Fluctuation-Dissipation Relations for Continuous Quantum Measurements

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The generating functional is derived for the fluctuation-dissipation relations which result from the unitarity and reversibility of microscopic dynamics and connect various statistical characteristics of many consecutive (continuous) observations in a quantum system subjected to external perturbations. Consequences of these relations in respect to the earlier suggested stochastic representation of interaction between two systems are considered.

PACS numbers: 72.70.+m, 05.60.Gg

I. INTRODUCTION

Both the classical and quantum mechanics is time-reversible and unitary (conserves classical phase volume and quantum probability). In thermodynamic systems (ensembles), the unitarity manifests itself in strong connections between noise and dissipation, while the reversibility in time symmetry of noise and reciprocity of transport processes. The famous examples are the Einstein relation 1, the Nyquist formula 2, and the Onsager reciprocity relations 3. All these are the relations between (i) second-order (quadratic) correlators of equilibrium noise and (ii) linear parts of complete, possibly nonlinear, responses to external perturbations (the expansion of the responses in a series over powers of “perturbing forces” is meant). Later, the fluctuation-dissipation theorem (FDT) proved by Callen and Welton 4 and the Green-Kubo formulas 5,6 exhausted this linear theory.

The first nonlinear generalizations were obtained by Efremov 7 who proved the quadratic FDT, which connects (i) quadratic parts of the responses and (ii) third-order (cubic) equilibrium correlators ((iii) linear responses of quadratic correlators also take a part here, but can be excluded). The next, fourth-order, relations connect together (i) cubic components of the responses, (ii) equilibrium fourth-order correlators and, besides, (iii) quadratic responses of quadratic non-equilibrium correlators ((iv) linear responses of cubic correlators are in play too, but can be excluded). Their investigation was started by Stratonovich 8 who found that “cubic FDT” does not exist (for details and more references see e.g. 9,10,11). Nevertheless, the fourth-order relations are much useful, for instance, when analysing low-frequency fluctuations in transport and relaxation rates, especially flicker fluctuations 12,13,14.

The producing formulas for the whole (infinite) chain of arbitrary-order fluctuation-dissipation and reciprocity relations (FDR) were obtained in 11. In the works 12,15 these results were extended to non-equilibrium steady states of open systems. In 10,12 the extension to the thermic perturbations was developed, that is perturbations of a probability distribution (density matrix) of the system, in addition to perturbations of its Hamiltonian which are termed dynamic ones. Examples of various applications of FDR can be found e.g. in 10,12,13,15,16,17,18,19.

It is desirable to combine all the infinite variety of arbitrary-order FDR into a compact visual generating FDR for the probabilistic functionals or corresponding characteristic functionals. In the framework of classical mechanics, this was realized in 11,15 (for review, see 10).

In the quantum theory, time-differed values of any interesting variable $X(t)$ (an operator in the Heisenberg representation) do not commute one with another, $X(t_1)X(t_2) \neq X(t_2)X(t_1)$. But its measurements in macroscopic devices are subjected to the classical description language, being thought as a commutative stochastic process, $x(t)$, whose values are usual $\mathbb{C}$-numbers. Hence, neither probabilistic nor characteristic functional of $x(t)$ has a sense, until a concrete definition of all $x(t)$-s correlators (statistical moments), in terms of the $X(t)$, is chosen.

In general, the two lowest-order correlators only seem unambiguously defined:

$$
\langle x(t) \rangle \equiv \text{Tr} X(t) \rho_0 ,
\langle x(t_1)x(t_2) \rangle \equiv \text{Tr} (X(t_1) \circ X(t_2)) \rho_0 ,
$$

with $\rho_0$ being the statistical operator (density matrix), and $\circ$ designating the symmetric product (Jordan product), $A \circ B \equiv (AB + BA)/2$. The subscript “0” of $\rho_0$ means that $\rho_0$ represents a quantum statistical ensemble at a fixed time moment, e.g. $t = 0$. The right-hand sides present definitions of the angle brackets on the left sides, that is effective statistics of classical image, $x(t)$, of the quantum variable $X(t)$.

What is for the higher-order correlators $\langle x(t_1)x(t_2)...x(t_N) \rangle$, unfortunately, at any $N > 2$, there are $N!/2 > 1$ different symmetrized (Hermitian) expressions produced by various permutations of $X(t_k)$ in $\frac{1}{N!} \text{Tr} (X(t_1)X(t_2)...X(t_N) + X(t_N)...X(t_2)X(t_1)) \rho_0$, plus uncountably many their weighted linear combinations.

