Riesz Potential and Maximal Function for Dunkl transform

D. V. Gorbachev · V. I. Ivanov · S. Yu. Tikhonov

Received: 8 May 2018 / Accepted: 9 July 2020 / Published online: 22 July 2020
© Springer Nature B.V. 2020

Abstract
We study weighted \((L^p, L^q)\)-boundedness properties of Riesz potentials and fractional maximal functions for the Dunkl transform. In particular, we obtain the weighted Hardy–Littlewood–Sobolev type inequality and weighted weak \((L^1, L^q)\) estimate. We find a sharp constant in the weighted \(L^p\)-inequality, generalizing the results of W. Beckner and S. Samko.

Keywords Dunkl transform · Generalized translation operator · Convolution · Riesz potential

Mathematics Subject Classification (2010) 42B10 · 33C45 · 33C52

1 Introduction

Let \(\mathbb{R}^d\) be the real Euclidean space of \(d\) dimensions equipped with a scalar product \((x, y)\) and a norm \(|x| = \sqrt{(x, x)}\). Let \(d\mu(x) = (2\pi)^{-d/2} dx\) be the normalized Lebesgue measure,
$L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, be the Lebesgue space with the norm $\|f\|_p = \left(\int_{\mathbb{R}^d} |f|^p \, d\mu\right)^{1/p}$, and $S(\mathbb{R}^d)$ be the Schwartz space. The Fourier transform is given by

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}^d} f(x) e^{-i(x,y)} \, d\mu(x).$$

Throughout the paper, we will assume that $A \lesssim B$ means that $A \leq C B$ with a constant $C$ depending only on nonessential parameters. For $p \geq 1$, $p' = \frac{p}{p-1}$ is the Hölder conjugate and $\chi_E$ is the characteristic function of a set $E$.

The Riesz potential operator or fractional integral $I_\alpha$ is defined by

$$I_\alpha f(x) = (\gamma_\alpha)^{-1} \int_{\mathbb{R}^d} f(y) |x - y|^{\alpha-d} \, d\mu(y) = (\gamma_\alpha)^{-1} \int_{\mathbb{R}^d} \tau^{-y} f(x) |y|^{\alpha-d} \, d\mu(y),$$

where $0 < \alpha < d$, $\gamma_\alpha = 2^{\alpha-d/2} \Gamma(\frac{\alpha}{2}) / \Gamma((d-\alpha)/2)$, and $\tau^{-y} f(x) = f(x + y)$ is the translation operator. Such operator was first investigated by O. Frostman [7]. Several important properties of the potential were obtained by M. Riesz [18].

The weighted $(L^p, L^q)$-boundedness of Riesz potentials is given by the following Stein–Weiss inequality

$$\| |x|^{-\gamma} I_\alpha f(x) \|_q \leq c(\alpha, \beta, \gamma, p, q, d) \| |x|^{\beta} f(x) \|_p (1.1)$$

with the sharp constant $c(\alpha, \beta, \gamma, p, q, d)$ and $1 < p \leq q < \infty$. Sufficient conditions for the finiteness of $c(\alpha, \beta, \gamma, p, q, d)$ are well known.

**Theorem 1.1** Let $d \in \mathbb{N}$, $1 \leq p \leq q < \infty$, $\gamma < \frac{d}{q}$, $\gamma + \beta \geq 0$, $0 < \alpha < d$, and $\alpha - \gamma - \beta = d(\frac{1}{p} - \frac{1}{q})$.

(a) If $1 < p \leq q < \infty$ and $\beta < \frac{d}{p}$, then $c(\alpha, \beta, \gamma, p, q, d) < \infty$.

(b) If $p = 1$, $1 < q < \infty$, $\beta \leq 0$, then, for $f \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda > 0$,

$$\int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} |I_\alpha f(x)| > \lambda\}} d\mu(x) \lesssim \left(\frac{\| |x|^{\beta} f(x) \|_1}{\lambda}\right)^q.$$

The part (a) in Theorem 1.1 was proved by G.H. Hardy and J.E. Littlewood [12] for $d = 1$, S. Sobolev [26] for $d > 1$ and $\gamma = \beta = 0$, E.M. Stein and G. Weiss [27] in the general case. The conditions for weak boundedness can be found in [9, 25].

The sharp constant $c(\alpha, 0, 0, p, q, d)$ in the non-weighted Sobolev inequality was calculated by E.H. Lieb [15] in any of the following cases: (1) $q = p'$, $1 < p < 2$, (2) $q = 2$, $1 < p < 2$, (3) $p = 2$, $2 < q < \infty$. Moreover, in these cases there exist maximizing functions. In the weighted Hardy–Littlewood–Sobolev inequality the constant $c(\alpha, \beta, \gamma, p, q, d)$ is known only for $q = p$.

**Theorem 1.2** If $d \in \mathbb{N}$, $1 < p < \infty$, $\gamma < \frac{d}{p}$, $\beta < \frac{d}{p}$, $\alpha > 0$, and $\gamma = \alpha - \beta$, then

$$c(\alpha, \beta, \gamma, p, p, d) = 2^{-\alpha} \frac{\Gamma\left(\frac{d}{p} - \alpha + \beta\right)}{\Gamma\left(\frac{d}{p} + \alpha - \beta\right)} \frac{\Gamma\left(\frac{d}{p'} - \beta\right)}{\Gamma\left(\frac{d}{p'} + \alpha - \beta\right)}.$$

Theorem 1.2 was proved by I.W. Herbst [14] for $\beta = 0$ and W. Beckner [4] and S. Samko [24] in the general case.

For $\alpha \in \mathbb{R}$, we define the Riesz potential in the distributional sense. Let $\Phi$ be the Lizorkin space [16], [23, p. 39], that is, a subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ which consists of
functions orthogonal to all polynomials:
\[
\int_{\mathbb{R}^d} x^n f(x) \, d\mu(x) = 0, \quad n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d.
\]
The subspace \( \Phi \) is invariant with respect to the operator \( I_\alpha \) and its inverse \( I_\alpha^{-1} = (-\Delta)^{\alpha/2} \):
\[
I_\alpha(\Phi) = (-\Delta)^{\alpha/2}(\Phi) = \Phi,
\]
where \( \Delta \) is the Laplacian. Note that \( \Phi \) is dense in \( L_p(\mathbb{R}^d, |x|^{\beta_p} \, d\mu) \) for \( 1 < p < \infty \) and \( \beta \in (-d/p, d/p) \) [23, p. 41].

It is worth mentioning that the Stein–Weisz inequality (1.1) on \( \Phi \) is equivalent to the Hardy–Rellich inequality
\[
\| |x|^{-\gamma} f(x) \|_q \leq c(\alpha, \beta, \gamma, p, q) \| |x|^{\beta} (-\Delta)^{\alpha/2} f(x) \|_p.
\]
Let \( D_j f(x) \) be the usual partial derivative with respect to a variable \( x_j \), \( j = 1, \ldots, d \), \( D = (D_1, \ldots, D_d) \), \( D^n f(x) = \prod_{j=1}^d D_j^n f(x), n \in \mathbb{Z}_+^d \). The subspace
\[
\Psi = \{ \mathcal{F}(f) : f \in \Phi \} = \{ f \in \mathcal{S}(\mathbb{R}^d) : D^n f(0) = 0, \, n \in \mathbb{Z}_+^d \}
\]
is invariant with respect to the operator \( \mathcal{F}(I_\alpha) \) and \( \mathcal{F}((-\Delta)^{\alpha/2}) \):
\[
\mathcal{F}(I_\alpha)(\Psi) = \mathcal{F}((-\Delta)^{\alpha/2})(\Psi) = \Psi.
\]
For a distribution \( f \in \Phi' \) and \( \alpha \in \mathbb{R} \) we set
\[
I_\alpha f = \mathcal{F}^{-1} \cdot |^{-\alpha} \mathcal{F}(f), \quad (-\Delta)^{\alpha/2} f = \mathcal{F}^{-1} \cdot |^{\alpha} \mathcal{F}(f).
\]
If \( \varphi \in \Phi \), then
\[
\langle I_\alpha f, \varphi \rangle = \langle f, \mathcal{F} \cdot |^{-\alpha} \mathcal{F}^{-1}(\varphi) \rangle, \quad \langle (-\Delta)^{\alpha/2} f, \varphi \rangle = \langle f, \mathcal{F} \cdot |^{\alpha} \mathcal{F}^{-1}(\varphi) \rangle.
\]
One of the generalizations of the Fourier transform is the Dunkl transform \( \mathcal{F}_k \) (see [6, 21]). Our main goal in this paper is to prove analogues of Theorems 1.1 and 1.2 for the Riesz potential associated with the Dunkl transform. We shall call it the D-Riesz potential.

Let a finite subset \( R \subset \mathbb{R}^d \setminus \{0\} \) be a root system, let \( R_+ \) be positive subsystem of \( R \), let \( G(R) \subset O(d) \) be finite reflection group, generated by reflections \( \{ \sigma_a : a \in R \} \), where \( \sigma_a \) is a reflection with respect to hyperplane \( \langle a, x \rangle = 0 \), let \( k : R \to \mathbb{R}_+ \) be \( G \)-invariant multiplicity function. Recall that a finite subset \( R \subset \mathbb{R}^d \setminus \{0\} \) is called a root system, if
\[
R \cap \mathbb{R}a = \{a, -a\} \quad \text{and} \quad \sigma_a R = R \quad \text{for all} \quad a \in R.
\]
Let
\[
v_k(x) = \prod_{a \in R_+} |\langle a, x \rangle|^{2k(a)}
\]
be the Dunkl weight. The normalized Macdonald–Metha–Selberg constant is given by
\[
c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2} v_k(x) \, dx.
\]
Let \( L^p(\mathbb{R}^d, d\mu_k) \) be the space of complex-valued Lebesgue measurable functions \( f \) such that
\[
\| f \|_{p, d\mu_k} = \left( \int_{\mathbb{R}^d} |f|^p \, d\mu_k \right)^{1/p} < \infty,
\]
where \( d\mu_k(x) = c_k v_k(x) \, dx \) is the Dunkl measure. Assume that
\[
T_j f(x) = D_j f(x) + \sum_{a \in R_+} k(a) \langle a, e_j \rangle \frac{f(x) - f(\sigma_a x)}{\langle a, x \rangle} \quad (1.2)
\]
are differential-differences Dunkl operators, \( j = 1, \ldots, d \), and \( \Delta_k = \sum_{j=1}^d T_j^2 \) is the Dunkl Laplacian \([10]\).

The Dunkl kernel \( E_k(x, y) \) is a unique solution of the system
\[
T_j f(x) = y_j f(x), \quad j = 1, \ldots, d, \quad f(0) = 1.
\]
Let \( e_k(x, y) = E_k(x, iy) \). It plays the role of a generalized exponential function. Its properties are similar to those of the classical exponential function \( e^{i(x,y)} \). Several basic properties follow from the integral representation given by M. Rösler \([20]\)
\[
e_k(x, y) = \int_{\mathbb{R}^d} e^{i(\xi, x)} d\mu_k(x),
\]
where \( \mu_k(x) \) is a probability Borel measure, whose support is contained in \( co(\{g x : g \in G(R)\}) \) the convex hull of the \( G \)-orbit of \( x \) in \( \mathbb{R}^d \). In particular, \( |e_k(x, y)| \leq 1 \) and \( \text{supp} \mu_k(x) \subset B_r \), where \( B_r \) is the Euclidean ball of radius \( r \) centered at 0.

For \( f \in L^1(\mathbb{R}^d, d\mu_k) \), the Dunkl transform is defined by the equality
\[
\mathcal{F}_k(f)(y) = \int_{\mathbb{R}^d} f(x) e_k(x, y) d\mu_k(x).
\]
If \( k \equiv 0 \), then \( \mathcal{F}_0 \) is the Fourier transform \( \mathcal{F} \). We note that \( \mathcal{F}_k(e^{-|\cdot|^2/2})(y) = e^{-|y|^2/2} \) and \( \mathcal{F}_k^{-1}(f)(x) = \mathcal{F}_k(f)(-x) \). The Dunkl transform is isometry in \( S(\mathbb{R}^d) \) and \( L^2(\mathbb{R}^d, d\mu_k) \) and \( \|f\|_2, d\mu_k = \|\mathcal{F}_k(f)\|_2, d\mu_k \).

