ON THE LOCAL HOMOLOGY OF ARTIN GROUPS OF FINITE AND AFFINE TYPE

GIOVANNI PAOLINI

Abstract. We study the local homology of Artin groups using weighted discrete Morse theory. In all finite and affine cases, we are able to construct Morse matchings of a special type (we call them “precise matchings”). The existence of precise matchings implies that the homology has a square-free torsion. This property was known for Artin groups of finite type, but not in general for Artin groups of affine type. We also use the constructed matchings to compute the local homology in all exceptional cases, correcting some results in the literature.

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1. Introduction

In this paper we study the local homology of Artin groups with coefficients in the Laurent polynomial ring $R = \mathbb{Q}[q^{\pm 1}]$, where each standard generator acts as a multiplication by $-q$. This homology has been already thoroughly investigated for groups of finite type [Pre88, DCPS99, DCPS01, Cal05, Sal15, PS18], also with integral coefficients [CS04, Cal06], and for some groups of affine type [CMS08a, CMS08b, CMS10, SV13, PS18]. This work is meant to be a natural continuation of [PS18], and is based on the combinatorial techniques developed in [SV13, PS18].

In [SV13] Salvetti and Villa introduced a new combinatorial method to study the homology of Artin groups, based on discrete Morse theory. They described the general framework, and made explicit computations for all exceptional affine groups. In [PS18] the author and Salvetti developed the theory further, and made explicit
computations for the affine family $\tilde{C}_n$ as well as for the (already known) families $A_n$, $B_n$ and $\tilde{A}_n$. A common ingredient emerged in all the cases considered there, namely precise matchings. It was shown that whenever an Artin group admits precise matchings, then its local homology has a square-free torsion. For Artin groups of finite type, the absence of higher powers in the torsion is a consequence of the isomorphism with the (constant) homology of the corresponding Milnor fiber [Cal05]. This geometric argument does not apply to Artin groups of infinite type, but precise matchings proved to be useful also beyond the finite case.

In the present work we show that precise matchings exist for all Artin groups of finite and affine type. As we said, this implies the square-freeness of the torsion in the local homology. The elegance of this conclusion seems to hint at some unknown deeper geometric reason. The main results, stated in Section 3, are the following.

**Theorem 3.1.** Every Artin group of finite or affine type admits a $\varphi$-precise matching for each cyclotomic polynomial $\varphi$.

**Corollary 3.2.** Let $G_W$ be an Artin group of finite or affine type. Then the local homology $H_*(X_W; R)$ has no $\varphi^k$-torsion for $k \geq 2$.

We are able to use precise matchings to carry out explicit homology computations for all exceptional finite and affine cases. In particular we recover the results of [DCPSS99, SV13], with small corrections. The matchings we find for $D_n$, $\tilde{B}_n$, and $\tilde{D}_n$ are quite complicated, so we prefer to omit explicit homology computations for these cases (the homology for $D_n$ and $\tilde{B}_n$ was already computed in [DCPSS99] and [CMS10], respectively). The remaining finite and affine cases, namely $A_n$, $B_n$, $\tilde{A}_n$, and $\tilde{C}_n$, were already discussed in [PS18].

We also provide a software library which can be used to generate matchings for any finite or affine Artin group, check preciseness, and compute the homology. Source code and instructions are available online [Pao17].

This paper is structured as follows. In Section 2 we review the general combinatorial framework developed in [SV13, PS18]. We introduce the local homology $H_*(X_W; R)$, which is the object of our study, together with algebraic complexes to compute it. We present weighted discrete Morse theory and precise matchings, and recall some useful results. In Section 3 we state and discuss the main results of this paper. Subsequent sections are devoted to the proof of the main theorem. In Section 4 we show that it is enough to construct precise matchings for irreducible Artin groups. In Section 5 we recall the computation of the weight of irreducible components of type $A_n$, $B_n$ and $D_n$, which is used later. In Sections 6-10 we construct precise matchings for the families $A_n$, $D_n$, $\tilde{B}_n$, $\tilde{D}_n$ and $I_2(m)$. Finally, in Section 11 we deal with the exceptional cases.

2. Local homology of Artin groups via discrete Morse theory

In this section we are going to recall the general framework of [SV13, PS18] for the computation of the local homology $H_*(X_W; R)$.

Let $(W, S)$ be a Coxeter system on a finite generating set $S$, and let $\Gamma$ be the corresponding Coxeter graph (with $S$ as its vertex set). Denote by $G_W$ the corresponding Artin group, with standard generating set $\Sigma = \{g_s \mid s \in S\}$. Define $K_W$ as the (finite) simplicial complex over $S$ given by

$$K_W = \{\sigma \subseteq S \mid \text{the parabolic subgroup } W_\sigma \text{ generated by } \sigma \text{ is finite}\}.$$
It is convenient to include the empty set $\emptyset$ in $K_W$. Let $X_W$ be the quotient of the Salvetti complex of $W$ by the action of $W$. This is a finite (non-regular) CW complex, with polyhedral cells indexed by $K_W$. It has $G_W$ as its fundamental group, and it is conjectured to be a space of type $K(G_W, 1)$ [Sal87, Sal94, Par14]. This conjecture is known to be true for all groups of finite type [Del72] and for some families of groups of infinite type, including the affine groups of type $\tilde{A}_n, \tilde{B}_n, \text{and } \tilde{C}_n$ [Oko79, CMS10].

Consider the action of the Artin group $G_W$ on the ring $R = \mathbb{Q}[q^{\pm 1}]$ given by $g_s \mapsto [\text{multiplication by } -q]$ for all $s \in S$.

We are interested in studying the local homology $H_*(X_W; R)$, with coefficients in the local system defined by the above action of $G_W = \pi_1(X_W)$ on $R$. Whenever $X_W$ is a $K(G_W, 1)$ space, this coincides with the group homology $H_*(G_W; R)$ with coefficients in the same representation.

The local homology $H_*(X_W; R)$ is computed by the algebraic complex

$$C_k = \bigoplus_{\sigma \in K_W \mid |\sigma| = k} R e_\sigma$$

with boundary

$$\partial(e_\sigma) = \sum_{\tau \subset \sigma} [\sigma : \tau] \frac{W_\sigma(q)}{W_\tau(q)} e_\tau,$$

where $W_\sigma(q)$ is the Poincaré polynomial of the parabolic subgroup $W_\sigma$ of $W$.

Let $C^0_k$ be the 1-shifted algebraic complex of free $R$-modules which computes the reduced simplicial homology of $K_W$ with (constant) coefficients in $R$. Namely:

$$C^0_k = \bigoplus_{\sigma \in K_W \mid |\sigma| = k} R e^0_\sigma$$

with boundary

$$\partial^0(e^0_\sigma) = \sum_{\tau \subset \sigma} [\sigma : \tau] e^0_\tau.$$

Then we have an injective chain map $\Delta: C_* \to C^0_*$ defined by $e_\sigma \mapsto W_\sigma(q) e^0_\sigma$.

Therefore there is an exact sequence of complexes:

$$0 \to C_* \xrightarrow{\Delta} C^0_* \xrightarrow{\pi} L_* \to 0,$$

where

$$L_k = \bigoplus_{\sigma \in K_W \mid |\sigma| = k} \frac{R}{(W_\sigma(q))} \bar{e}_\sigma,$$

with boundary induced by the boundary of $C^0_*$. The associated long exact sequence in homology then allows to compute the homology of $C_*$ in terms of the homology of $C^0_*$ and of $L_*$:

$$\ldots \to H_{k+1}(L_*) \xrightarrow{\delta} H_k(C_*) \xrightarrow{\Delta} H_k(C^0_*) \xrightarrow{\pi} H_k(L_*) \xrightarrow{\delta} H_{k-1}(C_*) \xrightarrow{\Delta} \ldots$$

In this paper we mostly focus on Artin groups of finite and affine type, for which $K_W$ is either the full simplex (in the finite case) or its boundary (in the affine case). In the former case $H_*(C^0_*)$ is trivial, and in the latter case the only non-trivial term
is \( H_{|S|-1}(C_0^0) \cong R \). Therefore the challenging part consists in understanding the homology of the complex \( L_* \), which encodes all the torsion.

