THE MUTUAL SINGULARITY OF HARMONIC MEASURE AND HAUSDORFF MEASURE OF CODIMENSION SMALLER THAN ONE

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ABSTRACT. Let $\Omega \subset \mathbb{R}^{n+1}$ be open and let $E \subset \partial \Omega$ with $0 < \mathcal{H}^s(E) < \infty$, for some $s \in (n, n+1)$, satisfy a local capacity density condition. In this paper it is shown that the harmonic measure cannot be mutually absolutely continuous with $\mathcal{H}^s$ on $E$. This answers a question of Azzam and Mourgoglou, who had proved the same result under the additional assumption that $\Omega$ is a uniform domain.

1. INTRODUCTION

In this paper we study the relationship between harmonic measure and Hausdorff measure of codimension smaller than 1 in $\mathbb{R}^{n+1}$. The importance of harmonic measure is mainly due to its connection with the Dirichlet problem for the Laplacian. Indeed, recall that given a domain $\Omega \subset \mathbb{R}^{n+1}$ and a point $p \in \Omega$, the harmonic measure with pole at $p$ is the measure $\omega^p$ satisfying the property that, for any function $f \in C(\partial \Omega) \cap L^\infty(\partial \Omega)$, the integral $\int f \, d\omega^p$ equals the value at $p$ of the harmonic extension of $f$ to $\Omega$.

The study of the metric and geometric properties of harmonic measure has been a classical subject in mathematical analysis since the Riesz brothers theorem [RR] asserting that harmonic measure is absolutely continuous with respect to arc length measure on simply connected planar domains with rectifiable boundary. In the plane, complex analysis plays a very important role in connection with harmonic measure, essentially because of the invariance of harmonic measure by conformal mappings. This fact makes the case of planar domains rather special.

In the plane it is known that the dimension of harmonic measure is at most 1 by a celebrated result of Jones and Wolff [JW]. This means that there exists a set $E \subset \partial \Omega$ of Hausdorff dimension at most 1 which has full harmonic measure. Furthermore, such set $E$ can be taken so that it has $\sigma$-finite length, as shown by Wolff [Wo1]. More precise results for simply connected planar domains had been obtained previously by Makarov [Mak1], [Mak2].

In higher dimensions one has to use real analysis techniques to study harmonic measure. The codimension 1 is still quite special, mainly because of relationship between harmonic measure and rectifiability. For instance, in [AHM'TV] it was shown that the mutual absolute continuity between harmonic measure and $n$-dimensional Hausdorff measure on a subset $E \subset \partial \Omega$, $\Omega \subset \mathbb{R}^{n+1}$, implies the $n$-rectifiability of $E$. Also, under the assumption that $\partial \Omega$ is AD-regular, that is $\mathcal{H}^n(B(x,r) \cap \partial \Omega) \approx r^n$ for all $x \in \partial \Omega$, $0 < r \leq \text{diam}(\partial \Omega)$, many recent works have been devoted to relate quantitative properties of harmonic measure and other analytic or geometric properties of the domain. See for example [Az1], [GMT], [HLMN], [HM1], [HM2], [HMM], [HMU], [MT], etc.

One of the main differences between the planar case and the higher dimensional case is that in $\mathbb{R}^{n+1}$, with $n \geq 2$, there exist domains where the dimension of harmonic measure is larger than $n$. This was proved by Wolff in [Wo2]. An important open problem consists of finding the sharp value for the upper bound of the dimension of harmonic measure in $\mathbb{R}^{n+1}$, $n \geq 2$. In [Bo] Bourgain showed that this sharp
value is strictly smaller than $n+1$. In [Jo] Jones conjectured that the sharp bound should be $n+1 - 1/n$. However, for the moment there have been no significative advances on this open problem. On the other hand, the techniques of Bourgain [Bo] play an important role in more recent results asserting that in some classes of sets (for example, in some self-similar sets) the dimension of harmonic measure is strictly smaller than the dimension of the set. See [Ba1], [Ba2], and [AZ2].

As mentioned above, the current paper deals with harmonic measure in the case of codimension less than 1. Although the main result of the paper is not directly related to the above Jones’ conjecture, I think that this contributes to a better understanding of the behavior of harmonic measure in this codimension.

To state precisely the main result, we need some additional notation. For $n \geq 2$, let $\Omega \subset \mathbb{R}^{n+1}$ be open and let $E \subset \partial \Omega$ be a non-empty set. We say that the local capacity density condition (or local CDC) holds in $E$ if there exists constants $r_E > 0$ and $c_E > 0$ such that

$$(1.1) \quad \text{Cap}(B(x, r) \cap \Omega^c) \geq c_E r^{n-1} \quad \text{for all} \quad x \in E \quad \text{and} \quad 0 < r \leq r_E,$$

where Cap stands for the Newtonian capacity (see Section 2.2 for the definition). We denote by $\omega$ the harmonic measure in $\Omega$.

The main result of this paper is the following.

**Theorem 1.1.** Given $n \geq 2$ and $s > n$, let $\Omega \subset \mathbb{R}^{n+1}$ be open and let $E \subset \partial \Omega$ be such $\mathcal{H}^s(E) < \infty$. Suppose that the harmonic measure $\omega$ and the Hausdorff measure $\mathcal{H}^s$ are mutually absolutely continuous in $E$ and that the local CDC holds in $E$. Then $\mathcal{H}^s(E) = \omega(E) = 0$.

In other words, harmonic measure cannot be mutually absolutely continuous with Hausdorff measure of codimension less than 1 in any subset of positive harmonic measure, under the local CDC assumption. Recall that the same result was proved in [AM] by Azzam and Mourgoglou under the additional assumption that $\Omega$ is a uniform domain. Recall that, roughly speaking, a domain is called uniform if it satisfies an interior porosity assumption (the so-called interior corkscrew condition), and a quantitative connectivity condition in terms of Harnack chains.

The methods in the current paper are very different from the ones used in [AM]. The new main tool is an identity obtained by integration by parts (see Section 3.1), whose application requires later some rather delicate stopping time arguments. On the other hand, the arguments in [AM] use blowups and tangent measures, and it seems that the uniformity assumption is important. In fact, in their work, Azzam and Mourgoglou leave as an open question the possibility of eliminating the uniformity assumption. They also ask the same questions about the CDC: can this be avoided? While Theorem 1.1 confirms that uniformity is not necessary, it is still an open problem to know if the CDC is required.

In fact, in [AM] a non-degeneracy condition weaker (at least, a priori) than the CDC is used. I think that, quite likely, in the arguments below one may be able to replace the local CDC assumption by the non-degeneracy condition of Azzam-Mourgoglou. However, I have preferred to state Theorem 1.1 in terms of the local CDC, which is closer to the usual CDC. On the other hand, the techniques in the current paper do not look very useful for codimensions larger than 1, unlike the arguments in [AM], which are applied by the authors to derive other related results.

The aforementioned integration by parts formula (see (3.1)) required for the proof of Theorem 1.1 is a generalization of a formula that has already been applied to some problems involving harmonic or elliptic measure and rectifiability in works such as [HLMN] or [AGMT], and it goes back to some work of Lewis and Vogel [LV], at least.
2. Preliminaries

In the paper, constants denoted by $C$ or $c$ depend just on the dimension and perhaps other fixed parameters (such as the constant $c_E$ involved the local CDC, for example). We will write $a \lesssim b$ if there is $C > 0$ such that $a \leq Cb$. We write $a \approx b$ if $a \lesssim b \lesssim a$.

2.1. Measures. The set of (positive) Radon measure in $\mathbb{R}^{n+1}$ is denoted by $\mathcal{M}_+(\mathbb{R}^{n+1})$. The Hausdorff $s$-dimensional measure and Hausdorff $s$-dimensional content are denoted by $\mathcal{H}^s$ and $\mathcal{H}^s_\infty$, respectively.

Given $\mu \in \mathcal{M}_+(\mathbb{R}^{n+1})$, its supper $s$-dimensional density at $x$ is defined by

$$\Theta^{s,*}(x, \mu) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{(2r)^s}.$$ 

Recall that, given an $\mathcal{H}^s$-measurable set $E \subset \mathbb{R}^{n+1}$ with $0 < \mathcal{H}^s(E) < \infty$, we have

(2.1) \[2^{-s} \leq \Theta^{s,*}(x, \mathcal{H}^s|_E) \leq 1\] for $\mathcal{H}^s$-a.e. $x \in E$.

See [Mat] Chapter 6, for example.

2.2. Newtonian capacity and harmonic measure. The fundamental solution of the minus Laplacian in $\mathbb{R}^{n+1}$, $n \geq 2$, equals

$$\mathcal{E}(x) = \frac{c_n}{|x|^n},$$

where $c_n = (n-1)\mathcal{H}^n(S^n)$, with $S^n$ being the unit hypersphere in $\mathbb{R}^{n+1}$.

