Risk-Sensitive Reinforcement Learning: Iterated CVaR and the Worst Path

Yihan Du
IIS, Tsinghua University
Beijing, China
duyh18@mails.tsinghua.edu.cn

Siwei Wang
CST, Tsinghua University
Beijing, China
wangsw2020@tsinghua.edu.cn

Longbo Huang∗
IIS, Tsinghua University
Beijing, China
longbohuang@tsinghua.edu.cn

Abstract

In this paper, we study a novel episodic risk-sensitive Reinforcement Learning (RL) problem, named Iterated CVaR RL, where the objective is to maximize the tail of the reward-to-go at each step. Different from existing risk-aware RL formulations, Iterated CVaR RL focuses on safety-at-all-time, by enabling the agent to tightly control the risk of getting into catastrophic situations at each stage, and is applicable to important risk-sensitive tasks that demand strong safety guarantees throughout the decision process, such as autonomous driving, clinical treatment planning and robotics. We investigate Iterated CVaR RL with two performance metrics, i.e., Regret Minimization and Best Policy Identification. For both metrics, we design efficient algorithms \textsc{ICVaR-RM} and \textsc{ICVaR-BPI}, respectively, and provide matching upper and lower bounds with respect to the number of episodes $K$. We also investigate an interesting limiting case of Iterated CVaR RL, called Worst Path RL, where the objective becomes to maximize the minimum possible cumulative reward, and propose an efficient algorithm with constant upper and lower bounds. Finally, the techniques we develop for bounding the change of CVaR due to the value function shift and decomposing the regret via a distorted visitation distribution are novel and can find applications in other risk-sensitive online learning problems.

1 Introduction

Reinforcement Learning (RL) Kaelbling et al. [1996], Szepesvári [2010], Sutton and Barto [2018] is a classic online decision making formulation, where an agent interacts with an unknown environment with the goal of maximizing the obtained reward. Despite the empirical success and theoretical progress of recent RL algorithms, e.g., Szepesvári [2010], Agrawal and Jia [2017], Azar et al. [2017], Jin et al. [2018], Zanette and Brunskill [2019], they focus mainly on the risk-neutral criterion, i.e., Regret Minimization and Best Policy Identification. For both metrics, we design efficient algorithms \textsc{ICVaR-RM} and \textsc{ICVaR-BPI}, respectively, and provide matching upper and lower bounds with respect to the number of episodes $K$. We also investigate an interesting limiting case of Iterated CVaR RL, called Worst Path RL, where the objective becomes to maximize the minimum possible cumulative reward, and propose an efficient algorithm with constant upper and lower bounds. Finally, the techniques we develop for bounding the change of CVaR due to the value function shift and decomposing the regret via a distorted visitation distribution are novel and can find applications in other risk-sensitive online learning problems.

∗Corresponding author.
We further study an interesting limiting case of Iterated CVaR RL when we design a simple yet efficient algorithm MaxWP with a delicate CVaR-adapted value iteration and exploration bonuses to allocate more attention on Agrawal and Jia [2017], Azar et al. [2017], Zanette and Brunskill [2019]. Iterated CVaR RL imposes where the performance is measured by the number of episodes used for identifying an optimal policy.

The contributions of this paper are summarized as follows.

- We study a novel Iterated CVaR RL formulation, where an agent interacts with an unknown environment, with the objective to maximize the tail of the reward-to-go at each step. This formulation enables us to model safety requirements throughout the decision process, and is most suitable for applications where such safety-at-all-time is critical.

- We investigate two important metrics of Iterated CVaR RL, i.e., Regret Minimization (RM) and Best Policy Identification (BPI). For both metrics, we propose efficient algorithms ICVaR-RM and ICVaR-BPI, and establish matching regret/sample complexity upper and lower bounds with respect to $K$. Our results reveal essential hardness of Iterated CVaR RL, i.e., any algorithm must suffer a regret for exploring risk-sensitive but hard-to-reach states.
We further investigate a limiting case of Iterated CVaR RL, called Worst Path RL, where the objective is to maximize the minimum possible cumulative reward. For Worst Path RL, we develop a simple and efficient algorithm \( \text{MaxWP} \), and provide constant regret upper and lower bounds (independent of \( K \)).

Due to space limit, we defer all proofs to Appendix.

2 Related Work

CVaR-based Markov Decision Process (MDP). Chow and Ghavamzadeh [2014] considers a CVaR-constrained MDP, which aims to minimize the expected total cost with a constraint on the CVaR of the total cost. Bäuerle and Ott [2011], Haskell and Jain [2015], Chow et al. [2015] investigate how to minimize the CVaR of the total cost in decision processes, i.e., CVaR MDPs, and propose approximate planning algorithms with convergence analysis (see Appendix A for a detailed comparison between our formulation and CVaR MDPs). Hardy and Wirch [2004] firstly defines Iterated CVaR, and demonstrates that it is a coherent and consistent dynamic risk measure, and applicable to equity-linked insurance. Osogami [2012], Chu and Zhang [2014], Bäuerle and Glauner [2022] investigate MDPs with general iterated coherent risk measures, including Iterated CVaR, and demonstrate the existence of Markovian optimal policies for these MDPs. Hardy and Wirch [2004], Coraluppi and Marcus [1997, 1999] study MDPs with the goal of minimizing the worst-case discounted cost, develop dynamic programming for value functions, and design heuristic algorithms without theoretical analysis. The above works mainly study mathematical properties (e.g., the existence of optimal policies), offline planning algorithms, and convergence guarantees for MDPs with known transition distribution, while our work investigates an online RL problem for unknown MDPs, designs online algorithms and derives regret/sample complexity guarantees.

Risk-Sensitive Reinforcement Learning. Chow et al. [2017] studies a risk-constrained RL problem where risk is represented by a constraint on the CVaR of the total cost, and designs policy gradient and actor-critic algorithms with convergence analysis. Fei et al. [2020] considers risk-sensitive RL with the exponential utility criterion, and proposes efficient algorithms with regret guarantees. Fei et al. [2021a] further improves the regret results of Fei et al. [2020] by developing an exponential Bellman equation and a Bellman backup analytical procedure. Fei et al. [2021b] extends the results in Fei et al. [2020, 2021a] from the tabular setting to function approximation, and designs algorithms with sub-linear regret guarantees. The above works study RL with the CVaR constraints or the exponential utility criterion, which greatly differ from our formulation, and their algorithms and results cannot be applied to our problem.

3 Problem Formulation

In this section, we present the problem formulations of Iterated CVaR RL and Worst Path RL.

Conditional Value-at-Risk (CVaR). We first introduce two risk measures, i.e., Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). Let \( X \) be a random variable with cumulative distribution function \( F(x) = \Pr[X \leq x] \). Given a risk level \( \alpha \in (0, 1) \), the VaR at risk level \( \alpha \) is the \( \alpha \)-quantile of \( X \), i.e., \( \text{Var}^{\alpha}(X) = \min \{ x \mid F(x) \geq \alpha \} \), and the CVaR at risk level \( \alpha \) is defined as Rockafellar et al. [2000]:

\[
\text{CVaR}^{\alpha}(X) = \sup_{x \in \mathbb{R}} \left\{ x - \frac{1}{\alpha} \mathbb{E}\left[(x - X)^{+}\right] \right\},
\]

where \( (x)^{+} = \max\{x, 0\} \). If there is no probability atom at \( \text{CVaR}^{\alpha}(X) \), CVaR can also be written as \( \text{CVaR}^{\alpha}(X) = \mathbb{E}[X \mid X \leq \text{Var}^{\alpha}(X)] \) Shapiro et al. [2021]. Intuitively, \( \text{CVaR}^{\alpha}(X) \) is a distorted expectation of \( X \) conditioning on its \( \alpha \)-portion tail, which depicts the average value when bad situations happen. When \( \alpha = 1 \), \( \text{CVaR}^{\alpha}(X) = \mathbb{E}[X] \), and when \( \alpha \downarrow 0 \), \( \text{CVaR}^{\alpha}(X) \) tends to min(\( X \)) Chow et al. [2015].

Iterated CVaR RL. We consider an episodic Markov Decision Process (MDP) \( \mathcal{M}(\mathcal{S}, \mathcal{A}, H, p, r) \). Here \( \mathcal{S} \) is the state space, \( \mathcal{A} \) is the action space, and \( H \) is the length of horizon in each episode. \( p \) is the transition distribution, i.e., \( p(s' | s, \alpha) \) gives the probability of transitioning to \( s' \) when starting from state \( s \) and taking action \( \alpha \). \( r : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1] \) is a reward function, so that \( r(s, \alpha) \) gives
the deterministic reward for taking action \( a \) in state \( s \). A policy \( \pi \) is defined as a collection of \( H \) functions, i.e., \( \pi = \{ \pi_h : \mathcal{S} \rightarrow \mathcal{A} \}_{h \in [H]} \), where \( [H] := \{ 1, 2, ..., H \} \).

The online episodic RL game is as follows. In each episode \( k \), an agent chooses a policy \( \pi^k \), and starts from an initial state \( s^k_1 \), which is arbitrarily picked by the environment and is the same for all \( k \) in the best policy identification setting, as in the literature Jin et al. [2018], Zanette and Brunsill [2019], Dann and Bruskill [2015], Ménard et al. [2021]. At each step \( h \in [H] \), the agent observes the state \( s^k_h \), takes an action \( a^k_h = \pi^k(s^k_h) \), receives a reward \( r(s^k_h, a^k_h) \), and then transitions to a next state \( s^k_{h+1} \) according to the transition distribution \( p(\cdot | s^k_h, a^k_h) \). After the agent takes an action and receives the reward at step \( H \), this episode ends, and she enters the next episode.

In Iterated CVaR RL, given a risk level \( \alpha \in (0, 1) \) and a policy \( \pi \), let \( V^\pi_h : \mathcal{S} \rightarrow \mathbb{R} \) and \( Q^\pi_h : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R} \) denote the value function and Q-value function at step \( h \), respectively Chu and Zhang [2014], Bäuerle and Glauner [2022]. Specifically, \( V^\pi_h(s) \) and \( Q^\pi_h(s,a) \) denote the cumulative reward that can be obtained when the worst \( \alpha \)-portion event happens (transitioning to the worst \( \alpha \)-percent states) at each step, starting from \( s \) and \( (s,a) \) at step \( h \), respectively. Formally, \( V^\pi_h(s) \) and \( Q^\pi_h(s,a) \) are recurrently defined as

\[
\begin{aligned}
    Q^\pi_h(s, a) &= r(s, a) + \text{CVaR}^\alpha_{s^h+\sim p(\cdot | s_h, \pi_h(s_h))}(V^\pi_{h+1}(s')) \\
    V^\pi_h(s) &= Q^\pi_h(s, \pi_h(s)) \\
    V^\pi_{H+1}(s) &= 0, \quad \forall s \in \mathcal{S},
\end{aligned}
\]

where \( \text{CVaR}^\alpha_{s^h+\sim p(\cdot | s_h, \pi_h(s_h))}(V^\pi_{h+1}(s')) \) denotes the CVaR value of random variable \( V^\pi_{h+1}(s') \) with \( s' \sim p(\cdot | s, a) \) at risk level \( \alpha \). Unfolding Eq. (1), \( Q^\pi_h(s,a) \) and \( V^\pi_h(s) \) can also be expressed as

\[
\begin{aligned}
    Q^\pi_h(s, a) &= r(s, a) + \text{CVaR}^\alpha_{s^h+\sim p(\cdot | s_h, \pi_h(s_h))}(r(s_{h+1}, \pi_{h+1}(s_{h+1}))) \\
    &\quad + \text{CVaR}^\alpha_{s^h+\sim p(\cdot | s_{h+1}, \pi_{h+1}(s_{h+1}))}(\cdots \text{CVaR}^\alpha_{s^H+\sim p(\cdot | s_{H-1}, \pi_{H-1}(s_{H-1}))}(r(s_{H}, \pi_{H}(s_{H})))) \\
    V^\pi_h(s) &= r(s, \pi_h(s)) + \text{CVaR}^\alpha_{s^h+\sim p(\cdot | s_h, \pi_h(s))}(r(s_{h+1}, \pi_{h+1}(s_{h+1}))) \\
    &\quad + \text{CVaR}^\alpha_{s^h+\sim p(\cdot | s_{h+1}, \pi_{h+1}(s_{h+1}))}(\cdots \text{CVaR}^\alpha_{s^H+\sim p(\cdot | s_{H-1}, \pi_{H-1}(s_{H-1}))}(r(s_{H}, \pi_{H}(s_{H})))) \\
\end{aligned}
\]

Since \( \mathcal{S}, \mathcal{A}, H \) are finite and the Iterated CVaR RL problem satisfies the optimal substructure property, there exists an optimal policy \( \pi^\ast \) which gives the optimal value \( V^\ast_h(s) = \max_{\pi} V^\pi_h(s) \) for all \( s \in \mathcal{S} \) and \( h \in [H] \) Chu and Zhang [2014], Bäuerle and Glauner [2022], Sutton and Barto [2018]. The Bellman equation is given in Eq. (1), and the Bellman optimality equation is as follows:

\[
\begin{aligned}
    Q^\ast_h(s, a) &= r(s, a) + \text{CVaR}^\alpha_{s^h+\sim p(\cdot | s_h, \pi_h(s_h))}(V^\ast_{h+1}(s')) \\
    V^\ast_h(s) &= \max_{a \in \mathcal{A}} Q^\ast_h(s, a) \\
    V^\ast_{H+1}(s) &= 0, \quad \forall s \in \mathcal{S}.
\end{aligned}
\]

We consider two performance metrics for Iterated CVaR RL, i.e., Regret Minimization (RM) and Best Policy Identification (BPI). In Iterated CVaR RL–RM, the agent aims to minimize the cumulative regret in \( K \) episodes, defined as

\[
\mathcal{R}(K) = \sum_{k=1}^{K} \left( V^\ast_1(s^k_1) - V^\pi_h(s^k_1) \right).
\]

In Iterated CVaR RL–BPI, given a confidence parameter \( \delta \in (0, 1) \) and an accuracy parameter \( \varepsilon > 0 \), the agent needs to use as few trajectories (episodes) as possible to identify an \( \varepsilon \)-optimal policy \( \hat{\pi} \), which satisfies

\[
V^\ast_1(s_1) \geq V^\ast_1(s_1) - \varepsilon,
\]

where \( s_1 \) denotes the fixed initial state in the BPI setting. The performance of BPI is measured by sample complexity, i.e., the number of trajectories used.
Worst Path RL. Furthermore, we investigate an interesting limiting case of Iterated CVaR RL when \( \alpha \downarrow 0 \), called Worst Path RL, in which case the goal just becomes to maximize the minimum possible reward. Note that this case cannot be simply solved by taking \( \alpha = 0 \) in Iterated CVaR RL, as the results there have a dependency on \( \frac{1}{\alpha} \) and changing from CVaR(\cdot) to min(\cdot) in Worst Path RL requires a brand-new algorithm design and analysis. For Worst Path RL, in this paper we mainly consider the regret minimization metric, and the definition of regret is the same as Eq. (3).

