ON q-ANALOGUES OF BOUNDED SYMMETRIC DOMAINS AND DOLBEAULT COMPLEXES

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1 Introduction

Consider an irreducible Hermitian symmetric space $X$ of non-compact type. Let $\mathfrak{g}$ and $\mathfrak{g}_0$ denote the complexifications of the Lie algebras of the automorphism group of $X$ and the stabilizer of a point $x \in X$ respectively. Then the center of $\mathfrak{g}_0$ is 1-dimensional ($Z(\mathfrak{g}_0) = \mathbb{C} \cdot H, \ H \in \mathfrak{g}_0$), and $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, where $\mathfrak{g}_\pm = \{ \xi \in \mathfrak{g} | [H, \xi] = \pm 2\xi \}$ (see, e.g., [8]).

It was shown by Harish-Chandra that there exists a natural embedding $i : X \hookrightarrow \mathfrak{g}_-$ with $iX$ being a bounded symmetric domain in $\mathfrak{g}_-$ [8].

Our purpose is to construct quantum analogues of the (prehomogeneous) vector space $\mathfrak{g}_-$, the bounded symmetric domain $iX \subset \mathfrak{g}_-$ and the differential calculus in $\mathfrak{g}_-$.

Normally we don’t dwell on describing the quantum algebras of functions and quantum exterior algebras in terms of generators and relations, although that could be done. (The case $\mathfrak{g} = \mathfrak{sl}_{m+n}$, $\mathfrak{g}_0 = \mathfrak{s}(\mathfrak{gl}_m \times \mathfrak{gl}_n)$ was partially considered in [21]).

The simplest homogeneous bounded domain is the unit disc: $U = \{ z \in \mathbb{C} | |z| < 1 \}$.

It was shown in [12, 14] that the Poisson brackets $\{ .., \}$ that agree with the action of the Poisson-Lie group $SU(1, 1)$ on $U$ are given by

$$\{ z, \overline{z} \} = i(1 - |z|^2)(a + b|z|^2), \quad a, b \in \mathbb{R}.$$  

Our construction (see Section 9) provides a quantization of this bracket with $b = 0$. This ”simplest” quantum disc was studied in [15, 23].

Most of the constructions of this paper originate from the works of V. G. Drinfeld [8], S. Z. Levendorski & Ya. S. Soibelman [16]. Specifically, we follow [8] in replacing the construction of algebras by forming the dual coalgebras; also our choice of a Poisson cobracket, together with the associated quantization procedure, is due to [16].

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2 Prehomogeneous vector spaces of a commutative parabolic type

Everywhere in the sequel C will be the ground field.

Let \( g \) be a simple complex Lie algebra, \( h \) its Cartan subalgebra and \( \alpha_i \in h^* \), \( i = 1, \ldots, l \) a simple root system of \( g \).

Choose an element \( \alpha_0 \in \{ \alpha_i \}_{i=1}^l \) and consider the associated \( \mathbb{Z} \)-grading

\[
\mathfrak{g} = \bigoplus_j \mathfrak{g}_j, \quad \mathfrak{g}_j = \{ \xi \in \mathfrak{g} \mid [H_0, \xi] = 2j\xi \},
\]

where \( H_0 \in h \), \( \alpha_0(H_0) = 2 \), \( \alpha_i(H_0) = 0 \) for \( \alpha_i \neq \alpha_0 \).

A subspace \( \mathfrak{g}_{-1} \) is called a prehomogeneous vector space of a commutative parabolic type if the above \( \mathbb{Z} \)-grading breaks off:

\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.
\]

The motives that justify this definition and the list of simple roots \( \alpha \in \{ \alpha_i \}_{i=1}^l \) with (2.1) being valid are given in [2, 19].

It is worthwhile to note that all the simple roots of series \( A_n \) Lie algebras possess the above property, and for the Lie algebra series \( B_n, C_n, D_n \), together with the exceptional Lie algebras \( E_6, E_7 \) the set of such roots is non-void.

Set \( p_+ := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), \( p_- := \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \). Our purpose is to construct a quantum analogue of the graded polynomial algebra \( \mathbb{C}[^{\mathfrak{g}_{-1}}] \) on the prehomogeneous vector space \( \mathfrak{g}_{-1} \). For this, it would be useful to have a definition of \( \mathbb{C}[^{\mathfrak{g}_{-1}}] \) in terms of the enveloping algebras \( U_{\mathfrak{g}} \supset U_{\mathfrak{p}_+} \supset U_{\mathfrak{g}_0} \) (but not the Lie algebras themselves).

We start with constructing the coalgebra \( V_- \) dual to \( \mathbb{C}[^{\mathfrak{g}_{-1}}] \). Consider the \( U_{\mathfrak{g}} \)-module \( V_- \) determined by its generator \( v \in V_- \) and the relations

\[
\xi v_- = \varepsilon(\xi)v_-, \quad \xi \in U_{\mathfrak{p}_+},
\]

where \( \varepsilon : U_{\mathfrak{p}_+} \to \mathbb{C} \simeq \text{End}(\mathbb{C}) \) is the trivial representation of \( U_{\mathfrak{p}_+} \). Equip \( V_- \) with a structure of a coalgebra \( \mathbb{F} \) by extending the map \( \Delta_- : v_- \mapsto v_- \otimes v_- \) to a morphism of \( U_{\mathfrak{g}} \)-modules. The existence and uniqueness of this extension are obvious, and the coassociativity of \( \Delta_- \) follows from

\[
(\Delta_- \otimes \text{id})\Delta_- v_- = (v_- \otimes v_-) \otimes v_-; \quad (\text{id} \otimes \Delta_-)\Delta_- v_- = v_- \otimes (v_- \otimes v_-).
\]

It is easy to verify that \( V_- = \bigoplus_j (V_-)_j \) with \( (V_-)_j = \{ v \in V_- \mid H_0v = 2jv \} \), and that the dual algebra \( \bigoplus_j ((V_-)_j)^* \) to the coalgebra \( V_- \) is canonically isomorphic to \( \mathbb{C}[^{\mathfrak{g}_{-1}}] \).

A replacement of '−' by '+' in the above construction leads to the algebra of antiholomorphic polynomials on \( \mathfrak{g}_{-1} \), which will be denoted by \( \mathbb{C}[^{\mathfrak{g}_{-1}}] \). We shall see in the sequel that these constructions can be transferred to the quantum case where they lead to the ”covariant” algebras \( \mathbb{C}[^{\mathfrak{g}_{-1}}]_q, \mathbb{C}[^{\mathfrak{g}_{-1}}]_q \).
3 Quantum universal enveloping algebras and their "real forms"

It is well known \cite{20} that a simple complex Lie algebra \( \mathfrak{g} \) admits a description in terms of generators \( \{X_i^+, H_i\}_{i=1} \) and relations

\[
[H_i, H_j] = 0; \quad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm; \quad [X_i^+, X_j^-] = \delta_{ij} H_i; \quad \text{ad}_{X_i^\pm}^{1-a_{ij}}(X_j^\pm) = 0. \tag{3.1}
\]

In the above \( i, j \in \{1, \ldots, l\} \), and \((a_{ij})\) is the Cartan matrix of the simple Lie algebra \( \mathfrak{g} \), i.e. \( a_{ij} = \alpha_i(H_j) \).

Let \( j_0 \) be the number of the simple root \( \alpha_0 \). The relations (2.2) can be rewritten in the form

\[
X_j^- v_+ = H_j v_+ = 0, \quad j = 1, 2, \ldots, l; \\
X_j^+ v_- = 0, \quad j \neq j_0.
\]

Consider the real Lie subalgebra \( \mathfrak{g}(\alpha_0) \subset \mathfrak{g} \) generated by the elements

\[
X_j^+ - X_j^-, \quad i(X_j^+ + X_j^-), \quad i H_j; \quad j \neq j_0
\]

\[
X^+_{j_0} - X^-_{j_0}, \quad i(X^+_{j_0} + X^-_{j_0}), \quad i H_{j_0},
\]

where \( i = \sqrt{-1} \). This subalgebra is interesting because it is the Lie algebra for the automorphism group of the corresponding bounded symmetric domain in \( \mathfrak{g}_{-1} \subset \mathfrak{g} \). We are seeking for the specific ways to distinguish \( U\mathfrak{g}(\alpha_0) \) inside \( U\mathfrak{g} \).

Recall that \( U\mathfrak{g} \) is a Hopf algebra \footnote{\textsuperscript{2}} whose comultiplication \( \Delta \), counit \( \varepsilon \) and antipode \( S \) are given by

\[
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^\pm) = X_i^\pm \otimes 1 + 1 \otimes X_i^\pm; \\
\varepsilon(H_i) = \varepsilon(X_i^\pm) = 0; \quad S(H_i) = -H_i, \quad S(X_i^\pm) = -X_i^\pm;
\]

\( i = 1, 2, \ldots, l \).

It is easy to verify that

\[
U\mathfrak{g}(\alpha_0) = \{\xi \in U\mathfrak{g} | \xi^* = S(\xi)\},
\]

with \( * \) being the antilinear involution which depends on \( \alpha \) and is determined by its values on generators \( X_j^\pm \), \( H_j \) as follows: \footnote{\textsuperscript{3}}

\[
H^*_{j_0} = H_{j_0}, \quad (X^+_{j_0})^* = -X^-_{j_0}; \\
H^*_j = H_j, \quad (X^+_j)^* = X^-_{j_0}, \quad j \neq j_0.
\]  \tag{3.2}

The Hopf algebra \( U\mathfrak{g}(\alpha_0) \) doesn’t survive under quantization; in the sequel it will be replaced by the pair \( (U\mathfrak{g}, \ast) \). Now let us consider the quantization of this Hopf \( \ast \)-algebra.

