In the last two decades, there has been a considerable amount of interest in understanding the transport properties of systems in the presence of both disorder and interactions. It is well-known that disordered systems described by quadratic Hamiltonians (e.g. noninteracting electrons in a disordered potential or disordered harmonic crystals) exhibit the phenomena of Anderson localization \[1\], whereby the single-particle eigenstates or normal modes (NMs) of the system form spatially localized states. This has a profound effect on transport — in particular for one dimensional systems all states are localized and one finds that the system is a thermal insulator. A question of great interest is to ask what happens when one introduces interactions in such a system: Does one need a nonzero critical strength of interactions to see an insulator-to-conductor transition? For quantum systems, this question has been extensively studied in the context of many-body localization (MBL) \[2\] \[3\]. It is now generally accepted that for one-dimensional quantum systems with sufficiently strong disorder, the localized insulating state persists up to a critical interaction strength. One can ask the same question in the context of classical systems and this has been addressed in a number of recent works \[4\] \[5\].

The work in \[1\] \[7\] leads one to believe that there is no classical analogue of an MBL phase while \[8\] provides evidence that such a phase might exist for a specially-designed realization of spring constants. Theoretical arguments \[11\] indicate that the thermal conductivity of a disordered nonlinear system goes to zero with decreasing temperature \(T\) at a rate faster than any power-law. The numerical study in \[9\] is consistent with this finding, however \[6\] found evidence for a power-law dependence.

Studies on energy transport in nonlinear disordered systems can be broadly classified into finite temperature studies such as those in \[4\] \[7\] \[9\] \[17\], and zero temperature studies studying wave-packet diffusion \[10\] \[12\] \[18\] \[20\] \[25\].

Numerically, it has been shown that nonlinearity gives rise to the subdiffusive spreading of a wave packet on an otherwise empty lattice, implying the destruction of Anderson localization. A theoretical explanation of the subdiffusive spreading is based on the fact that the nonlinearity results in non-integrability of the system, because of which the wave packet evolves chaotically and that leads to an incoherent spreading \[12\] \[18\] \[20\] \[25\]. A very interesting crossover from the weak chaos to strong chaos regime was proposed in \[26\]. The work in \[21\] \[24\] suggests that transport in the weak chaos regime is somewhat similar to that of variable range hopping for disordered interacting electron systems. The main aim of the present work is to look for signatures of the weak-strong chaos crossover at finite temperatures as one varies the disorder strength across a characteristic value and to understand the mechanism of transport.

Here we study energy transport across a one-dimensional chain of interacting oscillators, connected to two thermal reservoirs at different temperatures \(T_L\) and \(T_R\) at the left and right ends, respectively. The model studied is in the class introduced by Fröhlich, Spencer, and Wayne \[20\] and we refer to it as the FSW model. It is the strong disorder limit of the so called Klein-Gordon (KG) model \[6\] \[26\] \[28\].

We summarize here our most interesting findings: (i) We find that the conductivity \(\kappa_{\infty}\), in the thermodynamic limit, is nonzero at all \(T > 0\) and all \(\Delta\). To obtain \(\kappa_{\infty}\), we measured the thermal conductivity \(\kappa_N\) in nonequilibrium simulations for a chain of length \(N\) and then estimated \(\kappa_{\infty}\) using a finite size analysis. (ii) In the FSW model, for \(\Delta > 0\) in the limit \(T/\Delta \to 0\), the system essentially consists of a set of decoupled oscillators and \(\kappa_{\infty} \to 0\),
faster than any power law of $T$. Our data fits remarkably well to the scaling form $\kappa_\infty = b(T/\Delta)e^{-\Delta/T}$, where $b,c$ are constants. The leading factor $e^{-\Delta/T}$ can be argued on the basis of formation of chaotic islands. We show that a single resonant pair of oscillators is sufficient for the formation of these islands, in contrast to the double-resonance picture of Ref. [25]. (iii) We find that there is a temperature-dependent characteristic disorder line $\Delta_c(T)$ across which the form of the system-size scaling of the conductivity changes, suggesting different mechanisms of the leading finite-$N$ correction to the transport in the two regimes. For weak disorder $\Delta < \Delta_c(T)$, $\kappa_N$ first increases with $N$, before eventually saturating for large $N$. On the other hand, for $\Delta > \Delta_c(T)$, the conductivity first decays exponentially with $N$ before eventually saturating. We also find that $\Delta_c(T)$ depends on the system-bath coupling strength.

