On the Relation Between Quantum Mechanical and Classical Parallel Transport

J. Anandan
Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208

L. Stodolsky
Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, 80805 München, Germany

We explain how the kind of “parallel transport” of a wavefunction used in discussing the Berry or Geometrical phase induces the conventional parallel transport of certain real vectors. These real vectors are associated with operators whose commutators yield diagonal operators; or in Lie algebras those operators whose commutators are in the (diagonal) Cartan subalgebra.

In discussing the Berry or Geometrical phase one uses the concept of a “parallel transport” of a quantum mechanical state $\psi$, by which is meant

$$<\psi(t)|\dot{\psi}(t)> = 0$$

(1)

One may also consider a complete (orthonormal) basis set of states $|n>$ obeying this condition. If the time-dependent states $|n(t)>$ are obtained from a basis of initial states $|n>$ by a unitary transformation $U$ (as would be generated by a hamiltonian)

$$U(t)|n> = |n(t)>$$

(2)

we can say we have an orthonormal “frame” undergoing this kind of parallel transport. In this case the “parallel transport” condition, namely

$$<n(t)|\dot{n}(t)> = 0$$

(3)

for all $n$, can also be written as

$$<n(t)|\dot{U}U^\dagger|n(t)> = 0$$

(4)

(Quantities without a time argument refer to the fixed basis, while those with an argument (t) refer to the moving basis, thus $|n> = |n(0)>$.) Now the “parallel transport” and “moving frames” implied by these equations are not the same as those of usual differential geometry. Rather, there, in the viewpoint where one studies a euclidean frame moving in a higher dimensional space and then restricts to a submanifold, there is a set of real vectors $e_a$ instead of quantum mechanical state vectors, and parallel transport among a set of vectors $a, b, c...$ on the submanifold means

$$\dot{e}_a(t) \cdot e_b(t) = 0$$

(5)

for all pairs $a, b, c...$ in the submanifold. That is, the set $e_a$ are not a complete set, but rather form a moving subspace in a larger space. In this formulation the dot symbol means the ordinary derivative in the ambient space, while in the “intrinsic” formulation of differential geometry the dot symbol would mean the covariant derivative with a connection.

This condition Eq (5), which we might call “classical” parallel transport, looks quite different from Eq (3). What is the relationship between the two kinds of “parallel transport”, if any?

It seems there should be some such relationship. For example, in our treatment of the geometric phase in SU(2), where $U$ is an SU(2) group element, we could view the “quantum mechanical parallel transport” as inducing the “classical parallel transport” of the $x$ and $y$ vectors of a “dreibein” sliding, but not rotating, on the sphere.

Here we would like to briefly elucidate why this is and to indicate how to generalize the idea, including its application to higher groups. Briefly, we will show how the condition Eq (3) for a complex basis leads to a “classical” parallel transport, Eq (5), of certain vectors associated with the problem, such as the $x, y$ vectors of the “dreibein”.

Our first task is to identify the vectors $e_a$, which we do as follows. Consider a complete set of operators or matrices $\lambda_a$, like the generators of a Lie group, complete in the sense that they transform among each other under $U$. That is,
there are the time dependent operators $\lambda(t) = U(t)\lambda_a U^\dagger(t)$, which may be reexpressed in terms of the original, fixed, $\lambda$. These then generate the vectors $e_a$ via (summation convention)

$$\lambda_a(t) = U(t)\lambda_a U^\dagger(t) = e_a^\dagger(t)\lambda^a$$

(6)

If we choose the $\lambda_a$ such that $Tr(\lambda_a\lambda_b) = N\delta_{ab}$, where $N$ is a normalization factor, we can write explicitly

$$e_a(t)^2 = 1/N \ Tr[\lambda_a(t)\lambda^a] = Tr[\lambda_a(t)\lambda^a]$$

(7)

where we define $Tr$ to include the normalization factor. Furthermore, with hermitian $\lambda_a$ the $e_a(t)$ are real. The scalar product of two vectors is then given by the trace of the product of the corresponding $\lambda$, as in $e_a(t) \cdot e_b(t) = \lambda_\gamma \lambda^\gamma$. The definition of the $\lambda(t)$ is chosen so that $< n(t)|\lambda(t)|m(t) > = < n|\lambda|m >$.

