HYPERCONTRACTIVITY, NASH INEQUALITIES AND SUBORDINATION
FOR CLASSES OF NONLINEAR SEMIGROUPS

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Abstract. A suitable notion of hypercontractivity for a nonlinear semigroup \( \{T_t\} \) is shown to imply Nash–type inequalities for its generator \( H \), provided a subhomogeneity property holds for the energy functional \( (u, Hu) \). We use this fact to prove that, for semigroups generated by operators of \( p \)-Laplacian-type, hypercontractivity implies ultracontractivity. Then we introduce the notion of subordinated nonlinear semigroups when the corresponding Bernstein function is \( f(x) = x^\alpha \), and write an explicit formula for the associated generator. It is shown that hypercontractivity still holds for the subordinated semigroup and, hence, that Nash-type inequalities hold as well for the subordinated generator.

1. Introduction

Since the seminal papers of Gross [17], Nelson [20], Federbush [15], Simon and Høegh-Krohn [24], Davies and Simon [13], the relations among various types of contractivity properties of linear semigroups on the one hand and functional inequalities satisfied by their generators on the other hand, have been intensively investigated. In particular the notions of hypercontractivity, supercontractivity and ultracontractivity for linear Markov semigroups have been related to logarithmic Sobolev inequalities and/or Sobolev, Nash and Gagliardo-Nirenberg inequalities involving Dirichlet forms. See for example [1], [12], [18] for comprehensive discussions.

Moreover, generalizations of functional inequalities of Nash and Gagliardo-Nirenberg type have been considered in [2], and in particular it is shown there that, under suitable assumptions, the validity of a single Gagliardo-Nirenberg inequality implies the validity of a whole class of them.

Recently, for some classes of nonlinear parabolic partial differential equations, contractivity properties of their solutions have been proved as a consequence of the validity of suitable logarithmic Sobolev inequalities involving the nonlinear Dirichlet forms (in the sense of [10]) associated to their generator. See [11] and [14] for the evolution equation driven by the \( p \)-Laplacian and [15] for the porous media equation. We refer to the recent work of J.L. Vazquez [25] for an excellent discussion of the known smoothing and decay properties for classes of evolutions including the porous media equation and the evolution driven by the \( p \)-Laplacian, in the Euclidean case.

One of the aims of the present paper is to continue to investigate such relations in the nonlinear setting. In particular we shall concentrate ourselves on a sort of converse of what has been investigated in [11], [14] and [15], namely on the consequences which can be drawn assuming that a nonlinear semigroup is, in a sense to be defined later, hypercontractive. In the linear case hypercontractivity is equivalent to a logarithmic Sobolev inequality for the Dirichlet form associated to the generator, but in general it does not imply a Nash (or a Sobolev) inequality, as the Ornstein-Uhlenbeck example shows.

Concerning the methods, our starting point will be the ideas of Gross [17]. However, nonlinearity plays here a special role so that new phenomena occur: in particular it will be shown that nonlinear hypercontractivity implies functional inequalities of Nash-type for the generator \( H \) provided it satisfies the subhomogeneity property

\[
(\lambda u, H(\lambda u)) \leq M \lambda^p (u, Hu).
\]
for all positive $\lambda$, all $u$ in the $L^2$ domain of $H$, a suitable $M > 0$ and a suitable $p > 2$, where $(\cdot, \cdot)$ is the scalar product in $L^2$. Then we draw another surprising consequence of this fact. Consider the evolution equation driven by the subgradient of the functional

$$E_p(u) := \int_M |\nabla u|^p \, dm,$$

where $(M, g)$ is a complete Riemannian manifold, $\nabla$ is the Riemannian gradient and $m$ is a $\sigma$-finite nonnegative measure on $M$ with the property that $\nabla$ is closable as an operator from $L^2(M, m)$ into $L^2(TM, m)$ (notice that $m$ need not be the Riemannian measure). We call this operator a generalized Riemannian $p$-Laplacian. We shall show that, when $p > 2$, its associated semigroup is ultracontractive whenever it is hypercontractive, a property which has of course no linear analogue (i.e. no analogue in the case $p = 2$).

The second main goal of the paper is to introduce the process of subordination of a given nonlinear semigroup $\{T_t\}$ w.r.t. convolution semigroups of probability measures (see e.g. [19]). In the present paper we shall deal only with the subordination associated to the Bernstein functions $f(x) = x^\alpha$, $\alpha \in (0, 1]$, a choice for which we give an explicit description of the subordinated nonlinear generator in terms of the original semigroup. This procedure extends the classical Bochner’s one [3] when the starting semigroup is linear. Then we shall show that under some assumptions, satisfied in the case in which $\{T_t\}$ is the semigroup associated to a generalized $p$-Laplacian and hypercontractivity holds for $\{T_t\}$, the subordinated semigroup is hypercontractive and subhomogeneous and, hence, Nash-type inequalities hold for its generator as well.

The plan of the paper is as follows. In section 2 we prove Nash-type inequalities for nonlinear hypercontractive (or supercontractive) semigroups which are also subhomogeneous: see Theorem 2.9. In section 3 we prove that if the semigroup associated to a generalized Riemannian $p$-Laplacian is hypercontractive it is also ultracontractive: see Theorem 3.2. Section 4 is devoted to the construction of nonlinear subordinated semigroups. The main result there is an explicit formula for the right derivative $A_\alpha u$ at $t = 0$ of what we call the nonlinear subordinated family $S_t u := \int_0^{+\infty} T_s u \mu_t^{(\alpha)}(ds)$ for a certain choice of the subordinator $\mu_t^{(\alpha)}$, and the proof that the resulting operator is monotone, thus giving rise to a well defined nonlinear (subordinated) strongly continuous, nonexpansive semigroup: see Theorem 4.3. In section 5 we prove Nash-type inequalities for the subordinated generators associated to the Bernstein function $f(x) = x^\alpha$, $\alpha \in (0, 1]$ and to the semigroup driven by a generalized $p$-Laplacian, provided such semigroup is hypercontractive: see Theorem 5.6.

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2. Nonlinear hypercontractive and supercontractive semigroups

In the present section we shall deal with nonlinear hypercontractive semigroups. Before giving the appropriate definition, we recall that a (nonlinear) strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on a a Hilbert space $L^2(X, m)$ is said to be nonexpansive if, for all $u, v \in L^2(X, m)$, $t \geq 0$, one has $\|T_t u - T_t v\|_2 \leq \|u - v\|_2$. This is in general different from requiring that $\{T_t\}_{t \geq 0}$ is contractive, namely that $\|T_t u\|_2 \leq \|u\|_2$ for all $u \in L^2(X, m)$, $t \geq 0$.

**Definition 2.1.** A strongly continuous contractive semigroup $\{T_t\}_{t \geq 0}$, not necessarily linear, on a Hilbert space $L^2(X, m)$ (see [8], [23]) is said to be hypercontractive if there exist $\varepsilon > 0$ and continuously differentiable functions $r : [0, \varepsilon) \to [2, +\infty)$, $\alpha, k : [0, \varepsilon) \to (0, +\infty)$ with $r(0) = 2$, $r'(t) > 0$ for all $t$, $\alpha(0) = 1$, $k(0) = 1$ and such that, for all $u \in L^2(X, m)$ and all $t \in [0, \varepsilon)$ one has

$$\|T_t u\|_{r(t)} \leq k(t)\|u\|_{2}^{\alpha(t)}.$$

(2.1)

We shall also use the following definition.
Definition 2.2. A strongly continuous contractive semigroup \( \{T_t\}_{t \geq 0} \), not necessarily linear, on a Hilbert space \( L^2(X, m) \) is said to be \((\beta, s)\)-supercontractive if there exists \( \beta > 0 \), \( s > 2 \) and a continuous function \( k : (0, +\infty) \to [0, +\infty) \) such that \( T_t u \in L^s(X, m) \) for all \( t > 0 \) and all \( u \in L^2(X, m) \) and
\[
\|T_t u\|_s \leq k(t) \|u\|^\beta_2 \quad \forall t > 0, \forall u \in L^2(X, m).
\]

Example 2.3. Bounds of the form (2.1), (2.2) hold true, for example, for the solutions to the Euclidean \( p \)-heat equation \( \dot{u} = \Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) \) \((p > 2)\) or for the porous media equation \( \dot{u} = \Delta(u^m) := \Delta(|u|^{m-1} u) \) \((m > 1)\). See [9] [5] [6].