Moreover, in fact even the second correlator can be introduced in an alternative way. For example, let us
define the characteristic functional (CF) of \( x(t) \) by the identity
\[
\left\langle \exp \left( \int_0^t v(t')x(t'dt') \right) \right\rangle \equiv \text{Tr} \exp \left( \int_0^t v(t')X(t'dt' + \ln \rho_0 \right)
\]
Here \( t > 0 \), and \( v(t) \) is an “arbitrary probe function” (test function). Double differentiation of \( \mathcal{L} \) by \( v(t_1) \) and \( v(t_2) \) at \( v(t) \equiv 0 \) gives
\[
\langle x(t_1)x(t_2) \rangle = \text{Tr} \int_0^1 X(t_1)\rho_0^2 X(t_2)\rho_0^{1-\alpha} d\alpha ,
\]
and \( N \)-order moments include \( N \) different \( \rho_0 \)'s powers whose sum equals to unit. As pointed out in [10], under this specific definition of the CF all the FDR between the angle brackets look absolutely similar in both classical and quantum case.

Of course, to become practically useful, a choice of definition of the CF, i.e. definition of all statistical moments \( \langle x(t_1) \ldots x(t_N) \rangle \), should be based on analysis of real measurement procedures. The well known examples presents quantum theory of electromagnetic field fluctuations (see e.g. [21]). During last decade, in the original works by Levitov, Lesovik, Nazarov and others [21, 22, 23, 24] (see also references therein) the continuous measurements of the charge transport in electric devices were analyzed. The results of these works allow to conclude that frequently the adequate construction rule of the CF is the chronological symmetrized product:
\[
\langle \exp \left( \int_0^t v(t')x(t')dt' \right) \rangle \equiv \text{Tr} \\tilde{\exp} \left( \int_0^t v(t')X(t')dt' \right) \rho_0 \\text{exp} \left( \int_0^t v(t')X(t')dt' \right)
\]
Here \( \tilde{\exp} \) and \( \text{exp} \) are chronological and anti-chronological exponents, respectively. According to this rule (and to general properties of the trace operation \( \text{Tr} \)), the quadratic correlator remains as in [11]. For the higher-order correlators the rule [11] prescribes
\[
\langle x(t_1)x(t_2)x(t_3) \rangle = \text{Tr} (X(t_1) \circ (X(t_2) \circ X(t_3))) \rho_0 ,
\]
and so on, where, for definiteness, the inequalities \( t_N \geq \ldots t_2 \geq t_1 \) are presumed.

Below, we will derive generating FDR for so built statistical moments. All the more this is interesting because the same rule [11] naturally arose in the course of the so-called “stochastic representation of deterministic interactions” [22, 25, 26, 27] (the general method for correct construction of “Langevin equations” introducing thermodynamic noise and dissipation into quantum or classical dynamics).

The strong argument for the benefit of the rule [11] is that it can be deduced from the correspondence principle. To see this, firstly consider classical systems.

**II. CLASSICAL CHARACTERISTIC FUNCTIONALS**

Let a classical system has canonical variables (coordinates and momenta) \( \Gamma = \{q, p\} \). Generally, the system undergoes a dynamic perturbation from its outside, which means that its Hamiltonian, \( H_t = H_t(\Gamma) \), is time dependent. Introduce also the corresponding Liouville operator \( L_t \), and the evolution operator \( Z_t \):
\[
L_t = (\nabla_q H_t)\nabla_p - (\nabla_p H_t)\nabla_q , \quad Z_t = \tilde{\exp} \left( \int_0^t L_r dr \right)
\]
Eventually, we are interested in the evolution and fluctuations of variables \( X \) which represent definite functions of the phase space point currently occupied by the system: \( X = X(\Gamma) \) (briefly, “phase functions”).

The essential properties of the Liouville and evolution operators are as follow:
\[
\int A(\Gamma)L_t B(\Gamma) d\Gamma = -\int B(\Gamma)L_t A(\Gamma) d\Gamma ,
\int Z_t \rho(\Gamma) d\Gamma = \int \rho(\Gamma) d\Gamma ,
Z_t^{-1} = \Gamma_t(\Gamma) ,
Z_t^{-1} X(\Gamma)Z_t = X(Z_t^{-1} \Gamma) ,
\]
where \( A(\Gamma), B(\Gamma), X(\Gamma) \) and \( \rho(\Gamma) \) are any phase functions (such that \( A(\Gamma)B(\Gamma) \) and \( \rho(\Gamma) \) are integrable), and \( \Gamma_t(\Gamma) \) stand for the current values of the canonical variables (at time \( t \)) expressed through their initial values \( \Gamma \) (at the initial time moment \( t = 0 \)). In other words, \( \Gamma_t(\Gamma) \) is the solution of the Hamilton equations under initial condition \( \Gamma_0(\Gamma) = \Gamma \).

Importantly, the latter equality in [10] is operator equality, that is both \( X(\Gamma) \) on the left and \( X(Z_t^{-1} \Gamma) \) on the right have the sense of multiplication operators. Combination of this equality with the previous one implies
\[
Z_t^{-1} X(\Gamma)Z_t = X(t, \Gamma) \equiv X(\Gamma_t(\Gamma))
\]
Hence, the composite operator \( Z_t^{-1} X(\Gamma)Z_t \) reduces to operator of multiplication by the time-dependent number \( X(t, \Gamma) \), which is nothing but trajectory of the variable \( X \) under initial conditions \( \Gamma \).

