Interval-parametric synthesis of a robust controller on a base of characteristic polynomial with desired stability in a sector

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Abstract. In the paper a characteristic polynomial of a linear control system which coefficients include interval parameters of a control object and adjustable parameters of a controller is considered. In order to find polytope of controller parameters providing desired sector stability of a polynomial, values of polynomial coefficients limits are found on a base of desired oscillability degree of a system. On a base of these calculations a controller synthesis method is developed. Application example of the method proposed is also provided.

1. Introduction
One of the most important problems of modern industrial facilities is developing and implementing high-quality control systems with uncertain parameters. Most of control objects have uncertain parameters, which are defined inaccurately due to measurement errors, equipment aging or any other disturbances affecting the control object. There are also control systems, whose parameters may vary within certain ranges. In both cases such control systems can be considered as control systems with interval parameters. Problems of analysis and synthesis of such systems were considered in [1-21]. There is a vast variety of papers proposing to base analysis and synthesis of such systems on a base of modal approach [7-15], [21]. Also, some research propose methods of controller synthesis, which are based on choosing controller parameters by systems stability criteria and allow providing desired control quality despite parametric uncertainty of a system [1-21].

One of them is an approach based on interval characteristic polynomials (ICP) of a system which coefficients vary within certain intervals. There are methods allowing to find maximal width of ICP coefficients intervals on a base of desired control quality. Consequently, if dependencies between ICP coefficients and controller parameters are known, these methods can be inversed to perform controller synthesis. To do this, expressions linking desired values of control quality indices and interval coefficients of ICP are required. Such expressions can be derived on a base of coefficients methods [5] which allow estimating stability degree and oscillability degree of a system on a base of indices expressed from ICP coefficients.

2. Conditions of sector robust stability of an interval control system
Let us assume, that general form of a transfer function of a control object can be written as follows:

\[ W_g(s) = \frac{B(s)}{A(s)} = \sum_{j=0}^{n} \frac{[b_j] s^j}{\sum_{j=0}^{n} [a_j] s^j}, \]

where \( a_j \leq a_j \leq a_j^- \), \( b_j \leq b_j \leq b_j^- \) and general form of a transfer function
of a controller can be written as follows: \( W_r(s, \tilde{k}) = \frac{C(s, \tilde{k})}{s} \), where \( \tilde{k} \) – is a vector of intervals of controller parameters values. In this case, an ICP of the system considered has a following form:

\[
P(s, \tilde{k}) = B(s)C(s, \tilde{k}) + sA(s) = \sum_{i=0}^{\tilde{v}} [p_i(\tilde{k})] s^i.
\]

Let us now introduce parameters \( \lambda_i \) which can be calculated on a base of each four consequent coefficients of ICP via following expression: \( \lambda_i = \frac{[p_{i-1}][p_{i+2}]}{[p_i][p_{i+1}]} \), \( i = 1, n - 2 \). Parameters \( \lambda_i \) are stability indices [6]. On a base of \( \lambda_i \) calculation, a sufficient condition of robust stability of an ICS was formulated in [6].

Condition 1. ICS described with its ICP is robustly stable, if following inequalities are satisfied:

\[
\overline{\lambda} = \frac{p((\tilde{k}))_{i-1} p((\tilde{k}))_{i+2}}{p((\tilde{k}))_i p((\tilde{k}))_{i+1}} < \lambda^* \approx 0.465 \forall i = 1, n - 2.
\]

(2)

Examining coefficients method showed, that interval indices of oscillability \( \delta_i = \frac{[p_i^2]}{[p_{i-1}][p_{i+1}]} \), \( z = 1, n - 1 \) can be used to analyze ICP (1) roots allocation areas. On a base of \( \delta_i \) a sufficient condition of ICS robust stability in a sector was developed.

Condition 2. To provide allocation of ICP (1) roots in a desired sector according to defined value of \( \delta_i \) following inequalities must be satisfied:

\[
\delta_i = \frac{p((\tilde{k}))^2}{p((\tilde{k}))_{i-1} p((\tilde{k}))_{i+1}} \leq \delta_0, \ z = 1, n - 1,
\]

(3)

where parameters \( \delta_0 \) are acceptable oscillability indices [6] and can be defined on a base of table 1.

