GLOBAL BIFURCATION FOR THE HÉNON PROBLEM

ANNA LISA AMADORI

Dipartimento di Scienze Applicate, Università di Napoli “Parthenope”
Centro Direzionale di Napoli, Isola C4, 80143 Napoli, Italy

(Communicated by Shuangjie Peng)

Abstract. We prove the existence of nonradial solutions for the Hénon equation in the ball with any given number of nodal zones, for arbitrary values of the exponent $\alpha$. For sign-changing solutions, the case $\alpha = 0$ -Lane-Emden equation- is included. The obtained solutions form global continua which branch off from the curve of radial solutions $p \mapsto u_p$, and the number of branching points increases with both the number of nodal zones and the exponent $\alpha$. The proof technique relies on the index of fixed points in cones and provides information on the symmetry properties of the bifurcating solutions and the possible intersection and/or overlapping between different branches, thus allowing to separate them in some cases.

1. Introduction. The Hénon problem, introduced in the 70’s for the study of star clusters, is

\[\begin{cases}
-\Delta u = |x|^\alpha |u|^{p-1}u & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}\]

where $B$ stands for the unitary ball in $\mathbb{R}^N$ with $N \geq 2$ and the exponent $\alpha$ is positive. Here we have written the power-type nonlinearity in its odd formulation since we are interested in both positive and sign-changing solutions. For $\alpha = 0$ (1.1) gives back the Lane-Emden problem

\[\begin{cases}
-\Delta u = |u|^{p-1}u & \text{in } B, \\
u = 0 & \text{on } \partial B.
\end{cases}\]

Some of the results we present here are new also for the latter, and since our techniques allow to deal with both problems simultaneously we shall include the case $\alpha = 0$ in the reasoning.

It is well known that, for $\alpha > 0$ fixed, the Hénon problem (1.1) admits solutions, and in particular radial solutions, for every $p \in (1, p_\alpha)$, being

\[p_\alpha = \begin{cases}
\infty & \text{in dimension } N = 2, \\
\frac{N + 2 + 2\alpha}{N - 2} & \text{in dimension } N \geq 3.
\end{cases}\]

The same holds when $\alpha = 0$, i.e. for the Lane-Emden equation (1.2), and in this case the threshold exponent for the existence of solutions coincides with the critical

2020 Mathematics Subject Classification. 35J91, 35B05, 35B32.

Key words and phrases. Hénon problem, nodal solutions, bifurcation.

The author is member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
Sobolev exponent $p_0 = \frac{N+2}{N-2}$ in dimension $N \geq 3$. In that range of existence, for any given $m \geq 1$ there is exactly one couple of radial solutions of (1.1) which have exactly $m$ nodal zones, they are classical solutions and they are one the opposite of the other; see [10, 22].

Such radial solutions are the only possible ones only in the framework of positive solutions and Lane-Emden equation, where the celebrated symmetry result by Gidas, Ni and Nirenberg holds. It is well known that the Hénon problem in the ball has also nonradial positive solutions, and the literature on this subject is rich. First [24] showed that the minimal energy solution is nonradial when $\alpha$ is large and $p$ is subcritical. After multi-peak solutions have been constructed by finite-dimensional reduction methods under various incidental assumptions, we mention [16, 23] among others. Nonradial solutions have also been produced by variational methods as in [8, 1], after imposing some constrains on the symmetries of the solutions, and by bifurcation methods in [3, 17].

Coming to nodal solutions, considerations based on the Morse index yield that the minimal energy solution is nonradial for every $\alpha \geq 0$. Indeed the minimal energy nodal solution has Morse index 2, while the Morse index of nodal radial solutions is greater, see [6]. Sign-changing multi-bubble solutions have been produced by finite-dimensional reduction methods, we can quote [9] and the references therein for the Lane-Emden problem and [25] for the Hénon problem in the disk. They are very different from the radial ones since their nodal surfaces intersect the boundary of the ball. Another interesting paper by Gladiali and Ianni [19] showed the existence of solutions to the Lane-Emden equation which are nonradial but “quasi-radial”, in the sense that their nodal lines are the boundary of nested domains contained in the disc. Some of these quasi-radial solutions are produced as least energy nodal solutions in symmetric spaces, some others by bifurcation w.r.t. the parameter $\alpha$. The approach of least energy solutions in symmetric space has been extended also to the Hénon equation in [1, 2], always in dimension $N = 2$. Concerning the Hénon equation in dimension $N \geq 3$, in the subcritical case a very recent paper by Kübler and Weth [21] produced an infinite number of nonradial solutions by bifurcation w.r.t. the parameter $\alpha$, by a fine description of the profile of the radial solutions and of the distribution of their negative eigenvalues as $\alpha \to \infty$. Such nonradial solutions are called by the authors “almost radial” because their nodal surfaces are homeomorphic to spheres. Of course, also the solutions produced by bifurcation arguments in [19] are of the same kind.

Here we aim to obtain nonradial bifurcation w.r.t. the parameter $p \in (1, p_\alpha)$, for any given value of $\alpha > 0$ (and also $\alpha = 0$, as far as sign-changing solutions are concerned), so we must take into account also the supercritical case. The Morse index of radial solutions when the parameter $p$ approaches the supremum of the existence range has been recently computed in four different papers ([15, 14] concerning the Lane-Emden problem in dimension $N = 2$ and $N \geq 3$ respectively, and [1, 5] for the Hénon problem), while when $p$ is close to 1 it has been characterized in terms of the zeros of suitable Bessel functions in [2]. Starting from these computations we see that for the positive solution to the Hénon equation the Morse index for $p$ close to 1 is lower than at the supremum of the existence range, and the same holds for nodal solutions in dimension $N = 2$, while in dimension $N \geq 3$ the inequality is reversed. Although there are still nontrivial difficulties in deducing actual bifurcation: no variational structure can be used to handle supercritical values of $p$ and only an odd change in the Morse index can produce a bifurcation result. When dealing with
the positive solutions, the first eigenvalue alone plays a role and this ensures that
the kernel of the linearized operator contains exactly a one-dimensional subspace
of the $O(N-1)$-invariant functions, and this observation was crucial in [3, 17]. For
nodal solutions, instead, the structure of the kernel is highly nontrivial. We handle
this situation by turning to the notion of degree and index of fixed points in cones
introduced by Dancer in [11]. It can be applied to the Hénon equation because
the exact computations in aforementioned papers rely on a characterization of the
Morse index in terms of a singular Sturm-Liouville problem from [7], which allows
to describe in full details the kernel of the linearized operator. Furthermore this
tool provides a detailed bifurcation analysis also for positive solutions, and in the
subcritical case, since it gives informations about the symmetries of the bifurcating
solutions and the global properties of the branches.

This paper is organized as follows. In Section 2 we outline the positive cones that
we will use and the main bifurcation results that we are going to prove. Section
3 deals with the Morse index: after recalling its characterization by means of the
singular eigenvalues and the exact computations performed in the aforementioned
papers, we check that the Morse changes across the range $p \in (1, p^*_\alpha)$. Next in
Section 4 the main results are proved, by taking advantage of the previous discussion
on the Morse index and adapting that arguments to compute the index of fixed
points in cones.

2. Statement of the main results. We adopt the spherical coordinates in $\mathbb{R}^N$
given by $(r, \theta, \varphi)$ with $r = |x| \in [0, +\infty)$, $\theta \in [-\pi, \pi]$, $\varphi = (\varphi_1, \ldots, \varphi_{N-2}) \in (0, \pi)^{N-2}$ so that

$$
x_1 = r \cos \theta \prod_{h=1}^{N-2} \sin \varphi_h, \quad x_2 = r \sin \theta \prod_{h=1}^{N-2} \sin \varphi_h, \quad x_N = r \cos \varphi_{N-2},
$$

$$
x_k = r \cos \varphi_{k-2} \prod_{h=k-1}^{N-2} \sin \varphi_h \quad \text{as } k = 3, \ldots, N-1.
$$

In particular for any $x \neq 0$, $(\theta, \varphi)$ are the coordinates of $x/|x| \in S_{N-1}$. Next for
any natural number $n$ we introduce the spaces

$$
H^1_{0,n} := \{ u \in H^1_0(B) : u(r, \theta, \varphi) \text{ is even and } 2\pi/n \text{ periodic w.r.t. } \theta, \quad \text{for every } r \in (0, 1) \text{ and } \varphi \in (0, \pi)^{N-2} \},
$$

$$
X_n := H^1_{0,n} \cap C^{1,\gamma}(B),
$$

and the positive cones already used in [4], i.e.

$$
K_n := \{ u \in X_n : \text{is nonincreasing w.r.t. } \theta \in (0, \pi/n), \quad \text{for every } r \in (0, 1) \text{ and } \varphi \in (0, \pi)^{N-2} \}.
$$

Notice that radial functions belong to $K_n$ for every $n$. On the other side, only in
dimension $N = 2$ the intersection between two different cones reduces to the radial
functions alone. Instead in dimension $N \geq 3$ it contains also nonradial functions
that do not depend on the angle $\theta$.

Throughout the paper we will take the exponent $\alpha$ as fixed and write $S^m$ for the
curve of radial solutions to (1.1) with $m$ nodal zones, precisely

$$
S^m = \{ (p, u_p) \in (1, p_\alpha) \times C^{1,\gamma}(B) : u_p \text{ is the radial solution to (1.1)} \}.
$$
with \( m \) nodal zones and \( u_\alpha(0) > 0 \).