Poincaré polynomials of Coxeter groups are products of cyclotomic polynomials \( \varphi_d \) with \( d \geq 2 \). Thus the complex \( L_* \) decomposes as a direct sum of \( \varphi \)-primary components \( (L_*)_\varphi \), where \( \varphi = \varphi_d \) varies among the cyclotomic polynomials. Each component \( (L_*)_\varphi \) takes the following form:

\[
(L_k)_\varphi = \bigoplus_{\sigma \in K_W \text{ critical}} \frac{R}{(\varphi v_\varphi(\sigma))} \bar{e}_\sigma
\]

where \( v_\varphi(\sigma) \) is the multiplicity of \( \varphi \) in the factorization of \( W_\varphi(q) \). Again, the boundary in \( (L_*)_\varphi \) is induced by the boundary of \( C_0^0 \). The homology of \( L_* \) decomposes accordingly, so our goal is to study \( H_*(L_*)_\varphi \) for each cyclotomic polynomial \( \varphi \).

The main tool we are going to use is algebraic Morse theory for weighted complexes. We recall here the main points of the theory, referring to [For98 Cha00 BW02 Koz08 SV13] for more details. Let \( G \) be the incidence graph of \( K_W \), i.e. the graph having \( K_W \) as its vertex set and with a directed edge \( \sigma \to \tau \) whenever \( \tau < \sigma \). We still call simplices the vertices of \( G \), to avoid confusion with the vertices of \( K_W \). A matching \( M \) on \( G \) is a set of edges of \( G \) such that each simplex \( \sigma \in K_W \) is adjacent to at most one edge in \( M \). We say that \( \sigma \) is critical (with respect to the matching \( M \)) if none of its adjacent edges is in \( M \). We also call alternating path a sequence

\[
(\tau_0 <) \sigma_0 \triangleright \tau_1 \triangleleft \sigma_1 \triangleright \tau_2 \triangleleft \sigma_2 \triangleright \cdots \triangleright \tau_m \triangleleft \sigma_m (\triangleright \tau_{m+1})
\]

such that each pair \( \sigma_i \triangleleft \tau_i \) belongs to \( M \) and no pair \( \sigma_i \triangleright \tau_{i-1} \) belongs to \( M \). An alternating cycle is a closed alternating path. The following are key definitions of the theory.

- \( M \) is acyclic if all alternating cycles are trivial.
- \( M \) is \( \varphi \)-weighted if \( v_\varphi(\sigma) = v_\varphi(\tau) \) whenever \( (\sigma \to \tau) \in M \).

Notice that in general, by construction, one has \( v_\varphi(\sigma) \geq v_\varphi(\tau) \) for \( \tau < \sigma \).

**Theorem 2.1 (SV13 Theorem 2).** Fix a cyclotomic polynomial \( \varphi \). Let \( M \) be an acyclic and \( \varphi \)-weighted matching on \( G \). Then the homology of \( (L_*)_\varphi \) is the same as the homology of the Morse complex

\[
(L_*)_\varphi^M = \bigoplus_{\sigma \text{ critical}} \frac{R}{(\varphi v_\varphi(\sigma))} \bar{e}_\sigma
\]

with boundary

\[
\partial^M(\bar{e}_\sigma) = \sum_{\tau \text{ critical}} [\sigma : \tau]^M \bar{e}_\tau,
\]

where \( [\sigma : \tau]^M \in \mathbb{Z} \) is given by the sum over all alternating paths

\[
\sigma \triangleright \tau_1 \triangleleft \sigma_1 \triangleright \tau_2 \triangleleft \sigma_2 \triangleright \cdots \triangleright \tau_m \triangleleft \sigma_m \triangleright \tau
\]

from \( \sigma \) to \( \tau \) of the quantity

\[
(-1)^m [\sigma : \tau_1][\sigma_1 : \tau_1][\sigma_1 : \tau_2][\sigma_2 : \tau_2] \cdots [\sigma_m : \tau_m][\sigma_m : \tau].
\]

In [PS18] a special class of weighted matchings was introduced, namely precise matchings. We say that \( M \) is \( \varphi \)-precise (or simply precise) if \( M \) is acyclic and \( \varphi \)-weighted, and has the following additional property: \( v_\varphi(\sigma) = v_\varphi(\tau) + 1 \) whenever
\[ \sigma : \tau \mid M \neq 0 \] (here \( \sigma \) and \( \tau \) are critical simplices, so that \( [\sigma : \tau]_M \) is defined). This condition can be thought as a rigid (weight-consistent) maximality condition. It appears to arise naturally in the study of the local homology of Artin groups, as shown in [PS18] and in the present work.

We refer to [PS18, Section 4] for a general introduction to precise matchings. Here we are only going to briefly recall what we need to study Artin groups. Let \( C^0_*(\mathbb{Q}) \) be the 1-shifted algebraic complex of free \( \mathbb{Q} \)-modules which computes the reduced simplicial homology of \( K_W \) with coefficients in \( \mathbb{Q} \) (this is the same as \( C^0_* \), but with \( \mathbb{Q} \) instead of \( \mathbb{R} \) in the definition). An acyclic matching \( M \) on \( G \) can be used to compute a Morse complex \( C^0_*(\mathbb{Q})^M \) of \( C^0_*(\mathbb{Q}) \) as well. Call \( \delta^M_\ast \) the boundary of the Morse complex \( C^0_*(\mathbb{Q})^M \).

**Theorem 2.2** ([PS18, Theorem 5.1]). Fix a cyclotomic polynomial \( \varphi \). Let \( M \) be a \( \varphi \)-precise matching on \( G \). Then the \( \varphi \)-torsion component of the local homology \( H_\ast(X_W; R) \) in each dimension, as an \( R \)-module, is given by

\[
H_m(X_W; R)_{\varphi} \cong \left( \frac{R}{(\varphi)} \right) \oplus \text{rk} \delta^M_{m+1}.
\]

**Theorem 2.3** ([PS18, Theorem 5.1]). Suppose to have a \( \varphi \)-precise matching \( M_{\varphi} \) on \( G \) for every cyclotomic polynomial \( \varphi \). Then the local homology of \( X_W \) in each dimension, as an \( R \)-module, is given by

\[
H_m(X_W; R) \cong \left( \bigoplus_{\varphi} \left( \frac{R}{(\varphi)} \right) \oplus \text{rk} \delta^M_{m+1} \right) \oplus H_m(C^0_\ast).
\]

In particular the term \( H_\ast(C^0_\ast) \) gives the free part of the homology, whereas the other direct summands give the torsion part.

**Corollary 2.4** ([PS18, Corollary 5.2]). Suppose that \( G_W \) is an Artin group that admits a \( \varphi \)-precise matching for every cyclotomic polynomial \( \varphi \). Then the homology \( H_\ast(X_W; R) \) has no \( \varphi^k \)-torsion for \( k \geq 2 \).

The formula of Theorem 2.3 simplifies further when \( G_W \) is of finite type (the free part disappears) or of affine type (the free part only appears in dimension \( |S| - 1 \) and has rank 1).

### 3. The main theorem

As mentioned in the introduction, the main result of this paper is the following.

**Theorem 3.1.** Every Artin group of finite or affine type admits a \( \varphi \)-precise matching for each cyclotomic polynomial \( \varphi \).

By Corollary 2.4 this has the following immediate consequence.

**Corollary 3.2.** Let \( G_W \) be an Artin group of finite or affine type. Then the local homology \( H_\ast(X_W; R) \) has no \( \varphi^k \)-torsion for \( k \geq 2 \). \( \square \)

For Artin groups of finite type, the local homology \( H_\ast(X_W; R) \) coincides with the (constant) homology \( H_\ast(F_W; \mathbb{Q}) \) of the Milnor fiber of the associated hyperplane arrangement [Cal05]. The \( q \)-multiplication on the homology of \( X_W \) corresponds to the action of the monodromy operator on the homology of \( F_W \). The monodromy
operator has a finite order $N$, thus the polynomial $q^N - 1$ must annihilate the homology. Therefore there can only be square-free torsion.

The fact that the same conclusion holds for Artin groups of affine type is surprising, and might be due to some deeper geometric reasons which we still do not know.

The proof of Theorem 3.1 is split throughout the rest of this paper. In Section 4 we show that it is enough to construct precise matchings in the irreducible finite and affine cases. Case $A_n$ was done in [PS18]. However, we study it again in Section 6 as we need it for $D_n$, $B_n$ and $D_n$. Cases $B_n$, $A_n$ and $C_n$ were done as well in [PS18], so we do not treat them again here. Case $D_n$ is considered in Section 7, case $B_n$ in Section 8, case $D_n$ in Section 9 and case $I_2(m)$ in Section 10. In all remaining exceptional cases we construct precise matchings via a computer program, as discussed in Section 11.

We provide a software library to construct precise matchings for any given finite or affine Artin group, following [PS18] and the present paper. Source code and instructions can be found online [Pao17]. This library can be used to check preciseness and compute the homology.

