Fractional semilinear heat equations with singular and nondecaying initial data

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Abstract
We study integrability conditions for existence and nonexistence of a local-in-time integral solution of fractional semilinear heat equations with rather general growing nonlinearities in uniformly local $L^p$ spaces. Our main results about this matter consist of Theorems 1.4, 1.6, 5.1 and 5.3. We introduce a supersolution of an integral equation which can be applied to a nonlocal parabolic equation. When the nonlinear term is $u^p$ or $e^u$, a local-in-time solution can be constructed in the critical case, and integrability conditions for the existence and nonexistence are completely classified. Our analysis is based on the comparison principle, Jensen’s inequality and $L^p$-$L^q$ type estimates.

Keywords Local-in-time solution · Optimal singularity · Supersolutions · Fractional Laplacian

Mathematics Subject Classification Primary 35K55 · 35R11 · Secondary 35A01 · 46E30

1 Introduction and main results
We are interested in existence and nonexistence of a local-in-time solution of the Cauchy problem
\[
\begin{cases}
\partial_t u + (-\Delta)^{\theta/2} u = f(u) & \text{in } \mathbb{R}^N \times (0, T), \\
u(x, 0) = \phi(x) & \text{in } \mathbb{R}^N,
\end{cases}
\] (1.1)

where the domain is \(\mathbb{R}^N, N \geq 1\), the initial function \(\phi\) is nonnegative, \(\phi\) may be unbounded and nondecaying and \((-\Delta)^{\theta/2}, 0 < \theta \leq 2\), denotes the fractional power of the Laplace operator \(-\Delta\) on \(\mathbb{R}^N\). Throughout the present paper, we define \(F(u)\) by

\[F(u) := \int_u^\infty \frac{dt}{f(t)}.
\]

We impose the following assumptions on \(f\):

\[f \in C^1(0, \infty) \cap C[0, \infty), \ f(u) > 0 \text{ for } u > 0, \ f'(u) \geq 0 \text{ for } u \geq 0, \quad (F1)\]

\[F(u) < \infty \text{ for large } u > 0 \text{ and the limit } q := \lim_{u \to \infty} f'(u)/f(u) \text{ exists.}\]

By (F1) we see that \(F\) is defined on \((0, \infty)\) and \(0 < F(0) \leq \infty\). The inverse function \(F^{-1}(u)\) exists, because \(F'(u)\) is strictly decreasing. When \(f \in C^2\), by L’Hospital’s role we formally have

\[q = \lim_{u \to \infty} \frac{F(u)}{1/f'(u)'} = \lim_{u \to \infty} \frac{F(u)'}{(1/f'(u))''} = \lim_{u \to \infty} \frac{f''(u)}{2f(u)f''(u)}.
\]

The growth rate of \(f\) can be defined by \(p := \lim_{u \to \infty} uf'(u)/f(u)\). We formally obtain

\[p = \lim_{u \to \infty} \frac{(u)'}{(f(u)/f'(u))'} = \lim_{u \to \infty} \frac{1}{1 - \frac{f(u)f''(u)}{f'(u)^2}} = \frac{q}{q - 1}, \quad \text{and hence } \frac{1}{p} + \frac{1}{q} = 1.
\]

The exponent \(q := \lim_{u \to \infty} f'(u)/f(u)\) was introduced in [6], while Dupaigne-Farina [4] introduced the exponent \(\lim_{u \to \infty} f'(u)^2/f(u)f''(u)\) in a different viewpoint. The algebraic growth corresponds to \(q > 1\), while the exponential growth corresponds to \(q = 1\). We will show in Sect. 2 that \(q \geq 1\) if \(q\) exists.

Let us consider the classical case \(\theta = 2\), i.e.,

\[\begin{cases}
\partial_t u - \Delta u = f(u) & \text{in } \mathbb{R}^N \times (0, T), \\
u(x, 0) = \phi(x) & \text{in } \mathbb{R}^N.
\end{cases}\]

(1.2)

Weissler [24] started studying the solvability of (1.2) with possibly unbounded and sign-changing initial data \(\phi \in L^r(\mathbb{R}^N)\) and found the critical exponent \(N(p - 1)/2\) as described in Proposition 1.1. In the model case \(f(u) = |u|^{p-1}u, p > 1\), the solvability in [24] can be summarized as follows:

Proposition 1.1 Let \(N \geq 1\). Assume that \(f(u) = |u|^{p-1}u, p > 1\). The following hold:

(i) (Existence, subcritical case) Assume \(r \geq 1\) and \(r > N(p - 1)/2\). The problem (1.2) has a local-in-time solution for \(\phi \in L^r(\mathbb{R}^N)\).
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(ii) (Existence, critical case) Assume \( r = N(p-1)/2 > 1 \). The problem (1.2) has a local-in-time solution for \( \phi \in L^r(\mathbb{R}^N) \).

(iii) (Nonexistence, supercritical case) For each \( 1 \leq r < N(p-1)/2 \), there is \( \phi \in L^r(\mathbb{R}^N) \) such that (1.2) has no local-in-time nonnegative solution.

Let \( u_\lambda(x,t) := \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t) \). When \( u(x,t) \) satisfies the equation in (1.2) with \( f(u) = |u|^{p-1}u \), the function \( u_\lambda(x,t) \) also satisfies the same equation. We see that

\[
\|u_\lambda(\cdot,0)\|_{L^r(\mathbb{R}^N)} = \|u(\cdot,0)\|_{L^r(\mathbb{R}^N)}, \quad \lambda > 0
\]

if and only if \( r = N(p-1)/2 \). Proposition 1.1 indicates that \( L^{N(p-1)/2}(\mathbb{R}^N) \) becomes a borderline space for the solvability of the equation. The problem (1.2) has been studied by many authors. See [1,3,8,13,20,24,25] for example. The reader can consult Quittner-Souplet [19] and references therein.

Hereafter, let \( L^r(\Omega) \), \( 1 \leq r \leq \infty \), denote the usual Lebesgue space on the domain \( \Omega \). We write \( \|u\|_r = \|u\|_{L^r(\Omega)} \) for simplicity when \( \Omega = \mathbb{R}^N \). In order to deal with singular and nondecaying functions we define uniformly local \( L^r \) spaces:

\[
L^r_{ul,\rho}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) \left| \| u \|_{L^r_{ul,\rho}(\mathbb{R}^N)} < \infty \right. \right\}.
\]

Here, \( \rho > 0 \), \( B_\rho(x) := \{ x \in \mathbb{R}^N \mid |x-y| < \rho \} \) and

\[
\|u\|_{L^r_{ul,\rho}(\mathbb{R}^N)} := \begin{cases}
\sup_{y \in \mathbb{R}^N} \left( \int_{B_\rho(y)} |u(x)|^r dx \right)^{1/r} & \text{if } 1 \leq r < \infty, \\
\sup_{y \in \mathbb{R}^N} \|u\|_{L^\infty(B_\rho)} & \text{if } r = \infty.
\end{cases}
\]

We easily see that \( L^\infty_{ul,\rho}(\mathbb{R}^N) = L^\infty(\mathbb{R}^N) \) and that \( L^{r_1}_{ul,\rho}(\mathbb{R}^N) \subset L^{r_2}_{ul,\rho}(\mathbb{R}^N) \) if \( 1 \leq r_2 \leq r_1 \leq \infty \). We define \( L^r_{ul,\rho}(\mathbb{R}^N) \) by

\[
L^r_{ul,\rho}(\mathbb{R}^N) := \text{closure of the space of bounded uniformly continuous functions } BUC(\mathbb{R}^N) \text{ in the space } L^r_{ul,\rho}(\mathbb{R}^N).
\]

i.e., \( L^r_{ul,\rho}(\mathbb{R}^N) \) denotes the closure of the space of bounded uniformly continuous functions \( BUC(\mathbb{R}^N) \) in the space \( L^r_{ul,\rho}(\mathbb{R}^N) \). By working in \( L^r_{ul,\rho}(\mathbb{R}^N) \) (or \( L^r_{ul,\rho}(\mathbb{R}^N) \)) instead of \( L^r(\mathbb{R}^N) \) we can focus on the relationship between the singularity of \( \phi \) and the solvability, and eliminate the effect of the behavior of \( \phi \) near space infinity.

Fujishima-Ioku [6] studied the solvability of (1.2) in \( L^r_{ul,\rho}(\mathbb{R}^N) \) with nonnegative initial data \( \phi \). Specifically, they obtained the following:

**Proposition 1.2** Let \( N \geq 1 \) and \( \theta = 2 \). Assume that \( f \) satisfies (F1) with \( q \geq 1 \) and that \( \phi \geq 0 \). Then the following hold:

(i) (Existence, subcritical case) Assume that \( r > N/2 \) and \( r \geq q-1 \) and that \( f'(u)F(u) \leq q \) for large \( u > 0 \). If \( F(\phi)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N) \), then (1.1) has a local-in-time solution.
(ii) (Existence, critical case) Assume that \( r = N/2 > q - 1 \) and that \( f'(u)F(u) \leq q \) for large \( u > 0 \). If \( F(\phi)^{-r} \in L^1_{ul,p}(\mathbb{R}^N) \), then (1.1) has a local-in-time solution.

(iii) (Nonexistence, supercritical case) Assume that \( f \) is convex and \( q \in (1,r) \) for large \( u \). If \( r \in (q-1,N/2) \) if \( q > 1 \) or \( r \in (0,N/2) \) if \( q = 1 \), there is a nonnegative \( \phi \) such that \( F(\phi)^{-r} \in L^1_{ul,p}(\mathbb{R}^N) \) and (1.1) has no local-in-time nonnegative solution.

Note that \( F(u)^{-r} = 1/(F(u)^r) \), while \( F^{-1}(u) \) stands for the inverse function of \( F(u) \). Because of the existence of the \( q \) exponent in (F1), the equation (1.1) with \( \theta = 2 \) is approximately invariant under the quasi-scaling

\[
\tilde{u}_\lambda(x,t) := F^{-1}(\lambda^{-2}F(u(\lambda x,\lambda^2 t))) , \quad \lambda > 0
\]

and the invariance

\[
\int_{\mathbb{R}^N} F(u_\lambda(x,0))^{-N/2} dx = \int_{\mathbb{R}^N} F(u(x,0))^{-N/2} dx , \quad \lambda > 0
\]

gives the borderline set in Proposition 1.2, which may not be a space. The quasi-scaling (1.3) was introduced in [5], and (1.4) was pointed out in [6]. If \( f(u) = u^p \), then \( F(u)^{-N/2} = (p-1)^{N/2}u^{N(p-1)/2} \), and hence Proposition 1.2 is a generalization of Proposition 1.1 for nonnegative initial data. The leading term is not necessarily a pure power, and it can grow more rapidly than the single exponential function. Two nonlinearities \( f(u) = u^p(\log(u+1)) \) and \( e^{u^p} \), \( p \geq 1 \), are included as examples. See Example 2.2 of the present paper. In [6] the authors introduced interesting changes of variables [6, Eqs (1.18), (1.19)] which is denoted by \( \tilde{u} = T(u) \). Using these changes of variables, one can transform (1.2) not exactly but approximately into one of the canonical two equations:

\[
\partial_t \tilde{u} - \Delta \tilde{u} = f_q(\tilde{u}), \quad \text{where} \quad f_q(u) := \begin{cases} u^p, & \frac{1}{p} + \frac{1}{q} = 1, \quad \text{if } q > 1, \\ e^u, & \text{if } q = 1. \end{cases}
\]

See (1.7) for the exact equation for \( \tilde{u} \). In this paper we construct a supersolution of the integral equation of (1.1) which is based on these changes of variables. However, we do not directly use the changes of variables, and our method can be applied to a nonlocal equation.

