Vacua, walls and junctions in $G_{N_F,N_C}$

Sunyoung Shin *

Institute of Basic Science, Sungkyunkwan University, Suwon 16419, Republic of Korea

Abstract

We discuss vacua, walls and three-pronged wall junctions in the Grassmann manifold $G_{N_F,N_C} = \frac{SU(N_F)}{SU(N_C) \times SU(N_F-N_C) \times U(1)}$.

*e-mail:sihnsy@skku.edu
Introduction

The moduli matrix formalism is proposed to construct 1/2 BPS walls in non-Abelian
gauge theories [1]. In the infinite coupling limit, the model becomes a massive hyper-Kähler
nonlinear sigma model on the cotangent bundle over the Grassmann manifold $T^*G_{N_F,N_C}$
where $G_{N_F,N_C} = \frac{SU(N_F)}{SU(N_C) \times SU(N_F-N_C) \times U(1)}$. There is a bundle structure for nondegenerate
masses, so the vacua and the walls are on the Kähler manifold.

The 1/4 BPS states [2, 3] and domain wall webs, which contain two or more wall
junctions are obtained in Abelian gauge theories and non-Abelian gauge theories by the
moduli matrix formalism [4, 5]. The model is 3 + 1 dimensional $\mathcal{N} = 2$ supersymmetric
$U(N_C)$ gauge theory with $N_F(\geq N_C)$ massive hypermultiplets. $W_\mu$ ($\mu = 0, 1, 2, 3$) are
gauge fields, $\Sigma_\alpha$, ($\alpha = 1, 2$) are real adjoint scalars, and $M_\alpha$ ($\alpha = 1, 2$) are traceless
diagonal mass matrices, which are parameterized as $M_1 = \text{diag}(m_1, m_2, \cdots, m_{N_F})$ and
$M_2 = \text{diag}(n_1, n_2, \cdots, n_{N_F})$. The Fayet-Iliopoulos parameters are $c^a = (0, 0, c > 0)$. We
set $c = 1$ in this paper. The fields in the hypermultiplet, which parametrize the cotangent
space vanish for the BPS equations. All the solutions of the 1/4 BPS equations in $G_{N_F,N_C}$
can be solved by $S$ and $H_0$. $S$ are invertible $N_C \times N_C$ matrices and $H_0$ are $N_C \times N_F$
matrices.

The solution to the BPS equation for domain wall webs [4] is
\[
\phi = S^{-1} H_0 e^{M_1 x^1 + M_2 x^2},
\]
where
\[
W_1 - i\Sigma_1 = -i S^{-1} \partial_1 S, \quad W_2 - i\Sigma_2 = -i S^{-1} \partial_2 S,
\]
and
\[
SS^\dagger = H_0 e^{2(M_1 x^1 + M_2 x^2)} H_0^\dagger.
\]

There is the worldvolume symmetry
\[
H_0 \rightarrow H'_0 = VH_0, \quad S \rightarrow S' = VS
\]
with $V \in GL(N_C, \mathbb{C})$. Therefore the total moduli space is the complex Grassmann manifold
\[
\mathcal{M}^\text{tot} \simeq G_{N_F,N_C} = \{H_0|H_0 \sim VH_0, V \in GL(N_C, \mathbb{C})\},
\]
which includes 1/4 BPS states, 1/2 BPS walls and discrete SUSY vacua:
\[
\mathcal{M}^\text{tot} \simeq G_{N_F,N_C} = \mathcal{M}^\text{webs}_{1/4} \cup \mathcal{M}^\text{walls}_{1/2} \cup \mathcal{M}^\text{vacua}_{1/1}.
\]
The 1/4 BPS system reduces to the 1/2 BPS system when the $x^2$ dependence and the mass $M_2$ are turned off.