**Remark 3.3.** Not all Artin groups admit precise matchings for every cyclotomic polynomial. For example, consider the Coxeter system $(W, S)$ defined by the following Coxeter matrix:

$$
\begin{pmatrix}
1 & 3 & 3 & 2 & \infty & 4 \\
3 & 1 & 3 & 4 & 2 & \infty \\
3 & 3 & 1 & \infty & 4 & 2 \\
2 & 4 & \infty & 1 & \infty & \infty \\
\infty & 2 & 4 & \infty & 1 & \infty \\
4 & \infty & 2 & \infty & \infty & 1
\end{pmatrix}
$$

The simplicial complex $K_W$ consists of: three 2-simplices $\{\{1, 2, 4\}, \{2, 3, 5\}, \{1, 3, 6\}\}$, all having \(\varphi_2\)-weight equal to 3; six 1-simplices with \(\varphi_2\)-weight equal to 2 (\{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{1, 6\}, \{3, 6\}), and three 1-simplices with \(\varphi_2\)-weight equal to 1 (\{1, 2\}, \{2, 3\}, \{1, 3\}); six 0-simplices (\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}), all having \(\varphi_2\)-weight equal to 1: one empty simplex, with \(\varphi_2\)-weight equal to 0. A \(\varphi_2\)-weighted matching can only contain edges between simplices of weight 1. Since the three 1-simplices of weight 1 form a cycle, at least one of them (say \{1, 2\}) is critical. Then the incidence number between \{1, 2, 4\} and \{1, 2\} is non-zero, and their \(\varphi_2\)-weights differ by 2. Therefore the matching cannot be \(\varphi_2\)-precise.

We introduce here a few notations that will be used later. Given a simplex $\sigma \in K_W$, denote by $\Gamma(\sigma)$ the subgraph of $\Gamma$ induced by $\sigma$. We will sometimes speak about the connected components of $\Gamma(\sigma)$, which will be denoted by $\Gamma_i(\sigma)$ for some index $i$. Also, given a vertex $v \in S$, define

$$
\sigma \uplus v = \begin{cases} 
\sigma \cup \{v\} & \text{if } v \notin \sigma, \\
\sigma \setminus \{v\} & \text{if } v \in \sigma.
\end{cases}
$$

4. **Reduction to the irreducible cases**

Let $(W_1, S_1)$ and $(W_2, S_2)$ be Coxeter system, and consider the product Coxeter system $(W_1 \times W_2, S_1 \sqcup S_2)$. Suppose that the Artin groups $G_{W_1}$ and $G_{W_2}$ admit
The critical simplices \( \sigma \). The weights behave well with respect to this decomposition: projection onto be the set of alternating paths from \( \sigma \). If \( \sigma \) is not true for \( \sigma \) and \( \sigma \) takes the following form, for some fixed simplex \( \sigma \). This is because if an edge of the form \( \sigma \) is non-empty, so \( \dim(\tau) + 1 \) and so that \( v_\varphi(\tau_1 \cup \tau_2) = v_\varphi(\tau_1) + v_\varphi(\tau_2) \). The critical simplices \( \sigma \) and \( \sigma \) are critical in \( \mathcal{K}_{\mathbf{w_1}} \) and \( \mathcal{K}_{\mathbf{w_2}} \).

Any alternating path in \( \mathcal{K}_{\mathbf{w_1} \times \mathbf{w_2}} \) projects onto an alternating path in \( \mathcal{K}_{\mathbf{w_2}} \) via the map \( \sigma_1 \cup \sigma_2 \mapsto \sigma_2 \) (provided that multiple consecutive occurrences of the same simplex are replaced by a single occurrence). This is because an edge of the form \( \sigma_1 \cup \sigma_2 \rightarrow \sigma_1 \cup \tau_2 \). Therefore \( c \) takes the following form, for some fixed simplex \( \sigma_2 \) in \( \mathcal{K}_{\mathbf{w_2}} \): 

\[
\sigma_1 \cup \sigma_2 \cup \tau_1 \cup \sigma_2 \cup \ldots \cup \tau_1 \cup \sigma_2 \cup \sigma_1 \cup \sigma_2.
\]

If \( \sigma_2 \) is critical in \( \mathcal{K}_{\mathbf{w_2}} \), then also the projection of \( c \) onto \( \mathcal{K}_{\mathbf{w_1}} \) is an alternating cycle. By acyclicity of \( \mathcal{M}_1 \) such a projection must be the trivial cycle, so also \( c \) is trivial. On the other hand, if \( \sigma_2 \) is not critical, then none of the edges \( \sigma_1 \cup \sigma_2 \rightarrow \tau_1 \cup \sigma_2 \) is in \( \mathcal{M} \), thus \( c \) must be trivial as well.

By construction, and by additivity of the weight function \( v_\varphi \), the matching \( \mathcal{M} \) is \( \varphi \)-weighted.

Finally, suppose that \([\sigma_1 \cup \sigma_2 : \tau_1 \cup \tau_2]^{\mathcal{M}} \neq 0\), where \( \sigma_1 \cup \sigma_2 \) and \( \tau_1 \cup \tau_2 \) are critical simplices of \( \mathcal{K}_{\mathbf{w_1} \times \mathbf{w_2}} \) with \( \dim(\sigma_1 \cup \sigma_2) = \dim(\tau_1 \cup \tau_2) + 1 \). Let \( \mathcal{P} \neq \emptyset \) be the set of alternating paths from \( \sigma_1 \cup \sigma_2 \) to \( \tau_1 \cup \tau_2 \). Given any path \( p \in \mathcal{P} \), its projection onto \( \mathcal{K}_{\mathbf{w_2}} \) is an alternating path from \( \sigma_2 \) to \( \tau_2 \).

1. Suppose \( \sigma_2 = \tau_2 \). Then the projected paths are trivial in \( \mathcal{K}_{\mathbf{w_2}} \), so \( \mathcal{P} \) is in bijection with the set of alternating paths from \( \sigma_1 \) to \( \tau_1 \) in \( \mathcal{K}_{\mathbf{w_1}} \). Therefore \([\sigma_1 : \tau_1]^{\mathcal{M}_1} = \pm[\sigma_1 \cup \sigma_2 : \tau_1 \cup \tau_2]^{\mathcal{M}} \neq 0\). Since \( \mathcal{M}_1 \) is \( \varphi \)-precise, we conclude that \( v_\varphi(\tau_1) = v_\varphi(\sigma_2) = \dim(\tau_1 \cup \tau_2) + 1 \). Then

2. Suppose \( \sigma_2 \neq \tau_2 \). The projection of any \( p \in \mathcal{P} \) onto \( \mathcal{K}_{\mathbf{w_2}} \) is a non-trivial alternating path, and \( \mathcal{P} \) is non-empty, so \( \dim(\sigma_2) = \dim(\tau_2) + 1 \). Then
\[ \dim(\sigma_1) = \dim(\tau_1). \] For any alternating path \( p \in \mathcal{P} \), consider now its projection \( q \) onto \( K_{W_1} \). We want to prove that \( q \) is a trivial path (thus in particular \( \sigma_1 = \tau_1 \)). Suppose by contradiction that \( q \) is non-trivial. Then, since \( \sigma_1 \) and \( \tau_1 \) have the same dimension, one of the following three possibilities must occur.

- The path \( q \) begins with an upward edge \( \sigma_1 \triangleright \rho \). Then \( (\rho \rightarrow \sigma_1) \in M_1 \), which is not possible because \( \sigma_1 \) is critical.
- The path \( q \) ends with an upward edge \( \rho \triangleright \tau_1 \). Then \( (\tau_1 \rightarrow \rho) \in M_1 \), which is not possible because \( \tau_1 \) is critical.
- The path \( q \) begins and ends with a downward edge, so it must have two consecutive upward edges somewhere in the middle. This is also not possible by previous considerations.

We proved that the projection on \( K_{W_1} \) of any alternating path \( p \in \mathcal{P} \) is trivial, and thus in particular \( \sigma_1 = \tau_1 \) (because \( \mathcal{P} \) is non-empty). Then \( \mathcal{P} \) is in bijection with the set of alternating paths from \( \sigma_2 \) to \( \tau_2 \) in \( K_{W_2} \). We conclude as in case (1). \[ \square \]

In view of this lemma, from now on we only consider irreducible Coxeter systems.

