ON THE DIRICHLET PROBLEM FOR SPECIAL LAGRANGIAN CURVATURE POTENTIAL EQUATION

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Abstract. In this paper, we study a class of special Lagrangian curvature potential equations and obtain the existence of smooth solutions for Dirichlet problem. The existence result is based on a priori estimates of global $C^0$, $C^1$ and $C^2$ norms of solutions under the assumption of existence of a subsolution.

1. Introduction

It is well known that the Dirichlet problem is well posed for elliptic equations. The situation is complicated for fully nonlinear elliptic equations; see for instance the seminal papers of L. Caffarelli, L. Nirenberg and J. Spruck [1]-[4]. We will restrict our attention from now on to special Lagrangian equation and some related problems. R. Harvey and H.B. Lawson established the theory of calibrated geometry and introduced the special Lagrangian equation in [5] back in 1982. They obtained that its solutions $u$ have the property that the graph $p = \nabla u$ in $\mathbb{R}^n \times \mathbb{R}^n = C^n$ being a Lagrangian submanifold which is absolutely volume-minimizing and the linearization at any solution being elliptic. Let $x = (x_1, x_2, \cdots, x_n)$, $u = u(x)$ and $\lambda(D^2 u) = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of Hessian matrix $D^2 u$ according to $x$.

Yuan [6] proved that every classical convex solution of special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = c, \quad \text{in} \quad \mathbb{R}^n,$$

must be a quadratic polynomial. The second boundary value problem for special Lagrangian equation was considered by Brendle and Warren [7]. They obtained the existence and uniqueness of the solution by the elliptic method. Later Brendle-Warren’ theorem was generalized by several authors (see, e.g., [8]-[12]).

Assume that

(A): $\Omega$ is $C^4$ bounded domain in $\mathbb{R}^n$, $\varphi : \partial \Omega \rightarrow \mathbb{R}$ be in $C^4(\partial \Omega)$ and $h : \bar{\Omega} \rightarrow [(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2})$ be in $C^2(\bar{\Omega})$ where $\delta > 0$ small enough.

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In another case, Collins, Sebastien and Xuan [13] were concerned with Lagrangian phase equation

\[ \sum_{i=1}^{n} \arctan \lambda_i = h(x), \quad \text{in} \quad \Omega, \]

associated with Dirichlet problem

\[ u = \varphi, \quad \text{on} \quad \partial \Omega. \]

Let \( \alpha \) be any constant satisfying \( 0 < \alpha < 1 \). They proved that

**Theorem 1.1.** Suppose that \( \Omega, \varphi \) and \( h \) satisfy (A). If there exists a \( C^4(\overline{\Omega}) \) subsolution of (1.1) and (1.2), i.e. there exists a function \( u \in C^4(\overline{\Omega}) \) such that

\[
\begin{cases}
\sum_{i=1}^{n} \arctan \lambda_i \geq h(x), & \text{in} \quad \Omega, \\
u = \varphi, & \text{on} \quad \partial \Omega,
\end{cases}
\]

where \( \lambda_i \) are the eigenvalues of \( D^2u \). Then the Dirichlet problem (1.1) (1.2) admits a unique \( C^{3,\alpha}(\overline{\Omega}) \) solution. In addition, if \( \Omega, h, \varphi \) and \( u \) are smooth, then the solution \( u \) is smooth.

Here we noticed that they used the idea of B. Guan [14] that the curvature assumption on \( \partial \Omega \) is replaced by the assumption that there exists an admissible subsolution of (1.1) and (1.2) to obtain the global \( C^2 \) estimates for the admissible solutions.

The aim of this paper is to study special Lagrangian curvature potential equation

\[ \sum_{i=1}^{n} \arctan \kappa_i = h(x), \quad \text{in} \quad \Omega, \]

in conjunction with Dirichlet condition

\[ u = \varphi, \quad \text{on} \quad \partial \Omega. \]

where \( (\kappa_1, \cdots, \kappa_n) \) are the principal curvatures of the graph \( \Gamma = \{(x, u(x)) \mid x \in \Omega\} \).

It shows that a very nice geometric interpretation of special Lagrangian curvature operator \( F(\kappa_1, \cdots, \kappa_n) \triangleq \sum_{i=1}^{n} \arctan \kappa_i \) in [15]. The background of the so-call special Lagrangian curvature potential equation (1.3) can be seen in the literature [15] [16]. In addition, the first author and Sitong Li considered (1.3) with the second boundary condition

\[ Du(\Omega) = \tilde{\Omega}, \]

then they obtained the following result [17] for \( h(x) \) being some constant.

**Theorem 1.2.** Suppose that \( \Omega, \tilde{\Omega} \) are uniformly convex bounded domains with smooth boundary in \( \mathbb{R}^n \). Then there exist a uniformly convex solution \( u \in C^\infty(\overline{\Omega}) \) and a unique constant \( c \) with \( h(x) = c \) solving (1.3) and (1.5), and \( u \) is unique up to a constant.
It's interested in considering Dirichlet problem (1.3) and (1.4) with the same geometric conditions on \( \Omega \) as the above theorem. The central result of this paper is to find solution under the assumptions of the existence of the subsolution, which originate from a series of Bo Guan’s paper (cf. [14] [18] [19]).

Now we state our main result in the following which gives a generalization of Theorem 1.1.

**Theorem 1.3.** Suppose that \( \Omega, \varphi \) and \( h \) satisfy (A). If there exists a \( C^4(\bar{\Omega}) \) subsolution of (1.3) and (1.4), i.e. there exists a function \( u \in C^4(\bar{\Omega}) \) such that

\[
\begin{align*}
\sum_{i=1}^{n} \arctan \kappa_i & \geq h(x), & \text{in } \Omega, \\
\varphi &= u, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \kappa_i \) are the principal curvatures of the graph \( \Gamma = \{(x, u(x)) \mid x \in \Omega \} \). Then Dirichlet problem (1.3) (1.4) admits a unique \( C^{3,\alpha}(\bar{\Omega}) \) solution. In addition, if \( \Omega, h, \varphi \) and \( u \) are smooth, then the solution \( u \) is smooth.

This article is dived into several sections. In section two, we state some well known algebraic and inequalities according to the structure conditions for the operator \( F \). In section three, we can derive the zeroth and first order estimates for the admissible solutions. In section four, we exhibit how to estimate the second order derivatives of \( u \) by the assumption of the boundedness for them on \( \partial \Omega \). In section five, we establish the desired bound for the second order derivatives of \( u \) on \( \partial \Omega \) which is a key and difficult part in the paper. Finally we give the proof of Theorem 1.3 by the continuity method.

2. Preliminaries

In this section we collect various preliminary knowledge and auxiliary lemmas that used in the proof of Theorem 1.3.

At the beginning, we recall some necessary geometric quantities associated with the graph of a function \( u \in C^2(\Omega) \). We use the Einstein summation convention, if the indices are different from 1 and \( n \). \( u_i = D_i u, u_{ij} = D_{ij} u, u_{ijk} = D_{ijk} u, \cdots \) Denote the all derivatives of \( u \) according to \( x_i, x_j, x_k, \cdots \). In the coordinate system induced by the embedding \( x \mapsto (x, u(x)) \) the metric of graph \( u \) is given by

\[
g_{ij} = \delta_{ij} + u_i u_j
\]

which is called the first fundamental form and its inverse is

\[
g^{ij} = \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2}.
\]

The second fundamental form is given by

\[
h_{ij} = \frac{u_{ij}}{\sqrt{1 + |Du|^2}}.
\]

Using (2.1) and (2.2), we obtain

\[
h_{ik} g^{kj} = \frac{u_{ik}}{\sqrt{1 + |Du|^2}} \left( \delta_{kj} - \frac{u_k u_j}{1 + |Du|^2} \right).
\]
From [20] and [21], the principal curvatures of the graph of $u$ are the eigenvalues of $[h_{ij}]$ relative to $[g_{ij}]$, thus they are the eigenvalues of the generally nonsymmetric matrix $[h_{ik}g^{kj}]$. Equivalently they are the eigenvalues of the symmetric matrix

$$a_{ij} = b^{ik}h_{kl}b^{lj} = \frac{1}{w}b^{ik}u_{kl}b^{lj},$$

where $w = \sqrt{1 + |Du|^2}$ and $[b^{ij}]$ is the positive square root of $[g^{ij}]$, i.e. $b^{ik}b^{kj} = g^{ij}$, given by

$$b^{ij} = \delta_{ij} - \frac{u_iu_j}{w(1 + w)}.$$

Therefore

$$u_{ij} = wb_{ik}a_{kl}b_{lj},$$

where $[b_{ij}]$ is the inverse of $[b^{ij}]$, given by

$$b_{ij} = \delta_{ij} + \frac{u_iu_j}{1 + w}.$$

Explicitly we see

$$a_{ij} = \frac{1}{w}\left\{u_{ij} - \frac{u_iu_iu_{ij}}{w(1 + w)} - \frac{u_ju_ju_{ij}}{w(1 + w)} + \frac{u_iu_ju_ku_lu_{kl}}{w^2(1 + w)^2}\right\}.$$

Next we exhibit some properties of the function $F$, which can be found in [22] and [21]. As mentioned in the introduction, the function $F$ was defined by

$$F(A) = \sum_{i=1}^{n}\arctan\kappa_i,$$

where $(\kappa_1, \ldots, \kappa_n)$ are the eigenvalues of $A = [a_{ij}]$. The function $F$ satisfies

$$F^{ij}(A)\eta_i\eta_j > 0, \quad \eta \in \mathbb{R}^n - \{0\},$$

where

$$F^{ij}(A) = \frac{\partial F(A)}{\partial a_{ij}}.$$

Note that $[F^{ij}]$ is symmetric because $A$ is symmetric. Furthermore, $[F^{ij}]$ is diagonal if $A$ is diagonal.

As a matter of convenience, we use the notation

$$a_{ij} = \frac{1}{w}\left\{u_{ij} - \frac{u_iu_iu_{ij}}{w(1 + w)} - \frac{u_ju_ju_{ij}}{w(1 + w)} + \frac{u_iu_ju_ku_lu_{kl}}{w^2(1 + w)^2}\right\},$$

where $w = \sqrt{1 + |Du|^2}$.

