First-passage dynamics of linear stochastic interface models: weak-noise theory and influence of boundary conditions

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Abstract. We consider a one-dimensional fluctuating interfacial profile governed by the Edwards–Wilkinson or the stochastic Mullins-Herring equation for periodic, standard Dirichlet and Dirichlet no-flux boundary conditions. The minimum action path of an interfacial fluctuation conditioned to reach a given maximum height $M$ at a finite (first-passage) time $T$ is calculated within the weak-noise approximation. Dynamic and static scaling functions for the profile shape are obtained in the transient and the equilibrium regime, i.e. for first-passage times $T$ smaller or larger than the characteristic relaxation time, respectively. In both regimes, the profile approaches the maximum height $M$ with a universal algebraic time dependence characterized solely by the dynamic exponent of the model. It is shown that, in the equilibrium regime, the spatial shape of the profile depends sensitively on boundary conditions and conservation laws, but it is essentially independent of them in the transient regime.

Keywords: fluctuation phenomena, macroscopic fluctuation theory, extreme value, large deviations in non-equilibrium systems
1. Introduction

Let $h(x, t)$ be a one-dimensional interfacial height profile $h(x, t)$ subject to either the Edwards–Wilkinson (EW) equation \[ \partial_t h = \eta \partial_x^2 h + \zeta, \] (1.1) or the stochastic Mullins-Herring (MH) equation \[ \partial_t h = -\eta \partial_x^4 h + \partial_x \zeta. \] (1.2)

The white noise $\zeta$ is a Gaussian random variable with zero mean and correlations

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\[
\langle \zeta(x, t)\zeta(x', t') \rangle = 2D \delta(x - x') \delta(t - t').
\]

(1.3)

The friction coefficient \( \eta \) and the noise strength \( D \) are \textit{a priori} free parameters whose ratio can be fixed by requiring that the Gaussian steady-state distribution resulting from equations (1.1) and (1.2) is characterized by a certain temperature (see, e.g. [5, 6]). While \( h \) is locally conserved for equation (1.2), the noise term in equation (1.1) violates this property.

The EW equation describes surface growth caused by random deposition and relaxation. The Kardar–Parisi–Zhang equation [7] is a nonlinear extension of the EW equation accounting for the effect of lateral growth. The noiseless MH equation describes interfacial relaxation under the influence of surface diffusion [2]. If \( h \) represents a liquid interface, equation (1.2) can be understood as a linearized stochastic lubrication equation in the absence of disjoining pressure [8, 9]. Furthermore, the stochastic Cahn–Hilliard equation, which is used in the modeling of phase-separation, reduces deep in the super-critical phase to equation (1.2) [10].

Interfacial fluctuations typically exhibit long-ranged correlations and non-Markovian dynamics. Roughening of interfaces and the associated dynamic scaling behavior emerging from equations (1.1) and (1.2) has been extensively studied (see, e.g. [4, 11–22]). More recently, extreme events and first-passage properties of interfaces have been investigated [5, 6, 23–30]. The present study focuses on the time-evolution of a profile \( h(x, t) \) governed by equations (1.1) or (1.2), under the condition that \( h \) reaches a given height \( M \) for the first time at time \( T \),

\[
h(x_M, T) = M,
\]

(1.4)
given that, initially,

\[
h(x, t = 0) = 0.
\]

(1.5)

The location \( x_M \) where the height \( M \) is reached first depends on the specific model as well as on the boundary conditions. If \( T \) is larger than the relaxation time of the interface, the interfacial roughness (i.e. the one-point one-time variance of the height fluctuations) has saturated at the first-passage event [4, 14, 31] and the interface is accordingly governed by equilibrium dynamics (the precise meaning of this will be clarified further below). We consider profiles on a finite domain \([0, L] \) subject to either periodic boundary conditions (p),

\[
h^{(p)}(x, t) = h^{(p)}(x + L, t),
\]

(1.6)
or Dirichlet boundary conditions (D),

\[
h^{(D)}(0, t) = 0 = h^{(D)}(L, t).
\]

(1.7)

For the MH equation with Dirichlet boundary conditions, two further conditions are needed to completely determine the solution. We impose in this case a no-flux boundary condition (see also appendix B):

\footnote{We remark that, without a microscopic cutoff, the stochastic EW and MH equations yield a diverging variance of the one-point height distribution for spatial dimensions \( d \geq 2 \) [4, 42]. In the one-dimensional case considered here, the two models are well defined even without a regularization at small scales.}

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\[ \partial_z^3 h^{(D')}(0,t) = 0 = \partial_z^3 h^{(D')}(L,t), \tag{1.8} \]

and henceforth indicate equations (1.7) and (1.8) by a superscript \((D')\). We denote by the ‘mass’ \( \mathcal{A} \) the total area under the profile:

\[ \mathcal{A}([h],t) \equiv \int_0^L dx \, h(x,t). \tag{1.9} \]

For the EW equation with periodic boundary conditions, \( \mathcal{A}([h^{(p)}],t) \) is not constant in time, but instead behaves diffusively at large times [4]. In this case, we consider instead of \( h^{(p)} \) the relative height fluctuation

\[ \tilde{h}^{(p)}(x,t) \equiv h^{(p)}(x,t) - \mathcal{A}([h^{(p)}],t)/L, \tag{1.10} \]

which fulfills \( \mathcal{A}([\tilde{h}^{(p)}],t) = 0 \). We henceforth drop the tilde on \( \tilde{h}^{(p)} \) in order to simplify notation. Global conservation of the mass with

\[ \mathcal{A}([h],t) = 0 \tag{1.11} \]

holds also for the MH equation with either periodic or Dirichlet no-flux boundary conditions (given equation (1.5)). For the EW equation with standard Dirichlet boundary conditions, the mass vanishes only after averaging over time. Equation (1.10), which is rather artificial from a physical point of view, is imposed here mainly in order to compare the different models under the common mass constraint, equation (1.11). The basic situation and the relevant quantities considered in the present study are illustrated in figure 1. In passing, we introduce the dynamic index \( z \), which describes the dependence of the relaxation time \( \tau \) of a typical fluctuation governed by equations (1.1) or (1.2) on the system size \( L \) via \( \tau \propto L^z \), with

\[ z = 2 \quad \text{(EW equation),} \quad z = 4 \quad \text{(MH equation).} \tag{1.12} \]

Large deviations of stochastic processes are formally described by Freidlin–Wentzel theory [32–34], which is equivalent to a Martin–Siggia–Rose/Janssen/de Dominicis path-integral formulation [35–38] in the limit of weak noise [39–41]. This approach provides an action functional, the minimization of which yields the most probable (‘optimal’) path connecting two states (e.g. equations (1.4) and (1.5)). For an explicit derivation of the corresponding weak-noise theory (WNT) for the EW and MH equation see, e.g. [29, 42]. A related large deviation formalism in the context of lattice gases is reviewed in [43].

An important predecessor to the present work is [29], where the WNT of equation (1.2) with periodic boundary conditions has been solved. Here, we extend that study by discussing further aspects of the first-passage dynamics, focusing, in particular, on the effect of boundary conditions. Within the WNT of equations (1.1) and (1.2), we obtain minimum-action paths describing extremal fluctuations of the profile fulfilling equations (1.4) and (1.5), without conditioning on the first-passage. We

Results for the MH equation with standard Dirichlet boundary conditions are briefly summarized in appendix C.1.2.

For standard Dirichlet boundary conditions, the chemical potential \( \mu = -\partial^2 h \), instead of the flux \( -\partial_x \mu \), vanishes at the boundaries (see appendix B.1.3).
remark that the solution of WNT for Dirichlet no-flux boundary conditions (equations (1.7) and (1.8)) is technically involved since it requires the consideration of an adjoint eigenproblem (see appendix B.1.1). Predictions of WNT will be compared to Langevin simulations in an accompanying paper \[44\].

The first-passage problem for the MH equation discussed here and in \[44\] is relevant, \textit{inter alia}, for the rupture of liquid wetting films. In contrast to previous studies \[9, 45–58\], we focus here on the case where disjoining pressure is negligible and film rupture is solely driven by noise. A related WNT describing the noise-induced breakup of a liquid thread has been analyzed in \[59\]. Rare-event trajectories of the kind considered here are furthermore relevant for the understanding of chemical reaction pathways \[60–62\], phase transitions \[33, 63\] as well as for certain aspects in interfacial wetting (see \[64\] and references therein).

The main results of the present study are contained in sections 2 and 3, in which the necessary formalism of WNT for the EW and MH equation, respectively, is introduced and the exact analytical solution for the first-passage profile is discussed. The determination of the analytical solution as well as further mathematical details are deferred to appendices A to C. In the main part (sections 2.2 and 3.2), we focus on the time-evolution of the first-passage profile in the case of periodic and Dirichlet (no-flux) boundary conditions. For first-passage times \(T \ll \tau\) (transient regime) we find that the profile shape essentially depends only on the type of bulk dynamics, while the influence of boundary conditions and mass conservation is negligible. In contrast, at late times \(T \gg \tau\) (equilibrium regime), the profile evolves over the whole domain and strongly depends on the specific boundary conditions. In both temporal regimes, simple analytical expressions for the asymptotic dynamic and static scaling profiles are derived. These scaling forms indicate that, within WNT, the peak height \(h(x_M, t)\) of the profile approaches the first-passage height \(M\) in time with a \textit{universal} exponent \(1/\alpha\). Moreover, it is shown that, in the presence of a microscopic cutoff, the dynamic scaling exponent eventually crosses over to a value of 1 close to the first-passage event.

![Figure 1. Situation considered in the present study: an initially flat profile \(h(x, 0)\) on a domain \(0 \leq x \leq L\) reaches a given maximum height \(M\) for the first time at time \(T\). \(x_M\) denotes the location of this first-passage event. While the actual first-passage dynamics of the interface is stochastic (see equations (1.1) and (1.2)), we focus here on the weak-noise approximation as governed by the (non-stochastic) partial differential equations in equations (2.2) and (3.2).](image-url)
2. Edwards–Wilkinson equation

2.1. Macroscopic fluctuation theory

The Martin–Siggia–Rose field-theoretical action pertaining to equation (1.1) is given by [38, 42]

\[ S[h, p] = \int_0^T \int_0^L \, dt \, dx \, [p(\partial_t h - \eta \partial_x^2 h) - Dp^2], \tag{2.1} \]

where \( p \) is an auxiliary (‘conjugate’) field. The most-probable (optimal) path emerging from the stochastic dynamics is the one that minimizes \( S \):

\[ 0 = \frac{\delta S}{\delta p} = \partial_t h - \eta \partial_x^2 h - 2Dp, \tag{2.2a} \]

\[ 0 = \frac{\delta S}{\delta h} = -\partial_t p - \eta \partial_x^2 p. \tag{2.2b} \]

The field \( p \), which can be interpreted as the typical noise magnitude, is governed by an anti-diffusion equation (equation (2.2b)). This indicates that the creation of a rare event requires the local accumulation of noise intensity. We consider either periodic boundary conditions (equation (1.6)),

\[ h^{(p)}(x, t) = h^{(p)}(x + L, t), \quad p^{(p)}(x, t) = p^{(p)}(x + L, t), \tag{2.3} \]

or Dirichlet boundary conditions (equation (1.7)),

\[ h^{(D)}(0, t) = 0 = h^{(D)}(L, t), \quad p^{(D)}(0, t) = 0 = p^{(D)}(L, t). \tag{2.4} \]

Note that, since \( \partial_x^2 \) is self-adjoint on \([0, L]\) for the considered boundary conditions, \( p \) fulfills the same boundary conditions as \( h \) (see also appendices B and C). Inserting the mean-field equation (2.2) into the action in equation (2.1) yields the optimal action

\[ S_{\text{opt}} = D \int_0^T \int_0^L \, dt \, dx \, p^2. \tag{2.5} \]

Equation (2.2) admits a special solution which can be identified with thermal equilibrium. In equilibrium, the most-likely noise-activated trajectory \( h(x, t) \) is the time-reversed of the corresponding relaxation trajectory \( h_0(x, t) \)—a property known as Onsager–Machlup symmetry [65]. In order to exhibit this symmetry for the dynamics described by equation (2.2), consider the solution \( h_r(x, t) \) of the noise-free analog of equation (2.2a), i.e. the diffusion equation

\[ \partial_t h_r = \eta \partial_x^2 h_r, \tag{2.6} \]

with initial condition \( h_r(x, t = 0) = h_0(x) \), where \( h_0(x) \) is a given profile (e.g. the equilibrium first-passage profile \( h(x, T \to \infty) \), which can be determined independently, see equation (2.23) below). Then, the solution \( h(x, t), p(x, t) \) of equation (2.2), fulfilling \( h(x, T) = h_0(x) \) at some final time \( T \), is given by

\[ h(x, t) = h_r(T - t), \quad p(x, t) = -\frac{\eta}{D} \partial_x^2 h(x, t). \tag{2.7} \]
Indeed, it is readily checked that equations (2.7) solves (2.2), as
\[ \partial_t h = -\partial_t h_t = -\eta \partial_x^2 h = \eta \partial_x^2 h + 2D \rho, \] (2.8)
which is precisely equation (2.2a); furthermore \( \partial_t \rho = -(\eta/D)\partial_x^2 \rho h = (\eta^2 / D) \partial_x^2 h = -\eta \partial_x^2 \rho, \) which is equation (2.2b). According to equation (2.8), \( h \) effectively obeys an anti-diffusion equation in the equilibrium regime. Note that the ansatz in equation (2.7) implies that the time evolution starts at time \( t = 0 \) from the initial configuration \( h(x, 0) = h_0(T) \), which is flat only for \( T \rightarrow \infty \). Accordingly, under requirement of equation (1.5), the equilibrium regime corresponds to large first-passage times \( T \) as anticipated in section 1. The general solution of equation (2.2) fulfilling equations (1.4) and (1.5) for arbitrary \( T \) is presented below.

In the equilibrium regime, upon using equations (2.7) and (2.8), the optimal action in equation (2.5) reduces to
\[ S_{\text{opt,eq}} = \frac{\eta^2}{D} \int_0^T dt \int_0^L dx (\partial_x^2 h)^2 = -\frac{\eta^2}{D} \int_0^T dt \int_0^L dx (\partial_x^2 h)(\partial_x^2 h) \]
\[ = \frac{\eta}{D} \int_0^T dt \int_0^L dx (\partial_x^2 h)(\partial_x^2 h) = \frac{\eta}{D} \int_0^L dx (\partial_x^2 h)^2 \int_0^T dt \int_0^L dx (\partial_x^2 h)(\partial_x^2 h) \]
\[ = \frac{\eta}{2D} \int_0^L dx (\partial_x^2 h)^2 \int_0^T dt. \] (2.9)

In the partial integrations above we made use of the fact that the spatial boundary terms generally vanish for periodic and Dirichlet boundary conditions\(^4\). Equation (2.9) provides a fluctuation-dissipation relation, from which the temperature \( \Theta \) (in units of \( k_B \)) can be identified via \( \eta/(2D) = 1/(4\Theta) \).