In Abelian gauge theories, the scalar fields [4] are

$$\phi_{0A} \sim \frac{H_{0A}e^{m_Ax^1+n_Ax^2}}{\sqrt{\sum_{B=1}^{N_F} |H_{0B}|^2e^{2(m_Bx^1+n_Bx^2)}}}$$

(7)

The weight of the vacuum $\langle A \rangle$ is defined as

$$\left( H_0e^{M_1x^1+M_2x^2} \right)_A = e^{a_A+m_Ax^1+n_Ax^2}.$$  

(8)

The position of the wall which interpolates two vacua is determined by the condition of equal weights of the vacua. The position of the wall which connects $\langle A \rangle$ and $\langle B \rangle$ is

$$(m_A - m_B)x^1 + (n_A - n_B)x^2 + a_A - a_B = 0.$$  

(9)

Abelian three-pronged junctions divide sets of three vacua with different labels in one color component whereas non-Abelian three-pronged junctions divide sets of three vacua with different labels in two color components. Abelian junctions exist both in Abelian gauge theories and non-Abelian gauge theories while non-Abelian junctions exist only in non-Abelian gauge theories. In [5], Abelian junctions and non-Abelian junctions of the Grassmann manifold are studied by embedding $G_{N_F,N_C}$ into the complex projective space $\mathbb{C}P^{N_F,N_C-1}$ by the Plücker embedding. Therefore the wall separating $\langle \cdots A \rangle$ and $\langle \cdots B \rangle$ is on

$$(m_A - m_B)x^1 + (n_A - n_B)x^2 + a^{\langle \cdots A \rangle} - a^{\langle \cdots B \rangle} = 0,$$

(10)

which is similar to the wall positions (9) in the Abelian gauge theories.

As $SO(2N)/U(N)$ and $Sp(N)/U(N)$ are realized as quadrics of $G_{2N,N}$ in the moduli matrix formalism [6, 7], it is useful to examine wall junctions of the Grassmann manifold in terms of $N_C \times N_F$ moduli matrices.

The purpose of this paper is to discuss walls and single three-pronged wall junctions in $G_{N_F,N_C}$ by using $N_C \times N_F$ moduli matrices and diagrams which are similar to the tetrahedron for $\mathbb{C}P^3$ in [4]. We apply the pictorial representation proposed in [7] to the Grassmann manifolds and present the diagrams of $G_{4,2}, G_{5,2}, G_{5,3}, G_{6,2}, G_{6,3}$, and $G_{6,4}$. We discuss single three-pronged wall junctions in $G_{5,2}$ by reformulating the diagram for $G_{5,2}$ to make polyhedra.

**Vacua and elementary walls**

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Let $\langle A \rangle$ denote a vacuum and $\langle A \leftarrow B \rangle$ denote a wall which connects $\langle A \rangle$ and $\langle B \rangle$. The moduli matrix of elementary wall $\langle A \leftarrow B \rangle$ in $G_{N_F,N_C}$ is $H_{0(A\leftarrow B)} = H_{0(A)} e^{E_i(r)}$ where $E_i$ is a simple root generator of $SU(N_F)$. Therefore elementary walls can be identified with simple roots [8]. The simple root generators and the simple roots of $SU(N)$ are

$$E_i = e_{i,i+1}, \quad \alpha_i = \hat{e}_i - \hat{e}_{i+1}, \quad (i = 1, \ldots, N - 1).$$

The matrix $e_{i,j}$ is an $N \times N$ matrix whose $(i, j)$ component is one. The set of vectors $\{\hat{e}_i\}$ is the unit vectors $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$.

We apply the pictorial representation which is used in [7] to the Grassmann manifold. The diagram of the vacua and the elementary walls in $G_{4,2}$ is depicted in Figure 1. The vertices and the segments correspond to the vacua and the elementary walls of $G_{4,2}$. The parallelogram in the diagram presents two sets of penetrable walls, which interpolate $\{\langle 13 \rangle, \langle 23 \rangle, \langle 24 \rangle\}$ and $\{\langle 13 \rangle, \langle 14 \rangle, \langle 24 \rangle\}$. A pair of facing sides of the parallelogram are the same roots while a pair of adjacent sides of the parallelogram are orthogonal simple roots.

