ON THE ASYMPTOTIC CHARACTER OF A GENERALIZED RATIONAL DIFFERENCE EQUATION

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Abstract. We investigate the global asymptotic stability of the solutions of

\[ x_{n+1} = \frac{\beta x_{n-l} + \gamma x_{n-k}}{A + x_{n-k}} \]

for \( n = 1, 2, \ldots \), where \( l \) and \( k \) are positive integers such that \( l \neq k \). The parameters are positive real numbers and the initial
conditions are arbitrary non-negative real numbers. We find necessary and
sufficient conditions for the global asymptotic stability of the zero equilibrium.
We also investigate the positive equilibrium and find the regions of parameters
where the positive equilibrium is a global attractor of all positive solutions.

1. Introduction. Consider the difference equation

\[ x_{n+1} = \frac{\beta x_{n-l} + \gamma x_{n-k}}{A + x_{n-k}}, \quad n = 0, 1, \ldots, \quad (1) \]

where \( l \) and \( k \) are positive integers such that \( l \neq k \), the parameters \( \beta, \gamma, \) and \( A \)
are positive real numbers, and the initial conditions are arbitrary non-negative real
numbers.

A special case where \( l = 1 \) and \( k = 2 \) is studied in [7]. The case \( l = 0 \) was
investigated in [4]. In this case, without loss of generality, it was assumed that
\( \gamma = 1 \). It was shown that when \( A \geq \beta + 1 \) all solutions converge to the zero
equilibrium. When \( A < \beta + 1 \), two sufficient conditions for global attractivity of
the positive equilibrium \( \bar{x} = \beta + 1 - A \) are \( A > \beta - 1 \) and \( A \geq \frac{(k-1)\beta-1}{k} \). Some
other special cases of Eq.(1) has also been investigated. See [3] and [12] for related
work in some of these special cases. See also [2].

We consider here the most general case of double long delay recursion. The
linear fractional form of the recursion is maintained in the generalization in order

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to systematically study the effects of the long delays. Although, in some sections, the functional form of the recursion will be further generalized and not just specific to the linear fractional form.

The change of variables

\[ x_n = \gamma y_n \]

reduces Eq.(1) to the difference equation

\[ y_{n+1} = \frac{py_{n-1} + y_{n-k}}{q + y_n}, \quad n = 0, 1, \ldots, \]

(2)

with \( l \) and \( k \) positive integers such that \( l \neq k \) and with positive parameters and non-negative initial conditions.

Our goal is to investigate the global stability of the solutions of Eq.(2).

Note that, for initial conditions that are not all zero we have an eventually positive solution of Eq.(2). Without loss of generality, in the sequel we need to consider only the positive solutions of Eq.(2).

In the sequel we will assume \( l < k \) without loss of generality. Hence we obtain the recursion in \( k + 1 \) space. The proofs for the other case \( l > k \) follows in a similar manner.

2. Equilibria and invariant intervals. In this section we show that when \( q > p \), the interval \([0, 1]\) is an invariant interval for all solutions of Eq.(2). Furthermore, in this case, every solution of Eq.(2) is eventually bounded from above by the constant \( q/p \). We also show that when \( q < p \), the interval \((1, \infty)\) is an invariant interval for all solutions of Eq.(2) and every solution in this case is eventually bounded from below by the constant \( q/p \).

The equilibrium points of Eq.(2) are the non-negative solutions of the equation

\[ \bar{y} = \frac{p\bar{y} + \bar{y}}{q + \bar{y}}. \]

Clearly, zero is always an equilibrium point of Eq.(2). However, when \( q < p + 1 \), Eq.(2) also has the unique positive equilibrium point \( \bar{y} = p + 1 - q \). The following lemma, the proof of which is straightforward and will be omitted, exhibits two identities that will be useful in the study of Eq.(2).

**Lemma 2.1.** Let \( \{y_n\}_{n=-k}^{\infty} \) be a solution of Eq.(2). Then for \( n \geq 0 \) the following two identities hold:

\[ y_{n+1} - y_n = \frac{p(y_{n-1} - \frac{2}{p})}{q + y_{n-k}} \]

(3)

\[ y_{n+1} - y_{n-1} = \frac{(p-q)y_{n-1} + (1-y_{n-1})y_{n-k}}{q + y_{n-k}} \]

(4)

The following lemma establishes the existence of invariant intervals for solutions of Eq.(2).

