SPECIAL UNIPOTENT REPRESENTATIONS WITH HALF INTEGRAL INFINITESIMAL CHARACTERS

KOYUE DANIEL WONG

ABSTRACT. For any special nilpotent orbit, let $\frac{1}{2} h^\vee$ be one half of the semisimple element of a Jacobson-Morozov triple associated to the orbit. In 1985, Barbasch and Vogan defined the notion of special unipotent representations with infinitesimal character $(\frac{1}{2} h^\vee, \frac{1}{2} h^\vee)$. Some properties of such representations were discovered when $\frac{1}{2} h^\vee$ is integral. In this manuscript, we provide details on the proof of these properties when $\frac{1}{2} h^\vee$ is not integral.

1. INTRODUCTION

Let $G$ complex simple Lie group with Lie algebra $g$. For each $\lambda \in \mathfrak{h}^*$ in the dual of the Cartan subalgebra $\mathfrak{h}$ of $g$, there is a unique maximal primitive ideal in $I_\lambda \subset \mathcal{U}(g)$ such that $I_\lambda \cap \mathcal{Z}(\mathcal{U}(g)) = \ker(\chi_\lambda : \mathcal{Z}(\mathcal{U}(g)) \to \mathbb{C})$, where $\chi_\lambda$ is the central character of the universal enveloping algebra $\mathcal{U}(g)$ corresponding to $\lambda$.

Treating $G$ as a real Lie group with maximal compact subgroup $K$, we identify $g_C = g \times g$ with Cartan subalgebra $\mathfrak{h}_C = \mathfrak{h} \times \mathfrak{h}$, compact torus $t = \{(x, -x) \in \mathfrak{h}_C | x \in \mathfrak{h}\}$ and split torus $a = \{(x, x) \in \mathfrak{h}_C | x \in \mathfrak{h}\}$ with $\mathfrak{h} = t \oplus a$. Given $(\lambda, \mu) \in \mathfrak{h}_C^*$ so that $(\lambda - \mu)$ is a weight of a finite dimensional holomorphic representation of $G$, then the principal series representation with character $(\lambda, \mu)$ is the $(g_C, K_C)$-module $X_G(\lambda, \mu) := K$ – finite part of $\text{Ind}_B^G(e^{i \lambda, \mu})$.

For each $\lambda \in \mathfrak{h}^*$ and $w \in W$, one would like to study which irreducible quotient $X_G(\lambda, w\lambda)$ of $X_G(\lambda, w\lambda)$ satisfies $\text{LAnn}_{\mathcal{U}(g)} X_G(\lambda, w\lambda) = R\text{Ann}_{\mathcal{U}(g)} X_G(\lambda, w\lambda) = I_\lambda$. These are the irreducible modules with the smallest possible associated variety. In particular, the spherical module $X_G(\lambda, \lambda) = \mathcal{U}(g)/I_\lambda$ satisfies the above property.

In [BV], Barbasch and Vogan studied some special values of $\lambda$ as follows:

**Definition 1.1 ([BV], Definition 1.17).** Let $O^\vee$ be a special nilpotent orbit, and $\lambda_{O^\vee} := \frac{1}{2} h^\vee$ be one half of the semisimple element in a Jacobson-Morozov triple attached to $O^\vee$. Writing $O$ as the Lusztig-Spaltenstein dual of $O^\vee$, the special unipotent representations attached to $O^\vee$ are given by the set

$$
\Pi(O^\vee) := \{X_G(\lambda_{O^\vee}, w\lambda_{O^\vee}) \mid AV(X_G(\lambda_{O^\vee}, w\lambda_{O^\vee})) \subseteq \mathcal{O}\}.
$$

The main results in [BV] were proved under the condition that $\lambda_{O^\vee}$ integral, i.e. $O^\vee$ is even, where the representations in $\Pi(O^\vee)$ are called integral special unipotent representations. The aim of this manuscript is to provide details of their results for all $\lambda_{O^\vee}$.
Theorem 1.2. Let $O'$ be a special nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$. Then all elements in $\Pi(O')$ have associated variety equal to $\overline{O}$. Moreover, the cardinality of $\Pi(O')$ is equal to the number of irreducible representations of $\overline{A}(O') \cong \overline{A}$. If $O$ is not equal to one of the three exceptional orbits stated in Definition 4.5 of [BV3], then the character formulas for all $\overline{X}_G(\lambda_{O'}, w\lambda_{O'}) \in \Pi(O')$ are of the form given in Theorem III of [BV3].

