Universal nonlinear entanglement witnesses

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We give a universal recipe for constructing nonlinear entanglement witnesses able to detect non-classical correlations in arbitrary systems of distinguishable and/or identical particles for an arbitrary number of constituents. The constructed witnesses are expressed in terms of expectation values of observables. As such they are, at least in principle, measurable in experiments.

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I. INTRODUCTION

Nonclassical correlations among subsystems of a composite quantum system, known as entanglement, can be easily characterized mathematically but, at least in the case of mixed states, in a rather ineffective way. In general, given a mixed state of a composite system it is hard to decide whether it is separable (nonentangled) or not with respect to a given partition of the whole system into subsystems. In the case of many (more than two) subsystems even determination of separability of a pure state might pose a computational problem.

It is thus desirable to construct a ‘measure of entanglement’, i.e. a function from the set of quantum states into real numbers which vanishes on separable states and takes nonzero, say positive, values for the nonseparable ones. If we impose some further, natural and reasonable conditions to be fulfilled by such a function, we may obtain a useful, quantitative characterization of ‘the amount of entanglement’ in a given state. Evidently such a quantitative measure is not unique [1] and the choice of a particular measure is dictated by a particular case we want to analyze. In the following we will mostly focus our attention on discriminating between separable and nonseparable states. In terms of an entanglement measure it means that we are interested only if it takes zero or non-zero value on a state under investigation.

To be of practical use in experiments, an entanglement measure should be measurable, i.e. it should be possible to design an experiment in which a value of the measure for a particular state can be established. Since in quantum mechanics we can measure only observables, a measurable measure should be given as an expectation value of a Hermitian operator calculated in the investigated state. It is, however, straightforward to check that there is no observable such that its expectation values vanish exactly on pure separable states [2]. Such known measures of entanglement like, e.g., Wootters’s concurrence can be expressed in terms of expectation values of antilinear operators [3, 4], and as such are not directly measurable. It is however possible to find bilinear (i.e. acting on two copies of a state) Hermitian operators for which the condition of vanishing expectation values are fulfilled only by pure separable states [5–13].

In the following we show how to construct such measures for pure states in the two-partite as well as many-partite systems in an ‘algorithmic’ way. Moreover the constructed measures allow to estimate from below the amount of entanglement for mixed states and thus provide some effective (although not always decisive) criteria of entanglement. One of our main points is to stress the unifying character of the presented approach, allowing for a uniform treatment of arbitrary number of distinguishable and identical particles (bosons and fermions).

II. PURE SEPARABLE STATES

As it is customary in quantum information theory, for which entanglement is one of the most important resources, we will investigate quantum systems in finite-dimensional Hilbert spaces (one should think about various spin or spin-like system, multilevel atoms etc.). Thus with a quantum system we associate a Hilbert space $\mathcal{H}$ isomorphic to $\mathbb{C}^N$. Customarily, vectors from $\mathcal{H}$ are called (pure) states of the system. One should, however, keep in mind that in order to give a probabilistic interpretation to amplitudes we normalize states to unit norm (length) and moreover disregard an irrelevant phase of the vector. It is thus more appropriate to think of states as points in the projective space $\mathbb{P}(\mathcal{H})$. Alternatively and equivalently, it is convenient to treat a pure state as a one-dimensional projection operator $P_\psi := |\psi\rangle\langle\psi|/\langle\psi|\psi\rangle$, freeing ourselves from the normalization and phase problems and unifying the treatment of pure and mixed states by identifying both with positive-definite operators on $\mathcal{H}$ with unit trace, where pure states are distinguished by $\rho^2 = \rho$ exhibiting their projective character.
For composed systems the Hilbert space $\mathcal{H}_{\text{comp}}$ is a tensor product of the Hilbert spaces of subsystems, $\mathcal{H}_{\text{comp}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. The states represented by simple tensors, i.e. $\mathcal{H}_{\text{comp}} \ni |\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_n\rangle$ are called pure separable states. Identifying a pure state with a one-dimensional projection operator we call such a projection separable if it projects on the direction of a separable vector. We then define separable mixed states as the convex hull of the $\mathcal{H}$ state on $|\rangle$. If $|\rangle$ is the product form; the only exception is the state of bosons must be symmetric or antisymmetric with respect to relabeling of subsystems (particles). Indeed due to the quantum correlations by the number of nonvanishing terms when expanding a wave function in terms of elementary Slater determinants was proposed earlier [18, 19], employing various quantitative characterizations of the number of non-zero terms. In the case of bipartite systems an arbitrary pure state can be represented by a vector

$$|\psi\rangle = \sum_{i,j} c_{ij} |e_i\rangle \otimes |f_j\rangle,$$

(1)

where $\{|e_i\rangle\}_{i=1}^N$ and $\{|f_j\rangle\}_{j=1}^M$ are some orthonormal bases in $\mathcal{H}_1$ (of dimension $N$) and $\mathcal{H}_2$ (of dimension $M$), and $c = (c_{kl})$ is a complex matrix. By unitary changes of the bases $|e_i\rangle = \sum_k U_{ki} |e'_k\rangle$, $|f_j\rangle = \sum_l V_{lj} |f'_l\rangle$, which amounts to transformation $c \leftrightarrow UcV^T := c'$, with $T$ standing for transposition, one can bring (1) to its Schmidt form,

$$|\psi\rangle = \sum_k \lambda_k |e'_k\rangle \otimes |f'_k\rangle, \quad \lambda_k > 0.$$

(2)

A pure state is nonentangled if and only if only one of its Schmidt coefficients $\lambda_k$ does not vanish. The coefficients $\lambda_k$ are positive real numbers squares of which are non-zero eigenvalues of $c^*c$ (or, equivalently $cc^*$), hence are easily calculable for a given $|\psi\rangle$.

The definition of separability in terms of simple tensors lacks sense for systems of identical particles when states must be symmetric or antisymmetric with respect to relabeling of subsystems (particles). Indeed due to the antisymmetrization nearly all state vectors do not have the product form; the only exception is the state of bosons each occupying the same single particle state). For example, the simplest state vector of two identical fermions reads as $|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle - |\phi_2\rangle \otimes |\phi_1\rangle =: |\phi_1\rangle \wedge |\phi_2\rangle$ which according to the usual definition is entangled [33].

In a way, states of identical particles exhibit some a priori correlations and only additional amount of correlation should be classified as entanglement. Several, not necessarily equivalent ways were proposed to identify and quantify this phenomenon. In [15, 16] a correlation measure for states of two indistinguishable fermions was proposed. It is constructed in analogy with the distinguishable particles case by employing algebraic properties of the coefficient matrix in the expansion of a state in terms of basis states. To this end one observes that a pure states of two indistinguishable fermions in an $n$-dimensional single particle space $\mathcal{H}$ can be written as

$$|w\rangle = \sum_{i,j=1}^n w_{ij} f'_i f'_j |0\rangle,$$

(3)

where $w = (w_{ij})$ is a complex antisymmetric matrix fulfilling the normalization condition $\text{tr}(w^\dagger w) = \frac{1}{2}$. Here $f'_i$ are fermionic creation operators and $|0\rangle$ is the vacuum state. A unitary transformation $U$ in the single particle space changes $w$ to $w' = U w U^T$. By choosing an appropriate unitary $U$ we can transform $w$ to its canonical, block-diagonal form:

$$w = \text{diag}[Z_1, \ldots, Z_r, Z_0], \quad Z_i = \begin{bmatrix} 0 & z_i \\ -z_i & 0 \end{bmatrix}, \quad z_i > 0,$$

(4)

where $2r$ is the rank of $w$ and $Z_0$ is the null matrix of dimension $(n-2r) \times (n-2r)$. The squares of the coefficients $z_i$ are eigenvalues of $ww^\dagger$, hence again are easily calculable for a given $w$. A state is, by definition, nonentangled if only one of $z_i$ does not vanish, i.e. only a single, elementary $2 \times 2$ Slater determinant appears in the canonical decomposition [3]. In other words a state is nonentangled if it can be written as $|\psi\rangle = f''_1 f''_2 |0\rangle$ where $f''_i |0\rangle$, $i = 1, \ldots, n$ form an orthonormal basis in the single particle Hilbert space, i.e. $|\psi\rangle$ is the antisymmetrization of a product state. Measuring quantum correlations by the number of nonvanishing terms when expanding a wave function in terms of elementary Slater determinants was proposed earlier [18, 19], employing various quantitative characterizations of the number of non-zero terms.