Definition of FSW Model: We start by taking a chain of $N$ oscillators with masses $m$ and random spring constants $k_i = m\omega^2r_i$, with each $r_i$ chosen uniformly in the interval $[1 - \Delta, 1 + \Delta]$, where $\Delta$ defines the disorder strength. Nearest-neighbor oscillators are then coupled by a non-linear (quartic) interaction potential of strength $\nu$. The Hamiltonian of the system is given by

$$\mathcal{H} = \sum_{i=1}^{N} \left[ \frac{p_i^2}{2m} + k_i \frac{x_i^2}{2} \right] + \sum_{i=0}^{N-1} \nu \frac{(x_i - x_{i+1})^4}{4},$$

where $\{x_i, p_i\}$ are respectively the position and momentum of the oscillators in the chain and we set $x_0 = x_{N+1} = 0$. The limit $\Delta = 0$ represents the pure case and $\Delta = 1$ is the maximum disorder strength possible in our model.

Results from nonequilibrium simulations: The chain of oscillators is attached to two thermal reservoirs at unequal temperatures, so that a temperature gradient is generated and there is a heat current along the chain [29]. The two thermal reservoirs are modeled by Langevin equations. In dimensionless units, $t \to \omega t$ and $x \to \sqrt{\nu/(m\omega^2)}x$, the equations of motion for $1 \leq i \leq N$ are given by

$$\ddot{x}_i = -r_i x_i - \left[ (x_i - x_{i-1})^3 + (x_i - x_{i+1})^3 \right] - \gamma_i \dot{x}_i + \eta_i,$$

with $\eta_i = \eta_L \delta_i,1 + \eta_R \delta_i,N$ and $\gamma_i = \gamma (\delta_i,1 + \delta_i,N)$. The Gaussian white noise $\eta_{L,R}$ satisfy the fluctuation dissipation relation $\langle \eta_{L,R}(t)\eta_{L,R}(t') \rangle = 2\gamma T_{L,R}\delta(t - t')$ with $\langle \eta_{L,R} \rangle = 0$. Here the dissipation constant $\gamma$ is measured in units of $m\omega$ and temperature in units of $m^2\omega^3/(\nu k_B)$, where $k_B$ is the Boltzmann’s constant. The only relevant dimensionless parameters in the problem that remain with this scaling are the disorder strength $\Delta$, the temperature $T$ (which is equivalent to the nonlinearity $\nu$), dissipation constant $\gamma$, and the system size $N$.

Here we compute the heat current and the temperature profile in the nonequilibrium steady state (NESS), when $T_L > T_R$. The (scaled) heat current along the chain from left to right is given by $J = \langle J_N \rangle = \sum_{i=2}^{N} (f_{i,L-1}x_i)/(N - 1)$ where $f_{i,L-1} = (x_{i-1} - x_i)^3$ is the force exerted by the $(l - 1)^{th}$ particle on the $l^{th}$ particle. We define $T = (T_L + T_R)/2$. Then for a finite size we define a thermal conductivity $\kappa_N(\Delta, T) = J N/(T_L - T_R)$. For a diffusive system one expects a finite value for $\kappa_{\infty}(\Delta, T) = \lim_{N \to \infty} \kappa_N(\Delta, T)$. In all our numerical studies we set $(T_L - T_R)/T = 0.5$ (which implies $T_L = 1.25T$ and $T_R = 0.75T$) and explore the system properties as we vary $\Delta, T$ and $N$.

We perform numerical simulations by using velocity Verlet algorithm, adapted for Langevin dynamics [30], with $\gamma = 1.0$ in all cases. To speed up relaxation to the steady state, the initial conditions are chosen from a product form distribution corresponding to each disconnected oscillator being in equilibrium at a $T_i$ that varies linearly across the chain. The system is evolved up to times ranging from $2 \times 10^5 - 5 \times 10^7$ time steps of step size $dt = 0.005$, in order to reach its NESS and then NESS averages are obtained over another equal number of states. Relaxation times increase rapidly with increasing $N, \Delta$, and with decreasing $T$. We also average over 50 disorder samples and our error bars represent sample-to-sample variations. For $T \lesssim 0.01$, the conductivity becomes very small and reaching a steady state becomes difficult. In these cases we compute $\kappa$ by taking transient and averaging times of up to $10^{11}$ time steps, which has been possible for $N = 32$ and 64.

