Complete classification of spherically symmetric static spacetimes via Noether symmetries

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Abstract

In this paper we give a complete classification of spherically symmetric static spacetimes by their Noether symmetries. The determining equations for Noether symmetries are obtained by using the usual Lagrangian of a general spherically symmetric static spacetime which are integrated for each case. In particular we observe that spherically symmetric static spacetimes are categorized into six distinct classes corresponding to Noether algebra of dimensions 5, 6, 7, 9, 11 and 17. Using Noether’s theorem we also write down the first integrals for each class of such spacetimes corresponding to their Noether symmetries.

1 Introduction

Spherical symmetric static solutions of Einstein’s field equations are of great importance in general relativity for they possess the properties of being static and asymptotically flat which was proved by Birkhoff [1]. For a spherical symmetric spacetime there are exactly three rotational Killing vector fields that preserve the metric forming $SO(3)$ as the isometry group of symmetries of these spacetimes. The study of spherically symmetric spacetimes is interesting as it helps in giving the understanding of phenomena of gravitational collapse and black holes, a widely known subject in the literature. For example the Schwarzschild solution is a non-trivial exact solution of the Einstein’s field equations which is spherically symmetric that describes the gravitational field exterior to a static, spherical, uncharged mass without angular momentum and isolated from all other mass.
The search for spherically symmetric spacetimes is an important task and due to their significance in understanding the dynamics around black-holes it is crucial to classify them with respect to their physical properties. Hence it would be interesting to find out the general form of these spacetimes along with a detailed characterization of the first integrals of corresponding geodesic equations. Besides in order to obtain those quantities which remain invariant under the geodesic motions yield significant physical information. In [2], Isabel Cordero-Carrin provided a procedure based on geometric arguments to obtain maximal foliations of spherically symmetric spacetimes which they later use to calibrate numerical relativity complex codes. The classification of these spacetimes based on their local conformal symmetries is done in [10]. On the other hand classification of plane-, cylindrically- and spherically-symmetric spacetimes with respect to their Killing vectors, homotheties, Ricci collineations, curvature collineations have been done in [3, 15].

In a recent paper, Farhad Ali and Tooba Feroze [4] used the symmetry method approach to find the complete classification of plane-symmetric spacetimes from Noether symmetries and also present the first integral in each case. In this paper we employ symmetry methods to provide complete classification of spherically symmetric static spacetimes with respect to the Noether symmetries they possess. Later, we use famous Noether’s theorem to write down the first integrals of each spacetime. In this way we are not only be able to recover all the known cases but also list the corresponding first integrals of solutions of EFEs, which are given in standard gravitational units, $c = G = 1$ as,

$$ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\kappa T_{\mu\nu}. \quad (1) $$

The general form of a spherically symmetric static space-time is

$$ ds^2 = e^{\nu(r)} dt^2 - e^{\mu(r)} dr^2 - e^{\lambda(r)} d\Omega^2. \quad (2) $$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and both $\nu$ and $\mu$ are arbitrary functions of radial coordinate `$r$'. It is seen that $e^{\lambda(r)}$ can be one of the two forms (i) $\beta^2$ or (ii) $r^2$, where $\beta$ is some constant [5]. We write down the determining equations using the corresponding Lagrangian of above spacetime and study the complete integrability of those equations for each case.
The plan of the paper is as follows. In the section two we describe basic definitions and structure of Noether symmetries along. In section three we write down the determining equations for spherically symmetric static spacetimes which is a system of nineteen linear partial differential equations (PDEs). We obtain several cases for different forms of ‘ν’ and ‘µ’ while integrating the PDEs that give rise to a complete classification of spherically symmetric static spacetimes with Noether algebra of dimensions 5, 6, 7, 9, 11 and 17. The characterization of first integrals along geodesic motions is also carried out in the same section. The conclusion is given in the third section.

