The Henon-Heiles system defined on canonically deformed space-time

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Abstract

In this article we provide canonically deformed classical Henon-Heiles system. Further we demonstrate that for proper value of deformation parameter $\theta$ there appears chaos in the model.
1 Introduction

There exist a lot of papers dealing with physical models of which dynamics remains chaotic; the most popular of them are: Lorentz system [1], Henon-Heiles system [2], Rayleigh-Bernard system [3], Duffing equation [4], Double pendulum [4], [6], Forced damped pendulum [4], [6] and Quantum forced damped oscillator model [7]. The especially interesting seems to be Henon-Heiles system defined by the following Hamiltonian function

\[
H(p, x) = \frac{1}{2} \sum_{i=1}^{2} (p_i^2 + x_i^2) + x_1^2 x_2 - \frac{1}{3} x_2^3 ,
\]  

(1)

which in cartesian coordinates \(x_1\) and \(x_2\) describes the set of two nonlinearly coupled harmonic oscillators. In polar coordinates \(r\) and \(\theta\) it corresponds to the particle moving in noncentral potential of the form

\[
V(r, \varphi) = \frac{r^2}{2} + \frac{r^3}{3} \sin (3\varphi) ,
\]  

(2)

with \(x_1 = r \cos \varphi\) and \(x_2 = r \sin \varphi\). The above model has been inspired by the observational data indicating, that star moving in a weakly perturbed central potential should has apart of constant in time total energy \(E_{\text{tot}}\), the second conserved physical quantity \(I\). It has been demonstrated with use of so-called Poincare section method, that such a situation appears in the case of Henon-Heiles system only for small values of control parameter \(E_{\text{tot}}\). For high energies the trajectories in phase space become chaotic and the quantity \(I\) does not exist (see e.g. [8], [9]).

In this article we investigate the impact of the well-known (simplest) canonically deformed Galilei space-time [10]-[12] on the mentioned above Henon-Heiles system. Particularly, we provide the corresponding canonical equations of motion as well as we find the Poincare sections of the phase space trajectories of the model. In such a way we demonstrate that for proper value of deformation parameter \(\theta\) and for proper values of control parameter \(E_{\text{tot}}\) there appears chaos.

The paper is organized as follows. In second Section we briefly remaind the basic properties of classical Henon-Heiles system. In Section 3 we recall canonical noncommutative quantum Galilei space-time proposed in article [12]. Section 4 is devoted to the new canonically deformed Henon-Heiles model while the conclusions are discussed in Final remarks.

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1 The canonically noncommutative space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [10], [11]), for the canonically deformed quantum Galilei Hopf algebras \(U_\theta(G)\).

2 It should be noted that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see references [13], [14]), apart of canonical [10]-[12] space-time noncommutativity, there also exist Lie-algebraic [12]-[17] and quadratic [12], [17]-[19] type of quantum spaces.
2 Classical Henon-Heiles model

As it was already mentioned the Henon-Heiles system is defined by the following Hamiltonian function

\[ H(p, x) = \frac{1}{2} \sum_{i=1}^{2} (p_i^2 + x_i^2) + x_1^2 x_2 - \frac{1}{3} x_2^3, \]  

(4)

with canonical variables \((p_i, x_i)\) satisfying

\[ \{ x_i, x_j \} = 0 = \{ p_i, p_j \}, \quad \{ x_i, p_j \} = \delta_{ij}, \]  

(5)

i.e., it describes the system of two nonlineary coupled one-dimensional harmonic oscillator models. One can check that the corresponding canonical equations of motion take the form

\[ \dot{p}_1 = -\frac{\partial H}{\partial x_1} = -x_1 - 2x_1 x_2, \quad \dot{x}_i = \frac{\partial H}{\partial p_i} = p_i, \]  

(6)

\[ \dot{p}_2 = -\frac{\partial H}{\partial x_2} = -x_2 - x_1^2 + x_2^2, \]  

(7)

while the proper Newton equations look as follows

\[ \begin{cases} \ddot{x}_1 = -x_1 - 2x_1 x_2 \\ \ddot{x}_2 = -x_2 - x_1^2 + x_2^2 \end{cases}. \]  

(8)

Besides, it is easy to see that the conserved in time total energy of the model is given by

\[ E_{\text{tot}} = \frac{1}{2} \sum_{i=1}^{2} (\dot{x}_i^2 + \dot{x}_i^2) + x_1^2 x_2 - \frac{1}{3} x_2^3. \]  

(9)

In order to analyze the discussed system we find numerically the Poincare maps in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for five fixed values of total energy: \(E_{\text{tot}} = 0.03125, E_{\text{tot}} = 0.06125, E_{\text{tot}} = 0.10125, E_{\text{tot}} = 0.125, E_{\text{tot}} = 0.15125\) and \(E_{\text{tot}} = 0.16245\) respectively; the obtained results are summarized on Figures 1 - 6. We see that for \(E_{\text{tot}} = 0.03125\) the trajectories remain completely regular. However, for increasing values of control parameter \(E_{\text{tot}}\) they gradually become disordered until to the almost completely chaotic behavior of the system at \(E_{\text{tot}} = 0.16245\).³

