On the adiabatic expansion of the visible space in a higher dimensional cosmology

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Abstract

In the context of higher-dimensional cosmologies we study the conditions under which adiabatic expansion of the visible external space is possible, when a time-dependent internal space is present. The analysis is based on a reinterpretation of the four-dimensional stress-energy tensor in the presence of the extra dimensions. This modifies the usual adiabatic energy conservation laws for the visible Universe, leading to a new type of cosmological evolution which includes large-scale entropy production in four dimensions.

1 Introduction

The mathematical background for a non-linear gravitational lagrangian theory, free from metric derivatives of orders higher than the second, was formulated by Lovelock [1], who proposed that the most general gravitational lagrangian is of the form

$$\mathcal{L} = \sqrt{-g} \sum_{m=0}^{n/2} \lambda_m \mathcal{R}^{(m)}$$

(1)

where $\lambda_m$ are arbitrary constants, $n$ denotes the spacetime dimensions, $g$ is the determinant of the metric tensor and $\mathcal{R}^{(m)}$ are functions of the Riemann curvature tensor of the form

$$\mathcal{R}^{(m)} = \frac{1}{2m} \delta_{\alpha_1 \ldots \alpha_{2m}}^{\beta_1 \ldots \beta_{2m}} \mathcal{R}^{\alpha_1 \beta_2 \ldots \mathcal{R}^{\alpha_{2m-1} \beta_{2m}}}_{\beta_2 \ldots \beta_{2m}}$$

(2)

where $\delta_{\alpha_1 \ldots \alpha_{2m}}^{\beta_1 \ldots \beta_{2m}}$ is the generalized Kronecker symbol. In Eq. (2), $\mathcal{R}^{(1)} = \frac{1}{2} \mathcal{R}$ is the Einstein-Hilbert (EH) lagrangian, while $\mathcal{R}^{(2)}$ is a particular combination of the
quadratic terms, known as the Gauss-Bonnett (GB) combination, since in four dimensions it satisfies the functional relation

$$\delta \int \sqrt{-g} \left( R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} \right) d^4x = 0 \quad (3)$$

corresponding to the GB theorem [2]. Introduction of this term into the gravitational lagrangian will not affect the four dimensional field equations. From Eqs. (1) and (3) it becomes evident that if the gravitational lagrangian contains terms of the curvature tensor of orders higher than the second \((m \geq 2)\), then one needs to have a spacetime of more than four dimensions.

This idea has received much attention as a candidate for the unification of all fundamental interactions, including gravitation, in the framework of supergravity or in superstrings [3-10]. In most higher-dimensional theories of gravity the extra dimensions are assumed to form, at the present epoch, a compact manifold (internal space) of very small size compared to that of the three-dimensional visible space (external space) [11] and therefore they are unobservable at the energies currently available. This leads to the problem of compactification of the extra dimensions [12]. It has been recently suggested that compactification of the extra space may be achieved, in a natural way, by adding a square curvature term, \(R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}\), in the EH action for the gravitational field [13]. In this context, the higher-dimensional theories are closely related to those of non-linear lagrangians and their combination probably yields to a natural generalization of General Relativity (GR).

Many problems of the four-dimensional standard model attempted to be solved in the context of a higher-dimensional gravity theory [14]. Among them, many efforts have been made to explain the creation of the observed entropy content, as a result of the dynamic evolution of the internal space [5,7,8,15,16]. The four-dimensional Einstein equations are purely adiabatic and reversible and, consequently, can hardly provide, by themselves, an explanation relating to the origin of cosmological entropy. Therefore, to account for the entropy observed in the Universe one has either to assume it as an initial condition or to account for it through some dissipative mechanism [17]. This picture radically changes in the presence of a time-dependent internal space. Its dynamic evolution, preferably its contraction, could release energy in the external space, thus leading to an entropy change in four dimensions. This idea, although correct in principle, has been able to be applied only in spacetime models with a very large number of extra dimensions \((D \sim 40)\) [7] or after an unnatural fine-tuning of all the gauge coupling parameters [8].

