The Elliptic Solutions to the Friedmann equation and the Verlinde’s Maps.

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Abstract

We considered the solutions of the Friedmann equation in several setups, arguing that the Weierstraß form of the solutions leads to connections with some Conformal Field Theory on a torus. Thus a link with the Cardy entropy formula is obtained in a quite natural way. The argument is shown to be valid in a four dimensional radiation dominated universe with a cosmological constant as well as in four further different Universes.

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1 Introduction.

Some time ago Verlinde\cite{1} considered the holographic principle \cite{2} in a Friedmann-Robertson-Walker (FRW) universe in radiation dominated phase. Such a radiation is considered to be described by a Conformal Field Theory with a large central charge. In that work a very interesting relationship between the FRW equations controlling the cosmological expansion on the one hand and the formulas that relate the energy and the entropy of the CFT on the other hand was found. In particular three amazing relations mapping the $D$-dimensional Friedmann equation

$$H^2 = \frac{16\pi G}{(D-1)(D-2)} \frac{E}{V} - \frac{1}{R^2}$$ \hspace{1cm} (1)

into the Cardy formula\cite{3}

$$S = 2\pi \sqrt{\frac{c}{6}(L_0 - \frac{c}{24})},$$ \hspace{1cm} (2)

to the Cardy formula\cite{3} has been set up, that is

\begin{align*}
2\pi L_0 & \Rightarrow \frac{2\pi}{D-1} ER \\
2\pi \frac{c}{12} & \Rightarrow (D-2) \frac{V}{4GR} \\
S & \Rightarrow (D-2) \frac{HV}{4G} .
\end{align*} \hspace{1cm} (3)

The scenario considered in \cite{1} was that of a closed radiation dominated FRW universe with a vanishing cosmological constant. Such relations hold precisely when the holographic entropy bound is saturated. Within this context Verlinde proposed that the Cardy formula for 2D CFT can be generalized to arbitrary spacetime dimensions. Such a generalized entropy formula is known as the Cardy-Verlinde formula.

Later soon, this result was generalized and understood in several set ups \cite{4,5,6,7}. The CFT dominated universe has been described as a co-dimension one brane, in the background of various kinds of (A)dS black
holes. In such cases, when the brane crosses the black hole horizon, the entropy and temperature are expressed in terms of the Hubble constant and its time derivative. Verlinde’s proposal has inspired a considerable activity shedding further light on the various aspects of the Cardy-Verlinde formula. Nevertheless there is still no answer to the question about whether the merging of the CFT and FRW equations is a mere formal coincidence or, quoting Verlinde, whether this fact “strongly indicated that both sets of these equations arise from a single underlying fundamental theory”. Moreover it is very striking that the Cardy formula obtained in the very particular framework of two dimensional Conformal Field Theory gets generalized as directly as proposed. One thus wonders whether there is no essentially two-dimensional aspect overlooked in the whole calculation.

Aiming at a more complete picture of such a difficult task we seek, in this paper, new connections that can be established between the Friedmann equation and the Cardy formula. Our point of departure is the work of Kraniotis and Whitehouse [8], where those authors found solutions for the Friedmann equation that directly and naturally connects the Friedmann-Robertson-Walker cosmology with the theory of modular forms and elliptic curves [9].

2 Friedmann equation and Weierstraß form.

In this section we apply the procedure shown in [8] to the radiation-dominated four-dimensional universe with $k = 1$ and non vanishing cosmological constant. The Friedmann equation takes the form

$$H^2 = \frac{8\pi G}{3} \frac{C_1}{R^4} + \frac{\Lambda}{3} - \frac{1}{R^2}$$

(4)

where the density is $\rho = E/V = C_1 R^{-4}$.

