Sign non-reversal property for totally non-negative and totally positive matrices, and testing total positivity of their interval hull

Projesh Nath Choudhury, M. Rajesh Kannan and Apoorva Khare

Abstract

A matrix $A$ is totally positive (or non-negative) of order $k$, denoted $TP_k$ (or $TN_k$), if all minors of size $\leq k$ are positive (or non-negative). It is well known that such matrices are characterized by the variation diminishing property together with the sign non-reversal property. We do away with the former, and show that $A$ is $TP_k$ if and only if every submatrix formed from at most $k$ consecutive rows and columns has the sign non-reversal property. In fact, this can be strengthened to only consider test vectors in $\mathbb{R}^k$ with alternating signs. We also show a similar characterization for all $TN_k$ matrices — more strongly, both of these characterizations use a single vector (with alternating signs) for each square submatrix. These characterizations are novel, and similar in spirit to the fundamental results characterizing $TP$ matrices by Gantmacher–Krein (Compos. Math. 4 (1937) 445–476) and $P$-matrices by Gale–Nikaido (Math. Ann. 159 (1965) 81–93).

As an application, we study the interval hull $I(A, B)$ of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$. This is the collection of $C \in \mathbb{R}^{m \times n}$ such that each $c_{ij}$ is between $a_{ij}$ and $b_{ij}$. Using the sign non-reversal property, we identify a two-element subset of $I(A, B)$ that detects the $TP_k$ property for all of $I(A, B)$ for arbitrary $k \geq 1$. In particular, this provides a test for total positivity (of any order), simultaneously for an entire class of rectangular matrices. In parallel, we also provide a finite set to test the total non-negativity (of any order) of an interval hull $I(A, B)$.

1. Introduction and main results

Given an integer $k \geq 1$, a matrix is totally positive of order $k$ ($TP_k$) if all its minors of order at most $k$ are positive, and totally positive ($TP$) if all its minors are positive. Similarly, one defines totally non-negative ($TN$) and $TN_k$ matrices for $k \geq 1$. These classes of matrices have important applications in diverse areas in mathematics, including analysis, approximation theory, cluster algebras, combinatorics, differential equations, Gabor analysis, integrable systems, matrix theory, probability and statistics, and representation theory [3, 4, 7, 11, 17–19, 21, 23, 24, 28].

A property intimately linked with total positivity is variation diminution, which may be regarded as originating in the famous 1883 memoir of Laguerre [20]. Laguerre, following up on Descartes’ rule of signs [6], presented numerous results on the sign changes in the coefficients of power series — which he termed ‘variations’. One such result says that if $f(x)$ is a polynomial and $s \geq 0$, then the number $\text{var}(e^{sx}f(x))$ of variations in the Maclaurin coefficients of $e^{sx}f(x)$...
do not increase with s, hence are bounded above by \( \text{var}(f) < \infty \). In 1912, in correspondence with Pólya [9], Fekete reformulated and proved this result using (what are known today as) one-sided Pólya frequency sequences and their variation diminishing property. This led Pólya to coin the phrase ‘variation diminishing’ (or ‘variationsvermindernd’ in German) in the matrix-theoretic setting. We recall some of the fundamental contributions in this setting: in 1930, Schoenberg [26] showed that TN matrices (in fact sign-regular matrices) satisfy the variation diminishing property. The complete characterization of this property was achieved in 1936 by Motzkin in his thesis [22]. Subsequently, in their 1937 paper [13], Gantmacher–Krein showed that TP\(k\) and TN\(k\) matrices are characterized by the positivity (or non-negativity) of the spectra of all submatrices of size \(\leq k\) (see Theorem 1.9). In their 1950 book [14], Gantmacher–Krein made further fundamental contributions to total positivity and variation diminution. In particular, they characterized TN as follows (cited from a later source):

**Theorem 1.1** [23, Theorem 3.4]. Given a real \(m \times n\) matrix \(A\), the following statements are equivalent.

1. \(A\) is totally non-negative.
2. For all \(x \in \mathbb{R}^n\), \(S^{-}(Ax) \leq S^{-}(x)\). If moreover equality occurs and \(Ax \neq 0\), the first (last) non-zero component of \(Ax\) has the same sign as the first (last) non-zero component of \(x\). Here \(S^{-}(x)\) denotes the number of changes in sign after deleting all zero entries in \(x\).