Here is an outline of the proof:

(I) Let $\lambda := \lambda_{O'}$ and consider Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ with roots $\alpha$ in $\mathfrak{g}$ satisfying $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$. It turns out that there is a choice of simple roots of $\mathfrak{g}'$ such that their inner products with $\lambda$ is equal to either 0 or 1.

(II) Upon restricting to $\mathfrak{g}'$, the result in [BV3] applies, and we can obtain integral special unipotent representations with infinitesimal character $\lambda = \frac{1}{2}(h')^\vee$ for some even orbit $(O')^\vee \in L_{g'}$.

(III) More explicitly, let $O' \subset \mathfrak{g}'$ be the Lusztig-Spaltenstein dual of $(O')^\vee$, and $V^L(O')$ be the left cell representation of $W'$ denoted by $V^L(w_0w_{O'})$ in [BV3]. By Chapter 4 of [LL1], the irreducible representations in $V^L(O')$ are parametrized by $\{\sigma_{x}^f \mid x \in \overline{A}(O')\}$ (see Remark 3.2 below). By Theorem III of [BV3], the unipotent representations are parametrized by $\pi \in \overline{A}(O')^\vee$, and their character formulas are given by

$$
\overline{X}_G(\lambda, w_x \lambda) = \frac{1}{|\overline{A}(O')|} \sum_{x \in [\overline{A}(O')]} \text{tr}_x(x)|x| \sum_{w' \in W'} \text{tr}_{\sigma_x^f}(w')X_G(\lambda, w'\lambda).
$$

Under the above identification of elements in $V^L(O')$, the representation $\sigma_{sp}^f := \sigma_{[1]}^f$ is special, and is the Springer representation of $O'$. All other $\overline{X}_G(\lambda, w\lambda)$ have character formulas involving special representations $\tau_{sp}^f \in (W')^\wedge$, where $\text{deg}(\tau_{sp}^f) < \text{deg}(\sigma_{sp}^f)$.

(IV) By a version of the Kazhdan-Lusztig conjecture stated in Theorem 4.2 of [B1],

$$
\overline{X}_G(\lambda, w_x \lambda) = \frac{1}{|\overline{A}(O')|} \sum_{x \in [\overline{A}(O')]} \text{tr}_x(x)|x| \sum_{w' \in W'} \text{tr}_{\sigma_x^f}(w')X_G(\lambda, w'\lambda) \quad (1)
$$

are the character formulas of irreducible representations of $G$. And the same result goes for character formulas of other irreducible representations.

(V) Let $\text{Ind}^W_{W'}(\sigma_{sp}^f) = J^W_{W'}(\sigma_{sp}^f) + \ldots$, where $J^W_{W'}(\sigma_{sp}^f)$ is the $J$-induction defined in [LL1, Section 4.3]. It contains a unique special representation $\sigma_{sp}$ with $\text{deg}(\sigma_{sp}) = \text{deg}(\sigma_{sp}^f)$ (and similarly we have $\tau_{sp} \in J^W_{W'}(\tau_{sp}))$. By studying the $W \times W$-module $\mathbb{C}[W]$ (c.f. [BV3 Proposition 6.6]), these character formulas in $G$ contain expressions involving $\sigma_{sp}$ (or $\tau_{sp}$), and their associated varieties can be determined by the Springer correspondence of $\sigma_{sp}$ (or $\tau_{sp}$). Since $\text{deg}(\tau_{sp}) < \text{deg}(\sigma_{sp})$, the irreducible
Proposition 2.1 ([W2], Proposition 2.2)
We will make the following observations:

• For all orbits we are studying, \(\overline{A}(O) \cong A(O')\), and \(\sigma_{sp} = J^W_W(\sigma_{sp}')\) is irreducible. More precisely, \(\sigma_{sp}\) is the Springer representation attached to \(O\), the Lusztig-Spaltenstein dual of \(O'\).

