GENERALIZED THEORIES OF GRAVITY AND CONFORMAL CONTINUATIONS

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Many theories of gravity admit formulations in different, conformally related manifolds, known as the Jordan and Einstein conformal frames. Among them are various scalar-tensor theories of gravity and high-order theories with the Lagrangian \(f(R)\) where \(R\) is the scalar curvature and \(f\) is an arbitrary function. It may happen that a singularity in the Einstein frame corresponds to a regular surface \(S\text{trans}\) in the Jordan frame, and the solution is then continued beyond this surface. This phenomenon is called a conformal continuation (CC). We discuss the properties of vacuum static, spherically symmetric configurations of arbitrary dimension in scalar-tensor and \(f(R)\) theories of gravity and indicate necessary and sufficient conditions for the existence of solutions admitting a CC. Two cases are distinguished, when \(S\text{trans}\) is an ordinary regular sphere and when it is a Killing horizon. Two explicit examples of CCs are presented.

1. Introduction

High-order theories of gravity with the Lagrangian \(L = f(R)\) and scalar-tensor theories (STT) are well-known and important alternatives to Einstein’s general relativity. They are widely used, in particular, for describing inflation in the early Universe [1], for explaining its present-day accelerated expansion [2] and in many other applications. One can also mention that curvature-nonlinear corrections to the Einstein theory emerge due to quantum effects of material fields in curved space [3].

There is a conformal mapping from the manifold \(M_J\) with the metric \(g_{\mu\nu}\), where a theory (STT or \(f(R)\) gravity) is initially formulated (it is called the Jordan conformal frame, or Jordan picture), to the manifold \(M_E\) with the metric \(\bar{g}_{\mu\nu} = g_{\mu\nu}/F(x)\) (the Einstein picture), in which the equations of the original theory turn into the equations of general relativity with a scalar field \(\phi\) endowed with a certain potential \(V(\phi)\) (see, e.g., [4] and references therein). If the conformal factor \(F(x)\) is everywhere regular, then the basic physical properties of the manifolds \(M_J\) and \(M_E\) coincide since, in such transformations, a flat asymptotic in \(M_J\) maps to a flat asymptotic in \(M_E\), a horizon to a horizon, a centre to a centre. Of special interest are, however, the cases when a singularity in \(M_E\) maps (due to the properties of \(F(x)\)) to a regular surface in \(M_J\). Then \(M_J\) may be continued in a regular manner beyond this surface; this phenomenon has been termed conformal continuation [5]. In such cases the global properties of the manifold \(M_J\) may be much richer than those of \(M_E\). The new region

may, in particular, contain a horizon or another spatial infinity.

From a more general viewpoint, a possible existence of conformal continuations may mean that the observed Universe is only a region of a real, much larger Universe which should be described in another, more fundamental conformal frame. Detailed discussions of the physical meaning and role of different conformal frames may be found in Refs. [6, 7].

In this work we discuss necessary and sufficient conditions for the existence of conformal continuations (CC) in vacuum static, spherically symmetric space-times of arbitrary dimension \(D \geq 3\) in STT and \(f(R)\) theories and present two specific examples.

2. Field equations

\(f(R)\) gravity. Consider high-order gravity (HOG) with the action

\[
S_{\text{HOG}} = \int d^Dx \sqrt{|g|}f(R)
\]

where \(f\) is a function of the scalar curvature \(R\) calculated for the metric \(g_{\mu\nu}\) of a space-time \(M_J = M_J[g]\).

In accord with the weak field limit \(f \sim R\) at small \(R\), we assume \(f(R) > 0\) and \(f_R \equiv df/dR > 0\), at least in a certain range of \(R\) including \(R = 0\), but admit \(f_R < 0\) and maybe \(f < 0\) in general.