The Newtonian potential of a measure $\mu \in \mathcal{M}_+(\mathbb{R}^{n+1})$ is defined by

$$U_\mu(x) = \mathcal{E} * \mu(x),$$

and the Newtonian capacity of a compact set $F \subset \mathbb{R}^{n+1}$ equals

$$\text{Cap}(F) = \sup \{ \mu(F) : \mu \in \mathcal{M}_+(\mathbb{R}^{n+1}), \text{supp} \mu \subset F, \|U_\mu\|_\infty \leq 1 \}.$$

It is well known that

$$\|U_\mu\|_\infty = \|U_\mu\|_{\infty,F},$$

and that there exist a unique measure that attains the supremum in the definition of $\text{Cap}(F)$. If $\mu$ attains that supremum, then it turns out that $U_\mu(x) = 1$ for quasievery $x \in F$ (denoted also q.e. in $F$), i.e., for all $x \in F$ with the possible exception of a set of zero Newtonian capacity. The probability measure

$$\mu_F = \frac{1}{\text{Cap}(F)} \mu$$

is called equilibrium measure (of $F$), and so it holds that

$$U_{\mu_F}(x) = \frac{1}{\text{Cap}(F)}$$

for q.e. $x \in F$.

Recall that we denote by $\omega$ the harmonic measure on an open set $\Omega$. The associated Green function is denoted by $g(\cdot, \cdot)$. The following result is quite well known, but we prove it here for the reader’s convenience.

**Lemma 2.1.** Given $n \geq 2$, let $\Omega \subset \mathbb{R}^{n+1}$ be open and let $B$ be a closed ball centered at $\partial \Omega$. Then

$$\omega^s(B) \geq c(n) \frac{\text{Cap}(\frac{1}{2}B \cap \partial \Omega)}{r(B)^{n-1}}$$

for all $x \in \frac{1}{2}B \cap \Omega$,

with $c(n) > 0$. 
Proof. Let $\mu_{\frac{1}{2}B \cap \partial \Omega}$ be the equilibrium measure for $\frac{1}{2}B \cap \partial \Omega$, and let $\mu = \text{Cap}(\frac{1}{2}B \cap \partial \Omega, \mu_{\frac{1}{2}B \cap \partial \Omega})$, so that $\|U\mu\|_\infty \leq 1$ and $\|\mu\| = \text{Cap}(\frac{1}{2}B \cap \partial \Omega)$. Notice that, for all $x \in B^c$, 

$$U\mu(x) = \int \frac{c_n}{|x - y|^{n-1}} \, d\mu(y) \leq \frac{c_n\|\mu\|}{(\frac{3}{4}r(B))^{n-1}}.$$ 

Consider the function $f(x) = U\mu(x) - \frac{c_n\|\mu\|}{(\frac{3}{4}r(B))^{n-1}}$. Using that $f(x) \leq 0$ in $B^c$, $\|f\|_\infty \leq 1$, and that $f$ is harmonic in $\Omega$, by the maximum principle we deduce that, for all $x \in \Omega$,

$$\omega^x(B) \geq f(x).$$

In particular, for $x \in \frac{1}{2}B$ we have

$$\omega^x(B) \geq \frac{c_n}{|x - y|^{n-1}} \, d\mu(y) - \frac{c_n\|\mu\|}{(\frac{3}{4}r(B))^{n-1}} \geq \frac{c_n\|\mu\|}{(\frac{3}{4}r(B))^{n-1}} - \frac{c_n\|\mu\|}{(\frac{3}{4}r(B))^{n-1}} = c_n(2^{n-1} - (\frac{3}{4})^{n-1}) \frac{\text{Cap}(\frac{1}{2}B \cap \partial \Omega)}{r(B)^{n-1}},$$

which proves the lemma. \hfill \square

We recall also the following lemma, whose prove can be found in [AHM+TV].

**Lemma 2.2.** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^{n+1}$ be open. Let $B$ be a closed ball centered at $\partial \Omega$. Then, for all $a > 0$,

$$\omega^x(aB) \geq \inf_{x \in 2B \cap \partial \Omega} \omega^x(aB) \frac{r(B)}{n-1} g(x, y) \quad \text{for all } x \in \Omega \setminus 2B \text{ and } y \in B \cap \Omega,$$

with the implicit constant independent of $a$.

Combining the two preceding lemmas, choosing $a = 8$, we obtain:

**Lemma 2.3.** Let $n \geq 2$, $s > n - 1$, and $\Omega \subset \mathbb{R}^{n+1}$ be open. Let $B$ be a closed ball centered at $\partial \Omega$. Then,

$$\omega^x(8B) \geq n \frac{\text{Cap}(2B \cap \partial \Omega)}{r(B)^{n-1}} g(x, y) \quad \text{for all } x \in \Omega \setminus 2B \text{ and } y \in B \cap \Omega.$$

3. The key identity and the main idea

3.1. The key identity.

**Lemma 3.1 (Key identity).** Let $\Omega \subset \mathbb{R}^{n+1}$ be open, let $\psi \in C_c^\infty(\Omega)$, and let $u : \Omega \to \mathbb{R}$ be harmonic and positive in $\text{supp} \, \psi$. Then, for any $\alpha > 0$,

$$\int |\nabla^2 u|^2 |u^\alpha \psi| \, dx = \frac{1}{2} \alpha (\alpha - 1) \int |\nabla u|^4 u^{\alpha - 2} \psi \, dx - \frac{1}{2} \int \nabla(|\nabla u|^2) \cdot \nabla u^\alpha \psi \, dx + \frac{1}{2} \int |\nabla u|^2 \nabla(u^\alpha) \cdot \nabla \psi \, dx.$$

In the lemma we denoted

$$|\nabla^2 u|^2 = \sum_{i,j} (\partial_{i,j} u)^2.$$ 

The identity (3.1), in the particular case $\alpha = 1$, was already used in connection with harmonic measure in [LV] and [HLMN].
Proof. Notice that
\[ |\nabla^2 u|^2 = \sum_i |\nabla \partial_i u|^2. \]
So (3.1) follows by summing from \( i = 1 \) to \( n + 1 \) the following identity:
\[
(3.2) \quad \int |\nabla \partial_i u|^2 u^\alpha \psi \, dx = \frac{1}{2} \alpha (\alpha - 1) \int |\partial_i u|^2 |\nabla|^{\alpha-2} \psi \, dx
- \frac{1}{2} \int \nabla(|\partial_i u|^2) \cdot \nabla \psi \, u^\alpha \, dx + \frac{1}{2} \int |\partial_i u|^2 \nabla(u^\alpha) \cdot \nabla \psi \, dx.
\]
To prove this, we integrate by parts:
\[
\int \nabla \partial_i u \cdot \nabla (\partial_i u \, u^\alpha \, \psi) \, dx = \int \nabla \partial_i u \cdot \nabla \partial_i u \, u^\alpha \, \psi \, dx
= \int \nabla \partial_i u \cdot \nabla (\partial_i u \, u^\alpha \, \psi) \, dx - \int \nabla \partial_i u \cdot \nabla (u^\alpha) \, \partial_i u \, dx.
\]
The first integral on the right hand side vanishes because \( u \) is harmonic:
\[
\int \nabla \partial_i u \cdot \nabla (\partial_i u \, u^\alpha \, \psi) \, dx = - \int \Delta (\partial_i u) \, (\partial_i u \, u^\alpha) \, dx = 0.
\]
Using also \( \partial_i u \, \nabla \partial_i u = \frac{1}{2} \nabla (|\partial_i u|^2) \), we get
\[
(3.3) \quad \int |\nabla \partial_i u|^2 u^\alpha \psi \, dx = - \frac{1}{2} \int \nabla(|\partial_i u|^2) \cdot \nabla (u^\alpha \, \psi) \, dx
= - \frac{1}{2} \int \nabla(|\partial_i u|^2) \cdot \nabla (u^\alpha) \, \psi \, dx - \frac{1}{2} \int \nabla(|\partial_i u|^2) \cdot \nabla u^\alpha \, dx
= - \frac{1}{2} \int \nabla(|\partial_i u|^2) \psi \cdot \nabla (u^\alpha) \, dx + \frac{1}{2} \int |\partial_i u|^2 \nabla \psi \cdot \nabla (u^\alpha) \, dx
- \frac{1}{2} \int \nabla(|\partial_i u|^2) \cdot \nabla u^\alpha \, dx.
\]
Finally, integrating by parts and taking into account that \( \Delta (u^\alpha) = \alpha (\alpha - 1) |\nabla u^2| u^{\alpha-2} \), we deduce that the first term on the right hand side satisfies
\[
- \frac{1}{2} \int \nabla(|\partial_i u|^2) \psi \cdot \nabla (u^\alpha) \, dx = \frac{1}{2} \int |\partial_i u|^2 \psi \Delta (u^\alpha) \, dx = \frac{1}{2} \alpha (\alpha - 1) \int |\partial_i u|^2 \psi |\nabla u|^2 u^{\alpha-2} \, dx.
\]
Plugging this into (3.3), we get (3.2).

\[ \square \]

3.2. The strategy of the proof. Let \( s > n \) be as in Theorem 1.1. By Bourgain’s theorem [Bo], it is clear that we can assume \( s \in (n, n+1) \). Let \( a \in (0, 1) \) be such that \( s = n + a \), and let
\[
\alpha = \frac{1}{1+a},
\]
so that \( \alpha \in (0, 1) \) too. We will apply the identity (3.1) with \( u \) equal to the Green function \( g(\cdot, p) \) and a suitable function \( \psi \). The choice of the preceding value of \( \alpha \) is motivated by the fact that then the integrals that appear in (3.1) scale like
\[
\omega(\cdot) \left( \frac{\omega(\cdot)}{\ell s} \right)^{\alpha+1},
\]
under the assumption that that $u = g(\cdot, p)$ scales like $\omega(\cdot)\ell^{1-n}$.