**Lemma 2.2.** Let \( \{y_n\}_{n=-k}^{\infty} \) be a solution of Eq.(2). Then the following statements are true.

(a) Suppose that \( q > p \) and that there exists \( N \geq 0 \) such that \( y_{N-l}, y_{N-l+1}, \ldots, y_N \leq 1 \). Then \( y_n < 1 \) for all \( n > N \).

(b) Suppose that \( q < p \) and that there exists \( N \geq 0 \) such that \( y_{N-l}, y_{N-l+1}, \ldots, y_N \geq 1 \). Then \( y_n > 1 \) for all \( n > N \).
Proof. We will prove (a). The proof of (b) is similar and will be omitted. From Eq.(3) we see that \( y_{N-l} \leq 1 < \frac{q}{p} \) and so \( y_{N+1} < 1 \). The result now follows by induction.

The next result establishes an eventual bound for the solutions of Eq.(2) from above and below for the cases \( p < q \) and \( p > q \) respectively.

**Lemma 2.3.** Let \( \{y_n\}_{n=-k}^{\infty} \) be a solution of Eq.(2). Then the following statements are true.

(a) When \( q > p \) every solution of Eq.(2) is eventually bounded from above by the constant \( q/p \).
(b) When \( q < p \) every solution of Eq.(2) is eventually bounded from below by the constant \( q/p \).

**Proof.** (a) Suppose, for the sake of contradiction, that there exists \( N \) sufficiently large, such that

\[
y_{N+1} = \frac{py_{N-l} + y_{N-k}}{q + y_{N-k}} \geq \frac{q}{p}.
\]

Then clearly,

\[
y_{N-l} > \left( \frac{q}{p} \right)^2
\]

and similarly

\[
y_{N-2l-1} > \left( \frac{q}{p} \right)^3
\]

which eventually leads to a contradiction.

(b) Suppose for the sake of contradiction that there exists \( N \) sufficiently large, such that

\[
y_{N+1} = \frac{py_{N-l} + y_{N-k}}{q + y_{N-k}} \leq \frac{q}{p}.
\]

Then clearly,

\[
y_{N-l} < \left( \frac{q}{p} \right)^2
\]

and similarly

\[
y_{N-2l-1} < \left( \frac{q}{p} \right)^3
\]

which eventually leads to a contradiction and completes the proof.

3. **The stability character of the zero equilibrium.** Recall that zero is always an equilibrium point of Eq.(2). In this section we investigate the stability of the zero equilibrium of Eq.(2). The linearized equation of Eq.(2) with respect to the zero equilibrium is

\[
z_{n+1} - \frac{p}{q} z_{n-l} - \frac{1}{q} z_{n-k} = 0, \quad n = 0, 1, \ldots,
\]

with associated characteristic equation

\[
\lambda^{k+1} - \frac{p}{q} \lambda^{k-l} - \frac{1}{q} = 0.
\]

**Lemma 3.1.** When \( q > p+1 \), the zero equilibrium of Eq.(2) is locally asymptotically stable, while if \( q < p+1 \) it is unstable. In the case \( q = p+1 \) the zero equilibrium is stable.
Proof. It follows by Clark’s Theorem, see [5], that the zero equilibrium is locally asymptotically stable when

\[ q > p + 1. \]

Set

\[ f(\lambda) = \lambda^{k+1} - \frac{p}{q} \lambda^{k-l} - \frac{1}{q}. \]

Note that \( f(1) = \frac{q-(p+1)}{q} < 0 \) when \( q < p + 1 \). Thus, the zero equilibrium is unstable when \( q < p + 1 \).

