Global anomalies on physical space-times

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Abstract

We formulate an algebraic criterion for the presence of global anomalies on globally hyperbolic space-times in the framework of locally covariant field theory. We discuss some consequences and check that it reproduces the well-known global $SU(2)$ anomaly in four space-time dimensions.

1 Introduction and summary

Global anomalies are an interesting aspect of quantum field theory, as they constitute a non-perturbative effect and are thus one of the few aspects of this regime which are accessible with our current theoretical tools.

Global anomalies were first treated in a path integral formalism \cite{1} and manifest themselves as the non-invariance of the fermion path integral under large gauge transformations (in contrast to the well-known local anomalies, which occur for infinitesimal gauge transformations). Concretely, chiral fermions in four dimensional space-time, charged in the fundamental representation of $G = SU(2)$ were considered. As $\pi_4(SU(2)) = \mathbb{Z}_2$, there are compactly supported gauge transformations $g$ that cannot be deformed to the identity by compactly supported homotopies. But of course one may deform $A$ to its gauge transform $A^g$ via a path $A_\lambda$ of connections that are not gauge equivalent to $A$. By studying the flow of eigenvalues of the corresponding Dirac operator $D_{A_\lambda}$ along such a deformation, and using the mod 2 index theorem, it was shown that the fermion path integral

$$\left[ \int d\psi d\bar{\psi} \exp(\bar{\psi}iD_{A_\lambda}\psi) \right]^{1/2} = \left[ \det iD_{A_\lambda} \right]^{1/2}$$

changes sign as $A$ is varied to $A^g$ (note that the path integral for chiral fermions is defined as the square root of that for Dirac fermions). This implies that the full partition function

$$Z = \int dA \left[ \det iD_A \right]^{1/2} \exp \left( -\frac{i}{2g^2\gamma_5} \int \text{tr} F \wedge *F \right)$$

vanishes, as the contributions from $A$ and $A^g$ always cancel. The theory is thus inconsistent. We recall \cite{2} that the partition function is the formal Wick rotation of the transition

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amplitude $\langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle$ between the in- and the out-vacuum. Hence, one may expect that a global anomaly manifests itself in a Lorentzian setting in terms of vanishing transition probabilities between in- and out-states.

Apart from the ill-definedness of the path integral, the path integral formulation has the disadvantage that one considers fermions in non-trivial background fields, where the relation between the Euclidean and the Lorentzian setting is unclear.

There is also a Hamiltonian formulation \cite{1,3,4} of global anomalies. By choosing temporal gauge, a non-trivial element $g \in \pi_1(G)$ can be transformed into a non-trivial element of $\pi_1(\tilde{G})$ with $\tilde{G} = C_c^\infty(\mathbb{R}^3, G)$ the group of compactly supported spatial gauge transformations. A global anomaly occurs if the implementer of this non-trivial element is not the identity, as there is then no gauge invariant state. For the actual computation, one again treats the gauge field as a background field. But for time-dependent background fields, the Hamiltonian framework is not well suited as the time-evolution can in general not be implemented on a fixed (vacuum) Hilbert space \cite{5}.

The aim of the present paper is to give a criterion for global anomalies which mimics the above criteria but is properly formulated in a Lorentzian setting. We use the framework of locally covariant field theory \cite{6,7} which proved very fruitful for quantum field theory in curved space-times or other non-trivial backgrounds. Concretely, we use a generalization \cite{8} which also includes background gauge fields. Our formulation is based on the implementability of the relative Cauchy evolution \cite{6,9} and also relies on the concept of perturbative agreement \cite{10,11}.

As for the formulation in the path integral or the Hamiltonian framework, we define and compute global anomalies in the setting of free fermions in non-trivial background gauge fields. In this setting, a global anomaly does not spoil the consistency of the theory. But as in the path integral and the Hamiltonian formulation, one expects a global anomaly to spoil the consistency of the full theory, involving dynamical gauge fields. We will only briefly comment on how this happens in a locally covariant Lorentzian framework.

The article is structured as follows: In the next section, we recall the structures of locally (gauge) covariant field theory that are relevant for our discussion. In Section 3 we formulate our criterion for global anomalies. In order to make contact with the Hamiltonian formulation of global anomalies, we discuss their appearance in Hilbert space representations in Section 4. In particular, we sketch how global anomalies render a gauge theory inconsistent. In Section 5 we discuss a toy model where all the constructions and the actual computation can be performed explicitly, namely real fermions on the line. To prepare for the computation of global anomalies in higher dimensions, we clarify in Section 6 the relation between perturbative agreement and the Wess-Zumino consistency condition, a result that may be of interest independently of the current work. Finally, in Section 7 we compute the $SU(2)$ anomaly in four space-time dimensions, using arguments developed in \cite{12,13}.

**Notations and conventions:** The symbol $\doteq$ denotes the definition of the l.h.s. by the r.h.s.. The set of smooth compactly supported functions from $M$ to a Lie group $G$ are denoted by $C_c^\infty(M, G)$. For $B \to M$ a vector or Lie group bundle over $M$, $\Gamma_c^\infty(M,B)$ denotes its set of smooth compactly supported sections. By $\mathfrak{g}$ we denote the Lie algebra of $G$ and for a principal $G$-bundle $P$ we define $\mathfrak{p}$ to be the vector bundle associated to the adjoint representation of $G$ on $\mathfrak{g}$.

2 The framework

Let us recapitulate those aspects of the framework of locally covariant field theory \cite{6} which are relevant for our purposes. In contrast to studying quantum field theories on a fixed
background, the basic idea of locally covariant field theory is to work coherently over all of them. The collection of all possible backgrounds are the objects of the category \( \mathcal{B}_g \); its morphisms are used to define relations and consistency conditions between the theories on different backgrounds.

In the generalized setting of \( \mathcal{B}_g \), the admissible backgrounds are tuples \((SM, P, \bar{g}, A)\) consisting of a spin structure \( SM \) over an oriented, time-oriented, globally hyperbolic space-time \((M, \bar{g})\), and a principal \( G \)-bundle \( P \) over \( M \) with connection \( \bar{A} \). A morphism \( \chi : (SM, P, \bar{g}, A) \to (SM', P', \bar{g}', A') \) is a tuple \((\chi_{SM}, \chi_{P})\), where \( \chi_{SM} \) is a principal Spin\(_g\)-bundle morphism and \( \chi_{P} \) is a principal \( G \)-bundle morphism. Both, \( \chi_{SM} \) and \( \chi_{P} \) cover the same orientation, time-orientation and causality preserving isometric embedding \( \psi : M \to M' \). Furthermore, the connections are related by pull-back, i.e. \( \bar{A} = \chi_{P}^* \bar{A}' \). In the following, morphisms will often be isomorphisms of the form \((id, \chi_{P}) : (SM, P, \bar{g}, A) \to (SM, P', \bar{g}', A') \), which affect only the principal \( G \)-bundle. Typically, these are gauge transformations \((SM, P, \bar{g}, A) \to (SM, P, \bar{g}, A^g) \) where \( A^g = \text{Ad}_{g^{-1}} \bar{A} + g^* (\mu_G) \), with \( \mu_G \) the Maurer-Cartan form on \( G \) and \( g \in C^\infty_c(M, G) \).

