Time-varying parameter identification using orthogonal functions

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Abstract. The identification of unknown parameters of mechanical systems is a typical case of inverse problem treated in engineering. Depending on the application, the parameters can be either constant or varying with respect to time. The present contribution is dedicated to the use of orthogonal function to expand the excitation and the responses together with the expansion of the time-varying parameters in such a way that by using an operational matrix of integration, the equations of motion are written as an algebraic equation. It is expected that the proposed technique can be used for the identification of crack parameters in rotating machinery. For this aim, two case-studies are provided: the first one deals with the identification of a nonlinear time varying stiffness element of a 3 d.o.f. mechanical system; the second is focused on the identification of a cracked rotor.

1. Introduction
Techniques for constant parameter identification of mechanical systems using orthogonal function have been developed since 1975 by [1]. According to reference [2] an algorithm for parameter identification by means of Fourier orthogonal functions using signals of displacement, velocity and acceleration was developed. In [3] further developed this methodology aiming at working with non-linear system applications, by using various types of orthogonal functions. In a recent work by [4], they worked with Chebyshev polynomials for the identification of non-linear parameters of mechanical systems in the presence of noise in the measured signals.

All previous cited contributions used orthogonal functions for transforming differential equations (that represent the equations of motion) into algebraic equations. For this purpose, the so-called operational matrix is used to carry out the integration of the equations of motion. At the same time, the expansions of the input and output signals of the system are written by using orthogonal functions to perform the identification. In the present work, varying parameters identification is performed by using orthogonal functions.

Many types of mechanical systems for which the parameters are not constant with respect to time are found in different fields of engineering. For identification purposes, it is necessary to add to the previous methodology the expansion of the spatial parameters of the mechanical system with respect to the basis of the orthogonal vector function used. This procedure leads to the determination of the time-varying parameters, once the coefficients of the expansions have been obtained. In this work, the
The above-mentioned technique will be applied to damage detection problems. The case-studies will be devoted to crack detection both in structural systems and in rotating machinery. In the case of rotating machinery the crack opens and closes during the rotation performed by the system. The phenomenon of opening and closing the crack is known as breathing. Consequently, the parameters associated to the stiffness of the system change with respect to time. The main contribution of this paper is related to using the orthogonal function technique to identify stiffness time-varying parameters as found in the two case studies presented. To the best of the authors’ knowledge, the proposed technique has not yet been used in the context of time-varying parameters.

The development of transverse cracks in rotating shafts is very dangerous because they could cause catastrophic failure when they are not early detected. Comparing the experimental behaviour of the rotor with the simulated one can help significantly in detecting the crack. Therefore, reliable models are required to simulate the dynamic behaviour of cracked rotors. An open crack affects only the stiffness of the shaft and makes it axially asymmetrical. A closed crack leaves the stiffness unaffected. A rich bibliography about crack modelling in rotors has been produced in the past 30 years and different approaches have been proposed to evaluate the changes in the rotor stiffness over a complete rotation, which is the main effect in horizontal shafts loaded by gravity and constant loads. [5] gives further information regarding cracks in rotors.

Surveys on dynamics of cracked rotors have been published; see [6], [7], [8]. Generally either step-wise transition from open to closed crack has been assumed in the published investigations, or a predefined analytical sinusoidal law which is irrespective of the actual harmonic contents of the different horizontal and vertical components of the deflection variation of cracked horizontal shafts loaded by weight.

Many papers have been dedicated also in recent times to non linear analysis of small rotors emphasizing non linear effects, instabilities, side-effects like coupling of torsion or axial vibrations, but these effects are generally very small or are damped out by the oil film damping coefficients as soon as Jeffcott rotor models are substituted by more realistic rotor models, as can be seen in [9].

The true breathing mechanism can be simulated by means of 3D non-linear finite element (FE) models, or more simply by a simplified approximate 1D quasi-linear model which has been developed by [10].

2. Orthogonal Functions

A set of real functions \( \phi_k(t) \), \( k = 1, 2, 3... \) is said to be orthogonal in the interval \([a, b] \subset \mathbb{R} \), if:

\[
\int_a^b \phi_m(t)\phi_n(t)dt = K \quad \text{where:} \quad \begin{cases} 
K = 0 & \Rightarrow m \neq n \\
K \neq 0 & \Rightarrow m = n 
\end{cases} 
\]

(1)

The set of functions is said orthonormal if the relation below holds:

\[
\int_a^b \delta_{mn} dt = \delta_{mn}
\]

(2)

where \( \delta_{mn} \) is the Kronecker delta, i.e., \( \delta_{mn} = 0 \) if \( m \neq n \) or \( \delta_{mn} = 1 \) if \( m = n \).