A key fact in our arguments is that the first term on the right hand side of (3.1) is negative (because $\alpha(\alpha - 1) < 0$), while the left hand side is positive. These two terms should be considered as the main ones in (3.1), and the last two integrals should be considered as “boundary terms” because of the presence of $\nabla \psi$ in their integrands.

Writing $g(x) = g(x, p)$, from (3.1) we get

\begin{equation}
|\alpha(\alpha - 1)| \int |\nabla g|^4 g^{\alpha-2} \psi \, dx \\
\leq \left| \int \nabla (|\nabla g|^2) \cdot \nabla \psi \, g^\alpha \, dx \right| + \left| \int |\nabla g|^2 \nabla (g^\alpha) \cdot \nabla \psi \, dx \right| - 2 \int |\nabla^2 g|^2 g^\alpha \psi \, dx.
\end{equation}

Using this inequality and assuming the existence of a set $E \subset \partial \Omega$ with $\omega(E) > 0$ such that the harmonic measure and the Hausdorff measure $H^\alpha$ are absolutely continuous on $E$, we will get a contradiction. To this end, we will construct an appropriate function $\psi$ by some stopping time arguments involving the set $E$, and with this choice we will show that the integral on left hand side of (3.4) is much larger than the right hand side.

4. THE BALL $B_0$, THE STOPPING CONSTRUCTION, AND THE FUNCTION $\psi$

4.1. The ball $B_0$. From now we assume that we are under the assumptions of Theorem [11]. We consider a point $p \in \Omega$ and we denote by $\omega$ the harmonic measure for $\Omega$ with respect to the pole $p$. We also denote $\mu = H^\alpha|_E$ and we assume that $0 < \mu(E) < \infty$ and that $\mu$ is absolutely continuous with respect to $\omega$. Our objective is to find a contradiction.

By replacing $E$ by a suitable subset if necessary, by standard methods (taking into account the upper bound for the upper density of $\mu$ in (2.1)) we may assume that there exists some $\delta_0 > 0$ such that

$$\mu(B(x, r)) \leq 3^s \, r^s \quad \text{for all } x \in E \text{ and } 0 < r \leq \delta_0.$$  

Since $\mu \ll \omega$, there exists some non-negative function $h \in L^1(\omega)$ such that $\mu = h \omega$. We consider a point $x_0 \in E$ satisfying the following: $x_0$ is a Lebesgue point for $h$ with $h(x_0) > 0$ and a density point of $E$ (both with respect to $\omega$), and there exists a sequence of radii $r_k \to 0$ such that

\begin{equation}
\omega(B(x_0, 200r_k)) \leq (200)^{n+2} \omega(B(x_0, r_k)).
\end{equation}

For this last property, see, for example, Lemma 2.8 in [16]. Now, given some $\kappa_0 \in (0, 1/10)$, let $\delta_1 \in (0, \delta_0]$ be such that

\begin{equation}
\frac{1}{\omega(B(x, r))} \int_{B(x, r)} |h - h(x_0)| \, d\omega \leq \kappa_0 \, h(x_0) \quad \text{for all } r \in (0, \delta_1]
\end{equation}

and

\begin{equation}
\omega(B(x, r) \setminus E) \leq \kappa_0 \, \omega(B(x, r)) \quad \text{for all } r \in (0, \delta_1].
\end{equation}

The parameter $\kappa_0$ will be fixed below, and depends only on $n$. We take now a radius

$$\tilde{r} \in \left(0, \frac{1}{300} \min(r_E, \delta_1, |p - x_0|)\right)$$

such that (4.1) holds for $\tilde{r} = r_k$ (recall that $r_E$ is defined by local CDC in (1.1)), and we denote

$$B_0 = B(x_0, 2\tilde{r}).$$
From (4.2) we deduce that, for all \( r \in (0, 100r(B_0)) \),
\[
\mu(B(x_0, r)) = \int_{B(x_0, r)} h \, d\omega \leq h(x_0) \omega(B(x_0, r)) + \int_{B(x_0, r)} |h - h(x_0)| \, d\omega \leq 2 h(x_0) \omega(B(x_0, r)).
\]
Analogously,
\[
\mu(B(x_0, r)) \geq h(x_0) \omega(B(x_0, r)) - \int_{B(x_0, r)} |h - h(x_0)| \, d\omega \geq \frac{1}{2} h(x_0) \omega(B(x_0, r)).
\]

We collect some of the properties about \( B_0 \) in the next lemma.

**Lemma 4.1.** For all \( r \in (0, 100r(B_0)) \), we have
\[
(4.4) \quad \frac{1}{2} h(x_0) \omega(B(x_0, r)) \leq \mu(B(x_0, r)) \leq 2 h(x_0) \omega(B(x_0, r)).
\]
We also have
\[
(4.5) \quad \omega(100B_0) \leq (200)^{n+1} \omega(\frac{1}{2}B_0) \quad \text{and} \quad \mu(100B_0) \leq 4(200)^{n+1} \mu(\frac{1}{2}B_0).
\]

**Proof.** The estimates in (4.4) have been shown above, as well as the first inequality in (4.5). The second inequality follows from the preceding estimates:
\[
\mu(100B_0) \leq 2 h(x_0) \omega(100B_0) \leq 2 h(x_0) (200)^{n+1} \omega(\frac{1}{2}B_0) \leq 4(200)^{n+1} \mu(\frac{1}{2}B_0).
\]

\( \square \)

### 4.2. The bad balls and the function \( d(\cdot) \)

We consider the constant
\[
(4.6) \quad A = 4 \frac{\omega(5B_0)}{\mu(\frac{1}{2}B_0)},
\]
Notice that, by Lemma 4.1
\[
A \approx h(x_0)^{-1}.
\]

For each \( x \in \partial \Omega \cap 2B_0 \) and \( r \in (0, r(B_0)) \), we say that the ball \( B(x, r) \) is bad (and we write \( B(x, r) \in \text{Bad} \)) if
\[
\omega(B(x, r)) > A \mu(B(x, 10r)).
\]

Given some fixed parameter \( \rho_0 \in (0, \frac{1}{10}r(B_0)) \), if there exists some \( r \in (\rho_0, r(B_0)) \) such that \( B(x, r) \) is bad, we denote
\[
(4.7) \quad r_0(x) = \sup \{ r \in (\rho_0, r(B_0)) : B(x, r) \text{ is bad} \}.
\]

Otherwise, we set
\[
r_0(x) = \rho_0.
\]

Using the openness of the balls in the definition of \( r_0(x) \), it is easy to check that the supremum in (4.7) is attained and thus the ball \( B(x, r_0(x)) \) is bad if \( r_0(x) > \rho_0 \).

Next we define the following regularized version of \( r_0(\cdot) \):
\[
d(x) = \inf_{y \in 2B_0 \cap \partial \Omega} (r_0(y) + |x - y|), \quad \text{for } x \in \mathbb{R}^{n+1}.
\]

It is immediate to check that \( d(\cdot) \) is a 1-Lipschitz function. Further, since \( r_0(x) \leq r(B_0) \) for any \( x \in 2B_0 \cap \partial \Omega \), we infer that
\[
d(x) \leq r(B_0) \quad \text{for any } x \in 2B_0 \cap \partial \Omega.
\]
We need the following auxiliary result.

