Charged String-like Solutions of Low-energy Heterotic String Theory

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Abstract

Two string-like solutions to the equations of motion of the low-energy effective action for the heterotic string are found, each a source of electric and magnetic fields. The first carries an electric current equal to the electric charge per unit length and is the most general solution which preserves one half of the supersymmetries. The second is the most general charged solution with an event horizon, a ‘black string’. The relationship of the solutions to fundamental, macroscopic heterotic strings is discussed, and in particular it is shown that any stable state of such a fundamental string also preserves one half of the supersymmetries, in the same manner as the first solution.
1 Introduction

In the hope of understanding more about the structure of string theory by studying non-perturbative results, some interest has been focused recently on classical solutions to the theory which have the properties of solitons or instantons. Starting with the low-energy effective action for the massless, bosonic modes of the heterotic string, Dabholkar et al. found a singular string-like solution sharing many of the properties of supersymmetric solitons. In some sense, the solution represents the fields around a macroscopic fundamental string, since the singularities can be matched onto source terms from a sigma-model action describing the coupling of the fundamental string worldsheet to the spacetime fields. Starting with the same action, though restricted to a ten-dimensional spacetime, Strominger found a supersymmetric five-brane solution, which is essentially a Yang-Mills instanton in the four transverse dimensions, and so is a true soliton without singularities. He also conjectured that five-branes might be dual to heterotic strings, in the sense of Montenon and Olive, that string theory has two equivalent, ‘dual’ formulations, one a theory of strings where the five-branes appear as solitons, and one a theory of five-branes with the strings as solitons, and where weak coupling in one formulation corresponds to strong coupling in the other.

Giving a different perspective, Horowitz and Strominger have found that the solution of Dabholkar et al. is the extremal limit of a general ‘black string’ solution (one with an event horizon), suggesting that these solutions may prove to be useful models for studying black holes (or in general black p-branes) in the context of string theory.

The supersymmetric solution of Dabholkar et al. and the black string of Horowitz and Strominger both set the gauge fields in the low-energy effective action to zero. The purpose of this paper is to generalize these two solutions to include the presence of a non-zero $U(1)$ gauge field, and so find the most general charged string-like solutions in each case.

The paper is divided as follows. Section 2 describes, in terms of the usual first-quantized theory, the supersymmetries of a macroscopic heterotic string. It emerges that any stable state is invariant under half of the supersymmetries, a property which should be reflected in any string-like solution representing the fields around such a fundamental string. This constraint is then used in section 3 to find the most general charged version of the solution of Dabholkar et al.. The parameters of this solution are interpreted in terms of the properties of a fundamental string and a multiple-string solution is also given. Section 4 gives the most general charged black string solution, identifying both the extremal supersymmetric limit and the limit giving the uncharged solution of Horowitz and Strominger. Section 5 briefly summarizes the results.

2 The supersymmetries of macroscopic heterotic strings

It is a common property of most of the soliton solutions of supersymmetric theories that the solution preserves one half of the supersymmetries and so saturates a Bogomol’nyi bound. The broken supersymmetries generate at least some of the fermionic zero modes of the soliton. This property is useful when looking for soliton solutions since the equations for the supersymmetric variations of the fields are only first order, whereas the equations of motion are second order. For the solution
of \[2\] in ten dimensions, the supersymmetry transformations are described by a Majorana-Weyl spinor \(\epsilon\), and which supersymmetries are broken by the solution depends on the chirality of \(\epsilon\) in the eight-dimensional space transverse to the static string-like solution. Should the supersymmetries split in this way for a general string-like solution in ten dimensions?

The string-like solutions to the low-energy field theory for the heterotic string are taken to represent, at least to first order in the string coupling constant \(\alpha'\), the fields around a stable macroscopic heterotic string. Consequently the symmetries of the solution should be reflected in the symmetries of the fundamental, first-quantized string.

Fundamental macroscopic strings in \(D\)-dimensional spacetime are described in \[2\]. Heterotic strings exist in ten dimensions so first \(10 - D\) dimensions are compactified and replaced by a superconformal field theory. Then to construct a macroscopic string, one of the remaining spatial dimensions is also compactified onto a circle of large radius. Strings with non-zero winding number about this direction are designated as macroscopic states, since the string must be long enough to loop around the circle at least once. A tower of stable, purely bosonic, macroscopic states of increasing winding number can be constructed out of the product of a ground state of the right-moving, supersymmetric sector and various excited states from the left-moving, gauge sector.

For \(D = 10\) these stable states can be described in the light-cone gauge following \[3\]. Spacetime is considered as a product of a two-dimensional Minkowski space and an eight-dimensional transverse Euclidean space. Ten-dimensional Majorana-Weyl spinors, necessarily of a definite chirality, say \(+1\), then decompose under \(SO(9,1) \supset SO(1,1) \otimes SO(8)\) so that

\[
16 \to (\mathbf{1}_+, \mathbf{8}_+) \oplus (\mathbf{1}_-, \mathbf{8}_-), \quad (2.1)
\]

where the subscripts represent the spinor’s chirality in the subspaces. As ever the left- and right-moving modes of the string fully decouple, and for the latter, considered as comprising half a Green-Schwarz superstring, by a judicious use of the \(\kappa\)- and residual conformal symmetry, the bosonic and fermionic coordinates can be made to satisfy free wave equations and decomposed into Fourier modes \(\alpha^i_n\) and \(S^a_n\) respectively. (Here \(i\) refers to a vector index in the transverse \(SO(8)\) space, \(a\) to a \(\mathbf{8}_+\) spinor index and shortly \(\dot{a}\) will refer to a \(\mathbf{8}_-\) spinor index; \(n\) labels the mode.) The supersymmetry charge \(Q\) transforms as a Majorana-Weyl spinor, so for a general infinitesimal transformation parametrized by a Majorana-Weyl spinor \(\epsilon = (\zeta^a, \eta^{\dot{a}})\) the operator generating the field variations decomposes, so that

\[
\epsilon Q = \zeta^a Q^a + \eta^{\dot{a}} Q^{\dot{a}}, \quad (2.2)
\]

where the charges have the mode expansions

\[
Q^a = (2p^+)^{1/2} S^a_0, \quad Q^{\dot{a}} = (p^+)^{-1/2} \gamma^{i\dot{a}} \sum_{-\infty}^{\infty} S^a_{-n} \alpha^i_n. \quad (2.3)
\]

