On wall-crossing invariance of certain sums of Welschinger numbers

S. Finashin1 · V. Kharlamov2

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Abstract
We continue our quest for real enumerative invariants not sensitive to changing the real structure and extend the construction we uncovered previously for counting curves of anti-canonical degree \( \leq 2 \) on del Pezzo surfaces with \( K^2 = 1 \) to curves of any anti-canonical degree and on any del Pezzo surfaces of degree \( K^2 \leq 3 \).

Keywords Real del Pezzo surfaces · Pin-structures · Real enumerative geometry · Welschinger invariants · Wall-crossing

Mathematics Subject Classification Primary 14N10; Secondary 14P25 · 14N15 · 53D45

The obvious answer is always overlooked.
Known as Whitehead’s Law

1 Introduction

1.1 Problem formulation

Up to 2000 s it was a rather common opinion that there can not exist any reasonable integer valued enumerative geometry over the reals, and that it is a prerogative of geometry over the field of complex numbers and other algebraically closed fields. The situation is radically changed in 2003, when J.-Y. Welschinger invented an integer signed count of point-constrained real rational curves on real rational surfaces that respects the necessary invariance property to be preserved under equivariant deformations and equivariant isomorphisms.
Those Welschinger invariants, like most of their subsequent analogs, turned out to be sensitive to changes that a real structure experiences under wall-crossing. However, quite soon after Welschinger’s discovery, some examples appeared where under an appropriate counting scheme the invariance under wall-crossing holds. Initially, it was a signed count of real lines on real projective hypersurfaces (see [11, 22, 23, 26]), which includes as the starting case the count of real lines on real cubic surfaces in real projective 3-space.

It is the latter signed count on cubic surfaces that was recently extended by us to counting curves of anticanonical degree 1 and 2 on del Pezzo surfaces of degree $K^2 = 1$ (see [12, 13]). The invariance under wall-crossing was achieved there by combining the original Welschinger invariants with a certain intrinsic Pin$^-$-structure attributed to real loci of these surfaces.

This led us to a general question: What kind of real rational surfaces $X$ carry similar Pin$^-$-structures on $X_{\mathbb{R}}$ and what are (anticanonical) degrees for which the real rational curves in $X$ can be counted invariantly under wall-crossing?

In this paper we answer this question for del Pezzo surfaces of degree $K^2 \leq 3$.

1.2 Main results

Let $X$ be a real del Pezzo surface of degree $K^2 \leq 3$ and $\text{conj} : X \to X$ the complex conjugation. Denote by $\text{Eff}(X) \subset \text{Pic}(X) = H_2(X)$ the semigroup of effective divisor classes, and put $\text{Eff}_{\mathbb{R}}(X) = \text{Eff}(X) \cap \ker(1 + \text{conj}_*)$. If $X_{\mathbb{R}} \neq \emptyset$ (which is always the case if $K^2$ is odd), the latter semigroup coincides with the semigroup of divisor classes that can be realized by a real effective divisor.

For del Pezzo surfaces, $\text{Eff}(X)$ and $\text{Eff}_{\mathbb{R}}(X)$ are both finitely generated. For $K^2 = 2$ and 3, the semigroup $\text{Eff}(X)$ is generated by lines (which are, by definition, embedded $(-1)$-curves of genus 0), while for $K^2 = 1$ it has one additional generator, $-K$.

We split both $\text{Eff}(X)$ and $\text{Eff}_{\mathbb{R}}(X)$, into subsets called layers:

$$
\mathcal{L}^m(X) = \{ \alpha \in \text{Eff}(X) \mid -\alpha K = m \}, \quad \mathcal{L}_{\mathbb{R}}^m(X) = \{ \alpha \in \text{Eff}_{\mathbb{R}}(X) \mid -\alpha K = m \}.
$$

All the layers are finite sets. They are empty if $m$ is not a positive integer.

For each pair of integers $m, k$ with $0 \leq k \leq m - 1$, $k = m - 1 \mod 2$, and a collection $\mathbf{x} \subset X$ of $m - 1$ points including $k$ real ones and $\frac{1}{2}(m - k - 1)$ pairs of complex conjugate imaginary points, we consider the set $\mathcal{C}_{\mathbb{R}}(m, k, \mathbf{x})$ of real rational irreducible reduced curves $A$, $[A] \in \mathcal{L}_{\mathbb{R}}^m(X), \mathbf{x} \subset A$ that pass through $\mathbf{x}$. The sets $\mathcal{C}_{\mathbb{R}}(m, k, \mathbf{x})$ are all finite for a generic choice of $\mathbf{x}$.

The main object of this paper is the following double sequence of integers,

$$
N_{m, k} = \sum_{A \in \mathcal{C}_{\mathbb{R}}(m, k, \mathbf{x})} i\hat{q}([A]) - m^2 w(A), \quad (1.2.1)
$$

where $\hat{q} : \text{Eff}_{\mathbb{R}}(X) \to \mathbb{Z}/4$ is a certain quadratic function that $X$ inherits from its natural embeddings in an appropriate 3-fold (see Sect. 2) and $w(A)$ is the modified
Welschinger number \( w(A) = (-1)^c_A \) in which \( c_A \) denotes the number of cross-point real nodes in the real locus of \( A \).

As is known, for any \( \alpha \in \mathcal{L}^m_{\mathbb{R}}(X) \) a partial sum \( W_{\alpha,k} = \sum_{A \in \mathcal{C}_2(m,k,x), [A]=\alpha} w(A) \), which we call the modified Welschinger invariant, does not depend on a generic choice of \( x \) (see [2]) and is invariant under real deformations of \( X \) (see [27]). The same invariance of \( N_{m,k} \) now follows from the real deformation invariance of \( \tilde{q} \) (see Sect. 2).

Some precaution must be however taken in the case \( K^2 = 1, m = 1 \). Namely, to have such invariance for numbers \( W_{-K,0} \) and thus, for \( N_{1,0} \), we need, in addition to genericness of \( x \), to assume also genericness of \( X \) (see [17]).

Recall that over \( \mathbb{C} \) all del Pezzo surfaces with a fixed \( K^2 \) are deformation equivalent to each other, while over \( \mathbb{R} \) two real del Pezzo surfaces having the same \( K^2 \) are real deformation equivalent if, and only if, their real structures conj : \( X \to X \) are diffeomorphic.

The goal of this paper is to prove that the sequence \( N_{m,k} \) has the following strong invariance property.

**Theorem 1.2.1** The double sequence \( N_{m,k} \) is the same for all real del Pezzo surfaces \( X \) with \( X_{\mathbb{R}} \neq \emptyset \) having a given degree \( K^2 \leq 3 \).

Since for any \( k \geq 2 \) and any nonsingular real rational surface \( X \) with disconnected real locus \( X_{\mathbb{R}} \) the Welschinger invariants \( W_{\alpha,k} \) vanish (see [2]), and since for any \( K^2 \in \{1, 2, 3\} \) there exist real del Pezzo surfaces with disconnected \( X_{\mathbb{R}} \), Theorem 1.2.1 implies the following vanishing for \( N_{m,k} \).

**Corollary 1.2.2** For all real del Pezzo surfaces as in Theorem 1.2.1, each of the numbers \( N_{m,k} \) with \( k \geq 2 \) is equal to 0.

The numbers \( N_{m,k} \) happen to have remarkable properties. In particular, for \( k = 1 \) they turn out to be related in a “magic way” with Gromov–Witten invariants.

**Theorem 1.2.3** For any \( m \in \mathbb{Z}_{\geq 1} \) and any real surface \( X \) as in Theorem 1.2.1,

\[
N_{2m,1} = 2^{m-3} \sum_{\alpha \in \mathcal{L}^m(X)} (e\alpha)^2 GW_{\alpha}
\]

(1.2.2)

where \( e \) is an arbitrary class \( e \in K^1_{\mathbb{R}} \) with \( e^2 = -2 \) and \( GW_{\alpha} \) states for the Gromov–Witten invariant which counts rational irreducible reduced curves in class \( \alpha \in \mathcal{L}^m(X) \) passing through a generic collection of \( m - 1 \) points.

Besides giving a way to calculate the numbers \( N_{2m,1} \), this result plays a crucial role in deriving simple recursive formulas governing the both sequences, \( N_{2m,1} \) and \( N_{2m+1,0} \).
Theorem 1.2.4 For any $n \in \mathbb{Z}_{\geq 1}$ and any real surface $X$ as in Theorem 1.2.1,

$$mK^2 N_{m,0} = 2 \sum_{j=1}^{n} \binom{n-1}{n-j} j(m-2j)^2 N_{m-2j,0} N_{2j,1}, \text{ for } m = 2n + 1,$$

$$mK^2 N_{m,1} = 2 \sum_{j=1}^{n} \binom{n-1}{n-j} j(m-2j)^2 N_{m-2j,1} N_{2j,1}, \text{ for } m = 2n + 2.$$ 

These recursive relations allow to reconstruct the numbers $N_{m,k}$ from the initial values (see Sect. 5.5) and to observe their non-vanishing and positivity (except the case of $N_{2n+1,0} = 0$ for $K^2 = 2$).

Remark 1.2.5 In the case $K^2 = 2$, the numbers $N_{m,k}$ vanish for odd $m$ (see Proposition 2.1.2), and so in this case the first formula in Theorem 1.2.4 holds for trivial reasons.

In Theorem 5.4.1, we solve the above recurrence relations and get the following explicit formulas:

$$N_{2n+1,0} = \frac{1}{4} N_{1,0} b^n \left( n + \frac{1}{2} \right)^{n-2}, \quad N_{2n+2,1} = N_{2,1} b^n (n + 1)^{n-2}, \quad b = \frac{4N_{2,1}}{K^2}.$$ 

As an immediate consequence of these expressions, we evaluate the growth rate of the sequences $N_{2n+1,0}$ and $N_{2n+2,1}$ and compare it with the growth rate of an analogous sequence of Gromov–Witten numbers (see Corollary 6.2.2).

Remark 1.2.6 In the case of $X$ with $K^2 = 2$ and $X_{\mathbb{R}} = \emptyset$, one can prove that $C_{\mathbb{R}}(m, k, x) = \emptyset$ for any $m, k$, and so, for such $X$, all the numbers $N_{m,k}$ vanish. This shows that Theorem 1.2.1 can not be extended to this case.

The numbers $N_{m,k}$ fit perfectly into the general theory of open Gromov–Witten invariants as constructed by J. Solomon in a much more general setting (see [14, 24]). The heart of his construction is a choice of an arbitrary Pin$^{-}$-structure for orienting (relatively) the evaluation maps, and its ultimate result is a system of WDVV-like equations. The main novelty in our counting is a choice of the canonical Pin$^{-}$-structures and taking a combined count over all curves of a fixed canonical degree. Our sign-correction of Welschinger invariants is essentially the same as Solomon’s (see Sect. 4.1). And initially we tried to deduce from Solomon’s equations both wall-crossing invariance of our numbers and recursive formulas for them, using vanishing of our Pin$^{-}$-structures on real vanishing cycles, but our attempts failed. Instead, we separated these two tasks and deduced the invariance from a wall-crossing formula (3.1.2) established by Brugallé [2] on the base of a Brugallé–Puignau [3] real version of Abramovich-Bertram-Vakil formula. Having this invariance at hands we could apply Solomon’s equations together with the ”magic formula” of Theorem 1.2.3 to derive the recursive formulas of Theorem 1.2.4.
1.3 Plan of the paper

Section 2 starts from constructing of natural, basic for our counting scheme, \( \text{Pin}^- \)-structures and establishing their main properties. We conclude this preliminary section by a discussion of wall-crossing and precise the limit behavior of the curves involved into counting of \( N_{2m,1} \) in the case of contracting a spherical component of \( X_\mathbb{R} \). Section 3 is devoted to a proof of the central result, Theorem 1.2.1. Theorem 1.2.3 is proved in Sect. 1.2.3. In Sect. 5 we prove Theorems 1.2.4, 5.4.1 and discuss a few simple concrete applications. In Sect. 6 we are making few remarks on generating functions, on comparison of our count with a similar count for Gromov–Witten invariants, and on a situation with other, \( K^2 > 3 \), del Pezzo surfaces.

2 Preliminaries

From now on, we denote the anticanonical degree \( K^2 \) by \( d \).

