\textbf{Z}_k^2\text{-MANIFOLDS ARE ISOSPECTRAL ON FORMS.}

R. J. MIATELLO, R. A. PODESTÁ, AND J. P. ROSSETTI.

\textsc{Abstract.} We obtain a simple formula for the multiplicity of eigenvalues of the Hodge-Laplace operator, $\Delta_f$, acting on sections of the full exterior bundle $\Lambda(TM) = \bigoplus_{p=0}^{n} \Lambda^p(TM)$ over an arbitrary compact flat Riemannian $n$-manifold $M$ with holonomy group $\mathbb{Z}_k^2$, with $1 \leq k \leq n-1$. This formula implies that any two compact flat manifolds with holonomy group $\mathbb{Z}_k^2$ having isospectral lattices of translations are isospectral on forms, that is, with respect to $\Delta_f$. As a consequence, we construct a large family of pairwise $\Delta_f$-isospectral and nonhomeomorphic $n$-manifolds of cardinality greater than $2^{\frac{(n-1)(n-2)}{2}}$.

\textbf{Introduction}

In [\textsc{MR2,3,4}] the spectrum of the Hodge-Laplacian on $p$-forms on compact flat manifolds was studied, comparing $p$-isospectrality with other types of isospectrality. In particular, pairs of manifolds that are isospectral on $p$-forms for a fixed value of $p > 0$ were constructed, having different lengths of closed geodesics or different first eigenvalue of the Laplacian on functions. Most of the examples given belong to the class of $\mathbb{Z}_k^2$-manifolds, that is, flat Riemannian manifolds with holonomy group $\mathbb{Z}_k^2$. By the Cartan-Ambrose-Singer theorem, such manifolds are necessarily flat, hence of the form $M_{\Gamma} = \Gamma \backslash \mathbb{R}^n$, $\Gamma$ a Bieberbach group with translation lattice $\Lambda$ and with holonomy group $F := \Lambda \backslash \Gamma \cong \mathbb{Z}_k^2$.

The goal of this paper is to show that if we consider the full exterior bundle over a $\mathbb{Z}_k^2$-manifold $M_{\Gamma}$, there is a high degree of regularity in the spectrum of the Hodge Laplacian, $\Delta_f$, acting on sections of this bundle. Two manifolds having the same spectrum with respect to $\Delta_f$ will be called \textit{isospectral on forms}. We shall see that the spectrum of a flat manifold $M_{\Gamma}$ is completely determined by the spectrum of the covering torus $T_{\Lambda}$, and furthermore any two $\mathbb{Z}_k^2$-manifolds $M_{\Gamma}, M_{\Gamma'}$, with covering torus $T_{\Lambda}$ are isospectral on forms. They are also isospectral on even (resp. odd) forms, that is, with respect to the operator $\Delta_f$ restricted to even (resp. odd) forms. This allows to obtain very large families of $\Delta_f$-isospectral $n$-manifolds, pairwise nonhomeomorphic to each other. In particular we will describe a family of flat manifolds, the so called generalized Hantzsche-Wendt manifolds (see [\textsc{RS}]), having holonomy group $\mathbb{Z}_k^{n-1}$, whose cardinality is greater than $2^{\frac{(n-1)(n-2)}{2}}$. The proof of the main result uses the multiplicity formulae in [\textsc{MR2}] together with some symmetry properties of the Krawtchouk polynomials. We point out that the above isospectrality result is valid only for holonomy groups $F \simeq \mathbb{Z}_k^2$.

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Indeed, we shall see that it fails to hold for flat manifolds with holonomy group $F \simeq \mathbb{Z}_4$ and $F \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ (Example 3.5).

1. Preliminaries

We first recall from [Ch] or [Wo] some standard facts on compact flat manifolds. A Bieberbach group is a discrete, cocompact torsion-free subgroup $\Gamma$ of the isometry group $I(\mathbb{R}^n)$ of $\mathbb{R}^n$. Such $\Gamma$ acts properly discontinuously on $\mathbb{R}^n$, hence $M_\Gamma = \Gamma \backslash \mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group $\Gamma$. Furthermore, any such manifold arises in this way. Since $I(\mathbb{R}^n) \simeq O(n) \rtimes \mathbb{R}^n$, any element $\gamma \in I(\mathbb{R}^n)$ decomposes uniquely as $\gamma = BL_b$, with $B \in O(n)$ and $b \in \mathbb{R}^n$. The translations in $\Gamma$ form a normal maximal abelian subgroup of finite index $L_\Lambda$, $\Lambda$ a lattice in $\mathbb{R}^n$ which is $B$-stable for every $BL_b \in \Gamma$. The restriction to $\Gamma$ of the canonical projection $r : I(\mathbb{R}^n) \to O(n)$, given by $BL_b \mapsto B$, is a homomorphism with kernel $\Lambda$ and $r(\Gamma)$ is a finite subgroup of $O(n)$ isomorphic to $F := \Lambda \backslash \Gamma$, the linear holonomy group of the Riemannian manifold $M_\Gamma$.

We recall from [MR2] the multiplicity formula for the eigenvalues of the Hodge Laplace operator $-\Delta_p$ acting on smooth $p$-forms of a compact flat manifold $M_\Gamma$. For any $\mu \geq 0$, let

$$\Lambda_\mu^* = \{ \lambda \in \Lambda^*: \|\lambda\|^2 = \mu \}$$

where $\Lambda^*$ is the dual lattice of $\Lambda$. In [MR2], Theorem 3.1, it is shown that the multiplicity of the eigenvalue $4\pi^2 \mu$ of $-\Delta_p$ is given by

$$d_{p,\mu}(\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma = BL_b \in \Lambda \backslash \Gamma} \text{tr}_p(B) e_{\mu,\gamma}(\Gamma)$$

where $e_{\mu,\gamma} = \sum_{v \in \Lambda_\mu^* : Bv = v} e^{-2\pi iv \cdot b}$ and $\text{tr}_p$ is the trace of the $p$-exterior representation $\tau_p : O(n) \to \text{GL}(\Lambda^p(\mathbb{R}^n))$.

A Bieberbach group $\Gamma$ is said to be of diagonal type (see [MR3], Definition 1.3) if there exists an orthonormal $\mathbb{Z}$-basis $\{\lambda_1, \ldots, \lambda_n\}$ of the lattice $\Lambda$ such that for any element $BL_b \in \Gamma$, $BL_i = \pm \lambda_i$ for $1 \leq i \leq n$. These Bieberbach groups have holonomy group $\mathbb{Z}_2^k$ for some $1 \leq k \leq n - 1$. If $\Gamma$ is of diagonal type, after conjugation of $\Gamma$ by an isometry, it may be assumed that $\Lambda$ is the canonical (or cubic) lattice and, furthermore, that $b$ lies in $\frac{1}{2} \Lambda$ for any $\gamma = BL_b \in \Gamma$ (see [MR3], Lemma 1.4).

