REDUCTIONS, RESOLUTIONS AND THE COPOLARITY OF ISOMETRIC GROUP ACTIONS

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ABSTRACT. We present some results on reductions and the copolarity of isometric group actions, which we obtained in our thesis [Mag08]. We also describe a resolution construction for isometric actions with respect to a reduction and give examples.

1. Introduction

A reduction of an isometric action $(G, M)$ consists of a fat section $\Sigma$ and a fat Weyl group $W$ acting on $\Sigma$. Fat section were first introduced, under a different name, in [GOT04]. The motivation for this comes from polar actions and their sections. In this situation $\Sigma$ is a complete, connected and embedded submanifold which intersects every $G$-orbit and is perpendicular to them in every intersection point. Such actions have many fascinating properties and are well studied in the literature (see for instance [Dad85, PT88, BCO03, Kol07] for key results and further references). In [GOT04] Gorodski, Olmos and Tojeiro tried to measure the defect of an arbitrary isometric action from being polar. This led them to the notion of a fat section and the integer valued invariant copolarity of an isometric action. In this picture, polar actions are precisely the copolarity-0 actions. The authors obtained a classification of all irreducible copolarity-1 representations which in turn enabled them to characterize all irreducible orthogonal taut representations as those of copolarity 0 or 1.

Our paper is organized as follows: In Section 2 we define fat sections, fat Weyl groups, reductions and the copolarity of isometric actions and show some basic properties. In Section 3 the main result is that the orbit space of any reduction is isometric to the orbit space of the original action. Together with the implications obtained from this result this serves as a justification for our definition of a reduction.

In Section 4 we show that the copolarity of the slice representation in any point cannot exceed that of the global action. As consequences of this we show in Section 5 that a point on a fat section is $G$-regular if and only if it is $W$-regular with respect to the corresponding reduction, and that the copolarity does not change if we pass from an isometric action to any of its reductions.

In Section 6 we show that reductions also behave well with respect to variational completeness: an isometric action is variationally complete if and only if some/every reduction is variationally complete. At the end of this section we also generalize results on variational co-completeness from [GOT04].

In Section 7 we generalize the resolution construction from [GS00] using arbitrary fat sections. As an application one can construct from any $G$-space $M$ with sectional curvature bounded from below by some constant $\kappa \leq 0$ another $G$-space $\tilde{M}$ with the same curvature bound from below and such that both spaces have isomorphic orbit spaces.

Section 8 deals with Chevalley’s restriction theorem for reductions, which we are able to show under additional assumptions. A class of isometric actions on compact Lie
groups associated with Riemannian symmetric spaces where these conditions are met is explained in Section 10.

In Section 11 we investigate a class of affine isometric actions on Hilbert spaces related with the examples of Section 10. Using an adapted notion of copolarity, we are able to show an interesting dichotomy: Depending on whether the base action is hyperpolar or just polar, the copolarity of the infinite dimensional action is either 0 or $\infty$.

Finally, Section 9 describes how fat sections can be defined for singular Riemannian foliations.

2. Fat Sections, Fat Weyl Groups and the Copolarity of Isometric Actions

By an isometric action of a Lie group $G$ on a Riemannian manifold $M$ we mean a smooth and proper homomorphism $\Phi: G \to \text{Iso}(M)$. An action is also denoted by the associated map $\varphi: G \times M \to M, (g, q) \mapsto g \cdot q := \Phi(g)(q)$, or just by $(G, M)$. G-Regular points are points lying on principal orbits. Their collection is denoted by $M^{\text{reg}}$. All other points are called singular. Thus, points lying on exceptional orbits are also singular in our sense.

Definition 2.1. Let $M$ be a complete Riemannian manifold and let $(G, M)$ be an isometric action. A submanifold $\Sigma \subseteq M$ is called a fat section of $(G, M)$ if:

(A) $\Sigma$ is complete, connected, embedded and totally geodesic in $M$,
(B) $\Sigma$ intersects every orbit of the $G$-action,
(C) for all $G$-regular $p \in \Sigma$ we have $\nu_p(G \cdot p) \subseteq T_p \Sigma$,
(D) for all $G$-regular $p \in \Sigma$ and $g \in G$ such that $g \cdot p \in \Sigma$ we have $g \cdot \Sigma = \Sigma$.

In this situation, following [GOT04], we also call $\Sigma$ a $k$-section, where $k$ denotes the codimension of $\nu_p(G \cdot p)$ in $T_p \Sigma$ for any regular point $p \in \Sigma$. The integer \[\text{copol}(G, M) := \min\{k \in \mathbb{N} \mid \text{there is a } k\text{-section } \Sigma \subseteq M\}\]
called the copolarity of the $G$-action on $M$. If $\Sigma \subseteq M$ is a copol$(G, M)$-section, then we say that $\Sigma$ is minimal. If a submanifold $\Sigma \subseteq M$ satisfies only properties (A)-(C) above, it is called pre-section. Finally, if $M$ is a minimal section of $(G, M)$, we say that $(G, M)$ has trivial copolarity.

Remark 2.2.

(i) An isometric action $(G, M)$ is called polar if there exists a complete, connected and embedded submanifold $\Sigma$, called section, which intersects every orbit and such that in the intersection points the orbits are perpendicular to $\Sigma$. It follows that such a $\Sigma$ is totally geodesic and satisfies property (D) in the above definition. Hence, copol$(G, M) = 0$ and a section in the polar sense is a (minimal) 0-section in the sense of Definition 2.1. Conversely, an isometric action with copolarity zero is polar and all minimal sections are sections in the polar sense. The copolarity therefore measures the failure of an isometric action to be polar.

(ii) For a given Riemannian manifold $M$, one can define the copolarity of $M$ as:
\[\text{copol}(M) := \text{copol}(\text{Iso}(M), M).\]

Just like the symmetry rank, symmetry degree and the cohomogeneity of a Riemannian manifold (see for instance [Wil06] for the definitions), the copolarity is also a measure for the amount of symmetry a Riemannian manifold carries. For instance, homogeneous spaces and cohomogeneity one manifolds are manifolds of copolarity zero.
Situations in which the copolarity of an action is nontrivial and not equal to zero and where the minimal sections can be explicitly computed are described in Section [11] and [Mag08, Mag09]. To give some flavor:

**Example 2.3.** The k-fold direct sum of the standard representation of $\text{SO}(n)$ on $\mathbb{R}^n$ has nontrivial copolarity equal to $\frac{k(k-1)}{2}$ for $2 \leq k \leq n-1$ and a minimal section is given by $\mathbb{R}^{k^2}$, which is embedded into $\mathbb{R}^{kn}$ as block matrices with nonzero entries in the upper $(k \times k)$-block only.

**Example 2.4.** Consider the following action of $T^2 \times S(U(1) \times U(2))$ on $\text{SU}(3)$. The first factor acts by matrix multiplication from the left and the second factor by matrix multiplication from the right by the inverted matrix. The copolarity in this case is equal to 1 and a minimal section is given by $\text{SO}(3) \subset \text{SU}(3)$.

The following three lemmas are frequently used throughout the paper. Lemma 2.6 and 2.7 are [GOT04, Lemma 5.1 and Lemma 5.2]. Originally, the second of these was stated for orthogonal representations only, but its proof also works in the general case.

**Lemma 2.5.** Let $(G, M)$ be an isometric action and suppose that $M$ is connected and finite dimensional. If $p \in M^\text{reg}$, then $\exp_p(v_p(G \cdot p))$ intersects every $G$-orbit.

**Lemma 2.6.** Let $(G, M)$ be an isometric action and let $q \in M$ be arbitrary. For $v \in \nu_q(G \cdot q)$ the following assertions are equivalent:

(i) $v$ is $G_q$-regular.

(ii) There exists $\varepsilon > 0$ such that $\exp_q(tv)$ is $G$-regular for $0 < t < \varepsilon$.

(iii) $\exp_q(t_0v)$ is $G$-regular for some $t_0 > 0$.

**Lemma 2.7.** Let $\Sigma$ be a fat section of $(G, M)$. For all $q \in \Sigma$ there is a $G_q$-regular $v \in T_q\Sigma \cap \nu_q(G \cdot q)$. Furthermore, $v$ can be chosen such that $p = \exp_q v$ is $G$-regular and arbitrarily close to $q$.

The following proposition lists several properties related to the copolarity of an isometric action. They are either observations already made in [GOT04] or immediate consequences of them and Definition 2.1.

**Proposition 2.8.** Let $M, N$ be finite dimensional Riemannian manifolds and $G, H$ Lie groups which act smoothly and isometrically on $M$, resp. $N$. Let furthermore $p \in M$ be an arbitrary $G$-regular point.

(i) If $(G, M)$ and $(H, N)$ are orbit-equivalent (i.e. there is an isometry from $M$ onto $N$, mapping $G$-orbits onto $H$-orbits), then $\text{copol}(G, M) = \text{copol}(H, N)$.

(ii) $\text{copol}(G, M) = \text{copol}(G^c, M)$, where $G^c$ denotes the identity component.

(iii) For any two fat sections $\Sigma_1, \Sigma_2$ containing $p$, the connected intersection (i.e. the connected component of $p$ of the intersection $\Sigma_1 \cap \Sigma_2$) is again a fat section. Hence, a minimal section through $p$ is unique.

(iv) The minimal section through $p$ is the connected intersection of all fat sections containing $p$, and also the connected intersection of all pre-sections through $p$.

(v) The $G$-translates of a fat section $\Sigma$ foliate $M^\text{reg}$, the $G$-regular points of $M$.

(vi) $G$ is transitive on the set of all minimal sections of $(G, M)$.

(vii) The intersection of a principal orbit $G \cdot p$ with a fat section $\Sigma$ is an embedded submanifold of $M$. Moreover, if $N_G(\Sigma)$ denotes the normalizer of $\Sigma$ in $G$, then $\Sigma \cap (G \cdot p) = N_G(\Sigma) \cdot p$, if $p \in \Sigma$.

(viii) If $\Sigma$ is a fat section, then $\Sigma^\text{reg} := \Sigma \cap M^\text{reg}$ is open and dense in $\Sigma$. 

Clearly, $M$ itself is always a fat section of $(G, M)$ (hence, we speak of trivial copolarity if $M$ is the only fat section). More interesting fat sections can often be found using

**Proposition 2.9** ([GOT04, Section 3.2]). If $(G, M)$ is isometric and $p \in M^{\text{reg}}$, then $\Sigma := \text{Fix}(G_p, M)^o$, i.e. the connected component of $p$ of the fixed point set of $G_p$ is a $k$-section, where $k = \dim(T_p(G \cdot p)^G_r)$.

**Definition 2.10.** We call a fat section as in Proposition 2.9 a **canonical section**. Furthermore, we say a fat section is sufficiently small if it is contained in some canonical section. In particular, canonical sections and minimal sections are sufficiently small.

**Remark 2.11** ([GOT04], Section 3.2). Canonical sections need not be minimal sections. For instance, for $k = 2, n = 3$ in Example 2.3 the principal isotropy groups are trivial, but minimal sections are proper subspaces of the representation space. Nevertheless, for an isometric action $(G, M)$ the acting group $G$ can often be enlarged to a group $G'$, which also acts isometrically on $M$ with the same orbits as $G$, and such that $(G', M)$ has canonical minimal sections. By Proposition 2.8 (ii) both actions have the same copolarity and minimal sections. It is interesting to note that for every polar representation the sections can be obtained in this way ([Str94, Theorem 1.3]), and this is also the case for the representations in Example 2.3 (see also [Mag08, Chapter 7]).

**Definition 2.12.** Let $\Sigma$ be a fat section of the isometric action $(G, M)$. We put $W = W(\Sigma) := N_G(\Sigma)/Z_G(\Sigma)$ and call it the **fat Weyl group** of $\Sigma$. The isometric action $(W, \Sigma)$ is called a reduction of $(G, M)$ (induced by $\Sigma$). For minimal sections, $(W, \Sigma)$ is called a minimal reduction.

**Remark 2.13.**

(i) One can also define fat sections without requiring them to be embedded. For our purposes however it will be important that $\Sigma$ is closed in $M$, because then $N_G(\Sigma)$ is a Lie subgroup of $G$. Hence $W(\Sigma)$ is also a Lie group.

(ii) Every compact Lie group can be realized as a fat Weyl group. This generalizes [PT88, Remark 5.6.20] and is described at the end of Section 7.

**Example 2.14.** A minimal reduction of Example 2.3 is $(O(k), \mathbb{R}^k)$.

The next proposition is easy to check.

**Proposition 2.15.** Let $(G, M)$ be an isometric action and let $\Sigma \subseteq M$ be a sufficiently small section. Then $H := Z_G(\Sigma)$ is a principal isotropy group of $(G, M)$. In particular, all principal isotropy groups along $\Sigma$ coincide. It follows that $W = W(\Sigma)$ acts freely on $\Sigma^{\text{reg}}$, and if $\Sigma$ is a minimal section, then copol$(G, M) = \dim(W)$ and

$$
\dim \Sigma = \text{cohom}(G, M) + \text{copol}(G, M).
$$

3. **Properties of Reductions**

In this section we generalize several results of [GOT04, Section 5.2], where orthogonal representations are considered, to arbitrary isometric actions. Interestingly, we obtain the results in a reversed order than in loc. cit. We start with a metric observation concerning orbit spaces, which is a stronger result than [GOT04, Theorem 5.9]. In the following let $(G, M)$ be an isometric group action and let $\Sigma$ be a fat section with fat Weyl group $W = W(\Sigma)$. 
Theorem 3.1. The orbit spaces $W \backslash \Sigma$ and $G \backslash M$, both endowed with their respective orbital distance metric, are canonically isometric via the map 

$$\tilde{\iota} : W \backslash \Sigma \to G \backslash M, \ W \cdot q \mapsto G \cdot q.$$ 

Proof. First of all, $\tilde{\iota}$ is a well defined map: If $q, q' \in \Sigma$ are such that $W \cdot q = W \cdot q'$, then there exists $n \in N_G(\Sigma) \subseteq G$ such that $n \cdot q = q'$. Hence $G \cdot q = G \cdot q'$. Since $\Sigma$ intersects all $G$ orbits, it is clear that $\tilde{\iota}$ is surjective. Furthermore, the next diagram commutes:

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\tilde{\iota}} & M \\
\pi_W \downarrow & & \downarrow \pi_G \\
W \backslash \Sigma & \xrightarrow{\iota} & G \backslash M.
\end{array}
$$

As $\Sigma$ is embedded into $M$, the inclusion $\iota$ is continuous. The diagram then implies that $\tilde{\iota}$ is continuous, too. The distance between two points $G \cdot q$ and $G \cdot q'$ in $G \backslash M$ is the length of a minimal geodesic segment $\gamma$ in $M$ connecting the orbits $G \cdot q$ and $G \cdot q'$. Each such segment is perpendicular to both orbits. If $q, q'$ are both $G$-regular and $q \in \Sigma$, then by properties \ref{A} and \ref{C} of a fat section, $\gamma$ is a segment in $\Sigma$. We may thus further assume $q' \in \Sigma$, and thus $\gamma$ minimizes the distance between $W \cdot q$ and $W \cdot q'$. It follows that $\tilde{\iota}$ restricted to the open and dense subset $\Sigma^{reg}$ (see Proposition \ref{2,5} (viii)) is an isometry. By continuity and using that $W \backslash \Sigma$ and $G \backslash M$ are complete metric spaces, we see that $\tilde{\iota}$ is a surjective isometry. 

Corollary 3.2. The map $\iota^* : C^0(M)^G \to C^0(\Sigma)^W$, $f \mapsto f|_\Sigma$ is an isomorphism of Banach algebras, where both spaces are equipped with the corresponding $\| \cdot \|_\infty$-norm.