To each background one assigns an algebra \( \mathfrak{A}(SM, P, \bar{X}) \) of observables, where we subsumed the geometric data in \( \bar{X} = (\bar{g}, \bar{A}) \). The consistency of this assignment is encoded in the requirement that \( \mathfrak{A} : \mathcal{B}_g \to \text{Alg} \) is a functor with values in the category \( \text{Alg} \) of unital \( * \)-algebras (with injective unital \( * \)-homomorphisms as morphisms). This means that for each morphism \( \chi : (SM, P, \bar{X}) \to (SM', P', \bar{X}') \) there is an injective unital \( * \)-homomorphism

\[
\alpha_{\chi} : \mathfrak{A}(SM, P, \bar{X}) \to \mathfrak{A}(SM', P', \bar{X}'),
\]

which is an isomorphism if \( \chi \) is an isomorphism in \( \mathcal{B}_g \). When \((SM, P)\) are kept fixed, also the notation \( \mathfrak{A}(\bar{X}) \) will be used.

In order to relate particular observables on different backgrounds, one introduces the notion of fields, which are natural transformations from suitable functors \( \mathfrak{F} \) to \( \mathfrak{A} \). Important examples for \( \mathfrak{F} \) are the functors which assign to each background \((SM, P, \bar{X})\) the set of test tensors (of a fixed type) on \( M \), i.e., the set of smooth, compactly supported sections of a vector bundle associated to the bundle \( SM + P \to M \). A field \( \Phi \), restricted to a background \((SM, P, \bar{X})\), and smeared with a test tensor \( t \), is then denoted as \( \Phi_{(SM, P, \bar{X})}(t) \), or, if \((SM, P)\) are held fixed, also by \( \Phi_{\bar{X}}(t) \). Being a natural transformation amounts to

\[
\alpha_{\chi} \Phi_{(SM, P, \bar{X})}(t) = \Phi_{(SM', P', \bar{X}')}(\chi_{\bar{X}}) \chi_{\bar{X}}(t)
\]

for any \( t \) and any morphism \( \chi : (SM, P, \bar{X}) \to (SM', P', \bar{X}') \). A typical example is the current

\[
\tilde{j}_{(SM, P, \bar{X})}(A) = \int_{M} \bar{\psi} \bar{A} \psi
\]

of free fermions charged under \( G \). Here \( A \in \Gamma^\infty_c(M, \mathfrak{p} \otimes T^* M) \). For the explicit construction of the algebra \( \mathfrak{A}(SM, P, \bar{X}) \) (in terms of evaluation functionals) and fields, in particular non-linear ones, we refer to \[S\].

In the setting that we are considering, the background fields provide the hyperbolic (wave or Dirac) operator for the dynamical (matter) fields. There is then another way to relate the theories on different backgrounds. Let us keep \( SM \) and \( P \) fixed. For compactly supported perturbations \( X = (g, A) \) of the geometric data \( \bar{X} = (\bar{g}, \bar{A}) \), one defines the retarded and advanced Møller operators

\[
\tau_{\bar{X} + X, \bar{X}}^{t/a} : \mathfrak{A}(\bar{X}) \to \mathfrak{A}(\bar{X} + X),
\]

which are \( * \)-isomorphisms and act as the identity on observables localized in the past/future of \( \text{supp} X \), where the two backgrounds are identified via Cauchy surfaces. If the algebras
\(A\) are constructed as evaluation functionals, one may define the Møller operators by the pullback of the retarded/advanced scattering operator on the solution spaces. Alternatively, one may define \(\tau^{r/a}\) abstractly by using the time-slice axiom [6].

Remark 1. In principle, one could also allow for perturbations \(X\) that are not compactly supported, but only have a certain fall-off behavior. But this would require restrictions on the growth of the field configurations. However, the specific form of natural growth conditions depends both on the background \((SM, P, \bar{X})\) and on the specific form of the equations of motion operator. In order to work model-independently, we stick to perturbations with compact support.

For what follows, it is crucial that the Møller operators compose naturally, i.e.,

\[
\tau^{r/a}_{X+X', \bar{X}+X} \circ \tau^{r/a}_{X+X', \bar{X}} = \tau^{r/a}_{X+X', \bar{X}}.
\]

Furthermore, for \(\bar{X}\) and \(\bar{X} + X\) related by a morphism \(\chi : (SM, P, \bar{X}) \to (SM, P, \bar{X} + X)\) which acts as the identity outside of a compact set, we have

\[
\tau^{r/a}_{\bar{X} + X, \bar{X}} = \alpha_{\chi}.
\]

For generic \(X\), one also has [9, Prop. 3.7],

\[
\alpha_{\chi} \circ \tau^{r/a}_{\bar{X} + X, \bar{X}} = \tau^{r/a}_{\bar{X}' + \chi_{\star} X, \bar{X}} \circ \alpha_{\chi},
\]

where the morphism on the r.h.s. is \(\chi : (SM, P, \bar{X}) \to (SM, P, \bar{X}')\) and the one on the l.h.s. is \(\chi : (SM, P, \bar{X} + X) \to (SM, P, \bar{X}' + \chi_{\star} X)\). For a field \(\Phi\), we define the infinitesimal retarded/advanced variation as

\[
\delta^{r/a}_{\bar{X}}(X)\Phi(t) = \frac{d}{d\lambda} \tau^{r/a}_{\bar{X}, \bar{X} + \lambda X}(\Phi((SM, P, \bar{X} + \lambda X)(t))) \bigg|_{\lambda=0}.
\]

Using the Møller operators, one can define the relative Cauchy evolution [6] as the *-automorphism

\[
\text{rce}_{\bar{X}}(X) \doteq \tau^{r}_{\bar{X}, \bar{X} + X} \circ \tau^{a}_{\bar{X} + X, \bar{X}}
\]

of \(A(\bar{X})\). In particular, its derivative

\[
\text{rce}_{\bar{X}}(X) \doteq \frac{d}{d\lambda} \text{rce}_{\bar{X}}(\lambda X) \bigg|_{\lambda=0}
\]

is a derivation. As a consequence of [4] we have that, for \(\bar{X}\) and \(\bar{X} + X\) related by a morphism which acts as the identity outside of a compact set,

\[
\text{rce}_{\bar{X}}(X) = \text{id}.
\]

