Weyl semimetallic phase in an interacting lattice system

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Abstract

By using Wilsonian Renormalization Group (RG) methods we rigorously establish the existence of a Weyl semimetallic phase in an interacting three dimensional fermionic lattice system, by showing that the zero temperature Schwinger functions are asymptotically close to the ones of massless Dirac fermions. This is done via an expansion which is convergent in a region of parameters, which includes the quantum critical point discriminating between the semimetallic and the insulating phase.

1 Model and results

1.1 Introduction

There is an increasing interest in materials whose Fermi surface is not extended, as it is usually the case, but it consists of disconnected points. In several of such materials the charge carriers admit at low energies an effective description in terms of Dirac massless particles. This opens the exiting possibility that high energy phenomena have a counterpart at low energies in real materials. Graphene is probably the most known example of such systems; it was pointed out in \[1,2\] that fermions on the honeycomb lattice behave as massless Dirac fermions in 2+1 dimensions and indeed the experimental realization of graphene \[3\], a monolayer sheet of graphite, offered a spectacular physical realization of such a system.

As a next step, it is natural to look for materials with electronic bands touching in couples of points and with an emerging description in terms of 3 + 1 massless Dirac (or Weyl) particles, the same appearing in the standard model to describe quarks and leptons. Such systems have been called Weyl semimetals, and their existence has been predicted in several systems \[4,5,6,7,8\]; this has generated an intense experimental research, see for
instance [9], [10] (and the review [11]). It is of course important to understand the effect of the interactions, which are usually analyzed in effective relativistic models neglecting lattice effects [12], [13], [14], [15]. Perturbative considerations suggest that short range interactions can generate instabilities only at strong coupling, but in order to exclude non-perturbative effects one has to prove the convergence of the expansions. It is also known that in such class of systems the effective relativistic description misses important features; for instance in the case of graphene the universality of the optical conductivity emerges only taking into account the lattice [16].

In this paper we consider a three dimensional interacting fermionic lattice model with a point-like Fermi surface [8] (see also [7]), in presence of an Hubbard interaction. We construct the zero temperature correlations for couplings not too large, proving the persistence of the Weyl semimetallic phase in presence of interactions. In the non interacting case the semimetallic phase, in which the elementary excitations are well described in terms of Weyl fermions, is present in an extended region of the parameters; outside such a region an insulating behavior is present and a quantum critical point discriminates between the two phases. In the semimetallic phase the Fermi surface consists of two points; close to the critical point the two points are very close and the Fermi velocity is arbitrarily small (and vanishes at the boundary). The effective relativistic description coincides with a system of massless Dirac fermions in $3 + 1$ dimensions with an ultraviolet cut-off like the Gross-Neveu model or QED with massive photon: in such models the interaction is irrelevant and the convergence of the renormalized perturbative expansion has been established, see [17] and [18]. However the convergence radius in such models is vanishing with the particle velocity; therefore such results give essentially no information for lattice Weyl semimetals close to the boundary of the semimetallic phase where the Fermi velocity is very small. One may suspect that even an extremely weak interaction could produce some quantum instability close to the boundary of the semimetallic phase, where the parameters correspond to a strong coupling regime in the effective description. This is however excluded by the present paper: we can prove the persistence of the Weyl semimetallic phase in presence of interaction in all the semimetallic region, even arbitrarily close to the boundary where the Fermi velocity vanishes. This result is achieved writing the correlations in terms of a renormalized expansion with a radius of convergence which is independent from the Fermi velocity, and in order to get this one needs to exploit the non linear corrections to the dispersion relation due to the lattice. The proof is indeed based on two different multiscale analysis in two regions of the energy momentum space; in the smaller energy region the effective relativistic description is valid while in the larger energy region the quadratic corrections due to the lattice are dominating. In both regimes the interaction is irrelevant but the scaling dimensions are different; after the integration of the first regime one gets gain factors which compensate
exactly the velocities at the denominator produced in the second regime, so that uniformity is achieved. Such a phenomenon is completely absent in Graphene, in which the Fermi velocity is essentially constant. Another important phenomenon present here (and absent either in Graphene and in the effective relativistic description) is the movement of Weyl points due to the interaction.

The analysis is based on the Renormalization Group (RG) method of Wilson and its approach to the effective action [19], in the form implemented in [20] and [21] in the context of perturbative renormalization. It was realized in the eighties that such methods can be indeed used to get a full non-perturbative control of certain fermionic Quantum Field Theories in \( d = 1 + 1 \), [22], [23] using Gram bounds and Brydges formula for truncated expectation [24]. A very natural development was then to apply such techniques to condensed matter models, [25], [26] with the final aim at obtaining a full non perturbative control of the ground state properties of interacting systems. However, while the interaction in the models considered in [22] or [23] is marginally irrelevant or dimensionally irrelevant, this is not the case in interacting non relativistic fermionic models in one dimension, or in dimensions greater than one with extended Fermi surface. This is due to the fact that the ground state properties of the interacting system are generically different with respect to the non interacting case. In one dimension it was finally obtained a full control of the zero temperature properties of interacting fermions, in the spinless [27], [28] or repulsive spinning case [29]; this was achieved by combining RG methods with Ward Identities based on the emerging chiral symmetries. In systems in higher dimensions with extended symmetric Fermi surface, rigorous results were obtained, see [30], [31], for temperatures above an exponentially small scale setting the onset of (possible) quantum instabilities. Only in the case of an asymmetric Fermi surface a condition preventing the formation of Cooper pairs) the convergence of the renormalized expansion up to zero temperature for a interacting fermionic system was achieved [32], proving the existence of a Fermi liquid phase. In systems with point-like Fermi surfaces in two or three dimensions the interaction is irrelevant and this allows the proof of the convergence of the renormalized expansion up to zero temperature, as in the case of Graphene [33] or the case discussed in the present paper. In the case of Graphene, the combination of non perturbative bounds with lattice Ward Identities allows to establish remarkable physical conclusions, like the universality of the optical conductivity [16]. Similarly, Ward Identities combined with the results obtained in the present paper can be used to establish a weak form of universality for the optical conductivity in Weyl semimetals, see [34].
1.2 The model

We consider the interacting version of the tight binding model introduced in [8], describing fermions on a three dimensional lattice, with nearest and next to nearest neighbor hopping and with a properly defined magnetic flux density, whose effect is to decorate the hopping with phase factors [35].

We consider two cubic sublattices $\Lambda_A = \Lambda$ and $\Lambda_B = \Lambda + \vec{\delta}$, where $\Lambda_B = \{ n_1 \delta_1 + n_2 \delta_2 + n_3 \delta_3, n_1, n_2, n_3 = 0, 1, ..., L - 1 \}$ with $\delta_1 = (1, 0, 0)$, $\delta_2 = (0, 1, 0)$, $\delta_3 = (0, 0, 1)$ and $\delta_{\pm} = \frac{\delta_1 \pm \delta_2}{2}$. We introduce creation and annihilation fermionic operators for electrons sitting at the sites of the A- and B- sublattices; if $\vec{x} \in \Lambda$

$\hat{a}^\pm_{\vec{x}} = \frac{1}{|\Lambda|} \sum_{\vec{k} \in D_L} e^{\pm i \vec{k} \vec{x}} \hat{a}^\pm_{\vec{k}} \quad \hat{b}^\pm_{\vec{x} + \vec{\delta}} = \frac{1}{|\Lambda|} \sum_{\vec{k} \in D_L} e^{\pm i \vec{k} \vec{\delta}} \hat{b}^\pm_{\vec{k}} $ (1.1)

with $D_L = \{ \vec{k} = \frac{2\pi}{L} \vec{n}, \vec{n} = (n_1, n_2, n_3), n_i = (0, 1, ..., L - 1) \}$ and

$\{ \hat{a}^\epsilon_{\vec{k}}, \hat{a}^{-\epsilon'}_{\vec{k}} \} = |\Lambda| \delta_{\vec{k}, \vec{k}'} \delta_{\epsilon, \epsilon'} \quad \{ \hat{b}^\epsilon_{\vec{k}}, \hat{b}^{-\epsilon'}_{\vec{k}} \} = |\Lambda| \delta_{\vec{k}, \vec{k}'} \delta_{\epsilon, \epsilon'} $ (1.2)

and $\{ \hat{a}^\epsilon_{\vec{k}}, \hat{b}^\epsilon_{\vec{k}} \} = 0$.

The hopping (or non-interacting) Hamiltonian is given by, if $\hat{\psi}^\pm_{\vec{k}} = (\hat{a}^\pm_{\vec{k}}, \hat{b}^\pm_{\vec{k}})$

$$ H_0 = \frac{1}{|\Lambda|} \sum_{\vec{k} \in D_L} (\hat{\psi}^+_{\vec{k}}, E(\vec{k}) \hat{\psi}^-_{\vec{k}}) $$ (1.3)

where, if $k_{\pm} = k_0 \delta_{\pm}$

$$ E(\vec{k}) = t \sin(k_+ \sigma_1 + t \sin(k_-) \sigma_2 + \sigma_3 (\mu + t t' \cos k_3 - \frac{1}{2} t' (\cos k_1 + \cos k_2)) $$ (1.4)

and

$$ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} $$

The hopping parameters $t, t', t''$ are assumed $O(1)$ and positive; in coordinate space $t_{\perp}$ describes the hopping between fermions living in different horizontal layers, $t$ the nearest neighbor hopping in the same layer, $t'$ the next-to-nearest neighbor hopping, while $\mu$ the difference of energy between $a$ and $b$ fermions. The hopping terms are multiplied by suitable phases to take into account a magnetic flux pattern applied to the lattice.

The electrons on the lattice can interact through a short range (or Hubbard) two body interaction, so that the total Hamiltonian is

$$ H = H_0 + V $$ (1.5)
where

\[ V = U \sum_{\vec{x},\vec{y}} v(\vec{x} - \vec{y}) [a_x^+ a_x^- + b_{x+\delta_+}^+ b_{x+\delta_+}^-] [a_y^+ a_y^- + b_{y+\delta_+}^+ b_{y+\delta_+}^-] \] (1.6)

and \(|v(\vec{x})| \leq C e^{-\kappa |\vec{x}|}\) is a short-range interaction \((C, \kappa \text{ positive constants})\).

Defining \(\psi^\pm_x = (a_x^\pm, b_{x+\delta_+}^\pm)\), we consider the operators

\[
\psi^\pm_{x} = e^{-x_0 H} \psi^\pm_{x} e^{-x_0 H}
\] (1.7)

with \(x = (x_0, \vec{x})\) and \(0 < x_0 < \beta\) and \(\beta^{-1}\) is the temperature; on \(x_0\) antiperiodic boundary conditions are imposed. The 2-point Schwinger function is defined as

\[
S_U(x - y) = \text{Tr} \{ e^{-\beta H} \psi^+_x \psi^+_y \} \equiv \langle T \{ \psi^+_x \psi^+_y \} \rangle_{\beta,\Lambda} (1.8)
\]

where \(T\) is the fermionic time ordering operation.