In the present paper we explore the same idea under a different perspective, dealing with the thermodynamics of the visible space in the case where a time-dependent internal one is present. We propose a phenomenological macroscopic approach, using a four-dimensional reinterpretation of the higher-dimensional matter stress-energy tensor [18-21]. Accordingly, we show that both subspaces do not correspond to isolated thermodynamical systems but to closed ones [22], which allow for energy transfer between them. In this respect, a four-dimensional observer comoving with the matter
content will see an extra amount of "heat" received by the ordinary three-dimensional space, which is due entirely to the contraction of the extra dimensions. Now, adiabatic expansion of the external space occurs only under certain conditions, while the irreversible contraction of the internal space could lead to entropy production in three dimensions.

2 Four-dimensional thermodynamics in a higher-dimensional cosmology

We consider a spacetime of \( n = 1 + 3 + D \) dimensions which has been split topologically into three homogeneous and isotropic factors, \( T \times V^3 \times V^D \), where \( T \) is the time direction, \( V^3 \) is the three-dimensional external space, representing the visible Universe and \( V^D \) is the D-dimensional internal space which consists of the extra dimensions. The \( n \)-dimensional line-element is \((\hbar = c = 1)\)

\[
ds^2 = -dt^2 + R^2(t)ds^2_\text{ext} + S^2(t)ds^2_\text{int}
\]

in which the metrics of the factor spaces are of the form

\[
ds^2_\text{ext} = \frac{1}{(1 + k_{\text{ext}}r^2_{\text{ext}}/4)^2} \sum_{i=1}^{3} (dx^i_{\text{ext}})^2
\]

and

\[
ds^2_\text{int} = \frac{1}{(1 + k_{\text{int}}r^2_{\text{int}}/4)^2} \sum_{i=1}^{D} (dx^i_{\text{int}})^2
\]

with

\[
r^2_{\text{ext}} = \sum_{i=1}^{3} (x^i_{\text{ext}})^2 \quad \text{and} \quad r^2_{\text{int}} = \sum_{i=1}^{D} (x^i_{\text{int}})^2.
\]

In what follows we consider only models of an already compactified internal space, i.e. we examine the process of its contraction. The contraction of the inner dimensions presupposes their separation from the ordinary ones and strictly speaking starts immediately after compactification [10]. As far as the factor metric (6) is concerned, compactification may be achieved either in terms of a \( D \)-dimensional sphere, with \( k_{\text{int}} = 1 \) or in terms of a \( D \)-dimensional torus, with \( k_{\text{int}} = 0 \).

Euler variation of Eq. (1) gives the Lovelock tensor \( \mathcal{L}_{\mu\nu} \) [1] (Greek indices refer to the whole \( n \)-dimensional spacetime). \( \mathcal{L}_{\mu\nu} \) is the most general symmetric and divergenceless tensor which describes the propagation of the gravitational field and is a function of the metric tensor and its first and second order derivatives. In this case, the generalized gravitational field equations read

\[
\mathcal{L}_{\mu\nu} = -8\pi G_n T_{\mu\nu}
\]
where

\[ L_\mu = \frac{1}{2m} \sum_{m=0}^{n/2} \delta_{\mu \alpha_1...\alpha_{2m}} R_{\beta_1\beta_2...\alpha_{2m-1}} R^{\alpha_1...\alpha_{2m}}_{\beta_1\beta_2} \]  

is the Lovelock tensor, \( G_n \) is the \( n \)-dimensional gravitational constant and \( T_{\mu\nu} \) is the energy-momentum tensor, which is also included in the field equations through an action principle and contains all the matter and the energy present in the spacetime region \( V_n \). The maximally symmetric character of the two subspaces in the cosmological model (4) restricts the form of \( T_{\mu\nu} \) which, in this case, is diagonal and may be considered as representing a \( n \)-dimensional fluid. There exists one common energy density,

\[ T^{00} = \rho \]  

(9)
corresponding to the one time direction and two different pressures, \( p_{\text{ext}} \) and \( p_{\text{int}} \), associated with each factor space respectively. The spatial pressures do not necessarily coincide with the corresponding thermodynamical quantities. Both \( p_{\text{ext}} \) and \( p_{\text{int}} \) are isotropic in each factor space separately. In addition, since both subspaces are homogeneous and isotropic the energy density and the associated pressures are functions of the cosmic time only.