In Worst Path RL, the recurrent definition of Q-value/value functions and Bellman equations are

\[
\begin{align*}
Q_h^\pi(s, a) &= r(s, a) + \min_{s' \sim p(\cdot|s, a)} (V_{h+1}^\pi(s')) \\
V_h^\pi(s) &= Q_h^\pi(s, \pi_h(s)) \\
V_{h+1}^\pi(s) &= 0, \forall s \in S,
\end{align*}
\]

where \( \min_{s' \sim p(\cdot|s, a)} (V_{h+1}^\pi(s')) \) denotes the minimum value of random variable \( V_{h+1}^\pi(s') \) with \( s' \sim p(\cdot|s, a) \). From Eq. (4), we see that

\[
Q_h^\pi(s, a) = \min_{(s_t, a_t) \sim \pi} \left[ \sum_{t=h}^{H} r(s_t, a_t) \right] \left| s_h = s, a_h = a, \pi \right.,
\]

Thus, \( Q_h^\pi(s, a) \) and \( V_h^\pi(s) \) become the minimum possible cumulative reward under policy \( \pi \), starting from \( (s, a) \) and \( s \) at step \( h \), respectively. The optimal policy \( \pi^* \) maximizes the minimum possible cumulative reward (optimizes the worst path) for all starting states and steps. Formally, \( \pi^* \) gives the optimal value \( V_h^\pi(s) = \max_{\pi} V_h^\pi(s) \) for all \( s \in S \) and \( h \in [H] \).

4 Iterated CVaR RL with Regret Minimization

In this section, we investigate regret minimization (Iterated CVaR RL-RM), propose an algorithm ICVaR-RM with a CVaR-adapted exploration bonus, and demonstrate its sample efficiency.

4.1 Algorithm ICVaR-RM and Regret Upper Bound

We propose a value iteration-based algorithm ICVaR-RM, which adopts a Brown-type Brown [2007] (CVaR-adapted) exploration bonus and pays more attention to rare but risky states. Algorithm 1 illustrates the formal procedure of ICVaR-RM. Specifically, in each episode \( k \), ICVaR-RM computes the empirical CVaR for the values of next states CVaR(\( \tilde{p}(\cdot|s, a) \)) and a Brown-type Brown [2007] exploration bonus \( \frac{H}{\alpha} \sqrt{\frac{L}{n_h(s, a)}} \). Here \( n_h(s, a) \) is the number of times \( (s, a) \) was visited up to episode \( k \), and \( \tilde{p}^{k+1}(s'|s, a) \) is the empirical estimate of transition probability \( p(s'|s, a) \). Then, ICVaR-RM constructs optimistic Q-value function \( Q_h^k(s, a) \), value function \( V_h^k(s) \), and a greedy policy \( \pi^k \) with respect to \( Q_h^k(s, a) \). After calculating the value functions and policy, ICVaR-RM plays episode \( k \) with policy \( \pi^k \), observes a trajectory, and updates \( n_h(s, a) \) and \( \tilde{p}^{k+1}(s'|s, a) \). We summarize the regret performance of ICVaR-RM as follows.

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**Algorithm 1 ICVaR-RM**

1. **Input:** \( \delta, \alpha, \alpha' = \frac{\delta}{\alpha}, L := \log \left( \frac{KHS(H)}{\alpha} \right) \), \( V_{H+1}^k(s) = 0 \) for any \( k \geq 0 \) and \( s \in S \).
2. for \( k = 1, 2, \ldots, K \) do
3. for \( h = H, H - 1, \ldots, 1 \) do
4. for \( s \in S \) do
5. for \( a \in A \) do
6. \( \tilde{Q}_h^k(s, a) \leftarrow r(s, a) + \text{CVaR}_{s \sim \tilde{p}_h^k(s, a)} (V_{h+1}^k(s')) + \frac{H}{\alpha} \sqrt{\frac{L}{n_h(s, a)}} \)
7. end for
8. \( V_h^k(s) \leftarrow \max_{a \in A} \tilde{Q}_h^k(s, a) \).
9. end for
10. end for
11. Play the episode \( k \) with policy \( \pi^k \), and update \( n_{k+1}(s, a) \) and \( \tilde{p}^{k+1}(s'|s, a) \)
12. end for

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Worst Path RL. Furthermore, we investigate an interesting limiting case of Iterated CVaR RL when \( \alpha \downarrow 0 \), called Worst Path RL, in which case the goal just becomes to maximize the minimum possible reward. Note that this case cannot be simply solved by taking \( \alpha = 0 \) in Iterated CVaR RL, as the results there have a dependency on \( \frac{1}{\alpha} \) and changing from CVaR(\( \cdot \)) to min(\( \cdot \)) in Worst Path RL requires a brand-new algorithm design and analysis. For Worst Path RL, in this paper we mainly consider the regret minimization metric, and the definition of regret is the same as Eq. (3).

In Worst Path RL, the recurrent definition of Q-value/value functions and Bellman equations are

\[
\begin{align*}
Q_h^\pi(s, a) &= r(s, a) + \min_{s' \sim p(\cdot|s, a)} (V_{h+1}^\pi(s')) \\
V_h^\pi(s) &= Q_h^\pi(s, \pi_h(s)) \\
V_{h+1}^\pi(s) &= 0, \forall s \in S,
\end{align*}
\]

where \( \min_{s' \sim p(\cdot|s, a)} (V_{h+1}^\pi(s')) \) denotes the minimum value of random variable \( V_{h+1}^\pi(s') \) with \( s' \sim p(\cdot|s, a) \). From Eq. (4), we see that

\[
Q_h^\pi(s, a) = \min_{(s_t, a_t) \sim \pi} \left[ \sum_{t=h}^{H} r(s_t, a_t) \right] \left| s_h = s, a_h = a, \pi \right.,
\]

Thus, \( Q_h^\pi(s, a) \) and \( V_h^\pi(s) \) become the minimum possible cumulative reward under policy \( \pi \), starting from \( (s, a) \) and \( s \) at step \( h \), respectively. The optimal policy \( \pi^* \) maximizes the minimum possible cumulative reward (optimizes the worst path) for all starting states and steps. Formally, \( \pi^* \) gives the optimal value \( V_h^\pi(s) = \max_{\pi} V_h^\pi(s) \) for all \( s \in S \) and \( h \in [H] \).
Theorem 1 (Regret Upper Bound). With probability at least $1 - \delta$, the regret of algorithm ICVaR-RM is bounded by

$$O\left( \frac{HS\sqrt{KHA}}{\alpha} \log \left( \frac{KHS}{\delta} \right) \right),$$

where $w_{\pi,h}(s)$ denotes the probability of visiting state $s$ at step $h$ under policy $\pi$.

Remark 1. The factor $\min_{\pi,h,s: w_{\pi,h}(s)>0} w_{\pi,h}(s)$ stands for the minimum probability of visiting an available state under any feasible policy. Note that (i) The minimum operator is only taken over the feasible policies under which $s$ is available. Thus this factor will never be zero. (ii) The factor also exists in the lower bound (see Section 4.2), which characterizes the intrinsic hardness of Iterated CVaR RL, i.e., there exist critical states that are difficult to reach. Compared to existing risk-sensitive RL Chow et al. [2017], Fei et al. [2020, 2021a,b] which either consider the exponential utility criterion and have an exponential regret in $H$, or consider the CVaR constraint and do not have regret guarantees, for Iterated CVaR RL, we provide a polynomial regret in all problem parameters and demonstrate the sample efficiency.

Challenges and Novelty in Regret Analysis. The regret analysis for Iterated CVaR RL faces several unique challenges. First of all, in contrast to prior RL analysis Jaksch et al. [2010], Dann et al. [2017], Azar et al. [2017], Zanette and Brunskill [2019] where the contribution of a state to the regret is proportional to its visitation probability, the Iterated CVaR criterion places more emphasis on risky but hard-to-reach states, which are difficult to learn. Second, since the value function is measured by the tail reward rather than the expectation, the regret is not simply a summation of the estimation error over all state-action pairs under the standard visitation distribution. To tackle these obstacles, we develop novel analytical techniques, to bound the change of CVaR due to the value function shift and decompose regret into estimation error via a distorted visitation distribution. Below we present a proof sketch for Theorem 1.

Proof sketch of Theorem 1. First, we introduce a key inequality (Eq. (5)) to bound the change of CVaR when the true value function shifts to an optimistic one. Let $\beta^{\alpha,V}(\cdot|s,a) \in \mathbb{R}^S$ denote the assigned weights on $V(s')$ when computing $\text{CVaR}_{s' \sim \rho(\cdot|s,a)}(V(s'))$, which satisfies $\sum_{s' \in S} \beta^{\alpha,V}(s'|s,a) \cdot V(s') = \text{CVaR}_{s' \sim \rho(\cdot|s,a)}(V(s'))$. Intuitively, $\beta^{\alpha,V}(\cdot|s,a)$ is a distorted distribution of transition distribution $\rho(\cdot|s,a)$, which gives more weights to bad successor states. Then, for any $(s,a)$, optimistic and true value functions $V, \hat{V}$ such that $\hat{V}(s') \geq V(s')$ for any $s' \in S$, it holds that

$$\text{CVaR}_{s' \sim \rho(\cdot|s,a)}^\alpha(\hat{V}(s')) - \text{CVaR}_{s' \sim \rho(\cdot|s,a)}^\alpha(V(s')) \leq \beta^{\alpha,V}(\cdot|s,a)\top (\hat{V} - V).$$

Eq. (5) implies that the gap of CVaR between the optimistic and true value functions can be bounded by their value deviation under a distorted transition distribution, which serves as the basis of our recurrent regret decomposition.

Now, using Eq. (5) and the fact that $\hat{V}_h^k$ is an optimistic estimate of $V_h^\pi$, we decompose the regret as

$$\hat{V}_1^k(s^*_h) - V_1^\pi(s^*_h) = \frac{H}{\alpha} \sqrt{\frac{L}{n_k(s^*_h,a^*_h)}} + \text{CVaR}_{s' \sim \rho(\cdot|s^*_h,a^*_h)}(\hat{V}_2^k(s')) - \text{CVaR}_{s' \sim \rho(\cdot|s^*_h,a^*_h)}(V_2^k(s')) + \text{CVaR}_{s' \sim \rho(\cdot|s^*_h,a^*_h)}(\hat{V}_2^k(s')) - \text{CVaR}_{s' \sim \rho(\cdot|s^*_h,a^*_h)}(V_2^k(s'))$$

$$\leq \frac{H}{\alpha} \sqrt{\frac{L}{n_k(s^*_h,a^*_h)}} + \frac{H}{\alpha} \sqrt{\frac{SL}{n_k(s^*_h,a^*_h)}} + \beta^{\alpha,V}(\cdot|s^*_h,a^*_h)\top (\hat{V}_2^k - V_2^\pi)$$

$$\leq \sum_{h=1}^H \sum_{(s,a)} w_{k,h} \text{CVaR}_{\alpha,V}(\cdot|s,a) \cdot \frac{H\sqrt{L} + H\sqrt{SL}}{\alpha \sqrt{n_k(s,a)}}. \tag{6}$$

Here $w_{k,h} \text{CVaR}_{\alpha,V}(\cdot|s,a)$ denotes the distorted visitation probability of $(s,a)$ under the CVaR criterion. Inequality (a) uses the concentration of CVaR and Eq. (5), and inequality (b) follows from a recurrent
application of (a). Eq. (6) decomposes the regret into the estimation error over all state-action pairs under the distorted visitation distribution, which resolves the challenges due to additional focus on risky states under Iterated CVaR. Finally, summing Eq. (6) over all episodes $k$, and computing the ratio of distorted visitation probability over standard visitation probability, we obtain Theorem 1.

4.2 Regret Lower Bound

We now present a regret lower bound to demonstrate the optimality of algorithm ICVaR-RM.

