SOME CONDITIONS ON DOUGLAS ALGEBRAS THAT IMPLY THE INVARIANCE OF THE MINIMAL ENVELOPE MAP

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ABSTRACT. We give several conditions on certain families of Douglas algebras that imply that the minimal envelope of the given algebra is the algebra itself. We also prove that the minimal envelope of the intersection of two Douglas algebras is the intersection of their minimal envelope.

1. Introduction

Let $D$ denote the open unit disk in the complex plane, and $T$ the unit circle. By $L^\infty$ we mean the space of essentially bounded measurable functions on $T$ with respect to the normalized Lebesgue measure. We denote by $H^\infty$ the space of all bounded analytic functions in $D$. Via identification with boundary functions, $H^\infty$ can be considered as a uniformly closed subalgebra of $L^\infty$. Any uniformly closed subalgebra $B$ strictly between $H^\infty$ and $L^\infty$ is called a Douglas algebra. We denote by $M(B)$ the maximal ideal space of a Douglas algebra $B$. If $C$ is the set of all continuous functions on $T$, we set $H^\infty + C = \{h + g : h \in H^\infty, g \in C\}$. Then $H^\infty + C$ becomes the smallest Douglas algebra containing $H^\infty$ properly.

The function

$$q(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z - z_n}{1 - \bar{z}_n z}$$

is called a Blaschke product if $\sum_{n=1}^{\infty} (1 - |z_n|)$ converges. The set $\{z_n\}_n$ is called the zero set of $q$ in $D$. Here $|z_n|/z_n = 1$ is understood whenever $z_n = 0$. We call $q$ an interpolating Blaschke product if

$$\inf_n \prod_{m:m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_n z_m} \right| > 0.$$
An interpolating Blaschke product $q$ is called sparse (or thin) if

$$\lim_{n \to \infty} \prod_{m \neq n} \frac{|z_m - z_n|}{1 - \overline{z_n}z_m} = 1.$$  

The set $Z(q) = \{x \in M(H^\infty) \setminus D : q(x) = 0\}$ is called the zero set of $q$ in $M(H^\infty + C)$. Any function $h$ in $H^\infty$ with $|h| = 1$ almost everywhere on $T$ is called an inner function. Since $|q| = 1$ for any Blaschke product, Blaschke products are inner functions. Let $QC = (H^\infty + C) \cap (\overline{H^\infty + C})$ and for $x \in M(H^\infty + C)$, set

$$Q_x = \{y \in M(L^\infty) : f(x) = f(y) \text{ for all } f \in QC\}.$$  

Then $Q_x$ is called the QC-level set for $x$. For $x \in M(H^\infty + C)$, we denote by $\mu_x$ the representing measure for $x$, and its support set by $\text{supp } \mu_x$. By $H^\infty[\overline{\bar{q}}]$, we mean the Douglas algebra generated by $H^\infty$ and the complex conjugate of the function $q$. Since $M(L^\infty)$ is the Shilov boundary for every Douglas algebra, a closed set $E$ contained in $M(L^\infty)$ is called a peak set for a Douglas algebra $B$ if there is a function $f$ in $B$ with $f = 1$ on $E$ and $|f| < 1$ on $M(L^\infty) \setminus E$. A closed set $E$ is a weak peak set for $B$ if $E$ is the intersection of a family of peak sets. If the set $E$ is a weak peak set for $H^\infty$ and we define

$$H_E^\infty = \{f \in L^\infty : f|_E \in H^\infty|_E\},$$

then $H_E^\infty$ is a Douglas algebra. For a Douglas algebra $B$, we define $B_E$ similarly.

For an interpolating Blaschke product $q$ we put $N(\overline{\bar{q}})$ the closure of

$$\bigcup\{\text{supp } \mu_x : x \in M(H^\infty + C), |q(x)| < 1\}.$$  

Then $N(\overline{\bar{q}})$ is a weak peak set for $H^\infty$. By $N_0(\overline{\bar{q}})$ we denote the closure of $\bigcup\{\text{supp } \mu_x : x \in Z(q)\}$. In general $N_0(\overline{\bar{q}})$ does not equal $N(\overline{\bar{q}})$, but in this paper $N_0(\overline{\bar{q}}) = N(\overline{\bar{q}})$.

