ON DIVERGENCE FORM SPDES WITH VMO COEFFICIENTS

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Abstract. We present several results on solvability in Sobolev spaces $W^1_p$ of SPDEs in divergence form in the whole space.

1. Introduction

The theory of (usual) partial differential equations has two rather different parts depending on whether the equations are written in divergence or nondivergence form. Quite often the starting point is the same: equations with constant coefficients, and then one uses different techniques to treat different types of equations.

By now, one can say that the $L^p$-theory of evolutional second order SPDEs is quite well developed. The most advanced results of this theory can be found in the following papers and references therein: [1] (nondivergence type equations), [2] and [3] (divergence type equations). The results of the present paper are close to the corresponding results of [2]. However, unlike [2] we do not assume that the leading coefficients are continuous in the space variable. Instead we assume that the leading coefficients of the “deterministic” part of the equation are in VMO which is a much wider class than $C$. Still the leading coefficients of the “stochastic” part are assumed to be continuous in $x$.

The exposition in [2] and [3] is based on the theory of solvability in spaces $H^s_p = (1 - \Delta)^{-s/2}L^p$ of SPDEs with coefficients independent of $x$. Then the method of “freezing” the coefficients is applied as in the general framework set out in [6]. This method does not work if the coefficients are only in VMO and we use a different technique based on recent results from [8] on deterministic parabolic equations with VMO coefficients. In addition, our technique allows us to avoid using the $W^2_p$-theory of SPDEs, which is a starting point in the paper [6] and subsequent articles based on it.

One more difference of our approach from the one in [2] is that we represent the free term in the deterministic part in the form $D_if^i + f^0$ with $f^j \in L_p$ (see (1.1) below). Of course, this is just a general form of a distribution from $H^{-1}_p$. However, the spaces $H^s_p$ are most appropriate for equations

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in nondivergence form. One general inconvenience of these spaces is that the space or space-time dilations affect the norms in a way which is hard to control. For divergence form equations with low regularity of coefficients the most important space is $H^1_{p}$. This space coincides with the Sobolev space $W^1_p$ and the effect of dilations on the norm or on $D_i f^i + f^0$ can be easily taken into account.

The exposition here is self-contained apart from references to some very basic results of [6], [8], and [13] and is much more elementary than in [2], employing the derivatives instead of the powers of the Laplacian, and yet gives more information. In particular, the author intends to use Corollary 5.5 in order to largely simplify the theory in [2] of divergence form SPDEs in domains. It turns out that to develop this theory one need not first develop the theory of SPDEs in domains with coefficient independent of $x$, which in itself required quite a bit of work.

The author’s interest in divergence type equations and in simplifying the theory of them appeared after he realized that the corresponding results can be applied to filtering theory of partially observable diffusion processes, given by stochastic Itô equations, and proving that, under Lipschitz and nondegeneracy conditions only, the filtering density is almost Lipschitz in $x$ and almost Hölder $1/2$ in time. This is proved in [11] on the basis of Theorems 2.2 through 2.6 of the present article. The filtering density satisfies an SPDE usually written in terms of the operators adjoint to operators in nondivergence form with Lipschitz continuous coefficients. Writing these adjoint operators in divergence form makes perfect sense and allows us to obtain the above mentioned results (see [11]).

Our Theorem 2.2 is very close to Theorem 2.12 of [2]. Apart from weaker conditions on the coefficients, another important difference is the presence of the parameter $\lambda$ in (2.10). One of differences in the proofs is that we avoid proving the solvability on small consecutive time intervals and then gluing together the results.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete with respect to $(\mathcal{F}, P)$ $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$. Denote by $\mathcal{P}$ the predictable $\sigma$-field in $\Omega \times (0, \infty)$ associated with $\{\mathcal{F}_t\}$. Let $w^k_t$, $k = 1, 2, \ldots$, be independent one-dimensional Wiener processes with respect to $\{\mathcal{F}_t\}$.

We fix a stopping time $\tau$ and for $t \leq \tau$ in the Euclidean $d$-dimensional space $\mathbb{R}^d$ of points $x = (x^1, \ldots, x^d)$ we consider the following equation

$$
du_t = (L_t u_t - \lambda u_t + D_i f^i_t + f^0_t) \, dt + (\Lambda^k_t u_t + g^k_t) \, dw^k_t, \quad (1.1)$$

where $u_t = u_t(\omega, x)$ is an unknown function,

$$L_t \psi(x) = D_j (a^j_i(x) D_i \psi(x) + a^j_t(x) \psi(x)) + b^j_t(x) D_i \psi(x) + c_t(x) \psi(x),$$

$$\Lambda^k_t \psi(x) = \sigma^i_{jk} (x) D_i \psi(x) + \nu^k_t(x) \psi(x),$$

and $a^j_i(x)$, $b^j_t(x)$, $\sigma^i_{jk} (x)$, and $\nu^k_t(x)$ are given functions.
the summation convention with respect to \( i, j = 1, \ldots, d \) and \( k = 1, 2, \ldots \) is enforced and detailed assumptions on the coefficients and the free terms will be given later.

One can rewrite (1.1) in the nondivergence form assuming that the coefficients \( a^i_j \) and \( a^j_i \) are differentiable in \( x \) and then one could apply the results from \([6]\). It turns out that the differentiability of \( a^i_j \) and \( a^j_i \) is not needed for the corresponding counterparts of the results in \([6]\) to be true and showing this and generalizing the corresponding results of \([2]\) is one of the main purposes of the present article.

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2. Main results

Fix a number \( p \geq 2 \), and denote \( L_p = L_p(\mathbb{R}^d) \). We use the same notation \( L_p \) for vector- and matrix-valued or else \( \ell_2 \)-valued functions such as \( g_t = (g^k_t) \) in (1.1). For instance, if \( u(x) = (u^1(x), u^2(x), \ldots) \) is an \( \ell_2 \)-valued measurable function on \( \mathbb{R}^d \), then

\[
\|u\|^p_{L_p} = \int_{\mathbb{R}^d} |u(x)|^p_{\ell_2} \, dx = \int_{\mathbb{R}^d} \left( \sum_{k=1}^{\infty} |u^k(x)|^2 \right)^{p/2} \, dx.
\]

Introduce

\[
D_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, d, \quad \Delta = D_1^2 + \ldots + D_d^2.
\]

By \( Du \) we mean the gradient with respect to \( x \) of a function \( u \) on \( \mathbb{R}^d \).

As usual,

\[
W_1^p = \{ u \in L_p : Du \in L_p \}, \quad \|u\|_{W_1^p} = \|u\|_{L_p} + \|Du\|_{L_p}.
\]

Recall that \( \tau \) is a stopping time and introduce

\[
\mathbb{L}_p(\tau) := L_p((0, \tau], \mathcal{F}, L_p), \quad \mathbb{W}_p^1(\tau) := L_p((0, \tau], \mathcal{F}, W_1^p).
\]

We also need the space \( W_p^1(\tau) \), which is the space of functions \( u_t = u_t(\omega, \cdot) \) on \( \{ (\omega, t) : 0 \leq t \leq \tau, t < \infty \} \) with values in the space of generalized functions on \( \mathbb{R}^d \) and having the following properties:

(i) We have \( u_0 \in L_p(\Omega, \mathcal{F}_0, L_p) \);

(ii) We have \( u \in \mathbb{W}_p^1(\tau) \);

(iii) There exist \( f^i \in \mathbb{L}_p(\tau), i = 0, \ldots, d, \) and \( g = (g^1, g^2, \ldots) \in \mathbb{L}_p(\tau) \) such that for any \( \varphi \in C_0^\infty = C_0^\infty(\mathbb{R}^d) \) with probability 1 for all \( t \in [0, \infty) \) we have

\[
(u_{t \wedge \tau}, \varphi) = (u_0, \varphi) + \sum_{k=1}^{\infty} \int_0^t I_{s \leq \tau}(g^k_s, \varphi) \, dw^k_s
\]
In particular, for any \( \phi \in C_0^\infty \), the process \((u_{t\wedge \tau}, \phi)\) is \( \mathcal{F}_t \)-adapted and (a.s.) continuous.

The reader can find in [6] a discussion of (ii) and (iii), in particular, the fact that the series in (2.1) converges uniformly in probability on every finite subinterval of \([0, \tau]\). On the other hand, it is worth saying that the above introduced space \( W_{p,0}^1(\tau) \) is not quite the same as \( H_{p,0}^1(\tau) \) in [6] or in [2]. There are three differences. One is that there is an additional restriction on \( u_0 \) in [6] and [2]. But in the main part of the article we are going to work with \( W_{p,0}^1, 0(\tau) \) which is the subset of \( W_{p,0}^1(\tau) \) consisting of functions with \( u_0 = 0 \).

Another issue is that in [6] and [2] we have \( f_i = 0, i = 1, ..., d \), and \( f_0 \in H_{p,0}^{-1}(\tau) = L_p((0, \tau], \mathcal{P}, H_{p}^{-1}) \).

Actually, this difference is fictitious because one knows that any \( f \in H_p^{-1} \)

(a) has the form \( D_i f_i^j + f_0 \) with \( f_j^i \in L_p \) and

\[
\|f\|_{H_p^{-1}} \leq N \sum_{j=0}^{d} \|f_j^i\|_{L_p},
\]

where \( N \) is independent of \( f, f_j^i \), and on the other hand,

(b) for any \( f \in H_p^{-1} \) there exist \( f_j^i \in L_p \) such that \( f = D_i f_i^j + f_0 \) and

\[
\sum_{j=0}^{d} \|f_j^i\|_{L_p} \leq N \|f\|_{H_p^{-1}},
\]

where \( N \) is independent of \( f \).

The third difference is that instead of (i) the condition \( D^2 u \in H_p^{-1}(\tau) \) is required in [6] and [2]. However, as it follows from Theorem 3.7 of [6] and the boundedness of the operator \( D : L_p \rightarrow H_p^{-1} \), this difference disappears if \( \tau \) is a bounded stopping time.

To summarize, the spaces \( W_{p,0}^1(\tau) \) introduced above coincide with \( H_{p,0}^1(\tau) \) from [6] if \( \tau \) is bounded and we choose a particular representation of the deterministic part of the stochastic differential just for convenience. In the remainder of the article the spaces \( H_{p,0}^1(\tau) \) do not appear and none of their properties is used.

In case that property (iii) holds, we write

\[
du_t = (D_i f_i^j + f_0^j) \, dt + g_k^j \, dw_t^k
\]

(2.2)

for \( t \leq \tau \) and this explains the sense in which equation (1.1) is understood. Of course, we still need to specify appropriate assumptions on the coefficients and the free terms in (1.1).

**Assumption 2.1.** (i) The coefficients \( a_i^j, a_i^j, b_i^j, \sigma_i^k, c_t, \) and \( \nu_t^k \) are measurable with respect to \( \mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \), where \( \mathcal{B}(\mathbb{R}^d) \) is the Borel \( \sigma \)-field on \( \mathbb{R}^d \).
(ii) There is a constant $K$ such that for all values of indices and arguments

$$|a_i^1| + |b_i^1| + |c_i| + |\nu|_{\ell_2} \leq K, \quad c_i \leq 0.$$ 

(iii) There is a constant $\delta > 0$ such that for all values of the arguments and $\xi \in \mathbb{R}^d$

$$a_{ij}^1 \xi^i \xi^j \leq \delta^{-1} |\xi|^2, \quad (a_{ij}^1 - a_{ij}^0) \xi^i \xi^j \geq \delta |\xi|^2,$$

where $a_{ij}^0 = (1/2)(\sigma^i, \sigma^j)_{\ell_2}$. Finally, the constant $\lambda \geq 0$.

