Stochastic Contraction in Riemannian Metrics

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Abstract—Stochastic contraction analysis is a recently developed tool for studying the global stability properties of nonlinear stochastic systems, based on a differential analysis of convergence in an appropriate metric. To date, stochastic contraction results and sharp associated performance bounds have been established only in the specialized context of state-independent metrics, which restricts their applicability. This paper extends stochastic contraction analysis to the case of general time- and state-dependent Riemannian metrics, in both discrete-time and continuous-time settings, thus extending its applicability to a significantly wider range of nonlinear stochastic dynamics.

I. INTRODUCTION

Contraction theory provides a body of analytical tools to study the stability and convergence of nonlinear dynamical systems [8]. Based on a differential analysis of convergence, it allows global stability properties of a nonlinear system to be concluded from the system’s linearization at all points in some appropriate metric. Historically, basic convergence results on contracting systems can be traced back to the numerical analysis literature [7], [4], [3]. Recently, contraction theory has been extended to stochastic differential systems. Then, in section III, we address the contraction properties of discrete-time settings, thus extending its applicability to a significantly wider range of nonlinear stochastic dynamics.

II. DISCRETE STOCHASTIC CONTRACTION

We first state and prove a proposition (see also [1]), which makes explicit the original deterministic discrete contraction theorem (see section 5 of [8]).
Consider the curve \( \Gamma_t : [0, 1] \to \mathbb{R}^n \) defined by

\[ \Gamma_t(u) = \Gamma(u) + (1 - u)\sigma(a)\eta_1 + u\sigma(b)\eta_2. \]

It is clear that \( \Gamma_t \) is \( C^1 \)-continuous and verifies \( \Gamma_0(0) = a + \sigma(a)\eta_1 \)
and \( \Gamma_t(1) = b + \sigma(b)\eta_2 \). Thus, by the definition of the distance, one has

\[
d_\Gamma^2(a, b) = \int_0^1 \left( \frac{\partial \Gamma_t}{\partial u} \right)^\top M \left( \frac{\partial \Gamma_t}{\partial u} \right) du.
\]

Consider the curve \( \Gamma_t : [0, 1] \to \mathbb{R}^n \) defined by

\[
\forall u \in [0, 1], \quad \Gamma_t(u) = \Gamma(u) + (1 - u)\sigma(a)\eta_1 + u\sigma(b)\eta_2.
\]

Let \((a_k)_{k \in \mathbb{N}}\) and \((b_k)_{k \in \mathbb{N}}\) be two trajectories whose initial conditions are given by a probability distribution \( \rho(\xi, \xi') \). Then for all \( k \geq 0 \),

\[
\mathbb{E} \left[ d_\Gamma^2(a_k, b_k) \right] \leq \frac{2D}{1 - \mu} + \mu^k \int \left[ d_\Gamma^2(a_0, b_0) - \frac{2D}{1 - \mu} \right]^+ dp(a_0, b_0),
\]

(II.2)

where \( [\cdot]^+ = \max(0, \cdot) \). In particular, for all \( k \geq 0 \),

\[
\mathbb{E} \left[ \|a_k - b_k\|^2 \right] \leq \frac{2D}{\beta(1 - \mu)} + \mu^k \mathbb{E} \left[ d_\Gamma^2(\xi, \xi') \right].
\]

(III.3)

Proof: Taking the conditional expectation given \((a_0, b_0) = x\) and applying (HD2) and Proposition \(1\) one has

\[
\mathbb{E}_x \left[ d_\Gamma^2(a_{k+1}, b_{k+1}) \right] = \mathbb{E}_x \left[ d_\Gamma^2(f(a_k, k) + \sigma(a, k)w_k, f(b_k, k) + \tau(b, k)w_k) \right] 
\]

\[
\leq \mathbb{E}_x \left[ d_\Gamma^2(f(a_k), f(b_k)) \right] + 2D,
\]

where \( w_k \) has the same distribution as \( w_k \) but is independent of the latter.

