Nuclear polarizability in muonic atoms: Bayesian analysis of the $\eta$-expansion uncertainty

S. S. Li Muli, B. Acharya, O. J. Hernandez, and S. Bacca

1 Institut für Kernphysik and PRISMA Cluster of Excellence, Johannes Gutenberg-Universität Mainz, 55128 Mainz, Germany
2 Physics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA
3 Helmholtz-Institut Mainz, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany

Abstract.
Over the last decade, spectroscopy of muonic atoms has been used to extract nuclear charge radii with unprecedented precision. This extraction requires a careful analysis of the interplay between theoretical and experimental results, and the attainable precision is nowadays limited by the large uncertainties associated with the theoretical evaluation of the nuclear polarizability effects. To facilitate calculations, these polarizability corrections are conventionally expressed as expansions in a dimensionless parameter $\eta$. Based on the uncertainty principle, $\eta$ has been argued to hold a value of approximately 0.33 in light-nuclear systems. In this work, we check this claim by performing a Bayesian analysis of the nuclear-polarizability corrections to the Lamb shift in $\mu^2H$ and $\mu^3H$ atoms and in $\mu^3\text{He}^+$ and $\mu^4\text{He}^+$ ions at various orders in the $\eta$-expansion. We derive the Bayesian posterior probability distributions for the values of $\eta$, check for their sensitivity to several reasonable choices of the Bayesian priors, and estimate the uncertainties due to truncation of the $\eta$-expansion. This analysis supports the claim that $\eta \ll 1$ for these systems. Overall, the 68% credible intervals are in good agreement with the simpler truncation uncertainty estimates previously reported in the literature, with the only exception of the $\mu^3\text{He}^+$ ion, for which we find larger uncertainties.
1. Introduction

Light-muonic ions, bound systems composed of a muon and a nucleus, have attracted great interest over the last few years. With the muon being roughly 200 times heavier than the electron, the energy spectrum of the bound muon is much more sensitive to the details of the nuclear structure compared to a bound electron, allowing for such muonic systems to be used as an excellent precision probe for nuclear physics. Over the last decade, this feature has been extensively exploited to extract high-precision values of charge radii of light nuclei [1, 2, 3, 4]. The extraction is performed through a precise spectroscopic measurement of the Lamb shift in muonic ions and accurate theoretical calculations. The Lamb shift, $\delta_{\text{LS}}$, is related to the charge radius, $r_c$, by

$$\delta_{\text{LS}} = \delta_{\text{QED}} + r_c \delta_{\text{OPE}} + \delta_{\text{TPE}}.$$  \hspace{1cm} (1)

Here, $\delta_{\text{QED}}$ includes all the radiative effects that stem from a quantum electrodynamical (QED) description of the process [5], $\delta_{\text{OPE}}$ accounts for the nuclear finite-size corrections and can be pictured as a one-photon-exchange (OPE) diagram between the muon and the nucleus with a charge form factor insertion in the nuclear current, and lastly $\delta_{\text{TPE}}$ is the two-photon-exchange (TPE) correction. We note that Eq. (1) neglects both corrections arising from the exchange of three or more photons [6] and effects of the weak force. By measuring the Lamb shift and computing theoretically the terms on the right-hand side of Eq. (1), one is able to extract the charge radius of the nucleus under consideration. Given that the aim is to extract charge radii with the highest possible precision, it is crucial to rigorously assess uncertainties in the theoretical contributions. At present, there is a great interest in improving the precision of the calculation of the $\delta_{\text{TPE}}$ term because it is the dominant source of uncertainty in the extraction of $r_c$ from Eq. (1). This makes it worthwhile to further investigate the uncertainties that accompany the $\delta_{\text{TPE}}$ calculations.

Figure 1: Diagram of the polarizability correction, where the nucleus is excited between two photon vertexes. In the diagram, $\mu$ represents the muon state, $N$ and $N^*$ are correspondingly the ground and excited nuclear states, while $(r, r')$ and $(R, R')$ are the coordinates of the lepton-photon and nucleus-photon vertex, respectively.

It is common to split the total two-photon-exchange correction $\delta_{\text{TPE}}$ into an elastic contribution $\delta_{\text{Zem}}$, which accounts for corrections where the nucleus remains in the
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ground state, and a polarizability contribution $\delta_{\text{pol}}$, where the nucleus transitions to any of its excited states between the photon exchanges, see Fig. 1. We note that a full calculation of the polarizability correction requires a complete knowledge of all possible excited states of the system. Although excitation of the single nucleons can contribute, we neglect them in this work and restrict our study to the nuclear polarizability correction, which we denote by $\delta_{\text{pol}}^A$ to emphasize that we refer to the $A$-body nuclear dynamics [7]. In this paper we will analyze only the non-relativistic contributions of the polarizability corrections and, to simplify the notation, still denote them as $\delta_{\text{pol}}^A$, as opposed to $\delta_{\text{pol}}^{A,\text{NR}}$ which was used in Ref. [7]. Because our goal is to estimate the uncertainty in the $\eta$ expansion, this choice is justified as non-relativistic contributions are larger than relativistic ones, and the latter are expected to scale in $\eta$ in the same way as non-relativistic terms.

