Multibracket simple Lie algebras

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Abstract

We introduce higher-order (or multibracket) simple Lie algebras that generalize the ordinary Lie algebras. Their 'structure constants' are given by Lie algebra cohomology cocycles which, by virtue of being such, satisfy a suitable generalization of the Jacobi identity. Finally, we introduce a nilpotent, complete BRST operator associated with the $l$ multibracket algebras which are based on a given simple Lie algebra of rank $l$.

Given $[X, Y] := XY - YX$, the standard Jacobi identity (JI) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ is automatically satisfied if the product is associative. For a Lie algebra $\mathcal{G}$, $[X_i, X_j] = C_{ij}^k X_k$, the JI may be written in terms of $C_{ij}^k$ as

$$\frac{1}{2} \varepsilon^{i_1 j_1 i_2 j_2 i_3} C_{j_1 j_2}^\rho C_{\rho i_3}^\sigma = 0 .$$ (1)

Let $\mathcal{G}$ be simple and (for simplicity) compact. Then, the Killing metric $k$, with coordinates $k_{ij} = k(X_i, X_j)$, is non-degenerate and, after suitable normalization, can be brought to the form $k_{ij} = \delta_{ij}$. Moreover, $k$ is an invariant polynomial, i.e.

$$k([Y, X], Z) + k(X, [Y, Z]) = 0 .$$ (2)

We also know that $k$ defines the second order Casimir invariant. Using this symmetric polynomial we may always construct a non-trivial three-cocycle

$$\omega_{i_1 i_2 i_3} := k([X_{i_1}, X_{i_2}], X_{i_3}) = C_{i_1 i_2}^\rho k_{\rho i_3}$$ (3)

which is indeed skew-symmetric as consequence of (1) or (2).

In fact, it is known since the classical work of Cartan, Pontrjagin, Hopf and others that, from a topological point of view, the group manifolds of all simple compact groups are essentially equivalent to (have the [real] homology of) products of odd
spheres, that $S^3$ is always present in these products and that the simple Lie algebra cocycles are, via the ‘localization’ process, in one-to-one correspondence with bi-invariant de Rham cocycles on the associated compact group manifolds $G$. This is due to the intimate relation between the order of the ($\operatorname{rank} G$) primitive symmetric polynomials which can be defined on a simple Lie algebra, their $l$ associated generalized Casimir-Racah invariants [1] and the topology of the associated simple groups. Such a relation was a key fact in the eighties for the understanding of non-abelian anomalies in gauge theories [2].

The simplest (of order 3) higher-order invariant polynomial $d_{ijk} = d(X_i, X_j, X_k)$ appears for $su(3)$ (and only for $A_l$-type algebras, $l \geq 2$); it is given by the symmetric trace of three $su(3)$ generators. From $d_{ijk}$ we may construct

$$\omega_{i_1i_2i_3i_4i_5} := e^{j_2j_3j_4}d([X_{i_1}, X_{j_2}], [X_{j_3}, X_{j_4}], X_{i_5}) = e^{j_2j_3j_4}C^\rho_{i_1j_2}C^{\sigma}_{j_3j_4}d_{\rho\sigma i_5}$$

(cf. (3)), and it can be checked that (4) defines a fifth-order invariant form (the proof will be given in the general case). The existence of this five-form $\omega$ shows us that $su(3)$ is, from a topological point of view, equivalent to $S^3 \times S^5$. If we calculate in $su(3)$ the ‘four-bracket’ we find that

$$[X_{j_1}, X_{j_2}, X_{j_3}, X_{j_4}] = \sum_{s \in S_4} \pi(s)X_{s(j_1)}X_{s(j_2)}X_{s(j_3)}X_{s(j_4)} = \omega_{j_1j_2j_3j_4}X_{\sigma}$$

where the generators $X_i$ may be taken proportional to the Gell-Mann matrices, $X_i = \frac{\lambda_i}{2}$, and $\pi(s)$ is the parity sign of the permutation $s$. Thus, $\omega_{j_1j_2j_3j_4}$ is related to the four-bracket and a five-cocycle (five-form) in the same way as $C_{j_1j_2}$ is associated with the standard Lie bracket and a three-cocycle (three-form).

