Boundedness of pseudo-differential operators in subelliptic Sobolev and Besov spaces on compact Lie groups

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1. Introduction

In this work, Besov spaces associated to sub-Laplacians in the context of compact Lie groups are studied. We refer to them as subelliptic Besov spaces and we obtain some of their embedding properties in terms of the Hausdorff dimensions of compact Lie groups defined by the control distances associated to the sub-Laplacians. The novelty of our approach is that we use a description for these spaces in terms of the matrix-valued symbol associated to pseudo-differential operators, which we exploit particularly for sub-Laplacians. Our methods are inspired by the mapping properties of Fourier multipliers and global pseudo-differential operators consistently developed in the works [1, 2], as well as in the matrix-valued functional calculus established in [3] for the \((\rho, \delta)\)-Hörmander classes with \(0 \leq \delta < \rho \leq 1\). The new addressed question of this paper is, under which conditions the mapping properties for the \((\rho, \delta)\)-Hörmander classes can be improved to the critical case \(\delta = \rho\) adding also the subelliptic context.

The Besov spaces \(B^s_{p,q}(\mathbb{R}^n)\) arose from attempts to unify various definitions of several fractional-order Sobolev spaces. As it was pointed out by Stein [4], Taibleson [5] studied the generalised Hölder–Lipschitz spaces \(\Lambda^s_{p,q}(\mathbb{R}^n)\) and these spaces were called Besov spaces in honour of O. V. Besov who obtained a trace theorem and important embedding...
properties for them [6,7]. Dyadic decompositions (defined by the spectrum of the Laplacian) for Besov spaces on $\mathbb{R}^n$ were introduced by J. Peetre as well as other embedding properties were obtained [8,9]. We refer the reader to the works of Triebel [10,11] for a complete background on the subject as well as a complete description of the historical facts about Besov spaces and other function spaces.

Throughout this work, $G$ is a compact Lie group and the positive sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_n^2)$, will be considered in such a way that the system of vector fields $X = \{X_i\}$ satisfies the Hörmander condition. With a dyadic partition $\{\psi_\ell\}_{\ell \in \mathbb{N}}$ for the spectrum of the operator $(1 + \mathcal{L})^{\frac{s}{2}}$, Besov spaces $B^s_{p,q}(G)$ associated to $\mathcal{L}$ can be defined by the norm

$$
\|f\|_{B^s_{p,q}(G)} = \left\|2^{s\ell}\|\psi_\ell((1 + \mathcal{L})^{\frac{s}{2}}f)\|_{L^p(G)}\right\|_{\ell^q(\mathbb{N})}.
$$

The parameter $p$ in (1) measures the type of integrability of the function $f$, while the parameter $s$ is the maximal order of regularity of $f$. In terms of the sub-Laplacian, this regularity order can be measured by the action of the operator $(1 + \mathcal{L})^{\frac{s}{2}}$ on the information that the Fourier expansion of $f$ provides. This means that we can approximate the function $(1 + \mathcal{L})^{\frac{s}{2}}\psi_\ell((1 + \mathcal{L})^{\frac{s}{2}}f)$ by another more simple one, in this case given by $2^{s\ell}\psi_\ell((1 + \mathcal{L})^{\frac{s}{2}}f)$. This 'approximation' when $\ell \to \infty$ is understood in the $L^p$-norm and the $q$-parameter is a measure of the speed in which this approximation occurs. If it occurs with speed $O(\ell^{-\kappa})$ where $\kappa > 1/q$ then we classify $f$ to be in the space $B^s_{p,q}(G)$. If this approximation occurs with $O(1)$ speed, we can think that $f \in B^s_{p,\infty}(G)$. This general analysis can be applied to every (compact or non-compact) Lie group $G$ where Besov spaces are defined. The main feature of these spaces is that it allows us to better understand other functions spaces. It is the case of the Hilbert space $L^2(G)$ or another Sobolev spaces with integer or fractional order (see, e.g. Theorem 4.3).

If $\mathcal{L}_G = -(X_1^2 + \cdots + X_n^2)$, $n := \text{dim}(G)$, is the Laplacian on the group, the Fourier description for Besov spaces $B^s_{p,q}(G) \equiv B^s_{p,q}(G)$ was consistently developed by Nursultanov, Tikhonov and the second author in the works [12,13] for compact Lie groups and general compact homogeneous manifolds. For obtaining non-trivial embedding properties for these spaces, the authors developed the Nikolskii inequality, which was established in terms of the dimension of the group $n$ and the Weyl-eigenvalue counting formulae for the Laplacian. To develop similar properties in the subelliptic context, we obtain a suitable subelliptic version of the Nikolskii inequality but in our case this will be presented in terms of the Hausdorff dimension $Q$ defined by the control distance associated to the sub-Laplacian under consideration (see Equation 21).

In the subelliptic framework, there are important differences between subelliptic Besov spaces and the Besov spaces associated to the Laplacian. Indeed, if $\{X_i\}$ is a system of vector fields satisfying the Hörmander condition of order $\kappa$ (this means that their iterated commutators of length $\leq \kappa$ span the Lie algebra $\mathfrak{g}$ of $G$) the sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_n^2)$, $1 \leq k < n$, acting on smooth functions, derives in the directions of the vector fields $X_i$, $1 \leq i \leq n$, but the unconsidered directions $X_i$, $k + 1 \leq i \leq n$ provide a loss of regularity reflected in the following embeddings:

$$
B^s_{p,q}(G) \hookrightarrow B^{s,\mathcal{L}}_{p,q}(G) \hookrightarrow B^{s,\kappa}_{p,q}(G), \quad s_{\kappa,\kappa} := (s/\kappa) - \kappa (1 - (1/\kappa)) \left|\frac{1}{2} - \frac{1}{p}\right|, \quad (2)
$$
that we will prove in Theorem 5.11 for all \( s \), with \( 0 < s < \infty \). Here we have denoted by 
\[ \kappa := \lceil n/2 \rceil + 1, \]
the smallest integer larger than \( \frac{1}{2} \dim(G) \).

On the other hand, the last part of this work is dedicated to investigate the problem of the boundedness of pseudo-differential operators on subelliptic Sobolev and Besov spaces.

To motivate our results let us record the following sharp estimate due to C. Fefferman [14].

**Theorem 1.1 (Fefferman’s Theorem):** For \( 0 \leq \delta < \rho \leq 1 \), let us consider the pseudo-differential operator \( A : C^\infty(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n) \) with symbol \( \sigma \in S_{-m}^{-\rho,\delta}(\mathbb{R}^n) \). If \( m = \frac{n(1-\rho)}{2} \), then \( A \) extends to a bounded operator from \( L^\infty(\mathbb{R}^n) \) to \( \text{BMO}(\mathbb{R}^n) \), from \( H^1(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \), and is bounded on \( L^p(\mathbb{R}^n) \), for all \( 1 < p < \infty \). Moreover, for \( 1 < p < \infty \), and

\[
m \geq m_p := n(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|,
\]

the linear operator \( A \) extends to a bounded operator on \( L^p(\mathbb{R}^n) \).

The extension of Theorem 1.1 to pseudo-differential operators on compact Lie groups was obtained by J. Delgado and the second author in [1] (see also Theorem 5.6). Indeed, the following estimate was proved in [1]:

- Let \( G \) be a compact Lie group and \( n = \dim(G) \) its topological dimension. For \( 0 \leq \delta < \rho \leq 1 \), let us consider the pseudo-differential operator \( A : C^\infty(G) \rightarrow \mathcal{D}'(G) \) with symbol \( \sigma \in S_{-m}^{-\rho,\delta}(G) \). If \( m = \frac{n(1-\rho)}{2} \), then \( A \) extends to a bounded operator from \( L^\infty(G) \) to \( \text{BMO}(G) \), from \( H^1(G) \) to \( L^1(G) \), and is bounded on \( L^p(G) \), for all \( 1 < p < \infty \). Moreover, for \( 1 < p < \infty \), and

\[
m \geq m_p := n(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|,
\]

the linear operator \( A \) extends to a bounded operator on \( L^p(G) \).

The main results in this work are: the subelliptic Nikolskii inequality (Theorem 3.2), the embedding (2) and its dual embedding for \( s < 0 \) (Theorem 5.11), the embedding properties for subelliptic Besov and Sobolev spaces in Theorem 62, and the following Fefferman-type estimates on subelliptic Sobolev and Besov spaces. Here, for \( 0 \leq \delta \leq \rho \leq 1 \), \( \mathcal{S}_{\rho,\delta}^m(G) \) denotes the Hörmander class of symbols of order \( m \in \mathbb{R} \) and of type \( (\rho, \delta) \) and \( A \equiv \sigma(x, D) \) is the pseudo-differential operator associated with the symbol \( \sigma \in \mathcal{S}_{\rho,\delta}^{-1}(G) \) (see Section 5).

**Theorem 1.2 (Fefferman Subelliptic Sobolev Theorem):** Let \( G \) be a compact Lie group of dimension \( n \). Let us assume that \( \sigma \in \mathcal{S}_{\rho,\delta}^{-1}(G) \) and let \( 0 \leq \delta \leq \rho \leq 1 \), \( \delta \neq 1 \). Then \( A \equiv \sigma(x, D) \) extends to a bounded operator from \( L^{p,\mathcal{L}}_{\rho\delta}(G) \) to \( L^p(G) \) provided that

\[
n(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| - (\rho/\kappa) \leq \nu.
\]
In particular, if \( \sigma \in \mathcal{S}_{\rho,\delta}^0(G) \), the operator \( A \equiv \sigma(x,D) \) extends to a bounded operator from \( L^p_{\rho} \) to \( L^p(G) \) with
\[
n \kappa (1 - \min(\rho, 1/\kappa)) \left| \frac{1}{p} - \frac{1}{2} \right| \leq \vartheta.
\]

**Theorem 1.3 (Fefferman Subelliptic Besov Theorem):** Let us assume that \( \sigma \in \mathcal{S}_{\rho,\delta}^0(G) \) and let \( 0 \leq \delta \leq \rho \leq 1, \; \delta \neq 1 \). Then \( A \equiv \sigma(x,D) \) extends to a bounded operator from \( B_{p,q}^{s+\vartheta} \) to \( B_{p,q}^{s} \) provided that
\[
n(1 - \min(\rho, 1/\kappa)) \left| \frac{1}{p} - \frac{1}{2} \right| - (\vartheta / \kappa) \leq v.
\]

In particular, if \( \sigma \in \mathcal{S}_{\rho,\delta}^0(G) \), the operator \( A \equiv \sigma(x,D) \) extends to a bounded operator from \( B_{p,q}^{s+\vartheta} \) to \( B_{p,q}^{s} \) with
\[
n \kappa (1 - \min(\rho, 1/\kappa)) \left| \frac{1}{p} - \frac{1}{2} \right| \leq \vartheta.
\]

Now, let us discuss some consequences and the sharpness of our main results.

- The subelliptic Nikolskii inequality
\[
\| T_{\tilde{f}} \|_{L^q(G)} \leq C_{p,q,Q}^Q \| T_{\tilde{f}} \|_{L^p(G)}, \quad f \in C^\infty(G),
\]
where \( C = C_{p,q,Q} \) is independent on \( f \) and \( \tilde{f} \), obtained in Theorem 3.2 is sharp, in the sense that if we consider the situation of the Laplacian on \( G \), \( L_G \) instead of the sub-Laplacian \( L \), and we consider the topological dimension \( n \) instead of the Hausdorff dimension \( Q \), we recover the elliptic Nikolskii inequality in [13] for \( 1 < p \leq q < \infty \), which is sharp. Indeed, the exponent \( n(1/p - 1/q) \) is the best possible in several contexts, e.g. in the case of the \( n \)-torus \( G = \mathbb{T}^n \). We refer the reader to [12,13] for details. The subelliptic Nikolskii inequality will be applied to obtain the embedding \( B_{p_1,q_1}^{s_1,\vartheta} \) (\( G \) instead of \( G = \mathbb{T}^n \)) and \( 1 \leq p_1 \leq p_2 \leq \infty, \; 0 < q < \infty, \; r_1 \in \mathbb{R} \), and
\[
r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2}), \; \text{in Theorem 4.3.}
\]

- The index \( s_{\kappa,\vartheta} := (s/\kappa) - \vartheta(1 - (1/\kappa)) \) \( 1/2 - 1/p \) in the embedding (2) is sharp for the case of \( L^2 \)-subelliptic Sobolev spaces. Indeed, in such a particular case, the Sobolev embedding is equivalent to the inequality (85) which is known to be sharp from the classical work of Rothschild and Stein [15] (see also [16]).

- The critical order (4) is the best possible to assure the \( L^p \)-boundedness for operators with symbols in the Hörmander classes in several cases, \( \mathbb{R}^n \), graded Lie groups and arbitrary compact Lie groups (see [17] for details). If in Theorem 1.3 we consider \( \vartheta = 0, \; p = q = 2, \; \rho = 1 \), we re-obtain the classical Sobolev estimate for Hörmander classes on \( L^2(G) \), which is sharp for operators with order \( v = 0 \) from an argument via localizations. Indeed, it is also known that for \( \rho = \delta = 1 \), the class \( S^0_{1,1} (\mathbb{R}^n) \) begets
unbounded operators on $L^2(\mathbb{R}^n)$. On the other hand, if in Theorem 1.2, we consider $\vartheta = 0$, and $0 \leq \rho \leq 1/\kappa$, we re-obtain the condition

$$n(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right| \leq \nu,$$

(10)

which is the best possible to assure the $L^p$-boundedness of pseudo-differential operators, extending also Theorem 5.6 to the case $0 \leq \delta \leq \rho \leq 1/\kappa$. Observe that if we replace in the subelliptic theorems above the Laplacian $\mathcal{L}_G$ instead of the sub-Laplacian $\mathcal{L}$, then $\kappa = 1$, and our estimates are valid in the complete range $\delta \leq \rho \leq 1, \delta \neq 1$. We refer the reader to [17] and the work of the second author and J. Delgado [1] for details about the sharpness of the index $n(1 - \rho)/2$ where the Lebesgue space $L^1$ is replaced by the Hardy space $H^1$.

- As an application of the embeddings $B_{p,q}^s(G) \hookrightarrow B_{p,q}^{s,\mathcal{L}}(G)$ and $L_p^s(G) \hookrightarrow L_p^{p,\mathcal{L}}(G)$ for all $s \geq 0$ (see Theorem 5.9), and from Theorems 1.2 and 1.3, we deduce that the conditions $\sigma \in \mathcal{S}^{-\nu}_{\rho,\delta}(G)$ and $0 \leq \delta \leq \rho \leq 1, \delta \neq 1$, imply that $A \equiv \sigma(x,D)$ extends to a bounded operator from $B_{p,q}^{s+\vartheta,\mathcal{L}}(G)$ to $B_{p,q}^{s,\mathcal{L}}(G)$ and from $L_{p+\vartheta}^s(G)$ to $L_p^s(G)$ provided that

$$n(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| - (\vartheta/\kappa) \leq \nu,$$

(11)

with the optimal loss of regularity for $\vartheta = 0$ and

$$n(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| =: \nu_p \leq \nu,$$

being in the case of the Laplacian $\mathcal{L}_G$, $\kappa = 1$, and $\nu_p$ the critical order for the $L^p$-boundedness.