2.1 Basic Pin\(^-\) structures

Recall that each del Pezzo surface \( X \) of degree \( d = 3 \) has a natural anticanonical embedding \( X \hookrightarrow \mathbb{P}^3 \) representing it as a non-singular cubic surface. For del Pezzo surfaces of degree \( d = 2 \), the anticanonical map is a double covering \( X \to \mathbb{P}^2 \) branched along a non-singular quartic curve, and its deck transformation \( \gamma : X \to X \) is called Geiser involution. This covering lifts naturally to an embedding \( X \hookrightarrow \mathbb{P}(1, 1, 1, 2) \) into the weighted projective space \( \mathbb{P}(1, 1, 1, 2) \). For del Pezzo surfaces of degree \( d = 1 \), the bi-anticanonical map \( X \to \mathbb{P}^3 \) represents \( X \) as a double covering of a non-degenerate quadratic cone \( Q \subset \mathbb{P}^3 \) branched at its vertex and along a non-singular sextic \( C \subset Q \) (traced on \( Q \) by a transversal cubic surface). The deck transformation \( \tau : X \to X \) in this case is known as the Bertini involution. The above covering \( X \to Q \) lifts to an embedding \( X \hookrightarrow \mathbb{P}(1, 1, 2, 3) \) provided by the graded anti-canonical ring \( R = \sum_{m \geq 0} H^0(X; -mK) \). In each case \( d = 1, 2, 3 \), these embeddings of \( X \) are defined uniquely up to automorphisms of the ambient 3-space (see, for example, [10]).

These constructions are exhaustive, functorial, and work equally well over \( \mathbb{C} \) and \( \mathbb{R} \). In particular, over \( \mathbb{R} \) we obtain a natural embedding of the real locus \( X_\mathbb{R} \) into \( \mathbb{R}\mathbb{P}^3 \) if \( d = 3 \), into \( \mathbb{R}\mathbb{P}^2 \times \mathbb{R} = \mathbb{P}_\mathbb{R}(1, 1, 1, 2) \setminus v_q, v_q = (0, 0, 0, 1) \) if \( d = 2 \), and into \( \mathbb{R}\mathbb{P}^2 \times \mathbb{R} = \mathbb{P}_\mathbb{R}(1, 1, 2, 3) \setminus v_p, v_p = (0, 0, 1, 0) \) if \( d = 1 \) (see [12]).

Therefore, in each case \( d = 1, 2, 3 \), the real locus \( X_\mathbb{R} \) inherits from the ambient 3-space some natural \( \text{Pin}^- \)-structures. Namely, if \( d = 3 \), we have a pair of Spin-structures in \( \mathbb{R}\mathbb{P}^3 \) which differ by a shift by \( h \in H^1(\mathbb{R}\mathbb{P}^3; \mathbb{Z}/2), h \neq 0 \). They induce on \( X_\mathbb{R} \subset \mathbb{R}\mathbb{P}^3 \) a pair of \( \text{Pin}^- \)-structures, \( \theta^X \) and \( \theta^X + w_1 \), where \( w_1 = w_1(X_\mathbb{R}) \) is the pull-back of \( h \). If \( d = 1 \) or 2, the above embeddings \( X_\mathbb{R} \subset \mathbb{R}\mathbb{P}^2 \times \mathbb{R} \) are two-sided, and thus a pair of \( \text{Pin}^- \)-structures on \( \mathbb{R}\mathbb{P}^2 \times \mathbb{R} \) (which differ by a generator in \( H^1(\mathbb{R}\mathbb{P}^2 \times \mathbb{R}; \mathbb{Z}/2) \)) descends to a pair of \( \text{Pin}^- \)-structures on \( X_\mathbb{R} \), also denoted \( \theta^X \) and \( \theta^X + w_1 \) (since also differ by a shift by \( w_1 = w_1(X_\mathbb{R}) \)).
Recall that there is a canonical correspondence between $\text{Pin}^-\text{-structures}$ $\theta$ on $X_{\mathbb{R}}$ and quadratic functions $q_{\theta} : H_1(X_{\mathbb{R}}; \mathbb{Z}/2) \to \mathbb{Z}/4$, that is the functions satisfying $q_{\theta}(x + y) = q_{\theta}(x) + q_{\theta}(y) + 2(x, y)$ mod $4$. These functions can be viewed as $\mathbb{Z}/4$-liftings of $w_1$ seen as a homomorphism $w_1 : H_1(X; \mathbb{Z}/2) \to \mathbb{Z}/2$. In particular, it implies that $q_{\theta} + w_1 = -q_{\theta}$ (cf. [12, Lemma 2.4.1]).

Thus, in the case $d = 1$ or $3$, we distinguish the $\text{Pin}^-\text{-structure}$ $\theta^X$ from $\theta^X + w_1$ by requiring that $\theta^X$ is monic that is $q_{\theta^X}(w_1^*) = 1$, where $w_1^* \in H_1(X_{\mathbb{R}}; \mathbb{Z}/2)$ is dual to $w_1$ (then $q_{\theta^X} + w_1 = -q_{\theta^X}$ takes on $w_1^*$ value $-1$). We call such $\theta^X$ basic $\text{Pin}^-\text{-structure}$ on $X_{\mathbb{R}}$. In the case $d = 2$, there is no natural way to distinguish $\theta^X$ from $\theta^X + w_1$ and we call basic both of them. Note that our definition of basic structures is independent of the choice of a graded anticanonical embedding of $X_{\mathbb{R}}$.

As in [12], we consider also the function

$$\hat{q}_{\theta} : H_2^-(X) \to \mathbb{Z}/4, \quad \hat{q}_{\theta} = q_{\theta} \circ \Upsilon$$

where $H_2^-(X) = \ker(1 + \text{conj}_*) : H_2(X) \to H_2(X))$ and $\Upsilon : H_2^-(X) \to H_1(X_{\mathbb{R}}; \mathbb{Z}/2)$ is the Viro homomorphism (see [8, Chapter 1] or [12]). This homomorphism respects the intersection form, so that $\hat{q}_{\theta}$ inherits from $q_{\theta}$ the property

$$\hat{q}_{\theta}(x + y) = \hat{q}_{\theta}(x) + \hat{q}_{\theta}(y) + 2(x, y) \mod 4.$$ 

The Viro homomorphism $\Upsilon$ induces an isomorphism

$$H_1(X_{\mathbb{R}}; \mathbb{Z}/2) \cong H_2^-(X)/(1 - \text{conj}_*) H_2(X),$$

and has a simple interpretation in differential topology setting. Namely, for any real del Pezzo surface (and, more generally, for any compact complex surface $X$ with a real structure and $H_1(X; \mathbb{Z}/2) = 0$) each class $\alpha \in H_2^-(X)$ can be realized by a conj-invariant smoothly embedded oriented 2-manifold $F \subset X$ and then it is $F \cap X_{\mathbb{R}}$, which is a collection (may be empty) of smooth circles in $X_{\mathbb{R}}$, that realizes the class $\Upsilon(\alpha)$. Furthermore, if $\Upsilon(\alpha) = 0$ then $\alpha$ can be represented by a conj-invariant smoothly embedded oriented 2-manifold disjoint from $X_{\mathbb{R}}$, and if $\Upsilon(\alpha) \neq 0$ then $F \subset X$ can be chosen in such a way that each of the components of $F \cap X_{\mathbb{R}}$ represents a non-zero element in $H_1(X_{\mathbb{R}}; \mathbb{Z}/2)$.

Note also that $\Upsilon$ maps $c_1^*(X)$ to $w_1^*(X_{\mathbb{R}})$, so that with our choice of basic $\text{Pin}^-\text{-structures}$ we get the relation

$$\hat{q}_{\theta}(K) = q_{\theta}(w_1^*) = 1 \text{ for any real del Pezzo surfaces of degree } 1 \text{ and } 3.$$ 

**Theorem 2.1.1**  **The basic $\text{Pin}^-\text{-structures}$ have the following properties:**

1. Real automorphisms and real deformations preserve the basic $\text{Pin}^-\text{-structure}$ (resp. the pair of basic $\text{Pin}^-\text{-structures}$) of del Pezzo surfaces of degree 1 and 3 (resp. of degree 2).
2. The quadratic functions $q_{\theta^X}$ of basic $\text{Pin}^-\text{-structures}$ $\theta^X$ vanish on each real vanishing cycle in $H_1(X_{\mathbb{R}}; \mathbb{Z}/2)$.
Proof (1) holds because real automorphisms and real deformations preserve \(w_1\) and any real automorphism of \(X\) is induced by a real automorphism of the ambient 3-space. Property (2) is a special case of [12, Lemma 2.4.2]. 

**Proposition 2.1.2** For \(d = 2\) and any of the two basic \(\text{Pin}^-\)-structures \(\theta^X\), the following holds:

1. The functions \(\hat{q}\) and \(q\) associated with \(\theta^X\) are skew-symmetric with respect to \(\gamma\) and its restriction \(\gamma_R : X_R \to X_R\), correspondingly. That is
   \[
   q \circ (\gamma_R)_* = -q, \quad \hat{q} \circ (\gamma)_* = -\hat{q}.
   \]

2. For each pair \((m, k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}, k = m - 1 \mod 2\), the number \(N_{m,k}\) does not depend on the choice of \(\theta^X\). If, in addition, \(m\) is odd, then \(N_{m,k} = 0\).

3. \(\hat{q}(K) = q(w_1^*) = 0\).

**Proof** (1) The reflection map \(\mathbb{R}P^2 \times \mathbb{R} \to \mathbb{R}P^2 \times \mathbb{R}\) given by \((x, y, z, t) \mapsto (x, y, z, -t)\) shifts each of \(\text{Pin}^-\)-structures on \(\mathbb{R}P^2 \times \mathbb{R}\) by \(w_1(\mathbb{R}P^2 \times \mathbb{R})\) (cf. [18, Lemma 1.10]). Since \(\gamma_R\) is the restriction of this reflection to \(X_R\), it interchanges the two basic \(\text{Pin}^-\)-structures induced on \(X_R\), and hence transforms \(q\) into \(-q\). 

Skew-symmetry of \(\hat{q}\) follows from that of \(q\).

(2) Suppose first that \(m = 2n + 1\). Let \(x\) be a generic collection of \(k\) real points and \(\frac{1}{2}(m - k - 1)\) pairs of complex conjugate imaginary points and let \(A, [A] \in \mathcal{L}_R^m(X)\), be a real rational curve passing through \(x\). Note that \(A\) can not be \(\gamma\)-invariant, since otherwise \(\hat{q}([A]) = -\hat{q}([\gamma(A)]) = -\hat{q}([A]) \in \mathbb{Z}/4\), which implies that \(\hat{q}([A])\) is even and thus contradicts to the congruences \(\hat{q}([A]) \equiv [A]^2 \equiv -AK \equiv 2n + 1 \mod 2\). Therefore, the set of such curves \(A\) splits into a finite union of pairs, \([A, \gamma(A)]\), and there remains to notice that \(i\hat{q}([A]) - i\hat{q}([\gamma(A)]) = 0\) due to \(\hat{q}([A]) + \hat{q}([\gamma(A)]) = 0\), \(\hat{q}([A]) = \pm 1\).

In the case \(m = 2n\), it is sufficient to notice that \(\hat{q}(\alpha) = -\hat{q}(\alpha) \in \mathbb{Z}/4\) for any \(\alpha \in \mathcal{L}_R^m\), since (similar to the above) \(\hat{q}([A]) \equiv \alpha^2 \equiv -\alpha K \equiv 2n \mod 2\).

Property (3) is trivial in the case of orientable \(X_R\). Otherwise, \(-K\) can be represented by the pull-back \(L + \gamma(L)\) of a real bitangent to the quartic defining \(X\) as a double covering, and then the result follows from property (1), since \(\hat{q}(K) = \hat{q}(L + \gamma(L)) = 2(L, \gamma(L)) = 0 \mod 4\). 

**2.2 Three auxiliary surfaces**

To simplify proving Theorem 1.2.4 (see Sect. 5) we pick some particular real del Pezzo surface \(X\) for each of the degrees \(d = 1, 2, 3\).

**Proposition 2.2.1** (1) For \(d = 1\), pick \(X\) with \(X_R = \mathbb{R}P^2 \sqcup 3S^2\). Then:

- \(H_2^-(X) \cong \mathbb{Z}^2\) is generated by \(K\) and a real root vector \(e \in K^\perp\), while the first real layer \(\mathcal{L}_R^1(X)\) consists of \(-K\) and the divisor classes of 2 real lines, \(-K \pm e\).
- The Bertini involution acts on \(H_2^-(X)\) preserving the basic quadratic form \(\hat{q}\) and \(-K\), but permuting the divisor classes of lines.
(2) For \( d = 2 \), pick \( X \) with \( X_\mathbb{R} = \mathbb{K} \sqcup S^2 \) (\( \mathbb{K} \) denotes a Klein bottle). Then:

- \( H_2^-(X) \cong \mathbb{Z}^4 \) is generated by \( K \), three pairwise orthogonal real root vectors \( e_1, e_2, e_3 \in K^\perp \), and \( \frac{1}{2}(-K - e_1 - e_2 - e_3) \), while the first real layer \( \mathcal{L}_R^1(X) \) consists of divisor classes of 8 real lines, \( \frac{1}{2}(-K \pm e_1 \pm e_2 \pm e_3) \).
- \( X \) can be chosen so that its group of automorphisms contains \( \mathbb{Z}/2 \times \mathcal{D}_4 \), where \( \mathbb{Z}/2 \) is generated by the Geiser involution \( \gamma \) and \( \mathcal{D}_4 \) is the dihedral group of a square.
- \( \mathbb{Z}/2 \times \mathcal{D}_4 \) acts transitively on the above 8 lines, and, in particular, one of the \( \mathcal{D}_4 \)-orbits \( \{ \frac{1}{2}(-K - e_1 - e_2 - e_3), \frac{1}{2}(-K + e_1 + e_2 - e_3), \frac{1}{2}(-K + e_1 - e_2 + e_3), \frac{1}{2}(-K - e_1 + e_2 + e_3) \} \) is permuted by \( \gamma \)-action with the other \( \mathcal{D}_4 \)-orbit. The \( \mathcal{D}_4 \)-action preserves each of the basic quadratic functions \( \pm \hat{q} \), while \( \gamma \) acts as \((-1)^n \) on \( \hat{q} \vert \mathcal{L}_R^1 \).
- The multiples of \( K \) are the only \( \gamma \)-invariant classes and the only ones invariant under the \( \mathcal{D}_4 \)-action.