Next, let us be convinced that CF of any variable (phase function) \( X(t, \Gamma) \equiv X(\Gamma_t(\Gamma)) \) can be expressed [10] by the formula
\[
\left\langle \exp \left( \int_0^t v(\tau)x(\tau)d\tau \right) \right\rangle = \int \tilde{\exp} \left( \int_0^t [L_r + v(\tau)X(\Gamma)]d\tau \right) \rho_0(\Gamma)d\Gamma
\]
Here \( \rho_0 \) is the statistical operator, i.e. distribution function, of the system at time \( t = 0 \), and again \( v(t) \) is arbitrary probe function. In the angle brackets, \( x(t) \) means the internal variable \( X(t, \Gamma) \) as perceived by an outside observer (which knows nothing about \( \Gamma \)) and interpreted by him as a stochastic process.
To justify the solution of definite differential equation:

\[
\int \exp \left[ \int_0^t \left( L_\tau + v(\tau) X(\Gamma) \right) d\tau \right] \rho_0(\Gamma) d\Gamma = \\
= \int Z_t \exp \left[ \int_0^t v(\tau) Z^{-1}_\tau X(\tau, \Gamma) d\tau \right] \rho_0(\Gamma) d\Gamma = (9)
\]

Here, the latter transformation is made with taking into account the identity (11) and the second property from (10). Evidently, final expression in (9) coincides what was meant in the angle brackets in (8), i.e. the CF of the path \( X(t, \Gamma) \) whose action is defined by

\[
Z_t \rho = U(t) \rho U^{-1}(t) , \quad U(t) \equiv \exp \left[ \int_0^t H_\tau d\tau \right]
\]

The phase functions \( X(\Gamma) \) and \( X(\Gamma, \Gamma) \) are replaced by the operators of quantum variable (observable) in the Shroedinger and Heisenberg representation, respectively, \( X(\Gamma) \) and \( X(\Gamma, \Gamma) \). What is for the operation of multiplication by \( X(\Gamma) \) in (10), it should be replaced by super-operator of the symmetric product: \( X(\Gamma) \Phi(\Gamma) \Rightarrow X \circ \Phi \). The matter is that it is very hard to suggest something else. Under this extension, the equality (7) remains valid:

\[
Z_t^{-1}(X \circ (Z_t \Phi)) = X(t) \circ \Phi = (U^{-1}(t) X U(t)) \circ \Phi
\]

(but now, generally, the super-operator \( X(t) \circ \Phi \) in none base reduces to scalar multiplication).

Thus one comes to the construction of quantum CF as follows:

\[
\int \exp \left[ \int_0^t v(\tau) x(\tau) d\tau \right] = \text{Tr} \rho , \quad (11)
\]

\[
\frac{d\rho}{dt} = i \hbar [\rho, H_t] + v(t) X \circ \rho , \quad (12)
\]

with the initial condition \( \rho(t = 0) = \rho_0 \). One can easy verify that substitution of the formal direct solution of (12) into (11) produces just the formula (10), with \( X(t) \) standing for Heisenbergerian operator of the quantum variable and \( x(t) \) its effective commutative image.

The doubtless advantage of such definition, (11)-(12), of quantum CF is its differential nature, which highlights its automatic agreement with the causality principle too. The formulas demonstrate that because of the causality all the corresponding higher-order statistical moments appear asymmetric with respect to time inversion. It is natural: earlier observations (measurements) can affect later ones, but opposite influence is impossible.

Further, we want to extend to an arbitrary set of variables under simultaneous continuous observation. With this purpose, let us replace \( v(t) X \) in (12) with the operator

\[
V_t = \sum_{\mu \nu} x^\mu_{\tau} X_{\mu \nu} , \quad X_{\mu \nu} \equiv |\mu \rangle \langle \nu| , \quad (13)
\]

where \( |\mu \rangle \) are states, or vectors (in the Dirac’s designations), which constitute a complete orthonormal base, and \( \psi_{\mu}^\tau \) arbitrary probe functions. This is most general form of the observation. Analogously, most general external perturbation can be described as

\[
H_t = H_0 - F_t , \quad F_t = \sum_{\mu \nu} f_{\tau \mu}^\nu X_{\mu \nu} , \quad (14)
\]

where \( H_0 \) is Hamiltonian of the “free” system, and \( f_{\tau \mu}^\nu \) are “perturbing forces”.