2 Preliminaries

It is well-known that a general spherically symmetric static spacetime admits usual Lagrangian

\[ L = L(t, r, \theta, \phi) = e^{\nu(r)} \dot{t}^2 - e^{\mu(r)} \dot{r}^2 - e^\lambda(r) (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2), \]

where “\(\cdot\)” denotes differentiation with respect to arc-length parameter ‘\(s\)’. A symmetry \(X\) of the Lagrangian \(L\) that leaves the action of a spherically symmetric static spacetime invariant is a Noether symmetry if it satisfies the following equation,

\[ X^{(1)}L + D(\xi)L = DA, \]

where

\[ X^{(1)} = X + \eta_0 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial r} + \eta_2 \frac{\partial}{\partial \theta} + \eta_3 \frac{\partial}{\partial \phi}, \]

is the first order prolongation of

\[ X = \xi \frac{\partial}{\partial s} + \eta_0 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial r} + \eta_2 \frac{\partial}{\partial \theta} + \eta_3 \frac{\partial}{\partial \phi}. \]

\(D\) is the standard total derivative operator given by

\[ D = \frac{\partial}{\partial s} + \dot{t} \frac{\partial}{\partial t} + \dot{r} \frac{\partial}{\partial r} + \dot{\theta} \frac{\partial}{\partial \theta} + \dot{\phi} \frac{\partial}{\partial \phi}, \]

and \(A\) is some gauge function. The coefficients of Noether symmetry, namely, \(\xi, \eta^i\) and gauge function \(A\) are functions of \((s, t, r, \theta, \phi)\). The coefficients of prolonged operator \(X^{(1)}\), namely, \(\eta^i_s\), are functions of \((s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})\) and are determined by

\[ \eta_s^i = D(\eta^i) - \dot{u}^i D(\xi), \]
where \( u^i \) refers to the space of dependent variables \((t, r, \theta, \phi) \) for \( i = 0, 1, 2, 3 \). Using the same identification we state famous Noether’s theorem.

**Noether Theorem**

If \( X \) is a Noether symmetry of a given Lagrangian \( L \) with respect to the gauge function \( A \), then the quantity

\[
I = A - \left( \xi L + (\eta^0 - \xi \dot{u}^i) \frac{\partial L}{\partial \dot{u}^i} \right),
\]

is annihilated by the total derivative operator, i.e., \( DI = 0 \).

3 Classification Results and Computational Remarks

By substituting the value of Lagrangian (3) in equation (4) and comparing the coefficients of all monomials we obtain the following system of nineteen linear partial differential equations (PDEs)

\[
\begin{align*}
\xi_t &= 0, \quad \xi_r = 0, \quad \xi_\theta = 0, \quad \xi_\phi = 0, \\
A_s &= 0, \quad A_t - 2e^{\nu(r)} \eta^0_s = 0, \quad A_r + 2\eta^1_s = 0, \\
A_\theta + 2e^{\mu(x)} \eta^2_s = 0, \quad A_\phi + 2e^{\mu(r)} \eta^3_s = 0, \\
\xi_s - \mu_r(r) \eta^1_s - 2\eta^1_r &= 0, \quad \xi_s - \nu_r(r) \eta^1 - 2\eta^0_t = 0, \\
\xi_s - \frac{2}{r} \eta^1 - 2\eta^2_t &= 0, \quad \xi_s - \frac{2}{r} \eta^1 - 2 \cot \theta \eta^2 - 2\eta^3 \theta &= 0, \\
\eta^2_\theta + \sin^2 \theta \eta^3_\phi &= 0, \quad e^{\nu(r)} \eta^0_r - e^{\mu(x)} \eta^1_r &= 0, \\
e^{\mu(r)} \eta^1_\theta - r^2 \eta^2_r &= 0, \quad e^{\nu(r)} \eta^0_\theta - r^2 \eta^2_t &= 0, \\
e^{\nu(r)} \eta^0_\phi - r^2 \sin^2 \theta \eta^3_\theta &= 0, \quad e^{\mu(r)} \eta^1_\phi + r^2 \sin^2 \theta \eta^3_\phi &= 0.
\end{align*}
\]

We intend to classify all spherically symmetric static spacetimes with respect to their Noether symmetries by finding the solutions of above system of PDEs. We have used Computer Algebra System (CAS) Maple—17 to carry out the case-splitting with the help of an important algorithm ‘rifsimp’ which is essentially an extension of the Gaussian elimination and Groebner basis algorithms that is used to simplify overdetermined systems of polynomially nonlinear PDEs or ODEs and inequalities and bring them into a useful form. In the following section we enlist spherically symmetric static spacetimes, their Noether symmetries and relative first integrals.
We also present the Noether algebra of Noether symmetries in the cases that are not known in the literature.

