Skin effect in the time and frequency domain—comparison of power series and Bessel function solutions

Malcolm Stuart Raven
The Crayke, Bridlington, UK
E-mail: malcolmraven@talktalk.net

Keywords: transient currents, power series, skin effect, bessel functions, diffusion

Abstract
In a time dependent solution of electromagnetic field penetration into a conductor, it turns out that the impedance power series solution diverges sharply if the ratio conductor radius to skin depth $a/b$ exceeds 2.7, close to the mathematical irrational number $e$. This problem has previously been considered to be due to the role of inversion. However, in this paper we show that the power series solution may be derived analytically from Bessel function solutions where the latter does not show the divergence problem. Experimental results are presented and we find that the logarithmic slope of the ac resistance and reactance change close to the value $a/b = e$ in both models. Since both models are based on the solution of a diffusion-like equation then the divergence effect at $e$ may be expected to occur in other diffusion processes.

1. Introduction

When alternating currents flow in a conductor the current density does not penetrate fully into the conductor as it does in steady state direct current flow. The penetration becomes less as the frequency increases until at high frequencies the current mainly flows at the conductor surface. This results in extra losses in the conductor due to the increase in internal resistance which at high frequencies increases with frequency as $\sqrt{f}$. Conversely the internal inductance decreases with increasing frequency as $1/\sqrt{f}$. This is the well known Skin Effect which has been studied for well over a century, [1, 2]. Much of the interest in the early work concerned skin effect in the frequency domain mainly because of the dominance of analogue communications. Interest in skin effect in the time domain arises mainly because of the requirement for fast transition edge rates in digital electronics and particularly as broadband applications increase [3]. Also applications in environmental research [4] and critical applications including the physics of lightning and lightning effects on aircraft [5].

Although recent approaches to analyzing the skin effect in the time domain have started from the vector potential [6], James Clerk Maxwell used vector potentials in determining the electromotive force (emf) produced by a time varying current flowing in a cylindrical conductor [7]. This approach has been re-examined recently, the stability of the power series analyzed [8] and applications discussed [9]. However, even though this transient approach gives the correct results at low frequencies and is consistent with Bessel function solutions it turns out that the series solution diverges if the ratio conductor radius to skin depth $a/b$ exceeds 2.7. This coincidentally is close to the mathematical irrational number $e = 2.718...$ [1, 8]. This divergence has been considered previously to be a consequence of mathematical inversion but this has not yet been proved for the power series case. In addition to determining transient currents an interesting result arises from the power series solution. If the emf is integrated between constant time limits giving constant currents at these limits, then the average emf (integrated emf/time) is independent of the skin effect.

In this paper we give experimental impedance measurement results for a long copper wire meander showing that both power series and Bessel mathematical models produce a discontinuity in the impedance at $a/b \approx 2.7$. We first show that the time dependent solution can be derived from the Bessel function solution and also give an alternative method for multiplying the power series polynomials to obtain higher order terms. This, like in the
matrix method [1], has the advantage of using computer techniques to obtain higher order terms in the series and thus examine its stability. We then investigate the behaviour of the impedance in the vicinity of $a/b = 0$ both for the power series case and the Bessel function case and these theories compared with experimental measurements.

2. Time dependent and frequency dependent approaches

In the following analysis we assume a good conductor neglecting the displacement current due to distributed capacitance and any proximity effect i.e. coupling to other conductors. These practical considerations are considered in section 3.2. Impedance and related to the experimental results in figures 2 and 3. The problem refers to macroscopic EM fields and does not consider optical frequencies and the anomalous skin effect. We first consider briefly the vector potential procedure used by Maxwell and then the Bessel function approach.

2.1. Vector potential method

The specific problem considered is for current flow in a long cylindrical conducting wire circuit. The description of the circuit cannot be put any better than that given by Maxwell [7] Article 682:

‘In order to deal mathematically with such arrangements, we shall begin with the case in which the circuit consists of two very long parallel conductors with two pieces joining their ends, and we shall confine our attention to a part of the circuit which is far from the ends of the conductors that the fact that their not being infinitely long does not introduce any sensible change in the distribution of force.’