1.3 The non interacting case

The Hamiltonian in the non interacting \(U = 0\) case can be easily written in diagonal form

\[
H_0 = \frac{1}{|\Lambda|} \sum_{\vec{k} \in D_L} [\lambda(\vec{k}) \hat{\alpha}_{\vec{k}}^+ \alpha_{\vec{k}}^- - \lambda(\vec{k}) \hat{\beta}_{\vec{k}}^+ \beta_{\vec{k}}^-] (1.9)
\]

where

\[
\lambda(\vec{k}) = \sqrt{t^2 (\sin^2 k_+ + \sin^2 k_-) + (\mu + t_\perp \cos k_3 - \frac{1}{2} \nu' (\cos k_1 + \cos k_2))^2}
\] (1.10)

where \(\hat{\alpha}_{\vec{k}}^\pm, \hat{\beta}_{\vec{k}}^\pm\) are sitable linear combinations of \(\hat{a}_{\vec{k}}^\pm, \hat{b}_{\vec{k}}^\pm\). If \(-\beta < x_0 - y_0 \leq \beta\), the 2-point Schwinger function is given by

\[
\langle T \{ \alpha_x^- \alpha_y^+ \} \rangle_{\beta,\Lambda} = \frac{1}{|\Lambda|} \sum_{\vec{k} \in D} e^{-i\vec{k}(\vec{x} - \vec{y})} \left[ \chi(x_0 - y_0 > 0) \frac{e^{(x_0 - y_0)\lambda(\vec{k})}}{1 + e^{\beta \lambda(\vec{k})}} - \chi(x_0 - y_0 \leq 0) \frac{e^{-(x_0 - y_0+\beta)\lambda(\vec{k})}}{1 + e^{\beta \lambda(\vec{k})}} \right] (1.11)
\]

\[
\langle T \{ \beta_x^- \beta_y^+ \} \rangle_{\beta,\Lambda} = \frac{1}{|\Lambda|} \sum_{\vec{k} \in D} e^{-i\vec{k}(\vec{x} - \vec{y})} \left[ \chi(x_0 - y_0 > 0) \frac{e^{-(x_0 - y_0)\lambda(\vec{k})}}{1 + e^{-\beta \lambda(\vec{k})}} - \chi(x_0 - y_0 \leq 0) \frac{e^{-(x_0 - y_0+\beta)\lambda(\vec{k})}}{1 + e^{-\beta \lambda(\vec{k})}} \right] (1.12)
\]
and \( \langle T \{ \alpha_x^{+} \beta_y^{-} \} \rangle_{\beta,\Lambda} = \langle T \{ \beta_x^{-} \alpha_y^{+} \} \rangle_{\beta,\Lambda} = 0 \). A priori Eqs. (1.11) and (1.12) are defined only for \(-\beta < x_0 - y_0 \leq \beta\), but we can extend them periodically over the whole real axis; the periodic extension of the propagator is continuous in the time variable for \( x_0 - y_0 \notin \beta \mathbb{Z} \), and it has jump discontinuities at the points \( x_0 - y_0 \in \beta \mathbb{Z} \). Note that at \( x_0 - y_0 = \beta n \), the difference between the right and left limits is equal to \((-1)^n \delta_{\mathbf{x},\mathbf{y}}\), so that the propagator is discontinuous only at \( x - y = \beta \mathbb{Z} \times 0 \). For \( x - y \notin \beta \mathbb{Z} \times 0 \), we can write, defining \( \mathcal{D} = D_\beta \times D_\beta \), \( D_\beta = \{ k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2}), n_0 \in \mathbb{Z} \} \), \( k = (k_0, k) \)

\[
\langle T \{ \alpha_x^{+} \alpha_y^{+} \} \rangle_{\beta,\Lambda} = \frac{1}{\beta |A|} \sum_{k \in D_{\beta,L}} e^{-ik(x-y)} \frac{1}{-ik_0 - \lambda(k)}, \tag{1.13}
\]

\[
\langle T \{ \beta_x^{-} \beta_y^{-} \} \rangle_{\beta,\Lambda} = \frac{1}{\beta |A|} \sum_{k \in D_{\beta,L}} e^{-ik(x-y)} \frac{1}{-ik_0 + \lambda(k)}. \tag{1.14}
\]

If we now re-express \( \alpha_x^{\pm} \) and \( \beta_x^{\pm} \), in terms of \( a_x^{\pm} \) and \( b_{x+d}^{\pm} \), we get for \( x - y \notin \beta \mathbb{Z} \times 0 \)

\[
S_0(x-y) = \frac{1}{|A|\beta} \sum_{k \in D_{\beta,L}} e^{ik(x-y)} A^{-1}(k) \tag{1.15}
\]

where

\[
A(k) = -ik_0 I + t\sigma_1 \sin k_+ + t\sigma_2 \sin k_- + (\mu - t' + t_\perp \cos k_3 + E(k))\sigma_3 \tag{1.16}
\]

with

\[
E(k) = t'(\cos k_+ \cos k_- - 1) \tag{1.17}
\]

The Fermi surface is defined as the singularity of the Fourier transform of the 2-point function \( \tilde{S}_0(k) = A^{-1}(k) \) at zero temperature and \( k_0 = 0 \). Note that the functions \( \sin(k_+) \) and \( \sin(k_-) \) vanish in correspondence of two points \( (k_1, k_2) = (0, 0) \) and \( (k_1, k_2) = (\pi, \pi) \) and we will assume from now on

\[
\mu + t' > 2t_\perp \tag{1.18}
\]

so that the only possible singularities are when \( \mu - t' + t_\perp \cos k_3 = 0 \). Therefore if

\[
\frac{|\mu - t'|}{t_\perp} < 1 \tag{1.19}
\]

than \( \tilde{S}_0(k) \) is singular in correspondence of two points, called Weyl points and denoted by \( \pm \tilde{p}_F \), with

\[
\tilde{p}_F = (0, 0, \cos^{-1}(\frac{t' - \mu}{t_\perp})) \tag{1.20}
\]

Close to such points the 2-point function has the following form, if \( k = k' \pm \tilde{p}_F \) and \( |k'| << |\sin \tilde{p}_F| \)

\[
\tilde{S}_0(k' \pm \tilde{p}_F) \sim \left( \begin{array}{cc}
-i k_0 \pm v_{3,0} k_3' & v_0 (k_+ - i k_-) \\
v_0 (k_+ + i k_-) & -i k_0 - (\pm) v_{3,0} k_3'
\end{array} \right)^{-1} \tag{1.21}
\]
with
\[ v_0 = t, \quad v_{3,0} = t_\perp \sin p_F \] (1.22)

The two \( 2 \times 2 \) matrices \( \hat{S}_0(k' + p_F) \) and \( \hat{S}_0(k' - p_F) \) can be combined in a \( 4 \times 4 \) matrix coinciding with the propagator of a massless Dirac (or Weyl) particle in \( D = 3 + 1 \) dimension, with an anisotropic light velocity. In coordinate space, the 2-point function has a power law decay times an oscillating factor, denoting a metallic behavior (or semimetallic, as the conductivity computed via Kubo formula vanishes at zero frequency) under the conditions (1.1) and (1.19).

On the contrary for \( \frac{|\mu - t'|}{t_\perp} > 1 \) the 2-point function decays exponentially for large distances (\( \hat{S}_0(k) \) is non singular) and the system has an insulating behavior. Close to the boundaries of the semimetallic phase, the Fermi velocity \( t_\perp \sin p_F \) becomes arbitrarily small and the Weyl points are very close; the relativistic behavior (1.21) emerges only in a very small region \( |k'| < t_\perp \sin p_F \) around the Fermi points as the linear dispersion relation \( v_3,0 k_3' \) is dominating over the quadratic correction only in that region. There is a quantum critical point \( \frac{|\mu - t'|}{t_\perp} = 1 \) discriminating the metallic and the insulating region.

We ask now the question if Weyl semimetallic behavior, present under the conditions (1.1) and (1.19), survives to the presence of the interaction.

### 1.4 Grassmann Integral representation

The analysis of the interacting case is done by a rigorous implementation of RG techniques. The starting point is a functional integral representation of the Schwinger functions which is quite suitable for such methods. We want to establish the persistence of Weyl semimetallic behavior with Weyl points given by (1.20). However, even if the semimetallic phase persists in presence of interaction, there is no reason a priori for which the value of \( p_F \) should be the same in the free or interacting case. Therefore it is convenient to proceed in two steps. The first consists in writing \( \mu = \mu - \nu + \nu = \bar{\mu} + \nu \) and in proving that one can choose \( \nu = \nu(\bar{\mu}, \lambda) \) so that there is Weyl semimetallic behavior under the condition \( \frac{|\bar{\mu} - t'|}{t_\perp} < 1 \), and that in such region the Weyl points are given by \( (0,0,\pm p_F) \) with \( \cos p_F = \frac{|\bar{\mu} - t'|}{t_\perp} \); in this way the location of the singularity of the two point function does not move, and this is technically convenient as we construct the interacting function as series starting from the non interacting one. Once that this is (possibly) done the second step consists in solving the inversion problem \( \bar{\mu} + \nu(\bar{\mu}, \lambda) = \mu \), so that one can determine the location of the Weyl points as function of the initial parameters. We will deal here with the first step only, which is the substantial one; the inversion problem can be done via standard methods once the first step is done, see for instance Lemma 2.8 of [29] for a similar problem.
We will introduce a set of Grassmann variables \( \hat{\psi}_k^\pm = (\hat{a}_k^\pm, \hat{b}_k) \), \( k \in \mathcal{D}_{\beta,L} \) by the same symbol as the fermionic fields. We also define a “regularized” propagator \( g_M(x-y) \) (\( 2^M \) is an ultraviolet cut-off) with

\[
g_M(x-y) = \frac{1}{|A|^{1/2}} \sum_{k \in \mathcal{D}_{\beta,L}} e^{ik(x-y)} A^{-1}(k) \bar{\chi}(2^{-M}|k_0|) \quad (1.23)
\]

with \( \bar{\chi}(t) : \mathbb{R}^+ \to \mathbb{R} \) is a smooth compact support function equal to 1 for \( 0 < t < 1 \) and = 0 for \( t > 2 \). Note that, contrary to the function \( S_0(x-y) \), the sum \( \sum_{k \in \mathcal{D}_{\beta,L}} \) is restricted over a finite number of elements. Note also that for \( x - y \neq (0, n\beta), n \in \mathbb{Z} \) than

\[
\lim_{M \to \infty} g_M(x-y) = S_0(x-y) \quad (1.24)
\]

The above equality is however not true for \( x - y = (0, n\beta) \); indeed the r.h.s. of (1.24) is discontinuous while the l.h.s. is equal to \( \frac{1}{2}[S_0(0,0^+) + S_0(0,0^-)] \).