In order to solve system of PDEs (10), we note that the first equation simply implies that \( \xi \) can only be a function of arc-length parameter, i.e., \( \xi(s) \). To keep a distinction between Killing vector fields and Noether symmetries we use a different letter, namely, \( Y \) for those Noether symmetries which are not Killing vector fields. It is also remarked that a static spacetime always admit a time-like Killing vector field. Moreover, the Lagrangian (3) does not depend upon \('t'\) explicitly therefore the time-like Killing vector field appears as a Noether symmetry in each case. Furthermore, the Lagrangian (3) is spherically symmetric, therefore, the Lie algebra of Killing vector fields \( so(3) \) corresponding to the Lie group \( SO(3) \) is intrinsically admitted by each spacetime

\[
X_1 = \frac{\partial}{\partial \phi}, \quad X_2 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \quad X_3 = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}.
\]

The above algebra along with two Noether symmetries

\[
X_0 = \frac{\partial}{\partial t}, \quad Y_0 = \frac{\partial}{\partial s},
\]

form the basis of minimal 5–dimensional Noether algebra, in which \( Y_0 \) is not a Killing vector field of spherically symmetric spacetime (2). The Noether algebra of these five Noether symmetries is, [\( X_1, X_3 \)] = \( X_2 \), [\( X_2, X_3 \)] = −\( X_1 \), [\( X_i, X_j \)] = 0 and [\( X_i, Y_0 \)] = 0 otherwise, and is identified with the associated group \( SO(3) \times \mathbb{R}^2 \).

I - Spherically Symmetric Static Spacetimes with Five Noether Symmetries

Some examples of spacetimes that admit minimal set of Noether symmetries (five symmetries) appeared during the calculations are given in Table 1.
The Noether symmetries and corresponding first integrals are listed in the following Table 2

**Table 1: Forms of $\mu$ and $\nu$**

| No. | $\nu(r)$                      | $\mu(r)$                      |
|-----|-------------------------------|-------------------------------|
| 1.  | $\ln\left(\frac{r}{\alpha}\right)^2$ | arbitrary                     |
| 2.  | $\ln\left(1 - \left(\frac{r}{\alpha}\right)^2\right)$ | arbitrary                     |
| 3.  | $\ln\left(\frac{r}{\alpha}\right)^2$ | $- \ln\left(1 - \left(\frac{r}{\alpha}\right)^2\right)$ |
| 4.  | arbitrary                     | $- \ln\left(1 - \left(\frac{r}{\alpha}\right)^2\right)$ |

**Table 2: First Integrals**

| Gen | First integrals |
|-----|-----------------|
| $X_0$ | $\phi_0 = e^{\nu(r)} \dot{t}$ |
| $X_1$ | $\phi_1 = r^2 \sin^2 \theta \dot{\phi}$ |
| $X_2$ | $\phi_2 = r^2 \left(\cos \phi \dot{\theta} - \cot \theta \sin \phi \dot{\phi}\right)$ |
| $X_3$ | $\phi_3 = r^2 \left(\sin \phi \dot{\theta} + \cot \theta \cos \phi \dot{\phi}\right)$ |
| $Y_0$ | $\phi_4 = e^{\nu(r)} \dot{t}^2 - e^{\mu(r)} \dot{r}^2 - r^2 \left(\dot{\phi}^2 + \sin^2 \theta \dot{\phi}^2\right)$ |

with constant value of the gauge function, i.e., $A = \text{constant}$.