In Refs. [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], $\delta_{\text{pol}}^A$ values for different systems were computed by performing a Taylor expansion in the operator $\eta = \sqrt{2m_r\omega_N|\mathbf{R} - \mathbf{R}'|}$, where $m_r$ is the reduced mass of the muon-nuclear system, $\omega_N$ is the nuclear excitation energy and $|\mathbf{R} - \mathbf{R}'|$ can be pictured as the “virtual” distance that a nucleon travels inside the nucleus during the TPE process, see Fig. 1. In Ref. [7], $\eta$ was estimated to be approximately 0.33 for light nuclear systems by invoking the uncertainty principle to relate the operator $|\mathbf{R} - \mathbf{R}'|$ to the momentum scale corresponding to the excitation energy $\omega_N$. In this work, we put for the first time this claim to the testbench by performing a Bayesian analysis of the data of the nuclear polarizability corrections to the Lamb shift in $\mu^2\text{H}$ and $\mu^3\text{H}$ atoms and in $\mu^3\text{He}^+$ and $\mu^4\text{He}^+$ ions. Making different reasonable choices for the Bayesian priors, we derive probability distributions for values of $\eta$ and compute the uncertainties stemming from the truncation of the expansion. For this, we employ tools originally developed to study the truncation errors in nuclear effective field theories [18].

The paper is structured as follows. In Section 2, we review the theoretical framework for the $\eta$-expansion as well as the Bayesian formalism. In Section 3, we present the results of our statistical analysis and Section 4 is reserved for the concluding remarks.

2. Theoretical framework

In this section we introduce the $\eta$-expansion method and develop the tools used in the statistical analysis of the polarizability data. Following Ref. [7], we write the non-relativistic polarizability contribution to the Lamb shift in muonic ions as

$$\delta_{\text{pol}}^A = \sum_{N \neq N_0} \int d^3R \ d^3R' \rho_N^p(R) W(R, R', \omega_N) \rho_N^p(R'),$$

where $\rho_N^p(R) = \langle N| \frac{1}{2} \sum_{i=1}^A \delta(R - R_i) \hat{e}_i^p |N_0\rangle$ is the proton transition density, $\hat{e}_i^p$ is a proton-projection operator on the Hilbert space of the $i$-th nucleon, $N$ and $N_0$ are nuclear

† These effects have been labelled in Ref. [7] as $\delta_{\text{pol}}^N$ and the total polarizability is a sum of nuclear and pure nucleonic corrections $\delta_{\text{pol}} = \delta_{\text{pol}}^N + \delta_{\text{pol}}^A$. 

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quantum numbers with $N_0$ representing the ground state, $\omega_N$ is the excitation energy of the $N$-th nuclear state, $Z$ is the charge number of the nucleus, and $W(R, R', \omega_N)$ is the lepton matrix element. After carrying out the integral over the exchanged-momentum $q$, $W(R, R', \omega_N)$ can be written in terms of $\eta$ as

$$W(R, R', \omega_N) = -\frac{\pi}{m_r^2} (Z\alpha)^2 \phi^2(0) \left( \frac{2m_r}{\omega_N} \right)^\frac{3}{2} \frac{1}{\eta} \left( e^{-\eta} - 1 + \eta - \frac{1}{2} \eta^2 \right).$$  \hspace{1cm} (3)

Here, $\alpha$ is the fine-structure constant and $\phi^2(0)$ is the norm of the muonic 2S-state wave function. Next, we expand Eq. (3) in a Taylor series over $\eta$ obtaining

$$W(R, R', \omega_N) = \frac{\pi}{6m_r} (Z\alpha)^2 \phi^2(0) \left( \frac{2m_r}{\omega_N} \right)^\frac{3}{2} \left[ \eta^2 - \frac{1}{4} \eta^3 + \frac{1}{20} \eta^4 + \ldots \right].$$  \hspace{1cm} (4)

Inserting Eq. (4) into Eq. (2), we obtain the polarizability correction as an expansion in $\eta$

$$\delta^A_{\text{pol}} = \frac{\pi}{6m_r} (Z\alpha)^2 \phi^2(0) (2m_r)^\frac{3}{2} \sum_{N \neq N_0} \left( \frac{1}{\omega_N} \right)^\frac{3}{2} \times$$

$$\times \int d^3R \int d^3R' \rho^p_N(R) \left[ \eta^2 - \frac{1}{4} \eta^3 + \frac{1}{20} \eta^4 + \ldots \right] \rho^p_N(R') = D_2 + D_3 + D_4 + \ldots,$$

where the index in the “D” terms in the bottom line indicates the power of $\eta$ entering each term in the above line. Note that in Ref. [7] the leading ($D_2$), subleading ($D_3$) and sub-sub-leading ($D_4$) contributions were labelled as $\delta^{(0)}$, $\delta^{(1)}$ and $\delta^{(2)}$, respectively.