³The calculations are performed for single trajectory with initial condition \(x_1(0) = (2E_{\text{tot}})^{\frac{1}{2}}\) and \(x_2(0) = p_1(0) = p_2(0) = 0.\)
3 Canonically deformed Galilei space-time

In this section we very shortly recall the basic facts associated with the (twisted) canonically deformed Galilei Hopf algebra \( U_\theta(G) \) and with the corresponding quantum space-time \([12]\). Firstly, it should be noted that in accordance with Drinfeld twist procedure \([20]\) the algebraic sector of Hopf structure \( U_\theta(G) \) remains undeformed

\[
[K_{ij}, K_{kl}] = i (\delta_{ik} K_{jl} - \delta_{jk} K_{il}) ,
\]

or (10)

\[
[K_{ij}, V_k] = i (\delta_{jk} V_i - \delta_{ik} V_j) , \quad [K_{ij}, \Pi_\rho] = i (\eta_{j\rho} \Pi_i - \eta_{i\rho} \Pi_j) , \quad (11)
\]

\[
[V_i, V_j] = [V_i, \Pi_j] = 0 , \quad [V_i, \Pi_0] = -i \Pi_i , \quad [\Pi_\rho, \Pi_\sigma] = 0 ,
\]

where \( K_{ij}, \Pi_0, \Pi_i \) and \( V_i \) can be identified with rotation, time translation, momentum and boost operators respectively. Besides, the coproducts and antipodes of such algebra take the form

\[
\Delta_\theta(\Pi_\rho) = \Delta_0(\Pi_\rho) , \quad \Delta_\theta(V_i) = \Delta_0(V_i) ,
\]

or (13)

\[
\Delta_\theta(K_{ij}) = \Delta_0(K_{ij}) - \theta^{kl}[\delta_{ki} \Pi_j - \delta_{kj} \Pi_i] \otimes \Pi_l + \Pi_k \otimes (\delta_{li} \Pi_j - \delta_{lj} \Pi_i) ,
\]

or (14)

\[
S(\Pi_\rho) = -\Pi_\rho , \quad S(K_{ij}) = -K_{ij} , \quad S(V_i) = -V_i ,
\]

while the corresponding quantum space-time can be defined as the representation space, so-called Hopf modules (see e.g. \([10], [11]\)), for the canonically deformed Hopf structure \( U_\theta(G) \); it looks as follows

\[
[ t, \hat{x}_i ] = 0 , \quad [ \hat{x}_i, \hat{x}_j ] = i \theta_{ij} ,
\]

or (16)

and for deformation parameter \( \theta \) approaching zero it becomes commutative.

4 Classical Henon-Heiles system on canonically deformed space-time

Let us now turn to the Henon-Heiles model defined on quantum space-time \((16)\). In first step of our construction we extend the canonically deformed space to the whole algebra of momentum and position operators as follows (see e.g. \([21]-[24]\)]

\[
\{ \hat{x}_1, \hat{x}_2 \} = 2\theta , \quad \{ \hat{p}_i, \hat{p}_j \} = 0 , \quad \{ \hat{x}_i, \hat{p}_j \} = \delta_{ij} .
\]

or (17)

One can check that relations \((17)\) satisfy the Jacobi identity and for deformation parameter \( \theta \) approaching zero become classical.

\[\text{The correspondence relations are } \{ \cdot, \cdot \} = \frac{1}{\theta} [\cdot, \cdot].\]
Next, by analogy to the commutative case we define the corresponding Hamiltonian function by

$$H(\hat{p}, \hat{x}) = \frac{1}{2} \sum_{i=1}^{2} (\hat{p}_i^2 + \hat{x}_i^2) + \hat{x}_1 \hat{x}_2 - \frac{1}{3} \hat{x}_3^2 ,$$  \hspace{1cm} (18)

with the noncommutative operators \((\hat{x}_i, \hat{p}_i)\) represented by the classical ones \((x_i, p_i)\) as\cite{21}-\cite{26}

\[
\begin{align*}
\hat{x}_1 & = x_1 - \theta p_2 , \\
\hat{x}_2 & = x_2 + \theta p_1 , \\
\hat{p}_1 & = p_1 , \quad \hat{p}_2 = p_2 .
\end{align*}
\]

Consequently, we have

$$H(p, x) = \frac{1}{2M(\theta)} (\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2} M(\theta) \Omega^2(\theta) (x_1^2 + x_2^2) - S(\theta) L +$$  \hspace{1cm} (23)

$$+ (x_1 - \theta p_2)^2 \cdot (x_2 + \theta p_1) - \frac{1}{3} (x_2 + \theta p_1)^3 ,$$

where

\[
\begin{align*}
L & = x_1 p_2 - x_2 p_1 , \\
1/M(\theta) & = 1 + \theta^2 , \\
\Omega(\theta) & = \sqrt{(1 + \theta^2)} ,
\end{align*}
\]

and

$$S(\theta) = \theta .$$  \hspace{1cm} (27)