**II - Spherically Symmetric Static Spacetimes with Six Noether Symmetries**

There are two distinct classes of spherically symmetric static spacetimes admitting six Noether symmetries. In particular we get

$$(1): \quad ds^2 = \left(\frac{r}{a}\right)^\alpha dt^2 - dr^2 - r^2 d\Omega^2, \quad \alpha \neq 0, 2$$ (13)

which apart from minimal five-dimensional Noether algebra also admit an additional Noether
symmetry corresponding to the scaling transformation \((s, t, r) \rightarrow (\lambda s, \lambda^p t, \lambda^{1/2} r)\), given by

\[
Y_1 = s \frac{\partial}{\partial s} + pt \frac{\partial}{\partial t} + \frac{r}{2} \frac{\partial}{\partial r}, \quad p = \frac{2 - \alpha}{4}
\]

forming a 6–dimensional Noether algebra. This induces a scale-invariant spherically symmetric static spacetime. The corresponding first integral is

\[
\phi_0 = s \left( \left( \frac{r}{a} \right)^\alpha t^2 - r^2(\theta^2 + \sin^2 \theta \phi^2) \right) + \frac{\alpha-2}{2} \left( \frac{r}{a} \right)^2 t + rt
\]

The other spacetime with 6–dimensional Noether algebra is given by

\[
(2) : \quad ds^2 = dt^2 - e^{\mu(r)} dr^2 - r^2 d\Omega^2, \quad \mu(r) \neq \ln \left( 1 - \frac{r^2}{b^2} \right)^{-1}, \mu(r) \neq \text{constant}
\]

with an additional Noether symmetry relative to a non-trivial gauge term

\[
Y_1 = s \frac{\partial}{\partial t}, \quad A = 2t.
\]

The first integral corresponding to \(Y_1\) is \(\phi_0 = t - st\).

III - Spherically Symmetric Static Spacetimes with Seven Noether Symmetries

There arise four spacetimes with seven Noether symmetries in which three cases contains the group of six Killing vector fields whereas one case contain only the minimal group of Killing vectors while the other two symmetries are Noether symmetries. We discuss them separately.

The three metrics are given by

\[
\begin{align*}
(1) : \quad &ds^2 = e^{r/b} dt^2 - dr^2 - \beta^2 d\Omega^2, \quad b \neq 0, \quad \beta \neq 0 \\
(2) : \quad &ds^2 = \sec^2 \left( \frac{r}{a} \right) dt^2 - \sec^2 \left( \frac{r}{a} \right) dr^2 - \beta^2 d\Omega^2, \quad a \neq 0 \\
(3) : \quad &ds^2 = \left( 1 - \frac{r^2}{b^2} \right) dt^2 - \left( 1 - \frac{r^2}{b^2} \right)^{-1} dr^2 - \beta^2 d\Omega^2, \quad b \neq 0 \\
(4) : \quad &ds^2 = \frac{1}{r^2} dt^2 - \frac{1}{r^2} dr^2 - \beta^2 d\Omega^2,
\end{align*}
\]
have two additional symmetries along with the minimal set of symmetries

\[ X_{4.1} = t \frac{\partial}{\partial r} - b \left( e^{-r/b} + \frac{t^2}{4b^2} \right) \frac{\partial}{\partial t}, \]
\[ X_{5.1} = \frac{\partial}{\partial r} - \frac{t}{2b} \frac{\partial}{\partial t}, \]
\[ X_{4.2} = \sin \left( \frac{r}{a} \right) \cos \left( \frac{t}{a} \right) \frac{\partial}{\partial t} + \sin \left( \frac{r}{a} \right) \cos \left( \frac{r}{a} \right) \frac{\partial}{\partial r}, \]
\[ X_{5.2} = \cos \left( \frac{t}{a} \right) \cos \left( \frac{r}{a} \right) \frac{\partial}{\partial r} - \sin \left( \frac{r}{a} \right) \sin \left( \frac{t}{a} \right) \frac{\partial}{\partial t}, \]
\[ X_{4.3} = -\frac{rbe^{t/b}}{\sqrt{r^2 - b^2}} \frac{\partial}{\partial t} + \sqrt{r^2 - b^2} e^{-t/b} \frac{\partial}{\partial r}, \]
\[ X_{5.3} = \frac{rbe^{-t/b}}{\sqrt{r^2 - b^2}} \frac{\partial}{\partial t} + \sqrt{r^2 - b^2} e^{t/b} \frac{\partial}{\partial r}, \]
\[ X_{4.4} = \frac{t^2 + r^2}{2} \frac{\partial}{\partial t} + rt \frac{\partial}{\partial r}, \]
\[ X_{5.4} = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \]

respectively, where the new subscript refers to the case distinction and same will be followed from here after. The corresponding first integrals are mentioned in the Table below.