3 Global anomalies

In the following, we will focus on perturbations of the connection \(\bar{A}\). For a free theory in the presence of background fields, the infinitesimal version of the relative Cauchy evolution is given by the commutator with the current,

\[
\text{rce}_{\bar{A}}(A) = -\frac{i}{\hbar}[j_{\bar{A}}(A), \cdot].
\]
More generally, one may say that the theory $\mathcal{A} : Bg \rightarrow \mathbf{Alg}$ admits a current if there is a field $j$ such that the above holds. As a consequence of (9), $rce\bar{A}$ vanishes for an infinitesimal gauge transformation, i.e.,
\[ rce\bar{A}(\bar{d}c) = 0, \]  
where $\bar{d}$ is the covariant differential defined on sections of $p$ by
\[ (\bar{d}c)(\xi) = \bar{\nabla}_\xi c, \]
with $\xi$ a vector field and $\bar{\nabla}$ the covariant derivative induced on the associated bundle $p$ by the connection $\bar{A}$. The condition that the current $j$ is conserved can then be formulated as the requirement
\[ \bar{\delta}j\bar{A}(\bar{d}c) = 0, \]
where $\bar{\delta}$ is the covariant derivative defined on sections of $p$ by
\[ (\bar{d}c)(\xi) = \bar{\nabla}_\xi c, \]
with $\xi$ a vector field and $\bar{\nabla}$ the covariant derivative induced on the associated bundle $p$ by the connection $\bar{A}$.

The existence of a conserved current signifies the absence of infinitesimal (usually called local) anomalies and is assumed from now on. It is important to note that an anomaly can not be seen in a failure of (10), as a violation of (11) is a c-number.

In order to discuss global anomalies, we require that the $\ast$-automorphism $rce\bar{A}(A)$ is unitarily implemented by a field $V\bar{A}(A)$, i.e.,
\[ rce\bar{A}(A)(a) = Ad_{V\bar{A}(A)}(a) \equiv V\bar{A}(A) a V\bar{A}(A)^*, \]  
for all $a \in \mathfrak{A}(SM, P, \bar{g}, \bar{A})$, and
\[ V\bar{A}(A)^{-1} = V\bar{A}(A)^*. \]
In the present work, we are focusing on conceptual and structural aspects, not on functional analytic ones, so we allow the implementers to be formal power series in the perturbation $\bar{A}$. Whether this leads to a proper unitary representation in suitable Hilbert space representations is a very interesting open question.

1 This is related to the question whether the current $j\bar{A}(A)$ is represented by an essentially self-adjoint operator on $\mathcal{H}$. For the scalar field and Wick squares without derivatives, conditions ensuring this property were given in [14].

However, as discussed above, one should impose further constraints on the current than just (9). Hence, apart from the implementers being fields, we impose a further natural condition. To motivate it, we note that from (3) and (7) it follows that
\[ rce\bar{A}(A') = \tau_{\bar{A},\bar{A}+A}^\bar{A} \circ rce\bar{A} + A - A \circ \tau_{\bar{A},\bar{A}}^\bar{A}. \]
Using (12), it follows that
\[ rce\bar{A}(A') = Ad_{V\bar{A}(A')} = Ad_{\tau_{\bar{A},\bar{A}+A}^\bar{A} V\bar{A}(A' - A) V\bar{A}(A)}. \]

It thus seems natural to require that
\[ V\bar{A}(A') = \tau_{\bar{A},\bar{A}+A}^\bar{A} V\bar{A}(A' - A) V\bar{A}(A). \]

The discussion of Hilbert space representations in the following section provides further motivation for this requirement. If condition (14) is fulfilled, we say that $rce\bar{A}$ is unitarily implemented by $V$. 

\[ \]
It turns out that \( (14) \) is automatically fulfilled for free fields charged under \( G \) in space-time dimension \( d \leq 4 \) if 1.) the current \((13)\) is conserved and 2.) \( V_\lambda(0) = 1 \) is the unit. To see this, note that \( (14) \) can be equivalently written as

\[
\tau^{\lambda}_{\bar{A} + A}(V_{\bar{A} + A}(A'' - A)) V_\lambda(A) = \tau^{\lambda}_{\bar{A} + A'}(V_{\bar{A} + A'}(A'' - A')) V_\lambda(A').
\]  

(15)

As the space of connections is affine, there are no topological obstructions to fulfill this condition, but there may be local ones. Replacing \( A \) by \( \lambda A, A' \) by \( \eta A' \), and \( A'' \) by \( \lambda A + \eta A' \) in the above equation and evaluating the derivative w.r.t. \( \lambda \) and \( \eta \) at 0, we obtain

\[
E_\lambda(A, A') = \delta^\lambda_\bar{A}(A) j(A') - \delta^\lambda_\bar{A}(A') j(A) - i\hbar^{-1}[j_\lambda(A'), j_\lambda(A)] = 0,
\]  

(16)

where we used the definition \( (9) \). As will become clear in the following section, we may see \( E_\lambda \) as the curvature for “parallel transport” by \( \pi(V_\lambda) \) of vectors \( \Psi \in \mathcal{H} \) in any representation \( \pi : \mathfrak{A}(\bar{A}) \to \text{End}(\mathcal{H}) \). As shown in [11, Prop. 3.8], \( (16) \) is fulfilled for \( d \leq 4 \), whenever the current \( j \) is conserved.

Obvious consequences of \( (14) \) are

\[
V_\lambda(0) = 1,
\]  

\[
\frac{d}{d\lambda} V_\lambda(A + \lambda A')|_{\lambda = 0} = -\frac{i}{\hbar} \tau^{\lambda}_{\bar{A} + A}(j_{A + A'}(A')) V_\lambda(A).
\]  

(17)

(18)

Hence, in order to evaluate \( V_\lambda(A) \), we may choose any path \( A_\lambda \) such that \( A_0 = 0, A_1 = A \) and compute the path-ordered exponential integral

\[
V_\lambda(A) = P \exp \left( -\frac{i}{\hbar} \int_0^1 \tau^{\lambda}_{\bar{A} + A_\lambda}(j_{A + A_\lambda}(\dot{A}_\lambda)) d\lambda \right).
\]  

(19)

In particular, for any \( \bar{A} + A \) which is continuously connected by a path of compactly supported gauge transformations to \( \bar{A} \), we have that \( V_{\bar{A}}(A) = 1 \). Assuming that the center of \( \mathfrak{A}(SM, P, \bar{g}, \bar{A}) \) is trivial, we also know that \( V_\lambda(A) = e^{i\phi} 1 \) for some \( \phi \in \mathbb{R} \) whenever \( \bar{A}' = \bar{A} + A \) for a compactly supported gauge transformation \( g \). We say that the theory \( \mathfrak{A} \) has a global anomaly, if \( V_\lambda(A) \neq 1 \) in that case.