We henceforth consider time to be rescaled by the friction coefficient \( \eta \), i.e. \( \tilde{t} = \eta t \), and define new fields \( \tilde{h}, \tilde{\rho} \) via
\[ h(x, t) \equiv \tilde{h}(x, \eta t), \quad \rho(x, t) = (\eta/D)\tilde{\rho}(x, \eta t). \] (2.10)
The Euler–Lagrange equations in equation (2.2) can then be cast into the form
\[ \partial_t \tilde{h} = \partial_x^2 \tilde{h} + 2\tilde{\rho}, \] (2.11a)
\[ \partial_t \tilde{\rho} = -\partial_x^2 \tilde{\rho}. \] (2.11b)

Analogously, \( S_{\text{opt}} \) in equation (2.5) can be expressed in terms of the rescaled action
\[ \tilde{S}_{\text{opt}} = \int_0^{T} d\tilde{t} \int_0^L dx \tilde{\rho}(x, \tilde{t})^2 \] (2.12)
as
\[ S_{\text{opt}} = \frac{\eta}{D} \tilde{S}_{\text{opt}}, \] (2.13)
with \( T \equiv \eta T \). It is useful to remark that the dimension of \( \eta/D \) is the same as of \( L/M^2 \). Equation (2.13) makes it obvious that the saddle-point solution of the action dominates

\(^4\) Note that standard Dirichlet boundary conditions imply \( \partial_x^2 h(x) = 0 \) for \( x \in \{0, L\} \), as can be inferred from the series representation in equation (B.37).
the dynamics in the weak-noise limit $D \to 0$. We proceed with the analysis of equations (2.11) and (2.12) and henceforth drop the tilde in order to simplify the notation.

2.2. Exact solution

The solution of equation (2.11) subject to the initial and final conditions in equations (1.4) and (1.5) as well as to the boundary conditions in equations (2.3) or (2.4) can be determined exactly (see appendix C) and is summarized below. It turns out that initial and final conditions for $p$ do not have to be specified additionally, but instead implicitly follow from the ones imposed on $h$. Two characteristic regimes can be distinguished: a transient regime, corresponding to first-passage times $T \ll \tau$, and an equilibrium regime, corresponding to $T \gg \tau$. The relaxation time $\tau$ is given by

$$\tau(p) = \left( \frac{L}{2\pi} \right)^z$$  \hspace{1cm} (2.14a)

for periodic and by

$$\tau(D) = \left( \frac{L}{\pi} \right)^z$$  \hspace{1cm} (2.14b)

for Dirichlet boundary conditions. Within WNT, $\tau$ is in fact the characteristic time scale for the creation of a first-passage event. Asymptotically for $T \to \infty$, the profile in the equilibrium regime fulfills equation (2.7).

The optimal action (equation (2.12)) has the following formal scaling property (see appendix C):

$$S_{\text{opt}}(x_M, M, T, L) = \frac{M^2}{L} S_{\text{opt}}(\frac{x_M}{L}, 1, \frac{T}{L^z}, 1).$$  \hspace{1cm} (2.15)

Recalling equations (2.13) and (2.15) accordingly demonstrates that, within WNT, the weak-noise limit $D \to 0$ is equivalent to the limit of large heights $M^2/L \to \infty$. Furthermore, $S_{\text{opt}}$ determines the probability distribution of the first-passage coordinate $x_M$:

$$P_1(x_M) \sim \exp[-S_{\text{opt}}(x_M, M, T, L)],$$  \hspace{1cm} (2.16)

which is assumed to be normalized such that $\int_0^L dx_M P_1(x_M) = 1$. For the purpose of numerical evaluation it is convenient to use the relation $S_{\text{opt}}(x_M, M, T, L) = M^2/[2Q(x_M, T, L)]$, where the function $Q$ is reported in equation (C.29). Figure 2(a) displays $S_{\text{opt}}^{(D)}$ as a function of $x_M$ for Dirichlet boundary conditions in the asymptotic transient ($T \ll \tau(D)$) and equilibrium regimes ($T \gg \tau(D)$). In equilibrium, $S_{\text{opt}}$ generally simplifies to $S_{\text{opt,eq}}$ in equation (2.9). Minimization of $S_{\text{opt,eq}}^{(D)}$ yields (see appendix A)

$$x_M^{(D)} = L/2.$$  \hspace{1cm} (2.17)

Asymptotically for $T \to 0$ one has $S_{\text{opt}} \propto T^{-1/z}$ (see equation (C.57)). Specifically, for $T \to 0$ and Dirichlet boundary conditions, $S_{\text{opt}}^{(D)}$ becomes independent of $x_M$ for $0 < x_M < L$ and diverges for $x_M \in \{0, L\}$. For definiteness, we shall henceforth take for $x_M$ in the...
transient regime the same value as in equation (2.17). In fact, since the short-time profile is strongly localized for $T \to 0$ (see, e.g. figure 5(a)), its shape is independent of the precise value of $x_M$. In figure 2(b), the first-passage distribution in equation (2.16) is illustrated for Dirichlet boundary conditions, $M^2/L = 2$ (in units of $\eta/D$) and various values of $T/\tau^{(D)}$. The curves labeled by $T/\tau^{(D)} = 0$ and $\infty$ pertain to the asymptotic transient and the equilibrium regime, respectively, where $P_1^{(D)}$ is independent of $T$. Upon increasing $M^2/L$, the width of the curves (except the one corresponding to $T/\tau^{(D)} \to 0$) decrease and their peak height increases.

The profile $h(x, t)$ solving equation (2.11) can be brought into the following scaling form:

$$h(x, t, T, M, L) = M \tilde{h} \left( \frac{x}{L}, \frac{t}{\tau}, \frac{T}{\tau^{(D)}} \right),$$

where, for periodic boundary conditions, the scaling function $\tilde{h}$ is given by (see equations (C.34) and (C.35))

$$\tilde{h}^{(p)}(x, t, T) = \frac{1}{Q^{(p)}(T)} \sum_{k=1}^{\infty} \frac{1 - \exp(-2k^2T)}{k^2} \frac{\sinh(k^2t)}{\sinh(k^2T)} \cos(2\pi k(x - 1/2))$$

with

$$Q^{(p)}(T) \equiv \sum_{k=1}^{\infty} \frac{1 - \exp(-2k^2T)}{k^2}.$$
Although $S_{\text{opt}}^{(p)}$ (equation (2.12)) is manifestly independent of $x_M$ owing to translational invariance, for definiteness we choose $x_M^{(p)} = L/2$, which also simplifies the expressions for $h$ somewhat. As a consequence of explicitly enforcing the mass constraint (equation (1.11)) in this case, the zero-mode ($k = 0$) is absent from equations (2.19) and (2.20) (see equation (C.32)). Indeed, since $\int_0^L dx \cos(2\pi k(x/L - 1/2)) = 0$ for $k \geq 1$, the mass vanishes identically for $h^{(p)}$. For Dirichlet boundary conditions, using equation (2.17), one has (see equations (C.36)–(C.39))

$$h^{(D)}(x, t, T) = \frac{1}{Q^{(D)}(T)} \sum_{k = 1, 3, 5, \ldots}^{\infty} \frac{1 - \exp(-2k^2T)}{k^2} \frac{\sinh(k^2t)}{\sinh(k^2T)} \cos(\pi k(x - 1/2)) \tag{2.21}$$

with

$$Q^{(D)}(T) = \sum_{k = 1, 3, 5, \ldots}^{\infty} \frac{1 - \exp(-2k^2T)}{\lambda_k^2}. \tag{2.22}$$

Since $x_M^{(D)} = L/2$, the above sums run only over the odd eigenmodes $k = 1, 3, 5, \ldots$, which have nonzero mass, $\int_0^L dx \sin(k\pi x/L) = L/(k\pi)$ (eigenfunctions for even $k$ have vanishing mass). The general expression for the conjugate field $p(x, t)$ is provided in equation (C.30).

The typical spatio-temporal evolution of $h(x, t)$ is illustrated in figures 3 and 4 for periodic and Dirichlet boundary conditions, respectively. In the equilibrium regime ($T \gg \tau$), the profile at time $t = T \to \infty$ can be readily calculated from equations (2.19) and (2.21) (see equation (C.68) in appendix C.2.2): 

$$h^{(p)}(x, T)|_{T \to \infty}/M = 1 - 6 \left| \frac{x}{L} - \frac{1}{2} \right| + 6 \left( \frac{x}{L} - \frac{1}{2} \right)^2, \tag{2.23a}$$
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The same results are obtained via minimization of the equilibrium action in equation (2.9), using the fact that \( h(x, 0) = 0 \) (see appendix A). For times \( t = T - \delta t < T \) with \( \delta t \ll T \) and \( T \gg \tau \), equation (2.18) adopts a reduced dynamic scaling form (see equation (C.74)):

\[
h^{(D)}(x, T)\bigg|_{T\to\infty} = 1 - \left| 1 - \frac{2x}{L} \right|.
\]

(2.23)

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\[
h(x, T - \delta t)\bigg|_{T\gg\tau} \simeq M - M(\delta t)^{1/z}(1 - 1/z)\mathcal{H}\left(\frac{x - L/2}{\delta t^{1/z}}\right), \quad z = 2,
\]

(2.24)

with the scaling function

\[
\mathcal{H}(\xi) = \exp\left(-\frac{\xi^2}{4}\right) + \frac{1}{2}\sqrt{\pi}\xi\text{erf}\left(\frac{\xi}{2}\right),
\]

(2.25)

both for periodic and Dirichlet boundary conditions. It is convenient to carry along the dynamic index \( z \) (equation (1.12)) in these and the following expressions. Note that \( \eta \) has the same dimension as \( L^z/T \), such that, upon re-instating the unscaled quantities (see equation (2.10)), the argument of \( \mathcal{H} \) in equation (2.24) is seen to be dimensionless.

In the transient regime \( (T \ll \tau) \), the scaling profile at time \( t = T \) is given by (see equation (C.54)):

\[
h(x, T)\big|_{T\ll\tau} = M\mathcal{H}\left(\frac{x - L/2}{(2T)^{1/z}}\right), \quad z = 2,
\]

(2.26)

with the scaling function

\[
\mathcal{H}(\xi) = \exp\left(-\frac{\xi^2}{4}\right) + \frac{1}{2}\sqrt{\pi}\xi\text{erf}\left(\frac{\xi}{2}\right) - 1.
\]

(2.27)
Since there is no risk of confusion, we use the same symbol $\xi$ for the scaling variables in equations (2.25) and (2.27). For times $t = T - \delta t < T$ in the limit $\delta t / T \to 0$ (with $T \ll \tau$), a dynamic scaling profile follows as (see equation (C.61))

$$h(x, T - \delta t)\bigg|_{T < \tau, \delta t < T} = M - M \left(\frac{\delta t}{2T}\right)^{1/z} \tilde{H}\left(\frac{x - L/2}{\delta t^{1/z}}\right), \quad z = 2,$$

(2.28)

with the same scaling function as in equation (2.25). The above scaling profiles are independent of the specific boundary condition and apply for values of the scaling variable $|\xi| \lesssim \mathcal{O}(1)$, i.e. in an ‘inner’ region near the first-passage location $x_M$. The accuracy of the approximations involved in equation (2.28) is further illustrated in figure C2 in appendix C. (A short-time scaling profile for finite nonzero $\delta t \ll T$, which entails a scaling function different from $\tilde{H}$, is provided in equation (C.59).) Note that the final profile in the transient regime (equation (2.26)) still depends on $T$ via the scaling variable $\xi$, whereas the final profile in the equilibrium regime (equation (2.23)) is independent of $T$ for $T \gg \tau$. We remark that, in contrast to the exact expression in equation (2.19), $h^{(p)}$ as given in equation (2.26) has nonzero mass. This, however, constitutes a negligible error in the asymptotic limit $T \to 0$, as the profile becomes sharply peaked. The final profiles in the transient and the equilibrium regime are illustrated in figure 5.

According to equations (2.24) and (2.28) the maximum $h(x_M, t)$ of the profile approaches the height $M$ at the first-passage time $T$ via a power law,

$$1 - h(x_M, T - \delta t)/M \propto \delta t^{1/z}, \quad z = 2.$$

(2.29)

This behavior applies both in the transient and the equilibrium regime and is independent of the boundary conditions. If the system considered can accommodate only a finite number of modes—which, for instance, is the case when equations (1.1) and (1.2) are discretized on a lattice—the sums in equations (2.19) and (2.22) are bounded.

Figure 5. Asymptotic first-passage profiles $h(x, t = T)$ (normalized by $M$) obtained within WNT of the EW equation (1.1) in (a) the transient regime, $T \to 0$ (equation (2.26)), and (b) the equilibrium regime, $T \to \infty$ (equation (2.23)). In the transient regime, the profiles depend on the scaling variable $\xi \equiv (x - L/2)/(2T)^{1/2}$ and are identical for periodic and Dirichlet boundary conditions. In the equilibrium regime, the (normalized) profile is a function of $x/L$ and is specific to each boundary condition.
by a largest mode $k_x$. In this case, equation (2.29) is eventually superseded by a linear behavior,

$$1 - h(x_M, T - \delta t)/M \propto \delta t$$

for $\delta t \lesssim \tau_x = \tau/k_x$.

where $\tau_x$ denotes the corresponding cross-over time (see equation (C.76)). The time evolution of the peak $h(x_M, t)$ is illustrated in figure 6, where the time is rescaled by the characteristic relaxation time $\tau$ in equation (2.14). Note that, in the equilibrium regime, the evolution of the profile towards the first-passage event happens on a time scale of $\tau$, independently from the value of $T$. For times $t \ll T - \tau$ the equilibrium profile thus remains near its initial configuration (equation (1.5); see also panels (b) in figures 3 and 4). In the transient regime (dash-dotted lines in figure 6 and panels (a) in figures 3 and 4), the evolution proceeds over the whole time interval between 0 and $T$ (where, however, $T \ll \tau$).

According to equation (2.29), the distance $M$ is traversed within a time $\delta t^{1/2}$. Consequently, the requirement $\delta t \ll \tau$ for the transient regime implies $M/L \ll 1/(c\pi)$, with $c^{(p)} = 2$ and $c^{(D)} = 1$ (see equation (2.14)). Hence, in the transient regime, the weak-noise limit of equation (2.15) is obtained if $L \gg \frac{D}{\eta} \left( \frac{L}{M} \right)^2 \gg \frac{D}{\eta} (c\pi)^2$, where we re-instated all dimensional factors. Conversely, the equilibrium regime is realized if $M/L \gg 1/(c\pi)$, such that in this case the weak-noise limit requires $L \gg \frac{D}{\eta} \left( \frac{L}{M} \right)^2$ and $\left( \frac{L}{M} \right)^2 \ll (c\pi)^2$.

Figure 6. Time evolution of the peak of the profile, $h(x_M, t)$, which reaches the height $M$ at the first-passage time $T$, for the EW equation as a function of $T - t$. The solid curves correspond to $h^{(p,D)}(x_M, t)$ in the equilibrium regime, while the dash-dotted curves illustrate the time evolution of $h^{(p)}(x_M, t)$ in the transient regime for $T/\tau = 10^{-1}, 10^{-2}, 10^{-3}$ (from bottom to top). The corresponding behavior of $h^{(D)}(x_M, t)$ for $t \ll T$ is similar and not shown. Both in the transient and the equilibrium regime, a power law $M - h(x_M, T - \delta t) \propto \delta t^{1/2}$ is predicted (see equation (2.29)). In the presence of an upper bound to the number of (eigen-) modes in the system, a linear behavior in $\delta t$ emerges for times $\delta t \lesssim \tau_x = \tau/k_x$ (see equation (2.30)), where $k_x$ is the largest mode index ($k_x = \infty$ in the continuum limit). For illustrative purposes, we have chosen here $k_x = 1000$, corresponding to $\tau_x/\tau \approx 10^{-6}$. The fundamental time scale $\tau$ is defined in equation (2.14) for the respective boundary conditions.
3. Mullins-Herring dynamics

We now turn to the optimal first-passage dynamics emerging from the MH equation. The analysis in this section proceeds in essentially the same fashion as for the EW equation in section 2. However, at the expense of some redundancy, the subsequent discussion is kept largely self-contained.