Now assume \( q = p + 1 \). Let \( \epsilon > 0 \), and let \( \{y_n\}_{n=-k}^{\infty} \) be a nonnegative solution of Eq.(2). Letting \( \delta = \epsilon \), if the initial conditions are such that

\[ y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 < \delta, \]

we have

\[ 0 \leq y_1 = \frac{pq + y_{-k}}{p + 1 + y_{-k}} \leq \frac{pq + y_{-k}}{p + 1} < \delta = \epsilon. \]

The proof follows by induction.

The next result gives the global stability character of the zero equilibrium of Eq.(2).

**Theorem 3.2.** Assume that \( q \geq p + 1 \). Then the zero equilibrium of Eq.(2) is globally asymptotically stable.

Proof. It suffices to show that the zero equilibrium is a global attractor of all solutions of Eq.(2). Let \( \{y_n\}_{n=-k}^{\infty} \) be a solution of Eq.(2). Lemma (2.3) (a) implies that eventually, \( y_n < \frac{2}{p} \).

Observe that the function

\[ f(u, v) = \frac{pu + v}{q + v} \]

increases in \( u \) for all \( v \in [0, \infty) \) and increases in \( v \) for all \( u \in [0, \frac{2}{p}] \).

Set \( S = \lim \sup_{n \to \infty} y_n. \) Then clearly,

\[ S \leq \frac{(p + 1)S}{q + S} \]

from which it follows that

\[ S = 0. \]

The proof is complete.

4. **Stability of the positive equilibrium.** Recall that the unique positive equilibrium \( \bar{y} = p + 1 - q \) of Eq.(2) exists when \( q < p + 1 \). We first establish the local stability character of the positive equilibrium. The linearized equation of Eq.(2) with respect to the positive equilibrium is

\[ z_{n+1} = \frac{p}{p + 1} z_{n-l} + \frac{p - q}{p + 1} z_{n-k} = 0, \quad n = 0, 1, \ldots, \]

with associated characteristic equation

\[ \lambda^{k+1} - \frac{p}{p + 1} \lambda^{k-l} + \frac{p - q}{p + 1} = 0. \]

As a consequence of Clark’s Theorem, see [5], it follows that the positive equilibrium of Eq.(2) is locally asymptotically stable when \( q > p - 1 \).

We now discuss the global stability of the positive equilibrium of Eq.(2).
Theorem 4.1. Assume that $p - 1 < q < p + 1$. Then the positive equilibrium of Eq.(2) is globally asymptotically stable.

Proof. It suffices to show that the positive equilibrium is a global attractor of all solutions of Eq.(2).

Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(2). Observe that the function

$$f(u, v) = \frac{pu + v}{q + v}$$

increases in $u$ for all $v \in (0, \infty)$, increases in $v$ for all $u \in \left(0, \frac{q}{p}\right]$ and decreases in $v$ for all $u \in \left(\frac{q}{p}, \infty\right)$.

We divide the proof into the following three cases:

Case(i). $p < q < p + 1$.

By Lemma (2.3) (a) it follows that there exists $N \geq 0$, such that

$$y_n < \frac{q}{p} \quad \text{for all} \quad n \geq N.$$  

Choose $m > 0$ such that

$$m < \min(y_N, y_{N+1}, \ldots, y_{N+k}, p + 1 - q).$$

Then

$$y_{N+k+1} = \frac{py_{N+k} + y_N}{q + y_N} > \frac{(p + 1)m}{q + m} > m$$

and similarly by induction

$$y_n > m \quad \text{for all} \quad n \geq N + k + 1.$$  

Set

$$S = \limsup_{n \to \infty} y_n \quad \text{and} \quad I = \liminf_{n \to \infty} y_n.$$  

Then

$$S \leq \frac{(p + 1)S}{q + S} \quad \text{and} \quad I \geq \frac{(p + 1)I}{q + I}$$

and so

$$S = p + 1 - q = I.$$  

Case(ii). $p = q > 0$.