### 5. Weight of irreducible components

In order to compute the weight \( v_\varphi(\sigma) \) of a simplex \( \sigma \in K_W \), one needs to know the Poincaré polynomial of the parabolic subgroup \( W_\sigma \) of \( W \). Let \( \Gamma_1(\sigma), \ldots, \Gamma_m(\sigma) \) be the connected components of the subgraph \( \Gamma(\sigma) \subseteq \Gamma \) induced by \( \sigma \). Then the Poincaré polynomial of \( W_\sigma \) splits as a product of the Poincaré polynomials of irreducible components of finite type:

\[
W_\sigma(q) = \prod_{i=1}^{m} W_{\Gamma_i(\sigma)}(q), \quad \text{and therefore} \quad v_\varphi(\sigma) = \sum_{i=1}^{m} v_\varphi(\Gamma_i(\sigma)).
\]

In this section we derive formulas for the \( \varphi \)-weight of an irreducible component of type \( A_n \), \( B_n \) and \( D_n \) (see Figure 1).

![Figure 1](image_url)

**Figure 1.** Coxeter graphs of type \( A_n \), \( B_n \) and \( D_n \). All these graphs have \( n \) vertices.
Components of type $A_n$. The exponents of a Coxeter group $W_{A_n}$ of type $A_n$ are $1, 2, \ldots, n$. Then its Poincaré polynomial is $W_{A_n}(q) = [n + 1]_q$. If $\varphi_d$ is the $d$-th cyclotomic polynomial (for $d \geq 2$), the $\varphi_d$-weight is then

$$\omega_{\varphi_d}(A_n) = \left[\frac{n+1}{d}\right].$$

Components of type $B_n$. In this case the exponents are $1, 3, \ldots, 2n - 3, 2n - 1$, and the Poincaré polynomial is $W_{B_n}(q) = [2n]_q$. The $\varphi_d$-weight (for $d \geq 2$) is given by

$$\omega_{\varphi_d}(B_n) = \begin{cases} \left[\frac{n}{d}\right] & \text{if } d \text{ is odd;} \\ \left[\frac{n}{d/2}\right] & \text{if } d \text{ is even.} \end{cases}$$

Components of type $D_n$. Here the exponents are $1, 3, \ldots, 2n - 3, n - 1$, and the Poincaré polynomial is $W_{D_n}(q) = [2n - 2]_q$. The $\varphi_d$-weight (for $d \geq 2$) is given by

$$\omega_{\varphi_d}(D_n) = \begin{cases} \left[\frac{n}{d}\right] & \text{if } d \text{ is odd;} \\ \left[\frac{n-1}{d/2}\right] & \text{if } d \text{ is even and } d \nmid n; \\ \frac{n}{d/2} & \text{if } d \text{ is even and } d \mid n. \end{cases}$$

6. Case $A_n$ revisited

The construction of a precise matching for the case $A_n$ was thoroughly discussed in [PS18]. However, in order to better describe precise matchings for the cases $D_n$, $B_n$, and $D_n$, we need a slightly more general construction.

Throughout this section, let $(W_{A_n}, S)$ be a Coxeter system of type $A_n$ with generating set $S = \{1, 2, \ldots, n\}$, and let $K_n^A = kW_{A_n}$. See Figure 2 for a drawing of the corresponding Coxeter graph.

![A Coxeter graph of type $A_n$.](image)

Figure 2. A Coxeter graph of type $A_n$.

For integers $f, g \geq 0$, define $K_{n,f,g}^A \subseteq K_n$ as follows:

$$K_{n,f,g}^A = \{\sigma \in K_n \mid \{1, 2, \ldots, f\} \subseteq \sigma \text{ and } \{n - g + 1, n - g + 2, \ldots, n\} \subseteq \sigma\}.$$  

In other words, $K_{n,f,g}^A$ is the subset of $K_n^A$ consisting of the simplices which contain the first $f$ vertices and the last $g$ vertices. In general, $K_{n,f,g}^A$ is not a subcomplex of $K_n^A$. For any $d \geq 2$, we are going to recursively construct a $\varphi_d$-weighted acyclic matching on $K_{n,f,g}^A$. This matching coincides with the one of [PS18] Section 5.1] when $g = 0$ and $f \leq d - 1$. See also Table 1 for an example.

Matching 6.1 ($\varphi_d$-matching on $K_{n,f,g}^A$).

(a) If $f + g \geq n$ then $K_{n,f,g}^A$ has size at most 1, and the matching is empty. In the subsequent cases, assume $f + g < n$.

(b) If $f \geq d$, then $K_{n,f,g}^A \cong K_{n-d,f-d,g}^A$ via removal of the first $d$ vertices. Define the matching recursively, as in $K_{n-d,f-d,g}^A$. In the subsequent cases, assume $f \leq d - 1$. 


(c) Case $n \geq d + g$.
(c1) If $\{1, \ldots, d - 1\} \subseteq \sigma$, then match $\sigma$ with $\sigma \not\subseteq d$ (here the vertex $d$ exists and can be removed, because $n \geq d + g$). Notice that for $f = d - 1$ this is always the case, thus in the subsequent cases we can assume $f \leq d - 2$.
(c2) Otherwise, if $f + 1 \in \sigma$ then match $\sigma$ with $\sigma \cup \{f + 1\}$.
(c3) Otherwise, if $\{f + 2, \ldots, d - 1\} \not\subseteq \sigma$ then match $\sigma$ with $\sigma \cup \{f + 1\}$.
(c4) We are left with the simplices $\sigma$ such that $\{1, \ldots, f, f + 2, \ldots, d - 1\} \subseteq \sigma$ and $f + 1 \not\in \sigma$. If we ignore the vertices $1, \ldots, f + 1$ we are left with the simplices on the vertex set $\{f + 2, \ldots, n\}$ which contain $f + 2, \ldots, d - 1$; relabeling the vertices, these are the same as the simplices on the vertex set $\{1, \ldots, n - f - 1\}$ which contain $1, \ldots, d - 2 - f$. Then construct the matching recursively as in $K^A_{n - f - 1, d - 2 - f, g}$.

(d) Case $n < d + g$ (in particular, $f \leq d - 2$).
(d1) If $n \equiv -1, 0, 1, \ldots, f \pmod{d}$ and $\sigma$ is either $\{1, \ldots, n\}$ or $\{1, \ldots, f, f + 2, \ldots, n\}$, then $\sigma$ is critical.
(d2) Otherwise, match $\sigma$ with $\sigma \cup \{f + 1\}$.

| Simplices | $v_\varphi(\sigma)$ |
|-----------|---------------------|
| ![Diagram](Image) | 2                   |
| ![Diagram](Image) | 2                   |
| ![Diagram](Image) | 1                   |
| ![Diagram](Image) | (critical)           |
| ![Diagram](Image) | 2                   |
| ![Diagram](Image) | (critical)           |

Table 1. Matching $[6.1]$ on $K^A_{7,1,3}$ for $d = 3$.

Lemma 6.2. Matching $[6.1]$ is acyclic.

Proof. The proof is by induction on $n$, the case $n = 0$ being trivial. In case (a) the matching is empty and so it is acyclic. In case (b), the matching on $K^A_{n - d, f - d, g}$ is acyclic by induction, and therefore also the matching on $K^A_{n,f,g}$ is acyclic.

Consider now case (c). Let $\eta: K^A_{n,f,g} \to \{p_1 > p_4 > p_{2,3}\}$ be the poset map that sends $\sigma$ to: $p_1$, if subcase (c1) applies; $p_{2,3}$, if subcase (c2) or (c3) applies; $p_4$, if subcase (c4) applies. We have that $\eta(\sigma) = \eta(\tau)$ whenever $\sigma$ is matched with $\tau$. In addition, on each fiber $\eta^{-1}(p)$ the matching is acyclic (for $p = p_4$, this follows by induction). Therefore by the Patchwork Theorem [Koz08, Theorem 11.10] the entire matching is acyclic.

We are left with case (d). Here the matching is of the form $\{\sigma \leftrightarrow \sigma \cup (f + 1)\}$ (possibly leaving out a pair), so it is acyclic. □
Lemma 6.3. Matching 6.1 is $\varphi_d$-weighted.

Proof. This proof is also by induction on $n$. In case (b), removing the first $d$ vertices decreases all $\varphi_d$-weights by 1, so the matching is $\varphi_d$-weighted by induction.

Consider now case (c). In (c1) we have to check that $\omega_{\varphi_d}(A_{d-1} \cup A_k) = \omega_{\varphi_d}(A_{d+k})$ for all $k \geq 0$. This follows immediately from the formula for $\varphi_d$-weights (Section 5). In (c2)-(c3) we match simplices by adding or removing $f+1$, which alters the size of the leftmost connected component. However this component has always size $\leq d-2$, so it does not contribute to the $\varphi_d$-weight. Finally, in (c4) the matching is $\varphi_d$-weighted by induction.