We will show that a continuum of nonradial solutions in \( K_n \) detaches from the curve \( S^m \), for some integers \( n \) depending on the exponent \( \alpha \) and the number of nodal zones \( m \). To this aim we introduce the set

\[
\Sigma^m_n = \text{Cl}\{(p,u) \in (1,p_\alpha) \times K_n \setminus S^m : u \text{ solves } (1.1)\},
\]

where the closure is meant according to the natural norm in \((1,p_\alpha) \times C^{1,\gamma}(B)\). Remark that the set \( \Sigma^m_n \) contains also the curves of radial functions \( S^m \) with \( m' \neq m \), but of course \( S^m \) and \( S^{m'} \) are separated. So we say that a couple \((p_n,u_{p_n}) \in S^m \cap \Sigma^m_n \) is a nonradial bifurcation point, meaning that in every neighborhood of \((p_n,u_{p_n})\) in the product space \((1,p_\alpha) \times C^{1,\gamma}_0(B)\) there exists a couple \((q,v)\) such that \( v \) is a nonradial solution of \((1.1)\) related to the exponent \( q \). In this case we set

\[
C^m_n \text{ the closed connected component of } \Sigma^m_n \text{ containing } (p_n,u_{p_n})
\]

and we shall refer it as the “branch” departing from \((p_n,u_{p_n})\), with a little misuse of language. We will also write \([t]\) and \([\hat{t}]\), respectively, for the floor and the ceiling of a real number \( t \), i.e.

\[
[t] = \max \{n \in \mathbb{Z} : n \leq t\}, \quad [\hat{t}] = \min \{n \in \mathbb{Z} : n \geq t\}.
\]

Eventually the same reasoning enables us to prove several bifurcation results. First we produce \([\frac{\alpha}{2}]\) global branches of positive nonradial solutions, precisely

**Theorem 2.1.** (Bifurcation from positive solutions). *In any dimension \( N \geq 2 \) and for every \( \alpha > 0 \), there are at least \([\frac{\alpha}{2}]\) different points along the curve \( S^1 \) where a nonradial bifurcation occurs. More precisely for every \( n = 1, \ldots, [\frac{\alpha}{2}] \) there exists a nonradial bifurcation point \((p_n,u_{p_n}) \in S^1 \cap \Sigma^1_n \) and the respective branch \( C^1_n \) has the following global properties*

i) \( C^1_n \) is made up of positive solutions and unbounded, i.e. it contains a sequence \((p_k,u_k)\) with \( \|u_k\|_{C^{1,\gamma}} \to \infty \) or \( p_k \to p_\alpha \).

ii) In dimension \( N = 2 \) the branches are separated, in the sense that their intersection contains at most isolated points along the curve of positive radial solutions \( S^1 \).

iii) In dimension \( N \geq 3 \) two different branches can only have in common couples \((p,v)\), where \( v \) are positive solutions to \((1.1)\) which do not depend on the angle \( \theta \), and their overlapping can even make up a continuum.

In the disc solutions enjoying the same symmetry properties have been produced in [16] by the Lyapunov-Schmidt reduction method, and in [1] by minimizing the energy associated to \((1.1)\) in the space \( H^1_{0,n} \). In this last paper it has been proved that such “least energy \( n \)-invariant solutions” are nonradial and different one from another at least for \( p \in (p_n, +\infty) \), with \( p_n \) the same exponent appearing here. On the other hand, they are certainly radial for \( p \) close to one, thanks to the uniqueness result in [3]. It is therefore natural to think that the branches of bifurcating solutions shown by Theorem 2.1 are made up by these least energy \( n \)-invariant solutions, and so they do exist for every \( p \in (p_n, \infty) \), and are separated.

In higher dimension Theorem 2.1 improves the bifurcation result obtained in [3], which holds for \( \alpha \in (0,1] \) and produces only one branch of nonradial solutions. Nonradial solutions with similar symmetries have been produced by the finite-dimensional reduction method: in particular [23] concerns the slightly subcritical case and exhibits solutions which blow up when \( p \) approaches the critical
Sobolev exponent. Besides nonradial solutions do exist also for $p$ close to $p_n$, as showed in [17]. It is very likely that some of the nonradial solutions found in Theorem 2.1 coincide with the ones in [17], where the specular viewpoint (bifurcation w.r.t. $\alpha$) is adopted.

Coming to nodal solutions, the asymptotic Morse index and consequently the number of nonradial branches depend on the dimension. We therefore state the bifurcation results separately.

In the plane the set $\Sigma^2_n$ is nonempty at least for $n = \left\lceil \frac{2+\alpha}{2} \beta + 1 \right\rceil, \ldots, \left\lceil \frac{2+\alpha}{2} \beta - 1 \right\rceil$, where $\beta \approx 2.355$ and $\kappa \approx 5.1869$ are fixed numbers related to the computation of the Morse index at $p$ next to 1 and at infinity, respectively, whose characterization is recalled in (3.22) and (3.27). Remark that, since $\kappa > \beta + 2$, then $\left\lceil \frac{2+\alpha}{2} \beta - 1 \right\rceil - \left\lceil \frac{2+\alpha}{2} \beta + 1 \right\rceil > \alpha \geq 0$. We have

**Theorem 2.2** (Bifurcation from nodal solutions in dimension $N = 2$). Consider problem (1.1) in dimension $N = 2$. For every $\alpha \geq 0$ there are at least $\left\lceil \frac{2+\alpha}{2} \beta - 1 \right\rceil - \left\lceil \frac{2+\alpha}{2} \beta \right\rceil$ different points along the curve $\mathcal{S}^2$ where nonradial bifurcation occurs. More precisely for every $n = \left\lceil \frac{2+\alpha}{2} \beta + 1 \right\rceil, \ldots, \left\lceil \frac{2+\alpha}{2} \beta - 1 \right\rceil$ there exists a nonradial bifurcation point $(p_n, u_{p_n}) \in \mathcal{S}^2 \cap \Sigma^2_n$ and the respective branches $C^2_n$ have the following properties

i) There is a ball $B$ in $(1, \infty) \times C^{1,\gamma}(B)$ centered at $(p_n, u_{p_n})$ such that $C^2_n \cap B \{ (p_n, u_{p_n}) \}$ is made up of nonradial solutions with 2 nodal zones, one of which contains $x = 0$ and is homeomorphic to a disc.

ii) Every branch contains a sequence $(p_k, u_k)$ with either $\|u_k\|_{C^{1,\gamma}} \to \infty$, or $p_k \to \infty$, or possibly $p_k \to 1$ and $u_k$ converges to an eigenfunction of

$$\begin{align*}
-\Delta \omega &= \mu |x|^\alpha \omega \quad \text{in } B, \\
\omega &= 0 \quad \text{on } \partial B,
\end{align*}$$

which belongs to $K_n$.

iii) Two different branches can only have radial solutions in common. Precisely $C^2_n \cap \mathcal{S}^2 \cap \mathcal{S}^2$ contains at most isolated points, and if there is some $m \geq 3$ such that $C^2_n \cap C^2_n \cap \mathcal{S}^m$ is nonempty, then $\mathcal{S}^m \subset C^2_n \cap C^2_n$.

The possibility that $p_k \to 1$ but $u_k$ stays bounded remains open because the uniqueness of nodal solutions does not hold either in a neighborhood of $p = 1$, see [2, Theorem 1.3]. Concerning property iii), i.e. the possible overlapping of two different branches, we are not aware of any technique which enables to capture the formation of further nodal zones and/or a secondary bifurcation. Consequently a nonradial branch could, in principle, touch another radial curve $\mathcal{S}^m$ with $m \geq 3$, and then incorporate it because of the way in which $\Sigma^2_n$ and $C^2_n$ have been defined.

Theorem 2.2 applies also to $\alpha = 0$, i.e. to the Lane-Emden equation, giving back [19, Theorem 1.2] since in this particular case $\left\lceil \frac{2+\alpha}{2} \beta + 1 \right\rceil = 3$ and $\left\lceil \frac{2+\alpha}{2} \beta - 1 \right\rceil = 5$. For $\alpha > 0$ it is worth comparing this existence result with the ones in [2] and in [1], both concerning the least energy $n$-invariant nodal solutions, that we denote hereafter by $U_{p,n}$. For $n = 1, \ldots, \left\lceil \frac{2+\alpha}{2} \beta - 1 \right\rceil$, $U_{p,n}$ is nonradial for both $p$ close to 1 and large. It seems that in this case $U_{p,n}$ is nonradial for every $p > 1$ and the curve $p \mapsto U_{p,n}$ does not intersect the curve of radial solutions. This is certainly true for $n = 1$, i.e. the least energy nodal solution. Conversely for $n = \left\lceil \frac{2+\alpha}{2} \beta + 1 \right\rceil, \ldots, \left\lceil \frac{2+\alpha}{2} \beta - 1 \right\rceil$, [2, Proposition 4.10] and [1, Theorem 1.6] yield that $U_{p,n}$ are radial for $p$ close to 1, and then nonradial (and different one from another) when $p$ is large. Therefore the curves $p \mapsto U_{p,n}$ coincide with the one of
radial solutions for \( p \in (1, p_n) \), and then they give rise to the nonradial bifurcation stated by Theorem 2.2.

Only bifurcation from the curve \( S^2 \) is taken into account, since the behaviour of nodal solutions as \( p \to \infty \) is known only in the case of two nodal zones. When this paper was already finished we came to know that a very recent preprint by Ianni and Saldana [20] describes the asymptotic profile of every radial solutions. Starting from this it is possible, in principle, to compute exactly their Morse index and then the same arguments used here produce bifurcation also in the general case.