The first and second sentences in assertion (2) are known as the ‘variation diminishing property’ and the ‘sign non-reversal property’, respectively. Thus, these properties together characterize totally non-negative matrices. A similar result holds for TP matrices; see, for example, [23, Theorem 3.3].

In this short note, our goal is to show that the TP\(k\) and TN\(k\) properties are each equivalent to sign non-reversal alone. This provides characterizations — parallel to the above fundamental 20th-century results — that are equally simple, and remarkably, seem to our knowledge (and that of experts) to be novel.

To state these results, we isolate the following definitions, used below without further reference.

**Definition 1.2.** Let \(n \geq 1\) be an integer, and \(S \subseteq \mathbb{R}^n\) a subset.

1. Define the set \(\langle n \rangle := \{1, \ldots, n\}\) and the vector \(d^{(n)} := (1, -1, \ldots, (-1)^{n-1})^T \in \mathbb{R}^n\).
2. A matrix \(A \in \mathbb{R}^{n \times n}\) has the **sign non-reversal property** with respect to \(S\) if for all vectors \(0 \neq x \in S\), there is some coordinate \(i \in \langle n \rangle\) such that \(x_i(Ax)_i > 0\).
3. We will also need a non-strict version. A matrix \(A \in \mathbb{R}^{n \times n}\) has the **non-strict sign non-reversal property** with respect to \(S\) if for all vectors \(0 \neq x \in S\), there is some coordinate \(i \in \langle n \rangle\) such that \(x_i(Ax)_i \geq 0\).
4. Let \(\mathbb{R}^n_{\text{alt}} \subset \mathbb{R}^n\) comprise the vectors with all non-zero components and alternating signs.
5. Given \(z = (z_1, \ldots, z_n)^T \in \{\pm 1\}^n\), define \(D_z\) to be the diagonal matrix with \((i, i)\)th entry \(z_i\).
6. Finally, given two matrices \(A, B \in \mathbb{R}^{m \times n}\), and tuples of signs \(z \in \{\pm 1\}^m, z' \in \{\pm 1\}^n\), define the \(m \times n\) matrices \(|A|, I_{z,z'}(A, B), \text{ and } C^\pm (A, B)\) via:

\[
|A|_{ij} := |a_{ij}|, \quad I_{z,z'}(A, B) := \frac{A + B}{2} - D_z \frac{|A - B|}{2} D_{z'}, \quad C^\pm (A, B) := I_{d^{(m)}_z, \pm d^{(n)}_z} (A, B).
\]

Now our first main result characterizes total positivity in terms of increasingly weaker statements involving sign non-reversal. Here and below, we use the notion of a **contiguous submatrix**, that is, one whose rows and columns are indexed by sets of consecutive integers.
**Theorem A.** Let \( m, n \geq k \geq 1 \) be integers. Given \( A \in \mathbb{R}^{m \times n} \), the following statements are equivalent.

1. The matrix \( A \) is totally positive of order \( k \).
2. Every square submatrix of \( A \) of size \( r \leq k \) has the sign non-reversal property with respect to \( \mathbb{R}^r \).
3. Every contiguous square submatrix of \( A \) of size \( r \leq k \) has the sign non-reversal property with respect to \( \mathbb{R}^r_{\text{alt}} \).

In fact, this is equivalent to non-strict sign non-reversal at a single vector:

4. For every \( r \in \langle k \rangle \) and contiguous \( r \times r \) submatrix \( B \) of \( A \), define the vector \( z^B := \det(B) \text{adj}(B) d^{(r)} \),

where \( \text{adj}(B) \) is the adjugate matrix of \( B \).

(i) \( Bx \neq 0 \) for all \( x \in \mathbb{R}^r_{\text{alt}} \); and
(ii) \( B \) has the non-strict sign non-reversal property with respect to \( z^B \).

