Stationary Solutions to the Stochastic Burgers Equation on the Line

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Abstract: We consider invariant measures for the stochastic Burgers equation on \( \mathbb{R} \), forced by the derivative of a spacetime-homogeneous Gaussian noise that is white in time and smooth in space. An invariant measure is indecomposable, or extremal, if it cannot be represented as a convex combination of other invariant measures. We show that for each \( a \in \mathbb{R} \), there is a unique indecomposable law of a spacetime-stationary solution with mean \( a \), in a suitable function space. We also show that solutions starting from spatially-decaying perturbations of mean-\( a \) periodic functions converge in law to the extremal space-time stationary solution with mean \( a \) as time goes to infinity.

1. Introduction

The stochastic Burgers equation on the line. We consider strong solutions \( u(t, x) \) to the stochastic Burgers equation written formally as

\[
\partial_t u + \frac{1}{2} \partial_x (u^2) = \frac{1}{2} \partial_x^2 u + \partial_x \dot{V}, \quad t, x \in \mathbb{R}.
\] (1.1)

Here, the potential \( \dot{V} \) is a spatial smoothing, by a symmetric mollifier \( \rho \in \mathcal{C}^\infty(\mathbb{R}) \cap H^1(\mathbb{R}) \), of a space-time Gaussian white noise \( \dot{W} \):

\[
\dot{V}(t, x) = (\rho \ast \dot{W})(t, x),
\] (1.2)

where

\[
\mathbb{E}[\dot{W}(t, x)\dot{W}(t', x')] = \delta(t - t')\delta(x - x').
\]

In (1.2) and throughout the paper, \( \ast \) denotes spatial convolution. We will often use the notation \( \rho^{*2} = \rho \ast \rho \).

To be more precise, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a standard probability space and let \( W = W(t, x) \) be a cylindrical Wiener process on \( L^2(\mathbb{R}) \) whose covariance operator is the identity.
This is discussed, for example, in [22, Sect. 4.3.1]. Let \( \{ \mathcal{F}_t \}_{t \geq 0} \) be the usual filtration corresponding to \( W \), so that \( \mathcal{F}_t \subset \mathcal{F} \) is the \( \sigma \)-algebra generated by \( W_{[0,t] \times \mathbb{R}} \). We do not assume that \( \mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t \): we will freely define additional random variables throughout the paper which are independent of the noise \( W \), and will always assume that \( \Omega \) is large enough to include such random variables. The Itô time differential \( dW_t \) is thus a white noise on \( \mathbb{R} \times \mathbb{R} \). The random field \( V(t, x) = (\rho \ast W)(t, x) \) is a Gaussian process on \( \mathbb{R} \times \mathbb{R} \) with a continuous modification, which is in fact spatially smooth since \( \rho \) is smooth. We will always assume that we are working with this modification, and refer the reader to Sect. 2.1 below for more details. We interpret the equation (1.1) as

\[
\frac{du}{dt} = \frac{1}{2} \left[ \partial_x^2 u - \partial_x (u^2) \right] dt + d(\partial_x V), \quad t, x \in \mathbb{R}. \tag{1.3}
\]

The random Gaussian forcing \( V \) is not uniformly bounded in space, and so neither will be solutions to (1.3). Indeed, this would be the case even without the nonlinear term in (1.3). Thus, one needs to work with (1.3) in weighted function spaces that permit spatial growth. To the best of our knowledge, previous work has not considered the well-posedness of (1.3) in spaces allowing as much growth as we require, so let us first state the existence and uniqueness result we will need. We denote by \( \mathcal{X}_m \) the space of continuous functions on \( \mathbb{R} \) growing at infinity slower than \( |x|^{\ell} \) for all \( \ell > m \), equipped with an appropriately weighted topology described in Sect. 2.2. Our first result is that the equation (1.3) is well-posed in \( \mathcal{X}_m \) as long as \( m < 1 \). The restriction \( m < 1 \) is necessary: if solutions are allowed to grow linearly, then characteristics starting at infinity may reach the origin in a finite time, even in the absence of noise. This is related (by the Cole–Hopf transform, discussed below) to the familiar restriction that the initial condition for the standard heat equation should grow more slowly than \( \exp(|x|^2) \) at infinity.

In the following theorem, we assume that the initial condition for (1.3) is continuous. We will also need to consider discontinuous initial data, but as the spaces involved become more complicated, we defer the more general statement to Proposition 2.2 in Sect. 2.2.

**Theorem 1.1.** Let \( m \in (0, 1) \). With probability 1, there is a map

\[
\Psi : \mathcal{X}_m \to C_{\text{loc}}([0, \infty); \mathcal{X}_m)
\]

so that for each \( v \in \mathcal{X}_m \), \( u = \Psi(v) \) is the unique strong solution in \( C_{\text{loc}}([0, \infty); \mathcal{X}_m) \) to (1.3) with \( u(0, x) = v(x) \). The map \( \Psi \) is continuous almost surely. Finally, the semigroup \( P_t f(v) = \mathbb{E}[f(u(t, \cdot))] \) has the Feller property: if \( f \) is a bounded continuous function on \( \mathcal{X}_m \), then so is \( P_t f \) for any \( t > 0 \).

A proof of Theorem 1.1, as well as Proposition 2.2 handling discontinuous initial data, occupies Sect. 2.

**Space-time stationary solutions: existence and stability.** Our main interest is in solutions to (1.3) that are statistically stationary under both the time evolution and translations in space. We will need to consider invariant measures for ensembles of solutions \( u = (u_1, \ldots, u_N) = u(t, x) \) to (1.3), satisfying

\[
du_i = \frac{1}{2} \left[ \partial_x^2 u_i - \partial_x (u_i^2) \right] dt + d(\partial_x V), \quad t, x \in \mathbb{R}, \quad i = 1, \ldots, N. \tag{1.4}
\]
These equations are decoupled. The solutions \( u_1, \ldots, u_N \) have different initial conditions but are all subject to the same noise. It may often be convenient for the reader to think of the case \( N = 1 \), which corresponds to a single initial condition. We will use the \( N > 1 \) case to prove some statements for families of coupled solutions that we will need, in particular, for the ordering results below.

Let us first define precisely what we mean by invariant measures. Let \( \mathcal{P}(X^N_m) \) be the space of probability measures on \( X^N_m \), and for each \( v \in \mathcal{P}(X^N_m) \) and \( t \geq 0 \), let \( P_t^{*}v = \text{Law}(u(t, \cdot)) \). Here, \( u \) is a solution to (1.4) with initial condition \( u(0, \cdot) \sim v \). When we consider such solutions, we always assume that the noise \( V \) is independent of the random initial condition \( u(t, \cdot) \). The set of invariant measures under (1.4) is

\[
\mathcal{P}(X^N_m) = \{ v \in \mathcal{P}(X^N_m) : P_t^{*}v = v \}.
\]

To formulate the spatial translation invariance, we first define, for \( x \in \mathbb{R} \) and \( v = (v_1, \ldots, v_N) : \mathbb{R} \to \mathbb{R}^N \), the translation operator on \( X^N_m \) as

\[
\tau_x v(y) = (v_1(y-x), \ldots, v_N(y-x)),
\]

and the corresponding operator \((\tau_x)^{*} \) on \( \mathcal{P}(X^N_m) \). For \( G = \mathbb{R} \) or \( G = L\mathbb{Z} \) for some \( L > 0 \), we set

\[
\mathcal{P}_G(X^N_m) = \{ v \in \mathcal{P}(X^N_m) : (\tau_x)^{*}v = v \text{ for all } x \in G \},
\]

the space of probability measures on \( X^N_m \) that are invariant under the action of \( G \). We also define the corresponding invariant measures under (1.4):

\[
\overline{\mathcal{P}}_G(X^N_m) = \mathcal{P}(X^N_m) \cap \mathcal{P}_G(X^N_m).
\]

The space \( \overline{\mathcal{P}}_G(X^N_m) \) is the space of invariant measures corresponding to “space-time stationary solutions.” Here, spatial stationarity is understood as either with respect to all spatial translations if \( G = \mathbb{R} \), or as \( L \)-periodicity if \( G = L\mathbb{Z} \).

The space \( \overline{\mathcal{P}}_G(X^N_m) \) is convex, and we denote by \( \overline{\mathcal{P}}_G^e(X^N_m) \) the set of its extremal elements. We note that extremality is equivalent to the ergodicity property that if \( A \subset X^N_m \) is a Borel subset such that \( \tau_x A = A \) for all \( x \in G \) and \( P_t 1_A = 1_A \) \( \mu \)-a.s. for all \( t \geq 0 \), then either \( \mu(A) = 0 \) or \( \mu(A) = 1 \). The corresponding equivalence for measures that are invariant under a Markov semigroup can be found in Theorem 5.1 of [33]. Our measures are invariant both under a Markov semigroup and a group of translations. In that case, the equivalence of extremality and ergodicity can be proved using the corresponding statement for invariant measures under a set of maps in Proposition 12.4 of [49], along with the equivalence proved in Corollary 5.3 of [33]. On an intuitive level, both extremality and ergodicity are ways to formalize indecomposability – such invariant measures represent the “building blocks” of the possible long time behaviors.

To formulate our result on the existence, uniqueness and properties of the extremal invariant measures, we will use an auxiliary random variable \( X \), depending on \( G \). If \( G = \mathbb{R} \), we let \( X = 0 \) a.s., and if \( G = L\mathbb{Z} \) with some \( L > 0 \), then \( X \sim \text{Uniform}(0, L) \).

**Theorem 1.2.** Fix \( m \in [1/2, 1) \), \( N \in \mathbb{N} \), and \( G = \mathbb{R} \) or \( G = L\mathbb{Z} \) for some \( L > 0 \). The random variable \( X \) is independent of all other random variables. For each \( a \in \mathbb{R}^N \), there exists a unique \( v_a \in \overline{\mathcal{P}}_G^e(X^N_m) \) such that if \( v = (v_1, \ldots, v_N) \sim v_a \), then \( E[v(X)] = a \) and \( E|v(X)|^2 < \infty \). Moreover, the measures \( v_a \) satisfy the following properties.
(P1) Order: with probability one, we have
\[ \text{sgn}((v_j - v_k)(x)) = \text{sgn}(a_j - a_k) \]
for each \( j, k \in \{1, \ldots, N\} \) and all \( x \in \mathbb{R} \).

(P2) Shear invariance: if \( c \in \mathbb{R} \), then \( v_{a+c} = \text{Law}(v + (c, \ldots, c)) \), where \( v \sim v_a \).

(P3) G-independence: the measure \( v_a \) is also an element of \( \mathcal{P}_G'('X^N_1') \) for \( G' = \mathbb{R} \)
and \( G' = L^*\mathbb{Z} \), for all \( L' > 0 \).

(P4) If \( v \sim v_a \) then with probability 1, \( v \) is (spatially) smooth, and \( v \) and all of its
derivatives grow at most polynomially at infinity.

(P5) If \( v_a = (1 - q)\mu_0 + q\mu_1 \) for some \( q \in (0, 1) \) and measures \( \mu_0, \mu_1 \in \mathcal{P}_G('X^N_m') \),
and \( v \sim \mu_0 \), then \( \mathbb{E}v(x) = a \).

Properties (P1) and (P2) in the above theorem have a clear intuitive meaning: space-
time stationary solutions are ordered and have shear invariance. Property (P4) is simply
a reflection of the spatial parabolic smoothing of the viscous Burgers equation. It does
not, of course, extend to smoothness in time if \( a \) is allowed to evolve under the dynamics:
this is precluded by the presence of the temporally-rough forcing \( dV \). As for (P3), let
us note that, a priori, the extremal property of an element of \( \mathcal{P}_G('X^N_m') \) depends on
the group \( G \) of translations. Even if \( G' \subset G \) so that \( \mathcal{P}_G('X^N_m') \subset \mathcal{P}_G'('X^N_m') \), one might
worry that an extremal element of \( \mathcal{P}_G('X^N_m') \) is not necessarily extremal in \( \mathcal{P}_G'('X^N_m') \),
since the latter is potentially a larger set. Property (P3) rules out this situation. Finally,
property (P5) is a weak version of ergodicity under just \( G \) (not under the dynamics): if
\( \mu \in \mathcal{P}_G('X^N_m') \) can be decomposed into multiple measures that are \( G \)-invariant, even if
not \( P_1 \)-invariant, then those measures must have the same mean.

We emphasize that the properties (P1)–(P5) in Theorem 1.2 are not part of the unique-
ness statement. That is, to know that an extremal invariant measure \( v \in \mathcal{P}_G('X^N_m') \) is
equal to \( v_a \), we need only know that \( \mathbb{E}v(X) = a \) and \( \mathbb{E}|v(X)|^2 < \infty \) for \( v \sim v \). In
particular, for any \( v \in \mathcal{P}_G('X^N_m') \) such that \( \mathbb{E}|v(X)|^2 < \infty \) for \( v \sim v \), there is an \( a \in \mathbb{R}^N \)
so that \( v = v_a \).

Let us next discuss the long-time behavior of the dynamics (1.4). We denote by
\( L^\infty(\mathbb{R}/L\mathbb{Z}) \) the space of \( L \)-periodic functions in \( L^\infty(\mathbb{R}) \). We will prove a stability result
for initial conditions lying in spaces of functions that can be bounded above and below
by periodic functions whose averages can be made arbitrarily close to each other. The
following definition states this precisely. We will use the partial order \( \leq \) on \( \mathbb{R}^N \) defined by
\begin{equation}
(x_1, \ldots, x_N) \leq (y_1, \ldots, y_N) \iff x_i \leq y_i \text{ for each } i = 1, \ldots, N.
\end{equation}
Once again, it may be convenient for the reader to think of the case \( N = 1 \).

**Definition 1.3.** For \( a \in \mathbb{R}^N \), we denote by \( \mathcal{B}_a \) the set of all \( v \in L^\infty(\mathbb{R})^N \) such that
for every \( \varepsilon > 0 \), there exists an \( L_\varepsilon \in (0, \infty) \) and \( v_\varepsilon^-, v_\varepsilon^+ \in L^\infty(\mathbb{R}/L\mathbb{Z})^N \) so that
\( v_\varepsilon^- \leq v \leq v_\varepsilon^+ \) and
\begin{equation}
\frac{1}{L_\varepsilon} \int_0^{L_\varepsilon} v_\varepsilon^+(x) \, dx - (\varepsilon, \ldots, \varepsilon) \leq a \leq \frac{1}{L_\varepsilon} \int_0^{L_\varepsilon} v_\varepsilon^-(x) \, dx + (\varepsilon, \ldots, \varepsilon).
\end{equation}

The following proposition gives a reasonably general sufficient condition for a func-
tion to be in \( \mathcal{B}_a \).
Proposition 1.4. Suppose that a function $v \in L^\infty(\mathbb{R})^N$ can be written as $v = v_{\text{per}} + v_{\text{int}} + v_z$, where $v_{\text{per}} \in L^\infty(\mathbb{R}/\mathbb{Z})^N$ for some $L \in (0, \infty)$, $v_{\text{int}} \in (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$, and $v_z \in L^\infty(\mathbb{R})^N$ is such that

$$\lim_{|x| \to \infty} |v_z(x)| = 0.$$ 

Then $v \in \mathcal{B}_a$, where $a = \frac{1}{L} \int_0^L v_{\text{per}}(x) \, dx$.

We can now state our stability result.

Theorem 1.5. Let $m \in [1/2, 1)$, $a \in \mathbb{R}^N$, and $u$ be a solution to (1.4) with initial condition $v \in \mathcal{B}_a$. Then we have

$$\lim_{t \to \infty} \text{Law}(u(t, \cdot)) = v_a$$

in the sense of weak convergence of probability measures on $\mathcal{X}^N_m$.

We note that (1.9) can be upgraded to convergence as probability measures on spaces of higher regularity using parabolic regularity estimates. Because the norms involved become rather complicated, we direct the reader to Lemma 2.5 below.

Results similar to ours were obtained by Bakhtin and Li in [7], using completely different methods. In that paper, the authors considered (1.3) with driving noise $V$ that is not a Wiener process but rather a step process that jumps at integer times. This means that the solution only feels “kicks” at integer times, rather than white-in-time forcing. Their approach considers the question from the point of view of directed polymers. In addition to what we prove, they show that if the solution is started at a negative time $-T$, then as $T \to \infty$ the solution at time 0 converges almost surely to a stationary initial condition (the one-force-one-solution principle). They also prove somewhat larger basins of attraction than those described in Definition 1.3 (including, in particular, rarefaction waves). However, their proof uses the properties of the kick forcing in a serious way, and an adaptation to the white-in-time case is not clear. Our work extends many of the results of [7] to the white-in-time setting, and provides a completely different, PDE-based perspective on the problem.

The work [7] is part of a two-decade-long program to understand the attractors of the stochastic Burgers equation using the Lax–Oleinik formula in the inviscid case or directed polymers in the viscous case; see e.g. [3,4,6,16,26,40] and the reviews [5,8]. Part of the motivation for this program is the goal of understanding the KPZ universality phenomenon, as the equation is conjectured to lie in the KPZ universality class. We refer to [5] for more details and a fascinating discussion.

Our setting and PDE-based approach are closely related to those considered by Boritchev in [14] on the one-dimensional torus, with a multi-dimensional extension in [15], and a general review given in [13]. In particular, [14] establishes the existence and uniqueness of invariant measures for (1.3) on the torus. Existence of such measures was previously shown in [20]; see also [19] for the case when the noise is also white in space. As in the present paper, the classification of stationary solutions in [14] is based on contractive properties of the Burgers equation. However, [14] uses $L^1$-contraction and a maximum principle for $\partial_x u$ to establish a Doeblin-type condition and show that all mean-0 solutions must converge to the unique mean-0 stationary solution. This relies on the compactness of the domain in important ways. In the whole space, instead of using
Poincaré inequality-type ideas, we show that any invariant measures \( \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}_m) \) have a coupling \( \nu \in \mathcal{P}(\mathcal{X}_m^2) \), and, moreover, that if \( \mathbf{v} \sim \nu \), then the components of \( \mathbf{v} \) are ordered almost surely. This ordering allows us to classify the laws of extremal stationary solutions. It is here that using \( N > 1 \) in (1.4) becomes crucial.

The stochastic Burgers equation with unmollified spacetime white noise, or the spatial gradient of spacetime white noise, has also been the subject of significant interest in the literature. Much of this work, such as [9, 19, 31, 32, 34, 38], principally concerns well-posedness for the equation, which is of course a more difficult problem when the noise is spatially rough than when it is smooth. Well-posedness of the equation driven by the spatial gradient of spacetime white noise is essentially the same problem as the well-posedness of the KPZ equation driven by spacetime white noise, as considered in, for example, [11, 29, 35, 48]. Ergodicity properties for the stochastic Burgers equation with singular forcing on a compact domain are considered in [30, 50].

In a different direction, the papers [9, 17, 31, 32, 44, 48, 55–57, 59] consider the stochastic Burgers or KPZ equations on the whole space, but with initial conditions and/or noise that are constrained to be growing more slowly than we need to treat forcing by spacetime stationary Gaussian fields. Most of these works assume that the initial condition and/or noise are in some \( L^p(\mathbb{R}) \) space, which does not apply to the space-stationary setting. The works [9, 48, 59] assume that the integral of the Burgers solution grows at most linearly at infinity; this will be true for our stationary solutions by Birkhoff’s ergodic theorem, but we do not obtain the quantitative control on such growth that would be required to use results of this type. The work [55] considers the KPZ equation in spaces with “locally bounded averages,” a condition which again does not readily correspond to the estimates we obtain. The work [43] proves the existence of invariant measures for the stochastic Burgers equation with non-gradient-type noise but with a zero-order dissipation term to provide compactness and remove the potentially growing low frequencies.

The Cole–Hopf transform, connection to the KPZ equation, and compactness. In addition to the PDE arguments, the proof of the uniform bounds for the solutions of the stochastic Burgers equation requires one crucial application of the Feynman-Kac formula. By the Cole–Hopf transform [9, 18, 39]

\[
h = -\log \phi, \quad u = \partial_x h = -\partial_x \phi / \phi,
\]

the stochastic Burgers equation (1.3) is closely related to the KPZ equation [42]

\[
dh = \frac{1}{2} \left[ \partial_x^2 h - (\partial_x h)^2 + \| \rho \|_{L^2(\mathbb{R})}^2 \right] dt + dV
\]

and the multiplicative stochastic heat equation

\[
d\phi = \frac{1}{2} \partial_x^2 \phi - \phi dV,
\]
in which the last product is interpreted in an Itô sense. Here, \( \rho(x) \) is the mollifier in (1.2).

The fact that the results of the transformation (1.10) indeed satisfy the claimed PDEs is a computation using Itô’s formula (see e.g. [22, Theorem 4.17]). Note that, because we work with noise that is spatially smooth, the Cole–Hopf transform requires no infinite renormalization as in the white in time and space case [11, 35], but simply the finite Itô
correction given by the term \( \frac{1}{2} \| \rho \|_{L^2(\mathbb{R})}^2 \) in (1.11), which is half the derivative of the quadratic variation of the process \( t \mapsto V(t, x) \) for fixed \( x \). The Cole–Hopf transform is a common tool in the study of the stochastic heat, KPZ, and Burgers equations. In particular, the Cole–Hopf transform explains why it is natural to take the forcing in (1.3) to be the gradient of a random field, which is crucial for the existence of space-time stationary solutions.

Because of the close relationship between (1.3), (1.11), and (1.12), one might naively expect that stationary solutions for one of the equations induce stationary solutions for the others. However, this works only in one direction, because information is lost when taking the spatial derivative to pass from \( h \) to \( u \). That is, stationarity of \( u \) does not imply stationarity of its antiderivative \( h \). In one and two spatial dimensions, neither (1.11) nor (1.12) is expected to admit stationary solutions, as the pointwise statistics of solutions started from constant initial conditions diverge. The situation is different in three or more spatial dimensions, in which, if the noise \( V \) is sufficiently small, the multiplicative stochastic heat equation admits nonzero stationary solutions [24, 25, 47, 53].

The relationship with the KPZ equation is important in our proof strategy, as we now describe. To prove the existence of stationary solutions for the Burgers equation, we first establish a form of compactness. The proof of this starts by taking expectations in the KPZ equation (1.11). Since \( u = \partial_x h \), the nonlinear term in (1.11) is \( u^2 \), so second moments of solutions to (1.3) are related to the growth of \( E h \). Asymptotically, \( E h \) is \( t \) times the Lyapunov exponent for the stochastic heat equation (1.12), corresponding to the linear-in-time drift in the solution to the KPZ equation with white-noise forcing; see for instance [2, 12, 27, 51, 54]. More importantly for our purposes, \( E h \) can be shown to be increasing and concave, as a function of time, using the Feynman–Kac formula. We prove this in Proposition 5.2, the only part of our work that relies on the Feynman–Kac formula. At the moment, we do not know how to replace this use of the Feynman–Kac formula by a purely PDE argument. With the second moment bound in hand, we obtain a tightness statement that implies that \( u \) converges along subsequences of time-averaged laws of solutions to (1.4). Limits of such subsequences can be shown by the Krylov–Bogoliubov theorem (see Proposition 4.2) to be stationary in time.

While bounding the one-point variance of solutions to (1.3) is crucial to our proof, we do not say anything about the multipoint correlations of solutions. It is expected that stationary solutions to (1.3) should have correlation functions that are integrable in space, so that, when rescaled appropriately, the solutions approach a white noise process. To our knowledge, this question, which is related to KPZ universality, has not been resolved for the stochastic Burgers equation with any kind of spatially smooth noise. In [28, 37], a different regularization of the spacetime-white-noise-forced Burgers equation is considered for which this statement is clear.

**Shear-invariance, ordering and \( L^1 \)-contraction for the Burgers equation.** The three key ingredients to the classification of stationary solutions are the shear invariance, ordering, and \( L^1 \)-contraction properties of the Burgers equation. All three are analogues of well-known properties of the deterministic Burgers equation in the absence of random forcing.
The deterministic shear invariance simply says that if \( u(t, x) \) is a solution to (1.1) with \( V = 0 \) then \( u_c(t, x) = u(t, x + ct) \) is also a solution, for any \( c \in \mathbb{R} \). The shear invariance in law of (1.4) with a random forcing is the following property. Suppose that \( u = (u_1, \ldots, u_N) \) solves (1.4) and define \( \tilde{u}(t, x) = u(t, x + ct) - (c, \ldots, c) \). Then it is easy to see that

\[
\partial_t \tilde{u}_i = \frac{1}{2} \partial_x^2 \tilde{u}_i - \frac{1}{2} \partial_x (u_i^2) + d(\partial_x \tilde{V}),
\]

where \( \tilde{V}(t, x) = V(t, x + ct) \). Since \( d(\partial_x V) \) is white in time, informally speaking \( d(\partial_x \tilde{V}) \) and \( d(\partial_x V) \) have the same law. Therefore, \( \tilde{u} \) agrees in law with a solution to (1.4). This is made precise in Sect. 4.1. On the other hand, if \( u(t, \cdot) \) is space-stationary, then \( \tilde{u}(t, \cdot) \) has the same law as \( u(t, \cdot) - (c, \ldots, c) \). This directly leads to statement (P2) in Theorem 1.2. It also allows us, once we have constructed a single invariant measure, to construct many by vertical translation.

The ordering and \( L^1 \)-contraction properties for the random Burgers equation are closely related, as in the deterministic case. Informally speaking, in Theorem 3.8, we show that space-time stationary solutions to (1.3) are ordered. This is not an immediate consequence of the standard comparison principle because we cannot a priori pin down any fixed time when we would easily compare the two solutions and claim that this order propagates. A precise formulation of the ordering of the solutions is that the components of a space-time stationary solution to (1.4) must be ordered almost surely. In addition, we show in Proposition 4.3 that any two laws of space-time stationary solutions to (1.3) or (1.4) can be coupled to obtain another space-time stationary solution to (1.4), with more components. This implies that there cannot be two distinct elements of \( \mathcal{F}_R^e(\mathcal{X}_m) \) with the same mean. The ordering is a consequence of the comparison principle and \( L^1 \)-contraction for the Burgers equation, which we discuss in Sect. 3. To prove the ordering statement, we show that two components of a space-time-stationary solution to (1.4) cannot intersect transversely, as that would reduce an \( L^1 \)-norm. Then we use the strong maximum principle to rule out degenerate intersections.

To prove the convergence of the solutions to an invariant measure, we again use the \( L^1 \)-contraction property of the Burgers equation. Under appropriate conditions, two solutions evolving according to the same noise must get close to one another at many times. Then, intuitively, the \( L^1 \)-contraction forces them to stay close to each other for all times. Of course, the difference of two space-stationary solutions is generally not in \( L^1(\mathbb{R}) \), so here the \( L^1 \)-contraction is used on the probability space. The \( L^1 \) contraction property on the probability space is analogous to but different from the standard spatial \( L^1 \)-contraction and holds for spatially invariant solutions.

**Organization of the paper.** The paper is organized as follows. In Sect. 2, we show that the equation (1.1) is well-posed in certain weighted spaces spaces, as long as the growth at infinity is sublinear. The main result of that section is Proposition 2.2, from which Theorem 1.1 follows immediately. In Sect. 3 we prove the comparison principle and \( L^1 \)-contraction both in space and in probability. In Sect. 4 we prove some other useful basic properties of the solutions. In Sect. 5, we establish the tightness in \( \mathcal{X}_m \) of the solution to (1.3) started from a constant, for \( m \in [1/2, 1) \). This shows the existence of stationary solutions to (1.3). In Sect. 6, we complete the proof of Theorem 1.2 by classifying all extremal elements of \( \mathcal{F}_G(\mathcal{X}_m) \). In Sect. 7, we prove the stability result Theorem 1.5. The appendices contain the proofs of several auxiliary results. In Appendix A, we prove
Proposition 1.4, as its proof is elementary and unrelated to the rest of the paper. Appendix B includes some background on weighted spaces and estimates on the solutions to the heat equation in weighted spaces; these results are used extensively in Sect. 2. In Appendix C we show that classical solutions to the Burgers equation are mild (tying up a loose end from Sect. 2 that is not used in the rest of the paper), and in Appendix D, we prove some other technical lemmas that are used at various points throughout the paper.

2. Solutions to the Burgers Equation in Weighted Spaces

In this section, we construct solutions to the stochastic Burgers equation in weighted spaces that permit growth at infinity. A key preliminary step is a standard trick going back to [19]: by subtracting off a solution to the linearized version of (1.4), we reduce a stochastic partial differential equation (1.4) to a partial differential equation (2.6) with random coefficients coming from the solution to the linearized problem. This does not circumvent the need to work with solutions that may grow at infinity but it does allow us to work with classical solutions. The goal of this section is to show that (2.6) has classical solutions pathwise, in appropriate weighted spaces, so that we can treat the noise as a fixed object rather than a random one. Thus, the only genuine stochastic analysis required is to understand the Gaussian process solving the linearized problem. This step also allows us to avoid some of the minor additional technicalities involved with working directly with the strong solutions in the sense of [22].

The two main results of this section are, first, Proposition 2.2, which is a version of Theorem 1.1 that allows for discontinuous initial data, and stated in terms of the solutions to (2.6), and, second, the Feller property stated as Proposition 2.3.

To prove Proposition 2.2, we first consider the periodized version of the problem, with both the initial conditions and the noise periodized, and then pass to the limit as the periodization length is taken to infinity. The periodized problem is set up in Sect. 2.3. To solve it, we use the mild formulation of the problem, which we relate to the classical formulation in Sect. 2.4. We then solve the periodized problem, using a fixed-point argument similar to that of [19], in Sect. 2.5. To extend the solution theory to the whole space, we control the growth of the solutions in sublinearly weighted spaces in Sect. 2.6. It is here that the proof diverges significantly from the situation for the linear problem, as the sublinear weights are necessary for well-posedness. Finally, we pass to the limit of the periodization scales in Sect. 2.7 to prove Proposition 2.2 and Theorem 1.1.

2.1. From an SPDE to a PDE

We avoid working directly with the SPDE (1.3) by making use of the following trick introduced in [19]. Solving a linearization of (1.3), namely

\[ d\psi = \frac{1}{2}\partial_x^2 \psi dt + d(\partial_x V), \]

with initial condition \( \psi(0, \cdot) \equiv 0 \), is simple: the solution is given by the stochastic integral

\[ \psi(t, x) = \int_0^t [\partial_x G_{t-s} * dV(s, \cdot)](x) = \int_0^t \int_\mathbb{R} (\partial_x G_{t-s} * \rho)(x - y) dW(s, y), \]

where

\[ G_t(x) = (2\pi t)^{-1/2} \exp\{-x^2/(2t)\} \]
is the heat kernel. See [22, Chapter 5 and Theorem 5.2] for a detailed discussion of such stochastic integrals, but note also that $\psi$ is simply a mean-zero Gaussian process on $\mathbb{R} \times \mathbb{R}$ with covariance function

$$E\psi(t, x)\psi(t', x') = \int_0^{t \land t'} \int_\mathbb{R} (\partial_x G_{t-s} \ast \rho)(x - y)(\partial_x G_{t'-s} \ast \rho)(x' - y) \, dy \, ds$$

$$= -\int_0^{t \land t'} \partial_{xx} (G_{t+t'-2s} \ast \rho^*) (x - x') \, ds$$

$$= \int_0^{t \land t'} \frac{d}{ds} (G_{t+t'-2s} \ast \rho^*) (x - x') \, ds$$

$$= ([G_{t-t'} - G_{t+t'}] \ast \rho^*)(x - x').$$

(2.4)

In fact, $(\psi, V)$ is jointly Gaussian. A special case of (2.4) is

$$E\psi(t, x)\psi(t, x') = (\rho^* - G_{2t} \ast \rho^*)(x - x').$$

(2.5)

From this one can see that as $t \to \infty$, for fixed $x, x' \in \mathbb{R}$ we have

$$E\psi(t, x)\psi(t, x') \to \rho^*(x - x'),$$

and $\psi(t, \cdot)$ converges in law to a Gaussian process with covariance kernel $\rho^*$ in the topology of an appropriate weighted space. We discuss the necessary weights in Lemma 2.4 below.

Writing $u = \theta + \psi$, we see, as in [19], that a function $u$ is a strong solution to (1.3) if and only if $\theta = u - \psi$ is a classical solution to the PDE

$$\partial_t \theta = \frac{1}{2} \partial_x^2 \theta - \frac{1}{2} \partial_x (\theta + \psi)^2.$$  

(2.6)

Our analysis will start from this equation, rather than directly from (1.3). In particular, our first goal is to build strong solutions to (2.6) in certain weighted spaces. Going forward, we can treat $\psi$ pathwise, as if it were a deterministic object.

2.2. Solutions in weighted function spaces. We now introduce some weighted function spaces that we will use in constructing solutions to (2.6). This is necessary as the force $\psi$ and thus also the solution $\theta$ in (2.6) grow at infinity. Given a weight $w(x) > 0$, the weighted space $L^\infty_w(\mathbb{R})$ is the space of measurable functions $v : \mathbb{R} \to \mathbb{R}$ such that

$$\|v\|_{L^\infty_w(\mathbb{R})} = \text{ess sup}_{x \in \mathbb{R}} \frac{|v(x)|}{w(x)} < +\infty,$$

and $C^*_w(\mathbb{R}) \subset L^\infty_w(\mathbb{R})$ is the subspace of continuous functions in $L^\infty_w(\mathbb{R})$, with the same norm. For $\alpha \in (0, 1)$, the weighted Hölder space is the subspace of $C^*_w(\mathbb{R})$ of functions such that

$$\|v\|_{C^*_w(\mathbb{R})} = \|v\|_{C^*_w(\mathbb{R})} + \sup_{|x - y| \leq 1} \frac{|v(x) - v(y)|}{w(x)|x - y|^\alpha} < +\infty.$$
The higher order Hölder spaces $C^{k,\alpha}_w(\mathbb{R})$, with $k \in \mathbb{N}$ and $\alpha \in [0, 1)$, have the norms

$$\|v\|_{C^{k,\alpha}_w(\mathbb{R})} = \|v\|_{C^{k,\alpha}_w(\mathbb{R})} = \sum_{j=0}^{k} \|\partial_x^j v\|_{C^0_w(\mathbb{R})}.$$ 

Finally, for $p \in [1, \infty)$, the space $L^p_w(\mathbb{R})$ is equipped with the norm

$$\|v\|_{L^p_w(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(\frac{|v(x)|}{w(x)}\right)^p dx\right)^{1/p} < +\infty.$$ 

We will often use the weights

$$p_\ell(x) = \langle x \rangle^\ell \quad \text{with} \quad \langle x \rangle = \sqrt{4 + x^2} \quad (2.7)$$

and $\ell \in \mathbb{R}$. The constant 4 rather than 1 in the definition of $\langle x \rangle$ ensures that $\log \langle x \rangle > 0$ for all $x \in \mathbb{R}$, which will be convenient when we use logarithmic weights.

