Local symmetries in the Hamiltonian framework.

1. Hamiltonian form of the symmetries and the Noether identities.

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Abstract

We study in the Hamiltonian framework the local transformations \( \delta q^A(\tau) = \sum_{[k]} \frac{\partial}{\partial \tau^k} R_{(k)a}^A(q^B, q^C) \) which leave invariant the Lagrangian action: \( \delta S = \text{div} \). Manifest form of the symmetry and the corresponding Noether identities is obtained in the first order formalism as well as in the Hamiltonian one. The identities has very simple form and interpretation in the Hamiltonian framework. Part of them allows one to express the symmetry generators which correspond to the primarily expressible velocities through the remaining one. Other part of the identities allows one to select subsystem of constraints with a special structure from the complete constraint system. It means, in particular, that the above written symmetry implies an appearance of the Hamiltonian constraints up to at least \( ([k] + 1) \) stage. It is proved also that the Hamiltonian symmetries can always be presented in the form of canonical transformation for the phase space variables. Manifest form of the resulting generating function is obtained.

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1 Introduction

Formulation of the modern quantum field theory models involves necessarily an additional nonphysical variables. Their appearance is mostly due to our desire to incorporate the manifest Poincare invariance and locality as the leading principles of the formulation. To achieve this, the well known and the standard method is an appropriate extension of the physical variable space by the additional degrees of freedom, whose role is to supply the desired properties of a theory [1-5]. In simple cases, one or more variables of a transparent geometrical origin are needed (for example, for the case of massive relativistic particle it is sufficient to introduce only two variables). In contrast, the modern theories incorporate a lot of additional variables, whose nondynamical origin are supplied either by local symmetries presented in the Lagrangian action, or by algebraic character of equations of motion for these variables. In the Hamiltonian framework it manifest himself in appearance of the constraint system with higher nontrivial algebraic structure.

Presence of the additional variables leads to rather complicated problems on the classical as well as on the quantum level, and the Hamiltonian methods [1-12] turn out to be well adapted for investigation of a theory. Hamiltonization of a Lagrangian system can be formulated as procedure of rewrittening of the initial dynamics in an equivalent form in terms of extended (i.e. containing the Lagrangian multipliers) phase space [1]. The result of the procedure (besides the dynamical equations of motion in the Hamiltonian form) is some system of algebraic equations, which determines, in particular, physical sector of the theory. Advantage of the
Hamiltonian methods is, in particular, that phase space description allows one to separate automatically dynamical part of the equations of motion from the algebraic one as well as to analyse arbitrariness of the dynamics for the degenerated theories [4]. It crucially simplifies the problem and is an essential step for the selfconsistent transition to the quantum theory.

In general case, the above mentioned algebraic system is a mixture of the first and the second class constraints as well as of equations for determining of the Lagrangian multipliers. Also, it can be reducible (there are exist identities between the equations). The aim of this work is to study structure of the system in a general framework. We suppose that there is known local symmetry of the Lagrangian action, which implies appearance of the corresponding Noether identities among the equations of motion. Note that it seems to be natural formulation of the problem, since the symmetries are usually known for concrete models. Note also that we do not specify relation among rank of the Hessian and of the symmetry generators. The theory under consideration can has first and second class constraints of any stage. We use the Hamiltonization procedure with the aim to obtain Hamiltonian form of the symmetries and the corresponding identities. As it will be shown, the resulting Hamiltonian identities do not involve of the time derivatives. In other words, they contain information on the algebraic system under consideration. An opposite problem (the problem of restoring of gauge generators from the known constraint system) is discussed in [3, 13-22].

For our aims it turns out to be convenient to use the Hamiltonization procedure in the form developed in [4]. According to this method, starting from the Lagrangian action one obtains first an equivalent description for the system in the extended phase space \((q^A, p_A, v^A)\) (first order formalism). Equations of motion which follows from the first order action contain, in particular, the algebraic one. Part of them can be solved in the form
\[ v^i = v^i(q, p, v^\alpha). \] Then the Hamiltonian form of the dynamics [1] can be obtained by means of the direct substitution of this solution into all the quantities of the first order formalism.

In this work we repeat these steps for the local Lagrangian symmetry and for the corresponding Noether identities with the aim to obtain their form in the Hamiltonian framework. One advantage of this approach (as compare with discussion based on the Legendre transformation [24-26]) is that it turns out to be possible to obtain \textit{manifest form} of the quantities under discussion.

The work is organized as follows. In section 2 we review first steps of the Hamiltonization procedure [4] with the aim to introduce our notations. Then we obtain some relations among the Lagrangian and the Hamiltonian quantities which will be used systematically in the following sections. In section 3 we illustrate our tricks on example of the symmetry with at most one derivative acting on parameters (see also [23] for the case of the symmetry without derivatives).

In the following sections the case of a general local symmetry (see Eqs. (75), (76) below) is analysed. In subsections 4.1, 4.2 we obtain manifest form for the corresponding identities in the first order formalism as well as in the Hamiltonian one. It is shown that the Noether identities in the Hamiltonian form acquires very simple form, their meaning is discussed in subsection 4.3. In particular, they allows one to select some subsystem of constraints \( T \) of the complete constraint system (see Eqs. (95), (99)). We prove also that local symmetry with \([k]\) derivatives on parameters implies appearance of the Hamiltonian constraints up to at least \(([k] + 1)\) stage.

In subsections 5.1, 5.2 we obtain manifest form of the local symmetry in the first order formalism as well as in the Hamiltonian one. The first order action is invariant under the corresponding transformations as a con-
We prove also that the Hamiltonian symmetries can be presented (modulo trivial symmetries of the Hamiltonian action) in the form of canonical transformation for the phase space variables (see also [17-22]). The generating function is found in a manifest form (see Eq. (114) below) and is a combination of the above mentioned constraints $T$. Results of the work are enumerated in the Conclusion.

2 Hamiltonization procedure and some relations among the Lagrangian and the Hamiltonian quantities.

Let us consider dynamical system with the action

$$S = \int d\tau L(q^A, \dot{q}^A),$$

where $A = 1, 2, \ldots [A]$. The corresponding equations of motion are

$$\frac{\delta S}{\delta q^A} = \frac{\partial L}{\partial q^A} - \left( \frac{\partial L}{\partial \dot{q}^A} \right) \cdot = 0.$$  \hspace{1cm} (2)

If will be supposed that the Lagrangian $L$ is at most polynomial on $\dot{q}^A$ and is singular

$$\text{rank} \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \equiv \text{rank} M_{AB} = [i] < [A].$$  \hspace{1cm} (3)

According to this equation, it is convenient to express the index $A$ as $A = (i, \alpha)$, $i = 1 \cdots [i]$, $\alpha = 1 \cdots [\alpha]$, where $[\alpha] = [A] - [i]$. Without loss of generality [4], the matrix $M_{AB}(q, \dot{q})$ can be written as follows

$$M_{AB} = \begin{pmatrix} M_{ij} & M_{i\alpha} \\ M_{j\beta} & M_{\alpha\beta} \end{pmatrix},$$  \hspace{1cm} (4)

where $\text{det} M_{ij}(q, \dot{q}) \neq 0$. An opposite matrix will be denoted as $\tilde{M}^{ij}(q, \dot{q})$, one has $M_{ij} \tilde{M}^{jk} = \delta_i^k$. 

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As the first step of the Hamiltonization procedure, let us rewrite the theory (1) in terms of the first order action defined on the extended phase space \((q^A(\tau), p_A(\tau), v^A(\tau))\)

\[
S_v = \int d\tau \left[ \bar{L}(q^A, v^A) + p_A(\dot{q}^A - v^A) \right],
\]

where \(\bar{L} = \bar{L}(q, \dot{q})|_{\dot{q}^A \rightarrow v^A}\). All the variables are considered on equal footing. In particular, one writes equations of motion for all of them

\[
\dot{q}^A = v^A \equiv \{q^A, \bar{H}(q, p, v^A)\},
\]

\[
\dot{p}_A = \frac{\partial \bar{L}}{\partial q^A} \equiv \{p_A, \bar{H}(q, p, v^A)\},
\]

\[
p_\alpha - \frac{\partial \bar{L}}{\partial v^\alpha} = 0 \iff \frac{\partial \bar{H}}{\partial v^\alpha} = 0,
\]

\[
p_i - \frac{\partial \bar{L}}{\partial v^i} = 0 \iff \frac{\partial \bar{H}}{\partial v^i} = 0,
\]

where \(\{,\}\) is the Poisson bracket and it was denoted

\[
\bar{H}(q^A; p_A, v^A) \equiv p_A v^A - \bar{L}(q, v).
\]

Actions \(S\) and \(S_v\) describe the same dynamics in the following sense.

a) If \(q^A_0(\tau)\) is some solution of the problem (2), then the set of functions \(q^A_0, v^A_0 \equiv \dot{q}^A_0, p_{0A} \equiv \frac{\partial \bar{L}}{\partial v^A}|_{q_0v_0}\) will be solution for the system (6)-(8).