We now assume that $\delta^A_{\text{pol}}$ can be expressed as a power law in $\eta$

$$\delta^A_{\text{pol}} = X_{\text{ref}} \left[ c_2 \eta^2 + c_3 \eta^3 + c_4 \eta^4 + \ldots \right],$$  \hspace{1cm} (6)

where $c_i$ are dimensionless coefficients that may depend on mass number $A$ and charge number $Z$, and $X_{\text{ref}}$ is the natural scale of $\delta^A_{\text{pol}}$. Starting with this assumption we perform a Bayesian analysis of the convergence of the $\eta$-expansion. We use the data in Table 1 taken from Ref. [7] which correspond to calculations in chiral effective field theory ($\chi$-EFT) with an Hamiltonian that includes a two-body force at next-to-next-to-next-to-leading order [19] and a three-body force at next-to-next-to-leading order [20]. To ensure that the coefficients $c_i$ are dimensionless and natural, i.e. $O(1)$, the reference $X_{\text{ref}}$ should have the same dimensions and order of magnitude as the analyzed observable. We use as $X_{\text{ref}}$ the $\delta_{\text{TPE}}$ value computed in Ref. [7] with the phenomenological AV18+UIX interaction [21, 22].

We notice that going from Eq. (5) to Eq. (6) is analogous to what has been extensively done in $\chi$-EFT. There, the nuclear Hamiltonian is expanded in powers of a dimensionless ratio $Q/\Lambda$ where $Q$ is the typical low momentum scale of the process under study and $\Lambda$ is the breakdown scale. The calculated low-energy observables are
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then assumed to inherit this \( Q/\Lambda \) expansion and uncertainties are quantified by studying their order-by-order convergence [18, 23, 24, 25, 26].

As in \( \chi \)-EFT, the framework of Eq. (5) and (6) is only useful if the expansion parameter \( \eta \) has a value smaller than 1. In particular, since the higher-order terms become progressively more complicated to calculate [7], the practical utility of this expansion depends on a fast convergence so that meaningful estimates can be obtained by truncating the series at low orders.

In notation Ref. [7]

\[
\begin{array}{c|c|c|c|c}
D_2 & \delta_{D1}^{(0)} & -1.912 & -0.7848 & -6.633 & -4.701 \\
D_3 & \delta_{R3}^{(1)} + \delta_{Z3}^{(1)} & 0.359 & 0.1844 & -0.384 & 0.809 \\
D_4 & \delta_{Q}^{(2)} + \delta_{D1D3}^{(2)} + \delta_{R2}^{(2)} & -0.037 & -0.0247 & 0.83 & 0.101 \\
\sum_{k=2}^{4} D_k & - & -1.590 & -0.6251 & -6.187 & -3.791 \\
X_{\text{ref}} & - & -1.664 & -0.6986 & -14.564 & -8.220 \\
\end{array}
\]

Table 1: The nuclear polarizability correction to the Lamb shift (in meV) for several light muonic atoms at various orders in the \( \eta \)-expansion. The reference \( X_{\text{ref}} \) are the \( \delta_{\text{TPE}} \) values evaluated using the AV18+UIX interaction. Data taken from Ref. [7].

Next, we introduce the Bayesian method that allows us to compute the probability distribution for the value of \( \eta \) and to obtain the truncation error in the \( \eta \)-expansion therefrom. Given a data set \( \{D_m, \ldots, D_k\} \), Bayes’ theorem expresses the posterior probability distribution of \( \eta \) as

\[
pr(\eta|D_m, \ldots, D_k) = \frac{pr(D_m, \ldots, D_k|\eta) pr(\eta)}{pr(D_m, \ldots, D_k)}. \tag{7}
\]

Here, \( pr(D_m, \ldots, D_k|\eta) \) is known as the likelihood function and represents the probability of obtaining the data set given a particular value of \( \eta \). The prior probability distribution \( pr(\eta) \) should summarize the information we have about \( \eta \) before analyzing the data set. The denominator \( pr(D_m, \ldots, D_k) \) is the marginal likelihood, which, being independent of \( \eta \), can be taken as a normalization constant for our purpose. We make the assumption that the dimensionless coefficients \( c_i \), which are related to the data set \( \{D_m, \ldots, D_k\} \) by \( D_i = c_i \eta^i \), are uncorrelated among each other and their scale is set by a single scale parameter \( \bar{c} \) through a probability distribution \( pr(c_i|\bar{c}) \) for \( i = m, \ldots, k \). We then eliminate the dependence on this “nuisance” parameter \( \bar{c} \) by marginalizing over it as

\[
pr(D_m, \ldots, D_k|\eta) = \int dc_m \cdots dc_k pr(D_m, \ldots, D_k|c_m, \ldots, c_k, \eta) pr(c_m, \ldots, c_k|\eta)
\]

\[
= \int d\bar{c} dc_m \cdots dc_k pr(D_m, \ldots, D_k|c_m, \ldots, c_k, \eta) \ pr(c_m, \ldots, c_k|\bar{c}, \eta) \ pr(\bar{c}|\eta). \tag{8}
\]