It is worth emphasizing that we do not require the matrix $(a_{ij}^0)$ to be symmetric.

Assumption 2.1 guarantees that equation (1.1) makes perfect sense if $u \in \mathcal{W}_p^1(\tau)$. By the way, adding the term $-\lambda u_t$ with constant $\lambda \geq 0$ is one more technically convenient step. One can always introduce this term, if originally it is absent, by considering $v_t := u_t e^{\lambda t}$.

Let $\mathbb{B}$ denote the set of balls $B \subset \mathbb{R}^d$ and let $\rho(B)$ be the radius of $B \in \mathbb{B}$. For functions $h_t(x)$ on $[0, \infty) \times \mathbb{R}^d$ and $B \in \mathbb{B}$ introduce

$$h_t(B) = \frac{1}{|B|} \int_B h_t(x) \, dx,$$

where $|B|$ is the volume of $B$. Also let $Q$ denote the set of all cylinders in $[0, \infty) \times \mathbb{R}^d$ of type $Q = (s, t) \times B$, where $B \in \mathbb{B}$ and $t - s = \rho^2(B)$. For such $Q$ set $\rho(Q) = \rho(B)$. For $\rho \geq 0$, $0 < s < t$, a continuous $\mathbb{R}^d$-valued function $x_r, r \in [s, t]$, and a $Q = (s, t) \times B \in Q$, introduce

$$\text{Osc} \,(h, Q, x_r) = \frac{1}{t - s} \int_s^t (|h_r - h_r(B + x_r)|)_{(B + x_r)} \, dr,$$

$$\text{Osc} \,(h, Q, \rho) = \sup_{|x|_{\mathcal{C}}} \text{Osc} \,(h, Q, x_r), \quad \text{Osc} \,(h, Q) = \text{Osc} \,(h, Q, 0),$$

where $|x|_{\mathcal{C}}$ is the sup norm of $|x|$.

Observe that $\text{Osc} \,(h, Q, x_r) = 0$ if $h_t(x)$ is independent of $x$.

Denote by $B_\rho$ the open ball with radius $\rho > 0$ centered at the origin, define $Q_\rho = (0, \rho^2) \times B_\rho$ and for $t \geq 0$ and $x \in \mathbb{R}^d$ set $B_\rho(x) = B_\rho + x$, $Q_\rho(t, x) = Q_\rho + (t, x)$.

In the remaining two assumptions we use constants $\beta > 0$ and $\beta_1 > 0$ the values of which will be specified later.

Let $t_0 \geq 0$, $x_0 \in \mathbb{R}^d$, and constants $\varepsilon \geq \varepsilon_1 > 0$. We say that the couple $(a, \sigma)$ is $(\varepsilon, \varepsilon_1)$-regular at point $(t_0, x_0)$ if (for any $\omega$) either

(i) we have $\sigma_t^{nm}(x_0) = 0$ for $t \in (t_0, t_0 + \varepsilon_1^2)$ and all $n, m$ and

$$\text{osc} \,(a_{ij}^0, Q) \leq \beta, \quad \forall i, j,$$ 

for all $Q \in Q$ such that $Q \subset Q_{\varepsilon}(t_0, x_0)$, or

(ii) for all $Q \in Q$ such that $Q \subset Q_{\varepsilon}(t_0, x_0)$ we have

$$\text{Osc} \,(a_{ij}^0, Q, \varepsilon) \leq \beta, \quad \forall i, j.$$ 

Note that $(a, \sigma)$ is $(\varepsilon, \varepsilon_1)$-regular at any point $(t_0, x_0)$ for any $\beta > 0$ if, for instance, $a_{ij}^0$ depend only on $x$ and are of class VMO.
Assumption 2.2. There exist $\varepsilon \geq \varepsilon_1 > 0$ such that $(a, \sigma)$ is $(\varepsilon, \varepsilon_1)$-regular at any point $(t_0, x_0)$ and 

$$(a^{jk}_t (x) - a^{jk}_t (y)) \xi^j \xi^k \geq \delta |\xi|^2$$

for all $t, \xi, x, y$ satisfying $|x - y| \leq \varepsilon$.

Assumption 2.3. There exists an $\varepsilon_2 > 0$ such that

$$|\sigma_i \cdot t(x) - \sigma_i \cdot t(y)|_{l^2} \leq \beta_1$$

(2.6)

for all $i, t, x, y$ satisfying $|x - y| \leq \varepsilon_2$.

Needless to say that Assumptions 2.2 and 2.3 are satisfied with any $\beta, \beta_1 > 0$ and slightly reduced $\delta$ if (2.3) holds and $a^{ij}_t (x)$ and $\sigma^j_i (x)$ are uniformly continuous in $x$ uniformly with respect to $(\omega, t)$.

Finally, we describe the space of initial data. Recall that for $p \geq 2$ the Slobodetskii space $W^{1-2/p}_p = W^{1-2/p}_p (\mathbb{R}^d)$ of functions $u_0(x)$ can be introduced as the space of traces on $t = 0$ of (deterministic) functions $u$ such that

$$u \in L_p (\mathbb{R}_+, H^1_p), \quad \partial u / \partial t \in L_p (\mathbb{R}_+, H^{-1}_p),$$

where $\mathbb{R}_+ = (0, \infty)$. For such functions there is a (unique) modification denoted again $u$ such that $u_t$ is a continuous $L_p$-valued function on $[0, \infty)$ so that $u_0$ is well defined. Any such $u_t$ is called an extension of $u_0$.

The norm in $W^{1-2/p}_p$ can be defined as the infimum of

$$\|u\|_{L_p (\mathbb{R}_+, H^1_p)} + \|\partial u / \partial t\|_{L_p (\mathbb{R}_+, H^{-1}_p)}$$

over all extensions $u_t$ of elements $u_0$. It is also well known that an equivalent norm of $u_0$ can be introduced as

$$\|u\|_{L_p ((0,1), W^1_p)}$$

where $u = u_t$ is defined as the (unique) solution of the heat equation

$$\partial u_t(x) / \partial t = \Delta u_t(x)$$

with initial condition $u_0(x)$.

For $s \geq 0$ we introduce

$$\text{tr}_s W^1_p = L_p (\Omega, \mathcal{F}_s, W^{1-2/p}_p).$$

The following auxiliary result helps understand the role of $\text{tr}_s W^1_p$. We use spaces $W^1_p([S, T))$ and $\mathbb{W}^1_p((S, T))$, which are introduced in the same way as $W^1_p (\tau)$ and $\mathbb{W}^1_p (\tau)$ but the functions are only considered on $[S, T)$ and $(S, T)$, respectively.

Lemma 2.1. Let $s \geq 0$ be a fixed number and let $u_s$ be an $\mathcal{F}_s$-measurable function with values in the set of distributions over $\mathbb{R}^d$.

(i) We have $u_s \in \text{tr}_s W^1_p$ if and only if there exists a $v \in \mathbb{W}^1_p ([s, \infty))$ satisfying the equation

$$\partial v / \partial t = \Delta v - v, \quad t \geq s,$$

(2.7)
then equation (1.1) and is understood in the same sense) with initial data $u_s$. This $v$ is unique and satisfies
\[ \|v\|_{W^1_p((s,\infty))} \leq N\|u_s\|_{W^1_p((s,\infty))}, \quad \|u_s\|_{W^1_p((s,\infty))} \leq N\|v\|_{W^1_p((s,\infty))}, \] (2.8)
where the constants $N$ are independent of $s$, $u_s$, and $v$.

(ii) We have $u_s \in W^1_p$ if and only if there exists a $v \in W^1_p((s,s+1))$ such that $v_s = u_s$.

(iii) If such a $v$ exists and $dv_t = (D_t f^i_t + f^0_t)dt + g^k_t dw^k_t$, $t \geq s$, then
\[ \|u_s\|_{W^1_p((s,s+1))} \leq N\|v\|_{W^1_p((s,s+1))} + \sum_{j=0}^d \|f^j\|_{L_p((s,s+1))} + \|g\|_{L_p((s,s+1))}, \] (2.9)
where the constant $N$ is independent of $s$, $u_s$ and $v$.

(iv) If $s > 0$ and we have a $u \in W^1_p(s)$, then $u_s \in W^1_p$ and
\[ \|u_s\|_{W^1_p} \leq N\|u\|_{W^1_p(s)} + \sum_{j=0}^d \|f^j\|_{L_p(s)} + \|g\|_{L_p(s)}, \]
where $N$ is independent of $u$, and $f^j$ and $g^k$ are taken from (2.2).

We prove this lemma in Section 5.

Here are our main results concerning (1.1). The following theorem is very close to Theorem 2.12 of [2]. Important differences are the presence of the parameter $\lambda$ in (2.10) and weaker assumptions on the coefficients of the deterministic part of the equation.

**Theorem 2.2.** Let the above assumptions be satisfied with $\beta = \beta(d,p,\delta) = \beta_0/3$, where $\beta_0$ is the constant from Lemma 5.7, and $\beta_1 = \beta_1(d,p,\delta,\epsilon) > 0$ taken from the proof of Lemma 5.2. Let $\lambda \geq 0$, let $f^j, g \in L_p(\tau)$, and let $u_0 \in W^1_p$.

(i) Then equation (1.1) for $t \leq \tau \wedge T$ has a unique solution $u \in W^1_p(\tau \wedge T)$ with initial data $u_0$ and any $T \in (0,\infty)$. Moreover, if
\[ \lambda \geq \lambda_0(d,p,\delta,K,\epsilon,\epsilon_1,\epsilon_2) \geq 1, \]
then equation (1.1) for $t \leq \tau$ has a unique solution $u \in W^1_p(\tau)$ with initial data $u_0$.

(ii) Furthermore, if a $v \in W^1_p(\infty)$ is defined by equation (2.7) with initial condition $u_0$, then the above solution $u$ satisfies
\[ \lambda^{1/2}\|u\|_{L_p(\tau)} + \|Du\|_{L_p(\tau)} \]
\[ \leq N\left( \sum_{i=1}^d \|f^i\|_{L_p(\tau)} + \|g\|_{L_p(\tau)} + \|Dv\|_{L_p(\tau)} \right) + N\lambda^{-1/2}\|f^0\|_{L_p(\tau)} + N\lambda^{1/2}\|v\|_{L_p(\tau)} \]
(2.10)
provided that $\lambda \geq \lambda_0$, where the constants $N, \lambda_0 \geq 1$ depend only on $d, p, \delta, K, \epsilon, \epsilon_1, \text{and } \epsilon_2$. 
(iii) Finally, there exists a set $\Omega' \subset \Omega$ of full probability such that $u_{t\wedge T}I_{\Omega'}$ is a continuous $F_t$-adapted $L_p$-valued functions of $t \in [0, \infty)$.

Observe that estimate (2.10) shows one of good reasons for writing the free term in (1.1) in the form $D_t f^i + f^0$, because $f^i$, $i = 1, \ldots, d$, and $f^0$ enter (2.10) differently.

