Abstract. We discuss solution generating techniques treating stationary and axially symmetric metrics in the presence of a cosmological constant. Using the recently found extended form of Ernst’s complex equation, which takes into account the cosmological constant term, we propose an extension of spheroidal coordinates adapted to asymptotically de-Sitter and anti de-Sitter static spacetimes. In the absence of a cosmological constant we show in addition that any higher dimensional metric parametrised by a single angular momentum can be given by a 4 dimensional solution and Weyl potentials parametrising the extra Killing directions. We explicitly show how a stationary and a static axially symmetric spacetime solution in 4 dimensions can be added together to give a 5 dimensional stationary and axisymmetric solution.

1. Introduction
In recent years there has been an increasing effort in finding exact solutions of higher dimensional gravity [1], [2], [3], [4]. In particular, since the pioneering work of Maldacena [5] bringing into perspective the adS/CFT correspondence some effort has been devoted to understanding the effect of the cosmological constant term in Einstein gravity [7], [6], [8]. Such studies have been motivated by string theory and more recently braneworld gravity. Using the adS/CFT correspondence in the context of braneworlds [9], intriguing relations between bulk higher dimensional black holes and their 4-dimensional quantum versions (see also [10]) have been put forward. Exact solutions would be rather useful in this context giving straightforward checks for adS/CFT and possibly providing useful information about the quantum description of black holes. However not surprisingly they are very difficult to find, in particular in the presence of a cosmological constant term, which is vital for an adS/CFT description. These efforts in higher dimensional gravity also serve to advance our understanding and solution generating techniques of 4 dimensional general relativity where the presence of a cosmological constant term has not been studied in great detail.

In a recent paper [8], rotating spacetimes of axial symmetry were studied in the presence of a cosmological constant. Classical techniques, as that of Lewis-Papapetrou [15], where developed
to include the cosmological constant term. In this letter we will focus on the Ernst equation [16] as well as a novel solution generating method developed there. The Ernst equation, which was extended for a cosmological constant term [8], will permit us here to propose an extension of spheroidal coordinates to adS/dS static black holes. These coordinates for $\Lambda = 0$ have been shown to be very useful in the study of stationary metrics. In particular Ernst [16] showed how one could generate rather simply Kerr’s solution starting from Schwarzschild. The extension of asymptotically flat coordinate systems to asymptotically $\Lambda \neq 0$ coordinates maybe very important in order to find novel stationary solutions such as the adS version of the black ring solution [2].

The latter solution generating method on the other hand will allow us, using Weyl’s classical GR formalism, to extend 4 dimensional solutions to higher dimensional ones. In particular we will show how by literally adding together 4 dimensional metrics we can construct 5 dimensional ones.

In 4 dimensional general relativity Einstein’s equations in the vacuum $R_{AB} = 0$, guarantee that any locally static and axially-symmetric metric can be written as

$$ds^2 = -e^{2\lambda} dt^2 + e^{-2\lambda} [e^{2\chi}(dr^2 + dz^2)]_1$$

Since the field equation for $\alpha$ reads,

$$\Delta \alpha = 0$$

by a suitable conformal coordinate transformation we can set $\alpha = R$ thus obtaining the Weyl form [12]

$$ds^2 = -e^{2\lambda} dt^2 + e^{-2\lambda} [R^2 d\varphi^2 + e^{2\chi}(dR^2 + dZ^2)]_2$$

where $\lambda$, the Weyl potential, and $\chi$ depend on $R, Z$ and satisfy the field equations,

$$\frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2} \lambda = 0.$$ (4)

$$\frac{\partial}{\partial R} \chi = R \left[ \frac{1}{R} \frac{\partial}{\partial R} \lambda - \frac{\partial}{\partial Z} \lambda \right] \quad \text{and} \quad \frac{\partial}{\partial Z} \chi = 2 R \frac{\partial}{\partial R} \lambda$$ (5)

