States that are far from being stabilizer states

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Abstract
Stabilizer states are eigenvectors of maximal commuting sets of operators in a finite Heisenberg group. States that are far from being stabilizer states include magic states in quantum computation, MUB-balanced states, and SIC vectors. In prime dimensions the latter two fall under the umbrella of minimum uncertainty states (MUSs) in the sense of Wootters and Sussman. We study the correlation between two ways in which the notion of ‘far from being a stabilizer state’ can be quantified. Two theorems valid for all prime dimensions are given, as well as detailed results for low dimensions. In dimension 7 we identify the MUB-balanced states as being antipodal to the SIC vectors within the set of MUS, in a sense that we make definite. In dimension 4 we show that the states that come closest to being MUS with respect to all of the six stabilizer MUBs are the fiducial vectors for Alltop MUBs.

Keywords: mutually unbiased bases, Heisenberg groups, Clifford group

(Some figures may appear in colour only in the online journal)

1. Introduction
At the outset all vectors in a given Hilbert space are on the same footing, but in physics it frequently happens that a particular group of transformations is singled out for attention. Then the vectors are distinguished from each other by what the group does to them. In this paper we will look at a particularly interesting choice of the group: finite Heisenberg groups. Once this choice is made, an entire fauna of states springs into existence: stabilizer states, stabilizer
MUBs, magic states, Alltop MUBs, MUB-balanced states, SICs, and more. Our concern is to see how they relate to each other.

A stabilizer state, by definition, is an eigenvector of a maximal abelian subgroup of a finite Heisenberg group. In prime dimensions, there is a unique decomposition into $d + 1$ such subgroups [1], which each provide an eigenbasis composed of stabilizer states. In prime power dimensions, the decomposition is no longer unique [2]. The relationship between the eigenbases is rather special and each basis is itself an orbit under the whole Heisenberg group. These bases (or orbits) form what we shall refer to as stabilizer MUBs.

From the point of view of some yet-to-be-built quantum computers, stabilizer states play the role of ‘classical’ states [3, 4]. The idea is that unitary operations belonging to the automorphism group of the Heisenberg group can be performed in a fault-tolerant way. Such operations are known as Clifford operations. They permute the stabilizer states, and the computer cannot outperform a classical computer if it is confined to doing such permutations [5]. There is a clear analogy to quantum communication theory here. The Clifford operations are the analogues of the local unitaries, and the stabilizer states are the analogues of the separable states. An obvious question now poses itself: what are the analogues of the maximally-entangled states? In quantum communication theory, the path from separable to maximally-entangled states is clear. In quantum computation theory, the path leading away from the stabilizer states is not at all clear. In fact, we do not expect there is only one such path. We focus on two possible routes here and explore how the phrase ‘far from being stabilizer states’ can be given a precise meaning.

One class of states that suggest themselves are the SICs [6, 7] (symmetric informationally-complete positive operator-valued measures). A SIC is a collection of $d^2$ equiangular lines, that is $d^2$ vectors in a Hilbert space of dimension $d$ who have a constant absolute value of their overlap. Their claim to being far from stabilizer states comes from their group covariance. All known SICs (except one [9]) are orbits under a finite Heisenberg group [10]. So while the stabilizer states sit at one end of the scale as particularly short orbits, the SICs are distinguished longer orbits that sit at the other end. It is relevant to observe that, in entanglement theory, separable and maximally entangled states form two distinguished orbits under local unitaries.

Another class of states that could be called far from being stabilizer states are the magic states [11]. When used as a resource for quantum computing, they provide the quantum computer with its advantage over the computer that is limited to using stabilizer states and Clifford operations. Operationally, this puts the magic states as far as possible from the stabilizer states. If the stabilizer states can be considered as ‘classical’ from the point of view of quantum computation, the magic states are ‘non-classical’. In prime dimensions, one interesting class of magic states [12, 13] is provided by vectors in Alltop MUBs [14–16].

Minimum uncertainty states (MUSs) are a wide class of states with a geometrically natural definition that includes both SIC vectors and the recently observed MUB-balanced states. They are far from being stabilizer states in a geometrical sense, and they will be a key concern of this paper. The observation that SIC vectors are MUSs was made some time ago [17], and may have a bearing on the SIC existence problem in prime dimensions. Some further hints of a connection between SICs and MUBs in these dimensions have been found [18, 19].

For completeness, we mention that maximal mana states provide another meaning of far from being a stabilizer state [20]. They are SIC vectors if $d = 2, 3$ but if $d = 5$ they are not

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3 Definitions are given in the next section.

4 Regarding terminology, we use the name ‘SIC’ for the reasons given in [8].
even MUS [21]. They will not be discussed here, not because we do not find them interesting but because we have nothing new to say about them.

The paper is organized as follows: in section 2, we first provide definitions of the objects we will meet in the remainder of the paper. We then introduce two simple quantitative measures of how far a state is from being a SIC vector and how far it is from being a MUS, and establish some properties of these measures. In section 3, we report on computer calculations performed in dimensions 3, 5, and 7. They allow us to frame some precise conjectures concerning the correlation between the two measures that we introduced in section 2. In section 4, we prove some theorems inspired by the heuristic calculations. In section 5, we investigate the prime power dimension 4, again using computer calculations. The picture we arrive at is very different from that of prime dimensions. Section 6 holds our conclusions. In an appendix, we study a subspace which is of particular interest in the SIC existence problem. It provides a nice illustration of how our quantitative measures work.