Let us go back to fractional equations. We focus on the local-in-time solution with unbounded initial data. By [24, Theorem 3] we easily conclude an existence corresponding to Proposition 1.1 (i) and (ii) when \( 0 < \theta \leq 2 \). Li [14] considered (1.1) with \( 1 < \theta < 2 \) when \( f \) is continuous and nondecreasing and \( \phi \in L^r(\mathbb{R}^N) \) is nonnegative. He showed, among other things, that, for \( 1 < \theta < 2 \) and \( 1 < r < \infty \), (1.1) has a local-in-time solution if and only if \( \limsup_{s \to 0} f(s)/s < \infty \) and \( \lim_{s \to \infty} s^{-(1+r\theta/N)} f(s) < \infty \). Therefore, for \( 1 < r < \infty \), the problem (1.1) with \( f(u) = u^p \), \( p > 1 \), has a local-in-time solution if and only if \( r \geq N(p-1)/\theta \). His method was based on Laister et al. [13] which obtained necessary and sufficient conditions on \( f \) for the existence of a local-in-time nonnegative solution for (1.1)
with $\theta = 2$. For $1 < \theta \leq 2$ and $1 \leq r < \infty$, Li [15] constructed a nonnegative nondecreasing nonlinear term $f$ and a nonnegative initial data $\phi \in L^r(\mathbb{R}^N)$ such that $\int_1^\infty ds/f(s) = \infty$ and (1.1) has no local-in-time solution in $L^1_{\text{loc}}(\mathbb{R}^N)$. Hisa-Ishige [10] studied (1.1) with $f(u) = u^p$ when initial data $\phi$ is a Radon measure. They obtained necessary conditions and sufficient conditions for a local-in-time existence. Moreover, the authors obtained an optimal singularity which depends on $p > 1$. For example, they showed that for $\phi(x) = \gamma |x|^{-(\theta/(p - 1))}$ and $p > 1 + \theta/N$, there is $\gamma^* > 0$ such that (1.1) with $f(u) = u^p$ has (resp. does not have) a local-in-time solution if $0 \leq \gamma < \gamma^*$ (resp. $\gamma > \gamma^*$). Let $L^{N(p-1)/\theta, \infty}(\mathbb{R}^N)$ denote a Lorentz space, which we do not define in the present paper. It is known that

$$L^{N(p-1)/\theta}(\mathbb{R}^N) \subseteq L^{N(p-1)/\theta, \infty}(\mathbb{R}^N) \subseteq L^{N(p-1)/\theta - \varepsilon}_{\text{loc}}(\mathbb{R}^N)(\varepsilon > 0 \text{ is small}),$$

which describes the borderline property of the singularity of $|x|^{-(\theta/(p - 1))}$ in $L^{N(p-1)/\theta}(\mathbb{R}^N)$. In the case $f(u) \sim e^{ut^2}$ the existence and nonexistence of a local-in-time solution were studied by [11,12,21], and [7] studies the case $f(u) = |u|^{\alpha} ue^{ut'}$. In [7,12,21] the authors used Orlicz space $\exp L^r(\mathbb{R}^N)$. Note that $\exp L^r(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ for every $q \in [r, \infty)$.

Let us define a solution of (1.1) in $L^r_{ul, \rho}(\mathbb{R}^N)$. The linear fractional heat equation

$$\partial_t u + (-\Delta)^{\theta/2} u = 0 \quad (1.6)$$

admits a fundamental solution $G$. We recall various properties of $G$ in Section 2. Because of the decay estimate (2.1), the following definition of a solution of (1.6) with initial data $\phi$ becomes well-defined: For $\phi \in L^1_{ul, \rho}(\mathbb{R}^N),

$$S(t)[\phi](x) := \int_{\mathbb{R}^N} G(x - y, t) \phi(y) dy, \quad x \in \mathbb{R}^N, \quad t > 0.$$ 

For the optimal class of the positive initial data for $S(t)$, see Bonforte et al. [2].

**Definition 1.3** (Integral solution) Let $u$ and $\bar{u}$ be nonnegative measurable functions on $\mathbb{R}^N \times (0, T)$. We call $u$ a solution of (1.1) if $u$ satisfies the integral equation

$$\infty > u(t) = \mathcal{F}[u](t) \quad \text{a.e.} \ x \in \mathbb{R}^N, \quad 0 < t < T,$$

where

$$\mathcal{F}[u](t) := S(t)[\phi] + \int_0^t S(t - s) f(u(s)) ds.$$ 

We call $\bar{u}$ a supersolution of (1.1) if $\bar{u}$ satisfies the integral inequality

$$\mathcal{F}[\bar{u}](t) \leq \bar{u}(t) \quad \text{a.e.} \ x \in \mathbb{R}^N, \quad 0 < t < T.$$
Note that a mild solution is an integral solution defined by Definition 1.3. See [19, p. 78]. In this paper we do not pursue in which function space the solution \( u(t) \) converges to the initial data \( \phi \) as \( t \to 0 \).

The first main result is about a power case.

**Theorem 1.4 (Algebraic growth \( q > 1 \))** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Assume that \( f \) satisfies (F1) with \( q > 1 \) and that \( \phi \geq 0 \). Then the following hold: (i-1) (Existence, subcritical case 1) Assume that \( r > N/\theta \) and \( r > q - 1 \). If \( F(\phi)^{-r} \in L_{ul,\rho}^1(\mathbb{R}^N) \), then (1.1) has a local-in-time solution in the sense of Definition 1.3. (i-2) (Existence, subcritical case 2) Assume that \( r > N/\theta \) and \( r \geq q - 1 \) and that \( f'(u)F(u) \leq q \) for large \( u > 0 \). If \( F(\phi)^{-r} \in L_{ul,\rho}^1(\mathbb{R}^N) \), then (1.1) has a local-in-time solution in the sense of Definition 1.3. (ii) (Nonexistence, supercritical case) Assume that \( f \) is convex. For any \( r \in (0, N/\theta) \), there is \( \phi \geq 0 \) such that \( F(\phi)^{-r} \in L_{ul,\rho}^1(\mathbb{R}^N) \) and (1.1) has no local-in-time nonnegative solution in the sense of Definition 1.3.

**Remark 1.5** (i) If \( f(u) = u^p \), then \( F(u)^{-N/\theta} = (p - 1)^{N/\theta} u^{N(p - 1)/\theta} \), and hence the critical exponent becomes \( N(p - 1)/\theta \), which was presented in [10, 15, 24]. Hence, Theorem 1.4 is a generalization of Proposition 1.2 (i) and (ii) when \( q > 1 \). (ii) In [6, Theorem 1.1] the inequality \( f'(u)F(u) \leq q \) is assumed to guarantee the continuity at \( t = 0 \). The continuity at \( t = 0 \) is not imposed in our definition of a solution, and \( f'(u)F(u) \leq q \) is not assumed in Theorem 1.4 (i-1). (iii) The critical case \( r = N/\theta > q - 1 \), which corresponds to Proposition 1.2 (ii), is not included in Theorem 1.4. However, if \( f(u) = u^p \) or \( e^u \), then in Sect. 5 we construct a local-in-time solution in \( L_{ul,\rho}^r(\mathbb{R}^N) \). In particular, Theorem 5.1 is an \( L_{ul,\rho}^r(\mathbb{R}^N) \) version of Proposition 1.1 in the case \( \phi \geq 0 \). When \( f(u) = u^p \) and \( \phi \geq 0 \), Theorem 5.1 gives a complete classification in \( L_{ul,\rho}^r(\mathbb{R}^N) \).

(iv) In Theorem 1.4 (ii) we take \( \phi \geq 0 \) such that \( \phi(x) = F^{-1}(|x|^\theta) \), \( \theta < \alpha < N/r \), near the origin. We can choose \( \phi \) such that \( \phi \in L_{ul,\rho}^1(\mathbb{R}^N) \) if \( r \geq q - 1 \).

We consider the exponential growth case \( q = 1 \). In the next main theorem we assume the following:

\[
\text{\( f(u) \) is convex for large \( u > 0 \), and \( f'(u)F(u) \leq 1 \) for large \( u > 0 \). (F2)}
\]

**Theorem 1.6 (Exponential growth \( q = 1 \))** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Assume that \( f \) satisfies (F1) with \( q = 1 \) and that \( \phi \geq 0 \). Then the following hold:

(i) (Existence, subcritical case) Assume that \( r > N/\theta \) and that (F2) holds. If \( F(\phi)^{-r} \in L_{ul,\rho}^1(\mathbb{R}^N) \), then (1.1) has a local-in-time solution in the sense of Definition 1.3.

(ii) (Nonexistence, supercritical case) Assume that \( f(u) \) is convex for \( u \geq 0 \). For any \( r \in (0, N/\theta) \), there is \( \phi \geq 0 \) such that \( F(\phi)^{-r} \in L_{ul,\rho}^1(\mathbb{R}^N) \) and (1.1) does not have a local-in-time nonnegative solution in the sense of Definition 1.3.

**Remark 1.7** (i) The nonlinear terms \( \exp(u^p) \), \( p \geq 1 \), and \( \exp(\cdots \exp(u) \cdots) \) satisfy both (F1) and (F2). The two functions are convex. Therefore, we can apply Theorem 1.6 (i) and (ii). See Example 2.2.
The critical case is not included in Theorem 1.6. However, in Theorem 5.3 we construct a local-in-time solution for $f(u) = e^u$ in the critical case, as mentioned in Remark 1.5 (iii). When $f(u) = e^u$ and $\phi \geq 0$, Theorem 5.3 gives a complete classification in $L^r_{ul,\rho}(\mathbb{R}^N)$.

(iii) In Theorem 1.6 (ii) we take the same $\phi$ as in Theorem 1.4 (ii). We can choose $\phi$ such that $\phi \in L^1_{ul,\rho}(\mathbb{R}^N)$.

Let us explain technical details. The fixed point argument does not manage to construct a solution in $L^r(\mathbb{R}^N)$ or $L^r_{ul,\rho}(\mathbb{R}^N)$ when the nonlinear term grows exponentially. Then we use a monotonicity method (Lemma 2.6). See [20] for details of the method. The initial data has to be nonnegative, while we can deal with rapidly growing nonlinearities. In this method the existence of a supersolution is crucial. A feature of the present paper is supersolutions (3.5) and (3.17). They are inspired by [6,20]. We do not use changes of variables, and hence those supersolutions enable us to construct a solution of equations with nonlocal operators. Those supersolutions look natural in view of [20, Eq (13)] and [6, Eqs (1.18), (1.19)]. In particular, when $f(u) = u^p$ or $e^u$, the supersolutions (5.1), (5.6) and (3.8) are surprisingly simple. However, various estimates in Sect. 3 are nontrivial, and all estimates in the critical case, which are discussed in Sect. 5, are optimal in the sense where the exponent of the time variable $t$ becomes 0. When $\theta = 2$, the way of the construction of the supersolutions can be explained as follows: Assume that $\theta = 2$ and that $f$ satisfies (F1) with $q \geq 1$. Let $u$ be a solution of (1.1) with initial data $\phi$, and let $\tilde{u} := (F_q^{-1} \circ F)(u)$. Then, $\tilde{u}$ satisfies

$$\partial_t \tilde{u} - \Delta \tilde{u} = f_q(\tilde{u}) + q \frac{f'(u)F(u)}{f_q(u)F_q(\tilde{u})} |\nabla \tilde{u}|^2. \quad (1.7)$$

The solvability depends on the behavior of $u$ where $u$ is close to $\infty$. When $u$ is large, by (F1) we see that $f'(u)F(u) \approx q$, and hence $\partial_t \tilde{u} - \Delta \tilde{u} \approx f_q(\tilde{u})$ if $|\nabla \tilde{u}|^2$ is not large. In other words, the general equation (1.1) can be transformed into one of the canonical two equations (1.5) if $u$ is large. On the other hand, the solution of the canonical two equations is approximated by a solution of the linear heat equation in a short time. Therefore, $\tilde{u}(t) \approx (S(t) \circ F_q^{-1} \circ F)(\phi)$ for small $t > 0$. Then, the solution $u$ of the original equation can be approximated by the pullback of the approximate solution of $\partial_t \tilde{u} - \Delta \tilde{u} \approx f_q(\tilde{u})$, i.e., $u(t) \approx (F_q^{-1} \circ F_q \circ S(t) \circ F_q^{-1} \circ F)(\phi)$. See Fig. 1. Modifying this approximate solution, we obtain the supersolutions (3.5) and (3.17). These supersolutions work well even in the case $0 < \theta < 2$.

The proof of the nonexistence is rather standard. Specifically, the decay estimate (3.4) of the solution with the initial data (3.3) contradicts the necessary condition for...
the local-in-time existence given in Proposition 4.1. However, the exponential
decline of the heat kernel is not used in the proof. Sharp estimates, which are newly obtained
in this paper, are required. See Lemmas 4.2 and 4.3.

Let us mention $L^p_{ul,\rho}(\mathbb{R}^N)$ spaces. The $L^p_{ul,\rho} - L^q_{ul,\rho}$ estimate for convolution type
operators was obtained by Maekawa-Terasawa [16, Theorem 3.1]. The proof works
for the case $0 < \theta < 2$ with modifications.

This paper consists of six sections. In Sect. 2 we give examples of Theorems 1.4 and
1.6. We recall properties of the fundamental solution $G$. We prove basic results which
will be used in the proof of main theorems. In Sect. 3 we prove existence theorems,
i.e., Theorems 1.4 (i-1) (i-2) and 1.6 (i). In Sect. 4 we prove nonexistence theorems,
i.e., Theorems 1.4 (ii) and 1.6 (ii). In Sect. 5 we construct a local-in-time solution for
$f(u) = u^p$ or $e^u$ in the critical case. Sect. 6 is a summary and conjectures.

2 Examples and preliminaries

2.1 Example

The following Lemma 2.1 is a fundamental property about the exponent $q$.

Lemma 2.1 Let $f$ be a function such that (F1) with $q$ holds. Then $q \geq 1$.