Then, instead of (12) and (11), we have

\[
\int \exp \left[ \int_0^t \psi_{\tau \mu}^\nu x_{\mu \nu}(\tau) d\tau \right] H_{0, F} , \quad \text{Tr} \rho \equiv \Xi(V_F; H_0, F_t) \quad (16)
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\]

From here, the repeated indices \( \mu \) or \( \nu \) mean summation over them; \( x_{\mu \nu}(t) \) are effective classical images of quantum variables \( X_{\mu \nu}(t) = U^{-1}(t) X_{\mu \nu} U(t) \); \( \Xi(V_F; H_0, F_t) \) will be used as shortened designation of the CF, i.e. the angle brackets.

The second and third arguments of \( \Xi \), as well as the subscript under angle brackets, remind about eigen
Hamiltonian of the system and its perturbations. Direct solution of (15) yields
\[ Ξ(V_τ; H_0, F_τ) = \text{Tr} \exp\left( \int_{τ'}^t \left( -\frac{i}{\hbar} (H_0 - F_τ) + \frac{1}{2} V_τ \right) dτ \right) ρ_0 × \]
\[ × \exp\left( \int_{τ'}^t \left( \frac{1}{2} \bar{H}_0 - F_τ + \frac{1}{2} V_τ \right) dτ \right) = \]
\[ = \text{Tr} \exp\left[ \frac{1}{2} \int_{τ'}^t v_τ^{μν} X_μν(τ)dτ \right] ρ_0 × \]
\[ × \exp\left[ \frac{1}{2} \int_{τ'}^t v_τ^{μν} X_μν(τ)dτ \right] = \] (17)

The Ξ’s arguments \( V_τ \) and \( F_τ \) can be understood as either two sets, of the probe functions \( v_τ^{μν} \) and the forces \( f_τ^{μν} \), or the whole operators introduced in (13) and (14). Of course, the pair \( x_τ^{μν}(t) \) and \( x_{τp}(t) \) in (16) can be interpreted as two mutually conjugated complex-valued random processes (at least while \( F_τ \) assumed Hermitian).

**IV. THE GENERATING FDR**

In the present paper we confine ourselves to the assumption that the initial density matrix, \( ρ_0 \), is canonic thermodynamically equilibrium one: \( ρ_0 \propto \exp(-H_0/T) \) (in the classical theory, not only this case already was considered [11] [12] but also the case of non-equilibrium initial distributions [10] [12]).

Following the recipes of [11], let us make three transformations of the operator expression placed after the trace symbol in (17).

(i) Firstly, transpose this expression as a whole, with the help of the usual rule \( (AB...C) = C′...B′A′ \), where the prime means the transposition: \( A′ = (A^†)^* = (A^*)^† \) (symbols \( † \) and \( * \) will stand for the Hermitian and complex conjugation, respectively). As is well known, this operation does not change the trace.

(ii) Secondly, invert the direction of the time account, by means of rewriting chronological exponents as anti-chronological ones, and vice versa, as in the examples
\[ \exp\left[ \int_{τ'}^t A(τ)dτ \right] = \exp\left[ \int_{t}^T A(t − τ)dτ \right] \]

(iii) Thirdly, rewrite \( ρ_0 \) in the form \( \sqrt{ρ_0} \sqrt{ρ_0} \) and “drag” one of these multiplicands to the left and another to the right and after that again unify them, with the help of the trace property \( \text{Tr} ABC = \text{Tr} CAB \).

The result of these manipulations is an expression which looks quite identical to the initial one, but with modified operator of the observation instead of \( V_τ \) and modified operator of the perturbation instead of \( F_τ \). To write the result, of course, it is sufficient to choose the states \( |μ⟩ \) as a complete set of orthonormal eigenvectors of the free Hamiltonian \( H_0: H_0|μ⟩ = E_μ|μ⟩ \), with the eigenvalues (energies) \( E_μ \). Besides, introduce two operators
\[ L_0Φ = i[Φ, H_0]/\hbar \; \; ; \; \; U_0(t) = \exp[-iH_0 t/\hbar] \; \; , \]

the matrices (in terms of the chosen basis)
\[ S_{μν}, C_{μν} = \sinh \; \frac{E_μ}{2T}, \; Δ_{μν} = \frac{2T}{E_μ} \tanh \frac{E_μ}{2T} \; , \] (18)

and super-operators \( C \), \( S \) and \( Δ \) whose action is determined by these matrices:
\[ C = \cos \left( \frac{\hbar}{\Delta} L_0 \right), \; (CΦ)_{μν} = C_{μν}Φ_{μν} \; , \]
\[ S = \sin \left( \frac{\hbar}{\Delta} L_0 \right), \; (SΦ)_{μν} = S_{μν}Φ_{μν} \; , \] (19)

Then, finally, we can formulate the generating FDR. It is expressed by the formulas as follow:
\[ Ξ(V′_{τ−τ}; H_0, F_{τ−τ}) = Ξ(ν_bar; H_0, ν_bar) \; , \] (20)

where \( V′_{τ} = v_{τ}^{μν}X_{μν}′, \; ν_bar = ν_bar^{μν}X_{μν}, \; F′_{τ} = f_{τ}^{μν}X_{μν}′, \; F_bar = f_bar^{μν}X_{μν}, \; \]
\[ \left[ f_{τ}^{μν} f_bar^{μν} \right] = \left[ C_{μν} \frac{\hbar}{\Delta} S_{μν} \; C_{μν} \right] \left[ v_{τ}^{μν} v_bar^{μν} \right] \; , \] (21)

or, in another equivalent form,
\[ \left[ ρ_bar \; ν_bar \right] = \left[ C \frac{1}{T} S \; C \right] \left[ V′_τ \; V_bar \right] \] (22)

For convenience, the observation and perturbation variables are unified into the column vector.

