Existence and multiplicity of solutions for a class of Kirchhoff type 
\((\Phi_1, \Phi_2)\)-Laplacian system with locally super-linear condition in \(\mathbb{R}^N\)

Culing Liu, Xingyong Zhang

Faculty of Science, Kunming University of Science and Technology, Kunming, Yunnan, 650500, P.R. China.

Abstract: We investigate the existence and multiplicity of weak solutions for a nonlinear Kirchhoff type 
quasilinear elliptic system on the whole space \(\mathbb{R}^N\). We assume that the nonlinear term satisfies the locally super-\((m_1, m_2)\) condition, that is, \(\lim_{|u,v| \to +\infty} \frac{F(x,u,v)}{|u|^{m_1} |v|^{m_2}} = +\infty\) for a.e. \(x \in G\) where \(G\) is a domain in \(\mathbb{R}^N\), which is weaker than the well-known Ambrosseti-Rabinowitz condition and the naturally global restriction, \(\lim_{|u,v| \to +\infty} \frac{F(x,u,v)}{|u|^{m_1} |v|^{m_2}} = +\infty\) for a.e. \(x \in \mathbb{R}^N\). We obtain that system has at least one weak solution by using the classical Mountain Pass Theorem. To a certain extent, our theorems extend the results of Tang-Lin-Yu [Journal of Dynamics and Differential Equations, 2019, 31(1): 369-383]. Moreover, under the above naturally global restriction, we obtain that system has infinitely many weak solutions of high energy by using the Symmetric Mountain Pass Theorem, which is different from those results of Wang-Zhang-Fang [Journal of Nonlinear Sciences and Applications, 2017, 10(7): 3792-3814] even if we consider the system on the bounded domain with Dirichlet boundary condition.

Keywords: Orlicz-Sobolev spaces; Mountain Pass Theorem; Symmetric Mountain Pass Theorem; locally super-\((m_1, m_2)\) condition.

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1 Introduction

In this paper, we are dedicated to studying the existence and multiplicity of weak solutions for the following generalized nonlinear and non-homogeneous Kirchhoff type elliptic system in Orlicz-Sobolev spaces

\[
\begin{align*}
-M_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u|) \, dx \right) \Delta_{\Phi_1} u + V_1(x)\phi_1(|u|)u &= F_u(x,u,v), \quad x \in \mathbb{R}^N, \\
-M_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v|) \, dx \right) \Delta_{\Phi_2} v + V_2(x)\phi_2(|v|)v &= F_v(x,u,v), \quad x \in \mathbb{R}^N, \\
\end{align*}
\]

where \(\Delta_{\Phi_i}(u) = \text{div}(\phi_i(|\nabla u|)\nabla u), (i = 1, 2), \phi_i : (0, +\infty) \to (0, +\infty)\) are two functions which satisfy the following conditions:

*Corresponding author, E-mail address: zhangxingyong1@163.com
Yu investigated the existence of nontrivial solutions for the following semilinear Schr"{o}dinger equation:

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N,$$

where $\Omega$ is a domain in $\mathbb{R}^N$. Yu reduces to the following non-homogeneous and nonlocal quasilinear elliptic equation

$$\begin{aligned}
M_i &\leq \Phi_i(t) \leq C_i 2\lvert t \rvert^l_i, \\
\lim_{|z| \to +\infty} \text{meas}\{x \in \mathbb{R}^N : |x - z| \leq C_{i, 1}, V_i(x) \leq C_{i, 2}\} = 0 \text{ for every } C_{i, 2} > 0, i = 1, 2,
\end{aligned}$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Moreover, we introduce the following conditions on $F$, $V_i$ and $M_i$:

$(F_0)$ $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $C^1$ function such that $F(x, 0, 0) = 0$ for all $x \in \mathbb{R}^N$ and $F(x, u, v) \geq 0$ for all $(x, u, v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$;

$(V_0)$ $V_i \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V_i(x) > 1, \quad i = 1, 2$;

$(V_1)$ there exist constants $C_{i, 1} > 0$ such that

$$\text{meas}\{x \in \mathbb{R}^N : |x - z| \leq C_{i, 1}, V_i(x) \leq C_{i, 2}\} = 0 \text{ for every } C_{i, 2} > 0, i = 1, 2,$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;

$(M_0)$ $M_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $C_{i, 3} \leq M_i(t) \leq C_{i, 4}, \forall t \geq 0$ for some $C_{i, 3}, C_{i, 4} > 0, \quad i = 1, 2$;

$(M_1)$ $\widehat{M_i}(t) := \int_0^t M_i(s)ds \geq M_i(t)t, \quad i = 1, 2$.

Let $\phi_1 = \phi_2 = : \phi, v = u, M_1 = M_2 = : M, V_1 = V_2 = : V$ and $F(x, u, v) = F(x, v, u)$. Then the system (1.3) reduces to the following non-homogeneous and nonlocal quasilinear elliptic equation

$$\begin{aligned}
-\Delta (\int_\Omega \Phi(|\nabla u|)dx) \Delta \phi u + V(x)\phi(|u|)u = f(x, u), \quad x \in \Omega, \\
u \in W^{1,p}(\Omega),
\end{aligned}$$

where $\Omega$ is a domain in $\mathbb{R}^N$ and $f(x, u) = F_u(x, u, u)$.

In recent years, many authors are concerned with nonlocal problems like (1.2) which can been seen as a generalization of the second-order semilinear elliptic equations, $p$-Laplacian equations and $(p, q)$-Laplacian equations (see [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] and references therein).}

Next, we emphasize the results in [20] which mainly inspires our works in this paper. In [20], Tang, Lin and Yu investigated the existence of nontrivial solutions for the following semilinear Schrödinger equation:

$$\begin{aligned}
-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{aligned}$$

(1.3)
where the potential $V \in C(\mathbb{R}^N, \mathbb{R})$ is a sign-changing function which satisfies the periodic or coercive conditions. If $f$ satisfies a subcritical condition and the following locally super-quadratic condition:

\((f_1)\) there exists a domain $G \subset \mathbb{R}^N$ such that

$$
\lim_{t \to \infty} \frac{\int_0^t f(x,s) \, ds}{t^2} = +\infty, \text{ a.e. } x \in G,
$$

which is weaker than the following naturally super-quadratic condition:

\((f_1')\) \(\lim_{t \to \infty} \frac{F(x,t)}{|t|^2} = +\infty\) uniformly for $x \in \mathbb{R}^N$,

by using the linking geometry theorem, they obtained that the problem (1.3) has a nontrivial weak solution.

There have been some contributions devoted to the study of system (1.1) with $M_i = 1$ involving the existence and multiplicity of weak solutions. In [21], Wang, Zhang and Fang considered the following quasilinear elliptic system in Orlicz-Sobolev spaces:

\[
\begin{align*}
-\text{div}(\phi_1(|\nabla u|)\nabla u) &= F_u(x,u,v), \quad \text{in } \Omega, \\
-\text{div}(\phi_2(|\nabla v|)\nabla v) &= F_v(x,u,v), \quad \text{in } \Omega, \\
u = v = 0, \quad &\text{on } \partial \Omega,
\end{align*}
\]

(1.4)

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial \Omega$. When $F$ satisfies some appropriate conditions including $(\phi_1, \phi_2)$-superlinear and subcritical growth conditions at infinity as well as symmetric condition, by using the mountain pass theorem and the symmetric mountain pass theorem, they obtained that system (1.4) has a nontrivial weak solution and infinitely many weak solutions, respectively. Subsequently, more works were obtained for systems like (1.1) (see [22]-[26]). For example, in [24], Wang, Zhang and Fang considered the quasilinear elliptic system like (1.1) with $M_i(t) = 1$ in $\mathbb{R}^N$. When the potential functions are bounded, $F$ satisfies sub-linear growth condition, by using the least action principle, they obtained that system has at least one nontrivial weak solution. If $F$ also satisfies a symmetric condition, by using the genus theory, they obtained that system has infinitely many weak solutions. In [25]-[26], we developed the Moser iteration technique, and then by using the mountain pass theorem and cut-off technique, we obtain that system like (1.1) with $M_i(t) = 1$ and a parameter $\lambda$ has a nontrivial weak solution $(u_\lambda, v_\lambda)$ with $||(u_\lambda, v_\lambda)||_\infty \leq 2$ for every $\lambda$ large enough if the nonlinear term $F$ satisfies some growth conditions only in a circle with center $0$ and radius 4.

Inspired by [14, 19, 20, 21], in this paper, we shall investigate the existence and multiplicity of weak solutions for system (1.1) with locally super-$(m_1, m_2)$ growth in $\mathbb{R}^N$. We assume that $F$ satisfies the following local condition

$$
\lim_{||(u,v)|| \to +\infty} \frac{F(x,u,v)}{|u|^{m_1} + |v|^{m_2}} = +\infty \quad \text{for a.e. } x \in G,
$$

for a.e. $x \in G$. 

3
where $G$ is a domain in $\mathbb{R}^N$, by using the Mountain Pass Theorem, we obtain that system (1.1) has a nontrivial weak solution. Besides, if $F$ also satisfies a symmetric condition and the following naturally global restriction
\[
\lim_{|u|,|v| \to +\infty} \frac{F(x,u,v)}{|u|^{m_1} + |v|^{m_2}} = +\infty \text{ for a.e. } x \in \mathbb{R}^N,
\]
by using the Symmetric Mountain Pass Theorem, we obtain that system (1.1) has infinitely many weak solutions of high energy. We develop some results in some known references in the following sense:

(I) Our local condition extends the locally super-quadratic condition of [20] (see $(f_1)$ mentioned above);

(II) Different from those in [21]-[26], we consider the nonlocal Kirchhoff-type problems;

(III) Our conditions are weaker than the Ambrosetti-Rabinowitz (A-R for short) condition in [14, 23];

(IV) Different from that in [21], we work in the whole-space $\mathbb{R}^N$ rather than a bounded domain $\Omega \subset \mathbb{R}^N$.

Especially, we introduce a new $(\phi_1, \phi_2)$-superlinear condition (see $(F_4)$ below and Remark 3.4 for details), which is different from the following condition in [21] even if we restrict $(F_4)$ to the bounded domain $\Omega$:

\[
\text{there exists a continuous function } \gamma : [0, \infty) \to \mathbb{R} \text{ and it satisfies that } \Gamma(t) := \int_0^t \gamma(s)ds, \ t \in \mathbb{R} \text{ is an } N \text{-function with}
\]

\[
1 < l_\Gamma := \inf_{t > 0} \frac{t \gamma(t)}{\Gamma(t)} \leq \sup_{t > 0} \frac{t \gamma(t)}{\Gamma(t)} =: m_\Gamma < +\infty,
\]
such that

\[
\Gamma \left( \frac{F(x,u,v)}{|u|^{l_1} + |v|^{l_2}} \right) \leq d_1 \Phi_1(x,u,v) + \Phi_2(v), \quad x \in \Omega, \quad |(u,v)| \geq r_1,
\]

where constants $d_1, r_1 > 0$ and

\[
\Phi_i(x,u,v) := \frac{1}{m_i} F_i(x,u,v)u + \frac{1}{m_2} F_i(x,u,v)v - F_i(x,u,v), \quad \forall (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R},
\]

and the following $(\phi_1, \phi_2)$-superlinear growth conditions hold:

\[
\lim_{|(u,v)| \to +\infty} \frac{F(x,u,v)}{\Phi_1(u) + \Phi_2(v)} = +\infty \text{ uniformly for all } x \in \Omega,
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$;

(V) Similar to (IV), for the scalar equation, we also obtain a new $\phi$-superlinear condition which is different from those in [19, 21] even if we restrict it to the bounded domain $\Omega$. One can see the Remark 4.3 for details;

(VI) Because of the coupling relationship of $u$ and $v$ and the inhomogeneous properties of $\Phi_i, i = 1, 2$, our proofs become more difficult and complex than those in [20]. Especially, such difficulty and complexity can be embodied in the proofs of the compactness of Cerami sequence. Moreover, because the new condition
(F_4) below is different from (f_2) and we consider the problem (1.1) on the whole space \( \mathbb{R}^N \) rather than a bounded domain \( \Omega \), our proofs on the compactness of Cerami sequence are different from those in [21].

The remainder of this article focuses on some preliminaries, the main results of this paper and their proofs and an example which illustrates our results. Finally, a remark on semi-trivial solutions of (1.1) is given.

2 Preliminaries

In this section, to deal with such problem for system (1.1), we need to briefly list some fundamental definitions and essential properties of Orlicz and Orlicz-Sobolev spaces and introduce some classical results from variational methods. For a deeper understanding of these concepts, we refer readers for more details to the books [27, 28, 30].

Definition 2.1. [27] Let \( b : [0, +\infty) \to [0, +\infty) \) be a right continuous, monotone increasing function with

1. \( b(0) = 0; \)
2. \( \lim_{t \to +\infty} b(t) = +\infty; \)
3. \( b(t) > 0 \) whenever \( t > 0. \)

Then the function defined on \( \mathbb{R} \) by \( B(t) = \int_0^{|t|} b(s)ds \) is called as an \( N \)-function.

By the definition of \( N \)-function \( B \), it is obvious that \( B(0) = 0 \) and \( B \) is strictly convex. We call that an \( N \)-function \( B \) satisfies a \( \Delta_2 \)-condition globally (or near infinity) if

\[
\sup_{t>0} \frac{B(2t)}{B(t)} < +\infty \quad \text{or} \quad \lim_{t \to +\infty} \frac{B(2t)}{B(t)} < +\infty,
\]

which implies that there exists a constant \( K > 0 \) such that \( B(2t) \leq KB(t) \) for all \( t \geq 0 \) (or \( t \geq t_0 > 0 \)). We also state the equivalent form that \( B \) satisfies a \( \Delta_2 \)-condition globally (or near infinity) if and only if for any \( c \geq 1 \), there exists a constant \( K_c > 0 \) such that \( B(ct) \leq K_c B(t) \) for all \( t \geq 0 \) (or \( t \geq t_0 > 0 \)).

Definition 2.2. [27] For an \( N \)-function \( B \), we define

\[
\tilde{B}(t) = \int_0^{|t|} b^{-1}(s)ds, \quad t \in \mathbb{R},
\]

where \( b^{-1} \) is the right inverse of the right derivative \( b \) of \( B \). Then \( \tilde{B} \) is an \( N \)-function called as the complement of \( B \).

