OPTIMAL CONTROL FOR STOCHASTIC NONLINEAR SCHRODINGER EQUATION ON GRAPH*

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Abstract. We study the optimal control formulation for stochastic nonlinear Schrödinger equation (SNLSE) on a finite graph. By viewing the SNLSE as a stochastic Wasserstein Hamiltonian flow on density manifold, we show the global existence of a unique strong solution for SNLSE with a linear drift control or a linear diffusion control on graph. Furthermore, we provide the gradient formula, the existence of the optimal control and a description on the optimal condition via the forward and backward stochastic differential equations.

Key words. optimal control, density manifold, stochastic nonlinear Schrödinger equation on graph, Wasserstein Hamiltonian flow.

AMS subject classifications. 35R02, 30H05, 35Q55, 35Q93, 93E20

1. Introduction. The nonlinear Schrödinger equation (NLSE) given in the form

\[ \hbar i \frac{\partial}{\partial t} \Psi(t, x) = -\frac{\hbar^2}{2} \Delta \Psi(t, x) + \Psi(t, x)V(x) + \Psi(t, x)f(|\Psi(t, x)|^2) \]

has wide applications in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, and Bose-Einstein condensations (see, e.g., [34, 35, 10]). The unknown \( \Psi(t, x) \) represents a complex wave function for \( x \in \mathbb{R}^d, \hbar > 0 \) is the Planck constant, \( V(\cdot) \) and \( f(\cdot) \) are real-valued functions, referred as linear and nonlinear interaction potentials respectively. Considering the randomness in the propagation of nonlinear dispersive waves, the stochastic nonlinear Schrödinger equation (SNLSE)

\[ \hbar i d\Psi(t, x) = -\frac{\hbar^2}{2} \Delta \Psi(t, x)dt + \Psi(t, x)V(x)dt + \Psi(t, x)f(|\Psi(t, x)|^2)dt \]

\[ -iu(t, x)\mu(x)dt + u(t, x)dW(t, x), \]

has been introduced and studied in recent years (see, e.g., [21, 8, 7, 16, 19]). Here \( W \) is a colored Wiener process (see, e.g. [20]) defined by

\[ W(t, x) = \sum_{j=1}^{N} \mu_j e_j(x)\beta_j(t), \quad t \geq 0, \quad x \in \mathbb{R}^d, \]

and

\[ \mu(x) = \frac{1}{2} \sum_{j=1}^{N} |\mu_j|^2 |e_j(x)|^2, \quad x \in \mathbb{R}^d \]

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with $N \in \mathbb{N} \cup \infty$, $\mu_j \in \mathbb{C}$, $\epsilon_j$ real-valued function and $\beta_j$ independent Brownian motion on a complete filtrated probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Another physical significance of SNLSE is related to the theory of measurements continuous in time in quantum mechanics and open quantum system (see, e.g., [3, 4]).

In this paper, we focus on two types of SNLSEs on a finite graph $G = (V, E, w)$ and their related stochastic control problems. Here $V$ is the vertex set, $E$ is the edge set and $w_{jl}$ is the weight of the edge $(j, l) \in E$ satisfying $\omega_{jl} = \omega_{lj} > 0$ if there is an edge between nodes $j$ and $l$, and $\omega_{jl} = 0$ otherwise. Throughout this paper, we assume that $G$ is an undirected, connected graph with no self loops or multiple edges. The first type is the nonlinear Schrödinger equation with random perturbation,

\begin{equation}
\begin{aligned}
\mathbf{i} du_j &= \left(-\frac{1}{2}(\Delta_G u)_j + u_j V_j + u_j f_j(|u|^2)\right)dt + \sigma_j u_j \circ dW_t.
\end{aligned}
\end{equation}

Here $\Delta_G$ is a nonlinear discretization of Laplacian operator on $G$ introduced in [12] (see (13) for its formula), $f_j : \mathbb{R} \to \mathbb{R}$ is a continuous real-valued function, $V_j$ is a given linear potential on the node $j$, $\sigma_j \in \mathbb{R}$ represents the diffusion coefficient, and $\{W_t\}_{t \geq 0}$ is one dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The stochastic differential $\sigma dt W_j$ is understood in the Stratonovich sense. A typical example of the nonlinear function $f_j$ is that $f_j(|u|^2) = \sum_{k=1}^N \mathbb{W}_{jk} |u|^2$ with an interactive potential $\mathbb{W}_{jl} = \mathbb{W}_{lj}$ for any $(j, l) \in E$. We would like to remark that Eq. (2) can be viewed as a spatial discretization of Eq. (1) when $G$ is a lattice obtained by discretizing a continuous domain (see e.g. [15]). Another type is the nonlinear Schrödinger equation with white noise dispersion

\begin{equation}
\begin{aligned}
\mathbf{i} du_j &= \left(-\frac{1}{2}(\Delta_G u)_j \circ dW_t + (u_j V_j + u_j f_j(|u|^2))\right)dt.
\end{aligned}
\end{equation}

When $G$ is a lattice, (3) becomes a spatial discretization of NLSE with white noise dispersion [22], which describes the propagation of a signal in an optical fiber with dispersion management.

Our current investigation is motivated by several reasons. Firstly, the Schrödinger equation on graph and its control problem have their own interest and applications [11, 6, 32, 23]. Secondly, in contrast to the extensive literature on the optimal control problem and exact controllability of Schrödinger equations on continuous domain in both the deterministic and stochastic cases (see, e.g., [24, 26, 29, 27, 28, 2]), far fewer results are known when the problem settings are on graphs. One of the main difficulties lies on the weak regularization effect of free Schrödinger group and the difficulties on the weak regularization effect of free Schrödinger group and the difficulties on the weak regularization effect of free Schrödinger group.

Another one arises from the compact embedding theorem in probability space. Last but not least, both NLSE and SNLSE on a lattice graph can be viewed as a semi-discretization of NLSE and SNLSE on a continuous domain respectively [15], hence can be used as numerical schemes to compute (stochastic) optimal control problems involving SNLSEs in practice. However, many challenging questions remain open, such as the preservation of mass, energy, and symplectic structures, and the convergence analysis of semi-discretization of SNLSEs (see, e.g., [14] for more discussions).

Inspired by the optimal control of quantum mechanical system [33, 36], we shall study an optimal control problem associated with (2) or (3). Formally, we can view their solution $u = u(j, t, \omega), t \geq 0$, as the quantum state or the nonlinear wave at time $t$. The stochastic perturbation may represents an inaccurate measurement via the quantum observation or a dispersion management in optical fiber. The optimal control problem considered here is to find an input potential $V$ (or a diffusion...
coefficient $\sigma$) such that the state $u(T)$ is as close as possible to a target state $f^1(T)$ and a trajectory $Z^1$, and achieves the minimum cost (see sections 4 and 5 for more details). A different viewpoint for this problem is to recover the quantum mechanical potential $V$ or a diffusion coefficient $\sigma$ from the observation of the quantum state or the nonlinear wave $u(T)$ at the end of $[0, T]$. Despite many fruitful results on the continuous optimal control problems for NLSE and SNLSE [9, 5, 24, 26, 29, 27, 28, 2], a few exist for the problem defined on a graph. To the best of our knowledge, no result has been reported for stochastic control systems with (2) or (3).

In this work we study both linear drift and diffusion control. Our approach is based on two key ideas. One is used by Nelson in his derivation for NLSE [31]. The other is viewing SNLSE as a stochastic Wasserstein Hamiltonian flow [17]. By using the complex expression $u = \sqrt{\rho} e^{iS}$, we obtain the equivalent Madelung systems of SNLSE on graph (see, e.g., [12, 19]). Then by exploiting the properties of Madelung systems, we obtain the existence and uniqueness of the strong solution of (2) or (3) when the control $V$ or $\sigma$ is admissible. When the graph is taken as a lattice, we prove that the SNLSE on graph with the nonlinear Laplacian operator preserves the stochastic dispersion relationship, while any linear discretization does not. Furthermore, for a quadratic (or convex) cost functional, we provide the gradient formula and prove the existence of the optimal control by carefully studying the probability of a tail event of (2) or (3). When $\sigma$ is a constant potential on every node, we derive the adjoint equation of (2) or (3) which gives a forward-backward stochastic differential equation and characterizes the necessary optimal condition for the optimal control problem on graph.

Our paper is organized as follows. In section 2, we explain why we consider the nonlinear Laplacian for the stochastic Schrödinger equation on graph. In section 3, we present some useful properties of the stochastic Schrödinger equation on graph. In section 4, we prove the existence and uniqueness result for (2) or (3) with admissible control variables and prove the existence result of the optimal control. In section 5, we derive the gradient formula and present the necessary optimal condition by deriving a forward-backward stochastic differential equation.

2. Why nonlinear Laplacian for stochastic Schrödinger equation on graph? To explain the reason, we consider the stochastic linear Schrödinger equation

$$i\, du = -\frac{1}{2}\Delta u dt + \sigma u \circ dW_t.$$  \hfill (4)

and the white noise dispersion linear Schrödinger equation

$$i\, du = -\frac{1}{2}\Delta u \circ dW_t. \hfill (5)$$

One can directly verify that these equations possess the stochastic dispersion relationship by Itô’s formula.

**Lemma 2.1.** Let $\sigma \in \mathbb{R}$. Equation (4) (or (5)) admits infinitely many plane wave solutions given in the form of $u(x, t) = Ae^{i(k \cdot x - \mu W(t))}$ (or $Ae^{i(K \cdot x - \mu W(t))}$) with arbitrary $A \in \mathbb{R}^+$, any wave number $K \in \mathbb{R}^d$ and frequency $\mu$ satisfying $\mu = \frac{1}{2} |K|^2$.

From the above result, we see that the stochastic dispersion relationship $\mu = \frac{1}{2} |K|^2$ coincides with the classical dispersion relationship, and the argument of the plane wave contains all the information of the Wiener process. However, such a simple property may become problematic in discrete settings. To illustrate where the trouble is, let
us consider a lattice $G$ obtained by discretizing $\mathbb{R}^d$ or $\mathbb{T}^d$. Any linear discretizations of (4) and (5) can be stated

\begin{equation}
\dot{u}_j = \frac{1}{2} \sum_{l \in N(j)} C_{lj} u_l dt + \sigma u_j \circ dW_t
\end{equation}

and

\begin{equation}
\dot{u}_j = -\frac{1}{2} \sum_{l \in N(j)} C_{lj} u_l \circ dW_t,
\end{equation}

respectively. Here $\{C_{lj}\}_{(l,j) \in E}$ are chosen to approximate the Laplacian operator in (4) and (5). For simplicity, we assume that every node has the same number of adjacent nodes, and that the weight on each edge is uniformly given by $\Delta x$. We denote the coordinate of the node $j$ by $x_j = j \Delta x$. Regardless of how $\{C_{lj}\}_{(l,j) \in E}$ are selected, there are at most a finite discrete stochastic plane waves which satisfy the stochastic dispersion relationship.

