RIGID BODY EQUATIONS ON SPACES OF PSEUDO-DIFFERENTIAL OPERATORS WITH RENORMALIZED TRACE

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Abstract. We equip the regular Fréchet Lie group of invertible, odd-class, classical pseudodifferential operators $\text{Cl}_{\text{odd}}^0(M, E)$ —in which $M$ is a compact smooth manifold and $E$ a (complex) vector bundle over $M$— with pseudo-Riemannian metrics, and we use these metrics to introduce a large class of rigid body equations. We adapt to our infinite-dimensional setting Manakov’s classical observation on the integrability of Euler’s equations for the rigid body, and we show that our equations can be written in Lax form (with parameter) and that they admit an infinite number of integrals of motion. We also prove the existence of metric connections, we show that our rigid body equations determine geodesics on $\text{Cl}_{\text{odd}}^0(M, E)$, and we present rigorous formulas for the corresponding curvature and sectional curvature. Our main tool is the theory of renormalized traces of pseudodifferential operators on compact smooth manifolds without boundary.

Keywords: rigid body equations, pseudodifferential operators, renormalized traces, integrability.

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1. Introduction

In this paper we continue our work on Mathematical Physics themes posed on spaces built out of non-formal pseudodifferential operators. In [26] we introduced a Kadomtsev-Petviashvili hierarchy with the help of odd-class non-formal pseudodifferential operators, its importance being that our new KP hierarchy “covers” a KP hierarchy posed on spaces of symbols, that is, on equivalence classes of non-formal pseudodifferential operators; here we consider analogues of the rigid body equation on Fréchet Lie groups of non-formal pseudodifferential operators. We recall that the rigid body equation appears thus:

We fix a Lie group $G$ with Lie algebra $\mathcal{G}$. We recall that the functional derivative of a smooth function $f : \mathcal{G}^* \to \mathbb{R}$ at $\mu \in \mathcal{G}^*$ is the unique element $\delta f / \delta \mu$ of $\mathcal{G}$ determined by

$$\left< \nu, \frac{\delta f}{\delta \mu} \right> = \frac{d}{d \epsilon} \bigg|_{\epsilon=0} f(\mu + \epsilon \nu) = T_\mu f \cdot \nu$$

for all $\nu \in \mathcal{G}^* (= T_\mu \mathcal{G}^*)$, in which $\langle , \rangle$ denotes a natural pairing between $\mathcal{G}$ and $\mathcal{G}^*$, and that (see [3] p. 129 or [13] Chapter 9) the Lie-Poisson bracket on the dual space $\mathcal{G}^*$ is defined as follows: for all smooth functions $F, G : \mathcal{G}^* \to \mathbb{R}$ and $\mu \in \mathcal{G}^*$,

$$\{F, G\}(\mu) = \left< \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right>.$$
If $G^*$ is equipped with the Lie-Poisson structure $\langle \cdot, \cdot \rangle$, the Hamiltonian vector field corresponding to a function $H : G^* \to \mathbb{R}$ acts on smooth functions $F : G^* \to \mathbb{R}$ as

$$X_H(\mu) \cdot F = \{H, F\}(\mu) = \left\langle \mu, \left[ \frac{\delta H}{\delta \mu}, \frac{\delta F}{\delta \mu} \right] \right\rangle,$$

and it follows that the Hamiltonian equation of motion on $G^*$ is

$$\frac{d\mu}{dt} = X_H(\mu) = \left\langle \mu, \left[ \frac{\delta H}{\delta \mu}, \cdot \right] \right\rangle.$$

Now we assume that there exists a non-degenerate pairing $\langle \cdot, \cdot \rangle$ between $G^*$ and $G$. Then, we can write the equation of motion as an equation on $G$,

$$\left\langle \frac{dP}{dt}, \cdot \right\rangle = \left\langle P, \left[ \frac{\delta H}{\delta \mu}, \cdot \right] \right\rangle,$$

We call (4) the Euler equation in weak form. There are two paths we can take. First, if the pairing between $G^*$ and $G$ is, in addition, symmetric and infinitesimally $Ad$-invariant, this is, it satisfies

$$\langle P, [Q, R] \rangle = \langle [R, P], Q \rangle, \quad P, Q, R \in G,$$

then we can write Equation (4) in Lax form,

$$\frac{dP}{dt} = \left[ P, \frac{\delta H}{\delta \mu} \right].$$

Following Berezin and Perelomov, see [4], we call Equation (16) the Euler equation or, the rigid body equation posed on $G$. Second, if the pairing between $G^*$ and $G$ is symmetric, but not necessarily infinitesimally $Ad$-invariant, we define an adjoint map with respect to the pairing, $ad_A$, via the equation

$$\langle [P, Q], R \rangle = -\langle Q, ad_A(P) \cdot R \rangle.$$

Then, Equation (4) can be written as

$$\frac{dP}{dt} = -ad_A \left( \frac{\delta H}{\delta \mu} \right) \cdot P.$$

This equation is an evident generalization of (6): we also call it the Euler equation posed on $G$. If the pairing $\langle \cdot, \cdot \rangle$ is symmetric and positive-definite, then (7) determines geodesics on $G$, see for instance [35].

Now we can state our motivation for studying the rigid body equation (in their versions (4) and (7)) in the (non-commutative) setting of rigorous pseudo-differential operators: we are inspired by Arnold’s seminal work [2] and by the non-commutative version of the Korteweg-de Vries (KdV) equation considered by Berezin and Perelomov in [4]. In both cases they consider motion on infinite-dimensional Lie groups and Lie algebras, including geodesic motion. Now, previous research stemming from [2] has concentrated on providing rigorous analytic foundations for the beautiful results on Riemannian geometry of diffeomorphism groups appearing in [2], and on the study of some very interesting equations posed on these infinite-dimensional groups, see for instance [12, 3, 10] or the reviews [18, 35]. We wonder if we can study (geodesic) motion on groups which are natural alternatives to diffeomorphism groups in a fully rigorous way.

Indeed, we observe herein that there exist Fréchet Lie groups of non-formal pseudodifferential operators that can be equipped with (weak) pseudo-Riemannian metrics, and that it is very feasible to investigate rigid body equations posed on
them. Let us explain briefly our geometric setting: we fix a compact smooth manifold without boundary \( M \) and a (complex) vector bundle over \( M \), and we prove that the set \( G = \mathcal{C}^{0,\ast}_{\text{odd}}(M,E) \) of all zero order, invertible, and odd-class non-formal classical pseudodifferential operators acting on sections of \( E \), can be equipped with the structure of a regular Fréchet Lie group. Once we fix \( G \), we need to define a suitable linear functional that replaces standard trace, in order to consider formulas analogous to

\[
(A,B) \mapsto tr(A\mathcal{A}(B))
\]

which (depending on the operator \( \mathcal{A} \)) defines a metric on spaces of matrices. Now, the classical trace \( tr \) of (trace-class) operators on a Hilbert space is not defined on the entire Lie algebra of \( G \). We choose a linear extension of \( tr \), called the \( \zeta \)-renormalized trace or weighted trace, on the class \( \mathcal{C}^{0,\ast}_{\text{odd}}(M,E) \) of odd-class non-formal classical pseudodifferential operators introduced by Kontsevich and Vishik [21, 22] and fully described in [32, 34]. Properties of \( \zeta \)-renormalized traces allow us to endow the group \( G \) with the structure of an infinite-dimensional (weak) pseudo-Riemannian manifold by extending the results of [27] to our setting.

Our research differs from previous investigations carried out in the spirit of Arnold’s [2], such as [3, 10, 18, 35] or [17], in two ways. First, our rigid body equation is a bona fide ordinary differential equation (similar in this respect to the classical rigid body equation), due to the way the Lie bracket is defined in our context; it does not reduce to a partial differential equation as it happens if we study Euler’s equations on diffeomorphism groups, see [3, 10, 18, 17, 35]. Second, it is a non-commutative nonlinear equation, since it is a nonlinear equation for an unknown rigorous pseudodifferential operator. In this latter aspect, our Euler equation is similar in spirit to the non-commutative version of the Korteweg-de Vries (KdV) equation by Berezin and Perelomov, see [4], and to the equations considered by Olver and Sokolov in [31], although, in contradistinction with e.g. [31], our study belongs to global analysis rather than to formal geometry, because of the presence of non-formal (Fréchet) Lie groups of non-formal pseudo-differential operators and of traces which fully extend the trace of a finite rank operator.

Interestingly, our version of Euler’s equation shares with the classical rigid body equation the important properties of admitting a parameter-depending Lax formulation and integrals of motion. We check this claim by adapting Manakov’s seminal observation on the integrability of Euler’s equation appearing in [28], to our framework. For example, in the particular case in which the manifold \( M \) mentioned above is simply \( S^1 \), and the unknown pseudodifferential operator is a function \( X : S^1 \to \mathbb{C} \), we can check that the equation

\[
\frac{dX}{dt} = X(\Delta + \pi)X^*(\Delta + \pi)^{-1} - XX^*,
\]

where \( \pi \) is the \( L^2 \)-orthogonal projection on the kernel of the Laplacian, admits infinitely many independent integrals of motion (at least for a large class of initial conditions, see Subsection 6.2).

We organize this work as follows. In Section 2 we present a quick survey of the properties of pseudodifferential operators that we use, including the Fréchet structure of \( \mathcal{C}^{0,\ast}_{\text{odd}}(M,E) \). We also introduce the Wodzicki residue and renormalized traces, and we state the facts that make them interesting objects for geometry. In
Section 3 we show how to construct non-degenerate pairings on spaces of non-formal classical pseudodifferential operators using renormalized traces. In Section 4 we use these pairings to construct right-invariant pseudo-Riemannian metrics on $C^0_{\text{odd}}(M, E)$, we introduce our rigid body equations using variational methods, we show that they can be written as Lax equations depending on a parameter, and we present integrals of motion. In Section 5 we make some remarks on the pseudo-Riemannian geometry of the Fréchet Lie group $C^0_{\text{odd}}(M, E)$, and we prove that, as in more classical contexts, our rigid body equations determine geodesics. We finish in Section 6 with two examples: in 6.1 we present equations arising from a metric depending on the heat operator, and in 6.2 we analyse a class of equations which includes (8). In particular, by direct computations on renormalized traces, we show independence of integrals of motion.