Proof This is proved by [6, Remark 1.1] and [17, Lemma 2.1]. However, we show the
proof. Suppose the contrary, i.e., there is $q_0 \in (0, 1)$ such that $f'(u) F(u) \leq q_0$ for
$u \geq u_0$. Integrating $f'(u)/f(u) \leq q_0/(f(u) F(u))$ over $[u_0, u]$ twice, we have

$$\frac{u - u_0}{f(u_0)F(u_0)^{q_0}} \leq \frac{1}{1 - q_0} \left( F(u_0)^{1-q_0} - F(u)^{1-q_0} \right) \quad \text{for } u > u_0.$$ 

Then,

$$0 \leq F(u)^{1-q_0} \leq F(u_0)^{1-q_0} - \frac{(1 - q_0)(u - u_0)}{f(u_0)F(u_0)^{q_0}} \to -\infty \quad \text{as } u \to \infty.$$ 

We obtain a contradiction, and hence $q \geq 1$. □

Example 2.2 (i) Let $f(u):=(u + 1)^p \log(u + 1)$, $p > 1$. We have

$$\lim_{u \to \infty} f'(u) F(u) = \lim_{u \to \infty} \frac{f'(u)^2}{f(u) f''(u)} = \frac{p}{p - 1}.$$ 

Moreover, by direct calculation we see that $f'(u)^2/(f(u) f''(u)) \leq p/(p - 1)$. Integrating
$1/f(u) \leq p f''(u)/((p - 1) f'(u)^2)$ over $[u, \infty)$, we see that $f'(u) F(u) \leq p/(p - 1)$ for large $u > 0$. Hence, $f$ satisfies (F1) with $q = p/(p - 1)$ and
$f'(u) F(u) \leq q$ for large $u > 0$. Theorem 1.4 is applicable. The leading term of
$f$ is not necessarily $u^p$.
(ii) Let \( f(u) = \exp(u^p) \), \( p \geq 1 \). By a similar argument as in (i) we can show that \( \lim_{u \to \infty} f'(u) F(u) = 1 \). We can easily see that (F2) holds. Therefore, Theorem 1.6 is applicable.

(iii) Let \( f(u) = \exp(\cdots \exp(u) \cdots) \) be the \( n \)-th iterated exponential function. Then, \( f \) satisfies (F1) with \( q = 1 \) and (F2). Theorem 1.6 is applicable. See [17, Section 2.3].

(iv) Let \( f(u) = \exp(g(u)) \). If \( g \in C^2(0, \infty) \), \( g'(u) \geq 0 \) for \( u \geq 0 \), \( g''(u) \geq 0 \) for \( u \geq 0 \) and \( \lim_{u \to \infty} g''(u)/(g'(u)^2) = 0 \), then \( f \) satisfies (F1) with \( q = 1 \) and (F2). Theorem 1.6 is applicable.

### 2.2 Fundamental solution

Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). The fractional heat equation (1.6) admits a fundamental solution \( G \). We recall various properties of \( G \). The fundamental solution \( G(x, t) \) is expressed by

\[
G(x, t) = \begin{cases} 
(4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right) & \text{if } \theta = 2, \\
\int_0^t g_{t-s}(s)(4\pi s)^{-N/2} \exp\left(-\frac{|x|^2}{4s}\right) ds & \text{if } 0 < \theta < 2,
\end{cases}
\]

where \( g_{t-s}(s) \) is a nonnegative function on \([0, \infty)\) defined by

\[
g_{t-s}(s) := \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \exp\left(zs - tz^\theta\right) dz, \sigma > 0, t > 0.
\]

The fundamental solution \( G \) is a positive smooth function on \( \mathbb{R}^N \times (0, \infty) \). Moreover, the following hold:

\[
G(x, t) = t^{-\frac{N}{\theta}} G(t^{-\frac{1}{\theta}} x, 1),
\]

\[
C^{-1}(1 + |x|)^{-N-\theta} \leq G(x, 1) \leq C(1 + |x|)^{-N-\theta} \quad \text{if } 0 < \theta \leq 2,
\]

\[
G(x, 1) = (4\pi)^{-N/2} \exp(-|x|^2/4) \leq C(1 + |x|)^{-N-\theta} \quad \text{if } \theta = 2,
\]

\( G(\cdot, 1) \) is radially symmetric and \( G(x, 1) \leq G(y, 1) \) if \( |x| \geq |y| \),

\[
G(x, t) = \int_{\mathbb{R}^N} G(x - y, t - s)G(y, s)dy,
\]

\[
\int_{\mathbb{R}^N} G(x, t)dx = 1, \tag{2.1}
\]

for \( x, y \in \mathbb{R}^N \) and \( 0 < s < t \). See e.g., [22, Section 2] for the representation of \( G \) and above properties. These properties are summarized as follows:

**Proposition 2.3** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Then there is a function of one variable \( K(\cdot) \) such that the following hold: \( K(\cdot) \) is positive nonincreasing, \( G(x, t) = t^{-N/\theta} K(t^{-1/\theta}|x|) \) and there is \( C > 0 \) such that \( 0 \leq |x|^{N+\theta} K(|x|) \leq C \) for \( x \in \mathbb{R}^N \). In particular, \( K(|x|) = G(x, 1) \).
Proposition 2.4 Let $N \geq 1$ and $0 < \theta \leq 2$, and let $1 \leq \alpha \leq \beta \leq \infty$. Then there is $C > 0$ such that, for $\phi \in L^\alpha_{ul, \rho}(\mathbb{R}^N)$,
\[
\|S(t)\phi\|_{L^\beta_{ul, \rho}(\mathbb{R}^N)} \leq \left( C \rho^{-N \left( \frac{1}{2} - \frac{1}{\beta} \right)} + Ct^{-\alpha \left( \frac{1}{\alpha} - \frac{1}{\beta} \right)} \right) \|\phi\|_{L^\alpha_{ul, \rho}(\mathbb{R}^N)} \quad \text{for } t > 0. \tag{2.2}
\]

Hence, there are $C_0 > 0$ and $t_0 > 0$ such that
\[
\|S(t)\phi\|_{L^\beta_{ul, \rho}(\mathbb{R}^N)} \leq C_0 t^{-\alpha \left( \frac{1}{\alpha} - \frac{1}{\beta} \right)} \|\phi\|_{L^\alpha_{ul, \rho}(\mathbb{R}^N)} \quad \text{for } 0 < t < t_0. \tag{2.3}
\]

**Proof** The inequality (2.2) follows from the $L^\alpha$-$L^\beta$ inequality
\[
\|S(t)\phi\|_\beta \leq Ct^{-\alpha \left( \frac{1}{\alpha} - \frac{1}{\beta} \right)} \|\phi\|_\alpha \tag{2.4}
\]
and [16, Corollary 3.1] with minor modifications. See [7,9] for (2.4). The inequality (2.3) immediately follows from (2.2). \qed

We show that the constant $C_0 > 0$ in (2.3) can be chosen arbitrarily small if $\phi \in L^\alpha_{ul, \rho}(\mathbb{R}^N)$. This property is used in the critical case.

Proposition 2.5 Let $N \geq 1$ and $0 < \theta \leq 2$, and let $1 \leq \alpha < \beta \leq \infty$. For each $\phi \in L^\alpha_{ul, \rho}(\mathbb{R}^N)$ and each $C_0 > 0$, there is $t_0 = t_0(\phi, C_0) > 0$ such that
\[
\|S(t)\phi\|_{L^\beta_{ul, \rho}(\mathbb{R}^N)} \leq C_0 t^{-\alpha \left( \frac{1}{\alpha} - \frac{1}{\beta} \right)} \|\phi\|_{L^\alpha_{ul, \rho}(\mathbb{R}^N)} \quad \text{for } 0 < t < t_0. \tag{2.5}
\]

**Proof** We follow the proof of [3, Lemma 8]. Let $\gamma := \frac{N}{\alpha} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right)$. By (2.3) we see the following: For any $\psi \in L^\infty$, there is $t_0 > 0$ such that
\[
t^\gamma \|S(t)\phi\|_{L^\beta_{ul, \rho}(\mathbb{R}^N)} \leq t^\gamma \|S(t)(\phi - \psi)\|_{L^\beta_{ul, \rho}(\mathbb{R}^N)} + t^\gamma \|S(t)\psi\|_{L^\beta_{ul, \rho}(\mathbb{R}^N)} \\
\leq C \|\phi - \psi\|_{L^\alpha_{ul, \rho}(\mathbb{R}^N)} + Ct^\gamma \|\psi\|_{\infty}
\]
for $0 < t < t_0$. Then,
\[
\limsup_{t \to 0} t^\gamma \|S(t)\phi\|_{L^\beta_{ul, \rho}(\mathbb{R}^N)} \leq C \|\phi - \psi\|_{L^\alpha_{ul, \rho}(\mathbb{R}^N)}.
\]

Since $\phi \in L^\alpha_{ul, \rho}(\mathbb{R}^N)$, it follows from the definition of $L^\alpha_{ul, \rho}(\mathbb{R}^N)$ that we can choose $\psi \in BUC(\mathbb{R}^N) (\subset L^\infty(\mathbb{R}^N))$ such that $\|\phi - \psi\|_{L^\alpha_{ul, \rho}(\mathbb{R}^N)}$ is arbitrarily small. Thus, (2.5) holds. \qed
2.3 Preliminaries

First we recall the monotonicity method.

**Lemma 2.6** Let \(0 < T \leq \infty\) and let \(f\) be a continuous nondecreasing function such that \(f(0) \geq 0\). The problem (1.1) has a solution in the sense of Definition 1.3 if and only if (1.1) has a supersolution.

**Proof** This lemma is well known. See [20, Theorem 2.1] for details. However, we briefly show the proof for readers’ convenience.

If (1.1) has a solution, then the solution is also a supersolution. Thus, it is enough to show that (1.1) has a solution if it has a supersolution. Let \(\bar{u}\) be a supersolution in \(\mathbb{R}^N \times (0, T)\). Let \(u_1 = S(t)\phi\). We define \(u_n, n = 2, 3, \ldots\), by

\[ u_n = F[u_{n-1}]. \]

Then we can show by induction that

\[ 0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq \bar{u} < \infty \text{ a.e. } x \in \mathbb{R}^N, \ 0 < t < T. \]

This indicates that the limit \(\lim_{n \to \infty} u_n(x, t)\) which is denoted by \(u(x, t)\) exists for almost all \(x \in \mathbb{R}^N\) and \(0 < t < T\). By the monotone convergence theorem we see that

\[ \lim_{n \to \infty} F[u_{n-1}] = F[u], \]

and hence \(u = F[u]\). Then, \(u\) is a solution of (1.1) in the sense of Definition 1.3. \(\Box\)

Hereafter in this section we prove several useful lemmas.

**Lemma 2.7** Let \(f\) be a function such that (F1) with \(q \geq 1\) holds. Assume that there are \(\alpha \in [q, \infty)\) and \(u_0 > 0\) such that \(f'(u)F(u) \leq q_0\) for \(u \geq u_0\), then \(f(u) \leq f(u_0)^{q_0}/F(u)^{q_0}\) for \(u \geq u_0\).

**Proof** Integrating \(f'(u)/f(u) \leq q_0/(f(u)F(u))\) over \([u_0, u]\), we have that \(\log (f(u)/f(u_0)) \leq q_0 \log (F(u_0)/F(u))\), and hence we obtain the conclusion. \(\Box\)

Let \(f_q(u)\) be defined by (1.5). We define \(F_q(u)\) by

\[ F_q(u) := \int_u^\infty \frac{dt}{f_q(t)} = \begin{cases} \frac{1}{p-1}u^{-p+1}, & \frac{1}{p} + \frac{1}{q} = 1, \quad \text{if } q > 1, \\ e^{-u} & \text{if } q = 1. \end{cases} \quad (2.6) \]

**Lemma 2.8** Let \(f\) be a function such that (F1) with \(q \geq 1\) holds. Assume that there are \(\alpha \in [q, \infty)\) and \(u_0 > 0\) such that \(f'(u)F(u) \leq \alpha\) for \(u \geq u_0\). Let \(\Phi_\alpha(u)\) be defined by

\[ \Phi_\alpha(u) := F_\alpha^{-1}(F(u)) = \begin{cases} (\alpha - 1)^{-1}F(u)^{-\alpha}, & \text{if } \alpha > 1, \\ -\log F(u) & \text{if } \alpha = 1. \end{cases} \quad (2.7) \]

\(\Box\ Springer}
Then the following (i) (ii) and (iii) hold:

(i) \( \Phi_\alpha \in C^2(0, \infty) \),
(ii) \( \Phi'_\alpha(u) > 0 \) for \( u > 0 \),
(iii) \( \Phi''_\alpha(u) \geq 0 \) for \( u \geq u_0 \), and hence \( \Phi(u) \) is convex in \([u_0, \infty)\).