The above idea was thoroughly investigated in [15, 16, 20]. Generalizing the two-particle case, the relevant definition of pure nonentangled state of $n$ indistinguishable fermions (we will use the notion of `nonentanglement' rather than ‘separability' which in this context lacks its semantic sense), can be shortly summarized as follows. A fermionic state
is nonentangled if and only if it can be written as \( f_1^1 f_1^2 \cdots f_1^n |0\rangle \), i.e. it is the complete antisymmetrization of a product state. This coincides with definitions proposed in \cite{21,22}, where conclusions were reached by departing from slightly different starting points.

Similar ideas can be applied to bosons \cite{21,21,23}. A general, two-particle state in an \( n \)-dimensional single particle space,  

\[
|v\rangle = \sum_{i,j=1}^{n} v_{ij} b^\dagger_i b^\dagger_j |0\rangle, \tag{5}
\]

where \( b^\dagger_i \) are boson creation operators and \( v = (v_{ij}) \) is a complex symmetric matrix, can be transformed to a canonical form with \( v_{ij} = z_i \delta_{ij} \) by a unitary transformation in the single-particle space, amounting to \( v \mapsto UvU^T \) on the level of the coefficient matrix \[3,4]. Consequently a pure \( n \)-boson state is nonentangled if it can be written as \( |v\rangle = b_1^\dagger \cdots b_n^\dagger |0\rangle \). It should be pointed that in \cite{21} and \cite{24} (see also \cite{23}), in contrast to \cite{20,23}, a slightly broader definition of pure nonentangled boson states was proposed. In addition to the identified above, as nonentangled are also treated states which in some basis can be written as \( b_1^\dagger \cdots b_n^\dagger |0\rangle \), where all states \( b_k|0\rangle \) are orthogonal. We would like to make two comments concerning this point. Firstly, the extended class of states can be also easily described using methods proposed in this paper (as orbits of unitary groups - see below), although admittedly, characterization in terms of vanishing expectation value of some bilinear operator needs more efforts which we postpone to other occasion. Secondly, the real meaning of entanglement becomes important in particular experiments. Whether it can be exhibited (especially for indistinguishable particles) depends strongly of what and how we measure states in an occasion. We would like to conclude this section by stressing that there are no analogs of the Schmidt and Takagi decompositions in multipartite cases, therefore establishing nonentanglement demands considerably more elaborate methods \cite{16,20}.

### III. ACTIONS OF UNITARY GROUPS IN COMPOSITE SYSTEMS SPACES

In order to achieve the goals of characterizing separability \textit{via} observables let us look at the problem from a slightly different point of view. For the moment we restrict the attention to bipartite systems. It is obvious that separability of a pure state of distinguishable particles does not change if we individually evolve each subsystem in a quantum-mechanically allowed way i.e. \textit{via} a unitary transformation, \( |\phi_1\rangle \otimes |\phi_2\rangle \mapsto U_1|\phi_1\rangle \otimes U_2|\phi_2\rangle \). In the case of indistinguishable particles the same is true if we perform the same unitary transformation \( U \) in each one-particle space, \( |\phi_1\rangle \wedge |\phi_2\rangle \mapsto U|\phi_1\rangle \wedge U|\phi_2\rangle \) or \( |\phi_1\rangle \vee |\phi_2\rangle \mapsto U|\phi_1\rangle \vee U|\phi_2\rangle \) (where \( \wedge \) denotes the symmetrized tensor product). In fact we used this invariance to transform pure states to their canonical Schmidt or Takagi forms in the preceding section. We may assume that the matrices have determinant one, so they are elements of special unitary groups, since the phase of the state does not play any role. The action of the special unitary group on vectors from a Hilbert space \( \mathcal{H} \) translates in a natural way to an action on the projective space \( \mathbb{P}(\mathcal{H}) \). If we denote by \( |v\rangle \) the point in \( \mathbb{P}(\mathcal{H}) \) (the direction of \( v \)), or in other words the complex line through \( |v\rangle \), we have, by definition, \( U|v\rangle = |Uv\rangle \). This action of a unitary group on \( \mathbb{P}(\mathcal{H}) \) is transitive, i.e. any two points in \( \mathbb{P}(\mathcal{H}) \) can be connected by some unitary transformation. A straightforward conclusion is that any two nonentangled states in spaces of states of bipartite systems are connected by the above described unitary action of the direct product of two unitary groups in the case of distinguishable particles and a single unitary group for fermions and bosons. In more technical terms nonentangled state in all cases form a single orbit of an action of the group \( SU(N) \times SU(M) \) in \( \mathbb{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) for distinguishable particles or \( SU(N) \) in the case of bosons or fermions in \( \mathbb{P}(\mathcal{H} \vee \mathcal{H}) \) or \( \mathbb{P}(\mathcal{H} \wedge \mathcal{H}) \), respectively.

In order to identify uniquely the orbit in question we have to make a short excursion in the theory of representation of semisimple Lie groups (the special unitary group \( SU(N) \) is simple and the direct product of simple groups, such as \( SU(N_1) \times SU(N_2) \times \cdots \times SU(N_k) \) is semisimple \cite{24}). Let \( K \) be a semisimple real group and \( G \) its complexification, and denote by \( \mathfrak{g} \) the Lie algebra of \( G \). For \( K = SU(N) \) we have \( G = SL(N, \mathbb{C}) \) (special complex linear group in \( N \) dimensions) and \( \mathfrak{g} = \mathfrak{sl}_N(\mathbb{C}) \) (the algebra of complex \( N \times N \) matrices with vanishing trace). As a basis in \( \mathfrak{sl}_N \) we can choose \( N - 1 \) independent traceless diagonal matrices \( H_k \) which span its maximal commutative subalgebra and the matrices \( X_{ij} \) having a single nonvanishing element on in the \((i, j)\) position. The commutation relations among \( H_k \) and \( X_{ij} \) read as: \( [H_k,X_{ij}] = \alpha_{ij}(H_k)X_{ij} \) where \( \alpha_{ij} \) is some linear function on the set of diagonal matrices. An analogous construction exists for an arbitrary semisimple complex Lie algebra. We can distinguish in it a maximal
commutative subalgebra \( t \) of dimension \( r \) (called the rank of the algebra) and one-dimensional subspaces (root spaces) \( g_\alpha \) (spanned by \( X_\alpha \)) such that
\[
[H, X] = \alpha(H)X, \quad H \in t, \quad X \in g_\alpha.
\] (6)

If we choose a basis \( \{H_k\}_{k=1}^r \) in \( t \), the algebra is uniquely determined by the set of vectors \( \alpha = (\alpha(H_1), \ldots, \alpha(H_r)) \) where \( \alpha \) runs over all different root spaces. The linear form \( \alpha \) is called a root of \( g \), and the element \( X_\alpha \) corresponding to the root \( \alpha \) is a root vector. There always exists a natural symmetry: to each \( \alpha \) corresponds \( -\alpha \), as in the described above case of \( g_N(\mathbb{C}) \) treated as the algebra of \( N \times N \) traceless matrices, to each \( X_{ij} \) with, say, \( i < j \), i.e. with a single nonvanishing element in the upper-right triangle, there corresponds \( X_{ji} \) living in the lower-left triangle of the matrix for which the commutators with \( H_k \) have the opposite sign to those of the commutators of \( X_{ij} \). It means that to characterize an algebra we need only half of the root vectors - the ‘positive’ ones corresponding e.g., to upper triangular matrices which we will call positive root vectors and the corresponding roots \( \alpha \) - the positive roots \( (\alpha > 0) \).

The groups \( K \) and \( G \), as well as the Lie algebra \( g \) can be represented irreducibly in spaces of different dimensions, i.e. to each element of the group or algebra there corresponds a linear operator acting in some Hilbert space, say, \( \mathbb{C}^M \), such that the group multiplication and the Lie bracket (commutator) in the algebra are preserved. Hence, denoting by \( \pi(X) \) the representative of the Lie algebra element \( X \) we have from (1)
\[
[\pi(H), \pi(X)] = \alpha(H)\pi(X), \quad H \in t, \quad X \in g_\alpha.
\] (7)

Since the operators \( \{H_k\}_{k=1}^r \) commute, the same is true for their representatives \( \{\pi(H_k)\}_{k=1}^r \). It follows that \( \pi(H_k) \) have common eigenvectors. For each irreducible representation there exists a unique (up to a multiplicative constant) eigenvector \( |v_{\max}\rangle \) of all \( \pi(H_k) \) which is annihilated by all the representatives of the positive roots
\[
\pi(H_k)|v_{\max}\rangle = \lambda_k|v_{\max}\rangle, \quad k = 1, \ldots, r, \quad \pi(X_\alpha)|v_{\max}\rangle, \quad \alpha > 0.
\] (8)

The vector \( |v_{\max}\rangle \) is called the maximal weight-vector. An irreducible representation of \( g \) (and, in consequence, of \( G \) and \( K \)) is uniquely determined by the eigenvalues \( \lambda_k \) which we cast in a \( r \)-component vector \( \lambda = (\lambda_1, \ldots, \lambda_r) \).