2. Definitions

This section contains definitions of several mathematical objects we shall study. To readers familiar with MUBs and SICs, we emphasise the end of this section, where we define the less well known MUSs and MUB-balanced states. We end with two functions, \( f_{\text{SIC}} \) and \( f_{\text{MUS}} \), which occupy the remainder of the paper.

2.1. Mutually unbiased bases

A pair of orthonormal bases \([|e_i\rangle]_{i=0}^{d-1}\) and \([|f_j\rangle]_{j=0}^{d-1}\) are said to be unbiased if

\[
\left| \langle e_i | f_j \rangle \right|^2 = \frac{1}{d}
\]

for all values of \(i, j\). This means that a state prepared using a projective measurement in one basis has equal probability for all the outcomes of a projective measurement made using the second basis.

2.2. MUB

A set of \(d + 1\) mutually unbiased bases is said to form a complete set. We use the acronym MUB for such a set, which then contains the largest possible number of mutually unbiased bases. Complete sets have been constructed in all dimensions \(d\) equal to some power of a prime number [22, 23]. They may well not exist in other dimensions. See [24] for further details.

2.3. Weyl–Heisenberg group

In any dimension \(d\), the Weyl–Heisenberg group \(H(d)\) is generated by two elements \(X\) and \(Z\), which in themselves generate cyclic subgroups of order \(d\), and obey

\[
ZX = \omega XZ.
\]

where \(\omega = e^{2\pi i/d}\). Once the generator \(Z\) is given in diagonal form the unitary representation is unique. One finds that

\[
Z |e_i\rangle = \omega^i |e_i\rangle, \quad X |e_i\rangle = |e_{i+1}\rangle,
\]

\(3\)
where integers modulo $d$ are used to label the states. In odd prime dimensions it is convenient to work with the $d^2$ displacement operators

$$D_{ij} = \omega^{i j X'Z'},$$

(4)

where $1/2$ is the multiplicative inverse of 2 modulo $d$. The full story can be found in many places, say in [25].

### 2.4. Multipartite Heisenberg group

If $d = d_1 d_2$, where $d_1$ and $d_2$ are not relatively prime, the group $H(d_1) \times H(d_2)$ is not isomorphic to the group $H(d)$. We thus have a choice of groups when we leave prime dimensions. If $d = p^k$, where $p$ is a prime number and $k > 1$, we refer to the direct product of $k$ copies of $H(p)$ as the multipartite Heisenberg group. See [3] for a discussion. It is this choice of group that is relevant to MUBs (whereas the choice of $H(d)$ is relevant to SICs) and we shall refer back to it in section 5 when we investigate the situation in dimension 4.

### 2.5. Stabilizer states

A stabilizer state is an eigenvector of a maximal abelian subgroup of a finite Heisenberg group.

### 2.6. Stabilizer MUB

A stabilizer MUB is a MUB whose vectors are all stabilizer states. To see how stabilizer MUBs arise, note that finite Heisenberg groups are represented in $d$ dimensions by $d^2$ operators (not counting phase factors), and that these operators provide unitary operator bases in any dimension $d$. An important theorem states that a MUB exists if and only if there exists a unitary operator basis that can be split into $d^2$ commuting operators having only the unit element in common [1]. The mutually unbiased bases are the joint eigenbases of the operators in these sets.

In prime dimensions, the cyclic subgroups of the Weyl–Heisenberg group provide a unique partition of this kind [1]. In prime power dimensions, the multipartite Heisenberg group admits several such decompositions [2], which means that the number of stabilizer states is larger than the number of vectors in a single MUB. In either case, the individual vectors in the MUBs are stabilizer states, and we refer to a MUB that arises in this way as a stabilizer MUB. If the dimension is not a prime power, no such decomposition exists, for any version of the Heisenberg group [26].

### 2.7. Alltop MUB

An Alltop MUB consists of one basis of stabilizer states and a further $d^2$ vectors forming an orbit under the Weyl–Heisenberg group that can be collected into $d$ mutually unbiased bases. These bases are mutually unbiased to the basis of stabilizer states. We refer to the vectors in such an orbit as Alltop vectors.

The Alltop MUBs are unitarily equivalent to stabilizer MUBs. The original example found by Alltop [14] consists of the computational basis together with all the vectors obtained by acting with the Weyl–Heisenberg group on the fiducial vector $|\psi_0\rangle$, whose components are
This particular example assumes that the dimension \( d \) is an odd prime number greater than 3. Alltop MUBs are known to exist for all prime power dimensions \( p^k \) with \( p > 3 \) [15] and for \( d = 2, 3, 2^2 \). See [16] for further discussion.