Observe that for all $n \geq 0$,

$$0 \leq y_{n+1} - 1 \leq y_{n-1} - 1 \quad \text{or} \quad 0 \geq y_{n+1} - 1 \geq y_{n-1} - 1$$

which implies that the $l + 1$ subsequences $\{y_{(l+1)n}\}, \{y_{(l+1)n+1}\}, \ldots, \{y_{(l+1)n+l}\}$ converge to $l + 1$ finite limits $L_0, L_1, \ldots, L_l$ respectively. Therefore the sequence

$$\ldots, L_0, L_1, \ldots, L_l, L_0, L_1, \ldots, L_l,$$

is a period $l + 1$ solution of Eq.(2). But Eq.(2) has no prime period $l + 1$ solution and so

$$L_0 = L_1 = \ldots = L_l.$$  

Case(iii). $p - 1 < q < p$.

By Lemma (2.3) (b), we know that eventually,

$$y_n > \frac{q}{p}.$$  

Also from Lemma (2.2) (c), we know that eventually
\[ y_n > 1. \]
Hence eventually,
\[ y_{n+1} = \frac{py_{n-l} + y_{n-k}}{q + y_{n-k}} < \frac{py_{n-l} + 1}{q + 1} \]
and by the comparison principle
\[ \limsup_{n \to \infty} y_n \leq \frac{1}{q + 1 - p}. \]
Therefore the solution \( \{y_n\}_{n=-k}^{\infty} \) is bounded from above and below by positive constants. Set
\[ S = \limsup_{n \to \infty} y_n \quad \text{and} \quad I = \liminf_{n \to \infty} y_n. \]
Then
\[ S \leq \frac{pS + I}{q + I} \quad \text{and} \quad I \geq \frac{pI + S}{q + S} \]
from which it follows that
\[ (p - q)I + S \leq SI \leq (p - q)S + I \]
and so
\[ S = I. \]
The proof is complete. \( \square \)

5. Existence of prime period two solutions. In this section we establish the necessary and sufficient conditions for the existence of prime period two solutions of Eq. (2). It is interesting to note that the range of parameters for which such solutions exist, vary according to the parity of the delay terms \( l \) and \( k \). That is, the conditions under which prime period two solutions of Eq. (2) exist depends on the combination of \((l,k)\) being (odd, odd), (odd, even), (even, odd) or (even, even). The following theorem states the result.

**Theorem 5.1.** The necessary and sufficient conditions for the existence of prime period two solutions of Eq. (2) are the following respectively:

**Case(i).** If \( l, k \) both are odd then \( q < p + 1 \). Furthermore, \( \ldots, 0, p + 1 - q, 0, p + 1 - q, \ldots \) is the prime period two solution of Eq. (2).

**Case(ii).** If \( l \) is odd and \( k \) is even, then \( q = p - 1 \). In this case there exist infinitely many prime period two solutions of Eq. (2).

**Case(iii).** If \( l \) is even and \( k \) is odd, then \( p + q < 1 \). In this case there exist infinitely many prime period two solutions of Eq. (2). In particular, \( \ldots, 0, 1 - (p + q), 0, 1 - (p + q), \ldots \) is a prime period two solution of Eq. (2).

**Case(iv).** If \( l, k \) both are even then there does not exist any prime period two solutions of Eq. (2).

**Proof.** Let \( \ldots, \phi, \psi, \phi, \psi, \ldots, \phi \neq \psi \) be a prime period two solution of Eq. (2). We establish the necessary conditions in the proof. The sufficient conditions are straight
forward and are omitted in the proof. We divide the proof into the following four cases:

**Case(i).** \( l \) and \( k \) both odd. Then

\[
\phi = \frac{p\phi + \phi}{q + \phi} \\
\psi = \frac{p\psi + \psi}{q + \psi}
\]

Note that, if \( \phi \) and \( \psi \) are both positive or both zeroes, then there does not exist any prime period two solutions of Eq.(2). Otherwise, \( \ldots, 0, p + 1 - q, 0, p + 1 - q, \ldots \) is a prime period two solution of Eq.(2).