We start with V. G. Drinfeld – M. Jimbo formul\ae\ \footnote{\textsuperscript{4}} which determine a Hopf algebra \( U_h\mathfrak{g} \) over \( \mathbb{C}[[h]] \) complete in \( h \)-adic topology (\( \mathbb{C}[[h]] \) denotes the ring of formal series). First of all, choose an invariant scalar product in \( \mathfrak{g} \) in such a way that \( d_i = (\alpha_i, \alpha_i)/2 > 0 \). Now \( \{X_j^\pm, H_j\}_{j=1,\ldots,l} \) work as generators of the topological algebra \( U_h\mathfrak{g} \), and the list of relations is as follows:

\[
[H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \quad [X_i^+, X_j^-] = \delta_{ij} \frac{\text{sh}(d_j h H_j/2)}{\text{sh}(d_j h/2)}.
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1-a_{ij}}{k} \right] (X_i^\pm)^k X_j^\pm (X_i^\pm)^{(1-a_{ij}-k)} = 0.
\]

Here we use the notation
\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_h = \prod_{k=1}^n \frac{\text{sh}(kh/2)}{\text{sh}(h/2)} \left/ \left( \prod_{k=1}^m \frac{\text{sh}(kh/2)}{\text{sh}(h/2)} \cdot \prod_{k=1}^{n-m} \frac{\text{sh}(kh/2)}{\text{sh}(h/2)} \right) \right.,
\]

\[i,j = 1, \ldots, l.\]

Comultiplication \( \Delta \), counit \( \varepsilon \) and antipode \( S \) are determined by their values on the generators:
\[
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i; \quad \Delta(X_i^\pm) = X_i^\pm \otimes e^{hH_i^2/4} + e^{-hH_i^2/4} \otimes X_i^\pm;
\]
\[
\varepsilon(H_i) = \varepsilon(X_i^\pm) = 0; \quad S(H_i) = -H_i; \quad S(X_i^\pm) = -e^{\pm hH_i^2/2} \cdot X_i^\pm.
\]

An involution in \( \mathbb{C}[[h]] \) is introduced by setting \( h^* = h \). We equip \( U_h \mathfrak{g} \) with the structure of \( \ast \)-algebra over \( \mathbb{C}[[h]] \) defined by (3.2). The pair \( (U_h \mathfrak{g}, \ast) \) will be denoted by \( U_h \) for the sake of brevity.

A procedure of transition from algebras over \( \mathbb{C}[[h]] \) to algebras over \( \mathbb{C} \) is described in \[3\]; it allows one to ”fix the value of the formal parameter \( h^\prime \)”. Here we only remind the formulae which describe the ”change of variables” corresponding to the generators of the above algebra:
\[
q = e^{-h/2}, \quad K_i^{\pm 1} = e^{\pm hH_i^2/2}, \quad E_i = X_i^+ e^{-hH_i^2/4}, \quad F_i = e^{hH_i^2/4} X_i^-.
\]

We fix in the sequel the value of \( q \in (0,1) \). The Hopf algebra over \( \mathbb{C} \) given by the generators \( \{E_i, F_i, K_i^{\pm 1}\}_i \) and the relations deduced above from the relations in \( U_h \), will be denoted by \( U_q \mathfrak{g} \), and the Hopf \( \ast \)-algebra \( (U_q \mathfrak{g}, \ast) \) by \( U_q \mathfrak{g} \).

The defining relations for \( U_q \) are similar to (3.1), (3.2). We list a part of them (the quantum analogue of the last among the relations (3.2) can be found in \[3\]):
\[
K_i K_j = K_j K_i; \quad K_i K_j^{-1} = K_j^{-1} K_i = 1; \quad K_i E_j = q^{d_{ij}} \cdot E_j K_i; \quad K_i F_j = q^{-d_{ij}} F_j K_i;
\]
\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_{ij}} - q^{-d_{ij}}},
\]
\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i; \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i; \quad \Delta(K_i) = K_i \otimes K_i;
\]
\[
\varepsilon(E_i) = \varepsilon(F_i) = \varepsilon(K_i - 1) = 0;
\]
\[
S(E_i) = -K_i^{-1} E_i; \quad S(F_i) = -F_i K_i; \quad S(K_i) = K_i^{-1};
\]
\[
E_j^\ast = \begin{cases} K_j F_j & j \neq j_0 \\ -K_j F_j & j = j_0 \end{cases}, \quad F_j^\ast = \begin{cases} E_j K_j^{-1} & j \neq j_0 \\ -E_j K_j^{-1} & j = j_0 \end{cases}.
\]

\[K_j^\ast = K_j, \quad i,j \in \{1, \ldots, l\}.
\]

We equip the Hopf algebra \( U_q \mathfrak{g} \) with a grading as follows:
\[
\deg K_j = \deg E_j = \deg F_j = 0, \quad j \neq j_0
\]
\[
\deg K_{j_0} = 0, \quad \deg E_{j_0} = 1, \quad \deg F_{j_0} = -1.
\]

\[\ast\text{See the definition of a Hopf } \ast\text{-algebra in } [3].\]
4 Covariant algebras and involutions

Remind that \( \mathbb{C} \) is endowed with a structure of a \( U_q\mathfrak{g} \)-module by means of a counit \( \varepsilon : U_q\mathfrak{g} \to \mathbb{C} \cong \text{End}(\mathbb{C}) \).

Let \( \mathcal{F} \) be a unital algebra over \( \mathbb{C} \), which is also a \( U_q\mathfrak{g} \)-module. We call \( \mathcal{F} \) a \( U_q\mathfrak{g} \)-module algebra if the multiplication

\[
m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}; \quad m : f_1 \otimes f_2 \mapsto f_1 f_2, \quad f_1, f_2 \in \mathcal{F}
\]

and the unit

\[
1 : \mathbb{C} \to \mathcal{F}; \quad z \mapsto z \cdot 1, \quad z \in \mathbb{C}
\]

are morphisms of \( U_q\mathfrak{g} \)-modules.\(^3\)

Together with the term "\( U_q\mathfrak{g} \)-module algebra" we shall elaborate the substitute term "covariant algebra" for the sake of brevity in the cases when no confusion can occur.

Covariant modules and covariant bimodules over covariant algebras are defined in a similar way (see [1, 25]).

An involutive \((\mathcal{F}, \ast)\) algebra is said to be covariant [22] if it is a \( U_q\mathfrak{g} \)-module algebra and for all \( \xi \in U_q\mathfrak{g} \), \( f \in \mathcal{F} \) one has

\[
(\xi f) \ast = (S(\xi)) \ast f. \tag{4.1}
\]

A linear functional \( \nu : \mathcal{F} \to \mathbb{C} \) is called an invariant integral if

\[
\nu(\xi f) = \varepsilon(\xi) \nu(f), \quad \xi \in U_q\mathfrak{g}, \ f \in \mathcal{F}.
\]

The "compatibility condition" for involutions (4.1) is extremely important since it allows one to use the "positive" invariant integrals for producing \( \ast \)-representations of \( U_q\mathfrak{g} \) in the "Hilbert function spaces":

\[
(f_1, f_2) = \nu(f_2 \ast f_1), \quad f_1, f_2 \in \mathcal{F}.
\]

The problem of decomposing such \( \ast \)-representations is a typical one in harmonic analysis. On this way, for instance, the Plancherel measure for quantum \( SU(1, 1) \) was found (see [22]).

5 Generalized Verma modules

Choose a linear functional \( \lambda \in \mathfrak{h}^\ast \) so that \( m_j = \lambda(H_j) \) are non-positive integers for \( j \neq j_0 \).

Consider the graded \( U_q\mathfrak{g} \)-module determined by the single generator \( v_+(\lambda) \in V_+(\lambda) \) and the relations

\[
F_i v_+(\lambda) = 0, \quad K_i^{\pm} v_+(\lambda) = e^\pm d_i m_i h/2 v_+(\lambda), \quad i = 1, \ldots, l;
\]

\[
E_j^{-m_j+1} v_+(\lambda) = 0, \quad j \neq j_0;
\]

\[
\deg(v_+(\lambda)) = \frac{1}{2} \lambda(H_0).
\]
Note that $V_+(\lambda) = \bigoplus_j V_+(\lambda)_j$, with $V_+(\lambda)_j = \{ v \in V_+(\lambda) | \deg(v) = j \}$, and $\dim V_+(\lambda)_j < \infty$.

The finite dimensionality of the homogeneous component $V_+(\lambda)_j$ follows from the decomposition

$$V_+(\lambda)_j = \bigoplus_{\mu \in \mathfrak{h}^* | \mu(H_0) = 2j} V_+(\lambda)_\mu$$

into a finite sum of the finite dimensional weight subspaces

$$V_+(\lambda)_\mu = \{ v \in V_+(\lambda) | K_j v = e^{-d_j \mu(H_j) h/2} v, j = 1, \ldots, l \}.$$

The graded modules $V_-(\lambda)$ are defined in a similar way:

$$E_i v_-(\lambda) = 0, \quad K_i^\pm v_-(\lambda) = e^{\mp d_i \mu(H_i) h/2} v_-(\lambda), \quad i = 1, \ldots, l$$

$$F_j^{m_j+1} v_-(\lambda) = 0, \quad j \neq j_0; \quad \deg(v_-(\lambda)) = \frac{1}{2} \lambda(H_0).$$

Now suppose $m_{j_0} = \lambda(H_{j_0}) \in \mathbb{Z}$.

Consider the longest element $w_0$ of the Weyl group $W$ for a Lie algebra $\mathfrak{g}$. It is very well known from [3, 6] that one can associate to each reduced decomposition of $w_0$ a Poincaré-Birkhoff-Witt basis in $U_q \mathfrak{g}$. We demonstrate the reduced decompositions for which this basis “generates” the bases of weight vectors in generalized Verma modules.

Let $\mathfrak{g}' \subseteq \mathfrak{g}$ be a Lie subalgebra generated by $\{ X_j^\pm, H_j \}_{j \neq j_0}$, and let also $W' \subseteq W$ be a subgroup generated by simple reflections $s(\alpha_j), j \neq j_0$. Obviously, $W'$ is a Weyl group of the Lie algebra $\mathfrak{g}'$.

Denote by $U \subseteq W$ the subset of such elements $u \in W$ that

$l(s(\alpha_j)u) > l(u) \quad \text{for all} \quad j \neq j_0$.

It is known from [3, p. 19], that, firstly, each element $w \in W$ admits the unique decomposition $w = w' \cdot u$ with $w' \in W'$, $u \in U$. Secondly, if $w' \in W'$, $u \in U$, then one has

$l(w' \cdot u) = l'(w') + l(u)$,

with $l'(w')$ being the length of the element $w'$ in $W'$, and $l(u)$, $l(w' u)$ the lengths of $u$, $w' u$ in $W$.

That is, one can find in $U$ the unique element $u_0$ of maximum length such that $w_0 = w'_0 \cdot u_0$. ($w'_0$ here is the longest element of $W'$.) Now one can derive the desired reduced decompositions of $w_0$ by multiplication from the reduced decompositions of $w'_0$ and $u$.

### 6 From coalgebras to algebras

Let $U_q \mathfrak{g}^{op}$ stand for the Hopf algebra derived from $U_q \mathfrak{g}$ by replacing its comultiplication by the opposite one.