- The previous remark allows us to deduce that already in the case of the Laplacian, our main theorems provide improvements for the boundedness of pseudo-differential operators on subelliptic Sobolev and Besov spaces allowing the critical situation $1 \leq \delta \leq \rho \leq 1, \delta \neq 1$ extending the results in [18–20] where the restriction $\delta < \rho$ is imposed.

- In contrast with the Fefferman Theorem where the condition $0 \leq \delta < \rho \leq 1$ is imposed, we allow the borderline $\delta = \rho$. This improvement was first observed for graded Lie groups in [17]. Now, in the compact case, we observe that the critical order (4) presented above (and also in Theorem 5.6) can be improved to the critical situation $\rho = \delta$ (see Lemma 7.2) same as in the classical Calderón–Vaillancourt Theorem [21] (and also in the version of the Calderón–Vaillancourt Theorem by Fischer [22] on compact Lie groups).

- On $\mathbb{R}^n$, that the condition $\delta < \rho$ in Fefferman’s theorem can be relaxed to $\delta \leq \rho$, was first observed by C. Z. Li and R. H. Wang, [23]. Extensions of Fefferman’s theorem for Besov spaces in the case $\rho = \delta$ for Hörmander classes on $\mathbb{R}^n$ with the critical order $m_p = n(1 - \rho)|1/p - 1/2|$ have been obtained by Park in [24].

- The critical order $m_p = n(1 - \rho)|1/p - 1/2|$ remains sharp in different classes of pseudo-differential operators. In the general context of $S(m,g)$-classes for the Weyl–Hörmander calculus, we refer the reader to J. Delgado [25,26].
This paper is organised as follows. In Section 2, we present some preliminaries on sub-Laplacians and the Fourier analysis on compact Lie groups. In Section 3, we establish a subelliptic Nikolskii’s Inequality on compact Lie groups and consequently, in Section 4 we use it to deduce non-trivial embedding properties between subelliptic Besov spaces. In Section 5, we will use the global functional calculus and the \( L^p \)-mapping properties of pseudo-differential operators to deduce continuous embeddings between subelliptic Sobolev spaces (resp. subelliptic Besov spaces) and the Sobolev spaces (resp. Besov spaces) associated to the Laplacian. In Section 6, we introduce the subelliptic Triebel-Lizorkin spaces on compact Lie groups, their embedding properties and their interpolation properties in relation with subelliptic Besov spaces. Finally, with the analysis developed, we study the boundedness of pseudo-differential operators on subelliptic Sobolev and Besov spaces in Section 7.

2. Sub-Laplacians and Fourier analysis on compact Lie groups

Let us assume that \( G \) is a compact Lie group and let \( e = e_G \) be its identity element. For describing subelliptic Besov spaces in terms of the representation theory of compact Lie groups, we will use the Fourier transform. One reason for this is that the Fourier transform encodes the discrete spectrum of sub-Laplacians. If \( G \) is a compact Lie group, we denote by \( dx \equiv d\mu(x) \) its unique (normalised) Haar measure. We will write \( L^p(\mu) \), \( 1 \leq p \leq \infty \), for the Lebesgue spaces \( L^p(G, dx) \). The unitary dual \( \widehat{G} \) of \( G \) consists of the equivalence classes of continuous irreducible unitary representations of \( G \). Moreover, for every equivalence class \( [\xi] \in \widehat{G} \), there exists a unitary matrix-representation \( \phi_\xi \in [\xi] = [\phi_\xi] \), that is, we have a homomorphism \( \phi_\xi = (\phi_{\xi,ij})_{i,j=1}^{d_\xi} \) from \( G \) into \( U(d_\xi) \), where \( \phi_{\xi,ij} \) are continuous functions. So, we always understand a representation \( \xi = (\xi_{ij})_{i,j=1}^{d_\xi} \) as a unitary matrix representation.

The result that links the representation theory of compact Lie groups with the Hilbert space \( L^2(G) \) is the Peter–Weyl theorem, which asserts that the set

\[
B = \{ \sqrt{d_\xi} \xi_{ij} : 1 \leq i,j \leq d_\xi, [\xi] \in \widehat{G} \},
\]

is an orthonormal basis of \( L^2(G) \). So, every function \( f \in L^2(G) \) admits an expansion of the form,

\[
f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}[\xi(x)\widehat{f}(\xi)], \quad \text{a.e. } x \in G.
\]

Here, \( \operatorname{Tr}(\cdot) \) is the usual trace on matrices and, for every unitary irreducible representation \( \xi, \widehat{f}(\xi) \) is the matrix-valued Fourier transform of \( f \) at \( \xi \):

\[
\widehat{f}(\xi) := \int_G f(x)\xi(x)^* \, dx \in \mathbb{C}^{d_\xi \times d_\xi}.
\]

Taking into account the Fourier inversion formula (13), we have the Plancherel theorem

\[
\|f\|_{L^2(G)} = \left( \sum_{[\xi] \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 \right)^{\frac{1}{2}},
\]

where \( \|\cdot\|_{\text{HS}} \) denotes the Hilbert–Schmidt norm.
where \( \| A \|_{HS} = \text{Tr}(A^*A)^{\frac{1}{2}} \) is the Hilbert–Schmidt norm of matrices. Elements in the basis \( B \) are just the eigenfunctions of the positive Laplacian (defined as the Casimir element) \( \mathcal{L}_G \) on \( G \). This means that, for every \( [\xi] \in \hat{G} \), there exists \( \lambda_{[\xi]} \geq 0 \) satisfying

\[
\mathcal{L}_G \xi_{i,j} = \lambda_{[\xi]} \xi_{i,j}, \quad 1 \leq i, j \leq d_{\xi}.
\]  

(16)

We refer the reader to [27, Chapter 7], for the construction of the Laplacian through the Killing bilinear form and for the general aspects of the representation theory of compact Lie groups.

On the other hand, if \( X_1, X_2, \ldots, X_k \) are vector fields satisfying the Hörmander condition of order \( \kappa \) (see Hörmander [28])

\[
\text{Lie}\{X_j : 1 \leq j \leq k\} = T_e G,
\]

(17)

the (positive hypoelliptic) sub-Laplacian associated to the \( X_i \)'s is defined by

\[
\mathcal{L}_{\text{sub}} := -(X_1^2 + \cdots + X_k^2).
\]

(18)

The condition (17) is in turn equivalent to saying that at every point \( g \in G \), the vector fields \( X_i \) and the commutators

\[
[X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \ldots, [X_{j_1}, [X_{j_2}, \ldots, [X_{j_3}, \ldots, X_{j_\omega}]]], \quad 1 \leq \tau \leq \kappa,
\]

(19)

generate the tangent space \( T_g G \) of \( G \) at \( g \). A central notion in our work is that of the Hausdorff dimension, in this case, associated to the sub-Laplacian. Indeed, for all \( x \in G \), denote by \( H^\omega_x G \) the subspace of the tangent space \( T_x G \) generated by the \( X_i \)'s and all the Lie brackets

\[
[X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \ldots, [X_{j_1}, [X_{j_2}, \ldots, [X_{j_3}, \ldots, X_{j_\omega}]]],
\]

with \( \omega \leq \kappa \). The Hörmander condition can be stated as \( H^\kappa_x G = T_x G, x \in G \). We have the filtration

\[
H^1_x G \subset H^2_x G \subset H^3_x G \subset \cdots \subset H^{\kappa-1}_x G \subset H^\kappa_x G = T_x G, \quad x \in G.
\]

(20)

In our case, the dimension of every \( H^\omega_x G \) does not depend on \( x \) and we write \( \dim H^\omega G := \dim_x H^\omega G \) for any \( x \in G \). So, the Hausdorff dimension can be defined as (see, e.g. [29, p. 6]),

\[
Q := \dim(H^1 G) + \sum_{i=1}^{\kappa-1} (i+1)(\dim H^{i+1} G - \dim H^i G).
\]

(21)

Because the symmetric operator \( \mathcal{L}_{\text{sub}} \) acting on \( C^\infty(G) \) admits a self-adjoint extension on \( L^2(G) \), that we also denote by \( \mathcal{L}_{\text{sub}} \), functions of the sub-Laplacian \( \mathcal{L}_{\text{sub}} \) can be defined
according to the spectral theorem by
\[ f(\mathcal{L}_{\text{sub}}) = \int_0^\infty f(\lambda) \, dE_\lambda, \tag{22} \]
for every measurable function \( f \) defined on \( \mathbb{R}_0^+ := [0, \infty) \). Here \( \{dE_\lambda\}_{\lambda > 0} \) is the spectral measure associated to the spectral resolution of \( \mathcal{L}_{\text{sub}} \). In this case,
\[ \text{Dom}(f(\mathcal{L}_{\text{sub}})) = \left\{ f \in L^2(G) : \int_0^\infty |f(\lambda)|^2 \, d\|E_\lambda f\|_{L^2(G)}^2 < \infty \right\}, \tag{23} \]
where \( d\|E_\lambda f\|_{L^2(G)}^2 = d(E_\lambda f, f)_{L^2(G)} \) is the Riemann–Stieltjes measure induced by the spectral resolution \( \{E_\lambda\}_{\lambda > 0} \). In our further analysis, the functions \( \tilde{f}_\alpha(t) = t^{-\alpha} \) and \( f_\alpha(t) = (1 + t)^{-\alpha} \) will be useful, defining the operators
\[ (\mathcal{L}_{\text{sub}})^{-\alpha} := \tilde{f}_\alpha(\mathcal{L}_{\text{sub}}), \quad (1 + \mathcal{L}_{\text{sub}})^{-\alpha} := f_\alpha(\mathcal{L}_{\text{sub}}), \tag{24} \]
which are the homogeneous and inhomogeneous subelliptic Bessel potentials of order \( \alpha \).

### 3. Subelliptic Nikolskii’s inequality on compact Lie groups

Our main tool for obtaining embedding properties for Besov spaces in the subelliptic context is a suitable version of the Nikolskii inequality. We refer the reader to Bahouri et al. [30] for the proof of Bernstein inequalities and Besov embeddings in the Euclidean case. To present a subelliptic Nikolskii inequality, let us use the usual nomenclature. If \( t_{\tilde{c}} \) is a compactly supported function on \([0, \tilde{c}]\), for \( \tilde{c} > 0 \), we can define the operator
\[ T_{\tilde{c}} := t_{\tilde{c}}((1 + \mathcal{L}_{\text{sub}})^{\frac{1}{2}}) : C^\infty(G) \to C^\infty(G), \tag{25} \]
by the functional calculus. An explicit representation for \( T_{\tilde{c}} \) can be given by
\[ T_{\tilde{c}}f(x) = \sum_{[\xi] \in \hat{G}} \text{d}_\xi \text{Tr}[\xi(x) t_{\tilde{c}}((I_\xi + \hat{\mathcal{L}}_{\text{sub}}(\xi))^{\frac{1}{2}}\hat{f}(\xi))], \quad f \in C^\infty(G), \tag{26} \]
where \( \hat{\mathcal{L}}_{\text{sub}}(\xi) = \hat{k}_{\text{sub}}(\xi) \) is the Fourier transform of the right convolution kernel \( k_{\text{sub}} \) of the sub-Laplacian \( \mathcal{L}_{\text{sub}} \). In general, we write \( t_{\tilde{c}}(A) \) applied to continuous operators \( A \) on \( C^\infty(G) \), but we also write \( t_{\tilde{c}}(\hat{A}(\xi)) \) applied to \( \hat{A}(\xi) \), for its symbol \( \tilde{t}_c(A)(\xi) \). So, we present our subelliptic Nikolskii’s inequality. Our starting point is the following lemma proved in [31, p. 52].

**Lemma 3.1:** If \( 1 < p \leq 2 \leq q < \infty \), and \( f \in C^\infty(G) \), the estimate
\[ \|f\|_{L^q(G)} \leq C\|1 + \mathcal{L}_{\text{sub}}\|_{L^p(G)}^{\frac{q}{p}} \|f\|_{L^p(G)}, \tag{27} \]
holds true for all \( \gamma \geq Q(1/p - 1/q) \).

Here \( Q \) is the Hausdorff dimension of \( G \) associated to the Carnot–Caratheodory distance \( \rho \) associated to \( \mathcal{L}_{\text{sub}} \) (see, e.g. [29, p. 6]), which we have defined in (21).
Theorem 3.2 (Subelliptic Nikolskii Inequality): Let $G$ be a compact Lie group and let $t_{\tilde{\ell}}$ be a measurable function compactly supported in $[\frac{1}{2} \tilde{\ell}, \tilde{\ell}]$, $\tilde{\ell} > 0$. If $T_{\tilde{\ell}}$ is the operator defined by (26), then for all $1 < p \leq q < \infty$, we have

$$
\| T_{\tilde{\ell}}f \|_{L^q(G)} \leq C_{p,q}^{Q\frac{1}{p} - \frac{1}{q}} \| T_{\tilde{\ell}}f \|_{L^p(G)}, \quad f \in C^\infty(G),
$$

where $C = C_{p,q}^{Q}$ is independent on $f$ and $\tilde{\ell}$.