Hence, it is clear that Maxwell considered a complete circuit with forward and return currents and not a solitary conductor as assumed by Coufal [10].

For quasi-static fields and a good conductor with negligible displacement current, the fundamental equations relating the vector potential $A$, current density $J$ and electric potential $V$ are [11]

$$\nabla^2 A = -\mu_0 \mu_r J$$  \hspace{1cm} (1)

$$-\mathbf{E} = \nabla V + \frac{\partial \mathbf{A}}{\partial t}$$  \hspace{1cm} (2)

where the total electric field $\mathbf{E}$ includes the gradient of the electric potential $V$. Maxwell considered current flowing in the $z$-direction only in a solid cylindrical conductor and identified the emf due to other causes than the current induction on itself as

$$V_{\text{emf}} = -\frac{\partial V}{\partial z}$$  \hspace{1cm} (3)

where $l$ is the conductor length. Hence,

$$V_{\text{emf}} = -\frac{\rho l}{\mu_0 \mu_r} \nabla^2 A_z + \frac{1}{r} \frac{\partial A_z}{\partial r} \frac{\partial r}{\partial t}$$  \hspace{1cm} (4)

Consequently

$$V_{\text{emf}} = -\frac{\rho l}{\mu_0 \mu_r} \left( \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r} \frac{\partial A_z}{\partial r} \right) + \frac{1}{r} \frac{\partial A_z}{\partial t}$$  \hspace{1cm} (5)

Maxwell [7] solved this equation by assuming $A_z$ to be represented by the series

$$A_z = S + \frac{T_1}{r^2} + \frac{T_2}{r^4} + \frac{T_3}{r^6} + \ldots + \frac{T_m}{r^{2m}} + \ldots$$  \hspace{1cm} (6)

where $S$, $T_1$, $T_2$ ... etc are functions of time. After differentiating equation (6) and after some considerable algebra [9] which involves inversion of the power series for the current (brute force approach) [12] or using matrix inversion algebra [1, 10] the emf becomes

$$V_{\text{emf}} = R_o I + L_o I^{(1)} - \frac{\mu_0 H^2}{12 R_o} I^{(2)} + \frac{\mu_0 H^4}{48 R_o^2} I^{(3)} - \frac{\mu_0 H^6}{180 R_o^3} I^{(4)} + \ldots$$  \hspace{1cm} (7)

where $\mu = \mu_0 \mu_r / (4\pi)$ and $I^{(m)}$ is the $m$th derivative of the current. $I$ is the current, $R_o$ and $L_o$ are the low frequency resistance and inductance respectively. This inductance is the sum of the internal inductance of the conductor plus its external inductance i.e. $L_o = L_{\text{int}} + L_{\text{ext}}$ where in the vector potential approach it is derived rather than assumed, [9]. Equation (7) may be re-written as

$$V_{\text{emf}} = R_o I + L_o \frac{dI}{dt} - R_o \sum_{m=2}^{\infty} (-1)^m f_m \left( \frac{a}{d} \right)^{2m} I^{(m)}$$  \hspace{1cm} (8)
where

\[ d = \sqrt{\frac{4\rho}{\mu_0 \mu_r}} \text{ ms}^{-1/2}, \quad I^{(m)} = \frac{d^{m \ell}}{dt^{m \ell}}, \quad f_2 = \frac{1}{12}, \quad f_3 = \frac{1}{48}, \quad f_4 = \frac{1}{180} \ldots \text{ etc} \]  

(9)