b) If the set \((q^A_0, p_{0A}, v^A_0)\) is a solution of the system (6)-(8), then \(q^A_0\) obeys to Eq.(2). In accordance with the condition (3) one resolves Eq.(8) for the multipliers \(v^i\) algebraically

\[
v^i = v^i(q^A, p_j, v^\alpha).
\]

Hamiltonian form [1] of the initial dynamics can be obtained now by means of substitution of the Eq.(10) into all the quantities of the first order formalism. We use symbols with bar to denote quantities of the first order formalism and symbols without bar for the Hamiltonian quantities

\[
\bar{R}^A = R^A(q, \dot{q})|_{\dot{q}^A \rightarrow v^A},
\]
Consider first equations of motion (6)-(8). Substitution of Eq.(10) into Eq.(8) gives the identity
\[ \Phi_i \equiv p_i - \frac{\partial \bar{L}}{\partial v^i} \bigg|_{v^i} \equiv 0 \iff \frac{\partial H}{\partial v^i} \bigg|_{v^i} \equiv 0, \] (12)
while Eq.(7) acquire the form
\[ \Phi_\alpha(q, p) \equiv p_\alpha - \frac{\partial \bar{L}}{\partial v^\alpha} \bigg|_{v^i} = p_\alpha - f_\alpha(q^A, p_j) = 0. \] (13)
Note that Eq.(13) do not contains the multipliers \( v^\alpha \) (in other case one can use Eq.(13) and to express some of the multipliers \( v^\alpha' \) through the remaining one, in contradiction with the condition (3)). Eq.(13) determines the primary Hamiltonian constraints \( \Phi_\alpha \).

To find manifest form of equations (6) with the multipliers \( v^i \) substituted, one introduces Hamiltonian
\[ H(q^A, p_A, v^\alpha) \equiv \bar{H} \big|_{v^i} = (p_A v^A - \bar{L}(q, v)) \big|_{v^i}. \] (14)
It obeys the equation
\[ \frac{dH}{dv^\alpha} = \Phi_\alpha(q, p), \] (15)
which can be demonstrated as follows
\[ \frac{dH}{dv^\alpha} = \frac{\partial \bar{H}}{\partial v^\alpha} \bigg|_{v^i} + \frac{\partial \bar{H}}{\partial v^i} \bigg|_{v^i} \frac{\partial v^i}{\partial v^\alpha} = \Phi_\alpha(q, p), \] (16)
where Eqs.(12), (13) were used. General solution of Eq.(15) is
\[ H(q^A, p_A, v^\alpha) = H_0(q, p) + v^\alpha \Phi_\alpha(q, p), \] (17)
where \( H_0 \) can be find by comparison of equations (17) and (14)
\[ H_0(q^A, p_j) = \left( p_i v^i - \bar{L}(q, v) + v^\alpha \frac{\partial \bar{L}}{\partial v^\alpha} \right) \bigg|_{v^i(q^A, p_j, v^\alpha)}. \] (18)
Nontrivial part of this statement is that $H_0$ do not depends on the variables $v^\alpha$ (and $p_\alpha$). Thus we have obtained manifest form of the Hamiltonian (17), (18) for an arbitrary Lagrangian action (11). Using the same tricks as in Eq.(16) one finds also

$$\frac{\partial H}{\partial q^A} \bigg|_{v^i} = \frac{\partial H}{\partial q^A}, \quad \frac{\partial H}{\partial p_A} \bigg|_{v^i} = \frac{\partial H}{\partial p_A}. \quad (19)$$

It allows one to substitute the functions $v^i(q^A, p_j, v^\alpha)$ into Eq.(6). In the result, Hamiltonian form of the dynamics is

$$\dot{q}^A = \{q^A, H\}, \quad \dot{p}_A = \{p_A, H\}, \quad (20)$$

$$\Phi_\alpha \equiv p_\alpha - f_\alpha(q^A, p_j) = 0. \quad (21)$$

Note that to obtain Eq.(20), (21) we have combined in fact Eq.(8) with other equations of the system (6)-(8). It means that the systems (6)-(8) and (20), (21) are equivalent to each other. The first order action (3) can also be rewritten in the Hamiltonian form. Namely, one notes that the quantity

$$S_H \equiv S_v \big|_{v^i} = \int d\tau (p_A \dot{q}^A - H_0 - v^\alpha \Phi_\alpha), \quad (22)$$

reproduces the equations of motion (20), (21).

In the following sections we repeat these steps for the local symmetries and for the corresponding Lagrangian identities with the aim to obtain manifest form for these quantities in the Hamiltonian framework. It is now convenient to enumerate some relations among the Lagrangian and the Hamiltonian objects which will be used systematically in the subsequent sections.

Since Eq.(12) is identity, one has $\frac{\partial \Phi_\alpha}{\partial v^\alpha} \equiv 0$, which can be rewritten as

$$\frac{\partial v^i}{\partial v^\alpha} = -\tilde{M}^{ij} \tilde{M}_{j\alpha} \bigg|_{v^i} = -\tilde{M}^{ij} M_{j\alpha}. \quad \text{On other hand, from equations (3) and}$$
it follows $v^i(q^A, p_j, v^\alpha) = \{q^i, H\}$. It can also be used for computation of the derivative. Collecting these two results one has

$$\frac{\partial v^i}{\partial v^\alpha} = -\tilde{M}^{ij} M_{j\alpha} = \{q^i, \Phi_\alpha\}. \tag{23}$$

Other derivatives can be obtained in a similar fashion

$$\frac{\partial v^i}{\partial p_A} = \tilde{M}^{ij} \delta^A_j = \{q^i, \Phi^A\} \tag{24},$$

$$\frac{\partial v^i}{\partial q^A} = -\tilde{M}^{ij} \frac{\partial^2 \bar{L}}{\partial v^j \partial q^A}_{|v^i} = -\{p_A q^i, H\}. \tag{25}$$

We can use these relations for obtaining derivatives of the constraints (13) as follows

$$\frac{\partial \Phi_\alpha}{\partial v^\beta} \equiv 0 \implies M_{\alpha\beta} - M_{\alpha i} \tilde{M}^{ij} M_{j\beta} \equiv 0, \tag{26}$$

$$\frac{\partial \Phi_\alpha}{\partial p_A} = \delta^A_\alpha - M_{\alpha i} \tilde{M}^{ij} \delta^A_j, \tag{27}$$

$$\frac{\partial^2 \bar{L}}{\partial q^A \partial v^B}_{|v^i} = -\frac{\partial \Phi_\alpha}{\partial q^A} \delta^\alpha_B - M_{Bi} \frac{\partial v^i}{\partial q^A}. \tag{28}$$

Part of equations from (27), (28) is equivalent to Eqs. (23), (25). Also, for any function $\bar{T}(q^A, v^A)$ one finds

$$\frac{\partial \bar{T}}{\partial v^i}_{|v^i} = M_{ij} \frac{\partial \bar{T}}{\partial p_j} \equiv M_{ij} \{q^j, T\}, \tag{29}$$

$$\frac{\partial \bar{T}}{\partial v^\alpha}_{|v^\alpha} = \frac{\partial \bar{T}}{\partial v^\alpha} + \frac{\partial \bar{T}}{\partial v^i}_{|v^i} \tilde{M}^{ij} M_{j\alpha} = \frac{\partial T}{\partial v^\alpha} + \frac{\partial T}{\partial p_i} M_{i\alpha}, \tag{30}$$

and for a set of functions $C_A$

$$v^A|_{v^i} C_A = \{q^A, H\} C_A. \tag{31}$$
Further, from the condition (3) it follows, in particular, that the matrix $\bar{M}_{AB}|_{\nu^i}$ has $[\alpha]$ independent null vectors. Let us demonstrate that for an arbitrary theory they are

$$\frac{\partial \Phi_\alpha}{\partial p_A} \equiv \begin{pmatrix} \delta_{\alpha\beta} \\ -M_{\alpha j}\tilde{M}^{ji} \end{pmatrix}, \quad \alpha = 1, 2, \ldots, [\alpha],$$  \hspace{1cm} (32)$$

$$\bar{M}_{BA}|_{\nu^i} \frac{\partial \Phi_\alpha}{\partial p_A} = 0.$$  \hspace{1cm} (33)$$

Actually, from Eq.(27) one has

$$\bar{M}_{AB}|_{\nu^i} \frac{\partial \Phi_\alpha}{\partial p_B} = M_{A\alpha} - M_{A i} \tilde{M}^{ij} M_{j \alpha}.$$  \hspace{1cm} (34)$$

For the case $A = \beta$ the right hand side is zero according to Eq.(26), while for $A = k$ one has $M_{k\alpha} - M_{k i} \tilde{M}^{ij} M_{j \alpha} = M_{k\alpha} - M_{k\alpha} = 0$, from which it follows Eq.(33). From the right hand side of the equality (32) it follows that these null vectors are linearly independent.