For Bieberbach groups of diagonal type, the traces $\text{tr}_p(B)$ in (1.2) are given by integral values of the Krawtchouk polynomials of degree $p$

$$K_p^n(x) := \sum_{t=0}^{p} (-1)^t \binom{x}{t} \binom{n-x}{p-t}$$

(see [MR2], Remark 3.6 and [MR3]; also, see [KL] for more information on Krawtchouk polynomials). Indeed, we have

$$\text{tr}_p(B) = K_p^n(n - n_B), \quad \text{where } n_B := \dim (\mathbb{R}^n)^B = \dim \ker(B - \text{Id}).$$

The first Krawtchouk polynomials are $K_0^n(x) = 1$, $K_1^n(x) = -2x + n$, $K_2^n(x) = 2x^2 - 2nx + \binom{n}{2}$, $K_3^n(x) = -\frac{1}{3}x^3 + 2nx^2 - (n^2 - n + \frac{2}{3})x + \binom{n}{3}$. For later use we also give, in the following tables, the integral values of $K_p^n(x)$ for $0 \leq p, x \leq n$, $n = 3, 4$. 

\[\begin{array}{c|c|c|c|c|c|c}
 p & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
 x & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
 K_0^n(x) & 1 & 1 & 1 & 1 & 1 & 1 \\
 K_1^n(x) & 1 & n & n+1 & n+2 & n+3 & n+4 \\
 K_2^n(x) & 1 & n^2 & n^2+1 & n^2+2 & n^2+3 & n^2+4 \\
 K_3^n(x) & 1 & n^3 & n^3+1 & n^3+2 & n^3+3 & n^3+4 \\
\end{array}\]
The spectrum on forms of $\mathbb{Z}^k_\star$-manifolds.

Let $\bigoplus_{p=0}^n \Lambda^p (T(M_\Gamma))$ be the full exterior bundle of the compact flat manifold $M_\Gamma$ and let $\Delta_p$ be the Hodge Laplacian acting on $p$-forms. We shall denote by

$$\Delta_f := \sum_{p=0}^n \Delta_p, \quad \Delta_e := \sum_{p \text{ even}} \Delta_p, \quad \Delta_o := \sum_{p \text{ odd}} \Delta_p,$$

the Laplacian on forms, on even forms and on odd forms of $M_\Gamma$, respectively.

The multiplicity of the eigenvalue $4\pi^2 \mu$ for $\Delta_f$ is given by

$$d_{f,\mu}(\Gamma) = \sum_{p=0}^n d_{p,\mu}(\Gamma)$$

and similarly $d_{e,\mu}(\Gamma) = \sum_{p \text{ even}} d_{p,\mu}(\Gamma)$ and $d_{o,\mu}(\Gamma) = \sum_{p \text{ odd}} d_{p,\mu}(\Gamma)$, for $\Delta_e$ and $\Delta_o$ respectively. Thus, $\Delta_f = \Delta_e + \Delta_o$ and $d_{f,\mu}(\Gamma) = d_{e,\mu}(\Gamma) + d_{o,\mu}(\Gamma)$.

Clearly, $p$-isospectrality for all $p$ implies $\Delta_f$-isospectrality (as well as $\Delta_e$ and $\Delta_o$-isospectrality), but we shall see that the converse is far from being true.

**Theorem 2.1.** If $\Gamma$ is a Bieberbach group with translation lattice $\Lambda$ and holonomy group $\mathbb{Z}^k_\star$, then for any $\mu \geq 0$ the multiplicities of the eigenvalue $4\pi^2 \mu$ for $\Delta_f$, $\Delta_e$ and $\Delta_o$ are given respectively by

$$d_{f,\mu}(\Gamma) = 2^{n-k} |\Lambda^*|, \quad d_{e,\mu}(\Gamma) = d_{o,\mu}(\Gamma) = 2^{n-k-1} |\Lambda^*|.$$

Thus, if $M_\Gamma, M_{\Gamma'}$ are $\mathbb{Z}^k_\star$-manifolds with translation lattices $\Lambda, \Lambda'$, then $M_\Gamma$ and $M_{\Gamma'}$ are isospectral on forms (resp. on even or odd forms) if and only if $\Lambda$ and $\Lambda'$ are isospectral. In particular, for fixed $\Lambda$ and $k$, all $\mathbb{Z}^k_\star$-manifolds having covering torus $T_\Lambda$ are $\Delta_f$, $\Delta_e$ and $\Delta_o$-isospectral.

**Proof.** Let $M_\Gamma$ be a $\mathbb{Z}^k_\star$-manifold. Then $M_\Gamma = \Gamma \backslash \mathbb{R}^n$ with $\Gamma = \langle \gamma_1, \ldots, \gamma_k, \Lambda \rangle$, where $\Lambda$ is a lattice and $\gamma_i = B_i L_{b_i}$, $B_i \in O(n)$, $b_i \in \mathbb{R}^n$, $B_i \Lambda = \Lambda$, $B_i^2 = Id$, $B_i B_j = B_j B_i$ for each $1 \leq i, j \leq k$.

We know that if $B_i$ is diagonal then $\text{tr}_p(B_i) = K_p^\gamma(n-nB_i)$ (see [MR2], Remark 3.6). This is also valid for non-diagonal matrices $B$ of order 2. Indeed, $B$ has only eigenvalues of the form $\pm 1$, hence $B$ is conjugate in $\text{GL}_n(\mathbb{R})$ to the diagonal matrix $D_B := \begin{bmatrix} -1^{n-nB} & 0 \\ 0 & I_{nB} \end{bmatrix}$ where $I_m$ is the identity matrix in $\mathbb{R}^m$. Thus $\text{tr}_p(B) = \text{tr}_p(D_B) = K_p^\gamma(n-nB)$. 

