THE FOURTH ORDER NONLINEAR SCHRÖDINGER LIMIT FOR QUANTUM ZAKHAROV SYSTEM

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Abstract. This paper is concerned with the quantum Zakharov system. We prove that when the ionic speed of sound goes to infinity, the solution to the fourth order Schrödinger part of the quantum Zakharov system converges to the solution to quantum modified nonlinear Schrödinger equation.

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1. Introduction

We consider the quantum Zakharov system:

\[
\begin{align*}
    i\partial_t E_{\varepsilon,\lambda} + \Delta E_{\varepsilon,\lambda} - \varepsilon^2 \Delta^2 E_{\varepsilon,\lambda} &= n_{\varepsilon,\lambda} E_{\varepsilon,\lambda} & t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
    \lambda^{-2} \partial^2_t n_{\varepsilon,\lambda} - \Delta n_{\varepsilon,\lambda} + \varepsilon^2 \Delta^2 n_{\varepsilon,\lambda} &= \Delta |E_{\varepsilon,\lambda}|^2 & t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
    E_{\varepsilon,\lambda}(0, x) &= E_0(x), \ n_{\varepsilon,\lambda}(0, x) = n_0(x), \ \partial_t n_{\varepsilon,\lambda}(0, x) = n_1(x) & x \in \mathbb{R}^d,
\end{align*}
\]

where \( \varepsilon \in (0, 1], \lambda \in [1, \infty) \), \( E_{\varepsilon,\lambda} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) and \( n_{\varepsilon,\lambda} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) are unknown functions, and \( E_0 : \mathbb{R}^d \to \mathbb{C}, n_0 : \mathbb{R}^d \to \mathbb{R} \) and \( n_1 : \mathbb{R}^d \to \mathbb{R} \) are given functions.

This model was introduced by Garcia-Haas-Oliveira-Goedert [10] and Haas-Shukla [15] to describe the nonlinear interaction between high-frequency quantum Langmuir waves and low-frequency quantum ion-acoustic waves. The physical background of the system (1.1) can be found in [17].

The classical Zakharov system

\[
\begin{align*}
    i\partial_t E_{0,\lambda} + \Delta E_{0,\lambda} &= n_{0,\lambda} E_{0,\lambda} & t \in \mathbb{R}, \ x \in \mathbb{R}^d, \\
    \lambda^{-2} \partial^2_t n_{0,\lambda} - \Delta n_{0,\lambda} &= \Delta |E_{0,\lambda}|^2 & t \in \mathbb{R}, \ x \in \mathbb{R}^d
\end{align*}
\]

was proposed by Zakharov [30] as a model for describing the interaction between the Langmuir waves and ion-acoustic waves in a plasma. In the
system (1.2), $E_{0,\lambda}$ denotes the slowly varying envelope of the highly oscillatory electric field and $n_{0,\lambda}$ denotes the deviation of the ion density from the equilibrium, and $\lambda$ is the ionic speed of sound.

The classical Zakharov system (1.2) has been extensively studied from the point of view of local and global well-posedness [2, 3, 4, 11, 23, 24, 29], blow-up of solutions [12, 13, 25], scattering [15, 16, 19, 31, 33], and subsonic equilibrium, and $\lambda$ is the ionic speed of sound.

The classical Zakharov system (1.2) has been extensively studied from the point of view of the scaling of Sobolev spaces. Furthermore, for the point of view of the scaling of Sobolev spaces. Furthermore, for the point of view of local and global well-posedness [2, 3, 4, 11, 23, 24, 29], and [7] are based on the Fourier restriction method associated to the fourth order Schrödinger and the fourth order wave evolution groups.

Let us review the local and global well-posedness for the classical Zakharov system (1.2). We will mention the Cauchy problem of (1.2) in $\mathbb{R}^d$, $d = 1, 2, 3$ only. For the high dimensional case and the multi-dimensional torus case, see [11] and [24], respectively.

Bourgain-Colliander [2] proved the local well-posedness of (1.2) in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ for $d = 2, 3$ by using the Fourier restriction norm associated to the free Schrödinger and wave evolution groups. By refining the Fourier restriction norm used in [2], Ginibre-Tsutsumi-Velo [11] have shown the local well-posedness of (1.2) in $H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$ provided that $-1/2 < k - \ell \leq 1$ and $0 \leq \ell + 1/2 \leq 2k$ for $d = 1, 0 \leq k - \ell \leq 1$ and $1 \leq \ell + 1 \leq 2k$ for $d = 2, 3$.

For one dimensional case, Colliander-Holmer-Tzirakis [8] proved the global well-posedness of (1.2) in $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R}) \times H^{-3/2}(\mathbb{R})$. For the two dimensional case, Bejenaru-Herr-Holmer-Tataru [4] showed the local well-posedness of (1.2) in $L^2(\mathbb{R}^2) \times H^{-1/2}(\mathbb{R}^2) \times H^{-3/2}(\mathbb{R}^2)$ which is optimal from the point of view of the scaling of Sobolev spaces. Furthermore, for the three dimensional case, Bejenaru-Herr [3] have recently shown the local well-posedness of (1.2) in $H^k(\mathbb{R}^3) \times H^l(\mathbb{R}^3) \times H^{l-1}(\mathbb{R}^3)$ with $k > 0$ and $\ell > -1/2$.

Like the classical model (1.2), the quantum Zakharov system (1.1) possesses the conservation of mass

$$\|E_{\varepsilon,\lambda}(t)\|_{L_x^2}^2 = \text{constant}$$

and the conservation of the Hamiltonian

$$\|\nabla E_{\varepsilon,\lambda}(t)\|_{L_x^2}^2 + \varepsilon^2 \|\Delta E_{\varepsilon,\lambda}(t)\|_{L_x^2}^2 + \frac{1}{2}\lambda^{-2}\|\partial_t \nabla^{-1} n_{\varepsilon,\lambda}(t)\|_{L_x^2}^2$$

$$+ \frac{1}{2}\|n_{\varepsilon,\lambda}(t)\|_{L_x^2}^2 + \frac{1}{2}\|\nabla n_{\varepsilon,\lambda}(t)\|_{L_x^2}^2 + \int_{\mathbb{R}^d} |E_{\varepsilon,\lambda}|^2 dx = \text{constant.}$$

Compared to the classical Zakharov system (1.2), there are few results for the quantum Zakharov system (1.1). We summarize the known results for (1.1).

In one space dimension, the system (1.1) is studied from the point of view of the existence of exact solution and the local and global well-posedness. El-Wakil and Abdou [9] constructed the exact traveling solutions of (1.1) by using the improved tanh function method. Jiang-Lin-Shao [20] proved the local well-posedness of (1.1) in $H^k(\mathbb{R}) \times H^l(\mathbb{R}) \times H^{l-2}(\mathbb{R})$ provided that $-3/2 < k - \ell < 3/2$, $-3/2 < 2k - \ell$, $-3/2 < k + \ell$ and $k > -3/4$. Chen-Fang-Wang [7] have recently shown the global well-posedness of (1.2) in $L^2(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ with $-3/2 \leq \ell \leq 3/2$. The proofs given in [20] and [7] are based on the Fourier restriction method associated to the fourth order Schrödinger and the fourth order wave evolution groups.
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For space dimensions $d = 1, 2, 3$, Guo-Zhang-Guo [14] have proved the global well-posedness of (1.1) with initial data in $H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) \times (H^{k-3} \cap H^{-1})(\mathbb{R}^d)$ and in $H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) \times H^{k-3}(\mathbb{R}^d)$ for $d = 1, 2, 3$ and $k \geq 2$, respectively. Especially, they proved the global well-posedness of the quantum Zakharov system (1.1) in the energy space $H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \times (H^{-1} \cap H^{-1})(\mathbb{R}^d)$. However it is interesting that the classical Zakharov system (1.2) has the blow up solution in the energy space which is $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times (H^{-1} \cap H^{-1})(\mathbb{R}^d)$, see Glangetas-Merle [12] [13]. As pointed out in [14], this difference is caused by the strong dispersion stems from the quantum effect, namely the quantum effect stabilizes the solution.

In this paper, we consider the convergence of the solution to the quantum Zakharov system (1.1) as $\lambda \to 0$. Before stating the main result for (1.1), we review the convergence of the solution to the classical Zakharov system (1.2) as $\lambda \to 0$.

Let us formally take $\lambda \to 0$ for the second equation in (1.2). If $n_{0,\infty} + |E_{0,\infty}|^2$ vanishes at space infinity, we obtain the relation $n_{0,\infty} = -|E_{0,\infty}|^2$. Substituting this relation into the first equation in (1.2), we see that $E_{0,\infty}$ satisfies the focusing cubic nonlinear Schrödinger equation

$$i\partial_t E_{0,\infty} + \Delta E_{0,\infty} = -|E_{0,\infty}|^2 E_{0,\infty}. \quad (1.4)$$

The local existence and uniqueness of (1.2) is shown by Schütz-Weinstein [34] when the initial datum lies in $H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) \times (H^{k-2} \cap H^{-1})(\mathbb{R}^d)$ with $d = 1, 2, 3$ and $k \geq 2$. Furthermore, they proved that the existence time interval is independent of $\lambda \in [1, \infty)$ and the solution $(E_{0,\lambda}, n_{0,\lambda})$ of (1.2) converges to $(E_{0,\infty}, -|E_{0,\infty}|^2)$ as $\lambda \to \infty$, where $E_{0,\infty}$ is a solution to the focusing cubic nonlinear Schrödinger equation (1.4). Added-Added [1] studied the rate of convergence of solution. The optimal rates of convergence $E_{0,\lambda}$ is given by Ozawa-Tsutsumi [30], including some discussion of initial layer phenomenon. Notice that in [34] [1] [30], they imposed the assumption $n_1 \in H^{-1}(\mathbb{R}^d)$. Kenig-Ponce-Vega [24] obtained the optimal rates of convergence for $E_{0,\lambda}$ in the non-compatible case $n_0 + |E_0|^2 \neq 0$ without the assumption $n_1 \in H^{-1}(\mathbb{R}^d)$.

Let us turn to the quantum system (1.1). In analogy with the classical system (1.2), taking $\lambda \to \infty$ for the second equation in (1.1) and together with the assumption $n_{\varepsilon,\infty} + (1 - \varepsilon^2 \Delta)^{-1}|E_{\varepsilon,\infty}|^2$ vanishes at space infinity, we obtain the relation

$$n_{\varepsilon,\infty} = -(1 - \varepsilon^2 \Delta)^{-1}|E_{\varepsilon,\infty}|^2.$$

Substituting this relation into the first equation in (1.1), we see that $E_{\varepsilon,\infty}$ satisfies the quantum modified nonlinear Schrödinger type equation

$$i\partial_t E_{\varepsilon,\infty} - (-\Delta + \varepsilon^2 \Delta^2) E_{\varepsilon,\infty} = -(1 - \varepsilon^2 \Delta)^{-1}|E_{\varepsilon,\infty}|^2 E_{\varepsilon,\infty}. \quad (1.5)$$

The main purpose of this paper is to prove that the solution $E_{\varepsilon,\lambda}$ to the quantum Zakharov system converges to the solution $E_{\varepsilon,\infty}$ to the fourth order nonlinear Schrödinger type equation (1.3).

We introduce several function spaces and notations. For non-negative integers $m, n$, the Sobolev space $H^m$ and the weighted Sobolev space $H^{m,n}$
are defined by
\[
H^m(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d); \|u\|_{H^m} = \sum_{k=0}^{m} \|\nabla^k u\|_{L^2} < \infty \right\},
\]
\[
H^{m,n}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d); \|u\|_{H^{m,n}} = \sum_{k=0}^{m} \sum_{\ell=0}^{n} \|x^{\ell} \nabla^k u\|_{L^2} < \infty \right\},
\]
where the k-th derivative \(\nabla^k\) is defined by
\[
\nabla^k = \begin{cases} \Delta^{(k-1)/2} \nabla & \text{if } k \text{ is odd}, \\ \Delta^{k/2} & \text{if } k \text{ is even}. \end{cases}
\]
We also define the following notations
\[
\Delta_\varepsilon = \Delta - \varepsilon^2 \Delta^2, \quad |\nabla| = \sqrt{-\Delta}, \quad \text{and} \quad I_\varepsilon = (1 - \varepsilon^2 \Delta)^{-1}.
\]
Following Kishimoto-Maeda \[23\], we define the operator \(\nabla^{-1}\) via the Helmholtz decomposition. The homogeneous Sobolev space \(H^{-\sigma}\) is defined by
\[
H^{-\sigma}(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{H^{-\sigma}} = \|\xi^{-\sigma} \hat{f}(\xi)\|_{L^2_\xi} < \infty \}.
\]
Let \(m\) and \(M\) be integers throughout the paper. For the sake of convenience, we define
\[
C([0, \infty); X \times Y \times Z) = C([0, \infty); X) \times C([0, \infty); Y) \times C([0, \infty); Z),
\]
where \(X, Y, Z\) are Sobolev spaces. From now on, we drop the parameter \(\varepsilon\) of the solutions \(E_{\varepsilon, \lambda}\) and \(n_{\varepsilon, \lambda}\). We also denote the space, for \(d = 1, 2, 3, \)
\[
X_{M,d} := H^M(\mathbb{R}^d) \times H^{M-1}(\mathbb{R}^d) \times (H^{M-3}(\mathbb{R}^d) \cap \dot{H}^{-1}(\mathbb{R}^d)). \tag{1.6}
\]
Our main results are as follows.