**Theorem 2 (Regret Lower Bound).** There exists an instance of Iterated CVaR RL-RM, for which the regret of any algorithm is at least

$$\Omega \left( H \min_{\pi, h, s} \frac{AK}{\alpha} \wedge w_{n, h}(s) \right).$$

**Remark 2.** Theorem 2 demonstrates that the factor $\min_{\pi, h, s} \wedge w_{n, h}(s)$ in our regret upper bound (Theorem 1) is inevitable in general, and reveals the intrinsic hardness of Iterated CVaR RL, i.e., any algorithm must suffer a regret due to exploring risky but hard-to-reach states. This lower bound also validates that ICVaR-RM is optimal with respect to $K$.

**Hard Instances.** In the lower bound analysis, we construct an instance with a hard-to-reach bandit state (which has an optimal action and multiple sub-optimal actions), and show that this state is critical to minimizing the regret, but difficult for any algorithm to learn. As shown in Figure 1, we consider an MDP with $A$ actions, and $n$ regular states $s_1, \ldots, s_n$ and three absorbing states $x_1, x_2, x_3$. The reward function $r(s, a)$ depends only on the states, i.e., $s_1, \ldots, s_n$ generate zero reward and $x_1, x_2, x_3$ generate rewards 1, 0.8 and 0.2, respectively. Under all actions, state $s_1$ transitions to $s_2, x_1, x_2, x_3$ with probabilities $\alpha, 1 - 3\alpha, \alpha$ and $\alpha$, respectively, where $\alpha$ is the risk level, and state $s_i$ ($2 \leq i \leq n - 1$) transitions to $s_i+1, x_1$ with probabilities $\alpha$ and $1 - \alpha$, respectively. For the bandit state $s_n$, under the optimal action, $s_n$ transitions to $x_2, x_3$ with probabilities $1 - \alpha + \eta$ and $\alpha - \eta$, respectively. Under sub-optimal actions, $s_n$ transitions to $x_2, x_3$ with probabilities $1 - \alpha$ and $\alpha$.

In this MDP, under the Iterated CVaR criterion, $V_{n+1}$ mainly depends on the path $s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_n \rightarrow x_2/x_3$, and especially on the action choice in the bandit state $s_n$. However, when $\alpha$ is small, it is difficult for any policy to reach (and collect information from) $s_n$. Thus, to learn the optimal action in $s_n$, any algorithm must suffer a regret dependent on the probability of visiting $s_n$, which is exactly the minimum visitation probability of any state under a feasible policy $\min_{\pi, h, s} \wedge w_{n, h}(s) \geq 0 w_{n, h}(s)$.

5 Iterated CVaR RL with Best Policy Identification

In this section, we turn to best policy identification (Iterated CVaR RL-BPI). We design an efficient algorithm ICVaR-BPI, and establish rigorous sample complexity upper and lower bounds. To our best knowledge, these are the first sample complexity results for risk-sensitive RL.

5.1 Algorithm ICVaR-BPI and Sample Complexity Upper Bound

Algorithm ICVaR-BPI (Algorithm 2) constructs optimistic and pessimistic value functions, estimation error, and a hypothesized optimal policy in each episode, and returns the hypothesized optimal policy when the estimation error shrinks within $\varepsilon$. Specifically, in each episode $k$, ICVaR-BPI calculates the empirical CVaR for values of next states $\text{CVaR}_{\alpha}^{\pi^k}(V_{h+1}(s'))$, and exploration bonuses $H \frac{\beta k}{n}(s, a)$ and $\beta k \frac{\gamma}{n}(s, a)$, to establish the optimistic and pessimistic Q-value functions $Q^k_h(s, a)$ and $Q^k_h(s, a)$, respectively. ICVaR-BPI further maintains a hypothesized optimal policy $\pi^k$, which is greedy with respect to $Q^k_h$. Let $\beta k \frac{\gamma}{n}(V_{h+1}(s'))$ denote the assigned weights on $V_{h+1}(s')$ when computing $\text{CVaR}_{\alpha}^{\pi^k}(V_{h+1}(s'))$, which satisfies
We now present a lower bound for Iterated CVaR RL-BPI. We say algorithm which is the minimum probability of visiting an available state over all feasible policies. This sample complexity result also contains the factor (Theorem 3), and reveals intrinsic hardness of Iterated CVaR RL, i.e., one needs to spend a number

\[ \sum_{s' \in S} \beta^{k+1}(s'|s,a) \cdot \mathbb{V}_{k+1}(s') = \text{CVaR}_{s' \sim \rho^k(s,a)}(\mathbb{V}_{k+1}(s')) \]  

Then, ICVaR-BPI computes the estimation error \( G_{k}(s,a) \) and \( J_k(s) \) using the assigned weights \( \beta^{k+1}(s'|s,a) \) for next states under CVaR. Once the estimation error \( J_k(s) \) shrinks within the accuracy parameter \( \varepsilon \), ICVaR-BPI returns the hypothesized optimal policy \( \pi^k \).

The sample complexity of algorithm ICVaR-BPI is presented as follows.

**Theorem 3** (Sample Complexity Upper Bound). The number of trajectories used by algorithm ICVaR-BPI to return an \( \varepsilon \)-optimal policy with probability at least \( 1 - \delta \) is bounded by

\[
O \left( \frac{H^3 S^2 A}{\varepsilon^2 \alpha^2 \min_{\pi,h,s: w_{\pi,h}(s) > 0} w_{\pi,h}(s)} \log \left( \frac{SAH}{\delta} \right) \right).
\]

As in Theorem 1, this sample complexity result also contains the factor \( \min_{\pi,h,s: w_{\pi,h}(s) > 0} w_{\pi,h}(s) \), which is the minimum probability of visiting an available state over all feasible policies. This factor also exists in the lower bound and is indispensable in general (see Section 5.2). Theorem 3 corroborates the sample efficiency of ICVaR-BPI, i.e., uses only polynomial trajectories in \( K, H, S, A \) to identify a near-optimal policy.

### 5.2 Sample Complexity Lower Bound

We now present a lower bound for Iterated CVaR RL-BPI. We say algorithm \( \mathcal{A} \) is \((\delta, \varepsilon)\)-correct if \( \mathcal{A} \) returns an \( \varepsilon \)-optimal policy \( \hat{\pi} \) such that \( \mathbb{V}_{1}^*(s_1) \geq \mathbb{V}_{1}^*(s_1) - \varepsilon \) with probability \( 1 - \delta \).

**Theorem 4** (Sample Complexity Lower Bound). There exists an instance of Iterated CVaR RL-BPI, for which the number of trajectories used by any \((\delta, \varepsilon)\)-correct algorithm is at least

\[
\Omega \left( \frac{H^2 A}{\varepsilon^2 \alpha \min_{\pi,h,s: w_{\pi,h}(s) > 0} w_{\pi,h}(s)} \log \left( \frac{1}{\delta} \right) \right).
\]

Theorem 4 corroborates the tightness of the factor \( \min_{\pi,h,s: w_{\pi,h}(s) > 0} w_{\pi,h}(s) \) in the upper bound (Theorem 3), and reveals intrinsic hardness of Iterated CVaR RL, i.e., one needs to spend a number of trajectories on exploring critical but hard-to-reach states in order to identify an optimal policy.
We design a simple yet efficient algorithm \textbf{MaxWP}, in which case the agent aims to maximize the minimum possible cumulative reward.

**Algorithm 3 MaxWP**

1: **Input:** \( \delta, \delta' := \frac{\delta}{2}, L := \log\left(\frac{SA}{\delta}\right), \hat{V}\!_{H+1}^k(s) = 0 \) for any \( k > 0 \) and \( s \in \mathcal{S} \).
2: **for** \( k = 1, 2, \ldots, K \) **do**
3: **for** \( h = H, H - 1, \ldots, 1 \) **do**
4: **for** \( s \in \mathcal{S} \) **do**
5: **for** \( a \in \mathcal{A} \) **do**
6: \( \hat{Q}_h^k(s, a) \leftarrow r(s, a) + \min_{s' \sim \hat{p}^k(·|s, a)} (\hat{V}\!_{h+1}^k(s')) \)
7: **end for**
8: \( \hat{V}\!_h^k(s) \leftarrow \max_{a \in \mathcal{A}} \hat{Q}_h^k(s, a) \)
9: \( \pi_h^k(s) \leftarrow \arg\max_{a \in \mathcal{A}} \hat{Q}_h^k(s, a) \)
10: **end for**
11: **end for**
12: Play the episode \( k \) with policy \( \pi^k \), and update \( n_{k+1}(s, a) \) and \( \hat{p}^{k+1}(s'|s, a) \)
13: **end for**

6 Worst Path RL

In this section, we investigate an interesting limiting case of Iterated CVaR RL when \( \alpha \downarrow 0 \), called Worst Path RL, in which case the agent aims to maximize the minimum possible cumulative reward.

Note that the Worst Path RL problem has the \textit{unique feature} that the value function (Eq. (4)) concerns only the minimum value of successor states and are independent of transition probabilities. Therefore, as long as we learn the connectivity among states, we can perform a planning to compute the optimal policy. Yet, this feature does not make the Worst Path RL problem trivial, because it is difficult to distinguish whether a successor state is hard to reach or does not exist. As a result, a careful scheme is needed to both explore undetected successor states and exploit observations to minimize regret.

6.1 Algorithm MaxWP and Regret Upper Bound

We design a simple yet efficient algorithm \textbf{MaxWP} (Algorithm 3), which carefully combines the exploration of critical but hard-to-reach successor states and the exploitation of current best actions. Specifically, in each episode \( k \), MaxWP constructs empirical Q-value function \( \hat{Q}_h^k(s, a) \) and value function \( \hat{V}\!_h^k(s) \) using the lowest value of next states \( \min_{s' \sim \hat{p}^k(·|s, a)} (\hat{V}\!_{h+1}^k(s')) \), and maintains a greedy policy \( \pi_h^k(s) \) with respect to \( \hat{Q}_h^k(s, a) \). Then, MaxWP executes policy \( \pi_h^k(s) \) in episode \( k \), and updates the number of visitations \( n_{k+1}(s, a) \) and empirical transition distribution \( \hat{p}^{k+1}(s'|s, a) \).

The intuition behind MaxWP is as follows. Since the Q-value function for Worst Path RL use the min operator, if the Q-value function is not accurately estimated, it can only be over-estimated (not under-estimated). If over-estimation happens, MaxWP will be exploring an over-estimated action and urging its empirical Q-value to get back to its true Q-value. Otherwise, if the Q-value function is already accurate, MaxWP just selects the optimal action. In other words, MaxWP combines the exploration of over-estimated actions (which lead to undetected bad successor states) and exploitation of current best actions.

Below we provide the regret guarantee for algorithm MaxWP.

**Theorem 5.** With probability at least \( 1 - \delta \), the regret of algorithm MaxWP is bounded by

\[
O\left( \sum_{(s, a) \in \text{supp}(p(·|s, a))} \min_{\pi: V_\pi(s, a) > 0} v_\pi(s, a) \cdot \min_{s' \in \text{supp}(p(·|s, a))} p(s'|s, a) \log \left( \frac{SA}{\delta} \right) \right),
\]

where \( v_\pi(s, a) \) denotes the probability that \( (s, a) \) is visited at least once in an episode under policy \( \pi \).

**Remark 3.** The factor \( \min_{\pi: V_\pi(s, a) > 0} v_\pi(s, a) \) stands for the minimum probability of visiting \( (s, a) \) at least once in an episode over all feasible policies, and \( \min_{s' \in \text{supp}(p(·|s, a))} p(s'|s, a) \) denotes the minimum transition probability over all successor states of \( (s, a) \). We will show that these two factors also exist in the lower bound (discussed in Section 6.2), and are unavoidable in general. Theorem 5 exhibits that algorithm MaxWP enjoys a constant regret, since it effectively utilizes and adapts to the feature of Worst Path RL and efficiently explores the connectivity among states.
6.2 Regret Lower Bound

We now establish a regret lower bound for Worst Path RL. To exclude trivial instance-specific algorithms and formally state our lower bound, we first define an \( o(K) \)-consistent algorithm as an algorithm which guarantees an \( o(K) \) regret on any instance of Worst Path RL.

**Theorem 6.** There exists an instance of Worst Path RL, for which the regret of any \( o(K) \)-consistent algorithm is at least

\[
\Omega \left( \max_{(s,a)}: \exists h, a \neq \pi^*(s) \min_{\pi: v_\pi(s,a) > 0} \Delta_{\min} \cdot \min_{s' \in \text{supp}(p(\cdot|s,a))} p(s'|s,a) \right),
\]

where \( \max_{(s,a)}: \exists h, a \neq \pi^*(s) \) takes the maximum over all \((s,a)\) such that \(a\) is sub-optimal in state \(s\) at some step, and \( \Delta_{\min} \) denotes the minimum regret that a sub-optimal policy must suffer in an episode.

The insight behind this lower bound is as follows: For a critical but hard-to-reach state \(s\), any \(o(K)\)-consistent algorithm must explore all actions \(a\) in state \(s\), in order to detect their induced successor states \(s'\) and distinguish between the optimal and sub-optimal actions. Hence, this process incurs a regret dependent on factors \(\min_{\pi: v_\pi(s,a) > 0} v_\pi(s,a)\) and \(\min_{s' \in \text{supp}(p(\cdot|s,a))} p(s'|s,a)\).

7 Conclusion

In this paper, we investigate a novel Iterated CVaR RL problem with the Regret Minimization and Best Policy Identification. We design two efficient algorithms \( \text{ICVar-RM} \) and \( \text{ICVar-BPI} \), and provide matching regret/sample complexity upper and lower bounds with respect to \(K\). We also investigate a limiting case called Worst Path RL, and propose a simple and efficient algorithm \( \text{MaxWP} \) with rigorous regret guarantees. Novel techniques are developed to analyze the change of CVaR due to the value function shift and decompose the regret via a distorted visitation distribution.