We intend to use the generalized Verma modules for producing coalgebras dual to covariant algebras. To provide a precise correspondence between these two notions, we are going to replace $U_q \mathfrak{g}$ by $U_q \mathfrak{g}^{op}$ in tensor products of generalized Verma modules.
Consider the $U_q\mathfrak{g}$-modules $V_{\pm}(0)$. Evidently the maps
\[
\Delta_\pm : v_\pm(0) \mapsto v_\pm(0) \otimes v_\pm(0); \quad \varepsilon_\pm : v_\pm(0) \mapsto 1
\]
admit the unique extensions to morphisms of $U_q\mathfrak{g}$-modules:
\[
\Delta_\pm : V_{\pm}(0) \mapsto V_{\pm}(0) \otimes V_{\pm}(0); \quad \varepsilon_\pm : V_{\pm}(0) \rightarrow \mathbb{C}.
\]

Just as in the case $q = 1$, one can verify that the operations $\Delta_\pm$ are coassociative, and that $\varepsilon_\pm$ are the counits for coalgebras respectively with $\Delta_\pm$.

It follows that the vector spaces $(V_{\pm}(0))^* \overset{def}{=} \bigoplus_j (V_{\pm}(0)_j)^*$ are covariant algebras. Introduce the notation
\[
\mathbb{C}[\mathfrak{g}_{-1}]_q = V_-(0)^*, \quad \mathbb{C}[\overline{\mathfrak{g}}_{-1}]_q = V_+(0)^*.
\]
These covariant algebras may be treated as $q$-analogues of polynomial algebras (holomorphic or antiholomorphic identified by the sign) on the quantum prehomogeneous space $\mathfrak{g}_{-1}$.

7 Polynomial algebra

Consider the algebra $\text{Pol}(\mathfrak{g}_{-1}) = \mathbb{C}[\mathfrak{g}_{-1}] \otimes \mathbb{C}[\overline{\mathfrak{g}}_{-1}]$ of all polynomials on $\mathfrak{g}_{-1}$. Holomorphic and antiholomorphic polynomials admit the embeddings into this algebra as follows:
\[
\mathbb{C}[\mathfrak{g}_{-1}] \hookrightarrow \mathbb{C}[\mathfrak{g}_{-1}] \otimes \mathbb{C}[\overline{\mathfrak{g}}_{-1}], \quad f \mapsto f \otimes 1,
\]
\[
\mathbb{C}[\overline{\mathfrak{g}}_{-1}] \hookrightarrow \mathbb{C}[\mathfrak{g}_{-1}] \otimes \mathbb{C}[\overline{\mathfrak{g}}_{-1}], \quad f \mapsto 1 \otimes f.
\]

Our desire is to obtain that sort of algebra and similar embeddings in the quantum case ($q \neq 1$). For that, we intend to equip the $U_q\mathfrak{g}$-module $\text{Pol}(\mathfrak{g}_{-1})_q \overset{def}{=} \mathbb{C}[\mathfrak{g}_{-1}]_q \otimes \mathbb{C}[\overline{\mathfrak{g}}_{-1}]_q$ with a structure of covariant algebra in such a way that the maps $f \mapsto f \otimes 1$, $f \mapsto 1 \otimes f$ turn out to be algebra homomorphisms.

Our approach is completely standard \[\text{[1]}.\] Define the product of $\varphi_+ \otimes \varphi_-$, $\psi_+ \otimes \psi_- \in \text{Pol}(\mathfrak{g}_{-1})_q$ as follows:
\[
(\varphi_+ \otimes \varphi_-)(\psi_+ \otimes \psi_-) = m_+ \otimes m_- (\varphi_+ \otimes \hat{R}(\varphi_- \otimes \psi_+) \otimes \psi_-).
\]
Here $m_+$, $m_-$ are the multiplications in $\mathbb{C}[\mathfrak{g}_{-1}]_q$, $\mathbb{C}[\overline{\mathfrak{g}}_{-1}]_q$ respectively, and $\hat{R} : \mathbb{C}[\overline{\mathfrak{g}}_{-1}]_q \otimes \mathbb{C}[\mathfrak{g}_{-1}]_q \rightarrow \mathbb{C}[\mathfrak{g}_{-1}]_q \otimes \mathbb{C}[\overline{\mathfrak{g}}_{-1}]_q$ is the morphism of $U_q\mathfrak{g}$-modules defined below by V. G. Drinfeld’s universal R-matrix \[\overline{\text{[2]}}.\]

One can find in \[\overline{\text{[3, 4]}}\] the description of properties of the universal R-matrix which unambiguously determine it as an element of an appropriate completion of $\mathcal{U}_h \mathfrak{g} \otimes \mathcal{U}_h \overline{\mathfrak{g}}$. In particular,
\[
S \otimes S(R) = R, \quad R^{\ast \otimes \ast} = R^{21}.
\]  
(7.1)
The latter relation involves the element $R^{21}$ which is derived from $R$ by permutation of tensor multiples. The proof of this relation is completely similar to that of Proposition 4.2 in \[\overline{\text{[1]}}.\]

\[\text{[4]}\] Remind \[\overline{\text{[1]}}\] that the dual $U_q\mathfrak{g}$-module structure is given by $\xi f(v) \overset{def}{=} f(S(\xi)v)$, with $\xi \in U_q\mathfrak{g}$, $v \in V_{\pm}(0)$, $f \in V_{\pm}(0)^*$. 

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In [3] there is an explicit "multiplicative" formula for the universal R-matrix. More precisely, any reduced decomposition of the maximum length element \( w_0 \) possesses its own multiplicative formula. In the sequel we intend to restrict ourselves to those reduced decompositions which come from Section 5. (Note that the "multiplicative" formula was discovered in the papers of S. Z. Levendorskiǐ, Ya. S. Soibelman and also by A. N. Kirillov, N. Yu. Reshetikhin, see [3]. Its application should take into account the inessential differences in the choice of generators and deformation parameters in this work as compared with [3]. Specifically, one has to substitute \( X^+, X^-, H_i, K_i, q \) by \(-S(E_i), -S(F_i), -S(H_i), h/2, K_i^{-1}, q^{-1}\).)

It is easy to show that the universal R-matrix determines a linear operator in \( C[g^{-1}] \otimes C[g^{-1}] \).

Now we are in a position to define the operator \( \tilde{R} \) in a standard way: \( \tilde{R} = \sigma \cdot R \) with \( \sigma : a \otimes b \mapsto b \otimes a \) being a permutation of tensor multiples. Thus \( \tilde{R} \) becomes a morphism of \( U_q g \)-modules since \([6, 7, 3] \triangleq \). \( \Delta^{op}(\xi) = R\Delta(\xi)R^{-1}, \xi \in U_h g. \)

The associativity of the multiplication in \( Pol(g^{-1})_q \) can be easily derived by the standard argument \([10, 11] \Delta \otimes \text{id})(R) = R^{13}R^{23}, (\text{id} \otimes \Delta)(R) = R^{13}R^{12}. \) (Here \( R^{12} = i a_i \otimes b_i \otimes 1, R^{23} = i 1 \otimes a_i \otimes b_i, R^{13} = i a_i \otimes 1 \otimes b_i \) whenever \( R = i a_i \otimes b_i, \) see \([3, 4, 5]\)).

The existence of a unit and covariance of \( Pol(g^{-1})_q \) are evident.

8 Involution

Consider the antilinear operators \( * : V_+(0) \to V_-(0); \ * : V_-(0) \to V_+(0), \) which are determined by their properties as follows. Firstly, \( v_{\pm}(0)^* = v_{\mp}(0) \) and, secondly,

\[
(\xi v)^* = (S^{-1}(\xi))^* v^*
\]

(8.1) for all \( v \in V_{\pm}(0), \xi \in U_q g. \)

To rephrase the above, we set up

\[
(\xi v_{\pm}(0))^* = (S^{-1}(\xi))^* v_{\mp}(0)^*.
\]

It follows from the definition of \( V_{\pm}(0) \) that the involution as above is well defined. In particular, (8.1) can be easily deduced; it also follows from the relation \( (S^{-1}((S^{-1}(\xi))^*))^* = \xi \) that the operators constructed above are mutually converse.

The duality argument allows one to form the mutually converse antihomomorphisms \( * : C[g^{-1}] \to C[g^{-1}]; \ * : C[g^{-1}] \to C[g^{-1}] : \)

\[
f^*(v) \overset{\text{def}}{=} f(v^*), \quad v \in V_{\pm}(0), \quad f \in V_{\pm}(0)^*.
\]

(8.2)

Now we are in a position to define the antilinear operator \( * \) in \( Pol(g^{-1})_q \) by

\[
(f_+ \otimes f_-)^* \overset{\text{def}}{=} f_-^* \otimes f_+^*.
\]
for $f_+ \in \mathbb{C}[\mathfrak{g}_{-1}]_q$, $f_- \in \mathbb{C}[\mathfrak{g}_{-1}]_q$, and also to show that it equips $\text{Pol}(\mathfrak{g}_{-1})_q$ with a structure of covariant involutive algebra.

It remains to verify that $\ast$ is an antihomomorphism of $\text{Pol}(\mathfrak{g}_{-1})_q$. The best way to prove the relation

$$(f_1 f_2)^\ast = f_2^\ast f_1^\ast; \quad f_1, f_2 \in \text{Pol}(\mathfrak{g}_{-1})_q$$

is in applying (7.1) and the duality argument described in details in the concluding section of the present paper. (Note that it suffices to prove the relation $(f_1 f_2)^\ast(v) = f_2^\ast f_1^\ast(v)$ for the generator $v = v_-(0) \otimes v_+(0)$ of the $U_q\mathfrak{g}$-module $V_-(0) \otimes V_+(0)$ since the map $\xi \mapsto (S^{-1}(\xi))^\ast$ is an antiautomorphism of the coalgebra $U_q\mathfrak{g}$).

Verify (4.1); it suffices to consider the case $f \in V_+(0)^\ast$. An application of (8.2) and the relation $S((S(\xi))^\ast) = \xi^\ast$, $\xi \in U_q\mathfrak{g}$, yields for all $v \in V_\pm(0)$, $f \in V_\pm(0)^\ast$

$$f((\xi v)^\ast) = f((S^{-1}(\xi))^\ast v^\ast),$$

$$\overline{f(S(\xi)v^\ast)} = \overline{f((\xi^\ast v)^\ast)};$$

$$(\xi f)(v^\ast) = f^\ast(\xi^\ast v)$$

and

$$\overline{(\xi f)(v^\ast)} = f^\ast(S((S(\xi))^\ast)v)$$

$$\overline{(\xi f)^\ast(v)} = ((S(\xi))^\ast f^\ast)(v).$$

Thus, in the special case $f \in V_\pm(0)^\ast$ (4.1) is proved. Hence it is also valid for all $f \in \text{Pol}(\mathfrak{g}_{-1})_q$ since the antipode is an antiautomorphism of the coalgebra $U_q\mathfrak{g}$ and the involution $\ast$ is its automorphism. In fact, if $f = f_+ f_-$, $f_\pm \in (V_\pm(0))^\ast$ and $\Delta(\xi) = \sum_j \xi_j^\prime \otimes \xi_j''$; $\xi_j^\prime, \xi_j'' \in U_q\mathfrak{g}$ then one has

$$(\xi(f_+ f_-))^\ast = \sum_j (\xi_j'' f_-)^\ast(\xi_j' f_+)^\ast,$$

$$(S(\xi))^\ast(f_+ f_-)^\ast = (S(\xi))^\ast(f_-^\ast f_+^\ast) = \sum_j ((S(\xi_j''))^\ast f_-^\ast)((S(\xi_j'))^\ast f_+^\ast).$$

9 The simplest example

Let $\mathfrak{g} = \mathfrak{sl}_2$, then one has $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0$ and $\mathfrak{g}_{\pm 1}$ being Cartan and Borel subalgebras of $\mathfrak{sl}_2$ respectively. In particular, $\deg(\mathfrak{g}_{-1}) = 1$.