Proof: Note that the spectrum of $(1 + L_{\text{sub}})^{1/2}$ lies in $[1, \infty)$, so that for $0 \leq \tilde{\ell} < 1$,

$$
[\tilde{\ell}/2, \tilde{\ell}] \cap \text{Spec}[(I_{d_\xi} + \hat{L}_{\text{sub}}(\xi))^{1/2}] = \emptyset.
$$

Consequently, for $0 \leq \tilde{\ell} < 1$,

$$
T_{\tilde{\ell}} = \int_1^\infty t_{\tilde{\ell}}((1 + \lambda)^{1/2}) \, dE_{\lambda} = 0,
$$

is the null operator. In this case, we have nothing to prove. So, we will assume that $\tilde{\ell} \geq 1$. If $1 < p \leq 2 \leq q < \infty$, Lemma 3.1 gives

$$
\| T_{\tilde{\ell}}f \|_{L^q(G)} \leq C \| (1 + L_{\text{sub}})^{\gamma/2} T_{\tilde{\ell}}f \|_{L^p(G)},
$$

for $\gamma \geq Q(1/p - 1/q)$. In particular, if $\gamma = Q(1/p - 1/q)$, and $\{dE_{\lambda}^M \}_{1 \leq \lambda < \infty}$ is the spectral measure associated with $M = (1 + L_{\text{sub}})^{1/2}$, by the functional calculus, we can estimate

$$
\| (1 + L_{\text{sub}})^{\gamma/2} T_{\tilde{\ell}}f \|_{L^p(G)} = \left\| \int_0^\infty \lambda^\gamma t_{\tilde{\ell}}(\lambda) \, dE_{\lambda}^M f \right\|_{L^p(G)}
$$

$$
= \tilde{\ell}^\gamma \left\| \int_{\tilde{\ell}/2}^{\tilde{\ell}} \tilde{\ell}^{-\gamma} \lambda^\gamma t_{\tilde{\ell}}(\lambda) \, dE_{\lambda}^M f \right\|_{L^p(G)}
$$

$$
\times \tilde{\ell}^\gamma \left\| \int_{\tilde{\ell}/2}^{\tilde{\ell}} t_{\tilde{\ell}}(\lambda) \, dE_{\lambda}^M f \right\|_{L^p(G)}.
$$

Consequently

$$
\| (1 + L_{\text{sub}})^{\gamma/2} T_{\tilde{\ell}}f \|_{L^p(G)} \lesssim \tilde{\ell}^\gamma \left\| \int_{\tilde{\ell}/2}^{\tilde{\ell}} t_{\tilde{\ell}}(\lambda) \, dE_{\lambda}^M f \right\|_{L^p(G)} = \tilde{\ell}^\gamma \| T_{\tilde{\ell}}f \|_{L^p(G)}.\quad (30)
$$

So, we obtain

$$
\| T_{\tilde{\ell}}f \|_{L^q(G)} \leq C_{p,q}^{Q\frac{1}{p} - \frac{1}{q}} \| T_{\tilde{\ell}}f \|_{L^p(G)}, \quad 1 < p \leq 2 \leq q < \infty.
$$

Now, we generalise this result to the complete range $1 < p \leq q \leq 2$ by using interpolation inequalities. In this case from the inequality $1 < p \leq q \leq q' < \infty$, we get the following
Thus we obtain the estimate
\[ \| T_{\tilde{f}} \|_{L^q(G)} \leq C \tilde{\varepsilon}^Q(\frac{1}{p} - \frac{1}{q}) \| T_{\tilde{f}} \|_{L^p(G)} \] (32)
and
\[ \| T_{\tilde{f}} \|_{L^q(G)} \leq \| T_{\tilde{f}} \|_{L^p(G)}^{\theta} \| T_{\tilde{f}} \|_{L^{q'}(G)}^{1-\theta}, \quad 1/q = \theta/p + (1-\theta)/q', \] (33)
where we have used the interpolation inequality (see, e.g. Brezis [32, Chapter 4])
\[ \| h \|_{L^r(X, \mu)} \leq \| h \|_{L^0(X, \mu)}^{\theta} \| h \|_{L^{r1}(X, \mu)}^{1-\theta}, \quad 1/r = \theta/r_0 + (1-\theta)/r_1, \quad 1 \leq r_0 < r_1 \leq \infty. \] (34)

So, we have
\[ \| T_{\tilde{f}} \|_{L^q(G)} \leq \| T_{\tilde{f}} \|_{L^p(G)}^{\theta} \tilde{\varepsilon}^{Q(1-\theta)(\frac{1}{p} - \frac{1}{q})} \| T_{\tilde{f}} \|_{L^p(G)}^{1-\theta} = \tilde{\varepsilon}^{Q(1-\theta)(\frac{1}{p} - \frac{1}{q})} \| T_{\tilde{f}} \|_{L^p(G)}. \]
Since \( \theta \) satisfies \( (1-\theta)/q' = 1/q - \theta/p \), from the identity
\[ \tilde{\varepsilon}^{Q(1-\theta)(\frac{1}{p} - \frac{1}{q'})} = \tilde{\varepsilon}^{Q(\frac{1}{p} - \frac{1}{q})}, \]
we complete the proof for this case. Now, if \( 2 \leq p \leq q < \infty \), the proof will be based on the action of the linear operator \( T_{\tilde{\ell}} \) on the Banach spaces
\[ X_{s, \tilde{\ell}}(G) := \{ f \in L^s(G) : T_{\tilde{\ell}} f = f \}, \quad \tilde{\ell} \geq 1, \quad 1 < s < \infty, \] (35)
endowed with the norm
\[ \| f \|_{X_{s, \tilde{\ell}}(G)} := \| f \|_{L^s(G)} = \left( \int_G |f(x)|^s \, dx \right)^{\frac{1}{s}}. \] (36)
Putting \( s = p' \) and \( r = q' \) and taking into account that the parameters \( r, s \) satisfy the inequality, \( 1 < r \leq s \leq 2 \), we have the estimate
\[ \| f \|_{X_{s, \tilde{\ell}}(G)} = \| T_{\tilde{f}} f \|_{L^r(G)} \leq C \tilde{\varepsilon}^{Q(\frac{1}{s} - \frac{1}{r})} \| f \|_{X_{r, \tilde{\ell}}(G)}. \] (37)
Consequently, \( T_{\tilde{\ell}} : X_{s, \tilde{\ell}}(G) \to X_{s, \tilde{\ell}}(G) \) extends to a bounded operator with the operator norm satisfying
\[ \| T_{\tilde{\ell}} \|_{\mathcal{B}(X_{s, \tilde{\ell}}(G), X_{s, \tilde{\ell}}(G))} \leq C \tilde{\varepsilon}^{Q(\frac{1}{s} - \frac{1}{r})}. \] (38)

By using the identification, \( (X_{s, \tilde{\ell}}(G))^* = X_{q, \tilde{\ell}}(G) \) and \( (X_{s, \tilde{\ell}}(G))^* = X_{r, \tilde{\ell}}(G) \) where \( E^* \) denotes the dual of a Banach space \( E \), the argument of duality gives the boundedness of \( T_{\tilde{\ell}} \) from \( X_{s, \tilde{\ell}}(G) \) into \( X_{q, \tilde{\ell}}(G) \) with operator norm satisfying,
\[ \| T_{\tilde{\ell}} \|_{\mathcal{B}(X_{q, \tilde{\ell}}(G), X_{r, \tilde{\ell}}(G))} \leq C \tilde{\varepsilon}^{Q(\frac{1}{s} - \frac{1}{r})} = C \tilde{\varepsilon}^{Q(\frac{1}{q} - \frac{1}{r})} = C \tilde{\varepsilon}^{Q(\frac{1}{r} - \frac{1}{q})}. \] (39)
Thus we obtain the estimate
\[ \| T_{\tilde{f}} \|_{L^q(G)} \leq C \tilde{\varepsilon}^{Q(\frac{1}{p} - \frac{1}{q})} \| T_{\tilde{f}} \|_{L^p(G)}, \quad 2 \leq p \leq q < \infty. \] (40)
So, we finish the proof.
Corollary 3.3: Let $G$ be a compact Lie group and let $t_{\tilde{\ell}}$ be a measurable function compactly supported in $[\frac{1}{2}\tilde{\ell}, \tilde{\ell}]$, $\tilde{\ell} > 0$. If $T_{\tilde{\ell}}$ is the operator defined by (26), then $T_{\tilde{\ell}}$ from $X_{p,\tilde{\ell}}(G)$ into $X_{q,\tilde{\ell}}(G)$ extends to a bounded operator, with the operator norm satisfying

$$\|T_{\tilde{\ell}}\|_{\mathcal{B}(X_{q,\tilde{\ell}}(G), X_{p,\tilde{\ell}}(G))} \leq C\tilde{\ell}^{Q(\frac{1}{p} - \frac{1}{q})},$$

for all $1 \leq p \leq q < \infty$. Moreover, for all $1 \leq p < \infty$, $T_{\tilde{\ell}}$ from $X_{p,\tilde{\ell}}(G)$ into $X_{p,\tilde{\ell}}(G)$ extends to a bounded operator, with the operator norm satisfying

$$\|T_{\tilde{\ell}}\|_{\mathcal{B}(X_{p,\tilde{\ell}}(G), X_{p,\tilde{\ell}}(G))} \leq C.$$  (42)

4. Subelliptic Besov spaces

In this section, we introduce and examine some embedding properties for those Besov spaces associated to sub-Laplacians on compact Lie groups which we call subelliptic Besov spaces. Throughout this section, $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying the Hörmander condition of order $\kappa \in \mathbb{N}$, and by simplicity throughout this work we will use the following notation: $\mathcal{L} \equiv \mathcal{L}_{\text{sub}} = -\sum_{j=1}^{k} X_j^2$ for the associated sub-Laplacian. This condition guarantees the hypoellipticity of $\mathcal{L}$ [28].

4.1. Motivation and definition of subelliptic Besov spaces

The main tool in the formulation of the Besov spaces is the notion of dyadic decompositions. We say that the sequence $\{\psi_{\ell}\}_{\ell \in \mathbb{N}_0}$ of real functions is a dyadic decomposition, if it is defined as follows: we choose a function $\psi_0 \in C^\infty_0(\mathbb{R})$, $\psi_0(\lambda) = 1$, if $|\lambda| \leq 1$, and $\psi_0(\lambda) = 0$, for $|\lambda| \geq 2$. For every $\tilde{\ell} \geq 1$, let us define $\psi_{\tilde{\ell}}(\lambda) = \psi_0(2^{-\tilde{\ell}}\lambda) - \psi_0(2^{-\tilde{\ell}+1}\lambda)$. For $\psi(\lambda) := \psi_0(\lambda) - \psi_0(2\lambda)$, $\psi_{\tilde{\ell}}(\lambda) = \psi(2^{-\tilde{\ell}}\lambda)$. In particular, we have

$$\sum_{\tilde{\ell} \in \mathbb{N}_0} \psi_{\tilde{\ell}}(\lambda) = 1, \quad \text{for every } \lambda > 0. \quad (43)$$

We define the operator

$$\psi_{\tilde{\ell}}(D)f(x) := \sum_{[\xi] \in Q} d_{\xi} \text{Tr}[\xi(x)\psi_{\tilde{\ell}}((I_{d_k} + \tilde{\mathcal{L}}(\xi))\frac{1}{2})\hat{f}(\xi)], \quad f \in C^\infty(G). \quad (44)$$

If we denote by $\{E_{\lambda}\}_{0 \leq \lambda < \infty}$, $E_{\lambda} := E_{\{0,\lambda\}}$, the spectral measure associated to the subelliptic Bessel potential $B_{\mathcal{L}} = (1 + \mathcal{L})^{\frac{1}{2}}$, we have $\psi_{\tilde{\ell}}(D) = \int_{2^{\tilde{\ell}}}^{2^{\tilde{\ell}+1}} \psi_{\tilde{\ell}}(\lambda) \, dE_{\lambda}$. From the almost-orthogonality of the sequence $\{\psi_{\ell}\}_{\ell \in \mathbb{N}_0}$, for every $f \in L^2(G)$ we have

$$\|f\|_{L^2(G)} = \left(\sum_{\ell = 0}^{\infty} \psi_{\ell}(D)f \right)_{L^2(G)} \leq \left(\sum_{\ell = 0}^{\infty} \|\psi_{\ell}(D)f\|_{L^2(G)}^2\right)^{\frac{1}{2}}. \quad (45)$$

This simple analysis shows that we can measure the square-integrability of a function $f$ using the family of operators $\psi_{\tilde{\ell}}(D)$, $\tilde{\ell} \in \mathbb{N}_0$. To measure the regularity of functions/distributions, it is usual to use Sobolev spaces, in our case, associated to sub-Laplacians. But, if we want to measure at the same time, the regularity and the integrability
of functions/distributions, Besov spaces are the appearing spaces. By following [33], the subelliptic Sobolev space $H^{s,L}(G)$ of order $s \in \mathbb{R}$, is defined by the condition

$$f \in H^{s,L}(G) \text{ if and only if } \|f\|_{H^{s,L}(G)} := \|(1 + \mathcal{L})^{\frac{s}{2}}f\|_{L^2(G)} < \infty.$$  

(46)

So, in terms of the family $\psi_{\tilde{\ell}}(D)$, $\tilde{\ell} \in \mathbb{N}_0$, we can write

$$\|f\|_{H^{s,L}(G)} = \left\| \sum_{\tilde{\ell}=0}^{\infty} \psi_{\tilde{\ell}}(D)(1 + \mathcal{L})^{\frac{s}{2}}f \right\|_{L^2(G)} = \left( \sum_{\tilde{\ell}=0}^{\infty} \|\psi_{\tilde{\ell}}(D)(1 + \mathcal{L})^{\frac{s}{2}}f\|_{L^2(G)}^2 \right)^{\frac{1}{2}}.$$  

(47)

This allows us to write

$$\|\psi_{\tilde{\ell}}(D)(1 + \mathcal{L})^{\frac{s}{2}}f\|_{L^2(G)} := \left\| \int_{2^{\tilde{\ell}}}^{2^{\tilde{\ell}+1}} \psi_{\tilde{\ell}}(\lambda)(1 + \mathcal{L})^{\frac{s}{2}}f \right\|_{L^2(G)}$$

$$= \left\| \int_{2^{\tilde{\ell}}}^{2^{\tilde{\ell}+1}} \psi_{\tilde{\ell}}(\lambda)(1 + \lambda)^s \, dE_2f \right\|_{L^2(G)}$$

$$= 2^{st} \left\| \int_{2^{\tilde{\ell}}}^{2^{\tilde{\ell}+1}} \psi_{\tilde{\ell}}(\lambda)(2^{-\tilde{\ell}}(1 + \lambda))^s \, dE_2f \right\|_{L^2(G)}$$

$$\times 2^{s\tilde{\ell}} \left\| \int_{2^{\tilde{\ell}}}^{2^{\tilde{\ell}+1}} \psi_{\tilde{\ell}}(\lambda) \, dE_2f \right\|_{L^2(G)}$$

$$= 2^{s\tilde{\ell}} \|\psi_{\tilde{\ell}}(D)f\|_{L^2(G)}.$$  

So, we obtain the following estimate:

$$\|f\|_{H^{s,L}(G)} \asymp \left( \sum_{\tilde{\ell}=0}^{\infty} 2^{s\tilde{\ell}} \|\psi_{\tilde{\ell}}(D)f\|_{L^2(G)}^2 \right)^{\frac{1}{2}} =: \|f\|_{B^{s,L}_{2,2}(G)}.$$  

(48)

We then deduce that the norm appearing in the right-hand side of (48) defines an equivalent norm for the Sobolev space $H^{s,L}(G)$. With this discussion in mind, Besov spaces can be introduced by the varying of the parameters measuring the integrability and the regularity of functions/distributions in (48). So, for $s \in \mathbb{R}$, $0 < q < \infty$, the subelliptic Besov space $B^{s,L}_{p,q}(G)$ consists of those functions/distributions satisfying

$$\|f\|_{B^{s,L}_{p,q}(G)} = \left( \sum_{\tilde{\ell}=0}^{\infty} 2^{s\tilde{\ell}} \|\psi_{\tilde{\ell}}(D)f\|_{L^p(G)}^q \right)^{\frac{1}{q}} < \infty,$$  

(49)

for $0 < p \leq \infty$, with the following modification:

$$\|f\|_{B^{s,L}_{p,\infty}(G)} = \sup_{\tilde{\ell} \in \mathbb{N}_0} 2^{s\tilde{\ell}} \|\psi_{\tilde{\ell}}(D)f\|_{L^p(G)} < \infty,$$  