Proof. Consider the following commuting diagram of Banach algebras associated with the diagram from Theorem 3.1

$$
\begin{array}{ccc}
C^0(G \backslash M) & \xrightarrow{\iota^*} & C^0(W \backslash \Sigma) \\
\pi_G^* \downarrow & & \downarrow \pi_W^* \\
C^0(M)^G & \xrightarrow{\cdot} & C^0(\Sigma)^W.
\end{array}
$$

The top arrow is an isomorphism of Banach algebras, because $G \backslash M \approx W \backslash \Sigma$ and since the assignment $\iota^*(f) = f \circ \tilde{\iota}$ is clearly norm preserving. The vertical maps are Banach algebra isomorphisms by definition of the orbit space. Hence the bottom arrow is also an isomorphism of Banach algebras. 

Corollary 3.3. The fat Weyl group $W$ parameterizes intersections of $G$-orbits with $\Sigma$: For all $q \in \Sigma$ we have $W \cdot q = (G \cdot q) \cap \Sigma$. In particular, $(G \cdot q) \cap \Sigma$ is an extrinsic homogenous submanifold of the spaces $G \cdot q$, $\Sigma$ and $M$ for every $q \in \Sigma$.

Corollary 3.4. For every $q \in M$ the isotropy group $G_q$ is transitive on the set of all $G$-translates of $\Sigma$ containing $q$. In particular, $G_q$ is transitive on the set of minimal sections through $q$.

Proof. Since $\Sigma$ intersects every orbit we may assume $q \in \Sigma$. Let $g \in G$ be such that $q \in g \cdot \Sigma$. We have to show that there is some $\tilde{g} \in G_q$ such that $\tilde{g} \cdot \Sigma = g \cdot \Sigma$ holds. Since we have $q \in g \cdot \Sigma$, it follows that $g^{-1} \cdot q \in \Sigma$. By Corollary \ref{3,3} there is some $n \in N_G(\Sigma)$ such that $g^{-1} \cdot q = n \cdot q$ and it follows that $\tilde{g} := gn \in G_q$ and $\tilde{g} \cdot \Sigma = g \Sigma$. 

By property \ref{C} of a fat section $T_p(G \cdot p)$ decomposes orthogonally for every $G$-regular $p \in \Sigma$ into $(T_p(G \cdot p) \cap T_p \Sigma) \oplus \nu_p \Sigma$. More generally:
Proposition 3.5. In all points \( q \) of a fat section \( \Sigma \) the tangent space \( T_qM \) decomposes compatibly and orthogonally in two ways:
\[
T_qM = T_q\Sigma \oplus \nu_q\Sigma = T_q(G \cdot q) \oplus \nu_q(G \cdot q).
\]
This means that the following decompositions are orthogonal:
\[
T_q\Sigma = (T_q\Sigma \cap T_q(G \cdot q)) \oplus (T_q\Sigma \cap \nu_q(G \cdot q)),
\]
\[
\nu_q\Sigma = (\nu_q\Sigma \cap T_q(G \cdot q)) \oplus (\nu_q\Sigma \cap \nu_q(G \cdot q)).
\]

The proof is basically the same as of \([GOT04, \text{Lemma } 5.10]\).

Definition 3.6. For a given fat section \( \Sigma \) and for every \( q \in \Sigma \) we define
\[
\mathcal{D}_q := T_q(G \cdot q) \cap T_q\Sigma \quad \text{and} \quad \mathcal{E}_q := T_q(G \cdot q) \cap \nu_q\Sigma.
\]
Following \([GOT04]\) we extend \( \mathcal{D} \) and \( \mathcal{E} \) to \( G \)-invariant distributions on \( M^{\text{reg}} \) using property \([\mathcal{D}]\) of a fat section. This yields \( T_p(G \cdot p) = \mathcal{D}_p \oplus \mathcal{E}_p \) for all \( p \in M^{\text{reg}} \).

Remark 3.7. Due to Proposition 3.5, \( T_q(G \cdot q) = \mathcal{D}_q \oplus \mathcal{E}_q \) is an orthogonal decomposition for all \( q \in \Sigma \) and both \( \mathcal{D} \) and \( \mathcal{E} \) are \( W \)-invariant (singular) distributions along \( \Sigma \).

Theorem 3.8. Let \( \Sigma \) be a fat section of \((G,M)\). For every \( q \in \Sigma \), the submanifold \( W \cdot q \subseteq G \cdot q \) is totally geodesic in \( G \cdot q \). Furthermore, for every \( \eta \in \nu_q(G \cdot q) \cap T_q\Sigma \) the shape operator \( A_\eta \) of \( G \cdot q \) leaves the decomposition \( T_q(G \cdot q) = \mathcal{D}_q \oplus \mathcal{E}_q \) invariant.

Proof. \( W \cdot q \) is a submanifold of \( M \), and by Proposition 3.5 we have
\[
T_x(G \cdot q) = \mathcal{D}_x \oplus \mathcal{E}_x = (T_x(G \cdot x) \cap T_x\Sigma) \oplus (T_x(G \cdot x) \cap \nu_x\Sigma)
\]
for all \( x \in W \cdot q \). Therefore \( T_x(G \cdot q) \) is invariant under the orthogonal reflection on \( T_x\Sigma \).

Now the claim follows from the next Lemma, which is \([BCO03, \text{Exercise } 8.6.3]\). \( \square \)

Lemma 3.9. Let \( \Sigma, N \) and \( \Sigma \cap N \) be submanifolds of the Riemannian manifold \( M \) and suppose that \( \Sigma \) is totally geodesic. Suppose that \( T_pN \) is invariant under the orthogonal reflection at \( T_p\Sigma \) for all \( p \in \Sigma \cap N \), then \( \Sigma \cap N \) is totally geodesic as a submanifold of \( N \) and \( A_\eta \), the shape operator of \( N \), leaves \( T_p(\Sigma \cap N) \) invariant for all \( p \in \Sigma \cap N \) and \( \eta \in \nu_pN \cap T_p\Sigma \).

Remark 3.10. For a polar action the Weyl-group orbits are discrete sets of points and thus trivially totally geodesic. However, if the copolarity is positive and non-trivial, then the orbits of the fat Weyl group are proper positive-dimensional totally geodesic submanifolds in their ambient orbit. So one should expect that the theorem imposes certain restrictions on actions having non-trivial positive copolarity.

4. Copolarity and Reductions of the Slice Representation

We now generalize \([GOT04, \text{Theorem } 5.6]\) from representations to arbitrary isometric group actions, without making any further assumptions. Therefore, our proof follows a rather different approach than the one in loc. cit.

Lemma 4.1. Let \( \Sigma \) be a totally geodesic submanifold of the Riemannian manifold \( M \) and let \( \gamma \subseteq \Sigma \) be a geodesic. Then every Jacobi field \( J \) along \( \gamma \) splits uniquely into Jacobi fields \( Y \) and \( Z \) along \( \gamma \) such that \( Y \) is a Jacobi field in \( \Sigma \) and \( Z \) is perpendicular to \( \Sigma \). Furthermore, every derivative of \( Z \) is perpendicular to \( \Sigma \).
Proof. Consider the orthogonal decomposition

\[ J(t) = Y(t) + Z(t) \]

of \( J \). Then \( Y \) and \( Z \) defined in this way are smooth vector fields along \( \gamma \). Since \( J \) satisfies the Jacobi equation we have:

\[ 0 = J'' + R(J, \dot{\gamma}, \dot{\gamma}) = Y'' + R(Y, \dot{\gamma}, \dot{\gamma}) + Z'' + R(Z, \dot{\gamma}, \dot{\gamma}). \]  \((\Delta)\)

Clearly, \( Y'' \) is tangential to \( \Sigma \). Since \( \Sigma \) is totally geodesic, \( R(Y, \dot{\gamma}, \dot{\gamma}) \) is also tangential to \( \Sigma \). Since parallel transports of vectors normal to a totally geodesic submanifold stay perpendicular to the submanifold, it follows from the characterization of the covariant derivative by parallel transport that \( Z'' \) is perpendicular to \( \Sigma \). Finally, the expression \( R(Z, \dot{\gamma}, \dot{\gamma}) \) is perpendicular to \( \Sigma \), because for all \( v \in T\Sigma \) we have, using the symmetry properties of the curvature tensor,

\[ \langle R(Z, \dot{\gamma}, \dot{\gamma}), v \rangle = \langle R(v, \dot{\gamma}, \dot{\gamma}), Z \rangle = 0. \]

By \((\Delta)\), both \( Y \) and \( Z \) are Jacobi fields and \( Y \) is even a Jacobi field of \( \Sigma \). \( \square \)

**Theorem 4.2** (Slice Theorem). If \((G, M)\) is isometric, then for all \( q \in M \):

\[ \copol(G_q, \nu_q(G \cdot q)) \leq \copol(G, M). \]

More generally, if \( \Sigma \) is a fat section of \((G, M)\) and \( q \in \Sigma \), then \( V_q := \nu_q(G \cdot q) \cap T_q\Sigma \) is a fat section of \((G_q, \nu_q(G \cdot q))\). If \( W \) is the fat Weyl group of \( \Sigma \), then \( W_q \) projects canonically onto the fat Weyl group of \( V_q \).

**Proof.** Let \( \Sigma \) be a fat section through \( q \). Since \( V_q \) is a linear subspace of \( \nu_q(G \cdot q) \), property \( [\text{A}] \) of a fat section is already satisfied. Property \( [\text{B}] \) follows from \( [\text{C}] \). There exist \( G_q \)-regular points in \( V_q \) by Lemma 2.7. By property \( [\text{C}] \) and Lemma 2.5 it follows that \( V_q \) intersects every \( G_q \)-orbit.

We also have property \( [\text{D}] \). If \( v \in V_q \) is \( G_q \)-regular, then, after scaling if necessary, we may assume that \( p := \exp_q(v) \) lies in a slice \( S_q \) through \( q \). Let \( g \in G_q \) satisfy \( g \cdot v \in V_q \). Then \( g \cdot p \in \Sigma \). The \( G_q \) regular points in \( S_q \) are \( G \)-regular if viewed as points of \( M \). Hence, \( p \) is also \( G \)-regular and therefore \( g \cdot \Sigma = \Sigma \). It follows that

\[ g \cdot V_q = T_q(g \cdot (\Sigma \cap S_q)) = T_q(\Sigma \cap S_q) = V_q. \]

It remains to show property \( [\text{C}] \) of a fat section. Equivalent to \( [\text{C}] \) is

\[ V_q^\perp \subseteq T_v(G_q \cdot v) \]

for all \( G_q \)-regular \( v \in V_q \). Here \( V_q^\perp \) denotes the orthogonal complement of \( V_q \) in \( \nu_q(G \cdot q) \). As in the proof of property \( [\text{D}] \) assume that \( p = \exp_q(v) \) lies in a slice \( S_q \) through \( q \). Since \( p \) is a \( G \)-regular and in \( \Sigma \), property \( [\text{C}] \) of \( \Sigma \) implies \( \nu_p \Sigma \subseteq T_p(G \cdot p) \). Let \( w \in V_q^\perp \) be arbitrary. Then \( d \exp_q(v)(w) \in \nu_v \Sigma \). In fact, \( d \exp_q(v)(w) = J(1) \) for the Jacobi field \( J \) along \( \gamma_v(t) = \exp_q(t \cdot v) \) and initial values \( J(0) = 0 \) and \( J'(0) = w \in \nu_q \Sigma \). (see [Lan99, Chapter IX, Theorem 3.1]). By Lemma 4.4, \( J \) is perpendicular to \( \Sigma \). In particular,

\[ J(1) \in \nu_p \Sigma \subseteq T_p(G \cdot p). \]

Now let \( X \) be a \( G \)-Killing field with \( d \exp_q(v)(w) = X_p \). Since \( d \exp_q(v)(w) \in T_pS_q \) and \( (G \cdot p) \cap S_q = G_q \cdot p \) we may further assume that \( X \) is a \( G_q \)-Killing field. Therefore

\[ d \exp_q(v)(w) = X_p \in T_p(G_q \cdot p). \]
Since \( G_q \cdot p = \exp_q(G_q \cdot v) \), we get
\[
T_p(G_q \cdot p) = d\exp_q(v)(T_v(G_q \cdot v)).
\]
It follows that \( w \in T_v(G_q \cdot v) \), because \( d\exp_q(v) \) is bijective.

We have therefore proved that \( V_q \) is a fat section of \((G_q, \nu_q(G \cdot q))\). The actions \((G, M)\) and \((G_q, \nu_q(G \cdot q))\) have the same cohomogeneity, and \( \dim V_q \leq \dim \Sigma \). Choosing \( \Sigma \) as a minimal section, it therefore follows that the copolarity of the slice representation is less than or equal to the copolarity of the \( G \)-action on \( M \).

The fat Weyl group of \( V_q \) is given by
\[
W(V_q) = N_{G_q}(V_q)/Z_{G_q}(V_q).
\]
We first show \( N_{G_q}(V_q) = N_{G_q}(\Sigma) = (N_G(\Sigma))_q \). Let \( g \in N_G(\Sigma) \cap G_q = N_{G_q}(\Sigma) \) be arbitrary. Then \( g \) leaves both \( T_q \Sigma \) and \( \nu_q(G \cdot q) \) invariant. Therefore, \( T_q \Sigma \cap \nu_q(G \cdot q) = V_q \) is also left invariant and it follows that \( g \in N_{G_q}(V_q) \). Conversely, for \( g \in N_{G_q}(V_q) \), again as in the proof of property \( \Box \) it follows that \( g \cdot \Sigma = \Sigma \) and hence \( g \in N_{G_q}(\Sigma) \). Now it is easy to see that
\[
Z_G(\Sigma) = Z_{G_q}(\Sigma) \subseteq Z_{G_q}(V_q).
\]
The commuting diagram below implies that \( W(\Sigma)_q \) projects canonically onto \( W(V_q) \):
\[
\begin{array}{ccc}
N_{G_q}(\Sigma) & \longrightarrow & N_{G_q}(V_q) \\
pr & & pr \\
W(\Sigma)_q & \longrightarrow & W(V_q).
\end{array}
\]
\[\square\]

**Remark 4.3.** For a minimal section \( \Sigma \), we do not know whether \( V_q \) is necessarily a minimal section of the slice representation \((G_q, \nu_q(G \cdot q))\), or not. However, the above proof shows:

**Corollary 4.4.** If \( \Sigma \) is a pre-section of \((G, M)\), then \( V_q = \nu_q(G \cdot q) \cap T_q \Sigma \) is a pre-section of \((G_q, \nu_q(G \cdot q))\). If \( \Sigma \) is a sufficiently small section, then \( V_q \) is also sufficiently small and \( W(V_q) = W_q \).

5. **Stability of Copolarity under Reductions**

We next show that the copolarity of a reduction \((W, \Sigma)\) is equal to that of \((G, M)\). We start with a Lemma, which may be interesting in its own right.

**Lemma 5.1.** If \( \Sigma \) is a fat section of an isometric action \((G, M)\), then the \( G \)-regular points in \( \Sigma \) are \( W(\Sigma) \)-regular and vice versa.

**Proof.** According to Proposition \( \Box \) (viii) the set of \( G \)-regular points is open and dense in \( \Sigma \). If we can show that the \( G \)-regular points in \( \Sigma \) all have the same \( W(\Sigma) \)-orbit type, then they must be \( W(\Sigma) \)-regular. This is because the \( W(\Sigma) \)-regular points are open and dense in \( \Sigma \), too. Let \( p \in \Sigma \) be an arbitrary \( G \)-regular point. Property \( \Box \) of a fat section implies \( Z_G(\Sigma) \subseteq G_p \subseteq N_G(\Sigma) \), and thus \( (N_G(\Sigma))_p = G_p \). Let \( q \) be another \( G \)-regular point in \( \Sigma \). Connect \( q \) with \( G \cdot p \) by a \( G \)-transversal geodesic \( \gamma \). Then by properties \( \Box \) and \( \Box \) of a fat section, \( \gamma \) is a geodesic of \( \Sigma \). We may assume that \( \gamma(0) = q \) and \( \gamma(1) = g \cdot p \) for some \( g \in G \). By property \( \Box \) again we have \( g \in N_G(\Sigma) \).