For four-dimensional space-times with trivial topology \( \mathbb{R}^4 \), possible obstructions are thus classified by \( \pi_4(G) \). Given a non-trivial gauge transformation \( \bar{A} \mapsto \bar{A}' = \bar{A} + A \), \( (19) \) can in principle be used to decide whether a given theory is anomalous or not.

Apart from leading to the same obstructions as in the path integral framework, our formulation has the additional similarity that the computation of the anomaly proceeds by reaching the non-trivial gauge transformation via a path of gauge non-equivalent backgrounds.

Using \( (19) \), we can show that \( V \) is indeed a field. Let \( \chi : (SM, P, \bar{X}) \to (SM, P, \bar{X}') \) be a morphism. We compute

\[
\alpha_\chi V_{\bar{A}}(A) = P \exp \left( -\frac{i}{\hbar} \int_0^1 \alpha_\chi \circ \tau^{\lambda}_{\bar{A} + A_\lambda}(j_{\bar{A} + A_\lambda}(\dot{A}_\lambda)) d\lambda \right)
\]

\[
= P \exp \left( -\frac{i}{\hbar} \int_0^1 \tau^{\lambda}_{\bar{A}' + \chi + A_\lambda} \circ \alpha_\chi(j_{\bar{A}' + \chi + A_\lambda}(\dot{A}_\lambda)) d\lambda \right)
\]

\[
= P \exp \left( -\frac{i}{\hbar} \int_0^1 \tau^{\lambda}_{\bar{A}' + \chi + A_\lambda}(j_{\bar{A}' + \chi + A_\lambda}(\chi \cdot \dot{A}_\lambda)) d\lambda \right)
\]

\[
= V_{\bar{A}}(\chi \cdot A).
\]  

(20)

In the second equality we used \( (5) \) and in third equality \( (11) \).
As we argued above, $\phi_{\bar{A}}(g) = V_{\bar{A}}(\bar{A}^g - \bar{A})$ is a c-number for a compactly supported gauge transformation $g$. A natural question is now whether this c-number only depends on $g$, or also on the background $\bar{A}$. Using the fact that the implementer is a field, (20), it is straightforward to show that indeed

$$\phi_{\bar{A}}(g) = \phi_B(g)$$

if $\bar{B} - \bar{A}$ is compactly supported. We use (15) with $A = \bar{A}^g - \bar{A}$, $B = \bar{A} + A'$ and $\bar{B}^g = \bar{A} + A''$, obtaining

$$\tau_{\bar{A},\bar{A}^g}(V_{\bar{A}}(\bar{B}^g - \bar{A}^g))\phi_{\bar{A}}(g) = \phi_B(g)V_{\bar{A}}(\bar{B} - \bar{A}).$$

The equality (21) then follows from (4) and (20).

Remark 2. For space-times $M$ with non-trivial topologies, the possible obstructions for the absence of global anomalies are different. Technically, these obstructions are classified by the quotient set $\Gamma_c^\infty(M, P \times \text{Ad} G)/\sim$, where $P \times \text{Ad} G$ is the Lie group bundle associated to the adjoint action of the structure group $G$ on itself and $\sim$ is the equivalence relation given by compactly supported homotopy equivalences. In the case of $M \simeq \mathbb{R}^d$, this reduces to the $d$-th homotopy group $\pi_d(G)$.

The calculation of the anomaly amounts to the calculation of the implementers $V$. For free fermions, these are formally given by

$$V_{\bar{A}}(A) = T(e^{\frac{i}{\hbar} j_{\bar{A}}(A)})^{-1},$$

where $T$ denotes the time-ordered product, cf. [10]. To see this, we differentiate:

$$\frac{d}{d\lambda} V_{\bar{A}}(\lambda A) = -V_{\bar{A}}(\lambda A) \frac{d}{d\lambda} T(e^{\frac{i}{\hbar} j_{\bar{A}}(\lambda A)}) V_{\bar{A}}(\lambda A)$$

$$= -\frac{i}{\hbar} V_{\bar{A}}(\lambda A) \mathcal{R}(j_{\bar{A}}(A); e^{\frac{i}{\hbar} j_{\bar{A}}(\lambda A)}) V_{\bar{A}}(\lambda A)$$

$$= -\frac{i}{\hbar} \mathcal{R}(j_{\bar{A}}(A); e^{\frac{i}{\hbar} j_{\bar{A}}(\lambda A)}) V_{\bar{A}}(\lambda A).$$

In the third line, the retarded product defined by

$$\mathcal{R}(F; e^{iS}) = (e^{iS})^{-1} T(F \otimes e^{iS})$$

was introduced. In the last step, we used the integral form of perturbative agreement, cf. [11], and the fact that the current does not depend on the background connection, cf. (2). Hence, this implementer fulfills the initial condition (17) and the differential equation (18). However, it should be kept in mind that it is not clear whether the representation (22) converges (it is derived from a formal power series, but no longer has a formal parameter).

Leaving the convergence question aside for a moment, one could thus calculate the implementer by calculating the connected fermion loops with $n$ external legs. For simplicity, one may choose $\bar{A}$ as a trivial connection. From the fermion loops, only the non-local part may contribute, as the local terms are gauge invariants, and the field strength corresponding to $A$ vanishes. For the c-number part of the implementer, which is the relevant one for the
global anomaly, one then has

$$\langle \Omega | V_0(A)^{-1} | \Omega \rangle = \sum_{n=0}^{\infty} \frac{(i/\hbar)^n}{n!} \langle \Omega | T(j(A)^{\otimes n}) | \Omega \rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{k_2 + \cdots + nk_n = n; k_i \geq 0} \frac{n!}{k_2! \cdots k_n!} \prod_{j=2}^{n} \frac{1}{j k_j} I_j(A)^{k_j} = \exp \left( \sum_{k=2}^{\infty} \frac{i^k}{k} I_k(A) \right),$$

where $I_n(A)$ is the closed fermion loop Feynman diagram with $n$ external gauge boson lines, each smeared with $A$.

**Remark 3.** Analogously, one can define global gravitational anomalies [16], by replacing changes in the background connection by changes in the background metric, gauge transformations by diffeomorphisms, and the current by the stress-energy tensor.