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Appendix

A Comparison with CVaR MDP

In this section, we compare our Iterated CVaR MDP formulation with CVaR MDP Bäuerle and Ott [2011], Haskell and Jain [2015], Chow et al. [2015]. The objective of CVaR MDP is to maximize the CVaR value of the total reward, defined as

$$\max_{\pi} \text{CVaR}_\alpha^{\pi}(s_h, a_h) \sim p, \pi \left( \sum_{h=1}^{H} r(s_h, a_h) \right).$$

Below we present an illustrating example of financial investment to show that, compared to the CVaR MDP, our Iterated CVaR MDP has a stronger incentive to avoid getting into catastrophic states.

As shown in Figure 2, starting from an initial state $s_1$, we choose one of two multi-stage financial products to invest, denoted by actions $a_1, a_2$. In state $s_1$, with action $a_1, a_2$, we transition to $s_2, s_3$ deterministically, respectively. In states $s_2, \ldots, s_13$, actions have the same transition distribution. We receive a reward only at the final step, and the value of a reward stands for the ratio of current money over the invested money. The transition probabilities and rewards are specified in the figure. Let risk level $\alpha = 0.05$. We can see that, under the CVaR criterion, we will choose $a_2$ which has a higher CVaR value. However, under the Iterated CVaR criterion, we will select $a_1$ which generates a higher Iterated CVaR value. The intuition is that Iterated CVaR places more weights on the terrible states $s_8, s_11$, where we will lose all money. Hence, compared to the CVaR MDP, the Iterated CVaR MDP has a stronger effect on preventing us from getting into catastrophic states.

In addition, from the computation perspective, the CVaR MDP does not have Markovian optimal policies and is computationally intractable even for planning Bäuerle and Ott [2011]. Existing works for CVaR MDP Haskell and Jain [2015], Chow et al. [2015] mainly investigate how to design approximate planning algorithms with convergence analysis, rather than developing online RL algorithms with regret guarantees as in our work.

B Proofs for Iterated CVaR RL with Regret Minimization

In this section, we present the proofs of regret upper and lower bounds (Theorems 1, 2) for Iterated CVaR RL-RM.

B.1 Proofs for Regret Upper Bound

In order to prove the regret upper bound (Theorem 1), we first introduce several important lemmas (Lemmas 1-8) and define concentration event $E$.

**B.1.1 Concentration**

**Lemma 1** (Concentration for $V^*$). It holds that

$$\Pr \left[ \left| \text{CVaR}_{\alpha}^{\phi^k(\cdot|s,a)}(V^*_h(s')) - \text{CVaR}_{\alpha}^{\phi^k(\cdot|s,a)}(V^*_h(s')) \right| \leq \frac{H}{\alpha} \sqrt{\frac{\log(KHSA)}{n_k(s,a)}} \right] \geq 1 - 2\delta'. $$

$$\forall k \in [K], \forall h \in [H], \forall (s,a) \in S \times A \right] \geq 1 - 2\delta'. $$

**Proof of Lemma 1.** Using Brown’s inequality Brown [2007] (Theorem 2 in Thomas and Learned-Miller [2019]) and a union bound over $(s,a) \in S \times A$ and $n_k(s,a) \in [KH]$, we can obtain this lemma. \qed

\[\text{Some of prior works Bäuerle and Ott [2011], Chow et al. [2015] also interpret the reward as cost, and consider to minimize the CVaR value of the total cost.}\]
Lemma 2 (Concentration for any \( V \)). It holds that
\[
\left| \text{CVaR}_{s' \sim \hat{p}^k(s,a)}^\alpha \left( V(s') \right) - \text{CVaR}_{s' \sim p(s,a)}^\alpha \left( V(s') \right) \right| \leq \frac{2H}{\alpha} \sqrt{2S \log \left( \frac{KHSA}{\delta} \right) n_k(s,a)},
\]
\[\forall V(\cdot) : S \rightarrow [0, H], \forall k \in [K], \forall (s,a) \in S \times A \]
\[\geq 1 - 2\delta'.\]

Proof of Lemma 2. As shown in Figure 3, we sort all states \( s \in S \) by \( V(s) \) in ascending order (from the left to the right). Add a virtual line at the \( \alpha \)-quantile line. Let \( \mu(s'|s,a) \) and \( \hat{p}^k(s'|s,a) \) denote the truncated weights imposed on support \( V(s') \) when computing \( \text{CVaR}_{s' \sim p(s,a)}^\alpha \left( V(s') \right) \) and \( \text{CVaR}_{s' \sim \hat{p}^k(s,a)}^\alpha \left( V(s') \right) \), respectively, i.e.,
\[
\text{CVaR}_{s' \sim p(s,a)}^\alpha \left( V(s') \right) = \sum_{s' \in S} \mu(s'|s,a) \cdot V(s'),
\]
\[
\text{CVaR}_{s' \sim \hat{p}^k(s,a)}^\alpha \left( V(s') \right) = \sum_{s' \in S} \hat{p}^k(s'|s,a) \cdot V(s').
\]

Fix the support \( V(\cdot) \). When the transition probabilities changes from \( p(\cdot|s,a) \) to \( \hat{p}^k(\cdot|s,a) \), the \( \alpha \)-quantile line shifts to the left or right side. Denote the \( \alpha \)-quantile line before and after shift by Line 1 and Line 2, respectively. Then, denote the states on the left of Lines 1,2 by \( S_{left} \), denote the states between Lines 1,2 by \( S_{middle} \), and denote the states on the right of Lines 1,2 by \( S_{right} \).

It holds that
\[
\sum_{s' \in S_{left}} |\hat{p}^k(s'|s,a) - \mu(s'|s,a)| = \sum_{s' \in S_{middle}} |\hat{p}^k(s'|s,a) - \mu(s'|s,a)|,
\]
\[
\sum_{s' \in S_{left}} |\hat{p}^k(s'|s,a) - \mu(s'|s,a)| = \sum_{s' \in S_{right}} |\hat{p}^k(s'|s,a) - p(s'|s,a)|,
\]
\[
\hat{p}^k(s'|s,a) = \mu(s'|s,a) = 0, \ \forall s' \in S_{right}.
\]

Thus, we have
\[
\left| \text{CVaR}_{s' \sim \hat{p}^k(s,a)}^\alpha \left( V(s') \right) - \text{CVaR}_{s' \sim p(s,a)}^\alpha \left( V(s') \right) \right|
\]
\[
= \left| \sum_{s' \in S} (\hat{p}^k(s'|s,a) - \mu(s'|s,a)) \cdot V(s') \right|
\]
\[
\leq \sum_{s' \in S} \left| \hat{p}^k(s'|s,a) - \mu(s'|s,a) \right| \cdot H
\]
\[
= \frac{2 \sum_{s' \in S_{left}} \left| \hat{p}^k(s'|s,a) - \mu(s'|s,a) \right| \cdot H}{\alpha}
\]
By a union bound over $h$, we have that with probability at least $1 - 2\delta'$, for any $k \in [K]$ and $(s, a) \in S \times A$, 

$$\sum_{s' \in S} |\hat{p}^k(s'|s, a) - p(s'|s, a)| \leq \sqrt{\frac{2S \log \left( \frac{KHSA}{\delta'} \right)}{n_k(s, a)}}.$$  

(8)

Plugging Eq. (8) into Eq. (7), we have that with probability at least $1 - 2\delta'$, for any $k \in [K]$, $(s, a) \in S \times A$ and function $V : S \to [0, H]$, 

$$\left| \text{CVaR}_{\alpha}^{(\hat{V}^k)}(V(s')) - \text{CVaR}_{\alpha}^{(p(\cdot|s, a))}(V(s')) \right| \leq \frac{2H}{\alpha} \sqrt{\frac{2S \log \left( \frac{KHSA}{\delta'} \right)}{n_k(s, a)}}.$$ 

For any $k \in [K]$, $h \in [H]$ and $(s, a) \in S \times A$, let $n_{kh}(s, a)$ denote the number of times $(s, a)$ was visited at step $h$ up to episode $k$, and let $n_k(s, a) := \sum_{h=1}^{H} n_{kh}(s, a)$ denote the number of times $(s, a)$ was visited up to episode $k$.

**Lemma 3 (Concentration of Visitation).** It holds that 

$$\Pr \left[ n_k(s, a) \geq \frac{1}{2} \sum_{k' < k} H \log \left( \frac{HSA}{\delta'} \right), \forall k, (s, a) \in S \times A \right] \geq 1 - \delta'.$$

**Proof of Lemma 3.** Applying Lemma F.4 in Dann et al. [2017], we have that for a fixed $h \in [H]$ 

$$\Pr \left[ n_{kh}(s, a) \geq \frac{1}{2} \sum_{k' < k} w_{k'\cdot h}(s, a) - \log \left( \frac{HSA}{\delta'} \right), \forall k, (s, a) \in S \times A \right] \geq 1 - \frac{\delta'}{H}.$$ 

By a union bound over $h \in [H]$, we have 

$$\Pr \left[ n_k(s, a) \geq \frac{1}{2} \sum_{k' < k} \sum_{h=1}^{H} w_{k'\cdot h}(s, a) - H \log \left( \frac{HSA}{\delta'} \right) \right] \geq 1 - \delta'.$$

\[\square\]
Concentration Events. For ease of notation, we summarize the concentration events which will be used in our proof as follows:

\[
\mathcal{E}_1 := \left\{ \left| \text{CVaR}^\alpha_{s', \rho^h(\cdot | s, a)}(V^*_h(s')) - \text{CVaR}^\alpha_{s', \rho^h(\cdot | s, a)}(V^*_h(s')) \right| \leq \frac{H}{\alpha} \sqrt{\frac{\log(KHSA)}{n_k(s, a)}} \right\},
\]

\[
\forall k \in [K], \forall h \in [H], \forall (s, a) \in S \times A \right\}
\]

\[
\mathcal{E}_2 := \left\{ \left| \text{CVaR}^\alpha_{s', \rho^h(\cdot | s, a)}(V'(s')) - \text{CVaR}^\alpha_{s', \rho^h(\cdot | s, a)}(V(s')) \right| \leq \frac{2H}{\alpha} \sqrt{\frac{2S \log(KHSA)}{n_k(s, a)}} \right\},
\]

\[
\forall V(\cdot) : S \mapsto [0, H], \forall k \in [K], \forall (s, a) \in S \times A \right\}
\]

\[
\mathcal{E}_3 := \left\{ n_k(s, a) \geq \frac{1}{2} \sum_{k' < k} \sum_{h=1}^H w_{k' h}(s, a) - H \log \left( \frac{HSA}{\delta'} \right), \forall k, \forall (s, a) \in S \times A \right\}
\]

\[
\mathcal{E} := \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3
\]

**Lemma 4.** Letting \( \delta' = \frac{\delta}{S} \), it holds that

\[
\Pr[\mathcal{E}] \geq 1 - \delta.
\]

**Proof of Lemma 4.** This lemma can be obtained by combining Lemmas 1, 2, 3.

\[
\square
\]

**B.1.2 Optimism, Visitation and CVaR Gap**

Let \( L := \log \left( \frac{KHSA}{\delta'} \right) \).

**Lemma 5 (Optimism).** Suppose that event \( \mathcal{E} \) holds. Then, for any \( k \in [K], h \in [H] \) and \( s \in S \), we have

\[
\bar{V}^*_h(s) \geq V^*_h(s).
\]

**Proof of Lemma 5.** We prove this lemma by induction.

First, for any \( k \in [K], s \in S \), it holds that \( \bar{V}^*_h(s) = V^*_h(s) = 0 \).

Then, for any \( k \in [K], h \in [H] \) and \( (s, a) \in S \times A \),

\[
\bar{Q}^k_h(s, a) \geq r(s, a) + \text{CVaR}^\alpha_{s', \rho^h(\cdot | s, a)}(\bar{V}^k_{h+1}(s')) + \frac{H}{\alpha} \sqrt{\frac{L}{n_k(s, a)}}
\]

\[
(a) \geq r(s, a) + \text{CVaR}^\alpha_{s', \rho^h(\cdot | s, a)}(V^*_h(s')) + \frac{H}{\alpha} \sqrt{\frac{L}{n_k(s, a)}}
\]

\[
(b) \geq r(s, a) + \text{CVaR}^\alpha_{s', \rho^h(\cdot | s, a)}(V^*_h(s')) = Q^k_h(s, a),
\]

where (a) uses the induction hypothesis and (b) comes from Lemma 1.