The algebra $U_q\mathfrak{sl}_2$ is given by its generators $K^{\pm 1}$, $E$, $F$ and the relations

$$KK^{-1} = K^{-1}K = 1, \quad K^{\pm 1}E = q^{\pm 2}EK^{\pm 1}, \quad K^{\pm 1}F = q^{\mp 2}FK^{\pm 1},$$

$$EF - FE = (K - K^{-1})/(q - q^{-1}).$$

Remind that comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$ are defined on the above generators as follows:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1};$$

$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K^{\pm 1}) = 1;$$

$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K^{\pm}) = K^{\mp}.$$
In the notation

\[ q = e^{-h/2}, \quad K^\pm = e^{\pm hH/2}, \quad E = X^+ e^{-hH/4}, \quad F = e^{hH/4} X^- \]

the V. G. Drinfeld’s formula for the universal R-matrix \( R \) acquires the form

\[ R = \exp_q^2((q^{-1} - q) E \otimes F) \cdot \exp(H \otimes H \cdot h/4) \]

with \( \exp_t(x) = \sum_{n=0}^{\infty} x^n (\prod_{j=1}^{n} \frac{1 - t^j}{1 - t})^{-1} \).

The involution \( \ast \) in \( U_q \mathfrak{su}(1, 1) = (U_q \mathfrak{sl}_2, \ast) \) is defined on the generators \( E, F, K^\pm \) by

\[ E^\ast = -KF, \quad F^\ast = -EK^{-1}, \quad (K^\pm)^\ast = K^\mp \]

(equivalently, on the generators \( X^\pm, H \) of \( U_h \mathfrak{sl}_2 \) it is defined by \( (X^\pm)^\ast = -X^\mp, H^\ast = H \)).

Consider the \( U_q \mathfrak{sl}_2 \)-module \( V_+(0) \) determined by its single generator \( v_+(0) \in V_+(0) \) and the relations \( Fv_+(0) = 0, \quad K^\pm v_+(0) = v_+(0) \). This module admits the decomposition

\[ V_+(0) = \bigoplus_{j \in \mathbb{Z}_+} V_+(0)_j, \quad V_+(0)_j = \mathbb{C} \cdot E^j \cdot v_+(0). \]

Hence \( \{ E^j v_+(0) \}_{j \in \mathbb{Z}_+} \) is a basis in \( V_+(0) \).

Define a linear functional \( a_- \in V_+(0)^* = \mathbb{C}[[\mathfrak{g}]]_q \) by

\[ a_-(S(E^j)v_+(0)) = \left\{ \begin{array}{ll} 1 & j = 1 \\ 0 & j \neq 1 \end{array} \right. \]

Prove that \( a_- \) is a generator of \( \mathbb{C}[[\mathfrak{g}]]_q \), and that for any polynomial \( P \in \mathbb{C}[t] \)

\[ K^\pm : P(a_-) \mapsto P(q^{\mp 2}a_-), \quad (9.1) \]

\[ E : P(a_-) \mapsto (D_- P)(a_-), \quad (9.2) \]

\[ F : P(a_-) \mapsto -q \cdot a_-^2 \cdot (D_+ P)(a_-) \quad (9.3) \]

where \( (D_\pm P)(t) = (P(q^{\mp 2}t) - P(t))/(q^{\mp 2}t - t) \).

Note first that the relations

\[ Ka_-(S(E^j)v_+(0)) = \left\{ \begin{array}{ll} q^{-2} & j = 1 \\ 0 & j \neq 1 \end{array} \right. , \]

\[ Ea_-(S(E^j)v_+(0)) = \left\{ \begin{array}{ll} 1 & j = 0 \\ 0 & j \neq 0 \end{array} \right. \]

imply that

\[ Ka_- = q^{-2} a_-, \quad Ea_- = 1. \]

(9.4)

Now apply the covariance of \( \mathbb{C}[[\mathfrak{g}]]_q \) to obtain

\[ K^\pm(P_1(a_-)P_2(a_-)) = K^\pm(P_1(a_-)) \cdot K^\pm(P_2(a_-)), \]

\[ E(P_1(a_-)P_2(a_-)) = E(P_1(a_-)) \cdot P_2(a_-) + K(P_1(a_-)) \cdot E(P_2(a_-)) \]

for any polynomials \( P_1, P_2 \). This already allows one to deduce (9.1) (9.2) from (9.4).
It is worthwhile to note that \( a_j^j \neq 0 \) for all \( j \in \mathbb{Z}_+ \) since

\[
E^j a_j^j = \prod_{k=1}^{j} \left( \frac{(q^{-2k} - 1)}{(q^{-2} - 1)} \right) \neq 0.
\]

This implies that \((V_+(0)_j)^* = \mathbb{C} \cdot a_j^j\). Hence \( \{a_j^j\}_{j \in \mathbb{Z}_+} \) is a basis of the vector space \( \mathbb{C}[\mathfrak{g}_-]_q \).

That is, \( a_- \) is a generator of the algebra \( \mathbb{C}[\mathfrak{g}_-]_q \).

Now prove (9.3) in the special case \( P(a_-) = a_- \). Specifically, we are going to demonstrate

\[
Fa_- = -q \cdot a_-^2. \quad (9.5)
\]

Since \( Fa_- \in (V_+(0)_2)^* = \mathbb{C} a_-^2 \) we have \( Fa_- = \text{const} \cdot a_-^2 \). The fact that the constant in the latter relation is \( -q \) follows easily from

\[
E(Fa_-) = \frac{K - K^{-1}}{q - q^{-1}} a_- = -(q + q^{-1}) a_-,
\]

\[
E(a_-^2) = (q^{-2} + 1) a_- = q^{-1}(q^{-1} + q) a_-
\]

together with \( a_- \neq 0 \).

The passage from the special case \( P(a_-) = a_- \) to the general case can be performed (just as above) by a virtue of covariance. Specifically,

\[
F(P_1 P_2) = F(P_1) \cdot K^{-1}(P_2) + P_1 F(P_2)
\]

for any "polynomials" \( P_1(a_-), P_2(a_-) \).

Now turn to the description of the covariant algebra \( \mathbb{C}[\mathfrak{g}_-]_q \) for the same case \( \mathfrak{g} = \mathfrak{sl}_2 \).

One has \( V_- = \bigoplus_{-j \in \mathbb{Z}_+} V_-(-j) \), \( V_-(-j) = \mathbb{C} \cdot P_j v_-(-j) \). Define also the "coordinate function" \( a_+ \) by

\[
a_+(S(F^j v_-(-j))) = \begin{cases} 1 & j = 1 \\ 0 & j \neq 1 \end{cases}.
\]

Now one can prove in the same way as above that \( a_+ \) is the generator of \( \mathbb{C}[\mathfrak{g}_-]_q \) and

\[
K^{\pm 1} : P(a_+) \mapsto P(q^{\mp 2} a_+),
\]

\[
F : P(a_+) \mapsto (D_- P)(a_+),
\]

\[
E : P(a_+) \mapsto -q a_+^2 \cdot (D_+ P)(a_+)
\]

for any polynomial \( P \) of a single indeterminate.

In particular, one has

\[
K^{\pm 1} a_+ = q^{\mp 2} a_-, \quad Fa_+ = 1, \quad Ea_+ = -q a_+^2. \quad (9.6)
\]

Note that if \( \{f_i\} \) are the generators of a covariant algebra \( \mathcal{F} \) and \( \{a_j\} \) the generators of a Hopf algebra \( A \), then the action of \( A \) on \( \mathcal{F} \) can be unambiguously retrieved from the action of \( \{a_j\} \) on \( \{f_i\} \).

Turn to the description of the covariant algebra \( \text{Pol}(\mathfrak{g}_-)_q \) in terms of generators and relations.
By our construction, the covariant algebras \( \mathbb{C}[g^{-1}]_q \) and \( \mathbb{C}[\mathfrak{g}^{-1}]_q \) are embedded into \( \text{Pol}(g^{-1})_q \).

It follows from the explicit formula for the universal R-matrix and the definition of the action of \( \exp(H \otimes Hh/4) \) on the weight vectors that

\[
e^{H \otimes Hh/4} a_- \otimes a_+ = q^{-\frac{1}{2}} 2(-2) \cdot a_- \otimes a_+,
\]

\[
a_- a_+ = q^2(a_+ a_- + q^{-1}(1 - q^2) Fa_+ \cdot E a_-).
\]

Finally we have:

\[
a_- a_+ = q^2 a_+ a_- + q(1 - q^2).
\]

(9.7)

Since \( \text{Pol}(g^{-1})_q = \mathbb{C}[g^{-1}]_q \otimes \mathbb{C}[\mathfrak{g}^{-1}]_q \), we deduce that (9.7) gives a complete list of relations between the generators \( \{a_+, a_-\} \) of \( \text{Pol}(g^{-1})_q \), that is the natural map \( \mathbb{C}[a_+, a_-]/(a_- a_+ - (q^2 a_+ a_- + q(1 - q^2))) \to \text{Pol}(g^{-1})_q \) is injective. The action of the generators \( \{K^\pm, E, F\} \) of \( U_q \mathfrak{g} \) on the generators \( \{a_+, a_-\} \) of \( \text{Pol}(g^{-1})_q \) is given by (9.4) – (9.6).

It remains to describe the involution \( \ast \).

We start with proving that

\[
a^+_q = \text{const} \cdot a_-, \quad a^-_q = \text{const} \cdot a_+,
\]

(9.8)

and then find the constants by comparing the explicit expressions for \( E a^+_q \) and \( E a^-_q \). The relations (9.8) follow from the decompositions \( \mathbb{C}[g^{-1}]_q = \bigoplus (V_-(0)_i)^\ast \), \( \mathbb{C}[\mathfrak{g}^{-1}]_q = \bigoplus (V_+(0)_i)^\ast \), and \( (V_\pm(0)_{\pm 1})^\ast = \mathbb{C} \cdot a_\pm, \quad \ast : (V_\pm(0)_i)^\ast \to (V_\pm(0)_{-i})^\ast \).