**Remark 2.3.** As it follows from our proofs, if $p = 2$, Assumptions 2.2 and 2.3 are not needed for Theorem 2.2 to be true and mentioning $\varepsilon$, $\varepsilon_1$, and $\varepsilon_2$ can be dropped in the statement. Thus we provide a new way to prove the classical result on Hilbert space solvability of SPDEs (cf., for instance, [15]).

We prove Theorem 2.2 in Section 6 after we prepare necessary tools in Sections 3-5. In Section 3 we prove uniqueness part of Theorem 2.2 on the basis of Itô’s formula from [13]. Here Assumptions 2.2 and 2.3 are not used. In Section 4 we treat the case of the heat equation with random right-hand side and present a simplified version of the corresponding result from [6]. In Section 5 we prove an auxiliary existence theorem and derive some a priori estimates.

Here is a result about continuous dependence of solutions on the data.

**Theorem 2.4.** Assume that for each $n = 1, 2, \ldots$ we are given functions $a^{ij}_n, a^i_n, b^i_n, c_n, \sigma^{ik}_n, \nu^k_n, f^j_n, g^k_n$, and $u_{n0}$ having the same meaning as the original ones in Theorem 2.2 (with the same $\delta, K, \beta, \ldots$). Assume that for $i, j = 1, \ldots, d$ and almost all $(\omega, t, x)$ we have

\[
(a^{ij}_n, a^i_n, b^i_n, c_n) \rightarrow (a^{ij}, a^i, b^i, c_t),
\]

as $n \rightarrow \infty$. Also assume that

\[
\sum_{j=0}^d (\|f^j_n - f^j\|_{L_p(\tau)} + \|g_n - g\|_{L_p(\tau)} + \|u_{n0} - u_0\|_{W^1_p}) \rightarrow 0
\]

as $n \rightarrow \infty$. Take $\lambda \geq \lambda_0$, take the function $u$ from Theorem 2.2 and let $u_n \in W^1_p(\tau)$ be the unique solutions of equations (1.1) for $t \leq \tau$ constructed from $a^{ij}_n, a^i_n, b^i_n, c_n, \sigma^{ik}_n, \nu^k_n, f^j_n, g^k_n$ and having initial values $u_{n0}$.

Then, as $n \rightarrow \infty$, we have $\|u_n - u\|_{W^1_p(\tau)} \rightarrow 0$ and for any finite $T \in [0, \infty)$

\[
E \sup_{t \leq \tau \wedge T} \|u_{nt} - u_t\|^p_{L_p} \rightarrow 0. \tag{2.11}
\]

Proof. Set $v_t = u_{nt} - u_t$. Then

\[
dv_t = (Lu_{nt} + \lambda v_{nt} + D_t f^i_n + f^0) dt + (\Lambda^k v_{nt} + g^k) dw_t^k,
\]

where $L_n$ and $\Lambda^k_n$ are the operators constructed from $a^{ij}_n, a^i_n, b^i_n, c_n$ and $\sigma^{ik}_n, \nu^k_n$, respectively, and

\[
f^i_n = f^i - f^i_t + (a^{ij}_n - a^{ij}) D_j u_t + (a^i_n - a^i_t) u_t,
\]

and
where there is a unique solution for all

\[ E \text{ is standard (see, for instance, our Theorem 3.1) to derive from here the} \]

\[ \text{techniques was used. We give here a shorter proof based on a different idea.} \]

\[ \text{By Theorem 2.2 we know that } u \in W^1_p(\tau). \text{ This along with our assumptions and the dominated convergence theorem implies that} \]

\[ \sum_{j=0}^{d} \| \tilde{f}_n^j \|_{L_p(\tau)} + \| \tilde{g}_n \|_{L_p(\tau)} \to 0 \]

as \( n \to \infty \). After that by applying (2.10) to \( v_{nt} \) we immediately see that \( \| u_n - u \|_{W^1_p(\tau)} \to 0 \).

Assertion (2.11) is, actually, a simple corollary of the above. Indeed, by introducing \( \tilde{f}_n \) and \( \tilde{g}_n \) in an obvious way, we can write

\[ dv_{nt} = (D_i \tilde{f}_n^j + f^0_{nt}) dt + \tilde{g}_n^k dw^k_t, \tag{2.12} \]

and

\[ \sum_{j=1}^{d} \| \tilde{f}_n^j \|_{L_p(\tau)} + \| \tilde{g}_n \|_{L_p(\tau)} \to 0. \]

It is standard (see, for instance, our Theorem 3.1) to derive from here the estimate

\[ E \sup_{t \leq \tau \wedge T} \| u_{nt} - u_t \|_{L_p} \leq N \left( \sum_{j=1}^{d} \| \tilde{f}_n^j \|_{L_p(\tau \wedge T)} + \| \tilde{g}_n \|_{L_p(\tau \wedge T)} + E \| u_{n0} - u_0 \|_{L_p} \right), \]

where \( N \) is independent of \( n \). It is also well known that \( W^{1-2/p}_p \subset L_p \), that is

\[ \| u_{n0} - u_0 \|_{L_p} \leq N \| u_{n0} - u_0 \|_{W^{1-2/p}_p}. \]

By combining all this together we obtain (2.11) and the theorem is proved.

The following result could be proved on the basis of Theorem 2.4 in the same way as Corollary 5.11 of [6], where the solutions are approximated by solutions of equations with smooth coefficients and then a stopping time techniques was used. We give here a shorter proof based on a different idea.

**Theorem 2.5.** Let \( p_1, p_2 \in [2, \infty), p_1 < p_2 \), and let the above assumptions be satisfied with \( \beta \leq \beta(d, p, \delta) \) for all \( p \in [p_1, p_2] \) and \( \beta_1 \leq \beta_1(d, p, \delta, \varepsilon) \) for all \( p \in [p_1, p_2] \). Let \( \lambda \geq 0 \), and suppose that for \( p \in [p_1, p_2] \) we have \( f^j, g \in L_p(\tau) \), and \( u_0 \in \text{tr}_0 W^1_p \).

Then the solutions corresponding to \( p = p_1 \) and \( p = p_2 \) coincide, that is, there is a unique solution \( u \in W^1_{p_1}(\tau) \cap W^1_{p_2}(\tau) \) of equation (1.1) with initial data \( u_0 \).

Proof. Obviously, it suffices to concentrate on bounded \( \tau \). As is explained above in that case we may assume that \( \lambda \) is as large as we like. We take it so large that one could use assertion (ii) of Theorem 2.2 with any \( p \in [p_1, p_2] \).

Denote by \( u \) the solution corresponding to \( p = p_2 \) and observe that, owing to uniqueness of solutions in \( W^1_{p_1}(\tau) \), we need only show that \( u \in W^1_{p_1}(\tau) \).
Take a $\zeta \in C_0^\infty$ such that $\zeta(0) = 1$, set $\zeta_n(x) = \zeta(x/n)$, and notice that $u^n := u\zeta_n$ satisfies
\[
du^n_t = (L_tu^n_t) - \lambda u_t^n + D_i f_{nt}^i + f_{nt}^0) dt + (\Lambda^k u^n_t + g_{nt}^k) du_k^n,
\]
where
\[
f_{nt}^i = f_i^t \zeta_n - u a_{ij} D_j \zeta_n, \quad i \geq 1,
\]
\[
f_{nt}^0 = f_i^t \zeta_n - f_i^t D_i \zeta_n - (a_{ij} D_j u_t + \alpha_i \zeta_n) - b_i u_t D_i \zeta_n,
\]
\[
g_{nt}^k = g_i^k \zeta_n - \sigma_{i}^k u_t D_i \zeta_n.
\]
It follows that for $p_1 \leq p \leq p_2$ we have
\[
\|u^n\|_{W^1_p(\tau)} \leq N \left( \sum_{i=0}^{d} \|f_{nt}^i\|_{L_p(\tau)} + \|g_{nt}\|_{L_p(\tau)} + \|u_0 \zeta_n\|_{tr_0 W^1_{p_2}} \right).
\]
(2.13)

One knows that with constants $N$ independent of $n$
\[
\|u_0 \zeta_n\|_{tr_0 W^1_{p_2}} \leq N (\|u_0 \zeta_n\|_{tr_0 W^1_{p_1}} + \|u_0 \zeta_n\|_{tr_0 W^1_{p_2}}) \leq N (\|u_0\|_{tr_0 W^1_{p_1}} + \|u_0\|_{tr_0 W^1_{p_2}}).
\]

Similarly, and by Hölder’s inequality
\[
\|f_{nt}^i\|_{L_p(\tau)} \leq N \|u D \zeta_n\|_{L_2(\tau)} \leq N + \|u\|_{L_2(\tau)} \|D \zeta_n\|_{L_q(\tau)},
\]
where
\[
q = \frac{pp_2}{p - p_2}.
\]

Similar estimates are available for other terms in the right-hand side of (2.13). Since
\[
\|D \zeta_n\|_{L_q(\tau)} = N u^{-1+(p_2-p)d/(p_2p)} \to 0
\]
as $n \to \infty$ if
\[
\frac{1}{p} - \frac{1}{p_2} < \frac{1}{d},
\]
(2.14)
estimates (2.13) implies that $u \in W^1_{p_2}(\tau)$.

Thus knowing that $u \in W^1_{p_2}(\tau)$ allowed us to conclude that $u \in W^1_{p_1}(\tau)$ as long as $p \in [p_1,p_2]$ and (2.14) holds. We can now replace $p_2$ with a smaller $p$ and keep going in the same way each time increasing $1/p$ by the same amount until $p$ reaches $p_1$. Then we get that $u \in W^1_{p_1}(\tau)$. The theorem is proved.

In many situations the following maximum principle is useful.

**Theorem 2.6.** Let the above assumptions be satisfied with $\beta \leq \beta(d,q,\delta)$ for all $q \in [2,p]$ and $\beta_1 \leq \beta(d,q,\delta,\varepsilon)$ for all $q \in [2,p]$. Let $\lambda \geq 0$ and $f^0 \in L_p(\tau)$, $u_0 \in tr_0 W^1_p$, $f^i = 0$, $i = 1, \ldots, d$, $g = 0$ be such that $u_0 \geq 0$ and $f^0 \geq 0$. Then for the solution $u$ almost surely we have $u_t \geq 0$ for all finite $t \leq \tau$.

Proof. If $p = 2$ the result is proved in [9]. For general $p \geq 2$ take the same function $\zeta_n$ as in the preceding proof, introduce $f^{n_i} = f^i \zeta_n$, $g_n^k = 0$, and call $u^n$ the solution of (1.1) with so modified free terms and the initial data $u_0 \zeta_n$. By Theorem 2.5 we have $u^n \in W^1_{p_1}(\tau) \cap W^1_{p_2}(\tau)$. By the above, $u^n \geq 0$ and it only remains to use Theorem 2.4. The theorem is proved.
3. Itô’s formula and uniqueness

The following two “standard” results are taken from [13].