**Case(ii).** \( l \) is odd and \( k \) is even. Then

\[
\psi = \frac{p\psi + \phi}{q + \phi} \\
\phi = \frac{p\phi + \psi}{q + \psi} \\
q\psi + \phi\psi = p\psi + \phi \\
q\phi + \phi\psi = p\phi + \psi \\
q = p - 1
\]

Note that, in this case there exist infinite prime period two solutions \( \ldots, \phi, \psi, \phi, \ldots \) such that

\[
\phi = \frac{\psi}{\psi - 1}
\]

**Case(iii).** \( l \) is even and \( k \) is odd. Then

\[
\psi = \frac{p\phi + \psi}{q + \psi} \\
\phi = \frac{p\psi + \phi}{q + \phi} \\
q\psi + \psi^2 = p\phi + \psi \\
q\phi + \phi^2 = p\psi + \phi \\
\phi + \psi = 1 - (p + q)
\]

**Case(iv).** \( l \) and \( k \) both even. Then

\[
\psi = \frac{p\phi + \phi}{q + \phi} \\
\phi = \frac{p\psi + \psi}{q + \psi} \\
q\psi + \phi\psi = p\phi + \phi \\
q\phi + \phi\psi = p\psi + \psi
\]
which is not possible.

The proof is complete.

**Conjecture 5.1.** Suppose that \( l \) and \( k \) are both odd and \( q \leq p - 1 \). Then for positive initial conditions every solution of Eq.(2) converges to the positive equilibrium \( p + 1 - q \).

**Remark 5.1.** As the delay terms \( l \) and \( k \) increase in value, it appears from orbit plots, there exists solutions of Eq.(2) of varied periodicities. Very high periodicities are observed in such cases. The following orbit plots display some such higher order periodicities.

![Figure 1. Orbit plots with higher order periodicities.](image)

The value of \( l, k, p \) and \( q \) are given in each plot above in the Fig.1. Each of the above plot shows very clearly higher order periodicities.

**5.1. Local stability of prime period 2 solutions.** For simplicity, we show the local stability of prime period 2 solutions of the specialized equation where \( l = 0 \) and \( k = 1 \). Let \( \ldots, \phi, \psi, \phi, \psi, \ldots, \phi \neq \psi \) be a prime period two solution of Eq.(2) where \( l = 0 \) and \( k = 1 \). We set

\[
\begin{align*}
    u_n &= y_n \\
    v_n &= y_{n-1}
\end{align*}
\]

Then the equivalent form of the Eq.(2) with \( l = 0 \) and \( k = 1 \) is

\[
\begin{align*}
    u_{n+1} &= v_n \\
    v_{n+1} &= \dfrac{pu_n + v_n}{q + v_n}
\end{align*}
\]

Let \( T \) be the map on \((0, \infty) \times (0, \infty)\) to itself defined by

\[
T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{pu + v}{q + v} \end{pmatrix}
\]

Then \( \begin{pmatrix} \phi \\ \psi \end{pmatrix} \) is a fixed point of \( T^2 \), the second iterate of \( T \).

\[
T^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{pu + v}{q + v} \\ \dfrac{pv + pu + v}{q + v} \end{pmatrix}
\]
\[ T^2 \begin{pmatrix} u \\ v \end{pmatrix} = \left( \begin{array}{cc} g(u, v) \\ h(u, v) \end{array} \right) \]

where \( g(u, v) = \frac{pu+q}{q+u} \) and \( h(u, v) = \frac{pv+qw}{q+u} \). Clearly the two cycle is locally asymptotically stable when the eigenvalues of the Jacobian matrix \( J_{T^2} \), evaluated at \( \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \), lie inside the unit disk. We have,

\[
J_{T^2} \left( \begin{array}{c} \phi \\ \psi \end{array} \right) = \left( \begin{array}{cc} \frac{\delta g}{\delta u}(\phi, \psi) & \frac{\delta g}{\delta v}(\phi, \psi) \\ \frac{\delta h}{\delta u}(\phi, \psi) & \frac{\delta h}{\delta v}(\phi, \psi) \end{array} \right)
\]

where \( \frac{\delta g}{\delta u}(\phi, \psi) = \frac{p}{q+u} \) and \( \frac{\delta g}{\delta v}(\phi, \psi) = -\frac{v^2+pq}{(q+v)^2} + 1 \frac{\partial}{\partial v} \frac{\delta h}{\delta u}(\phi, \psi) \). And

\[
\frac{\delta h}{\delta u}(\phi, \psi) = \frac{\psi^2+pq}{(\psi+q+p+q)^2}
\]

