K-THEORY FOR THE TAME C*-ALGEBRA OF A SEPARATED GRAPH

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Abstract. A separated graph is a pair \((E, C)\) consisting of a directed graph \(E\) and a set \(C = \bigsqcup_{v \in E^0} C_v\), where each \(C_v\) is a partition of the set of edges whose terminal vertex is \(v\). Given a separated graph \((E, C)\), such that all the sets \(X \in C\) are finite, the K-theory of the graph C*-algebra \(C^* (E, C)\) is known to be determined by the kernel and the cokernel of a certain map, denoted by \(1_C - A_{(E,C)}\), from \(\mathbb{Z}(C)\) to \(\mathbb{Z}(E^0)\). In this paper, we compute the K-theory of the tame graph C*-algebra \(\mathcal{O}(E, C)\) associated to \((E, C)\), which has been recently introduced by the authors. Letting \(\pi\) denote the natural surjective homomorphism from \(C^* (E, C)\) onto \(\mathcal{O}(E, C)\), we show that \(K_1(\pi)\) is a group isomorphism, and that \(K_0(\pi)\) is a split monomorphism, whose cokernel is a torsion-free abelian group. We also prove that this cokernel is a free abelian group when the graph \(E\) is finite, and determine its generators in terms of a sequence of separated graphs \(\{(E_n, C^n)\}_{n=1}^\infty\) naturally attached to \((E, C)\). On the way to showing our main results, we obtain an explicit description of a connecting map arising in a six-term exact sequence computing the K-theory of an amalgamated free product, and we also exhibit an explicit isomorphism between \(\ker(1_C - A_{(E,C)})\) and \(K_1(C^* (E, C))\).

1. Introduction

A separated graph is a pair \((E, C)\) consisting of a directed graph \(E\) and a set \(C = \bigsqcup_{v \in E^0} C_v\), where each \(C_v\) is a partition of the set of edges whose terminal vertex is \(v\). Their associated C*-algebras \(C^* (E, C)\) ([4], [1]) provide generalizations of the usual graph C*-algebras (see e.g. [15]) associated to directed graphs, although these algebras behave quite differently from the usual graph algebras because the range projections corresponding to different edges need not commute. One motivation for their introduction was to provide graph-algebraic models for the C*-algebras \(\mathcal{U}_{m,n}^{\text{sh}}\) studied by L. Brown [7] and McClanahan [11], [12], [13]. Another motivation was to obtain graph C*-algebras whose structure of projections is as general as possible. The theory of [4] was mainly developed for finitely separated graphs, which are those separated graphs \((E, C)\) such that all the sets \(X \in C\) are finite.

Recall that a set \(S\) of partial isometries in a C*-algebra \(\mathcal{A}\) is said to be tame [9, Proposition 5.4] if every element of \(U = (S \cup S^*)\), the multiplicative semigroup generated by \(S \cup S^*\), is a partial isometry. As indicated above, a main difficulty in working with \(C^* (E, C)\) is that, in general, the generating set of partial isometries of these algebras is not tame. This is not the case for the usual graph algebras, where it can be easily shown that the generating set of
partial isometries is tame. In order to solve this problem, we introduced in [2] the tame graph C*-algebra $\mathcal{O}(E, C)$ of a separated graph. Roughly, this algebra is defined by imposing to $C^*(E, C)$ the relations needed to transform the canonical generating set of partial isometries into a tame set of partial isometries (see Section 2 for the precise definitions).

For a finite bipartite separated graph $(E, C)$, a dynamical interpretation of the C*-algebra $\mathcal{O}(E, C)$ was obtained in [2], and using this, a useful representation of $\mathcal{O}(E, C)$ as a partial crossed product of a commutative C*-algebra by a finitely generated free group was derived. This theory enabled the authors to solve ([2, Section 7]) an open problem on paradoxical decompositions in a topological setting, posed in [10] and [18]. It is worth mentioning here that the restriction to bipartite graphs in this theory is harmless, since by [2, Proposition 9.1], we can attach to every separated graph $(E, C)$ a bipartite separated graph $(\tilde{E}, \tilde{C})$ in such a way that the respective (tame) graph C*-algebras are Morita-equivalent.

One of the main technical tools in [2] is the introduction, for each finite bipartite separated graph $(E, C)$, of a sequence of finite bipartite separated graphs $\{(E_n, C_n)\}$ such that the graph C*-algebras $C^*(E_n, C_n)$ approximate the tame graph C*-algebra $\mathcal{O}(E, C)$, in the sense that $\mathcal{O}(E, C) \cong \varprojlim_n C^*(E_n, C_n)$, see [2, Section 5].

The main purpose of this paper is to compute the K-theory of the tame graph C*-algebras of finitely separated graphs. Concretely, we show the following result:

**Theorem 1.1.** Let $(E, C)$ be a finitely separated graph. Then

1. $K_0(\mathcal{O}(E, C)) \cong K_0(C^*(E, C)) \oplus H \cong \text{coker}(1 - A_{(E, C)}) \oplus H$, where $H$ is a torsion-free abelian group. The group $H$ is a free abelian group when $E$ is a finite graph.
2. The canonical projection map $\pi : C^*(E, C) \to \mathcal{O}(E, C)$ induces an isomorphism $K_1(\mathcal{O}(E, C)) \cong K_1(C^*(E, C)) \cong \ker(1 - A_{(E, C)})$.

The terms $\text{coker}(1 - A_{(E, C)})$ and $\ker(1 - A_{(E, C)})$ appearing in the above theorem come from [4, Theorem 5.2], where the K-theory of the graph C*-algebras of finitely separated graphs was computed. The formulas there are analogous to the ones previously known for non-separated graphs (see [16, Theorem 3.2]). The matrix $A_{(E, C)}$ is the incidence matrix of the separated graph, which encodes the number of edges between two vertices of $E$ belonging to the different sets $X \in C$. (See Section 6 for the precise definition of these matrices).

We first study the case of finite bipartite separated graphs. Under this additional hypothesis, we obtain the result for $K_0$ in Section 4 (Theorem 4.6) and the result for $K_1$ in Section 6 (Theorem 6.7). The proof of Theorem 6.7 involves a computation of the index map for certain amalgamated free products, which we develop in Section 5. As a byproduct of our approach, we also develop a concrete description of the isomorphism between $\ker(1 - A_{(E, C)})$ and $K_1(C^*(E, C))$, which we believe is of independent interest. Such a description was obtained by Carlsen, Eilers and Tomforde in [8, Section 3] for relative graph algebras of non-separated graphs, by using different techniques. Using these results and direct limit technology, we show Theorem 1.1 in Section 7 (see Theorems 7.3 and 7.13).
Contents. We now explain in more detail the contents of this paper. In Section 2 we recall the basic definitions needed for our work, coming from the papers [4], [5] and [2]. In Section 3, we recall the crucial concept of a multiresolution of a separated graph \((E, C)\) at a set of vertices of \(E\), and we determine the precise relation between the corresponding graph \(C^*\)-algebras (Lemma 3.4). This is a vital step for our results on \(K_0\). In Section 4, we show the isomorphism \(\hat{K_0} = K_0(C^*(E, C)) \oplus H\) for any finite bipartite separated graph \((E, C)\), where \(H\) is a free abelian group, generally of infinite rank. The generators of \(H\) are precisely determined in terms of the vertices of the graphs appearing in the canonical sequence \(\{E_n, C^n\}\) of finite bipartite separated graphs associated to \((E, C)\) (see Theorem 4.6). Section 5 contains the explicit calculation of the index map \(K_1(A_1 \ast_B A_2) \rightarrow K_0(B)\) for any finite bipartite separated graph \((E, C)\). We obtain indeed an enhanced version of this result (Theorem 5.7), which includes an explicit isomorphism of the above mentioned groups with the group \(\ker(1_{C^*} - A(E, C))\). We also show a corresponding result for the reduced tame graph \(C^*\)-algebra \(O_{red}(E, C)\) (Corollary 5.9). Finally, we extend the above results to (not necessarily bipartite) finitely separated graphs in Section 6. For this, we use the direct limit technology of [5] and [2, Proposition 9.1]. The result for \(K_1\) is easily derived using these techniques (Theorem 6.7). To obtain the result for \(K_0\), we need to refine some of the already developed tools, in particular we make use of the concrete information about the generators of the cokernel of the map \(K_0(\pi): K_0(C^*(E, C)) \rightarrow K_0(O(E, C))\) induced by the canonical surjection \(\pi: C^*(E, C) \rightarrow O(E, C)\) for finite bipartite separated graphs, see Theorems 7.12 and 7.13.

2. Preliminary definitions

The concept of separated graph, introduced in [5], plays a vital role in our construction. In this section, we will recall this concept and we will also recall the definitions of the monoid associated to a separated graph, the Leavitt path algebra and the graph \(C^*\)-algebra of a separated graph.

Regarding the direction of arrows in graphs, we will use the reverse notation as in [5] and [4], but in agreement with the one used in [3], and in the book [15].

Definition 2.1. ([5]) A separated graph is a pair \((E, C)\) where \(E\) is a graph, \(C = \bigcup_{v \in E^0} C_v\), and \(C_v\) is a partition of \(r^{-1}(v)\) (into pairwise disjoint nonempty subsets) for every vertex \(v\). (In case \(v\) is a source, we take \(C_v\) to be the empty family of subsets of \(r^{-1}(v)\).)

If all sets in \(C\) are finite, we say that \((E, C)\) is a finitely separated graph. This necessarily holds if \(E\) is column-finite (that is, if \(r^{-1}(v)\) is a finite set for every \(v \in E^0\).)

The set \(C\) is a trivial separation of \(E\) in case \(C_v = \{r^{-1}(v)\}\) for each \(v \in E^0 \setminus \text{Source}(E)\). In that case, \((E, C)\) is called a trivially separated graph or a non-separated graph.
Definition 2.2. [4, Definition 1.4] The Leavitt path algebra of the separated graph $(E, C)$ is the $*$-algebra $L_C(E, C)$ with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the following relations:

\begin{align*}
(V) \quad vv' &= \delta_{v,v'}v \quad \text{and} \quad v = v^* \quad \text{for all } v, v' \in E^0, \\
(E) \quad r(e)e &= es(e) = e \quad \text{for all } e \in E^1, \\
(SCK1) \quad e^*e' &= \delta_{e,e'}s(e) \quad \text{for all } e, e' \in X, X \in C, \text{ and} \\
(SCK2) \quad v &= \sum_{e \in X} ee^* \quad \text{for every finite set } X \in C_v, v \in E^0.
\end{align*}

We now recall the definition of the graph $C^*$-algebra $C^*(E, C)$, introduced in [4].

Definition 2.3. [4, Definition 1.5] The graph $C^*$-algebra of a separated graph $(E, C)$ is the $C^*$-algebra $C^*(E, C)$ with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the relations (V), (E), (SCK1), (SCK2). In other words, $C^*(E, C)$ is the enveloping $C^*$-algebra of $L_C(E, C)$.

In case $(E, C)$ is trivially separated, $C^*(E, C)$ is just the classical graph $C^*$-algebra $C^*(E)$. There is a unique $*$-homomorphism $L_C(E, C) \to C^*(E, C)$ sending the generators of $L_C(E, C)$ to their canonical images in $C^*(E, C)$. This map is injective by [4, Theorem 3.8(1)].

The $C^*$-algebra $C^*(E, C)$ for separated graphs behaves in quite a different way compared to the usual graph $C^*$-algebras associated to non-separated graphs, the reason being that the final projections of the partial isometries corresponding to edges coming from different sets in $C_v$, for $v \in E^0$, need not commute. In order to resolve this problem, a different $C^*$-algebra was considered in [2], as follows:

Definition 2.4. [2] Let $(E, C)$ be any separated graph. Let $U$ be the multiplicative subsemigroup of $C^*(E, C)$ generated by $(E^1) \cup (E^1)^*$ and write $e(u) = uu^*$ for $u \in U$. Then the tame graph $C^*$-algebra of $(E, C)$ is the $C^*$-algebra

$$
\mathcal{O}(E, C) = C^*(E, C) / J,
$$

where $J$ is the closed ideal of $C^*(E, C)$ generated by all the commutators $[e(u), e(u')]$, for $u, u' \in U$.

Observe that $J = 0$ in the non-separated case, so we get that $\mathcal{O}(E) = C^*(E)$ is the usual graph $C^*$-algebra in this case.

Recall that for a unital ring $R$, the monoid $\mathcal{V}(R)$ is usually defined as the set of isomorphism classes $[P]$ of finitely generated projective (left, say) $R$-modules $P$, with an addition operation given by $[P] + [Q] = [P \oplus Q]$. For a nonunital version, see [5, Definition 10.8].

For arbitrary rings, $\mathcal{V}(R)$ can also be described in terms of equivalence classes of idempotents from the ring $M_\infty(R)$ of all infinite matrices over $R$ with finitely many nonzero entries. The equivalence relation is Murray-von Neumann equivalence: idempotents $e, f \in M_\infty(R)$ satisfy $e \sim f$ if and only if there exist $x, y \in M_\infty(R)$ such that $xy = e$ and $yx = f$. Write $[e]$ for the equivalence class of $e$; then $\mathcal{V}(R)$ can be identified with the set of these classes. Addition in $\mathcal{V}(R)$ is given by the rule $[e] + [f] = [e \oplus f]$, where $e \oplus f$ denotes the block diagonal matrix $(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix})$. With this operation, $\mathcal{V}(R)$ is a commutative monoid, and it is conical, meaning that $a + b = 0$ in $\mathcal{V}(R)$ only when $a = b = 0$. Whenever $A$ is a $C^*$-algebra, the monoid $\mathcal{V}(A)$
agrees with the monoid of equivalence classes of projections in $M_\infty(A)$ with respect to the equivalence relation given by $e \sim f$ if and only if there is a partial isometry $w$ in $M_\infty(A)$ such that $e = w^*w$ and $f = w^*w$; see [6, 4.6.2 and 4.6.4] or [17, Exercise 3.11].

We will need the definition of $M(E, C)$ only for finitely separated graphs. The reader can consult [5] for the definition in the general case. Let $(E, C)$ be a finitely separated graph, and let $M(E, C)$ be the commutative monoid given by generators $a_v, v \in E^0,$ and relations $a_v = \sum_{e \in X} a_{s(e)}$, for $X \in C_v, v \in E^0$. Then there is a canonical monoid homomorphism $M(E, C) \rightarrow \mathcal{V}(L_C(E, C))$, which is shown to be an isomorphism in [5, Theorem 4.3]. The map $\mathcal{V}(L_C(E, C)) \rightarrow \mathcal{V}(C^*(E, C))$ induced by the natural $\ast$-homomorphism $L_C(E, C) \rightarrow C^*(E, C)$ is conjectured to be an isomorphism for all finitely separated graphs $(E, C)$ (see [4] and [11, Section 6]).

3. Multiresolutions

In this section, we will recall from [2] the concept of mutiresolution of a finitely separated graph $(E, C)$, which is closely related to the notion of resolution, studied in [5]. We will also establish the precise relation between the corresponding Grothendieck groups.

**Definition 3.1.** ([2]) Let $(E, C)$ be a finitely separated graph, and let $v$ be any given vertex. Let $C_v = \{X_1, \ldots, X_k\}$ with each $X_i$ a finite subset of $r^{-1}(v)$. Put $M = \prod_{i=1}^k |X_i|$. Then the multiresolution of $(E, C)$ at $v$ is the separated graph $(E_v, C^v)$ with

$$E^0_v = E^0 \sqcup \{v(x_1, \ldots, x_k) \mid x_i \in X_i, i = 1, \ldots, k\},$$

and with $E^1_v = E^1 \sqcup \Lambda$, where $\Lambda$ is a new set of arrows defined as follows. For each $x_i \in X_i$, we put $M/|X_i|$ new arrows $\alpha^{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k), x_j \in X_j, j \neq i$, with $r(\alpha^{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)) = s(x_i)$, and $s(\alpha^{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)) = v(x_1, \ldots, x_k)$.

For a vertex $w \in E^0$, define the new groups at $w$ as follows. These groups are indexed by the edges $x_i \in X_i, i = 1, \ldots, k$, such that $s(x_i) = w$. For each such $x_i$, set

$$X(x_i) = \{\alpha^{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \mid x_j \in X_j, j \neq i\}.$$

Then

$$(C^v)_w = C_w \sqcup \{X(x_i) \mid x_i \in X_i, s(x_i) = w, i = 1, \ldots, k\}.$$

The new vertices $v(x_1, \ldots, x_k)$ are sources in $E_v$.

**Definition 3.2.** ([2]) Let $V \subseteq E^0$ be a set of vertices such that, for each $u \in V$, $C_u = \{X^u_1, \ldots, X^u_{k_u}\}$, with each $X^u_i$ a finite subset of $r^{-1}(u)$. Then the multiresolution of $(E, C)$ at $V$ is the separated graph $(E_V, C^V)$ obtained by applying the above process to all vertices $u$ in $V$.

Hence

$$E^0_V = E^0 \sqcup \bigcup_{u \in V} \{v(x^u_1, \ldots, x^u_{k_u}) \mid x^u_i \in X^u_i, i = 1, \ldots, k_u\},$$
and $E^1_V = E^1 \sqcup \left( \bigsqcup_{u \in V} A_u \right)$, where $A_u$ is the corresponding set of arrows, defined as in Definition 3.1 for each $u \in V$. The sets $(C^V)_w$, for $w \in E^0_V$, are defined just as in Definition 3.1.

$$(C^V)_w = C_w \sqcup \{ X(x^u_i) \mid x^u_i \in X^u_i, s(x^u_i) = w, i = 1, \ldots, k_u, u \in V \}.$$  

The new vertices $v(x^u_1, \ldots, x^u_{k_u})$ are sources in $E_V$.

We will only need to consider multiresolutions at sets of vertices $V$ such that there are no edges between them. Observe that this implies that $r_E(v) = r_{E_V}(v)$ for all $v \in V$.

The notation used in the next Lemma will become clear when we prove Lemma 3.4.

**Lemma 3.3.** Let $X_1, \ldots, X_k$ be $k$ finite sets, with $X_i = \{ x^{(i)}_t \}_{t=1, \ldots, |X_i|}$ and let $G(M)$ be the abelian group generated by the $|X_1| + \cdots + |X_k|$ elements

$$\{ b(x^{(i)}_t) \mid t = 1, \ldots, |X_i|, i = 1, \ldots, k \}$$

subject to the relations $\sum_{t=1}^{|X_i|} b(x^{(i)}_t) - \sum_{s=1}^{|X_i|} b(x^{(j)}_s) = 0$, for $1 \leq i < j \leq k$. Let $G(F)$ be the free abelian group on the $|X_1| \cdot |X_2| \cdots |X_k|$ elements $a(x^{(1)}_{t_1}, x^{(2)}_{t_2}, \ldots, x^{(k)}_{t_k})$, for $t_i \in \{ 1, \ldots, |X_i| \}$, $i \in \{ 1, \ldots, k \}$. Let $G(\psi): G(M) \to G(F)$ be the group homomorphism given by

$$G(\psi)(b(x^{(i)}_t)) = \sum_{j \neq i} \sum_{t_j=1}^{|X_j|} a(x^{(1)}_{t_1}, \ldots, x^{(i-1)}_{t_{i-1}}, x^{(i)}_t, x^{(i+1)}_{t_{i+1}}, \ldots, x^{(k)}_{t_k})$$  

for $1 \leq t_i \leq |X_i|$, $1 \leq i \leq k$. Then $G(\psi)$ is injective, and $G(F)/G(\psi)(G(M))$ is a free abelian group of rank $|X_1| \cdot |X_2| \cdots |X_k| - |X_1| - \cdots - |X_k| + k - 1$, freely generated by the images in $G(F)/G(\psi)(G(M))$ of the elements of the form $a(x^{(1)}_{t_1}, x^{(2)}_{t_2}, \ldots, x^{(k)}_{t_k})$ such that $t_i > 1$ and $t_j > 1$ for at least two distinct indices $i, j \in \{ 1, \ldots, k \}$.