**Theorem 3.1.** Let \( u \in \mathcal{W}^1_p(\tau) \), \( f^j \in \mathbb{L}_p(\tau) \), \( g = (g^k) \in \mathbb{L}_p(\tau) \) and assume that (2.2) holds for \( t \leq \tau \) in the sense of generalized functions. Then there is a set \( \Omega' \subset \Omega \) of full probability such that

(i) \( u_{t \wedge \tau} I_{\Omega'} \) is a continuous \( \mathbb{L}_p \)-valued \( \mathcal{F}_t \)-adapted function on \([0, \infty)\);

(ii) for all \( t \in [0, \infty) \) and \( \omega \in \Omega' \) Itô’s formula holds:

\[
\int_{\mathbb{R}^d} |u_{t \wedge \tau}|^p \, dx = \int_{\mathbb{R}^d} |u_0|^p \, dx + p \int_0^{t \wedge \tau} \int_{\mathbb{R}^d} |u_s|^{p-2} u_s g^k_s \, dx \, dw^k_s
\]

\[
+ \int_0^{t \wedge \tau} \left( \int_{\mathbb{R}^d} \left[ p|u_t|^{p-2} u_t f^0_t - p(p-1)|u_t|^{p-2} f^1_t D_i u_t \right. \right.
\]

\[
+ \left. \left. (1/2)p(p-1)|u_t|^{p-2} |g^k_t|^2 \right] \, dx \right) \, dt. \tag{3.1}
\]

Furthermore, for any \( T \in [0, \infty) \)

\[
E \sup_{t \leq \tau \wedge T} \|u_t\|_{\mathbb{L}_p}^p \leq 2E\|u_0\|_{\mathbb{L}_p}^p + NT^{p-1}\|f^0\|_{\mathbb{L}_p(\tau)}^p
\]

\[
+ NT^{(p-2)/2} \left( \sum_{i=1}^d \|f^i\|_{\mathbb{L}_p(\tau)}^p + \|g\|_{\mathbb{L}_p(\tau)}^p + \|Du\|_{\mathbb{L}_p(\tau)}^p \right), \tag{3.2}
\]

where \( N = N(d, p) \).

Here is an “energy” estimate.

**Corollary 3.2.** Under the conditions of Theorem 3.1 assume that \( \tau < \infty \) (a.s.). Then

\[
E \int_{\mathbb{R}^d} |u_0|^p \, dx + E \int_0^\tau \left( \int_{\mathbb{R}^d} \left[ p|u_t|^{p-2} u_t f^0_t - p(p-1)|u_t|^{p-2} f^1_t D_i u_t \right. \right.
\]

\[
+ \left. \left. (1/2)p(p-1)|u_t|^{p-2} |g^k_t|^2 \right] \, dx \right) \, dt \geq EI_{\tau < \infty} \int_{\mathbb{R}^d} |u_\tau|^p \, dx. \tag{3.3}
\]

Furthermore, if \( \tau \) is bounded then there is an equality instead of inequality in (3.3).

The next result implies, in particular, uniqueness in Theorem 2.2.

**Lemma 3.3.** Under Assumption 2.1 there exist \( \lambda_0 \geq 0 \) and \( N \) depending only on \( d, p, k, \) and \( \delta \) such that, for any strictly positive \( \lambda \geq \lambda_0 \) and any solution \( u \in \mathcal{W}^1_{p,0}(\tau) \) of (1.1) for \( t \leq \tau \), we have

\[
\lambda\|u\|_{\mathbb{L}_p(\tau)} \leq N\lambda^{1/2} \left( \sum_{j=1}^d \|f^j\|_{\mathbb{L}_p(\tau)} + \|g\|_{\mathbb{L}_p(\tau)} \right) + N\|f^0\|_{\mathbb{L}_p(\tau)}. \tag{3.4}
\]

Furthermore, if \( a^i = b^i = \nu^k \equiv 0 \), then one can take \( \lambda_0 = 0 \).
Proof. We may assume that \( f^j \in L_p(\tau) \), \( g = (g^k) \in L_p(\tau) \), since otherwise the right-hand side of (3.4) is infinite.

If (3.4) is true for \( \tau \wedge T \) in place of \( \tau \) and any \( T \in (0, \infty) \), then it is obviously also true as is. Therefore, we may assume that \( \tau \) is finite. An advantage of this assumption is that we can use Corollary 3.2. Write (3.3)

with \( f_t^i, f_t^0, \) and \( \dot{g}_t^k \) in place of \( f_t^i, f_t^0, \) and \( g_t^k \), respectively, where

\[
\begin{align*}
\dot{f}_t^i &= a_{ij} D_j u_t + \dot{a}_{ij} u_t + f_t^i, \\
\dot{f}_t^0 &= b_i^k D_i u_t + (c_t - \lambda) u_t + f_t^0, \\
&\quad \dot{g}_t^k = \sigma_t^{ik} D_i u_t + \nu k u_t + g_t^k.
\end{align*}
\]

Then observe that inequalities like \( (a + b)^2 \leq (1 + \varepsilon) a^2 + (1 + \varepsilon^{-1}) b^2 \) show that for any \( \varepsilon \in (0, 1] \) we have

\[
|\dot{g}_t|_2^2 \leq (1 + \varepsilon) \left| \sum_{i=1}^d \sigma_t^{ij} D_i u_t \right|_2^2 + 2\varepsilon^{-1}|u_t|_2 + |g_t|_2^2 \\
\leq 2(1 + \varepsilon)\alpha_t^{ij} (D_i u_t)D_j u_t + N\varepsilon^{-1}(|u_t|^2 + |g_t|_2^2).
\]

Owing to (3.3), for \( \varepsilon = \varepsilon(\delta) > 0 \) small enough

\[
I_t := (1/2)|u_t|^{p-2}|\dot{g}_t|_2^2 - |u_t|^{p-2}\dot{f}_t^i D_i u_t + (p - 1)^{-1}|u_t|^{p-2}u_i^j D_i u_t \\
\leq -(\delta/2)|u_t|^{p-2}|Du_t|^2 + N|u_t|^{p-2}(|u_t|^2 + |g_t|_2^2) + |Du_t| |u_t| + |Du_t| \sum_{i=1}^d |f_t^i|.
\]

Next we use that for any \( \gamma > 0 \)

\[
|u_t|^{p-1}|Du_t| = (|u_t|^{(p-2)/2}|Du_t|)|u_t|^{p/2} \leq \gamma |u_t|^{p-2}|Du_t|^2 + \gamma^{-1}|u_t|^p, \\
|u_t|^{p-2}|Du_t| |f_t^i| \leq \gamma |u_t|^{p-2}|Du_t|^2 + \gamma^{-1}|u_t|^{p-2}|f_t^i|^2,
\]

and by choosing \( \gamma \) appropriately find from (3.5) that

\[
I_t \leq N|u_t|^p + N|u_t|^{p-2} \left( \sum_{i=1}^d |f_t^i|^2 + |g_t|_2^2 \right).
\]

After that Hölder’s inequality and (3.3), where the right-hand side is nonnegative, immediately lead to

\[
(\lambda - N_1)\|u\|_{L_p(\tau)}^{p-2} \leq N\|u\|_{L_p(\tau)}^{p-2} \left( \sum_{i=1}^d |f_t^i|_{L_p(\tau)}^2 + \|g\|_{L_p(\tau)}^2 \right) + N\|u\|_{L_p(\tau)}^{p-1}|f_t^0|_{L_p(\tau)}.
\]

Furthermore, simple inspection of the above argument shows that, if \( a^i = b^i = \nu^k \equiv 0 \), then the terms with \( |u_t|^2 \) and \( |u_t||Du_t| \) in (3.5) and the term with \( |u_t|^p \) in (3.6) disappear, so that we can take \( N_1 = 0 \) in this case (recall that \( c \leq 0 \)). Generally, for \( \lambda \geq 2N_1 \) we have \( \lambda - N_1 \geq (1/2)\lambda \) and

\[
\bar{U}^p \leq N\bar{U}^{p-2}\bar{G}^2 + N\bar{U}^{p-1}\bar{F},
\]

where

\[
\bar{U} = \lambda\|u\|_{L_p(\tau)}, \quad \bar{G} = \lambda^{1/2}\|f_t^i\|_{L_p(\tau)} + \|g\|_{L_p(\tau)}, \quad \bar{F} = \|f_t^0\|_{L_p(\tau)}.
\]
It follows that $\bar{U} \leq N(\bar{G} + \bar{F})$, which is (4.4) and the lemma is proved.

4. Case of the heat equation

To move further we need the following analytic fact established in [4] (see also [7] for a complete proof).

Lemma 4.1. Denote by $T_t$ the heat semigroup in $\mathbb{R}^d$ and let $p \geq 2$, $-\infty \leq a < b \leq \infty$, $g \in L_p((a, b) \times \mathbb{R}^d, \ell_2)$. Then

$$\int_{\mathbb{R}^d} \int_a^b \left[ \int_a^t |D(T_{t-s} g_s(x)|^2 \ell_2 \, ds \right] \, dt \, dx \leq N(d, p) \int_{\mathbb{R}^d} \int_a^b |g_t(x)|^2 \ell_2 \, dt \, dx.$$ 

In this section we deal with the following model equation

$$du_t = \Delta u_t \, dt + g_t^k \, dw_t^k. \quad (4.1)$$

Lemma 4.2. Assume that $\tau \leq T$, where the constant $T \in [0, \infty)$. Then for any $g = (g^1, g^2, \ldots) \in \mathbb{L}_p(\tau)$ there exists a unique $u \in W^{1,p}_0(\tau)$ satisfying (4.1) for $t \leq \tau$. Furthermore, for this solution we have

$$\|u_t\|^p_{\mathbb{L}_p(\tau)} \leq N(d, p) \|g\|^p_{\mathbb{L}_p(\tau)},$$

(4.2)

$$\|Du\|_{\mathbb{L}_p(\tau)} \leq N(d, p) \|g\|_{\mathbb{L}_p(\tau)}.$$

(4.3)

Proof. By replacing the unknown function $u_t$ with $v_t e^{\lambda t}$ we see that $v_t$ satisfies

$$dv_t = (\Delta v_t - \lambda v) \, dt + e^{-\lambda t} g_t^k \, dw_t^k.$$ 

Since $\tau$ is bounded the inclusions $u \in W^{1,p}_0(\tau)$ and $v \in W^{1,p}_0(\tau)$ are equivalent and our assertion about uniqueness follows from Lemma 3.3.

In the proof of existence we borrow part of the proof of Lemma 4.1 of [6]. As we have pointed out in the Introduction, the beginning of the theory of divergence and nondivergence type equations is the same. The only difference with that proof is that here we take $f \equiv 0$.

