Determination of the electrostatic potential produced by a uniformly charged ring

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Abstract. Finding the electrostatic potential produced by a point load is an easy task, when that point load is replaced by a load distribution this task is complicated, the difficulty of this procedure depends on the place where you want to know the potential, it is not the same to find the potential at a given point to find it in any place in space. In this case we will work with a uniformly charged two-dimensional ring, the problem that arises is the obtaining of the electrostatic potential at any point around the ring, for this we will look for the conditions that must be met to pose the problem, a theoretical solution for this problem, in which we find the Legendre polynomials and finally proceeds to make this same solution using the two dimensional finite element method.

1. Introduction
The finite element method is a method of approximation for the solution of differential equations [1,2], whether ordinary or partial, this method of solution is widely used in mathematics, physics and engineering, in many cases the analysis of dynamic and static systems leads to equations Differentials with a high complexity of theoretical solution, which implies the implementation of other solution methods, usually numerical methods. The civil and mechanical engineers frequently find themselves with the analysis of structures, in these problems they require to find the displacements and the total tensions in the structure. For many structures it is difficult to determine these deformations using conventional methods, and that is where the finite element method is required [3]. In the same way in many other branches of research, such as electronics, quantum mechanics, electricity, etc. It requires the solution of different equations that involve a respectable mathematical complexity, for this type of problems the finite element method represents a very useful tool. There are different methods of approximation, but for a method to be truly valid it must meet some requirements.

It must have a solid basis both mathematical and physical, it must not have limitations with the geometry used [4], the formulation of the equations must be independent of the domain and the boundary conditions, the method must be flexible, and must allow flexibility without need to reformulate the whole problem. It must contain a systematic procedure, such that it can be programmed in digital computers [5]. The finite elements method presents these characteristics and is adapted in such a way that there are specialized software’s for its implementation as FlexPDE, the program used for the realization of this document.

2. Finite element method
The finite element method is defined as an approximation method whose objective is to find an equation “y” such that it gives an approximate numerical solution on a body or domain (Any continuous medium),
dividing it into subsets of non-intersecting elements denominated “Finite elements”, these elements form a discretization within the domain of interest [6]. The construction of an algorithm to apply this method requires 4 key steps:

1. The problem must be formulated variationally.
2. The domain must be subdivided into subdomains, and associated with the previous partition a vector space is generated.
3. The variational problem is projected in the space of subsets obtained; the more partitions are made the better the numerical approximation of the solution.
4. The last step is the numerical calculation of the solution of the system of equations.

These previous steps allow you to construct a differential calculus problem in a linear algebra problem, as follows. We assume that it is necessary to solve the following one-dimensional Equation (1) [7]:

\[
- \frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = f(x) \tag{1}
\]

Subject to the following conditions Equation (2):

\[
y(0) = 0, y(x_n) = 0 \tag{2}
\]

It is possible to construct a potential function, based on the Equation (1) to solve, like this Equations (3) and Equation (4):

\[
Ly = \left[ - \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x) \right] y = f \tag{3}
\]

\[
y(0) = \frac{dy(x_n)}{dx} = 0 \tag{4}
\]

A solution for this equation is proposed in such a way that it is a linear combination of functions Equation (5) [5,6]:

\[
V_h(x) = \sum_{i=1}^{L} b_i N_i(x) \tag{5}
\]

With the weights \( b_i = K^{-1}F \) where the matrix \( K \) is a matrix of real and invertible coefficients and its terms are calculated by means of the energy integral of the finite elements, which for the case of the one-dimensional problem are straight defined in the nodes (partitions), as follows Equation (6):

\[
N^{(e)}_{i+1}(x) = \begin{cases} 
\frac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i \leq x \leq x_{i+1} \\
\frac{x - x_i}{x_{i+1} - x_i}, & x_{i+1} \leq x \leq x_{i+1}
\end{cases} \tag{6}
\]

\[
N^{(e)}_i(x) = \begin{cases} 
\frac{x_i - x_{i-1}}{x_{i-1} - x}, & x_{i-1} \leq x \leq x_i \\
\frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_i \leq x \leq x_{i+1} \\
0 & \text{in other case.}
\end{cases}
\]

In \( N^{(e)}_i(x) \), \( i \) refers to the node and \( e \) refers to the element Equation (7). And the terms of the vector \( F \) will depend on the values of the function \( f \) evaluated in each node of the elements [8,9].
\[
K_{ij} = \sum_{e=1}^{L} k_{ij}^e = \int_{e}^{e+1} \left( p(x) \frac{dN(x)^e}{dx} - \frac{dN(x)^e}{dx} + q(x)N(x)^eN(x)^e \right) dx
\]  

(7)

On the other hand, if the Equation (1) has a term that refers to the dynamics of the dependent variable then we want to solve the following differential Equation (8).

\[
-\frac{d}{dx} \left( p(x) \frac{dy}{dx} + q(x) \frac{dy}{dt} \right) = f(x)
\]  

(8)

Whose solution is given by Equation (9):

\[
V_h(x, t) = \sum_{i=1}^{L} b_i N_i(x)
\]  

(9)

3. Numerical results

The problem that arises is a ring loaded with a load density uniformly distributed around it, the electrostatic potential is sought at any point in space, as we saw in the previous point a theoretical solution with the Legendre polynomials is presented, to graph the behavior of this in finite function, an approximation was made with the first 8 polynomials [10]. We proceed with the analysis of the system, both theoretically and numerically, a square region with a concentric ring was proposed, as shown in Figure 1.

[Figure 1. Region of analysis with the initial condition in the center as a small potential difference.]

In the same program, the graph of the theoretical behavior of the problem was made, this behavior is presented as a contour plot (Figure 2) and as a distribution surface in space (Figure 3).

[Figure 2. Graph of the theoretical behavior of the system.  
Figure 3. Surface distribution chart.]
The behavior of the function is analyzed in the same way according to the data provided by the program used to carry out the implementation of the finite element method [11,12]. Obtaining the results presented in Figure 4 and Figure 5.

**Figure 4.** Behavior of the electrostatic potential obtained by the finite element method.

**Figure 5.** Distribution surface of the U function.

4. Conclusion
The behavior of the two functions differ significantly, the theoretical part has a slightly more regular distribution, both in the contour plot and in the surface graph compared to that obtained under the finite element method used to obtain the graphs of the U function.

The discretization made by the program used allows the function to travel more continuously the domain of the function, while the time to obtain the theoretical function is made a series of approaches in order to make it easier to obtain said equation.

The difference that occurs in the behavior is given by the coordinates, the coordinates used to perform the theoretical analysis were spherical coordinates, in the analysis performed an azimuthal symmetry is found, which is polar angle is discarded and only works with the radio and the axial angle, while the program when taking the system as a two-dimensional system takes polar coordinates, where the polar angle is not discarded but is at the reference angle. The best way to model this system is to define a three-dimensional region where the azimuthal symmetry of the system is taken into account; therefore, this approach is not efficient because it is defined in the incorrect space.

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