2. Preliminaries

2.1. Preliminaries on classical pseudodifferential operators. We introduce groups and algebras of non-formal pseudodifferential operators needed to set up our equations. Basic definitions are valid for real or complex finite-dimensional vector bundles $E$ over a compact manifold $M$ without boundary whose typical fiber is a finite-dimensional real or complex vector space $V$. We begin with the following definition after [5, Section 2.1].

**Definition 1.** The graded algebra of differential operators acting on the space of smooth sections $C^\infty(M, E)$ is the algebra $DO(E)$ generated by:

- Elements of $\text{End}(E)$, the group of smooth maps $E \to E$ leaving each fibre globally invariant and which restrict to linear maps on each fibre. This group acts on sections of $E$ via (matrix) multiplication;
- The differentiation operators $\nabla_X : g \in C^\infty(M, E) \mapsto \nabla_X g$ where $\nabla$ is a connection on $E$ and $X$ is a vector field on $M$.

Multiplication operators are operators of order 0; differentiation operators and vector fields are operators of order 1. In local coordinates, a differential operator of order $k$ has the form $P(u)(x) = \sum p_i \cdots \nabla_{x_i} \cdots \nabla_{x_r} u(x), \quad r \leq k$, in which $u$ is a (local) section and the coefficients $p_i \cdots \nabla_{x_i}$ can be matrix-valued. The algebra $DO(M, E)$ is filtered by order: we note by $DO^k(M, E), k \geq 0$, the differential operators of order less or equal than $k$.

Now we embed $DO(M, E)$ into the algebra of classical pseudodifferential operators $Cl(M, E)$. We need to assume that the reader is familiar with the basic facts on pseudodifferential operators defined on a vector bundle $E \to M$; these facts can be found for instance in [15], in the review [33, Section 3.3], and in the papers [6] and [37] in which the authors construct a global symbolic calculus for pseudodifferential operators showing, for instance, how the geometry of the base manifold $M$ furnishes an obstruction to generalizing local formulas of composition and inversion of symbols.

**Notations.** We note by $PDO(M, E)$ the space of pseudodifferential operators on smooth sections of $E$, see [33, p. 91]; by $PDO^o(M, E)$ the space of pseudodifferential operators of order $o$; and by $Cl(M, E)$ the space of classical pseudodifferential operators acting on smooth sections of $E$, see [33, pp. 89-91]. We also note by
Definition 3. A classical pseudodifferential operator $A$ in $\mathcal{M}$ is called the symbol $\sigma(A)(x, \xi)$ of $A$. We define

$$\sigma(A)(x, \xi) = \sum_{j=0}^{\infty} \sigma_{n-j}(A)(x, \xi), \quad (x, \xi) \in \mathcal{T}^* M,$$

in which each $\sigma_{n-j}(A)(x, \xi)$ satisfies the homogeneity condition

$$\sigma_{n-j}(A)(x, t\xi) = t^{n-j} \sigma_{n-j}(A)(x, \xi) \quad \text{for every } t > 0.$$  

The function $\sigma_0(A)(x, \xi)$ is the principal symbol of $A$. We define

Definition 3. A classical pseudodifferential operator $A$ in $\mathcal{M}$ is called

- **odd class** if and only if for all $n \in \mathbb{Z}$ and all $(x, \xi) \in \mathcal{T}^* M$ we have:

$$\sigma_n(A)(x, -\xi) = (-1)^n \sigma_n(A)(x, \xi),$$

and
• **even class** if and only if for all \( n \in \mathbb{Z} \) and all \((x, \xi) \in T^* M\) we have:
\[
\sigma_n(A)(x, -\xi) = (-1)^{n+1} \sigma_n(A)(x, \xi).
\]

Odd class pseudodifferential operators were introduced in [21][22]; they are called "even-even pseudodifferential operators" in the treatise [31]. For instance, recalling Definition 1, we see that all differential operators are odd class.

Hereafter, the subscript \( \text{odd} \) (resp. \( \text{even} \)) attached to a given space of (formal) pseudodifferential operators will refer to the set of all odd (resp. even) class (formal) pseudodifferential operators belonging to that space.

We need the following result, already essentially present in [21][34]:

**Lemma 4.** \( Cl_{\text{odd}}(M, E) \) and \( Cl^0_{\text{odd}}(M, E) \) are associative algebras.

**Proof.** We work locally. Let \( A, B \) be two odd class pseudodifferential operators of order \( m \) and \( m' \) respectively; the homogeneous pieces of the symbols of \( A, B, AB \) are related via (see [34, Section 1.5.2, Equation (1.5.2.3)])
\[
\sigma_{m+m'-j}(AB)(x, \xi) = \sum_{|\mu|+k+l=j} \frac{1}{\mu!} \partial_\xi^\mu \sigma_{m-k}(A)(x, -\xi) D_\xi^\mu \sigma_{m'-l}(B)(x, \xi),
\]
in which \(|\mu|\) is the length of the multi-index \( \mu \). We have, using the first equation appearing in Definition 3
\[
\partial_\xi^\mu \sigma_{m-k}(A)(x, -\xi) = (-1)^{m-k+|\mu|} \partial_\xi^\mu \sigma_{m-k}(A)(x, \xi)
\]
and
\[
D_\xi^\mu \sigma_{m'-l}(B)(x, -\xi) = (-1)^{m'-l} D_\xi^\mu \sigma_{m'-l}(B)(x, \xi),
\]
so that
\[
\sigma_{m+m'-j}(AB)(x, -\xi) = \sum_{|\mu|+k+l=j} \frac{1}{\mu!} (-1)^{m-k+|\mu|+m'-l} \partial_\xi^\mu \sigma_{m-k}(A)(x, \xi) D_\xi^\mu \sigma_{m'-l}(B)(x, \xi).
\]
Changing \(+|\mu|\) for \(-|\mu|\) in \((-1)^{m-k+|\mu|+m'-l}\) and using \(|\mu|+k+l=j\) we obtain
\[
\sigma_{m+m'-j}(AB)(x, -\xi) = (-1)^{m+m'-j} \sum_{|\mu|+k+l=j} \frac{1}{\mu!} \partial_\xi^\mu \sigma_{m-k}(A)(x, \xi) D_\xi^\mu \sigma_{m'-l}(B)(x, \xi)
\]
\[
= (-1)^{m+m'-j} \sigma_{m+m'-j}(AB)(x, \xi),
\]
which proves the first claim. That \( Cl^0_{\text{odd}}(M, E) \) is an associative algebra now follows from the standard fact that zero-order classical pseudodifferential operators form an algebra, see for instance the proof of Proposition 3 in [33]. \( \square \)

The next proposition singles out an interesting Lie group included in \( Cl_{\text{odd}}(M, E) \).

**Proposition 5.** The algebra \( Cl^0_{\text{odd}}(M, E) \) is a closed subalgebra of \( Cl^0(M, E) \). Moreover, \( Cl^0_{\text{odd}}(M, E) \) is
- an open subset of \( Cl^0_{\text{odd}}(M, E) \) and,
- a regular Fréchet Lie group with Lie algebra \( Cl^0_{\text{odd}}(M, E) \) and smooth Lie bracket \([A, B] = AB - BA\).

**Proof.** We denote by \( \sigma(A)(x, \xi) \) the total formal symbol of \( A \in Cl^0(M, E) \). We define the function
\[
\phi : Cl^0(M, E) \rightarrow \mathcal{F}Cl^0(M, E)
\]
as
\[ \phi(A) = \sum_{n \in \mathbb{N}} \sigma_{-n}(x, \xi) - (-1)^n \sigma_{-n}(x, -\xi). \]

This map is smooth, and
\[ Cl^0_{\text{odd}}(M, E) = \text{Ker}(\phi), \]
which shows that \( Cl^0_{\text{odd}}(M, E) \) is a closed subalgebra of \( Cl^0(M, E) \). Moreover, if \( H = L^2(M, E) \),
\[ Cl^0_{\text{odd}}(M, E) = Cl^0_{\text{odd}}(M, E) \cap GL(H), \]
which proves that \( Cl^0_{\text{odd}}(M, E) \) is open in the Fréchet algebra \( Cl^0_{\text{odd}}(M, E) \), and it follows that it is a regular Fréchet Lie group by arguing along the lines of [14, 30]. □

We finish our preliminaries on pseudodifferential operators noting that at a formal level we have the splitting
\[ \mathcal{F}Cl(M, E) = \mathcal{F}Cl_{\text{odd}}(M, E) \oplus \mathcal{F}Cl_{\text{even}}(M, E), \]
and the following composition rules for formal pseudodifferential operators \( A \circ B \):

|       | A odd class | A even class |
|-------|-------------|--------------|
| B odd class | A \circ B odd class | A \circ B even class |
| B even class | A \circ B even class | A \circ B odd class |

2.2. Renormalized traces. Hereafter we assume that the typical fiber of the bundle \( E \) is a complex vector space, and that \( E \) is equipped with an Hermitian product \( \langle \cdot, \cdot \rangle \). An excellent review of this geometric set-up appears in [36, Chapters III, IV]. This product allows us to define the following \( L^2 \)-inner product on sections of \( E \):
\[ \forall u, v \in C^\infty(M, E), \quad (u, v)_{L^2} = \int_M < u(x), v(x) > dx, \]
where \( dx \) is a fixed Riemannian volume on \( M \).

We need to use some further notions of the theory of pseudodifferential operators. First of all, we use the inner product just introduced to define self-adjoint and positive pseudodifferential operators. We also define elliptic pseudodifferential operators: a classical pseudodifferential operator \( P \) of order \( o \) is elliptic if its main symbol \( \sigma_o(P)(x, \xi) : E_x \rightarrow E_x \) is invertible, see for instance [36, Chapter IV, Section 4] or [34, p. 92]; these pseudodifferential operators are also discussed quickly in [32, Definitions 6.17, 6.31]. We denote by \( \text{Ell}(M, E) \) the space of all classical elliptic pseudodifferential operators.