The conformal mapping \(M_J \rightarrow M_E\) with

\[
g_{\mu\nu} = F(\psi)\bar{g}_{\mu\nu}, \quad F = |f_R|^{-2/(D-2)},
\]

where \(\psi\) is an arbitrary function. It may happen that a
transforms the “Jordan-frame” action (1) into the Einstein-frame action
\[ S = \int d^D x \sqrt{|g|} [R + (\partial \psi)^2 - 2V(\psi)], \tag{3} \]
where
\[ \psi = \pm \sqrt{\frac{D-1}{D-2}} \log |f_R|, \]
\[ 2V(\psi) = |f_R|^{-D/(D-2)}(R|f_R| - f). \tag{4} \]
The field equations due to (1) after this substitution turn into the field equations due to (3). Let us write them down for static, spherically symmetric configurations, taking the metric \( g_{\mu \nu} \) in the form
\[ ds_J^2 = g_{\mu \nu} dx^\mu dx^\nu = A(\rho) d\tau^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho) d\Omega_2^2, \tag{5} \]
where \( d\Omega_2^2 \) is the linear element on a sphere \( S^2 \) of unit radius, and \( \psi = \psi(\rho) \). Three independent combinations of the Einstein equations can be written as
\[ \frac{(A \rho r)}{\rho} = -\frac{1}{4\pi} J^\tau \tau, \tag{6} \]
\[ A(r^2)_{\rho \rho} - r^2 A_{\rho \rho} + (\tau^2 - 2) A_{\rho \rho} (2A_{\rho \rho} - A_{\rho r}) = 2(\tau^2 - 1); \tag{7} \]
The quantities in (9) and (5) are related by
\[ A(\rho) = F A(\rho), \quad r_s(\rho) = F r(\rho), \quad dq = \pm F d\rho. \tag{10} \]

In both metrics (5) and (9) we have chosen the “quasiglobal” radial coordinates \([8]\) \((\rho, q)\), respectively), which are convenient for describing Killing horizons: near a horizon \( \rho = \rho_h \), the function \( A(\rho) \) behaves as \((\rho - \rho_h)^k\) where \( k \) is the horizon order: \( k = 1 \) corresponds to a simple, Schwarzschild-type horizon, \( k = 2 \) to a double horizon, like that in an extremal Reissner-Nordström black hole etc. The function \( A(q) \) plays a similar role in the metric (9).

**General scalar-tensor theory.** The action in \( M_J \), instead of (1), has the form
\[ S_{STT} = \int d^D x \sqrt{|g|} [f(\phi)R + h(\phi)(\partial \phi)^2 - 2U(\phi)], \tag{11} \]
where \( f, h \) and \( U \) are functions of the real scalar field \( \phi \), \((\partial \phi)^2 = g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \).

The conformal mapping \( M_J \mapsto M_E \) with
\[ g_{\mu \nu} = F(\psi) \tilde{g}_{\mu \nu}, \quad F = |f|^{-2/(D-2)}, \tag{12} \]
transforms (11) into the same Einstein-frame action (3), where
\[ \frac{d\psi}{d\phi} = \pm \sqrt{\frac{|l(\phi)|}{f(\phi)}}, \quad l(\phi) \overset{\text{def}}{=} f h + \frac{D-1}{D-2} \left( \frac{df}{d\phi} \right)^2, \]
\[ V(\psi) = |f|^{-D/(D-2)}(\phi) U(\phi). \tag{13} \]
In \( M_E \) we have the same Eqs. (6)-(8).

### 3. Conformal continuations: conditions and properties
Consider the possible situation when the metric \( \tilde{g}_{\mu \nu} \) is singular at some value of \( q \) while \( g_{\mu \nu} \) at the corresponding value of \( q \) is regular. In such a case \( M_J \) can be continued in a regular manner through this surface (to be denoted \( S_{\text{trans}} \), i.e., by definition \([5, 9]\), we have a conformal continuation \( CC \).

In our case of spherical symmetry, the sphere \( S_{\text{trans}} \in M_J \) may be either an ordinary sphere, where both metric coefficients \( r_s^2 \) and \( A \) are finite (we label such a continuation \( CC-I \)), or a Killing horizon at which \( r_s^2 \) is finite but \( A = 0 \) (to be labelled \( CC-II \)).

Without loss of generality, we suppose for convenience that at \( S_{\text{trans}} \) the coordinate values are \( \rho = 0 \) and \( q = 0 \) and \( \rho > 0 \) in \( M_E \) outside \( S_{\text{trans}} \). According to (10), we must have, in terms of \( \tilde{g}_{\mu \nu} \),
\[ F^{-1} \sim r^2 \to 0 \quad \text{as} \quad \rho \to 0, \tag{14} \]
and, in addition, \( A(\rho) \sim r^2(\rho) \) for \( CC-I \), whereas for \( CC-II \) we must have in \( M_J \): \( A(q) \sim q^n \) at small \( q \), where \( n \in \mathbb{N} \) is the order of the horizon.