Since the technique in [13] that we will develop to prove Theorem 1.3 also works for our setting, then we will use certain properties of $F$ at various of points in the proof. These are summarized in the following lemmas, which was proved in [13].

**Lemma 2.1.** Let $0 < \delta < \frac{\pi}{2}$. Assume that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$ satisfy

$$\sum_{i=1}^{n}\arctan\kappa_i \geq (n - 2)\frac{\pi}{2} + \delta.$$
Then there hold
(i) \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_{n-1} > 0, |\kappa_n| \leq \kappa_{n-1}; \)
(ii) \( \sum_{i=1}^{n} \kappa_i \geq 0; \)
(iii) \( \kappa_n \geq -\frac{1}{\tan(\delta)}; \)
(iv) if \( \kappa_n < 0 \), then \( \sum_{i=1}^{n} \frac{1}{\kappa_i} \leq -\tan(\delta); \)
(v) For any \( \sigma \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2}) \),
\[
\Gamma^\sigma = \{ \kappa \in \mathbb{R}^n : \sum_{i=1}^{n} \arctan \kappa_i > \sigma \}.
\]
is a convex set, and \( \partial \Gamma^\sigma \) is a smooth convex hypersurface.

**Lemma 2.2.** Let \( \Omega, \varphi \) and \( h \) satisfy (A), \( u \in C^{3,\alpha}(\overline{\Omega}) \) satisfy the Dirichlet problem \( (1.3) \) and \( (1.4) \). Suppose that there exists a function \( u \) such that for each point \( x \in \Omega \) and each index \( i \), there holds
\[
\lim_{t \to \infty} F(\kappa + te_i) > h(x),
\]
where \( \kappa \) are the principal curvatures of the graph \( \Gamma = \{ (x, u(x)) | x \in \Omega \} \), \( \kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n) \), \( e_i \) is the \( i \)th standard basis vector and \( F(\kappa) = \sum_{i=1}^{n} \arctan \kappa_i \).

Then there are constants \( R_0, C_1 > 0 \) to arrive at the following inequalities. If \( |\kappa| \geq R_0 \), we either have
\[
\sum_{i,j=1}^{n} F^{ij}(A)[a_{ij} - a_{ij}] > C_1 \sum_{p=1}^{n} F^{pp}(A),
\]
or
\[
F^{ii}(A) > C_1 \sum_{p=1}^{n} F^{pp}(A)
\]
for each \( i \).

**Remark 2.3.** Without loss of generality, in the following we set \( C_1, C_2, \ldots \) to be positive constants depending only on the known data.

Introducing Corollary 3.2 in [13], we infer that

**Lemma 2.4.** Assume that \( \Omega, \varphi \) and \( h \) satisfy (A), \( u \) is a subsolution satisfying \( (1.6) \), \( u \in C^{3,\alpha}(\overline{\Omega}) \) is an admissible solution to \( (1.3) \) and \( (1.4) \). As before, let \( \kappa = (\kappa_1, \cdots, \kappa_n) \) are the principal curvatures of the graph \( \Gamma = \{ (x, u(x)) | x \in \Omega \} \). Then there exists \( R_0 \) depending only on \( u \) and \( \delta \), such that for any \( |\kappa| \geq R_0 \), we have
\[
F^{ij}(A)[a_{ij} - a_{ij}] \geq \tau > 0,
\]
where \( \tau \) is a constant depending on only \( u \) and \( \delta \).
3. Zeroth and first order estimates

In this section, we will prove the zeroth and first order estimates for Dirichlet problem (1.3) and (1.4). The two-sided bound for $u$ can be directly inferred from the existence of the sub-solution and super-solution.

To see this, it is enough to consider $u$ satisfying
\[
\begin{align*}
\Delta u &= 0, \quad \text{in } \Omega, \\
\bar{u} &= \varphi, \quad \text{on } \partial \Omega.
\end{align*}
\]

Review the identity: $u_{ij} = w b_{ik} a_{kl} b_{lj}$. Let $D^2 u = [u_{ij}]$, $B = [b_{ij}]$, $A = [a_{ij}]$, where $B^T = B$. By Lemma 2.3 we see that $\text{tr} A = \sum_{i=1}^{n} \kappa_i \geq 0$. Since $D^2 u = w B^T A B$ where $w = \sqrt{1 + |Du|^2}$, then we arrive at
\[
\text{tr} D^2 u = \text{tr}(w B^T A B) \geq 0.
\]

Next we have the following result.

**Lemma 3.1.** Suppose that $\Omega$, $\varphi$ and $h$ satisfy (A), $\underline{u}$ is a admissible subsolution defined by (1.6), $u$ satisfy (1.3) and (1.4). Let $u : \Omega \rightarrow \mathbb{R}$ be the function satisfying (3.1). Then we have
\[
\underline{u} \leq u \leq \bar{u}, \quad \text{in } \Omega.
\]

**Proof.** Note that $\Delta u = \text{tr} D^2 u \geq 0 = \Delta \bar{u}$ and $F(A) \leq F(A)$ where $A = [a_{ij}]$. Therefore, the conclusion is derived from the maximum principle. \qed

By Lemma 3.1 we have the following Hopf Lemma, due to Li-Wang [23].

**Lemma 3.2.** Suppose that $\Omega$, $\varphi$ and $h$ satisfy (A), $\underline{u}$ is a admissible subsolution defined by (1.6), $u$ satisfy (1.3) and (1.4). Suppose that $\hat{x} \in \partial \Omega$, $\hat{s} \in \mathbb{R}$, then we have
\[
\liminf_{\hat{s} \to 0^+} \frac{(u - \underline{u})(\hat{x} + \hat{s} \nu(\hat{x}))}{\hat{s}} > 0,
\]
where $\nu$ is the inner unit normal to $\partial \Omega$ at $\hat{x}$.

One of the obstacles in the present work is the lack of concavity of the operator $F$ appearing in (2.5). It is important to note that one of the key elements of this paper is to transform the equation (1.3) to find hidden concavity properties when $h(x) > (n - 2) \frac{\pi}{2}$, thus it allows us to obtain the gradient estimate. It is worthy of mention that we use the argument following from [13] and [22].

**Lemma 3.3.** Let $F(\kappa) = \sum_{i=1}^{n} \arctan \kappa_i$ be defined on
\[
\{ \kappa \in \mathbb{R}^n : \sum_{i=1}^{n} \arctan \kappa_i \geq (n - 2) \frac{\pi}{2} + \delta \}.
\]
Then there exists a sufficiently large $A$ depending only on $\delta$ such that $G(\kappa) = -e^{-AF(\kappa)}$ is a concave function.

Similarly, an consequence of Lemma 3.3 is
Lemma 3.4. Let $\Gamma := \{ M \in \text{sym}(n) : F(M) \geq (n-2)\frac{\pi}{2} + \delta \}$, then there exists $A = A(\delta)$ such that the operator
\[
G(A) = -e^{-AF(A)}
\]
is elliptic and concave on $\Gamma$.

The readers can see the proof of the above two lemmas in [13]. Now we use the concavity of the operator $G$ from Lemma 3.4. Without loss of generality, we assume that $A$ is diagonal. It is easy to verify that
\[
G^{ij}(A) = \frac{\partial G(A)}{\partial a_{ij}} = \frac{\partial G(A)}{\partial a_{ii}} \delta_{ij}, \quad G^{ij,kl}(A) = \frac{\partial^2 G(A)}{\partial a_{ij} \partial a_{kl}},
\]
then we have
\[
(3.2) \quad G^{ij,kl} M_{ij} M_{kl} \leq 0,
\]
for any real symmetric tensor $M_{ij}$. A direct computation shows that there exists a positive constant $\tilde{c}$ such that
\[
(3.3) \quad \sum_{i=1}^{\infty} \kappa_i \frac{\partial G(\kappa)}{\partial \kappa_i} = A e^{-AF(\kappa)} \sum_{i=1}^{\infty} \frac{\kappa_i}{1 + \kappa_i^2} \geq -\tilde{c}, \quad \text{in} \quad \Gamma^\sigma \setminus \{0\}.
\]
By the arguments in [13] one can conclude that there exists some constant $C_2$ such that
\[
(3.4) \quad |\kappa|^2 \sum_{i=1}^{\infty} \frac{\partial G(\kappa)}{\partial \kappa_i} \leq C_2 \sum_{i=1}^{\infty} \frac{\partial G(\kappa)}{\partial \kappa_i} \kappa_i^2, \quad \text{in} \quad \Gamma^\sigma \setminus \{0\}.
\]
After introducing the new operator $G$, then (1.3) and (1.4) are equivalent to the following Dirichlet problem
\[
(3.5) \quad \begin{cases} 
G(A) = -e^{-Ah} := \psi(x), & \text{in} \quad \Omega, \\
u = \varphi, & \text{on} \quad \partial\Omega.
\end{cases}
\]
It can be followed from Lemma 2.1 that
\[
(3.6) \quad \frac{\partial G(\kappa)}{\partial \kappa_n} = A e^{-AF(\kappa)} \frac{1}{1 + \kappa_n^2} \geq C_3 > 0,
\]
where $C_3$ is a positive constant depending only on $\delta$.

For the convenience of reader, we denote
\[
g_{\alpha} = \frac{\partial G(\kappa)}{\partial \kappa_\alpha}(1 \leq \alpha < n - 1), \quad g_{i} = \frac{\partial G(\kappa)}{\partial \kappa_i}(1 \leq i \leq n), \quad g_{n} = \frac{\partial G(\kappa)}{\partial \kappa_n}.
\]
After the corresponding transformation, inspired by literatures in [3] and [4], then we derive that

Lemma 3.5. Suppose that $\Omega$, $\varphi$ and $h$ satisfy (A). If there exists a $C^2$ admissible subsolution $u$ satisfying (1.6), $u \in C^3(\overline{\Omega})$ satisfy (1.3) and (1.4), then we have the following estimate
\[
(3.7) \quad \sup_{\Omega} |Du| \leq C_4,
\]
where $C_4$ is a positive constant that depends on $\Omega$, $\|u\|_{C^2(\Omega)}$, $\|h\|_{C^2(\Omega)}$, $\delta$.