### 2.2. Bessel function approach

In the Bessel function form of solution the voltage developed along the wire due to an alternating current \( I = I_0 e^{j\omega t} \) with amplitude \( I_0 \) and angular frequency \( \omega \) is given by

\[ V_{\text{emf}} = -R_o I \frac{x L_0(x)}{2 J_0(x)} + j\omega L_{\text{ext}} I, \quad J'_0(x) = \frac{dJ_0(x)}{dx} = -J_1(x) \]  

(10)

where \( R_o \) is the direct current resistance in Maxwell’s original analysis equation (7). \( J_0(x) \) and \( J_1(x) \) are zero order and first order Bessel functions of the first kind respectively. The argument of the Bessel functions is

\[ x = \beta^{1/2} \sqrt{2} (\alpha/\delta) \]  

(11)

where \( \beta = \sqrt{2\rho/(\mu_0 \mu_r \omega)} \) is the skin depth. The function \( J_0(x) \) is a solution of equation

\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0 \]  

(12)

that is

\[ \frac{d^2J_0(x)}{dx^2} + \frac{1}{x} \frac{dJ_0(x)}{dx} + J_0(x) = 0 \]  

(13)

This can be written as

\[ J''_0(x) + \frac{1}{x} J'_0(x) + J_0(x) = 0 \]  

(14)

or

\[ \frac{x J_0(x)}{J'_0(x)} = -1 - \frac{x^2}{J''_0(x)} = -1 - \frac{x}{J'_0(x)} \]  

(15)

Following Thomson [14] we can then show that (see Appendix)

\[ \frac{x L_0(x)}{2 J_0(x)} = 1 - \left( \frac{x^2}{8} + \frac{x^4}{192} + \frac{x^6}{3072} + \ldots \right) \]  

(16)

Equation (10) then becomes

\[ V_{\text{emf}} = R_o I \left[ 1 - \sum_{n=1}^{\infty} S_{2n} x^{2n} \right] + j\omega L_{\text{ext}} I \]  

(17)

where \( S_2 = 1/8, S_4 = 1/192, S_6 = 1/3072 \ldots \). The Bessel function argument \( x \) given in equation (11) can be written as

\[ x^{2n} = (-j\omega \mu_0 \mu_r a^2 / \rho)^n = (-j\omega \mu_0 \mu_r a / (R_o \pi))^n \]  

(18)

where for a cylindrical conductor \( R_o = \rho l/(\pi a^2) \). Equation (17) then becomes

\[ V_{\text{emf}} = R_o I \left[ 1 - \sum_{n=1}^{\infty} S_{2n} (-j\omega \mu_0 \mu_r a / (R_o \pi))^n \right] + j\omega L_{\text{ext}} I \]  

(19)

### 2.3. Comparison with Maxwell’s solution

For comparison with Maxwell’s time dependent solution, put \( j\omega I = d^n I/dt^n = I^{(n)} \). Also substitute \( \mu_0 \mu_r = 4\pi \mu_{\text{emu}} \). Hence,

\[ V_{\text{emf}} = R_o I - R_o \sum_{n=1}^{\infty} [S_{2n} (-1)^n (4\mu_{\text{emu}} / R_o)^n I^{(n)}] + L_{\text{ext}} I^{(1)} \]  

(20)

Substituting a few values of \( S_{2n}, S_2 = 1/8, S_4 = 1/192, S_6 = 1/3072 \), yields

\[ V_{\text{emf}} = R_o I + L_{\text{ext}} I^{(1)} + \frac{1}{2} I^{(2)} + \frac{1}{12} \left( \begin{array}{c} \mu_{\text{emu}} \mu_r \end{array} \right)^2 I^{(3)} + \frac{1}{48} \left( \begin{array}{c} \mu_{\text{emu}} \mu_r \end{array} \right)^3 I^{(4)} + \ldots \]  

(21)

This agrees with Maxwell’s equation (18) in reference [7] except we have included the magnetic permeability \( \mu_{\text{emu}} = \mu_{\text{SI}} / 4\pi \). This is an equation for the emf as a function of the arbitrary time dependent current function, \( I(t) \). For low frequencies this equation gives the well known result.
where $L_{int}$ is the internal inductance of the conductor. Steady state d.c. conditions lead to Ohm’s law $V_{emf} = R_oI$ as expected.

