ARITHMETIC AND GEOMETRY OF THE HECKE GROUPS

CHENG LIEN LANG AND MONG LUNG LANG

ABSTRACT. We study the arithmetic and geometry properties of the Hecke group $G_q$. In particular, we prove that $G_q$ has a subgroup $X$ of index $d$, genus $g$ with $v_\infty$ cusps, and $\tau_q$ (resp. $v_r$) conjugacy classes of elements that are conjugates of $S$ (resp. $R^{q/r}$) if and only if (i) $2g - 2 + \frac{\tau_2}{2} + \sum_{r=1}^{b} v_r (1 - 1/r) + v_\infty = d(1/2 - 1/q)$, and (ii) $v_m = 4g - 4 + \tau_2 + 2v_\infty + \sum_{r=1}^{b} v_r (2 - q/r) \geq 0$ is a multiple of $q - 2$, (iii) $m \geq 0$. In the case $q$ is odd, (ii) is a consequence of (i).

1. Introduction

1.1. The (inhomogeneous) Hecke group $G_q$ is defined to be the maximal discrete subgroup of $PSL(2,\mathbb{R})$ generated by $S$ and $T$, where $\lambda_q = 2\cos(\pi/q)$,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}. \quad (1.1)$$

Let $R = ST^{-1}$. Then $R$ has order $q$ and $\langle S, R \rangle$ is a set of independent generators of $G_q$. Equivalently, $G_q$ is a free product of $\langle S \rangle$ and $\langle R \rangle$. The main purpose of this article is to study the geometric and arithmetic properties of subgroups of finite index of $G_q$.

1.2. The set of cusps of $G_q$ is $\mathbb{Q}[\lambda_q] \cup \{\infty\}$ if and only if $q = 3, 5$. We will give an inductive procedure (induction on the depth of $q$-gons) that enables us to generate the set of cusps of $G_q$ (Lemma 3.2). As the index of $G_q$ in $PSL(2,\mathbb{Z}[\lambda_q])$ is infinite, it is important to characterise members of $G_q$. A simple algorithm that determines whether a matrix of $PSL(2,\mathbb{Z}[\lambda_q])$ belongs to $G_q$ can be found in Proposition 3.7. The algorithm can be implemented in a computer.

1.3. A set of generators $\{x_i\}$ of $X$ is called a set of independent generators if $X$ is a free product of the cyclic groups $\langle x_i \rangle$. $G_q$ is a free product of $\langle S \rangle$ and $\langle R \rangle$. By Kurosh’s Theorem, every subgroup $X$ of finite index of $G_q$ has a set of independent generators. Proposition 4.4 and Theorem 5.2 demonstrate how arithmetic and geometric properties can be combined to give an inductive procedure for finding a special polygon (fundamental domain) $M_X$ and a set of independent generators $I_X$ for $X$ (the case $q$ is a prime has been done in [LLT1]). In particular, this is applied to the principal congruence subgroup of level $2$, the commutator subgroup $G_q'$ and subgroups of index $2$ (subsection 5.4).

1.4. As a special case of the Hurwitz-Nielsen realisation problem, Millington [Mi] showed that as long as $d = 3\tau_2 + 4v_3 + 12q + 6q - 12$, then the modular group $G_3$ possesses a subgroup $X$ of index $d$, such that $X \setminus \mathbb{H}$ has $\tau_2$ (resp. $v_3$) elliptic points of order 2 (resp. 3), $t$ cusps, and genus $g$. We are able to generalise this result to $G_q$ by studying the Hecke-Farey symbols (see Section 6). As mentioned in [K1], the problem of recognising $X$ as a normal subgroup of certain geometric invariants has been left aside in the literature. Our study of this topic starts with some elementary observation of the permutation representations of $S$ and $R$ on the set of cosets $G_q/X$ which we will elaborate more in subsection 1.5.

2010 Mathematics Subject Classification. 11F06, 20H10.

Key words and phrases. Hecke groups, congruence subgroups, Kurosh’s Theorem, Hurwitz-Nielsen realisation problem, maps and map subgroups.
1.5. We prove that the action of $S$ and $R$ on $G_q/X$ is isomorphic to their action on $M_X$ (see Lemma 7.1) and that the permutation representations of $S$ and $R$ on $M_X$ can be obtained by a simple reading of the special polygon $M_X$ ((7.4) and (7.9)). These two representations $f(S)$ and $f(R)$ carry some important information about $X$. In particular,

(i) $f(S)$ and $f(R)$ can be used to determine whether $X$ is normal in $G_q$ and its normaliser in $PSL(2, \mathbb{R})$ (Proposition 8.1 and Discussion 8.2),

(ii) in the case $q = 3$, $f(S)$ and $f(R)$ can be used to determine whether $X$ is congruence (see Section 11 and [H]),

(iii) $f(S)$ and $f(R)$ can be used to study the geometric invariants of $X$ (Section 9) and Dessins d’enfants (see pp.12 of [HR]). Propositions 9.2-9.4 study the possible realisation of a group $Y$ as a normal subgroup of $G_q$.

1.6. It is well known that there is a correspondence between the set of maps and the set of subgroups of finite index of $G_q$ and that the maps are uniquely determined by their map subgroups (see [JS], [CS], [IS]). Let $X$ be a subgroup of $G_q$ of finite index. We give a detailed construction of the map $M(X)$ whose map subgroups are conjugates of $X$. Both $M(X)$ and $X$ are explicitly given. Aut $M(X)$ can be determined as well (see Section 10).

2. Tessellation of the upper half plane

Let $D^*$ denote the $(2, q, \infty)$ triangle with vertices $i$, $e^{\pi i/q}$ and $\infty$. $D^*$ is a fundamental domain of the Coxeter group $G_q^*$ generated by reflections along the sides of $D^*$. Hecke group $G_q$ is the subgroup of index 2 consists of all the orientation preserving isometries.

Let $\mathbb{H}$ be the union of the upper half plane and the set $\{g(\infty) : g \in G_q^*\}$. The $G_q^*$ translates of $D^*$ form a tessellation $\mathcal{T}^*$ of $\mathbb{H}$ (endowed with the hyperbolic metric) by $(2, q, \infty)$ triangles. The $G_q^*$ translates of $i$, $e^{\pi i/q}$ and $\infty$ are called even vertices, odd vertices and cusps (free vertices) of $\mathcal{T}^*$ respectively. The $G_q^*$ translates of the hyperbolic line joining $i$ to $\infty$ (resp. $e^{\pi i/q}$ to $\infty$) are called even edges (resp. odd edges) of $\mathcal{T}^*$. The $G_q^*$ translates of the hyperbolic line joining $i$ to $e^{\pi i/q}$ are called f-edges of $\mathcal{T}^*$. The hyperbolic line $(0, \infty)$ consists of two even edges. The $G_q^*$ translates of $(0, \infty)$ are called the even lines of $\mathcal{T}^*$. The hyperbolic line joining $x$ and $y$ is denoted by $(x, y)$.

The set of even lines give a tessellation of $\mathbb{H}$ into ideal $q$-gons, that is, hyperbolic $q$-gons with $q$ cusps. Note that their vertex angle is 0. Each $q$-gon contains a unique odd vertex.

The $f$-edges form a $q$-regular tree, where the odd vertices are considered as the set of vertices of this tree. We introduce a vertex of valence two to this $q$-regular tree at $i$, denoted by $v_0$. Each $q$-gon $P$ contains a unique vertex $v_P$ of this tree. The depth of $P$ denoted by $d(P)$ is defined to the the distance between $v_0$ and $v_P$ (the distance between adjacent vertices is 1).

3. Cusps and reduced forms

The main purpose of this section is to determine whether a matrix of $PSL(2, \mathbb{Z}[\lambda_q])$ belongs to $G_q$ (Proposition 3.7). Lemma 3.2 gives the set of cusps of $G_q$.

3.1. Reduced forms of cusps of $G_q$. The set of cusps of $G_q$ is a subset of $\mathbb{Q}(\lambda_q) \cup \{\infty\}$. Let $x$ be a cusp. We say $x = a/b$ is in reduced form if

(i) there exists $c/d$ such that $\left( \begin{array}{cc} a & c \\ b & d \end{array} \right) \in G_q$, and (ii) $b \geq 0$.

Let $g$ be given as in (i) of the above. Since $w = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ normalises $G_q$, $-wgw^{-1} \in G_q$. Hence $a/b$ is in reduced form if and only if $-a/b$ is in reduced form. In the case $a, b \geq 0$, it is also easy to see that $a/b$ is in reduced form if and only if $-b/a$ is in reduced form (study $S^{-1}g$) if and only if $b/a$ is in reduced form (study $w(Sg)w^{-1}$).
3.2. Construction of reduced forms. We give some basics about reduced forms.

Lemma 3.1. Suppose that \( w \neq \infty \). The reduced form of \( w \) is unique. The reduced form of \( \infty \) is either \(-1/0\) or \(1/0\).

Proof. Let \( a/b \) and \( a'/b' \) be the reduced forms of \( w \). It is clear that \( ab' = a'b \). Following our definition of reduced form, \( \Gamma_q \) contains the following two elements, \( \left( \begin{array}{cc} a & x \\ b & y \end{array} \right) \) and \( \left( \begin{array}{cc} a' & u \\ b' & v \end{array} \right) \).

An easy calculation gives \( \left( \begin{array}{cc} a & x \\ b & y \end{array} \right)^{-1} \left( \begin{array}{cc} a' & u \\ b' & v \end{array} \right) = \left( \begin{array}{cc} a'y - b'u \\ b'y - av \end{array} \right) \) \( \in \Gamma_q \). This element fixes \( \infty \). Since the stabilizer of \( \infty \) in \( \Gamma_q \) is generated by \( T \) (see (1.1)), we have the following.

\[
\left( \begin{array}{cc} a & x \\ b & y \end{array} \right)^{-1} \left( \begin{array}{cc} a' & u \\ b' & v \end{array} \right) = \left( \begin{array}{cc} 1 & m\lambda_q \\ 0 & 1 \end{array} \right).
\] (3.1)

It follows from (3.1) that \( a = a' \) and that \( b = b' \). This completes the proof of the lemma. \( \square \)

Lemma 3.2. Let \( P \) be an ideal \( q \)-gon with cusps \( \{c_1, c_2, \cdots, c_q\} \) arranged in increasing order (if \( \infty \) is a cusp, then \( c_1 = -\infty = -1/0 \) if \( P \) lies in the left half plane and \( c_q = \infty = 1/0 \) if \( P \) lies in the right half plane). Let \( a_i/b_i \) be the reduced form of \( c_i \) and \( \frac{a_0}{b_0} = -\frac{\pi}{q} \). Then

\[
a_i = \lambda_q a_{i-1} - a_{i-2} \quad \text{and} \quad b_i = \lambda_q b_{i-1} - b_{i-2} \quad \text{for} \quad 2 \leq i \leq q - 1.
\] (3.2)

Proof. Without loss of generality, we assume that \( P \) lies in the right half plane. Let \( A = T^{-1}S = \left( \begin{array}{cc} \lambda_q & 1 \\ 0 & 1 \end{array} \right) \in \Gamma_q \) and \( w_1 = \left( \begin{array}{cc} a_{i-1} & a_{i-2} \\ b_{i-1} & b_{i-2} \end{array} \right) \), \( 2 \leq i \leq q \). Note that \( \lambda e^{(q-1)\pi i/q} = e^{(q-1)\pi i/q} \) and that \( A \) is a counter-clockwise rotation about \( e^{(q-1)\pi i/q} \) of angle \( 2\pi/q \).

(A) Suppose that the depth of \( P \) is 1. Then \( (c_1, c_q) \) is \((0, \infty)\) and \( e^{\pi i/q} \) is the odd vertex of \( P \). We apply mathematical induction as follows.

(i) It is clear that \( w_2 = S^{-1} \in \Gamma_q \), \( w_2Aw_2^{-1} \) fixes \( e^{\pi i/q} \) and that \( w_2 = w_2A \in \Gamma_q \).

(ii) Suppose that \( w_{i-1} \in \Gamma_q \), \( w_{i-1}Aw_{i-1}^{-1} \) fixes \( e^{\pi i/q} \), and that \( w_i = w_{i-1}A \in \Gamma_q \).