Remark 2.4. Hereafter we shall denote by \( H \) the principal section, in the sense of Brezis [8], of the generator of the semigroup \( \{T_t\}_{t \geq 0} \).

Our first result consists in remarking that an immediate consequence of the definition of hypercontractivity is the validity of a suitable inhomogeneous logarithmic Sobolev inequality involving the functional \((u, H u)\).

Lemma 2.5. Let \( \{T_t\}_{t \geq 0} \) be a not necessarily linear hypercontractive semigroup in the sense of Definition 2.2. Then the logarithmic Sobolev inequality
\[
(2.3) \quad \int_X u^2 \log |u| dm - c_1 \|u\|^2_2 \log \|u\|_2 \leq c_2 (u, H u)_{L^2} + c_3 \|u\|^2_2
\]
holds for any \( u \) belonging to the \( L^2 \) domain of the \( L^2 \) generator \( H \) of \( \{T_t\}_{t \geq 0} \), where
\[
c_1 = \frac{\dot{r}(0)}{r(0)} + 2 \frac{\dot{\lambda}(0)}{r(0)}, \quad c_2 = \frac{1}{\dot{r}(0)}, \quad c_3 = \frac{2\ddot{k}(0)}{\dot{r}(0)}.
\]

Proof. By [18] Lemma 3.8 we have that, if \( r \) is a continuously differentiable function with values in \([2, +\infty)\) and \( r(0) = 2 \),
\[
\frac{d}{dt} \|T_t u\|_{r(t)} \bigg|_{t=0} = \|u\|^2_2 \left( \frac{\dot{r}(0)}{2} \int_X u^2 \log \|u\|_2 dm - (u, H u)_{L^2} \right).
\]

Notice that \( c_1 = 1 \) if and only if \( \dot{\alpha}(0) = 0 \). If this is the case the log-Sobolev inequality (2.3) involves the usual entropy functional.

Definition 2.6. We say that the generator \( H \) of the semigroup \( \{T_t\}_{t \geq 0} \) is subhomogeneous of degree \( p > 0 \) if \((u, H u)_{L^2} \geq 0\) for all \( u \in \text{Dom}(H) \) and there exists a positive \( M \) such that, for all positive \( \lambda \) and all \( u \in \text{Dom} H \) one has that \( \lambda u \in \text{Dom} H \) as well and moreover
\[
(\lambda u, H(\lambda u))_{L^2} \leq M \lambda^p (u, H u)_{L^2}.
\]

Example 2.7. Consider a domain \( D \) in the Euclidean space \( \mathbb{R}^n \) and the operator \( H \) defined on a suitable subset of \( L^2(D) \) and given, when the quantity below exists, by
\[
H u = -\nabla \cdot (a(x, u, \nabla u)|\nabla u|^{p-2} \nabla u),
\]
a : \( D \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty) \) being a positive differentiable function. Let us suppose that \( H \) can be extended as a densely defined maximally monotone operator: for conditions guaranteeing this facts see [23]. The subhomogeneity assumption of degree \( p \) is equivalent to:
\[
a(x, \lambda v, \lambda \xi) \geq M a(x, v, \xi) \quad \forall v, \xi \in \mathbb{R}, \lambda \in \mathbb{R}^n, \lambda > 0
\]
and for a suitable fixed positive \( M \). Of course this implies a similar lower bound on \( a(x, v, \xi) \), so that there exists \( C > 0 \) such that
\[
\frac{1}{C} a(x, \lambda v, \lambda \xi) \leq a(x, v, \xi) \leq Ca(x, \lambda v, \lambda \xi), \quad \forall \lambda > 0, x \in D, v \in \mathbb{R}, \xi \in \mathbb{R}^n.
\]
Letting $\lambda \to 0$ and using the continuity of $a$ yields that upper and lower bounds on $a(x,v,\xi)$ in terms of functions of the space variable $x$ only must hold.

**Lemma 2.8.** Let $\{T_t\}_{t \geq 0}$ be a nonlinear semigroup which is hypercontractive in the sense of Definition 2.1. Assume moreover that $H$ is strictly positive in the sense that

$$(u,Hu)_{L^2} > 0 \quad \forall u \in \text{Dom} \, H, \, u \neq 0,$$

that $H$ is subhomogeneous of degree $p$ in the sense of Definition 2.7 for a suitable $p > 2$, and that $\dot{\alpha}(0) < 0$, $\dot{r}(0) > 0$. Then the logarithmic Sobolev inequality

$$\int_X \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \, dm \leq k_1 \log \left( k_2 \frac{(u,Hu)_{L^2}}{\|u\|_2^2} \right)$$

(2.4)

holds true for any $u$ in the $L^2$ domain of $H$, where the positive constants $k_1, k_2$ are defined by

$$k_1 = -\frac{4\dot{\alpha}(0)}{(p-2)\dot{r}(0)}, \quad k_2 = -\frac{M(p-2)}{2\dot{\alpha}(0)} e^{1-|k(0)(p-2)/\dot{\alpha}(0)|}.$$

If in addition $X$ has finite measure, there exists $C > 0$ such that the coercive bound

$$\|Hu\|_2 \geq C\|u\|_2^{p-1}.$$ 

holds true.

Proof. Inequality (2.3), where $u$ is replaced by $\lambda u$ with $\|u\|_2 = 1$, $\lambda > 0$, implies by the subhomogeneity of $H$:

$$\int_X u^2 \log |u| \, dm + \bar{c}_1 \log \lambda \leq M c_2 \lambda^{p-2} (u,Hu)_{L^2} + c_3$$

where $\bar{c}_1 := 1 - c_1 = -\frac{2\dot{\alpha}(0)}{\dot{r}(0)} > 0$. We rewrite the above formula as

$$\bar{c}_1 \log \lambda - M c_2 \lambda^{p-2} (u,Hu)_{L^2} \leq c_3 - \int_X u^2 \log |u| \, dm.$$ 

(2.5)

The real function $a(\lambda) := A \log \lambda - B \lambda^{p-2}$ ($\lambda > 0$, $A,B > 0$) attains its minimum when $\lambda = \bar{\lambda} := [A/(B(p-2))]^{1/(p-2)}$ and one has

$$a(\bar{\lambda}) = \frac{A}{p-2} \log \left( \frac{A}{Be(p-2)} \right).$$

Then

$$\frac{\bar{c}_1}{p-2} \log \left( \frac{\bar{c}_1}{M c_2 (u,Hu)_{L^2} e^{(p-2)}} \right) \leq c_3 - \int_X u^2 \log |u| \, dm$$

which can be rewritten as

$$\int_X u^2 \log |u| \, dm \leq c_3 + \frac{\bar{c}_1}{p-2} \log \left[ \frac{M c_2}{c_1} e^{(p-2)} (u,Hu)_{L^2} \right]$$

$$= \frac{c_1}{p-2} \log \left[ e^{c_3(p-2)/\bar{c}_1} \frac{M c_2}{c_1} e^{(p-2)} (u,Hu)_{L^2} \right]$$

$$= \frac{k_1}{2} \log[k_2 (u,Hu)_{L^2}]$$

with $k_1, k_2$ as in the statement.

