A GENERALIZATION OF RADO’S THEOREM FOR ALMOST GRAPHICAL BOUNDARIES

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Abstract. In this paper, we prove a generalization of Rado’s Theorem, a fundamental result of minimal surface theory, which says that minimal surfaces over a convex domain with graphical boundaries must be disks which are themselves graphical. We will show that, for a minimal surface of any genus, whose boundary is “almost graphical” in some sense, that the surface must be graphical once we move sufficiently far from the boundary.

1. Introduction

One of the fundamental results of minimal surface theory is Rado’s Theorem, which is connected to the famous Plateau Problem. Rado’s Theorem (see [4]) states that if $\Omega \subset \mathbb{R}^2$ is a convex subset and $\sigma \subset \mathbb{R}^3$ is a simple closed curve which is graphical over $\partial \Omega$, then any minimal surface $\Sigma \subset \mathbb{R}^3$ with $\partial \Sigma = \sigma$ must be a disk which is graphical over $\Omega$, and hence unique by the maximum principle.

The proof of Rado’s Theorem begins by assuming there is a point at which $\Sigma$ is not graphical, i.e., where $\Sigma$ has a vertical tangent plane. One can then use the description of the local intersection of minimal surfaces as $n$-prong singularities (see, for example, [2, Section 4.6]) to derive a contradiction of the assumption on the boundary $\sigma$.

Our goal is to generalize Rado’s Theorem for the case in which $\sigma$ is not graphical, but satisfies some “almost graphical” condition. We will show that, although $\Sigma$ is obviously not graphical near its boundary, if we move far enough in from the boundary, $\Sigma$ will be graphical. We will prove the genus zero case first, and then generalize to higher genus surfaces.

Throughout this paper, we will use the topological fact that, if $\Sigma$ has genus $n$, any collection of $n + 1$ disjoint closed simple curves on $\Sigma$ must separate $\Sigma$ into at least two connected components. This can be seen as follows. If a compact connected orientable surface with boundary of genus $n$ has $k$ boundary components, its Euler characteristic is $2 - 2n - k$. By cutting the surface along a circle which does not separate the surface into at least two connected components, the number of
boundary components would increase by two, while the Euler characteristic would stay the same. Thus, the genus would decrease by one. Therefore, the maximum number of such cuts would be equal to the genus \( n \), so any collection of \( n + 1 \) disjoint closed simple curves must separate the surface into at least two connected components.

The “almost graphical” condition we will use will be as follows. Let \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) be a parametrization for \( \partial \Sigma \). Then, we say that \( \sigma \) is “\( C, h \)-almost graphical” if

1) \( \sigma \) has one connected component.
2) After possibly a rotation, \( |\sigma_3| < Ch \) (and thus, all of \( \Sigma \) lies in a narrow vertical slab).
3) \( \sigma \) is “\( h \)-almost monotone”, i.e., for any \( y \in \sigma \), \( B_{4h}(y) \cap \sigma \) has only one component which intersects \( B_{2h}(y) \). Therefore, any point \( b \in B_{2h}(y) \cap \sigma \) can be joined to \( y \) by a path in \( B_{4h}(y) \cap \sigma \). See Figure 1.

We now state our main result.

**Theorem 1.** There exists a \( C > 0 \) (not depending on \( \Sigma \)) such that if \( \Sigma \) is an embedded minimal surface of genus \( n \), \( n \geq 0 \), with \( C, h \)-almost graphical boundary \( \sigma = \partial \Sigma \subset \partial B_R \), then \( \Sigma \cap B_{R-(64n+30)h} \) is graphical.

2. **Catenoid foliations**

Although the proof of Rado’s Theorem utilizes intersections of minimal surfaces with planes (namely, vertical tangent planes), the proof of our generalization will require a greater degree of sophistication. We will be intersecting minimal surfaces at nongraphical points with carefully chosen catenoids.
Here, we provide some background and important results involving catenoid foliations. This material is covered in greater detail (including proofs) in [1, Appendix A]. Recall that in this paper we will be talking about minimal surfaces \( \Sigma \) which lie in a narrow vertical slab.