The combined transformation \( Φ_τ ≡ Φ_{τ−τ} \) represents the time reversal. Thus, (20) and (21) or (22) state invariance of the CF under (i) simultaneous time reversal of both the probe functions and external forces and (ii) their mutual mixing as described by (21) and (22).

Importantly, when dealing with (20) - (22) one should remember about the time translational invariance: any joint temporal shift of \( V_τ \) and \( F_τ \) does not change \( Ξ(V_τ; H_0, F_τ) \)’s value, since \( ρ_0 \) is invariant with respect to free (unperturbed and unobserved) evolution (\( |ρ_0, H_0⟩ = 0 \)).

It should be underlined also that by the very definition (see [13]) of the operators \( X_{μν} \) we have
\[ X′_{μν} = |μ⟩⟨μ| \; , \; H_0|μ⟩ = E_μ|μ⟩ \; , \] (23)

where \( |μ⟩ \) are eigenvectors of the transposed free Hamiltonian \( H_0′ \), with the same eigenvalues \( E_μ \) as that of \( |μ⟩ \) (one can write also \( |μ⟩ = |μ⟩^* = ⟨μ|′ \)). Hence, during the time reversed evolution the operator \( X′_{μν} \) represents those quantum variable which is represented by operator \( X_{μν} \) in the direct process. Consequently, in terms of their effective classical images, the left side of (20) looks as
\[ Ξ(V′_{τ−τ}; H_0, F′_{τ−τ}) = \]
\[ = \left\langle \exp \left[ \int_{τ'}^t v_τ^{μν} X_μν(τ)dτ \right] \right\rangle H_0, F′_{τ−τ} \] (24)
In other words, under the time reversal the matrix \( \{ \psi_{\mu}^{\nu} \} \) behaves like the operator \( V_{\tau} \), that is it undergoes transposition: \( \psi_{\nu}^{\mu} \leftrightarrow (\psi_{\tau}^{\mu})^{\nu} = \psi_{\tau}^{\mu} \) \( \{ \} \), similarly, \( f_{\mu}^{\nu} \leftrightarrow (f_{\tau}^{\mu})^{\nu} = f_{\tau}^{\mu} \) \( \{} \). 

In the classical limit (formally \( \hbar \to 0 \)), we have \( C \to 1 \), \( S \to 0 \), \( 2S/h \to T^{-1}L_0 \), and formula (22) takes the form

\[
\bar{F}_{\tau} = F_{\tau}, \quad \bar{V}_{\tau} = V_{\tau} + T^{-1}L_0F_{\tau} \tag{25}
\]

At that, the operators \( F_{\tau} \) and \( V_{\tau} \) turn into phase functions, \( L_0 = (\nabla_{q}H_{0})\nabla_{p} - (\nabla_{p}H_{0})\nabla_{q} \), and the prime in (20) means replacement of \( \{q,p\} \) with \( \{q,-p\} \). 

Comparison between (22) and (25) reveals that the observations anyway are influenced by the perturbations, but opposite effects exist in the quantum theory only. For illustration, let us choose \( F_{\tau} \equiv 0 \), that is the real direct-time observations (described by \( V_{\tau} \)) are made in equilibrium system. Then, according to (20)-(22), detection of exactly time-reversed results (described by \( V_{\tau}^{\prime} \)) in equilibrium system too, is in certain sense the same as detection of the direct results but under specific non-zero perturbations. This unpleasant peculiarity of the quantum theory is not surprising: since any measurement influences subsequent ones, but not vice versa, the mere rearrangement of their results, generally speaking, could not realize under same conditions.

Various particular FDR can be obtained from (20)-(22) by either some special choice of \( V_{\tau} \) and \( F_{\tau} \) or variational differentiations with respect to \( \psi_{\mu}^{\nu} \) and \( f_{\mu}^{\nu} \). Evidently, if both \( F_{\tau} \) and \( V_{\tau} \) are Hermitian, then both transforms \( \bar{V}_{\tau} \) and \( \bar{F}_{\tau} \) also are Hermitian. But if \( V_{\tau} \) is quite arbitrary, then one should be ready to deal with non-Hermitian perturbation \( \bar{F}_{\tau} \).