3.1. Macroscopic fluctuation theory

The Martin–Siggia–Rose action pertaining to the stochastic MH (equation (1.2)) is given by [38]

$$S[h, p] = \int_0^T dt \int_0^L dx \, p [\partial_t h + \eta \partial_x^4 h + D \partial_x^2 p].$$  \hfill (3.1)

The Euler–Lagrange equations describing the most-likely path of the profile $h$ and of the conjugate field $p$ follow as (see also [42])

$$0 = \frac{\delta S}{\delta h} = \partial_t h + \eta \partial_x^4 h + 2D \partial_x^2 p,$$  \hfill (3.2a)

$$0 = \frac{\delta S}{\delta p} = -\partial_t p + \eta \partial_x^4 p.$$  \hfill (3.2b)

We consider either periodic boundary conditions (equation (1.6)),

$$h^{(p)}(x, t) = h^{(p)}(x + L, t), \quad p^{(p)}(x, t) = p^{(p)}(x + L, t),$$  \hfill (3.3)

or Dirichlet boundary conditions with a no-flux condition (equations (1.7) and (1.8)),

$$h^{(D)}(0, t) = h^{(D)}(L, t), \quad \partial_x^3 h^{(D)}(0, t) = \partial_x^3 h^{(D)}(L, t).$$  \hfill (3.4)

In the latter case, the bi-harmonic operator $\partial_x^4$ is not self-adjoint on $[0, L]$, which renders the solution of equation (3.2) technically more involved than in the self-adjoint case (see appendix C). If Dirichlet no-flux boundary conditions are imposed on $h$, the conjugate field $p$ must fulfill the associated adjoint boundary conditions (see appendix B)

$$\partial_x p^{(D)}(0, t) = 0 = \partial_x p^{(D)}(L, t), \quad \partial_x^3 p^{(D)}(0, t) = 0 = \partial_x^3 p^{(D)}(L, t).$$  \hfill (3.5)

The mass-conserving property of the noise in equation (1.2) is reflected by the presence of a derivative of $p$ in equation (3.2a). Indeed, it is readily proven that the considered boundary conditions ensure conservation of the mass (equation (1.11)). Initial and final conditions on the profile $h$ are given in equations (1.4) and (1.5) and suffice to determine also the conjugate field $p$. Inserting equations (3.2) into (3.1) renders the optimal action

$$S_{opt} = -D \int_0^T dt \int_0^L dx \, p \partial_x^2 p,$$  \hfill (3.6)

which describes the most-likely activation dynamics [29, 42].

As was the case for the EW equation (see section 2.1), equation (3.2) admits, as a manifestation of the Onsager–Machlup time-reversal symmetry [65], a specific solution

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corresponding to thermal equilibrium. In fact, consider a profile \( h_r(x,t) \) obeying the (deterministic) fourth-order diffusion equation

\[
\partial_t h_r = -\eta \partial_x^4 h_r, \tag{3.7}
\]

with the initial condition \( h_r(x,t=0) = h_0(x) \), where \( h_0(x) \) is a given profile (e.g. \( h_0(x) = h(x,T \rightarrow \infty) \), where \( h(x,T \rightarrow \infty) \) is a known first-passage profile). Then the fields \( h, p \) defined by

\[
h(x,t) = h_r(x,T-t) \quad \text{and} \quad p(x,t) = -\frac{\eta}{D} \partial_x^2 h
\]

fulfill the relations

\[
\partial_t h = -\partial_t h_r = \eta \partial_x^4 h = -\eta \partial_x^4 h - 2D \partial_x^2 h
\]

as well as \( \partial_t p = -(\eta/D) \partial_t \partial_x^2 h = -(\eta^2/D) \partial_x^4 h_r = \eta \partial_x^4 p \), which coincide with equations (3.2a) and (3.2b), respectively. Accordingly, the fields defined in equations (3.8) solve (3.2) subject to the final condition \( h(x,T) = h_0(x) \). Equation (3.8a) implies that \( h(x,t=0) = h_0(x,T) \), which is generally nonzero for non-vanishing \( h_0(x) \) and finite \( T \). Hence, only for \( T \rightarrow \infty \), equilibrium dynamics is strictly compatible with the initial condition in equation (1.5). Using equations (3.8) and (3.9) in (3.6) renders the equilibrium action:

\[
S_{\text{opt,eq}} = -\frac{\eta^2}{D} \int_0^T dt \int_0^L dx \partial_x^2 (\partial_x^2 h)(\partial_x^2 h) = \int_0^T dt \left[ -\frac{\eta}{D} (\partial_x h)(\partial_x h) \right]_0^L + \frac{\eta^2}{D} \int_0^L dx (\partial_x^2 h)(\partial_x^2 h)
\]

\[
= -\frac{\eta}{D} \left[ h(\partial_x h) \right]_0^L + \frac{\eta^2}{D} \int_0^T dt h(\partial_x^2 h) \right]_0^L + \frac{\eta^2}{D} \int_0^L dx (\partial_x^2 h)(\partial_x^2 h)
\]

\[
= \frac{\eta}{D} \left[ \int_0^L dx (\partial_x^2 h)^2 \right]_0^L - \int_0^T dt \int_0^L dx (\partial_x^2 h)(\partial_x h)
\]

\[
= \frac{\eta}{2D} \int_0^L dx (\partial_x^2 h)^2 \right]_0^T,
\]

(3.10)

where we made use of the fact that the boundary terms vanish for the boundary conditions in equations (3.3) and (3.4). In equation (3.10) the temperature \( \Theta \) can be identified via \( \eta/(2D) = 1/(4\Theta) \). As expected, the final expression in equation (3.10) coincides with the one in equation (2.9) and shows that, in thermal equilibrium, the action essentially reduces to a free energy difference.

Upon rescaling time by \( \eta \) and redefining the fields \( h \) and \( p \) as in equation (2.10), equation (3.2) becomes

\[
\partial_t h = -\partial_x^4 h - 2\partial_x^2 p, \tag{3.11a}
\]

\[
\partial_t p = \partial_x^4 p. \tag{3.11b}
\]

We henceforth consider also \( S_{\text{opt}} \) to be rescaled as in equation (2.13) and proceed by analyzing equation (3.11).
3.2. Exact solution

The exact analytic solution of equation (3.11) subject to the the initial and final conditions in equations (1.4) and (1.5) as well as to the boundary conditions in equations (3.3) or (3.4) is determined in detail in appendix C and summarized below. The characteristic time scale for the creation of a rare event is given by (\( z = 4 \))

\[
\tau^{(p)} = \left( \frac{L}{2\pi} \right)^z
\]

(3.12a)

for periodic and by

\[
\tau^{(D')} = \left( \frac{L}{\omega_1} \right)^z
\]

(3.12b)

for Dirichlet no-flux boundary conditions, respectively, where \( \omega_1 \simeq 4.73 \) is the smallest positive solution of the eigenvalue equation \( \cos(\omega) \cosh(\omega) = 1 \) (see equation (B.18)). As was the case for the EW equation, the dynamics emerging from equation (3.11) is distinct in the transient \(( T \ll \tau )\) and the equilibrium \(( T \gg \tau )\) regime. In the latter case, equation (3.8) applies.

Analogously to equation (2.15), the optimal action (see equations (3.6) and (C.31); expressed in units of \( \eta/D \)) fulfills the formal scaling property

\[
S_{\text{opt}}(x_M, M, T, L) = \frac{M^2}{L} S_{\text{opt}} \left( \frac{x_M}{L}, 1, \frac{T}{Lz}, 1 \right).
\]

(3.13)

The value of the first-passage location \( x_M \) (see equation (1.4)) follows from minimizing \( S_{\text{opt}} \) evaluated on the general solution in equation (3.11). For periodic boundary conditions, one may simply set \( x_M^{(p)} = L/2 \) owing to translational invariance. For Dirichlet no-flux boundary conditions, the optimal action \( S_{\text{opt}}^{(D')} \) is shown as a function of \( x_M \) in figure 7(a). Figure 7(b) displays the corresponding (normalized) probability distribution of the first-passage location \( x_M \).

\[
\mathcal{P}_1(x_M) \sim \exp \left[ -S_{\text{opt}}(x_M, M, T, L) \right].
\]

(3.14)

For illustrative purposes, we have chosen \( M^2/L = 1 \) (in units of \( \eta/D \)) in the plot, and remark that, upon increasing \( M^2/L \), the peak height of the distribution grows and, correspondingly, its characteristic width decreases—except in the limit \( T \to 0 \), where the form of \( \mathcal{P}_1 \) is invariant. In the equilibrium regime \(( T \gg \tau )\), \( S_{\text{opt}} \) and hence also \( \mathcal{P}_1(x_M) \) are generally independent of \( T \) (see inset to figure 7(a)). For \( T \to \infty \), \( S_{\text{opt}} \) reduces to the expression in equation (3.10), which can be evaluated analytically (see appendix A). In the case of Dirichlet no-flux boundary conditions, \( S_{\text{opt,eq}} \) is minimal for the two values (see equation (A.14))

\[
x_M^{(D')} |_{T \gg \tau^{(D')}} = \frac{L}{2} \left( 1 \pm \frac{1}{\sqrt{3}} \right).
\]

(3.15)

Accordingly, \( \mathcal{P}_1^{(D')} \) shows two peaks, the sharpness of which increases with growing \( M \) according to equation (3.13). Asymptotically for \( T \to 0 \), \( S_{\text{opt}} \) scales \( \propto T^{-1/z} \), independently of the boundary conditions (see equation (C.57)). Furthermore, \( S_{\text{opt}}^{(D')} \) becomes
independent of \(x_M\) for \(0 < x_M < L\). The corresponding distribution \(P_1^{(D')}\) is thus flat and independent of \(M\) and \(T\) in this limit. One may therefore set \(x_M = L/2\) in order to evaluate the first-passage profile in this case. As illustrated in figure 7(b), \(P_1^{(D')}\) assumes rather intricate shapes between its asymptotic transient and equilibrium limits. In particular, as \(T/\tau^{(D')}\) grows from small values, \(P_1^{(D')}\) develops a pronounced peak in the central region. For \(T/\tau^{(D')} \gtrsim 0.1\), this peak diminishes while two maxima grow near the locations given in equation (3.15).

The profile solving equation (3.11) can be written in scaling form,

\[
h(x, t, T, M, L) = M h\left(\frac{x}{L}, \frac{t}{\tau}, \frac{T}{\tau}\right),
\]

where, for periodic boundary conditions (setting \(x_M = L/2\)) the dimensionless scaling function \(h\) is given by (see equations (C.34) and (C.35))

\[
h^{(p)}(x, t, T) = \frac{1}{Q^{(p)}(T)} \sum_{k=1}^{\infty} \frac{1 - \exp(-2k^4T)}{k^2} \frac{\sinh(k^4t)}{\sinh(k^4T)} \cos(2\pi k(x - 1/2))
\]

with

\[
Q^{(p)}(T) \equiv \sum_{k=1}^{\infty} \frac{1 - \exp(-2k^4T)}{k^2}.
\]

These expressions have been previously obtained in [29]. For Dirichlet no-flux boundary conditions, keeping \(x_M \equiv x_M/L\) general here, one has (see equations (C.36) and (C.37))
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\[ h^{(D)}(x, t, T) = \frac{1}{Q^{(D)}(T)} \sum_{k=1}^{\infty} \frac{1 - \exp \left( -2(\omega_k/\omega_1)^4 T \right) \sinh \left( (\omega_k/\omega_1)^4 t \right) \tilde{\sigma}^{(D)}_k(x_M) \tilde{\sigma}^{(D)}_k(x)}{\omega_k^2 \kappa_k} \]

with

\[ Q^{(D)}(T) \equiv \sum_{k=1}^{\infty} \left| \tilde{\sigma}^{(D)}_k(x_M) \right|^2 \frac{1 - \exp \left( -2(\omega_k/\omega_1)^4 T \right)}{\omega_k^2 \kappa_k}. \]

Here, \( \tilde{\sigma}^{(D)}_k(x) \equiv \sigma^{(D)}_k(xL) \) and the eigenfunctions \( \sigma^{(D)}_k \) are reported in equation (B.24) (see also equation (C.36) and table C1); furthermore \( \kappa_k = [1 - (-1)^k / \cosh(\omega_k)] / 3 \) and \( \omega_k \) denotes the \( k \)th positive solution of the equation \( \cos(\omega) \cosh(\omega) = 1 \) (see equation (B.19)). Since \( \int_0^L dx \cos(2\pi k(x/L - 1/2)) = 0 \) for \( k \geq 1 \), the profile for periodic boundary conditions in equation (3.17) exactly fulfills mass conservation (equation (1.11)). Note that, in contrast to the EW case, this property is not enforced explicitly (see equation (1.10)) but follows readily from the fact that equation (1.2) conserves \( \mathcal{H} \) locally. Global mass conservation applies, by construction, also to the profile for Dirichlet no-flux boundary conditions in equation (3.19) (see equation (B.27)). The general expression for the conjugate field \( p \) is reported in equation (C.30).

The spatio-temporal evolution of the optimal profile for periodic and Dirichlet no-flux boundary conditions is illustrated in figures 8 and 9, respectively. (For completeness, in figure C1 in appendix C also the profile obtained for the MH equation with standard Dirichlet boundary conditions is discussed.) In contrast to the EW equation, the transient first-passage profiles emerging from the MH equation show an oscillatory decay in space (see panels (a) of figures 8 and 9). In the equilibrium regime, the first-passage profile generally develops on a time scale of \( \mathcal{O}(\tau) \). In the case of periodic boundary conditions, the time-dependent equilibrium profiles are qualitatively similar for EW and MH dynamics (compare panels (b) of figures 3 and 8).

