On Minkowski-like and de Sitter-like space-times

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Abstract. This article summarises joint work with Juan A. Valiente-Kroon on de Minkowski-like and de Sitter-like space-times and was presented by the author at ERE 2009 in Bilbao. The existence and stability problem is re-examined using extended conformal field equations. In particular we make use of a gauge based on conformal geodesics to obtain a priori the location of the conformal boundary from the initial data.

1. The problem
The existence and stability of Minkowski-like and de Sitter-like space-times were originally investigated and proven by Friedrich in [1]. One of the central ideas is to exploit the conformal structure of these space-times. The analysis is carried out in the unphysical space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$, which is conformally related to $(\tilde{M}, \tilde{g}_{\mu\nu})$ with $g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}$ and $\tilde{M} \equiv \{ p \in M : \Theta(p) > 0 \} \subset M$. The Einstein conformal field equations developed in [2, 3] were used formulate the Cauchy problems, which are semi-global respectively global in time, as problems local in time. The conformal gauge chosen by Friedrich was based on the unphysical Ricci scalar.

Given the importance of the conformal geometry for the analysis in [1] we re-examined the problem [4] in a gauge based on two further conformal notions: general Weyl connections and conformal geodesics. The main motivation is the resulting simplification of the analysis of the space-times and their conformal boundaries both conceptually and computationally. In particular the location of the conformal boundary $\mathcal{I} = \{ p \in M : \Theta(p) = 0 \}$ is obtain directly from the initial data for the congruence and can thus in parts be fixed suitably before the evolution of the space-time.

In this article we focus mainly on the analysis for Minkowski-like space-times. However our setup will be more general so that the physical space-times $(\tilde{M}, \tilde{g}_{\mu\nu})$ satisfy the Einstein field equations with cosmological constant, i.e. $\tilde{R}_{\mu\nu} = \lambda \tilde{g}_{\mu\nu}$. The discussion of the initial data and the conformal boundary will focus first on the Minkowski-like case and the de Sitter-like scenario will be investigated later on.

The initial hypersurface $S$ will be space-like and diffeomorphic to $S^3$. Thus the resulting space-times have the form $M = I \times S$ where $I$ is an interval on $\mathbb{R}$. For Minkowski-like space-times we use hyperboloidal initial data. This type of data is obtained by solving the conformal constraints on a simply connected set $\tilde{S} \subset S$ with boundary $\mathcal{Z} = \partial \tilde{S}$ diffeomorphic to $S^2$. The conformal factor $\Omega$ on $\tilde{S}$ vanishes on $\mathcal{Z}$ and is positive on the interior. The data and the conformal factor are then extended smoothly to the rest of $S$. Note that the resulting space-time will only be conformally vacuum on the domain of dependence of $\tilde{S}$. 

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2. Conformal geometry and the conformal boundary

The evolution problem for \((M, g)\) resp. \((\tilde{M}, \tilde{g}_{\mu\nu})\) is formulated using Weyl connections and conformal geodesics. Given a 1-form \(b\) a Weyl connection is defined by

\[
\bar{\Gamma}_{\mu\rho}^\nu = \tilde{\Gamma}_{\mu\rho}^\nu + \left( \delta_{\mu}^\nu b_\rho + \delta_{\rho}^\nu b_\mu - g_{\mu\rho} \tilde{g}^{\nu\lambda} b_\lambda \right).
\]

Conformal geodesics are conformally invariant curves \(x(\tau)\) with velocity \(v(\tau)\) and an associated one form \(b(\tau)\) satisfying

\[
\begin{align*}
\tilde{\nabla}_v v^\mu + 2\langle b, v \rangle v^\mu - \tilde{g}(v,v)b^\mu &= 0, \quad (1) \\
\tilde{\nabla}_v b_\mu - \langle b, v \rangle b_\mu + \frac{1}{2} g_{\mu\nu} v^\nu \hat{g}(b,b) &= \tilde{L}_{\lambda\mu} v^\lambda. \quad (2)
\end{align*}
\]

where \(\tilde{L}_{\mu\nu} = \frac{1}{2} \left( \tilde{R}_{\mu\nu} - \frac{1}{6} \tilde{R} \tilde{g}_{\mu\nu} \right)\) is the Schouten tensor. The solutions (1) and (2) are preserved as point sets under conformal rescaling or reparametrisation of the conformal parameter \(\tau\) by a fractional linear transformation. Let \(\{e_k\}\) with \(e_0 = v\) be a conformally orthonormal frame that is Weyl propagated along the curves by \(\nabla_v e^k_\nu = 0\), where \(\nabla = \tilde{\nabla} + b\). The frame is used to define the unphysical metric by \(g^{\mu\nu} = \eta^{ij} e^\mu_i e^\nu_j\). The canonical conformal factor that gives \(g\) so that \(\{e_k\}\) is orthonormal is obtained by solving \(\tilde{\nabla}_v \Theta = \langle b, v \rangle \Theta\) along each curve. In particular, since \(\tilde{g}\) is a vacuum metric \(\Theta(\tau)\) automatically takes the form

\[
\Theta = \Theta_s + \tilde{\Theta}_s \tau + \frac{1}{2} \tilde{\Theta}_s \tau^2 \quad (3)
\]

Our calculations will use the 1-form \(d = \Theta b\), as its space-like components \(d_a\) in the frame \(\{e_k\}\) will be constant along each curve, while \(d_0 = \tilde{\Theta} = \hat{\Theta}_s + \tilde{\Theta}_s \tau\).