(3) For \( d = 3 \), pick \( X \) with \( X_\mathbb{R} = \mathbb{R}P^2 \sqcup S^2 \). Then:

- \( H_3^-(X) \) is generated by the divisor classes of the 3 real lines \( L_1, L_2, L_3 \subset X \).
- \( X \) can be chosen so that its automorphism group contains the symmetric group \( S_3 \) acting by permutations on the above lines.
- The \( S_3 \)-action in \( H_3^-(X) \) preserves the basic quadratic function \( \hat{q} \).
- The multiples of \( K \) are the only classes invariant under this action.

**Proof** In all 3 cases we use the 1–1 correspondence between real lines and divisor classes \( \alpha \in \mathcal{L}_R^1 \) with \( \alpha^2 = -1 \) and apply the Lefschetz fixed point theorem, \( \text{rk} \ H_3^+(X) - \text{rk} \ H_3^-(X) = \chi(X_\mathbb{R}) - 2 \), which together with \( \text{rk} \ H_2^+(X) + \text{rk} \ H_2^-(X) = \text{rk} \ H_2(X) = 10 - d \) calculates the ranks of \( H_3^\pm(X) = \ker(1 \mp \text{conj}_\alpha) \). We use also that the lattices \( H_3^\pm(X) = \ker(1 \mp \text{conj}_\alpha) \) have 2-periodic discriminant groups of rank \( \frac{1}{2}(10 - d - \text{rk} \ H^*(X_\mathbb{R})) \) (see, e.g., [8, Proposition 8.3.3]). Recall besides that both Bertini (for \( d = 1 \)) and Geiser (for \( d = 2 \)) involutions act on \( H_2(X) \) as a reflection against the line spanned by \( K \).

(1) In this case, \( K^\perp \cap H_2^- \cong \langle -2 \rangle \). The divisor classes of real lines \( \alpha \in \mathcal{L}_R^1 \) split as \( \alpha = -K \pm e \), where \( e \) is a generator of \( \langle -2 \rangle \), and the Bertini involution permutes these lines. Preserving of \( \hat{q} \) by its action is due to Theorem 2.1.1(1).

(2) In this case, \( \text{rk} \ H_2^\pm = 4 \) and \( \text{rk}(K^\perp \cap H_2^-) = 3 \). Since \( X \) is an \((M - 2)\)-surface of type \( I \), the discriminant form of \( H_2^+ \) is even 2-periodic of rank 2. Thus, we conclude that \( H_3^+ \cong D_4 \). By Nikulin’s gluing theorem [21], the discriminant group \( \text{discr}(K^\perp \cap H_2^-) \) is also 2-periodic and has rank 1 or 3 (since \( K^\perp \cap H_2^- \) is complementary to \( H_2^+ \) in \( K^\perp = E_7 \)). In the case of discriminant rank 1, the Brown invariant would be \( \pm 1 \in \mathbb{Z}/8 \), which contradicts to Brown’s congruence \( \text{Br}(\text{discr}(K^\perp \cap H_2^-)) = \sigma(K^\perp \cap H_2^-) = -3 \mod 8 \). Thus, the discriminant rank is 3, which implies \( K^\perp \cap H_2^- \cong 3A_1 \). Arithmetical description of the real lines is then straightforward.

For constructing \( X \) with the required \( \mathcal{D}_4 \)-symmetry, it is sufficient to pick a \( \mathcal{D}_4 \)-symmetric pair of conics, as shown on Fig. 1.
Note that this $D_4$-action is realized by projective transformations $\mathbb{P}^2 \to \mathbb{P}^2$, and hence has a “cylindrical” lifting to $\mathbb{P}(1, 1, 1, 2)$ which preserves $X$. Such transformations $\mathbb{P}(1, 1, 1, 2) \to \mathbb{P}(1, 1, 1, 2)$ preserve each of the two $\text{Pin}^+$-structures on $\mathbb{P}_R(1, 1, 1, 2)$ and send outward vector fields on $X_R$ to outward fields. Therefore, each of the $D_4$-symmetries preserves each of the basic $\text{Pin}^+$-structures on $X_R$ and thus preserves $\hat{q}$. Proposition 2.1.2 gives also the required skew-symmetry of the $\gamma$-action on $\hat{q}$.

(3) In this case, $\text{rk } H_2^-(X) = 3$ and its discriminant group has order 4. Since the intersection matrix $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ of the three real lines lying on $X$ (the only existing ones) has also determinant 4, these lines must generate the whole lattice $H_2^-(X)$.

An example of $X$ with a required $S_3$-action can be given by an equation of the form $x_0(x_1^2 + x_2^2 + x_3^2 - x_0^2) = \varepsilon x_1 x_2 x_3$. This action preserves $\hat{q}$ due to Theorem 2.1.1(1). The last claim about invariant classes follows from linear independence of divisor classes of the lines $L_1, L_2, L_3$ and the relation $[L_1] + [L_2] + [L_3] = -K$. \(\square\)

### 2.3 Wall-crossing

We call a complex analytic family $X(z), z \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ a Morse-Lefschetz family, if: $\mathcal{X} = \bigcup_{z} X(z)$ is a non-singular 3-fold; each $X(z)$ with $z \neq 0$ is non-singular while $X(0)$ is uninode; the projection $\pi : \mathcal{X} \to \mathbb{D}$ is a proper map and it is a submersion at every point except the nodal point of $X(0)$; at the nodal point of $X(0)$ in appropriate local coordinates $z_1, z_2, z_3$ the projection can be written as $z = \sum_i z_i^2$. If $\mathcal{X}$ is equipped with a real structure sending $X(z)$ to $X(\bar{z})$ for each $z \in \mathbb{D}$, the family is called real. Then the nodal point of $X(0)$ is real and with respect to appropriate real local coordinates at this point the map $\pi$ is given by

$$\pi(z_1, z_2, z_3) = a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2, \quad a_1, a_2, a_3 \in \mathbb{R}, \quad a_1 a_2 a_3 \neq 0. \quad (2.3.1)$$

Proposition 2.3.1 below is well known. For cubic surfaces it is straightforward. For del Pezzo surfaces with $K^2 = 1$, see, e.g., [13]. For del Pezzo surfaces with...
$K^2 = 2$, it follows similarly from their presentation as double covering branched along non-singular real plane quartic curves and the deformation classification of the latters (see, e.g., [8, A3.1.1]). To resolve the issue with non-unicity of the real structure in the double covering, it is sufficient to find a Morse-Lefschetz family $X(t)$ connecting $X_\mathbb{R}(t), t < 0$, homeomorphic to a Klein bottle with $X_\mathbb{R}(t), t > 0$, homeomorphic to a sphere. Such a one is provided by double planes branched along quartics $2y^4 - x^4 + y^2 - x^2 = t$, in affine coordinates, with $|t| < \frac{1}{8}$ (see Fig. 2).

**Proposition 2.3.1** Any pair of real del Pezzo surfaces of same degree $d \leq 3$ can be connected either by a real deformation, or a finite sequence of real Morse-Lefschetz families of del Pezzo surfaces embedded into the corresponding projective or weighted projective space. If the real locus of the both surfaces is non-empty the sequence of real Morse-Lefschetz families can be chosen not to involve surfaces with empty real locus.

Every real Morse-Lefschetz family $\pi : \mathcal{X} \to \mathbb{D}$ being restricted to the punctured upper half disc $\mathbb{D}^+ = \{ z \in \mathbb{D}, \text{Im} \, z \geq 0, \, z \neq 0 \}$ and considered as a fibration of smooth 4-manifolds is trivial. This provides a natural identification of lattices $H_2(X(t)), t \in \mathbb{R}^*, |t| < 1$, so that for these values of $t$ the involutions $c_t : H_2(X(t)) \to H_2(X(t))$ and $c_{-t} : H_2(X(-t)) \to H_2(X(-t))$ induced by the complex conjugation become related by the Picard-Lefschetz transformation:

$$c_t = s_e \circ c_{-t} \quad s_e(v) = v + (ev)e,$$

(2.3.2)

where $e \in H_2(X(t))$ is the vanishing class of the underlying nodal degeneration.

**Proposition 2.3.2** Let $\pi : \mathcal{X} \to \mathbb{D}$ be a real Morse-Lefschetz family of del Pezzo surfaces of degree $K^2 \leq 3$, embedded into the corresponding projective or weighted projective space, and let the direction of the wall-crossing be selected so that $\chi(X_\mathbb{R}(-t)) < \chi(X_\mathbb{R}(t))$ for $t > 0$. Then, $H_2^-(X(t))$ is naturally identified with $H_2^-(X(-t)) \cap e^\perp$ and with respect to this identification $\hat{q}_{\phi_X(t)}$ is the restriction of $\hat{q}_{\phi_X(-t)}$, where in the case $K^2 = 2$ we pick arbitrary Pin$^-$-structure on the ambient $\mathbb{P}(1,1,1,2)$ and choose that basic Pin$^-$-structures which are induced by means of outward vector fields.

**Proof** To identify the lattices $H_2(X(t)), t \in \mathbb{R}^*, |t| < 1$ we use, as it is described above, the trivialization of the family over the upper disc. Then, under assumption $\chi(X_\mathbb{R}(-t)) < \chi(X_\mathbb{R}(t))$ for $t > 0$, the identity $c_t = s_e \circ c_{-t}$ implies that $e \in H_2^-(X(-t))$ and $H_2^+(X(t)) = H_2^-(X(-t)) \cap e^\perp$. Also, $e \in H_2^{+}(X(t)) = \ker(1 -
This new family $\hat{Y}$ eliminated by a surgery of $F$ conjugate intersection points are of opposite intersection index and, hence, can be X surface defined a complex analytic fibered 3-fold $\hat{Y}$ contradict to smoothness of $F$ should be homologically non-trivial, while a presence of an isolated intersection point to an $S$ conjugate points, while the intersection number opposite eigen-spaces for the action of conj in $H$ $S$ on the direction of the wall crossing, $q X$ (note that $X$ from $F$ $\hat{Y}$ blow-down Lemma 2.3.3 $\hat{Y}$ up to a locally trivial conj-invariant family $F$ respecting conj-invariance, all geometric intersection points between $X$ generic conj and then take a small real perturbation of the (almost) symplectic setting. So, we endow $X$ spherical component, and let $x$ existence). $\hat{Y}$ $F$ $Q$ $\hat{Y}$ $S$, chosen conj-equivariant, is not contained in $F$ $t$ $0$, then we realize it $H_2(X(t_0)))$. Thus, we can eliminate, respecting conj-invariance, all geometric intersection points between $F(t_0)$ and $S^2(t_0)$ that might appear when $S^2(t_0) \cap X_R(t_0) \neq \emptyset$, since in such a case any 2 complex conjugate intersection points are of opposite intersection index and, hence, can be eliminated by a surgery of $F(t_0)$ along a conj-invariant path joining them along $S^2(t_0)$. As soon as $F(t_0) \cap S^2(t_0) = \emptyset$, there is no more obstruction for extending $F(t_0)$ up to a locally trivial conj-invariant family $F(t) \subset X(t)$, $t \in \mathbb{D}_R$, avoiding the node of $X(0)$. Now, the invariance of $q_{X_R(t)}(\{F(t) \cap X_R(t)\})$ follows from its continuity (note that $q^{X_R(0)}$ is well defined on $H_1$ of the smooth part of $X_R(0)$). 

