INVOLUTIVE DISTRIBUTIONS OF OPERATOR-VALUED
EVOLUTIONARY VECTOR FIELDS

ARTHEMY V. KISELEV AND JOHAN W. VAN DE LEUR

Abstract. Converting a pioneering idea of V. V. Sokolov et al. \[60\] to a geometric
object, we introduce a well-defined notion of linear matrix operators in total deriva-
tives, whose images in the Lie algebras \(g\) of evolutionary vector fields on jet spaces are
closed with respect to the commutation,
\[
[\text{im} A, \text{im} A] \subseteq \text{im} A.
\]
(*)
The images generate involutive distributions on infinite jet bundles \(J^\infty(\pi)\) over fibre
bundles \(\pi\), and the operators induce Lie algebra structures \([,]_A\) on \(\Omega = \text{dom} A/\ker A\).
We postulate that the operators are classified by the vector and covector transforma-
tion laws for their domains \(\text{dom} A\), which are related to sections of \(\pi\) by Miura substi-
tutions. The gauge of \(\text{dom} A\) may be independent from transformations of the images
\(\text{im} A\). This is a generalization of the classical theory with \(\text{dom} A = \text{im} A \subseteq \text{sym} E\) for
recursion operators for integrable systems \(E\), and with \(\text{dom} A = \hat{\text{im}} A \supseteq \text{cosym} E\) for
Hamiltonian operators (here the Miura substitutions are the identity mappings in both
cases, and the gauge transformations are uniquely correlated). In particular, recursion
operators \(A \in \text{End}_k \text{sym} E\) that satisfy (*) are solutions of the classical Yang–Baxter
equation \([A\varphi_1, A\varphi_2] = A([\varphi_1, \varphi_2]_A)\) for the Lie algebra \(\text{sym} E\).

If, for \(r\) linear differential operators \(A_i : \Omega \to g\) with a common domain \(\Omega\), pairwise
commutators of their images hit the sum of images again, we endow the spaces \(A = \bigoplus_{i=1}^r k \cdot A_i\) of the operators with a bilinear bracket
\[
[A_i, A_j] = \sum_{k=1}^r A_k \circ c^k_{ij}, \quad c^k_{ij} \in \text{Diff}(\Omega \times \Omega \to \Omega), \quad 1 \leq i, j \leq r
\]
that satisfies the Jacobi identity. A class of such algebras is given by operators that
generate Noether symmetries of hyperbolic Euler–Lagrange systems of Liouville type;
we calculate explicitly the operators \(A_i\) and the structural constants \(c^k_{ij}\). Thus we give
an exhaustive description of higher symmetries for all 2D Toda chains associated with
semi-simple Lie algebras of rank \(r\), completing the results of \[43, 48, 53, 56, 60\].

The bracket \([,]\) of operators is anti-symmetric for the domains \(\Omega\) composed by
vectors and is symmetric for \(\Omega\) with covector transformations. In the latter case, we
reveal a flat non-Cartan affine connection in the triples \(\Omega \xrightarrow{A} g\) such that symmetric
bi-differential Christoffel symbols \(\Gamma^k_{ij}\) are encoded by the structural constants \(c^k_{ij}\), and
such that completely integrable commutative hierarchies \(A \subset \text{im} A\) are the geodesics.

We demonstrate that the notion of Lie algebroids over infinite jet bundles \(J^\infty(\pi)\) does
not repeat the construction over finite-dimensional base manifolds, when condition \(\text{(*)}\)
is fulfilled by the anchors. To correlate them with Lie algebra homomorphisms \(A : \Omega \to g\), we introduce formal differential complexes over Lie algebras \((\Omega, [\, , ]_A)\).

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Address: Mathematical Institute, University of Utrecht, P.O.Box 80.010, 3508 TA Utrecht, The
Netherlands. E-mails: [A.V.Kiselev, J.W.vandeLeur]@uu.nl.
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Introduction. Let $M^n$ be a smooth $n$-dimensional real manifold and $\pi : E^{n+m} \to M^n$ be a fibre bundle. The space of linear differential operators over $\mathbb{R}$ on the infinite jet bundle $J^\infty(\pi) \xrightarrow{\pi_\infty} M^n$ contains operators $A$ in total derivatives whose images in the $C^\infty(J^\infty(\pi))$-module $\Gamma(\pi_\infty^*(\pi)) = \Gamma(\pi) \otimes C^\infty(M^n)$ $C^\infty(J^\infty(\pi))$ of evolutionary derivations are closed with respect to the commutation,

$$[\text{im } A, \text{im } A] \subseteq \text{im } A. \quad (1)$$

We say that such operators $A$ are Frobenius in view of the fact that their images constitute involutive distributions on $J^\infty(\pi)$. Each Frobenius operator transfers the Lie algebra structure of evolutionary vector fields to the bracket $[\cdot, \cdot]_A$ on the quotient of its domain by the kernel and determines an $\mathbb{R}$-homomorphism of Lie algebras. The Hamiltonian operators that correspond to the Poisson bi-vectors with vanishing Schouten bracket are an example, among many others; e.g., the 2D Toda systems $E_{\text{Toda}} = \{ u_{x_1} = \exp(K_{ij} u^j) \}$ associated with the Cartan matrices $K$ determine a class of Frobenius operators that take values in the spaces $\text{sym} E_{\text{Toda}}$ of their (Noether) higher infinitesimal symmetries.

The main problem we address is the coordinate-independent definition of operators that satisfy (1), and we perform the analysis of their algebraic and geometric properties. We conclude that, up to a special case, there are two principal classes of Frobenius operators, which generalize the recursions $\text{sym} \rightarrow \text{sym}$ and the Hamiltonian structures $\text{cosym} \rightarrow \text{sym}$ for integrable systems $\mathcal{E}$, respectively. The essential distinction of generic Frobenius operators $A$ from the recursion or Hamiltonian operators is that the coordinate transformations in the domains of $A$ can be not correlated with the transformations in their images. In this sense, the two arising classes of Frobenius operators correspond to vector and covector-type transformation laws for their domains $\Omega^1(\pi)$, which are defined by $\pi$ and another fibre bundle $\xi$ over $M^n$.

For example, if $\xi = \pi$ and hence $\Omega^1(\pi_x) = \Gamma(\pi_\infty^*(\pi))$, then Frobenius endomorphisms $R \in \text{End}_\mathcal{E} \Gamma(\pi_\infty^*(\pi))$ of generating sections $\varphi$ of evolutionary vector fields are not only recursions for the Lie algebra $(\mathfrak{g}(\pi), [\cdot, \cdot])$ of derivations. Indeed, they induce nontrivial deformations $[\cdot, \cdot]_R$ of the standard structure $[\cdot, \cdot]$ on $\Gamma(\pi_\infty^*(\pi))$ via the classical Yang–Baxter equation

$$[R\varphi_1, R\varphi_2] = R([\varphi_1, \varphi_2]_R), \quad \varphi_1, \varphi_2 \in \Gamma(\pi_\infty^*(\pi)). \quad (2)$$

Frobenius operators of second kind, which generalize the Hamiltonian structures, are assigned in this paper to hyperbolic Euler–Lagrange systems of Liouville type. To do that, we give an exhaustive description of higher symmetries for these integrable systems. In particular, our explicit formula (105) for the operators is valid for all 2D Toda chains associated with root systems of complex semi-simple Lie algebras of rank $r$. These operators determine commutative hierarchies of KdV-type evolution equations which are Noether symmetries of the Euler–Lagrange systems. The operators specify factorizations of higher Hamiltonian structures for the hierarchies.

We introduce a Lie-type algebra structure on spaces of linear differential operators in total derivatives. We consider the linear spaces $A = \bigoplus_{i=1}^r \mathbb{k} \cdot A_i$ of operators
$A_i : \Omega^1(\xi_\pi) \to \Gamma(\pi^*_\infty(\pi))$ with a common domain $\Omega^1(\xi_\pi)$ such that the sums of images of $A_i$ are closed under the commutation of vector fields,

$$\left[ \sum_i \text{im} A_i, \sum_j \text{im} A_j \right] \subseteq \sum_k \text{im} A_k. \quad (3)$$

By setting $[A_i, A_j](p, q) := [A_i(p), A_j(q)]$ for $p, q \in \Omega^1(\xi_\pi)$ and $1 \leq i, j \leq r$, we endow the linear spaces $\mathcal{A}$ with a Lie-type bracket

$$[A_i, A_j] = \sum_{k=1}^r A_k \circ \mathbf{c}_{ij}^k \quad (4)$$

that satisfies the Jacobi identity. The bi-differential structural constants $\mathbf{c}_{ij}^k : \Omega^1(\xi_\pi) \times \Omega^1(\xi_\pi) \to \Omega^1(\xi_\pi)$ are skew-symmetric for vector pre-images of evolutionary fields and are symmetric for covectors that constitute the domain $\Omega^1(\xi_\pi)$ of $A_i$. In the latter case, we calculate the constants $\mathbf{c}_{ij}^k$ explicitly. This algebraic problem and our approach are different from Sato’s formalism [20] of pseudodifferential operators because, e.g., the compositions of Frobenius operators $\Omega^1(\xi_\pi) \to \Gamma(\pi^*_\infty(\pi))$ are not defined even in the Hamiltonian case, and hence they do not constitute multiplicative associative algebras with unit. Therefore the Lie-type algebra structure relies heavily on the intrinsic geometry of the jets $J^\infty(\pi)$ as the base manifolds for all fibre bundles. The bracket we propose, in particular, for Frobenius recursions $R$ is different from the Richardson–Nijenhuis bracket $[,]$ that determines their Lie super-algebra structure [40, §5.3] by using the composition of operators, although we use similar geometric techniques.

In this paper, we pass from the (wide) category of fibre bundles over $M^n$ to a more narrow category of bundles and horizontal modules over the infinite jets $J^\infty(\pi)$. Then we investigate which objects survive and which there appear anew. We conclude that the operators that obey (1) are closely related to Lie algebroids over smooth finite-dimensional manifolds $M^n$. We show that Lie algebroids over $J^\infty(\pi)$, with the anchors given by Frobenius operators, can not be defined as straightforward generalizations of the finite-dimensional case, but this is achieved with a much more refine construction. To this end, for each Frobenius operator $A$, we generate a formal differential complex over Lie algebra $(\Omega^1(\xi_\pi), [\cdot, \cdot], A)$ and suggest to determine the Lie algebroids using representations of the differential in this complex through homological vector fields.

On the other hand, we discover that Frobenius operators determine a generalization of the affine geometry such that bi-differential Christoffel symbols $\Gamma^k_{ij} \odot A$ are encoded by the structural constants $\mathbf{c}_{ij}^k$ of the algebras $\mathcal{A}$. We prove that Frobenius operators $A$ determine flat connections in the triples $(\Omega^1(\xi_\pi), \Gamma(\pi^*_\infty(\pi)), A)$ consisting of two Lie algebras and a morphism, and we interpret commutative hierarchies in the images of Frobenius operators as the geodesics. This is different from the classical realization of integrable systems as geodesic flows on infinite-dimensional Lie groups [2].

Our main result is the following. We formulate a well-defined notion of linear differential operators whose images are closed w.r.t. the commutation, see (1), and we associate infinitely many such operators to Euler–Lagrange hyperbolic systems of Liouville type. Here, in particular, we solve an old problem [42, 43, 56] in geometry of Leznov–Saveliev’s 2D Toda chains related to semi-simple complex Lie algebras: We describe their infinitesimal symmetries by an explicit formula and calculate all commutation relations in these Lie algebras.
The paper is organized as follows. First in section 1 we outline our basic concept. Then in section 2 we summarize important properties of the Poisson structures for completely integrable evolutionary systems. We introduce the notion of nondegenerate operators in total derivatives and formulate a conjecture that simplifies the search for Poisson structures of PDE.

In section 3 we define the Frobenius operators, establish the chain rule for them, describe the differential Frobenius complex, and construct flat connections in the triples \( (\Omega^1(\xi), g(\pi), A) \) of two Lie algebras and a morphism. Also, we describe an inductive method for the reconstruction of the Sokolov brackets on the domains of nondegenerate Frobenius operators. We illustrate it using the dispersionless 3-component Boussinesq system \( (56) \), see \( [18, 24] \), that admits a family of Frobenius operators \( (57) \) and yields the Frobenius recursion \( (81) \).

In section 4 we propose the definitions of the linear compatibility and the strong compatibility of Frobenius operators. We endow the linear spaces of both linear and strong compatible Frobenius operators with the Lie-type algebra structure. We show
that the Magri schemes provide commutative examples of such algebras. The structural
constants encode the bi-differential Christoffel symbols, which are transformed under
reparametrizations by a direct analogue of the rules for the classical connection 1-forms.

In section 5 we assign the algebras of Frobenius operators $A$ to Liouville-type Euler–
Lagrange systems (in particular, to the 2D Toda chains associated with the root systems
of semi-simple Lie algebras [10 42 50]). To this end, we describe the generators $\varphi =
A(\cdot)$ of their higher symmetry algebras and calculate all commutation rules. As a by-
product, we find the Hamiltonian structures for KdV-type systems related to these
hyperbolic Darboux-integrable equations.

Finally, we discuss open problems in the theory of Frobenius structures.

Let us fix some notation; the language of jet bundles is contained, e.g., in [5 10 46
51]. In the sequel, everything is real and $C^\infty$-smooth. By $J^\infty(\pi)$ we denote the infinite
jet space over a fibre bundle $\pi: E^{n+m} \to M^n$, we set $\pi_\infty: J^\infty(\pi) \to M^n$,
and denote by $[u]$ the differential dependence on $u$ and its derivatives. Put $\mathcal{F}(\pi) =
C^\infty(J^\infty(\pi))$, which is understood as the inductive limit of filtered algebras [5] and hence each function from
$\mathcal{F}(\pi)$ depends on finitely many coordinates on $J^\infty(\pi)$. The $\pi$-vertical evolutionary
derivations $\mathcal{E}_\varphi = \sum_\sigma D_\sigma(\varphi) \cdot \partial/\partial u_\sigma$ are described by the sections $\varphi \in \Gamma(\pi_\infty(\pi)) =
\Gamma(\pi) \otimes_{C^\infty(M^n)} C^\infty(J^\infty(\pi))$ of the induced fibre bundle $\pi_\infty(\pi)$.
The shorthand notation for this $\mathcal{F}(\pi)$-module is $\mathcal{X}(\pi) \equiv \Gamma(\pi_\infty(\pi))$. For all $\psi$
such that $\mathcal{X}_\varphi(\psi)$ makes sense, the linearizations $t^{(u)}(\varphi)$ are defined by $t^{(u)}(\varphi) = \mathcal{E}_\varphi(\psi)$,
where $\varphi \in \mathcal{X}(\pi)$. Denote by $\bar{\Lambda}^n(\pi)$ the highest $\pi$-horizontal forms on $J^\infty(\pi)$.
For any $\mathcal{F}(\pi)$-module $\mathfrak{h} = \Gamma(\pi_\infty(\xi)) =
\Gamma(\xi) \otimes_{C^\infty(M^n)} C^\infty(J^\infty(\pi))$ of sections of an induced bundle over $M^n$, we use the notation
$\xi_\pi \equiv \pi^\ast_\infty(\xi)$ and denote by $\mathfrak{h} = \text{Hom}_{\mathcal{F}(\pi)}(\mathfrak{f}, \bar{\Lambda}^n(\pi))$ the dual module.
Examples of $\xi$ will be given later.

Frobenius operators map $\mathfrak{f} \to \mathcal{X}(\pi)$, where the module $\mathfrak{f}$ is one of the following
restrictions of $\mathfrak{h} \subseteq \Gamma(\xi_\pi)$ onto the image of a Miura differential substitution $w =
w[u]: J^\infty(\pi) \to \Gamma(\xi)$: We have that either

$$
\mathfrak{f} = \mathcal{X}(\pi)|_w : J^\infty(\pi) \to \Gamma(\xi) \quad \text{or} \quad \mathfrak{f} = \tilde{\mathcal{X}}(\pi)|_w : J^\infty(\pi) \to \Gamma(\xi).
$$

In particular, $\xi = \pi$ for recursion operators $R \in \text{End}_R \mathcal{X}(\pi)$, and $\mathfrak{f} = \tilde{\mathcal{X}}(\pi)$ for Hamiltonian operators;
here we set $w = \text{id}: \Gamma(\pi) \to \Gamma(\xi)$ in both cases.

Denote by $\mathfrak{g}(\pi) = (\mathcal{X}(\pi), [, , ])$ the Lie algebra of evolutionary vector fields $\mathcal{E}_\varphi$ with
the standard bracket of sections. Let $(\Omega^1(\xi_\pi), [ , , ]_A)$ denote the Lie algebra that is
isomorphic to $\mathfrak{f}/\ker A$ as a vector space and is endowed with the Lie bracket $[ , , ]_A$ by a
Frobenius operator $A: \mathfrak{f} \to \mathfrak{g}(\pi)$. Definitions are discussed in detail in sections 2 and 3.

1. Basic concept

The problem of construction and classification of operators in total derivatives that satisfy
(1) was suggested first in [60]. The operators $A$ were regarded there as non-
skew-adjoint generalizations of the Hamiltonian operators, whose images are closed
with respect to the standard Lie algebra structure on the modules of evolutionary
fields. The operators appeared in [60] in local coordinates in the context of the scalar
Liouville-type equations $\mathcal{E}_L = \{ u_{xy} = F(u, u_x, u_y; x, y), 1 \leq i \leq m \}$, whose infinite
groups of conservation laws \([f(x, [w]) \, dx] + [\bar{f}(y, [\bar{w}]) \, dy]\) are differentially generated by finitely many densities

\[ w_1, \ldots, w_r \in \ker D_y|_{\xi^L}, \quad \bar{w}_1, \ldots, \bar{w}_r \in \ker D_x|_{\xi^L}. \tag{5} \]

Let us recall the main motivating example that begins the theory of Frobenius operators.

**Example 1** (The Liouville equation). Consider the scalar Liouville equation

\[ \mathcal{E}_{\text{Liou}} = \{ \mathcal{U}_{xy} = \exp(2\mathcal{U}) \}. \tag{6} \]

The differential generators \(w, \bar{w}\) of its conservation laws \([\eta] = [f(x, [w]) \, dx] + [\bar{f}(y, [\bar{w}]) \, dy]\) are

\[ w = \mathcal{U}_x^2 - \mathcal{U}_{xx} \quad \text{and} \quad \bar{w} = \mathcal{U}_y^2 - \mathcal{U}_{yy} \tag{7} \]

such that \(D_y(w) = 0\) and \(D_x(\bar{w}) = 0\) by virtue (\(\dagger\)) of \(\mathcal{E}_{\text{Liou}}\) and its differential consequences. The operators

\[ \square = \mathcal{U}_x + \frac{1}{2}D_x \quad \text{and} \quad \bar{\square} = \mathcal{U}_y + \frac{1}{2}D_y \tag{8} \]

factor higher and Noether’s symmetries

\[ \varphi = \square(\phi(x, [w])), \quad \varphi_{\mathcal{L}} = \square\left( \frac{\delta \mathcal{H}(x, [w])}{\delta w} \right); \quad \bar{\varphi} = \square(\bar{\phi}(y, [\bar{w}])), \quad \bar{\varphi}_{\mathcal{L}} = \square\left( \frac{\delta \mathcal{\bar{H}}(y, [\bar{w}])}{\delta \bar{w}} \right) \]

of the Euler–Lagrange equation (\(\mathcal{L}\)) for any smooth \(\phi, \bar{\phi}\) and \(\mathcal{H}, \mathcal{\bar{H}}\). Note that the operator \(\square = \frac{1}{2}D_x^{-1} \circ (\mathcal{\bar{U}}_{w})^*\) is obtained using the adjoint linearization of \(w\), and similarly for \(\bar{\square}\).

Each of the images of (\(\mathcal{K}\)) is closed w.r.t. the commutation such that

\[ [\square(p), \square(q)] = \square(\mathcal{E}_{\square}(p) - \mathcal{E}_{\square}(q) + \{(p, q)\}, \quad \text{here} \quad \{(p, q)\} = D_x(\mathcal{U}_x) \cdot q - p \cdot D_x(q), \]

and same for \(\bar{\square}\): the evolutionary derivations \(\mathcal{E}_{\mathcal{L}}\) are given in (\(\mathcal{K}\)) on p. 13 in local coordinates. The symmetry algebra \(\text{sym} \mathcal{E}_{\text{Liou}} \simeq \text{im} \square + \text{im} \bar{\square}\) is the sum of images of (\(\mathcal{K}\)), and the two summands commute between each other, \([\text{im} \square, \text{im} \bar{\square}] = 0\) on \(\mathcal{E}_{\text{Liou}}\). Therefore,

\[ [\text{im} \square + \text{im} \bar{\square}, \text{im} \square + \text{im} \bar{\square}] \subseteq \text{im} \square + \text{im} \bar{\square}. \tag{9} \]

The Frobenius operator \(\square\) factors higher symmetries of the potential modified KdV equation

\[ \mathcal{E}_{\text{pmKdV}} = \{ \mathcal{U}_t = -\frac{1}{2}\mathcal{U}_{xxx} + \mathcal{U}_x^3 = \square(w) \}, \tag{10} \]

whose commutative hierarchy is composed by Noether’s symmetries \(\varphi_{\mathcal{L}} \in \text{im} \square \circ \delta / \delta w\) of the Liouville equation (\(\mathcal{L}\)). The operator \(\square\) factors the second Hamiltonian structure \(B_2 = \square \circ A_1 \circ \square^*\) for \(\mathcal{E}_{\text{pmKdV}}\), here \(A_1 = D_x^{-1}\).

The generator \(w\) of conservation laws for \(\mathcal{E}_{\text{Liou}}\) provides the Miura substitution (\(\mathcal{J}\)) from \(\mathcal{E}_{\text{pmKdV}}\) to the Korteweg–de Vries equation

\[ \mathcal{E}_{\text{KdV}} = \{ w_t = -\frac{1}{2}w_{xxx} + 3ww_x \}. \tag{11} \]

The second Hamiltonian structure for \(\mathcal{E}_{\text{KdV}}\) is factored to the product \(\dot{A}_2 = \square^* \circ B_1 \circ \square\), where \(\dot{B}_1 = D_x\) is the first Hamiltonian structure for the modified KdV (see diagram (\(\mathcal{L}\)) on p. 52). The domain of the Frobenius operator \(\square\) contains sections of the cotangent bundle \(\text{cosym} \mathcal{E}_{\text{KdV}}\) to the KdV equation. The bracket \(\{(\cdot, \cdot)\} \square\) on the domain of \(\square\) is equal to the bracket \(\{(\cdot, \cdot)\} \dot{A}_2\) induced on the domain of the operator \(\dot{A}_2\) (which is Hamiltonian and hence its image is closed under commutation) for \(\mathcal{E}_{\text{KdV}}\).
Example (1) is reproduced for all \( m \)-component 2D Toda chains

\[
\mathcal{E}_{\text{Toda}} = \left\{ u^i_{xy} = \exp \left( \sum_{j=1}^{m} K^i_j w^j \right) ; 1 \leq i \leq m \right\}
\]

associated with semi-simple complex Lie algebras \([12, 43]\). For example, all the 2D Toda chains (12) with the matrix \( K \) symmetrizable by a vector \( \vec{a} \) (that is, \( \kappa_{ij} := a_i k^j_i = \kappa_{ji} \), no summation over \( i \)) admit the conserved density \( w^1 = \langle \kappa u_x, u_x \rangle / 2 - \langle \vec{a}, u_{xx} \rangle \). At the same time, the chains (12) may admit other conserved densities. In the fundamental paper \([56]\), A. B. Shabat et al. proved the existence of maximal \((r = \vec{r} = m)\) sets (5) of conserved densities for \( \mathcal{E}_{\text{Toda}} \) if and only if the matrix \( K \) is the Cartan matrix of a root system for a semi-simple Lie algebra of rank \( r \). Further, Ref. \([55]\) contains an explicit procedure that yields special systems of jet coordinates for (12); with respect to them, all coefficients of the characteristic equation \( D_y(w) \equiv 0 \) on \( \mathcal{E}_{\text{Toda}} \) become linear, whence its first integrals \( w^1, \ldots, w^r \) are obtained. The differential orders of \( w, \bar{w} \) grow as \( r \) grows, and the formulas are big already for the Lie algebra \( G_2 \), see \([43, 31]\). We claim that the orders of \( w^i \) with respect to Shabat’s momenta \( m_j \) (see below) coincide with the gradations for the principal realizations of the basic (i.e., simplest nontrivial highest weight) representations of the corresponding affine Lie algebras.

The generators \( \phi = \square (\vec{\phi}(x, [w])) \) of higher symmetry algebras for Liouville-type equations are factored by matrix operators \( \square \) in total derivatives \([8, 29, 43]\). For the Euler–Lagrange Liouville-type systems, the operators provide Noether symmetries \( \phi_L \) with \( \vec{\phi} = \delta H(x, [w]) / \delta w \), see also \([53, 30, 25]\), whence the description of all symmetries follows. For this class of hyperbolic equations, which incorporates (12), commutative Lie subalgebras of higher Noether symmetry algebras yield completely integrable KdV-type hierarchies \([10, 28, 25]\). The generators \( w, \bar{w} \) of conservation laws for \( \mathcal{E}_L \) induce Miura’s transformations \( w = w[u] \) between the KdV- and modified KdV-type hierarchies upon \( w \) and \( u \), respectively. The operators \( \square \) factor higher Poisson structures for the evolutionary systems and, moreover, prescribe the nonlocalities that arise in these structures. For instance, the first integral \( w^1 \) for systems (12) with a symmetrizable matrix \( K \) yields the class \( \phi = (u_x + \Delta \cdot D_x)(\vec{\phi}(x, [w^1])) \) of symmetries for \( \mathcal{E}_{\text{Toda}} \), here \( \Delta = \sum_{j=1}^{m} k^j_j \); this always yields the second Poisson structure for KdV in the upper-left corner \( \langle A_k \rangle_1 \) of the Hamiltonian operators determined by the entire operator \( \square \) (see \([25]\) for details). Several formulas for the operators \( \square \) were known from \([13, 56]\) for 2D Toda chains (12) associated with semi-simple Lie algebras of low ranks, but not in the general case.

Under assumption that the densities (5) are known, we obtain the explicit formula (105) for these operators \( \square \). We derive all the commutation relations for sym \( \mathcal{E}_L \); they are encoded by bi-differential operators \( C^L_{ij} \) that act on \( \vec{\phi} \)'s. We prove that the \( r \)-tuples \( \vec{\phi} \) obey the variational covector transformation laws \( \vec{\phi} \mapsto \vec{\phi} = \left[ (u_{\hat{w}}^{(w)})^* \right]^{-1}(\vec{\phi}) \) under reparametrizations \( \hat{w} = \hat{w}[w] \) of (3). Therefore the operators \( \square \) assigned to Liouville-type systems \( \mathcal{E}_L \) are natural examples of well-defined Frobenius structures.

The second motivation to study linear differential operators subject to (11) is much more abstract, c.f. \([10]\). Let us compare geometry of ODE and of PDE from the following viewpoint. For a linear \( k \)-space \( \mathcal{V} \), one can study representations \( g \to \text{End}_k(\mathcal{V}) \) of Lie algebras \( g \) on \( V \) and thus endow the linear spaces of endomorphisms with Lie
algebra structures. On the other hand, let \( \mathcal{F} \) be a commutative associative algebra with unit, e.g., a \( k \)-algebra \( \mathcal{F}_{-\infty} = C^\infty(M^n) \) or an \( \mathcal{F}_{-\infty} \)-algebra \( \mathcal{F}(\pi) = C^\infty(J^\infty(\pi)) \), and let \( \partial : \mathcal{F} \to \mathcal{F} \) be a derivation. Consider two left \( \mathcal{F} \)-modules \( P, Q \) and the space of linear differential operators \( \text{Diff}(P \to Q) \). A question: Are there any natural algebraic structures on this linear space? If \( P = Q \) (e.g., \( P = Q = \text{sym} \mathcal{E} \) of a differential equation \( \mathcal{E} \), and we deal with recursion operators, see \([7, 24]\)), then one has the associative composition \( A \circ B \) and the formal commutation \( A \circ B - B \circ A \) for \( A, B \in P \), but what else? And what if \( P \neq Q \)? In this paper, we give an affirmative answer on the above question whenever \( \mathcal{F} = \mathcal{F}(\pi) \).

In geometry of PDE \([5, 40, 46]\), classical constructions such as the differentials of Hamiltonians, bi-vectors \( \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} \), etc., appear as zero-order terms in Taylor expansions of the variational derivatives, variational bi-vectors, or variational Poisson–Nijenhuis structures \([17, 22, 34]\). Likewise, we discover that the Lie-type brackets, which we introduce on linear spaces \( \mathcal{A} \subset \mathcal{CDiff}(j \to \mathcal{X}(\pi)) \) of Frobenius operators, encode bi-differential Christoffel symbols of flat connections. At the same time, we show by counterexample \([3]\) on p. \([23]\) that Frobenius operators are not the anchors of Lie algebroids over the jet spaces \( J^\infty(\pi) \).

We begin with the classical Hamiltonian formalism for ODE on finite-dimensional manifolds \( M \). Bearing it in mind, we pass to the PDE setting and formulate the assertions.

Let \( \mathcal{P} \in \Gamma(\wedge^2(TM)) \) be a bi-vector with vanishing Schouten bracket \([\mathcal{P}, \mathcal{P}] = 0\). Using the coupling \( \langle \cdot, \cdot \rangle \) on \( TM \times T^*M \) and a nondegenerate Poisson bi-vector \( \mathcal{P} \), one transfers the Lie algebra structure \([\cdot, \cdot]\) on \( \Gamma(TM) \) to \([\cdot, \cdot]\) on \( \Gamma(T^*M) \ni \psi_1, \psi_2 \) and obtains the Koszul–Dorfman–Daletsky–Karasëv bracket \([9]\)

\[
[\psi_1, \psi_2]_{\mathcal{P}} = L_{\mathcal{P}\psi_1}(\psi_2) - L_{\mathcal{P}\psi_2}(\psi_1) + d(\mathcal{P}(\psi_1, \psi_2)),
\]

(13)

here \( L \) is the Lie derivative. By \([35]\), the bracket \([13]\) is uniquely defined if, first,

- \([\cdot, \cdot]_{\mathcal{P}} \) is a derivation of \( C^\infty(M) \) with respect to the \( C^\infty(M) \)-module structure of \( \Gamma(T^*M) \), that is, if we have that

\[
[\psi_1, f \cdot \psi_2]_{\mathcal{P}} = L_{\mathcal{P}\psi_1}(f) \cdot \psi_2 + f \cdot [\psi_1, \psi_2]_{\mathcal{P}}
\]

(14)

for any \( f \in C^\infty(M) \) and \( \psi_1, \psi_2 \in \Gamma(T^*M) \), and second, if

- \([dh_1, dh_2] = -d\{h_1, h_2\}_{\mathcal{P}} \) holds for any \( h_1, h_2 \in C^\infty(M) \). Then \( \left( (T^*M, [, ]_{\mathcal{P}}) \rightarrow (TM, [, ]) \right) \) is a Lie algebroid \([59]\) with the morphism \( \mathcal{P} \) over the smooth manifold \( M \).

The de Rham differential \( d_{\mathcal{DR}} \) on \( \wedge^\bullet(T^*M) \) is defined in the complex over the Lie algebra \( (\Gamma(TM), [, ]) \) by using Cartan’s formula. If the Poisson bi-vector \( \mathcal{P} \) has the inverse symplectic two-form \( \mathcal{P}^{-1} \) such that \( \mathcal{P}^{-1}[x, y] = [\mathcal{P}^{-1} x, \mathcal{P}^{-1} y]_{\mathcal{P}} \), then the differential \( d_{\mathcal{DR}} \) is correlated with the Koszul–Schouten–Gerstenhaber bracket \([\cdot, \cdot]_{\mathcal{P}} \) on \( \wedge^\bullet(T^*M) \) by \( d_{\mathcal{DR}} = [\mathcal{P}^{-1}, \cdot]_{\mathcal{P}} \). The differential \( d_{\mathcal{DR}} \) on \( \wedge^\bullet(T^*M) \) is intertwined \([36, 37]\).

\(^1\)To facilitate the exposition, in appendix \([A]\) we summarize the notation for Hamiltonian evolutionary PDE and its analogy with the structures for ordinary differential equations. However, it is a very delicate matter to restore the full parallel between finite-dimensional manifolds \( M \) and horizontal bundles over \( J^\infty(\pi) \) with its own Cartan connection. Essentially, theorems either are converted to definitions or become false. This is the case of Lie algebroids, which we discuss in sections \([22, 3]\) and \([3]\).
with the Poisson differential $\partial_P = [\mathcal{P}, \cdot]$ on $\wedge^\bullet(TM)$ by
\[
(\bigwedge^{k+1} \mathcal{P})([P^{-1}, \Psi]_\mathcal{P}) + [\mathcal{P}, (\bigwedge^k \mathcal{P})(\Psi)] = 0, \quad \forall \Psi \in \bigwedge^k (T^*M).
\]

The trivial infinitesimal deformations
\[
[x, y]_N := [Nx, y] + [x, Ny] - N([x, y]) = \frac{d}{d\lambda} \bigg|_{\lambda=0} e^{-\lambda N} \left[ e^{\lambda N}(x), e^{\lambda N}(y) \right]
\]
of the standard Lie algebra structure $[\cdot, \cdot]$ on the tangent bundle $TM \ni x, y$ over smooth manifolds $M$ were described in [35] using the recursion operators $N: \Gamma(TM) \to \Gamma(TM)$. If the Nijenhuis torsion
\[
[N, N]^{fr}(x, y) = [Nx, Ny] - N([x, y]_N)
\]
for an endomorphism $N$ vanishes, then the Lie brackets $[\cdot, \cdot]_N$ obtained by iterations of the Nijenhuis recursion $N$ are pairwise compatible.

The Nijenhuis and Poisson structures $(N, \mathcal{P})$, whenever satisfying two compatibility conditions [17, 35], generate infinite hierarchies of pairwise compatible Poisson structures $N^k \circ \mathcal{P}$, $k \geq 0$, which means that linear combinations $\lambda_1 N^{k_1} \circ \mathcal{P} + \lambda_2 N^{k_2} \circ \mathcal{P}$ remain Poisson for any $\lambda_1: \lambda_2$. This assertion [35], which is valid for finite-dimensional manifolds $M$, admits a straightforward generalization to the infinite-dimensional case when the base manifold is the jet space $J^k M$ obtained by iterations of $J^1 M$ over a fibre bundle $E \to \mathcal{M}$, see [19, 22], such that the concept of Poisson–Nijenhuis’ structures is applicable verbatim to evolutionary PDE [17]. Also, this construction yields the Lie algebroids $\left( (\mathcal{H}^\infty(\pi), \{\cdot, \cdot\}_P) \xrightarrow{\mathcal{E}_{P, \xi}} (\mathcal{X}(\pi), [\cdot, \cdot]) \right)$.

Let $P$ be a Hamiltonian operator for a system of differential equations $\mathcal{E} = \{w^i = 0, \ldots, w^r = 0\}$. The left-hand sides $w^i$ belong to some $\mathcal{F}(\pi)$-module $\mathcal{F}$ of sections of an $r$-dimensional fibre bundle over $J^\infty(\pi)$, and the operator $P: \mathcal{F} \to \mathcal{X}(\pi)$ takes sections from the dual of $\mathcal{F}$ to evolutionary fields. In practice, the equations $u^i_t = f_i[u]$ in determined evolutionary systems $\mathcal{E}$ are labelled by the dependent variables $w^i$ in the bundle $\pi$, which establishes the isomorphism $\mathcal{F} \simeq \mathcal{X}(\pi)$. However, the system $\mathcal{E}$ as a geometric object in $\mathcal{J}^\infty(\pi)$ admits arbitrary reparametrizations $\tilde{w} = \tilde{w}[w]$ for its components. They are not anyhow correlated with admissible changes $\tilde{u} = \tilde{u}[u]$ of the dependent variables. Summarizing, we see that the Hamiltonian operators for PDE are well-defined under unrelated transformations of their domains $\mathcal{F}$ and images $\mathcal{X}(\pi)$. Let us formalize this property for a wider class of operators.

Frobenius linear differential operators in total derivatives, with images closed under the commutation, are well defined as follows. Let $\pi$ and $\xi$ be fibre bundles over $M^n$, let $w$ be a fibre coordinate in $\xi$, and construct the infinite jet bundle $\xi^\infty: J^\infty(\xi) \to M^n$. Consider the $\mathcal{F}(\xi)$-module of sections $\mathcal{X}(\xi) = \Gamma(\xi^\infty(\xi))$ of the induced fibre bundle $\xi^\infty(\xi)$, and denote its $\Lambda^n$-dual by $\hat{\mathcal{X}}(\xi)$. Suppose further that there is a Miura substitution $J^\infty(\pi) \to \Gamma(\xi)$ which embeds both $\mathcal{F}(\xi)$-modules into $\Gamma(\pi^\infty(\xi))$. (For instance, the substitution determines a system of differential equations $\mathcal{E} = \{w^i[u] = 0, \ldots, w^r[u] = 0\}$.) We denote the substitution by the same letter $w$, because from now on we take the restrictions of $\mathcal{X}(\xi)$ and $\hat{\mathcal{X}}(\xi)$ onto its image.