Initial data for \(v, d\) and \(\Theta\) on \(S\) satisfying the constraint

\[
2\Theta \tilde{\Theta} = \frac{1}{3} \lambda + g^4(d,d) \quad (4)
\]

is chosen such that

\[
\Theta_s = \Omega, \quad \langle d_s, v_s \rangle = \hat{\Theta}_s, \quad d_{sa} = \Omega b_{sa} = \hat{D}_a \Omega.
\]

The congruence \(\Gamma\) of time-like conformal geodesics developed in a neighbourhood of \(S\) is used to drag coordinates \(x^A\) from \(S\) along \(\Gamma\). In order for the coordinates \((\tau, x^A)\) to remain well defined throughout the evolution we need to guarantee that the congruence does not develop any caustics. This is done by studying the conformal Jacobi fields \(\eta\) along the congruence — see [5] for a definition. There are no conjugate points as long as the components of these fields orthogonal to the congruence do not vanish for \(\).

The initial data for \(\hat{\Theta}_s, \hat{\Theta}_a, \tilde{\Theta}_s\) and \(d_s\) has to satisfy (4) and we set \(v = n\), the normal of \(S\). On \(Z\) we choose \(\hat{\Theta} < 0\) so that the conformal boundary represents part of \(\mathcal{I}^+\). The value for \(\hat{\Theta}_s\) on \(Z\) is obtained by smooth extension from the interior. From this the location of \(\mathcal{I}\) can be computed explicitly prior to the evolution by solving for the roots \(\tau_\pm\) of (3). As long as \(\hat{\Theta} \neq 0\) there are always two roots and we can use the fractional linear parameter freedom of \(\tau\) to make sure that all curves starting on \(S\) reach \(\mathcal{I}^+\) in finite time. For Minkowski-space-times the existence and location of time-like infinity is of particular interest.

For hyperboloidal initial data sufficiently close to Minkowski data with respect to a suitable Sobolev norm, we can show that there exists a unique point \(p \in S \cap \{\Omega > 0\}\) such that \(d_a = 0\) at \(p\) for \(a = 1, 2, 3\). The conformal geodesic passing through \(p\) satisfies \(\Theta = 0\) and \(d\Theta = 0\) at the point \(i\) where \(\tau = \tau_+ = -\hat{\Theta}_s/\hat{\Theta}_s = -2 \Omega/\hat{\Theta}_s\). It can be shown that the Hessian of \(\Theta\) is non-degenerate at \(i\) and thus this is the unique point on the future conformal boundary that satisfies all the criteria of future time-like infinity of the Minkowski-like space-time.
3. The reference space-time

The Einstein cosmos is given by the manifold \( M' = \mathbb{R} \times S^3 \) with metric
\[
g_E = d^2t - d^2\chi - \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2). \tag{5}\]

The standard conformal embedding for Minkowski space is given by the conformal factor \( \Omega_M = \cos(t + \frac{\varphi}{2}) + \cos\chi \). By setting \( \tau = 2\tan\frac{t}{2} \) the \( t \)-curves of the Einstein cosmos can be reparametrised as conformal geodesics for which \( \Theta_E = 1 + \tau^2/4 \) and \( \theta_E = \tau/2dt \). For this congruence none of the conformal Jacobi fields vanish and the coordinate system \((\tau, \chi, \theta, \varphi)\) is well defined. We denote \( g' = \Theta_E^2 g_E \) and set \((M', g')\) as the reference space-time for our stability analysis. For Minkowski hyperboloidal data on \( t = 0 = \tau \) we set \( \Omega_M = \cos\chi \) and \( d_{M*} = -(dt + \sin\chi) \). Then (3) gives
\[
\Theta_{M*} = \frac{1}{\Theta_E} = \cos\chi \left( 1 - \frac{\tau}{\cos\chi} + \frac{\tau^2}{4} \right)
\]
which vanishes at \( \tau = 2\frac{1 + \sin\chi}{\cos\chi} \). The point \( i^+ \) lies on the central curve at \( \tau = 2 \).

From the conformal field equations we derive evolution equations along the conformal geodesic congruence. The space spinor formalism induced by the congruence can be used to show that we have a symmetric hyperbolic system for a variable \( u \) consisting of the frame, the connection, the curvature components and the conformal Jacobi fields. The difference between the solution \( u \) and the reference space-time \((M', g')\), represented by \( u' \), is denoted by \( \tilde{u}_0 \). By considering \( \tilde{u} \) as a variable over the Sobolev space \( (H^m(S^3, \mathbb{R}^N), \| \cdot \|_m) \) for \( m \geq 4 \) and some \( N \) we can apply a version of Kato’s theorem given in [1] to prove our existence and uniqueness results.