For example, let us choose \( V_{\tau} \equiv 0 \), i.e., there is no observation at all. According to (12) or (17), of course, in absence of observations the evolution of the statistical operator \( \rho \) is unitary under whatever perturbations. Therefore \( \text{Tr} \rho = \text{Tr} \rho_0 = 1 \), and \( \Xi(0;H_0,F_{\tau}) \equiv 1 \). Consequently, (20) and (22), together with (17), yield the equality

\[
1 = \Xi \left( \frac{1}{T} \Delta L_0 \bar{F}_{\tau} ; H_0, \bar{F}_{\tau} \right), \tag{26}
\]

or, in other equivalent designations,

\[
\left\langle \exp \left[ -\int_{0}^{t} \frac{2i}{\hbar} \tanh \frac{E_{\mu\nu}}{2T} \bar{f}_{\tau}^{\mu\nu} \delta x_{\mu\nu}(\tau) d\tau \right] \right\rangle_{H_0, \bar{F}_{\tau}} = 1 \tag{27}
\]

These relations are valid at arbitrary perturbation operator \( \bar{F}_{\tau} \). The subscript under the angle bracket reminds about the perturbation.

To make Eqs. (20,27) better transparent, combine them with the identities

\[
L_0 \bar{F}_{\tau} = \frac{i}{\hbar} [\bar{F}_{\tau}, H_0] = -\frac{i}{\hbar} [H_0 - \bar{F}_{\tau}, H_0] = -P_{\tau}
\]

where \( P_{\tau} \) is operator of the instant energy power being pumped by the perturbations (notice that \( dH_0(t)/dt = d(U^{-1}(t)H_0 U(t))/dt = U^{-1}(t)P_{\tau} U(t) \)). Hence, we can rewrite Eq. (26) also as

\[
\Xi \left( -\frac{1}{T} \Delta P_{\tau} ; H_0, \bar{F}_{\tau} \right) = 1 , \tag{28}
\]

with arbitrary \( \bar{F}_{\tau} \) and related \( P_{\tau} = \frac{i}{\hbar}[H_0, \bar{F}_{\tau}] \).

In the classical limit, \( \Delta \) disappears, in the sense that \( \Delta_{\mu\nu} \to 1 \). Therefore, (28) turns into equality \( \left\langle \exp(-A/T) \right\rangle = 1 \), where \( A \) is total work produced by perturbations during time interval \( (0,t) \). More generally, perhaps, \( \Delta P_{\tau}/T \) can be interpreted as operator of entropy production.

V. PROBABILITY FUNCTIONALS

Provided statistical moments of quantum variables \( X(t) \) and their CF are defined (by (15,17), in our case), we can introduce the probability functional (PF) of their classical (commutative) equivalents \( x(t) \). We will designate it by \( W(x(\tau) ; H_0, F_{\tau}) \). As usually, is represents functional Fourier transform of the CF:

\[
W(x(\tau) ; H_0, F_{\tau}) = \int \exp \left[ -i \int_{0}^{t} \psi_{\mu}^{\nu}(x)(\tau) d\tau \right] \times 
\times \Xi(\i V_{\tau} ; H_0, F_{\tau}) dV_{\tau} , \tag{29}
\]

where \( dV_{\tau} = \prod_{\mu\nu} (d\psi_{\mu}^{\nu}/2\pi) \).

Let us apply this transform to both sides of the FDR (20), again with the indices being related to the eigenstates of \( H_0 \). We omit rather bulky manipulations and write their result in two steps:

\[
W(x'(t - \tau) ; H_0', F_{\tau'-\tau}) = \tag{30}
\]

\[
= \exp \left[ -\int_{0}^{t} \frac{2i}{\hbar} \tanh \frac{E_{\mu\nu}}{2T} \bar{f}_{\tau}^{\mu\nu} \delta x_{\mu\nu}(\tau) d\tau \right] \times \n\times \Xi(\i V_{\tau} ; H_0, F_{\tau}) , \tag{31}
\]

\[
\times W(C^{-1}x(\tau) ; H_0, C^{-1}F_{\tau}) , \tag{32}
\]

where \( (C^{-1}x)(\mu) = \Phi_{\mu\nu}/C_{\mu\nu} \).

In these two formulas, both the exponents in middle rows result from the left bottom and right top non-diagonal elements of matrix (24) (or (22)), respectively, i.e. from mutual mixing of perturbation and observation.

Thus, the exponent in (31) reflects disturbing action of observations, which is quite unpleasant peculiarity of the quantum case. Of course, in fact this operator-valued
exponent acts as an integral operator. We leave it unwrapped till possible separate work (though example of similar operators can be found in [25]). For the present, confine ourselves by the sad conclusion that generally $\tilde{W}(x(\tau); H_0, F_\tau)$, and thus PF of the reversed process, $W'(x(t-\tau); H_{0}', F_{t-\tau}')$, relates to PF of the direct process $W(x(\tau); H_0, F_\tau)$ in some **non-local** way, with respect to both $f^{\mu\nu}$ and $x_{\nu}(t)$.