We may ask ourselves whether this construction could be extended to all the higher-order polynomials to define from them higher-order simple Lie algebras satisfying an appropriate generalization of the JI. The affirmative answer is given in [3]; we outline below the main steps that led to it. It is interesting to note that this construction may be used to produce examples of a generalization [4] of the Poisson structure different from that underlying Nambu mechanics [5].

a) Invariant polynomials on the Lie algebra $G$

Let $T_i$ be the elements of a representation of $G$. Then the symmetric trace $k_{i_1...i_m} \equiv \operatorname{sTr}(T_{i_1}...T_{i_m})$ (we shall only consider here $\operatorname{sTr}$ although not all invariant polynomials are of this form [1]; see [3]) verifies the invariance condition

$$\sum_{s=1}^m C^\rho_{i_1...i_{s-1}\rho i_{s+1}...i_m} = 0$$

Proof: By definition of $k$, the l.h.s. of (3) (cf. (3)) is

$$\operatorname{sTr}\left(\sum_{s=1}^m T_{i_1}...T_{i_{s-1}}[T_\nu, T_{i_s}]T_{i_{s+1}}...T_{i_m}\right) = \operatorname{sTr}(T_\nu T_{i_1}...T_{i_m} - T_{i_1}...T_{i_m} T_\nu) = 0$$

(7)
\textit{q.e.d.} The above symmetric polynomial is associated to an invariant symmetric tensor field on the group $G$ associated with $\mathcal{G}$, $k(g) = k_{i_1 \ldots i_m} \omega_i^1(g) \otimes \cdots \otimes \omega_i^m(g)$, where the $\omega_i^j(g)$ are left invariant one-forms on $G$. Since the Lie derivative of $\omega^k$ is given by $L_X \omega^k = -C^k_{ij} \omega^j$ for a LI vector field $X_i$ on $G$, the invariance condition is the statement

\[(L_X k)(X_{i_1}, \ldots, X_{i_m}) = - \sum_{s=1}^m k(X_{i_1}, \ldots, [X_{i_s}, X_{i_s}], \ldots, X_{i_m}) = 0 \tag{8}\]

\textit{c.f. (2)}. On forms, the invariance condition (8) may be written as

\[\epsilon_{i_1 \ldots i_q} C_{rj_1}^\rho \omega_{\rho j_2 \ldots j_q} = 0 \tag{9}\]

\textit{b) Invariant forms on the Lie group $G$}

Let $k_{i_1 \ldots i_m}$ be an invariant symmetric polynomial on $\mathcal{G}$ and let us define

\[\tilde{\omega}_{\rho j_2 \ldots j_{2m-2} \sigma} := k_{i_1 \ldots i_{m-1} \sigma} C_{i_1 j_1}^{i_2} \cdots C_{j_{2m-3} j_{2m-2}}^{i_{m-1}} \tag{10}\]

Then the odd order $(2m-1)$-tensor

\[\omega_{\rho l_2 \ldots l_{2m-2} \sigma} := \epsilon_{l_2 l_2 \ldots l_{2m-2}} \tilde{\omega}_{\rho j_2 \ldots j_{2m-2} \sigma} \tag{11}\]

is a fully skew-symmetric tensor. We refer to Lemma 8.1 in [4] for the proof.

Moreover, $\omega$ is an invariant form because for $q = 2m - 1$ the l.h.s. of (11) is

\[\epsilon_{i_1 \ldots i_{2m-1}} C_{ij_1}^\rho \omega_{j_2 \ldots j_{2m-1} \rho} = \epsilon_{i_1 \ldots i_{2m-1}} C_{ij_2}^\rho \omega_{l_3 l_3 \ldots l_{2m-1} \rho} \]

\[= (2m-3)! \epsilon_{i_1 \ldots i_{2m-1}} \sum_{s=2}^m k_{i_{l_2}, j_{s-1} l_s, j_2, \ldots, l_{2m-1}} C_{j_2 j_3}^{i_1} \cdots C_{j_{2m-2} j_{2m-1}}^{i_{m-1}} = 0 \tag{12}\]

This result follows recalling

\[\epsilon_{i_1 \ldots i_p j_{p+1} \ldots j_n} = (n-p)! \epsilon_{i_1 \ldots i_p j_{p+1} \ldots j_n} \tag{13}\]

in the second equality, using the invariance of $k$ [eq. (3)] in the third one and the JI in the last equality for each of the $(m-1)$ terms in the bracket.

This may be seen without using coordinates; indeed (11) is expressed as

\[\tilde{\omega}(X_\rho, X_{j_2}, \ldots, X_{j_{2m-2}}, X_\sigma) := k([X_\rho, X_{j_2}], \ldots, [X_{j_{2m-3}}, X_{j_{2m-2}}], X_\sigma) \tag{14}\]

and the $(2m-1)$-form $\omega$ is obtained antisymmetrizing (14) as in (11) (cf. (4)). Hence

\[(L_X \tilde{\omega})(X_{i_1}, \ldots, X_{i_{2m-1}}) = - \sum_{p=1}^{2m-1} \tilde{\omega}(X_{i_1}, \ldots, [X_\rho, X_{i_p}], \ldots, X_{i_{2m-1}})\]
Lie algebra cohomology cocycles on $G$ algebra a BRST nilpotent operator by to the odd $(2n)$ theorem.