We introduce the generating functional

\[
e^{\mathcal{W}_M(\phi)} = \int P(d\psi)e^{\mathcal{V}(\psi) + (\psi, \phi)} \quad (1.25)
\]

where \( P(d\psi) \) is the fermionic “measure” with propagator \( g_M(x-y) \) and \( \mathcal{V} \) is the interaction given by

\[
\mathcal{V} = (\nu + \nu_C)N + V \quad (1.26)
\]

where, if \( \int dx = \int dx_0 \sum_{\vec{x}} \)

\[
N = \int dx \psi_+^x \sigma_3 \psi^-_x \quad (1.27)
\]

\[
V = U \int dx dy (x-y)(\psi_+^x I \psi^-_x)(\psi_+^y I \psi^-_y)
\]

if \( v(x) = \delta(x_0)v(x) \). Moreover \( (\psi, \phi) = \int dx[\psi_+^x \sigma_0 \phi^-_x + \psi^-_x \sigma_0 \phi_+^x] \) and \( \nu_C = U\dot{\psi}(0)[S_0(0,0^+) - S_0(0,0^-)] \). We define

\[
S_2(x-y) = \lim_{M \to \infty} S_M(x-y) = \lim_{M \to \infty} \frac{\partial^2 \mathcal{W}_M}{\partial \phi_+^x \partial \phi^-_y} \bigg|_0 \quad (1.28)
\]

It is easy to check order by order in perturbation theory that \( \lim_{M \to \infty} S_{M,U}(x-y) \) coincides in the \( M \to \infty \) limit with \( S_U(x-y) \) by (1.5) with \( \mu \) replaced by \( \mu + \nu \). Note indeed that both functions can be expressed in terms of the same Feynman diagrams with propagator respectively \( S_0(x-y) \) and \( g_M(x-y) \). Therefore the equality is trivial except in the graphs containing a tadpole, involving a propagator computed at \( (0,0) \); the presence of the counterterm \( \nu_C \) ensures than the equality, see §2.1 of [29] for more details in a similar case.
One can prove more; if $S_M(x-y)$ given by (1.28) is analytic and bounded in $|U| \leq U_0$ with $U_0$ independent of $\beta, L$ and uniformly convergent as $M \to \infty$, then \( \lim_{M \to \infty} S_M(x-y) = S_U(x-y) \) where $S_U(x-y)$ is given by (1.8); the proof of this fact is rather standard (it is an application of Weierstrass theorem and of properties of analytic functions) and it will be not repeated here (see Lemma 1 of [33] or prop 2.1 of [29] for an explicit proof in similar cases). This ensures that one can study directly the Grassmann integral (1.28) to construct the Schwinger function (1.8).

1.5 Main results

Our main result is the following.

**Theorem 1.1** Let us consider $S_2(x)$ given by (1.28) with $\mu + t' > 2 t_\perp$ and $|\mu - t'| < \frac{3}{2} t_\perp$. There exists $U_0 > 0$, independent of $\beta, L$, such that if $|U| \leq U_0$, it is possible to find a $\nu$, analytic in $U$, such that $S_2(x)$ exists and is analytic uniformly in $\beta, L$ as $\beta \to \infty, L \to \infty$. Moreover the Fourier transform of $S_2(x)$ in the $\beta \to \infty, L \to \infty$ limit, denoted by $\hat{S}_2(k)$, in the case $\frac{|\mu - t'|}{t_\perp} < 1$ is singular only at $\pm p_F$, with $\cos p_F = \frac{|\mu - t'|}{t_\perp}$, $v_{3,0} = t_\perp \sin p_F$ and close to the singularity,

\[
\hat{S}_2(k' \pm p_F) = \frac{1}{Z} \left( \begin{array}{cc} -i k_0 \pm v_3 k_3' & v_0 (k_+ - i k_-) \\ v_0 (k_+ + i k_-) & -i k_0 - (\pm) v_3 k_3' \end{array} \right)^{-1} (1 + R(k')) \quad (1.29)
\]

with $|R(k)| \leq \frac{|k'|}{v_{3,0}}$ and

\[
Z = 1 + O(U), \quad \frac{v_3 - v_{3,0}}{v_{3,0}} = O(U), \quad v = v_0 + O(U). \quad (1.30)
\]

On the other hand for $\frac{|\mu - t'|}{t_\perp} > 1$ the 2-point function is bounded for any $k$.

**Remarks**

1. The above theorem establishes analyticity in $U$ for values of the parameters including either the semimetallic and the insulating phase, and proves for the first time the existence of a Weyl semimetallic phase in an interacting system with short range interactions. The effect of the interaction is to generically modify the location of the Weyl points (the counterterm $\nu$ takes this into account) and to change the parameters of the emerging relativistic description, like the wave function renormalization and the "light" velocity.

2. Note that close to the boundary of the semimetallic phase the (third component) of the Fermi velocity $v_3$ is small, and vanishes continuously at the *quantum critical point* $\frac{|\mu - t'|}{t_\perp} = 1$ discriminating between
insulating and semimetallic phase. The estimated radius of convergence is uniform in $v_3$; this is remarkable as small $v_3$ correspond to a strong coupling regime in the effective relativistic description. The main idea in order to achieve that is to perform a different multiscale analysis in two regions of the energy space, discriminated by an energy scale measuring the distance from the critical point.

3. The Renormalization Group analysis performed here to prove the above theorem can be used to determine the large distance behavior of the current-current correlations. As a consequence, in combination with Ward Identities, some universality properties of the optical conductivity in the semimetallic phase can be proved, see [34].

2 Renormalization Group analysis: First regime

We find convenient the introduction of a parameter measuring the distance from the boundary of the semimetallic phase; therefore we define

$$\frac{\mu - t^t}{t_\perp} = -1 + r$$

(2.31)

with $|r| \leq \frac{1}{2}$; the case $r = \frac{1}{2}$ corresponds, in the non interacting case, to the semimetal with the highest velocity $v_3$, while at $r = 0$ the Fermi velocity $v_3$ vanishes.

The starting point of the analysis of (1.28) is the decomposition of the propagator in the following way

$$g_M(x - y) = g^{(\leq 0)}(x - y) + g^{(> 0)}(x - y)$$

(2.32)

where

$$\hat{g}^{(\leq 0)}(k) = \tilde{\chi}(\gamma^{- M}|k_0|) \chi_0(k) \tilde{A}^{-1}(k)$$

(2.33)

$$\hat{g}^{(> 0)}(k) = \tilde{\chi}(\gamma^{- M}|k_0|)(1 - \chi_0(k)) \tilde{A}^{-1}(k)$$

(2.34)

and $\chi_{<0}(k) = \tilde{\chi}(a_0^{-1} |\det A(k)|^{1/2})$, with $a_0 = \frac{t_\parallel}{10}$. The above decomposition corresponds to a decomposition in the Grassmann variables $\psi = \psi^{(\leq 0)} + \psi^{(>0)}$ with propagators respectively $g^{(\leq 0)}(x)$ (the infrared propagator) and $g^{(> 0)}(x)$ (the ultraviolet propagator). We can write

$$e^{W(\phi)} = \int P(d\psi^{(> 0)}) P(d\psi^{(\leq 0)}) e^{V(\psi^{(> 0)} + \psi^{(\leq 0)} + (\psi^{(>0)} + \psi^{(\leq 0)},\phi)} =$$

$$= e^{\beta |A| E_0} \int P(d\psi^{(\leq 0)}) e^{V^{(0)}(\psi^{(\leq 0)},\phi)}$$

(2.35)

with

$$V^{(0)}(\psi, \phi) = \sum_{n,m \geq 0} \int dx \int dy \left[ \prod_{i=1}^n \psi_{xi}^{\varepsilon_i} \prod_{i=1}^m \phi_{xi}^{\varepsilon_i} \right] W_{n,m}(x, y)$$

(2.36)
with \[ \prod_{i=1}^{n} \psi_{x_i}^{\varepsilon_i} = 1 \] if \( n = 0 \) and \[ \prod_{i=1}^{m} \phi_{y_i}^{\varepsilon_i} = 1 \] if \( m = 0 \), and for \( U, \nu \) smaller than a constant (independent from \( L, \beta, M \))

\[
\frac{1}{|\Lambda|} \int dxdy |W_{n,m}^{(0)}(\mathbf{x}, \mathbf{y})| \leq |\Lambda| \max_1 |U|^{|\max_1, n-1|}
\]  

(2.37)

Moreover \( \lim_{M \to \infty} W_{n,m}^{(0)}(\mathbf{x}, \mathbf{y}) \) and is reached uniformly. The above properties follow from Lemma 2 of [29] (app. B) or Lemma 2.2 of [29]; the proofs in such papers are written for \( d = 2 \) or \( d = 1 \) lattice system, but the adaptation to the present case is straightforward (due to the presence of spatial lattice the ultraviolet problem is essentially independent from dimension).

The infrared negative scales are divided in two different regimes, which have to be analyzed differently as they have different scaling properties. They are discriminated by a scale

\[
h^* = [\min_0 (\log_2 a_0^{-1} |r|, 0)]
\]  

(2.38)

which discriminates the region where the non-linear corrections to the dispersion relation are dominating with the region where the energy is essentially linear; if \( h^* = 0 \) the first regime described here is absent. We describe now the integration of the scales \( h \geq h^* \) inductively. Assume that we have integrated already the scales \( 0, -1, \ldots, h^* + 1 \) showing that (1.25) can be written as (in the \( \phi = 0 \) for definiteness)

\[
e^{[|\Lambda| \beta E_h] \int P(d\psi^{|\leq h|}) e^{V(h)(\sqrt{Z_h \psi^{|\leq h|}})}}
\]  

where \( P(d\psi^{|\leq h|}) \) has propagator given by

\[
g^{|\leq h|}(x) = \int dke^{ikx} \frac{\chi_h(k)}{Z_h} A_h^{-1}(k)
\]  

(2.40)

where

\[
A_h(k) = \begin{pmatrix} -ik_0 + v_{3,h}(\cos k_3 - 1 + r + E(\mathbf{k})) & v_h(\sin k_+ - i \sin k_-) \\ v_h(\sin k_+ + i \sin k_-) & -ik_0 - v_{3,h}(\cos k_3 - 1 + r - E(\mathbf{k})) \end{pmatrix}
\]  

(2.41)

\[
\chi_h(k) = \bar{\chi} (a_0^{-1} 2^{-h} |\det A_h(k)|^\frac{1}{2})
\]

and

\[
V(h)(\psi) = \sum_{n \geq 1} \int dx_1 \ldots \int dx_n \prod_{i=1}^{n} \psi_{x_i}^{\varepsilon_i} W_{n}^{(h)}(\mathbf{x}) = \frac{1}{(|\Lambda| \beta)^n} \sum_{\varepsilon_{k_1} \ldots \varepsilon_{k_n}} \prod_{i=1}^{n} \hat{\psi}_{k_i}^{\varepsilon_i} \hat{W}_{n}^{(h)}(k_1, \ldots, k_{n-1}) \delta(\sum_{i=1}^{n} \varepsilon_i k_i)
\]
We introduce a localization operator acting on the effective potential as
\[ \mathcal{V}^{(h)} = \mathcal{E}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)} \] (2.42)
with \( \mathcal{R} = 1 - \mathcal{E} \) and \( \mathcal{E} \) is a linear operator acting on the kernels \( \tilde{W}_n^{(h)}(k_1, ..., k_{n-1}) \) in the following way:

1. \( \mathcal{E}\tilde{W}_n^{(h)}(k_1, ..., k_{n-1}) = 0 \) if \( n > 2 \).
2. If \( n = 2 \)
   \[
   \mathcal{E}\tilde{W}_2^{(h)}(k) = \tilde{W}_2^{(h)}(0) + k_0 \tilde{W}_2^{(h)}(0) + \sum_{i=+, -, \beta} \sin k_i \partial_i \tilde{W}_2^{(h)}(0) + (\cos k_3 - 1) \partial^2_W \tilde{W}_2^{(h)}(0) \] (2.43)

The definition of \( \mathcal{E} \) is written in the \( L = \beta = \infty \) limit for definiteness but its expression for \( L, \beta \) finite is straightforward. By symmetry
\[
\hat{W}_2^{(h)}(0) = \sigma_3 n_h \quad \partial_+ \hat{W}_2^{(h)}(0) = \sigma_1 b_{+h} \quad \partial_- \hat{W}_2^{(h)}(0) = \sigma_2 b_{-h} \\
\partial_3 \hat{W}_2^{(h)}(0) = 0 \quad \partial^2_3 \hat{W}_2^{(h)}(0) = \sigma_3 b_{3h} \] (2.44)
Note also that, by definition \( \mathcal{E}\mathcal{R} = 0 \). We can include the local part of the effective potential in the fermionic integration, so that (3.86) can be rewritten as
\[
e^{[\mathcal{A}^{(h)}]_{ij} E_h} \int \tilde{P}(d\psi^{(\leq h)}) e^{\mathcal{E}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}^{-1}\psi^{(\leq h)}}) + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}^{-1}\psi^{(\leq h)}})} \] (2.45)
where
\[
\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leq h)}) = 2^h \nu_h \int dx \psi^+_x(\leq h) \sigma_3 \psi^-_x(\leq h) \] (2.46)
and \( \tilde{P}(d\psi^{(\leq h)}) \) is the Grassmann integration with propagator similar to (3.93) with \( Z_{h-1}(k), v_{h-1}(k), v_{3,h-1}(k) \) replacing \( Z_h, v_h, v_{3,h} \), where
\[
Z_{h-1}(k) = Z_h[1 + \chi_h^{-1}(k)b_{0,h}] \\
v_{h-1}(k) = \frac{Z_h}{Z_{h-1}(k)}[v_h + \chi_h^{-1}(k)b_{+h}] \] (2.47)
\[
v_{3,h-1}(k) = \frac{Z_h}{Z_{h-1}(k)}[v_{3,h} + \chi_h^{-1}(k)b_{3,h}] 
\]
Now we write \( \tilde{P}(d\psi^{(\leq h)}) = P(d\psi^{(\leq h-1)})P(d\psi^{(h)}) \) where \( P(d\psi^{(h)}) \) has propagator similar to \( \hat{p}^{(h)} \) (3.93) with the following differences: a) \( A_h \) is replaced by \( A_{h-1} \), where \( Z_{h-1} = Z_{h-1}(0), v_{h-1} = v_{h-1}(0), v_{3,h-1} = v_{3,h-1}(0) \); b) \( \chi_h \) is replaced by \( f_h \), a smooth compact support function with support in \( c_1 2^{h-1} \leq |\det A_{h-1}(k)|^{\frac{1}{2}} \leq c_2 2^{h+1} \), with \( c_1 < c_2 \) positive constants.
Assuming that $Z_h, v_{3,h}, v_{\pm,h}$ are close $O(U)$ to their value at $h = 0$, one has for any $N$ the following bound

$$|g^{(h)}(x)| \leq \frac{1}{Z_h} \frac{2^{h}}{1 + \left[ 2^h (|x_0| + |x_+| + |x_-|) + 2^h |x_3| \right]^N}$$

(2.48)

where we have used that $a_0 \gamma - |r| \leq a_0 \gamma (1 - \frac{1}{10})$. Therefore $k_0, k_\pm = O(2^h)$, $k_3 = O(2^h)$ for large negative $h$ and the bound (2.48) follows by integration by parts. Finally we perform the integration over $\psi^{(h)}$ obtaining

$$e^{\beta |\Lambda|^2} (\psi^{(h)}(\sqrt{Z_h - 1} \psi^{(h-1)})) = \int P(d\psi^{(h)}) e^{L\psi^{(h)}(\sqrt{Z_h - 1} \psi^{(h-1)}) + \mathcal{R}\psi^{(h)}(\sqrt{Z_h - 1} \psi^{(h-1)})}$$

(2.49)

obtaining an expression identical to (3.86) with $h - 1$ replacing $h$, $E_{h-1} = \tilde{E}_h + \tilde{e}_h$, so that the procedure can be iterated.

2.1 Tree expansion for the effective potentials.

The effective potential $\mathcal{V}^{(h)}(\psi^{(\leq h)})$ can be written in terms of a tree expansion, defined as follows.

![Figure 1: A renormalized tree for $\mathcal{V}^{(h)}$](image)

1) Let us consider the family of all trees which can be constructed by joining a point $r$, the root, with an ordered set of $n \geq 1$ points, the endpoints of the unlabeled tree, so that $r$ is not a branching point. $n$ will be called the order of the unlabeled tree and the branching points will be called the non trivial vertices. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees
with \( n \) end-points is bounded by \( 4^n \). We shall also consider the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label \( h \leq -1 \) with the root and we denote \( T_{h,n} \) the corresponding set of labeled trees with \( n \) endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in \([h, 1]\), and we represent any tree \( \tau \in T_{h,n} \) so that, if \( v \) is an endpoint or a non trivial vertex, it is contained in a vertical line with index \( h_v > h \), to be called the *scale* of \( v \), while the root \( r \) is on the line with index \( h \). In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the vertices will be the union of the endpoints, of the trivial vertices and of the non trivial vertices; note that the root is not a vertex. Every vertex \( v \) of a tree will be associated to its scale label \( h_v \), defined, as above, as the label of the vertical line whom \( v \) belongs to. Note that, if \( v_1 \) and \( v_2 \) are two vertices and \( v_1 < v_2 \), then \( h_{v_1} < h_{v_2} \).

3) There is only one vertex immediately following the root, which will be denoted \( v_0 \) and cannot be an endpoint; its scale is \( h + 1 \).

4) Given a vertex \( v \) of \( \tau \in T_{h,n} \) that is not an endpoint, we can consider the subtrees of \( \tau \) with root \( v \), which correspond to the connected components of the restriction of \( \tau \) to the vertices \( w \geq v \). If a subtree with root \( v \) contains only \( v \) and an endpoint on scale \( h_v + 1 \), it will be called a *trivial subtree*.

5) With each endpoint \( v \) we associate one of the monomials contributing to \( \mathcal{R}^V(0)(\psi^{(\leq h_v - 1)}) \), corresponding to the terms in the r.h.s. of (2.36) (with \( \psi^{(\leq 0)} \) replaced by \( \psi^{(\leq h_v - 1)} \)) and a set \( x_v \) of space-time points (the corresponding integration variables in the \( x \)-space representation); or a term corresponding to \( \mathcal{L}V^{(h_v - 1)}(\psi^{(\leq h_v - 1)}) \).

6) We introduce a *field label* \( f \) to distinguish the field variables appearing in the terms associated with the endpoints described in item 5); the set of field labels associated with the endpoint \( v \) will be called \( I_v \); note that \( |I_v| \) is the order of the monomial contributing to \( \mathcal{R}^V(0)(\psi^{(\leq h_v - 1)}) \) or \( \mathcal{L}V^{(h_v - 1)}(\psi^{(\leq h_v - 1)}) \) and associated to \( v \).

Analogously, if \( v \) is not an endpoint, we shall call \( I_v \) the set of field labels associated with the endpoints following the vertex \( v \); \( x(f) \) will denote the space-time point of the Grassmann field variable with label \( f \).

In terms of these trees, the effective potential \( V^{(h)} \), \( h \leq -1 \), can be written as

\[
V^{(h)}(\psi^{(\leq h)}) + \beta |\Lambda| \tau_{k+1} = \sum_{n=1}^{\infty} \sum_{\tau \in T_{h,n}} V^{(h)}(\tau, \psi^{(\leq h)}) , \tag{2.50}
\]

where, if \( v_0 \) is the first vertex of \( \tau \) and \( \tau_1, \ldots, \tau_s \ (s = s_{v_0}) \) are the subtrees of \( \tau \) with root \( v_0 \), \( V^{(h)}(\tau, \psi^{(\leq h)}) \) is defined inductively as follows:
i) if \( s > 1 \), then
\[
\mathcal{V}(h)(\tau, \psi^{(\leq h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}_{h+1}^T \left[ \mathcal{V}(h+1) (\tau_1, \psi^{(\leq h+1)}); \ldots; \mathcal{V}(h+1) (\tau_s, \psi^{(\leq h+1)}) \right],
\]
where \( \mathcal{E}_{h+1}^T \) denotes the truncated expectation with propagator \( g^{(h)} \) and
\[
\mathcal{V}(h+1) (\tau_i, \psi^{(\leq h+1)}) \text{ is equal to } \mathcal{R} \mathcal{V}(h+1) (\tau_i, \psi^{(\leq h+1)}) \text{ if the subtree } \tau_i \text{ contains more than one end-point, or if it contains one end-point but it is not a trivial subtree; it is equal to } \mathcal{R} \mathcal{V}(0) (\psi^{(\leq h+1)}) \text{ or } \mathcal{L} \mathcal{V}(h+1) (\psi^{(\leq h+1)}) \text{ if } \tau_i \text{ is a trivial subtree;}
\]
ii) if \( s = 1 \), then \( \mathcal{V}(h+1) (\tau, \psi^{(\leq h)}) \) is equal to \( [\mathcal{R} \mathcal{V}(h+1) (\tau_1, \psi^{(\leq h+1)})] \) if \( \tau_1 \) is not a trivial subtree; it is equal to \( \mathcal{R} \mathcal{V}(0) (\psi^{(\leq h+1)}) \) or \( \mathcal{L} \mathcal{V}(h+1) (\psi^{(\leq h+1)}) \) if \( \tau_1 \) is a trivial subtree (and therefore its end-point \( v \) has scale \( h_v = h + 2 \)).