We take an integer $m \geq 1$, some bounded stopping times $\tau_0 \leq \tau_1 \leq \ldots \leq \tau_m \leq T$ and some (nonrandom) functions $g^{ij} \in C^\infty_0$, $i, j = 1, \ldots, m$. Then we define

$$g_t^k(x) = \sum_{i=1}^m g^{ik}(x) I_{(\tau_{i-1}, \tau_i]}(t),$$

$$v_t(x) = \sum_{k=1}^m \int_0^t g_s^k(x) \, dw^k_s = \sum_{i,k=1}^m g^{ik}(x)(w_{t \wedge \tau_i}^k - w_{t \wedge \tau_{i-1}}^k), \quad t \geq 0.$$ 

Obviously, for any $\omega$, the function $v_t(x)$ is continuous and bounded in $(t, x)$ along with any derivative in $x$. Furthermore, the function and its derivative in $x$ are Hölder $1/3$ continuous in $t$ uniformly with respect to $x$ (for almost any $\omega$). Also $v_t(x)$ has compact support in $x$. 
These properties of \( v_t(x) \) imply that for any \( \omega \) there exists a unique classical solution of the heat equation

\[
\frac{\partial}{\partial t} \bar{u}_t = \Delta \bar{u}_t + \Delta v_t, \quad t > 0,
\]

with zero initial data. Furthermore,

\[
\bar{u}_t(x) = \int_0^t T_{t-s} \Delta v_s(x) \, ds. \tag{4.4}
\]

This formula shows, in particular, that \( \bar{u}_t(x) \) is \( \mathcal{F}_t \)-adapted. Adding the fact that \( \bar{u}_t \) is continuous in \( t \) proves that \( \bar{u}_t(x) \) is predictable. The same holds for

\[
(\bar{u}_t, \phi) = \int_0^t (T_{t-s} \Delta v_s, \phi) \, ds
\]

with any \( \phi \in C_0^\infty \). The following corollary of Minkowski’s inequality

\[
\| \bar{u}_t \|_{L^p} \leq \int_0^t \| \Delta v_s \|_{L^p} \, ds \tag{4.5}
\]

shows that \( \bar{u}_t \) is \( L^p \)-valued. Since \( (\bar{u}_t, \phi) \) is predictable for any \( \phi \in C_0^\infty \), \( \bar{u}_t \) is weakly and hence strongly predictable as an \( L^p \)-valued process.

One can differentiate (4.4) with respect to \( x \) as many times as one wants and get similar statements about the derivatives of \( \bar{u}_t \). In particular, (4.5) implies that for any multi-index \( \alpha \)

\[
E \int_0^T \int_{\mathbb{R}^d} |D^\alpha \bar{u}_t|^p \, dx \, dt \leq T^p E \int_0^T \int_{\mathbb{R}^d} |D^\alpha \Delta v_t|^p \, dx \, dt < \infty,
\]

so that \( \bar{u}_t \in W^{1,p}_d(T) \).

Now, it is easily seen that

\[
u_t(x) := \bar{u}_t(x) + v_t(x)
\]

satisfies (4.1) pointwisely and by the above \( u_t \in W^{1,p}_d(T) \). The (deterministic) Fubini’s theorem also shows that \( u_t \) satisfies (4.1) in the sense of distributions.

Next, we use the same simple transformation as in the proof of Lemma 4.1 of [6] and conclude that for any \( t \) and \( x \) almost surely

\[
Du_t(x) = \sum_{k=1}^m \int_0^t T_{t-s} Dg_k(x) \, dw^k_s.
\]

Hence by Burkholder-Davis-Gundy inequality

\[
E|Du_t(x)|^p \leq NE \left[ \int_0^t |T_{t-s} Dg_s(x)|^2 \, ds \right]^{p/2},
\]

which along with Lemma 4.1 proves (4.3) for our particular \( g \). Theorem 3.1 shows that (4.2) follows from (4.3) and (4.1).

The rest is trivial since the set of \( g \)’s like the one above is dense in \( L^p(T) \) by Theorem 3.10 of [6]. The lemma is proved.
Next we introduce the parameter $\lambda$ into (4.1).

**Lemma 4.3.** Assume that $\tau \leq T$, where the constant $T \in [0, \infty)$. Let $\lambda > 0$. Then for any $g = (g^1, g^2, \ldots) \in L_p(\tau)$ there exists a unique $u \in \mathcal{W}_{p,0}^1(\tau)$ satisfying

$$
du_t = (\Delta u_t - \lambda u_t) \, dt + g^k_t \, dw^k_t.
$$

for $t \leq \tau$. Furthermore, for this solution we have

$$
\lambda^{p/2} \|u\|^p_{L_p(\tau)} \leq N(d, p) \|g\|^p_{L_p(\tau)},
$$

(4.7)

$$
\|D u\|^p_{L_p(\tau)} \leq N(d, p) \|g\|^p_{L_p(\tau)}.
$$

(4.8)

Proof. Uniqueness and estimate (4.7) follow from Lemma 3.3. The existence immediately follows from Lemma 4.2 and the result of transformation described in the beginning of its proof. To establish (4.8) consider the heat equation

$$
\frac{\partial}{\partial t} v_t = \Delta v_t - \lambda u_t.
$$

(4.9)

Since $u \in L_p(\tau)$, for almost any $\omega$ we have $u \in L_p((0, \tau) \times \mathbb{R}^d)$ and by a classical result (see, for instance, [12]) for almost any $\omega$ equation (4.9) with zero initial data has a unique solution in the class of functions such that along with derivatives in $x$ up to the second order they belong to $L_p((0, \tau) \times \mathbb{R}^d)$. Furthermore,

$$
\|D^2 v\|^p_{L_p((0,\tau)\times\mathbb{R}^d)} + \lambda^{p/2} \|D v\|^p_{L_p((0,\tau)\times\mathbb{R}^d)}
$$

$$
+ \lambda^p \|\nabla v\|^p_{L_p((0,\tau)\times\mathbb{R}^d)} \leq N \|\lambda u\|^p_{L_p((0,\tau)\times\mathbb{R}^d)}.
$$

(4.10)

The solution $v_t$ can be given by an integral formula, which implies that $v_t$ is $\mathcal{F}_t$-adapted. It is also continuous as an $L_p$-valued process, hence, is a predictable $L_p$-valued process. Taking expectations of both parts of (4.10) shows that $v \in \mathcal{W}_{p}^1(\tau)$.

Now observe that

$$
d(u_t - v_t) = \Delta(u_t - v_t) \, dt + g^k_t \, dw^k_t,
$$

which by Lemma 4.2 implies that

$$
\|D(u - v)\|^p_{L_p(\tau)} \leq N \|g\|^p_{L_p(\tau)}.
$$

Upon combining this with (4.10) we obtain

$$
\|D u\|^p_{L_p(\tau)} \leq N(\|g\|^p_{L_p(\tau)} + \lambda^{p/2} \|u\|^p_{L_p(\tau)}),
$$

which along with (4.7) yields (4.8). The lemma is proved.
5. A PRIORI ESTIMATES IN THE GENERAL CASE

First we deal with the case when $\sigma = \nu = 0$.

**Lemma 5.1.** Suppose that $\sigma^{jk} \equiv \nu^k \equiv 0$. Also suppose that Assumptions 2.1 and 2.2 are satisfied with $\beta \leq \beta_0$, where the way to estimate the constant $\beta_0(d, p, \delta) > 0$ is described in the proof. Let $f^j \in L_p(\tau)$ and $g \in L_p(\tau)$.

Then there exist constants $\lambda_0 \geq 1$ and $N$, depending only on $d, p, \delta, K$, and $\varepsilon$, such that for any $\lambda \geq \lambda_0$ there exists a unique $u \in W^{1, p}_0(\tau)$ satisfying (1.1) for $t \leq \tau$. Furthermore, this solution satisfies the estimate

$$\lambda^{1/2} \| u \|_{L_p(\tau)} + \| Du \|_{L_p(\tau)} \leq N \left( \sum_{i=1}^d \| f^i \|_{L_p(\tau)} + \| g \|_{L_p(\tau)} \right) + \lambda^{-1/2} \| f^0 \|_{L_p(\tau)}.$$

(5.1)

**Proof.** Uniqueness and part of estimate (5.1) follow from Lemma 3.3. In the rest of the proof we may assume that $\tau$ is bounded and split our argument into two parts.

**Case $g^k \equiv 0$.** First assume that the coefficients and $f^j$ are nonrandom.

We extend the coefficients of $L$ following the example $a_{ij}(x, t) = \delta_{ij}$, $t < 0$, and extend $f^j_t$ beyond $(0, \tau)$ arbitrary only requiring $f^j \in L_p(\mathbb{R}^{d+1})$.

Then by Theorem 4.5 and Remark 2.4 of [8] the equation

$$\frac{\partial}{\partial t} u_t = \theta L_t u_t - \lambda u_t + D_i f^i_t + f^0_t,$$

(5.2)

in $\mathbb{R}^{d+1}$ has a unique solution with finite norms

$$\| u \|_{L_p(\mathbb{R}^{d+1})} \quad \text{and} \quad \| Du \|_{L_p(\mathbb{R}^{d+1})}$$

provided that $\lambda \geq \lambda_0$. By Theorem 4.4 of [8]

$$\lambda^{1/2} \| u \|_{L_p(\mathbb{R}^{d+1})} + \| Du \|_{L_p(\mathbb{R}^{d+1})} \leq N \left( \sum_{i=1}^d \| f^i \|_{L_p(\mathbb{R}^{d+1})} + \lambda^{-1/2} \| f^0 \|_{L_p(\mathbb{R}^{d+1})} \right).$$

(5.3)

By Theorem 3.1 the function $u_t$ is a continuous $L_p$-valued function.

The proof of Theorem 4.4 of [8] is achieved on the basis of the a priori estimate (5.3) and the method of continuity by considering the family of equations

$$\frac{\partial}{\partial t} u_t = (\theta \lambda_t + (1 - \theta) \Delta) u_t - \lambda u_t + D_i f^i_t + f^0_t,$$

(5.4)

where the parameter $\theta$ changes in $[0, 1]$. We remind briefly the method of continuity because we want to show that certain properties of equation (5.4) which we know for $\theta = 0$ propagate from $\theta = 0$ to $\theta = 1$.

We fix a $\theta_0 \in [0, 1]$ and to solve (5.4) for given $f^j$ define a sequence of $u^n \in L_p(\mathbb{R}, W^{1, p}_0)$ by solving the equation

$$\frac{\partial}{\partial t} u^{n+1}_t = (\theta_0 \lambda_t + (1 - \theta_0) \Delta) u^{n+1}_t - \lambda u^{n+1}_t.$$
\[ D_t f_t^i + f_t^0 + (\theta - \theta_0)(L_t - \Delta)u^n, \quad n \geq 1, \quad u^0 = 0. \quad (5.5) \]

If we know that equation (5.4) is uniquely solvable with \( \theta_0 \) in place of \( \theta \) for arbitrary \( f^j \in L_p(\mathbb{R}^{d+1}) \), then the sequence \( u^n \) is well defined. Furthermore, estimate (5.3) easily shows that for \( \theta \) sufficiently close to \( \theta_0 \) the \( L_p(\mathbb{R}, W^1_p) \) norm of \( u^{n+1} - u^n \) goes to zero geometrically as \( n \to \infty \). In this way passing to the limit in (5.5) we obtain the solution of (5.4) for \( \theta \) close to \( \theta_0 \). Then we can repeat the procedure and starting from \( \theta = 0 \) and moving step by step eventually reach \( \theta = 1 \).

For \( \theta = 0 \) we are dealing with solvability of the heat equation which is proved by giving the solution explicitly by means of the heat semigroup. This representation formula has two important implications:

(i) For any constant \( T \in \mathbb{R} \), changing \( f_t^j \) for \( t \geq T \) does not affect \( u_t \) for \( t \leq T \);

(ii) If \( f^j \) are \( L_p(\mathbb{R}^{d+1}) \)-valued functions of a parameter, say \( \omega \) from a measurable space, say \((\Omega, \mathcal{F}_T)\), then the solution \( u \in L_p(\mathbb{R}, W^1_p) \), which now depends on \( \omega \) is also \( \mathcal{F}_T \)-measurable.

Property (i) is obtained by inspecting the representation formula. Property (ii) is true because the mapping \( L_p(\mathbb{R}^{d+1}) \ni f^j \to u \in L_p(\mathbb{R}, W^1_p) \) is continuous and hence Borel measurable.