(50)

when $q = \infty$. Clearly, from the earlier discussion, for every $s \in \mathbb{R}$, $H^{s,L}(G) = B^{s,L}_{2,2}(G)$, and particularly, $L^2(G) = B^{0,L}_{2,2}(G)$. Sobolev spaces modelled on $L^p$-spaces and associated to the
The positive Laplacian on a compact Lie group is given by

$$\mathcal{L}_G := -(X_1^2 + \cdots + X_n^2), \quad n = \text{dim}(G),$$

(52)

where $X = \{X_i : 1 \leq i \leq n\}$ is a basis for the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ of $G$. The global symbol associated to the Laplacian is given by

$$\hat{\mathcal{L}}_G(\xi) = \lambda_{[\xi]} I_{d_{\xi}},$$

(53)

where $\{\lambda_{[\xi]} : [\xi] \in \widehat{G}\}$ is the spectrum of the Laplacian. This sequence has the property that $\mathcal{L}_G \xi_{ij} = \lambda_{[\xi]} \xi_{ij}$, $1 \leq i, j \leq d_{\xi}$, where $d_{\xi}$ is the dimension of the representation $\xi$, for every $[\xi] \in \widehat{G}$. This means that $\lambda_{[\xi]}$ is an eigenvalue of the Laplacian, with the corresponding eigenspace $E_{\xi} := \{\xi_{ij}(x) : 1 \leq i, j \leq d_{\xi}\}$. Also, we note that the multiplicity of the eigenvalue $\lambda_{[\xi]}$ is $d_{\xi}^2$. Putting $\langle \xi \rangle := (1 + \lambda_{[\xi]})^{\frac{1}{2}}$, we have $(I_{\xi} + \hat{\mathcal{L}}_G(\xi))^\frac{1}{2} = \langle \xi \rangle I_{d_{\xi}}$. Consequently, for every $[\xi] \in \widehat{G}$ and $\ell \in \mathbb{N}_0$, the symbol of the operator $\psi_{\ell}(I + \mathcal{L}_G)^{\frac{1}{2}}$ is $\psi_{\ell}(\langle \xi \rangle) I_{d_{\xi}}$. This argument implies that the sequence of operators $\{\psi_{\ell}(D)\}_{\ell}$ is defined by

$$\psi_{\ell}(I + \mathcal{L}_G)^{\frac{1}{2}} f(x) := \sum_{2^\ell \leq \langle \xi \rangle < 2^{\ell+1}} d_{\xi} \text{Tr}[\xi(x) \psi_{\ell}(\langle \xi \rangle) \hat{f}(\xi)], \quad f \in C^\infty(G).$$

(54)

We conclude that the Besov spaces on compact Lie groups associated to the Laplacian are defined by

$$\|f\|_{B^{s,p}_{g,G}(G)} = \left( \sum_{\ell=0}^{\infty} \sum_{2^\ell \leq \langle \xi \rangle < 2^{\ell+1}} \langle \xi \rangle^{qs} d_{\xi} \text{Tr}[\xi(x) \psi_{\ell}(\langle \xi \rangle) \hat{f}(\xi)] \right)^{\frac{1}{q}} L^p(G) < \infty,$$

(55)

for $0 < p \leq \infty$, and

$$\|f\|_{B^{s,p,\infty}_{g,G}(G)} = \sup_{\ell \in \mathbb{N}_0} \sum_{2^\ell \leq \langle \xi \rangle < 2^{\ell+1}} \langle \xi \rangle^{qs} d_{\xi} \text{Tr}[\xi(x) \psi_{\ell}(\langle \xi \rangle) \hat{f}(\xi)] < \infty,$$

(56)

for $q = \infty$. The definitions (55) and (56) were introduced by E. Nursultanov, S. Tikhonov and the second author in [12] and consistently developed on arbitrary compact homogeneous manifolds in [13].
Remark 4.2 (Fourier description for subelliptic Besov spaces): Now, we discuss the formulation of subelliptic Besov spaces in terms of the Fourier analysis associated to the sub-Laplacian $\mathcal{L} = -\sum_{j=1}^{k} X_{j}^{2}$. The self-adjointness of this operator implies that the global symbol $\hat{\mathcal{L}}(\xi)$ is symmetric and it can be assumed diagonal for every $[\xi] \in \hat{G}$, under a suitable choice of the basis in the representation spaces. So, there exists a non-negative sequence $\{\nu_{ii}\}_{i=1}^{d}$, such that $\hat{\mathcal{L}}(\xi) = \text{diag}(\nu_{ii}(\xi)^{2})_{1 \leq i \leq d_{k}}$. (57)

So, the symbol $\hat{M}(\xi) := (M\xi)(e_{G})$ of the operator $(1 + \mathcal{L})^{1/2}$ is given by

$$\hat{M}(\xi) = \begin{bmatrix}
(1 + v_{11}(\xi)^{2})^{1/2} & 0 & 0 & \ldots & 0 \\
0 & (1 + v_{22}(\xi)^{2})^{1/2} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (1 + v_{d_{k}d_{k}}(\xi)^{2})^{1/2}
\end{bmatrix}. \quad (58)$$

We write $\hat{M}(\xi) = \text{diag}[(1 + v_{ii}(\xi)^{2})^{1/2}]_{1 \leq i \leq d_{k}}$. If $\psi_{\tilde{\ell}}(\xi)$ denotes the symbol of the operator $\psi_{\tilde{\ell}}(D)$, then we have

$$\psi_{\tilde{\ell}}(\xi) = \text{diag}[\psi_{\tilde{\ell}}((1 + v_{ii}(\xi)^{2})^{1/2})]_{1 \leq i \leq d_{k}}, \quad \tilde{\ell} \in \mathbb{N}_{0}, \quad [\xi] \in \hat{G}. \quad (59)$$

We conclude that the analogues of definitions (55) and (56) for subelliptic Besov spaces can be rewritten as

$$\|f\|_{q, L^{p}, q(G)}^{B_{p}^{s}} = \sum_{\tilde{\ell}=0}^{\infty} 2^{\tilde{\ell}s} \left\| \sum_{[\xi] \in \hat{G}} d_{\xi} \text{Tr}[\xi(x)\text{diag}[\psi_{\tilde{\ell}}((1 + v_{ii}(\xi)^{2})^{1/2})]\hat{f}(\xi)] \right\|_{L^{p}(G)}^{q} < \infty, \quad (60)$$

for $0 < p \leq \infty$, and

$$\|f\|_{p, \infty(G)}^{B_{p}^{s}} = \sup_{\tilde{\ell} \in \mathbb{N}_{0}} 2^{\tilde{\ell}s} \left\| \sum_{[\xi] \in \hat{G}} d_{\xi} \text{Tr}[\xi(x)\text{diag}[\psi_{\tilde{\ell}}((1 + v_{ii}(\xi)^{2})^{1/2})]\hat{f}(\xi)] \right\|_{L^{p}(G)} < \infty, \quad (61)$$

for $q = \infty$.

4.2. Embedding properties for subelliptic Besov spaces

In this section we study some embedding properties between Besov spaces and their interplay with Sobolev spaces modelled on $L^{p}$-spaces. To accomplish this, we will use the
Littlewood–Paley theorem,

$$\|f\|_{L^p(G)}^{\text{Br}_{p,2}^q} := \left\| \sum_{s \in \mathbb{N}_0} |\psi_s(D)f(x)|^2 \right\|_{L^p(G)}^{\frac{1}{2}} \lesssim \|f\|_{L^p(G)}.$$

A proof of this result can be found in Furioli et al. [34] where the Littlewood–Paley theorem for sub-Laplacians was proved for Lie groups of polynomial growth.

**Theorem 4.3:** Let G be a compact Lie group and let us denote by Q the Hausdorff dimension of G associated to the control distance associated to the sub-Laplacian \( \mathcal{L} = -(X^2_1 + \cdots + X^2_n) \), where the system of vector fields \( X = \{X_i\} \) satisfies the Hörmander condition. Then

1. \( B_{p, q_1}^{r, \varepsilon, \mathcal{L}}(G) \hookrightarrow B_{p, q_2}^{r, \varepsilon, \mathcal{L}}(G) \hookrightarrow B_{p, \infty}^{r, \mathcal{L}}(G), \ \varepsilon > 0, \ 0 < p \leq \infty, \ 0 < q_1 \leq q_2 \leq \infty. \)
2. \( B_{p, q_1}^{r, \varepsilon, \mathcal{L}}(G) \hookrightarrow B_{p, q_2}^{r, \varepsilon, \mathcal{L}}(G), \ \varepsilon > 0, \ 0 < p \leq \infty, \ 1 \leq q_2 < q_1 < \infty. \)
3. \( B_{p_1, q_1}^{r, \mathcal{L}}(G) \hookrightarrow B_{p_2, q_2}^{r, \mathcal{L}}(G), \ 1 \leq p_1 \leq p_2 \leq \infty, \ 0 < q < \infty, \ r_1 \in \mathbb{R}, \ \text{and} \ r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2}). \)
4. \( H^{r, \mathcal{L}}(G) = B_{2, 2}^{r, \varepsilon, \mathcal{L}}(G) \) and \( B_{p, q}^{r, \mathcal{L}}(G) \hookrightarrow L_r^{p, \mathcal{L}}(G) \hookrightarrow L_r^{p, \mathcal{L}}(G), \ 1 < p \leq 2. \)
5. \( B_{p, 1}^{r, \mathcal{L}}(G) \hookrightarrow L^q(G), \ 1 \leq p \leq q \leq \infty, \ r = Q(\frac{1}{p} - \frac{1}{q}) \) and \( L^q(G) \hookrightarrow B_{q, \infty}^{0, \mathcal{L}}(G) \) for \( 1 < q \leq \infty. \)

**Proof:** For the proof of (1) let us fix \( \varepsilon > 0, \ 0 < p \leq \infty, \) and \( 0 < q_1 \leq q_2 \leq \infty. \) We observe that

$$\|f\|_{B_{p, q_1}^{r, \mathcal{L}}(G)} = \sup_{s \in \mathbb{N}_0} 2^{rs} \|\psi_s(D)f\|_{L^p} \leq \|\{2^{sr}\|\psi_s(D)f\|_{L^p}\}_s \|_{\mathcal{H}^q(\mathbb{N}_0)} \equiv \|f\|_{B_{p, q_2}^{r, \mathcal{L}}(G)}. \ (63)$$

So, we have that \( B_{p, q_2}^{r, \mathcal{L}}(G) \hookrightarrow B_{p, \infty}^{r, \mathcal{L}}(G). \) From the estimates

$$\|f\|_{B_{p, q_2}^{r, \mathcal{L}}(G)} \leq \|\{2^{sr}\|\psi_s(D)f\|_{L^p}\}_s \|_{\mathcal{H}^q(\mathbb{N}_0)} \equiv \|f\|_{B_{p, q_1}^{r, \mathcal{L}}(G)}, \ (64)$$

and

$$\|f\|_{B_{p, q_1}^{r, \mathcal{L}}(G)} \leq \|\{2^{sr}\|\psi_s(D)f\|_{L^p}\}_s \|_{\mathcal{H}^q(\mathbb{N}_0)} \equiv \|f\|_{B_{p, q_1}^{r, \mathcal{L}}(G)}, \ (65)$$

we deduce the embeddings \( B_{p, q_1}^{r, \mathcal{L}}(G) \hookrightarrow B_{p, q_2}^{r, \mathcal{L}}(G) \hookrightarrow B_{p, \infty}^{r, \mathcal{L}}(G). \) For the proof of (2), we consider \( \varepsilon > 0, \ 0 < p \leq \infty, \ 1 \leq q_2 < q_1 < \infty. \) Now, by the Hölder inequality,

$$\|f\|_{B_{p, q_2}^{r, \mathcal{L}}(G)} = \|\{2^{sr}\|\psi_s(D)f\|_{L^p}\}_s \|_{\mathcal{H}^q(\mathbb{N}_0)} = \|\{2^{sr}\|\psi_s(D)f\|_{L^p}\}_s \|_{\mathcal{H}^q(\mathbb{N}_0)} \leq \|\{2^{sr}\|\psi_s(D)f\|_{L^p}\}_s \|_{\mathcal{H}^q(\mathbb{N}_0)} \frac{1}{q_2} \left( \sum_{s \in \mathbb{N}_0} 2^{-\frac{s \varepsilon q_1 q_2}{q_1 - q_2}} \right)^{\frac{1}{2}}$$

$$\lesssim \|f\|_{B_{p, q_1}^{r, \mathcal{L}}(G)}.$$
So, we obtain that $B_{p_1,q_1}^{r+\varepsilon,L}(G) \hookrightarrow B_{p_2,q_2}^{r,L}(G)$. To prove (3), we use from Theorem 3.2 the estimate
\[
\|\psi_s(D)f\|_{L^2} \leq C_2\|\psi_s(D)f\|_{L^1}, \quad 1 \leq p_1 \leq p_2 < \infty,
\] (66)
so that we have
\[
\left(\sum_{s \in \mathbb{N}_0} 2^{sr^2q_s} \|\psi_s(D)f\|_{L^2(G)}^q\right)^{\frac{1}{q}} \leq \left(\sum_{s \in \mathbb{N}_0} 2^{s[r_2+Q(\frac{1}{p_1} - \frac{1}{p_2})]q} \|\psi_s(D)f\|_{L^1(G)}^q\right)^{\frac{1}{q}}.
\]
This estimate proves that $B_{p_1,q}^{r,L}(G) \hookrightarrow B_{p_2,q}^{r,L}(G)$, for $1 \leq p_1 \leq p_2 \leq \infty$, $0 < q < \infty$, $r_1 \in \mathbb{R}$ and $r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2})$. Now we will prove (4), that is $B_{p,p}^{r,L}(G) \hookrightarrow L_{r,p}^{r,L}(G) \hookrightarrow B_{p_2}^{r,L}(G)$, for $1 < p \leq 2$. By the definition of the subelliptic $p$-Sobolev norm, $\|f\|_{L_{p,r}^{r,L}} \equiv \|(1 + \mathcal{L})^{\frac{r}{2}} f\|_{L^p}$, if $(E^M(\lambda))_{0 \leq \lambda < \infty}$ is the spectral measure associated to $\mathcal{M} = (1 + \mathcal{L})^{\frac{r}{2}}$, we have
\[
\|f\|_{L_{p,r}^{r,L}}^p = \|\mathcal{M}^r f\|_{L^p}^p = \left\|\int_0^\infty \lambda^r \, dE^M(\lambda)f\right\|_{L^p}^p
\]
\[
= \left\|\sum_{s \in \mathbb{N}_0} \int_{2^s}^{2^{s+1}} \lambda^r \, dE^M(\lambda)f\right\|_{L^p}^p \leq \sum_{s \in \mathbb{Z}} \left\|\int_{2^s}^{2^{s+1}} \lambda^r \, dE^M(\lambda)f\right\|_{L^p}^p
\]
\[
= \sum_{s \in \mathbb{Z}} 2^{sr^2} \left\|\int_{2^s}^{2^{s+1}} \lambda^r \, dE^M(\lambda)f\right\|_{L^p}^p
\]
\[
= \sum_{s \in \mathbb{Z}} 2^{sr^2} \left\|\int_{2^s}^{2^{s+1}} dE^M(\lambda)f\right\|_{L^p}^p \times \sum_{s \in \mathbb{N}_0} 2^{sr^2} \|\psi_s(D)f\|_{L^p}^p
\]
\[
= \|f\|_{B_{p,p}^{r,L}}^p.
\]
For the other embedding, we use the following version of the Minkowski integral inequality:
\[
\left(\sum_{j=0}^{\infty} \left(\int_X |f_j(x)| \, d\mu(x)\right)^\alpha\right)^{\frac{1}{\alpha}} \leq \int_X \left(\sum_{j=0}^{\infty} |f_j(x)|^\alpha\right)^{\frac{1}{\alpha}} \, d\mu(x), \quad f_j \text{ measurable}, \quad \alpha = \frac{\frac{1}{p}}{\frac{1}{p} - 1},
\]
with $\alpha = \frac{\frac{1}{p}}{\frac{1}{p} - 1}$. So, we get
\[
\|f\|_{B_{p_2}^{r,L}} = \left(\sum_{s \in \mathbb{N}_0} 4^{sr^2} \|\psi_s(D)f\|_{L^p}^p\right)^{\frac{1}{p}} \leq \left(\sum_{s \in \mathbb{N}_0} 4^{sr^2} \left[\int_G |\psi_s(D)f(x)|^p \, dx\right]^{\frac{1}{p}}\right)^{\frac{1}{p}}
\]
\[
\leq \left[\int_G \left(\sum_{s \in \mathbb{N}_0} 4^{sr^2} |\psi_s(D)f(x)|^{\frac{2p}{p}} \, dx\right)\right]^{\frac{1}{p}}
\]
\[
\left[ \int_G \left( \sum_{s \in \mathbb{N}_0} 4^{sr} |\psi_s(D)^s f(x)|^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} = \left[ \sum_{s \in \mathbb{N}_0} 4^{sr} |\psi_s(D)^s f(x)|^2 \right]^{\frac{1}{p}} \lesssim \left[ \sum_{s \in \mathbb{N}_0} |\psi_s(D)(1 + \mathcal{L})^{\frac{r}{2}} f(x)|^2 \right]^{\frac{1}{2}} \leq \| (1 + \mathcal{L})^{\frac{r}{2}} f \|_{L^p} = \| f \|_{H^{r,p,\mathcal{L}}},
\]

