Admissible Transformations and Normalized Classes of Nonlinear Schrödinger Equations

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The theory of group classification of differential equations is analyzed, substantially extended and enhanced based on the new notions of conditional equivalence group and normalized class of differential equations. Effective new techniques are proposed. Using these, we exhaustively describe admissible point transformations in classes of nonlinear (1 + 1)-dimensional Schrödinger equations, in particular, in the class of nonlinear (1+1)-dimensional Schrödinger equations with modular nonlinearities and potentials and some subclasses thereof. We then carry out a complete group classification in this class, representing it as a union of disjoint normalized subclasses and applying a combination of algebraic and compatibility methods. Moreover, we introduce the complete classification of (1 + 2)-dimensional cubic Schrödinger equations with potentials. The proposed approach can be applied to studying symmetry properties of a wide range of differential equations.

1 Introduction

Group classification is an efficient tool for choosing physically relevant models from parametrized classes of differential equations. In fact, invariance with respect to certain groups of transformations is a fundamental principle of many physical theories. Thus, all non-relativistic theories have to satisfy the Galilean principle of relativity, i.e., their underlying physical laws have to be invariant with respect to the Galilei group. In relativistic theories the Galilean principle is replaced by the principle of special relativity corresponding to invariance under the Poincaré group. Hence physical models are usually constrained by a priori requirements as to their symmetry properties. This naturally leads to the so-called inverse problems of group classification for systems of differential equations, which are generally formulated in the following way: Given a group of point transformations in a space of a fixed number of dependent and independent variables, describe those systems of differential equations admitting the given group as their symmetry group. The solution of such inverse problems typically involves methods of the theory of differential invariants.

Differential equations modelling real world phenomena often contain numerical or functional parameters (so-called arbitrary elements) which are determined experimentally and, therefore, are not fixed a priori. At the same time, the chosen models should be accessible to a systematic study. The symmetry approach offers a universal method for solving such nonlinear models. Moreover, a large symmetry group of a system of differential equations indicates the presence of further interesting properties of this system and paves the way for applying other more specific methods adapted to the respective concrete problem. Hence invariance properties of a model can serve for assessing its feasibility and, moreover, the strategy of choosing, among all possible ones, the model displaying the maximal symmetry group can be used to single out the ‘right’ values of the arbitrary elements in the modelling equations [50]. Within the framework of group analysis of differential equations, this strategy is called the (direct) problem of group classification. Its general formulation is the following: Given a parametrized class of systems
of differential equations in a space of a fixed number of dependent and independent variables, first determine the common symmetry group of all systems from the class and then describe those systems admitting proper extensions of this common group. This description is carried out up to the equivalence relation generated by the equivalence group of the class. (A rigorous definition of the problem of group classification for a class of system of differential equations is given in Section 2.4.) Some problems of group analysis (e.g., the selection of equations from a parametrized class which admit realizations of a fixed Lie algebra) display features of both direct and inverse group classification problems. A classical example of solving a group classification problem is Lie’s classification of second order ordinary differential equations \[41\] that allowed Lie to describe the equations from this class which can be integrated by quadratures.

The group classification in a class of differential equations is a much more complicated problem than finding the Lie symmetry group of a single system of differential equation since it usually requires the integration of a complicated overdetermined system of partial differential equations with respect to both the coefficients of the infinitesimal symmetry operators and the arbitrary elements (determining equations).

Summing up, group classification of differential equations is an important tool for real world applications and at the same time displays interesting mathematical structures in its own right.

Various methods for solving group classification problems have been developed over the past decades. The infinitesimal invariance criterion \[49, 50\] produces overdetermined systems of determining equations for these problems. These typically are analyzed with respect to compatibility and direct integrability up to the equivalence relation generated by the corresponding equivalence group. While this is the most common approach, its applicability is confined to classes of a relatively simple structure, e.g. those which have only few arbitrary elements of one or two arguments. A number of classification results obtained within the framework of this approach are collected in \[8, 20, 50\].

Another approach, more algebraic in character, focuses on the algebraic properties of the solution sets of the corresponding determining equations \[49, 50\]. Thus, for certain classes (now called normalized \[54, 55\]), the problem of group classification is reduced to the subgroup analysis of the corresponding equivalence groups.

Before the notion of normalization was defined in a rigorous and precise form, it had been used implicitly for a long time. The best known classical group classification problems such as Lie’s classifications of second order ordinary differential equations \[41\] and of second order two-dimensional linear partial differential equations \[40\] were solved essentially relying on the strong normalization of the above classes of differential equations. Similar classification techniques implicitly based on properties of normalized classes were recently applied by a number of authors to solving group classification problems of important classes of differential equations arising in physics and other sciences (see e.g. \[11, 25, 36, 37, 38, 39, 60, 59, 74, 75\]) although the underlying reason for the effectiveness of these techniques and the scope of their applicability long remained unclear. The so-called method of preliminary group classification involving subgroup analysis of equivalence groups and construction of differential invariants of the resulting inequivalent subgroups was independently developed \[2, 27\]. The question about when results of preliminary and complete group classifications coincide also remained open.

In the most advanced versions, the above approaches are combined with each other. To extend both their range of applicability and their efficiency, different notions and techniques (extended and generalized equivalence groups, additional equivalence transformations, conditional equivalence group, gauging of arbitrary elements by equivalence transformations, partition of a class to normalized subclasses etc.) were introduced \[26, 43, 54, 56, 69, 70\].

The purpose of this paper is twofold. First, we develop a systematic new approach to group classification problems which is based on the central notion of normalized classes of differential equations. To this end we build on the first presentation of this approach in \[54, 55\]. The
corresponding elements of the framework of group classification are analyzed, starting with the basic notion of classes of differential equations. A rigorous definition of sets of admissible point transformations is given. Special attention is paid to the recently proposed notion of conditional equivalence groups and its applications in different classification problems for classes of differential equations. A number of relevant notions (point-transformation image of a class, maximal conditional equivalence group, maximal normalized subclass, etc.) are introduced here for the first time.

Our second goal is to give a nontrivial example for the application of this new framework. For this purpose we choose the class $\mathcal{F}$ of $(1+1)$-dimensional nonlinear Schrödinger equations of the general form

$$i\psi_t + \psi_{xx} + F = 0,$$

where $\psi$ is a complex dependent variable of the two real independent variables $t$ and $x$ and $F = F(t, x, \psi, \psi^*, \psi_x, \psi^*_x)$ is an arbitrary smooth complex-valued function of its arguments. (The coefficient of $\psi_{xx}$ is assumed to be scaled to 1.) It is proved that this class is normalized, and a hierarchy of nested normalized subclasses of the class $\mathcal{F}$ is constructed. Then we describe the admissible point transformations in the class $\mathcal{V}$ of $(1+1)$-dimensional nonlinear Schrödinger equations with modular nonlinearities and potentials, i.e., the equations with $F = f(|\psi|)\psi + V\psi$, where $f$ is an arbitrary complex-valued nonlinearity depending only on $\rho = |\psi|, f_\rho \neq 0$, and $V$ is an arbitrary complex-valued potential depending on $t$ and $x$. Using the techniques proposed in this paper, we also carry out the complete group classification for the class $\mathcal{V}$. This encompasses and extends the results of [29, 55, 52, 59, 60] on group classification in subclasses of the class $\mathcal{V}$.