Now, set

\[
\chi = \frac{\delta g}{\delta u}(\phi, \psi) + \frac{\delta h}{\delta v}(\phi, \psi)
\]

\[
\lambda = \frac{\delta g}{\delta u}(\phi, \psi) \frac{\delta h}{\delta v}(\phi, \psi) - \frac{\delta g}{\delta v}(\phi, \psi) \frac{\delta h}{\delta u}(\phi, \psi)
\]

Then it follows from the Linearized Stability that both the eigenvalues of the \( J_{T^2} \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \) lie inside the unit disk if and only if \(|\chi| < 1 + \lambda < 2\). In other words, the equivalent inequalities are \( \chi < 1 + \lambda, -1 - \lambda < \chi \) and \( \lambda < 1 \).

In the present case, \( l = 0 \) and \( k = 1 \) and \( p + q < 1 \) then there exist infinitely many prime period two solutions of Eq. (2) with \( l = 0 \) and \( k = 1 \).

In particular, \( . . . , 0, 1 - (p + q), 0, 1 - (p + q), . . . \) is a prime period two solution of Eq. (2) with \( l = 0 \) and \( k = 1 \).

Under this case,\n
In particular, when \( \ldots, 0, 1 - (p + q), 0, 1 - (p + q), . . . \) is a prime period two solution of Eq. (2) with \( l = 0 \) and \( k = 1 \) then

\[
\chi = \frac{p}{1-p} + \frac{1 + p \left( 3 - 8p + 6p^2 \right)}{(2 - 3p)^2}
\]

\[
\lambda = \frac{2p^2}{2 - 3p}
\]

Here \( \chi \) and \( \lambda \) do not satisfy the linear stability criterion \(|\chi| < 1 + \lambda < 2\). For example if \( p = \frac{1}{2} \) and \( q < \frac{1}{2} \) then \( \chi = 6 \) and \( \lambda = 1 \). Hence by the Linear Stability Theorem the prime period 2 solution \( \ldots, 0, 1 - (p + q), 0, 1 - (p + q), . . . \) is not locally asymptotically stable.

6. Semi-cycle analysis. Definition 6.1. A positive semi-cycle of \( \{y_n\}_{n=-k}^\infty \) consists of a “string” of terms \( \{y_s, y_{s+1}, \ldots, y_m\} \) all greater than or equal to \( \bar{y} \), with \( s \geq -k \) and \( m \leq \infty \) such that either \( s = -k \) or \( s > -k \) and \( y_{s-1} < \bar{y} \), and either \( m = \infty \) or \( m < \infty \) and \( y_{m+1} < \bar{y} \).

A negative semi-cycle of \( \{y_n\}_{n=-k}^\infty \) consists of a “string” of terms \( \{y_s, y_{s+1}, \ldots, y_m\} \) all less than \( \bar{y} \), with \( s \geq -k \) and \( m \leq \infty \) such that either \( s = -k \) or \( s > -k \) and \( y_{s-1} \geq \bar{y} \), and either \( m = \infty \) or \( m < \infty \) and \( y_{m+1} \geq \bar{y} \).

The following two theorems given in [2] will be useful in establishing our main results in this section.
Theorem 6.1. Assume that \( f : (0, \infty) \times (0, \infty) \to (0, \infty) \) is a continuous function such that \( f(u,v) \) is increasing in \( u \) for fixed \( v \), and \( f(u,v) \) is increasing in \( v \) for fixed \( u \). Let \( \bar{y} \) be a positive equilibrium of Eq.(1). Then, except possibly for the first semi-cycle, every oscillatory solution of Eq.(2) has semi-cycle of length at least \( k \).

Theorem 6.2. Assume that \( f : (0, \infty) \times (0, \infty) \to (0, \infty) \) is a continuous function such that \( f(u,v) \) is increasing in \( u \) for fixed \( v \), and \( f(u,v) \) is decreasing in \( v \) for fixed \( u \). Let \( \bar{y} \) be a positive equilibrium of Eq.(1). Then, except possibly for the first semi-cycle, every oscillatory solution of Eq.(2) has semi-cycle of length at least \( k + 1 \) or of length most \( k - 1 \).