For $m \in \mathbb{R}$, we define the Fréchet space

$$L^\infty_{p_m}(\mathbb{R}) = \bigcap_{\ell > m} L^\infty_{p_\ell}(\mathbb{R})$$

equipped with the topology generated by all $L^\infty_{p_\ell}(\mathbb{R})$ norms for $\ell > m$. This space is metrizable, for example by the metric

$$d_{L^\infty_{p_m}(\mathbb{R})}(v_1, v_2) = \sum_{k=1}^\infty 2^{-k} \frac{\|v_1 - v_2\|_{L^\infty_{p_{m+1/k}}(\mathbb{R})}}{1 + \|v_1 - v_2\|_{L^\infty_{p_{m+1/k}}(\mathbb{R})}}.$$ 

A sequence $v_n$ converges to $v$ in $L^\infty_{p_m}(\mathbb{R})$ if and only if $v_n$ converges to $v$ in the topology of $L^\infty_{p_\ell}(\mathbb{R})$ for each $\ell > m$. Therefore, for any topological space $Z$, a map $f : Z \to L^\infty_{p_m}(\mathbb{R})$ is continuous if and only if $f$ is a continuous map $Z \to L^\infty_{p_\ell}(\mathbb{R})$ for each $\ell > m$. In particular, the inclusion maps $L^\infty_{p_m}(\mathbb{R}) \to L^\infty_{p_\ell}(\mathbb{R})$ for $\ell > m$ are continuous.

The key space for us is $\mathcal{X}_m$, which we define as the closed subspace of continuous functions in $L^\infty_{p_m}(\mathbb{R})$. The space of continuous, compactly-supported functions is dense in $\mathcal{X}_m$. This means that $\mathcal{X}_m$ is separable and hence a Polish space, unlike the spaces $C_w(\mathbb{R})$ and $L^\infty_{p_m}(\mathbb{R})$ which are not separable. We prefer to work with continuous functions when possible, since the separability of the space $\mathcal{X}_m$ will allow us to use probabilistic tools about random variables on Polish spaces. Solutions will be continuous at all positive times due to the smoothing effect of the Laplacian in (2.6). However, we will have occasion to solve (1.3) and (2.6) with discontinuous initial data. In particular, this will be relevant in the proof of the stability result in Theorem 1.5. Thus, we make the following definition. Here and henceforth, if $\mathcal{Y}_1$ is a metric space and $\mathcal{Y}_2$ is a topological vector spaces, we use the notation $C_0(\mathcal{Y}_1; \mathcal{Y}_2)$ to refer to the space of bounded continuous functions from $\mathcal{Y}_1$ to $\mathcal{Y}_2$.

**Definition 2.1.** We define $\mathcal{Z}_{m,T}$ to be the space of functions $u \in C_0((0, T]; \mathcal{X}_m)$ such that for each $\ell > m$ the limit

$$u(0, \cdot) := \lim_{t \downarrow 0} u(t, \cdot) \quad (2.8)$$
exists in the weak-* topology on $L^\infty_{p_\ell}(\mathbb{R})$, and the initial condition $u(0, \cdot) \in L^\infty_{p_m}(\mathbb{R})$. 

In particular, if \( u \in \mathcal{Z}_{m,T} \), then \( t \mapsto u(t, \cdot) \in L^\infty_{p_t}(\mathbb{R}) \) is continuous on \([0, T] \) if \( L^\infty_{p_t}(\mathbb{R}) \) is endowed with the weak-* topology. We endow \( \mathcal{Z}_{m,T} \) with the subspace topology inherited from the embedding
\[
\mathcal{Z}_{m,T} \ni u \mapsto (u(0, \cdot), u|_{(0,T] \times \mathbb{R}}) \in L^\infty_{p_m}(\mathbb{R}) \times C_b((0, T]; \mathcal{X}_m).
\]
We further define \( \mathcal{Z}_m = \mathcal{Z}_{m,\infty} \) to be the space of functions \( u : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) such that the restriction \( u|_{[0,T] \times \mathbb{R}} \in \mathcal{Z}_{m,T} \) for each \( T > 0 \), equipped with the weakest topology such that each restriction map \( u \mapsto u|_{[0,T] \times \mathbb{R}} \) is continuous.

To discuss classical solutions to (2.6), we define a smaller class \( \tilde{\mathcal{Z}}_m \) of functions \( \theta \in \mathcal{Z}_{m,T} \) that are twice-differentiable in space and once on time on \((0, T) \times \mathbb{R} \), and moreover are such that for every compact \( I \subset (0, T) \) and \( \varepsilon > 0 \), there exists \( C < \infty \) such that
\[
|\partial_t \theta(t, x)| \leq C e^{\varepsilon x^2} \quad \text{for all } (t, x) \in I \times \mathbb{R}.
\] (2.9)

The reason why we impose this bound on \( \partial_t \theta \) rather than just on \( \theta \) itself, as is done for the Cauchy problem for the heat equation, is explained in Lemma C.1 in Appendix C.

We also define \( \tilde{\mathcal{Z}}_m = \tilde{\mathcal{Z}}_{m,\infty} \subset \mathcal{Z}_m = \mathcal{Z}_{m,\infty} \). The following proposition implies the existence and uniqueness claims for the solutions to (1.3) in Theorem 1.1.

**Proposition 2.2.** Let \( m \in (0, 1) \). Almost surely, there is a map
\[
\Phi : L^\infty_{p_m}(\mathbb{R}) \to \tilde{\mathcal{Z}}_m,
\]
so that for each \( v \in L^\infty_{p_m}(\mathbb{R}) \), \( \theta = \Phi(v) \) is the unique strong solution to (2.6) with the initial condition \( \theta(0, \cdot) = v \). The map \( v \mapsto \Phi(v)|_{[0,T] \times \mathbb{R}} \) is measurable with respect to \( \mathcal{F}_T \). Moreover, also almost surely, the map \( \Phi \) is continuous and for any bounded set \( A \subset L^\infty_{p_m}(\mathbb{R}) \) and \( T > 0 \), the image \( \Phi(A)|_{[0,T] \times \mathbb{R}} \) is bounded in \( \mathcal{Z}_{m,T} \) and the restriction \( \Phi(A)|_{[T] \times \mathbb{R}} \) is compact in \( \mathcal{X}_m \).

The proof of this proposition occupies most of the rest of this section. Its immediate consequence is that we can define a solution map \( \Psi : L^\infty_{p_m}(\mathbb{R}) \to \mathcal{Z}_m \) so that for each \( v \in L^\infty_{p_m}(\mathbb{R}) \), \( u = \Psi(v) \) is the unique strong solution to (1.3) satisfying \( u(t, \cdot) = v \), given by
\[
\Psi(v)(t, x) = \Phi(v)(t, x) + \psi(t, x),
\] (2.10)
where \( \psi(t, x) \) is the solution to the linearized problem (2.1) given by (2.2). As with \( \Phi \), the map \( \Psi \) is continuous and for any bounded set \( A \subset L^\infty_{p_m}(\mathbb{R}) \) and \( T > 0 \), the image \( \Psi(A)|_{[0,T] \times \mathbb{R}} \) is bounded in \( \mathcal{Z}_{m,T} \) and \( \Psi(A)|_{[T] \times \mathbb{R}} \subset \mathcal{X}_m \) is compact in \( \mathcal{X}_m \).

This is the existence and uniqueness claim of Theorem 1.1 but, in addition, it allows discontinuous initial data. This result also generalizes immediately to the system (1.4) of \( N \) such decoupled equations.

Another consequence of Proposition 2.2 is that for any \( t > 0 \) and \( N \in \mathbb{N} \) there is a map \( P_t \) from the space of measurable functions on \( \mathcal{X}_m^N \) to the space of measurable functions on \( L^\infty_{p_m}(\mathbb{R})^N \) given by
\[
(P_t f)(v) = \mathbb{E} f(\Psi(v)(t, \cdot)),
\]
where \( v \) is the solution to the decoupled system (1.4). Because \( \Psi \) is almost surely continuous, we get the following Feller property.
**Proposition 2.3** (Feller property). If \( m \in (0, 1) \) and \( f \in C_b(\mathcal{X}_m^N) \), then \( P_t f \in C_b(L_{p,m}^\infty(\mathbb{R})^N) \) for all \( t > 0 \). In particular, \( P_t f \) (strictly speaking, \( P_t f|_{\mathcal{X}_m^N} \)) is an element of \( C_b(\mathcal{X}_m^N) \).

**Proof.** Let \( v^{(n)} \to v \) in \( L_{p,m}^\infty(\mathbb{R})^N \) and \( t > 0 \). Then for each component \( i = 1, \ldots, N \), we have that

\[
\Psi(v_i^{(n)}) (t, \cdot) \to \Psi(v_i) (t, \cdot) \quad \text{in} \quad \mathcal{X}_m^N,
\]

almost surely. It follows that

\[
\mathbb{E} f(\Psi(v_i^{(n)}) (t, \cdot), \ldots, \Psi(v_N^{(n)}) (t, \cdot)) \to \mathbb{E} f(\Psi(v_1) (t, \cdot), \ldots, \Psi(v_N) (t, \cdot)),
\]

by the bounded convergence theorem since \( f \) is bounded. \( \square \)

Proposition 2.3 implies the Feller property of (1.3) claimed in Theorem 1.1. Hence, the proof of Theorem 1.1 is reduced to that of Proposition 2.2.

### 2.3. The periodized problem

The proof of Proposition 2.2 proceeds in two steps. First, we show the existence of solutions to a periodized stochastic Burgers equation, and then we take the limit as the period tends to infinity. We use the notation \( C_\alpha(\mathbb{R}/L\mathbb{Z}) \) for functions in \( C_\alpha(\mathbb{R}) \) which are \( L \)-periodic, with the convention \( C_\alpha(\mathbb{R}/\infty\mathbb{Z}) = C_\alpha(\mathbb{R}) \). We emphasize that the use of periodicity in this section is fundamentally different from the use of periodicity in later sections. In this section, we periodize both the initial conditions and the driving noise, so that the solutions are periodic almost surely. In later sections, we will consider solutions to (1.4) with noise that is not periodic, but whose laws are periodic.

For \( L > 0 \), let \( \chi^{[L]} \) be a smooth, compactly-supported bump function, taking values in \([0, 1]\), such that

\[
\chi^{[L]}_{[-L/2,L/2]} \equiv 1, \quad \chi^{[L]}_{[-L/2-1,L/2+1]} \equiv 0, \quad \| \chi^{[L]} \|_{C^k(\mathbb{R})} \leq C_k \tag{2.11}
\]

for each \( k \), where \( C_k < \infty \) is a constant that may depend on \( k \) but not on \( L \). The \( L \)-periodized version of \( V \) is

\[
V^{[L]}(x) = \sum_{j \in \mathbb{Z}} (\chi^{[L]} V)(x - jL), \tag{2.12}
\]

For notational convenience, we define \( V^{[\infty]} = V \). For \( L \in (0, \infty] \), let

\[
\psi^{[L]}(t, \cdot) = \int_0^t \partial_x G_{t-s} * dV^{[L]}(s, \cdot), \tag{2.13}
\]

so that \( \psi^{[L]} \) is \( L \)-periodic and solves the SPDE

\[
d\psi^{[L]} = \frac{1}{2} \partial_x^2 \psi^{[L]} dt + d(\partial_x V^{[L]}),
\]

\[
\psi^{[L]}(0, \cdot) \equiv 0.
\]

The family \( \{ \psi^{[L]} \}_{L \in (0, \infty]} \) is coupled by taking the stochastic convolutions of the same realization of \( V \). We will always assume that we have taken modifications of \( \psi^{[L]} \) with continuous paths.
We will consider the $L$-periodic approximation to (2.6)

\[d\theta^{[L]} = \frac{1}{2} \partial_x^2 \theta^{[L]} - \frac{1}{2} \partial_x (\theta^{[L]} + \psi^{[L]})^2,\]

with some initial condition

\[\theta^{[L]}(0, \cdot) = v^{[L]},\]

(2.14)

(2.15)

This PDE has classical solutions, and can be solved pathwise in the noise. Indeed, the following lemma is the only fact about $\psi^{[L]}$ that we will use in this section.

**Lemma 2.4.** Define the weight $g(x) = (\log(x))^{3/4}$. For any $T < \infty$ and $j \in \mathbb{N}$, with probability 1 we have

\[
\sup_{L \in [1, \infty]} \|\psi^{[L]}\|_{C_b([0, T]; C_j^1(\mathbb{R}))} < \infty.
\]

(2.17)

**Proof.** Let us first recall a standard bound on the growth of $V(t, x)$ and its derivatives at infinity which implies that $\partial^j_t V \in C_b([0, T]; C_j^1(\mathbb{R}))$ almost surely. Since the noise $V$ is a convolution of a cylindrical Wiener process with a spatially smooth process, it is Gaussian, continuous in time, and smooth in space. In particular, we have, for any $j \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$ fixed, that

\[
\mathbb{E} \left( \sup_{x \in [k, k+1]} \sup_{t \in [0, T]} \left| \partial^j_t V(t, x) \right| \right) < \infty
\]

by Fernique’s inequality (see e.g. [1, Theorem 4.1]). Therefore, by the Borell–TIS inequality (see e.g. [1, Theorem 2.1]) there exist constants $c > 0$ and $C < \infty$, depending on $T$, so that for all $z > 0$ we have

\[
\mathbb{P} \left( \sup_{x \in [k, k+1]} \sup_{t \in [0, T]} \left| \partial^j_t V(t, x) \right| > z \right) \leq C e^{-cz^2}.
\]

By a union bound, this means that

\[
\mathbb{P} \left( \frac{1}{g(k)} \sup_{x \in [k, k+1]} \sup_{t \in [0, T]} \left| \partial^j_t V(t, x) \right| > z \right) \leq \sum_{k \in \mathbb{Z}} \mathbb{P} \left( \sup_{x \in [k, k+1]} \sup_{t \in [0, T]} \left| \partial^j_t V(t, x) \right| > zg(k) \right) \leq C \sum_{k \in \mathbb{Z}} e^{-cz^2 (\log(k))^{3/2}},
\]

and the sum on the right is finite and goes to 0 as $z \to \infty$. In particular,
is finite almost surely, and so \( \partial^j_x V \in C_b([0, T]; C^1_b(\mathbb{R})) \) almost surely, as we have claimed. In addition, it is clear from (2.12) that

\[
\sup_{L \in [1, \infty]} \| \partial^j_x V^{[L]} \|_{C_b([0, T]; C^1_b(\mathbb{R}))} \leq 3 \| \partial^j_x V \|_{C_b([0, T]; C^1_b(\mathbb{R}))} < \infty \tag{2.18}
\]

almost surely.

Now we can turn our attention to \( \psi^{[L]} \). We have

\[
\psi^{[L]}(t, x) = \int_0^t \partial_x G_{t-s} * dV^{[L]}(s, \cdot)(x) = -\int_0^t \partial_t x G_{t-s} * V^{[L]}(s, \cdot)(x) ds
\]

(2.19)

and so

\[
\partial^j_x \psi^{[L]}(t, x) = -\frac{1}{2} \int_0^t G_{t-s} * \partial^j_x V^{[L]}(s, \cdot)(x) ds.
\]

(2.20)

The conclusion (2.17) follows from (2.18)–(2.20) and Lemma B.4 in Appendix B.2. \( \square \)

2.4. Mild solutions. In this section, we show that the mild formulation of (2.14)–(2.15), namely

\[
\theta^{[L]}(t, \cdot) = G_t * v^{[L]} - \frac{1}{2} \int_0^t \partial_x G_{t-s} * (\theta^{[L]}(s, \cdot) + \psi^{[L]}(s, \cdot))^2 ds, \quad t > 0. \tag{2.21}
\]

is equivalent to its classical formulation. As part of the proof, we establish regularity estimates on mild solutions which will be important later on.

**Lemma 2.5.** Given \( \ell > 0, \alpha \geq 0, \beta \in [\alpha, \alpha + 1), \ell > 0, \) and \( S > 0, \) there exists \( C = C(\alpha, \beta, \ell, S) \) so that for all \( L \in (0, \infty) \) and \( v^{[L]} \) as in (2.16), if \( \theta^{[L]} \in C_b((0, S]; C^\alpha_{P_L}(\mathbb{R})) \) satisfies (2.21), then for all \( t \in (0, S) \) we have

\[
\| \theta^{[L]}(t, \cdot) \|_{C^\alpha_{P_L}(\mathbb{R})} \leq C t^{-\frac{\beta-\alpha}{2}} \| v^{[L]} \|_{L^\infty_{P_L}(\mathbb{R})} + C \int_0^t (t-s)^{-\frac{\beta-\alpha+1}{2}} \left[ \| \theta^{[L]}(s, \cdot) \|_{C^\alpha_{P_L}(\mathbb{R})} \| \theta^{[L]}(s, \cdot) \|_{C^\alpha_{P_L}(\mathbb{R})} \right] ds.
\]

**Proof.** This follows from applying Lemma B.4 to (2.21) and using the triangle inequality. \( \square \)

Lemma 2.5 can be iterated to obtain bounds on higher derivatives. For simplicity, we only state the results we use later on.

**Corollary 2.6.** Fix \( \ell > 0, \) \( L \in (0, \infty), \) and \( S > 0. \) If \( \theta^{[L]} \in C_b((0, S]; C_{P_L}(\mathbb{R})) \) satisfies (2.21) for some \( v^{[L]} \) satisfying (2.16), then for all \( t \in (0, S) \) there is a constant \( C(t) \) (depending also on \( \ell \)) so that

\[
\| \theta^{[L]}(t, \cdot) \|_{C^\ell_{P_L}(\mathbb{R})} \leq C(t) \left( \| \theta^{[L]} \|_{C^\alpha((0, t]; C_{P_L}(\mathbb{R}))}^4 + \| \psi^{[L]} \|_{C^\alpha((0, t]; C^1_{P_L}(\mathbb{R}))}^4 \right). \tag{2.22}
\]
Proof. Applying Lemma 2.5 on the time interval \([t/2, t]\) gives

\[
\|\theta^{[L]}(t, \cdot)\|_{C^1_{\mathcal{F}B}(\mathbb{R})} \leq C \|\theta^{[L]}\|_{C_0([t/2, t]; C^1_{\mathcal{P}_2}(\mathbb{R}))} + C \left( \|\theta^{[L]}\|_{C_{b}(0, t/2); C^1_{\mathcal{P}_2}(\mathbb{R}))}^2 + \|\psi^{[L]}\|_{C_{b}(0, t/2); C^1_{\mathcal{P}_2}(\mathbb{R}))}^2 + 1 \right),
\]

with a constant \(C\) depending on \(\ell\) and \(t\). Applying this lemma again to the terms with \(\theta^{[L]}\) in the right side, now on the time interval \([0, t/2]\), gives

\[
\|\theta^{[L]}(t, \cdot)\|_{C^1_{\mathcal{F}B}(\mathbb{R})} \leq C \left( \|\theta^{[L]}\|_{C_{b}(0, t/2); C^1_{\mathcal{P}_2}(\mathbb{R}))}^4 + \|\psi^{[L]}\|_{C_{b}(0, t/2); C^1_{\mathcal{P}_2}(\mathbb{R}))}^4 + 1 \right),
\]

which implies (2.22). \(\Box\)

We also have the following bound on the nonlinear term in (2.21): set

\[
D_t = -\frac{1}{2} \int_0^t \partial_x G_{t-s} * (\theta^{[L]}(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \, ds. \tag{2.23}
\]

Corollary 2.7. Fix \(m \in (0, 1)\), \(L \in (0, \infty)\), \(v^{[L]} \in L_{p_{r=1}}^\infty(\mathbb{R})\), and \(T > 0\). If \(\theta^{[L]} \in C_{b}((0, T]; \mathcal{X}_m)\) satisfies (2.21), then \(\lim_{t \downarrow 0} D_t = 0\) in \(\mathcal{X}_m\).

Proof. Fix \(m < \ell_1 < \ell\), so that \(v^{[L]} \in L_{p_{r=1}}^\infty(\mathbb{R})\). By Lemma B.4 with \(\alpha = \beta = 0\), we see that \(D_t \to 0\) as \(t \downarrow 0\) in the topology of \(C_{p_{r=1}}(\mathbb{R})\). In addition, we have

\[
D_t = \theta^{[L]}(t, \cdot) - G_t * v^{[L]}.
\]

But \(\theta^{[L]} \in C_{b}((0, T]; C_{p_{r=1}}(\mathbb{R}))\) by hypothesis, and \(G_t * v^{[L]}\) is bounded in \(C_{p_{r=1}}(\mathbb{R})\) uniformly in \(t\) by Lemma B.5. Therefore, by Proposition B.1 in Sect. B.1, we actually have

\[
\lim_{t \downarrow 0} D_t = 0 \tag{2.24}
\]

in the topology of \(C_{p_{r=1}}(\mathbb{R})\). Since \(\ell > m\) was arbitrary, \(D_t \to 0\) in \(\mathcal{X}_m\) as well. \(\Box\)

It follows that if \(\theta^{[L]} \in C_{b}((0, T]; \mathcal{X}_m)\) satisfies (2.21), and \(v^{[L]} \in \mathcal{X}_m\), then if we extend \(\theta^{[L]}\) to \(t = 0\) by setting \(\theta^{[L]}(0, \cdot) = v^{[L]}\), then the extension satisfies \(\theta^{[L]} \in C_{b}([0, T]; \mathcal{X}_m)\). We can now show that mild solutions to (2.14)–(2.15) are in \(\tilde{Z}_m\) and are in fact classical solutions.

Lemma 2.8 (Mild solutions are classical). Fix \(m \in (0, 1)\), \(L \in (0, \infty)\), \(T > 0\), and \(v^{[L]} \in L_{p_{r=1}}^\infty(\mathbb{R})\). If \(\theta^{[L]} \in C_{b}((0, T]; \mathcal{X}_m)\) satisfies (2.21), and we extend it to \(t = 0\) by \(\theta^{[L]}(0, \cdot) = v^{[L]}\), then \(\theta^{[L]} \in \tilde{Z}_{m, T}\) and \(\theta^{[L]}\) is a classical solution to the PDE (2.14).

Also, for any \(p \in [1, \infty)\) and any \(\ell' > m + 1/p\), we have

\[
\theta^{[L]} \in C_{b}([0, T]; L_{p_{r'=1}}^p(\mathbb{R})). \tag{2.25}
\]

Finally, if \(v^{[L]}\) is continuous, then

\[
\theta^{[L]} \in C_{b}([0, T]; \mathcal{X}_m). \tag{2.26}
\]
Conversely, we can show that classical solutions in \( \tilde{\psi} \) is smooth, we can iterate Lemma 2.5 to show that \( \theta^{[L]}(t, \cdot) \) is a smooth tempered distribution for each \( t > 0 \). Differentiating (2.21), we can easily check that \( \theta^{[L]} \) is differentiable in time and is a classical solution to (2.14).

Proof. Since \( \theta^{[L]}(t, \cdot) \) is a smooth tempered distribution for each \( t \geq 0 \), Differentiating (2.21), we can easily check that \( \theta^{[L]} \) is differentiable in time and is a classical solution to (2.14).

Now, fix \( \ell' > m + 1/p \) and \( \ell \in (m, \ell' - 1/p) \). It follows from Corollary 2.7 that \( D_t \to 0 \) as \( t \downarrow 0 \) in \( C^{\ell'}(\mathbb{R}) \). Since \( \ell' > \ell + 1/p \), this implies convergence in \( L_{p'}(\mathbb{R}) \) as well. In addition, Lemma B.5 shows that \( G_t \ast v^{[L]} \to v^{[L]} \) in \( L_p(\mathbb{R}) \), and (2.25) follows. Furthermore, Lemma B.5 implies that

\[
G_t \ast v^{[L]} \overset{w}{\to} v^{[L]}, \quad \text{as } t \downarrow 0 \text{ in } L_p(\mathbb{R}),
\]

and thus \( \theta^{[L]} \in \tilde{\mathcal{Z}}_{m,T} \). Similarly, (2.26) follows from Corollary 2.7 and Lemma B.5. □

Conversely, we can show that classical solutions in \( \tilde{\mathcal{Z}}_m \) are mild. The proof of this is quite standard and is presented in Lemma C.1 in Appendix C. Let us mention that one reason not to skip the proof completely is that it is there that the bound (2.9) is used.

2.5. Local-in-time existence for the periodized problem. We now show existence of solutions to the periodized problem (2.14)–(2.15) with \( L < \infty \). We use a fixed-point argument based on the mild formulation (2.21). For the moment, the solution we obtain exists only up to a time depending on \( L \) and the random forcing \( \psi^{[L]} \). This dependence will be eliminated in Proposition 2.11 below. Let \( \tilde{\mathcal{Z}}_T^{[L]} \subset \tilde{\mathcal{Z}}_{0,T} \) be the subspace of functions that are \( L \)-periodic in space.

**Proposition 2.9.** For each \( L \in (0, \infty) \) and \( v^{[L]} \in L_\infty(\mathbb{R} / \mathbb{Z}) \), there is a time \( S' \in (0, 2] \), depending on \( L, \psi^{[L]}, \) and \( v^{[L]} \), so that there exists a solution \( \theta^{[L]} \in \bigcup_{S' \in (0,S)} \mathcal{Z}_{S'}^{[L]} \) to (2.14)–(2.15). Moreover, if \( S < 2 \), then

\[
\lim_{t \to S} \| \theta^{[L]}(t, \cdot) \|_{L_\infty(\mathbb{R} / \mathbb{Z})} = \infty. \tag{2.27}
\]

**Proof.** In light of Lemma 2.8, it suffices to find a solution \( \theta^{[L]} \in \bigcup_{S' \in (0,S)} C_b((0, S'); C(\mathbb{R} / \mathbb{Z})) \) to (2.21). We use a fixed-point argument similar to that of [19, Lemma 2.1]. Let \( M \in (0, \infty) \) and \( \Xi_M \) be a smooth cutoff function so that \( \Xi_M(x) = x \) for \( |x| \leq M \) and \( |\Xi_M(x)| \leq M + 1 \) for all \( x \). Fix a time \( T \in (0, 2] \) to be chosen later and note that the operator

\[
\mathcal{A}_M \theta(t, \cdot) = G_t \ast v^{[L]} - \int_0^t \partial_s G_{t-s} \ast ((\Xi_M \circ \theta)(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \, ds
\]

maps \( C_B((0, T]; C(\mathbb{R} / \mathbb{Z})) \) into itself. By the triangle inequality, the maximum principle for the heat equation, and Lemma B.4, we have

\[
\| \mathcal{A}_M \theta(t, \cdot) \|_{C_B(\mathbb{R} / \mathbb{Z})} \leq \| G_t \ast v^{[L]} \|_{L_\infty(\mathbb{R} / \mathbb{Z})} + \int_0^t \| \partial_s G_{t-s} \ast ((\Xi_M \circ \theta)(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \|_{C_B(\mathbb{R} / \mathbb{Z})} \, ds \leq \| v^{[L]} \|_{L_\infty(\mathbb{R} / \mathbb{Z})} + C t^{1/2} \left( (M + 1)^2 + \| \psi^{[L]} \|^2_{C_B((0,2]; C(\mathbb{R} / \mathbb{Z}))} \right). \tag{2.28}
\]
with some universal constant $C$, so that
\[
\|A_M \theta \|_{C_b((0,T);C_b(\mathbb{R}/L\mathbb{Z}))} \\
\leq \|v^{[L]}\|_{L^{\infty}(\mathbb{R}/L\mathbb{Z})} + CT^{1/2} \left( (M + 1)^2 + \|\psi^{[L]}\|^2_{C_b((0,2);C(\mathbb{R}/L\mathbb{Z}))} \right) =: K_M.
\]

Let $B$ be the ball of radius $K_M$ about the origin in $C_b((0,T);C_b(\mathbb{R}/L\mathbb{Z}))$. If $\theta, \tilde{\theta} \in B$ and $t \in [0,T]$, then, once again, by Lemma B.4, we obtain
\[
\|A_M \theta(t, \cdot) - A_M \tilde{\theta}(t, \cdot)\|_{C_b(\mathbb{R}/L\mathbb{Z})} \\
\leq \int_0^t \| \partial_x G_{t-s} \ast \left[ ((\Xi_M \circ \theta)(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \\
- ((\Xi_M \circ \tilde{\theta})(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \right] \|_{C_b(\mathbb{R}/L\mathbb{Z})} ds \\
\leq \int_0^t (t-s)^{-1/2} \left\| ((\Xi_M \circ \theta)(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \\
- ((\Xi_M \circ \tilde{\theta})(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \right\|_{C_b(\mathbb{R}/L\mathbb{Z})} ds \\
\leq CT^{1/2} \left( M + 1 + \|\psi^{[L]}\|_{C_b((0,2);C(\mathbb{R}/L\mathbb{Z}))} \right) \|\theta - \tilde{\theta}\|_{C_b((0,T);C_b(\mathbb{R}/L\mathbb{Z}))},
\]

with another (possibly larger) constant $C$. Therefore, for all $\theta, \tilde{\theta} \in B$, we have
\[
\|A_M \theta - A_M \tilde{\theta}\|_{C_b((0,T);C_b(\mathbb{R}/L\mathbb{Z}))} \\
\leq CT^{1/2} \left( M + 1 + \|\psi^{[L]}\|_{C_b((0,2);C(\mathbb{R}/L\mathbb{Z}))} \right) \|\theta - \tilde{\theta}\|_{C_b((0,T);C_b(\mathbb{R}/L\mathbb{Z}))}.
\]

Thus, if
\[
T < \left( C \left( M + 1 + \|\psi^{[L]}\|_{C_b((0,2);C(\mathbb{R}/L\mathbb{Z}))} \right) \right)^{-2}, \quad (2.29)
\]

then the Banach fixed point theorem ensures the existence of a unique $\theta_M \in B$ so that $A_M \theta_M = \theta_M$, which is to say that
\[
\theta_M(t, \cdot) = G_t \ast v^{[L]} - \int_0^t \partial_x G_{t-s} \ast \left( ((\Xi_M \circ \theta_M)(s, \cdot) + \psi^{[L]}(s, \cdot))^2 \right) ds \quad (2.30)
\]

for all $t \in [0,T]$. Iterating this construction in the usual way, noting that the required bound (2.29) does not depend on the initial condition $v^{[L]}$, we conclude that $\theta_M \in C_b((0,2);C_b(\mathbb{R}/L\mathbb{Z}))$ and (2.30) holds for all $t \in [0,2]$.

To remove the $M$-cutoff, define
\[
S_M = \inf \{ t \in [0,2] : \|\theta_M(t, \cdot)\|_{L^{\infty}(\mathbb{R}/L\mathbb{Z})} \geq M \},
\]

or $S_M = 2$ if $\|\theta_M(t, \cdot)\|_{L^{\infty}(\mathbb{R}/L\mathbb{Z})} < M$ for all $t \in [0,2]$.

First, we claim that $\|\theta_M(t, \cdot)\|_{L^{\infty}(\mathbb{R}/L\mathbb{Z})}$ is continuous at $t = 0$, so that $S_M > 0$ for $M$ sufficiently large. The estimate on the integral in (2.28) shows that its contribution to
\( \theta_M \) vanishes strongly at \( t = 0 \). This leaves only the term \( G_t \ast v^\text{[L]} \) in (2.30) to estimate. By Lemma B.5, we have

\[
\| v^\text{[L]} \|_{L^\infty(\mathbb{R}/L\mathbb{Z})} \leq \lim \inf_{t \downarrow 0} \| G_t \ast v^\text{[L]} \|_{L^\infty(\mathbb{R}/L\mathbb{Z})}.
\]

On the other hand, the comparison principle implies

\[
\| v^\text{[L]} \|_{L^\infty(\mathbb{R}/L\mathbb{Z})} \geq \| G_t \ast v^\text{[L]} \|_{L^\infty(\mathbb{R}/L\mathbb{Z})}
\]

for all \( t \geq 0 \). Together, these imply that \( \| G_t \ast v^\text{[L]} \|_{L^\infty(\mathbb{R}/L\mathbb{Z})} \) are continuous at \( t = 0 \). Therefore if \( \| \theta_M(t, \cdot) \|_{L^\infty(\mathbb{R}/L\mathbb{Z})} < M \), we have \( S_M > 0 \).

Finally, it is clear from the uniqueness of \( \theta_M \) that \( S_M \) is increasing as a function of \( M \) and

\[
\theta_M|_{[0,S_M]} \equiv \theta_{M'}|_{[0,S_M]} \quad \text{whenever } M' \geq M.
\]

Therefore, there exists

\[
\theta^\text{[L]} \in \bigcup_{S' \in (0,S)} C_b((0, S'); C(\mathbb{R}/L\mathbb{Z}))
\]

satisfying (2.21) for all \( t \in (0, S) \), with

\[
S = \lim_{M \to \infty} S_M > 0.
\]

Moreover, if \( S < 2 \), then for each \( M \) we have \( \| \theta^\text{[L]} \|_{C_b((0,S_M);C_b(\mathbb{R}/L\mathbb{Z}))} \geq M \), and (2.27) follows. \( \Box \)

### 2.6. Global-in-time existence for the periodized problem

The results of Proposition 2.9 are insufficient to pass to the limit \( L \to \infty \) because the time of existence \( S \) is not uniform in \( L \). In addition, the existence time \( S \) is not uniform in \( v^\text{[L]} \) and \( v^\text{[L]} \). We now obtain a weighted bound on solutions that is independent of \( L \) and shows that the solutions \( \theta^\text{[L]} \) can always be extended up to time \( t = 1 \) (and thus by iteration to all positive times). This last point is a new ingredient, compared to, for example, the method of [19], since we need to control solutions on the whole space.