**Lemma 4.2.** Let \( x \in 2B_0 \cap \partial \Omega \). For all \( r \in [d(x), r(B_0)] \),
\[
\omega(B(x, r)) \leq A \mu(B(x, 32r)).
\]

(4.8)

**Proof.** Suppose first that \( r \geq r(B_0)/3 \). In this case, using just that \( B(x, r) \subset 3B_0, B_0 \subset B(x, 3r(B_0)) \), and the choice of \( A \) in (4.6), we infer that
\[
\omega(B(x, r)) \leq \omega(3B_0) \leq A \mu(B_0) \leq A \mu(B(x, 3r(B_0))) \leq A \mu(B(x, 9r)).
\]

Assume now that \( r < r(B_0)/3 \). Let \( y \in 2B_0 \cap \partial \Omega \) be such that
\[
2d(x) \geq r_0(y) + |x - y|.
\]
Using that \( B(x, r) \subset B(y, |x - y| + r) \subset B(y, 3r) \) (because \(|x - y| \leq 2d(x) \leq 2r|\)) and that
\[
3r \geq 3d(x) \geq \frac{3}{2} r_0(y) \quad \text{and} \quad 3r \leq r(B_0),
\]
we get
\[
\omega(B(x, r)) \leq \omega(B(y, 3r)) \leq A \mu(B(y, 30r)).
\]
Now we take into account that \( B(y, 30r) \subset B(x, |x - y| + 30r) \subset B(x, 32r) \) (again because \(|x - y| \leq 2r|\)), and we derive
\[
\omega(B(x, r)) \leq A \mu(B(y, 30r)) \leq A \mu(B(x, 32r)).
\]

Now we apply Vitali’s 5r-covering theorem to get a finite subfamily of balls
\[
\{B_i\}_{i \in I} \subset \{B(x, \frac{1}{2000} d(x))\}_{x \in 2B_0 \cap \partial \Omega}
\]
such that
- the balls \( B_i, i \in I \), are pairwise disjoint, and
- \( \bigcup_{x \in 2B_0 \cap \partial \Omega} B(x, \frac{1}{2000} d(x)) \subset \bigcup_{i \in I} 5B_i \).

In the next lemma we show some elementary properties of the family \( \{B_i\}_{i \in I} \).

**Lemma 4.3.** Let \( \{B_i\}_{i \in I} \) be the family of balls defined above. The following holds:

(a) For each \( i \in I \), \( r(B_i) \leq \frac{1}{2000} r(B_0) \) and \( 1000B_i \subset 3B_0 \).

(b) For all \( x \in 1000B_i \), with \( i \in I \), \( 1000 r(B_i) \leq d(x) \leq 3000 r(B_i) \).

(c) If \( 1000B_i \cap 1000B_j \neq \emptyset \), for \( i, j \in I \), then \( \frac{1}{7} r(B_i) \leq r(B_j) \leq 3r(B_i) \).

(d) The balls \( 1000B_i, i \in I \), have finite superposition. That is,
\[
\sum_{i \in I} \chi_{1000B_i} \leq C_1,
\]
for some absolute constant \( C_1 \).

**Proof.** Denote by \( x_i \) the center of \( B_i \), so that \( B_i = B(x_i, \frac{1}{2000} d(x_i)) \). The statement in (a) is due to the
fact that, for each \( i \in I \), we have
\[
r(B_i) = \frac{1}{2000} d(x_i) \leq \frac{1}{2000} r_0(x_i) \leq \frac{1}{2000} r(B_0),
\]
with \( x_i \in 2B_0 \).

On the other hand, notice that, for all \( x \in 1000B_i \),
\[
|d(x) - d(x_i)| \leq |x - x_i| \leq \frac{1000}{2000} d(x_i),
\]
and thus
\[ \frac{1}{2} d(x_i) \leq d(x) \leq \frac{3}{2} d(x_i), \]
which gives (b).

Concerning (c), given 1000B_i and 1000B_j with non-empty intersection, we consider \( x \in 1000B_i \cap 1000B_j \) and we deduce that
\[ \frac{1}{2} d(x_i) \leq d(x) \leq \frac{3}{2} d(x_j). \]
Together with the converse estimate, this shows that
\[ r(B_i) \leq 3r(B_j) \leq 9r(B_i). \]

To prove (d), let \( B_{i_1}, \ldots, B_{i_m} \) be such that
\[ \bigcap_{j=1}^m 1000B_{i_j} \neq \emptyset. \]
Suppose that \( B_{i_1} \) has maximal radius among the balls \( B_{i_1}, \ldots, B_{i_m} \), so that
\[ \bigcup_{j=1}^m 1000B_{i_j} \subset 3000B_{i_1}. \]
Since the balls \( B_{i_1}, \ldots, B_{i_m} \) are pairwise disjoint, by the properties (c), (b) and the usual volume considerations we deduce that
\[ \frac{m}{3^{n+1}} r(B_{i_1})^{n+1} \leq \sum_{j=1}^m r(B_{i_j})^{n+1} \leq (3000 r(B_{i_1}))^{n+1}, \]
and thus \( m \lesssim 1 \).

4.3. The function \( \psi \). Let \( \varphi \) be a radial \( C^\infty \) function such that \( \chi_{B(0,1,1)} \leq \varphi \leq \chi_{B(0,1,2)} \), and let
\[ \varphi_i(x) = \varphi \left( \frac{x - x_i}{5r_i} \right), \]
where \( x_i \) is the center of \( B_i \) and \( r_i \) its radius. Notice that \( \varphi \equiv 1 \) on \( 5.5B_i \) and vanishes out of \( 6B_i \). Next we need to define some auxiliary functions \( \theta_j \). First, by applying the \( 5r \)-covering theorem, we consider a covering of \( 3B_0 \setminus \bigcup_{i \in I} 1.1B_i \) with balls of the form \( B(z_j, 10^{-5}d(z_j)) \), with \( z_j \in 3B_0 \setminus \bigcup_{i \in I} 1.1B_i \), so that the balls \( \frac{1}{5} B(z_j, 10^{-5}d(z_j)) \) are disjoint. This implies that the dilated balls \( 1.2B(z_j, 10^{-5}d(z_j)) \) have finite superposition, by arguments analogous to the ones in Lemma 4.3. For each \( j \in J \), we define
\[ \theta_j(x) = \varphi \left( \frac{x - z_j}{10^{-5}d(z_j)} \right). \]
In this way, using the property (b) in the preceding lemma, for any \( j \in J \),
\[ \supp \theta_j \cap \bigcup_{i \in I} 5B_i = \emptyset. \]
We consider the functions
\[ \tilde{\varphi}_i = \frac{\sum_{j \in I} \varphi_j + \sum_{j \in J} \theta_j}{\sum_{j \in I} \varphi_j + \sum_{j \in J} \theta_j}, \quad i \in I. \]
Notice that the denominator above is bounded away from 0 in \(\text{supp} \varphi_i\), and thus \(\tilde{\varphi}_i \in C^\infty\), with \(\|\nabla \tilde{\varphi}_i\|_\infty \lesssim r_i^{-1}\). Further, by construction
\[
0 \leq \sum_{i \in I} \tilde{\varphi}_i \leq 1 \quad \text{in} \quad \mathbb{R}^{n+1}.
\]
Also, taking into account (4.11),
\[
\sum_{i \in I} \tilde{\varphi}_i \equiv 1 \quad \text{in} \quad \bigcup_{i \in I} 5B_i
\]
and, since \(\text{supp} \tilde{\varphi}_i \subset 6B_i\),
\[
\sum_{i \in I} \tilde{\varphi}_i \equiv 0 \quad \text{in} \quad \mathbb{R}^{n+1} \setminus \bigcup_{i \in I} 6B_i.
\]
We also denote
\[
\psi_0 = \left(1 - \sum_{i \in I} \tilde{\varphi}_i\right) \varphi\left(\frac{x - x_0}{r(B_0)}\right)
\]
(recall that \(x_0\) is the center of \(B_0\)). Finally, we let
\[
\psi = \psi_0^4.
\]

**Lemma 4.4.** The following holds:

(a) \(\text{supp} \psi_0 \subset 2B_0 \setminus \bigcup_{i \in I} 5B_i\) and \(\psi_0 \equiv 1\) in \(B_0 \setminus \bigcup_{i \in I} 6B_i\).

(b) \(\text{supp} \nabla \psi_0 \subset \bigcup_{i \in I} A(x_i, 5r_i, 6r_i) \cup A(x_0, r(B_0), 2r(B_0))\).

(c) \(|\nabla \psi_0(x)| \lesssim \frac{1}{r(B_0)}\) for all \(x \in 6B_i\).

(d) \(|\nabla \psi_0(x)| \lesssim \frac{1}{r(B_0)}\) for all \(x \in 2B_0 \setminus \bigcup_{i \in I} 6B_i\).

The same properties hold for \(\psi\).

The proof of the lemma follows easily from the construction above, and we leave it for the reader.

### 4.4. The sets \(V, \tilde{V},\) and \(F\).

By Vitali’s \(5r\)-covering theorem, there exists a subfamily \(\text{Bad}_V \subset \text{Bad}\) such that

- the balls from \(\text{Bad}_V\) are pairwise disjoint, and
- any ball from \(\text{Bad}\) is contained in some ball \(5B\), with \(B \in \text{Bad}_V\).

We denote
\[
V = \bigcup_{B \in \text{Bad}_V} 5B, \quad \tilde{V} = \bigcup_{B \in \text{Bad}_V} 10B.
\]
Notice that \(V \subset \tilde{V}\) and that all the bad balls are contained in \(V\) (not only the ones with radius larger than \(r_0\)).

In the next lemma we show that \(\tilde{V}\) is rather small, because of our choice of \(A\) above.