In case (d) most of the vertices belong to the rightmost connected component, which has size $\geq g$. If we add the vertex $f+1$ to a simplex $\sigma \in K_{n,f,g}^A$ not containing $f+1$, we either create a leftmost connected component of size $\leq d-2$, or we join the leftmost component with the rightmost component creating the full simplex $\{1,2,\ldots,n\}$. In the former case, the leftmost component is small enough not to contribute to the $\varphi_d$-weight. In the latter case we have $\sigma = \{1,\ldots,f,f+2,\ldots,n\}$; it is easy to check that $\omega_{\varphi_d}(\sigma) = \omega_{\varphi_d}(\sigma \cup \{f+1\})$ if and only if $n \equiv f+1,\ldots,d-2 \pmod{d}$. \hfill $\square$

Lemma 6.4. The critical simplices of Matching 6.1 are given by Table 2. In particular, the matching is $\varphi_d$-precise for $f = g = 0$.

| Case | # Critical | $|\sigma| - v_\varphi(\sigma)$ |
|------|------------|----------------------|
| $f > n$ or $g > n$ | 0 | - |
| $f, g \leq n$ and $f + g \geq n$ | 1 | $n - \left\lfloor \frac{n+1}{d} \right\rfloor$ |
| $n \equiv \max(d-1,f_0+g_0+1),\ldots,\min(f_0+d-1,g_0+d-1) \mod d$ | 2 | $n - \left\lfloor \frac{n-f}{d} \right\rfloor - \left\lfloor \frac{n-g}{d} \right\rfloor - 1$ |
| $f + g < n$ | 0 | - |

Table 2. Critical simplices of Matching 6.1

Here $f_0,g_0 \in \{0,\ldots,d-1\}$ are defined as $f \mod d$, and $g \mod d$, respectively. Moreover, the notation “$n \equiv a,\ldots,b \mod d$” means that $n$ is congruent modulo $d$ to some integer in the closed interval $[a,b]$.

Remark 6.5.

- The conditions in Table 2 are symmetric in $f$ and $g$, even if the definition of Matching 6.1 is not.
- The two intervals of Table 2 in the case $f + g < n$ are always disjoint.
For $g = 0$, Matching 6.1 coincides with the matching defined in [PSIS Section 5.1]. Table 2 simplifies a lot in this case, as both intervals contain only one element ($d - 1$ and $f_{0}$, respectively). See [PSIS Table 2].

- If $f \equiv -1 \pmod{d}$ or $g \equiv -1 \pmod{d}$, then the two intervals of Table 2 are empty, thus there is at most 1 critical simplex.
- If $\sigma \rightarrow \tau$ is in the matching, then $\sigma = \tau \cup \{v\}$ with $v \equiv 0 \pmod{v}$ or $v \equiv f + 1 \pmod{d}$. This can be easily checked by induction.

**Proof of Lemma 6.4** As a preliminary step, rewrite the two intervals of Table 2 as follows.

- The condition $n \equiv \max(d - 1, f_{0} + g_{0} + 1), \ldots, \min(f_{0} + d - 1, g_{0} + d - 1) \pmod{d}$ is equivalent to:

$$\begin{aligned}
&n \equiv -1, 0, \ldots, f_{0} - 1 \pmod{d}, \\
n - g \equiv f_{0} + 1, \ldots, d - 1 \pmod{d}.
\end{aligned}$$

(1)

- The condition $n \equiv \max(f_{0}, g_{0}), \ldots, \min(f_{0} + g_{0}, d - 2) \pmod{d}$ is equivalent to:

$$\begin{aligned}
&n \equiv f_{0}, \ldots, d - 2 \pmod{d}, \\
n - g \equiv 0, \ldots, f_{0} \pmod{d}.
\end{aligned}$$

(2)

Throughout the proof, we will refer to these two conditions as “Condition (1) (resp. (2)) for $K_{n,f,g}^{A}$.”

We prove the lemma by induction on $n$, the case $n = 0$ being trivial. If $f + g \geq n$ there is nothing to prove, thus we can assume $f + g < n$.

Suppose to be in case (b) of Matching 6.1, i.e. $f \geq d$. Conditions (1) and (2) for $K_{n,f,g}^{A}$ are equivalent to those for $K_{n-d,f-d,g}^{A}$. The critical simplices of $K_{n,f,g}^{A}$ are in one-to-one correspondence with the critical simplices of $K_{n-d,f-d,g}^{A}$ via removal of the first $d$ vertices. This correspondence decreases the size of a simplex by $d$, and decreases the $\varphi_{d}$-weight by 1. Adding $d - 1$ to the values of the last column of Table 2 for $K_{n-d,f-d,g}^{A}$, we exactly recover Table 2 for $K_{n,f,g}^{A}$.

Suppose to be in case (c), i.e. $f \leq d - 1$ and $n \geq d + g$. Critical simplices only arise from subcase (c4). Notice that $(d - 2 - f) + g < (n - f - 1)$, so by induction the critical simplices of $K_{n,f-1,d-2-g}^{A}$ are described by the last three rows of Table 2. Condition (1) (resp. (2)) for $K_{n,f-1,d-2-g}^{A}$ coincides with Condition (1) (resp. Condition (1)) for $K_{n,f,g}^{A}$. The critical simplices of $K_{n,f,g}^{A}$ are in one-to-one correspondence with the critical simplices of $K_{n,f-1,d-2-g}^{A}$ via removal of the first $f$ vertices. This correspondence decreases the size by $f$, and leaves the $\varphi_{d}$-weight unchanged. Adding $f$ to the values of the last column of Table 2 for $K_{n,f-1,d-2-g}^{A}$, we recover Table 2 for $K_{n,f,g}^{A}$.

Suppose to be in case (d), i.e. $f \leq d - 2$ and $n < d + g$. Since $f + g < n$, we must have $n - g \in \{f + 1, \ldots, d - 1\}$. In particular Condition (2) cannot hold, and the first part of Condition (1) holds. Case (d1) happens if and only if $n \equiv -1, 0, \ldots, f \pmod{d}$, i.e., if and only if the second part of Condition (1) holds. If this happens then there are two critical simplices, namely $\{1, \ldots, n\}$ and $\{1, \ldots, f, f + 2, \ldots, n\}$. The difference $|\sigma - v_{g}(\sigma)|$ is the same for these two simplices, and is given by $n - \left\lfloor \frac{n - f}{d} \right\rfloor = (n - 1) - \left\lfloor \frac{n - f}{d} \right\rfloor$. Since $\left\lfloor \frac{n - f}{d} \right\rfloor = 0$, we can rewrite it also as $n - \left\lfloor \frac{n - f}{d} \right\rfloor - \left\lfloor \frac{n - g}{d} \right\rfloor - 1$. $\square$
7. Case $D_n$

In [PS18] precise matchings were constructed for $A_n$ and $B_n$, but the third infinite family of groups of finite type, namely $D_n$, was left out (see Figure 3).

![Coxeter graph of type $D_n$.](image)

For $n \geq 4$ let $(W_{D_n}, S)$ be a Coxeter system of type $D_n$, with generating set $S = \{1, 2, \ldots, n\}$, and let $K_n^D = K_{W_{D_n}}$. We are going to construct a $\varphi_d$-precise matching on $K_n^D$. We actually split the definition according to the parity of $d$, and for $d$ even we construct a matching on each $K_n^D$. We will need this construction for $\tilde{B}_n$ and $\tilde{D}_n$.

**Matching 7.1** ($\varphi_d$-matching on $K_n^D$ for $d$ odd).

(a) If $1 \in \sigma$ then match $\sigma$ with $\sigma \cup \{2\}$.

(b) Otherwise, relabel the vertices $\{2, \ldots, n\}$ as $\{1, \ldots, n-1\}$ and construct the matching as in $K_{n-1}^A$.

**Matching 7.2** ($\varphi_d$-matching on $K_{n,g}^D$ for $d$ even).

(a) If $2 \notin \sigma$, relabel the vertices $\{1, 3, 4, \ldots, n\}$ as $\{1, \ldots, n-1\}$ and construct the matching as in $K_{n-1,0, g}^A$.

(b) Otherwise, if $d = 2$ and $\{1, 2, 3, 4\} \notin \sigma$, proceed as follows.