The \( \theta^2 \) term in (2.14) is dangerous from the perspective of global in time existence, and we need to use the fact that it is inside a gradient. This is not unrelated to our use (in Sect. 5 below) of the gradient on the noise to obtain uniform-in-time bounds for solutions to (1.3). However, the growth of the forcing and thus the solution at infinity requires the use of a weighted space, which breaks the symmetry used in [19,20] to eliminate the gradient term in (2.14) altogether. In a sense, the gradient term simply moves mass around, and in an unweighted space that does not affect the norm. In a weighted space, however, mass moving closer to zero causes the norm to grow. To control this growth, in the proof (but not the statement) of the following proposition we use a custom-built family of weights that grow at a similar rate to \( p\ell \) far from the origin, but close to zero are much flatter. This means that the effect of mass moving closer to the origin is reduced. The amount of flatness required depends on the initial condition and the noise, so the required weight is in fact random.
Proposition 2.10. Fix $0 < \ell < \ell' < 1$ and $A < \infty$. Then there is a constant $C = C(\ell, \ell', A) < \infty$ so that the following holds. Fix $L \in (0, \infty)$ and $v^{[L]} \in L_0^\infty(S; L\mathbb{Z})$ with $\|v^{[L]}\|_{L_0^\infty(S; L\mathbb{Z})} \leq A$ and suppose that $\|\psi^{[L]}\|_{C_b([0,1]; C_0^1(\mathbb{R}))} \leq A$, where $g$ is the weight defined in Lemma 2.4. If, for some $S \subseteq (0, 1]$, a function $\theta^{[L]} \in \mathcal{Z}_S^{[L]}$ solves (2.14)–(2.15), then

$$\sup_{t \in [0,S]} \|	heta^{[L]}(t, \cdot)\|_{C^{p_{\ell'}}(\mathbb{R})} \leq C.$$  

(2.31)

The constant $C$ does not depend on $L$. This will allow us to pass to a limit as $L \to \infty$.

Proof. Fix some $K \geq A$, to be chosen later, and $\ell < \ell_1 < \ell'$. Let $\varepsilon = \ell' - \ell_1$, and define

$$a(t, x) = K^{-1} \left( x^2 + K^{(1-\ell')/2} \right)^{(\ell_1+\varepsilon t)/2}.$$  

(2.32)

The function $z = a\theta^{[L]}$ satisfies

$$\|z(0, \cdot)\|_{L_0^\infty(\mathbb{R})} \leq 1$$  

(2.33)

and also, for $t > 0$,

$$\partial_t z = z\partial_t (\log a) + \frac{1}{2} \left[ \partial_x^2 z - z \left( \partial_x^2 (\log a) - (\partial_x (\log a))^2 \right) - 2(\partial_x z)\partial_x (\log a) \right]$$

$$- (a^{-1}z + \psi^{[L]})(\partial_x z - z\partial_x (\log a)) - (z + a\psi^{[L]})\partial_x \psi^{[L]}.$$  

(2.34)

In addition, since $\theta^{[L]}(t, \cdot)$ is periodic and hence bounded in space, we have

$$\lim_{|x| \to \infty} z(t, x) = 0$$  

(2.35)

for each $t \geq 0$.

We claim that the map $t \mapsto \|z(t, \cdot)\|_{L_0^\infty(\mathbb{R})}$ is continuous. Since $\theta^{[L]} \in \mathcal{Z}_S^{[L]}$, this is only in doubt at $t = 0$. We again write the mild formulation (2.21) as

$$\theta^{[L]} = G_t * v^{[L]} + D_t,$$

with $D_t$ as in (2.23). By Corollary 2.7, $D_t \to 0$ as $t \downarrow 0$ in $C^{p_{\ell'}}(\mathbb{R})$, so $aD_t \to 0$ in $L_0^\infty(\mathbb{R})$.

We thus need only consider $w(t, x) = a(t, x)[G_t * v^{[L]}(x)]$. We first show that

$$w(t, \cdot) \overset{w^*}{\longrightarrow} w(0, \cdot) \text{ in } L_0^\infty(\mathbb{R}).$$

To do so, fix $\phi \in L^1(\mathbb{R})$. By duality and the symmetry of the heat semigroup, it suffices to show that

$$G_t \ast [a(t, \cdot)\phi] \to a(0, \cdot)\phi$$

in $L^1(\mathbb{R})$ as $t \downarrow 0$. This follows from an approximation of the identity argument identical to that presented in the proof of Lemma B.5. Since $w$ is weak-$*$ continuous in $L_0^\infty(\mathbb{R})$, we obtain

$$\|w(0, \cdot)\|_{L_0^\infty(\mathbb{R})} \leq \liminf_{t \downarrow 0} \|w(t, \cdot)\|_{L_0^\infty(\mathbb{R})}.$$  

(2.36)
To establish the opposite bound, we use the comparison principle:

$$|G_t \ast v^{[L]}(x)| \leq \|a(0, \cdot) v^{[L]}\|_{L^\infty(R)} [G_t \ast a(0, \cdot)^{-1}](x)$$

$$= \|w(0, \cdot)\|_{L^\infty(R)} [G_t \ast a(0, \cdot)^{-1}](x).$$

We can easily check that $e^{\kappa t} a(0, \cdot)^{-1}$ is a supersolution to the heat equation for $\kappa > 0$ sufficiently large depending on $\ell'$, so that

$$[G_t \ast a(0, \cdot)^{-1}](x) \leq \frac{e^{\kappa t}}{a(0, x)}.$$

Thus, we have

$$|w(t, x)| \leq a(t, x) |G_t \ast v^{[L]}(x)| \leq a(t, x) \|w(0, \cdot)\|_{L^\infty(R)} [G_t \ast a(0, \cdot)^{-1}](x)$$

$$\leq e^{\kappa t} \|w(0, \cdot)\|_{L^\infty(R)} \frac{a(t, x)}{a(0, x)} \leq e^{\kappa t} \|w(0, \cdot)\|_{L^\infty(R)}.$$

so

$$\|w(0, \cdot)\|_{L^\infty(R)} \geq \limsup_{t \rightarrow 0} \|w(t, \cdot)\|_{L^\infty(R)}.$$

In concert with (2.36), this implies

$$\|w(0, \cdot)\|_{L^\infty(R)} = \lim_{t \downarrow 0} \|w(t, \cdot)\|_{L^\infty(R)}.$$

Since $a D_t \rightarrow 0$ in $L^\infty(R)$ and $z = w + a D_t$, we have shown that $t \mapsto \|z(t, \cdot)\|_{L^\infty(R)}$ is continuous.

In light of (2.33), this implies that

$$t^* := \max \{ t \in [0, S] : \|z(t, \cdot)\|_{L^\infty(R)} \leq 2 \}$$

exists and is positive. Moreover, if $t^* < S$, then $\|z(t^*, \cdot)\|_{L^\infty(R)} = 2$ and, by (2.35), there exists $x_* \in R$ so that

$$|z(t^*, x_*)| = 2 = \max_{[0, t^*] \times R} |z|.$$

As a consequence, we have

$$\text{sgn}(z(t^*, x_*)) \partial_t z(t^*, x_*) \geq 0, \quad \partial_x z(t^*, x_*) = 0, \quad \text{sgn}(z(t^*, x_*)) \partial_x^2 z(t^*, x_*) \leq 0.$$

Then (2.34) yields, at the point $(t^*, x_*)$,

$$0 \leq 2 \partial_t (\log a) + |a^2 (\log a)| + |a \partial_x (\log a)|^2 + 4|a^{-1} \partial_x (\log a)|$$

$$+ 2|\psi^{[L]}| |\partial_x (\log a)| + (2 + a |\psi^{[L]}|) |\partial_x \psi^{[L]}|.$$  \hfill (2.37)

Note that

$$|\partial_x (\log a)| = \left| -\frac{(\ell_1 + \epsilon t)x}{(x)^2 + K^2/(1-\ell')} \right| \leq 1.$$
Proposition 2.11. For each $L$, for all $t \leq 1$ and $\ell' < 1$. We further compute

\[
|\partial_x^2 (\log a)| = \left| -\frac{\ell_1 + \varepsilon t}{(x)^2 + K^{2/(1-\ell')}} + \frac{2(\ell_1 + \varepsilon t)x^2}{((x)^2 + K^{2/(1-\ell')})^2} \right| \leq 3 \tag{2.39}
\]

and

\[
\partial_t (\log a) = -\frac{\varepsilon}{2} \log ((x)^2 + K^{2/(1-\ell')}). \tag{2.40}
\]

Applying (2.32) and (2.38)–(2.40) to (2.37), and using the triangle inequality, we obtain

\[
0 \leq -\varepsilon \log ((x_s)^2 + K^{2/(1-\ell')}) + 8 + \|\psi [L]\|_{C_b([0,1];C^1_b(\mathbb{R}))} g(x_s) (4 + \|\psi [L]\|_{C_b([0,1];C^1_b(\mathbb{R}))} g(x_s) (x_s - \ell_1)). \tag{2.41}
\]

Now we can choose $K$ large enough, depending on $\ell_1$, $\ell'$, and $A$, so that the right side of (2.41) is guaranteed to be strictly negative regardless of $x_s$, which is a contradiction. Therefore, we have $|z| \leq 2$ on $[0, S] \times \mathbb{R}$, so

\[
|\theta [L](t, x)| \leq 2K ((x)^2 + K^{2/(1-\ell')})^{(\ell_1+\varepsilon t)/2}
\]

for all $t \in [0, S]$, $x \in \mathbb{R}$, and (2.31) follows. \(\square\)

Proposition 2.11. For each $L \in (0, \infty)$ and $v^{[L]} \in L^\infty(\mathbb{R}/L\mathbb{Z})$ there exists a solution $\theta^{[L]} \in Z^1_{[L]}$ to (2.14)–(2.15).

Proof. By Proposition 2.10 and the fact that there is a continuous embedding from the space of $L$-periodic functions in $C^\infty(\mathbb{R})$ into $L^\infty(\mathbb{R}/L\mathbb{Z})$ for every $L$ and $\ell$, (2.27) implies that $S > 1$ in Proposition 2.9, so $\theta^{[L]} \in Z^1_{[L]}$. \(\square\)

2.7. Solutions on the whole space. We now pass to the limit $L \to \infty$ to obtain global in time solutions to (2.14)–(2.15) on the whole space. In order to show that the sequence $\theta^{[L]}$ is Cauchy as $L \to \infty$, we will need some continuity of the solutions to the periodic problem with respect to the forcing and the initial conditions. The following proposition does this in a weaker topology, which uses weights growing superexponentially at infinity.

Proposition 2.12. Fix $\ell \in (0, 1)$, $T > 0$. Suppose $v_i \in L^\infty(\mathbb{R})$ and $\theta_i, \psi_i \in C_b((0, T]; C^\infty(\mathbb{R}))$ satisfy

\[
\theta_i(t, \cdot) = G_t * v_i - \int_0^t \partial_s G_{t-s} * (\theta_i(s, \cdot) + \psi_i(s, \cdot))^2 \, ds
\]

and

\[
|\partial_x (\log a)| = (\ell_1 + \varepsilon t)|x| K ((x)^2 + K^{2/(1-\ell')})(\ell_1 - \varepsilon t)/2 - 1
\]

\[
\leq \frac{|x|}{((x)^2 + K^{2/(1-\ell')})^{1/2}} \cdot K \leq 1, \tag{2.38}
\]

since $t \leq 1$ and $\ell' < 1$. We further compute

\[
|\partial_x^2 (\log a)| = \left| -\frac{\ell_1 + \varepsilon t}{(x)^2 + K^{2/(1-\ell')}} + \frac{2(\ell_1 + \varepsilon t)x^2}{((x)^2 + K^{2/(1-\ell')})^2} \right| \leq 3 \tag{2.39}
\]

and

\[
\partial_t (\log a) = -\frac{\varepsilon}{2} \log ((x)^2 + K^{2/(1-\ell')}). \tag{2.40}
\]
for all \( t \in [0, T] \) and \( i \in \{1, 2\} \). Fix \( \beta \in (2\ell \vee (3/2), 2) \) and define the weight \( q_\beta(x) = \exp(\lambda(x)\beta) \) as in Lemma B.6. Then there exists a constant \( C = C(\ell, \beta, T) < \infty \) so that

\[
\|\theta_1 - \theta_2\|_{C_b((0, T]; C_{q_{\beta}}(\mathbb{R}))} \leq e^{CX} \left( \|v_1 - v_2\|_{L^\infty_T(\mathbb{R})} + \|\psi_1 - \psi_2\|_{C_b((0, T]; C_{p_\beta}(\mathbb{R}))} \right),
\]

where

\[
X = 1 + \sum_{i=1}^2 \left[ \|v_i\|_{C_b((0, T]; C_{p_\beta}(\mathbb{R}))} + \|\psi_i\|_{C_b((0, T]; C_{p_\beta}(\mathbb{R}))} \right].
\]

Proof. The proof is similar in spirit to that of [36, Proposition 4.2]. We begin with

\[
\|\theta_1(t, \cdot) - \theta_2(t, \cdot)\|_{C_{q_{\beta}}(\mathbb{R})} 
\leq \|G_t * (v_1 - v_2)\|_{C_{q_{\beta}}(\mathbb{R})} 
+ \int_0^t \left\| \partial_x G_{t-s} * \left[ (\theta_1 + \psi_1)(s, \cdot)^2 - (\theta_2 + \psi_2)(s, \cdot)^2 \right] \right\|_{C_{q_{\beta}}(\mathbb{R})} \, ds. \tag{2.43}
\]

By Lemma B.4 and the fact that \( |x| \leq e^{|x|} \) for all \( x \in \mathbb{R} \), there is a constant \( C < \infty \) so that

\[
\|G_t * (v_1 - v_2)\|_{C_{q_{\beta}}(\mathbb{R})} \leq C\|v_1 - v_2\|_{C_{p_\beta}(\mathbb{R})}. \tag{2.44}
\]

We handle the integral term in (2.43) using Lemma B.6:

\[
\left| \partial_x G_{t-s} * \left[ (\theta_1 + \psi_1)(s, \cdot)^2 - (\theta_2 + \psi_2)(s, \cdot)^2 \right](x) \right|
\leq C(t-s)^{-\frac{1}{2}} e^{C(t-s)\langle x \rangle^{2(\beta-1)}} q_{1+\frac{1}{2}(s+\tau)}(x) 
\times \|\theta_1 + \psi_1\|_{C_{q_{1+\frac{1}{2}(s+\tau)}}(\mathbb{R})} \|\theta_2 + \psi_2\|_{C_{q_{1+\frac{1}{2}(s+\tau)}}(\mathbb{R})}. \tag{2.45}
\]

We note that, as \( 0 < \beta < 2 \), we have

\[
\sup_{x \in \mathbb{R}} \frac{e^{C(t-s)\langle x \rangle^{2(\beta-1)}} q_{1+\frac{1}{2}(s+\tau)}(x)}{q_{1+\frac{1}{2}(s+\tau)}(x)} = \exp \left\{ (t-s) \sup_{x \in \mathbb{R}} \left[ C\langle x \rangle^{2(\beta-1)} - \frac{1}{2}\langle x \rangle^\beta \right] \right\} \leq e^{C(t-s)}, \tag{2.46}
\]

for some new constant \( C \). We also have

\[
\|\theta_1 + \psi_1\|_{C_{q_{1+\frac{1}{2}(s+\tau)}}(\mathbb{R})}
\leq \|\theta_1 + \theta_2 + \psi_1 + \psi_2\|_{C_{q_{1+\frac{1}{2}(t-s)}}(\mathbb{R})} \|(\theta_1 - \theta_2 + \psi_1 - \psi_2)(s, \cdot)\|_{C_{q_{1+\frac{1}{2}}(\mathbb{R})}}, \tag{2.47}
\]

and

\[
\|\theta_1 + \theta_2 + \psi_1 + \psi_2\|_{C_{q_{1+\frac{1}{2}(t-s)}}(\mathbb{R})}
\leq \left( \sup_{y \in \mathbb{R}} \frac{p_\epsilon(y)}{q_{1+\frac{1}{2}(t-s)}(y)} \right) \|((\theta_1 + \theta_2 + \psi_1 + \psi_2)(s, \cdot))\|_{C_{p_\beta}(\mathbb{R})}.
\]


\[ \leq C(t-s)^{-\ell/p} \| (\theta_1 + \theta_2 + \psi_1 + \psi_2)(s, \cdot) \|_{C_{p_\ell}(\mathbb{R})} \]  

(2.48)

for another constant C. Using the bounds (2.46)–(2.48) in (2.45), we obtain

\[ \| \partial_x G_{t-s} * [(\theta_1 + \psi_1)(s, \cdot)^2 - (\theta_2 + \psi_2)(s, \cdot)^2] \|_{C_{q_{1+\sigma}}(\mathbb{R})} \]

\[ \leq C e^{C(t-s)}(t-s)^{-\frac{1}{2}-\frac{\ell}{p}} \| (\theta_1 + \theta_2 + \psi_1 + \psi_2)(s, \cdot) \|_{C_{p_\ell}(\mathbb{R})} \]

\[ \times \| (\theta_1 - \theta_2 + \psi_1 - \psi_2)(s, \cdot) \|_{C_{q_{1+\sigma}}(\mathbb{R})}. \]  

(2.49)

Using (2.44) and (2.49) in (2.43), we obtain

\[ \| \theta_1(t, \cdot) - \theta_2(t, \cdot) \|_{C_{q_{1+\sigma}}(\mathbb{R})} \]

\[ \leq A + CX \int_0^t e^{C(t-s)}(t-s)^{-\frac{1}{2}-\frac{\ell}{p}} \| (\theta_1 - \theta_2)(s, \cdot) \|_{C_{q_{1+\sigma}}(\mathbb{R})}, \]

where

\[ A = C \| v_1 - v_0 \|_{C_{p_\ell}(\mathbb{R})} + CX \| \psi_1 - \psi_2 \|_{C_b([0,T];C_{q_1}(\mathbb{R}))}. \]

Therefore, Grönwall’s inequality implies

\[ \| \theta_1(t, \cdot) - \theta_2(t, \cdot) \|_{C_{q_{1+\sigma}}(\mathbb{R})} \leq A \exp \left\{ CX \int_0^t e^{C(t-s)}(t-s)^{-\frac{1}{2}-\frac{\ell}{p}} ds \right\}, \]

and (2.42) follows. \( \square \)

We can now take \( L \to \infty \) and prove Proposition 2.2.

**Proof of Proposition 2.2.** It suffices to prove that there exists such a \( \theta \in Z_{m,1} \); then the result follows by iteration from \( t = 1 \) to \( t = 2 \), etc. Fix constants \( m < \ell_1 < \ell_2 < \ell \), and note that for \( v \in L_{p_{\ell_1}}^\infty(\mathbb{R}) \) we have

\[ \lim_{L \to \infty} \| v^{[L]} - v \|_{L_{p_{\ell_1}}^\infty(\mathbb{R})} = 0, \]

(2.50)

and

\[ \| v^{[L]} \|_{L_{p_{\ell_1}}^\infty(\mathbb{R})} \leq 3 \| v \|_{L_{p_{\ell_1}}^\infty(\mathbb{R})}. \]

(2.51)

Proposition 2.11 implies that there exists a solution \( \theta^{[L]} \in Z_{1}^{[L]} \) to (2.14)–(2.15), while Proposition 2.10, the bound (2.51), and Lemma 2.4 imply that there is a number \( Y < \infty \), depending on \( \| v \|_{L_{p_{\ell_1}}^\infty(\mathbb{R})}, \psi, \ell_1 \), and \( \ell_2 \), but not on \( L \), so that

\[ \| \theta^{[L]} \|_{C_b((0,1];C_{p_{\ell_2}}(\mathbb{R}))}, \| \psi^{[L]} \|_{C_b((0,1]C_{p_{\ell_2}}(\mathbb{R}))} \leq Y, \]

(2.52)

for each \( L \in [1, \infty) \). It is also clear from (2.13) that

\[ \lim_{L \to \infty} \| \psi^{[L]} - \psi \|_{C_b((0,1];C_{p_{\ell_1}}(\mathbb{R}))} = 0. \]

(2.53)

Hence by Proposition 2.12, there is a constant \( C = C(\ell_1) \) so that

\[ \| \theta^{[L]} - \theta^{[L']} \|_{C_b((0,1];C_{q_2}(\mathbb{R}))} \]

\[ \leq e^{CY} \left( \| v^{[L]} - v^{[L']} \|_{C_{p_{\ell_1}}(\mathbb{R})} + \| \psi^{[L]} - \psi^{[L']} \|_{C_b((0,1];C_{p_{\ell_1}}(\mathbb{R}))} \right), \]

(2.54)
where \( q_3 \) is defined as in the statement of Proposition 2.12 for some \( \beta \in ((2 \ell_1) \vee (3/2), 2) \) fixed. It follows from (2.50) and (2.53) that the right side of (2.54) goes to 0 as \( L, L' \to \infty \), so the left side does as well. Hence there is a \( \theta \in C_b((0, 1]; C_{q_2}(\mathbb{R})) \) such that

\[
\lim_{L \to \infty} \|\theta^{[L]} - \theta\|_{C_b((0, 1]; C_{q_2}(\mathbb{R}))} = 0.
\]

By (2.52) and Proposition B.1, this implies that in fact \( \theta \in C_b((0, 1]; C_{p_4}(\mathbb{R})) \) and

\[
\lim_{L \to \infty} \|\theta^{[L]} - \theta\|_{C_b((0, 1]; C_{p_4}(\mathbb{R}))} = 0.
\]  

We claim that \( \theta \) is a mild solution to (2.6), i.e. that \( \theta \) satisfies (2.21) with \( L = \infty \). By (2.50) and Lemma B.4, we have

\[
\lim_{L \to \infty} G_t * v^{[L]}(x) = G_t * v(x)
\]  

for each \((t, x) \in [0, 1] \times \mathbb{R}\). Also, we have

\[
\int_0^t \|\partial_x G_{t-s} \ast [(\theta(s, \cdot) + \psi(s, \cdot))^2 - (\theta^{[L]}(s, \cdot) + \psi^{[L]}(s, \cdot))^2]\|_{C_{p_2}(\mathbb{R})} ds \\
\leq \int_0^t (t-s)^{-\frac{1}{2}} \|\theta(s, \cdot) + \psi(s, \cdot)^2 - (\theta^{[L]}(s, \cdot) + \psi^{[L]}(s, \cdot))^2\|_{C_{p_2}(\mathbb{R})} ds \\
\to 0
\]

as \( L \to \infty \) by Lemma B.4, (2.53), and (2.55), which in particular means that for each \((t, x) \in [0, 1] \times \mathbb{R}\) we have

\[
\lim_{L \to \infty} \int_0^t \partial_x G_{t-s} \ast (\theta^{[L]}(s, \cdot) + \psi^{[L]}(s, \cdot))^2(x) ds = \int_0^t \partial_x G_{t-s} \ast (\theta(s, \cdot) + \psi(s, \cdot))^2(x) ds
\]

(2.57)

Since we also have \( \theta^{[L]}(t, x) \to \theta(t, x) \) as \( L \to \infty \), and each \( \theta^{[L]} \) satisfies (2.21), we have from (2.56) and (2.57) that \( \theta \) satisfies (2.21) with \( L = \infty \).

The measurability in the statement of Proposition 2.2 is obvious, and the uniqueness of \( \theta \) and the continuity of the map \( v \mapsto \theta \) follow immediately from Propositions 2.12 and B.1.

Now suppose \( A \subset L_{p_{1,1}}^\infty(\mathbb{R}) \) is bounded, and hence bounded in \( L_{p_{1,1}}^\infty(\mathbb{R}) \). Take \( v \in A \) and fix \( \ell_3 \in (\ell_2, \ell) \). By (2.52), we have

\[
\|\theta\|_{C_b((0, 1]; C_{p_{1,1}}(\mathbb{R}))} \leq Y < +\infty,
\]

with some \( Y \) that depends on \( \psi, \ell_1, \ell_2 \), and \( A \). In particular, \( \Phi(A)|_{(0, 1] \times \mathbb{R}} \) is bounded in the \( C_b((0, 1]; C_{p_4}(\mathbb{R})) \) norm. For \( t \in (0, 1), (2.22) \) yields

\[
\|\theta(t, \cdot)\|_{C_{p_4}(\mathbb{R}))} \leq C(t, \psi, Y) < \infty.
\]

Thus by Proposition B.2, \( \Phi(A)|_{[t] \times \mathbb{R}} \) is compact in \( C_{p_4}(\mathbb{R}) \). Since \( \ell > m \) was arbitrary, \( \Phi(A)|_{(0, 1] \times \mathbb{R}} \) is bounded in \( \mathcal{Z}_{m,1} \) and Lemma B.3 implies that \( \Phi(A)|_{[t] \times \mathbb{R}} \) is compact in \( \mathcal{X}_m \). \( \square \)

As we have mentioned, Theorem 1.1 follows from Propositions 2.2 and 2.3.
3. Comparison Principle and $L^1$-contraction

The uniqueness and stability results rely crucially on the comparison principle and $L^1$-contraction that are well known for the deterministic Burgers equation (1.4) with $V \equiv 0$; see for example [23,52] and references therein. Here, we establish these properties for the stochastic Burgers equation. The proofs are similar to those in the deterministic case, but some care is required to deal with the growth of solutions at infinity. Also, we will need the $L^1$-contraction with “$L^1$” interpreted separately as $L^1(\mathbb{R})$ and as $L^1(\Omega \times I)$ for an interval $I$. (Recall that $\Omega$ is the probability space.) These are different statements, the latter being irrelevant in the deterministic case.

Moreover, in Sect. 3.3 below we prove an ordering property for time-stationary solutions. This is a novel element here, as it holds for invariant measures and does not require a comparison at a fixed initial time.

Throughout, we rely on an equation for the difference of two solutions. If $u = (u_1, u_2) \in \mathcal{Z}_m^2$ is a solution to the system (1.4), and we define $\psi$ as in (2.2), then we have

$$\eta = u_1 - u_2 = (u_1 - \psi) - (u_2 - \psi),$$

and each of the two terms in parentheses satisfies the PDE (2.6), with the corresponding initial conditions. Subtracting these two copies of (2.6), we see that $\eta$ is differentiable in time and satisfies the partial differential equation

$$\partial_t \eta = \frac{1}{2} \partial^2_x \eta - \frac{1}{2} \partial_x (\eta \xi),$$  \hspace{1cm} (3.1)

$$\eta(0, x) = u_1(0, x) - u_2(0, x)$$  \hspace{1cm} (3.2)

almost surely, with $\xi = u_1 + u_2$.

3.1. Pathwise results. First we state the almost-sure comparison, contraction, and conservation properties, which involve the whole space $\mathbb{R}$.

**Proposition 3.1** (Comparison principle). Fix $m \in (0, 1)$. If $u = (u_1, u_2) \in \mathcal{Z}_m^2$ solves (1.4) and satisfies $u_1(0, x) \leq u_2(0, x)$ for all $x \in \mathbb{R}$, then $u_1(t, x) \leq u_2(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}$.

**Proposition 3.2** ($L^1(\mathbb{R})$-contraction). Fix $m \in (0, 1)$. If $u = (u_1, u_2) \in \mathcal{Z}_m^2$ solves (1.4), then for all $t \geq 0$ we have

$$\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_1(0, \cdot) - u_2(0, \cdot)\|_{L^1(\mathbb{R})}. \hspace{1cm} (3.3)$$

**Proposition 3.3** (Conservation of mass). Fix $m \in (0, 1)$. If $u = (u_1, u_2) \in \mathcal{Z}_m^2$ solves (1.4) and

$$\|u_1(0, \cdot) - u_2(0, \cdot)\|_{L^1(\mathbb{R})} < \infty, \hspace{1cm} (3.4)$$

then for all $t \geq 0$ we have

$$\int_{\mathbb{R}} [u_1(t, x) - u_2(t, x)] dx = \int_{\mathbb{R}} [u_1(0, x) - u_2(0, x)] dx. \hspace{1cm} (3.5)$$
Of course, (3.3) has no content unless the right side is finite (i.e. if (3.4) holds). Note that we will not use Proposition 3.2 or Proposition 3.3 in the sequel. They are included for completeness and because they can be proved very similarly to Proposition 3.1 (which we will indeed use). Lemma 3.5 below is the core of the argument. First, we make the following elementary observation.

**Lemma 3.4.** If $F : \mathbb{R} \to \mathbb{R}_+ \geq 0$ satisfies $F(0) = 0$ and $F'' = c_1 \delta_0$ with some $c_1 \geq 0$, then there is a family $F_\varepsilon \in C^2(\mathbb{R})$ of convex functions that has the following properties:

1. There is a constant $C < \infty$ so that for all $\varepsilon \in (0, 1]$ and $x \in \mathbb{R}$, we have
   \[F_\varepsilon(x) \leq C(|x| + \varepsilon),\] (3.6)
   \[|xF_\varepsilon'(x)| \leq CF_\varepsilon(x),\] (3.7)
   \[|F_\varepsilon'(x)| \leq C,\] (3.8)
   \[|x|F_\varepsilon''(x) \leq C1_{[0,\varepsilon]}(x).\] (3.9)

2. The restriction $F_\varepsilon|_{\mathbb{R}\setminus[-1,1]}$ is independent of $\varepsilon$.
3. We have
   \[\lim_{\varepsilon \to 0} \|F_\varepsilon - F\|_{C^b(\mathbb{R})} = 0.\] (3.10)

**Proof.** If $F$ is as in the statement of the lemma, then there is some $c_2 \in \mathbb{R}$ so that $F(x) = c_1|x| + c_2 x$. From this it is straightforward to construct such a family directly. \qed

Propositions 3.1 and 3.2 are special cases of the following lemma.

**Lemma 3.5.** Let $F : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfy $F(0) = 0$ and $F'' = c_1 \delta_0$ with some $c_1 \geq 0$. Assume that $m \in (0, 1)$, and $u = (u_1, u_2) \in \mathcal{Z}_m$ solves (1.4), and set $\eta = u_1 - u_2$, then for all $t \geq 0$, we have

\[
\int_\mathbb{R} [F \circ \eta](t, x) \, dx + \frac{c_1}{4} \int_0^t \left[ \sum_{y \in \eta(s, \cdot)^{-1}(0)} |\eta'(y)| \right] \, ds \leq \int_\mathbb{R} [F \circ \eta](0, x) \, dx. \tag{3.11}
\]

Before we prove Lemma 3.5, let us show how it implies Propositions 3.1 and 3.2.

**Proof of Propositions 3.1 and 3.2.** We use Lemma 3.5 with the nonnegative convex function

\[F(x) = x^+ := \max\{x, 0\}\]

that satisfies $F'' = \delta_0$. This gives

\[\|\eta(s, \cdot)^+\|_{L^1(\mathbb{R})} \leq \|\eta(0, \cdot)^+\|_{L^1(\mathbb{R})} = 0\]

for all $s \in [0, T]$, so $\eta(s, \cdot)^+ = 0$ almost everywhere, meaning $u_1 \leq u_2$. Similarly, using (3.11) with the function $F(x) = |x|$, which satisfies $F'' = 2\delta_0$, implies (3.3). \qed

We now prove Lemma 3.5.
Proof of Lemma 3.5. We first define an appropriate cutoff function. Fix $\ell \in (m, 1)$ and define 

$$
\zeta(x) = e^{21-\ell-\langle x \rangle_1-\ell},
$$

which is positive, decreasing in $|x|$, and in the Schwartz class. (The constant 2 corresponds to the constant 4 in (2.7) and is fixed to obtain (3.14) below.) Also, there is a constant $C$ so that

$$p_\ell(x)|\zeta'(x)| + |\zeta''(x)| \leq C\zeta(x)$$

for all $x \in \mathbb{R}$. For $\delta \in (0, 1]$, define the rescaled version

$$
\zeta_\delta(x) := \zeta(\delta x),
$$

which satisfies

$$p_\ell(x)|\zeta'_\delta(x)| = \delta p_\ell(x)|\zeta'(\delta x)| \leq \delta^{1-\ell} p_\ell(\delta x)|\zeta'(\delta x)| \leq C\delta^{1-\ell} |\zeta_\delta(x)|$$

and

$$|\zeta''_\delta(x)| = \delta^2 |\zeta''(\delta x)| \leq C\delta^2 |\zeta_\delta(x)|.$$  

Moreover, $\zeta_\delta(x)$ is decreasing in $\delta$ and, for each $x \in \mathbb{R}$, we have

$$\lim_{\delta \downarrow 0} \zeta_\delta(x) = 1.$$  

Let $\{F_\varepsilon\}_{\varepsilon \in (0,1]}$ be as in Lemma 3.4 and define

$$I_{\varepsilon,\delta}(t) := \int_{\mathbb{R}} (F_\varepsilon \circ \eta)(t, x))\zeta_\delta(x) \, dx.$$  

By Lemma B.5, $\eta$ is strongly continuous in $L^1_{p_2}(\mathbb{R})$. Since $\zeta_\delta$ decays much faster than $p_2$, (3.6) and (3.7) imply that $I_{\varepsilon,\delta}$ is continuous in $t$.

