Construction of Supersymmetric Nonlinear Sigma Models on Noncompact Calabi-Yau Manifolds with Isometry

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We propose a class of $\mathcal{N} = 2$ supersymmetric nonlinear sigma models on the noncompact Ricci-flat Kähler manifolds, interpreted as the complex line bundles over the hermitian symmetric spaces. Kähler potentials and Ricci-flat metrics for these manifolds with isometries are explicitly constructed by using the techniques of supersymmetric gauge theories. Each of the metrics contains a resolution parameter which controls the size of these base manifolds, and the conical singularity appears when the parameter vanishes.

1. Introduction

Nonlinear sigma models (NLσMs) in two dimensions are interesting for several reasons. They help us to understand various non-perturbative phenomena in four dimensional gauge theories such as confinement or dynamical mass generation. They also provide the description of strings propagating in the curved space-time. In the latter case, the consistency of strings requires the conformal invariance of the NLσMs. In the case of superstrings, furthermore, NLσMs have to possess the $\mathcal{N} = 2$ world sheet superconformal symmetry. Although all two-dimensional NLσMs with $\mathcal{N} = 2$ supersymmetry have $\mathcal{N} = 2$ superconformal symmetry at the tree level, they suffer from superconformal anomaly at the quantum level. Since these anomalies are proportional to the Ricci curvatures $R_{ab}$ of the background (target) manifolds, $\mathcal{N} = 2$ supersymmetric NLσMs taking values on the target manifolds with vanishing Ricci curvatures, provide the possible candidates of the consistent superstrings propagating in the curved space-time. They are also candidates of finite quantum field theories in two dimensions. $\mathcal{N} = 2$ supersymmetry in two dimensions requires the target manifolds of NLσMs are Kähler manifolds. The Kähler manifolds with vanishing Ricci curvatures are called the Calabi-Yau manifolds. Therefore, it is important to study the $\mathcal{N} = 2$ supersymmetric NLσMs defined on the Calabi-Yau manifolds.

To find the metric of the Ricci-flat manifolds, we have to solve the Einstein equation $R_{ab} = 0$ for the vacuum. If the manifold has enough symmetries, we can reduce the partial differential equation to an ordinary differential equation, which is usually easy to solve. In compact Calabi-Yau manifolds, however, there is no isometry and not a single explicit metric is known in this case. In some noncompact Calabi-Yau manifolds, the number of isometries is sufficient to reduce the Einstein equation to an ordinary differential equation. In this article, we report our recent progress to obtain the explicit metric for certain class of the noncompact Calabi-Yau manifolds.

2. NLσMs with $\mathcal{N} = 2$ Supersymmetry in Two Dimensions

$\mathcal{N} = 2$ supersymmetry requires that both scalars $A^a(x)$ and fermions $\psi^\alpha(x)$ are complex fields. The scalar fields $A^a(x)$ ($a = 1, \cdots, N$) are coordinates of the target manifold, whose
metric is given by the Kähler potential \( K(A, \bar{A}) \) which characterizes the theory:

\[
g_{ab}(A, \bar{A}) = \frac{\partial^2 K(A, \bar{A})}{\partial A^a \partial \bar{A}^b} = K_{ab}(A, \bar{A}). \quad (1)
\]

The lagrangian for \( \mathcal{N} = 2 \) supersymmetric NLσM is given by

\[
\mathcal{L}(x) = g_{ab}(A, \bar{A}) \partial_\mu A^a \partial_\mu \bar{A}^b + ig_{ab} \bar{\psi}^b (\bar{\partial} \psi)^a + \frac{1}{4} R_{abcd} \bar{\psi}^{a\bar{c}} \bar{\psi}^b \nu^d , \quad (2)
\]

where the covariant derivative and the Riemann curvature are defined by

\[
(D_\mu \psi)^a = \partial_\mu \psi^a + \partial_\mu A^b \Gamma^{a}_{bc} \psi^c \quad (3)
\]

\[
R_{abcd} = \partial_c \partial_d g_{ab} - g^{ef} \partial_d g_{af} \partial_c g_{eb} , \quad (4)
\]

with the connection being

\[
\Gamma^c_{ab} = g^{cd} g_{bd} = g^{cd} K_{ab} . \quad (5)
\]

Since all geometrical quantities are derived from the Kähler potential \( K \), all we have to do is to assume an appropriate symmetry of the scalar function \( K \). Symmetries of a scalar function is far simpler than the symmetries of the tensor \( g_{ab} \).

