Dynamical Spacetime and the Curvature of Projective State Space

P. Leifer

School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University, Tel-Aviv 69978, Israel

Abstract

If universal quantum interaction is really connected with the coset structure of deformations of quantum states then the curvature of projective Hilbert state space should be observable. I discuss some approach to the measurement of curvature-dependent values.

1 Introduction

In the beautiful popular book [1] Feynman discuss reflection of the light from the glass plate. “Phenomenologically” it may be described as a result of reflection from the front and from the rear surfaces of the plate. Infact the spacetime analysis of amplitudes behavior shows that one should take into account an emissions from all electrons of the plate (local event in spacetime has a nonlocal reason). That is behind very simple rule of addition of two amplitudes there is some geometric picture (arc of small amplitudes) on the complex plane $C^1$.

This example gives us some hint that on the fundamental level evolution of quantum state in the presence of a nonlocal spacetime interaction in a “field cloud” of any quantum particle may have some hidden geometry as well.

On the fundamental level the spacetime integration leads to major difficulties. But this is, as a matter of fact, the unfit task leading to different problem—many body problem, as we have in the case of the glass plate. Therefore, I think, this difficulty is merely artefack. If only fundamental aspect of interaction is really interesting for us then we ought to take into account not the spacetime omnipresence of scatterers, but rather entanglement of internal degrees of freedom. That is only “scattering on the elementary target—the field cloud” with the spatial diameter $r \sim 10^{-13}$cm and
the “defreezing” of internal degrees of freedom due to this “scattering”, one should
looking for.
This “entanglement of internal degrees of freedom” in fact has a geometric char-
acter but spacetime analysis is so restrictive that can not include specific rules for
different kinds of fundamental interaction of elementary particles. This geometry is
unacceptable for unification of quantum interaction and, therefore, for the consistent
foundation of quantum mechanics as well. However the unitary geometry of the pro-
jective Hilbert state space paves the way to some general approach to the dynamics
of quantum states [2, 3, 4, 5, 6, 7].
I argue that geometry of the projective Hilbert space or, maybe better, some spe-
cial case of the Kählerian geometry, has from the physical point of view a dynamical
meaning, namely: Fubini-Study metric induces quantum universal interac-
tion due to the positive holomorphic sectional curvature. This point of view
essentially differs from the statistical interpretation of this metric, say [8, 9, 10, 11].
These differences are as follows:
A. Quantum mechanics must be based on the natural (robust) results of QFT and
symmetries of “elementary particles”. It means that primordial quantum numbers
(integrals of motion) like electric charge, spin, color, beauty, etc, are only “rotated
charges” and entanglement of their amounts “shapes” states of “elementary particles”.

B. The trial process of “shaping” of states of “elementary particles” should be self-
consistent because the changing of numbers of “rotated charges” (creation and decay
of integrals of motion) have dynamical character [12]. Therefore the trial choice of
a basis in Hilbert space and the stationary choice of the superposition of the basis
states can not be identified with an “elementary particle” themselves.

C. Since a priori we know neither correct vacuum state nor appropriate set of an
immanent dynamical variables related to conservation and deformation of this vacuum
state, one should use a local trial variables. The deformations of a superposition
state of charges have coset structure [4, 5, 6, 7]. Therefore they may be labeled by
the points of the projective Hilbert space $CP(N)$ with Fubini-Study metric which
defines a fundamental interactions between charges. Local dynamical variables shape
a moving frame and some of them look like creation-annihilation of “elementary
particles”.

D. The holomorphic sectional curvature of $CP(N)$ is identified with the intensity
of fundamental interaction constant (fine structure constant, for instance) not with
the inverse Planck constant ($\kappa = 2/\hbar$) (to compare with [8], for example).

E. In this theory there is a natural affine connection which expresses as a func-
tion of the metric tensor of Fubini-Study. Therefore not the state vector itself (in
accordance with the ideology of Berry-Aharonov-Anandan [13, 14]) subjected to
comparison by the parallel transport, but those tangent vector fields on the state
manifold that take the place of dynamical variables.

F. Spacetime structure is a derivable entity. All paradoxical results like “fasten-
than-light-telegraph” [3] or “Everett phone” of [10] are rooted in the nonadequacy of assumptions about relationships between nonlinear quantum dynamics itself and
their spacetime presentation.

1.1 Affine Connection in CP(N), Setup Agreement and non-
Abelian Gauge Theory

I will try show that the root of difficulties in interpretation of both ordinary (linear) quantum mechanics and its nonlinear generalization [17] is the neglect of general properties of the comparison procedure of quantum dynamical variables.

The problem of the comparison of quantum states is not trivial one. As a matter of fact this lies in the basis of the measurement problem in quantum mechanics and closely connected with the EPR problem [18]. Let me use some passage from the article of Gisin [15]. ‘The experimental testing quantum mechanics against local hidden variables do not only violate the Bell inequality, but they also agree remarkably well with quantum mechanics. This supports the claim that if one spin of a singlet state pair is “found” to be in the up state, then the other spin is in the down state, for the same direction’ (it is my italization P.L.). The question is: what is ‘same direction’? This is the crucial point because this notion should have a physical meaning [18, 4, 5, 6]. The comparison of ‘z-direction’ at A and B is, as a matter of fact, the comparison of directions of physical fields. Since fields have indefinite numbers of degrees of freedom, a “parallel transport” has to be done in the projective Hilbert state [5, 6]. That is our credo in some “a priori spacetime geometry” must be subjected to verification and just result of such quantum measurement gives us a possibility to judge whether this is the “same direction” or not. Furthermore, we have not any a priori geometry of spacetime and should construct it basing on quantum setup [19].