Since \( G_q = G_{g.p} = gG_p g^{-1} \) we have that both \( p \) and \( q \) are of the same \( W(\Sigma) \)-orbit type.

Conversely, let \( q \in \Sigma \) be an arbitrary \( W(\Sigma) \)-regular point. By Theorem \( \Box \), \( V_q \) is a fat section of \((G_q, \nu_q(G \cdot q))\) and \( W_q \) projects canonically onto the fat Weyl group \( W(V_q) \) of \( V_q \). Proposition \( \Box \) shows that \( V_q \) is also the representation space for the slice
representation of \((W(\Sigma), \Sigma)\) in \(q\). By assumption, \(W_q\) acts trivially on \(V_q\). Since \(W(V_q)\) acts effectively on \(V_q\) by definition, the group \(W(V_q)\) must be trivial. In particular, \((G_q, \nu_q(G \cdot q))\) is a polar representation with generalized Weyl group \(W(V_q)\). According to [PT88, Corollary 5.6.22] the latter is a Weyl group in the classical sense. However, a polar representation with trivial Weyl group must be trivial itself. Thus \(G_q\) acts trivially on \(\nu_q(G \cdot q)\), and in conclusion \(q\) is \(G\)-regular. □

**Theorem 5.2 (Stability theorem).** Let \((G, M)\) be an isometric action and let \(\Sigma\) be an arbitrary fat section. Then a subset \(\Sigma' \subseteq \Sigma\) is a fat section of \((G, M)\) if and only if it is a fat section of \((W(\Sigma), \Sigma)\). It follows that

\[
\text{copol}(G, M) = \text{copol}(W(\Sigma), \Sigma).
\]

If \(\Sigma\) is a minimal section, then the copolarity of \((W(\Sigma), \Sigma)\) is trivial.

**Proof.** First of all, if \(\Sigma'\) is complete and connected, totally geodesic and embedded in \(\Sigma\), then it also has these properties as a submanifold of \(M\) and vice versa. If \(\Sigma'\) intersects every \(G\)-orbit, then it also intersects every \(W(\Sigma)\)-orbit, because the latter are the intersections of \(G\)-orbits with \(\Sigma\) and we have \(\Sigma' \subseteq \Sigma\) (Corollary 3.3). Conversely, if \(\Sigma'\) intersects every \(W(\Sigma)\)-orbit, then it also intersects every \(G\)-orbit, because every \(G\)-orbit contains a \(W(\Sigma)\)-orbit. Next, by Lemma 5.1 we need not distinguish between \(G\)-regular and \(W(\Sigma)\)-regular points in \(\Sigma'\). We have for every regular \(p \in \Sigma\):

\[
\nu_p(G \cdot p) = \nu_p^\Sigma(W(\Sigma) \cdot p).
\]

Therefore, \(\nu_p(G \cdot p) \subseteq T_p \Sigma'\) is equivalent to \(\nu_p^\Sigma(W \cdot p) \subseteq T_p \Sigma'\), for every regular \(p \in \Sigma'\). Finally, let \(p \in \Sigma'\) be regular and let \(g \in G\) be such that \(g \cdot p \in \Sigma'\). Since \(\Sigma' \subseteq \Sigma\), it follows that \(g \in N_G(\Sigma)\). Now it is clear that \(\Sigma'\) has property [D] of a fat section with respect to \((G, M)\) if and only if it has this property with respect to \((W(\Sigma), \Sigma)\). □

**6. A Remark on Variational Completeness and Co-Completeness**

A main result of this section is that variational completeness of an isometric action is inherited to every reduction of that action, and conversely variational completeness of a reduction extends to the variational completeness of the original action. As a slight excursion we also generalize [GOT04, Theorem 4.1] in such a way that we relax the condition that the fat section \(\Sigma\) has to be flat to the condition that \(\Sigma\) has no conjugate points. This applies to more general situations, like \(\text{sec}(\Sigma) \leq 0\).

**Definition 6.1.** Let \(N\) be a submanifold of \(M\). An \(N\)-geodesic \(\gamma : [0, \varepsilon) \to M\) is a geodesic of \(M\) which emanates perpendicularly from \(N\). An \(N\)-Jacobi field \(J\) is a Jacobi field (along an \(N\)-geodesic \(\gamma\)) which is induced by a variation of \(N\)-geodesics.

One can show that if \(\gamma(0) = p \in N\) and \(v = \gamma'(0)\), then \(J\) is an \(N\)-Jacobi field if and only if it is a Jacobi field satisfying \(J(0) \in T_p N\) and \(J'(0) + A_v J(0) \in \nu_p N\). Here \(A_v\) denotes the shape operator of \(N\) in the direction of \(v\). Furthermore, the vector space \(N(\gamma)\) of all \(N\)-Jacobi fields along \(\gamma\) is isomorphic to \(T_p M = T_p N \oplus \nu_p N\) via \(J \mapsto J(0) + (J'(0) + A_v J(0))\).

We fix a fat section \(\Sigma\) of \((G, M)\) and let \(N := G \cdot p\) denote a fixed principal orbit with \(p \in \Sigma\). For \(v \in \nu_p N\) let \(\gamma_v(t) := \exp_p(tv)\). The following lemmas as well as their proofs are [GOT04, Lemma 4.3 and Lemma 4.4]. The second one characterizes under which conditions an \(N\)-Jacobi field is perpendicular to a given fat section, whereas the first one shows that every \(N\)-Jacobi field, induced by a \(G\)-Killing field and with the proper initial values, always satisfies this condition. Note that \(\text{sec}(\Sigma)\) may be arbitrary.
**Lemma 6.2** ([GOT04, Lemma 4.3]). Let $J$ be an $N$-Jacobi field along $\gamma_v$ with $J(0) \in \mathcal{E}_p$. If $J$ is the restriction of a $G$-Killing field on $M$ to $\gamma_v$, then $J$ satisfies $J'(0) + A_v J(0) = 0$.

**Lemma 6.3** ([GOT04, Lemma 4.4]). Let $J$ be an $N$-Jacobi field along $\gamma_v$ such that $J(0) \in \mathcal{E}_p$. Then $J$ is always orthogonal to $\Sigma$ if and only if $J'(0) + A_v J(0) = 0$.

With these lemmas we get a refined decomposition of $\mathcal{J}^N(\gamma)$:

**Proposition 6.4.** Let $\tilde{N} := W(\Sigma) \cdot p$ and denote the $\tilde{N}$-Jacobi fields in $\Sigma$ by $\mathcal{J}^{\tilde{N}}(\gamma)$. Then

$$\mathcal{J}^N(\gamma) = \mathcal{J}_0^N(\gamma) \oplus \mathcal{J}_{D}^N(\gamma) \oplus \mathcal{J}_{E}^N(\gamma),$$

where

$$\mathcal{J}_0^N(\gamma) := \{ J \in \mathcal{J}^N(\gamma) \mid J(0) = 0, J'(0) \in \nu_p N \},$$

$$\mathcal{J}_{D}^N(\gamma) := \{ J \in \mathcal{J}^N(\gamma) \mid J(0) \in D_p, J'(0) = -A_v J(0) \},$$

$$\mathcal{J}_{E}^N(\gamma) := \{ X \in \mathcal{J}^N(\gamma) \mid J(0) \in \mathcal{E}_p, J'(0) = -A_v J(0) \}.$$

In particular, if $J = J_0 + J_D + J_E$ is an $N$-Jacobi field represented with respect to the above decomposition, then, in view of Lemma [4.4], $J_0 + J_D$ is the part of $J$ which is everywhere tangential to $\Sigma$ and $J_E$ is part of $J$ which is everywhere perpendicular to $\Sigma$.

**Proof.** The decomposition follows from the isomorphism $\mathcal{J}^N(\gamma) \simeq T_p N \oplus \nu_p N$ and because of $T_p (G \cdot p) = D_p \oplus \mathcal{E}_p$ (see Definition 3.9). Note that Theorem 3.8 implies that $A_v$ leaves $D_p$ invariant. This shows that every element $J_D$ of $\mathcal{J}_D^N(\gamma)$ is everywhere tangential to $\Sigma$, because $J_D(0) \in D_p$ and $J_D'(0) = -A_v J_D(0) \in D_p$. It is also clear that every element $J_0$ of $\mathcal{J}_0^N(\gamma)$ is tangent to $\Sigma$, because of $J_0(0) = 0$ and $J_0'(0) \in \nu_p N$.

We next show that $J_0$ and $J_D$ are $\tilde{N}$-Jacobi fields. First of all, $\gamma$ is a geodesic in $M$ which starts in $\Sigma$ and since $\gamma'(0) \in \nu_p N \subseteq T_p \Sigma$ it is also tangential to $\Sigma$. Since $\Sigma$ is totally geodesic in $M$, it follows that $\gamma$ is a geodesic of $\Sigma$ and furthermore, $\gamma$ is a $\tilde{N}$-geodesic. Using Lemma 4.1 we see that $J_0$ and $J_D$ are Jacobi fields on $\Sigma$. For $J_0$ we now have to show $J_0'(0) \in \nu_p \tilde{N}$, and this is clear since we have $\nu_p N = \nu_p \tilde{N}$. Concerning $J_D$, we have that $J_D(0) \in D_p = T_p \tilde{N}$ and if $\tilde{A}$ denotes the shape operator of $\tilde{N}$, then

$$J_D'(0) + \tilde{A}_v J_D(0) = J_D'(0) + A_v J_D(0) = 0,$$

where we have used that $\tilde{A}_v = A_v|_{D_p}$, because $\tilde{N}$ is totally geodesic in $N$, by Theorem 3.8 again. By tracing the previous arguments backwards, we obtain that in fact the equalities $\mathcal{J}_0^N(\gamma) = \mathcal{J}_{\tilde{N}}^N(\gamma)$ and $\mathcal{J}_D^N(\gamma) = \mathcal{J}_{\tilde{D}}^N(\gamma)$ hold. The statements concerning $\mathcal{J}_E^N(\gamma)$ are direct consequences of Lemma 6.2 and Lemma 6.3.

**Definition 6.5.** An isometric action $(G, M)$ is **variationally complete** if for every $G$-orbit $N$, every $N$-geodesic $\gamma$ and every $\tilde{N}$-Jacobi field along $\gamma$, which vanishes for some $t_0 > 0$, is the restriction of a $G$-Killing field to $\gamma$.

It suffices to consider principal orbits only in order to show that an isometric action is variationally complete. This fact seems to be known in the literature. For instance, in [GOT04] this is implicitly assumed in the characterization of variational completeness via $\text{covar}(G, M) = 0$ (see below). A proof can be found in [LT07a, Remark 5.5].

**Theorem 6.6.** An isometric action $(G, M)$ is variationally complete if and only if a minimal reduction $(W(\Sigma), \Sigma)$ is variationally complete.

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1I would like to thank Alexander Lytchak for giving me this reference.
Proof. In the following let \( p \in \Sigma \) be a regular point. Due to Lemma 5.1 \( G \) - and \( W \)-regular points are the same. Put 
\[
N := G \cdot p \quad \text{and} \quad \tilde{N} := W(\Sigma) \cdot p
\]
and let \( \gamma \) be an arbitrary \( \tilde{N} \)-geodesic starting in \( p \).

Suppose that \((G, M)\) is variationally complete. If \( J \in \mathcal{J}^{\tilde{N}}(\gamma) \) satisfies \( J(t_0) = 0 \) for some \( t_0 > 0 \), then we can view \( J \) as an \( N \)-Jacobi field along the \( N \)-geodesic \( \gamma \), according to Proposition 6.4. By variational completeness of \((G, M)\), there is a \( G \)-Killing field \( X \) such that \( J = X|_\gamma \). Let now \( \text{pr}_\Sigma X \) denote the orthogonal projection of \( X \) onto \( \Sigma \). By [Mag08, Theorem 1] this is a \( W \)-Killing field on \( \Sigma \) (here we use that \( \Sigma \) is a minimal section). Since \( X(\gamma(t)) = J(t) \in T_{\gamma(t)} \Sigma \) and therefore \( J(t) = \text{pr}_\Sigma X(\gamma(t)) \), we may conclude that \( J \) is the restriction of a \( W \)-Killing field to \( \gamma \).

For the converse direction, suppose now that \((W(\Sigma), \Sigma)\) is variationally complete. Let \( p \in M \) be an arbitrary regular point and \( \gamma \) an \( N \)-geodesic starting in \( p \). Without loss of generality, we may assume that \( p \in \Sigma \) and that \( \gamma \) is a \( \tilde{N} \)-geodesic (a suitable translate \( g \cdot \Sigma \) contains \( p \) and hence \( \gamma \), and the minimal reduction \((W(g \cdot \Sigma), g \cdot \Sigma)\) is also variationally complete). We decompose an arbitrary \( N \)-Jacobi field \( J \), which vanishes for some \( t_0 > 0 \), according to Proposition 6.4 into the three parts \( J = J_0 + J_D + J_E \). The proposition tells us that \( J_E \) is already induced by a \( G \)-Killing field. From 
\[
0 = J(t_0) = J_0(t_0) + J_D(t_0) + J_E(t_0) 
\]
and the variational completeness of \((W(\Sigma), \Sigma)\) it follows that \( J_0 + J_D \) is induced by an \( N(\Sigma) \)-Killing field. But such a field is also a \( G \)-Killing field and it follows that \( J \) is the restriction of a \( G \)-Killing field to \( \gamma \). \( \square \)

Corollary 6.7. An isometric action \((G, M)\) is variationally complete if and only if some (and hence any) reduction \((W(\Sigma), \Sigma)\) is variationally complete.

Proof. According to Theorem 5.2 \((G, M)\) and \((W(\Sigma), \Sigma)\) have a common minimal reduction \((W(\Sigma'), \Sigma')\) with \( \Sigma' \subseteq \Sigma \). Hence, we may apply Theorem 6.6 to \((G, M)\) and \((W(\Sigma'), \Sigma')\) and then to \((W(\Sigma), \Sigma)\) and \((W(\Sigma'), \Sigma')\) and vice versa. \( \square \)

Remark 6.8. Theorem 6.6 and Corollary 6.7 can also be deduced from [L.T07a, Theorem 1.3] and our Theorem 3.1. In fact, the first result states that variational completeness only depends on the metric properties of \( G \backslash M \), which by the second result is isometric to \( W \backslash \Sigma \) for any reduction \((W, \Sigma)\) of \((G, M)\).

Corollary 6.9. If \((G, M)\) is a polar and variationally complete action, then every section is free of conjugate points. In particular, if \( M \) is a Riemannian manifold of non-negative Ricci curvature or compact and of non-negative scalar curvature, then a variationally complete action on \( M \) is polar, if and only if it is hyperpolar.

Proof. Polarity implies that the generalized Weyl group \( W(\Sigma) \) of any section \( \Sigma \) is discrete. Furthermore, a Lie group acts variationally complete if and only if its identity component does. However, if the trivial group acts variationally complete, this only means that every Jacobi field which vanishes in two different points, vanishes entirely. Hence, there are no conjugate points in \( \Sigma \). By being totally geodesic, \( \Sigma \) inherits the curvature conditions of \( M \). By a result of Mendonca and Zhou, [M.Z00, Corollary 1], resp. Green [Gre58], we deduce from the above that \( \Sigma \) has to be flat. \( \square \)
Remark 6.10. Conlon proved in [Con72] that hyperpolar actions are variationally complete. In general, the converse is false. Take, for instance, the action of the trivial group on a non-flat space of non-positive curvature. This action is variationally complete and polar, but not hyperpolar. However, Lytchak and Thorbergsson proved in [LT07], that variationally complete actions on manifolds of non-negative curvature are hyperpolar.

