QUASI-SQUARES OF PSEUDOCONTINUABLE FUNCTIONS

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Abstract. For an inner function \( \theta \) on the unit disk, let \( K^p_\theta := H^p \cap \theta H^p_0 \) be the associated star-invariant subspace of the Hardy space \( H^p \). While the squaring operation \( f \mapsto f^2 \) maps \( H^p \) into \( H^{p/2} \), one cannot expect the square \( f^2 \) of a function \( f \in K^p_\theta \) to lie in \( K^{p/2}_\theta \). (Suffice it to note that if \( f \) is a polynomial of degree \( n \), then \( f^2 \) has degree \( 2n \) rather than \( n \).) However, we come up with a certain “quasi-squaring” procedure that does not have this defect. As an application, we prove an extrapolation theorem for a class of sublinear operators acting on \( K^p_\theta \) spaces.

1. Introduction

Let \( \mathbb{T} \) stand for the circle \( \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \) and \( m \) for the normalized Lebesgue measure on \( \mathbb{T} \); thus, \( dm(\zeta) = |d\zeta|/(2\pi) \). The spaces \( L^p := L^p(\mathbb{T}, m) \) are then defined in the usual way and equipped with the standard norm \( \| \cdot \|_p \). Also, for a nonnegative integer \( n \), we let \( P_n \) denote the space of polynomials (in one complex variable) of degree at most \( n \).

Consider the following problem, stated somewhat vaguely for the time being: Given \( f \in P_n \) (with \( n \in \mathbb{N} \)), find a polynomial \( g \in P_n \) that mimics \( f^2 \) in the sense that \( |g| \) and \( |f|^2 \) have the same order of magnitude on \( \mathbb{T} \). The exact meaning of this has yet to be specified, but once that is done, we would want our “quasi-squaring” procedure (leading from \( f \) to \( g \)) to be fairly explicit and applicable to all \( f \in P_n \).

Since \( f^2 \in P_{2n} \), whereas \( g \) is required to be in \( P_n \), there are, of course, limits to what can be expected. In particular, if \( f \) has precisely \( n \) zeros on \( \mathbb{T} \), then no \( g \in P_n \) will satisfy \( |g| = |f|^2 \) on \( \mathbb{T} \); nor can we hope for a two-sided estimate of the form

\[
|f(\zeta)|^2 \leq |g(\zeta)| \leq C|f(\zeta)|^2, \quad \zeta \in \mathbb{T},
\]

holding with a constant \( C > 0 \), because the right-hand inequality alone would force \( g \) to be null. It turns out, however, that a slightly weaker property can be achieved. To arrive at it, we replace the problematic inequality \( |g| \leq C|f|^2 \) in (1.1) by its \( L^p \) (or rather \( L^{p/2} \)) version

\[
\|g\|_{p/2} \leq C\|f\|_p^2,
\]

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while leaving the other (pointwise) inequality
\[(1.3) \quad |g(\zeta)| \geq |f(\zeta)|^2\]
as it stands. Here, the admissible values of \(p\) are those with \(2 < p < \infty\), as we shall see, and the constant \(C = C(p)\) in (1.2) is allowed to depend only on \(p\).

There is no chance (1.2) could hold with \(C = 1\), as long as (1.3) is also to be fulfilled, since otherwise it would follow that \(|g| = |f|^2\) on \(\mathbb{T}\), a condition we have already discarded as unrealistic. At the same time, our results imply the amusing fact that, for \(p\) as above, the two inequalities become compatible when \(C = C(p)\) is suitably large.

In fact, the polynomial case hitherto discussed—and intended as a prologue—is but a toy version of the more general situation to be dealt with, the context being that of pseudocontinuable functions in Hardy spaces. Recall, to begin with, that the Hardy space \(H^p\) (with \(0 < p < \infty\)) consists of all holomorphic functions \(f\) on the disk \(D := \{z \in \mathbb{C} : |z| < 1\}\) that satisfy
\[
\sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty,
\]

while \(H^\infty\) denotes the algebra of bounded holomorphic functions on \(D\). As usual, \(H^p\) functions are identified with their boundary traces on \(\mathbb{T}\), defined in the sense of nontangential convergence almost everywhere (cf. [14, Chapter II]), and \(H^p\) is viewed as a subspace of \(L^p\). Recall also that a function \(\theta \in H^\infty\) is said to be inner if \(|\theta| = 1\) a.e. on \(\mathbb{T}\). We use the notation \(\mathcal{I}\) for the set of nonconstant inner functions, and \(\mathcal{I}_0\) for the set of inner functions \(\theta\) with \(\theta(0) = 0\).

Now, for \(\theta \in \mathcal{I}\), the associated star-invariant (or model) subspace \(K^p_{\theta}\) is defined by
\[(1.4) \quad K^p_{\theta} := H^p \cap \overline{\theta H^p_0}, \quad 1 \leq p \leq \infty,
\]
where \(H^p_0 := zH^p = \{f \in H^p : f(0) = 0\}\) and the bar denotes complex conjugation. Equivalently, we have
\[K^p_{\theta} = \{f \in H^p : \overline{\theta f} \in H^p\},\]
with the understanding that the product \(\overline{\theta f}\) (and each of the three factors involved) is regarded as living a.e. on \(\mathbb{T}\). It is well known that each \(K^p_{\theta}\) is invariant under the backward shift operator
\[\mathfrak{B} : f \mapsto \frac{f - f(0)}{z},\]
and conversely, every closed and nontrivial \(\mathfrak{B}\)-invariant subspace in \(H^p\), with \(1 \leq p < \infty\), is of the form \(K^p_{\theta}\) for some \(\theta \in \mathcal{I}\); see, e.g., [3][6]. The functions belonging to some such subspace—i.e., those that are noncyclic for the backward shift—are known as pseudocontinuable functions, since they are indeed characterized by a certain “pseudocontinuation” property. Namely, the function in question must agree a.e. on \(\mathbb{T}\) with the boundary values of some meromorphic function of bounded characteristic in \(\mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})\); see [3] for details.