2.8. SIC

A SIC is a set of \( d^2 \) unit vectors where any two vectors obey

\[
\left| \langle \psi_i | \psi_j \rangle \right|^2 = \frac{1}{d + 1}
\]

in dimension \( d \) [6, 7]. It is also known as an equiangular tight frame [27]. This simple definition turns out to be very deep mathematically, and the existence of SIC vectors in all dimensions is conjectural only [25, 28]. With one exception, all known SICs form orbits of the Weyl–Heisenberg group [10]. In dimensions 2 and 3 it has been proved that this is the only possibility [29, 30].

2.9. Probability vectors

To define MUSs and MUB-balanced states, we need a few equations involving probability vectors. Given \( d + 1 \) mutually unbiased bases \( \{|e_i^{(z)}\}_{i=0}^{d-1} \), we define the \( d + 1 \) probability vectors \( \vec{p}_{z(} \) with components determined by the state vector \( |\psi\rangle \) through

\[
p_{z(} \equiv \left| \left\langle e_i^{(z)} | \psi \right\rangle \right|^2.
\]

Here \( 0 \leq z \leq d \) labels the \( d + 1 \) bases in a MUB, and \( i \) labels the individual elements in such a basis. One can prove that [31]

\[
\sum_{z=0}^{d} \sum_{i=0}^{d-1} p_{z(i)}^2 = 2.
\]

2.10. Minimum uncertainty state

Wootters and Sussman [32] defined a MUS as one for which

\[
\sum_{i=0}^{d-1} p_{i(}^2 = \frac{2}{d + 1}.
\]

for each basis individually. Now consider a state as a vector in Bloch space, the set of unit trace Hermitian matrices with its origin at the maximally mixed state. A probability vector arises, as in equation (7), from an orthogonal projection of a Bloch vector to a plane spanned by the states in a Hilbert space basis. Then equation (8) follows from Pythagoras’ theorem and equations (9) mean that the length of the Bloch vector when projected orthogonally onto the plane defined by a basis in a MUB is independent of which basis we pick [33]. It is understood that the MUB with respect to which the MUS is defined is a stabilizer MUB, and then it follows that there is a sense—at least in prime dimensions, where the stabilizer MUB is unique—in which a MUS is indeed far from being a stabilizer state. MUS are not all that exceptional though. In a Hilbert space of dimension \( d \) one expects, after having counted the number of parameters and the number of equations, that the set of MUS forms a continuous set of real dimension \( d – 2 \), while the set of all pure states has real dimension \( 2d – 2 \). (We have
strengthened the expectation by studying the dimension of the solution space in the neighbourhood of a handful of exemplary MUS. Section 4 contains a proof for \( d = 3 \).)

2.11. MUB-balanced state

This is a state for which the \( d_1 \) probability vectors \( \vec{p}_i \) are identical up to permutations of their components [35]. Such states form a distinguished discrete set inside the continuous set of MUS. They can arise as eigenvectors of so-called MUB cyclers, operators that cycle through all the \( d + 1 \) bases in a MUB. MUB-balanced states are known to exist if \( d = 2^n \) where unitary MUB-cyclers exist [32, 34], and if \( d = (\text{prime})^n = 3 \mod 4 \) where anti-unitary MUB-cyclers exist [35, 36].

These states have some intriguing and useful properties [35–37]. For odd \( d \) the parity of these states (in the language used in connection with discrete Wigner functions, say) is opposite to that of the stabilizer states, and there is a sense in which this property alone makes them far from being stabilizer states: they maximally violate a certain non-contextuality inequality used to demarcate states that behave ‘classically’ in quantum computing [38]. If they exist, and if the dimension of Hilbert space is a prime number, the SIC vectors form another distinguished discrete subset of the set of MUS [17].

2.12. SIC function

The SIC function quantifies how much a given state deviates from being a SIC vector. It is given by

\[
 f_{\text{SIC}}(\psi) = \sum_{(i,j) \neq (0,0)} \left( \left| \left\langle \psi | D_{ij} | \psi \right\rangle \right|^2 - \frac{1}{d + 1} \right)^2 . \tag{10}
\]

This octic expression in the components of the unit vector \( |\psi\rangle \) is also known as a frame potential [27] and is familiar from the study of 2-designs, except that we rescaled and then shifted it so that \( f_{\text{SIC}} = 0 \) if and only if \( |\psi\rangle \) is a SIC vector. The SIC itself is obtained by acting on such a vector with all the displacement operators—the absolute values squared of all the scalar products equal \( 1/(d + 1) \) in this case.

2.13. MUS function

Similarly, the MUS function tells us how much a given state deviates from being a MUS. It is given by

\[
 f_{\text{MUS}}(\psi) = \sum_{z=0}^{d} \left( \sum_{r=0}^{d-1} \left| e^{i2\pi rz/d} \right|^4 - \frac{2}{d + 1} \right)^2 . \tag{11}
\]
where $z$ labels the bases in a given MUB. This function is again an octic polynomial, and it vanishes if and only if $|\psi\rangle$ is a MUS.