It follows from Eq. (6) and from the assumption that the dimensionless coefficients \( c_i \)
are uncorrelated that

\[ \text{pr}(D_m, ..., D_k | c_m, ..., c_k, \eta) = \prod_{i=m}^{k} \text{pr}(D_i | c_i, \eta) = \prod_{i=m}^{k} \delta(D_i - \eta^i c_i) \]

\[ = \prod_{i=m}^{k} \frac{1}{\eta^i} \delta(D_i/\eta^i - c_i) = \frac{1}{\eta^{(k-m+1)(k+m)/2}} \prod_{i=m}^{k} \delta(D_i/\eta^i - c_i). \quad (9) \]

Using the delta functions to analytically perform the integrals over the dimensionless coefficients in Eq. (8), we obtain

\[ \text{pr}(D_m, ..., D_k | \eta) = \frac{1}{\eta^{(k-m+1)(k+m)/2}} \int \text{pr} \left( c_m = \frac{D_m}{\eta^m}, ..., c_k = \frac{D_k}{\eta^k} \mid \bar{c}, \eta \right) \text{pr} (\bar{c}) \, d\bar{c}, \quad (10) \]

where we further assumed that the scale-parameter \( \bar{c} \) is independent of \( \eta \). The posterior \( \text{pr}(\eta | D_m, ..., D_k) \) can then be written as

\[ \text{pr}(\eta | D_m, ..., D_k) = \frac{\mathcal{N}}{\eta^{(k-m+1)(k+m)/2}} \int \text{pr} \left( c_m = \frac{D_m}{\eta^m}, ..., c_k = \frac{D_k}{\eta^k} \mid \bar{c}, \eta \right) \text{pr} (\bar{c}) \, \text{pr}(\eta) \, d\bar{c}, \quad (11) \]

where the normalization constant is \( \mathcal{N} = \text{pr}(D_m, ..., D_k)^{-1} \).

It was argued in Ref. [7] that the parameter \( \eta \) has an approximate value of 0.33 for light-nuclear systems, and has, at most, a weak dependence on the exact system at hand. It is then interesting to explore the consequences of combining the data sets \( \{D_m, ..., D_k\} \) for different systems under the assumptions that they correspond to the same underlying probability distribution for \( \eta \) and for the expansion coefficients \( c_i \). To this end, we work with a new data set \( \{D^0_m, ..., D^0_k\} \equiv \{D^0_m, ..., D^0_k, ..., D^n_m, ..., D^n_k\} \) where \( \{D^0_m, ..., D^0_k\} \) is the data set of \( \mu^2 \text{H} \), \( \{D^1_m, ..., D^1_k\} \) is that of \( \mu^3 \text{H} \), \( \{D^2_m, ..., D^2_k\} \) that of \( \mu^3 \text{He}^+ \) and finally \( \{D^3_m, ..., D^3_k\} \) is that of \( \mu^4 \text{He}^+ \). Proceeding as before, the posterior for \( \eta \) can be written as

\[ \text{pr}(\eta | D^0_m, ..., D^n_k) = \frac{\mathcal{N}}{\eta^{(n+1)(k-m+1)(k+m)/2}} \int \text{pr} \left( c^0_m = \frac{D^0_m}{\eta^m}, ..., c^n_k = \frac{D^n_k}{\eta^k} \mid \bar{c}, \eta \right) \text{pr} (\bar{c}) \, \text{pr}(\eta) \, d\bar{c}, \quad (12) \]

where the normalization constant \( \mathcal{N} \) now denotes \( \text{pr}(D^0_m, ..., D^n_k)^{-1} \).

Next, we want to find a probability distribution for the truncation error of the \( \eta \)-expansion of Eq. (6), \( \Delta_k^{(1)} \), in the approximation that this is dominated by the first omitted term, \( \Delta_k^{(1)} \sim c_{k+1} \eta^{k+1} \). Below, we give a brief overview of the Bayesian methodology for quantification of truncation errors and refer the reader to Ref. [18] for further details.

Given the data set \( \{D_m, ..., D_k\} \), we begin by expressing the Bayesian posterior for the truncation error at a given value of \( \eta \), \( \text{pr}(\Delta_k^{(1)} | D_m, ..., D_k, \eta) \), as an integrated
Applying Bayes’ theorem on the last term of Eq. (14), we arrive at the expression

\[
\text{pr}(\Delta_k^{(1)} | D_m, ..., D_k, \eta) = \\
\int dc_{k+1} \text{pr}(\Delta_k^{(1)} | D_m, ..., D_k, \eta, c_{k+1}) \text{pr}(c_{k+1} | D_m, ..., D_k, \eta)
\]

\[
= \int dc_{k+1} \delta(\Delta_k^{(1)} - c_{k+1} \eta^{k+1}) \text{pr}(c_{k+1} | D_m, ..., D_k, \eta)
\]

\[
= \frac{1}{\eta^{k+1}} \text{pr}(c_{k+1} = \frac{\Delta_k^{(1)}}{\eta^{k+1}} | D_m, ..., D_k, \eta).
\]

We introduce again the scale parameter \(\bar{c}\) by marginalizing over it

\[
\text{pr}(\Delta_k^{(1)} | D_m, ..., D_k, \eta) = \frac{1}{\eta^{k+1}} \int d\bar{c} \text{pr}(c_{k+1} = \frac{\Delta_k^{(1)}}{\eta^{k+1}} | D_m, ..., D_k, \bar{c}) \text{pr}(\bar{c} | D_m, ..., D_k, \eta).
\]