• If \(O\) is not equal to one of the three exceptional orbits, \(J^W_W(\sigma_{sp}')\) is irreducible for all \(\sigma_{sp}' \in V^L(O')\). Since \(V^L(O) = J^W_W(V^L(O'))\) by [BV2, Proposition 3.18], the elements of \(V^L(O)\) can be parametrized by

\[ x \in [A(O)] \cong [A(O')] \iff \sigma_x := J^W_W(\sigma_{sp}') \in V^L(O). \]

This identification of elements in \(V^L(O)\) matches with that of [L1]. The first point guarantees that all \(\overline{X}_G(\lambda, w_\pi \lambda)\) have associated variety equal to \(\overline{O}\) (see also Theorem 5.2 of [McG]), and they have the same cardinality as the number of irreducible representations of \(\overline{A}(O)\). The second point and [BV3, Proposition 6.6] reformulate Equation (1) into

\[ \overline{X}_G(\lambda, w_\pi \lambda) = \frac{1}{|A(O)|} \sum_{x \in [A(O)]} \text{tr}_x(x) |x| \sum_{w \in W} \text{tr}_{\sigma_x}(w) X_G(\lambda, w \lambda). \]

In other words, the character formula of \(\overline{X}_G(\lambda, w_\pi \lambda)\) is given precisely by Theorem III of [BV3]. This finishes the proof of the main theorem.

2. Classical Lie Algebras

We will describe all special nilpotent orbits in \(\mathfrak{g}\) of type \(B, C\) and \(D\) as follows:

**Proposition 2.1** ([W2], Proposition 2.2).

- **\(\mathfrak{g}\) of Type \(B\):** Let \(O^\vee = [r_{2k} \geq r_{2k-1} \geq \cdots \geq r_0]\) be a special orbit. Separate all even rows (which must be of the form \(r_{2l-1} = r_{2l-2} = \alpha\)), along with odd row pairs of the form \(r_{2l} = r_{2l-1} = \beta\) and get

\[ O^\vee = [r_{2q}'' > r_{2q-1}'' \geq \cdots \geq r_2'' > r_1'' \geq r_0''] \cup \bigcup_i [\alpha_i, \alpha_i] \cup \bigcup_j [\beta_j, \beta_j], \]

- **\(\mathfrak{g}\) of Type \(C\):** Let \(O^\vee = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_1]\) be a special orbit. Separate all odd rows (which must be of the form \(r_{2l-1} = r_{2l-2} = \alpha\)), and even row pairs of the form \(r_{2l} = r_{2l-1} = \beta\) and get

\[ O^\vee = [r_{2q}'' > r_{2q-1}'' \geq \cdots > r_3'' \geq r_2'' \geq r_1''] \cup \bigcup_i [\alpha_i, \alpha_i] \cup \bigcup_j [\beta_j, \beta_j], \]
• **Lg of Type** D: Let \( \mathcal{O}^\vee = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_0] \) be a special, non-very even orbit. Separate all even rows (which must be of the form \( r_{2l-1} = r_{2l-2} = \alpha \)), and all odd row pairs \( r_{2l} = r_{2l-1} = \beta \) and get

\[
\mathcal{O}^\vee = \{ r_{2g+1}'' \geq r_{2g}' \geq \cdots \geq r_1'' > r_1' \geq r_0'' \} \cup \bigcup_i [\alpha_i, \alpha_i] \cup \bigcup_j [\beta_j, \beta_j],
\]

Then \( \overline{\mathcal{A}}(\mathcal{O}^\vee) = (\mathbb{Z}/2\mathbb{Z})^q \), regardless of the number of \( \alpha_i \)'s and \( \beta_j \)'s.

