Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle

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Abstract

Following the idea of T.A. Burton, of progressive contractions, presented in some examples (T.A. Burton, A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions, Fixed Point Theory, 20 (2019), No. 1, 107-113) and the forward step method (I.A. Rus, Abstract models of step method which imply the convergence of successive approximations, Fixed Point Theory, 9 (2008), No. 1, 293-307), in this paper we give some variants of contraction principle in the case of operators with Volterra property. The basic ingredient in the theory of step by step contraction is G-contraction (I.A. Rus, Cyclic representations and fixed points, Ann. T. Popoviciu Seminar of Functional Eq. Approxim. Convexity, 3 (2005), 171-178). The relevance of step by step contraction principle is illustrated by applications in the theory of differential and integral equations.

Keywords: Space of continuous function, operator with Volterra property, max-norm, Bielecki norm, contraction, G-contraction, fiber contraction, progressive contraction, step by step contraction, fixed point, Picard operator, weakly Picard operator, differential equation, integral equation, conjecture. 

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1. Introduction

Following an idea of T.A. Burton ([7], [8], [9], ...) of progressive contractions, and the forward step method ([21]), in this paper we give some variants of contraction principle in the case of operators with
Volterra property. The basic ingredient in our variant, step by step contraction principle, is $G$-contraction (20). Some applications to differential and integral equations are also given. In connection with our abstract results, a conjecture is formulated.

2. Preliminaries

2.1. $G$-contractions

Let $(X, d)$ be a metric space and $G \subset X \times X$ be a nonempty subset. An operator $f : X \to X$ is a $G$-contraction if there exists $l \in ]0, 1[$ such that,

$$d(f(x), f(y)) \leq ld(x, y), \forall (x, y) \in G.$$ 

Here are some examples of subsets $G \subset X \times X$:

1. $G := G(f)$, the graphic of the operator $f$. In this case, a $G$-contraction is a graphic contraction (17, 21, ...).

2. Let $A_i \subset X$, $i = 1, p$, be nonempty closed subsets such that:
   
   (i) $X = \bigcup_{i=1}^{p} A_i$;
   
   (ii) $f(A_i) \subset A_{i+1}$, $i = 1, p$, $(A_{p+1} = A_1)$.

For, $G := \bigcup_{i=1}^{p} (A_i \times A_{i+1})$, a $G$-contraction is a cyclic contraction of Kirk-Srinivasan-Veeramani (see the references in 20).

3. Let $a, b, c \in \mathbb{R}$, $a < c < b$ and $X := C[a, b]$ with $d(x, y) := \max_{t \in [a, b]} |x(t) - y(t)|$. For $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$, we consider the operator, $f : C[a, b] \to C[a, b]$, defined by,

$$f(x)(t) := \int_{a}^{c} K(t, s, x(s)) ds + \int_{a}^{t} H(t, s, x(s)) ds, \; t \in [a, b].$$

We suppose that there exists $L_H > 0$ such that

$$|H(t, s, u) - H(t, s, v)| \leq L_H |u - v|, \; \forall \; t, s \in [a, b], \; \forall \; u, v \in \mathbb{R}.$$

If, $L_H (b - c) < 1$ and if we take

$$G := \{(x, y) \in C[a, b] \times C[a, b] \mid x|_{[a, c]} = y|_{[a, c]}\},$$

then $f$ is a $G$-contraction.

For other examples of $G$-contractions see [20] and [24], pp. 282-284.

2.2. Weakly Picard operators

Let $(X, \to)$ be an $L$-space $((X, d), \overset{d}{\to}; (X, \tau), \overset{\tau}{\to}; (X, \|\cdot\|), \overset{\|\cdot\|}{\to}; \ldots)$. An operator $f : X \to X$ is weakly Picard operator (WPO) if the sequence, $(f^n(x))_{n \in \mathbb{N}}$, converges for all $x \in X$ and the limit (which generally depend on $x$) is a fixed point of $f$.

If an operator $f$ is WPO and the fixed point set of $f$, $F_f = \{x^*\}$, then by definition $f$ is Picard operator (PO).