2.4. Current impulse
Maxwell completed Article [690] in reference [7, 9] by considering a current impulse starting at a constant value $I_0$ and rising to a steady value $I_1$. The result applied to equation (21) gives: At $t = 0$ let $I(0) = I_0 = \text{constant}$ and at $t = t_1$ let $I(t_1) = I_1 = \text{constant}$. Then

$$
\int_0^{t_1} V_{emf} dt = R_o \int_0^{t_1} I(t) dt + L_o[I_1 - I_0] - \frac{\varepsilon^2_{emu}}{12R_o} \left[ \frac{dI_1}{dt} - \frac{dI_0}{dt} \right] + \frac{\varepsilon^3_{emu}}{48R_o^2} \left[ \frac{d^2I_1}{dt^2} - \frac{d^2I_0}{dt^2} \right] \ldots \text{etc}
$$

But since $I_0$ and $I_1$ are constants, the differentials are zero so we obtain

$$
\int_0^{t_1} V_{emf} dt = R_o \int_0^{t_1} I(t) dt + L_o[I_1 - I_0]
$$

Hence both the integrated emf and the average emf (integrated emf/T) are independent of the skin effect. This is discussed in more detail in reference [9].

3. Alternative solution
The previous derivation of equation (16) followed Thomson’s approach. The following provides an alternative solution. The Bessel functions are

$$
J_n(x) = 1 - a_2x^2 + a_4x^4 - a_6x^6 + a_8x^8 - \ldots = \sum_{n=0}^{\infty} (-1)^n a_{2n}x^{2n}
$$

where

$$
a_0 = 1, \quad a_2 = \frac{1}{2!}, \quad a_4 = \frac{1}{(2!)^22^3}, \quad \ldots, \quad a_{2n} = \frac{1}{(n!)^22^{2n}}
$$

Also

$$
\frac{J_n(x)}{x} = \frac{1}{2} - b_2x^2 + b_4x^4 - b_6x^6 + b_8x^8 - \ldots = \sum_{n=0}^{\infty} (-1)^n b_{2n}x^{2n}
$$

where

$$
b_0 = \frac{1}{2}, \quad b_2 = \frac{1}{2!2^3}, \quad b_4 = \frac{1}{2!3!2^5}, \quad b_6 = \frac{1}{3!4!2^7}, \quad \ldots, \quad b_{2n} = \frac{1}{n!(n+1)!2^{2n+1}}
$$

Now

$$
J_n(x) = \frac{J_n(x)}{x} \frac{J_0(x)}{J_0(x)}
$$

Substitute for equations (27) and (58) then

$$
J_n(x) = \left( \frac{1}{2} - b_2x^2 + b_4x^4 - b_6x^6 + b_8x^8 - b_{10}x^{10} + b_{12}x^{12} - b_{14}x^{14} + \ldots \right) \times 2(1 - S_2x^2 - S_4x^4 - S_6x^6 - S_8x^8 - S_{10}x^{10} - S_{12}x^{12} - S_{14}x^{14} - \ldots)
$$

Multiplying out gives

$$
J_n(x) = 1 - S_2x^2 - S_4x^4 - S_6x^6 - S_8x^8 - \ldots
\begin{align*}
&- 2[(b_2x^2 - b_2S_2x^4 - b_2S_4x^6 - b_2S_6x^8 - \ldots) + (b_4x^4 - b_4S_4x^4 - b_4S_6x^6 - b_4S_8x^8 - \ldots) - (b_6x^6 - b_6S_6x^6 - b_6S_8x^8 - b_6S_{10}x^{10} - \ldots) + (b_8x^8 - b_8S_8x^8 - b_8S_{10}x^{10} - b_8S_{12}x^{12} - \ldots)].
\end{align*}
$$