For $x \in M(H^\infty)$, we let $E_x = \{y \in M(H^\infty) : \text{supp } \mu_y = \text{supp } \mu_x\}$ and call $E_x$ the level set of $x$. Since the sets $\text{supp } \mu_x$ and $N(\overline{\bar{q}})$ are weak peak sets for $H^\infty$, both $H^\infty_{\text{supp } \mu_x}$ and $H^\infty_{N(\overline{\bar{q}})}$ are Douglas algebras. For the interpolating Blaschke product $q$, set

$$A = \bigcap_{x \in M(H^\infty + C)} \{H_{\text{supp } \mu_x}^\infty : |q(x)| < 1\}$$

and

$$A_0 = \bigcap \{H_{\text{supp } \mu_x}^\infty : x \in Z(q)\}.$$  

Our assumptions on $q$ throughout this paper imply that $H_{N(\overline{\bar{q}})}^\infty = A = A_0$ (see [10]).

For $x$ and $y$ in $M(H^\infty)$, the pseudohyperbolic distance $\rho$ is defined by

$$\rho(x, y) = \text{supp}\{|h(y)| : \|h\|_\infty \leq 1, h \in H^\infty, h(x) = 0\}.$$
For $x$ and $y$ in $D$, we have
\[
\rho(x, y) = \left| \frac{x - y}{1 - \bar{y}x} \right|.
\]
For $x \in M(H^\infty)$, we define the Gleason part $P_x$ of $x$ by
\[
P_x = \{ y \in M(H^\infty) : \rho(x, y) < 1 \}.
\]
If $P_x \neq \{ x \}$, then $x$ is said to be a nontrivial point. For the definition of those interpolating Blaschke products that are of type G and of finite type G, see [8].

Let $B$ be a Douglas algebra. The Bourgain algebra $B_h$ of $B$ relative to $L^\infty$ is the set of $f$ in $L^\infty$ such that $\|ff_n + B\|_\infty \to 0$ for every sequence $\{f_n\}$ in $B$ with $f_n \to 0$ weakly in $B$. An algebra $B$ is called a minimal superalgebra of an algebra $A$ if $A \subset B$ and $\text{supp} \mu_x = \text{supp} \mu_y$ for all $x, y \in M(A \setminus M(B)$. The minimal envelope $B_m$ of a Douglas algebra $B$ is defined to be the smallest Douglas algebra that contains all the minimal superalgebras of $B$. The mapping that assigns to $B$ the Douglas algebra $B_m$ is called the minimal envelope map.

2. Conditions for $B_m = B$

We begin with the following theorem. The case for the Bourgain algebra $B_h$ has been proven in [11]. The proof used here is quite different from theirs, and can be used to show that this result also holds for the Bourgain algebras.

**Theorem 1.** Let $A$ and $B$ be Douglas algebras. Then $(A \cap B)_m = A_m \cap B_m$.

**Proof.** Since $A \cap B$ is contained in both $A$ and $B$ by Proposition 6 of [11], $(A \cap B)_m$ is contained in both $A_m$ and $B_m$. Hence $(A \cap B)_m \subset A_m \cap B_m$.