The following property, related to the successive integration of the vectorial basis, holds for a set of \( r \) orthonormal functions in the interval \([0, T] \):

\[
\int_0^T \int_0^T \{\phi(t)\}(d\tau) \equiv [P]^r \{\phi(t)\}
\]

(3)

where \([P] \subset \mathbb{R}^{r,r}\) is a square matrix with constant elements, called operational matrix and \( \{\phi_m(t)\} = \{\phi_0(t), \phi_1(t), ..., \phi_r(t)\}^T \) is the vectorial basis of the orthonormal series. In fact, if the vectorial base is completed, i.e., if the series are not truncated, Eq. (3) is equality. However, in the practice truncated bases are used, in such a way that the order of matrix \([P] \) can be handled mathematically. In
the following sections, the vectorial basis and the operational matrix related to each type of orthogonal function considered in this paper can be found in [3].

In order to determine the behavior of the time-varying parameters applied to the identification of cracks in rotating machines, only the parameter of stiffness will be considered as varying with respect to time. Thus, the differential equation that governs the behavior of the dynamics of this system is given by:

\[
[M][\ddot{x}(t)] + [C][\dot{x}(t)] + [K(t)][x(t)] = \{f(t)\}
\]  

By integrating the equation above twice, one obtains:

\[
[M][x(t)] - \{x(0)\} - \{\dot{x}(0)t\} + [C] \int_0^t \{x(\tau)\} d\tau - \{x(0)t\} + \int_0^t [K(\tau)]\{x(\tau)\}d\tau^2 = \int_0^t \{f(\tau)\} d\tau^2
\]

Thus, the stiffness matrix expansion for a limited number of terms (necessary to have a good approximation of the parameters) is carried out. This way:

\[
[K(t)] = \hat{K}\Theta(t)
\]

where \(\hat{K}\) is the coefficient of expansion of the stiffness matrix and \(\Theta(t)\) is any truncated vectorial basis with “rp” terms used to expand all terms of the stiffness matrix.

The signals \(\{x(t)\}\) and \(\{f(t)\}\) are also expanded into sets of orthogonal functions truncated with “r” terms:

\[
\{x(t)\} = [X]\{\phi(t)\}
\]

\[
\{f(t)\} = [F]\{\phi(t)\}
\]

where:

\([X]\) is the matrix of the coefficients of expansion of \(\{x(t)\}\)

\([F]\) is the matrix of the coefficients of expansion of \(\{f(t)\}\)

Then, it is possible to obtain:

\[
[M][X]\{\phi(t)\} - \{x(0)\}[e]^T\{\phi(t)\} + \{\dot{x}(0)\}[e]^T[P]\{\phi(t)\} + [C] \int_0^t [X]\{\phi(t)\} d\tau - \{x(0)\}[e]^T[P]\{\phi(t)\} + \int_0^t \hat{K}\Theta(t)x(\tau)d\tau^2 = \int_0^t [F]\{\phi(\tau)\} d\tau^2
\]

Calling:

\[
g(t) = \Theta(t)x(t) = G\phi(t)
\]

where \(g(t)\) is a function given by the product between the vectorial basis (used to expand the varying parameter) and the signal response of the system, and \(G\) is the coefficient of expansion of \(g(t)\). This way:

\[
[M][X]\{\phi(t)\} - \{x(0)\}[e]^T\{\phi(t)\} + \{\dot{x}(0)\}[e]^T[P]\{\phi(t)\} + [C] \int_0^t [X]\{\phi(t)\} d\tau - \{x(0)\}[e]^T[P]\{\phi(t)\} + \hat{K}G\int_0^t \{\phi(t)\} d\tau^2 = [F]\int_0^t \{\phi(t)\} d\tau^2
\]

Applying the integral property given by Eq. (3), the following system of algebraic equations is obtained:

\[
[M][X]\{\phi(t)\} - \{x(0)\}[e]^T\{\phi(t)\} + \{\dot{x}(0)\}[e]^T[P]\{\phi(t)\} + [C] \int [X][P]\{\phi(t)\} - \{x(0)\}[e]^T[P]\{\phi(t)\} + \hat{K}G[P]^2\{\phi(t)\} = [F][P]^2
\]

Consequently:
The above equation can be rewritten on a compact form as follows:

\[
[H][J]=[E]
\]

By determining \( H \) it is possible to obtain the desired parameters, namely the parameters of mass, damping and \( \hat{K} \) (the coefficients of expansion of the stiffness matrix). As \( \Theta(t) \) is already known, it is possible to write:

\[
[K(t)]=\hat{K}\Theta(t)
\]

It was verified during the computational simulations that the methodology developed above only provides good results when all the degrees of freedom are externally excited, which is not applicable in most mechanical systems.

