Symmetry constraints for dispersionless integrable
equations and systems of hydrodynamic type

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Abstract
Symmetry constraints for (2+1)-dimensional dispersionless integrable equations are considered. It is demonstrated that they naturally lead to systems of hydrodynamic type which arise within the reduction method. One also easily obtains an associated complex curve (Sato function) and corresponding generating equations. Dispersionless KP and 2DTL hierarchy are considered as illustrative examples.

1 Introduction
Dispersionless integrable equations arise in various contexts and have variety of applications from complex analysis to topological field theory (see [1]-[22]). The reduction method seems to be the most developed method of construction of exact solutions of dispersionless equations [5, 8, 17, 23-34]. It reveals a close connection between such equations and certain systems of hydrodynamic type, emphasising a role of complex curves (Sato functions). The reduction method has been successfully applied to several dispersionless equations. However, an intrinsic background of its validity and efficiency seems to remain not very clear.

In the present paper it is shown that systems of hydrodynamic type arising within the reduction method are nothing but the symmetry constrained (2+1)-dimensional integrable equations. We will consider here the dispersionless KP and 2DTL hierarchies.

First, we find a general infinitesimal ‘nonisospectral’ symmetry transformation of the dispersionless $\tau$-function (free energy $F$). Then we impose symmetry constraint of the type ‘isospectral symmetry = nonisospectral symmetry’, similar to dispersionfull case [35-50]. It is demonstrated that

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such a constraint reduces (2+1)-dimensional dispersionless equation to a set of (1+1)-dimensional systems of hydrodynamic type with a finite number of dependent variables. We present also a simple way to find an associated complex curve and to construct generating equations.

Our approach starts with the \( \bar{\partial} \)-method applied to dispersionless equations. It provides us with a simple and natural way to consider symmetries and symmetry constraints. At the same time, an analysis of such constraints indicates a direction of a possible extension of the quasiclassical \( \bar{\partial} \)-method.

2 Quasiclassical \( \bar{\partial} \)-method

Dispersionless integrable hierarchies can be described in different forms within different approaches. In the papers [11, 12, 13] it was shown that such hierarchies can be introduced starting with the nonlinear Beltrami equation (quasi-classical \( \bar{\partial} \)-problem)

\[ S_{\bar{z}} = W(z, \bar{z}, S_z), \]

where \( z \in \mathbb{C} \), bar means complex conjugation, \( S_z = \frac{\partial S(z, \bar{z})}{\partial z} \), \( S_{\bar{z}} = \frac{\partial S(z, \bar{z})}{\partial \bar{z}} \), and \( W \) (quasi-classical \( \bar{\partial} \)-data) is an analytic function of \( S_z \),

\[ W(z, \bar{z}, S_z) = \sum_{p=0}^{\infty} w_p(z, \bar{z})(S_z)^p. \]

Applying the quasi-classical \( \bar{\partial} \)-dressing method based on equation (1), one can get dispersionless integrable hierarchies and the corresponding addition formulae in a very regular and simple way. Such an approach reveals also the connection of dispersionless hierarchies with the quasi-conformal mappings on the plane.

The specific choice of the hierarchy depends on the choice of domain \( G \) which is the support of the \( \bar{\partial} \)-data. Once the domain is fixed, we introduce the class of functions \( S_0(z) \), having the derivative \( \partial_z S_0(z) \) analytic in \( G \), and formulate the boundary problem in \( G \) in the following way. Let the function \( S_0(z) \) be given. The problem is to find the function \( S = S_0 + \bar{S} \), satisfying (1) in \( G \), with \( \bar{S} \) analytic outside \( G \) and decreasing at infinity.