### 4 Hilbert space representations

Let us discuss some consequences of our definitions for representations of our algebras on Hilbert spaces. Assume we have a representation $\bar{\pi} : \mathfrak{A}(\bar{A}) \to \text{End}(\bar{\mathcal{H}})$. Consider now arbitrary compactly supported perturbations $A$ of $\bar{A}$. The representation $\bar{\pi}$ naturally induces representations

$$\pi_A = \bar{\pi} \circ \tau^r_{\bar{A}, \bar{A} + A} : \mathfrak{A}(\bar{A} + A) \to \text{End}(\mathcal{H}_A),$$

where $\mathcal{H}_A = \mathcal{H}$. Physically, this means that we identify states that coincide in the past. In particular, using the canonical identification of $\mathcal{H}_A$ and $\mathcal{H}_{A'}$, the algebra homorphism $\tau^r_{\bar{A} + A', \bar{A} + A}$ is implemented by the identity, i.e.

$$\pi_{A'} \circ \tau^r_{\bar{A} + A', \bar{A} + A} = \pi_A.$$

However, there is also another natural map between the algebras $\mathfrak{A}(\bar{A} + A)$ and $\mathfrak{A}(\bar{A} + A')$, namely the advanced Møller operator. Let us see whether this can also be implemented:

$$\pi_{A'} \circ \tau^a_{\bar{A} + A', \bar{A} + A} = \pi_A \circ \tau^a_{\bar{A} + A, \bar{A} + A'} \circ \tau^a_{\bar{A} + A', \bar{A} + A} = \pi_A \circ \text{re}_A(A' - A) = \pi_A \circ \text{Ad}_V(\bar{A} + A).$$

Hence,

$$\pi_{A'} \circ \tau^a_{\bar{A} + A', \bar{A} + A} (\cdot) = U(A', A) \pi_A (\cdot) U(A', A)^*$$

with

$$U(A', A) = \pi_A (V_{\bar{A} + A}(A' - A)) = \bar{\pi} \left( \tau^a_{\bar{A}, \bar{A} + A}(V_{\bar{A} + A}(A' - A)) \right) = \bar{\pi}(V_{\bar{A}}(A') V_{\bar{A}}(A)^*),$$

where in the last step we used [13]. In particular, we obtain

$$U(A', A) U(A', A) = U(A', A),$$
$$U(A', A)^{-1} = U(A, A').$$

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Let us note that these natural properties are a direct consequence of \((14)\), which lends further support to the usefulness of this condition.

From \((21)\), it is clear that for a compactly supported gauge transformation \(g\) and \(A^g = \text{Ad}_g^{-1} A\)
\[
\rho(g) = U(A^g + A^g - \bar{A}, A)
\]
is a c-number independent of the choice of a compactly supported \(A\). In the case of a global anomaly, there are \(g\)'s such that \(\rho(g) \neq 1\). How can these turn a theory inconsistent? There is certainly no problem as long as the gauge fields are considered purely as background fields. As in the path integral approach, one expects inconsistencies in the non-perturbative interacting theory.

In the absence of a non-perturbatively defined theory, it is obviously difficult to state what precisely goes wrong in the presence of global anomalies. Let us therefore give a tentative answer. In the absence of matter fields (and after fixing a reference connection), one may describe the classical gauge field observables by the algebra \(C(C)\) of (smooth) functions on the space of gauge field configurations \(C = \Gamma^\infty(M, p \otimes T^*M)\). In the presence of matter quantum fields, with observables on a background \(A\) described by an algebra \(\mathfrak{A}_A\), a natural choice of observables for the combined system of matter and gauge fields are mappings
\[
\mathcal{C} \ni A \mapsto a(A) \in \mathfrak{A}_A,
\]
which assign to each gauge field \(A\) an algebra element \(a(A)\) in the matter observable algebra \(\mathfrak{A}_A\). (Notice that in this formulation the gauge field observables are not yet quantized.) Such observables can be represented on sections of the Hilbert space bundle \(H \to \mathcal{C}\) with fibers \(\mathcal{H}_A\), i.e. mappings
\[
\mathcal{C} \ni A \mapsto \Psi(A) \in \mathcal{H}_A.
\]
At least formally, there is a natural inner product
\[
\langle \Psi | \Psi' \rangle = \int_{\mathcal{C}} \langle \Psi(A) | \Psi'(A) \rangle_{\mathcal{H}_A} dA \quad (24)
\]
that involves an integral over the space of connections. If the representations \(\pi_A\) are constructed as above, then an \textit{in-state} would be defined as obeying \(\Psi(A) = \Psi(A')\) for any \(A, A'\) that only differ in a compact region. Similarly, for an \textit{out-state}, one would require \(\Psi(A) = U(A, A')\Psi(A')\). In the case of a global anomaly,\(^3\) it then follows that the inner product, i.e., the transition amplitude, of an in- and an out-state vanishes. As an example, consider the case where the relevant homotopy group is \(\mathbb{Z}_2\). Then the contributions from \(A\) and \(A^g\), with \(g\) the non-trivial element of the homotopy group, cancel in \((21)\). As anticipated in the introduction, we thus find that in the presence of a global anomaly, the transition amplitudes between in- and out-states vanish.

5 \hspace{1em} \textbf{A one-dimensional example}

As a simple example, where the anomalies can be computed straightforwardly and the background changes are properly unitarily implemented, we consider the case of real fermions on the line \(\mathbb{R}\). This was first studied in \([17]\) in the setting of Euclidean partition functions. Their results on the classification of anomalies are wrong, however, so the present section also serves to correct these.

\(^3\)Since global anomalies are of topological origin, one may assume that they are stable upon switching on the interaction.
Choosing a trivialization of the principal $G$-bundle $P \to \mathbb{R}$, or equivalently fixing the background connection $\bar{A} = 0$, the action for the fermions in the background $A + A$ can be written as

$$S_{A+A} = \frac{i}{2} \int_\mathbb{R} \psi_i (\delta_{ij} \partial_t + A_{ij}) \psi_j \, dt,$$

where $A_{ij}(t) = \lambda^m_{ij} A^m(t)$ (summation over $m$ understood) and $\lambda^m$ are the real antisymmetric generators of the Lie algebra in a representation $r$. In the background $\bar{A} = 0$, the $\psi$’s satisfy the equation of motion $i \partial_t \psi = 0$ and fulfill the anti-commutation relations $\{ \psi_i(t), \psi_j(t') \} = \hbar \delta_{ij}$. The corresponding current is

$$j(A) = \frac{i}{2} \int_\mathbb{R} \psi_i A'_{ij} \psi_j \, dt.$$

As there are no ordering ambiguities, this current is conserved, and in particular condition (16) is fulfilled. Moreover, the implementers $V(A)$ are given by, cf. (22),

$$V(A) = \mathcal{T} \exp \left( \frac{1}{2\hbar} \int_\mathbb{R} \psi_i A_{ij} \psi_j \, dt \right),$$

where $\mathcal{T}$ denotes anti time-ordering.