The same procedure above was applied by using the state-space formulation. This is very important, since this formulation leads to the unknown parameters without the necessity of exciting all the d.o.f. The only inconvenient is that a prior knowledge of the matrix of mass and the signals of displacement and velocity are necessary. After algebraic manipulation, it is possible to write the parameter estimation equation without the data regarding the velocity signal, as shown by Eq. (17).

\[
\begin{bmatrix}
M^{-1}\hat{K} & M^{-1}C
\end{bmatrix}
\begin{bmatrix}
G[P]\n
\end{bmatrix}
=\begin{bmatrix}
M^{-1}F[P]^2 - [X - (x(0)e)\n
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\]

Equation (17) can also be written on the compact form as represented by Eq. (15).

---

Figure 1. Stress field and cross section geometry of the shaft for various angular positions
3. Modelling of the breathing mechanism

As described by [9], the breathing mechanism is a result of the stress and strain distribution around the cracked area, which is due to static loads, like the weight, the bearing reaction forces and so on, and dynamical loads, like the unbalance and the vibration induced inertia force distribution. Accurate modeling of the breathing mechanism has been generally disregarded. When the static loads overcome the dynamical ones, the breathing is governed by the angular position of the shaft with respect to the stationary load direction, and the crack opens and closes again completely once each revolution. The transition from closed crack (full) stiffness to the open crack (weak) stiffness has been generally considered abrupt or represented by a given cosine function, but can be calculated step by step in an iterative procedure.

The breathing mechanism calculated with the described simplified approach, has been validated by numerical results obtained with a 3D model of a cracked cylindrical beam, clamped at one end and loaded mechanically at the other end with a rotating load. As an example, figure 1 shows the stress field for various angular positions and the cross section of the cracked shaft for the case in which the simplified methodology described above was used.

3.1. The Simplified FLEX Model

Details regarding the simplified method can be found in [9].

The stiffness matrix (square, symmetrical, 8×8 elements) is represented by Eq. (18):

\[
[K_s(\Omega)]\begin{bmatrix}
x_1 \\
\partial_1 \\
x_2 \\
\partial_2 
\end{bmatrix} = \begin{bmatrix}
a & c & p & q & -a & c & -p & q \\
e & -q & r & c & f & q & s & g \\
b & -d & -p & -q & -b & -d \\
h & -q & s & d & g & a & -c & p & -q \\
e & q & r & b & d & h \\
y_1 \\
\partial_1 \\
y_2 \\
\partial_2 
\end{bmatrix}
\]

(18)

where the coefficients are defined as follows:

\[
a = \frac{12J_yE}{(1+\phi)L_c^3}, b = \frac{12J_yE}{(1+\phi)L_c^3}, c = \frac{6J_yE}{(1+\phi)L_c^3}, d = \frac{6J_yE}{(1+\phi)L_c^3}, e = \frac{(4+\phi)J_yE}{(1+\phi)L_c^3},
\]

\[
f = \frac{(2-\phi)J_yE}{(1+\phi)L_c^3}, g = \frac{(2-\phi)J_yE}{(1+\phi)L_c^3}, h = \frac{(4+\phi)J_yE}{(1+\phi)L_c^3}, p = \frac{12J_mE}{(1+\phi)L_c^3},
\]

\[
q = \frac{6J_mE}{(1+\phi)L_c^3}, r = \frac{(4+\phi)J_mE}{(1+\phi)L_c^3}, s = \frac{(2-\phi)J_mE}{(1+\phi)L_c^3}, \phi = \frac{12EJ}{GS^2L_c^2}.
\]

The parameter \( \phi \) accounts for the shear effects, \( E \) and \( G \) are respectively the Young’s modulus and the shear modulus, \( S \) is the cross section area, \( J \) are the second moments of area.

4. Example 1 – Nonlinear hardening stiffness identification of a mechanical system

The methodology developed for time-varying parameter determination will be applied to the 3 d.o.f. system shown in Figure 2. One of the springs presents a non-linear behavior (cubic stiffness), as given by Eq. (19). As the stiffness increases as the spring’s deformation increases, the spring exhibits a hardening effect.