Writing now the latter inequality with $u$ replaced by $u/\|u\|_2$ and using again subhomogeneity yields (2.4).
As for the last statement, just notice that under the running assumption the bound
\[
\int_X \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \, dm \geq - \log m(X)
\]
holds true by Jensen's inequality. Therefore
\[
\log \left( k_2 \frac{(u, Hu)_{L^2}}{\|u\|_2^p} \right) \geq - \frac{\log m(X)}{k_1} = \log \left( (m(X))^{-1/k_1} \right)
\]
so that
\[
(u, Hu)_{L^2} \geq \frac{1}{k_2 m(X)^{1/k_1}} \|u\|_2^p.
\]
The final statement follows because \((u, Hu)_{L^2} \leq (u, Hu)_{L^2} \leq \|u\|_2 \|Hu\|_2\).
\[\square\]

We now prove that a Nash-type inequality follows straightforwardly from the logarithmic Sobolev inequality given in Lemma 2.8. We shall use this result in the final section, in which Nash-type inequalities will be proved for operators which are the generators of semigroups which are, in a sense to be defined, subordinated to the \(p\)-Laplacian semigroup.

We first define the Young functional
\[
J(p, u) = \int_X \frac{|u(x)|^p}{\|u\|_p^p} \log \left( \frac{|u(x)|}{\|u\|_p} \right) \, dm(x).
\]
Notice that, by definition, \(J(r, u) = \frac{1}{r} J(1, |u|^r)\).

**Theorem 2.9. (Nash-type inequalities).** Let \(\{T_t\}_{t \geq 0}\) be a nonlinear semigroup which is hypercontractive in the sense of Definition 2.1. Assume moreover that \(H\) is strictly positive in the sense that there exists \(H, Hu > 0\) \(\forall u \in \text{Dom } H, u \neq 0\), that \(H\) is subhomogeneous of degree \(p\) in the sense of Definition 2.6 for a suitable \(p > 2\), and that \(\hat{a}(0) < 0, \hat{r}(0) > 0\). Then, for any fixed \(m \in [1, 2)\) the inequality
\[
\|f\|_2 \leq C(H, f)_{L^2}^1 \|f\|_m^k \left( \frac{2 + m}{p + 2 + m} \right) \left( \frac{1}{p - 2 + 2 + m} \right) \|f\|_m^{1/p + 1} \|f\|_1^{1 - 1/p + 1}
\]
holds true for a suitable constant \(C\) and for all \(f \in \text{Dom } H \cap L^m\), where \(k_1 = -\frac{4\hat{a}(0)}{(p - 2)\hat{r}(0)} > 0\) is the constant appearing in Lemma 2.8. In particular the Nash inequality
\[
\|f\|_2 \leq C(H, f)_{L^2}^1 \|f\|_1^{k_1} \|f\|_2^{1 + k_1}
\]
holds true for all \(f \in \text{Dom } H \cap L^1\).

A similar conclusion holds if, keeping fixed the assumptions on \(H, \hat{a}(0)\) and \(\hat{r}(0)\), one replaces the hypercontractivity assumption by the assumption that the semigroup considered is \((\beta, s)\)-supercontractive in the sense of Definition 2.2, for suitable \(\beta < 1, s \geq \max(2, 2/\beta)\), and with \(k\) such that \(\lim_{t \to 0} k(t)^{\frac{1}{s}}\) is finite. In fact, the inequalities (2.7) and (2.8) hold if one replaces \(k_1\) above with \(k_1 = 2(1 - \beta)/[(\beta(p - 2) - 2)]\).

**Proof.** We rewrite the assertion of Lemma 2.8 as
\[
J(1, f^2) \leq k_1 \log \left( k_2 \frac{(Hf, f)_{L^2}}{\|f\|_2^2} \right).
\]
Next we recall that, if \(q > r\),
\[
\frac{\|u\|_q}{\|u\|_r} \leq e^{\frac{q - r}{r} J(1, |u|^r)}.
\]
See [7] for a proof of this fact. By using such inequality with \( q = 2, \ r = m < 2 \) we get
\[
\log \frac{\|f\|_2}{\|f\|_m} \leq \log \left\{ \left[ k_s(Hf,f)_{L^2} \right]^{k_s(2-m)/m} \right\}
\]
which is an equivalent form of our statement in the hypercontractive case.

To deal with the supercontractive case, first we notice that there exists a positive \( C > 0 \) such that, for all \( s \) sufficiently small we find that there is a constant \( C > 0 \) such that, for all \( \vartheta \) sufficiently small. Finally, for the same values of \( \vartheta \) we get:
\[
\|T_t u\|_{L^2}^{2(2(\vartheta+1-\vartheta))} \leq k(t)^{(1-\vartheta)/\vartheta} \|u\|_{L^2}^{2(1-\vartheta)}
\]
Choosing \( 1 - \vartheta = t \) for \( t \) small we get:
\[
\|T_t u\|_{L^2}^{p(t)a(t)} \leq k(t)^{2t} \|u\|_2^2
\]
with
\[
p(t) = \frac{2s(1-t)/\beta + t}{2t + s\beta(1-t)}; \quad a(t) = \frac{2s^2 + 2t + s\beta(1-t)}{s\beta}.
\]
By the assumption on the behaviour of the function \( k \) near \( t = 0 \) we have also
\[
\|T_t u\|_{L^2}^{p(t)a(t)} \leq A \|u\|_2^2
\]
for a suitable constant \( A \), where we can assume that \( A > 1 \) (note the strict inequality: if the function \( k \) is such that the above inequality holds with \( A = 1 \) the argument below can be simplified proceeding as in [17]). This is a hypercontractive bound similar to the ones studied above. In the l.h.s. of (2.11) we have a real function \( f \) of \( t \geq 0 \) which is differentiable at \( t = 0 \) and satisfies \( f(0) < A \|u\|_2^2 \). Then the tangent line to the graph of \( f \) at \( t = 0 \) lies below \( A \|u\|_2^2 \) for sufficiently small \( t \). Next, [18, Lemma 3.8] (see the proof of Lemma 2.5) and the chain rule imply that
\[
\left. \frac{d}{dt} \|T_t u\|_{L^2}^{p(t)a(t)} \right|_{t=0} = \frac{2(s-2)}{s} \int_X u^2 \log |u| \ dm - 2 \int_X u(Hu) \ dm - 2s\beta - 2 \frac{s\beta^2 - 2}{s^2} \|u\|_2^2 \log \|u\|_{L^2}.
\]
Therefore, for \( t \) sufficiently small,
\[
\|u\|_2^2 + t \left[ \frac{2(s-2)}{s} \int_X u^2 \log |u| \ dm - 2 \int_X u(Hu) \ dm - 2s\beta - 2 \frac{s\beta^2 - 2}{s^2} \|u\|_2^2 \log \|u\|_{L^2} \right] \leq A \|u\|_2^2
\]
or equivalently
\[
\frac{2(s-2)}{s} \int_X u^2 \log |u| \ dm - 2 \int_X u(Hu) \ dm - 2s\beta - 2 \frac{s\beta^2 - 2}{s^2} \|u\|_2^2 \log \|u\|_{L^2} \leq \frac{A - 1}{t} \|u\|_2^2
\]
again for \( t \) sufficiently small. Finally, for the same values of \( t \),
\[
\int_X u^2 \log |u| \ dm - \frac{s\beta - 2}{\beta(s-2)} \|u\|_2^2 \log \|u\|_2 \leq \frac{s}{s-2} (u, Hu)_{L^2} + \frac{s(A - 1)}{2t(s-2)} \|u\|_2^2.
\]
Fixing \( t \) sufficiently small we find that there is a constant \( C > 0 \) such that
\[
\int_X u^2 \log |u| \ dm - \frac{s\beta - 2}{\beta(s-2)} \|u\|_2^2 \log \|u\|_2 \leq \frac{s}{s-2} (u, Hu)_{L^2} + C \|u\|_2^2.
\]
To proceed further we use the same reasoning used in the proof of Lemma 2.8. In fact we can proceed exactly as in that proof provided the constant \( (s\beta - 2)/[\beta(s-2)] \), which takes here the role
of $c_1$ there, lies in the interval $(0,1)$. This is true under our assumptions on $s$ and $\beta$. Then we get that the logarithmic Sobolev inequality
\begin{equation}
\int_X \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \, dm \leq k_1 \log \left( \frac{k_2 (u,Hu)_{L^2}}{\|u\|_2^2} \right)
\end{equation}
holds true for any $u$ in the $L^2$ domain of $H$, where $k_2$ is a suitable positive constant and $k_1 = 2(1 - \beta)/(\beta(p - 2)(s - 2))]$. The final statement follows as in the hypercontractive case. \hfill \Box