Now a main point of Ref. 3 was that the information conveyed by the condition Eq [3] or Eq [4] could be interpreted, in the group theoretical context, by saying that in the “local frame” there was no rotation with respect to the subspace of diagonal generators, that is in the Cartan subspace. We can formulate this point in a general manner by viewing the evolution of the states as being determined by a hamiltonian $h(t)$, where $h(t) = iU\dot{U}^\dagger$. (We reserve the symbol $H$ for the more usual hamiltonian, which however is absent in the present considerations. $H$ includes the “dynamical phase” which usually Ref. 4 has been removed from the problem before we get to Eq [4]). Thus

$$i|\dot{n}(t) > = h(t)|n(t) >$$

(8)

and Eq [4] states that the diagonal elements of $h$ are zero in the moving basis:

$$< n(t)|h(t)|n(t) > = 0$$

(9)

We would now like to explain how Eq [3] can lead to $\dot{e}_a(t) \cdot e_b(t) = 0$ among some of the $e(t)$. The desired quantity $\dot{e}_a(t) \cdot e_b(t)$ may be found from

$$\dot{e}_a(t) \cdot e_b(t) = \dot{\lambda}_a(t)\lambda_b(t) = iTr[ [h(t), \lambda_a(t)]\lambda_b(t)]$$

(10)

where we use the equation of motion $i\dot{\lambda}_a(t) = [h(t), \lambda_a(t)]$ following from the definition of $\lambda_a(t)$. Rearranging the last expression we have

$$\dot{e}_a(t) \cdot e_b(t) = iTr[h(t)[\lambda_a(t), \lambda_b(t)]]$$

(11)

Now consider two $\lambda$’s such that their commutator gives a diagonal matrix, $[\lambda_a, \lambda_b] = (diag.)$. Applying $U$, the same holds in the moving basis for the $\lambda_a(t)$ namely

$$< m(t)|[\lambda_a(t), \lambda_b(t)]|n(t) > \sim \delta_{nm}$$

(12)

For such a pair in Eq [11], while $h$ has no diagonal elements, the commutator has only diagonal elements. But this gives zero for the trace, and hence $\dot{e}_a(t) \cdot e_b(t) = 0$.

We thus arrive at our conclusion: “classical parallel transport”, Eq [3], follows as a result of “quantum parallel transport” Eq [4] for those vectors $e_a(t)$ whose corresponding commutators among the $\lambda_a$ yield diagonal matrices. In an abbreviated language with a “matrix valued vector” $e_a^\dagger \lambda^a$, we could say “for those vectors whose mutual commutators are diagonal”.

In group theory this is the requirement that the commutator lie in the Cartan subalgebra, when the latter, as usual, has been chosen diagonal. Precisely this was the case in our SU(2) example Ref. 2 where the $S_x = \lambda_x$ and $S_y = \lambda_y$ generators are the two non-Cartan generators. Their commutator yields only the Cartan generator $S_z = \lambda_z$, which in the usual choice of basis is diagonal. This is why under a $U$ obeying Eq [4] they undergo parallel transport in the sense of Eq [3]. Note however, that it is necessary to explicitly take the Cartan operators diagonal.

**Comments**

A striking difference between the quantum Eq [3] and the classical Eq [4] is that in the classical case the concept is linear; if two vectors are parallel transported then their sum is also. However, for the parallel transport of Eq [4], as may be easily verified, this is not true.

This implies that Eq [3] is not in general preserved under linear transformation; a new “frame” $|n' > = \sum u_{n'n}|n >$ will not in general be parallel transported, even if the $|n >$ are. Thus a full statement of the problem involves a specification as to which set of vectors satisfy Eq [4], as reflected in the necessity to choose a definite basis, one in which the Cartan generators are diagonal. This helps in clarifying the following potential misunderstanding: We might be tempted to conclude that when dealing with real quantities, as with orthogonal rotations, that Eq [4]...
follows simply from unitarity and thus represents no further information. That is, given that all quantities are real,
\[ < n(t) | \dot{n}(t) > + < \dot{n}(t) | n(t) > = 2 < n(t) | \dot{n}(t) > = 0 \]
follows simply from preservation of the norm. Thus we would be led by our above arguments to the nonsensical result that any orthogonal transformation will automatically induce parallel transport. However, this argument would neglect the requirement that the Cartan operators be diagonal. In fact for real orthogonal representations the generators are antisymmetric, or in the above notation, the \( \lambda \) are pure imaginary. But we need the Cartan operators in diagonal form, and antisymmetric operators cannot be brought to diagonal form without introducing a complex basis. Thus complex numbers are reintroduced and Eq \[3\] does indeed represent a second condition, and not just simply unitarity or preservation of the norm.

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