- Frobenius operators of first kind are $A: \mathcal{X}(\xi)|_w \to \mathcal{X}(\pi)$. Under any diffeomorphisms $\tilde{u} = \tilde{u}[u]: J^\infty(\pi) \to \Gamma(\pi)$ and $\tilde{w} = \tilde{w}[w]: J^\infty(\xi) \to \Gamma(\xi)$, the operators $A$ of first
kind are transformed according to

\[ A \mapsto \tilde{A} = \ell_u^{(w)} \circ A \circ \ell_w^{(u)} \big|_{w=u[w]} \]  

(17)

- Frobenius operators of second kind are linear mappings \( A: \hat{\pi}(\xi) \big|_w \rightarrow \pi(\xi) \). For any differential changes of coordinates \( \tilde{u} = u[w] \) and \( \tilde{w} = w[w] \) in \( \pi \) and \( \xi \), respectively, the operators obey

\[ A \mapsto \tilde{A} = \ell_u^{(w)} \circ A \circ (\ell_w^{(u)})^* \big|_{w=u[w]} \]  

(18)

- Finally (the degenerate case), if no gauge of \( \Gamma(\xi) \) is allowed in a given setting, then Frobenius operators \( A: \hat{\pi}(\pi)\big|_w \rightarrow \pi(\pi) \) are transformed by \( A \mapsto \tilde{A} = \ell_u^{(w)} \circ A \big|_{u=u[w]} \) under \( \tilde{u} = u[w] \).

Frobenius operators of first and second kind generalize linear recursion operators \( R \in \text{End}_R \pi(\pi) \) and Hamiltonian operators \( P: \hat{\pi}(\pi) \rightarrow \pi(\pi) \), respectively. The domains of \( A \) may not be composed by sections of the tangent bundle \( \pi^*_\pi(\pi) \rightarrow J^\infty(\pi) \) or, respectively, of its dual, which was constructed in [41] (see also [19, 22]). Thence Frobenius operators are described neither by the Poisson bi-vectors nor by the recursion \((1, 1)\)-tensors.

For the same reason, the coupling \( (\cdot, \cdot) \) on \( g \times \Gamma(\xi) \) is missing. However, Frobenius operators \( A: \hat{\pi}(\pi) \big|_w \rightarrow \pi(\pi) \) transfer the Lie algebra structure \([\cdot, \cdot]\) for evolutionary vector fields to \( [\cdot, \cdot]_A \) on the quotients \( \Omega^1(\pi)_w = \Gamma(\pi)_w/\ker A \) of the respective \( \mathcal{F}(\pi)\)-submodules \( j \) of \( \Gamma(\xi) \). The Koszul bracket \( [\cdot, \cdot]_A \) is defined by \( A([\psi_1, \psi_2]_A) = [A\psi_1, A\psi_2] \) for any \( \psi_1, \psi_2 \in \Omega^1(\pi) \). Thus the operators determine the Lie algebra \( \mathbb{R}\)-homomorphisms

\[ A: (\Omega^1(\pi), [\cdot, \cdot]) \rightarrow (g(\pi), [\cdot, \cdot]). \]  

(19)

For any \( \pi \)-horizontal \( g \)-module \( K \) there is a flat connection

\[ \nabla^A: \text{Der}_K(\Omega^1(\pi), g(\pi)) \rightarrow \text{Der}(g(\pi), K) \]  

(20a)

that lifts inner derivations of \( \Omega^1(\pi) \) to \( K \)-valued derivations of \( g(\pi) \) by the formula

\[ \nabla^A_{\psi, \cdot} = [A(\psi), \cdot], \quad \psi \in \Omega^1(\pi). \]  

(20b)

The connection is \( \Omega^1(\xi) \)-linear, \( \nabla^A_{\phi \times \cdot} = A(\phi) \times \nabla^A_{\cdot} \), w.r.t. the Lie multiplications \( \phi \times \cdot = [\phi, \cdot]_A \) by any \( \phi \in \Omega^1(\pi) \), here \( A(\phi) \times \cdot = [A(\phi), \cdot] \). The connection \( \nabla^A \) is flat due to the Jacobi identity. Thence the commutative hierarchies whose flows belong to the image of \( A \) are the geodesics with respect to \( \nabla^A \).

The ‘variational’ analogue \( [\cdot, \cdot]_A \) of the Koszul–Dorfman bracket \((13)\) equals

\[ [\psi_1, \psi_2]_A = \mathcal{E}_{A\psi_1}(\psi_2) - \mathcal{E}_{A\psi_2}(\psi_1) + \{(\psi_1, \psi_2)\}_A, \quad \psi_1, \psi_2 \in \Omega^1(\pi), \]  

(21)

where the evolutionary derivations are \((28)\). Generally, the bi-differential term \( \{(\cdot, \cdot)\}_A \) is neither a Lie algebra structure nor a cocycle. The chain rule \((78)\) for the brackets \( \{(\cdot, \cdot)\}_A \) and \( \{(\cdot, \cdot)\}_{A \circ \Delta} \) determined by two Frobenius operators \( A \) and \( A \circ \Delta \) follows from the evolutionary summands in \((21)\).
An explicit formula for the Zhiber–Sokolov bracket \(\{\cdot,\}\) in the Hamiltonian case \(P: \mathcal{H}(\pi) \rightarrow \mathcal{H}(\pi)\) is given by (10). We reconstruct \(\{\cdot,\}\) for a class of Frobenius operators \(\square\): \(\text{cosym} \mathcal{E}_{\text{sym}} \rightarrow \text{sym} \mathcal{E}_{\text{sym}} \subset \text{sym} \mathcal{E}_{\text{Toda}}\) that factor higher symmetries of the 2D Toda chains (12) and the Hamiltonian structures for the associated KdV-type evolution equations [10]. Thus Zhiber–Sokolov’s pioneering idea in [60] to study operators with property (11) as generalizations of the Hamiltonian operators \(\mathcal{P}_{\text{sym}}\) was motivated by the examples \(\square\): \(J \rightarrow \mathcal{H}(\pi)\) of a different geometric origin with \(J \not\approx \mathcal{H}(\pi)\). The bracket \(\{\cdot,\}\) for Frobenius recursions \(R \in \text{End}_{\mathbb{R}} \mathcal{H}(\pi)\) is unknown. For Frobenius operators \(A\) that are nondegenerate in the sense of (59), we formulate an inductive procedure for reconstruction of \(\{\cdot,\}\) for Frobenius recursions.

The Leibnitz rule (13), which is the first axiom for the Lie algebroids, is lost for the Koszul brackets (21) even in the Hamiltonian case \(Q^2 = 0\) that encode the Frobenius complex (22) assign the Lie algebroid structures over the jet spaces \(J^n(\pi)\) to Frobenius operators using the approach of [59]. We say that the Frobenius operators are \(linear compatible\) if their arbitrary linear combinations retain the same property (11). Families of \(N\) linear compatible operators \(A_1, \ldots, A_N\) induce deformations of (19) such that the Sokolov brackets obey

\[
\left\{\cdot,\right\} \cong \sum_{i=1}^{N} \lambda_i \cdot \left\{\cdot,\right\} A_i R
\]

The linear compatibility extends the frames \(\bigcup_i \mathbb{R} \cdot A_i\) to the linear spaces \(A = \bigoplus_i \mathbb{R} \cdot A_i\). The operators \(A_1, \ldots, A_N\) are called \(strong compatible\) if, by (3), the commutators of evolutionary fields \(\mathcal{E}_{A_i}(\cdot)\) in their images belong to the sum of these images. That is, for any \(i, j \in 1, \ldots, N\) and \(p, q \in \Omega^1(\xi) = f/ \bigcap_i \ker A_i\) we have

\[
[A_i(p), A_j(q)] = A_j(A_i(p)(q)) - A_i(A_j(q)(p)) + \sum_{k=1}^{N} A_k(\Gamma_{ij}^k(p, q)) \in \bigcap_{\ell=1}^{N} \text{im} \ A_\ell.
\]
In this way, the Koszul brackets (21) in the domains of each \( \mathcal{A}_\ell \) merge to the collective decomposition (3) of the commutators. For example, the commutation closure (3) shows that the operators (8) are strong compatible and that
\[
\Gamma_{\Box}^{\Box} = D_x \otimes 1 - 1 \otimes D_x, \quad \Gamma_{\Box}^{\Box} = D_y \otimes 1 - 1 \otimes D_y, \\
\Gamma_{\Box}^{\Box} = D_y \otimes 1, \quad \Gamma_{\Box}^{\Box} = -1 \otimes D_x, \quad \Gamma_{\Box}^{\Box} = -1 \otimes D_y, \quad \Gamma_{\Box}^{\Box} = D_x \otimes 1, 
\]
where the notation is obvious. Note that \( \Gamma_{\Box}^{\Box}(p, q) \) is strong compatible and that the operators (8) are bi-differential symbols \( \Gamma_{ij}^{\Box} \) satisfy \( \Gamma_{ij}^{\Box}(p, q) = -\Gamma_{ji}^{\Box}(q, p) \) and other normalizations. They are transformed by a suitable variant of the classical formula \( \Gamma = g \Gamma g^{-1} + dg g^{-1} \) for connection 1-forms \( \Gamma \) and reparametrizations \( g \).

If the closure condition (3) is fulfilled for \( \mathcal{A}_1, \ldots, \mathcal{A}_N \), then we take \([A_i, A_j](p, q) := [A_i(p), A_j(q)]\) as the definition of a Lie-type bracket (4) between them. Each operator \( A_\ell \) spans the one-dimensional algebra that is described by \( \Gamma_{\Box}^{\Box} \) in equation (21). We extend the bracket of operators onto the linear span \( \bigoplus_{\ell=1}^{N} \mathbb{R} \cdot A_\ell \) if, in addition, the Frobenius operators are linear compatible. The constants \( \Gamma_{ij}^{\Box} \) are bi-differential Christoffel’s symbols for the flat affine connection \( \{ \nabla^A_\ell, 1 \leq \ell \leq N \} \) defined in (20). The Magri schemes for completely integrable systems [4] yield examples of commutative algebras of Frobenius operators.

Relaxing the normalization \( \Gamma_{\Box}^{\Box} = \{ \cdot, \cdot \} A_\ell \cdot \delta_\ell^k \) but retaining (3), we arrive at a wider class of linear differential operators \( A_\ell \) and Lie-type brackets (4) on the spaces \( \mathcal{A} = \bigoplus_{i=1}^{N} \mathbb{R} \cdot A_i \). Namely, the image of a particular operator may hit the images of other operators under the commutation. Now we formulate the assertion.

**Theorem.** Let \( \kappa \) be a real constant nondegenerate symmetric \((m \times m)\)-matrix. Consider a hyperbolic Euler–Lagrange system \( \mathcal{E}_L = \{ \delta \mathcal{L}/\delta u = 0 \} \) which, in a suitable system of coordinates, is determined by a Lagrangian \( \mathcal{L} = [L \, dx \, dy] \) with the density \( L = -\frac{1}{2} \langle \kappa u_x, u_y \rangle - H_L(u, x, y) \). Set \( m = \partial \mathcal{L}/\partial w_y \). Suppose further that the system \( \mathcal{E}_L \) is Liouville-type and admits \( r \) conserved densities \( w[\mathbf{m}] = (w^1, \ldots, w^r) \in \ker D_y|_{\mathcal{E}_L} \). Let them be minimal such that \( f \in \ker D_y|_{\mathcal{E}_L} \) implies \( f = f(x, [w]) \). Introduce the operator
\[
\Box = (e^{(\mathbf{m})})^*. 
\]

Then we claim the following:

(i) All (up to \( x \leftrightarrow y \)) Noether symmetries \( \varphi_{\mathcal{L}} \) of the Lagrangial \( \mathcal{L} \) for \( \mathcal{E}_L \) are
\[
\varphi_{\mathcal{L}} = \Box \left( \frac{\delta \mathcal{H}(x, [w])}{\delta w} \right) \quad \text{for any } \mathcal{H}.
\]

(ii) All (up to \( x \leftrightarrow y \)) symmetries \( \varphi \) of the system \( \mathcal{E}_L \) are
\[
\varphi = \Box (\phi(x, [w])) \quad \text{for any } \phi = (\phi^1, \ldots, \phi^r) \in \mathcal{F} = \chi^{(\xi)}|_{w=w[\mathbf{m}]}.
\]

(iii) In the chosen system of coordinates, the image of the operator \( \Box \) is closed with respect to the commutation in the Lie algebra \( \text{sym} \mathcal{E}_L \).

(iv) Under a diffeomorphism \( \tilde{w} = \tilde{w}[w] \), the \( r \)-tuples \( \phi \) are transformed by
\[
\phi \mapsto \tilde{\phi} = \left( (e^{(\mathbf{w})})^* \right)^{-1} (\phi).
\]

Therefore, under any reparametrization \( \tilde{u} = \tilde{u}[u] \) of the dependent variables \( \tilde{u} = \{u^1, \ldots, u^m\} \) in equation \( \mathcal{E}_L \), and under a simultaneous change \( \tilde{w} = \tilde{w}[w] \), the
operator \( \Box \) obeys (15). Consequently, the operator \( \Box \) satisfies (11) in any system of coordinates, and hence \( \Box : \tilde{\mathcal{X}}(\xi)|_{u=[u]} \to \text{sym} \mathcal{E}_L \) is a Frobenius operator of second kind.

(v) The operator

\[
\mathcal{P} = \Box^* \circ (f^u_m)^* \circ \Box : \tilde{\mathcal{X}}(\xi)|_{u=[u]} \to \mathcal{X}(\xi)|_{u=[u]}
\]

is Hamiltonian.

(vi) The bracket \( \{\ , \}_\Box \) on the domain \( \mathcal{f} \) of the operator \( \Box \) satisfies the equality

\[
\{\ , \}_\Box = \{\ , \}_\mathcal{P}.
\]

Its right-hand side is calculated explicitly by using formula (46) that is valid for Hamiltonian operators \( \mathcal{P} \). This yields the commutation relations in the Lie algebra \( \text{sym} \mathcal{E}_L \).

(vii) All coefficients of the operator \( \mathcal{P} \) and of the bracket \( \{\ , \}_\Box \) are differential functions of the minimal conserved densities \( w \) for \( \mathcal{E}_L \).

2. Hamiltonian operators for PDE

In this section we recall necessary definitions and introduce standard notation, which follows [51, 9, 11, 22, 40, 51].

2.1. Symmetries and conservation laws. Let \( \pi : E^{n+m} \to M^n \) be a vector bundle over an \( n \)-dimensional manifold \( M \) and \( J^\infty(\pi) \) be the infinite jets over this bundle. By definition, set \( \pi_\infty : J^\infty(\pi) \to M^n \). Put \( f[u] = f(u, u_x, \ldots, u_\sigma) \), here \( u \) denotes the components of a vector \( f(u^1, \ldots, u^m) \) and \( |\sigma| < \infty \).

On \( J^\infty(\pi) \), there is the Cartan distribution \( C \) of \( n \)-dimensional planes that project without degeneration onto \( TM^n \) under \( \pi_\infty \). The distribution is spanned by the total derivatives \( D_{x^i} \), \( 1 \leq i \leq n \), whose action on \( \mathcal{F}(\pi) = C^\infty(J^\infty(\pi)) \) is defined by restriction onto the jets \( j_\infty(s) \) of sections \( s \in \Gamma(\pi) \),

\[
j_\infty(s)(D_{x^i}(f)) = \frac{\partial}{\partial x^i}(j_\infty(s)(f)), \quad \forall s \in \Gamma(\pi), \ \forall f \in \mathcal{F}(\pi).
\]

In coordinates, we have

\[
D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma+1} \frac{\partial}{\partial u^j_\sigma}.
\]

The Cartan connection \( \tilde{\nabla} : \frac{\partial}{\partial x^i} \mapsto D_{x^i} \) is flat.

Denote by \( \mathcal{X}(\pi) \) the \( \mathcal{F}(\pi) \)-module \( \Gamma(\pi_\infty^*(\pi)) = \Gamma(\pi) \otimes_{C^0(M^n)} C^\infty(J^\infty(\pi)) \). The sections \( \varphi = (\varphi^1, \ldots, \varphi^m) \in \Gamma(\pi_\infty^*(\pi)) \) of the induced fibre bundle “look like” sections \( s(x) = (s^1(x), \ldots, s^m(x)) \in \Gamma(\pi) \), but their components \( \varphi^i(x, [u]) \in \mathcal{F}(\pi) \) are functions on the jet space \( J^\infty(\pi) \), see also [124]. In coordinates, the evolutionary derivations with sections \( \varphi \in \mathcal{X}(\pi) \) are

\[
\mathcal{E}_\varphi = \sum_{i,\sigma} D_\sigma(\varphi^i) \cdot \frac{\partial}{\partial u^i_\sigma}.
\]

5The notation \( \mathcal{E}_\varphi \) makes no confusion with \( \mathcal{E} = \{u_t = \varphi(x, [u])\} \), because almost always we identify such evolutionary systems with \( \pi \)-vertical derivations \( \mathcal{E}_\varphi \in D^v(J^\infty(\pi)) \). A synonymic notation \( \Theta_\varphi \) or \( \partial_\varphi \) is used for [28] in the literature.
The invariant definition is \([\mathcal{E}_\varphi, D_x] = 0 \) for \(\mathcal{E}_\varphi \in D^v(J^\infty(\pi))\). We shall use the permutability of evolutionary fields \((28)\) with total derivatives \((27)\) many times.

The commutators of evolutionary fields \((28)\) endow \(\mathcal{K}(\pi)\) with a Lie algebra structure,

\[
[\mathcal{E}_{\varphi_1}, \mathcal{E}_{\varphi_2}] = \mathcal{E}_{[\varphi_1, \varphi_2]}, \quad \text{where } [\varphi_1, \varphi_2] = \mathcal{E}_{\varphi_1}(\varphi_2) - \mathcal{E}_{\varphi_2}(\varphi_1)
\]

which is defined by the componentwise action. In what follows, we identify evolutionary vector fields \(\mathcal{E}_\varphi\) with their generating sections \(\varphi\). We denote by \(\mathfrak{g}(\pi)\) the Lie algebra \((\mathcal{K}(\pi), [\cdot, \cdot])\) with the bracket \((29)\), and we call it the standard Lie algebra structure on \(\mathcal{K}(\pi)\). By definition, a recursion \(R\) for \(\mathfrak{g}(\pi)\) is a linear differential operator \(\mathcal{K}(\pi) \to \mathcal{K}(\pi)\) in total derivatives.

The linearizations (the Frechét derivatives),

\[
\ell_\psi = \sum_{\sigma} \frac{\partial \psi^i}{\partial u_{\sigma}^i} \cdot D_{\sigma}, \quad \psi \in \Gamma(\pi^\infty(\xi)) \text{ for some bundle } \xi \text{ over } M^n,
\]

are correlated with \((28)\) by \(\mathcal{E}_\varphi(\psi) = \ell_\psi(\varphi)\).

Under a change \(\hat{u} = \tilde{u}[u]\) of fibre coordinates in \(\pi\), the generating sections \(\varphi\) of \((28)\) are transformed by \(\varphi \mapsto \tilde{\varphi} = \ell^{(u)}(\varphi)\). Hence the recursion operators obey

\[
R \mapsto \hat{R} = \ell^{(u)}(R \circ \ell^{(u)}) \bigg|_{u = \tilde{u}[u]}.
\]

Let \(d_h = \hat{d}_{\text{dR}(M^n)}\) be the Cartan lifting of the de Rham differential on \(M^n\),

\[
d_h = \sum_{i=1}^n dx^i \otimes D_{x^i}.
\]

Let \(\hat{\Lambda}^i(\pi)\) be the \(\mathcal{F}(\pi)\)-module of \(i\)-th horizontal forms on \(J^\infty(\pi)\). Take the cohomology \(\hat{\Lambda}^i(\pi)/\{\text{im } d_h\}\) w.r.t. the horizontal differential \(d_h\) and denote it by \(\tilde{H}^i(\pi)\). So, elements \(\{H(x, [\tilde{u}] \text{ d vol}(M^n)) \in \tilde{H}^n(\pi)\) are equivalent if they differ by exact terms \(d_h\eta\), where \(\eta \in \tilde{H}^{n-1}(\pi)\). The equivalence classes \(\mathcal{H} \in \tilde{H}^n(\pi)\) of the highest horizontal cohomology will be called the Hamiltonians or the Lagrangians, whenever it is appropriate.

The Cartan differential on \(J^\infty(\pi)\) is \(d_C = d_{\text{dR}}(J^\infty(\pi)) - d_h\). The pair of differentials \((d_h, d_C)\) generates the variational bi-complex and its \(\mathcal{C}\)-spectral sequence. Passing to the horizontal cohomology at each term of the variational bi-complex, construct the first term \(E_1(\pi)\) of the \(\mathcal{C}\)-spectral sequence. Choose a restriction of \(d_C\) onto \(\tilde{H}^n(\pi)\), which is a corner of the bi-complex, to be the Euler variational derivative. (That is, apply the Cartan differential \(d_C\) and then perform multiple integration by parts, which does not alter the equivalence class modulo \(d_h\).) Denote by \(E\) the Euler variational derivative \((\delta/\delta u^1, \ldots, \delta/\delta u^m)\) with respect to \(u = \{u^1, \ldots, u^m\}\). Recall that the operator of linearization for the image of the variational derivative is self-adjoint,

\[
\ell_{E(\mathcal{H})} = \ell^*_{E(\mathcal{H})}, \quad \mathcal{H} \in \tilde{H}^n(\pi).
\]

For any \(\mathcal{H} = [H \text{ d vol}(M^n)] \in \tilde{H}^n(\pi)\) we have \(E(\mathcal{H}) = \ell^*_H(1)\). Consequently, under any change of variables \(\hat{u} = \tilde{u}[u]: J^\infty(\pi) \to \Gamma(\pi)\), a section \(\psi \in E(\mathcal{H})\) in the image of the variational derivative is transformed by \(\psi \mapsto \tilde{\psi} = \left((\ell^{(u)}_\hat{u})^*\right)^{-1}(\psi)\bigg|_{u = \tilde{u}[u]}\).

For any fibre bundle \(\xi\) over \(M^n\), the \(\mathcal{F}(\pi)\)-modules \(\Gamma(\pi^\infty_{\pi}(\xi))\) will be called horizontal. For any horizontal module \(f\), denote by \(\mathfrak{f}\) the dual \(\mathcal{F}(\pi)\)-module \(\mathfrak{f} = \text{Hom}_{\mathcal{F}(\pi)}(f, \Lambda^n(\pi))\).
We study differential equations in jet bundles, considering them together with all their differential-algebraic consequences, which we assume to exist. Let $\mathcal{E} \subset J^\infty(\pi)$ be an equation, and let the differential ideal $I(\mathcal{E})$ of the equation be generated in coordinates by $\mathcal{E} = \{F^1 = 0, \ldots, F^r = 0\}$, here $F^i \in \mathcal{F}(\pi)$. In other words, the $r$-tuple $\ell(F^1, \ldots, F^r)$ is a section of a horizontal bundle $\ell = \Gamma(\pi_\infty^*(\pi))$ for $\xi$ with $r$-dimensional fibres over $M^n$. Note that $\ell_F : \pi(\pi) \rightarrow \ell$.

Remark 1. There are two particular examples of $\xi$ that, unfortunately, often lead to confusion and tempt us to ignore the general setting:

- Determined autonomous evolutionary systems $\mathcal{E} = \{u^i = \varphi^i(x^1, \ldots, x^n, |u|), 1 \leq i \leq m\}$. Then $\ell \simeq \pi(\pi)$.
- Euler–Lagrange systems $\mathcal{E}_{\text{EL}} = \{E(\mathcal{L}) = 0, \mathcal{L} \in \mathcal{L}^n(\pi)\}$. Then $\ell \simeq \pi(\pi) = \text{Hom}_{\mathcal{F}(\pi)}(\pi(\pi), \Lambda^n(\pi))$.

In both cases, the number of equations $r$ equals the number of unknowns $m$, which can be used misleadingly to index the equations. We emphasize that, in principle, the reparametrizations of $F \in \ell$ that determine the system $\mathcal{E} = \{F = 0\}$ and of the dependent variables $u$ in $J^\infty(\pi)$ are not correlated at all.

In this paper, we consider:

- Hamiltonian operators $P : \hat{\pi}(\mathcal{E}) \rightarrow \pi(\mathcal{E})$ for determined evolutionary systems $\mathcal{E}$ (see section 2.2),
- Noether non-skew-adjoint operators $A : \hat{\pi}(\mathcal{E}) \rightarrow \pi(\mathcal{E})$ for the same class of equations (see section 2.3) and inverse Noether operators $\omega : \pi(\mathcal{E}) \rightarrow \hat{\pi}(\mathcal{E})$ (see Example 14),
- Frobenius operators $\square : \hat{\pi}(\mathcal{E}_1) \rightarrow \text{sym} \mathcal{E}_2$ that factor symmetry flows $\mathcal{E}_2$ on Euler–Lagrange equations $\mathcal{E}_{\text{EL}}$ (see section 3),
- Frobenius recursion operators $R \in \text{End}_{\mathbb{R}} \pi(\pi)$ (see Example 14), and
- Frobenius operators $A : \ell \subseteq \Gamma(\pi_\infty^*(\xi)) \rightarrow \pi(\pi)$ of general nature (see section 3).

Let $\mathcal{E} = \{F = 0\}$ be a system of differential equations given by a section $F \in \ell$. Let a class $[\eta] \in \mathcal{H}^{n-1}|_{\mathcal{E}}$ be a conservation law. In other words, the divergence

$$d_h \eta = \bar{\nabla}(F)$$

vanishes on the ideal $J(\mathcal{E})$ of algebraic-differential consequences of $\mathcal{E}$, and this is realized by $r$ operators $\nabla = (\nabla^1, \ldots, \nabla^r)$ in total derivatives. The coefficients of $\bar{\nabla}$ belong to $C^\infty(\mathcal{E})$ and the operators take values in the module $\Lambda^n(\pi)$ of highest horizontal forms.

Represent the $n$-th horizontal form $d_h \eta$ as $\bar{\nabla}(F) = \langle 1, \nabla(F) \rangle$, where $\nabla$ coincides with $\bar{\nabla}$ in any local coordinates but takes values in $\ell$ instead of $\Lambda^n(\pi)$. Here 1 stands for the $r$-tuple $(1, \ldots, 1)$ and $\langle , \rangle : \ell \times \ell \rightarrow \Lambda^n(\pi)$ is the coupling. Integrating by parts and introducing the adjoint operator $\nabla^*$, we obtain $\langle 1, \nabla(F) \rangle = \langle \nabla^*(1), F \rangle \mod \text{im} \, d_h$. The section $\nabla^*(1) \in \ell$ is called the generating section of a conservation law $[\eta]$ for the equation $\mathcal{E} = \{F = 0\}$. By construction, $\psi = \nabla^*(1) \in \ell$ has as many components as there are equations in the system. Under a reparametrization $F = \Delta(\hat{F})$ of equations
that constitute the system $\mathcal{E}$, where $\Delta$ is a linear differential operator in total derivatives, the section $\psi \in \hat{\mathcal{F}}$ is transformed by $\psi \mapsto \hat{\psi} = \Delta^*(\psi)$. The generating sections $\psi$ of conservation laws are solutions of the equation $\ell_{\psi}^*(\psi) = 0$ on $\mathcal{E}$; this follows from the formulas $0 \equiv \mathbf{E}(d_t \eta) = \ell_{\psi}^*(\psi)(1) = \ell_{\psi}^*(\psi) + \ell_{\psi}^*(F) \equiv 0$, where the second summand is set to zero by restriction onto the consequences of $\mathcal{E} = \{F = 0\}$.

We conclude that $\psi \in \hat{\mathcal{F}}(\mathcal{E})$ for determined evolutionary systems (this is why the notation $\psi \in \cosym \mathcal{E}$ is used), and $\psi \in \hat{\mathcal{F}}(\mathcal{E})$ for Euler–Lagrange systems $\mathcal{E}_{EL}$.

**Lemma 1 ([40]).** Let $\mathcal{E} = \{F \equiv u_t - f(x, [u]) = 0\}$ be an evolutionary system. For any conservation law $[\eta] = [\rho \, dx + \cdots]$ such that $d_u(\eta) = \nabla(F) = \langle \nabla^*(1), F \rangle$, the generating section $\psi = \nabla^*(1)$ of $[\eta]$ is the ‘gradient’ $\psi = \mathbf{E}(\rho)$ of the conserved density $\rho$.

For the Euler–Lagrange equations $\mathcal{E}_{EL} = \{\mathbf{E}(\mathcal{L}) = 0\}$, the generating sections $\psi \in \hat{\mathcal{F}}(\mathcal{E}_{EL})$ of conservation laws have a geometric nature of symmetries. This is indeed so.

**Theorem 2 (Noether).** Let $\mathcal{E}_{EL} = \{\mathbf{E}(\mathcal{L}) = 0\}$ be the Euler–Lagrange equation for a Lagrangian $\mathcal{L} \in \bar{\mathcal{H}}^n(\pi)$. Then the evolutionary derivation $\mathcal{E}_{\varphi}$ is a Noether symmetry of the Lagrangian, $\mathcal{E}_{\varphi}(\mathcal{L}) = 0$, if and only if $\varphi$ is the generating section of a conservation law $[\eta]$ such that $d_u \eta = 0$ on $\mathcal{E}_{EL}$.

**Lemma 3 ([40], see a proof in [28]).** The relation
\[
\mathbf{E}(\mathcal{E}_{\varphi}(\rho)) = \mathcal{E}_{\varphi}(\mathbf{E}(\rho)) + \ell_{\varphi}^*(\mathbf{E}(\rho))
\]
holds for any $\varphi \in \mathcal{X}(\pi)$ and $\rho \in \bar{\Lambda}^n(\pi)$.

Consequently, any Noether symmetry $\varphi_L \in \text{sym}_L \mathcal{E}_{EL}$ of a Lagrangian is a symmetry of the Euler–Lagrange equation $\mathcal{E}_{EL} = \{\bar{F} = 0\}$ (and Lemma 3 shows also why the converse is not true). For this reason, mappings of generating sections of conservation laws to symmetries of differential equations will be called Noether in what follows. We have demonstrated that the identity transformation is Noether for Euler–Lagrange equations $\mathcal{E}_{EL} = \{\mathbf{E}(\mathcal{L}) = 0\}$.

In the sequel, we apply Lemma 3 for other purposes as well; namely, it will allow us to induce a bracket ([55b]) on $\hat{\mathcal{F}}$ starting with a bracket ([50a]) of conservation laws for evolutionary systems $\mathcal{E}$.

### 2.2. Poisson structures.

In the preceding section we have recalled that the image of Euler derivative is dual to the module $\mathcal{X}(\pi)$ of evolutionary fields. The standard concept of Hamiltonian dynamics for PDE (see [9, 22, 40]) is based on a class of Hamiltonian operators that map $\mathcal{X}(\pi) \to \mathcal{X}(\pi)$ and which induce a Poisson structure on $\bar{\mathcal{H}}^n(\pi)$ due to the existence of a coupling $\langle \cdot, \cdot \rangle : \mathcal{X}(\pi) \times \mathcal{X}(\pi) \to \bar{\Lambda}^n(\pi)$.

In this section we consider Hamiltonian operators. First we regard them as mappings of $\mathcal{F}(\pi)$-modules, and then as Lie algebra homomorphisms. Next, we restrict these homomorphisms onto evolutionary systems and, finally, to hierarchies of systems.

Now we recall necessary algebraic constructions.

---

6If the nondegenerate differential substitution $\Delta$ is nonlinear, e.g., $\bar{F} = \bar{F} + \bar{F}^2$, then $\psi$ becomes a nonlinear operator on the module $f$. This is a difficulty of the theory, see footnote 13 on p. 28. Hence only linear coordinate transformations are declared admissible in the module of equations; in the sequel, nonlinear changes of fibre coordinates will be allowed for horizontal bundles of other geometric nature.
For any vector space $V$ over $\mathbb{R}$, let $\Delta \in \text{Hom}_\mathbb{R}(\bigwedge V, V)$ and $\nabla \in \text{Hom}_\mathbb{R}(\bigwedge^1 V, V)$. Denote by $\Delta[\nabla] \in \text{Hom}_\mathbb{R}(\bigwedge^{k+l-1} V, V)$ the action $\Delta[\cdot] : \text{Hom}_\mathbb{R}(\bigwedge^N V, V) \to \text{Hom}_\mathbb{R}(\bigwedge^{N+k-1} V, V)$ of $\Delta$ on $\nabla$, which is given by the formula

$$\Delta[\nabla](a_1, \ldots, a_{k+l-1}) = \sum_{\sigma \in S_{k+l-1}} (-1)^{\sigma} \Delta(\nabla(a_{\sigma(1)}, \ldots, a_{\sigma(l)}), a_{\sigma(l+1)}, \ldots, a_{\sigma(k+l-1)}),$$

where $a_i \in V$ and $S_m^k \subset S_m$ denotes the unshuffles. The unshuffles are permutations such that $\sigma(1) < \sigma(2) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(m)$ for all $\sigma \in S_m^k$; note that $\sigma(i)$ is the index of the object placed onto the $i$-th position under the permutation, unlike in $[21]$. The Schouten (Richardson–Nijenhuis) bracket $[[\Delta, \nabla]] \in \text{Hom}_\mathbb{R}(\bigwedge^{k+l-1} V, V)$ of $\Delta$ and $\nabla$ is $[9, 27]$

$$[[\Delta, \nabla]] = \Delta[\nabla] - (-1)^{(k-1)(l-1)}\nabla[\Delta]. \quad (35)$$

Thus the commutator $[X, Y] = [X, Y]$ of two evolutionary vector fields $X, Y \in \mathfrak{g}(\pi)$ is skew-symmetric. The bracket $[[A_1, A_2]] \in \bigwedge^3 V$ of two bi-vectors is symmetric w.r.t. $A_1$ and $A_2$, but it is skew-symmetric w.r.t. its arguments $\psi_1, \psi_2, \psi_3 \in V^*$.

**Definition 1.** A linear skew-adjoint $(m \times m)$-matrix operator $A : \mathcal{K}(\pi) \to \mathcal{K}(\pi)$ in total derivatives is called Hamiltonian if, for any $H_1, H_2 \in \bar{H}^n(\pi)$, the bi-linear skew-symmetric bracket

$$\{\cdot, \cdot\}_A : \bar{H}^n(\pi) \times \bar{H}^n(\pi) \to \bar{H}^n(\pi), \quad \{H_1, H_2\}_A := \langle E(H_1), A(E(H_2)) \rangle$$

satisfies the Jacobi identity

$$\sum_{\emptyset} \{\{H_1, H_2\}_A, H_3\}_A = 0. \quad (36)$$

By construction of the Poisson bracket $\{\cdot, \cdot\}_A$, its equivalent definitions are

$$\{H_1, H_2\}_A = \langle \psi_1, A(\psi_2) \rangle \sim E_{A(\psi_2)}(H_1) \sim -E_{A(\psi_1)}(H_2) \quad (37)$$

modulo ($\sim$) exact terms, here $\psi_i = E(H_i)$. The bracket $\{\cdot, \cdot\}_A$ endows $\bar{H}^n(\pi)$ with a Lie algebra structure over $\mathbb{R}$. Two Hamiltonian operators $A_1$ and $A_2$ are called compatible if their linear combinations $\lambda_1 A_1 + \lambda_2 A_2$ are Hamiltonian as well.

We postpone examples of Hamiltonian operators to section [2.4], where Noether operators will be considered. In example [12] on p. [34] we investigate a wider class of Hamiltonian operators defined on $\mathfrak{a}$-modules over commutative $\mathfrak{k}$-algebras $\mathfrak{a}$.

**Lemma 4.** Suppose that evolution equations $\mathcal{E} = \{F = u_t - f[u] = 0, F \in \mathcal{F}\}$ are enumerated by the dependent variables $u$, which means that the isomorphism $f \simeq \mathcal{K}(\pi)$ is being used. Then, under a fibre coordinate change $\hat{u} = \hat{u}[u]$ in $\pi$ that preserves the evolutionary form of the equations $\mathcal{E}$, a Hamiltonian operator $A$ is transformed by

$$A \mapsto \hat{A} = \ell^{(u)}_{\hat{u}} \circ A \circ (\ell^{(u)}_{\hat{u}})^* \bigg|_{u = u[\hat{u}]} \quad (38)$$

Formula (38) is also valid for any Noether operator $A : \mathcal{K}(\pi) \to \mathcal{K}(\pi)$.

---

7The definition of Schouten bracket is valid over any field $\mathfrak{k}$ such that char $\mathfrak{k} \neq 2$.

8We respect Dirac’s notation. Our choice of the signs in (37) is such that we multiply by bra-covectors $\langle \psi \rangle$ from the left and have the ket-vectors $|\varphi\rangle$ standing on the right. However, sometimes we use the reversed coupling $\langle \cdot, \cdot \rangle : \mathfrak{f} \times \mathfrak{f} \to \Lambda^0(\pi)$ for convenience.
Hamiltonian operators $A$ can be regarded \([9,22]\) as the variational Poisson bi-vectors with vanishing Schouten brackets \([A, A]=0\) such that the Poisson bracket \(\{\mathcal{H}_1, \mathcal{H}_2\}_A = \{[A, \mathcal{H}_2], \mathcal{H}_1\}\) is a derived bracket in the sense of \([33]\). The compatibility condition for the Poisson bi-vectors is \([A_1, A_2]=0\).

