Non-SUSY Gepner Models
with Vanishing Cosmological Constant

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Abstract

In this article we discuss a construction of non-SUSY type II string vacua with the vanishing cosmological constant at the one loop level based on the generic Gepner models for Calabi-Yau 3-folds. We make an orbifolding of the Gepner models by $\mathbb{Z}_2 \times \mathbb{Z}_4$, which asymmetrically acts with some discrete torsions incorporated. We demonstrate that the obtained type II string vacua indeed lead to the vanishing cosmological constant at the one loop, whereas any space-time supercharges cannot be constructed as long as assuming the chiral forms such as ‘$Q_\alpha^L \equiv \int dz J^L_\alpha(z)$’. We further discuss possible generalizations of the models described above.

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1 Introduction

String theories on the non-geometric backgrounds would lead to interesting aspects not realized in the geometric ones. It is worthy of special mention that the vanishing cosmological constant (at least at the level of one-loop) could be realized with no helps of unbroken SUSY in such non-geometric string vacua. The studies on non-SUSY string vacua with vanishing cosmological constant have been initiated by [1–3] based on some non-abelian orbifolds, followed by studies e.g. in [4–9]. More recently, several non-SUSY vacua with this property have been constructed as asymmetric orbifolds [10] by simpler cyclic groups in [11, 12].

However, as far as we know, these attempts have been limited to the toroidal models, which are realized as some asymmetric orbifolds of tori at particular points of the moduli space. Therefore, it is interesting to try to construct the non-toroidal string vacua possessing these properties.

Another motivation of this study is to search the non-SUSY heterotic string vacua with an exactly vanishing cosmological constant at the one-loop. Notice that, in the known models of heterotic string vacua, one can only gain the small cosmological constant exponentially suppressed with respect to some moduli (e.g. the radii of tori of compactifications) in the early study [4], and also in closely related works given e.g. in [13–18].

In this paper, we start with the Gepner constructions for Calabi-Yau 3-folds, which are generic enough, and make an attempt to construct the type II string vacua by considering some asymmetric orbifolds, in which we have a vanishing cosmological constant at the one loop while we cannot compose the space-time supercharges with a reasonable form. We believe this work to be the first attempt to construct the non-SUSY string vacua with the properties mentioned above based on the Gepner constructions. We are still limited to the type II cases, but our approach could be extended to some heterotic string compactifications. We would like to report on the non-SUSY heterotic string vacua with the exactly vanishing cosmological constant as a future study.

This paper is organized as follows: We start with a very brief review of Gepner constructions [19] in section 2, mainly aiming at the preparations of notations. We also yield the definitions of orbifold actions utilized in our construction of string vacua. In section 3, we shall demonstrate our main results. Namely, we propose particular non-SUSY type II string vacua based on the asymmetric orbifolds of generic Gepner constructions of Calabi-Yau 3-folds, and show that the obtained vacua indeed realize vanishing cosmological constants at the one-loop, although any space-time supercharges with reasonable forms cannot be gained. We first consider the simplest case of $\mathbb{Z}_2 \times \mathbb{Z}_4$-orbifold, and then present possible generalizations of it. In section 4, we address the comparisons of the present models with the toroidal ones given in [11, 12] as discussions.
2 Preliminaries

In this preliminary section we summarize the Gepner construction [19] and the orbifold actions that we will utilize in the next section.

2.1 Gepner Models for CY$_3$

Let us consider the generic Gepner construction [19] for CY$_3$, that is, the superconformal system defined by

$$[\mathcal{M}_{k_1} \otimes \cdots \otimes \mathcal{M}_{k_r}]_{\mathbb{Z}_N\text{-orbifold}}, \quad \sum_{i=1}^r \frac{k_i}{k_i + 2} = 3, \quad (2.1)$$

where $\mathcal{M}_k$ denotes the $\mathcal{N} = 2$ minimal model of level $k$ ($\hat{c} \equiv \frac{c}{3} = \frac{k}{k+2}$), and we set

$$N := \text{L.C.M.} \{k_i + 2 ; i = 1, \ldots, r\}. \quad (2.2)$$

In order to realize the modular invariance manifestly, we start with simple products of the characters of $\mathcal{N} = 2$ minimal model [20, 21] in the NS-sector;

$$F^{(\text{NS})}_I(\tau, z) := \prod_{i=1}^r \text{ch}^{(\text{NS})}_{\ell_i, m_i}(\tau, z), \quad (I \equiv \{\ell_i, m_i\}, \quad \ell_i + m_i \in 2\mathbb{Z}, \forall i), \quad (2.3)$$

as the fundamental building blocks$^1$, and construct the ones for other spin structures by making the half spectral flows $z \mapsto z + \frac{r}{2} \tau + \frac{s}{2}$ ($r, s \in \mathbb{Z}_2)$:

$$F^{(\text{NS})}_I(\tau, z) := F^{(\text{NS})}_I(\tau, z + \frac{1}{2}), \quad (2.4)$$

$$F^{(\text{R})}_I(\tau, z) := q^{\frac{\hat{c}}{24}} y^{\frac{\hat{c}}{24}} F^{(\text{NS})}_I(\tau, z + \frac{\tau}{2}), \quad (2.5)$$

$$F^{(\tilde{\text{R}})}_I(\tau, z) := q^{\frac{\hat{c}}{24}} y^{\frac{\hat{c}}{24}} F^{(\text{NS})}_I(\tau, z + \frac{\tau + 1}{2}), \quad (2.6)$$

where we set $\hat{c} = 3$. Notice that the label $I \equiv \{\ell_i, m_i\}$ of the building blocks (and the spectral flow orbits introduced below) expresses the quantum numbers for the NS-sector even for $F^{(\text{R})}_I$ and $F^{(\tilde{\text{R}})}_I$.

Furthermore, we have to make the chiral $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifolding by $g_L \equiv e^{2\pi i j_L^0}$ and $g_R \equiv e^{2\pi i j_R^0}$, where $j_L^0$ ($j_R^0$) expresses the total $\mathcal{N} = 2$ $U(1)$-current in the left (right) mover acting over $\otimes_i \mathcal{M}_{k_i}$. Recall that the zero-mode $j_L^0$ takes the eigen-values in $\frac{1}{N} \mathbb{Z}$ for the NS sector. The