It holds that Young’s inequality (see [27, 30])

\[
st \leq B(s) + \tilde{B}(t), \quad s, t \geq 0
\] (2.1)
and the inequality (see [31, Lemma A.2])

\[ \tilde{B}(b(t)) \leq B(2t), \quad t \geq 0. \]  

(2.2)

Now, we recall the Orlicz space \( L^B(\Omega) \) associated with \( B \). The Orlicz space \( L^B(\Omega) \) is the vectorial space of the measurable functions \( u : \Omega \to \mathbb{R} \) satisfying

\[ \int_{\Omega} B(|u|) \, dx < +\infty, \]

where \( \Omega \subset \mathbb{R}^N \) is an open set. \( L^B(\Omega) \) is a Banach space endowed with Luxemburg norm

\[ \|u\|_B := \inf \left\{ \lambda > 0 : \int_{\Omega} B\left(\frac{|u|}{\lambda}\right) \, dx \leq 1 \right\}. \]

The fact that \( B \) satisfies \( \Delta_2 \)-condition globally implies that

\[ u_n \to u \text{ in } L^B(\Omega) \iff \int_{\Omega} B(u_n - u) \, dx \to 0. \]  

(2.3)

Moreover, a generalized type of Hölder’s inequality (see [27, 30])

\[ \left| \int_{\Omega} uv \, dx \right| \leq 2\|u\|_\Phi \|v\|_{\tilde{\Phi}}, \quad \text{for all } u \in L^\Phi(\Omega) \text{ and } v \in L^{\tilde{\Phi}}(\Omega) \]

can be gained by applying Young’s inequality (2.1).

The corresponding Orlicz-Sobolev space (see [27, 30]) is defined by

\[ W^{1,B}(\Omega) := \left\{ u \in L^B(\Omega) : \frac{\partial u}{\partial x_i} \in L^B(\Omega), i = 1, \cdots, N \right\} \]

with the norm

\[ \|u\|_{1,B} := \|u\|_B + \|\nabla u\|_B. \]

Consider the subspace \( X \) of \( W^{1,B}(\mathbb{R}^N) \),

\[ X = \left\{ u \in W^{1,B}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) B(|u|) \, dx < \infty \right\} \]  

(2.4)

with the norm

\[ \|u\|_X = \|\nabla u\|_B + \|u\|_{B,V}, \]  

(2.5)

where

\[ \|u\|_{B,V} = \inf \left\{ \alpha > 0 : \int_{\mathbb{R}^N} V(x) B \left( \frac{|u|}{\alpha} \right) \, dx \leq 1 \right\} \]

and \( \inf_{x \in \mathbb{R}^N} V(x) > 0 \). Then \( (X, \| \cdot \|) \) is a separable and reflexive Banach space (see [33]).

**Lemma 2.3.**[27, 31] If \( B \) is an \( N \)-function, then the following conditions are equivalent:
Lemma 2.4. If \( \xi_0(t) = \min\{t^l, t^m\} \) and \( \xi_1(t) = \max\{t^l, t^m\}, \ t \geq 0. \) B satisfies
\[
\xi_0(t)B(\rho) \leq B(\rho t) \leq \xi_1(t)B(\rho), \ \forall \rho, t \geq 0;
\]
(2) Let \( \xi_2(t) = \min\{t^l, t^m\} \) and \( \xi_3(t) = \max\{t^l, t^m\}, \ t \geq 0. \) B satisfies
\[
\xi_2(t)B(\rho) \leq B(\rho t) \leq \xi_3(t)B(\rho), \ \forall \rho, t \geq 0;
\]
(3) B satisfies a \( \Delta_2 \)-condition globally.

Lemma 2.5. \( \xi_0(\|u\|_B) \leq \int_{\Omega} B(u)dx \leq \xi_1(\|u\|_B), \ \forall u \in L^B(\Omega).

Lemma 2.6. \( \xi_0(\|u\|_{B^*}) \leq \int_{\Omega} B^*(u)dx \leq \xi_1(\|u\|_{B^*}), \ \forall u \in L^{B^*}(\Omega).

Lemma 2.7. If B is an N-function and (2.6) holds, then B satisfies
\[
1 \leq l = \inf_{t>0} \frac{tb(t)}{B(t)} \leq \sup_{t>0} \frac{tb(t)}{B(t)} = m < +\infty.
\]
(2) Let \( \xi_0(t) = \min\{t^l, t^m\} \) and \( \xi_1(t) = \max\{t^l, t^m\}, \ t \geq 0. \) B satisfies
\[
\xi_0(t)B(\rho) \leq B(\rho t) \leq \xi_1(t)B(\rho), \ \forall \rho, t \geq 0;
\]
(3) B satisfies a \( \Delta_2 \)-condition globally.

\[
B^{-1}(t) = \int_0^t \frac{B^{-1}(s)}{s^{\frac{\alpha}{\beta}}} ds \quad \text{for} \ t \geq 0 \quad \text{and} \quad B_*(t) = B_*(-t) \quad \text{for} \ t \leq 0.
\]
Next, we recall some embeddings. Let $\Psi$ be an $N$-function verifying $\Delta_2$-condition. If
\[
\lim_{t \to 0} \frac{\Psi(t)}{B(t)} < +\infty \quad \text{and} \quad \lim_{|t| \to +\infty} \frac{\Psi(t)}{B_\ast(t)} < +\infty,
\] (2.7)
then we have a continuous embedding $W^{1,B}(\mathbb{R}^N) \hookrightarrow L^\Psi(\mathbb{R}^N)$. Moreover, if
\[
\lim_{|t| \to 0} \frac{\Psi(t)}{B(t)} < +\infty \quad \text{and} \quad \lim_{|t| \to +\infty} \frac{\Psi(t)}{B_\ast(t)} = 0,
\] (2.8)
then the embedding $W^{1,B}(\mathbb{R}^N) \hookrightarrow L^\Psi_{loc}(\mathbb{R}^N)$ is compact and we call that such $\Psi$ satisfies the subcritical condition.

**Lemma 2.7.** Assume that $b : [0, \infty) \to [0, \infty) \in C^1$ and $V$ satisfies the following conditions:

(a) the function $t \to b(t)t$ is increasing in $(0, \infty)$;

(a) there exist $l, m \in (1, N)$ such that
\[
l \leq \frac{b(|t|)t^2}{B(t)} \leq m \quad \text{for all} \quad t \neq 0,
\]
where $l \leq m < l^*$ and $B(t) = \int_0^{|t|} b(s)ds$;

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0$;

(1) for all $C_0 > 0$, $\mu(V^{-1}(-\infty, C_0)) < \infty$, where $\mu$ is the Lebesgue measure in $\mathbb{R}^N$.

Then for any $N$-function $\Psi$ satisfying $\Delta_2$-condition and $\mathcal{P}$, the embedding from $X$ into $L^\Psi(\mathbb{R}^N)$ is compact. Specifically, $X$ into $L^B(\mathbb{R}^N)$ is compact, where $X$ is defined by (2.4).

**Remark 2.8.** By Lemma 2.3 and Lemma 2.5, assumptions $(\phi_1) - (\phi_3)$ show that $\Phi_i$ ($i = 1, 2$) and $\tilde{\Phi}_i$ ($i = 1, 2$) are $N$-functions satisfying $\Delta_2$-condition globally. Thus $L^{\Phi_i}(\mathbb{R}^N)$($i = 1, 2$) and $W^{1,\Phi_i}(\mathbb{R}^N)$($i = 1, 2$) are separable and reflexive Banach spaces (see [27, 30]).

By the end of this section, we recall the mountain pass theorem (see [28, Theorem 2.2]) and the symmetric mountain pass theorem (see [28, Theorem 9.12]) which will be used to prove Theorem 3.1 and Theorem 3.9 in Section 3, respectively.

We first recall that $I \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition ((PS)-condition for short) if any (PS)-sequence $\{u_n\} \subset E$ has a convergent subsequence, where (PS)-sequence $\{u_n\}$ means that
\[
I(u_n) \text{ is bounded,} \quad \|I'(u_n)\| \to 0, \quad \text{as} \quad n \to \infty,
\]
and we call that $I \in C^1(E, \mathbb{R})$ satisfies the Cerami-condition ((C)-condition for short) if any (C)-sequence $\{u_n\} \subset E$ has a convergent subsequence, where (C)-sequence $\{u_n\}$ means that
\[
I(u_n) \text{ is bounded and} \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0, \quad \text{as} \quad n \to \infty.
\] (2.9)
By the discussion in [34], the (PS)-condition can be substituted with (C)-condition in the following Lemma 2.9 and Lemma 2.10.

**Lemma 2.9.** [28, Theorem 2.2] (Mountain Pass Theorem) Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying (PS)-condition. Suppose $I(0) = 0$ and

1. there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$, and
2. there is an $e \in E \setminus B_\rho$ such that $I(e) \leq 0$.

Then $I$ possesses a critical value $c \geq \alpha$.

**Lemma 2.10.** [28, Theorem 9.12] (Symmetric Mountain Pass Theorem) Let $E$ be an infinite-dimensional Banach space and let $I \in C^1(E, \mathbb{R})$ be even, satisfy (PS)-condition, and $I(0) = 0$. If $E = V \oplus X$, where $V$ is finite dimensional, and $I$ satisfies

1. there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$, and
2. for each finite dimensional subspace $\tilde{E} \subset E$, there is an $R = R(\tilde{E})$ such that $I \leq 0$ on $\tilde{E} \setminus B_R(\tilde{E})$, where $B_R = \{u \in E : \|u\| < R\}$,

then $I$ possesses an unbounded sequence of critical values.

### 3 Main results and proofs

Define

$$W_i = \left\{ u \in W^{1, \Phi_i}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)\Phi_i(|u|)dx < \infty \right\}$$

with the norm

$$\|u\|_i = \|\nabla u\|_{\Phi_i} + \|u\|_{\Phi_i, V_i}, \quad i = 1, 2.$$ 

Throughout this paper, we work in the subspace $W = W_1 \times W_2$ of $W^{1, \Phi_1}(\mathbb{R}^N) \times W^{1, \Phi_2}(\mathbb{R}^N)$ with the norm

$$\|(u, v)\| = \|u\|_1 + \|v\|_2 = \|\nabla u\|_{\Phi_1} + \|u\|_{\Phi_1, V_1} + \|\nabla v\|_{\Phi_2} + \|v\|_{\Phi_2, V_2}.$$ 

Then $(W, \| \cdot \|)$ is a separable and reflexive Banach space.

Define the energy functional $I$ on $W$ corresponding to system (1.1) is

$$I(u, v) = \tilde{M}_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u|)dx \right) + \tilde{M}_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v|)dx \right) + \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u|)dx$$

$$+ \int_{\mathbb{R}^N} V_2(x)\Phi_2(|v|)dx - \int_{\mathbb{R}^N} F(x, u, v)dx, \quad (u, v) \in W.$$ (3.2)
Under the assumptions $(\phi_1)$–$(\phi_3)$, $(F_1)$, $(V_0)$, $(V_1)$, $(M_0)$ and $(M_1)$, by using the standard arguments as in [21, 39], we can prove that $I$ is well-defined and of class $C^1(W,\mathbb{R})$ with

$$
\langle I'(u,v),(\tilde{u},\tilde{v})\rangle = M_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u|)dx \right) \int_{\mathbb{R}^N} \phi_1(|\nabla u|)\nabla u \nabla \tilde{u} dx 
+ M_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v|)dx \right) \int_{\mathbb{R}^N} \phi_2(|\nabla v|)\nabla v \nabla \tilde{v} dx 
+ \int_{\mathbb{R}^N} V_1(x)\phi_1(|u|)u \tilde{u} dx + \int_{\mathbb{R}^N} V_2(x)\phi_2(|v|)v \tilde{v} dx 
- \int_{\mathbb{R}^N} F_u(x,u,v)\tilde{u} dx - \int_{\mathbb{R}^N} F_v(x,u,v)\tilde{v} dx 
$$

(3.3)

for all $(\tilde{u},\tilde{v}) \in W$. Then the critical points of $I$ on $W$ are the weak solutions of system (1.1).

### 3.1 Existence

In this subsection, we present the following existence result and prove it by using the Mountain Pass Theorem.

**Theorem 3.1.** Assume that $(\phi_1)$–$(\phi_4)$, $(F_0)$, $(V_0)$, $(V_1)$, $(M_0)$, $(M_1)$ and the following conditions hold:

1. **(F1)** there exist two continuous functions $\psi_i$ $(i = 1, 2) : [0, +\infty) \to \mathbb{R}$ and a constant $C_2 > 0$ such that
   
   \[
   \left\{ \begin{array}{l}
   |F_u(x,u,v)| \leq C_2 \left( |u|^{l_1 - 1} + \psi_1(|u|) + \Psi_1^{-1}(\Psi_1(|v|)) \right), \\
   |F_v(x,u,v)| \leq C_2 \left( |v|^{l_2 - 1} + \Psi_2^{-1}(\Psi_2(|u|)) + \psi_2(|v|) \right)
   \end{array} \right.,
   \]  
   (3.4)

   for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, where $\Psi_i(t) := \int_0^t \psi_i(s)ds, t \in \mathbb{R}$ $(i = 1, 2)$ are two $N$-functions satisfying

   \[
m_i < l_i := \inf_{t > 0} \frac{\psi_i(t)}{\Psi_i(t)} \leq \sup_{t > 0} \frac{\psi_i(t)}{\Psi_i(t)} =: m_i, < l_i^*,
   \]  
   (3.5)

   $\Psi_i$ denote the complements of $\Psi_i$ $(i = 1, 2)$, respectively;

2. **(F2)** there exists a constant $C_3 \in [0, 1)$ such that

   \[
   \limsup_{|u,v| \to 0} \frac{F(x,u,v)}{\Phi_1(|u|) + \Phi_2(|v|)} = C_3 \quad \text{uniformly in } x \in \mathbb{R}^N;
   \]

3. **(F3)** there exists a domain $G \subset \mathbb{R}^N$ such that

   \[
   \lim_{|u,v| \to +\infty} \frac{F(x,u,v)}{|u|^{m_1} + |v|^{m_2}} = +\infty, \quad \text{for a.e. } x \in G;
   \]

4. **(F4)** there exist a continuous function $\Upsilon : [0, \infty) \to \mathbb{R}^+$ and positive constants $\sigma_i \in \left[ \frac{l_i(m_r - 1)}{m_r}, \min \left\{ l_i, \frac{l_i^*(m_r - 1)}{l_r} \right\} \right), i = 1, 2, C_4, r > 0$ such that

   \[
   \Upsilon \left( \frac{F(x,u,v)}{|u|^{\sigma_1} + |v|^{\sigma_2}} \right) \leq C_4 F(x,u,v), \quad \text{for all } x \in \mathbb{R}^N \text{ and } (u,v) \in \mathbb{R}^2 \text{ with } |(u,v)| \geq r,
   \]  
   (3.6)
where \( \Gamma(t) := \int_0^{|t|} \overline{\gamma}(s)ds, \ t \in \mathbb{R} \), is an N-function with

\[
1 < l_\Gamma := \inf_{t > 0} \frac{\overline{\gamma}(t)}{t} \leq \sup_{t > 0} \frac{\overline{\gamma}(t)}{t} =: m_\Gamma < +\infty \tag{3.7}
\]

and

\[
\mathcal{F}(x,u,v) := \frac{1}{m_1} F_u(x,u,v)u + \frac{1}{m_2} F_v(x,u,v)v - F(x,u,v) \geq 0, \ \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}.
\]

Then system (1.1) possesses a nontrivial weak solution.