**Proof.** Observe that $G(\psi)$ is a well-defined homomorphism, since $G(\psi)$ sends $\sum_{t=1}^{|X_i|} b(x^{(i)}_t) - \sum_{s=1}^{|X_i|} b(x^{(j)}_s)$ to 0 for all $i, j$.

It is easy to check that

$$\mathcal{B} = \{ b(x^{(i)}_{t_i}) \mid 1 \leq t_i \leq |X_i| \} \cup \{ b(x^{(i)}_{t_i}) \mid 2 \leq t_i \leq |X_i|, 2 \leq i \leq k \}$$

is a family of generators for $G(M)$, with $|X_1| + \cdots + |X_k| - k + 1$ elements.

Write

$$B^{(1)}_i = G(\psi)(b(x^{(1)}_{t_1})), \quad B^{(i)}_{t_i} = G(\psi)(b(x^{(i)}_{t_i})), \quad 2 \leq t_i \leq |X_i|, 1 \leq i \leq k.$$  

Let $\mathcal{B}$ be the canonical basis of $G(F)$, and let $\mathcal{B}'$ be the subset of elements of $\mathcal{B}$ which are of the form $a(x^{(1)}_{t_1}, x^{(2)}_{t_2}, \ldots, x^{(k)}_{t_k})$ with $t_i > 1$ and $t_j > 1$ for at least two distinct indices $i, j \in \{ 1, \ldots, k \}$. By using the integer version of Steinitz’s Lemma, we see that

$$\{ B^{(1)}_i \} \cup \left( \mathcal{B} \setminus \{ a(x^{(1)}_{t_1}, x^{(2)}_{t_2}, \ldots, x^{(k)}_{t_k}) \} \right)$$

is a basis for $G(F)$. 
Lemma 3.4. Let $(E, C)$ be a separated graph and let $V \subseteq E^0$ be a finite set of vertices such that $|r^{-1}(u)| < \infty$ for all $u \in V$. Suppose that $s(r^{-1}(V)) \cap V = \emptyset$, that is, that there are no edges between elements of $V$. For $u \in V$, set $C_u = \{X_u^1, \ldots, X_u^{k_u}\}$. Let $\iota : (E, C) \to (E_V, C_V)$ denote the inclusion morphism, where $(E_V, C_V)$ is the multiresolution of $(E, C)$ at $V$. Then

\[ K_0(C^*(E_V, C_V)) \cong K_0(C^*(E, C)) \oplus \mathbb{Z}^W \]

where $W$ is the set of all vertices $v(x^{(1)}_i, \ldots, x^{(k_u)}_i)$, where $u \in V$, $x^{(i)}_i \in X_u^i$ for all $i$, and $t_i > 1, t_j > 1$ for at least two different indices $i$ and $j$. We have

\[ |W| = \sum_{u \in V} \left( \prod_{i=1}^{k_u} |X_u^i| - \sum_{i=1}^{k_u} |X_u^i| + k_u - 1 \right). \]

Proof. For a commutative monoid $M$, we denote by $G(M)$ the universal group of $M$. Given a monoid homomorphism $f : M_1 \to M_2$, there is an associated group homomorphism $G(f) : G(M_1) \to G(M_2)$. These assignments define a functor $G$ from the category of commutative monoids to the category of abelian groups.

Note that [4, Theorem 5.2] implies that, for every finitely separated graph $(E, C)$, the group $K_0(C^*(E, C))$ is isomorphic to the universal group of $M(E, C)$. More precisely, we have that the natural map $M(E, C) \to V(C^*(E, C))$ induces a group isomorphism $G(M(E, C)) \cong G(V(C^*(E, C)) = K_0(C^*(E, C))$).

Set $\mu = M(\iota)$, where $M(\iota) : M(E, C) \to M(E_V, C_V)$ is the natural map (see [5]). Note that, since $s(r^{-1}(V)) \cap V = \emptyset$, $(E_V, C_V)$ can be obtained as the last term of a finite sequence of separated graphs, each one obtained from the previous one by performing the multiresolution process with respect to a single vertex, with no new arrows in $r^{-1}(V)$ for all the graphs of the sequence. We may thus suppose that $V = \{v\}$ for a single vertex $v$ in $E^0$.

Set $C_v = \{X_1, \ldots, X_k\}$, and write $X_i = \{x^{(i)}_t\}_{t=1, \ldots, |X_i|}$. Let $F$ be the free commutative monoid on generators $a(x^{(1)}_1, x^{(2)}_1, \ldots, x^{(k)}_1)$, for $t_i \in \{1, \ldots, |X_i|\}$, $i \in \{1, \ldots, k\}$. Let $M$ be the commutative monoid given by generators

\[ \{b(x^{(i)}_t) \mid t = 1, \ldots, |X_i|, i = 1, \ldots, k\} \]

subject to the relations $\sum_{i=1}^{|X_i|} b(x^{(i)}_t) = \sum_{s=1}^{|X_j|} b(x^{(j)}_s)$, for $1 \leq i < j \leq k$. 

Now observe that for all $i \in \{1, \ldots, k\}$ and $t_i \in \{2, \ldots, |X_i|\}$, we have

\[ B^{(i)}_{t_i} \in a(x^{(1)}_1, \ldots, x^{(i-1)}_1, x^{(i)}_{t_i}, x^{(i+1)}_1, \ldots, x^{(k)}_1) + \langle B' \rangle. \]

Hence, the integer version of Steinitz’s Lemma gives immediately that

\[ \{B^{(1)}_{1} \} \cup \{B^{(i)}_{t_i} \mid 2 \leq t_i \leq |X_i|, 1 \leq i \leq k\} \cup B' \]

is a basis of $G(F)$. This shows in particular that $G(\psi)$ is injective and that the above generating family $B$ is a basis for $G(M)$. It also shows that $B'$ is a free basis for $G(F)/G(\psi)(G(M))$. \qed
There is a unique monoid homomorphism \( \eta : M \to M(E, C) \) sending \( b(x_t^{(i)}) \) to \([s(x_t^{(i)})]\) for \(1 \leq t \leq |X_i|\), and there is a unique homomorphism \( \eta' : F \to M(E_V, C^V) \) sending \( a(x_{t_1^{(i)}}, \ldots, x_{t_k^{(i)}}) \mapsto [v(x_{t_1^{(i)}}, \ldots, x_{t_k^{(i)}})]\) for \(1 \leq t_i \leq |X_i|, 1 \leq i \leq k\). There is a commutative diagram as follows:

\[
\begin{array}{c}
M & \xrightarrow{\psi} & F \\
\downarrow{\eta} & & \downarrow{\eta'} \\
M(E, C) & \xrightarrow{\mu} & M(E_V, C^V)
\end{array}
\]

where \( \psi \) is given by the formula (3.1) on the generators \( b(x_{t_i}^{(i)}) \) of \( M \). As noted in the proof of [2, 3.8], an easy adaptation of the proof of [5, Lemma 8.6] gives that (3.2) is a pushout in the category of commutative monoids. It is a simple matter to check that the functor \( G(\cdot) \) transforms a pushout diagram in the category of commutative monoids to a pushout diagram in the category of abelian groups. Since \( G(M(E, C)) \cong \mathbb{K}_0(C^*(E, C)) \) and \( G(M(E_V, C^V)) \cong \mathbb{K}_0(C^*(E_V, C^V)) \), we get a pushout diagram

\[
\begin{array}{c}
G(M) & \xrightarrow{G(\psi)} & G(F) \\
\downarrow{G(\eta)} & & \downarrow{G(\eta')} \\
\mathbb{K}_0(C^*(E, C)) & \xrightarrow{G(\mu)} & \mathbb{K}_0(C^*(E_V, C^V))
\end{array}
\]

By Lemma 3.3 the map \( G(\psi) \) is injective and we can write

\[
G(F) = G(\psi)(G(M)) \oplus H,
\]

where \( H \) is a free abelian group of rank \( \prod_{i=1}^k |X_i| - \sum_{i=1}^k |X_i| + k - 1 \). It follows easily from the usual description of pushouts in the category of abelian groups that

\[
\mathbb{K}_0(C^*(E_V, C^V)) \cong \mathbb{K}_0(C^*(E, C)) \oplus H.
\]

Indeed, we have that the mentioned pushout is computed as the quotient group

\[
(K_0(C^*(E, C)) \oplus G(F))/T,
\]

where \( T \) is the subgroup given by the elements of the form \((G(\eta)(x), -G(\psi)(x))\), for \( x \in G(M) \). It is quite easy to check that \( K_0(C^*(E, C)) \oplus G(F) = (K_0(C^*(E, C)) \oplus H) \oplus T \), from which the result follows.

\[\square\]

**Remark 3.5.** We may explicitly describe the Pontryagin dual of \( K_0(C^*(E_V, C^V)) \) using Lemma 3.4.
For any separated graph \((E, C)\), the Pontrjagin dual of the group \(K_0(C^*(E, C))\) can be thought of as the set of functions \(\lambda: E^0 \to \mathbb{T}\) which are invariant by the relations, that is, for every vertex \(v \in E^0\) and every \(X \in C_v\) we must have
\[
\lambda(v) = \prod_{x \in X} \lambda(s(x)).
\]
We get from Lemma 3.4 that
\[
K_0(C^*(E_V, C^V)) \cong K_0(C^*(E, C)) \oplus \mathbb{T}^W,
\]
that is, the character \(\lambda \in K_0(C^*(E_V, C^V))\) is determined by its values on the vertices of \(E^0\) and on the vertices \(v(x_1^{(1)}(u), \ldots, x_{k_u}^{(k)}(u))\), for \(u \in V\), where \(C_u = \{X_1^u, \ldots, X_{k_u}^u\}\), \(x_i^{(j)}(u) \in X_i^u\), and \(t_i > 1, t_j > 1\) for at least two different \(i\) and \(j\). We now indicate how to determine the values of \(\lambda\) at the remaining vertices of \(E_V\). Fix a vertex \(u\) in \(V\). To simplify notation, we will suppress the dependence on \(u\) in the notation. The elements of \(C_u\) will be denoted by \(X_1, \ldots, X_k\). For each index \(i\) and every \(t_i > 1\), we have
\[
\lambda(v(x_1^{(1)}, \ldots, x_1^{(i-1)}, x_i^{(i)}, x_1^{(i+1)}, \ldots, x_1^{(k)})) = \lambda(s(x_i^{(i)})).
\]
So all the values are determined except for \(\lambda(v(x_1^{(1)}, \ldots, x_1^{(k)}))\). Since \([u] = \sum_{t_1, \ldots, t_k} [v(x_1^{(1)}, \ldots, x_1^{(k)})]\) in \(K_0(C^*(E_V, C^V))\), we must have
\[
\lambda(v(x_1^{(1)}, \ldots, x_1^{(k)})) = \lambda(u) \cdot \left[ \prod_{(t_1, \ldots, t_k) \neq (1, 1, \ldots, 1)} \lambda(v(x_1^{(1)}, \ldots, x_1^{(k)})) \right]^{-1}.
\]
This is the way how all of the values of the character \(\lambda\) are determined from the given values.

4. **\(K_0\) for the tame \(C^*\)-algebra of a finite bipartite separated graph**

In this section, we will obtain a description of \(K_0(\mathcal{O}(E, C))\) for any finite bipartite separated graph \((E, C)\). This will be used in Section 7 to get a formula for general finitely separated graphs.

We first recall some basic terminology and our graph construction from [2].

**Definition 4.1.** ([2]) Let \(E\) be a directed graph. We say that \(E\) is a **bipartite directed graph** if \(E^0 = E^{0,0} \sqcup E^{0,1}\), with all arrows in \(E^1\) going from a vertex in \(E^{0,1}\) to a vertex in \(E^{0,0}\). To avoid trivial cases, we will always assume that \(r^{-1}(v) \neq \emptyset\) for all \(v \in E^{0,0}\) and \(s^{-1}(v) \neq \emptyset\) for all \(v \in E^{0,1}\).

A **bipartite separated graph** is a separated graph \((E, C)\) such that the underlying directed graph \(E\) is a bipartite directed graph.
Construction 4.2. ([2]) (a) Let \((E, C)\) be a finite bipartite separated graph. We define a nested sequence of finite separated graphs \((F_n, D^n)\) as follows. Set \((F_0, D^0) = (E, C)\). Assume that a nested sequence 
\[
(F_0, D^0) \subset (F_1, D^1) \subset \cdots \subset (F_n, D^n)
\]
has been constructed in such a way that for \(i = 1, \ldots, n\), we have \(F_i^0 = \bigcup_{j=0}^{i+1} F_i^0\) for some finite sets \(F_i^0\) and \(F_i^1 = \bigcup_{j=0}^{i} F_i^1\), with \(s(F_i^j) = F_i^0, j+1\) and \(r(F_i^j) = F_i^0, j\) for \(j = 1, \ldots, i\).

We can think of \((F_n, D^n)\) as a union of \(n\) bipartite separated graphs. Set \(V_n = F_{0,n}\), and let \((F_{n+1}, D^{n+1})\) be the multiresolution of \((F_n, D^n)\) at \(V_n\). (Note that there are no edges between elements of \(V_n\).) Then \(F_{0,n+1} = F_0^n \bigcup F_{0,n+2} = \bigcup_{j=0}^{n+2} F_0^j\) and \(F_{1,n+1} = F_1^n \bigcup F_{1,n+1} = \bigcup_{j=0}^{n+1} F_1^j\), with \(s(F_{1,n+1}) = F_{0,n+2}\) and \(r(F_{1,n+1}) = s(F_{1,n}) = F_{0,n+1}\).

(b) Let 
\[
(F_\infty, D_\infty) = \bigcup_{n=0}^{\infty} (F_n, D^n).
\]
Observe that \((F_\infty, D_\infty)\) is the direct limit of the sequence \(\{(F_n, D^n)\}\) in the category \(\text{FSGr}\) defined in [3] Definition 8.4]. We call \((F_\infty, D_\infty)\) the complete multiresolution of \((E, C)\).

(c) We define a canonical sequence \((E_n, C^n)\) of finite bipartite separated graphs as follows:

1. Set \((E_0, C^0) = (E, C)\).
2. \(E_{n+1}^0 = E_n^0, E_{n+1}^1 = E_{0,n+1}^0, \text{ and } E_1^1 = E_{1,n}^1\). Moreover \(C_v = D_v^0\) for all \(v \in E_{n,0}^0\) and \(C_v = \emptyset\) for all \(v \in E_{n,1}^0\).

We call the sequence \(\{(E_n, C^n)\}_{n \geq 0}\) the canonical sequence of bipartite separated graphs associated to \((E, C)\).

We will need the following Lemma, whose proof is contained in [2] Lemma 4.5].

Lemma 4.3. Let \((E, C)\) be a finite bipartite separated graph, let \((E_n, C^n)\) be the canonical sequence of bipartite separated graphs associated to \((E, C)\), and let \((F_\infty, D_\infty)\) be the complete multiresolution of \((E, C)\). Then the following properties hold:

(a) For each \(n \geq 0\), there is a natural isomorphism
\[
\varphi_n : M(E_{n+1}, C^{n+1}) \longrightarrow M((E_n)V_n, (C^n)V_n),
\]
where \(V_n = E_{0,n}^0, F_{0,n}^0\).

(b) For each \(n \geq 0\), there is a canonical embedding
\[
\iota_n : M(E_n, C^n) \rightarrow M(E_{n+1}, C^{n+1}).
\]

(c) The canonical inclusion \(j_n : (E_n, C^n) \rightarrow (F_n, D^n)\) induces an isomorphism
\[
M(j_n) : M(E_n, C^n) \rightarrow M(F_n, D^n).
\]

(d) We have \(M(F_\infty, D_\infty) \cong \varprojlim M(E_n, C^n), \iota_n)\).
Let \((E, C)\) be a finite bipartite separated graph, with \(r(E^1) = E^{0,0}\) and \(s(E^1) = E^{0,1}\). Let \(\{(E_n, C^n)\}_{n \geq 0}\) be the canonical sequence of bipartite separated graphs associated to it (see Construction 4.2(c)), and let \(B_n\) be the commutative C*-subalgebra of \(C^*(E_n, C^n)\) generated by \(E^n_0\).

**Theorem 4.4.** (cf. [2, Theorem 5.1]) With the above notation, for each \(n \geq 0\), there exists a surjective homomorphism

\[ \phi_n : C^*(E_n, C^n) \to C^*(E_{n+1}, C^{n+1}) . \]

Moreover, the following properties hold:

(a) \(\ker(\phi_n)\) is the ideal \(I_n\) of \(C^*(E_n, C^n)\) generated by all the commutators \([ee^*, ff^*]\), with \(e, f \in E^1_n\), so that \(C^*(E_{n+1}, C^{n+1}) \cong C^*(E_n, C^n)/I_n\).

(b) The restriction of \(\phi_n\) to \(B_n\) defines an injective homomorphism from \(B_n\) into \(B_{n+1}\).

(c) There is a commutative diagram

\[
\begin{array}{ccc}
G(M(E_n, C^n)) & \rightarrow & G(M(E_{n+1}, C^{n+1})) \\
\downarrow \cong & \downarrow \cong \\
K_0(C^*(E_n, C^n)) & \rightarrow & K_0(C^*(E_{n+1}, C^{n+1}))
\end{array}
\]

where the vertical maps are the canonical maps, which are isomorphisms by [4, Theorem 5.2].

Since we shall use it later, we recall here the definition of the map \(\phi_n\) appearing in Theorem 4.4(a) (see the proof of [2, Theorem 5.1]). The map \(\phi_n\) is defined on vertices \(u \in E^{0,0}_n\) by the formula

\[ \phi_n(u) = \sum_{(x_1, \ldots, x_{k_u}) \in \prod_{i=1}^{k_u} X_i^u} v(x_1, \ldots, x_{k_u}), \]

where \(C_u = \{X_1^u, \ldots, X_{k_u}^u\}\), and by \(\phi_n(w) = w\) for all \(w \in E^{0,1}_n\). For an arrow \(x_i \in X_i^u\), we have

\[ \phi_n(x_i) = \sum_{x_j \in X_j^u, j \neq i} (\alpha^{x_i}(x_1, \ldots, \bar{x}_i, \ldots, x_{k_u}))^* , \]

where \(\alpha^{x_i}(x_1, \ldots, \bar{x}_i, \ldots, x_{k_u}) = \alpha^{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k_u})\).