Note that (4) is a priori weaker than (3).

**Remark 1.3.** A ‘coordinate-based’ unpacking of the above characterization says: A matrix \( A \in \mathbb{R}^{m \times n} \) is TP \(_k\) if and only if the following holds.

Suppose \( y = \left( \begin{array}{c} y_0 \\ x \\ y_s \end{array} \right) \), where \( l \geq 0, s \geq 0, \) and \( 0 < r \leq \min\{m, n, k\} \) are integers such that \( l + r + s = n \), and \( x \in \mathbb{R}^r_{\text{alt}} \). Then for each \( j \in \langle m - r + 1 \rangle \), there exists \( i \in \langle r \rangle \) such that \( y_{i+l}(Ay)_{i+j-1} > 0 \).

To our knowledge (and that of experts), Theorem A is a novel characterization of total positivity of a given order \( k \) — as well as of TN \(_k\), stated and proved below. We now provide an application. The third assertion in the theorem helps to provide a test for not just one matrix but an entire interval hull of matrices (also termed ‘interval matrix’) to be TP \(_k\), by reducing it to two test matrices. Given matrices \( A, B \in \mathbb{R}^{m \times n} \), recall that their interval hull, denoted by \( I(A, B) \), is defined as follows:

\[
I(A, B) = \{ C \in \mathbb{R}^{m \times n} : c_{ij} = t_{ij}a_{ij} + (1 - t_{ij})b_{ij}, t_{ij} \in [0, 1] \}. \tag{1.2}
\]

If \( A \neq B \), their interval hull is an uncountable set. We say that \( I(A, B) \) is TP \(_k\) (TN \(_k\)) if every element in it is TP \(_k\) (TN \(_k\)). A natural question involves finding a minimal test set which would determine if \( I(A, B) \) is TP \(_k\). (See [16] for a recent survey of interval matrix results, including along these lines.) When (a) the interval consists of square matrices \( (m = n) \), and (b) the order of total positivity equals the dimension \( (k = n) \), this was answered by Garloff [15] in 1982. It is natural to ask what happens when these two equality-constraints are not imposed. Again, we could not find such a result in the literature. Our next result, an application of Theorem A, answers this question.

**Theorem B.** Let \( m, n \geq k \geq 1 \) be integers, and \( A, B \in \mathbb{R}^{m \times n} \). Then all matrices in \( I(A, B) \) are TP \(_k\) if and only if the two matrices \( C^\pm(A, B) \) are TP \(_k\).

**Remark 1.4.** Note that \( C^\pm(A, B) \) are independent of \( k \).

**Remark 1.5.** It is natural to ask if ‘totally non-negative’ analogues of Theorems A and B exist. We indeed provide these below — see Theorems C and D.
Our next — and immediate — application of Theorem A is a novel characterization of Pólya frequency sequences of order $k$; recall these are real sequences $(c_n)_{n \in \mathbb{Z}}$ such that for all integers $r \in \langle k \rangle$, $m_1 < \cdots < m_r$, $n_1 < \cdots < n_r$,

$$\det(c_{m_i-n_j})_{i,j=1}^{r} \geq 0.$$ 

the determinant $\det(c_{m_i-n_j})_{i,j=1}^{r} \geq 0$. If all such determinants are in fact positive, we say the sequence is a $TP_k$ Pólya frequency sequence.

**Corollary 1.6.** Let $k \geq 1$ be an integer. A real sequence $(c_n)_{n \in \mathbb{Z}}$ is a $TP_k$ Pólya frequency sequence, if and only if for all integers $r \in \langle k \rangle$ and $l \in \mathbb{Z}$, and all $x \in \mathbb{R}_\text{alt}^{r}$, there exists $j_0 \in \langle r \rangle$ such that $x_{j_0} \sum_{j=1}^{r} c_{l+j_0-j} x_j > 0$.