**Remark 3.2.** By Lemma 2.7 and Lemma 3.1 in [29], the assumptions (\(\phi_1\))–(\(\phi_4\)), (\(V_0\)), (\(V_1\)) and (3.8) imply that the following embeddings are compact:

\[
W_i \hookrightarrow L^{\Psi_i}(\mathbb{R}^N), \quad W_i \hookrightarrow L^{\Phi_i}(\mathbb{R}^N) \quad \text{and} \quad W_i \hookrightarrow L^{p_i}(\mathbb{R}^N), \quad i = 1, 2,
\]

where \(p_i \in [l_i, l_i^1] \). As a result, there exist some positive constants \(C_{i,5}, C_{i,6}, \ i = 1, 2\), such that

\[
\|u\|_{L^{p_i}} \leq C_{i,5}\|u\|_{\Psi_i}, \ \|u\|_{L^{\Psi_i}} \leq C_{i,6}\|u\|_{\Phi_i}, \tag{3.8}
\]

where \(p_i \in [l_i, l_i^1] \). In particular, we have \(\sigma_i l_\Gamma, \sigma_i m_\Gamma \in [l_i, l_i^1] \), where \(\sigma_i, i = 1, 2\), \(l_\Gamma = \frac{l_\Phi}{l_\Psi}\) and \(m_\Gamma = \frac{m_\Phi}{m_\Psi}\).

Hence, the following embeddings are compact:

\[
W_i \hookrightarrow L^{\sigma_i l_\Gamma}(\mathbb{R}^N) \quad \text{and} \quad W_i \hookrightarrow L^{\sigma_i m_\Gamma}(\mathbb{R}^N), \quad i = 1, 2.
\]

**Remark 3.3.** By Young’s inequality [23], [38] and \(F(x,0,0) = 0\), the fact

\[
F(x,u,v) = \int_0^u F_s(x,s,v)ds + \int_0^v F_t(x,0,t)dt + F(x,0,0), \quad \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}
\]

shows that there exists a constant \(C_5 > 0\) such that

\[
|F(x,u,v)| \leq C_5(|u|^{l_1} + |v|^{l_2} + \Psi_1(|u|) + \Psi_2(|v|)), \quad \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}. \tag{3.9}
\]

**Remark 3.4.** If we consider the system (1.1) on a bounded domain \(\Omega\) with Dirichlet boundary condition, then it is natural that we restrict those assumptions of Theorem 3.1 on the bounded domain \(\Omega\). Thus we can claim that (\(F_4\)) and (\(f_2\)) are complementary. Firstly, we claim that if \(l_\Gamma \geq m_\Gamma\), (\(F_4\)) and (\(F_1\)) imply that the condition (\(f_2\)) hold. To be specific, choosing \(r_1 \geq \sqrt{2r} \) with \(r > 1\) and taking the suitable values of \(l_\Gamma\) and \(m_\Gamma\) such that

\[
1 < l_\Gamma \leq m_\Gamma < +\infty, \ \max \left\{ 1, \left(\frac{m_\Psi - l_\Psi}{m_\Psi}, \frac{m_\Psi - l_\Psi}{m_\Psi} \right) \right\} < l_\Gamma < m_\Gamma < l_\Gamma \leq m_\Gamma \text{ and}
\]

\[
m_\Gamma \leq \min \left\{ \frac{l_1 m_\Phi (m_\Psi - l_1) + l_1 l_2}{m_\Phi (m_\Psi - l_2)}, \ \frac{l_1 m_\Phi (m_\Psi - l_2) + l_2 l_1}{m_\Phi (m_\Psi - l_1)}, \ \frac{l_1 l_2}{m_\Phi (m_\Psi - l_1)}, \ \frac{l_1 r m_\Phi (m_\Psi - l_1) + l_1 l_2}{l_1 r m_\Phi (m_\Psi - l_2)} \right\},
\]

we can see that (\(F_4\)) also hold for \(|(u,v)| \geq r_1\). Obviously, (\(F_1\)) implies that

\[
|F(x,u,v)| \leq d_2(|u|^{l_1} + |v|^{l_2} + |u|^{l_1} + |v|^{l_2}) \quad \text{for} \quad |(u,v)| \geq r, \tag{3.10}
\]
where \( d_2 > 0 \). Moreover, by \((3.10)\) and Young’s inequality, we have the following inequality

\[
(\|u\|^{\sigma_1} + \|v\|^{\sigma_2})^\tau \left[ d_2 (\|u\|^{l_1} + \|v\|^{l_2} + \|\sigma_1 u + |u|^{m+2}\|)^{m+\tau} \right]^{m+\tau-\tau} \\
\leq C_{m\tau - \tau} d_2^{m\tau - \tau} \left( \|u\|^{\sigma_1} + \|v\|^{\sigma_2} \right)^{\tau \min \left\{ m \|u\|^{l_1} + \|v\|^{l_2} + \|\sigma_1 u + |u|^{m+2}\| \right\}} \\
\quad + C_{m\tau - \tau} d_2^{m\tau - \tau} \left( \|u\|^{\sigma_1 + m+2} (m\tau - \tau) + \|v\|^{\sigma_2 + m+2} (m\tau - \tau) \right) \\
\quad + C_{m\tau - \tau} d_2^{m\tau - \tau} \left( \frac{1}{\xi_1} |\sigma_1 l\xi_1 + \xi_1 \frac{\sigma_1 u^{m+2}}{d_2} \xi_{l_1} + \frac{\xi_2 - 1}{\xi_2} \frac{\sigma_2 u^{m+2}}{d_2} \xi_{l_2} + \frac{1}{\xi_2} |v|^{\sigma_2 l\xi_2} \right) \\
\leq d_3 \left( (\|u\|^{\sigma_1} + \|v\|^{\sigma_2})^\tau \left[ d_2 (\|u\|^{l_1} + \|v\|^{l_2} + \|\sigma_1 u + |u|^{m+2}\|)^{m+\tau} \right]^{m+\tau-\tau} \\
\quad + |u|^{\xi u^{m+2} (m\tau - \tau)} + |v|^{\xi u^{m+2} (m\tau - \tau)} + |u|^{\xi |\sigma_2 u + |u|^{m+2}\| l\xi_1} + |v|^{\xi |\sigma_2 u + |u|^{m+2}\| l\xi_2} \right) \tag{3.11} \]

for some \( \xi_1 > \frac{l_2 m\tau}{l_1 m\tau - m\phi_1 (m\tau - \tau)} \), \( \xi_2 > \frac{l_1 m\tau}{l_1 m\tau - m\phi_1 (m\tau - \tau)} \), where \( C_{m\tau - \tau} = \begin{cases} 
2^{m\tau - \tau - 1}, & \text{if } m\tau - \tau > 1, \\
1, & \text{if } m\tau - \tau \leq 1,
\end{cases} \)

\( \tau > 1 \), \( d_3 > 0 \), \( \sigma_1 \in \left[ \frac{l_1 (m\tau - \tau)}{m\tau}, \min \left\{ l_1, \frac{l_1 m\tau - m\phi_1 (m\tau - \tau)}{l_1 m\tau - l_1 m\phi_1 (m\tau - \tau)}, \frac{l_1 (m\tau - \tau)}{m\tau} \right\} \right] \), and

\( \sigma_2 \in \left[ \frac{l_2 (m\tau - 1)}{m\tau} m\tau, \min \left\{ l_2, \frac{l_2 m\tau - m\phi_2 (m\tau - \tau)}{l_2 m\tau - l_2 m\phi_2 (m\tau - \tau)}, \frac{l_2 (m\tau - 1)}{m\tau} m\tau \right\} \right] \)

which is reasonable by our example in the last section with \( m\tau = 2 \). Hence, in virtue of \((F_1), (3.10), (3.11)\), Young’s inequality and Lemma 2.3, we have

\[
C_4 \bar{F}(x, u, v) \\
\geq \Gamma \left( \frac{F(x, u, v)}{|u|^{\sigma_1} + |v|^{\sigma_2}} \right) \\
\geq \Gamma \left( \frac{F(x, u, v)}{|u|^{l_1} + |v|^{l_2}} \right) \min \left\{ (\frac{F(x, u, v)}{|u|^{l_1} + |v|^{l_2}})^{m\tau}, (\frac{F(x, u, v)}{|u|^{l_1} + |v|^{l_2}})^{m\tau} \right\} \max \left\{ \left( \frac{F(x, u, v)}{|u|^{l_1} + |v|^{l_2}} \right)^{l\xi}, \left( \frac{F(x, u, v)}{|u|^{l_1} + |v|^{l_2}} \right)^{m\tau} \right\}
\]
\[ d_4 \overline{\Gamma}(1) \left( \frac{|F(x, u, v)|}{|u|^\sigma + |v|^\tau} \right) \]

for \(|u, v| \geq r_1\), where \(d_4 > 0\). To be specific, for \(|u|, |v| \geq r\), we have

\[
\frac{(|u|^\sigma_1 + |v|^\tau_2)^{m_\Gamma}}{(|u|^{\sigma_1} + |v|^\tau_2)^{m_\Gamma}(|u|^\sigma_1 + |v|^\tau_2)^{m_\Gamma - m_\Gamma}} \geq \min \left\{ r^{m_\Gamma(l_1-\sigma_1)}, r^{m_\Gamma(l_2-\sigma_2)} \right\},
\]

(3.12)

\[
\frac{(|u|^\sigma_1 + |v|^\tau_2)^{l_1}}{(|u|^{\sigma_1} + |v|^\tau_2)^{l_1}(|u|^\sigma_1 + |v|^\tau_2)^{l_1 - m_\Gamma}} \geq \min \left\{ r^{m_\Gamma(l_1-\sigma_1)}, r^{m_\Gamma(l_2-\sigma_2)} \right\},
\]

(3.13)

\[
\frac{(|u|^\sigma_1 + |v|^\tau_2)^{l_2}}{(|u|^{\sigma_1} + |v|^\tau_2)^{l_2}(|u|^\sigma_1 + |v|^\tau_2)^{l_2 - m_\Gamma}} \geq \min \left\{ r^{m_\Gamma(l_1-\sigma_1)}, r^{m_\Gamma(l_2-\sigma_2)} \right\},
\]

(3.14)

\[
\frac{|u|^{\sigma_1} + |v|^\tau_2}{(|u|^\sigma_1 + |v|^\tau_2)^{m_\Gamma}} \leq \frac{1}{\max \left\{ r^{m_1(l_1-\sigma_1)}, r^{m_1(l_2-\sigma_2)} \right\}},
\]

(3.15)

\[
\frac{|u|^\sigma_1 + |v|^\tau_2}{(|u|^\sigma_1 + |v|^\tau_2)^{m_\Gamma}} \leq \frac{1}{\max \left\{ r^{m_2(l_1-\sigma_1)}, r^{m_2(l_2-\sigma_2)} \right\}},
\]

(3.16)

\[
\frac{\xi_1^{m_\Gamma}|u|^\sigma_1 + \xi_2^{m_\Gamma}|v|^\tau_2}{(|u|^\sigma_1 + |v|^\tau_2)^{m_\Gamma}} \leq \frac{1}{\max \left\{ r^{m_1(l_1-\sigma_1)}, r^{m_1(l_2-\sigma_2)} \right\}},
\]

(3.17)

\[
\frac{\xi_1^{m_\Gamma}|u|^\sigma_1 + \xi_2^{m_\Gamma}|v|^\tau_2}{(|u|^\sigma_1 + |v|^\tau_2)^{m_\Gamma}} \leq \frac{1}{\max \left\{ r^{m_2(l_1-\sigma_1)}, r^{m_2(l_2-\sigma_2)} \right\}},
\]

(3.18)

\[
\frac{|u|^{\sigma_1} + |v|^\tau_2}{(|u|^\sigma_1 + |v|^\tau_2)^{m_\Gamma}} \leq \frac{1}{\max \left\{ r^{m_1(l_1-\sigma_1)}, r^{m_1(l_2-\sigma_2)} \right\}},
\]

(3.19)

For \(|u| \geq r, 0 \leq |v| < r\) and \(|v| \geq r, 0 \leq |u| < r\), the inequalities (3.13)-(3.19) also hold with different values on the right side of the inequalities. Hence, (3.12) holds for \(|(u, v)| \geq r_1\) and then it is easy to see that (f2) holds.

Next, we claim that (f2) and the following assumption (A) imply that condition (F4) holds if we take \(m_\Gamma \geq l_\Gamma \geq \frac{m_1 l_\Gamma}{m_\Gamma - l_\Gamma}\).

(A) \(F(x, u, v) \geq C(|u|^{\sigma_1} + |v|^{\sigma_2})\) for all \(|(u, v)| \geq r_1\), where \(g_i \in (0, m_i]\) with \(m_i > l_\Gamma m_\Gamma\).

In fact, choosing \(r = \sqrt{2}r_1\) with \(r_1 > 1\), we can see that (f2) also holds for \((u, v) \in \mathbb{R}^2\) with \(|(u, v)| \geq r\) and \(|u|^{\sigma_1} + |v|^{\sigma_2} \geq 1\). Next, there are four possible cases, that is, \(1. \frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} < 1, \frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} \geq 1, g_1 \geq \frac{l_\Gamma m_\Gamma}{m_1 - l_\Gamma};\)

(3.13)

\[
\frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} < 1, \frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} \geq 1, g_1 \geq \frac{l_\Gamma m_\Gamma}{m_1 - l_\Gamma};\]

(3.14)

\[
\frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} < 1, \frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} \geq 1, g_1 \geq \frac{l_\Gamma m_\Gamma}{m_1 - l_\Gamma};\]

(3.15)

\[
\frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} < 1, \frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} \geq 1, g_1 \geq \frac{l_\Gamma m_\Gamma}{m_1 - l_\Gamma};\]

(3.16)

Without loss of generality, we only focus on the proof of the first case. \(\frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} < 1, \frac{|F(x, u, v)|}{|u|^{\sigma_1} + |v|^{\sigma_2}} \geq 1\) and \(g_1 \geq \frac{l_\Gamma m_\Gamma}{m_1 - l_\Gamma}\) by (f2), (A) and Lemma 2.3, we have

\[
d_1 F(x, u, v) = d_1 \left( F(x, u, v) \right)^{m_\Gamma} (F(x, u, v))^{m_\Gamma - m_1}
\]

13
\[
\begin{align*}
&\geq d_1^{m_T-m_T} \left( \frac{F(x,u,v)}{|u|^a + |v|^2} \right)^{\frac{m_T}{m_T}} \left( \frac{F(x,u,v)}{|u|^a + |v|^2} \right)^{m_T-m_T} \\
&\geq d_1^{m_T-m_T} \Gamma(1) \left( \frac{F(x,u,v)}{|u|^a + |v|^2} \right)^{m_T} \left( |u|^a + |v|^2 \right)^{m_T-m_T} \\
&= d_1^{m_T-m_T} \Gamma(1) \left( \frac{F(x,u,v)}{|u|^a + |v|^2} \right)^{m_T} \left( |u|^a + |v|^2 \right)^{m_T-m_T} \\
&\geq d_1^{m_T-m_T} \Gamma(1) \left( \frac{F(x,u,v)}{|u|^a + |v|^2} \right)^{m_T} \left( |u|^a + |v|^2 \right)^{m_T-m_T} \\
&\geq \Gamma \left( \frac{F(x,u,v)}{|u|^a + |v|^2} \right) d_1^{m_T-m_T} \Gamma(1) \left( \frac{|u|^a + |v|^2}{|u|^a + |v|^2} \right)^{m_T-m_T} \\
&\geq d_5^{m_T-m_T} \Gamma \left( \frac{F(x,u,v)}{|u|^a + |v|^2} \right),
\end{align*}
\]

where \(d_5 > 0\) and \(C_{m_T} = 2^{m_T-1}\).