By (ii), \( w_iAw_i^{-1} = w_{i-1}Aw_{i-1}^{-1} \) fixes \( e^{\pi i/q} \) and \( w_iAw_i^{-1} \) is a counter-clockwise rotation about \( e^{\pi i/q} \) of angle \( 2\pi/q \). Hence \( w_iAw_i^{-1} \) sends \( (c_{i-1}, c_{i+1}) = (a_{i-1}/b_{i-1}, a_{i-2}/b_{i-2}) \) to \( \left( \begin{array}{cc} a_{i-1} & a_{i-2} \\ b_{i-1} & b_{i-2} \end{array} \right) \). By the first column of the following matrix

\[
w_iAw_i^{-1} = \left( \begin{array}{cc} a_{i-1} & a_{i-2} \\ b_{i-1} & b_{i-2} \end{array} \right) A = \left( \begin{array}{cc} \lambda_q a_{i-1} - a_{i-2} \\ \lambda_q b_{i-1} - b_{i-2} \end{array} \right) \in \Gamma_q.
\] (3.3)

To be more precise, \((\lambda_q a_{i-1} - a_{i-2})/(\lambda_q b_{i-1} - b_{i-2}) = a_i/b_i = c_i \). Since \( c_i > c_{i-1} \), det \( w_iA > 0 \), and \( a_k, b_k \geq 0 \) for all \( k \), one has \( \lambda_q b_{i-1} - b_{i-2} > 0 \). Hence \((\lambda_q a_{i-1} - a_{i-2})/(\lambda_q b_{i-1} - b_{i-2})\) in reduced form (subsection 3.1). Since the reduced form of \( c_i \) is unique, one has \( a_{i+1} = \lambda_q a_{i-1} - a_{i-2} \), \( b_{i+1} = \lambda_q b_{i-1} - b_{i-2} \) and \( w_{i+1} = w_iA \in \Gamma_q \). By induction, we conclude that if the depth of \( P \) is one, then \( a_i = \lambda_q a_{i-1} - a_{i-2} \) and \( b_i = \lambda_q b_{i-1} - b_{i-2} \). This completes the proof of the lemma. \( \square \)

Lemma 3.2 tells us how to generate the reduced forms of all positive cusps starting with \((0/1, 1/0)\) and all negative cusps starting with \((-1/0, 0/1)\). Namely,

\[
a_i/b_i = \frac{\lambda_q a_{i-1} - a_{i-2}}{\lambda_q b_{i-1} - b_{i-2}}, \quad i = 2, 3, \cdots, q - 1.
\] (3.4)

Equation (3.4) allows us to write the \( c_i \)’s \((1 \leq i \leq q)\) in terms of \( c_1 \) and \( c_q \) which generalises the construction of the Farey sequence. In the case \( q = 6 \), \( \lambda_6 = \sqrt{3} \), the reduced forms of
the $c_i$'s ($1 \leq i \leq 6$) in terms of $c_1 = a_1/b_1$ and $c_6 = a_6/b_6$ are given by
\[
\frac{a_1}{b_1}, \frac{\sqrt{3}a_1 + a_6}{\sqrt{3}b_1 + b_6}, \frac{2a_1 + \sqrt{3}a_6}{2b_1 + \sqrt{3}b_6}, \frac{\sqrt{3}a_1 + 2a_6}{\sqrt{3}b_1 + 2b_6}, \frac{a_1 + \sqrt{3}a_6}{b_1 + \sqrt{3}b_6}, \frac{a_6}{b_6}. \tag{3.5}
\]

The continuous solution of the equation $b_i = \lambda_q b_{i-1} + b_{i-2}$ ($b_i \geq 0$, $2 \leq i \leq q - 1$) is
\[
f(x) = \frac{b_1 + b_q \cos(\pi/q)}{\sin(\pi/q)} \sin(\pi x/q) - b_q \cos(\pi x/q) = A \sin(\pi x/q + w), \quad f(i) = b_i, \tag{3.6}
\]
for some $A$ and $w$. Note that $f(x)$ is a periodic function of period $2q$, $f(0) = -b_q < 0$, and $f(1) = b_1 > 0$. Further, $f(x)$ is concave down in the interval $[0, q]$. As a consequence,
\[
b_i \geq b_1 \text{ and } b_i \geq b_q \text{ for all } i. \tag{3.7}
\]

**Lemma 3.3.** Let $a/b$ be in reduced form. Suppose that $ab \neq 0$. Then $|a| \geq 1$ and $b \geq 1$.

**Proof.** Since $a/b$ is in reduced form, $a/b = x_i$ is a cusp of some $q$-gon $P$. Let $\{x_1, x_2, \cdots, x_q\}$ be the set of cusps of $P$. We prove that $b \geq 1$ by induction on the depth of $P$. In the case the depth of $P$ is $1$, the set of cusps of $P$ is either $\{0/1, 1/\lambda_q, \cdots, 0/1\}$ if $P$ lies in the right half plane or $\{-1/0, \cdots, -1/\lambda_q, 0/1\}$ if $P$ lies in the left half plane. By (3.7), $b \geq 1$. Suppose that our assertion holds when the depth of $P$ is $n$. We now consider the case $d(P) = n + 1$ and that $a/b$ is a cusp of $P$. In the case that $a/b$ is either $x_1$ or $x_q$, $a/b$ is also a cusp of a $q$-gon of depth $n$. By inductive hypothesis, $b \geq 1$. Hence we shall assume that $a/b = x_i$, where $2 \leq i \leq q - 1$. By (3.7), $b$ is larger than the denominators of the reduced forms of $x_1$ and $x_q$. As $x_1$ and $x_q$ are cusps of a $q$-gon of depth $n$, their denominators are at least $1$ by inductive hypothesis. Hence $b \geq 1$. Since $a/b$ is in reduced form, $b/|a|$ is also in reduced form (see subsection 3.1). Repeat the above argument, one has $|a| \geq 1$. \(\square\)

### 3.3. Pseudo Euclidean algorithm.
Let $x, y \in \mathbb{Z}[\lambda_q]$. Suppose that $y \neq 0$. There exists a unique integer $m \in \mathbb{Z}$ such that $x = y(m\lambda_q) + r$, $-|y\lambda_q|/2 < r \leq |y\lambda_q|/2$. We call such an algorithm **pseudo Euclidean (PEA)**. Let $a, b \in \mathbb{Z}[\lambda_q]$, where $ab \neq 0$. Apply the pseudo Euclidean algorithm repeatedly,
\[
a = b(m_0\lambda_q) + r_1, \quad b = r_1(m_1\lambda_q) + r_2, \quad \vdots \quad r_{k-1} = r_k(m_k\lambda_q) + r_{k+1}.
\]

Let $r_0 = b$. If the (PEA) terminates, that is $r_n \neq 0$, $r_{n+1} = 0$ for some $n$, we define
\[
(a, b)_q = |r_n|. \tag{3.9}
\]

If the (PEA) does not terminate, we define $(a, b)_q = 0$. Define further that $(a, 0)_q = (0, a)_q = |a|$. One sees easily that (i) $(a, b)_q = (b, a)_q$, (ii) $(a, b)_q = (-a, -b)_q = (-a, b)_q = (a, -b)_q$.

**Lemma 3.4.** Let $a/b$ be in reduced form. If $b = 1$, then $a = m\lambda_q$ for some $m \in \mathbb{Z}$.

**Proof.** Suppose that $a \neq 0$. Since $a/b$ is in reduced form, the (PEA) (see (3.8)) implies that $r_1/b$ is in reduced form and that $|r_1| < b\lambda_q/2 < b$. Since $b = 1$, one has $|r_1| < 1$. By Lemma 3.3, $r_1 = 0$. Hence $a = m\lambda_q$ for some $m \in \mathbb{Z}$. This completes the proof of the lemma. \(\square\)

**Remark.** The converse of Lemma 3.4 is not true as $G_4$ and $G_6$ possess infinitely many reduced forms $m\lambda_q/b$, where $m, b \in \mathbb{N}$, $b > 1$.

**Lemma 3.5.** Let $c/d$ and $a/b$ be the reduced forms of $x$ and $y$ respectively, where $x < y$. Then $g = \left( \begin{array}{cc} a & c \\ b & d \end{array} \right) \in G_q$ if and only if $(x, y)$ is an even line if and only if $ad - bc = 1$.

**Proof.** (i) Suppose that $g \in G_q$. Then $g(0, \infty)$ is an even line. Equivalently, $(x, y)$ is an even line. (ii) Suppose that $(x, y)$ is an even line. Then $A(0, \infty) = (x, y)$ for some $A \in G_q$. An easy study of Lemma 3.1 and the matrix form of $A(0, \infty) = (x, y)$ shows that $ad - bc = 1$. (iii) Suppose that $ad - bc = 1$. Since $a/b$ and $c/d$ are in reduced forms, $G_q$ contains elements
of the following forms, \((a \ b \ c \ d)\) and \((c \ d \ a \ b)\). An easy calculation shows that \((a \ b \ c \ d)^{-1}\) is the mirror image of \(\Phi\). By Lemma 3.4, one has \(cy - dx = m\lambda_q\) for some \(m \in \mathbb{Z}\). It follows that \((a \ b \ c \ d)^{-1} = (\begin{smallmatrix}a & c \\ b & d \end{smallmatrix})^{-1} (\begin{smallmatrix}1 & -m\lambda_q \\ 0 & 1 \end{smallmatrix}) \in G_q\). Hence \(g \in G_q\). □

**Lemma 3.6.** Suppose that \((a, b)_q = r \neq 0\). Then there exists some \(g \in G_q\) such that \(g(a/b) = \left(\begin{smallmatrix}a' \\ b' \end{smallmatrix}\right)\). In particular, \((a/r, b/r)_q = 1\) and \(g(a/r, b/r) = 1\) is in reduced form.

**Proof.** The lemma follows from (3.8), (3.9) and the observation that the matrix form of the equation \(x = y(m\lambda_q) + z\) is \(\begin{pmatrix}1 & -m\lambda_q \\ 0 & 1 \end{pmatrix}\). By Lemma 3.3, \(|a| \geq 1\) and \(|b| \geq 1\) whenever \(r_n|r_{n+1} \neq 0\). Since \(a\) and \(b\) are finite and \(\lambda_q/2 < 1\), an easy observation of (3.8) implies that there exists some \(m\) such that \(|k| < 1\) whenever \(k \geq m\). Hence there exists a \(d\) such that \(r_{d+1} = 0\) and that \(r_d \neq 0\). By Lemma 3.1, \(r_d = \pm 1\). Equivalently, \((a, b)_q = 1\). Since \(A \in G_q\), \(A^{-1}S \in G_q\). Since the transpose of \(S\) and \(T\) are members of \(G_q\), the transpose of \(A^{-1}S\) is also an element of \(G_q\), the first column of the transpose of \(A^{-1}S\) is \(\left(\begin{smallmatrix}a \\ b \end{smallmatrix}\right)\). This implies that \(c/|d|\) is in reduced form. Similar to the above, one can show that \((c, d)_q = 1\).

Conversely, suppose that \((a, b)_q = (c, d)_q = 1\). Replace \(A\) by \(-A\) if necessary, we may assume that \(d \geq 0\). Replace \(A\) by \(wAu^{-1}\) if necessary (see subsection 3.1 for \(w\)), we may assume that \(b \geq 0\) and that \(d = 0\). By Lemma 3.6, both \(a/b\) and \(c/d\) are in reduced forms. By Lemma 3.5, we have \(A \in G_q\). This completes the proof of the lemma. □

**Example 3.8.** Let \(q = 5\) and \(\lambda = \lambda_5\). Note that \(\lambda^2 = \lambda + 1\). \(\left(\begin{smallmatrix}4\lambda^2 - 1 \\ \lambda + 1 \end{smallmatrix}\right)\) is not an element of \(G_q\) as \((4\lambda - 1, 3)_5 = \lambda - 1 < 1\).

### 4. Hecke-Farey Symbols and Special Polygons

#### 4.1. \(r\)-clusters. Let \(\Phi\) be the hyperbolic triangle with vertices 0, \(e^{\pi i/q}\) and \(\infty\). \(\Phi\) is a fundamental domain of \(G_q\). The \(G_q\) translates of \(\Phi\) are called special triangles. For each divisor \(r\) of \(q\) (1 \(\leq r < q\)), set \(\Phi_r = \Phi \cup R\Phi \cup R^2\Phi \cup \cdots \cup R^{r-1}\Phi\). \(\Phi_r\) is a union of \(r\) copies of special triangles. These \(r\) special triangles meet at the odd vertex \(e^{\pi i/q}\). The \(G_q\) translates of \(\Phi_r\) are called the \(r\)-clusters (a 1-cluster is a special triangle). Let \(\Delta_r\) be an \(r\)-cluster. It is clear that (i) \(\Delta_r\) has \((r + 1)\) cusps and one odd vertex \(v\), (ii) the boundary of \(\Delta_r\) has \(r\) even lines and two odd edges, (iii) the two odd edges of \(\Delta_r\) meet each other at \(y\) with vertex angle \(2\pi r/q\). The cusps are called the free vertices of \(\Delta_r\).

Let \(\{v_1 = 0/v_2, \cdots, v_{q-1}, v_q = 1/0\}\) be the set of cusps of the depth 1 \(q\)-gon in the right half plane where the \(v_i\)s are arranged in increasing order. Following the definition of \(\Phi_r\), the set of free vertices (cusps) of \(\Phi_r\) is

\[
\Phi_r = \{0 = v_1, v_2, \cdots, v_r, v_q = \infty\}.
\]  

The odd vertex of \(\Phi_r\) is \(e^{\pi i/q}\). The odd edges are \((e^{\pi i/q}, v_r)\) and \((e^{\pi i/q}, v_q)\), where \(\Phi_r\) is the mirror image of \(\Phi_r\) (with respect to the \(y\)-axis). The set of free vertices of \(\Psi_r\) is

\[
\Psi_r = \{-\infty = -v_q, -v_r, \cdots, -v_2, -v_1 = 0\}.
\]

The odd vertex of \(\Psi_r\) is \(y = e^{(q-1)\pi i/q}\) and the odd edges of \(\Psi_r\) are \((y, -v_r)\) and \((y, -v_q)\). Let \(A\) be given as in Lemma 3.2. It is clear that \(\Psi_r = A^{4-r}S\Phi_r\).

**Example 4.1.** Let \(q = 6, \lambda = \lambda_6 = \sqrt{3}\) See Figure 1a for \(\Phi_2\) and Figure 1b for \(\Psi_2\).
4.2. Special polygons. A convex hyperbolic polygon $P$ of $\mathbb{H}$ is a union of some $q$-gons and a finite number of $r_i$-clusters ($r_i|q$, $1 \leq r_i < q$). The $q$-gons and the $r_i$-clusters of $P$ are called the tiles. The tiles intersect each other (if any) at either free vertices (cusps) or even lines. A special polygon $M_X = (P, I_X)$ of $\mathbb{H}$ is a convex hyperbolic polygon $P$ together with a set of side pairings $I_X$ satisfying the rules below.

(S1) An odd edge $e$ is always paired with an odd edge $f$ (in the same $r$-cluster) and makes an internal angle $2\pi r/q$ with $f$. The vertex where $e$ and $f$ meet is an odd vertex of $P$. Both $e$ and $f$ are considered as sides of $P$, and are called its odd sides.