Let \( \text{Cat}(y) \) be the vertical catenoid centered at \( y = (y_1, y_2, y_3) \in \mathbb{R}^3 \). In other words,

\[
\text{Cat}(y) = \{ x \in \mathbb{R}^3 | \cosh^2(x_3 - y_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 \}.
\]

For an angle \( \theta \in (0, \frac{\pi}{2}) \), we denote by \( \partial N_\theta(y) \) the cone

\[
\{ x \in \mathbb{R}^3 | (x_3 - y_3)^2 = |x - y|^2 \sin^2 \theta \}.
\]

Then, we see that \( \partial N_{\pi/4}(y) \cap \text{Cat}(y) = \emptyset \), since \( \cosh t > t \) for all \( t \geq 0 \). So, we set

\[
\theta_0 = \inf \{ \theta | \partial N_\theta(y) \cap \text{Cat}(y) = \emptyset \}.
\]

Thus, \( \partial N_{\theta_0}(y) \) and \( \text{Cat}(y) \) intersect tangentially in a pair of circles (one above the \( x_1 x_2 \)-plane and one below). Let \( \text{Cat}_0(y) \) be the component of \( \text{Cat}(y) \setminus \partial N_{\theta_0}(y) \) containing the neck

\[
\{ x | x_3 = y_3, (x_1 - y_1)^2 + (x_2 - y_2)^2 = 1 \}.
\]

If \( x \in \text{Cat}_0(y) \), then the line segment joining \( y \) and \( x \) intersects \( \text{Cat}_0(y) \) at exactly one point, namely \( x \). So, the dilations of \( \text{Cat}_0(y) \) about \( y \) are disjoint, and give us a minimal foliation (see Figure 2) of the solid

![Figure 2](image_url)

**Figure 2.** The catenoid foliation

(open) cone

\[
N_{\theta_0}(y) = \{ x \in \mathbb{R}^3 | (x_3 - y_3)^2 < |x - y|^2 \sin^2 \theta_0 \}.
\]

The leaves of this foliation all have boundary in \( \partial N_{\theta_0}(y) \) and are the level sets of the function \( f_y \) given by

\[
y + \frac{x-y}{f_y(x)} \in \text{Cat}_0(y).
\]
Choose $\beta_A > 0$ sufficiently small so that
\[
\{ x \mid |x_3 - y_3| \leq 2\beta_A h \} \setminus B_{h/8}(y) \subset N_{y_0}(y)
\]
and
\[
\{ x \mid f_y(x) = 3h/16 \} \cap \{ x \mid |x_3 - y_3| \leq 2\beta_A h \} \subset B_{7h/32}(y).
\]
Since the intersection of any two minimal surfaces (in particular, our given $\Sigma$ and any catenoid in our foliation) is locally given by an $n$-prong singularity, i.e., $2n$ embedded arcs which meet at equal angles (see Claim 1 of Lemma 4 in [3]), we get the following Lemma.

**Lemma 2.** If $z \in \Sigma \subset N_{y_0}(y)$ is a nontrivial interior critical point of $f_y|_\Sigma$, then $\{ x \mid f_y(x) = f_y(z) \}$ has an $n$-prong singularity at $z$ with $n \geq 2$.

As a consequence of this, we obtain a version of the Strong Maximum Principle.

**Lemma 3.** If $\Sigma \subset N_{y_0}(y)$, then $f_y|_\Sigma$ has no nontrivial interior local extrema.

Using this, we can show, using the foliation indexed by $f_y$, that a minimal surface in a narrow slab either stays near the boundary or comes near the center.

**Corollary 4.** If $\partial \Sigma \subset \partial B_h(y)$, $B_{3h/4}(y) \cap \Sigma \neq \emptyset$, and
\[
\Sigma \subset B_h(y) \cap \{ x \mid |x_3 - y_3| \leq 2\beta_A h \},
\]
then $B_{h/4}(y) \cap \Sigma \neq \emptyset$.

Iterating Corollary 4 along a chain of balls, we see that we will be able to extend curves out close to the boundary. Here, $T_h$ refers to a tubular neighborhood of radius $h$, and $\gamma_{p,q}$ is the line segment joining $p$ and $q$.

**Corollary 5.** If $\Sigma \subset \{ |x_3| \leq 2\beta_A h \}$, points $p, q \in \{ x_3 = 0 \}$ satisfy $T_h(\gamma_{p,q}) \cap \partial \Sigma = \emptyset$, and
\[
y_p \in B_{h/4}(p) \cap \Sigma,
\]
then a curve $\nu \subset T_h(\gamma_{p,q}) \cap \Sigma$ connects $y_p$ to $B_{h/4}(q) \cap \Sigma$.