It was pointed out before that \( E a_- = 1 \). Let’s compute \( E a^+_q \). First use the relation

\[
(S(F))^\ast = (-FK)^\ast = -K^\ast F^\ast = -K \cdot (-EK^{-1}) = q^2 E
\]

and the compatibility condition (4.1) for involutions to obtain

\[
q^2 E a^+_q = (S(F))^\ast a^+_q = (Fa_+)^\ast = 1^\ast = 1.
\]

Thus we have \( q^2 \cdot E a^+_q = E a_- \). Now (9.8) implies

\[
a^+_q = q^{-2} a_-; \quad a^-_q = q^2 a_+.
\]

(9.9)

The only shortcoming of the definition of the covariant \( \ast \)-algebra \( \text{Pol}(g^{-1})_q \) is that it is excessively abstract. In the example for \( U_q \mathfrak{su}(1, 1) \) we got another description of that covariant \( \ast \)-algebra. Specifically, its generators are \( \{a_+, a_-\} \), its complete list of relations reduces to (9.7), the action of \( U_q \mathfrak{su}(1, 1) \) is given by (9.4) – (9.7), and the involution is determined by (9.9).

Note that in the work [23] on the function theory in the unit disc the generator \( z = q^{1/2} \cdot a_+ \) was implemented instead of \( a_+ \). In this setting, (9.9) implies the relation \( z^\ast = q^{-3/2} a_- \), and (9.7) can be rewritten as

\[
z^\ast z - q^2 z^2 z^\ast = 1 - q^2.
\]

(9.10)

(The substitution \( q = e^{-h/2} \) and the formal passage to a limit as \( h \to 0 \) yield (cf. (1.1)):

\[
\lim_{h \to 0} \frac{[z, z^\ast]}{ih} = i(1 - zz^\ast).)
\]
Proceed with studying the *-algebra \( \text{Pol}(\mathfrak{g}_{-1})_q \) which was under investigation in the previous section. Evidently, the formulae

\[
T_\varphi(z) = e^{i\varphi}, \quad T_\varphi(z^*) = e^{-i\varphi}, \quad \varphi \in \mathbb{R}/2\pi\mathbb{Z}
\]

determine the one-dimensional representation of \( \text{Pol}(\mathfrak{g}_{-1})_q \). We shall also need a faithful infinitely dimensional *-representation \( T \) in the Hilbert space \( l^2(\mathbb{Z}_+^+) \) given by

\[
T(z)e_m = (1 - q^{2(m+1)})^{1/2}e_{m+1},
\]

\[
T(z^*)e_{m+1} = (1 - q^{2(m+1)})^{1/2}e_m,
\]

\[
T(z^*)e_0 = 0,
\]

with \( \{e_m\}_{m \in \mathbb{Z}_+} \) being the standard basis in \( l^2(\mathbb{Z}_+) \). An application of the standard techniques of operator theory in Hilbert spaces \([24]\) allows one to prove that any irreducible *-representation of the above algebra is unitarily equivalent to one of the representations \( \{T_\varphi\}_{\varphi \in \mathbb{R}/2\pi\mathbb{Z}}, T \).

Note that the spectrum of \( T(z) \) is the closure \( \overline{U} \) of the unit disc \( U \) in \( \mathbb{C} \). Just as in \([24]\), we use the notion "algebra of continuous functions in the quantum disc" for a completion of \( \text{Pol}(\mathfrak{g})_q \) with respect to the norm \( \|f\| = \sup \|\rho(f)\|. \) Here \( \rho \) varies inside the class of all irreducible *-representations up to unitary equivalence. One can easily deduce from the above that \( \|f\| = \|Tf\| \).

The enveloping von Neumann algebra \([3]\) of the above C*-algebra will be denoted by \( L^\infty(U)_q \) and called the algebra of continuous functions in the quantum disc. Certainly, \( L^\infty \) is worthwhile not alone, but only together with a distinguished dense covariant subalgebra \( \text{Pol}(\mathfrak{g}_{-1})_q \) (cf. \([25]\)).

Note that our quantum disc is only one among those described in \([12]\). Others can be derived from this one by a standard argument normally referred to as quantization by Berezin \([23]\).

Remark also that the definition of \( L^\infty(U)_q \) which implements a completion procedure and passage to an enveloping von Neumann algebra doesn’t use the specific features of the special case \( \mathfrak{g} = sl_2 \). That is, to any irreducible prehomogeneous vector space of commutative parabolic type we associate a pair constituted by a von Neumann algebra \( L^\infty(U)_q \) and its dense covariant subalgebra \( \text{Pol}(\mathfrak{g}_{-1})_q \).

### 11 Differential calculi: the outline

We follow G. Maltsiniotis \([17]\) in choosing the basic idea of producing the differential calculi. Specifically, we first construct differential calculi of order one, and then embed them into complete differential calculi by a simple argument described in \([17]\), the proof of theorem (1.2.3).

To outline the construction of order one differential calculi, we restrict ourselves to the simplest example of a quantum prehomogeneous vector space.
At the first step we consider the type (1,0) forms with holomorphic coefficients $f \cdot dz$, $f \in \mathbb{C}[\mathfrak{g}_{-1}]_q$, and type (0,1) forms with antiholomorphic coefficients $f \cdot dz^*$, $f \in \mathbb{C}[[\mathfrak{g}_{-1}]]_q$. We prove that
\[
dz \cdot z = q^2 z \cdot dz; \quad dz^* \cdot z^* = q^{-2} z^* \cdot dz^ *. \tag{11.1}
\]
At the second step we assume the consideration of all the forms of types (1,0) and (0,1): $f dz, f dz^*$, $f \in \text{Pol}(\mathfrak{g}_{-1})_q$. We prove that
\[
dz \cdot z^* = q^{-2} z^* \cdot dz; \quad dz^* \cdot z = q^2 z \cdot dz^ * . \tag{11.2}
\]
At the third step we turn to higher forms, which gives the additional relations
\[
dz \cdot dz = 0, \quad dz^* \cdot dz^* = 0, \quad dz^* \cdot dz = -q^2 dz \cdot dz^ *. \tag{11.3}
\]
Of course, the relations (11.1) – (11.3) are well known to the specialists (see, for instance, \[17\] and the references therein).

12 Differential calculi: step one

We follow the notation of sections 3, 5, 6.

Consider the linear functionals $\lambda_\pm \in \mathfrak{h}^*$ given by
\[
\lambda_\pm (H_i) = \pm a_{ij},
\]
jointly with the associated generalized Verma modules $V_\pm (\lambda_\pm)$. Just as in Section 8, define the "involutions" $^{*} : V_\pm (\lambda_\pm) \rightarrow V_\mp (\lambda_\mp)$ by (8.1) and $^{*} : v_\pm (\lambda_\pm) \mapsto v_\mp (\lambda_\mp)$.

It follows from the definitions that the maps
\[
v_+ (\lambda_+) \mapsto E_{j0} v_+(0), \quad v_+ (\lambda_+)^* \mapsto (E_{j0} v_+(0))^*
\]
admit the unique extensions to $U_q \mathfrak{g}$-module morphisms
\[
\delta_+ : V_+ (\lambda_+) \rightarrow V_+ (0), \quad \delta_- : V_- (\lambda_-) \rightarrow V_- (0).
\]

Consider the dual graded $U_q \mathfrak{g}$-modules:
\[
\bigwedge^1 (\mathfrak{g}_{-1})_q = \bigoplus_{j \in \mathbb{Z}_+} V_- (\lambda_-)^*_{-j}; \quad \bigwedge^1 (\mathfrak{g}_{-1})_q = \bigoplus_{j \in \mathbb{Z}_+} V_+ (\lambda_+)^*_{j}.
\]

Our definition of the graded components implies that
\[
\delta_+ V_+ (\lambda_+)_{-j} \subset V_+ (0)_j; \quad \delta_- V_- (\lambda_-)_{-j} \subset V_- (0)_j.
\]

Now the "adjoint" operators $\partial = \delta_-^*$, $\overline{\partial} = \delta_+^*$ are well defined and become $U_q \mathfrak{g}$-module morphisms:
\[
\partial : \mathbb{C}[\mathfrak{g}_{-1}]_q \rightarrow \bigwedge^1 (\mathfrak{g}_{-1})_q; \quad \overline{\partial} : \mathbb{C}[\overline{\mathfrak{g}}_{-1}]_q \rightarrow \bigwedge^1 (\overline{\mathfrak{g}}_{-1})_q.
\]
Evidently, the maps
\[
v_\pm (\lambda_\pm) \mapsto v_\pm (0) \otimes v_\pm (\lambda_\pm); \quad v_\pm (\lambda_\pm) \mapsto v_\pm (\lambda_\pm) \otimes v_\pm (0)
\]
admit the unique extension to $U_q\mathfrak{g}$-module morphisms

$$
\Delta^L_\pm : V_\pm(\lambda_\pm) \rightarrow V_\pm(0) \otimes V_\pm(\lambda_\pm), \quad \Delta^R_\pm : V_\pm(\lambda_\pm) \rightarrow V_\pm(\lambda_\pm) \otimes V_\pm(0).
$$

Pass again to the "adjoint" linear operators and observe that they are well defined and equip $\Lambda^1(\mathfrak{g}_-)q$ with a structure of a covariant bimodule over $\mathbb{C}[\mathfrak{g}_-]q$, and $\Lambda^1(\mathfrak{g}^-_1)q$ with a structure of a covariant bimodule over $\mathbb{C}[[\mathfrak{g}^-_1]q$. (The covariance here means that the actions $(\Delta^L_\pm)^*, (\Delta^R_\pm)^*$ of $\mathbb{C}[\mathfrak{g}_-]q$ and $\mathbb{C}[[\mathfrak{g}^-_1]q$ respectively are $U_q\mathfrak{g}$-module morphisms).