Indeed, this follows by applying Theorem A to the square submatrices

$$\begin{pmatrix} c_l & c_{l-1} & \cdots & c_{l-r+1} \\ c_{l+1} & c_l & \cdots & c_{l-r+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{l+r-1} & c_{l+r-2} & \cdots & c_l \end{pmatrix}, \quad l \in \mathbb{Z}.$$ 

Our final results are the counterparts of Theorems A and B for $TN$ matrices, promised above. In contrast to Theorem A(4) for $TP_k$ matrices, the $TN_k$ property turns out to be equivalent to all (small enough) square submatrices having the non-strict sign non-reversal property with respect to a single, well-chosen vector — which turns out to be either alternating or zero:

**Theorem C.** Let $m, n \geq k \geq 1$ be integers. Given $A \in \mathbb{R}^{m \times n}$, the following statements are equivalent.

1. The matrix $A$ is totally non-negative of order $k$.
2. Every square submatrix of $A$ of size $r \leq k$ has the non-strict sign non-reversal property with respect to $\mathbb{R}^{r}$.
3. Every square submatrix of $A$ of size $r \leq k$ has the non-strict sign non-reversal property with respect to $\mathbb{R}_\text{alt}^{r}$.
4. For every $r \in \langle k \rangle$ and $r \times r$ submatrix $B$ of $A$, the matrix $B$ has the non-strict sign non-reversal property with respect to the vector $z^B$, defined as in (1.1).

**Remark 1.7.** Theorems A and C are reminiscent of a classical result of Ky Fan of a similar nature [8, Theorem 5], shown in the context of proving Ostrowski-type inequalities. We briefly discuss it vis-à-vis the current results: firstly, Theorems A and C require the use of only a single vector $z^B$ as in (1.1), in contrast to an uncountable test set in [8]. Next, Ky Fan studies $TP$ matrices; we are able to account for both $TP$ and $TN$ matrices — and moreover, we characterize matrices that are $TP/TN$ of any order $k$. Finally, Ky Fan works with all submatrices, whereas the above results deduce total positivity (of order $k$) from working with just the contiguous submatrices (of size at most $k$).

Given Theorem C, which is a $TN_k$ analogue of Theorem A, a natural question is to seek a similar $TN_k$ analogue of Theorem B. Such a result was shown very recently (2020) by Adm et al. [2], for rectangular $TN$ matrices $A, B \in \mathbb{R}^{m \times n}$. (See also the related work [1], which resolves a longstanding conjecture from [15] involving (square) non-singular $TN$ interval matrices.) Returning to [2], the authors show that under certain technical constraints, a minimal test set of two matrices suffices to check the total non-negativity of the entire interval hull. Our final result removes the technical assumptions in [2], and holds for $TN_k$ interval hulls for arbitrary $k \geq 1$ — at the cost of working with a larger (but finite) test set:
Theorem D. Let \( m, n \geq k \geq 1 \) be integers, and \( A, B \in \mathbb{R}^{m \times n} \). Then all matrices in \( \mathbb{I}(A, B) \) are \( TN_k \) if and only if the matrices \( \{ I_{z,z'}(A, B) : z \in \{ \pm 1 \}^m, z' \in \{ \pm 1 \}^n \} \) are all \( TN_k \).

As in Theorem B, note that this test set is independent of \( k \).

Remark 1.8. Note that the two matrices \( C^\pm(A, B) \) in Theorem B do not always suffice as test matrices for the interval hull \( \mathbb{I}(A, B) \) to be \( TN_k \) as in Theorem D. For instance, let \( n \geq 4 \) and \( m, k \geq 3 \), and define

\[
A_{m \times n} = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{m \times n} = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad A' := \begin{pmatrix} 3 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad B' := \begin{pmatrix} 4 & 2 & 0 & 2 \\ 3 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}.
\]

It is easily verified that \( C^\pm(A, B) \) are both \( TN \), and the matrix \( C_{m \times n} = \begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix} \) lies in \( \mathbb{I}(A, B) \), where

\[
C' := \begin{pmatrix} 3 & 2 & 0 & 1 \\ 3 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix},
\]

yet \( C \) has a negative \( 3 \times 3 \) minor.