At last, by using of Eq.(25),(18),(19), some combinations of the Lagrangian derivatives can be presented in the Hamiltonian form as follows:

$$\frac{\partial \bar{L}}{\partial q^A}|_{\nu^i} = \{p_A, H\},$$  \hspace{1cm} (35)$$

$$-\frac{\partial^2 \bar{L}}{\partial q^B \partial v^A} v^B \bar{R}^A|_{\nu^i} = \frac{\partial H}{\partial p_A} \frac{\partial \Phi_\beta}{\partial q^A} \bar{R}^\beta + v^B \frac{\partial v^i}{\partial q^B} M_{iA} \bar{R}^A.$$  \hspace{1cm} (36)$$

Here $\bar{R}^A(q, v)$ is any function. In particular, if it is null vector of the matrix $\bar{M}_{AB} : \bar{M}_{AB} \bar{R}^B = 0$, one has the useful relation

$$\bar{R}^A \left( \frac{\partial \bar{L}}{\partial q^A} - \frac{\partial^2 \bar{L}}{\partial q^B \partial v^A} v^B \right)|_{\nu^i} = R^\alpha \{\Phi_\alpha, H\}.$$  \hspace{1cm} (37)$$
3 Symmetries with at most one derivative acting on parameters.

Hamiltonization of the local symmetries and the corresponding Noether identities for the general case implies sufficiently tedious algebraic manipulations. So, in this section we demonstrate all the necessary tricks on example of a symmetry with at most one derivative acting on parameters (symmetry without derivatives was considered in [23]). Namely, let us consider infinitesimal local transformations of the form

$$\delta c q^A = \epsilon^a R_{0a}^A(q, \dot{q}) + \epsilon^a R_{1a}^A(q, \dot{q}),$$

(38)

and suppose that the action (1) is invariant up to total derivative term

$$\delta c S = \int d\tau (\epsilon^a \omega_{0a} + \dot{\epsilon}^a \omega_{1a}),$$

(39)

where \(\omega_{0a}, \omega_{1a}\) are some functions. If Eq.(38) depends essentially on all the parameters \(\epsilon^a(\tau), a = 1, \ldots, [a]\) (namely, if \(\text{rank} R_{1a}^A = [a]\)), then \([a] \leq [\alpha]\), as it can be seen from Eq.(47) below.

3.1 Lagrangian identities in the first order formalism.

Real consequence of the property (39) is appearance of identities among the equations of motion for the theory. To obtain them let us write Eq.(39) in the form of a series on derivatives of \(\epsilon^a\)

$$\int d\tau \left[ \frac{\partial L}{\partial \dot{q}^A} R_{0a}^A + \frac{\partial L}{\partial \dot{q}^A} \dot{R}_{0a}^A \right] \epsilon^a + \left[ \frac{\partial L}{\partial q^A} R_{1a}^A + \frac{\partial L}{\partial q^A} (R_{0a}^A + \dot{R}_{1a}^A) \right] \dot{\epsilon}^a + \epsilon^a \frac{\partial L}{\partial q^A} R_{1a}^A = \int d\tau (\dot{\omega}_{0a} \epsilon^a + (\omega_{0a} + \dot{\omega}_{1a}) \dot{\epsilon}^a + \omega_{1a} \ddot{\epsilon}^a).$$

Since it is fulfilled for an arbitrary \(\epsilon^a(\tau)\), one has

$$\frac{\partial L}{\partial \dot{q}^A} R_{1a}^A = \omega_{1a},$$

(40)
\[
\frac{\partial L}{\partial q^A} R_{1A} + \frac{\partial L}{\partial \dot{q}^A} R_{0A} + \frac{\partial L}{\partial q^A} \dot{R}_{1A} = \omega_0 + \dot{\omega}_1,
\]  
(41)

\[
\frac{\partial L}{\partial q^A} R_{0A} + \frac{\partial L}{\partial \dot{q}^A} \dot{R}_{0A} = \dot{\omega}_0.
\]  
(42)

Substitution of Eq. (40) into (41) gives expression for \(\omega_0\)

\[
\frac{\delta S}{\delta q^A} R_{1A} + \frac{\partial L}{\partial \dot{q}^A} R_{0A} = \omega_0,
\]  
(43)

which can be used in Eq. (42) and gives the Noether identities in the form

\[
\left( \frac{\delta S}{\delta q^A} R_{1A} \right) - \frac{\delta S}{\delta q^A} R_{0A} \equiv 0.
\]  
(44)

Further, this expression can be presented in the form of a series on derivatives of \(q^A\). It is convenient to introduce the notation

\[
K_{ia}(q, \dot{q}) \equiv \left( \frac{\partial L}{\partial q^A} - \frac{\partial^2 L}{\partial q^B \partial \dot{q}^A} \dot{q}^B \right) R_{ia}^A,
\]  
(45)

where \(i = 1, 2\). Then the series looks as

\[
\left[ K_{0a} - \dot{q}^C \frac{\partial}{\partial q^C} K_{1a} \right] - \dot{q}^A \left[ M_{AB} R_{0a}^A + \frac{\partial}{\partial q^A} K_{1a} + \left( \dot{q}^C \frac{\partial}{\partial q^C} + \ddot{q}^C \frac{\partial}{\partial \dot{q}^C} \right) M_{AB} R_{1a}^B \right] + \frac{1}{2} \dot{q}^A \left[ M_{AB} R_{1a}^B \right] \equiv 0.
\]  
(46)

Since it is true for any \(q^A(\tau)\), the square brackets in Eq. (46) must be zero separately. It gives the final form of the Lagrangian identities for our theory. Since they are fulfilled for any \(q^A(\tau)\), they will remain identities after the substitution \(\dot{q}^A(\tau) \rightarrow v^A(\tau)\). In the result we obtain identities of the first order formalism

\[
\bar{M}_{AB}(q, v) \bar{R}_{1a}^B(q, v) \equiv 0,
\]  
(47)

\[
\bar{M}_{AB} \bar{R}_{0a}^B + \frac{\partial}{\partial v^B} \bar{K}_{1a} \equiv 0,
\]  
(48)
\[
\bar{K}_{0a} - v^B \frac{\partial}{\partial q_B} \bar{K}_{1a} \equiv 0,
\]

where \(\bar{K}\) is now function on the extended space

\[
\bar{K}_{ia}(q, v) \equiv \left( \frac{\partial \bar{L}(q, v)}{\partial q^A} - \frac{\partial^2 \bar{L}}{\partial q^B \partial v^A} v^B \right) \bar{R}_{ia}^A(q, v).
\]

Below we demonstrate that Eqs.(47)-(49) supply invariance of the first order action (5) under the corresponding local transformations (see Eq.(65)-(67)).

3.2 Hamiltonian form of the identities.

Let us obtain Hamiltonian form of the identities, i.e. we perform substitution of the multipliers \(v^i(q^A, p_j, \nu^\alpha)\) into Eqs.(47)-(49).

In accordance with our division of the index: \(A = (i, \alpha)\), Eq.(47) can be rewritten as

\[
\bar{R}_{1a}^i = -\tilde{M}^{ij} \tilde{M}_{j\alpha} \bar{R}_{1a}^\alpha,
\]

\[
(\bar{M}_{\alpha\beta} - \bar{M}_{ai} \tilde{M}^{ij} \tilde{M}_{j\beta}) \bar{R}_{1a}^\beta = 0,
\]

and substitution of the multipliers \(v^i\) gives

\[
R_{1a}^i = \{q^i, \Phi_\alpha\} R_{1a}^\alpha.
\]

Eq.(52) does not contain new information, see Eq.(26). Similarly, Eq.(48) is equivalent to the pair

\[
R_{0a}^i \equiv -\tilde{M}^{ij} M_{j\alpha} R_{0a}^\alpha - \frac{\partial}{\partial p_i} K_{1a},
\]

\[
\frac{\partial}{\partial \nu^\beta} K_{1a} \equiv 0,
\]
where Eqs. (51), (52), (29), (30) were used. By using of Eqs. (23), (37) we find finally

\[ R_{0a}^i \equiv \{ q^i, \Phi_\alpha \} R_{0a}^\alpha - \{ q^i, R_{1a}^\alpha \{ \Phi_\alpha, H \} \}, \]  

\[ \frac{\partial}{\partial v^\beta} (R_{1a}^\alpha \{ \Phi_\alpha, H \}) \equiv 0. \]  

To substitute the multipliers \( v^i(q^A, p^j, v^\alpha) \) into the first term of Eq. (49) we use Eqs. (35), (36), (54), with the result being

\[ K_{0a} = R_{0a}^\alpha \{ \Phi_\alpha, H \} + \frac{\partial H}{\partial q^A} \frac{\partial}{\partial p^A} (R_{1a}^\alpha \{ \Phi_\alpha, H \}) + v^B |_{v^i} \frac{\partial v^i}{\partial q_B} M_{iA} R_{0a}^A. \]  

For the second term of Eq. (49) one has after some algebra

\[ \left. - \left( v^B \frac{\partial}{\partial q_B} K_{1a} \right) \right|_{v^i} = -v^B \left. \frac{\partial}{\partial q_B} K_{1a} + v^B \frac{\partial v^i}{\partial q_B} \left( \frac{\partial}{\partial v^i} K_{1a} \right) \right|_{v^i} = -\frac{\partial H}{\partial p^A} \frac{\partial}{\partial q^A} (R_{1a}^\alpha \{ \Phi_\alpha, H \}) - v^B |_{v^i} \frac{\partial v^i}{\partial q_B} M_{iA} R_{0a}^A, \]  

where Eqs. (31), (37), (48) were used. Collecting equations (58) and (59) one finds finally the Hamiltonian form of Eq. (49)

\[ R_{0a}^\alpha \{ \Phi_\alpha, H \} - \{ R_{1a}^\alpha \{ \Phi_\alpha, H \}, H \} \equiv 0. \]  

