FUNDAMENTAL GROUP AND ANALYTIC DISKS

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ABSTRACT. Let $W$ be a domain in a connected complex manifold $M$ and $w_0 \in W$. Let $\mathcal{A}_{w_0}(W, M)$ be the space of all continuous mappings of a closed unit disk $\mathbb{D}$ into $M$ that are holomorphic on the interior of $\mathbb{D}$, $f(\partial \mathbb{D}) \subset W$ and $f(1) = w_0$. On the homotopic equivalence classes $\eta_1(W, M, w_0)$ of $\mathcal{A}_{w_0}(W, M)$ we introduce a binary operation $*$ so that $\eta_1(W, M, w_0)$ becomes a semigroup and the natural mappings $\iota_1 : \eta_1(W, M, w_0) \rightarrow \pi_1(W, w_0)$ and $\delta_1 : \eta_1(W, M, w_0) \rightarrow \pi_2(M, W, w_0)$ are homomorphisms.

We show that if $W$ is a complement of an analytic variety in $M$ and if $S = \delta_1(\eta_1(W, M, w_0))$, then $S \cap S^{-1} = \{e\}$ and any element $a \in \pi_2(M, W, w_0)$ can be represented as $a = bc^{-1} = d^{-1}g$, where $b, c, d, g \in S$.

Let $\mathcal{R}_{w_0}(W, M)$ be the space of all continuous mappings of $\mathbb{D}$ into $M$ such that $f(\partial \mathbb{D}) \subset W$ and $f(1) = w_0$. We describe its open dense subset $\mathcal{R}_{w_0}(W, M)$ such that any connected component of $\mathcal{R}_{w_0}(W, M)$ contains at most one connected component of $\mathcal{A}_{w_0}(W, M)$.

1. INTRODUCTION

An analytic disk in a complex manifold $M$ is a continuous mapping $f$ of the closed unit disk $\mathbb{D}$ into $M$ holomorphic on $\mathbb{D}$. We will denote the set of all such disks by $\mathcal{A}(M)$. For a domain $W$ in $M$ we introduce the space $\mathcal{A}(W, M)$ of all continuous mappings $f$ of the unit circle $T = \partial \mathbb{D}$ into $W$ such that $f$ extends to a mapping $\hat{f} \in \mathcal{A}(M)$. If $w_0 \in W$ then we denote by $\mathcal{A}_{w_0}(W, M)$ the subset of all $f \in \mathcal{A}(W, M)$ such that $f(1) = w_0$.

We let $\eta_1(W, M)$ to be the set of all connected components of $\mathcal{A}(W, M)$ and let $\eta_1(W, M, w_0)$ to be the set of all connected components of $\mathcal{A}_{w_0}(W, M)$. There is a natural mapping $\iota_1$ of the sets $\eta_1(W, M)$ or $\eta_1(W, M, w_0)$ into the sets $\pi_1(W)$ or $\pi_1(W; w_0)$ respectively.

In this paper we study the mapping $\iota_1$, its injectivity and its image. These questions originated in \cite{9}, where L. Rudolph showed that if $B$ is the braid group (the fundamental group of the complement of some set $A$ of hyperplanes in $\mathbb{C}^n$), then $S = \iota_1((\eta_1(\mathbb{C}^n \setminus A, \mathbb{C}^n, w_0)))$ is a semigroup, $S \cap S^{-1} = \{e\}$ and any element $a \in B$ can be represented as $a = bc^{-1} = d^{-1}g$, where $b, c, d, g \in S$. He called the elements of $S$ quasipositive.

In the recent paper \cite{5} J. Kollár and A. Némethi showed under additional assumptions that if $M$ is an algebraic variety with an isolated singularity $O$ and $W = M \setminus O$, then $\iota_1 : \eta_1(W, M) \rightarrow \pi_1(W)$ is an injection. They used this result to obtain more information about the singularity.

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These results do not hold in general. For example, when $M = \mathbb{CP}^2$ and $A$ is an algebraic variety in $M$ such that $\pi_1(M \setminus A, w_0) = \mathbb{Z}_p$ and $p$ is prime, then Rudolph’s result evidently fails and the result of Kollár and Némethi fails because the set $\eta_1(W, M)$ is infinite due to the homotopic invariance of the intersection index.

However Rudolph’s result stays true if we change ingredients. We introduce on the set $\eta_1(W, M, w_0)$ a binary operation $\ast$. With this operation $\eta_1(W, M, w_0)$ becomes a semigroup with unity. The natural mapping $\delta_1 : \eta_1(W, M, w_0) \to \pi_2(M, w_0)$ is a homomorphism and we show that its image $S$ has the same properties as in Rudolph’s result anytime when $W$ is a complement to an analytic variety in a connected complex manifold. When $\pi_1(M, w_0) = \pi_2(M, w_0) = 0$ we obtain a complete analogy but in more general settings.

The problem of injectivity is more interesting and looks more difficult. To advance in this direction we consider the space $\mathcal{R}(W, M, w_0)$ of continuous mappings $f$ of $\mathbb{T}$ into $M$ such that $f(\mathbb{T}) \subset W$ and $f(1) = w_0$. We show that there is an open dense set $\mathcal{R}_{w_0}^+(W, M)$ in $\mathcal{R}_{w_0}(W, M)$ such that the natural mapping $\delta_1$ of $\eta_1(W, M, w_0)$ into the set $\rho_1^+(W, M, w_0)$ of all connected components of $\mathcal{R}_{w_0}^+(W, M)$ is an injection.

For future purposes we need to consider not domains $W \subset M$ but Riemann domains $W$ over $M$. In Section 2 we prove basic facts about them. Since our constructions require more complicated compact sets than $\mathbb{T}$, in Section 3 we introduce an operator $I_{K, \gamma}$ that maps homotopic equivalence classes of holomorphic mappings of compact sets into homotopic equivalence classes of holomorphic mappings of the closed disk. The properties of this operator allows us in Section 4 to introduce on $\eta_1(W, M, w_0)$ the structure of a semigroup. In Section 5 we establish major algebraic properties of $\eta_1(W, M, w_0)$. In particular, we obtain the description of the set $\eta_1(W, M)$ as the set of all $\pi_1$-conjugacy classes in $\eta_1(W, M, w_0)$.

In Section 6 we introduce the group $\rho_1(W, M, w_0)$ and prove its basic properties. In Section 7 we consider the case when $W$ is the complement to an analytic variety in $M$ and $\Pi$ is an identity. The generalization of Rudolph’s result is one of theorems in this section. The last Section 8 is devoted to the problem of injectivity.

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2. Basic notions and facts

In this paper $\mathbb{D}(a, r)$ is an open disk of radius $r$ centered at $a$ and $\mathbb{T}(0, r)$ is its boundary. We let $\mathbb{D} = \mathbb{D}(0, 1)$ and $\mathbb{T} = \mathbb{T}(0, 1)$.

A Riemann domain over a complex manifold $M$ is a pair $(W, \Pi)$, where $W$ is a connected Hausdorff complex manifold and $\Pi$ is a locally biholomorphic mapping of $W$ into $M$. Let $\hat{d}$ be a Riemann metric on $M$ and let $d$ be its lifting to $W$.

Let $K$ be a connected compact set in $\mathbb{C}$ with connected complement. We denote by $\mathcal{A}(K, M)$ the set of all continuous mappings of $K$ into $M$ that are holomorphic on the interior $K^\circ$ of $K$. By $\mathcal{A}(K, W, M)$ we denote the set of all continuous mappings $f$ of $\partial K$ into $W$ such that there is a mapping $\hat{f} \in \mathcal{A}(K, M)$ coinciding with $\Pi \circ f$ on $\partial K$. The mapping $\hat{f}$ is unique. If $\zeta_0 \in \partial K$ and $w_0 \in W$ then the space $\mathcal{A}_{\zeta_0, w_0}(K, W, M)$ is the set of all $f \in \mathcal{A}(K, W, M)$ such that $f(\zeta_0) = w_0$. 

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The space $\mathcal{T}(W,M)$ consists of all pairs $(K,f)$, where $K \subset \mathbb{C}$ is a connected compact set with connected complement and $f \in \mathcal{A}(K,W,M)$. If $(K,f), (L,g) \in \mathcal{T}(W,M)$ we define the distance $d((K,f),(L,g))$ between $(K,f)$ and $(L,g)$ as the sum of the Hausdorff distances between the graphs of $f$ and $g$ on $\partial K$ and $\partial L$ respectively and between the graphs of $\hat{f}$ and $\hat{g}$ on $K$ and $L$ respectively. (The distance between points $(\zeta_1,w_1)$ and $(\zeta_2,w_2)$ on $\mathbb{C} \times M$ is defined as $\max\{|\zeta_1 - \zeta_2|, d(w_1,w_2)\}$.) Since the graphs are compact, $d$ is a metric on $\mathcal{T}(W,M)$ and the topology on $\mathcal{T}(W,M)$ is induced by this metric. Clearly, this topology does not depend on the choice of $\hat{d}$.

The set $\mathcal{T}_{\zeta_0,w_0}(W,M) \subset \mathcal{T}(W,M)$ consists of all pairs $(K,f)$ such that $f \in \mathcal{A}_{\zeta_0,w_0}(K,W,M)$. On this set and the sets $\mathcal{A}(K,W,M)$ and $\mathcal{A}_{\zeta_0,w_0}(K,W,M)$ we define the topology relative to the topology imposed on $\mathcal{T}(W,M)$.

Let $\mathcal{T}(M,M)$ be the set of pairs $(K,f)$, where $f \in \mathcal{A}(K,M)$. We define the mapping $\Pi_1$ of $\mathcal{T}(W,M)$ into the set $\mathcal{T}(M,M)$ as $\Pi_1(K,f) = (K,\hat{f})$. Clearly, $\Pi_1$ is open and locally isometric.

Suppose that $K \subset \mathbb{C}$ is a compact set, $f \in \mathcal{A}(K,M)$ and the graph $\Gamma_f$ of $f$ on $K$ has a Stein neighborhood $U$ in $\mathbb{C} \times M$. Let $F$ be an imbedding of $U$ into $\mathbb{C}^N$ as a complex submanifold. By [4] Theorem 8.3.8 there are an open neighborhood $V$ of $F(\Gamma_f)$ in $\mathbb{C}^N$ and a holomorphic retraction $P$ of $V$ onto $F(U)$.

Let $(L,g)$ be a pair, where $L \subset \mathbb{C}$ is a compact set, $g \in \mathcal{A}(L,M)$ and $\Gamma_g \subset U$. Then we let $\Phi(L,g)$ be the pair $(L,h)$, where $h(\zeta) = F(\zeta,g(\zeta))$. Conversely, if $(L,h)$ is a pair, where $L \subset \mathbb{C}$ is a compact set, $h \in \mathcal{A}(L,\mathbb{C}^N)$ and $\Gamma_h \subset V$, then we let $\Psi(L,h)$ to be the pair $(L,g)$, where $g = P_M \circ F^{-1} \circ P \circ h$ and $P_M$ is a projection of $\mathbb{C} \times M$ onto $M$. Clearly, the mappings $\Phi$ and $\Psi$ are continuous and $\Psi \circ \Phi$ is the identity.

This construction leads to the following lemma.

**Lemma 2.1.** Let $K$ be a connected compact set in $\mathbb{C}$ with connected complement. For every $\varepsilon > 0$ there is $\delta > 0$ such that:

1. If $f \in \mathcal{A}(K,W,M)$ and pairs $(L,g_0)$ and $(L,g_1)$ lie in the $\delta$-neighborhood of $(K,f)$ in $\mathcal{T}(W,M)$, then there is a continuous path $(L,g_t)$ in the $\varepsilon$-neighborhood of $(K,f)$ in $\mathcal{T}(W,M)$, $t \in [0,1]$, connecting $(L,g_0)$ and $(L,g_1)$. Moreover, if, additionally, a compact set $L' \subset L$ and $g_0|_{L'} = g_1|_{L'}$, then we can assume that $g_t|_{L'} = g_0|_{L'}$ for all $t \in [0,1]$.