To simplify the notation, we will write \(D_n = F^{0,n} = E^{0,0}_n\) for all \(n \geq 0\).

Note that, for \(n \geq 2\) we have a surjective map \(r_n : D_n \rightarrow D_{n-2}\) given by \(r_n(v(x_1, \ldots, x_{k_u})) = u\), where \(u \in D_{n-2}\) and \(x_i \in X_i^u\), and where, as usual, \(C_u^{n-2} = \{X_1^u, \ldots, X_{k_u}^u\}\). For \(n = 2m\), we thus obtain a surjective map \(\tau_{2m} = r_{2m} \circ r_{2m-2} \circ \cdots \circ r_2 : D_{2m} \rightarrow D_0\). Similarly, we have a map \(\tau_{2m+1} = r_{2m+1} \circ r_{2m-1} \circ \cdots \circ r_1 : D_{2m+1} \rightarrow D_1\). We call \(\tau(v)\) the root of \(v\). Observe that we have

\[
D_{2n} = \bigsqcup_{v \in D_0} \tau_{2n}^{-1}(v); \quad D_{2n+1} = \bigsqcup_{v \in D_1} \tau_{2n+1}^{-1}(v)
\]
Set $A_n = C^*(E_n, C^n)$. Then $\mathcal{O}(E, C) = \lim \rightarrow A_n$, and $B_\infty = \lim \rightarrow B_n = C(\Omega(E, C))$. We have a commutative diagram as follows:

$$
\begin{array}{cccccc}
B_0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & C(\Omega(E, C)) \\
\downarrow \phi_0 & & \downarrow \phi_1 & & \downarrow \phi_2 & & \\
A_0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & O(E, C)
\end{array}
$$

All the maps $B_n \rightarrow B_{n+1}$ are injective and all the maps $A_n \rightarrow A_{n+1}$ are surjective.

Let $\mathbb{F}$ be the free group on $E^1$. There is a natural partial action $\theta$ of $\mathbb{F}$ on $\Omega(E, C)$ so that

$$
\mathcal{O}(E, C) \cong C(\Omega(E, C)) \rtimes_\theta \mathbb{F}.
$$

Moreover $(\Omega(E, C), \theta)$ is the universal $(E, C)$-dynamical system (see [2]). Let us recall the definition:

**Definition 4.5.** An $(E, C)$-dynamical system consists of a compact Hausdorff space $\Omega$ with a family of clopen subsets $\{\Omega_v\}_{v \in E^0}$ such that

$$
\Omega = \bigsqcup_{v \in E^0} \Omega_v,
$$

and, for each $v \in E^{0,0}$, a family of clopen subsets $\{H_x\}_{x \in \varphi^{-1}(v)}$ of $\Omega_v$, such that

$$
\Omega_v = \bigsqcup_{x \in X} H_x
$$

for all $X \in C_v$,

and, for all $x \in E^1$.

Given two $(E, C)$-dynamical systems $(\Omega, \theta)$, $(\Omega', \theta')$, there is an obvious definition of equivariant map $f: (\Omega, \theta) \rightarrow (\Omega', \theta')$, namely $f: \Omega \rightarrow \Omega'$ is equivariant if $f(\Omega_w \subseteq \Omega'_w$ for all $w \in E^0$, $f(H_x) \subseteq H'_x$ for all $x \in E^1$ and $f(\theta_x(y)) = \theta'_x f(y)$ for all $y \in \Omega_s(x)$.

We say that an $(E, C)$-dynamical system $(\Omega, \theta)$ is universal in case there is a unique continuous equivariant map from every $(E, C)$-dynamical system to $(\Omega, \theta)$.

We write $\Omega(E, C) = \bigsqcup_{v \in E^0} \Omega(E, C)_v$, $\Omega(E, C)_v = \bigsqcup_{x \in X} H_x$ for all $X \in C_v$ ($v \in E^{0,0}$), and $\theta_x: \Omega(E, C)s(x) \rightarrow H_x$ for the structural clopen sets and homeomorphisms of the universal $(E, C)$-dynamical system.
We have
\[
\lim_{i\to \infty}(D_{2i}, \tau_{2i}) = \Omega^0 := \bigcup_{v \in D_0} \Omega(E, C)_v, \quad \lim_{i\to \infty}(D_{2i+1}, \tau_{2i+1}) = \Omega^1 := \bigcup_{v \in D_1} \Omega(E, C)_v.
\]

In the following \(\tau_{2k, \infty}: \Omega^0 \to D_{2k}\) and \(\tau_{2k+1, \infty}: \Omega^1 \to D_{2k+1}\) will denote the canonical projective limit surjections. The family \(\{\tau^{-1}_{k, \infty}(v) | v \in D_k, k = 0, 1, 2, \ldots\}\) is a basis of clopen sets for the topology of \(\Omega(E, C)\).

**Theorem 4.6.** Let \((E, C)\) be a finite bipartite separated graph, and let \(\pi: C^*(E, C) \to \mathcal{O}(E, C)\) be the natural projection map. Then \(K_0(\pi)\) is a split monomorphism, and its cokernel \(H\) is a free abelian group. Moreover, there are subsets \(W_k \subset D_k\), for \(k = 2, 3, \ldots\), such that \(H \cong \bigoplus_{k=2}^{\infty} \mathbb{Z}^{W_k}\). In particular, we have
\[
K_0(\mathcal{O}(E, C)) \cong K_0(C^*(E, C)) \oplus \left( \bigoplus_{k=2}^{\infty} \mathbb{Z}^{W_k} \right).
\]

**Proof.** We have \(K_0(\mathcal{O}(E, C)) \cong \varprojlim_k K_0(C^*(E_k, C^k))\).

By Theorem 3.4(c), it is enough to compute the limit \(\varprojlim(G(M(E_k, C^k)), G(\iota_k))\). Now the map \(\iota_k: M(E_k, C^k) \to M(E_{k+1}, C^{k+1})\) is the composition of the canonical map \(\iota_{V_k}: M(E_k, C^k) \to M((E_k)_{V_k}, (C^k)_{V_k})\) and the isomorphism \(\varphi^{-1}_k: M((E_k)_{V_k}, (C^k)_{V_k}) \to M(E_{k+1}, C^{k+1})\) (cf. [2 Lemma 4.5]).

By (the proof of) Lemma 3.4 there are subsets \(W_i\) of \(D_i\), for \(i = 2, 3, \ldots\), and isomorphisms
\[
\gamma_i: G(M((E_i)_{V_i}, (C^i)_{V_i})) \xrightarrow{\cong} G(M(E_i, C^i)) \oplus \mathbb{Z}^{W_{i+2}},
\]
such that \(\gamma_i([v]) = [v]\) for all \(v \in E^0_i, i = 0, 1, 2, \ldots\).

We construct by induction a family of group isomorphisms
\[
\theta_i: G(M(E_i, C^i)) \to K_0(C^*(E, C)) \oplus \mathbb{Z}^{W_2} \oplus \cdots \oplus \mathbb{Z}^{W_{i+1}}
\]
such that all the diagrams
\[
\begin{align*}
G(M(E_i, C^i)) & \xrightarrow{\theta_i} K_0(C^*(E, C)) \oplus \mathbb{Z}^{W_2} \oplus \cdots \oplus \mathbb{Z}^{W_{i+1}} \\
G(\iota_i) \downarrow & \quad \downarrow j_1^{(i)} \\
G(M(E_{i+1}, C^{i+1})) & \xrightarrow{\theta_{i+1}} K_0(C^*(E, C)) \oplus \mathbb{Z}^{W_2} \oplus \cdots \oplus \mathbb{Z}^{W_{i+2}}
\end{align*}
\]
are commutative, where \(j_1^{(i)}\) is the natural inclusion. The map \(\theta_0: G(M(E, C)) \to K_0(C^*(E, C))\) is defined to be the natural isomorphism. Assume that \(\theta_0, \ldots, \theta_k\) have been defined for some \(k \geq 0\). Define the map
\[
\tilde{\gamma}_k: G(M((E_k)_{C_k}, (C^k)_{V_k})) \to K_0(C^*(E, C)) \oplus \mathbb{Z}^{W_2} \oplus \cdots \oplus \mathbb{Z}^{W_{k+2}}
\]
by $\tilde{\gamma}_k = (\theta_k \oplus \text{id}_{\mathbb{Z}^{w_{k+2}}}) \circ \gamma_k$. Define $\theta_{k+1} = \tilde{\gamma}_k \circ G(\varphi_k)$. Then the two squares in the following diagram are commutative:

$$
\begin{array}{ccc}
G(M(E_k, C^k)) & \xrightarrow{\theta_k} & K_0(C^*(E, C)) \oplus \mathbb{Z}^{W_2} \oplus \cdots \oplus \mathbb{Z}^{W_{k+1}} \\
G(\iota_{V_k}) \downarrow & & \downarrow j^{(k)}_1 \\
G(M((E_k)_{V_k}, (C^k)_{V_k})) & \xrightarrow{\tilde{\gamma}_k} & K_0(C^*(E, C)) \oplus \mathbb{Z}^{W_2} \oplus \cdots \oplus \mathbb{Z}^{W_{k+2}} \\
G(\iota^{-1}_{V_k}) \downarrow \cong & & \downarrow \\
G(M(E_{k+1}, C^{k+1})) & \xrightarrow{\theta_{k+1}} & K_0(C^*(E, C)) \oplus \mathbb{Z}^{W_2} \oplus \cdots \oplus \mathbb{Z}^{W_{k+2}}
\end{array}
$$

(4.4)

Since $G(\iota_k) = G(\varphi_k)^{-1} \circ G(\iota_{V_k})$, we have completed the induction step.

We obtain

$$
K_0(\Omega(E, C)) \cong \lim_k (G(M(E_k, C^k)), G(\iota_k)) \cong K_0(C^*(E, C)) \oplus \left( \bigoplus_{k=2}^{\infty} \mathbb{Z}^{W_k} \right),
$$

as desired. \(\square\)

**Remark 4.7.** The Pontrjagin dual of $K_0(\Omega(E, C))$ can be identified with the set of $\mathbb{T}$-valued measures defined on the field $\mathbb{K}$ of clopen subsets of $\Omega(E, C)$, which are invariant under the action of $\mathbb{F}$. This is exactly the dual of the type semigroup $S(\Omega(E, C), \mathbb{F}, \mathbb{K})$, considered in [2 Section 7].

Let $\mathcal{B} = \{r^{-1}_{k,\infty}(v) \mid v \in W_k, k = 2, 3, \ldots \}$. Then $\mathcal{B}$ is a family of clopen subsets of $\Omega := \Omega(E, C)$, which together with $K_0(C^*(E, C))$, determine the Pontrjagin dual of $K_0(\Omega(E, C))$. Namely any character $\lambda$ on $K_0(\Omega(E, C))$ is determined by its values on the structural clopen sets $\Omega_v := \Omega(E, C)_v$, $v \in E^0$ (which have to fulfil the relations $\lambda(\Omega_v) = \prod_{x \in X} \lambda(\Omega_{s(x)})$ for every $v \in D_0$ and every $X \subset C_v$), and by the values $\lambda(U)$, for $U \in \mathcal{B}$, which can be arbitrary complex numbers of modulus one. The values of $\lambda$ on the other clopen sets of $\Omega$ are determined inductively by the rules indicated in Remark 3.5.

**Remark 4.8.** With suitable conditions of connectedness, the open set $\bigcup_{U \in \mathcal{B}} U$ is a dense subset of $\Omega$ (where $\mathcal{B}$ is as in Remark 4.7). For instance, we consider the separated graph $(E, C) = (E(m, n), C(m, n))$ appearing in [2 Example 9.3] (see also [3]), with $1 < m \leq n$. We have $D_0 = \{v_0\}$, $D_1 = \{v_1\}$, and $C_{v_0} = \{X^{v_0}, Y^{v_0}\}$, with $|X^{v_0}| = n$ and $|Y^{v_0}| = m$. Now, we consider the multiresolution of $(E, C)$, and we use the notation introduced before. We get $|C_v| = 2$ if $v \in D_{2k}$ and $|C_v| = n + m$ if $v \in D_{2k+1}$. We have $C_{v_i} = \{X_i^{v_1}, \ldots, X_i^{v_n}, Y_i^{v_1}, \ldots, Y_i^{v_m}\}$, with

$$
|X_i^{v_i}| = m, \quad |Y_i^{v_i}| = n \quad (1 \leq i \leq n, 1 \leq j \leq m).
$$

One checks inductively that, for $v \in D_{2k}$, $C_v = \{X^v, Y^v\}$, with $|X^v| = |X^{v'}|$ and $|Y^v| = |Y^{v'}|$ for all $v, v' \in D_{2k}$, and that, for $v \in D_{2k+1}$, $C_v = \{X_1^{v_1}, \ldots, X_n^{v_1}, Y_1^{v_1}, \ldots, Y_m^{v_1}\}$, with

$$
|X_i^{v_i}| = |X_i^{v_{i'}}|, \quad |Y_j^{v_j}| = |Y_j^{v_{j'}}|, \quad (1 \leq i, i' \leq n, 1 \leq j, j' \leq m, v, v' \in D_{2k+1}).
$$
Moreover, one has, for \( v \in D_{2k} \), $|X^v| = |X^v|^{-1}|Y^w|^m$, and $|Y^v| = |Y^w|^m|X^w|^{-1}$ where $w \in D_{2k-1}$, and, for $v \in D_{2k+1}$, $|X^v| = |Y^w|$ and $|Y^v| = |X^w|$ for any $w \in D_{2k}$. This clearly implies that $|C_v| \geq 2$, and $|X| \geq 2$ for all $v \in D_k$ and for all $X \in C_v$. Using these inequalities, and the constructions made in Lemmas 3.3 and 3.4, it follows that, if $v \in D_k$ for some $k$, then $r_{k+2}^{-1}(v) \cap W_{k+2} \neq \emptyset$. Therefore

$$r_{k,\infty}^{-1}(v) \cap r_{k+2}^{-1}(W_{k+2}) \neq \emptyset.$$ 

Since the family $\{r_{k,\infty}^{-1}(v) \mid v \in D_k, \ k = 0, 1, 2, \ldots\}$ is a basis for the topology of $\Omega$, we see that $\bigcup_{U \in B} U$ is a dense open set of $\Omega$.

5. Partial unitaries in amalgamated free products

In this section, we explicitly compute the image of certain partial unitary classes under the K-theory map

$$K_1(A_1 *_B A_2) \to K_0(B),$$

defined in [19], where $A_1 *_B A_2$ is an amalgamated free product. This will be used in Section 6 to compute $K_1(O(E,C))$ for any finite bipartite separated graph $(E,C)$.

Assume that $A_1, A_2, B$ are separable unital C*-algebras, with $B$ finite-dimensional, and that there are unital embeddings $\iota_k: B \to A_k$, $k = 1, 2$. Let $j_k: A_k \to A_1 *_B A_2$ be the canonical maps.

In our computations below we will use a special case of a main result by Thomsen, namely [19 Theorem 2.7].

Theorem 5.1. Let $B, A_1, A_2$ be separable C*-algebras. Assume that $B$ is finite-dimensional. Then there is a 6-term exact sequence:

$$K_0(B) \xrightarrow{(i_1, j_2)} K_0(A_1) \oplus K_0(A_2) \xrightarrow{j_1 - j_2} K_0(A_1 *_B A_2) \xrightarrow{i_1 - i_2} K_1(A_1 *_B A_2).$$

(5.2)

We will need an elementary Lemma, which is surely well known to specialists.

Lemma 5.2. Given a unital C*-algebra $A$, and a short exact sequence of C*-algebras

$$0 \to J \to A \xrightarrow{\pi} B \to 0,$$

let $u \in U_n(B)$, meaning the set of all unitary $n \times n$ matrices over $B$. Suppose that $v \in U_{2n}(A)$ is such that

$$\pi(v) = \begin{pmatrix} u & 0 \\ 0 & \bar{u}^* \end{pmatrix}$$

and $[\bar{u}]_1 = [u]_1$, in $K_1(B)$. Then

$$\delta([u]_1) = \left[ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^* \right] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right],$$
where $\delta: K_1(B) \to K_0(J)$ is the index map.

Proof. We should initially observe that, when $\tilde{u} = u$, then the conclusion of the Lemma is essentially the definition of $\delta$. Set

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and let $z \in U_{2n}(A)$ be such that

$$\pi(z) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix},$$

so we have by definition that

$$\delta([u]) = [zp^*] - [p]. \quad (\star)$$

By taking the direct sum of all unitary matrices in sight with a big enough identity matrix, one may suppose that there is a continuous path $u_t$ of unitaries such that $u_0 = u$, and $u_1 = \tilde{u}$. Therefore, the unitary matrix $u\tilde{u}^*$ lies in the connected component of $U_n(B)$, so there exists $x$ in $U_n(A)$, such that $\pi(x) = u\tilde{u}^*$. Setting

$$w = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix},$$

we then have that

$$\pi(zwv^*) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u\tilde{u}^* \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & \tilde{u} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

from where it follows that the element

$$y := zwv^*$$

lies in $U_{2n}(\tilde{J})$. Working within $K_0(\tilde{J})$ we then have that

$$[vpv^*]_0 = [y(vpv^*)y^*]_0 = [zwv^*z^*]_0 = [zp^*]_0,$$

so the conclusion follows immediately from $(\star)$. 

We start with an easy case. This case provides the motivation for the more sophisticated result that we need later.

**Lemma 5.3.** With the above notation, let $x$ and $y$ be partial isometries, with $x \in M_\infty(A_1)$ and $y \in M_\infty(A_2)$, such that

$$xx^* = e = yy^*, \quad x^*x = f = y^*y, \quad \text{with } e, f \in M_\infty(B).$$

Then the image of the partial unitary class $[j_2(y)j_1(x)^*]$ under the homomorphism $(5.1)$ is precisely $[f] - [e] \in K_0(B)$.

Proof. It suffices to deal with the case where $x \in A_1$ and $y \in A_2$.

Denote by $C$ the mapping cone of the map $B \to A_1 \oplus A_2$ sending $b$ to $(\iota_1(b), \iota_2(b))$, that is,

$$C = \{(b, g_1, g_2) : g_i \in C_0(0, 1) \otimes A_i, \ i = 1, 2, \ b \in B, \ g_1(1) = \iota_1(b), \ g_2(b) = \iota_2(b)\}.$$
Let $G : C \to S(A_1 \ast_B A_2) = C_0(0, 1) \otimes (A_1 \ast_B A_2)$ be Germain’s *-homomorphism, given by
\[
G(b, g_1, g_2)(t) = \begin{cases} 
  j_1(g_1(2t)), & t \in (0, \frac{1}{2}] \\
  j_2(g_2(2 - 2t)), & t \in [\frac{1}{2}, 1).
\end{cases}
\]

Then, by the proof of [19, Theorem 2.7], we have the following commutative diagram:
\[
\begin{array}{ccc}
  K_1(A_1 \ast_B A_2) & \longrightarrow & K_0(B) \\
  \delta \downarrow \cong & & \uparrow p_* \\
  K_0(S(A_1 \ast_B A_2)) & \xrightarrow{G_2^{-1}} & K_0(C).
\end{array}
\]

Here $p : C \to B$ is the natural map, which sends $(b, g_1, g_2)$ to $b$, and $\delta$ is the index map for the six-term exact sequence of K-groups obtained from the short exact sequence
\[
0 \to S(A_1 \ast_B A_2) \to C(A_1 \ast_B A_2) \to A_1 \ast_B A_2 \to 0,
\]
where $C(A_1 \ast_B A_2)$ is the cone of $A_1 \ast_B A_2$.