Remark 2.10. The restriction on $s$ in the result concerning supercontractive semigroups may seem strange since a similar result is proved in the same Theorem under the assumption of hypercontractivity only. However it is related to the additional request $\dot{\alpha}(0) < 0$ assumed in the hypercontractive case.

Remark 2.11. A whole family of Gagliardo-Nirenberg inequalities follows from the Nash inequality proved above if the functional $W(u) := (u,Hu)_{L^2}^{1/p}$ satisfies certain contractivity properties defined in [2]. This happens in particular in the case in which $H$ is the $p$-Laplacean on a Riemannian manifold, a situation which will be discussed further in the present paper.

3. FROM HYPERCONTRACTIVITY TO ULTRACONTRACTIVITY FOR EVOLUTIONS DRIVEN BY GENERALIZED $p$-LAPLACIANS

In this section we shall deal with a particular choice of the semigroup $\{T_t\}_{t \geq 0}$. In fact, let $(M,g)$ be a complete Riemannian manifold, whose Riemannian gradient is indicated by $\nabla$. Then we choose a $\sigma$-finite nonnegative measure $m$ on $M$, and assume that $\nabla$ is a closed operator from $L^2(M,m)$ to $L^2(TM,m)$. Notice that $m$ need not be the Riemannian measure. We shall then consider the strongly continuous contraction semigroup $\{T_t\}_{t \geq 0}$ associated to the subgradient of the convex lower semicontinuous functional given by
\begin{equation}
\mathcal{E}_p(u) := \int_M |\nabla u|^p \, dm
\end{equation}
for those $L^2(X,m)$ functions for which the integral is finite, and by $+\infty$ otherwise in $L^2(M,m)$. In fact, lower semicontinuity of the above functional is a consequence of the assumption that $\nabla$ is closed (see [11]), but could be alternatively assumed directly.

Hereafter the condition $p > 2$ will be necessary to apply the previous results. The generator of the semigroup that we consider in this section is the operator given formally by $H = -\nabla^* (|\nabla u|^{p-2} \nabla u)$, where $\nabla^*$ is for the formal adjoint of the gradient with respect to the inner products of $L^2(M,m)$ and of $L^2(TM,m)$.

We stress that we choose the above semigroup as a model case for our discussion to hold, but that much more general situations can be dealt with by identical methods: see [10, 11] for details.

The semigroup associated to the subgradient of $\mathcal{E}_p$ may or may not be hypercontractive. Our aim here is to show some consequences of hypercontractivity, if it holds: more precisely we shall show that under the same assumptions which allow to prove a homogeneous logarithmic Sobolev inequality of the type given in Lemma 2.8, ultracontractivity of the semigroup hold as well.

Before turning to this our first comment is that, as a consequence of our previous results, logarithmic Sobolev inequalities must hold.

Corollary 3.1. Let $\{T_t\}_{t \geq 0}$ be the semigroup associated to the functional $E$ given in (3.1), with $p > 2$. Assume that it is hypercontractive in the sense of Definition 2.7 with $\dot{\alpha}(0) < 0$, $\dot{\tau}(0) > 0$ and $\dot{\tau}(0) + 2\dot{\alpha}(0) > 0$. Let the dimension $d > 0$ of the hypercontractive semigroup considered be defined by
\begin{equation}
d = \frac{-4\dot{\alpha}(0)p}{(p-2)(\dot{\tau}(0) + 2\dot{\alpha}(0))}.
\end{equation}
Then the logarithmic Sobolev inequality
\[
\int_X u^p \log u \, dm \leq \frac{d}{p} \log \left( K \frac{\mathcal{E}_p(u)}{\|u\|_p^p} \right)
\]
holds true for a suitable constant $K$ and for any $u \in L^2(X, m)$.

**Proof.** It suffices to prove the claim for smooth, compactly supported functions. For such functions one has $(u, Hu)_{L^2} = \mathcal{E}_p(u)$ so that the positivity condition of Lemma 2.8 holds. Lemma 2.8 then implies that the logarithmic Sobolev inequality
\[
\int_X u^2 \log u \, dm \leq k_1 \log \left( k_2 \frac{\mathcal{E}_p(u)}{\|u\|_2^2} \right)
\]
holds true for any $u \in L^2(X, m)$, where the positive constants $k_1, k_2$ are those appearing in Theorem 2.8.
The thesis is then an immediate consequence of Theorem 10.2 of [2]. In fact, it suffices to use the known contraction properties of $\mathcal{E}$ discussed in [2] and inequality (3.3) above. \hfill \Box

It is remarkable that the above logarithmic Sobolev inequality has exactly the same form as the Euclidean one, proved first in [9] if $p < d$ and later on, with sharp constants, in [14], [16]. In particular the proportionality constant in front of the r.h.s. of (3.2) equals $d/p$ (with the present definition of $d$) as in the Euclidean case.

We are now ready to state the main result of this section: roughly speaking it says that, for the semigroups considered here, hypercontractivity implies ultracontractivity, a property which is clearly false in the linear case. Before stating the Theorem we comment that, by the results of [10], the semigroup considered has a well defined version acting as a strongly continuous contraction semigroup on all $L^p$ spaces, $p \in [1, +\infty)$.

**Theorem 3.2.** Let $\{T_t\}_{t \geq 0}$ be the semigroup associated to the generalized $p$-energy functional $\mathcal{E}$ given in (3.1), with $p > 2$. Assume that it is hypercontractive in the sense of Definition 2.1, with $\dot{\alpha}(0) < 0$, $\dot{\gamma}(0) > 0$. Then, for any $q \in [q, +\infty]$ the following supercontractive and ultracontractive bounds hold true:
\[
\|T_t u\|_q \leq C \frac{\|u\|_{\gamma}}{t^{\alpha}}
\]
for all $q \in L^q(M, m)$, where, for finite $q$,
\[
\alpha = \frac{d}{q} \frac{q - q}{pq + d(p - 2)}, \quad \gamma = \frac{q pq + d(p - 2)}{pq + d(p - 2)}
\]
whereas, for $q = \infty$:
\[
\alpha = \frac{d}{pq + d(p - 2)}, \quad \gamma = \frac{pq}{pq + d(p - 2)}
\]
The proof of the above Theorem can be done mimicking the discussion of [10], since the proofs in that paper did not depend either on the Euclidean setting discussed there or on the explicit, unweighted form of the generator, but only on the validity of (3.2) and on the fact that $\mathcal{E}_p$ is defined in terms of a suitable derivation.

4. **Subordination of nonlinear semigroups**

A well known method for defining, in the linear setting, a functional calculus for generators $A$ of strongly continuous nonexpansive semigroup $\{T_t\}_{t \geq 0}$ (relative to the class of the so called Bernstein functions) is to define a new semigroup $\{T_t^{(f)}\}_{t \geq 0}$ called the Bochner subordinated semigroup [3], and
then to consider its generator, which in fact is a possible definition of \( f(A) \). We briefly recall this construction.