For \( T \gg \tau \), the profile at time \( t = T \) minimizes the equilibrium action \( \mathcal{S}_{\text{opt,eq}} \) (equation (3.10)). Since the latter quantity is independent of the specific dynamics, the

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expression for the profile $h^{(p)}(x, T)|_{T \to \infty}$ subject to periodic boundary conditions coincides with the one in equation (2.23a). Alternatively, it can be directly derived from the expression in equation (3.17) (see equation (C.68a)). In contrast to standard Dirichlet boundary conditions (see equation (2.23b) as well as figure C1 in appendix C), for Dirichlet no-flux boundary conditions one has to additionally take into account the constraint of zero mass (equation (1.11)) in the minimization of $S_{\text{opt,eq}}$. Accordingly, using the fact that $h(x, 0) = 0$, one obtains (see equations (A.15) and (A.16))

$$h^{(D')}_{(x, T \to \infty)}/M = h^{(p)}(x + L/2 - x_M, T \to \infty)/M = \begin{cases} 6 \xi \left( \frac{x}{L} + \frac{1}{\sqrt{14}} \right), & x \leq x_M^{(D')} \\ 6 \left( \frac{x}{L} - 1 \right) \left( \frac{x}{L} - 1 + \frac{1}{\sqrt{14}} \right), & x > x_M^{(D')} \end{cases}$$

(3.21)

with $x_M^{(D')}$ given in equation (3.15) and the last expression in equation (3.21) applying to the smaller of the two possible values of $x_M^{(D')}$. Note that, while, at the time $t = T$, $h^{(D')}$ can be expressed in terms of $h^{(p)}$, this is not possible at arbitrary times $t < T$, as, e.g., a close inspection of figures 8(b) and 9(b) near $h \approx 0$ reveals. In the equilibrium regime for nonzero but small time differences $\delta t \equiv T - t \ll T$, equation (3.16) can be cast into a dynamic scaling form (see equation (C.74)):

$$h(x, T - \delta t)|_{T \gg \tau} \approx M - M(\delta t)^{1/z} \Gamma(1 - 1/z) \hat{\mathcal{H}} \left( \frac{x - x_M}{\delta t^{1/z}} \right), \quad z = 4,$$

(3.22)

with the scaling function

$$\hat{\mathcal{H}}(\xi) = _1F_3 \left( -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}; \frac{3}{256} \xi^4 \right) + \xi^2 \frac{\Gamma \left( \frac{1}{4} \right)}{8 \Gamma \left( \frac{3}{4} \right)} _1F_3 \left( \frac{1}{4}, \frac{5}{4}, \frac{3}{4}; \frac{3}{256} \xi^4 \right).$$

(3.23)

We recall that, in terms of the unscaled time variable, the argument of $\hat{\mathcal{H}}$ in equation (3.22) is given by $(x - x_M)/(\eta \delta t)^{1/z}$, which is dimensionless since $\eta$ and $L^z/T$ have

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the same dimensions. Asymptotically for $T \to 0$ in the transient regime, the profile at time $t = T$ is given by (see equation (C.54)):

$$h(x, T)|_{T \ll \tau} = M\mathcal{H}\left(\frac{x - L/2}{(2T)^{1/z}}\right), \quad z = 4,$$

(3.24)

with the scaling function

$$\mathcal{H}(\xi) = {}_1F_3\left(-\frac{1}{4}; \frac{1}{4}, \frac{3}{4}; \frac{\xi^4}{256}\right) + \frac{\xi^2}{8\Gamma\left(\frac{1}{4}\right)} {}_1F_3\left(-\frac{1}{4}; \frac{3}{4}, \frac{5}{4}; \frac{\xi^4}{256}\right) - \frac{\pi}{2\Gamma\left(\frac{3}{4}\right)} |\xi|.$$

(3.25)

For nonzero time differences $\delta t = T - t$ in the transient regime, a dynamic scaling profile follows at leading order in $\delta t/T \ll 1$ as (see equation (C.61))

$$h(x, T - \delta t)|_{T \ll \tau, \delta t \ll T} = M - M\left(\frac{\delta t}{2T}\right)^{1/z}\mathcal{H}\left(\frac{x - L/2}{\delta t^{1/z}}\right), \quad z = 4,$$

(3.26)

where the scaling function takes the same form as in equation (3.23). The scaling forms in equations (3.22), (3.24) and (3.26) apply to both periodic and Dirichlet boundary conditions and are valid for values of the scaling variable $|\xi| \lesssim O(1)$. (A comparison of the approximative profile in equation (3.26) with the exact one is provided in figure C2 in appendix C, while a scaling form improving equation (3.26) beyond leading order in $\delta t/T$ is reported in equation (C.59).) In the case of periodic boundary conditions, the expressions in equations (2.23a) and (3.25) have been previously obtained in [29].

Note that the static profile in the transient regime (equation (3.24)) still depends on $T$ via the scaling variable $\xi$, whereas the static profile in the equilibrium regime (equations (2.23a) and (3.21)) is independent of $T$ for sufficiently large $T$. The scaling profiles in equations (3.24) and (3.26) have (in contrast to the exact solution in equation (3.17)) nonzero mass (equation (1.9)), which, however, constitutes a negligible error in the asymptotic limit $T \to 0$, where the profiles become sharply peaked. The profiles at time $t = T$ in the transient and the equilibrium regime are illustrated in figure 10.

According to equations (3.22) and (3.26), noting that $\tilde{\mathcal{H}}(0) = 1$, the peak $h(x_M, t)$ of the profile approaches the maximum height $M$ via a power law

$$1 - h(x_M, T - \delta t)/M \propto \delta t^{1/z}, \quad z = 4.$$

(3.27)

This behavior applies to a continuum system both in the transient and the equilibrium regime and is independent of the specific boundary conditions. If, due to a microscopic cutoff, the mode spectrum of the system is bounded from above, equation (3.27) crosses over to a linear law,

$$1 - h(x_M, T - \delta t)/M \propto \delta t \quad \text{for} \quad \delta t \lesssim \tau_x,$$

(3.28)

where $\tau_x$ is the crossover time. For periodic boundary conditions, $\tau_x = \tau^{(p)}/k_x^z$ while for Dirichlet no-flux boundary conditions, $\tau_x^{(D')} = \tau^{(D')}\left(\omega_1/\omega_{k_x}\right)^z$, where $k_x$ is the maximum mode index and $\omega_k$ denotes the eigenvalues in equation (B.19). The time evolution of $h(x_M, t)$ is illustrated in figure 11, where the time is rescaled by the characteristic relaxation time $\tau$ defined in equation (3.12). As noted previously, in the equilibrium regime, the actual evolution of the profile towards the maximum occurs within a time interval $\tau$ before $T$. In the case of Dirichlet no-flux boundary conditions, the intermediate
asymptotic regime described by equation (3.27) is seen to be of somewhat smaller size than for periodic boundary conditions. In the transient regime, a condition determining the weak-noise limit of equation (3.13) follows from equation (3.27) as $L \gg \frac{P}{\eta} \left( \frac{\delta t}{M} \right)^{2} \gg \frac{P}{\eta} \omega_{1}^{2}$, with $\omega_{1}^{(p)} = 2\pi$ and $\omega_{1}^{(D')} = 4.73$ (see equation (3.12b)). In contrast, in the equilibrium regime, the weak-noise limit is realized for $L \gg \frac{P}{\eta} \left( \frac{\delta t}{M} \right)^{2}$ and $\left( \frac{\delta t}{M} \right)^{2} \ll \omega_{1}^{2}$.
4. Summary

In the present study, first-passage events of a one-dimensional interfacial profile \( h(x, t) \), subject to the Edwards–Wilkinson (EW) or the (stochastic) Mullins-Herring (MH) equation, have been investigated analytically. The approach here is based on the weak-noise approximation of a Martin–Siggia–Rose/Janssen/de Dominicis path integral formulation of the corresponding Langevin equations (equations (1.1) and (1.2)) [29, 38–41, 43]. A comparison to numerical solutions of the EW and MH equation beyond the weak-noise approximation will be provided in a separate paper. Minimization of the associated action yields the most-probable (‘optimal’) profile which, starting from a flat initial configuration (equation (1.5)), realizes the first-passage event \( h(x_M, T) = M \) at a specified time \( T \) and a location \( x_M \). Note that here the rare event dynamics is purely fluctuation-induced, i.e. there is no deterministic driving force involved—e.g. the classical problem [66] of determining noise-activated transitions between energy minima.

The first-passage problem of the MH equation for periodic boundary conditions has been studied previously in [29]. Extending that work, here we have investigated the influence of various boundary conditions on the spatio-temporal evolution of the optimal profile and discussed in detail its dynamic scaling behavior. Since the optimal profile is provided here in terms of a generic eigenfunction expansion (see appendix C), the corresponding expressions can be readily specialized to other boundary conditions. We point out that, in order to ensure mass conservation (equation (1.9)) for the MH equation with Dirichlet boundary conditions, a no-flux condition must be imposed (see equations (1.7) and (1.8)). This renders the solution of the corresponding WNT technically involved, as the bi-harmonic operator is not self-adjoint anymore. Standard Dirichlet boundary conditions, instead, do not conserve mass and are studied here mainly in conjunction with the EW equation.

The ensuing rare event dynamics is phenomenologically distinct for first-passage times \( T \ll \tau \) and \( T \gg \tau \), corresponding to the transient (non-equilibrium) and the equilibrium regime, respectively. \( \tau \) denotes the fundamental relaxation time of the model, which coincides with the characteristic time scale for the evolution of the first-passage event. In the equilibrium regime, the optimal profile at time \( t = T \) minimizes the equilibrium action and depends sensitively on the boundary conditions as well as on possible conservation laws. In contrast, in the transient regime, boundary conditions and mass conservation have a negligible influence and the optimal profile is strongly localized. In fact, in the transient regime, the profile shape close to the first-passage event (i.e. for \( t \to T \)) depends only on the type of bulk dynamics. The peak of the profile is predicted to approach the first-passage height \( M \) algebraically in time, \( M - h(x_M, t) \propto (T - t)^\alpha \), with an exponent \( \alpha = 1/z \), where \( z = 2 \) for the EW and \( z = 4 \) for the MH equation. Notably, this behavior applies both in the transient and the equilibrium regimes and is independent of the specific boundary conditions or conservation laws.

Appendix A. Equilibrium profiles

Here, we determine static profiles \( h(x) \) \((0 \leq x \leq L)\) which minimize the equilibrium action (see equations (2.9) and (3.10))

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\[ S_{eq}[h] = \frac{\eta}{2D} \int_0^L dx \left[ \partial_x h(x) \right]^2, \quad (A.1) \]

under the constraint of attaining a maximum height \( M \) at a certain location \( x_M \):

\[ M = h(x_M). \quad (A.2) \]

In certain cases, we additionally impose a mass constraint:

\[ \mathcal{A} = \int_0^L dx \, h(x). \quad (A.3) \]

The profile \( h \) is furthermore required to fulfill either periodic boundary conditions,

\[ h(x) = h(x + L), \quad (A.4a) \]

or Dirichlet boundary conditions,

\[ h(0) = 0 = h(L). \quad (A.4b) \]

Introducing Lagrange multipliers \( \lambda \) and \( \beta \), we obtain the augmented action

\[ \tilde{S}_{eq}(h, \lambda, \beta) \equiv S_{eq}[h] - \lambda \left[ \int_0^L dx \, h(x) - \mathcal{A} \right] - \beta \left[ \int_0^L dx \, h(x) \delta(x - x_M) - M \right], \quad (A.5) \]

the minimization of which results in the Euler–Lagrange equation

\[ 0 = \frac{\delta \tilde{S}_{eq}}{\delta h} = \frac{\eta}{D} \partial_x^2 h + \lambda + \beta \delta(x - x_M). \quad (A.6) \]

We remark that integration of equation (A.6) over an infinitesimal interval centered at \( x_M \) yields the relation \( h'(x_M^+) - h'(x_M^-) = \beta \), which, however, is not needed to determine the constrained profile. Instead, equation (A.6) is solved separately in the domains \( x \leq x_M \), subject to the boundary conditions in equation (A.4) and the requirement of continuity at \( x_M \) (see equation (A.2)), i.e.

\[ h(x_M^+) = h(x_M^-) = M. \quad (A.7) \]

Subsequently, the mass constraint in equation (A.3) is imposed. The expressions for the constrained profiles turn out to be independent of the factor \( \eta/2D \) present in equation (A.1).

For \( \mathcal{A} = 0 \) and periodic boundary conditions, setting \( x_M = L/2 \), one obtains the constrained profile [29]

\[ h^{(p)}(x)/M = 1 - 6 \left| \frac{x}{L} - \frac{1}{2} \right| + 6 \left( \frac{x}{L} - \frac{1}{2} \right)^2. \quad (A.8) \]

For Dirichlet zero-\( \mu \) boundary conditions (see appendix B.1.3), we do not enforce the mass constraint (equation (A.3)). Accordingly, the Lagrange multiplier \( \lambda \) is absent and one simply solves \( 0 = \partial_x^2 h \), subject to equations (A.2) and (A.4b), in each domain. The resulting solution still depends on \( x_M \); the associated action, which is displayed in figure 2(a) in the main text, follows as

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\[ \frac{2D}{\eta} \frac{L}{M^2} S_{\text{eq}}^{(D)}(x_M) = \frac{1}{\zeta_M} + \frac{1}{1 - \zeta_M}, \quad \text{with} \quad \zeta_M \equiv x_M/L. \quad (A.9) \]

\( S_{\text{eq}}^{(D)} \) is minimal for a value of \( x_M^{(D)} = \frac{L}{2} \), which finally leads to the constrained profile

\[ h^{(D)}(x)/M = 1 - \left| 1 - \frac{2x}{L} \right|. \quad (A.10) \]

For Dirichlet no-flux boundary conditions, instead, the mass constraint is respected and, for \( A = 0 \), one obtains

\[ h^{(D')}(x; x_M)/M = \begin{cases} \frac{\zeta(1+3\zeta M(\zeta-1))}{\zeta M(1+3\zeta M(\zeta-1))}, & x \leq x_M, \\ h^{(D)}(L-x, L-x_M), & x > x_M, \end{cases} \quad (A.12) \]

with \( \zeta \equiv x/L \) and \( \zeta_M \equiv x_M/L \). Inserting equations (A.12) into (A.1) results in

\[ \frac{2D}{\eta} \frac{L}{M^2} S_{\text{eq}}^{(D')}(x_M) = \frac{1}{\zeta_M} + \frac{1}{1 - \zeta_M} + \frac{3}{1 + 3\zeta M(\zeta_M - 1)}, \quad (A.13) \]

which is illustrated in figure 7. This free energy has two symmetric minima, located at

\[ x_M^{(D')} = \frac{L}{2} \left( 1 \pm \frac{1}{\sqrt{3}} \right). \quad (A.14) \]

The resulting optimal profile for Dirichlet no-flux boundary conditions and \( A = 0 \) is related to be \( h^{(p)} \) (equation (A.8)) via

\[ h^{(D')}(x) = h^{(p)}(x + L/2 - x_M^{(D')}). \quad (A.15) \]

Specifically, upon choosing the smaller value for \( x_M^{(D')} \), one obtains

\[ h^{(D')}(x)/M = \begin{cases} 6\zeta \left( \zeta + \frac{1}{\sqrt{3}} \right), & x \leq x_M^{(D')}, \\ 6(\zeta - 1) \left( \zeta - 1 + \frac{1}{\sqrt{3}} \right), & x > x_M^{(D')} \end{cases} \quad (A.16) \]

Note that, since the above constrained profiles are polynomials of at most second order, one has \( \partial^n h(x) = 0 \) for \( n \geq 3 \) in each domain \( x \leq x_M \), such that no-flux boundary conditions (see equation (1.8)) are indeed fulfilled by \( h^{(D')} \). In passing, we mention that, in the context of dewetting of thin films, related free-energy minimizing profiles have been considered in [45–48].

**Appendix B. Eigenvalue problem for the Mullins-Herring equation**

Consider the noiseless MH equation,

\[ \partial_t h(x, t) = -\partial_x^2 h(x, t), \quad (B.1) \]

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on the interval \([0,L]\) with

- periodic: \(h(x,t) = h(x + L,t)\), \((B.2a)\)
- Dirichlet: \(h(0, t) = h(L, t)\), \((B.2b)\)
- or Neumann: \(\partial_x h(0, t) = \partial_x h(L, t)\), \((B.2c)\)

boundary conditions. The separation ansatz

\[ h(x,t) = \sigma(x) \psi(t) \]  \((B.3)\)

leads to

\[ \partial_t \psi(t) = -\gamma \psi(t), \] \((B.4a)\)

\[ \partial^4_x \sigma(x) = \gamma \sigma(x), \] \((B.4b)\)

with a constant \(\gamma \geq 0\). While equation \((B.4a)\) is solved by

\[ \psi(t) \sim e^{-\gamma t}, \] \((B.5)\)

the general solution of the eigenvalue equation \((B.4b)\) is given by

\[ \sigma(x) = c_1 e^{x \gamma^{1/4}} + c_2 e^{-x \gamma^{1/4}} + c_3 \sin(x \gamma^{1/4}) + c_4 \cos(x \gamma^{1/4}) \] \((B.6)\)

with constants \(c_n\), which are determined below for the specific boundary conditions.

