Entangled Fock States for Robust Quantum Optical Metrology, Imaging, and Sensing

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(Dated: August 15, 2008, V1.0)

We propose a class of path-entangled photon Fock states for robust quantum optical metrology, imaging, and sensing in the presence of loss. We model propagation loss with beam-splitters and derive a reduced density matrix formalism from which we examine how photon loss affects coherence. It is shown that particular entangled number states, which contain a special superposition of photons in both arms of a Mach-Zehnder interferometer, are resilient to environmental decoherence. We demonstrate an order of magnitude greater visibility with loss, than possible with N00N states. We also show that the effectiveness of a detection scheme is related to super-resolution visibility.

PACS numbers: 42.50.St, 42.50.Ar, 42.50.Dv, 42.50.-p

Quantum states of light, such as squeezed states or entangled states, can be used for metrology, image production, and object ranging, with greater precision, resolution, and sensitivity than what is possible classically [11, 2, 3, 14]. In 2000, one of the authors introduced a path-entangled number state known as the N00N state, which is an entangled two mode object ranging, with greater precision, resolution, and sensitivity in a single detection event [7]. In 2004 the group of Steinberg demonstrated super-resolution for states with N00N states when loss is present. We find with these new states that super-sensitivity is defined as the ability of a particular quantum system to perform better than the shot-noise limit, and super-resolution as performing better than the Rayleigh diffraction limit. The super-resolution effect has been demonstrated for N = 2 in a proof-of-principle experiment by Y. Shih in 2001 [1]. In 2004 the group of Steinberg demonstrated super-resolution for N = 3, and the group of A. Zeilinger did so for N = 4 [3, 5, 10]. Finally in 2007 a joint Japanese-British collaboration demonstrated both super-resolution and sensitivity in a single N = 4 experiment [11]. A large amount of publications also investigated alternative states and detection schemes to obtain super-sensitivity and -resolution. N00N states served for many years as a standard model for the newly emerging fields of quantum optical metrology, imaging, and sensing. Consequently a few authors investigated the effects of loss on the performance of quantum interferometers with N00N states. It turns out that N00N states undergoing loss decohere very rapidly, making it difficult to achieve super-sensitivity and resolution in an environment with loss [12, 13, 14].

In this letter we address how environmental interaction brings about decoherence for a more generalized state with photons in both modes, and we have discovered a class of states that improve drastically on the performance of N00N states when loss is present. We find with these new states that while minimum sensitivity is slightly decreased, robustness against decoherence is greatly increased.

For practical purposes phase sensitivity is typically obtained by the linear error propagation method, (see however Ref. [6]), where \( \hat{O} \) represents the operator for the detection scheme being used,

\[
\delta \phi = \frac{\Delta \hat{O}}{|\partial (\hat{O})/\partial \phi|},
\]

and \( \Delta \hat{O}_N = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2} \). Eq. (1), for a N00N state with no loss, and a detection operator \( \hat{A}_N = |0, N \rangle \langle 0, N | + |N, 0 \rangle \langle N, 0 | \), which can be implemented with coincidence measurements [10], reduces to the Heisenberg limit, \( \delta \phi = 1/N \), which is a \( \sqrt{N} \) improvement over the shot-noise limit.

The state we now wish to examine is the following,

\[
|m : m' \rangle_{ab} = \frac{1}{\sqrt{2}} \left( |m, m' \rangle_{ab} + |m', m \rangle_{ab} \right),
\]

where we demand that \( m > m' \) (we refer to this as the M&M state). Such states can be produced, for example, by post-selecting on the output of a pair of optical parametric oscillators [15]. Our setup in Fig. 1 is a Mach-Zehnder or an equivalent Michelson interferometer where the details of our source and detection (such as beam-splitters, detectors, etc.) are contained in their respective boxes. Here we are concerned primarily with how the state evolves with respect to loss, which is typically modeled by additional beam-splitters coupled to the environment [16]. Similar to the approach of Ref. [13], we

![FIG. 1: Interferometer with loss modeled by beam-splitters in both arms. The reflectance of the beam-splitters determines how many photons one lost. An accumulated unknown phase \( \phi \) is obtained due to a path length difference between the arms. The unitary operator for the phase shift is given by \( \hat{U} = \exp(i \delta \phi) \). A simple proof shows that this operator commutes with the beam splitter operation. The placement of the beam splitter before the phase shift has been acquired therefore leads to the same result.](image-url)
model loss in the interferometer with fictitious beam-splitters, but in our case these are added to both arms of the interferometer. However we assume unit detection efficiency for the detectors. We develop the photon statistics as a function of beam-splitter transmittance as well as derive a reduced density matrix, which characterizes the propagation losses inside of the interferometer. Loss is represented by photons being reflected into the environment [17]. The beam-splitter transforms the modes according to [18].

