2-Representations of Lie 2-groups and 2-Vector Bundles

Zhen Huan

August 23, 2022

Abstract

Murray, Roberts and Wockel showed that there is no strict model of the string 2-group using the free loop group. Instead, they construct the next best thing, a coherent model for the string 2-group using the free loop group, with explicit formulas for all structure. Based on their expectations, we build a category of 2-representations for coherent Lie 2-groups and some concrete examples. We also discuss the relation between this category of 2-representations and the category of representations. In addition, we construct a model of equivariant 2-vector bundles. At the end, we discuss the adjoint action on the string 2-representations.

Contents

1 Introduction

2 Lie 2-groups and 2-representations
  2.1 Basic concepts
  2.2 2-Representations
  2.3 Examples of 2-representations
  2.4 Relation between 1-representations and 2-representations
  2.5 2-Representations of cross modules

3 Loop group and string 2-group
  3.1 Models of string Lie 2-group
  3.2 Positive energy representations of $LG$ and 2-representations of String($G$)

4 2-Vector bundles and Associated 2-vector bundles
  4.1 2-Vector bundles
  4.2 Principal $G$-bundle and the associated 2-vector bundles

5 A model of Equivariant 2-vector bundles
  5.1 Groupoid action on a category
  5.2 The $(3, 1)$-presheaf of equivariant 2-vector bundle
  5.3 The plus construction for $(3, 1)$-presheaves over Lie groupoids
  5.4 The 2-stack of equivariant 2-vector bundles
    5.4.1 The homotopy limit holim $2eqVectBd(G_\bullet)$
    5.4.2 The 2-stack of equivariant 2-vector bundles
1 Introduction

In this article, we explore the concept of 2-representation of a Lie 2-group on a more general type of 2-vector spaces. Since we wish our 2-representations to have fusion product eventually, we took some inspiration in [DR18], and consider the strict 2-category \( \mathbf{2Vect} \) of \( k \)-prelinear categories. The objects are \( k \)-prelinear categories, namely categories enriched in \( k \)-vector spaces. More precisely, a \( k \)-prelinear category is a category whose morphism sets are \( k \)-vector spaces such that compositions are \( k \)-bilinear. Here \( k \) is a field, which can be taken to be \( \mathbb{C} \) or \( \mathbb{R} \) from the angle of differential geometry especially. The 1-morphisms of \( \mathbf{2Vect} \) are \( k \)-linear functors and 2-morphisms are natural transformations between \( k \)-linear functors. Notice that direct sum of vector spaces is a biproduct, a category enriched in \( k \)-vector spaces is an additive category.

There are other two types of 2-vector spaces, that people might be familiar with. One is the 2-vector space via categorification [BDR04] by Baez, Dundas and Rognes, and the other is Kapranov-Voevodsky’s 2-vector space [Kap99]. Philosophically our 2-vector spaces may be viewed as a sort of unification of these two.

Technically, to adapt to the differential geometric setting, we do not require our object to be finite (i.e. to have finite simple objects) because of the following reason: to make our construction work for Lie 2-groups, and include some basic examples, we wish to include the representation category of a Lie group, which is not finite in general. We also do not require our objects to be semisimple, however, we do prove in the article when will certain desired category be semisimple. We choose our setting like this to make it on one hand general enough to contain important examples and on the other hand conditioned enough to expect nice properties. We also wish our choice can be flexible enough, that we may adjust to obtain desirable results in the future plan.

The representation of a groupoid can be interpreted as a functor from it to the category of vector spaces. Motivated by the categorical interpretation of group representations, we define the category of 2-representations of a 2-group \( \mathcal{G} \) to be the bicategory whose objects are the functors from the delooping \( B\mathcal{G} \) to the category of 2-vector spaces, and whose 1-morphisms are transformations between them and whose 2-morphisms are modifications. As indicated in [MRW17], the model of string 2-group which admits an action of the circle group on the loops cannot be provided by a strict 2-group. The next best model is given by a coherent Lie 2-group. This observation makes the 2-representations of a coherent 2-group more interesting to explore than strict 2-representations whereas the examples of coherent 2-representation can be much more complicated. We give the concrete and explicit construction of some examples afterwards. Our primary example of \( \mathcal{G} \)-representation sends the single object of \( B\mathcal{G} \) to the representation category of the underlying groupoid of \( \mathcal{G} \). Other than those examples for coherent 2-representation, we also give the construction of 2-representation of cross modules to make this part complete.

Motivated by the construction of string principal bundles in [SXZ17], we can build a model of 2-vector bundles. Given a manifold \( X \), we can define a bicategory \( \mathbf{2Vect}(X) \) of categorical vector bundles whose fibers are \( k \)-linear categories, with gauge transformations as 1-morphisms and families of natural transformations as 2-morphisms. This gives a presheaf of bicategories on the site of smooth manifolds. Just as (1-)vector bundles
form a 1-stack, 2-vector bundles are expected to form a 2-stack. We obtain the 2-stack from the presheaf mentioned above by applying the plus construction of Nikolaus and Schweigert \cite{NS11}. In addition, given a 2-vector bundle and a $G$-representation, we can define a $G$-principal bundle, which is a higher analogue of principal bundles.

In addition, as implied in \cite{NS11}, the plus construction can be generalized to presheaves of bicategories on Lie groupoids. We show explicitly in this article that this is true. We apply it in the construction of equivariant 2-vector bundles. Recall a $G$-vector bundle over a $G$-space is a vector bundle $p : E \to X$ with the projection $p$ a $G$-map and the $G$-action on the fibres linear. The fibre of a 2-vector bundle is a 2-vector space, i.e. a $k$-linear category.

To give a reasonable interpretation of equivariance, we define the action on a topological category by a topological groupoid based on the example of the groupoid action on another groupoid in \cite[Section 5.3, page 126]{MM03}. Therefore, we can define an $G$-equivariant 2-vector bundle with $G$ a groupoid. Locally, a $G$-equivariant (1-)vector bundle is a product vector bundle which may not still be $G$-equivariant but $H$-equivariant for some subgroup $H$ of $G$. In view of this, locally, an equivariant vector bundle should be the product of the base groupoid and a $k$-linear category, both of which have an action by a groupoid $G$ on it. Then, applying the plus construction for presheaves on Lie groupoids, we build the 2-stack of equivariant 2-vector bundles.

One of our principal goals studying 2-representation and 2-vector bundles is to explore string 2-group and its relation with physics. After sketching the construction of Murray, Roberts and Wockel’s coherent model of string 2-group, we apply the general constructions to the model of string 2-group \cite{MRW17}, define the string 2-representations and string 2-vector bundles. We give some examples of string 2-representations. As indicated in \cite[Section 4]{MRW17}, for a good bicategory of coherent 2-representations, there should be a functor from the category of positive energy representations of the loop group to the category of 2-representations of string 2-group which preserves the fusion product. We haven’t found a good way to define the fusion product on the category of string 2-representations. But we do explicitly construct a functor from the category of 1-representation to that of 2-representation of $G$ if the underlying groupoid of $G$ is an action groupoid. When we take $G$ to be the string 2-group, we get a functor described above.

Another expectation in \cite[Section 4]{MRW17} is that the category of positive energy representations of $\hat{LG}$ could be the ”fixed points” under the adjoint action of $G$ onto the category of Hilbert vector bundles on $S(QG)$. To explore the exact conclusion, we define the adjoint action by $G$ on the category of the string 2-representations.

Acknowledgements

The author is supported by the Young Scientists Fund of the National Natural Science Foundation of China (Grant No. 11901591) for the project “Quasi-elliptic cohomology and its application in geometry and topology”, and the research funding from Huazhong University of Science and Technology. This work was initiated while the author was visiting the University of Gottingen under the supervision of Professor Chenchang Zhu. The author thanks DAAD-K.C.Wong Postdoctoral Fellowship for hospitality and support and Professor Zhu for her kind and enlightening advice.
2 Lie 2-groups and 2-representations

2.1 Basic concepts

Lie 2-group is a well established concept.

**Definition 2.1.** [WZ05, Definition 2.8] Let $C$ be a bicategory with finite products. A group object in $C$ (or $C$-group, for brevity) consists of the following data:

- an object $G$ in $C$
- a list of 1-morphisms
  - $m : G \times G \to G$ (the multiplication)
  - $u : \ast \to G$ (the unit)

such that

\[(pr_1, m) : G \times G \to G \times G \quad (2.1)\]

is an equivalence in $C$.

- a list of invertible 2-morphisms
  - $a : m \circ (m \times 1) \Rightarrow m \circ (1 \times m)$ (the associator)
  - $l : m \circ (u \times 1) \Rightarrow 1$ (the left unit constraint)
  - $r : m \circ (1 \times u) \Rightarrow 1$ (the right unit constraint)

subject to the requirement that those coherence conditions [BL04, Definition 2.8, p. 37], that is, the pentagon identity for associator, the triangle identity for the left and right unit laws. Here $1$ is the identity morphism.

If $C = \text{Mfd}$ is the category of smooth manifold, then such a group object in $\text{Mfd}$ is a Lie 2-group.

We will also use $g \cdot h f$ to denote their multiplication $m(g, f)$ and use $I$ to denote the image of the unit $u$ in $G$.

**Remark 2.2.** As for the coherence conditions in Definition 2.1 as indicated in [BL04] Definition 2.8, p. 37], the associator $a$ satisfies a pentagon diagram. In addition, we should have three diagrams for unit laws, as shown below:

\[(1 \times u \times 1) \circ (m \times 1) \circ m \xrightarrow{(1 \times 1 \times 1) \circ a} (1 \times u \times 1) \circ (1 \times m) \circ m \quad (2.2)\]

\[(1 \times 1 \times u) \circ (m \times 1) \circ m \xrightarrow{(1 \times 1 \times u) \circ a} (1 \times 1 \times u) \circ (1 \times m) \circ m \quad (2.3)\]
However, as defined in the definition of coherent 2-group [BL04, Definition 2.8, p. 37], the diagram (2.2) is sufficient. The reason is, from the coherent conditions in the definition, the maps \( r : u \times u \to u \) and \( l : u \times u \to u \) are the same map. Then similar to the proof of [Mac71, Exercises 1, VII.1], we can show that the pentagon diagram of the associator \( a \) and the diagram (2.2) implies that (2.3) and (2.4) are both commutative (see the next lemma).

**Lemma 2.3.** The pentagon diagram of the associator \( a \) and (2.2) implies (2.3) and (2.4).

**Proof.** Let \( g \) and \( f \) be two objects in \( G \).

From the pentagon diagram for \( ((gf)I)I \) and the diagram (2.2), we have the commutative diagrams

\[
\begin{align*}
(gf)I & \xrightarrow{r_{(gf)I}} ((gf)I)I \xrightarrow{a_{gf,f.l,l}} (g(f)I)I \\
& \xrightarrow{1_{gf^1l_1}} (gf)(II) \xrightarrow{a_{gf,f.l,l}} g((f)II) \\
& \xrightarrow{a_{gf,f.l,l}} g((f)II) \xrightarrow{1_{g^1h_{f^1l^1}}} g(fI)
\end{align*}
\]  

Moreover, by naturality, we have

\[
\begin{align*}
(gf)(II) & \xrightarrow{a} g(f(II)) \\
& \xrightarrow{1_{g^1h_{f^1l^1}}} \quad \xrightarrow{1_{g^1h_{f^1l^1}}} \\
(gf)I & \xrightarrow{a} g(fI)
\end{align*}
\]  

Thus, (2.5) and (2.6) give us the commutative pentagon diagram below.

\[
\begin{align*}
((gf)I)I & \xrightarrow{a_{gf,f,l}h_{f^1l^1}} (g(f)I)I \\
& \xrightarrow{r_{(gf)I}} ((gf)I)I \xrightarrow{a_{gf,f,l,l}} (g(f)I)I \\
& \xrightarrow{r_{(gf)I}} \quad \xrightarrow{r_{(gf)I}} \quad \xrightarrow{r_{(gf)I}} g((f)II) \\
(gf)I & \xrightarrow{a_{gf,f,l,l}} g(fI) \xrightarrow{a_{g^1h_{f^1l^1}}} \xrightarrow{1_{g^1h_{f^1l^1}}} \\
& \xrightarrow{1_{g^1h_{f^1l^1}}} \quad \xrightarrow{1_{g^1h_{f^1l^1}}} \quad \xrightarrow{1_{g^1h_{f^1l^1}}}
\end{align*}
\]  

By naturality, the left square diagram commutes. Thus, the triangle diagram on the right commutes.

Finally, we apply the naturality of \( r \) we have the commutative diagram
2.2 2-Representations

Recall a representation of a group $G$ is a group morphism from $G \to \text{Aut}(V)$ for a vector space $V$. This is a functor from $BG$, i.e. the group $G$ viewed as a category with a single object $*$, to the category $\text{Vect}$ of vector spaces. The single object $pt$ maps to an object $V$ in $\text{Vect}$, and each element $g$ in $G$ maps to an automorphism of $V$. Thus a 2-functor (which is called a morphism, in the terminology of bicategories [Bén]) from $BG$, a 2-group $\mathcal{G}$ viewed as a 2-category with a single object $*$, to another 2-category $\mathcal{D}$ is called a 2-representation of $\mathcal{G}$ on $\mathcal{D}$, or we simply say that $\mathcal{G}$ represents on $\mathcal{D}$. These 2-representations form again a 2-category, that is

$$2\text{Rep}(BG, \mathcal{D}) = \text{Hom}(BG, \mathcal{D}),$$

is the 2-category of morphisms, transformations, and modifications.

We choose $\mathcal{D}$ to be the strict 2-category of $2\text{Vect}$, whose objects are $k$-prelinear categories, namely categories enriched in $k$-vector spaces. More precisely, a $k$-prelinear category is a category whose morphism sets are $k$-vector spaces such that compositions are $k$-bilinear. Here $k$ is a field (which is usually considered to be $\mathbb{C}$ or $\mathbb{R}$--for diff. geom, we need to see where it doesn’t pass for $\mathbb{R}$). The 1-morphisms of $2\text{Vect}$ are $k$-linear functors and 2-morphisms are natural transformations between $k$-linear functors. Notice that direct sum of vector spaces is a biproduct, a category enriched in $k$-vector spaces is an additive category.

**Example 2.4** (Category of groupoid representations). Notice that for a discrete groupoid $\Gamma = (\Gamma_1 \Rightarrow \Gamma_0)$, we have an associated category $\text{Rep}^\Gamma$--the category with objects functors $\Gamma \to \text{Vect}$, and 1-morphisms, natural transformations between functors. This is the prototype of 2-vector space [1] proposed in [MRW17]. This is not a set anymore, but rather a category: an object in $\text{Rep}^\Gamma$ is a functor $\rho$,

$$\rho : x \mapsto V_x, \quad (a : x \to y) \mapsto (V_x \xrightarrow{\rho(a)} V_y),$$

where $\rho(a)$ is a linear map, and $\rho(ab) = \rho(a)\rho(b)$. So it’s a representation of $\Gamma$ on $\text{Vect}$. If $\Gamma$ is further more a Lie groupoid, this category should be replaced by the representation category (let us use the same notation) $\text{Rep}^\Gamma$ of the Lie groupoid $\Gamma$. Let us recall that a representation of Lie groupoid $\Gamma$, that is a vector bundle $V$ over $\Gamma_0$, together with a Lie groupoid morphism $\Gamma \xrightarrow{\phi} \text{Gl}(V)$, the generalised linear Lie groupoid of $V$ (see [Mac00]): a morphism of $\text{Rep}^\Gamma$ is a $\Gamma$-equivariant morphism. The differential geometry aspect of Lie groupoid makes this category not abelian.

---

1It is denoted by $\text{Vect}^\Gamma$ therein, but to differentiate with $\Gamma$ graded vector spaces, when $\Gamma$ is a group, we use the notion $\text{Rep}^\Gamma$ and later we see that it is really the category of representation of $\Gamma$. 

6
Example 2.5 (Morphism in $2\text{Rep}G$). Let us write down explicit data for a morphism $T$ in $2\text{Rep}G$, based on the definition of the transformation between morphisms between bicategories [Lei98, Section 1.2].

$G$-2-representations are morphisms from $BG$ to $2\text{Vect}$. Let $(D_1, F_1, \eta_1, \phi_1, \psi_1)$ and $(D_2, F_2, \eta_2, \phi_2, \psi_2)$ denote such two objects, where, for $i = 1,2$,

- each $D_i$ is an object in $2\text{Vect}$, i.e. a $k$-prelinear category,
- each
  \[ F_i : G_0 \to \text{Aut}(D_i), \quad g_0 \mapsto F_{i,g_0} \]
  is a functor,
- each
  \[ \eta_i : G_1 \longrightarrow \{\text{Natural Transformations}\} \]
  \[ g_1 = (g_0 \xrightarrow{h} g'_0) \mapsto (F_{i,g_0} \xrightarrow{\eta_i} F_{i,g'_0}). \]
- \[ \phi_i : m_i' \circ ((F_i, \eta_i) \times (F_i, \eta_i)) \Rightarrow (F_i, \eta_i) \circ m \]
  and
- \[ \psi_i : u'_i \Rightarrow (F_i, \eta_i) \circ u_i \]
  are the natural transformations.

\[
\begin{array}{c c c c c}
G \times G & \xrightarrow{m} & G & \xrightarrow{u} & G \\
(F_i, \eta_i) \times (F_i, \eta_i) & \downarrow \phi_i & (F_i, \eta_i) & \downarrow \psi_i & (F_i, \eta_i) \\
\text{Auto}(D_i) \times \text{Auto}(D_i) & \xrightarrow{m_i'} & \text{Auto}(D_i) & \xrightarrow{u_i'} & \text{Auto}(D_i)
\end{array}
\]

(2.10)

A transformation $T : (D_1, F_1, \eta_1, \phi_1, \psi_1) \to (D_2, F_2, \eta_2, \phi_2, \psi_2)$ is defined by the following data and axioms.

- Data
  1. A morphism $T(*) : D_1 \to D_2$ in $2\text{Vect}$, i.e. a functor between the $k$-prelinear categories.
  2. A natural transformation

\[
\begin{array}{c c c c c}
G_0 & \xrightarrow{F_1} & \text{Auto}(D_1) \\
F_2 & \downarrow T & \text{Fun}(D_1, D_2)
\end{array}
\]

Above, the notation $T(*)_*$ denotes the functor

\[
\text{Auto}(D_1) \to \text{Fun}(D_1, D_2), \quad f \mapsto T(*) \circ f
\]

and the notation $T(*)^*$ denotes the functor

\[
\text{Auto}(D_2) \to \text{Fun}(D_1, D_2), \quad f \mapsto f \circ T(*)
\]

Thus, for any $g_0 \in G_0$, we have the natural transformation

\[ T_{g_0} : F_{2,g_0} \circ T(*) \Rightarrow T(*) \circ F_{1,g_0}. \]
• Axioms

The diagrams below commute.

1. \[(F_2 \circ T_{h_0}) \circ T(*) \xrightarrow{\phi_2 \circ h_0 \circ T_{h_0}} F_2 \circ h_0 \circ T(*) \quad (2.12)\]

2. \[I'_2 \circ T(*) \xrightarrow{l} T(*) \xrightarrow{r^{-1}} T(*) \circ I'_1 \quad (2.13)\]

**Lemma 2.6.** For a Lie groupoid \(\Gamma\), \(\text{Rep}^{\Gamma}\) is a category enriched in \(k\)-vector spaces. However, it is not an abelian category

**Proof.** We need to verify that \(\text{Rep}^{\Gamma}\) is a category whose morphism sets are \(k\)-vector spaces and compositions are \(k\)-bilinear. A morphism \(\rho \xrightarrow{f} \rho'\) in \(\text{Rep}^{\Gamma}\), is a \(\Gamma\)-equivariant morphism linear fiberwise, i.e. \(f_x \in \text{Hom}(V_x, V'_x)\) for every point \(x \in \Gamma_0\). Thus the linear structure of \(\text{Hom}(\rho, \rho')\) comes from the linear structure fiberwise.

It is not hard to verify that the composition is bilinear. We show that for \(f \in \text{Hom}(\rho, \rho')\) and \(f' \in \text{Hom}(\rho', \rho'')\), for any \(\alpha \in k\), \(x \in \Gamma_0\), any vector \(v\), \(f_x \circ (\alpha f'_x(v)) = \alpha (f_x \circ f'_x(v))\) and \((\alpha f_x) \circ f'_x(v) = \alpha (f_x \circ f'_x(v))\). Thus, the composition is bilinear.

This category is unfortunately not abelian, because the kernel \(\ker(f)\) for a morphism \(f\) without constant rank might not be a vector bundle again, therefore, not a representation of \(\Gamma\). Thus it is not a \(k\)-linear category, which is defined to be an abelian category enriched in \(k\)-vector spaces.

### 2.3 Examples of 2-representations

**Lemma 2.7.** A 2-group \(G = G_1 \Rightarrow G_0\) represents on \(2\text{Vect}\) in the following way
we defined $F$ is indeed a natural transformation, and i.e., the diagram

**Proof.** On the level of 1-morphisms, we need to prove that $F_g(\rho)$ is indeed a functor, $F_g(f)$ is indeed a natural transformation, and $F_g$ is $k$-linear, that is, $F_g(f + f') = F_g(f) + F_g(f')$ and $F_g(\lambda f) = \lambda F_g(f)$ for $\lambda \in k$. The fact that $F_g(\rho)$ is a functor is obvious, and the $k$-linearity of $F_g$ follows from that of Hom-space of $\text{Rep}^G$.

The fact that $F_g(f)$ is indeed a natural transformation follows from the fact that $f$ is a natural transformation. Indeed, given $x \xrightarrow{\rho} y \in G$, the diagram

$$
\begin{array}{ccc}
F_g(\rho)(x) & \xrightarrow{F_g(f)} & F_g(\rho')(x) \\
\downarrow F_g(\rho)(b) & & \downarrow F_g(\rho')(b) \\
F_g(\rho)(y) & \xrightarrow{F_g(f)g} & F_g(\rho')(y)
\end{array}
$$

i.e., the diagram

$$
\begin{array}{ccc}
\rho(x \cdot_h g) & \xrightarrow{\rho'(x \cdot_h g)} & \rho'(x \cdot_h g) \\
\downarrow \rho(b \cdot_h 1_g) & & \downarrow \rho'(b \cdot_h 1_g) \\
\rho(y \cdot_h g) & \xrightarrow{\rho'(y \cdot_h g)} & \rho'(y \cdot_h g)
\end{array}
$$

commutes because $f$ is a natural transformation.

Now on the level of 2-morphisms, we need to check that, given $(g_0 \xrightarrow{\eta_1} g_0') \in G_1$, what we defined $\eta_1(\rho) : F_{g_0}(\rho) \Rightarrow F_{g_0'}(\rho)$ is indeed a natural transformation. We need to check the commutativity of the following diagram for all $x \xrightarrow{b} y \in G_1$,

$$
\begin{array}{ccc}
F_{g_0}(\rho)(x) & \xrightarrow{\eta_1(\rho)_x} & F_{g_0'}(\rho)(x) \\
F_{g_0}(\rho)(b) & & \downarrow F_{g_0'}(\rho)(b) \\
F_{g_0}(\rho)(y) & \xrightarrow{\eta_1(\rho)_y} & F_{g_0'}(\rho)(y)
\end{array}
$$

Object level: the only object $pt$ in $BG$, maps to $\text{Rep}^G$. Here $G$ in $\text{Rep}^G$ also denotes the underlying Lie groupoid of the 2-group by abusing of notation.

1-Morphism level:

$$G_0 \rightarrow \text{Fun}(\text{Rep}^G, \text{Rep}^G), \quad g \mapsto F_g,$$

with

$$F_g(\rho) : x \mapsto \rho(x \cdot_h g)(= V_{x-hg}), \quad (x \xrightarrow{a} y) \mapsto (V_{x-hg} \xrightarrow{\rho(a \cdot_1 h)} V_{y-hg})$$

$$F_g(\rho f \Rightarrow \rho') = (F_g(\rho) \xrightarrow{F_g(f)} F_g(\rho')), \quad F_g(f) \text{ is a natural transformation with } F_g(f)_x = f_{x-hg},$$

for all $x, y \in G_0$, $(x \xrightarrow{a} y) \in G_1$, and 1-morphism $f$ in $\text{Rep}^G$, which is in turn a natural transformation $\rho \Rightarrow \rho'$. Here $\text{Fun}(-, -)$ denotes the set of functors.

2-Morphism level: given $(g_0 \xrightarrow{\eta_1} g_0') \in G_1$

$$G_1 \rightarrow \{\text{natural transformations} \}, \quad g_1 \mapsto \eta_{g_1},$$

$$\eta_{g_1}(\rho)_x : F_{g_0}(\rho)(x) \rightarrow F_{g_0'}(\rho)(x), \quad v \mapsto \rho(1_x \cdot_h g_1)(v), \quad \forall v \in F_{g_0}(\rho)(x), \quad (2.14)$$

where $1_x$ denotes the identity arrow in $G_1$ at $x \in G_0$.
which is in turn just
\[
V_{x,b_g} \xrightarrow{\rho(b \cdot 1_g)} V_{x,1_g}
\]
(2.18)
\[
\rho(b \cdot 1_g)
\]
\[
V_{y,b_g} \xrightarrow{\rho(1 \cdot b_g)} V_{y,1_g}
\]

This diagram follows from that fact that \( \rho \) is a functor and the following equation,

\[
(b \cdot 1_g) \cdot (1 \cdot b_g) = (b \cdot 1_g \cdot 1) = b \cdot 1_g = (1 \cdot b_g) \cdot b = b \\
\]
(2.19)

To show that the data in the above lemma gives us a morphism from \( B \mathcal{G} \) to \( 2 \text{Vect} \), we still need to verify the following (see [Lei98, 1.1])

1) \((F, \eta) : \mathcal{G} \to \text{Auto}(\text{Rep}_{\mathcal{G}})\) from the category \( \mathcal{G} \), (whose structure is the underlying Lie groupoid of the 2-group \( \mathcal{G} \)) to the category \( \text{Auto}(\text{Rep}_{\mathcal{G}}) \) of autofunctors from \( \text{Rep}_{\mathcal{G}} \) to itself and the natural transformations between them.