It is in the \(K^p_{\theta}\) setting that we actually consider our quasi-squaring problem. (The polynomial version, as discussed previously, is recovered by taking \(\theta(z) = z^{n+1}\), in
which case $K^p_\theta$ reduces to $P_n$.) Observe, first of all, that if $\theta \in \mathcal{I}$ and $p \geq 2$, then the square $f^2$ of any function $f \in K^p_\theta$ will belong to $K^{p/2}_\theta$; roughly speaking, this is the best we can say of it. Passing from $p$ to $p/2$ does not really bother us—after all, this is what happens when squaring an $H^p$ function—but passing from $\theta$ to $\theta^2$ is what we want to avoid. Rather, we insist on keeping $\theta$ intact. Accordingly, given an $f \in K^p_\theta$, we seek to replace the true square, $f^2$, by a suitable “quasi-square” $g \in K^{p/2}_\theta$ for which $|g|$ is approximately of the same size as $|f|^2$ on $T$. Precisely speaking, the properties our quasi-square should enjoy are (1.2), with a certain $C = C(p)$, and (1.3); the latter should hold for almost all $\zeta \in \mathbb{T}$. We then prove that such a quasi-square can indeed be constructed, provided that $2 < p < \infty$.

In order to describe our findings more accurately, we now introduce a bit of terminology. Suppose $E_1$ and $E_2$ are two vector spaces consisting of functions that are defined—possibly a.e. with respect to a certain measure—on a set $\mathcal{X}$. A (nonlinear) operator $S : E_1 \to E_2$ will be called superquadratic if it has the properties that
\begin{equation}
|S(\lambda f)| = |\lambda|^2 |Sf|
\end{equation}
and
\begin{equation}
|Sf| \geq |f|^2
\end{equation}
whenever $f \in E_1$ and $\lambda \in \mathbb{C}$; the two conditions should hold everywhere—or almost everywhere—on $\mathcal{X}$, depending on the context.

In what follows, the role of $\mathcal{X}$ will be played by either $\mathbb{D}$ or $\mathbb{T}$, with the “everywhere” or “almost everywhere” interpretation, respectively. In fact, for the spaces considered, either choice of $\mathcal{X}$ will do.

Our main result admits a neater formulation when the underlying class of inner functions is taken to be $\mathcal{I}_0$, and we now state the restricted version that arises. Namely, there exists a superquadratic map $S$ from $H^2$ to the weak Hardy space $H^1_w$ (a space slightly larger than $H^1$, to be defined in Section 3 below) for which the following holds whenever $2 < p < \infty$:
\begin{equation*}
S(K^p_\theta) \subset K^{p/2}_\theta \text{ for each } \theta \in \mathcal{I}_0, \quad S(H^p) \subset H^{p/2},
\end{equation*}
and
\begin{equation*}
\|Sf\|_{p/2} \leq B_p \|f\|_p^2 \text{ for all } f \in H^p,
\end{equation*}
where $B_p$ is a certain (explicit) constant depending only on $p$. In particular, if $f \in K^p_\theta$ with $2 < p < \infty$ and $\theta \in \mathcal{I}_0$, then the function $g := Sf$ is eligible as a quasi-square for $f$, since it belongs to $K^{p/2}_\theta$ and has the required properties (1.2) and (1.3).

In addition, our construction will ensure that the image $Sf$ of every $f \in H^2 \setminus \{0\}$ is an outer function. (By definition, a zero-free holomorphic function $F$ on $\mathbb{D}$ is outer if $\log |F|$ agrees with the harmonic extension of an $L^1$ function on $\mathbb{T}$.) Consequently, in our case it makes no difference whether (1.6) is supposed to hold a.e. on $\mathbb{T}$ or everywhere on $\mathbb{D}$, the two conditions being equivalent.

Our method relies on a preliminary result that describes the real parts of functions in $K^p_\theta$. The description, which may be of independent interest, is given in Section 2 along with another auxiliary fact, to be leaned upon later. In Section 3, we state and prove our main theorem in its entirety. This includes a more complete version
of the above statement involving the class $I_0$, plus its counterpart dealing with the case of a generic $\theta \in I$. In Section 4, we discuss the endpoint values of $p$ in our quasi-squaring theorem, the emphasis being on the case $p = 2$, where everything breaks down dramatically. In Section 5, we apply our quasi-squaring technique to derive an amusing extrapolation theorem for a class of sublinear operators acting on $K_\theta^p$ spaces. To be more precise, we prove that if $1 < p_0 < \infty$ and $1 \leq q_0 \leq \infty$, and if $T$ is an operator satisfying certain hypotheses that maps $K_\theta^{p_0}$ boundedly into $L^{q_0}(\mu)$ for some measure $\mu$, then $T$ is also bounded as an operator from $L^{q_0}(\mu)$ to $L^{q_0}(\mu)$, provided that the exponents involved are related by $p/q = p_0/q_0$ and $p_0 < p < \infty$.

Finally, this last theorem is discussed at some length in Section 6. In particular, we point out that our result extends a theorem of Aleksandrov from [1], where a similar extrapolation property was established in the context of Carleson-type measures for $K_\theta^p$.

We conclude this introduction by looking back at the case of $P_n$ and asking a question that puzzles us: Does our quasi-squaring construction carry over, in some form or other, to polynomials—or special classes of polynomials—in several real or complex variables?

2. Preliminaries

Given a function class $X$ on $T$, we write $\text{Re }X$ for the set of those (real-valued) functions $u$ on $T$ that have the form $u = \text{Re }f$ for some $f \in X$.