M. Rösler \([19]\) defined the generalized translation operator \( \tau^y \), \( y \in \mathbb{R}^d \), on \( L^2(\mathbb{R}^d, d\mu_k) \) by equality
\[
\mathcal{F}_k(\tau^y f)(z) = e_k(y, z) \mathcal{F}_k(f)(z),
\]
or
\[
\tau^y f(x) = \int_{\mathbb{R}^d} e_k(y, z) e_k(x, z) \mathcal{F}_k(f)(z) d\mu_k(z).
\]
It acts from \( L^2(\mathbb{R}^d, d\mu_k) \) to \( L^2(\mathbb{R}^d, d\mu_k) \) and \( \|\tau^y\|_2 \rightarrow 2 = 1 \).

If \( k \equiv 0 \), then \( \tau^y f(x) = f(x + y) \). If \( f \in S(\mathbb{R}^d) \), then \( \tau^y f(x) \in S(\mathbb{R}^d) \times S(\mathbb{R}^d) \) and equality \((1.4)\) holds pointwise. K. Trimèche extended \( \tau^y \) on \( C^\infty(\mathbb{R}^d) \) \([30]\). For example, \( \tau^y 1 = 1 \). In general, \( \tau^y \) is not positive operator and the question of its \( L_p \)-boundedness remains open.

First, we define the D-Riesz potential for distributions. Let
\[
\Phi_k = \left\{ f \in S(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^n f(x) d\mu_k(x) = 0, \quad n \in \mathbb{Z}_+^d \right\}
\]
be the weighted Lizorkin space,
\[
\Psi_k = \{ \mathcal{F}_k(f) : f \in \Phi_k \}.
\]

For \( \alpha \in \mathbb{R} \), we define the D-Riesz potential on \( \Phi_k \) by equality
\[
I^k_{\alpha} f = \mathcal{F}_k^{-1}|\cdot|^{-\alpha} \mathcal{F}_k(f).
\]
In Section 6, we will prove that \( \Psi_k = \Psi \) (see Theorem 6.1) and then \( I^k_{\alpha}(\Phi_k) = \Phi_k \) and \( \mathcal{F}_k(I^k_{\alpha}(\Psi_k)) = \Psi_k \). Therefore, we can define the D-Riesz potential \( I^k_{\alpha} \) for \( f \in \Phi_k \) and \( \alpha \in \mathbb{R} \) by the same equality \( I^k_{\alpha} f = \mathcal{F}_k^{-1}|\cdot|^{-\alpha} \mathcal{F}_k(f) \) as follows
\[
(I_{\alpha} f, \varphi) = (f, \mathcal{F}_k|\cdot|^{-\alpha} \mathcal{F}_k^{-1}(\varphi)), \quad \varphi \in \Phi_k.
\]
We will also prove (see Theorem 6.3) that $\Phi_k$ is dense in $L_p(\mathbb{R}^d, |x|^{-p} d\mu_k)$ for $1 < p < \infty$ and $\beta \in (-d_k/p, d_k/p)$, where

$$d_k = 2\lambda_k + 2, \quad \lambda_k = \frac{d}{2} - 1 + \sum_{a \in \mathbb{R}_+^k} k(a). \quad (1.5)$$

S. Thangavelu and Y. Xu defined [29] the D-Riesz potential on Schwartz space as follows

$$I_k^\alpha f(x) = (\gamma_k^\alpha)^{-1} \int_{\mathbb{R}^d} \tau^{-\gamma} f(x) |y|^{-d_k} d\mu_k(y), \quad (1.6)$$

where $0 < \alpha < d_k$ and $\gamma_k^\alpha = 2^{d_k/2}\Gamma(\alpha/2)/\Gamma((d_k - \alpha)/2)$.

We are interested in the Stein–Weiss inequality for the D-Riesz potential

$$|x|^{-\gamma} I_k^\alpha f(x) \leq c_k(\alpha, \beta, \gamma, p, d) |x|^{-\gamma} f(x)^p d\mu_k, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (1.7)$$

with the sharp constant $c_k(\alpha, \beta, \gamma, p, q, d)$ and $1 < p \leq q < \infty$.

Our main results read as follows.

**Theorem 1.3** If $d \in \mathbb{N}$, $1 < p < \infty$, $\gamma < \frac{d}{p}$, $\beta < \frac{d}{p}$, $\alpha > 0$, and $\alpha = \gamma + \beta$, then

$$c_k(\alpha, \beta, \gamma, p, q, d) = 2^{-\alpha} \frac{\Gamma\left(\frac{1}{2} \left(\frac{d_k}{p} - \gamma\right)\right) \Gamma\left(\frac{1}{2} \left(\frac{d_k}{p} + \gamma\right)\right)}{\Gamma\left(\frac{1}{2} \left(\frac{d_k}{p} + \beta\right)\right)} = c(\alpha, \beta, \gamma, p, q, d).$$

**Theorem 1.4** Let $d \in \mathbb{N}$, $1 \leq p \leq q < \infty$, $\gamma < \frac{d_k}{q}$, $\gamma + \beta \geq 0$, $0 < \alpha < d_k$, and $\alpha - \gamma - \beta = d_k(\frac{1}{p} - \frac{1}{q})$.

(a) If $1 < p \leq q < \infty$ and $\beta < \frac{d_k}{q}$, then $c_k(\alpha, \beta, \gamma, p, q, d) < \infty$.

(b) If $p = 1$, $1 < q < \infty$, $\beta \leq 0$, and $\lambda > 0$, then

$$\int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} I_k^\alpha f(x) > \lambda\}} d\mu_k(x) \lesssim \left(\| |x|^{-\gamma} f(x) \|_{1, d\mu_k}^p / \lambda\right)^q, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

For $k \equiv 0$, Theorems 1.3 and 1.4 become Theorems 1.1 and 1.2, therefore it is enough to consider the case $k \neq 0$, i.e., $\lambda_k = \frac{d}{2} - 1 + \sum_{a \in \mathbb{R}_+^k} k(a) > -1/2$ and $d_k = 2\lambda_k + 2 > 1$.

It is clear that $d_k$ plays the role of the generalized dimension of the space $(\mathbb{R}^d, d\mu_k)$.

For the reflection group $\mathbb{Z}_d^2$ and $\gamma = \beta = 0$, Theorem 1.4 was proved in [29]. For arbitrary reflection group $G$ and $\gamma = \beta = 0$, it was proved by S. Hassani, S. Mustapha and M. Sifi [13]. Following an idea from [29], we have recently given another proof in [11].

Regarding the weighted setting, part (a) was proved in [1] in the case $q = p$ under more restrictive conditions $1 < p < \infty$, $0 < \gamma < \frac{d_k}{p}$, $0 < \beta < \frac{d_k}{p}$, and $\alpha > 0$.

To estimate the $L^p$-norm of operator $I_k^\alpha$, S. Thangavelu and Y. Xu [29] used the maximal function, defined for $f \in \mathcal{S}(\mathbb{R}^d)$ as follows

$$M^k f(x) = \sup_{r > 0} \frac{\int_{\mathbb{R}^d} \tau^{-\gamma} f(x) \chi_{B_r}(y) d\mu_k(y)}{\int_{B_r} d\mu_k},$$

$$2 \text{ Springer}$$

Riesz Potential and Maximal Function for Dunkl transform
where $B_r = \{ x : |x| \leq r \}.$ They proved the strong $L^p$-boundedness of $M^k$ for $1 < p < \infty$ and the weak boundedness for $p = 1$ [28].

We will use Theorem 1.4 to obtain weighted boundedness of the fractional maximal function $M^k_\alpha f$, $0 \leq \alpha < d_k$, given by

$$M^k_\alpha f(x) = \sup_{r > 0} r^{\alpha - d_k} \left| \int_{\mathbb{R}^d} \tau^{-\gamma} f(x) \chi_{B_r}(y) d\mu_k(y) \right|$$

If $\alpha = 0$, then $M^k_0$ coincides with $M^k$ up to a constant. Since $\tau^y$ is a positive operator on radial functions [22, 28], and using

$$M^k_\alpha f(x) \leq M^k |f|(x) I^k_\alpha |f|(x),$$

Theorem 1.4 implies the boundedness conditions of the fractional maximal function.

**Theorem 1.5** Let $d \in \mathbb{N}$, $1 \leq p \leq q < \infty$, $\gamma < \frac{d_k}{p}$, $\gamma + \beta \geq 0$, $0 < \alpha < d_k$, $\alpha - \gamma - \beta = d_k \left( \frac{1}{p} - \frac{1}{q} \right)$, and $f \in \mathcal{S}(\mathbb{R}^d)$.

(a) If $1 < p \leq q < \infty$ and $\beta < \frac{d_k}{p}$, then

$$\left\| |x|^{-\gamma} M^k_\alpha f(x) \right\|_{p, d\mu_k} \lesssim \left\| |x|^\beta f(x) \right\|_{p, d\mu_k}.$$  

(b) If $p = 1$, $1 < q < \infty$, $\beta \leq 0$, and $\lambda > 0$, then

$$\int_{\{ x \in \mathbb{R}^d : |x|^{-\gamma} |M^k_\alpha f(x)| > \lambda \}} d\mu_k(x) \lesssim \left( \left\| |x|^\beta f(x) \right\|_{1, d\mu_k} \lambda \right)^q.$$  

In the case $\gamma = \beta = 0$ Theorem 1.5 was proved in [13].

The paper is organized as follows. In the next section, we obtain the sharp inequalities for Mellin convolution and investigate the following representation of the Riesz potential

$$I^k_\alpha f(x) = \int_{\mathbb{R}^d} f(y) \Phi(x, y) d\mu_k(y)$$

and basic properties of the kernel

$$\Phi(x, y) = \frac{2^{d_k/2 - \alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k - \alpha)/2 - 1} \tau^{-\gamma} (e^{-s|\%cdot|^2})(x) ds, \quad (x, y) \neq (0, 0).$$

In Section 3, we prove sharp $(L_p, L_p)$ Hardy’s inequalities with weights for the averaging operator $Hf(x) = \int_{|y|\leq|x|} f(y) d\mu_k(y)$. In the classical setting ($k = 0$), this result was proved by M. Christ and L. Grafakos [5] and Z.W. Fu, L. Grafakos, S.Z. Lu and F.Y. Zhao [8]. Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4 correspondingly. We finish with Section 6, which contains some important properties of the spaces $\Phi_k$ and $\Psi_k$.

2 Notations and Auxiliary Statements

Set as usual $\mathbb{R}_+ = [0, \infty), S^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \}$ and $x = rx' \in \mathbb{R}^d, r = |x| \in \mathbb{R}_+, x' \in S^{d-1}$. Let

$$dv(r) = r^{-1} dr, \quad dv_\lambda(r) = b_\lambda r^{2\lambda+1} dr, \quad b_\lambda^{-1} = 2^\lambda \Gamma(\lambda + 1), \quad \lambda \geq -1/2,$$
be the measures on $\mathbb{R}_+$,
\[ d\sigma_k(x') = a_kv_k(x') \, dx' \]
be the probability measure on $\mathbb{S}^{d-1}$, and
\[ d\mu_k(x) = c_kv_k(x) \, dx, \quad dm_k(x) = dv(r) \, d\sigma_k(x') \]
be the measures on $\mathbb{R}^d$.

Note that
\[ d\mu_k(x) = d\nu \lambda_k(r) d\sigma_k(x) = b_{\lambda_k} |x|^{d_k} \, dm_k(x), \quad (2.1) \]
where $\lambda_k$ and $d_k$ are defined in Eq. 1.5.