Applying Bayes’ theorem on the last term of Eq. (14), we arrive at the expression

\[
\text{pr}(\Delta_k^{(1)} | D_m, ..., D_k, \eta) = \frac{\int d\bar{c} \text{pr}(c_{k+1} = \frac{\Delta_k^{(1)}}{\eta^{k+1}} | \bar{c}) \left[ \prod_{i=m}^{k} \text{pr}(c_i = \frac{D_i}{\eta^i} | \bar{c}) \right] \text{pr}(\bar{c})}{\eta^{k+1} \int d\bar{c} \left[ \prod_{i=m}^{k} \text{pr}(c_i = \frac{D_i}{\eta^i} | \bar{c}) \right] \text{pr}(\bar{c})}.
\]

We now marginalize over the expansion parameter \(\eta\) to obtain

\[
\text{pr}(\Delta_k^{(1)} | D_m, ..., D_k) = \int d\eta \text{pr}(\Delta_k^{(1)} | D_m, ..., D_k, \eta) \text{pr}(\eta | D_m, ..., D_k),
\]

with \(\text{pr}(\eta | D_m, ..., D_k)\) given by Eq. (11).

The generalization for the combined data set \(\{D_{m}^{0}, ..., D_{n}^{n}\}\) is straightforward and yields

\[
\text{pr}(\Delta_k^{(1)} | D_{m}^{0}, ..., D_{k}^{n}, \eta) = \frac{\int d\bar{c} \text{pr}(c_{k+1} = \frac{\Delta_k^{(1)}}{\eta^{k+1}} | \bar{c}) \left[ \prod_{i=m}^{k} \prod_{j=0}^{n} \text{pr}(c_i^j = \frac{D_i^j}{\eta^i} | \bar{c}) \right] \text{pr}(\bar{c})}{\eta^{k+1} \int d\bar{c} \left[ \prod_{i=m}^{k} \prod_{j=0}^{n} \text{pr}(c_i^j = \frac{D_i^j}{\eta^i} | \bar{c}) \right] \text{pr}(\bar{c})}.
\]

After marginalization over \(\eta\) we obtain the posterior distribution of the truncation error,

\[
\text{pr}(\Delta_k^{(1)} | D_{m}^{0}, ..., D_{k}^{n}) = \int d\eta \text{pr}(\Delta_k^{(1)} | D_{m}^{0}, ..., D_{k}^{n}, \eta) \text{pr}(\eta | D_{m}^{0}, ..., D_{k}^{n}),
\]

with \(\text{pr}(\eta | D_{m}^{0}, ..., D_{k}^{n})\) given by Eq. (12).

2.1. Priors

To proceed with the statistical analysis, we need to specify the prior probability distributions, \(\text{pr}(\bar{c})\), \(\text{pr}(c_i | \bar{c})\) and \(\text{pr}(\eta)\). Whenever possible, we will be guided by the principle of maximum entropy [27] to find the least informative priors given some basic properties of the parameters \(c_i, \bar{c}\) and \(\eta\). Below we report a few considerations on the choice of the priors:
• The parameter $\bar{c}$ sets the overall scale of the dimensionless coefficients $c_i$. Our unbiased expectations about $\bar{c}$ under the only constraint that it is a positive definite quantity can be encoded by adopting the Jeffreys’ prior, $\text{pr}(\bar{c}) \propto 1/\bar{c}$ [28]. In order to work with a normalizable distribution, we introduce a slight modification and restrict the range of $\bar{c}$ from a minimum value of $c_\prec = 10^{-4}$ to a maximum value of $c_\succ = 10^4$ by multiplying with step functions $\theta(x)$.

• For $\text{pr}(c_i|\bar{c})$, we test two different reasonable choices. The extent to which the obtained posterior distributions are sensitive to these choices will tell us whether the data set is sufficiently informative to dominate the analysis. Our first choice, labelled prior A, assumes that the magnitude of the dimensionless coefficients can not be larger than $\bar{c}$. The maximum-entropy principle then leads to a uniform distribution in the range $-\bar{c} < c_i < \bar{c}$ with $i = m, \ldots, k$. Our second choice, labelled prior B, is a Gaussian distribution with zero mean and standard deviation $\bar{c}$. It was first adopted in nuclear effective field theories in Ref. [29] and is motivated by the maximum-entropy principle under the assumption of testable information over the means and standard deviations of the dimensionless coefficients.

• For $\text{pr}(\eta)$, we first note that this parameter is constrained to hold a positive value. In what follows, we assume that the simple estimate of $\eta \approx 0.33$ can be regarded as an information on the mean of $\text{pr}(\eta)$. The exponential distribution is the least-informative prior for a parameter which is positive definite and whose mean is known [30]. We label this choice as $\alpha_\eta$. To perform a check on how much the conclusions of the statistical analysis are stable with respect to reasonable modifications of the $\eta$-prior, we compare this choice with a $\beta$-distribution, $\beta(a,b)$, which constrains $\eta \in [0,1]$, and label this second prior choice as $\beta_\eta$. The parameters of $\beta_\eta$ are chosen such that the mean falls on 0.33. There is an infinite set of $\beta$-distributions that satisfy this constraint. We selected $a = 3.0$ and $b = 6.0$, because it holds a reasonable standard deviation for both the prior and the data to be informative.