Thus, we have

\[
\bar{V}^*_h(s) \geq Q^k_h(s, \pi^*_h(s)) \geq Q^k_h(s, \pi^*_h(s)) = V^*_h(s),
\]

which concludes the proof.

\[
\square
\]

For any episode \( k \), define the set of state-action pairs

\[
\mathcal{L}_k := \left\{ (s, a) \in S \times A : \frac{1}{H} \sum_{k' < k} \sum_{h=1}^H w_{k' h}(s, a) \geq H \log \left( \frac{HSA}{\delta'} \right) + H \right\}.
\]

Let \( \bar{n}_k(s, a) := \sum_{k' \leq k} \sum_{h=1}^H w_{k' h}(s, a) \).
Lemma 6 (Sufficient Visitation). Suppose that event $\mathcal{E}$ holds. Then, for any $k$ and $(s, a) \in \mathcal{L}_k$,

$$n_k(s, a) \geq \frac{1}{4} \sum_{k' \leq k} \sum_{h=1}^{H} w_{k'h}(s, a) = \frac{1}{4} \bar{n}_k(s, a).$$

Proof of Lemma 6. This proof is the same as that of Lemma 6 in Zanette and Brunskill [2019]. Using Lemma 3 and the definition of $\mathcal{L}_k$, we have

$$n_k(s, a) \geq \frac{1}{4} \sum_{k' \leq k} \sum_{h=1}^{H} w_{k'h}(s, a) - H \log \left( \frac{HSA}{\delta'} \right)$$

$$= \frac{1}{4} \sum_{k' \leq k} \sum_{h=1}^{H} w_{k'h}(s, a) + \frac{1}{4} \sum_{k' \leq k} \sum_{h=1}^{H} w_{k'h}(s, a) - H \log \left( \frac{HSA}{\delta'} \right)$$

$$\geq \frac{1}{4} \sum_{k' \leq k} \sum_{h=1}^{H} w_{k'h}(s, a) + H$$

$$\geq \frac{1}{4} \sum_{k' \leq k} \sum_{h=1}^{H} w_{k'h}(s, a) + \sum_{h=1}^{H} w_{kh}(s, a)$$

$$= \frac{1}{4} \sum_{k' \leq k} \sum_{h=1}^{H} w_{k'h}(s, a)$$

$$= \frac{1}{4} \bar{n}_k(s, a)$$

For any $k \in [K]$, $h \in [H]$ and $(s, a) \in \mathcal{S} \times \mathcal{A}$, let $w_{kh}^{\text{CVaR}, \alpha, V^k}(s, a)$ denote the distorted probability (weight) of visiting $(s, a)$ at step $h$ in episode $k$ under the CVaR metric. It holds that $\sum_{s, a} w_{kh}^{\text{CVaR}, \alpha, V^k}(s, a) = 1.$

Lemma 7 (Insufficient Visitation). It holds that

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{(s, a) \notin \mathcal{L}_k} w_{kh}^{\text{CVaR}, \alpha, V^k}(s, a) \leq \frac{8SAH}{\min_{\pi, h, s: w_{\pi, h, s}(s) > 0} w_{\pi, h, s}(s)} \log \left( \frac{HSA}{\delta'} \right),$$

Proof of Lemma 7. First, using the definition of $\mathcal{L}_k$, we have

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{(s, a) \notin \mathcal{L}_k} w_{kh}(s, a) = \sum_{k=1}^{K} \sum_{h=1}^{H} w_{kh}(s, a) \cdot \mathbb{1}\{(s, a) \notin \mathcal{L}_k\}$$

$$\leq \sum_{s, a} \left( 4H \log \left( \frac{HSA}{\delta'} \right) + 4H \right)$$

$$\leq 8SAH \log \left( \frac{HSA}{\delta'} \right)$$

Then, we have

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{(s, a) \notin \mathcal{L}_k} w_{kh}^{\text{CVaR}, \alpha, V^k}(s, a) = \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{(s, a) \notin \mathcal{L}_k} \frac{w_{kh}^{\text{CVaR}, \alpha, V^k}(s, a)}{w_{kh}(s, a)} \cdot w_{kh}(s, a) \cdot \mathbb{1}\{w_{kh}(s, a) \neq 0\}$$
We divide all states into three subsets $\mathcal{S}_{\text{asc}}, \mathcal{S}_{\text{des}}$, and $\mathcal{S}_{\text{unch}}$: $\delta_{\text{asc}} = \{s_1\}, \delta_{\text{des}} = \{s_2\}, \delta_{\text{unch}} = \{s_3\}$.

![Diagram](image)

**Figure 4:** Illustrating example for Lemma 8.

\[
\alpha \text{-quantile line}
\]

\[
\alpha \text{-quantile line}
\]

For any risk level $z \in (0, 1]$, function $V : \mathcal{S} \mapsto [0, H]$ and distribution $p(\cdot|s, a) \in \Delta_{\mathcal{S}}$, let $\beta_{\alpha, V}(\cdot|s, a)$ denote the distorted distribution (renormalized weights) on $\mathcal{S}$ when computing $\text{CVaR}_{s' \sim p(\cdot|s, a)}(V(s'))$, i.e.,

\[
\text{CVaR}_{s' \sim p(\cdot|s, a)}(V(s')) = \sum_{s' \in \mathcal{S}} \beta_{\alpha, V}(s'|s, a) \cdot V(s').
\]

**Lemma 8 (CVaR Gap due to Value Function Shift).** For any $(s, a) \in \mathcal{S} \times \mathcal{A}$ and functions $V, \bar{V} : \mathcal{S} \mapsto [0, H]$ such that $\bar{V}(s') \geq V(s')$ for any $s' \in \mathcal{S}$,

\[
\text{CVaR}_{s' \sim p(\cdot|s, a)}(\bar{V}(s')) - \text{CVaR}_{s' \sim p(\cdot|s, a)}(V(s')) \leq \beta_{\alpha, V}(\cdot|s, a) \top (\bar{V} - V)
\]

**Proof of Lemma 8.** As shown in Figure 4, let $\mu^V(s'|s, a), \mu^{\bar{V}}(s'|s, a)$ denote the truncated weights imposed on support $V(s'), \bar{V}(s')$ when computing $\text{CVaR}_{s' \sim p(\cdot|s, a)}(V(s'))$, $\text{CVaR}_{s' \sim p(\cdot|s, a)}(\bar{V}(s'))$, respectively. It holds that

\[
\text{CVaR}_{s' \sim p(\cdot|s, a)}(V(s')) = \sum_{s' \in \mathcal{S}} \frac{\mu^V(s'|s, a) \cdot V(s')}{\alpha} = \sum_{s' \in \mathcal{S}} \beta_{\alpha, V}(s'|s, a) \cdot V(s'),
\]

\[
\text{CVaR}_{s' \sim p(\cdot|s, a)}(\bar{V}(s')) = \sum_{s' \in \mathcal{S}} \frac{\mu^{\bar{V}}(s'|s, a) \cdot \bar{V}(s')}{\alpha} = \sum_{s' \in \mathcal{S}} \beta_{\alpha, V}(s'|s, a) \cdot \bar{V}(s').
\]

Add a virtual line at the $\alpha$-quantile for the shifted distribution (i.e., $\text{VaR}_{s' \sim p(\cdot|s, a)}(\bar{V}(s'))$), denoted by “the shifted $\alpha$-quantile line”.

We divide all states $\mathcal{S}$ into three subsets $\mathcal{S}_{\text{asc}}, \mathcal{S}_{\text{des}}$, and $\mathcal{S}_{\text{unch}}$ according to how $\mu^V(s'|s, a)$ changes to $\mu^{\bar{V}}(s'|s, a)$ when $V(s')$ shifts to $\bar{V}(s')$, as follows:
For any $s' \in S_{asc}$, $\mu^{\bar{V}}(s'|s,a) \leq \mu^{V}(s'|s,a)$, the rank of $s$ ascends, and $s$ lies at the middle or right of the shifted $\alpha$-quantile line.

For any $s' \in S_{des}$, $\mu^{\bar{V}}(s'|s,a) \geq \mu^{V}(s'|s,a)$, the rank of $s$ descends, and $s$ lies at the middle or left of the shifted $\alpha$-quantile line.

For any $s' \in S_{unch}$, $\mu^{\bar{V}}(s'|s,a) = \mu^{V}(s'|s,a)$, the rank of $s$ keeps unchanged, and $s$ lies at the right of the shifted $\alpha$-quantile line.

It holds that
\[
\sum_{s' \in S_{asc}} (\mu^{\bar{V}}(s'|s,a) - \mu^{V}(s'|s,a)) + \sum_{s' \in S_{des}} (\mu^{\bar{V}}(s'|s,a) - \mu^{V}(s'|s,a)) = 0. \tag{9}
\]

Then, we have
\[
\text{CVaR}^\alpha_{s' \sim p(\cdot|s,a)}(\bar{V}(s')) - \text{CVaR}^\alpha_{s' \sim p(\cdot|s,a)}(V(s'))
\]
\[
= \frac{1}{\alpha} \cdot \left( \sum_{s' \in S_{asc}} (\mu^{\bar{V}}(s'|s,a) \cdot \bar{V}(s') - \mu^{V}(s'|s,a) \cdot V(s')) + \sum_{s' \in S_{des}} (\mu^{\bar{V}}(s'|s,a) \cdot \bar{V}(s') - \mu^{V}(s'|s,a) \cdot V(s')) + \sum_{s' \in S_{unch}} (\mu^{\bar{V}}(s'|s,a) \cdot \bar{V}(s') - \mu^{V}(s'|s,a) \cdot V(s')) \right)
\]
\[
= \frac{1}{\alpha} \cdot \left( \sum_{s \in S} \mu^{V}(s'|s,a) \cdot (\bar{V}(s') - V(s')) - \sum_{s' \in S_{asc}} (\mu^{V}(s'|s,a) - \mu^{\bar{V}}(s'|s,a)) \cdot \bar{V}(s') + \sum_{s' \in S_{des}} (\mu^{V}(s'|s,a) - \mu^{\bar{V}}(s'|s,a)) \cdot \bar{V}(s') \right)
\]
\[
\leq \frac{1}{\alpha} \cdot \left( \sum_{s \in S} \mu^{V}(s'|s,a) \cdot (\bar{V}(s') - V(s')) - \min_{s' \in S_{asc}} \bar{V}(s') \cdot \sum_{s' \in S_{asc}} (\mu^{V}(s'|s,a) - \mu^{\bar{V}}(s'|s,a)) + \min_{s' \in S_{des}} \bar{V}(s') \cdot \sum_{s' \in S_{des}} (\mu^{V}(s'|s,a) - \mu^{\bar{V}}(s'|s,a)) \right)
\]
\[
= \frac{1}{\alpha} \cdot \sum_{s \in S} \mu^{V}(s'|s,a) \cdot (\bar{V}(s') - V(s'))
\]
\[
= \beta^{\alpha,V}(\cdot|s,a)^{\top} (\bar{V} - V),
\]

Here (a) is due to that for any $s' \in S_{asc}$, $\mu^{\bar{V}}(s'|s,a) \leq \mu^{V}(s'|s,a)$, and that for any $s \in S_{asc}$, $s' \in S_{des}$, it holds that $V(s) \geq V(s')$. (b) comes from Eq. (9).
B.1.3  Proof of Theorem 1

Proof of Theorem 1. Suppose that event \( \mathcal{E} \) holds. Then, for any \( k \in [K] \),

\[
V_1^*(s^k_1) - V_1^{\pi^k}(s^k_1) \leq (a) \Upsilon_1^*(s^k_1) - V_1^{\pi^k}(s^k_1)
\]

\[
= \text{CVaR}_{s' \sim \tilde{P}^k(|s^k_1, a^k_1)}^{\alpha} \left( \tilde{V}_2^k(s') \right) + \frac{H}{\alpha} \sqrt{\frac{L}{n_k(s^k_1, a^k_1)}} - \text{CVaR}_{s' \sim \tilde{P}^k(|s^k_1, a^k_1)}^{\alpha} \left( V_2^{\pi^k}(s') \right)
\]

\[
= \frac{H}{\alpha} \sqrt{\frac{L}{n_k(s^k_1, a^k_1)}} + \text{CVaR}_{s' \sim \tilde{P}^k(|s^k_1, a^k_1)}^{\alpha} \left( \tilde{V}_2^k(s') \right) - \text{CVaR}_{s' \sim \tilde{P}^k(|s^k_1, a^k_1)}^{\alpha} \left( V_2^{\pi^k}(s') \right)
\]

\[
+ \text{CVaR}_{s' \sim \tilde{P}^k(|s^k_1, a^k_1)}^{\alpha} \left( \tilde{V}_2^k(s') \right) - \text{CVaR}_{s' \sim \tilde{P}^k(|s^k_1, a^k_1)}^{\alpha} \left( V_2^{\pi^k}(s') \right)
\]

\[
\leq \frac{H}{\alpha} \sqrt{\frac{L}{n_k(s^k_1, a^k_1)}} + \frac{H}{\alpha} \sqrt{\frac{SL}{n_k(s^k_1, a^k_1)}} + \beta^{\alpha, V_2^{\pi^k}} \left( s^k_1, a^k_1 \right) \left( V_2^k - V_2^{\pi^k} \right)
\]

\[
\leq \sum_{h=1}^{H} \sum_{(s, a) \in \mathcal{L}_k} w_{kh}^{\text{CVaR}, \alpha, V_2^{\pi^k}} (s, a) \cdot \frac{H \sqrt{L} + H \sqrt{SL}}{\sqrt{z n_k(s, a)}} + \sum_{h=1}^{H} \sum_{(s, a) \notin \mathcal{L}_k} w_{kh}^{\text{CVaR}, \alpha, V_2^{\pi^k}} (s, a) \cdot 2H
\]

where (a) uses Lemma 5, (b) uses Lemmas 2, 8, and (c) is due to that the estimation error term \( \frac{H \sqrt{L} + H \sqrt{SL}}{\sqrt{z n_k(s, a)}} \) has a universal upper bound \( 2H \).