**Proof** It is clear that \( \Phi_\alpha \in C^2 \), since \( F(u) \in C^2 \). We have

\[
\Phi'_\alpha(u) = \begin{cases} \frac{(\alpha-1)^\alpha}{f(u)^\alpha} & \text{if } \alpha > 1, \\ \frac{1}{f(u)} & \text{if } \alpha = 1, \end{cases} \\
\Phi''_\alpha(u) = \begin{cases} \frac{(\alpha-1)^\alpha - f'(u)F(u)}{f(u)^{\alpha+1}} & \text{if } \alpha > 1, \\ \frac{1-f'(u)F(u)}{f(u)^2} & \text{if } \alpha = 1, \end{cases}
\]

The assertion (ii) follows from (2.8). If \( u \geq u_0, \) then \( \alpha - f'(u)F(u) \geq 0 \), and hence (iii) follows from (2.8).

\(\Box\)

**Proposition 2.9** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Let \( \Psi(\cdot) \in C[0, \infty) \) be a convex function, and let \( \psi(x) \) be a nonnegative function. Then the following (i) and (ii) hold:

(i) \( \psi \in L^\gamma_{ul, \rho}(\mathbb{R}^N) \) for some \( 1 \leq \gamma \leq \infty \), then

\[
\Psi(S(t)[\psi](x)) \leq S(t)[\Psi(\psi)](x) \text{ for } x \in \mathbb{R}^N, \ t > 0.
\]

(ii) Assume in addition that \( \Psi \geq 0 \) and \( \Psi \neq 0 \). If \( \Psi(\psi) \in L^\gamma_{ul, \rho}(\mathbb{R}^N) \) for some \( 1 \leq \gamma \leq \infty \), then (2.9) holds.

**Proof** Since we could not find a proof of Jensen’s inequality for \( \psi \in L^\gamma_{ul, \rho}(\mathbb{R}^N) \) in literature, we show the proof.

(i) Since \( L^\gamma_{ul, \rho}(\mathbb{R}^N) \subset L^1_{ul, \rho}(\mathbb{R}^N) \), we see that \( \psi \in L^1_{ul, \rho}(\mathbb{R}^N) \). Let \( \chi_{B(n)}(x) \) be the indicator function supported on \( B(n) := \{x \in \mathbb{R}^N | |x| \leq n\} \), and let \( \psi_n(x) := \psi(x)\chi_{B(n)}(x) \). We can easily see that \( \psi_n \in L^1(\mathbb{R}^N) \). Since

\[
G(x - \cdot, t) \geq 0, \quad \int_{\mathbb{R}^N} G(x - y, t) dy = 1 \quad \text{and} \quad \psi_n \in L^1(\mathbb{R}^N),
\]

by the classical Jensen’s inequality we have

\[
\Psi(S(t)[\psi_n](x)) \leq S(t)[\Psi(\psi_n)](x).
\]

Since \( \psi_n \leq \psi \), we see that

\[
S(t)[\Psi(\psi_n)](x) \leq S(t)[\Psi(\psi)](x).
\]

On the other hand, by the monotone convergence theorem we see that,

\[
\int_{\mathbb{R}^N} G(x - y, t) \psi_n(y) dy \to \int_{\mathbb{R}^N} G(x - y, t) \psi(y) dy \text{ as } n \to \infty.
\]
and hence
\[ S(t)[\psi_n](x) \to S(t)[\psi](x) \text{ as } n \to \infty. \] (2.12)

Since \( \Psi \) is continuous, by (2.10), (2.11) and (2.12) we have
\[ S(t)[\Psi(\psi)](x) \geq \Psi(S(t)[\psi_n])(x) \to \Psi(S(t)[\psi])(x) \text{ as } n \to \infty. \]

We obtain (2.9).

(ii) If we show that \( \psi \in \mathcal{L}_{1, \rho}^1(\mathbb{R}^N) \), then we can use (i), and the conclusion holds. Hereafter, we show that \( \psi \in \mathcal{L}_{1, \rho}^1(\mathbb{R}^N) \). Because of the assumption on \( \Psi \), there are \( a > 0 \) and \( b \in \mathbb{R} \) such that \( \Psi(u) \geq au - b \) for \( u \geq 0 \). When \( \gamma = 1 \) or \( \gamma = \infty \), there is \( C_0 > 0 \) such that
\[ a \int_{B_y(\rho)} \psi(x)dx \leq b|B_y(\rho)| + \int_{B_y(\rho)} \Psi(\psi(x))dx < C_0 \text{ uniformly for } y \in \mathbb{R}^N. \] (2.13)

We see that \( \psi \in \mathcal{L}_{1, \rho}^1(\mathbb{R}^N) \), and hence the proof is complete. When \( 1 < \gamma < \infty \), we have
\[ \int_{B_y(\rho)} \Psi(\psi(x))dx \leq \| \Psi(\psi) \|_{\mathcal{L}_\gamma(B_y(\rho))} \| 1 \|_{\mathcal{L}_{\gamma'}(B_y(\rho))}, \]
where \( \gamma' := \gamma/(1 - \gamma) \). By the same inequality as (2.13) we see that \( \psi \in \mathcal{L}_{1, \rho}^1(\mathbb{R}^N) \). The proof is complete.

\[ \square \]

3 Existence in the subcritical case

3.1 Algebraic growth case

In this subsection we mainly prove Theorem 1.4 (i-1). Let \( q > 1 \) and \( r > \max\{N/\theta, q - 1\} \) be given in Theorem 1.4 (i-1). Let
\[ 0 < \varepsilon < \min\left\{ \frac{\theta r}{N} - 1, r - q + 1, 2(q - 1) \right\} \text{ and } \delta := \frac{\varepsilon}{2}. \] (3.1)

Let \( f \) be a function such that (F1) with \( q > 1 \) holds. Then, there is \( u_1 > 0 \) such that
\[ f'(u)F(u) \leq q + \delta \text{ for } u \geq u_1. \] (3.2)

For simplicity we write \( q_0 := q + \delta \) and \( p_0 := q_0/(q_0 - 1) \). In particular, \( 1 < q < q_0 \).
Lemma 3.1 The following (i) and (ii) hold:

(i) There is $C > 0$ such that

$$F(u) \frac{1}{p_0 - 1} \leq Cu \quad \text{for} \quad u \geq 1.$$  

(ii) There is $u_2 > 0$ such that if $u \geq u_2$, then

$$F \left( \frac{u}{\sqrt{1 + \sigma}} \right) \leq (1 + \sigma)^{p_0 - 1} F(u) \quad \text{for} \quad 0 \leq \sigma \leq 1.$$

Proof (i) Let $\xi(u) := \log u - \log F(u)^{\frac{1}{p_0 - 1}} - \varepsilon$. Then,

$$\xi'(u) = \frac{f(u)F(u) - \left( \frac{1}{p_0 - 1} - \varepsilon \right) u}{uf(u)F(u)}.$$

Let $\eta(u) := f(u)F(u) - \left( \frac{1}{p_0 - 1} - \varepsilon \right) u$. Then,

$$\eta'(u) = f'(u)F(u) - 1 - \frac{1}{p_0 - 1} + \varepsilon$$
$$= q - 1 - (q_0 - 1) + \varepsilon \quad (u \to \infty)$$
$$= \varepsilon/2 > 0.$$

Thus, $\xi(u)$ is increasing for large $u$. Since $\xi(u)$ is continuous on $u \geq 1$, there is $C > 0$ such that $\xi(u) > \log C$ for $u \geq 1$. The conclusion of (i) holds.

(ii) Since $p_0 - 1 = \frac{1}{(q_0 - 1)}$, we define $\xi(\sigma) := \log F(u) + \frac{1}{q_0 - 1} \log(1 + \sigma) - \log F(\frac{u}{\sqrt{1 + \sigma}})$. Then,

$$\xi'(\sigma) = \frac{1}{(1 + \sigma)(q_0 - 1)} \frac{f(\frac{u}{\sqrt{1 + \sigma}})F(\frac{u}{\sqrt{1 + \sigma}}) - (q_0 - 1)u}{f(\frac{u}{\sqrt{1 + \sigma}})F(\frac{u}{\sqrt{1 + \sigma}})}.$$

Let $\eta(v) := f(v)F(v) - (q_0 - 1)v/2$. Then

$$\eta'(v) = f'(v)F(v) - 1 - \frac{q_0 - 1}{2} \to q - 1 - \frac{\delta}{2} \quad (> 0) \quad \text{as} \quad \sigma \to \infty.$$

Thus, $\eta(v) \to \infty$ as $\sigma \to \infty$, and hence there is $u_2 > 0$ such that $\eta(v) \geq 0$ for $v \geq u_2/\sqrt{2}$. If $u \geq u_2$, then

$$f(\frac{u}{\sqrt{1 + \sigma}})F(\frac{u}{\sqrt{1 + \sigma}}) - (q_0 - 1)u = \eta(\frac{u}{\sqrt{1 + \sigma}}) \geq 0 \quad \text{for} \quad 0 \leq \sigma \leq 1,$$

and hence $\xi'(\sigma) \geq 0$ for $0 \leq \sigma \leq 1$. Since $\xi(0) = 0$, we see that if $u \geq u_2$, then $\xi(\sigma) \geq 0$ for $0 \leq \sigma \leq 1$. The conclusion of (ii) holds. □

Hereafter, we define $u_0$ by

$$u_0 := \max\{u_1, u_2\},$$
where \( u_1 \) is given in (3.2) and \( u_2 \) is given in Lemma 3.1 (ii).

**Lemma 3.2** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Let \( \psi(x) \in L^\gamma_{ul,\rho} (\mathbb{R}^N), \ 1 \leq \gamma \leq \infty \), be a function such that \( \psi(x) \geq u_0 \). Then the following holds:

\[
S(t)[\psi](x) \leq F^{-1} \left( F_{q_0} \left( S(t) \left[ F_{q_0}^{-1} (F(\psi)) \right] \right) \right) (x) \quad \text{for} \quad x \in \mathbb{R}^N \text{ and } t > 0,
\]

where \( F_{q_0} \) is defined by (2.6) with \( q = q_0 \).

**Proof** Let \( \Phi_\alpha \) be defined by (2.7) with \( \alpha = q_0 \). Applying Lemma 2.8 (iii) with \( \alpha \), we see that \( \Phi_\alpha(u) \) is convex in \([u_0, \infty)\). Since \( \psi \geq u_0 \), by Proposition 2.9 we have that \( \Phi_\alpha(S(t)\psi) \leq S(t)\Phi_\alpha(\psi) \). It follows from (2.8) that \( \Phi'_\alpha > 0 \), and hence \( \Phi^{-1}_\alpha \) is increasing and

\[
S(t)\psi \leq \Phi^{-1}_\alpha(S(t)\Phi_\alpha(\psi)).
\]

Since \( \Phi_\alpha(u) = F^{-1}_\alpha(F(u)) \) and \( \Phi^{-1}_\alpha(u) = F^{-1}(F_\alpha(u)) \), the inequality (3.3) follows from (3.4).

Let us introduce the following function:

\[
\tilde{u}(t) := \left( F^{-1} \circ F_{q_0} \circ (1 + \sigma)S(t) \circ F_{q_0}^{-1} \circ F \right) (\phi_0)
\]

\[
= F^{-1} \left( \left( (1 + \sigma)S(t) \left[ F(\phi_0) \right]^{\frac{1}{p_0-1}} \right)^{-(p_0-1)} \right),
\]

where \( \phi_0(x) := \max \{ \phi(x), u_0 \}, \ 0 < \sigma \leq 1 \) and we define \((1 + \sigma)S(t)[u] = (1 + \sigma)(S(t)[u])\). By (2.8) we see that \( F_{q_0}^{-1}(F(u)) \) is increasing in \( u \). We easily see that

\[
\tilde{u} \geq u_0.
\]

Since \( \| \phi_0 \|_{L^\gamma_{ul,\rho}(\mathbb{R}^N)} \leq \| \phi \|_{L^\gamma_{ul,\rho}(\mathbb{R}^N)} + \| u_0 \|_{L^\gamma_{ul,\rho}(\mathbb{R}^N)} < \infty \), we see that \( \phi_0 \in L^\gamma_{ul,\rho}(\mathbb{R}^N) \). By (3.3) and (3.5) we have

\[
S(t)\phi_0 \leq F^{-1} \left( (1 + \sigma)^{p_0-1} F(\tilde{u}) \right).
\]

**Lemma 3.3** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Assume that \( r > N/\theta \), \( r > q - 1 \), that \( \phi \geq 0 \) and that \( f \) satisfies (F1) with \( q > 1 \). If \( F(\phi)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N) \), then there is \( T > 0 \) such that \( \tilde{u}(t) \) defined by (3.5) is a supersolution of (1.1) for \( 0 < t < T \).

**Proof** We show that \( \mathcal{F}[\tilde{u}] \leq \tilde{u} \). We note that \( q_0 \geq q > 1 \). Since \( \phi(x) \leq \phi_0(x), \) by (3.7) and Lemma 3.1 (ii) we have

\[
S(t)\phi \leq S(t)\phi_0 \leq F^{-1} \left( (1 + \sigma)^{p_0-1} F(\tilde{u}) \right) \leq \frac{\tilde{u}}{\sqrt{1 + \sigma}}.
\]
Because of (3.6), by Lemma 2.7 we have that

$$f(\tilde{u}) \leq \frac{f(u_0)F(u_0)^{q_0}}{F(\tilde{u})^{q_0}}. \quad (3.9)$$

By (3.1) we see that \((p_0 - 1)r = r/(q + \varepsilon - 1) \geq 1\). By (2.3) we see that there is \(T > 0\) such that

$$\|S(t)F(\phi_0)^{-1}\|_\infty \leq C t^{-\frac{N}{p(r(p_0 - 1))}} \left\| F(\phi_0)^{-1} \right\|_{L_{ul, \rho}^{(p_0 - 1)r} (\mathbb{R}^N)}$$

for \(0 < t < T\).