The final catenoid foliation result we will use shows that the vertical projection of $\Sigma$ cannot stray too far outside the vertical projection of $\partial \Sigma$.

**Corollary 6.** If $\Sigma \subset \{ |x_3| \leq 2\beta_A h \}$ and $E$ is an unbounded component of
\[
\mathbb{R}^2 \setminus T_{h/4}(\Pi(\partial \Sigma))
\]
then $\Pi(\Sigma) \cap E = \emptyset$. 
3. The genus zero case

Proof of Theorem 1 for \( n = 0 \). Let \( C = \beta_A \), where \( \beta_A > 0 \) is defined by (1). Let \( \Sigma \) be an embedded minimal disk with \( C \), \( h \)-almost graphical boundary \( \sigma = \partial \Sigma \subset \partial B_R \). Our proof begins with an argument which is similar to the first step of the proof of (1) Lemma I.0.11. Suppose \( \Sigma \cap B_{30h} \) is not graphical; let \( z \in \Sigma \cap B_{30h} \) be a point such that the tangent plane to \( \Sigma \) at \( z \) is vertical. Fix \( y \in \partial B_{4h}(z) \) so that the line segment \( \gamma_{y,z} \) is normal to \( \Sigma \) at \( z \). Then, \( f_y(z) = 4h \), where \( f_y \) is the function used to define the level sets of the catenoid foliation for catenoids centered at \( y \); see Section 2. Let \( y' \) be given such that \( y' \in \partial B_{10h}(y) \) and \( z \in \gamma_{y,y'} \).

Any simple closed curve \( \rho \subset \Sigma \setminus \{ f_y \geq 4h \} \) bounds a disk \( \Sigma_\rho \subset \Sigma \). By Lemma 3, \( f_y \) has no maxima on \( \Sigma_\rho \setminus \{ f_y > 4h \} \) so that \( \Sigma_\rho \setminus \{ f_y > 4h \} = \emptyset \). On the other hand, by Lemma 2 there is a neighborhood \( U_z \subset \Sigma \) of \( z \) such that \( U_z \cap \{ f_y = 4h \} \) is an \( n \)-prong singularity with \( n \geq 2 \); in other words, \( U_z \cap \{ f_y = 4h \} \} \) is the union of \( 2n \geq 4 \) disjoint embedded arcs meeting at \( z \). Moreover, \( U_z \setminus \{ f_y = 4h \} \) (i.e., the part of \( U_z \) inside the catenoid \( \{ f_y = 4h \} \)) has \( n \) components \( U_1, \ldots, U_n \) with

\[
\overline{U_i \cap U_j} = \{ z \} \quad \text{for} \ i \neq j.
\]

If a simple curve \( \widetilde{\rho}_z \subset \Sigma \setminus \{ f_y \geq 4h \} \) connects \( U_1 \) to \( U_2 \), connecting \( \partial \widetilde{\rho}_z \) by a curve in \( U_z \) gives a simple closed curve \( \rho_z \subset \Sigma \setminus \{ f_y > 4h \} \) with \( \widetilde{\rho}_z \subset \rho_z \) and \( \rho_z \cap \{ f_y = 4h \} = \{ z \} \). Hence, \( \rho_z \) bounds a disk \( \Sigma_{\rho_z} \subset \Sigma \setminus \{ f_y > 4h \} \). By construction,

\[
U_z \cap \Sigma_{\rho_z} \setminus \cup_i U_i \neq \emptyset.
\]

This is a contradiction, so \( U_1 \) and \( U_2 \) must be contained in components \( \Sigma_{4h}^1 \neq \Sigma_{4h}^2 \) of \( \Sigma \setminus \{ f_y \geq 4h \} \) with \( z \in \Sigma_{4h}^1 \setminus \Sigma_{4h}^2 \). For \( i = 1, 2 \), Lemma 3 and (1) give us \( y^a_i \subset B_{h/4}(y) \cap \Sigma_{4h}^i \). By Corollary 5 there exist curves \( \nu_i \subset T_h(\gamma_{y,y'}) \cap \Sigma \) with \( \partial \nu_i = \{ y^a_i, y^b_i \} \), where \( y^b_i \in B_{h/4}(y') \). There are two cases:

- If \( y^a_i \) and \( y^b_i \) do not connect in \( B_{4h}(y') \cap \Sigma \), take \( \gamma_0 \subset B_{5h}(y) \cap \Sigma \) from \( y^a_1 \) to \( y^a_2 \) and set \( \gamma_a = \nu_1 \cup \gamma_0 \cup \nu_2 \) and \( y_i = y^a_i \).
- Otherwise, if \( \gamma_0 \subset B_{4h}(y') \cap \Sigma \) connects \( y^a_1 \) and \( y^b_2 \), set \( \gamma_a = \nu_1 \cup \gamma_0 \cup \nu_2 \) and \( y_i = y^a_i \).