We consider a \( n \)-dimensional observer who is locally at rest, i.e. comoving with the fluid along the world lines with a tangent velocity vector of the form

\[ u^\mu = (1, 0, ..., 0) \]

Then, any \( 3 + D \)-dimensional proper comoving volume of fluid, \( V_{3+D} = R^3 S^D \), may be considered as an isolated thermodynamical system which, accordingly, evolves adiabatically [20,22]. Hence, there exist two equations of state in the form \( p_d = p_d(\rho) \), one for each subspace of dimensions \( d \) [23]. Once the overall matter-energy density \( \rho \) is properly defined, the form of the associated pressures, \( p_{\text{ext}} \) and \( p_{\text{int}} \), may be directly obtained in terms of those equations [10]. In four dimensions the general linear equation of state

\[ p = \left( \frac{m}{3} - 1 \right) \rho \]  

(10)
covers most of the matter components considered to fill the Early Universe, like quantum vacuum \((m = 0)\), gas of strings \((m = 2)\), dust \((m = 3)\), radiation \((m = 4)\) and Zel’dovich ultrastiff matter \((m = 6)\). A mixture of such components obeys the expansion law

\[ \rho = \sum_m \frac{M_m}{R^m} \]  

(11)
where \( M_m \) is constant if no transitions between the different components occur (there are no dissipative mechanisms) [10]. The generalization of Eq. (10) to multidimensional models which consist of factor spaces, requires [10]

\[ p_d = \left( \frac{m_d}{d} - 1 \right) \rho \]  

(12)
where $d$ is the number of dimensions of the corresponding factor space. For the cosmological model under consideration, $d = 3$ for the external space and $d = D$ for the internal one. In this case the evolution of $\rho$ results in

$$\rho = \sum_{m_3} \sum_{m_D} \frac{M_{m_3,m_D}}{R^{m_3} S^{m_D}}$$

where again, in the absence of dissipation, $M_{m_3,m_D}$ is constant.

A very interesting property of $\mathcal{L}_{\mu\nu}$ is that it is divergenceless, $\mathcal{L}_{\mu\nu,\mu} = 0$. This condition imposes, through the field equations (7), that the same is also true for the energy-momentum tensor of the fluid source

$$T_{\mu\nu,\mu} = 0$$

The specific form of the metric tensor together with the fact that $\rho, p_{\text{ext}}$ and $p_{\text{int}}$ are functions only of the cosmic time, restricts the number of components of Eq.(14) leaving only one which is not satisfied identically. Namely $T^{0\mu,\mu} = 0$. It leads to the condition

$$\dot{\rho} + 3(\rho + p_{\text{ext}}) \frac{\dot{R}}{R} + D(\rho + p_{\text{int}}) \frac{\dot{S}}{S} = 0$$

where a dot denotes derivative with respect to the cosmic time, $t$.

Eq.(15) describes the interchange of energy between matter and gravitation, in a curved spacetime [24] and corresponds to the energy conservation law for an observer who is comoving with the fluid. In this respect, $\rho, p_{\text{ext}}$ and $p_{\text{int}}$ are the matter-energy density and pressures locally measured by a $n$-dimensional observer inside the proper comoving volume $V_{3+D} = R^3 S^D$. Eq.(15) may be written in a more convenient form, as follows

$$\frac{d}{dt} \rho = \frac{1}{R^3} \left[ \frac{d}{dt} (\rho R^3) + p_{\text{ext}} \frac{d}{dt} (R^3) \right] + \frac{1}{S^D} \left[ \frac{d}{dt} (\rho S^D) + p_{\text{int}} \frac{d}{dt} (S^D) \right]$$

The question that arises now, is what will see a four-dimensional observer who is unaware of the existence of the extra dimensions probably due to the fact that the "physical size" of the internal space is very small. As regards the external space, the four dimensional observer may also choose a coordinate system in which he is locally at rest, comoving with a four-dimensional projection of the fluid element. Both comoving coordinate systems are of the same origin.