\[
\begin{align*}
\hat{a}' &= t_a \hat{a} + r_a' \hat{a}_2 , \\
\hat{b}' &= t_b \hat{b} + r_b' \hat{b}_2 ,
\end{align*}
\]

where \( t_a = \sqrt{T_a} \exp(i \varphi_a) \) and \( r_a = \sqrt{R_a} \exp(i \psi_a) \), \( u = a, b, \) are the complex transmission and reflectance coefficients, for mode \( a \) and \( b \), respectively. The input M&M state \( | m \rangle \otimes | m' \rangle \) acquires an unknown phase shift \( \phi \) and the beam splitter transformations are applied. We then trace over the environmental modes, to model the photons lost, and we obtain the reduced density matrix \( \hat{\rho}_{e,r} = \text{Tr}_{na} [ | \psi \rangle \langle \psi | ] \), which leads to

\[
\hat{\rho}_{e,r} = \sum_{k=0}^{m} \sum_{l=0}^{m'} | a_{k,l} |^2 | m - k, m' - l \rangle \langle m - k, m' - l | + | b_{k,l} |^2 | m' - l, m - k \rangle \langle m' - l, m - k | + a_{k,l}^* b_{l,k} | m' - l, m - l \rangle \langle m', m - l | + a_{k,l} b_{l,k}^* | m - l, m' - l \rangle \langle m - l, m' - l |. \]  

(4)

Here the \( a_{k,l} \) and \( b_{k,l} \) coefficients are defined as

\[
\begin{align*}
| a_{k,l} |^2 &= \gamma_{k,l}^2 T_{a}^{m-k} R_{b}^{m'-l} T_{b}^{m-k} R_{b}^{m'-l} , \\
| b_{k,l} |^2 &= \gamma_{k,l}^2 T_{b}^{m'-l} R_{b}^{m-k} R_{b}^{m'-l} T_{b}^{m-k} , \\
a_{k,l}^* b_{l,k} &= \gamma_{k,l} T_{a}^{m-k} R_{b}^{m'-l} R_{b}^{m-k} T_{b}^{m'-l} e^{-i(m-m')(\varphi_a - \varphi_b)} , \\
a_{k,l} b_{l,k}^* &= \gamma_{k,l} T_{b}^{m-k} R_{b}^{m'-l} R_{b}^{m-k} T_{b}^{m'-l} e^{-i(m-m')(\varphi_a - \varphi_b)} ,
\end{align*}
\]

(5)

\[
\gamma_{k,l} \equiv \frac{1}{\sqrt{2m!m'!}} \binom{m}{k} \binom{m'}{l} (m - k)! (m' - l)! l^{1/2}.
\]  

(6)

Without loss of generality we can set the transmission phases of the two beam splitters \( \varphi_a = \varphi_b = 0 \).

The reduced density matrix in Eq. (4) appears as an incoherent mixture plus interference terms, which survive with loss in either mode up to the limit of \( m' \). The surviving interference terms all carry amplified phase information in the quantity \( (m - m') \varphi \). Thus the best-case minimum phase sensitivity, under no loss, is reduced from the Heisenberg limit, \( \delta \varphi_{\text{Heis}} = 1/N \), to \( \delta \varphi_{\text{m,m'}} = 1/(m - m') \). Although this sensitivity is less than what N00N states are capable of achieving (in the absence of loss), the fact that many more interference terms survive than with N00N states suggests that these states are more robust against photon loss.