(b1) If $\{1, 2, 4\} \subseteq \sigma$: match $\sigma$ with $\sigma \cup 5$ if possible (i.e. if $n - g \geq 5$); else $\sigma$ is critical.

(b2) Otherwise: match $\sigma$ with $\sigma \cup 3$ if possible (i.e. if $n - g \geq 3$); else $\sigma$ is critical.

(c) Otherwise, if $d \geq 4$ and $3 \notin \sigma$, match $\sigma$ with $\sigma \cup \{1\}$.

(d) Otherwise, if $d = 4$ and $4 \notin \sigma$ (recall that at this point $\{2, 3\} \subseteq \sigma$), ignore vertex 1, relabel vertices $\{5, \ldots, n\}$ as $\{1, \ldots, n-4\}$, and construct the matching as in $K_{n-4,0, g}^A$.

(e) Otherwise, if $d \geq 6$ and $4 \notin \sigma$, match $\sigma$ with $\sigma \cup \{1\}$.

(f) Otherwise, if $d \geq 4$ and $1 \notin \sigma$, proceed as follows. Recall that at this point $\{2, 3, 4\} \subseteq \sigma$.

(f1) If $\{2, \ldots, d+1\} \subseteq \sigma$, relabel the vertices $\{2, \ldots, n\}$ as $\{1, \ldots, n-1\}$ and construct the matching as in $K_{n-1, \max(\frac{d}{2},3), g}^A$.

(f2) Otherwise, match $\sigma$ with $\sigma \cup \{1\}$.

(g) Otherwise, proceed as follows. Recall that at this point $\{1, 2, 3, 4\} \subseteq \sigma$. Let $k \geq 4$ be the size of the connected component $\Gamma_1(\sigma)$ of the vertex 1, in the
subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by $\sigma$. Write $k = \frac{q_d}{2} + r$ where:

\[
\begin{cases}
0 < r < \frac{d}{2} & \text{if } k \not\equiv 0 \pmod{\frac{d}{2}}; \\
r \in \{0, \frac{d}{2}\} & \text{if } k \equiv 0 \pmod{\frac{d}{2}}.
\end{cases}
\]

Define a vertex $v$ as follows:

\[
v = \begin{cases}
\frac{q_d}{2} + 1 & \text{if } q \text{ is even}; \\
\frac{q_d}{2} + 2 & \text{if } q \text{ is odd}.
\end{cases}
\]

It can be checked that $v = 1$ or $v \geq 5$. The idea now is that most of the times $\sigma \not\subseteq v$ has the same $\varphi_d$-weight as $\sigma$. Unfortunately there are some exceptions, so we still have to examine a few subcases.

(g1) Suppose $v \in \sigma$. If $v \leq n - g$, match $\sigma$ with $\sigma \setminus \{v\}$. Otherwise $\sigma$ is critical.

(g2) Suppose $v \not\in \sigma$. Match $\sigma$ with $\sigma \cup \{v\}$, unless one of the following occurs.

(g2.1) $v > n$ (i.e. the vertex $v$ doesn’t exist in $S$). Then $\sigma$ is critical.

(g2.2) $q$ is even, and $\{\frac{q_d}{2} + 2, \ldots, (q + 1)\frac{d}{2} + 1\} \subseteq \sigma$. In this case the connected components $\Gamma_1(\sigma)$ and $\Gamma_1(\sigma \cup \{v\})$ have a different $\varphi_d$-weight. Then ignore the vertices $\leq \frac{q_d}{2} + 1$, relabel the vertices $\{\frac{q_d}{2} + 2, \ldots, n\}$ as $\{1, \ldots, n - \frac{q_d}{2} - 1\}$ and construct the matching as in $K_{n-q_d-1, \frac{d}{2}, g}$.

(g2.3) $q$ is odd, and $\{\frac{q_d}{2} + 3, \ldots, (q + 1)\frac{d}{2}\} \subseteq \sigma$. Similarly to case (g2.2), relabel the vertices and construct the matching as in $K_{n-q_d-2, \frac{d}{2}-2, g}$.

Lemma 7.3. Matchings 7.1 and 7.2 are acyclic and $\varphi_d$-weighted. Critical simplices for these matchings on $K_n^D$ are given by Tables 3 and 4. In particular, both matchings on $K_n^D$ are $\varphi_d$-precise.

| Case | $|\sigma| - \nu_\varphi(\sigma)$ |
|------|-------------------------------|
| $n \equiv 0 \pmod{d}$ | $n - 2\frac{n}{d}$ |
| $n \equiv 1 \pmod{d}$ | $n - 2\frac{n-1}{d} - 1$ |
| else | - |

Table 3. Critical simplices of Matching 7.1 (case $D_n$, $d$ odd).

Sketch of proof. The proof is similar to those of Lemmas 6.2 and 6.3. For every $d \geq 2$, the quantity $|\sigma| - \nu_\varphi(\sigma)$ is constant among the critical simplices. Therefore the matchings are $\varphi_d$-precise.

8. Case $\tilde{B}_n$

Consider now, for $n \geq 3$, an affine Coxeter system $(\tilde{W}_{\tilde{B}_n}, S)$ of type $\tilde{B}_n$ (see Figure 4). Throughout this section, let $K_n = K_{\tilde{W}_{\tilde{B}_n}}$. We are going to describe a $\varphi_d$-precise matching on $K_n$. For $d$ even the matching is very simple, and has exactly one critical simplex. For $d$ odd the situation is more complicated.
$n \equiv 0 \pmod{d}$

(a), (b) for $n = 4$ and $d = 2$,
(d) for $d = 4$, (f) for $d \geq 6$
or $n = d = 4$, (g2.1), (g2.2), (g2.3)

$n \equiv 1 \pmod{d}$

(a)

$n - 2 \frac{n}{d} - 1$

$n \equiv \frac{d}{2} + 1 \pmod{d}$ for $d \geq 4$

(d) for $d = 4$, (f) for $d \geq 6$,
(g2.1), (g2.2), (g2.3)

$n - 2 \frac{n-1}{d}$

else

-

-

Table 4. Critical simplices of Matching 7.2 (case $D_n$, $d$ even) for $q = 0$. In the second column we indicate in which parts of Matching 7.2 the critical simplices arise.

Matching 8.1 ($\varphi_d$-matching on $K_n = K_{W_{B_n}}$ for $d$ odd). For $\sigma \neq \{1, 2, \ldots, n\}$, match $\sigma$ with $\sigma \not\subseteq 0$. Then $\{1, 2, \ldots, n\}$ is the only critical simplex.

Matching 8.2 ($\varphi_d$-matching on $K_n = K_{W_{B_n}}$ for $d$ even). For $\sigma \in K_n$, let $k$ be the size of the connected component $\Gamma_n(\sigma)$ of the vertex $n$, in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by $\sigma$. Let $k = q \frac{d}{2} + r$, with $0 \leq r < \frac{d}{2}$.

(a) If $r \geq 1$ match $\sigma$ with $\sigma \not\subseteq (n - q \frac{d}{2})$, unless $\sigma = \{0, 2, 3, \ldots, n\}$ and $r = 1$ (in this case $\sigma$ is critical).

(b) If $r = 0$ and $\{n - (q + 1)\frac{d}{2} + 1, \ldots, n - q \frac{d}{2} - 1\} \subseteq \sigma$: ignore vertices $\geq n - q \frac{d}{2}$, relabel vertices $\{0, 1, \ldots, n - q \frac{d}{2} - 1\}$ as $\{1, 2, \ldots, n - q \frac{d}{2}\}$, and construct the matching as in $K_{n-1-q \frac{d}{2}, \frac{d}{2}, -1}$.

(c) If $r = 0$ and $\{n - (q + 1)\frac{d}{2} + 1, \ldots, n - q \frac{d}{2} - 1\} \not\subseteq \sigma$, proceed as follows.

(c1) If $|\sigma| = n$ (i.e. $\sigma$ is either $\{0, 2, 3, \ldots, n\}$ or $\{1, 2, 3, \ldots, n\}$), then $\sigma$ is critical.

(c2) If $n = (q + 1)\frac{d}{2}$ and $\sigma = \{0, 2, 3, \ldots, n - q \frac{d}{2} - 1, n - q \frac{d}{2} + 1, \ldots, n\}$, then $\sigma$ is critical.

(c3) Otherwise, match $\sigma$ with $\sigma \not\subseteq (n - q \frac{d}{2})$.