Now we will discuss the procedure of the comparison of local (in $CP(N)$) dynamical variables. Let us assume we have the two separated in ordinary (spacetime) sense setups A, B (like spectrum analyzer of NMR or detectors, say, $K$-mesons). Their spacetime separation has explicit expression in internal (quantum) terms and they will be describe a little bit later. I will describe quantum dynamics of “spin” $S = \frac{N-1}{2}$ states in terms of relative amplitudes $\Pi_A$ and $\Pi_B$. In this case the $CP(N)$ projective Hilbert space takes the place of the base manifold of the tangent fiber bundle. If we have different states of “spins” in A setup and B setup, then we have $\Pi_A \neq \Pi_B$. But even if one has the coincidence of the “spin states” he has not degeneration since in our scheme A and B are not merely labels. They are sets of the physically distinguishable parameters \{A\} = \{U_A(1) \times U_A(N), SU_A(N + 1)/S[U_A(1) \times U_A(N)]\}, \{B\} = \{U_B(1) \times U_B(N), SU_B(N + 1)/S[U_B(1) \times U_B(N)]\} in fibers and coset transformations in the base manifold $CP(N)$. Therefore if even $\Pi_A = \Pi_B$, this means that one has different polarizations in the same fiber over general $\Pi^i$ because one should to
compare dynamical variables and this procedure is possible only after parallel transport these dynamical variables in, say, A setup. Of course, a priori there is no any physical connection between relative amplitudes \( \Pi_A^i \) and \( \Pi_B^i \). But a physical experience says us that a quantum transition in the setup A may induce a quantum transition in the setup B by some physical gauge field transferring an interaction. In our case this interaction related to deformation of quantum state [3, 5, 6, 7]. This is the problem of “internal quantum dynamics” and it should be solved now in the internal sense of “\( \{A\} - \{B\} \) spacetime separation”.

In both special and general relativity the clock synchronization is an important procedure. In our case we should agree of setups \( \{A\} \) and \( \{B\} \). This process includes the choice of “vacuum state” \( |\Psi^a\rangle \), for example, and the choice of the “axis of quantization”— direction of the field for the creation of inversion (field along the \( Z_A \), for example). Of course, in the \( \{B\} \) setup one can choose different “vacuum state” \( |\Psi^b\rangle \) and direction of the field along, say, \( X_B \). Then the relative amplitudes should be calculated in some single chart, say, in the chart \( U_a : \Psi^a \neq 0 \), where one has

\[
\Pi_i^{(a)} = W_a^b \Pi_i^{(b)},
\]

where \( W_a^b = \Pi_i^{(a)} \), and difference in the field directions should be taken into account under the comparison of the tangent vector fields over \( CP(N) \).

This means that in the framework of my model I intend to use the comparison of not quantum states (rays) themselves [13, 14] but dynamical variables which correspond their deformations [3, 5, 6, 7]. Therefore the natural connection in \( CP(N) \)

\[
\Gamma_{kl}^i = -2(\delta_k^i \Pi^k + \delta_l^i \Pi^k)(R^2 + \sum_s \Pi^2)^{-1}
\]

(1.2)
corresponding to the Fubini-Study metric (3.2) plays an important role in the process of the comparison of these dynamical variables. How the gauge field in “reference Minkowski spacetime” may arise under the local “gauge transformations” of the functional frame has been shown in [3]. It is akin the non-Abelian gauge potential of Wilczek-Zee [20]. Namely, we have shown that the connection (1.2) determines the natural intrinsic gauge potential of a local frame rotation in a tangent space of \( CP(N) \) and, therefore, modification of field dynamical variables. Relationships between the Goldsone and Higgs modes arise in an absolutely natural way.

Now I will build the tangent fiber bundle over \( CP(N) \) related to the process of the comparison of dynamical variables arising at two quantum transitions (events). For a simplicity we will compare dynamical variables associated with the transition
in the “vacuum” state

\[ |\Psi_0> = \left( e^{i\omega(|\Psi|)} \sum_{a=0}^{N} |\Psi_a|^2 \right) = Re^{i\omega(|\Psi|)} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{1.3} \]

and dynamical variables associated with the transition in the state

\[ |\Psi(f_1, ..., f_N; \tau) > = Re^{i\omega(|\Psi|)} \begin{pmatrix} \cos \Theta \\ \frac{f_1}{g} \sin \Theta \\ \frac{f_2}{g} \sin \Theta \\ \vdots \\ \frac{f_N}{g} \sin \Theta \end{pmatrix}, \tag{1.4} \]