We now establish the semi-cycle analysis for Eq.(2).

Theorem 6.3. Consider \( y_{n+1} = f(y_{n-1}, y_{n-k}) \) be the Eq.(2), for \( (u,v) \in (0, \frac{2}{p}) \times (0, \frac{2}{p}) \), every oscillatory solution of Eq.(2) has semi-cycle of length at least \( k \).

Proof. Observe that the function
\[
f(u, v) = \frac{pu + v}{q + v}
\]
increasing in \( u \) for all \( v \in (0, \infty) \), increasing in \( v \) for all \( u \in (0, \frac{2}{p}) \). Therefore, for \( (u,v) \in (0, \frac{2}{p}) \times (0, \frac{2}{p}) \), by Lemma (6.1) except possibly for the first semi-cycle, every oscillatory solution of Eq.(2) has semi-cycle of length at least \( k \).

This completes the proof.

Theorem 6.4. Consider \( y_{n+1} = f(y_{n-1}, y_{n-k}) \) be the Eq.(2), for \( (u,v) \in (\frac{2}{p}, \infty) \times (\frac{2}{p}, \infty) \), every oscillatory solution of Eq.(2) has semi-cycle of length at least \( k \).

Proof. Observe that the function
\[
f(u, v) = \frac{pu + v}{q + v}
\]
increasing in \( u \) for all \( v \in (0, \infty) \), decreasing in \( v \) for all \( u \in (\frac{2}{p}, \infty) \). Then for \( (u,v) \in (\frac{2}{p}, \infty) \times (\frac{2}{p}, \infty) \), by Lemma (6.2) except possibly for the first semi-cycle, every oscillatory solution of Eq.(2) has semi-cycle of length at least of length \( k + 1 \) or of length most \( k - 1 \).

This completes the proof.

7. Boundedness of solutions. Boundedness of solutions depends on the delay terms being even or odd. In this section we bring forward the cases for which every solution of Eq.(2) will be bounded and also the case where there will exist unbounded solutions. There exists unbounded solutions only when the delay terms \( l \) is odd and \( k \) is even. In that case, the existence of unbounded solutions occurs in the parameter space where \( q > p - 1 \). Otherwise, we have bounded solutions for all combinations of the delays \( l \) and \( k \).

Theorem 7.1. Every solution of Eq.(2) is bounded when \( l \) and \( k \) are both odd.

Proof. The proof follows from Theorem 2.3.4 in [2].

Conjecture 7.1. Every solution of Eq.(2) is bounded in the following cases:
(a) \( l \) and \( k \) are both even.
(b) \( l \) is even and \( k \) is odd.
Note that, when $q > p - 1$ we obtain the above result from Theorem (3.2) and Theorem (4.1). Thus the above conjecture needs to be shown only for $q \leq p - 1$.

In the remaining case of $l$ odd and $k$ even, the existence of unbounded solutions have been established in [8]. This is part of a period-two trichotomy result.

**Remark 7.1.** Suppose that $l$ is odd and $k$ is even. Then there exists a period two trichotomy for the solutions of Eq. (2):

$q > p - 1 \iff$ Every solution of Eq. (2) converges to the equilibrium.

$q = p - 1 \iff$ Every solution of Eq. (2) converges to a period two solution of Eq. (2) not necessarily prime.

$q < p - 1 \iff$ There exists unbounded solutions of Eq. (2)

This was established in [8].