**Lemma 4.5.** We have
\[
\mu(\tilde{V}) \leq \sum_{B \in \text{Bad}_V} \mu(10B) \leq \frac{1}{4} \mu(\frac{1}{2}B_0).
\]

**Proof.** By the definition of bad balls and the disjointness of the family \(\text{Bad}_V\), we get
\[
\mu(\tilde{V}) \leq \sum_{B \in \text{Bad}_V} \mu(10B) \leq \frac{1}{A} \sum_{B \in \text{Bad}_V} \omega(B) \leq \frac{1}{A} \omega(5B_0),
\]
where, in the last inequality, we took into account that the bad balls are centered at $2B_0$ and have radius at most $r(B_0)$. By the choice of $A$ in (4.6), we are done.

Next we need to consider another kind of bad set. We let $F$ be the subset of the points $x \in E \cap \frac{1}{2}B_0$ for which there exists some $r \in (0, \frac{1}{2}r(B_0)]$ such that

$$\omega(B(x, r)) \leq \kappa_0 h(x_0)^{-1} \mu(B(x, r))$$

(recall that $\kappa_0 \in (0, 1/10)$ will be fixed below).

**Lemma 4.6.** We have

$$\mu(F) \leq C \kappa_0 \mu(\frac{1}{2}B_0).$$

**Proof.** By the Besicovitch covering theorem, there exists a covering of $F$ by a family of balls $B(z_i, s_i)$, with $z_i \in F$, $0 < s_i \leq r(B_0)/4$, such that $\omega(B(z_i, s_i)) \leq \kappa_0 h(x_0)^{-1} \mu(B(z_i, r_i))$, and having finite superposition. That is, $\sum_i \chi_{B(z_i, s_i)} \lesssim 1$. Then we have:

$$\omega(F) \leq \sum_i \omega(B(z_i, s_i)) \leq \kappa_0 h(x_0)^{-1} \sum_i \mu(B(z_i, s_i))$$

$$\leq C \kappa_0 h(x_0)^{-1} \mu(B_0) \leq C \kappa_0 h(x_0)^{-1} \mu(\frac{1}{2}B_0),$$

taking into account that all the balls $B(z_i, s_i)$ are contained in $B_0$ and the finite superposition of the balls in the before to last inequality, and the fact that $\frac{1}{2}B_0$ is doubling with respect to $\mu$, by (4.5), in the last inequality. As a consequence,

$$\mu(F) = \int_F h \, d\omega \leq h(x_0) \omega(F) + \int_{\frac{1}{2}B_0} |h - h(x_0)| \, d\omega$$

$$\leq C \kappa_0 \mu(\frac{1}{2}B_0) + \kappa_0 h(x_0) \omega(\frac{1}{2}B_0) \leq C \kappa_0 \mu(\frac{1}{2}B_0).$$

□

5. PROOF OF THEOREM 1.1

Let $s > n$ be as in Theorem [14]. Recall that we assume $s \in (n, n + 1)$ and we denote $a = s - n$. Also, we take

$$\alpha = \frac{1 - a}{1 + a},$$

so that both $a, \alpha \in (0, 1)$. We will apply the identity (3.1) with $u$ equal to the Green function $g(\cdot, p)$, the function $\psi$ constructed in Section 4, and the preceding value of $\alpha$. Recall that $\psi$ supported in $2B_0$ and vanishes in a neighborhood of $\partial \Omega \cap 2B_0$. Thus, $g = g(\cdot, p)$ is harmonic in $\text{supp} \, \psi$. Recall also that we have

$$|\alpha(\alpha - 1)| \int |\nabla g|^4 g^{a-2} \psi \, dx$$

$$\leq \left| \int \nabla(|\nabla g|^2) \cdot \nabla \psi g^a \, dx \right| + \left| \int |\nabla g|^2 \nabla(g^a) \cdot \nabla \psi \, dx \right| - 2 \int |\nabla^2 g|^2 g^a \psi \, dx.$$

To achieve the desired contradiction to prove Theorem 1.1 we will show that the integral on the left hand side tends to $\infty$ as $\rho_0 \to 0$, while the right hand side is much smaller than the left hand side.
We denote

\[ I_0 = \int |\nabla g|^4 g^{\alpha - 2} \psi \, dx, \]
\[ I_1 = \int \nabla(|\nabla g|^2) \cdot \nabla \psi g^{\alpha} \, dx, \]
\[ I_2 = \int |\nabla g|^2 \nabla (g^{\alpha}) \cdot \nabla \psi \, dx, \]
\[ I_3 = \int |\nabla^2 g|^2 g^{\alpha} \psi \, dx. \]

5.1. **Estimate of** \( I_1 \). Using the fact that \(|\nabla(|\nabla g|^2)| \lesssim |\nabla^2 g| |\nabla g|\) and Hölder’s inequality and recalling that \( \psi = \psi_0^4 \), we get

\[ |I_1| \lesssim \int |\nabla^2 g||\nabla g| g^{\alpha} \psi_0^3 |\nabla \psi_0| \, dx \leq \left( \int |\nabla^2 g|^2 g^{\alpha} \psi_0^4 \, dx \right)^{1/2} \left( \int |\nabla g|^2 g^{\alpha} \psi_0^2 |\nabla \psi_0|^2 \, dx \right)^{1/2}. \]

Observe that the first integral on the right hand side coincides with \( I_3 \). To deal with the last one, we split it as follows:

\[ \int 6B_i \left( \int |\nabla g|^2 g^{\alpha} \psi_0^2 |\nabla \psi_0|^2 \, dx \right) \leq \sum_{i \in I} \int_{6B_i} \ldots + \int_{2B_0 \cup \bigcup_{i \in I} 6B_i} \ldots. \]

By Lemma \[4.4\](c) and Caccioppoli’s inequality, for each \( i \in I \) we obtain

\[ \int_{6B_i} |\nabla g|^2 g^{\alpha} |\nabla \psi_0|^2 \, dx \leq \frac{1}{r_i^2} \sup_{6B_i} g(x) \int_{6B_i} |\nabla g|^2 \, dx \leq \frac{1}{r_i^2} \sup_{6B_i} g(x) \int_{12B_i} |g|^2 \, dx \lesssim r_i^{n-3} \sup_{12B_i} g(x)^{\alpha+2}. \]

By (2.3), we have

\[ \sup_{12B_i} g(x) \lesssim \frac{\omega(96B_i)}{r_i^{n-1}}. \]

Therefore, by the choice of \( \alpha \),

\[ \int_{6B_i} |\nabla g|^2 g^{\alpha} |\nabla \psi_0|^2 \, dx \lesssim r_i^{n-3} \left( \frac{\omega(96B_i)}{r_i^{n-1}} \right)^{\alpha+2} = \omega(96B_i) \left( \frac{\omega(96B_i)}{r_i^s} \right)^{\alpha+1}. \]

By Lemma \[4.2\]

\[ \omega(96B_i) \leq \omega(B(x_i, d(x_i)) \leq A \mu(B(x_i, 32d(x_i)) \lesssim Ad(x_i)^s \approx Ar_i^s. \]

Thus,

\[ \int_{6B_i} |\nabla g|^2 g^{\alpha} |\nabla \psi_0|^2 \, dx \lesssim A^{\alpha+1} \omega(96B_i). \]

Finally, Lemma \[4.3\](a) and (d),

\[ \sum_{i \in I} \int_{6B_i} |\nabla g|^2 g^{\alpha} |\nabla \psi_0|^2 \, dx \lesssim A^{\alpha+1} \sum_{i \in I} \omega(96B_i) \lesssim A^{\alpha+2} \mu(\frac{1}{2}B_0). \]
Next we deal with the last integral on the right hand side of (5.2). We argue as in (5.2), (5.8), but now we use the fact that $|\nabla \psi_0| \lesssim 1/r(B_0)$ in $2B_0 \setminus \bigcup_{i \in I} 6B_i$ and we replace $6B_i$ by $2B_0$. Then, as in (5.5), we get

$$\int_{2B_0 \setminus \bigcup_{i \in I} 6B_i} |\nabla g|^2 g^\alpha |\nabla \psi_0|^2 \, dx \lesssim \omega(32B_0) \left( \frac{\omega(32B_0)}{r(B_0)^s} \right)^{\alpha+1}. \tag{5.9}$$

Using now (4.5) and (4.6), we derive

$$\int \int_{2B_0 \setminus \bigcup_{i \in I} 6B_i} |\nabla g|^2 g^\alpha |\nabla \psi_0|^2 \, dx \lesssim A^{\alpha+2} \mu(\frac{1}{2} B_0). \tag{5.10}$$

Altogether, we obtain

$$|I| \leq \left( CA^{\alpha+2} \mu(B_0) \right)^{1/2} I_3^{1/2} \lesssim CA^{\alpha+2} \mu(\frac{1}{2} B_0) + I_3. \tag{5.11}$$

5.2. Estimate of $I_2$. Using that $\nabla (g^\alpha) = \alpha g^{\alpha-1} \nabla g$, $\nabla \psi = 4\psi_0^3 \nabla \psi_0$, and Hölder’s inequality, we get

$$|I'_2| \lesssim \int |\nabla g|^3 g^{\alpha-1} \psi_0^3 |\nabla \psi_0|^3 \, dx \lesssim \left( \int |\nabla g|^4 g^{\alpha-2} \psi_0^4 \, dx \right)^{3/4} \left( \int g^{\alpha+2} |\nabla \psi_0|^4 \, dx \right)^{1/4}. \tag{5.12}$$