Proof. Our first goal is to estimate $|Du|$ in $\Omega$. Let

$$z = |Du|e^{\tilde{A}u},$$

where $\tilde{A} = \left[ \frac{2}{C_3^2} \max |D\psi(x)| \right]^{1/2}$, $C_3$ is the positive constant from (3.9). We assume that it achieves its maximum at a point $x_0$ in $\Omega$. Let $z = \frac{u_{n\alpha}}{u_n} + \tilde{A}u_i = 0$.

Thus for $1 \leq \alpha \leq n - 1$ we have

$$u_{n\alpha}(x_0) = -\tilde{A}u_n(x_0)u_n(x_0) = 0, \quad u_{nn}(x_0) = -\tilde{A}u_n^2(x_0).$$

By rotation, we may assume that $u_{ij}(x_0)$ is diagonal. For any $i$ we also get

$$u_{nij}(x_0) = \frac{u_{nii}}{u_n} - \frac{u_{nii}^2}{u_n^2} - \tilde{A}u_{ii}(x_0) \leq 0.$$  

From (3.1) we can find $a_{ij}(x_0)$ is also diagonal and

$$a_{nn} = \frac{u_{nn}}{w} \left( 1 - \frac{2u_n^2}{w(1 + w)} + \frac{u_n^4}{w^2(1 + w)^2} \right),$$

$$a_{\alpha\alpha} = \frac{u_{\alpha\alpha}}{w}, \quad 1 \leq \alpha \leq n - 1.$$  

Differentiating both sides of (3.5) with respect to $x_n$, one can deduced that

$$\psi_n = G^{ij}(A) \frac{\partial a_{ij}}{\partial x_n}.$$  

It follows from (3.11) that

$$a_{\alpha\alpha,n} = \left( \frac{1}{w} \right)_n u_{\alpha\alpha} + \frac{1}{w} u_{\alpha\alpha n} = \frac{u_n u_{nn} a_{\alpha\alpha} + u_{\alpha\alpha n}}{w^2},$$

$$a_{nn,n} = \left( \frac{u_{nn}}{w} \right)_n \left( 1 - \frac{2u_n^2}{w(1 + w)} + \frac{u_n^4}{w^2(1 + w)^2} \right) + \frac{u_{nn}}{w} \left( 1 - \frac{2u_n^2}{w(1 + w)} + \frac{u_n^4}{w^2(1 + w)^2} \right)_n$$

$$= \frac{u_{nnn}}{w^3} - \frac{3u_n u_{nn}^2}{w^3}.$$
Combining with (3.8), we can get that

\[
\begin{align*}
\text{By substituting (3.10) into (3.13), after some computation, we obtain} \\
(3.14) & \quad \frac{2g_n u_{nm}^2}{w^5} + \frac{u_n u_{mn}}{w^2} g_i a_{ii} - \frac{g_n u_{nm}^2}{w^2} + \tilde{A} u_n g_i a_{ii} \leq -\psi_n. \\
\text{Arranging (3.14), we get} \\
& \quad \frac{g_n u_{nn}^2(w^2 - 2)}{u_n w^5} + \frac{u_n u_{nn}}{w^2} g_i a_{ii} + \tilde{A} u_n g_i a_{ii} \leq -\psi_n. \\
\text{Combining with (3.8), we can get that} \\
(3.15) & \quad \frac{g_n A^2 u_n^3}{w^5}(w^2 - 2) + \tilde{A} \frac{A}{w^2} u_n g_i a_{ii} \leq -\psi_n.
\end{align*}
\]

It’s obvious that \( w = \sqrt{1 + u_n} \) at \( x_0 \). By Lemma 2.1, here we noted that there exists a positive constant \( \tau > 0 \) such that

\[
\sum_{1 \leq \alpha \leq n-1} g_{\alpha} a_{\alpha} = \sum_{1 \leq \alpha \leq n-1} g_{\alpha} \kappa_{\alpha} = A e^{-F(\kappa)} \sum_{1 \leq \alpha \leq n-1} \frac{\kappa_{\alpha}}{1 + \kappa_{\alpha}} \geq \tau > 0,
\]

therefore

\[
\frac{g_n A^2 u_n^3}{w^5}(u_n^2 - 1) + \tilde{A} \frac{A}{w^2} u_n g_{n} a_{nn} \leq \max |D\psi|.
\]

From (3.9) and (3.11) we obtain

\[
\frac{g_n A^2 u_n^3}{w^5}(u_n^2 - 1) - \frac{g_n A^2 u_n^3}{w^5} = \frac{g_n A^2 u_n^3}{w^5}(u_n^2 - 2) \leq \max |D\psi|.
\]

Using (3.6) we derive that

\[
\frac{w^3(u_n^2 - 2)}{w^5} = \frac{v_n^3(u_n^2 - 2)}{(1 + u_n^2)^{5/2}} \leq \frac{\max |D\psi|}{C_3 A^2} = \frac{1}{2}.
\]

Hence we conclude that \( \sup_{\Omega} |Du| \leq C_4 \).
Finally, we assume that \(x_0 \in \partial \Omega\) and \(\nu\) is the inner unit normal to \(\partial \Omega\) at \(x_0\) such that
\[
D_\nu u(x_0) = \sup_{\Omega} |Du|.
\]
We choose \(\tilde{p} > 0\) such that line segment \(\{x_0 + \tilde{p}\nu : 0 \leq \tilde{p} \leq \tilde{p}_0\}\) is contained in \(\Omega\).
Since \(u = u = \overline{u}\) on \(\partial \Omega\), it follows from Lemma 3.1 that we obtain
\[
\tilde{u}(x_0 + \tilde{p}\nu) - \overline{u}(x_0) \leq \tilde{p}.
\]
Therefore \(\tilde{D}_\nu u(x_0) \leq \overline{D}_\nu u(x_0) \leq \overline{D}_\nu \overline{u}(x_0)\). Thus this completes the proof of Lemma 3.5.

From (3.5), we define the equation
\[
\tilde{G}(D^2u, Du) \triangleq G(A) = \psi(x).
\]
We will use the notation
\[
\tilde{G}_{ij} = \frac{\partial \tilde{G}}{\partial r_{ij}}, \quad \tilde{G}_i = \frac{\partial \tilde{G}}{\partial p_i},
\]
where \(r\) represents for the second derivative and \(p\) represents for gradient variables.
By the property of the operator \(G\) and the boundedness of \(|Du|\), the operator \(\tilde{G}(\cdot, p)\) satisfies the structure conditions as same as Lemma 2.14 in [24]. Then another version of Lemma 2.4 in the following holds.

**Corollary 3.6.** Assume that \(\Omega, \varphi\) and \(h\) satisfy (A), \(\underline{u}\) is an admissible subsolution satisfying \((1.6)\), \(\underline{u} \in C^{3,\alpha}(\overline{\Omega})\) is an admissible solution to \((1.3)\) and \((1.4)\). As before, let \(\lambda(D^2\underline{u}) = (\lambda_1, \ldots, \lambda_n)\) are the eigenvalues of Hessian matrix \(D^2u\) according to \(x\). Then there exists \(R_0\) depending only on \(\underline{u}\) and \(\delta\), such that for any \(|\lambda| \geq R_0\), we have
\[
\tilde{G}_{ij}[\underline{u}_{ij} - \underline{u}_{ij}] \geq \tau > 0,
\]
where \(\tau\) is a constant depending only on \(\underline{u}\) and \(\delta\).

**4. Estimates for Second Derivatives from Bounds on the Boundary**

In this section we will show that how to estimate the second derivatives of \(u\) in \(\Omega\) if we know bounds for them on \(\partial \Omega\). Let us assume that

**Lemma 4.1.** Suppose that \(\Omega, \varphi\) and \(h\) satisfy (A). If there exists a \(C^4\) admissible subsolution \(\underline{u}\) of \((1.6)\), \(\underline{u} \in C^3(\overline{\Omega})\) satisfy \((1.3)\) and \((1.4)\), then we have the following estimate
\[
\sup_{\partial \Omega} |D^2u| \leq C_5.
\]
where \(C_5\) is a constant depending on \(\partial \Omega, \underline{u}, h, \delta\) and \(n\).

The proof of this statements will be deferred until the section 5. If (1.1) holds then we have
Lemma 4.2. Suppose that $\Omega$, $\varphi$, and $h$ satisfy (A). If there exists a $C^4$ admissible subsolution $u$ satisfying (1.6), $u \in C^3(\overline{\Omega})$ satisfy (1.3) and (1.4), then we have the following estimate

$$\sup_{\Omega} |D^2 u| \leq C_6,$$

where $C_6$ is a constant depending on $\Omega$, $\|u\|_{C^4(\overline{\Omega})}$, $\|h\|_{C^2(\overline{\Omega})}$, $\delta$.

We present useful knowledge for the proof of Lemma 4.2.

Lemma 4.3. There exists a positive constant $\Lambda$ only depending on $\delta$ such that

$$\sum_i g_i = Ae^{-AF(\kappa)} \sum_i \frac{1}{1 + \kappa_i^2} \geq \Lambda > 0.$$

Proof. Using Lemma 2.1 and $\sup_{\Omega} F < \frac{\pi}{2}$, we have an estimate for the smallest eigenvalue

$$|\kappa_n| \leq C_\delta,$$

where $C_\delta$ is a positive constant depending on $\delta$. Therefore there exist two positive constants $\Lambda_1$ and $\Lambda$ only depending on $\delta$ such that

$$\sum_i g_i = Ae^{-AF(\kappa)} \sum_i \frac{1}{1 + \kappa_i^2} \geq \Lambda_1 \frac{1}{1 + \kappa_n^2} \geq \Lambda_1 \frac{1}{1 + C_\delta} \geq \Lambda > 0.$$

□

In order to prove Lemma 4.2, we need to use the knowledge of [3] as follows. From the proof of Lemma 3.4, we have a bound for

$$C_7 = 2 \max_{\Omega} w,$$

where $w = \sqrt{1 + |Du|^2}$. Let

$$\tilde{\theta} \triangleq \frac{1}{w},$$

(4.4)

$$a = \frac{1}{k} \triangleq \frac{1}{2} \min_{\Omega} \tilde{\theta}.$$

(4.5)

Obviously we have

$$\frac{1}{\tilde{\theta} - a} \leq \frac{1}{a} = C_7.$$

(4.6)