which belongs to the geodesic emitted from the “vacuum” state. It is known that this geodesic is generated by the unitary matrix \( \hat{T}(\tau, g) = \exp(i\tau \hat{B}) = \)

\[ \begin{pmatrix} \cos \Theta & \frac{-f_1}{g} \sin \Theta & \cdots & \frac{-f_N}{g} \sin \Theta \\ \frac{f_1}{g} \sin \Theta & 1 + \left[ \frac{|f_1|}{g} \right]^2 (\cos \Theta - 1) & \cdots & \frac{f_N}{g} \sin \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_N}{g} \sin \Theta & \frac{f_1 f_N}{g^2} (\cos \Theta - 1) & \cdots & 1 + \left[ \frac{|f_N|}{g} \right]^2 (\cos \Theta - 1) \end{pmatrix}, \tag{1.5} \]

where \( g = \sqrt{\sum_{k=1}^{N} |f_k|^2}, \Theta = g\tau \) \{5, 6\}. It is clear that in the framework of the map \( U_0 : |\Psi^0\rangle \neq 0 \) all states with the norm \( R \) may be spanned by a geodesic of \( CP(N) \) emitted from \( (0, ..., 0) \) corresponding \{1.3\}. Now we have to have local dynamical variables subjected to parallel transport along this geodesic. In the linear fundamental representation of the action of \( SU(N + 1) \) one has

\[ |\Psi(s) > = \exp(-\frac{i}{\hbar} s \hat{P})|\Psi >, \tag{1.6} \]

where \( \hat{P}, ..., \hat{Q} \in AlgSU(N + 1) \) are “polarization operators” \( \hat{P} = \mu H^\sigma \hat{\lambda}_\sigma \in AlgSU(N + 1) \) which depend on external “multipole magnetic” or “gluon” field \( H^\sigma \), \( 1 \leq \sigma \leq N^2 + 2N \) and does not depend on the state of the quantum system. Under an appropriate choice of units, \( s \) is a proper time of the setup \( B \). Since \( \hat{\lambda}_\sigma \) matrices are “global”, they give the illusion of the omnipresence of a “spin” degrees of freedom. But in the nonlinear representation (realization) of the group symmetry the infinitesimal operators of the transformations depend on the state and thereby local
dynamical variables are not separable from the state. Then a real compound system will be in a self-consistent state. This property demolishes any (real, of course, not gedanken!) attempts to combine a compound system in a priori chosen states. Ordinary quantum ideology accepts this possibility and this leads to EPR paradox in linear quantum mechanics and to difficulties in its nonlinear versions. My theory seems to be very “rigid” construction which evidently contradicts our experience. **One can avoid this contradiction assuming that we have not spacetime background on quantum level with a priori structure.** A new construction of spacetime we will discuss in the paragraph 2.

Returning to the fiber bundle, we should obtain a “point” of the tangent bundle corresponding \( \tau \). Here \( \tau \) is the parameter of action which takes the place of the “universal time” of Horwitz [21]. In order to do it one must parallel transport a tangent space from (1.4) to (1.3). As a matter of fact we should parallel transport of rates of a state vector changing \( |v(s)\rangle = -(i/\hbar)\hat{P}|\Psi(s)\rangle \). \( \text{(1.7)} \)

The “descent” of the vector field \( |v(s)\rangle \) onto the base manifold \( CP(N) \) is a mapping by the two formulas:

\[
\begin{align*}
\Psi^0 &= \frac{R^2}{\sqrt{\sum_{s=1}^N |\Pi^s + \Delta \Pi^s|^2 + R^2}}, ..., \\
\Psi^i &= \frac{R}{\sqrt{\sum_{s=1}^N |\Pi^s + \Delta \Pi^s|^2 + R^2}}, \quad (\Pi^1, ..., \Pi^N), \quad (1.8)
\end{align*}
\]

and

\[
\tilde{\xi} = f_*(\Psi^0, ..., \Psi^N)|v(s)\rangle = \frac{d}{ds}(R \frac{\Psi^1}{\Psi^0}, ..., R \frac{\Psi^N}{\Psi^0}) |0\rangle = -(i/\hbar)[R P^1_0 - P^0_0 \Pi^1 + (P^1_k - (1/R)P^0_0 \Pi^k) \Pi^k, ..., \\
R P^N_0 - P^0_0 \Pi^N + (P^N_k - (1/R)P^0_0 \Pi^N) \Pi^k]. \quad (1.9)
\]

The *restriction* of these mappings onto the our geodesic is interesting for us. Now after a small shift along geodesic we should “lift” the new tangent vector \( \xi^i + \Delta \xi^i \) into the original Hilbert space \( \mathcal{H} \), that is, one needs to realize two inverse mappings:

\[
\begin{align*}
f^{-1} : CP(N) &\rightarrow \mathcal{H} \text{ at point } \Pi^i + \Delta \Pi^i \text{ by the formula} \\
\Psi^0 &= \frac{R^2}{\sqrt{\sum_{s=1}^N |\Pi^s + \Delta \Pi^s|^2 + R^2}}, ..., \\
\Psi^i &= \frac{R}{\sqrt{\sum_{s=1}^N |\Pi^s + \Delta \Pi^s|^2 + R^2}}, \quad (\Pi^1 + \Delta \Pi^1, ..., \Pi^N + \Delta \Pi^N) \rightarrow [\Psi^0 + \frac{\partial \Psi^0}{\partial \Pi^i} \Delta \Pi^i, ..., \Psi^N + \frac{\partial \Psi^N}{\partial \Pi^i} \Delta \Pi^i]. \quad (1.10)
\end{align*}
\]