Lemma 8.3. Matchings 8.1 and 8.2 are acyclic and $\varphi_d$-weighted. For $d$ odd, Matching 8.1 has exactly one critical simplex $\sigma$ which satisfies $|\sigma| - v_d(\sigma) = n - \left\lfloor \frac{n}{d} \right\rfloor$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{coxeter_graph_type_B_n}
\caption{A Coxeter graph of type $B_n$.}
\end{figure}
For $d$ even, all critical simplices $\sigma$ of Matching 8.2 satisfy $|\sigma| - v_{\varphi_d}(\sigma) = n - \left\lfloor \frac{n}{d^2} \right\rfloor$. In particular, both matchings are $\varphi_d$-precise.

**Sketch of proof.** The first part can be proved along the lines of Lemmas 6.2 and 6.3. The formulas for $|\sigma| - v_{\varphi_d}(\sigma)$ can be checked examining all critical simplices. Finally, since the quantity $|\sigma| - v_{\varphi_d}(\sigma)$ is constant among the critical simplices, both matchings are $\varphi_d$-precise. \qed

9. Case $\tilde{D}_n$

In this section we consider a Coxeter system $(W_{\tilde{D}_n}, S)$ of type $\tilde{D}_n$, for $n \geq 4$ (see Figure 5). Throughout this section, let $K_n = K_{W_{\tilde{D}_n}}$. We are going to describe a $\varphi_d$-precise matching on $K_n$. Again, this will be easier for $d$ odd and quite involved for $d$ even.

![Coxeter graph of type $\tilde{D}_n$](image)

**Figure 5.** A Coxeter graph of type $\tilde{D}_n$.

**Matching 9.1** ($\varphi_d$-matching on $K_n = K_{W_{\tilde{D}_n}}$ for $d$ odd).

(a) If $\sigma = \{1, 2, \ldots, n\}$, then $\sigma$ is critical.

(b) If $1 \not\in \sigma$, relabel vertices $\{n, n-1, \ldots, 3, 2, 0\}$ as $\{1, 2, \ldots, n\}$ and generate the matching as in $K_n^D$.

(c) In all remaining cases, match $\sigma$ with $\sigma \not\supseteq 0$.

**Matching 9.2** ($\varphi_d$-matching on $K_n = K_{W_{\tilde{D}_n}}$ for $d$ even). For $n = 4$ and $d \leq 6$, we construct the matching separately as follows.

- Case $n = 4$, $d = 2$. If $|\sigma| = 1$ and $2 \not\in \sigma$, or $|\sigma| = 2$, or $|\sigma| = 3$ and $2 \in \sigma$, then match $\sigma$ with $\sigma \not\supseteq 2$. Otherwise, $\sigma$ is critical.

- Case $n = 4$, $d = 4$. If $2 \not\in \sigma$ or $\sigma \cap \{1, 3, 4\} = \emptyset$, then match $\sigma$ with $\sigma \not\supseteq 0$. Otherwise, $\sigma$ is critical.

- Case $n = 4$, $d = 6$. Match $\sigma$ with $\sigma \not\supseteq 0$, except in the following two cases:
  - $2 \in \sigma$, $0 \not\in \sigma$ and $|\sigma| \geq 3$; or, $\{0, 2\} \subseteq \sigma$ and $|\sigma| = 4$.

In the remaining cases ($n \geq 5$ or $d \geq 8$), the matching is constructed as follows.

(a) If $1 \not\in \sigma$, relabel vertices $\{n, n-1, \ldots, 3, 2, 0\}$ as $\{1, 2, \ldots, n\}$ and construct the matching as in $K_n^D$.

(b) Otherwise, if $d = 2$ and $\{0, 1, 2, 3\} \not\subseteq \sigma$, proceed as follows.

(b1) If $\{0, 1, 3\} \subseteq \sigma$, match $\sigma$ with $\sigma \not\supseteq 4$ if $\{5, 6, \ldots, n\} \not\subseteq \sigma$, else $\sigma$ is critical.

(b2) Otherwise, if $\{1, 3, 4, \ldots, n\} \not\subseteq \sigma$ then match $\sigma$ with $\sigma \not\supseteq 2$, else $\sigma$ is critical.

(c) Otherwise, if $d \geq 4$ and $0 \not\in \sigma$, proceed as follows.

(c1) If $\{1, 2, \ldots, \frac{d+1}{2}\} \subseteq \sigma$, relabel vertices $\{n, n-1, \ldots, 2, 1\}$ as $\{1, 2, \ldots, n\}$ and construct the matching as in $K_n^{2d}$.

(c2) Otherwise, if $n = \frac{d+1}{2}$ and $\sigma = \{1, 2, \ldots, n-2, n\}$, then $\sigma$ is critical.
(c3) Otherwise, if $\sigma = \{1, 2, \ldots, n\}$, then $\sigma$ is critical.
(c4) Otherwise, match $\sigma$ with $\sigma \cup \{0\}$.
(d) Otherwise, if $d \geq 4$ and $2 \not\equiv \sigma$, match $\sigma$ with $\sigma \not\equiv 0$.
(e) Otherwise, if $d = 4$ and $3 \not\equiv \sigma$, ignore vertices $0, 1, 2$, relabel vertices $\{n, n-1, \ldots, 4\}$ as $\{1, 2, \ldots, n-3\}$ and construct the matching as in $K_D^{n-3}$.
(f) Otherwise, if $d \geq 6$ and $3 \not\equiv \sigma$, match $\sigma$ with $\sigma \not\equiv 0$.
(g) Otherwise, proceed as follows. Recall that at this point $\{0, 1, 2, 3\} \subseteq \sigma$. Let $k \geq 4$ be the size of the leftmost connected component $\Gamma_0(\sigma)$ of the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by $\sigma$. Notice that $\{0, 1, \ldots, k-1\} \subseteq \sigma$, unless $k = n$ and $\sigma = \{0, 1, \ldots, n-2, n\}$. Similarly to Matching $\text{9.2}$ write $k = q \frac{d}{2} + r$ where:
\[
\begin{cases}
0 < r < \frac{d}{2} & \text{if } k \not\equiv 0 \pmod{\frac{d}{2}}; \\
r \in \{0, \frac{d}{2}\} & \text{and } q \text{ even} \quad \text{if } k \equiv 0 \pmod{\frac{d}{2}}.
\end{cases}
\]

Define a vertex $v$ as follows:
\[
v = \begin{cases}
q \frac{d}{2} & \text{if } q \text{ is even}; \\
q \frac{d}{2} + 1 & \text{if } q \text{ is odd}.
\end{cases}
\]
(g1) If $d = 4$, $q$ odd and $r = 1$, proceed as follows.
(g1.1) If $k \leq n - 2$, ignore vertices $0, \ldots, k$, relabel vertices $\{n, n-1, \ldots, k+1\}$ as $\{1, 2, \ldots, n-k\}$, and construct the matching as in $K_D^{n-k}$.
(g1.2) Otherwise, $\sigma$ is critical.
(g2) Otherwise, if $\sigma = \{0, 1, \ldots, n-2, n\}$ and $v \geq n - 1$, then $\sigma$ is critical.
(g3) Otherwise, if $v \in \sigma$, match $\sigma$ with $\sigma \not\equiv v$.
(g4) Otherwise, if $v > n$, then $\sigma$ is critical.
(g5) Otherwise, proceed as follows. Let $c$ be the size of the (possibly empty) connected component $C = \Gamma_{v+1}(\sigma)$ of the vertex $v + 1$, in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by $\sigma$. Let
\[
\ell = \begin{cases}
q \frac{d}{2} & \text{if } q \text{ even}; \\
q \frac{d}{2} - 2 & \text{if } q \text{ odd}.
\end{cases}
\]
(g5.1) If $\{n-1, n\} \subseteq C$, then $\sigma$ is critical.
(g5.2) Otherwise, if $c < \ell$, match $\sigma$ with $\sigma \cup \{v\}$.
(g5.3) Otherwise, if $c = \ell$, $n-1 \not\in C$ and $n \in C$, then $\sigma$ is critical.
(g5.4) Otherwise, ignore vertices $0, 1, \ldots, v-1$, relabel vertices $\{n, n-1, \ldots, v+1\}$ as $\{1, 2, \ldots, n-v\}$ and construct the matching as in $K_D^{n-v-\ell}$.

**Lemma 9.3.** Matchings $\text{9.1}$ and $\text{9.2}$ are $\varphi_d$-precise. In addition, critical simplices of Matching $\text{9.1}$ are as in Table 3.

**Sketch of proof.** We only discuss the critical simplices of Matching $\text{9.1}$ (i.e. the case $d$ odd), and see why this matching is precise. The check for Matching $\text{9.2}$ is much more involved and will be omitted.