Our main aim is to investigate the correlation between these two measures. Some useful preliminary information is given in table 1. For comparison it is also useful to know the Fubini–Study averages over all Hilbert space, namely

$$
\langle f_{\text{SIC}} \rangle_{\text{FS}} = \begin{cases} 
\frac{d(d-1)}{(d+2)(d+1)} & \text{if } d \text{ is odd}, \\
\frac{d^2}{(d+3)(d+1)} & \text{if } d \text{ is even}, 
\end{cases}
$$

(12)

$$
\langle f_{\text{MUS}} \rangle_{\text{FS}} = \frac{4(d-1)}{(d+3)(d+2)(d+1)}.
$$

(13)

The latter tends to zero with growing dimension, as is reasonable. The first of these averages was computed in [39], to which we refer for the details of this computation.

### 3. Numerical calculations in dimensions 5, 7

Our target is to explore the correlation between ‘SICness’ and ‘MUSness’. This is turned into a well defined quantitative question by means of the functions $f_{\text{SIC}}$ and $f_{\text{MUS}}$, see equations (10) and (11) respectively. The first step to take is numerical: we choose vectors in Hilbert space at random according to the Fubini–Study measure, compute $f_{\text{MUS}}$ and $f_{\text{SIC}}$ for each, place them in a plot of one function against the other, and look for patterns. Figure 1 shows the results in dimensions 5 and 7. By inspection we see that the two plots look rather similar. It seems that the states end up in a bounded region whose lower boundary consists of a straight line connecting the SIC states (at the origin) to the stabilizer states. In dimension 3
all the states end up on this line, but dimension 3 is known to be very special from the point of view of the SICs that occur there [6, 25] and we ignore this dimension until section 4.

Before basing a concrete conjecture on these calculations it is necessary to explore higher dimensions \( d \), since there are many features of MUBs and SICs that depend on the number theoretic properties of \( d \), including its values modulo 3 and 4 (see the appendix and e.g. [25, 35], and [36] for more detail). We have therefore made similar plots for \( d = 11, 13, 17, \) and 19, but we then encounter the problem that the region in which we suspect the states to end up grows with \( d \) (see table 1), and a number of points of the order of \( 10^6 \) does not fill this region very densely towards its boundaries. We can counter this by choosing states in the neighbourhoods of special states—as we in fact did also in dimensions 5 and 7 before presenting the results shown in figure 1 (see the caption)—but the problem persists, and of course this strategy is open to the objection that there may be other corners of the large Hilbert spaces where the states behave in exceptional ways.

What we can say is that all the evidence collected is consistent with the following conjectures, which we state with reference to the sketch in figure 2:

- The boundary of the allowed region in a plot of \( f_{\text{SIC}} \) against \( f_{\text{MUS}} \) consists of four smooth segments meeting at four special points.
- The segment A is a straight line connecting SIC vectors (at \( \alpha \)) to stabilizer states (at \( \beta \)).
- The segment B consists of superpositions of two stabilizer states taken from the same basis.
- The point \( \gamma \) consists of equal weight superpositions of two stabilizer states.

We have been unable to frame a conjecture for segment C. Segment D evidently consists of the MUSs.

We will partly prove these conjectures in section 4. Before we turn to this we will discuss the special point \( \delta \) in some detail. It consists of those states that attain a maximum of \( f_{\text{SIC}} \) under the constraint that they are MUS. In this sense they are ‘antipodal’ to the SIC vectors (which are MUS that minimize \( f_{\text{SIC}} \)). We identified this point using the NMaximize routine in Mathematica. When \( d = 5 \) we find states that we do not recognize. However, when \( d = 7 \) we do recognize them. They are MUB-balanced states. Such states exist when \( d \) is an even prime
power [32] or an odd prime power equal to 3 modulo 4 [35, 36], and it has been conjectured that they do not exist otherwise.

We find it somewhat remarkable that MUB-balanced states are ‘antipodal’ to the SIC vectors within the set of MUSs when \( d = 7 \). Moreover we regard the fact that such states did not appear in \( d = 5 \) as circumstantial evidence for the conjecture that such states do not exist in this dimension. In dimension 7 we can say more: in this case the numerical calculation is consistent with the statement that

\[
   f_{\text{MUS}} = 0 \Rightarrow 0 \leq f_{\text{SIC}} \leq \frac{7}{8}.
\]

We do not have a proof that all MUS with \( f_{\text{SIC}} = 7/8 \) (the value attained by the MUB-balanced states) are indeed MUB-balanced, nor can we exclude the existence of MUB-balanced states not identical to those that have already been constructed [35, 36]. However, we generated 26 such states numerically with a precision of \( 10^{-24} \), and we identified all of them with a known MUB-balanced state.

As mentioned in section 2, MUB-balanced states in odd prime dimensions \( d \) have negative parity. There are \( d^2 \) negative parity eigenspaces altogether, related by the Weyl–Heisenberg group, and each negative parity eigenspace contains \( d(d - 1)/2 \) MUB-balanced states [36]. The orthogonality relations between 21 such vectors in \( d = 7 \) gives rise to a vertex transitive and perfect orthogonality graph, as shown in figure 3.