\[
\begin{array}{c|cccc}
  x & 0 & 1 & 2 & 3 \\
  \hline
  K^3_0(x) & 1 & 1 & 1 & 1 \\
  K^3_1(x) & 3 & 1 & -1 & -3 \\
  K^3_2(x) & 3 & -1 & -1 & 3 \\
  K^3_3(x) & 1 & -1 & 1 & -1 \\
\end{array} \\
\begin{array}{c|cccc}
  x & 0 & 1 & 2 & 3 \\
  \hline
  K^4_0(x) & 1 & 1 & 1 & 1 \\
  K^4_1(x) & 4 & 2 & 0 & -2 \\
  K^4_2(x) & 6 & 0 & -2 & 0 \\
  K^4_3(x) & 4 & -2 & 0 & -4 \\
  K^4_4(x) & 1 & -1 & 1 & -1 \\
\end{array}
\]
Hence, by (1.2), (1.4) and the fact that $K^n_p(0) = \binom{n}{p}$, we have
\[ d_{p,\mu}(\Gamma) = 2^{-k} \left( \binom{n}{p} |\Lambda^*_\mu| + \sum_{\gamma \in \Lambda \setminus \Gamma, \gamma \neq Id} K^n_p(n - n_B) e_{\mu,\gamma}(\Gamma) \right) \]
and, adding over $p$, we obtain
\[ d_{f,\mu}(\Gamma) = 2^n - k |\Lambda^*_\mu| + 2^{-k} \sum_{\gamma \in \Lambda \setminus \Gamma, \gamma \neq Id} \left( \sum_{p=0}^n K^n_p(n - n_B) \right) e_{\mu,\gamma}(\Gamma). \]
Now, we show that $\sum_{p=0}^n K^n_p(j) = 0$ for fixed $j \neq 0$. In fact,
\[
\begin{align*}
\sum_{p=0}^n K^n_p(j) &= \sum_{p=0}^n \sum_{t=0}^j (-1)^t \binom{j}{t} \binom{n-j}{p-t} \\
&= \sum_{t=0}^j (-1)^t \binom{j}{t} \sum_{p-t=0}^{n-j} \binom{n-j}{p-t} \\
&= 2^{n-j} \sum_{t=0}^j (-1)^t \binom{j}{t} = 0
\end{align*}
\]
Thus, since $n - n_B = 0$ if and only if $B = Id$, we obtain that $d_{f,\mu}(\Gamma) = 2^{n-k} |\Lambda^*_\mu|$, as claimed. The proofs for $d_{e,\mu}(\Gamma)$ and $d_{o,\mu}(\Gamma)$ are the same, except that we add over even and odd values of $p$, respectively. □

Remark 2.2. (i) We note that for the $n$-torus $T_\Lambda$, for each $0 \leq p \leq n$, we have $d'_{p,\mu}(\Lambda) = \binom{n}{p} |\Lambda^*_\mu|$, hence $0$-isospectrality is equivalent to $p$-isospectrality for any $p > 0$, and this in turn is equivalent to $\Delta_f$-isospectrality. However, there are many examples of pairs of compact flat manifolds that are $p$-isospectral for some $p > 0$ but are not isospectral on functions and also pairs of manifolds that are isospectral on functions but are not $p$-isospectral for any $0 < p < n$ (see [MR2, MR3]).

(ii) We shall see that Theorem 2.1 does not hold for general holonomy groups. For instance, Example 3.5 will show it fails to hold when $F$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$.

3. Examples and Counterexamples

Example 3.1. We now consider a family of $\mathbb{Z}_2$-manifolds, of cardinality quadratic in $n$, which are pairwise not isospectral on functions, but which isospectral on forms, according to Theorem 2.1.

Put $J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For each $0 \leq j, h < n$, define
\[ B_{j,h} := \text{diag}(J, \ldots, J, -1, \ldots, -1, 1, \ldots, 1) \]
where $n = 2j + h + l$, $j + h \neq 0$ and $l \geq 1$. Then $B_{j,h} \in O(n)$, $B_{j,h}^2 = Id$. Let $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ be the canonical lattice of $\mathbb{R}^n$ and for $j, h$ as before define the groups
\[ \Gamma_{j,h} := \langle B_{j,h} L e_n, \Lambda \rangle. \]
We have that $\Lambda$ is stable by $B_{j,h}$ and $(B_{j,h} + Id)^{2} = e_{n} \in \Lambda \setminus (B_{j,h} + Id)\Lambda$. It is easy to verify that $\Gamma_{j,h}$ is torsion-free, hence a Bieberbach group. In this way, if we set $M_{j,h} := \Gamma_{j,h} \backslash \mathbb{R}^{n}$, we have a family
\[(3.3) \quad F := \{M_{j,h} = \Gamma_{j,h} \backslash \mathbb{R}^{n} : 0 \leq j \leq \left[\frac{n-1}{2}\right], 0 \leq h < n - 2j, j + h \neq 0\}
\]
of compact flat manifolds with holonomy group $F \simeq \mathbb{Z}_{2}$.

Furthermore, the family $F$ gives a system of representatives for the diffeomorphism classes of $\mathbb{Z}_{2}$-manifolds of dimension $n$ (see [MP] for a proof). Also
\[(3.4) \quad H_{1}(M_{j,h}, \mathbb{Z}) \simeq \mathbb{Z}^{j+l} \oplus \mathbb{Z}_{2}^{h},\]
and if $1 \leq p \leq n$, then the Betti numbers are
\[(3.5) \quad \beta_{p}(M_{j,h}) = \sum_{i=0}^{\left\lfloor p \right\rfloor} \binom{j + h}{2i} \binom{j + l}{p - 2i}.
\]
Hence, if $\beta_{1}(M_{j,h}) = \beta_{1}(M_{j',h'})$, then $\beta_{p}(M_{j,h}) = \beta_{p}(M_{j',h'})$ for any $p \geq 1$.

Moreover,
\[(3.6) \quad \# F = (n - \left[\frac{n-1}{2}\right])\left(\left[\frac{n-1}{2}\right] + 1\right) - 1 = \begin{cases} \frac{n^{2} + 2n - 4}{4} & \text{if } n \text{ even} \\ \frac{n^{2} + 2n - 3}{4} & \text{if } n \text{ odd} \end{cases}\]

Now, since $B_{j,h}$ and $B_{0,j+h}$ are conjugate in $\text{GL}_{n}(\mathbb{R})$ we have that $\text{tr}(B_{j,h}) = \text{tr}(B_{0,j+h}) = K_{p}^{n}(j + h)$. Hence, by formula (1.2), the expression for the multiplicity of the eigenvalue $4\pi^{2}\mu$ of $-\Delta_{p}$ equals
\[(3.7) \quad d_{p,\mu}(\Gamma_{j,h}) = \frac{1}{2}\left(\binom{n}{p} |\Lambda_{\mu}| + K_{p}^{n}(j + h) e_{\mu,\gamma}(\Gamma_{j,h})\right),\]
where $e_{\mu,\gamma}(\Gamma) = \sum_{\nu \in \Lambda_{\mu}} e^{-2\pi i \nu \cdot \nu}$ and $\Lambda^{B}$ denotes the elements in $\Lambda$ fixed by $B$.

We claim that the manifolds in $F$ are pairwise not isospectral on functions. To see this, it will suffice to compare the multiplicities of the two smallest nonzero eigenvalues, namely $\mu = 1$ and $\mu = \sqrt{2}$.