The parameter identification process starts with the choice of the number of terms to be used for expanding all the unknown parameters of the system. These parameters can be constant or time-varying in the present case. In general, the number of terms to determine constant parameters is small; however, the number of terms in the expansion is higher for time-varying parameters. It is worth mentioning that a low number of expansion terms may be not enough to get a good determination of the parameters and a number well above the necessary leads to the accumulation of computational
errors due to the inversion of the matrix J, Eq. (15). This error occurs because, by increasing the number of expansion terms above the amount required to obtain a good approximation of the parameter, the singular values become close to zero, preventing a reliable inversion of matrix J. In the present case study the orthogonal Fourier series was used to expand the parameters in orthogonal series, considering 61 terms. For the excitation and response signals a basis of Legendre polynomials considering 450 expansion terms was used. It is not possible to use the same orthogonal function bases for expanding both the parameters and the signals (excitation and response) since matrix J would become linearly dependent. The system parameters considered for simulation purposes are: $M_1=1.0$ Kg; $M_2=1.5$ Kg; $M_3=2.0$ Kg; $K_1=1,100$ N/m; $K_2=950.0$ N/m; $K_3=800.0$ N/m; $C_1=10.0$ Ns/m; $C_2=15.0$ Ns/m; $C_3=20.0$ Ns/m; $C_4=25.0$ Ns/m. The stiffness of the nonlinear spring is represented by Eq. (19).

$$K_i(t)=500+10^{1.5}(x_i(t)-x_i(t))^3$$  \hspace{1cm} (19)

A harmonic excitation given by $F(t)=3\sin(\omega t)$ was applied to $M_1$ and the displacement responses were obtained for the 3 d.o.f. 2% noise was added to the output responses of the system. Figures 3 to 5 present some of the varying and constant identified stiffness parameters.

5. **Example 2 - Crack identification of a flexible rotor**

The simple rotor shown in Figure 6 was used in this case study. The depth (severity) of the crack is 0.004 m and its length is 0.0064 m. The material properties and the values of the bearing parameters used in the simulation are shown in table (1). Figure 5 illustrates the second moments of area used to calculate the cracked stiffness element matrix. The strategy implemented for the determination of the
varying parameters was to locate and quantify the variation of each element of the global stiffness matrix, as a function of time. As only the central element of the rotor modifies its parameters during rotation, it is expected to find a region with constant terms and another with varying terms in the identified matrix, as shown in Figure 7. Obviously, the region corresponding to time-varying parameters defines the crack position. 720 points were used along a complete rotation and data were stored during 20 complete laps, for parameter identification purposes (a total of 14,400 data points).

The speed of rotation used was 30 rad/s. The number of terms used in the expansion of the stiffness matrix was 91 and 650 terms were used in the vectorial basis corresponding in the input and output signals. Figures 8 and 9 illustrate some of the identification results obtained. The time-varying stiffness parameters permit to locate and quantify the crack, satisfactorily.

Table 1. Parameters used to simulate the rotor

| Parameter                    | Value          |
|------------------------------|----------------|
| $E$ [N/m$^2$]                | 2.00 E11       |
| Density [Kg/m$^3$]           | 7800           |
| Cracked element length [m]   | 0.0064         |
| $V$ (Poisson’s ratio)        | 0.3            |
| Bearing properties           |                |
| $K_{xx}$ [N/m]               | 5.00 E7        |
| $K_{zz}$ [N/m]               | 7.00 E7        |
| $K_{xz}$ = $K_{zx}$ [N/m]    | 0              |
| $C_{xx}$ [N s/m]             | 500.00         |
| $C_{zz}$ [N s/m]             | 700.00         |
| $C_{xz}$ = $C_{zx}$ [N s/m]  | 0              |

Figure 5. Second moments of area (used to calculate the stiffness matrix of the cracked rotor)

Figure 6. Cracked Rotor used in the simulation

Figure 7. Representation of the global stiffness matrix of the system
6. Conclusion
The present contribution used an orthogonal series identification technique to determine nonlinear time-varying parameters of mechanical systems. The main contribution of this paper has to do with the determination of stiffness time-varying parameters. Two case studies were analyzed, namely a three d.o.f. mechanical systems and a cracked rotor. It is worth mentioning that a crack in a rotor leads to modifications in the elements of the stiffness matrix. Besides, the position of the crack was included in the identification scheme. The proposed technique seems to be robust to noise, as demonstrated in the first case study. Further research will include the identification of cracks from real experimental data collected from a rotor test rig.

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