$^1$We summarize the explicit character formulas as well as the convention of theta functions in appendix A. We set $q := e^{2\pi i \tau}$, $y := e^{2\pi i z}$ through this paper.
chiral $\mathbb{Z}_N$-orbifolding (in the left-mover) is represented in a way respecting the good modular properties by considering the ‘spectral flow orbits’ [22] defined as follows:

$$\mathcal{F}_I^{(NS)}(\tau, z) := \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} q^{\frac{1}{2} a^2} y^{\frac{1}{2} a} F_I^{(NS)}(\tau, z + a \tau + b), \quad (2.7)$$

$$\mathcal{F}_I^{(NS)}(\tau, z) := \mathcal{F}_I^{(NS)}(\tau, z + \frac{1}{2})$$

$$\equiv \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} (-1)^{\tilde{\alpha}} q^{\frac{1}{2} a^2} y^{\frac{1}{2} a} F_I^{(NS)}(\tau, z + a \tau + b), \quad (2.8)$$

$$\mathcal{F}_I^{(R)}(\tau, z) := q^{\frac{1}{2}} y^{\frac{1}{2}} \mathcal{F}_I^{(NS)}(\tau, z + \frac{\tau + 1}{2})$$

$$\equiv \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} (-1)^{\tilde{\alpha}+b} q^{\frac{1}{2} a^2} y^{\frac{1}{2} a} F_I^{(R)}(\tau, z + a \tau + b). \quad (2.9)$$

We also use the abbreviated notation; $\mathcal{F}_I^{(\sigma)}(\tau) \equiv \mathcal{F}_I^{(\sigma)}(\tau, 0)$. See Appendix B for the explicit forms of $\mathcal{F}_I^{(\sigma)}(\tau, z)$ written in terms of the $\mathcal{N} = 2$ minimal characters.

The modular invariant partition function (for the transverse part) that describes the SUSY vacuum $\mathbb{R}^{3,1} \times CY_3$ is now written as

$$Z_{SUSY}(\tau, \tilde{\tau}) = \left( \frac{1}{\sqrt{\tau_2 |\eta|^2}} \right)^2 \cdot \frac{1}{4N} \sum_{\sigma_L, \sigma_R} \epsilon(\sigma_L) \epsilon(\sigma_R) \left( \frac{\theta_{|\sigma_L|}}{\eta} \right) \left( \frac{\bar{\theta}_{|\sigma_R|}}{\bar{\eta}} \right)$$

$$\times \sum_{I_L, I_R} N_{I_L, I_R} \mathcal{F}_{I_L}^{(\sigma_L)}(\tau) \mathcal{F}_{I_R}^{(\sigma_R)}(\bar{\tau}). \quad (2.11)$$

We assume the modular invariant coefficient $N_{I_L, I_R}$ to be diagonal through this paper:

$$N_{I_L, I_R} = \prod_{i=1}^r \frac{1}{2} \delta_{i, I_L} \delta_{i, I_R} \delta_{m_i, I_L} \delta_{m_i, I_R}, \quad (I_L \equiv \{(\ell_{i,L}, m_{i,L})\}, \quad I_R \equiv \{(\ell_{i,R}, m_{i,R})\}). \quad (2.12)$$

Here we set $\epsilon(\text{NS}) = -\epsilon(\text{NS}) = -\epsilon(\text{R}) = 1$ and $\theta_{|\text{NS}|} \equiv \theta_3(\tau, 0), \theta_{|\text{NS}|} \equiv \theta_4(\tau, 0), \theta_{|\text{R}|} \equiv \theta_2(\tau, 0), \theta_{|\text{R}|} \equiv -i \theta_1(\tau, 0) \equiv 0$ to describe the free fermion contributions.

We shall assume $k_1 + 2 \equiv 4K \in 4\mathbb{Z}_{>0}$ so as to make the $\mathbb{Z}_2 \times \mathbb{Z}_4$-orbifolding by $\gamma, \delta$ defined below well-defined.

### 2.2 Orbifold Actions

Let us clarify the orbifold actions that we will utilize in order to construct the string vacua.
(i) $\gamma_L \in \mathbb{Z}_2$ : 

We introduce an involution $\gamma_L$ which only acts on the left-mover of $\mathcal{M}_{k_1}$-sector as the sign factor $(-1)^{\ell_1,\ell}$ for the $'SU(2)_k$-quantum number' $\ell_{1,\ell}$ irrespective of the spin structures. Namely, $\gamma_L$ commutes with all the generators of superconformal algebra, and the primary states of $\mathcal{M}_{k_1}$ should be transformed as 

$$
\gamma_L : |\ell_{1,\ell}, m_{1,\ell}\rangle^{(\sigma_L)} \mapsto (-1)^{\ell_1,\ell} |\ell_{1,\ell}, m_{1,\ell}\rangle^{(\sigma_L)}.
$$

(2.13)

The twisted sector by $\gamma_L$ is slightly non-trivial: the primary states are of the forms as $|k_1 - \ell_{1,\ell}, m_{1,\ell}\rangle^{(\sigma_L)}$, and $\gamma_L$ acts on them as the different sign factor $(-1)^{\ell_1,\ell+1}$. This is required by the modular invariance. In fact, the modular invariant of the $\mathbb{Z}_2$-orbifold of affine $SU(2)_k$-theory by $\gamma_L \equiv (-1)^{\ell_1,\ell}$ is found to be 

$$
Z^{SU(2)_k}(\tau, \bar{\tau})|_{\gamma_L\text{-orb}} = \sum_{\ell_1=0, \ell_1 \in 2\mathbb{Z}}^{k_1} \left| \chi_{\ell_1}^{SU(2)_k}(\tau) \right|^2 + \sum_{\ell_1=1, \ell_1 \in 2\mathbb{Z}+1}^{k_1-1} \chi_{k_1-\ell_1}^{SU(2)_k}(\tau) \chi_{\ell_1}^{SU(2)_k}(\tau),
$$

(2.14)

which coincides with the $D_{k_1+2}$-type modular invariant for $k_1 \in 4\mathbb{Z} + 2$ [23, 24].

To summarize, $\gamma_L$ should act on the character of $\mathcal{M}_{k_1}$-sector as follows; 

$$
\gamma_{L(a,b)} \cdot \text{ch}_{\ell_1, m_1}^{(\sigma)} (\tau, z) := \begin{cases} 
(1)^{b\ell_1} \text{ch}_{\ell_1, m_1}^{(\sigma)} (\tau, z), & (a = 0), \\
(1)^{b(\ell_1+1)} \text{ch}_{k_1-\ell_1, m_1}^{(\sigma)} (\tau, z), & (a = 1),
\end{cases}
$$

(2.15)

where $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ labels the spatial and temporal twisting by $\gamma_L$. It is obvious that $\gamma_L$ preserves all the space-time SUSY, since it does not affect the integral spectral flows defining the spectral flow orbits $\mathcal{F}_L^{(\sigma)}$.

