The Relation between Almost Noetherian Module, Almost Finitely Generated Module and $T$-Noetherian Module

Faisol A$^{1,2}$, Surodjo B$^1$, and Wahyuni S$^1$

$^1$Department of Mathematics, Universitas Gadjah Mada
$^2$Department of Mathematics, Universitas Lampung
E-mail: $^1$ahmادfaisol@mail.ugm.ac.id, surodjo_b@ugm.ac.id, swahyuni@ugm.ac.id
$^2$ahmادfaisol@fmipa.unila.ac.id

Abstract. In this paper we study the relation between almost Noetherian modules, almost finitely generated (a.f.g. resp.) modules, and $T$-Noetherian modules. We show that if $R' = \{ r \in R | rM \neq M \}$ and $M$ is an almost Noetherian (a.f.g. resp.) $R$-module, then $M$ is an ($R'$)-Noetherian module. We also obtain that for any multiplicatively closed subset $T$ of a ring $R$ and $R' = \{ r \in R | rM \neq M \}$, if $M$ is an almost Noetherian (a.f.g. resp.) $R$-module and $T \cap R' \neq \emptyset$, then $M$ is a ($T \cap R'$)-Noetherian. Moreover, we show that if $M$ is an almost Noetherian (a.f.g. resp.) $R$-module and $T \cap R' \neq \emptyset$, then $M$ is a $T$-Noetherian module for every multiplicatively closed set $T \subseteq R$. Finally, we apply the results of this paper on the structure of Generalized Power Series Module (GPSM) $M[[S]]$.

1. Introduction

Armendariz [2] introduces the concepts of an almost Noetherian module, which is a generalization of Noetherian modules. An $R$-module $M$ is called almost Noetherian if every proper submodule of $M$ is finitely generated. One of the examples of an almost Noetherian $\mathbb{Z}$-module which is not Noetherian is $p$-quasicyclic group $\mathbb{Z}_{p^{\infty}}$. In fact, based on Gilmer and O’Malley [5], any almost Noetherian $\mathbb{Z}$-module is either Noetherian or isomorphic to $\mathbb{Z}_{p^{\infty}}$ for a suitable prime $p$.

Furthermore, Weakly [10] give the definition of an a.f.g. modules, i.e., a module that is not finitely generated but every its proper submodules are finitely generated. It is clear that every a.f.g. modules are almost Noetherian modules.

In [1], Anderson and Dumitrescu introduce the definition of $T$-Noetherian rings and modules. For any multiplicatively closed subset $T$ of a ring $R$, an $R$-module $M$ is called a $T$-Noetherian module if for each submodule $N$ of $M$, there exist an element $t \in T$ and a finitely generated submodule $F$ of $M$ such that $Nt \subseteq F \subseteq N$. Some properties of $T$-Noetherian modules studied by Baek, Lee, and Lim [3].

In a case of Noetherian modules, Varadarajan [8] give the necessary and sufficient conditions for the module of polynomials $M[X]$, the module of Laurent polynomials $M[X, X^{-1}]$, and the module of power series $M[[X]]$ to be Noetherian modules. As a generalization of these modules, Varadarajan [9] constructed the Generalized Power Series Modules (GPSM) $M[[S]]$, i.e. a module over Generalized Power Series Rings (GPSR) $R[[S]]$ whose constructed by Ribenboim.
[6]. Moreover, Varadarajan [9] determined the necessary and sufficient conditions for GPSM $M[[S]]$ to be a Noetherian module, which strengthens earlier results of Ribenboim [7].

Furthermore, Faisol, Surodjo, and Wahyuni [4] give the sufficient conditions for $R[X]$-module $M[X]$, $R[X, X^{-1}]$-module $M[X, X^{-1}]$, $R[[X]]$-module $M[[X]]$, and $R[[X, X^{-1}]]$-module $M[[X, X^{-1}]]$ to be $T[X]$-Noetherian, $T[X, X^{-1}]$-Noetherian, $T[[X]]$-Noetherian, and $T[[X, X^{-1}]]$-Noetherian, respectively, where $T$ is a multiplicatively and also additively closed subset of ring $R$.