To demonstrate the effectiveness of these new techniques also in the multidimensional case, we consider the normalized class $\mathcal{C}$ of $(1+2)$-dimensional cubic Schrödinger equations with potentials of the general form

$$i\psi_t + \psi_{11} + \psi_{22} + |\psi|^2\psi + V(t, x)\psi = 0.$$

Here $\psi$ is a complex dependent variable of the real independent variables $t$ and $x = (x_1, x_2)$ and $V$ is an arbitrary complex-valued potential depending on $t$ and $x$. The problem of group classification is solved for the class $\mathcal{C}$ by relying on the normalization property of this class and its subclasses.

Our paper is organized as follows. The first substantial part of this work (Sections 2 and 3) is devoted to the theoretical foundations of symmetry analysis in classes of (systems of) differential equations in general. First of all, in Section 2 we analyze notions and objects pertinent to the framework of group classification (classes of differential equations and their properties, admissible transformations, different kinds of equivalence and symmetry groups, etc.). The classical formulation of group classification problems is presented in a rigorous way and a number of possibilities for modifications and extensions are indicated. New notions associated with classes of differential equations naturally arise under these considerations, in particular as analogues of corresponding notions for single systems of differential equations. Thus, similarity of equations is extended to similarity and point-transformation imaging of classes of equations. The notion of conditional symmetry groups is a motivation for introducing the notion of conditional equivalence groups. Normalized classes of differential equations are studied in Section 3. We give rigorous definitions for (several variants of) normalized classes and provide examples for these notions. We also derive results which are necessary for concrete applications of normalization in group classification and for the determination of admissible transformations.

The class $\mathcal{F}$ of $(1+1)$-dimensional nonlinear Schrödinger equations of the general form (1) is the main subject of the second part of the paper (Sections 4–8). Known results on Lie symmetries of nonlinear Schrödinger equations are reviewed in Section 4. The strong normalization of the
class $\mathcal{F}$ is proved in Section 5. Ibidem we single out two important nested normalized subclasses of $\mathcal{F}$. The smaller subclass $\mathcal{S}$ still contains the class $\mathcal{V}$ of nonlinear $(1+1)$-dimensional Schrödinger equations with modular nonlinearities and potentials. That is why normalization properties of $\mathcal{S}$ are significant for the group classification in this class $\mathcal{V}$, carried out in Section 6. Since the class $\mathcal{V}$ is not normalized, for completing the classification we partition it into a family of normalized subclasses and then classify each of these subclasses separately. In order to demonstrate the effectiveness of the approach based on normalization properties in the case of more than two independent variables, in Section 7 we additionally carry out the group classification of the class $\mathcal{C}$ of $(1+2)$-dimensional cubic Schrödinger equations with potentials of the general form (2). To point out possible applications, in Section 8 the results of the second part of the paper are connected with previous results from the literature.

2 Point transformations in classes of differential equations

In this section we provide the necessary background for a rigorous exposition of the notion of normalized classes of differential equations in Section 4. A number of notions of group analysis are revised. We start by analyzing the basic notion of a general class (of systems) of differential equations. Then the notion of point transformations in such classes is formalized and different types of equivalence groups are defined. The problem of group classification in a class of differential equations as well as the problem of classification of admissible transformations are rigorously stated. This forms the basis for the development of group analysis methods based on the notion of normalized classes of differential equations and for the further investigation of classes of nonlinear Schrödinger equations.

The following conventions will be active throughout the paper: all transformations are assumed to act from the left-hand side. The term ‘point transformation group’ is used as an abbreviation for ‘local (pseudo)group of locally defined point transformations’. We do not explicitly indicate mapping domains and suppose that all functions are sufficiently smooth (as a rule, real analytic) on suitable open sets.

2.1 Classes of systems of differential equations

Let $\mathcal{L}_{\theta}$ be a system $L(x, u(p), \theta(x, u(p))) = 0$ of $l$ differential equations for $m$ unknown functions $u = (u^1, \ldots, u^m)$ of $n$ independent variables $x = (x_1, \ldots, x_n)$. Here $u(p)$ denotes the set of all the derivatives of $u$ with respect to $x$ of order not greater than $p$, including $u$ as the derivatives of order zero. $L = (L_1, \ldots, L_l)$ is a tuple of $l$ fixed functions depending on $x$, $u(p)$ and $\theta$. $\theta$ denotes the tuple of arbitrary (parametric) functions $\theta(x, u(p)) = (\theta^1(x, u(p)), \ldots, \theta^k(x, u(p)))$ running through the set $\mathcal{S}$ of solutions of the auxiliary system $S(x, u(p), \theta(q)(x, u(p))) = 0$. This system consists of differential equations with respect to $\theta$, where $x$ and $u(p)$ play the role of independent variables and $\theta(q)$ stands for the set of all the partial derivatives of $\theta$ of order not greater than $q$ with respect to the variables $x$ and $u(p)$. Often the set $\mathcal{S}$ is additionally constrained by the non-vanish condition $\Sigma(x, u(p), \theta(q)(x, u(p))) \neq 0$ with another tuple $\Sigma$ of differential functions. (This inequality means that no components of $\Sigma$ equal 0. For simplicity the tuple $\Sigma$ can be replaced by a single differential function coinciding with the product of its components.) In what follows we call the functions $\theta$ arbitrary elements. Also, we denote the class of systems $\mathcal{L}_{\theta}$ with the arbitrary elements $\theta$ running through $\mathcal{S}$ as $\mathcal{L}_{\theta}|_{\mathcal{S}}$.

Let $\mathcal{L}_{\theta}^i$ denote the set of all algebraically independent differential consequences of $\mathcal{L}_{\theta}$, which have, as differential equations, orders not greater than $i$. We identify $\mathcal{L}_{\theta}^i$ with the manifold determined by $\mathcal{L}_{\theta}^i$ in the jet space $J^{(i)}$. In particular, $\mathcal{L}_{\theta}$ is identified with the manifold determined by $\mathcal{L}_{\theta}^0$ in $J^{(0)}$. Then $\mathcal{L}_{\theta}|_{\mathcal{S}}$ can be interpreted as a family of manifolds in $J^{(p)}$, parametrized with the arbitrary elements $\theta \in \mathcal{S}$.
It should be noted that the above definition of a class of systems of differential equations is not complete. The problem is that the correspondence \( \theta \to L_\theta \) between arbitrary elements and systems (treated not as formal algebraic expressions but as real systems of differential equations or manifolds in \( J^{(p)} \)) may not be one-to-one. Namely, the same system may correspond to different values of arbitrary elements. A reason for this indeterminacy is that different values \( \theta \) and \( \tilde{\theta} \) of arbitrary elements can result after substitution into \( L \) in the same expression in \( x \) and \( u(p) \). Moreover, it is enough for \( L^p_\theta \) and \( L^p_{\tilde{\theta}} \) to coincide if the associated systems completed with independent differential consequences differ from one another by a nonsingular matrix being a function in the variables of \( J^{(p)} \).