Using (3.9), (3.10) and Lemma 3.1 (i), we have

$$\int_0^t S(t - s)f(\tilde{u}(s))ds \leq C \int_0^t S(t - s) \left[ F[\tilde{u}]^{-q_0} \right] ds$$

$$= C(1 + \sigma)^{p_0} \int_0^t S(t - s) \left[ \left( S(s) \left[ F(\phi_0)^{-1}\right] \right) \left( S(s) \left[ F(\phi_0)^{-1}\right] \right) \right]^{p_0 - 1} ds$$

$$\leq C(1 + \sigma)^{p_0} \left[ F(\phi_0)^{-1}\right] \int_0^t \left\| S(s) \left[ F(\phi_0)^{-1}\right] \right\|_\infty^{p_0 - 1} ds$$

$$= C(1 + \sigma)^{p_0} \left( S(t) \left[ F(\phi_0)^{-1}\right] \right)^{1 - (p_0 - 1)\varepsilon} \left\| S(t) \left[ F(\phi_0)^{-1}\right] \right\|_\infty^{(p_0 - 1)\varepsilon}$$

$$\times \int_0^t \left\| S(s) \left[ F(\phi_0)^{-1}\right] \right\|_\infty^{p_0 - 1} ds$$

$$\leq C(1 + \sigma)^{(p_0 - 1)(1 + \varepsilon)} F(\tilde{u})^{-1 + \varepsilon} \left( C t^{-\frac{N}{p(r(p_0 - 1))}} \left\| F(\phi_0)^{-1}\right\|_{L_{ul, \rho}^{(p_0 - 1)r} (\mathbb{R}^N)} \right)^{(p_0 - 1)\varepsilon}$$

$$\times \int_0^t \left( C s^{-\frac{N}{p(r(p_0 - 1))}} \left\| F(\phi_0)^{-1}\right\|_{L_{ul, \rho}^{(p_0 - 1)r} (\mathbb{R}^N)} \right)^{(p_0 - 1)} ds$$

$$\leq C\tilde{u}(1 + \sigma)^{(p_0 - 1)(1 + \varepsilon)} \left\| F(\phi_0)^{-1}\right\|_{L_{ul, \rho}^{(p_0 - 1)r} (\mathbb{R}^N)} \left( 1 + \varepsilon \right) \left( p_0 - 1 \right)$$

$$\times \int_0^t s^{-\frac{N}{p(r)\varepsilon}} ds$$

$$\leq C\tilde{u}(1 + \sigma)^{(p_0 - 1)(1 + \varepsilon)} \left\| F(\phi_0)^{-1}\right\|_{L_{ul, \rho}^{(p_0 - 1)r} (\mathbb{R}^N)} \left( t^{-\frac{N}{p(r)\varepsilon}} \int_0^t s^{-\frac{N}{p(r)}} ds \right)$$

Here, \(\int_0^t s^{-\frac{N}{p(r)}} ds\) in the above calculation is integrable. By (3.1) we see that \(1 - (1 + \varepsilon)N/(\theta r) > 0\), and hence there is a small \(T > 0\) such that if \(0 < t < T\), then

$$C(1 + \sigma)^{(p_0 - 1)(1 + \varepsilon)} \left\| F(\phi_0)^{-1}\right\|_{L_{ul, \rho}^{(p_0 - 1)r} (\mathbb{R}^N)} \left( t^{1 - \frac{(1 + \varepsilon)N}{\theta r}} \right) < \frac{\sqrt{1 + \sigma} - 1}{\sqrt{1 + \sigma}}. \quad (3.11)$$
Here, we can choose $T > 0$, which is still denoted by $T$, such that both (3.10) and (3.11) hold. Using (3.11) and (3.8), we have

$$\mathcal{F}[\bar{u}] = S(t)\phi + \int_0^t S(t-s) f(\bar{u}(s))ds$$

$$\leq \frac{\bar{u}}{\sqrt{1 + \sigma}} + \frac{(\sqrt{1 + \sigma} - 1)\bar{u}}{\sqrt{1 + \sigma}} = \bar{u} \text{ for } 0 < t < T.$$ 

Therefore, $\bar{u}$ is a supersolution.  

Lemma 3.4 Let $N \geq 1$ and $0 < \theta \leq 2$. Assume that $r > N/\theta$, $r \geq q - 1$, that $\phi \geq 0$, that $f$ satisfies $(F1)$ with $q > 1$ and that $f'(u)F(u) \leq q$ for large $u > 0$. If $F(\phi)^{-r} \in L^1_{u_0}(\mathbb{R}^N)$, then there is $T > 0$ such that $\bar{u}(t)$ defined by (3.5) is a supersolution of (1.1) for $0 < t < T$.

**Proof** Let $\bar{u}$ be given by (3.5) with $q_0 = q$. Because $f'(u)F(u) \leq q$ for large $u > 0$, we can show that $\bar{u}$ is a supersolution as follows: Lemma 3.1 (i) and (ii) hold if the proofs are slightly modified. Lemmas 3.2 holds without modification. In the proof of Lemma 3.3 we use $(p_0 - 1)r = r/(q + \varepsilon - 1) \geq 1$. Then, the conclusion of Lemma 3.4 holds. The details are omitted.  

\section{3.2 Exponential growth case}

We consider the case $q = 1$. Let $r$ be given in Theorem 1.6 (i).

Lemma 3.5 Assume that $f$ satisfies $(F1)$ with $q = 1$ and $(F2)$. For $\sigma > 0$, $\alpha > 0$ and $C_1 > 0$, there is $u_1 > 0$ such that

$$F(u - C_1 F(u)^{\alpha}) \leq e^{\sigma} F(u) \text{ for } u \geq u_1.$$ 

**Proof** It is enough to show that

$$\lim_{u \to \infty} \frac{F(u - C_1 F(u)^{\alpha})}{F(u)} = 1. \quad (3.12)$$

By L’Hospital’s rule we have

$$\lim_{u \to \infty} \frac{F(u - C_1 F(u)^{\alpha})}{F(u)} = \lim_{u \to \infty} \frac{f(u) + C_1 \alpha F(u)^{\alpha - 1}}{f(u - C_1 F(u)^{\alpha})}. \quad (3.13)$$

Since $f$ is convex for large $u > 0$, we have

$$f(u - C_1 F(u)^{\alpha}) \geq f(u) - C_1 f'(u) F(u)^{\alpha} \text{ for large } u > 0. \quad (3.14)$$

First, we consider the case $\alpha \geq 1$. We easily see that

$$f(u) \left(1 - C_1 f'(u) F(u) \frac{F(u)^{\alpha - 1}}{f(u)}\right) > 0 \text{ for large } u > 0. \quad (3.15)$$
By (3.14) and (3.15) we have
\[
1 \leq \lim_{u \to \infty} \frac{f(u) + C_1 \alpha F(u)^{\alpha - 1}}{f(u) - C_1 F(u)^{\alpha}} \leq \lim_{u \to \infty} \frac{f(u) + C_1 \alpha F(u)^{\alpha - 1}}{f(u) - C_1 f'(u) F(u)^{\alpha}}
\]
\[
= \lim_{u \to \infty} \frac{1 + C_1 \alpha \frac{F(u)^{\alpha - 1}}{f(u)}}{1 - C_1 f'(u) \frac{F(u)^{\alpha - 1}}{f(u)}} = 1.
\]

Thus, the limit in (3.13) is 1. We obtain (3.12).

Second, we consider the case \(0 < \alpha < 1\). We have
\[
(f(u) F(u)^{1-\alpha})' = \frac{f'(u) F(u) - 1 + \alpha}{F(u)^\alpha} \to \infty \quad \text{as} \quad u \to \infty.
\]
Therefore, \(f(u) F(u)^{1-\alpha} \to \infty\) as \(u \to \infty\). We easily see that
\[
f(u) \left(1 - C_1 \frac{f'(u) F(u)}{f(u) F(u)^{1-\alpha}}\right) > 0 \quad \text{for large } u > 0.
\]
(3.16)

By (3.14) and (3.16) we have
\[
1 \leq \lim_{u \to \infty} \frac{f(u) + C_1 \alpha F(u)^{\alpha - 1}}{f(u) - C_1 F(u)^{\alpha}} \leq \lim_{u \to \infty} \frac{f(u) + C_1 \alpha F(u)^{\alpha - 1}}{f(u) - C_1 f'(u) F(u)^{\alpha}}
\]
\[
= \lim_{u \to \infty} \frac{1 + C_1 \alpha \frac{F(u)^{\alpha - 1}}{f(u)}}{1 - C_1 f'(u) \frac{F(u)^{\alpha - 1}}{f(u)}} = 1.
\]

We see that the limit in (3.13) is 1. We obtain (3.12).

Because of (F2), there is \(u_2 > 0\) such that \(f(u)\) is convex on \([u_2, \infty)\) and that \(f'(u) F(u) \leq 1\) for \(u > u_2\).

In this subsection we define \(u_0\) by
\[
u_0 := \max\{u_1, u_2\},
\]
where \(u_1\) is given Lemma 3.5.

**Corollary 1** Let \(N \geq 1\) and \(0 < \theta \leq 2\). Assume that \(f\) satisfies (F1) with \(q = 1\) and (F2). Let \(\psi \in L^\gamma_{ul,\rho}(\mathbb{R}^N), 1 \leq \gamma \leq \infty,\) be a function such that \(\psi(x) \geq u_0\). Then the following holds:
\[
S(t)[\psi](x) \leq F^{-1} \left( F_1 \left( S(t) \left[ F^{-1}_1 \left( F(\psi) \right) \right] \right) \right)(x) \quad \text{for } x \in \mathbb{R}^N \text{ and } t > 0,
\]
where \(F_1\) is defined by (2.6) with \(q = 1\).

**Proof** The proof is the same as that of Lemma 3.2 with \(q_0 = 1\). We omit the details. □
Let us introduce the following function:

\[
\tilde{u}(t) = \left( F^{-1} \circ F_1 \circ (S(t) + \sigma) \circ F_1^{-1} \circ F \right)(\phi_0)
= F^{-1} \left( e^{-\sigma} \exp \left( S(t) \left[ \log F(\phi_0) \right] \right) \right),
\]

where \( \phi_0(x) := \max\{\phi(x), u_0\} \), \( \sigma > 0 \) and we define \((S(t) + \sigma)[u] := S(t)[u] + \sigma\). We easily see that

\[
\tilde{u} \geq u_0.
\] (3.18)

Since \( \|\phi_0\|_{L^r_{ul,\rho}(\mathbb{R}^N)} \leq \|\phi\|_{L^r_{ul,\rho}(\mathbb{R}^N)} + \|u_0\|_{L^r_{ul,\rho}(\mathbb{R}^N)} < \infty \), we see that \( \phi_0 \in L^r_{ul,\rho}(\mathbb{R}^N) \).

By Corollary 1 and (3.17), we have

\[
S(t)\phi_0 \leq F^{-1} \left( e^{\sigma} F(\tilde{u}) \right).
\] (3.19)

**Lemma 3.6** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Assume that \( r > N/\theta \), that \( \phi \geq 0 \) and that \( f \) satisfies \((F1)\) with \( q = 1 \) and \((F2)\) hold. If \( F(\phi)^{-1} \in L^r_{ul,\rho}(\mathbb{R}^N) \), then there is \( T > 0 \) such that \( \tilde{u}(t) \) defined by (3.17) is a supersolution of (1.1) for \( 0 < t < T \).