For a WPO, $f : X \to X$, we define the operator $f^\infty : X \to X$, by $f^\infty(x) := \lim_{n \to \infty} f^n(x)$.

We remark that, $f^\infty(X) = F_f$, i.e., $f^\infty$ is a set retraction of $X$ on $F_f$.

For the case of ordered $L$-spaces, we have some properties of WPO and PO.

**Abstract Gronwall Lemma.** Let $(X, \to, \leq)$ be an ordered $L$-space and $f : X \to X$ be an operator. We suppose that:
(1) $f$ is increasing;
(2) $f$ is WPO.

Then:

(i) $x \leq f(x) \Rightarrow x \leq f^\infty(x)$;
(ii) $x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

**Abstract Comparison Lemma.** Let $(X, \rightarrow, \leq)$ be an ordered $L$-space and $f, g, h : X \rightarrow X$ be such that:

(1) $f \leq g \leq h$;
(2) the operators $f, g, h$ are WPO;
(3) the operator $g$ is increasing.

Then:

$$x \leq y \leq z \Rightarrow f^\infty(x) \leq g^\infty(y) \leq h^\infty(z).$$

Regarding the theory of WPO and PO see [18], [19], [22], [23], [20], [17], [24], [2], ...

### 2.3. Fiber Contraction Principle

In order to present our results, we need the following theorems (see [22], [25], [26], [27], ...).

**Fiber Contraction Theorem.** Let $(X, \rightarrow)$ be an $L$-space, $(Y, \rho)$ be a metric space, $g : X \rightarrow X$, $h : X \times Y \rightarrow Y$ and $f : X \times Y \rightarrow X \times Y$, $f(x, y) := (g(x), h(x, y))$. We suppose that:

(1) $(Y, \rho)$ is a complete metric space;
(2) $g$ is WPO;
(3) $h(x, \cdot) : Y \rightarrow Y$ is $l$-contraction, $\forall x \in X$;
(4) $h : X \times Y \rightarrow Y$ is continuous.

Then, $f$ is WPO. Moreover, if $g$ is a PO, then $f$ is a PO.

**Generalized Fiber Contraction Theorem.** Let $(X, \rightarrow)$ be an $L$-space, $(X_i, d_i)$, $i = 1, m$, $m \geq 1$ be metric spaces. Let, $f_i : X_0 \times \ldots \times X_i \rightarrow X_i$, $i = 0, m$, be some operators. We suppose that:

(1) $(X_i, d_i)$, $i = 1, m$, are complete metric spaces;
(2) $f_0$ is a WPO;
(3) $f_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \rightarrow X_i$, $i = 1, m$, are $l_i$-contractions;
(4) $f_i$, $i = 1, m$, are continuous.

Then, the operator $f : X_0 \times \ldots \times X_m \rightarrow X_0 \times \ldots \times X_m$, defined by,

$$f(x_0, \ldots, x_m) := (f_0(x_0), f_1(x_0, x_1), \ldots, f_m(x_0, \ldots, x_m))$$

is a WPO.

If $f_0$ is a PO, then $f$ is a PO.
3. Operators with Volterra property with respect to a subinterval

Let \((B, +, \mathbb{R}, |.|)\) be a Banach space, \(a, b, c \in \mathbb{R}, a < c < b\). In what follows, we consider on \(C([a, b], B), C([a, c], B)\) norms of uniform convergence (max-norm, \(\|\cdot\|\), Bielecki norm, \(\|\cdot\|_\tau\)). In, \(C([a, b], B) \times C([a, b], B)\), we consider a subset defined by,

\[ G := \{(x, y) \mid x, y \in C([a, b], B), x|_{[a,c]} = y|_{[a,c]}\}, \]

and in, \(C([a, b], B)\), for each \(x \in C([a, c], B)\) we consider the subset,

\[ X_x := \{y \in C([a, b], B) \mid y|_{[a,c]} = x\}. \]

**Definition 3.1.** An operator, \(V : C([a, b], B) \rightarrow C([a, b], B)\), has the Volterra property with respect to the subinterval, \([a, c]\), if the following implication holds,

\[ x, y \in C([a, b], B), \ x|_{[a,c]} = y|_{[a,c]} \Rightarrow V(x)|_{[a,c]} = V(y)|_{[a,c]}. \]

**Definition 3.2.** An operator, \(V : C([a, b], B) \rightarrow C([a, b], B)\), has the Volterra property if it has the Volterra property with respect to each subinterval, \([a, t]\), for \(a < t < b\).