**Theorem 2.1.** For any linear discretization of (4) and (5), there exist at most a finite number of pairs $(\mu, \mathbb{K})$ with $\mu = \frac{1}{2} |\mathbb{K}|^2$ so that the discrete stochastic plane waves, i.e., $u_j = A e^{i(\mathbb{K} \cdot x_j - \mu t - \sigma W(t))}$ for (6) (or $A e^{i(\mathbb{K} \cdot x_j - \mu W(t))}$ for (7)), are the solutions.

**Proof.** Consider the discrete stochastic plane waves $u_j(t) = A e^{i(\mathbb{K} \cdot x_j - \mu t - \sigma W(t))}$ for (4) and $u_j(t) = A e^{i(\mathbb{K} \cdot x_j - \mu W(t))}$ for (5). Substituting them into (6) and (7), we get

\[\mu A e^{i(\mathbb{K} \cdot x_j - \mu t - \sigma W(t))} dt = \frac{1}{2} \sum_{l \in N(j)} C_{lj} A e^{i(\mathbb{K} \cdot x_j - \mu t - \sigma W(t))} dt,\]

and

\[\mu A e^{i(\mathbb{K} \cdot x_j - \mu W(t))} \circ dW(t) = \frac{1}{2} \sum_{l \in N(j)} C_{lj} A e^{i(\mathbb{K} \cdot x_j - \mu W(t))} \circ dW(t),\]

respectively. If $\mu = \frac{1}{2} |\mathbb{K}|^2$, we obtain

\[\mu = \frac{|\mathbb{K}|^2}{2} = \frac{1}{2} \sum_{l \in N(j)} C_{lj} e^{i(\mathbb{K} \cdot (x_l - x_j))}.\]

Since $\frac{|\mathbb{K}|^2}{2}$ is quadratic in $\mathbb{K}$ while the trigonometric polynomial on the right hand side is periodic and bounded in $K$, they intersect only in a bounded ball of the complex domain $|\mathbb{K}| \leq C_N$. Besides, it can be seen that the imaginary part of $\frac{1}{2} \sum_{l \in N(j)} C_{lj} \sin(\mathbb{K} \cdot (x_l - x_j)) = 0$ has at most finite zero point. Thus, we complete the proof.

To numerically preserve the stochastic dispersion relationship for any pair of $(\mu, \mathbb{K})$ with $\mu = \frac{1}{2} |\mathbb{K}|^2$, we decide to use the nonlinear Laplacian operator $\Delta_G$ constructed by using the Madelung transformation as shown in [12, 14].

**3. Stochastic nonlinear Schrödinger equation on graph.** Consider a graph $G = (V, E, \omega)$, let us denote the set of discrete probabilities on the graph by

\[\mathcal{P}(G) = \{(\rho_j)_{j=1}^N : \sum_{j=1}^N \rho_j = 1, \rho_j \geq 0, \text{ for } j \in V\},\]
and $\mathcal{P}_o(G)$ as its interior (i.e., all $\rho_j > 0$, for $j \in V$). $V_j$ is a linear potential on each node $j$, and $W_{ij} = W_{ji}$ is an interactive potential between nodes $j$ and $l$. We denote $N(i) = \{j \in V : (i, j) \in E\}$ the adjacency set of the node $a_i$ and $\theta_{ij}(\rho)$ a density dependent weight on the edge $(i, j) \in E$. More precisely, $\theta$ is defined by $\theta_{ij}(\rho) = \Theta(\rho_i, \rho_j)$, where $\Theta$ is a continuous differentiable function on $(0, 1)^2$ satisfying $\Theta(x, y) = \Theta(y, x)$, $\Theta(x, y) \geq 0$, and $\min(x, y) \leq \Theta(x, y) \leq \max(x, y)$ for any $x, y \in (0, 1)$. For example, we may take $\theta(\rho)$ as the averaged probability weight in [12], i.e., $\Theta(x, y) = \frac{1}{2}(x + y)$, or the logarithmic probability weight in [14], where $\Theta(x, y) = \frac{1}{x + 1} - \frac{1}{y + 1}$, or the harmonic probability weight in [30], i.e., $\Theta(x, y) = \frac{2}{x + 1}$.

In this section, we present the stochastic nonlinear Schrödinger equations on graph via the viewpoint of stochastic variational principle proposed in [17]. Define the total linear potential function $V$, interaction potential function $W$, and the entropy function $L$ by

$$V(\rho) = \sum_{i=1}^{N} V_i \rho_i, \quad W(\rho) = \frac{1}{2} \sum_{i,j=1}^{N} W_{ij} \rho_i \rho_j, \quad L(\rho) = \sum_{i=1}^{N} (\log(\rho_i) - \rho_i).$$

$I(\rho)$ is the discrete Fisher information on graph, i.e.,

$$I(\rho) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in N(i)} \overline{\omega}_{ij} \log(\rho_i) - \log(\rho_j)^2 \overline{\theta}_{ij}(\rho),$$

where $(\overline{\omega}, \overline{\theta})$ is another pair of weight and density dependent weight on the edges $G$. We remark that $(\overline{\omega}, \overline{\theta})$ may be selected the same as or differently from $(\omega, \theta)$. Throughout this paper, we take $\theta$ as the averaged probability weight, $\theta$ as the logarithmic probability weight, and $\omega_{ij} = \overline{\omega}_{ij}$ for simplicity.

As given in [18], the stochastic variational principle on graph is defined as

$$I(\rho^0, \rho^T) = \inf \{ S(\rho_t, \Phi_t) | \{ -\Delta_p \}^t P_t \in \mathcal{T}_p, P_o(M), \rho(0) = \rho^0, \rho(T) = \rho^T \},$$

whose action functional is expressed in the dual coordinates,

$$S(\rho_t, \Phi_t) = -\langle \rho(0), \Phi(0) \rangle + \langle \rho(T), \Phi(T) \rangle - \int_0^T \langle \partial_t \Phi(t), \rho_t \rangle - \mathcal{H}_0(\rho_t, \Phi_t) dt - \int_0^T \mathcal{H}_1(\rho_t, \Phi_t) \circ dW_t.$$

Here $(-\Delta_p)^t$ is the pseudo inverse of $\text{div}^\rho_G(\rho \nabla_G(\cdot))$ defined by

$$\left( \text{div}^\rho_G(\rho \nabla_G(\cdot)) \right)_i := \sum_{j \in N(i)} \theta_{ij}(\rho) \omega_{ij}(S_j - S_i)$$

for any potential function $S = \{S_i\}_{i \in V}$. The vector field $\nabla_G S$ induced by $S$ is defined by $\nabla_G S := \left( \sqrt{\omega_{ij}(S_i - S_j)} \right)_{ij \in E}$. With the above notation, one can also introduce the inner product for the vector fields on graph defined by

$$\langle u, v \rangle_{\theta(\rho)} := \frac{1}{2} \sum_{ij \in E} u_{ij} v_{ij} \theta_{ij}(\rho) \omega_{ij},$$

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for any two vector fields (skew-symmetric matrices) \( u, v \). The kinetic energy is defined by \( K(S, \rho) = \frac{1}{2} \langle \nabla_G S, \nabla_G S \rangle_{\theta(\rho)} \). Here ~\( \rho^0, \rho^T \) are \( \mathcal{F}_0 \) and \( \mathcal{F}_T \) measurable functions, the dominated energy \( \mathcal{H}_0 \) and perturbed energy \( \mathcal{H}_1 \) are given by

\[
\mathcal{H}_0(\rho, S) = K(S, \rho) + F(\rho) - \kappa L(\rho),
\]

\[
\mathcal{H}_1(\rho, S) = \eta_1 K(S, \rho) + \eta_2 I(\rho) + \eta_3 \Sigma(\rho) + \eta_4 \mathcal{W}(\rho) + \eta_5 L(\rho)
\]

with ~\( \kappa \in \mathbb{R} \), \( \Sigma \) defined by \( \Sigma(\rho) = \sum_{j=1}^N \sigma_j \rho_j \) for some \( \sigma_j \in \mathbb{R} \), and \( F(\rho) := \frac{1}{2} I(\rho) + \mathcal{V}(\rho) + \mathcal{W}(\rho) \). In particular, when \( \eta_1 = 0 \), (9) recovers the classical variational problem with random potential in Lagrangian formalism.

By finding the critical point of the stochastic variational principle (9), we achieve the following discrete stochastic Wasserstein Hamiltonian flow on the density manifold,

\[
d\rho = \frac{\partial}{\partial S} \mathcal{H}_0(\rho, S) + \frac{\partial}{\partial \rho} \mathcal{H}_1(\rho, S) \circ dW_t,
\]

\[
dS = - \frac{\partial}{\partial \rho} \mathcal{H}_0(\rho, S) - \frac{\partial}{\partial \rho} \mathcal{H}_1(\rho, S) \circ dW_t,
\]

Selecting different deterministic energy \( \mathcal{H}_0 \) and perturbed energy \( \mathcal{H}_1 \) results in various forms of stochastic nonlinear Schrödinger equations on graph. When \( \mathcal{H}_0(\rho, S) = K(S, \rho) + \mathcal{F}(\rho) - \kappa L(\rho), \mathcal{H}_1(\rho, S) = \Sigma(\rho) \), the Wasserstein Hamiltonian flow becomes

\[
d\rho_i + \sum_{j \in N(i)} \omega_{ij} (S_j - S_i) \theta_{ij}(\rho) = 0,
\]

\[
dS_i + \frac{1}{2} \sum_{j \in N(i)} \omega_{ij} (S_j - S_i)^2 \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i} dt + \frac{1}{8} \frac{\partial I(\rho)}{\partial \rho_i} dt + \mathcal{V}_i dt
\]

\[
+ \sum_{j=1}^N \mathcal{W}_{ij} \rho_j dt - \kappa \log(\rho_i) dt + \sigma_i dW_t = 0.
\]

Its complex formulation \( u(t) = \sqrt{\rho(t)} e^{iS(t)} \) gives the stochastic nonlinear Schrödinger on graph,

\[
ikdu = \left( -\frac{1}{2} \Delta_G u \right)_j + u_j \mathcal{V}_j + u_j \sum_{l=1}^N \mathcal{W}_{jl} |u_l|^2 - u_j \kappa \log(|u_j|^2) dt + \sigma_j u_j \circ dW_t.
\]

Here the nonlinear Laplacian on the graph is defined by

\[
(\Delta_G u)_j = -u_j \left( \frac{1}{|u_j|^2} \sum_{l \in N(j)} \omega_{jl} (\Im(\log(u_l))) - \Im(\log(u_j)) \theta_{jl} \right)
\]

\[
+ \sum_{l \in N(j)} \overline{\omega}_{lj} \bar{\theta}_{jl} (\Re(\log(u_l))) - \Re(\log(u_j)) \]

\[
+ \sum_{l \in N(j)} \omega_{jl} \frac{\partial \theta_{jl}}{\partial \rho_j} (\Im(\log(u_l))) - \Im(\log(u_j))^2
\]

\[
+ \sum_{l \in N(j)} \overline{\omega}_{lj} \frac{\partial \bar{\theta}_{jl}}{\partial \rho_j} (\Re(\log(u_l))) - \Re(\log(u_j))^2,\]

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where \( \Re \) and \( \Im \) are real and imaginary parts of a complex number. This is precisely the nonlinear graph Laplacian introduced in [14].