In view of diagram (5.3), it will be sufficient to find $z \in K_0(C)$ such that $p_*(z) = [f] - [e]$ and $\delta([yx^*]) = G_*(z)$. In order to simplify notation, we will suppress the reference to the maps $\iota_k, j_k$ in the rest of the proof.

Write
\[
u = \begin{pmatrix} 1 - e & x \\ x^* & 1 - f \end{pmatrix}, \quad \text{and} \quad \nu = \begin{pmatrix} 1 - e & y \\ y^* & 1 - f \end{pmatrix}.
\]

It is easy to see that $u$ and $v$ are self-adjoint unitary matrices over $A_1$ and $A_2$, respectively. Set $Q := \frac{1 - u}{2}$ and $Q' := \frac{1 - u}{2}$. Observe that $Q$ and $Q'$ are the spectral projections corresponding to the eigenvalue $-1$ of $u$ and $v$ respectively. Put
\[
P_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad P := \begin{pmatrix} 1 - e & 0 \\ 0 & f \end{pmatrix},
\]
which we view as $2 \times 2$ matrices over $B$. Consider the paths of unitaries
\[
u_t := (1 - Q) + e^{\pi it} Q, \quad \text{and} \quad \nu_t := (1 - Q') + e^{\pi it} Q'
\]
joining $I_2$ with $u$ and $v$ in $A_1$ and $A_2$, respectively. Consider the projection
\[
D = (P_1, g_1, g_2)
\]
in $M_2(\bar{C})$, where
\[
g_1(t) = u_t P_0 u_t^*, \quad g_2(t) = v_t P_0 v_t^*.
\]
(Observe that $g_1(1) = g_2(1) = P_1$ because $x^*x = f = y^*y.$) Set $z = [D] - [P_0] \in K_0(C)$. (Note that $z \in K_0(C)$ because the image of $D$ through the canonical map $M_2(\bar{C}) \to M_2(\mathbb{C})$ is the projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$.)

Note that
\[
p_*(z) = [P_1] - [P_0] = [f] - [e].
\]
It remains to show that $\delta([yx^*]_1) = G_*(z)$.

Notice that $yx^*$ is a partial isometry, with final projection

$$yx^*xy^* = y^fy^* = yy^*yy^* = e,$$

and initial projection

$$xy^*yx^* = xf^*x^* = xx^*x^* = e.$$  

So $yx^*$ is in fact a partial unitary, and hence $[yx^*]_1$ is defined to be the $K_1$-class of the unitary element

$$U := 1 - e + yx^*.$$ 

We will therefore show that

$$\delta([yx^*]_1) = \delta([U]_1) = G_*(z),$$

by applying Lemma 5.2 to the exact sequence (5.4).

Letting

$$w = \begin{pmatrix} x & 1 - e \\ 1 - f & x^* \end{pmatrix},$$

an easy computation shows that

$$w^* \begin{pmatrix} 1 - e + yx^* & 0 \\ 0 & 1 \end{pmatrix} w = \begin{pmatrix} 1 - f + x^*y & 0 \\ 0 & 1 \end{pmatrix},$$

so we see that the unitary element

$$\tilde{U} := 1 - f + x^*y$$

has the same $K_1$-class as $U$. Observe moreover that

$$vu = \begin{pmatrix} 1 - e + yx^* & 0 \\ 0 & 1 - f + y^*x \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & \tilde{U} \end{pmatrix}.$$ 

In order to apply Lemma 5.2 we thus need to find a lifting for the above matrix in the unitization of the cone over $A_1 \ast_B A_2$, namely a continuous path connecting the identity matrix to the above matrix. Such a path is not hard to find, it is enough to take

$$\gamma_t = \begin{cases} u_{2t}, & t \in [0, \frac{1}{2}] \\ ((1 - Q') + e^{(2t-1)\pi i}Q')u, & t \in [\frac{1}{2}, 1]. \end{cases}$$

By Lemma 5.2 we then have that

$$\delta([yx^*]_1) = \delta([U]_1) = [\gamma_t P_0 \gamma_t^*] - [P_0],$$

but now observe that

$$\gamma_t P_0 \gamma_t^* = \begin{cases} g_1(2t), & t \in [0, \frac{1}{2}] \\ g_2(2 - 2t), & t \in [\frac{1}{2}, 1], \end{cases}$$

where $g_1(t)$ and $g_2(t)$ are continuous functions.
Theorem 5.4. Let \( e, f \) be projections in \( M_\infty(B) \) and suppose that we have orthogonal decompositions
\[
e = e_1 \oplus e_2 = g_1 \oplus g_2, \quad f = f_1 \oplus f_2 = h_1 \oplus h_2,
\]
with \( e_i, g_i, f_i, h_i \in M_\infty(B) \), for \( i = 1, 2 \). Assume that \( x_1, y_1 \) are partial isometries in \( M_\infty(A_1) \), and \( x_2, y_2 \) are partial isometries in \( M_\infty(A_2) \) such that
\[
e_i = x_i x_i^*, \quad f_i = x_i^* x_i, \quad g_i = y_i y_i^*, \quad h_i = y_i^* y_i,
\]
for \( i = 1, 2 \). Set \( x := j_1(x_1) + j_2(x_2) \) and \( y := j_1(y_1) + j_2(y_2) \). Then the image of the partial unitary class \([yx^*] \) under the homomorphism (5.7) is precisely
\[
([f_1] - [e_1]) - ([h_1] - [g_1]) \in K_0(B).
\]

**Proof.** The proof is similar to the one of Lemma 5.3, but we need to solve some technical complications.

We will assume that \( e, f \in B \), and so \( x_i, y_i \in A_i \) as well. We will suppress any reference to the maps \( \iota_k \) and \( j_k \). As in the proof of Lemma 5.3, it suffices to find \( z \in K_0(C) \) such that \( p_*(z) = ([f_1] - [e_1]) - ([g_1] - [h_1]) \) and \( \delta([yx^*]) = G_*(z) \).

Write
\[
(5.7) \quad u_1 = \begin{pmatrix} 1 - e_1 & x_1 \\ x_1^* & 1 - f_1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 - h_1 & y_1^* \\ y_1 & 1 - g_1 \end{pmatrix}, \quad \text{and} \quad U = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}.
\]

Similarly, put
\[
(5.8) \quad v_1 = \begin{pmatrix} 1 - g_2 & y_2 \\ y_2^* & 1 - h_2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 - f_2 & x_2^* \\ x_2 & 1 - e_2 \end{pmatrix}, \quad \text{and} \quad V = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}.
\]

Note that \( U \) is a self-adjoint unitary in \( M_4(A_1) \) and \( V \) is a self-adjoint unitary in \( M_4(A_2) \). Consider the following projections in \( M_4(B) \):

\[
P_0 = \text{diag}(1, 0, 1, 0), \quad P_1 = U P_0 U^*, \quad P_2 = V P_0 V^*.
\]
Note that the projections $P_1$ and $P_2$ can be connected in $M_4(B)$. Indeed, consider

$$Z := \begin{pmatrix} 1 - e & 0 & 0 & e \\ 0 & 1 - f & f & 0 \\ 0 & f & 1 - f & 0 \\ e & 0 & 0 & 1 - e \end{pmatrix}. $$

Then $Z$ is a self-adjoint unitary in $M_4(B)$ and $ZP_1Z = P_2$. There exists a path $Z_t$ of unitaries in $M_4(B)$ such that $Z_0 = I_4$ and $Z_1 = Z$.

Set $Q := \frac{1 - U}{2}$ and $Q' := \frac{1 - V}{2}$. Consider the paths of unitaries

$$U_t := (1 - Q) + e^{\pi it}Q, \quad V_t := (1 - Q') + e^{-\pi it}Q'$$

joining $I_2$ with $U$ and $V$ in $A_1$ and $A_2$, respectively. Consider the projection

$$D = (P_2, g_1, g_2)$$

in $M_4(\mathcal{C})$, where

$$g_1(t) = \begin{cases} U_2P_0U_{2t}^* & t \in [0, \frac{1}{2}] \\ Z_{2t-1}P_1Z_{2t-1}^* & t \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad g_2(t) = V_tP_0V_t^*. $$

(Observe that $g_1(1) = g_2(1) = P_2$.) Set $z = [D] - [P_0] \in K_0(\mathcal{C})$. Note that

$$p_+(z) = [P_2] - [P_0] = [P_1] - [P_0] = ([f_1] - [e_1]) - ([h_1] - [g_1]).$$

It remains to show that $\delta([yx^*]_1) = G_+(z)$. A computation shows that

$$VZU = \begin{pmatrix} 1 - e & 0 & y & 0 \\ 0 & 1 - f & 0 & y^* \\ x^* & 0 & 1 - f & 0 \\ 0 & x & 0 & 1 - e \end{pmatrix}. $$

Observe that exchange of the second and third rows and columns of the matrix $VZU$ gives the unitary

$$W := \text{diag}\left(\begin{pmatrix} 1 - e & y \\ x^* & 1 - f \end{pmatrix}, \begin{pmatrix} 1 - f & y^* \\ x & 1 - e \end{pmatrix}\right),$$

and the two unitaries appearing in this formula are equivalent to $1 - e + yx^*$ and $1 - e + xy^*$ respectively. In particular, we have that $\Lambda(VZU)\Lambda = W$, where $\Lambda := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is a self-adjoint unitary scalar matrix. Therefore, Lemma 5.2 gives that

$$\delta([yx^*]_1) = \delta([1 - e + yx^*]_1) = \delta\left(\begin{pmatrix} 1 - e & y \\ x^* & 1 - f \end{pmatrix}\right)_1 = [\tilde{\gamma}_t \text{diag}(I_2, 0_2)\tilde{\gamma}_t^*] - [\text{diag}(I_2, 0_2)],$$

where $\tilde{\gamma}_t$ is a unitary $4 \times 4$ matrix over the unitization of the cone of $A_1 *_B A_2$, such that $\tilde{\gamma}_1 = W = \Lambda(VZU)\Lambda$. 
Define \( \gamma_t = \Lambda \tilde{\gamma}_t \Lambda \). Then \( \gamma_t \) is a unitary \( 4 \times 4 \) matrix such that \( \gamma_1 = VZU \).

Moreover, using the above computation, we get

\[
\delta([yx^*]_1) = [\Lambda \gamma_t(\Lambda \text{ diag}(I_2, 0_2) \Lambda)\gamma_t^* \Lambda] - [\text{diag}(I_2, 0_2)] \\
= [\Lambda \gamma_t P_0 \gamma_t^* \Lambda] - [P_0] \\
= [\gamma_t P_0 \gamma_t^*] - [P_0]
\]

in \( K_0(S(A_1 \ast_B A_2)) \). In conclusion, we get that \( \delta([yx^*]_1) = [\gamma_t P_0 \gamma_t^*] - [P_0] \), where \( \gamma_t \) is any unitary \( 4 \times 4 \) matrix over the unitization of the cone of \( A_1 \ast_B A_2 \) such that \( \gamma_1 = VZU \).

Consider the following unitary

\[
\gamma_t = \begin{cases} 
U_{4t}, & t \in [0, \frac{1}{4}] \\
Z_{4t-1}U, & t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\
[(1 - Q') + e^{(2t-1)\pi i}Q']ZU, & t \in \left[\frac{1}{2}, 1\right].
\end{cases}
\]

Then \( \gamma_0 = I_2 \), and \( \gamma_1 = VZU \), and so we can use \( \gamma_t \) to compute \( \delta([yx^*]_1) \).

Just as in \([5,3]\) we get \( G_s(z) = [\gamma_t P_0 \gamma_t^*] - [P_0] = \delta([yx^*]_1) \), as desired. \( \square \)

6. Computation of \( K_1(\mathcal{O}(E, C)) \).

In this section, we compute \( K_1(\mathcal{O}(E, C)) \) for any finite bipartite separated graph \((E, C)\).

Let \((E, C)\) be a finitely separated graph. For \( v, w \in E^0 \) and \( X \in C_v \), denote by \( a_X(w, v) \) the number of arrows in \( X \) from \( w \) to \( v \).

We denote by \( 1_C : \mathbb{Z}^C \rightarrow \mathbb{Z}^{(E^0)} \) and \( A_{(E, C)} : \mathbb{Z}^C \rightarrow \mathbb{Z}^{(E^0)} \) the homomorphisms defined by

\[
1_C(\delta_X) = \delta_v \quad \text{and} \quad A_{(E, C)}(\delta_X) = \sum_{w \in E^0} a_X(w, v) \delta_w \quad (v \in E^0, X \in C_v),
\]

where \((\delta_X)_{X \in C}\) and \((\delta_v)_{v \in E^0}\) denote the canonical basis of \( \mathbb{Z}^C \) and \( \mathbb{Z}^{(E^0)} \) respectively.

With this notation, the K-theory of \( C^*(E, C) \) has formulas which look very similar to the ones for the non-separated case ([16, Theorem 3.2]):

**Theorem 6.1.** ([4, Theorem 5.2]) Let \((E, C)\) be a finitely separated graph, and adopt the notation above. Then the K-theory of \( C^*(E, C) \) is given as follows:

\[
(6.1) \quad K_0(C^*(E, C)) \cong \text{coker} \left( 1_C - A_{(E, C)} : \mathbb{Z}^C \longrightarrow \mathbb{Z}^{(E^0)} \right),
\]

\[
(6.2) \quad K_1(C^*(E, C)) \cong \ker \left( 1_C - A_{(E, C)} : \mathbb{Z}^C \longrightarrow \mathbb{Z}^{(E^0)} \right).
\]

The non-appearance of the transpose in the matrix \( A_{(E, C)} \) in the above formulas, compared with the formulas given in [5], is due to our different convention on the direction of arrows. The isomorphism in (6.1) is given explicitly in [4], but this is not the case for the isomorphism in (6.2). We will obtain such an explicit isomorphism in this section.

Using this, we will show the following result:
Theorem 6.2. Let $(E, C)$ be a finite bipartite separated graph. The natural map $C^\ast(E, C) \to \mathcal{O}(E, C)$ induces an isomorphism $K_1(C^\ast(E, C)) \to K_1(\mathcal{O}(E, C))$. Consequently,

$$K_1(\mathcal{O}(E, C)) \cong \ker(1_C - A_{(E,C)}).$$

To show this result, it is enough to prove that, for any finite bipartite separated graph $(E, C)$, the natural map $\phi_0: C^\ast(E, C) \to C^\ast(E_1, C^1)$ induces an isomorphism

$$K_1(\phi_0): K_1(C^\ast(E, C)) \to K_1(C^\ast(E_1, C^1)),$$

where $(E_1, C^1)$ is the first of the infinite collection of separated graphs $(E_n, C^n)$ associated to $(E, C)$ (see Construction 3.2(c)).

We start by fixing some notation. Let $(F, D) := (E_1, C^1)$. Then $F^{0,0} = E^{0,1}$ and $F^{0,1} = \bigcup_{u \in E^{0,0}} F^{0,1}_u$, where $F^{0,1}_u$ is the set of all vertices $v(x^u_1, \ldots, x^u_{k_u})$ for $x^u_i \in X^u_i$, $i = 1, \ldots, k_u$, being $C_u = \{X^u_1, \ldots, X^u_{k_u}\}$, for any $u \in E^{0,0}$.

For $w \in F^{0,0} = E^{0,1}$, the set $D_w$ can be identified with $s^{-1}_E(w)$ (see Definition 3.2), so that the set $D$ can be identified with $E^1$:

$$Z^D = Z^{E^1} = \bigoplus_{u \in E^{0,0}} Z^{X^u_1 + \cdots + X^u_{k_u}}.$$  

We will denote by $\mathcal{D}_u = \{b(x^u_i) \mid x^u_i \in X^u_i, i = 1, \ldots, k_u\}$ a basis of $Z^{X^u_1 + \cdots + X^u_{k_u}}$, so that $\mathcal{D} = \bigcup_{u \in E^{0,0}} \mathcal{D}_u$ is a basis of $Z^D$.

On the other hand, $Z^{F_0} = Z^{F^{0,0}} \oplus Z^{F^{0,1}}$. We consider a basis

$$\{a(x^u_1, \ldots, x^u_{k_u}) \mid x^u_i \in X^u_i, u \in E^{0,0}\}$$

for $Z^{F^{0,1}}$.

For $u \in E^{0,0}$ and $i = 2, \ldots, k_u$, set $\gamma^u_i = \sum_{x^u_1 \in X^u_1} b(x^u_1) - \sum_{x^u_1 \in X^u_1} b(x^u_1)$. Let $Z_1$ be the subgroup of $Z^D$ generated by the elements $\gamma^u_i$, for $i = 2, \ldots, k_u$, $u \in E^{0,0}$. The map $\Psi: Z^D \to Z^{F^{0,1}}$ given by

$$\Psi(b(x^u_i)) = \sum_{X^u_1 \times \cdots \times X^u_{i-1} \times X^u_{i+1} \times \cdots \times X^u_{k_u}} a(x^u_1, \ldots, x^u_{i-1}, x^u_i, x^u_{i+1}, \ldots, x^u_{k_u})$$

is clearly related to the map $G(\psi)$ considered in Lemma 3.3.

By the proof of Lemma 3.3, we have that $Z_1 = \ker(\Psi)$, and that $Z_1$ is a free subgroup of $Z^D$ with free basis given by the elements $\gamma^u_i$, for $i = 2, \ldots, k_u$, $u \in E^{0,0}$. We have

$$Z^D = Z_1 \oplus Z_2,$$

and $\Psi$ induces an isomorphism from $Z_2$ onto its image.

Observe that the map $\Psi$ can be identified with the map $A_{(F,D)}$. We obtain:
Lemma 6.3. Let \((E, C)\) be a finite bipartite separated graph. With the above notation, we have
\[
\ker(1_D - A_{(F,D)}) = \ker(s_{Z_1}),
\]
where \(s_{Z_1} : Z_1 \to \mathbb{Z}^{E_0,1}\) is the restriction to \(Z_1\) of the map \(s_{Z^D} : \mathbb{Z}^D \to \mathbb{Z}^{F_0,1}\) defined by \(s_{Z^D}(b(x)) = \delta_{s(x)}\) for \(x \in E^1\).