Hence, \((F_1)\) and \((f_2)\) are complementary. We will give an example which satisfies \((F_4)\) but not satisfies \((f_2)\) in Section 5.

**Lemma 3.5.** Suppose that \((\phi_1)-(\phi_3), (M_0), (V_0)\) and \((F_1)\) hold. Then there are constants \(\rho, \alpha > 0\) such that \(I_{|\partial B_\rho \geq \alpha}\).

**Proof.** By \((\phi_1)-(\phi_3)\) and \((V_0)\), we have

\[
\begin{align*}
\min \left\{ \|u\|_{\Phi_1, V_1}, \|u\|_{\Phi_2, V_2} \right\} &\leq \int_{\mathbb{R}^N} V_1(x) \Phi_1(|u|) \, dx \leq \max \left\{ \|u\|_{\Phi_1, V_1}, \|u\|_{\Phi_2, V_2} \right\} \\
\min \left\{ \|v\|_{\Phi_1, V_1}, \|v\|_{\Phi_2, V_2} \right\} &\leq \int_{\mathbb{R}^N} V_2(x) \Phi_2(|v|) \, dx \leq \max \left\{ \|v\|_{\Phi_1, V_1}, \|v\|_{\Phi_2, V_2} \right\}
\end{align*}
\]

for all \(u \in W_1\) and \(v \in W_2\) (the details is in Lemma 2.1 of [33]). Moreover, by (3.9) and \((F_2)\), there exist constants \(C_6 \in (0, 1)\) and \(C_7 > 0\) such that

\[
|F(x,u,v)| \leq (1 - C_6)(\Phi_1(|u|) + \Phi_2(|v|)) + C_7(\Psi_1(|u|) + \Psi_2(|v|)), \quad \forall (x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}.
\]

Choosing \(\rho > 0\) such that \(\rho = \|u,v\| = \|u\|_1 + \|v\|_2 < \min \left\{ \frac{1}{\max(C_{1,6}, C_{2,6})}, 1 \right\}\). Then \(\|u\|_{\Psi_1} \leq C_{1,6}\|u\|_1 < 1\) and \(\|v\|_{\Psi_2} \leq C_{2,6}\|v\|_2 < 1\). By (3.20), (3.21), (3.22), \((M_0)\), Lemma 2.4 and Remark 3.2, we obtain

\[
I(u,v) \geq C_{1,3} \int_{\mathbb{R}^N} \Phi_1(\|\nabla u\|) \, dx + \int_{\mathbb{R}^N} V_1(x) \Phi_1(|u|) \, dx - (1 - C_6) \int_{\mathbb{R}^N} V_1(x) \Phi_1(u) \, dx - C_7 \int_{\mathbb{R}^N} \Phi_1(u) \, dx \\
+ C_{2,3} \int_{\mathbb{R}^N} \Phi_2(\|\nabla v\|) \, dx + \int_{\mathbb{R}^N} V_2(x) \Phi_2(|v|) \, dx - (1 - C_6) \int_{\mathbb{R}^N} V_2(x) \Phi_2(v) \, dx - C_7 \int_{\mathbb{R}^N} \Phi_2(v) \, dx
\]

\[
\geq C_{1,3} \min \left\{ \|\nabla u\|_{\Phi_1}, \|\nabla u\|_{\Phi_2} \right\} + C_6 \min \left\{ \|u\|_{\Phi_1, V_1}, \|u\|_{\Phi_2, V_1} \right\} - C_7 \max \left\{ \|u\|_{\Psi_1}, \|u\|_{\Phi_1} \right\} \\
+ C_{2,3} \min \left\{ \|\nabla v\|_{\Phi_2}, \|\nabla v\|_{\Phi_2} \right\} + C_6 \min \left\{ \|v\|_{\Phi_2, V_2}, \|v\|_{\Phi_2, V_2} \right\} - C_7 \max \left\{ \|v\|_{\Psi_2}, \|v\|_{\Phi_2} \right\}
\]

14
\[ \geq C_{1.3} \| \nabla u \|_{\Phi_1}^{m_1} + C_6 \| u \|_{\Phi_1}^{m_1} + C_7 C_{1.6} \| u \|_{l^{q_1}}^{l^{q_2}} + C_{2.3} \| \nabla v \|_{\Phi_2}^{m_2} + C_6 \| v \|_{\Phi_2}^{m_2} - C_7 C_{2.6} \| v \|_{l^{q_2}}^{l^{q_2}} \]
\[ \geq \| u \|_{\Phi_1}^{m_1} \left( \frac{\min\{C_{1.3}, C_6\}}{2^{m_1-1}} - C_7 C_{1.6} \| u \|_{l^{q_1}}^{l^{q_2}} \right) + \| v \|_{\Phi_2}^{m_2} \left( \frac{\min\{C_{2.3}, C_6\}}{2^{m_2-1}} - C_7 C_{2.6} \| v \|_{l^{q_2}}^{l^{q_2}} \right). \]

Since \( 1 < m_i < l_{\Phi_i} \), we can choose positive constants \( \rho \) and \( \alpha \) small enough such that \( I(u, v) \geq \alpha \) for all \( (u, v) \in W \) with \( \| (u, v) \| = \rho \).

\[ \square \]

**Lemma 3.6.** Suppose that \((\phi_1)\)–\((\phi_3)\), \((M_0)\), \((V_0)\) and \((F_3)\) hold. Then there is a point \((u, v) \in W \backslash B_\rho\) such that \(I(u, v) \leq 0\).

**Proof.** By \((F_3)\) and the continuity of \(F\), there exist two constants \( C_8 > 0 \) and \( C_9 > 0 \) such that

\[ F(x, u, v) \geq C_8 \| u \|_{\Phi_1}^{m_1} + \| v \|_{\Phi_2}^{m_2} - C_9, \forall (u, v) \in \mathbb{R} \times \mathbb{R} \text{ and a.e. } x \in G. \]  

Choose \( u_0 \in C_\infty^0(\mathbb{R}^N) \setminus \{0\} \) with \( 0 < u_0(x) \leq 1 \) and \( \text{supp}(u_0) \subset \subset G \). Obviously, \((tu_0, 0) \in W \) for all \( t \in \mathbb{R} \). By \((M_0)\), \((2)\) in Lemma 2.3, \((3.20)\), \((3.21)\) and \((3.23)\), when \( t > 1 \), we have

\[ I(tu_0, 0) = \tilde{M}_1 \left( \int_{\mathbb{R}^N} \Phi_1(t|\nabla u_0|)dx \right) + \int_{\mathbb{R}^N} V_1(x)\Phi_1(t|u_0|)dx - \int_G F(x, tu_0, 0)dx \]
\[ \leq C_{1.4} t^{m_1} \int_{\mathbb{R}^N} \Phi_1(|\nabla u_0|)dx + t^{m_1} \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u_0|)dx - C_8 \int_G t^{m_1}|u_0|^{m_1}dx + C_9|\text{supp}u_0| \]
\[ \leq t^{m_1} \left( C_{1.4} \| \nabla u_0 \|_{\Phi_1}^{m_1} + C_{1.4} \| \nabla u_0 \|_{\Phi_1}^{m_1} + \| u_0 \|_{\Phi_1, V_1}^{l_i} + \| u_0 \|_{\Phi_1, V_1}^{l_i} + \| u_0 \|_{\Phi_1, V_1}^{m_1} \right) + C_9|\text{supp}u_0|. \]

If we choose

\[ C_8 > \frac{C_{1.4} \| \nabla u_0 \|_{\Phi_1}^{m_1} + C_{1.4} \| \nabla u_0 \|_{\Phi_1}^{m_1} + \| u_0 \|_{\Phi_1, V_1}^{l_i} + \| u_0 \|_{\Phi_1, V_1}^{l_i} + \| u_0 \|_{\Phi_1, V_1}^{m_1}}{\| u_0 \|_{L^{1}(G)}} \]

then there exists \( t \) large enough such that \( I(tu_0, 0) \leq 0 \) and \( \| (tu_0, 0) \| > \rho \).

\[ \square \]

**Lemma 3.7.** Suppose that \((\phi_1)\)–\((\phi_4)\), \((V_0)\), \((V_1)\), \((M_1)\), \((F_1)\), \((F_3)\) and \((F_4)\) hold. Then \((C)\)-sequence in \( W \) is bounded.

**Proof.** Let \( \{ (u_n, v_n) \} \) be a \((C)\)-sequence of \( I \) in \( W \). Then, for \( n \) large enough, by \((\phi_3)\), \((M_1)\) and \((2.9)\), there exists a \( c > 0 \) such that

\[ c + 1 \geq I(u_n, v_n) - \left( I'(u_n, v_n), \left( \frac{1}{m_1}u_n, \frac{1}{m_2}v_n \right) \right) \]
\[ = \tilde{M}_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u_n|)dx \right) - \frac{1}{m_1}M_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u_n|)dx \right) \int_{\mathbb{R}^N} \phi_1(|\nabla u_n|)|\nabla u_n|^2dx \]
\[ + \tilde{M}_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v_n|)dx \right) - \frac{1}{m_2}M_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v_n|)dx \right) \int_{\mathbb{R}^N} \phi_2(|\nabla v_n|)|\nabla v_n|^2dx \]
\[ + \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u_n|)dx - \frac{1}{m_1}V_1(x)\phi_1(|u_n|)|u_n|^2dx \]
\[ + \int_{\mathbb{R}^N} V_2(x)\Phi_2(|v_n|)dx - \frac{1}{m_2}V_2(x)\phi_2(|v_n|)|v_n|^2dx \]

15
Arguing by contradiction, we assume that there exists a subsequence of \{\{(u_n, v_n)\}\}, still denoted by \{\{(u_n, v_n)\}\}, such that \|\{(u_n, v_n)\}\| = \|u_n\|_1 + \|v_n\|_2 \to +\infty. Then we discuss this problem in two situations.

**Case 1.** Suppose that \|u_n\|_1 \to +\infty and \|v_n\|_2 \to +\infty. Let \bar{u}_n = \frac{u_n}{\|u_n\|_1} and \bar{v}_n = \frac{v_n}{\|v_n\|_2}. Then \{\{(\bar{u}_n, \bar{v}_n)\}\} is bounded in \(W\). Passing to a subsequence \{(\bar{u}_n, \bar{v}_n)\}, by Remark 3.2, there exists a point \((\bar{u}, \bar{v})\) \(\in W\) such that

\[
\begin{align*}
\bar{u}_n &\to \bar{u} \text{ in } L^1(\mathbb{R}^N), \text{ in } L^{\sigma_1}(\mathbb{R}^N), \text{ and in } L^{\sigma_1}(\mathbb{R}^N), \\
\bar{v}_n &\to \bar{v} \text{ in } L^2(\mathbb{R}^N), \text{ in } L^{\sigma_2}(\mathbb{R}^N) \text{ and in } L^{\sigma_2}(\mathbb{R}^N).
\end{align*}
\]

To get the contradiction, we will first assume that both \([\bar{u} \neq 0] := \{x \in \mathbb{R}^N : \bar{u}(x) \neq 0\}\) and \([\bar{v} \neq 0] := \{x \in \mathbb{R}^N : \bar{v}(x) \neq 0\}\) have zero Lebesgue measure, that is, \(\bar{u} = 0\) a.e. in \(\mathbb{R}^N\) and \(\bar{v} = 0\) a.e. in \(\mathbb{R}^N\). By Lemma 2.4 and the inequality (66) in [38], we have

\[
\begin{align*}
\min\{C_{1.3, 1}\} \min\left\{\frac{\|u_n\|_{l_1}}{2^{m_1-1}}, \frac{\|u_n\|_{m_1}}{2^m_1}\right\} + \min\left\{\frac{\|v_n\|_{l_2}}{2^{m_2-1}}, \frac{\|v_n\|_{m_2}}{2^m_2}\right\} - \min\{C_{1.3, 1}, 1\} - \min\{C_{2.3, 1}\} \\
\leq C_{1.3} \min\left\{\frac{\|\nabla u_n\|_{\Phi_1}}{2^{m_1-1}}, \frac{\|\nabla u_n\|_{m_1}}{2^m_1}\right\} + C_{2.3} \min\left\{\frac{\|\nabla v_n\|_{\Phi_2}}{2^{m_2-1}}, \frac{\|\nabla v_n\|_{m_2}}{2^m_2}\right\} \\
+ \min\left\{\frac{\|u_n\|_{l_1}}{2^{m_1}}, \frac{\|u_n\|_{m_1}}{2^m_1}\right\} + \min\left\{\frac{\|v_n\|_{l_2}}{2^{m_2}}, \frac{\|v_n\|_{m_2}}{2^m_2}\right\} \\
\leq C_{1.3} \int_{\mathbb{R}^N} \Phi_1(|\nabla u_n|)dx + C_{2.3} \int_{\mathbb{R}^N} \Phi_2(|\nabla v_n|)dx + \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u_n|)dx + \int_{\mathbb{R}^N} V_2(x)\Phi_2(|v_n|)dx \\
\leq \tilde{M}_1(\int_{\mathbb{R}^N} \Phi_1(|\nabla u_n|)dx) + \tilde{M}_2(\int_{\mathbb{R}^N} \Phi_2(|\nabla v_n|)dx) + \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u_n|)dx + \int_{\mathbb{R}^N} V_2(x)\Phi_2(|v_n|)dx \\
= I(u_n, v_n) + \int_{\mathbb{R}^N} F(x, u_n, v_n)dx.
\end{align*}
\]