**Theorem 1.1** (Case \(d = 1\)). Let \(d = 1\) and \(M \geq 2\). Then for any \((E_0, n_0, n_1) \in X_{3M,1}\) and \(\lambda \in [1, \infty)\), there exist a unique solution to \((1.1)\) satisfying
\[
(E_\lambda, n_\lambda, \partial_t n_\lambda) \in C([0, \infty); X_{3M,1})
\]
and a unique solution to \((1.2)\) satisfying
\[
E_\infty \in C([0, \infty); H^{3M}(\mathbb{R})).
\]
(i) Assume \(n_0 + I_\varepsilon|E_0|^2 \neq 0\). Let \(m \geq 3\) and \(M \geq 3\) satisfy \(3M \geq m + 6\). If we further assume that \(E_0 \in H^{0,M}(\mathbb{R})\), \(x^j(n_0 + I_\varepsilon|E_0|^2) \in H^{m+j-3}\) with \(j = 0, 1, 2\), then for any \(T \in (0, \infty)\), \((E_\lambda, n_\lambda)\) satisfies
\[
\sup_{0 \leq t \leq T} \|E_\lambda(t) - E_\infty(t)\|_{H^m} \leq C \lambda^{-1} \tag{1.7}
\]
and
\[
\sup_{0 \leq t \leq T} \|n_\lambda(t) + I_\varepsilon|E_\lambda|^2(t) - Q_\lambda^{(0)}(t)\|_{H^m} \leq C \lambda^{-1} \tag{1.8}
\]
for any \(\lambda \in [1, \infty)\), where \(Q_\lambda^{(0)} = \cos(\lambda t \sqrt{-\Delta_\varepsilon})\{n_0 + I_\varepsilon|E_0|^2\}\) and the constant \(C\) depends on \(\varepsilon\) and \(T\), but independent of \(\lambda \in [1, \infty)\).
(ii) Assume \(n_0 + I_\varepsilon|E_0|^2 \equiv 0\). Let \(m \geq 6\) and \(M \geq 4\) satisfy \(3M \geq m + 5\). If we further assume that \(E_0 \in H^{0,M}(\mathbb{R})\) and \(x^j(\nabla^{-1}n_1 + 2 Im\{E_0 \nabla I_\varepsilon^{-1}E_0 +
\]
\[ \varepsilon^2 \nabla E_0 \Delta_{\varepsilon} E_0 \] \in H^{m+j-2} \) for \( j = 0, 1, 2 \), then for any \( T \in (0, \infty) \), \( (E_\lambda, n_\lambda) \) satisfies

\[
\sup_{0 \leq t \leq T} \| E_\lambda(t) - E_\infty(t) \|_{H^m} \leq C \lambda^{-2}
\]

for any \( \lambda \in [1, \infty) \).

**Theorem 1.2** (Case \( d = 2 \)). Let \( d = 2 \) and \( M \geq 2 \). Then for any \( (E_0, n_0, n_1) \in X_{3, M, 2} \) and \( \lambda \in [1, \infty) \), there exists a unique solution to (1.1) satisfying

\[
(E_\lambda, n_\lambda, \partial_t n_\lambda) \in C([0, \infty); X_{3, M, 2})
\]

and a unique solution to (1.5) satisfying

\[
E_\infty \in C([0, \infty); H^{3M}(\mathbb{R}^2))
\]

(i) Assume \( n_0 + I_\varepsilon |E_0|^2 \neq 0 \). Let \( m \geq 3 \) and \( M \geq 3 \) satisfy \( 3M \geq m + 6 \). If we further assume that \( E_0 \in H^{0, M} \), \( |x|^j (n_0 + I_\varepsilon |E_0|^2) \in H^{m+j-2} \) with \( j = 0, 1, 2 \), and \( n_0 + I_\varepsilon |E_0|^2 \in H^{-\sigma} \), with \( 0 < \sigma < 1 \), then for any \( T \in (0, \infty) \), \( (E_\lambda, n_\lambda) \) satisfies (1.7) and (1.8) for any \( \lambda \in [1, \infty) \), where the constant \( C \) depends on \( \varepsilon \) and \( T \).

(ii) Assume \( n_0 + I_\varepsilon |E_0|^2 \equiv 0 \). Let \( m \geq 5 \) and \( M \geq 4 \) satisfy \( 3M \geq m + 6 \). If we further assume that \( E_0 \in H^{0, M} \), \( |x|^j (n_0 + I_\varepsilon |E_0|^2) \in H^{m+j-2} \) with \( j = 0, 1, 2 \) and \( \nabla^{-1} n_1 + 2Im \{ E_0 \nabla I_\varepsilon^{-1} E_0 \} \in H^{-\sigma} \) with \( 0 < \sigma < 1 \), then for any \( T \in (0, \infty) \), \( (E_\lambda, n_\lambda) \) satisfies

\[
\sup_{0 \leq t \leq T} \| E_\lambda(t) - E_\infty(t) \|_{H^m} \leq C \lambda^{-2} \log \lambda
\]

for any \( \lambda \in [1, \infty) \), where the constant \( C \) depends on \( \varepsilon \) and \( T \).

**Theorem 1.3** (Case \( d = 3 \)). Let \( d = 3 \) and \( M \geq 2 \). Then for any \( (E_0, n_0, n_1) \in X_{3, M, 3} \) and \( \lambda \in [1, \infty) \), there exists a unique solution to (1.1) satisfying

\[
(E_\lambda, n_\lambda, \partial_t n_\lambda) \in C([0, \infty); X_{3, M, 3})
\]

and a unique solution to (1.5) satisfying

\[
E_\infty \in C([0, \infty); H^{3M}(\mathbb{R}^3))
\]

(i) Assume \( n_0 + I_\varepsilon |E_0|^2 \neq 0 \). Let \( m \geq 3 \) and \( M \geq 3 \) satisfy \( 3M \geq m + 6 \). If we further assume that \( E_0 \in H^{0, M} \), \( |x|^j (n_0 + I_\varepsilon |E_0|^2) \in H^{m+j-2} \) with \( j = 0, 1, 2 \), then for any \( T \in (0, \infty) \), \( (E_\lambda, n_\lambda) \) satisfies (1.7) and (1.8) for any \( \lambda \in [1, \infty) \), where the constant \( C \) depends on \( \varepsilon \) and \( T \).

(ii) Assume \( n_0 + I_\varepsilon |E_0|^2 \equiv 0 \). Let \( m \geq 5 \) and \( M \geq 4 \) satisfy \( 3M \geq m + 6 \). If we further assume that \( E_0 \in H^{0, M} \), \( |x|^j (\nabla^{-1} n_1 + 2Im \{ E_0 \nabla I_\varepsilon^{-1} E_0 \} \in H^{-\sigma} \) with \( 0 < \sigma < 1 \), then for any \( T \in (0, \infty) \), \( (E_\lambda, n_\lambda) \) satisfies

\[
\sup_{0 \leq t \leq T} \| E_\lambda(t) - E_\infty(t) \|_{H^m} \leq C \lambda^{-2}
\]

for any \( \lambda \in [1, \infty) \), where the constant \( C \) depends on \( \varepsilon \) and \( T \).
Remark 1.1. The function $Q_{\varepsilon,\lambda}^{(0)}$ is solution to the fourth order wave equation
\[
\begin{cases}
\lambda^{-2} \partial_t^2 n - \Delta_{\varepsilon} n = 0 & t \in \mathbb{R}, \; x \in \mathbb{R}^d, \\
n(0, x) = n_0 + I_{\varepsilon}|E_0|^2, \; \partial_t n(0, x) = 0 & x \in \mathbb{R}^d.
\end{cases}
\]

Theorems 1.1, 1.2 and 1.3 tell us that the term $Q_{\varepsilon,\lambda}^{(0)}$ represents the initial layer for (1.1). Note that $Q_{0,\lambda}^{(0)}$ coincides with the initial layer for the Zakharov system (1.2), see [1, 34, 30], with several modifications.

Let us give an outline of the proofs for Theorems 1.1, 1.2 and 1.3. The proofs follow from the arguments due to Ozawa-Tsutsumi [30] and Ukai [35].

From (1.11) and (1.5), the difference $E_{\lambda} - E_{\infty}$ satisfies
\[
\begin{cases}
i \partial_t (E_{\lambda} - E_{\infty}) + \Delta_{\varepsilon} (E_{\lambda} - E_{\infty}) \\
= \{-I_{\varepsilon}|E_{\lambda}|^2\} E_{\lambda} + \{I_{\varepsilon}|E_{\infty}|^2\} E_{\infty} + Q_{\lambda} E_{\lambda},
\end{cases}
\]
where $Q_{\lambda} = n_{\lambda} + I_{\varepsilon}|E_{\lambda}|^2$. To evaluate $E_{\lambda} - E_{\infty}$, we rewrite (1.12) into the integral equation
\[
E_{\lambda}(t) - E_{\infty}(t) = \int_0^t iU_{\varepsilon}(t-s) \left( \left\{ \{I_{\varepsilon}|E_{\lambda}|^2\} E_{\lambda} - \{I_{\varepsilon}|E_{\infty}|^2\} E_{\infty} \right\} - (Q_{\lambda} E_{\lambda}) \right) ds,
\]
where $U_{\varepsilon}(t) = \exp(it\Delta_{\varepsilon})$ is an $L^2$-unitary group generated by the differential operator $i\Delta_{\varepsilon}$. It is easy to evaluate the first term in the integral of (1.13) in $H_{x}^m$. To estimate the second term in the integral of (1.13) in $H_{x}^m$, we derive the relevant equation for $Q_{\lambda}$:
\[
\begin{cases}
\lambda^{-2} \partial_t^2 Q_{\lambda} - \Delta_{\varepsilon} Q_{\lambda} = \lambda^{-2} \partial_t^2 I_{\varepsilon}|E_{\lambda}|^2 & t \in \mathbb{R}, \; x \in \mathbb{R}^d, \\
Q_{\lambda}(0, x) = n_0(x) + I_{\varepsilon}|E_0|^2(x) & x \in \mathbb{R}^d, \\
\partial_t Q_{\lambda}(0, x) = n_1(x) + 2Im\{E_0 \Delta_{\varepsilon} \overline{E_0}\} & x \in \mathbb{R}^d,
\end{cases}
\]
which implies that $Q_{\lambda}(t)$ can be written as
\[
\cos(\lambda t \sqrt{-\Delta_{\varepsilon}}) \{n_0 + I_{\varepsilon}|E_0|^2\} + \frac{\sin(\lambda t \sqrt{-\Delta_{\varepsilon}})}{\lambda \sqrt{-\Delta_{\varepsilon}}} \{n_1 + 2Im\{E_0 \Delta_{\varepsilon} \overline{E_0}\}\}
+ \int_0^t \frac{\sin(\lambda(t-s)\sqrt{-\Delta_{\varepsilon}})}{\lambda \sqrt{-\Delta_{\varepsilon}}} \partial_s^2 I_{\varepsilon}|E_{\lambda}|^2(s) ds.
\]

The difficulty to evaluate the first term of (1.15) comes from the lack of the explicit representation for the unitary group $\cos(\lambda t \sqrt{-\Delta_{\varepsilon}})$. For the classical Zakharov system (1.2), the explicit formula is utilized for the unitary group $\cos(\lambda t \sqrt{-\Delta})$ and the Schrödinger part is localized near the origin, see [30].

To overcome this difficulty, we employ the method of stationary phase. The key point is that the wave part of (1.1) has no stationary point near the origin. Therefore the wave part of (1.1) decays faster than $(\lambda t)^{-1}$ as $\lambda \to \infty$, which guarantees that the interaction between $Q_{\lambda}$ and $E_{\lambda}$ are weak in the sense that the first term in (1.15) decays like $\lambda^{-1}$ in $H_{x}^m$ as $\lambda \to \infty$. 

The plan of this paper is as follows. In Section 2, we prove the solvability and uniform estimates for the solutions to (1.1) and (1.5). In Section 3, we prove Theorems 1.1, 1.2 and 1.3 and finally in Section 4, we derive the interaction estimate between $Q_{\lambda}$ and $E_{\lambda}$.

Throughout this paper, we use the notation $A \sim B$ to represent $C_1 A \leq B \leq C_2 A$ for some constants $C_1$ and $C_2$. We also use the notation $A \lessapprox B$ to denote $A \leq CB$ for some constant $C$.

2. Preliminaries

In this section we consider the solvability and uniform estimates for the solutions to (1.1) and (1.5). For the quantum Zakharov system (1.1), we have

**Proposition 2.1.** Let $d = 1, 2, 3$ and $M \geq 2$. Then for any $(E_0, n_0, n_1) \in X_{M,d}$ and $\lambda \in [1, \infty)$, there exists a unique solution to (1.1) satisfying

$$\begin{align*}
(E_{\lambda}, n_{\lambda}, \partial_{t} n_{\lambda}) & \in C([0, \infty) ; X_{M,d}).
\end{align*}$$

Furthermore, for any $T \in (0, \infty)$ and some $C > 0$, $(E_{\lambda}, n_{\lambda})$ satisfies

$$\begin{align*}
sup_{0 \leq t \leq T} (\|E_{\lambda}(t)\|_{H^M_x} + \|n_{\lambda}(t)\|_{H^{M-1}_x}) & \leq C \tag{2.1}
\end{align*}$$

for any $\lambda \in [1, \infty)$.

We need the following lemmas to show Proposition 2.1.

**Lemma 2.1 (Gagliardo-Nirenberg’s inequality).** Let $2 \leq p \leq \infty$ and $j, k \in \mathbb{Z}_+$ satisfy $	heta := \frac{d}{k} \left(\frac{1}{2} - \frac{1}{p} + \frac{j}{d}\right) \in (0, 1)$. Then the inequality

$$\|\nabla^j f\|_{L^p_x} \leq C \|f\|_{L^2_x}^{1-\theta} \|\nabla^k f\|_{L^2_x}^\theta$$

holds for any $f \in H^k(\mathbb{R}^d)$.

The next lemma is due to Brézis-Gallouët [5] which is needed to prove Proposition 2.1 for $d = 2$.

**Lemma 2.2 (Brézis-Gallouët inequality).** We have the following inequalities.

(i) Let $f \in H^2(\mathbb{R}^2)$. Then we have $f \in L^\infty(\mathbb{R}^2)$ and

$$\|f\|_{L^\infty_2} \leq C(\|f\|_{H^1_2} \sqrt{\log(e + \|\nabla^2 f\|_{L^2_2})} + 1).$$

(ii) Let $f \in H^1(\mathbb{R}^2)$. Then we have $f \in L^4(\mathbb{R}^2)$ and

$$\|f\|_{L^4_2} \leq C(\|f\|_{L^2_2}^{1/2} \|\nabla f\|_{L^2_2}^{1/2} \sqrt{\log(e + \|\nabla f\|_{L^2_2})} + 1).$$

**Proof of Lemma 2.2** See [5] Lemma 2. □

To prove Proposition 2.1 for $d = 3$, we employ the Strichartz estimates for the Schrödinger equations and the fourth-order Schrödinger equations

$$\begin{align*}
\begin{cases}
i \partial_t E - \Delta E + \Delta^2 E = h(t, x) & \text{for } t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
E(0, x) = E_0(x) & \text{for } t \in \mathbb{R}, \ x \in \mathbb{R}^3.
\end{cases}
\end{align*} \tag{2.2}$$
A pair \((q, r)\) is called (3 dimensional) Schrödinger admissible, if \(2 \leq q, r \leq \infty\), and
\[
\frac{2}{q} + \frac{3}{r} = \frac{3}{2}. \tag{2.3}
\]
A pair \((q, r)\) is called (3 dimensional) biharmonic admissible, if \(2 \leq q, r \leq \infty\), and
\[
\frac{4}{q} + \frac{3}{r} = \frac{3}{2}. \tag{2.4}
\]
We recall some known results.