### 3. Noncompact Calabi-Yau Manifolds

Although the NLσM defined by \([2]\) is conformally invariant at the tree level, it suffers from the conformal anomaly proportional to the Ricci curvature at the quantum level \([3]\). To obtain the conformally invariant theory at the quantum level, we impose the Ricci-flatness condition

\[
R_{ab} = 0 \quad (6)
\]

to the target manifold.

In order to reduce an Einstein equation to an ordinary differential equation of single variable, we assume our target manifold \( M_1 \) has a symmetry, which carries each point of the manifold to a certain point on a line, which is parameterized by a single variable. In other word, any point of our manifold \( M_1 \) can be reached from a point on a line by a suitable action of some symmetry group \( G \). If any point of the manifold can be reached from a single point, say the origin, by appropriate action of a symmetry group \( G \), the manifold is a homogeneous coset space \( G/H \) where \( H \) denotes a subgroup of \( G \). The local structure of our manifold is, therefore, the product of a homogeneous space \( M = G/H \) and a line. Since \( \mathcal{N} = 2 \) supersymmetry requires the manifold \( M_1 \) to be a Kähler manifold, we assume that the homogeneous space \( M = G/H \) is Kähler by itself and the line is a complex line. Furthermore, we assume the Kähler coset space \( M = G/H \) is a compact Einstein manifold, which satisfies the Einstein equation with a positive cosmological constant \( h > 0 \)

\[
R_{\bar{m}n} = hg_{\bar{m}n} , \quad (7)
\]

where we use the indices \( m, n \) for the compact Kähler manifold \( M = G/H \) whose complex dimension is \( d = \dim(G/H)/2 \). The complex dimension of the Ricci-flat manifold \( M_1 \) is \( D = 1 + d \).

Our total space \( M_1 \) is parameterized by the complex coordinates \( \phi_m \) \((m = 1, \cdots, d)\) and a complex number \( \rho \) that represents the complex line. We denote the Kähler potential of the compact Kähler-Einstein manifold \( M \) by \( \Psi(\phi, \bar{\phi}) \). Our assumption for the Kähler potential of the Ricci-flat manifold \( M_1 \) is as follows:

\[
K(A, \bar{A}) = K(X) , \quad (8)
\]

where

\[
X = \log |\rho|^2 + h \Psi(\phi, \bar{\phi}) . \quad (9)
\]

Now, if we notice

\[
R_{ab} = -g^{cd} R_{abcd} = -\partial_a \partial_b \log \det (g_{c\bar{d}}) \quad (10)
\]

the Ricci-flat condition \([3]\) takes a very simple form

\[
\det (g_{c\bar{d}}) = |\text{hol.}|^2 , \quad (11)
\]

where hol. is a holomorphic function of the coordinates \( \phi \) and \( \rho \).

Components of the metric for the whole space \( M_1 \)

\[
g_{a\bar{b}} = \begin{pmatrix}
    g_{\rho\bar{\rho}} & g_{\rho\bar{m}} \\
    g_{m\rho} & g_{m\bar{m}}
\end{pmatrix} , \quad (12)
\]

are given by

\[
g_{\rho\bar{\rho}} = K_{\nu} \frac{\partial X}{\partial \rho} \frac{\partial X}{\partial \bar{\rho}} , \quad (13)
\]
\[ g_{\rho \mu} = K^\mu \frac{\partial X}{\partial \rho} \frac{\partial X}{\partial \phi}, \quad (14) \]
\[ g_{\bar{m} \bar{n}} = K^\mu \frac{\partial X}{\partial \bar{m}} \frac{\partial X}{\partial \bar{n}} + K' \frac{\partial^2 X}{\partial \phi^m \partial \phi^n}, \quad (15) \]