When \( \mathcal{H}_0 = \mathcal{V}(\rho) + \mathcal{W}(\rho), \mathcal{H}_1 = K(\rho, S) + \frac{1}{8} I(\rho) \), the Wasserstein Hamiltonian flow becomes

\[
\begin{align*}
\sum_{j \in N(i)} \omega_{ij}(S_i - S_j)\theta_{ij}(\rho) & \circ dW_t; \\
\frac{1}{2} \sum_{j \in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}}{\partial \rho_i} + \frac{1}{8} \frac{\partial}{\partial \rho_i} I(\rho) & \circ dW_t + (V_i + \sum_{j=1}^{N} W_{ij} \rho_j) dt = 0,
\end{align*}
\]

whose complex formulation \( u(t) = \sqrt{\rho(t)e^{iS(t)}} \) satisfies the nonlinear Schrödinger equations with white noise dispersion on graph,

\[
\text{id}u_j = -\frac{1}{2}(\Delta_G u)_j \circ dW_t + (u_j V_j + u_j \sum_{l=1}^{N} W_{jl} |u_l|^2) dt.
\]

Both (12) and (15) can be viewed as spatial discretization of (2) and (3) respectively when \( G \) is a lattice graph.

Recall that in [12, 14], the global solution in deterministic case \( (\eta_1 = \cdots = \eta_4 = \eta_5 = 0, \kappa = 0) \) is obtained by using the energy conservation law if \( F(\rho) \) contains the Fisher information \( \beta I(\rho), \beta > 0 \). In the stochastic case, the existence of global solution has been studied in [18] by using the Poisson bracket \{\cdots\}. In particular, when \( \{\mathcal{H}_0, \mathcal{H}_1\} = 0 \), for example \( \mathcal{H}_0 \) is a multiple of \( \mathcal{H}_1 \), then \( \mathcal{H}_0 \) is an invariant of the stochastic Wasserstein Hamiltonian flow. Here we summarize some fundamental properties shared by the stochastic nonlinear Schrödinger equations on graph.

**Proposition 3.1.** Let \( T > 0, u(0) \) be \( \mathcal{F}_0 \)-measurable with any finite moment and \( u_j(0) \neq 0 \) for all \( j \in V \). Then (12) (or (15)) has a unique strong solution \( u(t) \) on \([0, T]\). Moreover, \( u(t) \) satisfies the following properties

(i) It conserves the total mass

\[
\sum_{j=1}^{N} |u_j(t)|^2 = 1, \text{ a.s. ;}
\]

(ii) The total energy satisfies

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \mathcal{E}^p(u(t)) \right] \leq C(\mathcal{E}(u(0)), T, p),
\]

where \( \mathcal{E} \) is defined by a combination of the discrete kinetic energy \( \mathcal{E}_{\text{kin}} \), linear potential \( \mathcal{E}_{\text{lin}} \), interaction potential \( \mathcal{E}_{\text{int}} \) and entropy \( \mathcal{E}_{\text{ent}} \), i.e.

\[
\mathcal{E}(u) = \mathcal{E}_{\text{kin}}(u) + \mathcal{E}_{\text{lin}}(u) + \mathcal{E}_{\text{int}}(u) + \mathcal{E}_{\text{ent}}(u).
\]

Here we have

\[
\mathcal{E}_{\text{kin}}(u) = \frac{1}{4} \sum_{(j,l) \in E} \{ |\Re(\log u_j - \log(u_l))|^2 \omega_{jl} \theta_{jl}(|u|^2) \\
+ |\Im(\log u_j - \log(u_l))|^2 \bar{\omega}_{jl} \theta_{jl}(|u|^2) \},
\]
\[ E_{\text{lin}}(u) = \sum_{j=1}^{N} V_j |u_j|^2, \quad E_{\text{int}}(u) = \frac{1}{2} \sum_{j,l=1}^{N} W_{jl} |u_j|^2 |u_l|^2, \]
\[ E_{\text{ent}}(u) = -\kappa \sum_{j=1}^{N} (\log(|u_j|^2)|u_j|^2 - |u_j|^2). \]

(iii) It is time transverse invariant when \( \mathbb{V} \) is independent of time: if \( u^\alpha(t) \) is the solution of (12) (or (15)), where \( \mathbb{V}^\alpha = (V_j + \alpha)_{j=1}^{N} \) with \( \alpha \) being a constant \( \mathcal{F}_0 \)-measurable random variable, then
\[ u^\alpha(t) = u(t)e^{i\alpha t}, \]
is also a solution.

(iv) It is time reversible when \( \mathbb{V} \) is independent of time in the following sense: for (12) (or (15)) with \( \tilde{W}(t) = W(t), t \geq 0 \) and \( \tilde{W}(t) = -W(-t), t < 0 \), then
\[ u(t) = \tilde{u}(-t). \]

**Proof.** We can show the existence and uniqueness of \( u \) by its complex representation for (15) and (12). Thanks to the complex formulation, we know that there always exists \( (\rho(0), S(0)) \) such that \( u(0) = \sqrt{\rho(0)}e^{iS(0)} \) with \( |u_i(0)|^2 = \rho_i(0) \) such that \( \rho_i > 0 \) for some \( i \in \mathbb{V} \). The potential \( S(0) \) in representation \( (\rho(0), S(0)) \) is unique up to a shift with \( 2\pi \). Let us fix and choose a potential \( S(0) \). Thus to prove the global existence of a unique solution \( u \), it suffices to prove that the equivalent systems (11) (or (14)) have a unique global solution. To this end, we can use the arguments in [18, Section 4] and obtain the global existence of the solution. The steps to check properties (i)-(iv) are similar to those to prove [Proposition 2.1][17].

Following the proof of [18, Theorem 4.1], one can also obtain the lower bounds for the density trajectories as stated in the next corollary.

**Corollary 3.1.** Let the conditions of Proposition 3.1 hold. For Eq. (11), there exists a positive random variable which is a lower bound of the density trajectory. For Eq. (14), there exists a positive constant which is a lower bound of the density trajectory.

To end this section, we demonstrate that the nonlinear discretization of (4) and (5) can preserve exactly the stochastic dispersion relationship. Consider the graph version of (4),
\[ i du_j = -\frac{1}{2} (\Delta_G u)_j dt + \sigma u_j \circ dW_t. \]
and that of (5),
\[ i du_j = -\frac{1}{2} (\Delta_G u)_j \circ dW_t. \]

**Proposition 3.2.** Given a lattice graph \( G \) with \( |x_j - x_l| = \Delta x \) for \( l \in N(j) \), \( \omega_{ij} = (\frac{\partial \theta_{ij}}{\partial \rho_j} N \Delta x^2)^{-1} \) where \( N \) is total number of nodes in \( N(j) \) and \( \theta_{ij} \) is the symmetric probability weight. The nonlinear discretizations of (16) and (17) preserve the stochastic dispersion relationship.
Proof. The discrete stochastic plane waves read \( u_j(t) = Ate^{i(Kx_j - \mu t - \sigma W(t))} \) for (4) and \( u_j(t) = Ate^{i(Kx_j - \mu t)} \) for (5) with \( \mu = \frac{1}{2}|K|^2 \). By the Madelung transformation \( u_j = \sqrt{\rho_j}e^{iS_j(t)} \), \( \rho_j = A \) is constant. As a consequence, the partial derivative of Fisher information \( \frac{\partial \mu_j(\rho)}{\partial \rho_i} = 0 \). On the other hand, since \( S_i = K \cdot x_i - \mu t - \sigma W(t) \), one can verify that \( \frac{1}{2} \sum_{j \in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i} = \frac{1}{2}|K|^2 = \mu \). This implies that

\[
\begin{align*}
&dS_i + \frac{1}{2} \sum_{j \in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i} dt + \frac{1}{8} \frac{\partial I(\rho)}{\partial \rho_i} dt + \sigma dW_i = 0
\end{align*}
\]

is satisfied. Thus (4) preserves all the stochastic dispersion relationship.

Similar calculations can show that (5) satisfy

\[
\begin{align*}
&dS_i + \frac{1}{2} \sum_{j \in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i} dt = 0, \quad \square
\end{align*}
\]

which implies that (5) preserves all the stochastic dispersion relationship.

4. Stochastic control problem on density manifold of finite graph. In this section, we propose two stochastic optimal control formulations corresponding to SNLSEs (2) and (3) on graph respectively.

4.1. Stochastic control problem with linear potential control. We first assume that the linear potential term \( \{V_j\}_{j \in N} \) is a control variable depending on \( t \). From the the proof of (3.1), this will not affect the well-posedness of (2) and (3). For convenience, we denote the corresponding solution by \( u_j^\nu \) in the complex function representation and \( (\rho_j^\nu, S_j^\nu) \) on Wasserstein manifold. The admissible control set \( \mathcal{U} \) is defined by

\[
\mathcal{U} := \left\{ \mathcal{V} : \Omega \times [0, T] \rightarrow \mathbb{R}^N \mid \mathcal{V}(t) \text{ is } \mathcal{F}_t\text{-adapted, } \forall j \in V \right\}
\]

with \( \gamma, \beta \geq 0 \). Our first optimal control problem is to minimize the cost functional

\[
J(\mathcal{V}) := \gamma \mathbb{E} \left[ \sum_{i=1}^{N} |u_i^\nu(T) - f_i^1|^2 \right] + \beta \mathbb{E} \left[ \int_{0}^{T} \sum_{i=1}^{N} |\mathcal{V}(t) - Z_j(t)|^2 dt \right],
\]

subject to the constraint given by either (11) or (14) with given \( (\rho(0), S(0)) \). Here \( f^1 \) is \( \mathcal{F}_T\text{-adapted satisfying } \|f^1\|_{L^2([0,T];\mathbb{R}^N)} < \infty \), and \( Z \in \mathcal{U} \). The above optimal control problem may be viewed as the graph version of the stochastic control problem in [1, 26, 27, 28].

The following lemma (see, e.g., [25, Chapter 3]) is very useful to show the existence and uniqueness of the optimal control.

**Lemma 4.1.** Let \( \mathcal{B} \) be a uniformly convex Banach space and \( \mathcal{S} \) a bounded closed subset of \( \mathcal{B} \). Furthermore, let \( F : \mathcal{S} \rightarrow \mathbb{R} \) be a lower semi-continuous functional which is bounded from below and \( p \geq 1 \). Then there exists a dense subset \( D \subset \mathcal{B} \) such that for each \( x \in D \), the functional \( F(s) + \|s - x\|_{\mathcal{B}}^p \) attains its minimum over \( \mathcal{S} \), which implies that there exists an \( s(x) \in \mathcal{S} \) such that

\[
F(s(x)) + \|s(x) - x\|_{\mathcal{B}}^p = \inf_{s \in \mathcal{S}} \{F(s) + \|s - x\|_{\mathcal{B}}^p\}.
\]
In particular, if \( p > 1 \), then \( s(x) \) is unique. Besides, each minimizing sequence converges strongly and the function \( x \mapsto s(x) \) is continuous in \( D \).