Let $L^p(X, d\mu)$, $1 \leq p \leq \infty$, be the Banach space with the norm
\[ \|f\|_{p,d\mu} = \begin{cases} \left( \int_X |f|^p \, d\mu \right)^{1/p}, & p < \infty, \\ \text{supvrai}_X |f|, & p = \infty. \end{cases} \]

Depending on the context, we assume that $L^p = L^p(X, d\mu)$ and $\|f\|_p = \|f\|_{p,d\mu}$.

### 2.1 Convolution Inequalities

The Mellin convolution is given by
\[ A_g f(r) = (f \ast g)(r) = \int_0^\infty f(r/t) g(t) \, dv(t). \]

We will frequently use the fact that
\[ (f \ast g)(r) = (g \ast f)(r). \quad (2.2) \]

**Lemma 2.1** Let $L^p = L^p(\mathbb{R}_+, dv)$, $1 \leq p \leq \infty$. If $f \in L^p$, $h \in L^{p'}$, $g \in L^1$, then
\[ \|f \ast g\|_p \leq \|g\|_1 \|f\|_p, \]
or
\[ \left\| \int_0^\infty \int_0^\infty h(r) f(t) g(r/t) \, dv(t) \, dv(r) \right\| \leq \|g\|_1 \|h\|_{p'} \|f\|_p. \]

If $g \geq 0$, then
\[ \|A_g\|_{p \rightarrow p} = \|g\|_1, \quad (2.3) \]
or
\[ \sup_{\|f\|_p \leq 1} \sup_{\|h\|_{p'} \leq 1} \left| \int_0^\infty h(r) A_g f(r) \, dv(r) \right| = \|g\|_1. \]

**Proof** For the classical convolution on the $\mathbb{R}$, see, e.g., [14]. We sketch the proof here for completeness of the exposition. For $1 < p < \infty$, using Hölder’s inequality, we obtain
\[ \left| \int_0^\infty f(r/t) g(t) \, dv(t) \right| \leq \left( \int_0^\infty |f(r/t)|^p |g(t)| \, dv(t) \right)^{1/p} \left( \int_0^\infty |g(t)|^p \, dv(t) \right)^{1/p'}, \]
and
\[ \|f \ast g\|_p \leq \left( \int_0^\infty \int_0^\infty |f(r/t)|^p |g(t)| \, dv(t) \, dv(r) \right)^{1/p} \|g\|_1^{1/p'}, \]
\[ = \left( \int_0^\infty |g(t)| \int_0^\infty |f(r/t)|^p \, dv(r) \, dv(t) \right)^{1/p} \|g\|_1^{1/p'} = \|f\|_p \|g\|_1. \]
Let \( g \geq 0 \). If \( p = 1 \), \( f \in L^1, f \geq 0 \), then
\[
\| A_g f \|_1 = \int_0^\infty \int_0^\infty f(r/t) g(t) \, d\nu(t) \, d\nu(r) = \int_0^\infty f(r) \, d\nu(r) \int_0^\infty g(t) \, d\nu(t) = \| g \|_1 \| f \|_1.
\]
which gives Eq. 2.3. If \( p = \infty \), we define \( f = \chi_{[\lambda, 1/\lambda]}, 0 < \lambda < 1 \). Then \( \| f \|_\infty = 1 \) and, for \( r \in [1, 2] \),
\[
A_g f(r) = \int_r^{r/\lambda} g(t) \, d\nu(t) \geq \int_{2\lambda}^{1/\lambda} g(t) \, d\nu(t) \rightarrow \| g \|_1, \quad \lambda \rightarrow 0.
\]
If \( 1 < p < \infty \), \( f = (2\lambda)^{-1/p} \chi_{[\lambda, e^\lambda]}, \) and \( h = (2\lambda)^{-1/p'} \chi_{[e^{-\lambda}, e^\lambda]} \), then \( \| f \|_p = \| h \|_p' = 1 \) and by the Lebesgue dominated convergence theorem
\[
\| A_g \|_{p \to p} \geq \lim_{\lambda \to \infty} \left\{ (2\lambda)^{-1} \int_{e^{-\lambda}}^{e^\lambda} \int_{e^{-\lambda}}^{e^\lambda} g(r/t) \, d\nu(r) \, d\nu(t) \right\} = \lim_{\lambda \to \infty} \left\{ (2\lambda)^{-1} \int_{e^{-\lambda}}^{e^\lambda} \int_{e^{-\lambda}}^{e^\lambda} g(r) \, d\nu(r) \, d\nu(t) \right\} = \lim_{\lambda \to \infty} (2\lambda)^{-1} \left\{ \int_{e^{-\lambda}}^{1} \int_{e^{-\lambda}/r}^{e^\lambda/t} g(r) \, dt \, g(r) \, dr + \int_{e^\lambda}^{2\lambda} \int_1^{e^\lambda/r} g(r) \, dt \, g(r) \, dr \right\} \geq \lim_{\lambda \to \infty} (2\lambda)^{-1} \left\{ \int_{e^{-\lambda}}^{1} g(r) (2\lambda + \ln r) \, dr + \int_{1}^{2\lambda} g(r) (2\lambda - \ln r) \, dr \right\} = \int_0^\infty g(r) \, dr = \| g \|_1.
\]

\[ \square \]

2.2 A Representation of the Riesz Potential

We will use the following representation (see [1]), which is different from the definition (1.6):
\[
I^x_\alpha f(x) = \int_{\mathbb{R}^d} f(y) \Phi(x, y) \, d\mu_k(y), \quad (2.4)
\]
where
\[
\Phi(x, y) = \frac{2^{dk/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty \frac{s^{(d_k-\alpha)/2-1} \tau^{-\gamma} e^{-s|\tau|^{2\alpha}}}{\pi} (x) \, ds, \quad (x, y) \neq (0, 0). \quad (2.5)
\]

To verify Eq. 2.4, we first remark that the convolution
\[
(f \ast_k g)(x) = \int_{\mathbb{R}^d} \tau^{-\gamma} f(x) g(y) \, d\mu_k(y)
\]
is commutative, i.e., \((f \ast_k g)(x) = (g \ast_k f)(x)\). Indeed, we have the following

\[ \square \]
Lemma 2.2 If \( f \in S(\mathbb{R}^d) \), \( g \in L^1(\mathbb{R}^d, d\mu_k) \), and \( f_t(x) = f(tx) \), then
\[
\int_{\mathbb{R}^d} \tau^{-y} f(x) g(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) \tau^{-y} g(x) d\mu_k(y),
\]
(2.6)
\[
\mathcal{F}_k(f_t)(z) = \frac{1}{t^{dk}} \mathcal{F}_k(f) \left( \frac{z}{t} \right), \quad \tau^y(f_t)(x) = \tau^y f(tx).
\]
(2.7)

Relation (2.6) has been recently proved in [11]. Equalities (2.7) can be verified by simple calculations.

Remark 2.1 It is worth mentioning that if the convolution is defined by
\[
(f *_k g)(x) = \int_{\mathbb{R}^d} \tau^x f(y) g(y) d\mu_k(y)
\]
(see [22]), it is not commutative:
\[
\int_{\mathbb{R}^d} \tau^x f(y) g(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) \tau^{-x} g(y) d\mu_k(y).
\]
Completing the proof of Eq. 2.4, we use Eqs. 1.6, 2.6 and the fact that (see [29])
\[
\frac{1}{|y|^{d_k-\alpha}} = \frac{1}{\Gamma((d_k-\alpha)/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s|y|^2} ds,
\]
(2.8)
to obtain
\[
I^k_{\alpha} f(x) = (y_k^\alpha)^{-1} \int_{\mathbb{R}^d} \tau^{-y} f(x)|y|^\alpha-dk d\mu_k(y)
\]
\[
= \frac{2^{dk/2-\alpha}}{\Gamma(\alpha/2)} \int_{\mathbb{R}^d} \tau^{-y} f(x) \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s|y|^2} ds d\mu_k(y)
\]
\[
= \frac{2^{dk/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} \int_{\mathbb{R}^d} \tau^{-y} f(x) e^{-s|y|^2} d\mu_k(y) ds
\]
\[
= \frac{2^{dk/2-\alpha}}{\Gamma(\alpha/2)} \int_{\mathbb{R}^d} f(y) \int_0^\infty s^{(d_k-\alpha)/2-1} \tau^{-y} (e^{-s|\cdot|^2})(x) ds d\mu_k(y).
\]
The interchange of the order of integration is legitimate, since, for any \( x \in \mathbb{R}^d \), the iterated integral
\[
\int_{\mathbb{R}^d} |\tau^{-y} f(x)| \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s|y|^2} ds d\mu_k(y)
\]
converges, where we have used the fact that \( \tau^y f(x) \in S(\mathbb{R}^d) \times S(\mathbb{R}^d) \) whenever \( f \in S(\mathbb{R}^d) \).

2.3 Properties of the Kernel \( \Phi(x, y) \)

We will need the following notation. Let \( \lambda \geq -1/2, J_\lambda(t) \) be the classical Bessel function of degree \( \lambda \) and
\[
j_\lambda(t) = 2^\lambda \Gamma(\lambda + 1)t^{-\lambda} J_\lambda(t)
\]
be the normalized Bessel function. The Hankel transform is defined as follows
\[
\mathcal{H}_\lambda(f_0)(r) = \int_0^\infty f_0(t) j_\lambda(rt) d\nu_\lambda(t), \quad r \in \mathbb{R}_+.
\]
It is a unitary operator in $L^2(\mathbb{R}_+, d\nu_{\lambda})$ and $\mathcal{H}_{\lambda}^{-1} = \mathcal{H}_{\lambda}$ [3, Chap. 7]. If $\lambda = \lambda_k$, the Hankel transform is a restriction of the Dunkl transform on radial functions. Recall that we assume that $\lambda_k > -1/2$.

For $\lambda > -1/2$, let us consider the Gegenbauer-type translation operator (see, e.g., [17])

$$G^s f_0(r) = c_\lambda \int_0^\pi f_0(\sqrt{r^2 + s^2 - 2rs \cos \varphi}) \sin^{2\lambda} \varphi \, d\varphi,$$

(2.9)

where $c_\lambda = \frac{\Gamma(2 + 1)}{\Gamma(1/2) \Gamma(\lambda + 1/2)}$. If $f_0 \in \mathcal{S}(\mathbb{R}_+)$, then

$$G^s f_0(r) = \int_0^\infty j_{\lambda}(rt) j_{\lambda}(st) \mathcal{H}_{\lambda}(f_0)(t) \, d\nu_{\lambda}(t).$$

(2.10)

We will also need the following partial case of the Funk–Hecke formula [31]

$$S_{d-1} e_k(x, ty') \, d\sigma_k(y') = J_{\lambda_k}(t|x|).$$

(2.11)

Let $x = rx'$, $y = ty'$, $r, t > \mathbb{R}_+$, and $x', y' \in S^{d-1}$.

**Lemma 2.3** The kernel $\Phi(x, y)$ satisfies the following properties

1. $\Phi(x, y) = \Phi(y, x)$;
2. $\Phi(rx', ty') = r^{\alpha-d_k} \Phi(x', (t/r)y')$;
3. $\int_{S^{d-1}} \Phi(rx', ty') \, d\sigma_k(x') = \Phi_0(r, t)$, where
   $$\Phi_0(r, t) := (\gamma_k^{1/2})^{-1} c_{\lambda_k} \int_0^\pi \left( r^2 + t^2 - 2rt \cos \varphi \right)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi \, d\varphi;$$
4. $\Phi(x, y) = (\gamma_k^{-1}) \tau^{-y} (|x|^{\alpha-d_k}) (x)$ or, equivalently,
   $$\Phi(x, y) = (\gamma_k^{-1}) \int_{\mathbb{R}_+} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{(\alpha-d_k)/2} \, d\mu_k^\alpha(\eta),$$

where $\mu_k^\alpha$ is a probability measure from Eq. 1.3.