For any $x_0 \in \Omega$, it is convenient to use the new coordinates by defining the surface by $\zeta(y)$, where $y$ are tangential coordinates to the surface at the point $(x_0, u(x_0))$. We choose a local orthonormal frame $\{e_i\}$ for $1 \leq i \leq n$ in neighborhood of $(x_0, u(x_0))$ in $\zeta(y)$, and $e_{n+1}$ is the normal. Introduce the new orthonormal vectors

$$\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1},$$

where $\varepsilon_{n+1}$ being the normal vector at $x_0$ with $\varepsilon_{n+1} = \frac{1}{w}(-u_1, \ldots, -u_n, 1)$ and $\varepsilon_1$ corresponding to the tangential direction at $x_0$ with largest principal curvature.
After proper rotation, we represent the surface near \((x^0, u(x^0))\) by tangential coordinates \(y_1, \ldots, y_n\) and \(\zeta(y)\) (summation is of the from 1 to \(n\))
\[
x_j e_j + u(x^0) e_{n+1} = x^0_j e_j + u(x^0) e_{n+1} + y_j \varepsilon_j + \zeta(y) \varepsilon_{n+1},
\]
so \(\nabla \zeta(0) = 0\) and \(x = x^0\) if and only if \(y = 0\). In this case we set that \(Y = \sqrt{1 + |\nabla \zeta|^2}\). Following [20], the first fundamental form of the graph of \(\zeta(y)\) is
\[
\tilde{g}_{ij} = \delta_{ij} + \zeta_i \zeta_j.
\]
The second fundamental form is given by
\[
\tilde{h}_{ij} = \frac{\zeta_{ij}}{\sqrt{1 + |\nabla \zeta|^2}}.
\]
The normal curvature in the \(\varepsilon_1\) direction is
\[
\kappa = \tilde{h}_{11} \frac{g_{11}}{g_{11}} = \frac{\zeta_{11}}{(1 + \zeta^2)} Y.
\]
In the \(y\)-coordinate we have the normal vector
\[
\theta = -\frac{1}{w} \zeta_j \varepsilon_j + \frac{1}{w} \varepsilon_{n+1},
\]
where \(a_j = \varepsilon_j \cdot e_{n+1}\), thus \(\sum a_j^2 \leq 1\). And the equation (3.5) locally reads as
\[
G(A) = \tilde{\psi}(y),
\]
where \(\tilde{\psi}(y) \triangleq \psi(x)\). By \(\nabla \zeta(0) = 0\), it follows from (4.8) that at \(y = 0\) we have
\[
\tilde{\theta}_i = -a_i \zeta_{ii}, \quad i = 1, \ldots, n.
\]
\[
\tilde{\theta}_{ii} = -a_j \zeta_{jii} - \frac{\zeta_i^2}{w}, \quad i = 1, \ldots, n.
\]
**Lemma 4.4.** We claim that \(\zeta_{1j}(x_0) = 0\) for \(j > 1\) at \(x_0 \in \partial \Omega\).

**Proof.** Let \(e_\theta = e_1 \cos \theta + e_j \sin \theta\). Then
\[
\zeta_{e_\theta e_\theta} = \zeta_{11} \cos^2 \theta + 2 \zeta_{1j} \cos \theta \sin \theta + \zeta_{jj} \sin^2 \theta
\]
has a maximum at \(\theta = 0\). It follows that
\[
\left. \frac{d}{d\theta} (\zeta_{e_\theta e_\theta}(x_0)) \right|_{\theta=0} = 0.
\]
This gives \(\zeta_{1j}(x_0) = 0\).

Using the knowledge of [3] as above, then we show the following

**Proof of Lemma 4.2.**

Set
\[
M = \max_{x \in \Omega, 1 \leq i \leq n} \frac{1}{\theta - a} \kappa_i(x),
\]
where the maximum is taken over all principal curvatures $\kappa_i$. We assume that the maximum is attained at some point $x^0 \in \Omega$ and $M > 0$. At this point corresponding to $y = 0$, the function

$$\frac{1}{\theta - a} \frac{\zeta_{11}}{1 + \zeta_i^2} \Upsilon$$

takes its maximum value equal to $M$ by the previous assumption. At the point $y = 0$, because $y_1$ direction is a direction of principal curvature, then by Lemma 4.4, we get that $\zeta_{1j} = 0$ for $j > 1$. By rotating the $\varepsilon_2, \ldots, \varepsilon_n$, without loss of generality we may have that $\zeta_{ij}(0)$ is diagonal. At $y = 0$, the function $\ln \frac{1}{\theta - a} \frac{\zeta_{11}}{1 + \zeta_i^2} \Upsilon$ reaches its maximum value, and taking the derivative twice in a row can be concluded as follows

\begin{equation}
\frac{\zeta_{11i}}{\zeta_{11}} - \frac{\widetilde{\theta}_i}{\theta - a} - \frac{2\zeta_{11\xi}}{1 + \zeta_i^2} - \frac{\Upsilon_i}{\Upsilon} = 0, \quad i = 1, \ldots, n, \tag{4.12}
\end{equation}

\begin{equation}
0 \geq \frac{\zeta_{11i}}{\zeta_{11}} - \frac{\zeta_{21i}}{\zeta_{11}} - \left( \frac{\widetilde{\theta}_i}{\theta - a} \right)^2 - 2\zeta_{11}^2 - \zeta_{ii}^2, \quad i = 1, \ldots, n. \tag{4.13}
\end{equation}

Observing (2.4), at the origin we also see that $a_{ij} = \zeta_{ij}$ is diagonal and

\begin{equation}
\frac{\partial a_{ij}}{\partial y_k} = a_{ij,k} = \zeta_{ijk}, \tag{4.14}
\end{equation}

\begin{equation}
\frac{\partial^2 a_{ij}}{\partial^2 y_1} = a_{ij,11} = \zeta_{ij11} - \zeta_{11}^2 \left( \zeta_{ij} + \delta_{i1} \zeta_{1j} + \delta_{j1} \zeta_{1i} \right). \tag{4.14}
\end{equation}

It should be noted that at a matrix $A = [a_{ij}]$ which is diagonal,

\begin{equation}
\frac{\partial G}{\partial a_{ij}} = \frac{\partial G}{\partial \kappa_i} \delta_{ij} = g_i \delta_{ij}. \tag{4.15}
\end{equation}

We proceed the differentiate the equation

$$G(A) = \tilde{\psi}(y),$$

then there holds

$$\frac{\partial G}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial y_k} = \tilde{\psi}_k(y).$$

Using (4.14) and (4.15), we find that at $y = 0$,

$$g_i \zeta_{iik} = \tilde{\psi}_k$$

for all $k$. Using the concavity of $G$ and (4.14), one can see that at $y = 0$,

$$\tilde{\psi}_{11} = \frac{\partial^2 G}{\partial a_{ij,11}} \frac{\partial a_{ij}}{\partial y_1} \frac{\partial a_{kl}}{\partial y_1} + \frac{\partial G}{\partial a_{ij}} a_{ij,11} \leq \frac{\partial G}{\partial a_{ij}} a_{ij,11}.$$
Thus we have, at $y = 0$,

$$\tilde{\psi}_{11} \leq \frac{\partial G}{\partial a_{ij}} a_{ij,11} = g_i (\zeta_{ii11} - \zeta_{ii} \zeta_{11}^2) - 2 g_1 \zeta_{11}^3$$

$$\leq g_i \left( \frac{\zeta_{11}^2}{\zeta_{11}} + \zeta_{11} \left( \frac{\tilde{\theta}_i}{\theta - a} \right) + 2 \zeta_{11} \zeta_{ii}^2 + \zeta_{11} \zeta_{ii}^2 - \zeta_{ii} \zeta_{11}^2 \right) - 2 g_1 \zeta_{11}^3$$

$$= \zeta_{11} g_i \left( \frac{\zeta_{11}^2}{\zeta_{11}} + \left( \frac{\tilde{\theta}_i}{\theta - a} \right) + \zeta_{ii}^2 - \zeta_{11} \zeta_{ii} \right)$$

$$= \zeta_{11} g_i \left[ \frac{1}{\theta - a} \left( -a_j \zeta_{jjii} - \zeta_{ii} \right) + \zeta_{ii}^2 - \zeta_{11} \zeta_{ii} \right]$$

$$= -\zeta_{11} a_j \tilde{\psi}_j \frac{1}{\theta - a} + \zeta_{11} g_i \left[ \left( 1 - \frac{1}{\theta - a} \right) \right] \zeta_{ii}^2 - \zeta_{11} \zeta_{ii} \right]$$

$$= -\zeta_{11} a_j \tilde{\psi}_j \frac{1}{\theta - a} - \zeta_{11} g_i \zeta_{ii}^2 \frac{a}{\theta - a} - \zeta_{11} \zeta_{ii}.$$

(4.16)

Here the first step follows from (4.14) while the second step comes from (4.13). Using (4.16) we have

$$\zeta_{11} g_i \zeta_{ii}^2 \frac{a}{\theta - a} + \zeta_{11} g_i \zeta_{ii} \leq -\zeta_{11} a_j \tilde{\psi}_j \frac{1}{\theta - a} - \tilde{\psi}_{11}. \quad (4.17)$$

Recall that $\tilde{\psi}(y) = -e^{-A h(y)}$, a routine computation gives rise to

$$\tilde{\psi}_{11} = -A^2 e^{-A h(y)} \left( \frac{\partial h}{\partial y_1} \right)^2 + A e^{-A h(y)} \frac{\partial^2 h}{\partial^2 y_1}.$$

Since $\sum a_j^2 < 1$, we now choose a sufficiently large constant $A$ so that

$$-\zeta_{11} a_j \tilde{\psi}_j \frac{1}{\theta - a} - \tilde{\psi}_{11} \leq C_8 (1 + \zeta_{11}),$$

where $C_8$ is a positive constant depending on $\Omega$, $\tilde{\psi}$, $\psi$ and $\tilde{\theta}$. Thus

$$\zeta_{11} g_i \zeta_{ii}^2 \frac{a}{\theta - a} + \zeta_{11} g_i \zeta_{ii} \leq C_8 (1 + \zeta_{11}).$$

Since (4.17) and the definition of $M$, we have

$$Ma \sum_i g_i \zeta_{ii}^2 + M^2 (\theta - a)^2 \sum_i g_i \zeta_{ii} \leq C_9 (1 + M). \quad (4.18)$$
where $C_9$ is a positive constant depending only on the known data. With the aid of (3.3), we can see that
\[ \sum_i g_i \zeta_{ii} = \sum_i g_i \kappa_i \geq -\bar{c}. \]
It follows that from (4.18) that
\[ Ma \sum_i g_i \zeta_{ii}^2 \leq \bar{c} M^2 (\bar{\theta} - a)^2 + C_9 (1 + M). \]
Using (3.3) and the definition of $M$, there exists a positive constant $C_{10}$ such that
\[ C_{10} M^3 \sum_i g_i \leq \bar{c} M^2 (\bar{\theta} - a)^2 + C_9 (1 + M). \]
By Lemma 4.3, we obtain that $\sum_i g_i \geq \Lambda > 0$, then
\[ M \leq C_6, \]
where a suitable constant $C_6$ depends on $\Omega$, $\|u\|_{C^4(\Omega)}$, $\|h\|_{C^2(\Omega)}$, $\delta$. The proof is completed.