In Fig. 3(a)-(c), we plot $\kappa$ against $N$ for different values of $\Delta$ and for temperature values $T = 0.01, 0.04, 0.08$. We observe that in all cases, the conductivity seems to converge with increasing system size to a nonzero $\kappa_{\infty}(\Delta, T)$. However, the approach to $\kappa_{\infty}$ with increasing $N$ is different for small and large disorder, demarcated by a temperature dependent characteristic disorder strength $\Delta_c(T)$. For $\Delta < \Delta_c(T)$, we find that $\kappa_N$ is an increasing function of $N$, while for $\Delta > \Delta_c(T)$, it is a decreasing function. At $\Delta = \Delta_c(T)$ we find that the conductivity is almost independent of system size. This is also illustrated in Fig. 4(d) which shows a plot of $\kappa$ vs $\Delta$ for different $N$ at $T = 0.01$, and where the curves for different $N$ intersect at $\Delta_c \simeq 0.2$. The variation of $\Delta_c(T)$ with temperature $T$ is shown as an inset in Fig. 4(d).

We now propose an explanation for the difference in the system-size dependence at weak and strong disorder. For weak disorder $\Delta < \Delta_c(T)$ we expect diffusive heat transport, but the resistance at small system sizes can be dominated by contributions from the boundaries. If we assume a boundary resistance $r$, then the bulk and boundary resistances add to give $R = (N/\kappa_{\infty}) + r$. Hence the effective finite-size conductivity is given by

$$\kappa_N(\Delta, T) = N/R = \frac{\kappa_{\infty}}{1 + (\kappa_{\infty}T/N)},$$

for $\Delta < \Delta_c(T)$. (3)

For system sizes $N \ll r\kappa_{\infty}$ the boundary resistance dominates and one has $\kappa_N \sim N$, while for larger $N > r\kappa_{\infty}$
Chaotic islands: The behavior of $\kappa_\infty$ with $T$ can be understood qualitatively from the following heuristic arguments based on the formation of chaotic islands (CI), which provide an effective channel for energy transport. It has been argued earlier \cite{25} that the formation of such CIs require three consecutive oscillators with resonant frequencies $(|\omega_{i+1} - \omega_i| \sim T)$ and thus occurs with probability $p \sim T^2/\Delta^2$. However, we have found that three consecutive oscillators with any neighboring pair in resonance is in fact sufficient to generate chaos and this implies $p \sim T/\Delta$. In Fig. \ref{fig3} we show the time-dependent Lyapunov exponent $\lambda(t)$ (see Supplemental Material \cite{31} for details) for closed systems with $N = 3$ for different parameter sets. For set (I) we set the two oscillators with a small frequency difference while the third oscillator has a very different frequency, while for set (II), all three oscillators have well separated frequencies. At sufficiently high temperature $(T = 0.2)$, the Lyapunov exponent seems to saturate to a finite value for both parameter sets. On the other hand at very low temperature $(T = 0.02)$, the Lyapunov exponent keeps decaying. Finally we consider an intermediate temperature $(T = 0.05)$, where for set (I) $|\omega_1 - \omega_2| \lesssim T$ but the third oscillator is out of resonance. In this case, we get chaos for set (I) and no chaos for set (II). For $N = 2$ particles we again see absence of chaos even when the resonance condition $|\omega_1 - \omega_2| \lesssim T$ is satisfied. Thus our numerics establishes that while we need at least three oscillators to generate chaos, it is only required that two neighboring oscillators are in resonance. This also suggests that, not surprisingly, a nonresonant nearby oscillator can enhance the chaos of a CI. Thus as we probe the chaos at lower $T/\Delta$ and thus longer time scales, larger distances, and more nearby nonresonant oscillators, the density of CIs may vanish slower than $T/\Delta$.

The CIs are the basis for the second parallel channel of transport leading to Eq. \ref{eq4}. Since the CIs form with probability $\sim T/\Delta$, they are separated on average by distance $d \sim \Delta/T$. They act as effective thermal reservoirs between which energy is transmitted via intermediate localized states. The current between two neighboring
CIs is given by $j_C \sim Ae^{-d/\xi}$, where $\xi$ is the localization length — this corresponds to a resistor with resistance $\sim j_C^{-1}$. For a chain of length $N$ we have $N/d$ such resistors in series, leading naively to an effective net resistance $(N/d)j_C^{-1}$. The resulting contribution to the conductivity from transport via CIs is given by $\kappa_\infty \sim Ad\exp(-d/\xi)$. This form is consistent with our numerics if we assume $d/\xi \sim \Delta/T$. In fact one can argue that $\xi$ should have a weak logarithmic dependence on temperature. At low temperatures, using a mean field approximation we can replace the nonlinearity $(x_i-x_{i+1})^4 \rightarrow T(x_{i+1}-x_i)^2$. This leads to a disordered harmonic chain with coupling constants $k \sim T$, for which we know that the localization length $\sim 1/\ln(\Delta/k)$. Hence we expect $\xi \sim 1/\ln(\Delta/T)$. But, as discussed in the previous paragraph, it is possible that the behavior $d \sim \Delta/T$ is also corrected by a logarithmic factor. Another aspect of this physics that we are leaving out is the contribution from the randomness in the positions of the CIs, which will make some of the thermal resistances much larger than others at low $T/\Delta$.