Since the second term in Eq. (10) can be bounded by Lemma 7, in the following, we analyze the first term. Summing the first term in Eq. (10) over \( k \in [K] \), we have

\[
\frac{K}{H} \sum_{h=1}^{H} \sum_{(s, a) \in \mathcal{L}_k} w_{kh}^{\text{CVaR}, \alpha, V_2^{\pi^k}} (s, a) \frac{H \sqrt{L} + H \sqrt{SL}}{\sqrt{z n_k(s, a)}}
\]

\[
\leq \left( \frac{H \sqrt{L} + H \sqrt{SL}}{\sqrt{\alpha}} \right) \sum_{h=1}^{H} \sum_{(s, a) \in \mathcal{L}_k} w_{kh}^{\text{CVaR}, \alpha, V_2^{\pi^k}} (s, a) \cdot \frac{H \sqrt{L} + H \sqrt{SL}}{\alpha} \cdot \sqrt{K H}
\]

\[
\leq \left( \frac{H \sqrt{L} + H \sqrt{SL}}{\sqrt{\alpha}} \right) \sqrt{K H} \sum_{h=1}^{H} \sum_{(s, a) \in \mathcal{L}_k} w_{kh}(s, a) \cdot \frac{w_{kh}(s, a)}{n_k(s, a)} \cdot \frac{H \sqrt{L} + H \sqrt{SL}}{\alpha} \cdot \sqrt{K H}
\]

\[
\leq \left( \frac{H \sqrt{L} + H \sqrt{SL}}{\sqrt{\alpha}} \right) \sqrt{K H} \sum_{h=1}^{H} \sum_{(s, a) \in \mathcal{L}_k} \min_{\tilde{w}_{\pi, h}(s, a) > 0} \frac{w_{\pi, h}(s, a)}{w_{\pi, h}(s)} \cdot \sqrt{SAL}
\]

\[
\leq \frac{2 H L \sqrt{KLH A}}{z} \sum_{h=1}^{H} \sum_{(s, a) \in \mathcal{L}_k} \min_{\tilde{w}_{\pi, h}(s) > 0} \frac{w_{\pi, h}(s)}{w_{\pi, h}(s)}
\]
The reward functions are as follows: For any state $S$, the state space is $\mathcal{L}$. Then, summing Eq. (10) over $k \in [K]$ and using Lemma 7, we have

$$R(K) = \sum_{k=1}^{K} \left( V^*(s^k) - V^\pi_k(s^k) \right)$$

$$\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{(s,a) \in \mathcal{L}_k} w_{k,h}^{\text{CVaR}}, V^\pi_k(s,a) \frac{H \sqrt{L} + H \sqrt{SL}}{z \sqrt{n_k(s,a)}}$$

$$+ \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{(s,a) \notin \mathcal{L}_k} w_{k,h}^{\text{CVaR}}, V^\pi_k(s,a) \cdot 2H$$

$$\leq \frac{2HS\sqrt{KHA}}{z} \log \left( \frac{KHA}{\delta'} \right) + \frac{16SAH^2}{z} \min_{\pi,h,s: w_{\pi,h}(s) > 0} \log \left( \frac{HSA}{\delta'} \right)$$

When $K$ is large enough, the first term dominates the bound, and thus we obtain Theorem 1.

### B.2 Proofs for Regret Lower Bound

In this subsection, we prove the regret lower bound (Theorem 2) for Iterated CVaR RL-RM.

**Proof of Theorem 2.** Consider the instance shown in Figure 5 (the same as Figure 1 in the main text):

The state space is $S = \{s_1, s_2, \ldots, s_n, x_1, x_2, x_3\}$, where $n = S - 3$ and $s_1$ is the initial state. Let $H > S$ and $0 < \alpha < \frac{1}{2}$.

The reward functions are as follows: For any $a \in \mathcal{A}$, $r(x_1, a) = 1$, $r(x_2, a) = 0.8$, $r(x_3, a) = 0.2$. For any $i \in [n]$ and $a \in \mathcal{A}$, $r(s_i, a) = 0$.

The transition distributions are as follows: For any $a \in \mathcal{A}$, $p(s_2 | s_1, a) = \alpha$, $p(x_1 | s_1, a) = 1 - 3\alpha$, $p(x_2 | s_1, a) = \alpha$ and $p(x_3 | s_1, a) = \alpha$. For any $i \in \{2, \ldots, n - 1\}$ and $a \in \mathcal{A}$, $p(s_i+1 | s_i, a) = \alpha$ and $p(x_1 | s_i, a) = 1 - \alpha$, $x_1$, $x_2$ and $x_3$ are absorbing states, i.e., for any $a \in \mathcal{A}$, $p(x_1 | x_1, a) = 1$, $p(x_2 | x_2, a) = 1$ and $p(x_3 | x_3, a) = 1$. Let $a_j$ be the optimal action in state $s_n$, which is uniformly drawn from $\mathcal{A}$. For the optimal action $a_j$, $p(x_2 | s_n, a_j) = 1 - \alpha + \eta$ and $p(x_3 | s_n, a_j) = \alpha - \eta$, where $\eta$ is a parameter which satisfies $0 < \eta < \alpha$ and will be specified later. For any suboptimal action $a \in \mathcal{A} \setminus \{a_j\}$, $p(x_2 | s_n, a) = 1 - \alpha$ and $p(x_3 | s_n, a) = \alpha$.
For any \( a_j \in \mathcal{A} \), let \( \mathbb{E}_j[\cdot] \) and \( \Pr_j[\cdot] \) denote the expectation and probability operators under the instance with \( a_j = a_j \). Let \( \mathbb{E}^{unif}[\cdot] \) and \( \Pr^{unif}[\cdot] \) denote the expectation and probability operators under the uniform instance where all actions \( a \in \mathcal{A} \) in state \( s_n \) have the same transition distribution, i.e., \( p(x_2|s_n, a) = 1 - \alpha \) and \( p(x_3|s_n, a) = \alpha \).

Fix an algorithm \( \mathcal{A} \). Let \( \pi^k \) denote the policy taken by \( \mathcal{A} \) in episode \( k \). Let \( N_{s_n, a_j} = \sum_{k=1}^{K} \mathbb{1} \{ \pi^k(s_n) = a_j \} \) denote the number of episodes that the policy chooses \( a_j \) in state \( s_n \). Let \( V_{s_n, a_j} \) denote the number of episodes that the algorithm \( \mathcal{A} \) visits \((s_n, a_j)\). Let \( w(s_n) \) denote the probability of visiting \( s_n \) in an episode (the probability of visiting \( s_n \) is the same for any policy). Then, it holds that \( \mathbb{E}[V_{s_n, a_j}] = w(s_n) \cdot \mathbb{E}[N_{s_n, a_j}] \).

According to the definitions of the value function and regret in CVaR RL, we have

\[
V_1^\ast(s_1) = \frac{(\alpha - \eta) \cdot 0.2(H - n - 1) + \eta \cdot 0.8(H - n - 1)}{\alpha} \\
V_1^\pi(s_1) = \frac{(\alpha - \eta) \cdot 0.2(H - n - 1) + \eta \cdot 0.8(H - n - 1)}{\alpha} \cdot \mathbb{1} \{ \pi(s_n) = a_j \} \\
+ 0.2(H - n - 1) \cdot (1 - \mathbb{1} \{ \pi(s_n) = a_j \})
\]

Thus,

\[
\mathbb{E}[\mathcal{R}(K)] = \frac{1}{A} \sum_{j=1}^{A} \sum_{k=1}^{K} \left( V_1^\ast(s_1) - V_1^\pi(s_1) \right) \\
= \frac{1}{A} \sum_{j=1}^{A} \frac{\eta}{\alpha} \cdot 0.6(H - n - 1) \left( K - \mathbb{E}_j[N_{s_n, a_j}] \right) \\
= 0.6(H - n - 1) \cdot \frac{\eta}{\alpha} \left( K - \frac{1}{A} \sum_{j=1}^{A} \mathbb{E}_j[N_{s_n, a_j}] \right)
\]  \quad (11)

For any \( j \in [A] \), using Pinsker’s inequality and \( 0 < \alpha < \frac{1}{2} \), we have that

\[
\text{KL}(p^{unif}(s_n, a_j)||p_j(s_n, a_j)) = \text{KL}(\text{Ber}(\alpha)||\text{Ber}(\alpha - \eta)) \leq \frac{\eta^2}{\alpha - \eta} \leq \frac{\alpha \eta^2}{\alpha}
\]

Then, using Lemma A.1 in \cite{auer2002}, we have that for any \( j \in [A] \),

\[
\mathbb{E}_j[N_{s_n, a_j}] \leq \mathbb{E}^{unif}[N_{s_n, a_j}] + \frac{K}{2} \sqrt{\mathbb{E}^{unif}[V_{s_n, a_j}] \cdot \text{KL}(p^{unif}(s_n, a_j)||p_j(s_n, a_j))} \\
\leq \mathbb{E}^{unif}[N_{s_n, a_j}] + \frac{K}{2} \sqrt{w(s_n) \cdot \mathbb{E}^{unif}[N_{s_n, a_j}] \cdot \frac{c_1 \eta^2}{\alpha}}
\]

Then, using \( \sum_{j=1}^{A} \mathbb{E}_j[N_{s_n, a_j}] = K \) and the Cauchy–Schwarz inequality, we have

\[
\frac{1}{A} \sum_{j=1}^{A} \mathbb{E}_j[N_{s_n, a_j}] \leq \frac{1}{A} \sum_{j=1}^{A} \mathbb{E}^{unif}[N_{s_n, a_j}] + \frac{K \eta}{2A} \sum_{j=1}^{A} \sqrt{\frac{c_1 \cdot w(s_n) \cdot \mathbb{E}^{unif}[N_{s_n, a_j}]}{\alpha A}} \\
\leq \frac{1}{A} \sum_{j=1}^{A} \mathbb{E}^{unif}[N_{s_n, a_j}] + \frac{K \eta}{2A} \sqrt{A \sum_{j=1}^{A} \frac{c_1 \cdot w(s_n) \cdot \mathbb{E}^{unif}[N_{s_n, a_j}]}{\alpha A}} \\
\leq \frac{K}{A} + \frac{K \eta}{2} \sqrt{\frac{c_1 \cdot w(s_n) K}{\alpha A}}
\]  \quad (12)

By plugging Eq. (12) into Eq. (11), we have

\[
\mathbb{E}[\mathcal{R}(K)] \geq 0.6(H - n - 1) \cdot \frac{\eta}{\alpha} \left( \frac{K}{A} + \frac{K \eta}{2} \sqrt{\frac{c_1 \cdot w(s_n) K}{\alpha A}} \right)
\]
Let $\eta = c_2 \sqrt{\frac{\alpha A}{w(s_n)K}}$ for a small enough constant $c_2$. We have

$$E[R(K)] = \Omega \left( H \sqrt{\frac{A}{\alpha \cdot w(s_n)K}} \cdot K \right)$$

$$= \Omega \left( H \sqrt{\frac{AK}{\alpha \cdot w(s_n)}} \right).$$

Since $\min_{\pi,h,s: w_{\pi,h}(s) > 0} w_{\pi,h}(s) = w(s_n)$ in the constructed instance (Figure 5), we have

$$E[R(K)] = \Omega \left( H \sqrt{\frac{AK}{\alpha \cdot \min_{\pi,h,s: w_{\pi,h}(s) > 0} w_{\pi,h}(s)}} \right).$$

\[\square\]

C Proofs for Iterated CVaR RL with Best Policy Identification

In this section, we give the proofs of sample complexity upper and lower bounds (Theorems 3,4) for Iterated CVaR RL-BPI.

C.1 Proofs for Sample Complexity Upper Bound

To prove the sample complexity upper bound (Theorem 3), we first introduce the following lemmas (Lemmas 9-13) and define concentration event $\mathcal{F}$.

C.1.1 Concentration

**Lemma 9** (Concentration for $V^*-\text{BPI}$). It holds that

$$\Pr \left[ \left| \text{CVaR}_{s' \sim \hat{p}(s',a)}(V^*_p(s')) - \text{CVaR}_{s' \sim p(s',a)}(V^*_p(s')) \right| \leq \frac{H}{\alpha} \sqrt{\log \frac{2k^3 HSA}{\delta'}} \frac{A}{n_k(s,a)} \right] \geq 1 - 2\delta'.$$

**Proof of Lemma 9.** Using the same analysis as Lemma 1, we have that for a fixed $k$,

$$\Pr \left[ \left| \text{CVaR}_{s' \sim \hat{p}(s',a)}(V^*_p(s')) - \text{CVaR}_{s' \sim p(s',a)}(V^*_p(s')) \right| \leq \frac{H}{\alpha} \sqrt{\log \frac{2k^3 HSA}{\delta'}} \frac{A}{n_k(s,a)} \right] \geq 1 - 2 \frac{\delta'}{2k^2}.$$ 

By a union bound over $k = 1, 2, \ldots$, we have

$$\Pr \left[ \left| \text{CVaR}_{s' \sim \hat{p}(s',a)}(V^*_p(s')) - \text{CVaR}_{s' \sim p(s',a)}(V^*_p(s')) \right| \leq \frac{H}{\alpha} \sqrt{\log \frac{2k^3 HSA}{\delta'}} \frac{A}{n_k(s,a)} \right] \geq 1 - 2 \sum_{k=1}^{\infty} \left( \frac{\delta'}{2k^2} \right)$$

23
Then, for any $k$

First, for any $V$ s

Lemma 12 (Optimism and Pessimism)

For any positive integer $C.1.2$ Optimism and Estimation Error

Proof of Lemma 12. This lemma can be obtained by combining Lemmas 9,10,3.

Concentration Events. For ease of notation, in the following, we summarize the concentration events which will be used in the proof for BPI, and recall the aforementioned event $E_3$.

Proof of Lemma 10. Using the same analysis as Lemma 2 and a union bound over $k = 1, 2, \ldots$, we can obtain this lemma. □

Let

Proof of Lemma 11. This lemma can be obtained by combining Lemmas 9,10,3.

C.1.2 Optimism and Estimation Error

For any positive integer $k$, let $\hat{L}(k) := \log \left( \frac{2HSAk^3}{\delta'} \right)$.