We first consider a \textit{convolution semigroup} of probability measures \( \{\mu_t\}_{t \geq 0} \) on \([0, +\infty)\). By this we mean that

\[
\begin{align*}
\mu_t \ast \mu_s &= \mu_{t+s} \quad \forall s, t > 0 \\
\mu_t &\rightarrow \delta_0 \quad \text{vaguely as } t \to 0
\end{align*}
\]

where \( \delta_0 \) is the Dirac delta at the origin. It is well known (see [19]), that there exists a function \( f \) such that

\[
(L\mu_t)(x) = e^{-tf(x)} \quad \forall t > 0,
\]

where \( L \) denotes the Laplace transform. Moreover \( f \) is well known to be a Bernstein function, i.e. a nonnegative \( C^\infty \) function on \((0, +\infty)\) with

\[
(-1)^k f^{(k)}(x) \leq 0 \quad \forall k \in \mathbb{N}, x > 0.
\]

Clearly any such \( f \) cannot diverge faster then linearly as \( x \to +\infty \).

Then we may define, given a (nonlinear) strongly continuous nonexpansive semigroup \( \{T_t\}_{t \geq 0} \) on a Hilbert space \( L^2(X, m) \) with generator \( A \), the subordinated family

\[
S_t u := \int_0^{+\infty} T_s u \, \mu_t(\,ds),
\]

for all positive \( t \) and all \( u \in L^2 \), provided the above Bochner integral is finite. A sufficient condition for this to hold is that \( \{T_t\}_{t \geq 0} \) is contractive. If \( \{T_t\}_{t \geq 0} \) is nonexpansive, a particularly important case since this property is satisfied by semigroups associated to convex and lower semicontinuous functionals, it suffices that one has in addition \( T_t 0 = 0 \) for all \( t \) to get contractivity as well. More generally (see [19]) one could assume that there exists a bounded orbit for the nonexpansive semigroup \( \{T_t\}_{t \geq 0} \), a condition which then implies that all orbits are bounded. In fact if \( u \) has a bounded orbit and \( v \in L^2 \) is arbitrary, the nonexpansivity of the semigroup yields

\[
\|T_t v\|_2 \leq \|T_t v - T_t u\|_2 + \|T_t u\|_2 \leq \|u - v\|_2 + C
\]

for a suitable \( C \) independent of \( t \). We shall anyway assume in the sequel without further comment that \( S_t u \) is well defined for all positive \( t \) and all \( u \in L^2 \).

Then we define the operator

\[
A_f u := \lim_{t \to 0^+} \frac{u - S_t u}{t} = \lim_{t \to 0} \int_0^{+\infty} T_s u \frac{\delta_0 - \mu_t}{t}(\,ds)
\]

for all those \( u \in L^2 \) for which the limit exists in \( L^2 \). Our aim will be to show for some particularly relevant choices of \( f \) that the limit exists for all \( u \in D(A) \), to prove that it is a monotone operator and to give an explicit formula for it.

We shall use the notation

\[
\nu_t := \frac{\delta_0 - \mu_t}{t}.
\]

Then \( \nu_t \) is a finite Radon measure on \([0, +\infty)\).

Hereafter, we shall also use the notation \( S([0, +\infty)) \) to indicate the space of restrictions to \([0, +\infty)\) of functions belonging to the Schwartz space \( \mathcal{S}(\mathbb{R}) \). Moreover \( \mathcal{S}'([0, +\infty)) \) will denote the space of all tempered distributions on the real line whose support is contained in \([0, +\infty)\). The Laplace transform \( Lu \) of an element \( u \in \mathcal{S}'([0, +\infty)) \) is the analytic function in the open right half-plane \( \mathbb{R} \mathbb{C} : \Re z > 0 \) given by \( Lu(z) = \hat{u}(iz) \), where \( \hat{u} \) denotes the Fourier transform of \( u \). In the sequel we shall sometimes simply write \( S \) and \( S' \) instead of \( S([0, +\infty)) \) and \( \mathcal{S}'([0, +\infty)) \) since no confusion can occur.

**Lemma 4.1.** There exists a tempered distribution \( \nu \in \mathcal{S}'([0, +\infty)) \) such that \( L\nu = f \).
Proof. A well known property of Bernstein functions shows that $f$ can be written as

$$f(x) = a + bx + \int_{(0, +\infty)} (1 - e^{-sx})\mu(ds)$$

where $a, b \geq 0$ and $\mu$ is a nonnegative measure on $(0, +\infty)$ such that

$$\int_{(0, +\infty)} \frac{s}{1 + s}\mu(ds) < +\infty.$$

The formula

$$f(z) = a + bz + \int_{(0, +\infty)} (1 - e^{-sz})\mu(ds)$$

extends $f$ over $\text{Re} z \geq 0$ to an analytic function on $\text{Re} z > 0$. We claim that

$$|f(z)| \leq C(1 + |z|) \text{ a.e. s.t. } \text{Re} z > 0.$$

Indeed, setting $z = x + iy$ for $x \geq 0$:

$$|1 - e^{-sz}|^2 = |1 - e^{-sx} \cos(sy) - ie^{-sx} \sin(sy)|^2$$

$$= (1 - e^{-sx} \cos(sy))^2 + e^{-2sx} \sin^2(sy)$$

$$= 1 - 2e^{-sx} \cos(sy) + e^{-2sx}$$

$$\leq 1 + 2e^{-sx} \left(\frac{s^2y^2}{2} - 1\right) + e^{-2sx}$$

$$= 1 - 2e^{-sx} + e^{-2sx} + e^{-sx}s^2y^2$$

$$\leq (1 - e^{-sx})^2 + s^2y^2 \leq s^2(x^2 + y^2)$$

i.e. $|1 - e^{-sz}| \leq s|z|$ for all $s \geq 0$, $\text{Re} z \geq 0$. Since moreover $|1 - e^{-sz}| \leq 2$ for such $s$, $z$ and then

$$\int_{(0, +\infty)} |1 - e^{-sz}|\mu(ds) \leq 2\mu((1, +\infty)) + |z| \int_{(0,1)} s\mu(ds)$$

(the latter integral being finite by the properties of $\mu$), the claim is proved.

This implies that $f$ is the Laplace transform of a tempered distribution $\nu \in S'([0, +\infty))$ by [22, page 306].

Lemma 4.2. With the above notations, the identification $\nu = \lim_{t \to 0} \nu_t$ holds true in the space $S'([0, +\infty))$.

Proof. By [22] page 307, Remarque 1 we know that a net $\xi_t$ converges to zero in $S'([0, +\infty))$ if:

- $(\mathcal{L}\xi_t)(z)$ converges to zero uniformly over the compact sets of the open right half-plane;
- for any compact interval $[a, b] \subset (0, +\infty)$ there exists a polynomial $p$ depending on $a, b$ such that

$$|(\mathcal{L}\xi_t)(z)| \leq p(y) \quad \forall z = x + iy \in [a, b] \times \mathbb{R}$$

for all $t$ sufficiently small.