2. If $t \leq t \leq 1$ and $(K,f_t), (L_t,g_t)$, $\eta_t$ and $\xi_t \in L_t$ are continuous paths in $\mathcal{T}(W,M)$, $\mathcal{T}(W,M)$, $W$ and $\mathbb{C}$ respectively and for all $0 \leq t \leq 1$ the pairs $(L_t,g_t)$ lie in the $\delta$-neighborhood of $(K,f_t)$ and $d(g_t(\xi_t),w_t) < \delta$, then there is another continuous path $(L_t,h_t)$ in $\mathcal{T}(W,M)$ such that $h_t(\xi_t) = w_t$ and the pairs $(L_t,h_t)$ lie in the $\varepsilon$-neighborhood of $(K,f_t)$ for all $0 \leq t \leq 1$. Moreover, if $g_t(\xi_t) = w_t$ for some $0 \leq t \leq 1$ then we can assume that $h_t = g_t$.

**Proof.** (1) It was shown in [4] Theorem 3.1] that the graph of $\hat{f}$ has a basis of Stein neighborhoods in $\mathbb{C} \times M$. If $M = \mathbb{C}^N$ then we connect $(L,\hat{g}_0)$ and $(L,\hat{g}_1)$ by the path $(L,\hat{g}_t)$, where

$$\hat{g}_t(\zeta) = (1-t)\hat{g}_0(\zeta) + t\hat{g}_1(\zeta).$$

In the general case, we take $\varepsilon > 0$ so that $\Pi_1^{-1}$ is defined on the $\varepsilon$-neighborhood of $(K,\hat{f})$. We choose $\delta > 0$ so small that if we take pairs $\Phi(L,\hat{g}_0)$ and $\Phi(L,\hat{g}_1)$,
connect them in $\mathbb{C}^N$ by $(L, h_t)$ as above and let $(L, g_t) = \Psi(L, h_t)$, then the path $(L, g_t)$ lies in the $\varepsilon$-neighborhood of $(K, \hat{f})$. Finally, we let $(L, g_t) = \Pi^{-1}(L, \tilde{g}_t)$.

(2) For the proof of the second part we note that by [7, Theorem 4.1] the set $\overline{\Gamma} = \{(t, \zeta, \hat{f}(\zeta)) : \zeta \in K, 0 \leq t \leq 1\}$ has a Stein neighborhood in $\mathbb{C} \times \mathbb{C} \times M$ and then the proof follows the same pattern as above.

By part (1) of this theorem the spaces $\mathcal{A}(W, M)$ and $\mathcal{A}_{\zeta_0, w_0}(W, M)$ are locally path-connected and, therefore, their connected components are path-connected.

3. Operator $I_{K, \gamma}$

Throughout this section $K$ will denote a connected compact set in $\mathbb{C}$ with the connected complement. Let $\zeta_0 \in \partial K$ and a base point $w_0 \in W$. We say that $f, g \in \mathcal{A}_{\zeta_0, w_0}(K, W, M)$ are $h$-homotopic or $f \sim_h g$ if there is a continuous path connecting $f$ and $g$ in $\mathcal{A}_{\zeta_0, w_0}(K, W, M)$. The relation $\sim_h$ is evidently an equivalence and we denote the equivalence class of $f$ by $[f]_{\zeta_0, w_0}$ or $[f]$ if $\zeta_0$ and $w_0$ are fixed. The set of equivalence classes will be denoted by $\mathcal{H}_{\zeta_0, w_0}[K, W, M]$ or $\mathcal{H}_{\zeta_0, w_0}[K]$. It follows from Lemma 2.1(1) that the equivalence classes are closed in $\mathcal{A}_{\zeta_0, w_0}(K, W, M)$.

Our goal is to construct a mapping of the set $\mathcal{H}_{\zeta_0, w_0}[K]$ into the set $\mathcal{H}_{1, w_0}[\overline{D}]$. Firstly we do it when $K$ is the closure of a Jordan domain, i.e., $K$ is bounded by a Jordan curve (a homeomorphic image of a circle). Let $e_1, \zeta_0$ be a conformal mapping of $\overline{D}$ onto $K$ that maps $1$ onto $\zeta_0$. We define the mapping $I_{K, \zeta_0}$ as $[f \circ e_{1, \zeta_0}]_{1, w_0}$.

Since the group of conformal automorphisms of the unit disk with a fixed point on the boundary is connected, this mapping does not depend on the choice of $e$.

To define the mapping of $\mathcal{H}_{\zeta_0, w_0}[K]$ into $\mathcal{H}_{1, w_0}[\overline{D}]$ for a general $K$ we will approximate $f \in \mathcal{A}_{\zeta_0, w_0}(K, W, M)$ by mappings on Jordan domains $\Omega$ containing $K$. To determine a point in $\partial \Omega$ where approximations are equal to $w_0$ we need an access curve to $K$ at $\zeta_0$, i.e., a continuous curve $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = \zeta_0$ and $\gamma(t) \in \mathbb{C} \setminus K$ when $t > 0$.

Let $\Omega$ be a smooth Jordan domain containing $K$ whose boundary meets $\gamma$. We let $\zeta_0, \gamma = \gamma(\Omega, \gamma)$, where $\epsilon_{\Omega, \gamma} = \inf\{t : \gamma(t) \in \partial \Omega\}$. A pair $(\overline{\Omega}, g) \in T(W, M)$ is an $\varepsilon$-approximation of $(K, f)$ in $T_{\zeta_0, w_0}(W, M)$ with respect to $\gamma$ if $K \subset \Omega$, $g(\zeta_0, \gamma) = w_0$ and $(\overline{\Omega}, g)$ lies in the $\varepsilon$-neighborhood of $(K, f)$.

The following proposition asserts the existence of $\varepsilon$-approximations for every $\varepsilon > 0$.

Proposition 3.1. Let $f \in \mathcal{A}_{\zeta_0, w_0}(K, W, M)$ and let $\gamma$ be an access curve to $K$ at $\zeta_0$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for any Jordan domain $\Omega$ containing $K$ and lying in the $\delta$-neighborhood of $K$ and any point $\zeta \in \gamma \cap \partial \Omega$ there is a mapping $g \in \mathcal{A}_{\zeta, w_0}(\overline{\Omega}, W, M)$ such that the pair $(\overline{\Omega}, g)$ lies in the $\varepsilon$-neighborhood of $(K, f)$.

Proof. Firstly, we prove this proposition when $f \in \mathcal{A}_{\zeta_0, w_0}(K, M, M)$ so $\hat{f} = f$. For the given $\varepsilon > 0$ we denote by $\eta$ the $\delta$ from Lemma 2.1(2). The mapping $f$ is uniformly continuous on $K$. So there is $\delta_1 > 0$ such that $d(f(\zeta_1), f(\zeta_2)) < \eta/2$ when $|\zeta_1 - \zeta_2| < \delta_1$. We assume that $\delta_1 < \eta$.

By Corollary 4.4 from [7] the compact sets $K \subset \mathbb{C}$ have the Mergelyan property, i.e., there are a neighborhood $U$ of $K$ and a holomorphic mapping $h : U \to M$ such that $d((K, f), (K, h)) < \delta_1/4$. Let us take $\delta > 0$ with the following properties:

1) any smooth Jordan neighborhood $\Omega$ of $K$ that lies in the $\delta$-neighborhood of $K$
that we denote also by \(\Omega\) lies in the \(\delta_1/2\)-neighborhood of \((K,f)\).

Let \(\zeta\) be any point in \(\gamma \cap \overline{\Omega}\). Since \((\overline{\Omega}, h)\) lies in the \(\delta_1/2\)-neighborhood of \((K,f)\) there is a point \(\xi \in K\) such that \(|\zeta - \xi| < \delta_1/2\) and \(d(h(\zeta), f(\xi)) < \delta_1/2\). Hence \(|\zeta_0 - \xi| < \delta_1\) because \(|\zeta - \zeta_0| < \delta_1/4\). Thus \(d(f(\zeta_0), f(\xi)) < \eta/2\) and \(d(h(\zeta), w_0) < \eta\).

By Lemma 2.1(2) we can shift \(h\) so that for the shifted mapping \(g\) we have \(g(\zeta) = w_0\) and \((\overline{\Omega}, g)\) lies in the \(\varepsilon\)-neighborhood of \((K,f)\).

If \(f \in \mathcal{A}_{\varepsilon_0, w_0}(K,W,M)\) then we take \(\varepsilon > 0\) so small that \(\Pi^{-1}\) is defined on the \(\varepsilon\)-neighborhood of \((K,f)\), approximate \((K,f)\) in this neighborhood and compose it with \(\Pi^{-1}\).

To continue we need the notion of Radó continuity. A family of Jordan domains \(\Omega_t \subset \mathbb{C}, 0 \leq t \leq 1\), is called Radó continuous at \(t_0 \in [0,1]\) if for some \(\varepsilon > 0\) a neighborhood of some point \(\zeta \in \mathbb{C}\) belongs to the intersection of all \(\Omega_t, t_0 - \varepsilon < t < t_0 + \varepsilon\), and the family of conformal mappings \(\phi_t\) of \(D\) onto \(\Omega_t\) such that \(\phi_t(0) = \zeta\) and \(\phi'_t(0) > 0\) converges uniformly on \(\overline{D}\) to \(\phi_{t_0}\) as \(t \to t_0\). (By a theorem of Carathéodory the mappings \(\phi_t\) extend to \(D\) as its homeomorphisms onto \(\overline{\Omega_t}\).) Such family is Radó continuous if it is Radó continuous at every \(t\). A result of Radó (see [8] or [3] Theorem II.5.2) claims, in particular, that a family of Jordan domains \(\Omega_t \subset \mathbb{C}\) is Radó continuous if and only if for every \(t_0 \in [0,1]\) there are homeomorphisms \(\Phi_t\) of \(\partial \Omega_t\) onto \(\partial \Omega_t\) converging uniformly to identity on \(\partial \Omega_{t_0}\) as \(t \to t_0\).

The significance of Radó continuity is in the following lemma.

**Lemma 3.2.** If the family of Jordan domains \(\Omega_t, t \in [0,1]\), is Radó continuous, \(\zeta_t \in \partial \Omega_t\) is a continuous path in \(\mathbb{C}\) and \((\overline{\Omega_t}, f_t)\) is a continuous path in \(T(W,M)\) such that \(f_t(\zeta_t) = w_0\), then \(I_{\overline{\Omega_t}, \zeta_t}(f_t) \equiv \text{const.}\)

**Proof.** Suppose that \(\zeta\) is a common point of all \(\Omega_t\) when \(t\) is near \(t_0\) and \(\phi_t\) be the conformal mappings from the definition of Radó continuity. Let \(\xi_t = \phi_t^{-1}(\zeta_t) \in T\) and let \(\alpha_t\) be the rotations of \(\overline{D}\) moving \(1\) to \(\xi_t\). Since the family \(\Omega_t\) is Radó continuous then \(\xi_t\) and \(\alpha_t\) are continuous in \(t\). If \(\psi_t = \phi_t \circ \alpha_t\) then \(\psi_t(1) = \zeta_t\) and \(\psi_t\) is also continuous on \(\overline{D} \times [0,1]\). Hence \((\overline{D}, f_t \circ \psi_t)\) is a continuous path in \(\mathcal{A}_{1,w_0}(\overline{D}, W, M)\) and \(I_{\overline{D}, \zeta_t}(f_t) \equiv \text{const.}\)

We need the following basic lemma.

**Lemma 3.3.** Let \(w_0 \in W\) and \((K,f) \in \mathcal{T}_{\varepsilon_0, w_0}(W,M)\). There is \(\delta > 0\) such that if:

1. \(\Omega_0 \subset \subset \Omega_1\) are smooth Jordan domains;
2. pairs \((\overline{\Omega_1}, g_1)\) and \((\overline{\Omega_0}, g_0)\) lie in the \(\delta\)-neighborhood of \((K,f)\) in \(T(W,M)\);
3. \(\zeta_0 \in \partial \Omega_0\) and \(\zeta_1 \in \partial \Omega_1\) and \(g_0(\zeta_0) = g_1(\zeta_1) = w_0\);
4. there is a continuous curve \(\gamma : [0,1] \to \overline{W}\) such that \(\gamma(t) \in \Omega_t\), \(0 \leq t < 1\), \(\gamma(0) = \zeta_0\), \(\gamma(1) = \zeta_1\) and \(\gamma \subset \mathbb{D}(\zeta_0, \delta)\),

then \(I_{\overline{\Omega_0}, \zeta_0}(g_0) = I_{\overline{\Omega_1}, \zeta_1}(g_1)\).

