Approximation to the Second Order Approximation of Einstein Field Equations with a Cosmological Constant in a Flat Background

C. J. de Matos*

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Abstract

Einstein field equations with a cosmological constant are approximated to the second order in the perturbation to a flat background metric. The final result is a set of Einstein-Maxwell-Proca equations for gravity in the weak field regime. This approximation procedure implements the breaking of gauge symmetry in general relativity. A brief discussion of the physical consequences (Pioneer anomalous deceleration) is proposed in the framework of the gauge theory of gravity.

1 Introduction

Einstein introduced by hand a Cosmological Constant (CC) $\Lambda$ in his field equations to compensate for the universe expansion, and predict a stationary universe. He made reference to the CC as being his biggest mistake, when Hubble discovered that our universe was in expansion. Modern observational cosmology reveals a flat universe in accelerated expansion, which might be explained through a positive CC different from zero. What Einstein considered to be his worst idea, might well become its smartest thought!

The only homogeneous and isotropic vacuum solution of the Einstein field equations (EFE) with CC is (anti) de Sitter space, not Minkowski space \cite{1}. However, since contemporary observational cosmology tend to measure a flat

*ESA-HQ, European Space Agency, 8-10 rue Mario Nikis, 75015 Paris, France, e-mail: Clovis.de.Matos@esa.int
universe in accelerated expansion [2], although Minkowski metric is not a solution of the vacuum EFE with CC, in the present work we impose the ansatz of a flat metric for the background instead of the (anti) de Sitter background metric, and we apply an approximation procedure, closely related to the standard linearization method [3] [4] [5], leading to ”massive” GravitoElectroMagnetism (GEM). The result is a set of Einstein-Maxwell-Proca type equations for GEM. The ansatz of a flat background metric to approximate EFE with a CC instead of the (anti) de Sitter background corresponds to the implementation of a spontaneous breaking of gauge invariance in general relativity [6]. As we will see in the conclusion, the Gauge Theory of Gravity (GTG) is particularly well suited to discuss the physical consequences of this symmetry breaking [13].

2 Second Order Approximation of Einstein Field Equations with a Cosmological Constant

In the following we propose an approximation of the second order approximation of Einstein Field Equations with cosmological constant. The following assumptions underly the approximation procedure:

1. The mass densities are normal (no dwarf stars), and correspond to local physical systems located in the Earth laboratory or in the solar system.

2. All motions are much slower than the speed of light, so that special relativity can be neglected. (Often special relativistic effects will hide general relativistic effects), \( v << c \).

3. The kinetic or potential energy of all the bodies being considered is much smaller than their mass energy, \( T_{\mu\nu} << \rho c^2 \).

4. The gravitational fields are always weak enough so that superposition is valid, \( \phi << c^2 \).

5. The distances between objects is not so large that we have to take retardation into account. (This can be ignored when we have a stationary problem where the fields have already been prescribed and are not changing with time.)
6. We consider a running cosmological constant which depends on the local density of mass of the physical system being considered

\[ \Lambda = \frac{4\pi G}{c^2} \rho \]

7. We consider that the proposed approximation is only valid in the following range of distances:

\[ \sqrt{|\frac{h_{\alpha\beta}}{\Lambda}|} << r << \sqrt{|\frac{1}{\Lambda}|} \]

defining the characteristic length scale for the physical system being considered. This restriction allows to neglect second order terms in the perturbation to Minkowsky’s metric $|h_{\alpha\beta}|^2$, but we do not neglect terms involving simultaneously the perturbation to the metric and the cosmological constant, $|\Lambda h_{\alpha\beta}|$.

8. The approximated second order EFE are solved, by approximation, using the solutions for the perturbation, $|h_{\alpha\beta}|$, to Minkowsky’s metric obtained in the linear approximation, in which only first order terms in $|h_{\alpha\beta}|$ are considered.