A Hilbert space representation of the algebra of smeared fields can be obtained by setting

$$\bar{\pi}(\psi(f)) = \Psi_i \int_\mathbb{R} f_i \, dt,$$

or, formally, $\bar{\pi}(\psi_i(t)) = \Psi_i$, where $\Psi_i$ are self-adjoint operators on a Hilbert space $\mathcal{H}$ fulfilling $\{ \Psi_i, \Psi_j \} = \hbar \delta_{ij}$. It follows that the implementers $U(A', A)$ are given by, cf. (22),

$$U(A', A) = \mathcal{T} \exp \left( \frac{1}{2\hbar} \Psi_i \Psi_j \int_\mathbb{R} A'_{ij} \, dt \right) \mathcal{T} \exp \left( -\frac{1}{2\hbar} \Psi_i \Psi_j \int_\mathbb{R} A_{ij} \, dt \right).$$

Following [17], let us now consider the representation $R$ on $\mathcal{H}$ that this induces. For $1 \leq m \leq \text{rank } G$, the generators $\lambda$ of the representation $r$ can be brought into the form

$$\lambda^m = \begin{pmatrix} 0 & \alpha^m_1 \\ -\alpha^m_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha^m_2 \\ -\alpha^m_2 & 0 \end{pmatrix} \ldots,$$

with $\alpha_k$ the weights of $r$. Defining annihilation and creation operators

$$a_k = \frac{1}{\sqrt{2}} (\Psi_{2k-1} + i \Psi_{2k}), \quad a^*_k = \frac{1}{\sqrt{2}} (\Psi_{2k-1} - i \Psi_{2k}),$$

the generators $\Lambda^m$ in the representation $R$, corresponding to the $\lambda^m$ in the representation $r$, are given by

$$\Lambda^m = \sum_{k=1}^{[d_r/2]} \alpha^m_k (a^*_k a_k - \frac{1}{2}),$$

where $d_r$ is the dimension of the representation $r$ and $[\cdot]$ denotes the integer part. The weights of $R$ thus have the form

$$\delta = \frac{1}{2} \sum \pm \alpha_k.$$
An anomaly occurs if there is a weight of $R$ which is non-integral for the gauge group $G$, as then $U(g^{-1}dg,0)$ is not the identity for $g \in \pi_1(G)$. A simple example is obtained taking the fermions in the fundamental representation of $G = SO(3)$.

The existence of integral weights of $\mathfrak{g}$ which are non-integral for $G$ is equivalent to a non-trivial $\pi_1(G)$, establishing the relation to the abstract result that a global anomaly is possible only for non-trivial $\pi_d(G)$, with $d$ being the space-time dimension.\footnote{In \cite{17}, the occurrence of a global anomaly was erroneously related to the question whether the weights of $R$ are in the same equivalence class as those of $r$ modulo the root lattice, leading to the conclusion that there are global anomalies for many groups with a trivial $\pi_1$.} Of course the example is somewhat artificial, as one may remove the anomaly by re-defining the gauge group to be the universal cover of $G$.

6 Perturbative agreement and the Wess-Zumino consistency condition

Let us consider again the condition (16) on the vanishing of the curvature $E_A$ for $A$, $A'$ being infinitesimal gauge transformations, i.e., $A = \delta \Lambda$ and $A' = \delta \Lambda'$. In that case the current evaluated at $\Lambda$ is a $c$-number, so that the commutator vanishes. For the expression on the l.h.s. of (16), we thus obtain

$$\delta^\prime_{A}(d\Lambda)j(d\Lambda') - \delta^\prime_{A}(d\Lambda')j(d\Lambda) = \delta^\prime_{A}([d\Lambda, \Lambda']) - \delta^\prime_{A}([d\Lambda', \Lambda]) - \delta_{A}([d\Lambda, \Lambda']) + j_{A}([d\Lambda', \Lambda])$$

$$= \delta j_{A}([\Lambda, \Lambda']) - \delta j_{A}([\Lambda', \Lambda]) - j_{A}([d\Lambda, \Lambda'])$$

where the notation introduced in (11) was employed. In the first step, we used an identity used in the proof of \cite{11}, Prop. 3.8. In the second step we used

$$\delta^\prime_{A}(d\Lambda)\Phi(t) = \Phi_{\Lambda}(\mathcal{L}_{A}t),$$

for any field $\Phi$, where $\mathcal{L}_{A}$ is the representation of $\Lambda \in \Gamma_{\infty}(M,p)$ on the test tensor. This is a straightforward consequence of (1) and (4). From (25), we see that for semisimple Lie algebras, the vanishing of the divergence of the current is a necessary condition for the fulfillment of (16). On the other hand, as shown in \cite{11}, the vanishing of the divergence of the current implies (16) in space-time dimension $d \leq 4$.

Now assume that there is a local anomaly, i.e., the divergence of the current does not vanish. Can we still fulfill (16) by giving up the requirement that the current is a field, i.e., by giving up covariance w.r.t. transformations of the background field? This means that we fix a reference connection and specify other connections by a Lie algebra valued one form $A$. For simplicity, we assume the reference connection to be flat. The current $j_{A}$ is then allowed to depend on $A$, i.e., we take the covariant current $j$ and add an $A$ dependent correction term $\Delta j_{A}$. In the calculation (26), we can then no longer use (24). Instead, in the expression on the r.h.s. of the first line of (24), we may use that $\delta j$ is a $c$-number, so that the retarded variation coincides with the functional derivative w.r.t. $A$. Hence, we obtain the condition

$$<\frac{\delta}{\delta A}(\delta j)A(A'), d_{A}A> - <\frac{\delta}{\delta A}(\delta j)A(A), d_{A}A'> = 0,$$

where $d_{A}$ is the covariant differential in the background $A$ and $(\delta j)A(A) \equiv j_{A}(d_{A}A)$. This is the \textit{Wess-Zumino consistency condition} \cite{18}, derived from the requirement that gauge transformations of the background field are represented on the vacuum functional. Hence, after giving up background gauge invariance, i.e., the requirement that physics should only
depend on the principal bundle connection, not on a particular choice of a trivialization, it may be possible to get rid of the path dependence in the definition of the implementers, even in the presence of local anomalies.