The following spacetime

\[ ds^2 = \left( \frac{r}{a} \right)^2 dt^2 - dr^2 - r^2 d\Omega^2 \]

contain two non-trivial Noether symmetries

\[ Y_1 = s \frac{\partial}{\partial s} + \frac{r}{2} \frac{\partial}{\partial r}, \]
\[ Y_2 = \frac{s^2}{2} \frac{\partial}{\partial s} + \frac{rs}{2} \frac{\partial}{\partial r}, \quad A = \frac{-r^2}{2}, \]

whose first integrals are also given in the same Table.
### Table 3: First Integrals

| Gen | First integrals |
|-----|-----------------|
| X\(_{4.1}\) | \( \phi_5 = b \left( 1 + \frac{t^2 r^2}{4b^2} \right) \dot{t} + 2 \dot{r} \) |
| X\(_{5.1}\) | \( \phi_6 = \frac{t r^2}{b} \dot{t} + 2 \dot{r} \) |
| X\(_{4.2}\) | \( \phi_5 = s \left( \left( \frac{r}{a} \right)^2 \dot{t}^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right) + r \dot{r} \) |
| X\(_{5.2}\) | \( \phi_6 = s^2 \left( \left( \frac{r}{a} \right)^2 \dot{t}^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right) + 2s r \dot{r} - r^2 \) |
| X\(_{4.3}\) | \( \phi_5 = - \dot{t} \sin \left( \frac{r}{a} \right) \cos \left( \frac{t}{a} \right) + \dot{r} \cos \left( \frac{r}{a} \right) \cos \left( \frac{t}{a} \right) \) |
| X\(_{5.3}\) | \( \phi_6 = \dot{t} \sin \left( \frac{r}{a} \right) \sin \left( \frac{t}{a} \right) + \dot{r} \cos \left( \frac{r}{a} \right) \cos \left( \frac{t}{a} \right) \) |
| X\(_{4.4}\) | \( \phi_5 = \frac{(t^2 + r^2) \dot{t}}{r^2} + \frac{b \dot{r}}{r} \) |
| X\(_{5.4}\) | \( 2 \left[ \frac{\dot{t}}{r^2} - \dot{r} \right] \) |
| Y\(_{1}\) | \( \phi_5 = \frac{r e^{t/b}}{\sqrt{r^2 - b^2}} \dot{t} + \frac{b^2 e^{t/b}}{\sqrt{r^2 - b^2}} \) |
| Y\(_{2}\) | \( \phi_6 = \frac{r e^{-t/b}}{\sqrt{r^2 - b^2}} \dot{t} - \frac{b^2 e^{-t/b}}{\sqrt{r^2 - b^2}} \) |

**IV - Spherically Symmetric Static Spacetimes with Nine Noether Symmetries**

This section contains some well known and important spacetimes. Here, we have five different cases of spacetimes in which three contain two additional Noether symmetries and one case contains one extra Noether symmetry besides others which are all Killing vector fields. We have the following results (spacetimes) with nine Noether symmetries:

\begin{align*}
(1) & : \quad ds^2 = dt^2 - dr^2 - \beta^2 d\Omega^2, \quad \beta \neq 0 \\
(2) & : \quad ds^2 = \frac{1}{r^2} dt^2 - \frac{1}{r^4} dr^2 - \beta^2 d\Omega^2, \\
(3) & : \quad ds^2 = \left( 1 + \frac{r}{b} \right)^2 dt^2 - dr^2 - \beta^2 d\Omega^2 \\
(4) & : \quad ds^2 = dt^2 - \frac{dr^2}{1 - \frac{r^2}{b^2}} - r^2 d\Omega^2.
\end{align*}
The first three space times correspond to the famous Bertotti-Robinson like solutions of Einstein’s field equations which describes a universe with uniform magnetic field whereas the last case is the Einstein universe. We first list the Killing vector fields which are also Noether symmetries