When \(n\) large enough, we have

\[
\|u_n\|_{l_1}^l + \|v_n\|_{m_2}^l \leq D_1 I(u_n, v_n) + D_1 \int_{\mathbb{R}^N} F(x, u_n, v_n)dx + D_2,
\]

where \(D_1 = \min\left\{\frac{\min\{C_{1.3, 1}\}}{\min\left\{\frac{\|u_n\|_{l_1}}{2^{m_1-1}}, \frac{\|u_n\|_{m_1}}{2^m_1}\right\}}, \frac{\min\{C_{2.3, 1}\}}{\min\left\{\frac{\|v_n\|_{l_2}}{2^{m_2-1}}, \frac{\|v_n\|_{m_2}}{2^m_2}\right\}}\right\}\) and \(D_2 = \min\left\{\frac{\min\{C_{1.3, 1}\} + \min\{C_{2.3, 1}\}}{\min\left\{\frac{\|u_n\|_{l_1}}{2^{m_1-1}}, \frac{\|u_n\|_{m_1}}{2^m_1}\right\}}, \frac{\min\{C_{2.3, 1}\}}{\min\left\{\frac{\|v_n\|_{l_2}}{2^{m_2-1}}, \frac{\|v_n\|_{m_2}}{2^m_2}\right\}}\right\}\). Then

\[
1 \leq \frac{D_1 I(u_n, v_n) + D_2}{\|u_n\|_{l_1}^l + \|v_n\|_{m_2}^l} + \left(\int_{|u_n - v_n| \leq R} + \int_{|u_n - v_n| > R}\right) \frac{D_1 F(x, u_n, v_n)}{\|u_n\|_{l_1}^l + \|v_n\|_{m_2}^l}dx
\]

\[
= o_n(1) + \left(\int_{|u_n - v_n| \leq R} + \int_{|u_n - v_n| > R}\right) \frac{D_1 F(x, u_n, v_n)}{\|u_n\|_{l_1}^l + \|v_n\|_{m_2}^l}dx,
\]

where \(R\) is a positive constant with \(R > r\). By \((F_2)\), there exists a constant \(0 < \delta < 1\) such that

\[
\frac{|F(x, u, v)|}{\Phi_1(|u|) + \Phi_2(|v|)} < C_3 + 1, \quad \forall x \in \mathbb{R}^N, \quad 0 < |(u, v)| \leq \delta.
\]
By \((\phi_4)\) and (3.28), we have
\[
\frac{|F(x,u,v)|}{|u|^{t_1} + |v|^{t_2}} = \frac{|F(x,u,v)|}{\Phi_1(|u|) + \Phi_2(|v|)} \cdot \frac{\Phi_1(|u|) + \Phi_2(|v|)}{|u|^{t_1} + |v|^{t_2}} < (C_3 + 1) \max\{c_{12}, c_{22}\}, \quad \forall x \in \mathbb{R}^N, \quad 0 < |(u,v)| \leq \delta.
\]
\[(3.29)\]

By the fact that \(F\) and \(\Phi_i\) are continuous, there exist two constants \(\overline{C}_{10} > 0\) and \(R > 0\) such that
\[
\frac{|F(x,u,v)|}{|u|^{t_1} + |v|^{t_2}} \leq \frac{\max_{\delta \leq |(u,v)| \leq R} |F(x,u,v)|}{\min_{\delta \leq |(u,v)| \leq R} |(u,v)|^{t_1} + |v|^{t_2}} = \overline{C}_{10}, \quad \forall x \in \mathbb{R}^N, \quad \delta \leq |(u,v)| \leq R,
\]
combining with (3.29), which implies that there exists a positive constant \(C_{10}\) such that
\[
\frac{|F(x,u,v)|}{|u|^{t_1} + |v|^{t_2}} \leq C_{10}, \quad \forall x \in \mathbb{R}^N, \quad 0 < |(u,v)| \leq R.
\]
\[(3.30)\]

Set
\[
\begin{align*}
\bar{B}_{n,R} &= \{x \in \mathbb{R}^N ||(u_n(x), v_n(x))| \leq R\}, \\
\Omega_{1n} &= \{x \in \bar{B}_{n,R}|v_n(x) = 0 \text{ and } u_n(x) = 0\}, \\
\Omega_{2n} &= \{x \in \bar{B}_{n,R}|v_n(x) \neq 0 \text{ and } u_n(x) = 0\}, \\
\Omega_{3n} &= \{x \in \bar{B}_{n,R}|v_n(x) = 0 \text{ and } u_n(x) \neq 0\}, \\
\Omega_{4n} &= \{x \in \bar{B}_{n,R}|v_n(x) \neq 0 \text{ and } u_n(x) \neq 0\}.
\end{align*}
\]

Then by \((F_0)\),
\[
\int_{\Omega_{1n}} \frac{F(x,u_n,v_n)}{||u_n||_1^{t_1} + ||v_n||_2^{t_2}} dx = 0.
\]
\[(3.32)\]

Note that \(v_n(x) = 0\) on \(\Omega_{2n}\). We have
\[
\int_{\Omega_{2n}} \frac{F(x,u_n,v_n)}{||u_n||_1^{t_1} + ||v_n||_2^{t_2}} dx \leq \int_{\Omega_{2n}} \frac{F(x,u_n,0)}{||u_n||_1^{t_1}} dx = \int_{\Omega_{2n}} \frac{F(x,u_n,0)}{||u_n||_1^{t_1}} \cdot |\bar{u}_n|^{t_1} dx \\
= \int_{\Omega_{2n}} \frac{F(x,u_n,v_n(x))}{||u_n||_1^{t_1} + ||v_n||_1^{t_1}} \cdot |\bar{u}_n|^{t_1} dx \\
\leq C_{10} \int_{\mathbb{R}^N} |\bar{u}_n|^{t_1} dx \to 0, \text{ as } n \to \infty.
\]
\[(3.33)\]

Similarly, we also have
\[
\int_{\Omega_{3n}} \frac{F(x,u_n,v_n)}{||u_n||_1^{t_1} + ||v_n||_2^{t_2}} dx \to 0, \text{ as } n \to \infty.
\]
\[(3.34)\]

Moreover,
\[
\begin{align*}
\int_{\Omega_{4n}} \frac{F(x,u_n,v_n)}{||u_n||_1^{t_1} + ||v_n||_2^{t_2}} dx &= \int_{\Omega_{4n}} \frac{F(x,u_n,v_n)}{||u_n||_1^{t_1} + ||v_n||_2^{t_2}} dx \\
&= \int_{\Omega_{4n}} \frac{F(x,u_n,v_n)}{||u_n||_1^{t_1} + ||v_n||_2^{t_2}} dx
\end{align*}
\]

17
\[
\int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + \|v_n\|_2^2} dx \\
\leq \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + \|v_n\|_2^2} dx \\
\leq \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + \|v_n\|_2^2} dx \\
\leq \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + \|v_n\|_2^2} dx \\
\leq C_{10} \int_{\mathbb{R}^N} (\|\tilde{u}_n\|_1 + \|\tilde{v}_n\|_2) dx \to 0, \text{ as } n \to \infty.
\]

Then by (3.32)–(3.35), we have

\[
\int_{B_{n,R}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + \|v_n\|_2^2} dx = \sum_{i=1}^4 \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + \|v_n\|_2^2} dx = o(1).
\]

Next, we set

\[
\begin{align*}
\Omega'_{2n} &= \{ x \in \mathbb{R}^N / B_{n,R} | u_n(x) \neq 0 \text{ and } v_n(x) = 0 \}, \\
\Omega'_{3n} &= \{ x \in \mathbb{R}^N / B_{n,R} | u_n(x) = 0 \text{ and } v_n(x) \neq 0 \}, \\
\Omega'_{4n} &= \{ x \in \mathbb{R}^N / B_{n,R} | u_n(x) \neq 0 \text{ and } v_n(x) \neq 0 \}.
\end{align*}
\]

Then for large \( n \), we have

\[
\int_{\Omega'_{4n}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + \|v_n\|_2^2} dx \\
\leq \int_{\Omega'_{4n}} \frac{\|F(x, u_n, v_n)\|}{\|u_n\|_1^2 + \|v_n\|_2^2} dx \\
\leq \int_{\Omega'_{4n}} \frac{\|F(x, u_n, v_n)\|}{\|u_n\|_1^2 + \|v_n\|_2^2} dx.
\]

If \( \frac{\|F(x, u_n, v_n)\|}{\|u_n\|_1^2 + \|v_n\|_2^2} \geq 1 \), then by Hölder’s inequality, (3.24) and Remark 3.2, we have

\[
\int_{\Omega'_{4n}} \frac{\|F(x, u_n, v_n)\|}{\|u_n\|_1^2 + \|v_n\|_2^2} (\|\tilde{u}_n\|_1 + \|\tilde{v}_n\|_2) dx \\
\leq \left( \int_{\Omega'_{4n}} \left( \frac{\|F(x, u_n, v_n)\|}{\|u_n\|_1^2 + \|v_n\|_2^2} \right)^{\frac{\sigma_1}{\tau}} dx \right)^{\frac{\tau}{\sigma_1}} \left( \int_{\Omega'_{4n}} (\|\tilde{u}_n\|_1^2 + \|\tilde{v}_n\|_2^2)^{\frac{\sigma_1}{\tau}} dx \right)^{\frac{\tau}{\sigma_1}} \\
\leq 2^{\frac{\sigma_1}{\tau} - 1} \left( \frac{1}{\Gamma(1)} \int_{\Omega'_{4n}} \frac{\|F(x, u_n, v_n)\|}{\|u_n\|_1^2 + \|v_n\|_2^2} dx \right)^{\frac{\tau}{\sigma_1}} (\|\tilde{u}_n\|_1^2 + \|\tilde{v}_n\|_2^2)^{\frac{\sigma_1}{\tau}} \\
\leq 2^{\frac{\sigma_1}{\tau} - 1} \left( \frac{C_4}{\Gamma(1)} \int_{\mathbb{R}^N} \frac{F(x, u_n, v_n)dx}{\|u_n\|_1^2 + \|v_n\|_2^2} \right)^{\frac{\tau}{\sigma_1}} (\|\tilde{u}_n\|_1^2 + \|\tilde{v}_n\|_2^2)^{\frac{\sigma_1}{\tau} - 1} \\
\leq 2^{\frac{\sigma_1}{\tau} - 1} \left( \frac{C_4 (c + 1)}{\Gamma(1)} \right)^{\frac{\tau}{\sigma_1}} (\|\tilde{u}_n\|_1^2 + \|\tilde{v}_n\|_2^2)^{\frac{\sigma_1}{\tau} - 1} \to 0, \text{ as } n \to \infty.
\]
\[
\leq 2 \frac{\hat{\gamma}}{\Gamma(1)} \left( \int_{\mathbb{R}^N} F(x, u_n, v_n) \, dx \right) \frac{1}{\Gamma} \left( ||\bar{u}_n||_{L^{\sigma_1}}^{\sigma_1} + ||\bar{v}_n||_{L^{\sigma_2}}^{\sigma_2} \right)
\leq 2 \frac{\hat{\gamma}}{\Gamma(1)} \left( \int_{\mathbb{R}^N} F(x, u_n, v_n) \, dx \right) \frac{1}{\Gamma(1)} \left( ||\bar{u}_n||_{L^{\sigma_1}}^{\sigma_1} + ||\bar{v}_n||_{L^{\sigma_2}}^{\sigma_2} \right) \rightarrow 0, \text{ as } n \to \infty. \tag{3.39}
\]

Hence, (3.37)–(3.39) imply that
\[
\int_{\Omega_{4n}} F(x, u_n, v_n) \, dx \to 0, \text{ as } n \to \infty. \tag{3.40}
\]

Note that \( v_n(x) = 0 \) on \( \Omega_{2n}^2 \). Then we have
\[
\int_{\Omega_{2n}^2} \frac{|F(x, u_n, v_n)|}{||u_n||_{1}^{\sigma_1} + ||v_n||_{2}^{\sigma_2}} \, dx \to 0, \text{ as } n \to \infty,
\]
which shows that
\[
\int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{||u_n||_{1}^{\sigma_1} + ||v_n||_{2}^{\sigma_2}} \, dx \to 0, \text{ as } n \to \infty. \tag{3.42}
\]

Similarly, we also get
\[
\int_{\Omega_{2n}^2} \frac{|F(x, u_n, v_n)|}{||v_n||_{2}^{\sigma_2}} = o(1). \tag{3.43}
\]

Hence, (3.40), (3.42) and (3.43) imply that
\[
\int_{\mathbb{R}^N / B_n / \bar{u}_n / \bar{v}_n} \frac{F(x, u_n, v_n)}{||u_n||_{1}^{\sigma_1} + ||v_n||_{2}^{\sigma_2}} \, dx = o(1). \tag{3.44}
\]

By combining (3.36), (3.44) with (3.27), we get a contradiction.

Next, we assume that \( \bar{u} \neq 0 \) or \( \bar{v} \neq 0 \) has nonzero Lebesgue measure. It is clear that
\[
|u_n| = ||u_n||_{1} \to +\infty \text{ for all } \bar{u} \neq 0
\]
or
\[
|v_n| = ||v_n||_{2} \to +\infty \text{ for all } \bar{v} \neq 0.
\]
Moreover, by (2) in Lemma 2.3, \( F(x, u, v) \geq 0 \) and the assumption \((F_3)\) show that

\[
\lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2} = \begin{cases} 
\lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |u|^\sigma_2}, & \text{if } |u| \geq 1, |v| \geq 1, (|u,v|) \to +\infty, \\
\lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |u|^\sigma_2} \geq \lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2}, & \text{if } 0 \leq |u| < 1, |v| \to +\infty, \\
\lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2} \geq \lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2}, & \text{if } 0 \leq |v| < 1, |u| \to +\infty
\end{cases}
\]

\[
\geq \begin{cases} 
\lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |u|^\sigma_2}, & \text{if } |u| \geq 1, |v| \geq 1, \\
\lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |u|^\sigma_2} \geq \frac{1}{2} \lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2}, & \text{if } 0 \leq |u| < 1, |v| > 1, \\
\lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2} \geq \frac{1}{2} \lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2}, & \text{if } 0 \leq |v| < 1, |u| > 1
\end{cases}
\]

\[= +\infty \quad \text{for a.e. } x \in G. \quad (3.45)\]

Hence, combining with \((F_4)\), it is easy to see that

\[
\lim_{|(u,v)| \to +\infty} \mathcal{F}(x, u, v) \geq \frac{1}{C_4} \lim_{|(u,v)| \to +\infty} \mathcal{T} \left( \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2} \right)
\]

\[
\geq \frac{1}{C_4} \lim_{|(u,v)| \to +\infty} \mathcal{T}(1) \max \left\{ \left( \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2} \right)^{\sigma_1}, \left( \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2} \right)^{\sigma_2} \right\}
\]

\[
\geq \frac{1}{C_4} \mathcal{T}(1) \max \left\{ \lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2}, \lim_{|(u,v)| \to +\infty} \frac{F(x, u, v)}{|u|^\sigma_1 + |v|^\sigma_2} \right\}
\]

\[= +\infty \quad \text{for a.e. } x \in G. \quad (3.46)\]

Then, by \((3.24)\), Fatou Lemma and the above formula, we have

\[c + 1 \geq \lim_{n \to +\infty} \int_{\mathbb{R}^N} F(x, u_n, v_n) dx \geq \lim_{n \to +\infty} \int_{G} F(x, u_n, v_n) dx \geq \int_{G} \lim_{n \to +\infty} F(x, u_n, v_n) dx = +\infty,\]

which is a contradiction. So, both \(|u_n|_1 \to +\infty\) and \(|v_n|_2 \to +\infty\) do not hold.

