A SURVEY ON THE ATOMICITY OF PUISEUX MONOIDS

SCOTT T. CHAPMAN, FELIX GOTTI, AND MARLY GOTTI

Abstract. A Puiseux monoid is an additive submonoid of the nonnegative cone of \( \mathbb{Q} \). Puiseux monoids exhibit, in general, a complex atomic structure. For instance, although various techniques have been developed in the past few years to identify subclasses of atomic Puiseux monoids, no characterization of atomic Puiseux monoids has been found so far. Here we survey some of the most relevant aspects of the atomicity of Puiseux monoids. The properties of being finitely generated, factorial, half-factorial, other-half-factorial, Prüfer, Krull, seminormal, root-closed, and completely integrally closed can be all characterized in the context of Puiseux monoids, and we will provide such characterizations. In the same context, on the other hand, there are not known characterizations for the properties of being atomic, satisfying the ACCP, being a BF-monoid, or being an FF-monoid. However, we shall construct classes of Puiseux monoids satisfying each of the latter properties.

Contents

1. Introduction 2
2. Preliminary 3
2.1. General Notation 3
2.2. Commutative Monoids 4
2.3. Factorizations 4
3. Closures and Conductor 5
3.1. The Grothendieck Group 6
3.2. Normal, Root, and Complete Integral Closures 6
3.3. Description of the Conductor 8
4. Homomorphisms 9
4.1. Homomorphisms Between Puiseux Monoids 9
4.2. Transfer Krull Characterization 10
4.3. Transfer Finite and C-monoid Characterizations 13
5. Atomic Structure 15
5.1. A Class of Atomic Puiseux Monoids 16
5.2. A Class of ACCP Puiseux Monoids 17

Date: August 27, 2019.

2010 Mathematics Subject Classification. Primary: 20M13; Secondary: 06F05, 20M14.

Key words and phrases. Puiseux monoids, atomicity, factorization theory, Krull monoids, transfer Krull monoids, C-monoids, Prüfer monoids, numerical monoids, ACCP, BF-monoids, FF-monoids.
5.3. Bounded Factorization Monoids
5.4. Finite Factorization Monoids
5.5. Factorial, Half-Factorial, and Other-Half-Factorial Monoids
References

1. Introduction

An integral domain is said to be atomic provided that every non-invertible element can be written as a product of irreducibles (also called atoms). Clearly, the property of being atomic is a natural relaxation of that one of being a UFD. Many relevant classes of integral domains consist of members that are atomic but not UFDs, including those of Dedekind/Krull domains (in particular, rings of algebraic integers), Noetherian domains, and primary Mori domains. For some of these families, the atomic structure of their members determines, up to certain extent, some of their algebraic properties. For instance, it was proved by Carlitz [7] that a ring of algebraic integers is half-factorial if and only if the size of its class group is at most 2. As a result, the atomicity structure of many classes of integral domains has been a focus of significant investigation through the last decades (see [26] and references therein).

It turns out that many problems involving factorizations of elements in an integral domain depend solely on its multiplicative structure. As a result, a significant part of the literature of the last forty years treating the phenomenon of non-unique factorizations takes place on the context of monoids. In particular, [5] and [15] first treated factorization problems in numerical monoids (i.e., additive submonoids of \( \mathbb{N}_0 \)). Since then many papers have been dedicated to the atomicity and factorization of numerical monoids [1, 8, 14, 30] and some of their higher-rank generalizations [16, 18, 20, 22]. More recently, a systematic study of the atomicity of Puiseux monoids (i.e., additive submonoids of \( \mathbb{Q}_{\geq 0} \)) has been initiated (see [12, 39] and references therein).

Although the atomicity of Puiseux monoids has earned attention only in the last few years, since the 1970s Puiseux monoids have been crucial in the construction of numerous examples in commutative ring theory. Back in 1974, A. Grams [44] used an atomic Puiseux monoid as the main ingredient to construct the first example of an atomic integral domain that does not satisfy the ACCP, and thus she refuted P. Cohn’s assumption that every atomic integral domain satisfies the ACCP. In addition, in [2], A. Anderson et al. appealed to Puiseux monoids to construct various needed examples of integral domains satisfying certain prescribed properties. More recently, Puiseux monoids have played an important role in [17], where J. Coykendall and the second author partially answer a question on the atomicity of monoid rings posed by R. Gilmer back in the 1980s (see [32, page 189]).
Puiseux monoids have also been important in factorization theory. For instance, the first (and only) examples known of primary atomic monoids with irrational elasticity are Puiseux monoids, which can be easily obtained via [42, Theorem 3.2]. A Puiseux monoid is a suitable additive structure containing simultaneously several copies of numerical monoids independently generated. This fact has been harnessed by A. Geroldinger and W. Schmid to achieve a nice realization theorem for the sets of lengths of numerical monoids [30]. Given the relevance of the class of Puiseux monoids in both commutative algebra and factorization theory as well as the substantial attention they have received in the last few years, we have decided to compile and organize some of the most significant developments on their atomicity. This is the main purpose of the present survey paper.

For the convenience and curiosity of the interested reader, we would like to highlight some connections and references related to previous studies of Puiseux monoids that we do not present in this survey. The elasticity of Puiseux monoids have been studied in [42, 43] while their systems of sets of lengths have received some attention in [39]. In addition, general factorization invariants of Puiseux monoids and numerical monoids have been compared and contrasted in [12]. The monoid algebras of Puiseux monoids have been considered in [3, 17, 34]. Finally, some connections between Puiseux monoids and music theory have been recently highlighted by M. Bras-Amoros in the Monthly article [6].

2. Preliminary

In this section we introduce most of the relevant concepts on commutative monoids and factorization theory required to follow our exposition. General references for background information can be found in [45] for commutative monoids and in [26] for atomic monoids and factorization theory.

2.1. General Notation. We let \( \mathbb{N} := \{1, 2, \ldots\} \) denote the set of positive integers and set \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). In addition, we let \( \mathbb{P} \) denote the set of all prime numbers. For \( a, b \in \mathbb{Z} \) we let \( [a, b] \) denote the set of integers between \( a \) and \( b \), i.e.,

\[
[a, b] := \{ z \in \mathbb{Z} \mid a \leq z \leq b \}.
\]

In addition, for \( X \subseteq \mathbb{R} \) and \( r \in \mathbb{R} \), we set

\[
X_{\geq r} := \{ x \in X \mid x \geq r \}
\]

and we use the notations \( X_{> r}, X_{\leq r}, \) and \( X_{< r} \) in a similar way. If \( q \in \mathbb{Q}_{> 0} \), then we call the unique \( n, d \in \mathbb{N} \) such that \( q = n/d \) and \( \gcd(n, d) = 1 \) the numerator and denominator of \( q \) and denote them by \( n(q) \) and \( d(q) \), respectively. Finally, for \( Q \subseteq \mathbb{Q}_{> 0} \), we set

\[
n(Q) := \{ n(q) \mid q \in Q \} \quad \text{and} \quad d(Q) := \{ d(q) \mid q \in Q \}.
\]
2.2. Commutative Monoids. Throughout this survey, the term monoid stands for a commutative and cancellative semigroup with identity. Unless we specify otherwise, monoids are written additively. Let $M$ be a monoid. We let $M^\bullet$ denote the set of nonzero elements of $M$ while we let $U(M)$ denote the set of invertible elements of $M$. When $M^\bullet = \emptyset$ we say that $M$ is trivial and when $U(M) = \{0\}$ we say that $M$ is reduced.

For $S \subseteq M$, we let $\langle S \rangle$ denote the smallest (under inclusion) submonoid of $M$ containing $S$, i.e., the submonoid of $M$ generated by $S$. The monoid $M$ is finitely generated if $M$ can be generated by a finite set. An element $a \in M \setminus U(M)$ is an atom provided that the equality $a = x + y$ for $x, y \in M$ implies that either $x \in U(M)$ or $y \in U(M)$. The set of atoms of $M$ is denoted by $\mathcal{A}(M)$. The monoid $M$ is atomic if $M = \langle \mathcal{A}(M) \rangle$ and antimatter if $\mathcal{A}(M) = \emptyset$. Every finitely generated monoid is atomic [26, Proposition 2.7.8(4)].

A subset $I$ of $M$ is an ideal of $M$ if $I + M \subseteq I$. An ideal $I$ is principal if $I = x + M$ for some $x \in M$, and $M$ satisfies the ascending chain condition on principal ideals (or ACCP) provided that every increasing sequence of principal ideals of $M$ eventually stabilizes. It is well known that every monoid satisfying the ACCP must be atomic [26, Proposition 1.1.4].

An equivalence relation $\rho \subseteq M \times M$ is a congruence if it is compatible with the operation of the monoid $M$, i.e., for all $x, y, z \in M$ with $(x, y) \in \rho$ it follows that $(zx, zy) \in \rho$. It can be readily verified that the set $M/\rho$ consisting of the equivalence classes of a congruence $\rho$ is a commutative semigroup with identity. For $x, y \in M$, we say that $x$ divides $y$ in $M$ and write $x|_M y$ provided that $x + x' = y$ for some $x' \in M$. Two elements $x, y \in M$ are associates if $y = ux$ for some $u \in M^\times$. Being associates defines a congruence on $M$ whose semigroup of classes is a monoid, which we denote by $M_{\text{red}}$. Observe that $M$ is reduced if and only if $M_{\text{red}} = M$.

The Grothendieck group $\mathbf{gp}(M)$ of $M$ is the abelian group (unique up to isomorphism) satisfying that any abelian group containing a homomorphic image of $M$ also contains a homomorphic image of $\mathbf{gp}(M)$. The rank of a monoid $M$ is the rank of the $\mathbb{Z}$-module $\mathbf{gp}(M)$ or, equivalently, the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{gp}(M)$.

A numerical monoid is a submonoid $N$ of $(\mathbb{N}_0, +)$ satisfying that $|\mathbb{N}_0 \setminus N| < \infty$. If $N \neq \mathbb{N}_0$, then $\max(\mathbb{N}_0 \setminus N)$ is the Frobenius number of $N$. Numerical monoids are finitely generated and, therefore, atomic with finitely many atoms. The embedding dimension of $N$ is the cardinality of $\mathcal{A}(N)$. For an introduction to numerical monoids, see [23], and for some of their many applications, see [4].

2.3. Factorizations. A multiplicative monoid $F$ is called free abelian with basis $P$ if every element $x \in F$ can be written uniquely in the form

$$x = \prod_{p \in P} p^{\nu_p(x)},$$

where \( v_p(x) \in \mathbb{N}_0 \) and \( v_p(x) > 0 \) only for finitely many elements \( p \in P \). The monoid \( F \) is determined by \( P \) up to isomorphism, so we will sometimes denote \( F \) by \( \mathcal{F}(P) \). By the fundamental theorem of arithmetic, the multiplicative monoid \( \mathbb{N} \) is free on the set of prime numbers. In this case, we can extend \( v_p \) to \( \mathbb{Q}_{\geq 0} \) as follows. For \( r \in \mathbb{Q}_{>0} \) let \( v_p(r) := v_p(n(r)) - v_p(d(r)) \) and set \( v_p(0) = \infty \).

Let \( M \) be a reduced monoid. The factorization monoid of \( M \), denoted by \( Z(M) \), is the free commutative monoid on \( A(M) \). The elements of \( Z(M) \) are called factorizations. If \( z = a_1 \ldots a_n \in Z(M) \), where \( a_1, \ldots, a_n \in A(M) \), then \( |z| := n \) is the length of \( z \). The unique monoid homomorphism \( \pi: Z(M) \to M \) satisfying that \( \pi(a) = a \) for all \( a \in A(M) \) is the factorization homomorphism of \( M \). For each \( x \in M \),

\[
Z(x) := \pi^{-1}(x) \subseteq Z(M) \quad \text{and} \quad L(x) := \{|z| : z \in Z(x)\}
\]

are the set of factorizations and the set of lengths of \( x \), respectively. Factorization invariants stemming from the sets of lengths have been studied for several classes of atomic monoids and domains; see, for instance, \([10, 11, 13, 15]\). In particular, the sets of lengths of numerical monoids have been studied in \([1, 9, 30]\). In \([30]\) the sets of lengths of numerical monoids were studied using techniques involving Puiseux monoids. An overview of sets of lengths and the role they play in factorization theory can be found in the Monthly article \([24]\).