**Proof** Recall that $E_k(x, y)$ is the Dunkl kernel. Using $E_k(\lambda x, y) = E_k(x, \lambda y)$, $\lambda \in \mathbb{C}$, we have from [19, Sec. 4.9] that

$$\int_{\mathbb{R}_+} e_k(x, z)e_k(-y, z)e^{-|z|^2/2} \, d\mu_k(\zeta) = e^{-|x+y|^2/2} E_k(x, y).$$

This, Eq. 2.7, and the fact that $F_k(e^{-|\cdot|^2/2})(y) = e^{-|x|^2/2}$ imply that

$$\tau^{-y} (e^{-s|\cdot|^2})(x) = e^{-s(|x|^2 + |y|^2)} E_k(\sqrt{2s} x, \sqrt{2s} y).$$

Since $E_k(x, y) = E_k(y, x)$, the property (1) follows and, moreover,

$$\Phi(x, y) = \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s(|x|^2 + |y|^2)} E_k(\sqrt{2s} x, \sqrt{2s} y) \, ds.$$  

Changing variables $s \to u/r^2$, we obtain the property (2):

$$\Phi(rx', ty') = r^{\alpha-d_k} \int_0^\infty u^{(d_k-\alpha)/2-1} e^{-u(1+(t/r)^2)} E_k(\sqrt{2u} x', \sqrt{2u} (t/r)y') \, du = r^{\alpha-d_k} \Phi(x', (t/r)y').$$
Since, by Eq. 2.7 and Eq. 1.4, we have
\[
\tau^{-y'} (e^{-s |\alpha \cdot \cdot |^2})(x') = \int_{\mathbb{R}^d} e_k(\sqrt{2sx'}) e_k(-\sqrt{2sty'}, z) e^{-|z|^2/2} d\mu_k(z),
\]
then taking into account Eqs. 2.1, 2.9, 2.10, and 2.11, we obtain
\[
\int_{\mathbb{R}^d} \tau^{-y'} (e^{-s |\alpha \cdot \cdot |^2})(x') d\sigma_k(x')
= \int_{\mathbb{R}^d} e_k(-\sqrt{2sty'}, z) e^{-|z|^2/2} \int_{\mathbb{S}^{d-1}} e_k(\sqrt{2sx'}) d\sigma_k(x') d\mu_k(z)
= \int_0^\infty j_{\lambda_k}(\sqrt{2sru}) e^{-u^2/2} \int_{\mathbb{S}^{d-1}} e_k(-\sqrt{2sty'}, uz') d\sigma_k(z') d\nu_{\lambda_k}(u)
= \int_0^\infty j_{\lambda_k}(\sqrt{2sru}) j_{\lambda_k}(\sqrt{2stu}) e^{-u^2/2} d\nu_{\lambda_k}(u)
= c_{\lambda_k} \int_0^\pi e^{-s(r^2+t^2-2rt\cos \varphi)} \sin^{d_k-2} \varphi d\varphi.
\]
This and Eq. 2.5 imply that
\[
\int_{\mathbb{S}^{d-1}} \Phi(x', ty') d\sigma_k(x')
= 2^{\frac{d_k}{2} - \alpha} \Gamma(\alpha/2)^{-1} c_{\lambda_k} \int_0^\infty s^{(d_k - \alpha)/2 - 1} \int_0^\pi e^{-s(r^2+t^2-2rt\cos \varphi)} \sin^{d_k-2} \varphi d\varphi ds.
\]
Finally, applying Eq. 2.8 gives
\[
\int_{\mathbb{S}^{d-1}} \Phi(x', ty') d\sigma_k(x')
= (\gamma_k^\alpha)^{-1} c_{\lambda_k} \int_0^\pi (r^2 + t^2 - 2rt\cos \varphi)^{(\alpha - d_k)/2} \sin^{d_k-2} \varphi d\varphi = \Phi_0(r, t),
\]
i.e., the property (3) follows.

Let us prove the property (4). Since for radial functions \( f(x) = f_0(|x|) \in \mathcal{S}(\mathbb{R}^d) \) [22, 28]
\[
\tau^{-y} f(x) = \int_{\mathbb{R}^d} f_0(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta \rangle}) d\mu_\alpha^k(\eta),
\]
where \( \mu_\alpha^k \) is the probability measure in Eq. 1.3, we derive
\[
\tau^{-y} (e^{-s |\alpha \cdot \cdot |^2})(x) = \int_{\mathbb{R}^d} e^{-s(|x|^2 + |y|^2 - 2\langle y, \eta \rangle)} d\mu_\alpha^k(\eta)
\]
and
\[
\Phi(x, y) = \frac{2^{\frac{d_k}{2} - \alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k - \alpha)/2 - 1} \int_{\mathbb{R}^d} e^{-s(|x|^2 + |y|^2 - 2\langle y, \eta \rangle)} d\mu_\alpha^k(\eta) ds
= 2^{\frac{d_k}{2} - \alpha} \Gamma(\alpha/2) \int_{\mathbb{R}^d} \int_0^\infty s^{(d_k - \alpha)/2 - 1} e^{-s(|x|^2 + |y|^2 - 2\langle y, \eta \rangle)} ds d\mu_\alpha^k(\eta)
= (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{(\alpha - d_k)/2} d\mu_\alpha^k(\eta),
\]
where we have used the Tonelli–Fubini Theorem for nonnegative functions.
3 Sharp Hardy’s Inequalities

Define the Hardy and Bellman operators as follows

\[ Hf(x) = \int_{|y| \leq |x|} f(y) \, d\mu_k(y) \]

and

\[ Bf(x) = \int_{|y| \geq |x|} f(y) \, d\mu_k(y). \]

Let \( 1 \leq p \leq \infty \). We are interested in the weighted Hardy inequalities of the form

\[ |x|^{-a} Hf(x) \leq c_H^k(a, b, p, d) \| |x|^b f(x) \|_{p, d\mu_k} \]

(3.1)

and

\[ |x|^{-a} Bf(x) \leq c_B^k(a, b, p, d) \| |x|^b f(x) \|_{p, d\mu_k} \]

(3.2)

with the sharp constants \( c_H^k(a, b, p, d) \) and \( c_B^k(a, b, p, d) \).

In the classical setting \( (k \equiv 0) \), the sharp constants were calculated by M. Christ and L. Grafakos [5] in the non-weighted case \((b = 0, a = d)\) and later by Z.W. Fu, L. Grafakos, S.Z. Lu and F.Y. Zhao [8] in the general case. We extend these results for the Dunkl setting.

Recall that

\[ \lambda_k = \frac{d}{2} - 1 + \sum_{a \in R_k} k(a), \quad d_k = 2\lambda_k + 2, \quad b_{k} = \frac{1}{2\lambda_k \Gamma(\lambda_k + 1)}. \]

**Theorem 3.1** Let \( d \in \mathbb{N} \) and \( 1 \leq p \leq \infty \). Inequality (3.1) holds with \( c_H^k(a, b, p, d) < \infty \) if and only if \( \frac{a}{p'} > \frac{b}{p} \) and \( a + b = d_k \). Moreover,

\[ c_H^k(a, b, p, d) = \frac{b_{k}}{\frac{a}{p'} - \frac{b}{p}}. \]

**Proof** Assume that \( \frac{a}{p'} > \frac{b}{p} \) and \( a + b = d_k \). We consider

\[ \tilde{H} f(x) = \int_{|y| \leq |x|} |y|^{d_k/p' - b} f(y) \, dm_k(y). \]

According to Eq. 2.1, inequality (3.1) is equivalent to the following estimate

\[ b_{k} \| |x|^{-a+d_k/p} \tilde{H} f(x) \|_{p, d\mu_k} \leq c_H^k(a, b, p, d) \| f \|_{p, d\mu_k}. \]

If \( x = rx', y = ty' \), then changing variables \( y \rightarrow (r/t)y' \) yields

\[ \tilde{H} f(x) = t^{d_k/p' - b} \int_{\mathbb{R}^d} f(((r/t)y')g_0(t) \, dm_k(ty'), \]

where

\[ g_0(t) = t^{b-d_k/p'} \chi_{[1,\infty)}(t). \]

Hence, by Eq. 2.2, we have

\[ |x|^{-a+d_k/p} \tilde{H} f(x) = \int_{\mathbb{R}^d} f((r/t)y')g_0(t) \, dm_k(ty') = \int_{\mathbb{R}^d} f(ty')g_0(r/t) \, dm_k(ty'). \]
Let us consider the integral
\[ J = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(rx') f(ty') g_0(r/t) \, d\mu_k(x) \, d\mu_k(y) \]
\[ = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty h(rx') f(ty') g_0(r/t) \, d\nu(t) \, d\nu(r) \, d\sigma_k(x') \, d\sigma_k(y'). \]
Using Hölder’s inequality and Lemma 2.1, we obtain
\[ |J| \leq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left( \int_0^\infty |h(rx')|^{p'} \, d\nu(r) \right)^{1/p} \left( \int_0^\infty |f(ty')|^{p'} \, d\nu(t) \right)^{1/p} \]
\[ \times \int_0^\infty g_0(r/t) \, d\nu(t) \, d\sigma_k(x') \, d\sigma_k(y') \]
\[ \leq \|g_0\|_1 \left( \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty |h(rx')|^{p'} \, d\nu(r) \, d\sigma_k(x') \, d\sigma_k(y') \right)^{1/p'} \]
\[ \times \left( \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty |f(ty')|^{p'} \, d\nu(t) \, d\sigma_k(x') \, d\sigma_k(y') \right)^{1/p} \]
\[ = \|g_0\|_1 \|h\|_{p',d\mu_k} \|f\|_{p,d\mu_k}. \]
Hence,
\[ c^H_k(a, b, p, d) \leq b_{\lambda_k} \|g_0\|_1 = b_{\lambda_k} \int_1^\infty t^{b-d_k/p} \, dt = \frac{b_{\lambda_k}}{\frac{a}{p'} - \frac{b}{p}}. \]
Considering radial functions \( f(x) = f_0(|x|) = f_0(r) \), we note that
\[ \mathcal{H} f(x) = r^{d_k/b - \frac{b}{p}} \int_{\mathbb{R}^d} f(r/t) g_0(t) \, dt \]
and
\[ |x|^{a-d_k/p} \mathcal{H} f(x) = \int_{\mathbb{R}^d} f(r/t) g_0(t) \, dt. \]
Thus, Lemma 2.1 yields that
\[ c^H_k(a, b, p, d) = b_{\lambda_k} \|g_0\|_1 = \frac{b_{\lambda_k}}{\frac{a}{p'} - \frac{b}{p}}. \]
Note that, in particular, this implies that the condition \( \frac{a}{p'} > \frac{b}{p} \) is necessary for \( c^H_k(a, b, p, d) < \infty \) to hold. Moreover, if \( f_1(x) = f(tx) \), then
\[ H f_1(x) = t^{-d_k} (H f)(x), \quad \|x|^b f_1(x)\|_{p,d\mu_k} = t^{-b-d_k/p} \|x|^b f(x)\|_{p,d\mu_k} \]
and inequality Eq. 3.1 can be written as
\[ t^{-d_k(1+1/p)+a} \|x|^{-a} H f(x)\|_{p,d\mu_k} \leq t^{-b-d_k/p} c^H_k(a, b, p, d) \|x|^b f(x)\|_{p,d\mu_k}, \]
which gives the condition \( a + b = d_k \).

Similarly, we prove the sharp Hardy’s inequality for Bellman transform.