By definition, put
\[
\ell_{A,\psi}(\varphi) := (E_\varphi(A))(\psi)
\]
for any $\varphi \in \mathcal{C}(\pi)$, $\psi \in \mathfrak{g} = \Gamma(\pi^*_\mathcal{H}(\xi))$ for some $\xi$, and an operator $A \in \text{CDiff}(\mathfrak{g}, \mathcal{C}(\pi))$ in total derivatives. Note that $\ell_A$ is an operator in total derivatives w.r.t. its argument $\varphi$ and w.r.t. $\psi$ (but not w.r.t. the coefficients of $A$), and hence the adjoint $\ell_A^*$ is well defined. We emphasize that the notation $\ell_{A,\psi}$ is not the same as the linearization $\ell_{A(\psi)}$, which was introduced in \((30)\).

**Lemma 5 (\([10]\)).** Let $A$ be a matrix operator in total derivatives as in \((39)\). Then one has
\[
\ell_{A,\psi_1}(\psi_2) = \ell_{A,\psi_2}(\psi_1)
\]
for any sections $\psi_1, \psi_2$ of a horizontal module over $J^\infty(\pi)$.

**Proposition 6** (A criterion of \([A, A]=0, \([10]\)).** A skew-adjoint operator $A \in \text{CDiff}(\mathfrak{g}(\pi), \mathcal{C}(\pi))$ in total derivatives is Hamiltonian if and only if the relation
\[
\ell_{A,\psi_1}(A(\psi_2)) - \ell_{A,\psi_2}(A(\psi_1)) = A(\ell_{A,\psi_2}(\psi_1))
\]
holds for all $\psi_1, \psi_2 \in \mathfrak{g}(\pi)$. The r.h.s. of formula \((40)\) is skew-symmetric w.r.t. $\psi_1, \psi_2$.

The proof is informative in itself. It amounts to a straightforward calculation of the value of variational Schouten bracket $[A, A]$ on three Hamiltonians $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \in \mathcal{H}^n(\pi)$. Let $\psi_i = E(\mathcal{H}_i)$ be the respective gradients. The Jacobi identity $[A, A](\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) = 0$ can be expressed as $\langle b(\psi_1, \psi_2), \psi_3 \rangle = 0$, where $b$ is a differential operator w.r.t. each argument and $\langle , , \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathcal{H}^n(\pi)$. Since $\psi_3 \in \mathfrak{g}$ is arbitrary, we have $b(\psi_1, \psi_2) = 0$ for all $\psi_1, \psi_2 \in \mathfrak{g}$. The calculation shows that $b(\psi_1, \psi_2)$ is equal to the left-hand side minus the right-hand side of \((40)\).

**Lemma 7** (\([10], \S 7.8\)). Consider an operator in total derivatives $J \in \text{CDiff}(\mathfrak{g}(\pi), P)$ which is skew-symmetric w.r.t. its $l$ arguments that belong to $\mathfrak{g}(\pi)$ and which takes values in an $\mathcal{F}(\pi)$-module $P$. If for all $\mathcal{H}_1, \ldots, \mathcal{H}_l \in \mathcal{H}^n(\pi)$ one has $J(E(\mathcal{H}_1), \ldots, E(\mathcal{H}_l)) = 0$, then $J \equiv 0$.

Lemma \([7]\) implies that the Jacobi identity $J(\psi_1, \psi_2, \psi_3) = 0$ can be verified for elements $\psi_i \in \text{im} E$ only (in particular, this is done in the proof of Proposition \([6]\)).

**Proof of Proposition \([6]\).** Let $\mathcal{H}_\alpha, \mathcal{H}_\beta$, and $\mathcal{H}_\gamma$ be Hamiltonians. The Jacobi identity is
\[
\{\{\mathcal{H}_\alpha, \mathcal{H}_\beta\}_A, \mathcal{H}_\gamma\}_A + \{\{\mathcal{H}_\beta, \mathcal{H}_\gamma\}_A, \mathcal{H}_\alpha\}_A + \{\{\mathcal{H}_\gamma, \mathcal{H}_\alpha\}_A, \mathcal{H}_\beta\}_A = - \sum \mathcal{E}_{A(\psi_\gamma)}(\langle A(\psi_\alpha), \psi_\beta \rangle)
\]
\[
- \sum [\mathcal{E}_{A(\psi_\gamma)}(A(\psi_\alpha))(\psi_\beta) + \langle A(\mathcal{E}_{A(\psi_\gamma)}(A(\psi_\alpha)))(\psi_\beta) + \langle A(\psi_\alpha), \mathcal{E}_{A(\psi_\gamma)}(A(\psi_\beta))\rangle] = 0. \tag{41}
\]
Consider the elements of the second sum,

\[ \langle A(\mathcal{E}_{A(\psi_1)}(\psi_\alpha)), \psi_\beta \rangle = \langle \psi_\beta, A(\mathcal{E}_{A(\psi_1)}(\psi_\alpha)) \rangle = -\langle A(\psi_\beta), \mathcal{E}_{A(\psi_1)}(\psi_\alpha) \rangle \]

Substituting this back in (41), we obtain

\[ -\langle A(\psi_\beta), \ell_{\psi_\alpha}(A(\psi_\gamma)) \rangle = -\langle \ell_{\psi_\alpha}(A(\psi_\beta)), A(\psi_\gamma) \rangle = (\text{by (33)}) \]

\[ = -\langle \ell_{\psi_\alpha}(A(\psi_\beta)), A(\psi_\gamma) \rangle = -\langle A(\psi_\gamma), \ell_{\psi_\alpha}(A(\psi_\beta)) \rangle. \]

Next, the second summand in (42) is equal to

\[ \langle A(\psi_3), \psi_2 \rangle - \langle A(\psi_2), \psi_1 \rangle - \langle A(\psi_2), \psi_3 \rangle. \]

Now set \( \alpha = 3, \beta = 2, \gamma = 1 \); thence we have

\[ 0 = -\langle (\mathcal{E}_{A(\psi_1)})(\psi_3), \psi_2 \rangle - \langle (\mathcal{E}_{A(\psi_1)})(\psi_2), \psi_1 \rangle - \langle (\mathcal{E}_{A(\psi_2)})(\psi_1), \psi_3 \rangle. \]

Consider the first summand,

\[ \langle (\mathcal{E}_{A(\psi_1)})(\psi_3), \psi_2 \rangle = \langle A(\psi_3), \mathcal{E}_{A(\psi_1)}(\psi_2) \rangle = (\text{by Lemma 5}) \]

\[ = \langle A(\psi_3), \mathcal{E}_{A,\psi_1}(\psi_2) \rangle = \langle A(\psi_3), \mathcal{E}_{A,\psi_1}(\psi_1) \rangle, \]

\[ = -\langle A(\psi_3), \mathcal{E}_{A,\psi_1}(\psi_1) \rangle = (43a). \]

Next, the second summand in (42) is equal to

\[ \langle (\mathcal{E}_{A(\psi_2)})(\psi_2), \psi_1 \rangle = \langle A(\psi_2), \mathcal{E}_{A,\psi_1}(\psi_1) \rangle = -\langle A(\mathcal{E}_{A,\psi_1}(\psi_1)), \psi_3 \rangle = (43b). \]

Now consider the third term in the right-hand side of (42),

\[ \langle (\mathcal{E}_{A(\psi_3)})(\psi_1), \psi_3 \rangle = \langle A(\psi_3), \mathcal{E}_{A,\psi_1}(\psi_3) \rangle. \]

(43c)

Substituting (43) in (42), we finally obtain

\[ \langle \mathcal{E}_{A,\psi_1}(A(\psi_1)), \psi_3 \rangle + \langle A(\mathcal{E}_{A,\psi_1}(A(\psi_1)), \psi_3) - \langle (\mathcal{E}_{A,\psi_1}(A(\psi_2)), \psi_3) = 0, \]

whence follows (40). The proof is complete. \( \square \)

Actually, Proposition 6 states that images of Hamiltonian operators are closed w.r.t. the commutation. This is readily seen from (45b) below, which contains the left-hand side of (40). Note that, by Lemma 7, the assertion holds for Hamiltonian evolutionary vector fields \( \mathcal{E}_{A(\psi)} \) which may not possess Hamiltonians \( \mathcal{H} \) such that \( \psi = E(\mathcal{H}) \). However, the Lie algebra of Hamiltonian evolutionary vector fields \( \mathcal{E}_{A(E(\mathcal{H}))} \in \mathfrak{g}(\pi) \) that do possess the Hamiltonians \( \mathcal{H} \) is correlated by a morphism of Lie algebras with the Lie algebra \( (H^n(\pi), \{ , \}, A) \) as follows.

**Proposition 8** ([9 §27]). The Euler derivative \( E \) and Hamiltonian operators \( A \) determine the Lie algebra morphisms

\[ (H^n(\pi), \{ , \}, A) \xrightarrow{E} (\hat{\mathcal{H}}(\pi), \{ , \}, A) \xrightarrow{\mathcal{E}_{A(\cdot)}} (\mathfrak{g}(\pi), \{ , \}), \]

such that

\[ A([\psi_1, \psi_2]_A) = [A\psi_1, A\psi_2], \quad \psi_1, \psi_2 \in \hat{\mathcal{H}}(\pi) \]
and
\[
[E(\mathcal{H}_1), E(\mathcal{H}_2)]_A = E(\{\mathcal{H}_1, \mathcal{H}_2\}_A), \quad \mathcal{H}_1, \mathcal{H}_2 \in \bar{H}^n(\pi).
\]
The correlation between the Poisson bracket \{ , \}_A on \bar{H}^n(\pi), the Koszul–Dorfman–Daletsky–Karasëv bracket \[ , \]_A on the quotient
\[
\Omega^1(\pi) := \mathcal{K}(\pi)/\ker A,
\]
and the standard commutator [ , ] of evolutionary fields \( E_{A(\cdot)} \), see (29), is
\[
[\psi_1, \psi_2]_A = E_{A(\psi_1)}(\psi_2) - E_{A(\psi_2)}(\psi_1) + \{\psi_1, \psi_2\}_A = E\left(\langle\psi_1, A(\psi_2)\rangle\right),
\]
where
\[
A\left(\{\psi_1, \psi_2\}_A\right) = \left(E_{A\psi_1}(A)\right)(\psi_2) - \left(E_{A\psi_2}(A)\right)(\psi_1).
\]
For \( A \) Hamiltonian, both \( \ker A \subset (\mathcal{K}(\pi), [ , ]_A) \) and \( \operatorname{im} A \subset (\mathfrak{g}(\pi), [ , ]) \) are ideals in the respective Lie algebras.

Formula (40) provides the expression for the Sokolov bracket \{ , \}_A,
\[
\{\psi_1, \psi_2\}_A = \ell^*_A(\psi_2), \quad \psi_1, \psi_2 \in \mathcal{K}(\pi),
\]
which is valid for Hamiltonian operators \( A \) if they are nondegenerate,\(^9\) in the sense of (59), see below. In coordinates, we have that for a Hamiltonian operator \( A = \| \sum_{\tau} \hat{A}^{\alpha\beta}_\tau D_\tau \| \) and \( \psi_1, \psi_2 \in \mathcal{K}(\pi), \)
\[
\{\psi_1, \psi_2\}_A = \sum_{\sigma, \alpha} (-1)^{\sigma} \left(D_\sigma \circ \left[ \sum_{\tau, \beta} \partial A^{\alpha\beta}_{\tau} \cdot \left(D_\tau(\psi^1_\beta) \cdot \partial \psi^2_\alpha\right)\right]\right)(\psi^1_\alpha).
\]
Another way of calculating Dorfman’s bracket \[ , \]_A for any Hamiltonian operator \( A \) is based on the equivalence (57) and further use of Lemma 3, see (55b) in section 2.4.

The property \( A: \mathcal{K}(\mathcal{E}) \rightarrow \mathcal{K}(\mathcal{E}) \) of Hamiltonian operators remains valid for their restrictions onto the equations they determine. We have it as follows.

**Lemma 9 (40).** Let \( A \) be a Hamiltonian operator and consider a Hamiltonian evolutionary system
\[
u_t = A(E(H)), \quad H \in \bar{H}^n(\pi).
\]
Then the operator \( A \) takes the generating sections \( \psi = E(\rho) \in \mathcal{K}(\mathcal{E}) \) of conservation laws \( [\eta] = [\rho \, dx + \cdots] \) for (47) to generating sections \( \varphi \in \mathcal{K}(\mathcal{E}) \) of symmetries \( E_\varphi \in \operatorname{sym} \mathcal{E} \) for this system.

Next, recall that the Schouten bracket (85) of variational bi-vectors satisfies the Jacobi identity
\[
[[A_1, A_2], A_3] + [[A_2, A_3], A_1] + [[A_3, A_1], A_2] = 0.
\]
Hence the original Jacobi identity \( [[A, A](\psi_1, \psi_2, \psi_3) = 0 \) for the arguments of \( A \) implies that \( \partial A = [A, \cdot] \) is a differential, giving rise to the Poisson cohomology \( H^k_A(\pi) \).

\(^9\) Hence the image of the Euler operator \( E \) is closed with respect to the commutation.
\(^10\) A coordinate-free definition of the nondegenerate Hamiltonian operators means that they have trivial kernels on the \( \ell^* \)-coverings over the jet bundle \( J^\infty(\pi) \), see (22) for the construction of \( \Delta \)-coverings over PDE.
Obviously, the Casimirs \( c \in \bar{H}^n(\pi) \) such that \([A, c] = 0\) for a Hamiltonian operator \( A \) constitute \( H^0_A(\pi) \).

**Theorem 10** (The Magri scheme \([9, 44]\)). Suppose \([A_1, A_2] = 0, H_0 \in H^0_{A_1}(\pi)\) is a Casimir of \( A_1 \), and \( H^1_{A_1}(\pi) = 0 \). Then for any \( k > 0 \) there is a Hamiltonian \( H_k \in \bar{H}^n(\pi) \) such that

\[
[A_2, H_{k-1}] = [A_1, H_k].
\]

Put \( \varphi_k := A_1(\mathbb{E}(H_k)) \). The Hamiltonians \( H_i, i \geq 0, \) pairwise Poisson commute w.r.t. either \( A_1 \) or \( A_2 \), the densities of \( H_i \) are conserved on any equation \( u_{t_k} = \varphi_k \), and the evolutionary derivations \( \mathcal{E}_{\varphi_k} \) pairwise commute for all \( k \geq 0 \).

We emphasize that Theorem 10 holds for the Hamiltonians which belong to the linear subspaces \( S \subset \bar{H}^n(\pi) \) spanned by the descendants of the Casimirs \( H_0 \in H^0_{A_1}(\pi) \). The commutativity of flows \( \varphi_k \) and formulas (47) imply that the phase volume \( \int_S \langle \psi_1, A(\psi_2) \rangle \) is conserved on the subspaces \( S \).

The Magri scheme starts from any Hamiltonians \( H_{k-1}, H_k \in \bar{H}^n(\pi) \) that satisfy (49), but we always intend to operate with maximal subspaces of \( H^n(\pi) \), and therefore we require \([A_1, H_0] = 0\) such that the sequences of Hamiltonians can not be extended with \( k < 0 \).

**Proof of Theorem 10.** The main homological equality (45) is established by induction on \( k \). Starting with a Casimir \( H_0 \), we obtain

\[
0 = [A_2, 0] = [A_2, [A_1, H_0]] = -[A_1, [A_2, H_0]] \quad \text{mod} \quad [A_1, A_2] = 0
\]

using the Jacobi identity (48). The first Poisson cohomology \( H^1_{A_1}(\pi) = 0 \) is trivial by assumption of the theorem, and hence the closed element \([A_2, H_0] \) in the kernel of \([A_1, \cdot] \) is exact: \([A_2, H_0] = [A_1, H_1] \) for some \( H_1 \).

For \( k \geq 1 \), we have

\[
[A_1, [A_2, H_k]] = -[A_2, [A_1, H_k]] = -[A_2, [A_2, H_{k-1}]] = 0
\]

using the Jacobi identity (48) for the Schouten bracket (35) and by \([A_2, A_2] = 0\). Hence \([A_2, H_k] = [A_1, H_{k+1}] \) by \( H^1_{A_1}(\pi) = 0 \), and thus we proceed infinitely. \( \square \)

Correlated Magri’s schemes for coupled hierarchies are further considered in section 5, see diagram (113).

**2.3. Lie algebroids.** Hamiltonian operators determine Lie algebra homomorphisms (44) that map the Poisson bracket (37) to the standard Lie structure (29) on \( \kappa(\pi) \). In the finite-dimensional case, this yields the Lie algebra homomorphisms \( P: \Gamma(T^*M) \to \Gamma(TM) \) to sections of the tangent bundles for smooth manifolds \( M \). Recall that a very important class of the Lie algebra homomorphisms to \( \Gamma(TM) \) is then provided by the Lie algebroids.

**Definition 2** ([59]). A **Lie algebroid** over an \( n \)-dimensional manifold \( M^n \) is a vector bundle \( \Omega \to M^n \) whose sections \( \Gamma\Omega \) are equipped with a Lie algebra structure \([\cdot, \cdot]_a\) together with a morphism of vector bundles \( a: \Omega \to TM \), called the **anchor**, such that the Leibnitz rule

\[
[A_1, f \cdot \psi_2]_a = f \cdot [A_1, \psi_2]_a + (a(\psi_1)f) \cdot \psi_2
\]

holds for any \( \psi_1, \psi_2 \in \Gamma\Omega \) and \( f \in C^\infty(M^n) \).
Equivalently, a Lie algebroid structure on $\Omega$ is a homological vector field $Q$ on $\Pi\Omega$ (take the fibres of $\Omega$, reverse their parities, and thus obtain the new super-bundle $\Pi\Omega$ over $M^n$). The homological vector fields, which are differentials on $C^\infty(\Pi\Omega) = \Gamma(\Lambda^*\Omega^*)$, equal

$$Q = \tilde{y}^i a_i^\alpha(x) \frac{\partial}{\partial x^\alpha} + \frac{1}{2} \tilde{y}^i \tilde{y}^j c_{ij}^k(x) \frac{\partial}{\partial \tilde{y}^k}, \quad Q^2 = 0,$$

(51)

where

- $(x^\alpha)$ is a system of local coordinates near a point $x \in M^n$,
- $(\tilde{y}^i)$ are local coordinates along the fibres of $\Omega$ and $(\tilde{y}^i)$ are the respective coordinates on $\Pi\Omega$, and
- $a(e_i) = a_i^\alpha(x) \cdot \partial/\partial x^\alpha$ is the image under the anchor $a$ and $[e_i, e_j]_a = c_{ij}^k(x)e_k$ give the structural constants for a local basis $(e_i)$ of sections $\Gamma$, respectively.

The proof of equivalence is straightforward, see [59, 33] for details and other equivalent definitions.

**Example 2.** Consider the Cartan connection (27), and set the anchor $a : C \to \Gamma(TM^n)$ to be $a = \pi_\infty \circ j_\infty(s_0)_*$, which is the restriction $j_\infty(s_0)_*(X)$ of horizontal fields $X$ onto the jet $j_\infty(s_0)$ of a fixed section $s_0 \in \Gamma(\pi)$ and further projection under $\pi_\infty : J^\infty(\pi) \to M^n$. The corresponding homological vector field is the restriction $j_\infty(s_0)^*(dh)$ of the horizontal differential (52), see [33].

**Example 3.** Let $P : \mathcal{X}(\pi) \to \mathcal{X}(\pi)$ be a Hamiltonian operator and $\{,\} : \mathcal{F}(\pi) \times \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ be the Poisson bracket. Consider the one-dimensional bundle over $J^\infty(\pi)$ with the fibre $dx^1 \wedge \ldots \wedge dx^n = d\text{vol}(M^n)$ at each point of $M^n$. The equivalence classes of sections constitute an $\mathcal{F}(\pi)$-module and are endowed with the Lie algebra structure $\{,\}_P$. Set the anchor $a := \mathcal{E}_{P,\mathcal{F}}$. By (37), the Leibnitz rule (50) holds:

$$\{\mathcal{H}_1, f \cdot \mathcal{H}_2\}_P = f \cdot \{\mathcal{H}_1, \mathcal{H}_2\}_P + a(\mathcal{H}_1)(f) \cdot \mathcal{H}_2, \quad \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{F}(\pi), \quad f \in \mathcal{F}(\pi).$$

(52)

The sections $\mathcal{H} \in H^n(\pi)$ are parameterized by the Hamiltonian densities $\hat{h} \in \mathcal{F}(\pi)$. Let $\hat{h}$ be the functional coordinates in $\Pi\hat{H}^n(\pi)$, whose fibre is parity-dual to $\hat{h}$. Then the homological vector field $Q$ that encodes the Lie algebroid $(\hat{H}^n(\pi), \{,\}_P) \xrightarrow{a} (\mathcal{X}(\pi), [\cdot,\cdot])$ is equal to

$$Q = \int_{\mathcal{F}(\pi)} \left[ \hat{h} \cdot \mathcal{E}_{P,\mathcal{F}}(\hat{h} d\text{vol}(M^n)) \right] + \frac{1}{2} \hat{h}_2 \hat{h}_1 \left\{ [\hat{h}_1 d\text{vol}(M^n), [\hat{h}_2 d\text{vol}(M^n)] \right\}_P \cdot \frac{\partial}{\partial \hat{h}} \right] d\mathcal{F}(\pi).$$

Proposition 8 states that $a$, which is defined by a Hamiltonian operator $P$, is a morphism of Lie algebras. This property was not included into the definition of Lie algebroids because, even in a general situation not related to Hamiltonian operators, the property is a consequence of the Leibnitz rule (50) and of the Jacobi identity for $[\cdot,\cdot]_a$.

**Proposition 11** (33). The anchors $a$ map the brackets $[\cdot,\cdot]_a$ between sections of bundles $\Omega$ over finite-dimensional manifolds $M^n$ to the Lie bracket $[\cdot,\cdot]$ on the tangent bundle to $M^n$. 
The converse is not true. Namely, there are morphisms of Lie algebra structures in the modules of sections of fibre bundles over the base manifold that do not respect the Leibnitz rule (50). Moreover, and this is our claim of crucial importance, the converse of Proposition 11 is not valid for Hamiltonian operators \( L \) of total derivatives and the morphisms \( a = \mathcal{E}_{P(\cdot)} : \Omega^1(\pi) \to \mathfrak{g}(\pi) \), where Dorfman’s bracket \([\cdot, \cdot]_P\) on \( \Omega^1(\pi) \) is (45a).

Indeed, there is no Leibnitz rule over morphisms \( \rho_1, \rho_2 \), and this is our claim of crucial importance, the converse of Dorfman’s bracket \([\cdot, \cdot]_P\) of 1-forms \( \psi_1, \psi_2 \in \Gamma(T^*M^n) \),

\[
[\psi_1, \psi_2]_P = L_{\rho_1 \psi_1} \psi_2 - L_{\rho_2 \psi_2} \psi_1 + d_{\text{dR}(M^n)}(\mathcal{P}(\psi_1, \psi_2)), \quad \mathcal{P} \in \Gamma\left( \bigwedge^2 TM^n \right), \tag{53}
\]
to Lie algebroid structures over the base manifolds \( J^\infty(\pi) \).

**Counterexample 4.** Consider the first Hamiltonian structure \( P_1 = D_x \) for the KdV equation (11), and consider two conserved densities \( \rho_1 = w \) and \( \rho_2 = \frac{1}{4} w^2 \). Obviously, \( \{\rho_1 \, dx, \rho_2 \, dx\} P_1 = 0 \), and hence Dorfman’s bracket \([\psi_1, \psi_2]_{P_1}\) of the gradients \( \psi_i = \mathcal{E}_w(\rho_i) \) also vanishes. Now multiply \( \rho_1 \) by any nonlinear \( f(x) \in C^\infty(M) \) and, applying Euler derivative \( \mathcal{E}_w \), get \( f \cdot \psi_1 \). First let us commute \( f \cdot \psi_1 \) with \( \psi_2 \) by \([\cdot, \cdot]_{P_1}\) and obtain the equivalence class in the quotient \( \hat{\mathcal{E}}(\pi)/\ker P_1 \),

\[
[f(x) \cdot 1, w]_{D_x} = f'(x) + \text{const.} \tag{54}
\]

On the other hand, the Leibnitz rule (50) prescribes that this is equal to

\[
f(x) \cdot [1, w]_{D_x} - \mathcal{E}_{D_x(w)}(f(x)) \cdot 1 = 0 - 0 = 0.
\]

This zero value can be achieved at any point \( x_0 \in M \) by choosing const := \( -f'(x_0) \) in (54), but not on the entire \( M \) at once.

Analogously, the Leibnitz rule does not hold for multiplication by \( f(x, [u]) \in \mathcal{F}(\pi) \), and we do not repeat the reasonings for the sake of brevity only.

This counterexample manifests the fundamental difference between jet bundles \( J^\infty(\pi) \) over \( M^n \) and smooth manifolds themselves, which can be regarded as the jet bundles over the point \( \{x_0\}, n = 0 \).

We conclude that, depending on a problem, we have the choice between, first, assuming that the Lie algebra homomorphism \( A : \Omega^1(\xi) \to \mathfrak{g}(\pi) \) is enough for further reasonings, and, second, postulating the Leibnitz rule (50). The former case (44) is realized by Frobenius operators. In this sense, they can be regarded as the anchors in the analogues of Lie algebroids over infinite jet bundles \( J^\infty(\pi) \).

However, if the Leibnitz rule is still needed, then we propose to resolve the difﬁculty as follows. First we assign formal differential complexes (12) to Frobenius operators, and the representations of the complexes through homological vector ﬁelds will determine the Lie algebroids (see (51) in the second part of deﬁnition (2)). This scheme can be further applied in the Batalin–Vilkovisky (BV) formalism to construction and analysis of the quantum Poisson manifolds (11).

### 2.4. The bracket of conservation laws

Let us convert Lemma 9 to a deﬁnition. Suppose \( \mathcal{E} = \{ F = 0 \mid F : J^\infty(\pi) \to \mathfrak{f} \} \) is a determined evolutionary system, hence \( \hat{\mathcal{E}}(\pi) \) is determined by \( \hat{\mathcal{E}}(\mathcal{E}) \) and the coupling \( \langle \cdot, \cdot \rangle : \hat{\mathcal{E}} \times \mathfrak{r} \to \Lambda^n(\pi) \) is well deﬁned. In this section we consider the class of Noether (or ‘pre-Hamiltonian’ (22) (24) operators \( A : \cosym \mathcal{E} \to \sym \mathcal{E} \) that map generating section of conservation laws for \( \mathcal{E} \) to symmetries but may not be skew-adjoint or, even if it is so, may not deﬁne a bracket that satisﬁes the Jacobi
identity (36). In all cases, if a Noether operator for an autonomous evolution equation $E = \{F = u_t - f(x, u) = 0\}$ does not depend on the time $t$, then $A$ satisfies the operator equation $\ell_F \circ A = A \circ \ell_F$ on $E$.

Using Noether operators, one can induce skew-symmetric brackets between conservation laws $\eta \in \hat{H}^{n-1}(E)$ and, second, between their generating sections $\hat{f}(E)$.

Let $\eta = \rho \, dx + \cdots \in \hat{A}^{n-1}(E)$ be a conserved current for an evolutionary system $E$. Then, by Lemma 1, its generating section $\psi_\eta = E(\rho) \in \hat{f}(E)$ is the Euler derivative of the conserved density. For any $\varphi \in \text{sym} \, E$, the current $E_\varphi(\eta)$ is obviously conserved on $E$ and its generating section $E(E_\varphi(\rho))$ equals $E_\varphi(\psi) + E_\varphi(\psi)$ by Lemma 3.

Given an operator $A$: $\text{cosym} \, E \rightarrow \text{sym} \, E$ and any two conserved currents $\eta_1, \eta_2 \in \hat{A}^{n-1}(E)$, set $^{11}$

$$
\eta_\eta = \left\langle \eta_1, \eta_2 \right\rangle_A \defeq \left\langle E_{A(\psi_1)}(\eta_2) - E_{A(\psi_2)}(\eta_1) + \frac{1}{2}(\psi_1, A(\psi_2)) - \frac{1}{2}(A(\psi_1), \psi_2), \psi \right\rangle,
$$

and (see (55), which is \[35\] Eq. (3.2'))

$$
\left\langle [], \right\rangle_A : \hat{f}(E) \times \hat{f}(E) \rightarrow \hat{f}(E).
$$

By (37), we have $\{H_1, H_2\}, A = -E_{E_{A(\psi_1)}(\eta_2)}(H_2)$ for Hamiltonian operators $A$, and therefore $\left\langle H_1, H_2 \right\rangle_A = -\{H_1, H_2\}, A$. Hence we conclude that formula (55b) gives an alternative way to calculate the bracket $\left\{\eta_1, \eta_2\right\}_A$, see (45a) and (46). Indeed, this is achieved by omitting the first two summands in (55b) and setting to zero the application of evolutionary derivatives to $\psi_1$ and $\psi_2$ in what remains. The equivalence between (46) and (55b) implies nontrivial identities even for the simplest scalar Hamiltonian operator $D_x$. Formula (46) is more elegant because it contains only the action by total derivatives onto $\psi_1, \psi_2 \in \hat{f}(E)$ (note that the partial derivatives $\partial/\partial u_\sigma^i$ are applied in it only to the coefficients of $A$). At the same time, the last four summands in (55b) incorporate an application of the partial derivatives to $\psi_1$ and $\psi_2$. However, the skew-symmetry $\left\{\psi_1, \psi_2\right\}_A = -\left\{\psi_2, \psi_1\right\}_A$ of the expression (46) is not obvious. Again, we recall that the notation $\ell_{A(\psi)}$, see (39), is not the linearization $\ell_{A(\psi)}$ given in (30).

**Remark 2.** Only skew-adjoint operators $T^* M^n \rightarrow TM^n$, which can be represented by bi-vectors $\Gamma(\bigwedge^2 T^* M^n)$, were considered in \[35\] §3.2. This was why the antisymmetrization was not performed in (53), which is the realization of (55b) in the case of finite-dimensional manifolds $M^n$.

Due to the antisymmetrization, brackets (55) are invariant under adding self-adjoint operators to $A$. This shows that the brackets $\left\langle \eta_1, \eta_2 \right\rangle_A$ and $\left\langle [], \right\rangle_A$ satisfy the Jacobi identity if and only if $A = P + S$, where $P$ is a Hamiltonian operator and $S^* = S$.

**Example 5.** Consider the dispersionless 3-component Boussinesq-type system \[18\]

$$
u_t = uu_x + v_x, \quad v_t = -uw_x + 3u_xw, \quad w_t = u_x,
$$

which is equivalent to the Benney–Lax equation. Note that (56) is scaling-invariant; e.g., the homogeneity weights are $|u| = 3/2, |v| = 2, |w| = 1, |D_x| = 1$, and $|D_t| = 3/2$. 

\[11\]The arising algebra $(\hat{H}^{n-1}(E), \left\langle \cdot, \cdot \right\rangle_A)$ of conservation laws converts the Lie algebra of Hamiltonians with a Poisson bracket to an algebra with the bracket specified by a Noether operator $A$: $\text{cosym} \, E \rightarrow \text{sym} \, E$. This algebra could be called the “current algebra” if the term were not already in use.
In [24], two compatible Hamiltonian operators $\hat{A}_0$ and $\hat{A}_2$, and a self-adjoint Noether operator $A_1$: $\cosym \mathcal{E} \rightarrow \sym \mathcal{E}$ were found for (56) by a calculation on the $\ell^\ast$-covering over $\mathcal{E}$ and using the scaling invariance of the system. The first Hamiltonian structure for (56) is given by the operator

$$\hat{A}_0 = \begin{pmatrix} D_x & 0 & 0 \\ 0 & -4wD_x + 2w_x & D_x \\ 0 & \frac{D_x}{D_x} & 0 \end{pmatrix}. \quad (57a)$$

Sokolov’s bracket $\{\cdot, \cdot\}_{\hat{A}_0}$ in the inverse image of $\hat{A}_0$ is obtained using formula (46), and we have that

$$\{\cdot, \cdot\}_{\hat{A}_0}^u = \{\cdot, \cdot\}_{\hat{A}_0}^v = 0, \quad \{\vec{p}, \vec{q}\}_{\hat{A}_0}^w = 2(p^w q^v - p^v q^w). \quad (57b)$$

The self-adjoint Noether operator $A_1$ is

$$A_1 = \begin{pmatrix} uw_x + v_x & -3u_x w - uw_x & u_x \\ -3u_x w - uw_x & -3u^2 w - 4v_x w - uu_x & v_x \\ u_x & v_x & w_x \end{pmatrix}. \quad (57c)$$

By Remark 2 the brackets $\langle \cdot, \cdot \rangle_{A_1}$ on $\hat{H}^1(\mathcal{E})$ and $[\cdot, \cdot]_{A_1}$ on $\hat{\mathcal{E}}(\mathcal{E})$ vanish identically. The second Hamiltonian operator for (56) is the linear combination

$$\hat{A}_2 = A_1 + A_2, \quad (57d)$$

where, following [24], the operator $A_2$: $\cosym \mathcal{E} \rightarrow \sym \mathcal{E}$ is equal to

$$A_2 = \begin{pmatrix} (2w^2 + 4v) D_x + uw_x + v_x & -11uw D_x - (2u_x w + 8uw_x) & 3u D_x \\ -11uw D_x - 3u_x w - uw_x & h_1 D_x + h_0 & 4v D_x \\ 3u D_x + u_x & 4v D_x + 2v_x & 2w D_x \end{pmatrix}, \quad (57e)$$

here we put $h_0 = -(2uu_x + 8vw_x + 4v_x w + 6w^2 w_x)$ and $h_1 = -(3u^2 + 16vw + 6w^3)$. The right-hand side of system (56) belongs to the image of each of the three operators $\hat{A}_0$, $A_1$, and $\hat{A}_2$:

$$\begin{pmatrix} u_1 \\ v_t \\ w_t \end{pmatrix} = \hat{A}_0 \begin{pmatrix} \frac{1}{2}w^2 + v \\ u \\ uw \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (57e)$$

Again, by Remark 2 the bracket $[\cdot, \cdot]_{A_2}$ coincides with Dorfman’s bracket $[\cdot, \cdot]_{\hat{A}_2}$ for the second Hamiltonian operator, and both of them satisfy the Jacobi identity. Thus Sokolov’s bracket $\{\cdot, \cdot\}_{\hat{A}_2}$ can be calculated not by (46), but using (55b) and omitting all terms with evolutionary derivations.

In what follows, the Noether operators $\hat{A}_0$, $A_1$, and $\hat{A}_2$: $\cosym \mathcal{E} \rightarrow \sym \mathcal{E}$ for (56) will be met again. We claim that the images of $A_1$ and $A_2$ are closed w.r.t. the commutation (29). The images of arbitrary linear combinations $\lambda_0 \hat{A}_0 + \lambda_1 A_1 + \lambda_2 \hat{A}_2$ are also closed under commutation, and hence the three Frobenius operators are linear compatible. Moreover, the image of a pre-symplectic operator $A_1^{-1}$ is closed w.r.t. either $[\cdot, \cdot]_{\hat{A}_0}$ or $[\cdot, \cdot]_{\hat{A}_2}$ on $\hat{\mathcal{E}}(\mathcal{E})$, and therefore both recursion operators $R_0 = A_0 \circ A_1^{-1}$ and $R_2 = \hat{A}_2 \circ A_1^{-1}$: $\sym \mathcal{E} \rightarrow \sym \mathcal{E}$ are Frobenius as well.
In the previous subsection we addressed the construction of Lie algebroids over $J^\infty(\pi)$ and noticed a very important property of (55): For operators $A = P + S$ in total derivatives, the Leibnitz rule is valid for the bracket (55a) but does not hold for (55b).

Not every skew-adjoint Noether operator is Hamiltonian since, in general, the brackets (37) and (55b) may not satisfy the Jacobi identity.

**Lemma 12** ([23]). Let $E = \{u_t = f(x, [u])\}$ be an evolutionary system of differential order $\text{ord } \ell_f > 1$, and let its symbol be nondegenerate on an open dense subset of $E$. Then any skew-adjoint Noether operator $A: \text{cosym } E \to \text{sym } E$ is Hamiltonian.

Unfortunately, the non-vanishing of the determinant of the symbol (which is the higher-order matrix $\sum_{|\sigma|=\max(\ell_f)\sigma}$) depends on the system of local coordinates.

**Counterexample 6.** Consider the Kaup–Boussinesq system

$$u_t = uu_x + v_x, \quad v_t = (uv)_x + \varepsilon u_{xxx},$$

whose symbol is degenerate and which extends the assertion of Lemma 12 see [23]. System (58) is transformed by $w = v - \varepsilon u_x$ to the second-order Kaup–Broer system

$$u_t = uu_x + w_x + \varepsilon u_{xx}, \quad w_t = (uw)_x - \varepsilon w_{xx}.$$

After the transformation, the determinant of the symbol does not vanish anywhere (of course, we assume $\varepsilon \neq 0$). At the same time, it is clear that the third-order linearization of the right-hand side of the Kaup–Boussinesq system (58) is nondegenerate in the following sense.