It is now easy to identify the orbit of \( SU(N) \times SU(M) \) in \( \mathbb{P}(H_1 \otimes H_2) \) of the nonentangled states as the orbit through the maximal weight vector for the representation of \( SU(N) \times SU(M) \) in \( H_1 \otimes H_2 \simeq \mathbb{C}^N \otimes \mathbb{C}^M \). Indeed, since the action of \( SU(N) \) on \( \mathbb{P}(\mathbb{C}^M) \) is transitive, each point \( |v\rangle \) is on the orbit \( \{[U|v_{\max}\rangle] : U \in SU(N)\} \). Thus each nonentangled state is on the orbit of \( SU(N) \times SU(M) \) of the state \( |w_{\max}\rangle := |v_{\max}\rangle \otimes |u_{\max}\rangle \) where \( |v_{\max}\rangle \) and \( |u_{\max}\rangle \) are the highest-weight vectors of the representations of \( SU(N) \) in \( \mathbb{C}^N \) and \( SU(M) \) in \( \mathbb{C}^M \). On the other hand, it is easy to see that \( |u_{\max}\rangle \) is exactly the highest-weight vector of the irreducible representation \( SU(N) \times SU(M) \) in \( H_1 \oplus H_2 \simeq \mathbb{C}^N \otimes \mathbb{C}^M \). Similar considerations show that analogous statements are true in the cases of many particles with and/or without assumptions about their distinguishability.

Summarizing, the nonentangled states form the orbit in the projective space \( \mathbb{P}(\mathcal{H}_{\text{comp}}) \) through the highest-weight vector of an irreducible representation of a unitary group \( K \) in \( \mathcal{H}_{\text{comp}} \). For \( n \)-distinguishable particles \( K = SU(N_1) \times \cdots \times SU(N_n) \) and \( \mathcal{H}_{\text{comp}} = H_1 \otimes \cdots \otimes H_n \), \( \text{dim } H_k = N_k \), whereas for indistinguishable particles \( K = SU(N) \) and \( \mathcal{H}_{\text{comp}} = H \wedge \cdots \wedge H \) (for bosons) or \( \mathcal{H}_{\text{comp}} = H \wedge \cdots \wedge H \) (for fermions), \( \text{dim } H = N \).

There exists a nice and simple method of characterizing the orbit through highest weight vector of an irreducible representation of a semisimple group \( G \) if and only if \( \mathcal{H}_{\text{comp}} \), first we define a second-order operator
\[
C_2 := \sum_{\alpha > 0} \left( \pi(X_\alpha)\pi(X_{-\alpha}) + \pi(X_{-\alpha})\pi(X_\alpha) \right) + \sum_{i=1}^r \pi(H_i)\pi(H_i).
\] (9)

It is called the (second-order) Casimir operator; using (7) one shows that it commutes with all operators of the representation of the algebra, and thus (for an irreducible representation) is proportional to the identity operator \( I \). \[35\]. In fact one can prove that \( C_2 = \langle \lambda, \lambda + 2\delta \rangle I \), where \( \delta = \frac{1}{2} \sum_{\alpha > 0} \alpha \) is the half-sum of the positive roots and \( \langle \cdot, \cdot \rangle \) is the Euclidean scalar product in the space of \( r \)-dimensional vectors to which \( \lambda, \lambda, \text{ and } \delta \) belong \( \mathcal{H}_{\text{comp}} \).

A vector \( |\psi\rangle \) belongs to the highest-weight orbit of the irreducible representation \( \pi \) with the highest weight \( \lambda \) of a semisimple group \( G \) if and only if \( \mathcal{H}_{\text{comp}} \)
\[
L|\psi\rangle \otimes |\psi\rangle = (2\lambda + 2\delta, 2\lambda)|\psi\rangle \otimes |\psi\rangle,
\] (10)

where
\[
L = C_2 \otimes I + I \otimes C_2 + 2 \sum_{\alpha > 0} \left( \pi(X_\alpha) \otimes \pi(X_{-\alpha}) + \pi(X_{-\alpha}) \otimes \pi(X_\alpha) \right) + 2 \sum_{i=1}^r \pi(H_i) \otimes \pi(H_i).
\] (11)

Relevant properties of the operator \( L \) for the cases of two distinguishable particles, fermions, and bosons are calculated in the Appendix. For general cases of \( n \) particles, distinguishable or not, similar calculations can be also performed (with considerably more effort).
IV. MEASURABLE ENTANGLEMENT MEASURES

The operator $L$ is Hermitian and positive semidefinite and its largest eigenvalue equals $l_{max} = (2\lambda + 2\delta, 2\lambda)$. Hence $A := l_{max}I - L$ is positive semidefinite and its expectation value $\langle \psi | (\psi | A | \psi) \otimes | \psi \rangle$ vanishes exactly for a nonentangled $| \psi \rangle$. This is the desired characterization of the nonentangled states in terms of an expectation value of an observable, i.e. $A$ can be regarded as a ‘nonlinear’ entanglement witness.

The operator $A$ acts in the tensor product $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{comp}}$. There is a natural isomorphism (the Jamiołkowski isomorphism [30]) between the set of operators on $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{comp}}$ (such as the operator $A$) and the set of operators acting on density matrices (or generally linear operators) on $\mathcal{H}_{\text{comp}}$ given by

$$\Lambda(\rho) = \text{tr}_1 \left( (\rho^T \otimes I)A \right),$$

where $^T$ denotes the transposition of a matrix and $\text{tr}_1$ is the trace over the first factor of the tensor product $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{comp}}$. The most important feature of (12) is, for our purposes, that for positively semidefinite $A$ the operator $\Lambda$ is completely positive. The actual definition of complete positiveness is not important here - what is crucial is that it is equivalent to the fact that the action of $\Lambda$ can be expressed in the so-called Kraus form [1]

$$\Lambda(\rho) = \sum_{\mu=1}^{s} T_{\mu} \rho T_{\mu}^\dagger.$$

The operators $T_{\mu}$ can be expressed in terms of eigenvectors of $A$. Let

$$A = \sum_{\mu=1}^{s} \nu_\mu |v_\mu \rangle \langle v_\mu|$$

be the spectral decomposition of $A$ and $s$ the rank of $A$. Since $A$ is positive semidefinite we have $\nu_\mu > 0$ and defining $|w_\mu\rangle = \sqrt{\nu_\mu} |v_\mu\rangle$ we obtain

$$A = \sum_{\mu=1}^{s} |w_\mu\rangle \langle w_\mu|.$$

Let us now choose in $\mathcal{H}_{\text{comp}}$ an orthonormal basis $|e_i\rangle$, $i = 1, \ldots, N = \text{dim } \mathcal{H}_{\text{comp}}$, and expand the unnormalized eigenvectors $|w_\mu\rangle$ of $A$ in this basis,

$$|w_\mu\rangle = \sum_{i,j=1}^{N} w_{ij}^{(\mu)} |e_i\rangle \otimes |e_j\rangle.$$

It is now easy to find that

$$T_{\mu} = \left( \sum_{k=1}^{N} (|e_k\rangle \otimes (|e_k\rangle \otimes I)) \left( I \otimes \left( \sum_{i,j=1}^{N} w_{ij}^{(\mu)} |e_i\rangle \otimes |e_j\rangle \right) \right) = \sum_{i,j=1}^{N} w_{ij}^{(\mu)} |e_i\rangle \langle e_j|.$$

Another straightforward calculation reveals the matrix elements of $A$ in terms of $T_{\mu}$ [31]

$$\langle \psi_2 \otimes \psi_4 | A | \psi_1 \otimes \psi_3 \rangle = \sum_{\mu=1}^{s} \langle \psi_2 | T_{\mu} | \psi_4^\ast \rangle \langle \psi_1^\ast | T_{\mu}^\dagger | \psi_3 \rangle.$$