2) There exist natural transformations
\[
\phi : m' \circ ((F, \eta) \times (F, \eta)) \Rightarrow (F, \eta) \circ m
\]
(2.20)
and
\[
\psi : u' \Rightarrow (F, \eta) \circ u.
\]
(2.21)

Here \( m'(-, -) \) denotes the composition of functors and horizontal concatenation of natural transformations in \( \text{Auto}(\text{Rep}_{\mathcal{G}}) \) and \( u' : * \to \text{Auto}(\text{Rep}_{\mathcal{G}}) \) is the unit in \( \text{Auto}(\text{Rep}_{\mathcal{G}}) \). Thus we write \( m'(F, F') \) also simply as \( F \circ F' \).

3) Moreover, \( \phi \) and \( \psi \) make the following diagrams commute. Here we use \( I' \) to denote \( u'(*) \).

\[
(F_g \circ F_h) \circ F_f \xrightarrow{\phi_{g \circ h} \circ \phi_f} F_{g \circ h \circ f} \xrightarrow{\eta_{g \circ h \circ f}} (F_g \circ F_h) \circ F_f
\]
(2.23)

\[
F_g \circ (F_h \circ F_f) \xrightarrow{\phi_{g \circ h, f}} F_{g \circ h \circ f} \xrightarrow{\eta_{g \circ h \circ f}} F_{g \circ h \circ f}
\]

\[
I' \circ F_g \xrightarrow{\psi_{I'}(F_g)} F_I \xrightarrow{\phi_{I, g}} F_{I \circ g}
\]
(2.24)
We now verify the three items one by one.

Verification of item 1): To verify 1), we need to see that $(F, \eta)$ preserves identities and compositions. That is, we need to verify that $\eta_{1_{g_0}} = \text{id}$ and $\eta_{g_1 \cdot h_{g_1}} = \eta_{g_1} \circ \eta_{h_{g_1}}$. These follow from some direct calculation. The morphism $g_0 \to g_0$ is sent to $\eta_{g_0}$, and $$\eta_{1_{g_0}}(\rho)_x : F_{g_0}(\rho)(x) \to F_{g_0}(\rho)(x), \quad v \mapsto \rho(1_x \cdot h_{1_{g_0}})(v) = \rho(1_x \cdot h_{g_0})(v) = v.$$ For $x \in G_0, g_1, g'_1 \in G_1$, and $\rho$ an object of $\text{Rep}^G$, $$\eta_{g'_1 \cdot g_1}(\rho)_x = \rho(1_x \cdot h_{g'_1 \cdot g_1}) = \rho((1_x \cdot h_{g'_1}) \cdot (1_x \cdot h_{g_1})) = \rho(1_x \cdot h_{g'_1}) \circ \rho(1_x \cdot h_{g_1}) = (\eta_{g'_1} \circ \eta_{g_1})(\rho)_x (2.28)$$

Verification of item 2): it is easy to see that $$F_{g_0 \cdot h_{g_0}}(\rho)(x) = \rho(x \cdot h_{g_0 \cdot h_{g_0}}) (2.26)$$

This shows that there is a natural transformation (in fact isomorphism) $$\phi_{g_0 \cdot h_{g_0}} : F_{g_0 \cdot h_{g_0}} \to F_{g_0} \circ F_{h_{g_0}}$$

between these two autofunctors, given by $$\phi_{g_0 \cdot h_{g_0}}(\rho) = \rho(a_{g_0 \cdot h_{g_0}}^{-1}) (2.29)$$

at each object $\rho$ in $\text{Rep}^G$,

Now we verify that $\phi_{g_0 \cdot h_{g_0}}$ defined above is indeed a natural transformation. Let $\alpha : \rho \Rightarrow \rho'$ be a 1-morphism in $\text{Rep}^G$. Then we need to show the diagram commutes

$$F_{g_0 \cdot h_{g_0}}(\rho) \xrightarrow{\rho(a_{g_0 \cdot h_{g_0}}^{-1})} (F_{g_0} \circ F_{h_{g_0}})(\rho) (2.30)$$

It suffices to show the diagram below commutes for any $x$

$$\rho(x \cdot h_{g_0 \cdot h_{g_0}}) \xrightarrow{\rho(a_{g_0 \cdot h_{g_0}}^{-1})} \rho((x \cdot h_{g_0}) \cdot h_{g_0}) (2.31)$$

$$\alpha(x \cdot h_{g_0 \cdot h_{g_0}}) \xrightarrow{\alpha((x \cdot h_{g_0}) \cdot h_{g_0})} \rho((x \cdot h_{g_0}) \cdot h_{g_0})$$

$$\rho'(x \cdot h_{g_0 \cdot h_{g_0}}) \xrightarrow{\rho'(a_{g_0 \cdot h_{g_0}}^{-1})} \rho'((x \cdot h_{g_0}) \cdot h_{g_0})$$
which commutes because of the naturality of the associator. Then \( \phi : m' \circ ((F, \eta) \times (F, \eta)) \Rightarrow (F, \eta) \circ m \), given by \( \phi_{(g_0, \tilde{g}_0)} \) at each object \((g_0, \tilde{g}_0) \in G_0 \times G_0\) is the natural transformation that we desire. The naturality of \( \phi \),

\[
\begin{align*}
F_{g_0, \tilde{g}_0} \xrightarrow{\phi_{(g_0, \tilde{g}_0)}} F_{g_0} & \circ F_{\tilde{g}_0} , \quad (2.32) \\
\eta_{g_1, \tilde{g}_1} & \circ \rho((1 - \tilde{g}_1, \tilde{g}_1)) \\
F_{g_0, \tilde{g}_0} \xrightarrow{\phi_{(g_0, \tilde{g}_0)}} F_{g_0} & \circ F_{\tilde{g}_0}
\end{align*}
\]

follows from the naturality of \( \rho \).

Now we construct the natural transformation \( \psi \) in (2.22) by

\[
\psi_x : I' = u'(\ast) \rightarrow (F, \eta)(\ast) = F_I, \quad \psi_x(\rho)(x) = \rho(r_x)^{-1}. \quad (2.33)
\]

The naturality of \( \rho \) and the functoriality of \( \rho \) implies that \( \psi_x \) is indeed a natural transformation between the two autofunctors \( I' \) and \( F_I \). Then the naturality of \( \psi \) is trivial to be shown because there is only identity morphism in \( \ast \).

Verification of item 3): The 6-arrow diagram (2.23) when evaluated on an object \( \rho \) in \( \text{Rep}^F \) becomes

\[
\begin{align*}
(F_{g_0} \circ F_{h_0}) \circ F_{f_0}(\rho) & \xrightarrow{\rho(a_{-g_0, h_0}) \circ \text{id}_{F_{f_0} \circ F_{h_0}}(\rho)} \rho(a_{-g_0, h_0}) \circ \rho(a_{-g_0, h_0}) \circ F_{f_0}(\rho) \\
F_{g_0} \circ (F_{h_0} \circ F_{f_0})(\rho) & \xrightarrow{\rho(a_{-g_0, h_0}) \circ \text{id}_{F_{f_0} \circ F_{h_0}}(\rho)} \rho(a_{-g_0, h_0}) \circ \rho(a_{-g_0, h_0}) \circ F_{f_0}(\rho)
\end{align*}
\]

\[
\begin{align*}
\eta_{g_0, h_0, f_0} & \circ \rho(a_{-g_0, h_0}) \circ \rho(a_{-g_0, h_0}) \circ F_{f_0}(\rho) \\
\eta_{g_0, h_0, f_0} & \circ (F_{g_0} \circ F_{h_0} \circ F_{f_0})(\rho)
\end{align*}
\]

Notice that \( \text{id}_{F_{f_0}}(\rho) = \rho(\text{id}_{f_0}) \), and when we further evaluate on \( x \in G_0 \), the natural transformation \( \rho(a_{-g_0, h_0}) \circ h \rho(\text{id}_{f_0}) \) at \( x \) as a morphism \( \rho((x \cdot g, h_0) \cdot h f_0) \rightarrow \rho((x \cdot (g_0 \cdot h_0) \cdot h f_0)) \), is exactly \( \rho(a_{x, g_0, h_0} \cdot h \text{id}_{f_0}) \). So the diagram follows from the pentagon condition of \( a \) and the functoriality of \( \rho \).

In the diagram (2.24) the composition

\[
(\eta_I \circ h \phi^{-1}_{(I, g_0)})(\rho)(x) = \rho(\text{id}_{x, I} \cdot h l_{g_0}) \circ a_{x, I, g_0}^{-1}
\]

\[
= \rho((\text{id}_{x, I} \cdot h l_{g_0}) \circ h a_{x, I, g_0}^{-1}) = \rho(r_x \cdot h l_{I})
\]

\[
= \rho(r_x) \circ h \text{id}_{I'}
\]

Thus, \( (\eta_I \circ h \phi^{-1}_{(I, g_0)})(\rho)(x) \cdot \psi(\rho)(x) \) is identity.

By Lemma (2.3) we also have the commutative diagram

\[
\begin{align*}
(x \cdot g_0) \cdot I & \xrightarrow{a_{x, g_0, I}} x \cdot (g_0 \cdot h I) \\
r_{g_0} & \circ (x \cdot h g_0) \\
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

Thus, in the diagram (2.25) the composition

\[
(\eta_I \circ h \phi^{-1}_{(g_0, I)})(\rho)(x) = \rho(\text{id} \cdot h r_{g_0}) \circ h a_{x, g_0, I}^{-1}
\]

\[
= \rho((\text{id} \cdot h r_{g_0}) \circ h a_{x, g_0, I})
\]

\[
= \rho(r_{x, h g_0})
\]

\[
\end{align*}
\]
Thus, the composition \((\eta \circ h^{-1} \circ g_0)(\rho)(x) \cdot h \cdot (\eta \circ h_0)(\rho)(x)\) is the identity. 

Suppose there is a Lie 2-group \(\mathcal{G} = (G_1 \Rightarrow G_0)\) whose underling Lie groupoid is an action groupoid, that is, \(G_0 = G, G_1 = H \ltimes G\) and \(s(h,g) = g, t(h,g) = h \cdot g, (h_1, g_1)(h_2, g_2) = (h_1 h_2, g_2), \) where \(H\) acts on \(G\) from the left with \(\cdot\). Let \(\text{Rep}_{tr}^G\) denote the full sub-category of \(\text{Rep}^G\) which consists of trivial vector bundles. Then a representation \(\rho \in \text{Rep}_{tr}^G\) reads for us a trivial vector bundle \(V \times G\) and \(\rho(h,g) \in \text{End}(V)\). Therefore, it is not hard to see that an \(H\)-representation \(\rho_0\) gives us an element \(\rho \in \text{Rep}_{tr}^G\), by
\[
\rho(h,g) := \rho_0(h).
\]

In fact, we may further see that \(\text{Rep}(H) \overset{i}{\to} \text{Rep}_{tr}^G\) is a full subcategory.

**Corollary 2.8.** For a Lie 2-group \(\mathcal{G}\), there is another 2-representation similar to what is given in Lemma 2.7 but restricted to the sub category \(\text{Rep}_{tr}^G\), that is, we replace \(\text{Rep}^G\) by \(\text{Rep}_{tr}^G\) everywhere. More precisely, there is a 2-representation of \(\mathcal{G}\) with

- **Object level:** the only object \(*\) in the 2-category corresponding to a 2-group, maps to \(\text{Rep}_{tr}^G\).

- **1-Morphism level:**
  \[
  F : G_0 \to \text{Fun}(\text{Rep}_{tr}^G, \text{Rep}_{tr}^G), \quad g \mapsto F_g,
  \]
  where
  \[
  F_g(\rho) : x \mapsto V, \quad (x \xrightarrow{\alpha} y) \mapsto (V \xrightarrow{\rho(\alpha \cdot h_1)} V),
  F_g(f) \circ F_g(\rho) = F_g(f \circ \rho)
  \]
  where \(\rho \in \text{Rep}_{tr}^G\) comes with the trivial bundle \(V \times G_0\), and \(f\) is a 1-morphism in \(\text{Rep}^G\), thus also in \(\text{Rep}_{tr}^G\).

- **2-Morphism level:** given \((g_0 \overset{g_1}{\Rightarrow} g_0') \in G_1\)
  \[
  G_1 \to \{\text{natural transformations}\}, \quad g_1 \mapsto \eta_{g_1},
  \eta_{g_0}(\rho)_x : F_{g_0}(\rho)(x) \to F_{g_0'}(\rho)(x), \quad v \mapsto \rho(1_x \cdot \eta_{g_1}(v)), \quad \forall v \in F_{g_0}(\rho)(x).
  \]

**Proof.** The 2-representation we describe above is what we have in Lemma 2.7 but restricting to \(\text{Rep}_{tr}^G\). Because the inclusion \(\text{Rep}_{tr}^G \subseteq \text{Rep}^G\) is fully faithful, this gives us another 2-representation of \(\mathcal{G}\).

### 2.4 Relation between 1-representations and 2-representations

Given a representation \((V, \zeta)\) of a Lie group \(H\), we form a category \(\{V\}/\zeta(H)\) with just one object \(V\). The morphisms form the set \(\zeta(H)\) with arrows from \(V\) to \(V\) given in the obvious way. Then given an action Lie groupoid \(G = H \ltimes G \Rightarrow G\), we have a \(k\)-linear category,
\[
C_V := \text{Fun}(G, \{V\}/\zeta(H)),
\]
and clearly \(C_V\) is a subcategory of \(\text{Rep}_{tr}^G\). In a differential geometric viewpoint, \(C_V\) is the category of special trivial representations of \(\mathcal{G}\) with objects \((G \times V, \rho)\) where \(\rho(h,g)\) is the arrow \((hg, \zeta(h)(v)) \leftrightarrow (g, v)\). Similarly, we have
Corollary 2.9. Let $\mathcal{G}$ be a Lie 2-group with underline Lie groupoid given by an action groupoid $H \ltimes G \Rightarrow G$, and let $(V, \zeta)$ be a representation of $H$. Then there is another 2-representation of $\mathcal{G}$ similar to what is given in Lemma 2.7 and Cor. 2.8 but restricted to the subcategory $C_V$, that is, we replace $\text{Rep}^G$ by $C_V$ everywhere. We denote this 2-representation of $G$ by $(f^V, \eta^V)$.

With these results, we have the following theorem.

Theorem 2.10. Let $\mathcal{G}$ be a Lie 2-group with underline Lie groupoid given by an action groupoid $H \ltimes G \Rightarrow G$, there is a functor $i : \text{Rep}H \rightarrow 2\text{Rep}\mathcal{G}$

defined as follows.

- On the level of objects: for $(V, \zeta) \in \text{Rep}H$, $i(V, \zeta) := (f^V, \eta^V)$ as in Cor. 2.9.

- On the level of morphisms: for each morphism

  $$(V_0, \zeta_0) \xrightarrow{\omega} (V_1, \zeta_1)$$

  in $\text{Rep}H$, we define a morphism $i(\omega)$

  $$i(V_0, \zeta_0) \xrightarrow{(T(\ast), T_{\omega})} i(V_1, \zeta_1)$$

  in $2\text{Rep}\mathcal{G}$ below. For each $h \in H$, we have the commutative diagram

  $$(V_0, \zeta_0) \xrightarrow{\omega} (V_1, \zeta_1)$$

  $$\downarrow \zeta_0(h) \quad \downarrow \zeta_1(h)$$

  $$(V_0, \zeta_0) \xrightarrow{\omega} (V_1, \zeta_1).$$

  For each $\omega$, we can define a functor

  $$F_\omega : \{V_0\}/\zeta_0(H) \rightarrow \{V_1\}/\zeta_1(H)$$

  by sending the object $V_0$ to $V_1$ and sending each morphism $\zeta_0(h)$ to $\zeta_1(h)$. Moreover, we can define a functor

  $$F_{\omega^*} : C_{V_0} \rightarrow C_{V_1}$$

  by sending a morphism

  $$\mathcal{G} \xrightarrow{\rho} \{V_0\}/\zeta_0(H)$$

  to

  $$\mathcal{G} \xrightarrow{\rho} \{V_0\}/\zeta_0(H) \xrightarrow{F_{\omega^*}} \{V_1\}/\zeta_1(H).$$

  Then we are ready to define the morphism $i(\omega)$

  $$i(V_0, \zeta_0) \xrightarrow{(T(\ast), T_{\omega})} i(V_1, \zeta_1)$$

  in $2\text{Rep}\mathcal{G}$ in the way below.
 definition \cite{Lur09, Example 1.2.4}. If \( \text{Vect}_k \) is the category of \( k \)-linear vector spaces, then \( \text{Rep}_G \) is a category enriched in \( k \)-linear category, thus, a 2-vector space by the definition \cite{Lur09, Example 1.2.4]. If \( \text{Vect} \) is the category of finite-dimensional vector spaces, then \( \text{Rep}_{G_\mathfrak{tr}} \) is semisimple. Let \( G = \langle G_1 \Rightarrow G_0 \rangle \) be a Lie 2-group, then \( \text{Rep}^G_{\mathfrak{tr}} \) is a category enriched in \( k \)-vector spaces and is a cocomplete \( k \)-linear category, thus, a 2-vector space by the definition \cite{Lur09, Example 1.2.4]. If \( \text{Vect} \) is the category of finite-dimensional vector spaces, then \( \text{Rep}^G_{\mathfrak{tr}} \) is semisimple.

\[ T(*) := F_{\mathfrak{tr}} : C_{V_0} \rightarrow C_{V_1} \]
\[ (G \xrightarrow{\rho} \{V_0\}/\zeta_0(H)) \mapsto (G \xrightarrow{\rho} \{V_0\}/\zeta_0(H) \xrightarrow{F_{\mathfrak{tr}}} \{V_1\}/\zeta_1(H)) \]

2. The natural transformation \( T_{\omega} \)

\[ \begin{array}{ccc}
G_0 & \xrightarrow{f_{V_0}} & \text{Auto}(C_{V_0}) \\
\downarrow f_{V_1} & & \downarrow \triangleright_{T_{\omega}} \\
\text{Auto}(C_{V_1}) & \xrightarrow{T(*)} & \text{Fun}(C_{V_0}, C_{V_1})
\end{array} \]

is defined to be identity.

\textbf{Proof.} The functor \( i \) defined in the theorem satisfies the axioms in Example \textbf{2.5}. \( \square \)

\textbf{Lemma 2.11.} Let \( G = \langle G_1 \Rightarrow G_0 \rangle \) be a Lie 2-group, then \( \text{Rep}^G_{\mathfrak{tr}} \) is a category enriched in \( k \)-vector spaces. Notice that direct sum of vector spaces is a biproduct, a category enriched in \( k \)-vector spaces is an additive category. Now we show that \( \text{Rep}^G_{\mathfrak{tr}} \) is abelian. We define biproduct \( \oplus \) in \( \text{Rep}^G_{\mathfrak{tr}} \) by

\[ \rho_1 \oplus \rho_2 : \mathcal{G} \rightarrow \text{GL}(V_1 \oplus V_2), \quad g \mapsto \rho_1(g) \oplus \rho_2(g), \quad \forall \rho_1, \rho_2 \in \text{Rep}^G_{\mathfrak{tr}}, \quad (2.37) \]

where \( V_i \) is the trivial vector bundle over \( G_0 \) on which \( \mathcal{G} \) represents by \( \rho_i \). Let \( \mathfrak{U} \) denote the 0-dimensional vector bundle over \( G_0 \). The zero object in \( \text{Rep}^G_{\mathfrak{tr}} \) is the morphism sending \( \mathcal{G} \) to \( \text{GL}(\mathfrak{U}) \). Let \( \rho_i \) be an object in \( \text{Rep}^G_{\mathfrak{tr}} \) whose image is a vector space \( V_i \), \( i = 1, 2 \). Let \( f : \rho_1 \rightarrow \rho_2 \) denote a morphism in \( \text{Rep}^G_{\mathfrak{tr}} \). Then it is an \( H \)-equivariant linear map from \( V_1 \) to \( V_2 \). (And given any \( H \)-equivariant linear map from \( V_1 \) to \( V_2 \), we can define a functor from \( \rho_1 \) to \( \rho_2 \).) The kernel of \( f \) is the functor ker \( f \) from \( \mathcal{G} \) to the vector space ker \( f \), which is an \( H \)-invariant subspace of \( V_1 \). We have the evident monomorphism from ker \( f \) to \( \rho_1 \). And the cokernel of \( f \) is the functor from \( \mathcal{G} \) to the vector space coker \( f \), which is an \( H \)-invariant subspace of \( V_2 \). We have the evident epimorphism from coker \( f \) to \( \rho_2 \).

Since the category of \( H \)-representations is an abelian category, for any \( H \)-equivariant linear map \( h : V_1 \rightarrow V_2 \), when it is a monomorphism, it is a kernel, and when it is an epimorphism, it is a cokernel. Thus, \( \text{Rep}^G_{\mathfrak{tr}} \) is also an abelian category. The cocompleteness of \( \text{Rep}^G_{\mathfrak{tr}} \) comes from the cocompleteness of the category of \( H \)-representations. An object in \( \text{Rep}^G_{\mathfrak{tr}} \) is simple if its image in \( \text{Vect} \) is simple. If \( \text{Vect} \) is the category of finite-dimensional vector spaces, it is semisimple, i.e. each object in it is the direct sum of finitely many simple objects. In this case, each object in \( \text{Rep}^G_{\mathfrak{tr}} \) is the direct sum of finitely many simple objects. Thus, \( \text{Rep}^G_{\mathfrak{tr}} \) is semisimple. \( \square \)

2.5 2-Representations of cross modules

When the Lie 2-group is strict, there is another sort of 2-representation for them. Even though we will not use this sort of 2-representation to later for String Lie 2-groups, we
will still demonstrate it here for some completeness. Let us first recall the definition of a
strict Lie 2-group through the concept of a crossed module of Lie groups. For convenience,
we use the version in [MRW17] with a right action because we will use their model on
string 2-groups.

**Definition 2.12.** A crossed module of Lie groups \((H \xrightarrow{\partial} G, \alpha)\) consists of a Lie group
morphism \(\partial : H \to G\) and a right action \(\alpha\) of \(G\) on \(H\), namely a Lie group morphism
\(\alpha : G^{op} \to \text{Aut}(H)\), such that

(Eq) \(\partial(\alpha(g)h) = g^{-1}\partial(h)g\) (Equivariance)

(Pf) \(\alpha(\partial(h))h' = h^{-1}h'\) (Pfeiffer identity).

Such a crossed module of Lie groups gives us a strict Lie 2-group with the simplicial
nerve

\[
X_0 = \text{pt}, \quad X_1 = G, \quad X_2 = G \times G \times H, \quad \ldots \quad X_k = G^{k \times G} \times H^{(k)},
\]

Or we may see it as a group structure on the action Lie groupoid

\(H \ltimes G \Rightarrow G\), where \(H\) acts from left on \(G\) through \(\partial\). More precisely, we have

\[
t(g_1, h) = g_2 = \partial(h)g_1, \quad s(g_1, h) = g_1,
\]

The groupoid multiplication maybe visualized as vertical product,

\[
(g_2, h_2) \cdot_v (g_1, h_1) = (g_1, h_2h_1),
\]

where \(g_1 = \partial(h_2)g_2 = \partial(h_2)\partial(h_1)g_1 = \partial(h_2h_1)g_1\).

The group structure give further a horizontal product,

\[
(g_1, h) \cdot_h (g_1', h') = (g_1g_1', h \cdot_h h'),
\]

where \(h \cdot_h h' = h\alpha(g_1^{-1})(h')\). Indeed,

\[
g_2g'_2 = \partial(h)g_1\partial(h')g'_1 = \partial(h)g_1\partial(h')g_1^{-1}g_1g'_1 = \partial(h\alpha'(g_1^{-1})(h'))g_1g'_1.
\]

Thus, \(H \ltimes G \Rightarrow G\) is a group object in the (strict) 2-category of Lie groupoid, Lie
groupoid morphisms, and 2-morphisms. All the 2-morphisms in Definition 2.1 are identity.
Therefore, it is a strict Lie 2-group.

From now on we do not distinguish strict Lie 2-groups and crossed modules.

**Lemma 2.13.** A crossed module \((H \xrightarrow{\partial} G, \alpha : G^{op} \to \text{Aut}(H))\) represents on \(\text{2Vect}\) in
the following way
- **Object level:** the only object \( * \) in the 2-category corresponding to a crossed module, which is a 2-group, maps to the category of \( H \)-representations, that is \( * \mapsto \text{Rep}H \).

- **1-Morphism level:**
  
  \[
  G \to \text{Fun}(\text{Rep}H, \text{Rep}H), \quad g \mapsto F_g,
  \]

  where

  \[
  F_g((V, \rho)) = (V, \rho \circ \alpha(g)), \quad F_g((V, \rho) \xrightarrow{f} (V', \rho')) = (V, \rho \circ \alpha(g)) \xrightarrow{f} (V', \rho' \circ \alpha(g)).
  \]

  Here \( \rho : H \to \text{Aut}(V) \) is a representation of \( H \) and we denote such a representation by \( (V, \rho) \).