The values \( \theta \) and \( \tilde{\theta} \) of arbitrary elements are called gauge-equivalent \( (\theta \sim \tilde{\theta}) \) if \( L_\theta \) and \( L_{\tilde{\theta}} \) are the same system of differential equations. For the correspondence \( \theta \to L_\theta \) to be one-to-one, the set \( S \) of arbitrary elements should be factorized with respect to the gauge equivalence relation. We formally consider \( L_\theta \) and \( L_{\tilde{\theta}} \) as different representations of the same system from \( L|_S \). It is often possible to realize gauge informally via changing the chosen representation of the class under consideration (changing the number \( k \) of arbitrary elements and the differential functions \( L \) and \( S \) although this may result in more complicated calculations).

Definition 1. The classes \( L|_S \) and \( L'|_{S'} \) are called similar if \( n = n' \), \( m = m' \), \( p = p' \), \( k = k' \) and there exists a point transformation \( \Psi: (x, u(p), \theta) \to (x', u'(p), \theta') \) which is projectable on the space of \( (x, u(q)) \) for any \( 0 \leq q \leq p \), and \( \Psi|_{(x, u(q))} \) being the \( q \)-th order prolongation of \( \Psi|_{(x, u)} \), \( \Psi S = S' \) and \( \Psi|_{(x, u(p))} L_\theta = L'|_{\tilde{\theta}} \) for any \( \theta \in S \).

Here and in what follows the action of such a point transformation \( \Psi \) in the space of \( (x, u(p), \theta) \) on arbitrary elements from \( S \) as \( p \)-th order differential functions is given by the formula:

\[
\tilde{\theta} = \Psi \theta \quad \text{if} \quad \tilde{\theta}(x, u(p)) = \Psi \theta \left( \Theta(x, u(p)), \theta(\Theta(x, u(p))) \right),
\]

where \( \Theta = (\text{pr}_p \Psi|_{(x, u)})^{-1} \) and \( \text{pr}_p \) denotes the operation of standard prolongation of a point transformation to the derivatives of orders not greater than \( p \).

Definition 1 is an extension of the notion of similar differential equations \([50]\) to classes of such equations. Moreover, similar classes consist of similar equations with the same similarity transformation.

The set of transformations used in Definition 1 can be extended via admitting different kinds of dependence on arbitrary elements as in the case of equivalence groups below. As a rule, similar classes of systems have similar properties from the group analysis point of view. In the case of point similarity transformations, the properties really are the same up to similarity. If \( \Psi \) is a point transformation in the space of \( (x, u(p), \theta) \) then these classes practically have the same transformational properties.

If the transformation \( \Psi \) is identical with respect to \( x \) and \( u \) then \( L'|_{\tilde{\theta}} = L_\theta \) for any \( \theta \in S \), i.e., in fact the classes \( L|_S \) and \( L'|_{S'} \) coincide as sets of manifolds in a jet space. We will say that the class \( L'|_{S'} \) is a re-parametrization of the class \( L|_S \), associated with the re-parametrizing transformation \( \Psi \). In the most general approach, \( \Psi \) can be assumed an arbitrary one-to-one mapping from \( S \) to \( S' \), satisfying the condition \( L'|_{\tilde{\theta}} = L_\theta \) for any \( \tilde{\theta} \in S' \). Note that the number of arbitrary elements in \( S' \) might not coincide with the one in \( S \). Transformational properties may be broken under generalized re-parametrizations.

An example of non-point re-parametrization often applied in group analysis is given by the classes \( \{ I = \theta(I', J') \} \) and \( \{ I' = \theta'(I, J) \} \), where \( I \) and \( I' \) (resp. \( J \)) are \( k \)-tuples (resp. an \( l \)-tuple) of fixed functionally independent expressions of \( x \) and \( u(p) \), and \( \theta \) and \( \theta' \) are arbitrary \((k+l)\)-ary \( k \)-vector functions having nonzero Jacobians with respect first \( k \) arguments, \( k, l \in \mathbb{N} \). The corresponding mapping between the sets of arbitrary elements is to take the inverse function to each set element. Fortunately, such re-parametrization preserves transformational properties of classes well.
Similarity of classes implies a one-to-one correspondence between the associated sets of arbitrary elements. If this feature is neglected, we arrive at the more general notion of mapping between classes of differential equations.

**Definition 2.** The class $L'|_{S'}$ is called a point-transformation image of the class $L|_S$ if $n = n'$, $m = m'$, $p = p'$, $k = k'$ and there exists a family of point transformations $\varphi_\theta: (x, u) \to (x', u')$ parametrized by $\theta \in S$ and satisfying the following conditions. For any $\theta \in S$ there exists $\theta' \in S'$ and, conversely, for any $\theta' \in S'$ there exists $\theta \in S$ such that $pr_p \varphi_\theta L_\theta = L'_\theta$.

A point-transformation image inherits certain transformational properties from its class-preimage. There is also a converse connection. For example, equations from the class-preimage are point-transformation equivalent iff their images are.

Subclasses are singled out in the class $L|_S$ with additional auxiliary systems of equations and/or non-vanish conditions which are attached to the main auxiliary system. Thus, the complement $S'\setminus S''$ is a subclass of $S'$ if the additional system of equations or the additional system of inequalities is empty. If $S'$ is defined by the inequalities $\Sigma'_1 \neq 0, \ldots, \Sigma'_i \neq 0$ then the additional auxiliary condition for $S'$ is $\Sigma'_1 \cdots \Sigma'_i = 0$.

The situation concerning complements, unions and differences of subclasses of $L|_S$ is more complicated. They will be subclasses of $L|_S$ in the above defined sense only under special restrictions on the additional auxiliary conditions. These difficulties arise from the simultaneous consideration of equations and inequalities as subclass constraints.

Thus, the complement $\overline{L|_S} = L|_{S'}$ of the subclass $L|_{S'}$ in the class $L|_S$ also is a subclass of $L|_S$ if the additional system of equations or the additional system of inequalities is empty. If $S'$ is determined by the system $S'_1 = 0, \ldots, S'_i = 0$ then the additional auxiliary condition for $\overline{S'}$ is $|S'_1|^2 + \cdots + |S'_i|^2 \neq 0$. If $S'$ is defined by the inequalities $\Sigma'_1 \neq 0, \ldots, \Sigma'_i \neq 0$ then the additional auxiliary condition for $\overline{S'}$ is $\Sigma'_1 \cdots \Sigma'_i = 0$.

