The cascade equivalent $A$-$H$-circuit of the salient-pole electric machine on the base of Laplace's equation in dimensionless Cartesian coordinates

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Abstract. For electromagnetic calculations of electric machines, the laminated models and the cascade equivalent circuits are often used. In the laminated model, each layer corresponds to the certain structural zone of the electric machine. Analyzing the solutions of the partial differential equations for each layer, we can conclude that these solutions may be reduced to the equations of the four-terminal circuit. Thereupon, the cascade equivalent circuit of the laminated model will be synthesized. In salient-pole machines, the solutions of partial differential equations are usually formed on the base of the piecewise continuous Sturm-Liouville eigenfunctions. Unfortunately, in this case, the cascade equivalent circuits cannot be synthesized since it needs many piecewise continuous functions in the zone of poles and many smooth functions in the zone of the air gap for ensuring the uniqueness of the solution. However, the author offers the approximate method on the base of the single piecewise continuous Sturm-Liouville eigenfunction in the zone of poles and many smooth functions in the zone of the air gap. This method allows synthesizing the cascade equivalent circuits of the salient-pole electric machines. In this paper, the synthesis of the cascade equivalent $A$-$H$-circuit of the synchronous salient-pole electric machine on the base of the Cartesian model is considered. In this model, the exciting field is generated with current sheets that are located on the rotor yoke. The active cell corresponding the exciting field is synthesized by means of Laplace's equation with the single piecewise continuous Sturm-Liouville eigenfunction.

1. Introduction
Since the 1960s, many papers appeared [1-6] where the electromagnetic calculations of electric machines on the base of laminated models were considered. In this model, each layer corresponds to the certain structural zone of the electric machine, for example the rotor yoke, the stator yoke, teeth and slots, air gaps etc. Analyzing the solutions of the partial differential equations for each layer, we can conclude that these solutions may be reduced to the equations of the four-terminal circuit and, on layer's boundaries, the tangential components of electric and magnetic intensities may be regarded as the voltage and the current. On the grounds of continuity of the tangential components of electric and magnetic intensities between layers, the cascade equivalent $E$-$H$-circuit of the laminated model will be synthesized. Thus, various $E$-$H$-, $B$-$H$- or $A$-$H$- equivalent circuits may be produced.

In salient-pole machines, the analytical solutions are formed on the base of the piecewise continuous Sturm-Liouville eigenfunctions when the zone containing ferromagnetic poles and non-magnetic spaces cannot be transformed into the homogeneous domain [7-9]. Unfortunately, in this
case, the cascade equivalent circuits cannot be synthesized since it needs many piecewise continuous functions in the zone of poles and many smooth functions in the zone of the air gap for ensuring the uniqueness of the solution. Though, the author had offered the approximate method on the base of the single piecewise continuous Sturm-Liouville eigenfunction in the zone of poles and many smooth functions in the zone of the air gap [10]. This method allows synthesizing the cascade equivalent circuits of the salient-pole electric machines [11].

In this paper, the synthesis of the cascade equivalent A-H-circuit of the synchronous salient-pole electric machine on the base of the Cartesian model is considered. In this model, the exciting field is generated with current sheets that are located on the rotor yoke. The active cell corresponding the exciting field is synthesized by means of Laplace’s equation with the single piecewise continuous Sturm-Liouville eigenfunction. Let us suppose that poles are fixed on the perfect ferromagnetic rotor yoke. The magnetic permeability of poles is constant. Let us suppose also, that the model has the unit length in the line of the machine axis.

2. The inductor of the salient-pole electric machine as the standard cell of the cascade equivalent A-H-circuit

Figure 1 shows the Cartesian model of the salient-pole electric machine. Because of symmetry, let us consider the problem upon half of the pole pitch i.e., with the x-coordinate varying from 0 to t.

![Figure 1. the Cartesian model of the salient-pole electric machine.](image)

Let us put dimensionless coordinates as

\[ x_0 = \frac{x}{\tau}; \quad y_0 = \frac{y}{\tau}; \quad b_0 = \frac{b}{\tau}; \quad t_0 = \frac{t}{\tau} \]

(1)

The exciting field is generated with the current \( I_0 \) that is located in air spaces between poles. Let us substitute the equivalent current sheets (that must be located on the rotor yoke between poles) for the exciting current. Fourier’s expansion turns the current sheets into

\[ H_0(x_0) = \sum_{k=1,3,5}^{\infty} \frac{2I_0 \sin k\pi b_0}{k\pi b_0 \tau} \cos k\pi x_0. \]

(2)

Magnetic permeability may be described by the piecewise continuous function in the whole zone, which corresponds to alternate poles and air spaces (in the range of half of the pole pitch):

\[ \mu(x_0) = \begin{cases} 
\mu_0, & 0 < x_0 < b_0, \\
\mu_f, & b_0 < x_0 < \frac{1}{2}.
\end{cases} \]

(3)