To proceed, it is useful to introduce the free energy functional

\[ F[h] = \int_0^L dx (\partial_x h)^4 \]

and the associated chemical potential \(\mu \equiv \delta F/\delta h = -\partial_x^2 h\), which allows one to rewrite equation \((B.1)\) as a ‘gradient-flow’ equation \([67]\):

\[ \partial_t h = \partial_x^2 \frac{\delta F}{\delta h} = \partial_x^2 \mu = -\partial_x [-\partial_x \mu]. \] \((B.7)\)

In the last step we have identified \(-\partial_x \mu\) as the flux, such that equation \((B.7)\) takes the form of a continuity equation. Being a fourth order differential equation \((B.1)\) requires two additional conditions on \(h\) beside those specified in equation \((B.2)\). Here, one typically chooses either a vanishing chemical potential at the boundaries:

\[ \mu(0, t) = 0 = \mu(L, t) \quad \Leftrightarrow \quad \sigma''(0) = 0 = \sigma''(L), \] \((B.8)\)

or a vanishing flux:

\[ \partial_x \mu(0, t) = 0 = \partial_x \mu(L, t) \quad \Leftrightarrow \quad \sigma'''(0) = 0 = \sigma'''(L). \] \((B.9)\)

In contrast to the zero-chemical potential boundary conditions in equation \((B.8)\), no-flux boundary conditions ensure mass conservation for the MH equation in a finite domain.

The type of boundary condition determines whether the operator \(\partial_x^4\) is self-adjoint on the interval \([0,L]\) (see, e.g. \([68–70]\)). Since, for two arbitrary functions \(\sigma(x)\) and \(\varphi(x)\), one has

\[ \int_0^L dx \sigma^{(4)}(x) \varphi(x) = [\sigma \varphi''']_0^L - [\sigma' \varphi'']_0^L + [\sigma'' \varphi']_0^L - [\sigma''' \varphi]_0^L + \int_0^L dx \sigma(x) \varphi^{(4)}(x), \] \((B.10)\)

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the operator $\partial_x^4$ is self-adjoint only if both $\sigma$ and $\varphi$ fulfill either (i) periodic boundary conditions (equation (B.2a)), (ii) Dirichlet zero-chemical potential boundary conditions (equations (B.2b) and (B.8)), or (iii) Neumann no-flux boundary conditions (equations (B.2c) and (B.9)). In these cases, the eigenfunctions $\sigma_m$ defined by equation (B.4b), with $m \in \mathbb{Z}$ enumerating the spectrum, are orthogonal:

$$\int_0^L dx \sigma_m^*(x)\sigma_n(x) = 0, \quad m \neq n. \quad \text{(B.11)}$$

In contrast, for Dirichlet no-flux boundary conditions (equations (B.2b) and (B.9)), the boundary terms in equation (B.10) do not vanish. Consequently, $\partial_x^4$ is not self-adjoint on $[0,L]$ and the ensuing eigenfunctions $\sigma_m$ are not guaranteed to be orthogonal. This issue can be dealt with by introducing a set of eigenfunctions $\varphi_m(x)$ which solve the associated adjoint eigenproblem [69]. In the case of Dirichlet no-flux boundary conditions, this is defined by the eigenvalue equation

$$\partial_x^4 \varphi(x) = \tilde{\gamma} \varphi(x) \quad \text{(B.12)}$$

and the boundary conditions

$$\varphi'(0) = 0 = \varphi'(L), \quad \text{(B.13a)}$$

$$\varphi''(0) = 0 = \varphi''(L). \quad \text{(B.13b)}$$

Note that these boundary conditions are indeed such that, upon using equation (B.9), all boundary terms in equation (B.10) vanish. In general, the (suitably ordered) proper and adjoint eigenvalues, $\gamma_m$ and $\tilde{\gamma}_m$, coincide [69],

$$\gamma_m = \tilde{\gamma}_m. \quad \text{(B.14)}$$

This result is proven explicitly in appendix B.1.1. Upon using this fact, equation (B.10) readily yields the mutual orthogonality of the proper and adjoint eigenfunctions $\sigma_m$, $\varphi_n$:

$$\int_0^L dx \sigma_m^*(x)\varphi_n(x) = 0, \quad m \neq n. \quad \text{(B.15)}$$

This equation replaces equation (B.11) in the non-self-adjoint case and is crucial in constructing the eigenfunction solution of equations (B.1) or (3.11) for Dirichlet no-flux boundary conditions. We now proceed by discussing the eigenproblem of the MH equation for various boundary conditions.

**B.1. Dirichlet boundary conditions**

**B.1.1. Vanishing flux.** We consider here the proper eigenproblem defined by equation (B.4b) and turn to the adjoint problem in the next subsection. Defining

$$\omega \equiv L \gamma^{1/4}, \quad \text{(B.16)}$$

the four conditions in equations (B.2b) and (B.9) result in the requirement
for the coefficients $c_i$ defined in equation (B.6). For a nontrivial solution of equation (B.17) to exist, the determinant of the coefficient matrix must vanish, which implies

$$\cos(\omega) \cosh(\omega) = 1.$$  \hfill (B.18)

In general, the solutions of equation (B.18) cannot be represented in a simple form. Numerically, one obtains

$$\omega_k = 0, \pm 4.7300, \pm 7.8532, \pm 10.9956, \ldots \quad (k = 0, \pm 1, \pm 2, \ldots).$$  \hfill (B.19)

For $k \geq 4$ the eigenvalues are well approximated by

$$|\omega_k| \simeq \pi \left( k + \frac{1}{2} \right),$$  \hfill (B.20)

which becomes exact in the limit $\omega \to \pm \infty$. Using equation (B.18), it can be shown that the eigenvalues $\omega_k$ fulfill the relation

$$\sin(\omega_k) = \text{sgn}(\omega_k) (-1)^k \sqrt{1 - \frac{1}{\cosh^2(\omega_k)}}.$$  \hfill (B.21)

Accordingly, equation (B.17) reduces to

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ e^\omega & e^{-\omega} & \sin(\omega) & \cos(\omega) \\ 1 & -1 & -1 & 0 \\ e^\omega & -e^{-\omega} & -\cos(\omega) & \sin(\omega) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$  \hfill (B.22)

which yields for the $c_i$ the nontrivial solutions

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}_k = \left( \text{sgn}(\omega_k) \right)^k \begin{pmatrix} \frac{(-1)^k}{\sqrt{3+3e^{2\omega_k}}} \\ -\frac{\sqrt{1+\tanh(\omega_k)}}{\sqrt{3+3e^{2\omega_k}}} \\ \frac{(-1)^k+e^{\omega_k}}{\sqrt{3+3e^{2\omega_k}}} \\ \frac{(-1)^k-e^{-\omega_k}}{\sqrt{3+3e^{2\omega_k}}} \end{pmatrix}.$$  \hfill (B.23)

The eigenfunctions $\sigma_k(x)$ (equation (B.6)) of the operator $\partial_x^4$ for Dirichlet no-flux boundary conditions thus result as

$$\sigma_k^{(D')} (x) = c_{1,k} x^{1/4} e^{\gamma_k x^{1/4}} + c_{2,k} x^{1/4} e^{-\gamma_k x^{1/4}} + c_{3,k} \sin(x^{1/4} \gamma_k) + c_{4,k} \cos(x^{1/4} \gamma_k),$$  \hfill (B.24)

with the $c_{i,k}$ given in equation (B.23). It is straightforward to show that $\sigma_k^{(D')} (x) = 0$ as well as $\sigma_k^{(D')} (x) = \sigma_k^{(-)} (x)$ (see equation (B.19)). Hence, we can restrict $k$ to strictly positive values, such that the general solution of equation (B.1) reads

$$h^{(D')} (x,t) = \sum_{k=1}^{\infty} a_k e^{-\gamma_k t} \sigma_k^{(D')} (x),$$  \hfill (B.25)
with constants $a_k$. It is furthermore useful to note that $\sigma_k(D')_{L/2} = 0$ for odd $k$. The eigenfunctions $\sigma_k(D')$ are not normalized here, but instead one has

$$\int_0^L dx \left[ \sigma_k(D')(x) \right]^2 = \frac{L}{3} \left( 1 + \frac{(-1)^k}{\cosh \omega_k} - \frac{2}{\omega} \tanh(\omega_k) \right).$$  \hspace{1cm} (B.26)

Upon using equations (B.18) and (B.21) it can be shown that the mass identically vanishes:

$$\int_0^L dx \sigma_k(D')(x) = 0.$$  \hspace{1cm} (B.27)

Consequently, the solution in equation (B.25) is only compatible with initial conditions having zero mass. (A nonzero mass can be trivially introduced by adding a constant to the r.h.s. of equation (B.25).) Moreover, it can be readily checked that, as a consequence of the non-self-adjoint character of $\partial_x^4$ for Dirichlet no-flux boundary conditions, the eigenfunctions $\sigma_k(D')(x)$ are in general not orthogonal. This is the reason for considering an additional adjoint set of eigenfunctions (see below). In figure B1(a), the first few eigenfunctions defined by equation (B.24) are illustrated.

**Figure B1.** (a) Eigenfunctions $\sigma_k(D')$ (equation (B.24)) for Dirichlet no flux boundary conditions (equations (B.2b) and (B.9)) for the four lowest modes $k = 1, \ldots, 4$. (b) Associated adjoint eigenfunctions $\varphi_k(D')$ given by equations (B.28) and (B.33). For $k = 0$ one has $\sigma_k(D')_{L/2} = 0$ and $\varphi_k(D')(x) = 2\sqrt{2/3}$.

**B.1.2. Vanishing flux: adjoint eigenproblem.** We now turn to the adjoint eigenvalue problem associated with Dirichlet no-flux boundary conditions, which is defined by equations (B.12) and (B.13). The ansatz for the solution of the adjoint eigenvalue equation (B.12) is of the same form as in equation (B.6), i.e.

$$\varphi(x) = \tilde{c}_1 e^{x_1^{1/4}} + \tilde{c}_2 e^{-x_1^{1/4}} + \tilde{c}_3 \sin(x_1^{1/4}) + \tilde{c}_4 \cos(x_1^{1/4}).$$  \hspace{1cm} (B.28)

The four conditions in equation (B.13) imply

$$\begin{pmatrix}
1 & -1 & 1 & 0 \\
\tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 & \tilde{c}_4 \\
\tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 & \tilde{c}_4 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.$$  \hspace{1cm} (B.29)

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for the coefficients $\tilde{c}_i$, where
\[ \tilde{\omega} \equiv L^\gamma 1/4. \] (B.30)

Existence of a nontrivial solution of equation (B.17) implies the following determinant condition:
\[ \cos(\tilde{\omega}) \cosh(\tilde{\omega}) = 1. \] (B.31)

As anticipated, this relation coincides with equation (B.18) and, consequently, also the adjoint and the proper eigenvalues (see equation (B.19)) coincide:
\[ \tilde{\omega}_k = \omega_k. \] (B.32)

Proceeding as in appendix B.1.1, one obtains the nontrivial solutions of equation (B.29) as
\[ (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4)_k = \left( \frac{(-1)^k}{\sqrt{3 + 3e^{2\omega_k}}}, \frac{1}{\sqrt{3}} \sqrt{1 + \tanh(\omega_k)}, \frac{-(-1)^k + e^{\omega_k}}{\sqrt{3 + 3e^{2\omega_k}}}, \frac{(-1)^k + e^{\omega_k}}{\sqrt{3 + 3e^{2\omega_k}}} \right). \] (B.33)

Since the eigenfunctions $\varphi_k(x)$ resulting from equations (B.28) and (B.33) are identical for $\pm \omega_k$, we consider henceforth only $\omega_k \geq 0$, i.e. $k \geq 0$. As a consequence of equation (B.32), the orthogonality property in equation (B.15) follows. Specifically, one has
\[ \int_0^L dx \sigma_m(x)\varphi_n(x) = \frac{L}{3} \left(1 - \frac{(-1)^n}{\cosh(\omega_n)}\right) \delta_{mn}. \] (B.34)

Furthermore, one readily proves the useful property
\[ \int_0^L dx \varphi_m(x)\varphi_n''(x) = -\frac{L}{3} \omega_n^2 \left(1 - \frac{(-1)^n}{\cosh(\omega_n)}\right) \delta_{mn}. \] (B.35)

In figure B1(b), the first few adjoint eigenfunctions $\varphi_k$ are illustrated.

**B.1.3. Vanishing chemical potential.** For completeness, we summarize here the solution of the eigenproblem for Dirichlet boundary conditions with a vanishing chemical potential at the boundaries (also called *Dirichlet zero-\(\mu\) boundary conditions). Following the same steps as in appendix B.1.1 renders the well-known normalized eigenfunctions
\[ \sigma_k(x) = \sqrt{\frac{2}{L}} \sin(x_{\gamma_k}^{1/4}), \quad \gamma_k = \left( \frac{\pi k}{L} \right)^4, \quad k = 0, 1, 2, \ldots \] (B.36)

Note that, since $\sigma_{k=0}(x) = 0$, $k = 0$ is not considered to be part of the actual eigenspectrum. In summary, the solution of equation (B.1) for Dirichlet zero-\(\mu\) boundary conditions takes the well-known form
\[ h^{(D)}(x, t) = \sum_{k=1}^\infty a_k e^{-\left(\frac{\pi k}{L}\right)^4 \sqrt{\frac{2}{L}} \sin \left( \frac{\pi k}{L} x \right)}, \] (B.37)

where the constants $a_k$ are determined by the initial conditions on $h^{(D)}$.

Requiring a constant chemical potential at the boundaries generally leads to a mass loss during the time evolution:
\[ \int_0^L dx h^{(D)}(x, t) = \sum_{k=1}^{\infty} a_k e^{-\left(\frac{\pi k}{L}\right)^2 t} \times \begin{cases} \frac{2L}{\pi k}, & \text{odd } k, \\ 0, & \text{even } k. \end{cases} \] (B.38)

One may wonder whether the coefficients \( a_k \) can be chosen such that \( h^{(D)} \) (equation (B.37)) satisfies no-flux boundary conditions (equation (B.9)): requiring a vanishing third derivative of \( h^{(D)} \) at the boundaries results in a relation involving the sum over all modes, e.g. for \( x = 0 \) one has \( 0 = \sum_{k=1}^{\infty} a_k \exp(-\pi k/L)^4 t)(\pi k/L)^3 \). As is readily seen, it is not possible to choose the coefficients \( a_k \) such that no-flux boundary conditions are ensured during the whole time evolution of \( h^{(D)} \). This requires, instead, a specific set of basis functions.

B.2. Periodic boundary conditions

In the case of periodic boundary conditions (equation (B.2a)), one has \( c_1 = c_2 = 0 \) in equation (B.6) and \( L \gamma_{n/4} = 2\pi n \) with \( n = 0, 1, 2, \ldots \). This yields the well-known series expansion

\[ h^{(p)}(x, t) = \sum_{k=-\infty}^{\infty} a_k e^{-\left(\frac{2\pi k}{L}\right)^2 t} \sqrt{\frac{1}{L}} e^{\frac{2\pi ik}{L} x}. \] (B.39)

The parameters \( a_k \) must fulfill \( a_{-k} = a_k^* \) in order to ensure that \( h^{(p)} \) is real-valued. Since \( \int_0^L dx h^{(p)}(x, t) = a_0 \), the mass (equation (1.9)) is conserved in time.