In our case, we take \( B := L^2(\Omega \times [0, T]; \mathbb{C}^N) \) which is uniformly convex, and choose \( S \) as the admission control set. The functional \( F = \gamma \mathbb{E} \left[ \sum_{i=1}^{N} |u^V_j(T) - f_j|^2 \right] \) is bounded from below and \( p = 2 \). According to Lemma 4.1, if we can verify the lower semi-continuity of \( F \), then there exists a dense subset \( D \) of \( B \) such that for each \( Z \in D \) the functional \( J(\mathbb{V}) = F(\mathbb{V}) + \beta \|\mathbb{V} - Z\|^2_B \) attains its unique minimum over \( U \). In other word, there exists a unique \( \mathbb{V}^* \in \mathcal{U} \) such that

\[
J(\mathbb{V}^*) = F(\mathbb{V}^*) + \beta \|\mathbb{V}^* - Z\|^2_B = \inf_{\mathbb{V} \in \mathcal{U}} J(\mathbb{V}).
\]

To prove the lower semi-continuity of \( u^V \) with respect to \( \mathbb{V} \), we show a strong convergence result first.

**Proposition 4.1.** Let \( u(0) \) be \( \mathcal{F}_0 \)-adapted with any finite moment satisfying \( u_j(0) \neq 0 \), \( j \leq N \). Let the sequence \( \{\mathbb{V}^n\}_{n \geq 1} \subset \mathcal{U} \) be convergent to \( \mathbb{V} \) and \( u^{V^n} \) be the corresponding solution of the stochastic nonlinear Schrödinger equation (11) (or (14)) with respect to the control \( \mathbb{V}^n \) and the initial value \( u^{V^n}(0) = u(0) \). Then the sequence \( \{u^{V^n}\} \in L^2(\Omega; \mathcal{C}([0, T]; \mathbb{C}^N)) \), \( n \geq 1 \), converges strongly to the solution of stochastic nonlinear Schrödinger equation (11) (or (14)) with respect to the control \( \mathbb{V} \in \mathcal{U} \).

**Proof.** In this proof, we only show the details when the constraint is (11). A similar argument can lead to the strong convergence result for the case of (14). By Proposition 3.1, the Itô formula, and the Burkholder’s inequality, we have the following a priori estimates,

\[
\sum_{i=1}^{N} |u^{V^n}_i(t)|^2 = \sum_{i=1}^{N} |u_i(0)|^2 = 1, \text{ a.s.}
\]

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left( \langle \nabla G_s^{V^n}(t), \nabla G_s^{V^n}(t) \rangle + \frac{1}{8} I(\nu^{V^n}(t)) \right)^p \right] \leq C(u(0), T, \alpha, p), \quad p \geq 1.
\]

To show the strong convergence of \( u^{V^n} \), we introduce a stopping time \( \tau_c \) defined by

\[
\tau_c^n := \inf\{t \in [0, T] : \|S^{V^n}\|_{C([0, t]; \mathbb{R}^N)} \geq c \} \wedge \inf\{t \in [0, T] : \min_{i=1}^{N} \min_{s \in [0, t]} \rho_i^{V^n}(s) \leq \frac{1}{c} \}.
\]

By Corollary 3.1, we have that \( \lim_{c \rightarrow \infty} \tau_c = T, \text{ a.s.} \). Introduce the truncated sample subspace \( \Omega^c_n \) defined by

\[
\Omega^c_n = \left\{ \sup_{t \in [0, T]} \|S^{V^n}\|_{C([0, t]; \mathbb{R}^N)} \leq c, \min_{i=1}^{N} \min_{s \in [0, T]} \rho_i^{V^n}(s) \geq \frac{1}{c} \right\}.
\]

Similarly, we denote \( \Omega_c \) as the truncated sample subspace with respect to \( u^V \). Our goal is to show the error estimate in \( \Omega^c_n \cap \Omega_c \) and \( \Omega/\{\Omega^c_n \cap \Omega_c \} \). First, we prove the convergence in \( \Omega/\{\Omega^c_n \cap \Omega_c \} \). Due to the mass conservation law (19) of the stochastic nonlinear Schrödinger equation, by applying the Chebyshev’s inequality, we get

\[
\|1_{\Omega^c_n \cap \Omega_c}(u^{V^n} - u^V)\|^2_B \]
When it suffices to prove all the above probabilities converges to 0 as $c \to \infty$. Indeed, since $G$ is connected, by applying the lower bound estimate in Section 3, there exists a positive random variable $C(\omega)$ such that

$$\inf_{t \geq 0} \min_{i \leq N} \rho^V_i (t) \geq c_2 \exp(-c_1 C(\omega)),$$

Here $c_2, c_1 > 0$ are constants depending on the structure of $G$, and $C(\omega)$ is the positive random variable in Corollary 3.1. More precisely, the positive random variable $C(\omega)$ is bounded by the upper bound of $V^n$ and $V$ plus

$$\sup_{t \in [0, T]} \left( \langle \nabla_G S^V_n (t), \nabla_G S^V_n (t) \rangle_{\theta (\rho^V_i (t))} + \frac{1}{8} I(\rho^V_i (t)) \right),$$

which possess any finite moment by (20). Thus, by (21), Chebyshev’s inequality and the monotonicity of the logarithmic function, we get

$$\mathbb{P} \left( \min_{s \in [0, T]} \min_{i \leq N} \rho^V_i (s) \leq \frac{1}{c} \right) \leq \mathbb{P} \left( c_2 \exp(-c_1 C(\omega)) \leq \frac{1}{c} \right) = \mathbb{P} (C(\omega) \geq \frac{1}{c_1} (\log(c) + \log(c_2))) \leq \frac{c_1^p \mathbb{E} \left[ C(\omega)^p \right]}{(\log(c) - \log(c_2))^p}, \; p \geq 1.$$

When $c \to \infty$, by the dominated convergence theorem, we have that

$$\lim_{c \to \infty} \left[ \mathbb{P} \left( \min_{i \leq N} \min_{s \in [0, T]} \rho^V_i (s) \leq \frac{1}{c} \right) + \mathbb{P} \left( \min_{i \leq N} \min_{s \in [0, T]} \rho^V_i (s) \leq \frac{1}{c} \right) \right] = 0.$$

For the tail estimate of $S^V_n$, we make use of the differential equation of $S^V_n$ and get that

$$|S^V_n (t)| \leq |S^V_n (0)| + \int_0^T \sum_{j \in N(i)} \frac{1}{4} |S_i - S_j|^2 \omega_{ij} + \frac{\partial}{\partial \rho_i} I(\rho) ds$$

$$+ \int_0^T |V^n_i| + \sum_{j=1}^N |W_{ij}| \rho_j ds + \sup_{t \in [0, T]} \int_0^t \sigma_i dW(s).$$

The Burkholder’s inequality yields that $\mathbb{E} \left[ \sup_{t \in [0, T]} |\int_0^t \sigma_i dW(s)|^p \right] \leq C(p, \sigma)$. Notice that (20) and (21) implies that

$$\max_{ij \in E} |S_i - S_j|^2 \leq \frac{2}{\min_{ij \in E} \omega_{ij} (\rho_i + \rho_j)} C(\omega) \leq \frac{1}{\min_{ij \in E} \omega_{ij} c_2} \exp(c_1 C(\omega)) C(\omega), \; \text{a.s.}$$
max \left| \frac{\partial}{\partial \rho_i} f(\rho) \right| \leq \max \omega_{ij} \max i \left[ \frac{2}{\rho_i} + 2|\log(\rho_i)| \right] \\
\leq \max \omega_{ij} 2(\frac{1}{c_2} \exp(c_1 C(\omega)) + |\log(c_2)| + c_1 C(\omega)) < \infty, \text{ a.s.}

Combining with the fact that $|\mathcal{V}_i^n(t)| \leq \alpha$, we conclude that for $c$ large enough,

$$
\mathbb{P}( \sup_{s \in [0,T]} |\mathcal{V}_i^n| \geq c) \\
\leq \mathbb{P}(\min_{ij \in E} \frac{1}{\omega_{ij} c_2} \exp(c_1 C(\omega)) C(\omega) \geq \frac{c}{4T}) \\
+ \mathbb{P}(\max_{ij \in E} \omega_{ij} 2(\frac{1}{c_2} \exp(c_1 C(\omega)) + |\log(c_2)| + c_1 C(\omega)) \geq \frac{c}{4T}) \\
+ \mathbb{P}(\sup_{i \leq N \cap \{\mathcal{W}_{ij} + \alpha T \geq \frac{c}{4T}\}} |\mathcal{V}_i^n(0)| + \max_{ij \in E} |\mathcal{W}_{ij} + \alpha T | \geq \frac{c}{4T}).
$$

Using the moment estimate of $C(\omega)$ and Chebyshev’s inequality, we obtain that

$$
\lim_{c \to \infty} \mathbb{P}( \sup_{s \in [0,T]} |\mathcal{V}_i^n| \geq c) = 0.
$$

Similarly, we can get $\lim_{c \to \infty} \mathbb{P}(\sup_{s \in [0,T]} |\mathcal{V}| \geq c) = 0.$

On $\Omega^n_e \cap \Omega_n$, we use the stopping time technique to show the strong convergence.

By the definition of $\tau^n_e$ and $\tau_e$, we can see that $\tau^n_e = T$ on $\Omega^n_e$ and $\tau_e = T$ on $\Omega_e$.

According to the complex form of $u^n = \sqrt{\rho^n} e^{iS^n}$, we have that

$$
\int_0^T \mathbb{E}[1_{\Omega^n_e \cap \Omega_n} |u^n - u|]^2 ds \\
\leq \int_0^T \sum_{i=1}^N 2 \left( \mathbb{E}[1_{\Omega^n_e \cap \Omega_n} |\sqrt{\rho_i^n} - \sqrt{\rho_i^n}|^2] + \mathbb{E}[1_{\Omega^n_e \cap \Omega_n} |\sqrt{\rho_i^n e^{iS^n} - e^{iS^n}}|^2] \right) ds \\
\leq C \int_0^T \sum_{i=1}^N 2 \left( \mathbb{E}[1_{\Omega^n_e \cap \Omega_n} |\sqrt{\rho_i^n} - \sqrt{\rho_i^n}|^2] + \mathbb{E}[1_{\Omega^n_e \cap \Omega_n} |S_i^n - S_i^n|^2] \right) ds.
$$