**Theorem 3.2** Let \( d \in \mathbb{N} \) and \( 1 \leq p \leq \infty \). Inequality (3.2) holds with \( c_k^B(a, b, p, d) < \infty \) if and only if \( \frac{a}{p'} < \frac{b}{p} \) and \( a + b = d_k \). Moreover,
\[ c_k^B(a, b, p, d) = \frac{b_{\lambda_k}}{\frac{b}{p} - \frac{a}{p'}}. \]
Proof We only sketch the proof. Considering
\[ Bf(x) = |y|^{d_k/p'} f(y) dm_k(y) \]
and Eq. 2.1, we rewrite inequality (3.2) as follows
\[ b_{\lambda_k} \|x|^{-a+\alpha} Bf(x)\|_{p, dm_k} \leq c_k^B(a, b, p, d) \|f\|_{p, dm_k}. \]
Then we have
\[ Bf(x) = r^{d_k/p'} \int_{\mathbb{R}^d} f((r/t)y') g_0(t) dm_k(ty'), \]
where
\[ g_0(t) = t^{b-d_k/p'} \chi_{[0,1]}(t). \]
Finally,
\[ c_k^B(a, b, p, d) = b_{\lambda_k} \|g_0\|_1 = b_{\lambda_k} \int_0^1 t^{b-d_k/p'} dt = \frac{b_{\lambda_k}}{p - \frac{a}{p'}}. \]

4 Proof of Theorem 1.3

Recall that we consider the case \( k \neq 0, \lambda_k > -1/2 \) and \( d_k > 1 \). Let \( 1 < p < \infty, \gamma < \frac{d_k}{p}, \beta < \frac{d_k}{p'}, \alpha > 0 \), and \( \alpha = \gamma + \beta \). Consider the modified operator
\[ \tilde{I}_k^\alpha f(x) = \int_{\mathbb{R}^d} f(y)|y|^{d_k/p'-\beta} \Phi(x, y) dm_k(y). \]
According to Eq. 2.1, inequality (1.7) for \( q = p \) is equivalent to
\[ b_{\lambda_k} \|x|^{-\gamma} \tilde{I}_k^\alpha f(x)\|_{p, dm_k} \leq c_k(\alpha, \beta, \gamma, p, d) \|f\|_{p, dm_k}. \]
If \( x = rx', y = ty' \), then using the change of variables \( y \to (r/t)y' \) and applying the properties (1), (2) in Lemma 2.3, we have
\[ \tilde{I}_k^\alpha f(x) = r^{\beta+a-d_k/p} \int_{\mathbb{R}^d} f((r/t)y') \Phi_1(t, x', y') dm_k(ty'), \]
where
\[ \Phi_1(t, x', y') = t^{d_k/p-a+\beta} \Phi(tx', y'). \]
Hence, by Eq. 2.2,
\[ |x|^{-\gamma+d_k/p} \tilde{I}_k^\alpha f(x) = \int_{\mathbb{R}^d} f(ty') \Phi_1(r/t, x', y') dm_k(ty'). \]
We set
\[ J := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(rx') f(ty') \Phi_1(r/t, x', y') dm_k(x) dm_k(y) \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty h(rx') f(ty') \Phi_1(r/t, x', y') d\nu(t) d\nu(r) d\sigma_k(x') d\sigma_k(y'). \]
In light of Lemma 2.1 and Hölder's inequality, we have

\[ |J| \leq \int_{\mathbb{R}^d-1} \int_{\mathbb{R}^d-1} \left( \int_0^\infty |h(rx')|^{p'} \, dv(r) \right)^{1/p'} \left( \int_0^\infty |f(ty')|^p \, dv(t) \right)^{1/p} \]

\[ \times \int_0^\infty \Phi_1(t, x', y') \, d\nu(t) \, d\sigma_k(x') \, d\sigma_k(y') \]

\[ = \int_{\mathbb{R}^d-1} \int_{\mathbb{R}^d-1} \left( \int_0^\infty |h(rx')|^{p'} \, dv(r) \right)^{1/p'} \left( \int_0^\infty \Phi_1(t, x', y') \, d\nu(t) \right)^{1/p} \]

\[ \times \left( \int_0^\infty |f(ty')|^p \, dv(t) \right)^{1/p} \left( \int_0^\infty \Phi_1(t, x', y') \, d\nu(t) \right)^{1/p} \]

\[ \leq \left( \int_{\mathbb{R}^d-1} \int_{\mathbb{R}^d-1} \left( \int_0^\infty |h(rx')|^{p'} \, dv(r) \right)^{1/p'} \left( \int_0^\infty \Phi_1(t, x', y') \, d\nu(t) \right)^{1/p} \]

\[ \left( \int_0^\infty |f(ty')|^p \, dv(t) \right)^{1/p} \left( \int_0^\infty \Phi_1(t, x', y') \, d\nu(t) \right)^{1/p} \]

Taking into account the properties (1) and (3) of Lemma 2.3, we have

\[ \int_{\mathbb{R}^d-1} \Phi_1(t, x', y') \, d\sigma_k(x') = t^{dk/p-\alpha+\beta} \int_{\mathbb{R}^d-1} \Phi(tx', y') \, d\sigma_k(x') = t^{dk/p-\alpha+\beta} \Phi_0(t, 1) \]

and

\[ \int_{\mathbb{R}^d-1} \Phi_1(t, x', y') \, d\sigma_k(y') = t^{dk/p-\alpha+\beta} \Phi_0(t, 1) = t^{dk/p-\alpha+\beta} \Phi_0(t, 1). \]

Then, changing the order of integration implies

\[ |J| \leq \left( \int_0^\infty \Phi_0(t, 1) \, d\nu(t) \right)^{1/p'} \left( \int_{\mathbb{R}^d-1} \left( \int_0^\infty |h(rx')|^{p'} \, dv(r) \right)^{1/p'} \left( \int_0^\infty |f(ty')|^p \, dv(t) \right)^{1/p} \right) \]

\[ \times \left( \int_{\mathbb{R}^d-1} \left( \int_0^\infty \Phi_1(t, x', y') \, d\nu(t) \right)^{1/p} \left( \int_0^\infty \Phi_1(t, x', y') \, d\nu(t) \right)^{1/p} \right) \]

\[ = \int_0^\infty \Phi_0(t, 1) \, d\nu(t) \left( \int_{\mathbb{R}^d-1} \left( \int_0^\infty |h(rx')|^{p'} \, dv(r) \right)^{1/p'} \left( \int_0^\infty |f(ty')|^p \, dv(t) \right)^{1/p} \right) \]

Thus,

\[ c_k(\alpha, \beta, \gamma, p, p, d) \leq b_h \int_0^\infty \Phi_0(t, 1) \, d\nu(t). \]

Since for radial functions \( f(x) = f_0(|x|) = f_0(r) \) we have that

\[ |x|^{-\gamma+dk/p} \int_0^\infty f_0(r/t) t^{dk/p-\alpha+\beta} \Phi_0(t, 1) \frac{dt}{t}, \]

Lemma 2.1 gives

\[ c_k(\alpha, \beta, \gamma, p, p, d) = b_h \int_0^\infty t^{dk/p-\alpha+\beta} \Phi_0(t, 1) \, d\nu(t). \]
Let us now prove that the conditions \( \gamma < \frac{d}{p}, \beta < \frac{d}{p}, \) and \( \gamma + \beta = \alpha > 0 \) guarantee that \( c_k(\alpha, \beta, \gamma, p, p, d) < \infty \). We have

\[
\begin{align*}
    c_k(\alpha, \beta, \gamma, p, p, d) & = (\gamma^k \alpha - 1) c_k b_{\lambda_k} \int_0^\infty t^{d_k/p - \alpha + \beta} \int_0^\pi \left( t^2 + 1 - 2t \cos \varphi \right)^{(\alpha - d_k)/2} \sin^{d_k - 2} \varphi \, d\varphi \, d\nu(t) \\
& = (\gamma^k \alpha - 1) c_k b_{\lambda_k} \int_0^\infty t^{d_k/p - \alpha + \beta} \int_0^\pi \left( 1 - 2t \cos \varphi \right)^{(\alpha - d_k)/2} \sin^{d_k - 2} \varphi \, d\varphi \, d\nu(t).
\end{align*}
\]

The integral with respect to \( t \) has singularities at \( t = 0, 1, \infty \). It converges at the origin if and only if \( \gamma = \alpha - \beta < \frac{d}{p} \). Moreover, the integral converges at \( \infty \) if and only if \( \beta < \frac{d}{p} \).

Concerning the point \( t = 1 \), we set \( r := \frac{2t}{1 + t^2} \) and note that, letting \( r \to 1 - 0 \),

\[
\psi(r) := \int_0^\pi (1 - r \cos \varphi)^{(\alpha - d_k)/2} \sin^{d_k - 2} \varphi \, d\varphi
\]

\[
\simeq \int_0^1 (1 - r + r^2 \varphi^2/2)^{(\alpha - d_k)/2} \varphi^{d_k - 2} \, d\varphi + 1
\]

\[
\simeq \int_0^{\sqrt{1-r}} (1 - r)^{(\alpha - d_k)/2} \varphi^{d_k - 2} \, d\varphi + \int_1^{1/r} r^{\alpha - 2} \, r \varphi^{d_k - 2} \, d\varphi + 1
\]

\[
\simeq \left\{
\begin{array}{l}
    (1-r)^{\frac{\alpha-1}{2}}, \quad 0 < \alpha < 1, \\
    -\ln (1-r), \quad \alpha = 1, \\
    1, \quad \alpha > 1.
\end{array}
\right.
\]

Therefore, letting \( t \to 1 \), we have

\[
\int_0^\pi \left( 1 - 2t \cos \varphi \right)^{(\alpha - d_k)/2} \sin^{d_k - 2} \varphi \, d\varphi \simeq \left\{
\begin{array}{l}
    |1 - t|^{\alpha-1}, \quad 0 < \alpha < 1, \\
    -\ln |1 - t|, \quad \alpha = 1, \\
    1, \quad \alpha > 1,
\end{array}
\right.
\]

which implies that the singularity at the point \( t = 1 \) is integrable.

It remains to calculate the integral \( \int_0^\infty t^{d_k/p - \alpha + \beta} \Phi_0(t, 1) \, d\nu(t) \). Let \( t \neq 1, r = 2t/(1 + t^2) \). The series

\[
(1 - r \cos \varphi)^{(\alpha - d_k)/2} = \Gamma \left( \frac{\alpha - d_k}{2} + 1 \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1) \Gamma(n+1)} r^n \cos^n \varphi
\]

\[
= \frac{1}{\Gamma \left( \frac{d_k - \alpha}{2} \right)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma \left( \frac{d_k - \alpha}{2} + n \right)}{\Gamma(n+1)} r^n \cos^n \varphi
\]

\[
\Phi_0(t, 1) = \int_0^\pi \left( 1 - 2t \cos \varphi \right)^{(\alpha - d_k)/2} \sin^{d_k - 2} \varphi \, d\varphi
\]

\[
\simeq \left\{
\begin{array}{l}
    |1 - t|^{\alpha-1}, \quad 0 < \alpha < 1, \\
    -\ln |1 - t|, \quad \alpha = 1, \\
    1, \quad \alpha > 1,
\end{array}
\right.
\]

\[
\int_0^\pi \left( 1 - 2t \cos \varphi \right)^{(\alpha - d_k)/2} \sin^{d_k - 2} \varphi \, d\varphi \simeq \left\{
\begin{array}{l}
    |1 - t|^{\alpha-1}, \quad 0 < \alpha < 1, \\
    -\ln |1 - t|, \quad \alpha = 1, \\
    1, \quad \alpha > 1.
\end{array}
\right.
\]

\[
\psi(r) := \int_0^\pi (1 - r \cos \varphi)^{(\alpha - d_k)/2} \sin^{d_k - 2} \varphi \, d\varphi
\]

\[
\simeq \int_0^1 (1 - r + r^2 \varphi^2/2)^{(\alpha - d_k)/2} \varphi^{d_k - 2} \, d\varphi + 1
\]

\[
\simeq \int_0^{\sqrt{1-r}} (1 - r)^{(\alpha - d_k)/2} \varphi^{d_k - 2} \, d\varphi + \int_1^{1/r} r^{\alpha - 2} \, r \varphi^{d_k - 2} \, d\varphi + 1
\]

\[
\simeq \left\{
\begin{array}{l}
    (1-r)^{\frac{\alpha-1}{2}}, \quad 0 < \alpha < 1, \\
    -\ln (1-r), \quad \alpha = 1, \\
    1, \quad \alpha > 1.
\end{array}
\right.
\]
converges uniformly on \([0, \pi]\) and
\[
\psi(r) = \int_0^\pi (1 - r \cos \varphi)^{\frac{\alpha - dk}{2}} \sin^{dk - 2} \varphi \, d\varphi
\]
\[
= \frac{1}{\Gamma \left( \frac{dk - \alpha}{2} \right)} \sum_{m=0}^\infty \frac{\Gamma \left( \frac{dk - \alpha}{2} + 2m \right)}{\Gamma (2m + 1)} r^{2m} \int_0^\pi \cos^{2m} \varphi \sin^{dk - 2} \varphi \, d\varphi
\]
\[
= \frac{1}{\Gamma \left( \frac{dk - \alpha}{2} \right)} \sum_{m=0}^\infty \frac{\Gamma \left( m + \frac{1}{2} \right) \Gamma \left( \frac{dk - \alpha}{2} + 2m \right) \Gamma \left( \frac{dk - 1}{2} \right)}{\Gamma (2m + 1) \Gamma \left( \frac{dk}{2} + m \right)} r^{2m}.
\]