We apply this result to the net $\xi_t = \nu_t - \nu$. In fact,

$$(\mathcal{L}\xi_t)(z) = \frac{1 - e^{-tf(z)}}{t} - f(z) = \frac{1 - tf(z) - e^{-tf(z)}}{t}$$
integrating by parts one has

\[
(1 - tx)^2 - 2(1 - tx)e^{-tx} \cos(ty) + e^{-2tx} \cos^2(ty)
+ t^2y^2 - 2tye^{-tx} \sin(ty) + e^{-2tx} \sin^2(ty)
\]

\[
\leq (1 - tx)^2 + 2(1 - tx)e^{-tx}\left(\frac{t^2y^2}{2} - 1\right) + e^{-2tx} + t^2y^2 - 2tye^{-tx} \sin(ty)
\]

\[
= \left[(1 - tx)^2 - 2(1 - tx)e^{-tx} + e^{-2tx}\right]
+ t^2y^2(1 - tx)e^{-tx} + t^2y^2 - 2tye^{-tx} \sin(ty)
\]

\[
= (1 - tx - e^{-tx})^2 + t^2y^2(1 - tx)e^{-tx} + t^2y^2 - 2tye^{-tx} \sin(ty)
\]

\[
\leq c_0 + c_1|y| + c_2y^2
\]

for suitable \(c_0, c_1, c_2 \in \mathbb{R}\) depending on the compact ranges of \(x\) and \(t\). Finally we notice that, setting \(z = e^{it\vartheta}, \vartheta \in (-\pi, \pi)\) and defining \(z^\alpha\) in the open right half-plane as \(z^\alpha := e^{i\vartheta \alpha}\), we have \(\Im(z^\alpha) = e^{i\alpha \vartheta}\) so that \(|\Im(z^\alpha)| \leq e^{a_0 \alpha} \leq a_0 + a_1|y|\).

We shall now specialize to a special and particularly relevant choice of the function \(f\). Namely we shall consider the case \(f(x) = x^\alpha\) for \(\alpha \in (0, 1)\). In this case it is easy to write down an explicit formula for \(\nu\).

**Lemma 4.3.** The function \(f_{\alpha}(x) = x^\alpha\) for \(\alpha \in (0, 1), x \geq 0\) is the Laplace transform of the tempered distribution \(\tau_{\alpha}\) given by

\[
\langle \tau_{\alpha}, \varphi \rangle := \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{\varphi(0) - \varphi(s)}{s^{1+\alpha}} \, ds
\]

for all test function \(\varphi\), where \(\Gamma\) indicates the Euler Gamma function. In particular the net \(\nu_t\) converges to \(\tau_{\alpha}\) in \(S'\) as \(t \to 0\).

**Proof.** By the definition of the Euler Gamma function:

\[
x^{\alpha-1} = \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty e^{-sx} \frac{ds}{s^\alpha}, \quad \forall x > 0.
\]

Thus the function \(x^{\alpha-1}\) is the Laplace transform of the tempered distribution \(\sigma_\alpha(s) = \Gamma(1 - \alpha)^{-1} s_+^{1-\alpha}\). Integrating by parts one has \(\tau_{\alpha} = \sigma_\alpha^\prime\). Thus

\[
\mathcal{L}(\tau_{\alpha})(x) = x\mathcal{L}(\sigma_\alpha)(x) = x^\alpha \quad \forall x > 0.
\]

It is then immediate to check that

\[
\langle (s^{-\alpha})^\prime, \varphi \rangle = \alpha \int_0^\infty \frac{\varphi(0) - \varphi(s)}{s^{1+\alpha}} \, ds.
\]

\[\square\]

Since we aim at making \(\tau_{\alpha}\) act on the function \(s \mapsto T_s u\) for \(u\) in the \(L^2\)-domain \(D(A)\) of the generator \(A\), we have to prove that the convergence of \(\nu_t\) to \(\tau_{\alpha}\) takes place in a stronger sense. To this end we introduce the space

\[
E = \left\{ \psi \in C_b([0, +\infty)) \;\text{s.t.} \; \lim_{t \downarrow 0} \frac{\psi(t) - \psi(0)}{t} \;\text{is finite} \right\}.
\]
Since $\nu_t$ are finite measures on $[0, +\infty)$, $\nu_t(\psi)$ makes sense for all bounded continuous functions $\psi$. Notice also that the r.h.s. of formula (4.6) still makes sense for all $\psi \in E$, thus defining a linear functional on $E$ still denoted by $\tau_\alpha$.

**Lemma 4.4.** For all $\psi \in E$ one has

$$\lim_{t \downarrow 0} \nu_t(\psi) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \psi(s) - \psi(s) \, ds := \tau_\alpha(\psi).$$

**Proof.** We denote by $g_t(x)$ the continuous function defining the density of $\mu_t$ with respect to the Lebesgue measure $[19]$. It is a standard fact that $g_t(x) = t^{-1/\alpha}g_1(xt^{-1/\alpha})$ for all $x \geq 0$, $t > 0$. We shall write $g$ instead of $g_1$ from now on. We shall need a property of $g$, namely the fact that $g(x) \sim c_1x^{-1-\alpha}$ as $x \to +\infty$, where $c_1$ is a suitable positive constant, whose explicit value is known but will not be useful in the sequel [21]. It follows that $\int_0^x yg(y)dy \sim c_2x^{1-\alpha}$ as $x \to +\infty$. This latter fact can also be proved directly by using only the expression of the Laplace transform, noticing that $\frac{d}{dt}\mathcal{L}(\mu_t)(x) = -\mathcal{L}(x\mu_t)(x)$ and using a Tauberian Theorem.

Take now $\psi \in E$, $\varphi \in \mathcal{S}$ and write

$$\int_0^{+\infty} \nu_t(dx)[\varphi(x) - \psi(x)] = \frac{1}{t} \left[ \varphi(0) - \psi(0) - \int_0^{+\infty} g_t(x)[\varphi(x) - \psi(x)]dx \right]$$

$$= \frac{1}{t} \int_0^{1} g_t(x)((\varphi(0) - \varphi(x)) - (\psi(0) - \psi(x))]dx$$

$$= \frac{1}{t} \int_0^{1} g_t(x)((\varphi(0) - \varphi(x)) - (\psi(0) - \psi(x))]dx$$

$$+ \frac{1}{t} \int_1^{+\infty} g_t(x)((\varphi(0) - \varphi(x)) - (\psi(0) - \psi(x))]dx.$$

Define the seminorm $p : E \to [0, +\infty)$ by:

$$p(\psi) := \sup_{x > 0} \left| \frac{\psi(x) - \psi(0)}{x \wedge x^{\alpha/2}} \right|^2.$$

Then we have, for a suitable constant $C_1 > 0$:

$$\left| \frac{1}{t} \int_0^{1} g_t(x)((\varphi(0) - \varphi(x)) - (\psi(0) - \psi(x))]dx \right|$$

$$\leq \left( \sup_{x \in (0,1]} \left| \frac{\varphi(x) - \varphi(0)}{x} \right| \frac{1}{t} \int_0^{1} g_t(x)dx \right) \frac{1}{t} \int_0^{1} g_t(x)dx$$

$$\leq p(\varphi - \psi) \frac{1}{t} \int_0^{1} g_t(x)dx$$

$$= p(\varphi - \psi) \frac{1}{t^{1+1/\alpha}} \int_0^{1} g(xt^{-1/\alpha})dx$$

$$= p(\varphi - \psi) \frac{1}{t^{1-1/\alpha}} \int_0^{1} g(y)dy$$

$$\leq C_1 p(\varphi - \psi).$$
for all $t$ sufficiently small. Similarly, for a suitable constant $C_2 > 0$:

\[
\left| \frac{1}{t} \int_1^{+\infty} g_t(x)[(\varphi(0) - \varphi(x)) - (\psi(0) - \psi(x))]dx \right|
\leq \left( \sup_{x \geq 1} \left| \frac{(\varphi(0) - \varphi(x)) - (\psi(0) - \psi(x))}{x^{\alpha/2}} \right| \right) \frac{1}{t} \int_1^{+\infty} g_t(x)x^{\alpha/2}dx
\leq p(\varphi - \psi) \frac{1}{t} \int_1^{+\infty} g(x)x^{\alpha/2}dx
\]

\[
= p(\varphi - \psi) \frac{1}{t} \int_1^{+\infty} g(x)x^{\alpha/2}dx
\]

\[
\leq p(\varphi - \psi) \frac{1}{t^{1+1/\alpha}} \int_1^{+\infty} g(x)x^{\alpha/2}dx
\]

\[
\leq p(\varphi - \psi) \frac{1}{t^{1/2}} \int_1^{+\infty} g(y)y^{\alpha/2}dy
\]

\[
\leq C_2 p(\varphi - \psi),
\]

again for all $t$ sufficiently small so that, for all such values of $t$:

\[
\left| \int_0^{+\infty} \nu_t(dx)[\varphi(x) - \psi(x)] \right| \leq C p(\varphi - \psi)
\]

for a suitable positive constant $C$. Proceeding in a very similar manner allows to prove the inequality

\[
|\tau_\alpha(\varphi(x) - \psi(x))| \leq C p(\varphi - \psi).
\]