Equation (4) is the linear Laplace equation written in three dimensional cylindrical coordinates. Given that (4) is a linear equation a general solution can be found by simple separation of variables and imposing adequate asymptotic boundary conditions [13]. In a more pictorial manner one can view Weyl potentials as Newtonian sources in three dimensions and can even superpose them creating new solutions from known ones. Once the Weyl potential has been specified, one evaluates the $\chi$ field by direct integration from (5). Equations (5) actually carry the full non-linearity of Einstein’s equations. Weyl potentials are not unique and are associated to the patch of coordinates we are using. Flat space for example has Weyl potentials (modulo a constant) given by,

$$\lambda = 0, \quad \lambda = \ln R, \quad \lambda = \frac{1}{2} \ln (\sqrt{R^2 + Z^2} + Z)$$ (6)

where in particular the last one is adapted to an accelerating Rindler patch. A Schwarzschild black hole of mass $M$ has Weyl potential given by

$$\lambda = \frac{1}{2} \ln \left( \frac{R_+ + R_- - 2M}{R_+ + R_- + 2M} \right)$$ (7)

where $r_\pm = R^2 + (z \pm m)^2$. As we mentioned one can superpose black hole Weyl potentials obtaining multiple black hole solutions [14]. Typically when sources are superposed conical
singularities for $R \rightarrow 0$ appear and are interpreted as struts holding, or strings pulling, the sources apart in a static equilibrium.

In essence, Weyl components adapt the problem of finding solutions to a three dimensional flat coordinate system. This system is convenient for analysing the solutions in parallel with their Newtonian sources but is not always tailored to the solutions themselves. Quite often it is more suitable to adapt the coordinate system to the Weyl potential $\lambda$ rather than $\alpha$. This is one of the ideas behind spheroidals although they were not initially introduced or defined this way. These coordinates were first discussed in the context of axial symmetric spacetimes by Zipoy\textsuperscript{1} [19] (see also [3] for $D \geq 4$). Consider polar-like coordinates $(u, \psi)$ but with hyperbolae as radial functions, that is

\[
Z = \cosh u \cos \psi, \\
R = \sinh u \sin \psi,
\]

so that in the $(R, Z)$ plane $\psi = \text{const}$ curves are hyperboloids and $u = \text{const}$ are ellipsoids. On setting $x = \cosh u$ and $y = \cos \psi$, the coordinate system becomes anew symmetric in $x$ and $y$.

Consider a Schwarzschild black hole: the standard metric

\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\Omega^2
\]

can be rewritten in Weyl coordinates $(r, z)$ of (1) setting $r^2/2M = \cosh^2(r/2)$ and $\theta = z$. The conformal transformation to (3) and (8) gives

\[
e^{2\lambda} = \frac{x - 1}{x + 1}
\]

so that in spheroidals the Weyl potential for (9) is rather simple. We will come back to this point in a moment but first let us switch on rotation.

Lewis and Papapetrou [15] generalised the approach of Weyl to stationary and axisymmetric solutions in vacuum. After a conformal transformation, the metric takes the Lewis-Papapetrou form

\[
ds^2 = -e^{2\lambda} (dt + A\varphi)^2 + e^{-2\lambda} \left[R^2d\varphi^2 + e^{2\lambda}(dR^2 + dZ^2)\right],
\]

which differs from the static form by the additional component $A = A(R, Z)$. Note that $\partial_t$ is no longer a static but rather a stationary locally timelike Killing vector field. For the metric (11), Ernst [16] pointed out an interesting reformulation of Einstein’s equations for $A$ and $\lambda$, which read respectively

\[
\partial_R \left(\frac{e^{A}}{R} \partial_R A\right) + \partial_Z \left(\frac{e^{A}}{R} \partial_Z A\right) = 0, \\
\left(\partial^2_R + \frac{1}{R} \partial_R + \partial^2_Z\right) \lambda = \frac{e^{4\lambda}}{2R^2} \left[(\partial_R A)^2 + (\partial_Z A)^2\right].
\]

Indeed introduce an auxiliary field, $\omega$, defined by

\[
(-\partial_Z \omega, \partial_R \omega) = \frac{e^{4\lambda}}{R} (\partial_R A, \partial_Z A),
\]

and the complex function

\[
\mathcal{E} = e^{2\lambda} + i\omega
\]

\[\text{According to [19] such coordinates were used to describe the exact Newtonian gravitational field of the Earth.}\]
then satisfies the differential equation

\[
\frac{1}{R} \nabla \cdot (R \nabla \mathcal{E}) = \frac{(\nabla \mathcal{E})^2}{\text{Re}(\mathcal{E})} \tag{15}
\]

where \( \nabla = (\partial_R, \partial_Z) \). This complex partial differential equation is known as the Ernst equation [16]. Its real and imaginary part are exactly (12). In this language, the Weyl potential \( \lambda \) is simply given by the real part of the Ernst potential \( \mathcal{E} \), whereas rotation is embodied by a non-trivial \( \omega \) (or \( A \)).