![Figure 1](https://example.com/figure1.png)

**Figure 1:** Vacua and elementary walls in $G_{4,2}$. The numbers indicate the subscripts of the simple roots $\alpha_i$.

The diagrams of the vacua and the elementary walls in $G_{5,2}$ and $G_{5,3}$ are depicted in Figure 2. Two diagrams are related by a $\pi$ rotation. This reflects the duality between $G_{N_F,N_C}$ and $G_{N_F,N_F-N_C}$. The configuration in Figure 1 appears in Figure 2 (a).

The diagrams of the vacua and the elementary walls in $G_{6,2}$, $G_{6,4}$ and $G_{6,3}$ are depicted in Figure 3. $G_{6,2}$ and $G_{6,4}$ are dual to each other so the diagram in Figure 3 (a) and the diagram in Figure 3 (b) are related by a $\pi$ rotation. $G_{6,3}$ is self-dual so the diagram in Figure 3 (c) is symmetric under a $\pi$ rotation. The configuration in Figure 2 (a) appears in Figure 3 (a).
Figure 2: Vacua and elementary walls in $G_{5,N}$. (a) $N = 2$ (b) $N = 3$.

Figure 3: Vacua and elementary walls in $G_{6,N}$. (a) $N = 2$ (b) $N = 4$ (c) $N = 3$. 
Three-pronged wall junctions

We study wall junctions in the moduli space $G_{N_F,N_C}$. The moduli matrices in $G_{N_F,N_C}$ can be parameterized by real parameters $a_{ij}$ and $b_{ij}$ as

$$(H_0)_{ij} := \exp(a_{ij} + ib_{ij}),$$

$$(i = 1, \cdots, N_C; j = 1, \cdots, N_F).$$

We study three-pronged junctions in the moduli space $G_{5,2}$. The moduli matrices in $G_{5,2}$ can be parameterized as

$$(H_0^{G_{5,2}})_{ij} := \exp(a_{ij} + ib_{ij}), \quad (i = 1, 2; j = 1, 2, \cdots 5),$$

with real parameters $a_{ij}$ and $b_{ij}$.

A single three-pronged junction is determined by three vacua which correspond to three vertices of a triangle that the 1/4 wall junction gets mapped onto. We choose two sets of triangles from Figure 2 (a) as shown in Figure 4 as an example. The diagram in Figure 4 (a) is an octahedron which is composed of eight triangles and the diagram in Figure 4 (b) is a pyramid which is composed of four triangles. The vertices, the edges and the triangular faces of the polyhedra correspond to the vacua, the 1/2 BPS walls and the three-pronged junctions in $G_{5,2}$.

![Figure 4: Polyhedra for $G_{5,2}$](image)

The moduli matrix for the configuration in Figure 4 (a) is the limit of (13) as $a_{i5} \to -\infty$, $(i = 1, 2)$. There are eight triangles in Figure 4 (a). \{\{12\}, \{13\}, \{14\}\}, \{\{12\}, \{23\}, \{24\}\}, \{\{13\}, \{34\}, \{23\}\}, and \{\{14\}, \{34\}, \{24\}\} are divided by Abelian junctions whereas \{\{12\}, \{14\}, \{24\}\},
\{(12), (13), (23)\}, \{(13), (34), (14)\}, and \{(23), (34), (24)\} are divided by non-Abelian junctions. Parallelogram \{(13), (14), (24), (23)\} presents two sets of penetrable walls.