Thus, we have obtained Hamiltonian form of the identities. They can be divided on two parts. The first part means that in arbitrary theory the generators \( R_{1a}^i |_{v^i} \), \( R_{0a}^i |_{v^i} \) can be expressed through the remaining one

\[ R_{1a}^i = \{ q^i, \Phi_\alpha \} R_{1a}^\alpha. \]  

\[ R_{0a}^i = \{ q^i, \Phi_\alpha \} R_{0a}^\alpha - \{ q^i, R_{1a}^\alpha \{ \Phi_\alpha, H \} \}. \]  

The second part involves only the generators \( R^\alpha \) and is

\[ \frac{\partial}{\partial v^\beta} (R_{1a}^\alpha \{ \Phi_\alpha, H \}) \equiv 0, \]  

\[ R_{0a}^\alpha \{ \Phi_\alpha, H \} - \{ R_{1a}^\alpha \{ \Phi_\alpha, H \}, H \} \equiv 0. \]  

Remind that \( R_{ia}^A \equiv \tilde{R}_{ia}^A(q^A, v^A) |_{v^i}. \)
3.3 Local symmetry of the first order action.

Let us return to discussion of the local symmetries structure. First we note that the following transformations
\[
\delta_c q^A = \epsilon^a \bar{R}_{0a}^A + \epsilon^a \bar{R}_{1a}^A, \tag{65}
\]
\[
\delta_c p_A = \frac{\partial^2 \bar{L}}{\partial q^A \partial v^B} \delta_c q^B + \epsilon^a \frac{\partial}{\partial q^A} \bar{K}_{1a}, \tag{66}
\]
\[
\delta_c v^A = (\delta_c q^A), \tag{67}
\]
leave invariant the first order action (5), as a consequence of the identities (47)-(49). Actually, variation of the action $S_v$ under Eqs.(65),(67) and under some $\delta p_A$ can be presented as (up to total derivative)
\[
\delta S_v = \int d\tau \left[ \epsilon^a \bar{K}_{0a} + \epsilon^a \bar{K}_{1a} - \dot{v}^A \bar{M}_{AB} (\epsilon^a \bar{R}_{0a}^B + \epsilon^a \bar{R}_{1a}^B) + \left( \delta p_A - \frac{\partial^2 \bar{L}}{\partial q^A \partial v^B} \delta_c q^B \right) (\dot{q}^A - v^A) \right] \tag{68}
\]
\[
= \int d\tau \left[ \epsilon^a \dot{v}^A \bar{M}_{AB} \bar{R}_{1a}^B + \epsilon^a \left( \bar{K}_{0a} - v^A \frac{\partial}{\partial q^B} \bar{K}_{1a} \right) - \epsilon^a \dot{v}^A \left( \bar{M}_{AB} \bar{R}_{0a}^B + \frac{\partial}{\partial v^A} \bar{K}_{1a} \right) + \left( \delta p_A - \frac{\partial^2 \bar{L}}{\partial q^A \partial v^B} \delta_c q^B - \epsilon^a \frac{\partial}{\partial q^A} \bar{K}_{1a} \right) (\dot{q}^A - v^A) \right], \tag{69}
\]
where integration by parts for the second term in Eq.(68) was performed. The first and the second lines in Eq.(69) are zero according to Eqs.(47)-(49). Then the variation $\delta S_v$ will be total derivative if we take $\delta p_A$ according to Eq.(66).
3.4 Local symmetry of the Hamiltonian action.

From the discussion in Section 2 one expects that the transformations (65)-(67) with the multipliers \( v^i \) substituted will be symmetry of the Hamiltonian action (22). Let us find their manifest form. Using Eqs.(62),(61) one has for the variation \( \delta \epsilon q^i |_{v^i} \)

\[
\delta \epsilon q^i |_{v^i} = (\epsilon^a R_{0a}^\beta + \dot{\epsilon}^a R_{1a}^\beta) \{q^i, \Phi_\beta\} - \epsilon^a \{q^i, R_{1a}^\beta \{\Phi_\alpha, H\}\}. \tag{70}
\]

The variation \( \delta \epsilon q^\alpha |_{v^i} \) can be identically rewritten in a similar form

\[
\delta \epsilon q^\alpha |_{v^i} = \epsilon^a R_{0a}^\alpha + \dot{\epsilon}^a R_{1a}^\alpha \equiv (\epsilon^a R_{0a}^\beta + \dot{\epsilon}^a R_{1a}^\beta) \{q^\alpha, \Phi_\beta\} - \epsilon^a \{q^\alpha, R_{1a}^\beta \{\Phi_\alpha, H\}\}, \tag{71}
\]

since \( \{q^\alpha, \Phi_\beta\} = \delta^\alpha_\beta \) and since the quantity \( R_{1a}^\beta \{\Phi_\beta, H\} \) do not depends on \( p_\alpha \). For the variation \( \delta p_A |_{v^i} \) one has

\[
\delta \epsilon p_A |_{v^i} = \frac{\partial \Phi_B}{\partial q^A} \delta \epsilon q^B |_{v^i} + \epsilon^a \frac{\partial}{\partial q^A} \bar{K}_{1a} |_{v^i} = - \frac{\partial \Phi_B}{\partial q^A} \delta \epsilon q^B |_{v^i} + \epsilon^a \frac{\partial}{\partial q^A} \bar{K}_{1a} |_{v^i} = - \frac{\partial \Phi_A}{\partial q^A} \delta \epsilon q^A |_{v^i} - M_{Bi} \frac{\partial v^i}{\partial q^A} (\epsilon^a R_{0a}^B + \dot{\epsilon}^a R_{1a}^B) + \epsilon^a \frac{\partial}{\partial q^A} (R_{1a}^\alpha \{\Phi_\alpha, H\}) + \epsilon^a M_{Bi} \frac{\partial v^i}{\partial q^A} R_{0a}^B = (\epsilon^a R_{0a}^\alpha + \dot{\epsilon}^a R_{1a}^\alpha) \{p_A, \Phi_\alpha\} - \epsilon^a \{p_A, R_{1a}^\alpha \{\Phi_\alpha, H\}\}. \tag{72}
\]

where Eqs.(12),(13),(37),(47),(48) were used. Thus we have found Hamiltonian form of the local symmetry (58)

\[
\delta \epsilon q^A = \{q^A, \Phi_\alpha\} \delta \epsilon q^\alpha - \epsilon^a \{q^A, R_{1a}^\alpha \{\Phi_\alpha, H\}\},
\]

\[
\delta \epsilon p_A = \{p_A, \Phi_\alpha\} \delta \epsilon q^\alpha - \epsilon^a \{p_A, R_{1a}^\alpha \{\Phi_\alpha, H\}\}, \tag{73}
\]

\[
\delta \epsilon v^\alpha = (\delta \epsilon q^\alpha)^{\prime},
\]
where
\[ \delta \epsilon q^\alpha \equiv \epsilon^a R_{0a}^\alpha + \dot{\epsilon}^a R_{1a}^\alpha. \]  

Hamiltonian action (22) is invariant under these transformations, as a consequence of the identities (63), (64). Up to total derivative, variation of the first term in Eq. (22) can be expressed as follows
\[ \delta (p_A \dot{q}^A) = \Phi_\alpha (\delta \epsilon q^\alpha) - \epsilon^a R_{1a}^\alpha \{ \Phi_\alpha, H \} - \epsilon^a v^\beta \frac{\partial}{\partial v^\beta} (R_{1a}^\alpha \{ \Phi_\alpha, H \}), \]
while for the second term one has
\[ \delta (-H) = -\Phi_\alpha (\delta \epsilon q^\alpha) + \epsilon^a R_{0a}^\alpha \{ \Phi_\alpha, H \} + \epsilon^a (R_{0a}^\alpha \{ \Phi_\alpha, H \} - \{ R_{1a}^\alpha \{ \Phi_\alpha, H \}, \}). \]

collecting these terms and using Eqs. (63), (64) one has \( \delta \epsilon S_H = div. \)

4 Hamiltonization of the identities for the general local symmetry.

This section is devoted to Hamiltonization of the Lagrangian identities which correspond to the local symmetry
\[ \delta \epsilon q^A = \sum_{k=0}^{[k]} (k) \epsilon^a R_{(k)a}^A (q, \dot{q}), \]  
where \((k) \epsilon^a \equiv \frac{\partial^k}{\partial \tau^k} \epsilon^a \equiv \partial^k \epsilon^a. \) It is supposed that the action (1) is invariant up to total derivative term
\[ \delta \epsilon S = \int d\tau \left( \sum_{k=0}^{[k]} (k) \epsilon^a \omega_{ka} \right), \]  
with some functions \( \omega_{ka}(q, \dot{q}). \)

\footnote{We show below that presence of the term \((k) \epsilon^a \) in Eq. (77) implies appearance of \( k \)-tiary Hamiltonian constraints. From this it follows that \([k] < \infty \) for a mechanical system with finite number of degrees of freedom. Also, with any transformation which involve variation of the evolution parameter: \( \hat{\delta} \tau, \hat{\delta} q^A \) one associates unambiguously the transformations of the form (72) as follows: \( \delta \tau = 0, \delta q^A = -\dot{q}^A \hat{\delta} \tau + \hat{\delta} q^A. \) If \( \hat{\delta} \) is a symmetry of the action, the same will be true for \( \delta. \) Thus, Eq. (72) incorporates this case also.}
4.1 Identites of the first order formalism.