In dimension \( N \geq 3 \) the set \( \Sigma^m_n \) is nonempty at least for \( n = 2 + \left\lfloor \frac{2}{3} \right\rfloor, \ldots, n^m_m \), where the number \( n^m_n \geq 2(m - 1) + \lfloor \alpha(m - 1) \rfloor \) is characterized later on in (3.42) and can be numerically computed.

**Theorem 2.3** (Bifurcation from nodal solutions in dimension \( N \geq 3 \)). Consider problem (1.1) in dimension \( N \geq 3 \). For every \( \alpha \geq 0 \) and \( m \geq 2 \), at least \( 2m - 3 + \lfloor \alpha(m - 1) \rfloor - \lfloor \alpha/2 \rfloor \) different nonradial bifurcations take place along the curve \( S^m \). More precisely for every \( n = 2 + \left\lfloor \frac{2}{3} \right\rfloor, \ldots, n^m_m \) there exists a nonradial bifurcation point \((p_n, u_{p_n}) \in S^m \cap \Sigma^m_n\) and the respective branches \( C^m_n \) have the following properties

i) There is a ball \( B \) in \((1, p_n) \times C^{1,\gamma}(B)\) centered at \((p_n, u_{p_n})\) such that \( C^m_n \cap B \setminus \{(p_n, u_{p_n})\}\) is made up of nonradial solutions with \( m \) nodal zones, one of which contains \( x = 0 \) and is homeomorphic to a ball, while the other ones are homeomorphic to spherical shells.

ii) Every branch contains a sequence \((p_k, u_k)\) with either \( \|u_k\|_{C^{1,\gamma}} \to \infty \), or \( p_k \to p_n \), or possibly \( p_k \to 1 \) and \( u_k \) converges to an eigenfunction of (2.7) which belongs to \( K_n \).

iii) The intersection between two different branches, if non-empty, is made up of nodal solutions which do not depend by the angle \( \theta \).

The branches of nodal bifurcating solution in dimension \( N \geq 3 \) can overlap along radial solutions with a different number of nodal zones, but also along nonradial solutions that do not depend by the angle \( \theta \).

The statement of Theorem 2.3 is new also in the simpler case \( \alpha = 0 \), to the author’s knowledge. For the reader’s convenience, we state separately the bifurcation result concerning the Lane-Emden equation.

**Theorem 2.4** (Bifurcation for the Lane-Emden equation in dimension \( N \geq 3 \)). Consider problem (1.2) in dimension \( N \geq 3 \). For every \( m \geq 2 \) the curve \( S^m \) bifurcates at \( 2m - 3 \) points, at least. More precisely for every \( n = 2, \ldots, n^m_m \geq 2(m - 1) \) there exists a nonradial bifurcation point \((p_n, u_{p_n}) \in S^m \cap \Sigma^m_n\) and the continuum detaching at \((p_n, u_{p_n})\), i.e. \( C^m_n\) has the following properties

- **Local property:** there is a ball \( B \) in \((1, p_n) \times C^{1,\gamma}(B)\) centered at \((p_n, u_{p_n})\) such that \( C^m_n \cap B \setminus \{(p_n, u_{p_n})\}\) is made up of nonradial solutions with \( m \) nodal zones, one of which contains \( x = 0 \) and is homeomorphic to a ball, while the other ones are homeomorphic to spherical shells.

- **Global property:** every branch contains a sequence \((p_k, u_k)\) with either \( \|u_k\|_{C^{1,\gamma}} \to \infty \), or \( p_k \to p_n \), or possibly \( p_k \to 1 \) and \( u_k \) converges to an eigenfunction of

\[
\begin{aligned}
-\Delta \omega &= \mu \omega & \text{in } B, \\
\omega &= 0 & \text{on } \partial B,
\end{aligned}
\]

which belongs to \( K_n \).
Separation property: the intersection between two different branches, if non-empty, is made up of nodal solutions which do not depend by the angle $\theta$.

There is numerical evidence that $n_0^m = 2(m - 1)$ in any dimension $N \geq 3$, so that Theorem 2.4 provides exactly $2m - 3$ branches of nonradial solutions. In particular, in the case of 2 nodal zones, there should be only one branch in dimension $N \geq 3$, while 3 different branches have been produced in dimension $N = 2$. The planar case indeed differs from the other ones, as already observed in several occasions.

Let us mention in passing that the number of nonradial branches produced in Theorems 2.1, 2.2 and 2.3 goes to infinity when $\alpha \to \infty$, which is consistent with the specular study (bifurcation w.r.t. $\alpha$) performed in [21].

3. Preliminaries on the computation of the Morse index. To emphasize the dependence on the exponent $p \in (1, p_n)$, we take the exponent $\alpha \geq 0$ and the number of nodal zones $m$ as fixed and denote by $u_p$ the unique radial solution to (1.1) with $m$ nodal zones which is positive at the origin. We also write

$$L_p \psi = -\Delta \psi - p|x|^\alpha |u_p|^{p-1} \psi, \quad (3.1)$$
$$Q_p(\psi) = \int_\Omega (|\nabla \psi|^2 - p|x|^\alpha |u_p|^{p-1} \psi^2) \, dx \quad (3.2)$$

for the linearized operator at $u_p$ and the related quadratic form, respectively. They will be considered on the space $H^1_0(B)$, or in one of its subspaces specified case-by-case.

The Morse index, that we denote hereafter by $m(u_p)$, is the maximal dimension of a subspace of $H^1_0(B)$ in which the quadratic form $Q_p$ is negative defined, or equivalently the number of the negative eigenvalues of

$$L_p \psi = \Lambda \psi, \quad \psi \in H^1_0(B). \quad (3.3)$$

For radial solutions one can also look at the radial Morse index, denoted by $m_{rad}(u_p)$, i.e. the number of the negative eigenvalues of for (3.3) whose relative eigenfunction is $H^1_{0,rad}(B)$, the subspace of $H^1_0(B)$ given by radial functions.

As explained in full details in [7], this matter can be regarded through a singular eigenvalue problem associated to the linearized operator $L_p$, which has to be handled in weighted Lebesgue and Sobolev spaces

$$\mathcal{L} = \{\omega : B \to \mathbb{R} : \omega/|x| \in L^2(B)\}, \quad \mathcal{H}_0 = H^1_0(B) \cap \mathcal{L}.$$ 

The Morse index (on $H^1_0(B)$ as well as on some of its subspaces) turns out to be equal to the number of the negative eigenvalues of

$$L_p \psi = \tilde{\Lambda} \psi/|x|^2, \quad \psi \in \mathcal{H}_0(B). \quad (3.4)$$

Concerning radial solutions to the Hénon problem, it turns helpful the transformation

$$t = r^{\frac{2+\alpha}{4}}, \quad w(t) = u(r), \quad (3.5)$$
or a slight variation of it

$$t = r^{\frac{2+\alpha}{4}}, \quad v(t) = \left(\frac{2}{2 + \alpha}\right)^{\frac{2}{p-1}} u(r), \quad (3.6)$$

which map radial solutions to (1.1) into solutions of one-dimensional problems.
\[
\begin{cases}
-\left(t^{M-1}w'\right)' = \left(\frac{2}{2+\alpha}\right)^2 t^{M-1}|w|^{p-1}w, & 0 < t < 1, \\
w'(0) = 0, \quad w(1) = 0
\end{cases}
\] (3.7)

or, respectively
\[
\begin{cases}
-\left(t^{M-1}v'\right)' = t^{M-1}|v|^{p-1}v, & 0 < t < 1, \\
v'(0) = 0, \quad v(1) = 0
\end{cases}
\] (3.8)

see [7, 6]. In both cases \(M\) is a real parameter given by
\[
M = M(N, \alpha) = \frac{2(N + \alpha)}{2 + \alpha} \in [2, N].
\] (3.9)

Both (3.7) and (3.8) can be regarded as generalized radial versions of the Lane-Emden problem (1.2), since for integer values of \(M\) the function \(v\) solves indeed a problem of type (1.2) settled in the unit \(M\)-dimensional ball. In general we should refer to \(M\) as the fictitious dimension of the associated generalized Lane-Emden problem. The natural generalization of the standard Lebesgue and Sobolev spaces from which regarding at (3.7), or (3.8) are
\[
\begin{align*}
\mathcal{L}^2_M &= \{ \varphi : (0, 1) \to \mathbb{R} : \varphi \text{ measurable and s.t. } \int_0^1 t^{M-1} |\varphi|^p dt < +\infty \}, \\
H^1_M &= \{ \varphi \in \mathcal{L}^2_M : \varphi \text{ has a first order weak derivative } \varphi' \text{ in } \mathcal{L}^2_M \}, \\
H^1_{0,M} &= \{ \varphi \in H^1_M : \varphi(1) = 0 \}
\end{align*}
\]

Similarly the weighted Lebesgue and Sobolev spaces related to (3.4) shall be
\[
\begin{align*}
\mathcal{L}^2_M &= \{ \varphi : (0, 1) \to \mathbb{R} : \varphi/t \in \mathcal{L}^2_M \}, \\
\mathcal{H}^1_{0,M} &= H^1_{0,M} \cap \mathcal{L}_M.
\end{align*}
\]

Of course \(\mathcal{L}_M\) is a Hilbert space with the obvious product which brings to the orthogonality condition
\[
\varphi \perp_M \psi \iff \int_0^1 t^{M-3} \varphi \psi dt = 0.
\]