By restricting the size of the sets of factorizations/lengths, one obtains subclasses of atomic monoids that have been systematically studied by many authors. We say that a reduced atomic monoid \( M \) is

1. a UFM (or a factorial monoid) if \( |Z(x)| = 1 \) for all \( x \in M \),
2. an HFM (or a half-factorial monoid) if \( |L(x)| = 1 \) for all \( x \in M \),
3. an FFM (or a finite-factorization monoid) if \( |Z(x)| < \infty \) for all \( x \in M \), and
4. a BFM (or a bounded-factorization monoid) if \( |L(x)| < \infty \) for all \( x \in M \).

3. Closures and Conductor

In this section we study some algebraic aspects of Puiseux monoids.

**Definition 3.1.** A Puiseux monoid is an additive submonoid of \( \mathbb{Q}_{\geq 0} \).

Note that Puiseux monoids are natural generalizations of numerical monoids. As numerical monoids, it is clear that Puiseux monoids are reduced. However, as we shall see later, Puiseux monoids are not, in general, finitely generated or atomic.
3.1. The Grothendieck Group. A monoid $M$ is torsion-free if for all $x, y \in M$ and $n \in \mathbb{N}$, the equality $nx = ny$ implies that $x = y$. Clearly, $M$ is a torsion-free monoid if and only if $\text{gp}(M)$ is a torsion-free group. Each Puiseux monoid $M$ is obviously torsion-free and, therefore, $\text{gp}(M)$ is a torsion-free group. Moreover, for a Puiseux monoid $M$, one can take the Grothendieck group $\text{gp}(M)$ to be a subgroup of $\mathbb{Q}$, specifically,

$$\text{gp}(M) = \{x - y \mid x, y \in M\}.$$  

Puiseux monoids can be characterized as follows.

Proposition 3.2. For a nontrivial monoid $M$ the following statements are equivalent.

1. $M$ is a rank-1 torsion-free monoid that is not a group.
2. $M$ is isomorphic to a Puiseux monoid.

Proof. To argue (1) $\Rightarrow$ (2), first note that $\text{gp}(M)$ is a rank-1 torsion-free abelian group. Therefore it follows from [21, Section 85] that $\text{gp}(M)$ is isomorphic to a subgroup of $(\mathbb{Q}, +)$, and one can assume that $M$ is a submonoid of $(\mathbb{Q}, +)$. Since $M$ is not a group, [32, Theorem 2.9] ensures that either $M \subseteq \mathbb{Q}_{\leq 0}$ or $M \subseteq \mathbb{Q}_{\geq 0}$. So $M$ is isomorphic to a Puiseux monoid. To verify (2) $\Rightarrow$ (1), let us assume that $M \subseteq \text{gp}(M) \subseteq \mathbb{Q}$. As $\text{gp}(M)$ is a subgroup of $(\mathbb{Q}, +)$, it is a rank-1 torsion-free abelian group. This implies that $M$ is a rank-1 torsion-free monoid. Since $M$ is nontrivial and reduced, it cannot be a group, which completes our proof.  

Puiseux monoids are abundant, as the next proposition illustrates.

Proposition 3.3. There are uncountably many non-isomorphic Puiseux monoids.

Proof. Consider the assignment $G \mapsto M_G := G \cap \mathbb{Q}_{\geq 0}$ sending each subgroup $G$ of $(\mathbb{Q}, +)$ to a Puiseux monoid. Clearly, $\text{gp}(M_G) \cong G$. In addition, for all subgroups $G$ and $G'$ of $(\mathbb{Q}, +)$, each monoid isomorphism between $M_G$ and $M_{G'}$ naturally extends to a group isomorphism between $G$ and $G'$. Hence our assignment sends non-isomorphic groups to non-isomorphic monoids. It follows from [21, Corollary 85.2] that there are uncountably many non-isomorphic rank-1 torsion-free abelian groups. As a result, there are uncountably many non-isomorphic Puiseux monoids.  

3.2. Normal, Root, and Complete Integral Closures. Given a monoid $M$ with Grothendieck group $\text{gp}(M)$, the sets

- $M' := \{x \in \text{gp}(M) \mid \text{there exists } N \in \mathbb{N} \text{ such that } nx \in M \text{ for all } n \geq N\}$,
- $\widetilde{M} := \{x \in \text{gp}(M) \mid nx \in M \text{ for some } n \in \mathbb{N}\}$, and
- $\hat{M} := \{x \in \text{gp}(M) \mid \text{there exists } c \in M \text{ such that } c + nx \in M \text{ for all } n \in \mathbb{N}\}$

are called the seminormal closure, root closure, and complete integral closure of $M$, respectively. It is not hard to verify that $M \subseteq M' \subseteq \widetilde{M} \subseteq \hat{M} \subseteq \text{gp}(M)$ for any monoid $M$. For Puiseux monoids, however, the three closures coincide.
Proposition 3.4. Let $M$ be a Puiseux monoid, and let $n = \gcd(n(M^*))$. Then
\begin{equation}
(3.1) \quad M' = \overline{M} = \tilde{M} = \gcd(M) \cap \mathbb{Q}_{\geq 0} = n\langle 1/d \mid d \in d(M^*) \rangle.
\end{equation}

Proof. The equality $M' = \tilde{M}$ follows from [29, Lemma 2.5.1]. In addition, it is easy to verify that $M^* = \gcd(M) \cap \mathbb{Q}_{\geq 0}$. Thus, $M' \subseteq \overline{M} \subseteq \gcd(M) \cap \mathbb{Q}_{\geq 0}$. On the other hand, if $x \in \gcd(M) \cap \mathbb{Q}_{\geq 0}$, then it readily follows that $nd(x)x \in M$ and, therefore, $\gcd(M) \cap \mathbb{Q}_{\geq 0} \subseteq M$. As a result, the first three equalities in (3.1) hold. So it suffices to argue that $M = n\langle 1/d \mid d \in d(M^*) \rangle$.

For $d \in d(M^*)$, take $k \in \mathbb{N}$ such that $k/d \in M$. As $\gcd(k, nd) = n$, there exist $\alpha, \beta \in \mathbb{N}_0$ satisfying that $n = \alpha k - \beta nd$, which implies that $n/d = \alpha(k/d) - \beta n \in \gcd(M)$. On the other hand, since $n \mid k$ and $(k/n)^{\alpha} = k/d \in M$, one finds that $n/d \in \overline{M}$. Thus, $n\langle 1/d \mid d \in d(M^*) \rangle \subseteq \overline{M}$. For the reverse inclusion we first verify that $d(M^*)$ is closed under taking positive divisors and least common multiples. Clearly, $d(M^*)$ is closed under taking positive divisors. To check that $d(M^*)$ is closed under taking least common multiples, take $q_1, q_2 \in M^*$ and then set $d = \gcd(d(q_1), d(q_2))$ and $\ell = \text{lcm}(d(q_1), d(q_2))$. As $\gcd(n(q_1), n(q_2))$ is the greatest common divisor of $n(q_1)d(q_2)/d$ and $n(q_2)d(q_1)/d$, there exist $N, c_1, c_2 \in \mathbb{N}_0$ such that
\[
\frac{(N\ell + 1)\gcd(n(q_1), n(q_2))}{\ell} = \frac{1}{\ell} \left( c_1 \frac{n(q_1)d(q_2)}{d} + c_2 \frac{n(q_2)d(q_1)}{d} \right) = c_1q_1 + c_2q_2 \in M.
\]

Since $\ell$ and $(N\ell + 1)\gcd(n(q_1), n(q_2))$ are relatively prime, $\ell \in d(M^*)$. Thus, $d(M^*)$ is closed under taking least common multiples, as desired.

One can easily see that $M \subseteq n\langle 1/d \mid d \in d(M^*) \rangle$. Then take $q \in \overline{M} \setminus M \subseteq \gcd(M)$, and then take $q_1, q_2 \in M^*$ such that $q = q_2 - q_1$. Clearly, $d(q)$ divides $\text{lcm}(d(q_1), d(q_2))$. As $d(M^*)$ is closed under taking positive divisors and least common multiples, one finds that $d(q) \in d(M^*)$. Hence $q = \left( \frac{n(q)}{n} \right) \frac{n}{d(q)} \in n\langle 1/d \mid d \in d(M^*) \rangle$, which implies the reverse inclusion.

A monoid $M$ is said to be root-closed provided that $\overline{M} = M$. In addition, $M$ is called a \textit{Prüfer monoid} if $M$ is the union of an ascending sequence of cyclic submonoids.

Corollary 3.5. For a Puiseux monoid $M$, the following statements are equivalent.

(1) $M$ is root-closed.
(2) $M = n\langle 1/d \mid d \in d(M^*) \rangle$, where $n = \gcd(n(M^*))$.
(3) \text{gp}(M) = M \cup -M.
(4) $M$ is a \textit{Prüfer monoid}.

Proof. The equivalences (1) $\iff$ (2) $\iff$ (3) follow from Proposition 3.4, while the equivalence (1) $\iff$ (4) follows from [32, Theorem 13.5].

We now characterize finitely generated Puiseux monoids in terms of its root closures.
Proposition 3.6. For a Puiseux monoid $M$ the following statements are equivalent.

1. $\widehat{M} \cong (\mathbb{N}_0, +)$.
2. $M$ is finitely generated.
3. $d(M^\bullet)$ is finite.
4. $M$ is isomorphic to a numerical monoid.

Proof. To prove (1) $\Rightarrow$ (2), suppose that $\widehat{M} \cong (\mathbb{N}_0, +)$. Proposition 3.4 ensures that $d(M^\bullet)$ is finite. Now if $\ell := \text{lcm} d(M^\bullet)$, then $\ell M$ is submonoid of $(\mathbb{N}_0, +)$ that is isomorphic to $M$. Hence $M$ is finitely generated. To argue (2) $\Rightarrow$ (3), it suffices to notice that if $S$ is a finite generating set of $M$, then every element of $d(M^\bullet)$ divides $\text{lcm} d(S^\bullet)$. For (3) $\Rightarrow$ (4), let $\ell := \text{lcm} d(M^\bullet)$. Then note that $\ell M$ is a submonoid of $(\mathbb{N}_0, +)$ that is isomorphic to $M$. As a result, $M$ is isomorphic to a numerical monoid. To prove (4) $\Rightarrow$ (1), assume that $M$ is a numerical monoid and that $\text{gp}(M)$ is a subgroup of $(\mathbb{Z}, +)$. By definition of $\widehat{M}$, it follows that $\widehat{M} \subseteq \mathbb{N}_0$. On the other hand, the fact that $\mathbb{N}_0 \setminus M$ is finite immediately implies that $\mathbb{N}_0 \subseteq \widehat{M}$. Thus, $\widehat{M} = (\mathbb{N}_0, +)$.

Corollary 3.7. A monoid $M$ is not finitely generated if and only if $\widehat{M}$ is antimatter.

Proof. Suppose first that $M$ is not finitely generated. Set $n = \gcd(n(M^\bullet))$. It follows from Proposition 3.4 that $\widehat{M} = \langle n/d \mid d \in d(M^\bullet) \rangle$. Fix $d \in d(M^\bullet)$. Since $d(M^\bullet)$ is an infinite set that is closed under taking least common multiples, there exists $d' \in d(M^\bullet)$ such that $d'$ properly divides $d$. As a consequence, $n/d'$ properly divides $n/d$ in $\widehat{M}$ and so $n/d' \notin A(\widehat{M})$. As none of the elements in the generating set $\{n/d \mid d \in d(M^\bullet)\}$ of $\widehat{M}$ is an atom, $\widehat{M}$ must be antimatter. The reverse implication is an immediate consequence of Proposition 3.6.

3.3. Description of the Conductor. Let $M$ be a monoid. The conductor of $M$, denoted by $(M : \widehat{M})$, is defined to be

$$(M : \widehat{M}) = \{ x \in \text{gp}(M) \mid x + \widehat{M} \subseteq M \}.$$ 

By Proposition 3.4, the conductor of a Puiseux monoid $M$ can be written as $(M : \widehat{M})$, which is more convenient for our purposes. The conductor of a Puiseux monoid was first considered in [25], where the following result was established.