Applying the chain rule to (3.1) and integrating by parts, we obtain

$$
\frac{d}{dt} I_{\varepsilon,\delta} = \int_{\mathbb{R}} \left[ (F_\varepsilon \circ \eta)'' - (F_\varepsilon' \circ \eta)\eta\xi' \\
- (F_\varepsilon'' \circ \eta)(\eta')^2 + (F_\varepsilon'' \circ \eta)(\xi\eta') \right] \zeta_\delta \, dx
\]

$$= \frac{1}{2} \int_{\mathbb{R}} \left[ (F_\varepsilon \circ \eta)\zeta''_\delta + (F_\varepsilon' \circ \eta)\eta\xi\zeta'_\delta - (F_\varepsilon'' \circ \eta)[(\eta')^2 - \xi \eta']\zeta_\delta \right] \, dx.$$  

The boundary terms vanish because $\zeta_\delta$ is in the Schwartz class. Now Young’s inequality yields

$$
\frac{d}{dt} I_{\varepsilon,\delta} \leq \int_{\mathbb{R}} \left\{ (F_\varepsilon \circ \eta)\zeta''_\delta + (F_\varepsilon' \circ \eta)\eta\xi\zeta'_\delta \\
+ \frac{1}{2}(F_\varepsilon'' \circ \eta) \left[-(\eta')^2 + \xi^2 \eta^2 \right] \zeta_\delta \right\} \, dx.
$$

We have by (3.13) that

$$
\int_{\mathbb{R}} |(F_\varepsilon \circ \eta)\zeta''_\delta| \, dx \leq C\delta^2 \int_{\mathbb{R}} (F_\varepsilon \circ \eta) \zeta_\delta \, dx.
$$
and by (3.7) and (3.12) that
\[ \int_{\mathbb{R}} |(F'_\varepsilon \circ \eta) \eta' \zeta_\delta'| \, dx \leq C \| \xi \|_{C_p(\mathbb{R})} \int_{\mathbb{R}} |(F_\varepsilon \circ \eta) p_\ell \zeta_\delta'| \, dx \]
\[ \leq C \delta^{1-\ell} \| \xi \|_{C_p(\mathbb{R})} \int_{\mathbb{R}} (F_\varepsilon \circ \eta) \zeta_\delta \, dx. \]

Also note that, by (3.9), we have
\[ \int_{\mathbb{R}} |(F''_\varepsilon \circ \eta) \xi_2^2 \eta_2 \zeta_\delta| \, dx \leq C \varepsilon \int_{\mathbb{R}} |\xi_2^2 \zeta_\delta| \, dx \leq C \varepsilon \| \xi \|_{C_p(\mathbb{R})}^2 \| p_\ell^2 \zeta_\delta \|_{L^1(\mathbb{R})}. \]

Substituting the last three displays into (3.15), we obtain
\[ \frac{d}{dt} I_{\varepsilon, \delta} \leq C (\delta^2 + \delta^{1-\ell} \| \xi \|_{C_p(\mathbb{R})} I_{\varepsilon, \delta} + \varepsilon \| \xi \|_{C_p(\mathbb{R})}^2 \| p_\ell^2 \zeta_\delta \|_{L^1(\mathbb{R})}) \]
\[ - \frac{1}{4} \int_{\mathbb{R}} (F''_\varepsilon \circ \eta)(\eta')^2 \zeta_\delta \, dx. \]

Integrating in time, the continuity of $I_{\varepsilon, \delta}$ implies
\[ I_{\varepsilon, \delta}(t) - I_{\varepsilon, \delta}(0) \leq B_t \delta^{1-\ell} \int_0^t I_{\varepsilon, \delta}(s) \, ds + t \varepsilon B_t^2 \| p_\ell^2 \zeta_\delta \|_{L^1(\mathbb{R})} \]
\[ - \frac{1}{4} \int_0^t \int_{\mathbb{R}} (F''_\varepsilon \circ \eta(\cdot))(\eta')^2(\cdot, \cdot) \zeta_\delta \, ds, \]

where we have defined
\[ B_t = C (\delta^{1+\ell} + \sup_{s \in [0, t]} \| \xi(s, \cdot) \|_{C_p(\mathbb{R})}). \]

Let us look at the last term in the right side of (3.16). By the coarea formula, we have
\[ \int_{\mathbb{R}} F''_\varepsilon(\eta(s, x)) \eta'(s, x)^2 \zeta_\delta(x) \, dx = \int_{\mathbb{R}} F''_\varepsilon(\lambda) \left[ \sum_{y \in \eta(s, \cdot)^{-1}(\lambda)} |\eta'(s, y)| \zeta_\delta(y) \right] d\lambda. \]

Using Lemma D.3 and the fact that $F''_\varepsilon$ converges weakly to $c_1 \delta_0$ as $\varepsilon \to 0$, we obtain
\[ \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\mathbb{R}} F''_\varepsilon(\eta(s, x)) \eta'(s, x)^2 \zeta_\delta(x) \, dx \, ds \]
\[ = c_1 \int_0^t \left[ \sum_{y \in \eta(s, \cdot)^{-1}(0)} |\eta'(y)| \zeta_\delta(y) \right] ds. \]

In addition, by (3.6), (3.10), the fact that $\eta \in \mathcal{C}_b((0, t]; C_p(\mathbb{R}))$, and the dominated convergence theorem, we have for each fixed $\delta \in (0, 1]$ that
\[ \lim_{\varepsilon \downarrow 0} I_{\varepsilon, \delta}(s) = \int_{\mathbb{R}} (F \circ \eta)(s, x) \zeta_\delta(x) \, dx \]
\[ \text{and} \]
\[ \lim_{\varepsilon \downarrow 0} \int_0^t I_{\varepsilon, \delta}(s) \, ds = \int_0^t \int_{\mathbb{R}} (F \circ \eta)(s, x) \zeta_\delta(x) \, dx \, ds. \]
We pass to the limit $\varepsilon \downarrow 0$ in (3.16) and apply (3.17) and (3.18) to obtain

\[
I_\delta(t) := \int_\mathbb{R} (F \circ \eta)(t, x) \zeta_\delta(x) \, dx = \lim_{\varepsilon \downarrow 0} I_{\varepsilon, \delta}(t)
\]

\[
\leq I_\delta(0) + B_t \delta^{1-\ell} \int_0^t I_\delta(s) \, ds - \frac{c_1}{4} \int_0^t \left[ \sum_{y \in \eta(s, \cdot)^{-1}(0)} |\eta'(y)| \zeta_\delta(y) \right] \, ds.
\]

As $B_t$ is increasing in $t$, it follows from the Grönwall inequality that

\[
I_\delta(t) \leq I_\delta(0) - \frac{c_1}{4} \int_0^t \left[ \sum_{y \in \eta(s, \cdot)^{-1}(0)} |\eta'(y)| \zeta_\delta(y) \right] \, ds + I_\delta(0)t B_t \delta^{1-\ell} \exp \left\{ \delta^{1-\ell} t B_t \right\}.
\]

By the monotone convergence theorem and (3.14), we conclude that

\[
I(t) := \int_\mathbb{R} (F \circ \eta)(t, x) \, dx
\]

satisfies

\[
I(t) + \frac{c_1}{4} \int_0^t \left[ \sum_{y \in \eta(s, \cdot)^{-1}(0)} |\eta'(y)| \right] \, ds \leq I(0)
\]

as claimed.  \(\square\)

We can use a similar argument to prove Proposition 3.3.

**Proof of Proposition 3.3.** Define $\zeta_\delta$ as in the proof of Lemma 3.5 and let

\[
I_\delta(t) = \int_\mathbb{R} \eta(t, x) \zeta_\delta(x) \, dx.
\]

Again, Lemma B.5 shows that $\eta$ is strongly continuous in $L^1_p(\mathbb{R})$, so $I_\delta$ is continuous. As in the proof of Lemma 3.5, we have

\[
\frac{d}{dt} I_\delta = -\frac{1}{2} \int_\mathbb{R} [\eta'' - (\eta \xi)'] \zeta_\delta \, dx = -\frac{1}{2} \int_\mathbb{R} [\eta \xi'' - \eta \xi \zeta'_\delta] \, dx,
\]

so that

\[
\left| \frac{d}{dt} I_\delta \right| \leq \frac{1}{2} \int_\mathbb{R} \left| \eta' \right| \left[ |\xi''| + |\xi| |\zeta'_\delta| \right] \, dx
\]

\[
\leq C(\delta^2 + \delta^{1-\ell} \|\xi(t, \cdot)\|_{L^1(\mathbb{R})}) \|\eta(t, \cdot)\|_{L^1(\mathbb{R})}
\]

\[
\leq C(\delta^2 + \delta^{1-\ell} \|\xi(t, \cdot)\|_{C_p(\mathbb{R})}) \|\eta(0, \cdot)\|_{L^1(\mathbb{R})},
\]

where in the last inequality we used Proposition 3.2 and (3.4). We now pass to the limit $\delta \to 0$ and use the dominated convergence theorem, Proposition 3.2 and (3.4), to obtain

(3.5).  \(\square\)
3.2. Conservation and $L^1$-contraction in the probability space. We now prove results similar to Propositions 3.3 and 3.2 for solutions stationary with respect to a group $G$ of spatial translations, either with $G = \mathbb{R}$ or by $G = L\mathbb{Z}$, with some $L > 0$. We define the fundamental domain $\Lambda_G = \{0\}$ if $G = \mathbb{R}$ and $\Lambda_G = \{0, L\}$ if $G = L\mathbb{Z}$, and let $\lambda_G$ be the unique translation-invariant measure on $\mathbb{R}$ such that $\lambda_G(\Lambda_G) = 1$. In other words, $\lambda_G$ is the counting measure and $\lambda_{L\mathbb{Z}}$ is $1/L$ times the Lebesgue measure. We recall that $\mathcal{P}_G(X)$ is the space of probability measures on $X$ that are invariant under the action of $G$, as in (1.6).

**Proposition 3.6** (Conservation of mass in the probability space). Let $u = (u_1, u_2) \in Z_m^2$ be a solution to (1.4) such that $\text{Law}(u(0, \cdot)) \in \mathcal{P}_G(L^\infty_{pm'}(\mathbb{R})^2)$, with some $m \in (0, 1)$, and set $\eta = u_1 - u_2$. Assume also that

$$
\sup_{i \in \{1, 2\}} \mathbb{E} \left[ \int_{\Lambda_G} u_i(t, x)^2 \, d\lambda_G(x) \right] < \infty. \tag{3.19}
$$

Then, for all $t \geq 0$ we have

$$
\mathbb{E} \left[ \int_{\Lambda_G} \eta(t, x) \, d\lambda_G(x) \right] = \mathbb{E} \left[ \int_{\Lambda_G} \eta(0, x) \, d\lambda_G(x) \right]. \tag{3.20}
$$

Strictly speaking, if $G = \mathbb{R}$, the integral in the right side of (3.20) is

$$
\int_{\Lambda_G} \eta(0, x) \, d\lambda_G(x) = \eta(0, 0),
$$

and is ill-defined because $u(0, \cdot) \in L^\infty_{pm'}(\mathbb{R})^2$ is not defined pointwise. This obstruction is merely formal. Here and in the sequel, we use the convention

$$
\mathbb{E} f(0) = \mathbb{E} \int_0^1 f(x) \, dx,
$$

whenever $\text{Law}(f) \in \mathcal{P}_\mathbb{R}(L^\infty_{pm'}(\mathbb{R})^2)$.

**Proof.** First, assume that $G = L\mathbb{Z}$ for some $L \in (0, \infty)$, so $\Lambda_G = \{0, L\}$. Note that

$$
\begin{aligned}
\frac{d}{dt} \int_0^L \eta(t, x) \, dx &= \frac{1}{2} \int_0^L \left[ \partial_x^2 \eta(t, x) - \partial_x (\eta \xi)(t, x) \right] \, dx \\
&= \frac{1}{2} \left[ \partial_x \eta(t, x) - (\eta \xi)(t, x) \right] \bigg|_{x=0}^{x=L}.
\end{aligned} \tag{3.21}
$$

for any $t > 0$.

Integrating in time, we have

$$
\int_0^L \eta(T, x) \, dx - \int_0^L \eta(0, x) \, dx = \frac{1}{2} \int_0^T \left[ \partial_x \eta(t, x) - (\eta \xi)(t, x) \right] \, dt \bigg|_{x=0}^{x=L}. \tag{3.22}
$$

The expected absolute value of the left side is finite by assumption (3.19), and thus so is the absolute expectation of the right side. Since the right side is the difference of two identically-distributed random variables, we can use Lemma D.1 to conclude that

$$
\mathbb{E} \int_0^L \eta(T, x) \, dx = \mathbb{E} \int_0^L \eta(0, x) \, dx,
$$

as claimed. The statement for $G = \mathbb{R}$ follows immediately. \(\Box\)
**Proposition 3.7** \((L^1\text{-contraction in the probability space})\). Suppose that \(u = (u_1, u_2) \in \mathbb{Z}_m^2\), with some \(m \in (0, 1)\), is a solution to (1.4) such that \(\text{Law}(u(0, \cdot)) \in \mathcal{P}_G(L^\infty_{\mathbb{P}_m}(\mathbb{R})^2)\) for \(G = L\mathbb{Z}\), with some \(L > 0\), and that

\[
A := \sup_{t \in [1, 2]} \mathbb{E} \left[ \int_{A(t)} u_i(t, x)^2 d\lambda_G(x) \right] < \infty. \quad (3.23)
\]

Let \(F : \mathbb{R} \to \mathbb{R}_{\geq 0}\) satisfy \(F'' = c \delta_0\) for some \(c > 0\), and set \(\eta = u_1 - u_2\). Then for all \(t \geq 0\) we have

\[
\mathbb{E} \int_0^t F(\eta(t, x)) \, dx + \frac{c}{4} \int_0^t \sum_{y \in [0, L]} |\partial_x \eta(s, y)| \, ds \leq \mathbb{E} \int_0^t F(\eta(0, x)) \, dx. \quad (3.24)
\]

Furthermore, if \(\text{Law}(u(0, \cdot)) \in \mathcal{P}_R(L^\infty_{\mathbb{P}_m}(\mathbb{R})^2)\) then for all \(T \geq 0\) and \(x \in \mathbb{R}\) we have

\[
\mathbb{E} F(\eta(T, x)) \leq \mathbb{E} F(\eta(0, x)). \quad (3.25)
\]

**Proof.** Define the approximants \(F_\varepsilon\) as in Lemma 3.4. Similarly to (3.21), we have

\[
\frac{d}{dt} \int_0^t F_\varepsilon(\eta(t, x)) \, dx = \int_0^t F_\varepsilon'(\eta(t, x)) \left[ \frac{1}{2} \partial_x^2 \eta(t, x) - \frac{1}{2} \partial_x (\eta \xi)(t, x) \right] \, dx
\]

\[
= \frac{1}{2} F_\varepsilon'((\eta(t, x))(\partial_x \eta(t, x) - (\eta \xi)(t, x)) \bigg|_{x = L} - \frac{1}{2} \int_0^t F_\varepsilon''(\eta(t, x)) \left[ (\partial_x \eta(t, x))^2 - ((\partial_x \eta) \eta \xi)(t, x) \right] \, dx,
\]

for all \(t > 0\). Integrating in time, we get

\[
\int_0^t F_\varepsilon(\eta(s, x)) \, dx \bigg|_{x = t} = \int_0^t \frac{1}{2} F_\varepsilon'(\eta(s, x))(\partial_x \eta(s, x) - (\eta \xi)(s, x)) \, ds \bigg|_{x = L}
\]

\[
- \frac{1}{2} \int_0^t \int_0^L F_\varepsilon''(\eta(s, x)) \left[ (\partial_x \eta(s, x))^2 - ((\partial_x \eta) \eta \xi)(s, x) \right] \, dx \, ds. \quad (3.26)
\]

The expectation of the left side is finite by assumption (3.23). We also have

\[
\int_0^t \int_0^L F_\varepsilon''(\eta(s, x)) \left[ -((\partial_x \eta(s, x))^2 + ((\partial_x \eta) \eta \xi)(s, x) \right] \, dx \, ds
\]

\[
\leq \int_0^t \int_0^L F_\varepsilon''(\eta(s, x)) \left[ -\frac{1}{2}((\partial_x \eta(s, x))^2 + \frac{1}{2}((\eta \xi)(s, x))^2 \right] \, dx \, ds \quad (3.27)
\]

\[
\leq -\frac{1}{2} \int_0^t \int_0^L F_\varepsilon''(\eta(s, x))(\partial_x \eta(s, x))^2 \, dx \, dt + \frac{\varepsilon}{2} \int_0^t \int_0^L \xi(s, x)^2 \, dx \, ds,
\]
where the first inequality is by Young’s inequality and the second is by (3.9). It follows that
\[
\mathbb{E} \left( \frac{1}{2} \int_0^t \int_0^L F''_\varepsilon(\eta(s, x)) \left[ (\partial_x \eta(s, x))^2 - ((\partial_x \eta)(\partial_x \xi))(s, x) \right] dx ds \right) 
\leq \mathbb{E} \left( \frac{\varepsilon}{2} \int_0^t \int_0^L \xi(s, x)^2 dx ds \right) < \infty.
\]

In addition, the absolute expectation of the left side of (3.26) is finite by assumption (3.23). Therefore, we can take expectations in (3.26) and apply Lemma D.1, using the fact that the first term in the right side of (3.26) is the difference of two identically-distributed random variables, to obtain, using (3.27)
\[
\mathbb{E} \int_0^L F_\varepsilon(\eta(s, x)) \, dx \bigg|_{s=0}^{s=t} = \frac{1}{2} \mathbb{E} \int_0^t \int_0^L F''_\varepsilon(\eta(s, x)) \left[ -((\partial_x \eta)(\partial_x \xi))(s, x) + ((\partial_x \eta)\eta \xi)(s, x) \right] dx ds 
\leq -\frac{1}{4} \mathbb{E} \int_0^t \int_0^L F''_\varepsilon(\eta(s, x))(\partial_x \eta \eta \xi)(s, x) \, dx ds + \varepsilon TLA. \tag{3.28}
\]

We would like to pass to the limit \( \varepsilon \downarrow 0 \) in (3.28). For the left side, using the bounded convergence theorem on \( F_\varepsilon \rangle \mathbb{R} \setminus [-1, 1] \), and the assumption that \( F_\varepsilon \rangle \mathbb{R} \setminus [-1, 1] \) is independent of \( \varepsilon \), we have
\[
\lim_{\varepsilon \downarrow 0} \mathbb{E} \int_0^L F_\varepsilon(\eta(s, x)) \, dx \bigg|_{s=0}^{s=t} = \mathbb{E} \int_0^L F(\eta(s, x)) \, dx \bigg|_{s=0}^{s=t}. \tag{3.29}
\]

Next, consider the first term in the right side of (3.28). Take a nonnegative function \( \zeta \in C^\infty_c(\mathbb{R}) \) with \( \|\zeta\|_{L^1(\mathbb{R})} = 1 \), and let
\[
\tilde{\zeta}(x) = \zeta \ast 1_{[0,L]}(x) = \int \zeta(x - y) 1_{[0,L]}(y) \, dy = \int_{x - L}^x \zeta(y) \, dy.
\]

The \( L^1 \)-invariance in law of \( \eta \) implies that
\[
\mathbb{E} \int_0^t \int_0^L F''_\varepsilon(\eta(s, x))(\partial_x \eta(s, x))^2 \, dx ds 
= \mathbb{E} \int_0^t \int_0^L \int_0^{L+y} F''_\varepsilon(\eta(s, x))(\partial_x \eta(s, x))^2 \zeta(y) \, dy \, ds (3.30)
= \mathbb{E} \int_0^t \int_0^L \int_{x-L}^x F''_\varepsilon(\eta(s, x))(\partial_x \eta(s, x))^2 \tilde{\zeta}(x) \, dx \, ds
= \mathbb{E} \int_0^t \int_0^L F''_\varepsilon(\eta(s, x))(\partial_x \eta(s, x))^2 \tilde{\zeta}(x) \, dx \, ds.
\]

By the co-area formula, we have
\[
\int_0^L \int_{\mathbb{R}} F''_\varepsilon(\eta(s, x))(\partial_x \eta(s, x))^2 \tilde{\zeta}(x) \, dx \, ds
= \int_0^T \int_{\mathbb{R}} F''_\varepsilon(\lambda) \sum_{y \in \eta(s, \cdot)^{-1}(\lambda)} |\partial_x \eta(s, y)| \zeta(y) \, d\lambda, \, ds.
\]
almost surely. Since $F''_{\epsilon}$ converges weakly to $c\delta_0$ as $\epsilon \downarrow 0$, and the map

$$\lambda \mapsto \sum_{y \in \eta(s, \cdot)^{-1}(\lambda)} |\partial_x \eta(s, y)| \bar{\zeta}(y)$$

is almost surely continuous by Lemma D.3, we have

$$\lim_{\epsilon \downarrow 0} \int_0^t \int_{\mathbb{R}} F''_{\epsilon}(\eta(s, x))(\partial_x \eta(s, x))^2 \bar{\zeta}(x) \, dx \, ds = c \int_0^T \sum_{y \in \eta(s, \cdot)^{-1}(0)} |\partial_x \eta(s, y)| \bar{\zeta}(y) \, ds,$$

almost surely. Again using the $L\mathbb{Z}$-invariance of $\eta$ we obtain, as in (3.30),

$$c \mathbb{E} \int_0^T \sum_{y \in [0, L]} |\partial_x \eta(s, y)| \, dt = c \mathbb{E} \int_0^T \sum_{y \in \eta(s, \cdot)^{-1}(0)} |\partial_x \eta(s, y)| \bar{\zeta}(y) \, dt$$

$$\leq \liminf_{\epsilon \downarrow 0} \mathbb{E} \int_0^t \int_{\mathbb{R}} F''_{\epsilon}(\eta(s, x))(\partial_x \eta(s, x))^2 \bar{\zeta}(x) \, dx \, dt$$

(3.31)

by Fatou’s lemma. Now we can take $\epsilon \downarrow 0$ in (3.28) and use (3.29) and (3.31) to obtain

$$\mathbb{E} \int_0^L F(\eta(s, x)) \, dx \bigg|_{s=t} \leq \frac{c}{4} \mathbb{E} \int_0^T \sum_{y \in [0, L]} |\partial_x \eta(s, y)| \, dt,$$

which is (3.24). Finally, (3.25) is a consequence of (3.24) and translation invariance. □

3.3. Ordering of stationary solutions. The key tool in the classification of space-time stationary solutions is an almost-sure ordering theorem for the components of spacetime stationary solutions to (1.4). This theorem is very similar to the comparison and $L^1$-contraction results proved in the previous two sections. The intuition is that if two solutions cross transversely, then the heat flow causes cancellation between the positive and negative parts of the difference at the crossing point, and so the $L^1$-norm of the difference decreases. But if two solutions are jointly time-stationary, the $L^1$-norm of their difference must be conserved in time. Of course, the $L^1$-norm must be taken with respect to the probability space and a compact interval, since the $L^1$-norm of the difference of two spatially-stationary solutions on the whole line is not generally finite.

**Theorem 3.8.** Let $m \in (0, 1)$ and $\nu \in \mathcal{P}(\lambda^2_m)$. Let $\nu = (\nu_1, \nu_2) \sim \nu$ and suppose that one of the following two conditions holds:

(H1) $\nu \in \mathcal{D}(\lambda^2_m)$ for $G = L\mathbb{Z}$ with some $L > 0$ or $G = \mathbb{R}$, and

$$A = \sup_{i \in \{1, 2\}} \mathbb{E} \int_{\lambda^G} \nu_i(x)^2 \, d\lambda^G(x) < \infty.$$  (3.32)

(H2) $||\nu_1 - \nu_2||_{L^1(\mathbb{R})} < \infty$ almost surely.
Then, almost surely, \( \text{sgn}(v_1(x) - v_2(x)) \) is a random constant independent of \( x \).

In the sequel, we will only apply Theorem 3.8 with hypothesis (H1), but we include (H2) for completeness. Theorem 3.8 is an immediate consequence of the following proposition.

**Proposition 3.9.** Let \( m \in (0, 1) \). Let \( u = (u_1, u_2) \in \mathcal{Z}^2_m \) solve (1.4) with random initial conditions independent of the noise, and let \( T > 0 \) be such that one of the following two conditions holds:

(H1') Law\((u(0, \cdot)) \in \mathcal{P}_G(L_{p_m+\infty}(\mathbb{R})^2)\) for \( G = L\mathbb{Z} \) with some \( L > 0 \) or \( G = \mathbb{R} \), (3.32) holds, and \( \int_{\Lambda G} \mathbb{E}|u_1(t, x) - u_2(t, x)| \, d\mu_G(x) \) is constant on \([0, T]\).

(H2') Almost surely, \( \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_1(t', \cdot) - u_2(t', \cdot)\|_{L^1(\mathbb{R})} < \infty \) for all \( t, t' \in [0, T] \).

Then, almost surely, \( \text{sgn}(u_1(t, x) - u_2(t, x)) \) is a random constant independent of \( x \in \mathbb{R} \) and \( t \in [0, T] \).

The key step in the proof of Proposition 3.9 is the following observation, which shows that \( u_1 \) and \( u_2 \) can only meet tangentially.

**Lemma 3.10.** Fix \( m \in (0, 1) \) and let \( u = (u_1, u_2) \in \mathcal{Z}_m^2 \) and \( T \) satisfy the assumptions of Proposition 3.9, including either (H1') or (H2').

Then, with probability 1, for all \( t \in [0, T] \) and \( x \in \mathbb{R} \) such that \( u_1(t, x) = u_2(t, x) \), we have \( \partial_x u_1(t, x) = \partial_x u_2(t, x) \).

**Proof.** If (H1') holds, then it is sufficient to assume that \( G = L\mathbb{Z} \) for some \( L \in (0, \infty) \), since if \( G = \mathbb{R} \) then (H1') holds for \( G = L\mathbb{Z} \) for every \( L \). In this case, by Proposition 3.7, with \( F(\eta) = |\eta| \), we have, for each \( x \in \mathbb{R} \) that

\[
\mathbb{E} \int_0^T \sum_{y \in [x, x+L]} \frac{|\partial_x \eta(t, y)|}{\eta(t, y) = 0} \, dt = 0,
\]

which means that

\[
\int_0^T \sum_{y \in \mathbb{R}} \frac{|\partial_x \eta(t, y)|}{\eta(t, y) = 0} \, dt = 0
\]

. Hence, with probability 1, for almost all \( t \in [0, T] \) we have \( \partial_x \eta(t, x) = 0 \) whenever \( \eta(t, x) = 0 \).

Let us now strengthen the conclusion to all \( t \geq 0 \). By Lemma 2.4 and an iteration of Lemma 2.5, we know that \( \theta_i = u_i - \psi \) and \( \psi \) are spatially smooth. Differentiating (1.3) in \( x \), we see that \( \partial_{tx} u_i \) exists. In particular, \( \partial_{tx} \eta \) is continuous in space-time. Now, suppose there exists \( (t_*, x_*) \in \mathbb{R}_+ \times \mathbb{R} \) such that \( \eta(t_*, x_*) = 0 \) but \( \partial_x \eta(t_*, x_*) \neq 0 \), and without loss of generality, assume that \( \partial_x \eta(t_*, x_*) > 0 \). Since \( \partial_x \eta \) is continuous, there exists a nonempty open rectangle \((t_1, t_2) \times (a, b)\) containing \((t_*, x_*)\) such that \( \partial_x \eta > 0 \) on \((t_1, t_2) \times (a, b)\) while

\[
\eta(t, a) < 0 \quad \text{and} \quad \eta(t, b) > 0, \quad \text{for all } t_1 < t < t_2,
\]
and thus \( \eta(t, x) \) vanishes in \((a, b)\) for all \( t \in (t_1, t_2)\), at some point \( x_\ast(t) \) such that \( \partial_t \eta(t, x_\ast(t)) > 0 \). This event has probability 0, and the proof under \((H1')\) is complete. The proof assuming \((H2')\) is analogous, using Lemma 3.5 instead of Proposition 3.7, also with \( F(\eta) = |\eta| \). □

**Proof of Proposition 3.9.** By Lemma 3.10, we may assume that \( \partial_x \eta = 0 \) whenever \( \eta = 0 \). This contradicts a parabolic Hopf lemma [45, Chapter 2] applied to the equation (3.1) for \( \eta \) unless \( u_1 \) and \( u_2 \) are ordered. To be precise, suppose that the set \( Z := \{ t > 0, \ x \in \mathbb{R} : \eta(t, x) = 0 \} \) is not empty and that there exists \( t_0 > 0 \) and \( x_0 \in \mathbb{R} \) so that \( \eta(t_0, x_0) = 0 \). For \( \lambda > 0 \), define the parabolic cone

\[
Q_\lambda = Q_\lambda(t_0, x_0) := \{ t > 0, \ x \in \mathbb{R} : (x - x_0)^2 < t - t_0, \ t_0 < t < t_0 + \lambda^2 \},
\]

and set

\[
\lambda_\ast := \inf\{ \lambda > 0 : Q_\lambda \cap Z \neq \emptyset \}.
\]

Suppose first for the sake of contradiction that \( \lambda_\ast \) is finite. As \( \eta \) is continuous, we have \( \lambda_\ast > 0 \) and \( \eta > 0 \) on \( Q_{\lambda_\ast} \), and moreover there is a point \( (t_0 + \lambda_\ast^2, x_\ast) \in Q_{\lambda_\ast} \cap Z \). The parabolic Hopf lemma [45, Lemma 2.8] implies that \( \partial_x \eta \neq 0 \) at \( (t_0 + \lambda_\ast^2, x_\ast) \), contradicting our hypothesis on \( \eta \). Strictly speaking, this formulation of the Hopf lemma only applies when \((t_0 + \lambda_\ast^2, x_\ast)\) is a corner, that is \(|x_\ast - x| = \lambda_\ast\). However, if \(|x_\ast - x_0| < \lambda_\ast\), we can apply the Hopf lemma to a smaller parabolic cone contained in \( Q_{\lambda_\ast} \), having \((t_0 + \lambda_\ast^2, x_\ast)\) as a corner. It follows that \( \lambda_\ast = \infty \) and \( \eta > 0 \) on \( Q_\infty \). In addition, if there exists \((t_1, x_1)\) such that \( \eta(t_1, x_1) < 0 \) then by a similar argument we have \( \eta < 0 \) on \( Q_\infty(t_1, x_1) \). As the intersection of \( Q_\infty(t_0, x_0) \) and \( Q_\infty(t_1, x_1) \) is not empty, this is a contradiction. Thus, \( \eta(t, x) \geq 0 \) for all \( t > 0 \) and \( x \in \mathbb{R} \). The parabolic strong maximum principle [45, Theorem 2.7] implies then that \( \eta > 0 \) for all \( t > 0 \) and \( x \in \mathbb{R} \).

Thus, almost surely, we have the following trichotomy: \( \eta > 0 \), \( \eta \equiv 0 \), or \( \eta < 0 \). □

### 4. Other Properties of Solutions

In this section, we establish a few other properties of solutions to (1.4). In Sect. 4.1, we formulate the shear-invariance property discussed in the introduction, in the context of solutions in weighted spaces. In Sect. 4.2, we use the Feller property in Proposition 2.3 and the standard Krylov–Bogoliubov argument to show that subsequential limits of time-averaged laws of solutions to (1.4) are stationary in time. We then use this to show that any two invariant measures for (1.4) can be coupled to create an invariant measure on a product space. Finally, in Sect. 4.3, we show how solutions to (1.3) can be used to build solutions to the KPZ equation (1.11) in weighted spaces.

#### 4.1. Shear-invariance

The shear-invariance of the Burgers equation mentioned in the introduction can be stated as follows.

**Proposition 4.1.** Suppose that \( u, u' \in Z_m \) are solutions to (1.4) such that

\[
u(0, \cdot) = u'(0, \cdot) + (a, \ldots, a).
\]

Then \( \text{Law}(u') = \text{Law}(u(t, x + at) - (a, \ldots, a)) \) as distributions on \( Z_m \).
Proposition 4.2. If \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N) = u(t, x + at) - (a, \ldots, a) \), then \( \tilde{u} \) is a strong solution to the SPDE

\[
d\tilde{u}_i = \frac{1}{2} [\partial_x^2 \tilde{u}_i - \partial_x (\tilde{u}_i^2)] dt + d(\partial_x \tilde{V}),
\]

where \( \tilde{V}(t, x) = V(t, x + at) \). Define

\[
\tilde{\psi}(t, x) = \int_0^t \partial_x G_{t-s} * d\tilde{V}(s, \cdot)(x) = \int_0^t \int_\mathbb{R} \partial_x G_{t-s} * \rho(x + at - y) dW(s, y),
\]

so that if we set \( \tilde{\psi} = (\tilde{\psi}, \ldots, \tilde{\psi}) \), then \( \tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_N) = \tilde{u} - \tilde{\psi} \) satisfies

\[
\partial_t \tilde{\theta}_i = \frac{1}{2} \partial_x^2 \tilde{\theta}_i - \frac{1}{2} \partial_x (\tilde{\theta}_i + \tilde{\psi}).
\]

Analogously, define \( \psi \) and \( \theta' = u' - \psi \), with \( \psi \) defined in (2.2). It satisfies

\[
\partial_t \theta'_i = \frac{1}{2} \partial_x^2 \theta'_i - \frac{1}{2} \partial_x (\theta'_i + \psi).
\]

By construction, \( \tilde{\theta}(0, \cdot) = \theta'(0, \cdot) \). Also, by computing the covariance structure, it is easy to see that \( \tilde{\psi} \) has the same law as \( \psi \). Therefore \( \tilde{\theta} \) and \( \theta' \) agree in law. It follows that \( \tilde{u} \) and \( u' \) also agree in law. \( \square \)

4.2. The Krylov–Bogoliubov theorem. The Feller property discussed in Theorem 1.1 allows us to apply the Krylov–Bogoliubov theorem. This is a standard argument in the ergodic theory of SPDEs (see e.g. [21, Theorem 3.1.1]) but since it is simple and central to our argument, we present a proof of the theorem and one consequence here.

**Proposition 4.2** (Krylov–Bogoliubov). Fix \( m \in (0, 1) \), \( N \in \mathbb{R} \), and \( G = L\mathbb{Z} \) with \( L > 0 \) or \( G = \mathbb{R} \). If \( s \geq 0 \), \( T_k \uparrow \infty \), and \( \nu_0 \in \mathcal{P}_G(L^{\infty}_{pm+}) \) are such that

\[
\nu := \lim_{k \to \infty} \frac{1}{T_k} \int_s^{s+T_k} P_t^* \nu_0 dt
\]

exists in the sense of weak convergence of measures on \( \chi^N_m \), then \( \nu \in \overline{\mathcal{P}}_G(\chi^N_m) \).