In this functional setting it is possible to look at a singular eigenvalue problem associated to (3.7), or equivalently (3.8), that is
\[
\begin{cases}
-\left(t^{M-1}\phi'\right)' - t^{M-1}a_p(t)\phi = t^{M-3}\nu \phi & \text{for } t \in (0, 1) \\
\phi \in \mathcal{H}^1_{0,M},
\end{cases}
\] (3.10)

where
\[
a_p(t) := p \left(\frac{2}{2+\alpha}\right)^2 |w_p(t)|^{p-1} = p|v_p(t)|^{p-1},
\] (3.11)

and \(w_p, v_p\) are the transformations of the radial solution \(u_p\) according to (3.5), (3.6), respectively. The definition of the singular eigenvalues \(\nu\) requests some care because of the singularity of the Sturm-Liouville problem (3.10) at the origin. It has been tackled in detail in [7, Section 3]\(^1\), by establishing and exploiting a variational

\(^{1}\)For the reader’s convenience, we remark that in the quoted paper the singular eigenvalues were denoted by \(\bar{\nu}\), while \(\nu\) stand for the classical eigenvalues associated with the linearization of (3.7)
characterization. Indeed, one can alternatively define
\[
\nu_1 := \inf \left\{ \frac{\int_0^1 t^{M-1} \left( \langle \phi' \rangle^2 - a_p \phi^2 \right) dt}{\int_0^1 t^{M-3} \phi^2 dt} : \phi \in \mathcal{H}_{0,M}, \ \phi \neq 0 \right\}
\]  
(3.12)
and see that, when the infimum stays below the threshold \((M - 2)/2\), then it is attained by a function \(\phi\) which solves \((3.10)\) in the weak sense, namely
\[
\int_0^1 t^{M-1} (\phi' \phi' - a_p \phi \phi) dt = \nu_1 \int_0^1 t^{M-3} \phi \phi \ dt
\]  
(3.13)
for every test function \(\varphi \in \mathcal{H}_{0,M}\). Such function \(\phi\) can therefore be called an eigenfunction related to the eigenvalue \(\nu_1\), and denoted by \(\phi_1\). Iteratively, if \(\nu_i < (M - 2)/2\), one can settle the minimization problem
\[
\nu_{i+1} := \inf \left\{ \frac{\int_0^1 t^{M-1} \left( \langle \phi' \rangle^2 - a_p \phi^2 \right) dt}{\int_0^1 t^{M-3} \phi^2 dt} : \phi \in \mathcal{H}_{0,M}, \ \phi \perp \phi_1, \ldots, \phi_i \right\}
\]  
(3.14)
Again, as far as \(\nu_{i+1} < (M - 2)/2\), it is attained by an eigenfunction \(\phi_{i+1}\) which solves \((3.10)\) in the weak sense. Next, the eigenfunctions related to these singular eigenvalues enjoy the same properties of the standard ones, in particular they are simple, mutually orthogonal, and the \(i\)th eigenfunction has exactly \(i\) nodal domains.

Eventually putting together [7, Proposition 1.4] and [6, Proposition 3.3, Theorem 1.3] we have

**Proposition 3.1.** Let \(\alpha \geq 0\) and \(u_p\) be a radial solution to \((1.1)\) with \(m\) nodal zones. Then the only nonnegative eigenvalues of \((3.10)\) are \(\nu_1(p) < \nu_2(p) < \cdots < \nu_m(p) < 0\) and satisfy
\[
\nu_i(p) < -\frac{2N - 2 + \alpha}{2 + \alpha} \quad \text{for } i = 1, \ldots, m - 1,
\]  
(3.15)
\[
-\frac{2N - 2 + \alpha}{2 + \alpha} < \nu_m(p) < 0.
\]  
(3.16)
Moreover the Morse index of \(u_p\) is given by
\[
m(u_p) = \sum_{i=1}^m \sum_{j=0}^{[J_{i-1}]} N_j,
\]  
(3.17)
where \([s] = \{ n \in \mathbb{Z} : n \geq s \}\) denotes the ceiling function and
\[
J_i(p) = \frac{2 + \alpha}{2} \left( \sqrt{\frac{N - 2}{2 + \alpha}} - \nu_i(p) - \frac{N - 2}{2 + \alpha} \right),
\]
being \(N_j = \frac{(N+2j-2)(N+j-2)!}{(N-3)!j!}\) the multiplicity of the eigenvalue \(\lambda_j = j(N + j - 2)\) for the Laplace-Beltrami operator in the sphere \(S_{N-1}\).

Afterward, the Morse index has been exactly computed at the ends of the existence range by computing the limits of the eigenvalues \(\nu_i(p)\). The paper [2] dealt with \(p\) close to 1, and we need to introduce some more notation to recall the obtained result. For every \(\beta \geq 0\) we write \(J_\beta\) for the Bessel function of first kind
\[
J_\beta(r) = r^\beta \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k + 1 + \beta)} \left( \frac{r}{2} \right)^{2k}, \quad r \geq 0,
\]
and \( z_i(\beta) \) for the sequence of its zeros on the interval \((0, \infty)\). Since the map \( \beta \mapsto z_i(\beta) \) is continuous and increasing, for every fixed integer \( m \) there exist a sequence \( \beta_i = \beta_i(\alpha, N, m) \) such that \( z_i(\beta_i) \) (the \( i^{th} \) zero of the Bessel function \( J_{\beta_i} \)) (3.18) coincides with \( z_m\left(\frac{N - 2}{2 + \alpha}\right) \) (the \( m^{th} \) zero of \( J_{\frac{N-2}{2+\alpha}} \)).

It is clear that

\[
\beta_1 > \beta_2 > \cdots > \beta_m = \frac{N - 2}{2 + \alpha}.
\]

In [2, Proposition 3.3 and Theorem 1.2] it is proved that

**Theorem 3.2.** Let \( \alpha \geq 0 \) and \( u_p \) be a radial solution to (1.1) with \( m \) nodal zones. Then

\[
\lim_{p \to 1} \nu_i(p) = \left( \frac{N - 2}{2 + \alpha} \right)^2 - \beta_i^2 \quad \text{as} \quad i = 1, \ldots, m.
\] (3.19)

After there exists \( \bar{p} = \bar{p}(\alpha) > 1 \) such that for \( p \in (1, \bar{p}) \) the Morse index of \( u_p \) is given by

\[
m(u_p) = 1 + \sum_{i=1}^{m-1} \sum_{j=0}^\left\lfloor \frac{(2+\alpha)\beta_i - N}{2} \right\rfloor N_j
\] (3.20)

if \( \alpha \neq \alpha_n = (2n + N)/\beta - 2 \), and it is estimated by

\[
1 + \sum_{i=1}^{m-1} \sum_{j=0}^\left\lfloor \frac{(2+\alpha)\beta_i - N}{2} \right\rfloor N_j \leq m(u_p) \leq 1 + \sum_{i=1}^{m-1} \sum_{j=0}^\left\lfloor \frac{(2+\alpha)\beta_i - N}{2} \right\rfloor N_j + \sum_{\ell} N_{1 + \left\lfloor \frac{(2+\alpha)\beta_{\ell} - N}{2} \right\rfloor}.
\] (3.21)

if \( \alpha = \alpha_n \) for some \( \ell \) and \( n \).

It is worth remarking that in dimension \( N = 2 \) the parameters \( \beta_i \), defined in (3.18) only depend by the total number of nodal zones \( m \), and not by \( \alpha \). In the case of two nodal zones one sees by numerical computation that

\[
\beta := \beta_1(\alpha, 2, 2) \approx 2.305,
\] (3.22)

and formulas (3.20) and (3.21), respectively, simplify into

\[
m(u_p) = 2 \left\lfloor \frac{2 + \alpha}{2} \beta \right\rfloor
\] (3.23)

if \( \alpha \neq \alpha_n = 2(n + 1)/\beta - 2 \), or

\[
(2 + \alpha)\beta \leq m(u_p) \leq (2 + \alpha)\beta + 2
\] (3.24)

if \( \alpha = \alpha_n \) for some for some integer \( n \).

The situation at the supremum of the existence range changes drastically depending if the dimension is \( N = 2 \) or greater. In dimension \( N = 2 \) only the Morse index of the least energy radial solution (i.e. the positive one) and of the least energy nodal radial solution (i.e. the one with two nodal zones) are known. They have both been computed in the paper [1], where it is shown that

**Theorem 3.3.** Let \( \alpha \geq 0 \) and \( u_p \) be a positive radial solution to (1.1) in dimension \( N = 2 \). Then

\[
\lim_{p \to \infty} \nu_1(p) = -1,
\] (3.25)
and there exists $p^* > 1$ such that for $p > p^*$ the Morse index of $u_p$ is given by

$$m(u_p) = 1 + 2 \left\lceil \frac{\alpha}{2} \right\rceil. \quad (3.26)$$

**Theorem 3.4.** Let $\alpha \geq 0$ and $u_p$ be a radial solution to (1.1) with 2 nodal zones in dimension $N = 2$, then

$$\lim_{p \to \infty} \nu_1(p) = -\kappa^2 \quad \text{with} \quad \kappa \approx 5.1869, \quad \lim_{p \to \infty} \nu_2(p) = -1. \quad (3.27)$$

Moreover there exists $p^* > 1$ such that for $p > p^*$ the Morse index of $u_p$ is given by

$$m(u_p) = 2 \left\lceil \frac{2 + \alpha}{2} \right\rceil + 2 \left\lceil \frac{\alpha}{2} \right\rceil \quad (3.28)$$

when $\alpha \neq \alpha'_n = 2(n/\kappa - 1)$, while when $\alpha = \alpha'_n$ it holds

$$(2 + \alpha)\kappa + 2 \left\lceil \frac{\alpha}{2} \right\rceil \leq m(u_p) \leq (2 + \alpha)\kappa + 2 \left\lceil \frac{\alpha}{2} \right\rceil + 2. \quad (3.29)$$

An analogous result for the radial solution to the Lane-Emden problem with two nodal zones has been obtained in [15]. The number $\kappa$ appearing in (3.27) is linked, through the formula $\kappa^2 = 2 + \ell^2/2$, to the constant $\ell$ used in that paper.