Using the symmetries of complex functions, several methods have been proposed to obtain solutions of the Ernst equation (15) and hence to generate new solutions (see [11], [16], [17] and references within). An elegant example appeared in Ernst's original paper [16], namely a simple method to obtain the Kerr solution from the Schwarzschild solution using spheroidal coordinates. Indeed, consider the Mobius transform,

\[
\mathcal{E} = \frac{1}{1 + \frac{1}{z}} \tag{16}
\]

defining a new potential \( \xi \) which solves

\[
\frac{1}{\alpha} \nabla \cdot (\alpha \nabla \xi) = \frac{2\xi^i (\nabla \xi)^2}{|\xi|^2 - 1}, \tag{17}
\]

where a star denotes complex conjugation. It follows immediately from (10) and (14) that \( \xi = x \) for (9). Hence our transformed Ernst potential \( \xi \) is now the new 'radial' coordinate \( x \). This is in contrast to (1) where \( \alpha = R \) and the Ernst potential is given by (7). In other words we have adapted the coordinate system to the real part of the black hole Ernst potential. We will be using this as our starting definition for extending spheroidal coordinates when we will switch on the cosmological constant term. Just in order to complete the discussion on the construction of Kerr given the \( x \leftrightarrow y \) symmetry, \( \xi = y \) is also solution of (15), as is \( \xi = x \sin \vartheta + iy \cos \vartheta \). It turns out that this is nothing other than the Ernst potential of the Kerr black hole, where \( \sin \vartheta = a/M \) is the ratio between the angular momentum parameter and the mass of the black hole [16].

When we consider a \( D \)-dimensional spacetime with a cosmological constant, a rotating metric of axial symmetry and with a single component angular momentum can be conveniently written in the form,

\[
d^2s = e^{2\nu} \alpha^{-\frac{D-3}{2}} (dt^2 + dz^2) + \alpha \frac{2}{\Lambda - 2} \left[ e^{-\sqrt{\frac{\Lambda - 2}{\Lambda - D + 2}}} \Psi^0 \left[ -e^{\frac{\nu}{2} (dt + \Lambda \varphi)^2} + e^{\frac{\nu}{2} d\varphi^2} \right] + e^{\sqrt{\frac{\Lambda - 2}{\Lambda - D + 2}}} \Psi^0 \sum_{i=1}^{D-1} e^{2\Psi_i (dx_i)^2} \right]. \tag{18}
\]

We have \( (D - 2) \) Killing vectors but as for \( D = 4, \Lambda = 0 \) only two of them do not commute, \( \partial_t \) and \( \partial_\varphi \). The fields \( \alpha, \nu, \Omega \) and \( \Psi_\mu \) again depend on \( r \) and \( z \). These metric components extend

\[ * \text{In this representation of the potential, (17) is invariant under the complex transformation } \xi \rightarrow \xi e^{i\vartheta} \text{ for any phase } \vartheta \in R \text{ and stationary solutions can be easily generated from static ones (see [16] and [8] for more details).} \]
the Lewis-Papapetrou form of the previous section [8]. Indeed the field equations take the form

$\Delta \alpha = -2\Delta \alpha e^{2\varphi}$  \hspace{1cm} (19)

$0 = \nabla \cdot \left( e^{2\Omega} \alpha \nabla A \right)$ \hspace{1cm} (20)

$\frac{1}{\alpha} \nabla \cdot \left( \alpha \nabla \Omega \right) = 2\epsilon \epsilon^\alpha \left( \nabla A \right)^2$  \hspace{1cm} (21)

$\nabla \cdot \left( \alpha \nabla \Psi_\mu \right) = 0, \hspace{0.5cm} \mu = 0..D-3$ \hspace{1cm} (22)

$2\varphi_{\alpha} \frac{\alpha_u}{\alpha} - \frac{\alpha_{uu}}{\alpha} = \frac{1}{4} \left( \Psi_{u}^2 + \frac{1}{2} \Omega_{u}^2 \right) + \frac{\epsilon}{2} \epsilon^\alpha \left( A_{uu} \right)^2 + \sum_{i=1}^{D-4} \Psi_{i,u}^2 (u \leftrightarrow v)$ \hspace{1cm} (23)