Let us use the field equations in \( M_E \) for further estimates. Eq. (8) may be put in the form
\[ \frac{d}{d\rho} \left( r^2 \frac{dB}{d\rho} \right) = -2(\tau^2 - 1)r^2 \tau, \tag{15} \]
where the function \( B(\rho) = A/r^2 = B(q) = A/r_s^2 \) is invariant under conformal transformations and should be finite at \( \rho = q = 0 \). Moreover, since \( S_{\text{trans}} \) is a regular sphere in \( M_J \), \( B(q) \) should be a smooth function near \( q = 0 \).

It can be shown \([10]\) that: 1) for \( D = 3 \) a CC can only exist if \( B = B_0 = \text{const} \); 2) for \( D > 3 \) the function \( B(q) \) behaves near \( S_{\text{trans}} \) as
\[ B(q) = B_0 + \frac{\tau}{2} B_2 q^2 + o(q^2), \quad B_2 \neq 0, \tag{16} \]
where \( B_2 < 0 \), i.e., the function \( B(q) \) has a maximum at \( q = 0 \).

All this was obtained by comparing the metrics \( g_{\mu \nu} \) and \( \tilde{g}_{\mu \nu} \), without specifying a theory in which the CC takes place, and for both kinds of transitions, \( CC-I \) and \( CC-II \). Both kinds of transitions are thus possible for
$D > 3$, and, in particular, in CC-II $S_{\text{trans}}$ is a double horizon connecting two $T$ regions (since $B = A/r^2$ is negative at both sides of $S_{\text{trans}}$).

In 3D gravity only CC-I can take place: a horizon, at which $B = 0$ but $B \neq 0$ in its neighbourhood, is inconsistent with the condition $B = \text{const.}$

3.1. Conformal continuations in $f(R)$ theories

Now, for the theory (1), the CC conditions can be made more precise. A transition surface $S_{\text{trans}}$ should correspond to values of $R$ at which the function $F(\psi)$ tends to infinity, i.e., where $f_{R} = 0$. In this case, according to (14), near $\rho = 0$ we have $f_{R}^{2/3} \sim r^{-2}$. Using the expression for $\psi$ in (4) and Eq. (7), we obtain: $r \approx \text{const} \cdot \rho^{1/D}$ as $\rho \to 0$. According to (10), we also obtain: $F \sim \rho^{-2/D}$ and the relation $\rho \sim \rho^{1/D}$ between the coordinates $\rho$ and $q$ at their small values. The results can be summarized in the following theorem [10]:

**Theorem 1.** For a static, spherically symmetric configuration in the theory (1) in $D \geq 3$ dimensions the following necessary conditions and properties of a CC (at $\rho = q = 0$) take place:

(a) $f(R)$ has an extremum, at which $f_{R} = 0$ and $f_{RR} \neq 0$;
(b) $dR/dq \neq 0$ at $q = 0$, hence the ranges of the curvature $R$ are different at the two sides of $S_{\text{trans}}$;
(c) in the Einstein frame, $r(\rho) \sim \rho^{1/D}$ as $\rho \to 0$;
(d) in the Jordan frame, $B(q)$ has a maximum at $q = 0$.
(e) For $D = 3$, $B(\rho) = B(q) = \text{const}$.
(f) A CC-II is only possible for $D \geq 4$, and $S_{\text{trans}}$ is then a double horizon connecting two $T$ regions.

One can prove [10] that the above necessary conditions are also sufficient for the existence of a CC. This is done by seeking the unknown functions in the field equations in $M_{J}$ in the form of Taylor expansions in $q$. Evidently, CC-I are of more general nature than CC-II since the existence of a double horizon is a very special condition for the metric, expressed in the initial condition $B(0) = 0$ in the field equations.

**Example.** Consider an example of an exact solution to the field equations with CC-I in $D = 4$-dimensional space-time. In the Jordan frame $M_{J}$ it is given by the functions

\[
\begin{align*}
  f &= -acR + 2c\sqrt{R} = 2c/q - ac/q^2, \\
  B &= (3q - 2a)/6q^3, \\
  R &= 1/q^2,
\end{align*}
\]

where $a, c - \text{const} > 0$. We take for convenience $a = 1, c = 1$ (choosing the appropriate units). Then $f_{R} = 0$ at $q = q_{\text{trans}} = 1$.