In [11], it was shown that in dimension $d \leq 4$, the vanishing of the divergence of the current implies that (16) is fulfilled also for variations $A, A'$ which are not gauge. We will prove an analogous statement for the modified current, i.e., that (27) implies (16) for non-gauge variations $A, A'$. However, we shall explicitly use the assumption that the gauge group is semisimple and that the (covariant) anomaly is of the form

$$\bar{\delta}j_A(\Lambda) = c \int_M \text{tr} (\Lambda F \wedge \bar{F}),$$

(28)

where the trace is taken in some representation. As a first step, we claim that

$$E_A(A, A') = c \int_M \text{tr} \left( (A \wedge A' - A' \wedge A) \wedge \bar{F} \right)$$

(29)

as this is the unique functional linear and antisymmetric in $A, A'$, invariant under $\delta \bar{A} = \bar{d} \Lambda$, $\delta A(\ell) = [\Lambda, A(\ell)]$ and of the correct scaling dimension, such that

$$E_A(\bar{d} \Lambda, \bar{d} \Lambda') = c \int_M \text{tr} \left( (\bar{d} \Lambda \wedge \bar{d} \Lambda' - \bar{d} \Lambda' \wedge \bar{d} \Lambda) \wedge \bar{F} \right) = \bar{\delta}j_A([\Lambda, A']).$$

Uniqueness follows from the fact that $E_A(\bar{d} \Lambda, \bar{d} \Lambda') = 0$ implies $E_A(A, A') = 0$ as proved in [11].

It is well-known [19] how to pass from the covariant anomaly (28) to the consistent anomaly fulfilling the Wess-Zumino consistency condition (27). One adds to the current the correction term

$$\Delta j_A(A') = \frac{1}{3} c \int_M \text{tr} \left( A' \wedge (A \wedge F_A + F_A \wedge A - \frac{1}{2} A \wedge A \wedge A) \right).$$

(30)

For the total anomaly, we thus obtain

$$\langle \delta j \rangle_A(\Lambda) = c \int_M \text{tr} \left( \Lambda F_A \wedge F_A + \frac{1}{3} d_A \Lambda \wedge (A \wedge F_A + F_A \wedge A - \frac{1}{2} A \wedge A \wedge A) \right) = \frac{1}{3} c \int_M \text{tr} \left( \Lambda (d A \wedge d A + \frac{1}{2} d(A \wedge A \wedge A)) \right),$$

which is the well-known expression for the consistent anomaly. Adding (30) modifies the curvature $E$ by

$$\Delta E_A(A_1, A_2) = \langle \frac{\delta}{\delta A} \Delta j_A(A_2), A_1 \rangle - \langle \frac{\delta}{\delta A} \Delta j_A(A_1), A_2 \rangle = -c \int_M \text{tr} \left( (A_1 \wedge A_2 - A_2 \wedge A_1) \wedge F_A \right).$$

Comparison with (29) shows that the curvature $E$ is indeed canceled.

Remark 4. In [10], the violation of perturbative agreement was related to a cohomological question by noting that $E$ is a cocycle w.r.t. a suitably defined differential. It was shown that a violation of perturbative agreement can be removed if $E$ is a coboundary. At least for the Yang-Mills case, we have seen that $E$ is always a coboundary, but in general of a background field functional depending on the choice of a trivialization of the background.
Remark 5. There is a subtle point here: For \( G = U(1) \) and on flat space-time, one could remove the anomaly by replacing the factor \( \frac{1}{3} \) in (30) by \( \frac{1}{2} \). In this way, one also fulfills the Wess-Zumino consistency condition (27). However, the curvature \( E \) does then not vanish (except when both arguments are exact). This shows that the condition (16) is stronger than the Wess-Zumino consistency condition (27) for non-semisimple \( G \) and non-generic space-times.

7 Computation of the anomaly

With the results obtained in the previous section, we can now actually compute the anomaly. Following the ideas of [12,13], we shall consider \( G = SU(2) \) as a subgroup of \( H = SU(3) \), whose fourth homotopy group is trivial. However, it has a local anomaly, given by

\[
\delta j(\Lambda) = \frac{i\hbar}{8\pi^2} \int_M \text{tr} \left( \Lambda \tilde{F} \wedge \tilde{F} \right)
\]

for a single generation of chiral fermions. Note that this can be computed in the Lorentzian setting [20], without recourse to Riemannian concepts. According to the above discussion, the corresponding consistent anomaly is given by

\[
(\delta j)_A(\Lambda) = \frac{i\hbar}{24\pi^2} \int_M \text{tr} \left( \Lambda d(A \wedge dA + \frac{1}{2} A \wedge A \wedge A) \right). \quad (31)
\]

We embed \( G \) in \( H \) in such a way that, in the fundamental representation, a \( G \)-gauge transformation \( g \) is given by

\[
g_H = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.
\]

The idea is to connect the non-trivial \( G \)-gauge transformation \( g \) (i.e. \([g] = -1 \in \pi_4(G) \simeq \mathbb{Z}_2 \)) by a path \( h(\lambda) \) of \( H \)-gauge transformations to the identity, i.e.,

\[
h(0) = \text{id}, \quad h(1) = g_H. \quad (32)
\]

The global anomaly of \( G \) can then be computed by integrating up the local anomaly of \( H \) along this path, using (19). For this to work, two requirements have to be fulfilled: First, it must be possible to deform the path \( h \) to a path with background connections solely in the \( G \) component, i.e., of the form

\[
A_H = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \quad (33)
\]

This is possible, since, as we have checked in the previous section, the curvature \( E \) vanishes for the modified current. Second, the implementer for perturbations of the form (33) must reduce to the implementer in the original theory. In particular, local covariance must be restored, i.e., it must be independent of the choice of a trivialization.

Let us sketch such a construction.\(^4\) Given a background connection \( \tilde{A} \) of the original theory, choose some trivialization, obtain the corresponding \( A \) and set the background connection of the extended theory to be \( A_H \). Given a two-point function \( \omega \) for the original background \( \tilde{A} \), construct a two-point function of the extended theory to be

\[
\omega_H(\Phi, \Psi) = \omega(\phi, \psi) + \omega_0(\Phi_3, \Psi_3),
\]

---

\(^4\)This paragraph is rather technical and deals with the details of how to construct the algebras \( \mathfrak{A} \) as evaluation functionals. A reader more interested in the structural aspects may safely skip it.
where $\Phi = (\phi, \Phi_3)$ in the decomposition above, $\Phi$ and $\Psi$ are test sections, and $\omega_0$ is some two-point function for the supplementary singlet with a vanishing current. We now require that for the implementers for changes of the background connection of the form (33), we have

$$V_A(A')_\omega(\phi) = V_{A_H}(A'_H)_{\omega_H}(\phi, 0).$$

(34)

Here we interpreted elements of the algebra as evaluation functionals (with $\phi$ a configuration) and used the “quantum functional” notation of [8], where in order to make local covariance explicit, one does not work with an algebra with a fixed $\star$ product derived from a two-point function $\omega$, but with flat sections over the set of all Hadamard two-point functions (flatness being defined by the canonical equivalence relation between different Hadamard two-point functions). The way to prove (34) is by noting that the implementers enter on both sides obey the same ODE if one reaches $A'$ by a path of background changes. Namely, by assumption, $\omega_0$ does not contribute to the current and also the correction term (30) does not contribute, as the original theory had no local anomalies (the constants $d_{abc}$ vanish).