Our current purpose is to characterize the functions $u$ in $\text{Re }K_\theta^p$ with $1 \leq p \leq \infty$. An obvious necessary condition to be imposed is that $u \in \text{Re }H^p$. When $1 < p < \infty$, the latter simply means that $u \in L^p_\mathbb{R}$ (where $L^p_\mathbb{R}$ is the set of real-valued functions in $L^p$), the equivalence between the two conditions being due to the M. Riesz theorem; see [13, Chapter III]. For $p = 1$, the assumption that $u \in \text{Re }H^1$ can be rephrased by saying that $u$ and its nontangential maximal function

$$u^*(\zeta) := \sup\{|Pu(z)| : z \in \mathbb{D}, |z - \zeta| \leq 2(1 - |z|)\}, \quad \zeta \in T$$

(where $Pu$ is the Poisson integral of $u$), are both in $L^1_\mathbb{R}$; the underlying result can also be found in [13, Chapter III].

**Theorem 2.1.** Let $u \in \text{Re }H^p$, where $1 \leq p \leq \infty$.

(a) Suppose that $\theta \in I_0$. Then $u \in \text{Re }K_\theta^p$ if and only if $\text{Re }u\theta \in H^p$.

(b) Suppose that $\theta \in I \setminus I_0$. Then $u \in \text{Re }K_\theta^p$ if and only if the following two conditions hold:

(2.1) $u\theta \in H^p$ and $\int_T u \left(\frac{\theta}{\theta(0)} - \frac{1}{2}\right) dm \in i\mathbb{R}$.

**Proof.** (a) If $u = \text{Re }f$ for some $f \in K_\theta^p$, then $u = \frac{1}{2}(f + \overline{f})$, whence

(2.2) $\overline{u}\theta = \frac{1}{2}f\theta + \frac{1}{2}\overline{f}\theta$.

The first term on the right is in $H^p$ (because $\overline{\theta} = \theta/z \in H^\infty$), and so is the second (because $f \in K_\theta^p$). This shows that $\overline{u}\theta \in H^p$, proving the “only if” part.
Conversely, assume that $zu\theta \in H^p$. Let $f \in H^p$ be the function satisfying $\text{Re} \, f = u$ (a.e. on $\mathbb{T}$) and $\text{Im} \, f(0) = 0$. Then (2.2) is again valid, or equivalently,
\begin{equation}
\overline{zf\theta} = 2\overline{zu\theta} - \overline{zf\theta}.
\end{equation}
(2.3)

Our current assumption on $u$, coupled with the fact that $\overline{zf\theta} \in H^p$, allows us to infer from (2.3) that $\overline{zf\theta} \in H^p$. This means that $f \in K^p_\theta$, so the “if” part is now established as well.

(b) Suppose that $u = \text{Re} \, f$ for some $f \in K^p_\theta$, and let $v := \text{Im} \, f$. (The functions $u$ and $v$, defined initially a.e. on $\mathbb{T}$, will be identified with their harmonic extensions into $\mathbb{D}$.) As before, we have (2.2) and hence also (2.3); yet another way of rewriting this identity is
\begin{equation}
u\theta = \frac{1}{2} f\theta + \frac{1}{2} \overline{f}\theta.
\end{equation}
Here, each of the two terms on the right-hand side is in $H^p$, and therefore $u\theta \in H^p$.

In addition, we use the fact that $\overline{zf\theta} \in H^p$ in conjunction with (2.3) to deduce that the function
\begin{equation}
g := 2u\theta - f\theta
\end{equation}
is in $zH^p (= H^p_0)$. In particular,
\begin{equation}
\int_{\mathbb{T}} g \, dm = 0.
\end{equation}
(2.5)

Now, because
\begin{equation}
\int_{\mathbb{T}} f\theta \, dm = f(0)\theta(0) = [u(0) + iv(0)] \cdot \theta(0),
\end{equation}
we may further rephrase (2.5) in the form
\begin{equation}
\frac{2}{\theta(0)} \int_{\mathbb{T}} u\theta \, dm = u(0) + iv(0).
\end{equation}
(2.6)

The quantity
\begin{equation}
\int_{\mathbb{T}} u \left( \frac{\theta}{\theta(0)} - \frac{1}{2} \right) \, dm
\end{equation}
is thus equal to the (purely imaginary) number $iv(0)/2$. The necessity of (2.1) is thereby verified.

Conversely, let $u \in \text{Re} \, H^p$ be a function satisfying (2.1). The value of the integral (2.7) being purely imaginary, say $ic$ for some $c \in \mathbb{R}$, we can find a function $f = u + iv \in H^p$ whose imaginary part, $v$, satisfies $v(0) = 2c$. This done, we have (2.6). Equivalently, the function $g$, defined by (2.4) as before, obeys (2.5). We also know that $g \in H^p$, since $u\theta$ and $f\theta$ are both in $H^p$, and together with (2.5) this means that $g$ actually belongs to $H^p_0$. Finally, we invoke the identity
\begin{equation}
\overline{zf\theta} = \overline{g}
\end{equation}
(which coincides with (2.3) and holds whenever $u = \text{Re} \, f$) to conclude that $\overline{zf\theta} \in H^p$ and consequently $f \in K^p_\theta$. □
Remarks. (1) When proving the “if” part in either (a) or (b), we had to produce a harmonic conjugate (say, $v$) of $u$ for which $u + iv \in K^p_\theta$. In (a), the normalization $v(0) = 0$ was used, but any other choice of $v$ (i.e., of the corresponding constant term) would also be fine; indeed, if $\theta \in \mathcal{I}_0$, then $K^p_\theta$ contains the constants. In (b), by contrast, the right choice is unique.

(2) In [6], we considered the natural analogues of $K^p_\theta$ spaces for the upper half-plane in place of the disk. In particular, the real parts of the functions that arise were characterized (on $\mathbb{R}$) by a condition similar to that in (a) above. However, the case of $\mathbb{D}$ turns out to be more sophisticated due to the privileged role of the point 0, and this accounts for the dichotomy that manifests itself in Theorem 2.1.