By applying the Itô formula before $\tau^n_e \cap \tau_e$ and Hölder’s inequality, we obtain that

$$
|\sqrt{\rho^n}(t) - \sqrt{\rho^n}(t)|^2 \\
= \int_0^t 2 \sum_{i=1}^N \sum_{j \in N(i)} \left( \frac{1}{\sqrt{\rho_i^n}} (S_i^n - S_j^n) \theta_{ij}(\rho^n) - \frac{1}{\sqrt{\rho_i^n}} (S_i^n - S_j^n) \theta_{ij}(\rho^n) \right) (\sqrt{\rho_i^n} - \sqrt{\rho_i^n}) ds \\
\leq \int_0^t C(1 + c) \sum_{i=1}^N (|S_i^n - S_i^n| \sqrt{\rho_i^n} - \sqrt{\rho_i^n} | + |\sqrt{\rho_i^n} - \sqrt{\rho_i^n}|^2 ds
$$

and that

$$
|S^n(t) - S^n(t)|^2
$$
\[
\begin{align*}
&= \int_0^t \sum_{i=1}^N \sum_{j \in N(i)} (-\frac{1}{4}(S_i^{V^n} - S_j^{V^n})^2 + \frac{1}{4}(S_i^V - S_j^V)^2)(S_i^{V^n} - S_j^V)ds \\
&+ \int_0^t \sum_{i=1}^N (-\mathcal{V}_i^n + \mathcal{V}_i)(S_i^{V^n} - S_j^V)ds \\
&+ \int_0^t \sum_{i=1}^N \sum_{j=1}^N (-\mathcal{W}_{ij}^n + \mathcal{W}_{ij}^V)(S_i^{V^n} - S_j^V)ds \\
&\leq \int_0^t C(1+c)\left(||S_i^{V^n} - S_j^V||^2 + ||\sqrt{\rho^n} - \sqrt{\rho^V}||^2 + ||V^n - V^V||^2\right)ds.
\end{align*}
\]

The Gronwall’s inequality, together with the above estimates, leads to
\[
\mathbb{E}\left[\sqrt{\rho^n}(t) - \sqrt{\rho^V(t)}^2 + |S_i^{V^n}(t) - S_i^V(t)|^2\right] \leq \exp\int_0^t C(1+c)ds \int_0^t \mathbb{E}[|V^n - V^V|^2]ds.
\]

Taking \(n \to \infty\) and then \(c \to \infty\), we achieve that
\[
\lim_{c \to \infty} \lim_{n \to \infty} \int_0^T \mathbb{E}[1\Omega_n \cap \Omega_c]|u^{V^n} - u^V|^2ds \\
\leq \lim_{c \to \infty} \lim_{n \to \infty} \int_0^T \exp\int_0^t C(1+c)ds \int_0^t \mathbb{E}[|V^n - V^V|^2] ds dt = 0.
\]

Combining the estimate on \(\Omega_n \cap \Omega_c\) and \(\Omega/(\Omega_n \cap \Omega_c)\), we obtain the desired result. Similarly, one could also obtain the strong convergence of \(u^{V^n}\) in the topology \(L^2(\Omega, \mathcal{C}([0,T]; \mathbb{C}^N))\).

**Theorem 4.1.** Let \(\beta \geq 0\). For the control problem (18) with the constraint (11) or (14), there always exists an optimal control \(\mathcal{V}^* \in \mathcal{U}\) which minimizes the objective functional \(J\).

**Proof.** By Lemma 4.1, to get the unique existence of an optimal control, it suffices to show the lower continuity of \(F\) if \(\beta > 0\), which can be obtained by using Proposition 4.1 and the Fatou lemma.

In the following, we show the existence of an optimal control when \(\beta = 0\). Since \(\gamma \sum_{i=1}^N |u_i^V(T) - f_i^V|^2\) is bounded from below and \(|V_i| \leq \alpha\) in \(\mathcal{U}\), the infimum of \(F\) exists. Let \((u^{V^n}, \mathcal{V}^n)\) be a minimizing sequence. By the a priori estimate in Proposition 3.1, there exists a subsequence, still denoted by \((u^{V^n}, \mathcal{V}^n)\), such that \(\mathcal{V}^n \to \mathcal{V}^*\) weakly in \(L^2(\Omega \times [0,T]; \mathbb{R}^N)\). By Mazur’s theorem, we have a sequence of convex combinations denoted by \(\bar{V}^m: \sum_{n \geq 1} \alpha_{nm}u_{n+m}\) with \(\alpha_{nm} \geq 0, \sum_{n \geq 1} \alpha_{nm} = 1\) such that
\[
\bar{V}^m \to \mathcal{V}^*, \text{ strongly in } L^2(\Omega \times [0,T]; \mathbb{R}^N).
\]

Using the fact that \(|\bar{V}^m_i| \leq \alpha\), it follows that \(\mathcal{V}^* \in \mathcal{U}\). By Proposition 4.1, we also have the strong convergence, \(u^{V^n} \to u^V\) in \(L^2(\Omega; \mathcal{C}([0,T]; \mathbb{C}^N))\). Therefore, \((u^V, \mathcal{V}^*)\) is admissible. By making use of the convexity of \(|u_i^V(T) - f_i^V|^2, i \leq N\) and the Fatou lemma, we conclude that
\[
J(u^V) \leq \lim_{m \to \infty} J(\bar{V}^m) \leq \lim_{m \to \infty} \sum_{n \geq 1} \alpha_{nm}J(\bar{V}^m) \leq \inf_{\mathcal{V} \in \mathcal{U}} J(\mathcal{V}),
\]

which completes the proof.  \(\Box\)
From the above procedures, it can be seen that all the results in this subsection still hold as long as the cost functional in (18) take the form of $J(\mathcal{V}) = \mathbb{E}[f(u^T(t))] + \beta \mathbb{E} \left[ \int_0^T \sum_{i=1}^N |V_j(t) - Z_j(t)|^2 dt \right]$, where the function $f$ has a lower bound and is lower semi-continuous convex. We also would like to remark that the second term $\mathbb{E} \left[ \int_0^T \sum_{i=1}^N |V_j(t) - Z_j(t)|^2 dt \right]$ could be extended to more general objective functional, like $\mathbb{E} \left[ \int_0^T |u^\gamma(t) - Z^1(t)|^2 dt \right]$ with an $\mathcal{F}_t$-adapted and $L^2$-integrable process $Z^1$, whose integrator is bounded from below and convex.

4.2. Stochastic control problem with diffusion control. Similar to the linear potential control problem on graph, we can also obtain the existence of an optimal control problem with diffusion control which has not been reported even in the continuous case. Since the proof is similar to that of Theorem 4.1, we omit the details and only present the main result here.

Consider the constraint (11) with the control variable $\sigma \in \mathbb{R}^N$. The admissible control set $\tilde{\mathcal{U}}$ is defined by

$$\tilde{\mathcal{U}} := \left\{ \sigma : \Omega \times [0, T] \to \mathbb{R}^N \mid \sigma(t) \text{ is } \mathcal{F}_t\text{-adapted, } \sigma \in L^2([0, T]), \right. $$

there exists $\alpha > 0$, such that $|\sigma_j| \leq \alpha$ a.s.\}

Here the optimal control problem is to minimize the cost functional

$$(24) \quad J(\sigma) := \gamma \mathbb{E} \left[ \sum_{i=1}^N |u^\sigma_i(T) - f_i^1|^2 \right] + \beta \mathbb{E} \left[ \int_0^T \sum_{i=1}^N |\sigma_i(t) - Z_i(t)|^2 dt \right],$$

where $\gamma, \beta \geq 0$, $f^1$ is $\mathcal{F}_T$-adapted satisfying $\|f^1\|_{L^2(\Omega; \mathbb{R}^N)} < \infty$, $Z \in \tilde{\mathcal{U}}$, $u^\sigma$ is the solution of (11) with the control $\sigma$.

**Theorem 4.2.** For the control problem (24) with the constraint (11), there always exists an optimal control $\sigma^* \in \tilde{\mathcal{U}}$ which minimizes the objective functional $J$.

**Proof.** By applying Proposition 3.1 and repeating the steps in the proof of Proposition 4.1, the lower continuity of $J$ when $\beta = 0$ can be established. Therefore, the existence of optimal control is ensured by the convexity of $|u^\sigma_i(T) - f_i^1|^2$. When $\beta > 0$, the existence of optimal control can be guaranteed by Lemma 4.1.

From the above procedures, it can be seen that all the results in this subsection still hold as long as the cost functional in (24) takes the form of $J(\sigma) = \mathbb{E}[f(u^\sigma(T))] + \beta \mathbb{E} \left[ \int_0^T \sum_{i=1}^N |\sigma_j(t) - Z_j(t)|^2 dt \right]$, where the function $f$ has a lower bound and is lower semi-continuous convex. Meanwhile, we can also have the existence of optimal potential and diffusion controls at the same time according to Theorems 4.1 and 4.2.

5. Optimal condition for the stochastic control on graph. As it has been pointed out in [27], compared to nonlinear Schrödinger equations driven by additive noise, it is more difficult to investigate multiplicative noise. Beyond that, for the nonlinear Schrödinger equation on graph, the appearance of the nonlinear Laplacian $\Delta_G$ makes it more challenging to characterize the optimal condition than the continuous control problem.
In this section we mainly consider the following control problem

\[(25) \quad J(\mathbb{V}) := \gamma \mathbb{E}\left[ \sum_{i=1}^{N} |u_i^T(t) - f_i^T|^2 \right] + \beta_1 \mathbb{E}\left[ \int_0^T \sum_{i=1}^{N} |u_i^T(t) - Z_i^T(t)|^2 dt \right] + \beta \mathbb{E}\left[ \int_0^T \sum_{i=1}^{N} |\mathbb{V}_i(t) - Z_i(t)|^2 dt \right]\]

with the constraint (11) to illustrate how to derive the optimal condition on graph. Here \(\gamma \geq 0, \beta_1 \geq 0, \beta \geq 0\), and \(Z^T\) is an \(\mathcal{F}_t\)-adapted and \(L^2\)-integrable process. When \(\beta_1 = 0\), (25) degenerates into (18). Our approach can be also extended to a more general smooth convex functional setting.

### 5.1. Gradient formula.

In section 4, we have shown the existence of optimal potential and diffusion controls. Furthermore, in this part we study the necessary optimal condition near the minimizer \(\mathbb{V}^*\) of (25) which is also called the gradient formula.

**Proposition 5.1.** Let \((u^{\mathbb{V}^*}, \mathbb{V}^*)\) be the solution and optimal control of (25). Then for

\[\sup_{t \in [0,T]} |\mathbb{V}^*(t) - \mathbb{V}^*(t)| \leq \epsilon, \quad \mathbb{V}^* \in \mathcal{U},\]

it holds that

\[(26) \quad \mathbb{E}\left[ 1_{\Omega_\epsilon} \sup_{t \in [0,T]} |u^{\mathbb{V}^*}(t) - u^{\mathbb{V}^*}(t)| \right] \leq C(c, u(0), T, p)\epsilon^p,\]

where \(p \geq 2\) and \(\Omega_\epsilon = \{ \sup_{t \in [0,T]} \sup_{s \in [0,T]} \frac{1}{\rho^T_1} + \sup_{t \in [0,T]} \sup_{s \in [0,T]} \frac{1}{\rho^T_2} \leq \epsilon \}\).