**Definition 6.** \( Q \) is a weight of order \( q \in \mathbb{N}^* \) on \( E \) if and only if \( Q \) is a classical, elliptic, self-adjoint and positive pseudodifferential operator acting on smooth sections of \( E \).
Under these assumptions, the weight $Q$ has a real discrete spectrum, and all its eigenspaces are finite dimensional. Moreover, for such a weight $Q$ of order $q$, we can define complex powers of $Q$, see e.g. [7] or [32, Section 7.1] for a quick overview of technicalities and further references: the powers $Q^{-s}$ of the weight $Q$ are defined for $\Re(s) > 0$ using a contour integral of the form

$$Q^{-s} = \int_{\Gamma} \lambda^s (Q - \lambda \text{Id})^{-1} d\lambda,$$

in which $\Gamma$ is a contour around the real positive axis that appears precisely identified in [32, Section 7.1]. The pseudodifferential operator $Q^{-s}$ is a classical pseudodifferential operator of order $-qs$.

Now we let $A$ be a log-polyhomogeneous pseudodifferential operator, that is, $A$ is a pseudodifferential operator such that its symbol is, locally, of the form

$$\sigma(x, \xi) \sim \sum_{j=0}^{a} \sum_{-\infty < k} \sigma_{j,k}(A)(x, \xi) \log(|\xi|)^j,$$

in which $\sigma_{j,k}(A)(x, \xi)$ are classical symbols, see [34, Section 2.6]. Within this general framework we introduce zeta functions and traces. The map

$$\zeta(A, Q, s) = s \in \mathbb{C} \mapsto \text{tr}(AQ^{-s}) \in \mathbb{C},$$

in which tr is the classical trace of trace-class pseudodifferential operators, see [34, Section 1.3.5.1], is well-defined for $\Re(s)$ large enough, and it extends to a meromorphic function on $\mathbb{C}$ with possibly a pole at $s = 0$ [32, 34]. When $A$ is classical, this pole is a simple pole, and when $A$ is classical, odd-class, and $M$ is odd dimensional, $\zeta(A, Q, s)$ has no pole at $s = 0$.

Gilkey [15, Section 1.12.2] treats zeta functions and their relation to the heat kernel in detail; Scott [34, Section 1.5.7] deals with zeta functions in a very general setting: he extends the computations of Kontsevich and Vishik [21].

When $A$ is a classical pseudodifferential operator, the Wodzicki residue, $\text{res}_W$, appearing in [38], see also [19], is directly linked with the simple pole of $\zeta(A, Q, .)$ at 0 by the residue formula

$$(10) \quad \text{res}_{s=0} \zeta(A, Q, s) = (1/q) \text{res}_W A .$$

The Wodzicki residue is a higher dimensional analog of the Adler trace on formal symbols introduced in [11], but we remark that the former fails to be a direct extension of the latter. For example, $(1 + \frac{d}{dx})$ is invertible in $\text{Cl}(S^1, \mathbb{C})$, and since $(1 + \frac{d}{dx})^{-1}$ is odd class, we have on one hand that

$$\text{res}_W (1 + \frac{d}{dx})^{-1} = 0 ,$$

see e.g. [34, Section 1.5.8.2], but on the other hand, the formal symbol of $(1 + \frac{d}{dx})^{-1}$ has a non-vanishing Adler trace.

Following [32, Chapter 7] and [34, Section 1.5.7], see also [7], we define renormalized traces of classical pseudodifferential operators as follows:

**Definition 7.** Let $A$ be a log-polyhomogeneous pseudo-differential operator and $Q$ a fixed weight of order $q$. The finite part of $\zeta(A, Q, s)$ at $s = 0$ is called the
renormalized trace of $A$. We denote it by $\text{tr}^QA$. If $A$ is a classical pseudodifferential operator, then

$$\text{tr}^QA = \lim_{s \to 0} \left( \text{tr}(AQ^{-s}) - \frac{1}{q s} \text{res}_W(A) \right).$$

If $A$ is a trace-class pseudodifferential operator acting on $L^2(M, E)$ then $\text{tr}^QA = \text{tr}(A)$, see e.g. [7]. However, generally speaking, the linear functional $\text{tr}^Q$ is not a trace, this is, it does not vanish on commutators, although the linear map $\text{res}_W$ determined by the Wodzicki residue does fulfil the trace property.

We state the main properties of $\text{res}_W$ and of renormalized trace in Propositions 8 and 9.

**Proposition 8.**
(i) The Wodzicki residue $\text{res}_W$ is a trace on the algebra of classical pseudodifferential operators $\text{Cl}(M, E)$, i.e. $\forall A, B \in \text{Cl}(M, E), \text{res}_W[A, B] = 0$.
(ii) If $m = \dim M$ and $A \in \text{Cl}(M, E)$,

$$\text{res}_W A = \frac{1}{(2\pi)^n} \int_M \int_{|\xi| = 1} \text{tr}_{-m}(x, \xi)d\xi dx$$

where $\sigma_{-m}$ is the $(-m)$ positively homogeneous part of the symbol of $A$, see [9]. In particular, $\text{res}_W$ does not depend on the choice of $Q$, in spite of what (10) may suggest.

**Proposition 9.** Let us fix a weight $Q$ of order $q$.
- Given two classical pseudo-differential operators $A$ and $B$,

$$(11) \quad \text{tr}^Q[A, B] = -\frac{1}{q} \text{res}(A[B, \log Q]).$$

- Let us consider a family $A_t$ of classical pseudo-differential operators of constant order, and a family $Q_t$ of weights of constant order $q$, both of which are differentiable with respect to the Kontsevich and Vishik Fréchet structure on $\text{Cl}(M, E)$. Then,

$$(12) \quad \frac{d}{dt} \left( \text{tr}^Q A_t \right) = \text{tr}^Q \left( \frac{d}{dt} A_t \right) - \frac{1}{q} \text{res} \left( A_t \frac{d}{dt} \log Q_t \right).$$

- If $C$ is a classical elliptic injective operator or a diffeomorphism, and $A$ is a classical pseudodifferential operator, $\text{tr}C^{-1}QC^* (C^{-1}AC)$ is well-defined and equals $\text{tr}^QA$.
- Finally,

$$\text{tr}^QA = \overline{\text{tr}^QA^*}.$$

In this proposition we have followed [7], and [24] for the third point.

We have stated that $\text{tr}^Q$ is not a true trace; however, the renormalized trace of the bracket satisfies some interesting properties which we state following [23].

**Definition 10.** Let $E$ be a vector bundle over $M$ let $Q$ a weight and let $a \in \mathbb{Z}$. We define :

$$A^Q_a = \{ B \in \text{Cl}(M, E) : [B, \log Q] \in \text{Cl}^0(M, E) \}.$$

**Theorem 11.**
(i) $A^Q_a \cap Cl^0(M, E)$ is an subalgebra of $Cl(M, E)$ with unit.
(ii) Let $B \in \text{Ell}^*(M, E)$, $B^{-1}A^Q_a B = A^{B^{-1}QB}_a$.
(iii) Let $A \in \text{Cl}^b(M, E)$, and $B \in \mathcal{A}_{b-2}^Q$, then $\text{tr}^Q[A, B] = 0$. As a consequence, 
\[ \forall (A, B) \in \text{Cl}^{-\infty}(M, E) \times \text{Cl}(S^1, V), \quad \text{tr}^Q[A, B] = 0. \]

We are ready to state the properties of $\text{tr}^Q$ that make odd-class pseudodifferential operators an interesting arena for infinite-dimensional mechanics.

**Theorem 12.** Let $A, B \in \text{Cl}(M, E)$ and let $Q$ be an odd-class weight of even order, e.g. $Q = \Delta$.

- If $(A, B) \in \text{Cl}_{\text{odd}}(M, E) \times \text{Cl}_{\text{odd}}(M, E)$, and if $M$ is odd dimensional, 
  \[ \text{tr}^Q([A, B]) = 0. \]
- If $(A, B) \in \text{Cl}_{\text{even}}(M, E) \times \text{Cl}_{\text{odd}}(M, E)$, and if $M$ is even dimensional, 
  \[ \text{tr}^Q([A, B]) = 0. \]

**Proof.** The first item is due to Kontsevich and Vishik, see [21, 22]. We sketch a proof of this theorem following [34]:

If $Q$ and $B$ are odd class, with $Q$ of even order, as in the statement of the theorem, $[B, \log Q] \in \text{Cl}_{\text{odd}}(M, E)$. Thus,

- If $A \in \text{Cl}_{\text{odd}}(M, E)$, then $A[B, \log Q] \in \text{Cl}_{\text{odd}}(M, E)$.
- If $A \in \text{Cl}_{\text{even}}(M, E)$, then $A[B, \log Q] \in \text{Cl}_{\text{even}}(M, E)$.

Symmetry properties show that in both cases
\[ \int_{|\xi|=1} \sigma_{-m}(A[B, \log Q]) = 0, \]
and the result follows by applying the local formula for the Wodzicki residue. \(\square\)

**Corollary 13.** Let $Q = f(\Delta)$ in which $f$ is any analytic function such that $Q$ is a weight, and assume that $A, B$ and $C$ are classical pseudodifferential operators either in the odd-class or in the even-class. If the product $ABC$ is odd class and $M$ is odd dimensional, or if the product $ABC$ is even class and $M$ is even dimensional, then
\[ \text{tr}^Q(ABC) = \text{tr}^Q(CAB) = \text{tr}^Q(BCA). \]

3. Renormalized traces determine non-degenerate pairings

In this short but crucial section we give an extension of a result from [27, Section 3.2] which connects the foregoing discussion with Hermitian geometry. We remark that we do not assume that $M$ is an odd dimensional manifold, so that $\text{tr}^Q$ is not a priori a true trace.