The coefficient \(p^+\) is one of the light-cone momenta, \(p^\pm = (p^0 \pm p^1) / \sqrt{2}\), where the superscripts refer to orthogonal time- and space-like directions in the two-dimensional Minkowski space, whilst
the coefficients $\gamma_{ia}$ are the building blocks for a set of real eight-dimensional Dirac matrices, or alternatively the Clebsch-Gordon coefficients relating the vector ($i$ index) and two spinor ($a$ and $\dot{a}$ indices) representations of $SO(8)$.

How does the supersymmetry act on the stable macroscopic states? First recall that these states were ground states of the right-moving sector, and further that by definition any annihilation operator — a mode with $n < 0$ — acting on a ground state gives zero. Thus since $\alpha^{i}_0$ is simply the transverse momentum $p^i$, the supersymmetry generator acting on a right-moving ground state $|\phi_0\rangle$ becomes a linear combination of the fermionic zero mode operators $S^{a}_0$, namely

$$\epsilon Q |\phi_0\rangle = \frac{1}{\sqrt{p^+}} \left( \sqrt{2} p^+ \zeta^a + p^i \eta^{\dot{a}} \gamma_{ia} \right) S^{a}_0 |\phi_0\rangle.$$  (2.4)

It would be natural to choose coordinates so that the string is at rest in the transverse space, that is, to choose one of the frames where $p^i = 0$, so that all dependence on $\eta^{\dot{a}}$ in (2.4) then disappears, implying that half the supersymmetries vanish. However since the ground state of the supersymmetric sector has $p^2 = 0$, then a frame where $p^i = 0$ has either $p^+ = 0$ or $p^- = 0$ and the light-cone gauge can not then be used.

Nevertheless, it is still possible to write the supersymmetry generator in terms of spinors of definite chirality in a transverse space in which the string is at rest. For instance, consider first some general frame used for the light-cone gauge. Clearly, simply making a Lorentz boost along $p^i$ to ‘catch up’ with the string gives a second frame with the string at rest in the transverse space. The spinor $\epsilon$ can be decomposed into parts of opposite chirality in the transverse space of the second frame, labelled $\zeta^i$ and $\eta^a$, so that in the first frame

$$\epsilon = \left( \begin{array}{c} \zeta^a \\ \eta^{\dot{a}} \end{array} \right) = \frac{1}{\sqrt{p^+ (p^+ - p^-)}} \left( \begin{array}{cc} p^+ & -p^i \gamma_{ia} / \sqrt{2} \\ -p^i \gamma^i_{\dot{a}} / \sqrt{2} & p^+ \end{array} \right) \left( \begin{array}{c} \zeta^{\dot{a}} \\ \eta^{\dot{a}} \end{array} \right) = U^{-1} \left( \begin{array}{c} \zeta^{\dot{a}} \\ \eta^{\dot{a}} \end{array} \right),$$  (2.5)

where $U$ is the Lorentz boost along $p^i$. Furthermore the supersymmetry generator acting on the ground state (2.4) can be written as

$$Q |\phi_0\rangle = \sqrt{2(p^+ - p^-)} U \left( \begin{array}{c} S^{a}_0 \\ 0 \end{array} \right) |\phi_0\rangle$$  (2.6)

so that, with $^T$ representing transposition,

$$\epsilon Q |\phi_0\rangle = \sqrt{2(p^+ - p^-)} \left( \begin{array}{c} \zeta^{\dot{a}} \\ \eta^{\dot{a}} \end{array} \right)^T U^{-1} U \left( \begin{array}{c} S^{a}_0 \\ 0 \end{array} \right) |\phi_0\rangle = \sqrt{2(p^+ - p^-)} \zeta^{\dot{a}} S^{a}_0 |\phi_0\rangle.$$  (2.7)

is independent of $\eta^{\dot{a}}$. The choice of frame for the light-cone gauge was quite arbitrary, so the conclusion is that for any 8-dimensional Euclidean subspace in which the string is at rest, if $\epsilon$ is of negative chirality in this subspace and so can be parametrized by $\eta^{\dot{a}}$, then the action of $\epsilon Q$ on the ground state is zero, and the ground state is invariant under these supersymmetries. However, if $\epsilon$ is of positive chirality in the subspace, and so is parametrized by $\zeta^{\dot{a}}$, the action of $\epsilon Q$ depends on the action of $S^{a}_0$ and so is generally non-zero, and these supersymmetries are broken. For a
macroscopic string it is natural to choose the subspace by first boosting to a frame where the string has motion only along its own length and then taking the space orthogonal to the string worldsheet.

In summary, it is indeed a generic property of stable macroscopic strings, or more generally of any state of the heterotic string which is a ground state of the right-moving sector, that half the supersymmetries are broken, and half preserved, depending on the chirality of the supersymmetry parameter $\epsilon$ in the space transverse to the string worldsheet in which the string is at rest. Any string-like soliton purported to approximately represent the fields around such a stable macroscopic string should reflect these symmetries.

### 3 A charged supersymmetric solution

This section describes a new string-like solution to the low-energy effective action for the massless modes of the heterotic string which includes a $U(1)$ gauge field. Starting with a suitable ansatz, a partial solution is obtained by imposing the condition that half the supersymmetries are broken. The solution is completed using the equations of motion.