Recall that untwisting of a given real Morse-Lefschetz family $X(z)$ induced by the substitution $z = t^2$ (see [13]) and followed by resolution of the nodal singularity defines a complex analytic fibered 3-fold $\hat{X} : \mathcal{X}^2 \to \mathbb{D}$ equipped with a real structure. This new family $\hat{X}(t) = \hat{\pi}^{-1}(t)$, $t \in \mathbb{D}$ is formed by $\hat{X}(t) = X(t^2)$ for $t \neq 0$ and a reducible fiber $\hat{X}(0) = \hat{X}(0) \cup \mathcal{E}$, where $\hat{X}(0)$ is the resolution of the nodal surface $X(0)$, $\mathcal{E}$ is a non-singular projective quadric surface, and their intersection $E = X(0) \cap \mathcal{E}$ can be considered at the same time as the exceptional divisor of the blow-down $\hat{X}(0) \to X(0)$ and as a hyperplane section of $\mathcal{E}$ viewed as a projective quadric.

The following lemma, which will be used in Sect. 4, requires a more flexible symplectic setting. So, we endow $\mathcal{X}$ and then $\hat{\mathcal{X}}$ with a Kähler structure $\omega$ respected by the complex conjugation conj and then take a small real perturbation of the (almost) complex structure in $\hat{\mathcal{X}}$ to make it generic in the class of admissible almost-complex structures on a symplectic fibration ($\hat{\pi} : \mathcal{X}^2 \to \mathbb{D}$, $\omega$) (see [25] for definitions and existence).

In this lemma we consider a very special type of wall-crossing, contraction of a spherical component, meaning by this that the node of $X_R(0)$ is solitary and give birth to an S2-component in $X_R(t)$, $t > 0$, or equivalently that $\mathcal{E}_R = S^2$ and $E_R = \emptyset$.

**Lemma 2.3.3** Let $\hat{\pi} : \hat{\mathcal{X}} \to \mathbb{D}$ be a described above real symplectic fibration (equipped with an admissible generic almost-complex structure) contracting a spherical component, and let $x(t) = (x(t), z_1(t), w_1(t), \ldots, z_{m-1}(t), w_{m-1}(t))$ be a generic conj-equivariant family of point-constraints such that $x(0)$ is real and belongs to an $S^2$-component in $X_R(t)$, $t > 0$.
to the sphere $S^2 = \mathcal{E}_\mathbb{R}$, while $w_1(0) = \text{conj} z_1(0), \ldots, w_{m-1}(0) = \text{conj} z_{m-1}(0)$ are imaginary and belong to $\hat{X}(0)$. Then:

1. For any selection $p_1 \in \{z_1(0), w_1(0)\}, \ldots, p_{m-1} \in \{z_{m-1}(0), w_{m-1}(0)\}$, each of $J$-holomorphic rational curves $C \subset \hat{X}(0)$ having $-KC = m$ and passing through $p_1, \ldots, p_{m-1}$ is irreducible nodal and intersects $E$ transversally. The number of such curves $C$ in a given divisor class does not depend on the selection of points $p_1, \ldots, p_{m-1}$.

2. For any continuous family $A(t) \in C_\mathbb{R}(2m, 1, x(t))$, the limit-curve $\lim_{t \to 0^+} A(t)$ is a curve-configuration

$$\bigcup \text{conj} C \cup L_1 \cup \text{conj} L_1 \cup \cdots \cup L_{n-1} \cup \text{conj} L_{n-1} \cup H_n \subset \hat{X}(0)$$

with the following properties: $C$ is a $J$-holomorphic rational curve which lies in $\hat{X}(0)$ and passes through a selection $p_1 \in \{z_1(0), w_1(0)\}, \ldots, p_{m-1} \in \{z_{m-1}(0), w_{m-1}(0)\}$; $-K \cdot C = m$; $n = C \cdot E$; $C \cap E$ consists of $n$ distinct points $q_1, \ldots, q_n$; $L_1, \ldots, L_{n-1}$ are $\mathbb{P}^1$-generators of $\mathcal{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and pass through a selection of $n-1$ points between $q_1, \ldots, q_n$; $H_n$ is a real hyperplane section of $\mathcal{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$ that passes through the remaining point and the point $x(0)$.

3. For any sufficiently small $t > 0$, passing to the limits as in item (2) establishes a bijection between the set $C_\mathbb{R}(2m, 1, x(t))$ and the set $I$ of curve-configurations described in item (2). The set $I$ is a disjoint union of $2^{m-2}$ subsets $I_{[p, \text{conj} p]}$ where $[p, \text{conj} p]$ is a pair of complex conjugate selections and $I_{[p, \text{conj} p]}$ consists of configurations with $p \subset C, \text{conj} p \subset \text{conj} C$. Each $I_{[p, \text{conj} p]}$ is itself a disjoint union of subsets $I_{[C, \text{conj} C]}$ consisting of configurations sharing the same pair $[C, \text{conj} C]$. Each $I_{[C, \text{conj} C]}$ contains $n2^{n-1}$ elements.

Proof Claim (2) and the first part of Claim (1) follow from [3, Proposition 3.7] (the option of tangency between $H$ and $E$ is excluded, since $E_\mathbb{R} = \emptyset$). Due to the transversality of curves $C$ in Claim (1), their number is nothing but a relative Gromov–Witten invariant of pair $(\hat{X}(0), E)$ (without constraints on $E$). The bijectivity in Claim (3) follows from Symplectic Sum Formula in its simplest, transversal intersection, case. The rest of Claim (3) is a straightforward consequence of bijectivity. \hfill $\square$

3 Proofs of Theorem 1.2.1

In this section we consider real del Pezzo surfaces $X$ of degree $d \leq 3$, with a basic Pin$^-$-structure $\theta^X$ (cf. Theorem 2.1.1) and the associated with it quadratic function $\tilde{q}^X : H_2^+(X) \to \mathbb{Z}/4$. In Introduction we omitted, for shortness, indicating $X$ and $\theta^X$. In this section we need to use a full notation, since $X$ will be varying. So, let us adjust our notation and rewrite our main definition (1.2.1) correspondingly:

$$N_{m,k}^X = \sum_{\alpha \in C_\mathbb{R}^m(X)} N_{\alpha,k}^X, \quad N_{\alpha,k}^X = \tilde{q}^X(\alpha) - m^2 W_{\alpha,k}^X.$$
\[ W_{\alpha,k}^X = \sum_{A \in C_{\mathbb{R}}(m,k,x), |A| = \alpha} w(A). \]

Recall that the traditional Welschinger invariants are defined similarly as

\[ \tilde{W}_{\alpha,k}^X = \sum_{A \in C_{\mathbb{R}}(m,k,x), |A| = \alpha} \tilde{w}(A), \quad \text{with} \quad \tilde{w}(A) = (-1)^{s_A} \]

where \( s_A \) stands for the number of solitary real nodal points of \( A \). Since \( w(A) = (-1)^{c_A} \) and \( c_A + s_A \) is congruent modulo 2 to the arithmetic genus \( g_a(\alpha) \), where \( \alpha = |A| \) (and \( c_A \) is the number of cross-point real nodes), we have

\[ \tilde{W}_{\alpha,k}^X = (-1)^{g_a(\alpha)} W_{\alpha,k}^X. \]

### 3.1 Push-forward formula

Assume that real del Pezzo surfaces \( X \) and \( Y \) are related by wall-crossing, and consider a real Morse-Lefschetz family \( \pi : \mathcal{X} \to \mathbb{D} \) such that \( X = X(-t_0) \) and \( Y = X(t_0) \) for a fixed \( 0 < t_0 < 1 \). Following the exposition in Sect. 2.3, we identify the lattices \( H_2(X) = H_2(Y) \) using a smooth trivialization of the fibration \( \pi : \mathcal{X} \to \mathbb{D} \) over the upper half disc \( \mathbb{D}^+ \).

We choose the direction of wall-crossing so that \( \chi(X_{\mathbb{R}}) < \chi(Y_{\mathbb{R}}) \), which implies

\[ H_2^{-}(Y) = \{ \alpha \in H_2^{-}(X) \mid \alpha e = 0 \} \quad \text{where} \quad e \in H_2^{-}(X) \quad \text{is the vanishing class}, \]

and consider the orthogonal projection

\[ p^e : H_2^{-}(X) \to H_2^{-}(Y) \otimes \mathbb{Q} \quad v \mapsto v + \frac{1}{2} (ev)e. \quad (3.1.1) \]

Our aim is to treat the push-forward by \( p^e \) of the function \( N_{-}^{X}(\alpha) : H_2^{-}(X) \to \mathbb{Z} \) defined by \( N_{-}^{X}(\alpha) = N_{-}^{X}(\alpha, k) \) if \( \alpha \) is an effective divisor class and \( N_{-}^{X}(\alpha, k) = 0 \) otherwise.

Note that, due to Theorem 2.1.1(1) and Proposition 2.1.2(2), none of the above ingredients, including \( p^e \) and \( N_{-}^{X}(\alpha) \), depends on a choice of \( t_0 \).

**Proposition 3.1.1** If \( X_{\mathbb{R}} \) and \( Y_{\mathbb{R}} \) are non-empty, then: (1) the push-forward \( p_{-}^{e}(N_{-}^{X}(\alpha)) \) is well-defined, (2) it is supported in the integer part \( H_2^{-}(Y) = H_2^{-}(Y) \otimes 1 \subset H_2^{-}(Y) \otimes \mathbb{Q} \), and (3) being restricted to this integer part, \( p_{-}^{e}(N_{-}^{X}(\alpha)) = \tilde{N}_{-}^{Y}(\alpha) \) on \( H_2^{-}(Y) \).

**Proof** By definition, \( p_{-}^{e}(N_{-}^{X}(\alpha)) = \sum_{n \in \mathbb{Z}} N_{-}^{X}(\alpha + n e) \), and (1) means finiteness of the sum. But if \( |n| \) is sufficiently large, then \( (\alpha + ne)^2 < -2 \) and thus, \( \alpha + ne \) can not be realized by a reduced irreducible rational curve, which gives \( N_{-}^{X}(\alpha + ne) = 0 \).

According to the projection formula (3.1.1), \( p_{-}^{e}(N_{-}^{X}(\alpha)) \) is supported in \( H_2^{-}(Y) \otimes \frac{1}{2} \mathbb{Z} \), and (2) means that

\[ \sum_{n \in \mathbb{Z}} N_{-}^{X}(\alpha + ne) = 0 \quad \text{for each} \quad \alpha \in H_2^{-}(X) \quad \text{with} \quad r = \alpha e \quad \text{odd.} \]
To prove that this sum is zero, we group the summands by pairs \( \alpha + ne, \alpha + (r - n)e \) and notice that the reflection \( s_e \) permutes the elements in each of these pairs. As is known (see for example [16, Theorem 4.3(1)]) the traditional Welschinger invariants are preserved under such a reflection, so that \( \tilde{\mathcal{W}}^X_{\alpha + ne, k} = \tilde{\mathcal{W}}^X_{\alpha + (r - n)e, k} \) As it follows from adjunction formula, the curves in the divisor class \( \alpha + ne \) and the curves in the divisor class \( \alpha + (r - n)e \) have the same parity of the arithmetic genus. Hence, \( \mathcal{W}^X_{\alpha + ne, k} \) and \( \mathcal{W}^X_{\alpha + (r - n)e, k} \) are also equal. Thus, there remains to notice, that

\[
\hat{\mathcal{q}}_{\theta X}(\alpha + ne) = \hat{\mathcal{q}}_{\theta X}(\alpha) + 2r(r - 2n) = -\hat{\mathcal{q}}_{\theta X}(\alpha + ne)
\]
due to \( \hat{\mathcal{q}}_{\theta X}(e) = 0 \) (see Theorem 2.1.1) and \( r = 1 \mod 2 \).