In particular, with the help of the expectation value of $A$ we can construct a pure-state entanglement measure (generalized concurrence)

$$c_A(\psi) := \langle \psi \otimes | A | \psi \otimes | \psi \rangle^{1/2} = \left( \sum_{\mu=1}^{s} |\langle \psi | T_{\mu} | \psi^\ast \rangle|^2 \right)^{1/2},$$

with the property $c_A(\alpha \psi) = |\alpha|^2 c_A(\psi)$ [6, 7].
Since the Kraus operators $T_\mu$ are calculable from the eigenvectors of $A$ and the latter are the same for $L$ which differs from $A$ by a multiple of the identity operator, we can find them by decomposing the image of $L$ under the Jamiołkowski isomorphism into the Kraus form. When calculating $T_\mu$ from eigenvectors of $L$ we should disregard those which correspond to the maximal eigenvalue $l_{\max}$ of $L$ since they belong to the zero eigenvalue of $A$ and as such do not contribute to the Kraus decomposition of its image under the Jamiołkowski isomorphism [cf. (14,15)]. In fact, we may disregard even more of the Kraus operators $T_\mu$ calculated from $L$. Indeed due to the appearance of the complex conjugate vector $|\psi^*\rangle$ in (19), the terms $\langle \psi | T_\mu | \psi^* \rangle$ vanish for antisymmetric $T_\mu$, hence for our purposes only symmetric $T_\mu$ are of interest when determining $c_A$. For example, as shown in the Appendix, only the subspace $H_1 \otimes H_2 \otimes H_2 \otimes H_1$ produced by antisymmetrizing separately the $H_1$ factors and the $H_2$ ones produces relevant Kraus operators and $A$ can be thus chosen as the projection on $H_1 \otimes H_2$ [36]. Similar analysis may be also performed in the case of bosons and fermions. In both cases $A$ is a projector on subspaces $H_B^B$ and $H_F^F$ described in details in the Appendix.

V. MIXED STATES

Every mixed state $\rho$ can be decomposed as a convex combination of pure states

$$\rho = \sum_{k=1}^K p_k |\psi_k\rangle \langle \psi_k|, \quad p_k > 0,$$

(20)

which, with $|\phi_k\rangle = \sqrt{p_k} |\psi_k\rangle$, can be rewritten as

$$\rho = \sum_{k=1}^K |\phi_k\rangle \langle \phi_k|.$$

(21)

A particular example of (20) is provided by the spectral decomposition of $\rho$,

$$\rho = \sum_{k=1}^R r_k |\eta_k\rangle \langle \eta_k|, \quad \rho |\eta_k\rangle = r_k |\eta_k\rangle, \quad R = \text{rank} \rho,$$

(22)

or, equivalently, as in (21)

$$\rho = \sum_{k=1}^R |\xi_k\rangle \langle \xi_k|, \quad |\xi_k\rangle = \sqrt{r_k} |\eta_k\rangle.$$  

(23)

In fact all other decompositions (21) can be obtained from (22) via

$$|\phi_k\rangle = \sum_{j=1}^R V_{kj} |\xi_j\rangle,$$

(24)

where $V$ is a $K \times R$ matrix fulfilling $V^\dagger V = I$ [32].

We may now define for a particular decomposition (21)

$$c_A(\{\phi_k\}) = \sum_{k=1}^K c_A(\phi_k),$$

(25)

where $c_A(\psi)$ is given by (19). It is now obvious that if minimizing $c_A(\{\xi_k\})$ over all decompositions (24) of $\rho$ gives zero then $\rho$ is nonentangled since it has a decomposition into a convex combination of nonentangled pure states. We obtain in this way a well defined measure of entanglement [3, 7] for mixed states

$$c_A(\rho) = \min \sum_k c_a(\phi_k),$$

(26)
where the minimum is taken over all decompositions \( \Phi_k \). Using \( \Phi_k \) and \( \Phi_k \) we obtain further
\[
C_A(\rho) = \min \sum_k C_A(\Phi_k) = \min \sum_k (\phi_k \otimes \phi_k | A | \phi_k \otimes \phi_k)^{1/2} \\
= \min \sum_k (V_{ki}^* V_{kj}^* V_{kl} V_{kl} \langle \xi_i \otimes \xi_j | A | \xi_i \otimes \phi_m \rangle)^{1/2} \\
= \min \sum_k \left( \sum_{\mu} (V_{ki}^* \tau_{\mu} V_{kj}^* \tau_{\mu} V_{kl}^* V_{kl}^* \langle \xi_i | T_{\mu}^* | \xi_j \rangle)^{1/2} \right) \\
= \min \sum_k \left( (V_{ki}^* \tau_{\mu} V_{kj}^* \tau_{\mu} V_{kl}^* V_{kl}^* \langle \xi_i | T_{\mu}^* | \xi_j \rangle)^{1/2} \right)
\]
where \( \tau_{\mu} \) are \( r \times r \) matrices,
\[
(\tau_{\mu})_{ij} = \langle \xi_i | T_{\mu} | \xi_j \rangle,
\]
constructed from easily calculated ingredients: the eigenvectors of \( \rho \) and the Kraus operators of \( A \) (or, equivalently, \( L \)), and the minimum is taken over all \( V \) fulfilling \( V^* V \).

We may further reduce the complexity of minimization \[3\] by employing the Cauchy-Schwartz inequality
\[
\left( \sum_{\mu} x_{\mu}^2 \right)^{1/2} \left( \sum_{\mu} y_{\mu}^2 \right)^{1/2} \geq \sum_{\mu} x_{\mu} y_{\mu}.
\]
With \( x_{\mu} = |(V_{ki}^* \tau_{\mu} V_{kj}^* \tau_{\mu} V_{kl}^* V_{kl}^* \langle \xi_i | T_{\mu}^* | \xi_j \rangle)^{1/2} \) and \( \sum_{\mu} y_{\mu} u = 1 \). We obtain thus
\[
c_A(\rho) \geq \min \sum_k \sum_{\mu} y_{\mu} |(V_{ki}^* \tau_{\mu} V_{kj}^* \tau_{\mu} V_{kl}^* V_{kl}^* \langle \xi_i | T_{\mu}^* | \xi_j \rangle)^{1/2} | \\
\geq \min \sum_k \left| V^* \left( \sum_{\mu} y_{\mu} \tau_{\mu} \right) V^* \right|_{kk}, \tag{30}
\]
where we used \( \sum_{\mu} |z_{\mu}| \geq \sum_{\mu} z_{\mu} \). The minimization over \( V \) can be now performed explicitly \[4\] giving,
\[
c_A(\rho) \geq \max \left\{ 0, \lambda_1 - \sum_{j>1} \lambda_j \right\}, \tag{31}
\]
where \( \lambda_j^2 \) are the singular values of the matrix \( T = \sum_{\mu} y_{\mu} \tau_{\mu} \). The matrix \( T \) still depends on the parameters \( \mu \) which can be chosen in an arbitrary way under the condition \( \sum_{\mu} y_{\mu}^2 = 1 \), leaving a large freedom to construct lower bounds for \( c_A(\rho) \).

VI. SUMMARY AND CONCLUSIONS

We have presented a method of discriminating pure nonentangled states for multipartite systems. The method is universal - it applies, at least in principle, to systems with arbitrary number of distinguishable as well as undistinguishable particles. Let us point at some other advantages of the proposed approach

- The defined measure of entanglement is expressed in terms of a Hermitian, albeit bilinear operator - “a nonlinear entanglement witness”, and as such is, in principle, a physically measurable quantity.

- The method, based solely on representation theory, can be easily adapted to more complicated situations, e.g. systems consisting of mixtures of bosons and fermions.

- Calculation of the generalized concurrence is made in an algorithmic way consisting of few steps: 1) identification of a relevant group of local transformation and its representation, 2) calculation of the Lichtenstein’s operator \( L \) given in terms of the operators of the Lie algebra of the group, 3) identification of relevant Kraus operators of the image of \( L \) under the Jamiołkowski isomorphism. To perform the last step one looks for the symmetric Kraus operators which are obtained from eigenvectors of \( L \) not belonging to its largest eigenvalue. The latter can be explicitly calculated from data about the group and representation in question.

The pure state generalized concurrence constructed here can be used as a basis for effective estimates of mixed-state entanglement. One example of such estimate has been presented in the last section of the paper.
Using the explicit representation (32)-(33) we obtain

\[ h_{l} = \frac{1}{\sqrt{l(l+1)}} \left( \sum_{k=1}^{l} |k\rangle|k| - l|l+1\rangle|l+1| \right) = \sum a_{lk} |k\rangle|k|, \quad l = 1, \ldots, N - 1. \tag{33} \]

The normalization of the basis elements was chosen to have \( \text{tr} H_{l}^2 = 1 = \text{tr} X_{ij} X_{ji} \). The positive roots correspond to \( i < j \) in (32).