\footnote{In the reference [8] the matter dominated universe is studied.}
Equation (4) can be recast into the form
\[(\dot{R})^2 = -1 + \frac{\Lambda}{3} R^2 + \frac{8\pi G C_1}{3} R^{-2}.\] (5)

For the time variable we define the elliptic integral
\[t = \int \left[ -1 + \frac{\Lambda}{3} R^2 + \frac{8\pi G C_1}{3} R^{-2} \right]^{-1/2} dR.\] (6)
\[= \int R \left[ -R^2 + \frac{8\pi G C_1}{3} R^{-2} + \frac{\Lambda}{3} R^4 \right]^{-1/2} dR.\] (7)

Now let us consider the auxiliary integral
\[u = \int \left[ -R^2 + C_2 + \frac{\Lambda}{3} R^4 \right]^{-1/2} dR,\] (8)
with \(C_2 = \frac{8\pi G C_1}{3}\). We have
\[t = \int R \, du.\] (9)

The integral (8) can be reduced to a cubic form upon introducing the variable \(X = \frac{1}{R^2}\), that is
\[u = \int \left[ -X^2 + C_2 X^3 + \frac{\Lambda}{3} X \right]^{-1/2} \left( \frac{1}{2} \right) dX.\] (10)

We introduce the variable \(X = \frac{\xi + \frac{1}{12}}{C_2^{1/4}}\), leading us to the relation
\[v = 2u = \int \left[ 4\xi^3 - \left( \frac{1}{12} - \frac{\Lambda C_2}{12} \right) \xi - \frac{1}{216} + \frac{\Lambda C_2}{12^2} \right]^{-1/2} d\xi.\] (11)

This integral can be identified with the Weierstraß normal form, arising from the well-known differential equation of the Weierstraß function \(\wp\) \[10, 11\], that is
\[(\wp')^2 = 4\wp^3 - g_2 \wp - g_3.\] (12)
The Weierstraß function \(\wp(u) = \wp(u|\omega_1, \omega_2) = \wp(u, \tau = \omega_1/\omega_2)\) is an even meromorphic elliptic function of semiperiods \(\omega_1, \omega_2\). Such semiperiods are given by the expressions
\[\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}, \quad \omega_2 = \int_{e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}.\] (13)
where \( e_1, e_2, e_3 \) are the roots of the cubic polynomial \( 4z^3 - g_2z - g_3 \) with discriminant
\[
\Delta = g_2^3 - 27g_3^2. \tag{14}
\]

Thus \( \xi = \wp(v + \epsilon) \), where \( \epsilon \) is a constant of integration. Moreover, the cubic invariants are
\[
g_2 = \frac{1}{12} - \frac{\Lambda C_2}{12} \quad \text{and} \quad g_3 = \frac{1}{216} - \frac{\Lambda C_2}{12^2}. \tag{15}
\]

Substituting back into \( R \), we find
\[
R = \sqrt{\frac{8\pi C_1 G}{12\wp(v + \epsilon) + 1}} \tag{16}
\]
and
\[
t = \frac{1}{2} \int R \, dv = \frac{1}{2} \int \sqrt{\frac{8\pi C_1 G}{12\wp(v + \epsilon) + 1}} \, dv. \tag{17}
\]

These interesting results open us another door to the understanding of the Verlinde’s maps. The fact that we can obtain the Weierstraß \( \wp \)-function from the Friedmann equation shows a new way of interpreting the relation between the Cardy formula and the Friedmann equation. This is due to the particular properties of this function. First, it is known from the theory of the elliptic curves [12] that the topological equivalence of an elliptic curve with a torus is given by an explicit mapping involving the Weierstraß \( \wp \)-function and its first derivative. This map is, in effect, a parametrization of the elliptic curve by points in a “fundamental parallelogram” in the complex plane. Another important property of this function is its non trivial modular transformation properties [12]. It is a meromorphic modular form of weight 2, that is,
\[
\wp \left( \frac{u}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \wp(u, \tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \tag{18}
\]

From (15) it is not hard to see that for \( \Lambda = 0 \) we get \( \Delta = 0 \). This case is not studied because the solutions are not given by elliptic functions and do not have modular properties.
We thus stress that the most important result arising out of this approach is the correspondence established between the given Friedmann equation and a particular torus through the parameter $\tau$. It is clear that $\tau$ is a function of the cosmological constant $\Lambda$ depending on the dimension as well (this will become evident in the next section).

On the other hand, we know that when we have a CFT theory, a partition function on a torus with modular parameter $\tau$ can be introduced, being given by the expression

$$Z(\tau) = \text{Tr} \, q^{L_0 - c/24},$$

where $q \equiv e^{2\pi i \tau}$ (for simplicity we have suppressed the $\bar{\tau}$ dependence).