(S2) Let $e$ and $f$ be two even edges in the boundary of $P$ forming an even line. Then either (i) $e$ is paired with $f$, both $e$ and $f$ are considered as sides of $P$, and are called its even sides, the point where $e$ and $f$ meet is an even vertex of $P$, or (ii) $e$ and $f$ form a free side of $P$, and this free side is paired with another free side of $P$.

(S3) $0$ and $\infty$ are vertices of $P$.

Let $M_X = (P, I_X)$ be a special polygon. The cusps in $P$ are called the free vertices of $M_X$.

4.3. Hecke-Farey sequences and symbols. A Hecke-Farey sequence is a finite sequence of cyclically arranged numbers in increasing order $\{-\infty, x_0, x_1, \ldots, x_n, \infty\}$ such that

(a) $x_i \in \mathbb{Q}[\lambda_q] = \mathbb{Q}(\lambda_q)$, $x_i = 0$ for some $i$, $0 \leq i \leq n$,

(b) $x_i = a_i/b_i$ is in reduced form for every $i$ $(x_{-1} = -\infty = -1/0$ and $x_{n+1} = \infty = 1/0$),

(c) if $a_{i+1}b_i - a_i b_{i+1} \neq 1$, then there exists an element $g \in G_q$ and an $r$-cluster $\Phi_r$ $(r|q$, $1 < r < q)$ such that $g(v_\nu) = x_i$, $g(v_\nu') = x_{i+1}$ (see (4.1a) for $v_\nu$ and $v_\nu'$),

(d) $\{x_i, x_{i+1}\}$ is called an interval. Intervals described as in (c) are called $r$-intervals.

$\{x_i, x_{i+1}\}$ is called an ordinary interval if $a_{i+1}b_i - a_i b_{i+1} = 1$. By Lemma 3.5, if $\{x_i, x_{i+1}\}$ is an ordinary interval, then the hyperbolic line $(x_i, x_{i+1})$ is an even line.

One can show easily that that the $g$'s and $r$'s in (c) of the above are unique (the only element of $G_q$ that fixes two or more points of $\mathbb{H}$ is the identity). A Hecke-Farey symbol is a Hecke-Farey sequence together with an additional side pairing on each consecutive pair of $x_i$'s. To avoid triviality, we insist that a Hecke-Farey symbol must have at least two distinct side pairings (see Discussion 4.5). In the case $a_{i+1}b_i - a_i b_{i+1} \neq 1$, the additional side pairing of the $r$-interval $\{x_i, x_{i+1}\}$ is denoted by

$$x_{i \cdots (g)} \quad x_{i+1},$$

where $g$ and $r$ are given as in (c) of the above. In the case $a_{i+1}b_i - a_i b_{i+1} = 1$, the additional side pairing of the interval $\{x_i, x_{i+1}\}$ can be either one of the following three types:

$$x_i \quad x_{i+1}, \quad x_{i+1}, \quad x_{i} \quad x_{i+1},$$

where $a$ is a natural number. Each natural number $a$ occurs exactly twice or not at all. The actual values of the $a$'s are unimportant, it is the pairing induced on the consecutive pairs that matters. We shall now give a detailed description of these side pairings.
(i) Let \( \{x_i, x_{i+1}\} \) be an \( r \)-interval and let \( g, \Phi \) be given as in (c) of the above. By (c), one has \( g(v_r) = x_i \) and \( g(v_q) = x_{i+1} \). Set \( g(e^{\pi i/2}) = y \).

\[
x_i \quad x_{i+1} \\
\uparrow_{v_r(g)}
\]

is the side pairing that fixes \( y \) and sends the odd edge \((y, x_i)\) to \((y, x_i)\). Since \( R \) is a counterclockwise rotation about \( e^{\pi i/2} \) of angle \( 2\pi/q \) and \((y, x_i)\) and \((y, x_{i+1})\) form an internal angle \( 2\pi/q \), this side pairing must be \( gR^q g^{-1} \in \mathbb{G}_q \) (the only element of \( \mathbb{G}_q \) that fixes two or more points of \( \mathbb{H} \) is the identity).

(ii) Let \( \{x_i, x_{i+1}\} \) be an ordinary interval. By (d) of the above, \((x_i, x_{i+1})\) is an even line. Hence there exists a unique \( A \in \mathbb{G}_q \) such that \( A(0) = x_i \) and \( A(\infty) = x_{i+1} \). Set \( A(e^{\pi i/2}) = y \). \((y, x_{i+1})\) and \((y, x_i)\) are odd edges.

\[
x_i \quad x_{i+1} \quad x_i
\]

is the side pairing that fixes \( y \) and sends the odd edge \((y, x_{i+1})\) to \((y, x_i)\). It is clear that this side pairing must be

\[
h = ARA^{-1} = \left( \begin{array}{cc} a_{i+1} & a_i \\ b_{i+1} & b_i \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & \lambda_q \end{array} \right) \left( \begin{array}{cc} a_{i+1} & a_i \\ b_{i+1} & b_i \end{array} \right)^{-1} \in \mathbb{G}_q. \tag{4.5a}
\]

(iii) Let \( \{x_i, x_{i+1}\} \) be an ordinary interval. By (d) of the above, \((x_i, x_{i+1})\) is an even line. Hence there exists a unique \( A \in \mathbb{G}_q \) such that \( A(0) = x_i \) and \( A(\infty) = x_{i+1} \). Set \( A(\sqrt{-1}) = y \). \((y, x_{i+1})\) and \((y, x_i)\) are even edges.

\[
x_i \quad x_{i+1} \quad x_i
\]

is the side pairing that fixes \( y \) and sends the even edge \((y, x_{i+1})\) to \((y, x_i)\). It is clear that this side pairing must be

\[
w = ASA^{-1} = \left( \begin{array}{cc} a_{i+1} & a_i \\ b_{i+1} & b_i \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} a_{i+1} & a_i \\ b_{i+1} & b_i \end{array} \right)^{-1} \in \mathbb{G}_q. \tag{4.6a}
\]

(iv) Let \( \{x_i, x_{i+1}\} \) and \( \{x_j, x_{j+1}\} \) be two ordinary intervals. By (d) of the above, \((x_j, x_{j+1})\) and \((x_i, x_{i+1})\) are even lines.

\[
x_i \quad x_{i+1} \quad x_j \quad x_{j+1}
\]

is the side pairing that sends \( x_{j+1} \) to \( x_i \) and \( x_j \) to \( x_{i+1} \) (equivalently, pairs the even lines \((x_j, x_{j+1})\) and \((x_i, x_{i+1})\)). It is clear that this side pairing must be

\[
k = \left( \begin{array}{cc} a_j & -a_{j+1} \\ b_j & -b_{j+1} \end{array} \right) \left( \begin{array}{cc} a_{i+1} & a_i \\ b_{i+1} & b_i \end{array} \right)^{-1} \in \mathbb{G}_q. \tag{4.7a}
\]

**Definition 4.2.** The intervals in (ii) are called **odd intervals**, the intervals in (iii) are called **even intervals**, and the intervals in (iv) are called **free intervals**.

**Example 4.3.** Let \( q = 6, \lambda = \lambda_6 = \sqrt{3}, g = \left( \begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array} \right) \). The following is a Hecke-Farey symbol (see Figure 2).

\[
M_X = \{-\infty, 0/1, 1/2\lambda, 1/\lambda, 1/2, \lambda, 2/\lambda, \lambda/2, \infty\}.
\tag{4.8}
\]

**Proof.** The set of cusps of \( \Phi_2 \) is \( \{0/1, 1/\lambda, \infty\} \). One must verify that the interval \( \{1/2\lambda, 1/\lambda\} \) satisfies (c) of the above. This can be checked easily as \( g(1/\lambda) = 1/2\lambda \) and \( g(\infty) = 1/\lambda \). Note that \( g\Phi_2 \) is the 2-cluster with odd vertex \( y \).

\( \square \)
4.4. Special polygons and Hecke-Farey symbols. We now relate Hecke-Farey symbols to special polygons. It is a generalisation of a theorem of [K3] and is proved in an analogous way (see Example 4.3 and Figure 2 for an example).

**Proposition 4.4.** There is a one to one correspondence between the set of special polygons and the set of Hecke-Farey symbols.

**Proof.** Let \( M_X = (P, I_X) \) be a special polygon. The free vertices (cusps) of \( P \) form a sequence \(-\infty = x_{-1}, x_0, x_1, \ldots, x_n, \infty = x_{n+1}\). Let \( a_k/b_k \) be the reduced form of \( x_k \) and let \( x_i, x_{i+1} \) be two consecutive terms. Suppose that \( a_{i+1}b_i - a_ib_{i+1} \neq 1 \). By Lemma 3.5, \( (x_i, x_{i+1}) \) is not an even line of \( P \). Following the definition of convex hyperbolic polygons (see (S1)-(S3) of subsection 4.2), one has (i) \( x_i \) and \( x_{i+1} \) are the end points of two odd edges \( e = (y, x_i) \) and \( f = (y, x_{i+1}) \) of \( P \) that meet at an odd vertex \( y \), (ii) \( e \) and \( f \) are the odd edges of an \( r \)-cluster \( g\Phi_r \) for some \( g \) and \( r \). As a consequence, \( x_i \) and \( x_{i+1} \) satisfy (c) of subsection 4.3 and gives an \( r \)-interval \( \{x_i, x_{i+1}\} \). Hence the sequence

\[
F = \{-\infty = x_{-1}, x_0, x_1, \ldots, x_n, \infty = x_{n+1}\} \quad (4.9)
\]

is a Hecke-Farey sequence. In the case \( \{x_i, x_{i+1}\} \) is an \( r \)-interval, following our notation above and (S1) of subsection 4.2, \( e = (y, x_i) \) and \( f = (y, x_{i+1}) \) are paired by an element of \( I_X \). As this side pairing must be unique, it has to be \( gR^tg^{-1} \) (see (4.4)). Hence the \( r \)-interval must carry the side pairing given by (4.4). In the case \( \{x_i, x_{i+1}\} \) is an ordinary interval of \( F \), similar study of (S1)-(S3) of subsection 4.2 implies that the side pairing in \( I_X \) that does the pairing for \( \{x_i, x_{i+1}\} \) is just the side pairing given by (4.3). This makes \( F \) into a Hecke-Farey symbol.

Conversely, given a Hecke-Farey symbol, one can construct a special polygon with free vertices corresponding to the points of the Hecke-Farey sequence, odd vertices where the symbols \( e, f(g) \) and \( \bullet \) occur and even vertices where the symbol \( o \) occurs and side pairings as determined by (i)-(iv) of subsection 4.3. \( \square \)

**Discussion 4.5.** Note that \( F_1 = \{-\infty, 0/1, \infty\} \) and \( F_2 = \{-\infty, 0/1, \infty\} \) are not Hecke-Farey symbols since a Hecke-Farey symbol must have at least two side pairings (see subsection 4.3) whereas \( F_1 \) has only one side pairing \( S \) and \( F_2 \) has only one side pairing \( I_2 \). It is clear that \( F_1 \) and \( F_2 \) do not correspond to any special polygon. As a consequence, Proposition 4.4 is no longer true if one does not insist that a Hecke-Farey symbol must carry at least two side pairings.

5. Poincaré’s Polygonal Theorem and Independent Generators

**Theorem 5.1.** If \( M_X = (P, I_X) \) is a special polygon, then the set of side pairings \( I_X \) generates independently a group \( G \subseteq G_q \) such that \( P \) is a fundamental domain of \( G \).

**Proof.** The stabilisers of the cusps of \( P \) in \( G \) are generated by conjugates of \( T^n \) for some \( n \in \mathbb{N} \). Let \( v \in P \) be an elliptic point of \( P \). The stabiliser \( G_v = \langle \tau \rangle \) of \( v \) in \( G \) is generated by either a conjugate of \( S \) or a conjugate of \( R^r \) for some \( r \), where \( r|q \). It is clear from our construction of \( P \) that (i) \( v \) is the intersection of two edges \( e_1 \) and \( e_2 \) of \( P \) (see (S1) and
(S2) of subsection 4.2), (ii) the sides \( e_1 \) and \( e_2 \) make an internal angle \( \pi \) or \( 2\pi/q \) at \( v \), and (iii) we may assume that \( \tau \) fixes \( v \) and sends \( e_1 \) to \( e_2 \). In summary,

(i) the stabiliser of a cusp of \( P \) in \( G \) is generated by a parabolic element,
(ii) \( e_1 \) and \( e_2 \) make an internal angle \( \pi \) or \( 2\pi/q \) at \( v \), where \( r|q \) and \( r < q \),
(iii) \( \tau \) is the side pairing that fixes \( v \) and sends \( e_1 \) to \( e_2 \).

Hence \( P \) is a Poincaré polygon. By Poincaré's polygonal theorem, \( P \) is a fundamental domain of \( G \) and \( I_X \) is a set of independent generators of \( G \) (see pp 223 of [Ma]).

Let \( X \) be a subgroup of finite index of \( G_q \). An admissible fundamental domain is a special polygon \( M_X = (P, I_X) \) such that \( P \) is a fundamental domain of \( X \) and \( I_X \) is a set of independent generators of \( X \).

**Theorem 5.2.** Let \( X \) be a subgroup of finite index of \( G_q \). Then \( X \) has an admissible fundamental domain \( M_X \).