- **2-Morphism level:**
  
  \[
  G \times H \to \{\text{natural transformations}\}, \quad (g_1, h) \mapsto \eta_{(g_1, h)},
  \]

  where

  \[
  \eta_{(g_1, h)}(V, \rho) : (V, \rho \circ \alpha(g_1)) \to (V, \rho \circ \alpha(g_2)), \quad v \mapsto \rho(\alpha(g_1)(h^{-1}))(v).
  \]

**Proof.** We first check whether \( F_g \) is indeed a functor. For this we need to see whether \( f : (V, \rho \circ \alpha(g)) \to (V', \rho' \circ \alpha(g)) \) is still \( H \)-equivariant thus still a morphism between \( H \)-representations. This follows from a direct calculation,

\[
 f(h \cdot v) = f(\rho(\alpha(g)(h))v) = \rho'(\alpha(g)(h))f(v) = h \cdot f(v).
\]

We now check whether \( \eta_{(g_1, h)}(V, \rho) \) is equivariant:

\[
\eta_{(g_1, h)}(V, \rho)(h \cdot v) = \rho(\alpha(g_1)(h^{-1}))(h \cdot v) = \rho(\alpha(g_1)(h^{-1}))\rho(\alpha(g_1)(\hat{h}))(v) = \rho(\alpha(g_1)(h^{-1})\alpha(g_1)(\hat{h}))(v),
\]

and

\[
\hat{h} \cdot \eta_{(g_1, h)}(V, \rho)(v) = \rho(\alpha(g_2)\hat{h})(\rho(\alpha(g_1)(h^{-1}))v)
\]

\[
= \rho(\alpha(\partial(h)g_1)\hat{h})(\rho(\alpha(g_1)(h^{-1}))v) = \rho(\alpha(g_1)\alpha(\partial(h))(\hat{h}\alpha(g_1)(h^{-1}))v)
\]

\[
= \rho(\alpha(g_1)(h^{-1})\hat{h}h\alpha(g_1)(h^{-1})\hat{h})(v) = \rho(\alpha(g_1)(h^{-1}\hat{h}))(v)
\]

\[
= \rho(\alpha(g_1)(h^{-1})\alpha(g_1)(\hat{h}))(v) = \eta(h \cdot v)
\]

where \( g_2 = \partial(h)g_1 \). Thus, \( \eta \) is equivariant and thus it defines a morphism between representations.

The following diagram

\[
\begin{array}{c}
F_{g_1}((V, \rho)) \xrightarrow{\eta_{(g_1, h)}(V, \rho)} F_{g_2}((V, \rho)) \\
\downarrow F_{g_1}(f) \\
F_{g_1}((V', \rho') \xrightarrow{\eta_{(g_1, h)}(V', \rho')} F_{g_2}((V', \rho'))
\end{array}
\]

(2.38)

commutes. In fact, this boils down to the fact that \( f(\rho(\alpha(g_1)(h^{-1}))v) = \rho'(\alpha(g_1)(h^{-1}))f(v) \), which is implied by the fact that \( f \) is a morphism of representations. Thus \( \eta \), that we construct, is a natural transformation.
Now to verify that these data gives us a morphism from $BG$ to $2\text{Vect}$, where $G$ is the strict Lie 2-group corresponds to our crossed module, we still need to verify three items as in Lemma 2.7. But the strictness simplifies a lot our proof. Now we check these items one by one:

**Verification of 1):** We need to check whether

\[
\eta_{(g_2,h_2)\circ\eta_{(g_1,h_1)}} = \eta_{(g_2,h_2)\circ\eta_{(g_1,h_1)}} = \eta_{(g_1,h_2h_1)}. \tag{2.39}
\]

Consider the composition

\[
\begin{array}{c}
\eta_{(g_2,h_2)}(V,\rho) \circ \eta_{(g_1,h_1)}(V,\rho)(v) \\
= \eta_{(g_2,h_2)}(V,\rho)(\rho(\alpha(g_1)(h_1^{-1}))(v)) = \rho(\alpha(g_2)(h_2^{-1}))\rho(\alpha(g_1)(h_1^{-1}))(v) \tag{2.40} \\
= \rho(\alpha(\partial(h_1)g_1)(h_2^{-1})\alpha(g_1)(h_1^{-1}))(v) = \rho(\alpha(g_1)\alpha(\partial(h_1))(h_2^{-1})\alpha(g_1)(h_1^{-1}))(v) \tag{2.41} \\
= \rho(\alpha(g_1)(h_1^{-1}h_2^{-1}h_1^{-1}))(v) = \rho(\alpha(g_1)(h_2h_1^{-1})(v)) \tag{2.42} \\
= \eta_{(g_1,h_2h_1)}(V,\rho)(v) = \eta_{(g_2,h_2)\circ\eta_{(g_1,h_1)}}(V,\rho)(v) \tag{2.43}
\end{array}
\]

**Verification of 2) and 3):** We see that $F_{gg} = F_g \circ F_{\bar{g}}$.

\[
F_{gg}(V,\rho) = (V,\rho \circ \alpha(g\bar{g})) = (V,\rho \circ \alpha(\bar{g}) \circ \alpha(g)) = F_g(V,\rho \circ \alpha(\bar{g})) = F_gF_{\bar{g}}(V,\rho).
\]

Therefore the natural transformation $\phi$ in (2.21) is id. It is then similar to show that $\psi$ in (2.21) is also id. Thus the coherence conditions in 3) automatically hold. \hfill \square

**Remark 2.14.** In the above setting, we see that there is a horizontal composition of the natural transformation. Let’s consider the functors $F_{g_1}$, $F_{g_1'}$, $F_{g_2}$, $F_{g_2'}$: $\text{Rep}H \to \text{Rep}H$ and the natural transformations

\[
\eta_{(g_1,h)} : F_{g_1} \Rightarrow F_{g_2}, \quad \eta_{(g_1',h')} : F_{g_1'} \Rightarrow F_{g_2'}.
\]

We will show the two maps

\[
F_{g_1'} \circ F_{g_1}(V,\rho) \xrightarrow{\eta_{(g_1,h)}(V,\rho)} F_{g_1'} \circ F_{g_2}(V,\rho) \xrightarrow{\eta_{(g_1',h')(V,\rho)}} F_{g_2'} \circ F_{g_2}(V,\rho) \tag{2.45}
\]

and

\[
F_{g_1'} \circ F_{g_1}(V,\rho) \xrightarrow{\eta_{(g_1',h')(V,\rho)}} F_{g_1'} \circ F_{g_2}(V,\rho) \xrightarrow{\eta_{(g_2,h)}(V,\rho)} F_{g_2'} \circ F_{g_2}(V,\rho) \tag{2.46}
\]

are the same. The map in (2.45) sends $v$ to $\rho(\alpha(g_1)(h_1^{-1})) \cdot v$ first and then to

\[
\rho \circ \alpha(g_2)(\alpha(g_1')(h_1^{-1})) \rho(\alpha(g_1)(h_1^{-1})) \cdot v,
\]

and

\[
\rho \circ \alpha(g_2)(\alpha(g_1')(h_1^{-1})) \rho(\alpha(g_1)(h_1^{-1})) \cdot v \tag{2.47}
\]

\[
= \rho(\alpha(\partial(h_1)g_1)(h_1^{-1})\alpha(g_1)(h_1^{-1}))(v) \tag{2.48}
\]

\[
= \rho(\alpha(g_1)\alpha(\partial(h))(h_1^{-1})\alpha(g_1)(h_1^{-1}))(v) \tag{2.49}
\]

\[
= \rho(\alpha(g_1)(h_1^{-1}h_1^{-1}h_1^{-1}))(v) = \alpha(g_1)(h_1^{-1}h_1^{-1}h_1^{-1}h_1^{-1}) \tag{2.50}
\]

\[
= \alpha(g_1)(h_1^{-1}h_1^{-1}h_1^{-1}). \tag{2.51}
\]
The map in (2.46) sends $v$ to $\rho \circ \alpha(g_1)(\alpha(g'_1)(h'^{-1}))(v)$ first and then to

$$\rho(\alpha(g_1)(h^{-1}))\rho(\alpha(g_1)(\alpha(g'_1)(h'^{-1}))(v)) = (\rho \circ \alpha(g_1))(h^{-1}\alpha(g'_1)(h'^{-1}))(v)$$

which is equal to (2.51).

Thus, the horizontal composition of natural transformations is well-defined, and the composition is $\eta_{\rho(g_1,h'^{-1})}(V, \rho)$.

### 3 Loop group and string 2-group

#### 3.1 Models of string Lie 2-group

We wish to construct 2-representations of String 2-group $\text{String}(G)$ from representations of loop group $LG$. To do this, we need a model of $\text{String}(G)$ involves $LG$.

There are two Lie groupoids in [MRW17], both are models for $\text{String}(G)$, $\hat{\Omega} \flat G \ltimes Q \flat$.

The letter $L$ stands for loops and $Q$ stands for quasi-periodic paths.

Then $\hat{\Omega}^\flat G$ is the standard $S^1$ central extension of $LG$ and $\hat{\Omega}^\flat G = LG|_{Q^\flat}$ is the restriction of this central extension. They fit in the following diagram

$$
\begin{array}{c}
\Omega^\flat G \times Q^\flat \xrightarrow{\xi} \hat{\Omega}^\flat G \xrightarrow{\pi^\flat} LG \times QG \\
\end{array}
$$

$Q^\flat := \{ \gamma \in C^\infty(\mathbb{R}, G) | \gamma(1)\gamma(0)^{-1} \text{ is constant w.r.t. to } t \}$

$Q^\flat_{b,*} G := \{ \gamma \in C^\infty(\mathbb{R}, G) | \gamma(0) = e, \gamma(t+1)\gamma(t)^{-1} = \gamma(1) \text{ is constant w.r.t. to } t, \gamma^{(n)}(1) = \gamma^{(n)}(0) = 0, \forall n \geq 1 \} \subset QG$

$LG := \{ \gamma \in C^\infty(\mathbb{R}, G) | \gamma(t+1)\gamma(t)^{-1} = e \} \subset QG$

$\Omega^\flat_{b,*} G := LG \cap Q^\flat_{b,*} G$

The letter $L$ stands for loops and $Q$ stands for quasi-periodic paths.

Then $\hat{\Omega}^\flat G$ is the standard $S^1$ central extension of $LG$ and

$$\Omega^\flat G = LG|_{Q^\flat_{b,*} G}$$

is the restriction of this central extension. They fit in the following diagram

$$
\begin{array}{c}
1 \longrightarrow S^1 \longrightarrow \hat{\Omega}^\flat G \longrightarrow \Omega^\flat G \longrightarrow 1 \\
\end{array}
$$

There is a natural group structure on $Q^\flat_{b,*} G$ defined by

$$\gamma_1 \gamma_2 := (\gamma_1|_{[0,1]}\gamma_2|_{[0,1]})^\sim,$$
where \( \sim \) denotes to extend a path in \([0,1]\) into a quasi-periodic path. \( \Omega_b G \) inherits this group structure and becomes its Lie sub-group. Moreover, we have a crossed module of Lie groups \( \tilde{\Omega}_b G \xrightarrow{\partial} Q_{b,*} G \) with

\[
\partial : \tilde{\Omega}_b G \xrightarrow{\partial} \Omega_b G \hookrightarrow Q_{b,*} G
\]

being the composition. This gives arise to a natural strict Lie 2-group structure on \( S(Q_{b,*} G) \) in (3.52), with \( s(\tilde{\eta}, \gamma) = \gamma \), and \( t(\tilde{\eta}, \gamma) = \eta \gamma \), where \( \eta = \pi(\tilde{\eta}) \). Here the product between \( \eta \) and \( \gamma \) is simply the one in \( Q_{b,*} G \). See [MRW17, Remark 3.4].

In (3.52), \( \iota \) is natural inclusion on both level, and by [MRW17, Theorem 3.6], it is a weak equivalence of Lie groupoids if \( G \) is simple, compact and 1-connected. We use the symbol \( \xi \) to denote the quasi-inverse of \( \iota \), which is the functor \( \rho \) in [MRW17]. They help to transfer the Lie 2-group structure to \( S(Q_{b,*} G) \), but this Lie 2-group is not strict any more. We recall below the explicit construction of the Lie 2-group structure on \( S(QG) \) in [MRW17].

First we use a relative standard trick to reparameterise with a smooth staircase cut-off function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \). \( \varphi \) is like a stair-walk, with

\[
\varphi(t + k) = \varphi(t) + k, \quad \varphi^{(n)}(k) = 0, \quad \varphi'(s) > 0,
\]

\( \forall n \geq 1, k \in \mathbb{Z}, s \notin \mathbb{Z}, t \in \mathbb{R} \). For example, we can take

\[
\varphi(t) = \tau(t - \lfloor t \rfloor),
\]

where \( \tau \) is the standard cut-off function on \([0,1]\) and \( \lfloor t \rfloor \) is the integer part of \( t \) such that \( t - \lfloor t \rfloor \in [0,1) \).

Then the formula of \( \xi \) on the level of \( QG \) is relatively easy:

\[
\xi(\gamma) = \gamma(0)^{-1} \gamma^\varphi,
\]

(3.55)

where a path \( g^\varphi(t) = g(\varphi(t)) \). Then clearly \( \xi(\gamma) \in Q_{b,*} G \), and \( \xi : LG \rightarrow \tilde{\Omega}_b G \). To see how \( \xi \) lifts to \( \xi : LG \rightarrow \tilde{\Omega}_b G \), and being defined on the level of arrows, we need to use the technique for classical quasi-inverses in category theory. The formula is also longer.

To lift \( \xi \) to the level of arrows in a groupoid, let us first observe a simple fact: suppose that we have an embedding of a full Lie subgroupoid,

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\iota} & G_1^c \\
\downarrow & & \downarrow \\
G_0 & \xrightarrow{\iota} & G_0^c
\end{array}
\]

This means that \( G_1 = G_1^c \times_{G_0^c,G_0} G_0 \), and \( \iota \) are embeddings on both levels. The Lie groupoid morphism \( \iota \) being essentially surjective, which means set-theoretically, that every element \( g^\varphi \in G^\varphi \) is isomorphic to an element \( g \in G \), is expressed in the world of Lie groupoids, as the following map being an surjective submersion,

\[
G_1^c \times_{G_0^c,G_0} G_0 \xrightarrow{\lambda} G_0^c.
\]

(3.56)

One may verify that if we don’t define it in \( Q_{b,*} G \) but in \( QG \), it will not be a good multiplication.
Then to construct a quasi-inverse, we can adapt the idea from (discrete case) category theory [Mac71 IV.4.1], namely we first find a map \( G^\varepsilon_0 \xrightarrow{\xi} G_0 \), such that \( \xi(g^\varepsilon) \cong g^\varepsilon \), and then given an arrow \( g^\varepsilon_1 : \hat{g}^\varepsilon_0 \xrightarrow{} g^\varepsilon_0 \), \( \xi(g^\varepsilon_1) \) is defined as the composed arrow \( \xi(\hat{g}^\varepsilon_0) \xrightarrow{g^\varepsilon_1} g^\varepsilon_0 \xrightarrow{} \xi(g^\varepsilon_0) \). This idea applied to the case of Lie groupoid, we need to find a section of \( 3.56 \)

\[
\sigma : G^\varepsilon_0 \rightarrow G^\varepsilon_1 \times_{s,G^\varepsilon_0} G_0.
\] (3.57)

Then \( \xi : G^\varepsilon_0 \rightarrow G_0 \) will be \( pr_{G_0} \circ \sigma \), which is exactly as in \( 3.55 \), and \( \xi : G^\varepsilon_1 \rightarrow G_1 \) will be \( \xi(g^\varepsilon_1) = (pr_{G^\varepsilon_0} \circ \sigma(s(g^\varepsilon_1)))^{-1} \cdot g^\varepsilon_1 \cdot pr_{G^\varepsilon_0} \circ \sigma(s(g^\varepsilon_1)). \) (3.58)

In our case,

\[
\begin{align*}
G^\varepsilon_0 &= QG, & G^\varepsilon_1 &= \hat{L}G \times QG, \\
G_0 &= Qb_\ast G, & G_1 &= \hat{\Omega}G \times Qb_\ast G.
\end{align*}
\]

Then we do have a section as in \( 3.57 \),

\[
\sigma : QG \rightarrow \hat{L}G \times QG Qb_\ast G, \quad \gamma \mapsto (\pi_2 \circ \sigma'(\gamma F_\ast, \gamma^{-1}), \gamma(0)^{-1} \gamma^\varphi),
\] (3.59)

where \( \sigma' : P_*LG \rightarrow P_*LG \times \hat{L}G \) is a section on “uniformly contractible loops” (or the space of paths of loops starting at constant loops) \( P_*LG \), and \( F : [0,1] \times [0,1] \rightarrow [0,1] \) is

\[
F_\ast(t) = (1 - s)t + s\varphi(t).
\]

Notice that \( \hat{L}G \rightarrow LG \) is a non-trivial \( S^1 \)-principal bundle in general, however, since the space \( P_*LG \) is contractible, we may always find a section.

\[
\begin{array}{ccc}
P_*LG \times_{LG} \hat{L}G & \xrightarrow{\sigma} & \hat{L}G \\
\downarrow & & \downarrow \\
QG & \xrightarrow{ev_1} & LG.
\end{array}
\] (3.60)

We give the construction of one such section below. \( P_*LG \) is the space of paths of loops starting at a certain constant loop. We fix this constant loop to be \( \varepsilon \). So \( P_*LG \) is contractible. Thus,

\[
P_*LG \times_{LG} \hat{L}G \cong P_*LG \times S^1
\]

and we will denote an element in \( P_*LG \times_{LG} \hat{L}G \) by a pair in \( P_*LG \times S^1 \). We can define the section \( \sigma' \) to be

\[
\sigma'(p) := (p, 1).
\] (3.61)

Recall the element \( C(l) \) in \( U(1) \) associated to each piecewise smooth loop \( l \) in \( LG \) in [PS86, Section 4.4] defined by

\[
C(l) := \exp(i \int_0^l \omega)
\] (3.62)
where $\omega$ is the integral closed 2-form defined \cite{PS86} (4.2.7), Section 4.4 and $\sigma$ is the surface in $LG$ bounded by $l$. Note that $\omega$ is the Kac-Moody cocycle defined in \cite{MRW17}, Remark 2.1.

The multiplication in $P_* LG \times S^1$ is given by the formula in \cite{PS86}, Section 4.4.

$$(p_1, u_1) \cdot (p_2, u_2) = (p_1 \cdot (\gamma_1 \cdot p_2), u_1 u_2) \quad (3.63)$$

where each $p_i$ is in $LG$ from $e$ to $\gamma_i$, $i = 1, 2$.

In (3.60), $\pi_2$ is the projection

$$\pi_2: P_* LG \times_{LG} \hat{LG} \rightarrow \hat{LG}$$

$$(p, u') \mapsto u'.$$ (3.64)

In other words, $\pi_2(p, u')$ is a lift of $p(1)$ to $\hat{LG}$.

Therefore, the Lie 2-group structure on $S(QG)$ is the following: on both levels of objects and morphisms, multiplication and inverse are defined by mapping to $S(Q_*, G)$ via $\xi$ and do the multiplication and inverse there. The horizontal unit is the constant path $1_\epsilon$ at the identity $e \in G$. We recall the construction below.

Let $\epsilon : \xi \Rightarrow \text{id}$ denote the counit and $\eta : \text{id} \Rightarrow \xi \epsilon$ denote the unit. Define the monoidal unit

$$I := \epsilon(1)$$

where $1$ is the monoidal unit in $S(Q_*, G)$. The multiplication in $S(QG)$ is defined by

$$f \otimes g = \epsilon(\xi(f)\xi(g)), \quad f, g \in QG. \quad (3.66)$$

The associator $a$ of the 2-group $S(QG)$ is defined as the composite

$$a_{f, g, h} : (f \otimes g) \otimes h = \epsilon[\xi(\epsilon(\xi(f)\xi(g)))] \xi(h)] \xrightarrow{\epsilon(\epsilon(f)\epsilon(g)\epsilon(h))} \epsilon(\xi(f)\epsilon(g)\epsilon(h))$$

$$\xrightarrow{\epsilon(\epsilon(f)\epsilon(g)\epsilon(h))} \epsilon(\xi(f)\xi(\epsilon(g)\xi(h))) = f \otimes (g \otimes h).$$

The right unitor $r$ is defined by

$$r_f : f \otimes I = \epsilon(\xi(f)\epsilon(1))) \xrightarrow{\epsilon(\epsilon(f)\epsilon(1))} \epsilon(\xi(f)) = \epsilon(\xi(f)) \xrightarrow{\eta_f^{-1}} f.$$ (3.67)

The left unitor $l$ is defined by

$$l_g : I \otimes g = \epsilon(\xi(1))\xi(g)) \xrightarrow{\epsilon(1\cdot \epsilon(g))} \epsilon(1\xi(g)) = \epsilon(\xi(g)) \xrightarrow{\eta_g^{-1}} g.$$ (3.68)

For more details, please see \cite{MRW17}, Section 3.

With the explicit formulas given above, we give some properties of $\sigma'$ and $\xi$ below.

Lemma 3.1 will be used in the proof of Lemma 6.6.

**Lemma 3.1.** The section $\sigma'$ defined by (3.31) has the property:

$$\sigma'(\alpha p^{-1}) = \alpha \sigma'(p) \alpha^{-1},$$

for any $\alpha \in G$ and any $p \in P_* LG$. 



22
Proof. For any $\alpha \in G$, let $\alpha$ denote the constant loop at $\alpha$. Let $\hat{\alpha}$ denote a lift of $\alpha$ in $\hat{LG}$. Let $p_{\alpha}$ denote a path in $P_*LG$ from $\underline{e}$ and $\alpha$. We may take $p_{\alpha}$ going through constant loops connecting $\underline{e}$ and $\alpha$.

For any path $p$ in $P_*LG$, we use $\overline{p}$ to denote the path $t \mapsto p(1-t)$ of the opposite direction. Define the path

$$p'_{\alpha} := \overline{p_{\alpha}}.$$

We have

$$(p_{\alpha}, 1) \cdot (p'_{\alpha}, 1) = (p_{\alpha} \ast (\overline{\alpha} \cdot p'_{\alpha}), 1) = (p_{\alpha} \ast \overline{p_{\alpha}}, 1)$$

by (3.63), which is equivalent to $(\underline{e}, 1)$ since the area $\sigma_0$ bounded by the loop

$$l = p_{\alpha} \ast \overline{p_{\alpha}}$$

is zero. Thus, the integral defined in (3.62) is zero.

In addition, by definition (3.63), we have

$$\pi_2(p_{\alpha}, 1) = \hat{\alpha}$$
$$\pi_2(p'_{\alpha}, 1) = \hat{\alpha}^{-1}$$

Now we are ready to compute

$$\alpha \cdot \pi_2(\sigma'(p)) \cdot \alpha^{-1} = \pi_2((p_{\alpha}, 1)(p, 1)(p'_{\alpha}, 1)) = \pi_2((p_{\alpha} \ast (\overline{\alpha} \cdot p), 1)(p'_{\alpha}, 1))$$

$$= \pi_2(p_{\alpha} \ast (\overline{\alpha} \cdot p) \ast (\overline{\alpha} \cdot p(1) \cdot p'_{\alpha}), 1)$$

Let $l_{\alpha}$ denote the loop consisting of $\alpha \cdot p \cdot \alpha^{-1}$, $\overline{\alpha} \cdot p(1) \cdot \overline{p_{\alpha}}$, $\alpha \cdot \overline{p_{\alpha}}$ and $\overline{p_{\alpha}}$. Let $\sigma_{\alpha}$ denote the surface in $LG$ bounded by $l_{\alpha}$.

By [MRW17] Remark 2.1] the integral closed 2-form $\omega$ is a continuous cycle on $Lg$ where $g$ here is the Lie algebra of $G$. And from this cocycle we can construct a central extension

$$U(1) \to \hat{LG} \to LG.$$ 

Then by [PS86] Theorem (4.4.1) (i)], the differential form $\omega$ represents an integral cohomology class on $LG$, i.e. its integral over every 2-cycle in $LG$ is an integer.

Since $\sigma_{\alpha}$ is a 2-cycle in $LG$, Thus, the integral $C(l_{\alpha})$ is an integer. So $(\alpha \cdot p \cdot \alpha^{-1})$ and $(p_{\alpha} \ast (\overline{\alpha} \cdot p) \ast (\overline{\alpha} \cdot p(1) \cdot p'_{\alpha}), 1)$ are equivalent. Thus,

$$\sigma'(\alpha \cdot p \cdot \alpha^{-1}) = \alpha \cdot \sigma'(p) \cdot \alpha^{-1}.$$

\[\square\]

Lemma 3.2. $\sigma'$ is a group homomorphism.

Proof. For any $p_1$, $p_2$ in $P_*LG$, let $p_i(1) = \gamma_i$ for $i = 1, 2$. We have

$$\sigma'(p_1p_2) = (p_1p_2, 1),$$
$$\sigma'(p_1) \sigma'(p_2) = (p_1 \ast (\gamma_1 \cdot p_2), 1).$$

Note that $p_1 \ast (\gamma_1 \cdot p_2)$ and $p_1p_2$ are both paths in $P_*LG$ from $\underline{e}$ to $\gamma_1 \gamma_2$.

Let $l$ denote the loop bounded by $p_1p_2$, $\gamma_1 \cdot \overline{p_2}$ and $\overline{p_1}$. Let $\sigma$ denote the surface in $LG$ bounded by $l$. Then by [PS86] Theorem (4.4.1) (i) ], the integral $C(l)$ is an integer. So $p_1 \ast (\gamma_1 \cdot p_2)$ and $p_1p_2$ are equivalent.

Thus, $\sigma'(p_1) \sigma'(p_2) = \sigma'(p_1p_2).$

\[\square\]
In addition, we have the property of $\xi$ below.

**Lemma 3.3.**

$$\xi(\xi(g_1)\xi(g_2)) = \xi(\xi(g_1))\xi(\xi(g_2)), $$

for any $g_1, g_2 \in G_1$ with $t(g_2) = s(g_1)$. And

$$\xi(\xi(\gamma_1)\xi(\gamma_2)) = \xi(\xi(\gamma_1))\xi(\xi(\gamma_2)),$$

for any $\gamma_1, \gamma_2 \in G_0$.