In this paper, we investigate the relation between almost Noetherian modules, a.f.g. modules, and $T$-Noetherian modules. As the main result of this paper, we obtain the sufficient condition for an $R$-module $M$ to be $T$-Noetherian related to almost Noetherian $R$-module and a.f.g. $R$-module. Furthermore, we apply the main result of this paper on the structure of GPSM $M[[S]]$.

2. Main Results

In this section, we investigate the relation between almost Noetherian modules, a.f.g. modules, and $T$-Noetherian modules.

Now, let $M$ be an $R$-module and $R' = \{ r \in R | rM \neq M \}$. It is easy to show that $R'$ is a multiplicatively closed subset of $R$. The following proposition shows that if $M$ is an almost Noetherian module, then $M$ is $(R')$-Noetherian.

**Proposition 2.1** Let $R$ be a ring, $M$ an $R$-module and $R' = \{ r \in R | rM \neq M \}$. If $M$ is almost Noetherian, then $M$ is $(R')$-Noetherian.

**Proof.** For any $r \in R'$ and every submodule $N$ of $M$,

$$rN \subseteq rM \subseteq M.$$  

Hence, $rN$ is a proper submodule. Since $M$ is almost Noetherian, $rN$ is a finitely generated submodule of $M$. Therefore, for any submodule $N$ of $M$, there exist an element $r \in R'$ and a finitely generated submodule $F = rN$ of $M$ such that

$$rN \subseteq F \subseteq N.$$  

So, $M$ is $(R')$-Noetherian. $\blacksquare$

The consequence of Proposition 2.1 is given by the following corollary.

**Corollary 2.2** Let $R$ be a ring, $M$ an $R$-module and $R' = \{ r \in R | rM \neq M \}$. If $M$ is an a.f.g. module, then $M$ is $(R')$-Noetherian.

Next, for any multiplicative subset $T$ of $R$, clearly $T \cap R'$ is also a multiplicative subset of $R$. The following proposition shows that if $M$ is almost Noetherian as an $R$-module and $(T \cap R') \neq 0$, then $M$ is $(T \cap R')$-Noetherian.

**Proposition 2.3** Let $R$ be a ring, $T$ a multiplicative subset of $R$, $M$ an $R$-module and $R' = \{ r \in R | rM \neq M \}$. If $M$ is an almost Noetherian module and $(T \cap R') \neq \emptyset$, then $M$ is $(T \cap R')$-Noetherian.

**Proof.** For any $a \in (T \cap R')$ and every submodule $N$ of $M$,

$$aN \subseteq aM \subseteq M.$$  

Hence, $aN$ is a proper submodule. Since $M$ is a.f.g., $aN$ is a finitely generated submodule of $M$. Therefore, for any submodule $N$ of $M$, there exist an element $a \in (T \cap R')$ and a finitely generated submodule $F = aN$ of $M$ such that

$$aN \subseteq F \subseteq N.$$  

So, $M$ is $(T \cap R')$-Noetherian. $\blacksquare$
The consequence of Proposition 2.3 is given by the following corollary.

**Corollary 2.4** Let $R$ be a ring, $T$ a multiplicative subset of $R$, $M$ an $R$-module and $R' = \{ r \in R | rM \neq M \}$. If $M$ is an a.f.g. module and $(T \cap R') \neq \emptyset$, then $M$ is $(T \cap R')$-Noetherian.

Next, we recall the properties of a Noetherian $R$-module related to two multiplicative subsets of ring $R$.

**Lemma 2.5** [3] Let $T_1, T_2$ are multiplicative subsets of a ring $R$. If $T_1 \subseteq T_2$, then any $T_1$-Noetherian $R$-module is $T_2$-Noetherian.

By using Lemma 2.5, we obtain the sufficient conditions for an $R$-module $M$ to be $T$-Noetherian related to almost Noetherian $R$-module and a.f.g. $R$-module as follows.

**Theorem 2.6** Let $R$ be a ring, $T$ a multiplicative subset of $R$, $M$ an $R$-module and $R' = \{ r \in R | rM \neq M \}$. If $M$ is an almost Noetherian $R$-module and $(T \cap R') \neq \emptyset$, then $M$ is $T$-Noetherian.