\[
\begin{align*}
X_{4,1} &= r \frac{\partial}{\partial t} + t \frac{\partial}{\partial r}, \\
X_{5,1} &= \frac{\partial}{\partial r}, \\
X_{4,2} &= -rse^{-t} \frac{\partial}{\partial t} + r^2se^{-t} \frac{\partial}{\partial r}, \\
X_{5,2} &= re^t \frac{\partial}{\partial t} + r^2e^t \frac{\partial}{\partial r}, \\
X_{4,3} &= \frac{b}{b + r} e^{-t} \frac{\partial}{\partial t} + e^{-t} \frac{\partial}{\partial r}, \\
X_{5,3} &= -\frac{b}{b + r} e^{(t/b)} \frac{\partial}{\partial t} + e^{(t/b)} \frac{\partial}{\partial r}, \\
X_{4,4} &= \sqrt{b^2 - r^2} \sin \phi \sin \theta \frac{\partial}{\partial r} - \frac{\sqrt{b^2 - r^2}}{r} \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \frac{\sqrt{b^2 - r^2}}{r} \cos \phi \frac{\partial}{\partial \phi}, \\
X_{5,4} &= \sqrt{b^2 - r^2} \cos \phi \sin \theta \frac{\partial}{\partial r} - \frac{\sqrt{b^2 - r^2}}{r} \cos \phi \frac{\partial}{\partial \theta} - \frac{\sqrt{b^2 - r^2}}{r} \sin \phi \frac{\partial}{\partial \phi}, \\
X_{6,4} &= \sqrt{b^2 - r^2} \cos \theta \frac{\partial}{\partial r} - \frac{\sqrt{b^2 - r^2}}{r} \sin \theta \frac{\partial}{\partial \phi}.
\end{align*}
\]

and the Noether symmetries corresponding to non-trivial guages are

\[
\begin{align*}
Y_{1,1} &= s \frac{\partial}{\partial t}, & A_{1,1} &= 2t, \\
Y_{2,1} &= s \frac{\partial}{\partial r}, & A_{2,1} &= 2r, \\
Y_{1,2} &= -rse^{-t} \frac{\partial}{\partial t} + r^2se^{-t} \frac{\partial}{\partial r}, & A_{1,2} &= \frac{2e^{-t}}{r}, \\
Y_{2,2} &= rse^t \frac{\partial}{\partial t} + r^2se^t \frac{\partial}{\partial r}, & A_{2,2} &= \frac{2e^t}{r}, \\
Y_{1,3} &= -\frac{bs}{2(b + r)} e^{(-t/b)} \frac{\partial}{\partial t} - \frac{s}{2} e^{(-t/b)} \frac{\partial}{\partial r}, & A_{1,3} &= (b + r)e^{(-t/b)}, \\
Y_{2,3} &= \frac{bs}{2(b + r)} e^{(t/b)} \frac{\partial}{\partial t} - \frac{s}{2} e^{(t/b)} \frac{\partial}{\partial r}, & A_{2,3} &= (b + r)e^{(t/b)}, \\
Y_{1,4} &= s \frac{\partial}{\partial t}, & A &= 2t.
\end{align*}
\]
\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Gen & First integrals \\
\hline
$X_{4.1}, \ X_{5.1}$ & $\phi_5 = t\dot{r} - r\dot{t}, \ \ \ \phi_6 = \dot{r}$ \\
\hline
$Y_{1.1}, \ Y_{2.1}$ & $\phi_7 = 2[t - s\dot{t}], \ \ \ \phi_8 = r - s\dot{r}$ \\
\hline
$X_{4.2}, \ X_{5.2}$ & $\phi_5 = t\frac{b^2}{2}\dot{r} - b^2 \ln(r/a)\dot{t}, \ \ \ \phi_6 = \dot{r}\frac{t^2}{r}$ \\
\hline
$Y_{1.2}, \ Y_{2.2}$ & $\phi_7 = 2se^{-t}\left[\frac{\dot{r}}{r} + \frac{\dot{t}}{r^2}\right] + \frac{2e^{-t}}{r}, \ \ \ \phi_8 = 2se^{t}\left[\frac{\dot{r}}{r} + \frac{\dot{t}}{r^2}\right] + \frac{2e^t}{r}$ \\
\hline
$X_{4.3}, \ X_{5.3}$ & $\phi_5 = e^{-t/b}\left(\frac{\dot{r}(b+r)}{b} + \dot{r}\right), \ \ \ \phi_6 = e^{t/b}\left(\frac{t(b+r)}{b} + \dot{r}\right)$ \\
\hline
$Y_{1.3}, \ Y_{2.3}$ & $\phi_7 = e^{-t/b}\left(\frac{t(b+r)}{b} + s\dot{r} + (b+r)\right), \ \ \ \phi_8 = e^{t/b}\left(\frac{s(b+r)}{b} - s\dot{r} + (b+r)\right)$ \\
\hline
$X_{4.4}$ & $\phi_5 = \frac{b^2\dot{r}\sin\phi\sin\theta}{\sqrt{b^2 - r^2}} - r\dot{\theta}\sqrt{b^2 - r^2}\cos\theta\sin\phi + r\dot{\phi}\sqrt{b^2 - r^2}\sin\theta\cos\phi$ \\
\hline
$X_{5.4}$ & $\phi_6 = \frac{b^2\dot{r}\cos\phi\sin\theta}{\sqrt{b^2 - r^2}} - r\dot{\theta}\sqrt{b^2 - r^2}\cos\theta\cos\phi + r\dot{\phi}\sqrt{b^2 - r^2}\sin\theta\sin\phi$ \\
\hline
$X_{6.4}, \ Y_{1.4}$ & $\phi_7 = \frac{b^2\dot{r}\cos\theta}{\sqrt{b^2 - r^2}} - r\dot{\theta}\sqrt{b^2 - r^2}\sin\theta, \ \ \ \phi_8 = t - s\dot{t}$ \\
\hline
\end{tabular}
\caption{First Integrals}
\end{table}