**Proof** We show that \( \mathcal{F}[\tilde{u}] \leq \tilde{u} \). Because of (3.18), by Lemma 2.7 we have

\[
f(\tilde{u}) \leq \frac{f(u_0)F(u_0)}{F(\tilde{u})}.
\] (3.20)

First, we consider the case \( r \geq 1 \). Using (3.20), Proposition 2.9 and (2.3), we have

\[
\int_0^t S(t-s)f(\tilde{u}(s))ds \leq C \int_0^t S(t-s)\left[ F(\tilde{u})^{-1} \right] ds
= C \int_0^t e^{\sigma} S(t-s) \left[ \exp \left( S(s) \left[ \log F(\phi_0)^{-1} \right] \right) \right] ds
\leq C e^{\sigma} \int_0^t S(t-s)\left[ S(s) \left[ F(\phi_0)^{-1} \right] \right] ds
\leq C e^{\sigma} S(t) \left[ F(\phi_0)^{-1} \right] \int_0^t ds
\leq C C_0 e^{\sigma} \left\| F(\phi_0)^{-1} \right\|_{L^r_{ul,\rho}(\mathbb{R}^N)} t^{1-\frac{N}{\theta r}}.
\] (3.21)

Using (3.19) and (3.21) and Lemma 3.5, we have

\[
\mathcal{F}[\tilde{u}] \leq S(t)\phi_0 + \int_0^t S(t-s)f(\tilde{u}(s))ds
\leq F^{-1} \left( e^{\sigma} F(\tilde{u}) \right) + C C_0 e^{\sigma} \left\| F(\phi_0)^{-1} \right\|_{L^r_{ul,\rho}(\mathbb{R}^N)}^{\frac{1}{r}} t^{1-\frac{N}{\theta r}}.
\]
\[
\leq \tilde{u} - C_1 F(\tilde{u})^\alpha + CC_0 e^{\sigma} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{\frac{1}{r}} t^{1 - \frac{N}{\theta r}}, \tag{3.22}
\]

where we define \(\alpha := \theta r / N - 1 > 0\) and \(C_1 := CC_0^{\theta r / N} e^{\sigma r / N} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{\theta r / N} \).

By Proposition 2.9 and (2.3) we have

\[
\frac{1}{F(\tilde{u})} = e^{\sigma} \exp \left( S(t) \left[ \log F(\phi_0)^{-1} \right] \right) \leq e^{\sigma} S(t) \left[ F(\phi_0)^{-1} \right] \\
\leq C_0 e^{\sigma} \left\| F(\phi_0)^{-1} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{\frac{\alpha}{r}} t^{\frac{\alpha N}{\theta r}}.
\]

Hence, \(F(\tilde{u})^\alpha \geq C_0^{-\alpha} e^{-\alpha \sigma} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{-\alpha/r} t^{\frac{\alpha N}{\theta r}} \). By (3.22) we have

\[
\mathcal{F}[\tilde{u}] \leq \tilde{u} - C_1 C_0^{-\alpha} e^{-\alpha \sigma} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{-\alpha/r} t^{\frac{\alpha N}{\theta r}} \\
+ C C_0 e^{\sigma} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{\frac{1}{r}} t^{1 - \frac{N}{\theta r}} \\
= \tilde{u} \text{ for } 0 < t < T,
\]

where we use \(\frac{\alpha N}{\theta r} = 1 - \frac{N}{\theta r} > 0\). Thus, \(\tilde{u}\) is a supersolution.

Second, we consider the case \(r < 1\). By Proposition 2.9 we have

\[
\frac{1}{F(\tilde{u})} = e^{\sigma} \left( \exp \left( S(t) \left[ \log F(\phi_0)^{-r} \right] \right) \right) t^{\frac{1}{r}} \leq e^{\sigma} \left( S(t) \left[ F(\phi_0)^{-r} \right] \right) t^{\frac{1}{r}}. \tag{3.23}
\]

By (3.23) and (2.3) we have

\[
\int_0^t S(t-s) f(\tilde{u}(s)) ds \\
\leq C \int_0^t S(t-s) \left[ F(\tilde{u})^{-1} \right] ds \\
\leq C e^{\sigma} \int_0^t S(t-s) \left[ S(s) \left[ F(\phi_0)^{-r} \right] \right] \left\| S(s) \left[ F(\phi_0)^{-r} \right] \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{-\frac{1}{r}} ds \\
= C e^{\sigma} S(t) \left[ F(\phi_0)^{-r} \right] \int_0^t \left\| S(s) \left[ F(\phi_0)^{-r} \right] \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{-\frac{1}{r}} ds \\
\leq C e^{\sigma} C_0 t^{-\frac{\alpha N}{\theta r}} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{\frac{1}{r}} \int_0^t \left( C_0 s^{-\frac{\alpha N}{\theta r}} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{\frac{1}{r}} \right) ds \\
\leq CC_0^{\frac{1}{r}} e^{\sigma} \left\| F(\phi_0)^{-r} \right\|_{L^1_{ul,\rho}(\mathbb{R}^N)}^{\frac{1}{r}} t^{1 - \frac{N}{\theta r}}.
\]
Here, \( \int_0^t s^{-\alpha t} (t - s)^{1-\frac{1}{\theta}} ds \) is integrable, since \(-N/\theta (1/\theta - 1) > -1\). We define \( \alpha := \theta r/N - 1 \) and \( C_1 = C\|F(\phi_0)^{-r}\|_{L_{ul,\rho}(\mathbb{R}^N)}^{\theta/N} \). Using (3.23) and (2.3), we have

\[
\frac{1}{F(\bar{u})} \leq e^\alpha \left( S(t) \left[ F(\phi_0)^{-r} \right] \right)^{1/(t - \alpha)} \leq e^\alpha \left( C_0 t^{-\frac{N}{\theta}} \|F(\phi_0)^{-r}\|_{L_{ul,\rho}(\mathbb{R}^N)}^{\theta/N} \right)^{1/(t - \alpha)}.
\]

We have

\[
\mathcal{F}[\bar{u}] \leq \bar{u} - C_1 C_0^{-\alpha} e^{-\alpha} \left\| F(\phi_0)^{-r} \right\|_{L_{ul,\rho}(\mathbb{R}^N)}^{\theta/N} t^{-\frac{N}{\theta}}
\]

\[
+ C C_0^{1/\theta} e^\alpha \left\| F(\phi_0)^{-r} \right\|_{L_{ul,\rho}(\mathbb{R}^N)}^{1/\theta} t^{1-\frac{N}{\theta}} = \bar{u}
\]

for \( 0 < t < T \). Thus, \( \bar{u} \) is a supersolution. The proof is complete. \( \square \)

**Proof** (Proof of Theorems 1.4 (i-1), (i-2) and 1.6 (i)) Theorem 1.4 (i-1) (resp. (i-2)) follows from Lemmas 2.6 and 3.3 (resp. Lemmas 2.6 and 3.4). Theorem 1.6 (i) follows from Lemmas 2.6 and 3.6. \( \square \)

### 4 Nonexistence in the supercritical case

We begin with a necessary condition for a local-in-time existence.

**Proposition 4.1** Let \( N \geq 1 \) and \( 0 < \theta \leq 2 \). Assume that \( f \) satisfies (F1) with \( q \geq 1 \) and that \( f(u) \) is convex for \( u \geq 0 \). Let \( \phi \in L_{ul,\rho}(\mathbb{R}^N) \) be a nonnegative initial data. If (1.1) has a nonnegative solution on \( \mathbb{R}^N \times (0, T) \) in the sense of Definition 1.3, then there is a small \( T > 0 \) such that

\[
\|S(t)\phi\|_{\infty} \leq F^{-1}(t) \text{ for } 0 < t < T.
\]

**Proof** When \( \theta = 2 \), the proof can be found in [6, Lemma 4.1]. When \( 0 < \theta < 2 \), the proof is also valid if the derivatives are understood in the weak sense. We omit the proof. \( \square \)

**Lemma 4.2** Assume that \( f \) satisfies (F1) with \( q > 1 \). For \( \beta > 1 \), there is \( s_0 > 0 \) such that

\[
F(s)^{\beta} \leq F(\beta s) \text{ for } s \geq s_0.
\]

**Proof** Let \( \xi(\gamma) := \log F(\gamma s) - \gamma \log F(s) \). Then,

\[
\gamma \xi'(\gamma) - \xi(\gamma) = \frac{-\gamma s}{f(\gamma s)F(\gamma s)} - \log F(\gamma s).
\]

\( \square \) Springer
Let \( \eta(\tau) := -\frac{\tau}{f(\tau)F(\tau)} - \log F(\tau) \). Since \( \lim_{\tau \to \infty} (f(\tau)F(\tau))' = \lim_{\tau \to \infty} (f'(\tau)F(\tau) - 1) = q - 1 > 0 \), we see that \( \lim_{\tau \to \infty} f(\tau)F(\tau) = \infty \). By L'Hospital's rule we have

\[
\lim_{\tau \to \infty} \left( -\frac{\tau}{f(\tau)F(\tau)} - \log F(\tau) \right) = \lim_{\tau \to \infty} \left( \frac{1}{f'(\tau)F(\tau)} - \log F(\tau) \right) = \infty,
\]

and hence there is \( s_0 > 0 \) such that \( \eta(\tau) \geq 0 \) for \( \tau \geq s_0 \). If \( \gamma s \geq s_0 \), then \( \gamma \xi'(\gamma) - \xi(\gamma) \geq 0 \). Therefore, when \( 1 \leq \gamma \leq \beta \) and \( s \geq s_0 \), we see that \( \gamma s \geq s_0 \), and hence \( \gamma \xi'(\gamma) - \xi(\gamma) \geq 0 \). Since \( (\xi(\gamma)/\gamma)' \geq 0 \) and \( \xi(1) = 0 \), we have that \( \xi(\gamma) \geq 0 \) for \( 1 \leq \gamma \leq \beta \) and \( s \geq s_0 \). The conclusion holds, since \( \xi(\beta) \geq 0 \).

**Lemma 4.3** Assume that \( f \) satisfies (F1) with \( q = 1 \) and that \( f(u) \) is convex for large \( u > 0 \). For \( \beta > 1, \gamma > 0 \) and \( C_1 > 0 \), there is \( s_1 > 0 \) such that

\[
F(s)^\beta \leq F(s + C_1 F(s)^\gamma) \quad \text{for} \quad s > s_1.
\]

**Proof** It is enough to show that

\[
\lim_{s \to \infty} \frac{F(s)}{F(s + C_1 F(s)^\gamma)} = 1,
\]

because

\[
\lim_{s \to \infty} \frac{F(s)^\beta}{F(s + C_1 F(s)^\gamma)} = \lim_{s \to \infty} \frac{F(s)^{\beta-1}}{F(s + C_1 F(s)^\gamma)} = 0.
\]

Since \( F \) is convex, we see that

\[
F(s + C_1 F(s)^\gamma) \geq F(s) - C \frac{F(s)^\gamma}{f(s)} \quad \text{for large} \quad s > 0.
\]

First, we consider the case \( \gamma \geq 1 \). Then \( F(s)(1 - C_1 F(s)^{\gamma-1}/f(s)) > 0 \) for large \( s > 0 \). Therefore, by (4.2) we have

\[
1 \leq \lim_{s \to \infty} \frac{F(s)}{F(s + C_1 F(s)^\gamma)} \leq \lim_{s \to \infty} \frac{1}{1 - C \frac{F(s)^{\gamma-1}}{f(s)}} = 1.
\]

Thus, we obtain (4.1).

Second, we consider the case \( 0 < \gamma < 1 \). Since

\[
(f(s)F(s)^{1-\gamma})' = \frac{f'(s)F(s) - 1 + \gamma}{F(s)^\gamma} \to \infty \quad \text{as} \quad s \to \infty,
\]

\[\square\] Springer
we see that \( f(s) F(s)^{1-\gamma} \to \infty \) as \( s \to \infty \). Then, \( F(s)(1 - C/(f(s) F(s)^{1-\gamma})) > 0 \) for large \( s > 0 \). Therefore, by (4.2) we have

\[
1 \leq \lim_{s \to \infty} \frac{F(s)}{F(s + C_1 F(s)^{\gamma})} \leq \lim_{s \to \infty} \frac{1}{1 - \frac{C_1}{f(s) F(s)^{1-\gamma}}} = 1.
\]

Thus, we obtain (4.1). The proof is complete. \( \square \)

**Proof** (Proof of Theorems 1.4 (ii) and 1.6 (ii)) Let \( r \in (0, N/\theta) \). Then one can take \( \alpha > 0 \) such that \( \theta < \alpha < N/r \). Let

\[
u_0(x) = \begin{cases} 
F^{-1}(|x|^\alpha) & \text{if } F(0) = \infty, \\
F^{-1}(\min\{|x|^\alpha, F(0)|) & \text{if } F(0) < \infty.
\end{cases}
\]

(3.3)

Then, \( F(u_0)^{-r} \in L^1_{ul,D}(\mathbb{R}^N) \). Let \( \varepsilon > 0 \) so that \( \alpha/\theta - \varepsilon \alpha > 1 \).