In either case, after possibly switching \( y \) and \( y' \), we get a curve

\[
\gamma_a \subset (T_h(\gamma_{y,y'}) \cup B_{5h}(y')) \cap \Sigma
\]

with \( \partial \gamma_a = \{ y_1, y_2 \} \subset B_{h/4}(y) \) and \( y_i \in S^a_i \) for components \( S^a_1 \neq S^a_2 \) of \( B_{4h}(y) \cap \Sigma \).
Let $y''$ be given so that $y'' \in \partial B_{\sqrt{R^2-1}}$ and $y \in \gamma_{y',y''}$ (that is, $y', z, y$, and $y''$ are all collinear). By Corollary \[ for $i = 1, 2$, there exist curves $\mu_i \subset T_h(\gamma_{y,y''}) \cap \Sigma$ connecting $y_i$ to points $z_i \in B_{h/4}(y') \cap \Sigma$. We then intersect $\Sigma$ with a plane connecting the points $z_1$ and $z_2$, and which is perpendicular to $\gamma_{y,y''}$ (such a plane exists since $z_i$ can be taken to be any point in $B_{h/4}(y') \cap \mu_i$). If $z_1$ and $z_2$ can be connected by a curve $\lambda$ in the intersection of $\Sigma$ with this plane, then $y_1$ and $y_2$ are connected by the curve $\mu_1 \cup \lambda \cup \mu_2$. This contradicts the following Claim.

**Claim:** Let $H$ be the half-space given by

$$H = \{ x | \langle y - y', x - y \rangle > 0 \}.$$ 

Then, $y_1$ and $y_2$ can not be connected by a curve in $T_h(H) \cap \Sigma$.

**Proof of Claim** \[ p. 11-12]. If $\eta_{1,2} \subset T_h(H) \cap \Sigma$ connects $y_1$ and $y_2$, then $\eta_{1,2} \cup \gamma_a$ bounds a disk $\Sigma_{1,2} \subset \Sigma$. Since $\eta_{1,2} \subset T_h(H)$, $\partial B_{h/4}(y') \cap \partial \Sigma_{1,2}$ consists of an odd number of points in each $S^a_i$ and hence $\partial B_{h/4}(y') \cap \Sigma_{1,2}$ contains a curve from $S^a_i$ to $S^a_2$. However, $S^a_1$ and $S^a_2$ are distinct components of $B_{h/4}(y) \cap \Sigma$, so this curve must contain a point

$$y_{1,2} \in \partial B_{h/4}(y) \cap \partial B_{h/4}(y') \cap \Sigma_{1,2}.$$ 

By construction, $\Pi(y_{1,2})$ is in an unbounded component of $\mathbb{R}^2 \setminus T_{h/4}(\Pi(\partial \Sigma_{1,2}))$, contradicting Corollary \[ . Therefore, $y_1$ and $y_2$ can’t be connected in $T_h(H) \cap \Sigma$. \]

Returning to the proof of the genus zero case of Theorem \[ , since there is no curve in the intersection of $\Sigma$ and this plane which connects $z_1$ to $z_2$, we get disjoint curves $\lambda_1$ and $\lambda_2$ in the intersection of $\Sigma$ with this plane, with $z_i \in \lambda_i$ for $i = 1, 2$. Neither $\lambda_1$ nor $\lambda_2$ can be closed, as if either were closed, it would bound a disk in the intersection of $\Sigma$ and the plane, violating the maximum principle. So, these curves must go to the boundary of $\Sigma$, i.e., there exist points $b_i \in \lambda_i \cap \sigma$ for $i = 1, 2$. By construction, $b_2 \in B_{2h}(b_1)$. By the $h$-almost monotonicity of $\sigma$, there is a curve $\alpha \subset B_{sh}(b_1) \cap \sigma$ connecting $b_1$ and $b_2$. Thus, $y_1$ is connected to $y_2$ by the curve $\mu_1 \cup \lambda_1 \cup \alpha \cup \lambda_2 \cup \mu_2$, contradicting that $y_1$ can not be connected to $y_2$ in $T_h(H) \cap \Sigma$. Therefore, $\Sigma \cap B_{R-30h}$ is graphical. \]