The Jordan and Einstein metrics are

\[
\begin{align*}
  ds_{J}^2 &= \left(\frac{1}{2} - \frac{1}{3q}\right) dt^2 - \left(\frac{1}{2} - \frac{1}{3q}\right)^{-1} dq^2 - q^2 d\Omega^2, \\
  ds_{E}^2 &= |q - 1| ds_{J}^2.
\end{align*}
\]

Thus the Jordan-frame metric has a form close to Schwarzschild’s, it is singular at the centre $q = 0$ and has a horizon at $q = 2/3$. Its asymptotic is non-flat due to a solid angle deficit equal to $2\pi$, i.e., it has the same nature as the asymptotic of a global monopole (as can be easily seen by changing the coordinates from $t$ and $q$ to $t = t/\sqrt{2}$ and $q = q\sqrt{2}$). In $M_{E}$, the metric is singular at $q = 0$ and $q = 1$ and contains a horizon at $q = 2/3$. The manifold $M_{J}$ has two Einstein counterparts $M_{E1}$ and $M_{E2}$, existing separately for $q > 1$ and $q < 1$. The first of them has a non-flat asymptotic as $q \to \infty$ and a naked singularity at the centre ($q = 1$), the other has two singular centres at $q = 0$ and $q = 1$, separated by a horizon at $q = 2/3$.

The scalar field and its potential in $M_{E}$ have the form

\[
\psi = \pm \sqrt{3/2} \ln |q - 1|, \\
V = -\frac{1}{2}q^{-1}(q - 1)^{-2}.
\]

This example is of methodological nature and demonstrates an essential distinction between the descriptions of the theory in the Jordan and Einstein pictures.

3.2. Conformal continuations in STT

Assume now that there is a STT (11) given in $M_{J}$. In this case a CC from $M_{E}$ into $M_{J}$ can occur at such values of the scalar field $\phi$ that the conformal factor $F$ is singular while the functions $f, h$ and $U$ in the action (11) are regular. This means that at $\phi = \phi_{0}$, corresponding to a possible transition surface $S_{\text{trans}}$, the function $f(\phi)$ has a zero of a certain order $n$. We then have in the transformation (13) near $\phi = \phi_{0}$ in the leading order of magnitude

\[
\phi(\psi) \sim \Delta \psi^{n}, \quad n = 1, 2, \ldots, \\
\Delta \psi \equiv \phi - \phi_{0}. \tag{19}
\]

One can notice, however, that $n > 1$ leads to $l(\phi_{0}) = 0$ (recall that by our convention $h(\phi) \equiv 1$). This generically leads to a curvature singularity in $M_{J}$, and though such a singularity can be avoided at some special choices of $f$ and $U$, we will ignore this possibility and simply assume $l > 0$ at $S_{\text{trans}}$.

Thus, according to (13), near $S_{\text{trans}}$ ($\phi = \phi_{0}$)

\[
\phi(\psi) \sim \Delta \psi \sim e^{-\psi \sqrt{d/(d+1)}}, \tag{20}
\]

where without loss of generality we choose the sign of $\psi$ so that $\psi \to \infty$ as $\Delta \psi \to 0$.

One can deduce from (20), (14) and (7) that near $S_{\text{trans}}$ it holds $r(\rho) \sim \rho^{1/D}$. It follows that both $\Delta \psi$ and $q$ behave as $r^{3}$ in the neighbourhood of $S_{\text{trans}}$ hence $d\psi/dq$ is finite.
1. **Continuation through an ordinary sphere (CC-I).** A CC-I can occur if \( F(\phi) = |f|^{-2/3} \sim 1/r^2 \sim 1/A \) as \( \psi \to \infty \), while the behaviour of \( f \) is specified by (20). The following theorem is valid [9]:

**Theorem 2.** Consider scalar-vacuum configurations with the metric (9) and \( \phi = \phi(q) \) in the theory (11) with \( h(\phi) \equiv 1 \) and \( l(\phi) > 0 \). Suppose that \( f(\phi) \) has a simple zero at some \( \phi = \phi_0 \), and \( |U(\phi_0)| < \infty \). Then:

(i) there exists a solution in \( M_1 \), smooth in a neighbourhood of the surface \( S_{\text{trans}} (\phi = \phi_0) \), which is an ordinary regular surface in \( M_1 \);

(ii) in this solution the ranges of \( \phi \) are different at different sides of \( S_{\text{trans}} \).

Thus, in fact, the CC necessary conditions turn out to be sufficient.