**Definition 3.** We say that a matrix operator $A$ in total derivatives is nondegenerate if

$$\bigcap_\sigma \ker A_\sigma = \{0\}, \quad \text{where } A = \sum_\sigma A_\sigma \cdot D_\sigma.$$ 

We shall consider nondegenerate Noether and recursion operators that are transformed using (31) and (38), respectively, under nondegenerate reparametrizations $\tilde{u} = \tilde{u}[u]$ of fibre coordinates. This makes the nondegeneracy (59) well defined.

**Example 7.** The Hamiltonian operator $(0_0 0_{D_x})$ is degenerate. The operator $\Box = u_x + \frac{1}{2} D_x$, which was introduced in (8) and which will be made well defined by (65) below, is nondegenerate, although its kernel is spanned by $\exp(-2u)$.

A conjecture that the assumptions of Lemma 12 can be weakened is already present in [23], but its proof [15], which is quoted there, hinted no way how it can be generalized.

**Conjecture 13.** The assertion of Lemma 12 holds for skew-adjoint Noether operators $A: \text{cosym } E \to \text{sym } E$ on evolutionary systems $E = \{u_t = f\}$ with $\text{ord } \ell_f > 1$, if the linearizations $\ell_f$ of the right-hand sides are nondegenerate, see (59), on open dense subsets of $E$.

3. Frobenius Operators

In this section we introduce the well-defined notion of Frobenius operators, revealing the nature of their domains $\Omega^1(\xi_\pi) \subseteq \Gamma(\pi_{\infty}^*(\xi))$, establish their properties, and give examples of Frobenius operators. In particular, we recognize Frobenius recursions.
χ(π) → χ(π) as higher differential order solutions of the classical Yang–Baxter equation for the Lie algebra g(π); here we construct examples of the projector solutions which are related to Liouville-type systems (e.g., to the 2D Toda chains). We establish the chain rule for factorizations of Frobenius operators and correlate the Sokolov brackets for Frobenius recursions that produce sequences of Lie algebra structures. We show that Frobenius operators A induce flat connections in triples (Ω¹(ξπ), g(π), A) of Lie algebras and their morphisms. Finally, we assign formal differential complexes over Ω¹(ξπ) to Frobenius operators.

The Frobenius theorem [6], which deals with involutive distributions over finite-dimensional manifolds, gave the proper name to the class of operators defined here.

3.1. Main definitions. By Proposition [8] the Poisson bracket { , }ₐ in the Lie algebra of Hamiltonians Hⁿ(π) is correlated with the commutator [ , ] in the Lie algebra g(π) of vector fields Eₐ. The skew-adjoint operator A in the Poisson bracket {H₁, H₂}ₐ = (E(H₁), A(E(H₂))) coincides with the operator A whose image is a subalgebra of evolutionary fields Eₐ(E(H)). Thus we have

\[ [Eₐ(E(H₁)), Eₐ(E(H₂))] = Eₐ(E((H₁, H₂)ₐ)), \]

This is readily seen from the Jacobi identity [13] for the Schouten bracket. Indeed, the commutator of two Hamiltonian vector fields equals

\[ [[A, H₁], [A, H₂]] = [A, [H₂, [A, H₁]]] + [H₂, [[A, H₁], A]], \]

whence the derived bracket [H₂, [A, H₁]] = {H₁, H₂}ₐ appears in the right-hand side of (60) and ∂ₐ(H₁) = 0 is contained in the second summand.

In this section we reverse the status of the two Lie algebra structures [ , ]|ₐ and { , }ₐ, giving the priority to the commutators of vector fields and thus considering involutive distributions of evolutionary vector fields Eₐ ∈ g(π) whose generators φ = A(·) ∈ χ(π) belong to images of matrix operators A in total derivatives.

**Definition 4.** The definition of a Frobenius operator consists of two parts; all notation is correlated with the previous material.

§1 (see [60]). Prohibit any changes of coordinates in all modules of sections. Consider a matrix linear differential operator A: f → χ(π) in total derivatives, where f ⊆ Γ(π⁺∞(ξ)) is a certain submodule that will be specified in §2 of this definition. The operator A is Frobenius if, first, its image in the F(π)-module χ(π) of generators of evolutionary vector fields is closed w.r.t. the commutation:

\[ [\text{im } A, \text{im } A] \subseteq \text{im } A. \]  

The commutator [ , ]|ₐ induces[12] a skew-symmetric Koszul bracket [ , ]ₐ in the quotient Ω¹(ξπ) = f/ ker A of the domains of A:

\[ [A(ψ₁), A(ψ₂)] = A([ψ₁, ψ₂]ₐ), \quad ψ₁, ψ₂ ∈ Ω¹(ξπ). \]  

[12]Two sets of summands appear in the bracket of evolutionary vector fields A(ψ₁), A(ψ₂) that belong to the image of a Frobenius operator A:

\[ [A(ψ₁), A(ψ₂)] = A(Eₐ(A(ψ₁))(ψ₂) − Eₐ(A(ψ₂))(ψ₁)) + (Eₐ(ψ₁)(A)(ψ₂) − Eₐ(ψ₂)(A)(ψ₁)). \]

In the first summand we have used the permutability of evolutionary derivations and operators in total derivatives. The second summand hits the image of A by construction.
\[ [\psi_1, \psi_2]_A = \mathcal{E}_{A(\psi_1)}(\psi_2) - \mathcal{E}_{A(\psi_2)}(\psi_1) + \{\{\psi_1, \psi_2\}\}_A. \] (62b)

It contains two standard summands and the Sokolov bracket \{ , \}_A.

§2. Let \( w \) be a fibre coordinate in the bundle \( \xi \). Allow coordinate reparametrizations, both in \( \xi \) and \( \pi \). Suppose that there is a differential substitution \( J^\infty(\pi) \to \Gamma(\xi) \) that yields the embedding \( \tilde{\mathbf{f}} \mapsto \Gamma(\pi^*_\xi(\xi)) \) of an \( \mathcal{F}(\xi) \)-module \( \mathbf{f} \). From now on, restrict \( \mathbf{f} \) onto the image of the substitution. Thence we denote the substitution by the same letter \( w[u] \) and retain the notation \( \mathbf{f} \) for the restriction of the \( \mathcal{F}(\xi) \)-module. Put \( \Omega^1(\xi_\pi) = \mathbf{f} \vert_{w=w[u]} / \ker A \).

We postulate that, under diffeomorphisms \( \tilde{\mathbf{u}} = \bar{\mathbf{u}}[u]: J^\infty(\pi) \to \Gamma(\pi) \) and \( \tilde{\mathbf{w}} = \bar{\mathbf{w}}[w]: J^\infty(\xi) \to \Gamma(\xi) \), the transformation rules for the operator \( A \) are uniquely defined by the fibre bundles \( \pi \) and \( \xi \). This implies that \( \mathbf{f} \) is one of the following:

\[ \mathbf{f} = \mathcal{K}(\xi) \vert_{w=w[u]}: \text{Frobenius operators of first kind are} \]

\[ A: \mathcal{K}(\xi) \vert_{w} \to \mathcal{K}(\pi). \] (63)

Under any diffeomorphisms \( \tilde{\mathbf{u}} = \bar{\mathbf{u}}[u]: J^\infty(\pi) \to \Gamma(\pi) \) and \( \tilde{\mathbf{w}} = \bar{\mathbf{w}}[w]: J^\infty(\xi) \to \Gamma(\xi) \), these operators are transformed according to

\[ A \mapsto \tilde{A} = \ell^{(u)}_\bar{\mathbf{u}} \circ A \circ \ell^{(\bar{\mathbf{w}})}_{w \vert u = u[u]} \vert_{u = u[u]}, \] (64)

\[ \mathbf{f} = \hat{\mathcal{K}}(\xi) \vert_{w=w[u]}: \text{Frobenius operators of second kind are linear mappings} \]

\[ A: \hat{\mathcal{K}}(\xi) \vert_{w} \to \mathcal{K}(\pi). \] (65)

For any differential changes of coordinates \( \bar{\mathbf{u}} = \hat{\mathbf{u}}[u] \) and \( \bar{\mathbf{w}} = \hat{\mathbf{w}}[w] \), these operators obey

\[ A \mapsto \tilde{A} = \ell^{(u)}_\bar{\mathbf{u}} \circ A \circ \ell^{(\bar{\mathbf{w}})}_{w \vert u = u[u]} \vert_{w = w[u]} \] (66)

Sections \( \psi \) that constitute the domains \( \Omega^1(\xi_\pi) \) of Frobenius operators \( \mathbf{f} \) of first kind are transformed as vectors \( \mathcal{E}_\psi = \psi \cdot \partial / \partial w + \cdots \); in the case \( \mathbf{f} \), the sections \( \psi \) satisfy the rules valid for variational covectors \( \psi = \delta(\cdot) / \delta w \).

If Frobenius operator \( A \) is a recursion \( \mathcal{K}(\pi) \to \mathcal{K}(\pi) \), then \( w = \text{id} \), and the transformations of the domain and image of \( A \) are uniquely correlated. If \( A \) is a Noether operator \( \hat{\mathcal{K}}(\pi) \to \mathcal{K}(\pi) \), then \( w \) determines the system \( \mathcal{E} = \{ w[u] = 0 \} \) of differential equations, and, by default, we use the misleading isomorphism \( \mathbf{f} \simeq \mathcal{K}(\pi) \) for evolution equations (see Remark on p. 13). Otherwise, there may be no constraint between
transformation of the domains and images. In particular, the gauge group of \( \mathfrak{f} \) can be trivial, meaning that transformations of \( w \) in \( \xi \) are still prohibited, as in §1:

\[
\mathfrak{f} = \Gamma(\pi_\infty^*(\xi)): \text{ These Frobenius operators } A: \Gamma(\pi_\infty^*(\xi)) \to \mathfrak{z}(\pi) \text{ are transformed }
\]

by \( A \mapsto \tilde{A} = \xi_u \circ A \big|_{u=\tilde{u}} \) under \( \tilde{u} = \tilde{u}[u] \).

Note that no substitution \( w[u] \) is needed in the third (degenerate) case.

**Remark 3** (On the “Frobenius theorem”). Frobenius operators \( A: \mathfrak{f} \to \mathfrak{z}(\pi) \) specify involutive distributions of vertical symmetries \( \mathcal{E}_{\mathcal{A}(\cdot)} \in \mathcal{D}(J^\infty(\pi)) \) of the Cartan distribution \( \mathcal{C} = (\partial/\partial x^i) \subset D(J^\infty(\pi)) \), see (27), which is itself Frobenius.

Assume that for such an involutive distribution \( \mathcal{E}_{\mathcal{A}(\cdot)} \) there is an integral manifold (typically, it would be infinite-dimensional), and suppose further that it is a differential equation \( \mathcal{E} \). (Note that by an equation we mean the infinite prolongation, which does not always exist.) The definition of Frobenius operators implies that \( \mathcal{E} \) admits infinitely many symmetries \( \varphi = A(\phi) \) which contain free functional parameters \( \phi \in \mathfrak{f} \). This property is close but not equivalent to Definition 8 of Liouville-type differential equations (see p. 177 and Remark 17 that follows).

**Lemma 14.** The bracket \([,]_{\mathcal{A}}\) is \( \mathbb{R} \)-bilinear, skew-symmetric, and transfers the Jacobi identity\(^{14}\) from the Lie algebra \( \mathfrak{g}(\pi) \) of evolutionary vector fields to \( \Omega^1(\xi_\pi) \), see (67). The kernel of \( \mathcal{A} \) is an ideal in \( \mathfrak{f} \). The image of a Frobenius operator may not be an ideal in the space of evolutionary derivations although it is a Lie subalgebra by definition.

**Remark 4.** The bi-differential skew-symmetric bracket \([\{,\}_{\mathcal{A}}] \subset C\text{Diff}(\wedge^2 \Omega^1(\xi_\pi), \Omega^1(\xi_\pi))\) does not generally satisfy the Jacobi identity. Indeed, for the Koszul bracket \([,]_{\mathcal{A}}\) we have

\[
0 = \sum_\circ \left[ \{\psi_1, \psi_2\}_{\mathcal{A}} \psi_3 \right]_{\mathcal{A}} = \sum_\circ \left[ \mathcal{E}_{\mathcal{A}(\psi_1)}(\psi_2) - \mathcal{E}_{\mathcal{A}(\psi_2)}(\psi_1) + \{\psi_1, \psi_2\}_{\mathcal{A}} \psi_3 \right]_{\mathcal{A}}
\]

\[
= \sum_\circ \left\{ \mathcal{E}_{\mathcal{A}(\psi_1)}(\psi_2) - \mathcal{E}_{\mathcal{A}(\psi_2)}(\psi_1) \right\} \psi_3 - \mathcal{E}_{\mathcal{A}(\psi_1)}(\mathcal{E}_{\mathcal{A}(\psi_2)}(\psi_1)) - \mathcal{E}_{\mathcal{A}(\psi_2)}(\mathcal{E}_{\mathcal{A}(\psi_1)}(\psi_1))
\]

\[
+ \{\{\psi_1, \psi_2\}_{\mathcal{A}} \psi_3 \} - \mathcal{E}_{\mathcal{A}(\psi_3)}(\{\psi_1, \psi_2\}_{\mathcal{A}}) - \{\{\psi_1, \psi_2\}_{\mathcal{A}} \psi_3 \} \right\} = 0.
\]

The underlined summand with a derivation of the coefficients of \( \{\psi_1, \psi_2\}_{\mathcal{A}} \), which belong to \( \mathcal{F}(\pi) \), may not vanish on the \( \mathcal{F}(\xi) \)-module \( \mathfrak{f} \), when the action of \( \mathcal{E}_{\mathcal{A}(\cdot)} \) onto the basis of sections \( \psi_i \) is set to zero (see Remark 20 on p. 62). Note that the Jacobi identity for \([,]_{\mathcal{A}}\) then amounts to the last line of (67), which equals \( \sum_\circ \left[ \{\psi_1, \psi_2\}_{\mathcal{A}} \psi_3 \right]_{\mathcal{A}} = 0 \) and which is a half of the equation for infinitesimal deformations of \([,]_{\mathcal{A}}\).

**Remark 5.** In this paper, we study only those operators whose images in \( \mathfrak{z}(\pi) \) are closed w.r.t. the standard bracket \([,]_\mathcal{A}\). Suppose, however, that an endomorphism \( R \in \text{End}_R \mathfrak{z}(\pi) \) is Frobenius and hence induces a new Lie algebra structure \([,]_R\) on the quotient \( \mathfrak{z}(\pi)/\ker R \) of its domain. Contrary to Proposition 8 the images of standard Hamiltonian operators are generally not closed w.r.t. the Koszul brackets \([,]_R\) induced by Frobenius recursions \( R \).

\(^{14}\)In what follows, the nondegeneracy condition \([59] \) specifies the case when the Jacobiator \( J(\psi_1, \psi_2, \psi_3) \) for \([,]_{\mathcal{A}}\) vanishes identically, not being a nonzero element of \( \ker A \) as in \([50] \).
Although one could repeat the whole construction for operators $\Omega^1(\xi_\pi) \to \mathcal{X}(\pi)$ whose images are closed w.r.t. $[,]_R$, this does not produce a new formalism. Indeed, the operators $A$, whose images in $\mathcal{X}(\pi)$ are closed w.r.t. $[,]_R$, generate standard Frobenius operators $R \circ A$, and vice versa. At the same time, the new formalism appears for operators $A$ whose images in $\mathcal{X}(\pi)$ are closed w.r.t. the $r$-brackets (76) given by $r$-matrices for the Lie algebra $\mathfrak{g}(\pi)$.

**Remark 6.** Specifying the Koszul bracket $[,]_A$ on $\Omega^1(\xi_\pi)$ by a Frobenius operator $A$, we define Frobenius generalizations $\omega: \mathcal{X}(\pi) \to \Omega^1(\xi_\pi)$ of symplectic structures $\mathcal{X}(\pi) \to \mathcal{X}(\pi)$. To this end, we require that the images of $\omega$ are closed w.r.t. $[,]_A$. Since all modules are already known, the adaptation of §2 of Definition [4] for $\omega$ is obvious.

3.1.1. **Flat connection.** Let $A: \mathfrak{g}(\pi) \to \mathcal{X}(\pi)$ be a Frobenius operator. It provides the homomorphism of Lie algebras

$$A: (\Omega^1(\xi_\pi), [,]_A) \to (\mathfrak{g}(\pi), [,]).$$

(68)

Let $K$ be a $\mathfrak{g}(\pi)$-module; typically, $K = \mathcal{F}(\pi)$ or any other horizontal $\mathcal{F}(\pi)$-module (e.g., $\mathfrak{g}(\pi)$ itself). By the homomorphism $A$, the $\mathfrak{g}(\pi)$-module $K$ is an $\Omega^1(\xi_\pi)$-module as well. The Jacobi identity implies that the adjoint action by an element of Lie algebra $\Omega^1(\xi_\pi)$ is a derivation. We further impose the condition of semi-simplicity: we claim that

$$[\psi_1, .]_A = [\psi_2, .]_A \implies \psi_1 = \psi_2,$$

for equivalence classes $\psi_1, \psi_2 \in \Omega^1(\xi_\pi) = \mathfrak{g}/\ker A$.

Now we define a connection $\nabla^A$ in the triple (68),

$$\nabla^A: \text{Der}_{\text{Int}}(\Omega^1(\xi_\pi), \mathfrak{g}(\pi)) \to \text{Der}(\mathfrak{g}(\pi), K).$$

The construction is analogous to the definition of connections in triples $(A, B, \iota)$, where $A$ is a $k$-algebra, $B$ is an $A$-algebra, and $\iota: A \to B$ is an algebra homomorphism [38]. In its turn, this generalizes the connections in fibre bundles $E^{n+m} \xrightarrow{\pi} M^n$, when $A = C^\infty(M^n)$, $B = C^\infty(E^{n+m})$, and $\iota: A \hookrightarrow B$. We set

$$\nabla^A: A \circ [\psi, .]_A \mapsto [A(\psi), .].$$

(69)

The above definition is $\Omega^1(\xi_\pi)$-linear. Indeed, for a derivation $\Delta = [\psi, .]_A$ we have that

$$\nabla^A_{f \times \Delta} = A(f) \times \nabla^A_{\Delta}, \quad f \in \Omega^1(\xi_\pi), \quad \Delta \in \text{Der}_{\text{Int}}(\Omega^1(\xi_\pi)).$$

(70)

where the multiplication $\times$ by $f$ and by its image under $A$ is the adjoint action. Note that the right-hand side of the analogue of (70) in a classical definition of $C^\infty(M^n)$-linearity of connections in fibre bundles $\pi$ does contain the image $\iota(f)$ of the identical embedding $A \hookrightarrow B$ and not $f \in A$ itself.

**Proposition 15.** Connection (69) is flat:

$$\left(\nabla^A_p \circ \nabla^A_q - \nabla^A_q \circ \nabla^A_p - \nabla^A_{[p,q],A}\right)(\varphi) = 0, \quad \forall p, q \in \Omega^1(\xi_\pi), \quad \varphi \in \mathfrak{g}(\pi).$$

(71)

**Proof.** The Jacobi identity for the bracket [29] of generating sections of evolutionary vector fields,

$$[A(p), [A(q), \varphi]] + [A(q), [\varphi, A(p)]] + [\varphi, [A(p), A(q)]] = 0$$

is the flatness condition (71). \qed
Corollary 16. If the flows of a commutative hierarchy $\mathfrak{A}$ belong to the image of a Frobenius operator $A$, then the hierarchy is a geodesic w.r.t. connection $\{\mathcal{E}_{\mathcal{A}(\cdot)}\}$.

Proof. Indeed, for any curve $\psi(\tau): \mathbb{R} \to \Omega^1(\mathfrak{A})$ located in the inverse image of $\mathfrak{A}$ under $A$, the covariant derivative $\nabla^A_{\psi(\tau)}A(\psi'(\tau))$ of the velocity $\psi'(\tau)$ vanishes along the curve. ∎

Remark 7. The Frobenius operators $[68]$ induce a morphism $\Omega^1(\xi_\pi) \to \Lambda^* g(\pi)$ to the Schouten algebra of evolutionary polyvector fields, which is endowed with the Schouten bracket $[35]$. Flat connections $[69]$ in triples $[68]$ are naturally extended to connections in $(\Omega^1(\xi_\pi), \Lambda^* g(\pi), A)$, which are flat in the graded sense.

Remark 8. The notion of connections for the triples (which generalize $A = C^\infty(M^n)$, $B = C^\infty(E^{n+m})$, and $\iota: A \to B$) was proposed by Krasil’shchik in [38]. It leads to the Cartan cohomology, see [40, Ch. 5], which allows to interpret symmetries and recursion operators as equivalence classes. Therefore we expect to encounter some cohomology for the connection $[69]$ in the triples $[68]$.

Of course, connection $[69]$ in the triples $[68]$ is not the Cartan connection $[27]$ on $J^\infty(\pi)$. This is readily seen from the fact that everything at hand is $\pi$-vertical and is projected to zero vector fields on $TM^n$ under $\pi_{\infty,*}$. (Everything would be projected to the point $x_0$ if the jet bundle were a finite-dimensional manifold $N^m$ for $\pi: N^m \to \{x_0\}$, see [54].) Therefore, instead of a dual description of the $n$-dimensional distribution $\mathcal{C} \subset \text{D}(J^\infty(\pi))$ by Cartan 1-forms, we face the problem of dual representation of the involutive distributions $\langle \mathcal{E}_{\mathcal{A}(\cdot)} \rangle$.

3.1.2. Frobenius complex. Now, using Cartan’s formula, we construct a differential complex on the Chevalley cohomology $\text{Hom}_\mathbb{R}(\Lambda^k \Omega^1(\xi_\pi), g(\pi))$ with values in an $\Omega^1(\xi_\pi)$-module $g(\pi)$. This is the Frobenius complex assigned to an operator $A: \Omega^1(\xi_\pi) \to g(\pi)$.

For any $k \geq 0$ and $\omega_k \in \text{Hom}_\mathbb{R}(\Lambda^k \Omega^1(\xi_\pi), g(\pi))$, define the differential $d: \omega_k \mapsto d\omega_k$ by setting

$$d\omega_k(\psi_0, \ldots, \psi_k) = \sum_i (-1)^i \nabla^A_{\psi_i} \omega_i(\psi_0, \ldots, \hat{\psi}_i, \ldots, \psi_k) + \sum_{i < j} (-1)^{i+j-1} \omega_k([\psi_i, \psi_j]_A, \psi_0, \ldots, \hat{\psi}_i, \ldots, \hat{\psi}_j, \ldots, \psi_k).$$

Hence we obtain the complex

$$g(\pi) \overset{\text{const}}{\to} \text{Hom}_\mathbb{R}(\Omega^1(\xi_\pi), g(\pi)) \overset{A\cdot[\cdot]_A}{\to} \text{Hom}_\mathbb{R}(\Lambda^2 \Omega^1(\xi_\pi), g(\pi)) \overset{\partial}{\to} \text{Hom}_\mathbb{R}(\Lambda^3 \Omega^1(\xi_\pi), g(\pi)) \to \cdots \ (72)$$

The first inclusion in $(72)$ consists of the commutations $\nabla^A_{\psi} \varphi_0$ with fixed elements $\varphi_0 \in g(\pi)$, the second arrow is the composition of the Koszul bracket $[\cdot, \cdot]_A$ with $A$, and the third arrow calculates the right-hand side of the Jacobi identity $(67)$. The Poisson complex is a special case of $(72)$, extending it from the left with the $\mathcal{F}(\pi)$-module of $n$-th horizontal cohomology $H^n(\pi)$ for $J^\infty(\pi)$. Again, we assume the semi-simplicity: no elements $\varphi_0 \in g(\pi)$ commute with the entire Lie algebra $g(\pi)$.
3.2. Examples of Frobenius operators. Throughout this paper, we consider only the commutative case when the base manifolds $M^n$ are not $\mathbb{Z}_2$-graded and all coordinates on the jet bundles $J^\infty(\pi)$ are permutable. Also, we confine ourselves to the local setting and consider Frobenius matrix differential operators which are polynomial in total derivatives. However, the search for nonlocal Frobenius (super-)operators can be performed using standard techniques, see Remark 20 on p. 62.

Let us have some examples of Frobenius operators of second kind and of Sokolov’s brackets on their domains. We start with Hamiltonian operators, which are the most well studied examples of Frobenius structures.

Example 8. Every Hamiltonian operator is Frobenius. Indeed, the criterion in Proposition 6 gives formula (46) for the bracket $\{\cdot, \cdot\}_A$ on the domains $\hat{\kappa}(\pi)$ of nondegenerate Hamiltonian operators $A$, which obey the law (38) under a change of coordinates. This demonstrates that the formalism of Frobenius operators is a true generalization of the Hamiltonian approach to nonlinear evolutionary PDE. The use of Frobenius operators can be a helpful intermediate step in the search for and classification of the Hamiltonian structures. An advantage of this approach is that it is easier to solve first equation (62) w.r.t. operators $A$ and filter out skew-adjoint solutions rather than to solve the Jacobi identity in the form of (46). Thus, by Proposition 6 a nondegenerate skew-adjoint Frobenius operator $A$ is Hamiltonian iff the Sokolov bracket $\{\cdot, \cdot\}_A$ is equal to the r.h.s. of (46) up to ker $A$. Also, we note that we do not require a restriction of the operators $A$ onto differential equations $E$ such that $A: \cosym E \to \sym E$, unlike in [22].

Remark 9. A recent version [16, 45] of the Darboux theorem for $(1 + 1)$D evolutionary systems implies that Hamiltonian operators $A$ for non-exceptional systems can be transformed to $\text{const} \cdot D_x$. The bracket $\{\cdot, \cdot\}_A$ is then zero, which is readily seen from (46). Hence the actual statement of the Darboux theorem for PDE is that the bracket $\{\cdot, \cdot\}_A$ can be trivialized for such Hamiltonian operators.

Remark 10. One bracket $\{\cdot, \cdot\}_A$ can correspond to several operators $A$ that satisfy (61). For example, the second structure $A_2^{KdV} = -\frac{1}{2} D_x^3 + 2w D_x + w_x$ for the KdV equation (11) determines the bracket $\{p, q\}_A^{KdV} = pq_x - p_x q$, which is also induced by the operators (73), see Example 10 below, and by $A^{(2)}_1 = D_x \circ(D_x + u)$, see Example 13. Actually, this Wronskian-based bracket (c.f. [27]) is scattered through the text, see Example 21 on p. 46. Hence there are fewer brackets $\{\cdot, \cdot\}_A$ than there are Frobenius operators $A$.

Formula (46) does not remain valid for arbitrary Frobenius operators, which are generally non-skew-adjoint. In appendix B we describe an inductive procedure that reconstructs the bracket $\{\cdot, \cdot\}_A$ for nondegenerate operators $A$, see (51). From now on, we consider only nondegenerate Frobenius operators.

Example 9. Noether operators (57) for the 3-component dispersionless Boussinesq-type system (56) are Frobenius. For the Frobenius operators $A_1, A_2: \cosym E \to \sym E$, the transformation formulas (38) classify them to the second kind; the substitutions $J^\infty(\pi) \to \Gamma(\xi)$ are the identity mappings. The same is obvious for Hamiltonian operators $\hat{A}_0$ and $\hat{A}_2$ for this system. However, unlike $[\cdot, A_1] = 0$ and $[\cdot, A_2] = [\cdot, A_2]$,
see [55] and Remark 2. The components of Sokolov’s brackets for $A_1$ and $A_2$ are obtained using the inductive algorithm, and the result is [128] on p. 63. Bi-differential representations for components of these brackets are formulated in appendix C.

In section 5 we shall consider a class of Frobenius operators $\square$: cosym $\mathcal{E}_1 \rightarrow \text{sym} \mathcal{E}_2$ of second kind that map (co-)tangent bundles for two hierarchies of evolutionary systems related by the substitution $w: \mathcal{E}_2 \rightarrow \mathcal{E}_1$.

Now we list several operators which are known in a fixed system of local coordinates and whose images are then closed under commutation. The transformation rules for their domains are unknown, although it may occur that these reparametrizations are uniquely determined by a change of coordinates in the images.

Example 10 (KdV scaling weights). Let us fix the weights $|u| = 2$, $|D_x| = 1$ that originate from the scaling invariance of the KdV equation $u_t = -\frac{1}{2} u_{xxx} + 3uu_x$, see also [11]; we have that $|D_t| = 3$. Using the method of undetermined coefficients, we performed the search for scalar Frobenius operators that are homogeneous w.r.t. the weights not greater than 7. We obtained two compatible Hamiltonian operators $A_{1\text{KdV}} = D_x$ and $A_{2\text{KdV}} = -\frac{1}{2} D_x^3 + 2u D_x + u_x$, the generalizations $D_x^{n+1}$ of $D_x$, and the operator

$$uu_{xxx} + 3uu_x D_x + 3uu_x D_x^2 + u^2 D_x^3.$$ 

Also, there are four non-skew-adjoint operators that satisfy (61).

$$A_4^{(6)} = u^3 - u_x^2, \quad A_3^{(6)} = 2u_x^2 - uu_x - 2uu_x D_x + u^2 D_x^2,$$

$${\{p, q\}}_{A_4^{(6)}} = 2u_x \cdot (pq_x - px_q), \quad {\{p, q\}}_{A_3^{(6)}} = -2u_x \cdot (pq_x - px_q) + u \cdot (pq_x - px_q);$$

$$A_8^{(7)} = u_x D_x - 2uu_{xx} D_x - 4uu_x D_x^2 - 4u^2 D_x^3, \quad {\{p, q\}}_{A_8^{(7)}} = u^2 \cdot (pq_x - px_q);$$

$$A_9^{(7)} = -2u_x u_{xx} - u_x^2 D_x, \quad {\{p, q\}}_{A_9^{(7)}} = 8u_{xx} \cdot (pq_x - px_q) + 2u_x \cdot (pq_x - px_q).$$

Finally, we have found the operators that contain arbitrary functions: $f(u) D_x^n$ and $f(u) u^2$ with vanishing brackets, and also we have got

$$A_3 = f(u) u_x, \quad {\{p, q\}}_{A_3} = f(u) (pq_x - px_q);$$

$$A_4 = f(u)(u D_x - u_x), \quad {\{p, q\}}_{A_4} = f(u) (pq_x - px_q). \quad (73)$$

Formula (46) is not valid for any of these non-skew-adjoint operators.

Example 11. There is a three-parametric family of scalar first-order Frobenius operators $A = a(u, u_x) D_x + b(u, u_x)$ with brackets $\{p, q\}_A = c(u, u_x) \cdot (pq_x - px_q)$. They are given by

$$a = \gamma(u), \quad b = \beta(u) \cdot u_x + \alpha \cdot \gamma(u), \quad c = -\beta(u), \quad (74)$$

where $\alpha$ is a constant and the functions $\beta, \gamma$ are arbitrary.

3.3. Frobenius recursion operators. Within this section, we set $\xi := \pi$ and $w = u$. Thus we consider Frobenius recursion operators $R: \pi(\pi) \rightarrow \pi(\pi)$ whose images are closed w.r.t. (29). The standard Lie algebra structure $[\cdot, \cdot]$ is transferred to $[\cdot, \cdot]_R$ on the quotients $\Omega^1(\pi) = \pi(\pi)/\ker R$ by the classical Yang–Baxter equation for $g(\pi)$,

$$[R\varphi_1, R\varphi_2] = R([\varphi_1, \varphi_2]_R), \quad \varphi_1, \varphi_2 \in \pi(\pi)/\ker R. \quad (75)$$
For \( R \neq \text{id} \), the new Lie bracket \([\cdot,\cdot]_R\) is different from the original commutation of evolutionary vector fields in the image.

**Remark 11.** Frobenius recursion operators provide higher differential order solutions of the classical Yang–Baxter equation for the algebra \( \mathfrak{g}(\pi) \). The parallel with the zero-order theory of \( r \)-matrices is as follows. Let \( \mathfrak{g} \) be a Lie algebra with the Lie bracket \([\cdot,\cdot]\). The classical \( r \)-matrix \( \mathfrak{g} \) is a linear map \( r: \mathfrak{g} \to \mathfrak{g} \) that endows \( \mathfrak{g}/\ker r \) with the second Lie product \([\cdot,\cdot]_r\); for any \( a,b \in \mathfrak{g} \) set

\[
[a,b]_r = [ra,b] + [a,rb].
\]

A sufficient condition for an operator \( r \) of differential order zero to be an \( r \)-matrix is that \( r \) satisfies the Yang–Baxter equation \( \text{YB}(\alpha) \):

\[
[ra,rb] = r([a,b]_r) - \alpha[a,b], \quad a,b \in \mathfrak{g}, \quad \alpha = 0 \text{ or } 1.
\]

Now let a recursion differential operator \( R \in \text{End}_R \mathfrak{z}(\pi) \) be Frobenius, then the Koszul bracket \((62)\) satisfies the Yang–Baxter equation \((75)\) with \( \alpha = 0 \).

We conclude that, for Lie algebra \( \mathfrak{g}(\pi) \) of evolutionary vector fields on jet spaces \( J^\infty(\pi) \), the classical Yang–Baxter equation \((75)\) admits solutions \((R,[\cdot,\cdot]_R)\) of form \((62)\) other than the standard \( r \)-brackets \((76)\). The two notions of \( r \)-matrices and Frobenius recursions \( R \in \text{End}_R \mathfrak{z}(\pi) \) as solutions of equation \((77)\) are very close. However, the distinction between them is expressed by the respective brackets \([\cdot,\cdot]_R\) and \([\cdot,\cdot]_r\) on their domain \( \mathfrak{z}(\pi) \): the Koszul bracket \((62)\) is calculated and the \( r \)-bracket \([\cdot,\cdot]_r\) is postulated. This explains why §1 of Definition 1 incorporates the form of the Koszul bracket.

**Example 12** (Projector solutions). Instead of the entire jet space \( J^\infty(\pi) \), consider a Darboux-integrable Liouville-type hyperbolic system \[56, 60\].

\[
\mathcal{E}_L = \{ u^m_{xy} = f^m(x,y;[u]), \ldots, u^m_{xxy} = f^m_0(x,y;[u]) \} \subset J^\infty(\pi),
\]

see Definition 8 on p. 47 and a comment on it in Remark 16. Essentially, \( \mathcal{E}_L \simeq \{ J^\infty(\pi_x) \oplus J^\infty(\pi_y) / \sim \}, \) plus the rules for calculating \( u_{x...xy...y} \), where \( \pi_x, \pi_y: \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} \) are trivial bundles over \( \mathbb{R} \) with the base coordinates \( x \) and \( y \), respectively, and \( \sim \) glues together their fibres with coordinates \( (u^1, \ldots, u^m) \).

In this notation, the symmetry algebra \( \text{sym} \mathcal{E}_L \) is decomposed along the \( x \)- and \( y \)-characteristics such that any \( \varphi \in \text{sym} \mathcal{E}_L \) is of the form \((100)\), see p. 48

\[
\varphi = \Box(\phi) + \Box(\bar{\phi}),
\]

where sections \( \phi \in \hat{\mathcal{X}}(\mathcal{E}_x) \) and \( \bar{\phi} \in \hat{\mathcal{X}}(\mathcal{E}_y) \) belong to certain restrictions of the modules and the operators \( \Box \) and \( \Box \) are Frobenius. This is discussed in section 5 in detail.

Without loss of generality (we consider the \( x \)-components of all structures, but one could replace \( x \leftrightarrow y \) in what follows), set

\[
R: \mathcal{E}_L \to \mathcal{E}_L \in \text{sym} \mathcal{E}_L.
\]

The Liouville-type systems are so special that \( R \) is a recursion on \( \text{sym} \mathcal{E}_L \), although \( \ker R \) is very big under this projection. The image of \( R \) is closed w.r.t. the restriction of \( \mathcal{E}_L \) onto \( \mathcal{E}_L \). Thus we obtain an example of a Frobenius recursion operator.

Surprisingly, we can apply here the algebraic construction of Poisson brackets, which was introduced in \[51\], to such projector recursions \( R \). We recall that a homomorphism
Π ∈ Hom_\(a(V, \text{End}_a V)\) is Hamiltonian for a commutative \(k\)-algebra \(a\) and an \(a\)-module \(V\) if the Poisson bracket \(\{v_1, v_2\}_\Pi := (\Pi(v_1))(v_2)\) of \(v_1, v_2 \in V\) satisfies the commutation closure condition \(\Pi(\{v_1, v_2\}_\Pi) = [\Pi(v_1), \Pi(v_2)]\). The latter reduces to (69) if we choose \(k = \mathbb{R}, a = \mathcal{F}(\pi), V = \tilde{H}^n(\pi)\), and \(\Pi = \mathcal{E}_{A(E(\pi))}\).