To show that $A_m \cap B_m \subset (A \cap B)_m$, let $\psi$ be an interpolating Blaschke product such that $\bar{\psi} \in A_m \cap B_m$. We show that $\bar{\psi} \in (A \cap B)_m$. By Theorem D of [11] we can assume that there is an $x \in M(A)$ and a $y \in M(B)$ such that $\{ \lambda \in M(A) : |\psi(\lambda)| < 1 \} = E_x$ and $\{ \mu \in M(B) : |\psi(\mu)| < 1 \} = E_y$. This implies that $M(A) = M(A[\bar{\psi}]) \cup E_x$ and $M(B) = M(B[\bar{\psi}]) \cup E_y$. Thus
\[
M(A \cap B) = M(A) \cup M(B) = M(A[\bar{\psi}]) \cup M(B[\bar{\psi}]) \cup E_x \cup E_y
\]
\[
= M(A[\bar{\psi}] \cap B[\bar{\psi}]) \cup E_x \cup E_y = M((A \cap B)[\bar{\psi}]) \cup E_x \cup E_y.
\]
Hence $\{ \omega \in M(A \cap B) : |\psi(\omega)| < 1 \} = E_x \cup E_y$. By Theorem 3 of [11] we have $\bar{\psi} \in (A \cap B)_m$. Our theorem follows.

The following corollaries are immediate consequences of the theorem.

**Corollary 1.** Let $A$ and $B$ be Douglas algebras with $A = A_m$ and $B = B_m$. Then $A \cap B = (A \cap B)_m$. 


Theorem 2. Let \( B_0 \) be a Douglas algebra with \((B_0)_m = B_0\) and \( B \) be a Douglas algebra such that there is an interpolating Blaschke product \( q \) with \( \{ \lambda \in M(B) : |q(\lambda)| = 1\} \subset M(B_0) \). If \( A \) has the property that \( A_m = A \), then \( B[q] \cap A = (B[q] \cap A)_m \). In particular, \( H^\infty[\tilde{\varphi}] \cap A = (H^\infty[q] \cap A)_m \) for every interpolating Blaschke product \( \varphi \).

Proof. The hypothesis \( \{ \lambda \in M(B) : |q(\lambda)| = 1\} \subset M(B_0) \) implies that \( B_0[q] = B[q] \), hence \( [B[q]]_m = B[q] \). So \( B[q] \cap A = (B[q] \cap A)_m \) follows from our theorem. By Theorem 1 of \( [2] \) we have \( H^\infty_m = H^\infty + C \), so \( \{ \lambda \in M(H^\infty) : |\psi(\lambda)| = 1\} \subset M(H^\infty + C) \) for every interpolating Blaschke product. The second part of the corollary follows. \( \square \)

Corollary 2. Let \( B_0 \) be a Douglas algebra with \((B_0)_m = B_0\) and \( B \) be a Douglas algebra such that there is an interpolating Blaschke product \( q \) with \( \{ \lambda \in M(B) : |q(\lambda)| = 1\} \subset M(B_0) \). For an interpolating Blaschke product \( q \) set \( B_x = B[q] \cap H^\infty_{\text{supp} \mu_x} \) for each \( x \in Z(q) \). Put \( B_e = \bigcap \{ B_x : x \in Z(q) \} \).

(i) If \( q \) is of finite type \( G \), then \((B_e)_b = B_e\).

(ii) If \( q \) is the product of a finite number of sparse interpolating Blaschke products, then \((B_e)_m = B_e\).

Proof. We show that \( B_e = B[\tilde{\varphi}] \cap H^\infty_{N(q)} \). By, an unpublished result of D. Sarason,

\[
M(B_e) = M\left( \bigcap_{x \in Z(q)} B_x \right) = \bigcup_{x \in Z(q)} M(B_x) = \bigcup_{x \in Z(q)} M(B[q] \cap H^\infty_{\text{supp} \mu_x}) = \bigcup_{x \in Z(q)} (M(B[q]) \cup M(H^\infty_{\text{supp} \mu_x})) = M(B[q]) = M(B[q] \cap H^\infty_{N(q)}).
\]

Now, if \( q \) is of finite type \( g \), Proposition 1 of \( [9] \) and Theorem 3.2(i) of \( [8] \) imply \( \bigcap_{x \in Z(q)} H^\infty_{\text{supp} \mu_x} = H^\infty_{N(q)} \). Thus

\[
\bigcup_{x \in Z(q)} M(H^\infty_{\text{supp} \mu_x}) = M(H^\infty_{N(q)}).
\]

and we get

\[
M(B_e) = M(B[q]) \cup M(H^\infty_{N(q)}) = M(B[q] \cap H^\infty_{N(q)}).
\]

So by the Chang–Marshall Theorem \( [1], [13] \) we have \( B_e = B[\tilde{\varphi}] \cap H^\infty_{\text{supp} \mu_x} \).