Proof. We have to compute the kernel of the map \(1_D - A_{(F,D)} : \mathbb{Z}^D = Z_1 \oplus Z_2 \to \mathbb{Z}^{F_0} = Z^{F_{0,0}} \oplus Z^{F_{0,1}}\). Note that \(1_D\) takes its values on \(Z^{F_0} = \mathbb{Z}^{E_0,1}\) and can be identified with the “source map” \(s_{Z^D}\). On the other hand the map \(A_{(F,D)}\) takes all its values on \(Z^{F_{0,1}}\) and can be identified with the map \(\Psi\) described above. Since \(Z_1 = \ker(\Psi)\), the map \(1_D - A_{(F,D)}\) decomposes as
\[
\left(\begin{array}{c}
s_{Z_1} \\
0 \\
\end{array}\right) : Z_1 \oplus Z_2 \to \mathbb{Z}^{F_{0,0}} \oplus \mathbb{Z}^{F_{0,1}},
\]
where \(s_{Z_2}\) is the restriction of \(s_{Z^D}\) to \(Z_2\). Since \(\Psi|_{Z_2}\) is injective we obtain that \(\ker(1_D - A_{(F,D)}) = \ker(s_{Z_1})\), as desired. \(\square\)

Lemma 6.4. Let \((E, C)\) be a finite bipartite separated graph. With the above notation, we have a natural isomorphism
\[
\ker(1_C - A_{(E,C)}) \cong \ker(1_D - A_{(F,D)}).
\]

Proof. By Lemma 6.3 it suffices to establish an isomorphism
\[
\Phi : \ker(1_C - A_{(E,C)}) \to \ker(s_{Z_1}).
\]
Recall that \(Z_1 = \bigoplus_{u \in E^{0,0}} \bigoplus_{i=2}^{k_u} \gamma_i^u \mathbb{Z}\).

For \(x = \sum_{u \in E^{0,0}} \sum_{i=1}^{k_u} n_i^u \delta_{X_i^u} \in \ker(1_C - A_{(E,C)})\), define
\[
(6.3) \quad \Phi(x) = \sum_{u \in E^{0,0}} \sum_{i=2}^{k_u} n_i^u \gamma_i^u.
\]
We show that this is well-defined, that is, that \(\sum_{u \in E^{0,0}} \sum_{i=2}^{k_u} n_i^u \gamma_i^u \in \ker(s_{Z_1})\). Since \(x\) belongs to \(\ker(1_C - A_{(E,C)})\), we have \(\sum_{i=1}^{k_u} n_i^u = 0\) for all \(u \in E^{0,0}\). We also have
\[
- \sum_{u \in E^{0,0}} \sum_{i=1}^{k_u} n_i^u a_{X_i^u}(w, u) = 0
\]
for all \(w \in E^{0,1}\). Substituting \(n_i^u\) by \(- \sum_{i=2}^{k_u} n_i^u\) gives
\[
\sum_{u \in E^{0,0}} \sum_{i=2}^{k_u} n_i^u \left( a_{X_i^u}(w, u) - a_{X_i^u}(w, u) \right) = 0
\]
for all \( w \in E^{0,1} \), which in turn gives that \( \sum_{u \in E^{0,0}} \sum_{i=2}^{k_u} n_i^u X_i^u \in \ker(s_{Z_1}) \). Clearly \( \Phi \) is a group homomorphism. The map \( \Upsilon: \ker(s_{Z_1}) \to \ker(1_C - A_{(E,C)}) \) defined by

\[
\Upsilon(\sum_{u \in E^{0,0}} \sum_{i=2}^{k_u} n_i^u X_i^u) = \sum_{u \in E^{0,0}} \sum_{i=1}^{k_u} n_i^u \delta_{X_i}^u,
\]

where \( n_i^u := -\sum_{i=2}^{k_u} n_i^u \) gives the inverse of \( \Phi \), so we have showed that \( \Phi \) is an isomorphism.

\[\Box\]

Observe that Lemma 6.4 and [4, Theorem 5.2] already give an isomorphism \( K_1(C^*(E,C)) \cong K_1(C^*(E_1,C^1)) \). (Recall that \( (F, D) = (E_1, C^1) \).) However, we need the fact that the natural surjection \( \phi_0: C^*(E,C) \to C^*(E_1,C^1) \) induces a \( K_1 \)-isomorphism. In order to obtain this, we are going to describe now an explicit isomorphism \( \lambda_{(E,C)}: \ker(1_C - A_{(E,C)}) \to K_1(C^*(E,C)) \) for any finite bipartite separated graph \( (E, C) \). This is interesting in its own sake, since it enables us to compute specific elements in \( K_1(C^*(E,C)) \).

Let \( (E, C) \) be a finite bipartite separated graph. Let

\[
(6.4) \quad x = \sum_{u \in E^{0,0}} \sum_{i=1}^{k_u} n_i^u \delta_{X_i}^u - \sum_{u \in E^{0,0}} \sum_{j=1}^{k_u} m_j^u \delta_{X_j}^u
\]

be an element in the kernel of \( 1_C - A_{(E,C)} \), where \( n_i^u, m_j^u \) are non-negative integers and \( n_i^u m_j^u = 0 \) for all \( u, i \). This means exactly that

\[
(6.5) \quad \sum_{i=1}^{k_u} n_i^u = \sum_{j=1}^{k_u} m_j^u \quad (\forall u \in E^{0,0})
\]

and

\[
(6.6) \quad \sum_{u \in E^{0,0}} \sum_{i=1}^{k_u} n_i^u a_{X_i^u}(w, u) = \sum_{u \in E^{0,0}} \sum_{j=1}^{k_u} m_j^u a_{X_j^u}(w, u) \quad (\forall w \in E^{0,1}).
\]

Let us denote by \( N_w \) the number appearing in (6.6), for \( w \in E^{0,1} \).

We define a matrix \( Z \), whose rows are labeled by the set

\[\mathcal{R}_1 = \bigsqcup_{u, i} \{X_i^u\} \times [1, n_i^u],\]

where \( [1, n] := \{1, \ldots, n\} \), and the indices range over all \( u \in E^{0,0}, i \in [1, k_u] \), such that \( n_i^u \geq 1 \) and whose columns are labeled by the set

\[\mathcal{C}_1 = \bigsqcup_{u, i, j} \{X_i^u\} \times [1, n_j^u] \times \{w\} \times [1, a_{X_j^u}(w, u)],\]

where the indices range over all \( w \in E^{0,1}, u \in E^{0,0}, i \in [1, k_u] \), such that \( n_i^u \geq 1 \) and \( a_{X_j^u}(w, u) \geq 1 \). The matrix \( Z \) is associated to “the positive part” \( \sum_{u, i} n_i^u \delta_{X_i}^u \) of \( x \). Given \( w \in E^{0,1}, u \in E^{0,0} \) and \( X_i^u \in C_u \), choose an ordering \( z_1, \ldots, z_{a_{X_i^u}(w, u)} \) of the set of arrows
from $X^u_i$ going from $w$ to $u$. The matrix $Z$ is the unique $R_1 \times C_1$-matrix such that, for each $w \in E^{0,0}$, $u \in E^{0,0}$ and $X^u_i \in C_u$, the column labeled by $(X^u_i, t, w, s)$ has a unique nonzero entry, and this nonzero entry is precisely $z_{u,i}$ in row $(X^u_i, t)$. In other words, the only nonzero entries of the row labeled $(X^u_i, t)$, for $t \in [1, n^u_i]$, are precisely the edges from $X^u_i$ and these are distributed in the columns corresponding to $(X^u_i, t)$, their source vertex $w$ and the ordering fixed on the sets of arrows from $X^u_i$ going from $w$ to $u$. With this description, it is clear that

\[
ZZ^* = \bigoplus_{u \in E^{0,0}} \left( \sum_{i=1}^{k_u} n^u_i \right) \cdot u,
\]

\[
Z^*Z = \bigoplus_{w \in E^{0,1}} N_w \cdot w.
\]

Similarly, we may associate a $R_2 \times C_2$-matrix $T$ to the “negative part” $\sum_{u \in E^{0,0}} \sum_{j=1}^{k_u} m^u_j \delta_{X^u_j}$ of $x$. The rows and columns do not match exactly, but they match after we apply a bijection. More concretely, we fix two bijections

\[
\sigma_1 : R_1 \to R_2, \quad \text{and} \quad \sigma_2 : C_1 \to C_2
\]

such that $\sigma_1$ restricts to a bijection from $\bigsqcup_i \{X^u_i \times [1, n^u_i]\}$ onto $\bigsqcup_j \{X^u_j \times [1, m^u_j]\}$ for all $u \in E^{0,0}$, and $\sigma_2$ restricts to a bijection from $\bigsqcup_{u,i} \{X^u_i \times [1, n^u_i] \times \{w\} \times \{w\} \times [1, \sigma(1, z_{u,i})] \}$ onto $\bigsqcup_{u,j} \{X^u_j \times [1, m^u_j] \times \{w\} \times \{w\} \times [1, \sigma(1, z_{u,i})] \}$ for all $u \in E^{0,1}$. Note that this is possible because of (6.5) and (6.6). Define a $R_1 \times C_1$ matrix $\sigma(T)$ by

\[
\sigma(T)_{r_i, c_1} = T_{\sigma_1(r_i), \sigma_2(c_1)}, \quad r_1 \in R_1, c_1 \in C_1.
\]

Finally we define the map $\lambda_{(E,C)} : \ker(1_C - A_{(E,C)}) \to K_1(C^*(E,C))$ by

\[
\lambda_{(E,C)}(x) = [U_x]_1, \quad \text{where} \quad U_x = Z \sigma(T)^*.
\]

It is easily checked that this map does not depend on the choices of orderings that we have made, and of the specific bijections $\sigma_1$ and $\sigma_2$. Similarly, we can use $[\sigma^{-1}(Z)T^*]_1$ to define $[U_x]_1$.

**Proposition 6.5.** With the notation above, the following diagram

\[
\begin{array}{ccc}
\ker(1_C - A_{(E,C)}) & \xrightarrow{\lambda_{(E,C)}} & K_1(C^*(E,C)) \\
\Phi \downarrow \cong & & \downarrow K_1(\phi_0) \\
\ker(1_D - A_{(F,D)}) & \xrightarrow{\lambda_{(F,D)}} & K_1(C^*(F,D))
\end{array}
\]

is commutative.

**Proof.** Recall that $(F, D) := (E_1, C^1)$. Let $x$ be an element in $\ker(1_C - A_{(E,C)})$, written as in (6.4). Note that $[U_x]_1 = [V_x]_1$ in $K_1(C^*(E,C))$, where $U_x = Z \sigma(T)^*$ and $V_x = \sigma(T)^*Z$. We now compute the image of $\sigma(T)^*Z$ under the map $\phi_0$. Consider a nonzero entry of this matrix, corresponding to row $\sigma^{-1}_2(X^u_j, r', w', t')$ and column $(X^u_i, r, w, t)$. The entry will be of the form $y^*z$ for some $y \in X^u_j$ and some $z \in X^u_i$, with $s(y) = w'$ and $s(z) = w$. Since the (nonzero) entry $y$ must be at position $((X^u_j, r'), (X^u_i, r', w', t'))$ in the matrix $T$,
and \( z \) must be at position \((\{X^u_p, r\}, \{X^u_r, w, t\})\) in the matrix \(Z\), we must necessarily have \(\sigma_1(X^u_i, r) = (X^u_1, r')\). In particular, by the choice of \(\sigma_1\), we must have \(u' = u\) and thus

\[
\sigma_1(X^u_i, r) = (X^u_j, r').
\]

Set \( y = y^u_j \) and \( z = z^u_j \). We have

\[
(6.9) \quad \phi_0(y^*z) = \left( \sum_{y^u_i \in X^u_i, i \neq j} \alpha^u_i(y^u_i, \ldots, \hat{y}^u_j, \ldots, y^u_k) \right) \left( \sum_{z^u_k \in X^u_k, k \neq i} \alpha^u_k(z^u_1, \ldots, \hat{z}^u_i, \ldots, z^u_k)^* \right)
\]

\[
= \sum_{z^u_i \in X^u_i, i \neq j} \alpha^u_i(z^u_1, \ldots, \hat{y}^u_j, \ldots, z^u_k) \alpha^u_k(z^u_1, \ldots, y^u_j, \ldots, \hat{z}^u_i, \ldots, z^u_k)^*.
\]

Now we wish to compute the image of \( x \) under \( \Phi \), where \( \Phi \) is the isomorphism defined in Lemma 6.4. Using (6.3), the definition of \( \gamma^u_i \), and the identification of \( b(x) \) with \( \delta_{X^u(x)} \), for \( x \in E^1 \), we obtain

\[
\Phi(x) = \sum_{u \in E^{0,0}} \sum_{i=2}^{k_u} n^u_i \left( \sum_{x^u_i \in X^u_i} \delta_{X^u(x^u_i)} - \sum_{x^u_i \in X^u_i} \delta_{X^u(x^u_i)} \right) - \sum_{u \in E^{0,0}} \sum_{j=2}^{k_u} m^u_j \left( \sum_{x^u_j \in X^u_j} \delta_{X^u(x^u_j)} - \sum_{x^u_j \in X^u_j} \delta_{X^u(x^u_j)} \right)
\]

\[
= \sum_{u \in E^{0,0}} \left( \sum_{i=2}^{k_u} n^u_i \right) \left( \sum_{x^u_i \in X^u_i} \delta_{X^u(x^u_i)} \right) + \sum_{u \in E^{0,0}} \left( \sum_{j=2}^{k_u} m^u_j \right) \left( \sum_{x^u_j \in X^u_j} \delta_{X^u(x^u_j)} \right)
\]

\[
- \left( \sum_{u \in E^{0,0}} \left( \sum_{j=2}^{k_u} m^u_j \right) \sum_{x^u_i \in X^u_i} \delta_{X^u(x^u_i)} \right) + \sum_{u \in E^{0,0}} \left( \sum_{i=2}^{k_u} n^u_i \right) \left( \sum_{x^u_i \in X^u_i} \delta_{X^u(x^u_i)} \right).
\]

From (6.5), we have

\[
\sum_{i=2}^{k_u} n^u_i - \sum_{j=2}^{k_u} m^u_j = m^u_i - n^u_i \quad (u \in E^{0,0}),
\]

and so we get from the above

\[
\Phi(x) = \sum_{u \in E^{0,0}} \sum_{j=1}^{k_u} m^u_j \left( \sum_{x^u_j \in X^u_j} \delta_{X^u(x^u_j)} \right) - \sum_{u \in E^{0,0}} \sum_{i=1}^{k_u} n^u_i \left( \sum_{x^u_i \in X^u_i} \delta_{X^u(x^u_i)} \right).
\]

Let \( Z_1 \) and \( T_1 \) be the matrices corresponding to the “positive part” and the “negative part” of \( \Phi(x) \), respectively. We will compute \( \tilde{\sigma}(Z_1)T^*_1 \), where \( \tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2) \) is defined later. We can identify the set \( R'_2 \) of rows of \( T_1 \) with \( C_1 \). Indeed the column labeled \((X^u_i, r, w, t)\) corresponds to the row \((X(z), r)\), where \( z \) is the \( t \)-th element in the list of elements from \( X^u_i \) which have source \( w \). Similarly, we can identify the set \( R'_1 \) of rows of \( Z_1 \) with \( C_2 \).

Note that, given elements \( x^u_p \in X^u_p, p = 1, \ldots, k_u \), there is only one arrow in \( X(x^u_i) \) with source \( v(x^u_1, \ldots, x^u_p) \), namely \( \alpha^{x^u_i}(x^u_1, \ldots, \hat{x}^u_i, \ldots, x^u_p) \). Therefore the labeling of the set \( C'_2 \) of columns of \( T_1 \) is given by

\[
C'_2 = \bigcup \{X(x^u_i)\} \times [1, n^u_i] \times \{v(x^u_1, \ldots, \hat{x}^u_i, \ldots, x^u_p)\},
\]
where the union is extended to all $x_i^u \in X_i^u$ such that $n_i^u > 0$ and to all choices of $(k_u - 1)$-tuples $(x_i^u, \ldots, x_{i-1}^u, x_i^{u+1}, \ldots, x_k^u) \in X_i^u \times \cdots \times X_{i-1}^u \times X_{i+1}^u \times \cdots \times X_k^u$. There is only one nonzero entry in the column of $T_1$ labeled $(X(x_i^u), r, v(x_i^u, \ldots, x_k^u))$, which is $\alpha x_i^u (x_i^u, \ldots, x_k^u)$ at row $(X(x_i^u), r)$.

The maps $\tilde{\sigma}_i$, $i = 1, 2$, are defined as follows. The map $\tilde{\sigma}_1: R'_2 \to R'_1$ is defined to be $\sigma_2$, with the identification of $R'_2$ and $R'_1$ with $C_1$ and $C_2$ outlined above, respectively. To define $\tilde{\sigma}_2: C'_2 \to C'_1$, put

$$\tilde{\sigma}_2(X(x_i^u), r, v(x_1^u, \ldots, x_k^u)) = (X(x_i^u), r', v(x_1^u, \ldots, x_k^u)),$$

where $\sigma_1(X_i^u, r) = (X_i^u, r')$ for $r' \in [1, m_i^u]$. That is, $x_i^u$ is determined as the unique element of $X_i^u$ which appears in the $k_u$-tuple $(x_1^u, \ldots, x_k^u)$.

Now, we wish to compute the $(\sigma_2^{-1}(X_j^u, r', w', t'), (X_i^u, r, w, t))$-entry of the matrix $\tilde{\sigma}(Z_1)T_1^*$, where $\sigma_1(X_i^u, r) = (X_i^u, r')$. Recall from the beginning of the proof that the edge corresponding to $(X_i^u, w, t)$ is denoted by $z = z_i^u$ and that the edge corresponding to $(X_i^u, w', t')$ is denoted by $y = y_i^u$, so we are dealing with the $(\tilde{\sigma}_1^{-1}(X(y_j^u), r'), (X(z_i^u), r))$-entry of $\tilde{\sigma}(Z_1)T_1^*$. Now consider a pair

$$\bigl((X(y_j^u), r'), (X(z_i^u), r', v(y_j^u, \ldots, y_k^u))\bigr), \quad \bigl((X(z_i^u), r), (X(z_i^u), r, v(z_i^u, \ldots, z_k^u))\bigr)$$

of positions in the matrices $Z_1$ and $T_1$ respectively, giving rise to a nonzero contribution to the $(\tilde{\sigma}_1^{-1}(X(y_j^u), r'), (X(z_i^u), r))$-entry of $\tilde{\sigma}(Z_1)T_1^*$. Then we must have

$$\tilde{\sigma}_2(X(z_i^u), r, v(z_1^u, \ldots, z_k^u)) = (X(y_j^u), r', v(y_1^u, \ldots, y_k^u)).$$

By the definition of $\tilde{\sigma}_2$ and the fact that $\sigma_1(X_i^u, r) = (X_j^u, r')$, this happens if and only if $y_i^u = z_i^u$ for all $l = 1, \ldots, k_u$. Hence, the contribution will be

$$\alpha y_j^u (z_1^u, \ldots, y_j^u, \ldots, z_k^u) \alpha z_i^u (z_1^u, \ldots, y_j^u, \ldots, z_k^u)^s.$$

So, the $(\sigma_2^{-1}(X_j^u, r', w', t'), (X_i^u, r, w, t))$-entry of the matrix $\tilde{\sigma}(Z_1)T_1^*$ is

$$\sum \alpha y_j^u (z_1^u, \ldots, y_j^u, \ldots, z_k^u) \alpha z_i^u (z_1^u, \ldots, y_j^u, \ldots, z_k^u)^s,$$

which is precisely [6,9]. Note that the argument we have just used gives that, if $\sigma_1(X_i^u, r) \neq (X_j^u, r')$ then the $(\sigma_2^{-1}(X_j^u, r', w', t'), (X_i^u, r, w, t))$-entry of the matrix $\tilde{\sigma}(Z_1)T_1^*$ is 0, for all $w, w', t, t'$. Thus, we obtain that $\phi_0(\sigma(T)^*Z) = \tilde{\sigma}(Z_1)T_1^*$, and so

$$\lambda_{(F,D)}(\Phi(x)) = [\tilde{\sigma}(Z_1)T_1^*]_1 = K_1(\phi_0)([\sigma(T)^*Z]_1) = K_1(\phi_0)(\lambda_{(E,C)}(x)),$$

as desired.