(ii) $\delta_R \in \mathbb{Z}_4$ : 

Nextly, we introduce an order 4 chiral operator $\delta_R$ which acts only on the right-mover of $\mathcal{M}_{k_1}$-sector as 

$$
\delta_R := e^{2\pi i \frac{k_1+2}{4} J^{(1)}_{R,\theta}},
$$

(2.16)

where $J^{(1)}_{R,\theta}$ denotes the right-moving $U(1)$-current of $\mathcal{N} = 2$ SCA in the $\mathcal{M}_{k_1}$-sector. $\delta_R$ obviously commutes with all the generators of SCA and induces a $\mathbb{Z}_4$-phase factor $e^{2\pi i \frac{m_1}{4} \frac{k_1+2}{4}}$ for the primary states $|\ell_{1,R}, m_{1,R}\rangle^{(NS)}$ ( $|\ell_{1,R}, m_{1,R}\rangle^{(R)}$ ).

The orbifold by $\delta_R$ is well described in terms of the spectral flow $\bar{z} \mapsto \bar{z} + \frac{k_1+2}{4} (\alpha \tau + \beta)$ ($\alpha, \beta \in \mathbb{Z}_4$). Indeed, the non-trivial part of the spatially and temporally twisted sector labeled by $(\alpha, \beta)$ is explicitly represented in terms of the $\mathcal{N} = 2$ minimal characters as 

$$
\delta_{R(\alpha,\beta)} \cdot \text{ch}_{\ell_1, m_1}^{(\sigma)} (\tau, z) := q^{\frac{4\ell_1(k_1+2)}{4} + 2 \alpha z} y^{\frac{k_1+2}{4}} e^{2\pi i \frac{4\ell_1(k_1+2)}{4} \alpha \beta} \text{ch}_{\ell_1, m_1}^{(\sigma)} (\tau, z + \frac{k_1+2}{4} (\alpha \tau + \beta)),
$$

(2.17)

$$
(\alpha, \beta) \in \mathbb{Z}_4 \times \mathbb{Z}_4,
$$

$$
\delta_{R(\alpha,\beta)} : \mathcal{M}_{k_1} \to \mathcal{F}_R^{(\sigma)}.
$$
irrespective of the spin structure $\sigma$. The phase factor $e^{i\frac{1}{32}(k_1^2+k_2^2)\alpha\beta}$ appearing in (2.17) is again required by the modular invariance, and will play a crucial role in our arguments given in the next section.

As is easily confirmed, the $\delta_R$-orbifolding completely breaks the right-moving space-time SUSY. For instance, the $\delta_R$-projection leaves the primary states with the quantum numbers $m_{1,R}$ of the same oddity both in the NS and R-sectors (in other words, $\ell_{1,R}$ with the opposite oddity). Thus, it is impossible to combine them into a supermultiplet by the action of half spectral flows.

(iii) $(-1)^{F_L}$:

$F_L$ denotes the ‘space-time fermion number’ in the left-mover, that is, $(-1)^{F_L}$ acts as the sign flip of the left-moving Ramond sector, which would often appear in the literatures of thermal superstring theory (see e.g. [25]). ($(-1)^{F_R}$ is defined in the same way for the right mover.) Denoting the spatial and temporal twisting by the operator $(-1)^{F_L}$ as $[(−1)^{F_L}]_{(a,b)}$, $((a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2)$, its action to the spectral flow orbit $F_L^{(\sigma)}(\tau)$ is summarized as follows (for $\sigma = \text{NS}, \bar{\text{NS}}, \text{R}$):

\[
\begin{cases}
\epsilon(R; 0, 1) = \epsilon(\bar{\text{NS}}; 1, 0) = \epsilon(\text{NS}; 1, 1) = -1, \\
\epsilon(\sigma; a, b) = 1 \text{ otherwise.}
\end{cases}
\] (2.18)

It is clearly compatible with the modular covariance. Namely, $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ behaves as the suitable doublet of $SL(2, \mathbb{Z})$ by modular transformations.

The $\gamma_L$ and $\delta_R$-orbifolding are obviously compatible and we can consider the $\mathbb{Z}_2 \times \mathbb{Z}_4$-orbifolds of the Gepner models generated by these operators. The corresponding modular invariant partition function is written as

\[
Z_{\text{chiral SUSY}}(\tau, \bar{\tau}) = \left(\frac{1}{\sqrt{2|\eta|}}\right)^2 \cdot \frac{1}{4N} \sum_{\sigma_L, \sigma_R} \epsilon(\sigma_L)\epsilon(\sigma_R) \left(\frac{\theta[\sigma_L]}{\eta}\right) \left(\frac{\theta[\sigma_R]}{\eta}\right)
\times \frac{1}{8} \sum_{a,b\in\mathbb{Z}_2} \sum_{\alpha,\beta\in\mathbb{Z}_4} \sum_{I_L,I_R} N_{I_L,I_R}\gamma_{L,(a,b)}\delta_{R,(\alpha,\beta)} \cdot F_L^{(\sigma_L)}(\tau)F_R^{(\sigma_R)}(\tau).
\] (2.19)

Since $\delta_R$ fully breaks the right-moving SUSY as mentioned above, we have an $\mathcal{N} = 1$ SUSY in 4-dim. which only comes from the left-mover in this string vacuum.

\[2\text{Though it is not necessary for our purpose, we also note } \epsilon(\bar{R}; 0, 1) = \epsilon(\bar{R}; 1, 0) = \epsilon(\bar{R}; 1, 1) = -1. \text{ (see e.g. [25]).}\]
3 Non-SUSY Models with Vanishing Cosmological Constant

In this section we present our main results. We shall demonstrate the construction of our proposals of non-SUSY string vacua based on some asymmetric orbifolding of Gepner models. We then show that the constructed vacua induce a vanishing cosmological constant (torus partition function), whereas any supercharges with the reasonable form cannot be made up.

3.1 Construction of the Non-SUSY Models

We consider the \( Z_2 \times Z_4 \)-orbifolding of the Gepner model (2.11) by the operators

\[
\hat{\gamma} := (-1)^{F_R} \gamma_L, \quad \hat{\delta} := (-1)^{F_L} \delta_R.
\]

We shall also assume the discrete torsion \([26–28]\) among the \( \hat{\gamma} \) and \( \hat{\delta} \)-actions, defined as

\[
\xi(a, \alpha; b, \beta) := (-1)^{(K - 1)(a\beta - b\alpha)},
\]

where the labels \( a, b \in Z_2 \) and \( \alpha, \beta \in Z_4 \) indicate respectively the \( \hat{\gamma} \), \( \hat{\delta} \) twisted sectors as presented in (2.19), for instance. \( (a, \alpha \) denote the spatial twistings, while \( b, \beta \) do the temporal ones.) Recall that we assumed \( k_1 + 2 \equiv 4K \in 4Z_{>0} \). The existence of this type torsion plays a crucial role to achieve the vanishing cosmological constant.