Using the modular properties of this partition function it is possible to find the density of states and consequently the Cardy entropy formula

$$S = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)}.$$  \hspace{1cm} (20)

These facts lead to the chain of connections: Friedmann equation $\rightarrow$ Weierstraß equation $\rightarrow$ Torus $\rightarrow$ CFT Partition Function $Z(\tau(\Lambda, D))$ $\rightarrow$ Cardy Formula. Therefore, in this context, Verlinde’s maps are less intriguing. We actually do not obtain a direct definition of the quantities appearing in Cardy-formula as a function of the quantities appearing in the Friedmann equation, but the very existence of a Cardy formula is natural, given the partition function on a torus.

3 Further examples.

In this section we extend the approach of the previous section to two further cases. First we study the case $D = 6, k = 1, \Lambda \neq 0$. Here the Friedmann equation takes the form

$$H^2 = \frac{16\pi G}{(6-1)(6-2)} \frac{C_1}{R^6} + \frac{2\Lambda}{(6-1)(6-2)} - \frac{1}{R^2}.$$  \hspace{1cm} (21)
The time integral and the auxiliary integral are now defined, respectively, as:

\[
t = \int \left[ -1 + \frac{\Lambda}{10} R^2 + \frac{4\pi G C_1}{5} R^{-4} \right]^{-1/2} dR \tag{22}
\]

\[
= \int R^2 \left[ -R^4 + \frac{4\pi G C_1}{5} + \frac{\Lambda}{10} R^6 \right]^{-1/2} dR \tag{23}
\]

and

\[
u = \int \left[ -R^4 + C_4 + \frac{\Lambda}{10} R^6 \right]^{-1/2} dR \tag{24}
\]

where \( C_4 = \frac{4\pi G C_1}{5} \).

Introducing \( X = \frac{1}{R^2} \) we get

\[
u = \int \left[ -X + C_4 X^3 + \frac{\Lambda}{10} \right]^{-1/2} \frac{1}{2} dX \tag{25}
\]

After some manipulations we arrive at the Weierstraß normal form

\[
v = C_4^{1/2} u = \int \left[ 4X^3 - \frac{4X}{C_4} + \frac{2\Lambda C_4}{5} \right]^{-1/2} dX \tag{26}
\]

with cubic invariants

\[
g_2 = \frac{4}{C_4} \quad \text{and} \quad g_3 = \frac{-2\Lambda}{5C_4} \tag{27}
\]

Substituting back into \( R \), we find

\[
R = \sqrt{\frac{1}{\wp(v + \epsilon)}} \tag{28}
\]

and

\[
t = \frac{1}{C_4^{1/2}} \int R^2 \, dv = \frac{1}{C_4^{1/2}} \int \frac{dv}{\wp(v + \epsilon)} \tag{29}
\]

In this case the solution keeps the modularity when \( \Lambda = 0 \) because \( \Delta \neq 0 \). From (21) it is clear that the parameters \( g_2, g_3 \) and \( \tau \) depends on the dimension. But here as in the four dimensional case we obtain a map from the six dimensional Friedmann equation to a particular torus with modular
parameter $\tau = \tau(\Lambda, D)$. This forces us to reexamine the argument appearing in the literature that the Cardy-Verlinde formula is actually D-dimensional. In fact we get a torus with a different $\tau$.

Similar results apply also to the eight-dimensional universe with radiation dominated and $\Lambda = 0$, where we find a solution in terms of the Weierstraß function, with

$$g_2^{(D=8)} = 0 \quad \text{and} \quad g_3^{(D=8)} = \frac{4}{C_6},$$

where $C_6 = \frac{8\pi G C_1}{21}$.

4 Conclusion.

The solution of the Friedmann equation in terms of the Weierstrass function $\wp(u)$ leads to a modular invariant partition function defined on a torus obtained from that cosmological equation. Thus the two dimensional character of the Cardy-Verlinde formula becomes evident, since there we can define a CFT partition function, computing the entropy following Cardy’s prescription. This work thus proposes a natural two dimensional space for Cardy-Verlinde formula, establishing a framework for Verlinde’s maps. Although the form of the map has different disguises in different dimensions and frameworks, we have obtained the results in at least four different and inequivalent situations. We think that this results lead us further in the comprehension of the physical implications of Verlinde’s proposal, implying new features about the mathematical structure of the cosmological equations.

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