For any special orbit \( \mathcal{O}^\vee \), consider \( \mathcal{O}^\vee = \mathcal{P}^\vee \cup \mathcal{Q}^\vee \), where \( \mathcal{P}^\vee \) is the orbit of the same type as \( \mathcal{O}^\vee \) without the \( \alpha_i \)'s, and \( \mathcal{Q}^\vee := \bigcup_i^{c} [\alpha_i, \alpha_i] \). Note that \( \mathcal{P}^\vee \) is even with \( \overline{\mathcal{A}}(\mathcal{P}^\vee) \equiv \overline{\mathcal{A}}(\mathcal{O}^\vee) \) and the results in [BY3] hold for \( \mathcal{P}^\vee \). More precisely, the coordinates of \( \lambda_{\mathcal{O}^\vee} \) consists of:

- **Lg of Type** B: integers coming from \( \mathcal{P}^\vee \), half-integers coming from \( \mathcal{Q}^\vee \).
- **Lg of Type** C: half-integers coming from \( \mathcal{P}^\vee \), integers coming from \( \mathcal{Q}^\vee \).
- **Lg of Type** D: integers coming from \( \mathcal{P}^\vee \), half-integers coming from \( \mathcal{Q}^\vee \).

Let \( Lg' \) be the Lie subalgebra of \( Lg \) whose roots are given by the roots \( \alpha^\vee \) in \( Lg \) satisfying \( (\alpha^\vee, \lambda_{\mathcal{O}^\vee}) \in \mathbb{Z} \). We will study integral special unipotent representations \( \Pi((\mathcal{O}^\vee)') \), where \( (\mathcal{O}^\vee)' \subset Lg' = Lg_1 + Lg_2' \) is an even orbit given by \( (\mathcal{O}^\vee)' = \mathcal{P}^\vee + \mathcal{Q}^\vee \), with

- **Lg of Type** B: \( \mathcal{P}^\vee \subset Lg_1 \) is of Type B; \( \mathcal{Q}^\vee \subset Lg_2' \) is of Type D.
- **Lg of Type** C: \( \mathcal{P}^\vee \subset Lg_1 \) is of Type C; \( \mathcal{Q}^\vee \subset Lg_2' \) is of Type C.
- **Lg of Type** D: \( \mathcal{P}^\vee \subset Lg_1 \) is of Type D; \( \mathcal{Q}^\vee \subset Lg_2' \) is of Type D.

Note that \( \overline{\mathcal{A}}(\mathcal{O}) = \overline{\mathcal{A}}(\mathcal{O}^\vee) = \overline{\mathcal{A}}(\mathcal{P}^\vee) = \overline{\mathcal{A}}(\mathcal{Q}^\vee) = \overline{\mathcal{A}}(\mathcal{O}) \).

We now study the unipotent representations \( \Pi(\mathcal{O}) \) and \( \Pi(\mathcal{P}) \) individually. For \( \mathcal{Q}^\vee \), it has trivial Lusztig quotient, and the special unipotent representation attached to it is \( \Pi(\mathcal{Q}^\vee) = \{ Ind_{G_2}^{GL_c(\alpha_1) \times \cdots \times GL_c(\alpha_2)}(triv \otimes \cdots \otimes triv) \} \). Moreover, the (unique) left cell representation is \( V(L(\mathcal{Q})) = J_{A_{\alpha_1-1} \times \cdots \times A_{\alpha_2-1}}(sgn) \).

On the other hand, suppose the left cell representation of \( \mathcal{P} \) is \( V(L(\mathcal{P})) = \bigoplus_{x \in \overline{\mathcal{A}}(\mathcal{P})} \sigma_x' \). Then

\[
V(L(\mathcal{O}')) = \bigoplus_{x \in \overline{\mathcal{A}}(\mathcal{O}')} \sigma_x' \otimes J_{A_{\alpha_1-1} \times \cdots \times A_{\alpha_2-1}}(sgn).
\]

For \( x \in \overline{\mathcal{A}}(\mathcal{O}') \equiv \overline{\mathcal{A}}(\mathcal{O}) \), let

\[
\sigma_x := J_{W(G)}(\sigma_x' \otimes J_{A_{\alpha_1-1} \times \cdots \times A_{\alpha_2-1}}(sgn)) = J_{W(G)}(\sigma_x' \otimes sgn).
\]