**Theorem 14.** We consider a weight $Q$ and a fixed classical pseudodifferential operator $Q_0 \in \text{Cl}(M, E)$.

1. The sesquilinear map
   \[ (\cdot, \cdot)_{Q, Q_0} : (A, B) \in \text{Cl}(M, E) \times \text{Cl}(M, E) \rightarrow \text{tr}^Q( AQ_0 B^*) \]
   is non-degenerate if and only if $Q_0$ is injective.
2. Moreover, if $Q_0$ is self-adjoint, then $(\cdot, \cdot)_{Q, Q_0}$ is Hermitian, this is,
   \[ (B, A)_{Q, Q_0} = (A, B)_{Q, Q_0}. \]
(3) As a consequence, the Hilbert-Schmidt positive definite Hermitian product

$$(A,B)_{HS} = \text{tr}(AB^*)$$

which determines a positive definite metric on $Cl^{-1-dimM}(M,E)$, extends to a Hermitian form

$$(\cdot,\cdot)_\Delta = (\cdot,\cdot)_{\Delta,Id}$$

- which is a non-degenerate form on $Cl(M,E)$
- whose real part defines a $(\mathbb{R}-)$ bilinear, symmetric non-degenerate form on $Cl(M,E)$.

The same properties hold true if we replace $Cl(M,E)$ by $Cl^0(M,E)$ in statements (1), (2), (3).

**Proof.**

(1) First, let us assume that $Q_0$ is not injective. Let $y \neq 0 \in \text{Ker}Q_0$ and let $A = p_y$ be $L^2$ orthogonal projection on the 1-dimensional vector space spanned by $y$. Then $AQ_0 = 0$, and $\forall B \in Cl(M,E)$ we have $(A,B)_{Q,Q_0} = 0$, so that $(\cdot,\cdot)_{Q,Q_0}$ is degenerate.

Let us now assume that $Q_0$ is injective. Then, $\forall A \neq 0 \in Cl(M,E)$, $AQ_0 \neq 0$. The formula $(A,B)_{Q,Q_0} = \text{tr}^Q(AQ_0B^*)$ certainly defines a sesquilinear form, let us prove that it is non-degenerate. Let $A \in Cl(M,E)$, and let $u \in C^\infty(M,E) \cap (\text{Im}AQ_0 - \{0\})$. We assume that $u$ is the image of a function $x$ such that $\|x\|_{L^2} = 1$, and we let $p_x$ be the $L^2-\text{orthogonal}$ projection on the $\mathbb{C}-\text{vector}$ space spanned by $x$. Finally, we also let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal base with $e_0 = x$. We will analyse $\phi(s) = \text{tr}(AQ_0B^*Q^{-s})$ when $s$ is “large enough”, and then we will pass to the meromorphic continuation. For $\Re(s) \geq (2\text{ord}(A) + 2\text{ord}Q_0 + 1 + \text{dim}M)/q$, we observe that the operators

$$AQ_0(AQ_0p_x)^*Q^{-s},$$
$$AQ_0(AQ_0p_x)^*Q^{-s/2},$$
$$Q^{-s/2},$$

and

$$Q^{-s/2}AQ_0$$

are Hilbert-Schmidt class. We recall that for Hilbert-Schmidt class operators $U$ and $V$, $UV$ is trace class and $\text{tr}(UV) = \text{tr}(VU)$. Thus, applying commutation relations of the usual trace of trace-class operators, we obtain the following:

$$\phi(s) = \text{tr}\left(AQ_0(AQ_0p_x)^*Q^{-s}\right) = \text{tr}\left(Q^{-s/2}AQ_0(AQ_0p_x)^*Q^{-s/2}\right)$$
$$= \text{tr}\left((AQ_0p_x)^*Q^{-s/2}.Q^{-s/2}AQ_0\right)$$
$$= \text{tr}\left((AQ_0p_x)^*Q^{-s}AQ_0\right).$$

Now we simplify this expression in order to show that the meromorphic continuation of $\phi(s)$ has no poles and a non-zero value at $s = 0$. The meromorphic continuation to $\mathbb{C}$ of $s \mapsto \text{tr}\left((AQ_0p_x)^*Q^{-s}AQ_0\right)$ exists and
it coincides with the meromorphic continuation of $s \mapsto \phi(s)$; in particular they coincide at $s = 0$. Moreover,

$$\text{tr} \left( (AQ_0 p_x)^* Q^{-s} AQ_0 \right) = \sum_{k \in \mathbb{N}} ((AQ_0 p_x)^* Q^{-s} AQ_0 e_k, e_k)_{L^2}$$

$$= \sum_{k \in \mathbb{N}} (Q^{-s} AQ_0 e_k, AQ_0 p_x e_k)_{L^2}$$

$$= (Q^{-s} AQ_0 x, AQ_0 x)_{L^2}$$

$$= (Q^{-s/2} u, Q^{-s/2} u)_{L^2}.$$

Since $\lim_{s \to 0} Q^{-s/2} = \text{Id}$ for weak convergence, the limit of the last term is $\|u\|_{L^2}^2 \neq 0$. The operator $AQ_0 p_x$ is a smoothing (rank 1) operator and hence it belongs to $\text{Cl}(M, E)$, which ends the proof.

(2) Let $(A, B) \in \text{Cl}(M, E)$. We calculate directly using Proposition 9:

$$\langle B, A \rangle_{Q_0} = \text{tr}^Q(BQ_0 A^*) = \text{tr}^Q((AQ_0 B^*)^*) = \text{tr}^Q(AQ_0 B^*) = \langle A, B \rangle_{Q_0},$$

(3) It follows from the two previous items.

We finish the proof by remarking that our foregoing arguments hold true when considering only bounded classical pseudodifferential operators.

Remark 15. We remark that if $Q_0$ is self-adjoint and injective, the polarization identity

$$\Re \langle A, B \rangle_{Q_0} = \frac{1}{4} \left[ (A + B, A + B)_{Q_0} - (A - B, A - B)_{Q_0} \right]$$

implies that $\Re \langle \cdot, \cdot \rangle_{Q_0}$ is a symmetric and non-degenerate real-valued bilinear form. This bilinear form is not positive-definite, see a direct calculation for $M = S^1$ in [27, Section 3].

4. Rigid body equations

In this and the next section we work with the regular Fréchet Lie group of odd-class pseudodifferential operators $\text{Cl}^{0,*}_{\text{odd}}(M, E)$ and its Lie algebra $\text{Cl}^{0}_{\text{odd}}(M, E)$. Our main claim is that this Lie group is a non-trivial differential geometric framework on which we can pose equations of mechanics in the spirit of Arnold, see [2]. Our main references for this section are [17, 18, 25] and [35].

We remark that in Subsection 4.2 (and also in Section 5) we consider pseudo-Riemannian metrics on $\text{Cl}^{0,*}_{\text{odd}}(M, E)$ induced by twisting the non-degenerate bilinear forms constructed in Section 3. Our metrics are defined using right translations, see e.g. [18]. This convention forces us to re-define the $\text{ad}_X$ morphism on $\text{Cl}^{0}_{\text{odd}}(M, E)$ as $\text{ad}_X Y = -[X, Y] = -(XY - YX) = YX - XY = [Y, X]$.

4.1. The Hamiltonian construction. We consider the trace $\text{tr}^A$ on the regular Lie algebra $\text{Cl}^{0}_{\text{odd}}(M, E)$ and the pairing

$$\langle A, B \rangle = \langle A, B \rangle_{\Delta, Q_0} = \text{tr}^A(AQ_0 B^*),$$

in which $Q_0$ is injective (and hence the pairing is non-degenerate) and self-adjoint (and hence the pairing is Hermitian), and we also consider its real part $\Re \langle A, B \rangle$. We also assume, here and hereafter, that the following constraints on $Q_0$ hold:
| Dimension       | Operator            |
|-----------------|---------------------|
| odd             | an odd-class operator |
| even            | an even-class operator |

In this way we are sure that the commutation relations for $\text{tr}^Q (A Q_0 B^*)$ appearing in Theorem 12 and Corollary 13 hold. Trivially, if $Q_0$ is injective, self-adjoint and smoothing (e.g. $Q_0 = e^{-\Delta}$), these conditions are fulfilled for any manifold $M$.

The next lemma is a direct consequence of Theorem 14:

**Lemma 16.** Let us assume that $Q_0$ is an injective and self-adjoint classical pseudodifferential operator.

1. The $\mathbb{C}$-valued pairing $\langle A, B \rangle = \text{tr}^\Delta (A Q_0 B^*)$ on $\text{Cl}^0_\text{odd} (M, E)$ is Hermitian and non-degenerate.
2. The real-valued pairing $\Re \langle A, B \rangle$ is bilinear, symmetric and non-degenerate for any choice of self-adjoint and injective operator $Q_0 \in \text{Cl}(M, E)$.

This lemma allows us to consider the regular dual space of $\text{Cl}^0_\text{odd} (M, E)$, namely, $\text{Cl}^0_\text{odd} (M, E)^\prime = \{ \mu \in L(\text{Cl}^0_\text{odd} (M, E), \mathbb{C}) : \mu = \langle A, \cdot \rangle \text{ for some } A \in \text{Cl}^0_\text{odd} (M, E) \}$. We can equip $\text{Cl}^0_\text{odd} (M, E)^\prime$ with a Fréchet structure simply by transferring the structure of $\text{Cl}^0_\text{odd} (M, E)$, since there is a bijection between $\text{Cl}^0_\text{odd} (M, E)^\prime$ and $\text{Cl}^0_\text{odd} (M, E)$.

We consider smooth polynomial functions on $\text{Cl}^0_\text{odd} (M, E)^\prime$ of the form

$$f(\mu) = \sum_{k=0}^{n} a_k \text{tr}^\Delta (P^k),$$

in which $a_k \in \mathbb{C}$ and $P$ is determined by the equation $\mu = \langle P, \cdot \rangle$.