B.3. Neumann boundary conditions

Imposing Neumann boundary conditions (equation (B.2c)) in conjunction with a no-flux condition (equation (B.9)) renders a solution of equation (B.1) in terms of standard Neumann eigenfunctions:

\[ h^{(N)}(x, t) = \sum_{k=0}^{\infty} a_k e^{-(\frac{\pi k}{L})^2 t} \sqrt{\frac{2 - \delta_{k,0}}{L}} \cos \left( \frac{\pi k x}{L} \right). \] (B.40)

We shall, however, not discuss Neumann boundary conditions further.

Appendix C. Solution of weak-noise theory for the optimal profile

Here, the general solution of equations (2.11) and (3.11) is determined, following the approach outlined in [29] for periodic boundary conditions. Recall that a flat profile is assumed at the initial time (equation (1.5)),

\[ h(x, t = 0) = 0, \] (C.1)

while the first-passage event at time \( T \) is defined by the condition that \( h \) attains its maximum height \( M > 0 \) at the location \( x_M \) (equation (1.4)),

\[ h(x_M, T) = M. \] (C.2)

However, for actually determining the solution of WNT, we neither explicitly enforce that \( h \) does not reach the height \( M \) before \( T \), nor that the profile stays below \( M \) for all
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Consequently, one has to check at the end of the calculation that the obtained solution fulfills these conditions. For sufficiently large $M$, this turns out to be the case.

We begin by casting equations (2.11) and (3.11) into the common form

$$ \partial_t h = (-\partial_x^2)^b [\partial_x^2 h + 2p], \quad (C.3a) $$

$$ \partial_t p = -(\partial_x^2)^b \partial_x^2 p, \quad (C.3b) $$

where $b = 0$ for EW dynamics and $b = 1$ for MH dynamics. The profile $h(x, t)$ is assumed to fulfill either periodic or Dirichlet boundary conditions (see equations (1.6) and (1.7)). For MH dynamics with Dirichlet boundary conditions, we additionally assume either a vanishing chemical potential (equation (B.8)) or a vanishing flux (equation (B.9)) at the boundaries. (In the main text, we focus only on the latter.) The profile is expanded into a set of eigenfunctions $\sigma_k$,

$$ h(x, t) = \sum_k h_k(t) \sigma_k(x), \quad (C.4) $$

which are determined by the associated eigenvalue problem (see appendix B),

$$ \partial_x^2 \sigma_k(x) = \gamma_k \sigma_k(x), \quad (C.5) $$

where the dynamic index $z = 2b + 2$. The conjugate field $p$ satisfies the boundary conditions of the associated adjoint eigenproblem (see appendix B) and is accordingly expanded in terms of the adjoint eigenfunctions $\varphi_k$ as

$$ p(x, t) = \sum_k p_k(t) \varphi_k(x). \quad (C.6) $$

The adjoint eigenfunctions $\varphi_k$ fulfill

$$ \partial_x^2 \varphi_k(x) = \gamma_k \varphi_k(x). \quad (C.7) $$

If the operator $\partial_x^2$ is self-adjoint on $[0, L]$, one has $\varphi_k = \sigma_k$. This is in particular the case for periodic or Dirichlet zero-\(\mu\) boundary conditions, such that

$$ \varphi_k^{(p,D)} = \sigma_k^{(p,D)}. \quad (C.8) $$

In contrast, for Dirichlet no-flux boundary conditions on $h$, the operator $\partial_x^4$ is not self-adjoint. In this case, the required adjoint eigenfunctions $\varphi_k^{(D')}$, which fulfill Neumann zero-\(\mu\) boundary conditions (see equation (B.13)), are provided in appendix B.1.25.

By construction, $\sigma_n$ and $\varphi_n$ are mutually orthogonal (see equation (B.15))

$$ \int_0^L dx \sigma_n^*(x) \varphi_n(x) = \kappa_n \delta_{mn}, \quad (C.9) $$

where the star denotes complex conjugation and $\kappa_n$ is a real number. Complex conjugation is necessary here in order to also take into account complex-valued eigenfunctions, which occur in the case of periodic boundary conditions (see equation (B.39)). We furthermore have

$$ ^5 $$

It turns out that the adjoint eigenmode $\varphi_k^{(D')}$ does not contribute to the dynamics for Dirichlet no-flux boundary conditions and will therefore be neglected henceforth in the corresponding expansion in equation (C.6).

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Table C1. Eigenfunctions and related properties of the operator $\partial_x^2$ on the interval $[0,L]$ for various boundary conditions. The proper and adjoint eigenfunctions are denoted by $\sigma_k$ and $\varphi_k$, respectively, and they coincide if $\partial_x^2$ is self-adjoint. The dynamic index $z$ is related to the parameter $b$ via $z = 2b + 2$, with $b = 0$ for EW dynamics and $b = 1$ for MH dynamics (see equation (C.3)).

| Boundary Condition | Periodic (equation (B.2a)) | Dirichlet zero-$\mu$ (equations (B.2b) and (B.8)) | Dirichlet no-flux (equations (B.2b) and (B.9)) $(b = 1)^a$ |
|--------------------|-----------------------------|-----------------------------------------------|--------------------------------------------------|
| $\partial_x^2$ self-adjoint | Yes | Yes | No |
| $\sigma_k$ | $\frac{1}{\sqrt{L}} \exp \left( \frac{2\pi ik}{L} x \right)$ | $\sqrt{\frac{2}{L}} \sin \left( \frac{Lk}{2} x \right)$ | $\sigma_k^{(D)}$ (equation (B.24)) |
| $\varphi_k$ | $\sigma_k$ | $\sigma_k$ | $\varphi_k^{(D)}$ (equation (B.28)) |
| $k$ | $0, \pm 1, \pm 2, \ldots^b$ | $1, 2, 3, \ldots$ | $1, 2, 3, \ldots$ |
| $\gamma_k$ (equations (C.5) and (C.7)) | $(-1)^{b+1} \left( \frac{\pi k}{L} \right)^z$ | $(-1)^{b+1} \left( \frac{Lk}{2} \right)^z$ | $(\omega_k/L)^4$ (equation (B.19)) |
| $\kappa_k$ (equation (C.9)) | 1 | 1 | $\frac{L}{\pi} \left( 1 - \frac{(-1)^b}{\cosh(L\gamma_k)} \right)$ (equation (B.34)) |
| $\epsilon_k$ (equation (C.10)) | $[-|\gamma_k|^{1/2}]^b \kappa_k, \epsilon_0 = 0$ | $[-|\gamma_k|^{1/2}]^b \kappa_k$ | $-\gamma_k^{1/2} \kappa_k$ (equation (B.35)) |

---

$^a$ Dirichlet no-flux boundary conditions are considered only for $b = 1$. Note that $\sigma_k^{(D)}$ and $\varphi_k^{(D)}$ are not normalized here, such that the system size $L$ appears in the corresponding expression for $\kappa_k$.

$^b$ Due to the mass constraint (equation (1.11)), the zero mode ($k = 0$) is absent from the actual solution for periodic boundary conditions (see equation (C.32) below).

$$
\int_0^L dx \varphi_m^*(x) \varphi_n''(x) = \epsilon_n \delta_{mn},
$$

with a real number $\epsilon_n$. The relevant properties of $\sigma_k$, $\varphi_k$ are summarized in Table C1.

To proceed, we insert the expansions given in equations (C.4) and (C.6) into (C.3), multiply equation (C.3a) by $\varphi_k^*$, equation (C.3b) by $\sigma_k^*$, and make use of the orthogonality properties in equations (C.9) and (C.10). This yields ordinary differential equations for the coefficients $h_k$ and $p_k$:

$$
\dot{h}_k = (-1)^b (\gamma_k h_k + 2p_k \epsilon_k),
$$

$$
\dot{p}_k = (-1)^{b+1} \gamma_k p_k,
$$

with

$$
\epsilon_k \equiv \begin{cases} 1, & b = 0 \\ \epsilon_k/\kappa_k, & b = 1. \end{cases}
$$

Equation (C.11b) is solved by

$$
p_k(t) = B_k \exp \left[ (-1)^{b+1} \gamma_k t \right],
$$

with integration constants $B_k$ determined below. The solution of equation (C.11a) follows as

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\[ h_k(t) = \begin{cases} 
A_k \exp \left[ \frac{1}{b} \gamma_k t \right] - p_k(t) \frac{\hat{\epsilon}_k}{\gamma_k}, & \gamma_k \neq 0 \\
A_k + \frac{1}{b} 2 \hat{\epsilon}_k B_k t, & \gamma_k = 0.
\end{cases} \]  \hspace{1cm} (C.14)

As can be inferred from table C1, the case \( \gamma_k = 0 \) is only relevant for \( k = 0 \) and periodic boundary conditions, where one obtains a linear dependence of \( h_0 \) on time for \( b = 0 \) (EW dynamics), whereas \( \hat{\epsilon}_0 = 0 \) for \( b = 1 \). Imposing the initial condition in equation (C.1) and using equations (C.14) and (C.13) yields

\[ B_k = \frac{\gamma_k}{\hat{\epsilon}_k} A_k, \quad (\gamma_k \neq 0), \]  \hspace{1cm} (C.15)

while for \( \gamma_k = 0 \) (\( k = 0 \)), one obtains \( A_0 = 0 \) and \( B_0 \) is left undetermined. Accordingly,

\[ h_k(t) = \begin{cases} 
2 A_k \sinh \left( \frac{1}{b} \gamma_k t \right), & \gamma_k \neq 0, \\
\frac{1}{b} 2 \hat{\epsilon}_0 B_0 t, & \gamma_k = 0,
\end{cases} \]  \hspace{1cm} (C.16)

from which readily follows that \( h_0(t) = 0 \) for periodic boundary conditions and MH dynamics. Expanding the profile at the final time \( T \) as

\[ h(x, T) = \sum_k H_k \sigma_k(x), \]  \hspace{1cm} (C.17)

provides the relations

\[ A_k = \frac{H_k}{2 \sinh \left( \frac{1}{b} \gamma_k T \right)}, \quad (\gamma_k \neq 0) \]  \hspace{1cm} (C.18)

as well as \( B_0 = \frac{1}{b} H_0 / (2 \hat{\epsilon}_0 T) \) (for \( \gamma_0 = 0 \) and if \( \hat{\epsilon}_0 \neq 0 \)). Summarizing, in terms of the (yet undetermined) coefficients \( H_k \), the solution of equation (C.11) is given, for \( \gamma_k \neq 0 \), by

\[ h_k(t) = H_k \frac{\sinh \left( \frac{1}{b} \gamma_k t \right)}{\sinh \left( \frac{1}{b} \gamma_k T \right)}, \]  \hspace{1cm} (C.19a)

\[ p_k(t) = H_k \frac{\gamma_k \exp \left( \frac{1}{b} \gamma_k t \right)}{2 \hat{\epsilon}_k \sinh \left( \frac{1}{b} \gamma_k T \right)}. \]  \hspace{1cm} (C.19b)

In the special case \( \gamma_0 = 0, \hat{\epsilon}_0 \neq 0 \) (\( k = 0 \)), corresponding to EW dynamics with periodic boundary conditions, one has

\[ h_0(t) = H_0 \frac{t}{T}, \]  \hspace{1cm} (C.20a)

\[ p_0(t) = -\frac{1}{b} \frac{H_0}{2 \hat{\epsilon}_0 T}, \]  \hspace{1cm} (C.20b)

whereas for \( \gamma_0 = 0, \hat{\epsilon}_0 = 0 \), corresponding to MH dynamics with periodic boundary conditions, one has

\[ h_0(t) = 0, \]  \hspace{1cm} (C.21a)

\[ p_0(t) = \text{const.} \]  \hspace{1cm} (C.21b)
In fact, performing the limit $\gamma_k \to 0$ in equation (C.19) leads to the expressions in equation (C.20). Furthermore, the fact that $h_0(t) = 0$ for periodic boundary conditions and MH dynamics (see equation (C.16)) implies $H_0 = 0$ in this case. This allows us to generally proceed by using equation (C.19), keeping in mind that $p_0(t) = 0$ for periodic boundary conditions and MH dynamics (as this result does not readily follow from a limit of equation (C.19b)).

The coefficients $H_k$ are determined by minimizing the (rescaled) action in equations (2.12) and (3.6),

$$S_{\text{opt}}[p] = (-1)^b \int_0^T dt \int_0^L dx p(\partial^2_x p),$$

subject to the constraint in equation (C.2). Inserting the expansion defined in equations (C.6) and (C.19b) into $S_{\text{opt}}$ and making use of the orthogonality property in equation (C.10) leads to

$$S_{\text{opt}} = \sum_k \frac{\gamma_k \tilde{\epsilon}_k}{2\tilde{\epsilon}_k^2} [\exp(2(-1)^b\gamma_k T) - 1]|H_k|^2 \equiv \sum_k N_k(T)|H_k|^2,$$

where

$$\tilde{\epsilon}_k \equiv \begin{cases} \epsilon_k, & b = 0, \\ \epsilon_k^*, & b = 1, \end{cases} \quad (C.24)$$

and the quantity $N_0(T)$ is introduced as a shorthand notation. Taking into account equation (C.17), the augmented action reads

$$\tilde{S}_{\text{opt}} = S_{\text{opt}} - \lambda[h(x_M, T) - M] = \sum_k N_k(T)|H_k|^2 - \lambda \left[ \sum_k H_k \sigma_k(x_M) - M \right],$$

where $\lambda$ is a Lagrange multiplier. Minimization of $\tilde{S}_{\text{opt}}$ with respect to $H_k$, i.e. requiring $0 = \delta \tilde{S}_{\text{opt}}/\delta H_k$, results in

$$H_k^* = \frac{\lambda \sigma_k(x_M)}{2N_k(T)}. \quad (C.26)$$

The complex conjugation in equation (C.26) is relevant only for periodic boundary conditions, where one has $H_k^* = H_{-k}$, $N_{-k} = N_k$, and $\varphi_{-k} = \varphi_k^*$ (which has also been used in equation (C.23)); for the other boundary conditions, $H_k^* = H_k$. Upon using equations (C.2) and (C.17), one obtains the constraint-induced value of the Lagrange multiplier,

$$\lambda(T) = \frac{M}{\sum_k |\sigma_k(x_M)/2N_k(T)|^2}. \quad (C.27)$$

The solution of equation (C.3) under the conditions in equations (C.1) and (C.2) is thus given by

$$h(x, t) = \frac{M}{Q(x_M, T, L)} \sum_k \frac{\tilde{\epsilon}_k^2}{\gamma_k \epsilon_k} \left[ \exp(2(-1)^b\gamma_k T) - 1 \right] \frac{\sinh((-1)^b\gamma_k T)}{\sinh((-1)^b\gamma_k T)} \sigma_k^*(x_M) \sigma_k(x) \quad (C.28)$$

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with

\[ Q(x_M, T, L) \equiv \sum_k \frac{\left| \sigma_k(x_M) \right|^2}{2N_k(T)} = \sum_k \left| \sigma_k(x_M) \right|^2 \frac{\hat{\epsilon}_k^2 \left[ \exp \left( 2(1-b)\gamma_k T \right) - 1 \right]}{\gamma_k \epsilon_k}. \] (C.29)

It is useful to note that \( H_k = \frac{M\sigma_k^*(x_M)}{2Q(x_M, T, L)N_k(T)} \). For the boundary conditions considered here and \( k \neq 0 \), one has \( \hat{\epsilon}_k^2/\epsilon_k = \epsilon_k^b/\kappa_k^2 \), \( \epsilon_k^b = \epsilon_k \) as well as \( \epsilon_k/\gamma_k < 0 \) (see table C1). We emphasize that in general \( Q(T/\tau) \) is only proportional to the function \( Q(T/\tau) \) defined in equations (2.20), (2.22), (3.18) and (3.20) in the main text, because the latter results from equation (C.28) after performing some simplifications. According to equations (C.6) and (C.19b), the conjugate field \( p \) is given by

\[ p(x, t) = \frac{M}{Q(x_M, T, L)} \sum_k \frac{\exp \left( (-1)^b \gamma_k T \right)}{\kappa_k \exp \left( (-1)^b \gamma_k t \right)} \sigma_k(x_M) \varphi_k(x). \] (C.30)

Notably, this result implies that the initial and final configurations of \( p(x, t) \) are fully determined by the corresponding ones for \( h \) specified in equations (C.1) and (C.2). The optimal action in equation (C.23) reduces to

\[ \mathcal{S}_{\text{opt}}(x_M, M, T, L) = \frac{M^2}{2Q(x_M, T, L)}, \] (C.31)

which is most easily proven by using equation (C.19b) and the expression for \( H_k \) stated after equation (C.29). Recall that the above results pertain to rescaled fields and time (see equation (2.10)). In particular, \( \mathcal{S}_{\text{opt}} \) in equation (C.31) gets multiplied by \( \eta/D \) upon returning to dimensional variables (see equation (2.13)).