The condition \( \{ \lambda \in M(B) : |q(\lambda)| = 1\} \subset M(B_0) \) implies that \( B[q] = B_0[q] \). Hence \( [B[q]]_m = [B_0[q]]_m = [B_0]_m \tilde{\varphi} = B_0[\tilde{\varphi}] = B[\tilde{\varphi}] \), where the middle equality
follows from Theorem 4 of [11]. By theorem 1 of [7] (and its proof) we have \((H^\infty_{N(q)})_b = H^\infty_{N(q)}\) if \(q\) is of finite type \(G\). Thus

\[(B_e)_b = (B[\varphi] \cap H^\infty_{N(q)})_b = B[\varphi] \cap H^\infty_{N(q)} = B_e.\]

This proves (i).

Now if \(q\) is the product of a finite number of sparse Blaschke products, then by Theorem 1 of [7] we have \((H^\infty_{N(q)})_m = H^\infty_{N(q)}\), and so \((B_e)_m = B_e\). This proves (ii).

\[\square\]

**Corollary 3.** Suppose the hypothesis of Theorem 2 holds if \(\varphi = q\). Then \(B_e = B\), and:

(i) \(B = (B)_b\) if \(q\) is of finite type \(G\);

(ii) \(B = B_m\) is the product of a finite number of sparse Blaschke products.

**Theorem 3.** Let \(B\) be a Douglas algebra and let \(E\) be a peak set for \(B\). Then \((B_E)_m = B_E\).

**Proof.** Since \(E\) is a peak set for \(B\), there is a function \(f \in B\) such that \(f = 1\) on \(E\) and \(|f| < 1\) on \(M(L^\infty) \setminus E\). By [3, p. 39] we have

\[M(B_E) = \{x \in M(B) : \text{supp} \mu_x \subseteq E\} \cup M(L^\infty) = \{x \in M(B) : f(x) = 1\} \cup M(L^\infty).\]

Let \(I\) be any interpolating Blaschke product such that \(Z(I) \cap M(B_E) \neq \emptyset\). We will show that there is an uncountable set \(\Gamma \subset Z(I) \cap M(B_E)\) such that \(E_\alpha \neq E_\beta\) for distinct \(\alpha, \beta \in \Gamma\). Let \(\{z_n\}\) be the zero sequence of \(I\) in \(D\) and \(y \in Z(I) \cap M(B_E)\).

Since \(y \in Z(I)\), there is a subsequence \(\{z_{n_k}\}\) of \(\{z_n\}\) such that \(z_{n_k} \to y\). Since \(y \in M(B_E)\), we have \(f(z_{n_k}) \to 1\). Let \(I_1\) be the factor of \(I\) such that \(\{z_{n_k}\}\) is the zero sequence of \(I_1\). Then

\[Z(I_1) = \overline{\{z_{n_k}\}} \setminus \{z_{n_k}\}\]

and since \(Z(I_1)\) is equivalent to the Čech compactification of the integers, \(Z(I_1)\) is an uncountable set. Since \(f(z_{n_k}) \to 1\), we have that \(f = 1\) on \(Z(I_1)\). Thus \(Z(I_1) \subset Z(I) \cap M(B_E)\), and so \(Z(I) \cap M(B_E)\) is an infinite set. To show that \(\Gamma\) exists, take a subsequence \(\{z_{n_{q_k}}\}\) of \(\{z_{n_k}\}\) such that \(\{z_{n_{q_k}}\}\) is a sparse Blaschke sequence in \(D\) (see [4]). Let \(I_0\) be the sparse Blaschke product whose zero sequence in \(D\) is \(\{z_{n_{q_k}}\}\). Then, by Theorem 1 of [12], we have