Now we notice that

\[
|\nu_t(\psi) - \tau_\alpha(\psi)| \leq |\nu_t(\varphi) - \tau_\alpha(\varphi)| + |\nu_t(\psi - \varphi)| + |\tau_\alpha(\psi - \varphi)|
\]

\[
\leq |\nu_t(\varphi) - \tau_\alpha(\varphi)| + C p(\psi - \varphi)
\]

so that, by Lemma 4.2

\[
\lim_{t \to 0} |\nu_t(\psi) - \tau_\alpha(\psi)| \leq C p(\psi - \varphi).
\]

Therefore we get $\limsup_{t \to 0} |\nu_t(\psi) - \tau_\alpha(\psi)| = 0$ for all $\psi \in E$ provided we prove that for all positive $\varepsilon$ there exists a function $\varphi \in S$ with $p(\varphi - \psi) < \varepsilon$. It suffices to consider only the case $\psi(0) = 0$ so that $\varphi(0) = 0$ can be assumed as well. Then, let $h \in C^\infty([0, +\infty))$ be such that $0 < h(x) \leq x \land x^{\alpha/2}$ for all $x > 0$ and $h(x) = x \land x^{\alpha}$ for $x \in [0, 1/2] \cup [2, +\infty)$. We have $p(\psi - \varphi) \leq g(\psi - \varphi) := \sup_{x \geq 0} h(x)^{-1}|\psi(x) - \varphi(x)|$ so that it suffices to show that $g(\psi - \varphi)$ is small for $\varphi \in S$ chosen appropriately. Notice that $h(x)^{-1}\psi(x)$ can be extended to a function belonging to $C_0([0, +\infty))$ since $\psi(0) = 0$ and the right derivative of $\psi$ at $t = 0$ exists. In turn this follows from the fact that $S([0, +\infty))$ is dense in $C_0([0, +\infty))$ (the space of continuous functions $g$ on $[0, +\infty)$ such that $\lim_{x \to +\infty} g(x) = 0$) in the uniform topology so that, if $f \in S$ is a function close to the function $h(x)^{-1}\psi(x) \in C_0([0, +\infty))$ in the uniform topology, the function $\varphi = fh$ belongs to $S$ and is close to $\psi$ in the topology associated to the norm $g$.

We have now all the ingredients to prove the following result.

**Theorem 4.5.** Let $\{T_t\}_{t \geq 0}$ be a (nonlinear) strongly continuous nonexpansive semigroup on a Hilbert space $L^2(X, m)$, with generator $A$. Suppose that $\{T_t\}_{t \geq 0}$ has a bounded orbit. Let $\mu_t^{(\alpha)} := t \geq 0$ be the convolution semigroup associated to the Bernstein function $f_\alpha(x) = x^\alpha$ for $x \geq 0$, where $\alpha \in (0, 1]$. Let finally $S_t$ be the subordination of $\{T_t\}_{t \geq 0}$ defined by

\[
S_t u := \int_0^{+\infty} T_s u \mu_t^{(\alpha)}(ds),
\]

Therefore \[
\lim_{t \to 0} \mu_t^{(\alpha)}(ds) = \]
for all \( t \geq 0 \) and all \( u \in L^2 \). Then the right derivative of \( S_t u \) exists at \( t = 0 \) for all \( u \in D(A) \) and, denoting it by \( A^\alpha u \), the formula
\[
A^\alpha u = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u - T_s u}{s^{1+\alpha}} \, ds
\]
holds for any \( u \in D(A) \). Moreover the operator \( A^\alpha : D(A) \to L^2 \) is monotone, so that it admits a maximally monotone extension which then defines a (nonlinear) strongly continuous nonexpansive semigroup \( \{ T^\alpha_t \}_{t \geq 0} \) on \( L^2 \), called the \( \alpha \)-subordinated semigroup.

**Proof.** It suffices to notice that the assumption that \( \{ T_t \}_{t \geq 0} \) has a bounded orbit and the nonexpansivity of \( \{ T_t \}_{t \geq 0} \) imply that all orbits are bounded, so that \( S_t \) is well defined and that, by [8, Theorem 3.1], the map \( s \to T_s u \) is Lipschitz continuous for any fixed \( u \in D(A) \) and has a right derivative at \( s = 0 \) for any fixed \( u \in D(A) \).

The monotonicity of \( A^\alpha \) is a consequence of:
\[
(A^\alpha u - A^\alpha v, u - v) = c \int_0^{+\infty} \frac{ds}{s^{1+\alpha}} \left( \| u - v \|_2^2 - (T_s u - T_s v, u - v) \right)
\]
and of the fact that
\[
(T_s u - T_s v, u - v) \leq \| T_s u - T_s v \|_2 \| u - v \|_2 \leq \| u - v \|_2^2
\]
by the nonexpansivity of \( \{ T_s \} \).

The latter statement follows by [8, Corollary 2.1]. □

**Corollary 4.6.** The inequality
\[
\| A^\alpha u \|_2 \leq \frac{\| A_\circ u \|_2 \sup_{s \geq 0} \| u - T_s u \|_2^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}
\]
where \( A_\circ \) is the principal section of \( A \) in the sense of [8].

**Proof.** One has, for all positive \( x \):
\[
\frac{\Gamma(1-\alpha)}{\alpha} \| A^\alpha u \|_2 \leq \int_0^\infty \frac{\| u - T_s u \|_2}{s^{1+\alpha}} \, ds
\]
\[
= \int_0^x \frac{\| u - T_s u \|_2}{s^{1+\alpha}} \, ds + \int_x^{\infty} \frac{\| u - T_s u \|_2}{s^{1+\alpha}} \, ds
\]
\[
\leq \| A_\circ u \|_2 \int_0^x s^{-\alpha} \, ds + \sup_{s \geq 0} \| u - T_s u \|_2 \int_x^\infty s^{-(1+\alpha)} \, ds
\]
\[
= \| A_\circ u \|_2 \frac{x^{1-\alpha}}{1-\alpha} + \sup_{s \geq 0} \| u - T_s u \|_2 \frac{x^{-\alpha}}{\alpha}.
\]
where we have used the fact (see [8, Theorem 3.1, item (2)]) that \( \| dT_s u / ds \|_{L^\infty(0, +\infty); L^2} \leq \| A_\circ u \|_2 \) for all \( u \in D(A) \). Optimizing over \( x > 0 \) we get the assertion. □

5. **Nash estimates for generators of subordinated semigroups.**

Our aim in this section will be to use the above construction of nonlinear subordinated semigroups in the specific setting of section 4, i.e. when \( \{ T_t \}_{t \geq 0} \) is the nonlinear semigroup generated by the generalized \( p \)-Laplacian introduced in such section. We shall show that, when such semigroup is hypercontractive, its subordination \( \{ T^\alpha_t \}_{t \geq 0} \) is hypercontractive as well, so that as a consequence we shall note that Nash-type inequalities hold for \( A^\alpha \) too.
We start with the following Lemma.