Now, we propose the string vacuum defined by the following modular invariant partition function:

\[
Z_{\text{non-SUSY}}(\tau, \bar{\tau}) = \left( \frac{1}{\sqrt{|\tau|}} \right)^2 \cdot \frac{1}{4N} \sum_{\sigma_L, \sigma_R} e(\sigma_L)e(\sigma_R) \left( \frac{\theta_{[\sigma_L]}}{\eta} \right) \left( \frac{\theta_{[\sigma_R]}}{\eta} \right) \times \frac{1}{8} \sum_{a,b \in Z_2} \sum_{\alpha, \beta \in Z_4} \sum_{I_L, I_R} N_{I_L, I_R} \xi(a, \alpha; b, \beta) \gamma_{L,(a,b)} \delta_{R,(a,\beta)} \cdot \mathcal{F}^{(\sigma_L)}_{I_L}(\tau) \mathcal{F}^{(\sigma_R)}_{I_R}(\tau). \tag{3.3}
\]

Because of the definitions of orbifold actions (3.1), it is obvious that the space-time SUSY is completely broken at least in the untwisted sector: the right-moving SUSY is already broken in the ‘chiral SUSY model’ (2.19), and furthermore, all the left-moving supercharges are removed due to the inclusion of factor \((-1)^{F_L}\) in \( \hat{\delta} \). We will later discuss more carefully why the space-time supercharges cannot be gained even if taking account of the degrees of freedom in the twisted sectors.
3.2 Vanishing Cosmological Constant

We next discuss that the torus partition function (3.3) actually vanishes in spite of the lack of space-time SUSY. To this aim we make use of the conventional notation: $\hat{\gamma} b \hat{\delta} a \square \hat{\gamma} a \hat{\delta} \alpha$ in order to express the contribution to the torus partition function from the $\hat{\gamma} a \hat{\delta} \alpha$, $\hat{\gamma} b \hat{\delta} \beta$-twisted sectors along the spatial and temporal directions respectively. In other words, we can schematically write

$$\xi(a, \alpha; b, \beta) = \text{Tr}_{\hat{\gamma} b \hat{\delta} \beta \text{-twisted}} \left[ \hat{\gamma} a \hat{\delta} \alpha q^{L_0 - \frac{c}{24}} q^{-L_0 + \frac{c}{24}} \right], \quad (3.4)$$

where $\xi(a, \alpha; b, \beta)$ denotes the discrete torsion mentioned above.

- **the sectors with even $\alpha$:**

  We first focus on the 'even sectors' in which both of $\alpha, \beta$ are even. Since $\hat{\delta}^2 = \delta_R^2$ holds, the left-moving SUSY is kept unbroken in these sectors, while the right-moving one is broken completely. We thus obtain

$$\hat{\gamma} a \hat{\delta} \alpha \square \hat{\gamma} a \hat{\delta} \alpha = 0, \quad (\forall \alpha, \beta \in 2\mathbb{Z}, \forall a, b). \quad (3.5)$$

However, $\hat{\delta} \beta$ includes $(-1)^{F_L}$ when $\beta$ is odd, and thus the left-moving SUSY is broken in this sector;

$$\hat{\gamma} a \hat{\delta} \alpha \square \hat{\gamma} a \hat{\delta} \alpha \neq 0, \quad (\forall \alpha \in 2\mathbb{Z}, \forall \beta \in 2\mathbb{Z} + 1, \forall a, b). \quad (3.6)$$

One can also confirm that

$$\sum_{b \in \mathbb{Z}_2} \sum_{\beta \in \mathbb{Z}_4} \hat{\gamma} a \hat{\delta} \alpha \square = \sum_{b \in \mathbb{Z}_2} \sum_{\beta \in \mathbb{Z}_2} \hat{\gamma} a \hat{\delta} \beta \alpha \square + 1 \neq 0, \quad (\forall \alpha \in 2\mathbb{Z}, \forall a). \quad (3.7)$$

Here we made use of (3.5) for the first equality. Moreover, the summation of $\beta' \in \mathbb{Z}_2$ leaves the states in the $M_{k_1}$-sector character $c_{k_1, R, m_1, R}(\tau)$ with $m_1, R \in 2\mathbb{Z}$ for each spin structure. At this point, it is a slightly non-trivial fact that $\delta_R(\alpha, \beta)$ includes the extra phase factor

$$e^{-2\pi i \frac{k_1(k_1+2)}{42} \alpha \beta} \equiv e^{-2\pi i \frac{k}{24} (2K-1) \alpha \beta}, \quad (3.8)$$

which ensures the modular invariance, as was mentioned around (2.17). However, since we are assuming $\alpha \in 2\mathbb{Z}$ here, this phase factor does not affect the oddity of $m_1, R$ survived by the $\hat{\delta}$-orbifolding. The discrete torsion (3.2) does not alter it, too.

On the other hand, $\hat{\gamma}$ acts as

$$\hat{\gamma} = \begin{cases} (-1)^{f_1, L} & (, \text{NS})\text{-sector}, \quad (\forall \alpha \in 2\mathbb{Z}, \forall a), \\ (-1)^{f_1, L+1} & (, \text{R})\text{-sector}, \quad (\forall \alpha \in 2\mathbb{Z}, \forall a), \end{cases} \quad (3.9)$$
because of (2.15), (2.18). Note that (3.2) does not affect it.

In this way, recalling that the modular invariant coefficients (2.12) are diagonal, we find that the states with even \( m_{1,R} \) finally survive after making the orbifold projections.

• the sectors with odd \( \alpha \):

We next focus on the sectors with odd \( \alpha \). Since \( \hat{\delta}^\alpha \) includes \((-1)^{FL}\), each contribution \( \hat{\gamma}^b \hat{\delta}^\alpha \big[ \big] \) does not vanish separately because of the lack of bose-fermi cancellation. However, we can show that these contributions totally vanish after summing over the temporal twisting \( \beta, b \);

\[
\sum_{b \in \mathbb{Z}_2} \sum_{\beta \in \mathbb{Z}_4} \hat{\gamma}^b \hat{\delta}^\alpha \big[ \big] = 0, \quad (\forall \alpha \in 2\mathbb{Z} + 1, \forall a). \tag{3.10}
\]

To be more precise, we can see

\[
\sum_{b \in \mathbb{Z}_2} \sum_{\beta' \in \mathbb{Z}_2} \hat{\gamma}^b \hat{\delta}^{2\beta'} \big[ \big] \text{each spin structure} = 0, \quad (\forall \alpha \in 2\mathbb{Z} + 1, \forall a). \tag{3.11}
\]

(3.10) obviously follows from the stronger one (3.11) by taking the modular \( T \)-transformation of it.