One can use Equation (4.6.5) of [L1] to check that the right hand side is irreducible, and hence the main theorem follows. More explicitly, \( \Pi(\mathcal{O}') \) is given by

\[
\Pi(\mathcal{O}') = \{ Ind_{G_2 \times GL_c(\alpha_1) \times \cdots \times GL_c(\alpha_2)}(\overline{X}(\lambda', w_\lambda \lambda') \otimes triv \otimes \cdots \otimes triv) \mid \overline{X}(\lambda', w_\lambda \lambda') \in \Pi(\mathcal{P}^\vee) \}.
\]
3. Exceptional Lie algebras

Let $O^\vee \subset L\mathfrak{g}$ be an exceptional, special nilpotent orbit with $\lambda_{O^\vee}$ not integral. Since there are no such orbits in $G_2$, we will only focus on exceptional Lie algebras of Type $E$ and Type $F$. As mentioned in the introduction, let $L\mathfrak{g}'$ be the Lie subalgebra of $L\mathfrak{g}$ whose roots are given by the roots $\alpha^\vee$ in $L\mathfrak{g}$ satisfying $\langle \alpha^\vee, \lambda_{O^\vee} \rangle \in \mathbb{Z}$. We will study $\Pi((O')^\vee)$ for $(O')^\vee \subset L\mathfrak{g}$ and $V^L(O')$ as in the classical case.

In the following tables, we list all $O^\vee$ with non-integral $\lambda_{O^\vee}$ and its Lusztig-Spaltenstein dual $O$. Then we give the orbit $O' \subset g'$ corresponding to $O$. In most cases, we can use \cite{Li2} to compute $V^L((O')^\vee)$ or $V^L(O')$ (see Example 3.1 below). Afterwards, we use Proposition 3.18 of \cite{BV2} to obtain $V^L(O) = \{J^{W(G)}_{W(G')} (\sigma') \mid \sigma' \in V^L(O')\}$, which is given in the second last column of the table (with $\sigma_{sp} \in V^L(O)$ always appears first in the list). And the last column records the degree $\deg(\sigma_{sp}) = \deg(\sigma'_{sp})$.

The computations below are carried out by \textsc{LiE} \cite{LiE} and \textsc{MATLAB}.

### A.2.1. $F_4$

The results for $F_4$ are as follows:

| $O^\vee$ | $O$ | $g'$ | $O'$ | $V^L(O)$ | $\deg(\sigma_{sp})$ |
|----------|----|-----|------|---------|------------------|
| $A_1$    | $F_4(a_1)$ | $B_4$ | $711$ | $4_2, 2_1$ | 1                |
| $A_1 + A_1$ | $F_4(a_2)$ | $C_3 + A_1$ | $[3^2] + [2]$ | $9_1$ | 2                |
| $C_3$    | $A_2$ | $C_3 + A_1$ | $[1^6] + [2]$ | $8_2$ | 9                |

### A.2.2. $E_6$

The results for $E_6$ are as follows:

| $O^\vee$ | $O$ | $g'$ | $O'$ | $V^L(O)$ | $\deg(\sigma_{sp})$ |
|----------|----|-----|------|---------|------------------|
| $A_1$    | $E_6(a_1)$ | $A_5 + A_1$ | $[6] + [1^4]$ | $6_p$ | 1                |
| $2A_1$   | $D_5$ | $D_5$ | $[71^4]$ | $20_p$ | 2                |
| $A_2 + A_1$ | $D_5(a_1)$ | $A_5 + A_1$ | $[41^2] + [1^2]$ | $64_p$ | 4                |
| $A_2 + 2A_1$ | $A_4 + A_1$ | $D_5$ | $[3^3] + [1^2]$ | $60_p$ | 5                |
| $A_3$    | $A_4$ | $D_5$ | $[51^3]$ | $81_p$ | 6                |
| $A_4 + A_1$ | $A_2 + 2A_1$ | $A_5 + A_1$ | $[21^4] + [1^2]$ | $60_p'$ | 11               |
| $D_5(a_1)$ | $A_2 + A_1$ | $D_5$ | $[2^4 1^6]$ | $64_p'$ | 13               |

### A.2.3. $E_7$

The results for $E_7$ are as follows:
explicitly, one can check that $L_g$ is of Type $E_7 + A_1$, and the coroots $\alpha^\vee$ such that $\langle \alpha^\vee, \frac{1}{2} h^\vee \rangle = 0$.