Suppose the contrary, i.e., (1.1) has a local-in-time nonnegative solution. Let \( K \) be given in Proposition 2.3. Then, by Propositions 4.1 and 2.3 we have

\[
F^{-1}(t) \geq \|S(t)u_0\|_{\infty} \\
\geq t^{-N/\theta} \int_{\mathbb{R}^N} K(t^{-1/\theta} |y|) F^{-1}(|y|^\alpha) dy \\
= \int_{\mathbb{R}^N} K(|z|) F^{-1}(t^{\alpha/\theta} |z|^\alpha) dz \\
\geq \int_{|z| \leq t^{-\varepsilon}} K(|z|) F^{-1}(t^{\alpha/\theta} |z|^\alpha) dz \\
\geq F^{-1}(t^\beta) \int_{|z| \leq t^{-\varepsilon}} K(|z|) dz \\
= F^{-1}(t^\beta) \left(1 - \int_{|z| > t^{-\varepsilon}} K(|z|) dz\right),
\]

(3.4)

where \( \beta = \alpha/\theta - \varepsilon \alpha > 1 \). Among other things, we used the fact that \( F^{-1} \) is decreasing. Now, we have

\[
\int_{|z| > t^{-\varepsilon}} K(|z|) dz = \omega_{N-1} \int_{t^{-\varepsilon}}^\infty \tau^{N-1} K(\tau) d\tau \\
= \omega_{N-1} \int_{t^{-\varepsilon}}^\infty \frac{1}{\tau^{\theta+1}} \left(\tau^{N+\theta} K(\tau)\right) d\tau,
\]

where \( \omega_{N-1} \) denotes the area of the unit sphere in \( \mathbb{R}^N \). By Proposition 2.3 we see that \( \tau^{N+\theta} K(\tau) \leq C \) for \( \tau \geq t^{-\varepsilon} \). Hence, as \( \theta > 0 \), we have

\[
\int_{|z| > t^{-\varepsilon}} K(|z|) dz \leq C t^{\varepsilon \theta}.
\]

(3.5)
Note that (3.5) also holds for \( \theta = 2 \). Therefore, if \( 0 < \theta \leq 2 \), then by (3.4) and (3.5) we have

\[
F^{-1}(t) \geq F^{-1}(t^\beta)(1 - Ct^\theta). \tag{3.6}
\]

First, we consider the case \( q > 1 \). Let \( t = F(s) \). By (3.6) and Lemma 4.2 we have

\[
F^{-1}(F(s)) \geq F^{-1}(F(s)^\beta)(1 - CF(s)^\theta)
\geq F^{-1}(F(\beta s))(1 - CF(s)^\theta).
\]

Therefore, we have

\[
0 \geq s(\beta - 1 - CF(s)^\theta).
\]

The above inequality does not hold for large \( s > 0 \). We obtain a contradiction, and hence the solution does not exist when \( q > 1 \). The proof of Theorem 1.4 (ii) is complete.

Second, we consider the case \( q = 1 \). Let \( t = F(s) \) and \( \gamma = \varepsilon \theta / 2 \). By (3.6) and Lemma 4.3 we have

\[
F^{-1}(F(s)) \geq F^{-1}\left(F(s)^\beta\right) (1 - CF(s)^\theta)
\geq F^{-1}(F(s + C_1 F(s)^\gamma))(1 - CF(s)^\theta)
= s + C_1 F(s)^\gamma - CS F(s)^\theta - CC_1 F(s)^{\theta+\gamma}.
\]

Then,

\[
0 \geq F(s)^\gamma(C_1 - CS F(s)^\theta - CC_1 F(s)^\theta). \tag{3.7}
\]

If we assume that

\[
s F(s)^{\theta/2} \to 0 \text{ as } s \to \infty, \tag{3.8}
\]

then we have a contradiction, because the right-hand side of (3.7) is positive for large \( s > 0 \).

It is enough to prove (3.8). Let \( \delta := \varepsilon \theta / 4 \). Then, \( f'(s) F(s) \leq 1 + \delta \) for large \( s \), because of (F1). Integrating \( f'(s)/f(s) \leq (1 + \delta)/(f(s)F(s)) \) over \([s_0, s]\) twice, we have

\[
\frac{s - s_0}{f(s_0)F(s_0)^{1+\delta}} \leq \frac{1}{\delta} \left(F(s)^{-\delta} - F(s_0)^{-\delta}\right).
\]

Thus,

\[
0 \leq s F(s)^{\varepsilon \theta / 2} \leq s \left(\frac{\delta(s - s_0)}{f(s_0)F(s_0)^{1+\delta}} + F(s_0)^{-\delta}\right)^{-\varepsilon \theta / 2} \to 0 \text{ as } s \to \infty,
\]
where we use $\varepsilon \theta / (2 \delta) = 2$. The proof of Theorem 1.6 (ii) is complete. \qed

5 Existence in the critical case

Theorems 1.4 and 1.6 do not cover the critical case $F(\phi)^{-N/\theta} \in L^1_{ul, \rho}(\mathbb{R}^N)$. In this section we prove a local-in-time existence in the critical case when $f(u) = u^p$ or $e^u$. We also give a simple generalization of the case $u^p$ or $e^u$.

5.1 Pure power case

Theorem 5.1 Let $N \geq 1$, $0 < \theta \leq 2$, $f(u) = u^p$, $p > 1$, and $\phi \geq 0$. Then the following hold:

(i) Assume that $r > N/\theta$ and $r \geq 1/(p - 1)$. If $\phi^{(p-1)r} \in L^1_{ul, \rho}(\mathbb{R}^N)$, then (1.1) has a local-in-time solution in the sense of Definition 1.3.

(ii) Let $r := N/\theta$ and $p > 1 + \theta/N$. If $\phi^{(p-1)r} \in L^1_{ul, \rho}(\mathbb{R}^N)$, then (1.1) has a local-in-time solution in the sense of Definition 1.3.

(iii) For any $r \in (0, N/\theta)$, there is $\phi \geq 0$ such that $\phi^{(p-1)r} \in L^1_{ul, \rho}(\mathbb{R}^N)$ has no local-in-time nonnegative solution in the sense of Definition 1.3.

Note that $r = N/\theta > q - 1$, since $p > 1 + \theta/N$.

Proof The assertions (i) and (iii) immediately follow from Theorem 1.4 (i-2) and (ii), respectively.

Hereafter, we prove (ii). Because of Lemma 2.6, it is enough to prove the existence of a supersolution. The idea of the calculation (5.4) below comes from [20, Section 4]. However, our supersolutions (5.1) and (5.6) are simpler than $w(t)$ given in the proof of [20, Theorem 4.4]. We divide the proof into two cases:

Case (1): $1 + \theta/N < p < N/(N - \theta)$,

Case (2): $p \geq N/(N - \theta)$.

Let $\alpha := N(p-1)/\theta$. Note that $p > \alpha$ if $p < N/(N - \theta)$, and $p \leq \alpha$ if $p \geq N/(N - \theta)$.

Case (1): We consider the case where $1 + \theta/N < p < N/(N - \theta)$. Let $\sigma > 0$ and

\[ \tilde{u}(t) := (1 + \sigma) (S(t)\phi^\alpha)^{1/\alpha}. \]  

(5.1)

We show that $\tilde{u}$ is a supersolution. By Proposition 2.9 we have $(S(t)\phi)^\alpha \leq S(t)\phi^\alpha$, and hence

\[ S(t)\phi \leq (S(t)\phi^\alpha)^{1/\alpha} = \frac{\tilde{u}}{1 + \sigma}. \]  

(5.2)

Since $\phi^\alpha \in L^1_{ul, \rho}(\mathbb{R}^N)$, it follows from Proposition 2.5 that there is $T > 0$ such that

\[ \| S(t)\phi^\alpha \|_{\infty} \leq C_0 t^{-N/\theta} \text{ for } 0 < t < T. \]  

(5.3)
Note that \( p/\alpha > 1 \) and \( \alpha > 1 \), since \( 1 + \theta/N < p < N/(N - \theta) \). Using (5.1) and (5.3), we have

\[
\int_0^t S(t-s)\tilde{u}(s)^p \, ds \leq (1 + \sigma)^p \int_0^t S(t-s) \left[ (S(s)\phi^\alpha)^{\frac{p}{\alpha}} \right] \, ds
\]

\[
\leq (1 + \sigma)^p \int_0^t S(t-s) \left[ S(s)\phi^\alpha \| S(s)\phi^\alpha \|_{p-1}^{\frac{p}{\alpha}} \right] \, ds
\]

\[
= (1 + \sigma)^p S(t)\phi^\alpha \int_0^t \| S(s)\phi^\alpha \|_{p-1}^{\frac{p}{\alpha}} \, ds
\]

\[
= (1 + \sigma)^p \left( S(t)\phi^\alpha \right)^{\frac{1}{\alpha}} \| S(t)\phi^\alpha \|_{p-1}^{\frac{1-\frac{1}{\alpha}}{p-1}} \int_0^t \| S(s)\phi^\alpha \|_{p-1}^{\frac{p}{\alpha}} \, ds
\]

\[
\leq \tilde{u}(1 + \sigma)^{p-1} \left( C_0 t - \frac{N}{\alpha} \right)^{\frac{1-\frac{1}{\alpha}}{p-1}} \int_0^t \left( C_0 S(s) - \frac{N}{\alpha} \right)^{\frac{p}{\alpha}-1} \, ds
\]

\[
= \tilde{u}(1 + \sigma)^{p-1} C_0^{\frac{p-1}{\alpha}} t^{-\frac{N(p-\alpha)}{\theta\alpha}} \int_0^t s^{-\frac{N(p-\alpha)}{\theta\alpha}} \, ds
\]

\[
= \tilde{u}(1 + \sigma)^{p-1} C_0^{\frac{p-1}{\alpha}} \frac{1}{1 - \frac{N(p-\alpha)}{\theta\alpha}}.
\]  \hspace{1cm} (5.4)

Since \( p > 1 + \theta/N \), we see that \(-N(p - \alpha)/(\theta\alpha) > -1\), and hence \( \int_0^t s^{-N(p-\alpha)/(\theta\alpha)} \, ds \) is integrable. Because of Proposition 2.5, we can choose \( C_0 > 0 \) and \( T > 0 \) such that

\[
(1 + \sigma)^{p-1} C_0^{\frac{p-1}{\alpha}} \frac{1}{1 - \frac{N(p-\alpha)}{\theta\alpha}} \leq \frac{\sigma}{1 + \sigma}.
\]  \hspace{1cm} (5.5)

Then, by (5.2), (5.4) and (5.5) we have

\[
\mathcal{F}[\tilde{u}] = S(t)\phi + \int_0^t S(t-s)\tilde{u}(s)^p \, ds
\]

\[
\leq \frac{\tilde{u}}{1 + \sigma} + \frac{\sigma}{1 + \sigma} \tilde{u} = \tilde{u} \text{ for } 0 < t < T.
\]

Since \( \mathcal{F}[\tilde{u}] \leq \tilde{u} \) for \( 0 < t < T \), \( \tilde{u} \) is a supersolution. It follows from Lemma 2.6 that (1.1) has a local-in-time solution.

**Case (2):** We consider the case where \( p \geq N/(N - 2) \). Let \( \sigma > 0 \) and

\[
\tilde{u}(t) := (1 + \sigma)(S(t)\phi^p)^{\frac{1}{p}}.
\]  \hspace{1cm} (5.6)

We show that \( \tilde{u} \) is a supersolution. Since \( S(t)\phi \leq (S(t)\phi^p)^{1/p} \), we have

\[
S(t)\phi \leq (S(t)\phi^p)^{\frac{1}{p}} = \frac{\tilde{u}}{1 + \sigma}.
\]
Note that $\phi \in L_{ul, \rho}^\alpha (\mathbb{R}^N)$ is equivalent to $\phi^p \in L_{ul, \rho}^{\alpha/p} (\mathbb{R}^N)$. Since $\phi^p \in L_{ul, \rho}^{\alpha/p} (\mathbb{R}^N)$, by Proposition 2.5 we have

$$\| S(t)\phi^p \|_\infty \leq C_0 t^{-\frac{Np}{\theta \alpha}}. \quad (5.7)$$

Note that $p > 1$. By (5.6) and (5.7) we have

$$\int_0^t S(t-s)\bar{u}(s)^p ds \leq (1 + \sigma)^p \int_0^t S(t-s) \left[ S(s)\phi^p \right] ds$$

$$= (1 + \sigma)^p S(t)\phi^p \int_0^t ds$$

$$\leq (1 + \sigma)^p \left( S(t)\phi^p \right)^{\frac{1}{p}} \| S(t)\phi^p \|_\infty^{\frac{p-1}{p}} t$$

$$\leq \bar{u}(1 + \sigma)^{p-1} \left( C_0 t^{-\frac{Np}{\theta \alpha}} \right)^{\frac{p-1}{p}} t$$

$$= \bar{u}(1 + \sigma)^{p-1} C_0^{\frac{p-1}{p}}.$$

Because of Proposition 2.5, we can choose $C_0 > 0$ and $T > 0$ such that

$$(1 + \sigma)^{p-1} C_0^{\frac{p-1}{p}} \leq \frac{\sigma}{1 + \sigma}.$$ 

The rest of the proof is the same as the case $1 + \theta / N < p < N/(N - \theta)$. We omit the details. \hfill \Box

**Remark 5.2** In the Laplacian case $\theta = 2$, the exponent $p = N/(N - 2)$ is called “doubly critical” in [3, Remark 5]. This exponent is obtained by the relation $N(p - 1)/2 = p$. It is known that the uniqueness of the solution to $\partial_t u = \Delta u + |u|^{2/(N-2)}u$ in $L^{N/(N-2)}(\mathbb{R}^N)$ does not hold. See [18,23].