On the other hand, from (HD1) and Proposition \(1\) one has

\[
\mathbb{E}_x \left[ d_\Gamma^2(f(a_k), f(b_k)) \right] \leq \mu \mathbb{E}_x \left[ d_\Gamma^2(a_k, b_k) \right].
\]

If one now sets \( u_k = \mathbb{E}_x [d_\Gamma^2(a_k, b_k)] \) then it follows from the above that

\[
\forall k \geq 0, \quad u_k \leq \mu u_k + 2D.
\]

(II.4)

Define next \( v_k = u_k - 2D/(1 - \mu) \). Then replacing \( u_k \) by \( v_k + 2D/(1 - \mu) \) in (II.4) leads to \( v_{k+1} \leq \mu v_k \). This implies that \( \forall k \geq 0, \quad v_k \leq v_0 \mu^k \leq \|v_0\|^+ \mu^k \). Replacing \( v_k \) by its expression in terms of \( u_k \) then yields

\[
\forall k \geq 0, \quad u_k \leq \frac{2D}{1 - \mu} + \mu^k \left[ u_0 - \frac{2D}{1 - \mu} \right]^+.
\]

(II.5)

Integrating the last inequality with respect to \( x \) leads to (II.2). Finally, (II.3) follows from (II.2) by remarking that

\[
\int \mathbb{E}_x \left[ d_\Gamma^2(a_0, b_0) - \frac{2D}{1 - \mu} \right]^+ dp(a_0, b_0) \leq \int d_\Gamma^2(a_0, b_0) dp(a_0, b_0) = \mathbb{E} \left[ d_\Gamma^2(\xi, \xi') \right],
\]

(II.5)

and that, \( \|a_k - b_k\|^2 \leq \frac{1}{\beta^2} d_\Gamma^2(a_k, b_k) \). 

(II.6)

Remark 2.1 [Relaxing the uniform bound on the noise]:

Assume that the initial conditions are contained in a region \( U \), then (HD2) can in fact be replaced by [5]

\[
\forall k \geq 0, \ a_k \in U \quad \mathbb{E} \left[ \left( \sigma(a_k, k) \right)^\top M(a_k, \sigma(a_k, k)) \right] \quad \mathbb{E} = a_0 = a \leq D.
\]

III. CONTINUOUS STOCHASTIC CONTRACTION

Based on the discrete stochastic contraction theorem just established, we can now state and prove the continuous stochastic contraction theorem in general Riemannian metrics.

Consider the Ito stochastic differential equation

\[
\begin{cases}
\dot{a} = f(a, t) + \sigma(a, k)W_k, \\
\dot{a}_0(0) = \xi,
\end{cases}
\]

(II.6)

To ensure existence and uniqueness of solutions to equation (II.6), we assume the following standard conditions on \( f \) and \( \sigma \):

Lipschitz condition: There exists a constant \( K_1 > 0 \) such that \( \forall t \geq 0 \), \( a, b \in \mathbb{R}^n \), \( ||f(t, a) - f(t, b)|| + ||\sigma(a, t) - \sigma(b, t)|| \leq K_1||a - b|| \).
Restriction on growth: There exists a constant $K_2 > 0$ such that
\[
\forall t \geq 0, \ a \in \mathbb{R}^n \quad \|f(a, t)\|^2 + \|\sigma(a, t)\|^2 \leq K_2 (1 + \|a\|^2).
\]

**Theorem 2:** Assume that system (I) verifies the following two hypotheses:

1. **(Hc1)** for all $t \geq 0$, the dynamics $f(a, t)$ is contracting in the time-$a$ and state-dependent metric $\|M(a, t)\| = \Theta_T(a, t)\Theta(a, t)$, with contraction rate $\lambda (\lambda > 0)$, i.e.
   \[
   \forall t \geq 0, \ a \in \mathbb{R}^n
   \lambda_{max}\left(\left(\Theta(a, t) + \Theta(a, t)\frac{\partial \Theta}{\partial a}\right)\Theta(a, t)^{-1}\right) \leq -\lambda,
   \]
   where $A_\ast = \frac{1}{2}(A^T + A)$ denotes the symmetric part of a given matrix $A$. Furthermore, the metric $M(a, t)$ is positive definite uniformly in $a$ and $t$, with lower bound $\beta$.

2. **(Hc2)** $\text{tr} (\sigma(a, t)^T M(a, t) \sigma(a, t))$ is uniformly upper-bounded by a constant $C$.