Obviously, both properties propagate from \( \theta = 0 \) to \( \theta = 1 \) by the above method of continuity. In particular, solutions of (5.2) on the time interval \((−\infty, T]\) depend only on the values of \( f_t^j \) for \( t \in (−\infty, T] \). It follows that with the same \( \lambda \) and \( N \), for any \( T \in \mathbb{R} \),

\[ \lambda^{1/2} \| u \|_{L_p((−\infty, T), L_p)} + \| Du \|_{L_p((−\infty, T), L_p)} \]

\[ \leq N \sum_{i=1}^{d} \| f_t^i \|_{L_p((−\infty, T), L_p)} + \lambda^{-1/2} \| f^0 \|_{L_p((−\infty, T), L_p)}. \quad (5.6) \]

From now on we allow the coefficients and \( f^j \) to be random, continue \( f^j \) as zero for \( t < 0 \) and solve (5.2) for each \( \omega \). By (5.6) with \( T = 0 \) we have that \( u_t = 0 \) for \( t \leq 0 \) and it makes sense considering equation (5.2) on \((0, T]\) for each \( T \in (0, \infty) \) with zero initial condition. In such situation properties (i) and (ii) still hold.

In particular, if \( f^j \) are measurable \( L_p((0, T), L_p) \)-valued functions of a parameter, say \( \omega \) from a measurable space, say \((\Omega, \mathcal{F}_T)\), then the solution \( u \in L_p((0, T), W^1_p) \) is also \( \mathcal{F}_T \)-measurable. Then from the equation itself it follows that \( (u_T, \phi) \) is \( \mathcal{F}_T \)-measurable for any \( \phi \in C_0^\infty \). Since \( u_T \) takes values in \( L_p \), it is an \( L_p \)-valued \( \mathcal{F}_T \)-measurable function.

If \( f_t^i \) are predictable \( L_p \)-valued function, the above conclusions are valid for any \( T \in [0, \infty) \). In particular, \( u_t \) is \( \mathcal{F}_T \)-adapted as an \( L_p \)-valued function and since it is continuous, \( u_t \) is a predictable \( L_p \)-valued function.

These properties and the fact that (5.6) holds for any \( T \in (0, \infty) \) and \( \omega \) prove the lemma in the particular case under consideration.
Obviously, \( v_t \in W^{1}_{p,0}(\tau) \) of (4.6). Observe that
\[
(L_t - \Delta)v_t = D_i \hat{f}_i^j + \hat{f}_t^0,
\]
where \( \hat{f}_i^j \) are function of class \( L_p(\tau) \) defined by
\[
\hat{f}_i^j = (a_t^{ij} - \delta^{ij})D_i v_t + a_t^j v_t, \quad j = 1, \ldots, d,
\]
\[
\hat{f}_t^0 = b_t^i D_i v_t + c_t v_t.
\]
By the above there is a unique solution \( u \in W^{1}_{p,0}(\tau) \) of
\[
\frac{\partial}{\partial t} u = L_t u_t - \lambda u_t + (L_t - \Delta)v_t + D_i f_i^j + f_t^0.
\]
Obviously, \( v_t + u_t \) is a solution of class \( W^{1}_{p,0}(\tau) \) of equation (1.1). By the particular case
\[
\lambda^{1/2} \| u \|_{L_p(\tau)} + \| Du \|_{L_p(\tau)} \leq N \left( \sum_{i=1}^{d} (\| f_i^j \|_{L_p(\tau)} + \| \hat{f}_i^j \|_{L_p(\tau)}) \right) + N \lambda^{-1/2} (\| f^0 \|_{L_p(\tau)} + \| f^0 \|_{L_p(\tau)})
\]
and to obtain (5.1) it only remains to use the estimates of \( v_t \) provided by Lemma 4.3. The lemma is proved.

Now we allow \( \sigma \neq 0 \).

**Lemma 5.2.** (i) Suppose that Assumptions [2.7] is satisfied with \( K = 0 \) and take \( \varepsilon \geq \varepsilon_1 > 0, \varepsilon_2 \in (0, \varepsilon/4] \), \( t_0 \geq 0 \), and \( x_0 \in \mathbb{R}^d \).

(ii) Let \( f^j \in L_p(\tau), g \in L_p(\tau), \) and \( u \in W^{1}_{p,0}(\tau) \) be such that (1.1) holds for \( t \leq \tau \). Assume that \( u_t(x) = 0 \) if
\[
(t, x) \notin \Gamma := (t_0, t_0 + \varepsilon^2_1) \times B_{\varepsilon_2}(x_0).
\]

(iii) Assume that the couple \( (a, \sigma) \) is \( (\varepsilon, \varepsilon_1) \)-regular at \( (t_0, x_0) \) with \( \beta = \beta_0/3 \) in (2.4) and (2.5), where \( \beta_0 \) is the constant from Lemma 5.1. Also assume that
\[
|\sigma^i_t(x) - \sigma^i_t(x_0)|_{\varepsilon_2} \leq \beta_1, \quad (a_t^{jk}(y) - a_t^{jk}(x_0))\xi^j \xi^k \geq \delta \| \xi \|^2
\]
for all values of indices and arguments such that \( (t, x) \in \Gamma \) and \( (t, y) \in Q_\varepsilon(t_0, x_0) \), where \( \beta_1 = \beta_1(d, \delta, p, \varepsilon) > 0 \) is a constant an estimate from below for which can be obtained from the proof.

Then there exist constants \( \lambda_0 \geq 1 \) and \( N \), depending only on \( d, p, \delta, \) and \( \varepsilon \), such that estimate (5.1) holds provided that \( \lambda \geq \lambda_0 \).

Proof. Without loss of generality we may and will assume that \( x_0 = 0 \). Also we modify, if necessary, \( a \) and \( \sigma \) in such a way that \( \sigma^{ik}_t(x) = 0 \) if \( t \not\in (t_0, t_0 + \varepsilon^2_1) \), and \( a^{ij}_t(x) = \delta^{-1} \delta^{ij} \) if \( t \not\in (t_0, t_0 + \varepsilon^2_1) \). Obviously, under this modification assumption (iii) is preserved and equation (1.1) remains unaffected due to assumption (ii). The rest of the proof we split into two cases.
Case \( \sigma_{ij}^k(x) = \sigma_{ij}^k(0) \) for \( |x| \leq \varepsilon_2 \) and \( t \geq 0 \). We want to apply Lemma 5.1 and for that, even if \( \sigma \equiv 0 \), we need \( a^{ij} \) to satisfy at least the condition \( \text{osc}(a^{ij}, Q) \leq \beta \) for all \( Q \in \mathbb{Q} \) with \( \rho(Q) \leq \varepsilon \). To achieve this we modify \( a_{ij}^k(x) \) for \( |x| \geq \varepsilon / 4 \) using the fact that such modifications have no effect on (1.1) since \( u_t(x) = 0 \) for \( |x| \geq \varepsilon_2 \) and \( \varepsilon_2 < \varepsilon / 4 \).

Take a \( \xi \in C_0^{\infty}((\mathbb{R}^d)^d) \) with support lying in the ball of radius \( \varepsilon / 2 \) centered at the origin and such that \( \xi(x) = 1 \) for \( |x| \leq \varepsilon / 4 \) and \( 0 \leq \xi \leq 1 \). Set

\[
\hat{a}_{ij}^k := \xi a_{ij}^k + \delta^{-1}(1 - \xi) \delta_{ij}.
\]

We can use \( \hat{a} \) in place of \( a \) in (1.1). It follows by Lemma 4.7 of [6] (Itô-Wentzell formula) that the function \( v_t(x) := u_t(x + x_t) \) satisfies the equation

\[
dv_t(x) = (\tilde{L}_t v_t(x) - \nu v_t + D_1 \tilde{f}^i_t + \tilde{f}^0_t) dt + \tilde{g}_t^k(x + x_t) dw_t^k,
\]

where

\[
\tilde{L}_t \phi := D_1 (\tilde{a}_t^{ij} D_1 \phi), \quad \tilde{a}_t^{ij}(x) = \hat{a}_t^{ij}(x + x_t) - a_t^{ij}(0),
\]

\[
\tilde{f}_t^i(x) := f_t^i(x + x_t) - \sigma_t^{ik}(0) g_t^k(x + x_t), \quad i = 1, \ldots, d,
\]

\[
\tilde{g}_t^k(x) := g_t^k(x + x_t),
\]

and the process \( x_t = (x_t^1, \ldots, x_t^d) \) is defined by

\[
x_t^i = -\int_0^t \sigma_s^{ik}(0) dw_s^k.
\]

This fact shows that the assertion of the present lemma is a direct consequence of Lemma 5.1 in case the latter is applicable to (5.7).

As is easy to see we will be able to apply Lemma 5.1 to (5.7) if we can find \( \varepsilon' = \varepsilon'(d, \delta, \varepsilon, p) > 0 \) such that

\[
\frac{1}{t - s} \int_s^t (|\tilde{a}_r^{ij} - \tilde{a}_r^{ij}(B)|)_B dr \leq \beta_0,
\]

whenever \( (s, t) \times B \in \mathbb{Q} \) and \( \rho(B) \leq \varepsilon' \).

Denote by \( N \), with or without subscripts, various (large) constants depending only on \( d, \delta, \varepsilon \) and \( \sigma \) and observe that \( |D\xi| \leq N \). It follows easily that for \( B \in \mathcal{B} \) we have

\[
(|\tilde{a}_r^{ij} - \tilde{a}_r^{ij}(B)|)_B \leq (|\xi a_r^{ij} - (\xi a_r^{ij}(B + x_r))|)_B + \delta^{-1} |\xi - (\xi (B + x_r))|_B + N_1 \rho =: I_r + N_1 \rho,
\]

where and below \( \rho = \rho(B) \).

Let \( z \) be the center of \( B \) and set

\[
y_r = (z + x_r)(\rho + \varepsilon/2)|z + x_r|^{-1}
\]

if \( |z + x_r| \geq \rho + \varepsilon/2 \) and \( y_r = z + x_r \) otherwise. Observe that \( y_r \) is continuous in \( r \) and

\[
|y_r| \leq \rho + \varepsilon/2.
\]

Next we claim that

\[
I_r \leq 2(|a_r^{ij} - a_r^{ij}(B_r \cup y_r)|)_B + N_2 \rho.
\]
If (5.11) is true, then by combining it with (5.9) and using (5.10) we find that the left-hand side of (5.8) is less than

\[(N_1 + N_2)\rho + 2 \sup_{|y| \leq \rho + \epsilon/2} \text{osc} (a^{ij}, Q_{\rho} + (0, y), 0)\]

if \(\sigma_t^{nm}(0) = 0\) for all \(t, n, m\) or, in general, less than

\[(N_1 + N_2)\rho + 2\text{Osc} (a^{ij}, Q_{\rho}, \rho + \epsilon/2),\]

where \(Q_{\rho} = (s, t) \times B_{\rho}\). Now (2.4) and (2.5) imply that (5.8) is satisfied for \(\rho \leq \epsilon'\) if we choose \(\epsilon' > 0\) so that

\[(N_1 + N_2)\epsilon' \leq \beta_0/3, \quad \epsilon' \leq \epsilon/4.\]

Therefore, it only remains to prove the claim. Obviously, if \(|z + x| \geq \rho + \epsilon/2\), then \(I_r = 0\) and (5.11) holds. In case \(|z + x| < \rho + \epsilon/2\) the estimates

\[\left(\|h_r - h_r(B')\|_{(B')}\right) \leq \frac{1}{|B'|^2} \int_{B'} \int_{B'} |h_r(y) - h_r(z)| dydz \leq 2\left(\|h_r - h_r(B')\|_{(B')}\right),\]

\[|\xi(y) a^{ij}_r(y) - \xi(z) a^{ij}_r(z)| \leq \xi(y)|a^{ij}_r(y) - a^{ij}_r(z)| + N|\xi(y) - \xi(z)|\]

show that

\[I_r \leq 2|a^{ij}_r - a^{ij}_{r(B+x)}|_{(B+x)} + N\rho,\]

which is equivalent to (5.11). This proves the lemma in the particular case under consideration.