**Proof.** The tessellation \( T^* \) (see subsection 2.1) induces a tessellation of the surface \( X \setminus \mathbb{H} \). Let \( v \) be an odd vertex of a tile \( T_0 \) of \( X \setminus \mathbb{H} \). The stabiliser \( X_v \subseteq X \) is cyclic of order \( d \), where \( d \) is a divisor of \( q \). In the case \( d > 1 \), \( v \) is the odd vertex of a \( q/d \)-cluster \( T_0 \). Note that the two odd sides of \( T_0 \) must be paired by some members of \( X_v \). In the case \( d = 1 \), \( v \) is the odd vertex of a \( q \)-gon \( T_0 \). In summary, the tiles of \( X \setminus \mathbb{H} \) are \( q \)-gons and \( q/d \)-clusters, where \( d|q \). If \( X \setminus \mathbb{H} \) has an elliptic point of order 2 on the boundary of \( T_0 \), then the two even edges incident to the elliptic point are paired by an element of \( X \) and forming an edge of \( X \setminus \mathbb{H} \). It is clear that these tiles intersect each other (if any) at either cusps or even lines as the two odd sides of a \( q/d \)-cluster must be paired by some elements in \( X_v \).

Finding a fundamental domain for \( X \) which is a special polygon amounts to cutting the surface \( X \setminus \mathbb{H} \) into its tiles (\( q \)-gons and \( q/d \)-clusters) and develop these tiles on \( \mathbb{H} \) so that \( 0 \) and \( \infty \) are vertices of the polygon (see (S3) of subsection 4.2).

(A) Let \( p : \mathbb{H} \rightarrow X \setminus \mathbb{H} \) be the projection map and let \((0, \infty)\) be the even line joining 0 and \( \infty \). \( p((0, \infty)) \) lies on the boundary of some tile \( T \) of \( X \setminus \mathbb{H} \). We develop \( T \) to \( \mathbb{H} \) so that \( p((0, \infty)) \) is developed to \((0, \infty)\). The other tiles of \( X \setminus \mathbb{H} \) are then developed onto \( \mathbb{H} \) in an inductive manner, piece by piece, where each new piece is adjacent to a tile that is already been developed.

(B) The determination of the first tile \( P_0 \).

(i) Let \( D_1 = \{ \Phi_d : \text{the two odd edges of } \Phi_d \text{ are paired by some elements of } X \} \) and \( D_2 = \{ \Psi_d : \text{the two odd edges of } \Psi_d \text{ are paired by some elements of } X \} \) (see subsection 4.1 for \( \Phi_d \) and \( \Psi_d \)). If \( D_1 \neq \emptyset \), let \( P_0 \) be the smallest \( \Phi_q \) (in area) of \( D_1 \). If \( D_1 = \emptyset \) and \( D_2 \neq \emptyset \), let \( P_0 \) be the smallest \( \Psi_q \) (in area) of \( D_2 \). If \( D_1 \cup D_2 = \emptyset \), let \( P_0 \) be the depth one \( q \)-gon that lies in the right half plane. Note that \( 0 \) and \( \infty \) are vertices of \( P_0 \) (see (S3) of subsection 4.2).

(ii) The reduced forms of the vertices of \( P_0 \) can be determined by Lemma 3.2. Note that the two odd edges of \( P_0 \) (if any) are paired by some elements of \( X \).

(C) Let \( e \) be a side (even line) of \( P_0 \).

(i) If there exists another side \( f \) of \( P_0 \) such that the element \( g \in G_q \) which pairs \( e \) and \( f \) is in \( X \), then we call \( e \) and \( f \) paired sides and add \( g \) to the generating set \( I_X \) of \( X \). The side pairing \( g \) can be determined by (4.7).

(ii) If the element \( g \) in \( G_q \) that pairs the two even edges of \( e \) is in \( X \), then we call \( e \) a paired side and \( g \) is put into the generating set \( I_X \) of \( X \). Note that \( g \) has order 2. Such \( g \) can be determined by (4.6).

(iii) Let \( D \) be the collection of all \( d \)-clusters \((d < q)\) attached to \( e \) and let \( D_e = \{ T \in D : \text{the two odd edges of } T \text{ are paired by some elements of } X \} \). If \( D_e \) is not empty, then we attach \( T_0 \) to \( P_0 \) (along \( e \)) to form a new polygon \( P_0 = P_0 \cup T_0 \), where \( T_0 \) is the smallest \( d \)-cluster (in area) of \( D_e \). The two odd edges of \( T_0 \) paired by \( g \in X \)
Two subgroups of the Hecke group $G_6$ of index 3.

(i) Let $\lambda = \lambda_6 = \sqrt{3}$ and let $M_X$ be a special polygon of $X$ given as follows (see Figure 3a).

$$M_X = \{-\infty \circ 0/1 \circ \lambda/2 \circ \lambda/\cdot \circ \ell_3(1) \circ \infty\}. \quad (5.1)$$

To determine the side pairing $\ell_3(1)$, we follow the notation of subsection 4.3. One has $r = 3$, $g = 1$, $x_i = \lambda/2$ and that $x_{i+1} = 1/0$. By (4.4), one has $\ell_3(1) = R^3$. $X$ is a normal subgroup of index 3 (see Discussion 8.2 for normality). The remaining side pairings can be determined easily by (4.6). In summary, a set of independent generators is given by

$$X = \left\langle S, \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array}\right), S \left(\begin{array}{cc} 1 & 0 \\ \lambda & 1 \end{array}\right)^{-1}, \left(\begin{array}{cc} \lambda & 1 \\ 2 & \lambda \end{array}\right), S \left(\begin{array}{cc} \lambda & 1 \\ 2 & \lambda \end{array}\right)^{-1}, \left(\begin{array}{cc} 0 & 1 \\ -1 & \lambda \end{array}\right)^{-3}\right\rangle. \quad (5.2)$$

(ii) Let $M_Y = \{-\infty \circ \infty \circ 1/\ell_3(1) \circ \lambda/2 \circ \lambda/2 \circ \infty\}$ (see Figure 3b). $Y$ is a subgroup of index 3 in $G_6$. An easy calculation of the side pairings shows that $Y = G_0(2) = \{(x_{ij}) \in G_6 : x_{12} \equiv 0 \pmod{2}\}$. Note that both $M_X$ and $M_Y$ are 3-clusters (with different side pairings).

5.2. Subgroups of index 2 of $G_q$. A special polygon of a subgroup of index 2 is either a 2-cluster or a union of two special triangles (1-clusters). In the case $q$ is odd, it has to be a union of two special triangles. As a consequence, its Hecke-Farey Symbol is given as follows.

$$M_1 = \{-\infty \circ \cdot \circ \infty\}. \quad (5.3)$$
A set of independent generators is \{ST^{-1}, T^{-1}S\}. In the case \(q\) is even, \(G_q\) has three subgroups of index 2, (5.3) and two more given as follows.

\[
M_2 = \{ -\infty \over \infty, 0 \over 1, 1/\lambda_q \over \infty \}, \quad M_3 = \{ -\infty \over \infty, 0 \over 1, \lambda_q \over \infty \}.
\]  

(5.4)

To determine \(e_2(1)\), we once again follow the notation of (i) of subsection 4.3. One has \(r = 2\), \(g = 1\), \(x_i = 1/\lambda_q\), and \(x_{i+1} = 1/0\). It follows from (4.4) that \(e_2(1) = R^2\). In particular,

\[
M_3 = \left\{ \frac{R^2 = \left( \begin{array}{cc} 0 & 1 \\ -1 & \lambda_q \end{array} \right)^2, \left( \begin{array}{cc} 1 & 0 \\ \lambda_q & 1 \end{array} \right) \right\}. 
\]

(5.5)

\(M_3\) is known as the even subgroup of \(G_q\) (\(q\) even). Note that subgroups of index 2 of \(G_q\) (\(q\) even) can be realised as Veech groups.

5.3. Commutator subgroups and principal congruence subgroups of level 2. Let \(P_0 = \{ x_1, x_2, \ldots, x_q-1, x_q \}\) be the cusps of the depth 1 \(q\)-gon in the right half plane. The reduced forms of the cusps of \(P_0\) can be determined by Lemma 3.2. Take note that \(x_i = 0\) and that \(x_q = \infty\). Throughout the subsection, the \(x_i\)'s are given as above and that \(x_i < x_{i+1}\).

**Proposition 5.3.** A special polygon for \(G_q\), the commutator subgroup of \(G_q\), is the union of two depth one \(q\)-gons forming a polygon with \(2(q-1)\) free sides as follows. \(G_q \setminus \mathbb{H}\) is a genus \([(q-1)/2]\) surface with one cusp (resp. two cusps) if \(q\) is odd (resp. even).

\[
\{ -\infty \over 1, x_{q-1} \over 2, \ldots, x_2 \over q-2, x_0 \over q-1, x_2 \over q-2, \ldots, x_{q-1} \over \infty \}.
\]

(5.6)

**Proof.** The special polygon in (5.6) contains 2\(q\) special triangles. Hence the side pairings given by (5.6) generates a subgroup of index \(2q\). As \([G_q : G_q'] = 2q\), it thus suffices to prove that the side pairings given in (5.6) are members in \(G_q\). This can be checked easily as the side pairing that pairs \((x_i, x_{i+1})\) and \((-x_{q-1}, -x_{q-1})\) is given by \(T^{-1}RTR^{-1}\).

**Proposition 5.4.** Let \(q \geq 3\) be a prime. A special polygon for \(G(2)\), the principal congruence subgroup of level 2, is the union of two depth one \(q\)-gons forming a polygon with \(2(q-1)\) free sides as follows. \(G(2) \setminus \mathbb{H}\) is a genus zero surface with \(q\) cusps.

\[
\{ -\infty \over 1, x_{q-1} \over 2, \ldots, x_2 \over q-2, x_0 \over q-1, x_2 \over q-2, \ldots, x_{q-1} \over \infty \}.
\]

(5.7)

**Proof.** Since \(q\) is a prime, \([G_q : G(2)] = 2q\). By (4.7), the side pairing that pairs \((x_i, x_{i+1})\) and \((-x_{q-1}, -x_{q-1})\) is in \(G(2)\) for every \(i\). This completes the proof of our assertion.

In the case \(q \geq 3\) is a prime, similar to Proposition 5.4, one can show that \(G_0(2) = G(2)\) has index \(q\) and admits the following special polygon \((P, Ix)\), where \(P\) is a \(q\)-gon.

\[
\{ -\infty \over 1, x_1 \over 2, x_2 \over q-2, \ldots, x_{(q-1)/2} \over q-1, x_{(q+1)/2} \over q-2, \ldots, x_{q-1} \over \infty \}.
\]

(5.8)

\(G_0(2) \setminus \mathbb{H}\) is a genus zero surface with one elliptic element of order 2 and \((q+1)/2\) cusps. Note that the above is not true if \(q\) is not a prime. For instance, the group in (ii) of subsection 5.1 gives \(G_0(2)\) of \(G_q\). Its special polygon is a 3-cluster, not a 6-gon.

5.4. Power subgroups. For each \(n \in \mathbb{N}\), the power subgroup \(G_q^n\) is the characteristic subgroup of \(G_q\) generated by \(\{x^n : x \in G_q\}\). Study of power subgroups has a long history back to Newman \[N\].

**Proposition 5.5.** Let \(q \geq 3\) be an odd integer. Then \(G_q^2\) is a normal subgroup of index 2. A special polygon of \(G_q^2\) is a union of two special triangles given as in (5.3). In the case \(q\) is even, a Hecke-Farey symbol of \(G_q^2\) is given as in \(M_3\) of (5.4).

**Proof.** We shall first assume that \(q\) is odd. It is clear that \(A = T^{-1}S\) (given as in Lemma 3.2) and \(R = ST^{-1}\), each has order \(q\), are elements of \(G_q^2\). Hence the special polygon of (5.3) is also a special polygon of \(G_q^2\).
In the case \( q \) is even, we consider the homomorphism \( \phi : G_q \to \mathbb{Z}_2 = \langle a \rangle \) defined by \( \phi(S) = \phi(R) = a \). It is clear that \( G_q^2 \) is the kernel of \( \phi \). Since \( G_q \) has exactly three subgroups of index 2 and the conjugates of \( S \) and \( R \) cannot be members of a set of independent generators of \( G_q^2 \), we conclude that \( M_1 \) and \( M_2 \) (see (5.3) and (5.4)) are not special polygons of \( G_q^2 \). As a consequence, a special polygon of \( G_q^2 \) is given as in \( M_3 \) of (5.4).

**Proposition 5.6.** Let \( \{x_1, x_2, \ldots, x_{q-1}, \infty\} \) be given as in subsection 5.3 and let \( r > 1 \) be an odd divisor of \( q \). Then \( [G_q : G_q^r] = r \) and a Hecke-Farey symbol of \( G_q^r \) is given by

\[
\{-\infty \ x_1 \ldots x_r \ x_{r+1} \ldots \infty\}.
\]

**Proof.** Define \( \phi : G_q \to \mathbb{Z}_r = \langle a \rangle \) by \( \phi(S) = 1, \phi(R) = a \). It is clear that \( \phi \) is a homomorphism and that \( G_q^r \subseteq \ker \phi \). Hence \( [G_q : G_q^r] \geq r \). Let \( G \) be the group generated by the side pairings of (5.8). The side pairings of (5.8) are conjugates of \( S = S' \) and \( e_r(1) = R' \). Hence \( G \subseteq G_q^r \). Since \( [G_q : G] = r \), one has \( [G_q : G_q^r] \leq r \). Hence \( G_q^r = G \). \( \square \)

### 5.5. Non-free normal subgroups

Let \( X \) be a proper normal subgroup of \( G_q \) that is not a free group. It follows that either \( R \notin X \) or \( S \notin X \). Suppose that \( R \notin X \). Since \( X \) contains all the conjugates of \( R \), both \( R = ST^{-1} \) and \( T^{-1}S \) are members of \( X \). Hence \( X \) has index 2 and a special polygon of \( X \) is given as in (5.3). Note that \( X \) is a free product of two copies of \( \mathbb{Z}_q \). In the case \( S \notin X \), \( G_q/X \) can be generated by \( RX \). Since \( R \) has order \( q \), \( G_q/X \) is a cyclic group of order \( q \), where \( (r|q) \).