In equations (25) and (31), equating coefficients of equal powers of $x$ yields

$$
x^2: \quad -S_2 = -2b_2 = -a_2
$$

$$
S_2 = a_2 - 2b_2 = 1/2^2 - 2/(2!2^3) = 1/8
$$

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$$
These agree with Thomson, equation (16), [16]. Extracting further values of $S_n$ exceeding $S_6$ is rather difficult due to the possibility of making mistakes in the cross multiplication of equation (30). For this case programmes were written using Matlab and values of $S_n$ up to $S_{24}$ were obtained. The results are listed in table 1 and plotted in figure 1.

A least squares fit to this data is given by

$$S_{2n} = q \times 10^{-rn}, \quad q = 1.27766, \quad r = 0.58945$$  \hspace{1cm} (35)

3.1 Series convergence or divergence
We can investigate equation (17) for convergence or divergence by using the ratio test on the summation term. This gives
Using \( r = 0.589 \) from the least squares fit, equation (35), then

\[
k = |0.06624x^2|
\]

(37)

The series converges if \( k < 1 \) and diverges if \( k > 1 \) or \( |x| > 3.885 \).

This can also be put in terms of the skin depth since from equation (11), \( x^2 = -2j(a/\delta)^2 \). Hence,

\[
k = 0.1325(a/\delta)^2
\]

(38)

The series therefore diverges if \( k > 1 \) or \( a/\delta > 2.747 \).

3.2. Impedance

From the power series equation (17) the impedance is

\[
Z = \frac{V_{emf}}{I} = R_o [1 - \sum_{n=1}^{\infty} S_{2n}x^{2n}] + j\omega L_{ext} = Z_i + Z_{ext}
\]

(39)

\( Z_i \) and \( Z_{ext} \) are the internal and external impedances respectively

\[
Z_i = R_o [1 - \sum_{n=1}^{\infty} S_{2n}x^{2n}], \quad Z_{ext} = X_{ext} = j\omega L_{ext}
\]

(40)

where in this case the external impedance is the same as the external reactance \( X_{ext} \).

The normalized internal impedance \( Z_{ni} = Z_i/(R_o) \) is

\[
Z_{ni} = 1 - \sum_{n=1}^{\infty} S_{2n}x^{2n}
\]

(41)

Substituting for \( x \) from equation (11)

\[
Z_{ni} = 1 - \sum_{n=1}^{\infty} (-1)^n (2j)^n S_{2n}x_1^{2n}, \quad x_1 = a/\delta
\]

(42)

Substituting for \( S_n \) from Table 1 gives

\[
Z_{ni} = 1 + \frac{j}{4} (a/\delta)^2 + \frac{1}{48} (a/\delta)^4 + \ldots
\]

(43)

\[
Re(Z_{ni}) = 1 + \frac{1}{48} (a/\delta)^4 - \frac{1}{2880} (a/\delta)^8 + \ldots
\]

(44)

\[
Im(Z_{ni}) = \frac{1}{4} (a/\delta)^2 - \frac{1}{384} (a/\delta)^6 + \ldots
\]

(45)

At zero frequency, equation (44) gives the dc resistance \( (R_o) \). The first expansion is a well known approximation for the low frequency ac resistance \( (R_{ac}) \) of the conductor [17, 18]. Using the first term only of equation (45) the internal inductance is

\[
L_i = X_i/\omega = \frac{1}{4\omega} (a/\delta)^2 R_o = \frac{\mu_0 H_i l}{8\pi}
\]

(46)

which corresponds to the low frequency internal inductance of the wire length \( l \).