**Proof.** On $G_1^c$,

$$\xi(g_1^c)\xi(g_2^c) = (pr_{G_1^c} \circ \sigma(t(g_1^c)))^{-1} \cdot g_1^c \cdot pr_{G_1^c} \circ \sigma(s(g_1^c)) \circ (pr_{G_1^c} \circ \sigma(t(g_2^c)))^{-1} \cdot g_2^c \cdot pr_{G_1^c} \circ \sigma(s(g_2^c))$$

$$= (pr_{G_1^c} \circ \sigma(t(g_1^c)))^{-1} \cdot g_1^c \cdot g_2^c \cdot pr_{G_1^c} \circ \sigma(s(g_2^c))$$

since $t(g_2^c) = s(g_1^c)$ and $s(g_2^c) = s(g_1^c)g_2^c$.

(3.69)

$$\xi(\xi(g_1^c))\xi(\xi(g_2^c)) = \xi(\xi(g_1^c)g_2^c)$$

because of the property of functors. Thus, $\xi(\xi(g_1^c))\xi(\xi(g_2^c)) = \xi(\xi(g_1^c))\xi(\xi(g_2^c))$.

On $G_0^c$,

$$\xi(\gamma_1)\xi(\gamma_2) = \gamma_1(0)^{-1}\gamma_2(0)^{-1}\gamma_1^c\gamma_2^c;$$

(3.72)

$$\xi(\xi(\gamma_1)\xi(\gamma_2)) = \gamma_1(0)^{-1}\gamma_2(0)^{-1}\gamma_1^c\gamma_2^c.$$ 

(3.73)

And

$$\xi(\xi(\gamma_1))\xi(\xi(\gamma_2)) = \xi(\gamma_1(0)^{-1}\gamma_1^c)\xi(\gamma_2(0)^{-1}\gamma_2^c) = \gamma_1(0)^{-1}\gamma_1^c\gamma_2(0)^{-1}\gamma_2^c.$$ 

Thus,

$$\xi(\xi(\gamma_1))\xi(\xi(\gamma_2)) = \xi(\xi(\gamma_1))\xi(\xi(\gamma_2)).$$

Moreover,

$$\xi(\gamma_1\gamma_2) = \gamma_1(0)^{-1}\gamma_2(0)^{-1}\gamma_1^c\gamma_2^c = \xi(\gamma_1)\xi(\gamma_2).$$

3.2 Positive energy representations of $LG$ and 2-representations of $String(G)$

Recall that a linear representation

$$\rho : LG \rightarrow Aut(V)$$

is of positive energy if its restriction to $S^1 \subset LG$, $\rho|_{S^1}$, is positive, i.e. $e^{i\theta} \in S^1$ acts by $\text{Exp}(iA\theta)$ where $A \in \text{End}(V)$ is a linear operator with positive spectrum.

We apply our lemmas to a model of String Lie 2-group $S(G) = LG \times QG \Rightarrow QG$.

We have a functor from representation category $\text{Rep}(LG)$ of $LG$ to 2-representation 2-category of $String(G)$. Since $\text{Rep}^+(LG)$, the category of representations of positive energy of $LG$, is a full subcategory of $\text{Rep}(LG)$, this further gives a functor from $\text{Rep}^+(LG)$ to the 2-representation 2-category of $String(G)$, we have the corollary of Theorem 2.10 below.
Corollary 3.4. We have a functor $i_{\text{string}}$ from $\text{Rep}^+(\hat{LG})$ to $2\text{Rep}(SQ(G))$. Explicitly, the functor is defined as follows.

- **On the level of objects:** for $(V, \zeta) \in \text{Rep}^+(\hat{LG})$, $i_{\text{string}}(V, \zeta) := (f_V, \eta_V)$ as in Cor. 2.9.

- **On the level of morphisms:** for each morphism $(V_0, \zeta_0) \xrightarrow{\omega} (V_1, \zeta_1)$ in $\text{Rep}^+(\hat{LG})$, we define a morphism $i_{\text{string}}(\omega)$ in $2\text{Rep}(SQ(G))$ below. For each $h \in H$, we have the commutative diagram

\[
\begin{array}{ccc}
(V_0, \zeta_0) & \xrightarrow{\omega} & (V_1, \zeta_1) \\
\zeta_0(h) & \downarrow & \zeta_1(h) \\
(V_0, \zeta_0) & \xrightarrow{\omega} & (V_1, \zeta_1).
\end{array}
\]

For each $\omega$, we can define a functor

$F_\omega : \{V_0\}/\zeta_0(H) \to \{V_1\}/\zeta_1(H)$

by sending the object $V_0$ to $V_1$ and sending each morphism $\zeta_0(h)$ to $\zeta_1(h)$. Moreover, we can define a functor

$F_{\omega*} : C_{V_0} \to C_{V_1}$

by sending a morphism

$G \xrightarrow{\rho} \{V_0\}/\zeta_0(H)$

to

$G \xrightarrow{\rho} \{V_0\}/\zeta_0(H) \xrightarrow{F_{\omega*}} \{V_1\}/\zeta_1(H)$.

Then we are ready to define the morphism $i(\omega)$

$i(V_0, \zeta_0) \xrightarrow{(T(\cdot), T_\omega)} i(V_1, \zeta_1)$

in $2\text{Rep}(SQ(G))$ in the way below.

1. $T(*) := F_{\omega*} : C_{V_0} \to C_{V_1}$

2. The natural transformation $T_\omega$

\[
\begin{array}{cccc}
G_0 & \xrightarrow{f_{V_0}} & \text{Auto}(C_{V_0}) \\
\downarrow{f_{V_1}} & & \downarrow{T(*)} \\
\text{Auto}(C_{V_1}) & \xrightarrow{T(\cdot)} & \text{Fun}(C_{V_0}, C_{V_1})
\end{array}
\]

is defined to be identity.
4 2-Vector bundles and Associated 2-vector bundles

4.1 2-Vector bundles

Inspired by the construction of 2-principal bundles in [SXZ17], we suggest the following model for 2-vector bundles.

The $(3,1)$-presheaf $\text{2VectBd}$ of 2-vector bundles: $\text{Mfd}^{op} \to \text{2Gpd}$ is constructed in the following way: first of all, on the level of object, the 2-groupoid $\text{2VectBd}(U)$ for $U$ a manifold, consists of the following data

- $\text{2VectBd}(U)_{0} = \{U \times V : V \text{ is an object in 2Vect.}\};$
- $\text{2VectBd}(U)_{1}$: a 1-simplex $U \times V \xrightarrow{\gamma_{01}} U \times V$ consists of a functor valued map $\gamma_{01} : U \to \text{Auto}(V)$\[^{3}\] In other words, $\gamma_{01}$ is a gauge transformation.
- $\text{2VectBd}(U)_{2}$: a 2-simplex with edges $\gamma_{ij}$ for $0 \leq i < j \leq 2$, or equivalently (in the model of bicategory), a 2-morphism between $\gamma_{01} \circ \gamma_{12}$ and $\gamma_{02}$, is given by a natural transformation (isomorphism) valued map $\phi : U \to \text{Auto}(V)$. Given a morphism $U \xrightarrow{\psi} V$, the associated functor $\text{2VectBd}(V) \to \text{2VectBd}(U)$ is given by pre-composition.

**Remark 4.1.** Note that for each hypercover $U \xrightarrow{\psi} V$, the pullback $\text{2VectBd}(V) \xrightarrow{\psi^{*}} \text{2VectBd}(U)$ is fully faithful.

For any 1-morphism $f$ and $f'$ in $\text{2VectBd}(V)$, let $\alpha : f \Rightarrow f'$ denote a natural transformation between them, which consists of a natural transformation valued map $V \to \text{Auto}(V)$, i.e. for each $v \in V$, we have an element

$$\alpha_v : f(v) \Rightarrow f'(v)$$

in $\text{Auto}(V)$. The pullbacks of the natural transformation $\alpha$ is $\text{id}_{\psi} \circ \alpha : \psi^{*}f \Rightarrow \psi^{*}f'$. Note that, by definition, a hypercover on the 0 level is surjective. Thus, $\psi^{*}$ on the Hom categories is faithful, since if $\alpha, \alpha' : f \Rightarrow f'$ are two different 2-morphisms, so are $\text{id}_{\psi} \circ \alpha$ and $\text{id}_{\psi} \circ \alpha'$. In addition, each natural transformation $f \circ \psi \Rightarrow f' \circ \psi$ can be expressed in the form $\text{id}_{\psi} \circ \alpha'$. Thus, $\psi^{*}$ is fully.

Thus, by Definition 5.10, the $(3,1)$-presheaf $\text{2VectBd}$ is a 2-prestack.

Now we do a plus construction as in [NS11] to obtain the $(3,1)$-sheaf $\text{2VectBd}^{\dagger}$ which describes the entire data of 2-vector bundles, their morphisms and 2-morphisms. To do this, first, let us write out $\text{holim} \text{2VectBd}(U(M)_)$ for a cover $\{U_i\}$ of $M$, where $U(M)_\bullet$ is the Čech nerve of $\{U_i\}$. An object in $\text{holim} \text{2VectBd}(U(M)_)$ consists of

- on each $U_i$, $U_i \times V$;
- on each $U_{ij}$, a functor valued map $\gamma_{ij} : U_{ij} \to \text{Auto}(V)$\[^{3}\]

$\text{Recall that } \text{Auto}(V)$ is the category of autofunctors from $V$ to itself and natural transformations between them. Thus $\text{Auto}(V)_{0}$ denotes the set of autofunctors, and $\text{Auto}(V)_{1}$ denotes the set of natural transformations.

26
• on each $U_{ijk}$, a natural transformation (isomorphism) valued map $\phi_{ijk} : U_{ijk} \to \text{Auto}(V)_1$ between $\gamma_{ij} \circ \gamma_{jk} \leftarrow \gamma_{ik}$;

• on each $U_{ijkl}$, a pentagon condition for $\phi_{ijk}$’s

\[
\begin{array}{c}
\gamma_{ij} \circ (\gamma_{jk} \circ \gamma_{kl}) \\
\gamma_{ij} \circ \gamma_{kl}
\end{array}
\]

\[
\begin{array}{c}
\text{ id} \circ \phi_{jkl} \\
\phi_{ijkl}
\end{array}
\]

\[\gamma_{ijkl} \circ \gamma_{ik} \circ \phi_{ijkl} \leftarrow \gamma_{ijk} \circ \gamma_{ik} \circ \gamma_{jkl} \]

A 1-morphism in $\text{holim}_2 \text{VectBd}(U(M)_\bullet)$ from $(U_{i} \times \tilde{V}, \tilde{\gamma}_{ij}, \tilde{\phi}_{ijk})$ to $(U_{i} \times V, \gamma_{ij}, \phi_{ijk})$ consists of

• $\gamma_{i} : U_{i} \to \text{Hom}(\tilde{V}, V)$ which is a 1-morphism between $U_{i} \times V \leftarrow U_{i} \times \tilde{V}$;

• an element $\phi_{ij} \in 2\text{VectBd}(\sqcup_{ij} U_{ij})_2$ which provides a 2-morphism making the following diagram 2-commute

\[
\begin{array}{c}
U_{j} \times V \\
\gamma_{ij} \circ \phi_{ij}
\end{array}
\]

\[
\begin{array}{c}
U_{i} \times V \\
\gamma_{i}
\end{array}
\]

• a higher coherence condition between 2-morphisms, namely compositions of the 2-morphisms on the five faces commute

A 2-morphism in $\text{holim}_2 \text{VectBd}(U(M)_\bullet)$ from $(\tilde{\gamma}_{i}, \tilde{\phi}_{ij})$ to $(\gamma_{i}, \phi_{ij})$ consists of

• $\phi_{i} \in 2\text{VectBd}(\sqcup_{ij} U_{ij})_2$ from $\tilde{\gamma}_{i}$ to $\gamma_{i}$;
• a coherence condition held on $U_{ij}$,

$$\phi_{ij} \circ \phi_i = \phi_j \circ \phi_{ij}.$$ 

Then the $(3, 1)$-sheaf $2\text{VectBd}^+ : \text{Mfd}^{\text{op}} \to \text{2Gpd}$ consists of

- $2\text{VectBd}^+(M)_0$: an object is a pair $\{(U_i), \mathcal{E}\}$, where $\{U_i\}$ is an open cover of $M$ and $\mathcal{E}$ is an element in holim $2\text{VectBd}(U(M)_\bullet)_0$;
- $2\text{VectBd}^+(M)_1$: a 1-morphism between $\{(U_i), \mathcal{E}\}$ and $\{(\bar{U}_i), \bar{\mathcal{E}}\}$ is a pair consisting of a common refinement $\{V_i\}$ of $\{U_i\}$ and $\{\bar{U}_i\}$ and an element $\alpha \in \text{holim} 2\text{VectBd}(V(M)_\bullet)_1$;
- $2\text{VectBd}^+(M)_2$: a 2-morphism between $\{(V_i), \alpha\}$ and $\{(\bar{V}_i), \bar{\alpha}\}$ consists of a common refinement $\{W_i\}$ of $\{V_i\}$ and $\{\bar{V}_i\}$ and an element $\beta \in \text{holim} 2\text{VectBd}(W(M)_\bullet)_2$. Moreover, $\{(W_i), \beta\}$ and $\{(\bar{W}_i), \bar{\beta}\}$ are identified if $\beta$ and $\bar{\beta}$ agree on a further common refinement of $\{W_i\}$ and $\{\bar{W}_i\}$.

Thus, with the plus construction in [NSTI], we describe what are the 2-vector bundles on manifolds, and their 1-morphisms and 2-morphisms. All is organised in the above $2\text{VectBd}^-$ model of principal 2-bundles and the associated 2-vector bundles.

### 4.2 Principal $G$-bundle and the associated 2-vector bundles

In this section we construct a model of principal 2-bundles and the 2-vector bundles associated to it.

Let $G = G_1 \to G_0$ be a Lie 2-group. The $(3, 1)$-presheaf, $B\mathcal{G} : \text{Mfd}^{\text{op}} \to \text{2Gpd}$ is constructed as following: first of all, $B\mathcal{G}(U)$ is a 2-groupoid made up by the following data:

- $B\mathcal{G}(U)_0$: an object is $U \times \mathcal{G}$;
- $B\mathcal{G}(U)_1$: a 1-simplex is a groupoid morphism $g_{01} : U \to \mathcal{G}$, which is simply a morphism $U \to G_0$ because $U$ is a manifold;
- $B\mathcal{G}(U)_2$: a 2-simplex with edges $g_{ij}$ for $0 \leq i < j \leq 2$, is given by a 2-morphism $f : U \to G_1$ from $g_{02}$ to $g_{01} \cdot g_{12}$.

Given a morphism $\mathcal{G} \to \mathcal{V}$, $\mathcal{V}$, the associated functor $B\mathcal{G}(V) \to B\mathcal{G}(U)$ is given by pre-composition. Then with plus construction we arrive at a $(3, 1)$-sheaf $B\mathcal{G}^+ : \text{Mfd}^{\text{op}} \to \text{2Gpd}$, which organises the information of $\mathcal{G}$-principal bundles, their morphisms and 2-morphisms together.

Then with a 2-representation of $\mathcal{G}$ on $2\text{Vect}$, $(F, \eta, \phi, \psi)$, we form a functor $A : B\mathcal{G} \to 2\text{VectBd}$, which is the process analogue to associate a $G$-principal bundle to an associated vector bundle via a representation of $G$. We call $A$ the 2-associating functor, and the result the associated 2-bundles.

It is relative easy to do this:

- $A_0 : B\mathcal{G}(U)_0 \to 2\text{VectBd}(U)_0$, $U \times \mathcal{G} \mapsto U \times \text{Rep}\mathcal{G}$;
- $A_1 : B\mathcal{G}(U)_1 \to 2\text{VectBd}(U)_1$, $g_{01} \mapsto \gamma_{01} := F \circ g_{01}$;
- $A_2 : B\mathcal{G}(U)_2 \to 2\text{VectBd}(U)_2$, $f \mapsto \phi := \eta \circ f$. 

28
Then, by the functoriality of the plus construction, we get the 2-associating functor $A^+ : B\mathcal{G}^+ \to 2\text{VectBd}^+$.

Remark 4.2.

5 A model of Equivariant 2-vector bundles

In this section we give the construction of a model of equivariant 2-vector bundle. And we show that (nonequivariant) 2-vector bundles and principal $\mathcal{G}$-bundle with $\mathcal{G}$ a coherent Lie 2-group are examples of equivariant 2-vector bundles.

5.1 Groupoid action on a category

Before we construct equivariant 2-vector bundles, we define the action of a topological groupoid on a topological category.

Example 5.1 (Groupoid action on a category). In [MM03, Section 5.3, page 126] there is a definition of a right action of a Lie groupoid on another Lie groupoid. This definition can be generalized to that of a right action of a topological groupoid $\Gamma$ on a topological category $A$. It is given by two right actions of $\Gamma$ on the space of morphisms $A_1$ and on the space of objects $A_0$, i.e. for $i = 1, 2$, the continuous maps

$$\epsilon_i : A_i \to \Gamma_0, \quad \mu_i : A_i \times_{\epsilon_i, \Gamma_0, t} \Gamma_1 \to A_i,$$

(5.75)

which satisfies the following identities:

$$\epsilon_0 \circ s = \epsilon_1 = \epsilon_0 \circ \iota,$$

(5.76)

$$(a_i \gamma) \gamma' = a_i (\gamma \gamma'),$$

(5.77)

$$a_i 1_{\epsilon_i(a_i)} = a_i,$$

(5.78)

$$\epsilon_i(a_i \gamma) = s(\gamma).$$

(5.79)

for any $a_i \in A_i$ and $\gamma_i \in \Gamma_i$ with $i = 1, 2$. We write $a_i \gamma$ for $\mu_i(a_i, \gamma)$ with $a_i \in A_i$.

For each $x \in \Gamma_0$, the fibre $A_x = \epsilon_i^{-1}(x)$ is a full subcategory of $A$ with objects $\epsilon_0^{-1}(x)$. So we get a family of subcategories indexed by the points $x \in \Gamma_0$. If $A$ is a groupoid, the groupoid $\Gamma$ acts on this family by a groupoid isomorphism $A_x \to A_{x'}$ for each arrow $x' \to x$ in $\Gamma_1$. However, if $A$ is not a groupoid and $\Gamma$ contains not only the identity morphisms, there is no well-defined action by $\Gamma$ on the family.

Remark 5.2. There are several examples of the definition of right action of a Lie groupoid on a topological category in Example 5.1. The construction of a right action of a Lie groupoid on another Lie groupoid and that on a smooth manifold in [MM03, Section 5.3, page 126] are definitely examples for it.

Another example is the action of a groupoid $\Gamma$ on a vector space $V$ given in [MRW17, Section 4], which can be viewed as a functor $F$ from $\Gamma$ to the category $\text{Vect}$ of vector spaces. Each object $x$ in $\Gamma_0$ is mapped to a vector space $V_x$ and each arrow $x \to y$ is mapped to a linear map $V_x \xrightarrow{F(a)} V_y$. This gives an action of $\Gamma$ on the disjoint union

$$A = \coprod_{x \in \Gamma_0} V_x,$$
whose objects are the $\Gamma$-equivariant maps between $A_j$,
whose morphisms are the $k$-linear maps between $V_x$ to itself, with

$$
\begin{align*}
\epsilon_0 : A_0 &\to \Gamma_0, \quad (v_x \in V_x) \mapsto x \\
\epsilon_1 : A_1 &\to \Gamma_0, \quad (f : V_x \to V_x) \mapsto x \\
\mu_0 : A_0 \times_{\epsilon_0, \Gamma_0, t} \Gamma_1 &\to A_0, \quad (v_x, \gamma_1) \mapsto s(\gamma_1) \\
\mu_1 : A_1 \times_{\epsilon_1, \Gamma_0, t} \Gamma_1 &\to A_1, \quad ((f : V_x \to V_x), \gamma_1) \mapsto \gamma_1^{-1} \circ f \circ \gamma_1
\end{align*}
$$

where $\gamma_1^{-1} \circ f \circ \gamma_1$ is the composition of $k$-linear maps.

**Example 5.3** (Groupoid equivariant maps). Let $A^1$ and $A^2$ denote two topological categories with a right action of the Lie groupoid $\Gamma$. A $\Gamma$-equivariant map $f : A^1 \to A^2$ between them is a functor commuting with the structure maps

$$
\epsilon^j_i : A^j_i \to \Gamma_0, \quad \mu^j_i : A^j_i \times_{\epsilon^j_i, \Gamma_0, t} \Gamma_1 \to A^j_i
$$

with $j = 1, 2$ and $i = 1, 2$. Explicitly,

$$
\begin{align*}
\epsilon^2_i \circ f &= \epsilon^1_i \\
f(a^1_i g) &= f(a^1_i)g
\end{align*}
$$

(5.80) for $i = 1, 2$ and $g \in \Gamma_1$ with $\epsilon^1_i(a^1_i) = t(g)$. Note that (5.80) implies $\epsilon^2_i(f(a^1_i)) = t(g)$.

Let $\text{Fun}_\Gamma(A^1, A^2)$ denote the full subcategory of the functor category $\text{Fun}(A^1, A^2)$ whose objects are the $\Gamma$-equivariant maps between $A^1$ and $A^2$.

The rest part of this section will not be used in the construction of equivariant 2-vector bundles. We keep this part here for completeness of the discussion.

**Example 5.4.** We know that if we have a $G$-space $X$, then a group homomorphism $\phi : H \to G$ induces an $H$-action on $X$. A natural question is: given an action of a topological groupoid $\Gamma$ on a topological category $A$ defined in Examples [5.1] and a groupoid homomorphism $\beta : \Gamma' \to \Gamma$, can we define an action of $\Gamma'$ on $A$?

A quick answer is below.

If $A$ has a right action of $\Gamma$ on it and $\beta : \Gamma' \to \Gamma$ is a topological groupoid homomorphism which is injective on the object level, then there are well-defined maps $\epsilon'_i : A_i \to \Gamma'_0$ such that $\epsilon_0 := \beta_0 \circ \epsilon'_0$ and $\epsilon' := \epsilon_0 \circ s = \epsilon'_0 \circ t$.

$\beta$ induces a right action of $\Gamma'$ on $A$ with the structure maps $\epsilon'_i$ and

$$
\mu'_i : A_i \times_{\epsilon'_i, \Gamma'_0, t} \Gamma'_1 \to A_i, \quad \mu'_i(a_i, \gamma'_1) := \mu_i(a_i, \beta(\gamma'_1)).
$$

It can be checked immediately that $\epsilon'_i$ and $\mu'_i$ define a $\Gamma'$-action on $A$.

However, this construction is not good enough. If $\beta$ is not injective, a $\Gamma'$-action on $A$ induced from the homomorphism $\beta$ does not always exist. In Example [5.5], we show we can always construct a topological category via pullback, which has a $\Gamma'$-action on it.

**Example 5.5.** Let $V$ denote a topological category with a right action of a Lie groupoid $\Gamma$ on it, which is given by the structure maps

$$
\epsilon_i : A_i \to \Gamma_0, \quad \mu_i : A_i \times_{\epsilon_i, \Gamma_0, t} \Gamma_1 \to A_i.
$$

Given a groupoid homomorphism $\beta : \Gamma' \to \Gamma$,
we can formulate a topological category

\[ \mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0 \Gamma'_0 \]

with objects

\[ \{(v_0, \gamma'_0) \in V_0 \times \Gamma'_0 \mid \epsilon_0(v_0) = \beta_0(\gamma'_0)\} \]

and the space of morphism

\[ Mor((v_0, \gamma'_0), (v'_0, \eta'_0)) := \{ (f : v_0 \to v'_0, 1_{\gamma'_0} : \gamma'_0 \to \gamma'_0) \mid f \text{ is a morphism in } \mathcal{V} \} \cong Mor(v_0, v'_0) \]

and

\[ Mor((v_0, \gamma'_0), (v'_0, \eta'_0)) \]

is empty if \( \gamma'_0 \) and \( \eta'_0 \) are different elements. Note that if \( \mathcal{V} \) is a \( k \)-linear category, so is \( \mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0 \Gamma'_0 \).

There is a right \( \Gamma' \)-action on the category

\[ \mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0 \Gamma'_0 \]

with the structure maps defined by

\[ \epsilon'_0 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow \Gamma'_0, \quad (v_0, \gamma'_0) \rightarrow \gamma'_0 \]

\[ \epsilon'_1 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow \Gamma'_0, \quad (f, 1_{\gamma'_0}) \rightarrow \gamma'_0 \]

\[ \mu'_0 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \times (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0), \quad ((v_0, \gamma'_0), (v'_0, \eta'_0)) \rightarrow (\mu_0(v_0, \beta'_0), s(\gamma'_0)) \]

\[ \mu'_1 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \times (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0), \quad ((f, 1_{\gamma'_0}), (f', 1_{\eta'_0})) \rightarrow (\mu_1(f, \beta'_1), 1_{s(\gamma'_0)}) \]

for \( i = 1, 2 \). These maps satisfy (5.76) (5.77) (5.78) (5.79).

In addition, there is a right \( \Gamma \)-action on the category

\[ \mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0 \Gamma'_0 \]

with the structure maps defined by

\[ \epsilon'_0 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow \Gamma'_0, \quad (v_0, \gamma'_0) \rightarrow \beta_0(\gamma'_0) \]

\[ \epsilon'_1 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow \Gamma'_0, \quad (f, 1_{\gamma'_0}) \rightarrow \beta_0(\gamma'_0) \]

\[ \mu'_0 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \times (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0), \quad ((v_0, \gamma'_0), (v'_0, \eta'_0)) \rightarrow (\mu_0(v_0, \gamma'_0), s(\gamma'_0)) \]

\[ \mu'_1 : (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \times (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0) \rightarrow (\mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0), \quad ((f, 1_{\gamma'_0}), (f', 1_{\eta'_0})) \rightarrow (\mu_1(f, \gamma'_1), 1_{s(\gamma'_0)}) \]

for \( i = 1, 2 \). These maps satisfy (5.76) (5.77) (5.78) (5.79).