In our case, we have \(a = \mathbb{R}\) and \(V = \text{sym} \mathcal{E}_L\). Define \(\Pi ∈ \text{Hom}_\mathbb{R}(\text{sym} \mathcal{E}_L, \text{End}_\mathbb{R} \text{sym} \mathcal{E}_L)\) by
\[
\Pi(\varphi) = \text{ad}_{\varphi} \circ R, \quad \varphi ∈ \text{sym} \mathcal{E}_L,
\]
whence we obtain \(\{A_1, A_2\}_\Pi = [R(\varphi_1), R(\varphi_2)]\) and
\[
[[R(\varphi_1), R(\varphi_2)], R(\varphi_3)] = \left[\left[\left[\left[R(\varphi_1), R\left[\left[R(\varphi_2), R(\varphi_3)\right]\right]\right]\right]\right] - [R(\varphi_2), R\left[\left[R(\varphi_1), R(\varphi_3)\right]\right]\right]
\]
for any \(\varphi_1, \varphi_2, \varphi_3 \in \text{sym} \mathcal{E}_L\). Clearly, the above equality holds by virtue of the Jacobi identity, since \(\text{im} R, \text{im} \mathcal{E}_L \subseteq \text{im} R\) and \(R^2 = R\).

This example describes a rare situation when a Poisson structure is introduced on \(\text{sym} \mathcal{E}_L\) and not on \(\tilde{H}^n(\pi)\). The price we pay for this is that the underlying algebra \(a = \mathbb{R}\) and an \(\mathbb{R}\)-module structure of \(\text{sym} \mathcal{E}_L\) are very poor in comparison with \(\mathcal{F}(\pi)\) for \(\tilde{H}^n(\pi)\), respectively.

Suppose a Frobenius operator is divisible by another Frobenius operator; then their Koszul brackets are correlated as follows.

**Proposition 17** (The chain rule, c.f. [54]). Suppose that the Frobenius operator \(A\) is nondegenerate, see (59), and the image of \(A' = A \circ \Delta\) is closed w.r.t. the commutation. Then the brackets \(\{\{\cdot, \cdot\}\}_A\) and \(\{\{\cdot, \cdot\}\}_{A'}\) are related by the formula
\[
\Delta(\{\{\xi_1, \xi_2\}\}_A) = \mathcal{E}_{A(\Delta(\xi_1))}(\Delta)(\xi_2) - \mathcal{E}_{A(\Delta(\xi_2))}(\Delta)(\xi_1) + \{\Delta(\xi_1), \Delta(\xi_2)\}_A
\]
for any sections \(\xi_1, \xi_2 \in \mathfrak{f} \subseteq \Gamma(\pi_\mathfrak{a}(\xi))\) of a horizontal fibre bundle in the domain of \(\Delta\).

This assertion provides a considerable simplification of the search for new Frobenius operators using known ones and allows to reconstruct Sokolov’s brackets in the inverse images of Frobenius recursions \(R \in \text{End}_\mathbb{R} \mathcal{N}(\pi)\), factored by Frobenius operators \(A: \Omega^1(\pi_x) → \mathcal{N}(\pi)\). A weaker version of Proposition 17 was formulated in [54] (where the operators were considered in a fixed system of local coordinates). This property is valid in the Hamiltonian case although is not well known.

**Proof.** Suppose \(\xi_1, \xi_2 \in \mathfrak{f}\) and \(\psi_i = \Delta(\xi_i), \varphi_i = A(\psi_i)\) for \(i = 1, 2\). We have
\[
\{\varphi_1, \varphi_2\} = (A \circ \Delta)(\mathcal{E}_{\varphi_1}(\xi_2) - \mathcal{E}_{\varphi_2}(\xi_1) + \{\{\xi_1, \xi_2\}\}_A).
\]  
(79a)

On the other hand, we recall that \(\psi_i = \Delta(\xi_i)\) and deduce
\[
\{\varphi_1, \varphi_2\} = A(\mathcal{E}_{\varphi_1}(\psi_2) - \mathcal{E}_{\varphi_2}(\psi_1) + \{\psi_1, \psi_2\}_A)
\]  
(79b)

Now subtract (79a) from (79b) and, using the nondegeneracy (59) and hence omitting the operator \(A\), we obtain the assertion. \(\square\)

**Example 13.** Historically, the class of scalar operators, first regarded in view of §1 in Definition 4 was studied in [60] using local coordinates. In [54], Gelfand’s symbolic
method \cite{13} was applied in that setting, and it was argued that the class is infinite and contains the (presumably for $A_1^{(2)}$ etc., well-defined Frobenius) operators

\[
\begin{align*}
A_1^{(1)} &= D_x; \\
A_1^{(2)} &= D_x \circ (D_x + u); \\
A_1^{(3)} &= A_1^{(2)} \circ (D_x + u); \\
A_1^{(4)} &= A_1^{(3)} \circ (D_x + u),
\end{align*}
\]

here the superscripts denote the differential order and the subscripts enumerate operators of equal order. Further, there are two operators

\[
A_1^{(n)} = A_1^{(n-1)} \circ (D_x + u), \quad A_1^{(n)} = A_1^{(n-1)} \circ (D_x + (n-2)u)
\]

for any odd $n \geq 5$, and there are four operators

\[
\begin{align*}
A_2^{(n)} &= A_2^{(n-1)} \circ (D_x + u), \\
A_2^{(n)} &= A_2^{(n-1)} \circ (D_x + 2u), \\
A_2^{(n)} &= A_2^{(n-1)} \circ (D_x + (n-3)u), \\
A_2^{(n)} &= A_2^{(n-1)} \circ (D_x + (n-2)u)
\end{align*}
\]

for any even $n \geq 6$. These operators are homogeneous w.r.t. the weights $|u| = |D_x| = 1$, hence the weights coincide with the differential orders.

None of these operators is Hamiltonian. The brackets $\{\{, \}\}_{A_i^{(n)}}$ are reconstructed from the vanishing bracket for $A_1^{(1)}$ by using the chain rule (see Proposition \cite{17}; the first four of them are

\[
\begin{align*}
\{\{p, q\}\}_{A_i^{(2)}} &= p_x q - pq_x, \\
\{\{p, q\}\}_{A_i^{(3)}} &= 2u(p_x q - pq_x) + p_{xx} q - pq_{xx}, \\
\{\{p, q\}\}_{A_i^{(4)}} &= 3(u^2 + u_x)(p_x q - pq_x) + 3u(p_{xx} q - pq_{xx}) + p_{xxx} q - pq_{xxx}, \\
\{\{p, q\}\}_{A_i^{(5)}} &= 6(u^2 + 2u_x)(p_x q - pq_x) + 6u(p_{xx} q - pq_{xx}) + 2(p_{xxx} q - pq_{xxx}) + p_{xxxx} q - px_{xx} q_x.
\end{align*}
\]

It remains unknown whether operators \cite{57}, which are factored to products of primitive first order operators with integer coefficients, exhaust all Frobenius operators with differential polynomial coefficients and homogeneous w.r.t. the weights $|u| = |D_x| = 1$. The transformation rules for these operators are not known and the substitutions $w$ for them, if they are not of the third (degenerate) kind, have not been constructed.

Consider an operator $R = A_2 \circ \omega$ factored by a Hamiltonian operator $A_2 : \mathcal{X}(\pi) \rightarrow \mathcal{X}(\pi)$ and a pre-symplectic structure $\omega : \mathcal{X}(\pi) \rightarrow \mathcal{X}(\pi)$ whose image is closed w.r.t. Dorfman’s bracket $[\cdot, \cdot]_{A_2}$. The recursion operator $R$ is Frobenius, but the bracket $\{\{, \}\}_R$, which can be reconstructed using \cite{73}, is generally nonlocal. The exception is given by operators $\omega$ of zero differential order if the determinants of their matrices do not vanish identically.

**Example 14.** Again, let us consider Hamiltonian and Noether operators \cite{57} for the dispersionless 3-component Boussinesq-type system \cite{56}, which is a true source of illustrative examples for the exposition \cite{13}. The image of the Hamiltonian operator $A_0 : \text{cosym} \mathcal{E} \rightarrow \text{sym} \mathcal{E}$ is closed w.r.t. commutation \cite{29}. Note that the determinant

\[15\text{The operator } A_2 \text{ was labelled ‘first’ and } A_1 \text{ was ‘second’ with the lexicographical order } w \prec u \prec v \text{ of their arguments } \psi = f(\psi^w, \psi^u, \psi^v) \in \mathcal{X}(\pi) \text{ and images } \varphi = f(\varphi^w, \varphi^u, \varphi^v) \in \mathcal{X}(\pi) \text{ in } \cite{24}.\]
of the zero-order operator \((57c)\) does not vanish identically, hence \(\omega = A_1^{-1}: \text{sym} \mathcal{E} \to \text{cosym} \mathcal{E}\) is an isomorphism of modules for an open dense subset of solutions \(s \in \Gamma(\pi)\) of \((56)\). Therefore \(A_1^{-1}\) is a Frobenius structure w.r.t. \([\cdot, \cdot]_{A_0}\). By this argument, define

\[
R_0 = \hat{A}_0 \circ A_1^{-1}: \text{sym} \mathcal{E} \to \text{sym} \mathcal{E}.
\] (81)

This recursion operator for \((56)\) is Frobenius. Indeed, its image is contained in the image of the Hamiltonian operator \(A_0\), whose Dorfman’s bracket \([\cdot, \cdot]_{A_0}\) is pushed forward by the zero-order operator \(A_1\) to \([\cdot, \cdot]_{R_0}\) on \(\text{sym} \mathcal{E}\).

The bracket \([\cdot, \cdot]_{R_0}\) can be calculated explicitly, although the resulting formulas are relatively long. For that, we use a MAPLE program for the Jets environment \([17]\); the listing is contained in appendix \(D\). Each component of Sokolov’s bracket \([\cdot, \cdot]_{R_0}\) for \((81)\) is a fraction of differential polynomials and contains about 15,000 summands in the numerator. The denominators are the cubes of the determinant of the matrix \(A_1\). Perhaps, these fractions are reducible.

In the same way, we construct the Frobenius recursion \(R_2 = \hat{A}_2 \circ A_1^{-1}\) for \((56)\) using its second Hamiltonian structure \((57c)\) and the bracket \([\cdot, \cdot]_{A_2} = [\cdot, \cdot]_{A_1} + [\cdot, \cdot]_{A_2}\), see \((128)\). The bracket \([\cdot, \cdot]_{R_2}\) is computed using a slight modification of the program that is contained in appendix \(D\). The modification amounts to a substitution of \(A_2\) and \([\cdot, \cdot]_{A_2}\) for \(A_0\) and \([\cdot, \cdot]_{A_0}\), respectively.

Finally, we specify a condition upon the brackets \([\cdot, \cdot]_{R_\ell}\) for sequences of Lie algebra structures \([\cdot, \cdot]_{R_\ell}\) induced on \(\mathcal{N}(\pi)\) by Frobenius iterations \(R_\ell\) of a recursion \(R \in \text{End}_\mathbb{R} \mathcal{N}(\pi)\). The condition \((82)\) is a consequence of Proposition \(17\).

**Proposition 18.** Let \(R: \mathcal{N}(\pi) \to \mathcal{N}(\pi)\) be a Frobenius operator and assume that its powers \(R^2, \ldots, R^k\) are Frobenius for some \(k > 1\). Then the brackets \([\cdot, \cdot]_{R_\ell}\) for each \(\ell \in [1, \ldots, k]\) satisfy the relation

\[
R^{\ell-1}(\{[\xi_1, \xi_2]\}_{R_\ell}) = \sum_{i=0}^{\ell-2} R^{\ell-i-2} \left[ \mathcal{E}_{R^i(\xi_1)}(R)(R^i\xi_2) - \mathcal{E}_{R^i(\xi_2)}(R)(R^i\xi_1) \right] + \{[R^{\ell-1}\xi_1, R^{\ell-1}\xi_2]\}_{R_\ell},
\] (82)

where \(\xi_1, \xi_2\) belong to the domain of \(R^\ell\).

Under assumption that \(R, \ldots, R^k\) are Frobenius for \(k > 1\), is there any condition for \(R^{k+1}, \ldots, R^{k+\ell}\) to be Frobenius as well? A well-known condition for Nijenhuis operators \([9, 35]\) originates from the standard cohomology theory for Lie algebras \([9, 12, 14]\). At the same time, the nontrivial finite deformations \([\cdot, \cdot]_{R^k}\) of the standard Lie algebra structure \([\cdot, \cdot]\) on \(\mathfrak{g}(\pi)\) are not the trivial infinitesimal deformations \([\cdot, \cdot]_{N^\ell}\), see \((153)\), which are obtained using powers of Nijenhuis recursion operators \(N \in \text{End}_\mathbb{R} \mathcal{N}(\pi)\) with vanishing Nijenhuis torsion \([N, N]^{[n]} = 0\).

4. **Compatibility of Frobenius Operators**

We introduce two types of compatibility for Frobenius operators \(A_1, \ldots, A_N: \Omega^1(\xi_\pi) \to \mathfrak{g}(\pi)\), where now \(\Omega^1(\xi_\pi) = f/\bigcap_{i=1}^N \ker A_i\). The linear compatibility means that linear

\footnote{In the next section we show why this compatibility occurs.}
combinations of operators remain Frobenius and hence the ‘individual’ Koszul brackets are correlated. The strong compatibility of $N$ Frobenius operators means that the sum of their images is an involutive distribution in the Lie algebra $\mathfrak{g}(\pi)$ of evolutionary vector fields. We endow the spaces of both linear and strong compatible Frobenius operators with a bi-linear bracket that satisfies the Jacobi identity, and then we relate the Lie-type algebras of operators to an affine geometry with bi-differential Christoffel symbols. Let us consider these notions in more detail.

4.1. The linear compatibility. First recall that any linear combination of two compatible Hamiltonian operators is Hamiltonian by definition and hence Frobenius. Example 10 shows that a Hamiltonian operator can be decomposed as a sum of operators that satisfy (61), e.g.,

\[ A_2^{KdV} = -\frac{1}{2} \cdot D_x^3 + 2 \cdot u D_x + u_x, \]

is a linear combination of $e_1 = D_x^2$, $e_2 = u D_x$, and $e_3 = u_x$. The decomposition may not be unique due to the existence of several linear dependent Frobenius operators that appear in the splitting; indeed, the operator $A_2^{KdV}$ can be also obtained using $A_4 = u D_x - u_x$, see (73).

**Definition 5.** Frobenius operators $A_1$, ..., $A_N$ are linear compatible if their arbitrary linear combinations $A_\tilde{\lambda} = \sum_{i=1}^N \lambda_i A_i$ are Frobenius for any $\tilde{\lambda} \in \mathbb{R}^N$. The operators are linear compatible at a point $\tilde{\lambda}_0 \in \mathbb{R}^N$ if $A_{\tilde{\lambda}_0}$ is Frobenius for a fixed linear combination.

**Example 15.** There are two classes of pairwise linear compatible Frobenius scalar operators [14] of first order. The first type of pairs is $\gamma_2(u) = \text{const} \cdot \gamma_1(u)$ with any $\alpha_1, \alpha_2 \in \mathbb{R}$ and arbitrary functions $\beta_1(u), \beta_2(u)$. The second class is given by letting $\alpha_1 = \alpha_2 \in \mathbb{R}$, while the functions $\beta_1, \beta_2, \gamma_1, \text{and} \gamma_2$ remain arbitrary.

**Proposition 19.** The Sokolov bracket induced on the domain of a linear combination $A_\tilde{\lambda} = \sum_{i=1}^N \lambda_i A_i$ of linear compatible Frobenius operators is

\[ \{\{ , \} \}_{A_\tilde{\lambda}}^{\times} = \sum_{i=1}^N \lambda_i \cdot \{\{ , \} \}_{A_i}. \]

**Proof.** This is readily seen by inspecting the coefficients of $\lambda_i^2$ in the quadratic polynomials in $\lambda_i$ that appear in both sides of the equality $[A_\tilde{\lambda}(p), A_\tilde{\lambda}(q)] = A_\tilde{\lambda}([p, q]_{A_\tilde{\lambda}})$ upon the Koszul bracket, here $p, q \in \Omega^1(\xi_\pi).

Consider the commutator $[\sum_i \lambda_i A_i(p), \sum_j \lambda_j A_j(q)]$. On one hand, it is equal to

\[ \sum_{i \neq j} \lambda_i \lambda_j [A_i(p), A_j(q)] + \sum_i \lambda_i^2 A_i(\mathcal{E}_{A_i(p)}(q) - \mathcal{E}_{A_i(q)}(p) + \{\{p, q\} \}_{A_i}). \quad (83) \]

On the other hand, the linear compatibility of $A_i$ implies

\[ A_\tilde{\lambda}(\mathcal{E}_{A_\tilde{\lambda}(p)}(q)) - A_\tilde{\lambda}(\mathcal{E}_{A_\tilde{\lambda}(q)}(p)) + A_\tilde{\lambda}([\{p, q\}]_{A_\tilde{\lambda}}). \]

The entire commutator is quadratic homogeneous in $\tilde{\lambda}$, whence the bracket $\{\{ , \} \}_{A_\tilde{\lambda}}$ is linear in $\tilde{\lambda}$. From (83) we see that the individual brackets $\{\{ , \} \}_{A_i}$ are contained in it. Therefore,

\[ \{\{p, q\} \}_{A_\tilde{\lambda}} = \sum \lambda_\ell \cdot \{\{p, q\} \}_{A_\ell} + \sum \lambda_\ell \cdot \gamma_\ell(p, q), \]
where \( \gamma_\ell : \Omega^1(\xi_\pi) \times \Omega^1(\xi_\pi) \to \Omega^1(\xi_\pi) \). We claim that all summands \( \gamma_\ell(\cdot, \cdot) \), which do not depend on \( \lambda \) at all, vanish. Indeed, assume the converse. Let there be \( \ell \in [1, \ldots, N] \) such that \( \gamma_\ell(p, q) \neq 0 \); without loss of generality, suppose \( \ell = 1 \). Then set \( \lambda = (1, 0, \ldots, 0) \), whence

\[
\left[ \sum_i \lambda_i A_i(p), \sum_j \lambda_j A_j(q) \right] = \left[ (\lambda_1 A_1)(p), (\lambda_1 A_1)(q) \right] = (\lambda_1 A_1)(\lambda_1 \gamma_1(p, q))
+ (\lambda_1 A_1)[E(\lambda_1 A_1)(p)(q) - E(\lambda_1 A_1)(q)(p) + \lambda_1 \{p, q\} A_1].
\]

Consequently, \( \gamma_\ell(p, q) \in \ker A_\ell \) for all \( p \) and \( q \). Since each \( A_\ell \) is nondegenerate by assumption, we have that \( \gamma_\ell = 0 \) for all \( \ell \), which concludes the proof.

**Corollary 20** (Infinitesimal deformations of Frobenius operators). Two Frobenius operators are linear compatible iff for any \( p, q \in \Omega^1(\xi_\pi) \) one has

\[
[B(p, A(q)) + [A(q), B(p)] = A([p, q]_B) + B([p, q]_A),
\]

which is equivalent to the relation

\[
E_{A(p)}(B)(q) + E_{B(p)}(A)(q) - E_{A(q)}(B)(p) - E_{B(q)}(A)(p) = A([p, q]_B) + B([p, q]_A).
\]

**Example 16.** The three Noether operators \( \hat{A}_0, A_1, \) and \( \hat{A}_2 \), see [57] on p. 25, are linear compatible Frobenius structures for system [56]. Any linear combination \( \lambda_0 \hat{A}_0 + \lambda_1 A_1 + \lambda_2 \hat{A}_2 \) is Frobenius again, and therefore Sokolov’s brackets for operators [57[57[57] are correlated by

\[
\{[\cdot, \cdot], A_2 = \{[\cdot, \cdot], A_1 + \{[\cdot, \cdot], A_2,
\]

which we claimed in Example [13].

**Remark 12.** The operators remain Frobenius when multiplied by a constant, therefore pass to the projective frame \( \lambda \in \mathbb{RP}^N \) of \( N \in \mathbb{N} \) Frobenius operators. Then in \( C_{Diff}(\Omega^1(\xi_\pi), \pi(\pi)) \) there is a basis of Frobenius operators which either are isolated points or which generate Frobenius cells with a nontrivial topology of attaching the simplexes together.

An illustration is given by Example [10]. For \( \bar{e}_1 = D_x^2, \bar{e}_2 = uD_x, \) and \( \bar{e}_3 = u_x, \) the curve \( A_2^{KdV} = (\lambda : 2 : 1) \) is Hamiltonian and the ray \( A_4 = f(u) \cdot (0 : 1 : -1) \) is Frobenius.

4.2. **The strong compatibility.** We impose an additional specification on the structure of the commutators of evolutionary vector fields whose generating sections belong to images of several Frobenius operators.

**Definition 6.** Frobenius operators \( A_1, \ldots, A_N : \Omega^1(\xi_\pi) \to \pi(\pi) \) are strong compatible if the commutators of evolutionary fields in the images of any two of them belong to the sum of the images of all the \( N \) operators such that, for any \( p, q \in \Omega^1(\xi_\pi) \) and \( 1 \leq i, j \leq N, \)

\[
[A_i(p), A_j(q)] = A_j(\mathcal{E}_{A_i(p)}(q)) - A_i(\mathcal{E}_{A_j(q)}(p)) + \sum_{k=1}^{N} A_k(\Gamma_{ij}^k(p, q)) \in \sum_{t=1}^{N} \text{im } A_\ell.
\]
Example 17. Consider the Liouville equation $U_{xy} = \exp(2U)$. Let $\zeta_x$ be the same projection onto the $x$-characteristics as in (77), and similarly for $\zeta$ and the coordinate $y$. Frobenius operators $\Box = (U_x + \frac{1}{2}D_x) \circ \zeta_x$, and $\Box = (U_y + \frac{1}{2}D_y) \circ \zeta_x$, c.f. [3], are strongly compatible. The bi-differential coefficients $\Gamma^k_{ij}$ are given in (24) on p. 12.

Definition (continued). The common domain $\Omega^1(\xi_\pi)$ of the operators $A_i$ is an $\mathcal{F}(\pi)$-submodule. Therefore, in view of the functional arbitrariness of sections $p$, $q \in \Omega^1(\xi_\pi)$, we say that the involutive distribution of evolutionary vector fields in the images of linear independent strongly compatible operators $A_1$, $\ldots$, $A_N$ has reduced dimension (the rank) $N$.

Example 18 (The Magri schemes). Completely integrable bi-Hamiltonian hierarchies determine commutative distributions of reduced dimension $N = 2$. All commutators vanish for the restrictions $A_1$, $A_2$ of Hamiltonian operators onto the hierarchies; here $1 \leq i, j, k \leq 2$.

In other words, the strong compatibility of Hamiltonian operators $A_k: \Omega^1(\pi) \rightarrow \mathcal{F}(\pi)$ is achieved on linear subspaces of $\Omega^1(\pi)$. This is valid for the linear span of the gradients $\psi_i = E(\mathcal{H}_i)$ of the Hamiltonians $\mathcal{H}_i \in H^0(\pi)$ which descend from the Casimirs $\mathcal{H}_0 \in H^0(\pi)$ in the Magri scheme, see Theorem 10. Indeed, one has $\text{im} A_2 \subset \text{im} A_1$ whenever both Hamiltonian operators are restricted onto the descendants of the Casimirs for $A_1$, and hence the commutators (however, which vanish by the same theorem) belong to the image of $A_1$. Thus the iteration (49) of the Magri scheme corresponds to involutive distributions of reduced dimension two.

Let us formulate the properties of the bi-differential symbols $\Gamma^k_{ij} \in CDiff(\otimes^2 \Omega^1(\xi_\pi)$, $\Omega^1(\xi_\pi)$), which are determined by strongly compatible Frobenius operators $A_1$, $\ldots$, $A_N$. Note that the symbols $\Gamma^k_{ij}$ depend on a point $\theta \in J^\infty(\pi)$.

Property 1. By construction, for any number $N$ of strongly compatible Frobenius operators, we have

$$\Gamma^k_{ii} = \delta^k_i \cdot \{ , \} A_i,$$

for each $i$, $1 \leq i \leq N$. (85)

Hence a Frobenius operator yields the involutive distribution of reduced dimension one.

Property 2. The symbols $\Gamma^k_{ij}$ are not uniquely defined. Indeed, they are gauged by the conditions

$$\sum_{k=1}^N A_k \left( E_{A_i(q)}(p) \delta^k_i - E_{A_i(p)}(q) \delta^k_j + \Gamma^k_{ij}(p, q) \right) = 0,$$

$p, q \in \Omega^1(\xi_\pi)$; (86)

again, we assume the semi-simplicity of all the images in $\mathfrak{g}(\pi)$: $[A_\ell(\psi), \mathfrak{g}(\pi)] = 0$ implies $\psi \in \ker A_\ell$.

\footnote{For example, the first and second Hamiltonian structures for the KdV equation [11] are not strongly compatible unless restricted onto some subspaces of their arguments. On the linear subspace of descendants of the Casimir $\int u dx$, we have $\text{im} A_2 \subset \text{im} A_1$ and, since the image of the Hamiltonian operator $A_1 = D_x$ is closed, we have $[\text{im} A_1, \text{im} A_2] \subset \text{im} A_1$. We emphasize that we do not exploit the commutativity of the flows.}
**Property 3.** If, additionally, two strong compatible Frobenius operators $A_i$ and $A_j$ are linear compatible, then their Sokolov’s brackets are

$$\{\{p, q\}\}_A = \Gamma^i_{ij}(p, q) + \Gamma^j_{ji}(p, q) \quad \text{and} \quad \{\{p, q\}\}_{A_j} = \Gamma^i_{ij}(p, q) + \Gamma^j_{ji}(p, q)$$

for any $p, q \in \Omega^1(\xi_\pi)$.

**Proof.** For brevity, denote $A = A_i$, $B = A_j$ and consider the Frobenius linear combination $\mu A + \nu B$. By Proposition 19 we have

$$\begin{align*}
(\mu A + \nu B)(\{\{p, q\}\}_A) & = \mu^2 A(\{\{p, q\}\}_A) + \mu \nu \cdot A(\{\{p, q\}\}_B) + \mu \nu \cdot B(\{\{p, q\}\}_A) + \nu^2 B(\{\{p, q\}\}_A).
\end{align*}$$

On the other hand,

$$\begin{align*}
(\mu A + \nu B)(\{\{p, q\}\}_B) & = \mu^2 [A(p), A(q)] + \mu \nu [A(p), B(q)] - \mu \nu [A(q), B(p)] + \nu^2 [B(p), B(q)].
\end{align*}$$

Taking into account (85) and equating the coefficients of $\mu \nu$, we obtain

$$A(\{\{p, q\}\}_B) + B(\{\{p, q\}\}_A) = A(\Gamma^A_{AB}(p, q)) + B(\Gamma^B_{AB}(p, q)) - A(\Gamma^A_{BA}(q, p)) - B(\Gamma^B_{BA}(q, p)).$$

Using the obvious formulas $\Gamma^A_{AB}(p, q) = -\Gamma^A_{BA}(q, p)$ and $\Gamma^B_{AB}(q, p) = -\Gamma^B_{BA}(q, p)$, see (86) below, we isolate the arguments of the operators and obtain the assertion. □

**Definition 7.** From now on, we consider Frobenius operators that are both linear and strong compatible. Such operators will be called **totally compatible**. By definition, totally compatible operators span linear spaces

$$\mathcal{A} = \bigoplus_{k=1}^N A_k \cdot \mathbb{R} \quad (87)$$

of Frobenius operators.

**Example 19.** The operators $\Box$ and $\hat{\Box}$, which were introduced in Example 17, are totally compatible. This construction admits a straightforward generalization (105) for other Euler–Lagrange Liouville-type systems, see Theorem 25.

**Remark 13.** Within the Hamiltonian approach [22], it is very productive to think that the cosymmetries $\psi \in \mathcal{H}(\pi)$ are odd. Indeed, in this particular situation the homomorphisms $\psi \in \text{Hom}_{\mathcal{F}(\pi)}(\mathcal{X}(\pi), \Lambda^n(\pi))$ are identified with Cartan 1-forms times the volume form $\text{d} \text{vol}(M^n)$ for the base of the jet bundle.

We preserve this understanding for domains $\Omega^1(\xi_\pi)$ of Frobenius operators (15) of second kind. The new $\mathbb{Z}$-grading must be never mixed with any (e.g., $\mathbb{Z}_2$) gradings of any variables on $J^\infty(\pi)$ and with the arising gradings of the sections $F \in \mathfrak{h}$ and $\psi \in \mathfrak{h}$ (here $\mathfrak{f} = \mathfrak{h}$). Hence if $\pi$ is a super-bundle with Grassmann-valued sections, then Frobenius operators $A$ are bi-graded [39]. Their proper $\mathbb{Z}$-grading is $|A|_\mathbb{Z} = -1$ because the images in $\mathfrak{g}(\pi)$ have degree zero; the $\mathbb{Z}_2$-degree of the operators $A$ can be arbitrary. In particular, Hamiltonian operators $P$ produce Cartan’s 0-forms

$$\mathcal{E}_P(\pi) : \Omega^1(\pi) \to \mathfrak{g}(\pi).$$

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18 We owe this remark to Yu. I. Manin (private communication).
In what follows, we assume for simplicity that all coordinates on \( J^\infty(\pi) \) are permutable, whence \( \Omega^1(\xi) \) is even w.r.t. the \( \mathbb{Z}_2 \)-grading.

**Property 4.** For any \( i, j, k \in [1, \ldots, N] \) and for arguments \( p, q \in \Omega^1(\xi) \) of \( \mathbb{Z} \)-degree 1 for strong compatible Frobenius operators of second kind, we have

\[
\Gamma^k_{ij}(p, q) = -\Gamma^k_{ji}(q, p) = (-1)^{|p|\cdot|q|\cdot|z|} \cdot \Gamma^k_{ji}(q, p) \tag{88}
\]

due to the skew-symmetry of the commutators \([24]\). Hence the symbols \( \Gamma^k_{ij} \) are symmetric w.r.t. the \( \mathbb{Z} \)-grading in the case \([25]\).

Now consider the space of flat connections \( \nabla^{A_k} \) defined by \([63]\) for each Frobenius operator \( A_k \). Let us reveal the standard behaviour of the Christoffel symbols \( \Gamma^k_{ij} \).

**Property 5** (Transformations of \( \Gamma^k_{ij} \)). Let \( \tilde{\omega} = \tilde{\omega}[w] \) be a non-degenerate change of fibre coordinates in the bundle \( \xi \). Recall that the sections \( p, q \in \mathfrak{f} \) are reparametrized by \( p \mapsto \tilde{p} = \Xi(p) \) and \( q \mapsto \tilde{q} = \Xi(q) \), where \( \Xi = \tilde{\omega}^{(w)} \) for Frobenius operators \([63]\) of first kind and \( \Xi = \left[(E_{\omega}^{(w)})^p\right]^{-1} \) for the operators of second kind. Consequently, Frobenius operators \( A_1, \ldots, A_N : \Omega^1(\xi) \to \mathfrak{g}(\pi) \) with a common domain \( \Omega^1(\xi) = \mathfrak{f}/\bigcap_i \ker A_i \) are transformed by \( A_i \mapsto \tilde{A}_i = A_i \circ \Xi^{-1}|_{\tilde{\omega}[w]} \). Then the bi-differential symbols \( \Gamma^k_{ij} \in \mathcal{CDiff}(\Omega^1(\xi) \times \Omega^1(\xi) \to \Omega^1(\xi)) \) obey

\[
\Gamma^k_{ij}(p, q) \mapsto \Gamma^k_{ij}(\tilde{p}, \tilde{q}) = (\Xi \circ \Gamma^k_{ij})(\Xi^{-1}\tilde{p}, \Xi^{-1}\tilde{q}) + \delta^k_i : \mathcal{E}_{\tilde{A}_j(\tilde{q})}(\Xi)(\Xi^{-1}\tilde{p}) - \delta^k_j : \mathcal{E}_{\tilde{A}_i(\tilde{p})}(\Xi)(\Xi^{-1}\tilde{q}). \tag{89}
\]

This is a direct analogue of the standard rules \( \tilde{\Gamma} = \Xi \Gamma \Xi^{-1} + d\Xi : \Xi^{-1} \) for the connection 1-forms \( \Gamma \) under reparametrizations \( \Xi \). Formula \((89)\) represents this rule in the case of connections over the infinite jet bundles.

**Proof.** Denote \( A = A_i \) and \( B = A_j \); without loss of generality, assume \( i = 1 \) and \( j = 2 \). Let us calculate the commutators of vector fields in the images of \( A \) and \( B \) using two systems of coordinates in the domain. Then we equate the commutators straightforwardly, because the fibre coordinate in the images of the operators is not touched at all. So, we have, originally,

\[
[A(p), B(q)] = B(\mathcal{E}_{A(p)}(q)) - A(\mathcal{E}_{B(q)}(p)) + A(\Gamma^A_{AB}(p, q)) + B(\Gamma^B_{AB}(p, q)) + \sum_{k=3}^N A_k(\Gamma^k_{AB}(p, q)).
\]

On the other hand, we substitute \( \tilde{p} = \Xi(p) \) and \( \tilde{q} = \Xi(q) \) in \([\tilde{A}(\tilde{p}), \tilde{B}(\tilde{q})]\), whence, by the Leibnitz rule, we obtain

\[
[\tilde{A}(\tilde{p}), \tilde{B}(\tilde{q})] = \tilde{B}(\mathcal{E}_{\tilde{A}(\tilde{p})}(\Xi)(q)) + (\tilde{B} \circ \Xi)(\mathcal{E}_{\tilde{A}(\tilde{p})}(q)) - \tilde{A}(\mathcal{E}_{\tilde{B}(\tilde{q})}(\Xi)(p)) + (\tilde{A} \circ \Xi)(\mathcal{E}_{\tilde{B}(\tilde{q})}(p))
\]

\[
+ (A \circ \Xi^{-1})(\Gamma^A_{AB}(\Xi p, \Xi q)) + (B \circ \Xi^{-1})(\Gamma^B_{AB}(\Xi p, \Xi q)) + \sum_{k=3}^N (A_k \circ \Xi^{-1})(\Gamma^k_{AB}(\Xi p, \Xi q)).
\]
Therefore,
\[
\Gamma^A_{AB}(p, q) = (\Xi^{-1} \circ \Gamma^A_{AB})(\Xi p, \Xi q) - (\Xi^{-1} \circ \mathcal{E}_{B(q)}(\Xi))(p),
\]
\[
\Gamma^B_{AB}(p, q) = (\Xi^{-1} \circ \Gamma^B_{AB})(\Xi p, \Xi q) + (\Xi^{-1} \circ \mathcal{E}_{A(p)}(\Xi))(q),
\]
\[
\Gamma^k_{AB}(p, q) = (\Xi^{-1} \circ \Gamma^k_{AB})(\Xi p, \Xi q) \quad \text{for } k \geq 3.
\]
Acting by $\Xi$ on these equalities and expressing $p = \Xi^{-1} \tilde{p}$, $q = \Xi^{-1} \tilde{q}$, we conclude the proof.

**Corollary 21.** The bi-differential symbols $\Gamma^k_{ij}$ constitute symmetric flat connections $\nabla(\tilde{\Lambda}) = \sum_k \lambda_k \nabla^{A_k}$ in the graded triples $(\Omega^1(\xi), g(\pi), A)$ determined by Frobenius operators of second kind.

**Remark 14.** A straightforward calculation shows that two operators (74) are strong compatible if and only if they are proportional. Next, consider operators (57) for the Boussinesq-type system (56). If these linear compatible Frobenius operators are not restricted to a subspace of $\Omega^1(\xi)$, and thus the arguments of $A_k$ are generic, then the commutators $[\im A_i, \im A_j]$, $0 \leq i < j \leq 2$, are not decomposed using (84) w.r.t. the images of any two operators $A_i$ and $A_j$. We argue that these negative examples have a very deep motivation, which has been indicated in Example 18.

Indeed, by definition, the commutator (84) always contains the standard first two terms in the r.h.s. and takes values in the entire sum of images $\sum_k \im A_k$ for generic $p, q \in \Omega^1(\xi)$. Hence the commutation relations, which determine the Lie-algebraic type of the involutive distribution in the sum of images of $A_1, \ldots, A_N$, depend on a linear subspace $S \subset \Omega^1(\xi)$ that contains $p$ and $q$. We may choose it ourselves in such a way that some of the operators become restricted on their kernels (up to the condition (80)).

Therefore we expect that the strong compatibility and the decompositions w.r.t. the three linear independent operators (57) are restored for restrictions of the Frobenius operators onto the spans of Hamiltonians for the dispersionless 3-component Boussinesq hierarchy. This will be discussed elsewhere.

### 4.3. Algebras of Frobenius operators.

Linear spaces of recursion operators $R \in \End_{\mathbb{R}} \mathcal{X}(\pi)$ are equipped with the associative composition $\circ$, and therefore the recursions constitute monoids. Their unit is the identity mapping, and there appear relations between the operators or between their restrictions onto differential equations. For instance, the structural relations for for recursion operators of the Krichever–Novikov equations are described by hyperelliptic curves, see [7]. Taking compositions of two recursions $R_i$ and $R_j$, we also obtain their formal commutators (35) by setting
\[
[R_i, R_j] = R_i \circ R_j - R_j \circ R_i.
\]

Nontrivial examples of relations for the algebra structures (90) are known, e.g., one has $[R_1, R_2] = R_1^2$ for the dispersionless 3-component Boussinesq system (56), see [24]. In this way, the Richardson–Nijenhuis bracket (35) endows the linear spaces of recursions with a graded Lie algebra structure [39, 40]. If the Nijenhuis torsion $[N, N]^{fn}$ vanishes for an operator $N: \mathcal{X}(\pi) \to \mathcal{X}(\pi)$ and the Frölicher–Nijenhuis bracket $[\cdot, \cdot]^{fn}$, then it produces trivial infinitesimal deformations (15) of the standard bracket $[\cdot, \cdot]$ on
another linear space, that is, on \( \mathcal{X}(\pi) \). By their turn, Frobenius recursion operators \((75)\) determine nontrivial finite deformations \([ , ]_R\) of the Lie algebra structure of \( \mathfrak{g}(\pi) \).