In Cartesian coordinates, the components of the magnetic intensity vector are determined as

\[ H_x(x_0, y_0) = \frac{1}{\mu(x_0)} \frac{\partial A(x_0, y_0)}{\partial y_0}; \quad H_y(x_0, y_0) = -\frac{1}{\mu(x_0)} \frac{\partial A(x_0, y_0)}{\partial x_0}. \]

(4)

where \( A \) is the z-component of the vector potential.
In consideration of expressions (4), Maxwell’s first equation \((\text{rot } \mathbf{H} = 0)\) turns into Laplace’s equation for the vector potential in the divergence form:

\[
\frac{\partial}{\partial (x_0, \tau)} \left[ \frac{1}{\mu(x_0)} \frac{\partial A(x_0, y_0)}{\partial (x_0, \tau)} \right] + \frac{\partial}{\partial (y_0, \tau)} \left[ \frac{1}{\mu(x_0)} \frac{\partial A(x_0, y_0)}{\partial (y_0, \tau)} \right] = 0.
\] (5)

Let the solution of the equation (5) has the single piecewise continuous Sturm-Liouville eigenfunction. In case of the exciting field, the vector potential is determined as

\[
A(x_0, y_0) = C_i \cosh n y_0 \begin{cases}
\cos n x_0, & 0 < x_0 < b_0, \\
K \sin n \left( \frac{1}{2} - x_0 \right), & b_0 < x_0 < \frac{1}{2}.
\end{cases}
\] (6)

where \(n\) is the first positive root of the transcendental equation

\[
\frac{\mu_i}{\mu_f} \cos n \left( \frac{1}{2} - b_0 \right) \cos nb_0 = \sin n \left( \frac{1}{2} - b_0 \right) \sin nb_0,
\] (7)

\[
K = \frac{\cos nb_0}{\sin n \left( \frac{1}{2} - b_0 \right)}.
\] (8)

The \(x\)-component of the magnetic intensity vector (excluding the exciting field source) is

\[
H_x(x_0, y_0) = C_i \frac{n}{\mu_0} \sinh n y_0 \begin{cases}
\cos n x_0, & 0 < x_0 < b_0, \\
K \sin n \left( \frac{1}{2} - x_0 \right), & b_0 < x_0 < \frac{1}{2}.
\end{cases}
\] (9)

If the rotor rotates (and the angular frequency is \(\omega\)), then a hypothetical observer would see the traveling waves of the functions (6) and (9), in which the \(k\)-th harmonic, in the complex plane, takes the form

\[
\hat{A}_k = j C_i Q_k \cosh n y_0, \\
\hat{H}_k \tau = j C_i \Psi_k \frac{n}{\mu_0} \sinh n y_0
\] (10, 11)

where \(k = 1, 3, 5...\) and

\[
Q_k = 4 \int_0^{b_0} \cos n x_0 \cos k \pi x_0 dx_0 + K \int_{b_0}^{1} \sin n \left( \frac{1}{2} - x_0 \right) \cos k \pi x_0 dx_0,
\] (12)

\[
\Psi_k = 4 \int_0^{b_0} \cos n x_0 \cos k \pi x_0 dx_0 + \frac{\mu_0}{\mu_f} K \int_{b_0}^{1} \sin n \left( \frac{1}{2} - x_0 \right) \cos k \pi x_0 dx_0.
\] (13)

Let the vector potential and the magnetic intensity vector are known at the boundaries of the piecewise zone with poles. If \(y_0 = 0\) (see Figure 1) then

\[
\hat{A}_k = j C_i Q_k, \\
\hat{H}_k \tau = 0.
\] (14, 15)

If \(y_0 = \frac{h}{\tau}\) (see Figure 1) then

\[
\hat{A}_k = j C_i Q_k \cosh \frac{nh}{\tau},
\] (16)
\[ H_{2k}\tau = jC_1\Psi_k \frac{n}{\mu_0} \sinh \frac{nh}{\tau}. \] (17)

Let the equivalent A-H-circuit corresponds to the zone of poles. Figure 2 shows this A-H-circuit in case of the first field harmonic. In this A-H-circuit, the vector potential is regarded as the voltage, and the \(x\)-component of the magnetic intensity vector multiplied by the pole pitch is regarded as the current. The ideal current source \(\dot{H}_0\tau\) corresponds to the first harmonic of the expression (2):

\[ \dot{H}_0\tau = \frac{2I_0\sin \pi b_0}{\pi b_0}. \] (18)

![Figure 2. The equivalent A-H-circuit of the zone of poles (in case of the first field harmonic).](image)

In the A-H-circuit, impedances \(Z_{12}\) and \(Z_{13}\) must not vary when operation conditions vary. Moreover, in expressions (14)-(17), the various values of the constant \(C_1\) correspond to various operating conditions. Analyzing expressions (14)-(17) in no-load conditions, we can determine impedances (for the first field harmonic) as

\[ Z_{12} = \frac{\mu_0}{n} \frac{Q_1}{\Psi_1} \frac{\tanh \frac{nh}{2\tau}}{\tau}, \] (19)

\[ Z_{13} = \frac{\mu_0}{n} \frac{Q_1}{\Psi_1} \frac{1}{\sinh \frac{nh}{\tau}}. \] (20)