The first step is to write Eq. (76) in the form of a series on derivatives of $\epsilon^a$

$$
\int d\tau \left[ \frac{\partial L}{\partial q^A} \sum_{k=0}^{[k]} \epsilon^a R_{(k)a}^A + \frac{\partial L}{\partial \dot{q}^A} \sum_{k=0}^{[k]} \left( (k+1)^a R_{(k)a}^A + \epsilon^a \dot{R}_{(k)a}^A \right) \right] = \\
\sum_{k=0}^{[k]} \left( (k+1)^a \omega_{ka} + (k)^a \dot{\omega}_{ka} \right).
$$

Since it is fulfilled for any $\epsilon^a(\tau)$, we can compare terms which are proportional to $\epsilon, \dot{\epsilon}, \ldots$, separately

$$
\frac{\partial L}{\partial q^A} R_{([k]+1-k)a}^A + \frac{\partial L}{\partial \dot{q}^A} \left( R_{([k]-k)a}^A + \dot{R}_{([k]+1-k)a}^A \right) = \\
\omega_{[k]-k,a} + \dot{\omega}_{[k]+1-k,a},
$$

where $k = 0, 1, \cdots, [k]$, and it is implied $R_{[k]+1} = \omega_{[k]+1} \equiv 0$. We can substitute first equation of the system (77) into the second one and so on, it gives manifest form of the functions $\omega_{ka}$. Let us denote

$$
S_{(i)a}(q, \dot{q}, \ddot{q}) \equiv \frac{\delta S}{\delta q^A} R_{(i)a}^A \equiv K_{(i)a} - \dot{q}^B M_{BA} R_{(i)a}^A, \\
K_{(i)a}(q, \dot{q}) \equiv \left( \frac{\partial L}{\partial q^A} - \frac{\partial^2 L}{\partial q^B \partial \dot{q}^A} \dot{q}^B \right) R_{(i)a}^A.
$$

Then one has expressions for $\omega$ as follows

$$
\sum_{i=0}^{k-1} (-)^{k-1-i} \partial^{k-1-i} S_{([k]-i)a} + \frac{\partial L}{\partial \dot{q}^A} R_{([k]-k)a}^A = \omega_{([k]-k)a},
$$

while the last equation of the system (77) gives the Noether identities

$$
\sum_{k=0}^{[k]} (-)^{[k]-k} \partial^{[k]-k} S_{([k]-k)a} \equiv 0, \quad a = 1, 2, \cdots, [a].
$$

Note that the $S_{(i)a}$ is at most linear on $\dot{q}^A$ and the maximum possible degree of the time derivative in Eq. (80) is $[k] + 2$. Further, Eq. (80) can be
presented in the form of a series which consist of the terms

\[ \partial^{k_1} q^A_1 \cdots \partial^{k_p} q^A_p X_{(A_1 \cdots A_p)a}(q, \dot{q}) \]

where the possible values for \( p, k_i \) are: \( p = 0, 1, \ldots, \left[ \frac{k}{2} \right] \), \( k_i \geq 2 \), \( \sum k_i \leq [k] + 2 \), and all the coefficients \( X_{(A_1 \cdots A_p)a} \) are symmetric on their indices \( A_i \). Since Eq. (80) is fulfilled for an arbitrary \( q^A(\tau) \) one concludes

\[ X_{(A_1 \cdots A_p)a}(q, \dot{q}) \equiv 0. \]  

(81)

Remarkable fact is that only \( a \cdot [A] \cdot ([k] + 1) + a \) functions \( X \) from Eq. (81) turns out to be independent. Namely, the direct and tedious calculation gives the following independent identities which follow form Eq. (80)

\[ M_{BA} R_{[k]a}^A \equiv 0, \]

\[ M_{BA} R_{([k]−1)a}^A + \frac{\partial}{\partial q^B} \left[ \sum_{i=1}^{k} (-)^{i−1} \partial^{i−1} S_{([k]−k+i)a} \right] \equiv 0, \quad k = 1, \ldots, [k], \]  

(82)

\[ \sum_{k=0}^{[k]} (-)^k \dot{q}^{C_1} \cdots \dot{q}^{C_k} \frac{\partial}{\partial q^{C_1}} \cdots \frac{\partial}{\partial q^{C_k}} K_{(k)a} \equiv 0. \]

This system can be expressed further in an equivalent form in terms of the quantities \( K_{(i)a} \) only. Actually, using the second equation of the system (82) in the third one and so on, one convinces that all the terms which are proportional to \( \ddot{q}^A \) disappears. This form of the identities was obtained also in [26] and used for analysis of constraint algebra in a theory without second class constraints. Since the resulting equations are satisfied for an arbitrary \( q^A(\tau) \), they will remain identities after the substitution \( \dot{q}^A(\tau) \rightarrow v^A(\tau) \). In the result, identities of the first order formalism are

\[ \tilde{M}_{BA} \tilde{R}_{([k])a}^A = 0, \]

\[ \tilde{M}_{BA} \tilde{R}_{([k]−1)a}^A + \frac{\partial}{\partial u^B} \tilde{K}_{([k])a} = 0, \]
\[ \bar{M}_{BA} R_{([k]-2)a}^A + \frac{\partial}{\partial v^B} \left[ \bar{K}_{([k]-1)a} + (-1)^{1} v^C \frac{\partial}{\partial q^C} \bar{K}_{([k])a} \right] = 0, \]

\[ \bar{M}_{BA} R_{([k]-k)a}^A + \frac{\partial}{\partial v^B} \left[ \sum_{i=1}^{k} (-)^{i-1} v^{C_1} \cdots v^{C_{i-1}} \frac{\partial}{\partial q^{C_1}} \cdots \frac{\partial}{\partial q^{C_{i-1}}} \bar{K}_{([k]-k+i)a} \right] \equiv 0, \]

\[ \bar{M}_{BA} R_{(0)a}^A + \frac{\partial}{\partial v^B} \left[ \sum_{i=1}^{[k]} (-)^{i-1} v^{C_1} \cdots v^{C_k} \frac{\partial}{\partial q^{C_1}} \cdots \frac{\partial}{\partial q^{C_k}} \bar{K}_{(k)a} \right] \equiv 0, \quad (83) \]

\[ \sum_{k=0}^{[k]} \sum_{i=1}^{k} \frac{\partial}{\partial q^{C_1}} \cdots \frac{\partial}{\partial q^{C_k}} \bar{K}_{(k)a} \equiv 0, \]

where

\[ \bar{K}_{(i)a} \equiv \left( \frac{\partial \tilde{L}}{\partial q^A} - \frac{\partial^2 \tilde{L}}{\partial q^B \partial v^A} \right) \tilde{R}_{(i)a}^A (q, v). \quad (84) \]

Similarly to the case which was discussed in section 3, these identities supply invariance of the first order action under the corresponding local transformations (see Eq. (102)-(104) below).

Eq. (83) prompts the following notation

\[ \tilde{T}_a^{(p)}(q, p, v) \equiv \sum_{i=1}^{p-1} (-)^{i-1} v^{C_1} \cdots v^{C_{i-1}} \frac{\partial}{\partial q^{C_1}} \cdots \frac{\partial}{\partial q^{C_{i-1}}} \bar{K}_{([k]+1-p+i)a}. \quad (85) \]

Note that the quantities \( \tilde{T}_a^{(p)} \) can be described as follows: let \( \tilde{T}_a^{(1)} \equiv 0 \), then

\[ \tilde{T}_a^{(p)} = \bar{K}_{([k]+2-p)a} - v^B \frac{\partial}{\partial q^B} \tilde{T}_a^{(p-1)}, \quad (86) \]

In this notation our first order identities are

\[ \bar{M}_{BA} R_{([k]+1-p)a}^A + \frac{\partial}{\partial v^B} \tilde{T}_a^{(p)} \equiv 0, \quad p = 1, 2, \ldots, ([k] + 1), \quad (87) \]
Motivation of the ”opposite” numeration is as follows: below we show that the quantities $T_a^{(p)} \equiv \tilde{T}_a^{(p)}|_{v^i}$ are some of the $p$-ary Hamiltonian constraints.