or in the first approximation

\[
f^{-1} : (\Pi^1 + \Delta \Pi^1, ..., \Pi^N + \Delta \Pi^N) \rightarrow [\Psi^0 + \frac{\partial \Psi^0}{\partial \Pi^i} \Delta \Pi^i, ..., \Psi^N + \frac{\partial \Psi^N}{\partial \Pi^i} \Delta \Pi^i]. \quad (1.11)
\]

and then

\[
f^{-1}_*(\xi + \Delta \xi) = [v^0 + \Delta v^0, v^1 + \Delta v^1, ..., v^N + \Delta v^N]
\]
\[ \frac{\partial \Psi_{0}}{\partial \Pi^{i}}(\xi^{i} + \Delta \xi^{i}), \frac{\partial \Psi_{1}}{\partial \Pi^{i}}(\xi^{i} + \Delta \xi^{i}), \ldots, \frac{\partial \Psi_{N}}{\partial \Pi^{i}}(\xi^{i} + \Delta \xi^{i}) \]. \tag{1.12} \]

It may be shown that under the parallel transport of the \( \vec{\xi} \) along a smooth curve, one has
\[ \Delta \xi^{i} = \xi^{i}(\tau) - \xi^{i}(0) = - \int_{0}^{\tau} \Gamma_{kl}^{i} \xi^{l} \frac{d\Pi^{k}}{dl} dl, \tag{1.13} \]
and, therefore, in the first approximation
\[ |\delta v(\tau)| = - \Gamma_{kl}^{i} \xi^{i}(\tau) \frac{\partial \Psi_{a}(\tau)}{\partial \Pi^{i}} |a| > . \tag{1.14} \]

This evolution effectively defines the map of the local vector field of dynamical variables \( \xi^{i} \) in \( \text{CP}(N) \) to the dynamically shifted states \( |\Psi + \Delta \Psi| \) in the original Hilbert space just along a geodesic (section of bundle).

Let me now to compare two dynamical variables
\[ D_{\sigma}(\hat{P}) = \Phi_{i}^{\sigma}(\Pi, P) \frac{\delta}{\delta \Pi^{i}} + \Phi_{i}^{\ast \sigma}(\Pi, P) \frac{\delta}{\delta \Pi^{i}}, \tag{1.15} \]
and
\[ D_{\sigma}(\hat{Q}) = \Phi_{i}^{\sigma}(\Pi, Q) \frac{\delta}{\delta \Pi^{i}} + \Phi_{i}^{\ast \sigma}(\Pi, Q) \frac{\delta}{\delta \Pi^{i}}, \tag{1.16} \]
(tangent vector fields) corresponding two quantum transitions in different quantum states by the parallel transport in the “vacuum” state. Here
\[ \Phi_{i}^{\sigma}(\Pi; P) = R \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \frac{[\exp(i\epsilon P_{\sigma})]_{m}^{i}}{[\exp(i\epsilon P_{\sigma})]_{m}^{i}} \Psi^{m} - \Psi^{i} \right\} = \lim_{\epsilon \to 0} \epsilon^{-1} \{ \Pi^{\prime}(\epsilon P_{\sigma}) - \Pi^{i} \} \tag{1.17} \]
are the local (in \( \text{CP}(N) \)) state-dependent components of the \( SU(N + 1) \) group generators, which are studied in \([4, 8, 11]\). The connection between \( \Phi_{i}^{\sigma}(\Pi, P) \) and \( \xi^{i} \) is simply \( \frac{\partial \Psi^{m}}{\partial \sigma} = \xi^{i} = \Phi_{i}^{\sigma}(\Pi, P) \omega^{\sigma} = \mu \Phi_{i}^{\sigma}(\Pi, P) H^{\sigma} \).

It is very important that there are transformations from the isotropy group of the “vacuum” state that leave (1.3) intact but rotate geodesic spanning (1.3) and (1.4). This fact seems to be paves the way to the introduction of dynamical spacetime of the ordinary dimension–4, since gives a possibility to truncate the multilevel amplitudes of traversal of the geodesic up to two-level. In the general case geodesic obeys equations
\[ \frac{d^{2}\Pi^{i}}{dl^{2}} + \Gamma_{kl}^{i} \frac{d\Pi^{k}}{dl} \frac{d\Pi^{l}}{dl} = 0, \text{c.c.,} \tag{1.18} \]
which in particular case CP(1) is simply
\[ \frac{d^{2}\Pi}{dl^{2}} = \frac{2\Pi^{\ast}}{R^{2} + |\Pi|^{2}} \left( \frac{d\Pi}{dl} \right)^{2}, \text{c.c.} \tag{1.19} \]
with the solution

\[ \Pi(l) = Re^{i\alpha} \tan(l). \]  