Furthermore, suppose there exists \(c(\epsilon) \to \infty\) such that the random variable \(C(\omega)\), defined by (22) with \(\mathbb{V} \in \mathcal{U}\), satisfies

\[(27) \quad \lim_{\epsilon \to 0} \left[ C(c(\epsilon), u(0), T, 2)\epsilon + \frac{1}{\epsilon} \mathbb{P}(C(\omega) \geq \frac{1}{c_1} (\log(c(\epsilon)) + \log(c_2))) \right] = 0,\]

then for any \(\mathbb{V} \in \mathcal{U}\), the following variational inequality holds:

\[(28) \quad \lim_{c(\epsilon) \to \infty} \mathbb{E}\left[ 1_{\Omega_{c(\epsilon)}} \Re\left\{ \int_0^T \sum_{i=1}^{N} \left( (u_i^{\mathbb{V}^*}(t) - Z_i^T(t))X_i(t) + (\mathbb{V}_i^*(t) - Z_i(t))(\mathbb{V}_i(t) - \mathbb{V}_i^*(t)) \right) dt \right. \right. \]

\[
\left. + \sum_{i=1}^{N} (u_i^{\mathbb{V}^*}(T) - f_i^T(T))\overline{\overline{X_i(T)}} \right\} \geq 0, \quad \square\]

where \(X\) is the solution of the following equation

\[(29) \quad dX_i(t) = \left\{ \frac{i}{2} \sum_{j \in N(i)} \frac{\partial (\Delta C u)_{ij}}{\partial u_j} \bigg|_{u = u^{\mathbb{V}^*}} X_j - i\mathbb{V}_i^* X_i - i \sum_{l=1}^{N} \mathbb{W}_l d|u_l^{\mathbb{V}^*}|^2 X_i \]

\[
- 2i \sum_{l=1}^{N} \mathbb{W}_l \Re(\bar{u}_l^{\mathbb{V}^*} X_l)u_l^{\mathbb{V}^*} \right\} dt + \left\{ - i u_i^{\mathbb{V}^*} (\mathbb{V}_i - \mathbb{V}_i^*) \right\} dt + \left\{ - i \sigma_i X_i \right\} \circ dW(t) \]

\(X(0) = 0.\)
Proof. Since the admission control set $\mathcal{U}$ is convex, we can use a convex perturbation to illustrate the procedures. Consider $V^\epsilon = (1 - \epsilon)V^* + \epsilon V$. Define two processes $\xi(t) := \frac{u^\epsilon - u^\nu}{\epsilon}$ and $\delta V := V - V^\epsilon$. Before the stopping time $\tau^\epsilon$, according to the proof of Proposition 4.1, the equation of $X_i$ is well-posed since the coefficients of (29) are globally Lipschitz. By the mean value theorem, $\xi$ will satisfy

$$d\xi_i(t) = \left\{ \frac{1}{2} \sum_{j \in N(i)} \int_0^1 \frac{\partial (\Delta_G u_i)}{\partial u_j} d\kappa \xi_j - \int_0^1 i(V^*_i + \epsilon \kappa \delta V_i) d\kappa \xi_i \right\} dt$$

$$- \sum_{l=1}^N i \mathbb{W} d \left( \int_0^1 |u^\nu_i + \epsilon \kappa \xi_i|^2 d\kappa \right) \xi_i$$

$$- 2 \sum_{l=1}^N i \Im \left( \overline{(u^\nu_i + \epsilon \kappa \xi_i)} (u^\nu_i + \epsilon \kappa \xi_i) d\kappa \right) dt$$

$$+ \int_0^1 \left\{ - i(u^\nu_i + \epsilon \kappa \xi_i)(V^*_i - V^*_i) \right\} d\kappa dt + \left\{ - i\sigma_i \xi_i \right\} \circ dW(t).$$

Using the similar steps in the proof of Proposition 4.1, on $\Omega_c$, it holds that for any $p \geq 2$,

$$\mathbb{E} \left[ 1_{\Omega_c} \sup_{t \in [0,T]} |\xi(t)|^p \right] + \mathbb{E} \left[ 1_{\Omega_c} \sup_{t \in [0,T]} |X(t)|^p \right] \leq C(c, u(0), T) \mathbb{E} \left[ \int_0^T |\delta V|^2 ds \right]$$

and that for $p \geq 2$,

$$\mathbb{E} \left[ 1_{\Omega_c} \sup_{t \in [0,T]} |\xi(t) - X(t)|^p \right] \leq C(c, u(0), T, p).$$

Thus, (26) follows. Here $C(c, u(0), T, p)$ is increasing with respect to $c$ satisfying $\lim_{c \to \infty} C(c, u(0), T, p) = +\infty$.

For convenience, let us denote

$$J_{\Omega_c}(V) := \gamma \mathbb{E} \left[ 1_{\Omega_c} \sum_{i=1}^N |u^\nu_i(T) - f^1_i|^2 \right] + \beta_1 \mathbb{E} \left[ 1_{\Omega_c} \int_0^T \sum_{i=1}^N |u^\nu(t) - Z^1(t)|^2 dt \right]$$

$$+ \beta \mathbb{E} \left[ 1_{\Omega_c} \int_0^T \sum_{i=1}^N |V_i(t) - Z_i(t)|^2 dt \right].$$

Due to the fact that $J(V^*) \leq J(V^\epsilon)$, we obtain that

$$0 \leq J(V^\epsilon) - J(V^*)$$

$$= J_{\Omega_c}(V^\epsilon) - J_{\Omega_c}(V^*) + J_{\Omega_c}(V^*) - J_{\Omega_c}(V^*).$$

Using the tail estimate of $1_{\Omega_c}$ by the arguments in the proof of Proposition 4.1, we get

$$\lim_{c \to \infty} \lim_{\epsilon \to 0} J_{\Omega_c}(V^\epsilon) - J_{\Omega_c}(V^*) = 0.$$

To derive a necessary optimal condition, we need consider the speed of the convergence for $c$ and $\epsilon$. By (23), we have that

$$J_{\Omega_c}(V^*) \leq C\mathbb{P}(C(\omega) \geq \frac{1}{c_1}(\log(c) + \log(c_2))).$$
By the Taylor expansion and (26), we have
\[
0 \leq \frac{1}{\epsilon} \left[ J_{\Omega_e}(V') - J_{\Omega_e}(V^*) \right] + \frac{1}{\epsilon} \left[ J_{\Omega/\Omega_e}(V') - J_{\Omega/\Omega_e}(V^*) \right]
\]
\[
\leq \mathbb{E} \left[ 1_{\Omega_e} \mathbb{R} \left\{ \int_0^T \sum_{i=1}^N \left( u_i^{V'}(t) - Z_i^1(t) \overline{X_i(t)} + (V_i^*(t) - Z_i(t))(V_i(t) - V_i^*(t)) \right) dt 
+ \sum_{i=1}^N (u_i^{V'}(T) - f_i^1(T)) \overline{X_i(T)} \right\} \right]
\]
\[
+ C(c, u(0), T, 2) \epsilon + \frac{1}{\epsilon} C(\mathbb{P}(C) \geq \frac{1}{c_1} (\log(c) + \log(c_2))).
\]
Using the condition (27), there exists \( c(\epsilon) \rightarrow \infty \) such that
\[
\lim_{c \rightarrow \infty} \mathbb{E} \left[ 1_{\Omega_e} \mathbb{R} \left\{ \int_0^T \sum_{i=1}^N \left( u_i^{V'}(t) - Z_i^1(t) \overline{X_i(t)} + (V_i^*(t) - Z_i(t))(V_i(t) - V_i^*(t)) \right) dt 
+ \sum_{i=1}^N (u_i^{V'}(T) - f_i^1(T)) \overline{X_i(T)} \right\} \right] \geq 0,
\]
which implies (28).

**Remark 5.1.** If \( V^* \) is in the interior of \( U \), then (28) becomes the equality. In general, the limit with respect to \( c \) in (28) does not commute with the expectation since the variational equation (28) may not have a global estimate in the expectation sense and the coefficient is singular near boundary of \( P(G) \).

Our approach is also applicable for the cost functional
\[
J(V) = \mathbb{E} \left[ \int_0^T g(u^V(t), V(t)) dt + h(u^V(T)) \right],
\]
where \( g \) and \( h \) are continuous convex and differentiable with bounded first derivatives. Here we only present the result since the proof is similar to that of Proposition 5.1.

**Proposition 5.2.** Assume that \( g \) and \( h \) are continuous differentiable with bounded first derivatives. Under the condition of Proposition 5.1 with the cost functional (30), it holds that
\[
\lim_{c(\epsilon) \rightarrow \infty} \mathbb{E} \left[ 1_{\Omega_{\epsilon,c}} \mathbb{R} \left\{ \int_0^T \sum_{i=1}^N g_{x_i}(u^{V'}, V^*) \overline{X_i(t)} + g_{u_i}(u^{V'}, V^*)(V_i(t) - V_i^*(t)) \right\} dt 
+ \sum_{i=1}^N h_{x_i}(u^{V'}, \overline{X_i(T)} \right\} \right] \geq 0,
\]
where \( X_i \) is the solution of (29).

Similarly, we could consider the diffusion control problem,
\[
J(\sigma) = \mathbb{E} \left[ \int_0^T g(u^\sigma(t), \sigma(t)) dt + h(u^\sigma(T)) \right].
\]
with the constraint (11) and \( \sigma \in \tilde{U} \). We state its optimal condition as follows and omit the detailed proof.
Proposition 5.3. Let \((u^{\sigma^*}, \sigma^*)\) be an optimal control of (31). Then for \(|\sigma^\epsilon - \sigma^*| \leq \epsilon, \sigma^\epsilon \in \mathcal{U}\), it holds that for \(p \geq 2\),

\[(32) \quad \mathbb{E} \left[ 1_{\Omega_{c}} \sup_{t \in [0,T]} |u^{\sigma^*}(t) - u^\epsilon(t)|^2 \right] \leq C(c, u(0), T, p)c^p,
\]

where \(\Omega_{c} = \left\{ \sup_{1 \leq N} \sup_{s \in [0,T]} \frac{1}{\rho^s} + \sup_{1 \leq N} \sup_{s \in [0,T]} \frac{1}{\rho^s} \leq c \right\} \).

Furthermore, suppose that the random variable \(C(\omega)\), defined by (22) with \(\sigma \in \mathcal{U}\), satisfies that there exists \(c(\epsilon) \to \infty\) such that

\[(33) \quad \lim_{\epsilon \to 0} \left[ C(c(\epsilon), u(0), T, 2) + \frac{1}{\epsilon} \mathbb{P}(C(\omega) \geq \frac{1}{c_1}(\log(c(\epsilon)) + \log(c_2))) \right] = 0.
\]

Suppose that \((u^{\sigma^*}, \sigma^*)\) is an optimal control of (31). Then for any \(\sigma \in \mathcal{U}\), the following variational inequality holds:

\[(34) \quad \lim_{\epsilon \to 0} \mathbb{E} \left[ 1_{\Omega_{c}} \mathbb{R} \left\{ \int_{0}^{T} \sum_{i=1}^{N} \left( g_{x_i}(u^{\sigma^*}, \sigma)X_{i}(t) + g_{y_i}(u^{\sigma^*}, \sigma^*)(\sigma_i(t) - \sigma^*_i(t)) \right) dt \right. \right.

\[+ \left. \sum_{i=1}^{N} h_{x_i}(u^{\sigma^*}(T))X_{i}(T) \right\} \geq 0,
\]

where \(X\) is the solution of the following equation

\[(35) \quad dX_{i}(t) = \left\{ \frac{i}{2} \sum_{j \in N(i)} \frac{\partial (\Delta G)}{\partial u_j} \bigg|_{u = u^{\sigma^*}} X_j - iV_i X_i - i \sum_{l=1}^{N} W_{il} |u^{\sigma^*}_i|^2 X_i - 2i \sum_{l=1}^{N} W_{il} \mathbb{R}(\bar{u}^{\sigma^*}_i X_l) u^{\sigma^*}_l \right\} dt

\[+ \left\{ - i\sigma^*_i X_i - iu^{\sigma^*}_i(\sigma_i - \sigma^*_i) \right\} \circ dW(t)
\]

\(X(0) = 0\).