We conclude the discussion of our main results by returning to the simplicity of the sign non-reversal condition in Theorems A and C to characterize total positivity/non-negativity. The equivalence of \( TP_k \) or \( TN_k \) to all square submatrices satisfying a property is reminiscent of the well-known, fundamental characterization of total positivity by Gantmacher–Krein [13] (or of \( P \)-matrices in [12]), which we now recall for the reader’s convenience:

Theorem 1.9. Given a rectangular matrix \( A \in \mathbb{R}^{m \times n} \), the following statements are equivalent.

1. The matrix \( A \) is totally positive of order \( k \).
2. Every square submatrix of \( A \) of size \( r \leq k \) has positive (and simple) eigenvalues.

In a sense, assertions (1) \( \iff \) (2) in Theorems A and C resemble this result in the structure of the second assertions and the simplicity of the statements. Similarly, recall the well-known paper by Fomin and Zelevinsky [10] about tests for totally positive matrices, as well as recent follow-ups such as [5]. Our results may be regarded as being similar in spirit.

2. The proofs

2.1. The sign non-reversal characterization of total positivity

We begin by proving Theorem A; this requires two preliminary results. The first, from 1965, establishes a sign non-reversal phenomenon for matrices with positive principal minors (these are known as \( P \)-matrices):

Theorem 2.1 (Gale–Nikaido, [12]). A matrix \( A \in \mathbb{R}^{n \times n} \) has all principal minors positive if and only if for all \( 0 \neq x \in \mathbb{R}^n \), there exists \( i \in \langle n \rangle \) such that \( x_i(Ax)_i > 0 \).

The next preliminary result is the well-known 1912 result of Fekete for \( TP \) matrices, subsequently extended in 1955 by Schoenberg to \( TP_k \) matrices.
Theorem 2.2 (Fekete [9], Schoenberg [27]). Let $m, n \geq k \geq 1$ be integers. Then $A \in \mathbb{R}^{m \times n}$ is TP$_k$ if and only if all contiguous submatrices of $A$ of size at most $k$ have positive determinant.

See also [7, p. 78]. For more details about totally positive matrices, we refer to [18, 23].

Proof of Theorem A. That (1) $\Rightarrow$ (2) follows from Theorem 2.1, while (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is immediate. To show (4) $\Rightarrow$ (1), by Theorem 2.2 it suffices to show all contiguous $r \times r$ minors of $A$ are positive, for $r \leq k$. We prove this by induction on $r$, with the $r = 1$ case immediate from (4)(i),(ii). Now suppose all contiguous minors of $A$ of size at most $(r - 1)$ are positive, and $B$ is a contiguous $r \times r$ submatrix of $A$. Then all proper minors of $B$ are positive, by Theorem 2.2.

We first claim that $B$ is invertible. Indeed, suppose $Bx = 0$, so that $x \not\in \mathbb{R}_{\text{alt}}^r$. Moreover, all components of $x$ are non-zero, else a proper minor of $B$ vanishes. Now partition $x \neq 0$ into ‘contiguous coordinates of like signs’:

$$(x_1, \ldots, x_{s_1}), \quad (x_{s_1 + 1}, \ldots, x_{s_2}), \quad \ldots \quad (x_{s_u + 1}, \ldots, x_r),$$

with all coordinates in the $i$th component having the same sign, which equals $(-1)^{i-1}$ without loss of generality. Set $s_0 := 0$ and $s_{u+1} := r$, and note that $u \leq r - 2$ since $x \not\in \mathbb{R}_{\text{alt}}^r$. Let $c^1, \ldots, c^u \in \mathbb{R}^r$ denote the columns of $B$, and define

$$y^j := \sum_{l = s_{j-1} + 1}^{s_j} |x_l| c^j, \quad j \in \{u + 1\}.$$

Let $Y := [y^1, y^2, \ldots, y^{u+1}] \in \mathbb{R}^{r \times (u+1)}$. We claim $Y$ is totally positive. Since no $x_j$ is zero, and all proper minors of $B$ are positive, for an integer $p \leq u + 1$ and $p$-element subsets $I \subset \langle r \rangle$, $J \subset \{u + 1\}$ one computes using standard properties of determinants:

$$\det Y_{I \times J} = \sum_{l_1 = s_{j_1 - 1} + 1}^{s_{j_1}} \cdots \sum_{l_p = s_{j_p - 1} + 1}^{s_{j_p}} |x_{l_1}| \ldots |x_{l_p}| \det B_{I \times L} > 0,$$

where $L = \{l_1, \ldots, l_p\}$, and $Y_{I \times J}$ denotes the submatrix of $Y$ with rows and columns indexed by $I, J$ respectively; a similar notation applies to $B_{I \times L}$. Hence $Y$ is totally positive. But $Y d^{(u+1)}(z^{(r)}) = Bx = 0$, a contradiction. Thus $B$ is invertible, as claimed.

Finally, we claim det $B > 0$. Define $z^B$ as in (1.1), then

$$z^B_i = (-1)^{i-1} (\det B) \sum_{j=1}^r \det B_{(\langle r \rangle \setminus \{j\}) \times (\langle r \rangle \setminus \{i\})}, \quad i \in \langle r \rangle \tag{2.1}$$

and the summation on the right is positive by assumption. Thus $z^B \in \mathbb{R}^{n_{\text{alt}}}$. So by (4)(ii) and (2.1), there exists $i \in \langle r \rangle$ such that

$$0 \leq z^B_i (Bz^B)_i = z^B_i (\det B)^2 (-1)^{i-1} (\det B)^3 \sum_{j=1}^r \det B_{(\langle r \rangle \setminus \{j\}) \times (\langle r \rangle \setminus \{i\})} \tag{2.2}$$

From this, it follows that det $B > 0$, and the induction step is complete.

2.2. A test for total positivity (of any order) of the interval hull of rectangular matrices

We next demonstrate the aforementioned test for total positivity of the interval hull $I(A, B)$, as in Theorem B. We first mention two preliminary results that are used in the proof. These require the following notation.
**Definition 2.3.** Fix integers $m, n \geq 1$ and matrices $A, B \in \mathbb{R}^{m \times n}$, with interval hull $I(A, B)$.

1. Define the $m \times n$ matrices $I_u, I_l$ via: $(I_u)_{ij} := \max\{a_{ij}, b_{ij}\}, (I_l)_{ij} := \min\{a_{ij}, b_{ij}\}$.
2. For matrices (or vectors) $A, B$, write $A \lessdot B$ if all entries of $B - A$ are non-negative.

The first preliminary result is a straightforward verification.

**Lemma 2.4.** Let $m, n \geq 1$ and $A, B \in \mathbb{R}^{m \times n}$. Then $I_u, I_l, C^\pm(A, B) \in I(A, B)$. If $m = n$, then $I_{z,z}(A, B) \in I(A, B)$ for all $z \in \{\pm1\}^n$.

The next lemma is precisely [25, Theorem 2.1]; as the proof is short, we include it.

**Lemma 2.5.** Fix $n \geq 1$ and $A, B \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$. Let the tuple $z \in \{\pm1\}^n$ be such that $z_i = 1$ if $x_i \geq 0$ and $z_i = -1$ if $x_i < 0$. If $C \in I(A, B)$, then

$$x_i(Cx)_i \geq x_i(I_{z,z}(A, B)x)_i, \quad \forall i \in \langle n \rangle.$$  

**Proof.** In this proof, we write $I_c := (A + B)/2$ and $\Delta := |A - B|/2$ for ease of exposition. Given $C \in I(A, B) = I(I_u, I_l)$,

$$I_c - \Delta = I_l \lessdot C \lessdot I_u = I_c + \Delta.$$

From this — and given $0 \neq x \in \mathbb{R}^n$ and $i \in \langle n \rangle$ — it follows via the triangle inequality that

$$|x_i((C - I_c)x)_i| \leq |x_i((C - I_c)\cdot x)|_i \leq |x_i|(|A - B|)_{i \in \langle n \rangle}.$$