Introducing parameterization of the class of functions \( S_0(z) \) in terms of infinite number of variables (times), and using the technique of quasiclassical \( \bar{\partial} \)-dressing method, it is possible to demonstrate that \( S(z, \bar{z}, t) \) is a solution of Hamilton-Jacobi equations of corresponding dispersionless hierarchy.
For dispersionless KP (dKP) hierarchy we take unit disc $D$ as $G$, and parameterization of $S_0(z)$ is given by

$$S_0(z, t) = \sum_{n=1}^{\infty} t_n z^n,$$

(3)

$S(z, \tilde{z}, t) = S_0 + \tilde{S}$, $\tilde{S}$ is analytic outside the unit disc, and at infinity it has an expansion $\tilde{S} = \sum_{i=1}^{\infty} S_i(t) z^{-i}$. The quantity $p = \frac{\partial S}{\partial t_1}$ is a basic homeomorphism \cite{42}. Important role in the theory of dKP hierarchy is played by the equation

$$p(z) - p(z_1) + z_1 \exp(-D(z_1)S(z)) = 0, \quad z \in \mathbb{C}, \quad z_1 \in \mathbb{C} \setminus D,$$

(4)

(where $D(z)$ is the quasiclassical vertex operator, $D(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \partial_{\bar{z}^n}$, $|z| > 1$) which generates Hamilton-Jacobi equations of the hierarchy by expansion into the powers of $z^{-1}$ at infinity (see, e.g., \cite{47}). This equation also implies existence of the $\tau$-function, characterized by the relation

$$\tilde{S}(z, t) = -D(z)F(t),$$

(5)

and provides the dispersionless addition formula.

For the dispersionless 2DTL (d2DTL) hierarchy the $\bar{\partial}$-data are localized on the domain $G$ which is an annulus $a < |z| < b$, where $a, b$ ($a, b \in \mathbb{R}, a, b > 0; b > a$) are arbitrary, $S_0$ can be represented as \cite{8,12,47}

$$S_0(t, x, y) = t \log z + \sum_{n=1}^{\infty} z^n x_n + \sum_{n=1}^{\infty} z^{-n} y_n.$$

(6)

We assume that $\tilde{S}(z) \sim \sum_{n=1}^{\infty} \frac{S_n}{z^n}$ as $z \to \infty$ and denote $\tilde{S}(0) = \phi$, $G_{+} = \{z, |z| > b\}$, $G_{-} = \{z, |z| < a\}$. The functions $p_{+} = \frac{\partial \tilde{S}}{\partial x_1}$ and $p_{-} = \frac{\partial \tilde{S}}{\partial y_1}$ have pole singularities while $p = \frac{\partial \tilde{S}}{\partial t_1}$ has a logarithmic singularity. Relations, characterizing the $\tau$-function $F$, are $\phi = DF$, $\tilde{S}(z_1) = -D_{+}(z_1)F(z_1 \in G_{+})$, $\tilde{S}(z_2) = \phi - D_{-}(z_2)F(z_2 \in G_{-})$, where $D_{+}(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \partial_{\bar{z}^n}$, $D_{-}(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} \frac{\partial}{\partial y_n}$. Hamilton-Jacobi equations of the hierarchy are generated by the relations \cite{8,12,47}

$$e^{\phi(z)} - e^{\phi(z_1)} + z_1 e^{-D_{+}(z_1)S(z)} = 0, \quad z \in \mathbb{C}, z_1 \in G_{+},$$

(7)

$$e^{\phi(z)} \left( 1 - e^{-D_{-}(z_2)S(z)} \right) = z_2 e^{(D_{-}(z_2))\phi}, \quad z \in \mathbb{C}, z_2 \in G_{-},$$

(8)

these relations also provide dispersionless addition formulae for $F$. 

3
3 \(\tau\)-function and symmetries

It was noted in [44] that equation (1) is a Lagrangian one. It can be obtained by variation of the action (for the boundary problem in \(G\))

\[
f = -\frac{1}{2\pi i} \int\int_G \left( \frac{1}{2} \tilde{S}_z \tilde{S}_\bar{z} - W_{-1}(z, \bar{z}, S_z) \right) dz \wedge d\bar{z},
\]

where

\[
W_{-1}(z, \bar{z}, S_z) = \sum_{p=0}^{\infty} w_p(z, \bar{z}) (S_z)^{p+1}, \quad \partial_{\eta} W_{-1}(z, \bar{z}, \eta) = W(z, \bar{z}, \eta).
\]

One should consider independent variations of \(\tilde{S}\), possessing required analytic properties (analytic outside \(G\), decreasing at infinity), keeping \(S_0\) fixed.