The next fact is well known and easy to prove; see, e.g., [11, Lemma 1]. When stating it, we use the standard notation $(H^\infty)^{-1}$ for the set $\{f \in H^\infty : 1/f \in H^\infty\}$.

Lemma 2.2. Suppose $\theta$ and $\varphi$ are two inner functions satisfying $\theta/\varphi = g/\overline{g}$ for some $g \in (H^\infty)^{-1}$. Then

$$K^p_\theta = gK^p_\varphi \left( = \{gh : h \in K^p_\varphi\} \right), \quad 1 \leq p \leq \infty.$$ 

In particular, this lemma applies (and will be applied) when $\varphi$ is a “Frostman shift” of $\theta$, i.e., has the form

$$(2.8) \quad \varphi = \frac{\theta - w}{1 - \overline{w}\theta}$$

for some $w \in \mathbb{D}$. In this case, we have $g = 1 - \overline{w}\theta$.

3. Main result

Given a function $f \in L^1$, we write $\mathcal{H}f$ for its (harmonic) conjugate, so that

$$(\mathcal{H}f)(\zeta) = \text{p.v.} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\zeta e^{-it}) \cot \frac{t}{2} \, dt$$

for almost all $\zeta \in \mathbb{T}$. Thus, dealing with a function $u \in L^1_{\mathbb{R}}$ (and using the same letter for its Poisson extension into $\mathbb{D}$), we may view $\mathcal{H}u$ as the boundary trace of the real harmonic function $v$ on $\mathbb{D}$ that vanishes at 0 and makes $u + iv$ holomorphic.

It is well known (see [13, Chapter III]) that the harmonic conjugation operator $\mathcal{H}$ maps $L^1$ into $L^1_{\mathbb{R}}$, the weak $L^1$-space, defined as the set of measurable functions $g$ on $\mathbb{T}$ with

$$\sup_{\lambda > 0} \lambda m \left( \{\zeta \in \mathbb{T} : |g(\zeta)| > \lambda\} \right) < \infty.$$ 

Another classical theorem (due to M. Riesz) asserts that $\mathcal{H}$ is bounded on $L^p$, and hence also on $L^p_{\mathbb{R}}$, when $1 < p < \infty$. Moreover, its norm has been computed. In fact, a result of Pichorides tells us that the quantity

$$A_p := \sup \{ \|\mathcal{H}u\|_p : u \in L^p_{\mathbb{R}}, \|u\|_p \leq 1\}$$

equals $\tan \frac{\pi}{2p}$ if $1 < p \leq 2$ and $\cot \frac{\pi}{2p}$ if $2 < p < \infty$; see [17, Theorem 3.7]. In what follows, we also need the constants

$$B_p := 1 + A_{p/2}, \quad 2 < p < \infty.$$
Finally, we recall that the Smirnov class $N^+$ is the set of all ratios $\varphi/\psi$, where $\varphi$ runs through $H^\infty$ and $\psi$ through the outer functions in $H^\infty$ (see [14, Chapter II]); and we define the weak Hardy space $H^1_w$ to be $N^+ \cap L^1_w$.

Our main result reads as follows.

**Theorem 3.1.** (A) There is a superquadratic map $S : H^2 \to H^1_w$ such that the image $Sf$ of every $f \in H^2 \setminus \{0\}$ is an outer function, and the following holds true whenever $2 < p < \infty$:

\begin{equation}
S(K_\theta^p) \subset K_\theta^{p/2} \quad \text{for each } \theta \in I_0, \quad S(H^p) \subset H^{p/2},
\end{equation}

and

\begin{equation}
\|Sf\|_{p/2} \leq B_p \|f\|^2_p
\end{equation}

for all $f \in H^p$.

(B) Given $\theta \in I$, there exists a superquadratic map $S_\theta : H^2 \to H^1_w$ such that the image $S_\theta f$ of every $f \in H^2 \setminus \{0\}$ is an outer function, and the following holds true whenever $2 < p < \infty$:

\begin{equation}
S_\theta(K_\theta^p) \subset K_\theta^{p/2}, \quad S_\theta(H^p) \subset H^{p/2},
\end{equation}

and

\begin{equation}
\|S_\theta f\|_{p/2} \leq B_p \left( \frac{1 + |\theta(0)|}{1 - |\theta(0)|} \right)^2 \|f\|^2_p
\end{equation}

for all $f \in H^p$.

We remark that, while (3.4) obviously reduces to (3.2) when $\theta \in I_0$, part (A) of the theorem is not really a special case of (B). The reason is that the “quasi-squaring” operator $S$ coming from (A) does not depend on $\theta$, whereas its counterpart $S_\theta$ from (B) does. At the same time, the operator $S_\theta$ produced by our construction does reduce to $S$ when $\theta \in I_0$.

**Proof of Theorem 3.1.** (A) Given a function $f \in H^2$, we define

\[ (Sf)(z) := \int_T (\frac{\zeta + z}{\zeta - z}) |f(\zeta)|^2 \, dm(\zeta), \quad z \in \mathbb{D}. \]

In terms of the boundary values, we have

\[ Sf = u + iv \quad \text{a.e. on } T, \]

where

\begin{equation}
(3.5) \quad u := |f|^2 |T|
\end{equation}

and $v := Hu$. Because $u \in L^1$, it follows that $v \in L^1_w$ and hence $Sf \in H^1_w$.