We now proceed to show that the map $\lambda_{(E,C)}$ is an isomorphism. Note that this allows us to explicitly compute generators for $K_1(C^* (E, C))$ (see below for some examples).

**Theorem 6.6.** Let $(E, C)$ be a finite bipartite separated graph. Then the map

$$\lambda_{(E,C)} : \ker(1_C - A(E,C)) \to K_1(C^*(E, C))$$

...
is a group isomorphism.

Proof. Set \( \lambda := \lambda_{(E,C)} \). It is easy to check that \( \lambda \) is a group homomorphism.

To show injectivity, suppose that \( \lambda(x) = 0 \), where

\[
x = \sum_{X \in C} n_X \delta_X - \sum_{Y \in D} m_Y \delta_Y,
\]

where \( C, D \subseteq C \), with \( C \cap D = \emptyset \), and \( n_X, m_Y > 0 \) for all \( X, Y \). It will be convenient to use the notations \( r(X) = u \) and \( s(X) = \sum_{x \in X} s(x) \), for \( X \in C_u \).

Choose a partition \( C = C_1 \sqcup C_2 \) such that \( C \subseteq C_1 \) and \( D \subseteq C_2 \). Then we have

\[
C^* (E, C) = C^* (E_1, C_1) \ast_B C^* (E_2, C_2),
\]

where \( B = C(E^0), E_1 \) is the restriction of \( E \) to \( C_1 \), that is \( (E^0)_1 = E^0_1 = \cup C_1 \), and similarly \( E_2 \) is the restriction of \( E \) to \( C_2 \). Now observe that from (6.5), (6.6) and (6.7) we get (6.10)

\[
ZZ^* = TT^* = \sigma(T) \sigma(T)^* = \bigoplus_{X \in C} n_X \cdot r(X), \quad Z^* Z = T^* T = \sigma(T)^* \sigma(T) = \bigoplus_{X \in C} n_X \cdot s(X),
\]

where \( Z \) and \( T \) are the matrices associated to the positive and negative parts of \( x \) respectively. Let \( \Delta \): \( K_1 (C^* (E_1, C_1) \ast_B C^* (E_2, C_2)) \to K_0 (B) \) be the homomorphism associated to this amalgamated free product, as in (5.2). By Lemma 5.3 and (6.10), we get

\[
\Delta([Z \sigma(T)^*]_1) = [Z^* Z] - [Z Z^*] = \sum_{X \in C} n_X [s(X)] - \sum_{X \in C} n_X [r(X)].
\]

Since the graph is bipartite, there are no cancellations in this sum, and therefore, if \( x \neq 0 \), then \( C \neq \emptyset \) and so \( \Delta(\lambda(x)) = \Delta([Z \sigma(T)^*]_1) \neq 0 \), showing that \( \lambda(x) \neq 0 \).

Finally we show that \( \lambda \) is surjective. First, we observe the naturality of the map \( \lambda \): If \( C' \subseteq C \) and \( E' \) is the restriction of \( E \) to \( C' \), then the following diagram

\[
\begin{array}{ccc}
\ker(1_{C'} - A_{(E', C')}) & \xrightarrow{\lambda_{(E', C')}} & K_1 (C^* (E', C')) \\
\downarrow & & \downarrow \iota' \\
\ker(1_{C} - A_{(E, C)}) & \xrightarrow{\lambda_{(E, C)}} & K_1 (C^* (E, C)),
\end{array}
\]

is commutative. This is clear from the definition. We assume by induction that for all \( C' \subseteq C \), we have that \( \lambda_{(E', C')} \) is an isomorphism. If there is \( C' \subseteq C \) such that \( \iota' (K_1 (C^* (E', C'))) = K_1 (C^* (E, C)) \), then by the commutativity of (6.11), we get that \( \lambda_{(E, C)} \) is surjective. So we can assume that \( \iota' (K_1 (C^* (E', C'))) \subseteq K_1 (C^* (E, C)) \) for all \( C' \subseteq C \).

Now let \( C' \) be such that \( C \setminus C' = \{X\} \), for \( X \in C \). The proof of [1, Theorem 5.2] gives that

\[
K_1 (C^* (E, C)) = K_1 (C^* (E', C')) \oplus H,
\]

where \( H \) is a cyclic group (see formula (5.9) in [1] and the comments below it). It is enough to show that the generator \( v \) of \( H \) belongs to the image of \( \lambda_{(E, C)} \). Let

\[
\Psi : K_1 (C^* (E, C)) \longrightarrow K_0 (B)
\]
be the connecting map corresponding to the decomposition
\[ C^*(E, C) = C^*(E', C') *_{B} C^*(E\{x\}, \{X\}) \]
of \(C^*(E, C)\) as an amalgamated free product, as in (5.2).

Following the notation in the proof of [4, Theorem 5.2], set \(A := 1_{C'} - A_{(E',C')}\) and \(B := 1_{\{x\}} - A_{(E\{x\},\{x\})}\). It is shown there that the map \(\Psi\) restricts to an isomorphism between \(H\) and \(\Psi(H)\), which is an infinite cyclic group. (Note that \(H \neq 0\) by our assumption.) Moreover,
\[ \Psi(H) = A(\mathbb{Z}^{C'}) \cap B(\mathbb{Z}\delta_{X}). \]
Let \(b = \Psi(v)\) be the generator of \(\Psi(H)\). It suffices to find an element \(g\) in the image of \(\lambda_{(E,C)}\) such that \(\Psi(g) = b\). Now write
\[ b = B(n_{X}\delta_{X}) = A(\sum_{Y \in C'} \lambda_{Y}\delta_{Y}), \]
where \(n_{X}, \lambda_{Y} \in \mathbb{Z}\). We may assume that \(n_{X} > 0\). Now we consider the element \([Z\sigma(T)^{\ast}]_{1} \in K_{1}(C^*(E, C))\) associated to the element
\[ x := n_{X}\delta_{X} - \sum_{Y \in C'} \lambda_{Y}\delta_{Y} \in \ker(1_{C} - A_{(E,C)}). \]
Then, with \(A_{1} = C^*(E', C')\) and \(A_{2} = C^*(E\{x\}, \{X\})\), we can decompose \(Z = Z_{1} \oplus Z_{2}\) with \(Z_{1}\) corresponding to the positive part of \(- \sum_{Y \in C'} \lambda_{Y}\delta_{Y}\) and \(Z_{2}\) corresponding to \(n_{X}\delta_{X}\). There is no contribution of \(A_{2}\) to the negative part of \(x\), so \(T = T_{1} \oplus 0\), where \(T_{1}\) corresponds to the negative part of \(- \sum_{Y \in C'} \lambda_{Y}\delta_{Y}\). We have
\[ e := Z_{1}Z_{1}^{\ast} + Z_{2}Z_{2}^{\ast} = T_{1}T_{1}^{\ast}, \quad f := Z_{1}^{\ast}Z_{1} + Z_{2}^{\ast}Z_{2} = T_{1}^{\ast}T_{1}. \]
Therefore, by Theorem 5.4 and (6.10), we get
\[ \Psi([Z\sigma(T)^{\ast}]_{1}) = ([Z_{1}Z_{1}^{\ast}] - [Z_{1}^{\ast}Z_{1}] - ([T_{1}^{\ast}T_{1}] - [T_{1}T_{1}^{\ast}]) \]
\[ = ([e] - [Z_{1}Z_{1}^{\ast}] - ([f] - [Z_{1}^{\ast}Z_{1}]) = [Z_{2}Z_{2}^{\ast}] - [Z_{2}^{\ast}Z_{2}] \]
\[ = n_{X}[r(X)] - n_{X}[s(X)] = B(n_{X}\delta_{X}) = b. \]
This shows that \(b = \Psi(\lambda_{(E,C)}(x))\), as wanted. The proof is complete. \(\square\)

We can now obtain a proof of an enhanced version of the main result of this section (Theorem 6.2).

**Theorem 6.7.** Let \((E, C)\) be a finite bipartite separated graph, and let \(\pi: C^*(E, C) \to \mathcal{O}(E, C)\) be the natural projection map. Then \(\pi\) induces an isomorphism
\[ \pi_{\ast}: K_{1}(C^*(E, C)) \xrightarrow{\cong} K_{1}(\mathcal{O}(E, C)). \]
Moreover, the map \(\pi_{\ast} \circ \lambda_{(E,C)}: \ker(1_{C} - A_{(E,C)}) \to K_{1}(\mathcal{O}(E, C))\) is an isomorphism.
Proof. It follows from Lemma 6.4, Theorem 6.6 and Proposition 6.5 that all the maps $K_1(\phi_n): K_1(C^*(E_n, C^n)) \to K_1(C^*(E_{n+1}, C^{n+1}))$ are isomorphisms. Since $K_1(O(E, C)) \cong \lim_{\to} K_1(C^*(E_n, C^n))$, with $K_1(\phi_n)$ as the connecting maps, the result follows.

The last statement follows from the first and Theorem 6.6.

Another possible method to compute the $K$-groups of $O(E, C)$ is by realizing it as a partial crossed product, and then using McClanahan’s generalized Pimsner-Voiculescu exact sequence for crossed products by semi-saturated partial actions of free groups [14, Theorem 6.2].

However the known groups in the above mentioned exact sequence turn out to be quite large and difficult to manage, making a concrete calculation rather difficult. Nevertheless, after having computed $K_*(O(E, C))$ by the methods employed in the present article, we may use McClanahan’s result to obtain the $K$-groups for the reduced version of $O(E, C)$, which we will now briefly discuss.

Recall from Section 4 that $O(E, C)$ is isomorphic to the full crossed product

$$C(\Omega(E, C)) \rtimes_{\theta} \mathbb{F},$$

where $(\Omega(E, C), \theta)$ is the universal $(E, C)$-dynamical system. The reduced version of $O(E, C)$ may then be defined as follows:

**Definition 6.8.** We shall denote by $O_{\text{red}}(E, C)$ the reduced crossed product

$$C(\Omega(E, C)) \rtimes_{\theta^*, \text{red}} \mathbb{F}.$$

**Corollary 6.9.** The natural map

$$\lambda: O(E, C) \to O_{\text{red}}(E, C)$$

induces an isomorphism on $K$-groups.

**Proof.** It is enough to notice that the arrow marked $\lambda_*$ in [14, Theorem 6.2] is an isomorphism by the Five Lemma.

**Example 6.10.** The algebra $U_{nc}^n$ is the $C^*$-algebra generated by the entries of a universal $n \times n$ unitary matrix $U = [u_{ij}]$, see [11]. This was generalized in [12], where the $C^*$-algebra $U_{m,n}^{nc}$ generated by a $m \times n$ unitary matrix was considered. The K-theory of $U_{nc}^n$ was found in [11, Corollary 2.4]. The K-theory of $U_{m,n}^{nc}$ was computed in [4], as a consequence of the computation of the K-theory of $C^*$-algebras of separated graphs, thus solving a conjecture raised by McClanahan in [12]. Recall from [4, Example 4.5] that

$$C^*(E(m,n), C(m,n)) \cong M_{n+1}(U_{m,n}^{nc}) \cong M_{m+1}(U_{m,n}^{nc}).$$

We now get from Theorem 6.7 and [4, Theorem 5.2]:

$$K_1(O_{\text{red}}^{m,n}) \cong K_1(O_{m,n}) \cong K_1(U_{m,n}^{nc}) \cong \ker \left( \begin{pmatrix} 1 & 1 \\ -n & -m \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2 \right) \cong \begin{cases} \mathbb{Z} & \text{if } n = m \\ 0 & \text{if } n > m. \end{cases}$$
For $m = n$, setting $E := E(m, n)$ and $C := C(m, n)$, we recover the fact that $K_1(U_n^{nc})$ is generated by the class of $U = (u_{ij})$. Indeed, Theorem 6.7 says that $K_1(C^*(E, C))$ is generated by $\lambda_{(E, C)}(x)$, where $x = \delta_X - \delta_Y$. Now $\lambda_{(E, C)}(\delta_X - \delta_Y) = [ZT^*]_1$, with $Z = (\alpha_1 \cdots \alpha_n), \ T = (\beta_1 \cdots \beta_n)$.

Thus $K_1(C^*(E, C))$ is generated by the class of the unitary $\sum_{i=1}^n \alpha_i \beta_i^* \in vC^*(E, C)v$. The unitary $T^*Z = (\beta_i^* \alpha_j)$ in $M_n(wC^*(E, C)w)$ represents the same element and corresponds to $(u_{ij})$ under the canonical isomorphism $wC^*(E, C)w \cong U_n^{nc}$ (see [1] Example 4.5 and [5] Proposition 2.12(1)). The images of these unitaries through the canonical projection maps $C^*(E, C) \to \mathcal{O}_{n,n} \to \mathcal{O}_{n,n}^{red}$ provide the generators of $K_1$ of these $C^*$-algebras.

**Example 6.11.** We now consider, for $p \geq 2$, the bipartite separated graph $(E, C)$ with $p + 1$ vertices $E^{0,0} = \{v\}$, $E^{0,1} = \{w_1, \ldots, w_p\}$ and $2p$ edges $E^1 = \{\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p\}$, with $s(\alpha_i) = s(\beta_i) = w_i$ and $r(\alpha_i) = r(\beta_i) = v$ for $i = 1, \ldots, p$, and with $C = \{X, Y\}$, $X = \{\alpha_i\}$, $Y = \{\beta_i\}$. It was observed in [1] Lemma 5.5(2)] that $vC^*(E, C)v \cong C^*((\ast_2\mathbb{Z}_p) \rtimes \mathbb{Z})$ and in [2] Example 9.7 that $v\mathcal{O}(E, C)v \cong C^*(\mathbb{Z}_p \wr \mathbb{Z})$, where $\mathbb{Z}_p \wr \mathbb{Z} = (\bigoplus \mathbb{Z}) \rtimes \mathbb{Z}$ is the wreath product of $\mathbb{Z}_p$ by $\mathbb{Z}$. (The latter groups are called the lamplighter groups.) Here we have

$$1_C - A_{(E, C)} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ \vdots & \vdots \\ -1 & -1 \end{pmatrix}$$

and so, by using a similar computation as in Example 6.10, we get that $K_1(C^*(E, C))$ is a cyclic group generated by $[\sum_{i=1}^n \beta_i \alpha_i^*]_1$, where $u := \sum_{i=1}^n \beta_i \alpha_i^*$ is a unitary in $vC^*(E, C)v$.

Observe that $u$ is the unitary corresponding to the generator of the copy of $\mathbb{Z}$ in $C^*((\ast_2\mathbb{Z}_p) \rtimes \mathbb{Z})$ under the canonical isomorphism between $vC^*(E, C)v$ and $C^*((\ast_2\mathbb{Z}_p) \rtimes \mathbb{Z})$. (Only the case $p = 2$ was considered in [1] Example 5.5(2)), but the case where $p > 2$ is completely analogous.)

Similarly we obtain that $K_1(C^*(\mathbb{Z}_p \wr \mathbb{Z}))$ is generated by the class of the unitary in $C^*(\mathbb{Z}_p \wr \mathbb{Z})$ corresponding to the generator of $\mathbb{Z}$.

### 7. Finitely separated graphs

In this section we develop some methods which allow us to extend our results for finite bipartite separated graphs to general finitely separated graphs. The methods combine the direct limit technology of [5] and [2] Proposition 9.1.

**Theorem 7.1.** Let $(E, C)$ be a finite separated graph. Then we have

1. The canonical map $\pi_{(E, C)}: C^*(E, C) \to \mathcal{O}(E, C)$ induces an injective split homomorphism $K_0(\pi): K_0(C^*(E, C)) \to K_0(\mathcal{O}(E, C))$. Moreover $K_0(\mathcal{O}(E, C)) \cong K_0(C^*(E, C)) \oplus H \cong \text{coker}(1_C - A_{(E, C)}) \oplus H$, where $H$ is a free abelian group.

2. The map $K_1(\pi_{(E, C)}): K_1(C^*(E, C)) \to K_1(\mathcal{O}(E, C))$ is an isomorphism.
Proof. For each separated graph \((E, C)\) there is a canonical finite bipartite separated graph \((\tilde{E}, \tilde{C})\) such that the following diagram is commutative

\[
\begin{array}{ccc}
M_2(C^*(E, C)) & \xrightarrow{=} & C^*(\tilde{E}, \tilde{C}) \\
M_2(\pi_{(E, C)}) & \downarrow & \downarrow \pi_{(\tilde{E}, \tilde{C})} \\
M_2(O(E, C)) & \xrightarrow{=} & O(\tilde{E}, \tilde{C})
\end{array}
\]

where the horizontal maps are isomorphisms [2 Proposition 9.1]. Apply \(K_i, i = 0, 1\), to this diagram and use Theorems 4.6 and 6.7. \(\square\)

Now we start the preparations to obtain the results for finitely separated graphs.

We first view the assignment \((E, C) \mapsto O(E, C)\) as a functor on a certain category. We will only consider finitely separated graphs in this paper. We believe that suitable generalizations should be possible for general separated graphs. The category \(\text{FSGr}\) of finitely separated graphs was defined in [5 Definition 8.4]. The objects of \(\text{FSGr}\) are all the finitely separated graphs. If \((E, C)\) and \((F, D)\) are finitely separated graphs, then a morphism \(\phi\) from \((E, C)\) to \((F, D)\) is a graph homomorphism \(\phi = (\phi^0, \phi^1): (E^0, E^1) \to (F^0, F^1)\) from \(E\) to \(F\) such that \(\phi^0\) is injective, and such that, for each \(X \in C\) there is (a unique) \(Y \in D\) such that \(\phi^1\) induces a bijection from \(X\) onto \(Y\).