To summarize, we encountered MUB-balanced states in dimension 7 but not in dimension 5, and all the MUB-balanced states we did find in dimension 7 were among those already known (which can be constructed as eigenvectors of MUB-cycling anti-unitaries belonging to the extended Clifford group [35, 36]). We regard this as strong circumstantial evidence for the conjecture that all MUB-balanced states in these dimensions have been identified already.

At this point it is tempting to conjecture that the special point \( d \) consists of MUB-balanced states in all prime dimensions \( d = 3 \) modulo 4. However, this is almost certainly false. In dimension 11 we numerically generated one vector maximizing \( f_{\text{SIC}} \) under the constraint \( f_{\text{MUS}} = 0 \). The resulting vector obeys the constraint to a precision of \( 10^{-20} \), but it has a value of \( f_{\text{SIC}} \) which is more than twice as large as that attained by the MUB-balanced states in this dimension. With less precision we also generated a MUS in dimension 19 whose \( f_{\text{SIC}} \) value exceeds that of the MUB-balanced states there.
4. Some theorems in prime dimensions

We begin this section by proving the first part of our conjecture:

**Theorem 1.** When the dimension is a prime number all states obey the inequality

\[ f_{\text{SIC}}(\psi) \geq \frac{d^2}{d-1} f_{\text{MUS}}(\psi), \]

with equality for all states if and only if \( d = 2, 3 \).

**Proof.** For \( d = 2 \) an explicit calculation shows that equality holds. Now let \( d \) be an odd prime. The idea of the proof is to split \( f_{\text{SIC}} \) into \( d + 1 \) terms, one for each cyclic maximal abelian subgroup. Focus on one such cyclic subgroup, denote its generator by \( Z \), and diagonalize. In this basis the state \( \psi \rangle \) is represented by

\[ \psi = \begin{pmatrix} \sqrt{p_0} \\ \sqrt{p_1} e^{i\theta_1} \\ \vdots \\ \sqrt{p_{d-1}} e^{i\theta_{d-1}} \end{pmatrix}, p_i \geq 0, \text{ } \sum_{i=0}^{d-1} p_i = 1. \]  

The calculation is exactly the same in all the \( d + 1 \) eigenbases. Therefore it will be enough to show that

\[ \sum_{j=1}^{d-1} \left( \left| \langle \psi | Z^j | \psi \rangle \right|^2 - \frac{1}{d+1} \right)^2 \geq \frac{d^2}{d-1} \left( \sum_{r=0}^{d-1} p_r^2 - \frac{2}{d+1} \right)^2. \]  

Using the standard representation of \( Z \) we observe that

\[ \left| \langle \psi | Z^j | \psi \rangle \right|^2 = \sum_{k=0}^{d-1} p_k^2 + \sum_{k \neq j} \omega^{j(k-1)} p_k p_j = \sum_{k=0}^{d-1} p_k^2 + \sum_{k=1}^{d-1} (\omega^{jk} + \omega^{-jk}) \Delta_k, \]  

where we defined

\[ \Delta_k = \sum_{m=0}^{d-1} p_m p_{m+k}. \]  

There holds

\[ 2 \sum_{k=1}^{d-1} \Delta_k = 1 - \sum_{k=0}^{d-1} p_k^2. \]  

We now insert the result (17) in the inequality (16) and use the fact that the primitive root of unity \( \omega \) obeys

\[ \sum_{j=1}^{d-1} (\omega^{jk} + \omega^{-jk})(\omega^{jr} - \omega^{-jr}) = -2. \]  

\[ \sum_{j=1}^{d-1} (\omega^{jk} + \omega^{-jk})(\omega^{jr} - \omega^{-jr}) = -2. \]
We then find that the inequality holds if and only if

\[
(d - 1) \sum_{k=1}^{d-1} \Delta_k^2 \geq \frac{1}{2} \left( 1 - \sum_{k=0}^{d-1} \sigma_k^2 \right)^2 = 2 \left( \sum_{k=1}^{d-1} \Delta_k \right)^2 \]

(21)

\[
\sum_{k=1}^{d-1} \sum_{l=1}^{d-1} (\Delta_k^2 + \Delta_l^2) \geq 2 \sum_{k=1}^{d-1} \sum_{l=1}^{d-1} \Delta_k \Delta_l \]

\[
\sum_{k=1}^{d-1} \sum_{l=1}^{d-1} (\Delta_k - \Delta_l)^2 \geq 0.
\]

The final inequality is manifestly obeyed, and it becomes an inequality for all states if \( d = 3 \) because the final sum then contains only one term.

This theorem has an immediate corollary:

**Corollary.** In dimension 3 the MUS form one dimensional smooth families.

**Proof.** Theorem 1 implies that every MUS is a SIC vector in this dimension. The converse holds. But it is known that SIC vectors belong to smooth one dimensional families when \( d = 3 \) [25]. The result follows.

Our second theorem gives a geometrical interpretation of what it means for a state to saturate this inequality:

**Theorem 2.** If a state saturates the inequality (14) it projects to regular simplices on all the \((d - 1)\)-planes spanned by the \(d + 1\) mutually unbiased bases in Bloch space.