We briefly recall the notion of variational co-completeness, which has been introduced in [GOT04]. Let \( N = G \cdot p \) denote an arbitrary principal orbit and consider the isomorphism \( \mathcal{J}^N(\gamma) \simeq T_pN + \nu_pN \). For a subspace \( U_p \subseteq T_pM \) consider the condition:

\[
(P) \quad \text{for every } N\text{-geodesic } \gamma \text{ and every } J \in \mathcal{J}^N(\gamma) \text{, vanishing in some } t_0 > 0 \text{ with } (J(0), J'(0) + A_\nu J(0)) \perp U_p \text{ it follows that } J \big|_\gamma \text{ for some } G\text{-Killing field } X.
\]

If \( U_p \) satisfies condition \((P)\), then \( g_p U_p \) satisfies this condition in \( g \cdot p \). Furthermore, \( U_p = T_pM \) always satisfies condition \((P)\).

Definition 6.11. We write \( \text{covar}_N(G, M) \leq \dim U_p \), if \( U_p \) satisfies condition \((P)\). We say that the variational co-completeness of \((G, M)\) is less than or equal to \( k \), if \( \text{covar}_N(G, M) \leq k \) holds for all principal orbits \( N \). We also write \( \text{covar}(G, M) \leq k \).

A canonical choice for \( U_p \) is always \( T_p \Sigma = \nu_pN \oplus D_p \), where \( \Sigma \) denotes a fat section through \( p \). This is due to Proposition 6.4. In particular, we always have

\[ \text{covar}(G, M) \leq \text{cohom}(G, M) + \text{copol}(G, M). \]

This estimate can sometimes be considerably improved as in the following result, which is a generalization of [GOT04] Theorem 4.1. We note however that one only has to replace the condition \( \text{sec}(\Sigma) = 0 \) in the proof of [GOT04] Lemma 4.2] by the condition that \( \Sigma \) has no conjugate points. This occurs, for instance, whenever \( \text{sec}(M) \leq 0 \).

Theorem 6.12. Let \((G, M)\) be an isometric action and \( \Sigma \subseteq M \) a \( k \)-section. If \( \Sigma \) is free of conjugate points in the induced metric, then \( \text{covar}(G, M) \leq k \). In particular,

\[ \text{covar}(G, M) \leq \text{copol}(G, M). \]

We even obtain Corollary 4.5 of loc. cit. under these relaxed conditions:

Corollary 6.13. Let \((G, M)\) be an isometric action and let \( \Sigma \) be a pre-section, with we assume to have no conjugate points. Let \( N \) be a principal orbit and let \( p \in N \cap \Sigma \). Then \( D_p = T_pN \cap T_p\Sigma \) has property \((P)\).

7. Global Resolutions of Isometric Actions with Respect to Fat Sections

In this section we define the (global) resolution \( M_\Sigma \) of an isometric action \((G, M)\) with respect to an arbitrary fat section \( \Sigma \). This is related to the core resolution construction of Grove and Searle in [GS00]. The reason, why \( M_\Sigma \) is called a resolution, is that it is a \( G \)-space whose isotropy groups are smaller than those of \((G, M)\). Roughly speaking, the \( G \)-orbits on \( M_\Sigma \) are less singular than the \( G \)-orbits on \( M \). In the following, let \((G, M)\) be an isometric action and let \( \Sigma \) be a fat section. Put \( N = N_G(\Sigma) \) and \( H = Z_G(\Sigma) \). Then \( W = N/H \) is the fat Weyl group of \( \Sigma \). Since \( \Sigma \) is a \( W \)-space, we may form the associated bundle \( G/H \times_W \Sigma \rightarrow G/N \) with fibre \( \Sigma \), where \( G/H \times_W \Sigma \) is the orbit space under the diagonal \( W \)-action on \( G/H \times \Sigma \) given by \( nH \cdot (gH, s) := (gn^{-1}H, n \cdot s) \). Its total space is a \( G \)-space with respect to the \( G \)-action \( l \cdot [gH, s] := [lgH, s] \).
Definition 7.1. The **resolution** of \((G, M)\) with respect to \(\Sigma\) is defined as
\[
M_\Sigma := G/H \times_W \Sigma.
\]
If \(\Sigma\) is a minimal section, we call \(M_\Sigma\) a **minimal resolution**.

We now list some features related to \(M_\Sigma\) (c.f. [GS00], Theorem 2.1):

**Theorem 7.2.** Let \(\varphi : G \times M \to M\) denote the group action \((G, M)\). Then

(i) The group action \(\varphi\) induces a smooth and surjective \(G\)-equivariant map:
\[
\tilde{\varphi} : M_\Sigma \to M, \left[gH, s\right] \mapsto g \cdot s.
\]

(ii) The isotropy group of a point \(\left[eH, s\right] \in M_\Sigma = G/H \times_W \Sigma\) is given by:
\[
G_{[eH, s]} = N \cap G_s = N_{G_s}(\Sigma).
\]

(iii) \(\Sigma\) is canonically \(N\)-equivariantly immersed into \(M_\Sigma\) via the map \(s \mapsto [eH, s]\).

The image \(\tilde{\Sigma}\) is embedded into \(M_\Sigma\) because it is a fibre of \(M_\Sigma \to G\setminus N\), and furthermore it intersects every \(G\)-orbit on \(M_\Sigma\). It follows that \(\tilde{\varphi}\) restricts to a \(W\)-equivariant diffeomorphism between \(\tilde{\Sigma}\) and \(\Sigma\).

(iv) The set of \(G\)-regular points \((M_\Sigma)^{\text{reg}}\) can be identified with \(G/H \times_W \Sigma^{\text{reg}}\), and \(\tilde{\varphi}\) restricts to a \(G\)-equivariant diffeomorphism from \((M_\Sigma)^{\text{reg}}\) onto \(M^{\text{reg}}\). This yields a bundle with structure group \(W\) and totally geodesic fibres \(g \cdot \Sigma^{\text{reg}}, g \in G\):
\[
\pi : M^{\text{reg}} \to G/N, \ g \cdot s \mapsto gN.
\]

(v) The orbit spaces \(G \setminus M_\Sigma\) and \(G \setminus M\) are canonically homeomorphic.

(vi) \(d\tilde{\varphi}_{[eH, s]} : T_{[eH, s]}M_\Sigma \to T_{s}M\) is a linear isomorphism if and only if
\[
T_s(G \cdot s) + T_s\Sigma = T_sM. \quad (*)
\]

This is furthermore equivalent to \(G_s \subseteq N\) and also to \((G_s)^{\circ} = (N \cap G_s)^{\circ}\). \(\tilde{\varphi}\) is a \(G\)-equivariant diffeomorphism if and only if \((*)\) is satisfied for all \(s \in \Sigma\).

(vii) The \(G\)-translates of \(\tilde{\Sigma}\) foliate \(M_\Sigma\).

**Proof.** (i): If \([gH, s] = [\tilde{g}H, \tilde{s}] \in M_\Sigma\), then there is some \(n \in N\) and \(h \in H\) with
\[
(\tilde{g}, \tilde{s}) = (gn^{-1}h, n \cdot s).
\]
It follows that
\[
\tilde{g} \cdot \tilde{s} = g \underbrace{\left(hn^{-1}hn\right)}_{\in H} \cdot s = g \cdot s
\]
and we have shown that \(\tilde{\varphi}\) is well defined. Since \(\Sigma\) intersects every orbit, it follows that \(\varphi\) restricted to \(G \times \Sigma\) maps onto \(M\). Furthermore, \(H\) acts trivially on \(\Sigma\), and thus
\[
G/H \times \Sigma^{\text{reg}} \to M^{\text{reg}}, \ (gH, s) \mapsto g \cdot s
\]
(again denoted by \(\varphi\)) is still surjective. The following diagram commutes:
\[
\begin{array}{ccc}
G/H \times \Sigma & \xrightarrow{\varphi} & M \\
pf \downarrow & & \downarrow \tilde{\varphi} \\
M_\Sigma & \xrightarrow{\tilde{\varphi}} & M.
\end{array}
\]
From this we can read off that \(\tilde{\varphi}\) is also surjective and \(G\)-equivariant, and since the vertical map is a surjective submersion, it follows that \(\tilde{\varphi}\) is smooth.
(ii): Let \( g \in G_{[eH,s]} \) be arbitrary. Then there exists some \( n \in N \) and \( h \in H \) such that \( (g, s) = (n^{-1}h, n \cdot s) \). This implies \( n \in G_s \) and therefore \( gh^{-1} \in G_s \). Since \( H \subseteq G_s \), it follows that \( g \in G_s \cap N \). If conversely \( g \in G_s \cap N \), then
\[
g \cdot [eH, s] = [gh, s] = [eH, g^{-1} \cdot s] = [eH, s],
\]
showing that \( g \in G_{[eH,s]} \).

(iii): This statement is easily verified.

(iv): The first part follows from (ii) and (iii). It remains to show that \( \varphi|_{(M_S)_{reg}} \) is injective with smooth inverse. Suppose that \( g \cdot s = \tilde{g} \cdot \tilde{s} \) for \( g, \tilde{g} \in G \) and \( s, \tilde{s} \in \Sigma_{reg} \). Then \( \tilde{s} = \tilde{g}^{-1}g \cdot s \) and property (D) of a fat section implies \( n := \tilde{g}^{-1}g \in N \). Hence,
\[
[gH, s] = [gn^{-1}H, n \cdot s] = [\tilde{g}, \tilde{s}].
\]

By property (C) of a fat section, \( \Sigma \) is transversal to every principal orbit. Using (vi) it follows that \( \varphi|_{(M_S)_{reg}} \) is a submersion and thus a diffeomorphism.

(v): The map \( f : G \setminus M_S \to G \setminus M \), \( G \cdot [eH, s] \mapsto G \cdot s \) is well defined and makes
\[
\begin{array}{ccc}
M_S & \xrightarrow{\varphi} & M \\
pr & & pr \\
G \setminus M_S & \xrightarrow{f} & G \setminus M
\end{array}
\]
commute. Hence \( f \) is continuous and surjective. It is also easy to see that \( f \) is injective. To show that \( f^{-1} \) is continuous, we write it as a composition of continuous maps:
\[
G \setminus M \xrightarrow{\tilde{t}^{-1}} W \setminus \Sigma \xrightarrow{} W \setminus \tilde{\Sigma} \xrightarrow{} G \setminus M_S,
\]
where \( \tilde{t} \) is the map from Theorem 3.1 and the other two maps are the continuous injections induced by the continuous maps \( \Sigma \hookrightarrow \tilde{\Sigma} \), resp. \( \Sigma \hookrightarrow M_S \), both of which appear in (iii).

(vi): From the diagram in the proof of (i) we see that \( d\tilde{\varphi}|_{[eH,s]} : T_{[eH,s]}M_S \to T_sM \) is surjective if and only if \( d\varphi|_{[eH,s]} : T_{[eH,s]}G/H \times \Sigma \to T_sM \) is surjective. We have
\[
d\varphi|_{[eH,s]}(X + \mathfrak{h}, v) = X_s + v,
\]
where \( X_s \) is the value of the Killing field induced by \( X \in \mathfrak{g} \) on \( M \) in \( s \). This yields
\[
\text{im}(d\varphi|_{[eH,s]}) = T_s(G \cdot s) + T_s\Sigma,
\]
and thus \( d\varphi|_{[eH,s]} \) is onto if and only if (**) holds.

By Proposition 3.5
\[
T_s(G \cdot s) + T_s\Sigma = T_s(G \cdot s) \oplus (T_s\Sigma \cap \nu_s(G \cdot s)).
\]

Since the decomposition on the right is orthogonal, (**) is equivalent to \( \nu_s(G \cdot s) \subseteq T_s\Sigma \). This in turn is equivalent to the statement that \( G_s \subseteq N \). In fact, since the \( G_s \)-regular points in \( \nu_s(G \cdot s) \) correspond to \( G \)-regular points in \( M \) under the exponential map, it follows from property (D) of a fat section that, if \( \nu_s(G \cdot s) \subseteq T_s\Sigma \) holds, then \( G_s \subseteq N \). Conversely, if \( G_s \subseteq N \) then, according to the Slice Theorem 4.2, \( \nu_s^{\Sigma}(W \cdot s) \) is a \( G_s \)-invariant subspace of \( \nu_s(G \cdot s) \). However, this just means \( \nu_s(G \cdot s) = \nu_s^{\Sigma}(W \cdot s) \subseteq T_s\Sigma \).

Again by Proposition 3.5 we have \( T_s(G \cdot s) + T_s\Sigma = T_s\Sigma \oplus (T_s(G \cdot s) \cap \nu_s(\Sigma)) \). Thus, (**) is furthermore equivalent to
\[
\dim M = \dim \Sigma + (\dim(G \cdot s) - \dim(W \cdot s))
\]
Let \((\cdot)_{\text{princ}}\) denote a principal isotropy group for the action in parentheses. Then
\[
\dim \Sigma = \text{cohom}(G, M) + \dim W - \underbrace{\dim(W, \Sigma)_{\text{princ}}}_{=\dim(G,M)_{\text{princ}}-\dim H}
\]
The right hand side of (*** *) is therefore equal to:
\[
\text{cohom}(G, M) + \dim W - \dim(W, \Sigma)_{\text{princ}} + (\dim G - \dim G_s - \dim W + \dim W_s) = \dim M + \dim H - \dim G_s + \dim W_s = \dim(G_s \cap N) - \dim H
\]
It follows that (*** *) is equivalent to \( \dim(G_s \cap N) = \dim G_s \), or \((G_s)^o = (G_s \cap N)^o\).

Suppose that \( \tilde{\varphi} \) is a local diffeomorphism. Since \( \tilde{\varphi} \) restricted to \( (M_2)_{\text{reg}} \) is a diffeomorphism onto \( M_{\text{reg}} \) and since the regular points form an open and dense subset of their surrounding space, it follows that \( \tilde{\varphi} \) is a diffeomorphism from \( M_2 \) onto \( M \).

(vii): Let \( q := [gH, s] \in M_2 \) be arbitrary. Due to Corollary 3.4, \( Gq = g(N \cap G_s)g^{-1} \) is transitive on the set of \( G \)-translates of \( \tilde{\Sigma} \) that contain \( q \). Clearly, \( g \cdot \tilde{\Sigma} \) contains \( q \). For an arbitrary \( gng^{-1} \in G_q \), where \( n \in N(\Sigma) \cap G_s \), we have \( (gng^{-1}) \cdot (g \cdot \tilde{\Sigma}) = (gn) \cdot \tilde{\Sigma} = g \cdot \tilde{\Sigma} \). Therefore, the only \( G \)-translate through \( q \) is \( g \cdot \tilde{\Sigma} \).

\[ \square \]

**Corollary 7.3.** If \((G, M)\) has only principal or exceptional orbits, then \( M_2 \cong M \).

**Proof.** (Compare with [GS00 Corollary 2.4]). Let \( q \in \Sigma \) be arbitrary. According to Lemma 2.7, there is some \( G \)-regular point \( p \in \Sigma \) in a slice around \( q \). We thus have \( G_p \subseteq G_q \), and by assumption \((G_q)^o = (G_p)^o\). Since \( p \in \Sigma \) is \( G \)-regular, property [D] of a fat section implies \( G_p \subseteq N \). This yields: \((G_p)^o \subseteq (N(\Sigma) \cap G_q)^o \subseteq (G_q)^o = (G_p)^o\), and the claim follows from Theorem 7.2 (vi).