We list in Table 2 the prior-choices for $\text{pr}(\bar{c})$ and $\text{pr}(c_i|\bar{c})$, while the prior choices for $\text{pr}(\eta)$ are listed in Table 3.

| Priors | $\text{pr}(c_i|\bar{c})$ | $\text{pr}(\bar{c})$ |
|--------|------------------------|------------------------|
| A      | $\frac{1}{2\pi\bar{c}}\theta(\bar{c} - |c_i|)$ | $\frac{1}{\ln(c_\succ/c_\prec)}\theta(\bar{c} - \bar{c}_\prec)\theta(\bar{c}_\succ - \bar{c})$ |
| B      | $\frac{1}{\sqrt{2\pi}\bar{c}}\text{Exp}\left(-\frac{c_i^2}{2\bar{c}^2}\right)$ | $\frac{1}{\ln(c_\succ/c_\prec)}\theta(\bar{c} - \bar{c}_\prec)\theta(\bar{c}_\succ - \bar{c})$ |

Table 2: Prior choices for the probability density distributions of the scale parameter $\bar{c}$ and for the dimensionless coefficients $c_i$. 

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| Priors | $\text{pr}(\eta)$ |
|--------|-------------------|
| $\alpha_\eta$ | $\frac{1}{\lambda} \exp \left( \frac{\eta}{\lambda} \right)$ |
| $\beta_\eta$ | $\beta(a, b)$ |

Table 3: Prior choices for the probability density distribution of the $\eta$ expansion parameter. Following the prediction in Ref. [9, 7] the mean of the exponential distribution is $\lambda = 0.33$, while for the $\beta$-distribution we take $a = 3.0$ and $b = 6.0$.

3. Results

Our first aim is to obtain posterior distributions for values of $\eta$ and check whether they depart from the priors motivated by the estimate of $\eta \sim 0.33$.

Figure 2: Analysis for the separate muonic systems. (Upper panel) Prior probability distributions $\text{pr}(\eta)$: $\alpha_\eta$ choice (left) and $\beta_\eta$ choice (right). (Lower panels) Corresponding posterior probability distributions $\text{pr}(\eta | D_2, \ldots, D_4)$ for the choice of $\text{pr}(\eta) = \alpha_\eta$ (left) and $\text{pr}(\eta) = \beta_\eta$ (right) and two given choices (A and B) of priors for $\text{pr}(\bar{c})$ and $\text{pr}(c_i | \bar{c})$. 
First, in Figure 2 we show the resulting posterior distributions for pr(η | D_2, ..., D_4) for different choices of the priors pr(η), pr(c_i | ¯c) and pr(¯c), when we do a separate analysis of each individual muonic system. Overall, the posterior distributions are quite stable under modifications of the priors, suggesting that there is enough information in the data sets to dominate the analysis. In particular for pr(η), we note that the exponential distribution only assumes η > 0, whereas the β-distribution constrains η ∈ [0, 1]. We find only weak deviations in the results with the two priors. This increases the significance of the assumption that η ∈ [0, 1], which in turns means that the expansion in Eq. (6) converges. Our Bayesian analysis indicates that the most likely η-value is smaller than the estimate obtained from the uncertainty principle for the μ^2H, μ^3H and μ^4He systems, because the maximum-likelihood value of η is approximately 0.15.

We then perform a combined analysis using all the muonic atoms and show the results in Figure 3. Also in this case we present results for two choices of the priors pr(η),
namely $\alpha_n$ and $\beta_n$, and for the choice A and B of the other two priors, as reported in Table 2. In this case, we note that the posterior probability distributions do not depend much on the $\eta$-priors $\alpha_n$ or $\beta_n$, while choices A and B give quite different results. This last feature occurs because in the process of combining the data sets it is assumed that all the expansion coefficients are drawn from the same underlying distribution. Although the assumption seems reasonable for the data sets of $\mu^2H$, $\mu^3H$ and $\mu^4He^+$, it is unclear whether the same applies to the $\mu^3He^+$ system where the expansion coefficients are – at fixed $\eta$-values – systematically larger compared to the other muonic atoms. The effect of the $\mu^3He^+$ system in the combined analysis impacts more the results with prior A than prior B, because calculations based on prior B are less sensitive to extreme values of the coefficients $c_i$. In particular, we find that choice B predicts a lower maximum-likelihood for $\eta$ than choice A, so that Eq. (6) will converge faster. Interestingly, the maximum-likelihoods of $\eta$ in the combined analysis are $\eta \sim 0.22$ (A) and $\eta \sim 0.20$ (B), which resemble a sort of “average” between the individual distributions of the separate analysis.