We can now formulate the following existence and stability result

**Theorem 1.** Let \( u_0 = u'_0 + \tilde{u}_0 \) be Minkowski-like initial data. Given \( T_0 > 2 \) there exists \( \varepsilon > 0 \) such that

(i) for \( \| \tilde{u}_0 \|_{m} < \varepsilon \) then there is a solution \( u' + \tilde{u} \) with minimal existence interval \( \tau \in [0, T_0] \) and \( u \in C^{m-2}([0, T_0] \times S) \);

(ii) the associated congruence of conformal geodesics contains no conjugate points in \([0, T_0] \);

(iii) for every \( \tilde{u}_0 \in B_{\varepsilon}(0) \) there is a unique point in \( S \setminus Z \) such that \( D_k \Omega = 0 \);

(iv) for all \( \tilde{u}_0 \in B_{\varepsilon}(0) \) we have \( \tau_i^+ \in [0, T_0] \).

The solution \( u' + \tilde{u} \) is unique on \( D^+(S) \), the domain of dependence of \( S \), and implies a \( C^{m-2} \) solution to the vacuum Einstein field equations with vanishing cosmological constant for which the set \( \mathcal{I}^+ \) represents future null infinity.

The point \( i^+ \) given by the conditions \( \tau = -\frac{2}{\langle b_*, v_* \rangle} \) and \( D_k \Omega = 0 \) represents timelike infinity.

For de Sitter-like space-times we can consider two different scenarios for the initial data on \( S \). In the standard Cauchy problem \( S \) lies entirely inside the physical space-time \((\tilde{M}, \tilde{g})\) and the fractional linear freedom of the conformal parameter \( \tau \) can be used to specify the initial data for \( \Theta \) such that \( \tau_{\pm} = \pm 2 \) on each curve. The other scenario that we investigate is the asymptotic Cauchy problem where \( S \) forms one half of the conformal boundary, say \( \mathcal{I}^- \), so that \( \Theta_* = \Omega = 0 \). In the case of asymptotic Cauchy data we set \( v_* = n \) so that (4) implies
\[
\Theta_* = \langle d, v \rangle_* = \sqrt{-\frac{\Lambda}{3}} \text{ and } \quad d_* = \langle d, v \rangle_* \cdot v_*. \]

Then the conformal boundary is given by \( \tau_* = 0 \) and \( \tau_{\pm} = -2\Theta_* / \tilde{\Theta}_* \). The initial data \( \Theta_* \) is freely specifiable and used to fix the location of the conformal boundary by setting \( \Theta_* = -\frac{1}{2} \tilde{\Theta}_* \), so that \( \tau_{\pm} = 4 \). Thus a suitable choice of data fixes the location of \( \mathcal{I}^\pm \) in both cases.

We can derive an analogue version of Theorem 1 for de Sitter-like space-times.
Theorem 2. Let \( I = [a, b] \) denote an interval in \( \mathbb{R} \) and let \( u_0 = u_0' + \tilde{u}_0 \) represent either standard Cauchy data \( (I= [-2,2]) \) or asymptotic Cauchy data \( (I= [0,4]) \). There exists \( \varepsilon > 0 \) such that if \( \| \tilde{u}_0 \|_m < \varepsilon \) then

(i) there is a unique solution \( u' + \tilde{u} \) with minimal existence interval \( \tau \in I \) with \( u \in C^{m-2}(I \times S) \)

(ii) the associated congruence of conformal geodesics contains no conjugate points in \( I \).

The fields \( u' + \tilde{u} \) imply a \( C^{m-2} \) solution to the vacuum Einstein field equations with negative cosmological constant. The sets \( \mathcal{I}^- = \{a\} \times S \) and \( \mathcal{I}^+ = \{b\} \times S \) represent, respectively, past and future null infinity.

The above approach of evolving vacuum space-times using conformal geodesics was further developed in [6] to prove semi-global existence and stability results for radiative space-times. Once more a priori knowledge of the location of the conformal boundary allowed proving that it is stable. In particular a new choice of initial data for the congruence in the perturbed space-time has further simplified the analysis of \( i^+ \) and the question of existence and uniqueness. In this new setup we write \( d = \nabla \Theta + \Theta f \), where \( f \) is another 1-form. The initial data for \( d_a \) is now set to be the same as for the reference space-time, i.e. as for \( d_M = -(dt + \sin \chi) \) above, and \( f \) is used to absorb the difference between \( d_a \) and \( D_a \Omega \) on \( S \). As a result \( i^+ \) is then clearly unique and lies on the same curve as in Minkowski.

It is currently being investigated whether similar approaches that use conformal geodesics or related concepts can help analysing stability problems for other matter models in this form.

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