For any two objects \((v_0, \gamma'_0)\) and \((v'_0, \eta'_0)\) in \( \mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0 \Gamma'_0 \), there is a one-to-one correspondence between the morphism spaces

\[ Mor((v_0, \gamma'_0), (v'_0, \eta'_0)) \text{ and } Mor(v_0, v'_0) \]

We can define an functor

\[ \mathcal{V} \times \epsilon_0, \Gamma_0, \beta_0 \Gamma'_0 \rightarrow \mathcal{V} \]

by sending an object \((v_0, \gamma'_0)\) to \( v_0 \) and sending a morphism \((f, 1_{\gamma'_0}) : (v_0, \gamma'_0) \rightarrow (v'_0, \eta'_0)\) to \( f : v_0 \rightarrow v'_0 \). It is fully faithful.
A $\Gamma'$-equivariant automorphism

\[
\mathcal{V} \times_{\epsilon_0, \Gamma_0, \beta_0} \Gamma'_0 \to \mathcal{V} \times_{\epsilon_0, \Gamma_0, \beta_0} \Gamma'_0
\]
is of the form

\[
(v_0, \gamma'_0) \mapsto (F(v_0), \gamma'_0)
\]
\[
(f, 1_{\gamma'_0}) \mapsto (F(f), 1_{\gamma'_0})
\]

where $F = (F_0, F_1) : \mathcal{V} \to \mathcal{V}$ is an im $\beta$-equivariant automorphism.

There is a well-defined functor

\[
\iota(\beta) : \text{Auto}_\Gamma(\mathcal{V}) \to \text{Auto}_{\Gamma'}(\mathcal{V} \times_{\epsilon_0, \Gamma_0, \beta_0} \Gamma'_0)
\]
sending a $\Gamma$-equivariant morphism

\[
F : \mathcal{V} \to \mathcal{V}
\]
to the functor which sends an object $(v_0, \gamma'_0)$ to

\[
(F(v_0), \gamma'_0)
\]
and sends a morphism $(v_0 \xrightarrow{f} v'_0, 1_{\gamma'_0})$ to

\[
(F(v_0) \xrightarrow{F(f)} F(v'_0), 1_{\gamma'_0})
\]

which is $\Gamma'$-equivariant. The functor $\iota$ is fully faithful and is injective on objects. If $\beta$ is surjective, $\iota(\beta)$ is an isomorphism.

For the category $\mathcal{V} \times_{\epsilon_0, \Gamma_0, \beta_0} \Gamma'_0$ and the functor $\iota(\beta)$, we have the conclusions below.

**Proposition 5.6.** Let $\mathcal{V}$ denote a topological category with a right action of a Lie groupoid $\Gamma$ on it. Let $\Gamma'' \xrightarrow{\phi} \Gamma'$ and $\Gamma' \xrightarrow{\psi} \Gamma$ be groupoid homomorphisms.

1. As shown in Example 5.5, there is a $\Gamma''$-action on the categories

\[
(\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0, \Gamma_0, \psi_0} \Gamma''_0
\]

and

\[
\mathcal{V} \times_{\epsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma''_0.
\]

These two categories are $\Gamma''$-isomorphic.

2. The composition

\[
\text{Auto}_\Gamma(\mathcal{V}) \xrightarrow{\iota(\phi)} \text{Auto}_\Gamma(\mathcal{V} \times_{\epsilon_0, \Gamma_0, \beta_0} \Gamma'_0) \xrightarrow{\iota(\psi)} \text{Auto}_{\Gamma''}(\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0, \Gamma_0, \psi_0} \Gamma''_0
\]

and

\[
\iota(\phi \circ \psi) : \text{Auto}_\Gamma(\mathcal{V}) \to \text{Auto}_{\Gamma''}(\mathcal{V} \times_{\epsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma''_0)
\]

are equal up to a $\Gamma''$-isomorphism.
Proof. 1. The objects of $(\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma_0') \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0$ are

$$\{((v_0, \gamma_0'), \gamma''_0) \in \mathcal{V}_0 \times \Gamma'_0 \times \Gamma''_0 \mid \epsilon_0(v_0) = \phi_0(\gamma_0'), \gamma_0' = \psi_0(\gamma''_0)\}$$

which is exactly

$$\{((v_0, \gamma''_0) \in \mathcal{V}_0 \times \Gamma''_0 \mid \epsilon_0(v_0) = (\phi \circ \psi)_0(\gamma''_0)\} = (\mathcal{V} \times_{\epsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma''_0)_{00}.$$ 

The space of morphisms

$$\text{Mor}(((v_0, \gamma_0'), \gamma''_0), ((v'_0, \eta'''_0), \eta''''_0))$$

is empty if $\gamma''_0$ and $\eta''''_0$ are different. In this case $\gamma'_0$ and $\eta'_0$ are also different. The space

$$\text{Mor}(((v_0, \gamma'_0), \gamma''_0), ((v'_0, \gamma_0'), \gamma''_0))$$

is

$$\{((f : v_0 \to v'_0, 1_{\gamma''_0}), 1_{\gamma''_0}) \mid f \text{ is a morphism in } \mathcal{V}.\}$$

It is the same as

$$\text{Mor}(((v_0, \gamma'_0), (v'_0, \gamma_0')).$$

In other words, we have an isomorphism

$$j : (\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0 \to (\mathcal{V} \times_{\epsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma''_0)_{00}$$

sending an object $((v_0, \gamma'_0), \gamma''_0)$ to $(v_0, \gamma''_0)$ and a morphism $((f : v_0 \to v'_0, 1_{\gamma''_0}), 1_{\gamma''_0})$ to $(f, 1_{\gamma''_0})$.

Next we show this isomorphism $j$ is $\Gamma''$-equivariant.

Apply the construction in Example 5.3 the structure maps of the $\Gamma''$-action on

$$(\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0$$

are

$$\epsilon''_0 : ((\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0)_0 \to \Gamma''_0,$$

$$\{((v_0, \gamma'_0), \gamma''_0) \to \gamma''_0\}$$

$$\epsilon''_1 : ((\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0)_1 \to \Gamma''_0,$$

$$\{(f, 1_{\gamma''_0}), 1_{\gamma''_0} \to \gamma''_0\}$$

$$\mu''_0 : ((\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0)_0 \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma''_0, \Gamma''_0 \to ((\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0)_0,$$

$$\{((v_0, \gamma'_0), \gamma''_0), \gamma''_0) \to ((\mu_0(v_0, \phi \circ \psi(\gamma''_0)), s(\gamma''_0)), s(\gamma''_0))\}$$

$$\mu''_1 : ((\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0)_1 \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_1 \to ((\mathcal{V} \times_{\epsilon_0, \Gamma_0, \phi_0} \Gamma'_0) \times_{\epsilon_0', \Gamma_0', \psi_0} \Gamma''_0)_1,$$

$$\{((f, 1_{\gamma''_0}), 1_{\gamma''_0}), \gamma''_0) \to ((\mu_1(f, \phi \circ \psi(\gamma''_0)), 1_{s(\gamma''_0)}), 1_{\gamma''_0})\}$$

for $i = 1, 2.$
And the $\Gamma^m$-action on the category

$$\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0$$

with the structure maps defined by

$$e''_0 : (\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0)_{0} \rightarrow \Gamma^m_0, \quad (v_0, \gamma''_0) \mapsto \gamma''_0$$

$$e''_1 : (\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0)_{1} \rightarrow \Gamma^m_0, \quad (f, 1_{\gamma''_0}) \mapsto \gamma''_0$$

$$\mu''_0 : (\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0)_{0} \times_{\varepsilon'_0, \Gamma_0, t} \Gamma^m_1 \rightarrow (\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0)_{0}, \quad ((v_0, \gamma''_0), \gamma''_1) \mapsto (\mu_0(v_0, (\phi \circ \psi)(\gamma''_1)), s(\gamma''_1))$$

$$\mu''_1 : (\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0)_{1} \times_{\varepsilon'_1, \Gamma^m_0} \Gamma^m_1 \rightarrow (\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0)_{1}, \quad ((f, 1_{\gamma''_0}), \gamma''_1) \mapsto (\mu_1(f, (\phi \circ \psi)(\gamma''_1)), 1_{s(\gamma''_1)})$$

for $i = 1, 2$.

We have

$$e''_{i} \circ j = e''_{i}, \quad j(a_i \gamma''_1) = j(a_i) \gamma''_1$$

for $i = 1, 2$ and any $a_i \in (\mathcal{V} \times_{\varepsilon_0, \Gamma_0, \phi_0} \Gamma^m_{0} \times_{\varepsilon'_0, \Gamma^m_0, \psi_0} \Gamma^m_0)_i$.

Thus, $j$ is $\Gamma^m$-equivariant.

2. From the $\Gamma^m$-isomorphism $j$, we can define an isomorphism

$$j_* : \text{Auto}^{-1}((\mathcal{V} \times_{\varepsilon_0, \Gamma_0, \phi_0} \Gamma^m_{0} \times_{\varepsilon'_0, \Gamma^m_0, \psi_0} \Gamma^m_0) \rightarrow \text{Auto}^{-1}(\mathcal{V} \times_{\varepsilon_0, \Gamma_0, (\phi \circ \psi)_0} \Gamma^m_0)$$

by sending an automorphism $F$ to $j \circ F \circ j^{-1}$.

It's straightforward to check the that

$$\iota(\phi \circ \psi) = j_* \circ \iota(\phi) \circ \iota(\psi).$$

$\square$

### 5.2 The $(3, 1)$-precover of equivariant 2-vector bundle

We start this section with a motivating example.

**Example 5.7.** Let $G$ denote a compact Lie group. Let $X$ denote a $G$-space and $V$ a $G$-representation. Then a trivial $G$-equivariant vector bundle over $X$ is the product $X \times V$ with the projection $X \times V \to X$.

Then we consider a trivial equivariant 2-vector bundle as below. Let $G$ denote a Lie groupoid. Let $G$ denote a Lie groupoid with a right action $\{\varepsilon_i, \mu_i\}^{i=1, 2}$ by $G$. Let $\mathcal{V}$ be a $k$-linear category with a right action $\{\varepsilon^Y_i, \mu^Y_i\}^{i=1, 2}$ of $G$ on it. We consider the category $\Gamma \times_{\varepsilon_0, \varepsilon_0, \phi_0} \mathcal{V}$ with the projection $\Gamma \times_{\varepsilon_0, \varepsilon_0, \phi_0} \mathcal{V} \to \Gamma$ which is a $G$-equivariant morphism. The fiber $F_{\gamma_0}$ at the object $\gamma_0$ of $\Gamma_0$ is the full subcategory $(\varepsilon^Y_0)^{-1}(\varepsilon_0(\gamma_0))$ of $\mathcal{V}$. 34
Note if $\epsilon_0^V(v_0)$ and $\epsilon_0^V(v'_0)$ are different, they don’t belong to the same fibre and there is no $\gamma_1$ in $\Gamma$ such that $v_0 = v'_0 \gamma_1$. Let’s consider a decomposition of the groupoid $\Gamma$,

$$\Gamma = \coprod_{\alpha} \Gamma_\alpha$$

where in each full subgroupoid $\Gamma_\alpha$ of $\Gamma$, the objects of it are all mapped to the same object under $\epsilon_0$. And in different subgroupoid $\Gamma_\alpha$ and $\Gamma_{\alpha'}$, their objects are mapped to different objects under $\epsilon_0$. Thus, for any objects $x, y$ in $\Gamma_\alpha$, the fibres $F_x$ and $F_y$ are the same full subcategory of $\mathcal{V}$ and we use $F_\alpha$ to denote it.

Then $\Gamma \times_{\epsilon_0, \epsilon_0^V} \mathcal{V}$ can be viewed as the disjoint union

$$\coprod_{\alpha} \Gamma_\alpha \times F_\alpha.$$

If we pick $\mathcal{G}$ to be the groupoid $BG$, $\Gamma$ a $G$-space and $\mathcal{V}$ a k-linear category with a single object, the category $\Gamma \times_{\epsilon_0, \epsilon_0^V} \mathcal{V}$ is exactly $\Gamma \times \mathcal{V}$. Thus, the above construction is a generalization of the trivial $G$-equivariant vector bundle.

Then we define a $(3,1)$-presheaf $2eq\text{VectBd} : \text{Gpd}^{op} \to \text{Bicat}$, which is a $(2,-)$-prestack and then apply the plus construction to $2eq\text{VectBd}$ in Section 5.4 and get a $(3,1)$-sheaf.

**Example 5.8.** The $(3,1)$-presheaf $2eq\text{VectBd}$ of equivariant 2-vector bundles: $\text{Gpd}^{op} \to 2\text{Gpd}$ is constructed in the following way. Let $\mathcal{G}$ be a Lie groupoid.

On the level of object, the 2-groupoid $2eq\text{VectBd}(\Gamma)$ for $\Gamma$ a Lie groupoid with a right action $\{\epsilon_i, \mu_i\}_{i=1,2}$ by $\mathcal{G}$, consists of the following data

- $2eq\text{VectBd}(\Gamma)_0 = \{\Gamma \times_{\epsilon_0, \epsilon_0^V} \mathcal{V} | \mathcal{V} \text{ is an object in } 2\text{Vect} \text{ with a right action } \{\epsilon^V_i, \mu^V_i\}_{i=1,2} \text{ of } \mathcal{G} \text{ on it};$

- $2eq\text{VectBd}(\Gamma)_1$: a 1-simplex

$$\Gamma \times_{\epsilon_0, \epsilon_0^V} \mathcal{V} \xrightarrow{\gamma_01} \Gamma \times_{\epsilon_0, \epsilon_0^V} \mathcal{V}$$

is a $\mathcal{G}$-bundle isomorphism covering the identity map on $\Gamma$. For each object $m$ in $\Gamma$, $(\gamma_01)_m$ gives an element in $\text{Auto}_\mathcal{G}(F_m)$. We still use the symbol $(\gamma_01)_m$ to denote this element.

For any morphism $g : m_1 \to m_2$ in $\Gamma$, any object $v_0$ in $\mathcal{V}$ with $\epsilon_0^V(v_0) = \epsilon_0(m_2)$, the fibers $F_{m_1}$ and $F_{m_2}$ are the same category.

$$\gamma_01((m_1, v_0)g) = \gamma_01(m_1, v_0)g$$

Thus,

$$(m_2, (\gamma_01)_m((v_0)g)) = (m_2, (\gamma_01)_m((v_0)g)).$$

Thus,

$$(\gamma_01)_m((v_0)g) = (\gamma_01)_m((v_0)g)^{-1}. \quad (5.82)$$

Thus, a 1-simplex $\gamma_01$ is a functor-valued map

$$\Gamma_0 \to \text{Auto}_\mathcal{G}(\mathcal{V})$$

which satisfies (5.82).
• $2\text{eqVectBd}(\Gamma)_2$: a 2-simplex with edges $\gamma_{ij}$ for $0 \leq i < j \leq 2$, or equivalently (in the model of bicategory), a 2-morphism between $\gamma_{01} \circ \gamma_{12}$ and $\gamma_{02}$, is given by a natural transformation valued map $\phi : \Gamma_0 \to (\text{Auto}_G(V))_1$.

Given a morphism $\Gamma' \xrightarrow{\phi} \Gamma$, the associated functor $2\text{eqVectBd}(\Gamma) \to 2\text{eqVectBd}(\Gamma')$ is given by pre-composition. Note that fiber $F_{\gamma_0}'$ with $\gamma_0'$ an object in $\Gamma$ and $F_{\phi(\gamma_0)}$ are the same subcategory of $V$.

**Remark 5.9.** The $(3,1)$-presheaf $2\text{VectBd}$ given in Section 4.1 is a special case of the $(3,1)$-presheaf $2\text{eqVectBd}$ of equivariant 2-vector bundle.

Each smooth manifold can be viewed as a Lie groupoid with only the identity morphisms. If we take the groupoid $G$ in Example 5.8 to be the trivial groupoid and the Lie groupoid $\Gamma$ to be a smooth manifold $U$, $2\text{VectBd}(U)$ is the same as $2\text{eqVectBd}(U)$.

In addition, the 2-vector bundle associated to the principal $G$-bundle given in Section 4.2 is a special case of the equivariant 2-vector bundles, if we take the Lie groupoid $\Gamma$ to be a smooth manifold and the groupoid $G$ to be the trivial groupoid and restrict to the case when the $k$-linear category $V$ is $\text{Rep}_G$.

The presheaf $2\text{eqVectBd}$ is in fact a 2-prestack. We first give the definition below.

**Definition 5.10.** Let $\mathcal{Y}$ be a presheaf in bicategories on $(n\text{Grpd},\tau)$.

1. A presheaf $\mathcal{Y}$ is called a $\tau$-prestack, if, for every hypercover $\Gamma_\bullet \to X_\bullet$ in $(n\text{Grpd},\tau)$, the functor of bicategories

   $\tau_\Gamma : \mathcal{Y}(X_\bullet) \to \mathcal{Y}(\Gamma_\bullet)$

   is fully faithful, i.e. all the functors on Hom categories are equivalences of categories.

2. A presheaf $\mathcal{Y}$ is called a $\tau$-stack, if, for every hypercover $\Gamma_\bullet \to X_\bullet$ in $(n\text{Grpd},\tau)$, the functor $\tau_U$ of bicategories is an equivalence of bicategories.

The definition of fully faithful bifunctors in the definition is given in [JY21, Definition 7.0.1], [GR04, Definition 2.4.9(ii)].

**Remark 5.11.** For any $1$-morphism $f$ and $f'$ in $2\text{eqVectBd}(\Gamma)$, let $\alpha : f \Rightarrow f'$ denote a natural transformation between them, which consists of a natural transformation valued map $\Gamma_0 \to (\text{Auto}_G(V))_1$, i.e. for each $\gamma_0' \in \Gamma_0$, we have an element

$\alpha_{\gamma_0'} : f(\gamma_0') \Rightarrow f'(\gamma_0')$

in $(\text{Auto}_G(V))_1$. The pullbacks of the natural transformation $\alpha$ is $\text{id}_\psi \circ \alpha_0 : \psi^*f \Rightarrow \psi^*f'$.

By definition, a hypercover on level 0 is surjective. Thus, if $\alpha, \alpha' : f \Rightarrow f'$ are different 2-morphisms, $\text{id}_\psi \circ \alpha_0$ and $\text{id}_\psi \circ \alpha'_0$ are different. So $\psi^*$ on the Hom categories is faithful. In addition, each natural transformation $f \circ \psi \Rightarrow f' \circ \psi$ can be expressed in the form $\text{id}_\psi \circ \alpha_0$. Thus, $\psi^*$ is fully.

Thus, by Definition 5.10 the $(3,1)$-presheaf $2\text{eqVectBd}$ is a 2-prestack.
5.3 The plus construction for \((3, 1)\)-presheaves over Lie groupoids

We show in this section how to obtain a 2-stack \(\tilde{X}^+\) on the category \(\text{Gpd}\) of Lie groupoids starting from a 2-prestack \(\tilde{X}\) on \(\text{Gpd}\). This is the procedure of higher sheafification. The main result in this section is a higher version of the plus construction in \([\text{NS11}]\) Theorem 3.3] which obtains a 2-stack on the category of differentiable manifolds starting from a 2-prestack on the category. As indicated in \([\text{NS11}]\), the plus construction can be generalized to 2-prestacks on the category \(\text{Gpd}\) of Lie groupoids. We show explicitly that this is true in this section.

In the construction of the descent category for the plus construction, instead of the Čech cover, we will use hypercover. Hypercovers were first introduced in \([\text{AGV72}]\) and have been fully studied and used afterwards throughout the homotopy theory of simplicial sheaves, \([\text{Bro73}]\), etc. Hypercovers for Lie \(n\)–groupoids play an important role in \([\text{Wol13}]\) and \([\text{Zhu09}]\). In addition, the cyclic fibrations in the Behrend-Getzler CFO structure for \(n\)–groupoids objects in a descent category are hypercovers.

Let \(\tau\) be a singleton Grothendieck pretopology on the category \(\text{Mfd}\) of differentiable manifolds.

**Definition 5.12.** For any Lie \(n\)–groupoid \(\Gamma_\bullet\) and \(X_\bullet\), a morphism \(\Gamma_\bullet \to X_\bullet\) is a hypercover if, for any \(0 \leq k \leq n - 1\), the natural map from \(\Gamma_k\) to the pull-back of the diagram

\[
\text{Hom}(\partial \Delta[k], \Gamma_\bullet) \to \text{Hom}(\partial \Delta[k], X_\bullet) \leftarrow X_k
\]

is a \(\tau\)–cover and it is an isomorphism for \(k = n\). More explicitly,

\[
\begin{align*}
\Gamma_0 & \to X_0, \\
\Gamma_1 & \to X_0 \times_{X_0} X_1 \times_{X_0} \Gamma_0 = (\Gamma_0 \times \Gamma_0) \times_{X_0 \times X_0} X_1 = \text{Hom}(\partial \Delta[1], \Gamma_\bullet) \times_{\text{Hom}(\partial \Delta[1], X_\bullet)} X_1, \\
\Gamma_2 & \to \text{Hom}(\partial \Delta[2], \Gamma_\bullet) \times_{\text{Hom}(\partial \Delta[2], X_\bullet)} X_2, \\
& \quad \vdots \\
\Gamma_{n-1} & \cong \text{Hom}(\partial \Delta[n-1], \Gamma_\bullet) \times_{\text{Hom}(\partial \Delta[n-1], X_\bullet)} X_{n-1} \text{ are all } \tau \text{–} \text{covers}, \\
\Gamma_n & \cong \text{Hom}(\partial \Delta[n], \Gamma_\bullet) \times_{\text{Hom}(\partial \Delta[n], X_\bullet)} X_n, \\
\Gamma_k & \cong \text{Hom}(\partial \Delta[k], \Gamma_\bullet) \times_{\text{Hom}(\partial \Delta[k], X_\bullet)} X_k, \text{ when } k \geq n.
\end{align*}
\]

A hypercover is always a weak equivalence.

In the rest of the section, we take the family \(\tau\) to be the surjective submersions.

**Definition 5.13** (descent category). Given a hypercover \(\Gamma_\bullet \to X_\bullet\) in \((\text{Gpd}, \tau)\), the descent category is defined by

\[\text{Desc}_{\tilde{X}}(\Gamma_\bullet) := \tilde{X}(\Gamma_\bullet).\]

Then we describe the bicategory \(\tilde{X}^+(X_\bullet)\) for a Lie groupoid \(X_\bullet\).

**Definition 5.14.** An object in \(\tilde{X}^+(X_\bullet)\) consists of a hypercover \(U_\bullet \to X_\bullet\) and an object \(G\) in the descent bicategory \(\text{Desc}_{\tilde{X}}(U_\bullet)\).

In order to define 1-morphisms and 2-morphisms between objects with possibly different hypercovers \(\pi : U_\bullet \to X_\bullet\) and \(\pi' : U'_\bullet \to X_\bullet\), and compare them there.

A common refinement of the hypercover \(\pi\) and \(\pi'\) is a hypercover. \(\xi : Z_\bullet \to X_\bullet\) if and only if there exist hypercovers \(s : Z_\bullet \to U_\bullet\) and \(s' : Z_\bullet \to U'_\bullet\) such that the diagram below commutes strictly.
We have the refinement functors $s^*$ and $s'^*$

\[
\text{Desc}_X(U) \xrightarrow{s^*} \text{Desc}_X(Z) \xrightarrow{s'^*} \text{Desc}_X(U')
\]

**Definition 5.15.**  
- A 1-morphism between objects $\mathcal{A} = (U, G)$ and $\mathcal{A}' = (U', G')$ of $\tilde{\mathcal{X}}(X)$ consists of a common refinement $Z \to X$ of the hypercovers $U \to X$ and $U' \to X$ and a 1-morphism $f : s^*(G) \to s'^*(G')$ in $\text{Desc}_X(Z)$.

- Let $m = (Z, f)$ and $m' = (Z', f')$ denote two 1-morphisms between objects $\mathcal{A} = (U, G)$ and $\mathcal{A}' = (U', G')$. A 2-morphism between $m$ and $m'$ consists of a common refinement $W \to X$ of the hypercovers $Z \to X$ and $Z' \to X$.

The diagram (5.84) commutes, i.e. $s \circ t = \tilde{s} \circ t'$ and $s' \circ t = \tilde{s}' \circ t'$.

Here $m_W$ is a 1-morphism between $\mathcal{A}$ and $\mathcal{A}'$ but with the common refinement $W$, and 1-morphism $t^*f : (s \circ t)^*(G) \to (s' \circ t)^*(G')$ in $\text{Desc}_X(W)$. Similarly, $m'_W$ is a 1-morphism between $\mathcal{A}$ and $\mathcal{A}'$ with the common refinement $W$ and 1-morphism $t'^*f' : (\tilde{s} \circ t')^*(G) \to (\tilde{s}' \circ t')^*(G')$, i.e. $t'^*f' : (s \circ t)^*(G) \to (s' \circ t)^*(G')$ in $\text{Desc}_X(W)$.

Thus, $\beta$ is a 2-morphism $\beta : t^*f \Rightarrow t'^*f'$. 

38
• In addition, two such 2-morphisms \((W_\bullet, \beta)\) and \((W'_\bullet, \beta')\) must be identified if and only if there exists a further common refinement \(V_\bullet \to X_\bullet\) of \(W_\bullet \to X_\bullet\) and \(W'_\bullet \to X_\bullet\) such that the refined 2-morphisms agree on \(V_\bullet\).