Figure 2 shows experimental measurements of resistance \( R \), reactance \( X \) and inductance \( L \) for a 1.3 mm diameter wire meander [19]. The slope of the ac resistance \( R \propto f^{1.2} \) tends to agree with theory for \( f < 400 \) kHz. Above this value the slope exceeds the theoretical calculations. This is considered to be due to resonance at the higher frequencies arising from the wire inductance and parasitic capacitance. The slope of \( X \propto f^{1.0} \) agrees with theory mainly because the internal inductance is insignificant in this wire

Figure 3 shows normalized \( (Z/R_o) \) real and imaginary values of the conductor internal impedance plotted against \( x_1 = a/\delta \).

The normalized internal impedance using Bessel functions was obtained from equation (10) and is given by

\[
Z_{ni} = \frac{V}{I R_o} = \frac{x L(x)}{2 H(x)} = Re(Z_{ni}) + jIm(Z_{ni})
\]

where \( x \) is given by equation (11). The Bessel function results are seen to be continuous whereas the power series results show a sharp increase at \( x_1 \geq 2.7 \). However the Bessel imaginary impedance does show a change in \( x_1 \) dependence from \( x_1^2 \) for \( x_1 \leq 2.55, Z_{ni} = 1.2 \) and \( x_1^4 \) for \( x_1 \geq 2.15, Z_{ni} = 1.2 \).
In the low frequency approximation, equations (44) and (45), \( Re(Z_{ni}) = Im(Z_{ni}) \) when

\[
Re(Z_{ni}) = Im(Z_{ni}) = 1 + \frac{1}{48}(a/\delta)^4 = \frac{1}{4}(a/\delta)^2
\]  

(48)

Solving gives for \( x_1 = a/\delta = 2.632 \). This is close to the above value of \( x_1 = 2.55 \) obtained from figure 3. However, the graphs are more accurate since they are based on many more terms in the series compared with the approximate equations (44) and (45). These impedance values are listed in table 2 and compared with low frequency calculations (LF) from equation (48).
From Poynting’s theorem, the power dissipation is given by

\[ P_{ac} = I_{rms}^2 \text{Re}(Z) = I_{rms}^2 R_\delta \text{Re} \left[ \frac{\lambda J_\delta(x)}{2 J_1(x)} \right] = I_{rms}^2 R_\delta \text{Re}(Z_{in}) \]  

(49)

The ratio of this to the low frequency or dc power dissipation \( I_{rms}^2 R_\delta \) is

\[ \frac{P_{ac}}{P_{dc}} = \text{Re}(Z_{in}) \]  

(50)

That is the normalized power dissipation is equal to the normalized internal impedance of the conductor.

Figure 3 also shows experimental measurements of the normalized ac resistance (\( R_N = \text{Re}(Z)/R_\delta \)) for the 1.3 mm diameter wire in figure 2. The slope of \( R_N \) tends to agree with theory for \( x_1 < 6 \). Above this value, corresponding to frequencies near 1 MHz, the slope of \( R_N \) exceeds the theoretical calculations. This is considered to be due to resonance at the higher frequencies arising from the wire inductance and parasitic capacitance as discussed for figure 2.

4. Summary and conclusions

We have seen from equations (17) and (32)–(34) that the ratio of the Bessel functions in equation (10) can be represented by summations of the Bessel function argument \( x^2n \) times numerical coefficients \( S_{2n} \). The first few of these numerical coefficients were determined individually. Values above \( n = 6 \) were determined using a Matlab programme and a general equation for \( S_{2n} \) obtained by curve fitting, equation (35). Substituting these values into the Bessel function equation (10) gives frequency and time dependent equations for the emf equations (19) finally leading to equation (21) which agrees with Maxwell’s equation (18) [7].

Although in this paper we do not give the reason for the divergence of Maxwell’s solution at \( a/\delta > 2.7 \), we do show that Thomson’s method and the alternative method used here both lead to divergence of the power series at the same value of \( a/\delta \), agreeing independently with Maxwell.