Further, using the formula \cite{23} one gets the following canonical Hamiltonian equations of motion

\[
\begin{align*}
\dot{x}_1 & = \frac{1}{M(\theta)} p_1 + S(\theta) x_2 + [(x_1 - \theta p_2)^2 - (x_2 + \theta p_1)^2] \theta , \\
\dot{x}_2 & = \frac{1}{M(\theta)} p_2 - S(\theta) x_1 - 2 (x_2 + \theta p_1) (x_1 - \theta p_2) \theta , \\
\dot{p}_1 & = -M(\theta) \Omega^2(\theta) x_2 + S(\theta) p_2 - 2 (x_2 + \theta p_1) (x_1 - \theta p_2) , \\
\dot{p}_2 & = -M(\theta) \Omega^2(\theta) x_1 + S(\theta) p_1 - (x_1 - \theta p_2)^2 + (x_2 + \theta p_2)^2 ,
\end{align*}
\]

\footnote{Such a construction of deformed Hamiltonian function (by replacing the commutative variables \((x_i, p_i)\) by noncommutative ones \((\hat{x}_i, \hat{p}_i)\)) is well-known in the literature - see e.g. \cite{21}, \cite{22} and \cite{23}.}
which for deformation parameter running to zero become classical.

Similarly to the undeformed case we find numerically the Poincare maps in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\). However, this time apart of parameter \(E_{\text{tot}}\) we take under consideration the parameter of deformation \(\theta\). Consequently, we derive the Poincare sections of phase space parameterized by pair \((E_{\text{tot}}, \theta)\) for \(\theta = 0.5, 1, 2\) and six values of total energy \(E_{\text{tot}}\). In such a way we detect chaos in the model only for \(\theta = 0.5\) and for \(E_{\text{tot}} = 0.160178, E_{\text{tot}} = 0.1607445\) and \(E_{\text{tot}} = 0.16245\) respectively (see for chaotic scenario Figures 7 - 12). In the case \(\theta = 1\) as well as \(\theta = 2\) the system remains ordered.

5 Final remarks

In this article we provide the canonically deformed Henon-Heiles system, i.e., we define the proper Hamiltonian function as well as we derive the corresponding equations of motion. We also demonstrate (with use of the Poincare section method) that for deformation parameter \(\theta = 0.5\) and for particular values of control parameter \(E_{\text{tot}}\) the analyzed model becomes chaotic.

As a next step of presented here investigations one can consider the canonical deformation of so-called generalized Henon-Heiles systems given by the following Hamiltonian function

\[
H(p, x) = \frac{1}{2} \left( p_1^2 + p_2^2 \right) + \delta x_1^2 + (\delta + \Omega) x_2^2 + \alpha x_1^2 x_2 + \alpha \beta x_2^3 ,
\]

with arbitrary coefficients \(\alpha, \beta, \delta\) and \(\Omega\) respectively. It should be noted that the properties of commutative models described by function (32) are quite interesting. For example, it is well-known (see e.g. [27]-[30] and references therein) that such systems remain integrable only in the Sawada-Kotera case: with \(\beta = 1/3\) and \(\Omega = 0\), in the KdV case: with \(\beta = 2\) and arbitrary \(\Omega\) as well as in the Kaup-Kupershmidt case: with \(\beta = 16/3\) and \(\Omega = 15\delta\). Besides there has been provided in articles [31] and [32] the different types of integrable perturbations of mentioned above (integrable) models such as, for example, \(q^{-2}\) perturbations, the Ramani series of polynomial deformations and the rational perturbations. Consequently the impact of the canonical deformation (3) on the above dynamical structures (in fact) seems to be very interesting. For this reason the works in this direction already started and are in progress.

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6As in the undeformed case the calculations are performed for single trajectory with initial condition \(x_1(0) = (2E_{\text{tot}})^{\frac{1}{2}}\) and \(x_2(0) = p_1(0) = p_2(0) = 0\).

7See also references therein.
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Figure 1: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.03125\). Trajectory is completely regular - there is no chaos in the system.
Figure 2: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for fixed value of total energy \(E_{\text{tot}} = 0.06125\). Trajectory is still regular - the system is chaos free.
Figure 3: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.10125\). Trajectory remains still regular.
Figure 4: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.125\). The system becomes mixed: chaotic and ordered simultaneously.
Figure 5: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.15125\). The system becomes chaotic.
Figure 6: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.16245\). The chaos increases.
Figure 7: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.15125\). Trajectory is completely regular - there is no chaos in the system.
Figure 8: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for fixed value of total energy \(E_{\text{tot}} = 0.1568\). Trajectory is still regular - the system is chaos free.
Figure 9: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.1596125\). Trajectory remains still regular.
Figure 10: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.160178\). The system becomes suddenly chaotic.
Figure 11: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.1607445\). The chaos increases.
Figure 12: Poincare map in two dimensional phase space \((x_2, p_2)\) for section \(x_1 = 0\) and for total energy \(E_{\text{tot}} = 0.16245\). The system becomes totally chaotic.