Let $a(t)$ and $b(t)$ be two trajectories whose initial conditions are independent of $W$ and given by a probability distribution $p(\xi, \xi')$. Then for all $T \geq 0$,

\[
\mathbb{E}\left[\mathbb{E}_x \left[ d_{2M(T)}^2(a(T), b(T)) \right] \right] \leq \frac{C}{\lambda} + \frac{e^{-2\lambda T}}{\beta} \mathbb{E}\left[ d_{2M(0)}^2(\xi, \xi') \right].
\]

In particular, for all $T \geq 0$,

\[
\mathbb{E}\left[\|a(T) - b(T)\|^2\right] \leq \frac{C}{\lambda} + \frac{e^{-2\lambda T}}{\beta} \mathbb{E}\left[ d_{2M(0)}^2(\xi, \xi') \right].
\]

**Proof:** Fix $a(0), b(0) = x \in \mathbb{R}^{2d}$ and $T \geq 0$. We first discretize the time interval $[0, T]$ into $N$ equal intervals of length $\delta = T/N$ and consider the two sequences $(a^k_{\delta})_{k \in \mathbb{N}}, (b^k_{\delta})_{k \in \mathbb{N}}$ defined by

\[
\begin{align*}
   a^k_{\delta+1} &= a^k_{\delta} + \delta f(a^k_{\delta}, \delta) + \sigma(a^k_{\delta}, \delta) w^k_{\delta} \\
   a^0_{\delta} &= a(0) \\
   b^k_{\delta+1} &= b^k_{\delta} + \delta f(b^k_{\delta}, \delta) + \sigma(b^k_{\delta}, \delta) w^k_{\delta} \\
   b^0_{\delta} &= b(0),
\end{align*}
\]

where $(w^k_{\delta})_{k \in \mathbb{N}}$ and $(w^k_{\delta})_{k \in \mathbb{N}}$ are two sequences of random variables defined by $w^k_{\delta} = W((k + 1)\delta) - W(k\delta)$ and $w^k_{\delta} = W^\prime((k + 1)\delta) - W^\prime(k\delta)$. Note that, since $W$ and $W^\prime$ are two independent Wiener processes, $(w^k_{\delta})_{k \in \mathbb{N}}$ and $(w^k_{\delta})_{k \in \mathbb{N}}$ are two sequences of independent Gaussian random variables with distribution $\mathcal{N}(0, \delta)$. Note also that, by the strong convergence of the Euler-Maruyama scheme (cf. [5], p. 342), one has

\[
\lim_{\delta \to 0} \mathbb{E}_x \left[\|a^N_{\delta} - a(T)\|^2\right] = 0.
\]

Hypothesis (Hc2) implies that system (I) satisfies (Hd2) with $D = \delta C$. To verify (Hc1), denote by $G_{\delta}(a)$ the generalized Jacobian matrix of (I) at step $k$. Denoting $t = k\delta$, one has

\[
G_{\delta}(a) = \Theta(a, t + \delta) \frac{\partial (a + \delta f(a, t))}{\partial a} \Theta(a, t)^{-1} = \Theta(t + \delta) \left(1 + \delta \frac{\partial \Theta}{\partial a} \right) \Theta(t)^{-1}.
\]

Remark that we have dropped the argument $a$ for convenience. One can next rewrite $G_{2\delta}^T G_{\delta} = A_0 + \delta A_1$, with

\[
A_0 = \begin{pmatrix}
\Theta(t)^{-1} & \Theta(t + \delta)^{-1} \\
\Theta(t) & \Theta(t + \delta) \frac{\partial \Theta}{\partial a} \Theta(t)^{-1}
\end{pmatrix};
\]

\[
A_1 = \delta \begin{pmatrix}
\Theta(t)^{-1} & \Theta(t + \delta)^{-1} \\
\Theta(t) & \Theta(t + \delta) \frac{\partial \Theta}{\partial a} \Theta(t)^{-1}
\end{pmatrix}
\]

Using the Taylor expansion $\Theta(t + \delta) = \Theta(t) + \delta \Theta(t) + O(\delta^2)$ leads to

\[
\begin{align*}
A_0 &= I + 2\delta \Theta(t) \Theta(t)^{-1} + O(\delta^2); \\
\delta A_1 &= \delta \Theta(t) \Theta(t)^{-1} \left(\Theta(t) \Theta(t)^{-1} \Theta(t) \Theta(t)^{-1} \right) + O(\delta^2) \\
&= \delta \Theta(t) \Theta(t)^{-1} + O(\delta^2) \\
&= 2 \delta \Theta(t) \Theta(t)^{-1} + O(\delta^2).
\end{align*}
\]