Observe that the first integral on the left hand side equals $I_0$. To estimate the second one we split it:

$$\int g^{\alpha+2} |\nabla \psi_0|^4 \, dx \leq \sum_{i \in I} \int_{6B_i} \ldots + \int_{2B_0 \setminus \bigcup_{i \in I} 6B_i} \ldots.$$ 

By Lemma 4.4(c), for each $i \in I$, we obtain

$$\int_{6B_i} g^{\alpha+2} |\nabla \psi_0|^4 \, dx \lesssim r_i^{n-3} \sup_{6B_i} g(x)^{\alpha+2}. \tag{5.13}$$

As in (5.4), we have

$$\sup_{6B_i} g(x) \leq \sup_{12B_i} g(x) \lesssim \frac{\omega(96B_i)}{r_i^{n-1}}. \tag{5.14}$$

Then, operating exactly as in (5.5)-(5.8), we derive

$$\sum_{i \in I} \int_{6B_i} g^{\alpha+2} |\nabla \psi_0|^4 \, dx \lesssim A^{\alpha+2} \mu(\frac{1}{2} B_0). \tag{5.15}$$

To estimate the last integral on the right hand side of (5.12) we use the fact that $|\nabla \psi_0| \lesssim 1/r(B_0)$ in $2B_0 \setminus \bigcup_{i \in I} 6B_i$ and we apply (2.3). Then we get

$$\int_{2B_0 \setminus \bigcup_{i \in I} 6B_i} g^{\alpha+2} |\nabla \psi_0|^4 \, dx \lesssim r(B_0)^{n-3} \sup_{2B_0} g(x)^{\alpha+2} \lesssim \omega(32B_0) \left( \frac{\omega(32B_0)}{r(B_0)^s} \right)^{\alpha+1}, \tag{5.16}$$

which is the same estimate as in (5.9). Then, as in (5.10), we deduce

$$\int_{2B_0 \setminus \bigcup_{i \in I} 6B_i} g^{\alpha+2} |\nabla \psi_0|^4 \, dx \lesssim A^{\alpha+2} \mu(\frac{1}{2} B_0). \tag{5.17}$$

Therefore,
\[ |I_2| \leq I_0^{3/4} (A^{\alpha+2} \mu(\frac{1}{2} B_0))^{1/4} \leq \frac{\alpha(1 - \alpha)}{2} I_0 + C(\alpha) A^{\alpha+2} \mu(\frac{1}{2} B_0). \]

5.3. **Lower estimate of** $I_0$. From the identity (5.1) and the estimates for $I_1$ and $I_2$ we derive

\[ |\alpha(\alpha - 1)| I_0 = |\alpha(\alpha - 1)| \int |\nabla g|^4 g^{\alpha - 2} \psi \, dx \]

\[ \leq I_1 + I_2 - 2I_3 \]

\[ \leq \left( C A^{\alpha+2} \mu(\frac{1}{2} B_0) + I_3 \right) + \left( \frac{\alpha(1 - \alpha)}{2} I_0 + C(\alpha) A^{\alpha+2} \mu(\frac{1}{2} B_0) \right) - 2I_3. \]

Hence,

\[ (5.13) \quad I_0 \leq C(\alpha) A^{\alpha+2} \mu(\frac{1}{2} B_0). \]

In this section, by estimating $I_0$ from below, we will contradict this inequality.

To get a lower estimate for $I_0$ we need to define some reasonably good set contained in $\frac{1}{2} B_0 \cap \partial \Omega$. To this end, we need first to introduce another type of balls. Let $I_0 \subset I$ be the subfamily of indices $i$ such that $r(B_i) > \frac{1}{2000} \rho_0$. Recall that $I$ is the set of indices in (4.9) and $r(B_i) = \frac{1}{2000} d(x_i) \leq \frac{1}{2000} r_0(x_i)$. So if $i \in I_0$, then $r_0(x_i) > \rho_0$ and thus $B(x_i, r_0(x_i))$ is a bad ball. We say that a ball $B$ is useless (and we write $B \in \text{Uss}$) if it is centered at $\frac{1}{2} B_0 \cap \partial \Omega \setminus \tilde{V}$ and

\[ (5.14) \quad \mu \left( \bigcup_{i \in I_0, B \subset B_i \neq \emptyset} 960 B_i \right) > \varepsilon_1 \mu(B) \quad \text{and} \quad \mu(B) \geq 3^{-s} r(\frac{1}{2} B)^s, \]

where $\varepsilon_1 \in (0, 1/10)$ is a small parameter to be fixed below that will depend only on $n$.

Recall now that, by Lemma 4.5,

\[ \sum_{B \in \text{Bad}_V} \mu(10 B) \leq \frac{1}{4} \mu(\frac{1}{2} B_0). \]

Hence there exists some $\rho_1 > 0$ such that

\[ (5.15) \quad \sum_{B \in \text{Bad}_V, r(B) \leq \rho_1} \mu(10 B) \leq \varepsilon_1^2 \mu(\frac{1}{2} B_0). \]

Notice that $\rho_1$ may depend here on the particular measure $\mu$, not only on $n$. We define

\[ U(\rho_1) = \bigcup_{B \in \text{Uss}: r(B) \leq \rho_1} B. \]

**Lemma 5.1.** We have \[ \mu(U(\rho_1)) \lesssim \varepsilon_1 \mu(\frac{1}{2} B_0). \]

**Proof.** Let $B \in \text{Uss}$ with $r(B) \leq \rho_1$ and let $B_i, i \in I_0$, be such that $6B_i \cap B \neq \emptyset$. Notice that $2000 B_i$ is contained in some bad ball (because $d(x_i) > \rho_0$), which in turn is contained in some ball $5 B'$, with $B' \in \text{Bad}_V$. Thus, $2000 B_i \subset 5 B'$. Now note that $B$ is centered at $\partial \Omega \setminus \tilde{V} \subset (10 B')^c$, and observe that the condition $6B_i \cap B \neq \emptyset$ implies that $B$ intersects $5 B'$. These two facts ensure that

\[ (5.16) \quad r(B) \geq r(5 B') \geq r(2000 B_i). \]

Then we deduce that

\[ 960 B_i \subset 2000 B_i \subset 3 B. \]
The first inequality in (5.16) also implies that \( r(B') \leq \rho_1 \), which in turn gives that
\[
960B_i \subset \bigcup_{B'' \in \text{Bad}_B : r(B'') \leq \rho_1} 5B'' \subset \bigcup_{B'' \in \text{Bad}_B : r(B'') \leq \rho_1} 10B'' =: \tilde{V}_0,
\]
with
\[
(5.17) \quad \mu(\tilde{V}_0) \leq \varepsilon_1^2 \mu(\frac{1}{2}B_0),
\]
by (5.15). From the first condition in (5.14), we deduce that
\[
\mu(3B \cap \tilde{V}_0) \geq \mu \left( \bigcup_{i \in I_k, 6B_i \cap B \neq \emptyset} 960B_i \right) > \varepsilon_1 \mu(B).
\]
Using also the fact that \( \mu(15B) \lesssim r(B)^s \lesssim \mu(B) \) (by the second condition in (5.14)), we get
\[
(5.18) \quad \mu(3B \cap \tilde{V}_0) \geq c \varepsilon_1 \mu(15B).
\]
Now we apply the 5\( r \)-covering theorem to get a subfamily \( I_U \) from the balls in \( \text{Uss} \) with radius not exceeding \( \rho_1 \) such that
- the balls \( 3B \) with \( B \in I_U \) are pairwise disjoint, and
- \( U(\rho_1) \subset \bigcup_{B \in I_U} 15B \).