For example, let \(K, H \in C([a, b] \times [a, b] \times B, B)\) and \(V : C([a, b], B) \rightarrow C([a, b], B)\) be defined by,

\[ V(x)(t) := \int_a^t K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \ t \in [a, b]. \]

This operator has the Volterra property with respect to the subinterval \([a, c]\), but \(V\) has not the Volterra property.

If, \(V : C([a, b], B) \rightarrow C([a, b], B)\), is an operator with Volterra property with respect to \([a, c]\), then the operator \(V\) induces an operator, \(V_1\), on \(C([a, c], B)\), defined by

\[ V_1(x) := V(\tilde{x})|_{[a,c]}, \] where \(\tilde{x} \in C([a, b], B)\) with, \(\tilde{x}|_{[a,c]} = x\).

**Remark 3.3.** If \(V\) has the Volterra property with respect to \([a, c]\) and \(V\) is a \(G\)-contraction (see section 2.1.), then the operator

\[ V|_{X_x} : X_x \rightarrow X_{V_1(x)}, \]

is a contraction for all \(x \in C([a, c], B)\). If \(x^* \in F_{V_1}\), then, \(V(X_{x^*}) \subset X_{x^*}\).

The first abstract result of our paper is the following.

**Theorem 3.4.** In terms of the above notations, we suppose that:

1. \(V\) has the Volterra property with respect to \([a, c]\);
2. \(V_1\) is a contraction;
3. \(V\) is a \(G\)-contraction.

Then:

1. \(F_V = \{x^*\}\);
2. \(x^*|_{[a,c]} = V_1^{\infty}(x), \ \forall \ x \in C([a, c], B)\);
3. \(x^* = V^{\infty}(x), \ \forall \ x \in X_{x^*}|_{[a,c]}\).
Proof. From (1) we have that, \( F_{V_1} = \{ x_1^* \} \), \( x_1^* \in C([a, c], B) \). From (3) and Remark 3.3, \( V|_{X_{x_1^*}} : X_{x_1^*} \to X_{x_1^*} \), is a contraction, i.e., it has a unique fixed point, \( x^* \), and \( x^*|_{[a,c]} = x_1^* \). From these we have (i), (ii) and (iii).

**Conjecture 3.5.** In the conditions of Theorem 3.4, the operator \( V \) is PO, i.e., \( x^* = V^\infty(x) \), \( \forall x \in C([a, b], B) \).

For a better understanding of Theorem 3.4 and Conjecture 3.5 in what follows, we present some examples.

**Example 3.6.** Let \( a, b, c \) be as above and \( B := \mathbb{R} \). For \( K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R}) \) we consider the following functional integral equation,

\[
x(t) = \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, \max_{\theta \in [a,s]} x(\theta))ds, \quad t \in [a, b].
\]  

(3.1)

We are looking for the solution of this equation in \( C[a, b] \). In addition, we suppose that:

(2') there exists \( L_K > 0 \) such that:

\[ |K(t, s, u) - K(t, s, v)| \leq L_K |u - v|, \quad \forall t \in [a, b], \quad \forall s \in [a, c], \quad \forall u, v \in \mathbb{R}; \]

(3') there exists \( L_H > 0 \) such that,

\[ |H(t, s, u) - H(t, s, v)| \leq L_H |u - v|, \quad \forall t, s \in [a, b], \quad \forall u, v \in \mathbb{R}. \]

In this case:

\[ V(x)(t) = \text{the second part of (3.1)}; \]

\[ V_1(x)(t) = \text{the second part of (3.1), for } t \in [a, c]. \]

It is clear that \( V \) has the Volterra property with respect to the subinterval \( [a, c] \).

We consider on \( C[a, c] \) and \( C[a, b] \) max-norms and if, \( (L_K + L_H)(c - a) < 1 \), the operator \( V_1 \) is a contraction and if, \( L_H(b - c) < 1 \), the operator \( V \) is a G-contraction.