Short calculations show that

\[ \sum_{i=1}^{N-1} a_{ik} a_{il} = \begin{cases} -\frac{1}{N} & \text{for } k \neq l \\ 1 - \frac{1}{N} & \text{for } k = l \end{cases} \tag{34} \]

Let us start with distinguishable particles. For simplicity we assume that \( H_{1} = H_{2} =: H \), \( \text{dim} \, H = N \). Remember that despite this we consider distinguishable particles, so we represent \( SU(N) \times SU(N) \) on \( H_{\text{comp}} = H_{1} \otimes H_{2} = H \otimes H \).

The corresponding representation of the Lie algebra \( \mathfrak{sl}(N) \otimes \mathfrak{sl}(N) \) is given by

\[ \pi((A,0)) = A \otimes I, \quad \pi((0,B)) = I \otimes B, \tag{35} \]

hence

\[ C_{2} = \sum_{a > 0} \left( (X_{\alpha} X_{\alpha} + X_{\alpha} X_{\alpha}) \otimes I + I \otimes (X_{\alpha} X_{\alpha} + X_{\alpha} X_{\alpha}) \right) + \sum_{j=1}^{N-1} \left( H_{j}^{2} \otimes I + I \otimes H_{j}^{2} \right). \tag{36} \]

Using the explicit representation (32)-(33) we obtain

\[ C_{2} = \left( 1 - \frac{1}{N^{2}} \right) I \otimes I. \tag{37} \]

Let \( |ijkl\rangle := |i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle \in H_{\text{comp}} \otimes H_{\text{comp}} = H \otimes H \otimes H \otimes H \) and let us introduce two operators \( S_{1}|ijkl\rangle = |kjil\rangle, \quad S_{2}|ijkl\rangle = |ilkj\rangle \). They define three orthogonal subspaces of \( H_{\text{comp}} \otimes H_{\text{comp}} \),

\[ H_{A} := \{ |\psi\rangle : S_{1} |\psi\rangle = -|\psi\rangle \} = \text{span}\{ |ijkl\rangle - |klji\rangle \}, \tag{38} \]
\[ H_{B}^{1} := \{ |\psi\rangle : S_{1} |\psi\rangle = S_{2} |\psi\rangle = -|\psi\rangle \} = \text{span}\{ |ijkl\rangle + |klji\rangle - |kjil\rangle - |ilkj\rangle \}, \tag{39} \]
\[ H_{B}^{2} := \{ |\psi\rangle : S_{1} |\psi\rangle = S_{2} |\psi\rangle = |\psi\rangle \} = \text{span}\{ |ijkl\rangle + |klji\rangle + |kjil\rangle + |ilkj\rangle \}, \tag{40} \]

with \( H_{A} \oplus H_{B}^{1} \oplus H_{B}^{2} = H_{\text{comp}} \otimes H_{\text{comp}} \). Applying \( L \) to the vectors spanning the subspaces written explicitly in (38)-(40) and using (37) and (38), we find that \( H_{A}, H_{B}^{1} \) and \( H_{B}^{2} \) are eigenspaces of \( L \) with the eigenvalues, respectively,

\[ \lambda_{A} = 2 - \frac{4}{N^{2}}, \quad \lambda_{B}^{1} = 2 - \frac{2}{N} - \frac{4}{N^{2}}, \quad \lambda_{B}^{2} = 2 + \frac{2}{N} - \frac{4}{N^{2}}. \tag{41} \]

The largest eigenvalue \( \lambda_{\text{max}} \) equals to \( \lambda_{B}^{2} \), the space \( H_{B}^{2} \) is thus in the kernel of the operator \( A \) and does not contribute Kraus operators to the generalized concurrence \( c_{A} \) (19). The Kraus operators constructed from the vectors in \( H_{A} \) take the form \( T_{ijkl} = |ij\rangle|kl\rangle - |kl\rangle|ij\rangle \) [cf. (17)], where \( |ij\rangle = |i\rangle \otimes |j\rangle \) etc. They are thus antisymmetric and, as explained in Section IV are also irrelevant to \( c_{A} \). We are left with the only ingredients given by the subspace \( H_{B}^{1} \), i.e. the Kraus operators \( T_{ijkl} = |ij\rangle|kl\rangle + |kl\rangle|ij\rangle - |kj\rangle|il\rangle - |il\rangle|kj\rangle \). Finally thus, in the definition (19) we can take as \( A \) the projection on \( H_{B}^{1} \), which is a subspace of \( H_{\text{comp}} \otimes H_{\text{comp}} = H \otimes H \otimes H \otimes H \) obtained by antisymmetrizing the first with the third factor as well as the second with the fourth ones. Calculations for \( H_{1} \neq H_{2} \) are only slightly more.
complicated and give the same result, i.e. $A$ as the projection on the subspace of $H_1 \otimes H_2 \otimes H_3 \otimes H_4$ obtained by separate antisymmetrizations of $H_1$ factors (i.e. the first and third ones) and $H_2$ (the second and fourth factor), which coincides with the results of \cite{5,7}.

In the case of two identical particles with $N$-dimensional single-particle spaces $H$ the relevant representation is that of $SU(N)$ on $H_{comp} = H \land H$ or $H_{comp} = H \lor H$. The corresponding representation of $\mathfrak{sl}_N(\mathbb{C})$ is given by

$$\pi(A) = A \otimes I + I \otimes A.$$  \hspace{1cm} (42)

As for the case of distinguishable particles we start with the calculation of the Casimir operator $C_2$. To shorten the considerations we may calculate $C_2$ on $H \otimes H$ for the representation \cite{42}, for which it is not proportional to the identity since the representation on the full tensor product is not irreducible. However, a further reduction to the irreducible invariant subspaces $H \land H$ and $H \lor H$ will lead to the desired results. Again from \cite{3} and \cite{42} we have

\[
C_2 = \sum_{\alpha > 0} \left( X_{\alpha} X_{-\alpha} \otimes I + I \otimes X_{\alpha} X_{-\alpha} + 2 X_{\alpha} \otimes X_{-\alpha} + X_{-\alpha} X_{\alpha} \otimes I + I \otimes X_{-\alpha} X_{\alpha} + 2 X_{-\alpha} \otimes X_{\alpha} \right) \nonumber \\
+ \sum_{i=1}^{N-1} \left( H_i^2 \otimes I + I \otimes H_i^2 + 2 H_i \otimes H_i \right). \hspace{1cm} (43)
\]

Using the explicit form of $X_{ij}$ and $H_i$ given by \cite{3} and \cite{43} with the help of \cite{34} we obtain

$$C_2 = \left( 1 - \frac{2}{N^2} \pm \frac{1}{N} \right) I \otimes I.$$ \hspace{1cm} (44)

where the upper sign is for the symmetric subspace, $H \lor H$, and the lower one for the antisymmetric one, $H \land H$. As it should, in each of these subspaces the Casimir operator is proportional to the identity, and in order to simplify calculations it is convenient to consider the operator

\[
L' = N(L - C_2 \otimes I - I \otimes C_2) \nonumber \\
= 2N \sum_{\alpha > 0} \left( \pi(X_{\alpha}) \otimes \pi(X_{-\alpha}) + \pi(X_{-\alpha}) \otimes \pi(X_{\alpha}) \right) + 2N \sum_{i=1}^{N} \pi(H_i) \otimes \pi(H_i). \hspace{1cm} (45)
\]

Substituting \cite{32} and \cite{33} we obtain after straightforward calculations

$$L'|ijkl = (1 - \delta_{ik})|kjl\rangle + (1 - \delta_{jl})|ljk\rangle + (1 - \delta_{jk})|ilk\rangle + (1 - \delta_{ij})|jil\rangle \nonumber \\
+ \left( -\frac{4}{N} + \delta_{ik} + \delta_{il} + \delta_{jk} + \delta_{jl} \right) |ijkl\rangle, \hspace{1cm} (46)$$

for $|ijkl\rangle \in H \otimes H \otimes H \otimes H$.

We are now in the position where we have to specify further calculations to $H_{comp} = H \lor H$ (bosons) or $H_{comp} = H \land H$ (fermions).