**Proof.** Since $M$ is an almost Noetherian $R$-module, based on Proposition 2.3, $M$ is $(T \cap R')$-Noetherian. Furthermore, since $(T \cap R') \subseteq T$, based on Lemma 2.5, $M$ is a $T$-Noetherian $R$-module. ■

**Theorem 2.7** Let $R$ be a ring, $T$ a multiplicative subset of $R$, $M$ an $R$-module and $R' = \{ r \in R | rM \neq M \}$. If $M$ is an a.f.g. module and $(T \cap R') \neq \emptyset$, then $M$ is $T$-Noetherian.

**Proof.** Since $M$ is an a.f.g. $R$-module, based on Corollary 2.4, $M$ is $(T \cap R')$-Noetherian. Furthermore, since $(T \cap R') \subseteq T$, based on Lemma 2.5, $M$ is a $T$-Noetherian $R$-module. ■

3. Application on Generalized Power Series Modules

In this section, we apply the results of the previous section on the structure of Generalized Power Series Modules (GPSM) $M[[S]]$.

Regarding ordered sets, strictly ordered monoids, Artinian and narrow sets, we will be following the terminology in [6] and [7].

An ordered set $(S, \leq)$ is Artinian if every strictly decreasing element of $S$ is finite, and $(S, \leq)$ is Noetherian if every strictly increasing element of $S$ is finite.

**Lemma 3.1** [6] Let $(S, \leq)$ be any ordered set.

1. If $S$ is Artinian (Noetherian) and $X \subseteq S$, then $X$ is Artinian (Noetherian).
2. If $X_1, X_2, ..., X_n$ are Artinian (Noetherian) subsets of $S$, then $\bigcup_{i=1}^{n} X_i$ is Artinian (Noetherian).

An ordered set $(S, \leq)$ is said to be narrow if $S$ does not contain an infinite subset consisting of pairwise incomparable elements.

**Lemma 3.2** [6] Let $(S, \leq)$ be any order set.

1. If $S$ narrow and $X \subseteq S$, then $X$ is narrow.
2. If $X_1, X_2, ..., X_n$ are narrow subsets of $S$, then $\bigcup_{i=1}^{n} X_i$ is narrow.

**Lemma 3.3** [7] If $X, Y$ are Artinian and narrow subsets of $(S, \leq)$, then $X + Y = \{ s + t | s \in X, t \in Y \}$ is also Artinian and narrow.
Now, we recall the construction of GPSR and GPSM as follows from Ribenboim [6] and Varadarajan [9].

Let \((S, \leq)\) be a strictly ordered monoid, \(R\) a commutative ring with an identity element and \(M\) an \(R\)-module. Let \(R^S = \{ f | f : S \to R \}\) and \(R[[S]] = \{ f \in R^S | \text{supp}(f) \text{ is Artinian and narrow } \}\),

where \(\text{supp}(f) = \{ s \in S | f(s) \neq 0 \}\).

For any \(f, g \in R[[S]]\), \(\text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g)\), \(\text{supp}(-f) = \text{supp}(f)\), and \(\text{supp}(fg) \subseteq \text{supp}(f) + \text{supp}(g)\). Therefore, under pointwise addition and convolution multiplication defined by

\[
(fg)(s) = \sum_{(x,y) \in \chi_s(f,g)} f(x)g(y),
\]

for all \(f, g \in R[[S]]\) where \(\chi_s(f,g) = \{(x, y) \in \text{supp}(f) \times \text{supp}(g) | xy = s \}\)
is finite, \(R[[S]]\) becomes a ring which is known as Generalized Power Series Ring (GPSR).

Next, let \(M^S = \{ \alpha | \alpha : S \to M \}\) and \(M[[S]] = \{ \alpha \in M^S | \text{supp}(\alpha) \text{ is Artinian and narrow } \}\),

where \(\text{supp}(\alpha) = \{ s \in S | \alpha(s) \neq 0 \}\).

For any \(\alpha, \beta \in M[[S]]\), \(\text{supp}(\alpha + \beta) \subseteq \text{supp}(\alpha) \cup \text{supp}(\beta)\), \(\text{supp}(-\alpha) = \text{supp}(\alpha)\), and \(\text{supp}(\alpha\beta) \subseteq \text{supp}(\alpha) + \text{supp}(\beta)\). Therefore, under pointwise addition and scalar multiplication defined by

\[
(f\alpha)(s) = \sum_{(x,y) \in \chi_s(f,\alpha)} f(x)\alpha(y),
\]

for all \(f \in R[[S]]\) and \(\alpha \in M[[S]]\) where \(\chi_s(f,\alpha) = \{(x, y) \in \text{supp}(f) \times \text{supp}(\alpha) | xy = s \}\)
is finite, \(M[[S]]\) acquires the structure of an \(R[[S]]\)-module. Next, this module is called Generalized Power Series Module (GPSM).