In [48] it was proved for dispersionless KP and 2DTL hierarchies that the action (9) evaluated on the solution of nonlinear Beltrami equation (1) gives a \(\tau\)-function of the corresponding hierarchy. Here we formulate first an auxiliary statement, valid for an hierarchy with arbitrary \(G\), and then recall the results of [48].

**Lemma 1** Let us evaluate the action (9) on the solution of the boundary problem for (1) and consider it as a functional of \(S_0\),

\[
F(S_0) = -\frac{1}{2\pi i} \int\int_G \left( \frac{1}{2} \tilde{S}_z \tilde{S}_\bar{z} - W_{-1}(z, \bar{z}, S_z) \right) dz \wedge d\bar{z}.
\]

Then

\[
\delta F = -\frac{1}{2\pi i} \int_{\partial G} \tilde{S} d\delta S_0.
\]

Using the formula (11), it is easy to reproduce the statements of the work [48].

**Proposition 1** The function

\[
F(t) = -\frac{1}{2\pi i} \int\int_D \left( \frac{1}{2} \tilde{S}_z(t) \tilde{S}_\bar{z}(t) - W_{-1}(z, \bar{z}, S_z(t)) \right) dz \wedge d\bar{z},
\]

i.e., the action (10) evaluated on the solution of the problem (1), with \(S_0\) given by (3), is a \(\tau\)-function of dKP hierarchy.
Proposition 2 The function

\[ F(t, x, y) = -\frac{1}{2\pi i} \int_G \left( \frac{1}{2} \tilde{S}_z(t, x, y) \tilde{S}_z(t, x, y) - W_{-1}(z, \bar{z}, S_z(t, x, y)) \right) dz \wedge d\bar{z}, \]

with \( S_0 \) given by (6), evaluated on the solution of the problem (7), is a \( \tau \)-function of dispersionless 2DTL hierarchy (\( G \) is an annulus defined above).

The function \( W \) is the \( \bar{\partial} \) data for the quasiclassical \( \bar{\partial} \)-problem. Its variations provide us with a wide class of variations of the function \( F \) (first we will use formula (10) defined for arbitrary \( G \)).

For the functions \( W \) of the form (2), varying \( w_n(z, \bar{z}) \), one has

\[ \delta W = \sum_{n=1}^{\infty} \delta w_n(z, \bar{z}) (S_z)^n, \quad \delta W_{-1} = \sum_{n=1}^{\infty} \frac{\delta w_n}{n+1} (S_z)^{n+1}, \]

and

\[ \delta F = \frac{\epsilon}{2\pi i} \int_G \int U(z, \bar{z}, S_z) dz \wedge d\bar{z}, \]

where \( U(z, \bar{z}, \eta) \) is an arbitrary function with the support in \( z \)-plane belonging to \( G \), analytic in \( \eta \). The formula (13) gives a general variation (infinitesimal symmetry transformation) of the \( \tau \)-function. Let us consider some special symmetry transformations.

Considering elementary variation \( \delta w_{n_0} = \epsilon \alpha_{n_0} \delta(z - z_0), \delta w_n = 0, n \neq n_0, \) one gets

\[ \delta F = \frac{\epsilon}{2\pi i} \frac{\alpha_{n_0}}{(n_0 + 1)} (S_z)^{n_0+1} \big|_{z=z_0}, \]

and, respectively,

\[ \delta \tilde{S} = -\frac{\epsilon}{2\pi i} \frac{\alpha_{n_0}}{(n_0 + 1)} D(z) (S_z(z_0))^{n_0+1}. \]

Taking superposition of elementary variations (14), we obtain a variation of the form

\[ \delta F = \epsilon f(S_z(z_0)), \]

where \( f \) is an arbitrary analytic function (summation over different points and integration over \( z_0 \) are also possible).

Another simple symmetry transformation is given by

\[ \delta F = \epsilon c(S(z_2) - S(z_1)), \]
where $z_1, z_2$ are arbitrary points belonging to the domain $G$, and $c$ is some constant. Indeed,

$$S(z_2) - S(z_1) = \int_{z_1}^{z_2} d(S) = \int_{z_1}^{z_2} S_z dz + \int_{z_1}^{z_2} S_{\bar{z}} d\bar{z}$$

$$= \int_{z_1}^{z_2} S_z dz + \int_{z_1}^{z_2} W(z, \bar{z}, S_z) d\bar{z},$$

thus the function $S(z_2) - S(z_1)$ corresponds to the special case of general variation (13). In the limit $z_2 \rightarrow z_1$ the transformation (17) is reduced to elementary symmetries (14).