Lemma 12 (Optimism and Pessimism). Suppose that event $F$ holds. Then, for any $k$, $h \in [H]$ and $s \in S$,

Proof of Lemma 12. The proof of $\hat{V}_h^k(s) \geq V_h^k(s)$ is similar to Lemma 5. Below we prove $\underline{V}_h^k(s) \leq V_h^k(s)$ by induction.

First, for any $k \in [K]$, $s \in S$, it holds that $\underline{V}_h^{k+1}(s) = V_h^{k+1}(s) = 0$.

Then, for any $k \in [K]$, $h \in [H]$ and $(s,a) \in S \times A$,

\[
Q_h^k(s,a) = r(s,a) + \text{CVaR}_{s',p(h^k)} \left( V_{h+1}^k(s') \right) - \frac{H}{\alpha} \sqrt{\frac{2S \log \left( \frac{2k^3HSA}{\delta'} \right)}{n_k(s,a)}}
\]
where (a) uses Lemma 8 and (b) comes from induction hypothesis.

Thus, for any \( k \), using Lemma 12, we have

\[
Q_h^k(s) - V_h^k(s) \leq Q_h^k(s, \pi_h^k(s)) - Q_h^k(s, \pi_h^k(s)) = V_h^k(s),
\]

de which concludes the proof.

**Lemma 13 (Estimation Error).** Suppose that event \( F \) holds. Then, for any \( k \),

\[
V^*_1(s_1) - V^*_1(s_1) \leq J^k_1(s_1)
\]

**Proof of Lemma 13.** We prove by induction that for any \( k, h \in [H] \) and \( s \in S \),

\[
\tilde{V}_h^k(s) - V_h^k(s) \leq J^k_h(s)
\]

First, for any \( k \), it holds that \( \tilde{V}_{h+1}^k(s) - V^k_{h+1}(s) = J^k_{h+1}(s) = 0 \).

Let \( \hat{\beta}^k; \alpha, \hat{\Lambda}_{h+1}^k(\cdot | s, a) \) denote the distorted distribution (re-normalized weights) when computing \( \text{CVaR}_{\alpha, \hat{\Lambda}_{h+1}^k(\cdot | s, a)}(V^k_{h+1}(s')) \), i.e.,

\[
\text{CVaR}_{\alpha, \hat{\Lambda}_{h+1}^k(\cdot | s, a)}(V^k_{h+1}(s')) = \sum_{s' \in S} \hat{\beta}^k; \alpha, \hat{\Lambda}_{h+1}^k(s'|s, a) \cdot V^k_{h+1}(s').
\]

Then, for any \( k, h \in [H] \) and \( (s, a) \in S \times A \),

\[
\begin{align*}
\tilde{Q}_h^k(s, a) - Q_h^k(s, a) & = \frac{H}{\alpha} \sqrt{\frac{L(k)}{n_k(s, a)}} + H \sqrt{\frac{S \hat{L}(k)}{n_k(s, a)}} \\
& + \text{CVaR}_{\alpha, \hat{\Lambda}_{h+1}^k(\cdot | s, a)}(\tilde{V}_{h+1}^k(s')) - \text{CVaR}_{\alpha, \hat{\Lambda}_{h+1}^k(\cdot | s, a)}(V_{h+1}^k(s')) \\
& \leq (a) \frac{H}{\sqrt{z \hat{L}(k)}} \frac{1 + \sqrt{S}}{\sqrt{n_k(s, a)}} + \hat{\beta}^k; \alpha, \hat{\Lambda}_{h+1}^k(\cdot | s, a) \hat{\Lambda}_{h+1}^k(\tilde{V}_{h+1}^k - V_{h+1}^k) \\
& \leq (b) \frac{H}{\sqrt{z \hat{L}(k)}} \frac{1 + \sqrt{S}}{\sqrt{n_k(s, a)}} + \hat{\beta}^k; \alpha, \hat{\Lambda}_{h+1}^k(\cdot | s, a) \hat{\Lambda}_{h+1}^k J_{h+1}^k \\
= G_h^k(s, a)
\end{align*}
\]

where (a) uses Lemma 8 and (b) is due to induction hypothesis.

Thus,

\[
\tilde{V}_h^k(s) - V_h^k(s) = \tilde{Q}_h^k(s, \pi_h^k(s)) - Q_h^k(s, \pi_h^k(s)) \leq G_h^k(s, \pi_h^k(s)) = J_h^k(s),
\]

which completes the proof.

Thus, for any \( k \),

\[
\tilde{V}_1^k(s_1) - V_1^k(s_1) \leq J_1^k(s_1).
\]

Using Lemma 12, we have

\[
V_1^k(s_1) - V^k_1(s_1) \leq \tilde{V}_1^k(s_1) - V_1^k(s_1) \leq J_1^k(s_1)
\]

\( \square \)
C.1.3 Proof of Theorem 3

Proof of Theorem 3. Suppose that event \( \mathcal{F} \) holds.

First, we prove the correctness. Using Lemma 13, when algorithm ICVaR-BPI stops, we have

\[
V^*_1(s_1) - V^\pi^k_1(s_1) \leq J^k_1(s_1) \leq \varepsilon.
\]

Thus, the output policy \( \pi^k \) is \( \varepsilon \)-optimal.

Next, we prove the sample complexity.

Let \( \tilde{w}_{kh} \) denote the distorted probability (weight) of visiting \((s, a)\) at step \( h \) in episode \( k \) under the CVaR transition metric. Let \( K \) denote the episode that algorithm ICVaR-BPI stops. Then, for any \( k \in [K - 1] \), we have \( \varepsilon < J^k_1(s_1) \). Summing over \( k \in [K - 1] \) and unfolding \( J^k_1(s_1) \), we have

\[
(K - 1) \cdot \varepsilon < \sum_{k=1}^{K-1} J^k_1(s_1)
\]

\[
\leq \sum_{k=1}^{K-1} \sum_{h=1}^{H} \sum_{(s,a) \in \mathcal{L}_k} \tilde{w}_{kh} \text{CVaR}_{\alpha, V^k_{h+1}}(s, a) \frac{H(1 + \sqrt{S}) \sqrt{L(k)}}{z \sqrt{n_k(s, a)}},
\]

\[
+ \sum_{k=1}^{K-1} \sum_{h=1}^{H} \sum_{(s,a) \notin \mathcal{L}_k} \tilde{w}_{kh} \text{CVaR}_{\alpha, V^k_{h+1}}(s, a) \cdot H
\]

\[
\leq H(1 + \sqrt{S}) \sqrt{L(K - 1)} \frac{1}{\alpha} \sum_{k=1}^{K-1} \sum_{h=1}^{H} \sum_{(s,a) \in \mathcal{L}_k} \tilde{w}_{kh} \text{CVaR}_{\alpha, V^k_{h+1}}(s, a) + \frac{8SAH^2}{\min_{\pi, h, s: \pi(s) > 0}} \log \left( \frac{HSA}{\delta'} \right)
\]

\[
\leq H(1 + \sqrt{S}) \sqrt{L(K - 1)} \frac{1}{\alpha} \sum_{k=1}^{K-1} \sum_{h=1}^{H} \sum_{(s,a) \in \mathcal{L}_k} \tilde{w}_{kh} \text{CVaR}_{\alpha, V^k_{h+1}}(s, a) \cdot \frac{w_{kh}(s, a)}{n_k(s, a)} \cdot \mathbb{1}\{w_{kh}(s, a) \neq 0\}
\]

\[
+ \frac{8SAH^2}{\min_{\pi, h, s: \pi(s) > 0}} \log \left( \frac{HSA}{\delta'} \right)
\]

\[
\leq (1 + \sqrt{S})H \sqrt{H \cdot L(K - 1) \cdot (K - 1)} \frac{1}{z \cdot \min_{\pi, h, s: \pi(s) > 0}} \sum_{k=1}^{K-1} \sum_{h=1}^{H} \sum_{(s,a) \in \mathcal{L}_k} \frac{w_{kh}(s, a)}{n_k(s, a)}
\]

\[
+ \frac{8SAH^2}{\min_{\pi, h, s: \pi(s) > 0}} \log \left( \frac{HSA}{\delta'} \right)
\]

\[
\leq 4SH \cdot L(K - 1) \cdot \sqrt{HA(K - 1)} \frac{1}{z \cdot \min_{\pi, h, s: \pi(s) > 0}} + \frac{8SAH^2}{\min_{\pi, h, s: \pi(s) > 0}} \log \left( \frac{HSA}{\delta'} \right),
\]
where (a) is due to that $J_1^J(s_1)$ has a universal upper bound $H$, (b) uses Lemma 7, and (c) uses Lemma 6.

Thus, we have

$$K - 1 \leq \frac{4SH\sqrt{HA}}{\varepsilon z \cdot \min_{\pi, h, s: w_{\pi, h}(s) > 0} w_{\pi, h}(s)} \cdot \sqrt{K - 1} \cdot \log \left( \frac{2HSA(K - 1)^3}{\delta'} \right) + \frac{8SAH^2}{\varepsilon} \cdot \min_{\pi, h, s: w_{\pi, h}(s) > 0} w_{\pi, h}(s) \log \left( \frac{HSA}{\delta'} \right)$$

Using Lemma 13 in Ménard et al. [2021] with $A = B = 1$, $E = 0$, $\tau = K - 1$, $\alpha = \frac{2HSA}{\delta'}$, $C = \frac{12SH\sqrt{HA}}{\varepsilon z \cdot \min_{\pi, h, s: w_{\pi, h}(s) > 0} w_{\pi, h}(s)}$ and $D = \frac{8SAH^2}{\varepsilon} \cdot \min_{\pi, h, s: w_{\pi, h}(s) > 0} w_{\pi, h}(s)$, we have that the number of used trajectories is bounded by

$$K - 1 = O \left( \frac{H^3S^2A}{\varepsilon^2\alpha^2} \cdot \log \left( \frac{HSA}{\delta \cdot \varepsilon z \cdot \min_{\pi, h, s: w_{\pi, h}(s) > 0} w_{\pi, h}(s)} \right) \right)$$

C.2 Proofs for Sample Complexity Lower Bound

In this subsection, we give the proof of sample complexity lower bound (Theorem 4) for Iterated CVaR RL-BPI.

**Proof of Theorem 4.** This proof uses a similar analytical procedure as Theorem 2 in Dann and Brunskill [2015].

Consider the same instance as the proof of Theorem 2 (Figure 5).

Fix an algorithm $A$. Define $E_{s_n} := \{\hat{\pi}(s_n) = a_J\}$ as the event that the output policy $\hat{\pi}$ of algorithm $A$ chooses the optimal action in state $s_n$.

Then, we have

$$V_1^*(s_1) - V_1^\pi(s_1) = 0.6(H - n - 1) \cdot \frac{\eta}{\alpha} \cdot (1 - \mathbb{I} \{E_{s_n}\})$$

For $\pi$ to be $\varepsilon$-optimal, we need

$$\varepsilon \geq V_1^*(s_1) - V_1^\pi(s_1) = 0.6(H - n - 1) \cdot \frac{\eta}{\alpha} \cdot (1 - \mathbb{I} \{E_{s_n}\}),$$

which is equivalent to

$$\mathbb{I} \{E_{s_n}\} \geq 1 - \frac{\varepsilon \alpha}{0.6(H - n - 1) \cdot \eta}$$

Let $\eta = \frac{8e^4 \alpha}{0.008(H - n - 1)}$ for some constant $c_0$. Then, for $\pi$ to be $\varepsilon$-optimal, we need

$$\mathbb{I} \{E_{s_n}\} \geq 1 - \frac{c_0}{8e^4}$$

Let $\phi := 1 - \frac{c_0}{8e^4}$. For algorithm $A$ to be $(\varepsilon, \delta)$-correct, we need

$$1 - \delta \leq \Pr[V^* - V^\pi \geq \varepsilon] \leq \Pr[\mathbb{I} \{E_{s_n}\} \geq \phi]$$

27
\[ \frac{\mathbb{E}[\mathcal{E}_{s_n}]}{\phi} \leq 1 \mathbb{P}[\mathcal{E}_{s_n}], \]

which is equivalent to

\[ \mathbb{P}[\mathcal{E}_{s_n}] = 1 - \mathbb{P}[\mathcal{E}_{s_n}] \leq 1 - \phi + \phi \delta \]

Let \( V_{s_n} \) be the number of times that algorithm \( A \) visited state \( s_n \).

To ensure \( \mathbb{P}[\bar{\mathcal{E}}_{s_n}] \leq 1 - \phi + \phi \delta \), we need

\[ \mathbb{E}[V_{s_n}] \geq \frac{c_1 \cdot \alpha A}{\eta^2} \log \left( \frac{c_2}{1 - \phi + \phi \delta} \right) \]

\[ = \frac{c_1 \cdot \alpha A \cdot 0.6^2 c_0^2 (H - n - 1)^2}{64 e^8 \varepsilon^2 \alpha^2} \log \left( \frac{c_2}{\frac{c_0}{8e^7} + \delta} \right), \]

for some constants \( c_1, c_2 \).