So far we have considered \( M_2 \) only as a smooth manifold without any Riemannian metric on it. It is natural to demand that \( G \) should act isometrically on \( M_2 \). Furthermore, the Riemannian metric on \( M_2 \) should be induced by a product metric on \( G/H \times \Sigma \). Hence, we consider \((G-W)\)-invariant metrics on \( G/H \) (cf. Section 12).

**Proposition 7.4.** Suppose that \( G/H \) carries a \((G-W)\)-invariant Riemannian metric and \( \Sigma \) the Riemannian metric induced by \( M \). Then \( M_2 \), endowed with the Riemannian metric submersed from \( G/H \times \Sigma \), has the following properties:

(i) \((G, M_2)\) is an isometric action.
(ii) If \( \Sigma \) is a \( k \)-section of \((G, M)\), then \( \tilde{\Sigma} = \{eH, s \mid s \in \Sigma\} \) is a \( k \)-section of \((G, M_2)\) and \( W(\tilde{\Sigma}) = W(\Sigma) \). In particular, the foliation of \( M_2 \) given by the \( G \)-translates of \( \tilde{\Sigma} \) has totally geodesic leaves.
(iii) \((M_2)_2 \tilde{\Sigma} \cong M_2 \) (G-equivalent).
(iv) If \( \Sigma \) is a minimal section of \((G, M)\), then \( \text{copol}(G, M_2) \leq \text{copol}(G, M) \).

**Proof.** (i) is clear by the assumptions made on the metric on \( G/H \).

(ii): By Theorem 7.2 (iii) we have that \( \tilde{\Sigma} \) is complete, connected and embedded into \( M_2 \) and intersects every \( G \)-orbit. Consider the principal bundle
\[
\psi : G/H \times \Sigma \to M_2, \ (gH, s) \mapsto [gH, s],
\]
which maps a point \((gH, s)\) to its \( W \)-orbit \([gH, s] = \{(gn^{-1}H, n \cdot s) \mid nH \in W\}\). By our choice of metric, \( \psi \) is a Riemannian submersion.

We claim that \( \tilde{\Sigma} \) is totally geodesic in \( M_2 \). In fact, \( \psi^{-1}(\tilde{\Sigma}) = W \times \Sigma \) and since \( W \) is totally geodesic in \( G/H \) by Corollary 12.4, it follows that \( W \times \Sigma \) is totally geodesic in \( G/H \times \Sigma \). Thus \( \tilde{\Sigma} = \psi(W \times \Sigma) \) is totally geodesic in \( M_2 \). This already yields properties (A) and (B) of a fat section. The fibre of \( \psi \) over \([eH, s]\) is
\[
\psi^{-1}(eH, s) = \{(nH, n^{-1} \cdot s) \mid nH \in W\}.
\]
In order to speak about metric relations in the tangent spaces of \( M_\Sigma \) we have to determine the vertical and horizontal distributions, \( \mathcal{V} \) and \( \mathcal{H} \), of \( \psi \) along \( \{ eH \} \times \Sigma \).

\[
\mathcal{V}_{(eH,s)} := T_{(eH,s)}\psi^{-1}(\{ eH, s \}] \quad \text{and} \quad \mathcal{H}_{(eH,s)} := (\mathcal{V}_{(eH,s)})^\perp.
\]

The definition of the fibre yields

\[
\mathcal{V}_{(eH,s)} = \{(X + h, -X_s) \mid X + h \in n/h \} \subseteq n/h \times T_s(W \cdot s),
\]

and a computation shows that

\[
\mathcal{H}_{(eH,s)} = ((n/h)^\perp \times \nu^\Sigma_s(W \cdot s)) \oplus A_s,
\]

where \( A_s := \mathcal{H}_{(eH,s)} \cap (n/h \times T_s(W \cdot s)) \). In fact, \( A_s \) corresponds to the tangent space of the \( W \)-orbit through \( [eH, s] \) (induced by the left action of \( G \)) and one can show that

\[
A_s = \{(f_s(v), v) \mid v \in T_s(W \cdot s)\},
\]

for some linear monomorphism \( f_s : T_s(W \cdot s) \to n/h \) (we do not need this fact in the following). By our assumptions on the Riemannian metric on \( G/H \times \Sigma \) and \( M_\Sigma \), we have that \( \psi \) is a Riemannian submersion. Hence, we may identify subspaces of \( T_{[eH,s]}M_\Sigma \) with certain subspaces of \( \mathcal{H}_{(eH,s)} \). More precisely,

\[
T_{[eH,s]}(G \cdot [eH, s]) \cong \mathcal{H}_{(eH,s)} \cap (T_{(eH,s)}(G \cdot (eH, s)) + \mathcal{V}_{(eH,s)}) = A_s \oplus ((n/h)^\perp \times \{0\}),
\]

and it follows that

\[
\nu_{[eH,s]}(G \cdot [eH, s]) \cong \{0\} \times \nu^\Sigma_s(W \cdot s) \subseteq (\{0\} \times \nu^\Sigma_s(W \cdot s)) \oplus A_s \cong T_{[eH,s]}\tilde{\Sigma}.
\]

We therefore have for all points \( [eH, s] \in \tilde{\Sigma} \) (and not just the \( G \)-regular ones) that

\[
\nu_{[eH,s]}(G \cdot [eH, s]) \subseteq T_{[eH,s]}\tilde{\Sigma}.
\]

This shows property \([C]\) of a fat section. We now come to property \([D]\). If \( [eH, s] \in \tilde{\Sigma} \) and \( g \in G \) with \( g \cdot [eH, s] = [gH, s] \in \tilde{\Sigma} \), it follows that \( g \in N \) (again this holds not only in the \( G \)-regular points). We have therefore shown that \( \tilde{\Sigma} \) is a \( k \)-section of \((G, M_\Sigma)\) if \( \Sigma \) is a \( k \)-section of \((G, M)\). It is also not difficult to show \( W(\tilde{\Sigma}) = W(\Sigma) \). In fact, \( N_G(\tilde{\Sigma}) = N_G(\Sigma) \) and \( Z_G(\tilde{\Sigma}) = Z_G(\Sigma) \).

(iii): Let \( \tilde{\varphi} : (M_\Sigma)_{\tilde{\Sigma}} \to M_\Sigma \) denote the canonical \( G \)-equivariant surjection. That is

\[
\tilde{\varphi} : G/H \times \tilde{\Sigma} \to M_\Sigma, \ [gH, [eH, s]] \mapsto [gH, s].
\]

If \( \tilde{\varphi}([gH, [eH, s]]) = \tilde{\varphi}([\tilde{g}H, [\tilde{e}H, \tilde{s}]]) \), then \( [gH, s] = [\tilde{g}H, \tilde{s}] \). Now \( \tilde{g}H = gn^{-1}H \) and \( \tilde{s} = n \cdot s \) for some \( n \in N \). But this implies

\[
[\tilde{g}H, [\tilde{e}H, \tilde{s}]] = [gn^{-1}H, [eH, n \cdot s]] = [gH, n \cdot [eH, n \cdot s]] = [gH, [nH, n \cdot s]] = [gH, [eH, s]].
\]

This shows that \( \tilde{\varphi} \) is injective. By Theorem 7.2(ii) we have \( G_{[eH,s]} \subseteq N \) for all \( s \in \Sigma \) and then (vi) of the same Theorem implies that \( \tilde{\varphi} \) is a submersion. It follows that the map is a \( G \)-equivariant diffeomorphism.

(iv) is an immediate consequence of (ii).
Remark 7.5.

(i) We do not know whether for a minimal section $\Sigma$ of $(G, M)$ it is actually possible that $\text{copol}(G, M_\Sigma) < \text{copol}(G, M)$, or not.

(ii) According to Proposition [12.2] (iv), the assumptions in the above Proposition above can be satisfied if $N$ is compact.

(iii) There are other natural ways to endow $M_\Sigma$ with a Riemannian metric such that $\Sigma$ is totally geodesic, see for instance [Bes87 Theorem 9.59]. We do not know if $G$ then still acts isometrically on $M_\Sigma$ though.

The next result generalizes [GS00, Proposition 2.6], basically using the same proof.

**Proposition 7.6.** Let $(G, M)$, $G$ compact, be an isometric action with fat section $\Sigma$. If $\sec(M) \geq k$ for a $k \leq 0$, then $\sec(M_\Sigma) \geq k$ for some Riemannian $G$-metric on $M_\Sigma$.

**Remark 7.7.** As a concluding remark of this section, we show that for every triple $H \subseteq N \subseteq G$, where $G$ is a lie group, $H$ and $N$ are closed subgroups of $G$ and such that $N$ is compact, there exists some manifold $\Sigma$ on which $W = N/H$ acts isometrically, with trivial principal isotropy group and such that $M := G/H \times_W \Sigma$ is a Riemannian $G$-manifold with fat section $\Sigma$ and fat Weyl group $W$. This generalizes the construction in [PT88, 5.6.20]. In fact, since $W$ is compact, it acts faithfully on some Euclidean vector space $V$. Then $W$ acts with trivial principal isotropy group on the $k$-fold inner direct sum $\Sigma := k \cdot V$ for some $k \in \mathbb{N}_{>0}$. If $G/H$ is endowed with a $(G-W)$-invariant Riemannian metric, then $M := G/H \times_W \Sigma$ with the submersed metric from $G/H \times \Sigma$ is a $G$-manifold. Similarly as in the proof of Proposition 7.3 (ii) one can show that $\tilde{\Sigma} := \{[eH, s] \mid s \in \Sigma\}$ is a fat section with fat Weyl group $W$.

### 8. On a Generalization of Chevalley’s Restriction Theorem

Recall that a smooth $p$-form $\omega \in \Omega(M)$ is called $G$-invariant, if for all $g \in G$ we have that $g^*\omega = \omega$. The set of all $G$-invariant $p$-forms on $M$ will be denoted by $\Omega^p(M)^G$. A $p$-form $\omega$ is called horizontal, if for all $X \in \mathfrak{g}$ we have $\iota_X(\omega) = 0$. Here $\iota_X$ denotes contraction by the Killing field generated by $X$. The set of all $G$-invariant horizontal $p$-forms is denoted by $\Omega^p_{\text{hor}}(M)^G$. These forms are also called basic forms.

If $\Sigma$ is a fat section with fat Weyl group $W$, then in view of Corollary 3.2 it is natural to ask whether the isomorphism $\iota^*$ also yields $C^\infty(M)^G \simeq C^\infty(\Sigma)^W$, or if we even have $\Omega^p_{\text{hor}}(M)^G \simeq \Omega^p_{\text{hor}}(\Sigma)^W$. In the polar case (i.e. $\text{copol}(G, M) = 0$) the first statement has been proved by Palais and Terng in [PT87] and the second statement by Michor in [Mic90, Mic97]. In the general case we note the following:

**Proposition 8.1.** The map $\iota^*: C^\infty(M)^G \to C^\infty(\Sigma)^W$, $f \mapsto f|_\Sigma$, is well defined and injective, and the $G$-invariant continuous extension $(\iota^*)^{-1}(f)$ of $f \in C^\infty(\Sigma)^W$ to $M$ is smooth on $M^{\text{reg}}$.

**Proof.** First note that $\iota^*: C^\infty(M)^G$ is well defined, because $\Sigma$ is an embedded submanifold of $M$. The injectivity is also clear, since $\iota^*$ as a map on $C^0(M)^G$ is already injective due to Corollary 3.2. Let now $f \in C^\infty(\Sigma)^W$ be arbitrary and denote its $G$-invariant extension to $M$ by $F$. Smoothness of $F$ is a local condition. Thus, let $p \in M$ be an arbitrary point and let $U$ be a tubular neighborhood of $G \cdot p$. Since $F$ is $G$-invariant, we may assume that $p \in \Sigma$. Let furthermore $S_p$ be a slice through $p$ such that $U = G \cdot S_p$. It is known that $F|_U$ is smooth in $p$ if and only if $F|_{S_p}$ is smooth in $p$. Since $\Sigma$ is a fat section we have $S_p \subseteq \Sigma$ in the case that $p$ is a $G$-regular point and $S_q$ is also a slice with respect to the $W$-action on $\Sigma$. Hence $F|_{S_p} = f|_{S_p}$ is smooth in $p$. \qed
Suppose that for every \( q \in \Sigma \) the relation \( \nu_q(G \cdot q) \subseteq T_q \Sigma \) holds. Then the arguments in the above proof show that the \( G \)-invariant continuation of a smooth \( W \)-invariant function on \( \Sigma \) is smooth on the whole of \( M \). In particular, if \( M_\Sigma \) is the resolution of \( (G, M) \) with respect to \( \Sigma \) and \( \tilde{\Sigma} = \{ [gH, s] \mid s \in \Sigma \} \) is the fat section induced by \( \Sigma \) (see Section 8), we have

**Corollary 8.2.** Let \( H = Z_G(\Sigma) \). If \( G/H \) carries a \((G-W)\)-invariant Riemannian metric, then \( i^* : C^\infty(M_\Sigma)^G \to C^\infty(\tilde{\Sigma})^W \) is an isomorphism.

**Proof.** Due to Proposition 7.3, \( \tilde{\Sigma} \) is a fat section of \( M_\Sigma \). Let \( q = [eH, s] \in \tilde{\Sigma} \) be an arbitrary point. According to the Slice Theorem 1.2 the set \( V_q := \nu_q(G \cdot q) \cap T_q \tilde{\Sigma} \) is a fat section of the slice representation in \( q \). Hence, for every \( v \in \nu_q(G \cdot q) \) there is some \( g \in G_q \) with \( v \in g \cdot V_q \). By Theorem 7.2 \( M_\Sigma \) is foliated by \( \{ g \cdot \tilde{\Sigma} \mid g \in G \} \). Therefore,

\[
g \cdot V_q = \nu_q(G \cdot q) \cap T_q(g \cdot \tilde{\Sigma}) = \nu_q(G \cdot q) \cap T_q(\tilde{\Sigma}) = V_q.
\]

Thus \( \nu_q(G \cdot q) \subseteq V_q \subseteq T_q \tilde{\Sigma} \) holds for every \( q \in \tilde{\Sigma} \). \( \square \)

**Proposition 8.3.** Let \( (G, M) \) be an isometric action and let \( \Sigma \) be a fat section with fat Weyl group \( W = W(\Sigma) \). Then the mapping \( i^* : \Omega^*_\text{hor}(M)^G \to \Omega^*_\text{hor}(\Sigma)^W \), which is obtained by restriction to \( \Sigma \), is injective.