C.1. Specialization to different boundary conditions

1.1. Periodic boundary conditions. In the case of EW dynamics with periodic boundary conditions, the mass constraint in equation (1.11) is explicitly imposed. Since \( \int_0^L dx \exp(ikx) = L\delta_{k,0} \) for \( k = 2\pi n/L \) with \( n \in \mathbb{Z} \), this constraint implies

\[ h_{k=0}(t) = 0 = H_{k=0} \] (C.32)

for the expansion coefficients defined in equations (C.4) and (C.17). Since the profile \( h(x, t) \) is real-valued, equation (C.4) yields \( h^* = \sum_{k=-\infty}^{\infty} h_k^* \exp(-2\pi ikx/L) = \sum_{k=-\infty}^{\infty} h_{-k} \exp(2\pi i(-k)x/L) = \bar{h} \) and thus

\[ h_k^* = \bar{h}_{-k}. \] (C.33)

Furthermore, we have the symmetry property \( N_k(T) = N_{-k}(T) \), as well as \( \sigma_k(L/2) = (-1)^k/\sqrt{L} \) and \( \sigma_{-k}(L/2)\sigma_{-k}(x) + \sigma_k(L/2)\sigma_k(x) = 2\cos(2\pi k(x/L - 1/2))/L \). Accordingly, equations (C.28) and (C.29) can be written as

\[ h^{(p)}(x, t) = \frac{2M}{LQ^{(p)}(T, L)} \sum_{k=1}^{\infty} \frac{1 - \exp \left( -2|\gamma_k|T \right) \sinh \left( |\gamma_k|t \right)}{|\gamma_k|^{1-k/2} \sinh \left( |\gamma_k|T \right)} \cos \left( 2\pi k(x/L - 1/2) \right) \] (C.34)
with

\[ Q_{(p)}(T, L) = \frac{2}{L} \sum_{k=1}^{\infty} \frac{1 - \exp(-2|\gamma_k|T)}{|\gamma_k|^{1-b/2}}. \]  

(C.35)

The factor 2 arises since the sum originally includes also negative \( k \). We have furthermore taken into account that, in the case of MH dynamics \((b = 1)\), the summand in equations (C.34) and (C.35) vanishes for \( k = 0 \) (which can be proven by carefully considering the limit \( \gamma_k \to 0 \)), such that the zero mode is absent from the solution. In fact, equation (C.34) agrees with the expression obtained for MH dynamics in [29]. In the case of EW dynamics without the mass constraint, the profile defined in equation (C.34) would superimpose onto a linear center-of-mass motion according to equation (C.20a).

C.1.2. Dirichlet boundary conditions. Both for standard and no-flux Dirichlet boundary conditions, equation (C.28) assumes the generic expression

\[ h^{(D)}(x, t) = \frac{M}{Q^{(D)}(x_M, T, L)} \sum_{k=1}^{\infty} \frac{1 - \exp(-2|\gamma_k|T) \sinh(|\gamma_k|T)}{|\gamma_k|^{1-b/2} \kappa_k} \sigma_k(x_M) \sigma_k(x) \]  

(C.36)

with

\[ Q^{(D)}(x_M, T, L) = \sum_{k=1}^{\infty} \sigma_k^2(x_M) \frac{1 - \exp(-2|\gamma_k|T)}{|\gamma_k|^{1-b/2} \kappa_k}. \]  

(C.37)

If a vanishing chemical potential is imposed at the boundaries, the eigenfunctions are given by the standard Dirichlet ones, \( \sigma_k^{(D)}(x) = \sqrt{2/L} \sin(\pi k x / L) \) with \( \gamma_k = (\pi k / L)^4 \). Taking \( x_M = L/2 \) (which is a convenient choice in the transient regime and minimizes the action in the equilibrium regime, see equation (A.10)), one has

\[ \sqrt{L/2} \sigma_k^{(D)}(L/2) = 1, 0, -1, 0, 1, \ldots \]  

(C.38)

for \( k = 1, 2, 3, \ldots \), implying that only the odd modes contribute to the evolution of the profile. Furthermore, we note the useful relation

\[ \sigma_k^{(D)}(L/2) \sigma_k^{(D)}(x) = \frac{2}{L} \cos(\pi k \left( x \frac{L}{x} \right) \left( x - \frac{L}{2} \right) ) , \quad k = 1, 3, 5, \ldots \)  

(C.39)

In the case of Dirichlet no-flux boundary conditions, the corresponding eigenfunctions \( \sigma_k^{(N)} \) are reported in equation (B.24). Here, one has \( \sigma_k^{(N)}(L/2) = 0 \) for odd \( k \). In the equilibrium regime, \( x_M \) as given in equation (A.14) has to be used instead of \( L/2 \).

The optimal profile \( h^{(D)}(x, t) \) for MH dynamics with Dirichlet no-flux boundary conditions is discussed in the main text (see equation (3.19)). As a byproduct of the present analysis, we readily obtain the optimal profile \( h^{(D)}(x, t) \) for MH dynamics with Dirichlet zero-\( \mu \) boundary conditions, which is illustrated in figure C1. Mass is in general not conserved in this case. Introducing the time scale

\[ \tau^{(D)} = \left( \frac{L}{\pi} \right)^4 , \]  

(C.40)

the scaling form in equation (3.16) applies with

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\[ h(D)(x, t, T) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k^2} \exp \left( -\frac{2k^4T}{k^2} \right) \frac{\sinh (k^4t)}{\sinh (k^4T)} \cos (\pi k (x - 1/2)) \]  
\hfill (C.41)

and

\[ Q(D)(T) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k^2} \exp \left( -\frac{2k^4T}{k^2} \right). \]  
\hfill (C.42)

The above expressions for \( h \) and \( Q \) in fact coincide with the corresponding ones in the EW case (equations (2.19) and (2.21)), except for the presence of \( k^2 \) instead of \( k^4 \).

C.2. Limiting cases

Introducing \( \delta t \equiv T - t \), \( \Gamma_k \equiv \epsilon_k / |\gamma_k| k_k^2 \) and using table C1, equation (C.28) can be simplified to

\[ h(x, \delta t) = \frac{M}{Q(T)} \sum_k \Gamma_k \left[ \exp (-|\gamma_k| (2T - \delta t)) - \exp (-|\gamma_k| \delta t) \right] \sigma_k^x(x_M) \sigma_k(x), \]  
\hfill (C.43)

where we suppressed further arguments of \( Q \) and note that \( \Gamma_k < 0 \) as well as \((-1)^k \gamma_k = -|\gamma_k|\). Here and in the following, \( h \) is considered to be a function of \( \delta t \) instead of \( t \). Specifically for \( \delta t = 0 \), equation (C.43) reduces to

\[ h(x, \delta t = 0) = \frac{M}{Q(T)} \sum_k \Gamma_k \left[ \exp(-2|\gamma_k| T) - 1 \right] \sigma_k^x(x_M) \sigma_k(x). \]  
\hfill (C.44)

Convenient analytical expressions for \( h \) can be derived by replacing the sum in equation (C.43) by an integral using the Euler–Maclaurin formula. The error caused by this approximation is small if the summands in equation (C.43) vary significantly only over a few values of \( k \). This is the case if \( \delta t \ll \tau \approx 1 / |\gamma_1| \) (or, equivalently, \( T \ll \tau \)), since then the variation occurs for large \( k \), where \( |\gamma_k| \sim k^z \). (For \( T \to \infty \), on the other hand, the first term in equation (C.43) can be neglected, see appendix C.2.2.)

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Figure C1. ((a) and (b)) Time evolution of the optimal profile for the MH equation with Dirichlet boundary conditions and a vanishing chemical potential at the boundaries (equation (C.36)) in (a) the transient regime \( (T = 10^{-3} \tau^{(D)}) \) and (b) the equilibrium regime \( (T = 100 \tau^{(D)}) \). The curves correspond from center top to bottom to (a) \( 1 - t/T = 0, 0.05, 0.3, 0.8 \) and (b) \( 1 - t/T = 0, 10^{-5}, 10^{-4}, 5 \times 10^{-4} \). The fundamental time scale is given by \( \tau^{(D)} = (L/\pi)^4 \).
C.2.1. Transient regime \((T \ll \tau)\).

Case \(\delta t = 0\). We first consider the case \(\delta t = 0\). For periodic boundary conditions, equation \((C.44)\) becomes

\[
\begin{align*}
\left. h^{(p)}(x, \delta t = 0) \right|_{T \ll \tau} &= \frac{2M}{LQ^{(p)}(T)} \left( \frac{L}{2\pi} \right)^2 \sum_{k=1}^{\infty} \frac{1 - \exp \left[- \left(2\pi k (2T)^{1/z}/L \right)^2 \right]}{k^2} \cos \left( \frac{2\pi k}{L} (x - L/2) \right) \\
&\simeq \frac{(2T)^{1/z} M}{\pi Q^{(p)}(T)} \int_0^{\infty} dy \frac{1 - e^{-y^2}}{y^2} \cos(y\xi),
\end{align*}
\]

with the fundamental integral

\[
\int_0^{\infty} dy \frac{1 - e^{-y^2}}{y^2} \cos(y\xi)
= \begin{cases} \\
\sqrt{\pi} \exp \left(- \frac{\xi^2}{4} \right) + \frac{1}{2\pi} |\xi| \left[ \text{erf} \left( \frac{\xi}{\sqrt{2}} \right) - 1 \right], & z = 2, \\
\Gamma \left( \frac{3}{2} \right) F_3 \left( -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}; \frac{5}{2} \right) + \frac{1}{8} \Gamma \left( \frac{1}{4} \right) \xi^2 F_3 \left( \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}; \frac{5}{2} \right) - \frac{1}{8} |\xi|, & z = 4,
\end{cases}
\]

and \(\xi \equiv (x - L/2)/(2T)^{1/z}\). Analogously, equation \((C.35)\) evaluates to

\[
\begin{align*}
Q^{(p)}(T \ll \tau) &= \frac{2}{L} \left( \frac{L}{2\pi} \right)^2 \sum_{k=1}^{\infty} \frac{1 - \exp \left(- \left[2\pi k (2T)^{1/z}/L \right]^2 \right)}{k^2} \\
&\simeq \frac{(2T)^{1/z}}{\pi} \int_0^{\infty} dy \frac{1 - \exp(-y^2)}{y^2} = \frac{(2T)^{1/z}}{\pi} \Gamma (1 - 1/z),
\end{align*}
\]

where, in the intermediate steps, the integration variable \(k\) has been substituted by \(y = 2\pi k (2T)^{1/z}/L\). The lower integration boundary has been sent to zero since we consider \(T \to 0\), noting that the associated error is negligible because the integrand vanishes for \(y \to 0\). Analogously, for Dirichlet zero-\(\mu\) boundary conditions, using equation \((C.39)\), we obtain from equations \((C.36), (C.37)\) and \((C.44)\):

\[
\begin{align*}
\left. h^{(D)}(x, \delta t = 0) \right|_{T \ll \tau} &= \frac{2M}{LQ^{(D)}(T)} \left( \frac{L}{\pi} \right)^2 \sum_{j=0}^{\infty} \frac{1 - \exp \left(- \left[\pi (2j+1) (2T)^{1/z}/L \right]^z \right)}{(2j+1)^2} \\
&\times \cos \left( \frac{(2j+1)\pi}{L} (x - L/2) \right) \\
&= \frac{(2T)^{1/z} M}{\pi Q^{(D)}(T)} \int_0^{\infty} dy \frac{1 - \exp(-y^2)}{y^2} \cos(y\xi),
\end{align*}
\]

with

\[
\begin{align*}
Q^{(D)}(T \ll \tau) &= \frac{2}{L} \left( \frac{L}{\pi} \right)^2 \sum_{j=0}^{\infty} \frac{1 - \exp \left(- \left[\pi (2j+1) (2T)^{1/z}/L \right]^z \right)}{(2j+1)^2} \\
&= \frac{(2T)^{1/z}}{\pi} \int_0^{\infty} dy \frac{1 - \exp(-y^2)}{y^2} = \frac{(2T)^{1/z}}{\pi} \Gamma (1 - 1/z).
\end{align*}
\]
In order to evaluate the sum in equation (C.44) for Dirichlet-no-flux boundary conditions, we assume a $k'$ such that, for $k \geq k'$, the eigenvalue $\gamma_k$ and the parameter $\kappa_k$ can be approximated by their respective asymptotic forms (see equation (B.20) and table C1)

$$\gamma_k^{(D')} \simeq \left(\frac{(k + 1/2)\pi}{L}\right)^4, \quad \kappa_k \simeq \frac{L}{3}.$$  (C.50)

In the transient regime, we set $x_M = L/2$ (see equation (C.57) for justification) and thus have $\sigma_k^{(D')}(L/2) = 0$ for odd $k$. For $T \ll (L/\omega_k)^4 \ll \tau^{(D')}$, terms with $k < k'$ in the sum in equation (C.44) are exponentially small and can be neglected. For even $k$ with $k \geq k'$, we approximate $\sigma_k^{(D')}$ by

$$\sigma_k^{(D')}(x) \simeq (-1)^{k/2} \frac{2}{3} \cos \left[\pi \left(\frac{k + 1/2}{2}\right) \left(x - \frac{L}{2}\right)\right].$$  (C.51)

While this approximation does not respect Dirichlet no-flux boundary conditions, it captures the oscillatory behavior of the actual $\sigma_k^{(D')}$ well. A numerical comparison of the resulting scaling profile with the exact one justifies the above approximations \textit{a posteriori}. Within the large $k$ approximation, we have $[\sigma_k^{(D')}(L/2)]^2 \simeq 2/3$ for even $k$. Accordingly, one obtains

$$h^{(D')}(x, \delta t = 0)\big|_{T \ll \tau} \simeq \frac{2M}{LQ^{(D')}(T)} \left(\frac{L}{\pi}\right)^2 \sum_{k \geq k'} 1 - \exp \left(-\frac{\pi(2j + 1)(2T)^{1/4}/L}{(2j + 1)^2}\right) \times \cos \left(\frac{(2j + 1)\pi}{L}(x - L/2)\right) \approx (2T)^{1/4} M \int_0^\infty \frac{dy}{\pi} \frac{1 - \exp(-y^4)}{y^2} \cos(y\xi)$$  (C.52)

and analogously

$$Q^{(D')}(T \ll \tau) = \frac{(2T)^{1/4}}{\pi} \Gamma(3/4).$$  (C.53)

As before, sending the lower integration boundary to zero is justified in the limit $T \to 0$. In summary, in the transient regime, the asymptotic expressions of the static profiles $h(x, \delta t = 0)$ for periodic and Dirichlet boundary conditions are identical and reduce to

$$h(x, \delta t = 0)\big|_{T \ll \tau} = M \mathcal{H} \left(\frac{x - L/2}{(2T)^{1/2}}\right),$$  (C.54)

with the scaling function

$$\mathcal{H}(\xi) = \begin{cases} 
\exp \left(-\frac{\xi^2}{4}\right) + \frac{1}{2} \sqrt{\pi} |\xi| \left[\text{erf} \left(\frac{\xi}{2}\right) - 1\right], & z = 2, \\
\text{I}_1F_3 \left(-\frac{1}{4}; \frac{1}{2}, \frac{1}{2}; \frac{2}{2}; \frac{\xi^2}{256}\right) + \frac{\xi^2}{2 \Gamma(3/4)} \text{I}_1F_3 \left(-\frac{1}{4}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{\xi^2}{256}\right) - \frac{\pi}{2 \Gamma(3/4)} |\xi|, & z = 4,
\end{cases}$$  (C.55)
which has the limits $H(0) = 1$ and $H(\xi \to \infty) = 0$. The expression of $H$ for $z = 4$ coincides with the result for periodic boundary conditions reported in [29]. The profile given by equation (C.54) does not respect mass conservation (equation (1.11)) for finite $T$. This can be readily shown by computing the mass using the last expression in equation (C.48) before performing the integral over $y$. However, as $T \to 0$, the resulting error becomes negligible since the width of the profile rapidly shrinks.