The Morse index in dimension $N \geq 3$ is computed in [5] extending some previous results on the Lane-Emden problem in [14]; precisely [5, Propositions 3.3, 3.10 and Theorem 1] state that

**Theorem 3.5.** Let $\alpha \geq 0$ and $u_p$ be a radial solution to (1.1) with $m$ nodal zones in dimension $N \geq 3$. Then

$$\lim_{p \to p_\alpha} \nu_i(p) = -\frac{2N - 2 + \alpha}{2 + \alpha} \quad \text{as} \quad i = 1, \ldots, m. \quad (3.30)$$

After there exists $p^* = p^*(\alpha) \in (1, p_\alpha)$ such that the Morse index of $u_p$ is given by

$$m(u_p) = \sum_{j=0}^{\left\lceil \frac{\alpha}{2} \right\rceil} N_j + (m - 1) \sum_{j=0}^{\frac{\alpha}{2}} N_j \quad (3.31)$$

for $p \in (p^*, p_\alpha)$.

Comparing (3.20) with (3.26) or (3.31) one sees that the positive radial solution has Morse index 1 when $p$ is close to 1, and greater than 1 when $p$ is at the opposite end of the existence range (for $\alpha > 0$).

It is not hard to see that in dimension $N = 2$ the solution with 2 nodal zones shares the same behaviour, for every $\alpha \geq 0$. Indeed in this case formulas (3.23) and (3.24) yield

$$m(u_p) \leq (2 + \alpha)\beta + 2 \quad \text{for} \quad p \in (1, \tilde{p})$$

for every $\alpha \geq 0$, since $[t] \geq t + 1$. Therefore taking $q > p^*$ we deduce from (3.28), (3.29) that

$$m(u_q) - m(u_p) \geq 2 \left\lceil \frac{2 + \alpha}{2} \kappa \right\rceil + 2 \left\lceil \frac{\alpha}{2} \right\rceil - (2 + \alpha)\beta - 2$$

and since clearly $\left\lceil t \right\rceil \geq t$ we have

$$\geq (2 + \alpha)(\kappa - \beta) + \alpha - 2 \geq 2(\kappa - \beta - 1) > 2,$$
	hanks{thanks to the approximated values of $\beta$ and $\kappa$ given in (3.22) and (3.27), respectively.}
In higher dimensions, the approximation of the parameters $\beta_i$ appearing in (3.20) can be numerically performed after having chosen a specific value for $\alpha$, which fixes the baseline Bessel function $J_{\frac{N-2}{2+\alpha}}$. To have an overall picture it can be useful to establish some estimate. We report here the elementary proof of an estimate of the Bessel zeros that contributes to this aim.

**Lemma 3.6.** For all $\beta \geq 0$ and $i, m$ integers with $i < m$ we have 

$$z_i(\beta + 2(m - i)) < z_m(\beta).$$  \hspace{1cm} (3.32)

**Proof.** It is known that the $m^{th}$ zero of $J_\beta$ lies in the $m^{th}$ nodal set of $J_{\beta+1}$, i.e. 

$$z_{m-1}(\beta + 1) < z_m(\beta) < z_{m}(\beta + 1),$$  \hspace{1cm} (3.33)

which also implies $(-1)^m J_{\beta+1}(z_m(\beta)) < 0$. On the other hand

$$z_{m-2}(\beta + 2) < z_m(\beta) < z_{m}(\beta + 2).$$  \hspace{1cm} (3.34)

Actually the first inequality is obtained by iterating (3.33) and the second one follows since the map $\beta \mapsto z_m(\beta)$ is increasing. Hence the $m^{th}$ zero of $J_\beta$ can only belong to the $(m-1)^{th}$ or to the $m^{th}$ nodal set of $J_{\beta+2}$. But by the recurrence relation

$$J_{\beta+2}(z_m(\beta)) = \frac{2(\beta + 1)}{z_m(\beta)} J_{\beta+1}(z_m(\beta)) - J_\beta(z_m(\beta)) = \frac{2(\beta + 1)}{z_m(\beta)} J_{\beta+1}(z_m(\beta)),$$

and therefore also $(-1)^m J_{\beta+2}(z_m(\beta)) < 0$, which means that the $m^{th}$ zero of $J_\beta$ lies in the $m^{th}$ nodal set of $J_{\beta+2}$. In particular

$$z_{m-1}(\beta + 2) < z_m(\beta).$$  \hspace{1cm} (3.35)

Applying iteratively (3.35) gives the claim. \hfill \Box

Inserting the estimate (3.32) inside the formulas (3.20) and (3.21) gives the following

**Proposition 3.7.** For every $\alpha \geq 0$ we have

$$\beta_i > \frac{N - 2}{2 + \alpha} + 2(m - i) \quad \text{as} \quad i = 1, \ldots, m - 1$$  \hspace{1cm} (3.36)

and there is $\bar{p} = \bar{p}(\alpha) > 1$ such that the Morse index of $u_p$ is estimated from below by

$$m(u_p) \geq 1 + \sum_{i=1}^{m-1} \sum_{j=0}^{[(2+\alpha)(m-i)]} N_j$$  \hspace{1cm} (3.37)

$$= m + \sum_{k=1}^{m-1} (m - k) \sum_{j=1+[(2+\alpha)(k-1)]}^{[(2+\alpha)k]} N_j$$  \hspace{1cm} (3.38)

$$\geq m + \sum_{k=1}^{m-1} (m - k) \sum_{j=1+[(2+\alpha)(k-1)]}^{(2+[\alpha])k} N_j$$  \hspace{1cm} (3.39)

for $p \in (1, \bar{p})$.

**Proof.** (3.36) is an immediate consequence of Lemma 3.6 since $\beta_m = \frac{N - 2}{2 + \alpha}$ and the map $\beta \mapsto z_i(\beta)$ is increasing. In particular the index $J_i(p)$ appearing in (3.17) satisfy $J_i(p) > (2 + \alpha)(m - i)$ in a right neighborhood of $p = 1$, and (3.37) follows.
Next,

\[
1 + \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} N_j = m + \sum_{i=1}^{m-1} \sum_{j=1}^i N_j = m + \sum_{i=1}^{m-1} \sum_{j=1}^i \sum_{k=1}^{(2+\alpha)(k-1)} N_j
\]

\[
= m + \sum_{k=1}^{m-1-m-k} N_j = m + \sum_{k=1}^{m-1} \sum_{j=1}^i (m - k) N_j
\]

which is (3.38).

Moreover, as clearly \((2+\alpha)(k) \geq 2 + [\alpha])\), we have

\[
(m - k) \sum_{j=1}^{(2+\alpha)(k-1)} N_j = (m - k) \sum_{j=1}^{(2+\alpha)(k-1)} N_j + (m - k) \sum_{j=1}^{[2+\alpha)(k-1)]} N_j
\]

\[
\geq (m - k) \sum_{j=1}^{[2+\alpha)(k-1)]} N_j + (m - k') \sum_{j=1}^{[2+\alpha)(k'-1)]} N_j
\]

Hence

\[
m(u_p) \geq m + \sum_{k=1}^{m-1} (m - k) \sum_{j=1}^{(2+\alpha)(k-1)} N_j
\]

\[
\geq m + \sum_{k=1}^{m-1} (m - k) \sum_{j=1}^{(2+\alpha)(k-1)} N_j + \sum_{k=2}^{m-1} (m - k) \sum_{j=1}^{(2+\alpha)(k-1)} N_j
\]

\[
= m + \sum_{j=1}^{2+\alpha]} N_j + \sum_{j=1}^{(2+\alpha)(k-1)} N_j
\]

which is (3.39).

We therefore see that, in dimension \(N \geq 3\), the Morse index of nodal radial solutions for \(p\) close to \(p_\alpha\) is smaller than the one for \(p\) is close to 1.

**Corollary 3.8.** In dimension \(N \geq 3\), for every value of \(\alpha \geq 0\) and \(m \geq 2\) there exist \(1 < \bar{p} < \bar{q} < p_\alpha\) such that we have

\[
m(u_p) > m(u_q) \quad \text{as} \quad 1 < \bar{p} < \bar{q} < p_\alpha.
\]

**Proof.** By (3.31) we know that for every \(\alpha \geq 0\)

\[
m(u_q) \leq m \sum_{j=0}^{[2+\alpha] / 2]} N_j
\]

as long as \(q\) is near \(p_\alpha\). So, thanks to the estimate (3.39), the claim follows after checking that

\[
h(m) := \sum_{i=1}^{m-1} (m - i) \sum_{j=1+(2+\alpha)(i-1]} N_j - m \sum_{j=1}^{1+[\alpha/2]} N_j > 0, \quad (3.40)
\]
(3.40) can be proved by induction on the number of nodal zones \( m \geq 2 \), taking advantage from the fact that in dimension \( N \geq 3 \) the multiplicity \( N_j \) increases with \( j \), i.e.

\[
N_{j+1} > N_j \quad \text{as} \quad j \geq 1.
\]  

(3.41)

We first check

\[
h(2) = 2 + \sum_{j=1}^{2+\lfloor \alpha \rfloor} N_j - 2 - \sum_{j=1}^{2+\lfloor \alpha/2 \rfloor} N_j = \sum_{j=1}^{2+\lfloor \alpha \rfloor} N_j - \sum_{j=1}^{2+\lfloor \alpha/2 \rfloor} N_j > 0.
\]

When \( \alpha \in [0, 2) \), \( \lfloor \alpha \rfloor \geq \lfloor \alpha/2 \rfloor = 0 \) and we have \( h(2) \geq N_2 - N_1 > 0 \).