For any \(r \in R\) and any \(s \in S\), we associated the maps \(c_r, e_s \in R[[S]]\), defined by

\[
c_r(t) = \begin{cases} 
r & \text{if } t = 1 \\
0 & \text{if } t \neq 1,
\end{cases}
\]

and

\[
e_s(t) = \begin{cases} 
1 & \text{if } t = s \\
0 & \text{if } t \neq s.
\end{cases}
\]

For any \(m \in M\) and any \(s \in S\), we define \(d^s_m \in M[[S]]\) by

\[
d^s_m(t) = \begin{cases} 
m & \text{if } t = s \\
0 & \text{if } t \neq s.
\end{cases}
\]
Then, it is clear that \( r \mapsto e_r \) is a ring embedding of \( R \) into \( R[[S]] \), and \( s \mapsto e_s \) is a monoid embedding of \( S \) into the multiplicative monoid of the ring \( R[[S]] \), and also \( m \mapsto d^0_m \) is a module embedding of \( M \) into \( M[[S]] \).

For any subset \( N \) of an \( R \)-module \( M \), we define

\[
N[[S]] = \{ \gamma \in M[[S]] | \gamma(s) \in N; \forall s \in S \}.
\]

The sufficient conditions of \( N[[S]] \) to be an \( R[[S]] \)-submodule of \( M[[S]] \) are given by the following lemma.

**Lemma 3.4** Let \( R \) be a ring, \( M \) an \( R \)-module, and \((S, \leq)\) a strictly ordered monoid. If \( N \) is an \( R \)-submodule of \( M \), then \( N[[S]] \) is an \( R[[S]] \)-submodule of \( M[[S]] \).

**Proof.** First, it is clear that \( N[[S]] \subseteq M[[S]] \). Next, we will show that for any \( f, g \in R[[S]] \) and any \( \alpha, \beta \in N[[S]] \), \( f\alpha + g\beta \in N[[S]] \).

It is clear that \( \text{supp}(f) \) and \( \text{supp}(\alpha) \) are Artinian and narrow. Since \( \text{supp}(f\alpha) \subseteq \text{supp}(f) + \text{supp}(\alpha) \) and \( \text{supp}(g\beta) \subseteq \text{supp}(g) + \text{supp}(\beta) \), based on Lemma 3.3, Lemma 3.1, and Lemma 3.2 \( \text{supp}(f\alpha) \) and \( \text{supp}(g\beta) \) are Artinian and narrow. Furthermore, since \( \text{supp}(f\alpha + g\beta) \subseteq \text{supp}(f\alpha) \cup \text{supp}(g\beta) \), based on Lemma 3.1 and 3.2 \( \text{supp}(f\alpha + g\beta) \) is Artinian and narrow. In other words, \( f\alpha + g\beta \in M[[S]] \).

Now, for any \( s \in S \), we will show that \( (f\alpha + g\beta)(s) \in N \). For any \( f, g \in R[[S]] \) and any \( \alpha, \beta \in M[[S]] \),

\[
(f\alpha)(s) = \sum_{xy=s} f(x)\alpha(y)
\]

and

\[
(g\beta)(s) = \sum_{xy=s} g(x)\beta(y).
\]

Since \( N \) is an \( R \)-submodule of \( M \), we have \( (f\alpha)(s) \in N \) and \( (g\beta)(s) \in N \). Hence, we obtain

\[
(f\alpha + g\beta)(s) = (f\alpha)(s) + (g\beta)(s) \in N.
\]

In other words, it is prove that \( N[[S]] \) is an \( R[[S]] \)-submodule of \( M[[S]] \). \( \blacksquare \)

It is clear that, if \( N \) is an \( R \)-submodule of \( M \), then \( rN \subseteq rM \) for any \( r \in R \). Therefore, according to Lemma 3.4, \( fN[[S]] \subseteq fM[[S]] \) for any \( f \in R[[S]] \).