4.2 Hamiltonian form of the identities.

Since Eq. (87), (88) are fulfilled for an arbitrary $v^A(\tau)$, they will remain identities after substitution of the multipliers $v^i(q^A, p_j, v^\alpha)$ according to Eq. (10). This procedure gives identities of the Hamiltonian formulation. In particular, they supply invariance of the Hamiltonian action (22) under the corresponding transformations (see Eq. (107)-(109) below) as well as contain information on the structure of the Hamiltonian constraint system. Let us obtain manifest form of the identities. The parts $B = i$ and $B = \alpha$ of Eq. (87) are

\[
\left( \tilde{M}_{ij} \tilde{R}_{([k]+1-p)a}^i + \tilde{M}_{i\beta} \tilde{R}_{([k]+1-p)a}^\beta + \frac{\partial}{\partial v^i} \tilde{T}_a^{(p)} \right)|_{v^i} = 0, \tag{89}
\]

\[
\left( \tilde{M}_{ij} \tilde{R}_{([k]+1-p)a}^j + \tilde{M}_{\beta\alpha} \tilde{R}_{([k]+1-p)a}^\beta + \frac{\partial}{\partial v^\alpha} \tilde{T}_a^{(p)} \right)|_{v^\alpha} = 0, \tag{90}
\]

We can find the generators $R^i$ from Eq. (89) and substitute into Eq. (90), the resulting equations are

\[
R_{([k]+1-p)a}^i = -\tilde{M}^{ij} M_{j\alpha} R_{([k]+1-p)a}^\alpha \quad \text{and} \quad \tilde{M}^{ij} \left( \frac{\partial}{\partial v^j} \tilde{T}_a^{(p)} \right)|_{v^i}, \tag{91}
\]

where Eq. (26) was used. Further, by using of Eqs. (23), (29), (30) one rewrites Eq. (91) as

\[
R_{([k]+1-p)a}^i \equiv \{ q^i, \Phi_\alpha \} R_{([k]+1-p)a}^\alpha - \{ q^i, T_a^{(p)} \}, \tag{92}
\]
\[
\frac{\partial}{\partial v^\alpha} T_a^{(p)} \equiv 0. \tag{93}
\]

Note that Eq. (93) means, in particular, that \( \bar{T} |_{v^i} \) do not depends on \( v^\alpha \): \( \bar{T}_a^{(p)} |_{v^i} = T_a^{(p)}(q^A, p_j) \). Thus we needs to find manifest form for the quantities \( \bar{T}_a^{(p)} |_{v^i} \). Starting from Eq. (86) one has

\[
\bar{T}_a^{(p)} |_{v^i} = \bar{K}_{([k]+2-p)a} |_{v^i} - v^B \frac{\partial}{\partial q^B} \left( \bar{T}_a^{(p-1)} |_{v^i} \right) - v^B \frac{\partial v^i}{\partial q^B} \left( \frac{\partial}{\partial v^i} \bar{T}_a^{(p-1)} \right) |_{v^i} =
\]

\[
\left( \frac{\partial \bar{L}}{\partial q^A} - \frac{\partial^2 \bar{L}}{\partial q^B \partial v^A u_i} u^B \right) R_{([k]+2-p)a}^A - v^B |_{v^i} \frac{\partial}{\partial q^B} T_a^{(p-1)} - v^B |_{v^i} \frac{\partial v^i}{\partial q^B} M_{i A} R_{([k]+2-p)a}^A,
\]

as a consequence of Eqs. (84), (87). The first term can be rewritten by means of Eqs. (35), (36), (92) which gives the expression (note that the function \( \bar{T}^{(p-1)} |_{v^i} \) do not depends on \( p_\alpha \))

\[
R_{([k]+2-p)a}^\beta \{ \Phi_\beta, H \} + \frac{\partial H}{\partial q^A} \frac{\partial}{\partial p_A} T_a^{(p-1)} + v^B |_{v^i} \frac{\partial v^i}{\partial q^B} M_{i A} R_{([k]+2-p)a}^A,
\]

while for the second term one finds

\[
- \frac{\partial H}{\partial p_A} \frac{\partial}{\partial q^A} T_a^{(p-1)},
\]

as a consequence of Eq. (31). Collecting this terms one obtain the recurrence relation for determining of the quantities \( T_a^{(p)} \)

\[
T_a^{(p)} = R_{([k]+2-p)a}^\beta \{ \Phi_\beta, H \} + \{ H, T_a^{(p-1)} \}, \tag{94}
\]

Since \( T_a^{(2)} \) is known from Eq. (34), one finds finally

\[
T_a^{(p)}(q^A, p_j) = \sum_{i=2}^{p} \{ H \ldots \{ H, R_{([k]+i-p)a}^\beta \{ \Phi_\beta, H \} \ldots \} \}. \tag{95}
\]
Thus, Hamiltonian form of the Lagrangian identities (82) which correspond to the general local symmetry (75) is

\[ R_{([k]+1-p)a}^i = \{q^i, \Phi_\alpha\} R_{([k]+1-p)a}^\alpha - \{q^i, T_a^{(p)}\}, \]

(96)

\[ \frac{\partial}{\partial v^\alpha} T_a^{(p)} = 0, \quad p = 2, \ldots, ([k] + 1), \]

(97)

\[ T_a^{([k]+2)} = 0, \]

(98)

with the quantities \( T_a^{(p)} \) given by Eq.(95). Remind that \( R_{(k)a} A \equiv \bar{R}_{(k)a} (q^A, v^A) |_{v^i} \). Note that the resulting expressions do not contain of the time derivatives. So, they can be compared with the Hamiltonian constraint system.

4.3 Hamiltonian identities and the complete constraint system.

The Hamiltonian identites (97) states, in particular, that the quantities \( T_a^{(p)} \), \( p = 2, 3, \ldots, ([k] + 1) \) are functions of the phase space variables \( (q^A, p_j) \) only. Their manifest form is

\[ T_a^{(2)} = R_{[k]a}^\alpha \{\Phi_\alpha, H\}, \]

\[ T_a^{(3)} = R_{([k]-1)a}^\alpha \{\Phi_\alpha, H\} + \{H, T_a^{(2)}\}, \]

\[ T_a^{(4)} = R_{([k]-2)a}^\alpha \{\Phi_\alpha, H\} + \{H, T_a^{(3)}\}, \]

(99)

\[ \ldots \]

Meaning of these quantities becomes clear if we compare Eq.(99) with the second, third \ldots stages of the Dirac-Bergmann procedure. The second stage is to investigate structure of the system \( \{H, \Phi_\alpha\} = 0 \). It can be
rewritten in the canonical form, one has schematically

\[ \{ H, \Phi_\alpha \} = Q(q, p) \begin{pmatrix} G(q, p, v) \\ \Phi^{(2)}(q, p) \\ 0 \end{pmatrix} = 0, \quad \text{det}Q \neq 0, \quad (100) \]

where the part \( G(q, p, v) = 0 \) contains (if any) equations for determining of the multipliers, while the part \( \Phi^{(2)} = 0 \) determines secondary constraints of the theory. From comparison of Eq.(100) with the first equation of the system (99) one concludes that the quantities \( T^{(2)}_a \) are some combinations of the secondary constraints \( \Phi^{(2)} \). Analogously, the third step is

\[ \{ H, \Phi^{(2)} \} \bigg|_{\Phi=\{\Phi,H\}=0} = Q' \begin{pmatrix} G'(q, p, v) \\ \Phi^{(3)}(q, p) \\ 0 \end{pmatrix} = 0 \quad (101) \]

which allows one to identity the quantities \( T^{(3)}_a \) with some of the tertiary constraints \( \Phi^{(3)} \), and so on.

Thus, the quantities \( T^{(p)}_a, \quad p = 2, 3, \ldots, ([k] + 1) \) which appears in the Hamiltonian identities (97) can be identified with some of the \( p \)-ary constraints. Then the identity (98) means that \( ([k] + 1) \)-ary constraints do not create neither \( ([k] + 2) \)-ary constraints nor equations for determining of the multipliers.

From this it follows, in particular, that a theory with the local symmetry (73) necessarily has the Hamiltonian constraints up to at least \( ([k] + 1) \)-stage.

Note that the subsystem of constraints \( T^{(p)}_a \) has very special structure. Actually, the expression \( \{ H, T^{(p)}_a \} \) gives the constraint \( T^{(p+1)}_a \) modulo the surface \( \Phi_\alpha = \{ \Phi_\alpha, H \} = 0 \) only. Namely this remarkable structure of the subsystem allows us to rewrite the Hamiltonian symmetries in the form of canonical transformation (see the subsection 5.2 below).

\footnote{Note that the primary constraints can not appear on the r.h.s. of this equation, see Eqs.(13),(14).}
5 Hamiltonian form for the general local symmetry.

5.1 The symmetry in the first order formalism.

In the subsection we prove that the following transformations

\[ \delta_\epsilon q^A = \sum_{k=0}^{[k]} \epsilon \tilde{\epsilon}^a (\tilde{K}_{(k)a} - \hat{\epsilon}^B \tilde{M}_{BA} \tilde{R}_{(k)a}^A), \]  

(102)

\[ \delta_\epsilon p_B = \frac{\partial^2 \tilde{L}}{\partial q^A \partial v^B} \delta_\epsilon q^B + \sum_{k=0}^{[k]-1} \epsilon \tilde{\epsilon}^a \frac{\partial}{\partial q^A} \tilde{T}_a^{([k]+1-k)}, \]  

(103)

\[ \delta_\epsilon v^A = (\delta_\epsilon q^A)^{\vphantom{\dagger}}, \]  

(104)

leave invariant the first order action (5) as a consequence of the first order identities (87),(88). Here \( \tilde{T}_a^{(p)} \) is given by Eq.(85). Note that the transformations (75); (102)-(104) and the transformations (107)-(109) below are equivalent sets of symmetries in the sense of definition which was given in [21].