(1.20)

One can render the solution of general equation (1.18) into solution of (1.19) by ansatz of the “squeezing” of full state vector (1.4) to the “two-level state” as followes. The first “squeezing” unitary matrix is

\[
\hat{G}_1^+ = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 1 & 0 \\
0 & 0 & \ldots & 0 & -e^{-i\alpha_1} \sin\phi_1 & \cos\phi_1
\end{pmatrix}
\]  

(1.21)

This matrix acts on the state vector (1.4) with the result

\[
\hat{G}_1^+ |\Psi> = \begin{pmatrix}
\cos\Theta \\
\frac{f_1}{g} \sin\Theta \\
\vdots \\
\frac{f_{N-1}}{g} \sin\Theta \cos\phi_1 + \frac{f_N}{g} \sin\Theta e^{i\alpha_1} \sin\phi_1 \\
\frac{f_{N-1}}{g} \sin\Theta e^{-i\alpha_1} \sin\phi_1 + \frac{f_N}{g} \sin\Theta \cos\phi_1
\end{pmatrix}
\]  

(1.22)

Now one has solve two “equations of annihilation” \[ \Re(-\frac{f_{N-1}}{g} \sin\Theta e^{-i\alpha_1} \sin\phi_1 + \frac{f_N}{g} \sin\Theta \cos\phi_1) = 0 \] and \[ \Im(-\frac{f_{N-1}}{g} \sin\Theta e^{-i\alpha_1} \sin\phi_1 + \frac{f_N}{g} \sin\Theta \cos\phi_1) = 0 \] in order to eliminate the last string and to find \(\alpha'_1\) and \(\phi'_1\). That is one will have a squeezed state vector

\[
\hat{G}_1^+ |\Psi> = \begin{pmatrix}
\cos\Theta \\
\frac{f_1}{g} \sin\Theta \\
\vdots \\
\frac{f_{N-1}}{g} \sin\Theta \cos\phi'_1 + \frac{f_N}{g} \sin\Theta e^{i\alpha'_1} \sin\phi'_1 \\
0
\end{pmatrix}
\]  

(1.23)

The next step is the action of the matrix with the shifted transformation block

\[
\hat{G}_2^+ = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 1 & 0 \\
0 & 0 & \ldots & 0 & -e^{-i\alpha_2} \sin\phi_2 & \cos\phi_2
\end{pmatrix}
\]  

(1.24)
on the vector (1.23) and the evaluation of $\alpha'_2$ and $\phi'_2$ and so on till the initial vector (1.4) will be reduced to the following form

$$|F(f^1, ..., f^N; \tau) > = \text{Re} i^{(\omega(\Psi)+\epsilon(f^1, ..., f^N))} \begin{pmatrix} \cos \Theta \\ \sin \Theta \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

(1.25)

That is $|F(f^1, ..., f^N; \tau) > = \hat{G}^{-1} |\Psi >$, where $\hat{G} = \hat{G}_1 \hat{G}_2 ... \hat{G}_N$. It is easy to see that the functions $(\Pi(\Theta), 0, ..., 0)$ being substituted into (1.18) where

$$\Pi(\Theta) = \text{Re} i^{(\omega(\Psi)+\epsilon(f^1, ..., f^N))} \tan(\Theta)$$

(1.26)

is solution of the equations (1.18). This set is reduced now to the single equation

$$\frac{d^2 \Pi(\Theta)}{d\Theta^2} - \frac{2\Pi'(\Theta)}{R^2 + |\Pi(\Theta)|^2} \left( \frac{d\Pi(\Theta)}{d\Theta} \right)^2, \text{c.c.}$$

(1.27)

This reduction to the single complex local coordinate $\Pi(\Theta)$ which accumulated a full information about initial and finite multilevel states is basis for integration over dynamical spacetime in order to take into account entanglement of internal degrees of freedom which have been mentioned above. The “direct” comparison of the rates of quantum transitions is possible only in the original Hilbert space by the compensation of the geodesic shift with the help of rotations of the functional frame $\{|a>\}$. This is equivalent to the variation of the “multipole magnetic” or “gluon” field $H^\sigma \rightarrow (H + \delta H)^\sigma$ in order to reduce to zero the difference between parallel transported from (1.4) to (1.3) dynamical variables $\xi$ and dynamical variables at the “vacuum” state (zero method of measurement which gives explicit answer “yes” or “no”). Here one finds that this variation is

$$\delta H = \frac{1}{\mu} \delta U = \frac{1}{\mu} A_m \delta \Pi^m = \frac{1}{\mu} \frac{\delta U}{\delta \Pi^m} \delta \Pi^m = -\frac{\hbar}{\mu} \Gamma_{km}^i \delta \Pi_i^a \delta \Pi^m |a>$$

(1.28)

may be connected with the “instantaneous” self-interacting potential associated with the infinitesimal gauge transformation of the local frame with the coefficients (1.2) [7]. Dynamical description of this gauge field requires the “internal” introduction of spacetime coordinates in pure quantum manner.