We start with Einstein Field Equations (EFE) with a cosmological constant.

\[ R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta} \tag{1} \]

The weak field approximation assumes small perturbations, $|h_{\alpha\beta}| << 1$, of Minkowsky’s metric $\eta_{\alpha\beta}(++,−−−)$ (Landau-Lifschitz ”timelike convention”). This approximation is deliberately kept also for the case of having a cosmological constant different from zero.

\[ g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta} \tag{2} \]

Doing Equ.(2) into Equ.(1) with the derivation indices obeying the same rule as the covariant indices, $f^\mu = \eta^{\mu\nu}f_{,\nu}$, we obtain:

\[ -\frac{1}{2}\left(h_{\alpha\beta,\mu} + \eta_{\alpha\beta}\tilde{h}_{\mu\nu} - \tilde{h}_{\alpha\mu,\beta} - \tilde{h}_{\beta\mu,\alpha}\right) - \Lambda \left(\eta_{\alpha\beta} + h_{\alpha\beta}\right) = \frac{8\pi G}{c^4}T_{\alpha\beta} \tag{3} \]

As usual in order to simplify the linearization procedure we have introduced the intermediate tensor:

\[ \tilde{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \tag{4} \]
where $h = h^\mu_\mu = \eta^{\mu\nu}h_{\mu\nu} = h_{00} - h_{11} - h_{22} - h_{33}$ is the trace of the perturbation tensor. Imposing the harmonic gauge condition
\[ \bar{h}_{\mu\nu} = 0 \] (5)
Equ.(5) reduces to
\[ \bar{h}_{\alpha\beta,\mu} + 2\Lambda(\eta_{\alpha\beta} + h_{\alpha\beta}) = -\frac{16\pi G}{c^4}T_{\alpha\beta} \] (6)
Equ.(6) can be written in function of the Dalemberian operator, $\triangle$. If $f$ is a given function, then
\[ \triangle f = f_{,\mu} = \eta^{\mu\nu}f_{,\mu\nu} = \left(\frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^i)^2}\right)f \] (7)
Where $x_0 = ct$. Therefore Equ.(6) becomes
\[ \triangle \bar{h}_{\alpha\beta} + 2\Lambda(\eta_{\alpha\beta} + h_{\alpha\beta}) = -\frac{16\pi G}{c^4}T_{\alpha\beta} \] (8)
This is the approximated second order form of EFE with a cosmological constant, assuming a flat background for the metric. We will now solve these equations, by approximation, using the solutions of the perturbation to Minkowsky’s metric we obtain in the case of linear EFE without CC.

To split spacetime into gravitoelectric and gravitomagnetic parts we consider respectively the energy-momentum tensor components:
\[ T_{00} = \rho c^2, \] (9)
and
\[ T_{0i} = -\rho cv_i. \] (10)
The solution of EFE without cosmological constant (with a flat background), $\triangle \bar{h}_{00} = -\frac{16\pi G}{c^4}T_{00}$, for the energy momentum tensor component given by Equ.(9) is:
\[ h_{00} = \frac{2\phi}{c^2} \] (11)
Where $\phi$ is the gravitational scalar potential. The solution of EFE without cosmological constant, $\triangle \bar{h}_{0i} = -\frac{16\pi G}{c^4}T_{0i}$, for the energy momentum tensor component of Equ.(10) is:
\[ h_{0i} = -\frac{4A_{gi}}{c} \] (12)
Where $A_{gi}$ are the three components of the gravitomagnetic vector potential.
Writing the Einstein tensor in function of the intermediate tensor $\bar{h}_{\alpha\beta}$, and using the gauge condition of Equ.(5), Einstein tensor reduces to the tensor $G_{\alpha\beta\mu}$.

$$G_{\alpha\beta\mu} = \frac{1}{4} \left( \bar{h}_{\alpha\beta,\mu} - \bar{h}_{\alpha\mu,\beta} \right)$$  \hspace{1cm} (13)

Using Equ.(13) one can re-write Equ(8) under the following form:

$$\frac{\partial G_{\alpha\beta\mu}}{\partial x^\mu} + \frac{1}{2} \Lambda \left( \eta_{\alpha\beta} + h_{\alpha\beta} \right) = - \frac{4\pi G}{c^4} T_{\alpha\beta}$$  \hspace{1cm} (14)