If $f$ is such a smooth function on $\text{Cl}^0_\text{odd} (M, E)^\prime$, we define the functional derivative of $f$, $\delta f / \delta \mu \in \text{Cl}^0_\text{odd} (M, E)$, as in (1), that is, via the equation

$$\left\langle \nu, \frac{\delta f}{\delta \mu} \right\rangle = (\delta f)_{\mu}(\nu) = \left. \frac{d}{dt} \right|_{t=0} f(\mu + t\nu),$$

and we equip $\text{Cl}^0_\text{odd} (M, E)^\prime$ with a Poisson bracket (which acts on polynomial functions), see Equation (2), (4), and also (13, 25). We set:

$$\{ f, g \}(\mu) = + \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle$$

for smooth functions $f, g : \text{Cl}^0_\text{odd} (M, E)^\prime \rightarrow \mathbb{C}$ and $\mu \in \text{Cl}^0_\text{odd} (M, E)^\prime$. The plus sign is due to our right translation convention, see (13, Remark 9.12). Next, let us fix a smooth function $H : \text{Cl}^0_\text{odd} (M, E)^\prime \rightarrow \mathbb{C}$. The bracket (14) determines a derivation $X_H$ on functions $f : \text{Cl}^0_\text{odd} (M, E)^\prime \rightarrow \mathbb{C}$ via a prescription as in (3), this is, $X_H(\mu) \cdot f = \{ H, f \} (\mu)$ for all $\mu \in \text{Cl}^0_\text{odd} (M, E)^\prime$; we can then pose Hamilton’s equations

$$\frac{d}{dt} (f \circ \mu) = X_H(\mu) \cdot f$$

on $\text{Cl}^0_\text{odd} (M, E)^\prime$. For $\mu(t) = \langle P(t), \cdot \rangle \in \text{Cl}^0_\text{odd} (M, E)^\prime$ they become

$$\left\langle \frac{d\mu}{dt}, \frac{\delta f}{\delta \mu} \right\rangle = + \left\langle \mu, \left[ \frac{\delta H}{\delta \mu}, \frac{\delta f}{\delta \mu} \right] \right\rangle,$$
this is,
\begin{equation}
\left\langle \frac{dP}{dt}, Q \right\rangle = \left\langle P, \left[ \frac{\delta H}{\delta \mu}, Q \right] \right\rangle
\end{equation}
for \( Q = \frac{\delta f}{\delta \mu} \in \text{Cl}_0^{\text{odd}}(M, E) \).

This is a “weak version” of the Euler equation appearing in Berezin and Perelomov’s paper \[4\]. In our Hermitian context we do not have infinitesimal Ad-invariance, and so we obtain (16) instead of a standard equation such as
\[
\frac{dP}{dt} = \left[ P, \frac{\delta H}{\delta \mu} \right],
\]
see \[4\, \text{Equation (8)}\]. For example, if we take \( \mu = \langle P, \cdot \rangle \) and \( H_k(\mu) = \text{tr} \Delta (P^k) \), \( k = 1, 2, 3, \cdots \), we can easily check that (assuming existence of \( Q^{-1}_0 \))
\[
\frac{\delta H_k}{\delta \mu} = kQ^{-1}_0(P^k)^{k-1},
\]
and Equation (16) on \( \text{Cl}_0^{\text{odd}}(M, E) \) become
\begin{equation}
\left\langle \frac{dP}{dt}, Q \right\rangle = k \left\langle P, [Q^{-1}_0(P^k)^{k-1}, Q] \right\rangle.
\end{equation}

**Remark 17.** The presence of the operator \( Q^{-1}_0 \) requires us to be careful. An injective self-adjoint pseudodifferential operator \( Q_0 \) has:

1. an inverse \( Q^{-1}_0 \) which is itself an injective self adjoint pseudodifferential operator if and only if \( Q_0 \) is not smoothing
2. an inverse \( Q^{-1}_0 \notin \text{Cl}(M, E) \) if and only if \( Q_0 \) is smoothing.

The second case is the one which needs more attention. Indeed, if \( Q_0 \) is smoothing, e.g. \( Q_0 = e^{-\Delta} \), then its formal symbol vanishes. This explains why its inverse cannot be a pseudodifferential operator. However, \( Q_0 \) is a self-adjoint injective compact operator. Hence, via spectral analysis, it is easy to define \( Q^{-1}_0 \) which is an unbounded operator with \( L^2 \)-dense domain in \( C^\infty(M, E) \). Therefore, Equation (17) is always well-stated.

The foregoing equations are equations on the (regular) dual of the Lie algebra \( \text{Cl}_0^{\text{odd}}(M, E) \). We can work directly on the Lie algebra \( \text{Cl}_0^{\text{odd}}(M, E) \) and we can use more general pairings if we proceed as follows.

We assume that there exists an operator \( \mathbb{A} : \text{Cl}_0^{\text{odd}}(M, E) \to \text{Cl}_0^{\text{odd}}(M, E) \) such that the new pairing
\[
\langle X, Y \rangle_\mathbb{A} = \langle X, \mathbb{A}(Y) \rangle
\]
is Hermitian and non-degenerate. We think of \( \mathbb{A} \) as a twist of our previous pairing or, motivated by \[2\, \text{[18\, [20]], see also [17], as an “inertia operator”. We also consider the real part of \( \langle \cdot, \cdot \rangle_\mathbb{A} \),
\[
\mathbb{R} \langle X, Y \rangle_\mathbb{A} = \mathbb{R} \langle X, \mathbb{A}(Y) \rangle
\]
for \( X, Y \in \text{Cl}_0^{\text{odd}}(M, E) \). Since \( \langle \cdot, \cdot \rangle_\mathbb{A} \) is Hermitian and non-degenerate, this new pairing is a symmetric and non-degenerate real-valued bilinear form which allows us to consider (in view of Remark 15) pseudo-Riemannian geometry. In order to do so, we define a new adjoint map as
\begin{equation}
\mathbb{R} \langle [X, Y], Z \rangle_\mathbb{A} = -\mathbb{R} \langle Y, \text{ad}_\mathbb{A}(X)Z \rangle_\mathbb{A} = -\mathbb{R} \langle \text{ad}_\mathbb{A}(X)Z, Y \rangle_\mathbb{A},
\end{equation}
so that $ad_h(X)$ is the adjoint of $ad_X$ in accordance with our sign convention, see also [17] Section 2]. We compute $ad_h$ explicitly as follows:

$$\begin{align*}
-\Re(Y, ad_h(X) Z)_h &= -\Re \langle ad_X Y, Z \rangle_h = \Re \langle [X, Y], h(Z) \rangle \\
&= \Re \tr \langle [X, Y] Q_0 \ h(Z)^* \rangle \\
&= \Re \tr \Delta(Y Q_0 h(Z)^* - Y X Q_0 h(Z)^*) \\
&= \Re \tr \Delta(Y Q_0 h(Z)^* X - YX Q_0 h(Z)^*) \\
&= \Re \tr \Delta(Y Q_0 h(Z)^* X - Q_0 h(Z)^*)) \\
&= \Re \tr \Delta(Y Q_0^{-1} [Q_0 h(Z)^*, X]) .
\end{align*}$$

We set

$$\begin{equation}
(\ref{eq:19}) - h(R)^* = Q_0^{-1} [Q_0 h(Z)^*, X] .
\end{equation}$$

Then, $-\Re(Y, ad_h(X) Z)_h = -\Re \tr \Delta(Y Q_0 h(R)^*) = -\Re \langle Y, R \rangle_h$, and therefore $ad_h(X) Z = R$. We compute $R$ quite easily. Equation (\ref{eq:19}) implies

$$h(R) = [X, Q_0 h(Z)^*] Q_0^{-1} = [h(Z) Q_0, X^*] Q_0^{-1} ,$$

and so we conclude that

$$\begin{equation}
(\ref{eq:20}) ad_h(X) Z = h^{-1} ([h(Z) Q_0, X^*] Q_0^{-1}) = -h^{-1} ([ad_h(Z) Q_0 X^*] Q_0^{-1}) .
\end{equation}$$

4.2. Euler-Lagrange equations. Now we use $ad_h$ and the bilinear form $\Re \langle \cdot, \cdot \rangle_h$ to write down equations of motion on the Lie group $Cl_0(M, E)$. Our equations are Euler-Lagrange equations arising from a natural action functional. We follow, roughly, Taylor’s lecture notes [35].

We set $\langle \cdot | \cdot \rangle = \Re \langle \cdot, \cdot \rangle_h$ just to simplify our notation. First of all, we extend the symmetric and non-degenerate bilinear form $\langle \cdot | \cdot \rangle_h$ to a pseudo-Riemannian metric on $G = Cl_0(M, E)$ via right translation:

$$g(P)(V, W) = \langle T_{P} R_{P^{-1}} V | T_{P} R_{P^{-1}} W \rangle ,$$

in which $P \in G$, $W, V \in T_PG$, and $R_{P^{-1}}$ is right translation. We simplify this expression using the identification $V = (P + \epsilon Q_1)'(0)$ and $W = (P + \epsilon Q_2)'(0)$ for $Q_1, Q_2 \in \text{Lie}(G)$; we obtain

$$g(P)(V, W) = \langle Q_1 P^{-1} Q_2 P^{-1} \rangle .$$

Now we set up the kinetic energy Lagrangian functional on curves in $G$,

$$I[P(t)] = \int_a^b g(P(t), \dot{P}(t)) dt = \int_a^b \langle \dot{P}(t) P(t)^{-1} | \dot{P}(t) P(t)^{-1} \rangle dt ,$$

in which $\dot{P}(t)$ is now considered an an element of $\text{Lie}(G)$ for each $t$, and we find the corresponding equation for critical points of $I$. We assume that $t \mapsto P(t)$ is a critical, and we deform this curve slightly via $P(t) \mapsto P(t) + \epsilon \eta(t) Q$, with $\eta(a) = \eta(b) = 0$ and $Q \in \text{Lie}(G)$ in such a way that that this deformed curve lies in $G$ equipped with its Fréchet topology (recall that $Cl_0^0(M, E)$ is open in $Cl_0(M, E)$). Because $t \mapsto P(t)$ is critical, we have

$$\frac{d}{dt} \bigg|_{\epsilon=0} I[P(t) + \epsilon \eta(t) Q] = 0 .$$
Hereafter we omit $t$ dependence for clarity. We have:

$$\frac{d}{de}\bigg|_{e=0} I[P(t)+\epsilon\eta(t)Q] = \int_a^b \frac{d}{de}\bigg|_{e=0} \left( (\dot{P} + \epsilon \dot{Q})(P + \epsilon \eta Q)(P + \epsilon \eta Q)^{-1} (P + \epsilon \eta Q)(P + \epsilon \eta Q)^{-1} \right) dt = 0,$$

this is,

$$\int_a^b \langle \ddot{\eta} Q P^{-1} - \dot{P} P^{-1} \eta Q P^{-1} | \dot{P} P^{-1} \rangle dt = \int_a^b \dot{\eta} \langle Q P^{-1} | \dot{P} P^{-1} \rangle dt - \int_a^b \eta \langle \dot{P} P^{-1} Q P^{-1} | \dot{P} P^{-1} \rangle dt = 0.$$