Variation \( \delta S_v \) under an arbitrary \( \delta q^A \), \( \delta p_A \) and \( \delta v^A = (\delta q^A)^{\dagger} \) has the form

\[ \delta S_v = \int d\tau \left( \frac{\partial \tilde{L}}{\partial q^A} - \frac{\partial^2 \tilde{L}}{\partial q^A \partial v^B} v^B \right) \delta q^A - \hat{\epsilon}^B \tilde{M}_{BA} \delta q^A + \] 

\[ \left( \delta p_A - \frac{\partial^2 \tilde{L}}{\partial q^A \partial v^B} \delta q^B \right) (\dot{q}^A - v^A). \]  

(105)

After substitution of \( \delta q^A \) from Eq.(102) into the first two terms, they can be written as

\[ \sum_{k=0}^{[k]-1} \epsilon \tilde{\epsilon}^a (\tilde{K}_{(k)a} - \hat{\epsilon}^B \tilde{M}_{BA} \tilde{R}_{(k)a}^A) + \epsilon \tilde{\epsilon}^a \tilde{T}_a^{(2)}, \]  

(106)

where the identity with \( p = 1 \) from Eq.(87) and Eq.(88) were used. After integration by parts of the last term one has

\[ \sum_{k=0}^{[k]-1} \epsilon \tilde{\epsilon}^a (\tilde{K}_{(k)a} - \hat{\epsilon}^B \tilde{M}_{BA} \tilde{R}_{(k)a}^A) - \epsilon \tilde{\epsilon}^a \left( \hat{\epsilon}^B \frac{\partial}{\partial v^B} + \dot{\epsilon}^B \frac{\partial}{\partial q^B} \right) \tilde{T}_a^{(2)}. \]
The intermediate terms form the identity with $p = 2$ from Eq.(87), while the remaining one can be presented as

$$\sum_{k=0}^{[k]-2} \epsilon^a \left( \hat{K}_{(k)a} - \dot{v} B M_{BA} \hat{R}_{(k)}) a^A \right) + \sum_{k=0}^{[k]-1} \epsilon^a \left( \hat{K}_{([k]-1)a} - v B \frac{\partial}{\partial q^B} \hat{T}_a^{(2)} \right) - \sum_{k=0}^{([k]-1)} \epsilon^a \left( \frac{\partial}{\partial q^B} \hat{T}_a^{(2)} \right) (\dot{q}^B - v^B),$$

or, equivalently

$$\sum_{k=0}^{[k]-2} \epsilon^a \left( \hat{K}_{(k)a} - \dot{v} B M_{BA} \hat{R}_{(k)}) a^A \right) + \sum_{k=0}^{[k]-1} \epsilon^a \left( \hat{K}_{([k]-1)a} - v B \frac{\partial}{\partial q^B} \hat{T}_a^{(2)} \right) - \sum_{k=0}^{([k]-1)} \epsilon^a \left( \frac{\partial}{\partial q^B} \hat{T}_a^{(2)} \right) (\dot{q}^B - v^B).$$

Except the last term this expression has the same structure as Eq.(106). Thus, we can repeat these calculations. After $[k]$ integrations by parts and using of the identities (87),(88) the first two terms in Eq.(105) acquire the form

$$- \sum_{k=0}^{[k]-1} \epsilon^a \left( \frac{\partial}{\partial q^B} \hat{T}_a^{([k]+1-k)} \right) (\dot{q}^B - v^B).$$

Thus, modulo of total derivative terms, one has

$$\delta S_v = \int d\tau \left[ \delta p_A - \frac{\partial \hat{L}}{\partial q^A} \delta q^B - \sum_{k=0}^{[k]-1} \epsilon^a \left( \frac{\partial}{\partial q^A} \hat{T}_a^{([k]+1-k)} \right) (\dot{q}^A - v^A) \right].$$

Then $\delta S_v = div$ if one takes $\delta p_A$ according to Eq.(103).

### 5.2 The symmetry in the Hamiltonian formalism.

In this subsection we present manifest form of the transformations (102)-(104) with the multipliers $v^i$ substituted according to Eq.(11). Then we prove that the resulting transformations is a symmetry of the Hamiltonian action (22).
By using of Eq.(96) one finds for the variation \( \delta q^i |_{v^i} \) the expression

\[
\delta q^i |_{v^i} = \sum_{k=0}^{[k]} (k) \epsilon^a R_{(k)\alpha}^\beta \left\{ q^i, \Phi_\beta \right\} - \left\{ q^i, T_a^{([k]+1-k)} \right\},
\]

where \( T_a^{(p)} \) is given in Eq.(95). The variation \( \delta q^\alpha |_{v^i} \) can be rewritten identically in a similar form

\[
\delta q^\alpha |_{v^i} = \sum_{k=0}^{[k]} (k) \epsilon^a R_{(k)\alpha}^\beta \left\{ q^\alpha, \Phi_\beta \right\} - \left\{ q^\alpha, T_a^{([k]+1-k)} \right\},
\]

since \( \{ q^\alpha, \Phi_\beta \} = \delta^\alpha_\beta \) and since \( T_a^{(p)} \) do not contains of the variable \( p_\alpha \).

Note also that the last term in both equations is absent for \( k = [k] \), since \( T_a^{(1)} \equiv 0 \).

Hamiltonization of the first term in Eq.(103) was already considered, see Eq.(72)

\[
\frac{\partial^2 \tilde{L}}{\partial q^A \partial v^B} \delta q^B |_{v^i} = -\frac{\partial \Phi_\alpha}{\partial q^A} \delta q^\alpha |_{v^i} - M_{Bi} \frac{\partial v^i}{\partial q^A} \delta q^B |_{v^i},
\]

while in accordance with Eq.(87), the second term of Eq.(103) can be presented as

\[
\sum_{k=0}^{[k]-1} (k) \epsilon^a \frac{\partial}{\partial q^A} T_a^{([k]+1-k)} |_{v^i} = \sum_{k=0}^{[k]-1} (k) \epsilon^a \left[ \frac{\partial}{\partial q^A} T_a^{([k]+1-k)} - \frac{\partial v^i}{\partial q^A} \left( \frac{\partial}{\partial v^i} T_a^{([k]+1-k)} \right) |_{v^i} \right] = \\
\sum_{k=0}^{[k]-1} (k) \epsilon^a \frac{\partial}{\partial q^A} T_a^{[k]+1-k} + M_{Bi} \frac{\partial v^i}{\partial q^A} \delta q^B |_{v^i} - \frac{\partial v^i}{\partial q^A} M_{Bi} \epsilon^a R_{[k]a}^\beta B.
\]

The last term is zero according to Eq.(87) with \( p = 1 \). Collecting these results one has finally

\[
\delta p_A |_{v^i} = \sum_{k=0}^{[k]} (k) \epsilon^a R_{(k)\alpha}^\beta \left\{ p_A, \Phi_\beta \right\} - \left\{ p_A, T_a^{([k]+1-k)} \right\}.
\]
Thus, the Hamiltonian form for the general local transformations (73) is

\[ \delta \epsilon q^A = \{ q^A, \Phi_\alpha \} \delta \epsilon q^\alpha - \sum_{k=0}^{[k]-1} (k) a \{ q^A, T_a^{([k]+1-\ell-k)} \}, \] (107)

\[ \delta \epsilon p_A = \{ p_A, \Phi_\alpha \} \delta \epsilon q^\alpha - \sum_{k=0}^{[k]-1} (k) a \{ p_A, T_a^{([k]+1-\ell-k)} \}, \] (108)

\[ \delta \epsilon v^\alpha = (\delta \epsilon q^\alpha) \cdot, \] (109)

where

\[ \delta \epsilon q^\alpha = \sum_{k=0}^{[k]} (k) a R_{(\ell)\alpha} \] (110)

and the quantities \( T^{(p)}_a \) are presented in Eqs.(94), (95).

The Hamiltonian identities supply invariance of the Hamiltonian action (22) under these transformations. Actually, modulo of total derivative terms, variation of the first term in Eq.(22) can be presented as

\[ \delta \epsilon (p_A \dot{q}^A) = (\delta q^\alpha) \cdot \Phi_\alpha - \sum_{k=1}^{[k]} (k) a T_a^{([k]+2-\ell-k)} - \sum_{k=0}^{[k]-1} (k) a v^\beta \frac{\partial}{\partial v^\beta} T_a^{([k]+1-\ell-k)}, \]

while for the second term in Eq.(22) one finds

\[ \delta \epsilon (-H) = -(\delta q^\alpha) \cdot \Phi_\alpha + \sum_{k=0}^{[k]} (k) a T_a^{([k]+2-\ell-k)}. \]

Collecting these terms one concludes \( \delta S_H = div \) as a consequence of Eqs.(97), (98).