using the Littlewood–Paley theorem (see Equation 62). We observe that in the embedding 
\[ B_{p,q}^{r,\mathcal{L}}(G) \hookrightarrow L_r^{p,\mathcal{L}}(G) \hookrightarrow B_{p,2}^{r,\mathcal{L}}(G), \]
if \( p = 2 \), then \( L_r^{2,\mathcal{L}}(G) = H^{r,\mathcal{L}}(G) = B_{2,2}^{r,\mathcal{L}}(G) \). This shows (4). Now, for the proof of (5) we choose \( f \in C^\infty(G) \). The Fourier inversion formula implies

\[
\| f \|_{L^q} = \left\| \sum_{[\xi] \in \hat{G}} \text{Tr}[\xi(x) \hat{f}(\xi)] \right\|_{L^q} = \left\| \sum_{s \in \mathbb{N}_0} \sum_{[\xi] \in \hat{G}} \text{Tr}[\xi(x) \psi_s(\xi)^s \hat{f}(\xi)] \right\|_{L^q} \leq \sum_{s \in \mathbb{N}_0} \left\| \sum_{[\xi] \in \hat{G}} \text{Tr}[\xi(x) \psi_s(\xi)^s \hat{f}(\xi)] \right\|_{L^q} \leq \sum_{s \in \mathbb{N}_0} \| \psi_s(D)^s f \|_{L^q} \leq 2^{Q(\frac{1}{p} - \frac{1}{q})} \| \psi_s(D)^s f \|_{L^p} = \| f \|_{B_{p,1}^{Q(\frac{1}{p} - \frac{1}{q}),\mathcal{L}}},
\]

where in the last line we have used Theorem 3.2. This show that \( B_{p,1}^{Q(\frac{1}{p} - \frac{1}{q}),\mathcal{L}}(G) \hookrightarrow L^q(G) \). The embedding \( L^q(G) \hookrightarrow B_{q,\infty}^{0,\mathcal{L}}(G) \) for \( 1 < q \leq \infty \) will be proved in Remark 5.12. This completes the proof.

5. Subelliptic Besov spaces versus Besov spaces associated to the Laplacian

In this section, we study embedding properties between Besov spaces associated to the sub-Laplacian and the Besov spaces associated to the Laplacian. For our further analysis, we require some properties of the global pseudo-differential operators as well as some Fourier multiplier theorems in the context of the matrix-valued quantization.

5.1. Pseudo-differential operators on compact Lie groups

A central tool for obtaining embedding properties between subelliptic Sobolev and Besov spaces are pseudo-differential operators. According to the matrix-valued quantisation
procedure developed in [27], to every continuous linear operator $A$ on $G$ mapping $C^\infty(G)$ into $\mathcal{D}'(G)$ one can associate a matrix-valued (full-symbol)

$$\sigma_A : G \times \hat{G} \to \bigcup_{[\xi] \in \hat{G}} \mathbb{C}^{d_\xi \times d_\xi}, \quad \sigma(x, \xi) := \sigma(x, [\xi]) \in \mathbb{C}^{d_\xi \times d_\xi}, \quad (67)$$

satisfying

$$Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x)\sigma_A(x, \xi)\hat{f}(\xi)]. \quad (68)$$

In terms of the operator $A$, we can recover the symbol $\sigma_A$ by the identity

$$\sigma_A(x, \xi) = \xi(x)^* (A\xi)(x), \quad (A\xi)(x) := [A\xi_{ij}(x)]_{i,j=1}^{d_\xi}.$$

To define the matrix-valued pseudo-differential calculus (68), it is necessary to have a difference structure on the unitary dual $\hat{G}$. Indeed, if we want to measure the decaying or increasing of matrix-valued symbols we need the notion of discrete derivatives (difference operators) $\Delta_\xi^\alpha$. According to [2], we say that $Q_\xi$ is a difference operator of order $k$, if it is defined by

$$Q_\xi \hat{f}(\xi) = \hat{qf}(\xi), \quad [\xi] \in \hat{G}, \quad (69)$$

for all $f \in C^\infty(G)$, for some function $q$ vanishing of order $k$ at the identity $e = e_G$. The set $\text{diff}^k(\hat{G})$ denotes the set of all difference operators of order $k$. For a such given smooth function $q$, the associated difference operator will be denoted by $\Delta_q := Q_\xi$. To define the Hörmander classes in the matrix-valued context, we will choose an admissible selection of difference operators [1,2],

$$\Delta_\xi^\alpha := \Delta_{q_1}^{\alpha_1} \cdots \Delta_{q_i}^{\alpha_i}, \quad \alpha = (\alpha_j)_{1 \leq j \leq i},$$

where

$$\text{rank}[\nabla q_j(e) : 1 \leq j \leq i] = \dim(G), \quad \text{and} \quad \Delta_{q_i} \in \text{diff}^1(\hat{G}). \quad (70)$$

**Remark 5.1:** Difference operators can be defined by using the representation theory on the group $G$. Indeed, if $\xi_0$ is a fixed irreducible representation, a particular matrix-valued difference operator is given by $D_{\xi_0} = (D_{\xi_0,i,j})_{i,j=1}^{d_{\xi_0}} = \xi_0(\cdot) - I_{d_{\xi_0}}$. If the representation is fixed we omit the index $\xi_0$ so that, from a sequence $D_1 = D_{\xi_0,j_1,i_1}, \ldots, D_n = D_{\xi_0,j_n,i_n}$ of operators of this type we define $D_\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $\alpha \in \mathbb{N}_0^n$.

**Remark 5.2:** By using the collection of fundamental representations, a set of difference operators $\{\Delta_\xi^\alpha : \alpha \in \mathbb{N}_0^n\}$ can be constructed [35]. We refer to Fischer [22] for an equivalent presentation of the calculus developed in [27] and [36].

The following step is to provide the definition of pseudo-differential operators on compact Lie groups associated to Hörmander classes (via localisations) and how this definition can be recovered from the notion of global symbol by using difference operators. So, if $U$
is an open subset of $\mathbb{R}^n$, we say that the function $a : U \times \mathbb{R}^n \to \mathbb{C}$, belongs to the Hörmander class $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, $0 \leq \rho, \delta \leq 1$, if for every compact subset $K \subset U$, we have the symbol inequalities,

$$|\partial^\beta_x \partial^\alpha_x a(x, \xi)| \leq C_{\alpha,\beta,\kappa}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

(71)

uniformly in $x \in K$ and $\xi \in \mathbb{R}^n$. In this case, a continuous linear operator $A$ from $C_0^\infty(U)$ into $C^\infty(U)$ is a pseudo-differential operator of order $m$, in the $(\rho, \delta)$-class, if there exists a function $a \in S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, such that

$$Af(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} a(x, \xi)(\mathcal{F}_{\mathbb{R}^n}f)(\xi) \, d\xi,$$

for all $f \in C_0^\infty(U)$, where $(\mathcal{F}_{\mathbb{R}^n}f)(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} f(x) \, dx$, is the Fourier transform of $f$ at $\xi$. The class $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$ is stable under coordinate changes if $\rho \geq 1 - \delta$, although a symbolic calculus is only possible for $\delta < \rho$ and $\rho \geq 1 - \delta$. In the case of a $C^\infty$-manifold $M$ (and consequently on every compact Lie group $G$), a linear continuous operator $A : C_0^\infty(M) \to C^\infty(M)$ is a pseudo-differential operator of order $m$, in the $(\rho, \delta)$-class, $\rho \geq 1 - \delta$, if for every local coordinate patch $\kappa : M_\kappa \subset M \to U \subset \mathbb{R}^n$, and for every $\phi, \psi \in C_0^\infty(U)$, the operator

$$Tu := \phi(\kappa^{-1})^* A \kappa^*(\phi u), \quad u \in C^\infty(U),$$

(72)

is a pseudo-differential operator with symbol $a_T \in S^m_{\rho,\delta}(U \times \mathbb{R}^n)$. The operators $\kappa^*$ and $(\kappa^{-1})^*$ are the pullbacks, induced by the maps $\kappa$ and $\kappa^{-1}$ respectively. In this case, we write $A \in \Psi^m_{\rho,\delta}(M, \text{loc})$. If $M = G$ is a compact Lie group, and $A \in \Psi^m_{\rho,\delta}(G, \text{loc})$, $\rho \geq 1 - \delta$, the matrix-valued symbol $\sigma_A$ of $A$ satisfies \cite{22,27,37},

$$\|\partial^\beta_x \partial^\gamma_x \sigma_A(x, \xi)\|_{\text{op}} \leq C_{\alpha,\beta,\kappa}(1 + |\xi|)^{m-\rho|\gamma|+\delta|\beta|}$$

(73)

for all $\beta$ and $\gamma$ multi-indices and all $(x, [\xi]) \in G \times \widehat{G}$. Now, if $0 \leq \delta, \rho \leq 1$, we say that $\sigma_A \in \mathcal{S}^m_{\rho,\delta}(G)$, if the global symbol inequalities (73) hold true. So, for $\sigma_A \in \mathcal{S}^m_{\rho,\delta}(G)$ and $A$ defined by (68), we write $A \in \text{Op}(\mathcal{S}^m_{\rho,\delta}(G))$. We have mentioned above that

$$\text{Op}(\mathcal{S}^m_{\rho,\delta}(G)) = \Psi^m_{\rho,\delta}(G, \text{loc}), \quad 0 \leq \delta < \rho \leq 1, \quad \rho \geq 1 - \delta.$$ (74)

However, a symbolic calculus for the classes $\text{Op}(\mathcal{S}^m_{\rho,\delta}(G))$, $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$ (without the condition $\rho \geq 1 - \delta$) has been established in \cite{27}. Next, we record some of the mapping properties of these classes on Lebesgue spaces. We start with the following $L^p$-Fourier multiplier theorem which was presented as Corollary 5.1 in \cite{2}.

**Theorem 5.3:** Let $0 \leq \rho \leq 1$, and let $\tilde{\nu}$ be the smallest even integer larger than $n/2$, $n := \dim(G)$. If $A : C^\infty(G) \to \mathcal{D}'(G)$ is left-invariant and its matrix symbol $\sigma_A$ satisfies

$$\|\partial^\beta_x \partial^\gamma_x \sigma_A(\xi)\|_{\text{op}} \leq C_{\gamma}(1 + |\xi|)^{-\rho|\alpha|}, \quad [\xi] \in \widehat{G}, \quad |\gamma| \leq \tilde{\nu},$$

(75)

then $A$ extends to a bounded operator from $L^p_{\nu(1-\rho)(\frac{1}{2}-\frac{1}{p})}(G)$ into $L^p(G)$ for all $1 < p < \infty$. 
Remark 5.4: Theorem 5.3 was obtained in [2] under the condition of evenness for \( \tilde{\kappa} \). However, if we replace the collection of differences operators \( \{D^\alpha\}_{\alpha \in \mathbb{N}_0^n} \) by the collection of differences operators \( \{D^\alpha\}_{\alpha \in \mathbb{N}_0^n} \) associated to fundamental representations of the group \( G \), \( \tilde{\kappa} \) can be replaced by \( \kappa := \lceil n/2 \rceil + 1 \), that is, the smallest integer larger than \( n/2 \), \( n = \dim(G) \) (see Remark 4.10 of [1]).

For non-invariant operators, Theorem 5.3 and Remark 5.4 allow us to obtain the following version of Theorem 1.1 presented in [1].

**Theorem 5.5:** Let \( 0 \leq \delta, \rho \leq 1 \), and let \( \kappa \) be the smallest integer larger than \( n/2 \), \( n = \dim(G) \) (see Remark 4.10 of [1]). If \( A : C^\infty(G) \to \mathcal{D}'(G) \) is a continuous operator such that its matrix symbol \( \sigma_A \) satisfies

\[
\| \partial_x^\beta \Delta^\gamma \sigma_A(x, \xi) \|_{op} \leq C_{\gamma, \beta} |\xi|^{-m_0 - \rho |\omega| + \delta |\gamma|}, \quad [\xi] \in \hat{G}, \quad |\gamma| \leq \kappa, \quad |\beta| \leq [n/p] + 1, \quad (76)
\]

with

\[
m_0 \geq \kappa(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right| + \delta([n/p] + 1). \quad (77)
\]

Then \( A \) extends to a bounded operator from \( L^p(G) \) into \( L^p(G) \) for all \( 1 < p < \infty \).

For operators with smooth symbols, in [1] it is proved the following \( L^p \)-pseudo-differential theorem.

**Theorem 5.6 (Fefferman \( L^p \) Theorem on compact Lie groups):** Let \( G \) be a compact Lie group of dimension \( n \). Let \( 0 \leq \delta < \rho \leq 1 \) and let

\[
0 \leq v < \frac{n(1 - \rho)}{2}, \quad (78)
\]

for \( 0 < \rho < 1 \) and \( v = 0 \) for \( \rho = 1 \). Let \( \sigma \in \mathcal{S}_{\rho, \delta}^{-v}(G) \). Then \( A \equiv \sigma(x, D) \) extends to a bounded operator on \( L^p(G) \) provided that

\[
v \geq n(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|. \quad (79)
\]

The sharpness of the order \( \frac{n(1 - \rho)}{2} \) in Theorem 5.6 was discussed in Remark 4.16 of [1].