We propose a reverse scheme that starts with the standard structure \((29)\) in the Lie algebra \( \mathfrak{g}(\pi) \) and then endows linear spaces of Frobenius operators with a Lie-type bracket.

First, let Frobenius operators \( A_1, \ldots, A_N : f \to \mathfrak{g}(\pi) \) be totally compatible such that each point of the linear space \((87)\) is a Frobenius operator. Consider the commutation closure relations \((84)\), which are specified for vector fields with generating sections that belong to the images of \( A_i \). These relations express the decomposition of the commutators in the left-hand side w.r.t. the images again. Note that formula \((84)\) is linear w.r.t. each operator from \( A \).

We suggest to take the decomposition \((84)\) and define the commutation rules
\[
[A_i, A_j](p, q) = [A_i(p), A_j(q)]
\]
on \( \mathcal{A} \), here \( 1 \leq i, j \leq N \) and \( p, q \in \Omega^1(\xi_\pi) = f/\bigcap_i \ker A_i \). The structural constants of the algebra \( \mathcal{A} \) are encoded by the bi-differential symbols \( \Gamma_{ij}^k \), whose properties were described in section 4.2. By construction, the Jacobi identity holds for the bracket \((91)\) which, under \( i \leftrightarrow j, \ p \leftrightarrow q \), is skew-symmetric for Frobenius operators \((63)\) of first kind and which is symmetric (in the graded sense \( |p|_\mathbb{Z} = |q|_\mathbb{Z} = 1 \), see Remark \((13)\) for Frobenius operators \((65)\) of second kind.

Consequently, each Frobenius operator \( A_i \in \mathcal{A} \) spans the one-dimensional algebra, and the Koszul bracket \([ , ]_R\) measures its noncommutativity. The assumption that the operators are linear compatible implies that each line \( \langle \vec{\lambda} \rangle \), where \( \langle \vec{\lambda} \rangle \overset{\text{def}}{=} \mathbb{R} \cdot \sum_{i=1}^N \lambda_i A_i \subset \mathcal{A} \), is a one-dimensional subalgebra of \( \mathcal{A} \).

By this argument, we obtain the third operation on the space \((87)\) of totally compatible Frobenius recursion operators \( \mathcal{X}(\pi) \to \mathcal{X}(\pi) \), in addition to the composition \( \circ \) and the commutation \((90)\). At the same time, if the Frobenius operators are not recursions, then the compositions of the operators \( A_i : f \to \mathcal{X}(\pi) \) are not defined, and the algebra structure \((91)\) on the linear space \( \mathcal{A} \) is the only one remaining.

**Example 20.** Proposition \((8)\) proves the existence of one-dimensional algebras of Hamiltonian Frobenius operators of second kind. Theorem \((10)\) yields the two-dimensional algebras \( \mathcal{A} \), see Example \((18)\). Restricting the Poisson pencils \((A_1, A_2)\) onto the hierarchies, we obtain analogues of the solvable two-dimensional Lie algebra with a relation \([a_1, a_2] = a_1 \). Moreover, the Magri scheme shows that these algebras \( \mathcal{A} \) are commutative.

Second, let us consider a wider class of operators \( A_i : f \to \mathcal{X}(\pi) \) that may not be Frobenius. Namely, let all the operators be transformed by either \((63)\) or \((65)\), which makes them well defined, and suppose that the commutation closure
\[
\left[ \sum_i \text{im} A_i, \sum_j \text{im} A_j \right] \subseteq \sum_k \text{im} A_k
\]
is valid for the images of the whole \( N \)-tuple \((A_1, \ldots, A_N)\). This incorporates the previous case of totally compatible Frobenius-tuple \((61)\). Now we introduce a Lie-type structure on the linear subspace \( \mathcal{A} \subseteq C\text{Diff}(f \to \mathcal{X}(\pi)) \).
Let \([A_i, A_j]\) be the same bracket \((91)\) of the operators \(A_1, \ldots, A_N \in \mathcal{A}\) at hand. In other notation, by \((92)\) we assume that

\[
[A_i, A_j] = \sum_{k=1}^{N} A_k \circ c_{ij}^k, \tag{93}
\]

where \(c_{ij}^k : f \times f \to f\) are the bi-differential structural constants of the new algebra \(\mathcal{A}\). Obviously, the Christoffel symbols \(\Gamma_{ij}^k\), which were introduced in \((84)\), are encoded by \(c_{ij}^k\) up to the gauge \((86)\). The structural constants are symmetric or skew-symmetric w.r.t. the lower indexes simultaneously with \(\Gamma_{ij}^k\). Admitting a slight abuse of language, we continue calling these algebras \(\mathcal{A}\) with relations \((93)\) by algebras of Frobenius operators, although the image of an operator \(A_\ell\) may be not closed under the commutation for some \(\ell \in [1, \ldots, N]\).

For instance, we have \(c_{ij}^k \equiv 0 \mod \ker A_k\) for the algebras \(\mathcal{A}\) of rank two that are generated by restrictions of Poisson pencils \((A_1, A_2)\) onto the Magri schemes, see Example 18. Other examples of such algebras \(\mathcal{A}\) will be given in the next section, where we assign Frobenius operators \((105)\) of second kind to Liouville-type integrable systems (in particular, to the 2D Toda chains associated with complex semi-simple Lie algebras, see \((12)\)).

Thus we arrive at the structural theory problem for the operator algebras \(\mathcal{A}\) with generators \(A_i : f \to \kappa(\pi)\) and relations \((93)\).

**Conjecture 22.** There are horizontal \(\mathcal{F}(\xi)\)-modules \(f \hookrightarrow \Gamma(\pi_\infty^*(\xi))\) and linear differential operators \(A_1, \ldots, A_N : f \to g(\pi)\) such that, by a choice of appropriate subspaces \(S \subset f\) in their domain, one recovers a bi-differential extension of the classical structural theory of (e.g., nilpotent, semi-simple, or solvable) Lie algebras of reduced dimension \(N\) for the linear spaces \(\mathcal{A}\), which are generated by the restrictions of the operators \(A_1, \ldots, A_N\) onto \(S\), and for the algebra structures \((93)\) on them.

**Remark** 15. Recursion operators for differential equations \(\mathcal{E} = \{ F = 0 \}\) can be understood as Bäcklund autotransformations between symmetries, which are solutions of the linearized systems \(\ell_F(\varphi) = 0\), see \([3, 39]\). Hence the structures \((93)\) of the algebras \(\mathcal{A}\) of Frobenius recursion operators for \(\mathcal{E}\) are inherited by these classes of Bäcklund autotransformations. This determines the brackets between non-abelian coverings over the equations \(\mathcal{E}\).

5. **Factorizations of symmetries of Liouville-type systems**

In this section we describe an infinite class of Frobenius operators \(\Box\) and calculate the brackets \(\{ \cdot, \cdot \}_{\Box}\) induced by them. These operators appear in the description of symmetries of the hyperbolic Liouville-type Euler–Lagrange nonlinear systems \([8, 56, 58, 60]\).

To start with, we extend the collection of known Frobenius operators with the one that factors point symmetries of the non-evolutionary \((2 + 1)\)-dimensional \(A_\infty\)-Toda equation.

---

\(^{19}\)Thus we postulate that the bi-differential dependence on the arguments in \((93)\) is entirely absorbed by the structural constants \(c_{ij}^k\).
Example 21 \((\ref{example:21})\). Generators of the point symmetry algebra for the ‘heavenly’ Toda equation \(u_{xy} = \exp(-u_{zz})\) have the form \(\varphi^x = \hat{\Box}^x(\phi(x))\) or \(\varphi^y = \hat{\Box}^y(\bar{\phi}(y))\), where \(\phi, \bar{\phi} \in C^\infty(\mathbb{R})\) and

\[
\hat{\Box}^x = u_x + \frac{1}{2}z^2 D_x, \quad \hat{\Box}^y = u_y + \frac{1}{2}z^2 D_y = (x \leftrightarrow y)(\hat{\Box}^x).
\]

Clearly, the commutator of any two point symmetries of \(x\)- or \(y\)-type is a point symmetry again such that the action of the operators \(\hat{\Box}\) on the spaces of the free functional parameters \(\phi, \bar{\phi}\) is given by the Wronskian, \(\{\{\phi_1, \phi_2\}\}_{\hat{\Box}} = \phi_1 \cdot (\phi_2)_x - (\phi_1)_x \cdot \phi_2\).

Operators \((\ref{example:21})\) factor the right-hand sides of the symmetry flows \(u_t = \hat{\Box}(\phi)\) on the heavenly equation. This means that a bigger differential equation is the representing object for a transformation group generated by the flows. This approach to constructing Frobenius operators is very productive. First, let us recall a procedure that assigns hyperbolic Euler–Lagrange systems to hierarchies which are Hamiltonian w.r.t. \(\hat{A}_1 = \text{const} \cdot D_x\), see \([\ref{example:21}, \ref{example:21}]\) and also Remark \([\ref{example:21}]\) on p. \(\ref{example:21}\). The method is based on the canonical coordinate-momenta formalism \([\ref{example:21}]\) for PDE and on a representation of the KdV-type hierarchies as commutative Lie subalgebras of Noether symmetries for the hyperbolic systems.

Example 22. The dispersionless 3-component Boussinesq-type system, see p. \(\ref{example:21}\)

\[
u_t = u w_x + v_x, \quad v_t = -u w_x - 3 u_x w, \quad w_t = u,\]

is not written here in the form of a conserved current only due to an unfortunate choice of local coordinates for it and for structures \((\ref{example:21})\) in \([\ref{example:21}]\). Indeed, let us return to the notation of \([\ref{example:21}]\) and let the new dependent variables \(m = (u, v, w)\) be

\[
u = v + w^2, \quad v = w, \quad w = u,
\]

which are densities of the Casimirs w.r.t. the first Hamiltonian structure \((\ref{example:21})\) for \((\ref{example:21})\). Thence we obtain the system

\[
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 \\1 & 0 & 0 \\0 & 0 & 1
\end{pmatrix} D_x \begin{pmatrix}\delta/\delta u \\\delta/\delta v \\\delta/\delta w
\end{pmatrix} (\mathcal{H}_1),
\]

where \(\mathcal{H}_1 = [(u w - \frac{1}{2}v^2) dx]\). Likewise, the dispersionless Benney equation \([\ref{example:21}]\) Eq. (19)\) acquires the same form \((\ref{example:21})\) in these coordinates.

Now we introduce the conjugate variables \(U = (U, V, W)\) such that

\[
\left(\ell^U_m\right)^* = - \begin{pmatrix}0 & 1 & 0 \\1 & 0 & 0 \\0 & 0 & 1
\end{pmatrix} \cdot D_x;
\]

in other words, the adjoint linearization \((\ell^U_m)^*\) of the canonical momenta \(m\) w.r.t. the canonical coordinates \(U\) is proportional to the first Hamiltonian structure for \((\ref{example:21})\). Thus we set

\[
u = V_x, \quad v = U_x, \quad w = W_x.
\]

The potential variables satisfy the system

\[
U_t = W_x, \quad V_t = -U_x W_x, \quad W_t = -\frac{1}{2} U_x^2 + V_x.
\]
Using the notation (95) and (97) together, we cast (96) and (96′) to the canonical form

\[
U_t = \frac{\delta H_1}{\delta U}, \quad V_t = \frac{\delta H_1}{\delta V}, \quad W_t = \frac{\delta H_1}{\delta W}, \quad \delta u = -\frac{\delta H_1}{\delta U}, \quad \delta v = -\frac{\delta H_1}{\delta V}, \quad \delta w = -\frac{\delta H_1}{\delta W},
\]

where \(H_1 = [(V_x W_x - \frac{1}{2}U_x^2 W_x) \, dx]\).

Next, let us find the hyperbolic Euler–Lagrange equation

\[
\mathcal{E}_{EL} = \left\{ E_U(\mathcal{L}) = 0 \mid \mathcal{L} = [L \, dx \, dy], \, L = -\frac{1}{2} \langle m, U_y \rangle - H_L(U) \right\}
\]

such that the bi-Hamiltonian hierarchy of commuting flows for (98) is composed by symmetries of \(\mathcal{E}_{EL}\). This is done straightforwardly. We note that system (98) is scaling-invariant w.r.t. the homogeneity weights \(|u| = 2, |w| = 1, |w| = 3/2; |U| = 0, |V| = 1, |W| = 1/2; and |D_x| = 1, |D_t| = 3/2\). The symmetries of (98) with time weights \(3/2 = |D_t|, 2, 3\) fix the Hamiltonian \(H_L(U)\) for \(\mathcal{E}_{EL}\) uniquely. The only system that is ambient w.r.t. the whole hierarchy of (98) is the wave equation

\[
\mathcal{E}_\varnothing = \{ U_{xy} = 0, \, V_{xy} = 0, \, W_{xy} = 0 \}.
\]

The wave equation \(\mathcal{E}_\varnothing\) is exactly solvable, because the conditions

\[
U_x, V_x, W_x \in \ker D_y |_{\mathcal{E}_\varnothing}, \quad U_y, V_y, W_y \in \ker D_x |_{\mathcal{E}_\varnothing}
\]

can be integrated immediately, and we obtain \(U = f(x) + g(y)\). At the same time, by the argument illustrated in this example, we have naturally arrived to the definition of a class of nonlinear hyperbolic systems such that conditions (99) become nontrivial.

**Definition 8** (56). A Liouville-type system \(\mathcal{E}_L\) is a system \(\{ u_{xy} = F(u, u_x, u_y; x, y) \}\) of hyperbolic equations which possesses the integrals \(w_1, \ldots, w_r; \bar{w}_1, \ldots, \bar{w}_r \in C^\infty(\mathcal{E}_L)\) such that the relations \(D_y |_{\mathcal{E}_L}(w_i) \equiv 0\) and \(D_x |_{\mathcal{E}_L}(\bar{w}_j) \equiv 0\) hold by virtue of (\(\varnothing\)) of \(\mathcal{E}_L\), and such that all conservation laws for \(\mathcal{E}_L\) are of the form \([f(x, [w]) \, dx] \oplus [g(y, [\bar{w}]) \, dy]\).

**Example 23.** The \(m\)-component 2D Toda chains (12) associated with semi-simple complex Lie algebras (42) constitute an important class of Liouville-type systems, here \(u = (u^1, \ldots, u^m)\). By (56), they possess the complete sets of \(2m\) integrals \(w_1, \ldots, w_m; \bar{w}_1, \ldots, \bar{w}_m\) and are integrable iff \(K\) is the Cartan matrix. This class is covered by the ansatz in Proposition 23 below.

**Remark 16.** Consider Liouville-type systems that possess complete sets of \(2m\) integrals: \(r = m\) and \(\bar{r} = m\). By slightly narrowing the class of such equations, let us consider the systems \(\mathcal{E}_L\) whose general solutions are parameterized by arbitrary functions \(f^1(x), \ldots, f^m(x)\) and \(g^1(y), \ldots, g^m(y)\). These equations are represented as the diagrams (26)

\[
\bigoplus_{i=1}^m J^\infty(\pi^x) \oplus J^\infty(\pi^y) \xrightarrow{\text{sol}} \mathcal{E}_L \xrightarrow{\text{int}} \bigoplus_{i=1}^m J^\infty(\pi^x) \oplus J^\infty(\pi^y),
\]

where \(\pi^x\) and \(\pi^y\) are the trivial fibre bundles \(\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) such that \(f, w, g, \bar{w}\) determine their sections, respectively; the first arrow is given by the formulas for exact solutions and the second arrow is determined by the integrals.

\[\text{For } |s| = 2, \text{ the symmetry of (40) is } U_s = V_s; \quad V_s = \frac{1}{2}U_x^3 - \frac{1}{4}W_x^2; \quad W_s = -U_x W_x; \quad \text{the flow with } |r| = 3 \text{ is given by } U_t = 2U_x V_x + W_x^2; \quad V_t = \frac{1}{2}U_x^2 + V_x^2 - 2U_x W_x^2; \quad W_t = 2V_x W_x - 2U_x^2 W_x. \text{ There is no symmetry of (40) with time weight 5/2.} \]
Remark 17. Consider now a class of hyperbolic quasilinear systems $\mathcal{E}_\infty$ whose symmetries $\varphi = \Box(\phi)$ are determined by Frobenius linear differential operators $\Box$, c.f. Remark 3 on p. 29. Definition 3 of the Liouville-type systems, which postulates the existence of the integrals, is not equivalent to the definition of $\mathcal{E}_\infty$ with the functional freedom in symmetries. However, the two notions are very close. In Corollary 24 see below, we prove that symmetries $\varphi$ of Euler–Lagrange Liouville-type systems $\mathcal{E}_L$ are factored by differential operators $\Box$, $\Box$ that originate from the integrals $w$, $\bar{w}$ for $\mathcal{E}_L$: we have

$$\varphi = \Box(\phi(x, [w])), \quad \bar{\varphi} = \Box(\bar{\phi}(y, [\bar{w}])), \quad (100)$$

where components of the sections $\phi$ and $\bar{\phi}$ are arbitrary smooth functions. We show that the operators $\Box$, $\Box$ are Frobenius, which yields decomposable operator algebras $\mathcal{A} = \Box \cdot \mathbb{R} \oplus \Box \cdot \mathbb{R}$ of rank two.

Let us remark on the history of the problem. Scalar Liouville-type equations were studied in [60] and their symmetries have been further analyzed in [8]. Operators $\Box$ that factor Noether symmetries of Euler–Lagrange Liouville-type systems $u_{xy} = F(u, x, y)$ were constructed in [25]. The general case $u_{xy} = F(u, u_x, u_y; x, y)$ of the Euler–Lagrange Liouville-type systems (see (101)) was considered in [58]; however, no method for reconstructing the brackets $\{\ {,\ }\} _{\Box}$ is described there.

The problem of construction of operators $\Box$ that assign (possibly, not all) symmetries of non–Euler–Lagrange Liouville-type systems to their integrals $w$, $\bar{w}$ is much less transparent. A considerable progress has been achieved here in a recent paper [57], see section 5.2, where it is shown that the existence of differential operators $\Box$ is based on the existence of $\bar{w}$, and respectively for $\Box$ and $w$. We analyse mainly the former case of Euler–Lagrange systems.

Example 24. Consider the parametric extension of the scalar Liouville equation (6),

$$u_{xy} = \exp(2u) \cdot \sqrt{1 + 4\varepsilon^2 u_x^2}. \quad (101)$$

This equation is ambient w.r.t. the hierarchy of Gardner’s deformation of the potential modified KdV equation, see [26]. The contraction $\mathcal{U} = \mathcal{U}(\varepsilon, [u(\varepsilon)])$ from (101) to the non-extended equation $\mathcal{U}_{xy} = \exp(2\mathcal{U})$ is $\mathcal{U} = u + \frac{1}{2}\arcsinh(2\varepsilon u_x)$; it determines the third order integral for (101) using the integral (7) at $\varepsilon = 0$, see Example 1. However, the regularized minimal integral of second order for (101) is

$$w = \left(1 - \sqrt{1 + 4\varepsilon^2 u_x^2}\right) / 2\varepsilon^2 + u_{xx} / \sqrt{1 + 4\varepsilon^2 u_x^2}; \quad (102)$$

such that all other $x$-integrals for (101) are differential functions of (102). The second integral for (101) is $\bar{w} = u_{yy} - u_y^2 - \varepsilon^2 \cdot \exp(4u) \in \ker D_x |_{\mathcal{E}}$. The operators $\Box = u_y + \frac{1}{2} D_y$ and

$$\Box = \frac{1}{2} (1 + 4\varepsilon^2 u_x^2 - 2\varepsilon^2 u_{xx}) \cdot D_x + u_x + 4\varepsilon^2 u_x^3 - 2\varepsilon^2 u_{xxx} + \frac{12\varepsilon^4 u_x u_{xx}}{1 + 4\varepsilon^2 u_x^2} \quad (103)$$

assign symmetries (100) of (101) to its integrals. We emphasize that operators in the family (103) assign higher symmetries $\varphi = \Box(\phi(x))$ of (101) to functions on the

\[\text{Remark 48 A. V. KISELEV AND J. W. VAN DE LEUR}\]

\[\text{Example 24.}\]

\[\text{Consider the parametric extension of the scalar Liouville equation (5),}\]

\[u_{xy} = \exp(2u) \cdot \sqrt{1 + 4\varepsilon^2 u_x^2}. \quad (101)\]

\[\text{This equation is ambient w.r.t. the hierarchy of Gardner’s deformation of the potential modified KdV equation, see [26]. The contraction } \mathcal{U} = \mathcal{U}(\varepsilon, [u(\varepsilon)]) \text{ from (101) to the non-extended equation } \mathcal{U}_{xy} = \exp(2\mathcal{U}) \text{ is } \mathcal{U} = u + \frac{1}{2}\arcsinh(2\varepsilon u_x)\text{; it determines the third order integral for (101) using the integral (7) at } \varepsilon = 0, \text{ see Example 1. However, the regularized minimal integral of second order for (101) is}\]

\[w = \left(1 - \sqrt{1 + 4\varepsilon^2 u_x^2}\right) / 2\varepsilon^2 + u_{xx} / \sqrt{1 + 4\varepsilon^2 u_x^2}; \quad (102)\]

\[\text{such that all other } x\text{-integrals for (101) are differential functions of (102). The second integral for (101) is } \bar{w} = u_{yy} - u_y^2 - \varepsilon^2 \cdot \exp(4u) \in \ker D_x |_{\mathcal{E}}. \text{ The operators } \Box = u_y + \frac{1}{2} D_y \text{ and}\]

\[\Box = \frac{1}{2} (1 + 4\varepsilon^2 u_x^2 - 2\varepsilon^2 u_{xx}) \cdot D_x + u_x + 4\varepsilon^2 u_x^3 - 2\varepsilon^2 u_{xxx} + \frac{12\varepsilon^4 u_x u_{xx}}{1 + 4\varepsilon^2 u_x^2} \quad (103)\]

\[\text{assign symmetries (100) of (101) to its integrals. We emphasize that operators in the family (103) assign higher symmetries } \varphi = \Box(\phi(x)) \text{ of (101) to functions on the}\]
base of the jet bundle whenever ε ≠ 0, while the operator □ always determines point symmetries \( \tilde{\varphi} = \square(\tilde{\phi}(y)) \).

Both operators □ and \( \square \) satisfy (61). The bracket \( \{\{p, q\}\} \square = pq - q-p \) for □ is familiar; the bracket induced in the inverse image of □ is calculated in appendix B. The surprisingly high differential orders of \( \{\{\cdot, \cdot\}\} \) with respect to its arguments and coefficients is motivated by the presence of higher order derivatives of \( u \) in (103).

**Remark 18.** Deformations of the algebras of Frobenius operators appear by virtue of the Gardner deformations for differential equations, see Example 24 and [26]. Let \( \mathcal{E}(\mu) \) and \( \mathcal{E}(\nu) \) be the extensions of an equation \( \mathcal{E}(0) \) and let \( m: \mathcal{E}(\varepsilon) \rightarrow \mathcal{E}(0) \) be the Miura contraction at \( \varepsilon \in I \subset \mathbb{R} \). Symmetries \( \varphi_\mu \in \text{sym} \mathcal{E}(\mu) \) and \( \varphi_\nu \in \text{sym} \mathcal{E}(\nu) \) induce the symmetries of \( \mathcal{E}(0) \) through the contraction, although the induced flows can be formal sums of infinitely many generators of higher orders. However, consider the commutator of the (formal) symmetries. For all \( \varepsilon(\mu, \nu) \) such that the commutator is lifted to a true symmetry of \( \mathcal{E}(\varepsilon) \), we define the product \( \diamond: (\mu, \nu) \mapsto \mu \diamond \nu = \varepsilon(\mu, \nu) \). Thus we obtain the multiplication \( M \diamond N = \{\varepsilon\} \subset \mathbb{R} \) on the sets \( M = \{\mu\} \), \( N = \{\nu\} \subset \mathbb{R} \). For the Liouville-type Gardner’s extension (101), we further obtain the product \( \diamond: (\square(\mu), \square(\nu)) \mapsto \square(\mu \diamond \nu) \) of the Frobenius operators (103).

5.1. **Frobenius operators for Lagrangian systems.** For Euler–Lagrange Liouville-type systems \( \mathcal{E}_{\text{EL}} = \{F \equiv \mathbb{E}(\mathcal{L}) = 0\} \), the existence of factorizations (100) for at least a part of symmetries is rigorous and can be readily seen as follows. For integrals \( w \) such that \( D_y(w) = \nabla(F) \) and for any \( I(x, [w]) \), the generating section \( \psi_I = \left[ \nabla^\ast \circ (\ell_{w}^{(u)})^* \circ ((\ell_{x}^{(w)})^*) \right] \) (1) for a conservation law \( [I \, dx] \) solves the equations \( \ell_{\mathbb{E}(\mathcal{L})}^* \psi_I = 0 \) on \( \mathcal{E}_{\text{EL}} \). The Helmholtz condition (33) for \( \ell_{\mathbb{E}(\mathcal{L})} \) implies that the vector

\[
\varphi[\phi] = \left[ \nabla^\ast \circ (\ell_{w}^{(u)})^* \right](\phi(x, [w])) \in \ker \ell_{\mathbb{E}(\mathcal{L})} |_{\mathcal{E}_{\text{EL}}} \quad (104)
\]

is a symmetry of \( \mathcal{E}_{\text{EL}} \) for any \( \phi = (\ell_{x}^{(w)})^* (1) = \mathbb{E}_w (I \, dx) \). A standard reasoning (see Lemma 7 or Corollary 24 below) shows that formula (104) yields symmetries of the system \( \mathcal{E}_{\text{EL}} \) even if sections \( \phi \) do not belong to the image of the variational derivative \( \mathbb{E}_w \).

In this section, we construct a class of Frobenius operators □ associated with symmetries of Euler–Lagrange Liouville-type systems. We express the operators □ and the brackets \( \{\{\cdot, \cdot\}\} \square \) in terms of minimal integrals \( w \). This is done by the following argument.

**Proposition 23** ([25]). Let \( \kappa \) be a nondegenerate symmetric constant real \( (m \times m) \)-matrix. Suppose that \( \mathcal{L} = [L \, dx \, dy] \) with the density \( L = -\frac{1}{2} \sum_{i,j} \kappa_{ij} w_i^j u_j^i - H_L(w; x, y) \) is the Lagrangian of a Liouville-type equation \( \mathcal{E}_L = \{\mathbb{E}(\mathcal{L}) = 0\} \). Let \( m = \partial L/\partial u_y \) be Dirac’s momenta [34] and \( w(m) = (w^1, \ldots, w^r) \) be the minimal set of integrals for \( \mathcal{E}_L \) that belong to the kernel of \( D_y |_{\mathcal{E}_L} \). Then the adjoint linearization

\[
\square = (\ell_{w}^{(m)})^* \quad (105)
\]

of the integrals w.r.t. the momenta factors all Noether symmetries \( \varphi_L \) of \( \mathcal{E}_L \), which are given by

\[
\varphi_L = \square(\delta \mathcal{H} / \delta w) \quad (106)
\]
for any $\mathcal{H} = [H(x,[w])] \, dx$.

Outline of the proof. The assertion follows from

- the structure $\psi = -D^{-1}_u(E_u(\mathcal{H}))$ of the generating sections of conservation laws for hyperbolic systems $u_{xy} = \kappa^{-1}(\delta H_u/\delta u)$ resolved w.r.t. the second-order derivatives,
- the correlation $\frac{\delta}{\delta u} = (\ell^w_u)\circ (\ell^m_u) \circ \frac{\delta}{\delta w}$

between the variational derivatives w.r.t. $u$ and $w$, and
- the correlation $\psi = \kappa \varphi_L$ between generating sections of conservation laws and Noether symmetries $\varphi_L$ of systems $u_{xy} = \kappa^{-1}(\delta H_u/\delta u)$, see Theorem 2. \hfill $\Box$

**Corollary 24.** Under the assumptions and notation of Proposition 23 the section

$$\varphi = \square (\bar{\phi}(x,[w]) \in \kappa(\pi)$$

is a symmetry of the Liouville-type equation $E_L$ for any $\bar{\phi} = \{\phi^1, \ldots, \phi^r\} \in \mathfrak{f}$.

**Proof.** The proof is standard and analogous to the one for Lemma 7 with the only alteration in the jet space at hand. Consider the jet bundle $J^\infty(\xi)$ over the fibre bundle

$$\xi: \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}$$

with the base $\mathbb{R} \ni x$ and the fibres $\mathbb{R}^r$ with coordinates $w^1, \ldots, w^r$. By Proposition 23 the statement of the theorem is valid for any $\bar{\phi} \in \kappa(\pi)$ in the image of the variational derivative $E_w$, where $w = w[m]$. Obviously, the image contains all $r$-tuples $\bar{\phi}$ whose components $\phi^i(x) \in C^\infty(\mathbb{R})$ are functions on the base of the new jet bundle. The substitution $w = w[m[u]]: \mathfrak{f} \to \kappa(\pi)$ converts sections $\bar{\phi} \in \mathfrak{f}$ to elements of $\Gamma(\pi^*_u(\xi))$. Now recall that $\square$ is an operator in total derivatives (27), whose action on $r$-tuples of functions on the jet space $J^\infty(\pi)$ is defined by (23) through their restriction onto the jets of sections $w = \phi(x)$, whence the assertion follows. \hfill $\Box$

**Theorem 25.** The operators (103) are Frobenius of second kind if the integrals $w$ are minimal.

This assertion is continued in Proposition 27 on p. 53.

**Proof of Theorem 25.** The commutator of two Noether symmetries $\varphi'_L, \varphi''_L$ is a Noether symmetry $\varphi_L$, and hence the conservation law corresponds to it. The geometry of the Euler–Lagrange Liouville-type equations $E_L \simeq \{\kappa^{-1} E_u(L) = 0\}$ is such that the conservation law is represented by an integral, $D_u(H) \equiv 0$ on $E_L$. By assumption, the integrals $w$ that specify the symmetries $\varphi'_L = \square (\bar{\phi}'[w])$ and $\varphi''_L = \square (\bar{\phi}''[w])$ are minimal, meaning that any integral is a differential function of them, hence $H = H(x,[w])$. Let the gauge of the minimal integrals be fixed. Then the factorization (106) for the new symmetry $\varphi_L$ follows from Proposition 23. Under differential reparametrizations $w = w[\hat{w}]$ of the integrals, the sections $\phi = \delta H/\delta w$ are transformed by $\phi = (\ell^w_w)\circ (\bar{\phi})$, hence $\square$ is a well defined Frobenius operator of second kind (65). \hfill $\Box$
Remark 19. Frobenius operators \( \square \) generate symmetries of Euler–Lagrange Liouville-type systems using arbitrary \( r \)-tuples \( \tilde{\phi} \) of integrals. Therefore let us consider columns of these \((m \times r)\)-matrix operators separately:

\[
\square = \left( (\square_1), \ldots, (\square_r) \right). \tag{108}
\]

Generally, the image of a \( k \)-th column \( \square_k \) is not closed under the commutation. This is the case when the \( k \)-th components \( \phi^k \) of sections \( \tilde{\phi} \in \tilde{\mathfrak{f}} \) are coupled not only in the \( k \)-th component of the bilinear bracket \( \{ , \} \square \). Likewise, the commutators of symmetries in the images of any two columns \( \square_i, \square_j \) are decomposed with respect to the images of other operators \( \square_1, \ldots, \square_r \) as well.

However, the decomposition (108) does not produce operator algebras \( \mathcal{A} \) of rank \( r \). Indeed, under a reparametrization \( \tilde{w} = \tilde{w}[w] \) in the domain of a Frobenius operator \( \square \) of second kind, which yields \( \square \circ (\xi(w)')^* \tilde{\phi} = \square(\tilde{\phi}) \), the columns \( \square^k \) are not transformed individually. Therefore they are not an \( r \)-tuple of objects, but they do constitute a single well-defined operator (108). By construction, it generates the algebra \( \mathcal{A} \) of rank 1.

Example 25. Consider the Euler–Lagrange 2D Toda system associated with the simple Lie algebra \( \mathfrak{sl}_3(\mathbb{C}) \), see \([42, 56]\),

\[
\mathcal{E}_{\text{Toda}} = \left\{ U_{xy} = \exp(2U - V), \ V_{xy} = \exp(-U + 2V), \quad K = \left( \begin{smallmatrix} 3 & -1 \\ 1 & 2 \end{smallmatrix} \right) \right\}. \tag{109}
\]

The minimal integrals for (109) are \([43, 56]\)

\[
w^1 = U_{xx} + V_{xx} - U_x^2 + U_x V_x - V_x^2, \quad w^2 = U_{xxx} - 2U_x U_{xx} + U_x V_{xx} + U_x^2 V_x - U_x V_x^2. \tag{110a, 110b}
\]

All symmetries (up to \( x \leftrightarrow y \)) of (109) are of the form \( \varphi = \square(\tilde{\phi}(x, [w^1], [w^2])) \), where \( \tilde{\phi} = t(\phi^1, \phi^2) \in \mathcal{Z}(\xi) \) and the \((2 \times 2)\)-matrix operator in total derivatives is \([22]\)

\[
\square = \left( \begin{array}{cc}
U_x + D_x & 2D_x^2 + 3U_x D_x + U_x^2 + 2U_x V_x - 2V_x^2 - U_{xx} + 2V_{xx} \\
V_x + D_x & D_x^2 - 2U_{xx} + V_{xx} + 2U_x^2 - 2U_x V_x - V_x^2
\end{array} \right). \tag{111}
\]

By Theorem 25, the image of operator (111) is closed w.r.t. the commutation. The image of the first column, of first order, is itself closed under commutation (see, e.g., [25]), but the image of the second column of \( \square = (\square_1, \square_2) \) is not. Indeed, the bracket \( \{ , \} \square \) in the domain of the entire operator \( \square \) is equal to

\[
\mathcal{E} \left( \{ \tilde{p}, q \} \square_1^1 \right) = \left( \begin{array}{c}
3p_1 q_1^1 - p_1^3 q_1^1 + 3p_2 q_1^1 - 3p_1^1 q_2^2 \\
+ 6p_2 q_2^2 - 6p_1^3 q_2^2 + 6w_1 \cdot (p_2^2 q_x^2 - p_2^2 q_y^2)
\end{array} \right), \tag{112}
\]

Here we box the individual bracket \( \{ , \} \square_1 \) for the \((2 \times 1)\)-matrix operator \( \square_1 \), and we underline the couplings of components in the domain of \( \square_2 \); under commutation,

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\[\text{For convenience, we have multiplied the second column of operator (111) by } -3, \text{ which of course does not preclude it from assigning symmetries of (109) to arbitrary arguments } \tilde{\phi}^2. \text{ However, this correction, which is introduced by hand, is the only reason why the operator } (A_{ij}) \text{ in Example 25 on p. 56 is not Hamiltonian; see [25, 31] for the bi-Hamiltonian pair of the Boussinesq hierarchy associated with the semi-simple Lie algebra } \mathfrak{sl}_3(\mathbb{C}).\]
they hit both images of $\square_1$ and $\square_2$. The remaining terms specify the bi-differential structural constants, which we denote by $\Gamma_{ij}^k$ with an obvious meaning. We have

$$\Gamma_{12}^1 = -3D_x^2 \otimes 1, \quad \Gamma_{21}^1 = 3 \cdot 1 \otimes D_x^2, \quad \Gamma_{12}^2 = -1 \otimes D_x + 2D_x \otimes 1, \quad \Gamma_{21}^2 = D_x \otimes 1 - 2 \cdot 1 \otimes D_x.$$  

In Proposition 27 we explain why the coefficient $6w_1$ in $\Gamma_{12}^1$ is expressed by a function of integrals for the Liouville-type system (109).

Examples of Frobenius operators associated with the Dynkin diagrams via the 2D Toda chains (12) are discussed in [31].

The integrals $w[m]$ of Euler–Lagrange Liouville-type systems $E_L$ determine the Miura substitutions from commutative mKdV-type Hamiltonian hierarchies $\mathfrak{B}$ of Noether symmetries on $E_L$ to completely integrable KdV-type hierarchies $\mathfrak{A}$ of higher symmetries of the multi-component wave equations $E_\sigma = \{ s_{xy} = 0 \}$, see below. The potential modified KdV equation (10), which is transformed to the KdV equation (11) by (7), gives a natural example. This is discussed in detail in [25].