where the prime denotes the differentiation with respect to the argument \( X \). Using equations
\[ \partial X / \partial \rho = 1 / \rho (\rho \neq 0), \quad \frac{\partial^2 X}{\partial \phi^m \partial \phi^n} = h g_{\bar{m} \bar{n}}, \quad (16) \]

we can express the determinant of the metric of \( M_1 \) in terms of the determinant of the metric of \( M = G/H \)
\[ \det g_{a \bar{b}} = g_{\rho \bar{m} \bar{n}} \cdot \det (g_{m \bar{m}} - g^{-1}_{m \bar{m}} g_{\rho \bar{m}} g_{\rho \bar{m}}) \quad (17) \]
\[ = \frac{1}{|\rho|^2} K''(K')^d \cdot \det (h g_{\bar{m} \bar{n}}). \quad (18) \]

Since we have assumed that the Kähler coset space is an Einstein space, we can easily calculate the determinant of its metric. By substituting the expressions of the Ricci curvature \([10]\) and the metric \( g_{m \bar{m}} \) in terms the Kähler potential \( \Psi \) \([4]\) to the Einstein equation \([1]\), we obtain
\[ R_{m \bar{n}} = - \partial_m \partial_{\bar{n}} \log \det (g_{k \bar{k}}) = h \frac{\partial^2 \Psi (\phi, \bar{\phi})}{\partial \phi^m \partial \bar{\phi}^n}, \quad (19) \]

which can be integrated to give a convenient formula for the determinant of the metric in Kähler-Einstein manifold \( M \)
\[ \det (g_{m \bar{m}}) = e^{-h \Psi} |\text{hol.}|^2. \quad (20) \]

Combining the formula \([13]\), \([21]\) and \([1]\), we obtain the Ricci-flatness condition for the Kähler potential of the manifold \( M_1 \)
\[ \det (g_{\bar{a} \bar{b}}) = e^{-X} K''(K')^d |\text{hol.}|^2 = (\text{const.}) \cdot |\text{hol.}|^2, \]

from which we find an ordinary differential equation for the Kähler potential:
\[ e^{-X} \frac{d}{dX} \left( K' \right) = a, \quad (21) \]

where \( a \) is a constant.

The equation \([21]\) is easily solved to yield
\[ K' = (\lambda e^X + b) \uparrow, \quad (22) \]

where \( \lambda \) is a constant related to \( a \) and \( D \), and \( b \) is an integration constant, interpreted as a resolution parameter. To calculate the metric \([13] \quad (14) \quad (15) \) of the Ricci-flat manifold \( M_1 \), the Kähler potential itself is not necessary, though it is easily obtained by integrating \([22]\).

Our method can be applied to any Kähler-Einstein coset manifold \( M \), however, we need the Kähler potential \( \Psi \) of \( M \) to obtain the explicit expression for the metric of the noncompact Calabi-Yau manifold \( M_1 \). Fortunately, the general prescription to calculate \( \Psi \) is given by Ito, Kugo and Kunitomo \([5]\). In this respect, our results gives the explicit realization of the method of Page and Pope, who constructed Ricci-flat manifolds by introducing a complex line as a fibre over a Kähler-Einstein manifold \([4]\).

Here, we further require that the Kähler coset space \( M \) is a symmetric space, called the hermitian symmetric space, then the Kähler potential can be explicitly obtained by the gauge theory technique \([2,7]\). Therefore, we find a new class of Ricci-flat manifolds corresponding to various hermitian symmetric spaces with \( h = C_2(G) / 2 \). For \( M = CP^{N-1}, M_1 \) reduces to the manifold obtained by Calabi \([3]\).

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