### 6. Hurwitz-Nielsen realisation problem

#### 6.1. Geometric invariants

An immediate application of the study of the special polygons is that the geometric invariants of \( X \setminus \mathbb{H} \) can be determined easily. Let \( M_X = (P, I_X) \) be a special polygon associated with \( X \subseteq G_q \).

(i) \( [G_q : X] = \) the number of special triangles in \( M_X = (P, I_X) \).

(ii) The subgroup \( X \) has \( r_2 \) (the number of the circles \( o \) in \( M_X \)) inequivalent classes of elliptic elements of order 2 that are conjugates of \( S \).

(iii) The subgroup \( X \) has \( r_4 \) (the number of the bullets \( \bullet \) in \( M_X \)) inequivalent classes of elliptic elements of order \( q \) that are conjugates of \( R = ST^{-1} \).

(iv) Let \( r (1 < r < q) \) be a divisor of \( q \). \( X \) has \( v_r \) (the number \( e_{q/r}(g)'s \) in \( M_X \)) inequivalent classes of elliptic elements of order \( r \) that are conjugates of \( R^{q/r} \).

(v) Suppose that the cusps of \( P \) is partitioned into \( v_\infty \) classes under the action of \( I_X \).

Then \( v_\infty = \) the number of cusps of \( X \setminus \mathbb{H} = \) the number of cycles of \( f(T) \), where \( f(T) \) is the permutation representation of \( T \) on the set of cosets \( G_q/X \).

(vi) Let \( \Delta = \{r : r \) is a divisor of \( q, 2 \leq r \leq q\} = \{r_1, r_2, \cdots, r_k = q\} \). The genus \( g \) of \( X \setminus \mathbb{H} \) is given by the following Riemann Hurwitz formula.

\[
2g - 2 + \frac{r_2}{2} + \sum_{r = 1}^{k} v_r (1 - 1/r_1) + v_\infty = [G_q : X](1/2 - 1/q).
\]

The terms in \( \{g, r_2, v_{r_1}, v_{r_2}, \cdots, v_{r_k}, v_\infty, [G_q : X]\} \) are called the geometric invariants of \( X \), where the \( v_r's \) are given as in (ii)-(iv).

**Proof.** (i) is clear. (ii)-(iv) follows from the observation that each conjugacy class of elliptic elements of \( X \) must have exactly one representative in \( I_X \) as \( I_X \) is a set of independent generators. (v) follows from the fact that the number of cusps is just the number of double cosets \( (T) \setminus G_q / X \).

**Lemma 6.1.** Suppose that a special polygon \( M_X = (P, I_X) \) of \( X \) consists of \( n_0 \) \( q \)-gons and \( v_{r_i} \) \( q/r_i \)-clusters, \( 1 \leq i \leq k \). Then the number of generators of infinite order in \( I_X \) is

\[
f = \frac{n_0(q - 2) + \sum_{i=1}^{k} v_{r_i}(q/r_i - 2) + 2 - r_2}{2}.
\]
Proof. Since the boundary of a \(q/r\)-cluster has \(q/r\) even lines, it follows that the boundary of \(P\) has \(n_0(q-2) + \sum_{i=1}^{k} v_{r_i}(q/r_i - 2) + 2\) even lines. Among these even lines, \(\tau_2\) of them are self paired by elements of \(I_X\). As a consequence, the remaining even lines are free sides and they are paired by side pairings of infinite order. In particular, the number of such side pairings is \(f = (n_0(q-2) + \sum_{i=1}^{k} v_{r_i}(q/r_i - 2) + 2 - \tau_2)/2\).

6.2. Construction of convex hyperbolic polygons. Let \(P_0\) be either the depth one \(q\)-gon in the right half plane or the \(r\)-cluster \(\Phi_r\) where \(r \geq 2\) is a divisor of \(q\) (see (4.1a) for \(\Phi_r\)). The Hecke-Farey sequence associated with \(P_0\) is given by

\[
F_0 = \{-\infty, 0, a_1, a_2, \ldots, a_k, a_k = \infty\}.
\]

Since \(r \geq 2\), \(\{0, a_1\}\) is an ordinary interval of \(F_0\). In the case \(P_0 = \Phi_r/\{a_{k-1}, a_k\}\) is an \(r\)-interval of \(F_0\) (see (d) of subsection 4.3 for the terms ordinary and \(r\)-intervals). Note that \(P_0\) is convex. Since \(\{0, a_1\}\) is an even line of \(P_0\) (see (d) of subsection 4.3), one may attach \(R_0\) to \(P_0\) along the even line \(\{0, a_1\}\) to get a new polygon \(P_0\), where \(R_0\) is either a \(q\)-gon or an \(s\)-cluster \((s \geq 2\) and \(s/q)\). The Hecke-Farey sequence associated with \(P_0\) takes the form

\[
F_1 = \{-\infty, 0, b_1, b_2, \ldots, b_{t-1}, a_1, a_2, \ldots, a_k\}.
\]

Since \(s \geq 2\), \(\{0, b_1\}\) is an ordinary interval of \(P_1\) and \(\{b_{t-1}, a_1\}\) is an \(s\)-interval if \(R_0\) is an \(s\)-cluster. Note that \(P_1\) is convex. Since \(\{0, b_1\}\) is an even line of \(P_1\), one may attach \(R_1\) to \(P_1\) along the even line \(\{0, b_1\}\) to get a new polygon \(P_2\), where \(R_1\) is either a \(q\)-gon or an \(u\)-cluster \((u \geq 2\) and \(u/q)\). Apply this procedure repeatedly, one admits a convex hyperbolic polygon \(P_n\) consists of \(n_0\) \(q\)-gons and \(v_{r_i}\), \(q/r_i\)-clusters for some \(n_0\) and \(v_{r_i}\), where \(r_i|q\) and \(q/r_i \geq 2\) (the key of our construction is that \(P_k\) always has an even line \(\{0, x_i\}\) and that \(R_i\) is always attached to \(P_i\) along \(\{0, x_i\}\)). Let \(\mathcal{F}\) be the Hecke-Farey sequence associated with \(P_n\). \(\mathcal{F}\) has \(v_{r_i}\), \(q/r_i\)-intervals, where \(q/r_i \geq 2\). The intervals of \(\mathcal{F}\) are divided into two classes (i) \(q/r_i\)-intervals, where \(q/r_i \geq 2\), and (ii) ordinary intervals. To count the number of ordinary intervals, we note that

(i) the boundary of a \(q\)-gon has \(q\) even line,

(ii) the boundary of a \(q/r_i\)-cluster has \(q/r_i\) even lines.

It follows that the boundary of \(P_n\) has \(n_0(q-2) + \sum v_{r_i}(q/r_i - 2) + 2\) even lines. As the even lines of \(P_n\) are associated with the ordinary intervals of \(\mathcal{F}\) (see (d) of subsection 4.3), the number of ordinary intervals of \(\mathcal{F}\) is given by

\[
n_0(q-2) + \sum v_{r_i}(q/r_i - 2) + 2, q/r_i \geq 2.
\]

6.3. Millington’s Theroem. Kulkarni [K2, K3] gives two proofs of Millington’s Theorem [Mi], one by Diagrams (see Section 4 of [K2]) and one by Farey symbols (see Section 7.6 of [K3]). We extend Millington’s result to \(G_q\) by studying Hecke-Farey symbols. Our proof is a simple generalisation of Kulkarni’s proof of Millington’s Theorem.

Theorem 6.2. Let \(g \geq 0\), \(\tau_2 \geq 0\), \(v_{r_i} \geq 0\), \(d \geq 1\), \(v_\infty \geq 1\) be integers and let \(r_i \in \Delta = \{r : r is a divisor of q, 2 \leq r \leq q\} = \{r_1, r_2, \ldots, r_k = q\}\). Then \(G_q\) has a subgroup \(X\) of index \(d\), genus \(g\) with \(v_\infty\) cusps, and \(\tau_2\) (resp. \(v_{r_i}\)) conjugacy classes of elements that are conjugates of \(S\) (resp. \(R_i/r_i\)) if and only if \(m_0 = 4g - 4 + \tau_2 + 2v_\infty + \sum_{i=1}^{k} v_{r_i}(2 - q/r_i) \geq 0\) is a multiple of \((q-2)\) and

\[
2g - 2 + \tau_2/2 + \sum_{i=1}^{k} v_{r_i}(1 - 1/r_i) + v_\infty = d(1/2 - 1/q).
\]

Note that if \(q\) is odd, then \(m_0/(q - 2) \in \mathbb{Z}\) is a consequence of (6.6).

Proof. Suppose that \(m_0 \geq 0\) is a multiple of \((q-2)\) and that \(g \geq 0\), \(\tau_2 \geq 0\), \(v_{r_i} \geq 0\), \(d \geq 1\), \(v_\infty \geq 1\) satisfy the Riemann-Hurwitz formula (6.6). Let \(r_i \in \Delta\). It follows that \(q/r_i \geq 2\) for \(1 \leq i \leq k - 1\). Let \(n_0 = m_0/(q - 2)\).
Case 1. \( n_0 = 0 \) and \( v_{r_i} = 0 \) for all \( i \leq k - 1 \). A simple calculation shows that either (i) \( d = 1, \tau_2 = v_q = 1, v_\infty = 1, g = 0, v_r = 0 \) for all \( i \leq k - 1 \) or (ii) \( d = 2, v_q = 2, v_\infty = 1, g = 0, v_r = 0 \) for all \( i \leq k - 1 \). In case (i), \( X = G_q \). In case (ii), \( X \) is given as in (5.3).

Case 2. \( n_0 > 0 \) or \( v_{r_i} > 0 \) for some \( i \leq k - 1 \). By (6.5), we have a polygon \( P_n \) and a Hecke-Farey sequence \( F \) with \( v_{r_i}, q/r_i \)-intervals \( (i = 1, 2, \cdots, k - 1) \) and \( n_0(q - 2) + \sum_{i=1}^{k-1} v_{r_i}(q/r_i - 2) + 2 = 4g - 2 + \tau_2 + v_q + 2v_\infty \geq 2 \) (6.7)

ordinary intervals. Since \( n_0 > 0 \) or \( v_{r_i} > 0 \) for some \( i \leq k - 1 \), \( F \) has at least three intervals. We make \( F \) into a Hecke-Farey symbol by declaring the first \( \tau_2 \) ordinary intervals even intervals (see Definition 4.2 for the terms even and odd intervals) and the next \( v_q \) ordinary intervals odd intervals (this is equivalent to adjoin \( v_q \) special triangles to \( P_n \)). The next \( 2(v_\infty - 1) \) ordinary intervals are declared to be free intervals and are divided into \((v_\infty - 1)\) consecutive pairs which are paired. The remaining \( 4g \) ordinary intervals are free intervals and are paired in the usual \( xyz^{-1}y^{-1} \) fashion (see Discussion 6.3). Let \( X \) be the group generated by the above side pairings and let \( I_X \) be the set of the side pairings.

(i) \( I_X \) possesses \( \tau_2 \) elements that are conjugates of \( S \) and \( v_{r_i} \) elements that are conjugates of \( R^{q/r_i} \) for \( 1 \leq i \leq k \).

(ii) An easy study of the side pairings shows that \( X \) has \( v_\infty \) cusps. The index of \( X \) is given by the number of special triangles of the special polygon \( M_X = (P, I_X) \), where \( P \) is the union of \( P_n \) and \( v_q \) special triangles. There are \( n_0 = (q - 2) \) \( q \)-gons, \( v_{r_i}, q/r_i \)-clusters \( (1 \leq i \leq k - 1) \) and \( v_q \) special triangles. Hence the index is \( n_0 + \sum_{i=1}^{k-1} v_{r_i}(q/r_i) + v_q = d \) (6.8).

The genus of \( X \) is \( g \) (see (6.1)). It follows that \( X \) is the required subgroup (Theorem 5.1).

Conversely, let \( X \) be given as in our theorem, the geometric invariants of \( X \) satisfy (6.6). Let \( M_X = (P, I_X) \) be a special polygon of \( X \). Then \( P \) has \( v_{r_i}, q/r_i \)-clusters \( (1 \leq i \leq k) \). Suppose that \( P \) has \( n_0 \) \( q \)-gons. By (i) of subsection 6.1, \( d = n_0q + \sum_{r_i=1}^{k} v_{r_i}q/r_i \). By (6.6), one has \( n_0 = (4g - 4 + \tau_2 + 2v_\infty + \sum_{r_i=1}^{k} v_{r_i}(2 - q/r_i))/q - 2) \). In particular, \( 4g - 4 + \tau_2 + 2v_\infty + \sum_{r_i=1}^{k} v_{r_i}(2 - q/r_i) \geq 0 \) is a multiple of \( q - 2 \).

Discussion. (i) A Hecke-Farey symbol must have at least two side pairings (subsection 4.3). (ii) If \( F \) has at least three intervals, our declaration gives at least two side pairings.

6.4. Kurosh’s Theorem. Let \( Y \) be a free product of \( F_f \) (a free group of rank \( f \)), \( \pi_2 \) copies of \( Z_2 \), and \( v_{r_i} \) copies of \( \mathbb{Z}_{r_i} \), where \( r_i \in \Delta_0 = \{ r : r \) is a divisor of \( q, 3 \leq r \leq q \} \). We say \( Y \) is realisable in \( G_q \) if \( Y \) is isomorphic to a subgroup \( X \) of finite index of \( G_q \). It is clear that \( \mathbb{Z}_2 * \mathbb{Z}_q \) and \( \mathbb{Z}_q * \mathbb{Z}_q \) are realisable (see (5.3) for \( \mathbb{Z}_q * \mathbb{Z}_q \)).