To first order in the string loop expansion, the effective action is simply that of ten-dimensional supergravity coupled to super Yang-Mills. For the bosonic modes, namely the graviton $g_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$, the dilaton $\phi$ and the gauge potential $A_\mu$, generalized to $D$ dimensions, it is given by

$$S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} e^{-\phi/\alpha} \left( R + \left( \partial (\phi/\alpha) \right)^2 - \frac{1}{3} H^2 - \frac{\alpha'}{30} \text{tr} F^2 \right),$$

(3.1)

where $\alpha = \sqrt{2/(D-2)}$, $\alpha'$ is the string coupling constant and $\kappa$ is the gravitational coupling constant. The notation follows [3] in using the so-called ‘sigma-model metric’ which arises naturally when the action is derived by considering a string moving in a classical background of the massless bosonic fields. The physical metric $\tilde{g}$, for which the action has the usual curvature term without the additional $e^{-\phi/\alpha}$ factor, is related by $\tilde{g}_{\mu\nu} = e^{-\phi/\alpha} g_{\mu\nu}$. $F_{\mu\nu}$ is the usual Yang-Mills field strength, here in the adjoint representation of $E_8 \otimes E_8$ or $SO(32)$, and $R$ is the usual Ricci scalar, but $H_{\mu\nu\rho}$ is a generalized field strength for the antisymmetric tensor field, modified by a Chern-Simons three-form so that, in terms of forms, $H = dB - \frac{\alpha'}{30} \text{tr} \left( AF - \frac{1}{3} g A^3 \right)$. The equations of motion following from this action can be written, after some rearrangement of the dilaton and curvature equations, as

$$e^{\phi/\alpha} \nabla_\mu \left( e^{-\phi/\alpha} g^{\mu\nu} \nabla_\nu (\phi/\alpha) \right) + \frac{2}{3} H^2 + \frac{\alpha'}{30} \text{tr} F^2 = 0,$$

$$e^{\phi/\alpha} \nabla_\mu \left( e^{-\phi/\alpha} H^{\mu\nu\rho} \right) = 0,$$

$$e^{\phi/\alpha} D_\mu \left( e^{-\phi/\alpha} F^{\mu\nu} \right) + H^{\mu\rho\sigma} F_{\rho\sigma} = 0,$$

(3.2)

$$R_{\mu\nu} = \kappa^2 S_{\mu\nu} = -\nabla_\mu \nabla_\nu (\phi/\alpha) + H_{\mu\rho\sigma} H^\rho_{\nu\sigma} + \frac{\alpha'}{15} \text{tr} F_{\mu\rho} F^{\rho \nu},$$

where $\nabla_\mu$ is the spacetime-covariant derivative and $D_\mu$ is the full spacetime- and gauge-covariant derivative.
The simplest string-like solution with gauge fields is that corresponding to a straight, static macroscopic string with its worldsheet covering, say, the $x^0-x^1$ plane, and which carries a $U(1)$ current, a source of ‘electromagnetism’. It is spherically symmetric in the space transverse to the worldsheet, and should be invariant under translations in either of the worldsheet directions. This implies that the metric has the form

$$ds^2 = \hat{g}_{\alpha\beta}(r)dx^\alpha dx^\beta + e^{B(r)}d\vec{x} \cdot d\vec{x}, \quad (3.3)$$

where $r = \sqrt{\vec{x} \cdot \vec{x}}$ and $\alpha$ and $\beta$ are worldsheet indices, taking values 0 or 1, whilst the vector notation refers to the transverse space, subsequently also denoted by the index $i$ or $j$. For $D > 4$ the only other non-zero fields are $B_{01}$, $\phi$ and $A_0$ and $A_1$, and again these depend only on the variable $r$. The gauge fields are normalized by the following convention: for $SO(32)$, and in terms of the $SO(16)$ subgroup of $E_8 \otimes E_8$, a trace in the fundamental representation $\text{tr}_{\text{fund}}$ is related to a trace in the adjoint representation by $\text{tr}_{\text{fund}} = \text{tr}_{\text{adj}}$, so since it is natural to normalize the $U(1)$ fields to match the fundamental representation, the electromagnetic kinetic term is taken to be simply $\frac{1}{2} \alpha'/F^2$.

This ansatz is partially solved by imposing the condition, applicable when $D = 10$, that the solution preserves half of the supersymmetries of the theory, namely those generated by a supersymmetry parameter of negative chirality in the transverse space, or equivalently of negative chirality on the worldsheet. Since the ansatz is purely bosonic, the infinitesimal supersymmetry variations of the bosonic fields are necessarily zero, and only the variations of the fermionic fields need be considered. They are

$$\delta_\epsilon \lambda = \frac{\sqrt{2}}{4\kappa} \left[ -\gamma^\mu \partial_\mu \phi + \frac{1}{6} H_{\mu\nu\rho} \gamma^{\mu\nu\rho} \right] \epsilon, $$

$$\delta_\epsilon \psi_\mu = \frac{1}{\kappa} \left[ \partial_\mu + \frac{1}{4} \left( \omega_\mu^{mn} - H_\mu^{mn} \right) \Gamma_{mn} \right] \epsilon, $$

$$\delta_\epsilon \chi = -\frac{1}{4g} F_{\mu\nu} \gamma^{\mu\nu} \epsilon, \quad (3.4)$$

where Greek indices refer to the spacetime-coordinate basis for the tangent space and Roman indices to an orthonormal basis, the two of which are related by zehnbems $e^m_\mu$, while $\omega^{mn}_\mu$ is the corresponding spin connection, and where $\Gamma^m$ are the ten-dimensional Dirac matrices satisfying $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$, $\gamma^\mu = e^\mu_m \Gamma^m$ and $\gamma^{\mu_1 \cdots \mu_n}$ is the antisymmetrized product $\gamma^{[\mu_1 \cdots \mu_n]}$ with unit weight, and finally $g$ is the gauge coupling, proportional to $\kappa/\sqrt{\alpha'}$ for the heterotic string.