Take $\mu = 1$. Then $\Lambda_{1} = \{\pm e_{1}, \ldots, \pm e_{n}\}$ and $\Lambda_{1}^{B_{j,h}} = \{\pm e_{2j+h+1}, \ldots, \pm e_{n}\}$ thus $|\Lambda_{1}| = 2n$ and $|\Lambda_{1}^{B_{j,h}}| = 2(n - (2j + h)) = 2l$. Now, one checks that $e_{1,\gamma}(\Gamma_{j,h}) = 2(l - 1) + 2(-1) = 2(l - 2)$ and hence we get from (3.7)
\[(3.8) \quad d_{p,1}(\Gamma_{j,h}) = \binom{n}{p} n + K_{p}^{n}(j + h)(l - 2).
\]

Now consider $\mu = \sqrt{2}$. Then $\Lambda_{\sqrt{2}} = \{\pm(e_{i} \pm e_{j}) : 1 \leq i < j \leq n\}$ and $\Lambda_{\sqrt{2}}^{B_{j,h}} = \{\pm(e_{2i-1} \pm e_{2i}) : 1 \leq i \leq j\} \cup \{\pm(e_{i} \pm e_{j}) : n - l + 1 \leq i < j \leq n\}$. Hence $|\Lambda_{\sqrt{2}}| = 4\binom{n}{2}$ and $|\Lambda_{\sqrt{2}}^{B_{j,h}}| = 2j + 4\binom{j}{2}$. One checks that $e_{\sqrt{2},\gamma}(\Gamma_{j,h}) = 2j + 4\left(\binom{j}{2} - 2(l - 1)\right) = 2j + 2(l - 1)(l - 4)$. In this way we obtain
\[(3.9) \quad d_{p,\sqrt{2}}(\Gamma_{j,h}) = 2\binom{n}{p} \binom{j}{2} + K_{p}^{n}(j + h)(j + (l - 1)(l - 4)).\]

In particular for $p = 0$, since $K_{0}^{n}(j + h) = 1$ for any $j$, we have
\[(3.10) \quad d_{0,1}(\Gamma_{j,h}) = n + l - 2\]
\[(3.11) \quad d_{0,\sqrt{2}}(\Gamma_{j,h}) = n(n - 1) + j + (l - 1)(l - 4).\]
This allows to distinguish the spectra on functions of the $\mathbb{Z}_2$-manifolds considered. Indeed, if $M_{j,h}, M_{j',h'}$ are isospectral then $l = l'$ by (3.10), thus $2j + h = 2j' + h'$. By (3.11), $j = j'$ and hence $h = h'$. This shows that all manifolds in $\mathcal{F}$ are pairwise not isospectral to each other.

To illustrate the compensations occurring in the sums in (2.2) we compute the individual multiplicities $d_{p,\mu}(\Gamma)$ corresponding to $\mu = 1, \sqrt{2}$, for manifolds in $\mathcal{F}$ in dimensions 3 and 4.

In dimension 3, there are only three $\mathbb{Z}_2$-manifolds up to diffeomorphism (see [Wo]): $M_{1,0}, M_{0,2}$ and $M_{0,1}$, with holonomy groups generated respectively by the matrices $[J_1], [-I_1]$ and $[-I_1]$ where $I$ is the $2 \times 2$ identity matrix. These manifolds are called dicosm ($c2$), first amphicosm ($+a1$) and second amphicosm ($-a1$) respectively, in [CR].

Using formulae (3.8) and (3.9) and the tables in (1.5) for the integral values of Krawtchouk polynomials we compute the following values of $d_{p,1}$ and $d_{p,\sqrt{2}}$, for $0 \leq p \leq 3$:

\[
\begin{array}{cccccc}
\mu = 1 & d_0 & d_1 & d_2 & d_3 & d_f \\
M_{1,0} & 2 & 8 & 26 & 16 & 3 \\
M_{0,2} & 2 & 10 & 10 & 2 \\
M_{0,1} & 3 & 9 & 9 & 3 \\
\end{array}
\quad
\begin{array}{cccccc}
\mu = \sqrt{2} & d_0 & d_1 & d_2 & d_3 & d_f \\
M_{1,0} & 7 & 19 & 17 & 5 \\
M_{0,2} & 6 & 18 & 18 & 6 \\
M_{0,1} & 4 & 16 & 20 & 8 \\
\end{array}
\]

In dimension 4 there are five nondiffeomorphic $\mathbb{Z}_2$-manifolds, $M_{1,1}, M_{1,0}, M_{0,3}, M_{0,2}$ and $M_{0,1}$, with holonomy group generated, respectively, by the matrices $[J_1], [J_1], [-I_1], [-I_1]$ and $[-I_1]$. Proceeding as before we get the tables:

\[
\begin{array}{cccccc}
\mu = 1 & d_0 & d_1 & d_2 & d_3 & d_f \\
M_{1,1} & 3 & 16 & 26 & 16 & 3 \\
M_{1,0} & 4 & 16 & 24 & 16 & 4 \\
M_{0,3} & 3 & 18 & 24 & 14 & 5 \\
M_{0,2} & 4 & 16 & 24 & 16 & 4 \\
M_{0,1} & 5 & 18 & 24 & 14 & 3 \\
\end{array}
\quad
\begin{array}{cccccc}
\mu = \sqrt{2} & d_0 & d_1 & d_2 & d_3 & d_f \\
M_{1,1} & 13 & 48 & 70 & 48 & 13 \\
M_{1,0} & 11 & 46 & 72 & 50 & 13 \\
M_{0,3} & 12 & 48 & 72 & 48 & 12 \\
M_{0,2} & 10 & 48 & 76 & 48 & 10 \\
M_{0,1} & 10 & 44 & 72 & 52 & 14 \\
\end{array}
\]

**Example 3.2.** Here we consider the $\mathbb{Z}_2^2$-manifolds of dimension 3 having the cubic lattice as lattice of translations (see [Wo], Section 3.5). There are three such manifolds, up to isometry. We shall see that they are not $p$-isospectral for any $0 \leq p \leq n$, showing for small eigenvalues how the compensations take place so that the sums of multiplicities for all $p$ become the same.

We consider the Hantzsche-Wendt manifold, $M_1 = \Gamma_1 \backslash \mathbb{R}^3$, and two nonorientable ones $M_2 = \Gamma_2 \backslash \mathbb{R}^3, M_3 = \Gamma_3 \backslash \mathbb{R}^3$, also called dicosm ($c2$), first amphicosm ($+a2$) and second amphicosm ($-a2$) respectively, in [CR].

The groups $\Gamma_i = \langle \gamma_1 = B_1L_{b_1}, \gamma_2 = B_2L_{b_2}, \Lambda \rangle$, are given in the table below, where $B_i = B_1, b_0 = B_2, b_1 = B_2b_1 + b_2 \mod \Lambda$, and $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ is the cubic lattice. All matrices $B_i$ are diagonal and are written as column vectors. We indicate the translation vectors $b_i$ also as column vectors, leaving out the coordinates that are equal to zero.
By substituting these values back in (3.12) we obtain:

\[ d_{p,\mu}(\Gamma_i) = \frac{1}{4} \left( \binom{3}{p} |\Lambda_{\mu}| + K_p^3(2) (e_{\mu,\gamma_1} + e_{\mu,\gamma_2} + e_{\mu,\gamma_3}) \right) \tag{3.12} \]

for \( i = 2, 3 \).