In order to answer this question we need to determine how the $3+D$-dimensionally defined quantities $\rho$ and $p_{\text{ext}}$ are ”projected” onto the three-dimensional spatial section of the external space. In the context of higher-dimensional cosmologies, when a compact internal space is present, the three-dimensional matter-energy density of the external space is defined [8,25] as the integral of the overall $3+D$-dimensional matter-energy density $\rho$ over the proper volume of the $D$-dimensional (closed and bounded) internal space. That is

$$\rho_3 = \int_{V_D} \rho \sqrt{g_D} d^D x$$
This statement is in complete agreement with the definition of the stress-energy tensor on a hypersurface considered in Ref. 26 (Eq. 21.163, p.552). It expresses the ability of a four-dimensional observer to measure, at each time, the total energy of the (multidimensional) Universe [25] which is justified by the fact that the extra dimensions usually manifest themselves in the four-dimensional energy-momentum tensor [27-30]. In this case, all the energy included in the extra space is somehow being "projected" onto the three-dimensional hypersurface which forms the external space. The total amount of matter-energy in a general 3 + D-dimensional proper comoving volume is given by

\[ E = \int_{V_{3+D}} \rho \sqrt{-g} \, d^{3+D}x \]  

(18)

where \( \hat{g} \) is the determinant of the overall metric tensor. Since the metric (4) is diagonal, Eq.(18) is decomposed to

\[ E = \int_{V_{3+D}} \rho \sqrt{-g} \cdot \sqrt{g_D} \, d^{3+D}x = \int_{V_3} d^3x \sqrt{-g} \left( \int_{V_D} \rho \sqrt{g_D} \, d^Dx \right) \]  

(19)

where \( g \) is the determinant of the four-dimensional external space and \( g_D \) is the corresponding determinant of the \( D \)-dimensional internal one. According to what previously stated, we define an effective three-dimensional total energy of the form

\[ \mathcal{E}_{\text{eff}}^3 = \int_{V_3} \rho_3 \sqrt{-\hat{g}} \, d^3x \]  

(20)

introducing \( \mathcal{E}_{\text{eff}}^3 = E \). Then, combination of Eqs. (19) and (20) implies that Eq.(17) holds. However, \( \rho \) depends only on the cosmic time and therefore \( \rho_3 = \rho V_D \). For the cosmological model under consideration, as regards any proper comoving volume of the internal space, \( V_D \sim S^D \), Eq.(17) may be written in the form

\[ \rho_3 = \rho S^D \]  

(21)

where in Eq.(21) we have ignored factors which appear in the formula for \( V_D \) of the form \( (2\pi)^{(D+1)/2} / \Gamma(D+1) \) a thing that amounts to a redefinition of \( S(t) \).

In this case, \( \rho_3 \) may be identified as the "phenomenological matter-energy density", measured inside a three-dimensional proper comoving volume of fluid in the visible space (\( V_3 \sim R^3 \)), when a compact time-dependent internal space is present. We see that the presence of the extra dimensions modifies the physical content of the ordinary Universe (a not unexpected result e.g. see also [8,18,19,25,27-30]). In the absence of the extra dimensions (\( D = 0 \)) we obtain \( \rho_3 = \rho \) corresponding to the energy density of a perfect fluid source in four dimensions [25].

When a corresponding analysis is carried out for the energy density of the internal space, from Eq.(19) we obtain

\[ \rho_D = \rho V_3 = \rho R^3 \]  

(22)
To account for the “phenomenological” expression of the three-dimensional pressure, \( p_3 \), in the presence of the inner dimensions, we see from Eq.(12) that the pressure associated to the external space is

\[
p_{\text{ext}} = \left( \frac{m_3}{3} - 1 \right) \rho
\]

which by virtue of Eq.(21) reads

\[
p_{\text{ext}} = \left( \frac{m_3}{3} - 1 \right) \frac{1}{S^D} p_3 \iff p_{\text{ext}} S^D = \left( \frac{m_3}{3} - 1 \right) p_3 \tag{23}
\]