Using its inductive definition, the right hand side of (2.50) can be further expanded, and in order to describe the resulting expansion we need some more definitions. We associate with any vertex \( v \) of the tree a subset \( P_v \) of \( I_v \), the external fields of \( v \). These subsets must satisfy various contraints. First of all, if \( v \) is not an endpoint and \( v_1, \ldots, v_{s_v} \) are the \( s_v \geq 1 \) vertices immediately following it, then \( P_v \subseteq \cup_i P_{v_i} \); if \( v \) is an endpoint, \( P_v = I_v \). If \( v \) is not an endpoint, we shall denote by \( Q_v \), the intersection of \( P_v \) and \( P_{v_1} \); this definition implies that \( P_v = \cup_i Q_{v_i} \). The union \( I_v \) of the subsets \( P_{v_i} \setminus Q_{v_i} \) is, by definition, the set of the internal fields of \( v \), and is non empty if \( s_v > 1 \). Given \( \tau \in T_{h,n} \), there are many possible choices of the subsets \( P_v \), \( v \in \tau \), compatible with all the constraints. We shall denote \( \mathcal{P}_\tau \) the family of all these choices and \( \mathcal{P} \) the elements of \( \mathcal{P}_\tau \).

With these definitions, we can rewrite \( \mathcal{V}(h)(\tau, \psi^{(\leq h)}) \) in the r.h.s. of (2.50) as:
\[
\mathcal{V}(h)(\tau, \psi^{(\leq h)}) = \sum_{\mathcal{P} \in \mathcal{P}_\tau} \mathcal{V}(h)(\tau, \mathcal{P}),
\]
\[
\mathcal{V}(h)(\tau, \mathcal{P}) = \int d\psi_{\nu_0} \tilde{\psi}^{(\leq h)} (P_{\nu_0}) K^{(h+1)}_{\tau,\mathcal{P}}(x_{\nu_0}),
\]
where
\[
\tilde{\psi}^{(\leq h)} (P_v) = \prod_{f \in P_v} \psi^{(\leq h)}(f)
\]
and \( K^{(h+1)}_{\tau,\mathcal{P}}(x_{\nu_0}) \) is defined inductively by the equation, valid for any \( v \in \tau \) which is not an endpoint,
\[
K^{(h_v)}_{\tau,\mathcal{P}}(x_v) = \frac{1}{s_v!} \prod_{i=1}^{s_v} [K^{(h_v+1)}_{v_i}(x_{v_i})] \mathcal{E}_{h_v}^T \left[ \tilde{\psi}^{(h_v)}(P_{v_1} \setminus Q_{v_1}); \ldots; \tilde{\psi}^{(h_v)}(P_{v_{s_v}} \setminus Q_{v_{s_v}}) \right],
\]
where \( \tilde{\psi}^{(h_v)}(P_v \setminus Q_v) \) has a definition similar to (2.53). Moreover, if \( v_i \) is an endpoint \( K_{v_i}^{(h_v+1)}(x_{v_i}) \) is equal to one of the kernels of the monomials contributing to \( \mathcal{R} \mathcal{V}^{(0)}(\psi^{(\leq h_v)}) \) or \( \mathcal{L} \mathcal{V}^{(h_v)}(\sqrt{\eta_h} \psi^{(\leq h_v)}) \); if \( v_i \) is not an endpoint, \( K_{v_i}^{(h_v+1)} = K_{\tau,\mathcal{P}_i}^{(h_v+1)} \), where \( \mathcal{P}_i = \{P_w, w \in \tau_i\} \).

We further decompose \( \mathcal{V}^{(h)}(\tau, \mathcal{P}) \), by using the following representation of the truncated expectation in the r.h.s. of (2.54). Let us put \( s = s_v \), \( P_i \equiv P_v \setminus Q_v \); moreover we order in an arbitrary way the sets \( P_i^{\pm} \equiv \{f \in P_i, \varepsilon(f) = \pm\} \), we call \( f_{ij}^{\pm} \) their elements and we define \( x^{(i)} = \bigcup_{f \in P_i} x(f), \ y^{(i)} = \bigcup_{f \in P_i} x(f) \), \( x_{ij} = x(f_{ij}^-), \ y_{ij} = x(f_{ij}^+) \). Note that \( \sum_{i=1}^s |P_i^-| = \sum_{i=1}^s |P_i^+| \equiv n \), otherwise the truncated expectation vanishes.

Then, we use the Brydges-Battle-Federbush formula [23] saying that, up to a sign, if \( s > 1 \),

\[
\mathcal{E}_{h}^T(\tilde{\psi}(P_1), \ldots, \tilde{\psi}(P_s)) = \sum_{T} \prod_{l \in T} \left[ g^{(l)}(x_l - y_l) \right] \int dP_T(t) \det G^{h,T}(t),
\]

where \( T \) is a set of lines forming an anchored tree graph between the clusters of points \( x^{(i)} \cup y^{(i)} \), that is \( T \) is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover \( t = \{t_{i,i'} \in [0,1], 1 \leq i,i' \leq s\} \), \( dP_T(t) \) is a probability measure with support on a set of \( t \) such that \( t_{i,i'} = u_i \cdot u_{i'} \) for some family of vectors \( u_i \in \mathbb{R}^s \) of unit norm. Finally \( G^{h,T}(t) \) is a \( (n-s+1) \times (n-s+1) \) matrix, whose elements are given by

\[
G_{i,j,i',j'}^{h,T} = t_{i,i'} g^{(h)}(x_{ij} - y_{i'j'}),
\]

with \( (f_{ij}^- , f_{i'j'}^+) \) not belonging to \( T \). In the following we shall use (2.53) even for \( s = 1 \), when \( T \) is empty, by interpreting the r.h.s. as equal to 1, if \( |P_1| = 0 \), otherwise as equal to \( \det G^h = \mathcal{E}_{h}^T(\tilde{\psi}(P_1)) \). It is crucial to note that \( G^{h,T} \) is a Gram matrix, i.e., the matrix elements in (2.56) can be written in terms of scalar products, and therefore it can be bounded by the Gram-Hadamard inequality.

If we apply the expansion (2.55) in each vertex of \( \tau \) different from the endpoints, we get an expression of the form

\[
\mathcal{V}^{(h)}(\tau, \mathcal{P}) = \sum_{T \in \mathcal{T}} \int dx_{v_0} \tilde{\psi}^{(\leq h)}(P_{v_0}) W_{\tau,h,\mathcal{P}}^{(h)}(x_{v_0}),
\]

where \( \mathcal{T} \) is a special family of graphs on the set of points \( x_{v_0} \), obtained by putting together an anchored tree graph \( T_v \) for each non trivial vertex \( v \). Note that any graph \( T \in \mathcal{T} \) becomes a tree graph on \( x_{v_0} \), if one identifies all the points in the sets \( x_v \), with \( v \) an endpoint. Given \( \tau \in \mathcal{T}_{h,n} \) and the labels \( \mathcal{P}, T \), calling \( I_R \) the endpoints of \( \tau \) to which is associated \( \mathcal{R} \mathcal{V}^{(0)} \) and \( I_v \) the end-points associated to \( \mathcal{L} \mathcal{V}^{(h_v - 1)} \), the explicit representation of
\( W_{\tau, P, T}(x_{v_0}) \) in (3.96) is

\[
W_{\tau, P, T}(x_{v_0}) = \left[ \prod_{v \in I_R} K_v^{(0)}(x_v) \right] \prod_{v \in I_v} 2^{h_v} \nu_{h_v} \cdot \left\{ \prod_{\text{not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(t_v) \det \tilde{G}_{h_v, T_v}(t_v) \prod_{l \in T_v} (x_l - y_l)^{\alpha_l} \frac{\partial^{\beta_l} \tilde{g}^{(h_v)}(x_l - y_l)}{\beta_l!} \right\},
\]

where \( K_v^{(0)}(x_v) \) are the kernels of \( RV^{(0)} \); the factors \((x_l - y_l)^{\alpha_l}\) and the derivatives \(\partial^{\beta_l}\) in the above expression are produced by the \( R \) operation and finally \( G_{h_v, T_v} \) differs from \( \tilde{G}_{h_v, T_v} \) for the presence of extra derivatives due to the \( R \) operation (see §3 of [36] for more details in a similar case). The functions appearing in the r.h.s. of (2.47), namely \( b_{0,h}, b_{\pm,h}, b_{3,h} \) can be written as derivatives of

\[
\bar{W}_{t}^{(h)} = \sum_{n=2}^{\infty} \sum_{\tau \in \bar{T}_{n,h}} \sum_{P \in P_{\tau}} \sum_{T \in T} \frac{1}{|\Lambda|^{\beta}} \int dx_{v_0} W_{\tau, P, T}(x_{v_0})
\]

where \( \bar{T}_{n,h} \) is the subset of \( T_{h,h} \) such that a) \( h_{v^*} = h + 1 \) where \( v^* \) is the first non trivial vertex; b) there is at least an end-point associated to \( V^{(0)} \).

Condition a) is due to the fact that, by construction, \( L\mathcal{R} = 0 \); condition b) is due to the fact that the contributions with only \( \nu \)-vertices are vanishing, as it can be easily verified by an explicit computation in momentum space (they are chain graphs and \( \tilde{g}^{(k)}(0) = 0 \)).

The next goal is the proof of the following result.

**Lemma 2.1** There exists a constant \( \varepsilon_0 \) independent of of \( \beta, L \) and \( r \), such that for \( |U| \leq \varepsilon_0 \) and \( \max_k h_k \geq |v_k|, |Z_{k-1}|, |v_{k,i} - v_{0,i}| \leq \varepsilon_0, i = \pm, 3 \) then for \( h \geq h^* \)

\[
\frac{1}{\beta|\Lambda|} \int dx_1 \cdots dx_l |\bar{W}_l^{(h)}(x_1, \ldots, x_l)| \leq 2^{h(7/2 - 5l/4)} (C\varepsilon_0)^{\max(1, l/2 - 1)}.
\]

Moreover, if \( \bar{W}_l^{(h)} \) is given by (2.60)

\[
\frac{1}{\beta|\Lambda|} \int dx_1 \cdots dx_l |\bar{W}_l^{(h)}(x_1, \ldots, x_l)| \leq 2^{h(7/2 - 5l/4)} 2^{\frac{1}{2}h} (C\varepsilon_0)^{\max(1, l/2 - 1)}.
\]

with \( C \) a suitable constant.

**Proof.** Using the tree expansion described above we find that the l.h.s. of
can be bounded from above by

\[
\sum_{n \geq 1} \sum_{\tau \in T_{\mathcal{R}}} \sum_{P \in P_{\tau}} \sum_{T \in \mathcal{T}} \left[ \prod_{l \in T^*} d(x_l - y_l) \left[ \prod_{v \in I_R} K_v^{(0)}(x_v) \right] \prod_{v \in I_v} 2^{h_v} |\nu_{h_v}| \right] (2.62)
\]

\[
\cdot \left[ \prod_{v \text{ not e.p.}} \frac{1}{s_v} \max_{t_v} \left| \det G^{h_v,T_v}(t_v) \right| \prod_{l \in T_v} \left[ (x_l - y_l)^{\alpha_l} |\partial^{\beta_l} g^{(h_v)}(x_l - y_l)| \right] \right]
\]

where \( T^* \) is a tree graph obtained from \( T = \bigcup_{\nu} T_{\nu} \), by adding in a suitable (obvious) way, for each endpoint \( v_i^* \), \( i = 1, \ldots, n \), one or more lines connecting the space-time points belonging to \( x_{v_i^*} \).