A fundamental tool in our further analysis is the following estimate for operators associated to compactly supported symbols in \( \xi \).

**Lemma 5.7:** Let \( G \) be a compact Lie group of dimension \( n \). Let \( \{A_j\}_{j \in \mathbb{N}} \) be a sequence of Fourier multipliers with full symbols \( \sigma_{A_j} \in \mathcal{S}_{1,0}^0(G) \), supported in

\[
\Sigma_j := \left\{ [\xi] \in \hat{G} : \frac{1}{2} j \leq \langle \xi \rangle < j \right\}. \quad (80)
\]

If \( \text{sup}_j \|A_j\|_{\mathcal{B}(L^2(G))} < \infty \), then every \( A_j \) extends to a bounded operator on \( L^p(G) \), \( 1 < p \leq \infty \), and \( \text{sup}_j \|A_j\|_{\mathcal{B}(L^p(G))} < \infty \).
Proof: The case $p = \infty$ is only a special case of Lemma 4.11 of [1] for $a = 0$. Now, we can use the Riesz–Thorin interpolation theorem between the estimates

$$\sup_j \|A_j\|_{\mathcal{H}(L^p)} < \infty, \quad i = 0, 1,$$

for $p_0 = 2$ and $p_1 = \infty$ to obtain Lemma 5.7 for $2 \leq p < \infty$. The case $1 < p \leq 2$ now follows by an argument of duality. The proof is complete. 

The following Lemma will be useful for our further analysis (see Lemma 4.5 of [37]).

Lemma 5.8: Let $m \geq m_0$, $0 \leq \delta < \rho \leq 1$, and let us fix difference operators $\{D_{ij}\}_{1 \leq i, j \leq d_{g_0}}$ corresponding to some representation $\xi_0 \in [\xi_0] \in \hat{G}$. Let the matrix symbol $a = a(x, \xi)$ satisfy

$$\|D^\gamma \partial^\beta x a(x, \xi)\|_{op} \leq C_{\gamma \beta} \langle \xi \rangle^{|m-\rho| |\gamma| + |\delta| |\beta|}$$

(81)

for all multi-index $\beta$ and $\gamma$. Assume that $a(x, \xi)$ is invertible for all $x \in G$ and $[\xi] \in \hat{G}$ and satisfies

$$\|a(x, \xi)^{-1}\|_{op} \leq C \langle \xi \rangle^{-m_0}, \quad x \in G, \quad [\xi] \in \hat{G},$$

(82)

and if $m_0 \neq m$, in addition that

$$\|a(x, \xi)^{-1} [D^\gamma \partial^\beta x a(x, \xi)]\|_{op} \leq C_{\gamma \beta} \langle \xi \rangle^{\rho |\gamma| + |\delta| |\beta|}.$$  (83)

Then the following inequalities

$$\|D^\gamma \partial^\beta x [a(x, \xi)^{-1}]\|_{op} \leq C'_{\gamma \beta} \langle \xi \rangle^{-m_0-\rho |\gamma| + |\delta| |\beta|}$$

(84)

hold true for all multi-indices $\beta$ and $\gamma$.

5.2. Embedding properties

Now, we compare subelliptic Besov spaces associated to the sub-Laplacian with Besov spaces associated to the Laplacian on the group. We first analyse it for Sobolev spaces modelled on $L^p$-spaces. Our starting point is the following estimate (see Proposition 3.1 of [33]):

$$c(\xi)^{\frac{1}{2}} \leq 1 + \nu_{ii}(\xi) \leq \sqrt{2} \langle \xi \rangle, \quad \langle \xi \rangle := (1 + \lambda_{[\xi]}^{\frac{1}{2}},$$

(85)

where $\kappa$ is the order in the Hörmander condition. We immediately have the following estimate:

$$\langle \xi \rangle^{\frac{1}{2}} \lesssim (1 + \nu_{ii}(\xi)^{2})^{\frac{1}{2}} \lesssim \langle \xi \rangle.$$  (86)

To compare subelliptic Besov spaces with Besov spaces associated to the Laplacian, we start by considering the problem for Sobolev spaces and later extend it to Besov spaces. So, we have the following theorem. Here,

$$\kappa := [n/2] + 1$$

is the smallest integer larger than $n/2$, $n = \text{dim}(G)$. 

Theorem 5.9: Let $G$ be a compact Lie group and let us consider the sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of order $\kappa$. Then we have the following embeddings

\[ L^p_s (G) \hookrightarrow L^{p,\mathcal{L}}_s (G) \hookrightarrow L^p_{\frac{s}{\kappa} + \frac{\kappa}{\kappa} - \frac{1}{2} - \frac{1}{p}} (G). \]  

(87)

More precisely, for every $s \geq 0$ there exist constants $C_a > 0$ and $C_b > 0$ satisfying

\[ C_a \|f\|_{L^p_{s \kappa \kappa} (G)} \leq \|f\|_{L^{p,\mathcal{L}}_s (G)}, \quad f \in L^{p,\mathcal{L}}_s (G), \]  

(88)

where $s_{\kappa,\kappa} := \frac{s}{\kappa} + \frac{\kappa}{\kappa} - \frac{1}{2} - \frac{1}{p}$, and

\[ \|f\|_{L^{p,\mathcal{L}}_s (G)} \leq C_b \|f\|_{L^p_s (G)}, \quad f \in L^p_s (G). \]  

(89)

Consequently, we have the following embeddings:

\[ L^p_{\frac{s}{\kappa} + \frac{\kappa}{\kappa} - \frac{1}{2} - \frac{1}{p}} (G) \hookrightarrow L^{p,\mathcal{L}}_{s - s} (G) \hookrightarrow L^p_{-s} (G). \]  

(90)

Proof: The proof will be based on the continuity properties of the operators

\[ \mathcal{R}_s := (1 + \mathcal{L}_G)^{\frac{s}{\kappa}} (1 + \mathcal{L})^{-\frac{s}{\kappa}} \quad \text{and} \quad \mathcal{R}'_s := (1 + \mathcal{L}_G)^{-\frac{s}{\kappa}} (1 + \mathcal{L})^{\frac{s}{\kappa}}, \quad s \geq 0. \]  

(91)

From Lemma 7.1, we deduce that $\mathcal{R}_s \in \text{Op} (\mathcal{S}^{1,0}_{1/\kappa,0} (G))$ and $\mathcal{R}'_s \in \text{Op} (\mathcal{S}^0_{1,0} (G))$. Let us observe that $\mathcal{R}'_s \in \text{Op} (\mathcal{S}^0_{1,0} (G))$ implies that for all $s \geq 0$, the operator $\mathcal{R}'_s$ extends to a bounded operator on $L^p_s (G)$ for all $1 < p < \infty$. So, for $C_b := \|\mathcal{R}'_s\|_{\mathcal{B} (L^p_s (G))}$, we have

\[ \| (1 + \mathcal{L}_G)^{-\frac{s}{\kappa}} (1 + \mathcal{L})^{\frac{s}{\kappa}} f \|_{L^p_s (G)} \leq C_b \|f\|_{L^p_s (G)}, \]  

(92)

which in turns implies

\[ \| (1 + \mathcal{L}_G)^{-\frac{s}{\kappa}} f \|_{L^p_s (G)} \leq C_b \|f\|_{L^p_s (G)}. \]  

(93)

So, we provide the embedding $L^{p,\mathcal{L}}_{s - s} (G) \hookrightarrow L^p_{s - s} (G)$ and by duality we have $L^p_s (G) \hookrightarrow L^{p,\mathcal{L}}_s (G)$. The last embedding is equivalent to the inequality

\[ \|f\|_{L^{p,\mathcal{L}}_s (G)} \leq C_b \|f\|_{L^p_s (G)}, \quad f \in L^p_s (G). \]  

(94)

For the second part of the proof, let us define

\[ T_s := (1 + \mathcal{L}_G)^{\frac{1}{2} (\frac{s}{\kappa} - s)} \mathcal{R}_s = (1 + \mathcal{L}_G)^{\frac{1}{2} (\frac{s}{\kappa})} (1 + \mathcal{L})^{-\frac{s}{\kappa}}. \]

Now, by Lemma 7.1 the operator $T_s$ belongs to the class $\text{Op} (\mathcal{S}^0_{1/\kappa,0} (G))$, $\rho = 1/\kappa$, and by Theorem 5.3 and Remark 5.4 we obtain its boundedness from $L^p_{\frac{s}{\kappa} + \frac{\kappa}{\kappa} - \frac{1}{2} - \frac{1}{p}} (G)$ into $L^p (G)$.
for all $1 < p < \infty$, where $\kappa := [n/2] + 1$. Consequently, we obtain

$$
C_a \|Tf\|_{L^p(G)} \leq \|f\|_{L^p}\kappa(1-\frac{1}{p})\frac{1}{2-\frac{1}{p}}(G), \quad C_a := 1/\|T\|_{\mathcal{B}(L^p(\kappa(1-\frac{1}{p})\frac{1}{2-\frac{1}{p}}(L^p))}. \tag{95}
$$

Similar to the first part of the proof, we have

$$
C_a \|(1 + L)^{-\frac{s}{2}}f\|_{L^p(G)} \leq \|f\|_{L^p}\kappa(1-\frac{1}{p})\frac{1}{2-\frac{1}{p}}(G), \tag{96}
$$

Indeed, by using that the operators $(1 + L_G)^{\frac{s}{2}}$ and $(1 + L)^{-\frac{s}{2}}$ commute, from (95), we have the inequality,

$$
C_a \|(1 + L)^{-\frac{s}{2}}(1 + L_G)^{\frac{s}{2}}f\|_{L^p(G)} \leq \|f\|_{L^p}\kappa(1-\frac{1}{p})\frac{1}{2-\frac{1}{p}}(G), \tag{97}
$$

which implies (96) if in (97) we change $f$ by $(1 + L_G)^{-\frac{s}{2}}f$. So, we conclude that

$$
L^p\kappa(1-\frac{1}{p})\frac{1}{2-\frac{1}{p}}(G) \hookrightarrow L^{p,\kappa}_s(G), \quad \text{and by duality we obtain the embedding } L^{p,\kappa}_s(G) \hookrightarrow L^p\kappa(1-\frac{1}{p})\frac{1}{2-\frac{1}{p}}(G),
$$

which implies the estimate,

$$
C_a \|f\|_{L^{p,\kappa}_s(G)} \leq \|f\|_{L^{p,\kappa}_s(G)}, \quad f \in L^{p,\kappa}_s(G), \tag{98}
$$

when $s_{\kappa,\kappa} = \frac{s}{\kappa} - \kappa(1-\frac{1}{\kappa})\frac{1}{2-\frac{1}{p}}$. Thus we finish the proof. \hfill \blacksquare

**Remark 5.10:** For $p = 2$, Theorem 5.9 was obtained in [33]. In this case for $s > 0$, we have $s_{\kappa,\kappa} = s/\kappa$. Consequently,

$$
H^s(G) \hookrightarrow H^{s,\kappa}_s(G) \hookrightarrow H^{\frac{s}{\kappa}}(G) \hookrightarrow L^2(G) \hookrightarrow H^{-\frac{s}{\kappa}}(G) \hookrightarrow H^{-s,\kappa}(G) \hookrightarrow H^{-s}(G).
$$

Also, as it was pointed out in [33], for integers $s \in \mathbb{N}$ the embedding $H^{s,\kappa}_s(G) \hookrightarrow H^{\frac{s}{\kappa}}(G)$, is in fact, Theorem 13 in [15].

Now, we prove the main theorem of this section.

**Theorem 5.11:** Let $G$ be a compact Lie group and let us consider the sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of order $\kappa$. Let $s \geq 0$, $0 < q \leq \infty$, and $1 < p < \infty$. Then we have the continuous embeddings

$$
B^s_{p,q}(G) \hookrightarrow B^{s,\kappa}_s(G) \hookrightarrow B^{\frac{s}{\kappa} - \kappa(1-\frac{1}{\kappa})\frac{1}{2-\frac{1}{p}}}_p(G). \tag{99}
$$

More precisely, for every $s \geq 0$ there exist constants $C_a > 0$ and $C_b > 0$ satisfying,

$$
C_a \|f\|_{B^s_{p,q}(G)} \leq \|f\|_{B^{s,\kappa}_s(G)}, \quad f \in B^{s,\kappa}_s(G), \tag{100}
$$

where $s_{\kappa,\kappa} := \frac{s}{\kappa} - \kappa(1-\frac{1}{\kappa})\frac{1}{2-\frac{1}{p}}$, and

$$
\|f\|_{B^{s,\kappa}_s(G)} \leq C_b \|f\|_{B^{s}_{p,q}(G)}, \quad f \in B^{s}_{p,q}(G). \tag{101}
$$

Consequently, we have the following embeddings:

$$
B^s_{p,q}(G) \hookrightarrow B^{s,\kappa}_s(G) \hookrightarrow B^{-s,\kappa}_s(G) \hookrightarrow B^{-s}_{p,q}(G). \tag{102}
$$
Proof:  First, let us observe that for every \( s \in \mathbb{R} \),

\[
\| f \|_{B^q_{p,q}(G)} \lesssim \left\| \left\{ \| \psi_j(D)f \|_{L^p_{s,q}(G)} \right\}_{j \in \mathbb{N}} \right\|_{L^q(\mathbb{N})}.
\]  (103)

This can be proved by the functional calculus. Indeed, if \( E^M_\lambda \), \( \lambda > 0 \), denotes the spectral resolution associated to \( M = (1 + L)^{1/2} \) and \( 0 < q < \infty \), we get

\[
2^{jsq} \| \psi_j(D)f \|_{L^p(G)}^q = \left\| \int_0^{2^{j+1}} \psi_j(\lambda) 2^s \, dE^M_\lambda f \right\|_{L^p(G)}^q \lesssim \left\| \int_0^{2^j} \lambda^s \, dE^M_\lambda f \right\|_{L^p(G)}^q \times \left\| \int_0^{2^{j+1}} \lambda^s \, dE^M_\lambda f \right\|_{L^p(G)}^q
\]

\[
\lesssim \left\| \int_0^{2^j} \lambda^s \, dE^M_\lambda f \right\|_{L^p(G)}^q \lesssim \int_0^\infty \lambda^s \, dE^M_\lambda \int_0^{2^j} \psi_j(\lambda) \, dE^M_\lambda f \right\|_{L^p(G)}^q
\]

\[
=: \| (1 + L)^{1/2} \psi_j(D)f \|_{L^p(G)}^q,
\]

which proves the desired estimate (103). In the previous lines, we have used the following property:

\[
\int_{[a,b]\cap[c,d]} f(\lambda) g(\lambda) \, dE^M_\lambda = \int_{[a,b]} f(\mu) \, dE^M_\mu \int_{[c,d]} g(\lambda) \, dE^M_\lambda,
\]

of the functional calculus (see, e.g. [38]). For the second part of the proof, we choose a \( S^1 \)-partition of unity (or Littlewood–Paley partition of the unity) \( \{ \tilde{\psi}_j \} \). So, we assume the following properties (see, e.g. Definition 2.1 of Garetto [16]):

- \( \tilde{\psi}_0 \in \mathcal{D}(\mathbb{R}) \), \( \tilde{\psi}_0(t) = 1 \) for \( |t| \leq 1 \), and \( \tilde{\psi}_0(t) = 0 \) for \( |t| \geq 2 \)
- For every \( j \geq 1 \), \( \tilde{\psi}_j(t) = \tilde{\psi}_0(2^{-j} t) - \tilde{\psi}_0(2^{-j+1} t) =: \tilde{\psi}_1(2^{-j+1} t) \)
- \( |\tilde{\psi}_j(\alpha)(t)| \leq C_\alpha 2^{-\alpha j}, \alpha \in \mathbb{N}_0 \)
- \( \sum_j \tilde{\psi}_j(t) = 1, t \in \mathbb{R} \).