Proposition 3.8. Let $M$ be a Puiseux monoid. Then the following statements hold.

1. If $M$ is root-closed, then $(M : \widehat{M}) = \widehat{M} = M$.
2. If $M$ is not root-closed and $\sigma = \sup \widehat{M} \setminus M$.
   (a) If $\sigma = \infty$, then $(M : \widehat{M}) = \emptyset$.
   (b) If $\sigma < \infty$, then $(M : \widehat{M}) = M_{\geq \sigma}$. 
Proof. To verify (1), note that $\widetilde{M} \subseteq \mathbb{Q}_{\geq 0}$ implies that $(M : \widetilde{M}) \subseteq \mathbb{Q}_{\geq 0}$. As a result, $(M : \widetilde{M}) \subseteq \text{gp}(M) \cap \mathbb{Q}_{\geq 0} = \widetilde{M}$. This, along with the obvious fact that $\widetilde{M} \subseteq (M : \widetilde{M})$, implies (1).

To show (2), suppose that $M$ is not root-closed. It follows from Proposition 3.4 that $(M : \widetilde{M}) \subseteq \text{gp}(M) \cap \mathbb{Q}_{\geq 0} = \widetilde{M}$.

Case (a): Take $\tilde{x} \in \widetilde{M}$. Since $\widetilde{M} \setminus M$ is unbounded, there exists $\tilde{x}_1 \in \widetilde{M} \setminus M$ such that $\tilde{x}_1 > \tilde{x}$. Then taking $\tilde{y} := \tilde{x}_1 - \tilde{x} \in \widetilde{M}$, we can see that $\tilde{x} + \tilde{y} = \tilde{x}_1 \notin M$. Therefore $\tilde{x} \notin (M : \widetilde{M})$. So we conclude that $(M : M) = \emptyset$.

Case (b): As in the above paragraph, we can argue that no element in $\widetilde{M}_{<\sigma}$ is in $(M : \widetilde{M})$ and then $(M : \widetilde{M}) \subseteq M_{\geq \sigma}$. For the reverse inclusion, take $x \in M_{\geq \sigma}$. If $\sigma \notin M$, then $x > \sigma$ and so $x + \widetilde{M} \subseteq (\widetilde{M} + \widetilde{M})_{>\sigma} \subseteq \widetilde{M}_{>\sigma} = M_{>\sigma} \subseteq M$. Thus, $x \in (M : \widetilde{M})$ when $\sigma \notin M$. If $\sigma \in M$, then $M_{>\sigma} = M_{\geq \sigma}$ and so $x + \widetilde{M} \subseteq (\widetilde{M} + \widetilde{M})_{>\sigma} \subseteq M_{>\sigma} \subseteq M_{\geq \sigma} \subseteq M$. Therefore $x \in (M : \widetilde{M})$ also when $\sigma \in M$. Hence $M_{>\sigma} \subseteq (M : \widetilde{M})$, as desired. \qed

Remark 3.9. With notation as in Proposition 3.8.2, although $\widetilde{M}_{>\sigma} = M_{>\sigma}$ holds, it can happen that $\widetilde{M}_{>\sigma} \neq M_{>\sigma}$. For instance, consider the Puiseux monoid $\{0\} \cup \mathbb{Q}_{>1}$.

4. Homomorphisms

4.1. Homomorphisms Between Puiseux Monoids. As we are about to show, homomorphisms between Puiseux monoids are given by rational multiplication, and one can use this to characterize Puiseux monoids that are transfer Krull as well as Puiseux monoids that are finitely generated.

Proposition 4.1. The homomorphisms between Puiseux monoids are given by rational multiplication.

Proof. Every rational-multiplication map is clearly a homomorphism. Suppose, on the other hand, that $\varphi: M \to M'$ is a homomorphism between Puiseux monoids. As the trivial homomorphism is multiplication by 0, one can assume without loss of generality that $\varphi$ is nontrivial. Let $\{n_1, \ldots, n_k\}$ be the minimal generating set of the additive monoid $N := M \cap \mathbb{N}_0$. Since $\varphi$ is nontrivial, $k \geq 1$ and $\varphi(n_j) \neq 0$ for some $j \in [1, k]$. Set $q = \varphi(n_j)/n_j$ and then take $r \in M^\bullet$ and $c_1, \ldots, c_k \in \mathbb{N}_0$ satisfying that $n(r) = c_1n_1 + \cdots + c_kn_k$. As $n_i\varphi(n_j) = \varphi(n_in_j) = n_j\varphi(n_i)$ for every $i \in [1, k]$, $\varphi(r) = \frac{1}{d(r)}\varphi(n(r)) = \frac{1}{d(r)}\sum_{i=1}^k c_i\varphi(n_i) = \frac{1}{d(r)}\sum_{i=1}^k c_i n_i \frac{\varphi(n_j)}{n_j} = rq.$

Thus, the homomorphism $\varphi$ is multiplication by $q \in \mathbb{Q}_{>0}$. \qed

As the set of homomorphisms between Puiseux monoids is very exclusive, one might wonder whether for every atomic Puiseux monoid $M$ the group of automorphisms $\text{Aut}(M)$ of $M$ is trivial. The answer is negative, as indicated by the next proposition.
Proposition 4.2. Let \( r \in \mathbb{Q}_{>0} \) such that \( n(r), d(r) > 1 \). If \( M = \langle r^k \mid k \in \mathbb{Z} \rangle \), then \( \text{Aut}(M) \cong \mathbb{Z} \).

Proof. Set \( A = \{ r^k \mid k \in \mathbb{Z} \} \). For \( n \in \mathbb{Z} \), the fact that \( r^n A = A \) implies that multiplication by \( r^n \) is an endomorphism of \( M \) whose inverse is given by multiplication by \( r^{-n} \). Thus, multiplication by any integer power of \( r \) is an automorphism of \( M \). To prove that these are the only elements of \( \text{Aut}(M) \), let us first argue that \( M \) is atomic with \( \mathcal{A}(M) = A \).

Assume first that \( r < 1 \), and fix \( k \in \mathbb{Z} \). To show that \( r^k \in \mathcal{A}(M) \), notice that the monoid \( \langle r^n \mid n \geq k \rangle \) is the isomorphic image (under multiplication by \( r^k \)) of the Puiseux monoid \( S_r := \langle r^n \mid n \in \mathbb{N}_0 \rangle \), which is atomic with set of atoms \( A = \{ r^n \mid n \in \mathbb{N}_0 \} \) (see Proposition 5.4). Since \( r \in \mathcal{A}(S_r) \), it follows that \( r \notin \langle r^n \mid n > 1 \rangle \). Then \( r^k \notin \langle r^n \mid n > k \rangle \). As \( r < 1 \), no atom in \( \{ r^n \mid n < k \} \) divides \( r^k \). Hence \( r^k \notin \langle A \setminus \{ r^k \} \rangle \) and, therefore, \( r^k \in \mathcal{A}(M) \). As a result, \( \mathcal{A}(M) = A \).

Now assume that \( r > 1 \) and, as before, fix \( k \in \mathbb{Z} \). As \( r > 1 \), proving that \( r^k \in \mathcal{A}(M) \) amounts to showing that \( r^k \notin \langle r^n \mid n < k \rangle \). Suppose, by way of contradiction, that \( r^k = c_1 r^{n_1} + \cdots + c_t r^{n_t} \) for some \( c_1, \ldots, c_t \in \mathbb{N} \) and \( n_1, \ldots, n_t \in \mathbb{N} \) such that \( k > n_1 > \cdots > n_t \). Then \( r^{k-n_1+1} \in \{ r^{n_1-n_t+1}, r^{n_2-n_t+1}, \ldots, r \} \), which contradicts that \( r^{k-n_1+1} \in \mathcal{A}(S_r) \). Hence \( \mathcal{A}(M) = A \).

By Proposition 4.1, any automorphism of \( M \) is given by rational multiplication. Take \( s \in \mathbb{Q}_{>0} \) such that \( \phi_s \in \text{Aut}(M) \), where \( \phi_s \) consists in multiplication by \( s \). Because \( \phi_s \) must send atoms to atoms, \( sr = \phi_s(r) \in A \). Therefore \( s \) must be an integer power of \( r \). Hence \( \text{Aut}(M) \) is precisely \( A \) when seen as a multiplicative subgroup of \( \mathbb{Q} \). As \( A \cong \mathbb{Z} \), the proof follows.

4.2. Transfer Krull Characterization. The concept of a transfer homomorphism will play a central role in this section.

Definition 4.3. A monoid homomorphism \( \theta : M \to N \) is said to be a transfer homomorphism if the following conditions hold:

\((T1)\) \( N = \theta(M) N^x \) and \( \theta^{-1}(N^x) = M^x \);
\((T2)\) if \( \theta(x) = y_1y_2 \) for \( x \in M \) and \( y_1, y_2 \in N \), then there exist \( x_1, x_2 \in M \) such that \( x = x_1x_2 \) and \( \theta(x_i) = y_i \) for \( i \in \{ 1, 2 \} \).

With notation as in Definition 4.3, when \( M \) and \( N \) are reduced, we can restate the first condition above as

\((T1')\) \( \theta \) is surjective and \( \theta^{-1}(1) = 1 \).

A transfer homomorphism allows us to pull the atomic structure and the arithmetic of lengths of factorizations from its codomain to its domain.

Proposition 4.4. [26, Proposition 1.3.2] If \( \theta : M \to N \) is a transfer homomorphism of atomic monoids, then the following conditions hold:
(1) \( a \in \mathcal{A}(M) \) if and only if \( \theta(a) \in \mathcal{A}(N) \);
(2) \( M \) is atomic if and only if \( N \) is atomic;
(3) \( L_M(x) = L_N(\theta(x)) \) for all \( x \in M \).

Our next goal is to prove that the atomic structure of Puiseux monoids almost never can be obtained by pulling back that one of Krull monoids via transfer homomorphisms.

**Definition 4.5.** A monoid \( K \) is called a **Krull monoid** if there exists a monoid homomorphism \( \varphi: K \to D \), where \( D \) is a free abelian monoid and \( \varphi \) satisfies the following two conditions:

1. if \( a, b \in K \) and \( \varphi(a) \mid_D \varphi(b) \), then \( a \mid_K b \);
2. for every \( d \in D \) there exist \( a_1, \ldots, a_n \in K \) with \( d = \gcd\{\varphi(a_1), \ldots, \varphi(a_n)\} \).

With notation as in Definition 4.5, it is easy to see that \( K \) is a Krull monoid if and only if \( K_{\text{red}} \) is a Krull monoid. The basis elements of \( D \) are called the **prime divisors** of \( K \). The abelian group \( \text{Cl}(K) := D/\varphi(K) \) is called the **class group** of \( K \) (see [27, Section 2.3] for more details). Clearly, Krull monoids are atomic.

Block monoids capture the essence of the arithmetic of lengths of factorizations in Krull monoids. Let \( G \) be an abelian group and let \( \mathcal{F}(G) \) be the free commutative monoid on \( G \). An element \( X = g_1 \ldots g_l \in \mathcal{F}(G) \) is called a **sequence over** \( G \). The **length** of \( X \) is defined as \( |X| = l = \sum_{g \in G} v_g(X) \). For every \( I \subseteq [1,l] \), the sequence \( Y = \prod_{i \in I} g_i \) is called a **subsequence** of \( X \). The subsequences are precisely the divisors of \( X \) in the free abelian monoid \( \mathcal{F}(G) \). The submonoid

\[
\mathcal{B}(G) := \left\{ X \in \mathcal{F}(G) \mid \sum_{g \in G} v_g(X)g = 0 \right\}
\]

of \( \mathcal{F}(G) \) is called the **block monoid** on \( G \), and its elements are referred to as **zero-sum sequences** or **blocks over** \( G \) (see [26, Section 2.5] for more details). Furthermore, if \( G_0 \) is a subset of \( G \), then the submonoid

\[
\mathcal{B}(G_0) := \{ X \in \mathcal{B}(G) \mid v_g(X) = 0 \text{ if } g \notin G_0 \}
\]

of \( \mathcal{B}(G) \) is called the **restriction** of the block monoid \( \mathcal{B}(G) \) to \( G_0 \). For \( X \in \mathcal{B}(G_0) \), the **support** of \( X \) in \( G_0 \) is defined to be

\[
\text{supp}_{G_0}(X) := \{ g \in G_0 \mid v_g(X) > 0 \}.
\]

One of the most relevant aspects of block monoids in factorization theory lies in the next result.