## 2 Introduction of Dynamical Spacetime

In my previous works the “reference Minkowski spacetime” has been introduced in order to connect the “internal dynamics” of the relative Fourier components of relativistic scalar field and their spacetime propagation [4, 5, 6]. This approach is not, of
course, logically consistent because my claim is “to forget about spacetime priority”. It means that quantum state contains in some sense a dynamical spacetime position of transition (event). Therefore dynamical spacetime coordinates ought to be the functions of relative amplitudes $\Pi^i$. Note that a necessity to build physics in the absence of a background spacetime geometry has already been discussed (see [19] and bibliography therein).

Now I intend to introduce dynamical spacetime which is based on the method of the “logical spin 1/2”. Briefly this method was mentioned at the end of the Section 6 of my article [4].

In order to unify quantum theory and relativity we have to have physical elements appropriate in both these theories. Event is undefinable primordial element in both special (SR) and general relativity (GR) [22] but in quantum case an event is a quantum transition. In SR and GR event is point of spacetime. In quantum theory a transition is not already pointwise element but it may be represented as the infinitesimal deformation of quantum generalized coherent states in projective Hilbert space by coset generators [3, 4, 7]. In SR and GR one can not say about absolute time interval or spatial length between events but they are fundamental notions in the framework of the theory. In quantum theory we have quite different situation: some pure quantum variables are fundamental notion and spacetime interval is a derivable entity. I think such the most appropriate variable is (observable) the dipole moment of transition, related to spatial length by a simplest but not unique way $\vec{d} = e \vec{x}$. It is well known that in atomic physics this dipole moment may be expressed, say, in terms of Einstein coefficients as follows

$$B_{if} = \frac{2\pi}{3\hbar^2} |ex_{if}|^2. \quad (2.1)$$

The dipole moment of transition may be expressed in terms of pure quantum relative amplitudes as well. However the dipole moment is only part of the matrix element of the Hamiltonian of interaction a quantum system with the quant of a gauge field, photon, for example, $e < f | \vec{u}_k \vec{v} (\Pi) | i >$ where the matrix elements of radius-vector

$$< f | x + iy | i > = \int_0^\infty drr^3\phi_i\phi_f \oint Y_{l,m}^*Y_{l,m} \sin \theta d\Omega, \quad (2.2)$$

$$< f | z | i > = \int_0^\infty drr^3\phi_i\phi_f \oint Y_{l,m}^*Y_{l,m} \cos \theta d\Omega, \quad (2.3)$$

are expressed in the terms of spherical functions and are used for the selection rules. This is the consequence of ordinary assumptions: the classical electrodynamical form of interaction energy and pseudo-Euclidean scalar product in four-dimensional space-time. In such way we can obtain only the photon-like dispersion law and ordinary Lorentz group which conserves the light cone. The question, however, is: are these assumptions really correct at an arbitrary short distances i.e. under a deep inelastic interaction? Metric of the dynamical spacetime certainly
depends on physical conditions of setup at the quantum level, i.e. the correspondence between these forms in spacetime and vectors of energy-momentum (spacetime metric) \[23\] and it presumably paves the way to the consistent theory of quantum gravity.

One should take into account that spherical functions depend only on two angles \(\theta\) and \(\phi\) corresponding a representation of spatial rotations in complex Hilbert space. But our approach is opposite: we try to represent unitary group \(SU(N + 1)\) by generalized Lorentz transformations with appropriate rates. A priori all parameters of “internal” unitary group \(SU(N + 1)\) have not spacetime sense like angles \(\theta\) and \(\phi\), etc., and only after establish of the generalized Lorentz transformation which lead to explict “answer”, one can restore spacetime picture of a pre-history of the event-quantum transition.

In ordinary quantum mechanics one has the wave function \(\Psi(x) = \langle x|\Psi \rangle\) describing spacetime (or space) distribution of quantum system. But what is \(|x\rangle\)? Physically it means that there is some more or less localizable (in macroscopic spacetime scale at distance “x” from “0”) quantum system –“detector”, which after interaction with our system may change its internal state. This interaction in general may change not only the internal state of the detector but the position “x” as well. However this is not so important for us. For us is very important only that the changing of the quantum state is essential fact, rather than any spacetime fixing of events. In some sense we can, however, to ascribe to a quantum transition (event) “spacetime coordinates” \(x, y, z, ct\). This procedure is based upon the identification of “coincidence of “answers” (“yes”=|1 \rangle, “no”=|0 \rangle) on “quantum question”. We must take into account this fact by introduction the space of coherent states \(|\alpha \rangle = \alpha^1|1 \rangle + \alpha^0|0 \rangle\) of the “logical spin 1/2” as a two-level system in the basis \(|1 \rangle, |0 \rangle\}. Then spinor \((\alpha^0, \alpha^1)\) defines a point \(\pi = \alpha^1 / \alpha^0\) of the space of coherent states \(CP(1)\). Therefore, under sharp tuning of the setup one can ascribe effective vector of polarization (dipole moment of transition)

\[
\begin{align*}
P_1(\pi) &= \frac{x}{ct} = \frac{\pi + \pi^*}{1 + |\pi|^2} \\
P_2(\pi) &= \frac{y}{ct} = -i \frac{\pi - \pi^*}{1 + |\pi|^2} \\
P_3(\pi) &= \frac{z}{ct} = \frac{1 - |\pi|^2}{1 + |\pi|^2}.
\end{align*}
\tag{2.4}
\]

and after that corresponding dynamical spacetime distance in appropriate normalization

\[
\hat{X} = \left(\begin{array}{cc}
ct + z & x - iy \\
0 & ct - z
\end{array}\right).
\tag{2.5}
\]