Proposition 6.4. Let \( q \geq 3 \) be an odd integer and let \( Y \) be given as above. Suppose that \( Y \) is not \( \mathbb{Z}_2 * \mathbb{Z}_q \) or \( \mathbb{Z}_q * \mathbb{Z}_q \). Then \( Y \) is isomorphic to a subgroup \( X \) of \( G_q \) if and only if \( 2f + \pi_2 + v_q - 2 \geq 0 \) and \( n_0 = 2f + \pi_2 + \sum_{r_i=1}^{k} v_{r_i}(2 - q/r_i) - 2 \geq 0 \) is a multiple of \( q - 2 \). The index of \( X \) is \( d = n_0q/(q - 2) + \sum_{r_i=1}^{k} v_{r_i}q/r_i \).

Proof. Suppose that \( Y \) is isomorphic to a subgroup \( X \) of \( G_q \). Let \( M_X = (P, I_X) \) be a special polygon of \( X \). Then \( I_X \) has \( f \) elements of infinite order, \( \pi_2 \) elements that are conjugates of \( S \), and \( v_{r_i} \) elements that are conjugates of \( R^{q/r_i} \). By Lemma 6.1, \( f, v_{r_i}, n_0 \geq 0 \), and \( \pi_2 \) satisfy the identity given in (6.2). This implies that \( n_0 = 2f + \sum_{r_i=1}^{k} v_{r_i}(2 - q/r_i) - 2 = n_0q/(q - 2) \geq 0 \). In particular, \( n_0 \geq 0 \) is a multiple of \( q - 2 \).

Note that \( n_0 = n_0q/(q - 2) \) is the number of \( q \)-gons of \( P \). Since \( P \) has \( n_0 q \)-gons and \( v_{r_i}, q/r_i \)-clusters, the index of \( X \) is \( d = n_0q/(q - 2) + \sum_{r_i=1}^{k} v_{r_i}q/r_i \) (see (i) of subsection 6.1). Finally, since \( m_0 = 2f + \pi_2 + v_q - 2 + \sum_{r_i=1}^{k} v_{r_i}(2 - q/r_i) = n_0(q - 2) \geq 0 \) and \( (2 - q/r_i) < 0 \)
for \( i \leq s - 1 \), one has \( 2f + \pi_2 + v_q - 2 \geq 0 \). Suppose that \( 2f + \pi_2 + v_q - 2 = 0 \). It follows easily that \( v_{r_i} = 0 \) for \( i \leq s - 1 \). A simple study shows that \( Y \) is isomorphic to a subgroup \( X \) of \( G_q \) only if \( Y \cong \mathbb{Z}_2 \ast \mathbb{Z}_q \) or \( \mathbb{Z}_4 \ast \mathbb{Z}_2 \). A contradiction. Hence \( 2f + \pi_2 + v_q - 2 > 0 \).

Conversely, suppose that \( 2f + \pi_2 + v_q - 2 > 0 \) and that \( n_0 = m_0/(q - 2) \in \mathbb{N} \cup \{0\} \). If \( n_0 = 0 \) and \( v_{r_i} = 0 \) for \( i \leq s - 1 \), then \( m_0 = 2f + \pi_2 + v_q - 2 = 0 \). A contradiction. Hence either \( n_0 > 0 \) or \( v_{r_i} > 0 \) for some \( i \leq s - 1 \). By (6.5), we have a polygon \( P_n \) and a Hecke-Farey sequence \( \mathbb{F} \) with \( v_{r_i}, q/r_i \) -intervals \( (q/r_i \geq 2 \) for \( i \leq s - 1 \) and \( n_0(q - 2) + \sum_{i=1}^{s-1} v_{r_i}(q/r_i - 2) + 2 = 2f + \pi_2 + v_q \geq 2 \) \( (6.9) \)

ordinary intervals. We make \( \mathbb{F} \) into a Hecke-Farey symbol by declaring the first \( \pi_2 \) ordinary intervals even intervals and the next \( v_q \) ordinary intervals odd intervals (this is equivalent to adjoin \( v_q \) special triangles to \( P_n \)). The last \( 2f \) ordinary intervals are declared to be free intervals and are divided into \( f \) consecutive pairs which are paired. Let \( X \) be the subgroup of \( G_q \) generated by the above side pairings and let \( I_X \) be the set of the side pairings. Then

(A) \( I_X \) possesses \( \pi_2 \) elements that are conjugates of \( S \) (of order \( 2 \)) and \( v_{r_i} \) elements that are conjugates of \( R^{r_i/2} \) (of order \( r_i \)) for \( 1 \leq i \leq s \).

(B) \( I_X \) has \( f \) elements of infinite order.

By Theorem 5.1, \( X \) is isomorphic to \( Y \). This completes the proof of the proposition. \( \square \)

In the case \( q \) is even, subgroups of \( G_q \) may possess two types of elliptic elements of order 2, namely, conjugates of \( S \) and \( R^{r_i/2} \). Note that \( 2 \notin \Delta_0 \). For our convenience, we set \( v_2 = 0 \).

**Proposition 6.5.** Let \( q \geq 4 \) be an even integer and let \( Y \) be given as above. Suppose that \( Y \) is not \( \mathbb{Z}_2 \ast \mathbb{Z}_q \) or \( \mathbb{Z}_4 \ast \mathbb{Z}_2 \). Then \( Y \) is isomorphic to a subgroup \( X \) of \( G_q \) if and only if

(i) \( m_0 = 2f + (\pi_2 - t) + t(2 - q/2) + \sum_{i=1}^{s} v_{r_i}(2 - q/r_i) - 2 \geq 0 \) is a multiple of \( q - 2 \) for some \( t \geq 0 \), where \( (\pi_2 - t) \) is nonnegative, and

(ii) \( 2f + (\pi_2 - t) + v_q - 2 \geq 0 \) and \( (2f + (\pi_2 - t) + v_q - 2, t, v_q/2) \neq (0, 0, 0) \).

Note that \( v_2 = 0 \). The index of \( X \) is \( d = m_0q/(q - 2) + tq/2 + \sum_{i=1}^{s} v_{r_i}q/r_i \).

**Proof.** Suppose that \( Y \cong X \subseteq G_q \). Let \( M_X = (P, I_X) \) be a special polygon of \( X \). Then \( I_X \) has \( f \) elements of infinite order, \( (\pi_2 - t) \) elements that are conjugates of \( S \), \( t \) elements that are conjugates of \( R^{r_i/2} \), and \( v_{r_i} \) elements that are conjugates of \( R^{r_i/2} \). Note that \( r_i \neq 2 \) \((r_i \in \Delta_0) \). The assertion now follows by applying the proof of Proposition 6.4. \( \square \)

7.

**Permutation representation of \( G_q \) on \( G_q/X \)**

The main purpose of this section is to give an easy and systematic method that determines the permutation representation of \( G_q \) on \( G_q/X \).

7.1.

**A commutative diagram.** Let \( \Phi \) be given as in subsection 4.1. \( \Phi \) is a fundamental domain of \( G_q \). For each coset \( Xg \in G_q/X \), there exists a unique \( x_g \in X \) such that \( x_g \Phi \in M_X = (P, I_X) \). Denoted by \( \Omega_X \) the set of all such special triangles \( x_g \Phi \). It follows that \( \bigcup_{x_g \Phi \in \Omega_X} x_g \Phi = M_X \). As a consequence, there is a one to one correspondence between \( G_q/X \) and \( \Omega_X \) defined by

\[
\tau(Xg) = x_g \Phi \in \Omega_X. \tag{7.1}
\]

An element \( g \) of \( G_q \) acts on \( G_q/X \) by \((Xh, g) \rightarrow Xhg \). We shall now study the action of \( X \) on \( \Omega_X \) as follows. Let \( g \in G_q \). For each \( g \Phi \in \Omega_X \), there exists a unique pair \((g_i \Phi, x_{ij}) \in \Omega_X \times X \) such that \( x_{ij}g_i \Phi = g_i \Phi \in \Omega_X \). The action of \( g \) on \( \Omega_X \) is defined by

\[
(g_i \Phi, g) \rightarrow x_{ij}g_i \Phi. \tag{7.2}
\]

**Lemma 7.1.** The action of \( G_q \) on \( G_q/X \) is isomorphic to the action of \( G_q \) on \( \Omega_X \).
Proof. We consider the following diagram, where the horizontal arrows represent the actions of $X$ on $G_q/X$ and $\Omega_X$.

\[
\begin{array}{ccc}
G_q/X \times G_q & \longrightarrow & G_q/X \\
(\tau, id) \downarrow & & \downarrow \tau \\
\Omega_X \times G_q & \longrightarrow & \Omega_X
\end{array}
\]

(7.3)

Let $G_q/X = \cup_X g_i$ be chosen such that $\cup g_i \Phi = M_X$. This implies that $\tau(Xg_i) = g_i \Phi$. As a consequence, (7.3) is a commutative diagram and the action of $G_q$ on $G_q/X$ is isomorphic to the action of $G_q$ on $\Omega_X$. \hfill \square

7.2. The permutation representation of $G_q$ on $\Omega_X$. The main purpose of this subsection is to give the permutation representations of $S$ and $R$ on $\Omega_X$.

Lemma 7.2. Let $\Omega_X = \{g_i \Phi : \cup g_i \Phi = M_X\}$. The permutation representation of $S$ on $\Omega_X$ is given by

\[
f(S) = \prod (g_i \Phi, g_j \Phi),
\]

where $g_i \Phi \in \Omega_X$ and $g_j \Phi \in \Omega_X$ share the same even line of $P$. $(g_i \Phi, g_j \Phi)$ is a one cycle if and only if $g_i S g_i^{-1} \in X$ if and only if $g_i (0, \infty)$ is paired with itself by $g_j S g_j^{-1} \in X$.

Proof. Let $g_i \Phi \in \Omega_X$. There exists a unique pair $(g_i \Phi, x_{ij}) \in \Omega_X \times X$ such that $x_{ij} g_i S \Phi = g_j \Phi \in \Omega_X$ (see (7.2)). By Lemma 7.1, the permutation representation of $S$ is given by

\[
\prod_{(\tau(Xg_i), \tau(Xg_jS))} = \prod (g_i \Phi, x_{ij} g_i S \Phi) = \prod (g_i \Phi, g_j \Phi).
\]

The even line of $g_j \Phi \in M_X$ is $g_j (0, \infty) = x_{ij} g_i S (0, \infty) = x_{ij} g_i (0, \infty) \in M_X$. Since $g_i (0, \infty) \in M_X$, $x_{ij} \in X$ and every even line has a unique $X$-image in $M_X$, we conclude that $g_i (0, \infty) = x_{ij} g_i (0, \infty) = g_i (0, \infty)$. Hence $g_j \Phi$ and $g_i \Phi$ share the same even line, $(g_i \Phi, g_j \Phi)$ is a one cycle if and only if $(Xg_i, Xg_j S)$ is a one cycle (see Lemma 7.1 and (7.1)) if and only if $g_i S g_i^{-1} \in X$ if and only if $g_i (0, \infty)$ is paired with itself by $g_j S g_j^{-1} \in X$. \hfill \square

Discussion 7.3. Let $E = E_1 \cup E_2$ be the set of even lines of $M_X = (P, I_X)$, where $E_1 = \{L_1, L_2, \cdots, L_m\}$ is the set of even lines that are not paired with itself by elements of $I_X$ of order 2, and $E_2 = \{L_{m+1}, \cdots, L_n\}$ is the set of even lines that are paired with itself by elements of $I_X$ of order 2. Consequently, an element $L_s$ of $E_1$ belongs to exactly one special triangle of $\Omega_X$. Lemma 7.2 suggests a simple way to list the special triangles in $\Omega_X$. Namely, the special triangles share the even line $L_r \in E_1$ are labeled as $r$ and there is no special triangle contains $L_s \in E_2$ is labeled as $s$.

Example 7.4. Let

\[
M_X = \{-\infty \quad \cdots \quad 0 \quad \begin{array}{c}1 \quad 2 \quad 3 \quad \infty \end{array}
\]

be the Hecke-Farey symbol of a subgroup $X$ of index 11 of $G_3 = \text{PSL}(2, \mathbb{Z})$ (see Figure 4). $M_X$ has 6 even lines, five of them are shared by two special triangles of $\Omega_X$, they are

\[
L_1 = (0, \infty), \quad L_2 = (1, \infty), \quad L_3 = (2, \infty), \quad L_4 = (3, \infty), \quad L_5 = (1, 2).
\]

(7.6)

Note that (1, 2) and (2, 3) give the same line as they are paired by the side pairing labeled by the natural number 1 (see (7.5)). The even line $L_6 = (0, 1)$ is paired to itself by an element of order 2 (see (7.5)). As a consequence, the special triangles of $\Omega_X$ are labeled as in Figure 4. By Lemma 7.2, $f(S)$ is given by

\[
f(S) = (1, 1)(2, 2)(3, 3)(4, 4)(5, 5)(6).
\]

(7.7)

The permutation representation of $R$ on $G_q/X$ that contains $Xg_i \in \Omega_X$ is given by the cycle $(Xg_i, Xg_i R, \cdots, Xg_i R^{r-1})$, where $r$ is the smallest positive integer such that $Xg_i = Xg_i R^r$. By Lemma 7.1, the permutation representation of $R$ on $\Omega_X$ that contains $g_i \Phi$ is given by the following $r$ cycle.

\[
(g_i \Phi = \tau(Xg_i), \tau(Xg_i R), \tau(Xg_i R^2), \cdots, \tau(Xg_i R^{r-1})).
\]

(7.8)
Proof. \(\Delta\) is a normal subgroup of \(G\). By Lemma 7.1, the action of \(G\) has only one member in (7.8). Let \(\tau(Xg, R^*) = x_{ij}g_i R^*\). The odd vertex of \(x_{ij}g_i R^*\) is ordered in the counter-clockwise manner. The permutation \(\tau(Xg, R^*) = x_{ij}g_i R^*\Phi\) is the same as the odd vertex of \(g_i R^*\Phi\) for every \(j\). Hence the special triangles in (7.8) are special triangles of a tile \(\Delta\) whose odd vertex is \(g_i(e^{\pi i/q})\). \(\Delta\) is either a \(q\)-gon or an \(r\)-cluster. Since \(R\) acts as a counter-clockwise rotation about \(e^{\pi i/q}\), the members in (7.8) are ordered following the orientation of \(M_X\). The following is clear.