**Lemma 2.3.** \((\text{Pausader} [32])\) Let \(E \in C([0, T], H^{-4}(\mathbb{R}^3))\) be a solution of \((2.2)\). For any biharmonic pairs \((q, r)\) and \((\tilde{q}, \tilde{r})\), it satisfies
\[
\|E\|_{L^q_t([0,T];L^r_x(\mathbb{R}^3))} \leq C \left( \|E_0\|_{L^q_x(\mathbb{R}^3)} + \|h\|_{L^r_t([0,T];L^q_x(\mathbb{R}^3))} \right), \tag{2.5}
\]
where \(C\) depends only on \(\tilde{q}'\), and \(\tilde{r}'\). Besides, for any Schrödinger admissible pairs \((q, r)\) and \((a, b)\), and any \(s \geq 0\), we have
\[
\||\nabla|^s E\|_{L^q_t([0,T];L^r_x(\mathbb{R}^3))} \leq C \left( \||\nabla|^{a-s} E_0\|_{L^q_x(\mathbb{R}^3)} + \||\nabla|^{b-s} h\|_{L^q_t([0,T];L^r_x(\mathbb{R}^3))} \right), \tag{2.6}
\]
where \(C\) depends only on \(a'\), and \(b'\).

**Proof of Proposition 2.1.** We prove this proposition by the induction argument on \(M\). For the simplicity, we abbreviate \((E_\lambda, n_\lambda)\) to \((E, n)\). By the density \(C^\infty_0(\mathbb{R}^d) \hookrightarrow H^M(\mathbb{R}^d)\), we may assume that \((E, n)\) is smooth. The global existence and uniqueness of the solution to \((1.1)\) is proved by Guo-Zhang-Guo [14, Theorem 1.1]. Hence we derive the uniform bound for solution \((2.1)\).

We first derive the \(L^2\) bound of \(E\). Taking the imaginary part of the inner product in \(L^2_x\) between the first equation of \((1.1)\) and \(E\), we have
\[
\frac{d}{dt}\|E(t)\|_{L^2} = 0. \tag{2.7}
\]
for any \(t \in (0, \infty)\).

Next we derive the \(H^2 \times H^1\) bound for \((E, n)\). Taking the real part of the inner product in \(L^2_x\) between the first equation of \((1.1)\) and \(\partial_t E\), we obtain
\[
\frac{d}{dt}(\|\nabla E(t)\|^2_{L^2} + \varepsilon^2\|\Delta E(t)\|^2_{L^2}) = -2\text{Re} \int_{\mathbb{R}^d} n\overline{E}\partial_t E dx. \tag{2.8}
\]
Applying the operator \(\nabla^{-1}\) to the second equation of \((1.1)\) and taking the inner product in \(L^2_x\) between the resultant equation and \(\nabla^{-1}\partial_t n\), we have
\[
\frac{d}{dt} \left( \frac{1}{2} \lambda^{-2}\|\partial_t \nabla^{-1} n(t)\|^2_{L^2} + \frac{1}{2}\|n(t)\|^2_{L^2} + \frac{\varepsilon^2}{2}\|\nabla n(t)\|^2_{L^2} \right) = -\text{Re} \int_{\mathbb{R}^d} \partial_t n|E|^2 dx. \tag{2.9}
\]
Let
\[
H_2(E, n)(t) = \|\nabla E(t)\|^2_{L^2} + \varepsilon^2\|\Delta E(t)\|^2_{L^2} + \frac{1}{2} \lambda^{-2}\|\partial_t \nabla^{-1} n(t)\|^2_{L^2} + \frac{1}{2}\|n(t)\|^2_{L^2} + \frac{\varepsilon^2}{2}\|\nabla n(t)\|^2_{L^2} + \int_{\mathbb{R}^d} n|E|^2 dx + C_2,
\]
where \(C_2\) is a constant.
where $C_2 = C'_2 \| E_0 \|_{L^2_x}^{\frac{16 - 2d}{2}}$. The positive constant $C'_2$ is chosen so that the inequality
\[
\| E(t) \|_{H^2_x}^2 + \| n(t) \|_{H^1_x}^2 \leq C H_2(E, n)(t)
\]
holds for some constant $C$ independent of $\lambda \in [1, \infty)$. Indeed, Lemma 2.1 and the Young inequality yield
\[
\int_{\mathbb{R}^d} n |E|^2 \, dx \leq C' \| n \|_{L^2_x} \| E \|_{L^\infty_x}^{2-\frac{d}{2}} \| \Delta E \|_{L^2_x}^{\frac{d}{2}}
\]
\[
\leq \frac{\varepsilon^2}{2} \| \Delta E \|_{L^2_x}^2 + \frac{1}{4} \| n \|_{L^2_x}^2 + C' \| E_0 \|_{L^2_x}^{\frac{16 - 2d}{2}}.
\]
Choosing $C'_2 = C' + 1$, we have
\[
\| E(t) \|_{H^2_x}^2 + \| n(t) \|_{H^1_x}^2 \leq \frac{4}{\varepsilon^2} H_2(E, n)(t).
\]
From (2.7), (2.8) and (2.9), we get $\frac{d}{dt} H_2(E, n)(t) = 0$. Hence
\[
H_2(E, n)(t) = H_2(E, n)(0).
\]
Furthermore from (2.11), we find
\[
H_2(E, n)(0) \leq C(\| E_0 \|_{L^2_x}^{\frac{16 - 2d}{2}} + \| E_0 \|_{H^2_x}^2 + \| n_0 \|_{H^1_x}^2 + \| n_1 \|_{H^{-1}_x}^2),
\]
where the constant $C$ is independent of $\lambda \in [1, \infty)$. Combining (2.10), (2.12) and (2.13), we have
\[
\| E(t) \|_{H^2_x}^2 + \| n(t) \|_{H^1_x}^2 \leq C(\| E_0 \|_{L^2_x}^{\frac{16 - 2d}{2}} + \| E_0 \|_{H^2_x}^2 + \| n_0 \|_{H^1_x}^2 + \| n_1 \|_{H^{-1}_x}^2)
\]
for any $t \in [0, \infty)$. This proves $H^2 \times H^1$ bound for $(E, n)$.

Finally we derive that for any integers $M \geq 3$, $H^{M-1} \times H^{M-2}$ bound for $(E, n)$ implies $H^M \times H^{M-1}$ bound for $(E, n)$. Now we assume that
\[
\sup_{0 \leq t \leq T} (\| E(t) \|_{H^{M-1}_x} + \| n(t) \|_{H^{M-2}_x}) \leq C.
\]
Applying $\nabla^{M-2}$ to the first equation of (1.1) and taking the real part of the inner product in $L^2_x$ between the resulting equation and $\partial_t \nabla^{M-2} E$, we obtain
\[
\frac{d}{dt} \left( \| \nabla^{M-1} E(t) \|_{L^2_x}^2 + \varepsilon^2 \| \nabla^M E(t) \|_{L^2_x}^2 \right)
\]
\[
= -2 \Re \int_{\mathbb{R}^d} \nabla^{M-2}(n \overline{E}) \partial_t \nabla^{M-2} E \, dx.
\]
Similarly, applying $\nabla^{M-3}$ to the second equation of (1.1) and taking the inner product in $L^2_x$ between the resulting equation and $\partial_t \nabla^{M-3} n$, we find
\[
\frac{d}{dt} \left( \frac{1}{2} \lambda^{-2} \| \partial_t \nabla^{M-3} n(t) \|_{L^2_x}^2 + \frac{1}{2} \| \nabla^{M-2} n(t) \|_{L^2_x}^2 + \frac{\varepsilon^2}{2} \| \nabla^{M-1} n(t) \|_{L^2_x}^2 \right)
\]
\[
= -\int_{\mathbb{R}^d} \nabla^{M-2} |E|^2 \partial_t \nabla^{M-2} n \, dx.
\]
Define
\[ H_M(E, n)(t) = \|\nabla^{M-1}E(t)\|_{L^2}^2 + \varepsilon^2\|\nabla^M E(t)\|_{L^2}^2 + \frac{1}{2}\lambda^{-2}\|\partial_t\nabla^{M-3}n(t)\|_{L^2}^2 \\
+ \frac{1}{2}\|\nabla^{M-2}n(t)\|_{L^2}^2 + \frac{\varepsilon^2}{2}\|\nabla^{M-1}n(t)\|_{L^2}^2 \\
+ \int_{\mathbb{R}^d} \nabla^{M-2}n\nabla^{M-2}|E|^2 \, dx + C_M. \]  
(2.17)

In fact, the positive constant \( C_M \) is chosen in a way such that \( H_M(E, n)(t) \) satisfies
\[ \|\nabla^M E(t)\|_{L^2}^2 + \|\nabla^{M-1}n(t)\|_{L^2}^2 + 1 \leq CH_M(E, n)(t) \]
for any \( t \in [0, T) \), where \( C \) is independent of \( \lambda \in [1, \infty) \). Indeed, by the Sobolev embedding \( H^1 \hookrightarrow L^4 \), we have
\[ \int_{\mathbb{R}^d} \nabla^{M-2}n\nabla^{M-2}|E|^2 \, dx \leq C'\|\nabla^{M-2}n\|_{L^2}^2 \|E\|_{H^{M-1}}^2, \]
and hence it suffices to choose \( C_M = C'\sup_{0 \leq t \leq T} \|n(t)\|_{H^{M-2}} \sup_{0 \leq t \leq T} \|E(t)\|_{H^{M-1}}^2 + 1 \). Note that \( C_M \) is independent of \( \lambda \in [1, \infty) \).

From (2.15) and (2.10), we obtain
\[ \frac{d}{dt} H_M(E, n)(t) \]
\[ = \int_{\mathbb{R}^d} \nabla^{M-2}\partial_t|E|^2\nabla^{M-2}n \, dx - 2\text{Re} \int_{\mathbb{R}^d} \nabla^{M-2}(n\overline{E})\partial_t\nabla^{M-2}E \, dx. \]

From (1.1) we see \( \partial_t|E|^2 = -2Im(\overline{E}\Delta E) + 2\varepsilon^2 Im(\overline{E}\Delta^2 E) \). Hence we have
\[ \frac{d}{dt} H_M(E, n)(t) = I_1 + I_2 + I_3 + I_4, \]  
(2.18)

where
\[ I_1 = -2Im \int_{\mathbb{R}^d} \nabla^{M-2}(\overline{E}\Delta E)\nabla^{M-2}n \, dx, \]
\[ I_2 = 2\varepsilon^2 Im \int_{\mathbb{R}^d} \nabla^{M-2}(\overline{E}\Delta^2 E)\nabla^{M-2}n \, dx, \]
\[ I_3 = +2Im \int_{\mathbb{R}^d} \nabla^{M-2}(n\overline{E})\nabla^M E \, dx, \]
\[ I_4 = -2\varepsilon^2 Im \int_{\mathbb{R}^d} \nabla^{M-2}(n\overline{E})\nabla^{M+2} E \, dx. \]  
(2.19)

For \( I_1 \) and \( I_3 \), we use the Sobolev inequality and the induction hypothesis (2.13) to obtain
\[ |I_1| \leq C\|E(t)\|_{H^M}^2 \|\nabla^{M-2}n(t)\|_{L^2} \]
\[ \leq C(\|\nabla^M E(t)\|_{L^2}^2 + 1), \]  
(2.20)
\[ |I_3| \leq \|n(t)\|_{H^{M-2}} \|E(t)\|_{W^{M-2,\infty}} \|\nabla^M E(t)\|_{L^2} \]
\[ \leq C(\|\nabla^M E(t)\|_{L^2}^2 + 1)\|\nabla^M E(t)\|_{L^2}. \]  
(2.21)
An integration by parts leads

\[ I_2 = -2\varepsilon^2 Im \int_{\mathbb{R}^d} E \nabla^{M+1} E \nabla^{M-1} n dx \]

\[-2\varepsilon^2 1_{M \geq 4}(M) \sum_{j=0}^{M-4} \binom{M-3}{j} Im \int_{\mathbb{R}^d} \nabla^{M-3-j} E \nabla^j E \nabla^{M-1} n dx,\]

where \(1_{M \geq 4}(M) = 0\) for \(M \leq 3\) and \(1_{M \geq 4}(M) = 1\) for \(M \geq 4\). Hence

\[ |I_2 + 2\varepsilon^2 Im \int_{\mathbb{R}^d} E \nabla^{M+1} E \nabla^{M-1} n dx| \]

\[ \leq C \|E(t)\|_{W^{M-3,\infty}} \|E(t)\|_{H^M} \|\nabla^{M-1} n\|_{L^2} \]

\[ \leq C(\|\nabla^M E(t)\|_{L^2} + 1)\|\nabla^{M-1} n\|_{L^2} \quad (2.22)\]

Again an integration by parts yields

\[ I_4 = 2\varepsilon^2 Im \int_{\mathbb{R}^d} E \nabla^{M-1} n \nabla^{M+1} E dx \]

\[-2\varepsilon^2 \sum_{j=1}^{M-1} \binom{M-1}{j} Im \int_{\mathbb{R}^d} \nabla^{M-j} n \nabla^j \nabla^{M} E dx \]

\[-2\varepsilon^2 \sum_{j=1}^{M-1} \binom{M-1}{j} Im \int_{\mathbb{R}^d} \nabla^{M-j} n \nabla^{j+1} \nabla^{M} E dx. \quad (2.23)\]

Now we discuss the cases \(d = 1, 2, 3\) separately.