The union $L|_{S'} \cup L|_{S''} = L|_{S' \cup S''}$ and the difference $L|_{S'} \setminus L|_{S''} = L|_{S' \setminus S''}$ of the subclasses $L|_{S'}$ and $L|_{S''}$ in the class $L|_S$ also are subclasses of $L|_S$ if the additional systems of equations or the additional system of inequalities of these subclasses coincide. The corresponding additional auxiliary conditions are respectively constructed from the additional auxiliary conditions for $S'$ ($S'_1 = 0, \ldots, S'_i = 0, \Sigma'_i \neq 0, \ldots, \Sigma'_i \neq 0$) and $S''$ ($S''_1 = 0, \ldots, S''_i = 0, \Sigma''_i \neq 0, \ldots, \Sigma''_i \neq 0$) in the following way (the coinciding part is preserved):

$$S' \cup S'': |\Sigma'_1 \cdots \Sigma'_i|^2 + |\Sigma''_1 \cdots \Sigma''_i|^2 \neq 0 \text{ resp. } S'_iS''_j = 0, \ i = 1, \ldots, s', \ j = 1, \ldots, s''$$

$$S' \setminus S'': \Sigma'_1 \cdots \Sigma'_i = 0, \Sigma''_1 \cdots \Sigma''_i = 0 \text{ resp. } S' = 0, |S'_1|^2 + \cdots + |S'_i|^2 \neq 0.$$
Definition 3. \( T(L|_S) = \{ (\theta, \tilde{\theta}, \varphi) \mid \theta, \tilde{\theta} \in S, \varphi \in T(\theta, \tilde{\theta}) \} \) is called the set of admissible transformations in \( L|_S \).

Note 1. The set of admissible transformations was first described by Kingston and Sophocleous for a class of generalized Burgers equations [32]. These authors call transformations of such type form-preserving [32] [33] [34]. The notion of admissible transformations can be considered as a formalization of their approach.

Note 2. Notions and results obtained in this and the following sections can be reformulated in infinitesimal terms by using the notions of vector fields, Lie algebras instead of point transformations, Lie groups etc. For instance, see [9] for the definition of “cones of tangent equivalences”, which is the infinitesimal analogue of the definition of \( T(\theta, L|_S) \). Ibidem a non-trivial example of semi-normalized classes of differential equations (see Definition 10) is investigated in the framework of the infinitesimal approach.

Note 3. In the case of one dependent variable \((m = 1)\) we can extend the previous and the subsequent notions to contact transformations.

An element \((\theta, \tilde{\theta}, \varphi)\) from \( T(L|_S) \) is called a gauge admissible transformation in \( L|_S \) if \( \theta \not\sim \tilde{\theta} \) and \( \varphi \) is the identical transformation.

Proposition 1. Similar classes have similar sets of admissible transformations. Namely, a similarity transformation \( \Psi \) from the class \( L|_S \) into the class \( L'|_S' \) generates a one-to-one mapping \( \Psi^T \) from \( T(L|_S) \) into \( T(L'|_S') \) via the rule \((\theta', \tilde{\theta}', \varphi') = \Psi^T(\theta, \tilde{\theta}, \varphi) \) if \( \theta' = \Psi \theta, \tilde{\theta}' = \Psi \tilde{\theta} \) and \( \varphi' = \Psi|_{(x,u)} \circ \varphi \circ \Psi|_{(x,u)}^{-1} \). Here \((\theta, \tilde{\theta}, \varphi) \in T(L|_S), (\theta', \tilde{\theta}', \varphi') \in T(L'|_S') \).

Proposition 2. A point-transformation mapping between classes of differential equations induces a mapping between the corresponding sets of admissible transformations. Namely, if the class \( L'|_S' \) is the point-transformation image of the class \( L|_S \) under the family of point transformations \( \varphi_\theta : (x, u) \rightarrow (x', u') \), \( \theta \in S \), then the image of any \((\theta, \tilde{\theta}, \varphi) \in T(L|_S)\) is \((\theta', \tilde{\theta}', \varphi') \in T(L'|_S')\), where \( L'_{\theta'} = \text{pr}_{\theta'} \varphi_\theta L\theta, L'_{\tilde{\theta}'} = \text{pr}_{\tilde{\theta}'} \varphi_\tilde{\theta} L\tilde{\theta} \) and \( \varphi' = \varphi_{\tilde{\theta}} \circ \varphi \circ (\varphi_\theta)^{-1} \).

Moreover, the set of admissible transformations of the initial class \( L|_S \) is reconstructed from the one of its point-transformation image \( L'|_S' \). Indeed, let \((\theta', \tilde{\theta}', \varphi') \in T(L'|_S')\) and let \( L\theta \) and \( L_{\tilde{\theta}} \) be some equations mapped to \( L'_{\theta'} \) and \( L'_{\tilde{\theta}'} \), respectively. Then \((\theta, \tilde{\theta}, \varphi) \in T(L|_S)\), where \( \varphi = (\varphi_{\tilde{\theta}})^{-1} \circ \varphi \circ \varphi_\theta \). Each admissible transformation of \( L|_S \) is obtainable in the above way.

Proposition 3. \( T(L|_S) \subset T(L|_{S'}) \) for any subclass \( L|_{S'} \) of the class \( L|_S \). If \( L|_{S''} \) is another subclass of \( L|_S \) then \( T(L|_{S'}) \cap T(L|_{S''}) = T(L|_{S' \cap S''}) \).

A number of notions connected with admissible transformations in classes of differential equations can be reformulated in terms of category theory [63]. Note that in addition to admissible transformations, the categories of parabolic partial differential equations constructed in [63] also include reduction mappings between equations with different numbers of independent variables.

2.3 Equivalence groups

Equivalence groups of different kinds, acting on classes of differential equations, are defined in a rigorous way via the notion of admissible transformations.

Thus, any element \( \Phi \) from the usual equivalence group \( G^\sim = G^\sim(L|_S) \) of the class \( L|_S \) is a point transformation in the space of \((x, u(p), \theta)\), which is projectable on the space of \((x, u(p'))\) for any \( 0 \leq p' \leq p \), so that \( \Phi|_{(x,u(p'))} \) is the \( p' \)-th order prolongation of \( \Phi|_{(x,u)} \), \( \forall \theta \in S; \Phi\theta \in S \), and \( \Phi|_{(x,u)} \in T(\theta, \Phi\theta) \). The admissible transformations of the form \((\theta, \Phi\theta, \Phi|_{(x,u)})\), where \( \theta \in S \) and \( \Phi \in G^\sim \), are called induced by the transformations from the equivalence group \( G^\sim \).
Let us recall that a point transformation \( \varphi: \tilde{z} = \varphi(z) \) in the space of the variables \( z = (z_1, \ldots, z_k) \) is called projectable on the space of the variables \( z' = (z_{i_1}, \ldots, z_{i_p}) \), where \( 1 \leq i_1 < \cdots < i_{k'} \leq k \), if the expressions for \( z' \) depend only on \( z' \). We denote the restriction of \( \varphi \) to the \( z' \)-space as \( \varphi|_{z'}: \tilde{z'} = \varphi|_{z'}(z') \).