**General case.** We rewrite the term \(\Lambda_1^k u_t + g_t^k\) in (1.1) as \(\sigma_t^{ijk}(0)\xi D_1u_t + \tilde{g}_t^k\) with \(\tilde{g}_t^k = g_t^k + (\sigma_t^{ijk} - \sigma_t^{ijk}(0))D_1u_t\) and use the above result to conclude that estimate (5.1) holds with \(N = N_1 = N_1(d, p, \delta, \epsilon)\) if we add to its right-hand side

\[N_2(d, p, \delta, \epsilon)\beta_1 \|Du\|_{L_p(\tau)}\]

By choosing \(\beta_1 = \beta_1(d, p, \delta, \epsilon)\) so that \(N_2 \beta_1 \leq 1/2\), we get (5.1) with \(2N_1\) in place of \(N_1\). The lemma is proved.

**Remark 5.3.** If Assumptions (2.1) is satisfied with \(K = 0\) and \(a_t^{ij}\) and \(\sigma_t^{ijk}\) depend only on \(\omega\) and \(t\), then the assertion of Lemma 5.2 is true with \(l_0 = 0\) and \(N = N(d, p, \delta)\) and without requiring \(u\) to have compact support. This fact can be obtained by following the arguments in Section 4.3 of [6]. Even though those arguments are much longer, they allow one to prove a very general result saying roughly speaking that “whatever estimate can be established for solutions of the heat equation in Banach function spaces with norms that are invariant under time dependent shifting of the \(x\) coordinate, the same estimate with the same constant also holds for solutions of the parabolic equations with no lower order terms and with the matrix of the second order coefficients depending only on \(t\) and dominating (in the matrix sense) the unit matrix” (see [5]).
Next step is to consider equations with lower order terms. The following lemma and its corollary are stated in a slightly more general form than it is needed in the present article. The point is that we intend to use them in a subsequent article about equations in half spaces.

**Lemma 5.4.** Let \( G \subset \mathbb{R}^d \) be a domain (perhaps, \( G = \mathbb{R}^d \)) and take \( \varepsilon \geq \varepsilon_1 > 0 \) and \( \varepsilon_2 \in (0, \varepsilon/4] \).

(i) Let \( f^t, g \in L_p(\tau) \) and let \( u \in \mathcal{W}^{1,0}_p(\tau) \) satisfy (1.1) for \( t \leq \tau \) and be such that \( u_t(x) = 0 \) if \( x \notin G \).

(ii) Suppose that Assumptions 2.1 is satisfied.

(iii) Suppose that assumption (iii) of Lemma 5.2 is satisfied for any \( t_0 \geq 0 \) and \( x_0 \) such that dist \((x_0, G)\) \( \leq \varepsilon_2 \).

Then there exist constants \( N, \lambda_0 \geq 0 \), depending only on \( d, p, K, \delta, \varepsilon, \varepsilon_1, \) and \( \varepsilon_2 \), such that in particular, the estimate (5.11) holds true whenever \( \lambda \geq \lambda_0 \).

Proof. As usual we will use partitions of unity. Take a nonnegative \( \xi \in C_0^\infty(B_{\varepsilon_2}) \) with unit \( L_p \)-norm and take a nonnegative \( \eta \in C_0^\infty((0, \varepsilon_1^2)) \) with unit \( L_p \)-norm. For \( s \in \mathbb{R} \) and \( y \in \mathbb{R}^d \) introduce

\[
\zeta(t, x) = \xi(x)\eta(t), \quad \xi^s, y(t, x) = \zeta(t - s, x - y), \quad u_t^s, y(x) = \xi^s, y(t, x)u_t(x)
\]

so that, in particular,

\[
|u_t(x)|^p = \int_{\mathbb{R}^{d+1}} |u_t^s, y(x)|^p dy ds. \quad (5.12)
\]

Observe that for each \( s, y \)

\[
du_t^s, y = (\sigma_t^{s,j} D_i u_t^s, y + \hat{g}_t^{s,j,k}) du_t^k + \left( D_j (a_t^{j,i} D_i u_t^s, y) - \lambda u_t^s, y + D_j \hat{f}_t^{s,j} + \hat{f}_t^{s,0} \right) dt \quad (5.13)
\]

for \( t \leq \tau \), where we dropped the argument \( x \) (and \( \omega \)) and

\[
\hat{g}_t^{s,j,k} = \xi^s, y (u_t^k - D_j \xi^s, y),
\]

\[
\hat{f}_t^{s,j} = \xi^s, y (a_t^{j,i} u_t + f_t^j) - a_t^{j,i} u_t D_j \xi^s, y, \quad j = 1, \ldots, d,
\]

\[
\hat{f}_t^{s,0} = \xi^s, y (b_t^i D_i u_t + c_t u_t) - f_t^j D_j \xi^s, y - (a_t^{j,i} D_i u_t + a_t^{j,i} u_t) D_j \xi^s, y + \xi^s, y u_t,
\]

and \( \xi^s, y(t, x) = \xi(x - y)\eta(t - s) \).

As is easy to see \( u_t^s, y(t, x) = 0 \) for \( (t, x) \notin (s_+, s_++\varepsilon_1^2) \times B_{\varepsilon_2}(y) \). Therefore, by Lemma 5.2 if dist \((y, G)\) \( \leq \varepsilon_2 \), then

\[
\lambda^{p/2} \|u_t^s, y\|^p_{L_p(\tau)} + \|Du_t^s, y\|^p_{L_p(\tau)} \leq N \left( \sum_{j=1}^d \|\hat{f}_t^{s,j}\|^p_{L_p(\tau)} + \|\hat{g}_t^{s,j}\|^p_{L_p(\tau)} \right) + N \lambda^{-p/2} \|\hat{f}_t^{s,0}\|^p_{L_p(\tau)} \quad (5.14)
\]

provided that \( \lambda \geq \lambda_0 \), where \( N \) and \( \lambda_0 \) depend only on \( d, \delta, p, \) and \( \varepsilon \). This estimate also, obviously, holds if dist \((y, G)\) \( \geq \varepsilon_2 \) since then \( u_t^s, y \equiv 0 \).

Next,

\[
|\hat{f}_t^{s,j}| \leq N \xi^s, y |u_t| + \xi^s, y |f_t^j|, \quad j = 1, \ldots, d,
\]
$$|f_t^{s,y,0}| \leq N\tilde{\zeta}^{s,y}(|Du_t| + |u_t|) + N\tilde{\zeta}^{s,y}\sum_{j=0}^{d}|f_j^t|,$$

$$|g_t^{s,y}|_{L_2} \leq N\tilde{\zeta}^{s,y}|u_t| + \zeta^{s,y}|g_t|_{L_2},$$

where $\tilde{\zeta} = \zeta + |D\zeta| + |\zeta_t|,$ $\tilde{\zeta}^{s,y}(t, x) = \zeta(t - s, x - y)$, and here and below we allow the constants $N$ to depend only on $d, p, \delta, K, \varepsilon, \varepsilon_1,$ and $\varepsilon_2$.

We also notice that $|\zeta^{s,y}Du_t| \leq |D(\zeta^{s,y}u_t)| + \tilde{\zeta}^{s,y}|u_t|$. Then we find that

$$\lambda^{p/2}\|\zeta^{s,y}u\|_{L^p(\tau)}^p + \|\zeta^{s,y}Du\|_{L^p(\tau)}^p$$

$$\leq N\left(\sum_{i=1}^{d}\|\tilde{\zeta}^{s,y}f^i\|_{L^p(\tau)}^p + \|\tilde{\zeta}^{s,y}g\|_{L^p(\tau)}^p + \|\tilde{\zeta}^{s,y}u\|_{L^p(\tau)}^p\right)$$

$$+ N\lambda^{-p/2}(\|\tilde{\zeta}^{s,y}f_0\|_{L^p(\tau)}^p + \|\tilde{\zeta}^{s,y}Du\|_{L^p(\tau)}^p).$$

We integrate through this estimate and use formulas like (5.12). Then we obtain

$$\lambda^{p/2}\|u\|_{L^p(\tau)}^p + \|Du\|_{L^p(\tau)}^p$$

$$\leq N_1\left(\sum_{i=1}^{d}\|f^i\|_{L^p(\tau)}^p + \|g\|_{L^p(\tau)}^p + \|u\|_{L^p(\tau)}^p\right) + N_1\lambda^{-p/2}(\|f_0\|_{L^p(\tau)}^p + \|Du\|_{L^p(\tau)}^p).$$

Finally, we increase $\lambda_0 \geq 0$, if necessary, in such a way that $N_1\lambda^{-p/2} \leq 1/2$ for $\lambda \geq \lambda_0$. Then we obviously arrive at (5.1) with $N = 2N_1$. The lemma is proved.

To the best of the author’s knowledge the following multiplicative estimate is new even in the deterministic case.

**Corollary 5.5.** Let $\lambda = 0$. Then under the assumptions of Lemma 4.1 we have

$$\|Du\|_{L^p(\tau)} \leq N\left(\sum_{i=1}^{d}\|f^i\|_{L^p(\tau)} + \|g\|_{L^p(\tau)} + \|f_0\|_{L^p(\tau)}^{1/2}\|u\|_{L^p(\tau)}^{1/2} + \|u\|_{L^p(\tau)}\right),$$

where $N$ depends only on $d, p, K, \delta, \varepsilon, \varepsilon_1,$ and $\varepsilon_2$.

Indeed, take a $\lambda > 0$ and add and subtract the term $(\lambda_0 + \lambda)u_t$ $dt$ on the right in (4.1), thus introducing $\lambda$ into the equation and modifying $f^0$ by including into it one of $(\lambda_0 + \lambda)u_t$. Then after applying (5.1), we see that

$$\|Du\|_{L^p(\tau)} \leq N\left(\sum_{i=1}^{d}\|f^i\|_{L^p(\tau)} + \|g\|_{L^p(\tau)}\right)$$

$$+ (\lambda_0 + \lambda)^{-1/2}\|f_0\|_{L^p(\tau)} + (\lambda_0 + \lambda)^{1/2}\|u\|_{L^p(\tau)}.$$}

Now it only remains to take the inf with respect to $\lambda > 0$.

**Proof of Lemma 2.1** By bearing in mind an obvious shifting of time we see that in the proof of assertions (i)-(iii) we may assume that $s = 0$.

(i) First of all observe that uniqueness of solution of (2.7) is well known even in a much wider class than $W^1_p(\infty)$. 