However, it is not hard to notice that a degree of the non-locality is proportional to $\tanh^2(E_{\mu\nu}/2T)$, hence, it is negligible in respect to low-energy quantum transitions. In the classical limit ($h \to 0$) the exponent in [31] turns into unit, $\tilde{W} \to W$, and [30] - [31] reduce to the purely **local** relation

\[ W(x'(t-\tau); H_{0}', F'_{t-\tau}') = \exp(-A/T) W(x(\tau); H_0, F_\tau) , \]

where $A$ is again the work of the external forces $f^{\mu\nu}$, and $\Phi(q, p) \equiv \Phi(q, -p)$. This FDR is equivalent to what was obtained in [11]. Curiously, if quantum CF is defined by Eq.2 instead of Eqs.15-17, then Eq.32 replaces [20] - [22] even in general quantum case.

### VI. STOCHASTIC REPRESENTATION OF RESPONSE TO PERTURBATIONS

The above consideration demonstrated a lot of formal symmetry between perturbations and observations. This symmetry suggests that the perturbing forces $f^{\mu\nu}$ can be treated as another test (probe) functions which correspond to watching for additional ghost variables. Let the latter be named as $y_{\mu\nu}(t)$. To define them, following [25, 27, 28], we rewrite Eqs.15-17 in the form

\[ \left\langle \exp \int_0^t [v^{\mu\nu}_{\tau} x_{\nu}(\tau) + f^{\mu\nu}_{\tau} y_{\mu\nu}(\tau)] d\tau \right\rangle_o = \Xi(V_\tau; H_0, F_\tau) = \] 

\[ = \text{Tr} \exp \int_0^t \left[ -\frac{i}{\hbar} (H_0 - F_\tau) + \frac{1}{2} V_\tau \right] d\tau \rho_0 \times \] 

\[ \times \exp \left( \int_0^t \left[ \frac{i}{\hbar} f^{\mu\nu}_{\tau} + \frac{1}{2} v^{\mu\nu}_{\tau} \right] X_{\mu\nu}(\tau) d\tau \right) , \]

Essentially, it is assumed here that an imaginative probability measure hidden behind the angle brackets is itself independent of the forces. In other words, all the random processes $x_{\nu}(t)$ and $y_{\mu\nu}(t)$ are meant be characteristics of the unperturbed dynamics governed by the Hamiltonian $H_0$. The subscript “o” serves to remind of this circumstance.

Therefore, it may be convenient to rewrite (33) once more, in the form

\[ \left\langle \exp \int_0^t [v^{\mu\nu}_{\tau} x_{\nu}(\tau) + f^{\mu\nu}_{\tau} y_{\mu\nu}(\tau)] d\tau \right\rangle_o = \Xi(V_\tau; H_0, F_\tau) = \] 

\[ = \text{Tr} \exp \int_0^t \left[ -\frac{i}{\hbar} (H_0 - F_\tau) + \frac{1}{2} V_\tau \right] d\tau \rho_0 \times \] 

\[ \times \exp \left( \int_0^t \left[ \frac{i}{\hbar} f^{\mu\nu}_{\tau} + \frac{1}{2} v^{\mu\nu}_{\tau} \right] X_{\mu\nu}(\tau) d\tau \right) , \]

where $X_{\mu\nu}(t)$ are the operators $X_{\mu\nu}$ considered in the interaction representation, thus representing free evolution:

\[ X_{\mu\nu}(t) = U_0^{-1}(t) X_{\mu\nu} U_0(t) = X_{\mu\nu} \exp (iE_{\mu\nu}t/h) \]

According to [33-34], if $V_\tau \equiv 0$ then

\[ \left\langle \exp \int_0^t f^{\mu\nu}_\tau y_{\mu\nu}(\tau) d\tau \right\rangle_o = \Xi(0; H_0, F_\tau) = \] 

\[ = \left[ \prod_{j} \delta \frac{\delta f(\tau_m)}{\delta g(\tau_m)} \right]_{G_\tau=0} \]

(indices $\mu, \nu$ are omitted). Hence, cross-correlation between $N$ copies of $x$ and $M$ copies of $y$ represents $M$-order response to perturbations of an $N$-order statistical moment of the $x$'s.

Interestingly, the relations (33) are valid also for the $x$'s and $y$'s cumulants (semiinvariants) whose generating function is $\ln \Xi(V_\tau; H_0, F_\tau)$. The proof is trivial:

\[ \ln \left\langle \prod_{j,m} x(t_j) y(\tau_m) \right\rangle_o^{c} = \left[ \prod_{j} \delta \frac{\delta f(\tau_m)}{\delta g(\tau_m)} \right]_{G_\tau=0} \]

again without indices. Consequently, instead of (33) we can write

\[ \left\langle \prod_{j,m} x(t_j) y(\tau_m) \right\rangle_o^{c} = \left[ \prod_{j} \delta \frac{\delta f(\tau_m)}{\delta g(\tau_m)} \right]_{G_\tau=0} \]

with the superscript “c” marking the cumulants.