To prove (3) we apply Theorem 2.1 from [2]. According to this theorem, if \( \alpha e = 0 \), then

\[
\tilde{\mathcal{W}}^Y_{\alpha, k} = \sum_{n \in \mathbb{Z}} (-1)^n \tilde{\mathcal{W}}^X_{\alpha + n e, k}.
\]

This implies

\[
\mathcal{W}^Y_{\alpha, k} = \sum_{n \in \mathbb{Z}} \mathcal{W}^X_{\alpha + n e, k},
\]

since, due to the adjunction formula, \( g_{\alpha}(\alpha + ne) = g_{\alpha}(\alpha) + n \mod 2 \) as soon as \( \alpha e = 0 \). Finally, it is left to notice that \( \hat{\mathcal{q}}_{\theta X}(\alpha + ne) = \hat{\mathcal{q}}_{\theta X}(\alpha) \) for all \( n \in \mathbb{Z} \), and that \( \hat{\mathcal{q}}_{\theta Y}(\alpha) = \hat{\mathcal{q}}_{\theta X}(\alpha) \) due to Proposition 2.3.2. \( \square \)

### 3.2 Proof of Theorem 1.2.1

Due to deformation invariance of \( \hat{\mathcal{q}} \) (see Theorem 2.1.1), and due to independence of \( \mathcal{N}^X_k \) from a choice of a basic Pin\(^-\) structure when \( K^2 = 2 \) (see Proposition 2.1.2(2)), it is left to consider sequences of real Morse-Lefschetz families as in Proposition 2.3.1 and to apply Proposition 3.1.1 (noticing that the projection \( p^n \) preserves the layers). \( \square \)

### 4 Proof of Theorem 1.2.3

Throughout this section we fix a real del Pezzo surface \( X \) of degree \( d = K^2 \leq 3 \) and use notation \( \hat{\mathcal{q}} : H_2(X) \to \mathbb{Z}/4 \) for the quadratic function associated to a basic Pin\(^-\) structure on \( X_\mathbb{R} \).
4.1 Switch to open Gromov–Witten invariants

In the computations below we essentially rely on Solomon’s recursion relations for open Gromov–Witten invariants (see [14]). In accordance with notation in [14], for each \( m \geq 1, 0 \leq k \leq m - 1, k = m - 1 \mod 2, \) and each \( v \in K^\perp \cap \ker (1 + \conj)_* \), we put

\[
N_{m,v,k} = \sum_{A \in C_{R}(m,k,x)} i \hat{q}([A]) - m^2 w(A),
\]

\[
\Gamma_{m,v,k} = -2^{1-l} \sum_{A \in C_{R}(m,k,x)} i \hat{q}([A]) - m^2 w(A), \quad l = \frac{1}{2} (m - k - 1)
\]

(4.1.1)

where \( x \) is a generic collection of \( k \) real points and \( l = \frac{1}{2} (m - k - 1) \) pairs of complex conjugate imaginary points, and define

\[
\Gamma_{m,k} = \sum_{v \in K^\perp \cap \ker (1 + \conj)} \Gamma_{m,v,k}.
\]

Note that:

- If \(-mK - v\) is not divisible by \( d\), then the sums involved in (4.1.1) are void and the numbers \( N_{m,v,k}, \Gamma_{m,v,k} \) are zero by definition.
- Numbers \( N_{m,v,k} \) and \( \Gamma_{m,v,k} \) do not depend on a choice of \( x \) and are preserved under equivariant deformations and equivariant isomorphisms (cf. discussion in Introduction).
- In accordance with (1.2.1) and above definitions we have

\[
N_{m,k} = -2^{l-1} \sum_{v \in K^\perp \cap \ker (1 + \conj)} \Gamma_{m,v,k} = -2^{l-1} \Gamma_{m,k},
\]

(4.1.2)

while Theorem 1.2.1 implies that both \( N_{m,k} \) and \( \Gamma_{m,k} \) do not depend even on a choice of a real del Pezzo surface of given degree.
- If \( d \) is odd, we have

\[
\hat{q}([A]) - m^2 = \hat{q}(d[A]) - m^2 = \hat{q}(-mK - v) - m^2 = \hat{q}(v),
\]

so that the above definitions can be rewritten as follows

\[
N_{m,v,k} = \sum_{A \in C_{R}(m,k,x)} i \hat{q}(v) w(A),
\]

\[
\Gamma_{m,v,k} = -2^{1-l} \sum_{A \in C_{R}(m,k,x)} i \hat{q}(v) w(A), \quad l = \frac{1}{2} (m - k - 1).
\]
Theorem 1.2.4 is reformulated as

\[
2md \Gamma_{m,0} + \sum_{j=1}^{n} \left( \frac{n-1}{n-j} \right) j(m-2j)^2 \Gamma_{m-2j,0} \Gamma_{2j,1} = 0 \quad \text{for } m = 2n + 1,
\]

\[
2md \Gamma_{m,1} + \sum_{j=1}^{n} \left( \frac{n-1}{n-j} \right) j(m-2j)^2 \Gamma_{m-2j,1} \Gamma_{2j,1} = 0 \quad \text{for } m = 2n + 2.
\]

(4.1.3)

4.2 Proof of Theorem 1.2.3

Lemma 4.2.1 For any \( n \in \mathbb{N} \), we have \( n2^n = \sum_{k=0}^{n} (n-2k)^2 \binom{n}{k} \).

Proof The identities \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \), \( \sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1} \), and \( \sum_{k=0}^{n} k^2 \binom{n}{k} = (n + n^2)2^{n-2} \) imply \( \sum_{k=0}^{n} (n-2k)^2 \binom{n}{k} = n^2 2^n - 4n^2 2^{n-1} + 4(n + n^2)2^{n-2} = n2^n \).

Proof Due to Theorem 1.2.1, it is enough to check the formula (1.2.2) for one particular del Pezzo surface \( X \) of each degree \( d \leq 3 \). We make choice by picking \( X = X(t_0), 0 < t_0 \ll 1 \), from a real Morse-Lefschetz family \( X(t) \) contracting a spherical component \( S^2 \subset X_{\mathbb{R}}(t_0) \) (see Sect. 2.3).

First, we interpret the right-hand side of (1.2.2) as a weighted count of complex rational curves \( C \) on the resolution \( \hat{X}(0) \) of the nodal surface \( X(0) \) that (1) belong to the \( m \)-th level \( \mathcal{L}^m(\hat{X}(0)) \), (2) pass through a fixed generic collection \( p \) of \( m - 1 \) points, and (3) have a non-trivial intersection, \( C \cdot E > 0 \), with the \((-2)\)-curve \( E \subset \hat{X}(0) \) representing the node. Namely, we observe that, in accordance with the Abramovich-Bertram-Vakil formula (see [16, Proposition 4.1] and [3, Theorem 2.5]), the input of each of such curves \( C \) into the right-hand side is equal to \( \sum_{k=0}^{n} (n-2k)^2 \binom{n}{k} \) where \( n = C \cdot E \). Thus, in accordance with Lemma 4.2.1, the right-hand side of (1.2.2) can be seen as the weighted count of the above curves \( C \subset \hat{X}(0) \) with weights \( n2^n \).

To treat the left-hand side we consider the untwisted family \( \hat{X}(t) \) and apply Lemma 2.3.3 choosing the constraint \( x(t) = (x(t), z_1(t), w_1(t), \ldots, z_{m-1}(t), w_{m-1}(t)) \) as indicated there. First, we note that the input of each of the curves \( A(t_0) \in \mathcal{C}(2m, 1, x(t_0)) \) into \( N_{2m,1} \) is equal to 1. This is because \( \hat{q} \) vanishes on the spherical component of \( X_{\mathbb{R}}(t_0) \) containing \( x(t_0) \) and curves \( A_{\mathbb{R}}(t) \) have no real cross-point nodes for all sufficiently small \( t > 0 \) (the latter follows from the explicit description of the limit curves \( \hat{X}_{\mathbb{R}}(0) \) in Lemma 2.3.3(2)). So, \( N_{2m,1} \) is just the cardinality of \( \mathcal{C}(2m, 1, x(t_0)) \), which in its turn coincides with the cardinality of the set \( I \) that counts the limit curve-configurations see Lemma 2.3.3(3).

To compare this cardinal count of the limit curve-configurations with the weighted count we made at the beginning, let us choose as \( p \) a selection \( p_1 \in \{z_1(0), w_1(0)\}, \ldots, p_{m-1} \in \{z_{m-1}(0), w_{m-1}(0)\} \). Restricting the cardinal count to any particular selection \( p \) is equivalent to dividing \( N_{2m,1} \) by \( 2^{m-2} \) (in accord with subsets \( I_{[p,\text{conj}]} \) in Lemma 2.3.3(3)). On the other hand, forgetting the line components in the curve
configuration is equivalent to counting with the weight \( n^{2^n - 1} \). Thus, we conclude that
\[
\sum_{\alpha \in \mathcal{L}^m} (e\alpha)^2 GW_{\alpha} = 2 \cdot 2^{-(m-2)} N_{2m,1} = 2^{3-m} N_{2m,1}.
\]

\( \Box \)

Due to (4.1.2), the result of Theorem 1.2.3 can be rewritten as follows.

**Corollary 4.2.2** In the same setting as in Theorem 1.2.3, we have
\[
-2\Gamma_{2m,1} = \sum_{\alpha \in \mathcal{L}^n} (e\alpha)^2 GW_{\alpha}.
\] (4.2.1)

\( \Box \)

### 5 Proof of Theorem 1.2.4

#### 5.1 Preparation for proving Theorem 1.2.4

Apart from Theorem 1.2.1 and Proposition 4.2.2, the proof is based on the following Solomon’s recursion rule (cf. [14, (OGW3)] and [4, Th.1.1(RWDVV3)]) which is a corollary of an analog for WDVV-equation designed by Solomon [24] in the framework of open strings.

**Theorem 5.1.1** 1 Let \( X \) be a real del Pezzo surface equipped with a basic \( \text{Pin}^- \)-structure, and let \( H_1, H_2, H_3 \in H_2^- (X) \) be fixed elements with \( H_1 H_3 = 0 \). Then, for each pair \((m, k) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}\) with \( l = \frac{1}{2}(m - 1 - k) \geq 1 \) and any \( B \in \mathcal{L}^m_{\mathbb{R}} \), the following relation holds:
\[
(H_1 H_2)(H_3 B)\Gamma_{B,k} = \Theta_{B,k}^{(1)} + \Theta_{B,k}^{(2)},
\]
where
\[
\Theta_{B,k}^{(1)} = \frac{1}{4} \sum_{B_1 + B_2 = B \atop k_1 + k_2 = k + 1} (H_1 B_1) ((H_3 B_1) (H_2 B_2) - (H_2 B_1) (H_3 B_2)) \left( \begin{array}{c} l - 1 \\ l_1 \end{array} \right) \left( \begin{array}{c} k \\ k_1 \end{array} \right) \Gamma_{B_1,k_1} \Gamma_{B_2,k_2},
\]
\[
\Theta_{B,k}^{(2)} = \frac{1}{2} \sum_{B_F - \text{conj}_u B_F + B_U = B} (B_U B_F) ((H_1 B_F) ((H_3 B_F) (H_2 B_U) - (H_2 B_F) (H_3 B_U)) \left( \begin{array}{c} l - 1 \\ l_U \end{array} \right) GW_{B_F} \Gamma_{B_U,k},
\]

\( \Box \)

1 In fact, this theorem, with appropriate definitions for the numbers \( \Gamma_{B,k} \), holds for all real rational surfaces \( X \) with any \( \text{Pin}^- \)-structure on \( X_{\mathbb{R}} \).
In this theorem, notation $\Gamma_{B,k}$ stands for $\Gamma_{m,v,k}$ such that $B = \frac{1}{d}(-mK - v)$ (and similar for $B_i$, $m_i$, $v_i$, etc.). We let also $l = \frac{1}{2}(m - 1 - k)$ (and similar for $m_i$, $k_i$, $l_i$, etc.). For shortness, we adopt the convention to put $\Gamma_{\alpha,k} = 0$ for any $\alpha \notin H_2^-(X)$.

Theorem 1.2.1 allows us to give the proof only for one particular real del Pezzo surface $X$ for each degree $d = 1, 2, 3$. We take $X$ as in Proposition 2.2.1 and denote by $G_d$ its automorphism group described there, that is:

$$G_d = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} \text{ generated by the Bertini involution } \tau, & \text{if } d = 1, \\
\mathbb{Z}/2\mathbb{Z} \times \mathcal{D}_4 \text{ where } \mathbb{Z}/2 \text{ is generated by the Geiser involution } \gamma, & \text{if } d = 2, \\
S_3, & \text{if } d = 3.
\end{cases}$$

To apply Theorem 5.1.1, we choose:

- $H_1 \in \mathcal{L}_{\mathbb{R}}^1 \setminus \{-K\}$, so that $H_1^2 = KH_1 = -1$,
- $H_2 = -K$, so that $H_1H_2 = 1$,
- $H_3 = H_1 + H_2$, so that $H_1H_3 = 0$ as required in Theorem 5.1.1,

which gives

$$(H_3B)\Gamma_{B,k} = \Theta_{B,k}^{(1)} + \Theta_{B,k}^{(2)}, \quad \text{where}$$

$$\Theta_{B,k}^{(1)} = \frac{1}{4} \sum_{m_1+m_2=m, m_1, m_2 \geq 1} \sum_{B_1+B_2=B, B_i \in \mathcal{L}_{\mathbb{R}}^{m_i}} \Gamma_{B_1,k_1} \Gamma_{B_2,k_2} \frac{(H_1B_1)(H_1B_2)(l_1-1)}{l_1} \frac{(H_1B_2)(H_2B_2)k}{k_2}$$

$$\Theta_{B,k}^{(2)} = \frac{1}{2} \sum_{2m_F+m_U=m, m_F, m_U \geq 1} \sum_{B_F \in \mathcal{L}_{\mathbb{R}}^{m_F}, B_U \in \mathcal{L}_{\mathbb{R}}^{m_U}, B_F-\text{conj}_k B_F+B_U=B} \left( (H_1B_F)(H_1B_F)m_U - m_F(H_1B_U) \right) \frac{(l_1-1)}{l_1} \frac{(H_1B_U)k}{k}$$