**A. Bosons**

We choose a basis in $H_{comp} = H \lor H$ consisting of vectors

\[
|\psi_i\rangle = |i\rangle \otimes |i\rangle, \quad j = 1, \ldots, N \hspace{1cm} (47) \\
|\psi_{ij}\rangle = |i\rangle \otimes |j\rangle + |j\rangle \otimes |i\rangle, \quad I, j = 1, \ldots, N, \quad i \neq j, \hspace{1cm} (48)
\]

and split the space $H_{comp} \otimes H_{comp} = H \lor H \otimes H \lor H$ into two parts, $H_B^B$ which is symmetric with respect to interchange of the two copies of $H_{comp}$ and $H_A^B$ which is antisymmetric with respect to this interchange. The space $H_A^B$ can be decomposed into invariant spaces of the operator $L'$,

1. $H_B^B$ spanned by vectors $|\psi_i\rangle \otimes |\psi_i\rangle$,
2. $H_A^B$ spanned by vectors $|\psi_i\rangle \otimes |\psi_j\rangle + |\psi_j\rangle \otimes |\psi_i\rangle$ and $|\psi_{ij}\rangle \otimes |\psi_{ij}\rangle, i \neq j$,
3. $H_A^A$ spanned by vectors $|\psi_i\rangle \otimes |\psi_{ij}\rangle + |\psi_{ij}\rangle \otimes |\psi_i\rangle, i \neq j$,
4. $\mathcal{H}_4^B$ spanned by vectors $|\psi_i\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_i\rangle$ and $|\psi_{ij}\rangle \otimes |\psi_{ik}\rangle + |\psi_{ik}\rangle \otimes |\psi_{ij}\rangle$, $i \neq j \neq k \neq i$.

5. $\mathcal{H}_4^B$ spanned by vectors $|\psi_{ij}\rangle \otimes |\psi_{kl}\rangle + |\psi_{kl}\rangle \otimes |\psi_{ij}\rangle$, $|\psi_{ik}\rangle \otimes |\psi_{jl}\rangle + |\psi_{jl}\rangle \otimes |\psi_{ik}\rangle$, and $|\psi_{il}\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_{il}\rangle$ with all $i, j, k, l$ different.

Using (46) we can diagonalize $L'$ in each of the above subspaces. In particular we have:

1. In the subspace $\mathcal{H}_4^B$

\[
L' |\psi_i\rangle \otimes |\psi_i\rangle = \left( 4 - \frac{4}{N} \right) |\psi_i\rangle \otimes |\psi_i\rangle = \lambda_+ |\psi_i\rangle \otimes |\psi_i\rangle.
\]

2. In the subspace $\mathcal{H}_2^B$

\[
L' \left( |\psi_i\rangle \otimes |\psi_j\rangle + |\psi_j\rangle \otimes |\psi_i\rangle \right) = -\frac{4}{N} \left( |\psi_i\rangle \otimes |\psi_j\rangle + |\psi_j\rangle \otimes |\psi_i\rangle \right) + 2|\psi_{ij}\rangle \otimes |\psi_{ij}\rangle
\]

\[
L' |\psi_{ij}\rangle \otimes |\psi_{ij}\rangle = 4 \left( |\psi_{ij}\rangle \otimes |\psi_j\rangle + |\psi_j\rangle \otimes |\psi_{ij}\rangle \right) + \left( 2 - \frac{4}{N} \right) |\psi_{ij}\rangle \otimes |\psi_{ij}\rangle.
\]

A consequent diagonalization of the $2 \times 2$ matrix,

\[
\begin{bmatrix}
-\frac{4}{N} & 2 - \frac{4}{N} \\
2 & 2 - \frac{4}{N}
\end{bmatrix},
\]

reveals that

\[
L' |\Psi_{ij}^1\rangle = \left( 4 - \frac{4}{N} \right) |\Psi_{ij}^1\rangle = \lambda_+ |\Psi_{ij}^1\rangle,
\]

\[
L' |\Psi_{ij}^2\rangle = \left( 2 - \frac{4}{N} \right) |\Psi_{ij}^2\rangle = \lambda_- |\Psi_{ij}^2\rangle,
\]

where

\[
|\Psi_{ij}^1\rangle = |\psi_i\rangle \otimes |\psi_j\rangle + |\psi_j\rangle \otimes |\psi_i\rangle + |\psi_{ij}\rangle \otimes |\psi_{ij}\rangle
\]

\[
|\Psi_{ij}^2\rangle = -2|\psi_i\rangle \otimes |\psi_j\rangle - 2|\psi_j\rangle \otimes |\psi_i\rangle + |\psi_{ij}\rangle \otimes |\psi_{ij}\rangle.
\]

3. In the subspace $\mathcal{H}_3^B$

\[
L' \left( |\psi_i\rangle \otimes |\psi_{ij}\rangle + |\psi_{ij}\rangle \otimes |\psi_i\rangle \right) = \left( 4 - \frac{4}{N} \right) \left( |\psi_i\rangle \otimes |\psi_{ij}\rangle + |\psi_{ij}\rangle \otimes |\psi_i\rangle \right)
\]

\[
= \lambda_+ \left( |\psi_i\rangle \otimes |\psi_{ij}\rangle + |\psi_{ij}\rangle \otimes |\psi_i\rangle \right)
\]

4. In the subspace $\mathcal{H}_3^B$

\[
L' \left( |\psi_i\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_i\rangle \right) = \left( -\frac{4}{N} \right) \left( |\psi_i\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_i\rangle \right)
\]

\[
+ 2 \left( |\psi_{ij}\rangle \otimes |\psi_{ik}\rangle + |\psi_{ik}\rangle \otimes |\psi_{ij}\rangle \right)
\]

\[
L' \left( |\psi_{ij}\rangle \otimes |\psi_{ik}\rangle + |\psi_{ik}\rangle \otimes |\psi_{ij}\rangle \right) = 4 \left( |\psi_{ij}\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_{ij}\rangle \right)
\]

\[
+ \left( 2 - \frac{4}{N} \right) \left( |\psi_{ij}\rangle \otimes |\psi_{ik}\rangle + |\psi_{ik}\rangle \otimes |\psi_{ij}\rangle \right).
\]

Hence, like in 2. above,

\[
L' |\Psi_{ijk}^3\rangle = \left( 4 - \frac{4}{N} \right) |\Psi_{ijk}^3\rangle = \lambda_+ |\Psi_{ijk}^3\rangle,
\]

\[
L' |\Psi_{ijk}^4\rangle = \left( 2 - \frac{4}{N} \right) |\Psi_{ijk}^4\rangle = \lambda_- |\Psi_{ijk}^4\rangle,
\]

where

\[
|\Psi_{ijk}^3\rangle = |\psi_i\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_i\rangle + |\psi_{ij}\rangle \otimes |\psi_{ik}\rangle + |\psi_{ik}\rangle \otimes |\psi_{ij}\rangle
\]

\[
|\Psi_{ijk}^4\rangle = -2|\psi_i\rangle \otimes |\psi_{jk}\rangle - 2|\psi_{jk}\rangle \otimes |\psi_i\rangle + |\psi_{ij}\rangle \otimes |\psi_{ik}\rangle + |\psi_{ik}\rangle \otimes |\psi_{ij}\rangle.
\]
5. In the subspace $\mathcal{H}_B^B$

\[
L'(|\psi_{ij}\rangle \otimes |\psi_{kl}\rangle + |\psi_{kl}\rangle \otimes |\psi_{ij}\rangle) = -\frac{4}{N}((|\psi_{ij}\rangle \otimes |\psi_{kl}\rangle + |\psi_{kl}\rangle \otimes |\psi_{ij}\rangle) + 2(|\psi_{ul}\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_{ul}\rangle)
\]

\[
L'(|\psi_{ik}\rangle \otimes |\psi_{jl}\rangle + |\psi_{jl}\rangle \otimes |\psi_{ik}\rangle) = 2(|\psi_{kl}\rangle \otimes |\psi_{ij}\rangle + |\psi_{ij}\rangle \otimes |\psi_{kl}\rangle) - \frac{4}{N}(|\psi_{ik}\rangle \otimes |\psi_{jl}\rangle + |\psi_{jl}\rangle \otimes |\psi_{ik}\rangle) + 2(|\psi_{il}\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_{il}\rangle)
\]

\[
L'(|\psi_{il}\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_{il}\rangle) = 2(|\psi_{kl}\rangle \otimes |\psi_{ij}\rangle + |\psi_{ij}\rangle \otimes |\psi_{kl}\rangle) + 2(|\psi_{ik}\rangle \otimes |\psi_{jl}\rangle + |\psi_{jl}\rangle \otimes |\psi_{ik}\rangle) - \frac{4}{N}(|\psi_{il}\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_{il}\rangle).
\]