### 5.2 Pure exponential case

**Theorem 5.3** Let $N \geq 1$, $0 < \theta \leq 2$, $f(u) = e^u$ and $\phi \geq 0$. Then the following hold:

(i) Assume that $r > N/\theta$. If $e^{r\phi} \in L_{ul, \rho}^1 (\mathbb{R}^N)$, then (1.1) has a local-in-time solution in the sense of Definition 1.3.

(ii) Let $r := N/\theta$. If $e^{r\phi} \in L_{ul, \rho}^1 (\mathbb{R}^N)$, then (1.1) has a local-in-time solution in the sense of Definition 1.3.

(iii) For any $r \in (0, N/\theta)$, there is $\phi \geq 0$ such that $e^{r\phi} \in L_{ul, \rho}^1 (\mathbb{R}^N)$ and (1.1) has no local-in-time nonnegative solution in the sense of Definition 1.3.

**Proof** The assertions (i) and (iii) immediately follow from Theorem 1.6 (i) and (ii), respectively.

Hereafter, we prove (ii). Because of Lemma 2.6, it is enough to prove the existence of a supersolution. Let $\sigma > 0$ and

$$\bar{u}(t) := S(t)\phi + \sigma. \quad (3.8)$$
Note that \((3.17)\) becomes \((3.8)\). We show that \((3.8)\) is a supersolution. The proof is divided into two cases: \(r \geq 1\) and \(r < 1\).

**Case (I):** We consider the case \(r \geq 1\). By \((3.8)\) we see that
\[
S(t) \phi = \tilde{u} - \sigma. \tag{3.9}
\]
Note that \(e^\phi \in \mathcal{L}^r_{ul, \rho} (\mathbb{R}^N)\) is equivalent to \(e^{r \phi} \in \mathcal{L}^1_{ul, \rho} (\mathbb{R}^N)\). Since \(e^\phi \in \mathcal{L}^r_{ul, \rho} (\mathbb{R}^N)\), it follows from Proposition 2.5 that there is \(T > 0\) such that
\[
\| S(t)e^\phi \|_\infty \leq C_0 t^{-1} \quad \text{for} \quad 0 < t < T. \tag{3.10}
\]
Using Proposition 2.9 and \((3.10)\), we have
\[
\int_0^t S(t-s)[e^{\tilde{u}(s)}] ds = e^\sigma \int_0^t S(t-s) \left[ \exp (S(s)\phi) \right] ds \\
\leq e^\sigma \int_0^t S(t-s) \left[ S(s)e^\phi \right] ds \\
= e^\sigma S(t)e^\phi \int_0^t ds \\
\leq e^\sigma \| S(t)e^\phi \|_\infty t \\
\leq e^\sigma C_0 t^{-1} t \\
= C_0 e^\sigma. \tag{3.11}
\]
Because of Proposition 2.5, we can choose \(C_0 > 0\) and \(T > 0\) such that
\[
C_0 e^\sigma \leq \sigma. \tag{3.12}
\]
Then, by \((3.9)\), \((3.11)\) and \((3.12)\) we have
\[
\mathcal{F}[\tilde{u}]:= S(t) \phi + \int_0^t S(t-s)e^{\tilde{u}(s)} ds \\
\leq \tilde{u} - \sigma + \sigma = \tilde{u} \quad \text{for} \quad 0 < t < T.
\]
Since \(\mathcal{F}[\tilde{u}] \leq \tilde{u}\) for \(0 < t < T\), \(\tilde{u}\) is a supersolution. It follows from Lemma 2.8 that \((1.1)\) has a local-in-time solution.

**Case (2):** We consider the case \(r < 1\). Since \(S(t) \phi = \tilde{u} - \sigma\), we obtain \((3.9)\). Since \(e^{r \phi} \in \mathcal{L}^1_{ul, \rho} (\mathbb{R}^N)\), it follows from Proposition 2.5 that there is \(T > 0\) such that
\[
\| S(t)e^{r \phi} \|_\infty \leq C_0 t^{-r} \quad \text{for} \quad 0 < t < T. \tag{3.13}
\]
Using Proposition 2.9 and \((3.13)\), we have
\[
\int_0^t S(t-s)e^{\tilde{u}(s)} ds = e^\sigma \int_0^t S(t-s) \left[ \left( \exp (S(s)[r \phi]) \right)^{\frac{1}{r}} \right] ds
\]
\[ \leq e^{\sigma} \int_0^t S(t-s) \left[ (S(s) [e^{r\phi}])^{\frac{1}{r}} \right] ds \]
\[ \leq e^{\sigma} \int_0^t S(t-s) \left[ S(s)e^{r\phi} \|S(s)e^{r\phi}\|_\infty^{\frac{1}{r}-1} \right] ds \]
\[ = e^{\sigma} S(t)e^{r\phi} \int_0^t \|S(s)e^{r\phi}\|_\infty^{\frac{1}{r}-1} ds \]
\[ \leq e^{\sigma} \|S(t)e^{r\phi}\|_\infty \int_0^t \|S(s)e^{r\phi}\|_\infty^{\frac{1}{r}-1} ds \]
\[ \leq e^{\sigma} C_0 t^{\frac{1}{r}} \int_0^t \left( C_0 s^{-r} \right)^{\frac{1}{r}-1} ds \]
\[ \leq e^{\sigma} C_0 t^{\frac{1}{r}} \int_0^t C_0^{-1} s^{-1+r} ds \]
\[ = e^{\sigma} C_0^{\frac{1}{r}}. \]

By Proposition 2.5 we can choose \( C_0 > 0 \) and \( T > 0 \) such that
\[ e^{\sigma} C_0^{\frac{1}{r}} \leq \sigma. \]

The rest of the proof is the same as the case \( r \geq 1 \). We omit the details. \( \square \)

### 5.3 Other nonlinearities

Modifying the proofs of Theorems 5.1 (i) and 5.3 (ii), we can easily prove the following:

**Corollary 2** Let \( N \geq 1, \ 0 < \theta \leq 2 \) and \( \phi \geq 0 \). Assume that \( f \) satisfies (F1) with \( q \geq 1 \). Then the following hold:

(i) Assume that there are \( p > 1 + \theta/N \) and \( C > 0 \) such that \( f(u) \leq Cu^p \) for large \( u > 0 \). If \( \phi \in L_{ul,\rho}^{(p-1)N/\theta}(\mathbb{R}^N) \), then (1.1) has a local-in-time solution in the sense of Definition 1.3.

(ii) Assume that there is \( C > 0 \) such that \( f(u) \leq Ce^{u} \) for large \( u > 0 \). If \( e^{N\phi/\theta} \in L_{ul,\rho}^{1}(\mathbb{R}^N) \), then (1.1) has a local-in-time solution in the sense of Definition 1.3.

The details of the proofs are omitted.

### 6 Summary and conjectures

We study integrability conditions for a local-in-time existence and nonexistence of positive solutions of (1.1) when the initial data is positive and in uniformly local \( L^p \) spaces. The exponent \( N(p-1)/\theta \) becomes a threshold, and we construct a local-in-time positive solution in the subcritical case (Theorems 1.4 (i-1), (i-2) and 1.6 (i)), and show that there is an initial data such that (1.1) has no solution in the supercritical case.
(Theorems 1.4 (ii) and 1.6 (ii)). For \( f(u) = u^p \) (resp. \( e^u \)), a local-in-time solution can be constructed in the critical case when \( \phi^{(p-1)N/\theta} \in L^1_{ul,\rho}(\mathbb{R}^N) \) (resp. \( e^{\phi} \in L^1_{ul,\rho}(\mathbb{R}^N) \)). The following conjectures are left open:

**Conjecture 6.1** *(Existence for general \( f \), critical case)* Assume that \( f \) satisfies \((F1)\) with \( q \geq 1 \), that \( f'(u)F(u) \leq q \) for large \( u > 0 \) and \( \phi \geq 0 \). Let \( r = N/2 > q - 1 \). If \( F(\phi)^{-r} \in L^1_{ul,\rho}(\mathbb{R}^N) \), then \((1.1)\) has a local-in-time solution.

**Conjecture 6.2** Theorem 1.6 (i) holds without the assumption \( f'(u)F(u) \leq 1 \) for large \( u > 0 \).

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**References**

1. Andreucci, D., DiBenedetto, E.: On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 18, 363–441 (1991)
2. Bonforte, M., Sire, Y., Vázquez, J.: Optimal existence and uniqueness theory for the fractional heat equation. Nonlinear Anal. 153, 142–168 (2017)
3. Brezis, H., Cazenave, T.: A nonlinear heat equation with singular initial data. J. Anal. Math. 68, 277–304 (1996)
4. Dupaigné, L., Farina, A.: Stable solutions of \(-\Delta u = f(u)\) in \( \mathbb{R}^N \). J. Eur. Math. Soc. 12, 855–882 (2010)
5. Fujishima, Y.: Blow-up set for a superlinear heat equation and pointedness of the initial data. Discrete Contin. Dyn. Syst. 34, 4617–4645 (2014)
6. Fujishima, Y., Ioku, N.: Existence and nonexistence of solutions for the heat equation with a superlinear source term. J. Math. Pures Appl. 118, 128–158 (2018)
7. Furioli, G., Kawakami, T., Ruf, B., Terraneo, E.: Asymptotic behavior and decay estimates of the solutions for a nonlinear parabolic equation with exponential nonlinearity. J. Differ. Equ. 262, 145–180 (2017)
8. Giga, Y.: Solutions for semilinear parabolic equations in \( L^p \) and regularity of weak solutions of the Navier-Stokes system. J. Differ. Equ. 62, 415–421 (1986)
9. Hayashi, H., Ogawa, T.: \( L^p-L^{q}_{\ell} \) type estimate for the fractional order Laplacian in the Hardy space and global existence of the dissipative quasi-geostrophic equation. Adv. Differ. Equ. Control Process. 5, 1–36 (2010)
10. Hisa, K., Ishige, K.: Existence of solutions for a fractional semilinear parabolic equation with singular initial data. Nonlinear Anal. 175, 108–132 (2018)
11. Ibrahim, S., Jrad, R., Majdoub, M., Saanouni, T.: Local well posedness of a 2D semilinear heat equation. Bull. Belg. Math. Soc. Simon Stevin 21, 535–551 (2014)
12. Ioku, N., Ruf, B., Terraneo, E.: Existence, non-existence, and uniqueness for a heat equation with exponential nonlinearity in \( \mathbb{R}^2 \). Math. Phys. Anal. Geom. 18, 19 (2015)
13. Laister, R., Robinson, J., Sierzega, M., Vidal-Lopez, A.: A complete characterisation of local existence for semilinear heat equations in Lebesgue spaces. Ann. Inst. H. Poincaré Anal. Non Linéaire 33, 1519–1538 (2016)
14. Li, K.: A characteristic of local existence for nonlinear fractional heat equations in Lebesgue spaces. Comput. Math. Appl. 73, 653–665 (2017)
15. Li, K.: No local \( L^1 \) solutions for semilinear fractional heat equations. Fract. Calc. Appl. Anal. 20, 1328–1337 (2017)
16. Maekawa, Y., Terasawa, Y.: The Navier-Stokes equations with initial data in uniformly local \( L^p \) spaces. Differ. Integral Equ. 19, 369–400 (2006)
17. Miyamoto, Y.: A limit equation and bifurcation diagrams of semilinear elliptic equations with general supercritical growth. J. Differ. Equ. 264, 2684–2707 (2018)
18. Ni, W., Sacks, P.: Singular behavior in nonlinear parabolic equations. Trans. Amer. Math. Soc. 287, 657–671 (1985)
19. Quittner, P., Souplet, P.: Superlinear parabolic problems. Blow-up, global existence and steady states, *Birkhäuser Advanced Texts: Basler Lehrbücher*. Birkhäuser Verlag, Basel, 2007. xii+584 pp. ISBN: 978-3-7643-8441-8

20. Robinson, J., Sierzega, M.: Supersolutions for a class of semilinear heat equations. *Rev. Mat. Complut.* 26, 341–360 (2013)

21. Ruf, B., Terraneo, E.: The Cauchy problem for a semilinear heat equation with singular initial data, Evolution equations, semigroups and functional analysis (Milano, 2000), 295–309, Progr. Nonlinear Differential Equations Appl., 50, Birkhäuser, Basel, (2002)

22. Sugitani, S.: On nonexistence of global solutions for some nonlinear integral equations. *Osaka J. Math.* 12, 45–51 (1975)

23. Terraneo, E.: Non-uniqueness for a critical non-linear heat equation. *Comm. Partial Differ. Equ.* 27, 185–218 (2002)

24. Weissler, F.: Local existence and nonexistence for semilinear parabolic equations in $L^p$. *Indiana Univ. Math. J.* 29, 79–102 (1980)

25. Weissler, F.: $L^p$-energy and blow-up for a semilinear heat equation, *Nonlinear functional analysis and its applications, Part 2* (Berkeley, Calif., 1983), 545–551, Proc. Sympos. Pure Math., 45, Part 2, Amer. Math. Soc., Providence, RI, (1986)

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