To compute the global anomaly on a trivial background connection, we proceed as follows: Given any path $h(\lambda)$ of $H$-gauge transformations as in (32), we set

$$A = h^{-1} dh,$$

(35)

where $d$ denotes the four-dimensional differential. Its derivative w.r.t. $\lambda$ is given by

$$\dot{A} = \partial_\lambda h^{-1} dh + h^{-1} d\partial_\lambda h = d\Lambda + [A, \Lambda]$$

(36)

with

$$\Lambda = h^{-1} \partial_\lambda h.$$  

(37)

Integrating the consistent anomaly (31) by using (19), we obtain the following candidate for the global $SU(2)$ anomaly

$$V_0(g_H^{-1} dg_H) = \exp \left( \frac{1}{48\pi^2} \int_0^1 d\lambda \int_M \text{tr} (\Lambda A \wedge A \wedge A \wedge A) \right)$$

$$= \exp \left( \frac{1}{240\pi^2} \int_{M \times [0,1]} h^*(\mu^5_H) \right),$$

(38)

where we used that the connection $A$ has vanishing curvature, i.e. $dA = -A \wedge A$. Moreover,

$$\mu^5_H = \text{tr} (\mu_H \wedge \mu_H \wedge \mu_H \wedge \mu_H \wedge \mu_H) \in \Omega^5(H)$$

is the Cartan 5-form on $H = SU(3)$ and $h^*(\mu^5_H) \in \Omega^5(M \times [0,1])$ denotes its pullback along our path of gauge transformations $h : M \times [0,1] \rightarrow H$.

It remains to show that (38) implies that $V_0(g_H^{-1} dg_H) = -1$, i.e. that the global $SU(2)$ anomaly may be detected in our approach. Recalling that our path of gauge transformations $h : M \times [0,1] \rightarrow H$ satisfies the boundary conditions in (32), it defines an element in the 5-th homotopy group $\pi_5(H/G)$ of the quotient $H/G = SU(3)/SU(2)$. The integral in (38) does not depend on the choice of representative, hence it defines a mapping

$$\pi_5(H/G) \rightarrow \mathbb{R}, \quad [h] \mapsto \frac{1}{240\pi^2} \int_{S^5} h^*(\mu^5_H),$$

(39)

which is easily seen to be a group homomorphism. Using as in [13] the following exact sequence of homotopy groups

$$\pi_5(H) \longrightarrow \pi_5(H/G) \longrightarrow \pi_4(G) \longrightarrow \pi_4(H),$$

(40)
together with the normalization

$$\frac{1}{240\pi^2} \int_{S^5} h_4^1(\mu_H^5) = 2\pi i$$

(41)

for the generator \([h_1]\) of \(\pi_5(H) \simeq \mathbb{Z}\), we obtain the desired result that \(V_0(g_H^{-1}dg_H) = -1\).

Let us explain this in more detail: Our non-trivial large \(G = SU(2)\)-gauge transformation \(g\) represents by definition the non-trivial homotopy class \([g] = -1 \in \pi_4(G) \simeq \mathbb{Z}_2\). Using the exact sequence \((40)\) and \(\pi_4(H) = 0\) for \(H = SU(3)\), we obtain a preimage of this class in \(\pi_5(H/G)\), which we may represent by a path \(h\) of \(H\)-gauge transformations as in \((32)\). As the corresponding homotopy class \([h] \in \pi_5(H/G) \simeq \mathbb{Z}\) is an odd number, the group homomorphism property of \((39)\) together with the normalization \((41)\) implies that the exponent in \((38)\) is an odd multiple of \(\pi i\). Hence, \(V_0(g_H^{-1}dg_H) = -1\) and we have detected the global \(SU(2)\) anomaly.

Remark 6. As discussed in Section 3, the global anomaly is not changed by compactly supported modifications of the background connection. Hence, it is conceivable that it depends on the asymptotic behavior of the background connection. This, however, is not the case for the global \(SU(2)\) anomaly, i.e., for a generic background connection \(\bar{A}\) we have \(V_{\bar{A}}(\bar{A}^0 - \bar{A}) = -1\), where \(g\) is a \(G\)-gauge transformation representing the non-trivial homotopy class \([g] = -1 \in \pi_4(G) \simeq \mathbb{Z}_2\). Instead of \((35)\), in the present situation we have to take

\[A = \bar{A}^h - \bar{A},\]

where \(h\) is a path of \(H\)-gauge transformations as in \((32)\). Instead of \((36)\), the \(\lambda\)-derivative then reads as

\[\dot{A} = d\Lambda + [\bar{A}^h, \Lambda]\]

with \(\Lambda\) given by \((37)\). It is convenient to regard \(\Lambda\) as the 5-th component of the 5-dimensional \(H\)-gauge potential on \(M \times [0,1]\) given by

\[A^h \equiv \bar{A}^h + \Lambda \ d\lambda = h^{-1}\dot{\bar{A}}h + h^{-1}dh,\]

(42)

where \(d\) is the 5-dimensional de Rham differential on \(M \times [0,1]\). Integrating the consistent anomaly \((31)\) by using \((19)\), we obtain after some simplifications

\[V_{\bar{A}}(\bar{A}^0 - \bar{A}) = \exp\left(\frac{1}{240\pi^2} \int_{M \times [0,1]} CS_5(A^h)\right),\]

where

\[CS_5(A^h) = \text{tr}\left( \mathbf{F}^h \wedge \mathbf{F}^h \wedge A^h - \frac{1}{2} \mathbf{F}^h \wedge A^h \wedge A^h + \frac{1}{10} A^h \wedge A^h \wedge A^h \wedge A^h \wedge A^h \right)\]

is the Chern-Simons 5-form and \(\mathbf{F}^h \equiv dA^h + A^h \wedge A^h\) is the 5-dimensional curvature. Using that by \((12)\) \(A^h\) is obtained by an \(H\)-gauge transformation of the 4-dimensional gauge potential \(A\), the transformation property of Chern-Simons forms together with \(CS_5(A) = 0\) (because \(\bar{A}\) is independent of \(\lambda\) and has no \(d\lambda\) component) implies that

\[\frac{1}{240\pi^2} \int_{M \times [0,1]} CS_5(A^h) = \frac{1}{240\pi^2} \int_{M \times [0,1]} h^5(\mu_H^5).\]

Hence, we obtain the same expression for the global \(SU(2)\) anomaly as in the case \(\bar{A} = 0\) above, cf. \((38)\).
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