If the arbitrary elements \( \theta \) explicitly depend on \( x \) and \( u \) only (one can always achieve this formally, introducing derivatives as new dependent variables), we can admit dependence of transformations of \( (x,u) \) on \( \theta \) and consider the generalized equivalence group \( G_{\text{gen}} = G_{\text{gen}}(\mathcal{L}|_{\mathcal{S}}) \). Any element \( \Phi \) from \( G_{\text{gen}} \) is a point transformation in \( (x,u,\theta) \)-space such that \( \forall \theta \in \mathcal{S}: \Phi \theta \in \mathcal{S} \) and \( \Phi(\cdot,\cdot,\theta(\cdot,\cdot))(x,u) \in T(\theta,\Phi \theta) \).

The action of \( \Phi \in G_{\text{gen}} \) on arbitrary elements as functions of \( (x,u) \) is given by the formula:

\[
\tilde{\theta} = \Phi \theta \text{ if } \tilde{\theta}(x,u) = \Phi^\theta(\Theta(x,u),\theta(\Theta(x,u))) \text{, where } \Theta = (\Phi(\cdot,\cdot,\theta(\cdot,\cdot)|_{(x,u)}))^{-1}.
\]

Roughly speaking, \( G_{\text{gen}} \) is the set of admissible transformations which can be applied to any \( \theta \in \mathcal{S} \) and \( G_{\text{gen}} \) is formed by the admissible transformations which can be separated to classes parametrized with \( \theta \) running through \( \mathcal{S} \).

It is possible to consider other generalizations of equivalence groups, e.g. groups with transformations which are point transformations with respect to independent and dependent variables and include nonlocal expressions with arbitrary elements \([30,69]\). Let us give the definitions of some generalizations.

**Definition 4.** The extended equivalence group \( \tilde{G}_{\text{gen}} = \tilde{G}_{\text{gen}}(\mathcal{L}|_{\mathcal{S}}) \) of the class \( \mathcal{L}|_{\mathcal{S}} \) is formed by the transformations each of which is represented by the pair \( \Phi = (\hat{\Phi}, \tilde{\Phi}) \). Here \( \hat{\Phi} \) is a one-to-one mapping in the space of arbitrary elements assumed as functions of \( (x,u_{(p)}) \) and \( \tilde{\Phi} = \Phi|_{(x,u)} \) is a point transformation of \( (x,u) \) belonging to \( T(\theta,\tilde{\Phi} \theta) \) for any \( \theta \) from \( \mathcal{S} \).

**Definition 5.** The extended generalized equivalence group \( \tilde{G}_{\text{gen}} = \tilde{G}_{\text{gen}}(\mathcal{L}|_{\mathcal{S}}) \) of the class \( \mathcal{L}|_{\mathcal{S}} \) consists of transformations each of which is represented by the tuple \( \Phi = (\hat{\Phi}, \{	ilde{\Phi}^\theta, \theta \in \mathcal{S}\}) \). Here \( \hat{\Phi} \) is a one-to-one mapping in the space of arbitrary elements assumed as functions of \( (x,u_{(p)}) \) and for any \( \theta \) from \( \mathcal{S} \) the element \( \tilde{\Phi}^\theta = \Phi|_{(x,u)}^\theta \) is a point transformation of \( (x,u) \) belonging to \( T(\theta,\tilde{\Phi} \theta) \).

The individual classes of transformations with respect to arbitrary elements should be specified depending on the investigated classes of systems of differential equations. Usually transformed arbitrary elements are smooth functions of independent variables, derivatives of dependent variables and arbitrary elements and integrals of expressions including arbitrary elements. Whenever possible we do not specify a fixed type of equivalence group, indicating that any of the above notions is applicable.

Similar classes of systems of differential equations have similar equivalence groups. More precisely, if classes are similar with respect to a transformation of a certain kind (e.g., a point transformation in the independent variables, the dependent variables, their derivatives and the arbitrary elements) then equivalence groups formed by the equivalence transformations of the same kind are similar with respect to this transformation.

The equivalence group generates an equivalence relation on the set of admissible transformations. Namely, the admissible transformations \( (\theta^1, \tilde{\theta}^1, \varphi^1) \) and \( (\theta^2, \tilde{\theta}^2, \varphi^2) \) from \( T(\mathcal{L}|_{\mathcal{S}}) \) are called \( G_{\text{gen}} \)-equivalent if there exist \( \Phi \in G_{\text{gen}} \) such that \( \theta^2 = \Phi \theta^1, \tilde{\theta}^2 = \Phi \tilde{\theta}^1 \) and \( \varphi^2 = \Theta \circ \varphi^1 \circ \Theta^{-1} \), where \( \Theta = \Phi|_{(x,u)} \) (or \( \Theta = \Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)} \) in case of \( G_{\text{gen}} \)).

**2.4 Group classification problems**

Let us recall that for a fixed \( \theta \in \mathcal{S} \) the maximal local (pseudo)group of point symmetries of the system \( \mathcal{L}_\theta \) coincides with \( T(\theta,\theta) \) and is denoted by \( G_\theta \). The common part \( G_\cap = G_\cap(\mathcal{L}|_{\mathcal{S}}) = \bigcap_{\theta \in \mathcal{S}} G_\theta \) of all \( G_\theta, \theta \in \mathcal{S} \), is called the kernel of the maximal point symmetry groups of systems.
from the class $L|S$. Note that $G^{\Gamma}$ can naturally be embedded into $G^{\sim}$ via trivial (identical) prolongation of the kernel transformations to the arbitrary elements. The associated subgroup $G^{\Gamma}$ of $G^{\sim}$ is normal.

The group classification problem for the class $L|S$ is to describe all $G^{\sim}$-inequivalent values of $\theta \in S$ together with the corresponding groups $G_{\theta}$, for which $G_{\theta} \neq G^{\Gamma}$. The solution of the group classification problem is a list of pairs $(S_{\gamma}, \{G_{\theta}, \theta \in S_{\gamma}\})$, $\gamma \in \Gamma$. Here $\{S_{\gamma}, \gamma \in \Gamma\}$ is a family of subsets of $S$, $\bigcup_{\gamma \in \Gamma} S_{\gamma}$ contains only $G^{\sim}$-inequivalent values of $\theta$ with $G_{\theta} \neq G^{\Gamma}$, and for any $\theta \in S$ with $G_{\theta} \neq G^{\Gamma}$ there exists $\gamma \in \Gamma$ such that $\theta \in S_{\gamma} \bmod G^{\sim}$. The structures of the $G_{\theta}$ are similar for different values of $\theta \in S_{\gamma}$ under fixed $\gamma$. In particular, $G_{\theta}, \theta \in S_{\gamma}$, display the same arbitrariness of group parameters.

Group classification problems in the above formulation are very complicated and, in the general case, are impossible to be solved since they lead to systems of functional differential equations. That is why one usually considers only the connected component $G_{\theta}^{\Gamma}$ of unity for each $\theta$ instead of the whole group $G_{\theta}$. $G_{\theta}^{\Gamma}$ is called the principal (symmetry) group of the system $L_{\theta}$. The generators of one-parameter subgroups of $G_{\theta}^{\Gamma}$ form a Lie algebra $A_{\theta}$ of vector fields in the space of $(x,u)$, which is called the maximal Lie invariance (or principal) algebra of infinitesimal symmetry operators of $L_{\theta}$. The kernel of principal groups of the class $L|S$ is the group $G^{p\Gamma} = G^{p\Gamma}(L|S) = \bigcap_{\theta \in S} G_{\theta}^{p\Gamma}$ for which the Lie algebra is $A^{\Gamma} = A^{\Gamma}(L|S) = \bigcap_{\theta \in S} A_{\theta}$.