A standard application of Gram–Hadamard inequality, combined with the dimensional bound on \( g^{(h)}(x) \) given by (2.48), implies that

\[
|\det G^{h_v,T_v}(t_v)| \leq e^{\sum_{i=1}^n |P_v| - |P_v| - 2(s_v - 1)} \cdot 2^{\delta_v} \left( \sum_{i=1}^n |P_v| - |P_v| - 2(s_v - 1) \right). \tag{2.63}
\]

By the decay properties of \( g^{(h)}(x) \) given by (2.48), it also follows that

\[
\prod_{v \text{ not e.p.}} \frac{1}{s_v} \int \prod_{l \in T_v} d(x_l - y_l) \left| g^{(h_v)}(x_l - y_l) \right| \leq c^n \prod_{v \text{ not e.p.}} \frac{1}{s_v} 2^{-\delta_v} \tag{2.64}
\]

The bound on the kernels produced by the ultraviolet integration implies that

\[
\int \prod_{l \in T^* \cup T_{\mathcal{R}}} d(x_l - y_l) \left[ \prod_{v \in I_R} K_v^{(0)}(x_v) \right] \prod_{v \in I_v} 2^{h_v} |\nu_{h_v}|
\]

\[
\leq C^n \varepsilon_0 \left[ \prod_{v \text{ e.p., } |I_v| = 2} 2^{h_v(1+\delta_v)} \right], \tag{2.65}
\]

where \( \delta_v = \frac{1}{2} \) if \( v \in I_R \) and \( |I_v| = 2 \) and \( \delta_v = 0 \) otherwise ; the factors \( \gamma^{(1+\delta_v)} h_v \) are due to fact that \( \mathcal{R} \) acts on the terms with \( |I_v| = 2 \). Therefore the l.h.s. of (2.65) can be bounded from above by

\[
\sum_{n \geq 1} \sum_{\tau \in T_{\mathcal{R}}} \sum_{P \in P_{\tau}} \sum_{T \in \mathcal{T}} C^n \varepsilon_0 \left[ \prod_{v \text{ not e.p.}} \frac{1}{s_v} 2^{h_v} \left( \sum_{i=1}^n |P_v| - \frac{5|P_v|}{2} - \frac{5}{2} (s_v - 1) \right) \right].
\]

\[
\cdot \left[ \prod_{v \text{ not e.p.}} 2^{-\delta(X_v - X_{v'})} \prod_{v \text{ e.p., } |I_v| = 2} 2^{h_v(1+\delta_v)} \right] \tag{2.66}
\]

where \( z(P_v) = \frac{3}{4} \) for \( |P_v| = 2 \); the factor \( \prod_{v \text{ not e.p.}} 2^{-\delta(X_v - X_{v'})} \) takes into account the presence of the \( \mathcal{R} \) operation on the vertices. Once that the bound (2.66) is obtained, we have to see if we can sum over the scales and the trees. Let us define \( n(v) = \sum_{v' : v_i^* > v} 1 \) as the number of endpoints following
we find that \((2.66)\) can be bounded above by
\[ C_v \]

\(\tau\) (note that \(|I_v|\) is the number of field labels associated to the endpoints following \(v\) on \(\tau\)) and using that
\[
\sum_{v \not\in \text{e.p.}} \left( \sum_{i=1}^{s_v} |P_{v_i}| - |P_v| \right) = |I_v| - |P_v| ,
\]
\[
\sum_{v \not\in \text{e.p.}} (s_v - 1) = n - 1
\]
(2.67)

\[\sum_{v \not\in \text{e.p.}} (h_v - h) \left[ \left( \sum_{i=1}^{s_v} |P_{v_i}| \right) - |P_v| \right] = \sum_{v \not\in \text{e.p.}} (h_v - h)(|I_v| - |P_v|) \]
\[
\sum_{v \not\in \text{e.p.}} (h_v - h)(s_v - 1) = \sum_{v \not\in \text{e.p.}} (h_v - h')(n(v) - 1) ,
\]
we find that \((2.66)\) can be bounded by
\[
\sum_{n \geq 1} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{p \in \mathcal{P}_v \cap |P_v| = l}} \sum_{T \in \mathbf{T}} \frac{\sum C_n \varepsilon_0 2^{\frac{n}{2} - \frac{3}{4} |P_v| + \frac{1}{2} |I_v| - \frac{7}{8} n}}{\prod_n \left( \frac{1}{s_v!} 2^{(h_v - h')(5 |P_v| - \frac{7}{2}) + \frac{1}{2} n(P_v) + z(P_v)} \right) \left( \prod_v 2^{h_v (1 + \delta_v)} \right)}
\]
Using the identities
\[
2^{h_n} \prod_{v \not\in \text{e.p.}} 2^{(h_v - h')(n(v))} = \prod_{v \in \text{e.p.}} 2^{h_v} ,
\]
\[
\gamma^{h|I_v|} \prod_{v \not\in \text{e.p.}} 2^{(h_v - h')(|I_v|)} = \prod_{v \in \text{e.p.}} 2^{h_v |I_v|} ,
\]
we finally obtain
\[
\frac{1}{\beta |\Lambda|} \int dx_1 \cdots dx_l |W_1^{(h)}(x_1, \ldots, x_l)| \leq \sum_{n \geq 1} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{p \in \mathcal{P}_v \cap |P_v| = l}} \sum_{T \in \mathbf{T}} \frac{C_n \varepsilon_0 2^{\frac{n}{2} - \frac{3}{4}}} {\prod_n \left( \frac{1}{s_v!} 2^{(h_v - h')(5 |P_v| - \frac{7}{2}) + \frac{1}{2} n(P_v) + z(P_v)} \right) \left( \prod_v 2^{h_v (1 + \delta_v)} \right)} (2.70)
\]
Note that, if \(v\) is not an endpoint, \(5 |P_v| - \frac{7}{2} + z(P_v) \geq 1/2\) by the definition of \(\mathcal{R}\). Now, note that the number of terms in \(\sum_{T \in \mathbf{T}}\) can be bounded by \(C_n \prod_{v \not\in \text{e.p.}} s_v\). Using also that \(5 |P_v| - \frac{7}{2} + z(P_v) \geq 1/2\) and \(|P_v| - 3 \geq
Finally, we find that the l.h.s. of (2.60) can be bounded as
\[
\frac{1}{\beta |\Lambda|} \int dx_1 \cdots dx_l |W_l^{(h)}(x_1, \ldots, x_l)| \leq 2^{h(\frac{2}{7} - \frac{2}{7})} \sum_{n \geq 1} C^n \varepsilon^n \sum_{\tau \in T_{h,n}} \tag{2.71}
\]
\[ \cdot \left( \prod_{v \text{ not e.p.}} 2^{-(h_v - h_{\nu})/4} \right) \sum_{\mathcal{P} \in \mathcal{P}_v} \left( \prod_{v \text{ not e.p.}} 2^{-|P_v|/8} \right). \]

The sum over \( \mathcal{P} \) is bounded using the following combinatorial inequality: let \( \{p_v, v \in \tau\} \), with \( \tau \in T_{h,n} \), a set of integers such that \( p_v \leq \sum_{i=1}^{n_v} p_{v_i} \) for all \( v \in \tau \) which are not endpoints; then \( \prod_{v \text{ not e.p.}} \sum_{p_v} 2^{-p_v/8} \leq C^n \).

Finally,
\[
\sum_{\tau \in T_{h,n}} \prod_{v \text{ not e.p.}} 2^{\frac{1}{2}(h_v - h_{\nu})} \leq C^n,
\]
as it follows by the fact that the number of non trivial vertices in \( \tau \) is smaller than \( n - 1 \) and that the number of trees in \( T_{h,n} \) is bounded by const\(^n\), and collecting all the previous bounds, we obtain (2.60). In order to derive (2.61) we note that, for any tree with no \( \nu \) end-points
\[
\left[ \prod_{v \text{ e.p.}} 2^{h_v(5|I_v|/4 - 7/2)} \right] \left[ \prod_{v \text{ e.p.}, |I_v| = 2} 2^{h_v(1 + \delta_v)} \right] \leq \prod_{v \text{ e.p.}} 2^{\frac{1}{2}h_v} \tag{2.72}
\]
so that we can replace (2.70) by

\[
\frac{1}{\beta |\Lambda|} \int dx_1 \cdots dx_l |W_l^{(h)}(x_1, \ldots, x_l)| \leq \sum_{n \geq 1} \sum_{\tau \in T_{h,n}} \sum_{\mathcal{P} \in \mathcal{P}_v} C^n 2^{h(\frac{2}{7} - \frac{2}{7})} C^n \varepsilon^n \sum_{\tau \in T_{h,n}} \prod_{v \text{ not e.p.}} 2^{\frac{1}{2}(h_v - h_{\nu})} \tag{2.73}
\]

and
\[
\left[ \prod_{v \text{ not e.p.}} 2^{-\frac{1}{2}(h_v - h_{\nu})} \right] \prod_{v \text{ e.p.}} 2^{\frac{1}{2}h_v} \leq 2^{h} \tag{2.74}
\]
This concludes the proof of Lemma 1. ■

### 2.2 The flow of the effective parameters

The previous lemma provides convergence of the renormalized expansion provided that the effective parameters remain close \( O(U) \) to their initial value and \( U \) is chosen small enough. The flow equation for \( \nu_h \) can be written as
\[
\nu_{h-1} = 2^h \nu_h + \beta^{(h)}_{\nu} \tag{2.75}
\]
with, from (2.61), \( \beta^{(h)}_{\nu} = O(U2^{\frac{h}{2}}) \). By iteration we get
\[
\nu_h = 2^{h+1}[\nu + \sum_{k=h+1}^{1} 2^{k-2}\beta^{(k)}_{\nu}] \tag{2.76}
\]
If we choose \( \nu \) so that
\[
\nu = - \sum_{k=h^*+1}^{1} 2^{k-2} \beta^{(h)}_\nu + 2^{h^*-1} \nu_{h^*}
\]
then
\[
\nu_h = -2^h \sum_{k=h^*+1}^{h} 2^{k-2} \beta^{(h)}_\nu + 2^{h^*-h} \nu_{h^*}
\]

By a fixed point argument one can prove, see for instance Lemma 4.2 of [36], that it is possible to find a sequence of \( \nu_h \) solving (2.78). Moreover,
\[
v_{i,h-1} - v_{i,h} = O(U^2 h/8),
\]
\[
Z_{h-1} - Z_h = 1 + O(U^2 h/8),
\]
from (2.61), so that
\[
v_{i,h} - v_{i,0} = O(U),
\]
\[
Z_h - 1 = 1 + O(U).
\]

3 Renormalization Group integration: the second regime

3.1 Tree expansion and convergence

While the analysis of the scales greater than \( h^* \) are insensitive to the sign of \( r \), the integration of the scales smaller than \( h^* \) depends on it. The case \( r < 0 \) corresponds to the insulating phase; all the scales \( \leq h^* \) can be integrated in a single step (setting \( \nu_{h^*} = 0 \)) as the propagator of \( \psi(\leq h^*) \) has the same asymptotic behavior of the single scale propagator for \( h \geq h^* \), that is
\[
|g^{(\leq h^*)}(x)| \leq \frac{1}{Z_{h^*} \left( \frac{2^{5h^*}}{2} + 2^{h^*} |x_0| + |x_+| + |x_-| + 2^{h^*} |x_3| \right)^N}
\]

Similarly the case \( r = 0 \) correspond to the case \( h^* = -\infty \) and it can be analyzed as in the previous section.