The moduli matrix of the Abelian junction which divides \{(12), (13), (14)\} is

\[
H_{0(121314)} = \begin{pmatrix}
ge^{a_{11}+ib_{11}} & 0 & 0 & 0 & 0 \\
0 & e^{a_{22}+ib_{22}} & e^{a_{23}+ib_{23}} & 0 & 0 \\
e^{a_{23}+ib_{23}} & e^{a_{24}+ib_{24}} & 0 & 0 & 0
\end{pmatrix} \tag{14}
\]

This is the limit of (13) as \(a_{1i} \to -\infty, (i = 2, \cdots 5)\), and \(a_{2j} \to -\infty, (j = 1, 5)\). The solutions is

\[
\phi = S_{(121314)}^{-1} H_{0(121314)} e^{M_1 x^1 + M_2 x^2} = \begin{pmatrix}
\frac{f_1}{\sqrt{\Delta_1}} & 0 & 0 & 0 & 0 \\
0 & \frac{f_2}{\sqrt{\Delta_2}} & \frac{f_3}{\sqrt{\Delta_2}} & \frac{f_4}{\sqrt{\Delta_2}} & 0
\end{pmatrix},
\]

\[
f_1 := \exp(m_1 x^1 + n_1 x^2 + a_{11} + ib_{11}), \\
f_n := \exp(m_n x^1 + n_n x^2 + a_{2n} + ib_{2n}), \quad (n = 2, 3, 4), \tag{15}
\]

with

\[
S_{(121314)} S_{(121314)}^\dagger = H_{0(121314)} e^{2M_1 x^1 + 2M_2 x^2} H_{0(121314)}^\dagger = \text{diag}(\Delta_1, \Delta_2), \\
\Delta_1 := e^{2m_1 x^1 + 2n_1 x^2 + 2a_{11}}, \\
\Delta_2 := \sum_{n=2}^4 e^{2m_n x^1 + 2n_n x^2 + 2a_{2n}} \tag{16}
\]

The wall dividing \langle 1A \rangle and \langle 1B \rangle, \((A, B = 2, 3, 4)\) is on

\[
(m_A - m_B)x + (n_A - n_B)y + a_{2A} - a_{2B} = 0. \tag{17}
\]

Therefore the junction position is

\[
(x^1, x^2) = \left( \frac{S_1}{S_3}, \frac{S_2}{S_3} \right),
\]

\[
S_1 := (-n_3 + n_4)a_{22} + (-n_4 + n_2)a_{23} + (-n_2 + n_3)a_{24}, \\
S_2 := (m_3 - m_4)a_{22} + (m_4 - m_2)a_{23} + (m_2 - m_3)a_{24}, \\
S_3 := (n_3 - n_4)m_2 + (n_4 - n_2)m_3 + (n_2 - n_3)m_4. \tag{18}
\]

In the same manner, the moduli matrix of the junction which divides \{(12), (23), (24)\} and the moduli matrix of the junction which divides \{(13), (34), (23)\} can be determined. The wall dividing \langle 2A \rangle and \langle 2B \rangle, \((A, B = 1, 3, 4)\) is on

\[
(m_A - m_B)x + (n_A - n_B)y + a_{1A} - a_{1B} = 0, \tag{19}
\]
and the wall dividing \( \langle 3A \rangle \) and \( \langle 3B \rangle \), \( A, B = 1, 2, 4 \) is on
\[
(m_A - m_B)x + (n_A - n_B)y + a_{1A} - a_{1B} = 0. \tag{20}
\]

Three vacua \{\langle 12 \rangle, \langle 13 \rangle, \langle 23 \rangle\} are divided by a non-Abelian junction. \( SS^t \) in (3) are diagonal for Abelian junctions so we can calculate junction positions by comparing weights. However, \( SS^t \) are not diagonal for non-Abelian junctions in general though they can be diagonalized. On the other hand, as three-pronged wall junctions are solitons which divide three vacua, junction positions can be determined by finding sets of three vacua connected by single \( 1/2 \) BPS walls, which correspond to the vertices of the triangles that junctions are mapped onto.