According to Eq.(10), a four-dimensional observer unaware of the existence of the extra dimensions would recognize the r.h.s. of Eq.(23) as representing the “physical pressure” in four dimensions. Therefore, combination of Eqs. (10) and (24) implies

\[
p_3 = p_{\text{ext}} S^D \tag{24}
\]

The quantity \( p_3 \) corresponds to the phenomenological “physical” pressure, measured by an observer inside a proper comoving volume of the external space, in the presence of a time-dependent internal space. However, as we will see later on, the expansion of the external space in this case is no longer adiabatic. Therefore, variation of entropy occur in four dimensions and, together with Eq.(10) one should impose an additional equation of state [23], of the form \( S_3 = S_3(S_3) \), where \( S_3 \) is the total entropy in the external space. In the absence of the extra dimensions \( S_3 = \text{constant} \), \( p_3 = p_{\text{ext}} \) and therefore \( T_{\mu\nu} \) would represent a four-dimensional perfect fluid.

As regards the corresponding “physical” pressure of the internal space, we obtain

\[
p_D = p_{\text{int}} R^3 \tag{25}
\]

By virtue of Eqs. (21), (24) and (25), Eq. (16) is written in the form

\[
d \left( \rho_3 R^3 \right) + p_3 d \left( R^3 \right) = -p_D d \left( S^D \right) \tag{26}
\]

Eq.(26), which represents the \( n \)-dimensional conservation law (15), is in complete correspondance with the first law of thermodynamics for a closed system in four-dimensions (any arbitrary comoving volume of fluid), dealing with changes in the energy content between succesive states of equilibrium [20,21,26,31,32]. This consideration leads to an extension of thermodynamics as associated with the four-dimensional cosmology.

To address this statement in detail let us, at first, consider the case of a static internal space, \( S = S_0 \), as it is imposed to be the case at the present epoch. Now, Eq.(26) becomes

\[
d \left( \rho_3 R^3 \right) + p_3 d \left( R^3 \right) = 0 \tag{27}
\]

Eq.(27) is used to describe the adiabatic evolution of a proper comoving volume element of the visible Universe [31], when the curvature is small enough, so that the
self-energy density of the gravitational field is much lower than the matter-energy density [32]. Equivalently, it corresponds to the first thermodynamical law for the adiabatic evolution of an \textit{isolated} system [20-22,26,28,31,32]. Then, we may interpret $\rho_3$ and $p_3$ as the \textit{true} thermodynamical energy-density and pressure [8,20,25]. The case in which the internal space is static corresponds to an \textit{adiabaticity condition} in four-dimensions.

The situation is not the same when a time-dependent internal space is present. Then the first thermodynamical law in four dimensions is given by Eq.(26) and consequently the expansion of a proper comoving volume of fluid in the external space is no longer adiabatic.

The same is also true for any proper comoving volume in the internal space. Indeed, inserting Eqs. (22), (24) and (25) into Eq.(16) we obtain

$$d \left( \rho_D S^D \right) + p_D d \left( S^D \right) = - p_3 d \left( R^3 \right)$$  \hspace{1cm} (28)

Eq.(28) represents the first law of thermodynamics as regards a proper comoving volume of fluid in $D$ dimensions. It is worth to note that in the case of a static internal space Eq.(28) is also reduced to Eq.(27).

We see that, although Eq.(16) states that the evolution of the $4 + D$-dimensional spacetime is isentropic and the total mass-energy is conserved, when it is reduced to the four-dimensional or $D$-dimensional expression, indicates that the two factor spaces do not constitute isolated systems but \textit{closed} ones, which permits for energy transfer between them [22]. In this case an extra amount of “heat” is received by the four-dimensional system, which is due entirely to the evolution of the internal space. Namely

$$d \left( Q_{ext} \right) = - p_D d \left( S^D \right)$$  \hspace{1cm} (29)

The same is also true for the corresponding system in $D$ dimensions

$$d \left( Q_{int} \right) = - p_3 d \left( R^3 \right)$$  \hspace{1cm} (30)

Since the $4 + D$-dimensional spacetime evolves adiabaticaly we impose

$$d \left( Q_{ext} \right) = - d \left( Q_{int} \right)$$  \hspace{1cm} (31)

Eq.(31) reduces to the condition

$$\frac{1}{R^m S^n D^D} R^3 S^D = \text{constant}$$  \hspace{1cm} (32)

which, according to Eq.(13), states that the total energy in a $4 + D$-dimensional comoving volume remains constant.