2. **Continuation through a horizon in \( M_1 \) (CC-II).** In this case we have near \( q = 0: f(\phi) \sim \Delta \phi, A(q) = AF \sim q^6, r^2(q) = Fr^2 = O(1) \). The following theorem describes the necessary and sufficient conditions for the existence of CC-II [9]:

**Theorem 3.** Consider scalar-vacuum configurations with the metric (9) and \( \phi = \phi(q) \) in the theory (11) with \( h(\phi) \equiv 1 \) and \( l(\phi) > 0 \). Suppose that \( f(\phi) \) has a simple zero at some \( \phi = \phi_0 \). There exists a solution in \( M_1 \), smooth in a neighbourhood of the surface \( S_{\text{trans}} (\phi = \phi_0) \), which is a Killing horizon in \( M_1 \), if and only if:

(a) \( D \geq 4 \),

(b) \( \phi_0 \) is a simple zero of \( U(\phi) \),

(c) \( dU/d\phi > 0 \) at \( \phi = \phi_0 \).

Then, in addition,

(d) \( S_{\text{trans}} \) is a second-order horizon, connecting two T-regions in \( M_1 \);

(e) the ranges of \( \phi \) are different at different sides of \( S_{\text{trans}} \).

Thus the only kind of STT configurations admitting CC-II is a \( D \geq 4 \) Kantowski-Sachs cosmology consisting of two T-regions in \( M_1 \), separated by a second-order horizon.

**Example: Conformal scalar field in GR.** Consider an explicit example of configurations with CC-I for \( D = 4 \). This example is well known [11] and is given here to illustrate the generic character of wormholes appearing due to CC.

The conformal scalar field in GR can be viewed as a special case of STT, such that in (11)

\[
f(\phi) = 1 - \phi^2/6, \quad h(\phi) = 1, \quad U(\phi) = 0.
\]

After the transformation \( g_{\mu\nu} = F(\phi)g_{\mu\nu} \) with \( \phi = \sqrt{\bar{\phi}} \tanh(\psi + \psi_0)/\sqrt{\bar{\phi}} \),

\[
F(\phi) = \cosh^2[(\psi + \psi_0)/\sqrt{\bar{\phi}}], \quad \psi_0 = \text{const},
\]

we obtain the action (3) with \( V \equiv 0 \). The corresponding static, spherically symmetric solution is well known: it is Fisher’s solution [12]. In terms of the harmonic radial coordinate \( u \in \mathbb{R}^+ \), specified by the condition \( g_{uu} = -g_{tt}(g_{\theta\theta})^2 \), the solution is [11]

\[
ds_E^2 = e^{-2mu} dt^2 - k^2 e^{2mu} \left[ \frac{k^2 du^2}{\sinh^2(ku)} + d\Omega^2 \right],
\]

\[
\psi = Cu,
\]

where \( m \) (mass), \( C \) (scalar charge), \( k > 0 \) and \( u_0 \) are integration constants, and \( k \) is expressed in terms of \( m \) and \( C \): \( k^2 = m^2 + C^2/2 \).

Another convenient form of the solution is obtained in isotropic coordinates: with \( y = \tanh(ku/2) \), Eqs. (23) are converted to

\[
ds_E^2 = A(y) dt^2 - \frac{k^2(1-y)^2}{y^4 A(y)} (dy^2 + y^2 d\Omega^2),
\]

\[
A(y) = \left| \frac{1-y}{1+y} \right|^{2m/k}, \quad \psi = C \frac{k}{6} \ln \left| \frac{1+y}{1-y} \right|.
\]

The solution is asymptotically flat at \( u \to 0 \) (\( y \to 0 \)), has no horizon when \( C \neq 0 \) and is singular at the centre \( u \to \infty, y \to 1 - 0, \psi \to \infty \).

The Jordan-frame solution for (21) is described by the metric \( ds^2 = F(\psi)ds_E^2 \) and the \( \phi \) field according to (22). It is the conformal scalar field solution [14, 15], its properties are more diverse and can be described as follows (putting, for definiteness, \( m > 0 \) and \( C > 0 \)):

1. \( C < \sqrt{6}m \). The metric behaves qualitatively as in Fisher’s solution: it is flat at \( y \to 0 \) (\( u \to 0 \)), and both \( g_{tt} \) and \( r^2 = |g_{\theta\theta}| \) vanish at \( y = 1 \) (\( u \to \infty \)) — a singular attracting centre. A difference is that here the scalar field is finite: \( \phi \to \sqrt{6} \).