**Case 2.** Suppose that \(|u_n|_1 \leq D_3\) or \(|v_n|_2 \leq D_3\) for some \(D_3 > 0\) and all \(n \in \mathbb{N}\). Without loss of generality, we assume that \(|u_n|_1 \to +\infty\) and \(|v_n|_2 \to D_3\) for some \(D_3 > 0\) and all \(n \in \mathbb{N}\). Let \(\bar{u}_n = \frac{u_n}{|u_n|_1}\) and \(\bar{v}_n = \frac{v_n}{|v_n|_2}\).

Then \(|\bar{u}_n|_1 = 1\) and \(|\bar{v}_n|_2 \to 0\). Passing to a subsequences \(\{(\bar{u}_n, \bar{v}_n)\}\), by Remark 3.2, there exist \(\bar{u} \in W_1\) and \(v \in W_2\) such that

\[
\begin{align*}
\bar{u}_n &\to \bar{u} \text{ in } W_1, \\
\bar{v}_n &\to \bar{v} \text{ in } L^1(\mathbb{R}^N), \text{ in } L^{\sigma_1}(\mathbb{R}^N) \text{ and in } L^{\sigma_1, \infty}(\mathbb{R}^N), \quad \bar{u}_n(x) \to \bar{u}(x) \text{ a.e. in } \mathbb{R}^N; \\
\bar{v}_n &\to 0 \text{ in } W_2, \\
v_n &\to v \text{ in } L^2(\mathbb{R}^N), \text{ in } L^{\sigma_2}(\mathbb{R}^N) \text{ and in } L^{\sigma_2, \infty}(\mathbb{R}^N), \quad \bar{v}_n(x) \to 0 \text{ a.e. in } \mathbb{R}^N;
\end{align*}
\]

\[
\begin{align*}
v_n &\to v \text{ in } W_2, \\
v_n &\to v \text{ in } L^2(\mathbb{R}^N), \text{ in } L^{\sigma_2}(\mathbb{R}^N) \text{ and in } L^{\sigma_2, \infty}(\mathbb{R}^N), \quad \bar{v}_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^N.
\end{align*}
\]

Firstly, we assume that \([\bar{u} \neq 0]\) has nonzero Lebesgue measure. We can see that

\[|u_n| = |\bar{u}_n||u_n|_1 \to +\infty \quad \text{for all } [\bar{u} \neq 0].\]
Then, being analogue to Case 1, we get a contradiction by
\[ c + 1 \geq \int_{\mathbb{R}^N} F(x, u_n, v_n) dx \rightarrow +\infty. \]

Next, we suppose that \([\bar{u} \neq 0]\) has zero Lebesgue measure, that is, \(\bar{u} = 0\) a.e. in \(\mathbb{R}^N\). By (3.24), we can see that
\[
\int_{\mathbb{R}^N} F(x, u_n, v_n) dx \leq c + 1.
\] (3.47)

Then when \(n\) large enough, we can choose two positive constants \(D_4, D_5\) such that (3.25) is changed into
\[
\|u_n\|_1^l \leq D_4 I(u_n, v_n) + D_4 \int_{\mathbb{R}^N} F(x, u_n, v_n) dx + D_5.
\] (3.48)

Similar to the arguments of (3.32)-(3.34), for any given
\[ D_6 > \max \left\{ 5C_{10}C_{2,5}^l D_3^l, 5 \left( \frac{C_4(c + 1)}{T(1)} \right)^{\frac{\Gamma(1)}{2}} C_{2,5}^\sigma D_3^\sigma, 5 \left( \frac{C_4(c + 1)}{T(1)} \right)^{\frac{\Gamma(1)}{2}} C_{2,5}^\sigma D_3^\sigma \right\}, \]
we obtain that
\[
\int_{\Omega_1n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^l + D_6} dx \leq \int_{\Omega_1n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^l} dx = 0,
\] (3.49)
\[
\int_{\Omega_2n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^l + D_6} dx \leq \int_{\Omega_2n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^l} dx \rightarrow 0, \text{ as } n \rightarrow \infty
\] (3.50)
and
\[
\int_{\Omega_3n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^l + D_6} dx \leq \int_{\Omega_3n} \frac{F(x, u_n, v_n)}{|v_n|^l} dx = \int_{\Omega_3n} \frac{F(x, u_n, v_n)}{|v_n|^l} \frac{|v_n|^l}{D_6} dx \leq \frac{C_{10}}{D_6} \int_{\mathbb{R}^N} |v_n|^l dx \leq \frac{C_{10}C_{2,5}^l D_3^l}{D_6} < \frac{1}{5}, \text{ as } n \rightarrow \infty.
\] (3.51)

Moreover, we have
\[
\int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^l + D_6} dx \
\leq \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{|u_n|^l + D_6 |v_n|^l} dx \
\leq \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{|u_n|^l + |v_n|^l} \min \left\{ \frac{1}{|u_n|^l}, \frac{D_6}{|v_n|^l} \right\} dx \
\leq \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{|u_n|^l + |v_n|^l} \max \left\{ |\bar{u}_n|^l, \frac{|v_n|^l}{D_6} \right\} dx \
\leq \int_{\Omega_{4n}} \frac{F(x, u_n, v_n)}{|u_n|^l + |v_n|^l} (|\bar{u}_n|^l + |v_n|^l) dx \
\leq C_{10} \int_{\mathbb{R}^N} |\bar{u}_n|^l dx + \frac{C_{10}}{D_6} \int_{\mathbb{R}^N} |v_n|^l dx
\]
\[
\leq C_{10} \int_{\mathbb{R}^N} \frac{|\bar{u}_n|^{\ell_1} \, dx}{1 + D_6} + \frac{C_{10} C_{10}^{l_2} D_3^{l_2}}{D_6} \\
\leq o(1) + \frac{1}{5}.
\] (3.52)

Hence, (3.49)-(3.52) imply that
\[
\int \frac{F(x, u_n, v_n)}{\|u_n\|_1^{\ell_1} + D_6} \, dx = \sum_{i=1}^{4} \int_{\Omega_n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^{\ell_1} + D_6} \, dx = o(1) + \frac{2}{5}.
\] (3.53)

Moreover,
\[
\int_{\Omega_n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^{\ell_1} + D_6} \, dx \leq \int_{\Omega_n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^{\ell_1} + D_6} \, dx \\
= \int_{\Omega_n} \frac{F(x, u_n, v_n)}{|u_n|^{\sigma_1} + \|v_n\|^{\sigma_2}} \, dx \\
\leq \int_{\Omega_n} \frac{|u_n|^{\sigma_1} + |v_n|^{\sigma_2}}{\|u_n\|_1^{\ell_1} + D_6} \, dx \\
= \int_{\Omega_n} \frac{F(x, u_n, v_n)}{|u_n|^{\sigma_1} + |v_n|^{\sigma_2}} \left( \frac{|\bar{u}_n|^{\sigma_1} + |v_n|^{\sigma_2}}{D_6} \right) \, dx \\
\leq \max \left\{ \left( \frac{C_4(c + 1)}{T(1)} \right)^{\frac{1}{\ell_1}} \|\bar{u}_n\|_L^{\sigma_1} \right\} \left( \frac{C_4(c + 1)}{T(1)} \right)^{\frac{1}{\ell_1}} \|\bar{u}_n\|_L^{\sigma_1} + \frac{1}{D_6} \int_{\Omega_n} \frac{|F(x, u_n, v_n)| \cdot |v_n|^{\sigma_2} \, dx}{\|u_n\|_1^{\ell_1} + D_6} \\
\leq \frac{1}{D_6} \sum_{i=1}^{4} \|F(x, u_n, v_n)\| \cdot |\bar{u}_n|^{\sigma_1} \, dx \rightarrow 0, \text{ as } n \rightarrow \infty.
\] (3.54)

Note that \( v_n(x) = 0 \) on \( \Omega_{2n} \). Similar to the arguments of (3.38) and (3.39), we can also get
\[
\int_{\Omega_n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^{\ell_1} + D_6} \, dx \\
\leq \int_{\Omega_n} \frac{|F(x, u_n, 0)|}{\|u_n\|_1^{\ell_1}} \, dx \\
\leq \int_{\Omega_n} \frac{|F(x, u_n, v_n)|}{\|u_n\|_1^{\ell_1} + |v_n|^{\sigma_2}} \cdot |\bar{u}_n|^{\sigma_1} \, dx \rightarrow 0, \text{ as } n \rightarrow \infty.
\] (3.55)

Note that \( \bar{u}_n(x) = \frac{v_n(x)}{|u_n|_1} \) and \( u_n(x) = 0 \) on \( \Omega_{3n} \). Similar to the argument of (3.54), we have
\[
\int_{\Omega_n} \frac{F(x, u_n, v_n)}{\|u_n\|_1^{\ell_1} + D_6} \, dx \leq \int_{\Omega_n} \frac{|F(x, 0, v_n)|}{D_6 |v_n|^{\sigma_2}} \, dx = \frac{1}{D_6} \int_{\Omega_n} \frac{|F(x, u_n, v_n)|}{\|u_n\|_1^{\ell_1} + |v_n|^{\sigma_2}} \cdot |v_n|^{\sigma_2} \, dx \leq \frac{1}{5}.
\] (3.56)
Hence, (3.54), (3.55) and (3.56) imply that
\[
\int_{\mathbb{R}^N/B_{n, R}} F(x, u_n, v_n)\, dx \leq o(1) + \frac{2}{5},
\] (3.57)

Then, by (3.48), (3.53) and (3.57), we have
\[
1 \leq \frac{D_1(u_n, v_n) + D_5 + D_6}{\|u_n\|_1^2 + D_6} + D_4 \int_{\mathbb{R}^N} F(x, u_n, v_n)\, dx
\]
\[
= o(1) + \int_{B_{n, R}} \frac{F(x, u_n, v_n)}{\|u_n\|_1^2 + D_6} + \int_{\mathbb{R}^N/B_{n, R}} F(x, u_n, v_n)\, dx
\]
\[
\leq o(1) + \frac{4}{5},
\]

which is a contradiction. Based on these advantages, we could get the conclusion of boundedness for sequence \{(u_n, v_n)\}. \hfill \Box

**Lemma 3.8.** Suppose that \((\phi_1)\)–\((\phi_4)\), \((V_0)\), \((V_1)\), \((M_1)\), \((F_1)\), \((F_3)\) and \((F_4)\) hold. Then \(I\) satisfies the \((C)\)-condition.

**Proof.** Let \{(u_n, v_n)\} be any \((C)\)-sequence of \(I\) in \(W\). Lemma 3.7 shows that \{(u_n, v_n)\} is bounded. Passing to a subsequence \{(u_n, v_n)\}, by Remark 3.2, there exists a point \((u, v)\in W\) such that
* \(u_n \to u\) in \(W_1\), \(u_n \to u\) in \(L^{\Phi_1}(\mathbb{R}^N)\), in \(L^{\Phi_2}(\mathbb{R}^N)\), in \(L^m(\mathbb{R}^N)\), \(u_n(x) \to u(x)\) a.e. in \(\mathbb{R}^N\);
* \(v_n \to v\) in \(W_2\), \(v_n \to v\) in \(L^{\Phi_2}(\mathbb{R}^N)\), in \(L^{\Phi_2}(\mathbb{R}^N)\), \(v_n(x) \to v(x)\) a.e. in \(\mathbb{R}^N\).

Now, we define the operators \(F : W_1 \to (W_1)^*\) by
\[
\langle F(u), \tilde{u} \rangle := M_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u|)\, dx \right) \int_{\mathbb{R}^N} \phi_1(|\nabla u|) \nabla u \nabla \tilde{u} \, dx + \int_{\mathbb{R}^N} V_1(x) \phi_1(|u|) u \tilde{u} \, dx, \quad u, \tilde{u} \in W_1
\]
and \(G : W_2 \to (W_2)^*\) by
\[
\langle G(v), \tilde{v} \rangle := M_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v|)\, dx \right) \int_{\mathbb{R}^N} \phi_2(|\nabla v|) \nabla v \nabla \tilde{v} \, dx + \int_{\mathbb{R}^N} V_2(x) \phi_2(|v|) v \tilde{v} \, dx, \quad v, \tilde{v} \in W_2.
\]

Then, we have
\[
\langle F(u_n), u_n - u \rangle = M_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u_n|)\, dx \right) \int_{\mathbb{R}^N} \phi_1(|\nabla u_n|) \nabla u_n \nabla (u_n - u) \, dx
\]
\[
+ \int_{\mathbb{R}^N} V_1(x) \phi_1(|u_n|) u_n (u_n - u) \, dx
\]
\[
= \langle I'(u_n, v_n), (u_n - u, 0) \rangle + \int_{\mathbb{R}^N} F_u(x, u_n, v_n)(u_n - u) \, dx.
\] (3.58)

and the boundness of \{(u_n)\} show that
\[
\langle I'(u_n, v_n), (u_n - u, 0) \rangle \leq \|I'(u_n, v_n)\|_{W^*} \|u_n - u\|_1 \to 0.
\] (3.59)

By \((F_1)\) and Hölder’s inequality, we get
\[
\left| \int_{\mathbb{R}^N} F_u(x, u_n, v_n)(u_n - u) \, dx \right|
\]
\[ \leq C_2 \int_{\mathbb{R}^N} (|u_n|^{l_1} + \psi_1(|u_n|)) + \tilde{\Psi}_1^{-1}(\Psi_2(|v_n|)))u_n - u|dx \\
= C_2 \int_{\mathbb{R}^N} |u_n|^{l_1}u_n - u|dx + C_2 \int_{\mathbb{R}^N} (\psi_1(|u_n|) + \tilde{\Psi}_1^{-1}(\Psi_2(|v_n|)))u_n - u|dx \\
\leq 2C_2||u_n|^{l_1-1}||\tilde{\Psi}_1||u_n - u||\psi_1 + 2C_2||\psi_1(|u_n|) + \tilde{\Psi}_1^{-1}(\Psi_2(|v_n|)))||\tilde{\Psi}_1||u_n - u||\psi_1. \quad (3.60) \]