We can also use the tensor Equ.(13) to express the gravitational field:

$$g_i = -c^2 G_{00i}.$$  \hspace{1cm} (15)

Which can also be written in terms of the gravitational scalar potential $\phi$ and of the gravitomagnetic vector potential $\vec{A}_g$.

$$\vec{g} = - \nabla \phi - \frac{\partial \vec{A}_g}{\partial t}$$  \hspace{1cm} (16)

Similarly we formulate the gravitomagnetic field as follows:

$$cG_{0ij} = - (A_{gi,j} - A_{gj,i})$$  \hspace{1cm} (17)

which obviously shows that the gravitomagnetic field $\vec{B}_g$ is generated by a vectorial potential vector $\vec{A}_g$.

$$\vec{B}_g = \nabla \times \vec{A}_g$$  \hspace{1cm} (18)

We have now everything we need to derive Proca-type equations for gravity. For the energy momentum tensor component of $T_{00}$ of Equ.(9), Equ.(14) reduces to:

$$\frac{\partial G_{00\mu}}{\partial x^\mu} + \frac{1}{2} \Lambda \left( \eta_{00} + h_{00} \right) = - \frac{4\pi G \rho}{c^2}$$  \hspace{1cm} (19)

Substituting the solution Equ(11) of EFE, into Equ.(19) we can approximate the divergent part of the gravitational field:

$$\nabla \cdot \vec{g} = -4\pi G \rho - \Lambda \phi - \frac{1}{2} c^2 \Lambda$$  \hspace{1cm} (20)

For the energy momentum tensor component $T_{0i}$ of Equ.(10), Equ.(14) reduces to:

$$\frac{\partial G_{0i\mu}}{\partial x^\mu} + \frac{1}{2} \Lambda \left( \eta_{0i} + h_{0i} \right) = \frac{4\pi G}{c^4} \rho v_i$$  \hspace{1cm} (21)
Substituting the solution Equ.(12) of EFE, into Equ.(21) we can approximate the rotational part of the gravitomagnetic field:

$$\nabla \times \vec{B}_g = -\frac{4\pi G}{c^2} \vec{j}_m + \frac{1}{c^2} \frac{\partial \vec{g}}{\partial t} - 2\Lambda \vec{A}_g$$

(22)

Where $\vec{j}_m = \rho \vec{v}$ is the mass current. The tensor $G_{\alpha\beta\mu}$, Equ.(13), has the following property:

$$G^{\alpha\beta\mu,\lambda} + G^{\alpha\lambda\beta,\mu} + G^{\alpha\mu\lambda,\beta} = 0.$$  

(23)

which are equivalent to the two other set of Maxwell like equations for gravity,

$$\nabla \cdot \vec{B}_g = 0$$

(24)

and

$$\nabla \times \vec{g} = -\frac{\partial \vec{B}_g}{\partial t}$$

(25)

Note also that Equ.(24) is a direct and trivial corollary of the definition of the gravitomagnetic field Equ.(18). As we see, Eqs. (24) and (25) are not affected by the cosmological constant.

In summary Equs. (20), (22), (24) and (25) form a set of Einstein-Maxwell-Proca equations for gravity in the weak field regime:

$$\nabla \cdot \vec{g} = -4\pi G \rho - \Lambda \phi - \frac{1}{2} c^2 \Lambda$$

(26)

$$\nabla \cdot \vec{B}_g = 0$$

(27)

$$\nabla \times \vec{g} = -\frac{\partial \vec{B}_g}{\partial t}$$

(28)

$$\nabla \times \vec{B}_g = -\frac{4\pi G}{c^2} \vec{j}_m + \frac{1}{c^2} \frac{\partial \vec{g}}{\partial t} - 2\Lambda \vec{A}_g$$

(29)

These equations are closely analogous to the ones derived by Argyris to investigate the consequences of massive gravitons in general relativity [10].