4. The Higher Genus Case

In this section we prove the higher genus case, i.e., Theorem \[ for $n \geq 1$. We begin by looking at the case $n = 1$, that is
**Theorem 7.** Let $\Sigma$ be an embedded minimal surface with genus 1 such that $\sigma = \partial \Sigma \subset \partial B_R$ is $C, h$-almost graphical. Then, $\Sigma \cap B_{R-94h}$ is graphical.

As in the genus 0 case, when we say $\partial \Sigma$ is $C, h$-almost graphical, we will be taking $C = \beta_A$.

In dealing with the genus zero case, each time we had a closed path we could say that it bounded a disk; that is, each closed path was homotopic to a point. This is no longer true when the genus is one or higher. However, given any genus one surface, any two disjoint closed paths divide the surface into at least two regions. In the following lemma, we show that, for our given minimal surface $\Sigma$, it is impossible to have two nontrivial closed paths (i.e., two closed paths which are not homotopic to a point) which are far apart.

**Lemma 8.** There can’t be two nontrivial closed simple paths $\gamma_1$ and $\gamma_2$ in $\Sigma$ so that

$$\text{dist}(T_{\frac{h}{4}}(\gamma_1), T_{\frac{h}{4}}(\gamma_2)) > 0.$$

**Proof.** Because of the topological properties of a genus one surface, $\gamma_1 \cup \gamma_2$ bounds a connected region $\Sigma' \subset \Sigma$. See Figure 3, which shows a closed torus (the same is true for any surface of genus one). However, this topological fact and \[2\] gives $\mathbb{R}^2 \setminus T_{h/4}(\Pi(\partial \Sigma')) \neq \emptyset$. This contradicts Corollary \[6\].

To apply Lemma \[8\] we need to be able to build a closed path which is not homotopic to a point and is contained in a fixed region. In the following lemma we build a path which is not homotopic to a point and is contained in a dumbbell-shaped region as shown in Figure 4.

**Lemma 9.** Let $\Sigma$ be as in Theorem \[1\] with genus $n \geq 1$, and let $z \in B_{R-30h} \cap \Sigma$ be such that $z$ is not graphical. Then, for any $z' \in \partial \Sigma$,
there exists a closed path contained in \((B_{8h}(z') \cup T_h(\gamma_{z,z'}) \cup B_{13h}(z)) \cap \Sigma\) which is not homotopic to a point.

Proof. The construction of this path starts in the same way as the construction at the beginning of the proof of the genus zero case of Theorem 1. Fix \(y \in \partial B_{4h}(z)\) so that the line segment \(\gamma_{y,z}\) is normal to \(\Sigma\) at \(z\). Then, \(f_y(z) = 4h\), where \(f_y\) is the function used to define the level sets of the catenoid foliation for catenoids centered at \(y\). Now, by Lemma 2, there is a neighborhood \(U_z \subset \Sigma\) of \(z\) such that \(U_z \cap \{f_y = 4h\}\) is an \(n\)-prong singularity with \(n \geq 2\); in other words, \(U_z \cap \{f_y = 4h\} \setminus \{z\}\) is the union of \(2n \geq 4\) disjoint embedded arcs meeting at \(z\). Moreover, \(U_z \setminus \{f_y \geq 4h\}\) (i.e., the part of \(U_z\) inside the catenoid \(\{f_y = 4h\}\)) has \(n\) components \(U_1, \ldots, U_n\) with

\[\overline{U_i} \cap \overline{U_j} = \{z\} \text{ for } i \neq j.\]