The union of the two sets of random processes, quite realistic $x$'s and indeed rather illusive $y$'s, gives stochastic representation of the system’s response to perturbations. If the latter are caused by interactions with some other dynamical system “D”, then we make first step towards the stochastic representation of deterministic (quantum or classical) interactions, which was suggested in [25] and developed in [26, 27, 28]. For instance, our system can serve as thermostat for “D”. General stochastic equations which describe “D” under influence by the thermostat inevitably include the $y$'s whose main effect is dissipation.

### VII. TIME REVERSAL AND GENERATING FDR IN THE STOCHASTIC REPRESENTATION

Combining Eqs.20-22 with Eq.25, one can simply reformulate the generating FDR in terms of $x$'s and $y$'s:

\[ \left\langle \exp \int_0^t [v^{\mu\nu}_{\tau} x_{\nu}(\tau - t) + f^{\mu\nu}_{\tau} y_{\mu\nu}(t - \tau)] d\tau \right\rangle_o = \]

\[ = \left[ \exp \int_0^t [v^{\mu\nu}_{\tau} x_{\nu}(\tau) + f^{\mu\nu}_{\tau} y_{\mu\nu}(\tau)] d\tau \right]_{o} , \]
where, of course, $x'_{\mu\nu} = x_{\nu\mu}$, $y'_{\mu\nu} = y_{\nu\mu}$, and $\bar{x}$, $\bar{y}$ relate to $x$, $y$ absolutely similar to (21):

$$
\frac{\bar{x}_{\mu\nu}(\tau)}{\bar{y}_{\mu\nu}(\tau)} = \left[ \frac{\bar{C}_{\mu\nu} + \frac{i}{2} \bar{S}_{\mu\nu}}{\bar{C}_{\mu\nu}} \right] \left[ x_{\mu\nu}(\tau) \right] \left[ y_{\mu\nu}(\tau) \right] \tag{39}
$$

Evidently, it the matrices $x = \{x_{\mu\nu}\}$ and $y = \{y_{\mu\nu}\}$ are Hermitian, $x^\dagger = x$ and $y^\dagger = y$, then the transformation (39) does not damage this their property.

Taking into account $\rho_0$’s invariance with respect to free evolution, the FDR (35) can be expressed also in the form

$$
x_{\nu\mu}(t_0 - t) \approx \bar{x}_{\nu\mu}(t), \quad y_{\nu\mu}(t_0 - t) \approx \bar{y}_{\nu\mu}(t), \tag{40}
$$

where $t_0$ is arbitrary time shift, and symbol $\approx$ means statistical equivalence of left- and right-handed random processes (i.e. equivalence in the sense of statistical moments, $(x(t_0 - t_1)...x(t_0 - t_N)) = (\bar{x}(t_1)...\bar{x}(t_N))$, and so on).

Following (25), it may be convenient to introduce another random processes, whose matrices are non-Hermitian:

$$
\xi(t) \equiv x(t) + \frac{\Im}{2} y(t),
$$

$$
\eta(t) \equiv \xi^\dagger(t) = x(t) - \frac{\Im}{2} y(t), \tag{41}
$$

In their terms the generating FDR look most simple:

$$
\xi_{\nu\mu}(t_0 - t) \approx \exp(\frac{E_{\mu\nu}}{2}T) \xi_{\nu\mu}(t), \tag{42}
$$

$$
\eta_{\mu\nu}(t_0 - t) \approx \exp(-\frac{E_{\mu\nu}}{2}T) \eta_{\mu\nu}(t) \tag{43}
$$

To some extent, the $\xi$’s and $\eta$’s can be thought like amplitudes of quantum jumps, hence, their squares, $|\xi|^2$ and $|\eta|^2$, like corresponding probabilities. Then, formulas (42) and (43) reduce to familiar relations between probabilities of mutually reversed jumps.

It is necessary to remember, of course, that generally (when $H_0 \neq H_0$) the left and right-hand processes in (40), (42) and (43) relate to somehow different systems.

Combining these statistical equalities with (35) (or (37)) and (40) (or (43)), and besides with the causality principle, as well as with independence of statistical moments on $t_0$, one can construct relatively simple algorithms for derivation of many particular FDR. However, that are tasks for separate work.

**VIII. CONCLUSION**

To resume, we obtained generating fluctuation-dissipation relations (FDR) for a dynamically perturbed quantum system, assuming special but theoretically and practically important definition of its quantum statistical moments (or corresponding characteristic functional) which describe consecutive or continuous measurements of the system.

In addition, short and expressive formulation of the FDR in terms of the stochastic representation of quantum interactions was done.

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