We have the commutative diagram below.

\[
\begin{array}{cccc}
W_\bullet & \xrightarrow{\alpha} & X_\bullet & \xrightarrow{\omega} V_\bullet \\
\downarrow{t} & & \downarrow{p} & \downarrow{\omega'} \\
Z_\bullet & \xrightarrow{\beta} & X_\bullet & \xrightarrow{p'} Z'_\bullet \\
\downarrow{p} & & \downarrow{\omega} & \downarrow{\omega'} \\
W'_\bullet & \xrightarrow{\beta'} & X_\bullet & \xrightarrow{p'} Z'_\bullet \\
\end{array}
\]

\[\alpha = \omega \circ p = \omega' \circ p'.\]

The refined 2-morphism of \((W_\bullet, \beta)\) is \((V_\bullet, \omega^* \beta)\) and the refined 2-morphism of \((W'_\bullet, \beta')\) is \((V_\bullet, \omega'^* \beta')\).

We have given the definition of objects, 1-morphisms and 2-morphisms in \(\tilde{X}^+(X_\bullet)\) above. Next we define the compositions and identities and check they are well-defined.

Let \(A = (U_\bullet, G_\cdot)\), \(A' = (U'_\bullet, G'_\cdot)\) and \(A'' = (U''_\bullet, G''_\cdot)\) be objects and \(m = (Z, f) : A \to A'\) and \(m' = (Z', f') : A' \to A''\) be morphisms.

Let \(Z'' := Z \times_{U'_\bullet} Z'\) denote the pullback of the diagram below.

\[
\begin{array}{cccc}
Z & & Z' & \\
\downarrow{Z} & & \downarrow{Z'} & \\
U_\bullet & \xrightarrow{f} & U'_\bullet & \xrightarrow{f'} U''_\bullet \\
\downarrow{Z} & & \downarrow{Z'} & \\
X & & X & \xrightarrow{f''} X \\
\end{array}
\]

This pullback exists in \(\text{Gpd}\) and is a common refinement of \(U_\bullet\) and \(U'_\bullet\). The composition \(m' \circ m\) is defined to be the tuple

\[(Z'', f''_Z \circ f''_{Z'})\]

where \(f''_Z \circ f''_{Z'}\) is the composition of the refined morphism.

By [Zhu09, Lemma 2.7], \(Z''\) is a Lie groupoid. And it is a common refinement of \(U_\bullet\) and \(U''_\bullet\). The composition \(m' \circ m := (Z'', f''_Z \circ f''_{Z'})\). This indeed defines the structure of a bicategory \(\tilde{X}^+(X)\).

In addition, a hypercover leads to the equivalence of bicategories, as shown in the lemma below. We need the conclusion in the proof of Theorem 5.17.

**Lemma 5.16.** Let \(X_\bullet \to Y_\bullet\) be a hypercover of Lie groupoid, then we have the following equivalence of bicategories:

\[
\tilde{X}(Y_\bullet) \xrightarrow{\sim} \text{holim}(\tilde{X}(X_\bullet) \Rightarrow \tilde{X}(X^{[2]}_\bullet) \ldots)
\]
\[
\tilde{\mathcal{X}}(X_\bullet) \xrightarrow{\sim} \text{holim}(\tilde{\mathcal{X}}(X_\bullet) \Rightarrow \tilde{\mathcal{X}}(X_\bullet^{[2]}) \ldots) 
\] (5.87)

Thus
\[
\tilde{\mathcal{X}}(Y_\bullet) \xrightarrow{\sim} \tilde{\mathcal{X}}(X_\bullet) 
\] (5.88)

**Proof.** (5.86) follows from \([\text{NS11}, \text{Theorem 7.5}]\) since hypercover of Lie groupoid especially implies a level-wise cover, which is a \(\tau\)-covering of simplicial manifolds. (5.87) follows from the fact that the diagonal map
\[(X_\bullet \leftarrow X_\bullet \ldots) \rightarrow (X_\bullet \leftarrow X_\bullet^{[2]} \ldots)\]
is a strong equivalence proven in Lemma 8.1 of the same article. Thus
\[
\text{holim}(\tilde{\mathcal{X}}(X_\bullet) \Rightarrow \tilde{\mathcal{X}}(X_\bullet^{[2]}) \ldots) \xrightarrow{\sim} \text{holim}(\tilde{\mathcal{X}}(X_\bullet) \Rightarrow \tilde{\mathcal{X}}(X_\bullet) \ldots) \xrightarrow{\cong} \tilde{\mathcal{X}}(X_\bullet) 
\]

Then we are ready for the main theorem in this section, which is a higher version of \([\text{NS11}, \text{Theorem 3.3}]\).

**Theorem 5.17.** If \(\tilde{\mathcal{X}}\) is a prestack, then \(\tilde{\mathcal{X}}^+\) is a stack. Furthermore the canonical embedding
\[\tilde{\mathcal{X}}(X_\bullet) \rightarrow \tilde{\mathcal{X}}^+(X_\bullet)\]
is fully faithful for each \(X_\bullet\).

**Proof.** For a hypercover \(\Gamma_\bullet \rightarrow X_\bullet\),
\[\text{Desc}_{\tilde{\mathcal{X}}^+} (\Gamma_\bullet) = \tilde{\mathcal{X}}^+ (\Gamma_\bullet)\]
To show \(\tilde{\mathcal{X}}^+\) is a \(\tau\)-stack, by Definition 5.10 we need to show, for every hypercover \(c : \Gamma_\bullet \rightarrow X_\bullet\), the functor
\[\tau_c : \tilde{\mathcal{X}}^+(X_\bullet) \rightarrow \text{Desc}_{\tilde{\mathcal{X}}^+} (\Gamma_\bullet)\]
of bicategories is an equivalence of bicategories.

By Definition 5.13 the objects of \(\tilde{\mathcal{X}}^+(\Gamma_\bullet) = \text{Desc}_{\tilde{\mathcal{X}}^+} (\Gamma_\bullet)\) are pairs consisting of a hypercover \(U_\bullet \rightarrow \Gamma_\bullet\) and an object \(G\) in the descent bicategory \(\text{Desc}_{\tilde{\mathcal{X}}} (U_\bullet)\). And the objects of \(\tilde{\mathcal{X}}^+(X_\bullet)\) are pairs consisting of a hypercover \(U_\bullet \rightarrow X_\bullet\) and an object \(G\) in the descent bicategory \(\text{Desc}_{\tilde{\mathcal{X}}} (U_\bullet)\).

And by Definition 5.14, the 1–morphisms and 2–morphisms of both bicategories are defined in the same way.

Thus, \(\tilde{\mathcal{X}}^+(\Gamma_\bullet)\) is the subbicategory of objects of \(\tilde{\mathcal{X}}^+(X_\bullet)\) which are defined on coverings \(U_\bullet \rightarrow \Gamma_\bullet \rightarrow X_\bullet\).

And by Lemma 5.10 this subbicategory is equivalent to the bicategory \(\tilde{\mathcal{X}}^+(X_\bullet)\).
5.4 The 2-stack of equivariant 2-vector bundles

In this section, we apply the plus construction in Section 5.3 to the 2-prestack \(2\text{-eqVectBd}\) defined in Section 5.2 and get the 2-stack \(2\text{-eqVectBd}^+\) of equivariant 2-vector bundles, which describes the entire data of equivariant 2-vector bundles, their 1-morphisms and 2-morphisms. To do this, we will first take the homotopy limit

\[ \text{holim} \, 2\text{-eqVectBd} \left( \Gamma_\bullet \right) \]

with \( \Gamma_\bullet \to X_\bullet \) a hypercover of \( X_\bullet \). Then we add the data of equivariant descent to \( 2\text{-eqVectBd} \) and get the 2-stack \( 2\text{-eqVectBd}^+ \).

5.4.1 The homotopy limit \( \text{holim} \, 2\text{-eqVectBd} \left( \Gamma_\bullet \right) \)

For a hypercover \( \Gamma_\bullet \to X_\bullet \), an object in \( \text{holim} \, 2\text{-eqVectBd} \left( \Gamma_\bullet \right) \) consists of

- on \( \Gamma_0 \), an object \( \alpha \) in \( 2\text{-eqVectBd}(\Gamma_0) \), i.e. an element in \( 2\text{-eqVectBd}(\Gamma_0)_0 \);
- on \( \Gamma_1 \), a 1-isomorphism
  \[ d^1_1 \alpha \xleftarrow{\alpha} d^1_0 \alpha \]
  in \( 2\text{-eqVectBd}(\Gamma_1) \), i.e. an element in \( 2\text{-eqVectBd}(\Gamma_1)_1 \);
- on \( \Gamma_2 \), a 2-isomorphism
  \[ d^1_1 \gamma \xleftarrow{\phi} d^2_2 \gamma \circ_h d^1_0 \gamma \]
  in \( 2\text{-eqVectBd}(\Gamma_2) \), i.e. an element in \( 2\text{-eqVectBd}(\Gamma_2)_2 \);
- on \( \Gamma_3 \), a pentagon condition for \( \phi \)'s

\[
\begin{align*}
(d^2_2 d^1_3 \gamma \circ_h d^1_0 d^1_0 \gamma) \circ_h d^1_0 d^1_1 \gamma & \xleftarrow{\text{d}^1_0 \phi \circ_h \text{id}} d^1_2 d^1_0 \gamma \circ_h d^1_0 d^1_1 \gamma \\
& \xleftarrow{\text{id} \circ_h d^1_0 \phi} d^2_2 d^1_1 \gamma
\end{align*}
\]

where \( a \) is the associator of the bicategory \( 2\text{-eqVectBd}(U_\bullet) \).

A 1-morphism in

\[ \text{holim} \, 2\text{-eqVectBd} \left( \Gamma_\bullet \right) \]

from \( (\tilde{\alpha}, \tilde{\gamma}, \tilde{\phi}) \) to \( (\alpha, \gamma, \phi) \) consists of

- A 1-morphism in \( 2\text{-eqVectBd}(\Gamma_0) \)
  \[ \alpha \xleftarrow{\beta} \tilde{\alpha} \]
- A 2-morphism \( \psi \) in \( 2\text{-eqVectBd}(\Gamma_1) \) making the following diagram 2-commutative

\[
\begin{align*}
d^1_1 \alpha & \xleftarrow{d^1_1 \beta} d^1_1 \tilde{\alpha} \\
& \xleftarrow{\gamma} \psi \\
& \xrightarrow{\gamma} \tilde{\gamma} \\
d^1_0 \alpha & \xleftarrow{d^1_0 \beta} d^1_0 \tilde{\alpha}
\end{align*}
\]
• A higher coherence condition between 2-morphisms in $2\text{eqVectBd}(\Gamma_2)$, namely compositions of the 2-morphisms on the five faces commute

$$d_0^1 \circ d_2^2 \alpha \rightsquigarrow d_0 \circ d_1^2 \tilde{\alpha}.$$ 

A 2-morphism in $\text{holim } 2\text{eqVectBd}(\Gamma_\bullet)$ from $\tilde{(\beta, \tilde{\psi})}$ to $(\beta, \psi)$ consists of

• on $2\text{eqVectBd}(\Gamma_0)$, a 2-morphism $\tilde{\beta} \xrightarrow{\xi} \beta$; 

• A coherence condition held on $2\text{eqVectBd}(\Gamma_1)$,

$$\psi \circ h \circ (\text{id}_h \circ d_0^i \xi) = (d_1^i \xi \circ h \text{id}) \circ h \tilde{\psi}.$$ 

This defines the structure of a bicategory. All the coherence conditions are still satisfied. And we have the pullback functors

$$\alpha^* : \text{holim } 2\text{eqVectBd}(Y_\bullet) \rightarrow \text{holim } 2\text{eqVectBd}(X_\bullet)$$

for all morphisms $\alpha : X_\bullet \rightarrow Y_\bullet$.

5.4.2 The 2-stack of equivariant 2-vector bundles

Then we add the descent data in the homotopy limit to the 2-prestack $2\text{eqVectBd}$. The (3, 1)-sheaf

$$2\text{eqVectBd}^+ : \text{Gpd}^{op} \rightarrow \text{Bicat}$$

consists of

• $2\text{eqVectBd}^+(X_\bullet)_0$: an object is a pair $(U_\bullet, \mathcal{E})$, where $U_\bullet$ is a hypercover of $X_\bullet$ and $\mathcal{E}$ is an element in $\text{holim } 2\text{eqVectBd}(U_\bullet)_0$; 

• $2\text{eqVectBd}^+(X_\bullet)_1$: a 1-morphism between $(U_\bullet, \mathcal{E})$ and $(\tilde{U}_\bullet, \tilde{\mathcal{E}})$ is a pair consisting of a common refinement $V_\bullet$ of $U_\bullet$ and $\tilde{U}_\bullet$ and an element $\alpha \in \text{holim } 2\text{eqVectBd}(V_\bullet)_1$; 

• $2\text{eqVectBd}^+(X_\bullet)_2$: a 2-morphism between $(V_\bullet, \alpha)$ and $(\tilde{V}_\bullet, \tilde{\alpha})$ consists of a common refinement $W_\bullet$ of $V_\bullet$ and $\tilde{V}_\bullet$ and an element $\beta \in \text{holim } 2\text{eqVectBd}(W_\bullet)_2$. Moreover, $(W_\bullet, \beta)$ and $(\tilde{W}_\bullet, \tilde{\beta})$ are identified if $\beta$ and $\tilde{\beta}$ agree on a further common refinement of $W_\bullet$ and $\tilde{W}_\bullet$.
6 Application to string 2-group

In this section we apply the construction in Section 4 to the model of string 2-group [MRW17].

6.1 2-Representations of string 2-group and the associated 2-vector bundles

By the general definition (2.10) of 2-representation, the bicategory of 2-representations of string 2-group \( S(QG) \) is defined to be

\[
\mathcal{2Rep}(B(S(QG)), \mathcal{2Vect}) = \text{Hom}(B(S(QG)), \mathcal{2Vect}),
\]

Applying the examples of 2-representations in Section 2.3 to the model of string 2-group \( S \), we can also define \( \text{Rep}^{S(QG)} \) with objects of the form \( S(QG) \rightarrow \ast \rightarrow \tilde{LG} \rightarrow \text{Rep}_{tr}^{S(QG)} \). There is a 2-representation of \( S(QG) \) with

- **Object level**: the only object \( \ast \) in the 2-category corresponding to a 2-group, maps to \( \text{Rep}_{tr}^{S(QG)} \).

\[\eta_{g_1}(\rho)_x : F_{g_0}(\rho)(x) \rightarrow F_{g_0}(\rho)(x), \quad v \mapsto \rho(1_x \cdot g_1)(v), \quad \forall v \in F_{g_0}(\rho)(x), \quad (6.90)\]

As shown in Corollary 2.8, we can also define \( \text{Rep}_{tr}^{S(QG)} \) and have the example below.

**Corollary 6.2.** We have the inclusion from \( \text{Rep}_{tr}^{S(QG)} \) to \( \text{Rep}^{S(QG)} \). \( \text{Rep}_{tr}^{S(QG)} \) is a subcategory of \( \text{Rep}^{S(QG)} \) with objects of the form \( S(QG) \rightarrow \ast \rightarrow \tilde{LG} \rightarrow \text{Rep}_{tr}^{S(QG)} \). There is a 2-representation of \( S(QG) \) with

- **Object level**: the only object \( \ast \) in the 2-category corresponding to a 2-group, maps to \( \text{Rep}_{tr}^{S(QG)} \).
we can get the associated 2-vector bundles and principal 2-bundles.

**B 2-stack from B**

Using the Lie groupoid presentation Corollary 6.3.

\[ \text{Rep}(\text{String}(G)) \] of principal \text{String}(G)-bundles. And we have the associating 2-functor of 2-vector bundles.

\[ F : QG \to \text{Fun}(\text{Rep}^+_\text{tr}(QG), \text{Rep}^+_\text{tr}(QG)), \ g \mapsto F_g, \]

where

\[ F_g(\rho) : x \mapsto V, \ (x \xrightarrow{a} y) \mapsto (V \xrightarrow{\rho(a+1)x} V) \]

\[ F_g(\rho \xrightarrow{f} \rho') = F_g(\rho) F_g(f)_x = F_g(\rho') \]

where \( \rho \in \text{Rep}^+_\text{tr}(QG) \) comes with the trivial bundle \( V \times QG \), and \( f \) is a 1-morphism in \( \text{Rep}^+_\text{tr}(QG) \).

**2-Morphism level:** given \( (g_0 \xrightarrow{g_1} g_0') \in \widehat{LG} \times QG \)

\[ (\widehat{LG} \times QG) \to \{\text{natural transformations }\}, \ g_1 \mapsto \eta_{g_1}, \]

\[ \eta_{g_1}(\rho)_x : F_{g_0}(\rho)(x) \to F_{g_0'}(\rho)(x), \ v \mapsto \rho(1_x \cdot \eta_{g_1})(v), \ \forall v \in F_{g_0}(\rho)(x). \quad (6.91) \]

In addition, by Corollary 2.8 we have the example below involving the positive energy representations of \( \text{Rep}^+(\widehat{LG}) \).

**Corollary 6.3.** Using the Lie groupoid presentation \( S(QG) = (\widehat{LG} \times QG) \) of \( \text{String}(G) \), the corresponding 2-representation is given by

- **The single object \(*\** is sent to the \( k \)-linear category \( \text{Rep}^+(\widehat{LG}) \)
- **1-Morphism level:**

\[ QG \to \text{Fun}(\text{Rep}^+(\widehat{LG}), \text{Rep}^+(\widehat{LG})), \ g \mapsto F_g, \]

where

\[ F_g(\rho) : (x \mapsto (V_{x, y}, \rho(x \cdot y, g))), \ (x \xrightarrow{a} y) \mapsto (\xrightarrow{\rho(a+1)x} V_{x, y, g}) \]

\[ F_g(\rho \xrightarrow{f} \rho') = (F_g(\rho) \xrightarrow{F_g(f)} F_g(\rho')) \]

where \( f \) is a 1-morphism in \( \text{Rep}^+(\widehat{LG}) \), that is a natural transformation \( \rho \Rightarrow \rho' \) with \( f_x \) a positive energy representation, and \( F_g(f) \) is a natural transformation, with \( F_g(f)_x = f_{x, \eta_{g_1}}. \) Note that \( F_g(f)_x \) is still a positive energy representation.

- **2-Morphism level:** given \( (g_0 \xrightarrow{g_1} g_0') \in \widehat{LG} \)

\[ \widehat{LG} \to \{\text{natural transformations }\}, \ g_1 \mapsto \eta_{g_1}, \]

\[ \eta_{g_1}(\rho)_x : F_{g_0}(\rho)(x) \to F_{g_0'}(\rho)(x), \ v \mapsto \rho(1_x \cdot \eta_{g_1})(v). \quad (6.92) \]

Then, apply the construction of 2-vector bundles in Section 4.2 to the case \( \mathcal{G} = S(QG) \), we can get the associated 2-vector bundles and principal 2-bundles.

In addition, As in Section 4.2 taking \( \mathcal{G} = S(QG) \), we can define the \( (3, 1) \)-presheaf \( B(S(QG)) \) over smooth manifolds. After applying the plus construction, we obtain the 2-stack \( B(S(QG))^+ \) of principal \( S(QG) \)-bundles. And we have the associating 2-functor from \( B(S(QG))^+ \) to the 2-stack \( \text{2VectBd}^+ \) of 2-vector bundles.
6.2 The Adjoint $G$–action on 2-representations of string 2-group

Murray, Roberts and Wockel conjectured in [MRW17, Section 4] that the inclusion from the category of positive energy representations of $\hat{LG}$ to the category of Hilbert vector bundles on $S(QG)$ preserves the fusion product of positive energy representations. And they conjectured at the end of [MRW17] that the category of positive energy representations of $\hat{LG}$ could be the "fixed points" under the adjoint action of $G$ onto the category of Hilbert vector bundles on $S(QG)$.

In this section we show there is a well-defined adjoint action of $G$ on $2\text{Rep}_{S}^+(\hat{LG})$.

Then we show that, When $G = S(QG)$ and $H = \hat{LG}$, by the inclusion $i : \text{Rep}H \to 2\text{Rep}G$ defined in Theorem 2.10, $\text{Rep}\hat{LG}$ is mapped to the fixed points of $2\text{Rep}_{S}^+(\hat{LG})$ by the adjoint action of $G$.

First, with the formula (3.55) of $\xi$ and the formula for the horizontal product [MRW17, Section 3] in $S(QG)$, we have the properties of the horizontal product. Here $e$ is the constant loop in $QG$.

Lemma 6.4. For $g \in QG$,

1. $\xi(e) = e$.
2. $\iota \circ \xi(e) = e$.
3. $g \circ_{h} e = \iota \circ \xi(g)$.
4. $\xi \circ_{h} g = \iota \circ \xi(g)$.
5. $e \circ_{h} e = e$.

Proof. 1. By (3.55), $\xi(e) = e(0)^{-1}e = e \cdot e = e$.

2. By the conclusion above, $\iota \circ \xi(e) = \iota(e) = e$.

3. By the formula for the horizontal product [MRW17, Section 3] and the conclusion above, $g \circ_{h} e = \iota(\xi(g) \xi(e)) = \iota(\xi(g) e) = \iota \circ \xi(g)$.

4. Similarly, $\xi \circ_{h} g = \iota(\xi(e) \xi(g)) = \iota(e \xi(g)) = \iota \circ \xi(g)$.

5. By the above conclusions, $\xi \circ_{h} e = \iota \circ \xi(e) = e$.

First we discuss the adjoint action of $G \subset LG$ on $\text{Rep}^+(\hat{LG})$. $G$ acts by conjugation on both $QG$ and $LG$. Let $\alpha \in G$ and $g_0, g'_0 \in QG$. Let

$$(g_0 \xrightarrow{g_1 = (g_0, h)} g'_0) \in \hat{LG} \times QG$$

where $g'_0 = h \cdot g_0$. We can define an adjoint action of $G$ on $LG$ by

$$(\alpha \cdot \hat{h})(t) := \alpha \hat{h}(t) \alpha^{-1}$$

45
for any $\alpha \in G$ and any $\overline{h} \in LG$. To define the adjoint action of $G$ on $\hat{LG}$, consider the central extension

$$U(1) \to \hat{LG} \to LG.$$  

For $\alpha \in G$, it defines a constant loop in $LG$ by $\alpha(t) = \alpha$. This loop $\alpha$ can lift to a loop $\overline{\alpha} \in \hat{LG}$. The adjoint action

$$(\alpha \cdot h)(t) := \overline{\alpha} h(t) \overline{\alpha}^{-1}$$

clearly does not depend on the lift since $U(1)$ is the center of $\hat{LG}$. When there is no confusion, we will still use $\alpha$ to denote this lift $\overline{\alpha}$ in $\hat{LG}$.

For any $g_0 \in QG$ and $h \in \hat{LG}$, the conjugation is defined by

$$\begin{align*}
(\alpha \cdot g_0)(t) &:= \alpha g_0(t) \alpha^{-1} \\
(\alpha \cdot h)(t) &:= \alpha h(t) \alpha^{-1} \\
\alpha \cdot g_1 &:= (\alpha \cdot g_0, \alpha \cdot h)
\end{align*}$$

(6.93) (6.94) (6.95)

We have the 1-morphism

$$(\alpha \cdot g_0 \xrightarrow{\alpha \cdot g_1} \alpha \cdot g_0') \in \hat{LG} \ltimes QG.$$  

The Lie 2-group identity $u = (u_0, u_1) : * \to S(QG)$ is defined by

$$\begin{array}{c}
\begin{array}{c}
* \\
\alpha \cdot g_0
\end{array}
\end{array}
\begin{array}{c}
\xrightarrow{u_1}
\xrightarrow{u_0}
\end{array}
\begin{array}{c}
\hat{LG} \ltimes QG \\
* \\
\alpha \cdot g_0
\end{array}$$

where $u_0(*) = \xi$ and $u_1(*) = 1_{\xi}$.

For the identity morphism $1_{g_0} = (g_0, \xi)$ at each object $g_0$, we have

$$\alpha \cdot 1_{g_0} = 1_{\alpha \cdot g_0}.$$  

(6.96)

**Lemma 6.5.** For any object $g_{0,1}, g_{0,2} \in QG$ of $S(QG)$, we have

$$(\alpha \cdot g_{0,1}) \circ_h (\alpha \cdot g_{0,2}) = \alpha \cdot (g_{0,1} \circ_h g_{0,2}).$$

**Proof.** Note that by the formula (3.55) of $\xi$, we have

$$\begin{align*}
\xi(\alpha \cdot g_0) &= \xi(\alpha g_0 \alpha^{-1}) = \alpha g_0(0)^{-1} \alpha^{-1} (\alpha g_0 \alpha^{-1})^x \\
&= \alpha g_0(0)^{-1} \alpha^{-1} (\alpha g_0 \alpha^{-1}) = \alpha g_0(0)^{-1} g_0^\alpha \alpha^{-1} \\
&= \alpha \xi(g_0) \alpha^{-1} = \alpha \cdot \xi(g_0).
\end{align*}$$

In addition,

$$\begin{align*}
(\alpha \cdot \xi(g_{0,1}))(\alpha \cdot \xi(g_{0,2})) &= (\alpha \xi(g_{0,1}) \alpha^{-1})(\alpha \xi(g_{0,2}) \alpha^{-1}) \\
&= \alpha (\xi(g_{0,1}) \xi(g_{0,2})) \alpha^{-1} = \alpha \cdot (\xi(g_{0,1}) \xi(g_{0,2}))
\end{align*}$$
Thus,
\[(\alpha \cdot g_{0,1}) \circ_{h} (\alpha \cdot g_{0,2}) = \iota(\xi(\alpha \cdot g_{0,1})\xi(\alpha \cdot g_{0,2}))\]
\[= \iota(\alpha \cdot (\xi(g_{0,1})\xi(g_{0,2}))) = \alpha \cdot (\iota(\xi(g_{0,1})\xi(g_{0,2}))) = \alpha \cdot (g_{0,1} \circ_{h} g_{0,2}).\]

\[\square\]

Recall the formulas (3.58) \(\xi : \hat{L}G \to \hat{\Omega}_{\xi}G\)
\[\xi(h) = (\text{pr}_{\hat{L}G} \circ \sigma(t(h)))^{-1} \cdot h \cdot \text{pr}_{\hat{L}G} \circ \sigma(s(h))\]
and (3.59) \(\sigma : QG \to \hat{L}G \times_{QG} Q_{\hat{L}G} G, \quad \gamma \mapsto (\pi_{2} \circ \sigma^{\prime}(\gamma^{F},\gamma^{-1}),\gamma(0)^{-1}\gamma^{\varpi}).\)

Lemma 6.6. We have the conclusions about \(\xi\) and \(\sigma\) below.