Summarizing the previous calculations, one has

\[
\begin{align*}
G_{2\delta}^T G_{\delta} &= I + 2 \delta \Theta(t) \Theta(t)^{-1} + O(\delta^2) \\
&= I + 2 \delta \Theta(t) \Theta(t)^{-1} + O(\delta^2).
\end{align*}
\]

Thus, the hypothesis (Hc1) that $f$ is contracting in the metric $M$ with rate $\lambda$ implies

\[
\lambda_{max}(G_{2\delta}^T G_{\delta} k) \leq 1 - 2\delta \lambda + \epsilon(\delta),
\]

with $\lim_{\delta \to 0} \epsilon(\delta) = 0$. Letting $\mu(\delta) = 1 - 2\delta \lambda + \epsilon(\delta)$, one then has that $\mu < 1$ for $\delta$ sufficiently small, which in turn means that system (I) satisfies (Hd1). Applying the discrete contraction theorem for $k = N$ leads to

\[
\mathbb{E}_x \left[ d_{2M_N}^2(a_N^0, b_N^0) \right] \leq \frac{25C}{1 - \mu(\delta)} + \frac{e^{-2\lambda T}}{\beta} \mathbb{E}\left[ d_{2M(0)}^2(\xi, \xi') \right].
\]

On the other hand, one can, by the triangle inequality,

\[
\mathbb{E}_x \left[ d_{2M(T)}^2(a(T), b(T)) \right] \leq \mathbb{E}_x \left[ d_{2M_N}^2(a_N^0, b_N^0) \right] + \mathbb{E}_x \left[ d_{2M_N}^2(a_N^0, a(T)) \right] + \mathbb{E}_x \left[ d_{2M_N}^2(b_N^0, b(T)) \right].
\]

From equation (III.5), the second and third terms of the right-hand side vanish when $\delta \to 0$. As for the first term, remark that

\[
\frac{25C}{1 - \mu(\delta)} \geq \frac{25C}{1 - 2\delta \lambda + \epsilon(\delta)} = \frac{25C}{1 + \epsilon(\delta)} \leq C\epsilon(\delta) \leq \frac{C}{\lambda} e^{-2\lambda T}.
\]

One can thus conclude, by letting $\delta \to 0$, that

\[
\mathbb{E}_x \left[ d_{2M(T)}^2(a(T), b(T)) \right] \leq \frac{C}{\lambda} + e^{-2\lambda T} \left[ d_{2M(0)}^2(\xi, \xi') - \frac{C}{\lambda} \right].
\]

Integrating with respect to $x$ then leads to the desired result (III.2). Finally, (III.3) follows from (III.2) by the same calculations as in (III.5) and (III.6).

**Remark 3.1 [Noisy and noise-free trajectories]:** If $(a, b)$ represent in fact a noisy and a noise-free trajectories then the bounds (III.2) and (III.3) are replaced by analogous bounds where $C$ is replaced by $C/2$ (cf. [9]).

**Remark 3.2 [Relaxing the uniform bound on the noise]:** As in Remark 2.1, if the initial conditions are contained in a region $U$, then (Hc2) can in fact be replaced by

\[
\forall a \in U \quad \forall k \geq 0
\]

\[
\mathbb{E}\left[ \text{tr} (\sigma(a, t)^T M(a, t) \sigma(a, t), t) \right] | a(0) = a \leq C.
\]
Remark 3.3 ["Optimality" of the mean square bound]: If $M$ is in fact state-independent, then the bound (III.2) is the same as that obtained in [9] (cf. Theorem 2 of that reference), which means that this bound is "optimal", in the sense that it can be attained (cf. section III-A of [9]). This contrasts with the bound obtained in [2] (cf. Lemma 2 of that reference), which has the same form as (III.3) but with different constants $\lambda_1$ and $C_1$, defined -- using our notations -- as follows:

$$\lambda_1 = \lambda - \frac{\epsilon}{\beta}; \quad C_1 = C + \frac{n\mu\sigma^2}{2\epsilon},$$

where $\sigma$ is a uniform upper-bound on the Frobenius norm of the matrix $\sigma(a, t)$, $\bar{m}$ is a uniform upper-bound on $\|M(a, t)\|$, and $\epsilon$ is a positive constant. Note that, for any choice of $\epsilon$, one has $\lambda_1 < \lambda$ and $C_1 > C$, which yield a strictly looser bound compared to (III.2). Moreover, if $\epsilon$ is small, $\lambda_1$ gets closer to $\lambda$, but $C_1$ becomes very small. On the other hand, if $\epsilon$ is large, $C_1$ gets closer to $C$, but $\lambda_1$ becomes very small. Thus, there is no value of $\epsilon$ for which $\lambda_1$ and $C_1$ are arbitrarily close to $\lambda$ and $C$ respectively -- and in practice, the difference between $C_1$ and $C$ can be extremely large because of the uniform upper-bounds $\sigma$ and $\bar{m}$.