To show (3.11), let us first recall the phase factor (3.8). We thus find that

\[
1 \big[ \big] + \hat{\gamma}^2 \big[ \big] \equiv \left\{
\begin{array}{ll}
(-1)^{\ell_1+K-1} & (\forall, \text{NS})-\text{sector}, \ (\forall \alpha \in 2\mathbb{Z} + 1, \forall a), \\
(-1)^{\ell_1+K} & (\forall, \text{R})-\text{sector}, \ (\forall \alpha \in 2\mathbb{Z} + 1, \forall a),
\end{array}
\right.
\]

\[
\tag{3.12}
\]

in place of (3.9). Therefore, after inserting the projection \( \frac{1+\hat{\gamma}}{2} \), no states survive in the \( \hat{\delta}^\alpha \)-twisted sectors with odd \( \alpha \), which proves (3.11).

In summary, we have found

• In each of the twisted sectors for \( \hat{\gamma}^a \hat{\delta}^\alpha \) with \( a = 0, 1, \alpha \in 2\mathbb{Z} \cap \mathbb{Z}_4 \), the space-time SUSY is completely broken and we obtain

\[
Z_{\hat{\gamma}^a \hat{\delta}^\alpha} (\tau, \bar{\tau}) \equiv \frac{1}{2} \cdot \frac{1}{4} \sum_{b \in \mathbb{Z}_2} \sum_{\beta \in \mathbb{Z}_4} \hat{\gamma}^b \hat{\delta}^\alpha \big[ \big] \neq 0.
\]
• In each of the twisted sectors for $\hat{\gamma}^a \hat{\delta}^\alpha$ with $a = 0, 1, \alpha \in (2\mathbb{Z} + 1) \cap \mathbb{Z}_4$, all the states are projected out by the orbifold projection.

Finally, we show that the total partition function (3.3) indeed vanishes:

$$Z_{\text{non-SUSY}}(\tau, \bar{\tau}) \equiv \sum_{a \in \mathbb{Z}_2} \sum_{\alpha \in 2\mathbb{Z} \cap \mathbb{Z}_4} Z_{\hat{\gamma}^a \hat{\delta}^\alpha}(\tau, \bar{\tau}) = 0. \quad (3.13)$$

In fact, the modular S-transformation of (3.11) yields

$$\sum_{a \in \mathbb{Z}_2} \sum_{\alpha \in \mathbb{Z}_2} \hat{\delta}^\beta \overset{□}{\hat{\gamma}^a \hat{\delta}^\alpha} = \sum_{a \in \mathbb{Z}_2} \sum_{\alpha \in \mathbb{Z}_2} \hat{\gamma}^\alpha \overset{□}{\hat{\gamma}^a \hat{\delta}^\alpha} = 0, \quad (\forall \beta \in 2\mathbb{Z} + 1). \quad (3.14)$$

Combining this result with (3.5) as well as (3.11), we immediately obtain the desired fact (3.13).

A few comments are in order.

1. We note that the GSO-projection operator for the right-moving NS-sector acts with the opposite sign in the $\hat{\gamma}$- and $\hat{\delta}^2$-twisted sectors, which would potentially lead to a tachyonic instability. However, the left-movers in these twisted sectors are correctly GSO-projected in each spin structure, and thus no tachyons appear after the level matching condition is imposed.

2. In the string vacuum constructed above, the bose-fermi cancellation occurs in the left-mover, after summing up over all the twisted sectors;

$$\sum_{a, \alpha} \left[ Z_{a, \alpha,(\text{NS}, \ast)}(\tau, \bar{\tau}) + Z_{a, \alpha,(\text{R}, \ast)}(\tau, \bar{\tau}) \right] = 0, \quad (3.15)$$

while it does not in the right-mover. One might thus wonder if the space-time SUSY would eventually survive in the left-mover. It is obvious by our orbifold construction that the space-time supercharges, if any, cannot originate from the untwisted sector. In other words it should belong to the twisted sector. Observing the aspect of bose-femi cancellation mentioned above, the expected supercharge has to be made up of the operators that intertwine the untwisted sector with the $\hat{\gamma} \hat{\delta}^2$-twisted sector, if it would exist anyway. However, it is not possible as long as assuming the chiral form such as $'Q_L' \equiv \oint dz J^\alpha_L(z)$ with some holomorphic currents $J^\alpha_L(z)$ for the expected supercharge, since the twist operator $\hat{\gamma} \hat{\delta}^2 \equiv \gamma_L \otimes \delta^2_R(-1)^{F_R}$ non-chirally acts on the Hilbert space at hand, changing the boundary conditions of fields in both of the left and right-movers. We believe that the chiral form of supercharges is a fairly reasonable assumption, because it physically means the conservation of supercharges to be searched on the world-sheet.

3. The unitarity of conformal system constructed above is readily confirmed, although it would often be subtle in models of asymmetric orbifolds, especially for the twisted sectors. Namely,
the partition sum for the each twisted sector $Z_{a,\alpha}(\tau, \bar{\tau})$ has a $q$-expansion with coefficients of positive integers. This fact is obvious because all the orbifolds actions, which are summarized in 2.2, are manifestly compatible with unitarity.

### 3.3 Generalization of the Non-SUSY Models

Here we shall present a generalization of the non-SUSY Gepner models constructed above. We again start with the Gepner models for CY$_3$ in which we have $N \equiv \text{L.C.M} \{k_i + 2\} \in 4\mathbb{Z}$. We would like to construct the orbifolds in the manner similar to the previous ones, but in which the orbifold operators (denoted as ‘$\hat{\gamma}$’, ‘$\hat{\delta}$’ again) act on the *multiple* factors of the $N' = 2$ minimal models.

Let us fix a subsystem of the minimal models $\otimes_{i \in S} \mathcal{M}_{k_i}$, $S \subset \{1, 2, \ldots, r\}$, on which the orbifold operators non-trivially act. We assume

$$N' \equiv \text{L.C.M} \{k_i + 2 : i \in S\} \in 4\mathbb{Z}. \tag{3.16}$$

It is obvious that the total central charge of the subsystem $S$ is written in the form;

$$\hat{c}_S \left(\equiv \sum_{i \in S} \frac{k_i}{k_i + 2}\right) = \frac{2M}{N'}, \quad \exists M \in \mathbb{Z}. \tag{3.17}$$

We also fix a positive integer $L$ dividing $\frac{N'}{4}$ and also define $S_1 \subset S$ by

$$S_1 := \left\{ i \in S : \frac{N'}{k_i + 2} \in 2\mathbb{Z} + 1 \right\}. \tag{3.18}$$

Note that $S_1 \neq \phi$, since $N'$ is the L.C.M. of $\{k_i + 2\}_{i \in S}$.