So, by Theorem 3.4, in the above conditions, equation (3.1) has in \( C[a, b] \) a unique solution, \( x^* \). Moreover, for \( t \in [a, c] \), \( x^*(t) = \lim_{n \to \infty} x_n(t) \), for each \( x_0 \in C[a, c] \), where \( \{x_n\}_{n \in \mathbb{N}} \) is defined by,

\[
x_{n+1}(t) = \int_a^c K(t, s, x_n(s))ds + \int_a^t H(t, s, \max_{\theta \in [a,s]} x_n(\theta))ds,
\]

and for \( t \in [a, b] \), \( x^*(t) = \lim_{n \to \infty} y_n(t) \), where \( \{y_n\}_{n \in \mathbb{N}} \), is defined by

\[
y_0 \in C[a, b], \text{ with } y_0|_{[a,c]} = x^*|_{[a,c]}, \text{ and } \\
y_{n+1}(t) = \int_a^c K(t, s, x^*(s))ds + \int_a^t H(t, s, \max_{\theta \in [a,s]} y_n(\theta))ds.
\]

**Remark 3.7.** In the case of operator \( V \), in this example, Conjecture 3.6 is a theorem. Indeed, let \( X_0 := C[a, c], X_1 := C[c, b] \) and \( C[a, b] \) be endowed with max-norms. We take, \( f_0 := V_1 \) and \( f_1(x, y) : C[a, c] \times C[c, b] \to C[c, b] \) be defined by

\[
f_1(x, y)(t) := \int_a^c K(t, s, x(s))ds + \int_a^c H(t, s, \max_{\theta \in [a,s]} x(\theta))ds + \\
+ \int_c^t H(t, s, \max_{\theta \in [a,s]} x(\theta), \max_{\theta \in [c,s]} y(\theta))ds.
\]
First, we remark that, \( f_0 \) is a PO, and \( f_1(x, \cdot) : C[c, b] \to C[c, b] \) is \( L_H (b - c) \)-contraction. By Fiber Contraction Theorem, in the conditions, \((L_K + L_H)(c - a) < 1 \) and \( L_H(b - c) < 1 \), the operator \( f \) is a Picard operator.

Let,

\[
x_0 \in C[a, c], \quad x_{n+1} = f_0(x_n), \quad n \in \mathbb{N},
\]

and

\[
y_0 \in C[c, b], \quad y_{n+1} = f_1(x_n, y_n), \quad n \in \mathbb{N}.
\]

Then, \( x_n \to x^*|_{[a, c]} \) as \( n \to \infty \), \( y_n \to x^*|_{[c, b]} \) as \( n \to \infty \).

We denote,

\[
u_n(t) = \begin{cases} 
  x_n(t), & t \in [a, c], \\
  y_n(t), & t \in [c, b].
\end{cases}
\]

Then, \( u_n \in C[a, b] \), for \( n \in \mathbb{N}^* \), and, \( u_{n+1} = V(u_n) \) with \( u_n \to x^* \) as \( n \to \infty \), i.e., \( V \) is a PO.

This result is very important because we can apply for \( V \), the Abstract Gronwall Lemma. So we have:

**Theorem 3.8.** Let us consider the equation \((3.1)\) in the following conditions: \((L_K + L_H)(c - a) < 1 \), \( L_H(b - c) < 1 \) and \( K(t, s, \cdot), H(t, s, \cdot) : \mathbb{R} \to \mathbb{R} \) are increasing functions, for all \( t, s \in [a, b] \). Let us denote by \( x^* \) the unique solution of \((3.1)\). Then the following implications hold:

(i) \( x \in C[a, b], \ x(t) \leq \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, \max_{\theta \in [a, s]} x(\theta))ds, \ t \in [a, b], \ \Rightarrow \ x \leq x^*; \)

(ii) \( x \in C[a, b], \ x(t) \geq \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, \max_{\theta \in [a, s]} x(\theta))ds, \ t \in [a, b], \ \Rightarrow \ x \geq x^*.

Also, from the Abstract Comparison Lemma we have a comparison result for equation \((3.1)\).