5. Proof of Lemma 4.1

The proof of Lemma 4.1 follows from the following steps.
First, we need to estimate the pure tangential second derivatives. Suppose that the point is the origin and the $x_n$-axis is the inner normal there. We may assume that the boundary near 0 is represented by
\[ x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} \rho_{\alpha \beta}(0) x_\alpha x_\beta + O(|x'|^3), \]
where $x' = (x_1, x_2, \cdots, x_{n-1})$. Using the Dirichlet boundary condition, we get
\[ (u - u)(x', \rho(x')) = 0. \]
Taking two derivatives for both sides of (5.2), we obtain
\[ (u - u)_{\alpha \beta}(0) = -(u - u)_{n}(0) \rho_{\alpha \beta}(0), \quad 1 \leq \alpha, \beta \leq n - 1. \]
From the boundary gradient estimate it follows that
\[ |u_{\alpha \beta}(0)| \leq C_{11}, \quad \alpha, \beta < n. \]

The next thing to do in the proof is to estimate the mixed normal-tangential derivative $u_{\alpha n}(0)$ for $\alpha < n$. Using the pre-knowledge in the second section, we can obtain
\[ \tilde{G}_{ij} = G^{kl} \frac{\partial a_{kl}}{\partial r_{ij}} = \frac{1}{w} b^{ik} G^{kl} b^{lj}, \]
and
\[ \tilde{G}_i = G^{kl} \frac{\partial a_{kl}}{\partial p_i} = G^{kl} \frac{\partial}{\partial p_i} \left( \frac{1}{w} b^{km} b^{nl} \right) u_{mq}. \]
Using a simple calculation, we have
\[ \tilde{G}_i = -\frac{u_i}{w^2} G^{kl} a_{kl} - \frac{2}{w} G^{kl} a_{lm} b^{ik} u_m \]
\[ = -\frac{u_i}{w^2} \sum_j g_j \kappa_j - \frac{2}{w} G^{kl} a_{lm} b^{ik} u_m. \]
Recalling (3.7), we see that $|Du|$ is bounded and the eigenvalues of $[b_{ij}]$ is bounded between two controlled positive constants. Since (5.4), we see that in fact

$$
\sum_i |\tilde{G}_i| \leq C_{12} \sum_i g_i |\kappa_i|.
$$

Moreover, according to Lemma 3.5, (2.3) and (5.3), there exist two positive constants $C_{13}, C_{14}$ such that

$$
C_{13} \sum_i G_{ii} \leq \sum_i \tilde{G}_{ii} \leq C_{14} \sum_i G_{ii}.
$$

We also have

$$
\sum_i \frac{\partial \tilde{G}}{\partial \lambda_i} \lambda_i = \tilde{G}_{ij} u_{ij} = \frac{1}{w} b_{ik} G^{kl} b_{lj} u_{ij} = G^{kl} A_{kl} = \sum g_i \kappa_i.
$$

where $(\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of Hessian matrix $D^2u$ of $x \in \Omega$. From Lemma 2.1, we have some positive constants $s_1, s_2$ and $s_3$ such that

$$
0 \leq \sum_i \frac{\partial G}{\partial \kappa_i} \leq s_1,
$$

and

$$
s_2 \leq \sum_i \frac{\partial G}{\partial \kappa_i} \kappa_i^2 = Ae^{-AF(\kappa)} \sum_i \frac{\kappa_i^2}{1 + \kappa_i^2} = Ae^{-AF(\kappa)} \sum_i \left(1 - \frac{1}{1 + \kappa_i^2}\right) \leq s_3.
$$

By taking (5.6) and (5.8) into consideration, then there exists a positive constant $C_{15}$ such that

$$
\sum \tilde{G}_{ii} \leq C_{15}.
$$

We consider the operator

$$
L = \tilde{G}_{ij} D_i D_j + \tilde{G}_i D_i.
$$

It is necessary to obtain a version of differential inequality for the later construction of auxiliary functions.

By the argument as before in this section, one can conclude that for any point $x_0 \in \partial \Omega$, we rotate the graph such that at $x_0$, $D(u - \bar{u}) = 0$. But the principle curvatures are invariant through such transformation. Then for any sufficiently small positive constant $\varepsilon > 0$, there exists $\delta' > 0$ depending only on $\varepsilon$, $\Omega$, $\bar{u}$ and $h$ such that in $\Omega \cap B_{\delta'}(x_0)$, we have

$$
|D(u - \bar{u})| \leq \varepsilon.
$$

Based on the above argument, we obtain the following useful differential inequality.

**Corollary 5.1.** There exists uniform positive constant $C_{16}$ such that for $|\lambda| \geq R_0$, we have

$$
L(u - \bar{u}) \leq -C_{16}, \quad \text{in} \quad \Omega \cap B_{\delta'}(x_0),
$$

where $R_0$ is a positive constant depending on $\tau$. 

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Proof. Using (5.10) and (5.11), we can see that there exists sufficiently small positive constant \( \varepsilon_1 \) with \( \varepsilon_1 \leq \frac{\lambda}{2} \) such that in \( \Omega \cap B_{\delta'}(0) \), we have

(5.12) \[ \tilde{G}_i(u - \bar{u})_i \leq \varepsilon_1. \]

From the Corollary 3.6 and (5.12), for \(|\lambda| \geq R_0\) we have

\[
L(u - \bar{u}) = \tilde{G}_{ij}(u - \bar{u})_{ij} + \tilde{G}_i(u - \bar{u}),
\]

where \( C_{16} = \frac{\lambda}{2} \).

**Lemma 5.2.** There exist some uniform positive constants \( \delta' \) sufficiently small depending only on \( u, h, \Omega \) and choose positive constants \( t_1, t_2, \varepsilon_2 \) depending on \( \delta' \) such that the function

\[
v = t_1(u - \bar{u}) + t_2d - d^2,
\]

satisfy

\[
Lv \leq -\varepsilon_2, \quad \text{in} \quad \Omega \cap B_{\delta'}(0),
\]

\[
v \geq 0, \quad \text{on} \quad \partial(\Omega \cap B_{\delta'}(0)).
\]

where \( d(x) = d(x, \partial\Omega) \) is the distance function.

Proof. Note that \( v = t_1(u - \bar{u}) + (t_2 - d)d \) and let \( t_2 = 2\delta' > d \), it is obvious that \( v \geq 0 \) on \( \partial(\Omega \cap B_{\delta'}(0)) \). Since

\[
\tilde{G}_{ij}D_iD_jd^2 = 2d\tilde{G}_{ij}D_iD_jd + 2\tilde{G}_{ij}D_iD_jd^2, \quad \text{in} \quad \Omega \cap B_{\delta'}(0),
\]

then we have

\[
Lv = t_1[\tilde{G}_{ij}(u_{ij} - \bar{u}_{ij}) + \tilde{G}_i(u_i - \bar{u}_i)] + t_2\tilde{G}_{ij}D_iD_jd + t_2\tilde{G}_iD_id
\]

\[
- 2d\tilde{G}_{ij}D_iD_j - 2\tilde{G}_{ij}D_iD_jd - 2d\tilde{G}_iD_id.
\]

(5.14)

\[
= t_1[\tilde{G}_{ij}(u_{ij} - \bar{u}_{ij}) + \tilde{G}_i(u_i - \bar{u}_i)] + (t_2 - 2d)\tilde{G}_{ij}D_iD_jd
\]

\[
+ (t_2 - 2d)\tilde{G}_iD_id - 2\tilde{G}_iD_idd.
\]

Since \( \partial\Omega \) is a \( C^4 \) bounded hypersurface, we can assume that \( d(x) \) is \( C^4 \) bounded in \( \Omega \cap B_{\delta'}(0) \). It can be verified using a result of G-T (see [26], Lemma 14.17) that \( d(x) \) satisfy

\[
Dd(x) = (0, \cdots, 0, 1),
\]

\[
D^2d(x) = d\text{diag}(\frac{\kappa_1}{1 - \kappa_1d}, \cdots, \frac{\kappa_n}{1 - \kappa_n-1d}, 0),
\]

in \( \Omega \cap B_{\delta'}(0) \), where \( \kappa_1, \cdots, \kappa_{n-1} \) are the principle curvature on \( \partial\Omega \). Combining the boundedness of \( \tilde{G}_i \) and (5.9), we obtain

\[
|\tilde{G}_{ij}D_iD_jd| \leq C_{17}, \quad |\tilde{G}_iD_id| \leq C_{18},
\]

where \( C_{17} \) and \( C_{18} \) are positive constants depending on \( \Omega, \delta', \delta \) and \( h \). There are two cases to consider: Firstly, we consider that \(|\lambda| \leq R_0\), where \( R_0 \) is a constant.
from Corollary 3.6. Then there holds $|\kappa| \leq \tilde{R}_0$ for some constant $\tilde{R}_0$ depending only on $R_0$ and the boundedness of $|Du|$. Without loss of generality, we assume that $[a_{ij}]$ is diagonal. By making use of (5.3) and the boundedness of $|Du|$, it is also easy to see that

$$\tilde{G}_{ij}D_i dD_j d = \frac{1}{1 + \kappa^2_i} \frac{1}{w} b^{i k} b^{j l} D_i dD_j d \geq \frac{C_{19}}{1 + R_0^2}.$$  