Case: \(d = 1\). By the embedding \(H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\), we find

\[ |I_4 - 2\varepsilon^2 Im \int_{\mathbb{R}^d} E \nabla^{M-1} n \nabla^{M+1} E dx| \]

\[ \leq C \sum_{j=1}^{2} \|\nabla^{M-j} n(t)\|_{L^2} \|\nabla^j E(t)\|_{L^\infty} \|\nabla^M E(t)\|_{L^2} \]

\[ + C \sum_{j=3}^{M} \|\nabla^{M-j} n(t)\|_{L^\infty} \|\nabla^j E(t)\|_{L^2} \|\nabla^M E(t)\|_{L^2} \]

\[ \leq C(1 + \|\nabla^M E(t)\|_{L^2} + \|\nabla^{M-1} n(t)\|_{L^2})\|\nabla^M E(t)\|_{L^2}. \]

Combining the above inequality, (2.20), (2.21) and (2.22), we have

\[ \frac{d}{dt} H_M(E, n)(t) \leq CH_M(E, n)(t). \]

The Gronwall lemma yields

\[ H_M(E, n)(t) \leq CH_M(E, n)(0)e^{Ct}. \]

Since

\[ \|\nabla^M E_0\|_{L^2}^2 + \|\nabla^{M-1} n_0\|_{L^2}^2 \leq H_M(E, n)(0) \]

\[ \leq C(\|E_0\|_{H^M} + \|n_0\|_{H^{M-1}} + \|n_1\|_{H^{M-3}} + \|E_0\|_{H^{M-1}} n_0\|_{H^{M-2}} + 1), \]

(2.24)

we have \(H^M \times H^{M-1}\) bound for \((E, n)\) with \(d = 1\).
Case: $d = 2$. By the Brézis-Gallouët inequality (Lemma 2.2), we have
\[
\left| I_4 - 2 \varepsilon^2 \partial_{x} \int_{\mathbb{R}^d} \sum_{E} n \left| \nabla E \right|^2 dx \right|
\leq C \| n(t) \|_{L^2}^2 \| E(t) \|_{L^2} + \left\{ \left( \| E(t) \|_{L^2} \right)^2 + 1 \right\}^{1/2} + C \| E(t) \|_{H^1} \left\{ \| \nabla^2 n(t) \|_{L^2} + 1 \right\}
\times \left\{ \| \nabla^2 E(t) \|_{L^2} + 1 \right\}.
\]
Combining the above inequality, (2.20), (2.21) and (2.22), we have
\[
\frac{d}{dt} H_M(E, n)(t) \leq C H_M(E, n)(t) \log(H_M(E, n)).
\]
The Gronwall lemma yields
\[
H_M(E, n)(t) \leq C(H_M(E, n)(0) e^{\varepsilon t}.
\]
By the above inequality and (2.24), we have $H^M \times H^{M-1}$ bound for $(E, n)$ with $d = 2$.

Case: $d = 3$. The conservation of mass and conservation of Hamiltonian (1.3) imply that
\[
\sup_{0 \leq t \leq T} (\| E(t) \|_{H^2} + \| \mathcal{N}(t) \|_{H^2}) \leq C,
\]
where
\[
\mathcal{N} = n + \frac{i}{\sqrt{(1 - \varepsilon^2 \Delta)(-\Delta)}} \frac{\partial_x n}{\lambda}.
\]
We first prove that
\[
\| \nabla E \|_{L_t^6 \cap L_x^\infty([0, T])} + \| \mathcal{N} \|_{L_t^2 \cap L_x^6([0, T])} \leq C,
\]
then we show that
\[
\| \nabla^2 E \|_{L_t^8 \cap L_x^6([0, T])} + \| \nabla \mathcal{N} \|_{L_t^6 \cap L_x^6([0, T])} \leq C.
\]
The Strichartz estimates (2.20) and (2.25) give that
\[
\| \nabla^2 E \|_{L_t^6 \cap L_x^6([0, T])} \lesssim \| E_0 \|_{H^2} + \| n E \|_{L_t^2 \cap L_x^{6/5}} \lesssim \| E_0 \|_{H^2} + T^{1/2} \| n \|_{L_t^6 \cap L_x^{6/5}} \| E \|_{L_t^6 \cap L_x^{6/5}} \lesssim T^{1/2}.
\]
From inequalities (2.23) and (2.25) we have
\[
\| E \|_{L_t^2 \cap L_x^6([0, T])} \lesssim T^{1/4} \| E \|_{L_t^4 \cap L_x^6([0, T])} \lesssim T^{1/4} \left( \| E_0 \|_{H^2} + \| n E \|_{L_t^{1/3} \cap L_x^{6/5}} \right) \lesssim T.
\]
Hence we obtain
\[
\| \nabla E \|_{L_t^2 \cap L_x^6([0, T])} \lesssim \| \nabla E \|_{L_t^2 \cap L_x^6([0, T])} \lesssim \| \nabla^2 E \|_{L_t^2 \cap L_x^6([0, T])} + \| \nabla^2 E \|_{L_t^2 \cap L_x^6([0, T])} \lesssim T.
\]
Therefore we obtain (2.20) for $E$. 

Next we estimate $\mathcal{N}$. Notice that $\mathcal{N}$ satisfies
\begin{equation}
    i\partial_t\mathcal{N} - \lambda\sqrt{I_\varepsilon^{-1}(-\Delta)}\mathcal{N} = \lambda\sqrt{I_\varepsilon(-\Delta)}|E|^2.
\end{equation}
(2.31)
To apply the Strichartz estimates to (2.31) we need the following proposition.

**Proposition 2.2.** Suppose that $(q, r)$ and $(\tilde{q}, \tilde{r})$ are Schrödinger admissible. If $u$ is a solution of
\begin{equation}
\begin{aligned}
    &\left\{ \begin{array}{ll}
        i\partial_t\mathcal{N} + \sqrt{I_\varepsilon^{-1}(-\Delta)}\mathcal{N} = h(t,x) & t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
        \mathcal{N}(0,x) = \mathcal{N}_0(x) & t \in \mathbb{R}, \ x \in \mathbb{R}^3
    \end{array} \right.
\end{aligned}
\end{equation}
for some data $u_0$, $h$ and time $0 < T < \infty$, then
\begin{equation}
    \|\mathcal{N}\|_{L_t^q(0,T);L_x^r} \lesssim \|\mathcal{N}_0\|_{L_x^q} + \|h\|_{L_t^q(0,T);L_x^r}.
\end{equation}
(2.33)

**Proof of Proposition 2.3.** Since $\sqrt{I_\varepsilon^{-1}(-\Delta)} \sim -\Delta$ in $L^2$, the proof for the Proposition 2.2 is analogous to that of the Strichartz estimate for the usual Schrödinger equation, see [22, Theorem 3.1] for instance. \hfill \square

**Proposition 2.3.** Let $\mathcal{N}$ be a solution of
\begin{equation}
\begin{aligned}
    &\left\{ \begin{array}{ll}
        i\partial_t\mathcal{N} + \lambda\sqrt{I_\varepsilon^{-1}(-\Delta)}\mathcal{N} = \lambda h(t,x) & t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
        \mathcal{N}(0,x) = \mathcal{N}_0(x) & t \in \mathbb{R}, \ x \in \mathbb{R}^3
    \end{array} \right.
\end{aligned}
\end{equation}
Then we have
\begin{equation}
    \|\mathcal{N}\|_{L_t^2H_{6/5}^6} \lesssim \lambda^{-1/2}\|\mathcal{N}_0\|_{H^s} + \|h\|_{L_t^2H_{6/5}^6}. \tag{2.35}
\end{equation}

**Proof of Proposition 2.3.** Set $\mathcal{N}(t) = \tilde{\mathcal{N}}(\lambda t)$ and $\tilde{h}(t) = h(\lambda t)$. Then the function $\tilde{\mathcal{N}}$ satisfies the equation (2.32). Invoking the Strichartz estimate (2.33), we obtain
\begin{equation}
    \|\tilde{\mathcal{N}}\|_{L_t^2H_{6/5}^6} \lesssim \|\mathcal{N}_0\|_{H^s} + \|\tilde{h}\|_{L_t^2H_{6/5}^6}. \tag{2.36}
\end{equation}
Thus we have (2.35) for $\mathcal{N}$. \hfill \square

For (2.31), we invoke the Strichartz estimates (Proposition 2.3) so that we can get
\begin{equation}
    \|\mathcal{N}\|_{L_t^2H_{6/5}^6} \lesssim \lambda^{-1/2}\|\mathcal{N}_0\|_{H^2} + \|\sqrt{I_\varepsilon(-\Delta)}|E|^2\|_{L_t^2H_{6/5}^6}
\end{equation}
\begin{equation}
\begin{aligned}
    &\lesssim \lambda^{-1/2}\|\mathcal{N}_0\|_{H^2} + \|\Delta E\bar{E}\|_{L_t^6L_x^{9/5}} + \|\nabla E\cdot \nabla \bar{E}\|_{L_t^2L_x^{3/5}} \tag{2.37}
\end{aligned}
\end{equation}
\begin{equation}
    \lesssim \lambda^{-1/2}\|\mathcal{N}_0\|_{H^2} + T^{1/2}\|E\|_{L_t^\infty L_x^2}^2.
\end{equation}
Notice that $\sqrt{I_\varepsilon(-\Delta)}$ is a bounded operator and $\sqrt{I_\varepsilon^{-1}(-\Delta)} \sim -\Delta$. Hence by (2.37), we obtain
\begin{equation}
    \|\mathcal{N}\|_{L_t^2[0,T];L_x^\infty} \lesssim \|\mathcal{N}\|_{L_t^2[0,T]H_{6/5}^6} \lesssim T^{1/2}. \tag{2.38}
\end{equation}

Thus we have proved (2.20) for $\mathcal{N}$.

To obtain (2.27), we interpolate between (2.25) and (2.28) so that we get
\begin{equation}
    \|\nabla^2 E\|_{L_t^{3/2}[0,T]L_x^2} \lesssim \|\nabla^2 E\|_{L_t^2[0,T]L_x^2}^{3/4} \|\nabla^2 E\|_{L_t^1[0,T]L_x^2}^{1/4} \lesssim T^{3/8}. \tag{2.39}
\end{equation}
Also we interpolate between (2.25) and (2.37) to derive

\[ \|\nabla N\|_{L^{s/3}_{t}[0,T]L^{4}_{x}} \lesssim \|\nabla N\|_{L^{3/4}_{t}[0,T]L^{5}_{x}}^{3/4} \|\nabla N\|_{L^{5}_{t}[0,T]L^{4}_{x}}^{1/4} \lesssim T^{-3/8}. \] (2.40)

Hence we obtain (2.24),

By (2.20), (2.21), (2.22) and (2.23), we obtain

\[ \frac{d}{dt} H^{1/2}_{3}(E, n)(t) \]
\[ \leq -4\varepsilon^{2} \int \nabla^{2} n \nabla E \nabla^{3} E dx - 6\varepsilon^{2} \int \nabla n \nabla^{2} E \nabla^{3} E dx + C H_{3}(E, n)(t) \]
\[ \leq 4\varepsilon^{2} \|\nabla E(t)\|_{L^{\infty}_{x}} H_{3}(E, n)(t) + 6\varepsilon^{2} \|\nabla n(t)\|_{L^{1}_{t}} \|\nabla^{2} E(t)\|_{L^{1}_{t}} H^{1/2}_{3}(E, n)(t) + C H_{3}(E, n)(t). \]

Hence

\[ \frac{d}{dt} H^{1/2}_{3}(E, n)(t) \]
\[ \leq (8\varepsilon^{2} \|\nabla E(t)\|_{L^{\infty}_{x}} + C) H^{1/2}_{3}(E, n)(t) + 12\varepsilon^{2} \|\nabla n(t)\|_{L^{1}_{t}} \|\nabla^{2} E(t)\|_{L^{1}_{t}}. \]

The Gronwall lemma yields

\[ H^{1/2}_{3}(E, n)(t) \]
\[ \leq H^{1/2}_{3}(E, n)(0) \exp(8\varepsilon^{2} \int_{0}^{t} \|\nabla E(\tau)\|_{L^{\infty}_{x}} d\tau + Ct) \]
\[ + 12\varepsilon^{2} \int_{0}^{t} \|\nabla n(\tau)\|_{L^{1}_{t}} \|\nabla^{2} E(\tau)\|_{L^{1}_{t}} \]
\[ \times \exp(8\varepsilon^{2} \int_{\tau}^{t} \|\nabla E(\sigma)\|_{L^{\infty}_{x}} d\sigma + C(t - \tau)) d\tau \]
\[ \leq H^{1/2}_{3}(E, n)(0) \exp(C t^{1/2} \|\nabla E\|_{L^{2}_{t}L^{\infty}_{x}} + Ct) \]
\[ + 12\varepsilon^{2} t^{1/4} \|\nabla n\|_{L^{1}_{t}L^{4}_{x}}^{3/4} \|\nabla^{2} E\|_{L^{1}_{t}L^{4}_{x}}^{1/4} \exp(8\varepsilon^{2} t^{3/4} \|\nabla E\|_{L^{1}_{t}L^{\infty}_{x}} + Ct). \]

By (2.26) and (2.27), we have

\[ H^{1/2}_{3}(E, n)(t) \leq H^{1/2}_{3}(E, n)(0) \exp(C t^{3/2}) + C \varepsilon^{2} t \exp(C t^{3/2}). \]

Combining the above inequality and (2.21), we have

\[ \sup_{0 \leq t \leq T} (\|\nabla^{3} E\|_{L^{2}_{t}} + \|\nabla^{2} n\|_{L^{2}_{t}}) \leq C \]
(2.41)

for any \( T \in (0, \infty) \) and \( \lambda \in [1, \infty) \). Further, combining (2.20), (2.21), (2.22), (2.23) and (2.41), we have for \( M \geq 4 \),

\[ \frac{d}{dt} H_{M}(E, n)(t) \leq C H_{M}(E, n)(t). \]

The Gronwall lemma and (2.24) yields \( H^{M} \times H^{M-1} \) bound (2.1) for \( M \geq 4 \). This completes the proof of Proposition 2.1 \( \square \)

**Proposition 2.4.** Let \( d = 1, 2, 3 \) and \( M \geq 2 \). Let \( T \) be given in Proposition 2.7. Then for any \((E_{0}, n_{0}, n_{1}) \in X_{M, d}\) and \( E_{0} \in H^{0, M}(\mathbb{R}^{d})\), the solution to
imaginary part of the inner product in \(L^2\), \(K\) hold for (inequality (2.43) with \((L, K)\)) for any \(\lambda \in [1, \infty)\).