The supersymmetry condition implies that the supersymmetry variations are zero for any spinor parameter of the form $\epsilon = e^\sigma \epsilon_0$ where $\sigma$ is some unknown function on the transverse space, and $\epsilon_0$ is a constant spinor of negative chirality on the worldsheet, that is $\Gamma^0 \Gamma^1 \epsilon_0 = -\epsilon_0$. Upon substituting the ansatz and using this form of $\epsilon$, each of the supersymmetry variations has a factor of the form $(a + b \Gamma^0 \Gamma^1) \epsilon_0$. If the supersymmetry condition holds, the $a$ and $b$ in each expression must be equal,
giving the relations,

\[ \sqrt{-\hat{g}} \partial_i \phi = -\frac{1}{2} \epsilon^{\alpha\beta} H_{\alpha\beta}, \]
\[ H_{\alpha\beta} = \frac{1}{2} \partial_i \sqrt{-\hat{g}} \epsilon_{\alpha\beta}, \]
\[ \sqrt{-\hat{g}} \hat{g}^{\alpha\beta} F_{\beta} = \epsilon^{\alpha\beta} F_{\beta}, \]
\[ \sqrt{-\hat{g}} \partial_i A = -\frac{1}{2} \left( \sqrt{-\hat{g}} \omega_0^1 + H_{i01} \right), \]
\[ \hat{g}_{00} = -\lambda^{-1} \left( \sqrt{-\hat{g}} + \hat{g}_{01} \right), \quad \hat{g}_{11} = \lambda \left( \sqrt{-\hat{g}} - \hat{g}_{01} \right), \quad e^{\beta} = \mu, \]

where \( \epsilon^{\alpha\beta} \) is an antisymmetric matrix with \( \epsilon^{01} = 1 \), and \( \lambda \) and \( \mu \) are constants. By choosing a particular coordinate system, for which the metric is asymptotically Minkowski, the constants can be set to unity, so that by further choosing a particular gauge for \( B_{\mu\nu} \) and taking \( \phi \) to be asymptotically zero, the ansatz, now generalized to \( D \)-dimensional spacetime, becomes

\[ B = 0, \]
\[ \hat{g}_{\alpha\beta} = e^{E(r)} \left( -\left( 1 + C(r) \right) C(r) \left( 1 - C(r) \right) \right), \]
\[ e^{\phi/\alpha} = -2B_{01} = e^{E(r)}, \]
\[ A_0 = -A_1 = M(r). \]

Furthermore \( \sigma = \frac{1}{2} E \) and the Chern-Simons three-form is zero, so that \( H = dB \). As for the ansatz in [2], the assumption is that the only dependence on \( D \) is through \( \alpha \) in the dilaton relation. In fact, apart from the presence of gauge fields and the off-diagonal terms in \( \hat{g} \), (3.6) is identical to the ansatz of [2]; furthermore, for \( D = 10 \), it is the most general ansatz consistent with the spacetime symmetries and supersymmetries of the simplest string-like configuration.

The solution is completed by substitution into the equations of motion (3.2). To allow for multiple-string solutions, \( E, C \) and \( M \) are now taken to be functions of all the transverse coordinates \( x^i \), rather than of \( r \) alone. The ansatz is consistent; the antisymmetric tensor and dilaton equations give

\[ \eta^{ij} \partial_i \partial_j e^{-E} = 0, \]

and the equation for the gauge field gives

\[ \eta^{ij} (\partial_i \partial_j M - 2\partial_i E \partial_j M) = 0, \]

whilst the graviton equation gives (3.7) again, together with

\[ \eta^{ij} (\partial_i \partial_j C - 2\alpha' e^{-E} \partial_i M \partial_j M) = 0. \]

Finally the substitutions \( M = e^E N \) and \( C = R + \alpha' e^E M^2 \) give the Laplace equations:

\[ \eta^{ij} \partial_i \partial_j e^{-E} = \eta^{ij} \partial_i \partial_j N = \eta^{ij} \partial_i \partial_j R = 0. \]

The solution corresponding to a single string, with the metric asymptotically flat and the gauge choice that \( A_\mu \) is asymptotically zero, has

\[ e^{-E} = 1 + 2m\kappa^2 \Lambda, \quad N = q \Lambda, \quad R = -2p\kappa^2 \Lambda, \]

\[ (3.11) \]
where \( m, q \) and \( p \) are constants and

\[
\Lambda(r) = \begin{cases} 
\frac{1}{(D-4)\omega_{D-3}r^{D-4}} & \text{if } D > 4 \\
\frac{1}{2\pi} \log r & \text{if } D = 4
\end{cases}
\]

with \( \omega_{D-3} \) the volume of the \((D-3)\)-sphere, so that the fields are given by

\[
\frac{1}{1 + 2m^2\Lambda} \left[ \eta_{\alpha\beta} + \left( \frac{2p^2\Lambda - \alpha'q^2\Lambda^2}{1 + 2m^2\Lambda} \right) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}_{\alpha\beta} \right] dx^\alpha dx^\beta + d\vec{x} \cdot d\vec{x},
\]

\[
e^{-\phi/\alpha} = 1 + 2m^2\Lambda,
\]

\[
F_{i0} = -F_{i1} = - \left[ \frac{x^i/r}{\omega_{D-3}r^{D-3}} \right] \frac{q}{(1 + 2m^2\Lambda)^2},
\]

\[
H_{i01} = - \left[ \frac{x^i/r}{2\omega_{D-3}r^{D-3}} \right] \frac{2m^2}{(1 + 2m^2\Lambda)^2}.
\]

The asymptotic forms of the gauge and antisymmetric tensor fields show the electric charge per unit length on the string equals the electric current along the string, both given by \( q \), and that the charge per unit length acting as a source for \( B_{\mu\nu} \), using a normalization compatible with \([2]\), is given by \( m \). The solution is unchanged under a Lorentz boost along \( x^1 \), except that the charge \( q \) is rescaled. This is as might be expected since the electric current is a null vector \((q, -q)\) on the worldsheet.