For $|\lambda| \leq R_0$, we have

$$|\tilde{G}_{ij}(u_{ij} - u_{ij})| \leq C_{20}, \quad |\tilde{G}_{i}(u_i - u_i)| \leq C_{21}.$$  

It follows from (5.14) that

$$Lv \leq t_1(C_{20} + C_{21}) + (t_2 - 2d)C_{17} + (t_2 - 2d)C_{18} - \frac{2C_{19}}{1 + R_0^2}$$  

$$\leq t_1C_{22} + (t_2 - 2d)C_{23} - \frac{2C_{19}}{1 + R_0^2}$$  

$$\leq t_1C_{22} + t_2C_{23} - \frac{2C_{19}}{1 + R_0^2}.$$  

Secondly, if $|\lambda| \geq R_0$, by using Corollary 5.1 we can obtain

$$Lv \leq -C_{16}t_1 + (t_2 - 2d)C_{17} + (t_2 - 2d)C_{18} - 2\tilde{G}_{ij}D_i dD_j d$$  

$$\leq -C_{16}t_1 + (t_2 - 2d)C_{23} - 2\tilde{G}_{ij}D_i dD_j d$$  

$$\leq -C_{16}t_1 + t_2C_{23}.$$  

We may fix $t_1$ and $t_2$ such that $C_{16}t_1 = 2t_2C_{23}$. Then for $|\lambda| \leq R_0$, we have

$$Lv \leq t_2C_{24} - \frac{2C_{19}}{1 + R_0^2} = 2\delta' - \frac{2C_{19}}{1 + R_0^2},$$  

and for $|\lambda| \geq R_0$, we get

$$Lv \leq -t_2C_{23} = -2\delta'C_{23}.$$  

Let

$$\delta' \leq \frac{C_{19}}{2(1 + R_0^2)}.$$  

Then the proof is completed.  

By making use of Lemma 2.1 (i), we find that it can’t be all negative for $\kappa$. Therefore, we can present

**Remark 5.3.** By the assumption, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0$. We claim that $\lambda_n < -R_0$ is not satisfied.

In order to estimate $|u_{\alpha n}(0)|$, we introduce the motivation for the vector field $T_{\alpha}$ by

$$T_{\alpha} = \frac{\partial}{\partial x_{\alpha}} + \sum_{\beta < n} \alpha_{\alpha \beta}(0)(x_{\beta} \frac{\partial}{\partial x_{\beta}} - x_{\alpha} \frac{\partial}{\partial x_{\beta}}), \quad \alpha \in \{1, 2, \ldots, k - 1\}.$$
Furthermore, it follows immediately from (5.1) that for \( \alpha < n \), on \( \partial \Omega \) near 0 we have see
\[
T_\alpha = \left[ \frac{\partial}{\partial x_\alpha} + \frac{\partial \rho}{\partial x_\alpha} \frac{\partial}{\partial x_n} \right] + O(|x'|^2) \frac{\partial}{\partial x_n} - \sum_{\beta < n} \rho_{\alpha \beta}(0) \rho(x') \frac{\partial}{\partial x_\beta}.
\]

Now, we can show that

**Lemma 5.4.** The function \( u - \underline{u} \) satisfies the following estimates
\[
|T_\alpha(u - \underline{u})| \leq C_{24} |x'|^2, \quad \text{on} \quad \partial(\Omega \cap B_\delta'(0)).
\]
(5.15)
\[
|LT_\alpha(u - \underline{u})| \leq C_{25}(1 + \sum_{i=1}^{n} \tilde{G}_{ii}) \leq C_{26}, \quad \text{in} \quad \Omega \cap B_\delta'(0).
\]
(5.16)

**Proof.** Since \( u = \underline{u} \) on \( \partial \Omega \), then the first inequality follows directly from \( C^1 \) estimate of Lemma 5.4. In the following we prove the last inequality. Since \( G(\kappa) \) is invariant under rotation, it follows that for the operator
\[
X_\beta = x_\beta \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial x_\beta},
\]
which is the infinitesimal generator of a rotation. We can apply the operator \( X_\beta \) to the equation \( \tilde{G}(D^2 u, Du) = \psi \). Consequently we have
\[
X_\beta \psi = \tilde{G}_{ij} D_i D_j(X_\beta u) + \tilde{G}_i D_i(X_\beta u),
\]
from which it follows that
\[
T_\alpha \psi = L(T_\alpha u).
\]
Since \( \tilde{G}_{ii} \) is bounded, we have
\[
|LT_\alpha(u - \underline{u})| \leq |LT_\alpha u| + |LT_\alpha \underline{u}|
= |T_\alpha \psi| + |LT_\alpha \underline{u}|
\leq C_{25} \left( 1 + \sum_i \tilde{G}_{ii} \right)
\leq C_{26}, \quad \text{in} \quad \Omega \cap B_\delta'(0).
\]

The function \( \varphi \) introduced in the next Lemma will be the main part of a barrier function which we shall construct in the following.

**Lemma 5.5.** There exist two positive constants \( \hat{A} \gg \hat{B} \gg 1 \) such that
\[
\varphi = \hat{A} v + \hat{B} |x|^2 \pm T_\alpha(u - \underline{u})
\]
satisfies the inequalities
\[
L \varphi \leq 0, \quad \text{in} \quad \Omega \cap B_{\delta'}(0),
\]
\[
\varphi \geq 0, \quad \text{on} \quad \partial(\Omega \cap B_{\delta'}(0)),
\]
where \( \delta' \) is a constant from Corollary 5.4.
Proof. First the condition \( \varphi \geq 0 \) on \( \partial(\Omega \cap B_{\delta'}(0)) \) follows if
\[
\hat{B}|x|^2 \pm T_\alpha(u - \underline{u}) \geq 0, \quad \text{on} \quad \partial(\Omega \cap B_{\delta'}(0)).
\]
In view of Lemma \( \ref{lemma5.4} \), this can be arranged by choosing \( \hat{B} \) sufficiently large. By making use of Lemma \( \ref{lemma5.2} \) and Lemma \( \ref{lemma5.4} \) we can see that the property \( L\varphi \leq 0 \) now follows from the inequality
\[
L\varphi = \hat{A}Lv + \hat{B}|x|^2 \pm T_\alpha(u - \underline{u}) \leq -\hat{A}\varepsilon_2 + \hat{B}(2\tilde{G}_{ii} + 2\tilde{G}_i x_i) + C_{26}
\]
\[
\leq -\tilde{\varphi}_2 + \hat{B}C_{27} + C_{26}
\]
with holds if \( \hat{A} \) is sufficiently large. \( \square \)

The maximum principle applied to Lemma \( \ref{lemma5.5} \) shows that \( \varphi \geq 0 \) in \( \Omega \cap B_{\delta'}(0) \). Since \( \left( \hat{A}v + \hat{B}|x|^2 \pm T_\alpha(u - \underline{u}) \right)(0) = 0 \), it follows that
\[
\frac{\partial}{\partial x_n}(\hat{A}v + \hat{B}|x|^2 \pm T_\alpha(u - \underline{u}))(0) \geq 0.
\]
Thus we obtain
\[
|\frac{\partial}{\partial x_n}(T_\alpha(u - \underline{u}))(0)| \leq |\frac{\partial}{\partial x_n}(\hat{A}v + \hat{B}|x|^2)(0)|,
\]
and this gives
\[
|u_{\alpha n}(0) - \underline{u}_{\alpha n}(0)| \leq |\hat{A}u_n(0)| + |\sum_{\beta < n} \rho_{\alpha\beta}(0)(u - \underline{u})_\beta(0)| \leq C_{28}.
\]
Hence we can obtain the mixed second-order derivative bound \( |u_{\alpha n}(0)| \leq C_{29} \) for all \( \alpha < n \).

Finally, estimating the remaining second derivative \( |u_{nn}| \) of the function \( u \) is somewhat complicated. We find that the principal curvatures admit a uniform lower bound, i.e. \( A = [a_{ij}] \geq -C_{30} \), since the Lemma \( \ref{lemma2.1} \) and \( F(\kappa) = \sum_i \arctan \kappa_i \geq (n - 2)\frac{\pi}{2} + \delta \) in \( \Omega \). Recall that \( a_{ij} = \frac{1}{w}b^k u_{kl}b^j \) and the gradient estimate, we can obtain that \( D^2u \geq -C_{31} \). It suffices to derive an upper bound
\[
u_{nn} \leq C_{32}, \quad \text{on} \quad \partial\Omega.
\]

For discussion purposes, we need to give some definition of the operator as follows. Assume that the \( (n - 1) \times (n - 1) \) unper block matrix \( u_{\alpha\beta} \) is diagonal. Before we deal with \( D^2u \), using the boundedness of \( Du \), we consider \( \hat{G} \) and \( \hat{\hat{G}} \) as functions through taking \( D^2u \) as variable all alone. In other word, we can define that
\[
\hat{G}[u_{ij}] := \hat{G}(D^2u, p),
\]
and
\[
\hat{\hat{G}}[u_{\alpha\beta}] = \lim_{t \to \infty} \hat{G}(D^2u + tq \otimes q, p),
\]
where \( q = (0, \ldots, 0, 1) \) and \( p \) represents for gradient variables.

Similar to Lemma \( \ref{lemma3.1} \), we can deduce that

**Corollary 5.6.** The operator \( \hat{G}[u_{\alpha\beta}] \) is concave on the set \( u_{\alpha\beta}(\partial\Omega) \).
We need to consider
\[ \Theta = \min_{x \in \partial \Omega} \hat{G}[u_{\alpha \beta}(x)] - \psi(x) \]
and
\[ c = \min_{x \in \partial \Omega} (\hat{G}[u_{\alpha \beta}] - \hat{G}[u_{ij}]) \]
that demonstrate \( u_{nn} \leq C_{32} \) on \( \partial \Omega \). It’s obvious that \( c > 0 \). According to an idea of Trudinger [25], we use the following Lemma that implies the boundary \( C^2 \) estimate.