The gradient formula characterizes the necessary optimal condition of the potential and diffusion control problems. However, such condition is not very useful in practice, because the variational solution depends on the control variable \(V\) or \(\sigma\).

5.2. Backward SDE. In this subsection, we aim to give a more in-depth description on the optimal condition via the forward and backward stochastic differential equations. To better illustrate the procedure while clearly explaining the main idea, we use the control problem (25) with \(\gamma = \beta = \beta_1 = 1\) as an example. To this end, we need a priori estimate of the variational solution \(X\) of (29) such that the limit with respect to \(\epsilon\) commutes with the expectation in (28).

Proposition 5.4. Let \(\sigma\) be a constant potential, i.e., \(\sigma_i = \sigma_j\), and \(p(0) \in \mathcal{P}_o(G)\), \(S(0) \in \mathbb{R}^N\). Assume that \(V \in \mathcal{U}\). Then it holds that for \(p \geq 2\),

\[(36) \quad \mathbb{E} \left[ \sup_{t \in [0,T]} \| u^\epsilon V(t) \|^p \right] \leq C(u(0), T, p, \alpha),
\]

\[\mathbb{E} \left[ \sup_{t \in [0,T]} \| X(t) \|^p \right] \leq C(u(0), T, p, \alpha).
\]

Proof. According to Proposition 3.1 and the proof of Proposition 4.1, it suffices to prove a uniform lower bound estimate of the density function \(\rho^V(t) = |u^V(t)|^2\).
Since $\sigma_i = \sigma_j$, we denote $\sigma_i = \tilde{\sigma}$. Introducing $\tilde{S}_i = S_i + \tilde{\sigma}W(t)$, (11) can be rewritten as

$$d\rho_i = \sum_{j \in N(i)} \omega_{ij}(\tilde{S}_i - \tilde{S}_j)\theta_{ij}(\rho)dt;$$

$$d\tilde{S}_i + \frac{1}{2} \sum_{j \in N(i)} \omega_{ij}(\tilde{S}_i - \tilde{S}_j)^2 \frac{\partial \theta_{ij}}{\partial \rho_i} + \frac{1}{8} \frac{\partial}{\partial \rho_i} I(\rho) + \mathcal{V}_j + \sum_{j \in N(i)} \mathcal{W}_{ij}\rho_j)dt = 0,$$

which is a nonlinear Schrödinger equation with random inputs. Thus it follows that

$$\mathcal{H}(\rho(t), \tilde{S}(t)) := \frac{1}{2} \langle \nabla_G \tilde{S}, \nabla_G \tilde{S} \rangle_{\theta(\rho(t))} + \mathcal{V}(\rho(t)) + \mathcal{W}(\rho(t)) + \frac{1}{8} I(\rho(t))$$

$$= \mathcal{H}(\rho(0), \tilde{S}(0)) < \infty, \text{ a.s.}$$

The property of Fisher information yields that there exists a constant $c_{low} > 0$ such that

$$\inf_{t \geq 0} \min_{i \leq N} \rho_i(t) \geq c_{low} > 0, \text{ a.s.}$$

Therefore we have $\Omega_{\text{low}} = \Omega$ and

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \| u^\rho(t) \|_P \right] \leq C(u(0), T, p, \alpha, c_{low}).$$

The lower bound of the density function also implies that the coefficient of (29) are bounded and Lipschitz. By repeating similar steps in the proof of Proposition 5.1, we complete the proof.

Thanks to the lower bound estimate of the density function, we are also able to derive the corresponding backward stochastic differential equation, which is also called the adjoint equation of (29).

**Corollary 5.1.** Let the condition of Proposition 5.1 hold. Let $(u^\rho, \mathcal{V}^\rho)$ be an optimal control of (25). Then there exists an adapted solution $(Y, Z)$ of the following system,

$$dY_i(t) = \left\{ \frac{1}{2} \sum_{j \in N(i)} \frac{\partial \langle \nabla_G u \rangle_j}{\partial u_i} \bigg|_{u = u^\rho} Y_j - \mathcal{W}_l|u^\rho|^2 Y_i - 2 \sum_{l=1}^N \mathcal{W}_l \Re(\mathcal{W}_l u^\rho \bar{Y}_l)u^\rho_i \right\} dt$$

$$+ \frac{1}{2} \sigma_i^2 Y_i(t) dt + i\sigma_i Z_i dt + 2(u_i^\rho - Z_i^1) dt + Z_i(t)dW(t),$$

$$Y(T) = -2u_i^\rho(T) + 2f_i(T).$$

**Proof.** Thanks to Proposition 5.4, the coefficients of (37) are Lipschitz and bounded. Then the standard arguments in [37, section 3] yield the well-posedness of the linear BSDE (37), that is, there exists a unique adapted solution $(Y, Z)$.

Based on the above results, we are ready to characterize the optimal condition by a coupled forward–backward SDE system.

**Theorem 5.1.** Let the condition of Proposition 5.4 hold. Then the optimal control pair $(u^\rho, \mathcal{V}^\rho)$ satisfies the generalized stochastic Hamiltonian system consisting
of (11), (37) with \( u(0) = \sqrt{\rho(0)} e^{iS(0)} \), \( Y(T) = -2u^V(T) + 2f_1(T) \) and the stationary condition, i.e., for arbitrary \( V \),

\[
\Re(-iu^V Y + 2(V^* - Z), V - V^*) \geq 0, \text{ a.e. } t \in [0, T], \text{ a.s.}
\]

Proof. For convenience, let us denote \( \Re(X, Y) := \Re(\sum_{i=1}^{N} \bar{X}_i Y_i) \) and \( \Re(a, b) = \Re(ab) \). Applying Itô’s formula, we obtain that

\[
d\Re\langle X(t), Y(t) \rangle
\]

\[
= \sum_{i=1}^{N} \left\{ \Re\left( \frac{1}{2} \sum_{j \in N(i)} \frac{\partial(\Delta_G u_i)}{\partial u_j} \bigg|_{u=u^V, X_j, Y_i} \right) - \Re(iV_i^* X_i, Y_i) - \Re(i \sum_{l=1}^{N} \Re(u_i^V |X_l|^2 X_l, Y_i) \right) dt
\]

\[
- 2\Re\left( \sum_{l=1}^{N} iW_l \Re(u_i^V X_l) u_i^V, Y_l \right) dt + \sum_{i=1}^{N} \left\{ \Re\left( -\frac{1}{2} \sigma_i^2 X_i, Y_i \right) + \sum_{l=1}^{N} \Re\left( -iu_i^V (V_i - V_i^*), Y_l \right) \right\} dt
\]

\[
+ \sum_{i=1}^{N} \left\{ \Re\left( \frac{1}{2} \sigma_i^2 Y_i, X_i \right) + \Re(i \sigma_i Z_i, X_i) + 2 \Re\left( u_i^V - Z_i^1, X_i \right) \right\} dt
\]

\[
+ \sum_{i=1}^{N} \left\{ \Re\left( Z_i(t), Y_i(t) \right) + \Re\left( -i \sigma_i X_i, Z_i \right) \right\} dW(t) + \sum_{i=1}^{N} \Re\left( -i \sigma_i X_i, Z_i \right) dt
\]

\[
= \sum_{i=1}^{N} \Re\left( -iu_i^V (V_i - V_i^*), Y_i \right) dt + \sum_{i=1}^{N} 2\Re\left( u_i^V - Z_i^1, X_i \right) dt + \sum_{i=1}^{N} \left\{ \Re\left( Z_i(t), Y_i(t) \right) \right\}
\]

\[
+ \Re\left( -i \sigma_i X_i, Z_i \right) \right\} dW(t).
\]

Taking expectation yields that

\[
- \mathbb{E}[2\Re\langle u^V(T) - f^1(T), X(T) \rangle] = \mathbb{E}[\Re(X(T), Y(T))] - \mathbb{E}[\Re(X(0), Y(0))]
\]

\[
= \int_{0}^{T} \mathbb{E}\left[ \Re\langle u^V (V - V^*), Y \rangle + 2 \Re(u^V - Z^1, X(t) \rangle dt.
\]

By using (28), Proposition 5.4, and Corollary 5.1, we obtain

\[
0 \leq \mathbb{E}\left[ \left\{ \int_{0}^{T} \sum_{i=1}^{N} 2\left( (u_i^V(t) - Z_i^1(t))X_i(t) + (V_i^*(t) - Z_i(t)) (V_i(t) - V_i^*(t)) \right) \right) dt
\]

\[
+ \sum_{i=1}^{N} 2(u_i^V(T) - f_1^i(T))X_i(T) \right\}
\]

\[
= \int_{0}^{T} \mathbb{E}\left[ - \Re\langle u^V Y, V - V^* \rangle + 2 \Re(V^* - Z, V - V^*) \right] dt.
\]

Thus for arbitrary \( V \), we conclude that

\[
\Re(-iu^V Y + 2(V^* - Z), V - V^*) \geq 0, \text{ a.e. } t \in [0, T], \text{ a.s.}
\]

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Theorem 5.1 can be also viewed as the Pontryagin’s maximum principle. Based on the above theorem, we propose the corresponding forward-backward stochastic differential equation (FBSDE) for (25),

\begin{equation}
\frac{dW_i}{dt} = -\frac{1}{2}(\Delta_G u)_j + u_j V_j + u_j \sum_{l=1}^{N} W_{jl}|u_l|^2 + \sigma_j u_j \circ dW_i,
\end{equation}

\begin{equation}
\begin{align*}
dY_i(t) &= -\left\{ \frac{1}{2} \sum_{i \in N(j)} \frac{\partial (\Delta_G u)_j}{\partial u_i} \Big|_{u=u^*} Y_j - iV_i Y_i - i \sum_{l=1}^{N} W_{il}|u_l|^2 Y_i - 2i \sum_{l=1}^{N} W_{il} \Re(u_l^* Y_i) u_i^* \right\} dt \\
&\quad + \frac{1}{2} \sigma_i^2 Y_i(t) dt + i \sigma_i Z_i dt + 2(u_i^* - Z_i) dt + Z_i(t)dW(t),
\end{align*}
\end{equation}

\begin{equation}
u(0) = \sqrt{\rho(0)}e^{iS(0)}, \quad Y(T) = -2u^*(T) + 2f_1(T), \quad \Re(u^* Y) + 2(V^* - Z), \quad V - V^* = 0.
\end{equation}

If the control problem (25) admits a unique optimal control, and the stochastic generalized FBSDE also admits a unique adapted solution \((u, Y, Z)\), then \(u\) is the optimal state process and the corresponding control \(V\) is optimal.