Given an object \((E, C)\) of \(\text{FSGr}\), a complete subobject of \((E, C)\) is a finitely separated graph \((F, D)\) such that \(F\) is a subgraph of \(E\) and \(D\) is a subset of \(C\). (In particular the edges of \(F\) are exactly all the edges of \(E\) which belong to some of the elements of the subset \(D\) of \(C\), i.e., \(F^1 = \cup_{Y \in D} Y\).) Note that a complete subobject corresponds essentially to the categorical notion of a subobject in the category \(\text{FSGr}\). By [4 Proposition 1.6], \(\text{FSGr}\) is a category with direct limits, and \((E, C) \mapsto C^*(E, C)\) defines a continuous functor from \(\text{FSGr}\) to the category \(C^*-\text{alg}\) of \(C^*\)-algebras. If \(\phi\) is a morphism from \((E, C)\) to \((F, D)\), then the associated \(*\)-homomorphism \(C^*(\phi): C^*(E, C) \to C^*(F, D)\) is given by \(C^*(\phi)(v) = \phi^0(v)\) and \(C^*(\phi)(e) = \phi^1(e)\), for \(v \in E^0\) and \(e \in E^1\).

Let \((E, C)\) is a finitely separated graph. Define a partial order on the set of complete subobjects of \((E, C)\) by setting \((F, D) \leq (F', D')\) if and only if \((F, D)\) is a complete subobject of \((F', D')\).

**Proposition 7.2.** The assignment \((E, C) \mapsto O(E, C)\) defines a continuous functor from the category \(\text{FSGr}\) of finitely separated graphs to the category of \(C^*\)-algebras. Moreover, for any finitely separated graph \((E, C)\), we have \(O(E, C) = \varprojlim O(F, D)\) where the limit is over the directed set of all the finite complete subobjects of \((E, C)\).

**Proof.** The second part follows from the first and the fact that every object in \(\text{FSGr}\) is the direct limit of the directed family of its finite complete subobjects ([5 8.4]).

For a finitely separated graph \((E, C)\), denote by \(J_{(E,C)}\) the closed ideal of \(C^*(E, C)\) generated by all the commutators \([e(u), e(u')]\), where \(u, u'\) belong to the multiplicative subgroup of \(C^*(E, C)\) generated by \(E^1 \cup (E^1)^*\). By definition, we have \(O(E, C) = C^*(E, C)/J_{(E,C)}\).
If \( \phi \) is a morphism from \((E, C)\) to \((F, D)\) in \(\mathbf{FSGr}\), then \( \phi \) induces a \(*\)-homomorphism \( C^*(\phi) : C^*(E, C) \to C^*(F, D) \). Clearly, we have \( C^*(\phi)(J_{(E,C)}) \subseteq J_{(F,D)} \), so that there is an induced map \( \mathcal{O}(\phi) : \mathcal{O}(E, C) \to \mathcal{O}(F, D) \), and we obtain a functor \( \mathcal{O} \) from \(\mathbf{FSGr}\) to \(C^*\)-alg.

To show that this functor is continuous, let \( \{(E_i, C^i), \varphi_{ji}, i \leq j, i, j \in I\} \) be a directed system in the category \(\mathbf{FSGr}\). By \([4, \text{Theorem 1.6}]\), we have \( C^*(E, C) = \lim_{i \in I} C^*(E_i, C^i) \), where \((E, C) = \lim_{i \in I} (E_i, C^i) \) in the category \(\mathbf{FSGr}\). Now it follows from the description of the direct limit in the category \(\mathbf{FSGr}\) that \( J_{(E,C)} = \lim_{i \in I} J_{(E_i,C^i)} \). Indeed, let \( u, u' \) belong to the multiplicative subsemigroup of \( C^*(E, C) \) generated by \( (E^1) \cup (E^1)^* \). Then there is \( i_0 \in I \) such that all the edges appearing in the expressions of \( u \) and \( u' \) belong to \( \varphi_{i_0}^1 \) (see \([5, \text{Definition 8.4 and Proposition 3.3}]\)). Here \( \varphi_{i,i_0} : (E_i, C^i) \to (E, C) \) are the canonical maps to the direct limit, for \( i \in I \). Hence there are \( v, v' \) in the multiplicative subsemigroup of \( C^*(E_{i_0}, C^{i_0}) \) generated by \( E_{i_0} \cup (E_{i_0})^* \) such that

\[
[e(u), e(u')] = C^*(\varphi_{i,i_0})([e(v), e(v')]),
\]

and this implies that \( J_{(E,C)} = \lim_{i \in I} J_{(E_i,C^i)} \). This in turn implies that

\[
\mathcal{O}(E, C) = C^*(E, C)/J_{(E,C)} = \lim_{i \in I} C^*(E_i, C^i)/J_{(E_i,C^i)} = \lim_{i \in I} \mathcal{O}(E_i, C^i),
\]
as desired. \(\square\)

With these preliminaries, we can already obtain the description of \( K_1 \) of tame graph \( C^*\)-algebras of finitely separated graphs. We still will need further work to obtain the corresponding result for \( K_0 \).

**Theorem 7.3.** Let \((E, C)\) be a finitely separated graph. Then the natural projection map \( \pi_{(E,C)} : C^*(E, C) \to \mathcal{O}(E, C) \) induces an isomorphism

\[
K_1(\mathcal{O}(E, C)) \cong K_1(C^*(E, C)) \cong \ker(1_C - A_{(E,C)}).
\]

**Proof.** By \([5, \text{Theorem 1.6}]\), \( C^*(E, C) = \lim_{\mathcal{C}} C^*(F, D) \), where \( \mathcal{C} \) is the directed system of the finite complete subobjects of \((E, C)\) in the category \(\mathbf{FSGr}\). By Proposition \([7, \text{2}]\) we have that \( \mathcal{O}(E, C) = \lim_{\mathcal{C}} \mathcal{O}(F, D) \). By using Theorem \([7, \text{12}]\) and the continuity of \( K_1 \), we get

\[
K_1(\mathcal{O}(E, C)) = \lim_{\mathcal{C}} K_1(\mathcal{O}(F, D)) \cong \lim_{\mathcal{C}} K_1(C^*(F, D)) = K_1(C^*(E, C)),
\]

with the mapping \( K_1(\pi_{(E,C)}) \) inducing the isomorphism. The last part follows from \([4, \text{Theorem 5.2}]\). \(\square\)

The correspondence \((E, C) \mapsto (\bar{E}, \bar{C})\) from \([2, \text{Proposition 9.1}]\) can be extended to a certain functor, which we describe below.

**Definition 7.4.** The objects of the category \(\mathbf{BFSGr}\) are all the bipartite finitely separated graphs. We stress here that this condition includes that \( r(E^1) = E^{0,0} \) and that \( s(E^1) = E^{0,1} \) (see Definition \([4, \text{11}]\)). For objects \((E, C)\) and \((F, D)\) of \(\mathbf{BFSGr}\), the morphisms from \((E, C)\) to \((F, D)\) are the morphisms \( \phi : E \to F \) of bipartite graphs (so that \( \phi^0(E^{0,0}) \subseteq F^{0,0} \) and...
\( \phi^0(E^{0,1}) \subseteq F^{0,1} \), such that \( \phi^0 \) is injective, and such that, for each \( X \in C \) there is (a unique) \( Y \in D \) such that \( \phi^1 \) induces a bijection from \( X \) onto \( Y \).

The category \( \text{BFSGr} \) is in fact a full subcategory of the category \( \text{FSGr} \). Indeed, if \((E, C), (F, D) \in \text{BFSGr} \) and \( \phi \) is a morphism in \( \text{FSGr} \) from \((E, C)\) to \((F, D)\), then, for \( v \in E^{0,0} \), there is \( e \in E^1 \) such that \( r_F(e) = v \) and so \( \phi^0(v) = r_F(\phi^1(e)) \in F^{0,0} \). Similarly, \( \phi^0(E^{0,1}) \subseteq F^{0,1} \). Hence, \( \text{BFSGr} \) is just the full subcategory of \( \text{FSGr} \) whose objects are the finitely separated graphs \((E, C)\) such that \( E^0 = s(E^1) \cup r(E^1) \).

We define the functor \( B : \text{FSGr} \rightarrow \text{BFSGr} \) by \( B((E, C)) = (\tilde{E}, \tilde{C}) \), where \((\tilde{E}, \tilde{C})\) is the bipartite separated graph associated to \((E, C)\) in \([2\text{ Proposition 9.1}] \). We have that \( \tilde{E}^{0,0} = V_0 \) and \( \tilde{E}^{0,1} = V_1 \), where \( V_0 \) and \( V_1 \) are disjoint copies of \( E^0 \), with bijections \( E^0 \rightarrow V^1 \), \( v \mapsto v_i \), and that \( \tilde{E}^1 \) is the disjoint union of a copy of \( E^0 \) and a copy of \( E^1 \):

\[
\tilde{E}^1 = \{ h_v \mid v \in E^0 \} \bigcup \{ e_0 \mid e \in E^1 \},
\]

with

\[
\tilde{r}(h_v) = v_0, \quad \tilde{s}(h_v) = v_1, \quad \tilde{r}(e_0) = r(e)_0, \quad \tilde{s}(e_0) = s(e)_1, \quad (v \in E^0, e \in E^1).
\]

For \( v \in E^0 \), and \( X \in C_v \) put \( \tilde{X} = \{ e_v : e \in X \} \). Then \( \tilde{C}_{v_0} := \{ \tilde{X} : X \in C_v \} \cup \{ h_v \} \), where \( h_v := \{ h_v \} \) is a singleton set.

For a morphism \( \phi : (E, C) \rightarrow (F, D) \) in \( \text{FSGr} \), the morphism \( B(\phi) : B(E, C) \rightarrow B(F, D) \) is defined by

\[
B(\phi)^0(v_i) = (\phi^0(v))_i, \quad B(\phi)^1(h_v) = h_{\phi^0(v)}, \quad B(\phi)^1(e_0) = \phi^1(e)_0, \quad (i = 0, 1, v \in E^0, e \in E^1).
\]

We leave to the reader the proof of the following result, which is a straightforward extension of the arguments in \([2\text{ Proposition 9.1}] \) and in Proposition \([7.2] \).

**Proposition 7.5.** (a) The category \( \text{BFSGr} \) is a full subcategory of \( \text{FSGr} \), closed under direct limits. Consequently the functors \( C^* : \text{BFSGr} \rightarrow C^*\text{-alg} \) and \( \mathcal{O} : \text{BFSGr} \rightarrow C^*\text{-alg} \) are continuous.

(b) There are natural isomorphisms of functors \( \text{FSGr} \rightarrow C^*\text{-alg} \), \( C^* \circ B \cong M_2 \circ C^*, \) and \( \mathcal{O} \circ B \cong M_2 \circ \mathcal{O}, \) where \( M_2 : C^*\text{-alg} \rightarrow C^*\text{-alg} \) is the functor defined by \( M_2(A) = A \otimes M_2(\mathbb{C}) \).

(c) Every object in \( \text{BFSGr} \) is the direct limit of its finite complete subobjects in \( \text{BFSGr} \).

In preparation for the next lemma, it is convenient to get a dynamical perspective on the \( \text{C}^*\)-algebra homomorphism \( \mathcal{O}(E, C) \rightarrow \mathcal{O}(F, D) \), when \((E, C)\) is a complete subobject of the finite bipartite separated graph \((F, D)\). Under this hypothesis, we are going to define an \((E, C)\)-dynamical system on \( \Omega := \bigsqcup_{v \in E^0} \Omega(F, D)_v \). For \( v \in E^0 \), set

\[
\Omega_v := \Omega(F, D)_v.
\]

The sets \( H_x \), for \( x \in E^1 \), are the corresponding structural sets for \((F, D)\) and the homeomorphisms

\[
\theta_x : \Omega_{s(x)} \rightarrow H_x, \quad x \in E^1.
\]
are also the structural homeomorphisms corresponding to \((F, D)\). Observe that \(\Omega\) is a clopen subset of \(\Omega(F, D)\). By the universal property of the \((E, C)\)-dynamical system \(\{\Omega(E, C)_v \mid v \in E^0\}\) there is a unique equivariant continuous map \(\gamma : \Omega \to \Omega(E, C)\). It is not difficult to describe this map in terms of the configurations used in [2, Section 8]. Namely a point in \(\Omega_v\), for \(v \in E^0\), is given by a certain subset of the free group \(F\) on \(F^1\), with property (c) of [2, page 783] at \(g = 1\) being satisfied with respect to the vertex \(v\). If \(\xi\) is such a configuration, then \(\gamma(\xi)\) is the configuration on the free group on \(E^1\), obtained by neglecting all the information which does not concern the graph \(E\). In terms of the Cayley graphs, the map \(\gamma\) consists of deleting all the vertices and arrows which do not belong to \(E^0\) and \(E^1\) respectively. This is a well-defined map by the fact that \((E, C)\) is a complete subgraph of \((F, D)\). The equivariant continuous map \(\gamma : \Omega \to \Omega(E, C)\) is surjective and induces an equivariant injective unital homomorphism \(C(\Omega(E, C)) \to C(\Omega) \subseteq C(\Omega(F, D))\), and thus a homomorphism

\[
O(E, C) = C(\Omega(E, C)) \times F(E^1) \to C(\Omega(F, D)) \times F(F^1) = O(F, D).
\]

Observe that this map is unital if and only if \(E^0 = F^0\).

The map \(\gamma : \Omega \to \Omega(E, C)\) induces a map \(K(\gamma) : K(\Omega(E, C)) \to K(\Omega)\), where \(K(\mathcal{X})\) denotes the field of open compact subsets on a topological space \(\mathcal{X}\), where \(K(\gamma)(K) = \gamma^{-1}(K)\). Since the vertices in the complete multiresolution graphs of \((E, C)\) and \((F, D)\) provide a basis of open compact subsets of the corresponding spaces \(\Omega(E, C)\) and \(\Omega(F, D)\), it is clear that the map \(K(\gamma)\) will have a significance with respect to these vertices. The exact connection is described below in Lemma 7.11.

To show this lemma we need first to introduce a new kind of maps between finite bipartite separated graphs, which is precisely the kind of maps that appear when we study the maps \((E_n, C^n) \to (F_n, D^n)\) induced by a complete subobject \((E, C) \to (F, D)\) in the category \(\text{BFSGr}\). (Here \(\{(E_n, C^n)\}_n\) and \(\{(F_n, D^n)\}_n\) denote the canonical sequences of finite bipartite separated graphs associated to \((E, C)\) and \((F, D)\), respectively; see Construction 4.2(c).) It is worth to observe that these maps also induce \(C^*\)-algebra homomorphisms between the respective separated graph \(C^*\)-algebras (see Lemma 7.7).

**Definition 7.6.** Let \((E, C)\) and \((F, D)\) be two finite bipartite separated graphs. A **locally complete** map \(\pi^* : (E, C) \to (F, D)\) consists of a complete subobject \((G, L)\) of \((F, D)\) and a graph homomorphism \(\pi = (\pi^0, \pi^1) : (G, L) \to (E, C)\), such that:

1. \(\pi^0 : G^0 \to E^0\) and \(\pi^1 : G^1 \to E^1\) are surjective maps.
2. For each \(X \in L\), we have \(\pi^1(X) \subseteq C\). In particular, \(\pi^1\) induces a (surjective) map \(\tilde{\pi} : L \to C\), by \(\tilde{\pi}(X) = \pi^1(X) \subseteq C\), for \(X \in L\).
3. For each \(w \in G^{0, 1}\), the map \(\pi^1|_w : s_E^{-1}(w) \to s_E^{-1}(\pi^0(w))\) is a bijection.
4. For each \(v \in G^{0, 0}\), the map \(\tilde{\pi}|_v : L_v \to C_{\pi^0(v)}\) is a bijection.

**Lemma 7.7.** Let \(\pi^* : (E, C) \to (F, D)\) be a locally complete map between finite bipartite separated graphs. Then there is an induced *-homomorphism \(C^*(\pi^*) : C^*(E, C) \to C^*(F, D)\). Moreover, there is a canonical locally complete map \(\rho^* : (E_1, C^1) \to (F_1, D_1)\) such that the
following diagram is commutative:

\[
\begin{array}{ccc}
C^*(E, C) & \xrightarrow{\phi(E, C)_0} & C^*(F, D) \\
\downarrow{\phi(E, C)_0} & & \downarrow{\phi(F, D)_0} \\
C^*(E_1, C^1) & \xrightarrow{C^*(\phi^*_1)} & C^*(F_1, D^1)
\end{array}
\]

where \(\phi(E, C)_0\) and \(\phi(F, D)_0\) are the canonical surjective maps (cf. Theorem 4.4).

Proof. Define \(C^*(\pi^*)\) as follows. For \(v \in E^0\) and \(e \in E^1\), set

\[
C^*(\pi^*)(v) = \sum_{w \in (\pi^0)^{-1}(v)} w, \quad C^*(\pi^*)(e) = \sum_{f \in (\pi^1)^{-1}(e)} f.
\]

It is easy to check that relations (V) and (E) are preserved by \(C^*(\pi^*)\). To show that relation (SCK1) is preserved, consider \(e, f \in X\), where \(X \in C_v\). Assume first that \(e \neq f\). Take \(g, h \in G^1\) such that \(\pi^1(g) = e\) and \(\pi^1(h) = f\). If \(r(g) \neq r(h)\), then \(g^*h = 0\). If \(r(g) = r(h)\), then \(g \neq h\) and \(g, h\) belong to the same element of \(L\), by condition (4) in Definition 7.6 (Indeed, if \(g \in Y \in L_{r(g)}\) and \(h \in Z \in L_{r(g)}\), then \(\pi|_{r(g)}(Y) = X = \pi|_{r(g)}(Z)\), and so \(Y = Z\) by the injectivity of \(\pi|_{r(g)}\).) Therefore \(g^*h = 0\). It follows that

\[
C^*(\pi^*)(e^*f) = (\sum_{\pi^1(g) = e} g^*) (\sum_{\pi^1(h) = f} h) = 0.
\]

Now assume that \(e = f\). By condition (3) in Definition 7.6 for each \(w \in (\pi^0)^{-1}(s(e))\) there is a unique \(w_w \in s^{-1}(w)\) such that \(\pi^1(h_w) = e\). If \(r(h_{w_1}) = r(h_{w_2})\) for \(w_1, w_2 \in (\pi^0)^{-1}(s(e))\), then it follows from the same argument as before that \(h_{w_1}\) and \(h_{w_2}\) belong to the same element of \(L\). It follows that \(h_{w_1}^* h_{w_2} = \delta_{w_1, w_2} w_1\) for all \(w_1, w_2 \in (\pi^0)^{-1}(s(e))\), and thus

\[
C^*(\pi^*)(e^*e) = \sum_{w_1 \in (\pi^0)^{-1}(s(e))} h_{w_1}^* (\sum_{w_2 \in (\pi^0)^{-1}(s(e))} h_{w_2}) = \sum_{w \in (\pi^0)^{-1}(s(e))} w = C^*(\pi^*)(s(e))
\]

as desired.