**Lemma 5.7.** There exist some positive constant \( \omega_0 > 0 \) depending only on the known data, such that
\[ (5.17) \Theta \geq \omega_0 > 0. \]

**Proof.** Choose proper coordinates in \( \mathbb{R}^n \) such that \( \Theta \) is achieved at \( 0 \in \partial \Omega \). By the Corollary 5.6 there exists a symmetric \( \{ \hat{G}^0_{\alpha \beta} \} \) such that
\[ (5.18) \hat{G}^0_{\alpha \beta}(u_{\alpha \beta}(x) - u_{\alpha \beta}(0)) \geq \hat{G}[u_{\alpha \beta}(x)] - \hat{G}[u_{\alpha \beta}(0)], \]
where \( \hat{G}^0_{\alpha \beta} = \frac{\partial^2 \hat{G}}{\partial u_{\alpha \beta}^2}(u_{\alpha \beta}(0)) \). Since (5.1) and (5.2), by calculating we obtain
\[ \begin{align*}
(u - u)_{\alpha} + (u - u)_n \rho_{\alpha} &= 0, \\
(u - u)_{\alpha \beta} + (u - u)_{\alpha n} \rho_{\beta} + (u - u)_{\beta n} \rho_{\alpha} + (u - u)_n \rho_{\alpha \beta} &= 0.
\end{align*} \]
Since \( \rho_{\alpha} = \rho_{\beta} = 0 \) on \( \partial \Omega \), we have
\[ (5.19) u_{\alpha \beta} - u_{\alpha \beta} = -(u - u)_n \rho_{\alpha \beta}, \quad \text{on} \quad \partial \Omega. \]
It follows that
\[ (u - u)_n(0) \hat{G}^0_{\alpha \beta}(0) \rho_{\alpha \beta}(0) = \hat{G}^0_{\alpha \beta}(u_{\alpha \beta}(0) - u_{\alpha \beta}(0)) \]
\[ \geq \hat{G}[u_{\alpha \beta}(0)] - \hat{G}[u_{\alpha \beta}(0)] \]
\[ = \hat{G}[u_{\alpha \beta}(0)] - \psi(0) - \Theta \]
\[ \geq c - \Theta. \]
Suppose now that
\[ (u - u)_n(0) \hat{G}^0_{\alpha \beta}(0) > \frac{1}{2}c. \]
Otherwise, taking (5.20) into consideration, we have \( \Theta \geq \frac{1}{2}c \). Hence we see that (5.17) holds.

Let \( \eta := \hat{G}^0_{\alpha \beta} \rho_{\alpha \beta} \). By Lemma 3.2 we obtain \( (u - u)_n > 0 \). Taking (5.7) and \( 0 < (u - u)_n \leq C_i \) into consideration, there exists a universal constant \( c_\tau > 0 \) depending on the \( C^1 \) estimate and \( u \) such that
\[ (5.21) \eta(0) \geq \frac{1}{2}c \cdot \frac{1}{(u - u)_n(0)} \geq 2c_\tau > 0, \]
where
\[ c_\tau = \frac{1}{4} \frac{c}{C_i}. \]
So we may assume that \( \eta \geq c_\tau \) on \( \Omega \cap B_{\delta'}(0) \) by choosing \( \delta' \) as in Lemma 5.2.
Consider the function

\[ \Phi = -(u - \bar{u})n + \frac{1}{\eta} \tilde{G}_0^{\alpha\beta}(\bar{u}_{\alpha\beta}(x) - u_{\alpha\beta}(0)) - \frac{\psi(x) - \psi(0)}{\eta}, \quad \text{in} \quad \Omega \cap B_{\delta}(0). \]

We have \( \Phi(0) = 0 \). Using (5.18) and (5.19), we see that

\[
\begin{align*}
\Phi &= \frac{1}{\eta} \left\{ -(u - \bar{u})n \tilde{G}_0^{\alpha\beta} \rho_{\alpha\beta} + \tilde{G}_0^{\alpha\beta} (\bar{u}_{\alpha\beta}(x) - u_{\alpha\beta}(0)) \right\} - \frac{1}{\eta} (\psi(x) - \psi(0)) \\
&= \frac{1}{\eta} \left\{ \tilde{G}_0^{\alpha\beta} (u_{\alpha\beta}(x) - \bar{u}_{\alpha\beta}(x) + \bar{u}_{\alpha\beta}(x) - u_{\alpha\beta}(0)) \right\} - \frac{1}{\eta} (\psi(x) - \psi(0)) \\
&= \frac{1}{\eta} \left\{ \tilde{G}_0^{\alpha\beta} (u_{\alpha\beta}(x) - u_{\alpha\beta}(0)) \right\} - \frac{1}{\eta} (\psi(x) - \psi(0)) \\
&\geq \frac{1}{\eta} \left\{ \tilde{G}[u_{\alpha\beta}(x)] - \tilde{G}[u_{\alpha\beta}(0)] - (\psi(x) - \psi(0)) \right\} \\
&= \frac{1}{\eta} \left\{ \tilde{G}[u_{\alpha\beta}(x)] - \psi(x) - \Theta \right\} \\
&\geq 0, \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

After direct computation, we have

\[
\begin{align*}
D_i \left[ \frac{1}{\eta} \tilde{G}_0^{\alpha\beta} (u_{\alpha\beta}(x) - u_{\alpha\beta}(0)) \right] &= \frac{1}{\eta} \tilde{G}_0^{\alpha\beta} \frac{\psi(x) - \psi(0)}{\eta} + \tilde{G}_i D_i \psi(x) - \psi(0) \\
D_i D_j \left[ \frac{1}{\eta} \tilde{G}_0^{\alpha\beta} (u_{\alpha\beta}(x) - u_{\alpha\beta}(0)) \right] &= \frac{1}{\eta} \tilde{G}_i D_j \psi(x) - \psi(0) + \tilde{G}_i D_i \psi(x) - \psi(0) \\
&= -\psi_n + \tilde{G}_i u_{\alpha\beta i} + \tilde{G}_i u_{\alpha\beta i} + \frac{1}{\eta} \tilde{G}_i \tilde{G}_0^{\alpha\beta} \frac{\psi(x) - \psi(0)}{\eta} + \tilde{G}_i D_i \psi(x) - \psi(0) \\
&\leq C_{33} (1 + \sum_i G_{ii}) \\
&\leq C_{34} (1 + \sum_i G_{ii}).
\end{align*}
\]

We can again use Lemma 5.2, Lemma 5.5 and take \( \tilde{A} \gg \tilde{B} \gg 1 \) large enough to get

\[
\begin{align*}
L(\tilde{A}v + \tilde{B}|x|^2 + \Phi) &\leq 0, \quad \text{in} \quad \Omega \cap B_{\delta}(0), \\
\tilde{A}v + \tilde{B}|x|^2 + \Phi &\geq 0, \quad \text{on} \quad \partial(\Omega \cap B_{\delta}(0)).
\end{align*}
\]
By the maximum principle, we see that
\[ \tilde{A}v + \tilde{B}|x|^2 + \Phi \geq 0, \quad \text{in} \quad \Omega \cap B_{\delta}'(0). \]

We have
\[ \tilde{A}v_n(0) + \Phi_n(0) = \frac{\partial}{\partial x_n}(\tilde{A}v + \tilde{B}|x|^2 + \Phi)(0) \geq 0, \]
since \( \tilde{A}v + \tilde{B}|x|^2 + \Phi = 0 \) at the origin. Thus one can deduce that \( \Phi_n(0) \geq -\tilde{A}v_n(0) \geq -C_{35}. \) Consequently we can conclude that
\[ u_{nn}(0) \leq C_{32}. \]
So that we obtain \( |D^2u(0)| \leq C_{36} \) for some positive constant. By letting \( t \) large enough, it yields that \( [D^2u + tq \otimes q] > [D^2u] \) uniformly. Thus we see that
\[ \tilde{G}[u_{\alpha\beta}(0)] > \psi(0). \]
As a result, (5.17) holds. \( \square \)

Now we need the following Lemma from [3].

**Lemma 5.8.** Consider the \( n \times n \) symmetric matrix
\[ M = \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 \\ 0 & d_2 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n-1} & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & a \end{pmatrix} \]
with \( d_1, \ldots, d_{n-1} \) fixed, \( a \) tending to infinity and \( |a_\alpha| \leq C_{37} \) for \( 1 \leq \alpha \leq n-1. \) Then the eigenvalues \( \lambda_i \) of \( M \) behave like
\[ \lambda_\alpha = d_\alpha + o(1), \quad 1 \leq \alpha \leq n-1, \]
\[ \lambda_n = a \left(1 + O\left(\frac{1}{a}\right)\right), \]
for \( 1 \leq i \leq n, \) where the \( o(1) \) and \( O\left(\frac{1}{a}\right) \) are uniform--depending only on \( d_1, \ldots, d_{n-1} \) and \( C_{37}. \)

Therefore, combining Lemma 5.7 with Lemma 5.8 we obtain that

**Lemma 5.9.** There exist suitable constants \( R_0, \omega_0 > 0 \) such that for all \( t \geq R_0 \) we have
\[ G(\kappa'(a_{\alpha\beta}), t) > \psi(x) + \omega_0. \]
where \( \kappa'(a_{\alpha\beta}) \) is the eigenvalue of the \( (n-1) \times (n-1) \) matrix \( a_{\alpha\beta}. \)

**Proof.** Since (5.17) holds, we have
\[ \lim_{t \to \infty} \tilde{G}(r + tq \otimes q, p) \geq \psi(x) + \omega_0. \]
Writing \( \mathcal{A}(r, p) = [a_{ij}] \) and \( \mathcal{A}(r + tq \otimes q, p) = [\tilde{a}_{ij}] \), we have
\[ \tilde{a}_{ij} = a_{ij} + tb^{ij}b^{kj}, \]
for \( 1 \leq i, j \leq n-1 \).
where \( b_{ij} = \delta_{ij} - \frac{D_i u D_j u}{w(1+w)} \). After an orthonormal transformation we may assume

\[
\tilde{a}_{ij} = a_{ij} + \frac{t}{\sqrt{1+|Du|^2}} \delta_{in} \delta_{nj}.
\]