**Remark 3.9.** For the functional integral equations with maxima, see \([1], [11], [10], [22], [13], \ldots\)

**Example 3.10.** Let \( a, b, c \in \mathbb{R}, \ a < b < c, \) and \((\mathbb{B}, +, \mathbb{R}, |.|)\) be a Banach space. For \( K, H \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B}) \) we consider the following integral equation,

\[
x(t) = \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \ t \in [a, b]. \tag{3.2}
\]

We are looking for solutions of these equations in \( C([a, b], \mathbb{B}) \). To do this, in addition, we suppose that:

(2") there exists \( L_K > 0 \) such that,

\[
|K(t, s, u) - K(t, s, v)| \leq L_K |u - v|, \ \forall \ t \in [a, b], \ \forall \ s \in [a, c], \ \forall \ u, v \in \mathbb{B} ;
\]

(3") there exists \( L_H > 0 \) such that,

\[
|H(t, s, u) - H(t, s, v)| \leq L_H |u - v|, \ \forall \ t, s \in [a, b], \ \forall \ u, v \in \mathbb{B}.
\]

In the case of equation \((3.2)\) we have:

\[
V(x)(t) = \text{the second part of } (3.2);
\]

\[
V_1(x)(t) = \text{the second part of } (3.2), \text{ for } t \in [a, c].
\]

First, we remark that \( V \) has the Volterra property with respect to the subinterval \([a, c]\).

If we consider on \((C[a, c], \mathbb{B})\) and \(C[a, b] \max\)-norms, then if \((L_K + L_H)(c - a) < 1 \), the operator \( V_1 \) is a contraction (i.e., PO) and if \( L_H(b - c) < 1 \), the operator \( V \) is a \( G \)-contraction. By Theorem
In these conditions, equation (3.2) has in \( C([a,b], \mathbb{B}) \) a unique solution, \( x^* \). Moreover, for \( t \in [a,c] \), 
\[
x^*(t) = \lim_{n \to \infty} x_n(t), \quad \text{where} \quad x_0 \in C([a,c]),
\]
and for \( t \in [a,b] \), 
\[
x^*(t) = \lim_{n \to \infty} y_n(t), \quad \text{where} \quad y_0 \in C([a,b], \mathbb{B}), \quad \text{with} \quad y_0|_{[a,c]} = x^*, \quad \text{and}
\]
\[
y_{n+1}(t) = \int_a^c K(t,s,x_n(s))ds + \int_a^t H(t,s,x_n(s))ds, \quad n \in \mathbb{N}
\]

Remark 3.11. In a similar way, as in the case of Example 3.6, the Conjecture 3.5 is a theorem for the operator \( V \) in Example 3.10.

Remark 3.12. We can work, in the case of Example 3.10 with max-norm on \( C([a,c], \mathbb{B}) \) and with a Bielecki norm on \( C([c,b], \mathbb{B}) \), i.e., on \( C([a,b], \mathbb{B}) \) with the norm, 
\[
\|x\| = \max_{t \in [a,c]} \left( \max_{t \in [c,b]} e^{-r(t-c)}|x(t)| \right).
\]
If \( \mathbb{B} := \mathbb{R}^m \), then we can work with vectorial max-norms and with vectorial Bielecki norms.

Remark 3.13. For example of integral operator like \( V \) in Example 3.10 which appear in differential equations, see: [3], [7], [4], [3] and the references in [3].

4. Operators with Volterra property

Let, \( V : C([a,b], \mathbb{B}) \to C([a,b], \mathbb{B}) \), be an operator with Volterra property. Let \( m \in \mathbb{N} \), \( m \geq 2 \), \( t_0 := a \), \( t_1 := t_0 + \frac{b-a}{m} \), \( \ldots \), \( t_k := t_0 + \frac{k(b-a)}{m} \), \( \ldots \), \( t_m := b \). We denote by \( V_k : C([t_0,t_k], \mathbb{B}) \to C([t_0,t_k], \mathbb{B}) \), \( k = 1, m - 1 \), the operators induced by \( V \) on \( [t_0, t_k] \) (see the definition of \( V_1 \) in section 3). We also consider the following sets,
\[
G_k := \{ (x, y) \mid x, y \in C([t_0, t_{k+1}], \mathbb{B}), \ x|_{[t_0, t_k]} = y|_{[t_0, t_k]} \}, \quad k = 1, m - 1.
\]
For, \( x_k \in C([t_0, t_k], \mathbb{B}) \), \( k = 1, m - 1 \), we denote,
\[
X_{x_k} := \{ y \in C([t_0, t_{k+1}], \mathbb{B}) \mid y|_{[t_0, t_k]} = x_k \}.
\]