### 4 Symmetry constraints

Given infinitesimal symmetries of the form

$$\delta F = c\Phi,$$

(cf., (13), (16), (17)), it is possible to introduce symmetry constraints

$$\frac{\partial}{\partial t_i} F = \Phi. \quad (18)$$

These constraints are preserved by the flows of the hierarchy and represent a reduction of the hierarchy. There is a lot of similarity between constraints (18) and symmetry constraints in the (2+1)-dimensional dispersionfull case, which are rather well studied [35]-[40]. Below we will demonstrate that constraints (18) lead to hydrodynamic reductions, and give a regular way to introduce basic objects connected with these reductions.

Technically, in the dispersionful case one starts with linear equations of the hierarchy (which give the hierarchy itself in the form of compatibility conditions). Symmetry constraints connect the potentials in these equations with some special wave functions, and in the end one obtains (1+1)-dimensional integrable system for the set of wave functions. The simplest symmetry constraint for KP hierarchy leads to nonlinear Schrödinger equation.

In the dispersionless case we will follow the same strategy. The role of linear equations of the hierarchy is played by Hamilton-Jacobi equations of the dispersionless hierarchy.
5 Zakharov reduction for the dKP hierarchy

We begin with a simple example of symmetry constraint for the dKP hierarchy. The first Hamilton-Jacobi equation of dKP hierarchy (dispersionless analogue of Lax equation) is

$$S_y = p^2 + 2u; \quad p = S_x, \quad u = -\partial_x \tilde{S}_1,$$

where $x = t_1$, $y = t_2$. Let us consider the symmetry constraint

$$F_x = f(S_z(z_0)),$$

which is equivalent to the condition (cf., (5))

$$u = \partial_x f(S_z(z_0)).$$

Differentiating (19) with respect to $z$ and evaluating the result at $z = z_0$, we get

$$\frac{\partial_y S_z(z_0)}{\partial_x S_z(z_0)} = 2p_0, \quad p_0 = p(z_0).$$

Using this equation and evaluating (19) at $z = z_0$, we obtain a system of hydrodynamic type

$$\begin{cases} 
\partial_y p_0 = \partial_x (p_0^2 + 2u), \\
\partial_y u = 2\partial_x (p_0 u).
\end{cases}$$

(21)

This system is well known [2], it represents a dispersionless limit of nonlinear Schrödinger equation. Thus a constraint (20) represents a proper dispersionless analogue of the simplest symmetry constraint of KP hierarchy.

It is possible to move further and demonstrate that the symmetry constraint (20) leads to explicit expression for the function $z(p)$, which plays a fundamental role in the picture of constrained hierarchy and allows to find Riemann invariants and hodograph equations. To do this, we will use generating Hamilton-Jacobi equation (4) rewritten as

$$D(z)S'(z') = -\log \frac{p(z) - p(z')}{z}, \quad |z| > 1,$$

which produces infinite set of Hamilton-Jacobi equations by expansion into the powers of $z^{-1}$ at infinity (see, e.g., [37]). Differentiating this equation by $z'$ and evaluating the result at $z' = z_0$, we get

$$D(z)S_z(z_0) = \frac{1}{p - p_0}.\)
Then, using the constraint (20) and simple formula
\[ D(z)F_x = z - p, \]
we obtain the relation
\[ z = p + \frac{u}{p - p_0}. \] (23)
The function \( z(p) \) (Sato function) plays a crucial role in the picture of dispersionless hierarchy. Dispersionless KP hierarchy (for general \( z(p) \)) can be written in the form \( \partial_n z(p) = \{ z^n_0, z \} \). Relation (23) defines the function \( z(p) \) for the constrained hierarchy, it is well known [2] and is usually called Zakharov reduction. Thus we have demonstrated that symmetry constraint (20) leads to Zakharov reduction (23).