As in \([2]\), the ADM prescription is used to define the mass and momentum per unit length of the solution, though with the physical metric \( \tilde{g} \), since this gives the usual form for the curvature term in the action. Far away from the string, the fields are small enough to use the linearized Einstein equations, with the metric written in terms of \( h_{\mu\nu} = \tilde{g}_{\mu\nu} - \eta_{\mu\nu} \) which is taken to be small. The total energy-momentum tensor \( \theta_{\mu\nu} \), which includes a contribution interpreted as the energy-momentum in the gravitational field, is then defined as

\[
k^2\theta_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R^{(1)}_{\rho\rho},
\]

where \( R^{(1)}_{\mu\nu} \) is the linearized Ricci tensor,

\[
R^{(1)}_{\mu\nu} = \frac{1}{2} \left( \partial_\rho \partial_\nu h^\rho_\mu + \partial_\mu \partial_\nu h^\rho_\rho - \partial_\rho \partial_\nu h^\rho_\mu - \partial_\mu \partial_\rho h^\rho_\nu \right),
\]

where all indices are raised and lowered with \( \eta_{\mu\nu} \). For the single string solution

\[
\theta_{\alpha\beta} = \frac{1}{2k^2} \partial^i \left[ -\partial_i h_{\alpha\beta} + \eta_{\alpha\beta} \left( \partial_i h^\gamma_\gamma + \partial_i h^j_\gamma - \partial_j h^\gamma_\gamma \right) \right],
\]

and integrating over the transverse space defines

\[
\Theta_{\alpha\beta} = \int d^{D-2}x \theta_{\alpha\beta} = \frac{1}{2k^2} \int_{S^D_{D-3}} d^dr \left[ -\eta_{\alpha\beta} \left( h^\gamma_\gamma + (D - 3)h_{ij} \right) \right] r^{D-3}d\Omega_{D-3},
\]

\[
= \begin{pmatrix} m + p & -p \\ -p & -m + p \end{pmatrix},
\]

7
where $S^\infty_{D-3}$ is the $(D - 3)$-sphere at spatial infinity in the transverse space, and there is no sum over the repeated $i$ index. This suggests interpreting the solution as a string with mass per unit length $m$, equal as usual to the tension, but with, in addition, momentum per unit length $p$ running down the string in the $-x^1$ direction at the speed of light.

How does this solution relate to the fundamental strings discussed in section 2? Exciting the gauge sector of the heterotic string introduces left-moving, current-carrying modes. However, the momentum and charge per unit length of these modes should be related, so the freedom to choose $p$ and $q$ independently in this solution probably corresponds to the freedom to excite additional neutral modes, which are not present for a heterotic string. To identify properly this solution as representing the fields around such a macroscopic string would require matching the field singularities with string source terms from, say, the sigma model action for the heterotic string, as in [2], with the expectation that this would then relate $p$ and $q$.

Finally, because of the linearity of Laplace’s equation, multiple-string solutions are also clearly allowed. To be explicit,

$$e^{-E} = 1 + 2\kappa^2 \sum_k m_k \Lambda(|\vec{x} - \vec{x}_k|),$$

$$N = 2\kappa^2 \sum_k q_k \Lambda(|\vec{x} - \vec{x}_k|),$$

$$R = -2\kappa^2 \sum_k p_k \Lambda(|\vec{x} - \vec{x}_k|),$$

(3.18)

gives the solution for a collection of parallel strings labelled by $k$, so that a given string has mass per unit length $m_k$, carries charge per unit length $q_k$ and momentum per unit length $p_k$, and is located at position $\vec{x}_k$ in the transverse space.

4 A charged black string solution

Starting with the same bosonic action, Horowitz and Strominger [5] have found a family of singular, string-like solutions, now each with an event horizon, for which the solution of Dabholkar et al. [2] is the extremal case where the event horizon coincides with the singularity. Given that the solution of the section 3 is simply a generalization of the solution of [4], similarly without a horizon, is there then a corresponding family of ‘black string’ solutions, each now also a source of gauge fields, for which it is the extremal case?

For a solution to have a horizon, the sigma-model metric on the transverse space cannot be flat, so by the discussion in the first part of section 3, black strings cannot preserve half the supersymmetries. (Rather the expectation is that only the particular extremal case, where the mass per unit length and the charge per unit length which acts as the source of the antisymmetric tensor field are set equal, saturates the relevant Bogomol’nyi inequality and allows the addition symmetry.) Retaining, however, the other symmetries of a straight, static, string-like solution, the
fields about a black string have the general form:

\[
\begin{align*}
\text{ds}^2 &= \hat{g}_{\alpha\beta}(r)\text{d}x^\alpha\text{d}x^\beta + e^{B(r)}\text{d}r^2 + r^2\text{d}\Omega_{D-3}^2, \\
\phi/\alpha &= E(r), \quad H_{rtx} = H(r), \quad A_\alpha = A_\alpha(r),
\end{align*}
\]

(4.1)

where \(\text{d}\Omega_{D-3}^2\) is the volume element on the \((D - 3)\)-sphere and otherwise the notation follows that of section 3.