The signatures of disorder and the strong temperature dependence of $\kappa_\infty$ can also be seen in the temperature profiles. In Fig. 4 we see that in the low-disorder regime [Fig. 4(a)], the boundary resistance is clearly seen and the profile slowly converges (with increasing system size) to an asymptotic form that is consistent with $\kappa \sim T$.

On the other hand, with somewhat stronger disorder [Fig. 4(b)], the profile converges quickly to the asymptotic form which is now consistent with $\kappa \sim T^{3.3}$. These two asymptotic forms are shown by a black dashed line in Figs. 4(a) and (4). At smaller temperatures and high disorder, a sufficiently small size system is effectively in the localized regime and we expect a step temperature profile [32]. There is some indication from our numerics that this is indeed the case [see Fig. 4(c)].

FIG. 2. (a) Plot of $\kappa_N$ vs. $T$ for $\Delta = 0.5$ and for the ordered case $\Delta = 0$. For the disordered case $\Delta = 0.5$ we see that the slope keeps increasing with decreasing $T$, and at low $T$ with increasing $N$. We find $\kappa \sim T^4$ in our lowest attained temperature range for $N = 64$. For the ordered case a dependence $\kappa_\infty \sim T$ is seen. (b) Plot of $\kappa_\infty$ as a function of $\Delta/T$ shows a good collapse except for the weakest disorder $\Delta = 0.1$. The solid line is the fit to the form $\kappa_\infty = (8T/\Delta) \exp(-c\Delta/T)$. Some finite size data at $\Delta = 0.5$ is also shown.

FIG. 3. Plot of the Lyapunov exponent $\lambda(t)$-vs-time for three oscillators, for different parameter sets. Set I: $\omega_1 = 0.707, \omega_2 = 0.742, \omega_3 = 1.225$, Set II: $\omega_1 = 0.707, \omega_2 = 1.0, \omega_3 = 1.225$. The $N = 2$ data corresponds to two oscillators with frequencies $\omega_1 = 0.707$ and $\omega_2 = 0.742$.

FIG. 4. (a) Temperature profile in the steady state with $T_L = 0.05$, $T_R = 0.03$ and $\Delta = 0.1$ for different system sizes. (b) Temperature profiles for $T_L, T_R$ as in (a) but with $\Delta = 0.5$. The analytical fits (black dashed lines) to the asymptotic profiles were obtained by solving $-\kappa(T)\partial_iT(i) = J$. With the form $\kappa \sim T^a$, it can be solved exactly for $T(i)$ and plotted as a function of $i/N$ using $a = 1$ in (a), and $a = 3.3$ in (b). (c) Temperature profile at a lower mean temperature $T = 0.01$ and $\Delta = 0.5$, which shows signatures of a step profile, as seen in MBL systems [32].
a strong signature for classical analogue of a MBL-like regime, which however does not survive in the thermodynamic limit.

**Conclusions:** We studied transport properties of a non-linear chain with strong disorder and looked for signatures of classical many-body localization. From our extensive numerical studies, we find an interesting crossover behaviour whereby the system-size scaling is qualitatively different above and below a characteristic disorder, that depends both on temperature and coupling strength to the baths. Our finite size scaling analysis leads to estimates of the thermodynamic limit conductivity $\kappa_\infty$ and we find evidence that for strong disorder $\kappa_\infty$ depends only on the combination $\Delta/T$. Our data is described well by the form $\kappa_\infty \sim (bT/\Delta)e^{-c\Delta/T}$ and we provide a heuristic theoretical derivation based on the idea of energy transfer between chaotic islands. Surprisingly we find that resonances due to a single pair of oscillators is sufficient to create chaotic islands. A complete understanding of the transport mechanism and the low temperature form of $\kappa_\infty$ at strong disorder remains an interesting open problem.