1. \(\sigma(\alpha \gamma \alpha^{-1}) = \alpha \sigma(\gamma) \alpha^{-1}\), for any \(\gamma \in QG\);
2. \(\xi(1_{g_{0}}) = 1_{\xi(g_{0})}\), for any \(g_{0} \in QG\);
3. \(\xi(\alpha \cdot g_{1}) = \alpha \cdot \xi(g_{1})\), for any \(g_{1} = (g_{0},h) \in \hat{L}G \times QG\);
4. \(1_{g_{0}} \circ_{h} g_{1} = (g_{0} \circ_{h} g_{0}, \xi(h))\) and \(g_{1} \circ_{h} 1_{g_{0}} = (g_{0} \circ_{h} g_{0}, \xi(h))\), for any \(g_{0} \in QG\) and any \(g_{1} = (g_{0},h) \in \hat{L}G \times QG\).

Proof. We prove them one by one.

1.
\[\sigma(\alpha \gamma \alpha^{-1}) = (\pi_{2} \circ \sigma^{\prime}((\alpha \gamma \alpha^{-1})^{F},(\alpha \gamma \alpha^{-1})^{-1}), (\alpha \gamma \alpha^{-1})(0)^{-1}(\alpha(\gamma^{\varpi}) \alpha^{-1}))\]
\[= (\pi_{2} \circ \sigma^{\prime}(\alpha \gamma \alpha^{-1} \alpha^{-1} \gamma^{-1} \alpha^{-1}), (\alpha \gamma(0)^{-1} \alpha^{-1} \gamma^{\varpi}) \alpha^{-1})\]
\[= (\pi_{2} \circ \sigma^{\prime}(\alpha \gamma \gamma^{-1}, (\alpha \gamma(0)^{-1} \gamma^{\varpi}) \alpha^{-1})\]
\[= (\alpha \pi_{2} \circ \sigma^{\prime}(\gamma^{F} \gamma^{-1}) \alpha^{-1}, \alpha(\gamma(0)^{-1} \gamma^{\varpi}) \alpha^{-1})\]
\[= (\alpha(\gamma \alpha^{-1}) \alpha^{-1} = \alpha \sigma(\gamma) \alpha^{-1}.\]

2. \(\xi(1_{g_{0}}) = (\xi(g_{0}),\xi(\xi)) \in \hat{L}G \times QG.\)
\[\xi(\xi) = (\text{pr}_{\hat{L}G} \circ \sigma(t(1_{g_{0}})))^{-1} \cdot \xi \cdot \text{pr}_{\hat{L}G} \circ \sigma(s(1_{g_{0}}))\]
\[= (\text{pr}_{\hat{L}G} \circ \sigma(g_{0}))^{-1} \cdot \xi \cdot \text{pr}_{\hat{L}G} \circ \sigma(g_{0})\]
\[= (\text{pr}_{\hat{L}G} \circ \sigma(g_{0}))^{-1} \cdot \text{pr}_{\hat{L}G} \circ \sigma(g_{0})\]
\[= \xi\]

Thus, \(\xi(1_{g_{0}}) = (\xi(g_{0}),\xi(\xi) = 1_{\xi(g_{0})}\) in \(\hat{\Omega}_{\xi}G \times Q_{\hat{L}G} G.\)

3. \(\xi(\alpha \cdot g_{1}) = (\xi(\alpha \cdot g_{0}),\xi(\alpha \cdot h)) = (\alpha \cdot (\xi(g_{0}),\xi(\alpha \cdot h)).\)
Lemma 6.7. The formulas (6.93) (6.94) (6.95) define a strict 2-functor
\[ \alpha_\bullet : BS(QG) \to BS(QG) \]

\[
\xi(\alpha \cdot h) = (\operatorname{pr}_{LG} \circ \sigma(t(ah\alpha^{-1})))^{-1} \cdot (ah\alpha^{-1}) \cdot \operatorname{pr}_{LG} \circ \sigma(s(ah\alpha^{-1})) \\
= (\operatorname{pr}_{LG} \circ \sigma(ag_0\alpha^{-1}))^{-1} \cdot (ah\alpha^{-1}) \cdot \operatorname{pr}_{LG} \circ \sigma(ahg_0\alpha^{-1}) \\
= \alpha(\operatorname{pr}_{LG} \circ \sigma(g_0))^{-1} \cdot (ah\alpha^{-1}) \cdot \alpha(\operatorname{pr}_{LG} \circ \sigma(hg_0))\alpha^{-1} \\
= \alpha(\operatorname{pr}_{LG} \circ \sigma(g_0))^{-1} h(\operatorname{pr}_{LG} \circ \sigma(hg_0))\alpha^{-1} \\
= \alpha h\alpha^{-1} \\
= \alpha \cdot \xi(h)
\]

Thus, \( \xi(\alpha \cdot g_1) = (\alpha \cdot \xi(g_0), \alpha \cdot \xi(h)) \).

4. Let \( g_1 = (g_0, h), g'_1 = (g'_0, h') \in \widehat{LG} \ltimes QG \)

Claim: The horizontal product
\[ g_1 \cdot h g'_1 = (g_0 \circ_h g'_0, h \cdot h') \]

is given by
\[ g_1 \cdot h g'_1 = (\iota(\xi(g_0)\xi(g'_0)), \iota(\xi(h)\xi(h'))). \] \hspace{1cm} (6.97)

Proof of the claim. The target of \( g_1 \cdot h g'_1 \) is
\[
\iota(\xi(h)\xi(h')) \cdot \iota(\xi(g_0)\xi(g'_0)) = \iota(\xi(h)\xi(h') \cdot \xi(g_0)\xi(g'_0)) \\
= \iota(\xi(h) \cdot \xi(g_0)) \cdot \xi(h') \cdot \xi(g'_0)) = \iota(\xi(h \cdot g_0)\xi(h' \cdot g'_0))
\]
is the horizontal product of the targets of \( g_1 \) and \( g'_1 \). \( \square \)

Let \( 1_{g'_0} = (g'_0, \xi) \in \widehat{LG} \ltimes QG \). Then \( g_1 \cdot h 1_{g'_0} = (g_0 \circ_h g'_0, h \cdot \xi) \).

\[
h \cdot h 1_{g'_0} = \iota(\xi(h)\xi(\xi)) \\
= \iota(\xi(h)\xi) = \iota(\xi(h)) \\
= \xi(h).
\]

Similarly, for \( 1_{g'_0} \cdot h g_1 = (g'_0 \circ_h g_0, \xi \cdot h) \), we have
\[
\xi \cdot h = \iota(\xi(\xi)\xi(h)) \\
= \iota(\xi(h)) = \xi(h).
\]

\( \square \)

Lemma 6.7. The formulas \( (6.93) (6.94) (6.95) \) define a strict 2-functor
\[ \alpha_\bullet : BS(QG) \to BS(QG) \]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\alpha_2} & G_1 \\
\downarrow & & \downarrow \\
G_0 & \xrightarrow{\alpha_1} & G_0 \\
\downarrow & \scriptstyle{\alpha_0} & \downarrow \\
* & \xrightarrow{\alpha_{0}} & *
\end{array}
\]

in the following way.

48
• On the object level, \( \alpha_0 \) sends the single object \( * \) to the single object \( * \);

• On the level of 1-morphisms, \( \alpha_1 \) sends \( g_0 \) to \( \alpha \cdot g_0 \);

• On the level of 2-morphisms, \( \alpha_2 \) sends a 2-morphism \( g_0 \xrightarrow{g_1} g_0' \) to \( \alpha \cdot g_0 \xrightarrow{\alpha \cdot g_1} \alpha \cdot g_0' \).

Proof. First we show \( \alpha_\bullet \) satisfies the conditions below.

1. \( \alpha_\bullet \) strictly preserves the Lie 2-group identity \( u \).
   \[ \alpha_1(u_0) = (\alpha \cdot u_0)(*) = \alpha \cdot u_0(*) = \alpha \cdot 1 = 1. \]
   And by Lemma 6.5, \( \alpha_2(u_1) = u_1 \).

2. \( \alpha_\bullet \) strictly preserves the horizontal products and vertical products.
   By Lemma 6.5, \( \alpha_1(g_{0,1}) \circ_h \alpha_1(g_{0,2}) = \alpha_1(g_{0,1} \circ_h g_{0,2}) \) for any \( g_{0,1}, g_{0,2} \in QG \).
   For \( g_{0,1} \xrightarrow{g_1} g_{0,2} \) and \( g_{0,1}' \xrightarrow{g_1'} g_{0,2}' \), by Lemma 6.5 and Lemma 6.6, we have
   \[
   \alpha_2(g_{1} \circ_h g_{1}') = \alpha \cdot (g_{1} \circ_h g_{1}') = \alpha \cdot (g_{0,1} \circ_h g_{0,1}', h \cdot h') \\
   = (\alpha \cdot (g_{0,1} \circ_h g_{0,1}'), h \cdot h') \\
   = (((\alpha \cdot g_{0,1}) \circ_h (\alpha \cdot g_{0,1}'), (\alpha \cdot \xi(h) \xi(h'))) \\
   = (((\alpha \cdot g_{0,1}) \circ_h (\alpha \cdot g_{0,1}'), (\alpha \cdot \xi(h) \xi(h')))^{-1}) \\
   = (((\alpha \cdot g_{0,1}) \circ_h (\alpha \cdot g_{0,1}'), (\alpha \cdot \xi(h) \xi(h')))^{-1}) \\
   = (\alpha \cdot (g_{0,1} \circ_h g_{0,1}'), h \cdot h') \\
   = \alpha_2(g_{1}) \circ_h \alpha_2(g_{1}')
   \]

In addition, \( \alpha_2 \) sends the vertical product, i.e. the composition of 2-morphisms,

\[
g_{0,1} \xrightarrow{g_1} g_{0,2} \xrightarrow{g_1'} g_{0,3}
\]

\[\xrightarrow{\alpha \cdot g_{0,1} \xrightarrow{\alpha \cdot g_1} \alpha \cdot g_{0,2} \xrightarrow{\alpha \cdot g_1'} \alpha \cdot g_{0,3}}\]

We have \( \alpha_2(g_1 \circ g_1') = \alpha_2(g_1') \circ \alpha_2(g_1) \).

3. \( \alpha_\bullet \) strictly preserves the horizontal inverses and vertical inverses.
   For \( g_{0,1}, g_{0,2} \in QG \), if \( g_{0,1} \circ_h g_{0,2} = 1 \), then, by Lemma 6.5,
   \[
   (\alpha \cdot g_{0,1}) \circ_h (\alpha \cdot g_{0,2}) = \alpha \cdot (g_{0,1} \circ_h g_{0,2}) = \alpha \cdot 1 = 1.
   \]
   Thus, \( \alpha_1(g_{0,2}) = \alpha \cdot g_{0,2} \) is the horizontal inverse of \( \alpha_1(g_{0,1}) \).

In addition, for \( g_{0,1} \xrightarrow{g_1} g_{0,2} \) and \( g_{0,1}' \xrightarrow{g_1'} g_{0,2}' \), by the proof above, we have

\[
\alpha_2(g_{1} \circ_h g_{1}') = \alpha_2(g_{1}) \circ_h \alpha_2(g_{1}').
\]

49
4. \( \alpha \) strictly preserve the left unit constraint.

By the definition [MRW17, Section 3, (15)] of the left unit constraint \( l \) of \( S(QG) \),
\( l_{\alpha \cdot g_0} \) is the composition

\[
l_{\alpha \cdot g_0} : u_0 \circ_h (\alpha \cdot g_0) = \iota(\xi(\iota(\varepsilon(\xi(\alpha \cdot g_0)))) \xrightarrow{\iota(\varepsilon(\iota(\varepsilon(\xi(\alpha \cdot g_0)))))} \iota(\xi(\alpha \cdot g_0)) \xrightarrow{\nu_{\alpha \cdot g_0}^{-1}} \alpha \cdot g_0
\]

and \( \alpha \cdot l_{g_0} \) is the composition

\[
\alpha \cdot l_{g_0} : \alpha \cdot (u_0 \circ_h g_0) = \alpha \cdot \iota(\xi(\iota(\varepsilon(\xi(g_0))))) \xrightarrow{\alpha \cdot (\iota(\varepsilon(\iota(\varepsilon(\xi(g_0))))) \xrightarrow{\alpha \cdot (\iota(\varepsilon(\xi(g_0)))) \xrightarrow{\alpha \cdot (\nu_{g_0}^{-1})} \alpha \cdot g_0
\]

Thus, if \( g_1 \cdot h \cdot g_1' = (g_{0,1} \cdot h \cdot g_{0,1}', \xi) \),
\( \alpha_2(g_1) \cdot \alpha_2(g_1') = \alpha_2(g_1 \cdot h \cdot g_1') = (\alpha \cdot (g_{0,1} \cdot h \cdot g_{0,1}'), \alpha \cdot \xi) = ((\alpha \cdot g_{0,1}) \circ_h (\alpha \cdot g_{0,2}), \xi).

Thus, \( \alpha_2(g_1) \) is the horizontal inverse of \( \alpha_2(g_1') \). \( \alpha \) preserves horizontal inverses.

For \( g_{0,1} \xrightarrow{g_{1,2}(g_{0,1}, h)} g_{0,2} \) and \( g_{0,2} \xrightarrow{g_{1,1}^{-1}(g_{0,2}, h^{-1})} g_{0,1} \), since

\[\alpha_2(g_1) \cdot \alpha_2(g_1') = \alpha_2(g_1 \cdot h \cdot g_1') = \alpha_2(1_{g_{0,1}} = 1_{\alpha \cdot g_{0,1}}
\]

thus, \( \alpha_2(g_1^{-1}) = \alpha_2(g_1)^{-1} \). We applied Lemma 6.6 (2) and Lemma 6.7 (2) in the proof above.

We have shown all the diagrams below commute. Next we will show that the left and right unit constraints \( l \) and \( r \) are also compatible with \( \alpha \).

4. \( \alpha \) strictly preserve the left unit constraint.
We want to show they are the same functor. In other words, for each $g_0 \in QG$ we have the commutative diagram

\[
\begin{array}{c}
u_0 \circ_h (\alpha \cdot g_0) \\ \downarrow \alpha \cdot \iota(e \cdot \iota(\alpha \cdot g_0)) \\ \alpha \cdot \iota(e \cdot \xi(g_0)) \end{array}
\begin{array}{c}
\alpha \cdot (e \cdot \xi(g_0)) \\ \downarrow \alpha \cdot \eta^{-1}_{\alpha \cdot g_0} \\ \alpha \cdot g_0
\end{array}
\]

(6.98)

Note that $\xi(\iota(e)) = \xi(\xi)$ and, by Lemma 6.9,

$\alpha \cdot \iota(e \cdot \iota(\alpha \cdot g_0)) = \iota(e \cdot \iota(\alpha \cdot g_0))$.

Next we show $\eta_{\alpha \cdot g_0}^{-1} = \alpha \cdot \eta_{g_0}^{-1}$ and $\iota(e \cdot \iota(\alpha \cdot g_0)) = \alpha \cdot \iota(e \cdot \iota(\alpha \cdot g_0))$ respectively.

In [MRW17, Section 3] Murray, Roberts and Wockel constructed the counit

$\epsilon : \xi \Rightarrow \text{id}$

after choosing the quasi-inverse $\xi$ of $\iota$ and they constructed the unit

$\eta : \iota \Rightarrow \text{id}$,

which at each $g_0 \in QG$ is the unique lift through $\xi$ of

$\epsilon_{\xi(g_0)}^{-1} : \xi(g_0) \rightarrow \xi(\iota(\xi(g_0)))$.

(6.99)

By Lemma 6.9 for each $g_0 \in Q_{s,s}G$ and $g_1 \in \Omega_2G \rtimes Q_{s,s}G$, we have

$\epsilon_{\alpha \cdot g_0} : \xi \iota(\alpha \cdot g_0) = \alpha \cdot (\xi \iota(g_0)) \rightarrow \alpha \cdot g_0$

$\epsilon_{\alpha \cdot g_1} : \xi \iota(\alpha \cdot g_1) = \alpha \cdot (\xi \iota(g_1)) \rightarrow \alpha \cdot g_1$

We have the formula from (3.59) that

$\epsilon_{g_0} = (\pi_2 \circ \sigma'((\alpha \cdot \gamma)F,\gamma^{-1}))^{-1}$,

where $g_0 = \gamma(t)$. And

$\epsilon_{\alpha \cdot g_0} = (\pi_2 \circ \sigma'((\alpha \cdot \gamma)F,\gamma^{-1}))^{-1} = (\pi_2 \circ \sigma'((\alpha \gamma)^{F^2},\alpha \gamma^{-1})^{-1} = (\alpha \cdot \epsilon_{g_0})^{-1}$

By the property of functors,

$\epsilon_{\alpha \cdot g_0}^{-1} = \alpha \cdot \epsilon_{g_0}^{-1}$.

By (6.99), $\eta_{\alpha \cdot g_0}$ is the unique lift through $\xi$ of

$\epsilon_{\alpha \cdot g_0}^{-1} = \epsilon_{\alpha \cdot \xi(g_0)}^{-1} = \alpha \cdot \epsilon_{\xi(g_0)}^{-1}$.

Thus,

$\eta_{\alpha \cdot g_0} = \alpha \cdot \eta_{g_0}$.
and
\[ \eta_{a \cdot g_0}^{-1} = \alpha \cdot \eta_{g_0}^{-1}. \]

By the definition of \( \alpha \) on the 2–morphisms, we have the equalities
\[
\begin{align*}
\epsilon_{a \cdot g_0} &= \alpha \cdot \epsilon_{g_0}, & \epsilon_{a \cdot g_0}^{-1} &= \alpha \cdot \epsilon_{g_0}^{-1} \\
\eta_{a \cdot g_0} &= \alpha \cdot \eta_{g_0}, & \eta_{a \cdot g_0}^{-1} &= \alpha \cdot \eta_{g_0}^{-1}
\end{align*}
\] (6.100)

Thus,
\[
\epsilon_{a \cdot \id_{a \cdot g_0}} = \epsilon_{a \cdot \id_{a \cdot g_0}} \text{ by } (6.100)
\]
\[
= \alpha \cdot (\epsilon_{a \cdot \id_{g_0}})
\]
\[
\eta(\epsilon_{a \cdot \id_{\xi(a \cdot g_0)}}) = \eta(\alpha \cdot (\epsilon_{a \cdot \id_{g_0}}))
\]
\[
= \alpha \cdot \eta(\epsilon_{a \cdot \id_{\xi(g_0)})}
\]

In fact they are both identity since \( \xi(\eta(\underline{L})) = \xi(g) = \underline{g} \).

Thus, \( l_{a \cdot g_0} \) and \( \alpha \cdot l_{g_0} \) are the same functor. The diagram commute.

5. Compatibility with the right unit constraint.

By the definition [MRW17 Section 3, (14)] of the left unit constraint \( l \) of \( S(QG) \), \( r_{a \cdot g_0} \) is the composition
\[
r_{a \cdot g_0} : (a \cdot g_0) \circ h u_0 = l(\xi(a \cdot g_0) \xi(\underline{g})) \xrightarrow{(\id_{\xi(a \cdot g_0)} \epsilon_{a \cdot g_0})} l(\xi(a \cdot g_0) \underline{g}) = l(\xi(a \cdot g_0) \xrightarrow{\eta_{a \cdot g_0}^{-1}} a \cdot g_0)
\]
and \( \alpha \cdot r_{g_0} \) is the composition
\[
\alpha \cdot r_{g_0} : \alpha \cdot (g_0 \circ h u_0) = \alpha \cdot l(\xi(g_0) \xi(\underline{g})) \xrightarrow{(\id_{\xi(g_0)} \epsilon_{g_0})} \alpha \cdot l(\xi(g_0) \underline{g}) = \alpha \cdot l(\xi(g_0) \xrightarrow{\eta_{g_0}^{-1}} \alpha \cdot g_0)
\]
Similarly, the first map of each composition, \( (\id_{\xi(a \cdot g_0)} \epsilon_{a \cdot g_0}) \) and \( \alpha \cdot (\id_{\xi(g_0)} \epsilon_{g_0}) \), are both the identity map. And by the analysis \( \alpha \cdot \eta_{g_0} = \alpha \cdot \eta_{g_0}^{-1}. \)

6. Compatibility with the associator.

For any 1-morphisms
\[
\begin{array}{cccc}
\ast & \xrightarrow{f} & \ast & \xrightarrow{g},
\end{array}
\]
By the definition [MRW17 Section 3, (13)] of the associator \( a \) of \( S(QG) \), \( a_{a \cdot f, a \cdot g, a \cdot h} \) is the composition
\[
a_{a \cdot f, a \cdot g, a \cdot h} : l(\xi(\xi(\alpha \cdot f) \xi(\alpha \cdot g))) \xrightarrow{\iota(\xi(\xi(\alpha \cdot f) \xi(\alpha \cdot g)))} l(\xi(\alpha \cdot f) \xi(\alpha \cdot g) \xi(\alpha \cdot h)) \xrightarrow{\iota(\xi(\alpha \cdot f) \xi(\alpha \cdot g) \xi(\alpha \cdot h))^{-1}} l(\xi(\alpha \cdot f) \xi(\alpha \cdot g) \xi(\alpha \cdot h)))
\]
We have
\[
\begin{align*}
\iota(\xi(\alpha \cdot f) \xi(\alpha \cdot g) \xi(\alpha \cdot h)) &= \iota(\epsilon_{a \cdot f} \xi(\alpha \cdot g)) (\alpha \cdot \xi(h)) \\
&= \iota((\alpha \cdot \epsilon_{a \cdot f} \xi(\alpha \cdot g))) (\alpha \cdot \xi(h)) \\
&= \iota(\alpha \cdot (\epsilon_{a \cdot f} \xi(\alpha \cdot g))) (\xi(h)) \\
&= \alpha \cdot \iota(\epsilon_{a \cdot f} \xi(\alpha \cdot g)) (\xi(h))
\end{align*}
\]
\[ u(\xi(\alpha \cdot f)e^{-1}_{\xi(\alpha)\xi(\alpha \cdot h)}) = u((\alpha \cdot \xi(f))e^{-1}_{\xi(\alpha)\xi(\alpha \cdot h)}) = \alpha \cdot (\xi(f)e^{-1}_{\xi(\alpha \cdot h)}) \]

Thus, \( a_{\alpha \cdot f,\alpha \cdot g,\alpha \cdot h} \) and

\[
\alpha_2(a_{f,g,h}) : \alpha \cdot u(\xi(\xi(f)\xi(g))\xi(h)) \overset{\alpha_1(u(\xi(\xi(f)\xi(g))\xi(h)))}{\longrightarrow} \alpha \cdot u(\xi(f)\xi(g)\xi(h))
\]

are the same functor.

Thus, \( \alpha_* \) is a well-defined strict 2-functor. \( \square \)

**Lemma 6.8.** For any \( \alpha, \alpha' \in G \), the composition \( \alpha_* \circ \alpha'_* \) and the multiplication \( (\alpha \alpha'_*)_* \) are the same 2-functor from \( BS(QG) \) to itself.

Especially, for each \( \alpha \in G \), \( \alpha_* \circ (\alpha^{-1})_* \) and \( (\alpha^{-1} \circ \alpha)_* \) are both identities. Thus, each \( \alpha \in G \) gives an equivalence of the bicategories \( \alpha_* : BS(QG) \rightarrow BS(QG) \).

**Proof.** Below we verify that \( \alpha_* \circ \alpha'_* = (\alpha \alpha'_*)_* \).

For any \( \alpha, \alpha' \in G \), we have the composition

\[
\alpha_* \circ \alpha'_* : BS(QG) \rightarrow BS(QG)
\]

defined by the data below.

- \( \alpha_* \circ \alpha'_* \) sends the single object \(*\) to \(*\).
- \( \alpha_* \circ \alpha'_* \) sends each 1-morphism \( g_0 \) in \( BS(QG) \) to

\[
(\alpha \alpha') \cdot g_0 = \alpha \cdot (\alpha' \cdot g_0).
\]

- It sends each 2-morphism \( g_1 \) to

\[
(\alpha \alpha') \cdot g_1 = \alpha \cdot (\alpha' \cdot g_1).
\]

- \( (\alpha_* \circ \alpha'_*) \) sends the unit \( u \) to itself since \( (\alpha_* \circ \alpha'_*)(u) = \alpha_* \circ (\alpha'_*(u)) = \alpha_*(u) = u \).

Thus, the 2-functor \( \alpha_1 \circ \alpha_2 = \alpha_1 \alpha_2 \).

\( \square \)

Moreover, we have the conclusion below.

**Lemma 6.9.** We have a \( G \)-action on the \( BS(QG) \) defined by the data below. (The definition of group actions on bicategories is from [Hes17, Section 2.2, Definition 2.14, Remark 2.16] and [BGM19, Section 2, page 7].)

- For each \( \alpha \in G \), we have the strict 2-functor \( \alpha_* : BS(QG) \rightarrow BS(QG) \) defined in Lemma 6.7, which is an equivalence of bicategories;
• For any \(\alpha, \alpha' \in G\), the natural equivalences

\[(\chi_{\alpha, \alpha'}, \chi_{\alpha, \alpha'}^0) : \alpha \circ \alpha' \Rightarrow (\alpha \alpha')\]

are defined to be identity.