\[N(I_0) = \bigcup_{x \in Z(I_0)} Q_x.\]
Since \( Z(I_0) \) is uncountable, Lemma 4 of [12] says that \( Q_x \neq Q_y \) for distinct \( x, y \in Z(I_0) \). This implies that \( \text{supp } \mu_x \cap \text{supp } \mu_y = \emptyset \) and \( E_x \neq E_y \). Set \( \Gamma = Z(I_0) \). Then

\[
\bigcup_{\alpha \in \Gamma} E_\alpha \subseteq \{ \lambda \in M(B_E) : |I(\lambda)| < 1 \}.
\]

By Theorem 3 of [11], \( \bar{I} \not\in (B_E)_m \). This shows that \( (B_E)_m = B_E \).

**Theorem 4.** Let \( A \) be a Douglas algebra and let \( \{q_n\} \) be a sequence of interpolating Blaschke products such that \( \overline{\sigma_n} \not\in A \) for every \( n \). Let \( B = A[\overline{q_n} ; n = 1, 2, 3 \ldots] \). Then \( B_m = B \).

**Proof.** Since \( M(B) = \{ \lambda \in M(A) : |q_n(\lambda)| = 1, n = 1, 2, \ldots \} \), we can define the function \( F \) on \( M(H^\infty) \) as

\[
F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} |q_n(x)|.
\]

Then \( F = 1 \) on \( M(B) \) and \( |F| < 1 \) on \( M(H^\infty) \setminus M(B) \). Let \( C \) be any interpolating Blaschke product such that \( Z(C) \cap M(B) \neq \emptyset \). Let \( \{z_n\} \) be the zero sequence of \( C \) in \( D \). As in the proof of Theorem 3 we can find a sparse subsequence \( \{z_{n_k}\} \) of \( \{z_n\} \) such that \( F = 1 \) on \( \overline{\{z_{n_k}\}} \setminus \{z_{n_k}\} \). If we set \( C_1 \) to be the factor of \( C \) whose zero sequence in \( D \) is \( \{z_{n_k}\} \), then as in the proof of Theorem 3 we can show that \( E_\alpha \neq E_\beta \) for distinct \( \alpha, \beta \in Z(C_1) \), and

\[
\bigcup_{\alpha \in Z(C_1)} E_\alpha \subset \{ \lambda \in M(B) : |C\lambda| < 1 \}.
\]

So, by Theorem 3 of [11], we have \( \bar{C} \not\in B_m \). This implies that \( B_m = B \).

**Theorem 5.** Let \( \{q_n\} \) be a sequence of interpolating Blaschke products not invertible in a Douglas algebra \( A \). Suppose that for each \( n \) there is a Douglas algebra \( A_n \) with \( (A_n)_m = A_n \) and

\[
\{ \lambda \in M(A) : |q_n(\lambda)| = 1 \} \subset M(A_n).
\]

Let \( B = \bigcap_{n=1}^{\infty} A[\overline{q_n}] \). Then \( B_m = B \).

**Proof.** Since \( B \subseteq A[\overline{q_n}] \) for \( n = 1, 2, \ldots \), by Proposition 6 of [11] we have \( B_m \subseteq (A[\overline{q_n}])_m \). Hence

\[
B_m \subseteq \bigcap_{n=1}^{\infty} (A[\overline{q_n}])_m.
\]
To show that $B_m = B$ it suffices to show that $(A[\tilde{q}_n])_m = A[\tilde{q}_n]$ for all $n$. Since 
$\{\lambda \in M(A) : |q_n(\lambda)| = 1\} \subset M(A_n)$, we have $A[\tilde{q}_n] = A_n[\tilde{q}_n]$. So using Theorem 4 of [11] we have

$$(A[\tilde{q}_n])_m = (A_n[\tilde{q}_n])_m = (A_n)_m[\tilde{q}_n] = A_n[\tilde{q}_n] = A[\tilde{q}_n].$$

Thus $B_m \subseteq \bigcap_{n=1}^{\infty} (A[\tilde{q}_n])_m = \bigcap_{n=1}^{\infty} A[\tilde{q}_n] = B$. \hfill \Box

**Corollary 4.** If $B = \bigcap_{n=1}^{\infty} H^{\infty}[\tilde{q}_n]$, then $B_m = B$.

**Theorem 6.** Let $B$ be a Douglas algebra. Then $B \subset B_m$ if and only if there exists a point $x \in M(B)$ whose level set $E_x$ is an open subset in $M(B)$.