Riemann invariants for the constrained hierarchy of equations of hydrodynamic type (the first of which is (21)), are given by the values of \( z(p) \) (23) at \( p_i \) where \( \frac{\partial z}{\partial p} \) vanishes, \( \lambda_i = z(p_i), \frac{\partial z}{\partial p}(p_i) = 0 \). It is rather straightforward to introduce hodograph transform for the solution of constrained hierarchy [15].

6 Generic symmetry constraint for the dKP hierarchy

Now we will use the symmetry (17) and consider the constraint
\[ F_x = \sum_{i=1}^{N} c_i (S_i - \tilde{S}_i), \] (24)
where \( S_i = S(z_i), \tilde{S}_i = S(\tilde{z}_i), z_i, \tilde{z}_i \) are some sets of points, and \( c_i \) are arbitrary constants. In equivalent form,
\[ u = \partial_x \sum_{i=1}^{N} c_i (S_i - \tilde{S}_i) = \sum_{i=1}^{N} c_i (p_i - \tilde{p}_i). \]
Evaluating Hamilton-Jacobi equation (19) and its higher counterpart
\[ S_t = p^3 + 3up - \frac{3}{2} \tilde{S}_1, \]
where \( t = t_3 \), at \( z \) equal to \( z_i, \tilde{z}_i \), we immediately get two systems of hydrodynamic type
\[
\begin{align*}
\partial_y p_k &= \partial_x \left( (p_k^2) + 2 \sum c_i(p_i - \tilde{p}_i) \right) \\
\partial_y \tilde{p}_k &= \partial_x \left( (\tilde{p}_k^2) + 2 \sum c_i(p_i - \tilde{p}_i) \right)
\end{align*}
\] (25)
and
\[
\begin{aligned}
\partial_t p_k &= \partial_x \left( (p_k^3) + 3p_k \sum_i c_i (p_i - \tilde{p}_i) + \frac{3}{2} \sum_i c_i (p_i^2 - \tilde{p}_i^2) \right), \\
\partial_t \tilde{p}_k &= \partial_x \left( (\tilde{p}_k^3) + 3\tilde{p}_k \sum_i c_i (p_i - \tilde{p}_i) + \frac{3}{2} \sum_i c_i (p_i^2 - \tilde{p}_i^2) \right).
\end{aligned}
\] (26)

These systems can be written in Hamiltonian form,
\[
\partial_t p = J \frac{\delta H_{n+1}}{\delta p}, \quad J = \begin{pmatrix} C^{-1} & 0 \\ 0 & -C^{-1} \end{pmatrix} \partial_x,
\]
where
\[
H_3 = \frac{1}{3} \int dx \left( \sum_i c_i (p_i^3 - \tilde{p}_i^3) + 3 \left( \sum_i c_i (p_i - \tilde{p}_i) \right)^2 \right),
\]
\[
H_4 = \frac{1}{4} \int dx \left( \sum_i c_i (p_i^4 - \tilde{p}_i^4) + 6 \sum_i c_i (p_i - \tilde{p}_i) \times \sum_i c_i (p_i^2 - \tilde{p}_i^2) \right),
\]

where \( p = (p_1, \ldots, p_N, \tilde{p}_1, \ldots, \tilde{p}_N)^t \), \( C \) is diagonal \( N \times N \) matrix with the entries \( c_{ik} = c_i \delta_{ik} \), \( i, k = 1, \ldots, N \).

Common solution of the systems (25), (26) generates a solution \( u = \sum_i c_i (p_i - \tilde{p}_i) \) of the dKP equation.

The Sato function \( z(p) \), which allows to construct Riemann invariants and higher Hamiltonians, is the central object for the constrained hierarchy. To find it, we, similar to the first example, use the generating Hamilton-Jacobi equation (22). Evaluating this equation at \( z' \) equal to \( z_i, \tilde{z}_i \) and combining the results, we get
\[
D(z) \sum_{i=1}^N c_i (S_i - \tilde{S}_i) = -\sum_{i=1}^N c_i \log \frac{p - p_i}{p - \tilde{p}_i},
\] (27)

Then, due to the constraint (24),
\[
D(z) F_x = -\sum_{i=1}^N c_i \log \frac{p - p_i}{p - \tilde{p}_i},
\]
and finally we obtain the function \( z(p) \),
\[
z = p - \sum_{i=1}^N c_i \log \frac{p - p_i}{p - \tilde{p}_i},
\] (28)