The quantity $Q$ has been evaluated above for the particular choice $x_M = L/2$. Analogous calculations can in fact be performed for arbitrary $x_M$ with $0 < x_M < L$, yielding

$$Q(x_M, T \ll \tau) = (2T)^{1/z} q(x_M/(2T)^{1/z}),$$

with a scaling function $q$ that has the property $q(\xi \to \infty) = \text{const}$. Accordingly, the action in equation (C.31) behaves as (see also [29])

$$S_{\text{opt}}(x_M)|_{T \to 0} \propto T^{-1/z},$$

and becomes independent of $x_M$ for $0 < x_M < L$ in the limit $T \to 0$. For $x_M \in \{0, L\}$, instead, Dirichlet boundary conditions imply $\sigma_k^{(D,D')} (x_M) = 0$ for all $k$, such that $Q^{(D,D')} (x_M)$ (equation (C.37)) vanishes identically at the boundaries, resulting in a divergence of $S_{\text{opt}}^{(D,D')} (x_M)$ for $x_M \in \{0, L\}$. The fact that $S_{\text{opt}}$ is independent of $x_M$ asymptotically in the transient regime justifies the choice $x_M = L/2$ made above.

**Case $\delta t > 0$.** In order to obtain dynamic scaling profiles for nonzero $\delta t$ with $\delta t \ll \tau$ and $T \ll \tau$, we rewrite equation (C.43) as

$$h(x, \delta t) = \frac{M}{Q(T)} \sum_k \Gamma_k \{\exp (-|\gamma_k| (2T - \delta t)) - 1\} + [1 - \exp (-|\gamma_k| \delta t)]\, \sigma_k^*(x_M)\sigma_k(x).$$

(C.58)

Performing calculations analogous to those leading from equations (C.44) to (C.54), the corresponding dynamic scaling profile in the transient regime follows as

$$h(x, \delta t \ll \tau)|_{T \ll \tau} = M \left(1 - \frac{\delta t}{2T}\right)^{1/z} H\left(\frac{x - L/2}{(2T - \delta t)^{1/z}}\right) - M \left(\frac{\delta t}{2T}\right)^{1/z} H\left(\frac{x - L/2}{(\delta t)^{1/z}}\right).$$

(C.59)

For $x = L/2$ and $\delta t \ll T$, equation (C.59) simplifies to $h(L/2, \delta t) \simeq M - [\delta t/(2T)]^{1/z}$. In order to obtain an analogous scaling form for $x \neq L/2$, we consider the expression

$$\left(\frac{2T}{\delta t}\right)^{1/z} [M - h(x, \delta t)] \simeq \left(\frac{2T}{\delta t}\right)^{1/z} M \left[1 - H\left(\frac{\delta t}{2T}\right)^{1/2}\right] + \left(\frac{\delta t}{2T}\right)^{1/z} H(\xi), \quad \delta t \ll T,$n

(C.60)

where we introduced $\xi \equiv (x - L/2)/\delta t^{1/2}$. Expanding the r.h.s. in equation (C.60) to leading (i.e. zeroth) order in $\delta t/T$, keeping $\xi$ fixed, yields the desired scaling form:
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\( h(x, \delta t \ll T) \big|_{T \ll \tau} \simeq M - M \left( \frac{\delta t}{2T} \right)^{1/z} \tilde{\mathcal{H}} \left( \frac{x - L/2}{(\delta t)^{1/z}} \right), \)  

(C.61)

with

\[ \tilde{\mathcal{H}}(\xi) = \begin{cases} 
\exp \left( -\frac{\xi^2}{4} \right) + \frac{1}{2} \sqrt{\pi} \xi \text{erf} \left( \frac{\xi}{2} \right), & z = 2, \\
1F3 \left( -\frac{1}{4}, \frac{3}{4}, 1; 1; \frac{3}{4}, \frac{1}{4}, \frac{\xi^4}{256} \right) + \xi^2 \frac{1}{8} (\frac{1}{4})_1F3 \left( \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{4}, \frac{\xi^4}{256} \right), & z = 4.
\end{cases} \]  

(C.62)

As shown in figure C2, the scaling form in equation (C.61) provides an accurate approximation to the full profiles (equations (C.34) and (C.36)) in a region around \( x_M \). The size of this region increases as \( \delta t/T \to 0 \).

C.2.2. Equilibrium regime (\( T \gg \tau \)). In the long-time limit, \( T \to \infty \), the first term in the square brackets in equation (C.43) can be neglected, as can the exponential function in equation (C.44). Accordingly, \( h \) becomes independent of \( T \) and equation (C.43) reduces to

Figure C2. Scaling behavior in the transient regime for a profile subject to (a) EW and (b) MH dynamics with periodic boundary conditions, and (c) MH dynamics with Dirichlet no-flux boundary conditions. The dashed black curve represents the scaling function equation (C.62), while the solid curves represent the full expression of the profile in equations (C.34) and (C.36), rescaled according to equation (C.61).

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\[ h_{eq}(x, \delta t) \equiv \bar{h}(x, \delta t) \big|_{T \to \infty} = -\frac{M}{Q_{eq}} \sum_k \frac{\epsilon_k}{\gamma_k \kappa_k^2} \exp \left( -|\gamma_k| \delta t \right) \sigma_k(x_M) \sigma_k(x), \quad (C.63) \]

with (see equation (C.29))
\[ Q_{eq} = -\sum_k |\sigma_k(x_M)|^2 \frac{\epsilon_k}{\gamma_k \kappa_k^2}. \quad (C.64) \]

**Case \( \delta t = 0 \).** For \( \delta t = 0 \), the expressions in equations (C.63) and (C.64) can be evaluated exactly in the case of periodic and Dirichlet zero-\( \mu \) boundary conditions; according to table C1, we have

\[ Q_{eq} = \sum_k |\sigma_k(x_M)|^2 |\gamma_k|^{b/2 - 1} \quad (C.65) \]

as well as
\[ h_{eq}(x, \delta t = 0) = \frac{M}{Q_{eq}} \sum_k |\gamma_k|^{b/2 - 1} \sigma_k(x_M) \sigma_k(x), \quad (C.66) \]

with \(|\gamma_k^{(p)}|^{b/2 - 1} = (2\pi k/L)^2\) for periodic and \(|\gamma_k^{(D)}|^{b/2 - 1} = (\pi k/L)^2\) for Dirichlet zero-\( \mu \) boundary conditions, independently of the value of \( b \in \{0, 1\} \). Specifically, one obtains, invoking known Fourier series representations (see, e.g. [71])

\[ Q_{eq}^{(p)} = 2L \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^2} = \frac{L}{12}, \quad (C.67a) \]
\[ Q_{eq}^{(D)} = 2L \sum_{k=1,3,5,...}^{\infty} \frac{1}{(\pi k)^2} = \frac{L}{4}, \quad (C.67b) \]

and analogously,
\[ h_{eq}^{(p)}(x, \delta t = 0) = \frac{2LM}{(2\pi)^2 Q_{eq}^{(p)}} \sum_{k=1}^{\infty} \cos(2\pi k(x/L - 1/2)) = M \left[ 1 - 6 \left| \frac{x}{L} - \frac{1}{2} \right| + 6 \left( \frac{x}{L} - \frac{1}{2} \right)^2 \right], \quad (C.68a) \]
\[ h_{eq}^{(D)}(x, \delta t = 0) = \frac{2LM}{\pi^2 Q_{eq}^{(D)}} \sum_{n=0}^{\infty} (-1)^n \frac{\sin((2n + 1)\pi x/L)}{(2n + 1)^2} = M - M \left| 1 - \frac{2x}{L} \right|, \quad (C.68b) \]

where we used equation (C.38). These expressions coincide with the ones in equations (A.8) and (A.11) for the respective boundary conditions. A direct proof of the equivalence between \( h_{eq}^{(D')}(x, \delta t = 0) \) and the expression in equation (A.16) is not available owing to the non-algebraic dependence of \( \omega \) on \( k \) (see equation (B.18)).

**Case \( \delta t > 0 \).** For \( T \to \infty \) and nonzero \( \delta t \ll \tau \), asymptotic scaling profiles can be derived from equation (C.63) analogously to the calculation leading from equations (C.44) to (C.54). In the conversion of the sum to an integral, however, possible
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divergences have to be taken care of. In the case of periodic boundary conditions one obtains, taking \( x_M = L/2 \),

\[
h^{(p)}(x, \delta t) \bigg|_{T \to \infty} = \frac{2M}{LQ^{(p)}_{\text{eq}}} \left( \frac{L}{2\pi} \right)^2 \sum_{k=1}^{\infty} \exp \left[ \frac{- (2\pi k \delta t^{1/z})^2}{2k^2} \right] \cos \left( \frac{2\pi k}{L} \left( x - L/2 \right) \right)
\]

\[
\approx \frac{(\delta t)^{1/z} M}{\pi Q^{(p)}_{\text{eq}}} \int_{Y_1}^{\infty} dy \frac{e^{-y^2}}{y^2} \cos(y\xi),
\]

(C.69)

where \( Y_1 \equiv 2\pi(\delta t)^{1/z}/L \) and \( \xi \equiv (x - L/2)/(\delta t)^{1/z} \) is a scaling variable. In order to take into account the singularity of the integral for \( Y_1 \to 0 \), we write

\[
\int_{Y_1}^{\infty} dy \frac{e^{-y^2}}{y^2} \cos(y\xi) = \int_{Y_1}^{\infty} dy \frac{e^{-y^2} - 1}{y^2} \cos(y\xi) + \int_{Y_1}^{\infty} dy \frac{\cos(y\xi)}{y^2}.
\]

(C.70)

In the first term on the r.h.s. the limit \( Y_1 \to 0 \) can be performed, yielding equation (C.46) up to a sign. For the second term, we obtain

\[
\int_{Y_1}^{\infty} dy \frac{\cos(y\xi)}{y^2} = \frac{\cos(\xi Y_1)}{Y_1} - \frac{1}{2\pi|\xi|} + \xi \text{Si}(\xi Y_1),
\]

(C.71)

where \( \text{Si} \) is the sine integral [72]. Since \( \xi Y_1 = 2\pi(x/L - 1/2) \), expanding to first order in \((x/L - 1/2)\), using \( \text{Si}(\zeta) \simeq \zeta + \mathcal{O}(\zeta^2) \), we obtain

\[
\int_{0}^{\infty} dy \frac{e^{-y^2}}{y^2} \cos(y\xi) \simeq \left\{ \begin{array}{ll}
\frac{1}{\xi Y_1} - \sqrt{\pi} \exp \left( -\frac{\xi^2}{4} \right) - \frac{\pi}{4} \xi \text{erf} \left( \frac{\xi}{2} \right), & z = 2, \\
\frac{1}{\xi Y_1} - \Gamma \left( \frac{3}{4} \right) F_3 \left( -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{2}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8} ; \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} ; \frac{1}{4} \right) & z = 4.
\end{array} \right.
\]

(C.72)

For consistency in the approximation, we calculate \( Q_{\text{eq}} \) in equation (C.67a) in an analogous fashion, obtaining

\[
Q^{(p)}_{\text{eq}} \simeq \frac{L}{2\pi^2} \int_{1}^{\infty} dk \ k^{-2} = \frac{L}{2\pi^2}.
\]

(C.73)

Inserting equations (C.72) and (C.73) in (C.69) yields

\[
h(x, \delta t) \bigg|_{T \to \infty} \simeq M - M(\delta t)^{1/z} \Gamma(1 - 1/z) \tilde{H} \left( \frac{x - L/2}{\delta t^{1/z}} \right),
\]

(C.74)

with the scaling function \( \tilde{H} \) given in equation (C.62). Hence, asymptotically, the scaling functions in the transient and the equilibrium regime are identical. The calculation proceeds analogously for Dirichlet boundary conditions, yielding for \( h^{(D)}_{\text{eq}} \) the same result as in equation (C.74). Moreover, equation (C.74) applies also to Dirichlet no-flux boundary conditions, since in the asymptotic regime, i.e. for \( \xi \lesssim \mathcal{O}(1) \) with \( \delta t \ll \tau \), the precise value of \( x_M \) is irrelevant, despite equation (3.15).
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C.2.3. Effect of an upper mode cutoff. Above results pertain to a continuum system, which can sustain an infinite number of eigenmodes. Conversely, the presence of a minimal length scale in the system (e.g. a lattice constant) gives rise to an upper bound on the mode spectrum. Accordingly, the sums in equations (C.28) and (C.29) are bounded by a maximum mode index $k_x$. Associated with this mode is a relaxation rate $\gamma_{k_x}$, which defines a cross-over time

$$\tau_x \equiv \frac{1}{\gamma_{k_x}}. \quad (C.75)$$

In a system with a mode cutoff, for times $\delta t \ll \tau_x$ and $\delta t \ll T$, equation (C.43) can be approximated as

$$h(x, \delta t \lesssim \tau_x) \simeq \frac{M}{Q(T)} \sum_{k}^{k_x} \Gamma_k \left[ \exp \left(-2|\gamma_k|T\right) - 1 + |\gamma_k|\delta t \right] \sigma_k(x_M)\sigma_k(x)$$

$$= h(x, 0) + \delta t \frac{M}{Q(T)} \sum_{k}^{k_x} \Gamma_k |\gamma_k|\sigma_k^*(x_M)\sigma_k(x), \quad (C.76)$$

where $h(x, 0)$ is the static profile defined in equation (C.44). Note that the second term in the last line of equation (C.76) is negative owing to the sign of $\Gamma_k$. Hence, for a bounded mode spectrum, the algebraic time evolution (with exponent 1/z) of the profile described by equations (C.61) and (C.74) crosses over to a linear one in $\delta t$ for small times, $\delta t \lesssim \tau_x$. This behavior applies both in the transient and the equilibrium regime, independently from the boundary conditions.

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