**Table 1.** Acceptable oscillability indices

| \( n \) | \( 65^\circ \) | \( 70^\circ \) | \( 75^\circ \) | \( 80^\circ \) | \( 85^\circ \) | \( 90^\circ \) |
|---|---|---|---|---|---|---|
| 3 | 1.846 | 1.684 | 1.518 | 1.348 | 1.175 | 1 |
| 4 | 1.75 | 1.696 | 1.609 | 1.515 | \( \sqrt{2} \) |
| 5 | | 1.52 | 1.465 |

In the table 1, parameter \( n \) corresponds to a degree of ICP (1).

3. **Algorithm of composing an interval characteristic polynomial on a base of desired degree of robust stability**

Let us assume, that at least two consecutive coefficients of an ICP (1) are known. By using conditions (2) and (3) a system of inequalities for finding acceptable intervals of unknown coefficients of ICP providing desired allocation of its roots in a sector:

\[
\begin{align*}
\left| \frac{p((\tilde{k}))_{i-1} p_{i+2}}{p_i p_{i+1}} \right| & \leq \lambda^*, \ i = 1, n - 2 \\
\left| \frac{p_i}{p_{i-1} p_{i+1}} \right| & \leq \delta_0, \ i = 1, n - 1,
\end{align*}
\]

(4)
From the system of inequalities (4) a set of expressions for finding limits of coefficients $p_i$ can be obtained:

$$ p(\bar{k})_v \leq \frac{p(\bar{k})^2}{\delta_d p(\bar{k})_{i+2}}, \quad v = n - g, $$  

$$ p(\bar{k})_v \leq \frac{\bar{p}(\bar{k})_i p(\bar{k})_{j+1}}{p(\bar{k})_{i+3}}, \quad i = v - 1...1, \quad j = v...2, \quad v = n - g; $$

$$ p(\bar{k})_v \leq \frac{p(\bar{k})^j}{\delta_d p(\bar{k})_{i+2}}, \quad i = v - 1...1, \quad j = v...2, \quad v = n - g. $$

where $g$ – is a number of unknown coefficients of ICP.

On a base of (5) and (6) an algorithm of calculating interval coefficients of an ICP providing desired allocation of its roots in an acceptable sector was developed:

1. Define $g$ of known interval coefficients $[p_i]$ of an ICP and values of an acceptable index $\delta_d$ of robust oscillability.
2. Find higher index $v = n - g$ of unknown ICP coefficient.
3. Compose and solve inequality (5) and choose $\underline{p_v}$ from the solution.
4. Solve an inequalities system (6) graphically and choose $\underline{p_i}, \underline{p_j}$ from the solution.
5. Compose the ICP on a base of steps 1-4 results.

### 4. Calculating intervals of controller parameters

#### 4.1. Synthesizing PI-controller

In order to formulate a synthesis algorithm for PI-controller $W_f(s, [k]) = \frac{[k_1]s + [k_0]}{s}$, let us consider two variants of transfer functions of control object. Let us assume, that general form of control object transfer function can be written as follows: $W_f(s) = \frac{b_0}{A(s)} = \frac{b_0}{\sum_{j=0}^{n} [a_j]s^j}$. In this case, intervals of ICP coefficients $[p(\bar{k})_v], [p(\bar{k})_i]$ should be found and then a system of inequalities should be composed and solved for finding intervals of controller parameters:

$$ p(k_1)_i \geq a_0 + b_0 \cdot \bar{k}_i; $$

$$ p(k_2)_i \leq a_0 + b_0 \cdot \bar{k}_i; $$

$$ p(k_2)_0 \geq b_0 \cdot \bar{k}_0; $$

$$ p(k_2)_0 \leq b_0 \cdot \bar{k}_0. $$

If transfer function of a control object has the following general form $W_f(s) = \frac{B(s)}{A(s)} = \frac{\sum_{j=0}^{n} [b_j]s^j}{\sum_{j=0}^{n} [a_j]s^j}$, then another system of inequalities should be composed and solved for finding controller parameters:
4.2. Synthesizing PID-controller