We will use the existence of an integer \( j_0 \in \mathbb{N} \) satisfying \( \tilde{\psi}_j \tilde{\psi}_{j'} = 0 \) for all \( j, j' \) with \( |j - j'| \geq j_0 \), as follows. If \( f \in C^\infty(G) \) and \( \ell \in \mathbb{N} \), then

\[
\psi_\ell(D)f = \sum_{j,j' \atop \ell+ j_0} \tilde{\psi}_j((1 + L)^{1/2}) \psi_\ell(D) \tilde{\psi}_{j'}((1 + L_G)^{1/2}) f
\]

\[
= \sum_{j,j' = \ell - j_0} \tilde{\psi}_j((1 + L)^{1/2}) \psi_\ell(D) \tilde{\psi}_{j'}((1 + L_G)^{1/2}) f.
\]
So, the following estimate:

\[ \| \psi_{\tilde{\ell}}(D) f \|_{L^p} \leq \sum_{j, j' = \tilde{\ell} - j_0}^{\tilde{\ell} + j_0} \| \tilde{\psi}_j ((1 + \mathcal{L})^{\frac{1}{2}}) \psi_{\tilde{\ell}}(D) \tilde{\psi}_{j'} ((1 + \mathcal{L}_G)^{\frac{1}{2}}) f \|_{L^p} \]  

(104)

holds true for every \( \tilde{\ell} \in \mathbb{N} \). From Lemma 5.7 applied to every operator \( \psi_{\tilde{\ell}}(D) \tilde{\psi}_{j'} ((1 + \mathcal{L})^{\frac{1}{2}}) \), we have

\[ \| \tilde{\psi}_{j'} ((1 + \mathcal{L})^{\frac{1}{2}}) \psi_{\tilde{\ell}}(D) \tilde{\psi}_{j'} ((1 + \mathcal{L}_G)^{\frac{1}{2}}) f \|_{L^p} \leq C \| \tilde{\psi}_{j'} ((1 + \mathcal{L}_G)^{\frac{1}{2}}) f \|_{L^p} \]  

(105)

and

\[ \| \tilde{\psi}_{j'} ((1 + \mathcal{L})^{\frac{1}{2}}) \psi_{\tilde{\ell}}(D) \tilde{\psi}_{j'} ((1 + \mathcal{L}_G)^{\frac{1}{2}}) f \|_{L^p} \leq C \| \tilde{\psi}_{j'} ((1 + \mathcal{L}_G)^{\frac{1}{2}}) f \|_{L^p}, \]  

(106)

where the constant \( C \) is independent of \( j, j' \in \mathbb{N} \). Now, if \( s \geq 0 \), we have

\[ \| f \|_{B^{s,\infty}_{p,q}(G)} = \left\{ \left\| 2^{j_0} \| \tilde{\psi}_{j'} (D) f \|_{L^p} \right\|_{\ell^q(\mathbb{N})} \right\} \lesssim (2j_0 + 1) \| f \|_{B^{s}_{p,q}(G)}. \]

This estimate proves the embedding \( B^{s}_{p,q}(G) \hookrightarrow B^{s,\infty}_{p,q}(G) \) which by duality implies \( B^{-s,\infty}_{p,q}(G) \hookrightarrow B^{-s}_{p,q}(G) \) for \( s \geq 0 \). Now, for \( s \geq 0 \) and \( s_{\kappa,\infty} := \frac{s}{\kappa} - \infty (1 - \frac{1}{\kappa}) \frac{1}{2} - \frac{1}{p} \), we have

\[ \| f \|_{B^{s,\infty}_{p,q}(G)} \lesssim \left\{ \left\| \psi_{\tilde{\ell}} ((1 + \mathcal{L}_G)^{\frac{1}{2}}) f \|_{L^p_{s,\infty}(G)} \right\|_{\ell^q(\mathbb{N})} \right\}, \]

which in turns implies

\[ \| f \|_{B^{s,\infty}_{p,q}(G)} \lesssim \left\{ \left\| \tilde{\psi}_{j'} ((1 + \mathcal{L})^{\frac{1}{2}}) f \|_{L^p_{s,\infty}(G)} \right\|_{\ell^q(\mathbb{N})} \right\}, \]

where we have used (88). The last estimate proves the embedding \( B^{s,\infty}_{p,q}(G) \hookrightarrow B^{-s}_{p,q}(G) \). The duality argument implies that \( B^{-s,\infty}_{p,q}(G) \hookrightarrow B^{-s}_{p,q}(G) \) for every \( s \geq 0 \). So, we finish the proof.
Remark 5.12: With the notation of the previous result, the embedding $L^q(G) \hookrightarrow B_{q,\infty}^0(G)$, $1 < q \leq \infty$, is a consequence of the following estimate:

$$\|f\|_{B_{q,\infty}^0} = \sup_{\ell \in \mathbb{N}} \|\tilde{\psi}_\ell(D)f\|_{L^q} \lesssim \sum_{\ell \geq j_0} \|\tilde{\psi}_j((1 + \mathcal{L})^{1/2})f\|_{L^q} \leq C(j_0 + 1)\|f\|_{L^q}.$$

6. Subelliptic Triebel–Lizorkin spaces and interpolation of subelliptic Besov spaces

The Littlewood–Paley theorem (62)

$$\|f\|_{\dot{F}^0_{p,2}(G)} := \left\| \left( \sum_{s \in \mathbb{N}_0} |\psi_s(D)f(x)|^2 \right)^{1/2} \right\|_{L^p(G)} \approx \|f\|_{L^p(G)}$$

suggests the subelliptic functional class defined by the norm

$$\|f\|_{\dot{F}^r_{p,q}(G)} := \left\| \left( \sum_{s \in \mathbb{N}_0} 2^{sqr}|\psi_s(D)f(x)|^q \right)^{1/q} \right\|_{L^p(G)}$$

for $0 < p \leq \infty$, and $0 < q < \infty$ with the following modification:

$$\|f\|_{\dot{F}^r_{p,\infty}(G)} := \left\| \sup_{s \in \mathbb{N}_0} 2^{srq}|\psi_s(D)f(x)| \right\|_{L^p(G)}$$

for $q = \infty$. By the analogy with ones presented in Triebel [10,11], we will refer to the spaces $F^r_{p,q}(G)$, as subelliptic Triebel–Lizorkin spaces. The main objective of this section is to present the interpolation properties for these spaces and their relation with the subelliptic Besov spaces. In the context of compact Lie groups, Triebel–Lizorkin spaces $F^r_{p,q}(G)$ associated to the Laplacian $\mathcal{L}_G$ have been introduced in [12,13], by using a global description of the Fourier transform through the representation theory of the group $G$. Taking these references as a point of departure, we have the following theorem.

Theorem 6.1: Let $G$ be a compact Lie group of dimension $n$. Then we have the following properties.

1. $F^r_{p,q_1}(G) \hookrightarrow F^r_{p,q_2}(G) \hookrightarrow F^r_{p,\infty}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $0 \leq q_1 \leq q_2 \leq \infty$.
2. $F^r_{p,q_1}(G) \hookrightarrow F^r_{p,q_2}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.
3. $F^r_{p,q}(G) = B^r_{p,p}(G)$, $r \in \mathbb{R}$, $0 < p \leq \infty$.
4. $F^r_{p,q}(G) \hookrightarrow F^r_{p,q}(G) \hookrightarrow F^r_{p,\max[p,q]}(G)$, $0 < p < \infty$, $0 < q \leq \infty$.

The proof is only an adaptation of the arguments presented in Triebel [10]. To clarify the relation between subelliptic Besov spaces and subelliptic Triebel-Lizorkin spaces, we
present their interpolation properties. We recall that if $X_0$ and $X_1$ are Banach spaces, intermediate spaces between $X_0$ and $X_1$ can be defined with the $K$-functional, which is defined by

$$K(f, t) = \inf \{ \|f_0\|_{X_0} + t \|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1 \}, \quad t \geq 0. \quad (110)$$

If $0 < \theta < 1$ and $1 \leq q < \infty$, the real interpolation space $X_{\theta,q} := (X_0, X_1)(\theta, q)$ is defined by those vectors $f \in X_0 + X_1$ satisfying

$$\|f\|_{\theta,q} = \left( \int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \quad \text{if } q < \infty, \quad (111)$$

and for $q = \infty$

$$\|f\|_{\theta,q} = \sup_{t > 0} t^{-\theta} K(f, t) < \infty. \quad (112)$$

Let us observe that if in Theorem 8.1 of [13] we consider the case of any sub-Laplacian $L$ instead of the Laplacian $L_G$, a similar result can be obtained by following the lines of the proof presented there. In our case, we have the following interpolation theorem.

**Theorem 6.2:** Let $G$ be a compact Lie groups of dimension $n$. If $0 < r_0 < r_1 < \infty$, $0 < \beta_0, \beta_1, q \leq \infty$, and $r = r_0(1 - \theta) + r_1 \theta$, then

1. $(B^{r_0,L}_{p,\beta_0}(G), B^{r_1,L}_{p,\beta_1}(G))(\theta,q) = B^{r,L}_{p,q}(G), 0 < p < \infty$.
2. $(L^{r_0,L}_{p,\beta_0}(G), L^{r_1,L}_{p,\beta_1}(G))(\theta,q) = B^{r,L}_{p,q}(G), 1 < p < \infty$.
3. $(F^{r_0,L}_{p,\beta_0}(G), F^{r_1,L}_{p,\beta_1}(G))(\theta,q) = B^{r,L}_{p,q}(G), 0 < p < \infty$.

**7. Boundedness of pseudo-differential operators associated to Hörmander classes**

In this section, we study the action of pseudo-differential operators on subelliptic Sobolev and Besov spaces. Throughout this section, $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying the Hörmander condition of order $\kappa$ (this means that their iterated commutators of length $\leq \kappa$ span the Lie algebra $\mathfrak{g}$ of $G$) and $\mathcal{L} \equiv \mathcal{L}_{\text{sub}} = - \sum_{j=1}^k X_j^2$ is the associated positive sub-Laplacian.

**7.1. Boundedness properties on subelliptic Sobolev spaces**

In this section, we consider the problem of the boundedness of pseudo-differential operators on subelliptic Sobolev spaces.

**Proof of Theorem 1.2:** We start the proof with the following Lemma.

**Lemma 7.1 (Order for negative powers of $1 + \mathcal{L})^{\frac{1}{2}}$:** For $1 \leq k < n := \dim G$ and $s > 0$, define the operator $\mathcal{M}_s := (1 + \mathcal{L})^{\frac{s}{2}}$. Then $\mathcal{M}_s \in \text{Op}(\mathcal{S}_{1,0}^s(G))$, $\mathcal{M}_{-s} := (1 + \mathcal{L})^{-\frac{s}{2}} \in \text{Op}(\mathcal{S}_{-1,0}^{-s}(G))$. \[\square\]
\[ \text{Op}(\mathcal{F}^{-s/\kappa}_{1,0}(G)), \text{ where the matrix-valued symbol associated to } \mathcal{M} = \mathcal{M}_1 \text{ which we denote by } \widehat{\mathcal{M}}_1(\xi) \text{ is given by} \]

\[ \widehat{\mathcal{M}}_1(\xi) = \text{diag}((1 + \nu_\mathfrak{s}(\xi)^2)^{\frac{1}{2}})_{1 \leq i \leq d}, \quad [\xi] \in \widehat{G}. \]  

\[ \text{Proof:} \] Because \( \mathcal{M}_2 \in \text{Op}(\mathcal{F}^2_{1,0}(G)) \), the matrix-valued functional calculus allows us to conclude that \( \mathcal{M}_1 \in \text{Op}(\mathcal{F}^1_{1,0}(G)) \) (see Corollary 4.3 of [3]). Therefore, we have that the matrix valued-symbol \( \widehat{\mathcal{M}}_1 \) satisfies estimates of the type,

\[ \| \partial^\gamma \widehat{\mathcal{M}}_1(\xi) \|_{op} \leq C_{\gamma}(\xi)^{1 - |\gamma|} \leq \| \partial^\gamma \mathcal{M}_1(\xi) \|_{op} \leq C_{\gamma}(\xi)^{1 - \rho |\gamma|}, \quad 0 < \rho \leq 1, \]  

for all multi-indices \( \beta \) and \( \gamma \). On the other hand, from (86), we have

\[ \| \widehat{\mathcal{M}}_1(\xi)^{-1} \|_{op} = \sup_{1 \leq i \leq d} (1 + \nu_\mathfrak{s}(\xi)^2)^{-\frac{1}{2}} \lesssim \langle \xi \rangle^{-\frac{1}{2}}. \]  

If \( |\gamma| = 1 \), observe that

\[ \| \widehat{\mathcal{M}}_1(\xi)^{-1} \|_{op} \lesssim \| \widehat{\mathcal{M}}_1(\xi)^{-1} \|_{op} \lesssim C_{\gamma}(\xi)^{-1/\kappa}. \]  

If we iterated this process for \( |\gamma| \geq 2 \), we can prove that

\[ \| \widehat{\mathcal{M}}_1(\xi)^{-1} \|_{op} \lesssim \| \widehat{\mathcal{M}}_1(\xi)^{-1} \|_{op} \lesssim C_{\gamma}(\xi)^{-|\gamma|/\kappa}. \]  

Indeed,

\[ \| \widehat{\mathcal{M}}_1(\xi)^{-1} \|_{op} \lesssim \langle \xi \rangle^{-1/\kappa} \langle \xi \rangle^{-(|\gamma| - 1)} \lesssim \langle \xi \rangle^{-1/\kappa - (|\gamma| - 1)/\kappa} = \langle \xi \rangle^{-|\gamma|/\kappa}. \]  

According to Lemma 5.8 and Theorem 4.2 of [3], the analysis above leads us to conclude that \( \mathcal{M}_{-s} = (1 + \mathcal{L})^{-\frac{s}{2}} \in \text{Op}(\mathcal{F}^{-s/\kappa}_{1,0}(G)) \) for all \( s > 0 \).

Now, we will present how to extend Theorem 5.6 to the critical case \( \rho = \delta \). For this, we will follow the approach developed in [17].