Next, for any subset \( T \) of a ring \( R \), we define the set

\[
T[[S]] = \{ f \in R[[S]] | f(s) \in T; \forall s \in S \}.
\]

It is clear that \( T[[S]] \subseteq R[[S]] \). The sufficient conditions of \( T[[S]] \) to be a multiplicatively closed subset of GPSR \( R[[S]] \) are given by the following lemma.

**Lemma 3.5** Let \( R \) be a ring, \( T \subseteq R \), \((S, \leq)\) a strictly ordered monoid, and \( R[[S]] \) a GPSR. If \( T \subseteq R \) is closed under the operations of \( R \), then \( T[[S]] \) is a multiplicatively closed subset of \( R[[S]] \).

**Proof.** For any \( f, g \in T[[S]] \), we will show that \( fg \in T[[S]] \). Based on the convolution multiplication in equation (1), for any \( s \in S \) we obtain

\[
(fg)(s) = \sum_{(x,y) \in x,y(f,g)} f(x)g(y)
\]

\[
= \sum_{xy=s} f(x)g(y).
\]
Since $T \subseteq R$ is closed under the operations of ring $R$, we have $\sum_{xy=s} f(x)g(y) \in T$. In other words, $fg \in T[[S]]$. Thus, $T[[S]]$ is a multiplicatively closed subset of $R[[S]]$. ■

As a direct result of Lemma 3.5 above, if we choose $S = \mathbb{N} \cup \{0\}$ with a trivial order $\leq$, we obtain [4, Lemma 6]. If we choose $S = \mathbb{Z}$ with a trivial order $\leq$, we obtain [4, Lemma 7(1)]. Next, if we choose $S = \mathbb{N} \cup \{0\}$ with a usual order $\leq$, we obtain [4, Lemma 7(2)]. Furthermore, if we choose $S = \mathbb{Z}$ with a usual order $\leq$, we obtain [4, Lemma 7(3)].

Now, let $M$ be an $R$-module and $R' = \{ r \in R | rM \neq M \}$, we defined the set

$$R'[[S]] = \{ g \in R[[S]] | g(s) \in R'; \forall s \in S \}.$$  

It is clear that $R'[[S]] \subseteq R[[S]]$, and it is easy to show that $R'[[S]]$ is a multiplicatively closed subset of $R[[S]]$.

The following lemma shows that if $M[[S]]$ is an $R[[S]]$-module, then $gM[[S]] \neq M[[S]]$ for any $g \in R'[[S]]$.

**Lemma 3.6** Let $R$ be a ring, $M$ an $R$-module, $R' = \{ r \in R | rM \neq M \}$, $(S, \leq)$ a strictly ordered monoid. Then for any $g \in R'[[S]]$, $gM[[S]] \neq M[[S]]$.

**Proof.** Suppose that for any $g \in R'[[S]]$ and for an $R[[S]]$-module $M[[S]]$, $gM[[S]] = M[[S]]$. In other words, $gM[[S]] \subseteq M[[S]]$ and $M[[S]] \subseteq gM[[S]]$. Therefore, for any $\alpha \in M[[S]]$, then $\alpha \in gM[[S]]$. Hence, for any $s \in S$ and $\beta \in M[[S]]$, we obtain

$$\alpha(s) = (g\beta)(s) = \sum_{xy=s} g(x)\beta(y) \in M. \quad (3)$$

In the other side, since $g(s) \in R'$ for all $s \in S$, we have $g(s)M \neq M$. Therefore, $\sum_{xy=s} g(x)\beta(y)$ in equation (3) not necessary in $M$. Thus, a contradiction. ■

Next, we apply Proposition 2.1 to the structure of GPSM $M[[S]]$ and we obtain the following proposition.

**Proposition 3.7** Let $R$ be a ring, $M$ an $R$-module, $N$ a proper $R$-submodule of $M$, $R' = \{ r \in R | rM \neq M \}$, and $(S, \leq)$ a strictly ordered monoid. If $M[[S]]$ is an almost Noetherian $R[[S]]$-module, then $M[[S]]$ is an almost Noetherian.

**Proof.** Based on Lemma 3.4, it is clear that for any $R$-submodule $N$ of $M$, $N[[S]]$ is an $R[[S]]$-submodule of $M[[S]]$. Therefore, based on Lemma 3.6,

$$gN[[S]] \subseteq gM[[S]] \subseteq M[[S]]$$

for any $g \in R'[[S]]$. Then, $gN[[S]]$ is a proper $R[[S]]$-submodule of $M[[S]]$.