Let us consider now the case \( r > 0 \), corresponding to the metallic phase: the Fourier transform of the propagator vanishes now in correspondence of two Fermi momenta and we need a multiscale decomposition. We write
\[
\hat{g}^{(\leq h^*)}(k) = \hat{g}^{(h^*)}(k) + \hat{g}^{(< h^*)}(k)
\]
where \( \hat{g}^{(< h^*)}(k) \) is equal to \( \hat{g}^{(\leq h^*)}(k) \) with \( \chi_{h^*-1}(k) \) replaced by
\[
\sum_{\omega=\pm} \theta(\omega k_3) \chi_{h^*-1}(k)
\]
and \( \chi_{h^*-1}(k) = \tilde{\chi}(b_0 2^{-h^*} \left| \det A_{h^*}(k) \right|^{1/2}) \) with \( b_0 \) chosen so that \( \chi_{h^*-1}(k) \) has support in two disconnected regions around \( \pm p_F \). The propagator \( \hat{g}^{(h^*)}(k) \), with support in \( \chi_{h^*} - \chi_{h^*-1} \), verifies the same bound as (2.48) with \( h = h^* \); in fact the denominator of \( \hat{g}^{(h^*)}(k) \) is \( O(r) \); moreover, if \( k = k' + \omega p_F \), one
has \( \cos(k' + \omega p_F) - \cos p_F = v_{3,0}k' + \frac{1}{2}k'^2 + O(k'^3) \), with \( v_{3,0} = O(\sqrt{r}) \) for small \( r \). Therefore each derivative with respect to \( k' \) produces an extra \( r^{-\frac{3}{2}} \).

We can decompose the Grassmann variables

\[
\psi_x^{\pm(h^*)} = \psi_x^{(h^*)} + \sum_{\omega=\pm} e^{\pm \omega p_F x} \psi_{\omega x}^{\pm(h^*)}
\]  

(3.82)

where \( \psi_x^{\pm(h^*)} \) has propagator

\[
g_{\omega}^{(h^*)}(x) = \int dx e^{ik'x} g^{(h^*)}(k' + \omega p_F)
\]  

(3.83)

We can therefore integrate \( \psi^{(h^*)} \) so that

\[
e^{\Lambda/\beta E_h} \int P(d\psi^{(h^*)}) \prod_{\omega=\pm} P(d\psi^{(h^*)}) e^{\mathcal{V}^{(h^*)}(Z_{h^*}\psi^{(h^*)})} =
\]  

(3.84)

\[
\int \prod_{\omega=\pm} P(d\psi^{(h^*)}) e^{\mathcal{V}^{(h^*)-1}(Z_{h^*+1}\psi^{(h^*)})}
\]

where

\[
\mathcal{V}^{(h^*)-1}(\psi) = \sum_{n \geq 1} \int dx_1... \int dx_n W_n^{(h^*)}(x) |\prod_{i=1}^n e^{i\omega_j p_F x_i} \psi^{(\leq h)}_{j i}|
\]  

(3.85)

and \( W_n^{(h^*)}(x) \) is translation invariant.

We describe the integration of the scales \( h < h^* \) inductively. Assume that we have integrate the scale \( h^*, ..., h + 1 \) showing that \( \mathcal{V}^{(h^*)} \) can be written as

\[
e^{\Lambda/\beta E_h} \int \prod_{\omega=\pm} P(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})}
\]  

(3.86)

where \( P(d\psi^{(\leq h)}) \) has propagator given by

\[
g^{(\leq h)}(x)_{\omega} =
\]  

(3.87)

\[
\int dk e^{ikx} \frac{\chi_h(k)}{Z_h} \begin{pmatrix}
-ik_0 + \omega v_{3,h} \sin k_3' + E'(k) \\
\omega v_h (\sin k_+ + i \sin k_-) \\
\omega v_h (\sin k_+ - i \sin k_-) \\
-ik_0 - \omega v_{3,h} \sin k_3' - E(kk)
\end{pmatrix}^{-1}
\]

where \( E'(k) = \cos p_F (\cos k_3 - 1) + E(k) \). and \( \mathcal{V}^{(h)}(\psi) \) is similar to \( \mathcal{V}^{(h^*)} \).

We introduce a localization operator acting on the effective potential as in \( (2.42) \) acting on the kernels \( \tilde{W}_n^{(h)}(k_1, ..., k_{n-1}) \) in the following way:

1. \( \mathcal{L} \tilde{W}_n^{(h)}(k_1, ..., k_{n-1}) = 0 \) if \( n > 2 \).

2. If \( n = 2 \)

\[
\mathcal{L} \tilde{W}_2^{(h)}(k) = \tilde{W}_2^{(h)}(\omega p_F) + k_0 \tilde{W}_2^{(h)}(\omega p_F) + \sum_{i=+,-,3} \sin k_i \partial_i \tilde{W}_2^{(h)}(\omega p_F)
\]  

(3.88)

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Note that, by symmetry
\[
\hat{W}_2^{(h)}(\omega p_F) = \sigma_3 n_h \quad \partial_+ \hat{W}_2^{(h)}(0) = \sigma_1 b_{+h},
\]
\[
\partial_- \hat{W}_2^{(h)}(\omega p_F) = \sigma_2 b_{-h} \quad \partial_3 \hat{W}_2^{(h)}(\omega p_F) = \sigma_3 b_{3h} \quad (3.89)
\]
We can include the quadratic part in the free integration; the single scale propagator verifies the following bound
\[
\left| g^h(x) \right| \leq \frac{1}{v_{3,h}} \frac{2^{3h}}{1 + 2^h(|x_0| + |x_+| + |x_-|) + v_{3,h}^{-1}|x|^N}.
\]
\[
\int dx |g^h(x)| \leq C 2^h \quad \max |g^h(x)| \leq \frac{2^h}{v_{3,h}} \quad (3.90)
\]
and \( v_{3,h^*} = O(\sqrt{r}) \). Note also that
\[
g^{(\leq h)}(x) = g^{(h)}_{rel,\omega}(x) + r^{(h)}(x) \quad (3.91)
\]
where
\[
g^{(\leq h)}(x) = \int \frac{d^k x}{Z_h} \chi_h(k) \left( \begin{array}{ccc}
-ik_0 + \omega v_{3,h} \sin k'_h & v_h (\sin k_+ - i \sin k_-) \\
 v_h (\sin k_+ + i \sin k_-) & -ik_0 - \omega v_{3,h} \sin k'_h
\end{array} \right)^{-1}.
\]
Lemma 3.1 If \( r > 0 \) there exists a constant \( \varepsilon_0 \) independent of \( \beta, L \) and \( r \), such that for \( |U| \leq \varepsilon_0 \) and \( \max_{k \geq h} |\nu_k|, |Z_k - 1|, |\frac{\nu_{k,i}}{\nu_{0,i}} - 1| \leq \varepsilon_0, i = \pm, 3 \) then for \( h \leq h^* \)
\[
\frac{1}{|\Lambda|} \int dx_1 \cdots dx_l |W_l^{(h)}(x_1, \ldots, x_l)| \leq 2^h (4\pi/2) (C \varepsilon_0)^{\max(1,l/2-1)} \quad . (3.93)
\]
and, if \( \bar{W}_l^{(h)} \) is given by \( 2.59 \)
\[
\frac{1}{|\Lambda|} \int dx_1 \cdots dx_l |\bar{W}_l^{(h)}(x_1, \ldots, x_l)| \leq 2^h (4\pi/2) 2^{4(h-h^*)} (C \varepsilon_0)^{\max(1,l/2-1)} \quad . (3.94)
\]
with \( C \) a suitable constant.

Proof. Again the effective potential can be written as a sum over trees similar to the previous ones ,but with some modifications ( see Fig 2).

The scales are \( \leq h^* \) and 5) in the previous definition is replaced by:

5) With each endpoint \( v \) we associate one of the monomials with four or more Grassmann fields contributing to \( R \psi^{(h^*)}(\psi^{(\leq h^*-1)}) \) and a set \( x_v \) of
space-time points (the corresponding integration variables in the $x$-space representation); or a term corresponding to $\mathcal{L}\mathcal{V}_{v,v-1}$.

In terms of these trees, the effective potential $\mathcal{V}(h)$, $h \leq -1$ is defined as

i) if $s > 1$, then

$$
\mathcal{V}(h)(\tau, \psi^{(\leq h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{L}_{h+1} \left[ \mathcal{V}^{(h+1)}(\tau_1, \psi^{(\leq h+1)}); \ldots; \mathcal{V}^{(h+1)}(\tau_s, \psi^{(\leq h+1)}) \right],
$$

(3.95)

where $\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)})$ is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\leq h+1)})$ if the subtree $\tau_i$ contains more than one end-point, or if it contains one end-point but it is not a trivial subtree; it is equal to $\mathcal{R}\mathcal{V}^{(h^*)}(\tau_i, \psi^{(\leq h+1)})$ or $\gamma^{h+1} v_{h+1} F_{\nu}(\psi^{(\leq h+1)})$ if $\tau_i$ is a trivial subtree;

ii) if $s = 1$, then $\mathcal{V}^{(h+1)}(\tau, \psi^{(\leq h)})$ is equal to $\left[ \mathcal{R}\mathcal{V}^{(h+1)}(\tau_1, \psi^{(\leq h+1)}) \right]$ if $\tau_1$ is not a trivial subtree; it is equal to $\left[ \mathcal{R}\mathcal{V}^{(h^*)}(\psi^{(\leq h+1)}) - \mathcal{R}\mathcal{V}^{(h^*)}(\psi^{(\leq h)}) \right]$ or if $\tau_1$ is a trivial subtree.