The hierarchies share the Hamiltonians $H_i[m] = H[w[m]]$ through the Miura substitution. The Hamiltonian structures for the Magri schemes of $\mathfrak{A}$ and $\mathfrak{B}$ are correlated by the diagram

$$\begin{align*}
\mathcal{H}_0 & \quad \mathcal{H}_1 & \quad \mathcal{H}_2 \\
\downarrow E_x & \quad \downarrow E_x & \quad \downarrow E_x \\
\Phi_0 \quad a_1 & \quad \Phi_1 & \quad \Phi_2 \\
\phi_0 & \quad \phi_1 & \quad \phi_2 \\
\square & \quad \square & \quad \square \\
\varphi_1 & \quad \varphi_2 & \quad \varphi_3 & \quad \ldots
\end{align*}$$

(hierarchy $\mathfrak{A}$)

$$\begin{align*}
\mathcal{H}_0 & \quad \mathcal{H}_1 & \quad \mathcal{H}_2 \\
\downarrow E_x & \quad \downarrow E_x & \quad \downarrow E_x \\
\varphi_0 & \quad \varphi_1 & \quad \varphi_2 \\
\psi_0 & \quad \psi_1 & \quad \psi_2 \\
\square & \quad \square & \quad \square \\
\psi_0 & \quad \psi_1 & \quad \psi_2 & \quad \ldots (hierarchy \mathfrak{B}).
\end{align*}$$

Frobenius operators $\square$ map cosymmetries $\phi_i$ for the hierarchy $\mathfrak{A}$ to symmetries $\varphi_{i+1}$ that belong to the modified hierarchy $\mathfrak{B}$. The first Hamiltonian structure $\hat{B}_1 = (\ell_m^{(u)})^*$ for $\mathfrak{B}$ originates from the differential constraint $m = \partial L/\partial u_y$ upon the coordinates $u$ and the momenta $m$ for $E_L$.

**Lemma 26.** Introduce the linear differential operator

$$\hat{A}_k = \square^* \circ \hat{B}_1 \circ \square, \quad (114)$$

which is completely determined by the Euler–Lagrange Liouville-type system $E_L$. The operator (114) is Hamiltonian and determines a higher (that is, $k = k(\square, m) \geq 2$) Poisson structure for the KdV-type hierarchy $\mathfrak{A}$.

**Proof.** By construction, the bracket (37) given by $\hat{A}_k$ satisfies the equality

$$\{ \mathcal{H}_1[w], \mathcal{H}_2[w] \}_{\hat{A}_k} = \{ \mathcal{H}_1[w[m]], \mathcal{H}_2[w[m]] \}_{\hat{B}_1}.$$  

Therefore the left-hand side of the above equality is bi-linear, skew-symmetric, and satisfies the Jacobi identity. Fourth, it measures the velocity of the integrals $w$ along a Noether symmetry of $E_L$. Since evolutionary derivations are permutable with the
total derivative \( D_y \), the velocity \( \{ H_1, H_2 \}_{A_k} \) lies in \( \ker D_y \big|_{E_L} \), and hence its density is a differential function of the minimal integrals \( w \). \( \square \)

**Proposition 27.** The bracket \( \{ \{ , \} \} \square \) in the inverse image of operator (105) is equal to the bracket \( \{ \{ , \} \}_{A_k} \), which is induced by the Hamiltonian operator \( A_k \) and which is calculated by formula (46),

\[
\{ \{ \phi', \phi'' \} \} \square = \{ \{ \phi', \phi'' \} \}_{A_k} = \left( \ell^{(w)}_{\phi', A_k} \right)^* \phi'' , \quad \phi', \phi'' \in \cosym A \subset \sym \mathcal{E}_\sigma . \tag{115}
\]

The coefficients of the Hamiltonian operator \( A_k \) and of the bilinear terms in the bracket \( \{ , \} \square \) are differential functions of the integrals \( w \).

The multi-component wave equation \( \mathcal{E}_\sigma = \{ s_{xy} = 0 \} \), whose symmetries contain the hierarchy \( \mathfrak{A} \) and such that \( A_1 = (\ell^{(s)})^* \) potentiates the image of the Miura substitution, is not a priori unique. Again, the constraint between the coordinates \( s \) and the momenta \( w \) for \( \mathfrak{A} \) determines the first Hamiltonian operator \( A_1 \) for \( \mathfrak{A} \), but the constraint appears apparently from nowhere, the shift of the field or the frozen point argument [2] are customary procedures here. Our paradoxal conclusion is that the first structure \( A_1 = A_1^{-1} \) for \( \mathfrak{A} \) is chosen such that \( A_1 \) factors the higher Hamiltonian structure for \( \mathfrak{B} \). We thus have

\[
B_{k'} = \square \circ A_1 \circ \square^* , \quad k' = k' \left( \square , (\ell^{(s)})^* \right) \geq 2 ,
\]

which specifies the required nonlocalities.

This means that Frobenius operators and the factorizations they provide are helpful in the bi-Hamiltonianity tests for integrable systems. It is likely that one can reveal a similar origin of the nonlocal first Hamiltonian structures for the Drinfeld–Sokolov hierarchies [10] associated with the Kac–Moody algebras, whose Cartan matrices are degenerate.

Reciprocally, formula (115) calculates the bi-differential brackets \( \{ \{ , \} \} \square \) induced by Frobenius operators (105). Setting to zero all but one components of sections \( \phi' \cdot 1_i , \phi'' \cdot 1_j \in \mathfrak{f} = \mathfrak{z}(\xi) \) in domains of operators \( \square \) (respectively, except \( i \)-th and \( j \)-th components, \( 1 \leq i, j \leq r \)), we obtain the coefficients in the bracket \( \{ \{ , \} \} \square \). Proposition 27 shows that the functional class for these coefficients is very narrow.

**Example 26** (The modified Kaup–Boussinesq equation). Consider an Euler–Lagrange extension of the scalar Liouville equation [26],

\[
A_{xy} = -\frac{1}{8} A \exp \left( -\frac{1}{4} B \right) , \quad B_{xy} = \frac{1}{2} \exp \left( -\frac{1}{4} B \right) . \tag{116}
\]

Denote the momenta by

\[
a = \frac{1}{2} B_x \quad \text{and} \quad b = \frac{1}{2} A_x . \tag{117}
\]

The minimal integrals of system (116) are

\[
w_1 = \frac{1}{4} a^2 - a_x , \quad w_2 = ab + 2b_x ,
\]

such that \( D_y (w_i) = 0 \) on (116), \( i = 1, 2 \). Hence the operator

\[
\square = \left( \ell_{w_1,w_2}^{(a,b)} \right)^* = \begin{pmatrix} \frac{1}{4} B_x & D_x & \frac{1}{2} A_x \\ 0 & \frac{1}{2} B_x - 2D_x \end{pmatrix} . \tag{118}
\]

\[23\]This difficulty of the theory was pointed out to us by B. A. Dubrovin (private communication).
factors (Noether) symmetries of (116). The bracket $\{\ , \\}$\textsuperscript{□} induced in the inverse image of the Frobenius operator □ is

$$\{\{\tilde{\psi}, \tilde{\chi}\}\}^{\square} = \frac{1}{2} \left( \psi_x^2 \chi^1 - \psi^1 \chi_x^2 + \psi^1 \chi_x^1 - \psi^2 \chi_x^1 \right),$$

where $\tilde{\psi} = \phi(\psi^1, \psi^2)$ and $\tilde{\chi} = \phi(\chi^1, \chi^2)$. Split the operator $\square = (\square_1, \square_2)$ to two columns. The bracket (119) determines the structural constants $\Gamma_{ij}^k$ for the commutation of images of $\square_1, \square_2$. For example, setting the components $\psi^2, \chi^2$ and $\psi^1, \chi^1$ in (119) to zero, respectively, we isolate the brackets

$$\{\ , \\}_{\square_1} = 0, \quad \{\ , \\}_{\square_2} = \frac{1}{2}(D_x \otimes 1 - 1 \otimes D_x)$$

for $\square_1$ and $\square_2$. Thus we have obtained an extension of the Wronskian-based bracket for the second Hamiltonian structure of KdV, see Remark 10. At the same time, bracket (119) contains all other structural constants for the rank one algebra generated by $\square = (\square_1, \square_2)$, e.g., $\Gamma_{12} = \frac{1}{2}(D_x \otimes 1 - 1 \otimes D_x)$, and $\Gamma_{12}^2 = 0$, whence $\Gamma_{21}^1$ and $\Gamma_{21}^2$ are deduced by the skew-symmetry.

Consider a symmetry of (116),

$$A_t = \frac{1}{2} A_x A_{xx} + \frac{1}{2} \left( \frac{4}{3} A_x^2 - 1 \right) B_x, \quad B_t = -2 A_{xxx} + \frac{1}{2} A_x B_x^2 - \frac{1}{4} A_x B_{xx}.$$  

Using the constraint (117) between the coordinates $A, B$ and the momenta $a, b$, we cast (120) to the canonical form

$$A_t = \frac{\delta H}{\delta a}, \quad a_t = -\frac{\delta H}{\delta A}, \quad B_t = \frac{\delta H}{\delta b}, \quad b_t = -\frac{\delta H}{\delta B},$$

where $H = \left[ \left( \frac{4}{3} A_x^2 B_x^2 + \frac{1}{4} A_x A_{xx} B_x + \frac{1}{4} A_x^2 - \frac{1}{8} B_x^2 \right) \mathrm{d}x \right]$.

Let us choose an equivalent pair of integrals $u = u_2, v = v_1 + \frac{1}{2} w_2$. It is remarkable that the evolution of $u$ and $v$ along (120), which equals (see p. 26)

$$u_t = uu_x + v_x, \quad v_t = (uv)_x + u_{xxx},$$

is the Kaup–Boussinesq system, and (120) is actually the potential twice-modified Kaup–Boussinesq equation. The right hand side of the integrable system (120) belongs to the image of the adjoint linearization $\tilde{\square} = (\tilde{\ell}_{(a,b)}^A)$. The Frobenius operator $\square$ factors the third Hamiltonian structure $\hat{A}_{3,\text{KB}} = \tilde{\square}^* \circ (\tilde{\ell}_{(a,b)}^A)^* \circ \tilde{\square}$ for (118); we have $k = 3$ and

$$\hat{A}_{3,\text{KB}} = \left( \begin{array}{c c}
\frac{1}{2} D_x + \frac{1}{2} u_x & D_x^3 + (\frac{1}{4} u + v) D_x + \frac{1}{4} (u^2 + 2v)_x \\
\frac{1}{2} u D_x + \frac{1}{8} u_x & \frac{1}{2} (2u D_x^2 + 3u_x D_x^2) + (3u_{xx} + 2uv)D_x + u_{xxx} + (uv)_x
\end{array} \right).$$

By Proposition 27, the bracket $\{\ , \\}$\textsuperscript{□} is equal to $\{\ , \\}_{\hat{A}_{3,\text{KB}}}$, which is given by formula (116). We obtain

$$\{\tilde{\psi}, \tilde{\chi}\}^{\square} = \{\tilde{\psi}, \tilde{\chi}\}_{\hat{A}_{3,\text{KB}}} = \left( \begin{array}{c}
\tilde{\psi} \cdot \nabla_1(\tilde{\chi}) - \nabla_1(\tilde{\psi}) \cdot \tilde{\chi} \\
\tilde{\psi} \cdot \nabla_2(\tilde{\chi}) - \nabla_2(\tilde{\psi}) \cdot \tilde{\chi}
\end{array} \right),$$

where $\nabla_1 = -\frac{1}{2} \left( \begin{array}{c}
D_x \\
u D_x + D_x^2\end{array} \right)$ and $\nabla_2 = -\frac{1}{2} \left( \begin{array}{c}
0 \\
D_x \end{array} \right)$. Here we use an alternative notation (122) for the components of the bracket $\{\ , \\}$\textsuperscript{□} that acts by the differential operators $\nabla_1$ and $\nabla_2$ on one factor in each coupling.
Finally, the operator $\hat{A}_1 = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}$ is the first Hamiltonian structure for (58); its inverse $A_1 = \hat{A}_1^{-1}$ factors the second Hamiltonian structure $B_2 = \square \circ A_1 \circ \square$ for (120).

We recall that the operators (105) determine recursions for the symmetry algebras sym $E_{\text{EL}}$ of Liouville-type systems.

**Proposition 28** (see [28, 29]). Let $E_{\text{EL}}$ be an Euler–Lagrange Liouville-type system that meets the assumptions of Proposition [23] $w[m]$ be the integrals, and $\square$ be Frobenius operators (105). Let the $r$-tuples $\vec{\phi} = (\phi^1, \ldots, \phi^r) \in \mathfrak{f} = \hat{\mathfrak{e}}(\xi)|_w$ be the same as in Corollary 24 see (107) on p. 50. Then the $m$-tuples

$$\omega_{\vec{\phi}} = \square \left( d\bar{\phi}(x, [w[m]]) \right)$$

of Cartan’s 1-forms are generating sections of local recursion operators $R_{\vec{\phi}}: \varphi \mapsto E_{\varphi, \mathfrak{f}} \omega_{\vec{\phi}}$ for $\varphi \in \text{sym} E_{\text{EL}}$.

**Conjecture 29.** Under the above assumptions and notation, the recursion operators $R_{\vec{\phi}}$ constitute infinite-dimensional Frobenius–Lie algebras by

$$[R_{\vec{\phi}}(\varphi'), R_{\vec{\phi}}(\varphi'')] = \int_{\bar{\mathfrak{f}}} R_{\vec{\phi}} \left( \mathcal{E}_{R_{\vec{\phi}}(\varphi')}(\varphi'') \delta(\bar{\phi} - \bar{\phi}') - \mathcal{E}_{R_{\vec{\phi}}(\varphi'')}(\varphi') \delta(\bar{\phi} - \bar{\phi}) + \Gamma_{\vec{\phi}, \varphi', \varphi''}(\varphi', \varphi'') \right) d\bar{\phi},$$

where $\bar{\phi}, \bar{\phi}' \in \mathfrak{f} = \hat{\mathfrak{e}}(\xi)|_w$ and $\varphi', \varphi'' \in \text{sym} E_{\text{EL}}$ are any symmetries of the Euler–Lagrangian Liouville-type systems $E_{\text{EL}}$.

5.2. **Non-Lagrangian Liouville-type systems.** Let $E_{\text{L}} = \{ F = 0 \}$ be a Liouville-type system; now it may not be Euler–Lagrange. Let $w \in \ker D_y|_{E_{\text{L}}}$ be a section composed by minimal integrals for $E_{\text{L}}$, thence $D_y(w) = \nabla(F)$, see (31) on p. 13. By construction of Liouville-type systems, hyperbolic equations in $E_{\text{L}}$ are independent from each other. Therefore $E_{\text{L}}$ is both normal and $\ell$-normal [5, 40], meaning that $\Delta(F) = 0$ or $\Delta \circ \ell_F = 0$ on $E_{\text{L}}$ implies $\Delta = 0$, respectively. By this argument, a section $\varphi \in \mathcal{X}(\pi)$ is a symmetry of a Liouville-type system $E_{\text{L}}$ if and only if the evolutionary vector field $\mathcal{E}_{\varphi}$ preserves the integrals,

$$D_y(\mathcal{E}_{\varphi}(w)) = \mathcal{E}_{\varphi}(\nabla(F) + \nabla(\mathcal{E}_{\varphi}(F))) = \nabla(\ell_F(\varphi)) \text{ on } E_{\text{L}}.$$

Consider the operator equation

$$D_y \circ \ell_w = \nabla \circ \ell_F \text{ on } E_{\text{L}}.$$

If, hypothetically, a differential operator $\square: \mathfrak{f} \rightarrow \mathcal{X}(\pi)$ such that $\ell_w \circ \square: \ker D_y|_{E_{\text{L}}} \rightarrow \ker D_y|_{E_{\text{L}}}$ were constructed, which thus resembles the right inverse of $\ell_w$, then it would assign symmetries of a Liouville-type system $E_{\text{L}}$ to arbitrary $r$-tuples of the integrals, see (100).

In a recent publication [57], the following fundamental result has been obtained:

*Suppose $E_{\text{L}} = \{ F = 0 \}$ is an $m$-component Liouville-type system that admits complete sets $w \in \ker D_y|_{E_{\text{L}}}$ and $\bar{w} \in \ker D_x|_{E_{\text{L}}}$ of $m$ integrals which are independent on a dense open subset of $E_{\text{L}}$. Let an integer*
Let \( k \gg 1 \) be sufficiently large, obeying an estimate in Lemma 4 of loc. cit. Consider the operator equation for \( \square: f \to \varphi(\pi) \),

\[
\ell_w \circ \square = 1_{m \times m} \cdot D^k_x \mod \text{CDiff}_{<k} \left( \ker D_y |_{\mathcal{E}_L} \to \ker D_y |_{\mathcal{E}_L} \right),
\]

which is not fixed a priori. Consider the approximation

\[
\square \approx \sum_{i=0}^{k-\text{ord}_x(w)-1} \alpha_i \cdot D^{k-\text{ord}_x(w)-i}_x + O(1)
\]

of a formal inverse to \( \ell_w^{(u)} \), and modify its columns following the constructive procedure of Lemma 4 in loc. cit. This yields the linear differential operator \( \square \) in total derivatives that solves equation (123) exactly.

The profound result quoted above establishes the truncation from below for the sequence of coefficients of \( \square \) that satisfy some recurrence relations; most remarkably, this is a consequence of the presence of a complete set \( \bar{w} \in \ker D_x |_{\mathcal{E}_L} \) of \( y \)-integrals for \( \mathcal{E}_L \).

Whenever a local operator in total derivatives \( \Box = (\Box^1, \ldots, \Box^r) \) is known, the differential order of its \( i \)-th column \( \Box^i \) can be decreased if the corresponding integral \( \bar{w}^i \) is not minimal. Indeed, the reduction \( \bar{w}^i = D_x(w^i) \) and the relation \( \ell_{\bar{w}^i} = D_x \circ \ell_{w^i} \) means that the order \( k \) of the right-hand side of the equation \( D_x \circ \ell_{w^i} \circ \Box^i = D^k_x \mod \text{CDiff} \left( \ker D_y |_{\mathcal{E}_L} \to \ker D_y |_{\mathcal{E}_L} \right) \) has been increased artificially.

Therefore the most general form of the determining equation (123) is

\[
\ell_w \circ \Box \in \text{CDiff} \left( \ker D_y |_{\mathcal{E}_L} \to \ker D_y |_{\mathcal{E}_L} \right),
\]

with an \textit{a priori} unfixed operator \( \sum_{i=0}^{k-1} \Omega_i \cdot [w^i] \cdot D^{k-i}_x \) in the right-hand side, here \( \Omega_i \in \ker D_y |_{\mathcal{E}_L} \) are \( (r \times r) \)-matrices. Likewise, the operator \( \Box \) itself is defined up to right multiplication by \( \text{CDiff} \)-operators from the same ring. As soon as the minimal integrals \( w \) are differentiated a suitable number of times such that their differential orders coincide in the end, and under assumption that the determinant \( \det \Omega_i \) does not vanish almost everywhere on \( \mathcal{E}_L \), the topmost coefficient in the r.h.s. of (123) can be set equal to unit, as in (123). Indeed, this is achieved by a solution \( \Box := \Box \circ \Omega^{-1}_k \) of (123). The coefficients \( (\Omega_i)_{\mu \nu} \) of the product \( \ell_{\bar{w}} \circ \Box \) are generally not equal to the Kronecker symbols \( \delta_{\mu \nu} \cdot \delta^k_x \). Thus the operator \( \Box \circ D^{-k}_x \) is not the right inverse of \( \ell_w \). We conclude that equation (123) describes the problem of “right-inverting” operators in a ring with nontrivial zero divisors. A constructive algorithm for solving it has been given in [57]. However, in most cases the image of the resulting operator \( \Box \) does not span the entire \( x \)-component of the Lie algebra sym \( \mathcal{E}_L \), and such operators are generally not Frobenius.

**Example 27.** Consider the Liouville equation \( \mathcal{E}_{\text{Liou}} = \{ u_{xy} = \exp(2u) \} \), see (1), its minimal integral \( w = u_{xx} - u_x^2 \), and the operator \( \Box = u_x + \frac{1}{2} D_x \) that assigns symmetries to integrals. A straightforward calculation yields that

\[
\ell_{2w} \circ \Box = D^2_x + 4w D_x + 2w_x = 1 \cdot D^2_x \mod \text{CDiff} \left( \ker D_y |_{\mathcal{E}_{\text{Liou}}} \to \ker D_y |_{\mathcal{E}_{\text{Liou}}} \right).
\]
Example 28. Similarly, consider the 2D Toda system (109), its minimal integrals (110), and the operator □ = (□1, □2), which is given in (111). Then we have
\[
\ell_{\frac{1}{2}w_1} \circ □^1 = D_x^3 + w^1 D_x + \frac{1}{2} w_x^1,
\]
\[
\ell_{\frac{1}{2}w_1} \circ □^2 = D_x^4 + w^1 D_x^2 + [2 w_x^1 - 3 w_x^2] \cdot D_x + [w_{xx}^1 - 2 w_x^2],
\]
\[
\ell_{w_2} \circ □^1 = D_x^4 + w^1 D_x^2 + 3 w_x^2 D_x + w_x^2,
\]
\[
\ell_{\frac{1}{2}w_2} \circ □^2 = D_x^5 + 2 w^1 D_x^3 + 3 w_x^1 D_x + [3 (w_{xx}^1 - w_x^2) - (w^1)_x] \cdot D_x + [w_{xxx}^1 - \frac{3}{2} w_{xx}^2 + w^1 w_x^1].
\]

After the correction \(\tilde{w}_1 = w^1_x, \tilde{□}^1 = □^1 \circ D_x\) of differential order for the first integral (110a), we obtain the matrix \(\Omega_5 = \left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right)\) with unit determinant at the top of the operator in the right-hand side of (123′). This yields a solution \(\tilde{□} \circ \Omega_5^{-1}\) of equation (123).

We refer to footnote 22 on p. 51 and to \([31]\) for further comments on this example, which is related to the Boussinesq hierarchy. Actually, the operator \(\ell_{w} \circ □\) is Hamiltonian whenever the second column, □2 in □, is taken in its original scaling (here it has been divided by \(-\frac{1}{3}\)).

Lemma 30. If a solution □ of (123′) for a Liouville-type system \(E_L\) is Frobenius, then all coefficients of the bracket \(\{\phi', \phi''\} □\) belong to \(\ker D_y|_{E_L}\).

Proof. By assumption, we have that \((D_y \circ \ell^{(u)}_{w} \circ □) (\{\phi', \phi''\} □) \equiv 0\) for \(\phi', \phi''(x, [w]) \in \mathfrak{f}\). This equals
\[
0 \equiv (D_y \circ \ell^{(u)}_{w} \circ □) \left( E □(\phi') (\phi'') - E □(\phi'') (\phi') + \{\phi', \phi''\} □ \right) \equiv (\ell^{(u)}_{w} \circ □) (D_y (\{\phi', \phi''\} □)),
\]
because the underlined composition satisfies (123′). Clearly, \(D_y(\phi')\) and \(D_y(\phi'')\) vanish on \(E_L\) for arbitrary \(\phi', \phi''\) ∈ \(\mathfrak{f}\). For the same reason, not only the whole bracket \(\{\phi', \phi''\} □\), but each particular coefficient standing at the bilinear terms lies in \(\ker D_y|_{E_L}\).
Conclusion

We introduced the coordinate-independent definition of Frobenius linear differential operators \( A : \Omega^1(\xi_\pi) \to g(\pi) \) in total derivatives, whose images constitute Lie subalgebras in the algebra \( g(\pi) \) of evolutionary vector fields. We demonstrated that, in presence of Miura-type substitutions \( w : J^\infty(\pi) \to \Gamma(\xi) \), there are two main classes of such operators. They generalize recursion and Hamiltonian operators, whenever sections of fibre bundles in their domains \( \Omega^1(\xi_\pi) \) obey vector and covector transformation laws under a change of coordinates, respectively. We defined the notions of linear and strong compatible Frobenius operators and gave examples of compatible Frobenius structures of each kind, indicating both the matrix operators \( A \) and the types of transformations \( \psi \mapsto \tilde{\psi} \) for pre-images \( \psi \) of vector fields \( E_A(\psi) \).

We solved a long-standing problem in geometry of integrable systems: We gave a complete description of higher symmetry algebras for hyperbolic Euler–Lagrange equations \( E_L \) of Liouville-type (in particular, for all 2D Toda chains associated with semi-simple complex Lie algebras of rank \( r \)). To this end, we obtained an explicit formula \( \Box \) for Frobenius differential operators \( \Box \) of second kind that assign (Noether) symmetries \( \varphi = \Box(\tilde{\varphi}) \) to arbitrary \( r \)-tuples \( \tilde{\varphi}(x,[w]) \) (respectively, to \( \tilde{\varphi} = \delta H / \delta w \) for any \( H(x,[w]) \)). Second, we calculated the commutation relations in symmetric \( E_L \), and we proved that the arising coefficients of the brackets \( \{ \cdot, \cdot \}_\Box \) are differential functions of the substitutions \( w \).

We suggested a new algebraic operation on the linear spaces \( A = \bigoplus_{i=1}^r \mathbb{R} \cdot A_i \) of operators \( A_i : \Omega^1(\xi_\pi) \to g(\pi) \) whenever the spaces \( \sum_i \text{im} A_i \) are closed under commutation. In particular, we considered the spaces of both linear and strong compatible Hamiltonian operators for the Magri schemes. Independently, we associated another class of such spaces to Liouville-type systems. Using the Lie algebra structure of vector fields \( E_\varphi \) with \( \varphi \in \text{im} A_i \), we defined a Lie-type bracket \( [A_i, A_j] = \sum_{k=1}^r A_k \circ c_{ij}^k \) of operators \( A_i, A_j \in A \). This operation on linear spaces of Frobenius operators extends the associative composition and the Schouten bracket of recursions \( g(\pi) \to g(\pi) \).

We have analysed the algebraic and geometric properties of Frobenius structures. We described flat connections in the triples \( (\Omega^1(\xi_\pi), g(\pi), A) \) such that bi-differential Christoffel symbols are encoded by the structural constants \( c_{ij}^k \) of the operator algebras \( A \). We derived a condition when Frobenius recursions, which are solutions of the classical Yang–Baxter equation \( (2) \) for the Lie algebra \( g(\pi) \), generate chains of non-trivial deformations for the standard structure on \( g(\pi) \). By our Counterexample \( \Box \) the concept of Lie algebroids over the infinite jet spaces does not repeat the case of finite-dimensional base manifolds, when the anchors are determined by the Lie algebra homomorphisms \( A \). Therefore we assigned formal differential complexes over Lie algebras \( \Omega^1(\xi_\pi) \) to Frobenius operators \( A \), such that the representations of the differentials in the complexes through homological vector fields restores one of equivalent constructions of Lie algebroids.

Our approach not only incorporates the standard Hamiltonian formalism for PDE, but also generates new nonlinear models. A rigorous parallel between the bi-differential and classical affine geometry allows to interpret known equations of string theory as the master equations in the setting over the jet spaces. The suggested algebraic technique
can be especially efficient whenever off-shell considerations are in order, without any restriction onto differential equations. With all this, we intend to contribute to a profound relation between integrable systems and Lie algebras, which is well acknowledged in mathematical and theoretical physics [1, 9, 10, 20, 49, 52].

Discussion. Open problems concerning with Frobenius structures can be assembled in two categories: questions about geometry of involutive distributions generated by evolutionary fields in the images of Frobenius operators, and application of the formalism to integrable systems.

Geometric aspects. Answering a motivating question of M. Kontsevich, who asked us whether Frobenius operators are the anchors in Lie algebroids over infinite jet spaces, we were stuck at Counterexample 4 for $A_{KdV}$, We proposed a way out, which is based on the formal complex (72) and the representation of its differential via the homological vector fields (in the spirit of the Koszul–Tate resolution for Lagrangian systems). At the same time, the difficulty means that a correct variational analogue of the Leibnitz rule (50) has not been recovered yet. Variational analogues on $J^\infty(\pi)$ are known from [22, 40] for vector fields, forms, commutators and Schouten brackets, for symplectic structures, etc. (see Table 1 on p. 61) but, to the best of our knowledge, the bare Leibnitz rule itself has never been under debate. Hence a profound analysis of Koszul–Schouten structures on $\Omega^k(M^n)$, which was performed in [35], remains without an analogue on

$$\Omega^k(\xi_\pi) \overset{\text{def}}{=} \bigotimes^k \Omega^1(\xi_\pi),$$

leaving our ad hoc construction of Lie algebroids over $J^\infty(\pi)$ so implicit.

We have already indicated that standard classification problems arise for the algebras $A$ of Frobenius operators with the brackets $[A_i, A_j] = \sum_k A_k \circ c_{ij}^k$. For Hamiltonian operators $P$, we know two ways, (46) and (55b), of calculating Dorfman’s brackets $[,]_P$ on $\Omega^1(\pi)$. Is there a similar construction that yields the bi-differential Christoffel symbols $\Gamma^k_{ij}: \Omega^1(\pi) \times \Omega^1(\pi) \to \Omega^1(\pi)$ for totally compatible Hamiltonian structures? More globally, is there any homological obstruction for the existence of compatible Hamiltonian or recursion operators that constitute non-commutative algebras? We expect to use here some techniques which will be more refine than the ones used in the proof of Lemma 12. We notice that the assertion of [23] was a reformulation of a much stronger result in [15] that states non-existence of nontrivial variational $k$-vectors and $k$-forms at all $k \geq 3$ under given assumptions. However, this is not our case of the Christoffel symbols $\Gamma^k_{ij}$, which are not skew-adjoint in each argument and which are not tensors, obeying the transformation rules for connection 1-forms. Let us indicate that, in these terms, the approach of geodesic motion is reproduced for the problems of optimization and control in the theory of partial differential equations.

The use of coordinate-independent constructions of geometry of integrable systems [5, 11, 22, 40], see section 2, allowed us to introduce the well-defined notion of Frobenius operators. This would have been impossible within a ‘simple’ approach of local coordinates. We emphasize that now the definition itself manifests the two main classes of the operators, whose domains are the analogues of tangent and cotangent bundles to smooth manifolds. The general case of Frobenius operators and their Lie-type algebras will be the object of another paper.
Field-theoretic viewpoint. The definition of Frobenius operators does not depend on the number \( n \) of independent variables. In this paper, we passed from examples of Frobenius operators for 1D evolutionary hierarchies (the KdV-type equations) to factorizations of symmetry generators for 2D Toda chains, which interpolate between KP and mKP (see [20]). We expect that Frobenius operators can serve a key to construction of integrable systems in multidimension. By this argument, we arrive at the problem of finding Frobenius structures for the flows of the KP hierarchy.

Another recent construction may have applications in this theory, as soon as examples of Frobenius operators are known (e.g., for \((1+n)\)-dimensional hydrodynamic chains). Namely, consider Frobenius complexes \([72]\) with the differentials \( d \), and study their strong homotopy deformations, which involve Schlessinger–Stasheff’s algebras with Lie-type brackets of many arguments (see \([1, 3, 27]\) and references therein). We know that in our situation, which is based on evolutionary fields on the jet spaces, the Wronskian determinants play a central role. Hence their homology-preserving generalizations for functions of many variables, which were defined in \([27]\), will be useful.

Returning one step back from the homotopy Lie algebras to homological vector fields, we recall their immanent presence in the Batalin–Vilkovisky formalism, see \([1]\). A description of its ingredients over the infinite jet spaces was our second motivation.

In this jet setting, Frobenius recursion operators are solutions of the classical Yang–Baxter equation \([2]\) for the Lie algebra \( g(\pi) \) of evolutionary vector fields. It is still not clear how far one can pattern upon the standard approach to the Yang–Baxter equations, what systems \( E \subset J^\infty(\pi) \) can be obtained, are there any reductions for them, and what is the physical significance of the models whose state functions are transformed by derivations that belong to \( g(\pi) \).

Finally, we note once again that we have always remained in the non-graded setting and that the differential operators were local. We are advised to apply the formalism of Frobenius structures towards the study of noncommutative differential equations, see Remark \([10]\) on p. 41, and, first, reveal these structures for integrable evolution equations on associative algebras \([50]\). In addition, the use of difference Frobenius operators can be a fruitful intermediate idea for discretization of integrable systems with free functional parameters in their symmetries (e.g., Liouville-type difference systems). The restriction \([11]\) produces narrow classes of difference operators such that the discrete systems are integral objects for the symmetry algebras.

We have described a generator of operator algebras \( \mathcal{A} \) with the commutation relations \([1]\). In section 5 we managed to calculate the bi-differential structural constants \( c^k_{ij} \) by using the fact that the integrals \( w \) for the Liouville-type systems \( \mathcal{E}_L \) induce Miura’s transformations \( w = w[u] : \mathcal{B} \rightarrow \mathcal{A} \) between bi-Hamiltonian hierarchies \( \mathfrak{A} \) and \( \mathfrak{B} \subset \text{sym} \mathcal{E}_L \). By \([44]\), many substitutions are obtained in this way. Similarly, the class of structures \([11]\) given by Dynkin diagrams via the 2D Toda chains may be standard in the following sense: Is it true that for all spaces \( \mathcal{A} \) of Frobenius operators \( A_i : \Omega^1(\xi) \rightarrow g(\pi) \) of second kind there are Hamiltonian operators \( P : \mathfrak{X}(\xi)|_w \rightarrow \mathfrak{X}(\xi)|_w \) such that \( \{\{, \}\}_A = \{\{, \}\}_P \)?

Final remark (M. Kontsevich, private communication). Algebras with bi-differential structural constants appear naturally in the BRST-formalism. Hence the algebras \( \mathcal{A} \) of Frobenius operators may have applications in the string theory.
Appendix A. Analogy between Hamiltonian ODE and PDE

In Table 1 we track the geometric correspondence between Hamiltonian ODEs and PDEs; we adapt the analogy to our needs, and therefore it is forced to remain incomplete. The distinction between the coordinates and momenta in the PDE framework, which is implemented in section 5 to (symmetries of) Euler–Lagrange systems, is addressed in [34, 25]. The concept of ∆-coverings over PDE, which is convenient in practical calculations that arise here, is developed in [22], see Remark 20 on p. 62.

Table 1. Hamiltonian ODE and PDE.

| Hamiltonian ODE | PDE |
|-----------------|-----|
| Smooth base manifold $M^n$ and a fibre bundle $\pi: E^{n+m} \to M^n$. | Infinite jet space $J^\infty(\pi) \to M^n$. |
| Sections $\varphi = (\varphi^1[u], \ldots, \varphi^n[u]) \in \mathcal{X}(\pi) = \Gamma(\pi_n^\ast(\pi))$ of the induced fibre bundle. | Evolutionary vector fields $\mathcal{E}\varphi \in \mathcal{D}(J^\infty(\pi))$. |
| Solutions of the autonomous equation $u_t = \varphi$. | Lie algebra $\mathfrak{g}(\pi)$ of evolutionary derivations. |
| No internal structure of a time point $t \in \mathbb{R}$; | The de Rham differential $d = d_h + d_C$ split to horizontal and vertical parts w.r.t. $\pi$. |
| Symplectic manifold $M^{2n} \ni (p, q)$; | The variational bi-complex and the $C$-spectral sequence. |
| Components $X^i$ of vector fields $X \in \Gamma(TM^{2n})$ on $M^{2n}$; | The highest horizontal cohomology $\bar{H}^n(\pi) \ni \mathcal{H}$. |
| Vector fields $X \in \Gamma(TM^{2n})$; | The dual module $\hat{\mathcal{H}}$ of the induced fibre bundle. |
| Integral trajectories $(p(t), q(t)) \subset M^{2n}$ of the field $X$; | Euler’s operator $\mathbf{E}$ as a restriction of $d_C$ to $H^0(\pi)$, the ‘gradient’ $\mathbf{E}(\mathcal{H}) \in \mathcal{X}$. |
| Lie algebra $(TM^{2n}, [,])$ of vector fields on $M^{2n}$; | Hamiltonian operator $A \in \mathcal{C}\text{Diff}(\hat{\mathcal{J}}, \mathcal{X})$ in total derivatives; $\hat{\mathcal{J}} \simeq \mathcal{X}(\pi)$ for evolution equations. |
| The de Rham differential $d$; | The de Rham complex $d = d_h + d_C$ split to horizontal and vertical parts w.r.t. $\pi$. |
| The de Rham complex; | The variational bi-complex and the $C$-spectral sequence. |
| The space of Hamiltonians $\mathcal{H} \in C^\infty(M^{2n})$; | The highest horizontal cohomology $\bar{H}^n(\pi) \ni \mathcal{H}$. |
| The cotangent bundle $T^*M^{2n}$ and its sections $\psi: TM^{2n} \to C^\infty(M^{2n})$; | The dual module $\hat{\mathcal{H}}$ of the induced fibre bundle. |
| The differential $d\mathcal{H}$ of a Hamiltonian $\mathcal{H}$; | Euler’s operator $\mathbf{E}$ as a restriction of $d_C$ to $H^0(\pi)$, the ‘gradient’ $\mathbf{E}(\mathcal{H}) \in \mathcal{X}$. |
| Symplectic 2-form $\omega \in \Omega^2(M^{2n})$, Poisson bi-vector $\mathcal{P} \in \Gamma\Lambda^2(TM^{2n})$; | Hamiltonian operator $A \in \mathcal{C}\text{Diff}(\hat{\mathcal{J}}, \mathcal{X})$ in total derivatives; $\hat{\mathcal{J}} \simeq \mathcal{X}(\pi)$ for evolution equations. |
| Hamiltonian vector field $X_{\mathcal{H}}$ such that $X_{\mathcal{H}} \cdot \omega = d\mathcal{H}$; | Sections $\varphi = A(\psi), \psi \in \mathcal{X}(\pi)$. |
| The Poisson bracket $\{\mathcal{H}_1, \mathcal{H}_2\} = \omega(X_{\mathcal{H}_1}, X_{\mathcal{H}_2}) = X_{\mathcal{H}_1} \cdot d\mathcal{H}_2$; | The Poisson bracket $\{\mathcal{H}_1, \mathcal{H}_2\}_A = \langle \mathbf{E}(\mathcal{H}_1), A(\mathbf{E}(\mathcal{H}_2)) \rangle = \mathcal{E}_{A(\mathbf{E}(\mathcal{H}_2))}(\mathcal{H}_1)$. |

Appendix B. Reconstruction of the brackets $\{\cdot, \cdot\}_A$

In this appendix we describe an inductive procedure that assigns the bracket $\{\cdot, \cdot\}_A$ to a nondegenerate Frobenius operator $A$, see [62]. The bracket $\{\cdot, \cdot\}_A$ may be not contained in our knowledge that $A$ is Frobenius if, e.g., the operator determines the factorization of symmetries of a Liouville-type system and has minimal differential
order. This is precisely the case of operator (108), which is used in Example 30 as an illustration.