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| $T$ | $\Delta$ | $r$ | $\xi$ | $A$ | $\kappa_\infty$ |
|-----|---------|-----|-----|-----|----------------|
| 0.01 | 0       | 762 ± 10 | -   | -   | 0.200(5) |
| 0.1  | 0       | 793 ± 36 | -   | -   | 0.0270(4) |
| 0.2  | 0       | 30 ± 31  | -   | -   | 0.00340(7) |
| 0.3  | -       | 11.3 ± 1.8 | 0.00022(8) | 0.00052(2) |
| 0.4  | -       | 8 ± 1    | 0.00017(7) | 0.00011(1) |
| 0.04 | 0.1     | 90 ± 2   | -   | -   | 0.263(1) |
| 0.2  | 0.1     | 84.6 ± 3.5 | -   | -   | 0.1440(9) |
| 0.3  | 0.2     | 76.3 ± 6.6 | -   | -   | 0.0761(4) |
| 0.4  | 0.3     | 19 ± 4   | -   | -   | 0.0380(4) |
| 0.5  | 0.4     | 14.1 ± 4.9 | 0.0009(6) | 0.0204(2) |
| 0.6  | 0.5     | 15 ± 4   | 0.0006(3) | 0.0111(2) |
| 0.08 | 0.3     | 33.5 ± 1.7 | -   | -   | 0.244(1) |
| 0.4  | 0.4     | 30.8 ± 2.5 | -   | -   | 0.1678(8) |
| 0.5  | 0.5     | 21.7 ± 4.2 | -   | -   | 0.1136(6) |
| 0.6  | 0.6     | 12 ± 8   | -   | -   | 0.0783(5) |
| 0.7  | 0.7     | 54.1 ± 27.2 | 0.00012(9) | 0.0536(5) |
| 0.8  | 0.8     | 29.4 ± 11.4 | 0.0003(2) | 0.0382(4) |

TABLE S1. A summary of exponents $r$, $\xi$, $A$, and $\kappa_\infty$ determined from the best non-linear fits of finite-size conductivities (see the main text). The numbers in parenthesis are the error estimates on the last significant figures.

SUPPLEMENTAL MATERIAL

S1. THERMAL CONDUCTIVITY WITHOUT DISORDER

For $\Delta = 0$, there is no mechanism for scattering of the heat carriers and the heat transport is ballistic, i.e., $\kappa \sim N$. With increasing system size, the harmonic oscillator potential part in the FSW model leads to a saturation of the conductivity $\kappa$. Thus, the behavior of finite size conductivity is given as

$$\kappa_N(T) = \frac{\kappa_\infty}{1 + \kappa_\infty r/N}$$  \hspace{1cm} (S1)

where $\kappa_\infty(T)$ is the saturated value of conductivity in the thermodynamic limit ($N \to \infty$) and $\ell$ is the mean free-path of the heat carriers. Fig. S1 shows a plot of finite size conductivities as a function of system size $N$ with varying temperature. At larger temperature, the anharmonic term starts dominating over the harmonic potential part and therefore conductivity saturates at smaller value of $N$. The point symbols are the simulated values of $\kappa$ whereas the solid lines are the best non-linear fits of Eq. (S1). Clearly, the Eq. (S1) fits accurately to the simulated data at all temperatures. From these fits, $\kappa_\infty$ can be extracted and plotted as a function of temperature $T$. At small $T$, $\kappa_\infty(T) \sim T$ and finally saturates at large temperature.

FIG. S1. Plot of $\kappa$ vs $N$ on a log-log scale for $\Delta = 0$. Point symbols are the simulated values of $\kappa$ whereas the solid lines are the best non-linear fits of Eq. (S1).

S2. FORMATION OF CHAOTIC ISLANDS

We here report on numerical results to verify the conditions required to get chaotic dynamics in a system of three oscullators described by the Hamiltonian

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2} + \omega_1 q_i^2 + \frac{1}{4} (q_{i+1} - q_i)^4 ,$$  \hspace{1cm} (S2)

where we consider periodic boundary conditions $q_4 = q_1$, $p_4 = p_1$. We consider initial conditions chosen from a Gibbs distribution at temperature $T$ and compute the time-dependent Lyapunov exponent defined as

$$\lambda(t) = \frac{1}{2t} \left< \ln \frac{\sum |\delta p_i(t)|^2}{\sum |\delta p_i(0)|^2} \right> ,$$  \hspace{1cm} (S3)

where $\delta p(0)$ is an infinitesimal initial perturbation, whose time evolution can be obtained from the linearized dynamics, and the average $\left< \cdots \right>$ is taken over thermal initial conditions.