Any operator $Q = \xi^{i}(x,u) \partial_{x^{i}} + \eta^{a}(x,u) \partial_{u^{a}}$ from $A_{\theta}$ satisfies the infinitesimal invariance criterion \[49, 50\] for the system $L_{\theta}$

$$Q(p)L(x,u_{(p)},\theta(x,u_{(p)}))|_{L_{\theta}^{p\Gamma}} = 0,$$

i.e., the result of acting by $Q(p)$ on $L$ vanishes on the manifold $L_{\theta}^{p\Gamma}$. In what follows we employ the summation convention for repeated indices. The indices $i$ and $a$ run from 1 to $n$ and from 1 to $m$, respectively. $Q(p)$ denotes the standard $p$-th prolongation of the operator $Q$,

$$Q(p) := Q + \sum_{0<|\alpha| \leq p} \left(D^{\alpha_{1}}_{1} \ldots D^{\alpha_{n}}_{n} (\eta^{a}(x,u) - \xi^{i}(x,u) u_{i}^{a}) + \xi^{i} u_{a,i}^{a}\right) \partial_{u^{a}}.$$  

$D_{i} = \partial_{i} + u_{a,i}^{a} \partial_{u^{a}}$ is the operator of total differentiation with respect to the variable $x_{i}$. The tuple $\alpha = (\alpha_{1}, \ldots, \alpha_{n})$ is a multi-index, $\alpha_{i} \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_{1} + \cdots + \alpha_{n}$. The variable $u_{a}^{a}$ of the jet space $J^{(n)}$ is identified with the derivative $\partial^{\alpha_{1}} u_{o}/\partial x_{1}^{\alpha_{1}} \ldots \partial x_{n}^{\alpha_{n}}$ and $u_{a,i}^{a} := \partial u_{a}^{a}/\partial x_{i}$. The infinitesimal invariance criterion implies the system of determining equations on the coefficients of the operator $Q$, where arbitrary elements are involved as parameters. So, Lie symmetry extensions are connected, as a rule, with extensions of solution sets of this system.

Knowing $A_{\theta}$, one can reconstruct $G_{\theta}^{p\Gamma}$. Then in the framework of the infinitesimal approach the problem of group classification is reformulated as finding all possible inequivalent cases of extensions for $A_{\theta}$, i.e., as listing all $G^{\sim}$-inequivalent values of the arbitrary parameters $\theta$ together with $A_{\theta}$ satisfying the condition $A_{\theta} \neq A^{\Gamma}$ \[2, 50\]. More precisely, in the infinitesimal approach the solution of the group classification problem is a list of pairs $(S_{\gamma}, \{A_{\theta}, \theta \in S_{\gamma}\})$, $\gamma \in \Gamma$. Here $\{S_{\gamma}, \gamma \in \Gamma\}$ is a family of subsets of $S$, $\bigcup_{\gamma \in \Gamma} S_{\gamma}$ contains only $G^{\sim}$-inequivalent values of $\theta$ with $A_{\theta} \neq A^{\Gamma}$, and for any $\theta \in S$ with $A_{\theta} \neq A^{\Gamma}$ there exists $\gamma \in \Gamma$ such that $\theta \in S_{\gamma} \bmod G^{\sim}$. The structures of the $A_{\theta}$ are similar for different values of $\theta \in S_{\gamma}$ under fixed $\gamma$. In particular, all $A_{\theta}, \theta \in S_{\gamma}$, have the same dimension or display the same arbitrariness of parameters in the infinite-dimensional case.

The procedure of group classification can be completed by finding explicit conditions (e.g., systems of differential equations) for the arbitrary elements, providing extensions of Lie symmetry. In other words, for any $\gamma \in \Gamma$ one should explicitly describe the subset $S_{\gamma} \subset S$ that consists of the arbitrary elements which are $G^{\sim}$-equivalent to arbitrary elements from the subset $S_{\gamma}$. Although this step is usually omitted, it may lead to nontrivial results (see, e.g., \[9, 10\]).
If the class $\mathcal{L}|_S$ is not semi-normalized, i.e., if $T(\mathcal{L}|_S)$ is not generated by $G^\sim$ and the point symmetry groups of the systems involved (see Section 3.1 for precise definitions), the classification list obtained may include equations which are mutually equivalent with respect to point transformations which do not belong to $G^\sim$. Knowledge of such additional equivalences allows one to substantially simplify the further investigation of $\mathcal{L}|_S$. Their explicit construction can be considered as one further step of the algorithm of group classification [56]. To carry out this step, one can use the fact that equivalent equations have equivalent maximal invariance algebras. A more systematic way is to describe the complete set of admissible transformations.

The classical statements of group classification problems can be extended in two main directions.

The first of these comprises possible variations in equivalence relations up to which the group classification is carried out. Different kinds of equivalence groups (generalized, extended, generalized extended) may be involved. In fact, all such groups consist of point transformations with respect to the independent and dependent variables. Modifications concern only the structure of the transformations with respect to the arbitrary elements. If a class admits a generalized/extended equivalence group essentially wider than the usual one, the group classification in this class with respect to the usual equivalence group is, as a rule, too cumbersome and the classification list is far from optimal or may even be unobtainable in a closed form [30, 69].

Another possibility for the modification of a group classification problem in the same direction is to partition the class under consideration into a family of subclasses with the following properties. Each of the subclasses is formed by equations inequivalent to equations from the other subclasses and possesses an equivalence group which is not contained in the equivalence group of the whole class (the so-called nontrivial conditional equivalence group, see Section 2.6). The final classification list for the whole class will be the union of the classification lists for the subclasses. The subclasses of the partition and their equivalence groups are necessary elements in the presentation of the result obtained. It is this approach that is used in the present paper for group classification of nonlinear Schrödinger equations with potentials and modular nonlinearities (Section 6).

The above inclusion of additional equivalence transformations into the group classification framework is also a way of changing the underlying equivalence relations. After all additional equivalences are found, the classification up to $G^\sim(\mathcal{L}|_S)$-equivalence is transmuted into the classification up to the $T(\mathcal{L}|_S)$-equivalence, i.e., up to the equivalence generated by all possible point transformations. Under the special property called semi-normalization, the class $\mathcal{L}|_S$ necessarily has no additional equivalences. Then both the classifications coincide (Corollary 1 below).

All the above equivalences are generated by point transformations in independent and dependent variables. In the case of one dependent variable ($m = 1$), contact transformations can be used instead of point transformations. Potential equivalence transformations arise for some classes with two independent variables [58]. In investigating approximate symmetries, it is natural to apply approximate equivalence transformations.