Proof of Proposition 2.4. We abbreviate \(E_\lambda\) to \(E\). Let \(L\) and \(K\) be integers satisfying \(0 \leq L \leq M\) and \(0 \leq K \leq 3M - 3L\). We prove that there exists a positive constant \(C\) such that for any \(\lambda \in [1, \infty)\),

\[
\sup_{0 \leq t \leq T} \sum_{\ell=0}^{3M-3\ell} \sum_{k=0}^{3M} \| |x|^\ell \nabla^K \epsilon(t) \|_{L^2} \leq C, \tag{2.42}
\]

for any \(\lambda \in [1, \infty)\).

\[
\sup_{0 \leq t \leq T} \| |x|^L \nabla^K E(t) \|_{L^2} \leq C, \tag{2.43}
\]

by induction on \((L, K)\). Notice that the inequalities (2.43) with \(L = 0\) and \(0 \leq K \leq 3M\) follow from (2.41).

We first prove that the inequalities (2.43) with \((L, K) = (0, 1)\) and \((L, K) = (0, 3)\) imply the inequality (2.43) with \((L, K) = (1, 0)\). We assume the inequalities (2.43) hold for \((L, K) = (0, 1)\) and \((L, K) = (0, 3)\). Taking the imaginary part of the inner product in \(L^2\) between the first equation of (1.1) and \(|x|^2 E\), we have

\[
\frac{d}{dt} \| |x| E(t) \|_{L^2}^2 = -2Im \int_{\mathbb{R}^d} |x|^2 \nabla |x|^2 \nabla E dx + 2\varepsilon^2 Im \int_{\mathbb{R}^d} |x|^2 \nabla^2 E dx. \tag{2.44}
\]

Notice that

\[
Im \int_{\mathbb{R}^d} |x|^2 \nabla E \cdot \nabla E dx = -Im \int_{\mathbb{R}^d} \nabla |x|^2 \cdot \nabla \nabla E dx. \tag{2.45}
\]

By an integration by parts, we have

\[
\frac{d}{dt} \| |x| E(t) \|_{L^2}^2 = 2Im \int_{\mathbb{R}^d} \nabla |x|^2 \cdot \overline{E} \nabla E dx - 4\varepsilon^2 Im \int_{\mathbb{R}^d} \nabla |x|^2 \cdot \nabla \nabla E dx. \tag{2.46}
\]

The Hölder inequality yields

\[
\frac{d}{dt} \| |x| E(t) \|_{L^2}^2 \leq C \| \nabla E(t) \|_{L^2} \| |x| E(t) \|_{L^2} + C \| \nabla^3 E(t) \|_{L^2} \| |x| E(t) \|_{L^2} \leq C \| |x| E(t) \|_{L^2}^2.
\]

Hence the Gronwall lemma implies

\[
\| |x| E(t) \|_{L^2} \leq \| |x| E_0 \|_{L^2} + C t \leq C
\]

for any \(t \in [0, T]\), where the constant \(C\) depends on \(T\) and independent of \(\lambda \in [1, \infty)\). This guarantees that the inequality (2.43) holds for \((L, K) = (1, 0)\).

Let \(1 \leq k \leq 3M - 3\). Next we show that the inequalities (2.43) with \((L, K) = (0, k + 1)\), \((0, k + 3)\) and \((1, k')\) \((k' = 0, \cdots, k - 1)\) imply the inequality (2.43) with \((L, K) = (1, k)\). Assume that the inequalities (2.43) hold for \((L, K) = (0, k + 1)\), \((0, k + 3)\) and \((1, k')\) \((k' = 0, \cdots, k - 1)\).

Applying the operator \(\nabla^k\) to the first equation in (1.1) and taking the imaginary part of the inner product in \(L^2\) between the resulting equation
and $|x|^2 \nabla^k E$, together with invoking integration by parts, we have

$$
\frac{d}{dt} \| |x| \nabla^k E(t) \|_{L_x^2}^2 = 2Im \int_{\mathbb{R}^d} -|x|^2 \nabla^k E \cdot \nabla^{k+2} E + \varepsilon^2 |x|^2 \nabla^k E \cdot \nabla^{k+4} E
+ |x|^2 \nabla^k E \cdot \nabla^k (nE) dx.
$$

(2.47)

The Hölder inequality and Sobolev inequality yield

$$
\frac{d}{dt} \| |x| \nabla^k E(t) \|_{L_x^2}^2 
\leq C \left( \| \nabla^{k+1} E(t) \|_{L_x^2} + \| \nabla^{k+3} E(t) \|_{L_x^2} + \sum_{k'=0}^{k-1} \| n(t) \|_{H_x^{3M-1}} \| |x| \nabla^{k'} E(t) \|_{L_x^2} \right)
\times \| |x| \nabla^k E(t) \|_{L_x^2}
\leq C \| |x| \nabla^k E(t) \|_{L_x^2}.
$$

Hence the Gronwall lemma implies

$$
\| |x| \nabla^k E(t) \|_{L_x^2} \leq C
$$

for any $t \in [0, T]$ with the constant $C$ depending on $T$ but independent of $\lambda \in [1, \infty)$. This shows that the inequality (2.43) holds for $(L, K) = (1, k)$.

Let $2 \leq \ell \leq M$. Next we prove that the inequalities (2.43) with $(L, K) = (\ell - 1, 1), (\ell - 1, 3)$ and $(\ell - 2, 2)$ imply the inequality (2.43) with $(L, K) = (\ell, 0)$. We assume the inequality (2.43) holds for $(L, K) = (\ell - 1, 1), (\ell - 1, 3)$ and $(L, K) = (\ell - 2, 2)$. By an argument similar to that in (2.44), we have

$$
\frac{d}{dt} \| |x|^\ell E(t) \|_{L_x^2}^2 = Im \int_{\mathbb{R}^d} \left( 2\nabla |x|^{2\ell} E \nabla E - 4\varepsilon^2 \nabla |x|^{2\ell} E \nabla^3 E - 4\varepsilon^2 (2\ell + d - 2) |x|^{2\ell-2} \nabla^2 E \Delta E \right) dx
\leq \left( \| |x|^{\ell-1} \nabla^2 E \|_{L_x^2} + \| |x|^{\ell-1} \nabla^3 E \|_{L_x^2} + \| |x|^{\ell-2} \nabla^2 E \|_{L_x^2} \right) \| |x|^{\ell} E \|_{L_x^2}
\leq \| |x|^{\ell} E(t) \|_{L_x^2},
$$

which implies the inequality (2.43) holds for $(L, K) = (\ell, 0)$.

Let $2 \leq \ell \leq M$ and $1 \leq k \leq 3M - 3\ell$. Finally we show that the inequalities (2.43) with $(L, K) = (\ell - 1, k+1), (\ell - 1, k+3), (\ell - 2, k+2)$ and $(\ell, k') (k' = 0, \cdots, k - 1)$ imply the inequality (2.43) with $(L, K) = (\ell, k)$. Assume that the inequality (2.43) hold for $(L, K) = (\ell - 1, k+1), (\ell - 1, k+3), (\ell - 2, k+2)$ and $(\ell, k') (k' = 0, \cdots, k - 1)$.
The similar argument as given in (2.48), we have
\[
\frac{d}{dt} \| |x|^\ell \nabla^k E(t) \|_{L^2_x}^2 = 2 \text{Im} \int_{\mathbb{R}^d} |x|^{2\ell} |x|^{2d} \nabla^k \nabla^{k+1} E \, dx - 4\varepsilon^2 \text{Im} \int_{\mathbb{R}^d} |x|^{2\ell} |x|^{2d} \nabla^k \nabla^{k+2} E \, dx \\
- 4\ell(2\ell + d - 2)\varepsilon^2 \text{Im} \int_{\mathbb{R}^d} |x|^{2\ell - 2} \nabla^k \nabla^{k+2} E \, dx \\
+ 2\text{Im} \int_{\mathbb{R}^d} |x|^{2d} \nabla^k \nabla^k (n E) \, dx \\
\lesssim \left( \| |x|^{\ell - 1} \nabla^{k+1} E \|_{L^2_x} + \| |x|^{\ell - 1} \nabla^{k+3} E \|_{L^2_x} + \| |x|^{\ell - 2} \nabla^{k+2} E \|_{L^2_x} \right) \\
\times \| |x|^\ell \nabla^k E \|_{L^2_x} \\
+ \sum_{k' = 0}^{k-1} \| \nabla^{k-k'} n \|_{L^\infty_x} \| |x|^\ell \nabla^{k'} E \|_{L^2_x} \| |x|^\ell \nabla^k E \|_{L^2_x} \\
\lesssim \| |x|^\ell \nabla^k E(t) \|_{L^2_x}.
\]
This shows that the inequality (2.43) holds for \((L, K) = (\ell, k)\). Collecting those estimates we obtain the inequality (2.42). \(\square\)

Next we consider the fourth order nonlinear Schrödinger type equation (1.5). For the existence and the uniform bound of solution for (1.5), we have the following.

**Proposition 2.5.** Let \(d = 1, 2, 3\) and \(M \geq 2\). Then for any \(E_0 \in H^M(\mathbb{R}^d)\) there exists a unique solution to (1.5) satisfying
\[
E_\infty \in C([0, \infty); H^M(\mathbb{R}^d)) .
\]
Furthermore, for any \(T \in (0, \infty)\) and some \(C > 0\), \(E_\infty\) satisfies
\[
\sup_{0 \leq t \leq T} \| E_\infty(t) \|_{H^M_x} \leq C . \tag{2.48}
\]

**Proof of Proposition 2.5.** The existence and uniqueness of solution to (1.5) follows from the combination of the Strichartz estimate for the unitary group \(U_\varepsilon(t)\) and the contraction mapping principle, see [6] for instance. The global existence of solution in \(H^2\) follows from the conservations of mass and Hamiltonian
\[
\dot{H}_2(E)(t) = \frac{1}{2} \| \nabla E(t) \|_{L^2_x}^2 + \frac{\varepsilon^2}{2} \| \Delta E(t) \|_{L^2_x}^2 - \frac{1}{4} \int_{\mathbb{R}^d} |E|^2 (1 - \varepsilon^2 \Delta)^{-1} |E|^2 \, dx
\]
and the Gagliardo-Nirenberg inequality (Lemma 2.1). For \(M \geq 3\), the global existence of solution in \(H^M\) follows from the usual energy method and the bound of \(L^\infty_t H^2_x\) norm of \(E\). Since the proof is almost similar to that of Proposition 2.1, we omit the detail. \(\square\)

## 3. Proofs of Theorems 1.1, 1.2 and 1.3

In this section we prove Theorems 1.1, 1.2 and 1.3. The initial value problem (1.12) can be rewritten as the integral equation (1.13). Hence we have
\[
\| E_\lambda(t) - E_\infty(t) \|_{H^n_x} \leq J_1 + J_2 , \tag{3.1}
\]
where
\[ J_1 = \int_0^t \| \{ I_x | E_\lambda \}^2 \} E_\lambda - \{ I_x | E_\infty \}^2 \} E_\infty \|_{H^m} ds \] (3.2)
and
\[ J_2 = \int_0^t \| (Q_\lambda E_\lambda)(s) \|_{H^m} ds. \] (3.3)

We first evaluate \( J_1 \). Since the operator \( I_x \) is bounded from \( H^m_x \) to \( H^{m-2}_x \), we obtain

\[
\| \{ I_x | E_\lambda \}^2 \} E_\lambda - \{ I_x | E_\infty \}^2 \} E_\infty \|_{H^m} \\
\leq \| I_x \{ | E_\lambda \}^2 - | E_\infty \}^2 \} \|_{H^m} \| E_\lambda \|_{H^m} \| E_\infty \|_{H^m} + \| I_x | E_\infty \}^2 \} \|_{H^m} \| E_\lambda - E_\infty \|_{H^m} \\
\leq C(\| E_\lambda \|_{H^m} \| + \| E_\infty \|_{H^{m-2}}) \| E_\lambda - E_\infty \|_{H^m}.
\]

Combining the above inequalities and (2.42), we have

\[ J_1 \leq C \int_0^t \| E_\lambda(s) - E_\infty(s) \|_{H^m} ds, \] (3.4)

where the constant \( C \) is independent of \( \lambda \in [1, \infty) \).

To evaluate \( J_2 \), we need to estimate \( \lambda \in [1, \infty) \) and thus rewrite it as

\[
Q_\lambda(t) = \cos(\lambda t \omega_\epsilon) f_0 + \frac{\sin(\lambda t \omega_\epsilon)}{\lambda \omega_\epsilon} \nabla \cdot f_1 + \int_0^t \frac{\sin(\lambda(t - s) \omega_\epsilon)}{\lambda \omega_\epsilon} \partial_t^2 I_x | E_\lambda \}^2 \} ds \] (3.5)

where
\[
f_0 = n_0 + I_x | E_0 \}^2, \quad \nabla \cdot f_1 = n_1 + 2Im \{ E_0 \Delta_x \bar{E}_0 \}, \quad \text{and} \quad \omega_\epsilon = \sqrt{-\Delta_\epsilon}. \] (3.6)

Notice that \( f_1 = \phi + 2Im \{ E_0 \nabla I_x \bar{E}_0 + \epsilon \nabla \Delta x \bar{E}_0 \}, \) where \( \nabla \cdot \phi = n_1 \).

For the three terms \( Q_\lambda^{(j)} \), \( j = 0, 1, 2 \), we have the following lemmas.

**Lemma 3.1.** Let \( d = 1, 2, 3 \), and \( m \geq 2 \). For any \( 0 \leq k \leq m \), the inequality

\[
| \partial_x^k Q_\lambda^{(j)}(t, x_0) | \\
\left\{ \begin{array}{l}
(1 + \lambda t)^{-2} (1 + |x_0|^2) ||f_0||_{H^{m+2 + |d/2|}} \quad \text{if } |x_0| \geq \lambda t/2 \text{ or } 0 \leq \lambda t \leq 1, \\
(1 + \lambda t)^{-d/2 - \sigma} \left( (d - 1) ||f_0||_{H^{m+2 + |d/2|}} + \sum_{j=0}^2 ||x|^j f_0||_{H^{m+2 + |d/2|}} \right) \\
\text{if } |x_0| \leq \lambda t/2, \lambda t > 1,
\end{array} \right.
\] (3.7)

holds for any \( t \in (0, \infty) \) and \( \lambda \in [1, \infty) \), where \( \sigma = 3/2 \) for \( d = 1 \), \( 0 \leq \sigma < 1 \) for \( d = 2 \), and \( 0 \leq \sigma < 1/2 \) for \( d = 3 \).