**Proof.** If \( f \in C_b(\chi^N_m) \) and \( r \geq 0 \), then by the Feller property (Proposition 2.3) we have \( P_r f \in C_b(\chi^N_m) \) as well, so

\[
\langle f, P_r^* \nu \rangle = \langle f, P_r f \rangle = \left\langle P_r f, \lim_{k \to \infty} \frac{1}{T_k} \int_s^{s+T_k} P_t^* \nu_0 dt \right\rangle
\]

\[
= \lim_{k \to \infty} \left\langle P_r f, \frac{1}{T_k} \int_s^{s+T_k} P_t^* \nu_0 dt \right\rangle
\]

\[
= \lim_{k \to \infty} \left\langle f, \frac{1}{T_k} \int_{s+r}^{s+r+T_k} P_t^* \nu_0 dt \right\rangle.
\]

On the other hand, we have

\[
\lim_{k \to \infty} \left\langle f, \frac{1}{T_k} \int_{s+r}^{s+r+T_k} P_t^* \nu_0 dt - \frac{1}{T_k} \int_s^{s+T_k} P_t^* \nu_0 dt \right\rangle = 0
\]
for all $f \in C_b(\mathcal{X}_m^n)$, so

$$\lim_{k \to \infty} \frac{1}{T_k} \int_{s+r}^{s+r+T_k} P_t^* v_0 \, dt = v,$$

so $\langle f, P_r^* v \rangle = \langle f, v \rangle$ for all $f \in C_b(\mathcal{X}_m^n)$, so $P_r^* v = v$. This holds for all $r \geq 0$, so $v \in \mathcal{P}(\mathcal{X}_m^n)$. □

An important application for the classification results, is that any two invariant measures for (1.4) can be coupled to form an invariant measure on the product space.

**Proposition 4.3.** Fix $m \in (0, 1)$ and $N_1, N_2 \in \mathbb{N}$. If $v_i \in \mathcal{P}(\mathcal{X}_m^{N_i})$, $i = 1, 2$, with either $G = L\mathbb{Z}$ with some $L > 0$, or $G = \mathbb{R}$, then there is a coupling $v \in \mathcal{P}(\mathcal{X}_m^{N_1+N_2})$ of $v_1$ and $v_2$.

**Proof.** First, let $v_0$ be a coupling of $v_1$ and $v_2$ such that the two factors are independent. Then $v_0$ is $G$-invariant if $v_1$ and $v_2$ are. However, $v_0$ is not expected to be invariant under the time evolution, since letting both factors evolve according to the same noise introduces dependence. To find a coupling that is invariant under the evolution, we observe that the family

$$\tilde{v}_T := \frac{1}{T} \int_0^T P_t^* v_0 \, dt$$

of measures on $\mathcal{X}_m^{N_1+N_2}$ is tight. Indeed, as $\mathcal{X}_m$ is a Polish space, by Prokhorov’s theorem both $v_1$ and $v_2$ are themselves tight. Fixing $\varepsilon > 0$, there exist compact sets $K_i \subset \mathcal{X}_m^{N_i}$ such that $v_i$ has at most $\varepsilon/2$ mass outside $K_i$, for each $i = 1, 2$. By the invariance of $v_i$ and a union bound, we have

$$\tilde{v}_T(K_1 \times K_2) = \frac{1}{T} \int_0^T (P_t^* v_0)(K_1 \times K_2) \, dt \geq 1 - v_1(K_1^c) - v_2(K_2^c) > 1 - \varepsilon.$$

It follows that the family $\tilde{v}_T$ is tight. Again by Prokhorov’s theorem, there is a sequence $T_k \uparrow \infty$ and a measure $v$ on $\mathcal{X}_m^{N_1+N_2}$ so that $\tilde{v}_{T_k} \to v$ weakly as $k \to \infty$. Note that $v$ is $G$-invariant if $v_0$ is, and by Proposition 4.2, $v$ is also invariant under the dynamics (1.4). Finally, marginals are preserved under weak limits, so $v$ couples $v_1$ and $v_2$. □

### 4.3. The solution to the KPZ equation

We will sometimes need to relate the solutions of the Burgers equation to the solution of the KPZ equation. In this section, we show how to obtain solutions to (1.11) from those to (1.3).

**Proposition 4.4.** Fix $m \in (0, 1)$ and a smooth, non-negative, compactly-supported function $\xi \in C^\infty(\mathbb{R})$ with $\|\xi\|_{L^1(\mathbb{R})} = 1$. Suppose that $u \in Z_m$ is a strong solution to (1.3), and let $\theta = u - \psi$, with $\psi$ defined as in (2.2). Then,

$$h(t, x) = \int_\mathbb{R} \xi(y) \int_y^x \theta(t, z) \, dz \, dy - \frac{1}{2} \int_0^t \int_\mathbb{R} \left[ \theta(s, y) \xi'(y) + (\theta(s, y) + \psi(s, y)) \xi(y) \right] \, dy \, ds - \int_0^t G_{t-s} * dV(s, \cdot)(x) + \frac{t}{2} \|\rho\|^2_{L^2(\mathbb{R})}$$

is a strong solution to (1.11).
Note that, in particular, we have
\[ \int_{\mathbb{R}} \zeta(x) h(0, x) \, dx = 0, \quad (4.2) \]
which fixes the “constant of integration” in taking the antiderivative of \( u(0, \cdot) \).

**Proof.** We write \( h(t, x) = \omega(t, x) + g(t, x) \), with
\[
\omega(t, x) = \int_{0}^{t} G_{t-s} * dV(s, \cdot)(x)
\]
and
\[
g(t, x) = \int_{\mathbb{R}} \zeta(y) \int_{y}^{x} \theta(t, z) \, dz \, dy
\]
\[
- \frac{1}{2} \int_{0}^{t} \int_{\mathbb{R}} \left[ \theta(s, y) \zeta'(y) + (\theta(s, y) + \psi(s, y))^2 \zeta(y) \right] dy \, ds
\]
\[
+ \frac{t}{2} \| \rho \|_{L^2(\mathbb{R})}^2.
\]

Note that \( \partial_x \omega = \psi \), and \( \omega \) is a strong solution to the SPDE
\[
d\omega = \frac{1}{2} \partial_x^2 \omega \, dt + dV. \quad (4.3)
\]
In addition, the function \( g \) satisfies \( \partial_x g = \theta \) and using (2.6), we see that
\[
\partial_t g(t, x) - \frac{1}{2} \| \rho \|_{L^2(\mathbb{R})}^2
\]
\[
= \int_{\mathbb{R}} \zeta(y) \int_{y}^{x} \partial_t \theta(t, z) \, dz \, dy
\]
\[
- \frac{1}{2} \int_{\mathbb{R}} \left[ \theta(t, y) \zeta'(y) + (\theta(t, y) + \psi(t, y))^2 \zeta(y) \right] dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} \zeta(y) \int_{y}^{x} \left[ \partial_x^2 \theta(t, z) - \partial_x (\theta(t, z) + \psi(t, z))^2 \right] dz \, dy
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}} [\partial_x \theta(t, y) - (\theta(t, y) + \psi(t, y))^2] \zeta(y) \, dy
\]
\[
= \frac{1}{2} \partial_x \theta(t, x) - \frac{1}{2} (\theta(t, x) + \psi(t, x))^2
\]
\[
= \frac{1}{2} \partial_x^2 g(t, x) - \frac{1}{2} (\partial_x g(t, x) + \partial_x \omega(t, x))^2.
\]

From this and (4.3), we see that \( h = g + \omega \) is a strong solution to (1.3). \( \square \)
5. Limits of Solutions Started from Bounded Initial Conditions

In this section, we consider solutions to (1.3) started at constant initial conditions. In Sect. 5.1, we show that the laws of solutions started at deterministic, globally-bounded initial conditions are tight as time goes to infinity. In Sect. 5.2, we show that, if the initial condition is in addition periodic in space, then the resulting subsequential limits are extremal.

5.1. Uniform-in-time bounds and tightness. In this section, we start with a bounded initial condition and establish tightness of the resulting family of solutions $u(t, \cdot)$, as $t \to +\infty$. This requires a priori bounds in weighted spaces, that we are only able to prove in $X^m$ for $m \geq 1/2$. In combination with the restriction $m < 1$ from Sect. 2, this explains the range of exponents $m$ in Theorem 1.2. The main result of this section is the following.

**Proposition 5.1.** Let $u \in Z^{N}_{1/2}$ solve (1.3) with bounded initial condition $u(0, \cdot) \in L^\infty(\mathbb{R})^N$. Then the family of random variables $\{u(t, \cdot)\}_{t \geq 1}$ is tight with respect to the topology of $X^{N}_{1/2}$, and

$$\sup_{t \geq 0, x \in \mathbb{R}} \mathbb{E}|u(t, x)|^2 < \infty. \quad (5.1)$$

Tightness arguments on compact domains [14,15,19,20] have generally relied on bounding the size of the solution in terms of the size of its derivative, using a Poincaré-type inequality. Such an argument cannot work on the whole space. Instead, we control the second moment of the Burgers solution $u$ by first controlling the expectation of the KPZ solution $h$. The following proposition will be key to the proof of Proposition 5.1.

**Proposition 5.2.** Let $h$ be a mild solution to (1.11) with initial condition $h(0, \cdot) \equiv 0$. Then, the function $\gamma(t) = \mathbb{E}h(t, 0)$ is increasing and concave for $t > 0$.

**Proof.** Using the Feynman–Kac formula as in [9,10], we have

$$h(t, 0) = -\log \mathbb{E}X_{t,0} \exp \left\{ -\int_0^t dV(s, X(s)) - \frac{t}{2} \|\rho\|_{L^2(\mathbb{R})}^2 \right\}. \quad (5.2)$$

Here, $\mathbb{E}X_{t,0}$ denotes expectation with respect to the measure in which $X$ is a two-sided Brownian motion satisfying $X(t) = x$, that is independent of the noise. Note that the right side of (5.2) is still a random quantity due to the randomness of the noise. As in [47], the time-reversed process

$$Z_t = \mathbb{E}X_{0,0} \exp \left\{ -\int_0^t dV(s, X(s)) - \frac{t}{2} \|\rho\|_{L^2(\mathbb{R})}^2 \right\},$$

satisfies

$$h(t, 0) \overset{\text{law}}{=} -\log Z_t.$$

The point of the time-reversal is that the process $Z_t$ is a martingale and satisfies

$$dZ_t = \mathbb{E}X_{0,0} \exp \left\{ -\int_0^t dV(s, X(s)) - \frac{t}{2} \|\rho\|_{L^2(\mathbb{R})}^2 \right\} dV(t, X(t)).$$
The convexity of \((-\log)\) implies that \(-\log Z_t\) is a submartingale, so
\[
\mathbb{E}(-\log Z_t) = \mathbb{E}h(t, 0) = \gamma(t)
\]
is increasing in \(t\).

To prove that \(\gamma\) is concave, we use Itô’s formula to compute an SDE for \(-\log Z_t\). First, for \(0 \leq q \leq t\), we introduce the polymer measure
\[
\hat{E}^{q,t}_{\hat{X}^t_0} F[X] = \frac{\mathbb{E}^{q,t}_{\hat{X}^t_0} \left[ F[X] \exp \left\{ -\int_0^t \frac{dV(s, X(s))}{\rho(s)^2} \right\} \right]}{\mathbb{E}^{q,t}_{\hat{X}^t_0} \exp \left\{ -\int_0^t \frac{dV(s, X(s))}{\rho(s)^2} \right\}}
\]
(5.3)

Then we have
\[
d(-\log Z_t) = -\frac{dZ_t}{Z_t} + \frac{d[Z]_t}{2}Z_t^2
\]
\[
= -\hat{E}^{0,t}_{\hat{X}^t_0}dV(t, X(t)) + \frac{1}{2} \hat{E}^{0,t}_{\hat{X}^t_0} \hat{E}^{0,t}_{\hat{X}^t_0} \rho^2(X(t) - \tilde{X}(t)) dt,
\]
(5.4)

where \(\tilde{X}\) denotes an independent copy of the Brownian motion \(X\).

Taking expectations in (5.4), we see that
\[
\gamma'(t) = \partial_t \mathbb{E}(-\log Z_t) = \frac{1}{2} \hat{E}^{0,t}_{\hat{X}^t_0} \hat{E}^{0,t}_{\hat{X}^t_0} \rho^2(X(t) - \tilde{X}(t)).
\]
(5.5)

We need to show that the right side of (5.5) is decreasing as a function of \(t\). Fix \(q \in [0, t]\) and write, for a fixed realization of the noise,
\[
\hat{E}^{0,t}_{\hat{X}^t_0} \hat{E}^{0,t}_{\hat{X}^t_0} \rho^2(X(t) - \tilde{X}(t)) = \hat{E}^{0,t}_{\hat{X}^t_0} \hat{E}^{0,t}_{\hat{X}^t_0} \int \rho(X(t) - \tilde{X}(t) - z) \rho(z) dz
\]
\[
= \hat{E}^{0,t}_{\hat{X}^t_0} \hat{E}^{0,t}_{\hat{X}^t_0} \int \rho(X(t) - z) \rho(z - \tilde{X}(t)) dz
\]
\[
= \hat{E}^{0,t}_{\hat{X}^t_0} \hat{E}^{0,t}_{\hat{X}^t_0} \int \rho(X(t) - z) \rho(\tilde{X}(t) - z) dz,
\]

since \(\rho(x)\) is even. Using the Markov property and splitting the time integrals over \([0, t]\) that appear in (5.3) into those over \([0, q]\) and \([q, t]\), we get
\[
\hat{E}^{0,t}_{\hat{X}^t_0} \hat{E}^{0,t}_{\hat{X}^t_0} \rho^2(X(t) - \tilde{X}(t))
\]
\[
= \hat{E}^{0,q}_{\hat{X}^t_0} \hat{E}^{0,q}_{\hat{X}^t_0} \int \hat{E}^{q,t}_{Y^q,X(q)} \rho(Y(t) - z) \hat{E}^{q,t}_{Y^q,X(q)} \rho(\tilde{Y}(t) - z) dz
\]
\[
\leq \frac{1}{2} \hat{E}^{0,q}_{\hat{X}^t_0} \int \left( \hat{E}^{q,t}_{Y^q,X(q)} \rho(Y(t) - z) \right)^2 dz
\]
\[
+ \frac{1}{2} \hat{E}^{0,q}_{\hat{X}^t_0} \int \left( \hat{E}^{q,t}_{Y^q,X(q)} \rho(\tilde{Y}(t) - z) \right)^2 dz
\]
\[
= \hat{E}^{0,q}_{\hat{X}^t_0} \int \left( \hat{E}^{q,t}_{Y^q,X(q)} \rho(Y(t) - z) \right)^2 dz
\]
\[
= \hat{E}^{0,q}_{\hat{X}^t_0} \hat{E}^{q,t}_{Y^q,X(q)} \hat{E}^{q,t}_{Y^q,X(q)} \rho^2(Y(t) - \tilde{Y}(t)).
\]
(5.6)
The inequality above follows from Young’s inequality. Now taking expectation over the noise, and using the fact that $V$ is white in time, we see that

$$2\gamma'(t) = \mathbb{E} E_{X_0,0}^{0, t} \tilde{E}_{X_0,0}^{0, t} \rho_{s^2}(X(t) - \tilde{X}(t))$$

$$\leq \mathbb{E} E_{X_0,0}^{0, q} \int_{X_0,0}^{t} \tilde{E}_{Y_0,0}^{q, t} \tilde{E}_{Y_0,0}^{q, t} \rho_{s^2}(Y(t) - \tilde{Y}(t)) \, dV |_{[0, q] \times \mathbb{R}}$$

$$= \mathbb{E} E_{Y_0,0}^{0, q} \tilde{E}_{Y_0,0}^{q, t} \rho_{s^2}(Y(t) - \tilde{Y}(t))$$

$$= \mathbb{E} E_{Y_0,0}^{0, t} \rho_{s^2}(Y(t) - \tilde{Y}(t))$$

$$= 2\gamma'(t - q).$$

As $q \in [0, t]$ is arbitrary, this shows that $\gamma'(t)$ is decreasing in $t$, and thus that $\gamma$ is concave, as claimed. \hfill \Box

An important application of Proposition 5.2 is the following $L^2$-bound, which is key to the compactness result.

**Lemma 5.3.** Suppose that $a \in \mathbb{R}$ and $u$ is a solution to (1.3) with initial condition $u(0, \cdot) \equiv a$. Then for all $t \geq 0$ and $x \in \mathbb{R}$, we have

$$\mathbb{E}(u(t, x) - a)^2 \leq \|\rho\|_{L^2(\mathbb{R})}^2. \quad (5.7)$$

**Proof.** By Proposition 4.1, it suffices to consider the case $a \equiv 0$. Let $h$ be a solution to (1.11) with initial condition $h(0, \cdot) \equiv 0$, so we can take $u = \partial_x h$. Averaging (1.11) in space, integrating it time over an interval $(t_1, t_2)$, and taking expectations, we get

$$\frac{1}{2}(t_2 - t_1)\|\rho\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \mathbb{E} \int_{t_1}^{t_2} \partial_x u(t, x + 1) - \partial_x u(t, x) \, dt = \frac{1}{2} \mathbb{E} \int_{t_1}^{t_2} u(t, x)^2 \, dt \quad (5.8)$$

for an arbitrary $x \in \mathbb{R}$ by space-stationarity of $u$ and $h$. Also by space-stationarity of $u$, the laws of

$$\int_{t_1}^{t_2} \partial_x u(t, x) \, dt \text{ and } \int_{t_1}^{t_2} \partial_x u(t, x + 1) \, dt$$

are the same. In addition, the right side of (5.8) is nonnegative by Proposition 5.2. Hence, by Lemma D.1, the expectation of the second term in the left side of (5.8) is well-defined and equals to 0. This gives (5.7) for almost all $t \geq 0$ and $x \in \mathbb{R}$. The statement for all $t \geq 0$ and $x \in \mathbb{R}$ follows from the almost-sure continuity of $u$ and Fatou’s lemma. \hfill \Box

Next, we get a bound on the derivative of the solution. The following lemma is based on an energy estimate similar to that leading to [14, formula (16)].

**Lemma 5.4.** Suppose that $a \in \mathbb{R}$ and $u$ is a solution to (1.3) with initial condition $u(0, \cdot) \equiv a$. Then for all $t \geq 0$ and $x \in \mathbb{R}$, we have

$$\mathbb{E}(\partial_x u(t, x))^2 \leq \|\partial_x \rho\|_{L^2(\mathbb{R})}^2. \quad (5.9)$$
Proof. Again using Proposition 4.1, we can assume that \( a = 0 \). By the Itô formula (see e.g. [22, Theorem 3.32]) applied to (1.3), we get that whenever \( t \geq q \geq 0 \),

\[
\begin{align*}
&u(t, x)^2 - u(q, x)^2 \\
&= \int_q^t 2u(s, x) \, d(\partial_x V)(s, x) + \int_q^t 2u(s, x) \left[ \partial_x^2 u - \partial_x (u^2) \right] (s, x) \, ds \\
&\quad + (t - q) \| \partial_x \rho \|_{L^2(\mathbb{R})}^2 \\
&= \int_q^t 2u(s, x) \, d(\partial_x V)(s, x) + 2 \int_q^t \left[ \frac{1}{2} \partial_x^2 (u^2) - (\partial_x u)^2 - \frac{1}{3} \partial_x (u^3) \right] (s, x) \, ds \\
&\quad + (t - q) \| \partial_x \rho \|_{L^2(\mathbb{R})}^2.
\end{align*}
\]

Let \( x_1 < x_2 \) and integrate in space to obtain

\[
\begin{align*}
&\int_{x_1}^{x_2} [u(t, x)^2 - u(q, x)^2] \, dx \\
&= \int_{x_1}^{x_2} \int_q^t 2u(s, x) \, d(\partial_x V)(s, x) \\
&\quad + 2 \int_{x_1}^{x_2} \int_q^t \left[ \frac{1}{2} \partial_x^2 (u^2) - (\partial_x u)^2 - \frac{1}{3} \partial_x (u^3) \right] (s, x) \, ds \, dx \\
&\quad + (t - q)(x_1 - x_2) \| \partial_x \rho \|_{L^2(\mathbb{R})}^2 \\
&= \int_{x_1}^{x_2} \int_q^t 2u(s, x) \, d(\partial_x V)(s, x) \\
&\quad + \int_q^t \left[ (\partial_x u^2) - \frac{2}{3} u^3 \right]_{x = x_1}^{x = x_2} - \int_{x_1}^{x_2} (\partial_x u)^2 (s, x) \, dx \, ds \\
&\quad + (t - q)(x_1 - x_2) \| \partial_x \rho \|_{L^2(\mathbb{R})}^2.
\end{align*}
\]

Taking expectations, we get

\[
\begin{align*}
&\int_{x_1}^{x_2} \mathbb{E}[u(t, x)^2 - u(q, x)^2] \, dx \\
&= \int_q^t \left[ \mathbb{E} \left[ \partial_x (u^2) - \frac{2}{3} u^3 \right]_{x = x_1}^{x = x_2} - \mathbb{E} \int_{x_1}^{x_2} (\partial_x u)^2 (s, x) \, dx \right] \, ds \\
&\quad + (t - q)(x_1 - x_2) \| \partial_x \rho \|_{L^2(\mathbb{R})}^2. \\
&\tag{5.10}
\end{align*}
\]

Since the first expectation in (5.10) is finite by Lemma 5.3 and the second term under the expectation in the right side has a sign, we can use Lemma D.1, to conclude that

\[
\mathbb{E} \left[ \partial_x (u^2) - \frac{2}{3} u^3 \right]_{x = x_1}^{x = x_2} = 0. \tag{5.11}
\]

On the other hand, we have by (5.8) that

\[
\frac{1}{2} (t_2 - t_1) \| \rho \|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \mathbb{E} \int_{t_1}^{t_2} u(t, x)^2 \, dt = [\mathbb{E} h(t_2, x) - \mathbb{E} h(t_1, x)]. \tag{5.12}
\]
and thus
\[ \mathbb{E}u(t, x)^2 = 2\partial_t \mathbb{E}(h(t, x)) + \|\rho\|_{L^2(\mathbb{R})}^2 \] (5.13)
is an increasing function by Proposition 5.2, so
\[ \mathbb{E}[u(t, x)^2 - u(q, x)^2] \geq 0. \] (5.14)
Plugging (5.11) and (5.14) into (5.10) yields
\[ \mathbb{E}\int_q^t \int_{x_1}^{x_2} (\partial_x u)^2(s, x) \, dx \, ds \leq (t - q)(x_1 - x_2)\|\partial_x \rho\|_{L^2(\mathbb{R})}^2. \]
Since this holds for all \( t \geq q \geq 0 \) and all \( x_1 > x_2 \), we can conclude (5.9) for almost every \( t \geq 0 \).

Before proceeding, we record a version of the derivative bound in Lemma 5.4 for solutions that start at a stationary distribution rather than a constant initial condition.

**Lemma 5.5.** Let \( G = L\mathbb{Z} \) or \( G = \mathbb{R} \), and let \( X \sim \text{Uniform}(\Lambda_G) \) be independent of all else. Suppose that \( \nu \in \mathcal{P}_G(\mathbb{R}) \) is such that \( \mathbb{E}\nu(X)^2 < \infty \) if \( \text{Law}(\nu) = \nu \). Then we have
\[ \mathbb{E}(\partial_x \nu(X))^2 = \|\partial_x \rho\|_{L^2(\mathbb{R})}^2. \]

**Proof.** Without loss of generality, we may assume that \( G = \mathbb{R}/L\mathbb{Z} \) for some \( L > 0 \). Let \( u \) solve (1.3) with random initial condition \( \nu \) independent of the noise \( V \). We again integrate (1.3) in space and time, and obtain (5.10). The time-invariance of \( \nu \) and the assumption \( \mathbb{E}\nu(X)^2 < \infty \) implies
\[ 0 = (t - q)\left\{ \mathbb{E}\left[ \partial_x (v^2) - \frac{2}{3}v^3 \right]_{x=x_1}^{x=x_2} - \mathbb{E} \int_{x_1}^{x_2} (\partial_x v)^2(x) \, dx + (x_1 - x_2)\|\partial_x \rho\|_{L^2(\mathbb{R})}^2 \right\}. \]
Furthermore, if we take \( x_1 = 0 \) and \( x_2 = L \), the \( G \)-invariance of \( \nu \) and Lemma D.1 imply
\[ \mathbb{E}\left[ \partial_x (v^2) - \frac{2}{3}v^3 \right]_{x=0}^{x=L} = 0. \]
Therefore, we have
\[ \frac{1}{L} \mathbb{E}\int_0^L (\partial_x v)^2(x) \, dx = \|\partial_x \rho\|_{L^2(\mathbb{R})}^2, \]
as desired. \( \square \)

Combining Lemma 5.3 and Lemma 5.4 yields the following uniform weighted bound. As a preliminary remark, we note that if \( v \in H^1_w(\mathbb{R}) \), then, by the Sobolev embedding theorem applied locally, \( v \) is continuous. Moreover, if \( w \) satisfies the condition \( w(x)/w(y) \leq C \) whenever \( |x - y| \leq 1 \), then
\[ \|v\|_{C_w(\mathbb{R})} \leq C \sup_{j \in \mathbb{Z}} \frac{\|v\|_{C^1([j,j+1])}}{w(j)} \leq C \sup_{j \in \mathbb{Z}} \frac{\|v\|_{H^1([j,j+1])}}{w(j)}. \] (5.15)
Proposition 5.6. Consider the weight \( w(x) = \langle x \rangle^{1/2} \log \langle x \rangle \). Suppose that \( a \in \mathbb{R} \) and \( u \) is a solution to (1.3) with initial condition \( u(0, \cdot) \equiv a \). Then there is a constant \( C < \infty \) so that for all \( t \geq 0 \), we have

\[
\mathbb{E} \| u(t, \cdot) - a \|_{C^2_{w}(\mathbb{R})}^2 \leq C.
\]

Proof. We have

\[
\mathbb{E} \| u(t, \cdot) - a \|_{C^2_{w}(\mathbb{R})}^2 \leq C \sum_{j \in \mathbb{Z}} \frac{\mathbb{E} \| u(t, \cdot) - a \|_{C^2_{w}((j,j+1))}^2}{w(j)^2} \leq C \sum_{j \in \mathbb{Z}} \frac{\mathbb{E} \| u(t, \cdot) - a \|_{H^1((j,j+1))}^2}{w(j)^2} \leq C \sum_{j \in \mathbb{Z}} w(j)^{-2} \leq C.
\]

The first two inequalities in (5.16) are by the first and second inequality in (5.15), respectively.

Proof of Proposition 5.1. We need to extend the above estimates to the case of bounded rather than constant initial data. Since in this case the solutions are not space-stationary, an analogue of Proposition 5.2 is less clear. However, as the initial condition is bounded, using the comparison principle Proposition 3.1, we can sandwich the solution between two solutions starting with constant initial conditions to achieve a bound on the size of the solution, and then use the parabolic regularity in Lemma 2.5 to control the derivatives.

Let us now describe the details. It suffices to consider the case \( N = 1 \), so let \( u \) be the sole component of \( u \), and let \( u_{\pm} \) solve (1.3) with initial condition \( u_{\pm}(0, \cdot) = \pm \| u(0, \cdot) \|_{L^\infty(\mathbb{R})} \), so that

\[
u_{\pm}(t, x) \leq u(t, x) \leq u_{\mp}(t, x) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}.
\]

Proposition 3.1 and Lemma 5.3 then imply (5.1).

To prove tightness, consider the weight \( \bar{w}(x) = \langle x \rangle^{1/2} \log \langle x \rangle \) as in Proposition 5.6. By (5.17) and Proposition 5.6, we have

\[
\mathbb{E} \| u(t, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 \leq \mathbb{E} \| u_{\mp}(t, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 + \mathbb{E} \| u_{\pm}(t, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 \leq C.
\]

It is not difficult to see from (2.5) and the Fernique and Borell–TIS inequalities (see e.g. [1, Theorems 4.1 and 2.1]) that \( \mathbb{E} \| \psi(s, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 \) is bounded uniformly in \( s \). Therefore, we can use Lemma 2.5 on a time interval \([t, t+1]\), with some \( \beta \in (0, 1) \) and \( \alpha = 0 \), to obtain, for all \( t \geq 0 \),

\[
\mathbb{E} \| u(t+1, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 \leq C \mathbb{E} \left( \| u(t, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 + \int_{t-1}^t (t-s)^{-(\beta+1)/2} \| u(s, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 \, ds \right) + C \leq C.
\]

We used (5.18) in the last inequality above. Now (5.18), (5.19), and Markov’s inequality imply that, for each \( t \geq 1 \),

\[
\mathbb{P} \left( \| u(t, \cdot) \|_{C^2_{\bar{w}}(\mathbb{R})}^2 > M \right) \leq C M^{-1}.
\]
By Proposition B.2, the set \( \{\|u(t, \cdot)\|_{C_w^{3}(\mathbb{R})} \leq M\} \) is compact in each \( C_{\beta}(\mathbb{R}), \) \( \ell > 1/2, \) hence compact in \( \mathcal{X}_{1/2} \) by Lemma B.3, and tightness of \( u(t, \cdot) \) in \( \mathcal{X}_{1/2} \) follows. \( \square \)

5.2. Extremality of the limits. In this section we show that if the initial condition is periodic, any subsequential limit of the tight family of laws considered in Proposition 5.1 is extremal.

We will need a couple of preparatory lemmas. The first is a simple calculation similar to Lemma 5.3.

Lemma 5.7. Let \( G = L\mathbb{Z} \) or \( G = \mathbb{R}, \) \( X \sim \text{Uniform}(\Lambda_G) \) be independent of everything else, and fix \( T > 0. \) Suppose that \( v \sim v \in \mathcal{F}_G(\mathcal{X}_{1/2}) \) is independent of the noise \( V \) and that \( \mathbb{E}v(X)^2 < \infty. \) Let \( h \) be a solution to (1.11) with initial condition such that \( \partial_x h(0, \cdot) = v. \) Then we have

\[
\mathbb{E}v(X)^2 = \|\rho\|_{L^{2}(\mathbb{R})}^2 - \frac{2}{T} \mathbb{E}[h(T, X) - h(0, X)].
\]

(5.21)

Proof. It is again sufficient to consider the case \( L > 0. \) Let \( u \) solve (1.3) with initial condition \( v. \) Integrating (1.11) in time from 0 to \( T \) and in space over \( \Lambda_G, \) taking expectations, and using (1.10), we have

\[
\mathbb{E} \int_{\Lambda_G} [h(T, x) - h(0, x)] d\lambda_G(x) = \frac{1}{2} \mathbb{E} \int_0^T \int_{\Lambda_G} \left[ \partial_x u(t, x) - u(t, x) \right]^2 + \|\rho\|_{L^{2}(\mathbb{R})}^2 \right) \right] d\lambda_G(x) dt.
\]

(5.22)

Note that

\[
\mathbb{E} \int_{\Lambda_G} \partial_x u(t, x) d\lambda_G(x) = \mathbb{E}[u((t, L) - u(t, 0)] = 0
\]

(5.23)

simply by the \( G \)-invariance in law of \( u \) and since \( \mathbb{E}(u(t, x))^2 < +\infty. \) (We do not have to use Lemma D.1 here.)

As \( v \) is time-stationary, (5.22) becomes

\[
\mathbb{E}[h(T, X) - h(0, X)] = -\frac{T}{2} \mathbb{E}v(X)^2 + \frac{T}{2} \|\rho\|_{L^{2}(\mathbb{R})}^2,
\]

which implies (5.21). \( \square \)

The next lemma compares the expectation of a KPZ evolution started with random initial condition to that of the KPZ equation started at deterministic bounded initial data.

Lemma 5.8. Fix \( m \in (0, 1) \) and \( G = L\mathbb{Z} \) or \( G = \mathbb{R}, \) and let \( X \sim \text{Uniform}(\Lambda_G) \) be independent of everything else. Let \( v \in L^\infty(\mathbb{R}) \) be \( G \)-invariant and let \( \tilde{v} \sim \tilde{v} \in \mathcal{F}_G(\mathcal{X}_m) \) satisfy \( \mathbb{E}\tilde{v}(X)^2 < \infty. \) Let \( u \) and \( \tilde{u} \) solve (1.3) with initial data \( v \) and \( \tilde{v}, \) respectively. Consider solutions \( h \) and \( \tilde{h} \) to (1.11) obtained from \( u \) and \( \tilde{u}, \) respectively, as in Proposition 4.4. Then for all \( t \geq 0 \) and \( x \in \mathbb{R} \) we have

\[
\mathbb{E}[\tilde{h}(t, x) - h(t, x)] \leq \sup_{x \in \mathbb{R}}[\mathbb{E}\tilde{h}(0, x) - h(0, x)].
\]

(5.24)
This is a version of the comparison principle for the KPZ equation. The difficulty comes from the fact that \( \tilde{h}(0, x) \) is not necessarily uniformly bounded, so the standard pathwise comparison principle would be vacuous, as the right side of (5.24) would be infinite without taking the expectation. Note that, as we use solutions coming from Proposition 4.4, the expectations \( E \tilde{h}(t, x) \) and \( E h(t, x) \) are finite under the assumptions of Lemma 5.8. We postpone the proof of this lemma until Sect. 5.3 and first explain how it is used to show the extremality of the limiting invariant measures. The first step is to show that a limiting solution started from deterministic bounded initial condition has minimal variance among all spacetime-stationary solutions.

**Proposition 5.9.** Fix \( m \in (0, 1) \) and let \( G = L \mathbb{Z} \) or \( G = \mathbb{R} \). Fix a deterministic, \( G \)-invariant function \( v \in L^\infty(\mathbb{R}) \) and let \( \delta_v \in \mathcal{P}_G(X_m) \) be the delta measure on \( v \). Suppose that \( T_k \to +\infty \), and
\[
\frac{1}{T_k} \int_0^{T_k} P_t^* \delta_v \, dt \to v \in \mathcal{P}(X_m),
\]
weakly.