By Lemma \([5.8]\) the eigenvalues of \( A(r + t q \otimes q, p) = [\tilde{a}_{ij}] \) are given by

\[
\kappa'_\alpha = \kappa_\alpha + o(1), \quad \tilde{\kappa}_n = t + \kappa_n + O(1),
\]

as \( t \to \infty \), where \( \kappa'_\alpha \) are the eigenvalues of \( [\tilde{a}_{\alpha\beta}]_{1 \leq \alpha, \beta \leq n-1} \). Through our observation, we have

\[
\tilde{G}(r + t q \otimes q, p) = G(\kappa_1(\tilde{a}_{ij})) = G(\kappa_\alpha + o(1), t + \kappa_n + O(1)),
\]

where \( \kappa_n \geq -\tan \delta \). Then there holds

\[
\lim_{t \to +\infty} G(\kappa'(a_{\alpha\beta}), t) = \hat{G}(u_{\alpha\beta}).
\]

Applying Lemma \([5.7]\) if \( t \to \infty \), we obtain

\[
G(\kappa'(a_{\alpha\beta}), t) > \psi(x) + \omega_0.
\]

Recall that

\[
G(\kappa) = -e^{-A \sum \arctan \kappa_i}, \quad \psi(x) = -e^{-Ah(x)}.
\]

Suppose that \( \kappa_1, \ldots, \kappa_n \) are the eigenvalues of \( a_{ij}(0) \) for \( x \in \partial \Omega \). Applying Lemma \([5.8]\) we see that the eigenvalues \( \kappa_1, \kappa_2, \ldots, \kappa_n \) behave like

\[
\kappa_\alpha = \frac{1}{w} \lambda_\alpha + o(1),
\]

\[
(5.24)
\kappa_n = \frac{1}{w^3} u_{nn}(0) \left( 1 + O\left( \frac{1}{u_{nn}(0)} \right) \right),
\]

as \( |u_{nn}(0)| \to \infty \), where \( \lambda_1, \ldots, \lambda_{n-1} \) are the eigenvalues of \( u_{\alpha\beta}(0) \).

Next we prove the following conclusion:

**Lemma 5.10.** We show that

\[
\max_{x \in \partial \Omega} |u_{nn}(x)| \leq C_{38}
\]

for all \( x \in \partial \Omega \), where \( C_{38} \) is positive constant depending on \( \partial \Omega, n, \delta, h \) and \( w \).

**Proof.** Suppose that \( |u_{nn}(x)| \) achieves its maximum at a point \( x_p \) on \( \partial \Omega \). We assume that \( u_{nn}(x_p) \geq K \) where \( K \) is a large constant to be chosen. Denote \( \kappa'_1 = (\kappa'_1, \ldots, \kappa'_{n-1}) \) eigenvalues of \( a_{ij}(x_p) \). By \((5.24)\), for every \( \delta > 0 \), there exists \( R(\delta) \gg 1 \) such that if \( u_{nn}(x_p) \geq R(\delta) \), then the eigenvalues of \( a_{ij}(x_p) \) satisfy

\[
\kappa_n \geq R_2,
\]

and

\[
(5.25) \quad |(\kappa_1, \ldots, \kappa_{n-1}, \kappa_n) - (\kappa'_1, \ldots, \kappa'_{n-1}, \kappa_n)| < \delta_1.
\]
Consequently, we find that there exists a positive constant $\delta_1 > 0$ depending on $\omega_0$ such that if $u_{nn}(x_p) \geq K \triangleq \max\{R(\delta_1), R_2\}$, then

$$\psi(x_p) = G(\kappa)(x_p) \geq G(\kappa_\alpha', \kappa_n)(x_p) - \frac{\omega_0}{2} \geq G(\kappa_\alpha', R_2) - \frac{\omega_0}{2} > \psi(x_p) + \frac{\omega_0}{2},$$

where the first step follows from the continuity of $G$, the second step follows from monotonicity of $G$, while the last comes from the fact that Lemma 5.9. So we obtain a contradictory result. Thus we complete the proof of Lemma 5.10. \qed

Hence the boundary $C^2$ estimate is complete. Then this completes the proof of Lemma 4.1.

6. PROOF OF THEOREM 1.3

The goal of this section is to prove Theorem 1.3. Since Lemma 4.1 and Lemma 4.2 have been established, by Evans-Krylov theorem and Schauder theory in [26], we get the priori bound for the $C^{3,\alpha}$ norm of $u$ for any $\alpha \in (0, 1)$ to Dirichlet problem (3.5), i.e.

$$\|u\|_{C^{3,\alpha}(\Omega)} \leq C_{39},$$

where $C_{39}$ is a positive constant depending on $\Omega, \varphi, \|u\|_{C^4(\Omega)}, \|h\|_{C^2(\Omega)}$ and $\delta$. In this section we explain how to make use of (6.1) and continuity method to prove the existence of solution to (1.3) and (1.4).

Following the same proof as Lemma 5.2 in [10], we can obtain

**Lemma 6.1.** If $\Omega, h, \varphi$ and $u$ are smooth satisfying (A), then the solution $u$ is smooth in $\overline{\Omega}$.

To solve the equation, we use the continuity method.

**Proof of Theorem 1.3.**

It’s enough for us to consider Dirichlet problem (3.5) as follows.

$$\begin{cases}
\tilde{G}(D^2u, Du) = \psi(x), & \text{in } \Omega, \\
u = \varphi, & \text{on } \partial\Omega.
\end{cases}$$

where $\tilde{G}(D^2u, Du) = G(A)$ and $\psi(x) = -e^{-Ah(x)}$. One can use the pre-knowledge in previous section. Assume that $u$ is $C^4(\overline{\Omega})$ subsolution satisfying (1.6). It yields that the classical solution to (1.3) and (1.4) is unique by maximum principle. For each $t \in [0, 1]$, set

$$J^t(D^2u, Du) = \tilde{G}(D^2u, tDu).$$

Consider

$$\begin{cases}
J^t(D^2u, Du) = \psi(x), & \text{in } \Omega, \\
u = \varphi, & \text{on } \partial\Omega.
\end{cases}$$

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Define the set
\[ S := \{ t \in [0, 1] : (6.2) \text{ has at least one admissible solution} \}. \]

By the main results in [13], (6.2) is solvable if \( t = 0 \). So \( S \) is not empty. We claim that \( S = [0, 1] \), which is equivalent to the fact that \( S \) is not only open, but also closed.

Define
\[ X := \{ u \in C^{3,\alpha}(\bar{\Omega}) : u |_{\partial\Omega} = \varphi \} \]
and
\[ Y := C^{\alpha}(\bar{\Omega}) \times C^{2,\alpha}(\partial\Omega). \]

Then \( X \) is the closed convex subset in Banach space \( C^{3,\alpha}(\bar{\Omega}) \) and \( Y \) is a Banach space. Define a map from \( X \times [0, 1] \) to \( Y \) as
\[ \mathcal{F}(u, t) := \left( J^t(D^2u, Du) - \psi(x), u \right). \]

Given \((u_0, t_0) \in X \times [0, 1]\). For any \( w \in C^{3,\alpha}(\bar{\Omega}) \) with \( w |_{\partial\Omega} = 0 \), we have
\begin{equation}
\frac{d}{d\varepsilon} \mathcal{F}(u_0 + \varepsilon w, t_0 + \varepsilon)|_{\varepsilon = 0} = \left( \tilde{G}_{ij}(D^2u_0, t_0Du_0)\partial_{ij}w + \tilde{G}_i(D^2u_0, t_0Du_0)t_0\partial_iw + \tilde{G}_i(D^2u_0, t_0Du_0)\partial_iu_0, 0 \right).
\end{equation}

Then the linearized operator of \( \mathcal{F}(u, t) \) at \((u_0, t_0)\) is given by
\[ D\mathcal{F}(u_0, t_0) = \left( \tilde{G}_{ij}(D^2u_0, t_0Du_0)\partial_{ij}w + \tilde{G}_i(D^2u_0, t_0Du_0)t_0\partial_iw + \tilde{G}_i(D^2u_0, t_0Du_0)\partial_iu_0, 0 \right). \]

One can see that \( \tilde{G}_{ij}(D^2u_0, t_0Du_0), \tilde{G}_i(D^2u_0, t_0Du_0) \in C^{1,\alpha}(\bar{\Omega}) \) and
\[ \mathcal{L} \triangleq \tilde{G}_{ij}(D^2u_0, t_0Du_0)\partial_{ij} + \tilde{G}_i(D^2u_0, t_0Du_0)t_0\partial_i \]
is a uniformly linear elliptic operator by calculation. Then making use of Schauder theory in [26], \( D\mathcal{F}(u_0, t_0) \) is invertible for any \( u_0 \in X \) and \( t_0 \in [0, 1] \) being the solution to (6.2). Then the fact that \( S \) is open follows from the invertibility of the linearized operator and the implicit function theorem [26].

To finish the proof, we need to prove the fact that \( S \) is a closed subset of \([0, 1]\). One can show that \( S \) is closed is equivalent to the fact that for any sequence \( \{t_k\} \subset S \), if \( \lim_{k \to \infty} t_k = t_0 \), then \( t_0 \in S \). For \( t_k, u_k \) is the solution of
\begin{equation}
\begin{cases}
J^{t_k}(D^2u_k, Du_k) = \psi(x), & \text{in } \Omega, \\
u_k = \varphi, & \text{on } \partial\Omega.
\end{cases}
\end{equation}

The operator \( J^t \) comes from \( \tilde{G} \). By the proof of the main theorem in [17], the structure conditions of \( J^t \) is as same as \( \tilde{G} \). Then for any admissible solution of (6.2), the estimate (6.2) also holds. So any solution \( u_k \) satisfies
\[ \|u_k\|_{C^{3,\alpha}(\bar{\Omega})} \leq C_40, \]

\[ \text{26} \]
where \( C_{40} \) depends only on the known data and also is independent to \( t_k \). Using Arzela-Ascoli Theorem, we know that there exist \( \hat{u} \in C^{3,\alpha}(\bar{\Omega}) \) and a subsequence of \( \{t_k\} \), which is still denoted as \( \{t_k\} \), such that letting \( k \to \infty \),

\[
\|u_k - \hat{u}\|_{C^3(\bar{\Omega})} \to 0.
\]

For equation (6.4), letting \( k \to \infty \), we have

\[
\begin{aligned}
J^0(D^2\hat{u}, D\hat{u}) &= \psi(x), \quad \text{in } \Omega, \\
\hat{u} &= \varphi, \quad \text{on } \partial\Omega.
\end{aligned}
\]

Therefore, \( t_0 \in S \), and thus \( S \) is closed. Consequently, \( S = [0, 1] \).

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