We calculate the junction position from the neighbouring \( 1/2 \) walls instead of inverting \( S_{12323} \) to obtain the solution (1) for the wall junction. The wall junction should be on the intersection point of the following linear equations:
\[
\begin{align*}
(m_2 - m_3)x + (n_2 - n_3)y + a_{22} - a_{23} &= 0, \\
(m_1 - m_3)x + (n_1 - n_3)y + a_{11} - a_{13} &= 0, \\
(m_1 - m_2)x + (n_1 - n_2)y + a_{11} - a_{12} &= 0,
\end{align*}
\tag{21}
\]
which are the positions of the walls interpolating \{\langle 12 \rangle, \langle 13 \rangle\}, \{\langle 12 \rangle, \langle 23 \rangle\} and \{\langle 13 \rangle, \langle 23 \rangle\} respectively. The condition for the existence of the solution is
\[
a_{12} - a_{13} = a_{22} - a_{23}. \tag{22}
\]

The non-Abelian junction position is
\[
(x, y) = \left( \frac{V_1}{V_3}, \frac{V_2}{V_3} \right)
\]
\[
V_1 := (n_2 - n_3)a_{11} + (n_3 - n_1)a_{12} + (n_1 - n_2)a_{13}, \\
V_2 := (-m_2 + m_3)a_{11} + (-m_3 + m_1)a_{12} + (-m_1 + m_2)a_{13}, \\
V_3 := (-n_2 + n_3)m_1 + (-n_3 + n_1)m_2 + (-n_1 + n_2)m_3. \tag{23}
\]

In [5], the wall webs in \( G_{4,2} \) are studied by embedding \( G_{4,2} \) to \( \mathbb{C}P^5 \) by the Plücker embedding and the junction positions are obtained from the wall webs. Since the sector described by (14) is in \( G_{4,2} \) we can compare the junction positions (18) and (23) with the results of [5] with mass parameters \([m_A, n_A] = \{-\sqrt{3}, -1], [\sqrt{3}, -1], [0, 2], [0, 0]\}\). The junction position (18) with these parameters is
\[
(x^1, x^2) = \left( \frac{-2a_{22} - a_{23} + 3a_{24}}{2\sqrt{3}}, \frac{-a_{23} + a_{24}}{2} \right), \tag{24}
\]

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\[
(x^1, x^2) = \left( \frac{-2a_{22} - a_{23} + 3a_{24}}{2\sqrt{3}}, \frac{-a_{23} + a_{24}}{2} \right), \tag{24}
\]
and the junction position (23) is
\[(x^1, x^2) = \left( \frac{a_{11} - a_{12}}{2\sqrt{3}}, \frac{a_{11} + a_{12} - 2a_{13}}{6} \right). \tag{25}\]

The moduli parameters are related to the moduli parameters of [5] by
\[a_{22} - a_{23} = a^{(12)} - a^{(13)},\]
\[a_{23} - a_{24} = a^{(13)} - a^{(14)},\]
\[a_{11} - a_{13} = a^{(12)} - a^{(23)},\]
\[a_{11} - a_{12} = a^{(13)} - a^{(23)}. \tag{26}\]

The junction positions are the same as the junction positions obtained in [5].

In Figure 4 (b), two triangles are divided by Abelian junctions and the other two triangles are divided by non-Abelian junctions. Parallelogram \(\{\langle 24 \rangle, \langle 25 \rangle, \langle 35 \rangle, \langle 34 \rangle\}\) is a two sets of penetrable walls. The same analysis can be done on the pyramid.

We have shown that the full configurations of vacua, 1/2 walls and singe three-pronged junctions in \(G_{N_F,N_C}\) can be determined by building polyhedra. We can always find the \(N_C \times N_F\) moduli matrices which correspond to the vertices, the edges and the faces of the polyhedra.