The second direction for modifying group classification problems is to apply selection criteria different from admitting a Lie symmetry extension. Transformations between equations (or classes of equations) preserve a number of their properties, induce transformations between objects related to them and, therefore, generate different equivalence relations on sets of pairs of the form (equation, object) resp. (classes of equations, object). As a result, specification of such objects or properties can be used as a selection criterion for equations (or classes of equations) up to the above equivalence relations in a way similar to Lie symmetries. The range of such objects is quite wide. It includes nonclassical (conditional) symmetries [53, 62], conservation laws, associated potential systems and potential or quasi-local symmetries [2, 30, 57, 58, 61], generalized (Lie–Bäcklund) symmetries [12], contact and approximate symmetries, admissible transformations [54] etc.
It is evident that similar classes have similar lists of Lie symmetries under group classification with respect to any of the above equivalence relations. Namely, if $\Psi$ is a transformation realizing the similarity of the class $L'|S'$ to the class $L|S$ (see Definition 1) and
\[
\{(\theta, A_\theta), \theta \in S_\gamma, S_\gamma \subset S, \gamma \in \Gamma\}
\]
is a classification list for the class $L|S$ then
\[
\{(\Psi \theta, (\Psi |_{(x,u)})_* A_\theta), \Psi \theta \in \Psi S_\gamma, \Psi S_\gamma \subset S', \gamma \in \Gamma\}
\]
is a classification list for the class $L'|S'$. Here $(\Psi |_{(x,u)})_*$ is the mapping induced by the transformation $\Psi$ in the set of vector fields on the space $(x,u)$ (push-forward of vector fields).

If the class $L'|S'$ is only a point-transformation image of the class $L|S$ in the sense of Definition 2 without the above similarity then the similarity of their classifications may be broken. Only in the case of classifications with respect to the entire sets of admissible transformations there always exists a one-to-one correspondence to hold between the classification lists for the class-image and the class-preimage. For such a correspondence to hold in the case of classifications with respect to the equivalence groups, we need to additionally require that the preimages of any arbitrary element from $S'$ are $G^\sim(L|S)$-equivalent. These facts can be applied for the simplification of solving group classification problems. If one of the classes is classified in a simpler way, possessing, e.g., a set of arbitrary elements (resp. equivalence group, resp. set of admissible transformations, etc.) of a simpler structure then its group classification can be carried out first and can subsequently be used to derive the classification of the other class [70].

2.5 Gauge equivalence groups

The equivalence group $G^\sim$ of the class $L|S$ may contain transformations which act only on arbitrary elements and do not really change systems, i.e., which generate gauge admissible transformations. In general, transformations of this type can be considered as trivial [42] (gauge) equivalence transformations and form the gauge subgroup $G^{g\sim} = \{\Phi \in G^\sim | \Phi x = x, \Phi u = u, \Phi \theta \equiv \theta\}$ of the equivalence group $G^\sim$. Moreover, $G^{g\sim}$ is a normal subgroup of $G^\sim$.

The application of gauge equivalence transformations is equivalent to rewriting systems in another form. Contrary to regular equivalence transformations, their role in group classification doesn’t amount to a choice of representatives in equivalence classes but to a choice of form of these representatives. It is quite common that the gauge equivalence relation on the set of arbitrary elements of a class of differential equations is generated by its gauge equivalence group.

We use the name “gauge equivalence transformation” since there exist rather trivial equivalence transformations which do not really transform even arbitrary elements. Such transformations arise if the auxiliary system implies functional dependence of arbitrary elements. They form normal subgroups in the corresponding equivalence groups and in the corresponding gauge equivalence groups. We will neglect these transformations and assume that equivalence groups coincide if they have the same factor group with respect to the trivial equivalence subgroups.

2.6 Conditional equivalence groups

The concept of conditional equivalence arises as an extension of the notion of conditional symmetry transformations of a single system of differential equations [17] to equivalence transformations in classes of systems. It is even more natural than the concept of conditional symmetry since the description of any class includes, as a necessary element, an auxiliary system (a condition) for the arbitrary elements. Imposing additional constraints on arbitrary elements, we may single out a subclass in the class under consideration whose equivalence group is not contained in the equivalence group of the whole class. Let $L|S'$ be the subclass of the class $L|S$, which is constrained
by the additional system of equations \( S'(x, u_{(p)}, \theta_{(q')}) = 0 \) and inequalities \( \Sigma'(x, u_{(p)}, \theta_{(q')}) \neq 0 \) with respect to the arbitrary elements \( \theta = \theta(x, u_{(p)}) \). \((\Sigma' \text{ can be the 0-tuple.})\) Here \( S' \subset S \) is the set of solutions of the united system \( S = 0, \Sigma \neq 0, S' = 0, \Sigma' \neq 0 \). We assume that the united system is compatible for the subclass \( L|S' \) to be nonempty.

**Definition 6.** The equivalence group \( G^\sim(L|S) \) of the subclass \( L|S \) is called a conditional equivalence group of the whole class \( L|S \) under the conditions \( S' = 0, \Sigma' \neq 0 \). The conditional equivalence group is called nontrivial iff it is not a subgroup of \( G^\sim(L|S) \).

Conditional equivalence groups may be trivial not with respect to the equivalence group of the whole class but with respect to other conditional equivalence groups. Indeed, if \( S' \subset S'' \) and \( G^\sim(L|S') \subset G^\sim(L|S'') \) then the subclass \( L|S' \) is not interesting from the conditional symmetry point of view. Therefore, the set of additional conditions on the arbitrary elements can be reduced substantially.

**Definition 7.** The conditional equivalence group \( G^\sim(L|S) \) of the class \( L|S \) under the additional conditions \( S' = 0, \Sigma' \neq 0 \) is called maximal if for any subclass \( L|S' \) of the class \( L|S \) containing the subclass \( L|S' \) we have \( G^\sim(L|S) \nsubseteq G^\sim(L|S') \).

In other words, only maximal conditional equivalence groups are interesting. It is evident that any maximal conditional equivalence group is nontrivial. The equivalence group \( G^\sim(L|S) \) of the class \( L|S \) is its conditional equivalence group associated with the empty additional condition.

The equivalence group \( G^\sim(L|S) \) generates an equivalence relation on the set of pairs of additional auxiliary conditions and the corresponding conditional equivalence groups. Namely, if a transformation from \( G^\sim(L|S) \) transforms the system \( S' = 0, \Sigma' \neq 0 \) to the system \( S'' = 0, \Sigma'' \neq 0 \) then the conditional equivalence groups \( G^\sim(L|S') \) and \( G^\sim(L|S'') \) are similar with respect to this transformation and will be called \( G^\sim \)-equivalent. If a conditional equivalence group is maximal then any conditional equivalence group \( G^\sim \)-equivalent to it is also maximal.