Otherwise if \( \alpha \geq 2 \) then \( \lfloor \alpha/2 \rfloor \geq 1 \) and (3.41) yields

\[
h(2) > (\lfloor \alpha \rfloor - \lfloor \alpha/2 \rfloor) N_{2+\lfloor \alpha/2 \rfloor} - \lfloor \alpha/2 \rfloor N_{1+\lfloor \alpha/2 \rfloor}
\]

and since \( \lfloor \alpha \rfloor \geq 2\lfloor \alpha/2 \rfloor \) we have

\[
\geq \lfloor \alpha/2 \rfloor (N_{2+\lfloor \alpha/2 \rfloor} - N_{1+\lfloor \alpha/2 \rfloor}) > 0
\]

by using (3.41) once more.

After we take that \( h(m) > 0 \) for some \( m \geq 2 \) and deduce that also \( h(m+1) > 0 \).

Let us compute

\[
\sum_{i=1}^{m} (m+1-i) \sum_{j=1}^{i(2+\lfloor \alpha \rfloor)} N_j = \sum_{i=1}^{m-1} (m-i) \sum_{j=1}^{i(2+\lfloor \alpha \rfloor)} N_j + \sum_{j=1}^{(2+\lfloor \alpha \rfloor)m} N_j,
\]

hence

\[
h(m+1) = h(m) + \sum_{j=1}^{(2+\lfloor \alpha \rfloor)m} N_j - \sum_{j=1}^{1+\lfloor \alpha/2 \rfloor} N_j > \sum_{j=2+\lfloor \alpha/2 \rfloor}^{(2+\lfloor \alpha \rfloor)m} N_j > 0,
\]

and this concludes the proof.

In the next section we will see that the changes in the Morse index caused by the first singular eigenvalue \( \nu_1(p) \) play a crucial role in establishing bifurcation results. Therefore the parameter \( \beta_1 \) defined in (3.18) deserves a special attention, in particular the integer number

\[
n^m_\alpha := \left\lfloor \frac{(2+\alpha)\beta_1 - N}{2} \right\rfloor,
\]

(3.42)

which is characterized by the double inequality

\[
z_1 \left( \frac{2n^m_\alpha + N - 2}{2 + \alpha} \right) < z_m \left( \frac{N - 2}{2 + \alpha} \right) \leq z_1 \left( \frac{2n^m_\alpha + N}{2 + \alpha} \right).
\]

(3.43)

Once that the dimension \( N \), the exponent \( \alpha \) and the number of nodal zones \( m \) have been fixed, the number \( n^m_\alpha \) can be easily computed by using iteratively the function \texttt{Besselzero} in MathLab, for instance. Besides it is already known by (3.36) that

\[
n^m_\alpha \geq 2(m-1) + \lfloor \alpha(m-1) \rfloor.
\]

(3.44)

For \( \alpha = 0 \) (Lane-Emden equation) there is numerical evidence that for every \( N \)

\[
z_1 \left( 2(m-1) + \frac{N - 2}{2} \right) < z_m \left( \frac{N - 2}{2} \right) \leq z_1 \left( 2(m-1) + \frac{N}{2} \right),
\]

so that \( n^m_0 = 2(m-1) \) indeed.
4. Global bifurcation. Here we prove the bifurcation results stated in Section 2. It is well known that if \((p,u_p)\) is a bifurcation point in the curve \(S^m\), then the solution \(u_p\) has to be degenerate, which means that the linearized operator \(L_p\) defined in (3.1) has nontrivial kernel in \(H_0^1(B)\), or equivalently \(\Lambda = 0\) is an eigenvalue for (3.3). In Section 3 we have noticed that the Morse index changes within the interval \((1,p_\alpha)\), so that degeneracy values do exist. Besides we can not rely on any variational structure, since we aim to include also supercritical values of \(p\), and bifurcation can be obtained only through an odd change of the Morse index. Hence a better knowledge of the kernel of \(L_p\) is needed. By [6, Theorem 1.3] the radial solutions are radially nondegenerate, i.e. the kernel of \(L_p\) does not contain radial functions. Moreover, the degeneracy has been characterized in [7, Proposition 1.5] in terms of the eigenvalues \(\nu_i(p)\) showing that

**Proposition 4.1.** Let \(u_p\) be a radial solution to (1.1) with \(m\) nodal zones. It is degenerate if and only if

\[
\nu_i(p) = -\left(\frac{2}{2+\alpha}\right)^2 j(N-2+j) \quad \text{for some } i = 1, \ldots, m \text{ and } j \geq 1. \tag{4.1}
\]

Besides any function in the kernel of \(L_p\) can be written according to the decomposition formula

\[
\phi(x) = \psi_{i,p}(|x|^{\frac{2\alpha}{2+\alpha} - 2})Y_j(x/|x|), \tag{4.2}
\]

where \(\psi_{i,p}\) is an eigenfunction for (3.10) related to an eigenvalue \(\nu_i(p)\) satisfying (4.1), and \(Y_j\) stands for an eigenfunction of the eigenvalue \(j(N-2+j)\) of the Laplace-Beltrami operator.

For a positive solution only the first eigenvalue \(\nu_1(p)\) plays a role and one can manage to obtain an odd change in the Morse index by restricting the attention to the subspace of \(O(N-1)\)-invariant functions, as in [3, 17]. For a nodal solution, instead, the equality (4.1) can hold for different values of \(i\) and \(j\) and (4.2) brings out that the kernel of \(L_p\) has a complex structure. This difficulty can be dealt with by turning to the notion of degree and index of fixed points in the positive cones introduced in Section 2.

Letting \(T\) be the operator

\[
T(p,v) : (1,p_\alpha) \times C_0^1 \gamma(B) \to C_0^1 \gamma(B), \quad T(p,v) = (-\Delta)^{-1} \left(|v|^{p-1}v\right),
\]

it is clear that \(S^m\) are curves of fixed points for \(T\) and more generally \(u\) solves (1.1) when \(u = T(p,u)\). Moreover minor variations on [4, Lemmas 2.2, 3.1] allow seeing that

**Lemma 4.2.** The operator \(T(p,\cdot)\) maps both \(X_n\) and \(K_n\) into themselves.

Denoting by \(T_p'(p,\cdot)\) the Fréchet derivative of \(T(p,\cdot)\) computed at \(u\), we say that \(u\) is an isolated fixed point for \(T(p,\cdot)\) w.r.t. \(X_n\) when \(I - T_p'(p,\cdot)\) is invertible in \(X_n\), which is assured by the nondegeneracy of \(u\). Starting from the characterization of degeneracy in Proposition 4.1 one can see that radial solutions \(u_p\) are isolated fixed points, except at most a discrete set of \(p\). It follows from a general regularity result.

**Lemma 4.3.** The maps \(p \mapsto \nu_i(p)\) are analytic in \(p\).

We do not report the details of the proof. For positive solutions to the Hénon equation it has been proved in [3, Proposition 4.1]. For sign changing solutions to
the Lane-Emden equation in dimension $N \geq 3$ the proof is contained in [13, Lemma 3.2], and it has been adapted to the case $N = 2$ in [19, Lemma 7.1].

One can now compute the index of $u_p$ relative to the cone $K_n$, see [11], which will be denote by $\text{index}_{K_n}(p, u_p)$. It is important to note that, also in the case of nodal solutions, the first singular eigenvalue determines by itself such index.

**Lemma 4.4.** Let $p$ be such that $u_p$ is nondegenerate. Then

$$\text{index}_{K_n}(p, u_p) = \begin{cases} 0 & \text{if } \nu_1(p) < -\left(\frac{2}{2 + \alpha}\right)^2 n(N - 2 + n), \\ \deg_{X_n}(I - T(p,)) = \pm 1 & \text{if } \nu_1(p) > -\left(\frac{2}{2 + \alpha}\right)^2 n(N - 2 + n). \end{cases}$$

Here the symbol $\deg_{X_n}(I - T(p,))$ stands for the Leray-Schauder degree of the operator $I - T(p,)$ restricted at $X_n$, computed in a neighborhood of $(p, u_p)$ which does not contain nonradial solutions (this choice is possible since $u_p$ is nondegenerate by assumption).

**Proof.** Theorem 1 in [11] states that for isolated fixed points

$$\text{index}_{K_n}(p, u) = \begin{cases} 0 & \text{if } T'(p, u) \text{ has the property } \alpha, \\ \deg_{X_n}(I - T(p,)) = \pm 1 & \text{otherwise.} \end{cases}$$

In this way, the prove reduces to show that the so-called property $\alpha$ holds if and only $\nu_1(p) + (2\pi^2)^2 n(N - 2 + n) < 0$. Several characterizations of the property $\alpha$ are provided in Lemma 3 and the following Remark in [11]. To state the one which will be used here we need the sets

$$W^+ := \{v \in X_n : u_p + \gamma v \in K_n \text{ for some } \gamma > 0\}, \quad W^0 := \{v \in W_{u_p}^+ : -v \in W_{u_p}^+\}.$$  

$V$ the orthogonal (in the $H_0^1$ sense) complement to $W^0$ in $X_n$.