Example: Following [10], consider the following system

$$\begin{align*}
\dot{x}_1 &= x_2 \sqrt{1 + x_1^2}; & \dot{x}_2 &= -x_1 x_2^2 \sqrt{1 + x_1^2}; & y &= x_1. \tag{III.7}
\end{align*}$$

Construct the observer

$$\begin{align*}
\dot{\tilde{x}}_1 &= \tilde{x}_2 - (\tilde{x}_1 - y); & \dot{\tilde{x}}_2 &= -x_1 x_2 \sqrt{1 + x_1^2}, \tag{III.8}
\end{align*}$$

$$\begin{align*}
\tilde{x}_1 &= \tilde{x}_2; & \tilde{x}_2 &= -x_1 \sqrt{1 + x_1^2}. \tag{III.9}
\end{align*}$$

Note that this observer differs from that of [10]: the denominator in (III.9) is $1 + x_1^2$ instead of $1 + y^2$. The observer of [10] is interesting in that it is contracting in no state-independent metric (cf. Example 2.5 of that reference). It can be shown that this property is shared by the modified version (III.8)-(III.9).

Differentiating (III.8) and replacing $\tilde{x}_1$ and $\tilde{x}_2$ by their expressions in terms of $\tilde{x}_1$, $\tilde{x}_2$, $y$, one obtains

$$\begin{align*}
\dot{\tilde{x}}_1 &= \tilde{x}_2 - (\tilde{x}_1 - y) = \tilde{x}_2 \sqrt{1 + \tilde{x}_1^2} - (\tilde{x}_1 - y); \\
\dot{\tilde{x}}_2 &= \frac{\tilde{x}_2 \tilde{x}_1 \tilde{x}_1}{\sqrt{1 + \tilde{x}_1^2}} - \tilde{x}_2 \tilde{x}_1 \sqrt{1 + \tilde{x}_1^2}, \tag{III.10}
\end{align*}$$

$$\begin{align*}
\frac{\tilde{x}_1 \tilde{x}_2}{\sqrt{1 + \tilde{x}_1^2}} &= \frac{x_1 x_2}{\sqrt{1 + x_1^2}} - x_1 x_2 \sqrt{1 + x_1^2}.
\end{align*}$$

Observe that $(\tilde{x}_1, \tilde{x}_2)$ is a particular solution of (III.10). To show the contraction behavior of (III.10), consider the following nonlinear transform

$$\begin{align*}
\tilde{x}_1 &= -3\tilde{x}_1 + 5\tilde{x}_2 \sqrt{1 + \tilde{x}_1^2}; & \tilde{x}_2 &= 3\tilde{x}_1 + 2\tilde{x}_2 \sqrt{1 + \tilde{x}_1^2}. \tag{III.11}
\end{align*}$$

From (III.11), one has

$$\begin{align*}
(\tilde{x}_1, \tilde{x}_2)^\top &= P \cdot (\tilde{x}_1, \tilde{x}_2)^\top,
\end{align*}$$

where $P$ is the $2 \times 2$ constant matrix $\begin{pmatrix} -3 & 5 \\ 3 & 2 \end{pmatrix}$. Thus

$$\begin{align*}
(\tilde{x}_1, \tilde{x}_2)^\top &= P \cdot (\tilde{x}_1, \tilde{x}_2)^\top = P Q \cdot (\tilde{x}_1, \tilde{x}_2)^\top = P Q P^{-1} \cdot (\tilde{x}_1, \tilde{x}_2)^\top,
\end{align*}$$

where the second inequality comes from (III.10) with $Q = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. A numerical computation shows that the eigenvalues of the symmetric part of $P Q P^{-1}$ are $(-0.24, -0.76)$, which means that system $(\tilde{x}_1, \tilde{x}_2)$ is contracting with rate 0.24 in the identity metric. From (III.9), one finally has that system $(\tilde{x}_1, \tilde{x}_2)$ is contracting with rate 0.24 in the metric

$$M = \Theta^\top P^\top P \Theta, \quad \text{where } \Theta = \begin{pmatrix} 1 & 0 \sqrt{1 + \tilde{x}_1^2} \\ 0 & 1 + \tilde{x}_1^2 \end{pmatrix}.$$