We integrate by parts and use the boundary conditions for $\eta$; we obtain

$$- \int_a^b \langle Q P^{-1} | \dot{P} P^{-1} \rangle dt - \int_a^b \eta \langle \dot{P} P^{-1} Q P^{-1} | \dot{P} P^{-1} \rangle dt = 0,$$

this is,

$$\int_a^b \eta \left\{ \langle Q P^{-1} | \dot{P} P^{-1} \rangle - \langle Q P^{-1} | (\dot{P} P^{-1})^\dagger \rangle - \langle \dot{P} P^{-1} Q P^{-1} | \dot{P} P^{-1} \rangle \right\} dt - \int_a^b \eta \langle \dot{P} P^{-1} Q P^{-1} | \dot{P} P^{-1} \rangle dt = 0.$$

Since $\eta(t)$ is arbitrary, we find the equation of motion

$$(22) \quad \langle Q P^{-1} \dot{P} P^{-1} | \dot{P} P^{-1} \rangle - \langle Q P^{-1} | (\dot{P} P^{-1})^\dagger \rangle - \langle \dot{P} P^{-1} Q P^{-1} | \dot{P} P^{-1} \rangle = 0$$

in which $Q$ is an arbitrary element of $\text{Lie}(G)$.

Since $\dot{P} P^{-1}$ and $Q P^{-1}$ belong to $\text{Lie}(G)$, we can write $\dot{P} P^{-1} = X$ and $Q P^{-1} = W$ for $X, W \in \text{Lie}(G)$. Equation (22) becomes

$$\langle WX | X \rangle - \langle W | \dot{X} \rangle - \langle X W | X \rangle = 0$$

for all $W \in \text{Lie}(G)$, this is,

$$\langle [W, X] | X \rangle = \langle W | \dot{X} \rangle$$

for all $W \in \text{Lie}(G)$. As pointed out in [35], if we solve for $X$ in (23), the curve $P(t)$ is recovered via $\dot{P}(t) = X(t) P(t)$. Thus, Equation (23) —an equation posed on $\text{Lie}(G)$— determines a family of curves on $G$. It remains to find a “strong” formulation of (23). We go back to the notation used in Subsection 4.1. Equation (23) becomes

$$\Re \langle ad_X W, X \rangle = \Re \langle \dot{W}, \dot{X} \rangle,$$

and therefore, using the operator $ad_\mathbb{H}$ we obtain

$$\Re \langle W, ad_\mathbb{H}(X)X \rangle = \Re \langle \dot{W}, \dot{X} \rangle.$$

Non-degeneracy of the inner product $\Re \langle \cdot, \cdot \rangle_{\mathbb{H}}$ implies that $X(t) \in C_{\text{odd}}^0(M, E)$ satisfies the non-linear equation

$$\frac{d}{dt}X = ad_\mathbb{H}(X(t))X(t).$$

We note the formal similarity between (23) and the Hamiltonian equation (16). Due to this fact, we naturally call (23), or (24), the Euler equation on $C_{\text{odd}}^0(M, E)$.

We have proven the following theorem:
Theorem 18. The Euler equation
\begin{equation}
\frac{d}{dt} X = \mathcal{A}^{-1} \left( [\mathcal{A}(X) Q_0, X^*] Q_0^{-1} \right)
\end{equation}
on $Cl^0_{odd}(M, E)$, is the Euler-Lagrange equation of the kinetic energy action functional on the Fréchet Lie group $Cl^0_{odd}(M, E)$ equipped with the pseudo-Riemannian metric \((21)\).

Equation \((24)\) is formally analogous to the Euler equation posed on a Lie group $G$ equipped with a Riemannian metric. In this Riemannian case, our foregoing computations translate \textit{mutatis mutandis} into the well-known fact that Euler equations determine geodesics on $G$, see for instance \cite{18, 20, 35} and references therein.

Remark 19. If we take $Q_0 = \text{Id}$ and we pose Equation \((25)\) on the subgroup of self-adjoint operators, we obtain
\[ \mathcal{A} \left( \frac{d}{dt} X \right) = [\mathcal{A}(X), X] = -ad_{\mathcal{A}(X)} X, \]
an equation that looks exactly as the classical Euler equation in \textit{so}(3), see \cite{18, Theorem 7.2}.

4.3. Remarks on integrability. Motivated by Manakov’s observation on the integrability of the rigid body equation, see \cite{28}, we state:

Proposition 20. The Euler equation
\begin{equation}
\frac{d}{dt} X = \mathcal{A}^{-1} \left( [\mathcal{A}(X) Q_0, X^*] Q_0^{-1} \right)
\end{equation}
on $Cl^0_{odd}(M, E)$ is equivalent to the Lax pair equation
\begin{equation}
\frac{d}{dt} (\mathcal{A}(X) Q_0 + \xi J^2) = [\mathcal{A}(X) Q_0 + \xi J^2, X^* + \xi J]
\end{equation}
in which $\xi$ is a complex parameter and $J$ is an operator satisfying $\mathcal{A}(X) Q_0 J = J \mathcal{A}(X) Q_0$ and $X^* J^2 = J^2 X^*$.

Proof. Equation \((26)\) can be written as
\[ \frac{d}{dt} (\mathcal{A}(X) Q_0) = [\mathcal{A}(X) Q_0, X^*], \]
and this equation is equivalent to Equation \((27)\) for arbitrary values of $\xi$. \hfill \square

We interpret this proposition as saying that our Euler equation \((26)\) posed on $Cl^0_{odd}(M, E)$ admits a Lax pair formulation and it is therefore integrable. We can also prove that
\[ I_k = tr^\Delta \left( (\mathcal{A}(X) Q_0 + \xi J^2)^k \right) \]
is conserved along solutions to \((27)\) for arbitrary values of $\xi$ and $k \geq 1$. Indeed, it is easy to check that
\[ \frac{d}{dt} \left( (\mathcal{A}(X) Q_0 + \xi J^2)^k \right) = tr^\Delta \left( [(\mathcal{A}(X) Q_0 + \xi J^2)^k, X^* + \xi J] \right) = 0 \]
on solutions to \((27)\), and therefore expansion of $I_k$ in powers of $\xi$ yields integrals of motion for \((26)\). Since the conditions on $J$ appearing in Proposition \((20)\) imply that the operators $\mathcal{A}(X) Q_0$ and $\xi J^2$ commute, we can easily obtain explicit expressions for these integrals by expanding $I_k$. We present an example in Section 6.
5. Pseudo-Riemannian Geometry on $Cl_{0,\ast}^0(M, E)$

In this section we review some basics facts of the pseudo-Riemannian geometry of the regular Fréchet group $Cl_{0,\ast}^0(M, E)$, motivated by Arnold’s classical paper [2]. We fix an inertia operator $A$ and we consider the pseudo-Riemannian metric on $Cl_{0,\ast}^0(M, E)$ induced by right translation of the non-degenerate and symmetric bilinear form $\Re \langle \cdot, \cdot \rangle_A$, see Equation (21).

We note that there exist some difficulties in describing the whole space of connection 1-forms

$$\Omega^1(Cl_{0,\ast}^0(M, V), Cl_{0,\ast}^0(M, V)).$$

Indeed, to our knowledge, the space of smooth linear maps acting on $Cl_{0,\ast}^0(M, V)$ is actually not well-understood. In the classical setting of a Riemannian (e.g. finite dimensional, or Hilbert) Lie group $G$ with Lie algebra $g$, the Levi-Civita connection 1-form (i.e. metric-compatible and torsion-free) reads as

$$\theta_X Y = \frac{1}{2} \{ ad_X Y - ad^*_X Y - ad^*_Y X \},$$

in which $ad^*$ is the adjoint of $ad$ with respect to the metric of $G$ and $X, Y$ are left invariant vector fields, see [14, Proposition 1.7]. It is possible to go beyond this well-known result, and extend it to (pseudo-)Riemannian right-invariant metrics, if an adjoint for $ad$ is known. Formal calculations have been already carried out, see for example the classical Arnold’s paper [2] or [17, Section 2] and references therein, but in the context of pseudodifferential operators, finding a rigorous (i.e. truly smooth) adjoint of the adjoint map, as described in [27], remains a difficult task. We can bypass this difficulty here, since we already have $ad_A$ at our disposal.

Theorem 21. Let $(X, Y) \in Cl_{0,\ast}^0(M, E)^2$. We define, using right invariance, the connection 1-form

$$\theta_X Y = \frac{1}{2} \{ ad_X Y - ad_A(X)Y - ad_A(Y)X \}.$$ 

Then we have that:

(a) $\forall (X, Y) \in Cl_{0,\ast}^0(M, E)^2$, $\theta_X Y - \theta_Y X = ad_X Y$ (Torsion-free)

(b) $\forall (X, Y, Z) \in Cl_{0,\ast}^0(M, E)^3$, $\Re \langle \theta_X Y, Z \rangle_A = -\Re \langle Y, \theta_X Z \rangle_A$ (Pseudo-Riemannian metric compatibility)

Moreover, $\theta : (X, Y) \mapsto \theta_X Y$ is the only bilinear map which satisfies these two properties.