Since a positive series can be integrated term-by-term, it follows that
\[
c_k(\alpha, \beta, \gamma, p, p, d) = b_{\lambda k} \int_0^\infty t^{d_k/p - \alpha + \beta} \Phi_0(t, 1) \, dv(t)
\]
\[
= \left( \gamma_k \right)^{-1} c_k b_{\lambda k} \sum_{m=0}^\infty 2^{2m} \frac{\Gamma \left( m + \frac{1}{2} \right) \Gamma \left( \frac{dk - \alpha}{2} + 2m \right) \Gamma \left( \frac{dk - 1}{2} \right)}{\Gamma (2m + 1) \Gamma \left( \frac{dk}{2} + m \right)} \int_0^\infty t^{d_k/p - \alpha + \beta + 2m - 1} (1 + t^2)^{(d_k - \alpha)/2 + 2m} \, dt.
\]

Taking into account that
\[
\int_0^\infty t^{d_k/p - \alpha + \beta + 2m - 1} (1 + t^2)^{(d_k - \alpha)/2 + 2m} \, dt = \frac{\Gamma \left( \frac{d_k}{2} + \frac{\beta - \alpha}{2} + m \right) \Gamma \left( \frac{d_k}{2p'} - \frac{\beta}{2} + m \right)}{2 \Gamma \left( \frac{d_k - \alpha}{2} + m \right)}
\]
and
\[
\gamma_k = \frac{2^\alpha - d_k/2 \Gamma \left( \frac{d_k}{2} \right)}{\Gamma \left( \frac{dk - \alpha}{2} \right)}, \quad c_k = \frac{\Gamma \left( \frac{d_k}{2} \right)}{\Gamma (1/2) \Gamma \left( \frac{dk - 1}{2} \right)}, \quad b_{\lambda k} = \frac{1}{2^{d_k/2 - 1} \Gamma \left( \frac{d_k}{2} \right)},
\]
we arrive at
\[
c_k(\alpha, \beta, \gamma, p, p, d) = 2^{-\alpha} \sum_{m=0}^\infty \frac{\Gamma \left( \frac{d_k}{2p} + \frac{\beta - \alpha}{2} + m \right) \Gamma \left( \frac{d_k}{2p'} - \frac{\beta}{2} + m \right)}{\Gamma (m + 1) \Gamma \left( \frac{dk}{2} + m \right)}.
\]

Letting
\[
a = \frac{d_k}{2p} + \frac{\beta - \alpha}{2}, \quad b = \frac{d_k}{2p'} - \frac{\beta}{2}, \quad c = \frac{d_k}{2},
\]
we write
\[
c_k(\alpha, \beta, \gamma, p, p, d) = 2^{-\alpha} \sum_{m=0}^\infty \frac{\Gamma (a + m) \Gamma (b + m)}{\Gamma (1 + m) \Gamma (c + m)}.
\]

Using now the hypergeometric function [2, Ch. II]
\[
F(a, b; c; z) = \sum_{m=0}^\infty \frac{(a)_m (b)_m}{m! (c)_m} z^m, \quad (a)_m = \frac{\Gamma (a + m)}{\Gamma (a)},
\]
we obtain that
\[
c_k(\alpha, \beta, \gamma, p, p, d) = 2^{-\alpha} \frac{\Gamma (a) \Gamma (b)}{\Gamma (c)} F(a, b; c; 1).
\]
Finally, since [2, Sect. 2.8, (46)]
\[ F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad c \neq 0, -1, -2, \ldots, \quad c > a + b, \]
we have
\[ c_k(\alpha, \beta, \gamma, p, p, d) = \frac{2^{-\alpha}\Gamma(a)\Gamma(b)\Gamma(c - a - b)}{\Gamma(\alpha/2)\Gamma(c - a)\Gamma(c - b)}, \]
where
\[ c - a - b = \frac{d_k}{2} - \left(\frac{d_k}{2p} + \frac{\beta - a}{2} + \frac{d_k}{2p^2} - \frac{\beta}{2}\right) = \frac{\alpha}{2}, \]
\[ c - a = \frac{d_k}{2} - \left(\frac{d_k}{2p} + \frac{\beta - a}{2}\right) = \frac{d_k}{2p} + \frac{\alpha - \beta}{2}, \]
\[ c - b = \frac{d_k}{2} - \left(\frac{d_k}{2p^2} - \frac{\beta}{2}\right) = \frac{d_k}{2p} + \frac{\beta}{2}, \]
or, equivalently,
\[ c_k(\alpha, \beta, \gamma, p, p, d) = 2^{-\alpha} \frac{\Gamma\left(\frac{1}{2}\left(\frac{d_k}{p} - \gamma\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{d_k}{p^2} - \beta\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{d_k}{p} + \gamma\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{d_k}{p^2} + \beta\right)\right)}. \]

Remark 4.1 It is clear that the condition \( \alpha = \gamma + \beta \) is necessary for \( c_k(\alpha, \beta, \gamma, p, p, d) < \infty \) to hold. Indeed, setting \( f_t(x) = f(tx) \), we have
\[ \mathcal{F}_k(f_t)(z) = t^{-d_k}\mathcal{F}_k(f)\left(\frac{z}{t}\right), \quad \tau^y f_t(x) = \tau^{ty} f(tx), \quad I_k^\alpha f_t(x) = t^{-\alpha}(I_k^\alpha f)_t(x), \]
\[ \|x|^\beta f_t(x)\|_{p, d\mu_k} = t^{-\beta - d_k/p}\|x|^\beta f(x)\|_{p, d\mu_k}. \]
Writing inequality Eq. 1.7 with \( q = p \) as follows
\[ t^{-\beta - d_k/p}\|x|^{-\gamma} I_k^\alpha f(x)\|_{p, d\mu_k} \leq t^{-\beta - d_k/p} c_k(\alpha, \beta, \gamma, p, p, d)\|x|^\beta f(x)\|_{p, d\mu_k} \]
implies \( \alpha = \gamma + \beta \).

5 Proof of Theorem 1.4

Part (a) Let \( 1 < p < q < \infty, \gamma < \frac{d_k}{q}, \beta < \frac{d_k}{p}, \gamma + \beta \geq 0, 0 < \alpha < d_k \), and \( \alpha - \gamma - \beta = d_k\left(\frac{1}{p} - \frac{1}{q}\right) \). Note that the case \( q = p \) was studied in Theorem 1.3. We will use the representation of the kernel \( \Phi(x, y) \) given in Lemma 2.3 and then essentially follow the ideas of [27].

We write
\[ \mathcal{F}^k_\alpha f(x) = \int_{\mathbb{R}^d} f(y)|y|^{-\beta}\Phi_\alpha(x, y)\ d\mu_k(y), \]
where
\[ f \in \mathcal{S}(\mathbb{R}^d), \quad \Phi_\alpha(x, y) = \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{(\alpha - d_k)/2}\ d\mu_k^\gamma(\eta), \]
and
\[ \text{supp } \mu_k^\gamma \subset B_{|x|} = \{\eta: |\eta| \leq |x|\}. \]

We define
\[ J := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)g(x) \frac{\Phi_\alpha(x, y)}{|x|^\gamma|y|^\beta} \ d\mu_k(y) \ d\mu_k(x). \]
It is sufficient to prove the inequality
\[ J \lesssim \|f\|_{p,d\mu_k}\|g\|_{q',d\mu_k} \] (5.1)
for \( f, g \geq 0 \).

Recall that in the case \( 1 < p < q < \infty \), \( \gamma = \beta = 0 \), and \( \alpha = d_k(\frac{1}{p} - \frac{1}{q}) \) inequality Eq. 5.1 holds (see [1, 11]). Let
\[ \mathbb{R}^d \times \mathbb{R}^d = E_1 \sqcup E_2 \sqcup E_3, \]
where
\[ E_1 = \{(x, y) : 2^{-1}|y| < |x| < 2|y|\}, \]
\[ E_2 = \{(x, y) : |x| \leq 2^{-1}|y|\}, \]
\[ E_3 = \{(x, y) : |y| \leq 2^{-1}|x|\}. \]

Then
\[ J = \iint_{E_1} + \iint_{E_2} + \iint_{E_3} = J_1 + J_2 + J_3. \]

Estimate of \( J_1 \). If \((x, y) \in E_1\), using \( |\eta| \leq |x|\), then by conditions \( \alpha - \beta - \gamma = d_k\left(\frac{1}{p} - \frac{1}{q}\right)\), \( \gamma + \beta \geq 0 \) we have
\[ \left(|x|^2 + |y|^2 - 2(y, \eta)\right)^{\frac{\gamma + \beta}{2}} \lesssim |x|^\gamma |y|^\beta \]
and
\[ \left(|x|^2 + |y|^2 - 2(y, \eta)\right)^{\frac{\gamma + \beta - d_k}{2}} \lesssim \left(|x|^2 + |y|^2 - 2(y, \eta)\right)^{\frac{\gamma + \beta - d_k}{2}} \]
\[ = \left(|x|^2 + |y|^2 - 2(y, \eta)\right)^{\frac{d_k\left(\frac{1}{p} - \frac{1}{q}\right) - d_k}{2}}. \]

Set \( \tilde{\alpha} = d_k\left(\frac{1}{p} - \frac{1}{q}\right)\). By Eq. 5.1 with \( \gamma = \beta = 0 \) and \( 0 < \tilde{\alpha} < d_k \), we have
\[ J_1 \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)g(x) \Phi_{\tilde{\alpha}}(x, y) \, d\mu_k(y) \, d\mu_k(x) \lesssim \|f\|_{p,d\mu_k}\|g\|_{q',d\mu_k}. \]

Estimate of \( J_2 \). If \((x, y) \in E_2\), then
\[ \sqrt{|x|^2 + |y|^2 - 2(y, \eta)} \geq \sqrt{|x|^2 + |y|^2 - 2|x||y|} \geq |y| - |x| \geq 2^{-1}|y|, \]
therefore
\[ \Phi_{\alpha}(x, y) = \int_{\mathbb{R}^d} \frac{1}{(\sqrt{|x|^2 + |y|^2 - 2(y, \eta)})^{d_k - \alpha}} \, d\mu_k^k(\eta) \]
\[ \lesssim |y|^\alpha \int_{\mathbb{R}^d} d\mu_k^k(\eta) = |y|^{\alpha - d_k}. \]

From here and since \( E_2 \subset \{(x, y) : |x| \leq |y|\}, \)
\[ J_2 \lesssim \iint_{|x| \leq |y|} \frac{f(y)g(x)}{|x|^\gamma |y|^\beta - d_k} \, d\mu_k(x) \, d\mu_k(y) \]
\[ = \int_{\mathbb{R}^d} f(y)|y|^\alpha \int_{|x| \leq |y|} g(x)|x|^{-\gamma} \, d\mu_k(x) \, d\mu_k(y) \]
\[ = \int_{\mathbb{R}^d} f(y)|y|^\alpha \, Vg(y) \, d\mu_k(y). \]
where

\[ V_g(y) = |y|^{\gamma - d_k} \int_{|x| \leq |y|} g(x) |x|^{-\gamma} \, d\mu_k(x). \]

Note that

\[ V_g(y) \leq |y|^{\gamma - d_k} \left( \int_{|x| \leq |y|} |x|^{-q' \gamma} \, d\mu_k(x) \right)^{1/q} \|g\|_{q',d\mu_k} \]
\[ \lesssim |y|^{\gamma - d_k} |y|^{d_k/q - \gamma} \|g\|_{q',d\mu_k} = |y|^{-d_k/q} \|g\|_{q',d\mu_k}. \]