**Lemma 3.2.** Let \( d = 1, 2, 3 \) and \( m \geq 2 \). Then the inequality

\[
||Q_\lambda^{(j)}(t)||_{H^m} \leq C \lambda^{-1} (||n_1||_{H^{m-1}} + ||n_1||_{H^{m-1}} + ||E_0||_{H^{m+2}}) \] (3.8)

holds for any \( t \in (0, \infty) \) and \( \lambda \in [1, \infty) \).
Lemma 3.3. Let $d = 1, 2, 3$, and $m \geq 2$. For any $0 \leq k \leq m$, the inequality
\[
|\partial_x^k Q^{(1)}(t, x)| \leq \begin{cases} 
\lambda^{-1}(1 + \lambda t)^{-2}(1 + |x_0|)^2\|f_1\|_{H^m_{\sigma+2|d/2|}} & \text{if } |x_0| \leq \lambda t/2, 0 \leq \lambda t \leq 1, \\
\lambda^{-1}(1 + \lambda t)^{-2}(d - 1)\|f_1\|_{H^m_{\sigma}} + \sum_{j=0}^{2} \|x|^j f_1\|_{H^{m+j+4|d/2|}} & \text{if } |x_0| \leq \lambda t/2, \lambda t > 1, 
\end{cases}
\] holds for any $t \in (0, \infty)$ and $\lambda \in [1, \infty)$, where $\sigma = 3/2$ for $d = 1, 0 \leq \sigma < 1$ for $d = 2$, and $0 \leq \sigma < \frac{1}{2}$ for $d = 3$.

Lemma 3.4. Let $d = 1, 2, 3$ and $m \geq 3$. Let $T^*$ be given by Proposition 2.7. Then for any $T \in (0, \infty)$ for $d = 1, 2$ and for any $T \in (0, T^*)$ for $d = 3$ and for any $(E_0, n_0, n_1) \in H^{m+1}(\mathbb{R}^d) \times H^{m+3}(\mathbb{R}^d) \times H^{m+1}(\mathbb{R}^d)$, the inequality
\[
\|Q^{(2)}_{\lambda}(t)\|_{H^m} \leq C\lambda^{-1}T \sup_{0 \leq t \leq T} (1 + \|n_\lambda(t)\|_{H^m}) \|E_\lambda(t)\|_{H^{m+4}}^2
\] holds for any $\lambda \in [1, \infty)$ where $(E_\lambda, n_\lambda)$ is the solution to (1.1) given in Proposition 2.7.

Lemma 3.5. Let $d = 1, 2, 3$ and $T^*$ be given by Proposition 2.7. Let $m \geq 6 - |d/2|$, and $M \geq 4$ be integers satisfying $3M \geq m + 6$. Then for any $T \in (0, \infty)$ for $d = 1, 2$ and for any $T \in (0, T^*)$ for $d = 3$, $(E_0, n_0, n_1) \in X_{3M,d}$, and $E_0 \in H^{0,M}(\mathbb{R}^d)$, the inequality
\[
|\partial_x^k Q^{(2)}_{\lambda}(t, x_0)| \leq C \begin{cases} 
\lambda^{-2}(1 + |x_0|) \sup_{0 \leq t \leq T} \left\{ (1 + \|n_\lambda(t)\|_{H^{m+2|d/2|}}) \|E_\lambda(t)\|_{H^{m+2|d/2|}}^2 \right\} & \text{if } |x_0| \geq \lambda t/2, 0 \leq \lambda t \leq 1, \\
F_d(\lambda) \sup_{0 \leq t \leq T} \left\{ (1 + \|n_\lambda(t)\|_{H^{m+2|d/2|}}) \left( \sum_{j=0}^{2} \|x|^j E_\lambda(t)\|_{H^{m+2|d/2|-j}} \right)^2 \right\} + \lambda^{-2}(1 + |x_0|) \sup_{0 \leq t \leq T} \left\{ (1 + \|n_\lambda(t)\|_{H^{m+2|d/2|}}) \|E_\lambda(t)\|_{H^{m+2|d/2|}}^2 \right\} & \text{if } |x_0| \leq \lambda t/2, \lambda t > 1
\end{cases}
\] holds for any $t \in (0, T)$, $0 \leq k \leq m$ and $\lambda \in [1, \infty)$, where $F_d(\lambda) = \lambda^{-2}$ for $d = 1$, $\lambda^{-2} \log \lambda$ for $d = 2$, and $\lambda^{-2}$ for $d = 3$.

We shall prove Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 in the next section. Now we prove Theorems 1.1 and 1.2 assuming that Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 hold.

Proof of Theorems 1.1, 1.2 and 1.3. We first consider the case when $n_0 + I_{\epsilon}|E_0|^2 \neq 0$. The inequality (1.8) follows from Lemmas 3.2 and 3.4
\[
\|Q_\lambda(t) - Q_\lambda^{(0)}(t)\|_{H^m} \leq \|Q_\lambda^{(1)}(t)\|_{H^m} + \|Q_\lambda^{(2)}(t)\|_{H^m} \leq C\lambda^{-1}.
\]
Let us show the inequality (1.7). From Lemma 3.1, Propositions 2.1 and 2.4 we have
\[
\|Q^{(0)}_\lambda E_\lambda(t)\|_{H^n_x} \leq C (1 + \lambda t)^{-\mu} \|E_\lambda(t)\|_{H^n_x} + C (1 + \lambda t)^{-\mu},
\]
(3.12)
where \(\mu = 2\) for \(d = 1\) and \(\mu = d/2 + \sigma\) for \(d = 2, 3\), and the constant \(C\) depends on \(E_0\) and \(n_0\). We deduce from Lemmas 3.2 and 3.4 that
\[
\|(Q^{(1)}_\lambda + Q^{(2)}_\lambda) E_\lambda(t)\|_{H^n_x} \leq C \lambda^{-1}. \tag{3.13}
\]
Therefore (3.12) and (3.13) together yield
\[
J_2 \leq C \int_0^t (1 + \lambda s)^{-\mu} ds + C \lambda^{-1}, \tag{3.14}
\]
where the constant \(C\) depends on \(T\) but is independent of \(\lambda \in [1, \infty)\). Combining (3.1), (3.4) and (3.14), we see
\[
\|E_\lambda(t) - E_\infty(t)\|_{H^n_x} \leq C \int_0^t \|E_\lambda(s) - E_\infty(s)\|_{H^n_x} ds + C \lambda^{-1}.
\]
The Gronwall lemma implies
\[
\|E_\lambda(t) - E_\infty(t)\|_{H^n_x} \leq C \lambda^{-1} \exp(CT).
\]
Hence we have (1.7) and (1.8). For the case where \(n_0 + I_\varepsilon|E_0|^2 \equiv 0\), the inequalities (1.9), (1.10) and (1.11) follow from the argument similar as above. Indeed it suffices to replace Lemmas 3.2 and 3.4 by Lemmas 3.3 and 3.5, respectively. This completes the proof of Theorems 1.1, 1.2 and 1.3. \(\square\)

### 4. Estimates for \(Q_\lambda\)

In this section we prove Lemmas 3.1 - 3.5. We first denote some notations which will be used throughout this section. Then we set the functions
\[
\varphi_\pm(t, \xi) = x_0 \cdot \xi \pm \lambda t \xi, \tag{4.1}
\]
where
\[
\xi = \begin{cases} 
\xi(1 + \varepsilon^2|x|^2)^{1/2} & \text{for } d = 1, \\
|\xi|(1 + \varepsilon^2|\xi|^2)^{1/2} & \text{for } d = 2, 3. \tag{4.2}
\end{cases}
\]
To estimate the integrals \(Q^{(k)}_\lambda(t, x_0)\) for \(k = 0, 1, 2\), we first compute the partial derivatives of \(\varphi_\pm\) over the variable \(\xi\) and we get, for \(d = 1,\)
\[
\varphi'_\pm = x_0 \pm \lambda t \xi, \quad \varphi''_\pm = \pm \lambda t \frac{2\xi^2}{(1 + \varepsilon^2\xi^2)^{3/2}}, \quad \varphi'''_\pm = \pm \lambda t \frac{3\xi^2}{(1 + \varepsilon^2\xi^2)^{5/2}}. \tag{4.3}
\]
We note the oscillatory integrals \(Q^{(k)}_\lambda(t, x_0)\) for \(k = 0, 1, 2\) have no stationary point in the region \(|x_0| < \lambda t\).
For $d = 1$, combining the identity
\[ e^{i\varphi_{\pm}(t,\xi)} = \frac{\partial_k e^{i\varphi_{\pm}(t,\xi)}}{i\varphi_{\pm}'(t,\xi)} \] (4.4)
and repeating the integration by parts twice, we have
\[
\int_{-\infty}^{\infty} e^{i\varphi_{\pm}(t,\xi)} \hat{g}(\xi)d\xi = \int_{-\infty}^{\infty} e^{i\varphi_{\pm}(t,\xi)} \left[ \frac{-1}{(\varphi_{\pm}')^2} \frac{\partial^2 \hat{g}}{\partial \xi^2} + \frac{3\varphi_{\pm}''}{(\varphi_{\pm}')^3} \frac{\partial \hat{g}}{\partial \xi} + \left( \frac{\varphi_{\pm}''}{(\varphi_{\pm}')^3} - \frac{3(\varphi_{\pm}')^2}{(\varphi_{\pm}')^4} \right) \hat{g}(\xi) \right] d\xi.
\] (4.5)

For $d = 2, 3$, the phase function is $\varphi_{\pm} = x_0 \cdot \xi \pm |\xi| \sqrt{1 + \varepsilon^2 |\xi|^2}$ whose second derivative $\Delta \phi_{\pm} \sim \pm \lambda t ((d-1)|\xi|^{-1} |\xi| + |\xi|(|\xi|)^{-1})$ is singular at the origin. This would require that the initial data lies in $H^{-2}$ which is more than that is required in the Hamiltonian (1.3). To avoid the difficulty, we modify the identity (4.4) as follows:
\[ e^{i\varphi_{\pm}(t,\eta)} = \frac{\partial_\eta (\eta e^{i\varphi_{\pm}(t,\eta)})}{1 + i\eta \varphi_{\pm}'(t,\eta)}. \] (4.6)

Combining the identity (4.6) and repeating the integration by parts twice, we have
\[
\int_{-\infty}^{\infty} e^{i\varphi_{\pm}(t,\eta)} g_k(\eta)d\eta = \int_{0}^{\infty} e^{i\varphi_{\pm}} q_1 \partial_\eta^2 \hat{g}_k(\eta)d\eta + \int_{0}^{\infty} e^{i\varphi_{\pm}} q_2 \partial_\eta \hat{g}_k(\eta)d\eta + \int_{0}^{\infty} e^{i\varphi_{\pm}} q_3 \hat{g}_k(\eta)d\eta
\]
\[= Q_{k,1}(t,x_0) + Q_{k,2}(t,x_0) + Q_{k,3}(t,x_0) + Q_{k,4}(t,x_0) + Q_{k,5}(t,x_0), \] (4.7)
where the functions $g_k$ for $k = 0, 1, 2$, will be specified later,
\[
q_1 = \frac{\eta^2}{(1 + i\eta \varphi_{\pm}'(t,\eta))^2}, \quad q_2 = \frac{3\eta^2(\varphi_{\pm}'(t,\eta) + \eta \varphi_{\pm}''(t,\eta))}{3\eta^2(\varphi_{\pm}'(t,\eta) + \eta \varphi_{\pm}''(t,\eta))}, \quad q_3 = \frac{-i(1 + i\eta \varphi_{\pm}'(t,\eta))}{(1 + i\eta \varphi_{\pm}'(t,\eta))^3}, \quad q_4 = \frac{3\eta^2(\varphi_{\pm}'(t,\eta) + \eta \varphi_{\pm}''(t,\eta))^2}{(1 + i\eta \varphi_{\pm}'(t,\eta))^4},
\] (4.8)
\[
q_5 = \frac{3\eta^2(\varphi_{\pm}'(t,\eta) + \eta \varphi_{\pm}''(t,\eta))^2}{(1 + i\eta \varphi_{\pm}'(t,\eta))^4}.
\]

For any $\eta \in [0, \infty)$ and in the region $|x_0| \leq (\lambda t)/2$, we have
\[
|\varphi_{\pm}'(t,\eta)| \sim \lambda t (1 + \varepsilon|\eta|), \quad |\varphi_{\pm}''(t,\eta)| \sim \lambda t \frac{\varepsilon^2 |\eta|}{1 + \varepsilon|\eta|},
\]
\[
|\varphi_{\pm}'''(t,\eta)| \sim \lambda t \frac{\varepsilon^2 |\eta|}{(1 + \varepsilon|\eta|)^5}, \quad |1 + i\eta \varphi_{\pm}'(t,\eta)| \sim 1 + |\eta| \lambda t (1 + \varepsilon|\eta|).
\] (4.9)
Invoking (4.8) and (4.9), we can obtain
\[
\begin{align*}
|q_1| & \sim \frac{\eta^2}{1 + (|\eta| \lambda t)^2(1 + \varepsilon |\eta|)^2}, \\
|q_2| & \sim \frac{|\eta|}{1 + (|\eta| \lambda t)^2(1 + \varepsilon |\eta|)^2}, \\
|q_3| & \sim \frac{\eta^2 \lambda t (1 + \varepsilon |\eta|)}{1 + (|\eta| \lambda t)^2(1 + \varepsilon |\eta|)^2}, \\
|q_4| & \sim \frac{|\eta| \lambda (1 + \varepsilon |\eta|)}{1 + (|\eta| \lambda t)^2(1 + \varepsilon |\eta|)^2}.
\end{align*}
\]

(4.10)

To estimate the quantities \(Q_{k,j}\) for \(k = 0, 1, 2\) and \(j = 1, \ldots, 5\), we split each of the integrals into three parts which are on the intervals \(I_1, I_2,\) and \(I_3\) given by
\[
I_1(t) = \{|\xi| \leq (\lambda t)^{-1}\}, \quad I_2(t) = \{(\lambda t)^{-1} < |\xi| \leq 1\}, \quad \text{and} \quad I_3(t) = \{|\xi| > 1\}.
\]