(5.1.1)

Our aim is to perform summation of these identities over all $B \in \mathcal{L}_{\mathbb{R}}^{m_i}$ and all $H_1 \in \mathcal{L}_{\mathbb{R}}^1 \setminus \{-K\}$. As a preliminary step, we prove a few auxiliary identities which then will be used repeatedly. In what follows (similarly to above) we write $H_1$ and $B_i \in \mathcal{L}_{\mathbb{R}}^{m_i}$ in a form

$$H_1 = \frac{1}{d}(-K - w), \quad B_i = \frac{1}{d}(-m_iK - v_i), \quad \text{where } w, v_i \in K^\perp,$$

so that $w^2 = -d(1 + d)$ and both $-K - w$, $-m_iK - v_i$ are divisible by $d$ in $H_2(X)$. We let also

$$\mathcal{M}_{\mathbb{R}} = \{w \in K^\perp \cap H_2^-(X) \mid w^2 = -d(1 + d) \text{ and } -K - w \in dH_2^-(X)\}.$$ 

**Proposition 5.1.2** For surfaces $X$ as in Proposition 2.2.1, the following holds.

\[ \Box \]
(1) If either \( d \in \{1, 3\} \), or \( d = 2 \) and \( m \) is even, then
\[
\Gamma_{m,v,k} = \Gamma_{m,gv,k} \quad \text{for any } g \in G_d \text{ and any } v \in K^\perp.
\]
If \( d = 2, m \) is odd, and \( v \in K^\perp \), then
\[
\Gamma_{m,v,k} = -\Gamma_{m,\gamma v,k} \quad \text{and } \Gamma_{m,v,k} = \Gamma_{m,gv,k} \quad \text{for any } g \in D_4.
\]

(2) \( \sum_{w \in \Omega^d} w = 0 \) and \( \sum_{H \in \mathcal{L}_1^d \setminus \{-K\}} H = -\frac{1}{d} \text{card}(\Omega^d)K \).

(3) If either \( d \in \{1, 3\} \) or \( m \) is even, then for any \( H \in \mathcal{L}_1^d \setminus \{-K\} \) we have
\[
\sum_{B \in \mathcal{L}_m^d} \Gamma_{B,k} B = -\frac{m}{d} \Gamma_{m,k} K, \quad \sum_{B \in \mathcal{L}_m^d} (HB)\Gamma_{B,k} = \frac{m}{d} \Gamma_{m,k}.
\]
If \( d = 2 \) and \( m \) is odd, then the above sums vanish.

(4) \(^2\) For any \( v \in K^\perp, v \neq 0 \), we have
\[
\sum_{B \in \mathcal{L}^d} B(vB)GW_B = \frac{1}{v^2} \sum_{B \in \mathcal{L}^d} v(vB)^2GW_B.
\]

(5) For every \( u, v \in K^\perp \), we have
\[
u^2 \sum_{B \in \mathcal{L}^d} (vB)^2GW_B = v^2 \sum_{B \in \mathcal{L}^d} (uB)^2GW_B.
\]

**Proof** Proof of the first claim is a direct combination of preservation of the arithmetic genus \( g_a \) and Welschinger numbers by the action of \( g \in G_d \) with the properties of their action on \( \hat{q} \) described in Proposition 2.2.1.

To prove Claims (2) and (3), except the vanishing statement in (3), it is sufficient to notice that the expressions \( \sum w, \sum H, \) and \( \sum \Gamma_{B,k} B \) are invariant under the action of \( G_d \), hence proportional to \( K \) (see Proposition 2.2.1), and that to determine the coefficient of proportionality it remains to take scalar product with \( K \). For proving the vanishing statement, we notice that, as it follows from Claim (1), if for an arbitrary chosen \( B \in \mathcal{L}_m^d \) the number \( \Gamma_{B,k} \) is not zero, then the orbit of \( B \) under the action of \( \mathbb{Z}/2 \times D_4 \) splits into 2 orbits of \( D_4 \) that are interchanged by the Geiser involution \( \gamma \). This implies (using the same arguments as above) that
\[
\sum_{g \in \mathbb{Z}/2 \times D_4} \Gamma_{gB,k} gB = \sum_{g \in D_4} \Gamma_{gB,k} gB + \sum_{g \in D_4} \Gamma_{\gamma gB,k} \gamma gB
\]
\[
= \Gamma_{B,k} \left( \sum_{g \in D_4} gB - \gamma (\sum_{g \in D_4} gB) \right) = \lambda K - \gamma \lambda K = 0.
\]

\(^2\) Properties (4) and (5) hold without any assumption on a real structure of \( X \).
To prove Claims (4) and (5) we notice that due to the invariance of Gromov–Witten numbers $GW_B$ under the Weyl group action on $K^\perp = E_{9-d}$ (which follows from to the monodromy invariance of Gromov–Witten numbers, and interpretation of the Weyl group as monodromy), these claims become straightforward consequences of the fact that the function $K^\perp \to \mathbb{R}$ given by $\alpha \mapsto \alpha^2$ is the only, up to scalar factor, quadratic function invariant under the action of the Weyl group (as it follows from the irreducibility of the action, see [1, 2.1, Prop. 1]). □

Next, we note that in the left-hand side of (5.1.1) we have $(H_3 B)\Gamma_{B,k} = ((\frac{1}{d} + 1)m - \frac{1}{d} wB)\Gamma_{B,k}$. Then, denoting by $\Theta^{(i)}_{m,k}$ the sum of $\Theta^{(i)}_{B,k}$ over all $H_1 \in \mathcal{L}_R^1 \setminus \{-K\}$ and $B \in \mathcal{L}_m^m$, we conclude that

$$\left(\frac{1}{d} + 1\right)\text{card}(\mathcal{L}_R^1 \setminus \{-K\})m \Gamma_{m,k} = \Theta^{(1)}_{m,k} + \Theta^{(2)}_{m,k}$$

(5.1.2)

since the terms $(wB)\Gamma_{B,k}$ vanish after summation due to Proposition 5.1.2(2).

5.2 Proof of Theorem 1.2.4 for odd $m = 2n + 1 \geq 3$

Due to Proposition 2.1.2(2), this part of Theorem 1.2.4 holds trivially for $d = 2$. Therefore, here we may assume (for simplicity) that $d \neq 2$.

We let $k = 0$ and have

$$\Theta^{(1)}_{m,0} = \frac{1}{2} \sum_{\substack{m_1 + m_2 = m, m_1 \geq 1 \\ B_1 \in \mathcal{L}_R^1, H_1 \in \mathcal{L}_R^1}} \left(\sum_{l_1 \geq 1} \frac{1}{l_1} (H_1 B_1)((H_1 B_1)m_2 - (H_1 B_2)m_1)\Gamma_{B_1,0} \Gamma_{B_2,1}\right)$$

$$= \frac{1}{2} \sum_{H_1 \in \mathcal{L}_R^1} \sum_{\substack{m_1 + m_2 = m \\ m_1, m_2 \geq 1}} \sum_{B_1 \in \mathcal{L}_R^m} \left(\sum_{l_1 \geq 1} \frac{1}{l_1} (H_1 B_1)(H_1 B_1)m_2 B_2 \Gamma_{B_1,0} (H_1 B_2)\Gamma_{B_2,1}\right)$$

$$= \frac{1}{2} \sum_{H_1 \in \mathcal{L}_R^1} \sum_{\substack{m_1 + m_2 = m \\ m_1, m_2 \geq 1}} \left(\sum_{B_1 \in \mathcal{L}_R^m} (H_1 B_1)\Gamma_{B_1,0} \sum_{B_2 \in \mathcal{L}_R^m} (H_1 B_2)\Gamma_{B_2,1}\right)$$

$$\Theta^{(2)}_{m,0} = \frac{1}{2} \sum_{\substack{m_F + m_U = m \\ m_F \geq 1}} \sum_{\substack{m_F + m_U = m \\ m_F, m_U \geq 1}} \left(\sum_{B_U \in \mathcal{L}_R^m} B_U (H_1 B_U)(H_1 B_U)(H_1 B_U)m_U (H_1 B_U)m_F \left(\frac{1}{l_U}\right)GW_B F \Gamma_{B_U,0}\right)$$

$$= \frac{1}{2} \sum_{\substack{m_F + m_U = m \\ m_F \geq 1}} \left(\sum_{B_U \in \mathcal{L}_R^m} B_U (H_1 B_U)(H_1 B_U)m_U \Gamma_{B_U,0} \sum_{B_F \in \mathcal{L}_R^m} B_F (H_1 B_F)^2 GW_B F\right)$$

$$- \frac{1}{2} \sum_{\substack{m_F + m_U = m \\ m_F \geq 1}} \left(\sum_{B_U \in \mathcal{L}_R^m} B_U (H_1 B_U)(H_1 B_U)m_U \Gamma_{B_U,0} \sum_{B_F \in \mathcal{L}_R^m} m_F B_F (H_1 B_F)GW_B F\right).$$
In the last relation, symbol “\(\cdot\)" is used to denote the intersection index in \(H_2(X)\).

Substituting \(H_1 = \frac{1}{d}(-K-w)\) and observing the vanishing (due to Proposition 5.1.2(2)) of summands where the factors \((wB_i)\) enter linearly, we obtain

\[
\Theta_{m,0}^{(1)} = \frac{1}{4d^2} \sum_{w \in \mathfrak{M}} \left( \sum_{m_1 + m_2 = m} \frac{(l - 1)}{l_1} \left( m_2(wB_1)^2 - m_1(wB_1)(wB_2) \right) \Gamma_{B_1,0} \Gamma_{B_2,1} \right).
\]

Next, we note that the terms with a factor \((wB_2)\) vanish after summation over \(B_2\), since by Proposition 5.1.2(3), \(\sum B_k \Gamma_{B_k,k}\) is collinear with \(K\), which is orthogonal to \(\omega\). Thus, we get

\[
\Theta_{m,0}^{(1)} = \frac{1}{4d^2} \sum_{m_1 + m_2 = m} \left( \frac{(l - 1)}{l_1} \right) m_2 \Gamma_{m_2,1} \sum_{w \in \mathfrak{M}} (wB_1)^2 \Gamma_{B_1,0}.
\] (5.2.1)

Similarly, after the same substitution for \(H_1\) into the first summand of \(\Theta_{m,0}^{(2)}\), we apply Proposition 5.1.2(3) to the factor containing \(\sum B_k \Gamma_{B_k,0}\) and obtain

\[
\frac{1}{2d^3} \sum_{w \in \mathfrak{M}} \left( \frac{l - 1}{l_U} \right) m_U^2 \Gamma_{m_U,0} \sum_{w} (m - wB_F)^2 G W_{B_F}.
\]

Then, cancelation in \((m - wB_F)^2\) of linear in \(w\) term (since \(\sum w \in \mathfrak{M} w = 0\), gives

\[
\frac{1}{2d^3} \sum_{w \in \mathfrak{M}} \left( \frac{l - 1}{l_U} \right) m_U^2 \Gamma_{m_U,0} (m^3 N_{mF}^{GW} + m F \sum_{B_F \in \mathcal{L}^m} (wB_F)^2 G W_{B_F}).
\]

In the second summand of \(\Theta_{m,0}^{(2)}\), after the same substitution for \(H_1\) and cancelation of linear in \(w\) terms, we apply Proposition 5.1.2(4) (with the choice \(v = w\)) and obtain

\[
-\frac{1}{2d^3} \sum_{w \in \mathfrak{M}} \left( \frac{l - 1}{l_U} \right) m_U^3 m_F^2 \Gamma_{m_U,0} N_{mF}^{GW}
\]

\[
+ \frac{1}{2d^3(1 + d)} \sum_{w \in \mathfrak{M}} \left( \sum_{B_F \in \mathcal{L}^m} (wB_U)^2 \Gamma_{B_U,0} \sum_{B_F \in \mathcal{L}^m} m_F (wB_F)^2 G W_{B_F} \right).
\]
Afterwards we cancel two opposite terms in the sum and conclude that

\[
\Theta_{m,0}^{(2)} = \frac{1}{2d} \sum_{w \in \mathcal{W}_R} \left( \binom{l-1}{l_U} m_F m_U^2 \Gamma_{m_U,0} \sum_{B_F \in \mathcal{L}_F^m} (w B_F^3)^2 G W_{B_F} \right)
\]

\[
+ \frac{1}{2d^3 (1+d)} \sum_{w \in \mathcal{W}_R, B_U \in \mathcal{L}_{m_U}^m} \left( \sum_{B_F \in \mathcal{L}_F^m} (l-1) (w B_U)^2 \Gamma_{B_U,0} \sum_{B_F \in \mathcal{L}_F^m} m_F (w B_F)^2 G W_{B_F} \right).
\]