The map $S : H^2 \to H^1_w$ that arises is sure to obey (1.5) and (1.6) (these hold a.e. on $T$, as well as everywhere on $\mathbb{D}$), so $S$ is superquadratic. In particular, the disk version of (1.6) is verified by noting that

\[ |(Sf)(z)| \geq \text{Re} \, (Sf)(z) = (P \, u)(z) \geq |f(z)|^2, \quad z \in \mathbb{D}, \]

where $P$ stands for the Poisson integral operator. In addition, the fact that $\text{Re} \, (Sf) > 0$ on the disk (unless $f$ is null) implies that $Sf$ is outer.
Now suppose that $\theta \in \mathcal{I}_0$ and $f \in K^p_{\theta}$, where $2 < p < \infty$. The corresponding function $u$, given by (3.5), will then satisfy

$$u \theta = f : \overline{\Omega} \theta \in H^{p/2},$$

since $f$ and $\overline{\Omega} \theta$ are both in $H^p$. By virtue of Theorem 2.1 part (a) (see also Remark (1) following that theorem’s proof), we readily deduce from (3.6) that $u \in \text{Re} K^p_{\theta}$ and therefore $Sf \in K^p_{\theta}$. Thus we arrive at the first inclusion in (3.1).

Finally, assuming that $f$ is merely in $H^p$ (with $2 < p < \infty$), we use the above-mentioned properties of the harmonic conjugation operator to obtain

$$\|Sf\|_{p/2} \leq \|u\|_{p/2} + \|v\|_{p/2} \leq (1 + A_{p/2}) \|u\|_{p/2} = B_p \|f\|^2.$$ 

This proves (3.2) and the second inclusion in (3.1).

(B) Given $\theta \in \mathcal{I}$, we write $w := \theta(0)$ and consider the function $g_\theta := 1 - \overline{w}\theta$. Note, in particular, that $g_\theta \in (H^\infty)^{-1}$. Moreover,

$$1 - |w| \leq |g_\theta| \leq 1 + |w|$$

on $\mathbb{D}$. Next, we define the map $S_\theta$ by putting

$$S_\theta f := (1 + |w|) g_\theta S(f/g_\theta), \quad f \in H^2,$$

where $S$ is the superquadratic operator coming from part (A) above. The facts that $S_\theta$ is superquadratic and maps $H^2$ into $H^2_{w}$ are easily deduced from the corresponding properties of $S$, coupled with (3.7). For example, to check that $|S_\theta f| \geq |f|^2$ (on $\mathbb{D}$) for each $f \in H^2$, one uses the estimate $|S(f/g_\theta)| \geq |f/g_\theta|^2$ and combines it with the right-hand inequality from (3.7).

We also have to verify that $S_\theta f$ is an outer function whenever $f \in H^2 \setminus \{0\}$. This is indeed true, because the functions $g_\theta$ and $S(f/g_\theta)$ are both outer, and so is their product.

Now let $2 < p < \infty$. To prove the first inclusion in (3.3), consider the inner function $\varphi$ given by (2.8) (with the current value of $w$) and note that $\varphi \in \mathcal{I}_0$. Given $f \in K^p_{\varphi}$, we may then invoke Lemma 2.2 (and the remark following it) to infer that $f/g_\theta \in K^p_{\varphi}$. Using part (A) above with $\varphi$ in place of $\theta$, we further deduce that $S(f/g_\theta) \in K^p_{\varphi}$, and another application of Lemma 2.2 ensures that $g_\theta S(f/g_\theta)$ is in $K^p_{\theta}$. This last function being a constant multiple of $S_\theta f$, we now see that $S_\theta f \in K^p_{\theta}$, and “half” of (3.3) is thereby established.

Finally, to prove the remaining part of (3.3) along with the norm estimate (3.4), we take an arbitrary $f \in H^p$ and proceed as follows:

$$\|S_\theta f\|_{p/2} \leq (1 + |w|) \|g_\theta\|_{\infty} \|S(f/g_\theta)\|_{p/2} \leq (1 + |w|)^2 B_p \|f/g_\theta\|^2 \leq B_p \left( \frac{1 + |w|}{1 - |w|} \right)^2 \|f\|^2.$$ 

Here, we have combined (3.7) and (3.2), the latter being applied with $f/g_\theta$ in place of $f$. The proof is complete. \qed
We conclude with a brief remark concerning the relation (3.6), which (in conjunction with Theorem 2.1) played a key role in the above proof. Namely, the condition \( \pi u_\theta \in H^{p/2} \) actually characterizes the nonnegative functions \( u \) on \( T \) that are writable as \( |f|^2 \) for some \( f \in K^p_\theta \); see [9] or [11, Lemma 5]. In the polynomial case (i.e., for \( \theta = z^{n+1} \)), this fact reduces to the classical Fejér–Riesz representation theorem for nonnegative trigonometric polynomials; see, e.g., [18, p. 26].

4. The endpoint cases

In light of the preceding result, which deals with the range \( 2 < p < \infty \), one may be curious about the endpoint cases \( p = 2 \) and \( p = \infty \). The operator \( S \) (or \( S_\theta \)) constructed in the proof admits no nice extension to the endpoints, but it is conceivable that some other map might do the job. However, we could scarcely expect to find a single superquadratic operator that obeys the required norm estimates for the whole extended range of \( p \)'s—that would probably be too much to hope for. Instead, we consider the two endpoints separately, asking in each case if there exists a superquadratic map \( S_\theta \) from \( K^p_\theta \) to \( K^{p/2}_\theta \) that satisfies the appropriate endpoint version of (3.2) and/or (3.4). The exponents in question are thus \( p = 2 \) and \( p = \infty \); our superquadratic operators are \( a \text{ priori} \) allowed to depend on \( \theta \) (even when \( \theta \in I_0 \)), but the constants replacing the \( B_p \)'s should be absolute.

The case of \( p = \infty \) is actually trivial, since the map \( S : H^\infty \to H^\infty \) defined by

\[
Sf = \|f\|\infty f
\]

is superquadratic, leaves \( K^\infty_\theta \) invariant (for each \( \theta \in I \)), and satisfies \( \|Sf\|\infty = \|f\|^2\infty \). In particular, (3.2) holds with \( p = \infty \) if we put \( B_\infty = 1 \). By contrast, things become really bad at the other extreme.