In what follows we are interested in studying the thermodynamics of the external space since it represents the ordinary Universe. Therefore, we mainly use Eqs. (26) and (29). A subsequent study may also be performed for any proper comoving volume of fluid in the internal space, if we use Eqs. (28) and (30) in the place of Eqs. (26) and (29).
3 Entropy production due to cosmological contraction of the extra dimensions

The extra amount of energy received by the external space will increase the random microscopic motions within a comoving volume element of fluid in the ordinary Universe. In this respect it modifies the four-dimensional energy density $\rho_3$ which now, according to Eqs. (13) and (21) in the case of one-component matter, reads

$$\rho_3 \sim \frac{1}{R m_3 S^{m_3 D - D}}$$

and a similar formula holds also for the corresponding pressure, $p_3$. Therefore, it also modifies the evolution of the visible Universe through the corresponding cosmological field equations [20-22].

However, when $p_{int} = 0$ (i.e. when $m_D = D$) the r.h.s. of Eq.(29) is zero. In this case, condition (27) for adiabatic expansion of the external space is satisfied, no matter which the dynamic behaviour of the internal space might be. Hence, the case of a pressureless internal space also corresponds to an adiabaticity condition in four dimensions. Since $dQ_{ext} = 0$ the existence of the internal space does not imply any energy contributions to the external one. No energy contributions means that there is not any change in the evolution of the external space which could be caused due to the existence of the extra dimensions [in connection see Eq.(33)]. This result was recently observed in the level of the field equations of a five-dimensional quadratic cosmology, which, for $p_{int} = 0$ decouple [33]. In this case, $\rho_3 \sim R^{-m_3}$ and the external space expands under its own laws of evolution. Therefore the two subspaces are completely disjoint.

When $p_{int} \neq 0$ the two factor spaces are not disjoint at all. If the internal space is filled with a conventional type of matter for which $m_D \geq D$, like dust ($m_D = D$), radiation ($m_D = D_4$) or ultrastiff matter ($m_D = 2D$), from Eq.(12) we obtain $p_{int} > 0$. In this case, Eq.(29) indicates that when the internal space contracts $dQ_{ext} > 0$, i.e. energy is transferred to the external space from the extra dimensions. In the same fashion, expansion of the internal space corresponds to $dQ_{ext} < 0$ and therefore, energy is extracted from the external space and transferred to the internal one in order to maintain its expansion. The opposite results are obtained if the internal space is filled with unconventional types of matter, for which $m_D < D$, like quantum vacuum ($m_D = 0$) or gas of strings ($m_D = \frac{2}{3}D$), since in this case $p_{int} < 0$. In what follows we always consider that $m_D \geq D$.

By virtue of Eq.(12), the first thermodynamical law (26) may be written in the form

$$d (E_3(t)) + p_3 d (R^3) = E_3(t) d (ln S^{D-m_D})$$

Eq.(34) indicates that the external space may be energy-supported by the evolution of the extra dimensions which acts as a source of internal energy in four dimensions. This energy transfer stops when the internal space becomes static ($dQ_{ext} = 0$). For
\( p_{\text{int}} \neq 0 \), the visible Universe evolves adiabatically only after stabilization of the inner dimensions. In any other case, the extra amount of "heat" on the r.h.s. of Eq.(34) corresponds to an entropy change, \( dS_3 \neq 0 \), in four dimensions.

\[
T_3(t) d(S_3(t)) = \mathcal{E}_3(t) d\left(\ln S^{D-m_D}\right)
\]  

(35)

The second law of thermodynamics states that \( dS_3 > 0 \). Therefore, according to Eq.(35), for \( m_D > D \) there is only one possible evolution of the internal space after compactification, that is, its contraction. In the context of higher-dimensional theories where only conventional matter is present, the contraction of the internal space appears to be an irreversible process, since the reverse one is thermodynamically forbidden.