Proof. As in the previous section, in this proof we set $\langle \cdot, \cdot \rangle = \Re \langle \cdot, \cdot \rangle_A$ for ease of notation.

We first check that $\theta$ satisfies (a) and (b). By direct computation, we have:

$$\theta_X Y - \theta_Y X = \frac{1}{2} \{ ad_X Y - ad_A(X)Y - ad_A(Y)X \}$$

$$- \frac{1}{2} \{ ad_Y X - ad_A(Y)X - ad_A(X)Y \}$$

$$= ad_X Y,$$
which proves (a). We now compute using Equation (18):

\[
2 \langle \theta_X Y | Z \rangle = \langle ad_X Y - ad_\theta(X) Y - ad_\theta(Y) X | Z \rangle \\
= \langle ad_X Y | Z \rangle - \langle ad_\theta(X) Y | Z \rangle - \langle ad_\theta(Y) X | Z \rangle \\
= \langle ad_\theta(X) Z | Y \rangle - \langle ad_X Z | Y \rangle - \langle ad_Y Z | X \rangle \\
= \langle ad_\theta(X) Z | Y \rangle - \langle ad_X Z | Y \rangle + \langle ad_Z Y | X \rangle \\
= \langle ad_\theta(X) Z | Y \rangle - \langle ad_X Z | Y \rangle + \langle ad_\theta(Z) X | Y \rangle \\
= -2 \langle Y | \theta_X Z \rangle ,
\]

which proves (b).

Now, let

\[ \Theta : (X, Y) \in Cl_{\text{odd}}^0(M, E)^2 \mapsto \Theta_X Y \in Cl_{\text{odd}}^0(M, E) \]

be a bilinear form satisfying (a) and (b). Then

\[
\langle \Theta_X Y | Z \rangle + \langle Y | \Theta_X Z \rangle = 0 , \\
\langle \Theta_Z X | Y \rangle + \langle X | \Theta_Z Y \rangle = 0 , \\
\langle \Theta_Y Z | X \rangle + \langle Z | \Theta_Y X \rangle = 0 .
\]

From the third line and (a) we get that

\[
\langle Z | \Theta_X Y \rangle = - \langle \Theta_Y Z | X \rangle - \langle Z | [X, Y] \rangle
\]

and from the first line we get that

\[
\langle \Theta_X Y | Z \rangle = - \langle Y | \Theta_X Z \rangle .
\]

Combining these two equalities, and exploiting properties (a) and (b), we have:

\[
2 \langle \Theta_X Y | Z \rangle = - \langle \Theta_Y Z | X \rangle - \langle Z | [X, Y] \rangle - \langle Y | \Theta_X Z \rangle \\
= - \langle -[Y, Z] + \Theta_Z Y | X \rangle - \langle Z | [X, Y] \rangle - \langle Y | -[X, Z] + \Theta_Z X \rangle \\
= \langle [Y, Z] | X \rangle - \langle Z | [X, Y] \rangle + \langle Y | [X, Z] \rangle \\
= 2 \langle \theta_X Y | Z \rangle .
\]

Since \( \langle \cdot | \cdot \rangle \) is non-degenerate, this equality ends the proof. \( \square \)

It is important for us to highlight the fact that the proof of Theorem 21 goes through because we can use the smooth adjoint \( ad_\theta \). Now, using \( \theta_X Y \) we can define the curvature operator and sectional curvature of the Lie group \( Cl_{\text{odd}}^0(M, E) \) as follows:

The curvature operator for the connection \( \theta \) is given, at the identity of \( Cl_{\text{odd}}^{0,\ast}(M, E) \), by

\[
R_H(X, Y) = [ \theta_X , \theta_Y ] - \theta_{[X,Y]} 
\]

for every \( X \) and \( Y \) in \( Cl_{\text{odd}}^0(M, E) \), see also [14] Equation (1.10)]. Hence, the sectional curvature associated to the biplane generated by \( X \) and \( Y \) is

\[
K_H(X, Y) = - \frac{\langle R_H(X, Y) X | Y \rangle}{|X \wedge Y|_H^2}
\]

whenever the area of the parallelogram spanned by \( X, Y, |X \wedge Y|_H \), is different from zero.

These constructions yield Theorem 5 of Arnold’s [2]. Using our foregoing notation this theorem reads as follows, see [17] Proposition 2.1:
Theorem 22. Let $\mathcal{A}$ be an inertia operator and set $\mathcal{N}(X,Y) = \frac{1}{2} (ad_\mathcal{A}(X)Y + \text{ad}_\mathcal{A}(Y)X)$. Given $X$ and $Y$ in $C^{0,\ast}_{\text{odd}}(M,E)$ we have the identity

$$\|X \wedge Y\|_A^2 K_\mathcal{A}(X,Y) = \frac{-3}{4} \left( [X,Y] \cdot [X,Y] \right) + \frac{1}{2} \left( [X,Y] \cdot \text{ad}_\mathcal{A}(X)Y - \text{ad}_\mathcal{A}(Y)X \right)$$

$$+ (\mathcal{N}(X,Y) \cdot \mathcal{N}(X,Y)) - (\mathcal{N}(X) \cdot \mathcal{N}(Y)).$$

We remark once again that this theorem is a rigorous statement on the sectional curvature of the Fréchet Lie group $C^{0,\ast}_{\text{odd}}(M,E)$, not a formal result as [17 Proposition 2.1]. We finish this section computing geodesics:

Let us set $G = C^{0,\ast}_{\text{odd}}(M,E)$ and $\text{Lie}(G) = C^{0,\ast}_{\text{odd}}(M,E)$. We recall that a spray over $G$ is a vector field $S : TG \to TTG$ satisfying $T\pi_G \circ S = \text{Id}_{TG}$, in which $\pi_G : TG \to G$ is the canonical projection, and the homogeneity condition

$$(T\mu_t) \cdot S(v) = \frac{1}{t} S(tv)$$

for $t \neq 0$, in which $\mu_t : TG \to TG$ is the smooth function $\mu_t(v) = tv$. In the present case we use $TG = G \times \text{Lie}(G)$ and $TTG = G \times \text{Lie}(G) \times \text{Lie}(G) \times \text{Lie}(G)$. Then

$$S : TG \to TTG, \quad S(g,X) = (g,X,X,\text{ad}_\mathcal{A}(X)X) = (g,X,X,\theta_X X)$$

for $g \in G$ and $X \in \text{Lie}(G)$, is a spray on $G$, as it can be easily checked (see [29 Section 1.21]). In actual fact, it can be proven that the spray $S$ is precisely the metric spray corresponding to our right-invariant metric [21], see [8 Section 6.2]. The integral curves of $S$ are the geodesics corresponding to the spray $S$. We obtain that $(g(t),X(t))$ is a geodesic for the spray $S$ if and only if

$$\frac{dg}{dt} = X$$

$$\frac{dX}{dt} = \text{ad}_\mathcal{A}(X)X.$$

The second equation is exactly the Euler-Lagrange equation (24).

6. Examples on the n-dimensional torus

In this section, we specialize to $M = T_n = (S^1)^n$, $n$ odd, equipped with its product metric, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We recall that the Laplace operator is

$$\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

6.1. When $Q_0$ is a heat operator. We set $\mathcal{A} = \text{Id}$. Let $s \in \mathbb{R}^\ast_+$. We define $Q_0 = e^{-s\Delta}$. This is an injective smoothing operator, hence it is both odd and even class. We apply our previous computations to $T_n$ for any $n \in \mathbb{N}^\ast$ and we obtain

$$\langle A, B \rangle = tr \left( A e^{-s\Delta} B \right) = tr \left( A e^{-s\Delta} B \right).$$

Now we need to define formally $Q_0^{-1} = e^{s\Delta}$ which is not a pseudodifferential operator but it can be rigorously defined, as we discussed in Remark [14].

We obtain the formulas:

$$\text{ad}_\mathcal{A}(X)Z = [Z e^{-s\Delta}, X^*] e^{s\Delta}$$
and
\[ \theta_X Y = \frac{1}{2} \left\{ [X, Y] + [Ye^{-s\Delta}, X^*]e^{s\Delta} + [Xe^{-s\Delta}, Y^*]e^{s\Delta} \right\}, \]
and the geodesic equation (24) reads
\[ \frac{dX}{dt} = [Xe^{-s\Delta}, X^*]e^{s\Delta}. \]
Let us now test these three equations taking \( X \in C^\infty(T_n, \mathbb{C}) \) and expanding it with respect to the Fourier basis. First, for \( n = 1 \), i.e. for \( T_n = S^1 \). Let \((l, m, p) \in \mathbb{Z}^3.\) We obtain
\[
(ad_A(z^l)m)z^p = z^m e^{-s\Delta} z^{-l}e^{s\Delta}z^{p} - z^{-l+m+p}
= \left(e^{s(p^2-(p-l)^2)} - 1 \right) z^{-l+m+p}
= \left(e^{s(2pl-l^2)} - 1 \right) z^{-l+m+p},
\]
and
\[
(\theta_X z^m)z^p = \frac{-1}{2} \left\{ \left(-e^{s(2pl-l^2)} + 1 \right) z^{-l+m+p} + \left(-e^{s(2pm-m^2)} + 1 \right) z^{l-m+p} \right\}.
\]
The same kind of relations can be implemented for \( n > 1 \), by considering tensor products.