**Proof.** Let such a state serve as the fiducial state in a Weyl–Heisenberg orbit. In Bloch space, project the \(d^2\) (pure) density matrices in the orbit orthogonally onto the plane spanned by an eigenbasis of a cyclic subgroup, that is to a plane defined by a basis in a MUB. Denote the generator of this cyclic subgroup with \(Z\). When this generator acts on a state it does not affect its image under the projection. Thus only \(d\) distinct points will appear when we project the entire orbit. Denote the complementary generator with \(X\). Its effect on the projection is to permute the entries of the probability vector cyclically, \(p_i \rightarrow p_{i-1}\). The state saturates the inequality if and only if \(\Delta_k\) takes the same value for all \(k\), for all eigenbases. But the \(\Delta_k\) are precisely the mutual scalar products of the probability vectors. Hence the \(d\) projections of the orbit sit at the vertices of a regular simplex for such a fiducial state. The same argument applies to all the \(d + 1\) eigenbases.

Some comments on these theorems: SIC vectors must saturate the inequality for all prime dimensions, and being MUS they have the additional property that all the simplices we see in the \(d + 1\) projections are of the same size [33]. Alltop vectors also saturate the inequality as
one can see from a formula given by Khaterinejad [40], and they have the additional property that \(d\) out of \(d+1\) projected simplices share the same size and the same orientation, as follows from the fact that every Alltop vector is left invariant by a unitary operator that cycles through \(d\) of the bases in the stabilizer MUB [41].

In dimensions 2 and 3 every MUS is a SIC vector (and the converse holds [29, 30]). In dimension 2 it is also true that each of the eight MUS is MUB-balanced. In dimension 3 it is known that every SIC is unitarily equivalent to one obtained by acting with the Weyl–Heisenberg group on a fiducial vector with components [6]

\[
\psi(\sigma) = \begin{pmatrix} 0 \\ 1 \\ -\omega^{\sigma} \end{pmatrix},
\]

where the parameter \(\sigma\) can be restricted [25] to lie in the interval \([0, 2\pi/6]\). This is a MUB-balanced state if and only if \(\sigma = 0\), and incidentally a maximal mana state if \(\sigma = 0\) or \(2\pi/6\) [20]. Since all states in these dimensions saturate the inequality (14) we see regular triangles in all projections of every Weyl–Heisenberg orbit onto the MUB simplices, but they have the same size if and only if the fiducial vector is a SIC vector. A glance at figure 4 may clarify the meaning of theorem 2.

We have not been able to prove the remaining parts of the conjecture we made in section 3. We can say something about the second part though. Suppose that segment B of the boundary of the allowed region of the \(f_{\text{MUS}}-f_{\text{SIC}}\) plot is indeed formed by states that are superpositions of two states from the same stabilizer basis, such as

\[
\begin{align*}
\psi(\sigma) &= \begin{pmatrix} 0 \\ 1 \\ -\omega^{\sigma} \end{pmatrix}, \\
\psi(\sigma) &= \begin{pmatrix} 0 \\ 1 \\ -\omega^{\sigma+\pi/3} \end{pmatrix},
\end{align*}
\]

Figure 4. For \(d = 3\) we show the orthogonal projections of the 9 vectors in a Weyl–Heisenberg orbit onto the four MUB simplices. It is always the case that three images coincide, so we see only 3 points in each projection. From top to bottom we see a MUB-balanced SIC, a generic SIC, an Alltop orbit, and an orbit whose fiducial vector was chosen at random.
Then a straightforward calculation shows that, for these states,

\[ f_{\text{MUS}} = \frac{(3d - 5 + (d + 1) \cos 2\theta)^2}{16d(d + 1)}, \]

\[ f_{\text{SIC}} = \frac{d}{8} \left( \cos^2 2\theta + 2 \cos 2\theta + \frac{5d - 11}{d + 1} \right) \]  

This is a parametric representation of a curve in the $f_{\text{MUS}}$-$f_{\text{SIC}}$-plane. We solve for $\cos^2 2\theta$ as a function of $f_{\text{MUS}}$ and insert the result in the expression for $f_{\text{SIC}}$. Finally we can state our conjecture about this part of the boundary as an inequality:

**Conjecture.** The inequality

\[ f_{\text{SIC}} \leq \frac{2d}{d + 1} \left( df_{\text{MUS}} - (d - 3) \sqrt{df_{\text{MUS}}} \right) + \frac{(d - 3)(d - 1)}{2(d + 1)} \]  

holds in all prime dimensions.

The inequality is saturated for the states (23). One checks that the graph of the corresponding function has a minimum at

\[ \theta = \frac{\pi}{2} \Rightarrow f_{\text{MUS}} = \frac{(d - 3)^2}{4d(d + 1)} \Rightarrow f_{\text{SIC}} = \frac{d(d - 3)}{2(d + 1)}. \]

At this point we conjecture that there is a cusp in the boundary. Another inequality comes into play there.