Since there are no other constraints on the form of the solution, all that is left is to do is to substitute the ansatz in the equations of motion (3.2), though with the physical conditions that asymptotically the metric should be flat and the fields should go to zero, that the spacetime should have a singularity surrounded by a horizon and that the fields must be finite everywhere away from the singularity. The algebra is slightly simplified by changing variables from \(r\) to \(\Lambda\) and introducing

\[
\dot{\hat{H}} = -2(\text{D} - 4)\omega_{\text{D}-3r}^{\bar{D}-3H}, \quad \dot{\hat{F}}_\alpha = -(\text{D} - 4)\omega_{\text{D}-3r}^{\bar{D}-3} \frac{\text{d}A_\alpha}{\text{d}r} = \dot{A}_\alpha,
\]

(4.2)

where the dot represents differentiation with respect to \(\Lambda\). In particular, the equations for \(\phi/\alpha\), \(\hat{g}^{\alpha\beta}R_{\alpha\beta}\), and the components of \(R_{\mu
u}\) parallel to the \((D - 3)\)-sphere are given by

\[
\begin{align*}
\dot{\hat{E}} + \left(\frac{1}{2}\log|\hat{g}| - \frac{1}{2}\hat{B} - \hat{E}\right)\dot{\hat{E}} - |\hat{g}|^{-1} \hat{H}^2 + \alpha' \hat{F}^{\bar{T}}\hat{g}^{-1}\hat{F} = 0, \\
\frac{1}{2}\log|\hat{g}| + \left(\frac{1}{2}\log|\hat{g}| - \frac{1}{2}\hat{B} + \hat{E}\right)\frac{1}{2}\log|\hat{g}| - |\hat{g}|^{-1} \hat{H}^2 + \alpha' \hat{F}^{\bar{T}}\hat{g}^{-1}\hat{F} = 0, \\
\Lambda \left(\frac{1}{2}\log|\hat{g}| - \frac{1}{2}\hat{B} + \hat{E}\right) + e^B - 1 = 0,
\end{align*}
\]

(4.3)

respectively, where \(|\hat{g}| = -\det\hat{g}_{\alpha\beta}\) and a matrix notation is used so that there is an implied summation over \(\alpha = 0, 1\) in expressions such as \(\hat{F}^{\bar{T}}\hat{g}^{-1}\hat{F}\), with the superscript \(\bar{T}\) representing transposition and \(\hat{g}^{-1} = \hat{g}^{\alpha\beta}\).

This set of equations can be partially solved. Taking the difference of the first two equations, together with the third gives a pair of equations in only the fields \(C = \log|\hat{g}| - 2E\) and \(B\),

\[
2\dot{C} + \left(\dot{C} - \dot{B}\right)\dot{C} = 0, \quad 2\left(e^B - 1\right) + \Lambda \left(\dot{C} - \dot{B}\right) = 0,
\]

(4.4)

which have the physical solution

\[
e^{-B} = 1 - a\Lambda, \quad \frac{1}{2}\log|\hat{g}| = E - \frac{1}{2}B,
\]

(4.5)

where \(a\) is a constant. The remaining unsolved equation, in terms of \(E\), is then

\[
\dot{\hat{E}} - \hat{B}\dot{\hat{E}} - |\hat{g}|^{-1} \hat{H}^2 + \alpha' \hat{F}^{\bar{T}}\hat{g}^{-1}\hat{F} = 0.
\]

(4.6)

Furthermore, the equation for \(\dot{\hat{H}}\),

\[
\dot{\hat{H}} + \left(\frac{1}{2}\log|\hat{g}| - \frac{1}{2}\hat{B} + \hat{E}\right)\dot{\hat{H}} = 0,
\]

(4.7)

then has the solution, with arbitrary constant \(c\),

\[
\hat{H} = ce^{2E},
\]

(4.8)
and taken together, the expressions (4.5) and (4.8) are unchanged from those for the black string without gauge fields of \( F \), though here the form of \( E \) is as yet unknown.

The remaining unsolved equations of motion are for \( F_{\alpha \beta} \), \( R_{\alpha \beta} \) and \( R_{rr} \) and in matrix notation are given by

\[
\begin{align*}
\dot{\hat{F}} - \hat{B} \dot{\hat{F}} + \hat{g} \hat{g}^{-1} \hat{F} + c e^B \hat{g} \epsilon \hat{F} &= 0, \\
\dot{\hat{g}} - \hat{B} \dot{\hat{g}} + \hat{g} \hat{g}^{-1} \dot{\hat{g}} - c^2 e^{2E+\hat{B}} + 2\alpha' \hat{F} \hat{T} = 0, \\
\frac{1}{2} B^2 + \frac{1}{2} \text{tr} \hat{g} \hat{g}^{-1} + c^2 e^{2E+\hat{B}} - 2\alpha' \hat{F} \hat{T} \hat{g}^{-1} \hat{F} &= 0,
\end{align*}
\] (4.9a, 4.9b, 4.9c)

respectively, where, as before, \( \epsilon \) is an antisymmetric matrix with \( \epsilon^{01} = 1 \). Clearly the unsolved equation for \( E \) is included, as can be seen by contracting (4.9b) with \( \hat{g}^{\alpha \beta} \) and using the relation (4.5) between \( \log |\hat{g}| \) and \( E \). Thus, what remains is to solve equations (4.9) for the fields \( \hat{g} \) and \( \hat{F} \). However the equations are nonlinear and coupled, and it appears difficult to make further progress.

The step to finding the full solution is to notice that the unknown fields enter the equations for \( E \) (4.4) and \( R_{rr} \) (4.9c) in only particular combinations — as \( E = \frac{1}{2} \log |\hat{g}| + \frac{1}{2} B, \hat{F} \hat{T} \hat{g}^{-1} \hat{F} \) and \( \text{tr} \hat{g} \hat{g}^{-1} \). Furthermore if (4.9a) is contracted with \( \hat{F}_\beta \hat{g}^{\beta \alpha} \) and (4.9b) with \( \hat{g}^{\alpha \beta} \) and \( \hat{F}^\alpha \hat{F}^\beta \), the same combinations of fields appear, though together with two new combinations, \( \hat{F} \hat{T} \hat{g}^{-1} \hat{F} \) and \( \hat{F} \hat{T} \hat{g}^{-1} \hat{F} \). In fact, a new set of variables can be introduced,