(4.11)

Observe that we have
\[
|\eta||q_5| \lesssim |\eta||q_4| \sim |q_3| \sim |q_2| = |\eta|^{-1}|q_1| \lesssim \left\{ \begin{array}{ll}
|\eta| & \text{for } \eta \in I_1(t), \\
(\lambda t)^{-2}|\eta|^{-1} & \text{for } \eta \in I_2(t), \\
(\lambda t)^{-2}|\eta|^{-3} & \text{for } \eta \in I_3(t).
\end{array} \right.
\]

(4.12)

**Proof of Lemma 3.1** We use the representation
\[
Q^{(0)}(t, x_0) = \left( \frac{1}{2\pi} \right)^{d/2} \frac{1}{2} Re \left[ \int_{\mathbb{R}^d} e^{i\xi \cdot x} (e^{i\lambda t \xi} + e^{-i\lambda t \xi}) \hat{f}_0(\xi) d\xi \right],
\]

(4.13)

where
\[
\xi = \xi \sqrt{1 + \varepsilon^2 \xi^2} \quad \text{for } d = 1, \quad \xi = |\xi| \sqrt{1 + \varepsilon^2 |\xi|^2} \quad \text{for } d = 2, 3.
\]

(4.14)

For the case \(0 \leq \lambda t \leq 1\), for any \(0 \leq k \leq m\), we have
\[
|\nabla^k Q^{(0)}(t, x_0)| \leq C \int_{\mathbb{R}^d} \langle \xi \rangle^m |\hat{f}_0(\xi)| d\xi \leq C \|f_0\|_{H^{m+1+[d/2]}}.
\]

Next we consider the case \(\lambda t > 1\) and \(|x_0| \geq \lambda t/2\). Then we have
\[
|\nabla^k Q^{(0)}(t, x_0)| \leq C \int_{\mathbb{R}^d} \langle \xi \rangle^m |\hat{f}_0(\xi)| d\xi \leq C(\lambda t)^{-2} |x_0|^2 \|f_0\|_{H^{m+1+[d/2]}}.
\]

Let us evaluate \(Q^{(0)}(t, x_0)\) in the region where \(|x_0| \leq \lambda t/2\). We rewrite (4.13) as
\[
\left( \frac{1}{2\pi} \right)^{d/2} \frac{1}{2} Re \int_{\mathbb{R}^{d-1}} \int_0^\infty \left( e^{i\varphi_+(t, \eta)} + e^{i\varphi_-(t, \eta)} \right) \hat{g}_0(\eta) d\eta d\sigma,
\]

where \(\varphi_\pm\) is given in (4.11), \(\xi = \eta \omega\), and
\[
\hat{g}_0(\eta) = \hat{f}_0(\eta \omega) \eta^{-d-1}.
\]

(4.15)
The partial derivatives of $\hat{g}_0$ are given by
\[
\partial_\eta \hat{g}_0(\eta) = \nabla \hat{f}_0(\eta_\omega) \cdot \omega \eta^{d-1} + (d-1) \hat{f}_0(\eta_\omega) \eta^{d-2},
\]
\[
\partial^2_{\eta} \hat{g}_0(\eta) = \sum_{j=1}^d \partial_j \nabla \hat{f}_0(\eta_\omega) \cdot \omega \eta_j \eta^{d-1} + 2(d-1) \nabla \hat{f}_0(\eta_\omega) \cdot \omega \eta^{d-2} + (d-1)(d-2) \hat{f}_0(\eta_\omega) \eta^{d-3}.
\]

(4.16)

**Case: $d = 1$**. Invoking (4.5), (4.9), (4.15), and (4.16), we have
\[
|Q^{(0)}_\lambda(t, x_0)| \leq C\lambda^2 \sum_{j=0}^2 \|x^j f_0\|_{H^j_{\lambda}}.
\]
Combining the above argument, we obtain
\[
|\nabla^k Q^{(0)}_\lambda(t, x_0)| \leq C\lambda^2 \sum_{j=0}^2 \|x^j f_0\|_{H^j_{\lambda}},
\]
for $0 \leq k \leq m$.

**Case: $d = 2$**. Invoking (4.7), (4.12), (4.15), and (4.16), we get
\[
|Q_{0,1}| \lesssim (\lambda t)^{-2}\left(\sqrt{\log(\lambda t)} \|(\eta)^{-1}\nabla \hat{f}_0 \eta^\frac{1}{2}\|_{L^2} + \sum_{j=1}^2 \|\partial_j \nabla \hat{f}_0 \eta^\frac{1}{2}\|_{L^2}\right),
\]
\[
|Q_{0,2} + Q_{0,3}| \lesssim (\lambda t)^{-1-\sigma}(\|(\eta)^{-\sigma} \hat{f}_0 \eta^\frac{1}{2}\|_{L^2} + \|\langle \eta \rangle^{-2} \hat{f}_0 \eta^\frac{1}{2}\|_{L^2})
\]
\[
+ \|\langle \eta \rangle^{-1} \nabla \hat{f}_0 \eta^\frac{1}{2}\|_{L^2},
\]
\[
|Q_{0,4} + Q_{0,5}| \lesssim (\lambda t)^{-1-\sigma}(\|(\eta)^{-\sigma} \hat{f}_0 \eta^\frac{1}{2}\|_{L^2} + \|\langle \eta \rangle^{-2} \hat{f}_0 \eta^\frac{1}{2}\|_{L^2}),
\]
where $0 \leq \sigma < 1$. Combining the above inequalities, we obtain
\[
|\nabla^k Q^{(0)}_\lambda(t, x_0)| \leq C(\lambda t)^{-1-\sigma}\left(\|f_0\|_{H^\sigma} + \sum_{l=0}^1 \|x^l f_0\|_{H^{\sigma+1-l}} + \sum_{j=1}^2 \|\langle \xi \rangle^m \mathcal{F}[x^j f_0]\|_{L^2}\right),
\]
for $0 \leq k \leq m$, where $0 \leq \sigma < 1$ and $\mu = \min\{\sigma + 1, 2\}$.

**Case: $d = 3$**. Analogously, we have
\[
|Q_{0,1}| \lesssim (\lambda t)^{-\frac{3}{2}-\sigma}(\|(\eta)^{-\sigma} \hat{f}_0 \eta\|_{L^2} + \|\langle \eta \rangle^{-2} \hat{f}_0 \eta\|_{L^2} + \|\langle \eta \rangle^{-1} \nabla \hat{f}_0 \eta\|_{L^2})
\]
\[
+ \sum_{j=1}^3 \|\partial_j \nabla \hat{f}_0 \eta\|_{L^2},
\]
\[
|Q_{0,2} + Q_{0,3}| \lesssim (\lambda t)^{-\frac{3}{2}-\sigma}(\|(\eta)^{-\sigma} \hat{f}_0 \eta\|_{L^2} + \|\langle \eta \rangle^{-2} \hat{f}_0 \eta\|_{L^2} + \|\langle \eta \rangle^{-1} \nabla \hat{f}_0 \eta\|_{L^2})
\]
\[
|Q_{0,4} + Q_{0,5}| \lesssim (\lambda t)^{-\frac{3}{2}-\sigma}(\|(\eta)^{-\sigma} \hat{f}_0 \eta\|_{L^2} + \|\langle \eta \rangle^{-2} \hat{f}_0 \eta\|_{L^2}),
\]
where $0 \leq \sigma < 1/2$. Therefore, we obtain
\[
|\nabla^k Q^{(0)}_\lambda(t, x_0)| \leq C(\lambda t)^{-\frac{3}{2}-\sigma}\left(\|f_0\|_{H^\sigma} + \sum_{l=0}^1 \|x^l f_0\|_{H^{\sigma+1-l}} + \sum_{j=1}^3 \|\langle \xi \rangle^m \mathcal{F}[x^j f_0]\|_{L^2}\right),
\]
for $0 \leq k \leq m$, where $0 \leq \sigma < 1/2$.

This completes the proof of Lemma 3.1 \hfill \Box

**Proof of Lemma 3.2** From (3.5) and (3.6), we have

$$\|Q^{(t)}_\lambda(t)\|_{H^m_x} \leq C\lambda^{-1}\|f_1\|_{H^m_x}$$
$$\leq C\lambda^{-1}(\|n_1\|_{H^{m-1}} + \|n_1\|_{H^{-1}} + \|E_0\|^2_{H^{m+3}}). \hfill \Box$$

**Proof of Lemma 3.3** The proof is analogous to that of Lemma 3.1. Now

$$Q^{(t)}_\lambda(t, x_0) = \left(\frac{1}{2\pi}\right)^{d/2} \frac{1}{2\lambda} Im\left[ \int_{\mathbb{R}^d} e^{i\varphi_+ (t, \eta)} (e^{i\lambda\xi} - e^{i\lambda\xi'}) \xi_\xi' \eta_\eta' \right]$$

where $\xi_e$ is given as in (4.14). For the case $0 \leq \lambda t \leq 1$, for any $0 \leq k \leq m$, we have

$$|\nabla^k Q^{(t)}_\lambda(t, x_0)| \leq C\lambda^{-1}\|f_1\|_{H^{m+|d|/2}}.$$ 

Next for the case $\lambda t > 1$ and $|x_0| \geq \lambda t/2$, we have

$$|\nabla^k Q^{(t)}_\lambda(t, x_0)| \leq C\lambda^{-1}(\lambda t)^{-2}|x_0|^2\|f_1\|_{H^{m+|d|/2}}.$$ 

Let us evaluate $Q^{(t)}_\lambda(t, x_0)$ in the region where $|x_0| \leq \lambda t/2$, and rewrite it as follows

$$\left(\frac{1}{2\pi}\right)^{d/2} \frac{1}{2\lambda} Im\int_{\partial \omega-1} \int_0^\infty \left( e^{i\varphi_+ (t, \eta)} - e^{i\varphi_- (t, \eta)} \right) \eta_\eta' d\eta d\sigma,$$

where $\varphi_\pm$ is given in (4.11) $\xi = \eta\omega$, and

$$\hat{g}_1(\eta) = \omega \cdot \hat{f}_1(\eta \omega)\eta^{d-1}/\sqrt{1 + \varepsilon^2 \eta^2}. \hfill (4.18)$$

For any $\eta \in [0, \infty)$ and in the region $|x_0| \leq \lambda t/2$, we estimate the quantities $Q_{1,j}$ for $j = 1, \cdots, 5$. We split the interval $[0, \infty)$ into three parts $I_1(t), I_2(t),$ and $I_3(t)$ which are given in (4.11). Let $f_1 = (f_{1,1}, \cdots, f_{1,d}).$ We estimate the partial derivatives of $\hat{g}_1$,

$$|\partial_\eta \hat{g}_1(\eta)| \lesssim \sum_{j=1}^d \left|\frac{\partial_\eta \hat{f}_{1,j}}{\eta}\right| \eta^{d-1} + (|d - 1| + |d - 2|\eta^2) \left|\frac{\hat{f}_{1,j}}{\eta}\right| \eta^{d-2},$$

$$|\partial_\eta^2 \hat{g}_1(\eta)| \lesssim \sum_{j, \ell=1}^d \left|\frac{\partial^2_\eta \hat{f}_{1,j}}{\eta}\right| \eta^{d-1} + (|d - 1| + |d - 2|\eta^2) \sum_{j=1}^d \left|\frac{\hat{f}_{1,j}}{\eta}\right| \eta^{d-2}$$
$$+ (|d - 1|d - 2| + \eta^2 + |d - 2||d - 3|\eta^4) \left|\frac{\hat{f}_{1,j}}{\eta}\right| \eta^{d-3}. \hfill (4.19)$$

**Case: $d = 1$.** Invoking (4.5), (4.9), (4.13) - (4.19), we have

$$|Q^{(t)}_\lambda(t, x_0)| \leq C\lambda^{-1}(\lambda t)^{-2} \sum_{j=0}^2 \|x^j f_1\|_{H_{-4}^k}.$$
Combining the above argument, we obtain

\[ |\nabla^k Q^{(1)}_\lambda(t, x_0)| \leq C\lambda^{-1}(\lambda t)^{-2} \sum_{j=0}^{2} \|x^j f_1\|_{H^{n+j-4}_\sigma}, \]

for \( 0 \leq k \leq m \).

**Case: \( d = 2 \)** Analogously we get

\[ |Q_{1,1}| \lesssim (\lambda t)^{-2} \left( \sum_{j, \ell=1}^{2} \|\langle \eta \rangle^{-1} \partial_t \nabla f_1 \eta \|_{L^2_\sigma} + \sum_{j=1}^{2} \|\langle \eta \rangle^{-2} \nabla f_1 \eta \|_{L^2_\sigma} \right), \]

\[ |Q_{1,2}| + |Q_{1,3}| \lesssim (\lambda t)^{-1-\sigma} \left( \sum_{j=1}^{2} \|\langle \eta \rangle^{-3} \nabla f_1 \eta \|_{L^2_\sigma} \right), \]

\[ |Q_{1,4}| + |Q_{1,5}| \lesssim (\lambda t)^{-1-\sigma} \left( \sum_{j=1}^{2} \|\langle \eta \rangle^{-3} \nabla f_1 \eta \|_{L^2_\sigma} \right), \]

where \( 0 \leq \sigma < 1 \). Combining the above inequalities, we obtain

\[ |\nabla^k Q^{(1)}_\lambda(t, x_0)| \leq C\lambda^{-1}(\lambda t)^{-1-\sigma} \left( \|f_1\|_{H^{-\sigma}} + \sum_{\ell=0}^{2} \|x^\ell f_1\|_{H^{n+\ell-3}_\sigma} \right), \]

for \( 0 \leq k \leq m \) and \( 0 \leq \sigma < 1 \).