Remark. With $\omega \in \Lambda^1(\mathfrak{g}_-)q$ or $\omega \in \Lambda^1(\mathfrak{g}^-_1)q$ one has $1 \cdot \omega = \omega \cdot 1 = \omega$, since

$$(\varepsilon \otimes \text{id})\Delta^L_\pm(v) = v, \quad (\text{id} \otimes \varepsilon)\Delta^R_\pm(v) = v, \quad v \in V_\pm(\lambda_\pm).$$

It is easy to show that $\partial$ and $\overline{\partial}$ are differentiations of the corresponding covariant bimodules:

$$
\partial(f_1f_2) = \partial f_1 \cdot f_2 + f_1\partial f_2; \quad f_1, f_2 \in \mathbb{C}[\mathfrak{g}_-]q,
$$

$$
\overline{\partial}(f_1f_2) = \overline{\partial} f_1 \cdot f_2 + f_1\overline{\partial} f_2; \quad f_1, f_2 \in \mathbb{C}[[\mathfrak{g}^-_1]q.
$$

For example, to prove the latter inequality, it suffices to pass in each its part to the adjoint operators

$$V_+(\lambda_+) \rightarrow V_+(0) \otimes V_+(0)$$

and then to apply both operators to the generator $v_+(\lambda_+)$ of the $U_q\mathfrak{g}$-module $V_+(\lambda_+)$. In both cases one obtains

$$E_{j_0}v_+(0) \otimes v_+(0) + v_+(0) \otimes E_{j_0}v_+(0).$$

In conclusion, let us prove one of the equalities (11.1). Another one can be derived in a similar way.

It follows from $z^*dz^* \in (V_+(\lambda_+)q)^*$, $dz^* \cdot z^* \in (V_+(\lambda_+)q)^*$, $\dim V_+(\lambda_+)q = 1$ that $z^* \cdot dz^* = \text{const} \cdot dz^* \cdot z^*$. Thus, it remains to compute the constant.

When applying the duality argument, we replace $f(v)$ by $\langle f, v \rangle$. Compare $\langle z^*dz^*, Ev_+(\lambda_+) \rangle$ and $\langle dz^* \cdot z^*, Ev_+(\lambda_+) \rangle$.

Firstly, one has

$$
\langle z^*dz^*, Ev_+(\lambda_+) \rangle = \\
\langle z^*dz^*, (1 \otimes E + E \otimes K)(v_+(0) \otimes v_+(\lambda_+)) \rangle = \\
\langle z^*dz^*, (E \otimes K)(v_+(0) \otimes v_+(\lambda_+)) \rangle = \\
\langle z^*, Ev_+(0) \rangle \langle dz^*, Kv_+(\lambda_+) \rangle = \\
q^2(z^*, Ev_+(0)) \langle dz^*, v_+(\lambda_+) \rangle = q^2(z^*, Ev_+(0))^2,
$$

and secondly

$$
\langle dz^* \cdot z^*, Ev_+(\lambda_+) \rangle = \\
\langle dz^* \otimes z^*, (1 \otimes E + E \otimes K)(v_+(\lambda_+) \otimes v_+(0)) \rangle = \\
\langle dz^*, v_+(\lambda_+) \rangle \langle z^*, Ev_+(0) \rangle = \langle z^*, Ev_+(0) \rangle^2.
$$

Since $\langle z^*, Ev_+(0) \rangle \neq 0$, we obtain finally

$$z^*dz^* = q^2dz^* \cdot z^*.$$
13 Differential calculi: step two

Consider the $U_q\mathfrak{g}$-module

$$\Omega^{(1,0)}(\mathfrak{g}_{-1})_q \overset{\text{def}}{=} \bigwedge^1 (\mathfrak{g}_{-1})_q \otimes \mathbb{C}[\mathfrak{g}_{-1}]_q.$$ 

Use the universal R-matrix in the same way as in Section 7 to equip $\Omega^{(1,0)}(\mathfrak{g}_{-1})_q$ with a structure of a covariant bimodule over $\text{Pol}(\mathfrak{g}_{-1})_q$.

There is a unique extension of the differentiation $\partial : \mathbb{C}[\mathfrak{g}_{-1}]_q \rightarrow \bigwedge^1 (\mathfrak{g}_{-1})_q$ to a differentiation $\partial : \text{Pol}(\mathfrak{g}_{-1})_q \rightarrow \Omega^{(1,0)}(\mathfrak{g}_{-1})_q$ such that $\partial \mathbb{C}[\mathfrak{g}_{-1}]_q = 0$. Clearly $\partial(f_+ \otimes f_-) = \partial f_+ \otimes f_- + f_+ \otimes \partial f_-$. 

Turn to the example $\mathfrak{g} = U_q\mathfrak{sl}_2$. Differentiation of both sides in (9.10) (with the properties $\partial : 1 \mapsto 0$, $\partial : z \mapsto dz$ being taken into account) yields $z^* \cdot dz - q^2 d\cdot z^* = 0$. This is just one of the relations (11.2).

Now consider the $U_q\mathfrak{g}$-module $\Omega^{(0,1)}(\mathfrak{g}_{-1})_q \overset{\text{def}}{=} \mathbb{C}[\mathfrak{g}_{-1}]_q \otimes \bigwedge^1 (\mathfrak{g}_{-1})_q$ together with the morphism of $U_q\mathfrak{g}$-modules

$$\overline{\partial} : \text{Pol}(\mathfrak{g}_{-1})_q \rightarrow \Omega^{(0,1)}(\mathfrak{g}_{-1})_q; \quad \overline{\partial} : f_+ \otimes f_- \mapsto f_+ \otimes \partial f_-,$$

where $f_+ \in \mathbb{C}[\mathfrak{g}_{-1}]_q$, $f_- \in \mathbb{C}[\mathfrak{g}_{-1}]_q$. Just as it was done before, one can equip $\Omega^{(0,1)}(\mathfrak{g}_{-1})_q$ with a structure of a covariant bimodule over $\text{Pol}(\mathfrak{g}_{-1})_q$ and prove that $\overline{\partial}$ is a differentiation. An application of $\overline{\partial}$ to both sides of (9.10) gives the second one of the relations (11.2).

Finally, set

$$\Omega^1(\mathfrak{g}_{-1})_q = \Omega^{(1,0)}(\mathfrak{g}_{-1})_q \oplus \Omega^{(0,1)}(\mathfrak{g}_{-1})_q, \quad d = \partial + \overline{\partial}.$$

14 Differential calculi: step three

Let $A$ be a unital algebra over $\mathbb{C}$.

**Definition.** Let $M$ be a bimodule over $A$ and $d : A \rightarrow M$ a linear operator. The pair $(M, d)$ is called a differential calculus of order one if

i) $d(a' \cdot a'') = da' \cdot a'' + a' \cdot da''$,

ii) $A \cdot (dA) \cdot A = M$.

In the case when $A$ is a covariant algebra, $M$ a covariant bimodule and $d : A \rightarrow M$ a $U_q\mathfrak{g}$-module morphism with conditions i), ii) being satisfied, the pair $(M, d)$ is called a covariant differential calculus of order one.

The results expounded in the appendix of this work imply that the five covariant differential calculi of order one:

$$(\bigwedge^1 (\mathfrak{g}_{-1})_q, \partial), \quad (\bigwedge^1 (\mathfrak{g}_{-1})_q, \overline{\partial})$$

$$(\Omega^{(1,0)}(\mathfrak{g}_{-1})_q, \partial), \quad (\Omega^{(0,1)}(\mathfrak{g}_{-1})_q, \overline{\partial}), \quad (\Omega^1(\mathfrak{g}_{-1})_q, d).$$

In the sequel we apply to each of those the "algorithm of constructing the full differential calculus" described in [17].

**Definition.** Let $\Omega = \bigoplus_{n \in \mathbb{Z}_+} \Omega_n$ be a $\mathbb{Z}_+$-graded algebra and $d$ a linear operator in $\Omega$ of order one. The pair $(\Omega, d)$ is called a differential graded algebra if
Let us describe the "algorithm" of construction of the pair \((\Omega, d)\) given the pair \((M, d)\). Let \(M_1 = dA \subset M\).

Equip the tensor algebra \(T = T(A, M_1)\) with a grading in which \(\deg a = 0\), \(\deg m = 1\), \(a \in A\), \(m \in M_1\). One has \(T_0 = T(A) = \mathbb{C} \oplus A \oplus A^\otimes 2 \oplus \ldots\), \(T_{j+1} = T(A) \otimes M_1 \otimes T_j\).

There exists a unique operator \(d : T \to T\) such that

i) \(d(t_1 t_2) = dt_1 \cdot t_2 + (\mathbf{1})^n t_1 \cdot dt_2\), \(t_1 \in T_n, t_2 \in T\),

ii) \(d\mid_A\) coincides with the differentiation in the initial calculus of order one,

iii) \(d\mid_{M_1} = 0\).

In fact, on \(T_0\) we have \(d1 = 0\), \(d(a_1 \otimes a_2 \otimes \ldots \otimes a_k) = \sum \limits_j a_1 \otimes \ldots \otimes a_{j-1} \otimes da_j \otimes \ldots \otimes a_k\).

From now on we proceed by induction:
\(d(a \otimes m \otimes t) = da \otimes m \otimes t - a \otimes m \otimes dt\), \(a \in T_0, m \in M_1, t \in T_j\).

(Note that \(d\) is well defined because of the multilinearity of the right-hand sides of the above identities in the "indeterminates" \((a_1, \ldots a_k)\) and \((a, m, t)\) respectively.)

Consider the least \(d\)-invariant bilateral ideal \(J\) of \(T\) which contains all the elements of the form

i) \(a_1 \otimes a_2 - a_1 a_2, a_1, a_2 \in A\)

ii) \(1 \otimes m - m \otimes 1 - m, m \in M_1\)

iii) \(\{\sum \limits_{ij} a'_{ij} \otimes m_{ij} \otimes a''_{ij} | a'_{ij}, a''_{ij} \in A, m_{ij} \in M_1, \sum \limits_{ij} a'_{ij} m_{ij} a''_{ij} = 0\}\)

(Note that the left hand side of the latter equality is a sum of elements of the \(A\)-bimodule \(M\).)

It follows from our construction that \(J\) is a graded ideal: \(J = \bigoplus \limits_j (J \cap T_j)\). Furthermore, \(J\) is a \(U_q\mathfrak{g}\)-submodule of \(T\) (due to the covariance of the algebra \(A\), the module \(M\) and the order one calculus \((M, d)\)).

Hence the quotient algebra \(\Omega = T/J\) with the differential \(d_J : t + J \mapsto dt + J\) is a covariant graded differential algebra. It is easy to show that \(A \simeq \Omega_0, M \simeq \Omega_1\), and the initial differential \(d : A \to M\) "coincides" with the restriction of \(d_J\) onto \(\Omega_0\).

The five order one differential calculi we have already produced lead to five covariant graded differential algebras

\[ (\bigwedge_q^{(0, -1)}, \partial), \quad (\bigwedge_q^{(0, -1)}, \overline{\partial}), \quad (\Omega_q^{(0, 0)}, \partial), \quad (\Omega_q^{(0, 0)}, \overline{\partial}), \quad (\Omega_q, d). \]

In the example \(\mathfrak{g} = \mathfrak{sl}_2\) the relations \(\partial^2(z^2) = \overline{\partial}^2((z^*)^2) = \partial(zdz^* - q^{-2}dz^*z) = 0\) imply (11.3).