**5. Dimension 4**

Since 4 is a prime power we have a choice between two non-isomorphic Heisenberg groups when $d = 4$. The one having a SIC as an orbit is the usual Weyl–Heisenberg group $H(4)$, while the one underlying the mutually unbiased bases is the bipartite direct product $H(2) \times H(2)$. In fact the bipartite group admits 15 maximal abelian subgroups having altogether 60 stabilizer states as eigenvectors. The latter can be organized into stabilizer
MUBs in 6 different but unitarily equivalent ways. This well known situation is summarized in figure 5, and elsewhere [42]. Since the SIC in this dimension is an orbit under $H(4)$, and no canonical identification of the computational bases of the two groups is known, it plays no role in this paper.

MUSs, and the function $f_{MUS}$, can be defined relative to any MUB. In dimension 4 we have 6 stabilizer MUBs to choose from. The definition of the frame function $f_{SIC}$ can be used in dimension 4 provided it is modified so that the sum in equation (10) runs over the non-trivial elements of the bipartite Heisenberg group. We then have

\textbf{Theorem 3.} When the dimension $d = 4$ all states obey the inequality

$$f_{SIC} \geq \frac{16}{3} f_{MUS},$$

provided that $f_{SIC}$ is defined using the bipartite Heisenberg group $H(2) \times H(2)$ and $f_{MUS}$ is defined with respect to one of the stabilizer MUBs.

The proof is by direct calculation and is omitted here. The maxima of the two functions are attained by the vectors in the relevant MUB. Moreover there holds, in general, that

$$0 < f_{SIC} \leq \frac{12}{5}.$$  \hfill (29)

The upper bound is saturated by the stabilizer states, but this time the lower bound cannot be reached because the bipartite Heisenberg group does not admit a SIC as an orbit. A plot is shown in figure 6.

In the rest of this section our argument is based on numerical rather than analytical calculations. However, since the dimension is low we are on much safer ground than we were in section 3.
There are MUB-balanced states in $d = 4$, and when the MUB is a stabilizer MUB they can be constructed as eigenvectors of an element in the Clifford group of order 5 [32]. Such unitaries cycle through the bases in one of the MUBs, and they move bases in one of the other five MUBs through these five. Since there are six MUBs altogether we are in fact dealing with six different MUS-functions $f^{(i)}_{\text{MUS}}$, where $i$ labels the particular MUB with respect to which the functions are defined. A MUB-balanced state is a MUS only with respect to one of them.

If the state is balanced with respect to the first MUB one finds

$$f_{\text{SIC}} = 0.32, f^{(1)}_{\text{MUS}} = 0, f^{(2)}_{\text{MUS}} = \ldots = f^{(6)}_{\text{MUS}} = 0.032. \quad (30)$$

In figure 6 these states end up well inside the allowed region.

Since each of the six stabilizer MUBs come with their own MUS function $f^{(i)}_{\text{MUS}}$ we can ask for the correlation between two of them. This is a new question that did not arise in prime dimensions. The answer begins to emerge in figure 7. Since there are points in this plot very close to the origin one may be led to believe that there exists a state which is MUS relative to two stabilizer MUBs, but the numerical results cited below strongly suggest that no such states exist. Still there will exist states that minimize the sum of the six different functions $f_{\text{MUS}}$ that we obtain from the six stabilizer MUBs. In this very sense, these states deserve to be regarded as being as far from stabilizer states as any state can be.

**Figure 7.** The correlation between the MUSness as defined with respect to any two different MUBs. The plot uses $5 \times 10^5$ random vectors.

**Table 2.** Minimizing the MUSness with respect to more than one MUB.

| Number of MUBs | Minimum($\sum f^{(i)}_{\text{MUS}}$) |
|---------------|-----------------------------------|
| 1             | 0.0000000000                      |
| 2             | 0.0041666666                      |
| 3             | 0.0102012357                      |
| 4             | 0.0187500000                      |
| 5             | 0.0468749999                      |
| 6             | 0.0749999999                      |

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We used Matlab to minimize, numerically, the sum of the $f_{\text{MUS}}$-functions for between 1 and 6 stabilizer MUBs. 10,000 random initializations were used in each case. The results are reported in table 2.

Numerically we find that the states that minimize the sum over 4 or more $f_{\text{MUS}}$ have the property that if we act on them with the bipartite Heisenberg group the resulting orbits form sets of four mutually unbiased bases. If this is so these states must be fiducial vectors for Alltop MUBs. Each orbit forms a MUB when taken together with one of the 15 stabilizer bases. Using an exact expression for such fiducials [16] we find that

$$f_{\text{MUS}} = \begin{cases} \frac{9}{320} & \text{if the Alltop and stabilizer MUBs share a basis,} \\ \frac{3}{640} & \text{otherwise.} \end{cases}$$

Since the stabilizer basis to which the Alltop orbit is unbiased occurs in two out of the four stabilizer MUBs (see figure 5) we can now predict that the minimum of $f_{\text{MUS}}$ summed over all six stabilizer MUBs is

$$\min \left( \sum_{i=1}^{6} f_{\text{MUS}}^{(i)} \right) = 4 \cdot \frac{3}{640} + 2 \cdot \frac{9}{320} = \frac{3}{40} = 0.075.$$ (32)

This agrees with the numerical evaluation in table 2. Thus we have clear-cut evidence that the prize for being as far as possible from the stabilizer states in $d = 4$ goes to the Alltop vectors.