**Theorem 4.1.** Suppose that to each \( \theta \in I_0 \) there corresponds a superquadratic map \( S_\theta : K^2_\theta \to K^1_\theta \). Then

\[
\sup_{\theta \in I_0} \sup \left\{ \|S_\theta f\|_1 / \|f\|_2^2 : f \in K^2_\theta \setminus \{0\} \right\} = \infty.
\]

**Proof.** If (4.1) were false, there would be an absolute constant \( C > 0 \) such that

\[
\|S_\theta f\|_1 \leq C \|f\|_2^2
\]

whenever \( \theta \in I_0 \) and \( f \in K^2_\theta \). Now let \( a \in \mathbb{D} \) be a point with \( |a| \geq \frac{1}{2} \), and put

\[
\theta_a(z) := z \frac{z - a}{1 - az},
\]

so that \( \theta_a \in I_0 \). Note also that the function

\[
f_a(z) := \frac{1}{1 - az}
\]

is in \( K^2_{\theta_a} \). In fact, this last subspace coincides with \( K^1_{\theta_a} \) and is two-dimensional; it is spanned by \( f_a \) and the constant function 1. Thus, writing \( h_a := S_{\theta_a} f_a \), we see that

\[
h_a(z) = \lambda_a + \frac{\mu_a}{1 - az}
\]

with certain coefficients \( \lambda_a, \mu_a \in \mathbb{C} \).
An application of (4.2) with $\theta = \theta_a$ and $f = f_a$ now yields
\begin{equation}
\|h_a\|_1 \leq C(1 - |a|^2)^{-1},
\end{equation}
and we are going to derive further information by estimating the left-hand side, $\|h_a\|_1$, from below. To this end, we invoke Hardy’s inequality
\begin{equation}
\|h\|_1 \geq \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{|\hat{h}(n)|}{n+1},
\end{equation}
valid for any $h \in H^1$ (see [14, p. 89]); here $\hat{h}(n)$ is the $n$th Taylor coefficient of $h$.
When $h = h_a$, (4.5) tells us that
\begin{equation*}
\|h_a\|_1 \geq \frac{1}{\pi} \left| \lambda_a + \mu_a \right| + \frac{1}{2\pi} |\mu_a| \log \frac{1}{1 - |a|},
\end{equation*}
Combining this with (4.4), we find that
\begin{equation}
|\lambda_a + \mu_a| \leq \frac{M}{1 - |a|}
\end{equation}
and
\begin{equation}
|\mu_a| \log \frac{1}{1 - |a|} \leq \frac{M}{1 - |a|}
\end{equation}
with an absolute constant $M > 0$; in fact, $M = 2\pi C$ would do.

On the other hand, because $S_{\theta_a}$ is superquadratic, we have $|h_a| \geq |f_a|^2$ on $T$, and so
\begin{equation}
\|h_a\|_2^2 \geq \|f_a\|_4^4.
\end{equation}
Parseval’s identity yields
\begin{equation*}
\|h_a\|_2^2 = |\lambda_a + \mu_a|^2 + |\mu_a|^2(|a|^2 + |a|^4 + \ldots)
\end{equation*}
\begin{equation*}
= |\lambda_a + \mu_a|^2 + \frac{|\mu_a|^2 |a|^2}{1 - |a|^2} \leq |\lambda_a + \mu_a|^2 + \frac{|\mu_a|^2}{1 - |a|},
\end{equation*}
while a simple computation reveals that
\begin{equation*}
\|f_a\|_4^4 \geq \frac{c}{(1 - |a|)^3}
\end{equation*}
with an absolute constant $c > 0$. Taking these estimates into account, we go back to (4.8) to deduce that
\begin{equation}
|\lambda_a + \mu_a|^2 + \frac{|\mu_a|^2}{1 - |a|} \geq \frac{c}{(1 - |a|)^3}.
\end{equation}
At the same time, (4.7) implies that
\begin{equation*}
|\mu_a|^2 \leq \frac{M^2}{(1 - |a|)^2} \left( \log \frac{1}{1 - |a|} \right)^2 \leq \frac{c}{2(1 - |a|)^2},
\end{equation*}
whenever $|a|$ is close enough to 1. Together with (4.9), this means that for such $a$’s we have

$$|\lambda_a + \mu_a|^2 \geq \frac{c}{2(1 - |a|)^3},$$

or equivalently,

$$|\lambda_a + \mu_a| \geq \frac{c_0}{(1 - |a|)^{3/2}}$$

with $c_0 := \sqrt{c/2}$. However, for small values of $1 - |a|$, this last estimate is obviously incompatible with (4.6). The contradiction completes the proof.\qed

A glance at the proof reveals that the class $I_0$ in the theorem’s statement can be actually replaced by a tiny subset thereof, namely, by the family of two-factor Blaschke products of the form (4.3). We now supplement Theorem 4.1 (and its refined version just mentioned) with another result in the same vein, which is essentially a consequence of Aleksandrov’s work in [2]. This time we produce a single inner function $\theta$ for which the estimate

$$\|Sf\|_1 \leq C\|f\|_2^2, \quad f \in K_2^\theta,$$

fails whenever $S : K_2^\theta \to K_1^\theta$ is a superquadratic map and $C$ a positive constant.

**Theorem 4.2.** There exists an inner function $\theta$ such that every superquadratic operator $S : K_2^\theta \to K_1^\theta$ satisfies

$$\sup \left\{ \frac{\|Sf\|_1}{\|f\|_2^2} : f \in K_2^\theta \setminus \{0\} \right\} = \infty.$$  

**Proof.** Results of [2, Section 4] imply that there exists an inner function $\theta$ and a positive Borel measure $\mu$ on $\mathbb{D}$ with the following properties: $K_1^\theta$ embeds in $L^1(\mu)$ (meaning that

$$\int_{\mathbb{D}} |g| \, d\mu \leq B\|g\|_1, \quad g \in K_1^\theta,$$

with some constant $B > 0$ independent of $g$), but $K_2^\theta$ does not embed in $L^2(\mu)$. Now, if for that $\theta$ we could find a superquadratic operator $S : K_2^\theta \to K_1^\theta$ satisfying (4.10) with some fixed $C > 0$, then it would follow that

$$\int_{\mathbb{D}} |f|^2 \, d\mu \leq \int_{\mathbb{D}} |Sf| \, d\mu \leq B\|Sf\|_1 \leq BC\|f\|_2^2$$

for each $f \in K_2^\theta$, leading to a contradiction. \qed

5. Extrapolation theorem: statement and proof

In this section, we apply our main result (namely, Theorem 3.1 above) to deduce an extrapolation theorem for a class of sublinear operators acting on $K_p^\theta$ spaces.

Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are two function spaces. More precisely, it will be assumed for $j = 1, 2$ that $\mathcal{E}_j$ is a vector space consisting of complex-valued functions that live on a certain set $X_j$. Recall that an operator $T : \mathcal{E}_1 \to \mathcal{E}_2$ is said to be sublinear if it satisfies

$$|T(f + g)| \leq |Tf| + |Tg|$$

(5.1)
and

\[(5.2) \quad |T(\lambda f)| = |\lambda| |T f| \]

whenever \( f, g \in \mathcal{E}_1 \) and \( \lambda \in \mathbb{C} \).

In the same context, we call an operator \( T : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \) **solid** if it has the following properties: first, there exists a constant \( \gamma > 0 \) such that

\[(5.3) \quad |T f|^2 \leq \gamma |T(f^2)| \]

for every \( f \in \mathcal{E}_1 \) satisfying \( f^2 \in \mathcal{E}_1 \), and secondly,

\[(5.4) \quad |T F| \leq |T G| \]

whenever \( F, G \in \mathcal{E}_1 \) are functions with \( |F| \leq |G| \) on \( X_1 \).

It is understood that conditions (5.1)–(5.4) above hold pointwise on \( X_2 \), either everywhere or almost everywhere (in the appropriate sense), depending on the context.

The statement of our extrapolation theorem below involves a general measure space \((X, \mathfrak{A}, \mu)\), where the three symbols have the usual meaning. We write \( L^p(\mu) \) for \( L^p(X, \mathfrak{A}, \mu) \); in particular, \( L^0(\mu) \) stands for the space of \( \mathfrak{A} \)-measurable functions on \( X \). The notation \( L^p \) (without specifying the measure) is, of course, retained for the case of \( m \), the normalized Lebesgue measure on \( \mathbb{T} \).

**Theorem 5.1.** Let \( 1 < \sigma < \infty \) and \( 1 \leq \tau \leq \infty \). Given an inner function \( \theta \) and a measure space \((X, \mathfrak{A}, \mu)\), suppose that \( T : K^\sigma_{\theta^2} \rightarrow L^0(\mu) \) is a solid sublinear operator. Assume also that \( T \) maps \( K^\sigma_{\theta^2} \) boundedly into \( L^\tau(\mu) \). Then \( T \) maps \( K^p_{\theta^2} \) boundedly into \( L^q(\mu) \) whenever \( \sigma < p < \infty \) and \( p/q = \sigma/\tau \).

**Proof.** The case where \( \tau = \infty \) is trivial, since the only possible value of \( q \) is then \( \infty \), and \( K^p_{\theta^2} \subset K^\sigma_{\theta^2} \) for \( p > \sigma \).

To deal with the case \( 1 \leq \tau < \infty \), we begin by showing that \( T \) acts boundedly from \( K^{2\sigma}_{\theta^2} \) to \( L^{2\tau}(\mu) \). Let \( S_\theta \) be the superquadratic map from Theorem 3.1. Given \( f \in K^{2\sigma}_{\theta^2} \), we have then \( S_\theta f \in K^\sigma_{\theta^2} \) and

\[(5.5) \quad \|S_\theta f\|_\sigma \leq C\|f\|_{2\sigma}^2, \]

where

\[ C = C(\sigma, \theta) := B_{2\sigma} \left( \frac{1 + |\theta(0)|}{1 - |\theta(0)|} \right)^2 ; \]

also, \( |S_\theta f| \geq |f|^2 \) on \( \mathbb{D} \) (and \( m \)-almost everywhere on \( \mathbb{T} \)). For \( f \) as above, we may invoke (5.3) and then (5.4), with \( F = f^2 \) and \( G = S_\theta f \), to find that

\[(5.6) \quad |T f|^2 \leq \gamma |T(f^2)| \leq \gamma |T(S_\theta f)| \]

\( \mu \)-almost everywhere on \( X \). (To justify the former step, note that \( f^2 \) is in \( K^\sigma_{\theta^2} \) and hence in \( K^{1\sigma}_{\theta^2} \).)

Raising the resulting inequality from (5.6) to the power \( \tau \) and integrating, we get

\[(5.7) \quad \int_X |T f|^{2\tau} d\mu \leq \gamma^\tau \int_X |T(S_\theta f)|^\tau d\mu. \]
On the other hand, we know by assumption that
\[ \int_X |Tg|^\tau d\mu \leq M^\tau \|g\|_\sigma^\tau, \quad g \in K_\sigma^\theta, \]
with some fixed \( M > 0 \). Applying this to \( g = S_\theta f \) gives
\[ (5.8) \quad \int_X |T(S_\theta f)|^\tau d\mu \leq (M\gamma)^\tau \|S_\theta f\|_\sigma^\tau, \]
and we now combine (5.7) with (5.8) to obtain
\[ \int_X |Tf|^2 \tau d\mu \leq (CM\gamma)^\tau \|f\|_{2\sigma}^2, \]
proving our claim that the operator (5.9)
\[ T : K_\sigma^\theta \to L^q(\mu) \]
is bounded when \( p = 2\sigma \) and \( q = 2\tau \).