15 Holomorphic bundles and Dolbeault complexes

Just as in Section 5, we choose a functional \(\mu \in \mathfrak{h}^*\) such that \(m_j = \mu(H_j) \in \mathbb{Z}_+\) for \(j \neq j_0\). The linear functional of this form \(\lambda_- \in \mathfrak{h}^*\) was already considered in Section 12.

Consider a \(U_q\mathfrak{g}\)-module \(V_-\(\mu\)\) and the associated "graded dual" module \(\Gamma_{\mu}\).
Recall that we use the comultiplication $\Delta^\text{op}$ to equip $V_-(0) \otimes V_-(\mu)$ and $V_-(\mu) \otimes V_-(0)$ with a structure of a $U_q\mathfrak{g}$-module. Also, the morphisms

$$\Delta_L : V_-(\mu) \rightarrow V_-(0) \otimes V_-(\mu); \quad \Delta_L : v_-(\mu) \mapsto v_-(0) \otimes v_-(\mu),$$

$$\Delta_R : V_-(\mu) \rightarrow V_-(\mu) \otimes V_-(0); \quad \Delta_R : v_-(\mu) \mapsto v_-(\mu) \otimes v_-(0),$$

(together with the adjoint linear maps $\Delta^*_L, \Delta^*_R$) are used to equip $\Gamma_\mu$ with a structure of a covariant bimodule over $\mathbb{C}[\mathfrak{g}^{-1}]$:

$$\Delta^*_L : \mathbb{C}[\mathfrak{g}^{-1}] \otimes \Gamma_\mu \rightarrow \Gamma_\mu; \quad \Delta^*_R : \Gamma_\mu \otimes \mathbb{C}[\mathfrak{g}^{-1}] \rightarrow \Gamma_\mu.$$  

It follows from the properties of the universal R-matrix over $U_q\mathfrak{g}$ that

$$\sigma R^{-1} v_-(0) \otimes v_-(\mu) = v_-(\mu) \otimes v_-(0),$$

where $\sigma : a \otimes b \mapsto b \otimes a$, and $R^{-1}$ is the universal R-matrix of the Hopf algebra $U_q\mathfrak{g}^\text{op}$. Hence $\sigma R^{-1} \Delta_L = \Delta_R, \quad \Delta_L = R \sigma \Delta_R,$

$$\Delta^*_L = \Delta^*_R \cdot \hat{R}. \quad (15.1)$$

Here $\hat{R} : \mathbb{C}[\mathfrak{g}^{-1}] \otimes \Gamma_\mu \rightarrow \Gamma_\mu \otimes \mathbb{C}[\mathfrak{g}^{-1}], \quad \hat{R} = \sigma R.$

(15.1) shows how to describe the covariant bimodule $\Gamma_\mu$ in terms of generators and relations.

The standard construction (see Section 7) allows one to equip the tensor product $M_\mu = \Gamma_\mu \otimes \mathbb{C}[\mathfrak{g}^{-1}]_q$ with a structure of a covariant bimodule over $\text{Pol}(\mathfrak{g}^{-1})_q = \mathbb{C}[\mathfrak{g}^{-1}]_q \otimes \mathbb{C}[\mathfrak{g}^{-1}]_q.$

Consider the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. Denote by $\gamma_\mu$ the lowest weight vector of the $U_q\mathfrak{g}$-module $\Gamma_\mu$ such that $\gamma_\mu(v_-(\mu)) = 1$. Clearly $m_\mu = \gamma_\mu \otimes 1$ is a generator of a covariant bimodule $M_\mu$. It is not hard to deduce the complete list of "commutation" relations using the explicit form of the universal R-matrix (see Section 9):

$$z \cdot m_\mu = q^{-\mu(H)} \cdot m_\mu \cdot z, \quad z^* \cdot m_\mu = q^{\mu(H)} \cdot m_\mu \cdot z^*. \quad (15.2)$$

It is easy to prove that

$$K^\pm m_\mu = q^{\pm \mu(H)} m_\mu, \quad Fm_\mu = 0, \quad Em_\mu = -q^{1/2} \cdot \frac{1 - q^{2\mu(H)}}{1 - q^{-1}} \cdot z m_\mu.$$  

(The last equality follows from the covariance of the bimodule $M_\mu$ and the relations $Em_\mu = \text{const} \cdot z m_\mu, \quad FEm_\mu = -(EF - FE)m_\mu = -q^{\mu(H)} \frac{q^{\mu(H)} - q^{-\mu(H)}}{q - q^{-1}} m_\mu.$)

The elements of $M_\mu$ could be treated as q-analogues of smooth sections of a holomorphic vector bundle. We are interested in differential forms whose coefficients are such "sections".

Consider the covariant bimodule $\Omega^{(0,*)}_{\mu,q} = M_\mu \otimes_{\text{Pol}(\mathfrak{g}^{-1})_q} \Omega^{(0,*)}_{q}$ over $\text{Pol}(\mathfrak{g}^{-1})_q$. Evidently

$$\Omega^{(0,*)}_{\mu,q} = \Gamma_\mu \otimes_{\mathbb{C}[\mathfrak{g}^{-1}]_q} \Omega^{(0,*)}_{q} = \Gamma_\mu \otimes_{\mathbb{C}} \Lambda(\mathfrak{g}^{-1}). \quad (15.3)$$

Apply the $U_q\mathfrak{g}$-module morphism $\hat{R} : \mathbb{C}[\mathfrak{g}^{-1}]_q \otimes \Gamma_\mu \rightarrow \Gamma_\mu \otimes \mathbb{C}[\mathfrak{g}^{-1}]_q, \quad \hat{R} = \sigma R$ derived from the universal R-matrix to equip $\Omega^{(0,*)}_{\mu,q}$ with a structure of a covariant bimodule over $\Omega^{(0,*)}_q$. In the example $\mathfrak{g} = \mathfrak{sl}_2$ one can readily describe this module: the relation list (15.2) should be completed with one more relation

$$d z^* \cdot m_\mu = q^{\mu(H)} \cdot m_\mu \cdot dz^*.$$  

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It follows from (15.3) and $\mathcal{J}_\mathbb{C}[\mathfrak{g}_{-1}]_q = 0$ that the operator

$$\tilde{\partial}_\mu = \text{id} \otimes_{\mathbb{C}[\mathfrak{g}_{-1}]_q} \tilde{\partial} : \Gamma_\mu \otimes_{\mathbb{C}[\mathfrak{g}_{-1}]_q} \Omega^{(0,s)}_q \to \Gamma_\mu \otimes_{\mathbb{C}[\mathfrak{g}_{-1}]_q} \Omega^{(0,s)}_q.$$

Certainly, $\Omega^{(0,s)}_{\mu,q}$ is a graded bimodule $\Omega^{(0,s)}_{\mu,q} = \bigotimes_j \Omega^{(0,j)}_{\mu,q}$ over $\Omega^{(0,s)}_q$, and $\tilde{\partial}_\mu$ is its differentiation of order one:

$$\tilde{\partial}_\mu(am) = (\partial a)m + (-1)^{\deg a} \cdot a \cdot \tilde{\partial}_\mu m,$$
$$\tilde{\partial}_\mu(ma) = (\partial m)a + (-1)^{\deg m} \cdot m \cdot \tilde{\partial}a$$

for all homogeneous elements $a \in \Omega^{(0,s)}_q$, $m \in \Omega^{(0,s)}_{\mu,q}$.

Evidently, the differentiation $\tilde{\partial}_\mu$ is determined unambiguously by its values on generators.

In the example $\mathfrak{g} = \mathfrak{sl}_2$ for the generator $m_\mu \in M_\mu \hookrightarrow \Omega^{(0,s)}_{\mu,q}$ as above we have:

$$\tilde{\partial}m_\mu = 0.$$

Now pass to the homogeneous components

$$\Omega^{(0,j)}_{\mu,q} = M_\mu \otimes_{\text{Pol}([\mathfrak{g}_{-1}]_q)} \Omega^{(0,j)}_{\mu,q} = \Gamma_\mu \otimes_{\mathbb{C}[\mathfrak{g}_{-1}]_q} \Omega^{(0,j)}_q$$

of the graded bimodule $\Omega^{(0,s)}_{\mu,q}$ to obtain the Dolbeault complex

$$0 \to M_\mu \otimes_{\text{Pol}([\mathfrak{g}_{-1}]_q)} \Omega^{(0,1)}_{\mu,q} \otimes \Omega^{(0,2)}_{\mu,q} \otimes \Omega^{(0,3)}_{\mu,q} \otimes \ldots.$$ 

Its terms are the covariant bimodules over $\text{Pol}([\mathfrak{g}_{-1}]_q)$, and the differentials are the $U_q\mathfrak{g}$-module morphisms which commute with the left and the right actions of $\mathbb{C}[\mathfrak{g}_{-1}]_q$.

16 Conclusion notes.

Let us now digress from involutions and differentiations and sketch our approach to the construction of $q$-analogues of Hermitian symmetric spaces of non-compact type (one can find more details in sections 2 – 10).

Let $q = 1$. Evidently, for all $\xi \in \mathfrak{g}_{\pm 1}$ the series $\exp(\xi) v_+(0)$ converge in some "completed" spaces $\tilde{V}_+(0) = \bigotimes_j V_+(0)_j$. This allows one to elaborate the Harish-Chandra method to produce embeddings $I_\pm : X \hookrightarrow \tilde{V}_+(0)$ of an irreducible Hermitian symmetric space $X$. The canonical embeddings can be obtained from $I_\pm$ by composing them from the right with the projections $\pi_\pm : \tilde{V}_+(0) \to V_+(0)_{\pm 1} \cong \mathfrak{g}_{\pm 1}$.

Our basic observation is that the topological $U\mathfrak{g}$-modules $\tilde{V}_+(0)$ and hence the subalgebras $\mathfrak{g}_{\pm 1}$ have the proper quantum analogues $\mathfrak{g}_{\pm 1}$ defined as $V_+(0)_\pm$.

This allows one to imitate the above Harish-Chandra embeddings $i_\pm = \pi_\pm I_\pm$ for $q \neq 1$.

There is a different exposition of our construction for $q$-analogues of bounded symmetric domains and prehomogeneous vector spaces. It provides more clear interplay between our

\[5\text{Remind that, unlike } U\mathfrak{g}, the algebra } \mathfrak{g} \text{ itself has no "good" quantum analogues.}\]
construction with the approach of V. G. Drinfeld [6] to quantum groups as well as with the interpretation of the quantum Weyl group described by S. Z. Levendorskiı and Ya. S. Soibelman [16].