Now, to show that \( M_1, M_2 \) and \( M_3 \) are not pairwise \( p \)-isospectral for any \( 0 \leq p \leq 3 \) we shall again use two eigenvalues, namely those corresponding to \( \mu = 1 \) and \( \mu = \sqrt{5} \).

Take \( \mu = 1 \). Then \( \Lambda_1 = \{ \pm e_1, \pm e_2, \pm e_3 \} \) and hence \( |\Lambda_1| = 6 \). Also, \( \Lambda_1^{B_1} = \{ \pm e_i \} \) for \( 1 \leq i \leq 3 \), \( \Lambda_1^{B_1} = \{ \pm e_3 \} \), \( \Lambda_1^{B_2} = \{ \pm e_1, \pm e_3 \} \), and \( \Lambda_1^{B_3} = \{ \pm e_2, \pm e_3 \} \). For \( \mu = \sqrt{5} \) we see that \( \Lambda_1^{\sqrt{5}} = \{ \pm (2e_i \pm e_j) : 1 \leq i, j \leq 3 \} \), so \( |\Lambda_1^{\sqrt{5}}| = 24 \). Now, we have \( \Lambda_1^{\sqrt{5}} = \Lambda_1^{B_1} = 0 \) for \( 1 \leq i \leq 3 \), \( \Lambda_1^{B_2} = \{ \pm (e_1 \pm e_3), \pm (e_1 \pm e_2) \} \), and \( \Lambda_1^{B_3} = \{ \pm (2e_2 \pm e_3), \pm (e_2 \pm e_3) \} \).

With this information one computes the following values of \( e_{\mu,\gamma}(\Gamma_i) \) for \( 1 \leq i \leq 3 \):

| \( e_{1,\gamma_1} \) | \( e_{1,\gamma_2} \) | \( e_{1,\gamma_3} \) | \( e_{\sqrt{5},\gamma_1} \) | \( e_{\sqrt{5},\gamma_2} \) | \( e_{\sqrt{5},\gamma_3} \) |
|---|---|---|---|---|---|
| \( M_1 \) | -2 | -2 | -2 | 0 | 0 | 0 |
| \( M_2 \) | -2 | 0 | 0 | 0 | 0 | 0 |
| \( M_3 \) | -2 | 0 | -4 | 0 | 0 | -8 |

By substituting these values back in (3.12) we obtain:

\[
\begin{array}{c|c|c}
\Gamma_1 & \frac{3}{2} \left( \binom{3}{p} - K_p^3(2) \right) & 6 \binom{3}{p} \\
\Gamma_2 & \frac{1}{2} \left( 3 \binom{3}{p} - K_p^3(2) \right) & 6 \binom{3}{p} \\
\Gamma_3 & \frac{1}{2} \left( 3 \binom{3}{p} - K_p^3(2) - 2K_p^3(1) \right) & 6 \binom{3}{p} - 2K_p^3(1) \\
\end{array}
\]

With this information and using (1.3), we are now in a position to give the multiplicities for the two eigenvalues we are considering.

| \( \mu = 1 \) | \( d_0 \) | \( d_1 \) | \( d_2 \) | \( d_3 \) | \( d_f \) | \( \mu = \sqrt{5} \) | \( d_0 \) | \( d_1 \) | \( d_2 \) | \( d_3 \) | \( d_f \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( M_1 \) | 0 | 6 | 6 | 0 | 12 | \( M_1 \) | 6 | 18 | 18 | 6 | 48 |
| \( M_2 \) | 1 | 5 | 5 | 1 | 12 | \( M_2 \) | 6 | 18 | 18 | 6 | 48 |
| \( M_3 \) | 0 | 4 | 6 | 2 | 12 | \( M_3 \) | 4 | 16 | 20 | 8 | 48 |
From these tables it is clear that the manifolds $M_1, M_2, M_3$ are pairwise not $p$-isospectral for any individual value of $p$, $0 \leq p \leq 3$.

**Example 3.3.** In [RT] a family $B_n$ of pairwise nonhomeomorphic $\mathbb{Z}_2^n$ manifolds of dimension $n$ with $\beta_1 = 0$, is given. This family, which includes Cobb’s family as a rather small subfamily, grows polynomially as $\frac{n^5}{2^3 \cdot \pi^2}$. By Theorem 2.1 all manifolds in $B_n$ are $\Delta_f$-isospectral.

**Example 3.4.** We first recall some facts from [MRT]. Let $n$ be odd. A Hantzsche-Wendt (or HW) group is an $n$-dimensional orientable Bieberbach group $\Gamma$ with holonomy group $F \simeq \mathbb{Z}_2^{n-1}$ such that the action of every $B \in F$ diagonalizes on the canonical $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of $\Lambda$. The holonomy group $F$ can thus be identified to the diagonal subgroup $\{B : Be_i = \pm e_i, 1 \leq i \leq n, \det B = 1\}$ and $M_1 = \Gamma \backslash \mathbb{R}^n$ is called a Hantzsche-Wendt manifold.

Denote by $B_i$ the diagonal matrix fixing $e_i$ and such that $B_i e_j = -e_j$ (if $j \neq i$), for each $1 \leq i \leq n$. Clearly, $F$ is generated by $B_1, B_2, \ldots, B_{n-1}$.

Any HW group has the form $\Gamma = \langle B_1 L_{b_1}, \ldots, B_{n-1} L_{b_{n-1}}, L_{\lambda} : \lambda \in \Lambda \rangle$, for some $b_i \in \mathbb{R}^n$, $1 \leq i \leq n-1$, where it may be assumed that the components $b_{ij}$ of $b_i$ satisfy $b_{ij} \in \{0, \frac{1}{2}\}$, for $1 \leq i, j \leq n$. Also, it is easy to see that $N^p(\mathbb{R}^n)^F = 0$ for any $1 \leq p \leq n-1$, hence all Betti numbers are zero for $1 \leq p \leq n-1$, thus HW manifolds are rational homology spheres.

We further recall that it is shown in [MRT] (by considering a rather small subfamily) that the cardinality $h_n$ of the family of all HW groups under isomorphism satisfies $h_n > \frac{2^{n-3}}{n-1}$. Moreover, the cardinality of the pairs of isospectral, nonisomorphic HW groups grows exponentially with $n$.

All HW manifolds form a family of pairwise nonhomeomorphic compact flat $n$-manifolds, of cardinality growing exponentially with $n$, which by Theorem 2.1 are mutually $\Delta_f$-isospectral.

**Example 3.5.** Here we show that Theorem 2.1 fails to hold without the assumption that $F \simeq \mathbb{Z}_2^k$, even when the manifolds are isospectral on functions.