A.2.4. $E_8$. The results for $E_8$ are as follows:

| $O^\vee$ | $O$ | $g'$ | $O'$ | $V^L(O)$ | $\deg(\sigma_{sp})$ |
|----------|-----|------|------|----------|-------------------|
| $A_1$    | $E_8(a_1)$ | $E_7 + A_1$ | $E_7 + [1^2]$ | $8_z$ | 1 |
| $2A_1$   | $E_8(a_2)$ | $D_8$ | $13, 1^3$ | $35_x$ | 2 |
| $A_2 + A_1$ | $E_8(a_3)$ | $E_7 + A_1$ | $E_7(a_3) + [1^2]$ | $210_{x}, 50_{x}$ | 4 |
| $A_2 + 2A_1$ | $E_8(b_1)$ | $D_8$ | $[93^4]_1$ | $560_{x}$ | 5 |
| $A_3$ | $E_7(a_1)$ | $D_8$ | $[11^3]$ | $567_{x}$ | 6 |
| $D_4(a_1) + A_1$ | $E_8(a_6)$ | $E_7 + A_1$ | $E_7(a_5) + [1^2]$ | $1400_{x}, 1050_{x}, 175_{x}$ | 8 |
| $A_3 + A_2$ | $D_7(a_1)$ | $D_8$ | $[73^1]_1$ | $3240_{x}$ | 9 |
| $A_4 + A_1$ | $E_6(a_1) + A_1$ | $E_7 + A_1$ | $D_5(a_1) + [1^2]$ | $4096_{x}$ | 11 |
| $A_4 + 2A_1$ | $D_7(a_2)$ | $D_8$ | $[53^1]_2$ | $4200_{x}, 840_{x}$ | 12 |
| $D_5(a_1)$ | $E_6(a_3)$ | $D_8$ | $[73^1]_1$ | $2800_{x}, 700_{xx}$ | 13 |
| $D_6(a_1) + A_1$ | $E_7(a_4)$ | $E_7 + A_1$ | $(A_3 + A_2 + A_2 + [2]$ | $6075_{x}$ | 14 |
| $A_4 + A_2 + A_1$ | $A_6 + A_1$ | $E_7 + A_1$ | $(A_3 + A_2 + A_1 + [1^2]$ | $2835_{x}$ | 14 |
| $D_6(a_1)$ | $E_6(a_3)$ | $D_8$ | $[53]_8$ | $5600_{x}, 3200_{x}$ | 21 |
| $A_6 + A_1$ | $A_4 + A_2 + A_1$ | $E_7 + A_1$ | $(A_2 + 3A_1 + [1^2]$ | $2835_{x}$ | 22 |
| $E_7(a_4)$ | $D_6(a_1) + A_1$ | $E_7 + A_1$ | $(A_2 + 2A_1 + [2]$ | $6075_{x}$ | 22 |
| $D_7(a_2)$ | $A_4 + 2A_1$ | $D_8$ | $[3^2^2]_1^4$ | $4200_{x}, 840_{x}$ | 24 |
| $E_6(a_1) + A_1$ | $A_4 + A_1$ | $E_7 + A_1$ | $(A_2 + A_1 + [1^2]$ | $4096_{x}$ | 26 |
| $E_7(a_3)$ | $A_3$ | $E_7 + A_1$ | $A_2 + [2]$ | $2268_{x}, 972_{x}$ | 30 |
| $E_7(a_1)$ | $A_3$ | $E_7 + A_1$ | $A_1 + [2]$ | $567_{x}$ | 46 |

Here is an example on how the above results are obtained:

**Example 3.1.** Let $O^\vee = D_4(a_1) + A_1$ be a nilpotent orbit of Type $E_8$. By calculating $\frac{1}{2} h^\vee$ explicitly, one can check that $L_g$ is of Type $E_7 + A_1$, and the coroots $\alpha^\vee$ such that $\langle \alpha^\vee, \frac{1}{2} h^\vee \rangle = 0$...
forms a Lie subalgebra \( L \) of Type \((A_5 + A_1) + 0\). Then \((O')^\vee \subset Lg\) is the orbit

\[
(O')^\vee = \text{Ind}_{Lg}^{Lg} \sigma = D_4(a_1) + [2].
\]

By looking at the tables of [Ca], one can check that Lusztig-Spaltenstein dual of \( O^\vee \) and \((O')^\vee \) are \( O = E_8(a_6) \) and \( O' = E_7(a_5) + [1^2] \) respectively. Also, one can check from [CM] that \( \Pi(O) = \Pi(O') = S_3 \).

We now study the left cell \( V^L(O')\): Firstly, note that \( E_7(a_5) = \text{Ind}_{Lg}^{E_8} D_4(a_1) \). By [L2], the special piece attached to \( D_4(a_1) \) is equal to \( \{D_4(a_1), A_3 + A_1, 2A_2 + A_1\} \), and their Springer representations constitute the left cell \( V^L(D_4(a_1)) = \{80_x, 60_x, 10_x\} \) (in fact, \( V^L(D_4(a_1)) \) can also be obtained directly from (4.11.2) of [L1], but this perspective is useful in determining left cells for classical orbits with large Lusztig quotient).

Let \([1^3], [21], [3]\) be the conjugacy classes of \( S_3 \), then by Proposition 4.14 of [BV3] (which is valid since \( O' \) is even),

\[
V^L(O') = \{\sigma'_{[1]} = J_{E_8 + A_1}^{E_7 + A_1} (80_x \boxtimes \text{sgn}), \quad \sigma'_{[21]} = J_{E_8 + A_1}^{E_7 + A_1} (60_x \boxtimes \text{sgn}), \quad \sigma'_{[3]} = J_{E_6 + A_1}^{E_7 + A_1} (10_x \boxtimes \text{sgn})\}.
\]

By Proposition 3.18 of [BV2] and (4.13.3) of [L1], we have

\[
V^L(O) = \{J_{E_8}^E (\sigma'_{[1]}), \quad J_{E_7 + A_1}^E (\sigma'_{[21]}), \quad J_{E_8}^E (\sigma'_{[3]})) = \{J_{E_8}^E (80_x \boxtimes \text{sgn}), \quad J_{E_8}^E (60_x \boxtimes \text{sgn}), \quad J_{E_8}^E (10_x \boxtimes \text{sgn})\} = \{1400_x, 1050_x, 175_x\}.
\]

Moreover, \( \sigma_{sp} = 1400_x \) is the special representation corresponding to the \( O \) under the Springer correspondence. This verifies our main theorem.

Alternatively, \( V^L(O) \) can also be obtained by the following: By [L2], the special piece attached to \( O^\vee \) is given by \( \{O^\vee, A_3 + 2A_1, 2A_2 + 2A_1\} \), and their corresponding Springer representations constitute the full left cell \( V^L(O^\vee) = \{1400'_x, 1050'_x, 175'_x\} \). Therefore, \( V^L(O) \) can be obtained easily by \( V^L(O) = V^L(O^\vee) \otimes \text{sgn} \).

**Remark 3.2.** In Section 4.3 of [L1], Lusztig defined a set \( \mathcal{M}(O) := \{(x, \psi) \mid x \in [\Pi(O)], \psi \in C_{[\Pi(O)]}(x)^\vee\} \) for each special nilpotent orbit \( O \), and there is an injective map \( V^L(O) \hookrightarrow \mathcal{M}(O) \).

If \( O \) is not equal to the three exceptional orbits \((A_4 + A_1) + (A_4 + A_1 + A_6 + a_1) + A_1 \) in \( E_7 \), then every element of the form \( (x, 1) \in \mathcal{M}(O) \) is the image of some \( \sigma_x \in V^L(O) \). One can check case-by-case that this identification of elements in \( V^L(O) \) matches with our identification \( \sigma_x := J_{W^x}(\sigma'_x) \) in Step (VI) of Section 1.