**Case: \( d = 3 \)** Analogously we have

\[ |Q_{1,1}| \lesssim (\lambda t)^{-\frac{3}{2}-\sigma} \left( \sum_{j, \ell=1}^{3} \|\langle \eta \rangle^{-1} \partial_t \nabla f_1 \eta \|_{L^2_\sigma} + \sum_{j=1}^{3} \|\langle \eta \rangle^{-2} \nabla f_1 \eta \|_{L^2_\sigma} \right), \]

\[ |Q_{1,2}| + |Q_{1,3}| \lesssim (\lambda t)^{-\frac{3}{2}-\sigma} \left( \sum_{j=1}^{3} \|\langle \eta \rangle^{-3} \nabla f_1 \eta \|_{L^2_\sigma} \right), \]

\[ |Q_{1,4}| + |Q_{1,5}| \lesssim (\lambda t)^{-\frac{3}{2}-\sigma} \left( \sum_{j=1}^{3} \|\langle \eta \rangle^{-3} \nabla f_1 \eta \|_{L^2_\sigma} \right), \]

where \( 0 \leq \sigma < 1/2 \). Note that the extra decay rate \( t^{1/2} \) compared to \( d = 2 \) is due to the increase of the space dimension. Combining the above inequalities, we obtain

\[ |\nabla^k Q^{(1)}_\lambda(t, x_0)| \leq C\lambda^{-1}(\lambda t)^{-\frac{3}{2}-\sigma} \left( \|f_1\|_{H^{-\sigma}} + \sum_{\ell=0}^{2} \|x^\ell f_1\|_{H^{n+\ell-3}_\sigma} \right), \]

for \( 0 \leq k \leq m \), where \( 0 \leq \sigma < 1/2 \).

This completes the proof of Lemma \[3.3\]. \hfill \Box

**Proof of Lemma \[3.4\]** For the sake of convenience, we drop the indices of \( E_\lambda \) and \( n_\lambda \).
From (1.1), we see
\[
\partial_t^2 |E|^2 = -2\partial_t \text{Im}[E \Delta_x E]
\]
\[= -2\partial_t \text{Im}[\nabla \cdot \{ \overline{E}_x \nabla E \}] - 2\varepsilon^2 \partial_t \sum_{k=1}^{d} \text{Im}[\nabla \cdot (\partial_k \overline{E} \nabla \partial_k E)]
\]
\[= 2 \nabla \cdot \text{Re}[\{ \Delta_x \overline{E} - n \overline{E} \}] \overline{E}_x \nabla E + E \Delta_x \nabla \{ -\Delta_x E + n E \}
\]
\[+ 2 \varepsilon^2 \nabla \cdot \text{Re} \sum_{k=1}^{d} \left[ \partial_k \{ \Delta_x \overline{E} - n \overline{E} \} \nabla \partial_k E + \partial_k \overline{E} \nabla \partial_k \{ -\Delta_x E + n E \} \right]
\]
\[\equiv \nabla \cdot f_2.
\] (4.20)

The Sobolev embedding yields
\[
\|(-\Delta)^{-1/2} I_{\varepsilon^{-3/2}} \partial_t^2 |E|^2\|_{H^m} \leq C \|f_2\|_{H^m-3} \leq C(1 + \|n\|_{H^m}) \|E\|^2_{H^m+4},
\]
for \(m \geq 3\). Hence we have
\[
\|Q^{(2)}_{\lambda}\|_{H^m} \leq C \int_0^{T^*} \lambda^{-1} \|(-\Delta)^{-1/2} I_{\varepsilon^{-3/2}} \partial_t^2 |E|^2\|_{H^m} ds
\]
\[\leq C \lambda^{-1} T \sup_{0 \leq t \leq T} \{(1 + \|n(t)\|_{H^m}) \|E(t)\|^2_{H^m+4}\},
\]
where the constant \(C\) depends on \(T^*\) but is independent of \(\lambda \in [1, \infty)\). This completes the proof of Lemma 3.5. □

**Proof of Lemma 3.5** Now we rewrite \(Q^{(2)}_{\lambda}\) as follows.
\[
Q^{(2)}_{\lambda}(t, x_0) = \lambda^{-1} \text{Im} \left[ \int_0^{t} \int_{\mathbb{R}^d} e^{i \lambda(t-s)x_\varepsilon} \left( e^{i \lambda(t-s)x_\varepsilon} - e^{-i \lambda(t-s)x_\varepsilon} \right) \frac{\xi \cdot \hat{f}_2(s, \xi)}{\xi_\varepsilon (1 + \varepsilon^2 \xi_\varepsilon^2)} d\xi ds \right],
\] (4.21)
where \(\xi_\varepsilon\) is given in (4.14) and \(f_2\) is given in (4.20). For the case \(0 \leq t \leq \lambda^{-1} \max\{1, 2|x_0|\}\), for any \(0 \leq k \leq m\), we have
\[
|\nabla^k Q^{(2)}_{\lambda}(t, x_0)|
\]
\[\leq C \lambda^{-2} (1 + |x_0|) \sup_{0 \leq t \leq T} \|f_2(t)\|_{H^{m-2+|d/2|}}
\]
\[\leq C \lambda^{-2} (1 + |x_0|) \sup_{0 \leq t \leq T} \left\{(1 + \|n_\lambda(t)\|_{H^{m+4+|d/2|}}) \|E_\lambda(t)\|^2_{H^{m+5+|d/2|}}\right\}.
\] (4.22)

In the region \(t \geq \lambda^{-1} \max\{1, 2|x_0|\}\), we split \(Q^{(2)}_{\lambda}\) into the following two pieces
\[
Q^{(2)}_{\lambda}(t, x_0)
\]
\[\sim \lambda^{-1} \text{Im} \left[ \left( \int_0^{t-\lambda^{-1}b} + \int_{t-\lambda^{-1}b}^t \right) \int_{\mathbb{R}^d} e^{i \lambda(t-s)x_\varepsilon} \left( e^{i \lambda(t-s)x_\varepsilon} - e^{-i \lambda(t-s)x_\varepsilon} \right) \frac{\xi \cdot \hat{f}_2(s, \xi)}{\xi_\varepsilon (1 + \varepsilon^2 \xi_\varepsilon^2)} d\xi ds \right]
\]
\[\equiv Q^{(2, 1)}_{\lambda}(t, x_0) + Q^{(2, 2)}_{\lambda}(t, x_0),
\]
where \( b = \max\{1, 2|x_0|\} \). For \( Q^{(2,2)}_\lambda \), we can easily see that

\[
|\nabla^k Q^{(2,2)}_\lambda(t, x_0)| \\
\leq C\lambda^{-2}(1 + |x_0|) \sup_{0 \leq t \leq T} \|f_2(t)\|_{H^{m-2+|d/2|}} \\
\leq C\lambda^{-2}(1 + |x_0|) \sup_{0 \leq t \leq T} \left\{ (1 + \|\eta\lambda(t)\|_{H^{m+4+|d/2|}}) \|E_\lambda(t)\|_{H^{m+5+|d/2|}}^2 \right\},
\]

(4.23)

where \( 0 \leq k \leq m \).

We rewrite \( Q^{(2,1)}_\lambda(t, x_0) \) as

\[
C\lambda^{-1} \text{Im} \int_0^{t-\lambda^{-1}b} \int_{\mathbb{R}^{d-1}} \int_0^\infty \left( e^{i\varphi_+((t-s),\eta)} - e^{i\varphi_-(t-s,\eta)} \right) \hat{g}_2(\eta) d\eta d\sigma ds,
\]

where \( \varphi_{\pm} \) is given in (4.11), \( \xi = \eta \omega \), and

\[
\hat{g}_2(\eta) = \eta^{-d-1}(1 + \varepsilon^2\eta^2)^{-3/2} \omega \cdot \hat{f}_2(s, \eta \omega).
\]

(4.24)

For any \( \eta \in [0, \infty) \) and in the region \( |x_0| \leq \lambda(t-s)/2 \), we estimate the quantities \( Q_{2,j} \) for \( j = 1, \ldots, 5 \). We split the interval \([0, \infty)\) into three parts \( I_1(t-s), I_2(t-s), \) and \( I_3(t-s) \) which are given in (4.11). We then estimate the partial derivatives of \( \hat{g}_2 \),

\[
\begin{align*}
|\partial_\eta \hat{g}_2(\eta)| &\lesssim \frac{\eta^{d-1}|\nabla \hat{f}_2|}{\langle \eta \rangle^3} + \left( |d-1| + \eta^2 \right) \eta^{d-2} \frac{|\hat{f}_2|}{\langle \eta \rangle^3}, \\
|\partial_\eta^2 \hat{g}_2(\eta)| &\lesssim \sum_{j=1}^d \frac{\eta^{d-1}|\partial_j \nabla \hat{f}_2|}{\langle \eta \rangle^3} + \left( |d-1| + \eta^2 \right) \eta^{d-2} \frac{|\nabla \hat{f}_2|}{\langle \eta \rangle^3} \\
&\quad + \left( |d-1||d-2| + \eta^2 + \eta^4 \right) \eta^{d-3} \frac{|\hat{f}_2|}{\langle \eta \rangle^3}.
\end{align*}
\]

(4.25)

Case: \( d = 1 \). Invoking (4.15), (4.9), (4.24) - (4.25), we have

\[
|Q^{(2,1)}_\lambda(t, x_0)| \leq C\lambda^{-1} \int_0^{t-\lambda^{-1}b} (1 + \lambda(t-s))^{-2} \sum_{\ell=0}^2 \|x^\ell f_2(s)\|_{H_x^{\ell-6}} ds \\
\leq C\lambda^{-2}T^{1/2} \sup_{0 \leq t \leq T} \sum_{\ell=0}^2 \|x^\ell f_2(t)\|_{H_x^{\ell-6}}.
\]

(4.26)

Combining the above argument, for \( 0 \leq k \leq m \), we obtain

\[
|\nabla^k Q^{(2,1)}_\lambda(t, x_0)| \leq C\lambda^{-1} \int_0^{t-\lambda^{-1}b} (1 + \lambda(t-s))^{-2} \sum_{\ell=0}^2 \|x^\ell f_2(s)\|_{H_x^{m+\ell-6}} ds \\
\leq C\lambda^{-2}(T)^{1/2} \sup_{0 \leq t \leq T} \sum_{\ell=0}^2 \|x^\ell f_2(t)\|_{H_x^{m+\ell-6}}.
\]

(4.27)
Case: $d = 2$. Analogously we get

$$|Q_{2,1}| \lesssim (\lambda(t-s))^{-2} \sqrt{\log(\lambda(t-s))} \left( \sum_{j=1}^{2} \| \langle \eta \rangle^{-3} \partial_j \hat{\nabla} \hat{f}_2 \eta^\sharp \|_{L^2_\eta} + \| \langle \eta \rangle^{-4} \hat{\nabla} \hat{f}_2 \eta \|_{L^2_\eta} \| \langle \eta \rangle^{-5} \hat{f}_2 \eta^\sharp \|_{L^2_\eta} \right),$$

$$|Q_{2,2}| + |Q_{2,3}| \lesssim (\lambda(t-s))^{-1} \left( \| \langle \eta \rangle^{-4} \hat{\nabla} \hat{f}_2 \eta \|_{L^2_\eta} + \| \langle \eta \rangle^{-5} \hat{f}_2 \eta \|_{L^2_\eta} \right),$$

$$|Q_{2,4}| + |Q_{2,5}| \lesssim (\lambda(t-s))^{-1} \| \langle \eta \rangle^{-5} \hat{f}_2 \eta \|_{L^2_\eta}.$$

Combining the above inequalities, for $0 \leq k \leq m$, we obtain

$$\| \nabla^k Q^{(2,1)}_\lambda(t,x_0) \| \leq C \lambda^{-1} \int_0^{t-\lambda^{-1}b} (1 + \lambda(t-s))^{-1} \sum_{\ell=0}^{2} \| |t|^\ell f_2(s) \|_{H^{m+\ell-5}_x} ds \leq C \lambda^{-2} \log(1 + \lambda T) \sup_{0 \leq t \leq T} \left( \sum_{\ell=0}^{2} \| |t|^\ell f_2(t) \|_{H^{m+\ell-5}_x} \right). \quad (4.28)$$

Case: $d = 3$. Analogously we have

$$|Q_{2,1}| \lesssim (\lambda(t-s))^{-3/2} \left( \sum_{j=1}^{3} \| \langle \eta \rangle^{-3} \partial_j \hat{\nabla} \hat{f}_2 \eta \|_{L^2_\eta} + \| \langle \eta \rangle^{-4} \hat{\nabla} \hat{f}_2 \eta \|_{L^2_\eta} \right),$$

$$|Q_{2,2}| + |Q_{2,3}| \lesssim (\lambda(t-s))^{-3/2} \left( \| \langle \eta \rangle^{-4} \hat{\nabla} \hat{f}_2 \eta \|_{L^2_\eta} + \| \langle \eta \rangle^{-5} \hat{f}_2 \eta \|_{L^2_\eta} \right),$$

$$|Q_{2,4}| + |Q_{2,5}| \lesssim (\lambda(t-s))^{-3/2} \| \langle \eta \rangle^{-5} \hat{f}_2 \eta \|_{L^2_\eta}.$$

Combining the above estimates, for $0 \leq k \leq m$, we get

$$\| \nabla^k Q^{(2,1)}_\lambda(t,x_0) \| \leq C \lambda^{-1} \int_0^{t-\lambda^{-1}b} (1 + \lambda(t-s))^{-3/2} \sum_{\ell=0}^{2} \| |t|^\ell f_2(s) \|_{H^{m+\ell-5}_x} ds \leq C \lambda^{-2} \sup_{0 \leq t \leq T} \left( \sum_{\ell=0}^{2} \| |t|^\ell f_2(t) \|_{H^{m+\ell-5}_x} \right). \quad (4.29)$$

On the other hand, the Sobolev embedding yields

$$\sup_{0 \leq t \leq T} \left( \sum_{\ell=0}^{2} \| |t|^\ell f_2(t) \|_{H^{m+\ell-6+[d/2]}_x} \right) \lesssim \sup_{0 \leq t \leq T} \left( 1 + \| n(t) \|_{H^{m+2+[d/2]}_x} \right) \left( \sum_{\ell=0}^{2} \| |t|^\ell E(t) \|_{H^{m+3+[d/2]-3\ell}_x} \right)^2 \quad (4.30)$$

for $m \geq 6 - [d/2]$ if $d = 1, 2, 3$. From (4.22), (4.23), (4.27) and (4.30), we obtain (3.11) for $d = 1$. From (4.22), (4.23), (4.28), and (4.30), we obtain (3.11) for $d = 2$. Further, from (4.22), (4.23), (4.29), and (4.30), we obtain (3.11) for $d = 3$. This completes the proof of Lemma 3.5. \qed
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