Under the preparations given above, we define the orbifold actions as well as the discrete torsion that generalize those given in the previous subsection. Note that the previous one just corresponds to the case $S = S_1 = \{i = 1\}$, $N' = k_1 + 2 = 4K$, $L = \frac{k_1 + 2}{4} = K$ and $M = \frac{k_1}{2} = 2K - 1$;

**(i) $\hat{\gamma} \in \mathbb{Z}_2$**:

We define

$$\gamma_L := \prod_{i \in S_1} (-1)^{F_{iL}}. \tag{3.19}$$

Namely, $\gamma_L$ acts on the left-moving characters of each minimal model $\mathcal{M}_{k_i}$, $\forall i \in S_1$ as the $\mathbb{Z}_2$-twisting (2.15). We then set

$$\hat{\gamma} := \begin{cases} (-1)^{F_{iL}}\gamma_L & (\#S_1 \in 2\mathbb{Z} + 1), \\ \gamma_L & (\#S_1 \in 2\mathbb{Z}). \end{cases} \tag{3.20}$$
(ii) \( \hat{\delta} \in \mathbb{Z}_{N'/L} : \)

We define
\[
\delta_R := e^{2\pi i L \sum_{i \in S} J_{i,0}^{(i)}}, \quad \hat{\delta} := (-1)^{F_r} \delta_R,
\]
where \( J_{R}^{(i)} \) is the right-moving \( U(1) \)-current in \( \mathcal{M}_{k_i}, \forall i \in S \). In other words, \( \delta_R \) acts on the right-moving characters of \( \mathcal{M}_{k_i}, \forall i \in S \) as the integral spectral flow \( \bar{z} \mapsto \bar{z} + L(\alpha \bar{\tau} + \beta) \);
\[
\delta_{R(\alpha, \beta)} : ch_{i,m_i}^{(\sigma)}(\tau, z) := q^{\frac{k_i}{k_i+2} L^2 \alpha^2} \frac{y^{k_i+2} L^2 \alpha \beta}{y^{k_i+2} L^2} e^{2\pi i \frac{k_i}{k_i+2} L^2 \alpha \beta} \cdot \pi \delta_{(\sigma)}(\tau, z + L(\alpha \bar{\tau} + \beta)),
\]

\[(\alpha, \beta) \in \mathbb{Z}_{N'/L} \times \mathbb{Z}_{N'/L}, \quad (3.22)\]
as in (2.17).

(iii) discrete torsion :

We also introduce the discrete torsion with respect to the \( \hat{\gamma} \) and \( \hat{\delta} \)-actions as in (3.2);

\[
\xi(a, a; b, b) := (-1)^{(LM-1)(a, b)} \cdot \chi_{i,m_i}^{(\sigma)}(\tau, z) := q^{\frac{k_i}{k_i+2} L^2 \alpha^2} \frac{y^{k_i+2} L^2 \alpha \beta}{y^{k_i+2} L^2} e^{2\pi i \frac{k_i}{k_i+2} L^2 \alpha \beta} \cdot \pi \delta_{(\sigma)}(\tau, z + L(\alpha \bar{\tau} + \beta)),
\]

\[(a, b) \in \mathbb{Z}_2, \quad (3.23)\]

We can now define the \( \mathbb{Z}_2 \times \mathbb{Z}_{N'/L} \)-orbifold of the Gepner model at hand, which describes a non-SUSY string vacuum. We can further show that the torus partition function vanishes;

\[
Z_{\text{non-SUSY}}(\tau, \bar{\tau}) \equiv \sum_{a, b \in \mathbb{Z}_2} \sum_{\alpha, \beta \in \mathbb{Z}_{N'/L}} \pi \delta_{(\sigma)} \chi_{i,m_i}^{(\sigma)}(\tau, \bar{\tau}) = 0,
\]

\[(3.24)\]

while

\[
Z_{a, \alpha}(\tau, \bar{\tau}) \equiv \sum_{b \in \mathbb{Z}_2} \sum_{\beta \in \mathbb{Z}_{N'/L}} \pi \delta_{(\sigma)} \chi_{i,m_i}^{(\sigma)}(\tau, \bar{\tau}) \neq 0,
\]

\[(3.25)\]

for each twisted sector with \( a \in \mathbb{Z}_2 \) and \( \alpha \in \mathbb{Z}_{N'/L} \cap 2\mathbb{Z} \).

To show it, the next fact plays a crucial role;

\[
\sum_{b \in \mathbb{Z}_2} \sum_{\beta \in \mathbb{Z}_{N'/L} \cap 2\mathbb{Z}} \pi \delta_{(\sigma)} \chi_{i,m_i}^{(\sigma)}(\tau, \bar{\tau}) \neq 0,
\]

\[(3.26)\]

for any ‘odd sectors’ with \( \forall \alpha \in \mathbb{Z}_{N'/L} \cap (2\mathbb{Z} + 1) \) and \( \forall a \in \mathbb{Z}_2 \), which is the analogue of (3.11).

In fact, the summation over \( \beta \in \mathbb{Z}_{N'/L} \cap 2\mathbb{Z} \) imposes the constraint

\[
N' \sum_{i \in S} \left( \frac{m_{i,R} - 2n_R}{k_i + 2} + \frac{c_s}{2} L \alpha \right) = \sum_{i \in S} d_i (m_{i,R} - 2n_R) + LM \alpha \in \frac{N'}{2L} \mathbb{Z} \quad (d_i = \frac{N'}{k_i + 2}),
\]

\[(3.27)\]
on the characters $c_{\ell_{i,R,m_i,R-2n_R}}^{(\sigma)}(\tau, z) \ (i \in S)$. \ (\(n_R \in \mathbb{Z}_N\) denotes the spectral flow parameter appearing in the right-moving orbit $F_{I,R}^{(\sigma)}(\tau)$.) Recalling our assumption that $d_i \in 2\mathbb{Z} + 1$ iff $i \in S_1$ and $\frac{N'}{2L} \in 2\mathbb{Z}$, this relation implies
\[
\sum_{i \in S_1} m_{i,R} \equiv LM \mod 2,
\] (3.28)
in each spin structure. In other words,

- for $\#S_1 \in 2\mathbb{Z} + 1$, we obtain
  \[
  \sum_{i \in S_1} \ell_{i,R} \equiv \begin{cases} 
  LM \mod 2, & (\ast, \text{NS})\text{-sector}, \\
  LM + 1 \mod 2, & (\ast, \text{R})\text{-sector},
  \end{cases}
  \]
  (3.29)

- for $\#S_1 \in 2\mathbb{Z}$, we obtain
  \[
  \sum_{i \in S_1} \ell_{i,R} \equiv LM \mod 2,
  \]
  (3.30)
irrespective of the spin structure.