**Proof.** Let $B \subset B_m$. By Theorem D of [11] there exists an interpolating Blaschke product $\varphi$ such that $\{\lambda \in M(B) : |\varphi(\lambda)| < 1\} = E_x$ for some $x \in M(B)$. By Lemma 9(ii) of [11], $E_x$ is an open subset of $M(B)$.

Suppose that $B = B_m$. Then for every $y \in M(B)$ and every interpolating Blaschke product $q$ with $q(y) = 0$ the set $Z(q) \cap M(B)$ contains an infinite subset $\Gamma$ such that $E_\alpha \neq E_\beta$ for all distinct $\alpha, \beta \in \Gamma$, and

$$\bigcup_{\alpha \in \Gamma} E_\alpha = \{\lambda \in M(B) : |q(\lambda)| < 1\}.$$ 

We’ll show that there is a subset $\{x_\sigma\}_{\sigma \in A}$ of $\Gamma$ such that $y \notin \overline{\{x_\sigma\}_{\sigma \in A}} \setminus \{y\}$, that is,

$$E_y \subseteq \overline{\{E_\alpha\}_{\alpha \in \{\Gamma \setminus \{y\}\}}}.$$ 

Suppose to the contrary that $E_y \not\subset \overline{\{E_\alpha\}_{\alpha \in \{\Gamma \setminus \{y\}\}}}$ and $y$ are two closed disjoint subsets of $M(H^{\infty})$. Choose neighborhoods $V$ of $\overline{\{E_\alpha\}_{\alpha \in \{\Gamma \setminus \{y\}\}}}$ and $U$ of $\{y\}$ such that $\overline{U} \cap \overline{V} = \emptyset$. Let $q_1$ be the factor of $q$ with zeros in $U \cap \overline{D}$. Then we see that $Z(q_1) \cap M(B) \subset E_y$ and $\{\lambda \in M(B) : |q_1(\lambda)| < 1\} = E_y$. Thus $\tilde{q}_1 \in B_m \setminus B$, which is a contradiction. Hence

$$E_y \subseteq \overline{\{E_\alpha\}_{\alpha \in \{\Gamma \setminus \{y\}\}}}.$$ 

Thus there is a subset $\{x_\sigma\}_{\sigma \in A} \subset \Gamma$ such that $x_\sigma \to y$, and $x_\sigma \notin E_y$ for all $\sigma \in A$. This implies that $M(B) \setminus E_y$ is not a closed subset of $M(H^{\infty})$. Thus $E_y$ is not an open subset of $M(B)$. This proves our theorem. \hfill \Box

I have been unable to answer the following two questions, which I close with.

**Question 1.** Find an $x \in M(H^{\infty} + C) \setminus M(L^{\infty})$ such that $H^{\infty}_{\text{supp} \mu_x} = (H^{\infty}_{\text{supp} \mu_x})_m$ (if such an $x$ exists).
Question 2. Find a Douglas algebra $B$ such that $B \varsubsetneq B_b \varsubsetneq B_m$ (if such a Douglas algebra exists).

An immediate consequence of Theorem 6 above and Theorem 7 of [14], related to Question 2, is the following:

Theorem 7. Let $B$ be a Douglas algebra. Then $B = B_b \subsetneq B_m$ if and only if for all $x \in M(B)$ the set $P_x$ is not open in $M(B)$ but for some $x \in M(B)$ the set $E_x$ is an open subset in $M(B)$.

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