**Proof.** Let us show that if \((K,f) \in \mathcal{T}_{\varepsilon_0, w_0}(M,M)\) then for any \(\varepsilon > 0\) the \(\delta\) can be chosen in such a way that we can connect the pairs \((\overline{\Omega_0}, g_0)\) and \((\overline{\Omega_0}, g_1)\) in the \(\varepsilon\)-neighborhood of \((K,f)\). Let us fix \(\varepsilon > 0\) and find \(0 < \delta_1 < \varepsilon\) such that Lemma 2.1(1) holds with \(\delta = \delta_1\). Let us denote by \(\eta\) the \(\delta\) in Lemma 2.1(2) for which this lemma holds when \(\varepsilon = \delta_1\). The mapping \(f\) is uniformly continuous on \(K\). So
there is \( \delta > 0 \) such that \( d(f(\zeta), f(\xi)) < \eta/2 \) when \( |\zeta - \xi| < 2\delta \). We assume that \( \delta < \eta/2 < \delta_1 \).

Since \((\Omega_1, g_1)\) lies in the \( \delta \)-neighborhood of \((K, f)\) for any \( \zeta \in \gamma \) there is a point \( \xi \in K \) such that \( |\zeta - \xi| < \delta \) and \( d(g_1(\zeta), f(\xi)) < \delta \). Hence \( |\xi - \zeta| < 2\delta \) because \( \gamma \subset \mathbb{D}(\zeta_0, \delta) \). Thus \( d(f(\zeta_0), f(\xi)) < \eta/2 \) and \( d(g_1(\zeta_0), w_0) < \eta/2 + \delta < \eta \).

Let \( \Theta \) be a conformal mapping of \( \overline{\Omega}_1 \setminus \Omega_0 \) onto an annulus \( A(r_0, 1) = \{ \zeta \in \mathbb{C} : r_0 \leq |\zeta| \leq 1 \} \) that maps \( \partial \Omega_1 \) onto the unit circle. We define the intermediate domains \( \Omega_t, 0 \leq t \leq 1 \), as bounded domains with boundaries equal to \( \Theta^{-1}(\{ |\zeta| = (1 - r_0)t + r_0 \}) \). The domains \( \Omega_t \) are simply connected and the family \( \Omega_t \) is Radó continuous because as homeomorphisms \( \Psi_t \) of \( \partial \Omega_0 \) onto \( \partial \Omega_0 \) we can take preimages under the mapping \( \Theta \) of the radial correspondences between circles in \( A(r_0, 1) \).

We will reparameterize this family letting \( G_t := \Omega_{e_t} \gamma(t) \in \partial \Omega_2, t \in [0, 1] \). Then the new family is still Radó continuous. For \( t \in [0, 1] \) we define the pairs \((G_t, h_t)\), where \( h_t \) is the restriction of \( g_t \) to \( G_t \). This family still lies in the \( \delta \)-neighborhood of \((K, f)\). Now \( h_t(\gamma(t)) = g_t(\gamma(t)) \) so \( d(h_t(\gamma(t)), w_0) < \eta \). By Lemma 2.112 we can shift \( h_t \) to get mappings \( p_t \) so that \( p_t(\gamma(t)) = w_0 \) and pairs \((G_t, p_t)\) lie in the \( \delta_1 \)-neighborhood of \((K, f)\). Note that \( G_1 = \Omega_1, G_0 = \Omega_0 \) and by the same lemma we can assume that \( p_1 = g_1 \).

The pairs \((\overline{\Omega}_0, p_0)\) and \((\overline{\Omega}_0, g_0)\) are in the \( \delta_1 \)-neighborhood of \((K, f)\) and by our choice of \( \delta_1 \) we can connect them by a continuous path in the intersection of the \( \epsilon \)-neighborhood of \((K, f)\) with \( \mathcal{A}_{\overline{\Omega}_0, w_0}(\overline{\Omega}_0, W, M) \). Consequently we can connect the pairs \((\overline{\Omega}_0, g_0)\) and \((\Omega_1, g_1)\) in the \( \epsilon \)-neighborhood of \((K, f)\).

If \((K, f) \in \mathcal{T}_{\overline{\Omega}_0, w_0}(W, M)\) then we take \( \epsilon > 0 \) such that \( \Pi_{\overline{\Omega}_0, w_0}(\epsilon, W, M) \) is defined and continuous on the \( \epsilon \)-neighborhood of \((K, f)\) in \( \mathcal{T}(W, M) \). We find \( \delta > 0 \) for \((K, f)\) such that the pairs \((\overline{\Omega}_0, g_0)\) and \((\Omega_1, g_1)\) can be connected by a continuous path \((\overline{\Omega}_t, h_t)\), \( 0 \leq t \leq 1 \), in the \( \epsilon \)-neighborhood of \((K, f)\) and the family of Jordan domains \( \Omega_t \) is Radó continuous. Then the continuous path \( \Pi_{\overline{\Omega}_0, w_0}(\epsilon, W, M) \) connects \((\overline{\Omega}_0, g_0)\) and \((\Omega_1, g_1)\). Hence by Lemma 3.2 \( I_{\overline{\Pi}, \zeta_1}(g_1) = I_{\overline{\Pi}, \zeta_0}(p_0) \).

Now we prove that close approximations have the same homotopic type.

**Proposition 3.4.** Let \( f \in \mathcal{A}_{\overline{\Omega}_0, w_0}(K, W, M) \) and let \( \gamma \) be an access curve to \( K \) at \( \zeta_0 \). There is \( \delta > 0 \) such that if \((\overline{\Omega}_1, g_1)\) and \((\overline{\Omega}_2, g_2)\) are \( \delta \)-approximations of \((K, f)\) with respect to \( \gamma \), then \( I_{\overline{\Pi}, \zeta_1}(g_2) = I_{\overline{\Pi}, \zeta_1}(g_1) \).

**Proof.** We take as \( \delta = \delta_1 \) in Proposition 3.3. By Lemma 3.1 there is a Jordan domain \( \Omega_0 \) containing \( K \) such that \( \overline{\Omega}_0 \subset \Omega_1 \cap \Omega_2 \) and, given any point \( \zeta_1 \in \gamma \cap \partial \Omega_0 \), a mapping \( g_0 \in \mathcal{A}(\Omega_0, W, M) \) such that the pair \((\overline{\Omega}_0, g_0)\) lies in the \( \delta \)-neighborhood of \((K, f)\) and \( g_0(\zeta_1) = w_0 \). Let \( t_0 = \sup\{ t : t < s_{\zeta_1, \gamma}, t \in \Omega_0 \} \) and let \( \zeta_1 = \gamma(t_0) \). By Lemma 3.3 \( I_{\overline{\Pi}, \zeta_1}(g_1) = I_{\overline{\Pi}, \zeta_1}(g_0) \) and by the same argument \( K_{\overline{\Pi}, \zeta_1}(g_2) = K_{\overline{\Pi}, \zeta_1}(g_0) \).

Let \( \gamma \) be an access curve to \( K \) at \( \zeta_0 \). We define the mapping

\[
I_{K, \gamma} = I_{\gamma} : \mathcal{H}_{\overline{\Omega}_0, w_0}[K, W, M] \to \mathcal{H}_{\overline{\Omega}_0, w_0}[W, M] = \eta(W, M, w_0)
\]

as \( I_{K, \gamma}(f) = I_{\overline{\Pi}, \zeta_1}(g) \), where \((\overline{\Pi}, g)\) is a sufficiently close approximation of \((K, f)\). By Proposition 3.4 this mapping is well defined.

If \( f \in \mathcal{A}_{w_0}(W, M) \) let \( \iota(f) \) be the loop \( f|_T \) in \( W \). Clearly, if \( [f]_{1, w_0} = [g]_{1, w_0} \) in \( \eta_1(W, M, w_0) \), then \( \iota(f) \) and \( \iota(g) \) are homotopic in \( \pi_1(W, w_0) \). Hence the mapping

\[
\iota_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0)
\]
Proof. Let $\delta_1$ be the $\delta$ from Lemma 3.3. Let $(\overline{\Omega}, g_1)$ be a $\delta_1$-approximation of $(K, f)$ such that the restriction of $\gamma$ to $[0, s_{\delta_1, \gamma}]$ lies in $D(\zeta_0, \delta_1)$. Let $r$ be the minimal distance between points on $\partial \Omega_1$ and $K$. We take $\delta = \min\{r, \delta_1\}/2$.

If a pair $(L, g)$ lies in the $\delta$-neighborhood of $(K, f)$ then $L \subset \Omega_1$. We take a $\delta$-approximation $(\overline{\Omega}_0, g_0)$ of $(L, g)$ such that $\Omega_0 \subset \Omega_1$ and $I_{\overline{\Omega}_0, \zeta_0, \gamma}(g_0) = I_{L, \gamma}(g)$. The pair $(\overline{\Omega}_0, g_0)$ lies in the $\delta_1$-neighborhood of $(K, f)$ so by Lemma 3.3

$$I_{K, \gamma}(f) = I_{\overline{\Omega}_1, \zeta_1, \gamma}(g_1) = I_{\overline{\Omega}_0, \zeta_0, \gamma}(g_0) = I_{L, \gamma}(g).$$

□

The following technical lemma will be used later several times.

Lemma 3.6. Suppose that $K$ consists of a simple curve $\alpha$ connecting $\zeta_0$ and $\zeta_1$ and the closure of a smooth Jordan domain $\Omega_1$ such that $\overline{\Omega}_1 \cap \alpha = \{\zeta_1\}$. Let $(K, f) \in T_{\zeta_0, w_0}(W, M)$ and let $\gamma$ be an access curve to $K$ at $\zeta_0$. Let $\Omega_0 \subset \Omega_1$ be another smooth Jordan domain such that $\partial \Omega_1 \cap \partial \Omega_0 = \{\zeta_1\}$ and let $L = \alpha \cup \overline{\Omega}_0$. Then there is a mapping $g \in A_{\zeta_0, w_0}(L, W, M)$ such that $g = f$ on $\alpha$, $I_{\overline{\Omega}_1, \zeta_1}(f) = I_{\overline{\Omega}_0, \zeta_1}(g)$ and $I_{K, \gamma}(f) = I_{L, \gamma}(g)$.

Proof. We take a conformal mapping $\Phi$ of $\Omega_1 \setminus \overline{\Omega}_0$ onto the strip $\{0 < \text{Re} \zeta < 1\}$. This mapping extends smoothly to the boundary and we assume that $\Phi(\partial \Omega_1) = \{\text{Re} \zeta = 1\}$ and $\Phi(\partial \Omega_0) = \{\text{Re} \zeta = 0\}$. Since $\Phi^{-1}(\zeta)$ converges to $\zeta_1$ when $\text{Re} \zeta \to \pm \infty$ the domains $\Omega_t$ bounded by curves $\Phi^{-1}(\{\text{Re} \zeta = t\})$ and $\zeta_1$ are Jordan domains. Moreover the family $\{\Omega_t\}$ is Radó continuous because as homeomorphisms of $\partial \Omega_1$ onto $\partial \Omega_0$ we can take preimages of mappings $x + it \to x + it_0$.

Let $\Psi_t$ be a continuous family of conformal mappings of $\Omega_t$ onto $\Omega_1$ such that $\Psi_t(\zeta_1) = \zeta_1$ and let $K_t = \alpha \cup \overline{\Omega}_0$. We define $f_t$ as $f$ on $\alpha$ and as $f \circ \Psi_t$ on $\overline{\Omega}_0$. Thus we obtain a continuous path in $T_{\zeta_0, w_0}(W, M)$ and letting $g = f_0$ we get our lemma. □

Two access curves $\gamma_1$ and $\gamma_2$ are equivalent if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < t_1, t_2 < \delta$ then the points $\gamma_1(t_1)$ and $\gamma_2(t_2)$ can be connected by a continuous curve $\alpha$ in $D(\zeta_0, \varepsilon) \setminus K$. In the terminology of the prime ends theory (see 11) it means that curves $\gamma_1$ and $\gamma_2$ determine the same prime end.

The following result shows that $I_{K, \gamma_1} = I_{K, \gamma_2}$ when $\gamma_1$ and $\gamma_2$ are equivalent. In particular, if $K$ is the closure of a Jordan domain then by a theorem of Carathéodory all access curves at any point of $\partial K$ are equivalent and $I_{K, \gamma}$ is determined only by $\zeta_0$ so we can write $I_{K, \gamma} = I_{K, \zeta_0}$.