2. \( C > \sqrt{6}m \). Instead of a singular centre, at \( y \to 1 \) (\( u \to \infty \)) one has a singularity of infinite radius: \( g_{tt} \to \infty \) and \( r^2 \to \infty \). Again \( \phi \to \sqrt{6} \).

3. \( C = \sqrt{6}m, k = 2m \). Now the metric and \( \phi \) are regular at \( y = 1 \); it is \( S_{\text{trans}} \), and the coordinate \( y \) provides a continuation. The solution acquires the form

\[
ds^2 = \frac{(1+y y_0)^2}{1-y_0} \left[ \frac{dt^2}{(1-y)^2} - \frac{m^2(1+y)^2}{y^4} (dy^2 + y^2 d\Omega^2) \right],
\]

\[
\phi = \sqrt{6} \frac{y + y_0}{1 + y y_0},
\]

where \( y_0 = \tanh(\psi_0/\sqrt{6}) \). The range \( u \in \mathbb{R}^+ \), describing the whole manifold \( M_E \) in Fisher’s solution, corresponds to the range \( 0 < y < 1 \), describing only a region \( M_1 \) of the manifold \( M_1 \) of the solution (25). The properties of the latter depend on the sign of \( y_0 \) [11]. In all cases, \( y = 0 \) corresponds to a flat asymptotic, where \( \phi \to \sqrt{6}y_0, |y_0| < 1 \).

3a: \( y_0 < 0 \). The solution is defined in the range \( 0 < y < 1/|y_0| \). At \( y = 1/|y_0| \), there is a naked attracting central singularity: \( g_{tt} \to 0, r^2 \to 0, \phi \to \infty \).
3b: $y_0 > 0$. The solution is defined in the range $y \in \mathbb{R}_+$. At $y \to \infty$, we find another flat spatial infinity, where $\phi \to \sqrt{b}/y_0$, $r^2 \to \infty$ and $g_{tt}$ tends to a finite limit. This is a wormhole solution found in Ref. [11] and recently discussed in Ref. [13].

3c: $y_0 = 0$, $\psi = \sqrt{b}y$, $y \in \mathbb{R}_+$. In this case it is helpful to pass to the conventional coordinate $r = m/(r - m)$. The solution is the well-known black hole with a conformal scalar field [14,15].

The whole manifold $\mathcal{M}_J$ can be represented as the union $\mathcal{M}_J = \mathcal{M}_J^\prime \cup \mathcal{S}_{\text{trans}} \cup \mathcal{M}_J^\prime\prime$ where $\mathcal{M}_J^\prime$ is the region $y < 1$, which is, according to (22), in one-to-one correspondence with the manifold $\mathcal{M}_E$ of the Fisher solution (23). The “antigravitational” ($f(\psi) < 0$) region $\mathcal{M}_J^\prime\prime$ ($y > 1$) is in similar correspondence with another “copy” of the Fisher solution, where, instead of (22),

$$\phi = \sqrt{6}\coth(\psi/\sqrt{6}), \quad F(\psi) = \sinh^2(\psi/\sqrt{6}).$$

(26)

4. Concluding remarks

Studying static, spherically symmetric vacuum solutions in two important classes of theories of gravity, $f(R)$ theories and STT, we have demonstrated that conformal continuations (CC) are quite a widespread phenomenon. It has also been shown [9] that one of generic types of configurations in the Jordan picture in STT, existing due to CC, are traversable wormholes. In $f(R)$ theory, one can also expect the existence of non-singular vacuum solutions of physical interest.

The results presented here may be extended to electrovacuum solutions of the same class of theories as well as to solutions of a more general class of theories, unifying these two, with the action

$$S = \int d^3x \sqrt{|g|} \left[ f(R, \phi) \pm (\partial \phi)^2 \right],$$

(27)

where $f$ is an arbitrary (sufficiently smooth) function of two variables. Such theories also admit a transition to the Einstein picture, but with two scalar fields combined to a kind of sigma model [6]. Different conformal relations between STT and $f(R)$ theories are also discussed in Ref. [16].

It has been shown that many wormhole solutions with CC in STT are unstable under monopole perturbations, but the instability weakens with growing electric charge [17]. Many questions are yet to be answered before one could judge whether or not configurations with CC can describe the space-times able to exist in nature.

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