It is clear in figure 3 that the real part of the impedance i.e. the ac resistance for both Maxwell and Bessel only begins to increase as \( a/\delta \) approaches 2.7. Also the imaginary part of the Bessel impedance changes from \( \text{Im}(Z) \propto x_2^2 \) to \( \text{Im}(Z) \propto x_1^2 \) for \( a/\delta > 2.7 \). It is also interesting to note that the Bessel functions \( I_1 \) and \( J_1 \) when plotted against \( a/\delta \) show peaks and troughs at \( a/\delta = e \) as shown in figure 4. The significance here is that both Maxwell and Bessel mathematical models for electromagnetic diffusion in a conductor show discontinuities in the impedance at \( a/\delta \approx 2.7 \). The fact that this occurs close to the mathematical constant \( e \) may be a coincidence and the proof that the impedance change is due to \( a/\delta \) becoming equal to \( e \) has yet to be determined.

Acknowledgments

The author would like to thank the following for discussions about this research: Professor J A Brandao Faria, Technical University of Lisbon, Dr David W Knight Technical Director, Cameras Underwater Ltd., Devon, Mrs M A Raven and ex-colleagues at the University of Nottingham, U K.

Appendix

Following from equation (15) use the identity

\[ \frac{d \log y}{dx} = \frac{d \log y}{dy} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} \]  

(51)

Thus

\[ \frac{d J_1(x)}{dx} = \frac{1}{J_1(x)} \frac{d J_1(x)}{dx} = \frac{J_1'(x)}{J_1(x)} \]  

(52)
so that
\[
x J_n(x) = -1 - x \frac{d}{dx} \log J_n(x)
\]
where we have used \( J'_n(x) = - J_n(x) \). Now let \( x_0, x_1, x_2, x_3 \ldots \) be the positive roots of \( J_1(x) = 0 \). Then
\[
d \frac{d}{dx} \log J_1(x) = \frac{1}{x} - \frac{2x^2}{x^2(1-x^2/x_1^2)} - \frac{2x^2}{x^2(1-x^2/x_2^2)} - \ldots
\]
\[
x \frac{d}{dx} \log J_n(x) = 1 - 2x^2 \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots \right) - \ldots
\]
Hence, we can put using equation (15)
\[
x J_n(x) = -1 - x \frac{d}{dx} \log J_n(x) = -2 + 2x^2 S_2 + 2x^4 S_4 + 2x^6 S_6 + \ldots
\]
where
\[
S_2 = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots, \quad S_4 = \frac{1}{x_1^4} + \frac{1}{x_2^4} + \ldots, \quad S_6 = \frac{1}{x_1^6} + \frac{1}{x_2^6} + \ldots
\]
The Bessel series for \( J_1(x)/x = 0 \) is
\[
J_1(x) = \frac{1}{x} \left[ 1 - \frac{(x/2)^2}{1!2!} + \frac{(x/2)^4}{2!3!} - \frac{(x/2)^6}{3!4!} + \ldots \right] = \sum_{n=0}^\infty (-1)^n \frac{(x/2)^{2n}}{n!(n+1)!} = 0
\]
\[
J_1(x) = \frac{1}{x} \left[ 1 - \frac{x^2}{2.4} + \frac{x^4}{2.4.6.6.8} - \frac{x^6}{2.4.6.6.8.10} + \ldots \right] = 0
\]
Using Newton’s method [15]
\[
S_2 = \frac{1}{4} \times \frac{1}{2}, \quad S_4 = \frac{1}{4} \times \frac{1}{12}, \quad S_6 = \frac{1}{4} \times \frac{1}{48}, \quad S_8 = \frac{1}{4} \times \frac{1}{180}, \quad S_{10} = \frac{1}{4} \times \frac{13}{8640} \ldots
\]
This leads to equation (16) as required

\[
\frac{x J_0(x)}{2 J_1(x)} = 1 - \left(\frac{x^2}{8} + \frac{x^4}{192} + \frac{x^6}{3072} + \ldots\right)
\]  

(63)

ORCID iDs

Malcolm Stuart Raven @ https://orcid.org/0000-0002-8294-3760

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