Equation \( \frac{\partial z}{\partial p} = 0 \) is an algebraic one, having \( 2N \) roots \( p_\alpha \), and Riemann invariants for the constrained hierarchy are \( \lambda_\alpha = z(p_\alpha) \).
Expansion of $z(p)$ at $p \to \infty$ looks like

$$z = p + \sum_{n=1}^{\infty} \frac{v_n}{p^n}; \quad v_n = \sum_{i=1}^{N} \frac{c_i (p^n_i - \tilde{p}_n^i)}{n}. \quad (29)$$

Equation (22) allows us to obtain the generating equation for the whole hierarchy of systems of hydrodynamic type associated with the constraint (24). Indeed, differentiating (22) with respect to $x$ and evaluating it at the points $z_i, \tilde{z}_i$, one gets

$$\begin{cases}
D(z)p_k = -\partial_x \log(p - p_k) \\
D(z)\tilde{p}_k = -\partial_x \log(p - \tilde{p}_k)
\end{cases} \quad k = 1, \ldots, N, \quad (30)$$

where $p$ is a function of $z$, $(p_1, \ldots, p_N, \tilde{p}_1, \ldots, \tilde{p}_N)$, defined by the relation (28). Expanding both sides of $z$, $(p_1, \ldots, p_N, \tilde{p}_1, \ldots, \tilde{p}_N)$, one gets the systems (25), (26) and their higher counterparts. Introducing to $S_0(z, t)$ dependence on extra time $t_{\log}$, $S_0(z') \to S_0(z') - t_{\log} \log(1 - \frac{z'}{z})$, $\frac{\partial}{\partial t_{\log}} = D(z)$, we may consider (30) as a system of hydrodynamic type with the time $t_{\log}$. Hamiltonian for this system is given by the expression

$$H_{\log} = \int dx \left( \frac{1}{2} p^2 - \sum_{i=1}^{N} c_i (\tilde{p}_i - p_i + p_i \log(p - p_i) - \tilde{p}_i \log(p - \tilde{p}_i)) \right).$$

Differentiating (30) by $z$, we will get another form of generating system,

$$\begin{cases}
D(z)p_k = -\partial_z \frac{p_z}{p - p_k} \\
D(z)\tilde{p}_k = -\partial_z \frac{\tilde{p}_z}{p - \tilde{p}_k}
\end{cases} \quad k = 1, \ldots, N, \quad (31)$$

where $p, p_z$ are functions of $z$, $(p_1, \ldots, p_N, \tilde{p}_1, \ldots, \tilde{p}_N)$, defined by the relation (28). This system can be considered as a system of hydrodynamic type with the time $t_{\text{pole}}$, $S_0(z') \to S_0(z') + t_{\text{pole}} (z - z')^{-1}$, $\frac{\partial}{\partial t_{\text{pole}}} = D_z(z)$. Hamiltonian for the generating system (31) is remarkably simple,

$$H_{\text{pole}} = \int p \, dx,$$

and its expansion into the powers of $z^{-1}$ gives a general formula for the Hamiltonians of constrained hierarchy,

$$H_n = \frac{1}{n} \int \text{res}_{\infty} (z(p)^n) \, dx = \partial_x \tilde{S}_n, \quad (32)$$

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which is in agreement with general results [1].

In a similar way one can treat the symmetry constraints

$$\frac{\partial F}{\partial t_n} = \sum_{i=1}^{N} c_i (S_i - \tilde{S}_i).$$  \hspace{1cm} (33)

In particular, since

$$\frac{\partial S}{\partial t_n} = p^n + u_n - 2p^{n-2} + \cdots + u_0,$n

one obtains a corresponding Sato function

$$E(p) = z^n = p^n + u_n - 2p^{n-2} + \cdots + u_0 - \sum_{i=1}^{N} c_i \log \frac{p - p_i}{p - \tilde{p}_i}. \hspace{1cm} (34)$$

**Remark.** It is easy to check that in the limit \( \tilde{z}_i = z_i + \epsilon_i \rightarrow 0, c_i \epsilon_i = \text{const} \), the Sato function (34) reproduces the curve

$$z^n = p^n + u_n - 2p^{n-2} + \cdots + u_0 + \sum_{i=1}^{N} \frac{a_i}{p - p_i}, \hspace{1cm} (35)$$

which has been discussed in [15,17]. At \( n = 1 \) one gets a general Zakharov reduction.