If $\Lambda \neq 0$, $\alpha$ is no longer a harmonic function. Furthermore, $\alpha$ fixes $\nu$ from (19) which is no longer directly given by (23) as for $\Lambda = 0$. Pictorially $\alpha$ is a messenger component relating all the equations together. This is in contrast to the case when $\Lambda = 0$ and equations (22) are decoupled from (20) and (21) (see the geometric interpretation in [8]). The components $\Psi_\mu$ and $\Omega$ play a similar role to the Weyl potential $\lambda$ of (3). When $D = 4$ the pole at $D = 4$ (18) is artificial since then $\Psi_\mu = 0$. It will be useful here to rewrite the field equations in terms of the dual potential $\omega$,

$$(-\partial_\omega \omega, \partial_\omega \omega) = e^{2\Omega} (\partial_\nu A, \partial_\nu A).$$ \hspace{1cm} (24)

As it was demonstrated in [8], using (24), we can rewrite (20), (21) in terms of a single complex differential equation with respect to the complex potential

$$\mathcal{E} = e^{\frac{\Omega}{2}} \alpha + i\omega,$$ \hspace{1cm} (25)

In fact, we can go one step further and rewrite the field equations with respect to $\mathcal{E}$. We get

$$\text{Re}(\mathcal{E}) \Delta \alpha = -2\Delta \alpha e^{2\varphi} e^{\frac{D-2}{4}}$$ \hspace{1cm} (26)

$$\frac{1}{\alpha} \nabla \cdot \left( \alpha \nabla \mathcal{E} \right) = \frac{\left( \nabla \mathcal{E} \right)^2}{\text{Re}(\mathcal{E})} + \text{Re}(\mathcal{E}) \frac{\Delta \alpha}{\alpha}$$ \hspace{1cm} (27)

$$\nabla \cdot \left( \alpha \nabla \Psi_\mu \right) = 0, \hspace{0.5cm} \mu = 0..D-4$$ \hspace{1cm} (28)

$$2\varphi_{\alpha} \frac{\alpha_u}{\alpha} - \frac{\alpha_{uu}}{\alpha} = \frac{1}{4} \Psi_{u}^2 + 2 \frac{\mathcal{E}_{\alpha} \mathcal{E}_{\alpha}}{\text{Re}(\mathcal{E})^2} + \sum_{i=1}^{D-4} \Psi_{i,u}^2 (u \leftrightarrow v)$$ \hspace{1cm} (29)

where we have redefined for convenience $2\nu = 2\varphi - \frac{1}{2} \ln \alpha - \frac{\Omega}{2}$. Equation (27) extends the Ernst equation [16] to the presence of a cosmological constant—indeed the extra term in $\Delta \alpha$ drops out if $\Lambda = 0$ and we get (15).

We stress here that Weyl coordinates (3) cannot be used once $\Lambda \neq 0$. We cannot therefore set $\alpha = R$ and the equations (26-29) are no longer integrable. We noticed that if we consider (16) then the Ernst potential of the Schwarzschild black hole (9) is simply $\xi = x$ in spheroidal coordinates. Given that we have the extension of Ernst’s equation for $\Lambda \neq 0$ we can now consider doing the same trick for $\Lambda \neq 0$. Consider first the four dimensional Kottler black hole with line element

$$ds^2 = r^2 \left( \frac{dr^2}{r^2 V(r)} + d\theta^2 \right) - V(r) dt^2 + r^2 \sin^2 \theta d\psi^2$$ \hspace{1cm} (30)
and \( V(r) = 1 + k^2 r^2 - \frac{2M}{r} \). Note that we will bypass Weyl coordinates by setting \( \frac{dr}{r V(r)} = dx \) and not explicitly do the integral which involves elliptic functions. Comparing with (18) we now read off the metric components in \((r, \theta)\),

\[
\alpha = r \sin \theta \sqrt{V}, \quad \mathcal{E} = V, \quad e^{2 \nu} = r^2 \alpha^{1/2}.
\]

and therefore we set

\[
x = 1
\]

\[
x + 1 = V(r) = > x = -1 - \frac{2}{k^2 r^2 - \frac{2M}{r}}
\]