As before, we get

$$
\mathcal{V}^{(h)}(\tau, \mathcal{P}) = \sum_{T \in \mathcal{T}} \int dx_0 \psi_0^{(\leq h)}(P_{x_0}) W^{(h)}_{\tau, \mathcal{P}, T}(x_0) = \sum_{T \in \mathcal{T}} \mathcal{V}^{(h)}(\tau, \mathcal{P}, T),
$$

(3.96)

where, given $\tau \in \mathcal{T}_{h,n}$ and the labels $\mathcal{P}, T$, calling $I_R$ the endpoints of $\tau$ to which is associated $\mathcal{R}\mathcal{V}^{(h^*)}$ and $I_\nu$ the end-points associated to $\mathcal{L}\mathcal{V}_{v,v-1}$, the explicit representation of $W^{(h)}_{\tau, \mathcal{P}, T}(x_{vo})$ in (3.96) is

$$
W_{\tau, \mathcal{P}, T}(x_{vo}) = \left[ \prod_{v \in I_R} K_v^{(h^*)}(x_v) \right] \prod_{v \in I_\nu} \gamma^{h_v} v_{h_v}; \left\{ \prod_{v \not\in \text{e.p.}} \frac{1}{s_v!} \int dP_v(t_v) \right\}
$$

$$
\det \tilde{G}^{h_v, T}(t_v) \left[ \prod_{l \in T_v} \delta_{\omega_l - \omega_l'} \left[ (x_l - y_l)^{\alpha_l} \partial^{\beta_l} g_{\omega_l}(x_l - y_l) \right] \right]
$$

(3.97)
where $K^{(h^*)}_v(x_v)$ are the kernels of $\mathcal{R}^{(h^*)}$. By using the bounds obtained in the previous regime (2.60)

$$
\int \prod_{l \in T^* \setminus \cup_s T_s} d(x_l - y) \left[ \prod_{v \in I_R} K^{(h^*)}_v(x_v) \right] \prod_{v \in I_v} 2^{h^*|\nu_{h_v}|} \leq C^n \varepsilon_0^n
$$

$$
\sum_{n \geq 1} \left\{ \sum_{\tau \in T_{h,n}} \sum_{P \in \tau} \sum_{P_{\nu}(|P_{\nu}| = 1)} \sum_{v \not \in e} \left[ \prod_{v \not \in e} \frac{1}{s_v!} \left[ 2^{h^*} \left( \sum_{i=1}^{3|P_{\nu}|} \frac{|P_{\nu}|}{2} - 3|P_{\nu}| - 4(s_v - 1) \right) \right] \right] \left[ \prod_{v \in I_v, |I_v| = 2} 2^{h^*+\delta_v(h_v-2h^*)} \right] \left[ \prod_{v \in I_v, |I_v| \geq 4} 2^{h^*\left(\frac{7}{2} - \frac{5|I_v|}{4}\right)} \right]
$$

where $\delta_v = 1$ if $v \in I_R$ (again if $v \in I_R$ the factor $2^{2(h_v-h^*)}$ comes from the definition of $\mathcal{R}$). Therefore

$$
\frac{1}{|\Lambda|} \int dx_1 \cdots dx_l |W^{(h)}_l(x_1, \ldots, x_l)| \leq C^n \varepsilon_0^n \hspace{1cm} (3.99)
$$

$$
\sum_{n \geq 1} \left\{ \sum_{\tau \in T_{h,n}} \sum_{P \in \tau} \sum_{P_{\nu}(|P_{\nu}| = 1)} \sum_{v \not \in e} \left[ \prod_{v \not \in e} \frac{1}{s_v!} \left[ 2^{h^*} \left( \sum_{i=1}^{3|P_{\nu}|} \frac{|P_{\nu}|}{2} - 3|P_{\nu}| - 4(s_v - 1) \right) \right] \right] \left[ \prod_{v \in I_v, |I_v| = 2} 2^{h^*+\delta_v(h_v-2h^*)} \right] \left[ \prod_{v \in I_v, |I_v| \geq 4} 2^{h^*\left(\frac{7}{2} - \frac{5|I_v|}{4}\right)} \right]
$$

and by using (67)

$$
\frac{1}{|\Lambda|} \int dx_1 \cdots dx_l |W^{(h)}_l(x_1, \ldots, x_l)| \leq C^n \varepsilon_0^n \hspace{1cm} (3.100)
$$

$$
\sum_{n \geq 1} \left\{ \sum_{\tau \in T_{h,n}} \sum_{P \in \tau} \sum_{P_{\nu}(|P_{\nu}| = 1)} \sum_{v \not \in e} \left[ \prod_{v \not \in e} \frac{1}{s_v!} \left[ 2^{h^*} \left( \sum_{i=1}^{3|P_{\nu}|} \frac{|P_{\nu}|}{2} - 3|P_{\nu}| - 4(s_v - 1) \right) \right] \right] \left[ \prod_{v \in I_v, |I_v| = 2} 2^{h^*(1+\delta_v(h_v-2h^*))} \right] \left[ \prod_{v \in I_v, |I_v| \geq 4} 2^{h^*\left(\frac{7}{2} - \frac{5|I_v|}{4}\right)} \right]
$$

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and finally using (2.69)

\[
\frac{1}{|\Lambda|} \int dx_1 \ldots dx_l |W_1^{(h)}(x_1, \ldots, x_l)| \leq \quad (3.101)
\]

\[
\sum_{n \geq 1} \sum_{\tau \in T_n} \sum_{p \in T_p} \sum_{T \in T} C^n \varepsilon_0 h(4 - \frac{3|P_0|}{2}) \cdot \prod_{v \not\in \text{e.p.}} \frac{1}{s_v} \left(2(h_v - h_\tau)(4 - \frac{3|P_0|}{2})\right) \cdot \prod_{v \in \text{e.p.} \mid |I_v| = 2, v \in I_R} 2^{2(h_v - h_\tau)}
\]

\[
\prod_{v \not\in \text{e.p.}} \frac{1}{v_3,0} \left(\sum_{v \not\in \text{e.p.}} (|v_3|, |I_v|-2, (s_v-1))\right) \cdot \prod_{v \in \text{e.p.}} v_3,0^{(h_v - h_\tau)(4 - \frac{3|P_0|}{2})} \cdot \prod_{v \in \text{e.p.} \mid |I_v| \geq 4} 2^{2(h_v - h_\tau)}
\]

By writing

\[
\prod_{v \in \text{e.p.} \mid |I_v| = 2, v \in I_R} 2^{2(h_v - h_\tau)} = \prod_{v \in \text{e.p.} \mid |I_v| = 2, v \in I_R} 2^{2(h_v - h_\tau)(-4 + \frac{3|P_0|}{2})} = \prod_{v \in \text{e.p.} \mid |I_v| = 2, v \in I_R} 2^{2(h_v - h_\tau)(4 - \frac{3|P_0|}{2})}
\]

and using that \(2^{2(h_v - h_\tau)(4 - \frac{3|P_0|}{2})} \leq C(v_3,0)^{-1 + \frac{|I_v|}{2}}\) we get

\[
\frac{1}{|\Lambda|} \int dx_1 \ldots dx_l |W_1^{(h)}(x_1, \ldots, x_l)| \leq C^n \varepsilon_0 \quad (3.103)
\]

\[
\sum_{n \geq 1} \sum_{\tau \in T_n} \sum_{p \in T_p} \sum_{T \in T} C^n \varepsilon_0 h(4 - \frac{3|P_0|}{2}) \cdot \prod_{v \not\in \text{e.p.}} \frac{1}{s_v} \left(2(h_v - h_\tau)(4 - \frac{3|P_0|}{2})\right) \cdot \prod_{v \in \text{e.p.} \mid |I_v| = 2, v \in I_R} 2^{2(h_v - h_\tau)}
\]

\[
\prod_{v \not\in \text{e.p.}} \frac{1}{v_3,0} \left(\sum_{v \not\in \text{e.p.}} (|v_3|, |I_v|-2, (s_v-1))\right) \cdot \prod_{v \in \text{e.p.}} v_3,0^{(h_v - h_\tau)(4 - \frac{3|P_0|}{2})} \cdot \prod_{v \in \text{e.p.} \mid |I_v| \geq 4} 2^{2(h_v - h_\tau)}
\]

Using that

\[
\prod_{v \in \text{e.p.} \mid |I_v| = 2, v \in I_R} v_3,0^{-1 + \frac{|I_v|}{2}} = \prod_{v \in \text{e.p.}} v_3,0^{-1 + \frac{|I_v|}{2}} \quad (3.104)
\]

which follows from the fact that for \(v \in I'\) one has \(|I_v| = 2\) so that \(v_3,0^{-1 + \frac{|I_v|}{2}} = 1\), we can write

\[
\prod_{v \in \text{e.p.}} v_3,0^{-1 + \frac{|I_v|}{2}} \leq v_3,0^{-n + \sum_{v \not\in \text{e.p.}} |I_v|/2} \quad (3.105)
\]
and using \( \sum (s_v - 1) = n - 1 \) where \( n \) is the number of end-points we get

\[
\prod_{v \in \text{e.p.}} v_{3,0}^{-1} \prod_{v \not\in \text{e.p.}} \left[ \frac{1}{v_{3,0}} \right]^{-\sum (s_v - 1)} \leq C\nu_{3,0}^n v_{3,0}^{-1} \leq C v_{3,0}^{-1}
\]  

(3.106)

Moreover \( \sum_{v \in \text{e.p.}} |I_v| = l + \sum_v \left| \sum_{i=1}^{\nu_v} |P_{v_i}| - |P_v| \right| \)

\[
\left[ \prod_{v \in \text{e.p.}} (v_{3,0})^{\frac{|I_v|}{l_v}} \right] \prod_{v \not\in \text{e.p.}} \left[ \frac{1}{v_{3,0}} \right]^{\left| \sum_{i=1}^{\nu_v} \frac{|P_{v_i}|}{l_v} - \frac{|P_v|}{l_v} \right]} \right] \leq C(v_{3,0})^{l/2}
\]  

(3.107)

so that in total we get \( v_{3,0}^{-1/2} \) in agreement with (3.93). Note that the small divisors proportional to \( v_{3,0}^{-1} \), which could in principle spoil convergence, are exactly compensated from the factors due to the different scaling of the two regions.

The flow of the effective coupling can be analyzed as before, noting that the beta function is \( O(U^2 h - h^*) \) by the above estimate and we get

\[
Z \to h \to -\infty \quad Z = 1 + O(U^2)
\]

(3.108)

\[
v_{3,h} \to h \to -\infty \quad v_3 = t_{\perp} \sin(p_F) + a_3 U + O(U^2)
\]

\[
v_{\pm, h-1} \to h \to -\infty \quad v_{\pm} = t + a_{\pm} U + O(U^2)
\]

(3.109)

where

\[
a_3 \sigma_3 = \int dk \tilde{v}(k) \partial_3 \hat{g}(k) \quad a_+ \sigma_1 = \int dk \tilde{v}(k) \partial_+ \hat{g}(k)
\]

(3.110)

Moreover

\[
\nu = U v(0) \tilde{S}_0(0, 0^-) + \int dk v(k) \hat{g}(k)
\]

(3.111)

and this concludes the proof of Lemma 2.

From Lemma 1 and Lemma 2 the proof of the main theorem follows easily.

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