Let \( c_0 \) be a small constant such that \( \frac{c_0}{8e^7} < \delta \). Let \( w(s_n) \) be the probability of visiting \( s_n \) in an episode, and this probability is the same for any policy. Then, the number of required trajectories for \( A \) to be \((\varepsilon, \delta)\)-correct is at least

\[ K \geq \frac{c_1 A \cdot 0.6^2 c_0^2 (H - n - 1)^2}{64 e^8 \varepsilon^2 \alpha \cdot w(s_n)} \log \left( \frac{c_2}{\frac{c_0}{8e^7} + \delta} \right) = \Omega \left( \frac{H^2 A}{\varepsilon^2 \alpha \cdot w(s_n)} \log \left( \frac{1}{\delta} \right) \right). \]

Since \( \min_{\pi, h, s: w_{\pi, h, s}(s) > 0} w_{\pi, h}(s) = w(s_n) \) in the constructed instance (Figure 5), we have

\[ K = \Omega \left( \frac{H^2 A}{\varepsilon^2 \alpha \cdot \min_{\pi, h, s: w_{\pi, h, s}(s) > 0} w_{\pi, h}(s)} \log \left( \frac{1}{\delta} \right) \right). \]

**D Proofs for Worst Path RL**

In this section, we provide the proofs of regret upper and lower bounds (Theorems 5,6) for Worst Path RL.

**D.1 Proofs for Regret Upper Bound**

In order to prove the regret upper bound, we first introduce the following lemmas (Lemmas 14-16) and define concentration event \( G \).

**D.1.1 Concentration**

Recall that \( n_k(s, a) \) is the number of times that the algorithm visited \((s, a)\) up to episode \( k \). For any \( k \) and \((s', s, a) \in S \times S \times A\), let \( n_k(s', s, a) \) denote the number of times that the algorithm visited \((s, a)\) and transitioned to \( s' \) up to episode \( k \).

For any policy \( \pi \) and \((s, a) \in S \times A\), let \( v_{\pi}(s, a) \) be the probability that the algorithm visited \((s, a)\) at least once in an episode under policy \( \pi \).

**Lemma 14.** It holds that

\[ \mathbb{P} \left[ n_k(s, a) \geq \frac{1}{2} \sum_{k' < k} v_{\pi, k'}(s, a) - \log \left( \frac{SA}{\delta'} \right), \forall k, \forall (s, a) \in S \times A \right] \geq 1 - \delta' \]
Proof of Lemma 14. For any \( k \) and \((s, a) \in \mathcal{S} \times \mathcal{A}\), conditioning on the filtration of episodes 1, \ldots, \( k - 1 \), whether the algorithm visited \((s, a)\) at least once in episode \( k \) is a Bernoulli random variable with success probability \( \psi_{\pi_k}(s, a) \).

Then, using Lemma F.4 in Dann et al. [2017], we can obtain this lemma.

**Lemma 15.** It holds that

\[
\Pr \left\{ n_k(s', s, a) \geq \frac{1}{2} \cdot n_k(s, a) \cdot p(s'|s, a) - 2 \log \left( \frac{SA}{\delta'} \right) \right\} \geq 1 - \delta'
\]

\( \forall k, \forall (s', s, a) \in \mathcal{S} \times \mathcal{S} \times \mathcal{A} \)

Proof of Lemma 15. For any \( k, h \in [H] \) and \((s, a) \in \mathcal{S} \times \mathcal{A}\), conditioning on the event \( \{s_h^k = s, a_h^k = a\}\), the indicator \( \mathbb{1} \{s_{h+1}^k = s'\} \) is a Bernoulli random variable with success probability \( p(s'|s, a) \).

Then, using Lemma F.4 in Dann et al. [2017], we can obtain this lemma.

**Concentration Events.** For ease of notation, we summarize the concentration events which will be used in the proof for algorithm MaxWP as follows:

\( \mathcal{G}_1 := \left\{ n_k(s, a) \geq \frac{1}{2} \sum_{k' < k} \psi_{\pi_{k'}}(s, a) - \log \left( \frac{SA}{\delta'} \right), \forall k, \forall (s, a) \in \mathcal{S} \times \mathcal{A} \right\} \)

\( \mathcal{G}_2 := \left\{ n_k(s', s, a) \geq \frac{1}{2} \cdot n_k(s, a) \cdot p(s'|s, a) - 2 \log \left( \frac{SA}{\delta'} \right), \forall k, \forall (s', s, a) \in \mathcal{S} \times \mathcal{S} \times \mathcal{A} \right\} \)

\( \mathcal{G} := \mathcal{G}_1 \cap \mathcal{G}_2 \)

**Lemma 16.** Letting \( \delta' = \frac{\delta}{2} \), it holds that

\[
\Pr [\mathcal{G}] \geq 1 - \delta.
\]

**Proof of Lemma 16.** This lemma can be obtained by combining Lemmas 14,15.

**D.1.2 Proof of Theorem 5**

**Proof of Theorem 5.** Suppose that event \( \mathcal{G} \) holds.

Let

\[
\bar{T} := \sum_{(s, a)} \min_{\nu_\pi(s, a) > 0} \psi_\pi(s, a) \cdot \frac{1}{\min_{s' \in \text{supp}(p(\cdot|s, a))} p(s'|s, a)} \cdot 8 \left( 2 \log \left( \frac{SA}{\delta} \right) + 1 \right).
\]

For any \((s, a) \in \mathcal{S} \times \mathcal{A}\), let

\[
T(s, a) := \min_{\nu_\pi(s, a) > 0} \psi_\pi(s, a) \cdot \frac{1}{\min_{s' \in \text{supp}(p(\cdot|s, a))} p(s'|s, a)} \cdot 8 \left( 2 \log \left( \frac{SA}{\delta} \right) + 1 \right).
\]

It holds that \( \bar{T} = \sum_{s, a} T(s, a) \).

To prove Theorem 5, it suffices to prove that algorithm MaxWP will not take a sub-optimal policy after episode \( \bar{T} \). In the following, we prove this statement by contradiction.

Suppose that in some episode \( k > \bar{T} \), algorithm MaxWP takes a sub-optimal policy \( \pi^k \).

Note that, under the \( \min \) metric, if a Q-value/V-value is not accurately estimated, it can only be overestimated (not underestimated). In addition, the overestimation of a Q-value/V-value comes from
either of the following reasons: (i) Algorithm MaxWP has not detected a bad successor state; (ii) The V-value of the successor state is over-estimated.

According to the supposition, we have that in episode \(k\), algorithm MaxWP chooses some \((s, a_{sub})\) with an overestimated Q-value at some step \(h\), i.e., \(Q_k^h(s, a_{sub}) > Q_k^h(s, a_{sub})\) and \(v_{n^+}(s, a_{sub}) > 0\). Then, there exists some \((s', a')\) with an overestimated Q-value and an accurate future V-value at step \(h' \geq h\), i.e., \(Q^h_k(s', a') > Q^h_k(s', a')\), \(V^h_{k+1}(\cdot) = V^h_{k+1}(\cdot)\) point-wise, and \(v_{n^+}(s', a') > 0\). In other words, the overestimation of \(\hat{Q}^k_h(s', a')\) is purely due to that algorithm MaxWP has not detected a bad successor state of \((s', a')\).

For any \((s, a) \in S \times A\), let \(T^k(s, a) = \{k' < k : v_{n^+}(s, a) > 0\}\) denote the subset of episodes \(1, \ldots, k - 1\) in which \((s, a)\) is available.

**Case (1):** If \(|T^k(s', a')| \geq T(s', a')\), using Lemma 14, we have
\[
n_k(s', a') \geq \frac{1}{2} \sum_{k' < k} v_{n^+}(s', a') - \log \left(\frac{SA}{\delta}\right)
\[
\geq \frac{1}{2} \cdot T(s', a') \cdot \min_{\nu_{n^+}(s', a') > 0} v_{n^+}(s', a') - \log \left(\frac{SA}{\delta}\right)
\[
= \frac{4 \left(2 \log \left(\frac{SA}{\delta}\right) + 1\right)}{\min_{s' \in \text{supp}(p(|s, a|))} p(s'|s, a)} - \log \left(\frac{SA}{\delta}\right)
\[
\geq \frac{2 \left(2 \log \left(\frac{SA}{\delta}\right) + 1\right)}{\min_{s' \in \text{supp}(p(|s, a|))} p(s'|s, a)}
\]

Then, using Lemma 15, we have that for any \(s \in \text{supp}(p(|s', a'|))\),
\[
n_k(s, s', a') \geq \frac{1}{2} \cdot n_k(s', a') \cdot \min_{s' \in \text{supp}(p(|s', a'|))} p(s'|s, a') - 2 \log \left(\frac{SA}{\delta}\right)
\[
\geq 1
\]

which contradicts that \(\hat{Q}^k_h(s', a')\) is overestimated.

**Case (2):** If \(|T^k(s', a')| < T(s', a')\), then we exclude all the episodes in \(T^k(s', a')\), and consider the last episode \(k < k\) where the algorithm took a sub-optimal policy. Note that the excluded state-action pair \((s', a')\) cannot be visited in episode \(k\).

Then, we repeatedly apply the argument in this proof. Once Case (1) happens, we derive a contradiction and complete the proof.

Otherwise, Case (2) repeatedly happens and we exclude the episodes in \(T^k(s, a)\) for all \((s, a) \in S \times A\). Since \(\sum_{(s, a)} |T^k(s, a)| < \sum_{(s, a)} T(s, a) = T < k\), there exists an episode \(k_1 < k\) where \(v_{n^+}(s, a) = 0\) for any \((s, a) \in S \times A\), which gives a contradiction.

\(\square\)

**D.2 Proofs for Regret Lower Bound**

In this subsection, we prove the regret lower bound (Theorem 6) for Worst Path RL.

**Proof of Theorem 6.** Consider the instance \(\mathcal{I}\) as shown in Figure 6:

The action space contains two actions, i.e., \(A = \{a_1, a_2\}\). The state space is \(S = \{s_1, s_2, \ldots, s_8, x_1, x_2, x_3\}\), where \(n = S - 3\) and \(s_1\) is the initial state. Let \(H > S\) and \(0 < \alpha < \frac{1}{3}\).

The reward functions are as follows: For any \(a \in A\), \(r(x_1, a) = 1\), \(r(x_2, a) = 0.8\) and \(r(x_3, a) = 0.2\). For any \(i \in [n]\) and \(a \in A\), \(r(s_i, a) = 0\).
A we have that \( A \)

Thus, any

The transition distributions are as follows: For any \( a \in A \), \( p(s_2|s_1, a) = \alpha \), \( p(x_1|s_1, a) = 1 - 3\alpha \), \( p(x_2|s_1, a) = \alpha \) and \( p(x_3|s_1, a) = \alpha \).

For any \( i \in \{2, \ldots, n-1\} \) and \( a \in A \), \( p(s_{i+1}|s_i, a) = \alpha \) and \( p(x_1|s_i, a) = 1 - \alpha \).

\( x_1, x_2 \) and \( x_3 \) are absorbing states, i.e., for any \( a \in A \), \( p(x_1|x_1, a) = 1 \), \( p(x_2|x_2, a) = 1 \) and \( p(x_3|x_3, a) = 1 \).

The state \( s_n \) is a bandit state, which has an optimal action and a suboptimal action. Let \( a_\star \) denote the optimal action in state \( s_n \), which is uniformly drawn from \{\( a_1, a_2 \}\}, and let \( a_{\text{sub}} \) denote the other suboptimal action in state \( s_n \). For the optimal action \( a_\star \), \( p(x_2|s_n, a_\star) = 1 \). For the suboptimal action \( a_{\text{sub}}, p(x_2|s_n, a_{\text{sub}}) = 1 - \alpha \) and \( p(x_3|s_n, a_{\text{sub}}) = \alpha \).

Fix an \( O(K) \)-consistent algorithm \( A \), which guarantees a sub-linear regret on any instance of Worst Path RL. We have that \( A \) needs to observe the transition from \((s_n, a_{\text{sub}})\) to \( x_3 \) at least once. Otherwise, \( A \) cannot distinguish whether the suboptimal action in state \( s_n \) is \( a_1 \) or \( a_2 \). Specifically, no matter \( A \) chooses \( a_1 \) or \( a_2 \) in state \( s_n \), it will suffer a linear regret in the counter case where the unchosen action is optimal.

Thus, any \( O(K) \)-consistent algorithm must observe the transition from \((s_n, a_{\text{sub}})\) to \( x_3 \) at least once, and needs at least

\[
\frac{1}{v_{\pi_{\text{sub}}}(s_n, a_{\text{sub}}) \cdot p(x_3|s_n, a_{\text{sub}})}
\]

episodes with sub-optimal policies. Here \( \pi_{\text{sub}} \) denotes a policy which chooses \( a_{\text{sub}} \) in state \( s_n \), and \( v_{\pi_{\text{sub}}}(s_n, a_{\text{sub}}) \) denotes the probability that \((s_n, a_{\text{sub}})\) is visited at least once in an episode under policy \( \pi_{\text{sub}} \).

Therefore, \( A \) needs to suffer at least

\[
\Omega \left( \frac{1}{v_{\pi_{\text{sub}}}(s_n, a_{\text{sub}}) \cdot p(x_3|s_n, a_{\text{sub}}) \cdot \Delta_{\text{min}}} \right)
\]

regret in expectation.

Since in the constructed instance (Figure 6)

\[
\max_{(s, a): \exists h, a \neq \pi_h^*(s)} \min_{v_e(s, a) > 0} v_{\pi}(s, a) \cdot \min_{s' \in \text{supp}(p(\cdot|s, a))} p(s'|s, a) = \frac{1}{v(s_n, a_{\text{sub}}) \cdot p(x_3|s_n, a_{\text{sub}})}
\]

we have that \( A \) needs to suffer at least

\[
\Omega \left( \max_{(s, a): \exists h, a \neq \pi_h^*(s)} \min_{v_e(s, a) > 0} v_{\pi}(s, a) \cdot \min_{s' \in \text{supp}(p(\cdot|s, a))} p(s'|s, a) \cdot \Delta_{\text{min}} \right)
\]

regret.