Notice that the functions in $W^0$ do not depend by the angle $\theta$. Next, $T'$ has the property $\alpha$ if there exists $t \in (0, 1)$ such that the problem

$$\begin{cases} -\Delta v = tp|x|^\alpha |u_p|^{p-1}v & \text{in } B, \\ v \in \overline{W^+} \setminus W^0 \end{cases}$$  

has a solution. We follow the proof of [12, Theorem 1] and look at the family of eigenvalue problems

$$\begin{cases} -\Delta v - tp|x|^\alpha |u_p|^{p-1}v = \Lambda v & \text{in } B, \\ v \in V \end{cases}$$  

and let $\Lambda_1$ be its first eigenvalue. When $t = 0$ (4.4) reduces to an eigenvalue problem for the Laplacian and certainly $\Lambda_0 > 0$. When $t = 1$, instead, (4.4) gives back the eigenvalue problem (3.3), but only eigenfunctions in $V$ matter. Furthermore the variational characterization yields that the first eigenvalue $\Lambda_t$ is strictly decreasing w.r.t. $t$.

If $T'$ has the property $\alpha$, then $\Lambda_t \leq 0$ for some $t < 1$ and therefore $\Lambda_1 < 0$. This in turn means that the eigenvalue problem (3.3) has a negative eigenvalue with related eigenfunction in $V$ and then $\nu_1(p) + (2\pi^2)^2 j(N - 2 + j) < 0$ for some $i = 1, \ldots, m$ and $j$ such that the related spherical harmonic belongs to $V$, by the characterization in [7, Theorem 1.4]. Taking advantage from the description of the spherical harmonics given in the proof of Theorem 1.1 in [4], one sees that $j$ must be a multiple of $n$ and so, in particular, $\nu_1(p) + (2\pi^2)^2 n(N - 2 + n) < 0$. 


Hence the first eigenvalue $\Lambda_1$ and as using the change of variable $t$ radial eigenfunction for (3.10) and $Y_\psi$ belonging such that $\Lambda_1$ with defined according to (2.6) Proposition 4.5. If for some integer $n \in \mathbb{N}$ or $p = 2 + \alpha$ we let $\psi$ be the first radial eigenfunction for (3.10) and $Y_n$ the spherical harmonic related to $n(N-2+n)$ belonging $V$ (which does exist for what we have said before). Now $v(r,\theta,\phi) = \psi(r2^{1+\alpha})Y_n(\theta,\phi)$ is in $V$ and an easy computation shows that

$$
\int_B \left( (\nabla v)^2 - p|x|^2|u_p|^{p-1}v^2 \right) \nu
= \int_0^1 \int_{S^{N-1}} \left( \left( \frac{d}{dr}\psi(r^{2+\alpha}) \right)^2 + \frac{pr^\alpha}{u_p|u_p|^{p-1}} \left( \psi(r^{2+\alpha}) \right)^2 \right) d\sigma d\theta d\phi
+ \int_0^1 \int_{S^{N-1}} \left| \nabla Y_n \right|^2 d\sigma d\theta d\phi
= \int_0^1 \int_{S^{N-1}} \left( \frac{2+\alpha}{2} \left( \frac{\psi'(r^{2+\alpha})}{r^{2+\alpha}} \right)^2 - p|u_p|^{p-1} \left( \psi(r^{2+\alpha}) \right)^2 \right) d\sigma d\theta d\phi
+ \int_0^1 \int_{S^{N-1}} \left| \nabla Y_n \right|^2 d\sigma d\theta d\phi
$$

using the change of variable $t = r^{2+\alpha}$ and the notation in (3.11)

$$
= \frac{2+\alpha}{2} \int_0^1 t^{M-1} \left( \psi'(t)^2 - a_p(t) \left( \psi(t) \right)^2 \right) dt \int_{S^{N-1}} Y_n^2 d\sigma d\theta d\phi
+ \frac{2+\alpha}{2} \int_0^1 t^{M-3} \psi^2(t) dt \int_{S^{N-1}} \left| \nabla Y_n \right|^2 d\sigma d\theta d\phi
$$

and as $\psi$ solves (3.10) and $Y_n$ is a spherical harmonics we get

$$
= \frac{2+\alpha}{2} \nu_1(p) \int_0^1 t^{M-3} \psi^2(t) dt \int_{S^{N-1}} Y_n^2 d\sigma d\theta d\phi
+ \frac{2+\alpha}{2} \int_0^1 t^{M-3} \psi^2(t) dt \int_{S^{N-1}} Y_n^2 d\sigma d\theta d\phi
= \frac{2+\alpha}{2} \left( \nu_1(p) + \left( \frac{2}{2+\alpha} \right)^2 n(N-2+n) \right) \int_0^1 t^{M-3} \psi^2(t) dt \int_{S^{N-1}} Y_n^2 d\sigma d\theta d\phi
= \left( \nu_1(p) + \left( \frac{2}{2+\alpha} \right)^2 n(N-2+n) \right) \int_B \frac{v^2}{|x|^2} dx < 0.
$$

Hence the first eigenvalue $\Lambda_1$ is negative, and since $\Lambda_0 > 0$ there exists $t \in (0,1)$ such that $\Lambda_t = 0$, which means that $T^*$ has the property $\alpha$.

Relying on Lemma 4.4 one can see that a sufficient condition for bifurcation is

$$
\lim_{p \to p_n} \nu_1(p) + \left( \frac{2}{2+\alpha} \right)^2 n(N-2+n) \left( \lim_{p \to p_n} \nu_1(p) + \left( \frac{2}{2+\alpha} \right)^2 n(N-2+n) \right) < 0, \quad (4.5)
$$

for some integer $n$.

**Proposition 4.5.** If $n$ is an integer which fulfills (4.5), then there exists at least one $p_n \in (1, p_\alpha)$ such that $(p_n, u_{p_n})$ is a nonradial bifurcation point and the branch $C_n$ defined according to (2.6) is global, in the sense that it contains a sequence $(p_k, u_k)$ with

i) either $\left\| u_k \right\|_{C^{1,1}(B)} \to +\infty$,  
ii) or $p_k \to p_\alpha$,  
iii) or $p_k \to 1.$
Proof. Under assumption (4.5), and thanks to Lemma 4.3, there exists at least one (and an odd number of) \( \bar{p} \in (1, p_\alpha) \) and \( \delta > 0 \) such that

\[
\nu_1(\bar{p}) = -\left(\frac{2}{2 + \alpha}\right)^2 n(N - 2 + n), \quad \nu_1(p) \neq -\left(\frac{2}{2 + \alpha}\right)^2 \frac{\alpha}{2 + \alpha}, \quad (\nu_1(\bar{p} - \delta) + \left(\frac{2}{2 + \alpha}\right)^2 n(N - 2 + n)) (\nu_1(\bar{p} + \delta) + \left(\frac{2}{2 + \alpha}\right)^2 n(N - 2 + n)) < 0,
\]

for every \( i = 1, \ldots, m, j \geq 0, \) and \( p \in (\bar{p} - \delta, \bar{p} + \delta), p \neq \bar{p}. \)

Lemma 4.4 then implies that the Leray Schauder degree in the cone \( K_n \) changes and the remaining of the proof follows as in [4, Theorem 1.2]. See also [18], where a more detailed proof is given in the case of positive solutions.

We are now ready to prove the bifurcation results stated in Section 2. Concerning positive solutions, we have already pointed out that the Morse index near at \( 1, \) for every \( \alpha > 0. \) Since only the first eigenvalue \( \nu_1(p) \) is negative and gives a contribution to the Morse index, it is clear that there exists at least one value of \( n \) such that (4.5) holds. Let us complete the proof of Theorem 2.1.

Proof of Theorem 2.1. First, we check that (4.5) is fulfilled for every integer \( n = 1, \ldots, \lfloor \alpha/2 \rfloor. \) Recalling that \( \lim_{p \to p_\alpha} \nu_1(p) = 0 \) by (3.19) (since for positive solutions \( \beta_1 = \frac{N-2}{2+\alpha} \), it is equivalent to see that

\[
\lim_{p \to p_\alpha} \nu_1(p) < -\left(\frac{2}{2 + \alpha}\right)^2 n(N - 2 + n)
\]

for \( 1 \leq n < \frac{2+\alpha}{2}, \) i.e.

\[
\lim_{p \to p_\alpha} \nu_1(p) \geq -\frac{2N - 2 + \alpha}{2 + \alpha}.
\]

But (3.30) and (3.25) state that equality holds in any dimension \( N \geq 2. \)

Therefore Proposition 4.5 gives the first part of the claim. As for property \( i), \) every branch \( C_n \) must be composed of nonnegative solutions by continuity, so that maximum principle ensures that or they are positive, or they are identically zero. But this last occurrence is not allowed since the trivial solution is isolated. Besides the same Proposition 4.5 states that \( C_n \) contains a sequence \( (p_k, u_k) \) such that either \( \|u_k\| \to \infty, \) or \( p_k \to p_\alpha, \) or \( p_k \to 1. \) Moreover the occurrence \( p_k \to 1 \) is forbidden by the uniqueness of positive solutions for \( p \) close to 1 in [3, Theorem 3.1] (which can be easily extended also to dimension \( N = 2). \)

As for the possible intersection between two branches \( C_n \) and \( C_n', \) it has to be composed by \( (p, v) \) such that \( v \in K_n \cap K_n' \) is a positive solution to (1.1). In dimension \( N = 2 \) \( K_n \cap K_n' \) reduces to radial function, and therefore \( v = u_p \) is a radial positive solution to (1.1), which has to be degenerate and therefore isolated. In dimension \( N \geq 3, \) instead, \( K_n \cap K_n' \) contains also functions which are nonradial, but do not depend by the angle \( \theta \).