Let \( w \sim v, \tilde{w} \sim \tilde{v} \in \mathcal{P}_G(X_m) \), and \( X \sim \text{Uniform}(\Lambda_G) \) be independent of everything else. Then we have
\[
\text{Var}[w(X)] \leq \text{Var}[\tilde{w}(X)].
\]

**Proof.** If \( \text{Var} \tilde{w}(X) = \infty \) then there is nothing to show, so we may assume that \( \text{Var} \tilde{w}(X) < \infty \). By Proposition 4.1, it suffices to consider the case
\[
E v(X) = \int_{\Lambda_G} v(x) d\lambda_G(x) = 0.
\]

Let \( u \in Z_m \) be a solution to (1.3) with initial condition \( u(0, \cdot) = v \). By Proposition 3.6 and the uniform integrability coming from Lemma 5.3, we have that
\[
E w(X) = \int_{\Lambda_G} w(x) d\lambda_G(x) \leq \lim_{k \to \infty} \int_{\Lambda_G} E u(S_{T_k}, x) d\lambda_G(x) = 0,
\]
where \( S_T \sim \text{Uniform}([0, T]) \) is independent of everything else. Let \( h \) be a solution to (1.11) such that
\[
\partial_x h(0, \cdot) = v(0, \cdot) \quad \text{and} \quad \int_{\Lambda_G} h(0, x) d\lambda_G(x) = 0
\]
so that \( \partial_x h(t, \cdot) = u(t, \cdot) \). We see from (5.26) and the second condition in (5.28) that
\[
\|h(0, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|v\|_{L^\infty(\mathbb{R})}.
\]

Hence, the comparison principle implies that
\[
|h(t, x) - h_0(t, x)| \leq \|v\|_{L^\infty(\mathbb{R})}.
\]
Here, \( h_0(t, x) \) is the solution to (1.11) with the initial condition \( h_0(0, x) \equiv 0 \). As a consequence, we know that \( E h(t, x) \) is finite for all \( t \geq 0 \).
By Fatou’s lemma and (5.25), we have
\[
\text{Var } w(X) = \mathbb{E} \left[ \int_{\Lambda} w(x)^2 \, d\lambda_G(x) \right] \leq \lim_{k \to \infty} \inf \mathbb{E} \left[ \int_{\Lambda} u(S_{T_k}, x)^2 \, d\lambda_G(x) \right].
\]
(5.30)

The expectation in the right side is finite by Proposition 5.1. Recalling (5.22), noting that (5.23) relies only on the spatial stationarity of \( u \), and using (5.29), we see that
\[
\text{Var } w(X) \leq \lim_{k \to +\infty} \sup \frac{2}{T_k} \mathbb{E} \left[ \int_{\Lambda} h(T_k, x) \, d\lambda_G(x) \right].
\]
(5.31)

Next, by Proposition 4.1, we can also assume without loss of generality that
\[
\int_{\Lambda} \mathbb{E} \tilde{w}(y) \, d\lambda_G(y) = 0,
\]
(5.32)
since we can subtract off the appropriate mean without changing the variance. Let \( \tilde{u} \) solve (1.3) with initial condition \( \tilde{w} \), and construct a solution \( \tilde{h} \) from \( \tilde{u} \) using Proposition 4.4, so that, in particular, \( \partial_x \tilde{h} = \tilde{u} \). As \( \tilde{h}(0, x) \) satisfies (4.2), there exists a random point \( x_0 \in \text{supp} \xi \) such that \( \tilde{h}(0, x_0) = 0 \). As \( \mathbb{E} \tilde{w} \) is \( G \)-invariant and satisfies (5.32), we may write
\[
\left| \mathbb{E} \tilde{h}(0, x) \right| = \left| \mathbb{E} \int_{x_0}^x \tilde{w}(y) \, dy \right| \leq \mathbb{E} \left| \int_{x_0}^x \tilde{w}(y) \, dy \right| \leq \mathbb{E} \int_0^L |\tilde{w}(y)| \, dy < L^{1/2}(\mathbb{E}(\tilde{w}(X)^2))^{1/2} < \infty.
\]
(5.33)

In addition, directly from (4.1) we get
\[
\mathbb{E} |\tilde{h}(t, x)| < \infty \text{ for all } t > 0, x \in \mathbb{R}.
\]

As \( \mathbb{E} \tilde{u}(0, X) = \mathbb{E} \tilde{w}(X) = 0 \), Lemma 5.7 implies that for all \( t > 0 \) we have
\[
\text{Var } \tilde{w}(X) = \mathbb{E} \tilde{u}(t, X)^2 = \|\rho\|^2_{L^2(\mathbb{R})} - \frac{2}{t} \mathbb{E} [\tilde{h}(t, X) - \tilde{h}(0, X)].
\]
(5.34)

Also, it follows from Lemma 5.8 that
\[
\mathbb{E} \tilde{h}(t, X) \leq \mathbb{E} h(t, X) + \|\mathbb{E} \tilde{h}(0, \cdot)\|_{C_b(\mathbb{R})} + \|h(0, \cdot)\|_{C_b(\mathbb{R})}.
\]
(5.35)

Therefore, we have
\[
\text{Var } w(X) \leq \|\rho\|^2_{L^2(\mathbb{R})} - \lim_{k \to \infty} \sup \frac{2}{T_k} \mathbb{E} h(T_k, X)
\]
\[
\leq \|\rho\|^2_{L^2(\mathbb{R})} - \lim_{k \to \infty} \sup \frac{2}{T_k} \left( \mathbb{E} \tilde{h}(T_k, X) - \|\mathbb{E} \tilde{h}(0, \cdot)\|_{C_b(\mathbb{R})} - \|h(0, \cdot)\|_{C_b(\mathbb{R})} \right).
\]
\[
\limsup_{n \to \infty} \left( \| \rho \|_{L^2(\mathbb{R})}^2 - \frac{2}{T_k} \left[ \mathbb{E} \tilde{h}(T_k, X) - \mathbb{E} \tilde{h}(0, X) \right] \right) = \text{Var} \tilde{w}(X),
\]
where the first inequality is (5.31); the second inequality is by (5.35); the first equality is by (5.29); and the second equality is by (5.34).

**Proposition 5.10.** Suppose that \( G = L^2 \) with \( L > 0 \) or \( G = \mathbb{R} \), and that a deterministic function \( v \in L^\infty(\mathbb{R}) \) is \( G \)-invariant. Let \( \delta_v \) be the measure on \( L^2_{P_1/2^+}(\mathbb{R}) \) with a single atom at \( v \). If \( s \geq 0 \), \( T_k \uparrow \infty \), and \( v \in \mathcal{P}(\mathcal{X}_{1/2}) \) are such that
\[
v = \lim_{k \to \infty} \frac{1}{T_k} \int_{s+T_k} P_t^* \delta_v \, dt
\]
in the sense of weak convergence of probability measures on \( \mathcal{X}_{1/2} \), then \( v \in \overline{\mathcal{P}}_G(\mathcal{X}_{1/2}) \).

**Proof.** The fact that \( v \in \overline{\mathcal{P}}_G(\mathcal{X}_{1/2}) \) is Proposition 4.2, so we only need to show that \( v \) is extremal. Suppose that we can decompose \( v \) as
\[
v = (1 - q) \mu_0 + q \mu_1,
\]
for some \( q \in (0, 1) \) and \( \mu_0, \mu_1 \in \overline{\mathcal{P}}_G(\mathcal{X}_{1/2}) \). By Proposition 4.3, there exists a coupling \( \mu \in \overline{\mathcal{P}}_G(\mathcal{X}_{1/2}^2) \) of \( \mu_0 \) and \( \mu_1 \). Let \( (v_0, v_1) \sim \mu \) and consider \( v_I \), where \( I \sim \text{Bernoulli}(q) \) is a random variable, independent of everything else. Then, \( v_I \) is distributed according to \( v \):
\[
\text{Law}(v_I) = (1 - q) \mu_0 + q \mu_1 = v.
\]
By Theorem 3.8, the sign \( \chi := \text{sgn}(v_0(x) - v_1(x)) \) \( \mu \)-almost surely does not depend on \( x \), and by the comparison principle in Proposition 3.1 we know that \( \chi \) is invariant under the dynamics (1.4). Hence, the restrictions of \( \mu \) onto the sets \( \{ \chi = b \} \subset \mathcal{X}_{1/2}^2 \) are invariant, for each \( b \in \{-1, 0, 1\} \), as are
\[
v_{i,b} := \text{Law}(v_I | I = i, \chi = b) \in \overline{\mathcal{P}}_G(\mathcal{X}_m)
\]
for all \( i \in \{0, 1\} \) and \( b \in \{-1, 0, 1\} \) such that \( \mathbb{P}[\chi = b] > 0 \).

Let now \( X \sim \text{Uniform}(\Lambda_G) \) be independent of everything else. By the law of total variance, we have
\[
\text{Var}(v_I(X)) = \mathbb{E} \text{Var}(v_I(X) | I, \chi) + \text{Var}(\mathbb{E}[v_I(X) | I, \chi]).
\]
As \( \text{Law}(v_I) = v \), and \( v_{i,b} \) are invariant, by Proposition 5.9 and (5.36), we have
\[
\text{Var}(v_I(X) | I, \chi) \geq \text{Var}(v_I(X)), \quad \text{a.s.}
\]
In light of (5.37), this means that
\[
\text{Var}(\mathbb{E}[v_I(X) | I, \chi]) = 0,
\]
so that \( \mathbb{E}[v_I(X) | I, \chi] \) is constant almost surely. On the other hand, if \( \chi = 1 \), then \( v_0(x) > v_1(x) \) almost surely, thus
\[
\mathbb{E}[v_0(X) | \chi = 1] \geq \mathbb{E}[v_1(X) | \chi = 1].
\]
Similarly, if \( \chi = -1 \), then \( v_0(x) < v_1(x) \) almost surely, and
\[
\mathbb{E}[v_0(X) | \chi = -1] \leq \mathbb{E}[v_1(X) | \chi = -1].
\]
The only way these facts can be consistent with the almost-surely-constant nature of \( \mathbb{E}[v_I(X) | I, \chi] \) is that \( \chi = 0 \) almost surely. Therefore, \( v_0 = v_1 \) almost surely, which means that \( \mu_0 = \mu_1 \). This implies that \( v \) is extremal. □
5.3. Proof of Lemma 5.8.

Proof. Let us recall again that the claim of Lemma 5.8 would be just the comparison principle for the KPZ equation, and would hold without taking the expectation, if only it were true that the initial conditions \( \tilde{h}(0, x) \) and \( h(0, x) \) are bounded. Indeed, the difference \( H(t, x) = h(t, x) - \tilde{h}(t, x) \) satisfies

\[
\partial_t H + \frac{1}{2} (\partial_x h + \partial_x \tilde{h}) \partial_x H = \frac{1}{2} \partial_x^2 H,
\]

and the maximum principle would imply

\[
H(t, x) \leq \sup_x H(0, x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R},
\]

if the supremum in the right side were finite.

As usual, we assume without loss of generality that \( G = L \mathbb{Z} \) for some \( L > 0 \), and let \( X \) be uniformly distributed on \([0, L]\). Note that, as spatial integrals of a periodic functions, \( h(0, x) \) and \( \tilde{h}(0, x) \) satisfy

\[
h(0, x) \sim [\mathbb{E}v(X)]x, \quad \tilde{h}(0, x) \sim [\mathbb{E}\tilde{v}(X)]x, \quad \text{as } |x| \to \infty.
\]

Thus we can assume \( \mathbb{E}v(X) = \mathbb{E}\tilde{v}(X) \), since otherwise the supremum on the right side of (5.24) is infinite and the statement is vacuous. By Proposition 4.1, we may assume that \( \mathbb{E}v(X) = \mathbb{E}\tilde{v}(X) = 0 \), so that \( \mathbb{E}\tilde{h}(0, x) \) is continuous and \( L \)-periodic in \( x \).

Let \( E \) denote expectation in which only the initial condition \( \tilde{h}(0, \cdot) \) and \( X \) are treated as random and the noise is considered to be deterministic. That is, \( E[Y] = \mathbb{E}[Y | \mathcal{F}_\infty] \), where \( \mathcal{F}_\infty \) is the \( \sigma \)-algebra generated by the noise. Taking formally the \( E \) expectation of (1.11) gives

\[
d(E\tilde{h}) \leq \frac{1}{2} [\partial_x^2 (E\tilde{h}) - (\partial_x (E\tilde{h}))^2] + \|\rho\|_{L^2(\mathbb{R})}^2] dt + dV, \quad (5.38)
\]

Subtracting another copy of (1.11) for \( h \), we find

\[
\partial_t (E\tilde{h} - h) \leq \frac{1}{2} [\partial_x^2 (E\tilde{h} - h) - \partial_x (E\tilde{h} - h) \cdot \partial_x (E\tilde{h} + h)]. \quad (5.39)
\]

Once again, if the comparison principle could be applied to (5.39), we would be essentially done. However, we need to justify that \( E\tilde{h} \) exists and is sufficiently regular, so that the formal manipulations are justified. This propagation of moments does not follow from the regularity results we have obtained so far, and we will use stationarity of \( \tilde{v} \) instead.

We denote \( \tilde{u} = \partial_x \tilde{h} \). The time-invariance and Lemma 5.5 imply that

\[
\mathbb{E}\|\tilde{u}(t, \cdot)\|_{H^1(\Lambda_G)}^2 \leq \mathbb{E}\|\tilde{u}(0, \cdot)\|_{H^1(\Lambda_G)}^2 = \mathbb{E}\|\tilde{v}\|_{H^1(\Lambda_G)}^2 = L \mathbb{E}\tilde{v}(X)^2 + L\|\partial_x \rho\|_{L^2(\mathbb{R})}^2 < \infty \quad (5.40)
\]

for any \( t \in [0, T] \). Here, \( \Lambda_G \) is a fundamental domain for \( G \), and as usual we use the notation \( \mathbb{E} \) for expectation on the largest probability space involving all objects. Now fix \( \ell \in (m \vee 1/2, 1) \). Morrey’s inequality implies

\[
\|\tilde{u}(t, \cdot)\|_{C^{\ell/2} \nu^j(\mathbb{R})} \leq C \sup_{j \in \mathbb{Z}} \frac{\|\tilde{u}(t, \cdot)\|_{C^{\ell/2}(\Lambda_G + Lj)}}{\langle Lj \rangle^\ell} \leq C \sup_{j \in \mathbb{Z}} \frac{\|\tilde{u}(t, \cdot)\|_{H^\ell(\Lambda_G + jL)}}{\langle j \rangle^\ell}
\]
with some constant $C$ depending on $L$. Arguing as in the proof of Proposition 5.6, and using (5.40), we obtain

$$\mathbb{E}\|E\tilde{u}(t, \cdot)\|_{C^{1/2}_{pl}(\mathbb{R})}^2 < C,$$

uniformly in $t$. Integrating in $t$, we find that for any $T \geq 0$, we have

$$E \tilde{u} \in L^2([0, T]; C^{1/2}_{pl}(\mathbb{R})) \quad (5.41)$$

almost surely in the noise. Next, recall that

$$E(\partial_x \tilde{h})^2 = E\tilde{u}^2,$$

and

$$E\tilde{u}(t, X)^2 = \mathbb{E}(\tilde{h})(X)^2 < \infty.$$

Also, Lemma 5.5 implies

$$E|\partial_x (\tilde{u})^2(t, X)| \leq 2E[|\tilde{u}(t, X)||\partial_x \tilde{u}(t, X)|] \leq \mathbb{E}\|\tilde{v}\|_{H^1(A_G)}^2 < \infty.$$

It follows that

$$E\|\tilde{u}^2(t, \cdot)\|_{W^{1,1}(A_G)} < C,$$

uniformly in $t$. Since $W^{1,1}(A_G)$ embeds continuously into $C_b(A_G)$, we have

$$E\|\tilde{u}^2(t, \cdot)\|_{C^{\ell+1/2}_{pl+1/2}(\mathbb{R})} \leq C \sum_{j \in \mathbb{Z}} \frac{\mathbb{E}\|\tilde{u}^2(t, \cdot)\|_{W^{1,1}(A_G+Lj)}}{\langle Lj \rangle^{\ell+1/2}} \leq C \sum_{j \in \mathbb{Z}} \langle j \rangle^{-(\ell+1/2)} < \infty.$$

Thus,

$$E\tilde{u}^2 = E(\partial_x \tilde{h})^2 \in L^1([0, T]; C^{\ell+1/2}_{pl+1/2}(\mathbb{R})) \quad (5.42)$$

almost surely in $V$. Hence, it makes sense to take the expectation $E$ of the mild formulation

$$\tilde{h}(t, x) = G_t * \tilde{h}(0, x) - \frac{1}{2} \int_0^t \left[ G_{t-s} * (\partial_x \tilde{h}(s, \cdot))^2 \right](x) \, ds + \frac{t}{2} \|\rho\|_{L^2(\mathbb{R})}^2$$

$$(5.43)$$

of (1.11). As in the proof of Proposition 4.4, define

$$\omega(t, x) = \int_0^t \int_{\mathbb{R}} (G_{t-s} * \rho)(x-y) \, dW(s, y).$$

Then $\tilde{g} = E\tilde{h} - \omega$ satisfies the mild equation

$$\tilde{g}(t, \cdot) = G_t * E\tilde{h}(0, \cdot) - \frac{1}{2} \int_0^t G_{t-s} * [E\tilde{u}(s, \cdot)^2] \, ds + \frac{t}{2} \|\rho\|_{L^2(\mathbb{R})}^2. \quad (5.44)$$
By the fact that $\tilde{E}h(0, \cdot) \in C(\mathbb{R}/L\mathbb{Z})$ and (5.42), we have
\[
\tilde{g} \in L^\infty([0, T]; L^\infty_{p+1/2}(\mathbb{R}))
\]
and $\tilde{g}$ is continuous in time. Standard Gaussian process estimates show that $\omega(t, \cdot)$ is a continuous process in $C_{p'}(\mathbb{R})$ for any $p' > 0$, so
\[
E\tilde{h} \in L^\infty([0, T]; L^\infty_{p+1/2}(\mathbb{R})).
\tag{5.45}
\]
As a spatial integral, $E\tilde{h}$ is differentiable in space. Since $\tilde{g}$ is continuous in time, $E\tilde{h}$ is in fact spacetime-continuous.

We may now apply Jensen’s inequality to (5.43) to obtain (5.38) and (5.39) in the mild sense. The next natural step is to apply the maximum principle to (5.39) to conclude that $E\tilde{h} - h$ never exceeds the maximum of its initial data. However, (5.39) need only hold in the mild sense, and, in addition, the drift coefficient $\partial_x (E\tilde{h} + h)$ in (5.39) is potentially irregular in time and grows polynomially in space. We therefore directly verify that (5.39) obeys the maximum principle. Let
\[
z(t, x) = E\tilde{h}(t, x) - h(t, x) - \sup_{y \in \mathbb{R}}[E\tilde{h}(0, y) - h(0, y)]
\]
and $b(t, x) = \partial_x (E\tilde{h} + h)(t, x)$, so that (5.39) can be written as
\[
\partial_t z \leq \frac{1}{2} \partial_x^2 z - b \partial_x z,
\tag{5.46}
\]
but this only holds in the mild sense. In addition, we have $z(0, x) \leq 0$ for all $x \in \mathbb{R}$. As for regularity of $z$ and $b$, by (5.41) and (5.45), we know that $z \in L^\infty([0, T]; L^\infty_{p+1/2}(\mathbb{R}))$ is continuous in time and differentiable in space. Moreover, by (5.41), both $\partial_x z$ and $b$ lie in $L^2([0, T]; C^{1/2}_{p'}(\mathbb{R}))$.

To obtain an honest differential inequality, we mollify (5.46) in space. Let $\zeta \in C^\infty(\mathbb{R})$ be non-negative with $\|\zeta\|_{L^1(\mathbb{R})} = 1$. Define $\zeta_\delta(x) = \delta^{-1}\zeta(\delta^{-1}x)$ for $\delta > 0$, and write $z_\delta = \zeta_\delta \ast z$, so
\[
\partial_t z_\delta \leq \frac{1}{2} \partial_x^2 z_\delta - b \partial_x z_\delta + [b(\zeta_\delta \ast \partial_x z) - \zeta_\delta \ast (b \partial_x z)].
\tag{5.47}
\]
This inequality is now valid pointwise. To make use of this inequality, we must show that the bracketed commutator
\[
E := b(\zeta_\delta \ast \partial_x z) - \zeta_\delta \ast (b \partial_x z)
\]
in (5.47) is sufficiently small. Note that
\[
|E(t, x)| \leq \int_{\mathbb{R}} |b(t, x) - b(t, y)||\partial_x z(t, y)||\zeta_\delta(x - y)\, dy
\leq CF_1(t)\delta^{1/2}\langle x \rangle^\ell \int_{\mathbb{R}} \langle y \rangle^\ell \zeta_\delta(x - y)\, dy
\leq C_{E}\delta^{1/2}F_1(t)\langle x \rangle^{2\ell},
\tag{5.48}
\]
where
\[ F_1(t) = \|b(t, \cdot)\|_{C^{1/2}_{p\ell}(\mathbb{R})} \|\partial_x z(t, \cdot)\|_{C^{1/2}_{p\ell}(\mathbb{R})} \in L^1([0, T]). \]

Hence, the commutator can be made arbitrarily small in \( L^1([0, T]; L^\infty_{p\ell}(\mathbb{R})) \).

We now construct a super-solution to (5.47). The function
\[
H(t, x) = \sqrt{\frac{3T}{3T-t}} \exp \left[ \frac{x^2}{4(3T-t)} \right].
\]
satisfies
\[ \partial_t H = \partial_x^2 H, \]
and there exists \( C_H \in [1, \infty) \) such that
\[ \partial_x^2 H \geq C_H^{-1} \langle x \rangle^2 H \quad \text{and} \quad |\partial_x H| \leq C_H \langle x \rangle H \]
on \([0, 2T] \times \mathbb{R}\). Define
\[ F_2(t) = \|b(t, \cdot)\|_{C_{p\ell}(\mathbb{R})} \in L^2([0, T]) \subset L^1([0, T]) \]
and let
\[
\eta(t) = t + c \int_0^t \left[ F_1(s) + F_2(s) \right] ds
\]
with \( c > 0 \) chosen so that \( \eta(T) \leq 2T \). We use \( \eta \) as a time-change in \( H \):
\[
\partial_t \left[ H(\eta(t), x) \right] = \frac{\dot{\eta}}{2} \partial_x^2 H + \frac{\dot{\eta}}{2} \partial_x^2 H \\
\geq \frac{1}{2} \partial_x^2 H - b \partial_x H + \left[ \frac{c}{2C_H} \langle x \rangle^2 - C_H \langle x \rangle^{1+\ell} \right] F_2(t) H.
\]
Since \( \ell < 1 \), the term in brackets is positive when \( x \) is large. To handle small \( x \), we multiply by a time-dependent factor. There exists \( \kappa > 0 \) such that
\[
\partial_t \left[ e^{\kappa \eta(t)} H(\eta(t), \cdot) \right] > \frac{1}{2} \partial_x^2 (e^{\kappa \eta(t)} H) - b \partial_x (e^{\kappa \eta(t)} H) + \kappa c F_1(t) e^{\kappa \eta(t)} H.
\]
Now let
\[
\varepsilon = \frac{C_E \delta^{1/2}}{c \kappa} \sup_{(t, x) \in [0, T] \times \mathbb{R}} \frac{\langle x \rangle^2}{H(t, x)}
\]
so that \( \varepsilon c \kappa F_1 H \geq |E| \) by (5.48). Then
\[
\tilde{z}(t, x) = z_{\delta}(t, x) - \varepsilon e^{\kappa \eta(t)} H(\eta(t), x)
\]
satisfies \( \tilde{z}(0, \cdot) \leq -\varepsilon \) and
\[
\partial_t \tilde{z} < \frac{1}{2} \partial_x^2 \tilde{z} - b \partial_x \tilde{z}. \tag{5.49}
\]
In addition, by (5.45), $H$ grows much faster than $z_\delta$ in space. The standard maximum principle then implies that $\tilde{z}(t,x) < 0$ for all $t > 0$ and $x \in \mathbb{R}$. Thus, we have

$$z_\delta(t,x) \leq \varepsilon e^{\kappa \eta(t)} H(\eta(t), x) \leq C \delta^{1/2} \exp \left( \frac{x^2}{4T} \right)$$

(5.50)

for some $C$ depending on $T$, $m$, and $L$, but not on $\delta$.

We now take $\delta \to 0$. Since $z$ is continuous, $z_\delta \to z$ pointwise. Therefore (5.50) implies $z \leq 0$, as desired. It follows that

$$\sup_{t \geq 0, x \in \mathbb{R}} (\mathbb{E} \tilde{h} - h)(t,x) \leq \sup_{x \in \mathbb{R}} (\mathbb{E} \tilde{h} - h)(0,x).$$

Taking expectations with respect to the noise, we obtain (5.24). $\square$

6. Classification of Extremal Invariant Measures

In this section we identify all elements of $\mathcal{P}_G(\mathcal{X}_m)$, the extremal invariant measures, proving Theorem 1.2. Here is the key step.

**Proposition 6.1.** Let $m \in (0,1)$, $G = L\mathbb{Z}$ with $L > 0$ or $G = \mathbb{R}$, and $\Lambda_G$ be a fundamental domain for $G$. Suppose that $v \in \mathcal{P}_G(\mathcal{X}_m^{2N})$, and let

$$v = (v_1, v_2) = ((v_{1,1}, \ldots, v_{1,N}), (v_{2,1}, \ldots, v_{2,N})) \sim v,$$

and $v_i = \text{Law}(v_i)$, so that $v_1, v_2 \in \mathcal{P}_G(\mathcal{X}_m^{N})$, and let $v_{i,j}$ be the marginal law of $v_i$ on the $j$th component. Suppose that for each $i = 1, 2$, $j = 1, \ldots, N$, we have $v_{i,j} \in \mathcal{P}_G(\mathcal{X}_m)$, and

$$\mathbb{E} \left[ \int_{\Lambda_G} v_{i,j}(x)^2 d\lambda_G(x) \right] < \infty. \quad (6.1)$$

If, in addition, we have

$$\mathbb{E} \left[ \int_{\Lambda_G} v_{1,j}(x) d\lambda_G(x) \right] = \mathbb{E} \left[ \int_{\Lambda_G} v_{2,j}(x) d\lambda_G(x) \right], \text{ for each } j = 1, \ldots, N, \quad (6.2)$$

then $v_1 = v_2$ almost surely.

**Proof.** By Theorem 3.8 with hypothesis (H1), the random variables $\chi_j := \text{sgn}(v_{1,j}(x) - v_{2,j}(x))$ do not depend on $x$. For given $i$, $j$, and $b \in \{-1, 0, 1\}$, (i) if $\mathbb{P}(\chi_j = b) \neq 0$, we let $v_{i,j,b}$ be the conditional marginal distribution of $v_{i,j}$ on the event $\chi_j = b$, and (ii) if $\mathbb{P}(\chi_j = b) = 0$, we set $v_{i,j,b} = \text{Law}(v_{i,j})$. By Proposition 3.1, the Burgers evolution preserves ordering, so $v_{i,j,b}$ is invariant under the evolution. Moreover, $v_{i,j,b}$ is $G$-invariant since it is the distribution of a $G$-invariant measure conditional on a $G$-invariant event. Therefore, we have $v_{i,j,b} \in \mathcal{P}_G(\mathcal{X}_m)$. As we can decompose $\text{Law}(v_{i,j})$ as

$$\text{Law}(v_{i,j}) = \sum_{b \in \{-1,0,1\}} \mathbb{P}(\chi_j = b)v_{i,j,b},$$
and \( \text{Law}(v_{i,j}) \in \overline{\mathcal{P}}_G(\mathcal{X}_m^N) \), it follows that \( v_{i,j} = \text{Law}(v_{i,j}) \) for all \( b \). Thus if \( X \sim \text{Uniform}(\Lambda) \) is independent of all else, we have, by (6.2), that

\[
\mathbb{E}[v_{1,j}(X) - v_{2,j}(X) | \chi_j = b] = \mathbb{E}[v_{1,j}(X) | \chi_j = b] - \mathbb{E}[v_{2,j}(X) | \chi_j = b] = \mathbb{E}v_{1,j}(X) - \mathbb{E}v_{2,j}(X) = 0.
\]

(6.3)

On the other hand, if \( b \in \{\pm 1\} \) and \( \mathbb{P}(\chi_j = b) > 0 \), then

\[
\mathbb{E}[v_{1,j}(X) - v_{2,j}(X) | \chi_j = b] \neq 0,
\]

contradicting (6.3). Therefore, \( \mathbb{P}(\chi_j = 0) = 1 \), and thus \( v_1 = v_2 \) almost surely. \( \square \)

We have the following corollary.

**Corollary 6.2.** Let \( m \in (0, 1), G = L\mathbb{Z} \) with \( L > 0 \) or \( G = \mathbb{R} \). Suppose that \( v_1, v_2 \in \overline{\mathcal{P}}_G(\mathcal{X}_m^N) \), with some \( N \in \mathbb{N} \), and let \( v_{i,j} \) be the marginal law of \( v_i \) on the \( j \)th component, and \( v_{i,j} \sim v_{i,j} \). Suppose that for each \( i = 1, 2, j = 1, \ldots, N \), we have \( v_{i,j} \in \overline{\mathcal{P}}_{G}(\mathcal{X}_m) \), and both (6.1) and (6.2) hold. Then \( v_1 = v_2 \).

**Proof.** By Proposition 4.3, we have a coupling \( \nu \in \overline{\mathcal{P}}_G(\mathcal{X}_m^2) \) of \( v_1 \) and \( v_2 \). Then Proposition 6.1 applies, so \( v_1 = \text{Law}(v_1) = \text{Law}(v_2) = v_2 \). \( \square \)

Using Corollary 6.2, we prove Theorem 1.2.

**Proof of Theorem 1.2.** We first prove the existence claim. Fix \( m \in [1/2, 1) \), \( N \in \mathbb{N} \), and \( a \in \mathbb{R}^N \). By Proposition 5.1 applied with \( N = 1 \) and \( u(0, \cdot) \equiv 0 \), Prokhorov’s theorem, and Proposition 5.10, there exists \( \nu_0 \in \overline{\mathcal{P}}_{G}(\mathcal{X}_m) \) satisfying the moment condition (H1). Proposition 5.10 implies also that \( \nu_0 \in \overline{\mathcal{P}}_{G}(\mathcal{X}_m) \) for \( G = L\mathbb{Z} \) for any \( L > 0 \). Moreover, if \( v_0 \sim v_0 \), then \( \mathbb{E}v_0(x) = 0 \) for all \( x \in \mathbb{R} \).

Next, take some \( a \in \mathbb{R} \). By the shear-invariance proved in Proposition 4.1, if \( v_0 \sim v_0 \), then

\[
v_a := \text{Law}(v_0 + a) \in \overline{\mathcal{P}}_G(\mathcal{X}_m)
\]

as well.

By Proposition 4.3 (applied inductively), for any \( N \in \mathbb{N} \) and \( a = a_1, \ldots, a_N \in \mathbb{R} \), there is a coupling \( v_a \in \overline{\mathcal{P}}_G(\mathcal{X}_m^N) \) of \( v_{a_1}, \ldots, v_{a_N} \), both for \( G = \mathbb{R} \) and \( G = L\mathbb{Z} \) for all \( L > 0 \). We claim that \( v_a \) are in \( \overline{\mathcal{P}}_G(\mathcal{X}_m^N) \).

Suppose that

\[
v_a = (1 - q)\mu_0 + q\mu_1,
\]

for some \( q \in (0, 1) \) and \( \mu_0, \mu_1 \in \overline{\mathcal{P}}_G(\mathcal{X}_m^N) \). For each \( j = 1, \ldots, N \), the extremality of \( v_{a_j} \) means that the \( j \)th marginals of \( \mu_0 \) and \( \mu_1 \) must both be \( v_{a_j} \). Then Corollary 6.2 implies that \( \mu_0 = \mu_1 \). So indeed \( v_a \in \overline{\mathcal{P}}_G(\mathcal{X}_m^N) \). We have thus shown existence in Theorem 1.2.

For uniqueness, fix \( G = L\mathbb{Z} \) or \( G = \mathbb{R} \) and a fundamental domain \( \Lambda_G \) for \( G \). Let \( X \sim \text{Uniform}(\Lambda_G) \) be independent of all else. Suppose that \( \nu \in \overline{\mathcal{P}}_G(\mathcal{X}_m^N) \) satisfies \( \mathbb{E}v(X) = a \) and \( \mathbb{E}|v(X)|^2 < \infty \) for \( v \sim \nu \). We will show that \( \nu = \nu_a \). Using Proposition 4.3, let \( \tilde{v} \in \overline{\mathcal{P}}_G(\mathcal{X}_m^{N+1}) \) couple \( v \) and \( v_{a_1} \), and take \( \tilde{v} \sim \nu \). By Theorem 3.8 with hypothesis
(H1), \( \chi = \text{sgn}(\bar{v}_{N+1} - \bar{v}_1) \) is independent of \( x \) almost surely. Suppose for the sake of contradiction that \( \mathbb{P}[\chi \neq 0] > 0 \). Since

\[
\mathbb{E}\bar{v}_{N+1}(X) = a_1 = \mathbb{E}\bar{v}_1(X),
\]
this would imply that

\[
q := \mathbb{P}[\chi \geq 0] \in (0, 1).
\]

Hence, we may define the laws \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) as \( \bar{v} \) conditioned on the \( G \)-invariant events \( \{\chi = -1\} \) and \( \{\chi \geq 0\} \), respectively, so that

\[
\bar{v} = (1-q)\tilde{\mu}_1 + q\tilde{\mu}_2,
\]
and if \((w_{i,1}, \ldots, w_{i,N}, w_{i,N+1}) \sim \tilde{\mu}_i\), then \( w_{1,1} \geq w_{1,N+1} \) and \( w_{2,1} < w_{2,N+1} \) almost surely. In addition, as \( \nu_{a_1} \) is extremal, we know that both \( w_{1,N+1} \) and \( w_{2,N+1} \) have the law \( \nu_{a_1} \). It follows that

\[
\mathbb{E}w_{1,1}(X) \geq \mathbb{E}w_{1,N+1} = \mathbb{E}w_{2,N+1} > \mathbb{E}w_{2,1}(X).
\]

By Proposition 3.1, \( \tilde{\mu}_i \in \mathcal{P}_G(\mathcal{X}_m^{N+1}) \) for \( i = 1, 2 \).