Then the unitary transformations of this coordinate matrix

\[
\hat{X}' = \hat{L}X\hat{L}^*
\tag{2.6}
\]
we will interpret as a two-sheeted covering of **Lorentz group** which conserves a light cone
\[ \det \hat{X}' = \det \hat{L} \hat{X} \hat{L}^* = \det \hat{X} = c^2 t^2 - x^2 - y^2 - z^2, \] (2.7)
and which say how one should orient oneself own setup and the velocity with which it must move in order to get an explicite answer “yes”=|1 > or “no”=|0 >. The formulas (2.4) are an analog of the well known matrix elements of radius-vector (2.2) and (2.3) which are expressed in terms of the relative amplitudes of transition. The connection of de-Broglie wave phase and their surfaces (form) in spacetime for motion of a “quantum particle” and the method of spacetime introduction infact has already been described [23]. But we should remember that only a two-level approximation has been used and there are different degrees of freedom that are now outside of our coherent state space \( CP(1) \) of the “logical spin 1/2” which is a “support” of the quantum dynamical spacetime. That is under the “defreezing” of a multipole interaction (it is possible only by taking into account higher three- etc.-level approximations [24]), description in dynamical spacetime is very pale. **Therefore behind Lorentz group there is more wide group structure.** The Lorentz group is only an “inverse representation” of this structure in four-dimensional spacetime: coset transformations in projective Hilbert space \( CP(N) \) and isotropy group transformations in fiber bundle over \( CP(N) \) should be represented in \( CP(1) \) and only after that in the momentum or coordinate spaces. There is a very interesting consequence of this structure: the magnitude of a distance in the dynamical spacetime (defined by the dipole moment of transition) depends not only on relative motions of setups but on the dynamics of multipole moments–it may be subjected to the analog of the Fitzgerald-Lorentz contraction with the increasing of quadrupole, octupole etc. multipole components. This is some justification of title “Super-relativity” but I admit that the prefix “super” is misleading.

### 3 Testing of the Metric Nonlinearity

Our construction based on the assumption of physically important role of the nonlinearity of the curved Kähler state space (projective Hilbert space \( CP(N) \)). There was attempts to find some evidence of the nonlinearity of Weinberg’s form [17]. Such kind of deviations from linearity have not been found [25]. I propose to check different kind of nonlinearity (metric) which connected with the curvature of the Kähler state space.

One can express infinitesimal invariant interval in the original Hilbert space (chord) as followes
\[ \delta L^2 = \delta_{ab} \delta \Psi^a \delta \Psi^b = G_{ik*} \delta \Pi^i \delta \Pi^{*k} = \sum_a \frac{\partial \Psi^a}{\partial \Pi^i} \frac{\partial \Psi^{*a}}{\partial \Pi^{*k}} \delta \Pi^i \delta \Pi^{*k} \] (3.1)
That is the generalized metric tensor of the original flat Hilbert space in the local coordinates $\Pi$ is

$$G^{H}_{ik\ast} = \sum_{a=0}^{N} \frac{\partial \Psi^a}{\partial \Pi^i} \frac{\partial \Psi^{\ast a}}{\partial \Pi^{\ast k}} = R^2 \frac{(\sum_{s=1}^{N} |\Pi^s|^2 + R^2)\delta_{ik} - \frac{3}{4}\Pi^s\Pi^k}{(\sum_{s=1}^{N} |\Pi^s|^2 + R^2)^2}. \quad (3.2)$$

I propose to compare in a physical experiment the full invariant interval under deformations $\Pi^i$ of the initial state $|\Psi_0 > \delta L^2$ in original Hilbert space and the interval

$$dl^2 = R^2 \frac{(\sum_{s=1}^{N} |\Pi^s|^2 + R^2)\delta_{ik} - \Pi^s\Pi^k}{(\sum_{s=1}^{N} |\Pi^s|^2 + R^2)^2} \delta \Pi^i \delta \Pi^{\ast k} = G^{P}_{ik\ast} \delta \Pi^i \delta \Pi^{\ast k} \quad (3.3)$$

in the projective Hilbert space $CP(N)$ (arc).

In accordance with our approach one should to compare local dynamical variables i.e. tangent fields associated with the deformations of quantum state $|\Psi >$. We should to compare these fields in respect with the affine connection (1.2). The scalar products of two rates has the sense of a frequensy “correlation”

$$f^2_P = \frac{1}{4\pi^2} G^{P}_{ik\ast} \xi^i \eta^{\ast k}. \quad (3.4)$$

The maximum of the difference between (3.4) and

$$f^2_H = \frac{1}{4\pi^2} G^{H}_{ik\ast} \xi^i \eta^{\ast k}. \quad (3.5)$$

lies in $|\Pi| = R$. That is under traversing of the relative amplitudes one can see this difference if, of course, there is a possibility to realize physically a comparable dynamics in ordinary (flat) Hilbert space and in projective Hilbert space.