V- Spherically Symmetric Static Spacetimes with Eleven Noether Symmetries

The famous de-Sitter metric turns out to be the only case with eleven Noether symmetries. Except $Y_0$ all others are the Killing vectors.

\begin{equation}
1: \quad ds^2 = \left(1 - \frac{r^2}{b^2}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{r^2}{b^2}\right)} - r^2 d\Omega^2. \tag{19}
\end{equation}

We have the following Noether symmetries along with the minimal set of Noether symmetries
for the metric given by \( [19] \)

\[
\begin{align*}
X_4 &= \frac{br \sin \phi \sin \theta \cos (t/b)}{\sqrt{b^2 - r^2}} \frac{\partial}{\partial t} + \sin(t/b) \sqrt{b^2 - r^2} \left( \sin \theta \sin \phi \frac{\partial}{\partial r} + r \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \right), \\
X_5 &= \frac{br \cos \phi \sin \theta \cos (t/b)}{\sqrt{b^2 - r^2}} \frac{\partial}{\partial t} + \sin(t/b) \sqrt{b^2 - r^2} \left( \sin \theta \cos \phi \frac{\partial}{\partial r} + \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \frac{\partial}{\partial \phi} \right), \\
X_6 &= -\frac{br \sin \phi \sin \theta \sin (t/b)}{\sqrt{b^2 - r^2}} \frac{\partial}{\partial t} + \cos(t/b) \sqrt{b^2 - r^2} \left( \sin \theta \sin \phi \frac{\partial}{\partial r} + \sin \theta \sin \theta \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \right), \\
X_7 &= -\frac{br \cos \phi \sin \theta \sin (t/b)}{\sqrt{b^2 - r^2}} \frac{\partial}{\partial t} + \cos(t/b) \sqrt{b^2 - r^2} \left( \sin \theta \cos \phi \frac{\partial}{\partial r} + \sin \theta \frac{\partial}{\partial \theta} - \sin \phi \frac{\partial}{\partial \phi} \right), \\
X_8 &= \frac{br \cos \theta \cos (t/b)}{\sqrt{b^2 - r^2}} \frac{\partial}{\partial t} + \sin(t/b) \sqrt{b^2 - r^2} \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right), \\
X_9 &= -\frac{br \cos \theta \sin (t/b)}{\sqrt{b^2 - r^2}} \frac{\partial}{\partial t} + \cos(t/b) \sqrt{b^2 - r^2} \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right).
\end{align*}
\]