Let us formulate the method of finding intervals of PID-controller parameters in the same way as for PID-controller: by considering two types of general form of control object transfer function. If polynomial $B(s)$ includes only one coefficient $[b_0]$, then it is enough to solve the following system of inequalities in order to find intervals of PID-controller parameters:

\[
\begin{align*}
    p(k_2) & \geq a_j + b_z \cdot k_2; \\
    p(k_2) & \leq a_j + b_z \cdot k_2; \\
    p(k_1)_{j+1} & \geq a_{j-1} + \sum_{r=0}^{\infty} b_r \cdot (k_2 s^2 + k_1 s + k_0) \cdot s'; \\
    p(k_1)_{j+1} & \leq a_{j-1} + \sum_{r=0}^{\infty} b_r \cdot (k_2 s^2 + k_1 s + k_0) \cdot s'; \\
    p(k_0)_{j+1} & \geq b_0 \cdot k_0; \\
    p(k_0)_{j+1} & \leq b_0 \cdot k_0.
\end{align*}
\]  

(8)

If polynomial $B(s)$ includes more than one coefficient, then it is enough to solve another system of inequalities to find intervals of PID-controller parameters:

\[
\begin{align*}
    p(k_2) & \geq a_j + b_z \cdot k_2; \\
    p(k_2) & \leq a_j + b_z \cdot k_2; \\
    p(k_1) & \geq a_0 + b_z \cdot k_1; \\
    p(k_1) & \leq a_0 + b_z \cdot k_1; \\
    p(k_0)_{j+1} & \geq b_0 \cdot k_0; \\
    p(k_0)_{j+1} & \leq b_0 \cdot k_0.
\end{align*}
\]  

(9)

4.3. Method of interval-parametric synthesis of a linear robust controller

An algorithm of synthesizing a linear robust controller providing desired allocation of ICP roots in a sector can be formulated as follows:

1. Choosing a controller type and finding coefficients of transfer function of a control object.
2. Deriving ICP (1) and finding limits of ICP coefficients $[p_i]$ not including controller parameters.
3. Finding limits of ICP coefficients including controller parameters $[p_i([k])]$ via algorithm described in chapter 3.
4. Composing an inequalities system (7) or (9) if polynomial $B(s)$ includes one coefficient $[b_0]$. Solving the system and finding limits of acceptable intervals of controller parameters.
5. Composing systems of inequalities (8) or (10) if polynomial \( B(s) \) includes more than one coefficient. Solving the system and finding limits of acceptable values of controller parameters.

5. Example of application

Let us define an ICP of a closed-loop control system with a PID-controller as follows:

\[
P(s,k) = [p_4]s^4 + [p_3]s^3 + [p_2](k_2)s^2 + [p_1](k_1)s + [p_0](k_0),
\]

where \([p_4] = [0.002; 0.005] \), \([p_3] = [0.5; 1] \), \([p_2](k_2) = [13.3; 16] + [1.7; 2] \cdot [k_2] \), \([p_1](k_1) = [53; 70] + [1.7; 2] \cdot [k_1] \), \([p_0](k_0) = [1.7; 2] \cdot [k_0] \).

The problem is to find intervals of PID-controller parameters providing ICP roots allocation in a sector \( \pm 60^\circ \) (\( \delta_2 = 2 \)). To do this on a base of coefficients \([p_1] \) and \([p_4] \) via expressions (4)-(5) following intervals were found:

\[
[p_2](k_2) = [15; 20], \quad [p_1](k_1) = [70; 100], \quad [p_0](k_0) = [100; 120].
\]

Then, according to synthesis algorithm, a system of inequalities (5) was derived:

\[
\begin{align*}
15 & < 13.3 + 1.7 \cdot k_2 \\
20 & > 16 + 2 \cdot k_2 \\
70 & < 53 + 1.7 \cdot k_1 \\
100 & > 70 + 2 \cdot k_1 \\
100 & < 1.7 \cdot k_0 \\
120 & > 2 \cdot k_0
\end{align*}
\]

By solving it, acceptable intervals of PID-controller parameters providing desired allocation of ICP roots were found: \( k_0 \in [58; 60], \quad [k_1] \in [10; 15], \quad [k_2] \in [1; 2] \).

6. Conclusion

The paper proposes a method of interval-parametrical synthesis of a controller providing acceptable oscillability degree for a control system with interval parameters. The method is based on an algorithm of deriving a desired interval characteristic polynomial with acceptable stability degree. The algorithm is based on coefficient indices of robust stability and desired robust oscillability. The synthesis method was tested by solving a problem of synthesizing a PID-controller placing poles of the synthesized system in a desired sector of a complex plane.

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7. References

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