In conclusion of this subsection let us note that Eq.(107)-(108) can be presented in the form of canonical transformations. Let us rewrite them as follows

\[ \delta \epsilon q^A = \left\{ q^A, \delta \epsilon q^\alpha \Phi_\alpha - \sum_{k=0}^{[k]-1} (k) a T_a^{([k]+1-\ell-k)} \right\} - \{ q^A, \delta \epsilon q^\alpha \} \Phi_\alpha, \]
\[ \delta \epsilon p_A = \left\{ p_A, \delta \epsilon q^\alpha \Phi_\alpha \right\} - \sum_{k=0}^{[k]-1} \epsilon^{(k)} a T_a^{([k]+1-k)} \right\} - \left\{ p_A, \delta \epsilon q^\alpha \right\} \Phi_\alpha, \quad (111) \]

\[ \delta \epsilon v^\alpha = (\delta \epsilon q^\alpha)^\prime. \]

It can be accompanied by the trivial transformation of the Hamiltonian action (see [23])

\[ \bar{\delta} \epsilon q^A = \{ q^A, \delta \epsilon q^\alpha \} \Phi_\alpha, \]

\[ \bar{\delta} \epsilon p_A = \{ p_A, \delta \epsilon q^\alpha \} \Phi_\alpha, \quad (112) \]

\[ \bar{\delta} \epsilon v^\alpha = - (\delta \epsilon q^\alpha)^\prime + \{ H, \delta \epsilon q^\alpha \}. \]

Combination of the equations (111) and (112) has the desired canonical form.

Thus we have proved that the Hamiltonian symmetries which correspond to a general local symmetry (75) of the Lagragian formalism can be presented in the form of canonical transformation for the phase space variables \( q^A, p_A \)

\[ \delta \epsilon q^A = \{ q^A, G \}, \quad \delta \epsilon p_A = \{ p_A, G \}, \]

\[ \delta \epsilon v^\alpha = \{ H, \delta \epsilon q^\alpha \}, \quad (113) \]

where the generating function is

\[ G = \sum_{k=0}^{[k]} \epsilon^{(k)} a R(k) a^\alpha \Phi_\alpha - \sum_{k=0}^{[k]-1} \epsilon^{(k)} a T_a^{([k]+1-k)}. \quad (114) \]

6 Conclusion

Let us enumerate results of this work.
1. Starting from the Lagrangian theory with the local symmetry of a general form
\[ \delta \delta q^A = \sum_{k=0}^{[k]} \langle k \rangle a R_{(k)a} A(q, \dot{q}), \] (115)
we have obtained manifest form of the symmetry and of the corresponding Noether identities in the first order formalism (Eqs.(102)-(104),(87);(88)) as well as in the Hamiltonian one (Eqs.(107)-(109), (96)-(98)). The identities supply invariance of the first order action and of the Hamiltonian one under the corresponding transformations.

2. The Hamiltonian identities consist of two parts. The first part allows one to express the generators \( R^i \) through the others (remind that now \( R_{(k)a} A = \bar{R}_{(k)a} A(q^A, v^A)|_{v^i} \) and \( R^A = (R^i, R^\alpha) \) in accordance with Eq.(3))
\[ R_{([k]+1-p)a} \equiv \{ q^i, \Phi_\alpha \} R_{([k]+1-p)a} \alpha - \{ q^i, T_{a}^{(p)} \}. \] (116)
Manifest form of the quantities \( T \) is known
\[ T_{a}^{(p)}(q^A, p_j) = \sum_{i=2}^{p} \{ H \ldots \{ H, R_{([k]+i-p)a} \beta \{ \Phi_\beta, H \} \ldots \}. \] (117)

3. The second part of the Hamiltonian identities
\[ \frac{\partial}{\partial v^\alpha} T_{a}^{(p)} = 0, \quad p = 2, \ldots, ([k] + 1), \] (118)
\[ T_{a}^{([k]+2)} = 0, \] (119)
has the following interpretation. Eq.(118) means, in particular, that the quantities \( T_{a}^{(p)} \) do not depend on \( v^\alpha \). We have demonstrated also that the quantities \( T_{a}^{(p)}(q^A, p_j) \) are some part of the \( p \)-ary Hamiltonian constraints. Then Eq.(118) states that in a theory with the local symmetry (115) the
complete constraint system contains subsystem of constraints $T_a^{(p)} \approx 0$ of the special structure (117). Equation (119) means that $([k] + 1)$-ary constraints $T$ do not create neither new constraints nor equations for determining of the multipliers.

4. Local symmetry (113) for the Lagrangian theory implies appearance of the Hamiltonian constraints up to at least $([k] + 1)$ stage.

5. We have proved that the Hamiltonian symmetry (107)-(109), which corresponds to the Lagrangian one (113), can be presented in the form of canonical transformation (for the phase space variables $q^A, p_A$)

$$
\delta_c q^A = \{q^A, G\}, \quad \delta_c p_A = \{p_A, G\},
$$

$$
\delta_c v^\alpha = \{H, \delta_c q^\alpha\},
$$

(120)

where the generating function is the following combination of the primary constraints $\Phi_\alpha$ and the constraints $T_a^{(p)}$

$$
G = \sum_{k=0}^{[k]} \epsilon^a R_{(k)a}^\alpha \Phi_\alpha - \sum_{k=0}^{[k]-1} \epsilon^a T_a^{([k]+1-k)}. \quad (121)
$$

Difference among the equations (107)-(109) and the canonical transformations (120) is the trivial symmetry of the Hamiltonian action

$$
\tilde{\delta}_c q^A = \{q^A, \delta_c q^\alpha\} \Phi_\alpha, \quad \tilde{\delta}_c p_A = \{p_A, \delta_c q^\alpha\} \Phi_\alpha,
$$

$$
\tilde{\delta}_c v^\alpha = -(\delta_c q^\alpha) + \{H, \delta_c q^\alpha\}. \quad (122)
$$

From the previous discussion one expects that the constraints $T_a^{(p)}$ are of first class in the complete constraint system. Also, from Eqs. (117)-(119) it follows that for the simple cases (symmetry without derivative or with one derivative only) the gauge generators $R$ can be easily restored starting from the Hamiltonian formulation. We hope that the results obtained allows one to formulate a simplified procedure (as compare with [14, 21]) for the general case also. These problems will be discussed in a forthcoming paper.
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References

[1] P.A.M. Dirac, Can. J. Math. 2 (1950) 129; Lectures on Quantum Mechanics (Yeshiva Univ., New York, 1964).

[2] J.L. Anderson and P.G. Bergmann, Phys. Rev. 83 (1951) 1018.

[3] P.G. Bergmann and I. Goldberg, Phys. Rev. 98 (1955) 531.

[4] D. M. Gitman and I. V. Tyutin, Quantization of Fields with Constraints (Berlin: Springer-Verlag, 1990).

[5] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems (Princeton: Princeton Univ. Press, 1992).

[6] K. Kamimura, Nuovo Cimento 68B (1982) 33.

[7] C. Batlle, J. Gomis, J.M. Pons and N. Roman-Roy, J. Math. Phys. 27 (1986) 12.

[8] A. Cabo and D. Louis-Martinez, Phys. Rev. D42 (1990) 2726.

[9] R. Banerjee and J. Barcelos-Neto, Ann. Phys. 265 (1998) 134.

[10] J. Barcelos-Neto, Phys. Rev. D55 (1997) 2265.

[11] C. Wotzasek, Ann. Phys. 243 (1995) 76.

[12] J.M. Pons and J.Antonio García, hep-th/9908151.
[13] J.Gomis, K. Kamimura and J.M. Pons, Eurphys. Lett., 2 (1986) 187.

[14] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B332 (1990) 169.

[15] V. Mukhanov and A. Wipf, Int. J. Mod. Phys., A10 (1995) 579.

[16] R. Sugano and T. Kimura, Phys. Rev. D41 (1990) 41.

[17] R. Sugano and T. Kimura, J. Math. Phys. 31 (1990) 2337.

[18] J. Gomis, M. Henneaux and J.M. Pons, Class. Quantum. Grav. 7 (1990) 1089.

[19] S.A. Gogilidze, V.V. Sanadze, Yu.S. Surovtsev and F.G. Tkebuchava, Theor. Math. Phys. 102 (1995) 47.

[20] V.A. Borovkov and I.V. Tyutin, Physics of Atomic Nuclei 61 (1998) 1603.

[21] V.A. Borovkov and I.V. Tyutin, Physics of Atomic Nuclei 62 (1999) 1070.

[22] R. Banerjee, H.J. Rothe and K.D. Rothe, hep-th/9907217; hep-th/9909039.

[23] A.A. Deriglazov, hep-th/9412244.

[24] Kh.S. Nirov and A.V. Razumov, Int. J. Mod. Phys. 7 (1992) 5549.

[25] Kh.S. Nirov and A.V. Razumov, J. Math. Phys. 34 (1993) 3933.

[26] Kh.S. Nirov, Int. J. Mod. Phys. A 10 (1995) 4087.