The first integrals are given in the following Table 6.

| Gen | First integrals |
|-----|----------------|
| \( X_5 \) | \( \phi_5 = -\frac{r}{r} \sqrt{b^2 - r^2} \sin \phi \sin \theta \cos (t/b)t + \frac{b^2}{\sqrt{b^2 - r^2}} \sin \phi \sin \theta \sin (t/b)r + r \sqrt{b^2 - r^2} \cos \theta \sin \phi \sin (t/b)\dot{\theta} + r \sqrt{b^2 - r^2} \sin \theta \cos \phi \sin (t/b)\dot{\phi} \) |
| \( X_6 \) | \( \phi_6 = -\frac{r}{r} \sqrt{b^2 - r^2} \cos \phi \sin \theta \cos (t/b)t + \frac{b^2}{\sqrt{b^2 - r^2}} \cos \phi \sin \theta \sin (t/b)r + r \sqrt{b^2 - r^2} \cos \theta \cos \phi \sin (t/b)\dot{\theta} - r \sqrt{b^2 - r^2} \sin \theta \sin \phi \sin (t/b)\dot{\phi} \) |
| \( X_7 \) | \( \phi_8 = -\frac{r}{r} \sqrt{b^2 - r^2} \sin \phi \sin \theta \sin (t/b)t + \frac{b^2}{\sqrt{b^2 - r^2}} \sin \phi \sin \theta \cos (t/b)r + r \sqrt{b^2 - r^2} \cos \theta \sin \phi \cos (t/b)\dot{\theta} + r \sqrt{b^2 - r^2} \sin \theta \cos \phi \cos (t/b)\dot{\phi} \) |
| \( X_8 \) | \( \phi_9 = -\frac{r}{r} \sqrt{b^2 - r^2} \cos \phi \sin \theta \cos (t/b)t + \frac{b^2}{\sqrt{b^2 - r^2}} \cos \phi \sin \theta \sin (t/b)r + r \sqrt{b^2 - r^2} \cos \theta \cos \phi \sin (t/b)\dot{\theta} - r \sqrt{b^2 - r^2} \sin \theta \sin \phi \sin (t/b)\dot{\phi} \) |
| \( X_9 \) | \( \phi_7 = -\frac{r}{r} \sqrt{b^2 - r^2} \cos \theta \cos (t/b)t + \frac{b^2}{\sqrt{b^2 - r^2}} \cos \theta \sin (t/b)r - r \sqrt{b^2 - r^2} \sin \theta \sin (t/b)\dot{\theta} \) |
| \( X_{10} \) | \( \phi_{10} = \frac{r}{r} \sqrt{b^2 - r^2} \cos \theta \sin (t/b)t + \frac{b^2}{\sqrt{b^2 - r^2}} \cos \theta \cos (t/b)r - r \sqrt{b^2 - r^2} \sin \theta \cos (t/b)\dot{\phi} \) |

**Table 5:** First Integrals

V - Spherically Symmetric Static Spacetimes with Seventeen Noether Symmetries
It is the famous Minkowski metric that represents a flat spacetime and admits seventeen Noether symmetries. The list of all symmetries is given in [14], while the first integrals are mentioned in [4] in Cartesian coordinate system.

4 Conclusion

In this paper a complete list of classification of spherically symmetric static spacetimes is given. It is seen that spherically symmetric static spacetimes may have 5, 6, 7, 9, 11, or 17 Noether symmetries. A few examples of spacetimes having minimal (i.e. 5) Noether symmetries are given in Table 1. Briefly, there appear two cases in which spacetimes admit six, three cases having seven, five cases admitting nine (including the Bertotti-Robinson and the Einstein metrics) whereas there appear only one case of eleven (which is the famous de-Sitter spacetime) and seventeen (Minkowski spacetime) Noether symmetries. Just like plane symmetric static spacetimes, for spherically symmetric static spacetimes the minimum number of Noether symmetries are five and the maximum number of Noether symmetries are seventeen, while the minimum number of isometries are four and the maximum number of isometries are ten. Classification of cylindrically symmetric static spacetimes is in progress.

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