From these properties and (5.18) and (5.17), we obtain
\[
\mu(U(\rho_1)) \leq \sum_{B \in I_U} \mu(15B) \lesssim \frac{1}{\varepsilon_1} \sum_{B \in I_U} \mu(3B \cap \tilde{V}_0) \leq \frac{1}{\varepsilon_1} \mu(\tilde{V}_0) \leq \varepsilon_1 \mu(\frac{1}{2}B_0).
\]

\[ \square \]

Now we are ready to define the aforementioned reasonably good set contained in \( \frac{1}{2}B_0 \cap \partial \Omega \). First denote
\[
G_0 = \frac{1}{2}B_0 \cap \partial \Omega \setminus (F \cup \tilde{V} \cup U(\rho_1)),
\]
and recall that, by Lemmas 4.6, 4.5, and 5.1
\[
\mu(G_0) \geq \mu(\frac{1}{2}B_0) - C\kappa_0 \mu(\frac{1}{2}B_0) - \frac{1}{4} \mu(\frac{1}{2}B_0) - C\varepsilon_1 \mu(\frac{1}{2}B_0).
\]
We assume \( \kappa_0 \) to be an absolute constant small enough so that \( C\kappa_0 \leq 1/4 \) and also \( \varepsilon_1 \) small enough so that \( C\varepsilon_1 \leq 1/4 \), and then we obtain
\[
\mu(G_0) \geq \frac{1}{4} \mu(\frac{1}{2}B_0).
\]

Next we need to define some families of balls centered at \( G_0 \) inductively. Let \( G \) be the subset of those \( x \in G_0 \) such that \( \Theta^{s,s}(x, \mu) \geq 2^{-8} \). Note that, by (2.1), \( \mu(G_0 \setminus G) = 0 \). By definition, for each \( \eta_k \in (0, r(B_0)/10] \), for \( \mu \)-a.e. \( x \in G \) there exists a ball \( B^i_x \) centered at \( x \) with radius \( r(B^i_x) \leq \eta_k \) such that
\[
(5.19) \quad \mu(B^i_x) \geq 3^{-8}r(B^i_x)^s.
\]
Hence, by the 5\( r \)-covering theorem, we can extract a subfamily \( \tilde{F}_k \subset \{2B^i_x\}_{x \in G} \) such that
(a) \( G \subset \bigcup_{B \in \tilde{F}_k} 80B \), and
(b) the balls 16\( B \), \( B \in \tilde{F}_k \), are disjoint.
Further, we can still extract a finite subfamily $\mathcal{F}_k \subset \tilde{\mathcal{F}}_k$ such that
\begin{equation}
\mu\left(\bigcup_{B \in \mathcal{F}_k} 80B\right) \geq \frac{1}{2} \mu(G) \geq \frac{1}{8} \mu(\frac{1}{2}B_0).
\end{equation}

Now we fix inductively the parameters $\eta_k$ as follows: first we take $\eta_1 = r(B_0)/10$, and next we set
\[ \eta_k = \varepsilon_0 \min_{B \in \mathcal{F}_{k-1}} r(B), \]
where $\varepsilon_0 \in (0, 1/100)$ is some small constant to be chosen below. Notice that this choice ensures that the balls from the family $\mathcal{F}_k$ are much smaller than the ones of the preceding families $\mathcal{F}_1, \ldots, \mathcal{F}_{k-1}$.

Remark that the balls $B(x, r)$ centered at $G$ with radius $r \in (0, r(B_0)/4]$ (like the balls from the families $\mathcal{F}_k$) satisfy
\begin{equation}
\omega(B(x, r)) \geq \kappa_0 h(x_0)^{-1} \mu(B(x, r)),
\end{equation}
and the ones with radius $r \in [\rho_0, r(B_0)]$,\begin{equation}
\omega(B(x, r)) \leq A \mu(B(x, 10r)) \leq CA r^s \approx h(x_0)^{-1} r^s,
\end{equation}
by (4.8) and the choice of $A$.

**Lemma 5.2.** Let $B \in \mathcal{F}_k \setminus \text{Uss}$ and suppose that $\rho_0 \leq \frac{1}{2} \eta_{k+1}$. Denote
\begin{equation}
\tilde{B} = B \setminus \left( \bigcup_{B' \in \mathcal{F}_{k+1}} B' \cup \bigcup_{i \in I} 6B_i \right).
\end{equation}

Then,
\begin{equation}
\int_B |\nabla g|^4 g^{\alpha-2} \psi \, dx \gtrsim A^{\alpha+2} \mu(B).
\end{equation}

**Proof.** Denote by $x_B$ the center of $B$ and let
\[ \varphi_B(y) = \varphi\left(\frac{y - x_B}{2r(B)}\right), \]
where $\varphi$ is the radial $C^\infty$ function appearing in (4.10), so that $\text{supp} \varphi_B \subset B$, $\varphi \equiv 1$ in $\frac{1}{2} B$, and $\|\nabla \varphi_B\|_{\infty} \lesssim 1/r(B)$. Then we have
\[ \frac{1}{r(B)} \int_B |\nabla g| \, dx \gtrsim \int B |\nabla g \cdot \nabla \varphi_B| \, dx = \int B \varphi_B \, d\omega \geq \omega(\frac{1}{2}B) \geq \kappa_0 h(x_0)^{-1} \mu(\frac{1}{2}B), \]
taking into account (5.21) for the last inequality.

Next we will show that $\frac{1}{r(B)} \int_{\tilde{B} \setminus B} |\nabla g| \, dx$ is small if the parameters $\varepsilon_0$ and $\varepsilon_1$ above are small.

To this end, given any ball $B'$ intersecting $B$ and centered at $x_{B'} \in B_0 \cap \partial\Omega$ with radius $r(B') \in [d(x_{B'}), r(B)]$, we write:
\[ \int_{B \cap B'} |\nabla g| \, dx \lesssim r(B')^n \sup_{2B'} g(x), \]
applying Hölder’s inequality and Cacciopoli’s inequality. By (2.3), the last supremum can be bounded above by $\omega(16B') r(B')^{1-n}$, and then we get
\begin{equation}
\int_{B \cap B'} |\nabla g| \, dx \lesssim r(B') \omega(16B').
\end{equation}
We split
\[
\int_{B^c \setminus B} |\nabla g| \, dx \leq \int_{B^c \cup \bigcup_{B' \in \mathcal{F}_{k+1}} B'} |\nabla g| \, dx + \int_{B^c \cup \bigcup_{i \in I \setminus I_b} 6B_i} |\nabla g| \, dx + \int_{B^c \cup \bigcup_{i \in I_b} 6B_i} |\nabla g| \, dx.
\]

To deal with the first integral on the right hand side, recall that the balls $16B'$, with $B' \in \mathcal{F}_{k+1}$, are disjoint and their radius is at most $\varepsilon_0 r(B)$, by construction. Thus,
\[
\frac{1}{r(B)} \int_{B^c \cup \bigcup_{B' \in \mathcal{F}_{k+1}} B'} |\nabla g| \, dx \lesssim \frac{1}{r(B)} \sum_{B' \in \mathcal{F}_{k+1}} r(B') \omega(16B') \lesssim \varepsilon_0 \omega(2B).
\]

The second integral on the right hand side of (5.26) is estimated analogously. In this case we use that all the balls $B_i$ with $i \in I \setminus I_b$ have radius equal to $r(B)$, and that the balls $96B_i$, $i \in I \setminus I_b$, have bounded overlap, by Lemma 4.3(d). Then, by (5.25), we deduce
\[
\frac{1}{r(B)} \int_{B^c \cup \bigcup_{i \in I \setminus I_b} 6B_i} |\nabla g| \, dx \lesssim \frac{1}{r(B)} \sum_{i \in I \setminus I_b} r(B_i) \omega(96B_i) \lesssim \varepsilon_0 \omega(2B).
\]

Regarding the last integral on the right hand side of (5.26), we will use the fact that
\[
\mu \left( \bigcup_{i \in I_b, 6B_i \cap B \neq \emptyset} 960B_i \right) \leq \varepsilon_1 \mu(B),
\]

because $B$ is assumed to be not useless. Note that given $i \in I_b$ such that $B \cap 6B_i \neq \emptyset$, we have $r(6B_i) \leq r(B)$. Otherwise, $B \subset 18B_i$, which violates the condition (5.27). Then, applying (5.25) again, we deduce
\[
\frac{1}{r(B)} \int_{B^c \cup \bigcup_{i \in I_b} 6B_i} |\nabla g| \, dx \lesssim \frac{1}{r(B)} \sum_{i \in I_b, 6B_i \cap B \neq \emptyset} r(B_i) \omega(96B_i) \lesssim \sum_{i \in I_b, 6B_i \cap B \neq \emptyset} \omega(96B_i) \lesssim A \sum_{i \in I_b, 6B_i \cap B \neq \emptyset} \mu(960B_i) \lesssim A \mu \left( \bigcup_{i \in I_b, 6B_i \cap B \neq \emptyset} 960B_i \right) \lesssim A \varepsilon_1 \mu(2B),
\]

by the finite superposition of the balls $960B_i$ and (5.27).