**Proposition 4.6.** [26, Theorem 3.4.10.3] Let \( K \) be a Krull monoid with class group \( G \), and let \( G_0 \) be the set of classes of \( G \) having prime divisors. Then \( \mathcal{L}(K) = \mathcal{L}(\mathcal{B}(G_0)) \).

As a result, understanding the lengths of factorizations in Krull monoids amounts to understanding the same in block monoids.
Definition 4.7. A Puiseux monoid $M$ is transfer Krull if there exist an abelian group $G$, a subset $G_0$ of $G$, and a transfer homomorphism $\theta : M \to \mathcal{B}(G_0)$.

We will use the next lemma to characterize transfer Krull (and Krull) Puiseux monoids.

Lemma 4.8. If $(a_n)_{n \in \mathbb{N}}$ is a sequence of positive integers, then there exists $m \in \mathbb{N}$ such that $a_{m+1} \in \langle a_1, \ldots, a_m \rangle$.

Proof. If $(a_n)_{n \in \mathbb{N}}$ is bounded there is a term that repeats infinitely many times, making the conclusion of the lemma obvious. Thus, suppose that $(a_n)_{n \in \mathbb{N}}$ is not bounded. Let $(a_n)_{j \in \mathbb{N}}$ be a subsequence of $\{a_n\}$ satisfying that

$$a_{n_{j+1}} > \prod_{i=1}^{j} a_n$$

for every $j \in \mathbb{N}$. Now set $d_j = \gcd(a_n, \ldots, a_{n_{j+1}})$ for every $j \in \mathbb{N}$, and notice that $d_{j+1} | d_j$ for every $j \in \mathbb{N}$. Therefore $d_{k+1} = d_k$ holds for some $k \in \mathbb{N}$. In particular, $d_k \nmid a_{n_{k+1}}$. On the other hand, condition (4.1) ensures that $a_{n_{k+1}}/d_k$ is greater than the Frobenius number of the numerical monoid $\langle a_{n_1}/d_k, \ldots, a_{n_k}/d_k \rangle$. This implies that $a_{n_{k+1}} \in \langle a_{n_1}, \ldots, a_{n_k} \rangle$. The lemma follows by taking $m = n_{k+1} - 1$. \qed

We are in a position to characterize those Puiseux monoids that are, up to isomorphism, Krull monoids.

Theorem 4.9. For a nontrivial Puiseux monoid $M$, the following statements are equivalent.

1. $M$ is isomorphic to a Krull monoid.
2. $M$ is a transfer Krull monoid.
3. $M$ is isomorphic to $(\mathbb{N}_0, +)$.

Proof. Clearly, (3) $\Rightarrow$ (1) $\Rightarrow$ (2). To prove (2) $\Rightarrow$ (3), let $M$ be a nontrivial Puiseux monoid that is transfer Krull. As Krull monoids are atomic, $M$ is atomic by Proposition 4.4. Let $G$ be an abelian group, and let $\theta : M \to \mathcal{B}(G_0)$ be a transfer homomorphism, where $G_0$ is a subset of $G$. Because both $M$ and $\mathcal{B}(G_0)$ are reduced, $\theta^{-1}(\emptyset) = \{0\}$. Assume, by way of contradiction, that $M$ is not isomorphic to a numerical monoid. Take $X \in \mathcal{B}(G_0)^*$ and $r, s \in M^*$ such that $\theta(r) = \theta(s) = X$. Taking $m, n \in \mathbb{N}$ such that $mr = ns$, one obtains

$$\prod_{g \in G_0} g^{mv_g(X)} = \theta(r)^m = \theta(s)^n = \prod_{g \in G_0} g^{nv_g(X)}.$$  \hspace{1cm} (4.2)

Since $|X| \geq 1$ and $mv_g(X) = nv_g(X)$ for every $g \in G_0$, it follows that $m = n$, which yields $r = s$. Hence the preimage under $\theta$ of each element of $\mathcal{B}(G_0)^*$ is a singleton. This, along with the fact that $\theta^{-1}(\emptyset) = \{0\}$, implies that $\theta$ is injective. In addition, the same equality (4.2) implies that $\text{supp}_{G_0}(\theta(a)) = \text{supp}_{G_0}(\theta(a'))$ for all $a, a' \in \mathcal{A}(M)$. As
a consequence, any two elements of $\theta(M^*)$ have the same support, and we can assume, without loss of generality, that $G_0$ is finite. Let $G_0 =: \{g_1, \ldots, g_l\}$ be the common support. List the set $A(M)$ as a sequence $\{a_n\}$, and let $A_n = \theta(a_n)$ for every $n \in \mathbb{N}$. Because $\theta$ is injective, $A_i \neq A_j$ when $i \neq j$. Now, for any pair $(i, j) \in \mathbb{N}^2$, there exist $c_i, c_j \in \mathbb{N}$ such that $c_i a_i = c_j a_j$. For each $n \in [1, t]$, we can apply $\nu_{g_n} \circ \theta$ to the equality $c_i a_i = c_j a_j$ to get $c_i \nu_{g_n}(A_i) = c_j \nu_{g_n}(A_j)$. Thus,

$$\frac{\nu_{g_n}(A_i)}{\nu_{g_n}(A_j)} = \frac{c_j}{c_i} = \frac{\nu_{g_1}(A_i)}{\nu_{g_1}(A_j)}$$

for each $n \in [1, t]$. On the other hand, notice that Lemma 4.8 guarantees the existence of $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}_0$ satisfying that

$$\nu_{g_1}(A_{m+1}) = \sum_{i=1}^{m} \alpha_i \nu_{g_1}(A_i).$$

It follows from (4.3) that the equality (4.4) holds when we replace $g_1$ by any other element of $G_0$ (exactly with the same $\alpha_i$’s). As a result,

$$A_{m+1} = \prod_{j=1}^{G_0} g_j^{\nu_{g_1}(A_{m+1})} = \prod_{j=1}^{G_0} \prod_{i=1}^{m} g_j^{\alpha_i \nu_{g_1}(A_i)} = \prod_{i=1}^{m} \left( \prod_{j=1}^{G_0} g_j^{\nu_{g_1}(A_i)} \right)^{\alpha_i} = \prod_{i=1}^{m} A_i^{\alpha_i}.$$

This contradicts that $A_{m+1}$ is an atom of the block monoid $B(G_0)$. Therefore $M$ is isomorphic to a numerical monoid, and it follows from [31, Theorem 5.5.2] that $M \cong (\mathbb{N}_0, +)$. \hfill $\Box$

4.3. **Transfer Finite and C-monoid Characterizations.** After Krull monoids, perhaps the most systematically studied classes of atomic monoids in the context of factorization theory are those consisting of C-monoids. We proceed to characterize Puiseux monoids whose atomicity can be pulled from finitely generated monoids (i.e., transfer finite Puiseux monoids) in terms of C-monoids and numerical monoids.

**Definition 4.10.** We say that a Puiseux monoid $M$ is transfer finite if there exists a transfer homomorphism from $M$ to a monoid that is finitely generated up to units.

By the fundamental structure theorem of finitely generated abelian groups, every finitely generated monoid $F$ is a submonoid of a group $T \times \mathbb{Z}^\beta$ for some finite abelian group $T$ and $\beta \in \mathbb{N}_0$. If $F$ is reduced, then it can be embedded into $T \times \mathbb{N}_0^\beta$.

Condition (T2) in the definition of a transfer homomorphism $\theta: M \rightarrow F$ is crucial to pull back certain factorization properties from $F$ to $M$. However, how much will $\text{Hom}(M, F)$ increase if we drop condition (T2)? Surprisingly, the set of homomorphisms will remain the same as long as $F$ is reduced. We use this to classify transfer finite Puiseux monoids. First, note that a monoid homomorphism $\phi: M \rightarrow N$ naturally induces a monoid homomorphism $\phi_{\text{red}}: M_{\text{red}} \rightarrow N_{\text{red}}$. 


Let \( D \) be a multiplicative monoid with Grothendieck group \( \text{gp}(D) \), and let \( M \) be a submonoid of \( D \). Two elements \( x, y \in D \) are \( M \)-equivalent if \( x^{-1}M \cap D = y^{-1}M \cap D \). Being \( M \)-equivalent defines a congruence relation on \( D \). For each \( x \in D \), let \( [x]_M^D \) denote the congruence class of \( x \). The set
\[
\mathcal{C}^*(M, D) := \{ [x]_M^D \mid x \in (D \setminus D^\times) \cup \{1\} \}
\]
is a commutative semigroup with identity, which is called the reduced class semigroup of \( M \) in \( D \).

**Definition 4.11.** A monoid \( M \) is called a C-monoid if it is a submonoid of a factorial monoid \( F \) such that \( F^\times \cap M = M^\times \) and \( \mathcal{C}^*(M, F) \) is finite.

With notation as in Definition 4.11, we say that \( M \) is a C-monoid defined in \( F \) (there is a canonical way of choosing \( F \) (see [26, Theorem 5.6.A.3])). Clearly, C-monoids are atomic. The class of C-monoids has been crucial to study the arithmetic of non-integrally closed Noetherian domains (see, for instance, [28, 46, 47]).

**Theorem 4.12.** Let \( M \) be a nontrivial Puiseux monoid. Then the following statements are equivalent.

1. \( M \) is isomorphic to a C-monoid.
2. \( M \) is finitely generated.
3. \( M \) is transfer finite.

**Proof.** For (1) \( \Rightarrow \) (2), suppose that \( M \) is isomorphic to a C-monoid. It has been proved in [26, Theorem 2.9.11(2)] that the complete integral closure of a C-monoid is a Krull monoid, so \( \overline{M} \) is a Krull monoid. In particular, it is transfer Krull. Now Theorem 4.9 ensures that \( \overline{M} \) is isomorphic to \( (\mathbb{N}_0, +) \). Since \( M \) is a submonoid of its complete integral closure, \( M \) must be finitely generated.

Obviously, (2) \( \Rightarrow \) (3).

To prove that (3) \( \Rightarrow \) (1), assume that \( \theta: M \to F \) is a transfer homomorphism, where \( F \) is finitely generated up to units. As \( \theta_{\text{red}}: M_{\text{red}} = M \to F_{\text{red}} \) is also a transfer homomorphism, one can assume, without loss of generality, that \( F \) is reduced. As both \( M \) and \( F \) are reduced, condition (T1’) yields \( \theta^{-1}(0) = \{0\} \). Suppose that \( F \) is a submonoid of \( T \times \mathbb{N}_0^\beta \), where \( T \) is a finite abelian group and \( \beta \in \mathbb{N}_0 \). First, assume, by way of contradiction, that \( \beta = 0 \). In this case, it is not hard to verify that \( \theta(M) \) must be a subgroup of \( T \). If \( \alpha = |\theta(M)| \) and \( r \in M^\star \), then \( \theta(\alpha r) = \alpha \theta(r) = 0 \). This contradicts that \( \theta^{-1}(0) = \{0\} \). Thus, \( \beta \geq 1 \). Define \( \pi: T \times \mathbb{N}_0^\beta \to \mathbb{N}_0^\beta \) by \( \pi(t, v) = v \) for all \( t \in T \) and \( v \in \mathbb{N}_0^\beta \). To check that \( \pi(\theta(M)) \) is finitely generated, take \( x = (x_1, \ldots, x_\beta) \in \pi(\theta(M))^\star \) and let \( d = \gcd(x_1, \ldots, x_\beta) \). We shall verify that \( \pi(\theta(M)) \subseteq (x/d) \). To do so, consider \( y = (y_1, \ldots, y_\beta) \in \pi(\theta(M))^\star \). Now take \( r, s \in M^\star \) such that \( \pi(\theta(r)) = x \) and \( \pi(\theta(s)) = y \), and take \( m, n \in \mathbb{N} \) satisfying that \( \gcd(m, n) = 1 \).
It is also known that, in general, none of the implications in (5.1) holds. As \( \gcd(m, n) = 1 \), it follows that \( n \) divides each \( x_i \), i.e., \( d/n \in \mathbb{N} \). So

\[
y = \frac{m}{n}x = \left( \frac{md}{n} \right) \frac{x}{d} \in \left\langle \frac{x}{d} \right\rangle.
\]

Hence \( \pi(\theta(M)) \subseteq \langle x/d \rangle \). Because \( \langle x/d \rangle \) is isomorphic to \( \mathbb{N}_0 \), it follows that \( \pi(\theta(M)) \) is finitely generated.