Now we check that (SCK2) is preserved by \(C^*(\pi^*)\). Take \(v \in E^{0,0}\) and \(X \in C_v\). Let \(g, h \in G^1\) be such that \(\pi^1(g) = e = \pi^1(h)\), where \(e \in X\). If \(s(g) \neq s(h)\), then \(gh^* = 0\). If \(s(g) = s(h)\), then by condition (3) in Definition 7.6 we have that \(g = h\). It follows that \(C^*(\pi^*)(ee^*) = \sum_{g \in (\pi^1)^{-1}(e)} gg^*\). Now, it follows from conditions (2) and (4) in Definition 7.6 that for each \(w \in (\pi^0)^{-1}(v)\) there is a unique \(Y_w \in L_w\) such that \((\pi^1)^{-1}(X) \cap r^{-1}(w) = Y_w\). Hence, we get

\[
C^*(\pi^*)\left(\sum_{e \in X} ee^*\right) = \sum_{e \in X} \sum_{g \in (\pi^1)^{-1}(e)} gg^* = \sum_{g \in (\pi^1)^{-1}(X)} gg^*
\]

\[
= \sum_{w \in (\pi^0)^{-1}(v)} \left(\sum_{g \in Y_w} gg^*\right) = \sum_{w \in (\pi^0)^{-1}(v)} w = C^*(\pi^*)(v),
\]

as desired.
We now show the statement about the associated separated graphs \((E_1, G^1)\) and \((F_1, D^1)\). We first define a complete subobject \((G_1, L^1)\) of \((F_1, D^1)\). Set \(G_0^{0,0} = G_0^{0,1}\) and \(G_1^{0,1} = r_2^{-1}(G^{0,0})\). In other words, \(v \in G_1^{0,1}\) if and only if there is \(u \in G_0^{0,0}\) such that \(v = v(x_1, \ldots, x_i)\), where \(x_i \in X_i\) and \(D_u = \{X_1, \ldots, X_i\}\). Now for \(w \in G_0^{0,0} = G_0^{0,1}\), define
\[
L^1_w = \{X(x) \mid x \in G_1^1\}, \quad G_1^1 = \bigsqcup_{w \in G_0^{0,0}} L^1_w.
\]
Clearly \((G_1, L^1)\) is a complete subobject of \((F_1, D^1)\).

Now we define the graph homomorphism \(\rho = (\rho^0, \rho^1): G_1 \to E_1\). Define \(\rho^0(w) = \pi^0(w)\) for \(w \in G_0^{0,0} = G_0^{0,1}\). Now, for \(u \in G_0^{0,0}\), set \(D_u = \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_l\}\), where \(L_u = \{X_1, \ldots, X_k\}\). Then define \(\rho^0\) on an element \(v = v(x_1, \ldots, x_k, x_{k+1}, \ldots, x_l)\), with \(x_i \in X_i, i = 1, \ldots, l\), by
\[
\rho^0(v(x_1, \ldots, x_k, x_{k+1}, \ldots, x_l)) = v(\pi^1(x_1), \ldots, \pi^1(x_k)) \in E_1^{0,1}.
\]
Note that this is well-defined because, by conditions (2) and (4), we have that \(C_{\pi^0(u)} = \{\pi^1(X_1), \ldots, \pi^1(X_k)\}\).

Now we define \(\rho^1\). An element in \(G_1^1\) is of the form \(\alpha^{x_i}(x_1, \ldots, x_i, x_k, x_{k+1}, \ldots, x_l)\), where \(x_1, x_l, x_l (X_1, \ldots, X_l)\) are as above. For such an element, put
\[
\rho^1(\alpha^{x_i}(x_1, \ldots, x_i, x_k, x_{k+1}, \ldots, x_l)) = \alpha^{\pi^1(x_i)}(\pi^1(x_1), \ldots, \pi^1(x_l)).
\]
Clearly \(\rho\) is a graph homomorphism. Finally, we have to check conditions (1)-(4) in Definition 7.6 for \(\rho\).

(1) Let \(v(x_1, \ldots, x_k) \in E_1^{0,1}\), where \(x_i \in X_i\) and \(C_u = \{X_1, \ldots, X_k\}\) for some \(u \in E_0^{0,0}\). Since \(\pi^0\) is surjective, there is \(u' \in G_0^{0,0}\) such that \(\pi^0(u') = u\). Now, by conditions (2) and (4) (for \(\pi\)), we can write \(D_u = \{Y_1, \ldots, Y_k, Y_{k+1}, \ldots, Y_l\}\) and \(L_u = \{Y_1, \ldots, Y_k\}\), with \(\pi^1(Y_i) = X_i\) for \(i = 1, \ldots, k\). Take \(y_i \in Y_i\) such that \(\pi^1(y_i) = x_i, i = 1, \ldots, k\); and take any \(y_j \in Y_j\) for \(j = k + 1, \ldots, l\). Then
\[
\rho^0(v(y_1, \ldots, y_k, y_{k+1}, \ldots, y_l)) = v(\pi^1(y_1), \ldots, \pi^1(y_k)) = v(x_1, \ldots, x_k).
\]
This shows that \(\rho^0\) is surjective. For \(i = 1, \ldots, k\), we also get
\[
\rho^1(\alpha^{y_i}(y_1, \ldots, y_k, y_{k+1}, \ldots, y_l)) = \alpha^{\pi^1(y_i)}(x_1, \ldots, x_i, x_k, x_{k+1}, \ldots, x_l),
\]
which shows that \(\rho^1\) is also surjective.

(2) If \(X \in L_1\), then there is \(u \in G_0^{0,0}\) with \(D_u = \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_l\}\) and \(L_u = \{X_1, \ldots, X_k\}\) such that \(X = X(x_i)\) for some \(i\) with \(1 \leq i \leq k\). By the definition of \(\rho^1\) and conditions (2) and (4) for \(\pi^1\), we get that \(\rho^1(X) = X(\pi^1(x_i))\).

(3) Let \(v = v(x_1, \ldots, x_k, x_{k+1}, \ldots, x_l)\) be a vertex in \(G_1^{0,1}\), where the notation is as before. Then
\[
s^{-1}_{G_1^1}(v) = \{\alpha^{x_i}(x_1, \ldots, x_i, x_k, x_{k+1}, \ldots, x_l) \mid i = 1, \ldots, k\},
\]
so that it is clear that \(\rho^1\) induces a bijection \(\rho^1|: s^{-1}_{G_1^1}(v) \to s^{-1}_{E_1}(\rho^0(v))\).

(4) Let \(w \in G_0^{0,0} = G_0^{0,1}\). Then the elements of \(L^1_w\) are in bijective correspondence with the elements of \(s^{-1}_{G_1^1}(w)\). If \(x \in G_1^1\) is one of such vertices, then the corresponding element of
$L_w^1$ is $X(x)$, and $\tilde{\rho}(X(x)) = X(\pi^1(x))$. Since $\pi^1$ establishes a bijection between $s^{-1}_E(\pi^0(w))$, we see that $\tilde{\rho}$ establishes a bijection from $L_w^1$ onto $C^1_{\rho(w)}$, as desired. \hfill $\square$

**Corollary 7.8.** Let $(F, D)$ be a finite bipartite separated graph, and let $(E, C)$ be a complete subobject of $(F, D)$ in $\text{BFSGr}$. Let $\{(E_n, C^n)\}$ and $\{(F_n, D^n)\}$ be the canonical sequences of finite bipartite separated graphs associated to $(E, C)$ and $(F, D)$ respectively. Then there are canonical locally complete maps $\pi_n^*: (E_n, C^n) \to (F_n, D^n)$ such that $C^*(\pi_{n+1}^*) \circ \phi(E, C)_n = \phi(F, D)_n \circ C^*(\pi_n^*)$ for all $n \geq 0$. Consequently, if $\iota: (E, C) \to (F, D)$ is the inclusion map, and $\mathcal{O}(\iota): \mathcal{O}(E, C) \to \mathcal{O}(F, D)$ is the induced $*-\text{homomorphism}$, then $\mathcal{O}(\iota) = \lim_n C^*(\pi_n^*)$.

**Proof.** Use Lemma 7.7 and induction, starting with the natural map $\iota: (E, C) \to (F, D)$, which is obviously a locally complete map. \hfill $\square$

Using suitable orderings we will be able to determine a canonical complement $H_{(E, C)}$ of $K_0(C^*(E, C))$ in $K_0(\mathcal{O}(E, C))$, for each finite bipartite separated graph $(E, C)$.

**Definition 7.9.** Let $(E, C)$ be a bipartite finitely separated graph. An order in $(E, C)$ is given by the following data:

1. A total order in each of the sets $C_v$, for $v \in E^{0,0}$.
2. A total order in each of the sets $s^{-1}_E(w)$, for $w \in E^{0,1}$.
3. A total order in each of the sets $X$, for $X \in C$.

It is clear that every bipartite finitely separated graph can be endowed with an order. When this is given we refer to $(E, C)$ as an ordered separated graph. If $(E, C)$ is ordered, each complete subobject $(F, D)$ of $(E, C)$ in $\text{BFSGr}$ inherits an order, defined by restricting the corresponding total orderings.

**Notation 7.10.** Let $(E, C)$ be an ordered finite bipartite separated graph. Then the proof of Theorem 4.6 and Lemma 3.4 give a canonical complement of $K_0(C^*(E, C))$ in $K_0(C^*(E_1, C^1))$, namely the group $\mathbb{Z}^{W_2}$, where $W_2$ is the set of vertices of $E_i^{1,1}$ of the form $v(x_1, \ldots, x_k)$, where $x_i \in X_i$, $C_u = \{X_1, \ldots, X_k\}$ for some $u \in E^{0,0}$, and at least two different elements $x_i$ and $x_j$ are not the first elements in the respective sets $X_i$ and $X_j$ in the given order on them. The choice of a given order in each of the sets $X \in C^n$, for all the sets $C^n$ appearing in the canonical sequence of finite bipartite separated graphs $\{(E_n, C^n)\}$ associated to $(E, C)$ will thus, by Theorem 4.6 give a canonical complement $H_{(E, C)}$ of $K_0(C^*(E, C))$ in $K_0(\mathcal{O}(E, C))$. Indeed, we can inductively define an order on each of the finite bipartite separated graphs $(E_n, C^n)$, as follows. Assume that, for some $n \geq 0$, an order has been defined on $(E_n, C^n)$, and let us define the order on $(E_{n+1}, C^{n+1})$. For $v \in E_{n+1}^{0,0} = E_{n+1}^{0,1}$, we have that $C^{n+1}_v$ is in bijective correspondence with $s^{-1}_{E_n}(v)$ (through $X(x) \leftrightarrow x$). Define the total order in $C_{n+1}$ as the order induced by this bijection. For $v \in E_{n+1}^{0,1}$, we have $v = (x_1, \ldots, x_k)$, where $u \in E_{n+1}^{0,0}$ with $C^u_n = \{X_1, \ldots, X_k\}$, and $x_i \in X_i$ for $i = 1, \ldots, k$. (Here we are assuming that $X_1 < X_2 < \cdots < X_k$ in the given total order on $C^u_n$.) Now note that

$s^{-1}_{E_{n+1}}(v) = \{a^{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k): i = 1, \ldots, k\}$. 


We define the total order in $s_{E_n^{m+1}}(v)$ by setting $\alpha^{x_i}(x_1, \ldots, \hat{x}_i, \ldots, x_k) < \alpha^{x_j}(x_1, \ldots, \hat{x}_j, \ldots, x_k)$ if and only if $i < j$. Finally, let $X$ be an element of $C^{m+1}$. Then there is $u \in E_n^{m+1}$, with $C_u = \{X_1, \ldots, X_k\}$, and $x_i \in X_i$ for some $i = 1, \ldots, k$, such that $X = X(x_i)$. Recall that $X(x_i) = \{\alpha^{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) : x_j \in X_j, j \neq i\} \cong X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$, so we take the left lexicographic order on $X(x_i)$.

This gives a canonical choice of sets $W_2, W_3, \ldots$ and thus a canonical choice of a complement $H_{(E,C)} := \bigoplus_{k=2}^{\infty} \mathbb{Z}^{W_k}$ of $K_0(C^*(E,C))$ in $K_0(O(E,C))$, so that

$$K_0(O(E,C)) = K_0(C^*(E,C)) \oplus H_{(E,C)}.$$  

**Lemma 7.11.** Let $(F,D)$ be an ordered finite bipartite separated graph, and let $(E,C)$ be a complete subobject of $(F,D)$ in $\text{BFSGr}$, endowed with the induced order. Let $\varphi: K_0(O(E,C)) \to K_0(O(F,D))$ denote the map induced by the inclusion $i: (E,C) \to (F,D)$. Then the restriction of $\varphi$ to $H_{(E,C)}$ is injective, and $\varphi(H_{(E,C)}) \subseteq H_{(F,D)}$.

**Proof.** By the proof of Theorem 4.6 and Corollary 7.8, it suffices to show inductively that, for each $n \geq 1$, the induced map $C^*(\pi_n^*): C^*(E_n, C^n) \to C^*(F_n, D^n)$ sends each projection coming from $W_{n+1}$ to an orthogonal sum of projections coming from $W_{n+1}'$, where $W_{n+1}$ corresponds to $(E_n, C^n)$ and $W_{n+1}'$ corresponds to $(F_n, D^n)$. The injectivity of $\varphi|_{H_{(E,C)}}$ follows then from the fact that $C^*(\pi_n^*)$ sends projections corresponding to distinct vertices of $E_n$ to orthogonal projections of $C^*(F_n, D^n)$ (see Lemma 7.7). In order to show this, it is enough to show, by Lemma 7.7 and induction, that the result holds for the first terms $(E_1, C^1)$, $(F_1, D^1)$ of the canonical sequences of finite bipartite separated graphs associated to $(E,C)$ and $(F,D)$ respectively, where $\pi^*: (E,C) \to (F,D)$ is a certain locally complete map. Concretely we will show the following statement:

**Claim:** Let $\pi^*: (E,C) \to (F,D)$ be a locally complete map, and let $\rho^*: (E_1, C^1) \to (F_1, D^1)$ be the corresponding locally complete map, as defined in the proof of Lemma 7.7. Assume that the following condition holds: $\pi^1$ sends the first element of each $Y$ in $L$ to the first element of $\pi(Y') \in X$. Then $C^*(\rho^1)$ sends each projection coming from $W_2$ to a projection in $C^*(F_1, D^1)$ which is an orthogonal sum of projections coming from $W_2'$. Moreover, the map $\rho^*$ has the same property as $\pi^*$, that is, it sends the first element of each $Y \in L_1$ to the first element of $\rho(Y) \in C_1$.

**Proof of Claim:** The set $W_2$ above is the set of projections of the form $v = v(x_1, \ldots, x_k)$, where $x_i \in X_i$, $C_u = \{X_1, \ldots, X_k\}$, and at least for two different indices $j, t$ we have that $x_j$ and $x_t$ are not the first elements of $X_j$ and $X_t$ respectively (see the proofs of Theorem 4.6 and Lemma 3.4). The set $W_2'$ is the analogous set of projections in $C^*(F,D)$.

For $v = v(x_1, \ldots, x_k) \in W_2$, we have

$$C^*(\rho^*)(v) = \sum v(y_1, \ldots, y_k, y_{k+1}, \ldots, y_l),$$

where the sum is extended over all $(y_1, \ldots, y_l) \in Y_1 \times \cdots \times Y_l$, where $D_u = \{Y_1, \ldots, Y_l\}$ and $L_u' = \{Y_1, \ldots, Y_k\}$, where $u'$ ranges over all the vertices in $G$ such that $\pi^1(u') = u$, and $\pi^1(y_i) = x_i$ for all $i = 1, \ldots, k$. (Note that here the index $l$ may depend on $u'$.)
Now by the hypothesis on \( \pi^1 \), we have that \( y_j \) is not the first element of \( Y_j \) and \( y_t \) is not the first element of \( Y_t \), showing that each \( v(y_1, \ldots, y_k, y_{k+1}, \ldots, y_t) \) belongs to \( W_2' \).

Finally we check that \( \rho^1 \) has the same property as \( \pi^1 \). Take \( Y \in L^1 \). Then there is \( u \in G^{0,0} \), with \( D_u = \{X_1, \ldots, X_k, X_{k+1}, \ldots, X_l \} \) and \( L_u = \{X_1, \ldots, X_k \} \) such that \( Y = X(x_i) \) for some \( x_i \in X_i \) with \( 1 \leq i \leq k \). The first element of \( Y \) is thus the element
\[ e = \alpha^x(x_1, \ldots, \hat{x}_i, \ldots, x_k, x_{k+1}, \ldots, x_l), \]
where, for each \( j \neq i \), \( x_j \) is the first element of \( X_j \). Consequently, by the hypothesis on \( \pi^1 \), the element \( \pi^1(x_j) \) is the first element of \( \pi(X_j) \), for \( j \neq i \) and \( j \in \{1, \ldots, k\} \). Therefore
\[ \rho^1(e) = \alpha^{\pi^1(x_i)}(\pi^1(x_1), \ldots, \pi^1(x_i), \ldots, \pi^1(x_k)), \]
which is the first element of \( X(\pi^1(x_i)) = \rho(Y) \).

Note that the hypothesis on \( \pi^1 \) is trivially satisfied in the base case, that is, in the case where \( (E, C) \) is a complete subobject of \( (F, D) \). Indeed, in that case \( (G, L) = (E, C) \) and \( \pi \) is the identity. Therefore, the Claim gives the desired result by induction, using Lemma 7.11. \( \square \)

**Theorem 7.12.** Let \( (E, C) \) be an ordered bipartite finitely separated graph and let \( C \) be the directed set of finite complete subobjects of \( (E, C) \) in \( BFSGr \). For complete subobjects \( (F, D), (F', D') \) of \( (E, C) \), with \( (F, D) \leq (F', D') \), let \( \varphi_{(F', D'), (F, D)}: K_0(\mathcal{O}(F, D)) \to K_0(\mathcal{O}(F', D')) \) be the natural map. Write \( K_0(\mathcal{O}(F, D)) = K_0(C^*(F, D)) \oplus H_{(F, D)} \) for each \( (F, D) \in C \), where \( H_{(F, D)} \) is the canonical complement associated to the induced order on \( (F, D) \), as defined in Notation 7.10. Then the following properties hold:

1. For \( (F, D), (F', D') \in C \) with \( (F, D) \leq (F', D') \), the map \( \varphi_{(F', D'), (F, D)} \) induces an injective homomorphism from \( H_{(F, D)} \) to \( H_{(F', D')} \).

2. We have
\[ K_0(\mathcal{O}(E, C)) \cong K_0(C^*(E, C)) \bigoplus H \cong \ker(1 - A_{(E, C)}) \bigoplus H, \]
where \( H = \lim_{\rightarrow}(H, D) \in C \). In particular \( H \) is a torsion-free group, and the maps \( \varphi_{(E, C), (F, D)}|_{H_{(F, D)}} \) are injective for all \( (F, D) \in C \).

**Proof.** The decomposition \( K_0(\mathcal{O}(F, D)) = K_0(C^*(F, D)) \oplus H_{(F, D)} \) for each \( (F, D) \in C \) is described in Notation 7.10. (1) follows from Lemma 7.11 and (2) follows from Proposition 7.5 the continuity of \( K_0 \) and (1). \( \square \)

**Theorem 7.13.** Let \( (E, C) \) be a finitely separated graph. Then \( K_0(\mathcal{O}(E, C)) = K_0(C^*(E, C)) \oplus H \), where \( H \) is a torsion-free group.

**Proof.** This follows from [2] Proposition 9.1 and Theorem 7.12. \( \square \)

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