**Lemma 7.5.** Let \(\Delta\) be a tile of \(M_X\) (\(\Delta\) is either a \(q\)-gon or an \(r\)-cluster). Denoted by \(c_\Delta\) the cycle of special triangles of \(\Delta\) ordered in the counter-clockwise manner. The permutation representation of \(R\) on \(\Omega_X\) is

\[
f(R) = \prod c_\Delta.
\]

**Example 7.6.** Let \(M_X\) be given as in (7.5). Then \(f(R) = (1, 6, \overline{2})(2, 5, \overline{3})(3, \overline{5}, 4)(\overline{1})(4)\).

8. NORMALISER OF \(X\) IN \(PSL(2, \mathbb{R})\)

Let \(G_X = \langle f(S), f(R) \rangle\). Then \(G_X\) is a subgroup of \(S_n\), where \([G_q : X] = n\). The main purpose of this section is to determine the normaliser \(N(X)\) of \(X\) in \(PSL(2, \mathbb{R})\). In particular, we are able to determine whether \(X\) is a normal subgroup of \(G_q\).

**Proposition 8.1.** Let \(X\) be a subgroup of \(G_q\) of index \(n\). Then \(N_{G_q}(X)/X \cong C_{S_n}(G_X)\). \(X\) is a normal subgroup of \(G_q\) if and only if the order of \(G_X\) is \([G_q : X] = n\).\(\square\)

**Proof.** Consider the action of \(G_X\) on \(\Omega_X\). It is clear that the action is transitive. Let \(G_0\) be a one point stabiliser. By Lemma A1 of Appendix A, \(C_{S_n}(G_X) \cong N_{G_X}(G_0)/G_0\). By Lemma 7.1, the action of \(G_q\) on \(G_q/X\) is isomorphic to the action of \(G_X\) on \(\Omega_X\). Hence \(N_{G_q}(X)/X \cong N_{G_X}(G_0)/G_0 \cong C_{S_n}(G_X)\).

Consider the action of \(G_q\) on \(G_q/X\), one has \(X \triangleleft G_q\) if and only if \(Y = \cap gXg^{-1} = X\) if and only if \(X \subseteq G_q\). By \(G_q\) has order \(n\).

**Discussion 8.2.** (i) Since \(f(S)\) and \(f(R)\) can be obtained easily by studying the special polygon \(M_X\) and the order of \(G_X\) can be determined by \textbf{GAP}, whether \(X\) is normal becomes an easy issue. In the case \(q \neq 3, 4, 6\), Margulis’ characterisation of arithmeticity in terms of the commensurator implies that \(N(X) = N_{G_q}(X)\). Hence \(C_{S_n}(G_X) \cong N(X)/X\). (ii) By Proposition 8.1, the group in (i) of subsection 5.1 is normal.

**Example 8.3.** Let

\[
M_X = \{-\infty, 0, 1, 3/2, 2, 3, \ldots, \infty\}
\]
Note that $(3, \infty)$ is paired with $L_1$, $(1, 3/2)$ is paired with $L_2$, and that $(2, 3)$ is paired with $L_3$ (see (8.1) for the side pairings). One sees easily that all the even lines are shared by 2 special triangles. Following Discussion 7.3, the special triangles are labeled as in Figure 5.

By Lemmas 7.2 and 7.5, we have

$$f(R) = (1, 2, 4)(4, 6, 5)(5, 3, 1)(6, 2, 3), \quad f(S) = (1, 1)(2, 2)(3, 3)(4, 4)(5, 5)(6, 6).$$

(8.3)

One has $|G_X| = |\langle f(S), f(R) \rangle| = [G_3 : X] = 12$. By Proposition 8.1, $X$ is normal.

**Example 8.4.** Let $\lambda = \lambda_4 = \sqrt{2}$ and let

$$MX = \{-\infty, 0, 1/2, \lambda, 3/2, 2\lambda, \infty\}$$

be the Hecke-Farey symbol of $X \subseteq G_4$ of index 8 (see Figure 6). $M_X$ has four even lines, $L_1 = (0, \infty), L_2 = (0, 1/\lambda), L_3 = (\lambda, 3/\lambda),$ and $L_4 = (\lambda, \infty).$ One sees from Figure 6 that

$$f(R) = (1, 2, 1, 4)(4, 3, 2, 3), \quad f(S) = (1, 1)(2, 2)(3, 3)(4, 4).$$

(8.5)

One has $|G_X| = 16 > 8 = [G_4 : X].$ By Proposition 8.1, $X$ is normal.

**Example 8.5.** Let $P = \{x_1 = 0, x_2, \ldots, x_q = \infty\}$ be the depth 1 $q$-gon in the right half plane and let $r$ ($1 \leq r < q$) be a divisor of $q$. Let

$$M_A = \{-\infty, x_1, x_2, \ldots, x_{q-1}, x_q\} \quad \text{and} \quad M_B = \{-\infty, x_1, x_2, \ldots, x_r, \cdots, x_q, x_1, \cdots, x_{q-1}\}.$$

$f(S)$ is the identity permutation for both $A$ and $B$. $f(R)$ is a $q$-cycle (resp. $r$-cycle) for $A$ (resp. $B$). By Proposition 8.1, both $A$ and $B$ are normal subgroups of $G_q$.

9. Hurwitz-Nielsen Realisation Problem for Normal Subgroups

A transitive subgroup $G$ of $S_d$ is called regular if every non-identity element of $G$ is fixed point free. The result in this section reveals the connection between normal subgroups of index $d$ of $G_q$ and regular subgroups of $S_d$.

9.1. Geometric Invariants Revisited. Let $M_X = (P, I_X)$ be a special polygon of $X$. The geometric invariants of $X$ in terms of $f(S)$ and $f(R)$ are given as follows.

(i) $I_X$ has $r_2$ elliptic elements of order 2 that are conjugates of $S$, where $r_2$ is the number of one cycles of $f(S)$.

(ii) $P$ has $r_v$, $q/r_v$-clusters. The number $n_0$ of $q$-gons of $M_X = (P, I_X)$ is the number of $q$-cycles of $f(R)$. The index $[G_q : X]$ is given by $n_0q + \sum v, q/r$.
(iv) The number of cusps is the number of cycles of \( f(T) \), where \( f(T) \) is the permutation representation of \( T \) on \( \Omega _X \) (see (v) of subsection 6.1).

(v) \( I_X \) has \( f = (n_0(q - 2) + \sum _{r \mid q} v_r(q/r - 2) + 2 - \tau _2)/2 \) side pairings of infinite order.

Proof. By Lemma 7.5, it is clear that (a) an \( s \)-cycle \( (s < q) \) of \( f(R) \) corresponds to an \( s \)-cluster of \( P \) as well as a side pairing which is a conjugate of \( R^s \), (b) a \( q \)-cycle of \( f(R) \) corresponds to a \( q \)-gon of \( P \). Similarly, each one cycle of \( f(S) \) corresponds to a side pairing which is a conjugate of \( S \). (i)-(iv) of the above is clear. See Lemma 6.1 for (v). \( \square \)

9.2. Geometric invariants for normal subgroups. Suppose that \( X \) is normal of index \( d \).

By Proposition 8.1, \( G_X = \langle f(R), f(S) \rangle \) is a transitive subgroup of order \( d \) of the symmetric group \( S_d \). Since every nonidentity element acts freely, \( f(R) \) is a product of \( d/u \) \( u \)-cycles for some \( u \mid q \). Similarly, \( f(S) \) is either 1 or a product of \( d/2 \) 2-cycles. The geometric invariants can be determined by subsection 9.1. In particular,

(vi) \( I_X \) has \( f = d(u - 2)/u + 2 - \tau _2)/2 \) side pairings of infinite order \( (\tau _2 = 0 \text{ if } S \notin X) \).

The following fact is well known. Unlike most of the results in the literature, our study gives the sets of generators of such \( X \)'s in matrix forms (see Example 8.5).

Lemma 9.1. Let \( A \) and \( B \) be given as in Example 8.5 and let \( X \) be a normal subgroup of index \( d \) of \( G_q \) that contains \( S \). Then \( X \) is isomorphic to either \( A \) or \( B \).

Proof. Let \( M_X = (P, I_X) \) and let \( G_X = \langle f(S), f(R) \rangle \). Since \( X \) is normal, \( G_X \) is regular of order \( d \). Since \( S \in X \) and \( X \) is normal, \( f(S) = 1 \) (see (i) of subsection 9.1). Hence \( G_X = \langle f(R) \rangle \). Since \( G_X \) is transitive, \( f(R) \) must be a \( d \)-cycle. It follows that \( P \) is either a \( q \)-gon (if \( d = q \)) or a \( d \)-cluster \( \Phi _d \) (if \( d < q \)). As \( S \in X \) and \( X \) is normal, the even lines of \( P \) are self paired by conjugates of \( S \). This completes the proof of the proposition. \( \square \)

9.3. Realisation of groups as normal subgroups. Throughout the subsection, \( Y \) is a free product of a free group \( F_f \) of rank \( f \), \( \pi _2 \) copies of \( \mathbb{Z} _2 \), and \( v_r \) copies of \( \mathbb{Z} _{r'} \), where \( r_i \in \Delta _0 = \{ r : r \mid q, 3 \leq r \leq q \} \). Proposition 9.2 is an immediate consequence of Lemma 9.1.

Proposition 9.2. Let \( q \geq 3 \). Suppose that \( \pi _2 \neq 0 \). Then \( Y \) can be realised as a normal subgroup of \( G_q \) that contains \( S \) if and only if \( Y \) is a free product of \( q \) copies of \( \mathbb{Z} _2 \) or a free product of \( \mathbb{Z} _{q/r} \) and \( r \) copies of \( \mathbb{Z} _2 \), where \( r \mid q \) and \( r < q \).

Proposition 9.3. Let \( q \geq 3 \). Suppose that \( \pi _2 \neq 0 \). Then \( Y \) can be realised as a normal subgroup of index \( d \) of \( G_q \) that does not contain \( S \) if and only if (i) \( Y \) is a free product of \( 2d/q \) copies of \( \mathbb{Z} _2 \) and a free group of rank \( f \), where \( 2f = 2d(q/2 - 2)/q + 2 \), (ii) \( S_d \) has a regular subgroup \( G = \langle \alpha , \beta \rangle \) of order \( d \), where \( \alpha \) is a product of 2-cycles and \( \beta \) is a product of \( q/2 \)-cycles. In particular, \( q \) is even. Note that \( d \) is determined by \( f \) and \( q \).

Proof. Let \( Y \) and \( G \) be given as in (i) and (ii). Since \( G_q \) is a free product of \( S \) and \( R \), the map \( \chi : G_q \to G \) defined by \( \chi (S) = \alpha , \chi (R) = \beta \) is a homomorphism. Let \( X \) be the kernel of \( \chi \) and let \( M_X = (P, I_X) \) be a special polygon of \( X \). One sees that \( f(S) \) (resp. \( f(R) \)) and \( \alpha \) (resp. \( \beta \)) have the same cycle decomposition. Hence \( f(S) \) is a product of 2-cycles, \( f(R) \) is
a product of $q/2$-cycles, and $P$ is a union of $q/2$-gons ($2d/q$ of them). By (vi) of subsection 9.2, $2f = 2d(q/2 - 2)/q + 2$. Hence $I_X$ has $2d/q$ elements that are conjugates of $R^{d/2}$ and $f$ elements of infinite order. In particular, $Y \cong X \leq G_q$.

Conversely, suppose that $Y \cong X \leq G_q$, where $[G_q : X] = d$, $S \notin X$. Let $G_X = (f(S), f(R))$. By (i)-(vi) of subsections 9.1 and 9.2, $G_X$ is regular, $f(S)$ (resp $f(R)$) is a product of 2-cycles (resp. $q/2$-cycles). Further, $X$ is a free product of $2d/q$ copies of $Z_2$ and a free group $F_f$, where $2f = 2d(q/2 - 2)/q + 2$. Hence (i) and (ii) holds.

The following proposition can be proved by applying the proof of Proposition 9.3.

**Proposition 9.4.** Let $q \geq 3$. Suppose that $\pi_2 = 0$. Then $Y$ can be realised as a normal subgroup of index $d$ of $G_q$ if and only if

(i) $Y$ is a free group of rank $f$, where $2f = d(q - 2)/q + 2$, $S_d$ has regular subgroup $\langle \alpha, \beta \rangle$ of order $d$, where $\alpha$ is a product of 2-cycles and $\beta$ is a product of $q$-cycles, or

(ii) $Y$ is a free product of $d/r$ copies of $Z_{q/r}$, and a free group of rank $f$, where $2f = d(r - 2)/r + 2$, $S_d$ has regular subgroup $\langle \alpha, \beta \rangle$ of order $d$, where $\alpha$ is a product of 2-cycles and $\beta$ is a product of $r$-cycles. Note that $d$ is determined by $f, q$ and $r$.