On the other hand setting as before \( y = \cos \theta \) the metric (30) takes the form

\[
ds^2 = \frac{x-1}{x+1} dt^2 + r^2 \left[ \frac{dx^2}{(x^2-1)(x+1)^2} + \frac{1}{(r^2 k^2 + \frac{2M}{r})^2} + \frac{dy^2}{1-y^2} + (1-y^2)d\phi^2 \right]
\]

and \( r \) is the real positive root of the third order polynomial, \( k^2 r^3 + \frac{2}{r^2} r - 2M = 0 \) with respect to the new radial coordinate \( x \). If we set \( k = 0 \) we recover the usual spheroidal patch of (30) [3]. It is of more interest here to set \( M = 0 \) in order to study adS and dS. For the former case we start with the global patch of adS and setting \( X = -x \) we get,

\[
ds^2 = \frac{X+1}{X-1} dt^2 + \frac{dX^2}{2k^2(X+1)(X-1)^2} + \frac{1}{X-1} \left[ \frac{dy^2}{1-y^2} + (1-y^2)d\phi^2 \right]
\]

The coordinate range is \( X > 1 \), \(-1 < y < 1 \) and only covers half of anti-de-Sitter space with the boundary sitting at \( X = -1 \). The coordinate transformation for planar or hyperbolic slicings of adS goes through the same way. For \( \Lambda = 0 \) we saw that in the \((R, Z)\)-plane the integral curves \( x \) constant are hyperbolas. So what happens for adS? To see this set \( kr = \cosh ku \) which takes us to the usual global adS patch and note from (1) that the Weyl coordinate \( r \) reads,

\[
r = \ln \left| \tanh \frac{ku}{2} \right|
\]

which means that after a conformal transformation according to (8) that \( x = \coth(ku) \) and \( y = \alpha \cos(\theta) \) a very similar geometrical set of integral curves. For de-Sitter set \( k^2 = -a^2 \) starting from the locally static patch (30) to get:

\[
ds^2 = -\frac{x-1}{x+1} dt^2 + \frac{dx^2}{2a^2(x+1)(x^2-1)} + \frac{1}{1+x} \left( \frac{dy^2}{1-y^2} + (1-y^2)d\phi^2 \right)
\]

The coordinate range is \( x > -1 \) and the usual horizon at \( r = 1/a \) is not in this range. Therefore the spheroidals cover a yet smaller patch than the usual static de-Sitter (9).

For \( D = 5 \) there is an interesting twist. Proceeding as before we get

\[
\mathcal{E} = V(r)(r \cos \theta)
\]

\[\text{3} \quad \text{In the presence of a negative cosmological constant the generalised metric reads, } ds^2 = -V dt^2 + \frac{\phi^2}{r^2} + r^2 d\Omega_{D-2}^2 \]

with \( V(r) = \kappa + k^2 r^2 - \frac{2M}{r} \), adS curvature scale \( 2\Lambda = -(D-1)(D-2)k^2 \) and constant curvature of the \( D-2 \) compact space \( \kappa = 0, 1, -1 \). Note that in the presence of a negative cosmological constant we can obtain a black hole geometry with a flat (compact) horizon-the cosmological constant providing the necessary curvature scale. Furthermore, switching-off \( M \) gives us different slicings of constant curvature spacetime.
and now $E$ is a function of $r$ and $\theta$. Note however that $E_0 = r \cos \theta$ is simply the Ernst potential for flat spacetime (i.e. when we set $k = 0$ and $M = 0$) and we can therefore neglect it keeping the Ernst potential modulo flat spacetime (this also agrees with [18] for $D = 5$ and $\Lambda = 0$). In other words we once more have (32) (for $D = 5$) which yields with,

$$x + 1 = -\frac{2}{k^2 r^2} - \frac{2M}{r^2}$$

and $y = \cos 2\theta$,

$$ds^2 = -\frac{x - 1}{x + 1}dt^2 + \sqrt{1 + 2M k^2 (x + 1)^2 - 1} \left[ \frac{dx^2}{16(x^2 - 1)(1 + 2M k^2 (x + 1)^2)} + \frac{dy^2}{4(1 - y^2)} + (1 - y) d\phi^2 + (y + 1) d\psi^2 \right]$$

(39)

An interesting open question, regarding the extension of spheroids we have undertaken here, is if they can give the rotating generalisation of (9) as so happens for $\Lambda = 0$ [16], [18].