**Proof.** The mapping \( i^* \) is well defined, since \( \Sigma \) is an embedded submanifold and due to Corollary 3.3. Suppose now that \( i^* \omega = 0 \) for some \( p \)-form \( \omega \in \Omega^*_\text{hor}(M)^G \). Let \( q \in \Sigma \cap M_{\text{reg}}^\text{hor} \) be an arbitrary \( G \)-regular point in \( \Sigma \). By property [C] of a fat section, we have a (not necessarily direct) decomposition of \( T_q M = T_q \Sigma + T_q(G \cdot q) \). Let \( X_1, \ldots, X_p \) be arbitrary vectors in \( T_q M \). According to the above decomposition we can write \( X_i = Y_i + Z_i \), where \( Y_i \in T_q \Sigma \) and \( Z_i \in T_q(G \cdot q) \) for all \( i = 1, \ldots, p \). Now \( \omega_q(X_1, \ldots, X_p) \) decomposes into a sum where each summand contains either \( Y_i \) or \( Z_i \) for all \( i = 1, \ldots, p \). If a summand contains at least one \( Z_i \), then it vanishes, since \( \omega \) is horizontal. Otherwise, the summand is \( \omega_q(Y_1, \ldots, Y_p) \) and vanishes because \( i^* \omega = 0 \). All in all we thus have that \( \omega_q = 0 \). Since \( \omega \) is \( G \)-invariant, this holds along the whole orbit through \( q \). Now \( q \in M_{\text{reg}}^\text{hor} \) was arbitrary, so \( \omega \) vanishes on the \( G \)-regular set of \( M \), and since the regular set is dense in \( M \), we finally conclude that \( \omega = 0 \) on all of \( M \). \( \square \)

One would expect that \( i^* \) should also be surjective in general. However, we can show this only under strong assumptions:

**Theorem 8.4.** Let \( (G, M) \) be an isometric action and let \( \Sigma \subseteq M \) be a minimal section. Put \( W = W(\Sigma) \). Suppose that the slice representation \((G_q, \nu_q(G \cdot q))\) is polar for every \( q \in \Sigma \) and that \( V_q = \nu_q(G \cdot q) \cap T_q \Sigma \) is a 0-section. Then \( \Omega^*_\text{hor}(M)^G \simeq \Omega^*_\text{hor}(\Sigma)^W \). In particular, \( C^\infty(M)^G \simeq C^\infty(\Sigma)^W \) and the isomorphism in both cases is given by the map \( i^* \) from proposition 8.3.

**Proof.** All that is left to show is the surjectivity of \( i^* \). The proof is basically the same as Michor’s in [Mic96 4.2] (see also [Mag08 Section 2.7]). Sketch of proof: Given a form \( \tilde{\omega} \in \Omega^*_\text{hor}(\Sigma)^W \), we have to construct a form \( \omega \in \Omega^*_\text{hor}(M)^G \) with \( i^* (\omega) = \tilde{\omega} \). In a first step, we locally construct \( \omega \) using that the slice representation is polar in every point and in combination with [Mic96 Corollary 3.8 and Lemma 4.1]. The corollary states that basic forms correspond to Weyl-invariant forms for polar representations, and the lemma states that basic forms on a slice can be extended to basic forms on the corresponding tube. Finally, the various local forms are glued up via a \( G \)-invariant partition of unity. \( \square \)
Remark 8.5.

(i) The assumption of polarity of the slice representation in the theorem above enters in the step where Corollary 3.8 is used. For this to work we need that for a polar representation \((G,V)\) with \(G\) compact and section \(\Sigma\) with Weyl group \(W\), the restriction \(p \mapsto p|_{\Sigma}\) induces an isomorphism \(R[V]^G \simeq R[\Sigma]^W\).

(ii) For examples where the assumptions of Theorem 8.4 hold see Theorem 10.4.

(iii) The cohomology of the complex \(\Omega^*_{\text{hor}}(M^G)\) is basic cohomology \(H^*_G(M^G) \simeq H^*_{W-\text{basic}}(\Sigma)\). 

Koszul observed in [Kos53] that for \(M\) compact, basic cohomology is isomorphic to the singular cohomology of \(G\backslash M\). Hence, using \(G\backslash M \approx W\backslash \Sigma\) from Theorem 3.1, we obtain the isomorphism of basic cohomology under the weaker assumption of \(M\) being compact.

9. Copolarity of Singular Riemannian Foliations

Since pre-sections are purely geometrical objects and since minimal sections can be expressed as connected components of the intersections of certain pre-sections (see Proposition 2.8 (iv)), there is a meaningful way to define these notions for singular Riemannian foliations. This also leads to the notion of copolarity for the latter. A reference for the following notions is [Mo88, Chapter 6]. A transnormal system \(\mathcal{F}\) on a Riemannian manifold \(M\) is a partition of \(M\) into complete connected immersed submanifolds of \(M\) such that every geodesic perpendicular to one leaf is perpendicular to all other leaves it meets. A singular Riemannian foliation (SRF) is a transnormal system such that the module \(\Xi_{\mathcal{F}}\) of all vector fields, which are tangent to all leaves in \(\mathcal{F}\), spans for every \(p \in M\) the tangent space \(T_p F\) of the leaf \(F \in \mathcal{F}\) through \(p\). A leaf \(F\) is called regular if it has maximal dimension, otherwise it is called singular.

The partition of a \(G\)-manifold \(M\) into the \(G\)-orbits is a transnormal system. Since the tangent space of every orbit is spanned by the \(G\)-Killing fields, this partition is also an SRF. Note that principal and exceptional orbits are both considered as regular leaves for the singular foliation.

Pre-sections for an SRF with locally closed leaves can be defined as for \(G\)-manifolds:

**Definition 9.1.** Let \(M\) be a Riemannian manifold and let \(\mathcal{F}\) be singular Riemannian foliation with locally closed leaves on \(M\). A submanifold \(\Sigma \subseteq M\) is called a pre-section for \(\mathcal{F}\) if the following three conditions are satisfied:

(i) \(\Sigma\) is complete, connected, embedded and totally geodesic in \(M\),

(ii) \(\Sigma\) intersects every leaf of \(\mathcal{F}\),

(iii) for every regular leaf \(F \in \mathcal{F}\) and all points \(p \in \Sigma \cap F\) we have \(\nu_p(F) \subseteq T_p \Sigma\).

If \(p \in M\) is a point which lies on a regular leaf, then a pre-section of least dimension which contains \(p\) is called a minimal section through \(p\).

The properties of a singular Riemannian foliation together with the assumption that the leaves are locally closed in \(M\) yield the following generalization of Lemma 2.5.

**Lemma 9.2.** If \(F \in \mathcal{F}\) is an arbitrary leaf, then for every \(q \in F\) the set \(\exp_q(\nu_q(F))\) intersects any other leaf of \(\mathcal{F}\).

Let \(p\) be a point in some regular leaf. Using the above lemma, it is easy to see that, if \(\Sigma_1\) and \(\Sigma_2\) are two pre-sections through \(p\), then the connected component of \(\Sigma_1 \cap \Sigma_2\) which contains \(p\) is again a pre-section. Hence, through every regular point \(p\) passes

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I thank Peter W. Michor for this information.
a unique minimal section. From here on it is seems quite natural to assume that all results on minimal sections of isometric group actions should carry over in one way or another to the case of minimal sections of SFRs with locally closed leaves.

However, a noteworthy point is that a corresponding definition of canonical fat sections (Definition 2.10) or cores ([GS00]) makes no sense for general singular Riemannian foliations with locally closed leaves. Hence, the minimal sections we defined above serve as a generalization of canonical fat sections.

10. Copolarity of Actions induced by Polar Actions on Symmetric Spaces

In this section, our aim is to compute the copolarity of actions on compact Lie groups which are associated to certain polar actions on symmetric spaces of compact type.

We first recall some notions for symmetric spaces in order to fix our notation (for the details we refer to Helgason’s monograph [He101]). A symmetric pair $(G, K)$ consists of a Lie group $G$ and a closed subgroup $K$ such that an involutive automorphism $\sigma : G \to G$ exists with $\text{Fix}(\sigma)^\circ \subseteq K \subseteq \text{Fix}(\sigma)$. If in addition $\text{Ad}_G(K)$ is compact, then the pair is called Riemannian. The involution $\sigma$ induces an involution of the Lie algebra $\mathfrak{g}$ of $G$ (also denoted by $\sigma$). This yields the so called Cartan-decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the $(+1)$- and $\mathfrak{p}$ the $(-1)$-eigenspace of $\sigma$. Note that $\mathfrak{k}$ is at the same time the Lie algebra of $K$. If $\pi : G \to G/K$ denotes the canonical projection, then $T_eK G/K$ is identified with $\mathfrak{p}$ via $d\pi(e)$.

It is well known that the complete connected totally geodesic submanifolds $\Sigma$ of $G/K$ correspond bijectively to the Lie triple systems $\mathfrak{m}$ of $\mathfrak{p}$. Furthermore, $\mathfrak{s} := [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$ and its corresponding Lie subgroup $S$ of $G$ together with $L := S_{eK} = S \cap K$ form a Riemannian symmetric pair. We have $\Sigma = \pi(S) \simeq S/L$, and $S$ is the smallest subgroup of $G$ that acts transitively on $\Sigma$.

Although natural, we could not find a reference for the following statement.

**Lemma 10.1.** Let $(G, K)$ be a Riemannian symmetric pair with $G$ compact. Suppose that $\Sigma \subseteq G/K$ is a complete, connected and totally geodesic submanifold. Then $\Sigma$ is embedded in $G/K$ if and only if $S$ is closed in $G$.

**Proof.** If $S$ is closed in $G$, then $S$ acts isometrically on $G/K$. Therefore, its orbit $S \cdot eK = \Sigma$ is an embedded submanifold of $G/K$.

Conversely, $\mathfrak{s} := [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is a compact Lie algebra because $\mathfrak{g}$ is. Let $\mathfrak{s} = \mathfrak{z}(\mathfrak{s}) \oplus [\mathfrak{s}, \mathfrak{s}]$ denote the decomposition of $\mathfrak{s}$ into its center $\mathfrak{z}(\mathfrak{s})$ and its semisimple part $[\mathfrak{s}, \mathfrak{s}]$. It follows that $\exp([\mathfrak{s}, \mathfrak{s}])$ is closed in $G$ ([Mos50], p. 615) and hence compact. The same holds for $\exp([\mathfrak{m}, \mathfrak{m}]) = (\exp([\mathfrak{s}, \mathfrak{s}]) \cap \text{Fix}(\sigma))^\circ$. Since $\Sigma$ is embedded in $G/K$, its image under $\phi : G/K \to G$, $gK \mapsto g\sigma(g)^{-1}$ yields the compact submanifold $\exp(\mathfrak{m})$ of $G$. Note that $\exp(\mathfrak{m})$ is closed under forming rational powers of elements. Applying $\sigma$ to an element of $\exp(\mathfrak{m})$ has the same effect as forming its inverse. Clearly, $\exp(\mathfrak{m})$ projects onto $\Sigma$ under $\pi$.

We next claim that every element $s \in S$ can be written as a product $s = xy$ where $x \in \exp(\mathfrak{m})$ and $y \in \exp([\mathfrak{m}, \mathfrak{m}])$. In fact, let $s \in S$ be arbitrary and let $s_t$ be a path from $e$ to $s$. Let then $x_t$ be a path in $\exp(\mathfrak{m})$ which starts in $e$ and satisfies

\[ x_t^2 = s_t\sigma(s_t)^{-1} = \phi \circ \pi(s_t) \in \exp(\mathfrak{m}) \]
for all \( t \). We claim that \( y_t := x_t^{-1}s_t \) is a path in \( S \), which is fixed by \( \sigma \). In fact,
\[
\sigma(y_t) = \sigma(x_t^{-1})\sigma(s_t) = x_t\sigma(s_t)x_t^{-1}s_t \\
= x_t(x_t^{-1}s_t) = x_t^{-1}s_t = y_t.
\]
This shows \( y \in \exp([m,m]) \). It follows that \( S = \exp(m) \exp([m,m]) \) is closed in \( G \). \( \square \)

Now let \( G \) be a compact Lie group equipped with a bi-invariant metric. Viewed as a symmetric space, \( G \) can be identified with \((G \times G)/\Delta(G)\), where \( \Delta(G) = \{(g,g) \mid g \in G\} \).
So \( g \in G \) is identified with the coset \([g,e] = \{(gh,h) \mid h \in G\}\). Let \( N \subseteq G \) be a totally geodesic submanifold of \( G \). Then \( n := T_eN \) is a Lie triple system of \( g = L(G) \). As before, a transitive group of isometries of \( N \) can be realized as a subgroup of \( G \times G \): Let \( \tilde{n} := \{(X,-X) \mid X \in n\} \subset g \times g \). Obviously, \( \tilde{n} \) is a Lie triple system, hence we may consider the Lie subalgebra \( s := [\tilde{n},\tilde{n}] \oplus \tilde{n} = \Delta([n,n]) \oplus \tilde{n} = \langle ([X,Y] + Z, [X,Y] - Z) \mid X,Y,Z \in n \rangle \).

**Lemma 10.2.** Let \( S \subseteq G \times G \) be the connected Lie subgroup of \( G \times G \) with \( L(S) = s \).
Then \( S \) is a group of isometries of \( N \) and we have for all \((g,h) \in S : g \cdot N \cdot h^{-1} = N \) and therefore
\[
T_{gh^{-1}}N = g \cdot T_eN \cdot h^{-1} = g \cdot n \cdot h^{-1}.
\]
In particular, \((\exp(X),\exp(-X)) \in S \) for all \( X \in n \), and hence
\[
T_{\exp(2X)}N = \exp(X) \cdot n \cdot \exp(X).
\]

Let now \((G,K)\) be a Riemannian symmetric pair with \( G \) compact. The reason for all the preliminary work is the following: Whenever \( H \) is a subgroup of \( G \), the action \( \psi \) of \( H \) on \( G/K \) by left translation lifts to an action \( \varphi \) of \( H \times K \) on \( G \) in the following way: \( (h,k) \cdot g := hkg^{-1} \). If \( \operatorname{pr}_H : H \times K \rightarrow H \) denotes the projection onto the first factor, then the situation fits into the following commutative diagram:

\[
\begin{array}{ccc}
(H \times K) \times G & \xrightarrow{\varphi} & G \\
\downarrow \operatorname{pr}_H \times \pi & & \downarrow \pi \\
H \times G/K \xrightarrow{\psi} G/K.
\end{array}
\]

The lift \( \varphi \) has certain distinctive features:

**Proposition 10.3.**

(i) \( \pi \) maps \( \varphi \)-orbits onto \( \psi \)-orbits: \( \pi(HgK) = H \cdot (gK) \). The orbit spaces \( (H \times K) \backslash G \) and \( H \backslash G/K \) are canonically homeomorphic via \( HgK \leftrightarrow H \cdot (gK) \).

(ii) For the isotropy subgroups of both actions we have
\[
(H \times K)_g = \{(h,g^{-1}h) \mid h \in H \cap gKg^{-1}\} \quad \text{and} \quad H_{gK} = H \cap gKg^{-1}.
\]

Therefore, both groups are isomorphic via \( \operatorname{pr}_H : (H \times K)_g \rightarrow H_{gK} \), \( (h,k) \mapsto h \).

(iii) The actions \( \psi \) and \( \varphi \) have the same cohomogeneity. More precisely, the slice of \( \varphi \) through \( g \in G \) is given by \( \nu_g(HgK) = g \cdot (\operatorname{Ad}_{g^{-1}}(k^\perp) \cap \kappa^\perp) \). The \( \varphi \)-orbits contain the fibres of \( \pi \) and since they are mapped onto the orbits of \( \psi \), the slice through \( g \cdot p \) is given by \( \nu_{gK}(H \cdot (gK)) = d\pi(g)(\nu_g(HgK)) \). Furthermore, the slice representation \( ((H \times K)_g, \nu_g(HgK)) \) of \( \varphi \) is equivariantly isomorphic to the slice representation \( (H_{gK}, \nu_{gK}(H \cdot (gK))) \) of \( \psi \).

For the details we refer to \([GT02]\).
Theorem 10.4. Let \((G, K)\) be a Riemannian symmetric pair with compact \(G\). Let \(H\) be a closed subgroup of \(G\). If \((H, G/K)\) is polar and \(\Sigma\) is a section through \(eK\) with \(m := T_{eK}\Sigma\), then
\[
\text{copol}(H \times K, G) = \dim([m, m]).
\]
A minimal section through \(e\) is given by the connected Lie subgroup \(S\) corresponding to the Lie subalgebra \(s := [m, m] \oplus m\).