**Lemma 5.1.** Let \( \{ \mu_t : t \geq 0 \} \) be the convolution semigroup associated to the Bernstein function \( f_\alpha(x) = x^\alpha \) for \( x \geq 0 \), where \( \alpha \in (0,1] \) is fixed. Then, for any \( \beta > 0 \) and all \( t > 0 \) the integral \( \int_0^{+\infty} \mu_t(ds)s^{-\beta} \) is finite.

**Proof.** It is clear, since each \( \mu_t \) is a probability measure, that it suffices to prove the claim for \( \beta \geq 1 \). For such \( \beta \) an elementary induction argument and the fact that \( e^{-tx^\alpha} \) is the Laplace transform of \( \mu_t \) show that, if \([\cdot]\) denotes the integer part of a real number:
\[
\int_0^{+\infty} dx x^{\beta-1}e^{-tx^\alpha} = \left( \int_0^{+\infty} dy y^{\beta-\left[\beta\right]}e^{-y} \right) \int_0^{+\infty} \mu_t(ds)\frac{\left[\beta-1\right]!}{s^{\beta}}
\]
so that in particular the latter integral in the r.h.s. is finite. \( \square \)

As a consequence of the above Lemma and of the results of section 2 we have:

**Lemma 5.2.** Let \( \{ T_t \}_{t \geq 0} \) be the semigroup associated to the \( p \)-energy functional \( E \) given in (2.1), with \( p > 2 \). Assume that it is hypercontractive in the sense of Definition 2.1, with \( \delta(0) < 0 \), \( \dot{r}(0) > 0 \). Then the subordinated family \( \{ S_t \} \) satisfies, for all \( \varrho > 2 \), the supercontractive bound
\[
\| S_t u \|_\varrho \leq k_\varrho(t)\| u \|_2^{\gamma(\varrho)}
\]
where
\[
\gamma(\varrho) = \frac{2pq + d(p-2)}{q} \frac{2p + d(p-2)}{2p + d(p-2)}.
\]

**Proof.** This is an immediate consequence of the bound
\[
\| S_t u \| \leq \int_0^{+\infty} \| T_s u \|_\varrho \mu_t(ds) \leq C\| u \|_2^{\gamma(\varrho)} \int_0^{+\infty} s^{-\beta} \mu_t(ds)
\]
for a suitable positive \( \beta \), valid because of the results of Theorem 3.2 and of the above Lemma. \( \square \)

The results of section 2 then can be used directly to prove logarithmic Sobolev inequalities for the right derivative at \( t = 0 \) of \( S_t \), although such map does not give rise to a semigroup. In fact we made no use of the semigroup property there. Noticing in addition that the explicit expression of \( \gamma \) shows immediately that \( \varrho \gamma(\varrho) > 2 \) for any \( \varrho > 2 \), one therefore has, proceeding as in the proof of Theorem 2.1.

**Lemma 5.3.** Let, for \( \alpha \in (0,1) \), \( A^\alpha \) be the maximally monotone operator (see Theorem 4.3) associated to the right derivative at \( t = 0 \) of the subordinated family \( S_t \) defined in the previous Lemma. Then, under the assumptions of such Lemma, the inequality
\[
\frac{\varrho - 2}{\varrho} \int_X u^2 \log |u|dm - \frac{\varrho \gamma(\varrho)}{\varrho} \| u \|_2^2 \log \| u \|_2 \leq (u, A^\alpha u)_{L^2} + C\| u \|_2^2
\]
holds true for each \( u \) in \( D(A) \) and \( \varrho > 2 \).

It may be useful to recall again that \( D(A_\alpha) \supset D(A) \).

To proceed further we now prove subhomogeneity for the functional \( (u, A^\alpha u) \). In fact:

**Lemma 5.4.** With the above notations and assumptions, the operator \( A^\alpha \) enjoys the following property: for any \( u \in D(A) \) and for all positive \( \lambda \) one has
\[
A^\alpha(\lambda u) = \lambda^{1+\alpha(p-2)} A^\alpha u.
\]
Proof. We use the explicit expression for $A_\alpha$ when acting on functions belonging to the domain of $A$ and the fact that

$$T_s(\lambda u) = \lambda T_{\lambda s}^{-2} u,$$

a property which can be verified directly from the differential equation satisfied by $T_s$. In fact:

$$A^\alpha(\lambda u) = c \int_0^{+\infty} \frac{T_s(\lambda u) - \lambda u}{s^{1+\alpha}} \, ds = c\lambda \int_0^{+\infty} \frac{T_{\lambda s}^{-2} u - u}{s^{1+\alpha}} \, ds$$

$$= c\lambda^{1+\alpha(p-2)} \int_0^{+\infty} \frac{T_s u - u}{s^{1+\alpha}} \, ds = \lambda^{1+\alpha(p-2)} A^\alpha u.$$ 

\[\square\]

The following Proposition then follows along the same line of proof given in Theorem 2.9

**Proposition 5.5.** Under the assumptions of Lemma 5.2, the logarithmic Sobolev inequality

$$\int_X \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \, dm \leq A_1 \log \left( A_2 \frac{\langle u, A^\alpha u \rangle_{L^2}}{\|u\|_2^{2+\alpha(p-2)}} \right)$$

holds for any $u \in D(A)$, with $A_1 = 2(1 - \gamma(q))/(\gamma(q)(p-2)(p-2))$ and $A_2$ a suitable positive constant.

**Proof.** First we notice that $\gamma(q) < 1$ for all $q > 2$ and that, as already stated, $q^2\gamma(q) > 2$ for all such $q$. The fact that $(u, A^\alpha u)$ is nonnegative follows from the explicit expression (4.11) of $A^\alpha$ and from the fact that $T_s$ is contractive in $L^2$ because $T_s 0 = 0$ for all $s$ as $A_0 = 0$. Finally, the functional $(u, A^\alpha u)$ is, by the previous Lemma, homogeneous of degree $2 + \alpha(p-2)$ which is then always strictly larger than two.

\[\square\]

We are ready to state the final result of this paper, whose proof is the same given in Theorem 2.9

**Theorem 5.6.** (Nash-type inequalities for generators subordinated to the $p$-Laplacian). Let $(M, g)$ be a complete Riemannian manifold, whose Riemannian gradient is indicated by $\nabla$. Let $m$ be a $\sigma$-finite nonnegative measure on $M$, and assume that $\nabla$ is a closed operator from $L^2(M, m)$ to $L^2(TM, m)$. Consider the strongly continuous contraction semigroup $\{T_t\}_{t \geq 0}$ associated to the subgradient of the convex l.s.c. functional given by

$$E_p(u) := \frac{1}{M} \int_M |\nabla u|^p \, dm.$$ 

Assume that it is hypercontractive in the sense of Definition 2.7 with $\lambda(0) < 0$, $\dot{\lambda}(0) > 0$. Let $A^\alpha$ be the generator of the subordinated semigroup associated to the Bernstein function $f(x) = x^\alpha$ with $\alpha \in (0, 1)$. Then, for any $\theta \in (0, 1)$ the inequality

$$\|f\|_2 \leq c(A^\alpha f, f) L_2^{\frac{\theta}{p-1}} \|f\|_2^{\frac{1-p}{p-1}}$$

holds true for a suitable constant $c$ and for all $f \in \text{Dom} A \cap L^2$, where $A_1 > 0$ is the constant appearing in Proposition 5.5. In particular the Nash inequality

$$\|f\|_2 \leq c(A^\alpha f, f) L_2^{\frac{\alpha}{p-1}} \|f\|_1^{\frac{p-1}{p-1}}$$

holds true for all $f \in \text{Dom} A \cap L^1$.

**Remark 5.7.** An elementary, although tedious, calculation shows that the dimension $d_\alpha$ of the subordinated semigroup is, if $d$ is the dimension of the original semigroup (in the sense given in the statement of Corollary 3.1),

$$d_\alpha = \frac{d[2 + \alpha(p-2)]}{\alpha p}.$$
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