• The natural equivalence

\[\kappa : \text{id} \Rightarrow e\]

is defined to be the identity, where

\[e : BS(QG) \to BS(QG)\]

is the functor corresponding to the identity element \(e \in G\).

• For any 1–morphisms \(g, h, f\) in \(S(QG)\), the invertible modifications

\[\omega_{g, h, f} : \chi_{g, h, f} \circ (\chi_{g, h} \cdot h 1_f) \Rightarrow \chi_{g, h, f} \circ (1_g \cdot h \chi_{h, f})\]

\[\kappa_g : \chi_{g, g} \circ (\kappa \cdot h 1_g) \Rightarrow 1_g\]

\[\xi_g : \chi_{g, g} \circ (1_g \cdot h \kappa) \Rightarrow 1_g\]

are all defined to be the identity.

Proof. Since the invertible 2-morphisms and modifications in the definitions are all identity, it’s straightforward to check that they satisfy the coherence conditions, i.e. [Hes17 Section 2.2, HTA1, HTA2] and [HGM19 Section 2, (2.1)(2.2)].

\[\square\]

Lemma 6.10. For each \(\alpha \in G\), the 2-functor

\[\alpha : BS(QG) \to BS(QG)\]

defines a 2-functor

\[\alpha^* : 2\text{Rep}(QG) \to 2\text{Rep}(QG)\]

in the following way.

• On the object level, sending an \(S(QG)\)–representation

\[A = (V, F, \eta)\]

to

\[\alpha^*(A) = (V, \alpha \cdot F, \alpha \cdot \eta)\]

where

\[\alpha \cdot F_{g_0} := F_{\alpha^{-1} \cdot g_0} ;\]

\[\alpha \cdot \eta_{g_1} := \eta_{\alpha^{-1} \cdot g_1} ;\]

for any \(g_0 \in QG\) and \(g_1 \in \hat{L}G \ltimes QG\).
Let $A_i := (V_i, F_i, \eta_i)$ with $i = 1, 2, \cdots$. On the level of 1-morphisms, with the definition of 1-morphisms in the bicategory of 2-representations in Example 2.5, sending

$$f = (T(*), T) : A_1 \longrightarrow A_2$$

to

$$\alpha_1^*(f) = (T(*), \alpha_2^*(T)) : \alpha_0^*(A_1) \longrightarrow \alpha_0^*(A_2)$$

where

$$\alpha_2^*(T) : T(*)^* \circ (\alpha \cdot F_2) \Rightarrow T(*)^* \circ (\alpha \cdot F_1)$$

is given by the composition

$$T(*)^* \circ (\alpha \cdot F_2) = T(*)^* \circ (F_2 \circ \alpha) \xrightarrow{\alpha} (T(*)^* \circ F_2) \circ \alpha \xrightarrow{T \circ \alpha} (T(*)_* \circ F_1) \circ \alpha \xrightarrow{\alpha} T(*)_* \circ (F_1 \circ \alpha) = T(*)^* \circ (\alpha \cdot F_1)$$

where $\alpha$ is the associator and $T \circ 1_\alpha$ is below.

\[ \begin{array}{ccc}
G_0 & \xrightarrow{\alpha} & \text{Auto}(D_1) \\
\downarrow F_1 & & \downarrow T(*)_* \\
G_0 & \xrightarrow{\alpha} & \text{Auto}(D_2) \\
\downarrow F_2 & & \downarrow T(*)^* \\
& & \text{Fun}(D_1, D_2)
\end{array} \tag{6.102} \]

\textbf{Proof.} We show $\alpha_\bullet$ satisfies the conditions below.

1. $\alpha_\bullet^*$ strictly preserves the Lie 2-group identity $\underline{0}$ in $\text{2RepS}(QG)$, which is defined by the data below.
   - On the level of objects, sends $*$ to the category consisting of only one object 0 and one morphisms at 0, the identity morphism.
   - On the level of 1-morphisms, all the elements in $QG$ is mapped to the identity morphism at 0.
   - On the level of 2-morphisms, all the elements in $\hat{LG} \ltimes QG$ are sent to the identity natural transformation from the identity morphism to itself.

   $\alpha_0^*(\underline{0}) = \underline{0}$ by definition.

   The identity morphism in $\text{2RepS}(QG)$ from $\underline{0}$ to itself

   $\text{id} = (\text{id}, \text{id}) : \underline{0} \longrightarrow \underline{0}$

   is sent to

   $\alpha_1^*(\text{id}) = (\text{id}, \alpha_2^*(\text{id})) = (\text{id}, \text{id}) = \text{id}.$

2. $\alpha_\bullet$ strictly preserves the horizontal products and vertical products.

For $S(QG)$-representations $A_i = (V_i, F_i, \eta_i)$
where \( i = 1, 2, 3, 4 \), the 1−morphisms
\[
f_1, f_2 : A_1 \to A_2; \quad f_3, f_4 : A_3 \to A_4
\]
and the 2−morphisms
\[
\beta_1 : f_1 \Rightarrow f_2 \text{ and } \beta_2 : f_3 \Rightarrow f_4,
\]
\[
\alpha^*_2(\beta_1 \circ_h \beta_2) = \alpha^*_2(\beta_1) \circ_h \alpha^*_2(\beta_2).
\]
If we have the composition of 1−morphisms in \( \text{2Rep}(S(QG)) \)
\[
A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3
\]
\( \alpha^*_\bullet \) sends it to
\[
\alpha^*_0(A_1) \xrightarrow{\alpha^*_1(f)} \alpha^*_0(A_2) \xrightarrow{\alpha^*_1(g)} \alpha^*_0(A_3)
\]
3. \( \alpha^*_\bullet \) strictly preserves the inverses.
For \( S(QG) \)−representations
\[
A_i = (V_i, F_i, \eta_i)
\]
where \( i = 1, 2 \), and the 1−morphisms
\[
f : A_1 \to A_2; \quad f^{-1} : A_2 \to A_1,
\]
\[
\text{id} = \alpha^*_1(f \circ f^{-1}) = \alpha^*_1(f) \circ \alpha^*_1(f^{-1}).
\]
4. Since \( \text{2Vect} \) is a strict 2−category, \( \text{2Rep}(S(QG)) \) is a strict 2-category as well. So the associator, left and right unit constraints in it are all identity.

We have the corollary of Lemma 6.8 and Lemma 6.10 below, which can be checked straightforwards.

**Lemma 6.11.** For any \( \alpha, \alpha' \in G \), \( \alpha^*_\bullet \circ \alpha'^*_\bullet \) and \( (\alpha \alpha')^*_\bullet \) are the same 2-functor from \( \text{2Rep}(S(QG)) \) to itself.

Especially, for each \( \alpha \in G \), \( \alpha \alpha^{-1} \) and \( \alpha^{-1} \alpha \) are both identities. Thus, each \( \alpha \in G \) gives an equivalence of the bicategories \( \alpha^*_\bullet : \text{2Rep}(S(QG)) \to \text{2Rep}(S(QG)) \).

**Lemma 6.12.** The \( \text{G−action on the BS}(QG) \) in Lemma 6.14 induces a \( \text{G−action on 2Rep}(S(QG)). \)

- For each \( \alpha \in G \), we have the strict 2-functor
  \[
  \alpha^*_\bullet : \text{2Rep}(S(QG)) \to \text{2Rep}(S(QG))
  \]
defined in Lemma 6.10.
- For any \( \alpha, \alpha' \in G \), the natural equivalences
  \[
  \chi^*_\bullet : \alpha^*_\bullet \circ \alpha'^*_\bullet \Rightarrow (\alpha \alpha')^*_\bullet
  \]
is defined to be the identity.
• The natural equivalence
\[ \kappa^* : \text{id} \Rightarrow \epsilon^* \]
is defined to be the identity, where \( \epsilon^* : 2\text{Rep}(S(QG)) \to 2\text{Rep}(S(QG)) \) is the functor corresponding to the identity element \( e \in G \).

• For any \( g, h, f \in G \), the invertible modifications
\[
\begin{align*}
\omega^*_{g,h,f} : & \chi^*_{g,h} \circ (\chi^*_h \cdot 1_f) \Rightarrow \chi^*_{g,hf} \circ (1_g \cdot h_1 f) \\
\kappa^*_{g} : & \chi^*_{g} \circ (\kappa^* \cdot 1_g) \Rightarrow 1_g \\
\xi^*_{g} : & \chi^*_{g} \circ (1_g \cdot \kappa^*) \Rightarrow 1_g
\end{align*}
\]
are all defined to be the identity.

**Proof.** Since the invertible 2-morphisms and modifications in the definitions are all identity, it’s straightforward to check that they satisfy the coherence conditions, i.e. [Hes17 Section 2.2, HTA1, HTA2] and [HGM19 Section 2, (2.1)(2.2)].

**Example 6.13.** Below is the comparison of the 2-Representation
\[ A = (C_V, f_V, \eta_V) \]
in Corollary 2.9 A and
\[ \alpha \cdot A = (C_V, \alpha \cdot f_V, \alpha \cdot \eta_V) \]
For \( \alpha \cdot f_V \), it sends an object \( g_0 \) of \( QG \) to \( \alpha \cdot f_{V,g_0} \) which is defined by
\[
\alpha \cdot f_{V,g_0} : C_V \to C_V, \quad \alpha \cdot f_{V,g_0}(\rho) : x \mapsto \rho(x \circ (\alpha^{-1} \cdot g_0)) = V_0,
\]
\[
(x \xrightarrow{a} y) \mapsto (V_0 \xrightarrow{\rho(a \cdot \alpha^{-1} \cdot g_0)} V_0)
\]
Let \( a = (x, h) \). By the property of horizontal product in Lemma 6.6
\[
a \cdot h_1 \alpha^{-1} \cdot g_0 = (x \cdot h (\alpha^{-1} \cdot g_0), \xi(h))
\]
Thus,
\[
\rho(a \cdot h_1 \alpha^{-1} \cdot g_0) = \rho(a \cdot h_1 g_0).
\]
And
\[
\alpha \cdot f_{V,g_0}(\rho \xrightarrow{b} \rho') = (f_{V,\alpha^{-1} \cdot g_0}(\rho) \xrightarrow{f_{V,\alpha^{-1} \cdot g_0}(b)} f_{V,\alpha^{-1} \cdot g_0}(\rho')), \quad f_{V,\alpha^{-1} \cdot g_0}(b) = b
\]
for any \( \rho \in C_V \), \( x \in QG \), \( (x \xrightarrow{a} y) \in L\mathbb{G} \times QG \), and \( (\rho \xrightarrow{b} \rho') \) a morphism in \( C_V \).
Thus,
\[
\alpha \cdot f_V = f_V. \quad (6.103)
\]
And, given \( g_1 = (g_0 \xrightarrow{h} g'_0) \in L\mathbb{G} \times QG \), \( \alpha \cdot \eta_V \) is defined by
\[
\eta_V : G_1 \to \{ \text{natural transformations} \}, \quad g_1 \mapsto \eta_{V,\alpha^{-1} \cdot g_1},
\]
\[
\eta_{V,\alpha^{-1} \cdot g_1} : f_{V,\alpha^{-1} \cdot g_0} \to f_{V,\alpha^{-1} \cdot g'_0},
\]
57
with
\[ \eta_{V,\alpha^{-1}g_1}(\rho)_x = \rho(1_x \cdot \eta(h)), \]
component-wise. By \[6.103\], both \( \eta_{V,g_1}(\rho)_x \) and \( \eta_{V,\alpha^{-1}g_1}(\rho)_x \) are functors from \( f_{V,g_0}(\rho)_x \) to \( f_{V,g_0}(\rho)_x \), i.e., from \( V_0 \) to \( V_0 \). By the property of horizontal product in Lemma 6.6

\[ 1_x \cdot \eta(g_1) = (x \circ_h g_0, \xi(h)) \]  
\[ 1_x \cdot \eta(\alpha^{-1} \cdot g_1) = (x \circ_h (\alpha^{-1} \cdot g_0), \xi(\alpha^{-1} \cdot h)) = (x \circ_h (\alpha^{-1} \cdot g_0), \alpha^{-1} \cdot \xi(h)). \]  

(6.104)  

(6.105)

Thus, \( \rho(1_x \cdot \eta(g_1)) \) may not always equal to \( \rho(1_x \cdot \eta(g_1)) \) for any \( \rho \in C_V \). So \( \alpha \cdot \eta_V \) may not be the same natural transformation as \( \eta_V \).

Then we show that, for any \( \alpha \in QG \), for \( A = (C_V, f_V, \eta_V, \phi, \psi) \) in \( 2\text{RepS}(QG) \), there is a 1-equivalence in \( S(QG) \)
\[ T_\alpha : A \to \alpha \cdot A \]
where
\[ \alpha \cdot A = (C_V, \alpha \cdot f_V, \alpha \cdot \eta_V, \alpha \cdot \phi, \alpha \cdot \psi). \]

As we show above, \( \alpha \cdot f_V = f_V \). And, as for the natural transformations \( (\psi, \phi) \)

\[ S(QG) \times S(QG) \overset{m}{\longrightarrow} S(QG), \quad \overset{u'}{\longrightarrow} S(QG), \]

\[ \text{Auto}(D) \times \text{Auto}(D) \overset{m'}{\longrightarrow} \text{Auto}(D), \quad \overset{u'}{\longrightarrow} \text{Auto}(D), \]

where \((F, \eta)\) here is \((f_V, \eta_V)\) or \((\alpha \cdot f_V, \alpha \cdot \eta_V)\).

By the formula (2.29), the functor
\[ \alpha \cdot \psi_{(g_0, \hat{g}_0)} : (\alpha \cdot f)_{V, g_0 \circ_h \hat{g}_0} \to (\alpha \cdot f)_{V, g_0} \circ (\alpha \cdot f)_{V, \hat{g}_0} \]
is defined by
\[ \alpha \cdot \psi_{(g_0, \hat{g}_0)}(\rho)(x) : (\alpha \cdot f)_{V, g_0 \circ_h \hat{g}_0}(\rho)(x) \to (\alpha \cdot f)_{V, g_0} \circ (\alpha \cdot f)_{V, \hat{g}_0}(\rho)(x), \quad \alpha \cdot \psi_{(g_0, \hat{g}_0)}(\rho)(x) = \rho(\alpha x_{g_0 \cdot \hat{g}_0}). \]  

(6.107)

By the definition of the associator \( a_{x,g_0} : (x \circ_h g_0) \circ_h \hat{g}_0 \to x \circ_h (g_0 \circ_h \hat{g}_0) \) in \([13]\), \( a_{x,g_0} \) is the composition
\[ a_{x,g_0} : (x \circ_h (\alpha \cdot g_0)) \circ_h (\alpha \cdot \hat{g}_0) = \iota(\xi(x)(\alpha \cdot g_0))(\alpha \cdot \hat{g}_0) \]
\[ \overset{\iota(\xi(x)(\alpha \cdot g_0))(\alpha \cdot \hat{g}_0)}{\longrightarrow} \iota(\xi(x)(\alpha \cdot g_0))\xi(\alpha \cdot \hat{g}_0) \]
\[ \overset{\iota(\xi(x)(\alpha \cdot g_0))(\alpha \cdot \hat{g}_0)}{\longrightarrow} \iota(\xi(x)(\alpha \cdot g_0))\xi(\alpha \cdot \hat{g}_0) \]
\[ = x \circ_h ((\alpha \cdot g_0) \circ_h (\alpha \cdot \hat{g}_0)). \]

Let \( y \) denote \( \alpha^{-1} \cdot x \). By the proof of Lemma [6.3] and the property of \( \alpha \cdot \epsilon \) and \( \alpha \cdot \epsilon^{-1} \),
\[ \iota(\xi(x)(\alpha \cdot g_0))\xi(\alpha \cdot \hat{g}_0) = \iota(\epsilon(\xi(x))(\alpha \cdot g_0))(\alpha \cdot \xi(\hat{g}_0)) = \iota(\epsilon(\xi(\alpha \cdot g_0))(\alpha \cdot \xi(\hat{g}_0))) \]
\[ = \iota(\epsilon(\xi(\alpha \cdot g_0))(\alpha \cdot \xi(\hat{g}_0))) = \iota(\epsilon(\xi(y))(\alpha \cdot \xi(\hat{g}_0))) = \iota(\epsilon(\xi(y))\xi(\alpha \cdot \hat{g}_0)) = \alpha \cdot \iota(\epsilon(\xi(y))(\alpha \cdot \xi(\hat{g}_0))). \]

\[ \iota(\xi(x)(\alpha \cdot g_0))(\alpha \cdot \hat{g}_0) = \iota(\epsilon(\xi(y))(\alpha \cdot \xi(\hat{g}_0))(\alpha \cdot \xi(\hat{g}_0))) = \iota(\epsilon(\xi(y))(\alpha \cdot \xi(\hat{g}_0))(\alpha \cdot \xi(\hat{g}_0))) = \alpha \cdot \iota(\epsilon(\xi(y))(\alpha \cdot \xi(\hat{g}_0))). \]
Thus, as the composition of
\[
\alpha \cdot \iota(\xi(g_0)\xi(\hat{g}_0)) \text{ and } \alpha \cdot \iota(\xi(y)\xi^{-1}(\xi(g_0)\xi(\hat{g}_0))).
\]
\[a_{x,\alpha \cdot g_0, \alpha \cdot \hat{g}_0}\] is the same functor as \(\alpha \cdot a_{x, g_0, \hat{g}_0}\). In fact we have proved above the general conclusion that for any \(g_0, 1, g_0, 2, g_0, 3\) in \(QG\),
\[
a_{\alpha \cdot g_0, 1, \alpha \cdot g_0, 2, \alpha \cdot g_0, 3} = \alpha \cdot a_{\alpha \cdot g_0, 1, \alpha \cdot g_0, 2, \alpha \cdot g_0, 3}.
\]
(6.108)

Thus,
\[
\alpha \cdot \psi_{(g_0, \hat{g}_0)}(\rho) = \rho(\alpha^{-1} \cdot a^{-1}_{\alpha \cdot g_0, \hat{g}_0})
\]

By the formula (2.33), we have
\[
\alpha \cdot \psi^* : I' = u'(\ast) \rightarrow (\alpha \cdot f_V, \alpha \cdot \eta_V)(\ast) = f_V \cdot I, \quad (\alpha \cdot \psi)(\rho) = \rho(r_x)^{-1} = \psi(\rho)(x).
\]

Note that \(\alpha^{-1} \cdot I = I\). Thus,
\[
\alpha \cdot \psi^* = \psi^*.
\]

As the Definition of 1–morphisms in \(2\text{Rep}_S(QG)\), \(T_\alpha\) consists of the data below.

- A morphism \(T_\alpha(\ast) : D \rightarrow D\) in \(\text{2Vect}\), which is defined to be the identity.
- A natural transformation
\[
\begin{array}{ccc}
QG & \xrightarrow{\alpha \cdot f_V} & \text{Auto}(C_V) \\
\downarrow f_V & & \downarrow \gamma_T & & \downarrow \text{id} \\
\text{Auto}(C_V) & \xrightarrow{\text{id}} & \text{Auto}(C_V)
\end{array}
\]
which is defined to be
\[
\alpha : f_V \mapsto \alpha \cdot f_V
\]

with \(\alpha \cdot f_V\) defined in Lemma 6.10.

Thus, it’s straightforward to check that the diagram (2.12) (2.13) commutes. The vertical maps are all identity and the horizontal maps are the same.

Similarly, we have the \(T_{\alpha^{-1}} : \alpha \cdot A \rightarrow \alpha^{-1} \cdot (\alpha \cdot A) = A\). And \(T_\alpha \circ T_{\alpha^{-1}}\) and \(T_{\alpha^{-1}} \circ T_\alpha\) are both identities.

Thus, \(T_\alpha\) is a 1–equivalence.

**Proposition 6.14.** When \(G = S(QG)\) and \(H = \hat{L}G\), the inclusion
\[
i : \text{Rep}_H \rightarrow 2\text{Rep}_G
\]
defined in Theorem 2.10 sends \(\text{Rep}_H\) to the fixed points of \(2\text{Rep}_S(QG)\) by the adjoint action of \(G\).

Here we use the definition of the bicategory of fixed points in [Hes17, Section 2.2, Definition 2.17, Remark 2.20].

**Proof.** Here I prove that, for any \(\alpha \in QG\), for any object \(A = (D, F, \eta, \phi, \psi)\) in \(2\text{Rep}_S(QG)\), there is a 1–equivalence in \(S(QG)\)
\[
T_\alpha : A \rightarrow \alpha \cdot A
\]
where
\[
\alpha \cdot A = (D, \alpha \cdot F, \alpha \cdot \eta, \alpha \cdot \phi, \alpha \cdot \psi).
\]

As the Definition of 1–morphisms in \(2\text{Rep}_S(QG)\), \(T_\alpha\) consists of the data below.
• A morphism $T_\alpha(\ast) : D \to D$ in $\mathbf{2Vect}$, which is defined to be the identity.

• A natural transformation

$$
\begin{array}{ccc}
QG & \xrightarrow{\alpha \cdot F} & \text{Auto}(D) \\
\downarrow F & \uparrow \gamma & \downarrow \text{id} \\
\text{Auto}(D) & \xrightarrow{\text{id}} & \text{Auto}(D)
\end{array}
$$

(6.110)

which is defined to be

$$
\alpha : F \mapsto \alpha \cdot F
$$

with $\alpha \cdot F$ defined in Lemma 6.10.

Note that, as for the natural transformations $(\psi, \phi)$

$$
\begin{array}{ccc}
S(QG) \times S(QG) & \xrightarrow{m} & S(QG) \\
\downarrow (F,\eta) \times (F,\eta) & \uparrow \phi & \downarrow (F,\eta) \\
\text{Auto}(D) \times \text{Auto}(D) & \xrightarrow{\text{id} \times \text{id}} & \text{Auto}(D)
\end{array}
$$

(6.111)

By the formula (2.29) (2.33) we have $\alpha \cdot \psi = \psi$ and $\alpha \cdot \phi = \phi$. Thus, it’s straightforward to check that the diagram commutes.

Similarly, we have the $T_{\alpha^{-1}} : \alpha \cdot A \to \alpha^{-1} \cdot (\alpha \cdot A) = A$. And $T_\alpha \circ T_{\alpha^{-1}}$ and $T_{\alpha^{-1}} \circ T_\alpha$ are both identities.

Thus, $T_\alpha$ is a 1–equivalence.

References

[2ve13] 2-vector bundle, 2013. (available from the n-Lab, https://ncatlab.org/nlab/show/2-vector+bundle).

[AGV72] M. Artin, A. Grothendieck, and J. L. Verdier. Théorie des topos et cohomologie étale des schémas. Tome 2. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 270.

[B67] Jean Bénabou. Introduction to bicategories. In Reports of the Midwest Category Seminar, pages 1–77. Springer, Berlin, 1967.

[Bar] Toby Bartels. Higher gauge theory i: 2-bundles. arXiv:math/0410328v3 [math.CT].

[BDR04] Nils A. Baas, Bjørn Ian Dundas, and John Rognes. Two-vector bundles and forms of elliptic cohomology. In Topology, geometry and quantum field theory, volume 308 of London Math. Soc. Lecture Note Ser., pages 18–45. Cambridge Univ. Press, Cambridge, 2004.

[BGM19] E. Bernaschini, C. Galindo, and M. Mombelli. Group actions on 2-categories. manuscripta math, 159:81–115, 2019.
[BL04] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra. V. 2-groups. *Theory Appl. Categ.*, 12:423–491 (electronic), 2004.

[Bro73] Kenneth S Brown. Abstract homotopy theory and generalized sheaf cohomology. *Transactions of the American Mathematical Society*, 186:419–458, 1973.

[DR18] Christopher L. Douglas and David J. Reutter. Fusion 2-categories and a state-sum invariant for 4-manifolds. *arXiv: Quantum Algebra*, 2018.

[GR04] Ofer Gabber and Lorenzo Ramero. Foundations for almost ring theory – release 7.5. *arXiv: Algebraic Geometry*, 2004.

[Hes17] Jan Hesse. *Group Actions on Bicategories and Topological Quantum Field Theories*. PhD thesis, U. Hamburg (main), 2017.

[JY21] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford Scholarship Online, 2021.

[Kap99] M. Kapranov. Rozansky-Witten invariants via Atiyah classes. *Compositio Math.*, 115(1):71–113, 1999.

[Lei98] Tom Leinster. Basic bicategories, 1998.

[Lur09] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[Mac71] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[Mac00] K. C. H. Mackenzie. Double Lie algebroids and second-order geometry. II. *Adv. Math.*, 154(1):46–75, 2000.

[MM03] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.

[MRW17] Michael Murray, David Michael Roberts, and Christoph Wockel. Quasiperiodic paths and a string 2-group model from the free loop group. *J. Lie Theory*, 27(4):1151–1177, 2017.

[NS11] Thomas Nikolaus and Christoph Schweigert. Equivariance in higher geometry. *Adv. Math.*, 226(4):3367–3408, 2011.

[PS86] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1986. Oxford Science Publications.

[SXZ17] Yunhe Sheng, Xiaomeng Xu, and Chenchang Zhu. String principal bundles and courant algebroids. *International Mathematics Research Notices, rnz017*, https://doi.org/10.1093/imrn/rnz017IMRN, 01 2017.

[Wol13] Jesse Wolfson. Descent for n-bundles. 08 2013.
[WZ05] Alan Weinstein and Marco Zambon. Variations on prequantization. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., XVI, pages 187–219. Univ. Luxemb., Luxembourg, 2005.

[Zhu09] Chenchang Zhu. Lie $n$-groupoids and stacky groupoids. *Int. Math. Res. Not. IMRN*, 2009(21):4087–4141, 2009.