As observed by the authors of [8] all equations (27-29) apart from (26) are independent of $D$ whereas of course the metric (18) depends on the dimension. Therefore starting from a $D = 4$ solution with $\Lambda = 0$ we can construct an infinite number of $(D + n)$-dimensional solutions parametrised by the $n$ extra Weyl potentials $\Psi_{\mu}$-solutions of (28). In fact we have

$$2(\nu|_{D}, u - \nu|_{D+1}, u) \frac{\alpha}{\alpha} = -\frac{1}{4} \Psi^2, \quad (u \leftrightarrow v).$$

(40)

Setting $\sigma = \nu|_{D}, u - \nu|_{D+1}, u$ we rewrite the above equation in terms of $R$ and $Z$ in Weyl coordinates:

$$4\sigma_{,z} = -2r \Psi_{,R} \Psi_{,Z}$$

$$8\sigma_{,R} = R(\Psi_{,R}^2 - \Psi_{,Z}^2)$$

(41)

In [8] examples were given where one started from a higher dimensional solution and found its seed $D = 4$ solution. Here we will do the opposite and uplift solutions making use of the Weyl formalism [12]. Consider as our seed metric in $D = 4$ that of Kerr,

$$ds^2_4 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2} (adt - (r^2 + a^2) d\phi)^2$$

$$+ \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right),$$

(42)

where $M$ is the black hole mass, $a$ the angular momentum parameter and

$$\Delta = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2 \cos^2 \theta.$$

(43)

Comparing with (18) we now read off the relevant components

$$\alpha = \sin \theta, \quad e^{2\nu} = \rho^2 \alpha^{1/2} \sqrt{\Delta},$$

$$A = \frac{2Ma \sin^2 \theta}{\rho^2 - 2M}, \quad e^\Omega = \frac{(\rho^2 - 2M)^2}{\Delta \rho^4 \sin^2 \theta}$$

(44)-(46)
In order to uplift the solutions we will use Weyl coordinates (3) which are related to (42) by the relations
\[
R = \sqrt{\Delta} \sin \theta = \alpha, \quad Z = (r - M) \cos \theta.
\] (47)
Note that if we set \( \lambda = \Psi/\sqrt{\delta} \) and \( \chi = -\sigma/3 \) in (41) we obtain the 4 dimensional Weyl equations, (4) and (5) solutions of metric (3). We have thus demonstrated the following: for each seed solution of \( \Lambda = 0 \), (26-29), for our example here (42), we can take any Weyl solution in 4 dimensions (in the form (3)), use the coordinate transform relating it to the seed coordinate system (here (47)) and thus obtain a \( D = 5 \) solution given by,
\[
ds^2 = \rho^2 R^{-1} e^{\frac{3}{2}(R)} (dr^2 + dz^2) + R^{2/3} e^{-\frac{\lambda}{R}} \left[ -e^{\frac{2}{3}} (dt + Ad\phi)^2 + e^{\frac{2}{3}} d\phi^2 \right] + R^{2/3} e^{2\lambda(R)} d\psi^2
\] (48)
We emphasize that \( R \) and \( Z \) are functions of \( r \) and \( z \) as verify (47).

Let us consider a flat potential (6) as an example. Start by noting that (4) has the obvious symmetry, \( \lambda \rightarrow \lambda l, l \in R \). As it was pointed out in [8], generically, when uplifting solutions asymptotic flatness is not guaranteed. Indeed in 5 dimensions, the extra dimensional \( d\psi^2 \) component in (48) reads \( R^{2/3} e^{2\lambda} \) and given the form of \( R \) in (41) an obvious problem will be to get this extra direction asymptotically flat. So let us start with \( \lambda = l \ln R \) where \( l \in R \). It is straightforward to integrate and we get
\[
ds^2 = \rho^2 R^{\frac{2}{3}} e^{\frac{1}{2}(R)} (dr^2 + dz^2) + R^{\frac{2}{3} - l} \left[ -e^{\frac{2}{3}} (dt + Ad\phi)^2 + e^{\frac{2}{3}} d\phi^2 \right] + R^{\frac{2}{3} + 2l} d\psi^2
\] (49)
This is a solution to Einstein’s equations for all \( l \) where one uses (47). For example, choosing \( l = -1/3 \) we get \( R^{2/3} e^{2\lambda} = 1 \). It is then easy to check that this is the Kerr string solution in 5 dimensions. One can follow the same technique using this time the Weyl potential for the static black hole, 7 and adjusting the free constants in such a way as to obtain an asymptotically flat spacetime. It would be particularly interesting to extend this method to the case where \( \Lambda \neq 0 \). By analogy to what we have done here, it is possible that a starting point for such a generalisation would be to find how the additional fields of (26-29) transform upon keeping the Ernst potential \( \mathcal{E} \) fixed.

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