If in the zone of poles, we have to calculate the quadrature-axis armature reaction then the vector potential must be determined as

\[ A(x_0, y_0) = C_1 \cosh ny_0 \begin{cases} \sin nx_0, & 0 < x_0 < b_0 \\ K \cos n \left(\frac{1}{2} - x_0\right), & b_0 < x_0 < \frac{1}{2} \end{cases} \] (21)

where the parameter \(n\) is the first (or second) positive root of the transcendental equation

\[ \cos n \left(\frac{1}{2} - b_0\right) \cos nb_0 = \frac{\mu_0}{\mu_f} \sin n \left(\frac{1}{2} - b_0\right) \sin nb_0, \] (22)

\[ K = \frac{\sin nb_0}{\cos n \left(\frac{1}{2} - b_0\right)}. \] (23)

Moreover, in the case of the quadrature-axis armature reaction, the ideal current source \(\dot{H}_0\tau\) must be removed from the A-H-circuit and

\[ Q_1 = 4 \left[ \int_0^b \sin nx_0 \sin \pi x_0 dx_0 + \frac{1}{b_0} \int_{\frac{1}{2}} 0 \cos n \left(\frac{1}{2} - x_0\right) \sin \pi x_0 dx_0 \right], \] (24)
\[
\Psi_1 = 4 \left[ \int_0^{b_0} \sin n x_0 \sin \pi x_0 dx_0 + \frac{\mu_0}{\mu_f} \frac{1}{b_0} \int_0^{b_0} \cos n \left( \frac{1}{2} - x_0 \right) \sin \pi x_0 dx_0 \right]. 
\]

(25)

3. The cascade equivalent A-H-circuit of the synchronous salient-pole electric machine

Figure 3 shows the cascade equivalent A-H-circuit of the synchronous salient-pole electric machine.

![A-H-circuit of the synchronous salient-pole electric machine.](image)

Figure 3. The A-H-circuit of the synchronous salient-pole electric machine.

The first cell \((Z_{12} \text{ and } Z_{13})\) corresponds to the rotor. The second cell \((Z_{21}, Z_{22}, Z_{23})\) corresponds to the air gap. Constants of this cell are given in [6]. For the first field harmonic

\[
Z_{21} = Z_{22} = \frac{\mu_0}{\pi} \tanh \frac{\pi \Delta}{2 \tau},
\]

(26)

\[
Z_{23} = \frac{\mu_0}{\pi} \frac{1}{\sinh \frac{\pi \Delta}{\tau}}.
\]

(27)

For taking into account the air gap irregularity, it is reasonable to multiply the air gap by \(\frac{k_\delta}{\alpha_\delta}\), where \(k_\delta\) is Carter’s factor, and \(\alpha_\delta\) is the pole overlap factor.

If magnetic permeability of stator teeth is infinite then the zone of stator teeth may be removed from the A-H-circuit and the ideal current source \(I_0\) (that corresponds to the armature reaction) should be connected to the output of the second cell. In compliance with Ampere’s circuital low, this current source is determined as

\[
I_0 = \delta h_z \tau \frac{t_z - b_z}{t_z}
\]

(28)

where \(h_z\) is the tooth height; \(b_z\) is the tooth width; \(t_z\) is the tooth pitch; \(\delta\) is a current density in stator slots (the first harmonic amplitude).

4. Test results

For verifying the cascade equivalent A-H-circuit, tests are carried out: calculations by means of both the cascade equivalent A-H-circuit and the numerical calculation (Elcut 5.1). While the cascade equivalent A-H-circuit is based on the Cartesian model, the numerical simulation is based on the cylindrical model.

Calculation data are given. The number of poles is 8; the rotor diameter is 632 mm; the pole height is 83.4 mm; the pole pitch is 248.2 mm; \(b_0\) is 0.25; the relative pole permeability is 500; the air gap is 4 mm; the tooth height is 53 mm; the tooth width is 10.34 mm; the tooth pitch is 20.68 mm; Carter’s factor is 1.22; the pole overlap factor is 0.73; the exciting current is 4582.5 A; the current density in stator slots is 2 A/mm² (the first harmonic amplitude with the quadrature-axis armature reaction).

Figure 4 shows the normal component of the magnetic induction vector on the stator’s surface (the first field harmonic). It is obvious that the calculation of the cascade equivalent A-H-circuit and the numerical simulation give well correlated results. That indicates correctness of modeling.
5. Conclusion

The cascade equivalent A-H-circuit of the synchronous salient-pole electric machine (on the base of the Cartesian model) is considered. In this A-H-circuit, the exciting field is generated with current sheets that are located on the rotor yoke. Depending on the field configuration (the exciting field or the quadrature-axis armature reaction), the constants of the A-H-circuit change. For taking into account the air gap irregularity, it is reasonable to increase the air gap. As follows from tests, the cascade equivalent A-H-circuit and the numerical simulation give well-correlated results. That indicates the correctness of modeling.

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