## 4 Discussion

It is very interesting to proof that 2-level restriction of whole N+1-level state in general case leads not only to dynamical spacetime but to the “probability” as well. This maybe because there are a degrees of freedom for the arbitrary “orientation” of the quantum setup relative to, say, vector energy-momentum and relative to surfaces of form in the local dynamical spacetime and, therefore, the process of the restoration of the pre-history of quantum event–transition is not unique.

In order to clarify my approach to this problem I should make a short explanation [2, 3, 6].

This concerns a simple fact that quantum state of any 2-level system may be presented by points of the $CP(1)$ or its realization as Poincaré sphere $S^2$ [13]. Physically coordinates of each point of $S^2$ determine the shape of the ellipse of polarization and
its orientation. Any “evolution” of quantum state (including the changing polarization character) may be labeled by points of $CP(1)$. There are only two elementary kinds of the state “evolution”:

1. Motion of the ellipse of polarization along one of the parallel of latitude without deformation (only rotation with the shape conservation);

2. Motion of the ellipse of polarization along one of the meridian with arbitrary strong deformations of the shape—from the right circuit polarization through right elliptic, linear, left elliptic, to the left circuit polarization.

These very well known facts closely connected with the invariant properties of $Z_2$-graduated algebra $AlgSU(2)$ and geometry of the projective Hilbert space. I intend to generalize this picture in the case of $SU(N+1)$ [2, 3, 6] because this generalization presumably paves the way to the consistent quantum formalism and to the “internal” manner of arising spacetime from the pure internal degrees of freedom.

That is I do not intend here to solve the Pauli problem [26]. In opposite,–my goal is to formulate and to solve inverse “Pauli problem”, namely:

In the case of N+1-level quantum system from the minimal set of immanent local dynamical variables related to the dynamical group $SU(N+1)$ and the coset structure $SU(N+1)/SU(1) \times SU(N)$ of state deformation to find “spacetime orientation” of this system in absence of a background spacetime structure. This topic will be discuss elsewhere.

ACKNOWLEDGEMENTS

I sincerely thank Yuval Ne’eman and Larry Horwitz for useful discussions and critical notes.

References

[1] R.P.Feynman, *QED. The Strange Theory of Light and Matter*, (Princeton University Press, 1985).

[2] P.Leifer, Quantum theory Requires Gravity and Superrelativity, Preprint gr-qc/9610043.

[3] P.Leifer, Why we can not see the curvature of the quantum state space? Preprint gr-qc/9701006.

[4] P.Leifer, Nonliner modification of quantum mechanics. Preprint hep-th/9702160.

[5] P.Leifer, Found.Phys. 27 (2) 261 (1997).
[6] P.Leifer, Int.J.Theor.Phys, in press.

[7] P.Leifer, Inertia as the “Threshold of Elasticity” of Quantum States, Preprint gr-qc/9706056.

[8] R. Cirelli, A. Mania’, L. Pizzocchero, Int.Mod.Phys., 6, (12) 2133 (1991).

[9] Wei-Min Zhang, Da Hsuan Feng, Phys.Rep.252, 1 (1995).

[10] A.Ashtekar and T.A.Schilling, Geometrical Formulation of Quantum Mechanics, Preprint gr-qc/9706069.

[11] J.S.Anandan, A.K.Pati, Phys.Lett.A 231, 29 (1997).

[12] H.Umezawa and H.Matsumoto, M. Tachiki, Thermo Field Dynamics and Condensed States, (North-Holland Publishing Company, Amsterdam-New York-Oxford, 1982).

[13] M.V. Berry et. al. in Geometry Phases in Physics, Ed. Alfred Shapere, Frank Wilczek, World Scientific, Singapore (1988).

[14] J.Anandan, Y.Aharonov, Phys.Rev.D 38 (6) 1863 (1988).

[15] N.Gisin,Phys.Lett.A 143 1 (1990).

[16] J.Polchinski,Phys.Rev.Lett., 66 397 (1991).

[17] S. Weinberg, Ann. Phys. (N.Y.), 194, 336 (1989).

[18] Y.Ne’eman, Proc.Natl.Acad.Sci.USA 80 7051 (1983).

[19] A.Ashtekar and J.Lewanowski, Quantum Field Theory of Geometry, Preprint hep-th/9603083.

[20] F.Wilczek and A.Zee, Phys.Rev.Lett. 52 (24) 2111 (1984).

[21] L.P.Horwitz, Found.Phys. 22 491 (1992).

[22] A.Einstein, Ann.Phys., 49 769 (1916).

[23] C.W.Misner, K.S.Thorne, J.A. Wheeler, Gravitation, (W.H.Freeman and Company, San Francisco, 1973).

[24] V.S.Ostrovskii, J.Exp.Theor.Phys.(SU),91 5(11), 1690 (1986).

[25] J.J.Bollinger et al Phys.Rev.Lett. 63 2261 (1989).

[26] S.Weigert, Phys.Rev.A 53 (4) 2078 (1996).