In principle we can integrate Eq.(35) to obtain the total amount of entropy produced inside a causal volume in four dimensions. The limits of integration range from an initial time \( t_{\text{in}} \) at which compactification (i.e. separation of the extra dimensions from the ordinary ones) begins, up to the final stage \( t_f \) at which stabilization of the internal space is achieved. However, there still remains the question of how to calculate the unknown function \( T_3(t) \), since for \( S_3 \neq 0 \) we can not use the definition of thermodynamic temperature (e.g. see [31,34]) for a fluid source in four dimensions. It is more convenient to find the form of \( S_3(t) \) as a function of \( S(t) \). We do so by noting that, when the entropy in the external space varies, Eq.(10) is not by itself sufficient to determine completely the state of the matter content [22, 23]. We also need a second equation of state in the form \( \mathcal{E}_3 = \mathcal{E}_3(S_3) \). When a closed thermodynamical system evolves non-adiabatically, the free energy of Helmholtz \( F = \mathcal{E} - TS \) equals to the generalized thermodynamical potential \( \Omega = -pV \) [35]. For the corresponding system inside a proper comoving volume of the external space we obtain [31]

\[
S_3(t) = \frac{R^3}{T_3(t)} (\rho_3 + p_3)
\]  

(36)

since \( \rho_3 \) and \( p_3 \) are functions of the cosmic time only. Eq.(36) gives the entropy associated to the measured thermodynamical content of the external space at each time. For adiabatic expansion of the external space the r.h.s. of Eq.(36) is constant. Combining Eqs. (23) and (36), for \( m_3 \neq 0 \), we obtain

\[
\mathcal{E}_3(t) = \frac{3}{m_3} T_3(t) S_3(t)
\]  

(37)

Eq.(37) corresponds to the second equation of state which, together with \( p = p(\rho) \), is appropriate for the description of a thermodynamical system during non-adiabatic procedures [23,31,34,35]. Inserting Eq.(37) into Eq.(35) we finally obtain

\[
S_3(t) = S_{30} \left( \frac{S}{S_{0}} \right)^{\frac{3}{m_3}(D-m_D)}
\]  

(38)
where $S_{30}$ is the constant value of entropy of the external space from the moment at which $S(t)$ becomes static and afterwards, while $S_0$ represents the value of the internal scale factor at stabilization. Eq.(38) determines the entropy produced in the external space due to the dynamical evolution of the extra dimensions. $S_3$ depends on three free parameters of the theory, namely (i) the equation of state in the external space through $m_3$, (ii) the equation of state in the internal space through $m_D$ and (iii) the number of the extra dimensions. For $m_D > D$ the entropy of the external space increases as the internal space contracts.

Once the form of $S_3(t)$ is found as a function of the internal scale factor $S(t)$, the corresponding expression of $T_3(t)$ inside a proper comoving volume of fluid in four dimensions results from the combination of Eqs. (37) and (38). For $m_3 \neq 0$, it can be cast into the form

$$T_3(t) = T_{30} \frac{1}{R^{m_3-3}} \left( \frac{S}{S_0} \right)^{-\frac{1}{m_3}(m_D-D)(m_3-3)}$$  \hspace{1cm} (39)

where $T_{30}$ is constant. Eq.(39) indicates that for any of the adiabaticity conditions: (i) the internal space is static, $S = S_0$ or (ii) $m_D = D$ (i.e. $p_{int} = 0$), the functional form of the external temperature is reduced to the corresponding expression of the four-dimensional FRW cosmology

$$T_3(t) \sim \frac{1}{R^{m_3-3}}$$  \hspace{1cm} (40)

For non-adiabatic expansion of the external space, when both subspaces are filled with conventional types of matter, i.e. $m_3 \geq 3$ and $m_D \geq D$, contraction of the internal space results in an increase of the four-dimensional temperature and therefore, "heat" is received by the external space. This result verifies the corresponding considerations of Abbott et al. [7] and Kolb et al. [8].