Since $M[[S]]$ is almost Noetherian, we have $gN[[S]]$ is a finitely generated $R[[S]]$-submodule of $M[[S]]$. Therefore, for any $R[[S]]$-submodule $N[[S]]$ of $M[[S]]$, there exist an element $g \in R'[[S]]$ and a finitely generated submodule $K = gN[[S]]$ of $M[[S]]$ such that

$$gN[[S]] \subseteq K \subseteq N[[S]].$$

So, $M[[S]]$ is an $R'[[S]]$-Noetherian $R[[S]]$-module. ■

Now, we apply Proposition 2.3 on the structure of GPSM $M[[S]]$, and we obtain the following proposition.
Proposition 3.8 Let $R$ be a ring, $T$ an additively and a multiplicatively closed subset of ring $R$, $M$ an $R$-module, $N$ a proper $R$-submodule of $M$, $R' = \{r \in R | rM \neq M\}$, and $(S, \leq)$ a strictly ordered monoid. If $M[[S]]$ is an almost Noetherian $R[[S]]$-module and $T[[S]] \cap R'[[S]] \neq \emptyset$, then $M[[S]]$ is $(T[[S]] \cap R'[[S]])$-Noetherian.

Proof. Based on Lemma 3.4, it is clear that for any $R$-submodule $N$ of $M$, $N[[S]]$ is an $R[[S]]$-submodule of $M[[S]]$. Based on Lemma 3.5, $T[[S]]$ is a multiplicative subset of $R[[S]]$. Therefore, based on Lemma 3.6, $gN[[S]] \subseteq gM[[S]] \subset M[[S]]$ for any $g \in (T[[S]] \cap R'[[S]]).$ Then, $gN[[S]]$ is a proper $R[[S]]$-submodule of $M[[S]]$.

Since $M[[S]]$ is almost Noetherian, we have $gN[[S]]$ is a finitely generated $R[[S]]$-submodule of $M[[S]]$. Since $T[[S]] \cap R'[[S]] \neq \emptyset$, there exist an element $g \in T[[S]] \cap R'[[S]]$ and a finitely generated submodule $K = gN[[S]]$ of $M[[S]]$ such that $gN[[S]] \subseteq K \subseteq N[[S]]$, for every $R[[S]]$-submodule $N[[S]]$ of $M[[S]]$. So, $M[[S]]$ is an $(T[[S]] \cap R'[[S]])$-Noetherian $R[[S]]$-module.

Finally, we apply Theorem 2.6 on the structure of GPSM $M[[S]]$, and we get the following theorem.

Theorem 3.9 Let $R$ be a ring, $T$ an additively and a multiplicatively closed subset of ring $R$, $M$ an $R$-module, $N$ a proper $R$-submodule of $M$, $R' = \{r \in R | rM \neq M\}$, and $(S, \leq)$ a strictly ordered monoid. If $M[[S]]$ is an almost Noetherian $R[[S]]$-module and $T[[S]] \cap R'[[S]] \neq \emptyset$, then $M[[S]]$ is $T[[S]]$-Noetherian.

Proof. Since $M[[S]]$ is an almost Noetherian $R[[S]]$-module, based on Proposition 3.8 $M[[S]]$ is an $(T[[S]] \cap R'[[S]])$-Noetherian $R[[S]]$-module. Furthermore, since $(T[[S]] \cap R'[[S]]) \subseteq T[[S]]$, then based on Lemma 2.5, we obtain $M[[S]]$ is $T[[S]]$-Noetherian.

4. Conclusion
An $R$-module $M$ is $T$-Noetherian if $M$ is almost Noetherian (a.f.g. resp.) and $T \cap R' \neq \emptyset$, for any multiplicatively closed subset $T$ of ring $R$ and $R' = \{r \in R | rM \neq M\}$.

On the structure of GPSM, $M[[S]]$ is $T[[S]]$-Noetherian if $M[[S]]$ is almost Noetherian and $T[[S]] \cap R'[[S]] \neq \emptyset$, for an additively and a multiplicatively closed subset $T$ of ring $R$ and $R' = \{r \in R | rM \neq M\}$.

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