\[
E = \frac{1}{2} \log |\hat{g}| + \frac{1}{2} B, \quad \alpha = \frac{1}{2} e^{-4E} \text{tr} \hat{g} \epsilon \hat{g} \epsilon, \\
\beta = \alpha' e^{-4E} \hat{F} \epsilon \hat{g} \epsilon \hat{F}, \quad \gamma = \alpha' e^{-4E} \hat{F} \epsilon \hat{g} \epsilon \hat{F}, \quad \delta = \alpha' e^{-4E} \hat{F} \epsilon \hat{g} \epsilon \hat{F},
\] (4.10)

which are related to the field combinations already mentioned, (it is helpful to recall that, since \( \hat{g} \) is symmetric, \( \hat{g}^{-1} = |\hat{g}|^{-2} \epsilon \hat{g} \epsilon \)), and in terms of which the equations for \( E \) (4.4) and \( R_{rr} \) (4.9c) become

\[
\begin{align*}
e^{-2E-\hat{B}} \left( \hat{E} - \hat{B} \hat{E} \right) &= -\beta + c^2, \\
e^{-2E-\hat{B}} \left( \hat{E}^2 - \hat{B} \hat{E} \right) &= -\frac{1}{2} \beta + \beta - \frac{1}{2} c^2,
\end{align*}
\] (4.11)

whilst the contraction of (4.9a) with \( \hat{F}_\beta \hat{g}^{\beta \alpha} \) and (4.9b) with \( \hat{g}^{\alpha \beta} \) and \( \hat{F}^\alpha \hat{F}^\beta \) give

\[
\begin{align*}
\dot{\beta} + \gamma &= 0, \\
\dot{\alpha} + \left( \alpha - c^2 \right) \left( 2\hat{E} - \hat{B} \right) + 4\gamma &= 0, \\
\delta - 2\hat{E} \gamma + \left( \alpha - c^2 \right) e^{2E+\hat{B}} \beta &= 0,
\end{align*}
\] (4.12)

respectively.

Eliminating \( \alpha, \beta \) and \( \gamma \) between (4.11) and (4.12) gives

\[
\left( e^{-E} \right) = \text{constant} e^{2B} e^{-E}.
\] (4.13)

However, there is the physical condition that the fields \( e^{-\phi/\alpha} = e^{-E} \) and \( H_{101} = -c e^{2E}/2\omega_{D-3} r^{D-3} \) are non-singular on the horizon at \( e^{-B} = 0 \), which implies that the only physical solution of (4.13) is

\[
e^{-E} = 1 + b\Lambda,
\] (4.14)
in fact, as was the case in all previous solutions. Solving the remaining equations gives the simple relations
\[
\begin{align*}
\alpha &= \frac{1}{2} e^{-4E} \dot{\epsilon} \dot{\epsilon} \epsilon = c^2, \\
\beta &= e^{-4E} \hat{F}^T \epsilon \epsilon \hat{F} = c^2 - (a + b) b, \\
\gamma &= e^{-4E} \hat{F}^T \epsilon \epsilon \hat{F} = 0, \\
\delta &= e^{-4E} \hat{F}^T \epsilon \epsilon \hat{F} = 0.
\end{align*}
\] (4.15)

It is then easy to solve for \(\dot{g}\) and \(\hat{F}\) in these relations, and since, except in the case \(\beta = 0\), the equations (4.11) and (4.12) are equivalent to the original unsolved equations for \(\dot{g}\) and \(\hat{F}\) (13), doing so gives the most general black string solution,

\[
d s^2 = \frac{1}{1 + b\Lambda} \left( -1 + \frac{q_0^2 a + (q_0^2 + q_1^2) b + 2q_0q_1 c}{q_0^2 - q_1^2} \Lambda - \frac{\alpha' q_0^2 \Lambda^2}{1 + b\Lambda} \right) (d\epsilon^0)^2 \\
+ \frac{2}{1 + b\Lambda} \left( \frac{q_0q_1 a + 2q_0q_1 b + (q_0^2 + q_1^2) c}{q_0^2 - q_1^2} \Lambda - \frac{\alpha' q_0q_1 \Lambda^2}{1 + b\Lambda} \right) dx^0 dx^1 \\
+ \frac{1}{1 + b\Lambda} \left( 1 + \frac{q_1^2 a + (q_0^2 + q_1^2) b + 2q_0q_1 c}{q_0^2 - q_1^2} \Lambda - \frac{\alpha' q_1^2 \Lambda^2}{1 + b\Lambda} \right) (dx^1)^2 \\
+ \frac{dr^2}{1 - a\Lambda} + r^2 d\Omega_D^2 - 3,
\] (4.16)

\(e^{-\phi/\alpha} = 1 + b\Lambda,\)

\(F_{r\alpha} = -\left[ \frac{1}{\omega_{D-3} r^{D-3}} \right] q_0 \frac{q_0}{(1 + b\Lambda)^2},\)

\(H_{r01} = -\left[ \frac{1}{2\omega_{D-3} r^{D-3}} \right] c \frac{c}{(1 + b\Lambda)^2}.\)

The constants \(a, b, c, q_0\) and \(q_1\) are related by

\[c^2 = \alpha' \left( q_0^2 - q_1^2 \right) + (a + b) b,\] (4.17)

and from the asymptotic forms of the fields, \(q_0\) is the electric charge per unit length, \(q_1\) is the electric current and \(c\) is the antisymmetric tensor field charge per unit length. Again the Chern-Simons three-form vanishes for this solution.