On the other hand, taking account of the discrete torsion (3.23), we find that $\hat{\gamma}$ effectively acts as

- for $\#S_1 \in 2\mathbb{Z} + 1$,
  \[
  \hat{\gamma} = \begin{cases} 
  (-1)^{\sum_{i \in S_1} \ell_{i,L} + LM - 1} & (\forall \alpha \in 2\mathbb{Z} + 1, \forall a), \\
  (-1)^{\sum_{i \in S_1} \ell_{i,L} + LM} & (\forall \alpha \in 2\mathbb{Z} + 1, \forall a),
  \end{cases}
  \]
  (3.31)
as in (3.12), while

- for $\#S_1 \in 2\mathbb{Z}$,
  \[
  \hat{\gamma} = (-1)^{\sum_{i \in S_1} \ell_{i,L} + LM - 1}, \quad (\forall \alpha \in 2\mathbb{Z} + 1, \forall a),
  \]
  (3.32)
irrespective of the spin structure.

By comparing (3.29), (3.30) with (3.31), (3.32) one can show that the contributions in question vanish separately in each spin structure after making $\hat{\gamma}$-projection, proving the fact (3.26). We thus obtain the desired result (3.24) according to the same argument as given in the previous subsection.
4 Discussions

In this paper, we have studied some asymmetric orbifolds of the Gepner models for Calabi-Yau 3-folds, aiming at the construction of non-SUSY type II string vacua with the vanishing cosmological constant at one-loop.

We would like to compare several aspects of the present model with those of the ones given in [11, 12], which are constructed as asymmetric orbifolds of tori.

In the ones adopted in [11, 12], the asymmetric orbifold actions that generate the ‘SUSY-breaking factors’ \((-1)^{F_L}, (-1)^{F_R}\) have been combined with the translation along some direction in the compactification space. This would be an analogue of Scherk-Schwarz type compactification [29]. It is a characteristic feature of this model that the bose-fermi cancellation occurs at the each sector corresponding to (3.4) in this paper. Indeed, the left-moving bose-fermi cancellation occurs in the sectors with even winding numbers along the ‘Scherk-Schwarz circle’, whereas we have the right-moving bose-fermi cancellation in the odd winding sectors. This aspect prevents us from constructing any supercharges defined over the total Hilbert space.

In the present model the orbifold actions \(\hat{\gamma}, \hat{\delta}\) likewise include \((-1)^{F_L}, (-1)^{F_R}\). On the other hand, we did not assume the Scherk-Schwarz circle in any direction of compactification. Indeed, we started with a Gepner model for CY\(_3\) and the Scherk-Schwarz type compactification seems to be hard to make, since the translational invariance is generically broken.

Another crucial difference is that we do not have the bose-fermi cancellation in each twisted sector in the present model. Namely \(Z_{a,\alpha} \equiv \sum_{b,\beta} \hat{\gamma}^b \hat{\delta}^\beta \square \hat{\gamma}^a \hat{\delta}^\alpha\) for fixed \(a, \alpha\) does not necessarily vanish. Nevertheless, the total partition function vanishes after summing up over all the twisted sectors:

\[
Z \equiv \sum_{a,\alpha} Z_{a,\alpha} \equiv \sum_{a,\alpha} \sum_{b,\beta} \hat{\gamma}^b \hat{\delta}^\beta \square \hat{\gamma}^a \hat{\delta}^\alpha = 0.
\]

This feature is in a sharp contrast with the previous ones. To be more specific, we have the bose-fermi cancellation only in the left-mover, as was noted around (3.15). Nonetheless we cannot gain the left-moving supercharges because of the non-chirality of the orbifold actions \(\hat{\gamma}, \hat{\delta}\).

We would also like to point out that the unitarity is rather simple to confirm in the present model, though it was non-trivial whether the torus partition functions are \(q\)-expanded in the way compatible with the unitarity in the models given in [11, 12].

Since the right-mover does not play any role in achieving the vanishing cosmological constant, the present construction could be applicable to the heterotic string vacua, too, whereas it was difficult for the previous ones in [11, 12], in which both of the left and right-moving bose-fermi cancellations are necessary for realizing the desired non-SUSY vacua. We would like to make the detailed studies of extensions to the heterotic string vacua in a future work.
Appendix A: Summary of Conventions

We summarize the notations and conventions adopted in this paper. We set \( q \equiv e^{2\pi i \tau}, \ y \equiv e^{2\pi i z} \).

1. Theta Functions

\[
\theta_1(\tau, z) := i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} = 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - y q^m)(1 - y^{-1} q^m), \tag{A.1}
\]

\[
\theta_2(\tau, z) := \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} = 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + y q^m)(1 + y^{-1} q^m), \tag{A.2}
\]

\[
\theta_3(\tau, z) := \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n = \prod_{m=1}^{\infty} (1 - q^m)(1 + y q^{m-1/2})(1 + y^{-1} q^{-m-1/2}), \tag{A.3}
\]

\[
\theta_4(\tau, z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n = \prod_{m=1}^{\infty} (1 - q^m)(1 - y q^{m-1/2})(1 - y^{-1} q^{-m-1/2}). \tag{A.4}
\]

\[
\Theta_{m,k}(\tau, z) := \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2})^2} y^{k(n+\frac{m}{2})}, \tag{A.5}
\]

\[
\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{A.6}
\]

Here, we have set \( q := e^{2\pi i \tau}, \ y := e^{2\pi i z} \ (\forall \tau \in \mathbb{H}^+, \forall z \in \mathbb{C}) \), and used abbreviations, \( \theta_i(\tau) \equiv \theta_i(\tau, 0) \) (\( \theta_1(\tau) \equiv 0 \)), \( \Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0) \).

2. Character Formulas for \( \mathcal{N}=2 \) Minimal Model

The character formulas of the level \( k \mathcal{N}=2 \) minimal model \( (\hat{c} = k/(k+2)) \) \cite{20,21} are described as the branching functions of the Kazama-Suzuki coset \cite{30} \( SU(2)_k \times U(1)_2 \) defined by

\[
\chi^{(k)}_{\ell}(\tau, w) \Theta_{s,2}(\tau, w-z) = \sum_{m \in \mathbb{Z}_{2(k+2)}} \chi^{\ell,s}_{m}(\tau, z) \Theta_{m,k+2}(\tau, w-2z/(k+2)) , \tag{A.7}
\]

\[
\chi^{\ell,s}_{m}(\tau, z) \equiv 0 , \quad \text{for } \ell + m + s \in 2\mathbb{Z} + 1 ,
\]

where \( \chi^{(k)}_{\ell}(\tau, z) \) is the spin \( \ell/2 \) character of \( SU(2)_k \);

\[
\chi^{(k)}_{\ell}(\tau, z) = \frac{\Theta_{\ell+1,k+2}(\tau, z) - \Theta_{\ell-1,k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} = \sum_{m \in \mathbb{Z}_{2k}} c^{(k)}_{\ell,m}(\tau) \Theta_{m,k}(\tau, z) . \tag{A.8}
\]