Following the definition of Matching $\text{9.1}$ one critical simplex is always given by $\bar{\sigma} = \{1, 2, \ldots, n\}$. It has size $|\bar{\sigma}| = n$ and weight $w_v(\bar{\sigma}) = \left\lfloor \frac{n}{2} \right\rfloor$. The other critical simplices arise from the matching on $K_D^0$, and therefore from the matching on $K_{n-1}^A$. There are two of them (which we denote by $\bar{\tau}_1$ and $\bar{\tau}_2$) when $n \equiv 0, 1 \pmod{d}$, and zero otherwise.
These two paths give opposite contributions to the incidence number $|\bar{\tau}| - v_\phi(\bar{\tau})$ between forward consequence we also obtain the homology groups $H_w$. The polynomial is given by $\omega(I_2(m)) = [2]_{q}[m]_q$. Then the $\varphi_d$-weight is

$$\omega_{\varphi_d}(I_2(m)) = \begin{cases} 2 & \text{if } d = 2 \text{ and } m \text{ even;} \\ 1 & \text{if } d = 2 \text{ and } m \text{ odd;} \\ 1 & \text{if } d \geq 3 \text{ and } d \mid m; \\ 0 & \text{if } d \geq 3 \text{ and } d \nmid m. \end{cases}$$

$$\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}$$

Figure 6. A Coxeter graph of type $I_2(m)$.

In this case $\varphi_d$-precise matchings are easy to construct by hand. As a straightforward consequence we also obtain the homology groups $H_*(X_W; R)$.

Matching 10.1 ($\varphi_d$-matching on $K_{I_2(m)}$).
To summarize, the local homology is given by:
\[
H_0(X_W; R) \cong \frac{R}{\varphi_2}, \quad H_1(X_W; R) \cong \bigoplus_{d|m} \frac{R}{\varphi_d}
\]

This result corrects the one given in [DCPSS99], where proper divisors of \( m \) were not taken into account.

| \( \tilde{D}_n \) | \( \tilde{D}_5 \) | \( \tilde{D}_6 \) | \( \tilde{D}_7 \) | \( \tilde{D}_8 \) | \( \tilde{D}_9 \) |
|---|---|---|---|---|---|
| \( H_0 \) | \{2\} | \{2\} | \{2\} | \{2\} | \{2\} |
| \( H_1 \) | \{3\} | 0 | 0 | 0 | 0 |
| \( H_2 \) | \{4\} \text{ or } \{6\} | \{3\} | 0 | 0 |
| \( m_{\tilde{D}_4} \) | \{5\} | \{5\} | 0 | 0 | \{3\} |
| \( H_4 \) | \( R \) | \( m_{\tilde{D}_6} \) | \{4\} \text{ or } \{6\} | \{4\} | \{4\} |
| \( H_5 \) | \( R \) | \( m_{\tilde{D}_6} \) | \{4\} \text{ or } \{7\} | \{4\} \text{ or } \{7\} |
| \( H_6 \) | \( R \) | \( m_{\tilde{D}_8} \) | \{4\} \text{ or } \{6\} \text{ or } \{8\} \text{ or } \{10\} |
| \( H_7 \) | \( R \) | \( m_{\tilde{D}_8} \) | \{6\} \text{ or } \{9\} |
| \( H_8 \) | \( R \) | \( m_{\tilde{D}_8} \) | \( R \) |
| \( H_9 \) | \( R \) | \( m_{\tilde{D}_8} \) | \( R \) |

Table 6. Homology in the case \( \tilde{D}_n \) for \( n \leq 9 \).

- If \( d = 2 \) and \( m \) is even, every simplex is critical. Critical simplices are then: \{1, 2\} (size 2, weight 2), \{1\}, \{2\} (size 1, weight 1), and \( \varnothing \) (size 0, weight 0). By Theorem 2.3, the homology groups are: \( H_0(X_W; R)_{\varphi_2} \cong \frac{R}{\varphi_2} \) and \( H_1(X_W; R)_{\varphi_2} \cong \frac{R}{\varphi_2} \).
- If \( d = 2 \) and \( m \) is odd, match \{1, 2\} with \{1\} (both simplices have weight 1). The critical simplices are: \{2\} (size 1, weight 1), and \( \varnothing \) (size 0, weight 0). The homology groups are: \( H_0(X_W; R)_{\varphi_2} \cong \frac{R}{\varphi_2} \) and \( H_1(X_W; R)_{\varphi_2} \cong \frac{R}{\varphi_2} \).
- If \( d \geq 3 \) and \( d \mid m \), match \{2\} with \( \varnothing \) (both simplices have weight 0). The critical simplices are: \{1, 2\} (size 2, weight 1), and \{1\} (size 1, weight 0). The homology groups are: \( H_0(X_W; R)_{\varphi_2} \cong \frac{R}{\varphi_2} \) and \( H_1(X_W; R)_{\varphi_2} \cong \frac{R}{\varphi_2} \).
- If \( d \geq 3 \) and \( d \nmid m \), match \{1, 2\} with \{1\} and \{2\} with \( \varnothing \) (all simplices have weight 0). There are no critical simplices, and all homology groups are trivial.
Table 7. Homology in the exceptional finite cases.

\[
\begin{array}{cccccc}
   & H_3 & H_4 & F_4 & E_6 & E_7 & E_8 \\
H_0 & \{2\} & \{2\} & \{2\} & \{2\} & \{2\} & \{2\} \\
H_1 & 0 & 0 & \{2\} & 0 & 0 & 0 \\
H_2 & m_{H_3} & 0 & \{2\} \oplus \{3\} \oplus \{6\} & 0 & 0 & 0 \\
H_3 & & m_{H_4} & m_{F_4} & 0 & 0 & 0 \\
H_4 & & & \{6\} \oplus \{8\} & \{6\} & \{4\} & \\
H_5 & & & m_{E_6} & \{6\} & 0 & \\
H_6 & & & m_{E_7} & \{8\} \oplus \{12\} & & \\
H_7 & & & & m_{E_8} & & \\
\end{array}
\]

\[m_{H_3} = \{2\} \oplus \{6\} \oplus \{10\}\]
\[m_{H_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\} \oplus \{30\}\]
\[m_{F_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{6\} \oplus \{8\} \oplus \{12\}\]
\[m_{E_6} = \{3\} \oplus \{6\} \oplus \{9\} \oplus \{12\}\]
\[m_{E_7} = \{2\} \oplus \{6\} \oplus \{14\} \oplus \{18\}\]
\[m_{E_8} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\}\]
\[\oplus \{24\} \oplus \{30\}\]

11. Exceptional cases

In all exceptional finite and affine cases (see for example \cite{BB05} Appendix A1 for a classification), we constructed precise matchings by means of a computer program. The explicit description of these matchings, together with proof of preciseness and homology computations, can be obtained through the software library available online \cite{Pao17}.

The homology groups can be computed using Theorem 2.3. They are described in Tables 7 and 8, where again we employ the notation \(d = R/\langle \varphi_d \rangle\). We recover the results of \cite{DCPSS99} (for the finite cases) and \cite{SV13} (for the affine cases), except for minor corrections in the cases \(E_8\) and \(\tilde{E}_8\).

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| $H_0$ | $I_1$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|------|------|------|------|------|------|------|
|      | {2}  | {2}  | {2}  | {2}  | {2}  | {2}  |
| $H_1$ | $R$   | {2}  | {2}  | 0    | 0    | 0    |
| $H_2$ | $R$   | {2}  | {2}  | 0    | 0    | 0    |
| $H_3$ |       |      |      | $m_{E_4}$ | {3} | 0    |
| $H_4$ | $R$   |      |      |      | {5}  | 0    | {4}  |
| $H_5$ |       |      |      | $m_{E_6}$ | 0 | 0    |
| $H_6$ | $R$   |      |      |      |      | $m_{E_7}$ | {5}  | {8}  |
| $H_7$ | $R$   |      |      |      |      | $m_{E_8}$ |      |      |
| $H_8$ | $R$   |      |      |      |      |      |      |      |

$m_{E_4} = {2}^2 \oplus {3} \oplus {4} \oplus {8}$

$m_{E_6} = {2} \oplus {3}^3 \oplus {6}^2 \oplus {9}^2 \oplus {12}^2$

$m_{E_7} = {2}^3 \oplus {3} \oplus {4} \oplus {6} \oplus {8} \oplus {10} \oplus {14} \oplus {18}$

$m_{E_8} = {2}^2 \oplus {3} \oplus {4} \oplus {5} \oplus {8} \oplus {9} \oplus {14}$

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