6.2. When \( Q_0 \) is a power of the Laplacian. We set \( A = Id. \) We now investigate \( Q_0 = (\Delta + \pi)^{(n+1)/2} \), where \( \pi \) is the \( L^2 \)-orthogonal projection on the kernel of the Laplacian. We remark that \( Q_0 \) is injective and self-adjoint of order \( n+1 \). Moreover, if \( n \) is odd, then \( Q_0 \) is odd class, and if \( n \) is even, then \( Q_0 \) is even class. In this class of examples, we get an operator \( Q_0^{-1} \) which is a pseudo-differential operator of order \(-n-1\), in the same class as \( Q_0. \) Hence the following formulas are fully valid in \( Cl^{-1}_{odd}(T_n, \mathbb{C}) :\)
\[
ad_A(X)Z = [ZQ_0, X^*]Q_0^{-1} \in Cl^{-1}_{odd}(T_n, \mathbb{C})
\]
\[
\theta_X Y = \frac{1}{2} \left\{ [X, Y] - [YQ_0, X^*]Q_0^{-1} - [XQ_0, Y^*]Q_0^{-1} \right\}.
\]
and the geodesic equation reads
\[
\frac{dX}{dt} = [XQ_0, X^*]Q_0^{-1} = XQ_0X^*Q_0^{-1} - XX^*,
\]
where the right-hand side is an operator of order \(-1\) (and hence compact). Let \( M = S^1 \) and let us restrict ourselves to \( X \in C^\infty(S^1, \mathbb{C}). \) Then, for \( p \in \mathbb{Z} \) we obtain
\[
\frac{dX}{dt}(z^p) = X(\Delta + \pi)X^*(\Delta + \pi)^{-1}(z^p) - XX^*z^p.
\]
Interestingly, in this case we can say more about the integrals of motion \( I_k \) considered in Subsection 4.3. We obtain
\[
I_k = tr^{\Delta}([X(\Delta + \pi) + \xi J^k]k) = \sum_{j=0}^{k} \binom{h}{j} tr^{\Delta}([X(\Delta + \pi)]^{k-j} J^2j) \xi^{j}.\]
Some integrals are trivial (for example, if \( j = k \), we obtain the integral of motion \( tr^{\Delta}(J^{2k}) \) which does not depend on the variable \( X \)) but non-vanishing integrals constructed with different \( k \)'s cannot be all dependent, since the symbols of the pseudodifferential operators \([X(\Delta + \pi)]^{k-j} J^{2j}\) are all of different order. Thus, it
follows that non-vanishing integrals $\text{tr}^\Delta (|X(\Delta + \pi)|^{k-j}J^{2j})$ are also independent functions. We can easily prove that indeed there exists a countable family of such non-vanishing functions, at least for a large family of initial conditions:

We take $J = \text{Id}$ and we evaluate $\text{tr}^\Delta (|X(\Delta + \pi)|^{k-j}J^{2j})$ on the initial condition $X = \text{Id} + M_{a(x)}e^{-\Delta}M_{\pi(x)}$ in which $M_{a(x)}$ is the multiplication operator by the complex valued function $a \in C^\infty(S^1, \mathbb{C})$. For the sake of simplicity, we take $a(x) = e^{inx}$ for $n \in \mathbb{Z}$. We then have

$$\text{tr}^\Delta (|X(\Delta + \pi)|^{k-j}J^{2j}) = \text{tr}^\Delta ((\text{Id} + M_{a(x)}e^{-\Delta}M_{\pi(x)})^{k-j}(\Delta + \pi)^{k-j})$$

The second term is strictly positive, while the first term can be computed as the limit of

$$1 + \sum_{k \in \mathbb{Z}^*} (k^2)^{k-j-s}$$

which is equal to $2\zeta(-2k+2j+1) = \frac{2B_{2k}(k-j)}{2k-2j+1} + 1$ following computations of [7, 27] —in which the authors expand the renormalized trace $\text{tr}^\Delta$ in the Fourier basis $(x \mapsto e^{inx})_{n \in \mathbb{Z}}$ — and the well-known formulas for the $\zeta$–function in terms of the Bernoulli numbers [9].

The rigid body equation (29) is therefore an example of a non-commutative nonlinear differential equation admitting a Lax pair formulation and an infinite number of independent (at least for a large number of initial conditions) integrals of motion. It is an integrable equation posed on the Lie algebra of the regular Lie group $Cl^{0,*}(S^1, E)$.

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References

[1] Adler, M.; On a trace functional for formal pseudodifferential operators and the symplectic structure of Korteweg-de Vries type equations Inventiones Math. 50 219-248 (1979)
[2] Arnold, V.; Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits. Ann. Institut Fourier Vol 16, 1, 319–361 (1966).
[3] M. Bauer, M. Bruveris, E. Cismas, J. Escher and B. Kolev, Well-posedness of the EPDiff equation with a pseudo-differential inertia operator. J. Differential Equations 269 (2020), 288–325.
[4] Berezin, F.A. and A.M. Perelomov, Group–Theoretical Interpretation of the Korteweg-de Vries Type Equations. Commun. Math. Phys. 74 (1980), 129–140.
[5] Berline, N.; Getzler, E.; Vergne, M.; Heat Kernels and Dirac Operators Springer (2004)
[6] Bokobza-Haggiag, J.; Opérateurs pseudo-différentiels sur une variété différentiable; Ann. Inst. Fourier, Grenoble 19,1 125-177 (1969).
[7] Cardona, A.; Ducourtioux, C.; Magnot, J-P.; Paycha, S.; Weighted traces on pseudodifferential operators and geometry on loop groups; Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5 no4 503-541 (2002)
[8] Cismas, E.-C.; A spray theory for the geometric method in hydrodynamics. Preprint arXiv:1909.13383v1 (2019).
[9] Colmez, P.; Arithmétique de la fonction Zêta, in: Actes des journées X-UPS 2002 http://www.math.polytechnique.fr/xups/xups02-02.pdf
[10] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach. Annales de l’Institut Fourier, Tome 50 (2000), 321–362.
[11] Ducourtioux, C.; Weighted traces on pseudodifferential operators and associated determinants. Ph.D thesis, Université Blaise Pascal, Clermont-Ferrand, France (2000)
[12] D.G. Ebin and J. Marsden, Groups of Diffeomorphisms and the Motion of an Incompressible Fluid. Annals of Mathematics 92 (1970), 102–163.
[13] Eslami Rad, A.; Reyes, E. G.; The Kadomtsev-Petviashvili hierarchy and the Mulase factorization of formal Lie groups J. Geom. Mech. 5, no 3 (2013) 345–363.
[14] Freed, D.; The Geometry of loop groups. J. Diff. Geom. 28 (1988) 223–276
[15] Gilkey, P; Heat equation and the Atiyah-Singer index theorem. Second Edition CRC (1995).
[16] Glöckner, H; Algebras whose groups of the units are Lie groups Studia Math. 153 (2002), 147–177.
[17] Gorka, P.; Pons, D.J.; Reyes, E.G.; Equations of Camassa–Holm type and the geometry of loop groups. Journal of Geometry and Physics 87 (2015), 190–197.
[18] Holm, D.D.; Schmah, T.; Stoica, C.; Geometric Mechanics and Symmetry. Oxford University Press, England, (2009).
[19] Kassel, Ch.; Le résidu non commutatif (d’après M. Wodzicki) Séminaire Bourbaki, Vol. 1988/89. Astérisque 177-178, Exp. No. 708, 199-229 (1989)
[20] Khesin, B.; Wendt, R.; The Geometry of Infinite-Dimensional groups. Springer-Verlag, 2009.
[21] Kontsevich, M.; Vishik, S.; Determinants of elliptic pseudodifferential operators Max Plank Institut fur Mathematik, Bonn, Germany, preprint n. 94-30 (1994)
[22] Kontsevich, M.; Vishik, S.; Geometry of determinants of elliptic operators. Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), Progr. Math. 131,173-197, (1995)
[23] Magnot, J-P.; Chern forms on mapping spaces, Acta Appl. Math. 91, no. 1, 67-95 (2006).
[24] Magnot, J-P.; On Diff(M)–pseudodifferential operators and the geometry of non linear grassmannians. Mathematics 4, 1; doi:10.3390/math4010001 (2016)
[25] Magnot, J-P. and Reyes, E.G.: Well-posedness of the Kadomtsev-Petviashvili hierarchy, Mulase factorization, and Frölicher Lie groups. Ann. H. Poincaré 21 no 6 (2020). https://doi.org/10.1007/s00023-020-00896-3.
[26] Magnot, J-P. and Reyes, E. G.; On the Cauchy problem for a Kadomtsev-Petviashvili hierarchy on non-formal operators and its relation with a group of diffeomorphisms. Submitted, 2021. Preprint available at https://hal.archives-ouvertes.fr/hal-01857150v2
[27] Magnot, J-P.; On the geometry of Diff(S^1)–pseudodifferential operators based on renormalized traces. Proceedings of the International Geometry Center 14 (1) (2021), 19-47 . https://doi.org/10.15673/tmgc.v14i1.1784
[28] Manakov; Note on the integration of Euler’s equations of the dynamics of an n-dimensional body. Functional Analysis and Its Applications 10 (1976), 328–329.
[29] Michor, P.W.; Manifolds of Differentiable Mappings. Shiva Publishing Limited, UK. (1980).
[30] Neeb, K-H.; Towards a Lie theory of locally convex groups Japanese J. Math. 1 (2006), 291–468.
[31] P.J. Olver and V.V. Sokolov, Integrable Evolution Equations on Associative Algebras. Comm. Math. Phys. 193 (1998), 245–268.
[32] Paycha, S; Regularised integrals, sums and traces. An analytic point of view. University Lecture Series 59, AMS (2012).
[33] Paycha, S.; Paths towards an extension of Chern-Weil calculus to a class of infinite dimensional vector bundles. Geometric and topological methods for quantum field theory, 81–143, Cambridge Univ. Press, Cambridge, (2013).
[34] Scott, S.; Traces and determinants of pseudodifferential operators; OUP (2010).
[35] Taylor, M.; Finite and Infinite Dimensional Lie Groups and Evolution Equations. Symmetries, Conservation Laws, and Integrable Systems. https://www.ams.org/open-math-notes/omn-view-listing?listingId=110678
[36] Wells, Jr., R.O.; Differential Analysis on Complex Manifolds. Third Edition. Springer, 2008.
[37] Widom, H.; A complete symbolic calculus for pseudodifferential operators; Bull. Sc. Math. 2e serie 104 (1980) 19-63
[38] Wodzicki, M.; Local invariants in spectral asymmetry Inv. Math. 75, 143-178 (1984)
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