However, unlike in the genus zero case, we can not say that that \(U_1\) and \(U_2\) must be contained in distinct components \(\Sigma_{4h}^1 \neq \Sigma_{4h}^2\) of \(\Sigma \setminus \{f_y \geq 4h\}\) with \(z \in \overline{\Sigma_{4h}^1} \cap \overline{\Sigma_{4h}^2}\). There are two possibilities: either the components are distinct (as in the genus zero case), or they coincide, i.e., \(\Sigma_{4h}^1 = \Sigma_{4h}^2\). First, we consider the case where the components coincide. Then, as in the proof of the genus zero case, we would have a simple curve \(\tilde{\rho}_z \subset \Sigma \setminus \{f_y \geq 4h\}\) connecting \(U_1\) to \(U_2\), and connecting \(\partial \tilde{\rho}_z\) by a curve in \(U_z\), we can build a simple closed curve \(\rho_z \subset \Sigma \setminus \{f_y > 4h\}\) with \(\tilde{\rho}_z \subset \rho_z\) and \(\rho_z \cap \{f_y \geq 4h\} = \{z\}\). We saw in the proof of the genus zero case that \(\rho_z\) can not bound a disk. Thus, the curve \(\rho_z\) is not homotopic to a point, and so the lemma is proved in this case, since \(\rho_z \subset \Sigma \cap \{f_y \leq 4h\} \subset \Sigma \cap B_{13h}(z)\).
If instead the components \( \Sigma_{1h} \) and \( \Sigma_{2h} \) are distinct (as in the genus zero case), then let \( y' \) be given such that \( y' \in \partial B_{10h}(y) \) and \( z \in \gamma_{y,y'} \). Then, as in the proof of the genus zero case, we get a curve

\[
\gamma_a \subset (T_h(\gamma_{y,y'}) \cup B_{5h}(y')) \cap \Sigma
\]

with \( \partial \gamma_a = \{y_1, y_2\} \subset B_{h/4}(y) \) and \( y_i \in S_i^a \) for components \( S_1^a \neq S_2^a \) of \( B_{4h}(y) \cap \Sigma \).

Let \( S \) be the annulus given by \( B_{11h}(z) \setminus B_{10h}(z) \), let \( z^a \in \partial B_{11h}(z) \) such that \( y \in \gamma_{z^a} \) and let \( z^b \in \partial B_{11h}(z) \) such that \( z^b \in \gamma_{z^a} \). Let \( \bar{\gamma} \) be the shortest polygonal path in \( S \) connecting \( z^a \) and \( z^b \) then, using Corollary \( \mathbb{C} \) we can first build two paths \( \gamma_i^a \) connecting \( y_i \) to \( z_i^a \in B_{h/4}(z^a) \) with \( \gamma_i^a \subset T_h(\gamma_{y_i,z^a}) \). Then we can build another two paths \( \gamma_i^b \) from \( z_i^a \) to \( z_i^a \in B_{h/4}(z^b) \) with \( \gamma_i^b \subset T_h(\bar{\gamma}_i) \). Eventually, we build two paths \( \gamma_i^b \) connecting \( z_i^a \) to \( z_i^a \in B_{h/4}(z^b) \) with \( \bar{\gamma}_i^b \subset T_h(\gamma_{y_i,z^b}) \), where \( \gamma_i^b \) is in \( \partial B_{\sqrt{R^2-h^2}} \) such that \( \gamma_i^b \subset \gamma_{z_i^a} \). So far we have a path \( \gamma = \gamma_1^b \cup \gamma_1^a \cup \gamma_2^a \cup \gamma_2^b \) whose ends \( z_i^a \), \( z_i^b \), are contained in \( B_{4h}(z') \). As in the end of the proof of the genus zero case, taking the intersection of \( \Sigma \) with a plane connecting the points \( z_i^a \) and \( z_i^b \), and which is perpendicular to \( \gamma_{z_i^a} \), we can connect \( z_i^a \) and \( z_i^b \) with a path, \( \gamma_i^b \), that is contained in \( B_{4h}(z') \cap \partial \Sigma \).

We are left to show why \( \gamma = \gamma_1^b \cup \gamma_1^a \cup \gamma_2^a \cup \gamma_2^b \cup \gamma_{z_i^a} \cup \gamma_{z_i^b} \) is not homotopic to a point.

In the proof of the genus zero case, in the claim, we proved that if a path \( \eta_{1,2} \subset T_h(H) \cap \Sigma \) where \( H = \{x|\langle y-y', x-y \rangle > 0\} \) connects \( y_1 \) and \( y_2 \), the ends of \( \gamma_a \), then \( \gamma_a \cup \eta_{1,2} \) cannot bound a disk. However, a closer look at the proof of that shows that as long as \( \partial B_{4h}(y) \cap \partial B_{5h}(y') \cap \Sigma \) is in the unbounded component of \( \mathbb{R}^2 \setminus T_{h/4}(\Pi(\gamma_a \cup \eta_{1,2})) \) then \( \gamma_a \cup \eta_{1,2} \) cannot bound a disk. Since this is the case, \( \gamma \) cannot bound a disk.