Proof of Theorem 2.2. First, we compute the values of the integer \( n \) for which (4.5) holds. Thanks to (3.19) and (3.27), it means that

\[
-\beta^2 = \lim_{p \to 1} \nu_1(p) > -\left(\frac{2n}{2 + \alpha}\right)^2 > \lim_{p \to \infty} \nu_1(p) = -\kappa^2,
\]
which is clearly equivalent to $\frac{2+\alpha}{2} \beta < n < \frac{2+\alpha}{2} \kappa$, i.e. $n = \left\lceil \frac{2+\alpha}{2} \beta + 1 \right\rceil, \ldots, \left\lceil \frac{2+\alpha}{2} \kappa - 1 \right\rceil$. Here $\beta$ and $\kappa$ are respectively given by (3.22) and (3.27).

So Proposition 4.5 yields that, for any of such values of $n$, there exist a nonradial bifurcating point $(p_n, u^{p_n})$ and a bifurcating branch in $K_n^\ast$. The local property mentioned $i)$ is a plain consequence of the continuity of the branch. As for property $ii)$, Proposition 4.5 states that the branch $C_n^\ast$ contains a sequence $(p_k, u_k)$ such that either $\|u_k\|_{C^{1,\gamma}} \to \infty$, or $p_k \to \infty$, or $p_k \to 1$. If $p_k \to 1$ but $\|u_k\|_{C^{1,\gamma}}$ stays bounded, then [2, Lemma 2.1] ensures that $u_k$ converges to an eigenfunction of (2.7). Coming to property $iii)$, since $K_n \cap K_n^\ast$ is the set of radial functions, then the intersection point between two branches $C_n^\ast$ and $C_n^\ast'$ should be another nonradial bifurcation point $(p, u^m)$. Here possibly $m \neq 2$, because the number of nodal zones of the bifurcating solutions could become larger that 2, far from the bifurcation point. If this happens, then the very definition of $C_n^\ast$ yields that $S^m \subset C_n^\ast$ and this concludes the proof.

The proof of the bifurcation in higher dimensions is quite similar.

Proof of Theorem 2.3. Let us check that property (4.5) holds as long as $\frac{2+\alpha}{2} < n < \frac{2+\alpha}{2} \beta_1 - \frac{N-2}{2}$, where $\beta_1$ is defined in (3.18). We have already noticed that now the first singular eigenvalue close to $p = p_n$ is greater than close to $p = 1$, therefore it is needed that

$$\lim_{p \to p_n} \nu_1(p) > - \left( \frac{2}{2 + \alpha} \right)^2 n(N - 2 + n) > \lim_{p \to 1} \nu_1(p).$$

By (3.19) and (3.30) it means that

$$\frac{2N - 2 + \alpha}{2 + \alpha} < \left( \frac{2}{2 + \alpha} \right)^2 n(N - 2 + n) < \beta_1^2 - \left( \frac{N - 2}{2 + \alpha} \right)^2,$$

which can be rearranged into

$$1 + 2 \frac{N - 2}{2 + \alpha} < \left( \frac{2n}{2 + \alpha} \right)^2 + 2 \frac{2n}{2 + \alpha} + \frac{N - 2}{2 + \alpha} < \beta_1^2 - \left( \frac{N - 2}{2 + \alpha} \right)^2,$$

which in turn, after adding the term $(\frac{N - 2}{2 + \alpha})^2$ to every member and extracting square roots, becomes $\frac{N + \alpha}{2 + \alpha} < \frac{N - 2 + 2n}{2 + \alpha} < \beta_1$, i.e.

$$\frac{2 + \alpha}{2} < n < \frac{2 + \alpha}{2} \beta_1 - \frac{N - 2}{2},$$

as claimed. It means that (4.5) is fulfilled by $n = 2 + \left\lceil \frac{\alpha}{2} \right\rceil, \ldots, n^m_\alpha$, where $n^m_\alpha$ is defined in (3.42). Remembering that $n^m_\alpha \geq 2(m - 1) + \lceil \alpha(m - 1) \rceil$, see (3.44), one can see that (4.5) holds for at least $2m - 3 + \lceil \alpha(m - 1) \rceil - \left\lceil \frac{\alpha}{2} \right\rceil$ different values of $n$.

Eventually the conclusion follows by Proposition 4.5, arguing as in the proof of Theorem 2.2. The only difference stands in the possible overlapping between branches, i.e. property $iii)$, which in higher dimension can also contain nonradial solutions which do not depend by the angle $\theta$.

REFERENCES

[1] A. L. Amadori and F. Gladiali, The Hénon problem with large exponent in the disc, J. Differ. Equ., 268 (2020), 5892–5944.
[2] A. L. Amadori, On the asymptotically linear Hénon problem, to appear, Commun. Contemp. Math., 2019.
[3] A. L. Amadori and F. Gladiali, Bifurcation and symmetry breaking for the Hénon equation, Adv. Differ. Equ., 19 (2014), 755–782.
4816 ANNA LISA AMADORI

[4] A. L. Amadori and F. Gladiali, Nonradial sign changing solutions to Lane-Emden problem in an annulus, *Nonlinear Anal.*, **155** (2017), 294–305.

[5] A. L. Amadori and F. Gladiali, Asymptotic profile and morse index of nodal radial solutions to the Hénon problem, *Calc. Var. Partial Differ. Equ.*, **58** (2019), 1–47.

[6] A. L. Amadori and F. Gladiali, On a singular eigenvalue problem and its applications in computing the morse index of solutions to semilinear PDE’s: II, *Nonlinearity*, **33** (2020), 2541–2561.

[7] A. L. Amadori and F. Gladiali, On a singular eigenvalue problem and its applications in computing the morse index of solutions to semilinear PDE’s, *Nonlinear Anal. Real World Appl.*, **55** (2020), Art. 103133.

[8] M. Badiale and E. Serra, Multiplicity results for the supercritical Hénon equation, *Adv. Nonlinear Stud.*, **4** (2004), 453–467.

[9] T. Bartsch, T. D’Aprile and A. Pistoia, On the profile of sign-changing solutions of an almost critical problem in the ball, *Bull. Lond. Math. Soc.*, **45** (2013), 1246–1258.

[10] T. Bartsch and M. Willem, Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.*, **124** (1993), 261–276.

[11] E. N. Dancer, On the indices of fixed points of mappings in cones and applications, *J. Math. Anal. Appl.*, **91** (1983), 131–151.

[12] E. N. Dancer and P. Hess, Global breaking of symmetry of positive solutions on two-dimensional annuli, *Differ. Integral Equ.*, **5** (1992), 903–913.

[13] E. N. Dancer and J. Wei, Sign-changing solutions for supercritical elliptic problems in domains with small holes, *Manuscr. Math.*, **123** (2007), 493–511.

[14] F. De Marchis, I. Ianni and F. Pacella, A Morse index formula for radial solutions of Lane-Emden problems, *Adv. Math.*, **322** (2017), 682–737.

[15] F. De Marchis, I. Ianni and F. Pacella, Exact Morse index computation for nodal radial solutions of Lane-Emden problems, *Math. Ann.*, **367** (2017), 185–227.

[16] P. Esposito, A. Pistoia and J. Wei, Concentrating solutions for the Hénon equation in $\mathbb{R}^2$, *J. Anal. Math.*, **100** (2006), 249–280.

[17] P. Figueroa and S. L. N. Neves, Nonradial solutions for the Hénon equation close to the threshold, *Adv. Nonlinear Stud.*, **19** (2019), 757–770.

[18] F. Gladiali, A global bifurcation result for a semilinear elliptic equation, *J. Math. Anal. Appl.*, **369** (2010), 306–311.

[19] F. Gladiali and I. Ianni, Quasi-radial solutions for the Lane-Emden problem in the ball, *NoDea Nonlinear Differ. Equ. Appl.*, **27** (2020), Art. 13.

[20] I. Ianni and A. Saldana, Sharp asymptotic behavior of radial solutions of some planar semilinear elliptic problems, preprint, *arXiv:1908.10503*.

[21] J. Kubler and T. Weth, Spectral asymptotics of radial solutions and nonradial bifurcation for the Hénon equation, *Discrete Contin. Dyn. Syst.*, **40** (2019), Art. 3629.

[22] W. M. Ni and R. D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$, *Commun. Pure Appl. Math.*, **38** (1985), Art. 67.

[23] A. Pistoia and E. Serra, Multi-peak solutions for the Hénon equation with slightly subcritical growth, *Math. Z.*, **256** (2007), Art. 75.

[24] M. Willem, D. Smets and J. Su, Non-radial ground states for the Hénon equation, *Commun. Contemp. Math.*, **4** (2002), Art. 467.

[25] Y. B. Zhang and H. T. Yang, Multi-peak nodal solutions for a two-dimensional elliptic problem with large exponent in weighted nonlinearity, *Acta Math. Appl. Sin.*, **31** (2015), 261–276.

Received September 2019; revised May 2020.

E-mail address: annalisa.amadori@uniparthenope.it