To maximize phase information we choose a detection operator of the form

\[
\hat{A} = \sum_{r,s=0}^{m'} \langle m' - r, m - s | (m - r, m' - s) + | m - r, m' - s | (m - r, m - s) ,
\]  

(7)

which can be implemented with number-resolving photo-detectors [19]. This operator is a more general summation over all possible cases up to \( m' \) photons in either arm than the \( \hat{A}_N \) operator (traditionally used for N00N states [1]). The reduced density matrix for a N00N state is easily obtained by setting \( m = N \) and \( m' = 0 \) in Eq. (4). We then obtain for the expectation value of \( \hat{A}_N \)

\[
\langle \hat{A}_N \rangle = \text{Tr}[\hat{A}_N \hat{\rho}_{e,r}] = 2 \text{Re}(a_{1,0}^* b_{0,0}) = (T_{a} T_{b})^{\frac{1}{2}} \cos(N \varphi) .
\]

(8)

The expectation value of the operator \( \hat{A} \) given in Eq. (7) for the M&M state shows the benefit of having many more interference terms compared to the N00N state

\[
\langle \hat{A} \rangle = \text{Tr}[\hat{A} \hat{\rho}_{e,r}] = 2 \text{Re}(a_{1,0}^* b_{0,1}) = 2 \sum_{l=0}^{m'} \langle a_{l,0}^* b_{l,1} \rangle \cos(m - m') \varphi .
\]

(9)

The visibility of an attenuated mixed state in an interferometer may be expressed as a function of the off-diagonal terms in the reduced density matrix from Eq. (4) [22].

\[
V_{1} = 2 \langle \hat{\rho}_{1,2} \rangle = 2 \sum_{l=0}^{m'} \langle a_{l,0}^* b_{l,1} \rangle = 2 \sum_{l=0}^{m'} | a_{l,0}^* b_{l,1} | ,
\]

(10)

where we call \( V_{1} \) the fundamental visibility and \( \hat{\rho}_{1,2} \) is taken from one of the off-diagonal terms in the density matrix in Eq. (4). From Eqs. (9) and (10) we see that the expectation value of \( \hat{A} \) may be written as \( \langle \hat{A} \rangle = V_{1} \cos(m - m') \varphi \). For a general detection operator \( \hat{O} \) the amplitude of the cosine may be smaller than \( V_{1} \), i.e., \( V_{1} \geq \langle \hat{O} \rangle_{\Phi=0} \). If we use a detection operator \( \hat{O} \) so that \( V_{1} > \langle \hat{O} \rangle_{\Phi=0} \), we know our detection scheme is inefficient and we are not retrieving full phase information. We call the visibility of a particular detection scheme the detection visibility, \( V_{\text{det}} = \langle \hat{O} \rangle_{\Phi=0} \). We see that the \( \hat{A}_N \) operator, and its more general form \( \hat{A} \) in Eq. (7), are both optimal for N00N and M&M states, respectively, and give a detection visibility equivalent to the fundamental visibility. The fundamental visibility for a N00N state is simply \( V_{1} = (T_{a} T_{b})^{N/2} \), which is just the probability the N00N state arrives at the detector with no loss.

The M&M states, with \( m - m' = N \), are capable of producing the same resolution as a \( N \) photon N00N state, but at the cost of requiring \( m' \) more photons to do so, and thus they operate at a smaller shot-noise limit. As we will show, in the
presence of loss, however, many M&M states operate below their own shot-noise limit, while N00N states of the same resolving power do not.

To compare a certain M&M to a N00N state we choose the state such that \( m - m' = N \), so the amount of phase information is the same for either state. This way our minimum phase sensitivity also starts from the same point, \( 1/(m - m') = 1/N \).

The true Heisenberg-limit for a M&M state however is determined by the total photon number in the state and is therefore given by \( 1/(m + m') \). The shot-noise limit for a M&M state is \( 1/\sqrt{m + m'} \), while the N00N state is the usual \( 1/\sqrt{N} \).

![FIG. 2: Phase sensitivity \( \delta \phi \) for a \(|20 : 10\rangle\) M&M state (curved black solid line) versus a \(|10 : 0\rangle\) N00N state (curved blue dashed line) undergoing loss. Loss is 40% in the long arm and zero in the delay arm. Bottom black solid line is the Heisenberg limit for \(|20 : 10\rangle\), \( 1/(m + m') \). The red solid line is the Heisenberg limit for the \(|10 : 0\rangle\) N00N state and lossless limit for \(|20 : 10\rangle\), \( 1/(m - m') \). The black dotted line is the shot-noise limit for \(|20 : 10\rangle\), while the blue dashed line is the shot-noise limit for \(|10 : 0\rangle\). The N00N state is no longer below its shot-noise limit while the minimum phase sensitivity for the M&M state \(|20 : 10\rangle\) is at its respective shot-noise limit.