\[ \text{Theorem} \quad \text{Let } G \text{ be a simple compact algebra, and let } \omega \text{ be the non-trivial Lie algebra } (2p + 1) \text{-cocycle obtained from the associated } p \text{ invariant symmetric tensor on } G. \text{ Then } \omega \text{ verifies the generalized Jacobi condition (GJC)} \]

\[
\epsilon^{j_1 \ldots j_{2p-1} \rho}_{i_1 \ldots i_{2p-1}} \omega_{j_1 \ldots j_{2p-1}} \omega_{\rho j_{2p-1} \ldots j_{4p-1}} = 0.
\]

**Proof:** Using (11), (10) and (13), the l.h.s. of (16) is equal to

\[
-(2p - 3)! \epsilon^{j_1 \ldots j_{4p-1}} k_{i_1 \ldots i_p} \omega_{j_1 \ldots j_{2p-1}} \omega_{j_{2p} \ldots j_{4p-1}}
= -(2p - 3)! \epsilon^{j_1 \ldots j_{4p-1}} k_{i_1 \ldots i_p} \omega_{j_1 \ldots j_{2p-1}} \omega_{j_{2p} \ldots j_{4p-1}} = 0,
\]

where the invariance of $\omega$ (eq. (13)) has been used in the last equality, q.e.d.

c) The generalized Jacobi condition

Now we are ready to check that the tensor $\omega$ introduced above verifies a generalized Jacobi condition that extends eq. (1) to multibracket algebras.

**Theorem** Let $G$ be a simple compact algebra, and let $\omega$ be the non-trivial Lie algebra $(2p + 1)$-cocycle obtained from the associated $p$ invariant symmetric tensor on $G$. Then $\omega$ verifies the generalized Jacobi condition (GJC)

\[ \epsilon^{j_1 \ldots j_{2p-1} \rho}_{i_1 \ldots i_{2p-1}} \omega_{j_1 \ldots j_{2p-1}} \omega_{\rho j_{2p} \ldots j_{4p-1}} = 0. \]

**Proof:** Using (11), (10) and (13), the l.h.s. of (16) is equal to

\[
-(2p - 3)! \epsilon^{j_1 \ldots j_{4p-1}} k_{i_1 \ldots i_p} \omega_{j_1 \ldots j_{2p-1}} \omega_{j_{2p} \ldots j_{4p-1}}
= -(2p - 3)! \epsilon^{j_1 \ldots j_{4p-1}} k_{i_1 \ldots i_p} \omega_{j_1 \ldots j_{2p-1}} \omega_{j_{2p} \ldots j_{4p-1}} = 0,
\]

where the invariance of $\omega$ (eq. (13)) has been used in the last equality, q.e.d.

d) Multibrackets and higher-order simple Lie algebras

Eq. (10) now allows us to define higher-order simple Lie algebras based on $G$ using $\omega$ the Lie algebra cocycles $\omega$ on $G$ as generalized structure constants:

\[ [X_{i_1}, \ldots, X_{i_{2n-2}}] = \omega_{i_1 \ldots i_{2n-2}} X_{\sigma}. \]

The GJC (16) satisfied by the cocycles is necessary since for even $n$-brackets of associative operators one has the generalized Jacobi identity

\[ \frac{1}{(n - 1)!n!} \sum_{\sigma \in S_{2n-1}} (-)^{\pi(\sigma)} [[X_{\sigma(1)}, \ldots, X_{\sigma(n)}], X_{\sigma(n+1)}, \ldots, X_{\sigma(2n-1)}] = 0. \]

This establishes the link between the $G$-based even multibracket algebras and the odd Lie algebra cohomology cocycles on $G$ (note that for $n$ odd the l.h.s is proportional to the odd $(2n-1)$-multibracket $[X_1, \ldots, X_{2n-1}]$).

Finally we comment that just in the same way that we can introduce for a Lie algebra a BRST nilpotent operator by

\[ s = -\frac{1}{2} \epsilon^{i j} C_{i j k} \frac{\partial}{\partial C_{k}} \equiv s_2, \quad s^2 = 0, \]

(20)
with $c^i c^j = -c^j c^i$, the set of invariant forms $\omega$ associated with a simple $G$ allows us to complete this operator in the form

$$s = -\frac{1}{2} c^{j_1} c^{j_2} \omega_{j_1 j_2} \frac{\partial}{\partial c^\sigma} - \ldots - \frac{1}{(2m_i - 2)!} c^{j_1} \ldots c^{j_{2m_i-2}} \omega_{j_1 \ldots j_{2m_i-2}} \frac{\partial}{\partial c^\sigma} - \ldots$$

$$- \frac{1}{(2m_l - 2)!} c^{j_1} \ldots c^{j_{2m_l-2}} \omega_{j_1 \ldots j_{2m_l-2}} \frac{\partial}{\partial c^\sigma} \equiv s_2 + \ldots + s_{2m_i-2} + \ldots + s_{2m_l-2}. \quad (21)$$

This new nilpotent operator $s$ is the complete BRST operator $[3]$ associated with $G$.

For the relation of these constructions with the strongly homotopy algebras $[6]$, possible extensions and connections with physics we refer to $[3]$ and references therein.

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