4 Discussion and Conclusions

In the present paper we have examined the conditions under which adiabatic evolution of the visible space is possible in the context of a higher-dimensional non-linear theory of gravity. We have considered a $n$-dimensional cosmological model ($n = 1 + 3 + D$), consisting of one time direction and two homogeneous and isotropic factor spaces, the external space and the internal one. The Universe is filled with one-component matter, in the form of fluid. In this model there is one common energy density, $\rho$ and two different pressures, $p_{ext}$ and $p_{int}$, associated to each factor space respectively.

To impose the conditions on the adiabatic evolution of the external space we have used a reinterpretation of the $n$-dimensional stress-energy tensor from the point of view of a four-dimensional observer who is unaware of the existence of the extra dimensions. For a compact internal space, the three-dimensional energy-density, $\rho_3$, has been defined considering that all the energy included in the extra space is projected, at
each time, onto the spatial section of the visible Universe [8,25,26]. Then, \( \rho_3 \) is given by Eq.(21), while similar expressions hold for the pressures, Eqs. (24) and (25). Using these expressions we have written the \( n \)-dimensional first thermodynamical law (15) in the four-dimensional point of view (26). In this case, the two subspaces correspond to closed thermodynamical systems which allows for energy transfer between them. As regards the external space, in contrast to four-dimensional cosmology, \( dQ_{\text{ext}} \neq 0 \) and therefore its evolution is no longer adiabatic.

According to Eq.(29), for \( p_{\text{int}} > 0 \), contraction of the inner dimensions implies \( dQ_{\text{ext}} > 0 \), i.e. extra energy is received by the external space, while expansion of the inner dimensions implies \( dQ_{\text{ext}} < 0 \), i.e. energy is extracted from the external space. Since the \( n \)-dimensional spacetime corresponds to an isolated thermodynamical system this energy amount is subsequently received by the internal space in order to maintain its expansion. There are only two cases at which the evolution of the visible space is adiabatic: (i) The case of a static internal space, \( S(t) = S_0 \) and (ii) the case of a pressureless internal space, \( p_{\text{int}} = 0 \). Both cases correspond to adiabaticity conditions since \( dQ_{\text{ext}} = 0 \). Now, the two subspaces are completely disjoint and the cosmological field equations decouple [33].

For non-adiabatic evolution of the visible Universe, the extra amount of energy received by the external space results to a large-scale entropy production \( (dS_3 \neq 0) \) in four dimensions. Taking into account the second thermodynamical law \( dS_3 > 0 \), we see that for \( p_{\text{int}} > 0 \) is true only when the internal space contracts. In this context, contraction of the extra dimensions appears to be an irreversible process, since the reverse one is thermodynamically forbidden. Both the entropy produced in the visible space due to the contraction of the extra dimensions \( S_3(t) \) and the corresponding external temperature \( T_3(t) \), have been obtained as functions of the internal scale factor, Eqs. (38) and (39). These expressions increase as the physical size of the internal space decreases and they depend on three free parameters of the theory: (i) The equation of state in the external space, through \( m_3 \), (ii) the equation of state in the internal space, through \( m_D \) and (iii) the number of the extra dimensions, \( D \). It is probable that a fine-tunning of these parameters could lead to production of a considerably large entropy amount, inside a causal volume in four dimensions, to match up with the observational data [7,8,31].

In order to maintain the treatment as general as possible we have not imposed anything, throughout this article, about the form of the "external heat" \( dQ_{\text{ext}} \neq 0 \), i.e. the amount of energy which is received by the visible Universe due to the dynamic evolution of the internal space. Recent developments indicate that the cosmological contraction of one extra dimension could lead to massless particle production (radiation) in the ordinary space [36]. In this case, the energy received by the external space probably corresponds to that of the produced radiation which, in turn, is extracted from the anisotropic gravitational field [37-40]. Moreover, in the context of the open thermodynamical systems [22] it has been also shown that irreversible particle production in four dimensions is closely related to entropy creation, \( dS_3 > 0 \), in the ordinary space [20,40]. An extension of these results to higher-dimensional
cosmologies, together with a detailed study of the possible relation between them, would be very interesting and it will be the scope of a future work.

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