Remark 20. The algorithm we suggest is based on the use of the $\Delta$-coverings \([22]\) over the jet spaces $J^\infty(\pi)$. In our case, it amounts to ‘forgetting’ the $\mathcal{F}(\pi)$-(sub)module structure of $\Omega^1(\xi_\pi)$ and treating it as a jet (super-)bundle over $J^\infty(\pi)$, see \([22]\) for details. Hence, instead of calculating $D_x(\psi[u])$ for $\psi \in \Omega^1(\xi_\pi)$, one introduces the variable $\psi_x$ and so on\([24]\) setting the derivatives $\xi_x(\psi)$ in the Koszul bracket to zero, see \([22]\). Consequently, only the derivatives of coefficients of the operator $A$ contribute to the left-hand side of (103), which is used in Example 30 as an illustration.

Also, the nature of the assumption (59) becomes clear. Indeed, this construction tells us that there are no nontrivial kernels for restrictions of the nondegenerate operators onto the new jet bundles with sections $\psi_1, \psi_2$. Note that the same operators $A$ may have kernels which are spanned by certain sections $\psi[u] \in \Gamma(\pi^*_n(\xi))$ of the induced fibre bundles, see Example 7 on p. 26.

For brevity, let us technically assume that $A = \| \sum_k A^i_k D^k_x \|$ is a matrix operator in $D_x$, where $A^i_k \in \mathcal{F}(\pi)$. We use the notation $1_i$ for the basic sections $\psi = t(0_1, \ldots, 0_{i-1}, 1_i, 0_{i+1}, \ldots, 0_r)$ of $\Omega^1(\xi_\pi)$, whence

$$
\psi = \sum_{i=1}^r \psi^i \cdot 1_i, \quad \psi^i \in \mathcal{F}(\pi).
$$

Suppose further that the operator $A$ is nondegenerate, see p. 26

$$
\bigcap_k \ker A_k = \{0\} \quad \text{for} \quad A = \sum_k A_k \cdot D^k_x.
$$

Hence we encounter no difficulties when resolving inhomogeneous equations $A(\{\{\psi_1, \psi_2\}\}_A) = \varphi$ w.r.t. the brackets $\{\{ , \}_A \in C\text{Diff}(A^2 \Omega^1(\xi_\pi), \Omega^1(\xi_\pi))$, which are bilinear in $\psi_1, \psi_2 \in \Omega^1(\xi_\pi)$.

Let the (yet unknown) bracket be

$$
\{\{\psi_1, \psi_2\}\}_A = \sum_{i,j,k=1}^r c^{\alpha\beta}_{ijk} \cdot (\psi^1_i)_\alpha (\psi^2_\beta)_\beta \cdot 1_k,
$$

where $c^{\alpha\beta}_{ijk} \in \mathcal{F}(\pi)$ and the condition $c^{\alpha\beta}_{ijk} = -c^{\beta\alpha}_{jik}$ follows from the skew-symmetry of the bracket $\{\{ , \}_A$. The coefficients $c^{00}_{ijk}$ can be nontrivial if the dimension $r$ of the fibres of $\xi$ is $r > 1$, see Example 29 below.

The base of the algorithm is given by the Jacobi bracket of the sections $1_i, 1_j$:

$$
[A(1_i), A(1_j)] = A(c^{00}_{ijk} \cdot 1_k).
$$

The choice $1 \leq i \leq j \leq r$ yields $\tau(r - 1)/2$ compatible systems of $m$ equations. The components of sections in domains of $A$ in the right-hand side of (125) are enumerated

\[24\] Note that an additional relation $\psi_0 = 0$ can be introduced for the new variables $\psi$ that imitate the integrals. These integrals are the sections in domains of Frobenius operators (see (100) and (107)) for Liouville-type systems.
by $k$. Since $A$ is nondegenerate, the equations are solvable. Actually, these systems are overdetermined whenever the differential order of $A$ is positive and hence the left-hand sides of (125) and (126), see below, contain higher order derivatives of $u$ that are not present among the arguments of $c_{ijk}^{\alpha \beta}[u]$.

The inductive step is made by using the sections $x^\alpha \cdot 1_i$ and $x^\beta \cdot 1_j$. We obviously have

$$[A(x^\alpha \cdot 1_i), A(x^\beta \cdot 1_j)] = A(\sum_{0 \leq \alpha' + \beta' < \alpha + \beta} (x^\alpha)^{(\alpha')}(x^\beta)^{(\beta')} c_{ijk}^{\alpha' \beta'} \cdot 1_k), \quad (126)$$

whence the coefficients $c_{ijk}^{\alpha \beta}$ on the diagonal $\alpha + \beta = \text{const}$ are obtained one by one. Having passed through the diagonal $0 \leq \alpha + \beta = \text{const}$, with $\alpha \geq \beta$ or $\alpha \leq \beta$ in view of the relation $c_{ijk}^{\beta \alpha} = -c_{ijk}^{\alpha \beta}$, we check the condition

$$[A(\psi), A(\chi)] = A(\sum_{\alpha + \beta \leq \text{const}} c_{ijk}^{\alpha \beta} \cdot \psi^i \chi^j)$$

that terminates the algorithm when holds. The differential order of the bracket $\{ \ , \ \} A$ with respect to its arguments is estimated by calculating the Lie bracket $[A(\psi), A(\chi)]$ and taking into account the Leibniz rule in the right-hand side of (52a). We remark that the representation of jet coordinates $u_\sigma$ using powers $x^\alpha$ of base variables is standard in geometry of differential equations [5].

**Example 29.** The operator $A_1$ for the dispersionless 3-component Boussinesq system [50] has order zero and its matrix (57c) is nondegenerate almost everywhere, hence the condition (59) is valid. We reconstruct the bracket $\{ \ , \ \} A_1$ for this operator performing two steps of the above algorithm.

The first step involves six combinations of $\psi_1 = 1_i$ and $\psi_2 = 1_j$ with $1 \leq i \leq j \leq 3$. The second step repeats the first, but now $\psi_2$ is multiplied by $x$, and we have nine combinations $\psi_1 = 1_i$ and $\psi_2 = x \cdot 1_j$, $1 \leq i, j \leq 3$. Then the terminal check (127) is fulfilled. This proves that the components of the bracket $\{ \ , \ \} A_1$ are

$$\{p, q\}^a_{A_1} = p_x^u q^a - p_x^w q^a + 3w(p_x^u q^a - p_x^w q^a) + 3w(p_x^u q^a - p_x^w q^a)$$

$$+ p_x^w q^a - p_x^w q^a + 2w_x(p_x^a q^a - p_x^w q^a) + u(p_x^w q^a - p_x^w q^a), \quad (128a)$$

The first order operator (57c) is also nondegenerate, and we obtain

$$\{p, q\}^u_{A_2} = 3w(p_x^u q^a - p_x^w q^a) + 2w(p_x^u q^a - p_x^w q^a) + 2w_x(p_x^a q^a - p_x^a q^a)$$

$$- p_x^w q^a - p_x^w q^a + 2u(p_x^w q^a - p_x^w q^a), \quad (128b)$$

$$\{p, q\}^w_{A_2} = 4w(p_x^w q^a - p_x^w q^a) + p_x^u q^a - p_x^u q^a + 2(p_x^u q^a - p_x^w q^a),$$

$$\{p, q\}^w_{A_2} = 8w(p_x^w q^a - p_x^w q^a) + 8u(p_x^u q^a - p_x^w q^a) + 6w^2(p_x^u q^a - p_x^w q^a)$$

$$+ w(p_x^w q^a - p_x^w q^a) + 2u_x(p_x^w q^a - p_x^w q^a) + u(p_x^w q^a - p_x^w q^a).$$

$$\{p, q\}^u_{A_2} =$$
Sokolov's bracket for Hamiltonian operator (57d) is equal to the sum of brackets (128), because the operators are linear compatible. The result agrees with (30).

**Example 30.** Following the above algorithm and using the package [47], we obtain the bracket on the domain of the Frobenius operator (103): for any \( p, q \in \Omega^1(\xi_t^\prime) \), we have

\[
\{(p, q)\} = \varepsilon^2 \cdot (p_xx q_x - p_x q_xx) - 2\varepsilon^2 \cdot (p_x q_x q - p_{xxx}) \\
- 12\varepsilon^4 \cdot (\varepsilon^2 u^3 u_{xx} - 4\varepsilon^2 u^2 u_{xxx} + 12\varepsilon^2 u_x u^2 + 2u_x u_{xx} - u_{xxx}) \\
\times [1 + 8\varepsilon^2 u^2 - 2\varepsilon^2 u_x - 8\varepsilon^2 u^2 u_{xx}]^{-1} \cdot (p_{xx} q - p_{xx}) \\
+ (\mathbb{1} + 288\varepsilon^4 u_x^4 - 288\varepsilon^4 u^2 u_x + 28\varepsilon^2 u_x^2 - 16\varepsilon^2 u_x u_{xx} - 288\varepsilon^6 u_{xx} u_{xxx} \\
- 96\varepsilon^6 u_{xx}^3 + 3072\varepsilon^4 u_x^6 + 24\varepsilon^6 u_{xxx} + 24\varepsilon^4 u_x u_x + 1408\varepsilon^6 u_x^6 + 3328\varepsilon^8 u_x^8 \\
- 768\varepsilon u_x u_x u_x^4 - 384\varepsilon^4 u_x^4 u_x^2 u_x - 2304\varepsilon^8 u_x^3 u_{xx} u_{xx} + 384\varepsilon^8 u_{xx} u_x u_{xx} \\
- 4608\varepsilon^8 u_x^5 u_{xx} u_{xxx} + 16\varepsilon^2 u_x^2 - 5632\varepsilon^8 u_x^6 u_{xx} - 1920\varepsilon^6 u_x u_x u_x^4 + 3328\varepsilon^8 u_x u_x^2 \\
+ 512\varepsilon^6 u_x u_x^2 + 384\varepsilon^4 u_x^4 u_x^2 - 960\varepsilon^6 u_x^4 u_x^2 - 48\varepsilon^4 u_x u_{xxx} - 3072\varepsilon^6 u_x^7 u_{xxx} \\
+ 3072\varepsilon^{10} u_x^3 u_x^4 - 2304\varepsilon^8 u_x^5 u_{xxx} - 576\varepsilon^6 u_x^3 u_{xxx} + 288\varepsilon^6 u_x u_x^2 + 384\varepsilon^8 u_x^3 u_x^2 \\
+ 6144\varepsilon^6 u_x^2 u_x^6 - 6144\varepsilon^6 u_x^6 u_x^8 + 1152\varepsilon^8 u_x u_x^4 + 1536\varepsilon^6 u_x u_x^6 + 192\varepsilon^8 u_x^2 u_x^2 \\
+ 240\varepsilon^8 u_x u_x + 1536\varepsilon^8 u_x^2 u_x^3 u_{xxx} - 48\varepsilon^6 u_x u_x] \\
\times [\mathbb{1} + 96\varepsilon^4 u_x^4 + 256\varepsilon^6 u_x^6 + 256\varepsilon^8 u_x^8 + 4\varepsilon^4 u_x^2 - 48\varepsilon^4 u_x u_x^2 + 32\varepsilon^6 u_x^2 u_x^2 \\
- 4\varepsilon^2 u_x^2 - 256\varepsilon^6 u_x u_x + 64\varepsilon^8 u_x^4 u_x^2 - 192\varepsilon^6 u_x u_x^4 + 16\varepsilon^2 u_x^2]^{-1} \cdot (p_x q - p_{xx}).
\]

The two underlined units correspond to the bracket \( p_x q - p_{xx} \) on the domain of the operator \( \Box = \mathcal{U}_t + \frac{1}{2} D_x \) that factors symmetries of the Liouville equation \( \mathcal{U}_{xy} = \exp(2\mathcal{U}) \) at \( \varepsilon = 0 \). In agreement with Lemma 30 the non-constant coefficients of bilinear terms \( p_x q - p_{xx} \) and \( p_x q - p_{xx} \) belong to \( \ker D_y | \xi_t \).

**Remark 21.** The classification problem for Frobenius operators \( A \) and the task of reconstruction of the associated brackets \( \{\cdot, \cdot\}_A \) can be performed using any software capable for calculation of the commutators, e.g., [32, 47] designed for symmetry analysis of evolutionary (super-)PDE. The implementation of technique of the \( \Delta \)-coverings [22] is extremely productive here; see [32] for numerous examples. In this paper, we considered not only the \( \ell^* \) and \( \ell^\prime \)-coverings, which correspond to Frobenius recursions [17] for symmetries of PDE and to Noether operators for determined evolutionary systems, respectively.

**APPENDIX C. BI-DIFFERENTIAL REPRESENTATIONS OF THE BRACKETS \( \{\cdot, \cdot\}_A \)**

The components of Sokolov’s brackets \( \{\cdot, \cdot\}_A \in \mathcal{CDiff}(\wedge^2 \Omega^1(\xi_t), \Omega^1(\xi_t)) \) are matrix bi-differential operators in total derivatives w.r.t. the components of sections that belong to \( \Omega^1(\xi_t) \). We illustrate this for the brackets (128), which were calculated in Example 29 on p. 65 for Frobenius operators (57d) and (57e).

The notation means that the differential operators standing in the first and second tensor factors act, respectively, on the first and second arguments of the coupling \( \langle \psi_1 | \{\cdot, \cdot\}_A | \psi_2 \rangle \).
The $u$-, $v$-, and $w$-components of the bracket $\{\ , \\}_A_1$ on the domain $\mathcal{A}(\pi)$ of $A_1$ are the bi-differential operators

$$
\begin{pmatrix}
0 & 3w \cdot (1 \otimes D_x - D_x \otimes 1) + 2w_x \cdot 1 \otimes 1 & u \cdot (1 \otimes D_x - D_x \otimes 1) & 0 \\
3w \cdot (1 \otimes D_x - D_x \otimes 1) - 2w_x \cdot 1 \otimes 1 & u \cdot (1 \otimes D_x - D_x \otimes 1) & 0 & 0 \\
D_x \otimes 1 - 1 \otimes D_x & 0 & 4w \cdot (1 \otimes D_x - D_x \otimes 1) & 0 \\
0 & D_x \otimes 1 - 1 \otimes D_x & 0 & 4w \cdot (1 \otimes D_x - D_x \otimes 1) \\
\end{pmatrix},
$$

The components of the bracket $\{\ , \\}_A_2$ associated with the operator $A_2$ are

$$
\begin{pmatrix}
0 & 3w \cdot 1 \otimes D_x - 2w D_x \otimes 1 + 2w_x \cdot 1 \otimes 1 & -1 \otimes D_x & 0 \\
2w \cdot 1 \otimes D_x - 3w D_x \otimes 1 - 2w_x \cdot 1 \otimes 1 & u \cdot (1 \otimes D_x - D_x \otimes 1) & 0 & 0 \\
D_x \otimes 1 - 1 \otimes D_x & 0 & 4w \cdot (1 \otimes D_x - D_x \otimes 1) & 0 \\
0 & D_x \otimes 1 & 0 & 2D_x \otimes 1 \\
\end{pmatrix},
$$

$$
\begin{pmatrix}
w \cdot (D_x \otimes 1 - 1 \otimes D_x) & u \cdot 1 \otimes D_x - 8u D_x \otimes 1 - 2u_x \cdot 1 \otimes 1 & 0 \\
2u_x \cdot 1 \otimes 1 + 8u \cdot 1 \otimes D_x - u D_x \otimes 1 & (8v + 6w^2) \cdot (1 \otimes D_x - D_x \otimes 1) & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
$$

Both bi-differential matrix representations are skew-symmetric in their arguments.

**APPENDIX D. CALCULATION OF $\{\ , \\}_R$ FOR FROBENIUS RECURSIONS $R$**

The following program for Jets environment [47] under Maple calculates Sokolov’s bracket $\{\ , \\}_R$ for Frobenius recursion $R_0 = A_0 \circ A_1^{-1}$, see [51], which is factored by the first Hamiltonian operator (57a) and the inverse of Noether operator (57c) for the dispersionless 3-component Boussinesq-type system (50).

```maple
> read 'Jets.s';
> coordinates([x], [u, v, w, p1, p2, p3, q1, q2, q3], 5):
> A := Matrix(3, 3): ia := Matrix(3, 3): M := Matrix(3, 3): N := Matrix(3, 3):
> A := \{-3*w*u_x+u*w_x,-3*w*u_x-u*w_x,u_x\|
> \{-3*w*u_x-u*w_x,-3*w^2*w_x-4*w*v_x-u*u_x,v_x\|
> \{-u_x,v_x,w_x\}:
> We have assigned $A = A_1$, see (57c).
```

> with(LinearAlgebra):
> ia := MatrixInverse(A):
> This yields $ia = A_1^{-1}$: sym $\mathcal{E} \rightarrow cosym \mathcal{E}$, which we also denote by $\omega$. Now put $m = A_1^{-1}(\xi_1)$ and $n = A_1^{-1}(\xi_2)$.
> m1 := simplify(ia[1,1]*p1+ia[1,2]*p2+ia[1,3]*p3):
> m2 := simplify(ia[2,1]*p1+ia[2,2]*p2+ia[2,3]*p3):
> m3 := simplify(ia[3,1]*p1+ia[3,2]*p2+ia[3,3]*p3):
>
> n1 := simplify(ia[1,1]*q1+ia[1,2]*q2+ia[1,3]*q3):
> n2 := simplify(ia[2,1]*q1+ia[2,2]*q2+ia[2,3]*q3):
> n3 := simplify(ia[3,1]*q1+ia[3,2]*q2+ia[3,3]*q3):

We have computed \( \psi_i = \omega(\xi_i) \); recall that \( \psi \in \cosym \mathcal{E} \).

Next, we apply the Hamiltonian operator \( \hat{A}_0 \) and obtain symmetries of (56), \( k = \hat{A}_0(m) = \hat{A}_0(A^{-1}_1(\xi_1)) \) and \( 1 = \hat{A}_0(n) = \hat{A}_0(A^{-1}_1(\xi_2)) \).

> k1 := simplify(evalTD(TD(m1,x))):
> k2 := simplify(evalTD(-4*w*TD(m2,x)-2*w_x*m2+TD(m3,x))):
> k3 := simplify(evalTD(TD(m2,x))):
>
> l1:=simplify(evalTD(TD(n1,x))):
> l2:=simplify(evalTD(-4*w*TD(n2,x)-2*w_x*n2+TD(n3,x))):
> l3:=simplify(evalTD(TD(n2,x))):

Now we act by evolutionary derivations on the coefficients of the operator \( \omega = A^{-1}_1 \), see (78). We set \( M = E_{R_0(\xi_1)}(\omega) \); note that the matrix \( M \) is symmetric.

> M[1,1]:=simplify(evalTD(k1*pd(ia[1,1],u)+TD(k1,x)*pd(ia[1,1],u_x)+
k2*pd(ia[1,1],v)+TD(k2,x)*pd(ia[1,1],v_x)+k3*pd(ia[1,1],w)+
TD(k3,x)*pd(ia[1,1],w_x))):
> M[1,2]:=simplify(evalTD(k1*pd(ia[1,2],u)+TD(k1,x)*pd(ia[1,2],u_x)+
k2*pd(ia[1,2],v)+TD(k2,x)*pd(ia[1,2],v_x)+k3*pd(ia[1,2],w)+
TD(k3,x)*pd(ia[1,2],w_x))):
> M[1,3]:=simplify(evalTD(k1*pd(ia[1,3],u)+TD(k1,x)*pd(ia[1,3],u_x)+
k2*pd(ia[1,3],v)+TD(k2,x)*pd(ia[1,3],v_x)+k3*pd(ia[1,3],w)+
TD(k3,x)*pd(ia[1,3],w_x))):
> M[2,2]:=simplify(evalTD(k1*pd(ia[2,2],u)+TD(k1,x)*pd(ia[2,2],u_x)+
k2*pd(ia[2,2],v)+TD(k2,x)*pd(ia[2,2],v_x)+k3*pd(ia[2,2],w)+
TD(k3,x)*pd(ia[2,2],w_x))):
> M[2,3]:=simplify(evalTD(k1*pd(ia[2,3],u)+TD(k1,x)*pd(ia[2,3],u_x)+
k2*pd(ia[2,3],v)+TD(k2,x)*pd(ia[2,3],v_x)+k3*pd(ia[2,3],w)+
TD(k3,x)*pd(ia[2,3],w_x))):
> M[3,3]:=simplify(evalTD(k1*pd(ia[3,3],u)+TD(k1,x)*pd(ia[3,3],u_x)+
k2*pd(ia[3,3],v)+TD(k2,x)*pd(ia[3,3],v_x)+k3*pd(ia[3,3],w)+
TD(k3,x)*pd(ia[3,3],w_x))):

In the same way, we define the symmetric matrix \( N = E_{R_0(\xi_2)}(\omega) \).

> N[1,1]:=simplify(evalTD(11*pd(ia[1,1],u)+TD(11,x)*pd(ia[1,1],u_x)+
12*pd(ia[1,1],v)+TD(12,x)*pd(ia[1,1],v_x)+13*pd(ia[1,1],w)+
TD(13,x)*pd(ia[1,1],w_x))):
> N[1,2]:=simplify(evalTD(11*pd(ia[1,2],u)+TD(11,x)*pd(ia[1,2],u_x)+
12*pd(ia[1,2],v)+TD(12,x)*pd(ia[1,2],v_x)+13*pd(ia[1,2],w)+
TD(13,x)*pd(ia[1,2],w_x)):
> N[1,3]:=simplify(evalTD(l1*pd(ia[1,3],u)+TD(l1,x)*pd(ia[1,3],u_x)+
12*pd(ia[1,3],v)+TD(l2,x)*pd(ia[1,3],v_x)+13*pd(ia[1,3],w)+
TD(13,x)*pd(ia[1,3],w_x))):
> N[2,2]:=simplify(evalTD(l1*pd(ia[2,2],u)+TD(l1,x)*pd(ia[2,2],u_x)+
12*pd(ia[2,2],v)+TD(l2,x)*pd(ia[2,2],v_x)+13*pd(ia[2,2],w)+
TD(13,x)*pd(ia[2,2],w_x))):
> N[2,3]:=simplify(evalTD(l1*pd(ia[2,3],u)+TD(l1,x)*pd(ia[2,3],u_x)+
12*pd(ia[2,3],v)+TD(l2,x)*pd(ia[2,3],v_x)+13*pd(ia[2,3],w)+
TD(13,x)*pd(ia[2,3],w_x))):
> N[3,3]:=simplify(evalTD(l1*pd(ia[3,3],u)+TD(l1,x)*pd(ia[3,3],u_x)+
12*pd(ia[3,3],v)+TD(l2,x)*pd(ia[3,3],v_x)+13*pd(ia[3,3],w)+
TD(13,x)*pd(ia[3,3],w_x))):

We act by the operators $M, N$ on $\xi_2$ and $\xi_1$, respectively, and calculate the difference $e = E_{R_0}(\xi^1)\omega(\xi^2) - E_{R_0}(\xi^2)\omega(\xi^1)$.

> e1:=simplify(M[1,1]*q1+M[1,2]*q2+M[1,3]*q3-N[1,1]*p1-N[1,2]*p2-N[1,3]*p3):
> e2:=simplify(M[1,2]*q1+M[2,2]*q2+M[2,3]*q3-N[1,2]*p1-N[2,2]*p2-N[2,3]*p3):
> e3:=simplify(M[1,3]*q1+M[2,3]*q2+M[3,3]*q3-N[1,3]*p1-N[2,3]*p2-N[3,3]*p3):

Next, we recall that Sokolov’s bracket for $\hat A_0$ is equal to (57b), and substitute $\psi_i = \omega(\xi^i)$ in it. Thus we put $s = e + \{(m,n)\}$, and

> s1:=simplify(e1):
> s2:=simplify(e2):
> s3:=simplify(e3+evalTD(2*(m2*TD(n2,x)-TD(m2,x)*n2))):

We can now check that formula (57b) is correct.

> J:=Jacobi([k1,k2,k3,0,0,0,0,0,0],[l1,l2,l3,0,0,0,0,0,0]):
> D1:=simplify(evalTD(TD(s1,x))):
> D2:=simplify(evalTD(-4*w*TD(s2,x)-2*w_x*s2+TD(s3,x))):
> D3:=simplify(evalTD(TD(s2,x))):
> simplify(evalTD(J[1]-D1));
0
> simplify(evalTD(J[2]-D2));
0
> simplify(evalTD(J[3]-D3));
0

Finally, we act onto the right-hand side of (78) by the operator $\omega^{-1} = A_1$, and thus we obtain the components $Z = A_1(s)$ of Sokolov’s bracket $\{., .\}_{R_0}$.

> Z1:=simplify((w*w_x+v_x)*s1+(-3*w*u_x-u*w_x)*s2+u_x*s3):
> Z2:=simplify((-3*w*u_x-u*w_x)*s1+(-3*w*w_x-4*w*v_x-u_u_x)*s2+v_x*s3):
> Z3:=simplify(u_x*s1+v_x*s2+w_x*s3):

The result is somewhat a surprise: the output contains more than 15,000 lines. Be that as it may, the operator $R_0 = \hat A_0 \circ \omega$ is the first known example of a well-defined Frobenius recursion for an integrable system.
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References

[1] Alexandrov M., Schwarz A., Zaboronsky O., Kontsevich M. (1997) The geometry of the master equation and topological quantum field theory, Int. J. Modern Phys. A 12:7, 1405–1429.
[2] Arnol’d V. I., Khesin B. A. (1998) Topological methods in hydrodynamics. Applied Mathematical Sciences, 125. Springer–Verlag, NY.
[3] Barannikov S., Kontsevich M. (1998) Frobenius manifolds and formality of Lie algebras of polyvector fields, Int. Math. Res. Notices 4, 201–215.
[4] Błaszak M. (1998) Multi-Hamiltonian theory of dynamical systems, Springer, Berlin.
[5] Bocharov A. V., Chetverikov V. N., Duzhin S. V. et al. (1999) Symmetries and conservation laws for differential equations of mathematical physics. (Krasil’shchik I. S. and Vinogradov A. M., eds.) AMS, Providence, RI.
[6] Bryant R. L., Chern S. S., Gardner R. B. et al. (1991) Exterior differential systems. Mathematical Sciences Research Institute Publications, 18. Springer-Verlag, NY.
[7] Demskoi D. K., Sokolov V. V. (2008) On recursion operators for elliptic models, Nonlinearity 21:6, 1253–1264.
[8] Demskoi D. K., Startsev S. Ya. (2006) On construction of symmetries from integrals of hyperbolic partial differential systems, J. Math. Sci. 136:6 Geometry of integrable models, 4378–4384.
[9] Dorfman I. Ya. (1993) Dirac structures, J. Wiley & Sons.
[10] Drinfel’d V.G., Sokolov V.V. (1985) Lie algebras and equations of Korteweg–de Vries type, J. Sov. Math. 30, 1975-2035.
[11] Dubrovin B. A. (1996) Geometry of 2D topological field theories, Lect. Notes in Math. 1620 Integrable systems and quantum groups (Montecatini Terme, 1993), Springer, Berlin, 120–348. arXiv:hep-th/940708
[12] Fuchs D. B. (1986) Cohomology of infinite-dimensional Lie algebras. Contemporary Soviet Mathematics. Consultants Bureau, NY.
[13] Gelfand I. M., Dikii L. A. (1975) Asymptotic properties of the resolvent of Sturm–Liouville equations, and the algebra of Korteweg–de Vries equations, Russ. Math. Surveys 30:5, 77–113.
[14] Gerstenhaber M., Schack S.D. (1988) Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications (M. Gerstenhaber and M. Hazewinkel, eds.) Kluwer, Dordrecht, 11–264.
[15] Gessler D. (1997) On the Vinogradov C-spectral sequence for determined systems of differential equations, Diff. Geom. Appl. 7:4, 303–324.
[16] Getzler E. (2002) A Darboux theorem for Hamiltonian operators in the formal calculus of variations, Duke Math. J. 111:3, 535–560.
[17] Golovko V. A., Krasil’shchik I. S., Verbovetsky A. M. (2008) Variational Poisson–Nijenhuis structures for partial differential equations, Theor. Math. Phys. 154:2, 227–239.
[18] Gümral H., Nutku Y. (1994) Bi-Hamiltonian structures of d-Boussinesq and Benney–Lax equations, J. Phys. A: Math. Gen. 27, 193–200.

[19] Igonin S., Verbovetsky A., Vitolo R. (2004) Variational multivectors and brackets in the geometry of jet spaces, Symmetry in nonlinear mathematical physics (Kiev, 2003). Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos. 50, 1335–1342.

[20] Kac V. G., van de Leur J. W. (2003) The n-component KP hierarchy and representation theory. Integrability, topological solitons and beyond, J. Math. Phys. 44:8, 3245–3293.

[21] Kassel C. (1995) Quantum groups. NY, Springer–Verlag.

[22] Kersten P., Krasil’shchik I., Verbovetsky A. (2004) Variational multivectors and brackets in the geometry of jet spaces, Symmetry in nonlinear mathematical physics (Kiev, 2003). Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos. 50, 1335–1342.

[23] Kersten P., Krasil’shchik I., Verbovetsky A. (2004) On the integrability conditions for some structures related to evolution differential equations, Acta Appl. Math. 83:1-2, 167–173.

[24] Kersten P., Krasil’shchik I., Verbovetsky A. (2006) A geometric study of the dispersionless Boussinesq type equation, Acta Appl. Math. 90:1–2, 143–178.

[25] Kiselev A. V. (2005) Hamiltonian flows on Euler-type equations, Theor. Math. Phys. 144:1, 952-960. arXiv:nlin.SI/0409061

[26] Kiselev A. V. (2007) Algebraic properties of Gardner’s deformations for integrable systems, Theor. Math. Phys. 152:1, 96–112. arXiv:nlin.SI/0610072

[27] Kiselev A. V. (2007) On associative Schlessinger–Stasheff’s algebras and the Wronskians, J. Math. Sci. 141:1, 1016–1030. arXiv:math.RA/0410185

[28] Kiselev A. V. (2007) On conservation laws for the Toda equations, Acta Appl. Math. 83:1-2, 175–182.

[29] Kiselev A. V. (2004) On Noether symmetries of the Toda equations, Vestnik Mosk. Univ. (ser. 3: Phys. Astr.) n.2, 16–18 (in Russian).

[30] Kiselev A. V., van de Leur J. W. (2009) A geometric derivation of KdV-type hierarchies from root systems, in: Proc. 4-th Int. workshop ‘Group analysis of differential equations and integrable systems’ (Protaras, Cyprus, 2008), 19 p. arXiv:nlin.SI/0901.4866

[31] Kiselev A. V., Wolf T. (2007) Classification of integrable super-systems using the SsTools environment, Comput. Phys. Commun. 177:3, 315–328.

[32] Kosmann-Schwarzbach Y. (2004) Derived brackets, Lett. Math. Phys. 69, 61–87.

[33] Kosmann-Schwarzbach Y. (1981) Hamiltonian systems on fibered manifolds. Poisson and vertical brackets in field theory, Lett. Math. Phys. 5:3, 229–237; Kosmann-Schwarzbach Y. (1985) On the momentum mapping in field theory. Differential geometric methods in mathematical physics (Clausthal, 1983), Lecture Notes in Math. 1139, (H. D. Doebner, J. D. Hennig, eds.), Springer, Berlin, 25–73.

[34] Kosmann-Schwarzbach Y., Magni F. (1990) Poisson–Nijenhuis structures, Ann. Inst. H. Poincaré, ser. A: Phys. Théor. 53:1, 35–81.

[35] Koszmider J.-L. (1985) Crochet de Schouten–Nijenhuis et cohomologie. The mathematical heritage of Élie Cartan (Lyon, 1984), Astérisque, hors serie, 257–271.

[36] Krasil’shchik I. S. (1988) Schouten bracket and canonical algebras. Global analysis studies and applications III, Lecture Notes in Math. 1334, Springer, Berlin, 79–110.

[37] Krasil’shchik I. S. (1998) Algebras with flat connections and symmetries of differential equations, Lie Groups and Algebras: Their Representations, Generalizations and Applications (B. P. Komrakov, I. S. Krasil’shchik, G. L. Litvinov, and A. B. Sossinsky, eds.) Kluwer, Dordrecht, 425–434.

[38] Krasil’shchik I. S., Kersten P. H. M. (2000) Symmetries and recursion operators for classical and supersymmetric differential equations, Kluwer, Dordrecht etc.
[40] Krasil’schik I., Verbovetsky A. (1998) Homological methods in equations of mathematical physics. Open Education and Sciences, Opava. arXiv:math.DG/9808130
[41] Kupershmidt B. A. (1980) Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms. Geometric methods in mathematical physics. Lecture Notes in Math. 775, Springer, Berlin, 162–218.
[42] Leznov A. N., Saveliev M. V. (1979) Representation of zero curvature for the system of nonlinear partial differential equations $x_{\alpha,zz} = \exp(Kx)_\alpha$ and its integrability, Lett. Math. Phys. 3, 489–494.
[43] Leznov A. N., Smirnov V. G., Shabat A. B. (1982) Internal symmetry group and integrability conditions for two-dimensional dynamical systems, Theor. Math. Phys. 51:1, 322–330.
[44] Magri F. (1978) A simple model of the integrable equation, J. Math. Phys. 19:5, 1156–1162.
[45] Magri F., Casati P., Falqui G., and Pedroni M. (2004) Eight lectures on integrable systems, Lect. Notes in Phys. 638 (Y. Kosmann–Schwarzbach, B. Grammaticos, K.M. Tamizhmani, eds.), 209–250.
[46] Manin Yu. I. (1978) Algebraic aspects of nonlinear differential equations. Current problems in mathematics 11, AN SSSR, VINITI, Moscow, 5–152 (in Russian).
[47] Marvan M. (2003) Jets. A software for differential calculus on jet spaces and diffieties, Opava. http://diffiety.org/soft/soft.htm
[48] Meshkov A. G. (1985) Symmetries of scalar fields III: Two-dimensional integrable models, Theor. Math. Phys. 63:3, 539–545.
[49] Mikhailov A. V., Shabat A. B., Sokolov V. V. (1991) The symmetry approach to classification of integrable equations. What is integrability? (V. E. Zakharov, ed.) Series in Nonlinear Dynamics, Springer, Berlin, 115–184.
[50] Mikhailov A. V., Sokolov V. V. (2000) Integrable ODEs on associative algebras, Commun. Math. Phys. 211, 231–251.
[51] Olver P. J. (1993) Applications of Lie groups to differential equations, Grad. Texts in Math. 107 (2nd ed.), Springer–Verlag, NY.
[52] Reyman A. G., Semenov–Tian-Shansky M. A. (1994) Group-theoretical methods in the theory of finite dimensional integrable systems, in: Dynamical systems VII (V. I. Arnold and S. P. Novikov, eds.), Encyclopaedia of Math. Sci. 16, Springer, Berlin, 116–225; Reyman A. G., Semenov–Tian-Shansky M. A. (2003) Integrable systems: group-theoretic approach. Inst. Comp. Stud., Moscow etc. (in Russian).
[53] Sakovich S. Yu. (1994) On conservation laws and zero-curvature representations of the Liouville equation, J. Phys. A 27:5, L125–L129.
[54] Sanders J. A., Wang J. P. (2002) On a family of operators and their Lie algebras, J. Lie Theory 12:2, 503–514.
[55] Shabat A. B. (1995) Higher symmetries of two-dimensional lattices, Phys. Lett. A 200, 121–133.
[56] Shabat A. B., Yamilov R. I. (1981) Exponential systems of type I and the Cartan matrices, Prepr. Bashkir division Acad. Sci. USSR, Ufa, 22 p.
[57] Sokolov V. V., Startsev S. Ya. (2008) Symmetries of nonlinear hyperbolic systems of Toda chain type, Teor. Matem. Fiz. 155:2, 344–355 (in Russian).
[58] Startsev S. Ya. (2006) On the variational integrating matrix for hyperbolic systems, Fun-dam. Appl. Math. 12.7 Hamiltonian & Lagrangian systems and Lie algebras, 251–262.
[59] Vantrob A. Yu. (1997) Lie algebroids and homological vector fields, Russ. Math. Surv. 52:2, 428–429.
[60] Zhiber A.V., Sokolov V.V. (2001) Exactly integrable hyperbolic equations of Liouvillean type, Russ. Math. Surveys 56:1, 61–101.