(5.2.2)

Now, we substitute the expressions obtained in (5.2.1) and (5.2.2) into (5.1.2) and observe that, as it follows from Proposition 4.2.2 where we transform \(e\) into \(w\) in accordance with Proposition 5.1.2(5), the second term in (5.2.2) cancels (5.2.1). In this way we get

\[
\frac{1+d}{d} \text{ card}(\mathcal{L}_1^R \setminus \{-K\}) m \Gamma_{m,0}
\]

\[
= \frac{d(1+d)}{4d^3} \text{ card} \mathcal{W}_R m \sum_{m_F + m_U = m} (m_F m_U^2 \binom{l-1}{l_U} \Gamma_{m_U,0} \sum_{B_F \in \mathcal{L}_F^m} (e B_F)^2 G W_{B_F}).
\]

Finally, division by \(\frac{1+d}{2d^3} \text{ card}(\mathcal{L}_1^R \setminus \{-K\}) = \frac{1+d}{2d^3} \text{ card} \mathcal{W}_R\) and using once more Proposition 4.2.2 gives

\[
2d m \Gamma_{m,0} = \frac{1}{2} \sum_{m_F + m_U = m} (m_F m_U^2 \binom{l-1}{l_U} \Gamma_{m_U,0} (-2 \Gamma_{2m,F,1}))
\]

which proves the first relation of Theorem 1.2.4 (cf. its reformulation (4.1.3)). \(\square\)

### 5.3 Proof of Theorem 1.2.4 for \(m = 2n + 2 \geq 4\)

We let \(k = 1\) and have:

\[
\Theta_{m,1}^{(1)} = \frac{1}{4} \sum_{m_1 + m_2 = m, m_i \geq 1} \left( \binom{l-1}{l_1} (H_1 B_1)((H_1 B_1) m_2 - (H_1 B_2) m_1) \Gamma_{B_1,1} \Gamma_{B_2,1} \right)
\]

\[
= \frac{1}{4d^2} \sum_{m_1 + m_2 = m, m_i \geq 1} \left( \binom{l-1}{l_1} m_2 \Gamma_{m_2,1} \sum_{w \in \mathcal{W}_R, B_1 \in \mathcal{L}_B^m} (w B_1)^2 \Gamma_{B_1,1} \right).
\]

(5.3.1)
\[ \Theta_{m,1}^{(2)} = \frac{1}{2} \sum_{B_F \in L^{m_1}, B_U \in L^{m_2}} \sum_{m,F,m_U \geq 1} \left( l - \frac{1}{l_U} \right) G W_{B_F} \Gamma_{B_U,1} \]

\[ \Theta_{m,1}^{(2)} = \frac{d}{2d^6} \sum_{m,F,m_U \geq 1} \left( l - \frac{1}{l_U} \right) m_m U^2 \Gamma_{m_U,1} \sum_{w \in T, v_F \in K^\perp} (wv_F)^2 G W_{B_F} - \]

\[ - \frac{1}{2d^6} \sum_{m,F,m_U \geq 1} \left( l - \frac{1}{l_U} \right) (v_F v_U) (wv_F) (wv_U) G W_{B_F} \Gamma_{B_U,1}. \]

We substitute (5.3.2) and (5.3.3) into (5.3.1), apply Proposition 5.1.2(5) where we choose \( u = w \) and \( v = e \) with \( e^2 = -2 \), \( e \in K^\perp \) to perform a transformation

\[ \sum_{v_F \in K^\perp} (wv_F)^2 G W_{B_F} = d^2 \sum_{B_F \in L^{m_F}} (wB_F)^2 G W_{B_F} \]

\[ = \frac{d^3(1 + d)}{2} \sum_{B_F \in L^{m_F}} (eB_F)^2 G W_{B_F} \overset{\text{Prop.4.2.2}}{=} -d^3(1 + d) \Gamma_{2m,F,1}, \]

cancel similar terms in \( \Theta_{m,1}^{(1)} + \Theta_{m,1}^{(2)} \) and get

\[ \frac{1 + d}{d} m \Gamma_{m,1} = -d^3(1 + d) \sum_{2m,F,m_U \geq 1} \left( l - \frac{1}{l_U} \right) m_m U^2 \Gamma_{m_U,1} \Gamma_{2m,F,1} \]  

wherefrom the required recursion relation (cf. (4.1.3))

\[ 2m d \Gamma_{m,1} + \sum_{j=1}^{n} \left( \begin{array}{c} n - 1 \n j \end{array} \right) j (m - 2j)^2 \Gamma_{m-2j,1} \Gamma_{2j,1} = 0 \quad \text{for } m = 2n + 2. \]

5.4 Explicit formulas

The recursive formulas of Theorem 1.2.4 have a surprisingly simple explicit solution.
Theorem 5.4.1  For each $d = 1, 2, 3$ and any $n \in \mathbb{Z}_{\geq 0}$, we have

$$N_{2n+1,0} = \frac{1}{4} N_{1,0} b^n \left( n + \frac{1}{2} \right)^{n-2}, \quad N_{2n+2,1} = N_{2,1} b^n (n+1)^{n-2}$$

with $b = \frac{4N_{2,1}}{d}$.

Proof  Both relations stated in Theorem 1.2.4 hold trivially for these values if $n = 0$. To check the second of these two relations for $n > 0$ we substitute there the given values of $N_{2k,1}, k \leq n+1$ and get

$$(2n+2)d N_{2,1} b^n (n+1)^{n-2}$$

$$= 2 \sum_{j=1}^{n} \left( \frac{n-1}{n-j} \right) j (2n+2-2j) N_{2,1} b^{n-j} (n-j+1)^{n-j-2} N_{2,1} b^{j-1} j^{-3}.$$  

After division by $2d N_{2,1} b^n = 8N_{2,1}^{2} b^{n-1}$ it gives

$$(n+1)^{n-1} = \sum_{j=1}^{n} \left( \frac{n-1}{n-j} \right) (n-j+1)^{n-j} j^{-2}$$

which is a special case of Abel’s binomial theorem (see, for example, [6])

$$\frac{(x+y)^m}{x} = \sum_{k=0}^{m} \binom{m}{k} (x-kz)^{k-1} (y+kz)^{m-k}$$

where we put $x = -z = 1$, $y = n$, $m = n-1$, $k = j-1$.

Substituting the given values into the first relation we obtain

$$(2n+1)d \frac{1}{4} N_{1,0} b^n \left( n + \frac{1}{2} \right)^{n-2}$$

$$= 2 \sum_{j=1}^{n} \left( \frac{n-1}{n-j} \right) j (2n+1-2j) \frac{1}{4} N_{1,0} b^{n-j} \left( n-j+\frac{1}{2} \right)^{n-j-2} N_{2,1} b^{j-1} j^{-3}.$$  

Division by $\frac{1}{2} d N_{1,0} b^n = 2N_{1,0} b^{n-1} N_{2,1}$ turns it into

$$\left( n + \frac{1}{2} \right)^{n-1} = \sum_{j=1}^{n} \left( \frac{n-1}{n-j} \right) \left( n-j+\frac{1}{2} \right)^{n-j} j^{-2}$$

which is Abel’s binomial relation for $x = -z = 1$, $y = n - \frac{1}{2}$, $m = n - 1$, $k = j - 1$.  $\Box$
Corollary 5.4.2  For real del Pezzo surfaces $X$ of degree $d = 1$ or $d = 2$ whose real locus $X_\mathbb{R}$ has the maximal number of connected components, the invariant $N_{dm,k}$ coincides with the Welschinger invariant $W_{-mK,k}$. In particular, for $d = 1$ and $X_\mathbb{R} = \mathbb{RP}^2 \sqcup 4S^2$ we have

$$W_{-(2n+1)K,0} = 2 (120)^n \left( n + \frac{1}{2} \right)^{n-2},$$

$$W_{-(2n+2)K,1} = 30 (120)^n (n + 1)^{n-2},$$

and for $d = 2$ and $X_\mathbb{R} = 4S^2$

$$W_{-(n+1)K,1} = 6 (12)^n (n + 1)^{n-2}.$$

Proof  This is immediate from Theorem 5.4.1 and knowledge of values of $N_{1,0}, N_{2,1}$ (see Table below), since under the assumptions imposed on $X_\mathbb{R}$ the eigenspace $\ker (1 + \conj) \subset H_2(X)$ is generated by $K$ and, in addition, $\hat{q}(mK) - m^2 = m^2 \hat{q}(K) - m^2 = 0$. 

\[ \square \]

5.5 Few first values and simple applications

To begin, we list the values of $N_{m,k}$ for $m \leq 6$ obtained by means of the formulas established above. The initial values $N_{1,0}, N_{2,1}$ we use there:

- for $K^2 = 1$, were obtained in [12, 13] via some “lattice calculation”,
- for $K^2 = 2$, can be obtained in a similar way,
- for $K^2 = 3$, the value $N_{1,0} = 3$ was explained in [11], and $N_{2,1} = N_{1,0}$ follows from a natural one-to-one correspondence $\mathcal{L}_1^1 \mathcal{R} \rightarrow \mathcal{L}_2^2$ assigning to a line the divisor class of residual conics of hyperplane sections containing this line.

Proposition 5.5.1  For any nonsingular real cubic surface $X$:

- The signed count of 2-points-constrained real twisted cubics is equal to $3 - \chi(X_\mathbb{R})$, independently on the number of real points in the points-constraint. When the surface is maximal, it contains $h = 40$ hyperbolic (with $\hat{q} = 1$) and $e = 32$ elliptic (with $\hat{q} = -1$) twisted cubics.
- The signed count of 3-points-constrained real non-singular rational quartic curves is equal to $9 - 3\chi(X_\mathbb{R})$, independently on the number of real points in the points-constraint. When the surface is maximal, it contains $h = 120$ hyperbolic (with $\hat{q} = 0$) and $e = 96$ elliptic (with $\hat{q} = 2$) quartics.

Proof  The third layer is formed by $-K$ and divisor classes of twisted cubics. Therefore, the first statement follows from $N_{3,k} = 2 - k$ and $W_{-K,k} = \chi(X_\mathbb{R}) - (k + 1)$ (2-points-constrained hyperplane sections form an elliptic pencil with $k + 1$ real fixed points, so that the latter equality can be proved, as usual, by integration over Euler characteristic, c.f. [9], Proposition 4.7.3). For a separate count of hyperbolic and elliptic twisted cubics, it is sufficient to notice, in addition, that their combined number is equal to the number of roots, 72, in $E_6$.  

\[ \square \]
The fourth layer is formed by divisor classes representable by non-singular rational quartic curves (that is by divisor classes of form $-2K - L' - L''$ where $L', L''$ is any pair of disjoint lines) and genus 1 quartic curves (that is by divisor classes $-K + L$ where $L$ is any line). Therefore, the second statement follows from $N_{4,k} = 9 - 3k$, $W_{-K + L,k} = \chi(X_{\mathbb{R}}) - k$ and $\hat{q}(-K + L) = \hat{q}(L) - 1$ (similar to above, the 3-points-constrained curves in the divisor class $-K + L$ form an elliptic pencil with $k + 1$ real fixed points and the second equality follows from integration over Euler characteristic, where one takes into account that the pencil contains a reducible real curve consisting of $L$ and the plane section through the 3 constrained points). For a separate count of hyperbolic and elliptic quartics, we note that the number of pairs of disjoint real lines on a maximal cubic surface is equal to $27 \cdot 16 = 216$.

Remark 5.5.2 In the case of cubic surfaces, there are natural bijections: between the set of divisor classes of twisted cubics (resp. non-singular rational quartic curves) and the set of isomorphism classes of presentations of the surface as 6-blowup of $\mathbb{P}^2$ (resp. as 5-blowup of $\mathbb{P}^1 \times \mathbb{P}^1$). Therefore, cited counts can be interpreted as signed counts of the corresponding blow-up models.

Considering real del Pezzo surfaces $X$ with $K^2 = 2$ as double coverings of $\mathbb{P}^2$ branched along real non-singular quartic curves $A \subset \mathbb{P}^2$ and using skew-invariance of $\hat{q}$ under the deck transformation, we translate our signed count of curves belonging to even layers $\mathcal{L}^n_{2\mathbb{R}}$ into a the signed count of real rational point-constrained curves of degree $n$ tangent to $A$ at $2n - 1$ points plus a signed count of real rational point-constrained curves of degree $2n$ tangent to $A$ at $4n$ points. In particular, such a point-constrained curve of degree $2n$ is called hyperbolic (resp. elliptic), if $\hat{q}$ takes value 0 (resp. 2) on its lifts to $X$.