We take into account now that, by (5.22), the $s$-growth of $\mu$, and (5.19),
\[
\omega(2B) \lesssim h(x_0)^{-1} r(2B)^s \approx h(x_0)^{-1} r(\frac{1}{2}B)^s \lesssim h(x_0)^{-1} \mu(\frac{1}{2}B).
\]

Therefore,
\[
\frac{1}{r(B)} \int_{B^c \cup \bigcup_{B' \in \mathcal{F}_{k+1}} B'} |\nabla g| \, dx + \frac{1}{r(B)} \int_{B^c \cup \bigcup_{i \in I \setminus I_b} 6B_i} |\nabla g| \, dx \lesssim \varepsilon_0 h(x_0)^{-1} \mu(\frac{1}{2}B).
\]

Then, using also that $\mu(\frac{1}{2}B) \gtrsim r^s \gtrsim \mu(2B)$, $h(x_0)^{-1} \approx A$, and recalling the splitting (5.26), we deduce
\[
\frac{1}{r(B)} \int_{B \setminus \tilde{B}} |\nabla g| \, dx \geq (\kappa_0 - C\varepsilon_0 - C\varepsilon_1) h(x_0)^{-1} \mu(\frac{1}{2}B)
\geq \frac{\kappa_0}{2} h(x_0)^{-1} \mu(\frac{1}{2}B) \approx \kappa_0 A \mu(\frac{1}{2}B) \approx \kappa_0 A \mu(B),
\]

assuming $\varepsilon_0$ and $\varepsilon_1$ small enough.
Next, applying Hölder’s inequality, we get
\[
\int_{B \setminus \tilde{B}} |\nabla g| \, dx \leq \left( \int_{B \setminus \tilde{B}} |\nabla g|^4 \, dx \right)^{1/4} \left( \int_{\tilde{B}} g^{(2-\alpha)/3} \, dx \right)^{3/4}.
\]
To estimate the last integral on the right hand side we take into account that, by (2.3) and (5.22),
\[
g(x) \lesssim \frac{\omega(8B)}{r(B)^{n-1}} \lesssim A \frac{\mu(80B)}{r(B)^{n-1}} \approx A \frac{\mu(B)}{r(B)^{n-1}}
\]
for all \( x \in B \). Hence,
\[
\int_B g^{(2-\alpha)/3} \, dx \lesssim A^{(2-\alpha)/3} \mu(B)^{2-\alpha/3} r(B)^{n+1-(2-\alpha)(n-1)/3}.
\]
Therefore,
\[
\int_{B \setminus \tilde{B}} |\nabla g|^4 \, dx \geq \left( \int_{B \setminus \tilde{B}} |\nabla g| \, dx \right)^4 \left( A^{(2-\alpha)/3} \mu(B)^{(2-\alpha)/3} r(B)^{n+1-(2-\alpha)(n-1)/3} \right)^{-3}
\]
\[
\geq \left( \kappa_0 A r(B) \mu(B) \right)^4 \left( A^{(2-\alpha)/3} \mu(B)^{(2-\alpha)/3} r(B)^{n+1-(2-\alpha)(n-1)/3} \right)^{-3}
\]
\[
= \kappa_0^4 A^4 \mu(B) \left( \frac{A \mu(B)}{r(B)^3} \right)^{1+\alpha} \approx \kappa_0 A^{2+\alpha} \mu(B).
\]
To finish the proof of the lemma it just remains to notice that \( \kappa_0 \) is some absolute constant depending just on \( n \) and that \( \psi = 1 \) on \( B \setminus \tilde{B} \).

Now we are ready to obtain the lower estimate for \( I_0 \) required to complete the proof of Theorem 1.1. Let \( N > 1 \) be an arbitrarily large integer, and choose \( \rho_0 \in (0, \frac{1}{2} \eta_{N+1}] \) such that \( \rho_0 \ll \rho_1 \) too. Let \( k_0 \) be the minimal integer such that \( 2 \eta_{k_0} \leq \rho_1 \). If \( B \in \mathcal{F}_k \) with \( k \in [k_0, N] \), then \( r(B) \leq \rho_1 \) by construction and so \( B \notin \text{Uss} \) (since \( B \) is centered in \( U(\rho_1)^\circ \)). Then we can apply Lemma 5.2 to deduce that
\[
\int_{\tilde{B}} |\nabla g|^4 \, dx \geq A^{\alpha+2} \mu(B),
\]
with \( \tilde{B} \) defined in (5.23). Then we get
\[
\int |\nabla g|^4 \, dx \geq \sum_{k=k_0}^N \sum_{B \in \mathcal{F}_k} \int_B |\nabla g|^4 \, dx \geq \sum_{k=k_0}^N \sum_{B \in \mathcal{F}_k} A^{\alpha+2} \mu(B),
\]
using the fact that the regions \( \tilde{B} \) above do not overlap. Now, by the doubling property of the balls from \( \mathcal{F}_k \) and (5.20), for each \( k \) we have
\[
\sum_{B \in \mathcal{F}_k} \mu(B) \approx \sum_{B \in \mathcal{F}_k} \mu(80B) \geq \frac{1}{2} \mu(G) \geq \frac{1}{8} \mu(\frac{1}{2} B_0).
\]
Thus,
\[
I_0 = \int |\nabla g|^4 \, dx \geq (N - k_0) A^{\alpha+2} \mu(\frac{1}{2} B_0).
\]
Taking \( N \) big enough, we contradict (5.13), as wished.
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REFERENCES

[AHM³TV] J. Azzam, S. Hofmann, J.M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg. Rectifiability of harmonic measure. Geom. Funct. Anal. (GAFA), 26 (2016), no. 3, 703–728.

[AGMT] J. Azzam, J. Garnett, M. Mourgoglou and X. Tolsa. Uniform rectifiability, elliptic measure, square functions, and ε-approximability via an ACF monotonicity formula. Preprint arXiv:1612.02650 (2016).

[AM] J. Azzam and M. Mourgoglou. Tangent measures and absolute continuity of harmonic measure. Vol. 34(1), 2018, 305–330.

[Az1] J. Azzam. Semi-uniform domains and a characterization of $A_{\infty}$ property for harmonic measure. Preprint arXiv:1711.03088v3 (2017).

[Az2] J. Azzam. Dimension drop for harmonic measure on Ahlfors regular boundaries. Preprint arXiv:1811.03769 (2018).

[Ba1] A. Batakis. Harmonic measure of some Cantor type sets. Ann. Acad. Sci. Fenn. Math. 21(2):255-270 (1996).

[Ba2] A. Batakis. Dimension of the harmonic measure of non-homogeneous Cantor sets. Ann. Inst. Fourier (Grenoble) 56(6):1617-1631 (2006).

[Bo] J. Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. Invent. Math. 87 (1987), no. 3, 477–483.

[GMT] J. Garnett, M. Mourgoglou, and X. Tolsa. Uniform rectifiability in terms of Carleson measure estimates and ε-approximability of bounded harmonic functions. To appear in Duke Math. J. arXiv:1611.00264 (2016).

[HLMN] S. Hofmann, P. Le, J. M. Martell and K. Nyström. The weak-$A_{\infty}$ property of harmonic and $p$-harmonic measures implies uniform rectifiability. Anal. PDE 10 (2017), no. 3, 653–694.

[HM1] S. Hofmann and J.M. Martell. Uniform Rectifiability and Harmonic Measure I: Uniform rectifiability implies Poisson kernels in $L^p$, Ann. Sci. École Norm. Sup. 47 (2014), no. 3, 577–654.

[HM2] S. Hofmann and J.M. Martell. A sufficient geometric criterion for quantitative absolute continuity of harmonic measure. Preprint arXiv:1712.03696v1 (2017).

[HMM] S. Hofmann, J.M. Martell, and S. Mayboroda. Uniform rectifiability, Carleson measure estimates, and approximation of harmonic functions. Duke Math. J. 165 (2016), no. 12, 2331–2389.

[HMU] S. Hofmann, J.M. Martell and I. Uriarte-Tuero. Uniform rectifiability and harmonic measure, II: Poisson kernels in $L^p$ imply uniform rectifiability. Duke Math. J. 163 (2014), no. 8, p. 1601–1654.

[Jo] P.W. Jones. On scaling properties of harmonic measure. Perspectives in analysis, 73?81, Math. Phys. Stud., 27, Springer, Berlin, 2005.

[JW] P. Jones and T. Wolff. Hausdorff dimension of harmonic measures in the plane. Acta Math. 161 (1988), no. 1-2, 131–144.

[LV] J. L. Lewis and A. Vogel. Symmetry theorems and uniform rectifiability. Boundary Value Problems Vol. 2007 (2007), article ID 030190, 59 pages.

[Mak1] N.G. Makarov. On the distortion of boundary sets under conformal mappings. Proc. London Math. Soc. (3) 51 (1985), no. 2, 369-384.

[Mak2] N.G. Makarov. Harmonic measure and the Hausdorff measure. (Russian) Dokl. Akad. Nauk SSSR 280 (1985), no. 3, 545-548.

[Mat] P. Mattila. Geometry of sets and measures in Euclidean spaces. Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.

[MT] M. Mourgoglou and X. Tolsa. Harmonic measure and Riesz transform in uniform and general domains. To appear in J. Reine Angew. Math. arXiv:1509.08386 (2015).

[RR] F. and M. Riesz. Über die randwerte einer analtischen funktion. Compte Rendues du Quatrième Congrès des Mathématiciens Scandinaves, Stockholm 1916, Almqvists and Wilkels, Upsala, 1920.

[To] X. Tolsa. Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory. Birkhäuser, 2014.

[Wo1] T.H. Wolff. Plane harmonic measures live on sets of σ-finite length. Ark. Mat. 31 (1993), no. 1, 137–172.

[Wo2] T. Wolff. Counterexamples with harmonic gradients in $\mathbb{R}^3$. In: Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton Math. Ser. 42, Princeton Univ. Press, 321–384 (1995).
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