Let us proceed to show that \( M \) is also finitely generated, which amounts to proving that \( \pi \circ \theta : M \to \mathbb{N}_0^\beta \) is injective. First, let us verify that \( \pi \) is injective when restricted to \( \theta(M) \). Clearly, \( \theta(M) \) is reduced. Suppose that \( (t_1, v), (t_2, v) \in \theta(M) \). If \( v = 0 \), then \( t_1 = t_2 = 0 \) because \( \theta(M) \) is reduced. Otherwise, there exist \( r, s \in M^\bullet \) such that \( \theta(r) = (t_1, v) \) and \( \theta(s) = (t_2, v) \). Take \( m, n \in \mathbb{N} \) such that \( mr = ns \). Since \( m(t_1, v) = n(t_2, v) \), one finds that \( m = n \) and, therefore, \( r = s \). This, in turn, implies that \( t_1 = t_2 \). Hence the restriction of \( \pi \) to \( \theta(M) \) is injective.

We finally show that \( \theta \) is also injective. Let \( r, s \in M \) such that \( \theta(r) = \theta(s) \neq 0 \). Taking \( m, n \in \mathbb{N} \) satisfying \( mr = ns \), we have \( m\theta(r) = \theta(mr) = \theta(ns) = n\theta(s) \). Since \( \theta(M) \) is reduced, the element \( \theta(r) \) must be torsion-free in \( T \times \mathbb{N}_0^\beta \). Thus, \( m = n \), which implies that \( r = s \). As \( \theta^{-1}(0) = \{0\} \), it follows that \( |\theta^{-1}(a)| = 1 \) for all \( a \in \theta(M) \). Therefore \( \theta \) is injective, leading us to the injectivity of \( \pi \circ \theta \). Now that fact that \( \pi(\theta(M)) \) is finitely generated implies that \( M \) is also finitely generated and, by Proposition 3.6, isomorphic to a numerical monoid \( N \). Thus any two natural numbers greater than the Frobenius number of \( N \) are \( N \)-equivalent, which implies that \( C^\ast(N, \mathbb{N}_0) \) is finite. Hence \( M \) is isomorphic to a \( C \)-monoid.

\[ \square \]

5. Atomic Structure

It is well known that in the class consisting of all monoids, the following chain of implications holds.

\[ \text{UFM} \Rightarrow \text{HFM} \Rightarrow \text{FFM} \Rightarrow \text{BFM} \Rightarrow \text{ACCP} \Rightarrow \text{atomic monoid} \]

It is also known that, in general, none of the implications in (5.1) is reversible (even in the class of integral domains [2]). In this section, we provide various examples to illustrate that none of the above implications, except the first one, is reversible in the class of Puiseux monoids. We characterize the Puiseux monoids belonging to the first two classes of the chain of implications (5.1). For each of the last four classes, we find a family of Puiseux monoids belonging to such a class but not to the class right before.
5.1. A Class of Atomic Puiseux Monoids. In Theorem 4.12, we offered a few characterizations of finitely generated Puiseux monoids. We begin this section collecting another such a characterization, now in terms of the atomicity.

**Proposition 5.1.** A Puiseux monoid $M$ is finitely generated if and only if $M$ is atomic and $\mathcal{A}(M)$ is finite.

**Proof.** The direct implication follows immediately from the fact that finitely generated Puiseux monoids are isomorphic to numerical monoids via Proposition 3.6. The reverse implication is also obvious because the atomicity of $M$ means that $M$ is generated by $\mathcal{A}(M)$, which is finite. 

Corollary 3.7 yields, however, instances of non-finitely generated Puiseux monoids containing no atoms. As the next example shows, for every $n \in \mathbb{N}$ there exists a non-finitely generated Puiseux monoid containing exactly $n$ atoms.

**Example 5.2.** Let $m \in \mathbb{N}$, and take distinct primes numbers $p$ and $q$ with $q > m$. Consider the Puiseux monoid $M = \langle \llbracket m, 2m - 1 \rrbracket \cup \{ qp^{-m-i} \mid i \in \mathbb{N} \} \rangle$. To verify that $\mathcal{A}(M) = \llbracket m, 2m - 1 \rrbracket$, write

$$a = a' + \sum_{n=1}^{N} c_n \frac{q}{p^{n+n}}.$$  

where $a' \in \{0\} \cup \llbracket m, 2m - 1 \rrbracket$ and $c_n \in \mathbb{N}_0$ for every $n \in \llbracket 1, N \rrbracket$. After cleaning denominators in (5.2), one finds that $q \mid a - a'$. So $a = a'$ and $c_1 = \cdots = c_N = 0$, which implies that $a \in \mathcal{A}(M)$. Thus, $\llbracket m, 2m - 1 \rrbracket \subseteq \mathcal{A}(M)$. Clearly, $qp^{-m-i} \notin \mathcal{A}(M)$ for any $i \in \mathbb{N}$. Hence $\mathcal{A}(M) = \llbracket m, 2m - 1 \rrbracket$, and so $|\mathcal{A}(M)| = m$. As $d(M^*)$ is not finite, it follows from Proposition 3.6 that $M$ is not finitely generated.

Perhaps the class of non-finitely generated Puiseux monoids that has been most thoroughly studied is that one consisting of cyclic rational semirings [12].

**Definition 5.3.** For $r \in \mathbb{Q}_{>0}$, we call $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$ a cyclic semiring Puiseux monoid.

Although $S_r$ is, indeed, a rational cyclic semiring, we shall only be concerned here with its additive structure. The atomicity of cyclic semiring Puiseux monoids was studied in [40, Section 6] while several factorization aspects were recently investigated in [12].

**Proposition 5.4.** For $r \in \mathbb{Q}_{>0}$, consider the cyclic semiring Puiseux monoid $S_r$. Then the following statements hold.

(1) If $r \geq 1$, then $S_r$ is atomic and
   - either $r \in \mathbb{N}$ and so $S_r = \mathbb{N}_0$,
   - or $r \notin \mathbb{N}$ and so $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.

(2) If $r < 1$, then
Proof. To argue (1), suppose that \( r \geq 1 \). If \( r \in \mathbb{N} \), then it easily follows that \( S_r = \mathbb{N}_0 \). Then we assume that \( r \notin \mathbb{N} \). Clearly, \( A(S_r) \subseteq \{ r^n \mid n \in \mathbb{N}_0 \} \). To check the reverse inequality, fix \( j \in \mathbb{N}_0 \) and write \( r^j = \sum_{i=0}^{N} \alpha_i r^i \) for some \( N \in \mathbb{N}_0 \) and \( \alpha_i \in \mathbb{N}_0 \) for every \( i \in \llbracket 0, N \rrbracket \). As \( \{ r^n \} \) is an increasing sequence, one can assume that \( N \leq j \). Then, after cleaning denominators in \( r^j = \sum_{i=0}^{N} \alpha_i r^i \) we obtain \( N = j \) as well as \( \alpha_j = 1 \) and \( \alpha_i = 0 \) for every \( i \neq j \). Hence \( r^j \in A(S_r) \) for every \( j \in \mathbb{N}_0 \), which yields the second statement of (1).

Now suppose that \( r < 1 \). If \( n(r) = 1 \) then \( r^n = d(r) r^{n+1} \) for every \( n \in \mathbb{N}_0 \), and so \( S_r \) is antimatter, which is the first statement of (2). Finally, suppose that \( n(r) > 1 \). Fix \( j \in \mathbb{N} \), and notice that \( r^i \nmid r^j \) for any \( i < j \). Then write \( r^j = \sum_{i=j}^{j+k} \beta_i r^i \), for some \( k \in \mathbb{N}_0 \) and \( \beta_i \in \mathbb{N}_0 \) for every \( i \in \llbracket j, j+k \rrbracket \). Notice that \( \beta_j \in \{ 0, 1 \} \). Suppose by a contradiction that \( \beta_j = 0 \). In this case, \( k \geq 1 \). Let \( p \) be a prime dividing \( n(r) \), and let \( \alpha \) be the maximum power of \( p \) dividing \( n(r) \). From \( r^j = \sum_{i=j}^{j+k} \beta_i r^i \) one obtains

\[
\alpha j = v_p (r^j) = v_p \left( \sum_{i=1}^{k} \beta_{j+i} r^{j+i} \right) \geq \min_{i \in [1,k]} \{ v_p(\beta_{j+i} r^{j+i}) \} \geq \alpha (j+m),
\]

where \( m = \min \{ i \in [1,k] \mid \beta_{j+i} \neq 0 \} \). The inequality (5.3) yields the desired contradiction. Hence \( r^j \in A(S_r) \) for every \( j \in \mathbb{N}_0 \), which implies the second statement of (2).

\[\square\]

**Corollary 5.5.** For each \( r \in \mathbb{Q} \cap (0,1) \) with \( n(r) \neq 1 \), the monoid \( S_r \) is an atomic monoid that does not satisfy the ACCP.

\[Proof.\] Proposition 5.4 guarantees the atomicity of \( S_r \). To verify that \( S_r \) does not satisfy the ACCP, consider the sequence of principal ideals \( (n(r)n + S_r)_{n \in \mathbb{N}_0} \). Since

\[
n(r)n = d(r)n^{n+1} = (d(r) - n(r))n^{n+1} + n(r)n^{n+1},
\]

\( n(r)n^{n+1} \mid S_r, n(r)n^{n} \) for every \( n \in \mathbb{N}_0 \). Therefore \( (n(r)n + S_r)_{n \in \mathbb{N}_0} \) is an ascending chain of principal ideals. In addition, it is clear that such a chain of ideals does not stabilize. Hence \( S_r \) does not satisfy the ACCP, which completes the proof.
\[\square\]

5.2. *A Class of ACCP Puiseux Monoids.* We proceed to present a class of ACCP Puiseux monoids containing a subclass of monoids that are not BFMs.

**Theorem 5.6.** Every Puiseux monoid of the form \( \langle a_p/p \mid p \in \mathbb{P} \text{ and } a_p \in \mathbb{N}_0 \rangle \) satisfies the ACCP.
Proof. Let \((p_n)_{n \in \mathbb{N}}\) be a strictly increasing sequence of prime numbers, and let \((a_n)_{n \in \mathbb{N}}\) be a sequence of positive integers such that \(p_n \nmid a_n\). Now set \(M = \langle a_n/p_n \mid n \in \mathbb{N} \rangle\). It is not hard to verify that for all \(x \in M\) there exist \(k, n \in \mathbb{N}_0\) and \(\alpha_i \in \llbracket 0, p_i \rrbracket\) such that

\[
(5.4) \quad x = n + \sum_{i=1}^{k} \alpha_i \frac{a_i}{p_i}.
\]

Let us now check that the sum decomposition in \((5.4)\) is unique. To do this, take \(k', m \in \mathbb{N}_0\) and \(\beta_i \in \llbracket 0, p_i \rrbracket\) such that

\[
(5.5) \quad n + \sum_{i=1}^{k} \alpha_i \frac{a_i}{p_i} = m + \sum_{i=1}^{k'} \beta_i \frac{a_i}{p_i}.
\]

Suppose, without loss of generality, that \(k = k'\). After isolating \((\alpha_i - \beta_i) a_i / p_i\) in \((5.5)\) and applying the \(p_i\)-adic valuation, we obtain that \(p_i \mid \alpha_i - \beta_i\), which implies that \(\alpha_i = \beta_i\) for each \(i \in \llbracket 1, k\rrbracket\). As a consequence, \(n = m\) and we can conclude that the uniqueness of the decomposition in \((5.4)\) holds.