(i) First, we consider the pair $M, M'$ of manifolds of dimension 6, having holonomy group $\mathbb{Z}_4 \times \mathbb{Z}_2$, studied in [MR2], Example 5.1. Take $\tilde{J} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Let $\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}, \Lambda \rangle$ and $\Gamma' = \langle B'_1 L_{b'_1}, B'_2 L_{b'_2}, \Lambda \rangle$ where $\Lambda$ is the canonical lattice in $\mathbb{R}^6$ and

\[
B_1 = \begin{bmatrix} j & 1 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b_1 = \frac{\alpha_4}{4}, \quad b_2 = \frac{\alpha_6}{2},
\]

\[
B'_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B'_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad b'_1 = \frac{\alpha_6}{4}, \quad b'_2 = \frac{\alpha_4 + \alpha_6}{2}.
\]

In [MR2] it is shown that $M = \Gamma \backslash \mathbb{R}^6, M' = \Gamma' \backslash \mathbb{R}^6$ are isospectral on functions –and hence 6-isospectral, by orientability– but they are not $p$-isospectral for any $1 \leq p \leq 5$. We claim they are not $\Delta_f$-isospectral. To see this, it will be sufficient to look at $\mu = 0$. Indeed, since $d_{p,0}(\Gamma) = \beta_p(M)$ we have that $d_{f,0}(M) = \sum_{p=0}^{6} \beta_p(M)$. The Betti numbers for $M, M'$ are
forms. In particular, for each nonhomeomorphic to each other, which by Theorem 2.1 will be isospectral on $B$ while for are, in this case, respectively given by 1, 2, 5, 8, 5, 2, 1 and 1, 2, 3, 4, 3, 2, 1. For instance, to verify the values in the case of $\beta_2$, we note that a basis of vectors fixed by $B_1$ on $\Lambda^2(\mathbb{R}^6)$ is given by $e_1 \wedge e_2, e_3 \wedge e_4, e_5 \wedge e_6, e_1 \wedge e_4, e_2 \wedge e_3$ while for $B_1'$ a basis of fixed vectors is $e_1 \wedge e_2, e_3 \wedge e_6, e_4 \wedge e_5$.

Thus, in this case we have $d_{f,0}(M) = 24$ while $d_{f,0}(M') = 16$, hence $M$ and $M'$ are not isospectral on forms.

4. LARGE FAMILIES OF MANIFOLDS ISOSPECTRAL ON FORMS

In this section we will exhibit large families of $\mathbb{Z}_2^k$-manifolds, pairwise nonhomeomorphic to each other, which by Theorem 2.1 will be isospectral on forms. In particular, for each $n$, we shall construct a family of $n$-dimensional $\mathbb{Z}_2^{n-1}$-manifolds with cardinality of order approximately $(\sqrt{2})^{n^2}$, for $n$ large. Consider the subgroup of $I(\mathbb{R}^n)$ with set of generators $\{C_i, L_{c_i}, L_{e_j} : 1 \leq i \leq n-1, 1 \leq j \leq n\}$, where $C_i := \text{diag}(1, \ldots, 1, -1, 1, \ldots, 1)$, and $c_i := \frac{\sqrt{2}+1}{2} + \sum_{j=1}^i c_{ij} e_j$ for some choice of $c_{ij} \in \{0, \frac{1}{2}\}$. Here $\{e_1, e_2, \ldots, e_n\}$ denotes the canonical basis of $\Lambda = \mathbb{Z}^n$. Similarly as in Example 3.2 above, we show in (4.1) such a group in column notation, placing the coordinates of the translation vectors $c_i$ as subindices in each column.

\[
\begin{array}{cccccc}
& C_1 & C_2 & C_3 & \cdots & C_{n-1} & C_n \\
\hline
-1 & 1 & 1 & 1 & \cdots & 1 & -1 \\
1 & -1 & 1 & 1 & \cdots & 1 & -1 \\
1 & 1 & 1 & -1 & \cdots & 1 & -1 \\
1 & 1 & 1 & -1 & \cdots & 1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\end{array}
\]

(4.1)

where each $*$ is 0 or $\frac{1}{2}$, depending on the choice of the $c_{ij}$'s. We have added an extra column $C_n$ corresponding to the product $C_n := C_1 C_2 \cdots C_{n-1}$ and we take the respective column vector $c_n \equiv c_1 + \cdots + c_{n-1}$ mod $\Lambda$ and having coordinates in $\{0, \frac{1}{2}\}$.

In (4.1), in the case when all $*$'s in the first $n-1$ columns equal zero (and thus the $*$'s in the $n$-th column are $\frac{1}{2}$'s, except for the one in the entry $(1, n)$ which is zero), the corresponding group, which we will denote by $K_n$ (see Figure 1), was introduced in [LS] and is known to be torsion-free, i.e., a Bieberbach group. Here, we will prove that this is true in the
more general case above. We shall denote by $K_n$ the family consisting of all groups constructed in this manner.

**Proposition 4.1.** All groups in $K_n$ are Bieberbach groups.

*Proof.* It is clear that the groups are Euclidean crystallographic groups. Hence, we need only show that they are torsion-free. Every element in each group is either a translation $L_\lambda$ with $\lambda \in \Lambda$, or it is of the form

\[ \gamma := C_{i_1}L_{c_{i_1}} \cdots C_{i_k}L_{c_{i_k}}L_\lambda, \quad i_1 < \cdots < i_k, \quad 1 \leq k \leq n - 1. \]

The translations $L_\lambda$, $\lambda \neq 0$, are clearly not elements of finite order. Concerning the remaining elements, we observe that on the $(i_k+1)$-th coordinate the product $\gamma$ in (4.2) acts as the translation by $\frac{1}{2} + \lambda_{i_k+1}$, with $\lambda_{i_k+1} \in \mathbb{Z}$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$. This is so since the action of $C_{i_j}$ on $e_{i_k+1}$ is trivial for $j = 1, 2, \ldots, k$, and the $(i_k+1)$-th coordinates of the translational parts $c_{i_j}$ are equal to zero, for $j = 1, \ldots, k - 1$, and $\frac{1}{2}$ for $j = k$.

Now, $\gamma^2$ is a translation and its $(i_k+1)$-th coordinate is $2\left(\frac{1}{2} + \lambda_{i_k+1}\right) = 2\lambda_{i_k+1} + 1$. In general, the $(i_k+1)$-th coordinate of the $m$-th power of $\gamma$ in (4.2) equals $m\left(\frac{1}{2} + \lambda_{i_k+1}\right)$, hence $\gamma^m \neq Id$ for every $m \neq 0$. \hfill $\square$

**Remark 4.2.** Manifolds of dimension $n$ and holonomy group $\mathbb{Z}_n^{n-1}$ have been called in [RS] generalized Hantzsche-Wendt manifolds, or GHW manifolds for short. They necessarily have diagonal holonomy representation. The family $K_n$ is properly contained in this larger class. It is not difficult to see that the holonomy representation in (4.1) is the only possible one for GHW manifolds with first Betti number one (see [RS]).