The second basic result of this paper is the following.

Theorem 4.1 (Theorem of step by step contraction). We suppose that:

1. \( V \) has the Volterra property;
2. \( V_1 \) is a contraction;
3. \( V_k \) is a \( G_{k-1} \)-contraction, for \( k = \frac{2}{m} \).

Then:

(i) \( F_V = \{ x^* \} \);
(ii)
\[
x^*|_{[t_0, t_1]} = V_1^\infty(x), \quad \forall \ x \in C([t_0, t_1], \mathbb{B}),
\]
\[
x^*|_{[t_0, t_2]} = V_2^\infty(x), \quad \forall \ x \in X_{x^*}|_{[t_0, t_1]},
\]
\[
\vdots
\]
\[
x^*|_{[t_0, t_{m-1}]} = V_{m-1}^\infty(x), \quad \forall \ x \in X_{x^*}|_{[t_0, t_{m-2}]}.
\]
\[ (iii) \ x^* = V^\infty(x), \ \forall \ x \in X_{x^*}^{\mid [a,t_{m-1}]} . \]

Proof. It follows from successive (step by step!) application of Theorem 3.4 for the pairs, \((V_{k+1}, V_k)\), \(k = 1, m - 1\), with \(V_{k+1}\) as \(V\) and \(V_k\) as \(V_1\).

\[ \textbf{Conjecture 4.2.} \text{ In the condition of Theorem 4.1 the operator } V \text{ is PO, with respect to uniform convergence on } C([a,b], \mathbb{R}). \]

\[ \textbf{Example 4.3.} \text{ For } K \in C([a,b] \times [a,b] \times \mathbb{R}) \text{ we consider the following functional integral equation with maxima,} \]

\[ x(t) = \int_a^t K(t, s, \max_{\theta \in [a,s]} x(\theta))ds, \ t \in [a,b] \quad (4.1) \]

By step by step contraction principle we shall prove that, if there exists \(L_K > 0\) such that,

\[ |K(t, s, u) - K(t, s, v)| \leq L_K|u - v|, \ \forall \ t, s \in [a, b], \ \forall \ u, v \in \mathbb{R}, \]

then the equation (4.1) has in \([a,b]\) a unique solution.

Indeed, let \(m \in \mathbb{N}^*\) be such that, \(\frac{L_K(b-a)}{m} < 1\). Let, \(V : C[a, b] \to C[a, b]\) be defined by,

\[ V(x)(t) := \text{ the second part of (4.1)}. \]

First, we remark that \(V\) has the Volterra property. In this case:

\[ V_1 : C[t_0, t_1] \to C[t_0, t_1], \ V_1(x)(t) = \int_{t_0}^{t_1} K(t, s, \max_{\theta \in [t_0,s]} x(\theta))ds, \ t \in [t_0, t_1]. \]

A Lipschitz constant for \(V_1\) is, \(\frac{L_K(b-a)}{m}\). So, \(V_1\) is a contraction with respect to max-norm.

In a similar way, \(V_2\) is a \(G_1\)-contraction, \(V_k\) is a \(G_{k-1}\)-contraction and \(V\) is \(G_{m-1}\)-contraction.