As a next step, we address the evaluation of the truncation errors in the $\eta$-expansion of $\delta^A_{pol}$ with a Bayesian analysis. Using the posterior distributions $pr(\eta|D_2, ..., D_4)$ of Figure 2, we calculate the distributions for the truncation uncertainty $pr(\Delta_k^{(1)}|D_2, ..., D_4)$ by marginalization as shown in Eq. (16). We show first the results of the separate analysis in Figure 4 and we tabulate the confidence interval (CI) at the 68% and 95% level in Table 4. Again, the distributions show good stability when the prior choices get modified and we find a fairly good agreement between the 68% CI uncertainties of this work and the estimates in Refs. [7] as shown in Table 4, with the exception of the truncation uncertainties in the $\mu^3He^+$ system, which we found to be roughly a factor of 4 larger. In the specific case of this muonic atom, we see that choices A and B yield different posterior distributions but similar 68% CI. We note that the 95% CIs in Table 4 are much larger than twice the size of the corresponding 68% CIs for all systems and all prior choices, reflecting the large tails of the posterior distributions compared to the Gaussian (see Figure 4).

When performing the analogous analysis of the combined data, reported in Figure 5 and in Table 5, we see that the results of the posterior distributions of $pr(\Delta_k^{(1)}|D_0, ..., D_4)$ depend on the prior choice A and B, but only marginally on $\alpha_\eta$ and $\beta_\eta$. As for the posterior distributions of the truncation uncertainty, we find that the truncation errors in $\mu^2H$ do not change by much while in $\mu^4He^+$ they become larger and for $\mu^3He^+$ and $\mu^3H$ they are reduced upon combining the data sets. Again, the calculated posterior distributions have much larger tails that the Gaussian distribution for all prior choices.

The $\eta$-prior $\alpha_\eta$ efficiently summarizes, through the maximum-entropy principle, the information available on the parameter $\eta$ before the calculation of the data sets. The analysis with prior $\beta_\eta$, on the other hand, makes additional assumptions about the value of $\eta$, e.g., it constrains $\eta$ to a value smaller than 1. While this assumption is well supported by both prior and posterior information available to us, it does not allow us to diagnose divergences of the expansion in Eq. (6). Furthermore, the analysis with
Figure 4: Posterior distributions for the truncation uncertainties of the $\eta$-expansion in a Bayesian analysis of the separate muonic systems. Results with $\text{pr}(\eta) = \alpha_\eta$ (left) and $\text{pr}(\eta) = \beta_\eta$ (right). Solid lines (dotted lines) represent posterior distributions obtained with choices A (B) for priors of Table 2. The 68% and 95% CI are reported as dark and light shaded areas for the prior choice A, respectively, while for the prior choice B they are reported as vertical dashed lines.

prior B for $\text{pr}(c_i|\bar{c})$ is more sensitive to details of the distributions of the coefficients $c_i$ compared with the analysis with prior A. We therefore consider the uncertainty estimates obtained with the combinations of priors $\alpha_\eta$ and $B$ to be better calibrated than the other choices explored in this work.

We would like to comment on the fact that we get an $\eta$-expansion uncertainty larger than previously estimated for $\mu^3\text{He}^+$. As already pointed out in Ref. 13, this muonic system does not display the expected scaling in $\eta$, where, as one can see from Table 1, $D_3$ is unusually small and $D_4$ unusually large. In Ref. 7, the $\eta$ parameter has been estimated by taking an average of the ratio between the first terms appearing in the Taylor expansion of Eq. (5), namely $\eta \sim 4|D_3/D_2|$ and $\eta \sim \sqrt{20}|D_4/D_2|^{1/2}$, and the final uncertainty has been calculated as $\frac{1}{120}\eta^3$, which is the first omitted term in Eq. (5). Here, instead, we do not take any coefficients from the Taylor expansion, but leave the constants $c_i$ of Eq. (6) free to float, requiring that they are natural in our analysis. Hence, the Bayesian analysis is on the one hand sensitive to the unusually large ratio $D_4/D_3$ and on the other hand insensitive to the suppressing factor $1/120$, which explains why

$\S$ Note that here relativistic terms were included.
Figure 5: Posterior distributions for the leading order truncation uncertainties of the $\eta$-expansion in a Bayesian analysis for the combined muonic systems. Results with $\text{pr}(\eta) = \alpha_\eta$ (left) and $\text{pr}(\eta) = \beta_\eta$ (right). Solid lines represent posterior distributions obtained with choices A for priors $c_k$’s and $\bar{c}$, while dotted lines correspond to the choice B. The 68% and 95% CI are reported as dark and light shaded areas for the prior choice A, respectively, while for the prior choice B they are reported as vertical dashed lines.

we get a larger uncertainty. In general, the Bayesian analysis gives more conservative uncertainty estimates for a given value of $\eta$. However, since the posterior probability distributions of $\eta$ for $\mu^2H$, $\mu^3H$, and $\mu^4He^+$ peak below $\eta = 0.33$, our 68% CI are in good agreement with the estimates from Ref. [7]. For $\mu^3He^+$ the posterior probability distribution peaks slightly above 0.33 leading to larger uncertainties compared to Ref. [7].