For the three exceptional orbits, \( \Pi(O) = \{1, s\} \) and \( V^L(O) = \{\sigma_{sp}\} \), which maps to \((1, 1) \in \mathcal{M}(O) \) under the inclusion map. It turns out that

\[
J_{W^x}(\sigma'_{[1]}) = J_{W^x}(\sigma'_{sp}) = \sigma_{sp}, \quad J_{W^x}(\sigma'_{[1]}) = 0
\]
in all cases. Even though we cannot express the character formulas for these orbits in the form of Theorem III of [BV3], they can be obtained by simply using Step (IV) of Section 1.

We end this manuscript by correcting a few typos found in [L1]:

- In (4.11.3), $64_p = \gamma_{A_2 \times A_1}^{W}(\text{sign})$.
- In (4.13.3), $840^x = J_{E_8}D_7(234013)$, and $4096^x = J_{E_8}E_7(512a)$.

REFERENCES

[A] D. Alvis, Induce/restrict matrices for exceptional Weyl groups, arXiv:0506377
[AL] D. Alvis, G. Lusztig, On Springer’s correspondence for simple groups of type $E_n$ ($n=6,7,8$), Math. Proc. Camb. Phil. Soc. (1982) 92, 65-72
[B1] D. Barbasch, The unitary dual for complex classical Lie groups, Invent. Math. 96 (1989), 103-176
[B2] D. Barbasch, Representations with maximal primitive ideal, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math. 92 (1990), 317-331
[B3] D. Barbasch, Unipotent representations and the dual pair correspondence, Representation Theory, Number Theory and Invariant Theory: In honor of Roger Howe on the occasion of his 70th birthday (2017)
[BV1] D. Barbasch, D. Vogan, Primitive ideals and orbital integrals in complex classical groups, Math. Ann. 259 (1982), 153-199
[BV2] D. Barbasch, D. Vogan, Primitive ideals and orbital integrals in complex exceptional groups, J. Algebra 80 (1983), 350-382
[BV3] D. Barbasch, D. Vogan, Unipotent representations of complex semisimple Lie groups, Ann. of Math. 121 (1985), 41-110
[Ca] R. Carter, Finite Groups of Lie Type, Wiley & Sons (1985)
[CM] D. Collingwood, W. M. McGovern, Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold Mathematics Series (1993)
[dGE] W. A. de Graff, A. Elashvili, Induced Nilpotent Orbits of the Simple Lie Algebras of Exceptional Type, Georgian Mathematical Journal 16 (2009), 257-278
[LiE] M. A. A. van Leeuwen, A. M. Cohen and B. Lisser, LiE, A Package for Lie Group Computations, Computer Algebra Nederland, Amsterdam (1992)
[L1] G. Lusztig, Characters of reductive groups over a finite field, Ann. Math. Studies 107 (1984), Princeton University Press
[L2] G. Lusztig, Notes on unipotent classes, Asian J. Math. 1 (1997), 194-207
[L3] G. Lusztig, Unipotent classes and special Weyl group representations, J. Algebra. 321 (2009), 3418-3449
[LS] G. Lusztig, N. Spaltenstein, Induced Unipotent Classes, J. London Math. Soc. (2) 19 (1979), no. 1, 41-52
[McG] W. M. McGovern, Completely prime maximal ideals and quantization, Mem. Amer. Math. Soc. 519 (1994)
[W1] K. D. Wong, Quantization of Special Symplectic Nilpotent Orbits and Normality of their Closures, J. Algebra 462 (2016), 37-53
[W2] K. D. Wong, Some Calculations of the Lusztig-Vogan Bijection for for Classical Nilpotent Orbits, J. Algebra 487 (2017), 317-339

School of Science and Engineering, the Chinese University of Hong Kong, Shenzhen, Longgang District, Shenzhen, Guangdong 518172, China

E-mail address: kayue.wong@gmail.com