Let us now study the convergence properties of the observer when the measure $y_p$ is corrupted by white noise as $y_p = y + \xi$, where $y = x_1$ is the unperturbed measure, $\xi$ is a "white noise" of variance 1 and $S$ is the noise intensity. Using the formal rule $dW = \xi dt$, equations (III.3) become

$$\begin{align*}
\dot{\tilde{x}}_1 &= -\tilde{x}_1 - y - \tilde{x}_2 \sqrt{1 + \tilde{x}_1^2} + (\tilde{x}_1 - y)\; dt + SdW \\
\dot{\tilde{x}}_2 &= -\tilde{x}_2 \sqrt{1 + \tilde{x}_1^2} + (\tilde{x}_1 - y)\; dt + SdW. \tag{III.12}
\end{align*}$$

The observer equations (III.10) become

$$\begin{align*}
\dot{\tilde{x}}_1 &= \tilde{x}_2 \sqrt{1 + \tilde{x}_1^2} - (\tilde{x}_1 - y)\; dt + SdW \\
\dot{\tilde{x}}_2 &= -\tilde{x}_2 \sqrt{1 + \tilde{x}_1^2} + (\tilde{x}_1 - y)\; dt + SdW.
\end{align*}$$

One is now in the settings of Theorem 2 with

$$\sigma(\tilde{x}_1, \tilde{x}_2) = \sqrt{S} \tilde{x}_1 \tilde{x}_2 \sqrt{1 + \tilde{x}_1^2}.$$

From the above expression, it can be shown algebraically that $\sup_{a,b} \sigma(a, b)^\top M(a, b) \sigma(a, b) = 15.2S^2$.

We now make the assumption that $\|\tilde{x}\|$ is uniformly upper-bounded by a constant $B$ (which can indeed be shown using an independent method, see also simulations in Fig. 1). Then, it can be shown that, uniformly,

$$\|\Theta^\top P^\top P \Theta x\|^2 \geq \gamma(B)\|x\|^2.$$

One thus can apply Theorem 2 and obtain the bound (III.3) with $\lambda = 0.24$, $C = 15.2S^2$ and $\beta = \gamma(B)$. Note that, for $t \to \infty$, one has $\tilde{x}_2 \to 0$, such that one has the bound $B = 0$, which in turn corresponds to $\gamma(B) = 12.95$. The bound after exponential transients is then given by (cf. Fig. 1 for numerical simulations)

$$\frac{C}{2\lambda} = 2.45S^2. \tag{III.13}$$

IV. CONCLUSION

We have established the stochastic contraction theorems in the case of general time- and state-dependent Riemannian metrics. In the limit when the metric becomes linear (state-independent), the bounds we derived are the same as those obtained in [9], which means that they are "optimal", in the sense that they can be attained. This development allows extending the applicability of contraction analysis to a significantly wider range of nonlinear stochastic dynamics, such as stochastic observers or networks of noisy nonlinear oscillators.

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Fig. 1. Simulations for the observer studied in the text. A: evolution of the systems for $t \in [0 \text{s}, 5 \text{s}]$. Equations (III.7) were integrated using the Euler method with time step $\Delta t = 0.01 \text{s}$ (red line: $x_1$; blue line: $x_2$). Equations (III.12) were integrated using the Euler-Maruyama scheme (cf. [6]) with the same time step $\Delta t = 0.01 \text{s}$. We plotted 20 sample trajectories for noise intensity $S = 1$ starting from the same deterministic initial values $(\hat{x}_1(0), \hat{x}_2(0))$ (magenta lines: $\hat{x}_1$; cyan lines: $\hat{x}_2$). B: evolution of the systems for $t \in [5 \text{s}, 15 \text{s}]$. Note that, for clarity, the values of $x_2$ and $\hat{x}_2$ were multiplied by 400 in this plot. To assess the theoretical bounds, we plotted the sample mean square error $(x_1 - \hat{x}_1)^2 + (x_2 - \hat{x}_2)^2$ (plain green line) and the theoretical bound after transients given by equation (III.13) (dashed green line). For clarity, these values were multiplied by 10.

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