**Lemma 7.2:** Let \( G \) be a compact Lie group. For \( 0 \leq \delta \leq \rho \leq 1 \), \( \delta \neq 1 \), let us consider the pseudo-differential operator \( \mathcal{A} : C^\infty(G) \rightarrow \mathcal{D}'(G) \) with symbol \( \sigma \in S_{\rho,\delta}^{-s/\kappa}(G) \). If \( m = \frac{n(1-\rho)}{2} \), then \( \mathcal{A} \) extends to a bounded operator from \( L^\infty(G) \) to \( \text{BMO}(G) \), from \( H^1(G) \) to \( L^1(G) \), and is bounded on \( L^p(G) \), for all \( 1 < p < \infty \). Moreover, for \( 1 < p < \infty \), and

\[ m \geq m_p := n(1-\rho) \left| \frac{1}{p} - \frac{1}{2} \right|, \]  

the linear operator \( \mathcal{A} \) extends to a bounded operator on \( L^p(G) \).

**Proof:** If \( m = \frac{n(1-\rho)}{2} \), and \( 0 \leq \delta \leq \rho \leq 1 \), \( \delta \neq 1 \), then \( \mathcal{A} \) extends to a bounded operator from \( L^\infty(G) \) to \( \text{BMO}(G) \). This can be deduced from Theorem 4.12 of [1]. Indeed, although this theorem was announced for \( \delta < \rho \), this condition is only necessary in (4.29) of [1, p. 22]. The condition \( \delta < \rho \) can be removed allowing the condition \( \delta = \rho \), if instead of
the $L^2$-boundedness of pseudo-differential operators with symbols in the class $\mathcal{S}_\rho^{0,\delta}(G)$, $0 \leq \delta' < \rho' \leq 1$, we use the Calderón–Vaillancourt Theorem proved in Proposition 8.1 of [22]. Indeed, Proposition 8.1 in [22] assures the $L^2$-boundedness of pseudo-differential operators with symbols in the class $\mathcal{S}_\rho^{0,\delta}(G)$, $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$. By using the duality argument, and the fact that the pseudo-differential calculus is stable under adjoints for $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$ (see [22, Corollary 7.6]), we deduce that $A$ is bounded from $H^1(G)$ to $L^1(G)$. On the other hand, if $m \geq m_p := n(1 - \rho)\left(\frac{1}{p} - \frac{1}{2}\right)$, the $L^p$-boundedness of $A$ can be deduced by the complex interpolation via the Fefferman–Stein interpolation Theorem. Let us write $a := 1 - \rho$. We only need to prove the theorem for the critical order $m = m_p$ in view of the inclusion $\mathcal{S}_{\rho,\delta}^{-m}(G) \subset \mathcal{S}_{\rho,\delta}^{-m_p}(G)$ for $m > m_p$. Let us define the complex family of pseudo-differential operators indexed by $z \in \mathbb{C}$, with $\Re(z) \in [0, 1]$, and given by

$$T_z := \text{Op}(\sigma_z), \quad \sigma_z(x, \xi) := e^{x^2} \sigma(x, \xi) (\xi)^{m + \frac{Q_a}{2}}(z-1).$$

The family of operators $\{T_z\}$ defines an analytic family of operator from $\Re(z) \in (0, 1)$, (resp. continuous for $\Re(z) \in [0, 1]$), into the algebra of bounded operators on $L^2(G)$. Let us observe that $\sigma_0(x, \xi) = \sigma(x, \xi) (\xi)^{-m + \frac{Q_a}{2}}$, and $\sigma_1(x, \xi) = e\sigma(x, \xi)$. Because $T_0$ is bounded from $L^\infty(G)$ into $\text{BMO}(G)$ and $T_1$ is bounded on $L^2(G)$, the Fefferman–Stein interpolation theorem implies that $T_t$ extends to a bounded operator on $L^p(G)$, for $p = \frac{2}{t}$ and all $0 < t \leq 1$. Because $0 \leq m \leq \frac{Q_a}{2}$, there exist $t_0 \in (0, 1)$ such that $m = m_p = \frac{Q_a}{2}(1 - t_0)$. So, $T_{t_0} = e^{\frac{Q_a}{2}}A$ extends to a bounded operator on $L^2(G)$. The Fefferman–Stein interpolation theorem, the $L^2(G)$-boundedness and the $L^\frac{2}{t_0}$-boundedness of $A$ is given the $L^p(G)$-boundedness of $A$ for all $2 \leq p \leq \frac{2}{t_0}$, and interpolating the $L^\frac{2}{t_0}(G)$-boundedness with the $L^\infty(G)$-$\text{BMO}(G)$ boundedness of $A$ we obtain the boundedness of $A$ on $L^p(G)$ for all $\frac{2}{t_0} \leq p < \infty$. So, $A$ extends to a bounded operator on $L^p(G)$ for all $2 \leq p < \infty$. The $L^p(G)$-boundedness of $A$ for $1 < p \leq 2$ now follows by the duality argument.

Returning to the proof of Theorem 1.2, let us define the operator $T := \mathcal{A}\mathcal{M}_{-\vartheta}$ where $\mathcal{M} = (1 + \mathcal{L})^{\frac{1}{2}}$ and $\mathcal{M}_{-\vartheta} = (1 + \mathcal{L})^{-\frac{\vartheta}{2}}$. From Lemma 7.1, $\mathcal{M}_{-\vartheta} \in \Psi_{1/k,0}^{-\vartheta/k-\nu}$ and consequently $T \in \Psi_{\min\{\rho,1/k\},\delta}^{-\vartheta/k-\nu}$. Lemma 7.2 implies that $T$ extends to a bounded operator on $L^p(G)$ provided that

$$n(1 - \min\{\rho, 1/k\}) \left| \frac{1}{p} - \frac{1}{2} \right| \leq \nu + (\vartheta/k). \quad (119)$$

In this case, there exists $C > 0$ independent of $g \in L^p(G)$, satisfying

$$\|Tg\|_{L^p(G)} = \|A\mathcal{M}_{-\vartheta}g\|_{L^p(G)} \leq C\|g\|_{L^p(G)}. \quad (120)$$

In particular, if we replace $g$ in (120) by $\mathcal{M}_\vartheta f, f \in C^\infty(G)$, we obtain

$$\|Af\|_{L^p(G)} \leq C\|\mathcal{M}_\vartheta f\|_{L^p(G)} = C\|f\|_{L^p_{\vartheta,c}(G)}. \quad (121)$$

Thus we end the proof.
7.2. Boundedness properties on subelliptic Besov spaces

With the machinery developed above we are ready in order to prove our subelliptic Besov estimate allowing the sharp situation $\delta = \rho$.

**Proof of Theorem 1.3:** Let us choose a $S^1_0$-partition of unity (or Littlewood–Paley partition of the unity) $\{\tilde{\psi}_j\}$. So, we assume the following properties (see, e.g. Definition 2.1 of Garetto [16]),

- $\tilde{\psi}_0 \in \mathcal{D}(-\infty, \infty), \tilde{\psi}_0(t) = 1$ for $|t| \leq 1$, and $\tilde{\psi}_0(t) = 0$ for $|t| \geq 2$
- For every $j \geq 1$, $\tilde{\psi}_j(t) = \tilde{\psi}_0(2^{-j}t) - \tilde{\psi}_0(2^{-j+1}t) := \tilde{\psi}_1(2^{-j+1}t)$
- $|\tilde{\psi}^{(\alpha)}_j(t)| \leq C\alpha 2^{-\alpha j}$, $\alpha \in \mathbb{N}_0$
- $\sum_j \tilde{\psi}_j(t) = 1, t \in \mathbb{R}$.

If $\sigma \in S_{\rho, \delta}^{-\nu}(G), 0 \leq \delta < \rho \leq 1$, to study the subelliptic Besov boundedness of the pseudo-differential operator $A \equiv \sigma(x, D)$, for every $\tilde{\ell} \in \mathbb{N}$, and $f \in C^\infty(G)$ we will decompose $\psi^{\tilde{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af$ as

$$
\psi^{\tilde{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af = \sum_{j,j' = 0}^\infty \psi^{\tilde{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A\psi_j(D)f = \sum_{j,j' = 0}^\infty T_{\tilde{\ell}, j, j'}f,
$$

with

$$
T_{\tilde{\ell}, j, j'} := \psi^{\tilde{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A\psi_j(D).
$$

By observing that the matrix symbol $\sigma_{T_{\tilde{\ell}, j, j'}}$ of every operator $T_{\tilde{\ell}, j, j'}$ satisfies the identity

$$
\sigma_{T_{\tilde{\ell}, j, j'}}(x, \xi) = \psi^{\tilde{\ell}}((\xi))\tilde{\psi}_{j'}((\xi))\sigma_{A\psi_j(D)}(x, \xi)
= \sigma_{\psi^{\tilde{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A}(x, \xi),
$$

for all $x \in G$ and $[\xi] \in \hat{G}$, where

$$
\sigma_{A\psi_j(D)}(x, \xi) \quad \text{and} \quad \sigma_{\psi^{\tilde{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A}(x, \xi)
$$

are the corresponding symbols to the operators

$$
A\psi_j(D) \quad \text{and} \quad \psi^{\tilde{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A
$$

respectively, we deduce that

$$
\text{supp}(\sigma_{T_{\tilde{\ell}, j, j'}}(x, \xi)) \subset \text{supp}(\psi^{\tilde{\ell}}((\xi))Id_{\tilde{\ell}}) \cap \text{supp}(\psi_j(\xi)) \cap \text{supp}(\tilde{\psi}_{j'}((\xi))Id_{\tilde{\ell}}), \quad (122)
$$
for all \( x \in G \), and \([\xi] \in \hat{G} \). From the existence of an integer \( j_0 \in \mathbb{N} \) satisfying \( \tilde{\psi}_j \psi_{j'} = 0 \) for all \( j, j' \) with \(|j - j'| \geq j_0 \), we have the estimate

\[
\| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af \|_{L^p(G)}
\]

\[
\leq \sum_{j, j' = \bar{\ell} - j_0} 2^j \| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A\psi_{j}(D)f \|_{L^p(G)},
\]

for all \( \bar{\ell} \in \mathbb{N} \). From Lemma 4.11 of [1] applied to every operator \( \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A\psi_{j}(D)f \|_{L^p(G)} \leq C\|A\psi_{j}(D)f \|_{L^p(G)} \), where the positive constant \( C > 0 \) is independent of \( \bar{\ell}, j, j' \) and \( f \). So, if additionally we use Theorem 1.2, we can estimate the \( B^\nu_{p,q} \)-Besov norm of \( Af \) as follows:

\[
\|Af\|_{B^\nu_{p,q}(G)} \leq \left\{ 2^{j_0} \| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af \|_{L^p(G)} \right\}_{\bar{\ell} \in \mathbb{N}} \| \ell^q(\mathbb{N}) \}
\]

\[
\leq \left\{ \sum_{j, j' = \bar{\ell} - j_0} 2^j \| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})\tilde{\psi}_{j'}((1 + \mathcal{L}_G)^{\frac{1}{2}})A\psi_{j}(D)f \|_{L^p(G)} \right\}_{\bar{\ell} \in \mathbb{N}} \| \ell^q(\mathbb{N}) \}
\]

\[
\leq 2^{j_0} \| A\|_{\mathcal{B}(L^p_{\rho,q},L^p(G))} \| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af \|_{L^p(G)} \}
\]

\[
\leq 2^{(s+\theta)} \| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af \|_{L^p(G)} \}
\]

\[
\leq (2j_0 + 1) \| 2^{(s+\theta)} \| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af \|_{L^p(G)} \}
\]

\[
(123)
\]

\[
\|Af\|_{B^\nu_{p,q}(G)} \leq (2j_0 + 1) \| 2^{(s+\theta)} \| \psi_{\bar{\ell}}((1 + \mathcal{L}_G)^{\frac{1}{2}})Af \|_{L^p(G)} \}
\]

provided that

\[
n(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| - (\theta/\kappa) \leq \nu.
\]

\[
\textbf{Remark 7.3:} \text{ If we consider the Laplacian on } G, \mathcal{L}_G, \text{ instead of any sub-Laplacian, and we put } \kappa = 1, \text{ a similar argument to the used in the proof of Theorem 1.3 leads us to prove a similar result. We record it as follows.}
\]
Theorem 7.4: Let us assume that $\sigma \in \mathcal{S}_{\rho,\delta}^{-\nu}(G)$ and $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$. Then $A \equiv \sigma(x,D)$ extends to a bounded operator from $B^{\delta+\vartheta}_{p,q}(G)$ to $B^{\vartheta}_{p,q}(G)$ provided that

$$n(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right| - \vartheta \leq \nu, \quad (124)$$

for $0 < \rho < 1$, and $\nu = -\vartheta$, for $\rho = 1$. In particular, if $\sigma \in \mathcal{S}_{\rho,\delta}^{0}(G)$, the operator $A \equiv \sigma(x,D)$ extends to a bounded operator from $B^{\delta+\vartheta}_{p,q}(G)$ to $B^{\vartheta}_{p,q}(G)$ with

$$n(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right| \leq \vartheta, \quad (125)$$

for $0 < \rho < 1$, and $\vartheta = 0$, for $\rho = 1$.

Remark 7.5: Theorem 7.4 shows that, for $0 \leq \delta \leq \rho \leq 1$, $\delta \neq 1$, if we apply an operator $A$ to a function/distribution $f \in B^{\rho}_{p,q}(G)$, the function $Af$ has regularity order $s + \vartheta$, $\vartheta = n(1-\rho)\left|\frac{1}{p}-\frac{1}{2}\right|$. So, this result improves to ones recently reported in [18,19] and [20] where the restriction $\delta < \rho$ is imposed. Other properties of pseudo-differential operators related to the nuclearity notion on Besov spaces (associated to the Laplacian) are proved in [39]. The work [40] by the authors provide a systematic investigation of the Besov spaces associated to Rockland operators on graded Lie groups, taking as a starting point the Sobolev spaces associated to Rockland operators as developed by the second author and V. Fischer in [41]. The case of the Heisenberg group was studied in Bahouri et al. [42].

Notes

1. $A$ is defined by $f \mapsto Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \sigma(x,\xi) \hat{f}(\xi) \, d\xi$, $\hat{f}$ is the Fourier transform of $f$, and $\sigma$ satisfies the $(\rho,\delta)$-conditions $|\partial^\beta_x \partial^\alpha_\xi \sigma(x,\xi)| = O((1 + |\xi|)^{-m-\rho|\alpha|+\delta|\beta|})$.

2. $A$ is defined by $f \mapsto Af(x) = \sum_{[\xi] \in \hat{G}} \hat{f}(\xi) Tr[\xi(x) \sigma(x,\xi)]$, $\hat{f}$ is the Fourier transform of $f$, and $\sigma$ satisfies the $(\rho,\delta)$-conditions $\|\partial^\beta_x \Delta^\alpha_\xi \sigma(x,\xi)\|_{op} = O((\xi)^{-m-\rho|\alpha|+\delta|\beta|})$.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The first author was supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations. The second author was supported in parts by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations, EPSRC grant EP/R003025/1 and by the Leverhulme Grant RPG-2017-151. This is a revised version of the submitted one to the arxiv as arXiv:1901.06825 on 21-January-2019.

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