Proposition 3.7. Let $f \in A_{\zeta_0, w_0}(K, W, M)$ and $\zeta_0 \in \partial K$. If $\gamma_1$ and $\gamma_2$ are equivalent access curves to $K$ at $\zeta_0$, then $I_{K, \gamma_1}(f) = I_{K, \gamma_2}(f)$.

Proof. We take $\delta > 0$ that is less than $\delta$’s in Lemma 3.3 and Propositions 3.1 and 3.4. Then we find $\delta$-approximations $(\overline{\Omega}, g_1)$ of $(K, f)$ with respect to $\gamma_1$ and $(\overline{\Omega}, g_2)$ of $(K, f)$ with respect to $\gamma_2$. The domain $\Omega$ has been chosen so that the restrictions
of curves $\gamma_1$ and $\gamma_2$ to $[0, s_{\Omega, \gamma_1}]$ and $[0, s_{\Omega, \gamma_2}]$ lie in $\mathbb{D}(\zeta_0, \delta)$. By Proposition 3.3 $I_{K, \gamma_1}(f) = I_{\overline{K}, \zeta_0, \gamma_1}(g_1)$ and $I_{K, \gamma_2}(f) = I_{\overline{K}, \zeta_0, \gamma_2}(g_2)$. We take $0 < \sigma < \delta$ such $\mathbb{D}(\zeta_0, \sigma) \subset \Omega$ and find $t_1, t_2 > 0$ such that $\gamma_1(t_1)$ and $\gamma_2(t_2)$ can be connected by a continuous path $\gamma_3$ in $\mathbb{D}(\zeta_0, \sigma) \setminus K$.

We take a Jordan domain $\Omega_0 \subset \subset \Omega$ such that $K \subset \Omega_0$ and $\gamma_3 \subset \Omega \setminus \overline{\Omega_0}$. Let $t_3 = \sup \{t : \gamma_2(t) \in \Omega_0 \}$. By Proposition 3.4 there is a $\delta$-approximation $(\overline{\Omega_0}, h)$ of $(K, f)$ such that $g(\gamma_2(t_3)) = w_0$. Let $\gamma$ be a curve in $\overline{\Omega_0} \setminus \Omega_0$ that follows $\gamma_1$ to $\gamma_1(t_1)$, then $\gamma_3$ until $\gamma_2(t_2)$ and then $\gamma_2$ to $\zeta_0$. Since $\gamma_3 \subset \mathbb{D}(\zeta_0, \delta)$ by Lemma 3.3 $I_{\overline{\Omega_0}, \zeta_0, \gamma_1}(g_1) = I_{\overline{\Omega_0}, \zeta_0, \gamma_2}(t_3)(h)$. By the same lemma $I_{\overline{\Omega_0}, \zeta_0, \gamma_2}(g) = I_{\overline{\Omega_0}, \zeta_0, \gamma_2}(t_3)(h)$ and we are done.

4. Holomorphic fundamental semigroup of Riemann domains

If $f \in \mathcal{A}_{w_0}(W, M)$ we will denote $[f]_{1, w_0}$ by $[f]$. To introduce on $\eta_1(W, M, w_0)$ a semigroup structure compatible with $t_1$ we need an additional construction since in the standard definition the concatenation of two loops cannot be realized as a boundary of an analytic disk.

Suppose that $f_1, f_2 \in \mathcal{A}_{w_0}(W, M)$ are representatives of equivalence classes $[f_1]$ and $[f_2]$ respectively in $\eta_1(W, M, w_0)$. Let $K \subset \mathbb{C}$ be the union of $K_1 = \{\zeta : |\zeta - 1| \leq 1\}$ and $K_2 = \{\zeta : |\zeta + 1| \leq 1\}$ and let $\gamma(t) = -it$, $0 \leq t \leq 1$. Then $\gamma$ is an access curve for $K$ to 0. We define the mapping

$$h_{f_1, f_2}(\zeta) = \begin{cases} f_1(1 - \zeta), & \zeta \in \partial K_1, \\
f_2(1 + \zeta), & \zeta \in \partial K_2 \end{cases}$$

of $\partial K$ into $W$. The mapping $h_{f_1, f_2}$ maps $K$ into $M$ so $h_{f_1, f_2} \in \mathcal{A}_{0, w_0}(K, W, M)$.

We let $[f_1] \ast [f_2] = I_{K, \gamma}(h_{f_1, f_2})$. If $f_1$ and $f_2$ are $h$-homotopic to $g_1$ and $g_2$ respectively in $\mathcal{A}_{w_0}(W, M)$, then evidently $h_{f_1, f_2}$ is $h$-homotopic to $h_{g_1, g_2}$ in $\mathcal{A}_{0, w_0}(K, W, M)$. Hence the class $[f_1] \ast [f_2]$ is well defined.

The following notion of stars is similar to the notion of stars in [10]. A star is a plane compact set $K$ that consists of $n$ simple curves $\alpha_j : [0, 1] \to \mathbb{C}$ starting at the same point $\zeta_0$ called the center of the star and $n$ disjoint closed disks $D_j$ such that $\zeta_j = \alpha_j(1) \in \partial D_j$. It is required that curves $\alpha_j$ and $\alpha_i$ meet only at $\zeta_0$ when $i \neq j$ and $D_j \cap (\bigcup_{i=1}^n \alpha_i) = \{\zeta_j\}$ for all $j$. We let $K_j = \alpha_j \cup D_j$ and call them the arms of a star. Note that for any star there is a homeomorphism of the plane that transforms this star into a straight star, i.e., a star where all curves $\alpha_j$ are intervals.

The conformal mapping $\phi : \mathbb{D} \to \mathbb{C}^1 \setminus K$ extends continuously to the boundary. This happens because the natural mapping of the prime ends space of $\mathbb{C}^1 \setminus K$ onto $\partial K$ is continuous. If $\gamma$ is an access curve to $K$ at $\zeta_0$ and $\phi(1) = \zeta_0$, then the numeration of $K_j$ is chosen in such a way that as $\zeta$ travels by $T$ clockwise starting at 1 the point $\phi(\zeta)$ travels first by $\partial K_1$, then $\partial K_2$ and so on.

**Proposition 4.1.** Suppose that $K$ is a star with arms $K_j = \alpha_j \cup D_j$, $1 \leq j \leq n$, $\zeta_0$ is center of $K$ and $\gamma$ is an access curve to $K$ at $\zeta_0$. Let $L_2$ be the star with arms $K_j$, $2 \leq j \leq n$, and let $L_1$ be the star with arms $K_j$, $1 \leq j \leq n - 1$. If $f \in \mathcal{A}_{w_0}(W, M)$ and $f_1 = f|_{K_1}$, then

$$I_{K, \gamma}(f) = I_{K_1, \gamma}(f_1) \ast I_{L_2, \gamma}(f|_{L_2}) = I_{L_1, \gamma}(f|_{L_1}) \ast I_{K_n, \gamma}(f_n).$$

**Proof.** We assume that $\zeta_0 = 0$. Take some small $t_0 > 0$ and redefine $f$ on each $\alpha_j$ letting it to be $w_0$ on $\alpha_j([0, t_0])$ while on $[t_0, 1]$ we set the mappings as $f(\alpha_j(t -
t_0) / (1 - t_0)). We don’t change f on disks and preserve for new mapping the same notation. Since this operation can be achieved by a continuous family of deformations by Theorem 3.5 the h-homotopic classes of (K, f) and all (K_j, f_j) will not change.

In the next step for s ∈ [0, 1] we squeeze intervals of curves α_2, . . . , α_n considering a continuous family of continuous curves α_i^2, . . . , α_n^2 on [0, 1] such that each of these curves is simple, α_i^2 = α_i, α_j^2(t) = α_j(t) for t > t_0 and α_i^2(t) = α_i^1(t) for all 2 ≤ i, j ≤ n and t ∈ [0, t_1] for some 0 < t_1 < t_0. We set f_j^2(α_j^2(t)) = f_j(α_j(t)) and do not change f_j on D_j. It is easy to see that such family can be found for straight stars and, consequently, for all stars. Let N_2 be the union of curves α_i^2 and disks D_j, 2 ≤ j ≤ n, and let g_2 be the mapping of N_2 equal to f_j^1 on α_i^1 ∪ D_j. Let N = K_1 ∪ N_2 and let g ∈ A_{t_0, w_0}(N, W, M) be equal to f_1 on K_1 and to g_2 on N_2. It follows from Theorem 3.5 that I_{K, γ}(f) = I_{N, γ}(g) and I_{L, γ}(f) = I_{N, γ}(g).

Let β_1 = α_1|[0, t_0], N'_1 = K_1 \ β_1, and g_1 = f_1|N'_1. Let β_2 = α_2|[0, t_1], N'_2 = N_2 \ β_2, and h_2 = g_2|N'_2. We take close approximations of (N'_j, f_j) by (U_j, p_j), j = 1, 2, where U_j are smooth Jordan domains. Proposition 3.1 gives us a lot of freedom for choices of U_j so we may assume that U_j meet β_j only once at points η_j = β_j(s_j) and U_1 ∩ U_2 = ∅. Let γ_j = β_j|[0, s_j], A_j = γ_j ∪ U_j, A = A_1 ∪ A_2 and q_j = p_j on U_j and w_0 on γ_j. By our choice I_{A, γ}(g_j) = I_{N_j, γ}(g_j) and if approximations are close enough I_{A, γ}(g) = I_{N_j, γ}(g), where q = q_j on A_j.

Using Lemma 3.3 we replace Jordan domains U_j with small disjoint disks V_j ⊂ U_j attached to γ_j at η_j and the mapping q on them with a mapping r preserving all involved h-homotopy classes. Then we take continuous deformations γ_j^s of γ_j, s ∈ [0, 1], so that γ_j^s(0) = η_j = 0 and γ_j^s(1) = t while γ_j^1(0) = 0. We may assume that if V_j^s = V_j + γ_j^s(1) − η_j, M_j^s = γ_j^s ∪ V_j^s and M^s is the union of M_j^s, then M^s are stars for all s. Let r_j^s(γ_j^s(t)) = r_j(t) and r_j^s(η_j) = r_j(η_j − γ_j^s(1) + η_j) on V_j^s. We let r^s to be equal to q_j^s on M_j^s.

Applying a continuous family of rotation and dilations we can make the disks V_j^s perpendicular to the real axis and of radius 1. The mapping r^s will follow these changes as compositions with rotations and dilations.

The obtained compact set L that consist of intervals [0, 1] and [-1, 0] and disks B(2, 1) and B(-2, 1) has two prime ends at 0: one of them is equivalent to [-1, 0] and another to [0, i]. The chosen numeration tells us that our access curve is γ = [-i, 0]. Now we shrink intervals [0, 1] and [-1, 0] to 0 simultaneously translating disks V_j. We can do this because r_j^1 ≡ w_0 on these intervals. As the result we obtain figure form the definition of the * operation and we see that I_{K, γ}(f) = I_{K_1, γ}(f_1) * I_{L_2, γ}(f_1)_{L_1}.

The second equality in the proposition is proved similarly. □

This proposition leads to the following theorem.

**Theorem 4.2.** The operation * induces on η_1(W, M, w_0) the structure of a semi-group with unity.

**Proof.** The unity is the class of the constant mapping equal to w_0 on T. If, say, in the definition of the * operation f_1 ≡ w_0 then continuously shrinking K_1 to the origin leaving the functions equal to w_0 we will get a continuous path in T_{t_0, w_0}(W, M) which ends at (K_2, f_2(1 + ζ)). By Theorem 3.5 I_{T}(h_{f_1, f_2}) = [f_2].

To prove that the operation * is associative we consider a compact set L consisting of three intervals I_1 = [0, 1], I_2 = [0, i], I_3 = [-1, 0] and three closed disks
\[ D_1 = \{|\zeta - 2| \leq 1\}, \quad D_2 = \{|\zeta - 2i| \leq 1\} \quad \text{and} \quad D_3 = \{|\zeta + 2| \leq 1\}. \]

The access curve \( \gamma = [-i, 0] \). Given \( f_1, f_2, f_3 \in A_{1,w_0}(D,W,M) \) we define the mapping \( f \) on \( L \) to be equal to \( w_0 \) on intervals \( I_1, I_2, I_3 \) and \( f_1(2 - \zeta) \) on \( D_1 \), \( f_2(2 + i\zeta) \) on \( D_2 \) and \( f_3(2 + \zeta) \) on \( D_3 \).