As a consequence of Lemma \( \mathbb{K} \) and Lemma \( \mathbb{L} \) we have the following lemma that says that if the interior of the surface fails to be graphical at two points, then these two points have to be close. In other words, the interior minus a smaller ball is graphical.

**Lemma 10.** Let \( \Sigma \) be as in Theorem \( \mathbb{J} \) with genus \( n \geq 1 \), and let \( z_1, z_2 \in B_{R-30h} \cap \Sigma \) such that \( z_1 \) and \( z_2 \) are not graphical. Then, \( |z_1 - z_2| < 31h \).

**Proof.** Assume \( |z_1 - z_2| \geq 31h \) and let \( \pi \) be the plane perpendicular to \( \gamma_{z_1,z_2} \) through its midpoint. Fix \( z'_1, z'_2 \in \partial \Sigma \) so that \( z'_i \) is in the half space containing \( z_i \) and \( |z'_i - z'_2| > 17h \). Then applying Lemma \( \mathbb{D} \) creates two paths \( \gamma_1 \) and \( \gamma_2 \) as in Lemma \( \mathbb{K} \) see Figure \( \mathbb{S} \) giving the contradiction. \( \square \)
We are now ready to prove Theorem 7.

Proof of Theorem 7. Lemma 10 says that if \( z_1 \) and \( z_2 \) are two non-graphical points in \( B_{R-30h} \) then \( (B_{R-30h} \setminus B_{31h}(z_1)) \cap \Sigma \) is graphical. If \( z_1 \in B_{R-30h-32h} \) then the annulus \( (B_{R-30h} \setminus B_{R-31h}) \cap \Sigma \) is graphical and applying Rado’s theorem gives that \( B_{R-30h} \cap \Sigma \) is graphical. If instead \( z_1 \) is not in \( B_{R-62h} \) then \( B_{R-94h} \cap \Sigma \) is graphical. In either case, the theorem follows.

Now we begin to prove the general case of Theorem 1.

Let \( A_i = B_{R-64ih-30h} \setminus B_{R-64ih-31h} \) for \( i = 0, \ldots, n \). These \( n+1 \) annuli have width \( h \), and the distance between \( A_i \) and \( A_{i+1} \) is \( 64h \). Theorem 1 will clearly follow once we have proved the following proposition.

**Proposition 11.** There exists an \( i = 0, \ldots, n \) such that \( A_i \cap \Sigma \) is graphical.

The proof of Proposition 11 uses the equivalent of Lemma 8 for the genus \( n \) case.

**Lemma 12.** There can’t be \( n+1 \) nontrivial closed simple paths \( \gamma_i \in \Sigma \), \( i = 0, \ldots, n \), so that

\[
\text{dist}(T_h(\gamma_i), T_h(\gamma_j)) > 0, \quad \text{for } i \neq j.
\]

Proof of Lemma 12. The proof uses the same idea that it is used to prove Lemma 8. In the genus \( n \) case we use the topological property that \( n+1 \) disjoint closed simple paths bound at least one connected region with more than one boundary component.
Proof of Proposition 11. Let us assume the proposition is false and let $z_i \in A_i \cap \Sigma$ be nongraphical points. Working as we did in the proof of Lemma 9, we can find $z'_i \in \partial \Sigma$ and polygonal paths $\gamma_i$, and build $n + 1$ nontrivial paths $\gamma_i \subset B_{31h}(z_i) \cup T_h(\gamma_i) \cup B_{30h}(z'_i)$ (where the $n + 1$ boundary points $z'_i$ are chosen in such a way that the corresponding dumbbell-shaped regions are pairwise disjoint) such that (3) in Lemma 12 holds. See Figure 6.

This contradicts Lemma 12, proves Proposition 11 and therefore proves Theorem 1. Note that because of the way we constructed the annuli we have $|z_i - z_j| > 64h$ for $i \neq j$ and therefore $B_{31h}(z_i)$ and $B_{31h}(z_j)$ are always a distance of more than $h$ apart. □

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