Now suppose, by way of contradiction, that \(M\) does not satisfy the ACCP. Then there exists a strictly decreasing sequence \(\{q_n\}\) of elements in \(M\) such that

\[
(5.6) \quad q_1 + M \subsetneq q_2 + M \subsetneq \cdots.
\]

In the unique sum decomposition of \(q_1\) as in \((5.4)\), let \(n\) be the integer and let \(p_{n_1}, \ldots, p_{n_k}\) be the distinct prime denominators of the atoms with nonzero coefficients. By \((5.6)\), for every \(n \in \mathbb{N}\) there exists \(r_{n+1} \in M\) such that \(q_n = q_{n+1} + r_{n+1}\) and, therefore,

\[
\sum_{i=1}^{n} r_{i+1} \leq q_{n+1} + \sum_{i=1}^{n} r_{i+1} = q_1
\]

for every \(n \in \mathbb{N}\). Thus, \(\lim_{n \to \infty} r_n = 0\). Then, for any finite subset \(A\) of \(\mathcal{A}(M)\) there exist a large enough \(\ell \in \mathbb{N}\) and \(a \in \mathcal{A}(M)\) such that \(a \mid M \ r_t\) and \(a \notin A\). Hence we can find \(t \in \mathbb{N}\) and (possibly repeated) \(p_{m_1}, \ldots, p_{m_t} \in \mathbb{P}\) satisfying that \(|\{p_{m_1}, \ldots, p_{m_t}\}| > N + k\) and \(a_{m_i}/p_{m_i} \in \mathcal{A}(r_{i+1})\) for each \(i \in \llbracket 1, t\rrbracket\). Take \(z_i \in \mathbb{Z}(r_i)\) containing the atom \(a_{m_i}/p_{m_i}\), and take \(z_{t+1} \in \mathbb{Z}(q_1).\) As \(q_1 = q_{t+1} + \sum_{i=1}^{t} r_{i+1}\), we have \(z = z_{t+1} + \sum_{i=1}^{t} z_i \in \mathbb{Z}(q_1).\) By the uniqueness of the sum decomposition in \((5.4)\), the factorization \(z\) contains at least \(d(a)\) copies of each atom \(a\) for which \(d(a) \in P := \{p_{m_1}, \ldots, p_{m_t}\} \setminus \{p_{n_1}, \ldots, p_{n_k}\}\). Since \(|\{p_{m_1}, \ldots, p_{m_t}\}| > N + k\), it follows that \(|P| > N\). Thus,

\[
N \geq \sum_{a \in \mathcal{A}(M) \mid d(a) \in P} d(a) a \geq |P| > N,
\]

which is a contradiction. Hence \(M\) satisfies the ACCP. \(\Box\)

**Corollary 5.7.** There are Puiseux monoids satisfying the ACCP that are not BFMs.
Proof. Consider the Puiseux monoid $M = \langle 1/p \mid p \in \mathbb{P} \rangle$. We have seen in Theorem 5.6 that $M$ satisfies the ACCP. However, it is clear that $p \in L(1)$ for every $p \in \mathbb{P}$. As $|L(1)| = \infty$, the monoid $M$ is not a BFM. □

5.3. Bounded Factorization Monoids. Our next main goal is to find a huge class of Puiseux monoids that are BFMs. This amounts to prove the following result.

Theorem 5.8. Let $M$ be a Puiseux monoid. If 0 is not a limit point of $M^\bullet$, then $M$ is a BFM.

Proof. It is clear that $A(M)$ consists of those elements of $M^\bullet$ that cannot be written as the sum of two positive elements of $M$. Since 0 is not a limit point of $M$ there exists $\epsilon > 0$ such that $\epsilon < x$ for all $x \in M^\bullet$. Now we show that $M = \langle A(M) \rangle$. Take $x \in M^\bullet$. Since $\epsilon$ is a lower bound for $M^\bullet$, the element $x$ can be written as the sum of at most $\lfloor x/\epsilon \rfloor$ elements of $M^\bullet$. Take the maximum natural $m$ such that $x = a_1 + \cdots + a_m$ for some $a_1, \ldots, a_m \in M^\bullet$. By the maximality of $m$, it follows that $a_i \in A(M)$ for every $i \in \llbracket 1, m \rrbracket$, which means that $x \in \langle A(M) \rangle$. Hence $M$ is atomic. We have already noticed that every element $x$ in $M^\bullet$ can be written as the sum of at most $\lfloor x/\epsilon \rfloor$ atoms, i.e., $|L(x)| \leq \lfloor x/\epsilon \rfloor$ for all $x \in M$. Thus, $M$ is a BFM. □

The converse of Theorem 5.8 does not hold. The following example sheds some light upon this observation.

Example 5.9. Let $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ be two strictly increasing sequence of prime numbers such that $q_n > p_n^2$ for every $n \in \mathbb{N}$. Set $M := \langle q_n \mid n \in \mathbb{N} \rangle$. It follows from [40, Corollary 5.6] that $M$ is atomic, and it is easy to verify that $A(M) = \{p_n/q_n \mid n \in \mathbb{N}\}$. To argue that $M$ is indeed a BFM, take $x \in M^\bullet$ and note that since both sequences $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ are strictly increasing, there exists $N \in \mathbb{N}$ such that $q_n \nmid d(x)$ and $p_n > x$ for every $n \geq N$. As a result, if $z \in Z(x)$, then none of the atoms in $\{p_n/q_n \mid n > N\}$ can appear in $z$. From this, one can deduce that $Z(x)$ is finite. Then $L(x)$ is finite for any $x \in M$, and so $M$ is a BFM. However, $q_n > p_n^2$ for every $n \in \mathbb{N}$ implies that 0 is a limit point of $M^\bullet$.

As we have seen in Corollary 5.7, not every ACCP Puiseux monoid is a BFM. However, under a mild assumption on conductors, each of these atomic conditions is equivalent to having 0 as a limit point.

Theorem 5.10. If $M$ is a nontrivial Puiseux monoid with nonempty conductor, then the following conditions are equivalent.

1. 0 is not a limit point of $M^\bullet$.
2. $M$ is a BFM.
3. $M$ satisfies the ACCP.

1Theorem 5.10 has been borrowed from [25], which is a manuscript under preparation.
Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from Theorem 5.8 and [26, Corollary 1.3.3], respectively. Note that such implications do not depend on whether the conductor of $M$ is nonempty.

We proceed to show that (3) $\Rightarrow$ (1). Suppose, by way of contradiction, that 0 is a limit point of $M^\bullet$. Since $M$ satisfies the ACCP, it is atomic. Thus, there exists a sequence of atoms $(a_n)_{n \in \mathbb{N}}$ such that $a_n < 1/2^n$ for every $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges to a certain limit $\ell \in (0, 1)$ and, as a consequence, $s_n := \sum_{i=1}^{n} a_i \in M \cap (0, 1)$ for every $n \in \mathbb{N}$. Since the conductor of $M$ is nonempty, Proposition 3.8 guarantees that $\hat{M} \setminus M$ is bounded. Take $x \in M$ such that $x > 1 + \sup \hat{M} \setminus M$. It follows from Proposition 3.4 that $x - s_n \in \hat{M}$ for every $n \in \mathbb{N}$. As $M$ has nonempty conductor and $x - s_n > \sup \hat{M} \setminus M$, Proposition 3.8 guarantees that $x - s_n \in M$. Consider the sequence of principal ideals $(x - s_n + M)_{n \in \mathbb{N}}$ of $M$. Since $x - s_n = x - s_{n+1} + (s_{n+1} - s_n) = (x - s_{n+1}) + a_{n+1}$, the sequence $(x - s_n + M)_{n \in \mathbb{N}}$ is an ascending chain of principal ideals. Because $x - s_{n+1} < x - s_n$, the ascending chain of principal ideals $(x - s_n + M)_{n \in \mathbb{N}}$ does not stabilize. This contradicts that $M$ satisfies the ACCP, which completes the proof. \qed

As Corollary 5.7 and Example 5.9 indicate, without the nonempty-conductor condition, none of the last two statements in Theorem 5.10 implies its predecessor. In addition, even inside the class of Puiseux monoids with nonempty conductor neither being atomic nor being an FFM is equivalent to being a BFM (or satisfying the ACCP).

Example 5.11. Consider the Puiseux monoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$. It is clear that the conductor of $M$ is nonempty. In addition, it follows from Theorem 5.10 that $M$ is a BFM. Note that $\mathcal{A}(M) = [1, 2)$. However, $M$ is far from being an FFM; for instance, the formal sum $(1 + 1/n) + (x - 1 - 1/n)$ is a length-2 factorization in $\mathbb{Z}(x)$ for all $x \in (2, 3] \setminus \mathbb{Q}$ and $n \geq \lceil \frac{1}{x-2} \rceil$, which implies that $|\mathbb{Z}(x)| = \infty$ for all $x \in M_{\geq 2}$.

Example 5.12. Now consider the Puiseux monoid $M = \langle 1/p \mid p \in \mathbb{P} \rangle \cup \mathbb{Q}_{\geq 1}$. Since the monoid $\langle 1/p \mid p \in \mathbb{P} \rangle$ is atomic by Theorem 5.6, it is not hard to check that $M$ is also atomic. It follows from Proposition 3.8 that $M$ has nonempty conductor. Since 0 is a limit point of $M^\bullet$, Theorem 5.10 ensures that $M$ does not satisfy the ACCP.

5.4. Finite Factorization Monoids. Our next task is to introduce a class of Puiseux monoids that are FFMs. This class consists of all Puiseux monoids that can be generated by an increasing sequence of rationals.

Definition 5.13. A Puiseux monoid $M$ is called increasing (resp., decreasing) if it can be generated by an increasing (resp., decreasing) sequence. A Puiseux monoid is monotone if it is increasing or decreasing.

Not every Puiseux monoid is monotone, as the next example shows.
Example 5.14. Let $p_1, p_2, \ldots$ be a strictly increasing enumeration of $\mathbb{P}$. Consider the Puiseux monoid $M = \langle A \cup B \rangle$, where

$$A = \left\{ \frac{1}{p_{2n}} \mid n \in \mathbb{N} \right\} \quad \text{and} \quad B = \left\{ \frac{p_{2n-1}-1}{p_{2n-1}} \mid n \in \mathbb{N} \right\}.$$ 

It follows immediately that both $A$ and $B$ belong to $\mathcal{A}(M)$. So $M$ is atomic and $\mathcal{A}(M) = A \cup B$. Every generating set of $M$ must contain $A \cup B$ and so will have at least two limit points, namely, 0 and 1. Since every monotone sequence of rationals can have at most one limit point in the real line, we conclude that $M$ is not monotone.

The next proposition offers a first insight into the atomicity of increasing monoids.

**Proposition 5.15.** Every increasing Puiseux monoid is atomic. Moreover, if $(r_n)_{n \in \mathbb{N}}$ is an increasing sequence of positive rationals generating a Puiseux monoid $M$, then $\mathcal{A}(M) = \{ r_n \mid r_n \notin \langle r_1, \ldots, r_{n-1} \rangle \}$.

**Proof.** The fact that $M$ is atomic follows from observing that $r_1$ is a lower bound for $M^*$ and so 0 is not a limit point of $M^*$. To prove the second statement, set

$$A = \{ r_n \mid r_n \notin \langle r_1, \ldots, r_{n-1} \rangle \}.$$ 

Note that $A$ is finite if and only if $M$ is finitely generated, in which case it is clear that $A = \mathcal{A}(M)$. Then suppose that $A$ is not finite. List the elements of $A$ as a strictly increasing sequence, namely, $(a_n)_{n \in \mathbb{N}}$. Note that $M = \langle A \rangle$ and $a_n \notin \langle a_1, \ldots, a_{n-1} \rangle$ for any $n \in \mathbb{N}$. Since $a_1 = \min M^*$, we have that $a_1 \in \mathcal{A}(M)$. Take $n \in \mathbb{Z}, \mathbb{A}]$. Because $(a_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence and $a_n \notin \langle a_1, \ldots, a_{n-1} \rangle$, one finds that $a_n$ cannot be written as a sum of elements in $M$ in a non-trivial manner. Hence $a_n \in \mathcal{A}(M)$ for every $n \in \mathbb{N}$, and we can conclude that $\mathcal{A}(M) = A$. \hfill $\square$

Let us collect another characterization of finitely generated Puiseux monoids, now in terms of monotonicity.

**Proposition 5.16.** A nontrivial Puiseux monoid $M$ is finitely generated if and only if $M$ is both increasing and decreasing.