By Proposition 4.1
\[
I_{L,\gamma}(f) = I_{L,\gamma}(f_1) \star (I_{L,\gamma}(f_2) \star I_{L,\gamma}(f_3)) = (I_{L,\gamma}(f_1) \star I_{L,\gamma}(f_2)) \star I_{L,\gamma}(f_3).
\]

Using induction and Proposition 4.1 and the previous theorem we get

**Theorem 4.3.** In assumptions of Proposition 4.1
\[
I_{K,\gamma}(f) = I_{K,\gamma}(f_1) \cdots I_{K,\gamma}(f_n).
\]

We finish this section with two examples of the semigroup \( \eta_1 \) when \( W = A_{s,r} = \{s < |z| < r\} \), \( 0 < s < 1 < r \), and \( M = \mathbb{C}P^1 \) or \( M = \mathbb{D}(0, x) \), where \( r \leq x \leq \infty \). We fix \( \Pi(z) = z \) and \( w_0 = 1 \). The examples below show that the mapping \( t_1 : \eta_1(W, M, w_0) \to \pi_1(W, w_0) \) need not to be injective or surjective.

If \( f \in A_{w_0}(A_{s,r}, \mathbb{C}P^1) \) then we can write it as \( \hat{f} = gB_1B_2^{-1} \), \( B_1 \) and \( B_2 \) are Blaschke products and \( B_1 \) contains all zeros of \( f \) while \( B_2 \) contains all poles. The function \( g \) has no zeros and poles, \( g(1) = 1 \) and \( s < |g| < r \) on \( T \). Hence \( g \) maps \( \mathbb{D} \) into \( A_{s,r} \) and is \( h \)-homotopic to the constant mapping equal to \( 1 \).

We change \( B_1 \) by dragging its zeros to \( 0 \) by continuous curves and then change \( B_2 \) by dragging its zeros to some \( a \neq 0 \) at \( \mathbb{D} \). Thus \( f \) is \( h \)-homotopic to
\[
h(\zeta) = (-a)^n \zeta^m \left( \frac{1 - \bar{a}\zeta}{\zeta - a} \right)^n
\]

Thus we obtained a mapping \([f] \to (m, n)\) and, clearly, it is a homomorphism and it is injective because continuous deformations of analytic disks do not change the numbers of zeros and poles. Hence the semigroup \( \eta_1(A_{s,r}, \mathbb{C}P^1, w_0) \) is isomorphic to \( N_0 \oplus N_0 \), where \( N_0 \) is the semigroup by addition of non-negative integers.

A similar argument shows that the semigroup \( \eta_1(A_{s,r}, \mathbb{D}(0, x), w_0) \) is isomorphic to \( N_0 \). The isomorphism is given by the mapping \([f] \to m\), where \( m \) is the number of zeros of \( \hat{f} \) counted with multiplicity.

**5. Properties of holomorphic fundamental semigroups**

Let \((W_1, \Pi_1)\) and \((W_2, \Pi_2)\) be two Riemann domains over two complex manifolds \( M_1 \) and \( M_2 \) respectively. Suppose \( w_1 \in W_1 \), \( w_2 \in W_2 \) and there are holomorphic mappings \( \phi : W_1 \to W_2 \) such that \( \phi(w_1) = w_2 \) and \( \psi : M_1 \to M_2 \) which satisfy \( \psi \circ \Pi_1 = \Pi_2 \circ \phi \). Then for any \( f \in \mathcal{A}(K,W_1,M_1) \) we have \( \Pi_2 \circ \phi \circ f = \psi \circ \Pi_1 \circ f = \psi \circ \hat{f} \) and we get a continuous mapping from \( T(W_1, M_1) \) to \( T(W_2, M_2) \) which maps a pair \((K, f)\) to \((K, \phi \circ f)\). Hence, firstly, the mapping from \( \mathcal{A}_{w_1}(W_1, M_1) \) to \( \mathcal{A}_{w_2}(W_2, M_2) \) induces a well defined mapping \( \phi_* \) from \( \eta_1(W_1, M_1, w_1) \) to \( \eta_1(W_2, M_2, w_2) \) given by \( \phi_*([f]) = [\phi \circ f] \). Secondly, if \( \gamma \) is an access curve to \( K \), then \( \phi_* (I_{K,\gamma}(f)) = I_{K,\gamma}(\phi \circ f) \). In particular, if \((K, h_{f_1,f_2})\) is the pair in the definition of the \(*\) operation, then
\[
\phi_*([f_1] \star [f_2]) = \phi_* (I_{K,\gamma}(h_{f_1,f_2})).
\]

This leads us to the following proposition.
Proposition 5.1. The induced mapping $\phi_* : \eta_1(W_1, M_1, w_1) \to \eta_1(W_2, M_2, w_2)$ is a homomorphism.

Clearly $(W_1 \times W_2, (\Omega_1, \Omega_2))$ is a Riemann domain over $M_1 \times M_2$. As in the classical homotopy theory Proposition 5.1 leads to the following corollary.

Corollary 5.2. If $(W_1, \Omega_1)$ and $(W_2, \Omega_2)$ are two Riemann domains over two complex manifolds $M_1$ and $M_2$ respectively, then

$$\eta_1(W_1 \times W_2, M_1 \times M_2, (w_1, w_2)) \cong \eta_1(W_1, M_1, w_1) \times \eta_1(W_2, M_2, w_2).$$

Another corollary describes the powers in $\eta_1(W, M, w_0)$.

Corollary 5.3. Let $f \in \mathcal{A}_{w_0}(W, M)$ and let $[f]^k$ be the product of $k$ classes $[f]$. Then $[f]^k = [f(\zeta^k)]$.

Proof. We may assume that $\hat{f}$ is defined on $\mathbb{D}(0, r)$, $r > 1$, and $f$ maps $A_{r-1, r} = \{\zeta \in \mathbb{C} : r^{-1} < |\zeta| < r\}$ into $W$. Set $W_1 = A_{r-1, r}$, $M_1 = \mathbb{D}(0, r)$ and $w_1 = 1$. Let $\phi = f$. By Proposition 5.1 and an example at the end of the previous section we have

$$[f(\zeta^k)] = \phi_*(\zeta^k) = \phi_*(\zeta)^k = \phi_*(\zeta^k)^k = [f]^k.$$

\[\square\]

Let $\alpha(t), t \in [0, 1]$, be a continuous curve in $W$ with $\alpha(0) = w_0$ and $\alpha(1) = w_1$. Let $L$ be a compact set on the plane consisting of the interval $I = [0, 1]$ and the disk $D = \{|\zeta| \leq 1\}$. Given a mapping $f \in \mathcal{A}_{w_0}((W, M))$ we define a mapping $\tilde{f}$ on $L$ to be equal to $\alpha$ on $I$ and to $f(2 - \zeta)$ on $\partial D$. Clearly, $\tilde{f} \in \mathcal{A}_{w_0}(L, W, M)$.

We take the access curve $\gamma(t) = -it$, $0 \leq t \leq 1$, to $L$ at the origin. Clearly, if $[f]_1, w_0 = [g]_1, w_0$, then $I_{L, \gamma}(\tilde{f}) = I_{L, \gamma}(\tilde{g})$. Hence we have a well-defined mapping

$$F_{\alpha}(f) = F_{\alpha}([f]) = I_{L, \gamma}(\tilde{f})$$

from $\eta_1(W, M, w_1)$ into $\eta_1(W, M, w_0)$.

By Theorem 3.3 any curve connecting $w_0$ to $w_1$ which is homotopic to $\alpha$ will give us the same mapping $F_{\alpha}$. Thus $F_{\alpha}$ depends only on the homotopy class $\{\alpha\}$ of $\alpha$ in $\pi_1(W, w_0, w_1)$.

We let $\alpha^{-1}$ to be the curve $(\alpha^{-1})(t) = \alpha(1 - t)$ for $0 \leq t \leq 1$ and, if a curve $\beta$ connects $w_1$ and $w_2 \in W$, denote by $\alpha \beta$ the curve on $[0, 1]$ defined as $\alpha(2t)$ when $0 \leq t \leq 1/2$ and as $\beta(2t - 1)$ when $1/2 \leq t \leq 1$.

Theorem 5.4. Let $w_0, w_1, w_2$ be points of $W$, continuous curves $\alpha$ and $\beta$ connect $w_0$ with $w_1$ and $w_1$ with $w_2$ respectively. Then:

1. $F_{\alpha \beta} = F_{\alpha} \circ F_{\beta}$;
2. $F_{\alpha}$ is an isomorphism of $\eta_1(W, M, w_1)$ onto $\eta_1(W, M, w_0)$.

Proof. (1) Let $K = [0, 2] \cup \mathbb{D}(3, 1)$ and let $f \in \mathcal{A}_{w_0}(W, M)$. We define the mapping $g$ on $K$ as $\alpha$ on $[0, 1]$, $\beta(t - 1)$ on $[1, 2]$ and $f(3 - \zeta)$ on $\mathbb{D}(3, 1)$. All access curves to $K$ at 0 are equivalent, we take any such $\gamma$ and $I_{K, \alpha}(g) = F_{\alpha \beta}(f)$.

We take a Jordan domain $\Omega$ containing $K_1 = [1, 2] \cup \mathbb{D}(3, 1)$ in its closure such that $1 \in \partial \Omega$ and a close approximation $h_1$ of $g|_{K_1}$ on $\Omega$ so that $I_{\mathbb{D}, h_1} = I_{K, 1}(g|_{K_1}) = F_{\beta}(f)$. If the mapping $h$ is defined on $K_2 = [0, 1] \cup \Omega$ as $h_1$ on $\Omega$ and $\alpha$ on $[0, 1]$, then also $I_{K, \gamma}(f) = I_{K, 0}(h)$.

Then using Lemma 3.3 we deform $\Omega$ to a small disk attached to 1 and then to the disk $\mathbb{D}(2, 1)$ while $h$ changes to $p_1$. If the mapping $p$ is defined on $K_3 = \mathbb{D}(2, 1)$ we take one such $\gamma$ and apply the previous argument.

Consider a compact set \( \mathcal{A}_{w_0}(W, M) \) then we let the mapping \( F_g = F_{\alpha} \), where \( \alpha(t) = g(e^{2\pi it}) \), and if \( \alpha \) is a loop in \( W \) starting at \( w_0 \) we denote by \( \{\alpha\} \) the equivalence class of \( \alpha \) in \( \pi_1(W, w_0) \).

**Theorem 5.5.** The mapping \( \Phi : \{\alpha\} \rightarrow F_{\alpha} \) establishes a homomorphism of \( \pi_1(W, w_0) \) into \( \text{Aut}(\mathcal{N}(W, M, w_0)) \). Moreover, if \( g \in \mathcal{A}_{w_0}(\overline{W}, M) \), then \( F_g([f]) \ast [g] = [g] \ast [f] \) and \( F_g([g]) = [g] \).

**Proof.** The first part of the theorem is a direct consequence of Theorem 5.3. Let us show that \( F_g([f]) \ast [g] = [g] \ast [f] \).

Consider a compact set \( K \) consisting of the disks \( I \times D^1 = \{(\zeta, 1) \leq 1 \} \) and \( D^2 = \{1 \times \zeta \leq 1 \} \) and the interval \( I = [0, 1] \). Define a mapping \( h(\zeta) = g(e^{2\pi i\zeta}) \) on \( I \) and \( D^2 \) as \( f(3 - \zeta) \). Let \( \gamma = [-i, 1, 1] \) be an access curve to \( K \) at \( 1 \). Then \( I_{K, \gamma}(h) = F_g([f]) \ast g \).