### 7 Symmetry constraint for the d2DTL hierarchy

Now we will consider symmetry reduction of d2DTL hierarchy defined by the constraint

$$F_i = \sum_{j=1}^{N} c_i (S_i - \tilde{S}_i).$$  \hspace{1cm} (36)

Technically this case is very similar to that considered in the previous section, and we will omit some details. The first Hamilton-Jacobi equations of d2DTL hierarchy are

$$\begin{align*}
\partial_x S(z) &= e^p - \partial_t \tilde{S}_1, \\
\partial_y S &= e^{\phi_1 - p},
\end{align*} \hspace{1cm} (37)$$

where \( x = x_1, y = y_1 \). Using the constraint (36), from these equations we get two systems of hydrodynamic type, namely

$$\begin{align*}
\partial_x p_i &= \partial_t \left( e^{p_i} + \sum_k c_k (e^{p_k} - e^{\tilde{p}_k}) \right) \\
\partial_x \tilde{p}_i &= \partial_t \left( e^{\tilde{p}_i} + \sum_k c_k (e^{p_k} - e^{\tilde{p}_k}) \right), \\
\quad i = 1, \ldots, N
\end{align*} \hspace{1cm} (39)$$
and

\[
\begin{align*}
\partial_y p_i &= \partial_t \exp(-p_i + \sum_k c_k(p_k - \tilde{p}_k)) \\
\partial_y \tilde{p}_i &= \partial_t \exp(-\tilde{p}_i + \sum_k c_k(p_k - \tilde{p}_k))
\end{align*}
\]

\[i = 1, \ldots, N\quad (40)\]

Both systems can be written in Hamiltonian form,

\[
\partial_x p = J \frac{\delta H_1^+}{\delta p}, \quad \partial_y p = J \frac{\delta H_1^-}{\delta p}, \quad J = \begin{pmatrix} E^{-1} & 0 \\ 0 & -C^{-1} \end{pmatrix} \partial_t,
\]

where \(E\) is \(2N \times 2N\) matrix with all entries equal to 1, \(E_{ik} = 1, i, k = 1, \ldots, 2N\), \(C\) is diagonal \(N \times N\) matrix with the entries \(c_i, C_{ik} = c_i \delta_{ik}\), \(i, k = 1, \ldots, N\), and the Hamiltonians are, respectively,

\[
H_1^+ = \int dt \sum_k c_k(e^{p_k} - e^{\tilde{p}_k}),
\]

\[
H_1^- = \int dt \exp\left(\sum_k c_k(p_k - \tilde{p}_k)\right) \sum_k c_k(e^{-\tilde{p}_k} - e^{-p_k}).
\]

Using generating equation (7), we obtain the function \(z(p)\) for the constrained hierarchy,

\[
z = e^p \prod_{k=1}^{N} \left(\frac{e^p - e^{p_k}}{e^p - e^{\tilde{p}_k}}\right)^{-c_k}.\]

(43)

Starting with second generating equation (8), we get equivalent form of this relation,

\[
z = \exp(p - \sum_{k=1}^{N} c_k(p_k - \tilde{p}_k)) \prod_{k=1}^{N} \left(\frac{e^{-p} - e^{-p_k}}{e^{-p} - e^{-\tilde{p}_k}}\right)^{-c_k}.
\]

(44)

Equation \(\frac{\partial z}{\partial p} = 0\) is algebraic (with respect to the variable \(\xi = e^p\)), having \(2N\) roots \(\xi_\alpha\), and Riemann invariants for the constrained hierarchy are \(\lambda_\alpha = z(\xi_\alpha)\).