**Proof.** The direct implication is obvious. For the reverse implication, suppose that $M$ is a nontrivial Puiseux monoid that is increasing and decreasing. Proposition 5.15 implies that $M$ is atomic and, moreover, $\mathcal{A}(M)$ is the underlying set of an increasing sequence. Assume by contradiction that $|\mathcal{A}(M)| = \infty$. Then $\mathcal{A}(M)$ does not contain a largest element. As $M$ is decreasing, there exists $D := \{ d_n \mid n \in \mathbb{N} \} \subseteq \mathbb{Q}_{>0}$ such that $d_1 > d_2 > \cdots$ and $M = \langle D \rangle$. Let $m = \min \{ n \in \mathbb{N} \mid d_n \in \mathcal{A}(M) \}$, which must exist because $\mathcal{A}(M) \subseteq D$. Now the minimality of $m$ implies that $d_m$ is the largest element of $\mathcal{A}(M)$, which is a contradiction. Hence $\mathcal{A}(M)$ is finite. As $M$ is atomic, it must be finitely generated. \hfill $\square$

**Definition 5.17.** An increasing Puiseux monoid is said to be strongly increasing if $\mathcal{A}(M)$ is unbounded.
Not every increasing Puiseux monoid is strongly increasing; to confirm this, consider the monoid \( \langle \frac{p-1}{p} \mid p \in \mathbb{P} \rangle \). We will show that the strongly increasing property is hereditary. First, let us argue the following lemma.

**Lemma 5.18.** Let \( R \) be an infinite subset of \( \mathbb{Q}_{\geq 0} \). If \( R \) does not contain any limit points, then it is the underlying set of an increasing and unbounded sequence.

**Proof.** For any \( r \in R \) and \( S \subseteq R \), the interval \([0, r]\) must contain only finitely many elements of \( S \); otherwise there would be a limit point of \( S \) in \([0, r]\). Therefore every nonempty subset of \( R \) has a minimum element. So the sequence \( (r_n)_{n \in \mathbb{N}} \) recurrently defined by \( r_1 = \min R \) and \( r_n = \min R \setminus \{r_1, \ldots, r_{n-1}\} \) is strictly increasing and has \( R \) as its underlying set. Since \( R \) is infinite and contains no limit points, the increasing sequence \( (r_n)_{n \in \mathbb{N}} \) must be unbounded. Hence \( R \) is the underlying set of the increasing and unbounded sequence \( (r_n)_{n \in \mathbb{N}} \). \( \square \)

**Proposition 5.19.** A nontrivial Puiseux monoid \( M \) is strongly increasing if and only if every submonoid of \( M \) is increasing.

**Proof.** If \( M \) is finitely generated, then the statement of the theorem follows immediately. So we will assume for the rest of this proof that \( M \) is not finitely generated. Suppose that \( M \) is strongly increasing. Let us start by verifying that \( M \) does not have any real limit points. By Proposition 5.15, the monoid \( M \) is atomic. As \( M \) is atomic and non-finitely generated, \( |\mathcal{A}(M)| = \infty \). Let \( (a_n)_{n \in \mathbb{N}} \) be an increasing sequence whose underlying set is \( \mathcal{A}(M) \). Since \( M \) is strongly increasing and \( \mathcal{A}(M) \) is an infinite subset contained in every generating set of \( M \), the sequence \( (a_n)_{n \in \mathbb{N}} \) is unbounded. Therefore, for every \( r \in \mathbb{R} \), the interval \([0, r]\) contains only finitely many elements of \( (a_n)_{n \in \mathbb{N}} \).

There is no loss in assuming that such elements are \( a_1, \ldots, a_k \) for some \( k \in \mathbb{N} \). Since \( \langle a_1, \ldots, a_k \rangle \cap [0, r] \) is finite, \( M \cap [0, r] \) is also finite. Because \( |[0, r] \cap M| < \infty \) for all \( r \in \mathbb{R} \), it follows that \( M \) does not have any limit points in \( \mathbb{R} \).

Now suppose that \( N \) is a nontrivial submonoid of \( M \). Being a subset of \( M \), the monoid \( N \) cannot have any limit points in \( \mathbb{R} \). Thus, by Lemma 5.18, the set \( N \) is the underlying set of an increasing and unbounded sequence of rationals. Hence \( N \) is a strongly increasing Puiseux monoid, and the direct implication follows.

For the reverse implication, suppose that \( M \) is not strongly increasing. We will check that, in this case, \( M \) contains a submonoid that is not increasing. If \( M \) is not increasing, then \( M \) is a submonoid of itself that is not increasing. Suppose, therefore, that \( M \) is increasing. By Proposition 5.15, the monoid \( M \) is atomic, and we can list its atoms increasingly. Let \( (a_n)_{n \in \mathbb{N}} \) be an increasing sequence with underlying set \( \mathcal{A}(M) \).
Because \( M \) is not strongly increasing, there exists a positive real \( \ell \) that is the limit of the sequence \( (a_n)_{n \in \mathbb{N}} \). Since \( \ell \) is a limit point of \( M \), which is closed under addition, \( 2\ell \) and \( 3\ell \) must be limit points of \( M \). Let \( (b_n)_{n \in \mathbb{N}} \) and \( (c_n)_{n \in \mathbb{N}} \) be sequences in \( M \) having infinite underlying sets such that \( \lim_{n \to \infty} b_n = 2\ell \) and \( \lim_{n \to \infty} c_n = 3\ell \). Furthermore,
assume that for every $n \in \mathbb{N}$,

\[(5.7) \quad |b_n - 2\ell| < \frac{\ell}{4} \quad \text{and} \quad |c_n - 3\ell| < \frac{\ell}{4}.\]

Take $N$ to be the submonoid of $M$ generated by the set $A := \{b_n, c_n \mid n \in \mathbb{N}\}$. Note that $A$ contains at least two limit points. Let us verify that $A(N) = A$. The inequalities (5.7) immediately imply that $A$ is bounded from above by $3\ell + \ell/4$. On the other hand, proving that $A(N) = A$ amounts to showing that $A$ and $A + A$ are disjoint. To verify this, it suffices to note that

\[
\inf(A + A) = \inf \{b_m + b_n, b_m + c_n + c_n \mid m, n \in \mathbb{N}\} \geq \min \left\{4\ell - \frac{\ell}{2}, 5\ell - \frac{\ell}{2}, 6\ell - \frac{\ell}{2}\right\} > 3\ell + \frac{\ell}{4} \geq \sup A.
\]

Thus, $A(N) = A$. Since every increasing sequence has at most one limit point in $\mathbb{R}$, the set $A$ cannot be the underlying set of an increasing rational sequence. As every generating set of $N$ contains $A$, we conclude that $N$ is not an increasing Puiseux monoid, which completes the proof. $\square$

The next result is important as it provides a large class of Puiseux monoids that are FFM.

**Theorem 5.20.** Every increasing Puiseux monoid is an FFM.

**Proof.** Let $M$ be an increasing Puiseux monoid. Since 0 is not a limit point of $M^\bullet$, Theorem 5.8 ensures that $M$ is a BFM. Suppose, by way of contradiction, that $M$ is not an FFM. Consider the set $S = \{x \in M : |Z(x)| = \infty\}$. As $M$ is not an FFM, $S$ is not empty. In addition, $s = \inf S \neq 0$ because 0 is not a limit point of $M^\bullet$. Note that $M^\bullet$ contains a minimum element because $M$ is increasing. Set $m = \min M^\bullet$ and fix $\epsilon \in (0, m)$. Now take $x \in S$ such that $s \leq x < s + \epsilon$. Each $a \in A(M)$ shows in only finitely many factorizations of $x$; otherwise $x - a$ would be an element of $S$ satisfying that $x - a < s$, contradicting that $s = \inf S$. Since $L(x)$ is finite, there exists $\ell \in L(x)$ such that the set

\[Z = \{z \in Z(x) : |z| = \ell\}\]

has infinite cardinality. Fix $z_0 = a_1 \ldots a_\ell \in Z(x)$, where $a_1, \ldots, a_\ell \in A(M)$, and set $A = \max\{a_1, \ldots, a_\ell\}$. Because every atom shows in only finitely many factorizations in $Z$ and $|Z| = \infty$, there exists $z_1 = b_1 \ldots b_\ell \in Z$, where $b_1, \ldots, b_\ell \in A(M)$ and $b_n > A$ for every $n \in [1, \ell]$ (here we are using the fact that $A(M)$ is the underlying set of an increasing sequence). But now, if $\pi : Z(M) \to M$ is the factorization homomorphism of $M$,

\[x = \pi(z_0) = \sum_{n=1}^{\ell} a_n \leq A\ell < \sum_{n=1}^{\ell} b_n = \pi(z_1) = x,
\]

which is a contradiction. Hence $M$ is an FFM. $\square$
On the other hand, the converse of Theorem 5.20 does not hold; the following example sheds some light upon this observation.

Example 5.21. Let \((p_n)_{n \in \mathbb{N}}\) be a strictly increasing sequence of primes, and consider the Puiseux monoid of \(\mathbb{Q}\) defined as follows:

\[
M = \langle A \rangle, \text{ where } A = \left\{ \frac{p_{2n} + 1}{p_{2n}}, \frac{p_{2n+1} + 1}{p_{2n+1}} \mid n \in \mathbb{N} \right\}.
\]

Since \(A\) is an unbounded subset of \(\mathbb{R}\) having 1 as a limit point, it cannot be increasing. In addition, as \(d(a) \neq d(a')\) for all \(a, a' \in A\) such that \(a \neq a'\), every element of \(A\) is an atom of \(M\). As each generating set of \(M\) must contain \(A\) (which is not increasing), \(M\) is not an increasing Puiseux monoid.

To verify that \(M\) is an FFM, fix \(x \in M\) and then take \(D_x\) to be the set of prime numbers dividing \(d(x)\). Now choose \(N \in \mathbb{N}\) such that \(N > \max\{x, d(x)\}\). For each \(a \in A\) with \(d(a) > N\), the number of copies \(\alpha\) of the atom \(a\) appearing in any \(z \in \mathbb{Z}(x)\) must be a multiple of \(d(a)\) because \(d(a) \notin D_x\). Then \(\alpha = 0\); otherwise, we would have that \(x \geq \alpha a \geq d(a)a > d(a) > x\). Thus, if an atom \(a\) divides \(x\) in \(M\), then \(d(a) \leq N\). As a result, only finitely many elements of \(A(M)\) divide \(x\) in \(M\) and so \(|\mathbb{Z}(x)| < \infty\). Hence \(M\) is an FFM that is not increasing.

Remark 5.22. For an ordered field \(F\), a positive monoid of \(F\) is an additive submonoid of the nonnegative cone of \(F\). As for Puiseux monoids, a positive monoid is increasing if it can be generated by an increasing sequence. Increasing positive monoids are FFMs [35, Theorem 5.6], but the proof of this general version of Theorem 5.20 is much more involved.

5.5. Factorial, Half-Factorial, and Other-Half-Factorial Monoids. The only Puiseux monoid that is a UFM (or even an HFM) is, up to isomorphism, \((\mathbb{N}_0, +)\). The following proposition formalizes this observation.

Proposition 5.23. For a nontrivial atomic Puiseux monoid \(M\), the following statements are equivalent.

1. \(M\) is a UFM.
2. \(M\) is a HFM.
3. \(M \cong (\mathbb{N}_0, +)\).
4. \(M\) contains a prime element.

Proof. Clearly, (3) \(\Rightarrow\) (1) \(\Rightarrow\) (2). To argue (2) \(\Rightarrow\) (3), assume that \(M\) is an HFM. Since \(M\) is an atomic nontrivial Puiseux monoid, \(A(M)\) is not empty. Let \(a_1\) and \(a_2\) be two atoms of \(M\). Then \(z_1 := n(a_2)d(a_1)a_1\) and \(z_2 := n(a_1)d(a_2)a_2\) are two factorizations of the element \(n(a_1)n(a_2) \in M\). As \(M\) is an HFM, it follows that \(|z_1| = |z_2|\) and so \(n(a_2)d(a_1) = n(a_1)d(a_2)\). Then \(a_1 = a_2\), which implies that \(|A(M)| = 1\). As a consequence, \(M \cong (\mathbb{N}_0, +)\). Because (3) \(\Rightarrow\) (4) holds trivially, we only need to argue
(4) ⇒ (3). Fix a prime element $p \in M$ and take $a \in \mathcal{A}(M)$. Since $p \mid_M n(p)d(a)a$, one finds that $p \mid_M a$. This, in turn, implies that $a = p$. Hence $\mathcal{A}(M) = \{p\}$, and so $M \cong (\mathbb{N}_0, +)$.