As was the case in section 3, the solution is unchanged under Lorentz boosts along the \(x^1\) axis, except that \(q_0\) and \(q_1\) are replaced throughout by the boosted pair

\[
\begin{pmatrix}
q_0' \\
q_1'
\end{pmatrix} = \begin{pmatrix}
\cosh\phi & \sinh\phi \\
\sinh\phi & \cosh\phi
\end{pmatrix} \begin{pmatrix}
q_0 \\
q_1
\end{pmatrix},
\] (4.18)

where \(\phi\) is the angle of the Lorentz boost. Thus if \(q_0^2 > q_1^2\), by boosting to a frame where \(q_0' = 0\), the solution can be written in the simpler form:

\[
d s^2 = - (d\epsilon^0)^2 - \frac{2c\Lambda}{1 + b\Lambda} dx^0 dx^1 + \frac{1 - a\Lambda - c^2 \Lambda^2}{(1 + b\Lambda)^2} (dx^1)^2 + \frac{dr^2}{1 - a\Lambda} + r^2 d\Omega_D^2 - 3,
\]

\(e^{-\phi/\alpha} = 1 + b\Lambda,\)

\(F_{r0} = 0, \quad F_{r1} = -\left[ \frac{1}{\omega_{D-3} r^{D-3}} \right] \frac{q_0'}{(1 + b\Lambda)^2},\)

\(H_{r01} = -\left[ \frac{1}{2\omega_{D-3} r^{D-3}} \right] \frac{c}{(1 + b\Lambda)^2},\) (4.19)
where now \( c^2 + \alpha' q_1^2 = (a + b) b \). (Of course if \( q_0^2 > q_1^2 \) then the solution can be written in a similar form by boosting so \( q_1' = 0 \).)

Finally the extremal limit should be identified, as well as the limit giving the black string solution of Horowitz and Strominger [3]. For the former, taking independently \( a \to 0 \), \( q_0 \to q \) and \( q_1 \to -q \) gives

\[
\begin{align*}
\text{d}s^2 &= \frac{1}{1 + b\Lambda} \left[ \left( -1 + q^2 \Lambda / b \right) \left( \text{d}x^0 \right)^2 - \left( 2q^2 \Lambda / b \right) \text{d}x^0 \text{d}x^1 + \left( 1 + q^2 \Lambda / b \right) \left( \text{d}x^1 \right)^2 \right] \\
&\quad + \text{dr}^2 + r^2 \Omega_{D-3}^2, \\
\text{e}^{-\phi/\alpha} &= 1 + b\Lambda, \\
F_{r0} &= -F_{r1} = - \left[ \frac{1}{\omega_{D-3} r^{D-3}} \right] \frac{q}{(1 + b\Lambda)^2}, \\
H_{r01} &= - \left[ \frac{1}{2\omega_{D-3} r^{D-3}} \right] \frac{b}{(1 + b\Lambda)^2},
\end{align*}
\]

which is the supersymmetric solution of section 3, with the identification \( m = b/2\kappa^2 \) and \( p = \alpha' q^2 / 2b\kappa^2 \). Note however that it is not quite the general solution which has \( p \) and \( q \) arbitrary. This is because the black string solution was constrained by physical conditions which no longer necessarily hold in the extremal case. In particular, the condition that the fields are finite on the horizon is no longer applicable when the horizon coincides with the singularity, and it is this additional freedom which allows an extra parameter to enter the extremal solution.

The black string solution without gauge fields of Horowitz and Strominger is easily obtained by writing \( q_0 = \sqrt{a + b}\epsilon \) and \( q_1 = -\sqrt{b}\epsilon \) and taking \( \epsilon \to 0 \) and \( D = 10 \). This gives

\[
\begin{align*}
\text{d}s^2 &= - \frac{1 - a\Lambda}{1 + b\Lambda} \left( \text{d}x^0 \right)^2 + \frac{\left( \text{d}x^1 \right)^2}{1 - a\Lambda} + \frac{\text{dr}^2}{1 - a\Lambda} + r^2 \Omega_{D-3}^2, \\
\text{e}^{-2\phi} &= 1 + b\Lambda, \\
H_{r01} &= - \left[ \frac{3}{r' (1 + b\Lambda)^2} \right] c,
\end{align*}
\]

together with \( c = \sqrt{(a + b)b} \) so that in terms of the parameters of [3] \( a = 6\omega_7 C \), \( b = 6\omega_7 r^6 \) and \( c = 2\omega_7 Q \).

5 Conclusion

Two new, charged, string-like solutions of the equations of motion for the low-energy effective field theory of the heterotic string have been described. The first was the most general solution which preserves one half of the supersymmetries, which is found to have equal electric current and electric charge per unit length, a mass per unit length equal to the antisymmetric-tensor-field charge per unit length, as in [4], and furthermore allows multi-string solutions. The second was the most general solution with a horizon, a charged black string, for which the former solution is identified as an extremal limit where the horizon coincides with the singularity.

To complete the picture, some other limits of these solutions should briefly be mentioned. Taking the gauge charges, \( q_0 \) and \( q_1 \), arbitrarily to zero in the charged black string solution gives
the black string of Horowitz and Strominger \cite{5}, though in general a Lorentz-boosted version. For the supersymmetric solution, taking $q$ to zero gives the solution of Dabholkar \textit{et al.} \cite{2}, though now modified to include momentum running down the string, a case which is in fact the most general uncharged, horizonless, string-like solution. Finally simply taking $a = 0$ in the charged black string solution gives a horizonless solution with arbitrary current and charge per unit length, and which does not preserve half the supersymmetries. Thus, unlike the uncharged case, for a charged string it is possible to have a solution without a horizon which nonetheless is not supersymmetric.

Throughout, the correspondence between these solutions and fundamental strings has been stressed. Section 2 demonstrated that any macroscopic heterotic string which is a ground state of the right-moving sector (a case which includes the stable states) preserves one half of the supersymmetries, in the same manner as the supersymmetric solution. It would be interesting to strengthen this correspondence and identify the singularities of the solutions with suitable fundamental-string source terms, as was done in \cite{2}, perhaps not only for the supersymmetric case but in general, when the fundamental string is no longer in the right-moving ground state.

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\textbf{Note}

During the process of writing this paper I was made aware of a preprint by A. Sen \cite{8}, who was able to obtain essentially the same solutions by making a ‘twist’ transformation on the known uncharged supersymmetric and black string solutions.

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