Now let \( \mu_i \) denote the marginal of \( \tilde{\mu}_i \) on the first \( N \) components. Then \( \mu_1 \in \mathcal{P}_G(\mathcal{X}_m^N) \) and \( v = (1-q)\mu_0 + q\mu_1 \). Since \( v \) is extremal, we must have \( \mu_0 = \mu_1 \). However, (6.4) shows that the first marginals of \( \mu_0 \) and \( \mu_1 \) have different means, a contradiction. We conclude that \( \chi = 0 \) almost surely, so the marginal of \( v \) on its first component is \( \nu_{a_1} \).

By an identical argument, all of the one-component marginals of \( v \) and \( \nu_{a} \) agree. Using Corollary 6.2, we conclude that \( v = \nu_{a} \). Thus \( \nu_{a} \) is the unique element of \( \mathcal{P}_G(\mathcal{X}_m^N) \) with mean \( a \) and finite variance.

We now establish properties (P1)–(P5) for \( \nu_{a} \). The first four properties have been essentially proved already: (P1) follows from the ordering proved in Theorem 3.8, (P2) follows from the shear-invariance proved in Proposition 4.1 and the uniqueness claim, (P3) follows from the uniqueness statement, and (P4) follows from Lemma 2.8, Lemma 2.5, and the smoothness and sub-polynomial growth at infinity of \( \psi \).

To prove (P5), suppose that there exists a \( q \in (0, 1) \) and \( \mu_0, \mu_1 \in \mathcal{P}_G(\mathcal{X}_m^N) \) so that

\[
\nu_{a} = (1-q)\mu_0 + q\mu_1.
\]

We claim that for \( i = 0, 1 \), the family \( \{P_t^*\mu_i\}_{t \geq 0} \) is tight with respect to the topology of \( \mathcal{X}_m^N \). Indeed, note that

\[
\{P_t^*((1-q)\mu_0 + q\mu_1)\}_{t \geq 0} = \{P_t^*\nu_{a}\}_{t \geq 0} = \{\nu_{a}\}
\]
is tight with respect to the topology of \( \mathcal{X}_m^N \). Hence, for each \( \varepsilon > 0 \) there is a compact set \( \mathcal{K} \subset \mathcal{X}_m^N \) so that for all \( t \geq 0 \), we have

\[
(1-q)\varepsilon > P_t^*((1-q)\mu_0 + q\mu_1)(\mathcal{X}_m^N \setminus \mathcal{K})
\]
\[
= (1-q)P_t^*\mu_0(\mathcal{X}_m^N \setminus \mathcal{K}) + qP_t^*\mu_1(\mathcal{X}_m^N \setminus \mathcal{K})
\]
\[
> (1-q)P_t^*\mu_0(\mathcal{X}_m^N \setminus \mathcal{K}),
\]
which means that \( P^*_t \mu_0 (\mathcal{X}_m^N \setminus \mathcal{K}) < \varepsilon \) for all \( t \geq 0 \), so \( \{ P^*_t \mu_0 \}_{t \geq 0} \) is tight. By an identical argument, so is \( \{ P^*_t \mu_1 \}_{t \geq 0} \). Therefore, a subsequential limit

\[
\bar{\mu}_i := \lim_{T_k \to \infty} \frac{1}{T_k} \int_0^{T_k} P^*_t \mu_i \, dt
\]

exists (in the sense of weak convergence of measures on \( \mathcal{X}_m^N \)) for each \( i \), and by Proposition 4.2, we know that \( \bar{\mu}_i \in \mathcal{P}_G (\mathcal{X}_m^N) \). In addition, it follows from (6.5) that

\[
(1 - q) \bar{\mu}_0 + q \bar{\mu}_1 = \nu_a.
\]

As \( \nu_a \in \mathcal{P}^e_G (\mathcal{X}_{1/2}^N) \), we deduce that \( \bar{\mu}_0 = \bar{\mu}_1 = \nu_a \). Recall that by Proposition 3.6 the Burgers dynamics (1.4) preserves the expectation of a space-stationary solution. We conclude that if \( \nu_i \sim \mu_i \) and \( \nu_i \sim \bar{\mu}_i = \nu_a \), then \( \mathbb{E}_i (X) = \mathbb{E}_i (X) = a \). \( \square \)

7. Stability of the Extremal Invariant Measures

In this section we prove Theorem 1.5, establishing convergence to the extremal invariant measures \( \nu_a \) constructed in Theorem 1.2. As a first step, we upgrade the convergence along subsequences of time-averaged laws of solutions starting at deterministic initial data (implied by Proposition 5.1) to true convergence by identifying the limits using Proposition 5.10 and Theorem 1.2.

**Proposition 7.1.** Let \( G = L\mathbb{Z} \), with \( L > 0 \) or \( G = \mathbb{R} \), and \( \nu \in L^\infty (\mathbb{R})^N \) be a deterministic \( G \)-invariant function. Let \( \Lambda_G \) be a fundamental domain for \( G \) and let

\[
a = \int_{\Lambda_G} \nu (x) \, d\lambda_G (x).
\]

(7.1)

Let \( \delta_\nu \) be the measure on \( \mathcal{X}_{1/2}^N \) with a single atom at \( \nu \). Then

\[
\lim_{T \to \infty} \frac{1}{T} \int_1^{T+1} P^*_t \delta_\nu \, dt = \nu_a
\]

in the sense of weak convergence of probability measures on \( \mathcal{X}_{1/2}^N \).

**Proof.** By Propositions 5.1 and 4.2, for any sequence \( T_k \uparrow \infty \) there is a subsequence \( T_k \) and \( \nu \in \mathcal{P}_G (\mathcal{X}_{1/2}^N) \) so that

\[
\lim_{k \to \infty} \frac{1}{T_k} \int_1^{T_k+1} P^*_t \delta_\nu \, dt = \nu,
\]

in the sense of weak convergence of probability measures on \( \mathcal{X}_{1/2}^N \). By Proposition 3.6, (5.1) and (7.1), if \( w \sim \nu \) we have

\[
\mathbb{E} \left[ \int_{\Lambda} \nu (x) \, d\lambda_G (x) \right] = a.
\]

Then, Proposition 5.10 and Theorem 1.2 imply that \( \nu = \nu_a \). Since the topology of weak convergence of probability measures on \( \mathcal{X}_{1/2}^N \) is metrizable, the uniqueness of the subsequential limit implies (7.2). \( \square \)
The next step is to use the $L^1$-contraction in Proposition 3.7 to eliminate the time averaging in the statement of Proposition 7.1. Let us first introduce some definitions. For $G = \mathbb{L}$ or $G = \mathbb{R}$, recall the definition of the measure $\lambda_G$ from the beginning of Sect. 3.2, and set $\tilde{\lambda}_G = 1_{\Lambda G} \lambda_G$. We also define the weight
\[
p_G(x) = \frac{1}{(\frac{1}{p_2} * \tilde{\lambda}_G)(x)},
\]
and the Banach space $Y_G^N = L_{p_G}^1(\mathbb{R})^N$, equipped with the norm
\[
\| (f_1, \ldots, f_N) \|_{Y_G^N} = \sum_{i=1}^N \| f_i \|_{L_{p_G}^1(\mathbb{R})}.
\]
We note that $X_{1/2}^{N}$ is continuously included in $Y_G^N$ for any $G$, since (as it is sufficient to check $N = 1$), if we fix $\ell \in (1/2, 1)$, then
\[
\| f \|_{Y_G^1} = \int_{\mathbb{R}} | f(x) | \left( \frac{1}{p_2} * \lambda_G \right)(x) \, dx \leq \| f \|_{C_p(\mathbb{R})} \int_{\mathbb{R}} p_\ell(x) \left( \frac{1}{p_2} * \tilde{\lambda}_G \right)(x) \, dx
\]
and the integral is finite, so $C_p(\mathbb{R})$ embeds continuously into $Y_G^1$ and of course $X_{1/2}$ embeds continuously into $C_p(\mathbb{R})$ for any $\ell > 1/2$.

Let $d_{G,N}$ denote the Wasserstein-1 (or Kantorovich–Rubinstein) metric between measures on $Y_G^N$. That is, if $\nu_1, \nu_2$ are probability measures on $Y_G^N$, then
\[
d_{G,N}(\nu_1, \nu_2) = \inf_{\mu} \int_{Y_G^2} \| f_1 - f_2 \|_{Y_G^N} \, d\mu(f_1, f_2)
\]
with the supremum taken over all couplings $\mu \in \mathcal{P}(Y_G^2)$ of $\nu_1$ and $\nu_2$, and $\text{Lip}(Y_G^N)$ is the Lipschitz seminorm. The second equality in (7.3) is the Kantorovich–Rubinstein duality, see e.g. [58, (6.3)]. We will use the fact that for probability measures $\{\mu_t\}_{t \in [0,T]}$ and $\nu$, we have
\[
d_{G,N} \left( \frac{1}{T} \int_0^T \mu_t \, dt, \nu \right) = \sup_{\| f \|_{\text{Lip}(Y_G^N)} = 1} \left| \frac{1}{T} \int_0^T \int f \, d\mu_t \, dt - \int f \, d\nu \right|
\]
and
\[
\leq \sup_{\| f \|_{\text{Lip}(Y_G^N)} = 1} \left| \frac{1}{T} \int_0^T \int f \, d\mu_t - \int f \, d\nu \right| dt
\]
and
\[
\leq \frac{1}{T} \int_0^T d_{G,N}(\mu_t, \nu) \, dt.
\]
Indeed, the metric $d_{G,N}$ comes from a norm on the space of all finite signed measures [41], from which (7.4) follows immediately.

It is well-known (see e.g. [58, Theorem 4.1]) that if $d_{G,N}(\nu_1, \nu_2)$ is finite, then the infimum in the second expression in (7.3) is actually achieved by some coupling $\mu$. In the case of translation-invariant measures, we will need the stronger statement that the coupling can be chosen to also be translation-invariant. We record this in the following lemma, which is an application of the results in [46].
Lemma 7.2 [46]. Suppose \( m \in (0, 1) \), \( G = L\mathbb{Z} \) with some \( L > 0 \) or \( G = \mathbb{R} \), and \( \mu_1, \mu_2 \in \mathcal{P}_G(\mathcal{X}_m^N) \). Then there is a coupling \( \mu \in \mathcal{P}_G(\mathcal{X}_m^{2N}) \) of \( \mu_1 \) and \( \mu_2 \) so that
\[
\int_{\mathcal{Y}_G^N} \| f_1 - f_2 \|_{\mathcal{Y}_G^N} \, d\mu(f_1, f_2) = d_{G,N}(\mu_1, \mu_2).
\] (7.5)

Proof. If \( G = L\mathbb{Z} \) for some \( L > 0 \), then this is immediate from [46, Corollary 1]. If \( G = \mathbb{R} \), then we note that by [46, Theorem 1.6], there exists a measure \( \mu \) satisfying (7.5) that is invariant under spatial translations by elements of \( \mathbb{Q} \). We can extend this to elements of \( \mathbb{R} \) by continuity as in the proof of [46, Corollary 1]. \( \square \)

The next observation is that the Burgers dynamics (1.4) is a contraction in the \( d_{G,N} \) metric, which is a simple consequence of the \( L^1 \)-contraction in the probability space in Proposition 3.7.

Proposition 7.3. If \( m \in (0, 1) \), \( N \in \mathbb{N} \), \( G = L\mathbb{Z} \) with \( L > 0 \) or \( G = \mathbb{R} \), and \( \mu_1, \mu_2 \in \mathcal{P}_G(\mathcal{X}_m^N) \), then for all \( t \geq 0 \),
\[
d_{G,N}(P_t^\ast \mu_1, P_t^\ast \mu_2) \leq d_{G,N}(\mu_1, \mu_2).
\]

Proof. Let \( \mu \in \mathcal{P}_G(\mathcal{X}_m^{2N}) \) be a coupling of \( \mu_1 \) and \( \mu_2 \) so that
\[
\int_{\mathcal{Y}_G^N} \| v_1 - v_2 \|_{\mathcal{Y}_G^N} \, d\mu(v_1, v_2) = d_{G,N}(\mu_1, \mu_2),
\]
that exists by Lemma 7.2. Proposition 3.7 then implies that if
\[
\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) = ((u_{1,1}, \ldots, u_{1,N}), (u_{2,1}, \ldots, u_{2,N}))
\]
solves (1.4) with initial condition \( \mathbf{u}(0, \cdot) \sim \mu \) (independent of the noise), then for any \( t \geq 0, y \in \mathbb{R} \), and \( j = 1, \ldots, N \), we have
\[
\mathbb{E} \left( \int_{y+\Lambda_G} |u_{1,j}(t, x) - u_{2,j}(t, x)| \, d\lambda_G(x - y) \right) \leq \mathbb{E} \left( \int_{y+\Lambda_G} |u_{1,j}(0, x) - u_{2,j}(0, x)| \, d\lambda_G(x - y) \right).
\] (7.6)

We note that for any \( f : \mathbb{R} \to \mathbb{R} \), we have
\[
\| f \|_{\mathcal{Y}_G} = \int_{\mathbb{R}} \frac{|f(y)|}{p_G(y)} \, dy = \int_{\mathbb{R}} \left( \frac{1}{p_2} * \tilde{\lambda}_G \right)(y) |f(y)| \, dy = \int_{\mathbb{R}} \frac{1}{p_2(y)} \int_{y+\Lambda_G} |f(x)| \, d\lambda_G(x - y) \, dy.
\] (7.7)

Therefore, integrating (7.6) in \( y \) against \( 1/p_2 \) yields
\[
\mathbb{E}\|u_{1,j}(t, \cdot) - u_{2,j}(t, \cdot)\|_{\mathcal{Y}_G} \leq \mathbb{E}\|u_{1,j}(0, \cdot) - u_{2,j}(0, \cdot)\|_{\mathcal{Y}_G},
\]
and adding up the components we obtain
\[
d_{G,N}(P_t^\ast \mu_1, P_t^\ast \mu_2) \leq \mathbb{E}\|\mathbf{u}_1(t, \cdot) - \mathbf{u}_2(t, \cdot)\|_{\mathcal{Y}_G^N} \leq \mathbb{E}\|\mathbf{u}_1(0, \cdot) - \mathbf{u}_2(0, \cdot)\|_{\mathcal{Y}_G^N} = d_{G,N}(\mu_1, \mu_2),
\]
as claimed. \( \square \)
We now show that if two solutions with the same periodic initial conditions are started at different times, then their laws become close in the Wasserstein distance as they evolve.

**Proposition 7.4.** Let \( N \in \mathbb{N} \) and \( G = L\mathbb{Z} \) with \( L > 0 \) or \( G = \mathbb{R} \). Given a deterministic \( G \)-invariant function \( v \in L^\infty(\mathbb{R})^N \), let \( \delta_v \) be a delta measure on \( v \). Then for any fixed \( s \geq 0 \), we have

\[
\lim_{t \to \infty} d_{G,N}(P_t^s \delta_v, P_{t+s}^s \delta_v) = 0. \tag{7.8}
\]

**Proof.** Let \( u = (u_1, u_2) \) solve (1.4) with initial condition

\[
u(0, \cdot) \sim (P_s^t \delta_v) \otimes \delta_v.
\]

For each \( T \in (0, \infty) \), let \( S_T \sim \text{Uniform}([0, T]) \) be independent of everything else. As in the proof of Proposition 5.1, there is a sequence \( T_k \uparrow \infty \) and a measure \( \omega_* \in \mathcal{P}_G(X^N_{1/2}) \) so that

\[
\text{Law}(u(S_T, \cdot)) \to \omega_*
\]

weakly. By Proposition 7.1, each of the \( 2N \) marginals of \( \omega_* \) is extremal, i.e. an element of \( \mathcal{P}_G(X^N_{1/2}) \). By Proposition 6.1, this means that if \( v = (v_1, v_2) \sim \omega_* \) then \( v_1 = v_2 \) almost surely since

\[
\mathbb{E} \left[ \int_{\Lambda_G} v_1(x) \, d\lambda_G(x) \right] = \mathbb{E} \left[ \int_{\Lambda_G} v_2(x) \, d\lambda_G(x) \right] = a,
\]

as the expectation is preserved in the limit). In particular, the difference \( u_1(S_{T_k}, \cdot) - u_2(S_{T_k}, \cdot) \) converges to zero in distribution with respect to the topology of \( X^N_{1/2} \), hence with respect to the topology of \( L^1_p(\mathbb{R}) \) as \( k \to \infty \). Since the limit is constant, it also converges in probability. Furthermore, (5.1) provides uniform integrability, so in fact

\[
u_1(S_{T_k}, \cdot) - u_2(S_{T_k}, \cdot) \to 0 \text{ in mean. Therefore, we have}
\]

\[
0 = \lim_{k \to \infty} \mathbb{E} \| u_1(S_{T_k}, \cdot) - u_2(S_{T_k}, \cdot) \|_{L^1_{p,1,G}(\mathbb{R})^N} = \lim_{k \to \infty} \frac{1}{T_k} \int_0^{T_k} \mathbb{E} \| u_1(t, \cdot) - u_2(t, \cdot) \|_{L^1_{p,1,G}(\mathbb{R})^N} \, dt \geq \lim_{k \to \infty} \frac{1}{T_k} \int_0^{T_k} d_{G,N}(\text{Law}(u_1(t, \cdot)), \text{Law}(u_2(t, \cdot))) \, dt = \lim_{k \to \infty} \frac{1}{T_k} \int_0^{T_k} d_{G,N}(P_t^{s+t} \delta_v, P_t^s \delta_v) \, dt \geq \lim_{k \to \infty} d_{G,N}(P_{T_k+s}^s \delta_v, P_{T_k}^s \delta_v),
\]

with the last inequality by Proposition 7.3. Using Proposition 7.3 again proves (7.8), since if a subsequence of a nonnegative decreasing sequence converges to zero, then so does the whole sequence. \( \Box \)

The following proposition establishes Theorem 1.5 for periodic initial conditions, or, in the notation of Proposition 1.4, when \( v_{\text{int}} = v_z \equiv 0 \).
Proposition 7.5. Let $N \in \mathbb{N}$ and $G = L\mathbb{Z}$ with $L > 0$ or $G = \mathbb{R}$. Given a deterministic $G$-invariant $v \in L^{\infty}(\mathbb{R})^N$, let $\delta_v$ be the delta measure on $v$, and set

$$a = \int_{\Lambda_G} v(x) \, d\lambda_G(x).$$

With $v_a$ defined as in Theorem 1.2, we have

$$\lim_{t \to \infty} d_{G,N}(P_t^* \delta_v, v_a) = 0 \quad (7.9)$$

and

$$\lim_{t \to \infty} P_t^* \delta_v = v_a, \quad (7.10)$$

weakly with respect to the topology of $\mathcal{X}^N_{1/2}$.

Proof. Let

$$\mu_{t,T} = \frac{1}{T} \int_t^{t+T} P_s^* \delta_v \, ds.$$

By Proposition 7.1, $\mu_{0,T}$ converges to $v_a$ weakly with respect to the topology of $\mathcal{X}^N_{1/2}$, hence with respect to the topology of $\mathcal{Y}^N_G$, as $T \to \infty$. By Lemma 5.3, if $v \sim \mu_{0,T}$, then $E\|v\|^2_{\mathcal{Y}^N_G}$ is bounded uniformly in $T$, so in fact we have

$$\lim_{T \to \infty} d_{G,N}(\mu_{0,T}, v_a) = 0. \quad (7.11)$$

Now $P_t^* \mu_{0,T} = \mu_{t,T}$ and $P_t^* v_a = v_a$, so Proposition 7.3 implies that $d_{G,N}(\mu_{t,T}, v_a)$ is decreasing in $t$. Hence we can upgrade (7.11) to

$$\lim_{T \to \infty} \sup_{t \geq 0} d_{G,N}(\mu_{t,T}, v_a) = 0. \quad (7.12)$$

Now fix $T > 0$ and consider

$$\lim_{t \to \infty} \int_t^{t+T} d_{G,N}(P_t^* \delta_v, P_s^* \delta_v) \, ds = \lim_{T \to \infty} \int_0^T d_{G,N}(P_t^* \delta_v, P_{t+s}^* \delta_v) \, ds.$$

By Proposition 7.3, the integrand is at most $d_{G,N}(\delta_v, P_s^* \delta_v)$, which is uniformly bounded in $s$ by Proposition 3.1 and Proposition 5.6. Furthermore, Proposition 7.4 shows that the integrand converges to 0 pointwise in $s$. Thus by dominated convergence, we have

$$\lim_{t \to \infty} \int_t^{t+T} d_{G,N}(P_t^* \delta_v, P_s^* \delta_v) \, ds = 0. \quad (7.13)$$

Now the triangle inequality and (7.4) yield

$$d_{G,N}(P_t^* \delta_v, v_a) \leq d_{G,N}(P_t^* \delta_v, \mu_{t,T}) + d_{G,N}(\mu_{t,T}, v_a) \leq \frac{1}{T} \int_t^{t+T} d_{G,N}(P_t^* \delta_v, P_s^* \delta_v) \, ds + d_{G,N}(\mu_{t,T}, v_a).$$
Now for any $\varepsilon > 0$, (7.12) let us choose $T$ so large that the second term is less than $\varepsilon/2$ for any $t$, and then (7.13) lets us choose $t$ so large that the first term is less than $\varepsilon/2$. This proves (7.9).

To prove (7.10), note that by Proposition 5.1, for every sequence $t_k \to \infty$ we have a subsequence $t_{kn} \to \infty$ and a $v \in \mathcal{P}(\chi_{1/2}^N) \mathcal{P}$ so that $P^*_{t_{kn}} \delta_v$ converges weakly in $\mathcal{P}(\chi_{1/2}^N)$ to $v$ as $n \to \infty$. Since the topology of weak convergence is metrizable, to complete the proof it suffices to show that $v = v_a$. Since $P^*_{t_{kn}} \delta_v$ converges weakly to $v$ in $\mathcal{P}(\chi_{1/2}^N)$, it also converges weakly to $v$ with respect to the topology of $L_1^1(\mathbb{R})$ due to the continuous inclusion $\chi_{1/2}^N \subset L_1^1(\mathbb{R})$. On the other hand, (7.9) implies that $P^*_{t_{kn}} \delta_v$ converges to $v_a$ weakly with respect to the topology of $L_1^1(\mathbb{R})$. This means that $v = v_a$, so the proof is complete. \(\square\)

We now consider initial conditions satisfying Definition 1.3 to prove Theorem 1.5. Initial conditions satisfying Definition 1.3 can be sandwiched between periodic initial conditions whose means are very close to each other. Solutions started with such periodic initial conditions converge to the stationary solutions with the corresponding means. Then the solutions started from the aperiodic initial conditions are stuck in the middle by the comparison principle, and must therefore converge to the stationary solution with the appropriate mean.

**Proof of Theorem 1.5.** Proposition 4.1 implies that may assume that $a = (0, \ldots, 0)$. Consider the solution $u = (u_0, u) \in \mathbb{Z}_{1/2}^{2N}$ to (1.4) with initial condition

$$u(0, \cdot) = ((0, \ldots, 0), v).$$

By Proposition 5.1, for every sequence $t_k \uparrow \infty$, there is a subsequence $t_{kn} \uparrow \infty$ and a limit measure $v \in \mathcal{P}(\chi_{1/2}^N)$ so that

$$\text{Law}(u(t_{kn}, \cdot)) \to v$$

weakly in $\mathcal{P}(\chi_{1/2}^N)$ as $k \to \infty$. Since the topology of weak convergence is metrizable, it suffices to prove that the law of the marginal of $v$ on the last $N$ components is $v_0$. Let $\varepsilon > 0$ and pick $L$, $v_-, v_+$ as in Definition 1.3. Let $\bar{u} = (u_-, u_+, u_0, u) \in \mathbb{Z}_{1/2}^{4N}$ solve (1.4) with initial condition $\bar{u}(0, \cdot) = (v_-, v_+, 0, v)$. By the comparison principle Proposition 3.1, we have

$$u_-(t, \cdot) \leq u_0(t, \cdot) \leq u_+(t, \cdot) \quad \text{and} \quad u_-(t, \cdot) \leq u(t, \cdot) \leq u_+(t, \cdot)$$

almost surely for all $t \geq 0$. Here, $\leq$ refers to the partial order defined in (1.7). Define

$$a_\pm = (a_{\pm, 1}, \ldots, a_{\pm, N}) = \frac{1}{L} \int_0^L v_\pm(x) \, dx,$$

so that

$$\sum_{j=1}^N (a_+, j - a_-, j) < 2N \varepsilon$$

by (1.8). By Proposition 5.1, there is a further subsequence $t_{kn_i}$ of $t_{kn}$ and a measure $\bar{v} \in \mathcal{P}(\chi_{1/2}^N)$, so that

$$\lim_{i \to \infty} \text{Law}(\bar{u}(t_{kn_i}, \cdot)) = \bar{v}$$

(7.17)
weakly in the topology of $P(X_{1/2}^{4N})$. By (7.14),
the marginal of $v$ on the last $2N$ coordinates is $v$. By Proposition 7.5,
the marginal of $v$ on the first $3N$ components is $v_{a_-,a_+}$. On the other hand, we have by (7.15) that for each $t \geq 0$,
\[
\|u(t,\cdot) - u_0(t,\cdot)\|_{Y^N_G} \leq \|u_+(t,\cdot) - u_-(t,\cdot)\|_{Y^N_G}
\]  
(7.18)
almost surely. By (5.1), the family $\{\|u_+(t,\cdot) - u_-(t,\cdot)\|_{Y^N_G}\}_{t \geq 0}$ is uniformly integrable, so by the Skorokhod representation theorem and (7.17), if
\[
(w_-,w_+) = ((w_-,1, \ldots, w_-,N), (w_+,1, \ldots, w_+,N)) \sim v_{a_-,a_+},
\]
then (P1) of Theorem 1.2, (7.7), and (7.16) imply
\[
\lim_{t \to \infty} \mathbb{E}\|u_+(t,\cdot) - u_-(t,\cdot)\|_{Y^N_G} = \mathbb{E}\|w_+ - w_-\|_{Y^N_G}
\]
\[
= \left(\int_{\mathbb{R}} \frac{dx}{p_2(x)} \right) \sum_{j=1}^{N} (a_{+,j} - a_{-,j}) \leq 2N \epsilon \int_{\mathbb{R}} \frac{dx}{p_2(x)}.
\]
(7.19)
Using Fatou’s lemma, (7.18), and (7.19), we have that if $(w_0,w) \sim v$, then
\[
\mathbb{E}\|w_0 - w\|_{Y^N_G} \leq \liminf_{t \to \infty} \mathbb{E}\|u(t,\cdot) - u_0(t,\cdot)\|_{Y^N_G}
\]
\[
\leq \lim_{t \to \infty} \mathbb{E}\|u_+(t,\cdot) - u_-(t,\cdot)\|_{Y^N_G} \leq 2N \epsilon \int_{\mathbb{R}} \frac{dx}{p_G(x)}.
\]
But this holds for all $\epsilon > 0$, so in fact $\mathbb{E}\|w_0 - w\|_{Y^N_G} = 0$, so $w_0 = w$ almost surely. This means that $\text{Law}(w) = \text{Law}(w_0) = v_0$, which is what we needed to show. □

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A. Proof of Proposition 1.4

We now prove Proposition 1.4. The proof is elementary and independent of the rest of the paper. Let $\epsilon > 0$. For each $j = 1, \ldots, N$, define $w_j \in L^\infty(\mathbb{R})$ by
\[
w_j(x) = \sup_{|y| \geq |x|} |v_{x,j}(y)|,
\]
so $|v_{x,j}(x)| \leq w_j(x)$ for all $x \in \mathbb{R}$. Then, $w_j(x)$ is decreasing in $|x|$, and
\[
\lim_{|x| \to \infty} w_j(x) = 0.
\]
Therefore, we can find a $K \in \mathbb{N}$ so large that
\[
\frac{1}{KL} \max_{j=1}^{N} \| v_{\text{int},j} \|_{L^1(\mathbb{R})} < \varepsilon / 2, \quad \frac{1}{KL} \max_{j=1}^{N} \int_{0}^{KL} w_j(x) \, dx < \varepsilon / 2.
\]

Let us define
\[
v_{\pm,j}(x) = v_{\text{per},j}(x) \pm w_j(x) \pm \sup_{m \in \mathbb{Z}} |v_{\text{int},j}(x + mKL)| \tag{A.1}\]
and $v_- = (v_{-,1}, \ldots, v_{-,N})$ and $v_+ = (v_{+,1}, \ldots, v_{+,N})$, so $v_-, v_+ \in L^\infty(\mathbb{R})^N$ and $v_- \leq v \leq v_+$. Then we have
\[
\frac{1}{KL} \int_{0}^{KL} v_- (x) \, dx - a
\]
\[
= \frac{1}{KL} \int_{0}^{KL} [v_-(x) - v_{\text{per}}(x)] \, dx
\]
\[
= \frac{1}{KL} \int_{0}^{KL} \left[ -w_j(x) + \inf_{i \in \mathbb{Z}} v_{\text{int},j}(x + iKL) \right] \, dx
\]
\[
\geq -\frac{\varepsilon}{2} (1, \ldots, 1) - \frac{1}{KL} \max_{j=1}^{N} \sum_{i \in \mathbb{Z}} |v_{\text{int},j}(x + iKL)| \, dx
\]
\[
= -\frac{\varepsilon}{2} (1, \ldots, 1) - \frac{1}{KL} \max_{j=1}^{N} \| v_{\text{int},j} \|_{L^1(\mathbb{R})} \geq -(\varepsilon, \ldots, \varepsilon).
\]

A similar argument shows that
\[
\frac{1}{KL} \int_{0}^{KL} v_+(x) \, dx - a \leq (\varepsilon, \ldots, \varepsilon).
\]

Since this is possible for every $\varepsilon > 0$, we have $v \in \mathcal{B}_a$. □

B. Weighted Spaces

We now record several useful results on weighted spaces. We begin with some basic lemmas that are used throughout the paper. Then we will prove bounds on the heat kernel on weighted spaces.

B.1. Basic properties of weighted spaces. Our first lemma allows us to upgrade convergence from one weight to another.

**Proposition B.1.** Let $T$ be a metric space and $w_1, w_2, w_3$ be weights on $\mathbb{R}$ so that
\[
\lim_{|x| \to \infty} \frac{w_1(x)}{w_2(x)} = 0.
\]
Suppose that $u_n \in C_b(T; C_{w_1}(\mathbb{R}))$, and $u \in C_b(T; C_{w_3}(\mathbb{R}))$ satisfy
\[
\lim_{n \to \infty} \| u_n - u \|_{C_b(T; C_{w_3}(\mathbb{R}))} = 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \| u_n \|_{C_b(T; C_{w_1}(\mathbb{R}))} < \infty.
\]
Then $u \in C_b(T; C_{w_2}(\mathbb{R}))$, and $\lim_{n \to \infty} \| u_n - u \|_{C_b(T; C_{w_2}(\mathbb{R}))} = 0$ as well.
Proof. Fix $\epsilon > 0$ and define

$$K = \sup_{n \in \mathbb{N}} \|u_n\|_{\mathcal{C}_b(T;C_{w_1}(\mathbb{R}))} < \infty.$$ 

Then choose $M$ so that

$$\left| \frac{w_1(x)}{w_2(x)} \right| \leq \frac{\epsilon}{2K} \quad \text{if} \quad |x| \geq M,$$

and $N$ so large that if $n \geq N$ then

$$\|u_n - u\|_{\mathcal{C}_b(T;C_{w_3}(\mathbb{R}))} \leq \epsilon \inf_{|x| \leq M} \frac{w_2(x)}{w_3(x)}.$$ 

Now, for any $n \geq N$, if $|y| \leq M$, then for all $t \in T$ we have

$$|(u_n - u)(t, y)| \leq \epsilon w_3(y) \inf_{x \in [-M, M]} \frac{w_2(x)}{w_3(x)} \leq \epsilon w_2(y)$$

while if $|y| \geq M$ we have

$$|(u_n - u)(t, y)| \leq \epsilon w_3(y) \inf_{x \in [-M, M]} \frac{w_2(x)}{w_3(x)} \leq \epsilon w_2(y).$$

Therefore, $\|u_n - u\|_{\mathcal{C}_b(T;C_{w_2}(\mathbb{R}))} < \epsilon$. This proves that $\lim_{n \to \infty} \|u_n - u\|_{\mathcal{C}_b(T;C_{w_2}(\mathbb{R}))} = 0$, as claimed. \(\square\)

We next establish a form of the Arzelà–Ascoli theorem in weighted spaces.