The branching function \( \chi^{\ell,s}_{m}(\tau, z) \) is explicitly calculated as follows;

\[
\chi^{\ell,s}_{m}(\tau, z) = \sum_{r \in \mathbb{Z}_k} c^{(k)}_{\ell,m-s+4r}(\tau) \Theta_{2m+(k+2)(-s+4r),2k(k+2)}(\tau, z/(k+2)) . \tag{A.9}
\]
Then, the character formulas of unitary representations are written as
\[
\begin{align*}
\chi^{(\text{NS})}_{\ell,m}(\tau, z) &= \chi^\ell_m(\tau, z) + \chi^{\ell,2}_m(\tau, z), \\
\chi^{(\tilde{\text{NS}})}_{\ell,m}(\tau, z) &= \chi^\ell_m(\tau, z) - \chi^{\ell,2}_m(\tau, z), \\
\chi^{(R)}_{\ell,m}(\tau, z) &= \chi^\ell_m(\tau, z) + \chi^{\ell,3}_m(\tau, z), \\
\chi^{(\tilde{R})}_{\ell,m}(\tau, z) &= \chi^\ell_m(\tau, z) - \chi^{\ell,3}_m(\tau, z).
\end{align*}
\] (A.10)

**Appendix B: Explicit Forms of Spectral Flow Orbits and Their Orbifold Twistings**

In Appendix B, we summarize the explicit expressions of spectral flow orbits (2.7)-(2.10), which play the role of building blocks of relevant modular invariants, and their twistings by the orbifold actions $\gamma_L$, $\delta_R$ introduced in section 2.2.

We make use of the abbreviated index $I \equiv \{(\ell_i, m_i)\} (\ell_i + m_i \in 2\mathbb{Z})$ again, and set
\[
Q(I) \equiv Q((\ell_i, m_i)) := \sum_{i=1}^{r} \frac{m_i}{k_i} + 2 \left( \frac{1}{N} \right),
\] (B.1)
for the convenience. $F_\sigma^{(\sigma)}(\tau, z)$ obviously vanishes for $Q(I) \notin \mathbb{Z}$ by the definitions (2.7)-(2.10), and we obtain the following expressions in the case of $Q(I) \in \mathbb{Z}$.

\[
\begin{align*}
F_\sigma^{(\text{NS})}(\tau, z) &= \sum_{n \in \mathbb{Z}_N} \prod_{i=1}^{r} \chi^{(\text{NS})}_{\ell_i, m_i - 2n \mathbb{Z}}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} F_{s_n(I)}^{(\text{NS})}(\tau, z), \\
F_{\tilde{\sigma}}^{(\tilde{\text{NS}})}(\tau, z) &= (-1)^{Q(I)} \sum_{n \in \mathbb{Z}_N} (-1)^{\hat{c} - r} n \prod_{i=1}^{r} \chi^{(\tilde{\text{NS}})}_{\ell_i, m_i - 2n \mathbb{Z}}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} (-1)^{\hat{c} - r} n F_{s_n(I)}^{(\tilde{\text{NS}})}(\tau, z), \\
F_\sigma^{(R)}(\tau, z) &= \sum_{n \in \mathbb{Z}_N} \prod_{i=1}^{r} \chi^{(R)}_{\ell_i, m_i - 2n - 1 \mathbb{Z}}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} F_{s_n(I)}^{(R)}(\tau, z), \\
F_{\tilde{\sigma}}^{(\tilde{R})}(\tau, z) &= (-1)^{Q(I)+r} \sum_{n \in \mathbb{Z}_N} (-1)^{\hat{c} - r} n \prod_{i=1}^{r} \chi^{(\tilde{R})}_{\ell_i, m_i - 2n - 1 \mathbb{Z}}(\tau, z) \equiv \sum_{n \in \mathbb{Z}_N} (-1)^{\hat{c} - r} n F_{s_n(I)}^{(\tilde{R})}(\tau, z),
\end{align*}
\] (B.2)-(B.5)

where we introduced the notation
\[
s_n(I) := \{(\ell_i, m_i - 2n)\} \quad \text{(for } I \equiv \{(\ell_i, m_i)\}).
\]
Then, the explicit actions of $\gamma$ and $\delta$-twisting\(^3\) onto $\mathcal{F}_I^{(\sigma)}(\tau)$ are evaluated as follows; ($\mathcal{F}_I^{(\sigma)}(\tau) \equiv \mathcal{F}_I^{(\sigma)}(\tau, 0), a, b \in \mathbb{Z}_2, \alpha, \beta \in \mathbb{Z}_4, k_1 + 2 = 4K \in 4\mathbb{Z}_{\geq 0}$)

$$
\gamma(a, b) \cdot \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau) = \begin{cases} (-1)^b \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau), & (a = 0) \\ (-1)^b(\ell + 1) \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau), & (a = 1) \end{cases} (B.6)
$$

$$
\delta(\alpha, \beta) \cdot \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau) = \zeta_{K-\frac{1}{2}}(\sigma; \alpha, \beta) e^{2\pi \frac{iK}{4}(2K-1)\alpha\beta} 	imes \sum_{n \in \mathbb{Z}_N} e^{2\pi \frac{i(m_i - 2n + 2K\alpha)}{2} \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau)}, (B.7)
$$

where we introduced the phase factor

$$
\zeta_{\kappa}(\text{NS}; \alpha, \beta) = 1, \quad \zeta_{\kappa}(\tilde{\text{NS}}; \alpha, \beta) = e^{-i\pi\kappa\alpha}, \quad \zeta_{\kappa}(\text{R}; \alpha, \beta) = e^{i\pi\kappa\beta}, \quad \zeta_{\kappa}(\tilde{\text{R}}; \alpha, \beta) = e^{-i\pi\kappa(\alpha - \beta)}. (B.8)
$$

For the ones given in section 3.3, we can also summarize as follows;

$$
\gamma(a, b) \cdot \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau) = \begin{cases} (-1)^b \sum_{S_1} \ell' \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau), & (a = 0) \\ (-1)^b \sum_{S_1} (\ell' + 1) \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau), & (a = 1) \end{cases} (B.9)
$$

where we set

$$
\ell'_i := \begin{cases} k_i - \ell_i & i \in S_1, \\ \ell_i & \text{otherwise.} \end{cases}
$$

$$
\delta(\alpha, \beta) \cdot \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau) = \zeta_2 L(\sigma; \alpha, \beta) e^{2\pi \frac{i\alpha \beta}{2} \sum_{n \in S} e^{\frac{2\pi i\sum_{n \in S} \frac{L(m_i - 2n)}{m_i - 2n}}} \mathcal{F}^{(\sigma)}_{\{(\ell, m_i)\}}(\tau)}, (B.10)
$$

where we set

$$
m''_i := \begin{cases} m_i - 2L\alpha & i \in S, \\ m_i & \text{otherwise.} \end{cases}
$$

\(^3\)Here we omit the subscripts ‘L’ and ‘R’ used in the main text.
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