As would be the case in a practical quantum sensor, we assume loss in the long arm \( b \) of the interferometer to be much greater than that of the delay arm \( a \), which we assume to be under controlled loss conditions. Figure 2 is an example of a M&M state showing more robustness to loss in phase sensitivity than an equivalent N00N state. A N00N state of \( N = 10 \) degrades to the shot-noise limit at approximately 26% loss in the long arm (zero loss in the delay arm), whereas a \(|20 : 10\rangle\) M&M state degrades to its respective shot-noise limit at larger loss, 40% loss in the long arm (zero loss in the delay arm).

Also important is to note how \( \langle A \rangle \), and by extension, the visibility, evolve with loss. Under lossless conditions the visibility of a N00N or M&M state is always one, and hence so is the amplitude of \( \langle A \rangle \). Figure 3 shows a comparison of \( \langle A \rangle \) for \(|20 : 10\rangle\) and \(|10 : 0\rangle\) under 50% = 3dB loss in the long arm (zero in the delay arm). We can examine the visibility as a function of loss in both arms directly with contour plots. Figures 4(a) and 4(b) show an order of magnitude increase in visibility for the \(|20 : 10\rangle\) M&M state over the \(|10 : 0\rangle\) N00N state. The improvement in visibility is greater than that seen in minimum phase sensitivity in Figure 2. This suggests that the M&M states are much better suited than N00N states for resolving interference fringes under loss.

A heuristic way to understand the improvement of M&M states over N00N states is to consider “which-path” information available to the environment after photon loss. For example, even a single photon lost in mode \( b \) projects the N00N
state from \(|N,0\rangle + e^{i\delta \phi} |0,N\rangle \rightarrow e^{i\delta \phi} |0,N-1\rangle\). That is, a single photon in environmental mode \(b\) provides complete which-path information—the environment “knows” with certainty it cannot have the \(|0\rangle\) component of the N00N state, which collapses the state into the separable state \(|0,N-1\rangle\). In contrast, an M&M state may lose up to \(m'\) photons to the environment without complete knowledge of whether the \(m\) or \(m'\) component was present, and hence complete “which-path” information is not available, and a great deal of coherence is hence preserved.

In comparing the M&M states to N00N states there emerges a delicate tradeoff in sensor performance from adding \(m'\) photons to increase the number of available output states, which contain phase information. Add too few \(m'\) photons, and there will not be significant improvement. Add too many \(m'\) photons, and the total number of photons required to carry the phase information for an equivalent N00N state rise, causing the shot-noise limit to be lowered further and reached quicker under conditions of loss (see Tab. I).

| \(m\) | \(m'\) | Visibility | \(\delta \phi_{\text{min}}\) | HL | SNL |
|---|---|---|---|---|---|
| 10 | 0 | 3.13% | 2.264 | 0.100 | 0.316 |
| 11 | 1 | 6.74% | 1.051 | 0.083 | 0.289 |
| 12 | 2 | 10.96% | 0.652 | 0.071 | 0.267 |
| 14 | 4 | 19.85% | 0.372 | 0.056 | 0.236 |
| 16 | 6 | 28.11% | 0.279 | 0.045 | 0.213 |
| 18 | 8 | 35.19% | 0.238 | 0.038 | 0.196 |
| 20 | 10 | 41.11% | 0.254 | 0.033 | 0.183 |

We have shown that the class of entangled Fock states with photons in both modes, M&M states, is more robust to loss than N00N states possessing all photons in either mode. The visibility for a M&M state under loss may be an order of magnitude or more greater than N00N states, as well as having attenuated minimum phase sensitivities that are lower and more likely to be less than the shot-noise limit than a N00N state. While the M&M states are not capable of reaching the Heisenberg limit of \(1/N\), it seems unlikely that any state is capable of reaching this precision in the limit of practical sensing with appreciable photon loss. While M&M states are more robust, they do appear to have loss-induced limitations as well. For many M&M states visibility drops to approximately 10\% around the 70\% loss level in one arm, assuming perfect transmission in the other.

Another issue that needs consideration is how to produce M&M states. As of yet there is no efficient, on demand, Fock number state generator. However the output from a optical parametric amplifier (OPA) is essentially a summation of many M&M states as well as several N00N states. We are currently analyzing the sensing capabilities for the entire output state of an OPA, as well as schemes for generating M&M states from an OPA output with post-selection.

We would like to acknowledge support from the Defense Advanced Research Projects Agency, the Army Research Office, the Intelligence Advanced Research Projects Activity, as well as helpful discussions with H. Lee and G. A. Durkin.

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