Proof. We first show that \(S\) contains a minimal section. In a second step we show that each minimal section contains \(S\). Without loss of generality we may assume that \(e\) is regular with respect to the \((H \times K)\)-action.

Clearly, \(S\) is totally geodesic and complete as it is a Lie subgroup of \(G\). Since \(\Sigma\) is embedded in \(G/K\), Lemma [10.1] shows that \(S\) is embedded in \(G\). Furthermore, since \(S\) maps under the projection \(\pi : G \to G/K\) onto \(\Sigma\), it intersects every orbit. Now suppose that \(g \in S\) is regular with respect to the action \(\varphi\). Then \(\pi(g) = gK\) is regular with respect to \(\psi\) and the normal space \(\nu_g(HgK)\) to the orbit \(HgK\) in \(g\) is given by
\[
(\mathfrak{h}^\perp \cdot g) \cap (g \cdot \mathfrak{p}) = g \cdot (\text{Ad}_{g^{-1}}(\mathfrak{h}^\perp) \cap \mathfrak{p}).
\]
However, since the \(H\)-action on \(G/K\) is polar, we know that \(\text{Ad}_{g^{-1}}(\mathfrak{h}^\perp) \cap \mathfrak{p} = \mathfrak{m}\) (see [Cor04 p. 195]). Since \(S\) is a Lie subalgebra of \(G\), its tangent space in \(g\) is given by left translation of \(s\) with \(g\), i.e. \(T_gS = g \cdot s\). Combining this with the above, we obtain:
\[
\nu_g(HgK) = g \cdot (\text{Ad}_{g^{-1}}(\mathfrak{h}^\perp) \cap \mathfrak{p}) = g \cdot \mathfrak{m} \subseteq g \cdot s = T_gS.
\]
We have therefore established that any minimal section is contained in \(S\).

Now assume that \(N \subseteq S\) is a minimal section through \(e\) and write \(n := T_{eN}\). In particular we have the inclusion \(\nu_g(HgK) = g \cdot \mathfrak{m} \subseteq T_gN\) for all regular \(g \in N\) and therefore \(\mathfrak{m} \subseteq n\). Since the set of regular points of the \(H \times K\)-action on \(G\) is open and dense in \(G\) and \(e\) is assumed to be a regular point, there is a small \(\varepsilon > 0\), such that for all \(t \in (-\varepsilon, \varepsilon)\) and \(X \in \mathfrak{m}\) with unit length, the value of \(g^2 = \exp(t \cdot X)\) is regular. Applying the tangent space formula from lemma [10.2] it follows that \(g^2 \cdot \mathfrak{m} \subseteq T_{g^2}N = g \cdot n \cdot g\), or in other words:
\[
\text{Ad}_g(\mathfrak{m}) = \text{Ad}_{\exp(t/2,X)}(\mathfrak{m}) \subseteq n.
\]
Since \(\text{Ad}_{\exp(X)} = e^{ad_X}\), it follows for all \(Y \in \mathfrak{m}\) and \(t \in \mathbb{R}\):
\[
\text{Ad}_{\exp(t/2,X)}(Y) = e^{t/2 \cdot ad_X}(Y) \in n.
\]
Differentiating in \(t = 0\) yields that \(ad_X Y = [X, Y] \in n\). By linearity of the Lie bracket we may thus conclude that \([\mathfrak{m}, \mathfrak{m}] \subseteq n\) and therefore \(s \subseteq n\) which in turn implies \(S \subseteq N\). \(\square\)

Remark 10.5. We have proved along the lines that even if the action of \(H\) on \(G/K\) is not polar, the following inequality still holds:
\[
\text{copol}(H \times K, G) \leq \text{copol}(H, G/K) + \dim([m, m]).
\]
Here \(m\) is the tangent space of a minimal section through \(eK\). To be more precise, if \(\Sigma \subseteq G/K\) denotes a minimal section with respect to the action \(\psi\) and \(m = T_{eK}\Sigma\), then \(S := \exp([m, m] \oplus m)\) contains a minimal section of the action \(\varphi\).

Corollary 10.6. With the assumptions and notation as in Theorem [10.4]:

(i) Assuming that \(\psi\) is polar, then \(\varphi\) is polar if and only if it is hyperpolar.

(ii) If \(H = \{e\}\), then \(\text{copol}(K, G) = \dim([p, p])\) (the action is by right translation), and the copolarity is trivial.
We can also describe the relation between the generalized Weyl group of $\Sigma$ and the fat Weyl group of $S$:

**Proposition 10.7.** In addition to the assumptions of Theorem [10.4] let $e$ be regular.

(i) $N_{H \times K}(S) = \{(h, k) \in H \times K \mid hk^{-1} \in S\}$ and $Z_{H \times K}(S) = \Delta(H \cap K)$.

(ii) $\text{pr}_H(N_{H \times K}(S)) = N_H(\Sigma)$ and $\text{pr}_H(Z_{H \times K}(S)) = Z_H(\Sigma)$.

(iii) The following diagram is commutative

$$
\begin{array}{ccc}
N(S) & \longrightarrow & N(\Sigma) \\
\downarrow p_1 & & \downarrow p_2 \\
W(S) & \longrightarrow & W(\Sigma),
\end{array}
$$

where $\text{pr}_W$ denotes the homomorphism induced by $p_2 \circ \text{pr}_H$. Hence, $W(S)$ is mapped canonically onto $W(\Sigma)$ and has at least as many connected components as the latter.

(iv) $N(\Sigma) \simeq N(S)/\{e\} \times (K \cap S)$ and $W(\Sigma) \simeq W(S)/p_1(\{e\} \times (K \cap S))$.

**Proof.** The description of the normalizer in (i) follows from property $(D)$ of a fat section. The centralizer of a minimal section coincides with the isotropy group of any $(H \times K)$-regular point of $S$. Since $e$ is a regular point, $Z_{H \times K}(S) = \Delta(H \cap K)$ follows from Proposition [10.3] (ii).

Let $(h, k) \in N_{H \times K}(S)$ be arbitrary. If we apply $\pi$ to the equation $hSk^{-1} = S$ we obtain $h \in N_H(\Sigma)$. Conversely, assume that $h \in N_H(\Sigma)$. Since $e$ is a regular point, $hK \in \Sigma$. Since $\pi(S) = \Sigma$, we can find an element $s \in S$ with $hK = sK$. It follows that $k := s^{-1}h \in K$, which we rewrite as $hk^{-1} = s \in S$. Since $e$ is a regular point for the action $\varphi$, by assumption, we conclude that $(h, k) \in N_{H \times K}(S)$ by property $(D)$ of a fat section. This completes the proof that $\text{pr}_H$ maps $N_{H \times K}(S)$ onto $N_H(\Sigma)$.

The statement in (iii) is easily verified. The same is true in the case of (iv). In fact, the kernel of $\text{pr}_H$ is given by

$$\ker(\text{pr}_H) = \{(h, k) \in N(S) \mid h = 1, k \in S\} = \{e\} \times (K \cap S).$$

**Remark 10.8.** With the assumptions of Theorem [10.4] Proposition [10.3] (iii) shows that the assumptions made in Theorem [8.4] are satisfied. I.e. the basic forms on $S$ and $G$ are naturally isomorphic to each other. In particular, the smooth $(H \times K)$-invariant functions on $G$ correspond to the smooth $N(S)$-invariant functions on $S$.

The polar, non-hyperpolar actions on compact rank one symmetric spaces yield interesting examples where Theorem [10.4] is applicable. These actions have been classified in [PT99]. As an example, consider the action of the torus $T^2 \subset \text{SU}(3)$ on $\text{P}_2(C) = \text{SU}(3)/S(\text{U}(1) \times \text{U}(2))$. It is polar with Weyl group $\mathbf{Z}_2$, but not hyperpolar as it has $\text{P}_2(\mathbb{R})$ as a section. Its lift to the action of $T^2 \times S(\text{U}(1) \times \text{U}(2))$ on $\text{SU}(3)$ has nontrivial copolarity 1 with minimal section given by $\text{SO}(3)$. The fat Weyl group is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{O}(2)$ in this case.

11. **An Infinite Dimensional Isometric Action**

In [GOT04] it is shown that one may easily construct actions with prescribed fat sections in the following way: Take a polar action $(G, M)$ with section $\Sigma_1$ and any
action \((G_2, M_2)\) whose principal orbit has dimension \(k\). Then
\[
(G, M) := (G_1 \times G_2, M_1 \times M_2)
\]
has \(\Sigma := \Sigma_1 \times M_2\) as a \(k\)-section. If \((G_1, M_1)\) is an infinite dimensional isometric Fredholm action\(^3\) and \(G_2\) and \(M_2\) are finite dimensional, then it follows that \(\Sigma_1 \times M_2\) has finite dimension. Hence, \(\text{copol}(G, M)\) is also finite in this case. Besides these constructed examples, one might ask if there exist isometric Fredholm actions of infinite dimensional Lie groups on infinite dimensional manifolds with finite dimensional minimal sections. A natural candidate is the action by gauge transformation, which we describe in the following (see [PT88, TT95]). Let \(G\) be a compact Lie group with a bi-invariant Riemannian metric and let \(H\) and \(K\) be closed subgroups of \(G\). The action by\textbf{ gauge transformation} is defined as:

\[
* : \mathcal{P}(G, H \times K) \times V \to V, \quad (g, u) \mapsto \text{Ad}_g(u) - g'g^{-1} = gug^{-1} - g'g^{-1}.
\]

Here \(\mathcal{P}(G, H \times K)\) is the Hilbert-Lie group of \(H^1\) paths \(g : I \to G, \ (g(0), g(1)) \in H \times K\), and let \(V = H^0(I; \hat{g}) = L^2(I; \hat{g})\) be the Hilbert space of \(L^2\) integrable paths \(u : I \to \hat{g}\) in \(\hat{g} = L(G)\), equipped with the inner product

\[
\langle u, v \rangle_0 := \int_0^1 \langle u(t), v(t) \rangle_1 \, dt \quad \text{with} \quad \langle \cdot, \cdot \rangle_1 \text{ Ad}_G\text{-invariant}.
\]

We briefly summarize some facts concerning the gauge transformation without proofs:

(i) \(*\) is a smooth isometric Fredholm action by affine transformations.

(ii) The action of \(\mathcal{P}(G, e \times G)\) on \(V\) is simply transitive. In other words, the orbit map \(\alpha : \mathcal{P}(G, e \times G) \to V, \ g \mapsto g * \hat{0} = -g'g^{-1}\) is a diffeomorphism.

(iii) The map \(\phi : V \to G, \ u \mapsto \alpha^{-1}(u)(1)\), obtained by mapping \(u\) into \(\mathcal{P}(G, e \times G)\) and then evaluating in \(t = 1\), is a surjective Riemannian submersion.

(iv) The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{P}(G, H \times K) \times V & \xrightarrow{*} & V \\
\pi \times \phi \downarrow & & \downarrow \phi \\
(H \times K) \times G & \xrightarrow{\phi} & G,
\end{array}
\]

where \(\pi\) denotes the map \(\pi : \mathcal{P}(G, H \times K) \to H \times K, \ g \mapsto (g(0), g(1))\). Thus, \(\phi\) is equivariant with respect to \(\pi\). Furthermore, the isotropy subgroups of both actions are isomorphic via \(\pi\).

(v) For \(u \in V\) we have that \(\mathcal{P}(G, H \times K) * u = \phi^{-1}((H \times K) \cdot \phi(u))\).

(vi) The fibres of \(\phi\) coincide with the orbits of \(\Omega_e(G) = \mathcal{P}(G, e \times e)\). That is

\[
\phi^{-1}(\phi(u)) = \Omega_e(G) * u, \quad \text{for all } u \in V.
\]

In particular, we have \(\phi^{-1}(\exp(Y)) = \Omega_e(G) * \hat{Y}\) for all \(Y \in \hat{g}\).

(vii) For \(u \in V\) let \(\tilde{M} := \mathcal{P}(G, H \times K) * u\). The tangent space on \(\tilde{M}\) in \(u\) is:

\[
T_u(\tilde{M}) = \{[\xi, u] - \xi' \mid \xi \in H^1(I; \hat{g}), \xi(0) \in \hat{b}, \xi(1) \in \hat{t}\}.
\]

(viii) If \(h \in \mathcal{P}(G, H \times K)\) with \(u = h * \hat{0}\) and \(x = \phi(u)\), then:

\[
\nu_u(\tilde{M}) = \{hx^{-1}h^{-1} \mid b \in \nu_x(HxK)\} = \text{Ad}_{hx}(\text{Ad}_x^{-1}(\hat{b}^\perp) \cap \hat{t}^\perp).
\]

Hence, \(\nu_u(\tilde{M})\) is the set of constant paths in \(\text{Ad}_x^{-1}(\hat{b}^\perp) \cap \hat{t}^\perp = \nu_x(HxK)\).

Lemma 2.3 also holds for the action by gauge transformation:

\(^3\)A (proper) isometric action \((G, M)\) is called \textbf{Fredholm} if \(\text{cohom}(G, M) < \infty\).
Lemma 11.1. $\nu_0(\mathcal{P}(G, H \times K) \ast \hat{0})$ intersects all orbits of $(\mathcal{P}(G, H \times K), V)$.

Proof. Let $\mathcal{P}(G, H \times K) \ast u$ be an arbitrary orbit and put $x := \phi(u)$. Now consider $X \in \text{Ad}_{\nu_0}^{-1}(\nu_0) \cap \mathfrak{t}^\perp = \nu_x(HxK)$ such that $(H \times K) \cdot x = (H \times K) \cdot \exp(X)$. Such an $X$ exists, because $\exp(\nu_x(HxK))$ intersect every $(H \times K)$-orbit on $G$. Using (v) above we obtain

$$\mathcal{P}(G, H \times K) \ast u = \phi^{-1}((H \times K) \cdot \phi(u)) = \phi^{-1}((H \times K) \cdot \exp(X)).$$

It follows, using (vi) above, that $\phi^{-1}(\exp(X)) = \Omega_e(G) \ast \hat{X} \subseteq \mathcal{P}(G, H \times K) \ast u$. ☐

In the following, we assume that $(G, K)$ is a Riemannian symmetric pair with compact $G$ and that $H \subseteq G$ is a closed subgroup. As usual, we identify $T_eK G/K$ with $\mathfrak{p}$ from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Our aim is to show that if $H$ acts polarly on $G/K$, then the action by gauge transformation is either polar (and hence hypopolar), or it has infinite dimensional minimal sections and hence infinite copolarity. This gives a partial negative answer to the question we asked at the beginning of this section.

If $I \subseteq \mathfrak{g}$ is an arbitrary subset of $\mathfrak{g}$, we denote by $\hat{I} \subseteq V$ the set of constant paths in $V$ with value in $I$. It is clear that if $I$ is a subspace (or subalgebra) of $\mathfrak{g}$, then $\hat{I}$ is a subspace (resp. subalgebra) of $V$ which is canonically isomorphic to $I$. In particular, $\mathfrak{g}$ is embedded into $V$ via $\hat{\mathfrak{g}}$.

Lemma 11.2. Suppose that $H$ acts polarly on $G/K$ and let $\mathfrak{m} \subseteq \mathfrak{p}$ be the tangent space of a section through $eK$. If $eK$ is $H$-regular, then every fat section $S \subseteq V$ of the $\mathcal{P}(G, H \times K)$-action on $V$ through $\hat{0}$ contains the linear subspace

$$\text{span}\{t \mapsto e^{(1-t)\text{ad}X}(Y) \mid X \in \mathfrak{m} \text{ regular}, Y \in \mathfrak{m}\}.$$ 

Here we call an element $X \in \mathfrak{g}$ regular, if $\exp(X) \in G$ is regular with respect to the $(H \times K)$-action on $G$.