Proposition 5.5.3 Let $A \subset \mathbb{P}^2$ be a real non-singular quartic curve and $\Omega$ one of two halves of $\mathbb{RP}^2$ bounded by $A_{\mathbb{R}}$. Then:

\begin{align*}
\begin{array}{c|c|c|c}
(m, k) & N_{m,k} & K^2 = 1 & K^2 = 2 \\
\hline
(1, 0) & 8 & 0 & 3 \\
(2, 1) & 30 & 6 & 3 \\
(3, 0) & 160 & 0 & 2 \\
(4, 1) & 1800 & 36 & 6 \\
(5, 0) & 28800 & 0 & 12 \\
(6, 1) & 432000 & 864 & 48 \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c|c|c|c}
(m, k) & |\Gamma_{m,k}| & K^2 = 1 & K^2 = 2 \\
\hline
(1, 0) & 16 & 0 & 6 \\
(2, 1) & 60 & 12 & 6 \\
(3, 0) & 160 & 0 & 2 \\
(4, 1) & 1800 & 36 & 6 \\
(5, 0) & 14400 & 0 & 6 \\
(6, 1) & 216000 & 432 & 24 \\
\end{array}
\end{align*}
The signed count of real conics 4-tangent to \( A \) and constrained by a point in \( \Omega \) gives \( 8 - 2\chi(\Omega) \). If \( A \) is maximal and \( \Omega \) is non-orientable, then this count involves \( h = 70 \) hyperbolic and \( e = 56 \) elliptic conics.

If \( A \) is maximal and \( \Omega \) is non-orientable, then the signed count of real non-singular rational quartic curves 8-tangent to with \( k \) real point-constraints chosen in \( \Omega \) gives 336 if \( k = 1 \) and 896 if \( k = 3 \).

**Proof** The second layer, \( L^2 \), is formed by \( -K \) and the divisor classes of lifts of 4-tangent conics. Therefore, the first statement follows from \( N_{2,1} = 6 \) and \( W_{-K,1} = \chi(X_R) - 2 = 2\chi(\Omega) - 2 \). In the maximal case, \( 8 - 2\chi(\Omega) = 14 \) and the number of 4-tangent conics is the number of roots, 126, in \( E_7 \).

The fourth layer is formed by \( -2K \) and the divisor classes of lifts of 8-tangent quartics. Here we obtain the required number of quartics, \( 4 \cdot 224 \) if \( k = 3 \) and \( 2(132 + 36) \) if \( k = 1 \), due to \( N_{4,k} = 0 \), \( W_{-2K,k} = -224 \) for \( k = 3 \) and \( N_{4,k} = 36 \), \( W_{-2K,k} = -132 \) for \( k = 1 \) (see [16] for these values of \( W_{-2K,k} \)).

Considering real del Pezzo surfaces \( X \) with \( K^2 = 1 \) as double coverings \( \pi : X \to Q \) of a real quadric cone \( Q \subset \mathbb{P}^2 \) branched along real non-singular sextics \( C \subset Q \) and using invariance of \( \hat{q} \) under the deck transformation, we translate our signed count of curves belonging to the second layer \( L^2_R \) into a signed count of real rational 1-point-constrained hyperplane sections tangent to \( C \) at 2 points plus a signed count of real 1-point-constrained quartics tangent to \( C \) at 6 points (and obtained as transversal sections of \( Q \) by quadrics). In particular, such a point-constrained quartic is called hyperbolic (resp. elliptic), if \( \hat{q} \) takes value 0 (resp. 2) on its lifts to \( X \).

**Proposition 5.5.4** For a real non-singular sextic \( C \subset Q \), the signed count of real quartics 6-tangent to \( C \) constrained by a point in \( \pi(X_R) \) gives \( 6 + \frac{\chi^2(X_R) - 1}{2} \). If \( X \) is maximal and \( X_R \) is connected, then this count involves \( h = 1192 \) hyperbolic and \( e = 1208 \) elliptic quartics.

**Proof** Follows from \( N_{2,1} = 30 \) and \( W_{-2K,1} = 6 + \frac{\chi^2(X_R) - 1}{2} \) (see [13]).

### 6 Concluding remarks

#### 6.1 Generating functions

Recall the tree function \( T(x) = \sum_{n \geq 1} n^{n-1} \frac{x^n}{n!} \).

**Proposition 6.1.1** For each \( d = 1, 2 \) and 3, the functions

\[
N^{\text{even}}(x) = \sum_{n \geq 0} N_{2n+2,1} \frac{x^n}{n!}, \quad N^{\text{odd}}(x) = \sum_{n \geq 0} N_{2n+1,0} \frac{x^n}{n!}
\]

can be expressed through \( T(x) \) as

\[
N^{\text{even}}(x) = N_{2,1}(bx)^{-1}(T(bx) - \frac{1}{2} T^2(bx)),
\]
\[ N^{\text{odd}}(x) = N_{1,0}(bx)^{-\frac{1}{2}} \left( T^{\frac{1}{2}}(bx) - \frac{1}{3} T^{\frac{2}{3}}(bx) \right), \]

where \( b \) is 120, 12 and 4 for \( d = 1, 2, 3 \) respectively.

**Proof** After substitution of the values of \( N_{2n+1} \) from Theorem 5.4.1 we obtain

\[ N^{\text{even}}(x) = N_{2,1} \sum_{n \geq 0} (n+1)^{n-1} \frac{(bx)^{n}}{(n+1)!} = N_{2,1} G(bx), \]

where

\[ G(x) = \sum_{n \geq 1} n^{n-2} x^{n-1} \frac{x^{n}}{n!} = \frac{1}{x} \left( T(x) - \frac{1}{2} T^{2}(x) \right). \]

For the latter identity, see, e.g., [20]. For values of \( b = \frac{4N_{2,1}}{d} \) see Sect. 5.5.

A similar substitution for \( N_{2n+1} \) gives

\[ N^{\text{odd}}(x) = \sum_{n \geq 0} \frac{1}{4} N_{1,0} b^{n} \left( n+\frac{1}{2} \right)^{n-2} x^{n} \frac{x^{n}}{n!} = N_{1,0} Q(bx) \]

with

\[ Q(x) = \sum_{n \geq 0} \frac{1}{4} \left( n+\frac{1}{2} \right)^{n-2} x^{n} \frac{x^{n}}{n!} = x^{-\frac{1}{2}} \int_{0}^{x} \frac{1}{2} x^{-\frac{1}{2}} e^{\frac{1}{2} T(x)} dx \]

\[ = x^{-\frac{1}{2}} \left( T^{\frac{1}{2}}(x) - \frac{1}{3} T^{\frac{2}{3}}(x) \right) \]

where the last equalities follow from the following relations (see, f.e., [7])

\[ \sum_{n \geq 0} \frac{1}{2} \left( n+\frac{1}{2} \right)^{n-1} x^{n} \frac{x^{n}}{n!} = e^{\frac{1}{2} T(x)} = \left( \frac{T(x)}{x} \right)^{\frac{1}{2}}. \]

\[ \square \]

**Remark 6.1.2** Note that Theorem 1.2.4 can be reformulated as a differential equation

\[ K^{2}(F_{tt} - F_{t} - y F_{ty}) = 2 F_{tt} F_{ty}, \]

where

\[ F(t, y) = \sum_{m=2n+1} N_{m,0} \frac{e^{mt}}{n!} + y \sum_{m=2n+2} N_{m,1} \frac{e^{mt}}{n!} = x^{\frac{1}{2}} N^{\text{odd}}(x) + xy N^{\text{even}}(x), \]

\[ x = e^{2t}. \]

**6.2 On Gromov–Witten side**

It is thought-provoking that over the complex field, for any del Pezzo surface, the sums of genus-0 Gromov–Witten invariants over layers \( \mathcal{L}^{m} \) satisfy essentially the same recursion relation as genus-0 Gromov–Witten invariants of projective plane.
Proposition 6.2.1  For every del Pezzo surface of degree $1 \leq d \leq 6$ and every $m \geq 4$,

$$d^2 N_m^{GW} = \sum_{m_1 + m_2 = m \atop m_1, m_2 \geq 1} N_{m_1}^{GW} N_{m_2}^{GW} m_1^2 m_2 \left( \frac{m - 4}{m_1 - 2} - m_1 \left( \frac{m - 4}{m_1 - 1} \right) \right) \quad (6.2.1)$$

where $N_m^{GW}$ stands for $\sum_{\alpha \in L_m(X)} GW(\alpha)$. The initial values of $N_m^{GW}$ for $m = 1, 2, 3$ and $1 \leq d \leq 6$ are shown in the table below.

| $d$ = 1 | $d$ = 2 | $d$ = 3 | $d$ = 4 | $d$ = 5 | $d$ = 6 |
|---------|---------|---------|---------|---------|---------|
| $N_1^{GW}$ | 252     | 56      | 27      | 16      | 10      | 6       |
| $N_2^{GW}$ | 5130    | 138     | 27      | 10      | 5       | 3       |
| $N_3^{GW}$ | 446,400 | 344     | 84      | 16      | 5       | 2       |

It is not difficult to show that this implies the following square root rule.

Corollary 6.2.2  For any $1 \leq d \leq 6$ one has $\log N_{2m}^{GW} = 2m \log m + O(m)$, while $\log N_{2m,1}$ for $d = 1, 2, 3$ and $\log N_{2m+1,0}$ for $d = 1, 3$ are of type $m \log m + O(m)$ (recall that $N_{2m+1,0} = 0$ for $d = 2$).

In the proof of Proposition 6.2.1 we use (in a similar, but much simpler, way as in our proof of recursion relations for $N_m,k$) the fact that in the case of $d \leq 6$ the action of the Weyl group on $K^\perp$ has no any nonzero invariant element. Note that, as an elementary check shows, the relation (6.2.1) does not hold for $d = 7$ and 8. When $d = 9$, it turns (after replacing $m_i$ by $3d_i$) into celebrated Kontsevich-Manin recursion relation [20].

The next formula is an immediate consequence of Proposition 4.2.2 combined with Theorem 5.4.1.

Proposition 6.2.3  $\sum_{\alpha \in L^n(ea)} (ea)^2 GW_\alpha = 2da^n n^{n-3}$ where $e$ is an arbitrary class $e \in K^\perp$ with $e^2 = -2$ and $a$ equals 60 if $d = 1$, 6 if $d = 2$, and 2 if $d = 3$.

6.3 Other del Pezzo surfaces

There are two more kinds of del Pezzo surfaces to which our approach applies almost literally. One of them is real nonsingular quadrics $X \subset \mathbb{P}^3$ with $X_\mathbb{R} \neq \emptyset$. Such non-empty quadrics form two real deformation classes: one with $X_\mathbb{R} = S^2$ and another with $X_\mathbb{R} = S^1 \times S^1$. Pick a Spin-structure on $X_\mathbb{R}$ induced from a Spin-structure on $\mathbb{RP}^3$, and let $q : H_1(X_\mathbb{R}; \mathbb{Z}/2) \to \mathbb{Z}/2$ denote the associated quadratic function. We put in this case $N_{m,k} = \sum_{A \in C_{m,k}} (-1)^{q(A_\mathbb{R})} w(A)$ and obtain the following results.

- The numbers $N_{m,k}$ are independent of the choice of a real quadric $X$ with $X_\mathbb{R} \neq \emptyset$ and a Spin-structure on $\mathbb{RP}^3$. 

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The numbers $N_{m,k}$ vanish unless $m = 4n$, while $N_{4n,k}$ is equal to the Welschinger invariant $W_{nh,k}$, $h = -\frac{1}{2}K$, of $X$ with $X_\mathbb{R} = S^2$ (see [3, 5, 15, 28] for various methods of calculation of $W_{nh,k}$).

The relation $N_{4n,1} = 2^{2n-3}\sum_{\alpha \in L^2}(\alpha e)^2GW_{\alpha}$ (where $e$ is an arbitrary class $e \in K_\perp$ with $e^2 = -2$) holds for any $n \geq 1$.

Another example is provided by del Pezzo surfaces $X$ of degree 4 presented as double coverings of real non-singular quadrics $Q \subset \mathbb{P}^3$ branched along real non-singular curves representing the doubled hyperplane section class. The covering $X \to Q$ is naturally embedded into the quadratic cone $Z \subset \mathbb{P}^4$ over $Q$. Thus, we can pick a Pin$^-\text{-structure}$ on $X_\mathbb{R}$ induced from a Pin$^-\text{-structure}$ on the non-singular part of $Z_\mathbb{R}$ and put $N_{m,k} = \sum_{A \in C_\mathbb{R}(m,k)} \hat{q}([A]) - m^2 w(A)$, as usual.

The numbers $N_{m,k}$ are preserved under change of the branching curve.

The numbers $N_{m,k}$ vanish if either $m$ is odd or $k > 1$.

The relation $N_{2n,1} = 2^{n-3}\sum_{\alpha \in L^n}(\alpha e)^2GW_{\alpha}$ (where $e$ is an arbitrary class $e \in K_\perp$ with $e^2 = -2$) holds for any $n \geq 1$.

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