From this, we compute using that $|x| = D_2x$:

$$x_i(Cx)_i \geq x_i(I_cx)_i - z_ix_i(\Delta D_2x)_i = x_i(I_cx)_i - x_i(D_2\Delta D_2x)_i = x_i(I_{z,z}(A, B)x)_i.$$  

With these preliminaries at hand, we have:

**Proof of Theorem B.** If $I(A, B)$ is $TP_k$, then so are $C^\pm(A, B)$ by Lemma 2.4. Conversely, suppose $C^\pm(A, B)$ are $TP_k$, and let $C \in I(A, B)$. Fix $r \in \langle k \rangle$ and a vector $x \in \mathbb{R}^{\langle k \rangle}_{\text{alt}}$. Now let $C'$ be an $r \times r$ contiguous submatrix of $C$, say $C' = C_{J \times K}$ for contiguous sets of indices $J, K \subseteq \langle k \rangle$ with $|J| = |K| = r$. It suffices to show by Theorem A(3) that $x_i(C'x)_i > 0$ for some $i \in \langle r \rangle$.

To proceed, we require some notation. Let $A', B'$ be contiguous submatrices of $A, B$ respectively, consisting of the entries in the same positions as $C'$. Now since $C' \in I(A', B')$,

**Lemma 2.5** implies for some $i \in \langle r \rangle$

$$x_i(C'x)_i \geq x_i(I_{z^{(r)},z^{(r)}}(A', B')x)_i,$$

where the vector $z^{(r)} \in \{\pm1\}^r \cap \mathbb{R}^{\langle r \rangle}_{\text{alt}}$ is given by $z_j^{(r)} := x_j/|x_j| \quad \forall j$. Since $C' = C_{J \times K}$, we can embed the vector of signs $z^{(r)}$ in contiguous positions $J \subseteq \langle m \rangle$ and $K \subseteq \langle n \rangle$, and uniquely extend to alternating $\pm1$-valued vectors of lengths $m, n$, respectively. Formally, there exist unique signs $\varepsilon_J, \varepsilon_K \in \{\pm1\}$ such that $z^{(r)}$ is the restriction to positions $J$ (respectively, $K$) of $\varepsilon_Jd^{(m)} \in \mathbb{R}^{\langle m \rangle}_{\text{alt}}$ (respectively, $\varepsilon_Kd^{(n)} \in \mathbb{R}^{\langle n \rangle}_{\text{alt}}$). But then $I_{z^{(r)},z^{(r)}}(A', B')$ is a contiguous $r \times r$ submatrix of

$$I_{\varepsilon_Jd^{(m)}, \varepsilon_Kd^{(n)}}(A, B) = \frac{A + B}{2} - \varepsilon_J\varepsilon_KD_{d^{(m)}}\frac{|A - B|}{2}D_{d^{(n)}} \in \{C^+(A, B), C^-(A, B)\}.$$  

By assumption, this matrix is $TP_k$, so $I_{z^{(r)},z^{(r)}}(A', B')$ is $TP$. Using Theorem A(3) and (2.3), we have

$$x_i(C'x)_i \geq x_i(I_{z^{(r)},z^{(r)}}(A', B')x)_i > 0.$$
Thus $C'$ has the sign non-reversal property with respect to all $x \in \mathbb{R}^n_{alt}$. By Theorem A, $C$ is $TP_k$. \hfill \Box

Remark 2.6. We remark that Garloff’s results for $n \times n$ matrices in [15] (see also [23, Chapter 3]) are stated with respect to the checkerboard ordering $\preceq^*$, in which

$$A \preceq^* B \iff D_{d(n)}(B - A)D_{d(n)} \succeq 0_{n \times n}.$$ 

Now an easy computation shows that the hull $I(A, B)$ itself does not depend on whether one uses the entrywise ordering or the checkerboard ordering. More precisely,

$$I(A, B) = \{ C : I_1 \leq C \leq I_0 \} = \{ C : C^-(A, B) \preceq^* C \preceq^* C^+(A, B) \}.$$ 

In particular, the test set of $\{C^+(A, B), C^-(A, B)\}$ works regardless of which ordering is used — and works for all rectangular matrices and for testing the $TP_k$ property for any $k \geq 1$.