Summary

We have presented diagrams for the Grassmann manifolds \(G_{N_F,N_C}\) in the pictorial representation which is proposed in [7]. We have observed that the duality between \(N_C\) and \(N_F - N_C\) is realized as a \(\pi\) rotational symmetry in the representation. We have reformulated the diagrams to make polyhedra whose vertices, edges and triangular faces correspond to vacua, 1/2 BPS walls and single three-pronged junctions.

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References

[1] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. 93, 161601 (2004) doi:10.1103/PhysRevLett.93.161601 [hep-th/0404198]; Y. Isozumi,
M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 70, 125014 (2004) doi:10.1103/PhysRevD.70.125014 [hep-th/0405194].

[2] E. R. C. Abraham and P. K. Townsend, Nucl. Phys. B 351, 313 (1991). doi:10.1016/0550-3213(91)90093-D; G. W. Gibbons and P. K. Townsend, Phys. Rev. Lett. 83, 1727 (1999) doi:10.1103/PhysRevLett.83.1727 [hep-th/9905196]; S. M. Carroll, S. Hellerman and M. Trodden, Phys. Rev. D 61, 065001 (2000) doi:10.1103/PhysRevD.61.065001 [hep-th/9905217].

[3] H. Oda, K. Ito, M. Naganuma and N. Sakai, Phys. Lett. B 471, 140 (1999) doi:10.1016/S0370-2693(99)01355-6 [hep-th/9910095]; M. A. Shifman and T. ter Veldhuis, Phys. Rev. D 62, 065004 (2000) doi:10.1103/PhysRevD.62.065004 [hep-th/9912162]; A. Gorsky and M. A. Shifman, Phys. Rev. D 61, 085001 (2000) doi:10.1103/PhysRevD.61.085001 [hep-th/9909015]; P. K. Townsend, Class. Quant. Grav. 17, 1267 (2000) doi:10.1088/0264-9381/17/5/336 [hep-th/9911154]. J. P. Gauntlett, G. W. Gibbons, C. M. Hull and P. K. Townsend, Commun. Math. Phys. 216, 431 (2001) doi:10.1007/s002200000341 [hep-th/0001024]; S. Nam and K. Olsen, JHEP 0008, 001 (2000) doi:10.1088/1126-6708/2000/08/001 [hep-th/0002176]; J. P. Gauntlett, G. W. Gibbons and P. K. Townsend, Phys. Lett. B 483, 240 (2000) doi:10.1016/S0370-2693(00)00582-7 [hep-th/0004136]; J. P. Gauntlett, D. Tong and P. K. Townsend, Phys. Rev. D 63, 085001 (2001) doi:10.1103/PhysRevD.63.085001 [hep-th/0007124]; K. Kakimoto and N. Sakai, Phys. Rev. D 68, 065005 (2003) doi:10.1103/PhysRevD.68.065005 [hep-th/0306077].

[4] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 72, 085004 (2005) doi:10.1103/PhysRevD.72.085004 [hep-th/0506135].

[5] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Lett. B 632, 384 (2006) doi:10.1016/j.physletb.2005.10.017 [hep-th/0508241].

[6] M. Arai and S. Shin, Phys. Rev. D 83, 125003 (2011) doi:10.1103/PhysRevD.83.125003 [arXiv:1103.1490 [hep-th]]; M. Eto, T. Fujimori, S. B. Gudnason, Y. Jiang, K. Konishi, M. Nitta and K. Ohashi, JHEP 1112, 017 (2011) doi:10.1007/JHEP12(2011)017 [arXiv:1108.6124 [hep-th]].

[7] B. H. Lee, C. Park and S. Shin, Phys. Rev. D 96, no. 10, 105017 (2017) doi:10.1103/PhysRevD.96.105017 [arXiv:1708.05243 [hep-th]]; M. Arai, A. Golubtsova B. C, C. Park and S. Shin, arXiv:1803.09275 [hep-th].

[8] N. Sakai and D. Tong, JHEP 0503, 019 (2005) doi:10.1088/1126-6708/2005/03/019 [hep-th/0501207].