Condition \((F_1)\) shows that functions \(\Psi_1\) and \(\tilde{\Psi}_1\) are \(N\)-functions satisfying \(\Delta_2\)-condition globally, which together with the convexity of \(N\)-function, \((\Phi_4)\), Lemma 2.4, Remark 2.8, Remark 3.2, inequality (A.9) in [31], inequality (2.2) and the boundedness of \(\{(u_n, v_n)\}\), imply that the boundedness of the following integrals

\[ \int_{\mathbb{R}^N} \tilde{\Phi}_1(|u_n|^{l_1-1})dx = \int_{|u_n|=0} \tilde{\Phi}_1(|u_n|^{l_1-1})dx + \int_{0<|u_n|<1} \tilde{\Phi}_1(|u_n|^{l_1-1})dx + \int_{|u_n|\geq 1} \tilde{\Phi}_1(|u_n|^{l_1-1})dx \\
\leq \int_{\mathbb{R}^N} \tilde{\Phi}_1(\frac{\Phi_1(|u_n|)}{c_{1,1}|u_n|}) dx + \tilde{\Phi}_1(1) \int_{\mathbb{R}^N} |u_n|^{l_1} dx \\
\leq \int_{\mathbb{R}^N} \left( \frac{1}{c_{1,1} + K\gamma_{l_1}} \right) \Phi_1(|u_n|) dx + \tilde{\Phi}_1(1) \int_{\mathbb{R}^N} |u_n|^{m_3} dx \\
\leq \left( \frac{1}{c_{1,1} + K\gamma_{l_1}} \right) (\|u_n\|_{\Phi_1} + \|u_n\|_{\psi_1}^{m_2}) + \tilde{\Phi}_1(1)\|u_n\|_{L^{m_2}}^{m_2}. \quad (3.61) \]

and

\[ \int_{\mathbb{R}^N} \tilde{\Psi}_1(\psi_1(|u_n|) + \tilde{\Psi}_1^{-1}(\Psi_2(|v_n|)))dx \\
\leq \tilde{K}_2 \int_{\mathbb{R}^N} \left( \Phi_1(|u_n|) + \tilde{\Psi}_1^{-1}(\Psi_2(|v_n|)) \right) dx \\
\leq \tilde{K}_2 \int_{\mathbb{R}^N} \left( \tilde{\Psi}_1(\psi_1(|u_n|)) + \tilde{\Psi}_1 \left( \tilde{\Psi}_1^{-1}(\Psi_2(|v_n|)) \right) \right) dx \\
\leq \tilde{K}_2 \int_{\mathbb{R}^N} \left( \Psi_1 (2|u_n|) + \Psi_2(|v_n|) \right) dx \\
\leq \tilde{K}_2 \int_{\mathbb{R}^N} \left( K_2 \Psi_1(|u_n|) + \Psi_2(|v_n|) \right) dx \\
\leq D_5 \int_{\mathbb{R}^N} \left( \Psi_1(|u_n|) + \Psi_2(|v_n|) \right) \right) dx \\
\leq D_5 \left( \|u_n\|_{\psi_1} + \|u_n\|_{\psi_1}^{m_3} + \|v_n\|_{\Psi_2}^{m_3} + \|v_n\|_{\Psi_2}^{m_3} \right), \quad (3.62) \]

where \(D_5 = \tilde{\Phi}_{\max} \{K_2, 1\},\ \tilde{K}_2 > 0\) and \(K_2 > 0\), which shows that

\[ |||u_n|^{l_1-1}||_{\tilde{\Phi}_1} \leq D_6 \quad (3.63) \]

and

\[ ||\psi_1(|u_n|) + \tilde{\Psi}_1^{-1}(\Psi_2(|v_n|))||_{\tilde{\Phi}_1} \leq D_7 \quad (3.64) \]

for some \(D_6, D_7 > 0\). Moreover, * shows that

\[ \|u_n - u\|_{\psi_1} \to 0 \text{ and } \|u_n - u\|_{\psi_1} \to 0. \quad (3.65) \]
Then, combining (3.59), (3.60), (3.63)-(3.65) with (3.58), we obtain
\[ \langle F(u_n), u_n - u \rangle \to 0 \quad \text{as } n \to \infty. \]

By [19, Proposition A.3], $F$ is of the class $(S_+)$, that is, if a sequence $\{u_n\} \subset W_1$ satisfying
\[ u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n \to \infty} \langle F(u_n), u_n - u \rangle \leq 0, \]
then $u_n \to u$ in $W_1$. Similarly, we can also obtain that $v_n \to v$ in $W_2$. Therefore, $\{(u_n, v_n)\} \to (u, v)$ in $W$. \hfill \Box

**Proof of Theorem 3.1.** It is obvious that $I(0) = 0$. By Lemma 3.5, Lemma 3.6 and Lemma 3.8, all conditions of Lemma 2.9 hold. Then system (1.1) possesses a nontrivial weak solution which is a critical point of $I$. \hfill \Box

### 3.2 Multiplicity

In this section, by using the Symmetric Mountain Pass Theorem, we can obtain the following multiplicity result.

**Theorem 3.9.** Assume that $(\phi_1)-(\phi_4)$, $(V_0)$, $(V_1)$, $(M_0)$, $(M_1)$, $(F_0)$, $(F_1)$, $(F_4)$ and the following conditions hold:

1. $F''(u, v) \to +\infty$ uniformly in $x \in \mathbb{R}^N$;
2. $F(x, -u, -v) = F(x, u, v)$ for all $(x, u, v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$.

Then system (1.1) possesses infinitely many weak solutions $\{(u_k, v_k)\}$ such that
\[ I(u_k, v_k) := \tilde{M}_1 \left( \int_{\mathbb{R}^N} \Phi_1(\nabla u_k)dx \right) + \tilde{M}_2 \left( \int_{\mathbb{R}^N} \Phi_2(\nabla v_k)dx \right) + \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u_k|)dx \]
\[ + \int_{\mathbb{R}^N} V_2(x)\Phi_2(|v_k|)dx - \int_{\mathbb{R}^N} F(x, u_k, v_k)dx \to +\infty \quad \text{as } k \to \infty. \]

To apply the Symmetric Mountain Pass Theorem (i.e., Lemma 2.10), we need the following knowledge. One can see the details in [21, 36, 37]. Since $W$ is a reflexive and separable Banach spaces, there exist two sequences $\{e_{ij} : j \in \mathbb{N}\} \subset W_i$ ($i = 1, 2$) and $\{e_{ij}^* : j \in \mathbb{N}\} \subset W_i^*$ ($i = 1, 2$) such that
\[ W_i = \text{span}\{e_{ij} : j = 1, 2, \ldots\}, \quad W_i^* = \text{span}\{e_{ij}^* : j = 1, 2, \ldots\}, \quad i = 1, 2, \]
and
\[ e_{in}^*(e_{im}) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases} \quad i = 1, 2. \]
Let $Y_{i(k)}$ and $Z_{i(k)}$ be the subsets of $W_i$ defined by

$$Y_{i(k)} := \operatorname{span}\{\epsilon_{ij} : j = 1, \ldots, k\}, \quad Z_{i(k)} := \operatorname{span}\{\epsilon_{ij} : j = k + 1, \ldots\}, \quad i = 1, 2.$$ 

Then

$$W_i = Y_{i(k)} \oplus Z_{i(k)}, \quad i = 1, 2, \quad k \in \mathbb{N}.$$ 

Moreover, since the embeddings $W_i \hookrightarrow L^{{p_i}}(\mathbb{R}^N)$ ($i = 1, 2$) and $W_i \hookrightarrow L^{s_i}(\mathbb{R}^N)$ ($i = 1, 2$) are compact, with a similar discussion as [21, 37], we can get

$$\alpha_{i(k)} := \sup \left\{ \|z\|_{\Psi_i} : \|z\|_i = 1, z \in Z_{i(k)} \right\} \to 0 \quad (3.66)$$

and

$$\beta_{i(k)} := \sup \left\{ \|z\|_{L^{s_i}} : \|z\|_i = 1, z \in Z_{i(k)} \right\} \to 0, \quad i = 1, 2, \text{ as } k \to \infty. \quad (3.67)$$

In addition, for Banach space $W = W_1 \times W_2$, there exists a sequence $\{\eta_{(j)}\} \subset W$ defined by

$$\eta_{(j)} = \begin{cases} (e_{1n}, 0) & \text{if } j = 2n - 1, \\ (0, e_{2n}) & \text{if } j = 2n, \quad \text{for } n \in \mathbb{N}, \end{cases}$$

such that

1. $W = \operatorname{span}\{\eta_{(j)} : j = 1, 2, \ldots\}$,

2. $W = Y_k \oplus Z_k,$

where

$$Y_k := \operatorname{span}\{\eta_{(j)} : j = 1, \ldots, k\} \quad \text{and} \quad Z_k := \operatorname{span}\{\eta_{(j)} : j = k + 1, \ldots\}.$$ 

**Lemma 3.10.** Suppose that $(\phi_1)$–$(\phi_3)$, $(M_0)$, $(V_0)$ and $(F_1)$ hold. Then there are two constants $\rho, \alpha > 0$ and $k \in \mathbb{N}$ such that $I_{|\partial B_\rho \cap Z_{2k}} \geq \alpha$.

**Proof.** For $(u, v) \in Z_{2k}$, in view of the inequality (3.39), (3.20), (3.21), (3.66), (3.70), (M_0), Young’s inequality, Lemma 2.4, Remark 3.2 and the inequality (66) in [38], we have

$$I(u, v) = \widetilde{M}_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u|)dx \right) + \widetilde{M}_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v|)dx \right) + \int_{\mathbb{R}^N} (V_1(x)\Phi_1(|u|) + V_2(x)\Phi_2(|v|)) - F(x, u, v) \right) dx$$

\[ \geq C_{1,3} \int_{\mathbb{R}^N} \Phi_1(|\nabla u|)dx + C_{2,3} \int_{\mathbb{R}^N} \Phi_2(|\nabla v|)dx + \int_{\mathbb{R}^N} V_1(x)\Phi_1(|u|)dx + \int_{\mathbb{R}^N} V_2(x)\Phi_2(|v|)dx \]

\[ - C_3 \int_{\mathbb{R}^N} |u|^{\alpha_1}dx - C_5 \int_{\mathbb{R}^N} |u|^{\alpha_2}dx - C_5 \int_{\mathbb{R}^N} \Psi_1(u)dx - C_5 \int_{\mathbb{R}^N} \Psi_2(v)dx \]

= 26
\[ I(u, v) = \widehat{M}_1 \left( \int_{\mathbb{R}^N} \Phi_1(|\nabla u|) dx \right) + \widehat{M}_2 \left( \int_{\mathbb{R}^N} \Phi_2(|\nabla v|) dx \right) + \int_{\mathbb{R}^N} V_1(x) \Phi_1(|u|) dx \]
\[ + \int_{\mathbb{R}^N} V_2(x) \Phi_2(|v|) dx - \int_{\mathbb{R}^N} F(x, u, v) dx \]
\[ \leq \left( C_{1,4} \int_{\mathbb{R}^N} \Phi_1(|\nabla u|) dx + \int_{\mathbb{R}^N} V_1(x) \Phi_1(|u|) dx \right) - L \int_{\mathbb{R}^N} |u|^{m_1} dx + C(L) \int_{\mathbb{R}^N} |u|^{l_1} dx \]
\[ + \left( C_{2,4} \int_{\mathbb{R}^N} \Phi_2(|\nabla v|) dx + \int_{\mathbb{R}^N} V_2(x) \Phi_2(|v|) dx \right) - L \int_{\mathbb{R}^N} |v|^{m_2} dx + C(L) \int_{\mathbb{R}^N} |v|^{l_2} dx \]
Note that I and 4 Results for the scalar equation

By (4.1) Proof of Theorem 3.9.

Thus, there exists an $R$ such that all conditions of Lemma 2.10 hold. Then system (1.1) possesses finitely many weak solutions $f$.

Moreover, we introduce the following conditions on $\phi$

In this section, we study the existence and multiplicity of solutions for the following generalized Kirchhoff elliptic equation in Orlicz-Sobolev spaces:

$$
\begin{aligned}
-M \left( \int_{\mathbb{R}^N} \Phi(|\nabla u|) \right) \Delta u + V(x)\phi(|u|)u &= f(x,u), \quad x \in \mathbb{R}^N, \\
u \in W^{1,\Phi}(\mathbb{R}^N),
\end{aligned}
$$

(4.1)

where $\phi : (0, +\infty) \to (0, +\infty)$ is a function which satisfies:

$(\phi_1)' \phi \in C^1(0, +\infty), \quad t\phi(t) \to 0$ as $t \to 0$, $t\phi(t) \to +\infty$ as $t \to +\infty$;

$(\phi_2)' \quad t \to t\phi(t)$ are strictly increasing;

$(\phi_3)' \quad 1 < l := \inf_{t>0} \frac{t^2\phi(t)}{\phi(t)} \leq \sup_{t>0} \frac{t^2\phi(t)}{\phi(t)} =: m < \min\{N,l^*\}$, where $\Phi(t) := \int_0^t s\phi(s)ds$, $t \in \mathbb{R}$, and $l^* = \frac{N}{N-1}$;

$(\phi_4)' \quad$ there exist positive constants $c_1$ and $c_2$ such that

$$c_1|t|^l \leq \Phi(t) \leq c_2|t|^l, \quad \forall |t| < 1;$$

Moreover, we introduce the following conditions on $f$, $V$, and $M$:

4 Results for the scalar equation

In this section, we study the existence and multiplicity of solutions for the following generalized Kirchhoff elliptic equation in Orlicz-Sobolev spaces:

$$
\begin{aligned}
-M \left( \int_{\mathbb{R}^N} \Phi(|\nabla u|) \right) \Delta u + V(x)\phi(|u|)u &= f(x,u), \quad x \in \mathbb{R}^N, \\
u \in W^{1,\Phi}(\mathbb{R}^N),
\end{aligned}
$$

(4.1)

where $\phi : (0, +\infty) \to (0, +\infty)$ is a function which satisfies:

$(\phi_1)' \phi \in C^1(0, +\infty), \quad t\phi(t) \to 0$ as $t \to 0$, $t\phi(t) \to +\infty$ as $t \to +\infty$;

$(\phi_2)' \quad t \to t\phi(t)$ are strictly increasing;

$(\phi_3)' \quad 1 < l := \inf_{t>0} \frac{t^2\phi(t)}{\phi(t)} \leq \sup_{t>0} \frac{t^2\phi(t)}{\phi(t)} =: m < \min\{N,l^*\}$, where $\Phi(t) := \int_0^t s\phi(s)ds$, $t \in \mathbb{R}$, and $l^* = \frac{N}{N-1}$;

$(\phi_4)' \quad$ there exist positive constants $c_1$ and $c_2$ such that

$$c_1|t|^l \leq \Phi(t) \leq c_2|t|^l, \quad \forall |t| < 1;$$

Moreover, we introduce the following conditions on $f$, $V$, and $M$:
\( (F_0)' \) \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function such that \( f(x, 0) = 0, \ x \in \mathbb{R}^N \);

\( (V_0)' \) \( V \in C(\mathbb{R}^N, \mathbb{R}) \) and \( \inf_{\mathbb{R}^N} V(x) > 1 \);

\( (V_1)' \) there exist constants \( c_3 > 0 \) such that

\[
\lim_{|z| \to \infty} \text{meas}\{x \in \mathbb{R}^N : |x - z| \leq c_3, V(x) \leq c_4\} = 0 \quad \text{for every } c_4 > 0,
\]

where \( \text{meas}(\cdot) \) denotes the Lebesgue measure in \( \mathbb{R}^N \);

\( (M_0)' \) \( M \in C(\mathbb{R}^+, \mathbb{R}^+) \) and \( c_5 \leq M(t) \leq c_6, \ \forall \ t \geq 0 \) for some \( c_5, c_6 > 0 \);

\( (M_1)' \) \( \hat{M}(t) := \int_0^t M(s) ds \geq M(t) t \).