Generating systems for the constrained hierarchy read

\[
\begin{align*}
D_+(z)p_k &= -\partial_t \log(e^p - e^{p_k}) \\
D_+(z)\tilde{p}_k &= -\partial_t \log(e^p - e^{\tilde{p}_k})
\end{align*}
\]

\[k = 1, \ldots, N,\quad (45)\]

\[
\begin{align*}
D_-(z)p_k &= -\partial_t \log(1 - e^p - p_k) \\
D_-(z)\tilde{p}_k &= -\partial_t \log(1 - e^p - \tilde{p}_k)
\end{align*}
\]

\[k = 1, \ldots, N,\quad (46)\]
(‘logarithmic’ form) and
\[
\begin{aligned}
D_{z^\pm}(z)p_k &= \partial_t \frac{p_z}{e^{p_z} - p - 1} \quad k = 1, \ldots, N, \\
D_{\bar{z}^\pm}(z)\bar{p}_k &= \partial_t \frac{p_z}{e^{p_z} - p - 1}
\end{aligned}
\]

(‘pole’ form). Hamiltonian for the ‘pole’ form of generating equations is
\[
H_\pm(z) = \frac{1}{z} \int p \, dt.
\]
Expansion of this Hamiltonian at \(z = \infty\) and \(z = 0\) gives general expressions for the Hamiltonians of constrained hierarchy,
\[
\begin{aligned}
H^+_n &= \frac{1}{n} \int \text{res}_\infty (z(\xi)^n \xi^{-1}) \, dt, \\
H^-_n &= \frac{1}{n} \int \text{res}_0 (z(\xi)^{-n} \xi^{-1}) \, dt,
\end{aligned}
\]
where \(\xi = e^p\).

In more detail, the Hamiltonian and bi-Hamiltonian structure of equation considered above will be considered elsewhere.

Symmetry constraints of the type \(F_{x_n} = \sum_{i=1}^N c_i (S_i - \tilde{S}_i)\) and \(F_{y_n} = \sum_{i=1}^N c_i (\tilde{S}_i - S_i)\) are treated analogously. The corresponding Sato function is given by (34) with substitutions \(p \to e^p, \bar{p} \to e^{-\bar{p}}\), respectively.

8 Conclusion

As we have seen, the systems of hydrodynamic type and associated complex curves (Sato functions) are direct consequences of the generating Hamilton-Jacobi equations (4), (7,8) under the symmetry constraint.

We would like to note here that complex curves discussed above are connected with the \(\bar{\partial}\)-equation in a very natural way. Let us consider the dKP hierarchy as an example. The function \(p = \partial_z S\) solves a linear Beltrami equation \(p_z = W'p_z\), and under certain conditions is a basic homeomorphism. The inverse function \(z(p, \bar{p})\) is analytic in some neighborhood of infinity. The Cauchy-Green formula implies that
\[
z = p + \frac{1}{2\pi i} \int \int \frac{dp' \wedge d\bar{p}'}{p' - p} \frac{\partial z}{\partial \bar{p}'}.
\]
In the neighborhood of infinity one has
\[
z = p + \sum_{n=1}^{\infty} \frac{v_n}{p^n},
\]
where
\[ v_n = -\frac{1}{2\pi i} \int \int p' \frac{n}{\partial} \partial \bar{p}' dp' \wedge d\bar{p}'. \] (52)

Let us choose a special \( \frac{\partial z}{\partial \bar{p}} \) with a support along the curve \( \Gamma = \bigcup_i \Gamma_i \) composed of \( N \) disconnected pieces \( \Gamma_i \) with the ends at the points \( z_i \) and \( \tilde{z}_i \) and such that \( \frac{\partial z}{\partial \bar{p}} = -\sum_{i=1}^N c_i \delta_{\Gamma_i} \). For such ‘constrained’ \( \bar{\partial} \)-data formula (50) gives
\[ z = p - \sum_{i=1}^N c_i \log \frac{p - p_i}{p - \tilde{p}_i}, \]
while from (52) we obtain
\[ v_n = \sum_{i=1}^N c_i \left( p^n_i - \tilde{p}_i^n \right), \]
that coincides with (28, 29). For the \( \bar{\partial} \)-data given by \( \frac{\partial z}{\partial \bar{p}} = -\sum_{i=1}^N a_i \delta(p - p_i) \), equation (50) immediately leads to the Sato function associated with Zakharov reduction (see (35)). Thus, the Cauchy-Green formula is a generator of complex curves associated with equations of hydrodynamic type.

Equation (50) is the two-dimensional extension of the integral equation discussed in [2], [3].

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