Next we shall show that all the Bieberbach groups in $K_n$ are pairwise not isomorphic. In [MRT], we have attached a directed graph to any orientable GHW manifold, with diffeomorphism of manifolds corresponding to isomorphism of graphs, essentially. It is also possible to do the same for arbitrary GHW manifolds, i.e., to associate a directed graph with $n$ vertices to any $n$-dimensional GHW manifold. We will do this in the case of the family $K_n$. This graph will be helpful to better understand the elements in our family.

Firstly, we replace the array in (4.1) by an $n \times n$ array $A$ of 0’s and $\frac{1}{2}$’s by keeping just the translational parts mod $\mathbb{Z}^n$. We observe that the total number of $\frac{1}{2}$’s in each row must be even, since the last column in $A$ is the sum mod $\mathbb{Z}^n$ of the others. Thus

\begin{equation}
A = \begin{bmatrix}
0 & * & * & \cdots & * \\
\frac{1}{2} & 0 & * & \cdots & * \\
0 & \frac{1}{2} & 0 & \cdots & * \\
0 & 0 & \frac{1}{2} & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & \frac{1}{2}
\end{bmatrix}
\end{equation}

where each * can be equal to 0 or $\frac{1}{2}$.

We recall that these arrays are in a one-to-one correspondence with groups in $K_n$. We will associate to each such array (or group in $K_n$) a directed graph
having a fixed set of $n$ vertices \{v_1, \ldots, v_n\} and so that there is an arrow issuing from vertex $v_i$ into vertex $v_j$ if and only if the entry $(i, j)$ in $A$ equals $\frac{1}{2}$.

**Remark 4.3.** Note that similar matrices $A$ were used in [MR1] to describe orientable GHW manifolds (there called HW manifolds), however the translational parts, shown as columns of the matrix $A$ here, are shown as rows in [MR1]. We also observe that another option to define these graphs would have been to ‘colour’ vertex $v_n$ leaving out the arrow joining $v_n$ to itself. In [MR1] these loops were omitted since all the vertices were of the same kind.

To illustrate the definition above, we display some GHW groups and their graphs.

In Figure 1, we show the array and the graph corresponding to the group $K_n$. Note that the arrows going from right to left in the figure of the graph will be present in every graph corresponding to a group in $K_n$.

\[
\begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & 0 \\
\frac{1}{2} & 0 & \cdots & \cdots & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & \frac{1}{2} \\
0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

**Figure 1.** The matrix and the graph of the group $K_n$.

Next, we will show the graphs of GHW groups in dimensions 2 and 3. In dimension 2, the Klein bottle group (which is isomorphic to $K_2$) belongs to $K_2$.

\[
\begin{pmatrix}
0 & 0 \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

**Figure 2.** The graph of the Klein bottle.

In dimension 3, out the three existing GHW groups, two of them, the first amphidicosm $+a_2$ and the second amphidicosm $-a_2$, belong to $K_3$ while the other one, the didicosm $c_22$ (or Hantzsche-Wendt manifold), does not (see figures 3 and 4).

In dimension 4, there are twelve GHW manifolds (see [CS] for instance), ten having first Betti number equal to one, i.e. $\beta_1 = 1$ (see figures 5 and 6) and two having $\beta_1 = 0$. Out of these, eight manifolds are in $K_4$. They are given by the array

\[
\begin{pmatrix}
0 & x & y & x + y \\
\frac{1}{2} & 0 & z & z + \frac{1}{2} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]
where $x, y, z \in \{0, \frac{1}{2}\}$ and the sums are taken mod $\mathbb{Z}$. If we choose the eight different possibilities for $x, y, z \in \{0, \frac{1}{2}\}$, we obtain the eight groups in $K_4$. The eight graphs corresponding to these manifolds have some common features, as can be seen in Figure 5.

**Proposition 4.4.** Let $\Gamma$ and $\Gamma'$ be groups in $K_n$ corresponding to arrays $A$ and $A'$ as in (4.3), respectively. Then $\Gamma \simeq \Gamma'$ if and only if $A = A'$, or equivalently, if and only if their associated graphs $G$ and $G'$ are isomorphic.

**Proof.** We first check that if two directed graphs, $G, G'$, attached to arrays as in (4.3) are isomorphic (as directed graphs) then $G = G'$. We will do this by showing that each vertex $v_i$, $1 \leq i \leq n$, is completely determined by the isomorphism class of the graph. The vertex $v_n$ is determined since it is the only vertex with an arrow going to itself. Now, there is only one more arrow issuing from $v_n$ and it goes to $v_{n-1}$, so this determines $v_{n-1}$. Also, there is only one arrow issuing from $v_{n-1}$ and going to vertices different from $v_n$. This arrow goes to $v_{n-2}$, so this determines $v_{n-2}$. Continuing in this way we get $v_2$ determined; furthermore $v_2$ is the only vertex which has an arrow going to $v_1$, hence determining $v_1$. Thus, all the vertices are determined. In other words, an isomorphism $\phi$ between two of these graphs $\phi : G \to G'$ (recall that the set of vertices is the same in both graphs) satisfies $\phi(v_i) = v_i$ for every $i = 1, 2, \ldots, n$, hence $\phi = \text{Id}$ and thus $G = G'$.

By the definition of the graph, one has that $G = G'$ if and only $A = A'$. Thus, we will be done if we prove that $\Gamma \simeq \Gamma'$ implies $A = A'$.

Let $\Gamma' = \langle C_1 L_{c_1} : 1 \leq i \leq n - 1; L_A \rangle$. By Bieberbach’s second theorem, an isomorphism between $\Gamma$ and $\Gamma'$ must be given by conjugation by an affine motion $\delta = DL_d, D \in \text{GL}_n(\mathbb{R}), d \in \mathbb{R}^n$, i.e. $\Gamma' = \delta \Gamma \delta^{-1}$. This implies, in
particular, that

\[ \delta C_i L_{c_i} \delta^{-1} = D C_i D^{-1} L_{D(c_i + (C_i - I) d)} . \]