A dual notion of being an HFM was introduced in [19] by Coykendall and Smith.

**Definition 5.24.** An atomic monoid $M$ is an OHFM (or an other-half-factorial monoid) if for all $x \in M \setminus U(M)$ and $z, z' \in \mathbb{Z}(x)$ with $|z| = |z'|$, we have that $z = z'$.

Clearly, every UFM is an OHFM. Although the multiplicative monoid of an integral domain is a UFM if and only if it is an OHFM [19, Corollary 2.11], OHFMs are not always UFMs or HFMs, even in the class of Puiseux monoids.

**Proposition 5.25.** For a nontrivial atomic Puiseux monoid $M$, the following conditions are equivalent.

1. $M$ is an OHFM.
2. $|\mathcal{A}(M)| \leq 2$.
3. $M$ is isomorphic to a numerical monoid with embedding dimension in $\{1, 2\}$.

**Proof.** To prove (1) ⇒ (2), let $M$ be an OHFM. If $M$ is factorial, then $M \cong (\mathbb{N}_0, +)$, and we are done. Then suppose that $M$ is not factorial. In this case, $|\mathcal{A}(M)| \geq 2$. Assume for a contradiction that $|\mathcal{A}(M)| \geq 3$. Take $a_1, a_2, a_3 \in \mathcal{A}(M)$ satisfying that $a_1 < a_2 < a_3$. Let $d = d(a_1)d(a_2)d(a_3)$, and set $a'_i = da_i$ for each $i \in [1, 3]$. Since $a'_1, a'_2, a'_3$ are integers satisfying that $a'_1 < a'_2 < a'_3$, there exist $m, n \in \mathbb{N}$ such that

$$m(a'_2 - a'_1) = n(a'_3 - a'_2).$$

Clearly, $z_1 := ma_1 + na_3$ and $z_2 := (m + n)a_2$ are two distinct factorizations in $\mathbb{Z}(M)$ satisfying that $|z_1| = m + n = |z_2|$. In addition, after dividing both sides of the equality (5.9) by $d$, one obtains $ma_1 + na_3 = (m + n)a_2$, which means that $z_1$ and $z_2$ are factorizations of the same element. However, this contradicts that $M$ is an OHFM. Hence $|\mathcal{A}(M)| \leq 2$, as desired.

To show that (2) ⇒ (3), suppose that $|\mathcal{A}(M)| \leq 2$. By Proposition 3.6, $M$ is isomorphic to a numerical monoid $N$. As $|\mathcal{A}(M)| \leq 2$, the embedding dimension of $N$ belongs to $\{1, 2\}$, as desired.

To show (3) ⇒ (1), suppose that either $M \cong (\mathbb{N}_0, +)$ or $M \cong \langle a, b \rangle$ for $a, b \in \mathbb{N}_{>2}$ with $\gcd(a, b) = 1$. If $M \cong (\mathbb{N}_0, +)$, then $M$ is factorial and, in particular, an OHFM. On the other hand, if $M \cong \langle a, b \rangle$, then it is an OHFM by [19, Example 2.13].

**Remark 5.26.** There are Puiseux monoids that are FFMs but neither HFMs nor OHFMs. As a direct consequence of Theorem 5.20 and Propositions 5.23 and 5.25, one finds that $\langle \frac{p-1}{p} \mid p \in \mathbb{P} \rangle$ is one of such monoids.
References

[1] J. Amos, S. T. Chapman, N. Hine, and J. Paixao: *Sets of lengths do not characterize numerical monoids*, Integers 7 (2007) A50.
[2] D. D. Anderson, D. F. Anderson, and M. Zafrullah: *Factorizations in integral domains*, J. Pure Appl. Algebra 69 (1990) 1–19.
[3] D. D. Anderson, J. Coykendall, L. Hill, and M. Zafrullah: *Monoid domain constructions of antimatter domains*, Comm. Algebra 35 (2007) 3236–3241.
[4] A. Assi and P. A. García-Sánchez: *Numerical Semigroups and Applications*, RSME Springer Series, Springer, New York, 2016.
[5] C. Bowles, S. T. Chapman, N. Kaplan, and D. Reiser: *On delta sets of numerical monoids*, J. Algebra Appl. 5 (2006) 695–718.
[6] M. Bras-Amorós: *Increasingly enumerable submonoids of $\mathbb{R}$: Music theory as a unifying theme*, Amer. Math. Monthly (to appear).
[7] L. Carlitz: *A characterization of algebraic number fields with class number two*, Proc. Amer. Math. Soc. 11 (1960) 391–392.
[8] S. T. Chapman, M. Corrales, A. Miller, C. Miller, and D. Patel: *The catenary degrees of elements in numerical monoids generated by arithmetic sequences*, Comm. Algebra 45 (2017) 5443–5452.
[9] S. T. Chapman, J. Daigle, R. Hoyer, and N. Kaplan: *Delta sets of numerical monoids using nonminimal sets of generators*, Comm. Algebra 38 (2010) 2622–2634.
[10] S. T. Chapman, J. I. García-García, P. A. García-Sánchez, and J. C. Rosales: *Computing the elasticity of a Krull monoid*, Linear Algebra Appl. 336 (2001) 191–200.
[11] S. T. Chapman, P. A. García-Sánchez, Z. Tripp, and C. Viola: *Measuring primality in numerical semigroups with embedding dimension three*, J. Algebra Appl. 15 (2016) 1650007.
[12] S. T. Chapman, F. Gotti, and M. Gotti, *Factorization invariants of Puiseux monoids generated by geometric sequences*, Comm. Algebra (2019), https://doi.org/10.1080/00927872.2019.1646269
[13] S. T. Chapman, F. Gotti, and R. Pelayo: *On delta sets and their realizable subsets in Krull monoids with cyclic class groups*, Colloq. Math. 137 (2014) 137–146.
[14] S. T. Chapman, R. Hoyer, and N. Kaplan: *Delta sets of numerical monoids are eventually periodic*, Aequationes Math. 77 (2009) 273–279.
[15] S. T. Chapman, M. T. Holden, and T. A. Moore: *Full elasticity in atomic monoids and integral domains*, Rocky Mountain J. Math. 36 (2006) 1437–1455.
[16] S. T. Chapman, U. Krause, and E. Oeljeklaus: *On Diophantine monoids and their class groups*, Pacific J. Math. 207 (2002) 125–147.
[17] J. Coykendall and F. Gotti: *On the atomicity of monoid algebras*, J. Algebra 539 (2019) 138–151.
[18] J. Coykendall and G. Oman: *Factorization theory of root closed monoids of small rank*, Comm. Algebra 45 (2017) 2795–2808.
[19] J. Coykendall and W. W. Smith: *On unique factorization domains*, J. Algebra 332 (2011) 62–70.
[20] M. D’Anna, P. A. García-Sánchez, V. Micale, and L. Tozzo: *Good subsemigroups of $\mathbb{N}^n$*, Internat. J. Algebra Comput. 28 (2018) 179–206.
[21] L. Fuchs: *Infinite Abelian Groups II*, Academic Press, 1973.
[22] P. A. García-Sánchez, I. Ojeda, and A. Sánchez-R.-Navarro: *Factorization invariants in half-factorial affine semigroups*, Internat. J. Algebra Comput. 23 (2013) 111–122.
[23] P. A. García-Sánchez and J. C. Rosales: *Numerical Semigroups*, Developments in Mathematics Vol. 20, Springer-Verlag, New York, 2009.
[24] A. Geroldinger: *Sets of lengths*, Amer. Math. Monthly 123 (2016) 960–988.
[25] A. Geroldinger, F. Gotti, and S. Tringali: *On strongly primary monoids with a focus on Puiseux monoids*, Manuscript.
[26] A. Geroldinger and F. Halter-Koch: Non-unique Factorizations: Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics Vol. 278, Chapman & Hall/CRC, Boca Raton, 2006.

[27] A. Geroldinger and F. Halter-Koch: Non-unique factorizations: a survey. In: J. W. Brewer, S. Glaz, W. J. Heinzer, B. M. Olberding (eds) Multiplicative Ideal Theory in Commutative Algebra. Springer, Boston, MA, 2006.

[28] A. Geroldinger, S. Ramacher, and A. Reinhart: On v-Marot Mori rings and C-rings, J. Korean Math. Soc. 52 (2015) 1–21.

[29] A. Geroldinger and M. Roitman: On strongly primary monoids and domains. In arXiv: https://arxiv.org/abs/1807.10683.

[30] A. Geroldinger and W. Schmid: A realization theorem for sets of lengths in numerical monoids, Forum Math. 30 (2018) 1111–1118.

[31] A. Geroldinger, W. Schmid, and Q. Zhong: Systems of sets of lengths: transfer Krull monoids versus weakly Krull monoids, In: M. Fontana, S. Frisch, S. Glaz, F. Tartarone, P. Zanardo (eds): Rings, Polynomials, and Modules, Springer, Cham, 2017, pp. 191–235.

[32] R. Gilmer: Commutative Semigroup Rings, The University of Chicago Press, 1984.

[33] R. Gilmer and T. Parker: Semigroup rings as Prüfer rings, Duke Math. J. 41 (1974) 219–230.

[34] R. Gipson and H. Kulosman: For which additive submonoids $M$ of nonnegative rationals is $F[X; M]$ AP? In arXiv: https://arxiv.org/pdf/1805.10373.pdf

[35] F. Gotti: Increasing positive monoids of ordered fields are FF-monoids, J. Algebra 518 (2019) 40–56.

[36] F. Gotti: Irreducibility and factorizations in monoid rings, Springer INdAM Series: Proceedings of the IMNS (to appear). In arXiv: https://arxiv.org/pdf/1905.07168.pdf

[37] F. Gotti: On the atomic structure of Puiseux monoids, J. Algebra Appl. 16 (2017) 1750126.

[38] F. Gotti: Puiseux monoids and transfer homomorphisms, J. Algebra 516 (2018) 95–114.

[39] F. Gotti: Systems of sets of lengths of Puiseux monoids, J. Pure Appl. Algebra 223 (2019) 1856–1868.

[40] F. Gotti and M. Gotti: Atomicity and boundedness of monotone Puiseux monoids, Semigroup Forum 96 (2018) 536–552.

[41] F. Gotti and M. Gotti: On the molecules of numerical monoids, Puiseux monoids, and Puiseux algebras, Springer INdAM Series: Proceedings of the IMNS (to appear). In arXiv: https://arxiv.org/pdf/1702.08270.pdf

[42] F. Gotti and C. O’Neil: The elasticity of Puiseux monoids, J. Commut. Algebra (2019), https://projecteuclid.org/euclid.jca/1523433696

[43] M. Gotti: On the local k-elasticities of Puiseux monoids, Internat. J. Algebra Comput. 29 (2019) 147–158.

[44] A. Grams: Atomic domains and the ascending chain condition for principal ideals. Math. Proc. Cambridge Philos. Soc. 75 (1974) 321–329.

[45] P. A. Grillet: Commutative Semigroups, Advances in Mathematics Vol. 2, Kluwer Academic Publishers, Boston, 2001.

[46] F. Kainrath: Arithmetic of Mori domains and monoids: the global case, Multiplicative Ideal Theory and Factorization Theory. Springer, 2016. 183–218.

[47] A. Reinhart: On integral domains that are C-monoids, Houston J. Math. 39 (2013) 1095–1116.

[48] A. Zaks: Half-factorial domains, Bull. Amer. Math. Soc. 82 (1976) 721–723.
Department of Mathematics, Sam Houston State University, Huntsville, TX 77341
E-mail address: scott.chapman@shsu.edu

Department of Mathematics, UC Berkeley, Berkeley, CA 94720
E-mail address: felixgotti@berkeley.edu

Department of Mathematics, University of Florida, Gainesville, FL 32611
E-mail address: marlycormar@ufl.edu