The relevant matrix to diagonalize reads as

\[
\begin{bmatrix}
-\frac{11}{4} & 2 & 2 \\
2 & -\frac{1}{4} & 2 \\
2 & 2 & -\frac{1}{4}
\end{bmatrix},
\]

which leads to

\[
L'|\Psi_{ijkl}^1\rangle = (4 - \frac{4}{N})|\Psi_{ijkl}^1\rangle = \lambda_+|\Psi_{ijkl}^1\rangle,
\]

\[
L'|\Psi_{ijkl}^2\rangle = (-2 - \frac{4}{N})|\Psi_{ijkl}^2\rangle = \lambda_-|\Psi_{ijkl}^2\rangle,
\]

\[
L'|\Psi_{ijkl}^3\rangle = (-2 - \frac{4}{N})|\Psi_{ijkl}^3\rangle = \lambda_-|\Psi_{ijkl}^3\rangle,
\]

with

\[
|\Psi_{ijkl}^1\rangle = |\psi_{ij}\rangle \otimes |\psi_{kl}\rangle + |\psi_{kl}\rangle \otimes |\psi_{ij}\rangle + |\psi_{ik}\rangle \otimes |\psi_{jl}\rangle + |\psi_{jl}\rangle \otimes |\psi_{ik}\rangle
\]

\[
|\Psi_{ijkl}^2\rangle = |\psi_{ij}\rangle \otimes |\psi_{kl}\rangle + |\psi_{kl}\rangle \otimes |\psi_{ij}\rangle + |\psi_{ul}\rangle \otimes |\psi_{jk}\rangle + |\psi_{jk}\rangle \otimes |\psi_{ul}\rangle
\]

\[
|\Psi_{ijkl}^3\rangle = |\psi_{ij}\rangle \otimes |\psi_{kl}\rangle + |\psi_{kl}\rangle \otimes |\psi_{ij}\rangle - |\psi_{ul}\rangle \otimes |\psi_{jk}\rangle - |\psi_{jk}\rangle \otimes |\psi_{ul}\rangle.
\]

To finish the calculations in the bosonic case we have to consider the antisymmetric part $\mathcal{H}_A^B = \mathcal{H} \wedge \mathcal{H} \wedge \mathcal{H} \wedge \mathcal{H}$ of the space $\mathcal{H}_{comp} \otimes \mathcal{H}_{comp}$. It decomposes into invariant subspaces

1. $\mathcal{H}_6^B$ spanned by vectors $|\tilde{\Psi}_{ij}^1\rangle = |\psi_{ij}\rangle \otimes |\psi_{j}\rangle - |\psi_{j}\rangle \otimes |\psi_{i}\rangle$, $i \neq j$,

2. $\mathcal{H}_7^B$ spanned by vectors $|\tilde{\Psi}_{ij}^2\rangle = |\psi_{ij}\rangle \otimes |\psi_{j}\rangle - |\psi_{j}\rangle \otimes |\psi_{i}\rangle$, $i \neq j$,

3. $\mathcal{H}_8^B$ spanned by vectors $|\tilde{\Psi}_{ij}^3\rangle = |\psi_{ij}\rangle \otimes |\psi_{j}\rangle - |\psi_{j}\rangle \otimes |\psi_{i}\rangle$ and $|\tilde{\Psi}_{ij}^4\rangle = |\psi_{ij}\rangle \otimes |\psi_{j}\rangle - |\psi_{j}\rangle \otimes |\psi_{i}\rangle$, $i \neq j \neq k$,

4. $\mathcal{H}_9^B$ spanned by vectors $|\tilde{\Psi}_{ij}^5\rangle = |\psi_{ij}\rangle \otimes |\psi_{j}\rangle - |\psi_{j}\rangle \otimes |\psi_{i}\rangle$, $|\tilde{\Psi}_{ij}^6\rangle = |\psi_{ik}\rangle \otimes |\psi_{jl}\rangle - |\psi_{jl}\rangle \otimes |\psi_{ik}\rangle$, and $|\tilde{\Psi}_{ij}^7\rangle = |\psi_{ul}\rangle \otimes |\psi_{jk}\rangle - |\psi_{jk}\rangle \otimes |\psi_{ul}\rangle$ with all $i, j, k, l$ different.

From [10] we obtain straightforwardly

\[
L'|\tilde{\Psi}\rangle = -\frac{4}{N}|\tilde{\Psi}\rangle = \lambda|\tilde{\Psi}\rangle
\]

for all the subspaces, i.e. for $|\tilde{\Psi}\rangle = |\tilde{\Psi}_{ij}^{1,2}\rangle$, $|\tilde{\Psi}_{ij}^{1,2}\rangle$, and $|\tilde{\Psi}_{ij}^{1,2,3}\rangle$.

Furthermore, we easily see that if we do not assume that all $i, j, k, l$ are different in the definitions of $|\Psi_{ijkl}^{1,2,3}\rangle$ and $|\tilde{\Psi}_{ijkl}^{1,2,3}\rangle$ (adopting the notation $|\psi_{ii}\rangle = 2|ii\rangle$), we can express all vectors spanning the subspaces $\mathcal{H}_i^B$, $i = 1, \ldots, 9$, in terms of $|\Psi_{ijkl}^{1,2,3}\rangle$ and $|\tilde{\Psi}_{ijkl}^{1,2,3}\rangle$ (for example, $|\psi_{i}\rangle \otimes |\psi_{ij}\rangle + |\psi_{ij}\rangle \otimes |\psi_{i}\rangle = \frac{1}{\lambda}|\Psi_{iiij}\rangle$ etc.).
To sum it up, the Hilbert space $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{comp}} = \mathcal{H} \vee \mathcal{H} \otimes \mathcal{H} \vee \mathcal{H}$ splits into three eigenspaces of $L'$,

$$\mathcal{H}^0_+ = \text{span}\{\vert \Phi_{ij}^1 \rangle\}.$$  

[see Equations (49), (53), (55), (57), (60), (62), (68), and (71)],

$$\mathcal{H}^B = \text{span}\{\vert \Phi_{ij}^2 \rangle, \vert \Phi_{ij}^3 \rangle\},$$  

[see Equation (74)], and

$$\mathcal{H}^B_- = \text{span}\{\vert \Phi_{ij}^2 \rangle, \vert \Phi_{ij}^3 \rangle\},$$  

for all $i, j, k, l$, not necessarily different - see the remark above, [see Equations (54), (56), (61), (63), (69), (70), (72), and (73)].

They correspond to three different eigenvalues of $L'$, respectively, $\lambda_+ = 4 - \frac{4}{N}$, $\lambda = -\frac{4}{N}$, and $\lambda_- = -2 - \frac{4}{N}$. The subspaces (75-77) are also eigenspaces of $L = \frac{1}{2}L' + 2 \left(1 - \frac{2}{N} + \frac{1}{N^2}\right)I$, and the corresponding eigenvalues of $L$ are $\lambda_+ = 2 - \frac{8}{N^2} + \frac{1}{N^2}$, $\lambda = 2 - \frac{8}{N^2} + \frac{2}{N^2}$, and $\lambda_- = 2 - \frac{8}{N^2}$. The largest one, $l_{\text{max}} = \lambda_+^L$, corresponds to the kernel of the operator $A = l_{\text{max}}I - L$, and vectors in the subspace $\mathcal{H}^B$ give antisymmetric Kraus operators $T_{ij}$. Hence as $A$ we may take the projection on $\mathcal{H}^B$ and construct the relevant Kraus operators from vectors in this space.

## B. Fermions

Calculations for fermions closely follow the bosonic case, so we omit most of details. In the space $\mathcal{H}_{\text{comp}} = \mathcal{H} \wedge \mathcal{H}$ we choose a basis $\{\vert \phi_{ij} \rangle \} = \{i \otimes j - j \otimes i, i \neq j\}$. As in the bosonic case we decompose $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{comp}}$ into the symmetric $\mathcal{H}_{\text{sym}}^F = \mathcal{H} \wedge \mathcal{H} \vee \mathcal{H} \wedge \mathcal{H}$, and antisymmetric, $\mathcal{H}_{\text{asym}}^F = \mathcal{H} \wedge \mathcal{H} \wedge \mathcal{H} \wedge \mathcal{H}$, parts. The symmetric part can be further split into invariant subspaces of $L'$.