Considering the case of an universe empty of material gravitational sources, $\rho = 0$ and $\phi = 0$, Equ. (26) reduces to:

$$\nabla \vec{g} = -\frac{1}{2} c^2 \Lambda$$

(30)

Integrating this equation over a volume bounded by a sphere of radius $R$ we obtain a fundamental "Machian-type" accelerated contraction of that volume, which only depends on its radius and on the value of the CC, $\Lambda$.

$$g = \frac{1}{6} c^2 \Lambda R$$

(31)
This acceleration is directed inwards on the boundary of the sphere. Doing the radius of the observable universe expressed in function of the CC

\[ R = R_U = \sqrt{\frac{3}{\Lambda}} \] and using \( \Lambda = 1.29 \times 10^{-52}[m^{-2}] \) derived from the value of \( 71[Km.S^{-1}.Mpc^{-1}] \) for the Hubble constant \( H = c\sqrt{\frac{\Lambda}{3}} \) assumed by Nottale [7], into Eq. (31) we obtain a fundamental cosmic deceleration.

\[ g = \frac{1}{2\sqrt{3}}c^2\Lambda^{1/2} = 2.9 \times 10^{-10}m.s^{-2} \quad (32) \]

So the linearized acceleration is a deceleration, it is interesting to note that this value is in good agreement with the Pioneer anomalous deceleration, whose current measured value is \( a_{Pio} = (8.5 \pm 1.3) \times 10^{-10}m.s^{-2} \)[8]. If this is a correct contribution to the Pioneer anomaly, this would imply that this deceleration would be independent of the origin at which the physical observer, measuring the Pioneer deceleration, is located. The deceleration should always be directed towards the observer, because the universe has no preferred center, and the deceleration only depends on the universe radius. Similarly the acceleration of expansion of the universe is independent of the origin at which the astronomer measuring it is located. The Pioneer anomalous deceleration would be a kind of local ”Machian-type back reaction” to the accelerated cosmological expansion.

It might seem odd that the linearized analysis predicts an inward acceleration \( \vec{g} \) when \( \Lambda > 0 \), as a positive value of \( \Lambda \) is known to cause the universe to accelerate outwards. The acceleration in the second case is not \( \vec{g} \) as expressed in Eq. (31), but the acceleration \( \ddot{R}/R \) of the scale factor in the Friedmann Robertson Walker (FRW) metric:

\[ ds^2 = c^2d\tau^2 - R^2(\tau)(dx^2 + dy^2 + dz^2) \]

According to the Friedmann equation (the Einstein equation for \( R_{00} \), with non-negligible isotropic pressure \( p \)),

\[ \ddot{R}/R = \frac{1}{3}c^2\Lambda - \frac{4}{3}\pi G(\rho + 3pc^{-2}) \]

and so a positive \( \Lambda \) increases the value of \( \ddot{R}/R \). The two accelerations \( g \) and \( \ddot{R}/R \) are not directly comparable, as the FRW metric is not written in harmonic coordinates.
3 Spontaneous Breaking of Gauge Invariance in General Relativity

General Relativity is founded on the principle of equivalence, which rests on the equality between the inertial and the gravitational mass of any physical system, and formulates that at every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravity. In other words, The inertial frames, that is, the "freely falling coordinate systems", are indeed determined by the local gravitational field, which arises from all the matter in the universe, far and near. However, once in an inertial frame, the laws of motion are completely unaffected by the presence of nearby masses, either gravitationally or in any other way.

Following Steven Weinberg, the Principle of General Covariance (PGC) is an alternative version of the principle of equivalence, which is very appropriate to investigate the field equations for electromagnetism and gravitation. It states that a physical equation holds in a general gravitational field, if two conditions are met:

1. The equation holds in the absence of gravitation; that is, it agrees with the laws of special relativity when the metric tensor $g_{\alpha\beta}$ equals the Minkowsky tensor $\eta_{\alpha\beta}$ and when the affine connection $\Gamma^\gamma_\beta_\gamma$ vanishes.

2. The equation is generally covariant; that is, it preserves its form under a general coordinate transformation $x \rightarrow x'$.