So, we are in the conditions of Theorem 4.1. From this theorem we have that:

The equation (4.1) has in \([a,b]\) a unique solution, \(x^*\). Moreover,

- for \(t \in [t_0, t_1]\), \(x^*(t) = \lim_{n \to \infty} x_n(t)\), where \(x_0 \in C[t_0, t_1]\), \(x_{n+1}(t) = \int_{t_0}^{t_1} K(t, s, \max_{\theta \in [t_0,s]} x_n(\theta))ds;\)

- for \(t \in [t_0, t_2]\), \(x^*(t) = \lim_{n \to \infty} x_n(t)\), where \(x_0 \in C[t_0, t_2]\) with \(x_0|_{[t_0, t_1]} = x^*|_{[t_0, t_1]}\), and \(x_{n+1}(t) = \int_{t_0}^{t_1} K(t, s, \max_{\theta \in [t_0,s]} x_n(\theta))ds, n \in \mathbb{N};\)

- for \(t \in [t_0, t_m]\), \(x^*(t) = \lim_{n \to \infty} x_n(t)\), where \(x_0 \in C[t_0, t_m]\) with \(x_0|_{[t_0, t_{m-1}]} = x^*|_{[t_0, t_{m-1}]}\), and \(x_{n+1}(t) = \int_{t_0}^{t_1} K(t, s, \max_{\theta \in [t_0,s]} x_n(\theta))ds.\)

\[ \textbf{Remark 4.4.} \text{ In a similar way as in the Example 3.6, by Generalized fiber contraction theorem, we have that, for } V \text{ in Example 4.3 the Conjecture 4.2 is a theorem.} \]

\[ \textbf{Example 4.5.} \text{ For } f \in C([a, b] \times \mathbb{R}), \text{ we consider the following Cauchy problem} \]

\[ x'(t) = f(t, \max_{\theta \in [a,t]} x(\theta)), \ t \in [a, b] \quad (4.2) \]

\[ x(a) = 0 \]

This problem with \(x \in C^1[a,b]\) is equivalent with the following functional integral equation with maxima, in \(C[a,b]\),

\[ x(t) = \int_a^t f(s, \max_{\theta \in [a,s]} x(\theta))ds, \quad (4.3) \]
From the result, in Example 4.3, we have that, if there exists $L_f > 0$ such that,

$$|f(t,u) - f(t,v)| \leq L_f|u - v|, \forall \ t \in [a, b], \forall \ u, v \in \mathbb{R},$$

then the equation (4.3) has in $C[a, b]$ a unique solution, i.e., the Cauchy problem (4.2) has in $C^1[a, b]$ a unique solution.

**Remark 4.6.** For functional differential equations see: [1], [6], [11], [12], [16], [22], . . .

**Remark 4.7.** For operators with Volterra property see: [10], [21], [15] and the references therein.

5. Step by step generalized contraction principles

There is a large class of generalized contraction principle (see, for example, [24], [2], [17]). As an example in what follows, we consider the case of $\varphi$-contractions.

Let $(X, d)$ be a metric space, $G \subset X \times X$ a nonempty subset and $f : X \rightarrow X$ be an operator.

**Definition 5.1.** Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a comparison function. By definition, $f$ is a $(G, \varphi)$-contraction if,

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \forall \ x, y \in G.$$

In the terms of notations in section 4, in a similar way as in the case of Theorem 4.1, we have:

**Theorem 5.2** (Theorem of step by step $\varphi$-contraction). We suppose that:

(1) $V$ has the Volterra property;

(2) $V_1$ is a $\varphi$-contraction;

(3) $V_k$ is a $(G_{k-1}, \varphi)$-contraction, for $k = 2, m$.

Then:

(i) $F_V = \{x^*\};$

(ii)

$$x^*|_{[t_0, t_1]} = V_1^\infty(x), \forall \ x \in C([t_0, t_1], \mathbb{B}),$$

$$x^*|_{[t_0, t_2]} = V_2^\infty(x), \forall \ x \in X_{x^*}|_{[t_0, t_1]},$$

$$\vdots$$

$$x^*|_{[t_0, t_{m-1}]} = V_{m-1}^\infty(x), \forall \ x \in X_{x^*}|_{[t_0, t_{m-2}]}.$$

(iii) $x^* = V^\infty(x), \forall \ x \in X_{x^*}|_{[t_0, t_{m-1}]}.$

**Problem 5.3.** For which generalized contractions we have step by step corresponding result ? If such generalized contractions are found, then the problem is to give relevant applications of such result.
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