1. $\mathcal{H}^F_1$ spanned by vectors $\vert \Phi_{ij}^1 \rangle = \vert \phi_{ij} \rangle \otimes \vert \phi_{ij} \rangle$ for which

$$L' \vert \Phi_{ij}^1 \rangle = \left(2 - \frac{4}{N}\right) \vert \Phi_{ij}^1 \rangle = \lambda' \vert \Phi_{ij}^1 \rangle$$

2. $\mathcal{H}^F_2$ spanned by vectors $\vert \Phi_{ij}^2 \rangle = \vert \phi_{ij} \rangle \otimes \vert \phi_{ik} \rangle + \vert \phi_{ik} \rangle \otimes \vert \phi_{ij} \rangle$, $i \neq j$, $j \neq k$ for which again

$$L' \vert \Phi_{ij}^2 \rangle = \lambda' \vert \Phi_{ij}^2 \rangle$$

3. $\mathcal{H}^F_3$ spanned by vectors $\vert \phi_{ij} \rangle \otimes \vert \phi_{kl} \rangle + \vert \phi_{kl} \rangle \otimes \vert \phi_{ij} \rangle$, $\phi_{ik} \otimes \phi_{jl} + \phi_{jl} \otimes \phi_{ik}$, $\phi_{il} \otimes \phi_{jk} + \phi_{jk} \otimes \phi_{il}$ with all $i, j, k, l$ different.

Here we diagonalize a set of equations

$$L' \left(\vert \phi_{ij} \rangle \otimes \vert \phi_{kl} \rangle + \vert \phi_{kl} \rangle \otimes \vert \phi_{ij} \rangle\right) = -\frac{4}{N} \left(\vert \phi_{ik} \rangle \otimes \vert \phi_{jl} \rangle + \vert \phi_{jl} \rangle \otimes \vert \phi_{ik} \rangle\right)$$

$$+ 2 \left(\vert \phi_{il} \rangle \otimes \vert \phi_{jk} \rangle + \vert \phi_{jk} \rangle \otimes \vert \phi_{il} \rangle\right)$$

$$L' \left(\vert \phi_{ik} \rangle \otimes \vert \phi_{jl} \rangle + \vert \phi_{jl} \rangle \otimes \vert \phi_{ik} \rangle\right) = 2 \left(\vert \phi_{il} \rangle \otimes \vert \phi_{jk} \rangle + \vert \phi_{jk} \rangle \otimes \vert \phi_{il} \rangle\right)$$

$$- \frac{4}{N} \left(\vert \phi_{ik} \rangle \otimes \vert \phi_{jl} \rangle + \vert \phi_{jl} \rangle \otimes \vert \phi_{ik} \rangle\right)$$

$$L' \left(\vert \phi_{il} \rangle \otimes \vert \phi_{jk} \rangle + \vert \phi_{jk} \rangle \otimes \vert \phi_{il} \rangle\right) = 2 \left(\vert \phi_{il} \rangle \otimes \vert \phi_{jk} \rangle + \vert \phi_{jk} \rangle \otimes \vert \phi_{il} \rangle\right)$$

$$+ 2 \left(\vert \phi_{il} \rangle \otimes \vert \phi_{jk} \rangle + \vert \phi_{jk} \rangle \otimes \vert \phi_{il} \rangle\right)$$

With all $i, j, k, l$ different.
producing

\[ L'\Phi_{ijkl} = \left( -4 - \frac{4}{N} \right) \Phi_{ijkl} = \lambda'_- \Phi_{ijkl}, \tag{83} \]
\[ L'\Phi_{ijkl} = \left( 2 - \frac{4}{N} \right) \Phi_{ijkl} = \lambda'_+ \Phi_{ijkl}, \tag{84} \]
\[ L'\Phi_{ijkl} = \left( 2 - \frac{4}{N} \right) \Phi_{ijkl} = \lambda'_+ \Phi_{ijkl}, \tag{85} \]

with

\[ |\Phi_{ijkl}\rangle = -|\phi_{ij} \rangle \otimes |\phi_{kl}\rangle - |\phi_{kl} \rangle \otimes |\phi_{ij}\rangle + |\phi_{ik} \rangle \otimes |\phi_{jl}\rangle + |\phi_{il} \rangle \otimes |\phi_{jk}\rangle + |\phi_{il} \rangle \otimes |\phi_{jk}\rangle \tag{86} \]
\[ |\Phi_{ijkl}\rangle = |\phi_{ij} \rangle \otimes |\phi_{kl}\rangle + |\phi_{kl} \rangle \otimes |\phi_{ij}\rangle + |\phi_{il} \rangle \otimes |\phi_{jk}\rangle + |\phi_{il} \rangle \otimes |\phi_{jk}\rangle \tag{87} \]
\[ |\Phi_{ijkl}\rangle = |\phi_{ij} \rangle \otimes |\phi_{kl}\rangle + |\phi_{kl} \rangle \otimes |\phi_{ij}\rangle + |\phi_{il} \rangle \otimes |\phi_{jk}\rangle + |\phi_{il} \rangle \otimes |\phi_{jk}\rangle \tag{88} \]

The antisymmetric part \( \mathcal{H}^F \) splits into invariant subspaces

1. \( \mathcal{H}_{+}^F \) spanned by vectors \( \tilde{\Phi}_{ijkl}^1 = |\phi_{ij} \rangle \otimes |\phi_{kl}\rangle - |\phi_{ik} \rangle \otimes |\phi_{ij}\rangle, j \neq k, \)

2. \( \mathcal{H}_{-}^F \) spanned by vectors \( \tilde{\Phi}_{ijkl}^2 = |\phi_{ij} \rangle \otimes |\phi_{kl}\rangle - |\phi_{ik} \rangle \otimes |\phi_{ij}\rangle, \tilde{\Phi}_{ijkl}^3 = |\phi_{ik} \rangle \otimes |\phi_{jl}\rangle - |\phi_{jl} \rangle \otimes |\phi_{ik}\rangle, \) and \( \tilde{\Phi}_{ijkl}^4 = |\phi_{il} \rangle \otimes |\phi_{jk}\rangle - |\phi_{jk} \rangle \otimes |\phi_{il}\rangle \) with all \( i, j, k, l \) different.

In both subspaces \( L' \) acts as the multiplication by \( \lambda' = -\frac{4}{N} \).

As in the bosonic case, we can express all vectors spanning subspaces \( \mathcal{H}_{+}^F, \mathcal{H}_{-}^F, i = 1, \ldots, 5, \) using only \( |\Phi_{ijkl}^{1,2,3} \rangle \) and \( |\Phi_{ijkl}^{1,2,3} \rangle \) (here we have \( |\phi_{il}\rangle = 0 \)).

Therefore \( L' \) has three different eigenvalues \( \lambda'_+, \lambda'_- \) corresponding to three eigenspaces,

\[ \mathcal{H}_{+}^F = \text{span}\{ |\Phi_{ijkl}^2 \rangle, |\Phi_{ijkl}^3 \rangle \}, \tag{89} \]
\[ \mathcal{H}_{-}^F = \text{span}\{ |\Phi_{ijkl}^1 \rangle, |\Phi_{ijkl}^2 \rangle, |\Phi_{ijkl}^3 \rangle \}, \tag{90} \]

and

\[ \mathcal{H}^F = \text{span}\{ |\Phi_{ijkl}^1 \rangle \}, \tag{91} \]

for all \( i, j, k, l \), not necessarily different. These are also the eigenspaces of \( L = \frac{1}{N} L' + 2 \left( 1 - \frac{4}{N^2} \right) I \). The subspace \( \mathcal{H}_{+}^F \) corresponding to the largest eigenvalue of \( L \), \( I_{\text{max}} = \lambda'^+ = 2 - \frac{8}{N^2} \), is in the kernel of \( A \), whereas \( \mathcal{H}_{-}^F \) corresponding to the middle eigenvalue \( \lambda'^- = 2 - \frac{8}{N^2} - \frac{2}{N} \) of \( L \) gives rise to antisymmetric Kraus operators. The operator \( A \) can be thus chosen as the projection on \( \mathcal{H}_{-}^F \).

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[33] Here and in the following we will not pay attention to the normalization of vectors since it does not play any crucial role, moreover is taken into account if we pass to the projective space.
[34] This algebraic fact is known as the Takagi factorization theorem, see [17].
[35] We use the same symbol $I$ to denote the identity operator in an arbitrary space. From the context it is usually obvious in which space $I$ actually acts.
[36] If we use the equivalence $H_1 \otimes H_2 \otimes H_1 \otimes H_2 \simeq H_1 \otimes H_1 \otimes H_2 \otimes H_2$, the subspace $H_S^1$ can be identified with $H_1 \otimes H_1 \otimes H_2 \otimes H_2$. 