Since \( \bar{C}_n \) is the only matrix in the holonomy group with exactly \( n - 1 \) diagonal elements equal to \(-1\), we must have \( D \bar{C}_n D^{-1} = \bar{C}_n \). Similarly, there is a permutation \( \sigma \in S_{n-1} \) such that \( D C_i D^{-1} = C_{\sigma(i)} \) for \( 1 \leq i \leq n-1 \). Also,

\[ c'_{\sigma(i)} \equiv D(c_i + (C_i - I) d) \mod \Lambda . \]
Now we take into account that \( c_i \) and \( c'_i \) are of the form \( \sum_{j=1}^{i+1} \epsilon_{ji}c_j \) where \( \epsilon_{ii} = 0 \) and \( \epsilon_{i+1,i} = \frac{1}{2} \) for \( i = 1, 2, \ldots, n-1, \epsilon_{ji} = 0 \) or \( \frac{1}{2} \) for \( j < i \) and \( \epsilon_{nn} = \frac{1}{2} \). We note that equation (4.6) allows to change modulo \( \mathbb{Z} \) only the coordinates in \( c_i \) in which \( C_i \) acts by \( -1 \), while the other coordinates cannot change (modulo \( \mathbb{Z} \)). In particular, since the entries \( (ji) \) with \( j > i \) in \( A \) correspond to the action of \( C_i \) as the identity, they remain unchanged after conjugation by \( \delta \). Then we see that \( C_{n-1} \) must be matched (via the isomorphism) to \( C'_{n-1} \) (since they are the only \( C_i \)'s with \( \frac{1}{2} \) in the \( n \)th coordinates of their translation vectors). This implies that \( \sigma(n-1) = n-1 \). In the same way, we see that for each \( i, 1 \leq i \leq n-2 \), \( C_i \) must be matched with \( C'_i \), thus \( \sigma(i) = i \) for \( 1 \leq i \leq n-1 \). Therefore, it follows that the permutation \( \sigma \) must be the identity.

This implies that \( D \) is diagonal with eigenvalues \( \pm 1 \). By taking into account that we have chosen the coefficients in the main diagonal in \( A \) and in \( A' \) to be zero for the first \( n-1 \) entries, we must have \( d \equiv 0 \mod \mathbb{Z} \). Hence, conjugation by \( \delta = DL_d \) produces an automorphism of \( \Gamma \). Thus, \( \Gamma = \Gamma' \), and hence we have \( A = A' \) for the corresponding arrays, which completes the proof.

Now, it is easy to compute the cardinality of \( \mathcal{K}_n \), since there are two choices for each entry \((i,j)\) with \( 1 \leq i < j < n \):

**Corollary 4.5.** There are \( 2^{(n-1)(n-2)/2} \) Bieberbach groups in \( \mathcal{K}_n \), all of them pairwise nonisomorphic to each other.

If we put this corollary together with Theorem 2.1 we have:

**Corollary 4.6.** There exists a family of \( 2^{(n-1)(n-2)/2} \) compact n-manifolds, isospectral on forms and pairwise nonhomeomorphic to each other.

**Remark 4.7.** (i) It is easy to see that there are larger families with similar properties as \( \mathcal{K}_n \). Indeed, in [RS] it was shown that, for a given \( n \), there are \( [(n+1)/2] \) different integral holonomy representations for GHW manifolds. For each of these representations, one can define a family of Bieberbach groups in a similar way as for \( \mathcal{K}_n \) above, and all the resulting flat manifolds will be isospectral on forms by Theorem 2.1 yet pairwise nonhomeomorphic. Thus, this procedure should allow to multiply the number in Corollaries 4.5 and 4.6 by a factor of \( [(n+1)/2] \), approximately. However, this does not improve the result significantly.

(ii) If one considers families of Bieberbach groups with holonomy group \( \mathbb{Z}_2^k \) for some \( k \) with \( \frac{n}{2} \leq k < n-1 \), one should obtain larger families of manifolds than in the case \( k = n-1 \), pairwise nonhomeomorphic to each other and again isospectral on forms. A support for this claim is given by the classification of low dimensional Bieberbach groups (see [CS]).

(iii) The manifolds in \( \mathcal{K}_n \) are all nonorientable. By using a duplication method (see for instance [BDM]) applied to the manifolds in \( \mathcal{K}_n \) one obtains \( 2^{(n-1)(n-2)/2} \), orientable, nonhomeomorphic manifolds of dimension \( 2n \) isospectral on forms.

(iv) Using the methods in [MR3] (see Thm. 3.12) and in [MR4] (see Prop. 4.7) one can show that the manifolds in \( \mathcal{K}_n \) are, generically, not \( p \)-isospectral for any value of \( p, 0 \leq p \leq n \).
REFERENCES

[BDM] Barberis M.L., Dotti I., Miatello R., Clifford structures on certain locally homogeneous manifolds, Ann. Global Anal. Geom. 13 (1995), 289-301.

[CS] Cid, C.; Schulz, T., Computation of Five and Six dimensional Bieberbach groups, Experiment Math. 10 (2001), 109–115.

[CR] Conway J.H., Rossetti J.P. Describing the platycosms, preprint 2003, math.DG/0311476

[Ch] Charlap L., Bieberbach groups and flat manifolds, Springer Verlag, Universitext, 1988.

[DM] Dotti I., Miatello R., Isospectral compact flat manifolds, Duke Math. J. 68 (1992), 489-498.

[KL] Krasikov I., Litsyn S. On integral zeros of Krawtchouk polynomials, J. Combin. Theory A 74 (1996), 71-99.

[LS] Lee, R.; Saccarba, R.H., On the integral Pontrjagin classes of a Riemannian flat manifold, Geom. Dedicata 3 (1974), 1-9.

[MP] Miatello R.J., Podestá R.A. Spin structures and spectra of $\mathbb{Z}_2^k$-manifolds, Math. Z., to appear. Also, arXiv:math.DG/0311354

[MR1] Miatello R.J., Rossetti J.P. Isospectral Hantzsche-Wendt manifolds, J. Reine Angew. Math. 515 (1999), 1-23.

[MR2] Miatello R.J., Rossetti J.P. Flat manifolds isospectral on $p$-forms, Jour. Geom. Anal. 11 (2001), 649-667.

[MR3] Miatello R.J., Rossetti J.P. Comparison of twisted Laplace $p$-spectra for flat manifolds with diagonal holonomy, Ann. Global Anal. Geom. 21 (2002), 341-376.

[MR4] Miatello R.J., Rossetti J.P. Length spectra and $P$-spectra of compact flat manifolds, Jour. Geom. Anal. 13 (2003), 631-657. Also, arXiv:math.DG/0110325

[RT] Rossetti J.P., Tirao P.A. Compact flat manifolds with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, Proc. Am. Math. Soc. 124 (1996), 2491-2499.

[RS] Rossetti J.P., Szczepanski A., Generalized Hantzsche-Wendt manifolds, Rev. Mat. Iberoamericana, to appear.

[Wo] Wolf J., Spaces of constant curvature, Mc Graw-Hill, NY, 1967.

FAAMAF–CIEM, UNIVERSIDAD NACIONAL DE CÓRDOBA, 5000 CóRDOBA, ARGENTINA.
E-mail address: miatello@mate.uncor.edu

E-mail address: podesta@mate.uncor.edu

E-mail address: rossetti@mate.uncor.edu