Similar to the results in section 3, we can get the following results.

**Theorem 4.1.** Assume that \( \phi \) and \( f \) satisfy \( (\phi_1)'-(\phi_4)', (F_0)', (M_0)', (M_1)', (V_0)', (V_1)' \) and the following conditions:

\( (F_1)' \) there exist a constant \( c_7 > 0 \) and a continuous function \( \psi : [0, +\infty) \to \mathbb{R} \) such that

\[
|f(x, u)| \leq c_7(|u|^{l-1} + \psi(|u|))
\]

for all \( (x, u) \in \mathbb{R}^N \times \mathbb{R} \), where

\[
\Psi(t) := \int_0^t \psi(s) ds, \quad t \in \mathbb{R}
\]

is an \( N \)-function satisfying

\[
m < l_\psi := \inf_{t > 0} \frac{t \psi(t)}{\Psi(t)} \leq \sup_{t > 0} \frac{t \psi(t)}{\Psi(t)} =: m_\psi < l^* := \frac{ln}{N-l};
\]

\( (F_2)' \) there exists a constant \( c_8 \in [0, 1) \) such that

\[
\lim \sup_{u \to 0} \frac{F(x, u)}{\Phi(u)} = c_8 \quad \text{uniformly in } x \in \mathbb{R}^N,
\]

where \( F(x, u) = \int_0^u f(x, s) ds \) for all \( (x, u) \in \mathbb{R}^N \times \mathbb{R} \);

\( (F_3)' \) there exists a domain \( G \subset \mathbb{R}^N \) such that

\[
\lim_{u \to \infty} \frac{F(x, u)}{|u|^m} = +\infty, \quad \text{for a.e. } x \in G;
\]

\( (F_4)' \) there exists a continuous function \( \gamma : [0, +\infty) \to \mathbb{R} \) and constants \( \sigma \in \left[ \frac{l(m-1)}{m}, \min \left\{ l, \frac{l^*(m-1)}{m} \right\} \right], c_9, r_2 > 0 \) such that

\[
\overline{\gamma} \left( \frac{F(x, u)}{|u|^\sigma} \right) \leq c_9 F(x, u), \quad \text{for all } x \in \mathbb{R}^N \text{ and all } u \in \mathbb{R} \text{ with } |u| \geq r_2, \quad (4.2)
\]
where \( \Gamma(t) := \int_0^{|t|} \gamma(s)ds, \ t \in \mathbb{R}, \) is an \( N \)-function with
\[
1 < l_\Gamma := \inf_{t > 0} \frac{\gamma(t)}{\Gamma(t)} \leq \sup_{t > 0} \frac{\gamma(t)}{\Gamma(t)} =: m_\Gamma < +\infty
\] (4.3)
and
\[
\overline{F}(x,u) := \frac{1}{m} f(x,u)u - F(x,u), \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.
\]

Then the equation (4.1) has a nontrivial weak solution.

**Theorem 4.2.** Assume that \((\phi_1)',(\phi_2)', (F_0)', (M_0)', (M_1)', (V_0)', (V_1)', (F_4)'\) and the following conditions hold:

\[ (F_3)' \quad \lim_{t \to +\infty} F(x,u) \frac{1}{|u|^m} = +\infty \text{ uniformly in } x \in \mathbb{R}^N; \]

\[ (F_5)' \quad F(x,-u) = F(x,u), \text{ for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \]

Then the equation (4.1) possesses infinitely many weak solutions \( \{u_k\} \) such that
\[
I(u_k) := \widehat{M} \left( \int_{\mathbb{R}^N} \Phi(|\nabla u_k|)dx \right) + \int_{\mathbb{R}^N} V(x)\Phi(|u_k|)dx - \int_{\mathbb{R}^N} F(x,u_k)dx \to +\infty, \ \text{as} \ k \to \infty.
\]

**Remark 4.3.**

(I) It is clear that the conditions \((F_3)'\) and \((F_4)\) extend the conditions \((F_3)\) and \((F_5)\) in [20];

(II) Comparing Theorem 1.1 in [33] and Theorem 1.5 in [14] with Theorem 4.1, it is easy to see that the condition \((F_3)'\) is weaker than the following global (A-R) condition:

(A-R) there exist \( \theta > m \) such that for all \( u \in \mathbb{R}/\{0\}, \)
\[
0 < F(u) := \int_0^u f(x,s)ds \leq \frac{1}{\theta} uf(x,u).
\]

(III) If we consider the system (4.1) on a bounded domain \( \Omega \) with Dirichlet boundary condition, then it is natural that we restrict those assumptions of Theorem 4.1 on the bounded domain \( \Omega \). Then the condition \((F_4)\)' is different from the condition \((f_4)\) in Theorem 5.1 of [21] and \((f_2)\) in [19]. To exemplify this, let
\[
F(x,t) = (\sin(2\pi x_1) + |\sin(2\pi x_1)|) \left(|t|^{10} \ln(1+|t|)\right). \text{ If we choose } l_\Gamma = \frac{6}{5}, \text{ then the condition } (F_4) \text{ holds.}
\]
But in [19, 21], the constant \( l_\Gamma \) must satisfy \( l_\Gamma > \frac{N}{5}, \) i.e., \( l_\Gamma > \frac{3}{2} > \frac{6}{5}. \) So the condition \((f_4)\) in [21] and \((f_2)\) in [19] do not hold.
\[ \left\{ \begin{array}{l}
-M_1 \left( \int_{\mathbb{R}^6} (|\nabla u|^4 + |\nabla u|^5) \, dx \right) \text{div} [(4|\nabla u|^2 + 5|\nabla u|^3) \nabla u] + V_1(x) (4|u|^2 + 5|u|^3) u = F_u(x, u, v), \; x \in \mathbb{R}^6, \\
-M_2 \left( \int_{\mathbb{R}^6} (|\nabla v|^4 \ln(e + |\nabla v|)) \, dx \right) \text{div} \left[ \left( 4|\nabla v|^2 \ln(e + |\nabla v|) + \frac{|\nabla v|^3}{e + |\nabla v|} \right) \nabla v \right] + V_1(x) \left( 4|v|^2 \ln(e + |v|) + \frac{|v|^3}{e + |v|} \right) v = F_v(x, u, v), \; x \in \mathbb{R}^6,
\end{array} \right. \]  

(5.1)

where

\[ M_1(t) = 2 + \frac{1}{e^t - 1}, \; t \geq 0, \quad M_2(t) = 3 + \frac{1}{e^{\frac{2}{3}t} + 3t}, \; t \geq 0, \]

(5.2)

\[ F(x, t, s) = (\sin(2\pi x) + |\sin(2\pi x)|) \left( |t|^5 \ln(1 + |t|) + |s|^5 \ln(1 + |s|) + |t|^3 |s|^3 \right). \]

(5.3)

Let \( N = 6, \phi_1(t) = 4|t|^2 + 5|t|^3 \) and \( \phi_2(t) = 4|t|^2 \ln(e + |t|) + 5|t|^3 \). Then \( \phi_i (i = 1, 2) \) satisfy (\( \phi_1 \))-(\( \phi_4 \)), \( M_i \) satisfy (\( M_0 \))-(\( M_1 \)) by (5.2), \( l_1 = l_2 = 4, m_1 = m_2 = 5 \) and \( \Phi_1(t) = |t|^4 + |t|^5, \Phi_2(t) = |t|^4 \ln(e + |t|) \). So, \( l_1^* = l_2^* = 12 \).

Let \( V_1(x) = \sum_{i=1}^6 x_i^2 + 1 \) and \( V_2(x) = \sum_{i=1}^6 x_i^3 + 2 \) for all \((x, t, s) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \). Then it is obvious that \( V_i, i = 1, 2 \) satisfy (V0) and (V1).

By (5.3), we have

\[ F_i(x, t, s) = (\sin(2\pi x) + |\sin(2\pi x)|) \left( 5|t|^3 t \ln(1 + |t|) + \frac{|t|^4 t}{1 + |t|} + 3|t|^3 |s|^3 t \right), \]

(5.4)

\[ F_i(x, t, s) = (\sin(2\pi x) + |\sin(2\pi x)|) \left( 5|s|^3 s \ln(1 + |s|) + \frac{|s|^4 s}{1 + |s|} + 3|t|^3 |s|^3 s \right). \]

(5.5)

Hence

\[ \mathcal{F}(x, t, s) = (\sin(2\pi x) + |\sin(2\pi x)|) \left( \frac{|t|^6}{5(1 + |t|)} + \frac{|s|^6}{5(1 + |s|)} + \frac{1}{5} |t|^3 |s|^3 \right) \]

\[ \geq \left( \begin{array}{l}
\sin(2\pi x) + |\sin(2\pi x)| \\
\sin(2\pi x) + |\sin(2\pi x)| \\
\sin(2\pi x) + |\sin(2\pi x)| \\
\sin(2\pi x) + |\sin(2\pi x)| \\
\sin(2\pi x) + |\sin(2\pi x)|
\end{array} \right) \left( \frac{|t|^5 + |s|^5}{5(1 + |t|)} + \frac{|t|^6 + |s|^6}{5(1 + |s|)} \right), \]

if \( |t| \geq 1, |s| \geq 1, \)

if \( 0 \leq |t| < 1, |s| \geq 1, \)

if \( |t| \geq 1, 0 \leq |s| < 1, \)

if \( 0 \leq |t| < 1, 0 \leq |s| < 1. \)

(5.6)

It is easy to see that conditions (F0) and (F5) hold. Since

\[ \lim_{|t, s| \to 0} \frac{F(x, t, s)}{|t|^{m_1} + |s|^{m_2}} = 0, \quad \text{and} \quad \lim_{|t, s| \to +\infty} \frac{F(x, t, s)}{|t|^{m_1} + |s|^{m_2}} = +\infty, \]
by (2) of Lemma 2.3, we can see that $(F_2)$ and $(F_3)$ hold with $G = (1/8, 3/8) \times \mathbb{R}^5$. Choose $\Psi_1(t) = \Psi_2(t) = t^\delta, \Gamma(t) = |t|^\frac{\delta}{2}$ and $\sigma_1, \sigma_2 = \frac{11}{12}$. Then

$$\limsup_{|(u,v)|\to\infty} \left( \frac{|F(x,u,v)|}{|u|^\frac{11}{12} + |v|^{\frac{11}{12}}} \right)^{\frac{\delta}{2}} \frac{1}{F(x,u,v)} \leq \begin{cases} 
\limsup_{|(u,v)|\to\infty} \frac{10(\sin(2\pi x_1) + |\sin(2\pi x_1)|)^{\frac{11}{12}} |v|^6 \ln(1+|u|) + |v|^5 \ln(1+|v|) + |u|^3 |v|^3}{(|u|^\frac{11}{12} + |v|^\frac{11}{12})^{\frac{\delta}{2}}}, & \text{if } |u| \geq 1, |v| \geq 1, \\
\limsup_{|(u,v)|\to\infty} \frac{10(\sin(2\pi x_1) + |\sin(2\pi x_1)|)^{\frac{11}{12}} |v|^6 \ln(1+|u|) + |v|^5 \ln(1+|v|) + |u|^3 |v|^3}{(|u|^\frac{11}{12} + |v|^\frac{11}{12})^{\frac{\delta}{2}}}, & \text{if } 0 \leq |u| < 1, |v| \geq 1, \\
\limsup_{|(u,v)|\to\infty} \frac{10(\sin(2\pi x_1) + |\sin(2\pi x_1)|)^{\frac{11}{12}} |v|^6 \ln(1+|u|) + |v|^5 \ln(1+|v|) + |u|^3 |v|^3}{(|u|^\frac{11}{12} + |v|^\frac{11}{12})^{\frac{\delta}{2}}}, & \text{if } |u| \geq 1, 0 \leq |v| < 1 \\
\limsup_{|(u,v)|\to\infty} \frac{40(\sin(2\pi x_1) + |\sin(2\pi x_1)|)^{\frac{3}{2}} |v|^6 \ln(1+|u|) + |v|^5 \ln(1+|v|) + |u|^3 |v|^3}{(|u|^\frac{11}{12} + |v|^\frac{11}{12})^{\frac{\delta}{2}}}, & \text{if } |u| \geq 1, |v| \geq 1, \\
\limsup_{|(u,v)|\to\infty} \frac{40(\sin(2\pi x_1) + |\sin(2\pi x_1)|)^{\frac{3}{2}} |v|^6 \ln(1+|u|) + |v|^5 \ln(1+|v|) + |u|^3 |v|^3}{(|u|^\frac{11}{12} + |v|^\frac{11}{12})^{\frac{\delta}{2}}}, & \text{if } 0 \leq |u| < 1, |v| \geq 1, \\
\limsup_{|(u,v)|\to\infty} \frac{40(\sin(2\pi x_1) + |\sin(2\pi x_1)|)^{\frac{3}{2}} |v|^6 \ln(1+|u|) + |v|^5 \ln(1+|v|) + |u|^3 |v|^3}{(|u|^\frac{11}{12} + |v|^\frac{11}{12})^{\frac{\delta}{2}}}, & \text{if } |u| \geq 1, 0 \leq |v| < 1 \\
< +\infty. 
\end{cases}$$

So the condition $(F_4)$ holds. Then by Theorem 3.1, system (5.1) has at least one nontrivial weak solution. If we let

$$F(x,t,s) \equiv |t|^5 \ln(1 + |t|) + |s|^5 \ln(1 + |s|) + |t|^3 |s|^3 \text{ for all } x \in \mathbb{R}^N \text{ and } (t,s) \in \mathbb{R}^2,$$

then by Theorem 3.9, system (5.1) has infinitely many nontrivial weak solutions of high energy.

6 Remark on the semi-trivial solutions of (1.1)

In Theorem 3.1 and Theorem 3.9, we do not exclude the possibility of semi-nontrivial solutions. Hence, it is possible that the solutions of system are $(u,v) = (0,v)$ or $(u,v) = (u,0)$ which are called as semi-nontrivial solutions. In general, it is not a simple work to rule out the semi-nontrivial solutions and some extra assumptions have to be added. We refer readers to [29], [10] and [41] for related work. If we make the extra assumption $F(x,u,v) = F(x,|u|,|v|)$, by using Corollary 2.6 in [41] and combing with the proofs of Theorem 3.9, it is easy to exclude the semi-trivial solutions, that is, system (1.1) possesses infinitely many non semi-trivial solutions. Especially, we can obtain that system (6.1) with $F$ satisfying (5.7) has infinitely many non semi-trivial solutions.
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