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Let \( u_0 \in \text{tr}_0 W^3_p \), then \( u_0 \in W^{1-2/p} \) for almost each \( \omega \) and there is a unique solution of the heat equation
\[
dv_t = \Delta v_t \, dt
\]
of class \( L_p((0, 1), W^1_p) \) with initial condition \( u_0 \). Furthermore,
\[
\|v\|_{L_p((0, 1), W^1_p)} \sim \|u_0\|_{W^{1-2/p}_p}.
\]

Next take a \( \zeta \in C_0^\infty(\mathbb{R}) \) such that \( \zeta_0 = 1 \) and \( \zeta_t = 0 \) for \( t \geq 1/2 \) and define \( \psi_t(x) = e^{-t} u_t(x) \zeta_t \) for \( t \in [0, 1] \) and as zero if \( t \geq 1/2 \). Notice that (a.s.)
\[
\psi \in L_p(\mathbb{R}^+, W^1_p),
\]
and
\[
\frac{\partial}{\partial t} \psi_t = \Delta \psi_t - \psi_t + e^{-t} \zeta_t' v_t
\]
Then it is a classical result that there exists a unique \( \phi \in L_p(\mathbb{R}^+, W^2_p) \) which solves the equation
\[
d\phi_t = (\Delta \phi_t - \phi_t + e^{-t} \zeta_t' v_t) \, dt
\]
with zero initial condition. In addition,
\[
\|\phi\|_{L_p(\mathbb{R}^+, W^2_p)} \leq N \|\zeta' v\|_{L_p(\mathbb{R}^+, L_p)} \leq N \|u_0\|_{W^{1-2/p}_p},
\]
where the constants \( N \) depend only on \( d \) and \( p \). Owing to these estimates and uniqueness, the operators mapping \( u_0 \) into \( v \) and \( \phi \) are continuous (and nonrandom). Since \( u_0 \) is \( \mathcal{F}_0 \)-measurable, the same is true for \( \psi \), \( \phi \), and \( u = \psi - \phi \), which is of class \( L_p((0, 1), W^1_p) \), satisfies (2.7) and equals \( u_0 \) for \( t = 0 \). Also for each \( \omega \)
\[
\|u\|_{L_p(\mathbb{R}^+, W^1_p)} \leq \|\psi\|_{L_p(\mathbb{R}^+, W^1_p)} + \|\phi\|_{L_p(\mathbb{R}^+, W^1_p)} \leq N \|u_0\|_{W^{1-2/p}_p},
\]
where \( N \) depends only on \( d \) and \( p \). By raising the extreme terms to the \( p \)th power and taking expectations we get the first inequality in (2.8) and also finish proving the “only if” part of (i).

To prove the “if” part assume that we have a \( v \in \mathcal{W}_p^1(\infty) \) satisfying (2.7) and equal \( u_0 \) at \( t = 0 \). Then \( u_t = v_t e^t \) satisfies \( \partial u_t / \partial t = \Delta u_t \) and is of class \( \mathcal{W}_p^1(1) \). It follows that almost all \( \omega \) we have \( u \in L_p((0, 1), W^1_p) \), \( u_0 \in W^{1-2/p}_p \), and
\[
\|u_0\|_{W^{1-2/p}_p} \leq N \|u\|_{L_p((0, 1), W^1_p)} \leq N \|v\|_{L_p(\mathbb{R}^+, W^1_p)}.
\]
By raising all expressions to the power \( p \) and taking expectations we arrive at the second estimate in (2.8). Assertion (i) is proved.

The “only if” part in (ii) is, actually, proved above. To prove the “if” part write
\[
dv_t = (D_t f_t + f^0_t) \, dt + g^k_t \, dw^k_t = (\Delta v_t - \lambda v_t + D_t \hat{f}^0_t + \hat{f}^0_t) \, dt + g^k_t \, dw^k_t,
\]
where the constant \( \lambda > 0 \) will be chosen later, \( \hat{f}^i_t = f^i_t - D_i v_t, \ i = 1, \ldots, d, \)
\( \hat{f}^0_t = f^0_t + \lambda v_t, \) and \( \hat{f}^j, g \in \mathbb{L}_p(1). \) Next, take the function \( \zeta \) as above, set \( u = v \zeta, \) and observe that

\[
d u_t = (\Delta u_t - \lambda u_t + D_i \hat{f}^i_t + \hat{f}^0_t) dt + \hat{g}^k_t \, du^k_t, \tag{5.15}
\]

where \( \hat{f}^0 = \zeta f^0 + v \zeta, \ \hat{f}^i_t = \zeta \hat{f}^i_t, \ i = 1, \ldots, d, \ \hat{g}^k = \zeta g^k \) and \( \hat{f}^j, \hat{g} \in \mathbb{L}_p(\infty) \) and \( u \in \mathcal{W}_p^1(\infty). \)

By Lemma 5.11 for \( \lambda \) fixed and large enough (actually, one can take \( \lambda = 1, \) which is shown by using dilations), equation (5.15) with zero initial condition admits a unique solution \( \psi \in \mathcal{W}_p^1(\infty) \) and

\[
\|\psi\|_{\mathcal{W}_p^1(\infty)} \leq N \left( \sum_{j=0}^d \|\hat{f}^j\|_{\mathbb{L}_p(\infty)} + \|g\|_{\mathbb{L}_p(\infty)} \right)
\]

\[
\leq N \left( \sum_{j=0}^d \|f^j\|_{\mathbb{L}_p(1)} + \|g\|_{\mathbb{L}_p(1)} + \|v\|_{\mathcal{W}_p^1(1)} \right).
\]

Then the difference \( \phi = u - \psi \) satisfies (2.7), is of class \( \mathcal{W}_p^1(\infty), \) and \( \phi_0 = u_0. \) By assertion (i) we have \( u_0 \in \text{tr}_0 \mathcal{W}_p^1, \) which proves the “if” part in (ii). Furthermore,

\[
\|u_0\|_{\text{tr}_0 \mathcal{W}_p^1} \leq N\|\phi\|_{\mathcal{W}_p^1(\infty)} \leq N\|u\|_{\mathcal{W}_p^1(\infty)} + N\|\psi\|_{\mathcal{W}_p^1(\infty)}
\]

\[
\leq N\|v\|_{\mathcal{W}_p^1(1)} + N\|\psi\|_{\mathcal{W}_p^1(\infty)} \leq N \left( \sum_{j=0}^d \|f^j\|_{\mathbb{L}_p(1)} + \|g\|_{\mathbb{L}_p(1)} + \|v\|_{\mathcal{W}_p^1(1)} \right).
\]

This proves assertion (iii).

To prove (iv) observe that obvious dilations of the \( t \) axis allow us to assume that \( s = 1. \) Then write (2.2) for \( t \in [0, 1] \) and notice that \( tu_t \) admits representation (2.2) with new \( f^j \) and \( g^k \) having simple relations with \( u_t \) and the original \( f^j \) and \( g^k. \) It follows that in the rest of the proof we may assume that \( u_0 = 0. \)

In that case take a sufficiently large \( \lambda > 0 \) and consider the equation

\[
dv_t = (\Delta v_t - \lambda v_t + D_i \hat{f}^i_t + \hat{f}^0_t) dt + \hat{g}^k_t \, dv^k_t
\]

for \( t \geq 0 \) with zero initial condition, where

\[
\hat{f}^i_t = f^i_t I_{(0,1)}(t) - D_i u_t I_{(0,1)}(t), \quad i = 1, \ldots, d,
\]

\[
\hat{f}^0_t = (f^0_t + \lambda u_t) I_{(0,1)}(t), \quad \hat{g}^k_t = g^k_t I_{(0,1)}(t).
\]

By uniqueness, \( v_t = u_t \) for \( t \in [0, 1] \) and by assertion (iii) we have \( v_1 \in \text{tr}_1 \mathcal{W}_p^1. \) This fact combined with already known estimates of \( v \) proves assertion (iv). The lemma is proved.
Therefore, in the rest of the proof of assertions (i) and (ii) we assume that which is established in Lemma 5.4. For the sentence we will rely on the method of continuity and the a priori estimate (5.1) in Lemma 5.4. In that case uniqueness follows from Lemma 3.3. In the proof of existence we will rely on the method of continuity and the a priori estimate (5.1), which is established in Lemma 5.4. For \( \lambda \geq \lambda_0 \) and \( \theta \in [0,1] \) we consider the equation

\[
du_t = (\theta L_t + (1 - \theta)\Delta)u_t - \lambda u_t + D_t f^j_t + f^0_t \, dt + (\theta \Lambda^k_t u_t + \bar{g}^k_t) \, dw^k_t, \tag{6.1}
\]
We call a $\theta \in [0,1]$ “good” if the assertions of the theorem hold for equation (6.1). Observe that 0 is a “good” point by Lemma 5.1. Now to prove the theorem it suffices to show that there exists a $\gamma > 0$ such that if $\theta_0$ is a good point then all points of the interval $[\theta_0 - \gamma, \theta_0 + \gamma] \cap [0,1]$ are “good”. So fix a “good” $\theta_0$ and for any $v \in W^{1}_p(\tau)$ consider the equation

$$
\frac{du_t}{dt} = \left[(\theta_0 L_t + (1 - \theta_0)\Delta)u_t - \lambda u_t + (\theta - \theta_0)(L_t - \Delta)v_t + D_i f^i_t + f^0_t\right] dt
+ (\theta_0 \Lambda^k_t u_t + (\theta - \theta_0)\Lambda^k v_t + g^k_t) dw^k_t.
$$

(6.2)

Observe that

$$(L_t - \Delta)v_t = D_j \left((a^{ij} - \delta^{ij})D_i v_t + a^j_i v_t\right) + b^i D_i v_t + c v_t$$

and recall that $v \in W^{1}_p(\tau)$. It follows by assumption that equation (6.2) has a unique solution $u \in W^{1}_{p,0}(\tau) \subset W^{1}_p(\tau)$.

In this way, for $f^j$ and $g$ being fixed, we define a mapping $v \rightarrow u$ in the space $W^{1}_p(\tau)$. It is important to keep in mind that the image $u$ of $v \in W^{1}_p(\tau)$ is always in $W^{1}_{p,0}(\tau)$. Take $v', v'' \in W^{1}_p(\tau)$ and let $u', u''$ be their corresponding images. Then $u := u' - u''$ satisfies

$$
\frac{du_t}{dt} = \left[(\theta_0 L_t + (1 - \theta_0)\Delta)u_t - \lambda u_t + (\theta - \theta_0)(L_t - \Delta)v_t\right] dt
+ (\theta_0 \Lambda^k_t u_t + (\theta - \theta_0)\Lambda^k v_t) dw^k_t,
$$

where $v = v' - v''$. It follows by Lemma 5.4 that

$$
\|u\|_{W^{1}_p(\tau)} \leq N|\theta - \theta_0| \|v\|_{W^{1}_p(\tau)}
$$

with a constant $N$ independent of $v'$, $v''$, $\theta_0$, and $\theta$. For $\theta$ sufficiently close to $\theta_0$, our mapping is a contraction and, since $W^{1}_p(\tau)$ is a Banach space, it has a fixed point. This fixed point is in $W^{1}_{p,0}(\tau)$ and, obviously, satisfies (6.1). This proves assertion (i) of the theorem.

Estimate (2.10) is proved above in Lemma 5.4 and assertion (iii) follows from Theorem 3.1. The theorem is proved.

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