8. Chaotic solutions. The method of Lyapunov characteristic exponents serves as a useful tool to quantify chaos. Specifically Lyapunov exponents measure the rates of convergence or divergence of nearby trajectories. Negative Lyapunov exponents indicate convergence, while positive Lyapunov exponents demonstrate divergence and chaos. The magnitude of the Lyapunov exponent is an indicator of the time scale on which chaotic behavior can be predicted or transients decay for the positive and negative exponent cases respectively. In this present study, the largest Lyapunov exponent is calculated for a given solution of finite length numerically [14]. For the chaotic solution the largest Lyapunov exponent is positive. Here some examples are given in the following table with figures.

| Parameters | Delay Terms | Estimated Interval of Lyapunov Exponent |
|------------|-------------|----------------------------------------|
| $p = 83, q = 2$ | $l = 23, k = 39$ | $(1.2047, 2.6210)$ |
| $p = 11, q = 2$ | $l = 5, k = 7$ | $(1.5999, 2.8415)$ |
| $p = 64, q = 57$ | $l = 13, k = 29$ | $(1.8484, 3.0188)$ |
| $p = 9, q = 4$ | $l = 9, k = 17$ | $(0.782, 1.1173)$ |
| $p = 70, q = 34$ | $l = 5, k = 9$ | $(1.8142, 2.8784)$ |
| $p = 61, q = 20$ | $l = 9, k = 17$ | $(0.2173, 1.4842)$ |

**Table 1.** Chaotic Solutions: The parameters, delay terms and corresponding Lyapunov exponent for about 5000 solutions.

The orbit (trajectory) plots corresponding to the 1st, 3rd and 6th cases in the Table-1 are given in the following Fig. 2. In each row different number of iterations of orbit plots are shown for the same set of initial of values.

In each of the above cases the computed Lyapunov exponents happen to be positive which ensured the chaoticity of the solutions listed in the Table-1.

**REFERENCES**

[1] R. M. Abu-Saris and R. DeVault, Global Stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, *Applied Mathematics Letters*, 16 (2003), 173–178.

[2] E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations; With Open Problems and Conjectures*, Chapman & Hall/CRC Press, 2008.

[3] E. Camouzis, E. Chatterjee and G. Ladas, On the dynamics of $x_{n+1} = \frac{\delta x_{n-2} + x_{n-3}}{A + x_{n-3}}$, *Journal of Mathematical Analysis and Applications*, 331 (2007), 230–239.

[4] E. Chatterjee, R. DeVault and G. Ladas, On the Global Character of $x_{n+1} = \frac{\beta x_n + \delta x_{n-k}}{A + x_{n-k}}$, *International Journal of Applied Mathematical Sciences*, 2 (2005), 39–46.

[5] C. W. Clark, A delayed recruitment model of population dynamics with an application to baleen whale populations, *J. Math. Biol.*, 3 (1976), 381–391.
Figure 2. Chaotic solutions for different cases as adumbrated in the Table-1.

[6] R. DeVault, G. Ladas and S. W. Schultz, On the recursive sequence $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$, Proc. Amer. Math. Soc., 126 (1998), 3257–3261.

[7] E. A. Grove, G. Ladas, M. Predescu and M. Radin, On the global character of $x_{n+1} = \frac{px_{n-1} + x_{n-2}}{q + x_{n-2}}$, Math. Sci. Res. Hot-line, 5 (2001), 25–39.

[8] E. A. Grove, G. Ladas, M. Predescu and M. Radin, On the global character of the difference equation $x_{n+1} = \frac{\alpha + \gamma x_{n-(2k+1)} + \delta x_{n-2l}}{A + x_{n-2l}}$, Journal of Difference Equations and Applications, 9 (2003), 171–199.

[9] V. L. Kocic and G. Ladas, Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, Holland, 1993.

[10] V. L. Kocic, G. Ladas and I. W. Rodrigues, On Rational Recursive Sequences, J. Math. Anal. Appl. 173 (1993), 127–157.

[11] M. R. S. Kulenović and G. Ladas, Dynamics of Second Order Rational Difference Equations; With Open Problems and Conjectures, Chapman & Hall/CRC Press, 2002.

[12] M. R. S. Kulenović, G. Ladas and N. R. Prokup, On a rational difference equation, Computers and Mathematics with Applications, 41 (2001), 671–678.

[13] V. G. Papanicolaou, On the Asymptotic Stability of a Class of Linear Difference Equations, Mathematics Magazine, 69 (1996), 34–43.

[14] A. Wolf, J. B. Swift, H. L. Swinney and J. A. Vastano, Determining Lyapunov exponents from a time series, Physica D, 16 (1985), 285–317.

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