Consider the continuous family of compact sets \( K_s \), \( 0 \leq s \leq 1/2 \), consisting of the disk \( I_s = \{e^{\pi is}, (2-s)e^{3\pi is}\} \) and the closed unit disk \( D^2_s \) attached normally to \( (2-s)e^{3\pi is} \). The mapping \( h_s \) on \( K_s \) is defined as \( h_s \) on \( D^1 \) and as \( h(s + \zeta) \) when \( \zeta \in I_s \). The mapping \( h_s \) on \( D^2_s \) is defined as a composition of \( h \) on \( D^2 \) and a conformal mapping that maps \( D^2_s \) onto \( D^2 \) moving \( (2-s)e^{3\pi is} \). Similarly, we rotate \( I \times D^2 \) around \( D^1 \) leaving one end of \( I_s \) attached normally to \( D^1 \). Clearly, the pairs \( (K_s, h_s) \) form a continuous path and \( I_{K_{s}, \gamma}(h_s) = F_g([f]) \ast [g] \).

When \( s = 1/2 \) the set \( K_{1/2} \) consists of \( D^1, I_{1/2} = [-1, -3/2] \) and the disk \( D^2_{1/2} \). Since all access curves to \( K_{1/2} \) at 1 are equivalent we replace \( \gamma \) with \( \gamma' = [i + 1, 1] \). Still \( I_{K_{1/2}, \gamma'}(h_{1/2}) = F_g([f]) \ast [g] \).

Then we continue the process described above for \( 1/2 \leq s \leq 1 \). Finally, \( K_1 \) will consists of \( D^1 \) and \( D^2_1 = \{1 \times \zeta \leq 1 \} \). The mapping \( h_1 \) is equal to \( g \) on \( D^1 \) and to \( f(2 - \zeta) \) on \( D^2_1 \). Now it is clear that \( I_{K_1, \gamma'}(h_1) = [g] \ast [f] \).

To show that \( F_g([g]) = [g] \) we start with the compact set \( K_1 \) consisting of the interval \( I = [0, 1] \) and the unit disk \( D_1 = \{1 \times \zeta \leq 1 \} \). The mapping \( f_1 \) on \( K_1 \) is defined as \( g(e^{3\pi i\zeta}) \) on \( I \) and as \( g(2 - \zeta) \) on \( D_1 \). If the access curve \( \gamma = [-i, 0] \), then \( I_{K_1, \gamma}(f_1) = F_{g([g])} \).

For \( 0 \leq s \leq 1 \) we define compact sets \( K_s \) consisting of the intervals \( I_s = [0, s] \) and the disks \( D_s = \{1 \times (s + 1) \leq 1 \} \). The mapping \( f_s \) is defined as \( g(e^{2\pi is}) \) on \( I_s \)
and as \( g(e^{2\pi i(1 + s - \zeta)}) \) on \( D_s \). The pairs \((K_\alpha, f_s)\) form a continuous path and \( I_{K_\alpha, \zeta}(f_s) = F_\alpha([g]). \) Since \( K_0 \) consists of the disk \( \{ \zeta - 1 \leq 1 \} \) and the mapping \( f_0(\zeta) = g(1 - \zeta) \) we see that \( I_{K_1, \zeta}(f_1) = [g] \).

We remark that a semigroup \( S \) is cancellative if \( ab = ac \) or \( ba = ca \) imply \( b = c \) for any \( a, b, c \in S \) and it is right(left)-reversible if \( Sa \cap Sb \neq \emptyset \) (\( aS \cap bS \)) for any \( a, b \in S \) and reversible if it is both right- and left- reversible.

**Corollary 5.6.** Let \( f, g \in A_{w_0}(W, M) \). Then:

1. if \( \iota_1([f]) = \iota_1([g]) \) then \([f]*[g] = [g]*[f] \).
2. The semigroup \( \eta_1 \) is reversible.
3. The semigroup \( \eta_1 \) is embeddable into a group if and only if it is cancellative.
4. The image of \( \eta_1(W, M, w_0) \) in \( \pi_1(W, w_0) \) under the mapping \( \iota_1 \) is invariant with respect to the inner automorphisms in \( \pi_1(W, w_0) \).

**Proof.** Since \( F_1 = F_g \), \([f]*[g] = [g]*[f] \) and we get (1) by the second part of Theorem 5.5. (2) follows because \( F_g([f])*[g] = [g]*[f] \) and \([f]*[g] = [g]*F_{g^{-1}}([f]) \). (3) follows from Ore’s theorem (2. 1.10) which says that any right-reversible cancellative semigroup can be embedded into a group.

To show that the image is invariant with respect to the inner automorphisms we take \( f \in A_{w_0}(W, M) \) and the representative \( \alpha \) of \( \{ \alpha \} \in \pi_1(W, w_0) \). Since \( \{\alpha\}\{f\}\{\alpha\}^{-1} = \iota_1(F_\alpha([f])) \) the invariance follows.

We say that elements \( [f_0] \) and \( [f_1] \) in \( \eta_1(W, M, w_0) \) are \( \pi_1 \)-conjugate if \( F_\alpha([f_0]) = [f_1] \) for some \( \alpha \in \pi_1(W, w_0) \). The sets of all elements of \( \eta_1(W, M, w_0) \) that are \( \pi_1 \)-conjugate of \( [f] \) is said to be the \( \pi_1 \)-conjugacy class of \( [f] \). We denote the set of all \( \pi_1 \)-conjugacy classes by \( C(W, M, w_0) \).

**Proposition 5.7.** Mappings \( f_0, f_1 \in A_{w_0}(W, M) \) belong to the same connected component of \( A(W, M) \) if and only if they are \( \pi_1 \)-conjugate.

**Proof.** If mappings \( f_0, f_1 \in A_{w_0}(W, M) \) belong to the same connected component of \( A(W, M) \) then there is a continuous curve \( f_t \), \( 0 \leq t \leq 1 \), in \( A(W, M) \) that connects \( f_0 \) and \( f_1 \). For \( 0 \leq t \leq 1 \) define \( \alpha(t) = f_t(1), K_t = [0, t] \cup \overline{D}(1 + t, 1) \) and the mappings \( g_t \) to be equal to \( \alpha \) on \( [0, t] \) and to \( f_t(1 + t - \zeta) \) on \( \overline{D}(1, t + 1) \). The access curve \( \gamma = [-1, 0] \). Note that the curve \( \alpha \) is closed, \( I_{K_1, \zeta}(g_1) = F_\alpha([f_1]) \) and \( I_{K_0, \zeta}(g_0) = [f_0] \). By Theorem 5.5 \( F_\alpha([f_1]) = [f_0] \).

If \( F_\alpha([f_1]) = [f_0] \) for some \( \alpha \in \pi_1(W, w_0) \) then we let \( K = [0, 1] \cup \overline{D}(2, 1) \) and let \( g \) to be equal to \( \alpha \) on \( [0, 1] \) and to \( f_1(2 - \zeta) \) on \( \overline{D}(2, 1) \). The access curve \( \gamma = [-i, 0] \). Then \( I_{K, \gamma}(g) = [f_0] \) and \( I_{(2, 1), \gamma}(g) = [f_1] \).

Let \( (\Omega, h) \) be a close approximation of \( (K, g) \) with the following properties: \( \Omega \) is symmetric with respect to the real axis and intersects the real axis only at 0 and \( x_0 > 3 \), \( h(0) = h(1) = w_0 \), \( I_{\Omega, 0}(h) = [f_0] \) and \( I_{(2, 1), \gamma}(h) = [f_1] \). We take a conformal mapping \( \Phi \) of \( \Omega \cap \overline{D}(2, 1) \) onto an annulus \( A = \{ s \leq |\zeta| \leq r \} \) such that \( \Phi(\zeta) = \Phi(\zeta) \) and \( \Phi \) moves \( [0, 1] \) to \([s, r]\). Define domains \( \Omega_t \) as domains bounded by \( \Phi^{-1}(\Omega(0, t)), s \leq t \leq r \). The domains \( \Omega_t \) are Jordan and if \( e_t \) are conformal mapping of \( \overline{D} \) onto \( \Omega_t \) such that \( e_t(1) = e_t^{-1}(t) \) and \( e_t(0) > 0 \), then the mappings \( h \circ e_t \) form a continuous path in \( A(W, M) \) while \( h \circ e_r = [f_1] \) and \( h \circ e_s = [f_0] \).

It follows from this proposition that the mapping \( \Psi \) assigning to each class in \( C(W, M, w_0) \) the connected component of \( A(W, M) \) containing representatives of elements in this class is well defined.
Theorem 5.8. The mapping $\Psi$ is a bijection.

Proof. By Proposition 5.7 $\Psi$ is an injection. If $U$ is a connected component of $\mathcal{A}(W, M)$ then there are a point $w_1 \in W$ and a mapping $f \in \mathcal{A}_{w_1}(W, M)$ such that $f \in U$. Let $\alpha$ be a path in $W$ that connects $w_0$ to $w_1$. By Theorem 5.3 $F_\alpha([f]) \in \eta_1(W, M, w_0)$ and the same argument as in the proof of “only if” part in Proposition 5.7 shows that the representatives of $F_\alpha([f])$ are in $U$. So $\Psi$ is surjective.

6. The group $\rho_1(W, M, w_0)$

By the analogy with the complex case we introduce the space $\mathcal{T}^R(W, M)$ of pairs $(K, f)$, where $K$ is a connected compact set on the plane with connected complement and $f$ is a continuous mapping of $\partial K$ into $W$ such that $\tilde{f} = \Pi \circ f$ extends to a continuous mapping of $K$ into $M$. If $(K, f)$ and $(L, g)$ are in $\mathcal{T}^R(W, M)$ we define the distance between $(K, f)$ and $(L, g)$ similar to the definition of the distance $d$ on $\mathcal{T}(W, M)$. That makes the imbedding of $\mathcal{T}(W, M)$ into $\mathcal{T}^R(W, M)$ an isometry.

For a such compact set $K$, a point $\zeta_0 \in \partial K$ and a point $w_0 \in W$ let us denote by $\mathcal{R}_{\zeta_0, w_0}(K, W, M)$ the subset of all pairs $(K, f) \in \mathcal{T}(W, M)$ such that $f(\zeta_0) = w_0$. If $K = \mathbb{D}$ then $\mathcal{R}_{w_0}(W, M) = \mathcal{R}_{1, w_0}(\mathbb{D}, W, M)$. We say that $f_0, f_1 \in \mathcal{R}_{\zeta_0, w_0}(K, W, M)$ are equivalent if they belong to the same connected component of $\mathcal{R}_{\zeta_0, w_0}(K, W, M)$ and denote the set of all equivalence classes by $\mathcal{H}^R_{\zeta_0, w_0}(K, W, M)$ and let $[f]_{\rho}$ be the equivalence class containing $f$.

If $\gamma$ is an access to $K$ at $\zeta_0$ then, similar to the complex case, we can introduce the mapping

$$I^R_{K, \gamma} : \mathcal{H}^R_{\zeta_0, w_0}(K, W, M) \to \mathcal{H}^R_{1, w_0}(\mathbb{D}, W, M) = \rho_1(W, M, w_0).$$

Similar to the $*$ operation introduced earlier we can define the $*$ operation on $\rho_1(W, M, w_0)$ and a similar but simpler reasoning shows that $\rho_1(W, M, w_0)$ with the $*$ operation is a semigroup with unity. All properties of the operator $I^R_{K, \gamma}$ and the semigroup $\eta_1$, proved in the previous sections, stay true for their analogs $I^R_{K, \gamma}$ and $\rho_1$ but $\rho_1$ is a group.

Theorem 6.1. The operation $*$ induces on $\rho_1(W, M, w_0)$ the structure of a group: if $[f]_{\rho} \in \rho_1(W, M, w_0)$ then $[f(\zeta)]_{\rho} = [f]_{\rho}^{-1}$.

Proof. Let $f_1 \in \mathcal{R}_{w_0}(W, M)$ and let $f_2(\zeta) = f_1(\tilde{\zeta})$. Let $K_1 = \{\zeta - 1 \leq 1\}$ and $K_2 = \{\zeta + 1 \leq 1\}$ be the sets from the definition of the $*$ operation. Let $K'_1 = K_1 - t$ and $K'_2 = K_2 + t$ and let $L_t = K'_1 \cup K'_2$ when $0 \leq t \leq 1$ and $L_t = K'_1 \cap K'_2$ when $1 \leq t \leq 2$. Since $\tilde{f}_t(1 + \zeta) = \tilde{f}_1(1 + \zeta) = \tilde{f}_1(1 - (\tilde{\zeta}))$ the mappings $\tilde{f}_t(1 - \zeta)$ and $\tilde{f}_t(1 + \zeta)$ are symmetric with respect to the imaginary axis. Hence the mappings $g_t$ equal $\tilde{f}_t(1 - \zeta + t)$ on $\partial L_t \cap K'_1$ and $\tilde{f}_t(1 + \zeta - t)$ on $\partial L_t \cap K'_2$ are continuous and $\tilde{g}_t$ extends to the continuous mapping of $L_t$.