10. CONSTRUCTION OF ALL MAPS ON COMPACT ORIENTABLE SURFACES

10.1. Known results. A map $\mathcal{M}$ on a compact orientable surface $X$ is an embedding of a finite connected graph $\mathcal{G}$ in $X$ such that the connected components (faces of $\mathcal{M}$) of $X \setminus \mathcal{G}$ are simply connected. An edge $e$ of $\mathcal{M}$ is called a segment if $e$ has two vertices. An edge homeomorphic to the circle one vertex is called a free edge. Segments and loops are also called non-free edges. The darts of $\mathcal{M}$ are directed edges. Each non-free edge gives two darts whereas a free edge gives only one dart (see pp. 276 of [JS]). The two darts associated with a non-free edge $e$ travel along $e$ in opposite directions and the only dart associated with a free edge $e$ (with vertex $v$) is the directed edge that points towards $v$. Denoted by $\Omega$ the set of darts of $\mathcal{M}$. $\mathcal{M}$ can be characterised completely by two permutations $r_1$ and $r_2$ on $\Omega$. $r_1$ is the permutation that (i) fixes the darts associated with the free edges, and (ii) transposes the two darts associated with the non-free edges. $r_2$ is the permutation whose cycles correspond to the faces of $\mathcal{M}$ (following the orientation of $X$). Consequently, the map $\mathcal{M}$ can be represented by

$$(G, \Omega, r_1, r_2),$$

where $G = \langle r_1, r_2 \rangle$. Suppose that the order of $r_2$ is $q$. Then $\theta : G_q \to G = \langle r_1, r_2 \rangle \subseteq S_\Omega$ defined by $\theta(S) = r_1$ and $\theta(R) = r_2$ is a homomorphism. Since the graph $\mathcal{G}$ is connected, $\theta$ induces a transitive action of $G_q$ on $\Omega$ via $A(w) = \theta(A)(w)$, for all $A \in G_q$, $w \in \Omega$. Let $X$ be a one point stabiliser of the action. It is clear that the action of $G_q$ on $G_q/X$ is isomorphic to the action of $G$ on $\Omega$ and that $(G, \Omega, r_1, r_2)$ is isomorphic to $(G_q/X_0, G_q/X, S, R)$, where $X_0 = \cap_{g \in G_q} \varrho g X g^{-1}$ (see pp. 283 of [JS]). Equivalently,

$$\mathcal{M} \cong (G, \Omega, r_1, r_2) \cong (G_q/X_0, G_q/X, S, R).$$

$(10.2)$

$X$ is known as a map subgroup of $\mathcal{M}$. Since the action is transitive, the map subgroups are conjugate to each other. See pp.63 of [CS] for more detail.

10.2. Maps associated with subgroups of $G_q$. Let $M_X = (P, I_X) = X \setminus \mathbb{H}$ be a special polygon of $X \subseteq G_q$ given as in Section 4. Define $M(X)$ as follows.

(i) $V$ set of vertices of $M(X)$ = the set of equivalence classes of cusps of $M_X$.

(ii) $E$ = set of edges of $M(X)$ = the set of equivalence classes of even lines of $M_X$.

(iii) $F$ = set of faces of $M(X)$ = the set of clusters and $q$-gons of $M_X$.

As the faces of $M(X)$ are simply connected, $M(X) = (V, E, F)$ is a map.

**Lemma 10.1.** Suppose that $[G_q : X] < \infty$. Then $M(X) \cong (G_X, \Omega_X, f(S), f(R))$. 


Proof. Let $E_1$ and $E_2$ be given as in Discussion 7.3. Let $e \in E_1$. Then $e$ is not paired with itself by elements of order 2. Hence $e$ is either a loop or a segment. Consequently, $e$ is a non-free edge. In the case $e \in E_2$, $e$ is paired with itself by an element of order 2. Hence $e$ is homeomorphic to $[0, 1]$ with only one vertex. It follows that $e$ is a free edge. Note that every non-free edge $e$ is shared by two special triangles and every free edge $e$ belongs to exactly one special triangle. This allows us to construct a one to one correspondence $\nu$ between the set of special triangles $\Omega_X$ and the set of darts $\Omega$ such that $\nu(f(S)) = r_1(\nu(g_1))$ and that $\nu(f(R)) = r_2(\nu(g_2))$. Hence the following is a commutative diagram.

\[
\begin{array}{ccc}
\Omega_X \times G_X & \longrightarrow & \Omega_X \\
\downarrow \nu & & \downarrow \nu \\
\Omega \times G & \longrightarrow & \Omega 
\end{array}
\] (10.3)

where the horizontal arrows represent the group actions and $\chi(f(S)) = r_1$, $\chi(f(R)) = r_2$.

As a consequence, $M(X) \cong (G, \Omega, r_1, r_2) \cong (G_X, \Omega_X, f(S), f(R))$. \hfill $\square$

Discussion 10.2. Let $\mathcal{M}$ be a map. One knows very little about the map subgroups of $\mathcal{M}$ except that they are the one point stabilisers. Conversely, for each subgroup $X$ of $G_q$, one cannot really visualise $(G_q/X_0, G_q/X, S, R)$ as a map as it is not very easy to describe the incidence relations of this map. The following proposition implies that every map (with $o(r_2) = q$) takes the form $M(X)$, where $X$ is a subgroup of finite index of $G_q$. Both the map $M(X)$ and its map subgroups can be described explicitly as $M(X)$ can be described by its special polygon and the map subgroups of $M(X)$, which are conjugates of $X$, can be described by the set of independent generators $I_X$.

Proposition 10.3. Let $\mathcal{M}$ be a map, where $o(r_2) = q$. Then $\mathcal{M} \cong M(X)$ for some $X \subseteq G_q$.

Further, the map subgroups of $M(X)$ are conjugates of $X$.

Proof. By the results in subsection 10.1, $\mathcal{M}$ is isomorphic to $(G_q/X_0, G_q/X, S, R)$. By Lemmas 7.1, 7.2, and 7.5, $(G_q/X_0, G_q/X, S, R)$ is isomorphic to $(G_X, \Omega_X, f(S), f(R))$. By Lemma 10.1, one has $\mathcal{M} \cong (G_X, \Omega_X, f(S), f(R)) \cong M(X)$. $g \in G_q$ fixes $g_1\Phi \in \Omega_X$ if and only if there exists some $x \in X$ such that $xg_1g_2 = g_1\Phi$ (see (7.2)). Since the action of $G_q$ on $\Omega_X$ is fixed point free, $g$ fixes $g_1\Phi$ if and only if $g_1^{-1}xg = 1$. This completes the proof of the proposition. \hfill $\square$

Proposition 10.4. The automorphism group of $M(X)$ is $C_{S_n}(G_X)$, where $\mathcal{M} = [G_q : X]$.

Proof. Let $\sigma \in S_n$. The incidence relations of $M(X)$ is determined by $G_X = \langle f(S), f(R) \rangle$. Hence $\sigma \in Aut(M(X)$ if and only if $[\sigma, G_X] = 1$. This completes the proof of the lemma. \hfill $\square$

11. Congruence Subgroup Problem for $\Gamma = PSL(2, \mathbb{Z})$

Let $\Gamma = G_3 = PSL(2, \mathbb{Z})$. The principal congruence subgroup of $\Gamma$ of level $r \in \mathbb{N}$ is

$$\Gamma(r) = \{ x \in \Gamma : x \equiv \pm 1 \pmod{r} \}. \quad (11.1)$$

Let $X \subseteq \Gamma$ is a congruence subgroup if $\Gamma(r) \subseteq X$ for some $r$. Consider the action of $T$ on $\Gamma/X$, the order of $T$ on $\Gamma/X$ is called the level of $X$. By a result of Wohlfahrt [W], $X \subseteq \Gamma$ is a congruence subgroup if and only if $\Gamma(n) \subseteq X$. To the best of our knowledge, the congruence test developed by Tim Hsu [H] is the most effective one in the literature (see [LLT2] for another test). His test can be implemented as long as the permutation representations of $T$ and $U$ on $\Gamma/X$ can be determined (Theorem 3.1 of [H]), where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (11.2)$$

Incidently, an algorithm that determines the permutation representations of $T$ and $U$ on $\Gamma/X$ is not included in [H]. By Lemma 7.1, such action is isomorphic to the action of $T$ and $U$ on $\Omega_X$. Following Lemmas 7.2 and 7.5, $f(T) = f(R^{-1}S)$ and $f(U) = f(RS)$
can be determined easily as long as \( X \) is given in terms of a special polygon and a set of independent generators. As a consequence, Hsu’s algorithm can be implemented with ease.

**Example 11.1.** Let

\[
M_X = \{ -\infty, 0, 1, 2, 3, \infty \}
\]

be the Hecke-Farey symbol of a subgroup \( X \) of index 11 of \( PSL(2, \mathbb{Z}) \) (see Figure 7). Then \( f(S) = (1, 1)(2, 2)(3, 3)(4, 4) \) and \( f(R) = (1, 5, 2)(2, 6, 3)(3, 7, 4)(4) \). Hence

\[
f(T) = (1, 2, 3, 4, 7, 3, 6, 2, 5, 1), \quad f(U) = (1, 5, 2, 6, 3, 7, 4, 3, 2, 1).
\]

By the algorithm given in Section 3 of [H], \( X \) is non-congruence. Note that \( \cap gXg^{-1} \cong (f(T), f(U)) \cong A_{11} \) is the alternating group on 11 letters. The group \( \cap gXg^{-1} \) was first studied by Magnus [M] as part of his study of non-congruence subgroups of \( \Gamma \).

**Appendix A**

In this appendix, \( \Omega \) is a finite set, \( S_\Omega \) is the symmetric group on \( \Omega \), and \( G \subseteq S_\Omega \).

**Lemma A1.** Suppose that \( G \) acts transitively on \( \Omega \). Then \( C_{S_\Omega}(G) \cong N_G(G_d)/G_d \), where \( G_d \) is the one point stabiliser of \( d \in \Omega \).

**Proof.** Since \( G \) is transitive, the action of \( G \) on \( \Omega \) is isomorphic to the action of \( G \) on the set of cosets \( G/G_d \). Without loss of generality, we may assume that \( \Omega = G/G_d \) and that \( x(G_d) = xG_d \) for \( x \in G \). Let \( x \in C_{S_\Omega}(G) \). Then \( x(G_d) = e_xG_d \in G/G_d \) for some \( e_x \in G \). For each \( g \in G_d \), one has \( gx(G_d) = xg(G_d) = x(G_d) \). Hence \( g \cdot e_xG_d = e_xG_d \). This implies that \( e_xG_d \in N_G(G_d)/G_d \). As a consequence, one can show that \( C_{S_\Omega}(G) \cong N_G(G_d)/G_d \) by studying the homomorphism \( \Phi : C_{S_\Omega}(G) \to N_G(G_d)/G_d \) defined by \( \Phi(x) = e_x^{-1}G_d \). Note that for each \( r \in N_G(G_d) \), the permutation defined by \( rG_d = gr^{-1}G_d \) commutes with \( G \) which implies that \( \Phi \) is surjective \( (e_x = r^{-1} \) and \( \Phi(x) = rG_d) \). \( \square \)

**References**

[CS] I. N. Cangül, D. Singerman, Normal subgroups of Hecke groups and regular maps, Math. Proc. Camb. Phil. Soc. 123, (1998), 59-74.

[G] GAP, Groups, Algorithms, Programming, A system for Computational Discrete Algebra, http://www.gap-system.org

[HR] Y. H. He, J. Read, Hecke groups, Dessins d’Enfants and the Archimedean solids, arXiv:math/1309.2326v1, [math.NT], 2013.

[H] T. Hsu, Identifying congruence subgroups of the modular group, Proc. Amer. Mat. Soc. 124, no 5 (1996), 1351-1359.

[IS] I. Ivrissimtzis, D. Singerman, Regular maps and principal congruence subgroups of Hecke groups, European J. of Comb. 26 (2005), 437-456.

[JS] G. A. Jones, D. Singerman, Theory of maps on orientable surfaces, Proc. London Math. Soc. (3) 37 (1978), 273-307.

[K1] R.S. Kulkarni, An extension of a theorem of Kurosh and applications to Fuchsian groups, Michigan Math. J. 30 (1983), 259-272.
[K2] R.S. Kulkarni, A new proof and extension of a theorem of Millington on the modular group, Bull. London Math. Soc. 17 (1985), 458-462.

[K3] R.S. Kulkarni, An arithmetic-Geometric method in the study of subgroups of the modular group, Amer. J Math. 113 (1991), 1053-1134.

[LLT1] M. L. Lang, C. H. Lim, S. P. Tan, Independent generators for congruence subgroups of Hecke groups, Math. Z. 220 (1995), 569 – 594.

[LLT2] M. L. Lang, C. H. Lim, S. P. Tan, An algorithm for determining if a subgroup of the modular group is congruence, J. of London Math. Soc. (2) 51 (1995) 491-502.

[M] W. Magnus, Non-Euclidean tessellations and their groups, Academic Press 1974.

[Ma] B. Maskit, On Poincaré’s theorem for fundamental polygons, Advances in Math. 7. (1971), 219-230.

[Mi] M. H. Millington, Subgroups of the classical modular group, J. Lon. Math. Soc. 1 (1969), 351-357.

[N] M. Newman, The structure of some subgroups of the Modular Group, Illinois J. Math. (1962), 480-487.

[W] K. Wohlfahrt, An extension of F. Klein’s level concept, Illinois J. of Math. 8 (1964), 529 – 535.

Cheng Lien Lang
Department of Mathematics, I-Shou University, Kaohsiung, Taiwan.
cilang@isu.edu.tw

Mong Lung Lang
Singapore 669608, Singapore.
lang2to46@gmail.com