It should be stressed that general covariance by itself is empty of physical content. The significance of the principle of general covariance lies in its statement about the effects of gravitation, that a physical equation by virtue of its general covariance will be true in a gravitational field if it is true in the absence of gravitation. The PGC is not an invariance principle, like the principle of Galilean or special relativity, but is instead a statement about the effects of gravitation, and about nothing else. In particular general covariance does not imply Lorentz invariance. Any physical principle such as the PGC, which takes the form of an invariance principle but whose content is actually limited to a restriction on the interaction of one particular field, is called a dynamic symmetry. Local gauge invariance, which governs the electromagnetic interaction is another important dynamical symmetry. We can actually say that the Principle of General Covariance in general relativity is the analogous of the Principle of Gauge Invariance in electrodynamics.
Spontaneous breaking of gauge invariance in general relativity would therefore correspond to a breaking of the PGC.

In contrast to the Einstein-Maxwell type theory of linear gravitation [4], in the Einstein-Maxwell-Proca type theory, Equations (26)-(29), the potentials $\phi$ and $\vec{A}_g$ are directly measurable quantities so that gauge invariance is not possible, and the Lorentz gauge condition

$$\nabla \cdot \vec{A}_g + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$$

is required in order to conserve energy [10]. Since $\Lambda \neq 0$ is not consistent with gauge invariance, Proca generalization of gravitoelectromagnetism could be aesthetically defective in the eyes of many theoretical physicists. However, the only certain statements about the value of $\Lambda$ that can be made must be based on experiment, and cosmological observations.

It should be noted that Einstein-Maxwell-Proca equations, Equations (26)-(29), form a good phenomenological base to investigate the GEM properties of superconductors [6]. They offer also an interesting perspective on Mach’s principle as formulated in the framework of relational mechanics [11].

### 4 Conclusion

The set of Einstein-Maxwell-Proca equations derived from EFE with CC assuming a flat background, can be understood as the result of spontaneous breaking of gauge invariance in general relativity, which is physically revealed through the violation of the principle of general covariance. This is elegantly expressed in the framework of gauge theory of gravity. The Gauge Theory of Gravity (GTG) is formulated in the language of Geometric Calculus (GC), initially discovered by Clifford and further developed by Hestenes [13], Doran, Lasenby [12]. GTG is fully compliant with all classical tests of the standard formulation of General Relativity (GR). However it is not based on the principle of equivalence. Gauge symmetry plays a more fundamental role in the theory than the spacetime metric. The following two gauge principles for gravitation form the base of the theory:

1. The Displacement Gauge Principle (DGP), which states that the equations of physics must be invariant under arbitrary smooth remappings of events onto spacetime.

2. The Rotation Gauge Principle (RGP), which formulates that the equations of physics must be covariant under local Lorentz rotations.
DGP is a vast generalization of "translational invariance" in special relativity, so it has a comparable physical interpretation. Accordingly, the DGP can be interpreted as asserting that "spacetime is globally homogeneous". In other words, with respect to the equations of physics all spacetime points are equivalent. DGP throws new light on Einstein’s Principle of General Covariance (PGC). The problem with the PGC, as we saw above, is that it is not a true symmetry principle [9]. For a transformation group to be a physical symmetry group, there must be a well defined "geometric object" that the group leaves invariant. Thus the "displacement group" of the DGP is a symmetry group, because it leaves the flat spacetime background invariant. Following Noether’s theorem, homogeneity of spacetime is associated with the conservation of 4-linear momentum.

In special relativity, Lorentz transformations are passive rotations expressing equivalence of physics with respect to different inertial reference frames. In RGP, however, covariance under active rotations expresses local physical equivalence of different directions in spacetime. In other words, RGP asserts that spacetime is locally isotropic. Thus "passive equivalence" is an equivalence of observers, while "active equivalence" is an equivalence of states. Noether’s theorem establishes the conservation of the 4-angular momentum. Therefore a violation of gauge symmetry in general relativity is associated with a violation of energy-momentum conservation, which naturally takes place in a non-homogeneous and anisotropic universe! This is indeed what tends to be confirmed by the latest cosmological observations [14], which would also implicitly confirm the validity of Einstein-Maxwell-Proca equations.

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