Hence

\[ |V_g(y)|^{\nu - q'} |y|^{(\alpha - \beta - \gamma) p'} \lesssim |y|^{-d_k (p' - q')/q' + (\alpha - \beta - \gamma) p'} \|g\|_{q',d\mu_k}. \]

Since

\[ -\frac{d_k (p' - q')}{q'} + (\alpha - \beta - \gamma) p' = \frac{d_k (p' - q')}{q'} + (\alpha - \beta - \gamma) \left( \frac{1}{q'} - \frac{1}{p'} \right) = 0, \]

it follows that

\[ |V_g(y)|^{\nu - q'} |y|^{(\alpha - \beta - \gamma) p'} \lesssim \|g\|_{q',d\mu_k}. \quad (5.2) \]

On the other hand, by Theorem 3.1 with \( a = d_k - \gamma, b = \gamma, p = q' \), and \( \frac{a}{q} > \frac{b}{q} \) (or, equivalently, \( \gamma < \frac{d_k}{q} \)), we see that

\[ \|V_g\|_{q',d\mu_k} \lesssim \|g\|_{q',d\mu_k}. \quad (5.3) \]

Using Eqs. 5.2 and 5.3, we have

\[ \int_{\mathbb{R}^d} |V_g(y)|^{\nu - q'} |y|^{(\alpha - \beta - \gamma) p'} \, d\mu_k(y) = \int_{\mathbb{R}^d} |V_g(y)|^{q'} |V_g(y)|^{\nu - q'} |y|^{(\alpha - \beta - \gamma) p'} \, d\mu_k(y) \]
\[ \lesssim \int_{\mathbb{R}^d} |V_g(y)|^{q'} \, d\mu_k(y) \|g\|_{q',d\mu_k} \lesssim \|g\|_{q',d\mu_k}. \]

This gives

\[ J_2 \lesssim \|f\|_{p,d\mu_k} \|y|^{\alpha \beta - \gamma} V_g(y)\|_{p',d\mu_k} \lesssim \|f\|_{p,d\mu_k} \|g\|_{q',d\mu_k}. \]

Note that, for \( p = 1 \), a similar result is valid as well, i.e.,

\[ J_2 \lesssim \|f\|_{1,d\mu_k} \|g\|_{q',d\mu_k}, \quad \gamma < \frac{d_k}{q}, \beta \leq 0, \quad (5.4) \]

since \( \alpha - \beta - \gamma = d_k/q' \) and

\[ |y|^{\alpha \beta - \gamma} V_g(y) \lesssim |y|^{\alpha \beta - \gamma} |y|^{-d_k/q'} \|g\|_{q',d\mu_k} = \|g\|_{q',d\mu_k}. \]

Estimate of \( J_3 \). If \((x, y) \in E_3\), we similarly have

\[ \Phi_\alpha(x, y) = \int_{\mathbb{R}^d} \frac{1}{(\sqrt{|x|^2 + |y|^2} - 2(y, \eta))^d - \alpha} d\mu_k^b(\eta) \lesssim |x|^{-d_k} \]

and

\[ J_3 \lesssim \int_{|y| \leq |x|} \frac{Y(y) g(x)}{|x|^\gamma \alpha + d_k |y|^{\beta}} \, d\mu_k(x) \, d\mu_k(y) = \int_{\mathbb{R}^d} g(x) |x|^{\alpha \beta - \gamma} Vf(x) \, d\mu_k(x). \]

\[ \text{D.V. Gorbachev et al.} \]
where
\[ Vf(x) = |x|^\beta - d_k \int_{|y| \leq |x|} f(y)|y|^{-\beta} d\mu_k(y). \]

Since
\[ Vf(x) \leq |x|^\beta - d_k \left( \int_{|y| \leq |x|} |y|^{-p'} d\mu_k(y) \right)^{1/p'} \|f\|_{p, d\mu_k} \]
\[ \leq |x|^\beta - d_k |x|^{d_k/p' - \beta} \|f\|_{p, d\mu_k} = |x|^{-d_k/p} \|f\|_{p, d\mu_k}, \]
we obtain
\[ |Vf(x)|^q - p |x|^{(\alpha - \beta - \gamma)q} \leq |x|^{-d_k(q - p)/p + (\alpha - \beta - \gamma)q} \|f\|^{q - p}_{p, d\mu_k} = \|f\|^{q - p}_{p, d\mu_k}. \]

Taking into account Theorem 3.1 with \( a = d_k - \beta, \ b = \beta, \ p = p, \ a/p > b/p \) (or, \( \beta < d_k/p \)),
we obtain
\[ Vf_{p, d\mu_k}, \]
which implies
\[ \int_{\mathbb{R}^d} |Vf(x)|^q |x|^{(\alpha - \beta - \gamma)q} d\mu_k(x) = \int_{\mathbb{R}^d} |Vf(x)|^p |Vf(x)|^{q - p} |x|^{(\alpha - \beta - \gamma)q} d\mu_k(x) \]
\[ \leq \int_{\mathbb{R}^d} |Vf(x)|^p d\mu_k(x) \|f\|^{q - p}_{p, d\mu_k} \leq \|f\|^{q}_{p, d\mu_k}. \]

We finally have
\[ J_3 \leq \|g\|_{q', d\mu_k} \|x|^{\alpha - \beta - \gamma} Vf(x)\|_{q, d\mu_k} \leq \|f\|_{p, d\mu_k} \|g\|_{q', d\mu_k}. \]
Again, the estimate \( J_3 \leq \|f\|_{p, d\mu_k} \|g\|_{q', d\mu_k} \) also holds for \( p = 1, \gamma < \frac{d_k}{q}, \) and \( \beta < 0. \)

This completes the proof of part (a).

**Part (b)** Let us prove the weak (\( L^q, L^1 \))-boundedness of D-Riesz potential for \( 1 < q < \infty, \ \gamma < \frac{d_k}{q}, \ \beta < 0, \ \gamma + \beta \geq 0, \ 0 < \alpha < d_k. \) We will use the notations and assumptions of part (a).

It is sufficient to prove the inequality
\[ S = \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} \tilde{f}_{\alpha}^i (x) > \lambda/3\}} d\mu_k(x) \leq \left( \frac{\|f\|_{1, d\mu_k}}{\lambda} \right)^q. \]  

(5.5)

Let us consider the operators
\[ A_{\alpha}^i f(x) = \int_{\mathbb{R}^d} f(y)|y|^{-\beta} \Phi_{\alpha}(x, y) \chi_{E_i}(x, y) \ d\mu_k(y), \quad i = 1, 2, 3. \]

We have
\[ \tilde{f}_{\alpha}^i = \sum_{i=1}^{3} A_{\alpha}^i, \]
and
\[ S = \sum_{i=1}^{3} \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} A_{\alpha}^i f(x) > \lambda/3\}} d\mu_k(x) = S_1 + S_2 + S_3. \]

Estimate of \( S_1. \) Applying the estimate of \( J_1 \) and inequality (5.5) with \( 1 < q < \infty, \ \gamma = \beta = 0, \) and \( \alpha = \tilde{\alpha} = \frac{d_k}{q} \) (see [1, 11]), we derive
\[ A_{\alpha}^1 f(x) \lesssim \int_{\mathbb{R}^d} f(y)|y|^{-\beta} \Phi_{\tilde{\alpha}}(x, y) \ d\mu_k(y) = \tilde{f}_{\alpha}^1 f(x), \]
and
\[
S_1 = \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma}A_1^\delta f(x) > \lambda/3\}} d\mu_k(x)
\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma}T_0^\delta f(x) > \lambda\}} d\mu_k(x) \lesssim \left( \frac{\|f\|_1 \mu_k}{\lambda} \right)^q.
\]

Estimate of \(S_2\). Applying the obtained estimate of \(J_2\), we get
\[
A_2^\alpha f(x) \lesssim \int_{|y| \geq |x|} |y|^{\alpha-\beta-d_k} f(y) d\mu_k(y) = B_1 f(x).
\] Since
\[
\int_{\mathbb{R}^d} g(x) |x|^{-\gamma} \int_{|y| \geq |x|} |y|^{\alpha-\beta-d_k} f(y) d\mu_k(y) d\mu_k(x)
\]
\[
= \int_{\mathbb{R}^d} f(y) |y|^{\alpha-\beta-d_k} \int_{|x| \leq |y|} g(x) |x|^{-\gamma} d\mu_k(x) d\mu_k(y),
\]
in light of Eq. 5.4 with \(p = 1\), \(\gamma < \frac{d_k}{q}\), and \(\beta \leq 0\), we have
\[
\| |x|^{-\gamma} B_1 f(x) \|_{q,d\mu_k} \lesssim \| f \|_{1,d\mu_k}.
\]
Hence,
\[
S_2 = \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma}A_1^\delta f(x) > \lambda/3\}} d\mu_k(x)
\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma}B_1 f(x) > \lambda\}} d\mu_k(x) \lesssim \left( \frac{\|f\|_{1,d\mu_k}}{\lambda} \right)^q.
\]

Estimate of \(S_3\). Applying the estimate of \(J_3\), we obtain
\[
A_3^\alpha f(x) \lesssim |x|^{\alpha-d_k} \int_{|y| \leq |x|} |y|^{-\beta} f(y) d\mu_k(y) = H_1 f(x).
\] Using the estimate \(J_3 \lesssim \|f\|_{1,d\mu_k} \|g\|_{q',d\mu_k}\) with \(\gamma < \frac{d_k}{q}\) and \(\beta < 0\) yields
\[
\| |x|^{-\gamma} H_1 f(x) \|_{q,d\mu_k} \lesssim \| f \|_{1,d\mu_k}.
\]
Thus,
\[
S_3 = \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma}A_1^\delta f(x) > \lambda/3\}} d\mu_k(x)
\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma}H_1 f(x) > \lambda\}} d\mu_k(x) \lesssim \left( \frac{\|f\|_{1,d\mu_k}}{\lambda} \right)^q. \tag{5.6}
\]
If \(\beta = 0\), \(\alpha - \gamma = d_k(1 - \frac{1}{q})\), then
\[
|x|^{-\gamma} H_1 f(x) = |x|^{\alpha-\gamma-d_k} \int_{|x| \leq |y|} f(y) d\mu_k(y) = |x|^{-d_k/q} \int_{|y| \leq |x|} f(y) d\mu_k(y).
\]
Since inequality (5.6) is homogeneous, we can assume that \( \|f\|_{1,d\mu_k} = 1 \). Therefore,

\[
S_3 \lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-d_k/q} f(y) d\mu_k(y) \geq \lambda\}} d\mu_k(x)
\]

\[
\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-d_k/q} \geq \lambda\}} d\mu_k(x) \lesssim \lambda^{-q} = \left( \frac{\|f\|_{1,d\mu_k}}{\lambda} \right)^q,
\]

completing the proof.

Let us mention that for the so-called B-Riesz potentials the results that are similar to Theorem 1.4 were established in [9].

### 6 Properties of the Spaces \( \Phi_k \) and \( \Psi_k \)

Recall that \( T_j, j = 1, \ldots, d, \) are differential-differences Dunkl operators given by (1.2),

\[
T^n = \prod_{j=1}^d T_j^{n_j}, n \in \mathbb{Z}_+^d,
\]

\[
\Phi_k = \left\{ f \in S(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^n f(x) d\mu_k(x) = 0, \ n \in \mathbb{Z}_+^d \right\},
\]

and

\[
\Psi_k = \{ \mathcal{F}_k(f) : f \in \Phi_k \}.
\]

If \( f \in S(\mathbb{R}^d) \), then from the definition of the generalized exponential function \( e_k(x, y) \) and

\[
f(x) = \int_{\mathbb{R}^d} e_k(x, y) \mathcal{F}_k(f)(y) d\mu_k(y),
\]

we obtain that

\[
T^n f(x) = i^n \int_{\mathbb{R}^d} y^n e_k(x, y) \mathcal{F}_k(f)(y) d\mu_k(y), \ n \in \mathbb{Z}_+^d,
\]

and

\[
T^n f(0) = i^n \int_{\mathbb{R}^d} y^n \mathcal{F}_k(f)(y) d\mu_k(y).
\]

Therefore,

\[
\Psi_k = \left\{ f \in S(\mathbb{R}^d) : T^n f(0) = 0, \ n \in \mathbb{Z}_+^d \right\}.
\]

Note that in the classical case \( k \equiv 0 \) we have

\[
\Psi = \Psi_0 = \{ \mathcal{F}(f) : f \in \Phi \} = \left\{ f \in S(\mathbb{R}^d) : D^n f(0) = 0, \ n \in \mathbb{Z}_+^d \right\}.
\]

#### Theorem 6.1

We have \( \Psi_k = \Psi \).