We also present the Pontryagin’s maximum principle for (31) with the constraint (11) and the diffusion control \(\sigma_i = \sigma_j, \ i, j \leq N\).

**Theorem 5.2.** Let the condition of Proposition 5.4 hold. Then the optimal control pair \((u^*, \sigma^*)\) satisfies the generalized stochastic Hamiltonian system consisting of (11), and

\begin{equation}
\begin{align*}
dY_i(t) &= -\left\{ \frac{1}{2} \sum_{i \in N(j)} \frac{\partial (\Delta_G u)_j}{\partial u_i} \Big|_{u=u^*} Y_j - iV_i Y_i - i \sum_{l=1}^{N} W_{il}|u_l|^2 Y_i - 2i \sum_{l=1}^{N} W_{il} \Re(u_l^* Y_i) u_i^* \right\} dt \\
&\quad + \frac{1}{2} \sigma_i^2 Y_i(t) dt + i \sigma_i Z_i dt + 2(u_i^* - Z_i) dt + Z_i(t)dW(t),
\end{align*}
\end{equation}

\begin{equation}
Y(T) = -2u^*(T) + 2f_1(T)
\end{equation}

with \(u(0) = \sqrt{\rho(0)}e^{iS(0)}, \ Y(T) = -2u^*(T) + 2f_1(T)\) and the stationary condition

\[\Re(-\sigma u^* Y - iu^* Z + 2(\sigma - Z), \sigma - \sigma^*) \geq 0 \ a.e. \ t \in [0, T], \ a.s.\]

**Proof.** The proof is similar to that of Theorem 5.1. By applying Propositions 5.4 and 5.3, we can apply Itô formula to \(\Re(X(t), Y(t))\). Using the similar steps in the proof of Theorem 5.1 and utilizing (34), we can get that

\[\int_0^T \mathbb{E}\left[\Re(-\sigma u^* Y - iu^* Z, \sigma - \sigma^*) + 2\Re(\sigma^* - Z, \sigma - \sigma^*)\right] dt \geq 0,
\]

which completes the proof. □

In general, if the cost functional is (30) or (31), analogous analysis leads to the following results.

**Theorem 5.3.** Let the condition of Proposition 5.4 hold. There exists an adapted solution \((Y_i, Z)\) of

\begin{equation}
\begin{align*}
dY_i(t) &= -\left\{ \frac{1}{2} \sum_{i \in N(j)} \frac{\partial (\Delta_G u)_j}{\partial u_i} \Big|_{u=u^*} Y_j - iV_i Y_i - i \sum_{l=1}^{N} W_{il}|u_l|^2 Y_i - 2i \sum_{l=1}^{N} W_{il} \Re(u_l^* Y_i) u_i^* \right\} dt \\
&\quad + \frac{1}{2} \sigma_i^2 Y_i(t) dt + i \sigma_i Z_i dt + 2(u_i^* - Z_i) dt + Z_i(t)dW(t),
\end{align*}
\end{equation}
which correspond to the stochastic control problems with the cost (30) such that the stationary condition

\[ \Re(-iu^{V^*}Y + g_y(u^{V^*}, V - V^*)) \geq 0, \ a.e. \ t \in [0, T], \ a.s. \]

hold. For the stochastic control problem of (31), there exists an adapted solution \((Y, Z)\) of

\[
dY_i(t) = -\left\{ \frac{1}{2} \sum_{j \in N(i)} \left( \frac{\partial (\Delta G_{\sigma})}{\partial u_i} \right)_{u=\sigma^*} Y_j - i\bar{V}_i Y_i - i \sum_{l=1}^{N} \Re(E(u_i \sigma^* Y_l) u_i^*) \right\} dt \\
+ \frac{1}{2} \sigma_i^2 Y_i(t) dt + i\bar{\sigma}_i Z_i dt + g_x(u^{V^*}, \sigma^*) dt + Z_i(t) dW(t),
\]

such that the stationary condition

\[ \Re(-u^{V^*}Z - g_y(u^\sigma, \sigma - \sigma^*) \geq 0, \ a.e. \ t \in [0, T], \ a.s. \]

hold.

It can be seen that if the \(V^*\) (or \(\sigma^*\)) is achieved in the interior of \(\mathcal{U}\) (or \(\tilde{\mathcal{U}}\)), then the stationary condition could be simplified to an equality.

6. Conclusion. In this paper, we propose the stochastic control problem subject to stochastic nonlinear Schrödinger equation on graph with either a linear potential or diffusion control. From the numerical viewpoint, we demonstrate that the particular features such as the stochastic dispersion relationship, mass conservation law, moment bounds of energy of stochastic nonlinear Schrödinger on graph. Furthermore, we provide the gradient formula and the Pontryagin’s maximum principle for stochastic nonlinear Schrödinger equation on graph driven by multiplicative noise. These may serve as a foundation of the numerical computation for stochastic control of stochastic nonlinear Schrödinger equation in a continuous domain as well (see, e.g., [13]).

There are many interesting questions that remain to be tackled. For instance, it will be more difficult to investigate the nonlinear potential and diffusion controls of the stochastic nonlinear Schrödinger equation driven by general multiplicative noise. Given the solutions of the FBSDEs, can this stationary condition uniquely determine the optimal control for stochastic nonlinear Schrödinger equation on graph? The stochastic control problem over density manifold, such as the mean-field game involved with the Fisher information or non-monotone coefficient, is challenging. Besides, the numerical computation has not been addressed in the current work. We plan to explore these issues in the future work.

REFERENCES

[1] V. Barbu, M. Röckner, and D. Zhang. Stochastic nonlinear Schrödinger equations. Nonlinear Anal., 136:168–194, 2016.

[2] V. Barbu, M. Röckner, and D. Zhang. Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise. Ann. Probab., 46(4):1957–1999, 2018.

This manuscript is for review purposes only.
[3] A. Barchielli and M. Gregoratti. Quantum trajectories and measurements in continuous time: the diffusive case, volume 782. Springer, 2009.

[4] A. Barchielli and A. S. Holevo. Constructing quantum measurement processes via classical stochastic calculus. Stochastic Process. Appl., 58(2):293–317, 1995.

[5] A. Borzì and U. Hohenester. Multigrid optimization schemes for solving Bose-Einstein condensate control problems. SIAM J. Sci. Comput., 30(1):441–462, 2007/08.

[6] J. Bourgain and A. Klein. Bounds on the density of states for Schrödinger operators. Invent. Math., 194(1):41–72, 2013.

[7] Z. Brzeźniak, F. Hornung, and L. Weis. Martingale solutions for the stochastic nonlinear Schrödinger equation in the energy space. Probab. Theory Related Fields, 174(3-4):1273–1338, 2019.

[8] Z. Brzeźniak and A. Millet. On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. Potential Anal., 41(2):269–315, 2014.

[9] A. G. Butkovskiy and Yu. I. Samoilenko. Control of quantum-mechanical processes and systems, volume 56 of Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1990.

[10] T. Cazenave. Semilinear Schrödinger equations, volume 10 of Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[11] S. Chatterjee and K. Kirkpatrick. Probabilistic methods for discrete nonlinear Schrödinger equations. Comm. Pure Appl. Math., 65(5):727–757, 2012.

[12] S. Chow, W. Li, and H. Zhou. A discrete Schrödinger equation via optimal transport on graphs. J. Funct. Anal., 276(8):2449–2469, 2019.

[13] J. Cui, L. Dieci, and H. Zhou. A continuation multiple shooting method for Wasserstein geodesic equation. arXiv:2105.09502, SIAM J. Sci. Comput., 2021.

[14] J. Cui, L. Dieci, and H. Zhou. Time discretizations of Wasserstein-Hamiltonian flows. Math. Comp., 91(no. 335):1019–1075, 2022.

[15] J. Cui, J. Hong, and Z. Liu. Strong convergence rate of finite difference approximations for stochastic cubic Schrödinger equations. J. Differential Equations, 263(7):3687–3713, 2017.

[16] J. Cui, J. Hong, and L. Sun. On global existence and blow-up for damped stochastic nonlinear Schrödinger equation. Discrete Contin. Dyn. Syst. Ser. B, 24(12):6837–6854, 2019.

[17] J. Cui, S. Liu, and H. Zhou. Stochastic Wasserstein Hamiltonian flows. arXiv:2111.15163, 2021.

[18] J. Cui, S. Liu, and H. Zhou. Wasserstein Hamiltonian flow with common noise on graph. arXiv:2204.01185, 2022.

[19] J. Cui and L. Sun. Stochastic logarithmic Schrödinger equations: energy regularized approach. arXiv:2102.12607.

[20] G. Da Prato and J. Zabczyk. Stochastic equations in infinite dimensions, volume 44 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.

[21] A. de Bouard and A. Debussche. A stochastic nonlinear Schrödinger equation with multiplicative noise. Comm. Math. Phys., 205(1):161–181, 1999.

[22] A. de Bouard and A. Debussche. The nonlinear Schrödinger equation with white noise dispersion. J. Funct. Anal., 259(5):1300–1321, 2010.

[23] J. Eilbeck and M. Johansson. The discrete nonlinear Schrödinger. 2003.

[24] D. Feng and D. Zhao. Optimal bilinear control of Gross-Pitaevskii equations with Coulombian potentials. J. Differential Equations, 260(3):2973–2993, 2016.

[25] C. Groetsch. The theory of Tikhonov regularization for Fredholm equations of the first kind. P1 1984.

[26] K. Hito and K. Kunisch. Optimal bilinear control of an abstract Schrödinger equation. SIAM J. Control Optim., 46(1):274–287, 2007.

[27] D. Keller. Optimal control of a linear stochastic Schrödinger equation. Discrete Contin. Dyn. Syst., (Dynamical systems, differential equations and applications. 9th AIMS Conference. Suppl.):437–446, 2013.

[28] D. Keller. Optimal control of a nonlinear stochastic Schrödinger equation. J. Optim. Theory Appl., 167(3):862–873, 2015.

[29] Qi Lü. Exact controllability for stochastic Schrödinger equations. J. Differential Equations, 255(8):2484–2504, 2013.

[30] J. Maas. Gradient flows of the entropy for finite Markov chains. J. Funct. Anal., 261(8):2250–2292, 2011.

[31] E. Nelson. Derivation of the Schrödinger equation from Newtonian mechanics. Physical Review,
150(4):1079–1085, 1966. cited By 1075.

[32] D. Noja. Nonlinear Schrödinger equation on graphs: recent results and open problems. 372, 2013.

[33] A. P. Peirce, M. A. Dahleh, and H. Rabitz. Optimal control of quantum-mechanical systems: existence, numerical approximation, and applications. Phys. Rev. A (3), 37(12):4950–4964, 1988.

[34] E. Schrödinger. Uber die umkehrung der naturgesetze. (144):144–153, 1931.

[35] C. Sulem and P. Sulem. The nonlinear Schrödinger equation, volume 139 of Applied Mathematical Sciences. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

[36] J. Werschnik and E. K. U. Gross. Quantum optimal control theory. J. Phys. B, 40(18):R175–R211, 2007.

[37] J. Yong. Stochastic optimal control-a concise introduction. Mathematical Control and Related Fields, 0, 2020.