To preserve the base points we shift $L_t$ and $f_t$ upward by $i\alpha(t)$, where $\alpha(t) = \cos^{-1}(1 - t)$, and let $L'_t = (L_t + i\alpha(t)) \cup [0, i\alpha(t)]$. We set $g'_t$ as $g_t$ shifted upward and also let $g'_t(\zeta) = f_1(e^{i\alpha(\zeta)})$ for $0 \leq \zeta \leq \alpha(t)$. Note $g_t(0) = f_t(1) = w_0$ and $g_t(\alpha(t)) = f_t(e^{i\alpha(t)})$.

The path $(L'_t, g'_t)$, $0 \leq t \leq 2$, is continuous in $\mathcal{T}^R(W, M)$ and by the real analog of Theorem 3.6, we see that $[f_1]_{\rho} \ast [f_2]_{\rho} = [g'_2]_{\rho}$. But $L'_2$ is the interval $[0, i\pi]$ so $[g'_2]_{\rho} = e$.  \hfill \Box
There are natural homomorphisms \( \delta_1 : \eta_1(W, M, w_0) \to \rho_1(W, M, w_0) \) and \( \delta_2 : \rho_1(W, M, w_0) \to \pi_1(W, M, w_0) \) such that \( \delta_2 \circ \delta_1 = \iota_1 \).

**Theorem 6.2.** Let \( W \subset M \). Then in notation above:

1. If \( \pi_1(M, w_0) = 0 \) then \( \delta_2 \) is onto;
2. If \( \pi_2(M, w_0) = 0 \) then \( \delta_2 \) one-to-one.

**Proof.** (1) is evident. To show (2) we take an element \([f]_\rho \in \ker \delta_2\) and let \( \alpha = f|_\eta \). Then \( \{\alpha\} = \delta_2([f]_\rho) \). There is a continuous mapping \( g : \overline{D} \to W \) such that \( g|_\eta = \alpha \).

It means that \([g]_\rho = e\). If we realize \( \hat{f} \) as a mapping of the upper hemisphere of the unit ball in \( \mathbb{R}^3 \) and \( g \) as the mapping of the lower one then we obtain the mapping \( h \) of the sphere \( S^2 \) into \( M \) equal to \( \alpha \) on the equator. We may assume that \( h(1, 0, 0) = w_0 \). Since \( \pi_2(M, w_0) = 0 \) the mapping \( h \) can be continuously extended to the ball as a mapping into \( M \). Thus \([f]_\rho = [g]_\rho = e\). \( \square \)

As simple consequences of Corollary 5.6(1) and Theorem 6.2 we obtain

**Corollary 6.3.** Let \( W \subset M \). Then

1. The kernel of \( \delta_1 \) is a commutative semigroup.
2. If \( \pi_1(M, w_0) = \pi_2(M, w_0) = 0 \) then \( \rho_1(W, M, w_0) = \pi_1(W, w_0) \).

If \( W \subset M \) and \( \Pi \) is an inclusion map then \( \rho_1(W, M, w_0) \) is, of course, the relative homotopy group \( \pi_2(M, W, w_0) \). So for examples in Section 4 we get that \( \rho_1(A_{s, r}, \mathbb{CP}^1, w_0) = \mathbb{Z} \oplus \mathbb{Z} \) while \( \rho_1(A_{s, r}, \mathbb{C}, w_0) = \mathbb{Z} \).

7. **COMPLEMENTS TO ANALYTIC VARIETIES**

Let \( A \) be an analytic set in a connected complex manifold \( M \) and \( W = M \setminus A \).

We assume that \( A \) is the union of irreducible components \( A_j \) of pure codimension 1. (Analytic sets of codimension 2 and higher do not influence groups \( \eta_1 \) and \( \rho_1 \).)

If \( f \in \mathcal{R}_{w_0}(W, M) \) and the set \( A(f) = \{ \zeta \in \mathbb{D} : f(\zeta) \in A \} \) is finite, then we define the index \( \text{ind}(f, A_j) \) as the intersection index of \( f(\overline{D}) \) and \( A_j \). If \( \zeta \in A_j(f) \) and \( \phi \) is a defining function of \( A_j \), then we define a local index \( \text{ind}_{\zeta}(f, A_j) \) of \( f \) at \( \zeta \) as the index of \( \phi \circ f \) at \( \zeta \).

\[
\text{ind}(f, A_j) = \sum_{f(\zeta_j) \in A_j} \text{ind}_{\zeta_j}(f, A_j).
\]

A general \( f \in \mathcal{R}_{w_0}(W, M) \) can be approximated by such mappings and close approximations have the same indexes so \( \text{ind}(f, A_i) \) is defined for all \( f \in \mathcal{R}_{w_0}(W, M) \).

The index is a homotopic invariant so if \( f_0, f_1 \in \mathcal{R}_{w_0}(W, M) \) and \([f_0]_\rho = [f_1]_\rho\), then \( \text{ind}(f_0, A_j) = \text{ind}(f_1, A_j) \). Thus the mapping \( \text{ind} \) is well defined on \( \rho_1 \). Also \( \text{ind}(f, A_j) = 0 \) for all \( j \) if \([f]_\rho = e\).

It follows directly from the definition of the \( \ast \) operation that \( \text{ind}([f]_\rho \ast [g]_\rho, A_j) = \text{ind}([f]_\rho, A_j) + \text{ind}([g]_\rho, A_j) \). In particular, the group \( \rho_1 \) has no idempotents.

Suppose that \( f \in \mathcal{R}_{w_0}(W, M) \) and the set \( A(f) \) is finite. Let \( K \) be a star in \( \overline{D} \) with its center at 1 such that its arms \( K_j \) consist of simple curves \( \alpha_j \in \overline{D} \setminus A(f) \) that meet \( \partial \overline{D} \) only at 1 and closed disjoint disks \( D_j \subset \mathbb{D}, 1 \leq j \leq k \), such that the set \( \partial D_j \setminus A(f) \) is empty, each \( D_j \) contains exactly one point of \( A(f) \) and \( A(f) \) is covered by disks \( D_j \). Let \( \tilde{f} \) be the restriction of \( f \) to \( K \). We will call \((K, \tilde{f})\) the factorization of \( f \).
**Theorem 7.1.** Suppose that \( f \in \mathcal{R}_{w_0}(W,M) \) and \( f(\mathbb{D}) \) meets \( A \) only at finitely many points \( \zeta_1, \ldots, \zeta_k \). Let \( K = \bigcup_{j=1}^{k} K_j \) be a factorization of \( f \). Then

\[
[f]_\rho = I^R_{K,1}(f) = \prod_{j=1}^{k} I^R_{K_j,1}(f).
\]

**Proof.** Since in this case \( \rho_1(W,M,w_0) = \pi_2(M,W,w_0) \) and \( K \) is a homotopic retract of \( \overline{\mathbb{D}} \), the first equality follows from the real analog of Theorem \( \ref{thm:real-analog} \). \( \square \)

Now we can prove the general analog of the result of L. Rudolph in [9]. Let \( S = \delta_1(\eta_1(W,M,w_0)) \) and \( S^{-1} \) is the semigroup consisting of all \( a \in \rho_1(W,M,w_0) \) such that \( a = b^{-1} \) and \( b \in S \).

**Theorem 7.2.** The semigroup \( S \) has the following properties:

1. \( S \) is reversible;
2. \( S \cap S^{-1} = \{ e \} \);
3. any element \( a \in \rho_1(W,M,w_0) \) is expressible in the form \( bc^{-1} \), \( b,c \in S \);
4. any element \( a \in \rho_1(W,M,w_0) \) is expressible in the form \( d^{-1}f \), \( d,f \in S \).

**Proof.** (1) holds because \( \eta_1(W,M,w_0) \) is reversible. To show (2) we suppose that \( a = b^{-1} \), \( a = \delta_1([f_0]) \neq e \), and \( b = \delta_1([f_1]) \). \( f_0, f_1 \in \mathcal{A}_{w_0}(W,M) \). If the set \( A(f_0) \) is empty then \([f_0] = e \) and \([f_1] = e \). So we assume that \( \text{ind}(f_0, A_j) > 0 \) for some \( j \). Let \( f_2(\zeta) = f_1(\zeta) \). By Theorem \( \ref{thm:factorization} \) \([f_2]_\rho = [f_1]_\rho^{-1} = [f_0]_\rho \). But \( \text{ind}(f_0, A_j) > 0 \) while \( \text{ind}(f_2, A_j) < 0 \) and we came to a contradiction.

To show (3) we take \( f \in \mathcal{R}_{w_0}(W,M) \) that is smooth and transverse to \( A \) and \([f]_\rho = a \). In this case the set \( A(f) \) is finite and consists of points \( \zeta_1, \ldots, \zeta_k \) in \( \mathbb{D} \) such that \( \text{ind}_{\zeta_j}(f, A) = \pm 1 \). We assume that points \( \zeta_j \) are enumerated in such a way the this local index is 1 when \( 1 \leq j \leq n \) and -1 when \( n < j \leq k \) and change \( f \) slightly near these points so it become holomorphic when \( 1 \leq j \leq n \) and antiholomorphic when \( n < j \leq k \).

Then we form a factorization \((K,g) \) of \( f \) with arms \( K_j = \alpha_j \cup D_j \), where disks \( D_j \) are so small that \( f \) is either holomorphic or antiholomorphic on them. By Theorem \( \ref{thm:factorization} \) \([f_j]_\rho = [h_j]_\rho^{-1} \) where \( h_j \in \mathcal{A}_{w_0}(K_j, W,M) \) when \( n < j \leq k \) and by Theorem \( \ref{thm:factorization} \)

\[
[f]_\rho = \prod_{j=1}^{n} \delta_1([f_j]) \prod_{j=n+1}^{k} (\delta_1([h_j]))^{-1}.
\]

The part (4) has the same proof. \( \square \)

By Theorem \( \ref{thm:factorization} \) if \( \pi_1(M,w_0) = \pi_2(M,w_0) = 0 \) then the group \( \rho_1 \) in Theorem \( \ref{thm:factorization} \) can be replaced by \( \pi_1(W,w_0) \).

**8. Connected components of \( \mathcal{A}(W,M) \) and \( \mathcal{R}(W,M) \)**

There are natural mappings of the set \( \eta_1(W,M) \) of connected components of \( \mathcal{A}(W,M) \) into the set \( \rho_1(W,M) \) of connected components of \( \mathcal{R}(W,M) \) and \( \pi_1(W) \). We will denote these mapping also by \( \delta_2 \) and \( \epsilon_1 \) respectively. The mapping \( \epsilon_1 \) need not to be an injection especially when the group \( \pi_2(M) \) is non-trivial. For example, if \( M = \mathbb{C}P^2 \) and \( A \) is an algebraic variety in \( M \) such that \( \pi_1(W) \) is finite then \( \epsilon_1 \) is not an injection because \( \eta_1(W,M) \) is always infinite due to the invariance of index.
There is hope that $\delta_2$ is an injection at least when $M = \mathbb{C}^n$. To advance in this direction we introduce the set $\mathcal{R}_{w_0}^\pm(W, M)$ of mappings $f \in \mathcal{R}_{w_0}(W, M)$ such that the set $A(f)$ is finite and the points in $A(f)$ have non-zero local indexes. This set is open and dense in $\mathcal{R}_{w_0}(W, M)$.

**Lemma 8.1.** Let $M$ be a complex manifold and let $A$ be an analytic variety in $M$ and $W = M \setminus A$. Suppose that mappings $f_0, f_1 \in \mathcal{R}_{w_0}^\pm(W, M)$ are smooth and transverse to $A$ and can be connected by a continuous curve $f_t$, $0 \leq t \leq 1$, in $\mathcal{R}_{w_0}^\pm(W, M)$. Then there is a smooth path $g_t$, $0 \leq t \leq 1$, connecting $f_0$ and $f_1$ in $\mathcal{R}_{w_0}^\pm(W, M)$ such that:

1. the mapping $G(t, \zeta) = g_t(\zeta)$ of $[0, 1] \times \mathbb{D}$ into $M$ is transverse to $A$;
2. the set $\mathcal{A}_G$ consists of finitely many disjoint smooth curves $\{a_j(t)\}, t \in [0, 1], \text{ such that } a_j(t) = (t, \zeta_j(t))$.