An alternate approach to introducing $\text{Pol}(\mathfrak{g}^{-1})_q$ is in producing a covariant involutive coalgebra and further passage to the dual covariant involutive algebra. This approach requires more detailed exposition of the "duality theory" for $U_q\mathfrak{g}$-module algebras and $U_q\mathfrak{g}^{op}$-module coalgebras. Specifically, we need to equip our algebras with the strongest locally compact topologies. The dual coalgebras are the completions of coalgebras considered in this work above, with respect to $W^*$-weak topologies, and their tensor products are replaced by the completed tensor products $\hat{\otimes}$ (see [13]). We describe here the topological covariant $^\ast$-coalgebra dual to $\text{Pol}(\mathfrak{g}^{-1})_q$. Remind (see section 6) that we replace $U_q\mathfrak{g}$ by $U_q\mathfrak{g}^{op}$ in tensor products of generalized Verma modules.

It is easy to show that the vector $v_0 = v_-(0) \otimes v_+(0)$ is a generator of the topological $U_q\mathfrak{g}$-module $V_0 = V_-(0) \otimes V_+(0)$. The structure of a covariant coalgebra in $V_0$ is imposed by introducing a $U_q\mathfrak{g}$-module morphism $\Delta : V_0 \to V_0 \hat{\otimes} V_0$ given by an application of a universal $R$-matrix:

$$\Delta v_0 = R v_0 \otimes v_0.$$  

The coassociativity of $\Delta$ follows from the quasitriangularity of the Hopf algebra $U_q\mathfrak{g}$. Impose an involution in $V_0$ by

$$(\xi v_0)^\ast = (S^{-1}(\xi))^\ast v_0, \quad \xi \in U_q\mathfrak{g},$$

which already implies

$$(\xi v)^\ast = (S^{-1}(\xi))^\ast v^\ast, \quad \xi \in U_q\mathfrak{g}, \ v \in V_0.$$  

(Note that (7.1) provides $^\ast$ to be an antilinear coalgebra antihomomorphism of $V_0$).

Consider the maps

$$\varepsilon_- \otimes \text{id} : V_0 \to V_+(0); \text{id} \otimes \varepsilon_+ : V_0 \to V_-(0),$$

with $\varepsilon_\pm$ being the counits of the coalgebras $\overline{V_\pm(0)}$.

It follows from the relations

$$(\varepsilon \otimes \text{id})(R) = (\text{id} \otimes \varepsilon)(R) = 1$$

that these maps are the morphisms of covariant coalgebras (dual to the embeddings $\mathbb{C}[\mathfrak{g}^{-1}]_q \hookrightarrow \text{Pol}(\mathfrak{g}^{-1})_q, \, \mathbb{C}[\mathfrak{g}^{-1}]_q \hookrightarrow \text{Pol}(\mathfrak{g}^{-1})_q$).

The relation

$$R(v_0 \otimes v_0) = v_-(0) \otimes R\sigma(v_-(0) \otimes v_+(0)) \otimes v_+(0)$$

with $\sigma : a \otimes b \mapsto b \otimes a$, demonstrates that the comultiplication $\Delta$ agrees with the multiplication in $\text{Pol}(\mathfrak{g}^{-1})_q$ introduced in Section 7.

Finally, let us note that the commutation relations between the elements of $(\mathbb{C}[\mathfrak{g}^{-1}]_q)_+$ and $(\mathbb{C}[\mathfrak{g}^{-1}]_q)_-$ are of degree at most two, as it follows from the properties of the universal $R$-matrix.
Appendix. Images of the differentials $\partial, \overline{\partial}$

Consider the coalgebra $V_-(0)$ and the comodule $V_-(\lambda_-)$ and disregard for a moment their $U_q\mathfrak{g}$-module structures.

**Lemma 1.** The left comodule $V_-(\lambda_-)$ over the coalgebra $V_-(0)$ is isomorphic to a direct sum of $m$ copies of $V_-(0)$, with $m = \dim V_-(\lambda_-)_1$.

**Proof.** Remind the decomposition $w_0 = w'_0 \cdot u$, with $w_0$ and $w'_0$ being the maximum length elements in the Weyl group of Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ respectively (see Section 5). It was our agreement to consider only those reduced decompositions of $w_0$ which are given by concatenation of reduced decompositions for $w'_0$ and $u$.

Choose such a decomposition. Just as in [4, Proposition 1.7(c)], associate to it the bases in $U_q\mathfrak{g}'$, $U_q\mathfrak{g}$ and the bases of the vector spaces

$$V_-(0) = U_q\mathfrak{g}v_-(0); \quad V_-(\lambda_-)_1 = U_q\mathfrak{g}'v_-(\lambda_-); \quad V_-(\lambda_-) = U_q\mathfrak{g}v_-(\lambda_-).$$

(One can verify that $(V_-(\lambda_-))_1$ is a simple $U_q\mathfrak{g}'$-module). The above bases are of the form

$$\{\xi_i v_-(0)\}, \quad \{\eta_j v_-(\lambda_-)\}, \quad \{\xi_i \eta_j v_-(\lambda_-)\},$$

with $i \in \mathbb{Z}_+$, $j \in \{1, \ldots, m\}$, and $\xi_i \in U_q\mathfrak{g}$, $\eta_j \in U_q\mathfrak{g}'$ can be derived from the bases of $U_q\mathfrak{g}$ and $U_q\mathfrak{g}'$ described explicitly in [4]. It remains to prove that for each $j$ the map

$$\pi_j : \xi_i v_-(0) \mapsto \xi_i \eta_j v_-(\lambda_-), \quad i \in \mathbb{Z}_+$$

is a morphism of left comodules. So

$$\Delta^L \pi_j(\xi_i v_-(0)) = \Delta(\xi_i) \Delta^L(\eta_j v_-(\lambda_-)) = \Delta(\xi_i)(v_-(0) \otimes \eta_j v_-(\lambda_-)) = \text{id} \otimes \pi_j \Delta(\xi_i v_-(0)).$$

**Lemma 2.** Let $v \in V_-(0)$. Then $\Delta_-(v) \in v_-(0) \otimes V_-(0)$ if and only if $v \in \mathbb{C}v_-(0)$.

**Proof.** If $v = \text{const} \cdot v_-(0)$, then $\Delta_-(v) = v_-(0) \otimes \text{const} \cdot v_-(0) \in v_-(0) \otimes V_-(0)$. Conversely, if $\Delta_-(v) = v_-(0) \otimes v_1$, $v_1 \in V_-(0)$, then

$$v = (\text{id} \otimes \varepsilon_-) \Delta_-(v) = (\text{id} \otimes \varepsilon_-)(v_-(0) \otimes v_1) = \varepsilon_-(v_1) \cdot v_-(0) \in \mathbb{C} \cdot v_-(0).$$

**Lemma 3.** Let $v \in V_-(\lambda_-)$. Then $\Delta^L_-(v) \in v_-(0) \otimes V_-(\lambda_-)$ if and only if $v \in V_-(\lambda_-)_1$.

**Proof.** Let $L = \{v \in V_-(\lambda_-) \mid \Delta^L_-(v) \in v_-(0) \otimes V_-(\lambda_-)\}$. Evidently, $L \supset V_-(\lambda_-)_1$, and by lemmas 1,2, $\dim L = \dim V_-(\lambda_-)_1$. It follows that $L = V_-(\lambda_-)_1$.

**Remark 4.**

If $\Delta^L_-(v) = v_-(0) \otimes v_1$, then $v_1 = v$ since $v_1 = (\varepsilon \otimes \text{id})(v_-(0) \otimes v_1) = (\varepsilon \otimes \text{id}) \Delta^L_-(v)$.

**Lemma 5.** The restriction of $\delta_-$ onto $(V_-(\lambda_-))_1$ is an injective linear operator.

**Proof.** This operator is non-zero and is a $U_q\mathfrak{g}$-module morphism $(V_-(\lambda_-))_1 \to V_-(0)$, where $(V_-(\lambda_-))_1$ is a simple $U_q\mathfrak{g}'$-module.
Proposition 6. $\mathbb{C}[g^{-1}]_q \cdot \partial \mathbb{C}[g^{-1}]_q = \wedge^1 (g^{-1})_q$.

Proof. Assume the contrary. Let $V' = \{ v \in V_-(\lambda_{-}) | \langle f_1 \partial f_2, v \rangle = 0, \forall f_1, f_2 \in \mathbb{C}[g^{-1}]_q \}$. Then $V' = \bigoplus_{i \in \mathbb{Z}_+} V'_{-i} \neq 0$. It follows from the definitions that $\Delta_L(V') \subset V_-(0) \otimes V'$.

Let $i'$ be the least such $i \in \mathbb{Z}_+$ that $V'_{-i} \neq 0$. We have

$$\Delta^L(V'_{-i'}) \subset (\mathbb{C}v_{-}(0) \bigoplus_{k>0} V_{-}(0)_{-k}) \otimes V' \subset v_{-}(0) \otimes V_{-}(\lambda_{-}) + (\bigoplus_{k>0} V_{-}(0)_{-k}) \otimes V'.$$

On the other hand, $(\bigoplus_{k>0} V_{-}(0)_{-k}) \otimes V'_{-i'} = 0$, and hence $\Delta^L(V'_{-i'}) \subset v_{-}(0) \otimes V_{-}(\lambda_{-})$. Now we deduce from lemma 3 that $V'_{-i'} = V_{-}(\lambda_{-})_{-1} \cap V'$.

Let $v' \in V_{-}(\lambda_{-})_{-1} \cap V'$. It follows from the definition of $V'$ that $(\text{id} \otimes \delta_{-})\Delta^L(v') = 0$. On the other hand, by lemma 3 and remark 4 we observe that $\Delta_{L}(v') = v(0) \otimes v'$. Hence $(\text{id} \otimes \delta_{-})(v(0) \otimes v') = 0$. That is, $\delta_{-}(v') = 0$ and hence, by lemma 5, $v' = 0$.

Thus we have proved that $V'_{-i'} = 0$ which makes a contradiction to the contrary of proposition 6.

Remark 7.

One can prove in a similar way that

$$\overline{\mathbb{C}[g^{-1}]_q} \cdot \mathbb{C}[g^{-1}]_q = \mathbb{C}[g^{-1}]_q \cdot \overline{\mathbb{C}[g^{-1}]_q} = \mathbb{C}[g^{-1}]_q \cdot \wedge^1 (g^{-1})_q,$$

$$\partial \mathbb{C}[g^{-1}]_q \cdot \mathbb{C}[g^{-1}]_q = \wedge^1 (g^{-1})_q.$$

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