Proposition B.2. Suppose that $w_1, w_2, w_3$ are weights so that $\lim_{|x| \to \infty} \frac{w_1(x)}{w_2(x)} = 0$ and fix $\alpha > 0$. Then the embedding

$$\mathcal{C}_{w_1}(\mathbb{R}) \cap \mathcal{C}_{w_3}^{\alpha}(\mathbb{R}) \hookrightarrow \mathcal{C}_{w_2}(\mathbb{R})$$

is compact, where $\mathcal{C}_{w_1}(\mathbb{R}) \cap \mathcal{C}_{w_3}^{\alpha}(\mathbb{R})$ is equipped with the norm $\|u\|_{\mathcal{C}_{w_1}(\mathbb{R}) \cap \mathcal{C}_{w_3}^{\alpha}(\mathbb{R})} = \|u\|_{\mathcal{C}_{w_1}(\mathbb{R})} + \|u\|_{\mathcal{C}_{w_3}^{\alpha}(\mathbb{R})}$. \(\square\)

Proof. It suffices to show that the unit ball of $\mathcal{C}_{w_1}(\mathbb{R}) \cap \mathcal{C}_{w_3}^{\alpha}(\mathbb{R})$ is precompact in $\mathcal{C}_{w_2}(\mathbb{R})$. Fix a sequence $(v_n)_n$ of elements of this unit ball. On any compact subset of $\mathbb{R}$, $(v_n)$ is uniformly bounded and Hölder. Thus by Arzelà–Ascoli and diagonalization, there exists a subsequence $(v_{n_k})_k$ which converges locally uniformly to some $v \in \mathcal{C}(\mathbb{R})$. As noted in the proof of Lemma B.4, this is equivalent to convergence in some weighted space. Since $(v_n)$ is uniformly bounded in $\mathcal{C}_{w_1}(\mathbb{R})$, Proposition B.1 implies that $v \in \mathcal{C}_{w_2}(\mathbb{R})$ and $\lim_{k \to \infty} v_{n_k} = v$ in $\mathcal{C}_{w_2}(\mathbb{R})$. \(\square\)

Finally, we record a compactness criterion in $X_m$.

Lemma B.3. If $K \subset X_m$ is such that $K$ is compact in the topology of $\mathcal{C}_{p_\ell}(\mathbb{R})$ for each $\ell > m$, then $K$ is compact in the topology of $X_m$ as well.

Proof. Let $(v_n)_n$ be a sequence of elements in $K$. By a diagonal argument, there is a subsequence $(v_{n_k})_k$ which converges in the topology of $\mathcal{C}_{p_\ell}(\mathbb{R})$ for all $\ell > m$, and hence in the topology of $X_m$. \(\square\)
B.2. Heat kernel bounds in weighted spaces. Here, we prove some weighted estimates for the heat kernel.

Lemma B.4. Fix $a$ weight $w \in \{(\log(\cdot))^{3/4}\} \cup \{p_\ell : \ell \in \mathbb{R}\}$, $\beta \geq \alpha \geq 0$, and $T < \infty$. There is a constant $C = C(w, \alpha, \beta, T) < \infty$ so that for all $t \in (0, T]$ and $f \in C^\alpha_w(\mathbb{R})$ we have

$$\|G_t * f\|_{C^\alpha_w(\mathbb{R})} \leq Ct^{-\frac{\beta - \alpha}{2}} \|f\|_{C^\alpha_w(\mathbb{R})} \quad (B.1)$$

In particular,

$$\|\partial_x G_t * f\|_{C^\alpha_w(\mathbb{R})} \leq Ct^{-\frac{\beta - \alpha + 1}{4}} \|f\|_{C^\alpha_w(\mathbb{R})}. \quad (B.2)$$

In the case $\alpha = 0$, it is only necessary to assume that $f \in L^\infty_w(\mathbb{R})$ and the norm $\|f\|_{C^\alpha_w(\mathbb{R})}$ can be replaced by $\|f\|_{L^\infty_w(\mathbb{R})}$ on the right-hand sides of (B.1) and (B.2).

The proof of this lemma is word-for-word the same as that of [36, Lemma 2.8]. There, only exponential weights (since a uniformity statement in the weight is needed) and continuous functions are considered, but there is no difference in the treatment given the Gaussian decay of the heat kernel. The essence of the argument is that only singularity in the heat kernel is at $t = 0$, $x = 0$, so the part of the heat kernel that is exposed to the growth of $f$ at infinity is smooth, and moreover decays quickly enough not to pose any difficulty for these estimates.

Lemma B.5. Fix $m \in \mathbb{R}$. If $p \in [1, \infty)$ and $f \in L^p_{pm}(\mathbb{R})$, then $G_t * f \to f$ in $L^p_{pm}(\mathbb{R})$ as $t \downarrow 0$. If $f \in L^\infty_{pm}(\mathbb{R})$, then $G_t * f \xrightarrow{w^*} f$ in $L^\infty_{pm}(\mathbb{R})$ as $t \downarrow 0$. Finally, if $f \in C_{pm}(\mathbb{R})$, then $G_t * f \to f$ in $\mathcal{X}_m$ as $t \downarrow 0$.

Proof. First fix $p \in [1, \infty)$ and $f \in L^p_{pm}(\mathbb{R})$. We provide a simple variant of a standard “approximation of the identity” argument to deal with the weighted spaces. Using the scaling symmetry of $G$, we can write

$$G_t * f(x) - f(x) = \int_\mathbb{R} [f(x - \sqrt{t}y) - f(x)]G_1(y) \, dy,$$

so by the triangle inequality,

$$\|G_t * f - f\|_{L^p_{pm}(\mathbb{R})} \leq \int_\mathbb{R} \|\tau_{\sqrt{t}y} f - f\|_{L^p_{pm}(\mathbb{R})} G_1(y) \, dy. \quad (B.3)$$

We will use the dominated convergence theorem, so we first establish an integrable majorant. Assume $t \in (0, 1]$. We can easily verify that there exists $C = C(m) < \infty$ such that

$$p_{-m}(a + b) \leq C p_{-m}(a)p_{|m|}(b) \quad (B.4)$$

for all $a, b \in \mathbb{R}$. In the sequel, we permit $C$ to change from line to line. Then

$$\|\tau_{\sqrt{t}y} f\|_{L^p_{pm}(\mathbb{R})} = \left(\int_\mathbb{R} |f(x)|^p p_m(x + \sqrt{t}y)^{-p} \, dx\right)^{\frac{1}{p}} \leq C p_{|m|}(y) \|f\|_{L^p_{pm}(\mathbb{R})}.$$

Since $p_{|m|}$ is integrable against the Gaussian $G_1$, this is a suitable majorant.
By the dominated convergence theorem, it now suffices to prove pointwise (in $y$) convergence to 0 as $t \downarrow 0$ in (B.3). We can therefore fix $y \in \mathbb{R}$ and consider $t > 0$ such that $\sqrt{t} y \leq 1$. For fixed $\varepsilon > 0$, we can find a compactly-supported continuous function $\xi$ on $\mathbb{R}$ so that $\| \xi - f \|_{L^p_{\text{pm}}(\mathbb{R})} < \varepsilon$. Then we have

$$\| \tau_{\sqrt{t} y} f - f \|_{L^p_{\text{pm}}(\mathbb{R})} \leq \| \tau_{\sqrt{t} y} f - \tau_{\sqrt{t} y} \xi \|_{L^p_{\text{pm}}(\mathbb{R})} + \| \tau_{\sqrt{t} y} \xi - \xi \|_{L^p_{\text{pm}}(\mathbb{R})} + \| \xi - f \|_{L^p_{\text{pm}}(\mathbb{R})}.$$  

By (B.4) and $\sqrt{t} y \leq 1$, the first and third terms are each less than a constant times $\varepsilon$ and the second term goes to 0 as $t \to 0$. Therefore $G_t \ast f \to f$ in $L^p_{\text{pm}}(\mathbb{R})$ as $t \downarrow 0$.

Now suppose $f \in L^\infty_{\text{pm}}(\mathbb{R})$, and fix $\phi$ in the dual space $L^1_{\text{pm}}(\mathbb{R})$. We must show that

$$\langle \phi, G_t \ast f \rangle \to \langle \phi, f \rangle \quad \text{as } t \downarrow 0. \tag{B.5}$$

Since $G_t$ is symmetric, we have $\langle \phi, G_t \ast f \rangle = \langle G_t \ast \phi, f \rangle$. But we have just shown that $G_t \ast \phi \to \phi$ in $L^1_{\text{pm}}(\mathbb{R})$, so (B.5) follows.

Finally, suppose that $f \in C^\infty_{\text{pm}}(\mathbb{R})$. Fix $\varepsilon > 0$ and $x \in \mathbb{R}$. We write

$$G_t \ast f(x) - f(x) = \int_\mathbb{R} [f(y) - f(x)]G_t(x - y) \, dy.$$  

Now $|f(y)| \leq \|f\|_{C^\infty_{\text{pm}}(\mathbb{R})} p_m(y)$ and $p_m G_t(x - \cdot) \in L^1(\mathbb{R})$. When $t \in (0, 1]$, $G_t$ is decreasing in $t$ outside a compact set. Thus there exists a compact set $K \subset \mathbb{R}$ containing $x$ so that

$$\int_{\mathbb{R} \setminus K} |f(y) - f(x)||G_t(x - y)| \, dy < \varepsilon$$  

for all $n \geq 0$. On $K$, $f$ is uniformly continuous and bounded. Thus there exists a $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ when $|y - x| < \delta$. Since $G_t$ has unit mass, this implies

$$\int_{B_\delta(x)} |f(y) - f(x)||G_t(x - y)| \, dy < \varepsilon$$

for all $t \in (0, 1]$. Finally, $G_t(x - y) \to 0$ uniformly on $K \setminus B_\delta(x)$ as $t \downarrow 0$, so there exists $\delta > 0$ such that

$$\int_{K \setminus B_\delta(x)} |f(y) - f(x)||G_t(x - y)| \, dy < \varepsilon$$

for all $t < \delta$. Together, these bounds show that $|G_t \ast f(x) - f(x)| \to 0$ as $n \to 0$.

In fact, the convergence is locally uniform in $x$. But locally uniform convergence is equivalent to the existence of a weight $w$ such that

$$\lim_{n \to \infty} \| G_t \ast f - f \|_{C_w(\mathbb{R})} = 0.$$  

Combining this with the uniform bound (B.1) with $\alpha = \beta = 0$ and $w = p_m$, Proposition B.1 implies

$$\lim_{n \to \infty} \| G_t \ast f - f \|_{C_{p_\ell}(\mathbb{R})} = 0$$

for any $\ell > m$. That is, $G_t \ast f \to f$ in $X_m$. \qed
Next, we show an estimate with super-exponential weights. The restriction \( \beta < 2 \) is needed in the following lemma simply because the heat equation is not well-posed for initial conditions growing like \( \exp(cx^2) \) with \( c > 0 \).

**Lemma B.6.** Fix \( \beta \in (3/2, 2) \) and define, for \( \lambda \geq 0 \), \( q_\lambda(x) = e^{\lambda \langle x \rangle^\beta} \). For any \( \Lambda > 0 \) and \( T > 0 \), there exists a \( C < \infty \) so that for all \( \lambda \in [0, \Lambda] \), \( t \in (0, T) \), \( f \in C_q(\mathbb{R}) \), and \( x \in \mathbb{R} \), we have

\[
|\partial_x G_t * f(x)| \leq C t^{-\frac{1}{2}} e^{Ct(x)^{2(\beta-1)}} q_\lambda(x) \| f \|_{C_q(\mathbb{R})}.
\]

(B.6)

**Remark B.7.** This also holds for \( \beta \in (1, 3/2) \). The argument is similar but not identical, and is not needed in this paper, so we omit it.

**Proof.** Throughout the proof, \( C \) denotes a positive constant that depends only on \( \Lambda \) and \( T \). It may change from line to line. We may assume without loss of generality that \( \| f \|_{C_q(\mathbb{R})} = 1 \). We begin by noting that \( |y| \leq \exp(y^2/4) \) for all \( y \in \mathbb{R} \), so

\[
|\partial_x G_t| \leq \sqrt{2t^{-\frac{1}{2}}} G_{2t}.
\]

Therefore, we have

\[
|\partial_x G_t * f(x)| \leq \sqrt{2t^{-\frac{1}{2}}} G_{2t} * q_\lambda(x),
\]

(B.7)

so it remains to bound \( |G_{2t} * q_\lambda(x)| \). The function \( w(s, \cdot) = G_s * q_\lambda \) solves the heat equation with initial condition \( q_\lambda \), so we can bound it from above by constructing a supersolution \( v \) with the same initial condition. Set

\[
v(s, x) = \exp\left(As\langle x \rangle^{2(\beta-1)} + Bs^{-\frac{\beta}{2-\beta}}\right) q_\lambda(x)
\]

(B.8)

for constants \( A, B > 0 \) to be determined. Then we have

\[
\partial_s v \geq \left(A\langle x \rangle^{2(\beta-1)} + Bs^{-\frac{\beta}{2-\beta}}\right) v
\]

and

\[
\partial_{xx} v \leq 8 \left(\lambda^2 + \lambda + A(\lambda + 1)s\langle x \rangle^{\beta - 2} + A^2 s^2\langle x \rangle^{2(\beta - 2)}\right) \langle x \rangle^{2(\beta - 1)} v.
\]

Comparing these, in order for \( v \) to be a supersolution for the heat equation, i.e. to satisfy

\[
\partial_s v \geq \frac{1}{2} \partial_{xx} v,
\]

(B.9)

it suffices to choose \( A \) and \( B \) so that

\[
4 \left(\lambda^2 + \lambda + A(\lambda + 1)s\langle x \rangle^{\beta - 2} + A^2 s^2\langle x \rangle^{2(\beta - 2)} \right) \leq A + Bs^{-\frac{2(\beta - 1)}{2-\beta}} \langle x \rangle^{-2(\beta - 1)}.
\]

(B.10)

To accomplish this, let \( A = 4(\lambda^2 + \lambda) + 2 \) and \( \xi = s\langle x \rangle^{\beta - 2} \). Then for (B.10) to hold, it suffices to choose \( B \) so that

\[
4(\lambda + 1)A\xi + 4A^2\xi^2 \leq 2 + B\xi^{-\frac{2(\beta - 1)}{2-\beta}}.
\]

(B.11)
for all \( \xi \geq 0 \). When \( \xi \leq 1/(4A(\lambda + 1)) \), the left side of (B.11) is bounded by 2, and the inequality holds regardless of \( B \). Moreover, \( 2(\beta - 1)/(2 - \beta) \geq 2 \) since \( \beta \geq 3/2 \), so the right side of (B.11) grows at least as fast as the left as \( \xi \to +\infty \). Thus, there exists \( B = B(\Lambda) \) sufficiently large that (B.11) holds also when \( \xi \geq 1/(4A(\lambda + 1)) \). With these values of \( A \) and \( B \), \( v \) satisfies (B.9) and \( v(0, \cdot) \equiv q_2 \equiv v(0, \cdot) \). Thus \( w(s, x) \leq v(s, x) \) for all \( s \geq 0 \) and \( x \in \mathbb{R} \) by the comparison principle for the heat equation. The bound (B.6) then follows from (B.7) and (B.8). \( \square \)

C. Classical Solutions are Mild

In this appendix we prove a converse to Lemma 2.8.

Lemma C.1 (Classical solutions are mild). Suppose that \( m \in (0, 1) \), \( L \in (0, \infty) \), \( T > 0 \) and \( \theta[L] \in \mathcal{Z}_m,T \) is a classical solution to (2.14). Then \( \theta[L] \) satisfies (2.21).

Proof. Let \( f = -\partial_x (\theta[L] + \psi[L])^2 \) denote the nonlinear forcing in (2.14). Fix \( t \in (0, T] \) and define, for \( s \in [0, t] \),

\[
\Theta(s, x) = G_s \ast \theta[L](t - s, x), \quad F(s, x) = G_s \ast f(t - s, x).
\]

It is clear that \( \Theta \) is twice-differentiable in space. We claim that it is also continuous in \( s \), pointwise in \( x \). Fix \( \ell > m \), \( (s, x) \in [0, t] \times \mathbb{R} \), and a sequence \( (s_n)_{n \in \mathbb{N}} \subset (0, t) \) converging to \( s \). If \( s > 0 \), we use

\[
|\Theta(s_n, x) - \Theta(s, x)| \leq |(G_{s_n} - G_s) \ast \theta[L](t - s_n, \cdot)(x)|
+ |G_s \ast [\theta[L](t - s_n, \cdot) - \theta[L](t - s, \cdot)](x)|. \tag{C.1}
\]

Then \( \|G_{s_n} - G_s\|_{L_{1-p}^1(\mathbb{R})} \to 0 \). Since \( \theta[L] \) is uniformly bounded in \( L_{p,t}^\infty(\mathbb{R}) \), we have

\[
|(G_{s_n} - G_s) \ast \theta[L](t - s_n, \cdot)(x)| \leq \|G_{s_n} - G_s\|_{L_{1-p}^1(\mathbb{R})} \|\theta[L](t - s_n, \cdot)\|_{L_{p,t}^\infty(\mathbb{R})} \to 0.
\]

On the other hand,

\[
G_s \ast [\theta[L](t - s_n, \cdot) - \theta[L](t - s, \cdot)](x) = \int_\mathbb{R} G_s(x - y)[\theta[L](t - s_n, y) - \theta[L](t - s, y)] dy \to 0
\]

by the weak-\( \ast \) continuity of \( \theta[L] \), since \( G_s \in L_{p,1}^1(\mathbb{R}) \). By (C.1), \( |\Theta(s_n, x) - \Theta(s, x)| \to 0 \) as \( n \to \infty \).

Now suppose \( s = 0 \). Then \( G_0 = \delta_0 \) is singular, so we must argue differently. In this case, we are considering \( \theta[L] \) near a time \( t > 0 \), where it is continuous. Fix \( \varepsilon > 0 \), and consider the opposite decomposition

\[
|\Theta(s_n, x) - \Theta(s, x)| \leq |G_{s_n} \ast [\theta[L](t - s_n, \cdot) - \theta[L](t, \cdot)](x)|
+ |(G_{s_n} - \delta_0) \ast \theta[L](t, \cdot)(x)|. \tag{C.2}
\]

Since \( \theta[L] \in \mathcal{C}_b((0, S]; C_{p,1}(\mathbb{R})) \), there exists a \( \delta > 0 \) such that

\[
\|\theta[L](t - s_n, \cdot) - \theta[L](t, \cdot)\|_{C_{p,1}(\mathbb{R})} < \varepsilon
\]
when $s_n < \delta$. Now $G_{s_n} * p_\ell \leq C_\ell p_\ell$, so

$$|G_{s_n} * [\theta^{[L]}(t - s_n, \cdot) - \theta^{[L]}(t, \cdot)](x)| \leq C_\ell \varepsilon p_\ell(x).$$

For the second term of (C.2), we use the fact that $\theta^{[L]}(t, \cdot) \in C_p(\mathbb{R})$, so by Lemma B.5, we have pointwise convergence and $|\{G_{s_n} - \delta_0\} * \theta^{[L]}(t, \cdot)(x)| \to 0$. Thus by (C.2), $|\Theta(s_n, x) - \Theta(0, x)| \to 0$ as $n \to \infty$, as desired.

Next, (2.9) implies that $\Theta$ is differentiable in $s$ on $(0, t)$ and that

$$\partial_t \Theta(s, x) = (\partial_t G_s) * \theta^{[L]}(t - s, x) - \partial_t \theta^{[L]}(t - s, x) = \frac{1}{2} \partial_{xx} G_s * \theta^{[L]}(t - s, x) - \frac{1}{2} G_s * \partial_{xx} \theta^{[L]}(t - s, x) - F(s, x).$$

Now $\theta^{[L]}(t - s, \cdot)$ is a tempered distribution, so we may exchange differentiation and convolution to find

$$\partial_s \Theta(s, x) = -F.$$

The continuity of $\Theta$ in time ensures that

$$\Theta(t, x) - \Theta(0, x) = \int_0^t \partial_s \Theta(s, x) \, ds = -\int_0^t F(s, x) \, ds$$

for all $x \in \mathbb{R}$. After a change of variables in the integral, this is simply (2.21). □

D. Elementary Probabilistic and Analytic Lemmas

In this appendix we prove some elementary technical lemmas that were deferred until this point to avoid disrupting the flow of the paper.

Several symmetry arguments in the paper relied on the following lemma.

Lemma D.1. Let $X_1$ and $X_2$ be random variables such that $X_1 \overset{law}{=} X_2$ and $\mathbb{E}(X_2 - X_1)^- \geq -\infty$. Then $\mathbb{E}(X_2 - X_1) < \infty$ and $\mathbb{E}(X_2 - X_1) = 0$.

Proof. Of course the claim is obvious if $\mathbb{E}|X_j| < \infty$, but for our applications in the paper it will be convenient to not have to assume this. Define $f_M(x) = \max\{\min\{x, M\}, -M\}$ and note that $f_M$ is bounded and 1-Lipschitz. The function $g_M(x) = f_M(x) - x$ is also 1-Lipschitz. Fix $R \in (0, \infty]$ and compute

$$\mathbb{E}(|g_M(X_2) - g_M(X_1)| \cdot 1\{X_2 - X_1 \leq R\})$$

$$\leq \mathbb{E}(1\{X_2 - X_1 \leq R\})$$

$$\leq \mathbb{E}(1\{X_2 - X_1 \leq R\}) \leq M \mathbb{E}(\max\{|X_1|, |X_2|\} \geq M \mathbb{E}(X_2 - X_1 \leq R)). \quad (D.1)$$

For each $0 < R < \infty$ fixed, the last expression in (D.1) goes to 0 as $M \to \infty$ by the dominated convergence theorem, since $\mathbb{E}(1\{X_2 - X_1 \leq R\})$ is finite since $(X_2 - X_1)^+ 1\{X_2 - X_1 \leq R\}$ is bounded and $\mathbb{E}(X_2 - X_1)^-$ is finite by assumption. Therefore, we have

$$0 = \lim_{M \to \infty} \mathbb{E}(|g_M(X_2) - g_M(X_1)| \cdot 1\{X_2 - X_1 \leq R\})$$

$$= \lim_{M \to \infty} \mathbb{E}(|f_M(X_2) - f_M(X_1) - (X_2 - X_1)| \cdot 1\{X_2 - X_1 \leq R\})$$
Lemma D.2. Let \( \text{Stationary Solutions to the Stochastic Burgers Equation} \)

\[
\geq \limsup_{M \to \infty} \left| \mathbb{E}( (f_M(X_2) - f_M(X_1)) \mathbf{1}[X_2 - X_1 \leq R])
- \mathbb{E}( (X_2 - X_1) \mathbf{1}[X_2 - X_1 \leq R]) \right|.
\] (D.2)

Since \( X_1 \overset{\text{law}}{=} X_2 \), we have \( \mathbb{E}( f_M(X_2) - f_M(X_1)) = 0 \). Also, \( f_M(x) \) is monotone in \( x \), so

\[
\mathbb{E}( (f_M(X_2) - f_M(X_1)) \mathbf{1}[X_2 - X_1 \leq R])
= -\mathbb{E}( (f_M(X_2) - f_M(X_1)) \mathbf{1}[X_2 - X_1 > R]) \leq 0
\]

for all \( M, R \in (0, \infty) \). Using this in (D.2) gives

\[
\mathbb{E}( (X_2 - X_1) \mathbf{1}[X_2 - X_1 \leq R]) \leq 0
\]

for all \( R \in (0, \infty) \). Taking \( R \to +\infty \) and using the monotone convergence theorem and the assumption \( \mathbb{E}(X_2 - X_1) > -\infty \) yields \( \mathbb{E}(X_1 - X_2) \leq 0 \), so in particular \( \mathbb{E}|X_1 - X_2| < \infty \). Thus we can take \( R = +\infty \) in (D.1) and again use the dominated convergence theorem to obtain (D.2) with \( R = +\infty \), namely

\[
0 \geq \limsup_{M \to \infty} \left| \mathbb{E}( f_M(X_2) - f_M(X_1)) - \mathbb{E}(X_2 - X_1) \right| = |\mathbb{E}(X_2 - X_1)|,
\]

and so \( \mathbb{E}(X_2 - X_1) = 0 \) as claimed. \( \square \)

The following lemma will be used in the proof of Lemma D.3 below.

**Lemma D.2.** Let \( \eta \in C^3(I) \) for some closed interval \( I \subset \mathbb{R} \) and define \( A = \| \eta \|_{C^3(I)} \).

If \( x_1 \neq x_2 \in I \) satisfy \( \eta(x_1) = \eta(x_2) \) and \( \eta'(x_1), \eta'(x_2) > \epsilon \) or \( \eta'(x_1), \eta'(x_2) < -\epsilon \), then \( |x_1 - x_2| \geq \sqrt{2\epsilon}/A \).

**Proof.** Without loss of generality, we may assume that \( x_2 > x_1 \) and \( \eta'(x_1), \eta'(x_2) > \epsilon \).

Let \( \delta = x_2 - x_1 \). By Rolle’s theorem, there is a \( y \in (x_1, x_2) \) so that \( \eta'(y) = 0 \). By the mean value theorem, there exist \( z_1 \in (x_1, y) \) and \( z_2 \in (y, x_2) \) so that

\[
\eta''(z_1) \leq -\delta^{-1}\epsilon, \quad \eta''(z_2) \geq \delta^{-1}\epsilon.
\]

By another application of the mean value theorem, there exists a \( w \in (z_1, z_2) \) so that

\[
\eta'''(w) \geq 2\delta^{-2}\epsilon.
\]

This means that \( 2\delta^{-2}\epsilon \leq A \), so \( \delta \geq \sqrt{2\epsilon}/A \), as claimed. \( \square \)

The following lemma was used in the proof of Lemma 3.5.

**Lemma D.3.** Suppose that \( \zeta \geq 0 \) is in the Schwartz class and \( \eta \) is a smooth function all of whose derivatives have at most polynomial growth at infinity. Define

\[
g(\lambda) = \sum_{y \in \eta^{-1}(\lambda), \eta'(y) \neq 0} |\eta'(y)| \zeta(y).
\]

Then \( g \) is a continuous function of \( \lambda \).
Proof. Let $\xi_k \geq 0$ be smooth such that $\text{supp} \xi_k \subset [k - 1, k + 2]$ and $\sum_{k \in \mathbb{Z}} \xi_k = \xi$. Let

$$g_k(\lambda) = \sum_{y \in \eta^{-1}(\lambda)} |\eta'(y)|\xi_k(y). \quad (D.3)$$

We first show that $g_k$ is continuous. Let $A_k = \|\eta\|_{C^3([k-1,k+2])} + \|\xi_k\|_{C^0(\mathbb{R})} + \|\eta'\xi_k\|_{C^1(\mathbb{R})} + 1$. Define

$$S_k,\ell(\lambda) = \{y \in (k - 1, k + 2) \mid \eta(y) = \lambda, |\eta'(y)| \in [2^{-\ell}, 2^{-\ell + 1}]\}.$$

Lemma D.2 implies

$$|S_k,\ell(\lambda)| \leq 2^3 A_k^{1/2} 2^{\ell/2}, \quad (D.4)$$

so

$$\sum_{y \in S_k,\ell(\lambda)} |\eta'(y)|\xi_k(y) \leq 2^{-\ell + 1} \|\xi_k\|_{C^0(\mathbb{R})}|S_k,\ell(\lambda)| \leq 2^4 A_k^{3/2} 2^{-\ell/2} \quad (D.5)$$

for all $\lambda$.

Now fix $\varepsilon > 0$ and choose $\ell$ so large that $2^9 A_k^{3/2} 2^{-\ell/2} < \varepsilon$. Define

$$T_{k,\ell}^+(\lambda_1) = \{y \in (k - 1, k + 2) \mid \eta(y) = \lambda_1, \eta'(y) \geq 2^{-\ell}\},$$

$$T_{k,\ell}^-(-\lambda_1) = \{y \in (k - 1, k + 2) \mid \eta(y) = \lambda_1, \eta'(y) \leq -2^{-\ell}\}.$$

Suppose that $\lambda_1 < \lambda_2$ satisfy $\lambda_2 - \lambda_1 < 2^{-2\ell - 2} A_k^{-1}$. Take $x \in T_{k,\ell}^+(\lambda_1)$. On the interval $[x - 2^{-\ell - 1} A_k^{-1}, x + 2^{-\ell - 1} A_k^{-1}]$, we must have

$$\eta' \geq \eta' - ||\eta''||_{C^0([k-1,k+2])} 2^{-\ell - 1} A_k^{-1} \geq 2^{-\ell - 1}.$$

Thus

$$\eta(x + 2^{-\ell - 1} A_k^{-1}) \geq \lambda_1 + 2^{-\ell - 1} 2^{-\ell - 1} A_k^{-1} > \lambda_2.$$

Since $\eta$ is continuous, there exists $y \in T_{k,\ell}^+(\lambda_2) \cap (x, x + 2^{-\ell - 1} A_k^{-1}]$. We say that $y$ is “paired to $x$.” Notice that $y - x < 2^{-\ell - 1} A_k^{-1} \geq 2^{-\ell - 1}$ while $\eta \leq \lambda_1$ on $[x - 2^{-\ell - 1} A_k^{-1}, x]$, so $y$ is not paired to any other $x \in T_{k,\ell}^+(\lambda_1)$. Thus to each $x \in T_{k,\ell}^+(\lambda_1)$ we have paired a unique $y \in T_{k,\ell}^+(\lambda_2)$. Also,

$$|\eta'(x)|\xi_k(x) - |\eta'(y)|\xi_k(y) \leq ||\eta'\xi_k||_{C^1(\mathbb{R})} 2^{-\ell - 1} A_k^{-1} \leq 2^{-\ell - 1}.$$

Now consider the difference

$$\sum_{x \in T_{k,\ell}^+(\lambda_1)} |\eta'(x)|\xi_k(x) - \sum_{y \in T_{k,\ell}^+(\lambda_2)} |\eta'(y)|\xi_k(y). \quad (D.6)$$

Each $x \in T_{k,\ell}^+(\lambda_1)$ is paired to a unique $y \in T_{k,\ell}^+(\lambda_2)$, and the corresponding terms’ difference is at most $2^{-\ell - 1}$. On the other hand, if $x \in T_{k,\ell}^+(\lambda_1) \setminus T_{k,\ell}^+(\lambda_2)$, we have

$$|\eta'(x)|\xi_k(x) \leq 2^{-\ell + 1} A_k.$$
Decomposing (D.6) into these two cases, we obtain

\[ \sum_{x \in \mathcal{T}_{k,\ell}^+(\lambda_1)} |\eta'(x)| \zeta_k(x) - \sum_{y \in \mathcal{T}_{k,\ell}^-(\lambda_2)} |\eta'(y)| \zeta_k(y) \leq \sum_{x \in \mathcal{T}_{k,\ell}^+(\lambda_1)} (2^{-\ell-1} + 2^{\ell+1} A_k) \leq 2^2 A_k 2^{-\ell} |\mathcal{T}_{k,\ell}^+(\lambda_1)|. \]

Now (D.4) implies

\[ |\mathcal{T}_{k,\ell}^+(\lambda_1)| \leq \sum_{\ell' \leq \ell} |\mathcal{S}_{k,\ell'}(\lambda_1)| \leq 2^3 A_k^{1/2} 2^{\ell/2} \sum_{m \geq 0} 2^{-m/2} \leq 2^5 A_k^{1/2} 2^{\ell/2}. \]

Therefore,

\[ \sum_{x \in \mathcal{T}_{k,\ell}^+(\lambda_1)} |\eta'(x)| \zeta_k(x) - \sum_{y \in \mathcal{T}_{k,\ell}^-(\lambda_2)} |\eta'(y)| \zeta_k(y) \leq 2^2 A_k 2^{-\ell} |\mathcal{T}_{k,\ell}^+(\lambda_1)| \leq 2^7 A_k^{3/2} 2^{-\ell/2}. \]

Symmetric arguments show that in fact

\[ \left| \sum_{x \in \mathcal{T}_{k,\ell}^-(\lambda_1)} |\eta'(x)| \zeta_k(x) - \sum_{y \in \mathcal{T}_{k,\ell}^+(\lambda_2)} |\eta'(y)| \zeta_k(y) \right| \leq 2^7 A_k^{3/2} 2^{-\ell/2}. \quad (D.7) \]

Now consider \( g_k(\lambda_1) - g_k(\lambda_2) \). We divide (D.3) into terms in \( \mathcal{T}_{k,\ell}^+(\lambda_1) \) or \( \mathcal{T}_{k,\ell}^+(\lambda_2) \) for \( \ell' > \ell \) and \( i \in \{1, 2\} \). We pair terms in \( \mathcal{T}_{k,\ell}^+(\lambda_1) \) with those in \( \mathcal{T}_{k,\ell}^+(\lambda_2) \) to take advantage of the cancellation in (D.7). We treat the terms in \( \mathcal{S}_{k,\ell'}(\lambda_i) \) as error and use (D.5). Then

\[ |g_k(\lambda_1) - g_k(\lambda_2)| \leq 2^8 A_k^{3/2} 2^{-\ell/2} + 2^5 A_k^{3/2} \sum_{\ell' > \ell} 2^{-\ell'/2} \leq 2^9 A_k^{3/2} 2^{-\ell/2} < \varepsilon. \]

It follows that \( g_k \) is uniformly continuous.

We now wish to sum over \( k \) to conclude the same for \( g \). To do so, we bound \( g_k \). Let \( \ell_k = -\log_2 A_k + 1 \). Then \( \mathcal{S}_{k,\ell}(\lambda) = 0 \) for all \( \lambda \in \mathbb{R} \) and \( \ell < \ell_k \). Hence (D.4) implies

\[ g_k(\lambda) = \sum_{\ell \geq \ell_k} \sum_{y \in \mathcal{S}_{k,\ell}(\lambda)} |\eta'(y)| \zeta_k(y) \leq 2 \| \zeta_k \|_{C^0(\mathbb{R})} \sum_{\ell \geq \ell_k} 2^{-\ell} |\mathcal{S}_{k,\ell}(\lambda)| \leq 2^4 \| \zeta_k \|_{C^0(\mathbb{R})} A_k^{1/2} 2^{-\ell_k/2} \sum_{m \geq 0} 2^{-m/2}. \]

Using the definition of \( \ell_k \), this yields

\[ g_k(\lambda) \leq 2^5 \| \zeta_k \|_{C^0(\mathbb{R})} A_k. \]

By hypothesis, there exists \( C \geq 1 \) independent of \( k \) such that \( A_k \leq C \langle k \rangle^C \) for all \( k \in \mathbb{Z} \). But \( \zeta_k \leq \zeta \) decays super-polynomially as \( |k| \to \infty \). It follows that \( \| g_k \|_{C^0(\mathbb{R})} \) itself decays super-polynomially in \( k \). Therefore \( g = \sum_{k \in \mathbb{Z}} g_k \) is an absolutely and uniformly convergent sum of uniformly continuous functions and hence is (uniformly) continuous. \( \square \)
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