**Proof** Let \( f \in \Psi, D = (D_1, \ldots, D_d) \), and let \( \partial_a f(x) = (D f(x), \frac{a}{|a|}) \) be the directional derivative with respect to a vector \( a \). Taking into consideration

\[
\frac{f(x) - f(\sigma_a x)}{(a, x)} = \frac{2}{|a|} \int_0^1 \partial_a f \left( x - \frac{2t(a, x)}{|a|^2} a \right) dt
\]

and

\[
T_j f(x) = D_j f(x) + \sum_{a \in \mathbb{R}_+} \frac{2k(a)(a, e_j)}{|a|} \int_0^1 \partial_a f \left( x - \frac{2t(a, x)}{|a|^2} a \right) dt \quad (6.1)
\]
we obtain that $T_j f(0) = 0$, $j = 1, \ldots, d$. By Eq. 6.1, we derive that $\Psi \subset \Psi_k$. In addition, if $D^n f(0) = 0$ for $|n| = \sum_{j=1}^{d} n_j \leq m$, then $T^n f(0) = 0$ for $|n| \leq m$.

Let $m \in \mathbb{Z}_+$ and $f \in \Psi_k$ be a real function. Using the Taylor formula, we write

$$f(x) = p(x) + r(x),$$

where $p(x)$ is a polynomial of degree $\deg p \leq m$, and $D^n r(0) = 0$ for $|n| \leq m$. Since $T^n f(0) = T^n r(0) = 0$ for $|n| \leq m$, it follows that $T^n p(0) = 0$ for $|n| \leq m$ and, in particular, $p(0) = 0$. By [21],

$$0 = p(T) p(0) = \int_{\mathbb{R}^d} \left( e^{-\Delta k/2} p(x) \right)^2 e^{-|x|^2/2} d\mu_k(x)$$

and $e^{-\Delta k/2} p(x) = 0$. Since $e^{-\Delta k/2}$ is a bijective operator on the set of all polynomials [21], we obtain that $p(x) \equiv 0$, and $D^n f(0) = D^n p(0) = 0$ for $|n| \leq m$. Thus, $\Psi_k \subset \Psi$. \qed

Theorem 6.1 immediately implies the following

**Corollary 6.2** We have $I^k_{\alpha} (\Phi_k) = \Phi_k$ and $\mathcal{F}_k (I^k_{\alpha}) (\Psi_k) = \Psi_k$.

Let $f \in S(\mathbb{R}^d)$. Using the positive $L^p$-bounded generalized translation operator

$$T^i f(x) = \int_{\mathbb{R}^d} \tau^y f(x) d\sigma_k(y'),$$

Gorbachev et al. [11], we can write the D-Riesz potential and the convolution with a radial function $g_0(|y|)$ as follows

$$I^k_{\alpha} f(x) = (\gamma^k_{\alpha})^{-1} \int_0^{\infty} T^i f(x) t^{\alpha-dk} d\nu_{\lambda_k}(t) \quad (6.2)$$

and

$$\int_{\mathbb{R}^d} \tau^{-y} f(x) g_0(|y|) d\mu_k(y) = \int_0^{\infty} T^i f(x) g_0(t) d\nu_{\lambda_k}(t). \quad (6.3)$$

The proof of the following result is based on Theorem 1.3.

**Theorem 6.3** If $1 < p < \infty$, $-\frac{dk}{p} < \beta < \frac{dk}{p}$, then $\Phi_k$ is dense in $L^p(\mathbb{R}^d, |x|^{\beta p} d\mu_k)$.

**Proof** Let $\eta \in S(\mathbb{R}^d)$ be such that $\eta(x) = 1$ if $|x| \leq 1$, $\eta(x) > 0$ if $|x| < 2$, and $\eta(x) = 0$ if $|x| \geq 2$. We can assume that $f \in S(\mathbb{R}^d)$.

Set

$$\psi_0(|y|) = \mathcal{F}_k (\eta)(y), \quad \psi_{0N}(|y|) = \frac{1}{N^{dk}} \psi_0 \left( \frac{|y|}{N} \right) = \mathcal{F}_k (\eta(N \cdot))(y),$$

$$\varphi_N(x) = f(x) - \int_{\mathbb{R}^d} \tau^{-y} f(x) \psi_{0N}(|y|) d\mu_k(y).$$

Since (see [11])

$$\mathcal{F}_k (\varphi_N)(z) = (1 - \eta(Nz)) \mathcal{F}_k (f)(z) \in \Psi_k,$$

it follows that $\varphi_N \in \Phi_k$ and, by Eq. 6.3,

$$\| |x|^\beta (f(x) - \varphi_N(x)) \|_{p,d\mu_k} \leq \| |x|^\beta \int_0^{\infty} T^i f(x) \psi_{0N}(t) d\nu_{\lambda_k}(t) \|_{p,d\mu_k}. \quad (6.4)$$
For any $\alpha \in (0, d_k)$, we have
\[ |\psi_0(t)| \lesssim t^{\alpha - d_k} \quad \text{and} \quad |\psi_0^N(t)| \lesssim N^{-\alpha} t^{\alpha - d_k}. \]
Hence, by positivity of the operator $T^t$ and Eq. 6.2,
\[
\left| \int_0^\infty T^t f(x) \psi_0^N(t) \, d\nu_{\lambda_k}(t) \right| \leq \int_0^\infty T^t |f(x)| \psi_0^N(t) \, d\nu_{\lambda_k}(t) \lesssim N^{-\alpha} \int_0^\infty T^t |f(x)| t^{\alpha - d_k} \, d\nu_{\lambda_k}(t) = N^{-\alpha} I^k \alpha |f(x)|. 
\]
This, Eq. 6.4, and Theorem 1.3 imply
\[
\left\| |x|^\beta \left( f(x) - \varphi_N(x) \right) \right\|_{p,d\mu_k} \lesssim N^{-\alpha} \left\| |x|^\beta I^k \alpha |f(x)| \right\|_{p,d\mu_k} \lesssim N^{-\alpha} \left\| |x|^\delta f(x) \right\|_{p,d\mu_k} \lesssim N^{-\alpha},
\]
where $\alpha > 0$ is chosen so that $\delta = \alpha + \beta < \frac{d_k}{p'}$. \hfill \Box

References

1. Abdelkefi, C., Rachdi, M.: Some properties of the Riesz potentials in Dunkl analysis. Ricerche Mat. 4, 195–215 (2015)
2. Bateman, G., Erdélyi, A., et al.: Higher Transcendental Functions. I. McGraw Hill Book Company, New York (1953)
3. Bateman, H., Erdélyi, A.: Higher Transcendental Functions, vol. 2. MacGraw-Hill, New York (1953)
4. Beckner, W.: Pitt’s inequality with sharp convolution estimates. Proc. Am. Math. Soc. 5, 1871–1885 (2008)
5. Christ, M., Grafakos, L.: Best constants for two nonconvolution inequalities. Proc. Am. Math. Soc. 6, 1687–1693 (1995)
6. Dunkl, C.F.: Hankel transforms associated to finite reflections groups. Contemp. Math. 138, 123–138 (1992)
7. Frostman, O.: Potentiel d’équilibre et capacite des ensembles avec quelques applications a la theorie des fonctions. These, Commun. Semin. Math. de l’Univ. de Lund., vol. 3 (1935)
8. Fu, Z.W., Grafakos, L., Lu, S.Z., Zhao, F.Y.: Sharp bounds for $m$-linear Hardy and Hilbert operators. Houst. J. Math. 1, 225–244 (2012)
9. Gadjiev, A.D., Guliyev, V.S., Serbetci, A., Guliyev, E.V.: The Stein–Weiss type inequalities for the B-Riesz potentials. J. Math. Ineq. 5(1), 87–106 (2011)
10. Gaczyk, P., Luks, T., Rösler, M.: On the green function and poisson integrals of the Dunkl Laplacian. Potential Anal. 48, 337–360 (2018)
11. Gorbachev, D.V., Ivanov, V.I., Tikhonov, S.Yu.: $L^p$-bounded Dunkl-type generalized translation operator and its applications. Constr. Approx. 49(3), 555–605 (2019)
12. Hardy, G.H., Littlewood, J.E.: Some properties of fractional integrals, I. Math. Zeit. 27, 565–606 (1928)
13. Hassani, S., Mustapha, S., Sifi, M.: Riesz potentials fractional maximal function for the Dunkl transform. J. Lie Theory 4, 725–734 (2009)
14. Herbst, I.W.: Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. Commun. Math. Phys. 53, 285–294 (1977)
15. Lieb, E.H.: Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. Math. 2, 349–374 (1983)
16. Lizorkin, P.I.: Generalized Liouville differentiation and function spaces $L^p(E_n)$. Embedding theorems. Sb. Math. 60(3), 325–353 (in Russian) (1963)
17. Platonov, S.S.: Bessel harmonic analysis and approximation of functions on the half-line. Izv. Math. 71(5), 1001–1048 (2007)
18. Riesz, M.: L’integrale de Riemann–Liouville et le probleme de Cauchy. Acta Math. 1, 1–222 (1949)
19. Rösler, M.: Generalized Hermite polynomials and the heat equation for Dunkl operators. Commun. Math Phys. 192, 519–542 (1998)
20. Rösler, M.: Positivity of Dunkl’s intertwining operator. Duke Math. J. 98, 445–463 (1999)
21. Rösler, M.: Dunkl Operators. Theory and Applications, in Orthogonal Polynomials and Special Functions, Lecture Notes in Math, vol. 1817, pp. 93–135. Springer, Berlin (2003)
22. Rösler, M.: A positive radial product formula for the Dunkl kernel. Trans. Am. Math. Soc. 355, 2413–2438 (2003)
23. Samko, S.G.: Hypersingular Integrals and Their Applications. Series Analytical Methods and Special Functions, vol. 5. Taylor, Francis, London–New York (2005)
24. Samko, S.: Best constant in the weighted Hardy inequality: the spatial and spherical version. Fract. Calc. Anal. Appl. 8, 39–52 (2005)
25. Sawyer, E.: A two weight weak type inequality for fractional integrals. Trans. Am. Math. Soc. 281, 339–345 (1984)
26. Soboleff, S.: On a theorem in functional analysis. Rec. Math. Mat. Sb. N.S. 4(46), 3 (1938). 471–497, Am. Math. Soc. Transl. no. 2(34) (1963), 39-68
27. Stein, E.M., Weiss, G.: Fractional integrals on n-dimensional Euclidean space. J. Math. Mech. 4, 503–514 (1958)
28. Thangavelu, S., Xu, Y.: Convolution operator and maximal function for Dunkl transform. J. d’Analyse. Math. 97, 25–55 (2005)
29. Thangavelu, S., Xu, Y.: Riesz transform and Riesz potentials for Dunkl transform. J. Comput. Appl. Math. 199, 181–195 (2007)
30. Trimèche, K.: Paley-Wiener Theorems for the Dunkl transform and Dunkl translation operators. Integral Transform. Spec. Funct. 13, 17–38 (2002)
31. Xu, Y.: Dunkl operators: FunkHecke formula for orthogonal polynomials on spheres and on balls. Bull. London Math. Soc. 32, 447–457 (2000)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.