Iterating the above argument, we arrive at a similar boundedness result for the operator (5.9) whenever \( p = 2^n\sigma (=: p_n) \) and \( q = 2^n\tau (=: q_n) \) for some integer \( n \geq 0 \).

The remaining cases can now be proved by interpolation. Indeed, for \( 1 < p < \infty \), the operator \( P_\theta \) defined by
\[ P_\theta h := P_\theta h - \theta P_\theta(\overline{h}), \quad h \in L^p, \]
where \( P_\theta : L^p \to H^p \) is the Riesz projection (see [14] Chapter III), is bounded on \( L^p \). Moreover, \( P_\theta \) is a bounded projection from \( L^p \) onto \( K_\sigma^\theta \). Consequently, the (already established) boundedness property of the map (5.9) with \( p = p_n \) and \( q = q_n \) can be rephrased by saying that the sublinear operator \( TP_\theta : L^p \to L^q(\mu) \) is bounded for any such pair of exponents. The Riesz–Thorin convexity theorem, or rather its extension to sublinear operators due to Calderón and Zygmund (see [3] or [19]), now guarantees that \( TP_\theta \) maps \( L^p \) boundedly into \( L^q(\mu) \) whenever the point \((1/p, 1/q)\) in \( \mathbb{R}^2 \) belongs to the line segment that joins \((1/p_n, 1/q_n)\) to \((1/p_{n+1}, 1/q_{n+1})\), for some (any) nonnegative integer \( n \). In other words, \( TP_\theta \) is bounded as an operator from \( L^p \) to \( L^q(\mu) \) provided that the exponents involved satisfy \( \sigma < p < \infty \) and \( p/q = \sigma/\tau \). This, in turn, is equivalent to the desired conclusion. \( \square \)

6. Extrapolation theorem: discussion

In connection with our last theorem, a few comments and examples seem to be appropriate.

(1) First, we observe that Theorem 5.1 would break down if we removed the word “solid” from its formulation. An example can be furnished as follows. Assume that \( \theta \) has infinitely many zeros, say \( a_n \) \((n = 1, 2, \ldots)\), and take the (linear) differentiation operator \( f \mapsto f' \) as \( T \); finally, define the measure \( \mu \) by
\[ d\mu(z) = (1 - |z|) dA(z), \quad z \in \mathbb{D}, \]
where $A$ is area measure on $\mathbb{D}$. With this choice, the operator (5.9) becomes bounded for $p = q = 2$, but no such thing is true for $p = q = 3$. Indeed, on the one hand, the classical Littlewood–Paley inequality
\[
\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|) \, dA(z) \leq C \|f\|_2^2
\]
holds, with an absolute constant $C > 0$, for all $f \in H^2$ (see, e.g., [13] or [14]) and hence for all $f \in K_2^\theta$. On the other hand, let $f_n(z) := (1 - |a_n z|)^{-1}$ and note that $f_n \in K_3^\theta$; a straightforward computation then shows that the quantity
\[
\|f_n\|^{-3}_3 \int_{\mathbb{D}} |f_n'(z)|^3 (1 - |z|) \, dA(z)
\]
behaves as a constant times $(1 - |a_n|)^{-1}$ and therefore blows up as $n \to \infty$.

(2) To see an example where Theorem 5.1 does apply, suppose that $T$ is the identity (or inclusion) map, and $\mu$ a suitable measure on the closed disk. Precisely speaking, let $\mu$ be a finite Borel measure on $\mathbb{D} \cup \mathbb{T}$ such that the singular component of $\mu|_\tau$ assigns no mass to the set of boundary singularities for $\theta$. The values of $K_\theta^p$ functions are then well defined $\mu$-almost everywhere, and the boundedness issue for the operator (5.9) amounts to asking whether $K_\theta^p$ embeds (continuously) in $L^q(\mu)$. The problem of characterizing such measures $\mu$ for a given $\theta$ was posed, initially for $p = q = 2$, by Cohn [4] and has attracted quite a bit of attention. Among the many papers that treat it, chiefly in the “diagonal” case where $p = q$, we mention [1, 2, 8, 9, 10, 20] and [15, pp. 80–81]. See also [7, 12], where the off-diagonal case $p < q$ was discussed (for some special measures on $\mathbb{T}$) in connection with the multiplicative structure of holomorphic Lipschitz spaces.

The identity operator is obviously solid (in particular, (5.3) holds with $\gamma = 1$), so Theorem 5.1 is indeed applicable in this situation. Applying it with $\tau = \sigma$, we recover a result of Aleksandrov (namely, [1, Theorem 1.5]): If $1 < \sigma < p < \infty$ and if $K_\theta^\sigma$ embeds in $L^q(\mu)$, then $K_\theta^p$ embeds in $L^q(\mu)$. Furthermore, we know from [11] that the restriction $\sigma > 1$ in the preceding statement cannot be replaced with $\sigma \geq 1$. Consequently, our Theorem 5.1 would also become false if the endpoint $\sigma = 1$ were included.

(3) Finally, we remark that Theorem 5.1 applies to certain maximal operators $T$. To give an example, let us associate to each point $\zeta \in \mathbb{T}$ a set $\Omega_\zeta \subset \mathbb{D}$ and define
\[
(Tf)(\zeta) := \sup \{ |f(z)| : z \in \Omega_\zeta \}, \quad f \in K_\theta^1,
\]
so that $T$ is sublinear (but not linear) and solid. Setting $\mu = m$ or perhaps considering more general measures on $\mathbb{T}$, we may be curious about the boundedness properties of the operator (5.9) for various values of $p$ and $q$, as soon as the sets $\Omega_\zeta = \Omega_\zeta(\theta)$ are chosen appropriately. A good choice would be one where the $\Omega_\zeta$’s are reasonably nice, $\Omega_\zeta$ touches the circle at $\zeta$, and the order of contact is controlled in terms of the distance from $\zeta$ to the boundary singularities of $\theta$. 
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