Interestingly, the Alltop vectors also minimize $f_{\text{SIC}}$ as defined using the bipartite Heisenberg group. Again we are missing an analytical proof, but it does emerge clearly from the insert in figure 6. To strengthen the case we performed a numerical minimization of $f_{\text{SIC}}$. We made 10,000 trials and ended up, each time, with vectors having the values of $f_{\text{SIC}}$ and $f_{\text{MUS}}$ that obtain for the Alltop vectors.

It is perhaps worth noticing that the situation in dimension 8 must be different, since there does exist a SIC which is an orbit under $H(2) \times H(2) \times H(2)$ in this dimension [9]. But this is an exceptional case [24]. In dimensions $d = 2^n$ the number of ways in which the stabilizer states can be collected into MUBs grows quickly with $n$, which is one reason why we have not tried to go further.

6. Conclusions

We have been concerned with states classified by a Heisenberg group into stabilizer states, Alltop vectors, MUB-balanced states, SIC vectors, and more. The stabilizer states are peaceful ‘classical’ states from the point of view of some quantum computers, while the others can be regarded as essentially ‘quantum’ in various ways that we have specified. Our special concern has been with quantitative measures of how far a state deviates from being a SIC state on the one hand, and a MUS [32] on the other. Two sharp theorems valid in all prime dimensions were given in section 5, and a more detailed conjecture in section 3.

Things are fairly simple in dimension 3 where every MUS is a SIC, one of which is composed of MUB-balanced states. In dimension 5 MUB-balanced states presumably do not exist, but in dimension 7 they do, and turn out to be—in a sense we made precise—antipodal to the SIC vectors within the set of MUSs. We regard the results reported here as circumstantial evidence that the set of all MUB-balanced states coincides with the set of those that are already known [32, 35, 36].
In dimension 4 an interlocking system of six MUBs can be constructed from the stabilizer states. We presented a strong argument to the effect that there is a definite sense in which the Alltop vectors are the states that are as far from being stabilizer states, and as close to being SIC vectors under the relevant group, as any state can be.

With the caveat that some of our discussion has been confined to low dimensions, we believe that we have introduced a certain amount of order into the question of providing meaning to the expression ‘far from being a stabilizer state’. The caveat is of course an important one. Many things are unknown in higher dimensions, including even the existence of Alltop MUBs in even prime power dimensions larger than four, and the existence of SICs in three digit dimensions and larger. Moreover a truly satisfactory picture requires further study also for the dimensions we do study. We have not discussed maximal mana states, or more generally how the states we have discussed sit relative to the set of mixed states with positive Wigner function. This set plays a special role in quantum computation. We hope to address some of these issues in the future.

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Appendix: a look at a Zauner subspace

What do the level surfaces of the function $f_{\text{SIC}}$ actually look like? The question may be naive, but one would like to know. Unfortunately the dimension of the Hilbert space is typically too large for visualization. However, there are some circumstances when the behaviour of $f_{\text{SIC}}$ on 2-real-dimensional subspaces is of interest. They arise because of an unexplained property shared by every Weyl–Heisenberg SIC so far constructed [28], namely that—as conjectured by Zauner [6]—every vector in such an orbit is left invariant by an element of the unitary automorphism group of the Weyl–Heisenberg group, having order 3. Thus they sit in special subspaces, known as Zauner subspaces.

If the Hilbert space has dimensions 4 or 5 the Zauner subspace has dimension 2 only, and this can be visualized as a Bloch sphere. The resulting pictures of $f_{\text{SIC}}$ are quite complex [43], which begins to explain why finding SIC vectors using numerical methods is a difficult art [28]. When $d = 5$ the one-parameter family of MUSs in the Zauner subspace can be solved for exactly [43].

In Hilbert space of dimension 7 the Zauner subspace is three dimensional but contains a real subspace of considerable interest, and this real subspace defines a two dimensional real projective space. Figure 8 is a map of the real Zauner subspace in dimension 7, making use of the fact that real projective 2-space can be viewed as a sphere with antipodal points identified, or equivalently as the upper hemisphere of a sphere. Stereographic coordinates are used. By solving the polynomial equations that define MUSs we have verified that there are exactly 6 such states within this real subspace. They are marked on the map. Only two of them are SIC vectors [25]. Interestingly this subspace also contains 6 Alltop vectors, with a specific value of $f_{\text{SIC}}$ and their positions are given as well. In fact Alltop vectors do occur in the analogous subspace in all prime dimensions $d = 1$ modulo 3 [19], while SIC vectors occur there only in a few cases [40].
Figure 8. Stereographic projection of a hemisphere, or equivalently of the real subspace of the Zauner subspace, in dimension 7. The positions of 6 Alltop vectors (at latitude 22°) and 6 MUS (at latitude 42°) are shown against a background of contour curves for \( f_{SIC} \). Maxima \( f_{SIC} = 5.25 \) occur at two stabilizer states (one of them sits at the pole), there are several local minima, and nothing special happens at the MUS unless they are also SIC vectors (as happens for two out of six).

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