Proof. Since $S \subseteq V$ is supposed to be a fat section through $\hat{0}$, it is complete, connected, and totally geodesic in $V$. Hence, $S$ has to be a linear subspace of $V$.

From $\hat{\mathfrak{m}} = \nu_0(\mathcal{P}(G, H \times K) \ast \hat{0}) \subseteq T_0S = S$ and property (C) of a fat section we may conclude that for all regular $\hat{X} \in \hat{\mathfrak{m}}$ we have $\nu_{\hat{X}}(\mathcal{P}(G, H \times K) \ast \hat{X}) \subseteq S$. Let $h \in \mathcal{P}(G, e \times G)$ be the path defined by $h(t) := \exp(-t \cdot X)$. Then $X = h \ast \hat{0}$ and $\phi(\hat{X}) = h(1)^{-1} = \exp(X)$ is a regular element for the $(H \times K)$-action on $G$. Since the action of $H$ on $G/K$ is polar, it follows that $\text{Ad}_{\exp(-X)}(h^\perp) \cap \mathfrak{p} = \mathfrak{m}$. From (viii) above we thus conclude that

$$\nu_{\hat{X}}(\mathcal{P}(G, H \times K) \ast \hat{X}) = \text{Ad}_{h \exp(X)}(\text{Ad}_{\exp(-X)}(h^\perp) \cap \mathfrak{p}) = \text{Ad}_{h \exp(X)}(\mathfrak{m}).$$

Since $h \exp(X) = \exp(-t \cdot X) \exp(X) = \exp((1-t)X)$ and $\text{Ad}_{\exp(X)} = e^{\text{ad}X}$, we obtain $\text{Ad}_{h \exp(X)}(\mathfrak{m}) = \{t \mapsto e^{(1-t)\text{ad}X}(Y) \mid Y \in \mathfrak{m}\}$. This fact together with $S$ being linear completes the proof. ☐

Theorem 11.3. Let $(G, K)$ be a Riemannian symmetric pair with compact $G$ and let $H \subseteq G$ be a closed subgroup. Suppose that the action of $H$ on $G/K$ is polar, then the following are equivalent:

(i) $\text{copol}(\mathcal{P}(G, H \times K), V) < \infty$.
(ii) $\text{copol}(\mathcal{P}(G, H \times K), V) = 0$.

(iii) The action of $\mathcal{P}(G, H \times K)$ on $V$ is hypopolar.

(iv) The action of $H \times K$ on $G$ is hypopolar.

(v) The action of $H$ on $G/K$ is hypopolar.
Proof. The equivalence of (iii), (iv) and (v) is well known. Furthermore, since sections in $V$ are automatically flat and copol = 0 implies that an action is polar, (iii) is equivalent to (ii). Certainly, (ii) implies (i).

Let $\Sigma$ be a section of $(H, G/K)$ through $eK$ and assume that $eK$ is $H$-regular. Then $e$ is $H\times K$-regular and $0$ is regular with respect to the action by gauge transformation. Put $m := T_{eK}\Sigma$. We now show that if copol$(\mathcal{P}(G, H\times K), V) \neq 0$ then the copolarity must already be infinite. Let $X, Y \in m$ be elements with

$$(\text{ad}_X)^2(Y) = -\delta Y \neq 0 \text{ and } ||Y|| = 1.$$ 

Such elements exist, since $m$ is a Lie triple system and $m$ is not Abelian. Otherwise, $\hat{m}$ would be a 0-section, which contradicts our assumption. Recalling that $0$ is regular, there is a ball of regular elements around $0 \in m$. In fact, this is clear since $e \in G$ is regular and the set of regular points is open in $G$. We may thus further assume that $\varepsilon \cdot X$ is regular for all $\varepsilon \in [0, 1]$. By lemma 11.2, every minimal section $S$ of $(\mathcal{P}(G, H\times K), V)$ contains the infinite family $M := \{t \mapsto e^{(1-t)p_n \cdot \text{ad}X}(Y) \mid n \in \mathbb{N}\} \subseteq V$,

where $p_n$ denotes the $n$-th odd prime number. We claim that $M$ is linearly independent.

Since every equivalence class in $L^2(I; \mathfrak{g})$ has at most one continuous representative, it suffices to show that the family $M$ is linearly independent as a subset of $C(I; \mathfrak{g})$. Furthermore, all members of $\mathcal{M}$ are analytic maps which can be extended analytically to $\mathbb{R}$ and so we only need to show that they are linearly independent when viewed as functions on $\mathbb{R}$.

Now assume there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$t \mapsto \sum_{k=1}^{n} \lambda_k e^{(1-t)p_k \cdot \text{ad}X}(Y) = 0.$$ 

For any $s := (1-t) \in \mathbb{R}$, we then have:

$$0 = \left\langle \sum_{k=1}^{n} \lambda_k e^{s p_k \cdot \text{ad}X}(Y), Y \right\rangle = \sum_{k=1}^{n} \lambda_k \langle e^{s p_k \cdot \text{ad}X}(Y), Y \rangle$$

$$= \sum_{k=1}^{n} \lambda_k \sum_{l=0}^{\infty} \frac{s^l}{p_k^l \cdot l!} \langle (\text{ad}_X)^l(Y), Y \rangle$$

$$= \sum_{k=1}^{n} \lambda_k \sum_{l=0}^{\infty} \frac{s^l}{p_k^l \cdot (2l)!}\delta^{2l}(-1)^l$$

$$= \sum_{k=1}^{n} \lambda_k \cos \left( \frac{s \delta}{p_k} \right),$$

where we made use of the continuity of the Ad-invariant inner product $\langle \cdot, \cdot \rangle$ and the fact that

$$\langle (\text{ad}_X)^l(Y), Y \rangle = \begin{cases} 0 & \text{for } l \text{ odd} \\ (-1)^{l/2} \delta^l & \text{for } l \text{ even} \end{cases}.$$ 

By choosing

$$s_k := \pi \cdot \left( \prod_{l=1}^{n} p_l \right)/(2\delta p_k)$$

for $k = 1, \ldots, n$, we obtain that $\lambda_k = 0$. We have thus shown, that for any $n \in \mathbb{N}$ the first $n$ members of $\mathcal{M}$ are linearly independent, which completes our proof. □
12. Appendix - Invariant Metrics

We are interested in left-$G$-invariant metrics on a homogeneous space $G/H$ which are also right-invariant under a certain group $W$. This concept generalizes that of a $G$-invariant metric on $G/H$ and is used in Section 7. First recall that any triple $(H \leq N \leq G)$, where $G$ is a Lie group, $H$ and $N$ are closed subgroups of $G$ and $H$ is normal in $N$, gives rise to a $W$-principal bundle:

$$W \hookrightarrow G/H \twoheadrightarrow G/N,$$

where $W = N/H$. In this situation $G$ acts on $G/H$ from the left and $W$ acts properly and freely on $G/H$ from the right by $(gH, nH) \mapsto gnH$. We are interested in the case that these actions are isometric, so we are lead to consider Riemannian metrics on $G/H$ which are left-$G$- and right-$W$-invariant.

**Definition 12.1.** A Riemannian metric on $G/H$ which is both left-$G$- and right-$W$-invariant is called $(G-W)$-invariant.

**Proposition 12.2.**

(i) The $(G-W)$-invariant Riemannian metrics on $G/H$ are in $1-1$ correspondence with the $\text{Ad}_G(N)$-invariant scalar products on $\mathfrak{g}/\mathfrak{h}$.

(ii) If $N$ is connected, then a scalar product $\langle \cdot | \cdot \rangle$ on $\mathfrak{g}/\mathfrak{h}$ is $\text{Ad}_G(N)$-invariant if and only if $\text{ad}_n$ is skew-symmetric with respect to $\langle \cdot | \cdot \rangle$.

(iii) If $\mathfrak{g}/\mathfrak{h}$ admits a decomposition $\mathfrak{g}/\mathfrak{h} = \mathfrak{n}/\mathfrak{h} \oplus \mathfrak{p}$ with $\text{Ad}_G(N)(\mathfrak{p}) \subseteq \mathfrak{p}$, then the $\text{Ad}_G(N)$-invariant scalar products on $\mathfrak{g}/\mathfrak{h}$, which satisfy $(\mathfrak{n}/\mathfrak{h}) \perp \mathfrak{p}$, are in $1-1$ correspondence with pairs $(\langle \cdot | \cdot \rangle_{\mathfrak{n}/\mathfrak{h}}, \langle \cdot | \cdot \rangle_{\mathfrak{p}})$ of $\text{Ad}_W$-invariant scalar products on $\mathfrak{n}/\mathfrak{h}$, resp. $\text{Ad}_G(N)$-invariant scalar products on $\mathfrak{p}$.

Such a pair exists if and only if $W$ is covered by a product of a compact Lie group with a vector group and if the image of $N$ under $n \mapsto \text{Ad}_n|_{\mathfrak{p}}$ in $\text{GL}(\mathfrak{p})$ is relatively compact.

Conversely, if $\langle \cdot | \cdot \rangle$ on $\mathfrak{g}/\mathfrak{h}$ is $\text{Ad}_G(N)$-invariant, then $\langle \cdot | \cdot \rangle|_{\mathfrak{n}/\mathfrak{h}}$ is $\text{Ad}_W$-invariant. If $\mathfrak{p} := (\mathfrak{n}/\mathfrak{h})^\perp$, then $\text{Ad}_G(N)(\mathfrak{p}) \subseteq \mathfrak{p}$ and $\langle \cdot | \cdot \rangle|_{\mathfrak{p}}$ is $\text{Ad}_G(N)$-invariant.

(iv) If $N$ is compact, then $G/H$ admits a $(G-W)$-invariant Riemannian metric.

**Proof.** (i): Let $h$ be a Riemannian metric on $G/H$ and put $\langle \cdot | \cdot \rangle := h_{eH}$. Then it is well known that $h$ is left-$G$-invariant if and only if $\langle \cdot | \cdot \rangle$ is $\text{Ad}_G(H)$-invariant. If additionally $h$ is right-$W$-invariant, then $r_n^*h = h$ for all $n \in N$. Hence, we have for all $X, Y \in T_gH G/H$:

$$h_{gnH}(X \cdot n, Y \cdot n) = h_{gH}(X, Y).$$

Using the $G$-invariance we obtain

$$h_{eH}(n^{-1}g^{-1} \cdot X \cdot n, n^{-1}g^{-1} \cdot X \cdot n) = h_{eH}(g^{-1} \cdot X, g^{-1} \cdot Y).$$

Under the natural identification $T_{eH}G/H \simeq \mathfrak{g}/\mathfrak{h}$ this is equivalent to

$$\langle \text{Ad}_n(X) | \text{Ad}_n(Y) \rangle = \langle X | Y \rangle,$$

for all $X, Y \in \mathfrak{g}/\mathfrak{h}$.

Conversely, if we are given an $\text{Ad}_G(N)$-invariant scalar product on $\mathfrak{g}/\mathfrak{h}$ then, in particular, it is $\text{Ad}_G(H)$-invariant. It therefore gives rise to a left-$G$-invariant Riemannian metric on $G/H$. Furthermore, it is easy to see, that it is right-$W$-invariant.

(ii): This is a standard consideration.

(iii): If $\langle \cdot | \cdot \rangle$ is an $\text{Ad}_G(N)$-invariant scalar product on $\mathfrak{g}/\mathfrak{h}$ satisfying $(\mathfrak{n}/\mathfrak{h}) \perp \mathfrak{p}$, then its restriction to $\mathfrak{n}/\mathfrak{h}$ resp. $\mathfrak{p}$ clearly yields the stated pair of invariant scalar products.
Conversely, we may patch such a pair of invariant scalar products together to form an \( \text{Ad}_G(N) \)-invariant scalar product \( \langle \cdot | \cdot \rangle \) on \( \mathfrak{g}/\mathfrak{h} \) by defining:

\[
\langle X + Y | Z + W \rangle := \langle X | Z \rangle_{n/\mathfrak{h}} + \langle Y | W \rangle_{p}, \quad \text{for all} \ X, Z \in n/\mathfrak{h}, \ Y, W \in p.
\]

The \( \text{Ad}_W \)-invariance of \( \langle \cdot | \cdot \rangle_{n/\mathfrak{h}} \) is equivalent to the existence of a bi-invariant Riemannian metric on \( W \). Using [CE75, Proposition 3.34] yields that this is the case if and only if \( W \) is covered by the product of a compact Lie group and a vector group. Also, if \( \langle \cdot | \cdot \rangle_p \) is \( \text{Ad}_G(N) \)-invariant, then the image of \( N \) under \( f : N \rightarrow \text{GL}(p), \ n \mapsto \text{Ad}_n|_p \) is contained in the compact group \( \text{O}(p) \) and therefore relatively compact. Conversely, if \( K := f(N) \subseteq \text{GL}(p) \) is compact, then we may define by an averaging process a \( K \)-invariant scalar product on \( p \), which in turn is \( \text{Ad}_G(N) \)-invariant.

(iv) follows from (iii) and the fact that a representation of a compact Lie group is completely reducible. \( \square \)

The following Proposition shows that the concept of a \((G-W)\)-invariant metric on \( G/H \) is actually the same as that of a left-\( G \)-invariant metric on \( G/N \).

**Proposition 12.3.** The \((G-W)\)-invariant metrics on \( G/H \) correspond to the left-\( G \)-invariant metrics on \( G/N \).

**Proof.** If we are given a \((G-W)\)-invariant metric on \( G/H \), then the submersed metric on \( G/N \) under the canonical \( G \)-equivariant mapping \( gH \mapsto gN \) is left-\( G \)-invariant. Conversely, if we start with a left-\( G \)-invariant metric on \( G/N \), then \( G \) admits a left invariant metric which is right-\( N \)-invariant. This induces an \( \text{Ad}_G(N) \)-invariant scalar product on \( \mathfrak{g} \) and since \( \mathfrak{h} \) is \( \text{Ad}_G(N) \)-invariant, the induced scalar product on \( \mathfrak{g}/\mathfrak{h} \) is \( \text{Ad}_G(N) \)-invariant. Using Proposition 12.2 (i), this yields a \((G-W)\)-invariant Riemannian metric on \( G/H \). \( \square \)

The next result is basically [Bes87, Theorem 9.80].

**Corollary 12.4.** If \( G/H \) carries a \((G-W)\)-invariant Riemannian metric, then the principal fibre bundle \( G/H \rightarrow G/N \) is a Riemannian submersion, where \( G/N \) is endowed with the quotient metric. Its fibres are totally geodesic. In particular \( W \), viewed as a subset of \( G/H \), is totally geodesic in \( G/H \). Furthermore, the map

\[
(n/\mathfrak{h})^\perp \rightarrow \mathfrak{g}/\mathfrak{n}, \ X + \mathfrak{h} \mapsto X + \mathfrak{n}
\]

is a linear isometry.

**Proof.** By left-\( G \)-invariance, the fibres of the principal bundle \( G/H \rightarrow G/N \) are all isometric to the fibre \( W \) over \( eN \). Now \( W \) is the image of \( N \) under the canonical projection \( G \rightarrow G/H \), which is a Riemannian submersion if \( G \) is endowed with a left-invariant metric that is right-\( N \)-invariant. Such a metric exists due to, [CE75, Proposition 3.16]. By the following lemma, \( N \) is a totally geodesic submanifold of \( G \). Hence, its image \( W \) under the Riemannian submersion \( G \rightarrow G/H \) is totally geodesic in \( G/H \). \( \square \)

**Lemma 12.5.** Let \( G \) be a Lie group and \( H \subseteq G \) a closed subgroup. If \( G \) carries a left-invariant metric which is right-\( H \)-invariant, then the induced metric on \( H \) is bi-invariant and \( H \) is a totally geodesic submanifold of \( G \).

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