**Proof.** We may assume that the mapping $F : [0, 1] \times \mathbb{D} \to M$ defined as $F(t, \zeta) = f_t(\zeta)$ is smooth. Let $A_{\text{sing}} = A^1_{\text{sing}}$ be the set of singular points of $A$. Define by induction the sets $A^k_{\text{sing}} = (A^{k-1}_{\text{sing}})^{\text{sing}}$. For some $k \leq n + 1$ the set $A^k_{\text{sing}}$ is empty and therefore the set $A^n_{\text{sing}}$ is a manifold. By the Thom Transversality Theorem we can approximate $F$ by a smooth mapping $F_k$ transverse to $A^{k-1}_{\text{sing}}$. By the definition of transversality $F_k([0, 1] \times \mathbb{D})$ never meets the set $A^{k-1}_{\text{sing}}$ if $\dim A^{k-1}_{\text{sing}} \leq n - 2$. Now we let $M_{k-1} = M \setminus A^{k-1}_{\text{sing}}$ and apply the transversality theorem to $M_{k-1}$ and $A^{k-2}_{\text{sing}}$ to find $F_{k-2}$. By induction we obtain an approximation $H$ of $F$ that never meets the set $A_{\text{sing}}$ and is transverse to $A$. Let $h_t(\zeta) = H(t, \zeta)$. Since $M$ admits a real analytic embedding into some $\mathbb{R}^N$ we can choose $H$ to be real analytic and since the set $\mathcal{R}_{w_0}^\pm(W, M)$ is open we may assume that $h_t \in \mathcal{R}_{w_0}^\pm(W, M)$ for all $t \in [0, 1]$.

The set $A_M$ is a compact set in $[0, 1] \times \mathbb{D}$ and a smooth submanifold, i.e., it is a collection $\Gamma$ of finitely many disjoint smooth curves $\{\gamma_j\}, 1 \leq j \leq m$.

Suppose that $(t_0, \zeta_0) \in \gamma_j$ and $H(t_0, \zeta_0) = w_0 \in A$. The point $w_0$ is a regular point of $A$ and there is a neighborhood $U$ of $w_0$ such that in appropriate coordinates $(z_1, \ldots, z_n)$ the set $A \cap U = \{z_1 = 0\}$. In coordinates $(z_1, \ldots, z_n)$ the mapping $H(t, \zeta) = (H_1(t, \zeta), \ldots, H_n(t, \zeta))$ and the functions $H_k$ are real analytic. Since the rank of $dH_1$ is 2 and the curve $\gamma_j = \{H_1(t, \zeta) = 0\}$, by the Implicit Function Theorem the curve $\gamma_j$ admits a real analytic parametrization $\gamma_j(s) = (t(s), \zeta_j(s))$ near $(t_0, \zeta_0)$ with $t(0) = t_0$. The mapping $h_{t_0}$ is not transverse to $A$ at $\zeta_0$ if and only if either $t(s) = t_0$ near 0 or $t(s) = t_0 + as^p + o(s^p)$, $a \neq 0$ and $p > 1$. But the former case is excluded because $h_{t_0} \in \mathcal{R}_{w_0}^\pm(W, M)$ and the set $A(h_{t_0})$ is finite. By real analyticity in the latter case there are only finitely many points where $t'(s) = 0$. Hence the set $E$ of those $t$ where $h_t$ is not transverse to $A$ is finite.

Let $t_0 \in E$. If $p$ is even and $a > 0$ then $t(s)$ has a strict local minimum at 0 and if $a < 0$ then it has a strict local maximum there. In both cases $\text{ind}_{\zeta_0}(h_{t_0}, A) = 0$ and this contradicts the assumption that $h_{t_0} \in \mathcal{R}_{w_0}^\pm(W, M)$. If $p > 1$ is odd then $t(s)$ is either strictly increasing or decreasing near 0 and the set $h_t([\zeta_0]) \cap A$ has only one point in a small neighborhood of $\zeta_0$ for $t$ sufficiently close to $\zeta_0$. Hence we can choose a real analytic parametrization $\gamma_j(s)$ such that $s \in [0, 1]$, the function $t_j(s)$ is strictly increasing and $t_j'(s) = 0$ only at finitely many points.

Finally, we take smooth diffeomorphism $\Phi(t, \zeta)$ of $[0, 1] \times \mathbb{D}$ such that $\Phi(t, \zeta) = (t, \phi(t, \zeta))$ and the functions $t(\Phi(\gamma_j(s)))$ have strictly positive derivatives. Let $s_j(t)$
be the inverse of the later function. The mapping \( G = H \circ \Phi^{-1} \) and the curves \( \alpha_j(t) = \Phi(\gamma_j(s_j(t))) \) have all required properties. \( \square \)

The proof of the lemma below follows the same line of argument as that in the proof of Assertion 2 in the proof of [12, Lemma 2.1].

**Lemma 8.2.** Let \( \zeta_k(t) \), \( 1 \leq k \leq n \), are smooth mappings of \([0, 1]\) into \( \mathbb{D} \) such that \( \zeta_i(t) \neq \zeta_j(t) \) when \( i \neq j \) and \( 0 \leq t \leq 1 \). Then there is a \( C^\infty \) mapping \( \Phi: \mathbb{D} \times [0, 1] \to \mathbb{D} \) such that \( \Phi_t(\zeta) = \Phi(\zeta, t) \) is a diffeomorphism of \( \mathbb{D} \) onto itself for each \( t \), \( \Phi_t(\zeta) = \zeta \) when \( |\zeta| = 1 \) and \( \Phi_t(\zeta_j(0)) = \zeta_j(t) \) for \( j = 1, \ldots, n \).

**Proof.** By Whitney extension theorem (see [6, Theorem 1.5.6]) we can find a complex valued \( C^\infty \)-function \( F(t, \zeta) \) on \([0, 1] \times \mathbb{C} \) such that \( F(t, \zeta_j(t)) = \partial \zeta_j(t)/\partial t \) for \( 0 \leq t \leq 1 \), \( j = 1, \ldots, n \). Replacing \( F \) with the product \( F\phi \), where \( \phi \) is a \( C^\infty \)-function with \( \phi = 1 \) on \( \cup_{j=1}^{n} \{ (t, \zeta_j(t)) : 0 \leq t \leq 1 \} \) and \( \phi = 0 \) for \( |\zeta| \geq 1 \), we can make \( F(t, \zeta) = 0 \) for \( |\zeta| \geq 1 \). Then by standard existence and uniqueness theorems for ordinary differential equations, the initial value problem \( \partial x/\partial t(t) = F(t, x(t)) \), \( x(0, \zeta_j) = \zeta_j \), \( 0 \leq t \leq 1 \), has a unique solution \( x(t, \zeta_j) \). Since \( F(t, \zeta) \) is smooth, this solution is smooth on \([0, 1] \times \mathbb{C} \).

Now define a mapping \( \Phi: \mathbb{C} \times [0, 1] \to \mathbb{C} \) by \( \Phi(\zeta, t) = x(t, \zeta_j) \). Then for each \( 0 \leq t \leq 1 \), \( \Phi \) is a diffeomorphism and since the related initial value problem has a unique solution we have \( \Phi(\zeta_j(0), t) = \zeta_j(t) \) for \( j = 1, \ldots, n \). Also note that \( \Phi(\zeta_j, t) = \zeta_j \) for all \( 0 \leq t \leq 1 \) when \( |\zeta| \geq 1 \). So, the restriction of \( \Phi \) to \( \mathbb{D} \times [0, 1] \) has desired properties. \( \square \)

Let \( \gamma: [0, 1] \to M \) be a continuous curve connecting the base point \( w_0 \) with a point \( w \) in a regular part of some component \( A_i \) and such that \( \gamma([0, 1]) \subset W \). Given a neighborhood \( U \) of \( w \) and \( \varepsilon > 0 \) consider continuous mappings \( f_{U, \varepsilon} \) of \([0, 1] - \varepsilon \cup \mathbb{D}(2 - \varepsilon, 1) \) such that \( f(t) = \gamma(t) \) when \( 0 \leq t \leq 1 - \varepsilon \) and the restriction of \( f \) to \( \mathbb{D}(2 - \varepsilon, 1) \) is an analytic disk in \( U \) transversal to \( A_i \) and whose index with respect to \( A_i \) is 1. Rephrasing the definition from [11] we call such mappings lassos \( \lambda_{\gamma} \) around \( A_i \). Clearly, there are \( U \) and \( \varepsilon_0 \) such that all mappings \( f_{V, \varepsilon} \) are equivalent in \( H_{(0, w_0)}(\mathbb{D}, W, M) \) when \( V \subset U \) and \( \varepsilon < \varepsilon_0 \).

**Lemma 8.3.** Let \( \lambda_{\gamma_0} \) and \( \lambda_{\gamma_1} \) be lassos around \( A_i \). Suppose that there is a continuous mapping \( \phi: [0, 1]^2 \to M \) such that for all \( t \) we have \( \phi(0, t) = \gamma_0(t) \), \( \phi(1, t) = \gamma_1(t) \), \( \phi(t, 0) = w_0 \), \( \phi(t, 1) \in A_i^{reg} \) and \( \phi(\varepsilon, s) \in W \) when \( s \neq 1 \). Then \( \lambda_{\gamma_0} \) and \( \lambda_{\gamma_1} \) are equivalent in \( H_{(0, w_0)}(\mathbb{D}, W, M) \).

**Proof.** For some small \( \varepsilon > 0 \) we can construct a continuous family \( g_t \) of analytic disks transversal to \( A_i \) and of index 1, centered at \( \phi(t, 1) \) and such that \( g_t(1) = \phi(t, 1 - \varepsilon) \). Then define \( K = [0, 1 - \varepsilon] \cup \mathbb{D}(2 - \varepsilon, 1) \) and \( \lambda_{\gamma_1} : K \to M \) as \( \phi(t, s) \) on \([0, 1] - \varepsilon \) and as \( g_t(\zeta - 2 + \varepsilon) \) on \( \mathbb{D}(2 - \varepsilon, 1) \). The path \( \lambda_{\gamma_1} \) is continuous in \( T(W, M) \) and by Theorem [12, Lemma 2.3] \( \lambda_{\gamma_0} |_{[0, w_0]} = \lambda_{\gamma_1} |_{[0, w_0]} \).

**Theorem 8.4.** Let \( M \) be a complex manifold and let \( A \) be an analytic set in \( M \) and let \( W = M \setminus A \). If \( f_0 \) and \( f_1 \) in \( A_{1, w_0}(W, M) \) belong to the same connected component of \( R_{w_0}^{\pm}(W, M) \) then \( [f_0]_{1, w_0} = [f_1]_{1, w_0} \).

**Proof.** We may assume that \( f_0 \) and \( f_1 \) are transverse to \( A \). Since they belong to the same connected component of \( R_{w_0}^{\pm}(W, M) \) there is a smooth mapping \( G(t, \zeta) : \)
$[0,1] \times \mathbb{F} \to M$ satisfying conclusions of Lemma 8.1. Let us apply Lemma 8.2 to the curves $\gamma_j(t)$ in Lemma 8.1 to get the diffeomorphisms $\Phi_t$ of $\mathbb{F}$.

Let us take a factorization $(K, g_0)$ of $f_0$ consisting of arms $(K_j, g_j)$, $1 \leq j \leq k$, such that $f_0(K_j)$ are lassos $\alpha_j$ so that

$$[f_0] = \prod_{j=1}^{k} [\alpha_j].$$

Let $L_j = \Phi_1(K_j)$. Then $L_j$ form a star and mappings $f_1(L_j)$ are also lassos. Moreover, the lassos $\alpha_j$ and $\beta_j$ are $h$-homotopic. The needed $h$-homotopy is achieved by the family $f_t(\Phi_t(K_j))$. Thus $[\alpha_j] = [\beta_j]$. Since

$$[f_1] = \prod_{j=1}^{k} [\beta_j]$$

the theorem is proved. \hfill \Box

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