Remark 2.7. Garloff has pointed out to us that Theorem B can also be proved by reducing to the case of square $TP$ matrices (which is his result in [15]) as follows: if $C^\pm(A, B) \in \mathbb{R}^{m \times n}$ are both $TP_k$, then every contiguous square submatrix of $C^\pm(A, B)$ of size $r \in \langle k \rangle$ is $TP$. Now given $C \in I(A, B)$, every contiguous square submatrix of $C$ of size $r \in \langle k \rangle$ lies in between square $TP$ submatrices of $C^+(A, B)$ and $C^-(A, B)$, hence is $TP$ by [15]. Now use the Fekete–Schoenberg Theorem 2.2.

2.3. The sign non-reversal characterization and interval test of total non-negativity

We conclude by proving our results involving the $TN_k$ property. Theorem C, which characterizes the $TN_k$ property, requires a classical density result by Whitney:

Theorem 2.8 (Whitney, [29]). Given integers $m, n \geq k \geq 1$, the set of $m \times n$ $TP_k$ matrices is dense in the set of $m \times n$ $TN_k$ matrices.

This helps to show the promised characterization:

Proof of Theorem C. We prove a cyclic chain of implications. First suppose $A \in \mathbb{R}^{m \times n}$ is $TN_k$. By Theorem 2.8, there exists a sequence $A^{(l)} \to A$ of $TP_k$ matrices. Fix $r \in \langle k \rangle$ and an $r \times r$ submatrix $B$ of $A$, and let $B^{(l)}$ be the submatrix of $A^{(l)}$ in the same positions as $B$. Also fix a vector $0 \neq x \in \mathbb{R}^r$, and let $J \subseteq \langle r \rangle$ index the non-zero components of $x$. Now $B^{(l)}$ is $TP$, so by Theorem A(2) there exists $j_l \in J$ such that $x_{j_l}(B^{(l)}x)_{j_l} > 0$. Hence there exists an (increasing) subsequence $l_p, p \geq 1$ of positive integers such that all $j_{l_p}$ equal the same entry in $J$, say $j_0$. But then,

$$x_{j_0}(Bx)_{j_0} = \lim_{p \to \infty} x_{j_{l_p}}(B^{(l_p)}x)_{j_{l_p}} \geq 0, \quad x_{j_0} \neq 0.$$

Thus $(1) \Rightarrow (2)$. Clearly $(2) \Rightarrow (3) \Rightarrow (4)$. Now assume $(4)$; we show that $\det B \geq 0$ for all $r \times r$ submatrices $B$ of $A$, by induction on $r \in \langle k \rangle$. Indeed, the base case $r = 1$ is immediate. For the induction step, if $\det B = 0$, then we are done; else $\det(B) \neq 0$. Now we once again have (2.1), and the sum on the right is non-negative. But since $B$ is invertible, no column of $\text{adj}(B)$ is zero, so the sum on the right is in fact positive by the induction hypothesis. This implies $z^B \in \mathbb{R}^r_{alt}$, so we may repeat the calculation in (2.2) and deduce that $\det B > 0$. This proves (1) by induction. \hfill \Box

We conclude by showing our final remaining result:
Proof of Theorem D. First verify that \( I_{z,z'}(A, B) \in \mathbb{I}(A, B) \) for all \( z, z' \). Now repeat the proof of Theorem B, but working with arbitrary \( r \times r \) submatrices \( C' \) of \( C \in \mathbb{I}(A, B) \), where \( r \in (k) \). Once again obtaining (2.3), we note that \( I_{z,z'}(A', B') \) is a submatrix of \( I_{z,z'}(A, B) \) for some \( z, z' \), and the latter is \( T_{N_k} \) by assumption. The remainder of the proof is unchanged, except for the use of Theorem C instead of Theorem A.

\[ \square \]

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