4. Conclusion

In this work, we have performed a Bayesian analysis of the polarizability data sets of the nuclear structure corrections to the Lamb shift in $\mu^2H$, $\mu^3H$, $\mu^3He^+$ and $\mu^4He^+$. For the $\mu^2H$, $\mu^3H$ and $\mu^4He^+$ systems we find that the maximum-likelihood value for $\eta$ is about half the value of 0.33 estimated using the uncertainty principle. Despite the smaller value of the expansion parameter, the Bayesian analysis gives 68% CIs for the truncation uncertainty that are in good agreement with the estimates in Ref. [7]. When compared to the other muonic systems, both the value of $\eta$ and the value of the truncation uncertainty in $\mu^3He^+$ are anomalously large. Most likely this is the consequence of the very large sub-sub-leading correction ($D_4$) to $\delta_{\text{pol}}^A$ in this system. From our analysis, we find that the
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| Atom   | Prior | 68% CI $(\alpha_\eta)$ | 68% CI $(\beta_\eta)$ | 95% CI $(\alpha_\eta)$ | 95% CI $(\beta_\eta)$ | Ref. |  |
|--------|-------|-------------------------|------------------------|-------------------------|------------------------|------|---|
| $\mu^2$H | A     | 0.60                    | 1.03                   | 2.81                    | 3.72                   | 0.4  |  |
|         | B     | 0.79                    | 1.38                   | 3.46                    | 4.07                   |  |
| $\mu^3$H | A     | 1.18                    | 1.73                   | 5.44                    | 6.67                   | 1.3  |  |
|         | B     | 1.52                    | 2.29                   | 6.88                    | 7.73                   |  |
| $\mu^3$He$^+$ | A   | 4.95                    | 4.90                   | 14.47                   | 13.05                  | 1.1  |  |
|         | B     | 4.81                    | 4.74                   | 14.99                   | 13.87                  |  |
| $\mu^4$He$^+$ | A | 0.63                    | 1.02                   | 3.32                    | 4.86                   | 0.8  |  |
|         | B     | 0.89                    | 1.55                   | 4.70                    | 5.80                   |  |

Table 4: Bayesian analysis of the separated muonic systems: truncation uncertainties in the $\eta$-expansion of $\delta_{\text{pol}}^A$ expressed as confidence interval % for various prior choices.

| Atom   | Prior | 68% CI $(\alpha_\eta)$ | 68% CI $(\beta_\eta)$ | 95% CI $(\alpha_\eta)$ | 95% CI $(\beta_\eta)$ | Ref. |  |
|--------|-------|-------------------------|------------------------|-------------------------|------------------------|------|---|
| $\mu^2$H | A     | 1.06                    | 1.09                   | 1.81                    | 1.91                   | 0.4  |  |
|         | B     | 0.87                    | 1.01                   | 2.43                    | 2.77                   |  |
| $\mu^3$H | A     | 1.12                    | 1.17                   | 1.94                    | 2.05                   | 1.3  |  |
|         | B     | 0.93                    | 1.09                   | 2.69                    | 3.14                   |  |
| $\mu^3$He$^+$ | A | 2.37                    | 2.44                   | 4.07                    | 4.31                   | 1.1  |  |
|         | B     | 1.96                    | 2.30                   | 5.65                    | 6.58                   |  |
| $\mu^4$He$^+$ | A | 2.18                    | 2.25                   | 3.74                    | 3.95                   | 0.8  |  |
|         | B     | 1.79                    | 2.07                   | 4.84                    | 5.42                   |  |

Table 5: Bayesian analysis of the combined muonic systems: truncation uncertainties in the $\eta$-expansion of $\delta_{\text{pol}}^A$ expressed as confidence interval % for various prior choices.

The maximum-likelihood value of $\eta$ is 0.35, whereas the 68% CI of the truncation uncertainty is $\sim 5\%$ of the total non-relativistic polarizability contribution. When combining the data sets we obtain an “average” of the results coming from the individual analysis, with the uncertainties in $\mu^2$H being overall unchanged, the uncertainties in $\mu^4$He$^+$ becoming slightly larger, the uncertainties in $\mu^3$H slightly smaller and the uncertainties in $\mu^3$He$^+$ becoming considerably smaller.

The analysis indicates that the $\eta$-expansion in $\mu^3$He$^+$ might converge slower than
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previously expected. Our updated value for the truncation uncertainty of the $\eta$-expansion in $\mu^3\text{He}^+$ is as large as other contributions, such as the Coulomb term (see Table 8 in Ref. [7]). A possible solution could be found by using the $\eta$-less method [31, 32] for the evaluation of the nuclear polarizability effects in $\mu^3\text{He}^+$, that completely avoids the expansion in $\eta$, but requires the more cumbersome calculation of the longitudinal and transverse response functions. This method has been so far successfully implemented and used only for the $\mu^2\text{H}$ system.

Acknowledgments

We would like to thank Daniel Phillips, Nir Barnea and Chen Ji for useful discussions. This work was supported by the Deutsche Forschungsgemeinschaft (DFG) with the Collaborative Research Center 1044 and through the Cluster of Excellence “Precision Physics, Fundamental Interactions, and Structure of Matter” (PRISMA+ EXC 2118/1) funded by the DFG within the German Excellence Strategy (Project ID 39083149).

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