Building on the concept of conditional equivalence, we can formulate the problem of describing \( T(L|S) \) analogously to the usual group classification problem. Nontrivial additional auxiliary conditions for arbitrary elements naturally arise when studying \( T(L|S) \). Typically, the following steps have to be carried out:

1. Construction of \( G^\sim(L|S) \) (or \( G^\sim_{\text{gen}}(L|S) \), etc.).
2. Description of conditional equivalence transformations in \( L|S \), i.e., searching for a complete family of \( G^\sim \)-inequivalent additional auxiliary conditions \( S_\gamma, \gamma \in \Gamma \), such that \( G^\sim(L|S_\gamma) \) is a maximal conditional equivalence group of the class \( L|S \) for any \( \gamma \in \Gamma \).
3. Finding admissible transformations which do not belong to any conditional equivalence groups.

Actually, the proposed procedure is far from optimal. We will return to this point after presenting more elaborate techniques.

### 3 Normalized classes of differential equations

Solving group classification problems is considerably simpler if the class \( L|S \) of systems of differential equations under consideration has the additional property of normalization with respect to point transformations. In addition, the investigation of \( T(L|S) \) can be deepened by considering conditional equivalence groups for subclasses possessing this property.
3.1 Definition of normalized classes of differential equations

**Definition 8.** The class $\mathcal{L}|_{\mathcal{S}}$ is called normalized if $\forall (\theta, \bar{\theta}, \varphi) \in T(\mathcal{L}|_{\mathcal{S}}) \exists \Phi \in G^{\sim}: \bar{\theta} = \Phi\theta$ and $\varphi = \Phi|_{(x,u)}$. It is called normalized in the generalized sense if $\forall (\theta, \bar{\theta}, \varphi) \in T(\mathcal{L}|_{\mathcal{S}}) \exists \Phi \in G_{gen}^{\sim}: \bar{\theta} = \Phi\theta$ and $\varphi = \Phi(\cdot, \cdot, \theta(\cdot, \cdot))|_{(x,u)}$.

**Proposition 4.** If the class $\mathcal{L}|_{\mathcal{S}}$ is normalized (in the usual or generalized sense) and its subclass $\mathcal{L}|_{\mathcal{S}'}$ is closed under the action of $G^{\sim}$ (or $G_{gen}^{\sim}$) then the subclass $\mathcal{L}|_{\mathcal{S}'}$ is normalized in the same sense.

**Definition 9.** The class $\mathcal{L}|_{\mathcal{S}}$ is called strongly normalized if it is normalized and $G^{\sim}|_{(x,u)} = \prod_{\theta \in S} G_{\theta}$. It is called strongly normalized in the generalized sense if it is normalized in the generalized sense and $\forall \theta^{0} \in S$: $G_{gen}^{\sim|_{(x,u)}} = \prod_{\theta \in S_{\theta^{0}}} G_{\theta}$, where $S_{\theta^{0}} = \{ \theta' \in S | G_{gen}^{\sim|_{(x,u)}}(\theta = \theta') \}$. The intersection of normalized subclasses of the class $\mathcal{L}|_{\mathcal{S}}$ is generated by the transformations from the equivalence group $\mathcal{G}_{0}$, which admits the maximal point symmetry group $G$. Indeed, let $\mathcal{L}|_{\mathcal{S}'}$ and $\mathcal{L}|_{\mathcal{S}''}$ be normalized subclasses of the class $\mathcal{L}|_{\mathcal{S}}$ and $G^{\sim}(\mathcal{L}|_{\mathcal{S}'}) = G^{\sim}(\mathcal{L}|_{\mathcal{S}'}) = G_{0}$. If $\Phi \in G_{0}$ then $(\theta, \bar{\theta}, \varphi) = 0$. The proof in the case of normalization in the generalized sense is analogous.

**Definition 10.** The class $\mathcal{L}|_{\mathcal{S}}$ is called semi-normalized if $\forall (\theta, \bar{\theta}, \varphi) \in T(\mathcal{L}|_{\mathcal{S}}) \exists \varphi \in G_{\theta}, \exists \Phi \in G^{\sim}: \varphi = \Phi|_{(x,u)} \circ \varphi$, i.e.,

$$T(\mathcal{L}|_{\mathcal{S}}) = \{(\theta, \Phi\theta, \Phi|_{(x,u)} \circ \varphi) | \theta \in S, \varphi \in G_{\theta}, \Phi \in G^{\sim}\}.$$ 

(Proposition 5. If the class $\mathcal{L}|_{\mathcal{S}}$ is normalized/semi-normalized (in the usual or generalized sense) and the subclass $\mathcal{L}|_{\mathcal{S}'}$ is closed under the action of $G^{\sim}$ (or $G_{gen}^{\sim}$) then the subclass $\mathcal{L}|_{\mathcal{S}'}$ is normalized/semi-normalized in the same sense.)

3.2 Examples of normalized classes

There exist a number of obvious examples of normalized classes. Thus, it is intuitively understandable that the extreme cases of classes formed by either a single system of differential equations or all systems having a fixed number of independent variables, unknown functions and differential equations with or without restriction of order are normalized. Let us demonstrate this within the framework of the above formal approach.

Consider a system $L(x, u(\varphi)) = 0$ of $l$ differential equations for $m$ unknown functions $u$ of $n$ independent variables $x$, which admits the maximal point symmetry group $G$. We assume that the tuple $\theta$ consists of a single arbitrary element denoted also as $\theta$ and $L$ depends on $\theta$.
constantly. The auxiliary system \( S \) for the arbitrary element \( \theta \) can be chosen in different ways. Here we discuss two possibilities.

The first one is to constrain \( \theta \) with a single (algebraic or differential) equation, for example, \( \theta = 0 \). Hence, \( S \) is a singleton consisting of the function identically vanishing on \( f^{(p)} \), \( T(L|S) = \{ (0, 0, \varphi) \mid \varphi \in G \} \) and \( G^\sim = \{ (\tilde{x}, \tilde{u}) = \varphi(x, u), \tilde{\theta} = F(x, u(p), \theta) \mid \varphi \in G, F(\cdot, \cdot, 0) \neq 0 \} \), i.e., in view of definition 1 the class \( L|S \) is normalized. It possesses the nonempty trivial equivalence group \( G^\text{triv} = \{ (\tilde{x}, \tilde{u}) = (x, u), \tilde{\theta} = F(x, u(p), \theta) \mid F(\cdot, \cdot, 0) \neq 0 \} \) which should be disregarded, and \( G^\sim / G^\text{triv} = \{ (\tilde{x}, \tilde{u}) = \varphi(x, u), \tilde{\theta} = \theta \mid \varphi \in G \} \).

The second possibility is to impose no constraints on \( \theta \), so \( S \) is the whole set of \( p \)-th order differential functions of \( (x, u) \), \( T(L|S) = \{ (\theta, \tilde{\theta}, \varphi) \mid \theta, \tilde{\theta} \in S, \varphi \in G \} \) and \( G^\sim = \{ (\tilde{x}, \tilde{u}(p)) = \text{pr}_p \varphi(x, u(p)), \tilde{\theta} = F(x, u(p), \theta) \mid \varphi \in G, \partial F / \partial \theta \neq 0 \} \). Therefore, the class \( L|S \) is normalized. This class provides an example of classes without one-to-one correspondence between arbitrary elements and systems of differential equations.

The class of all systems of \( l \) differential equations for \( m \) unknown functions of \( n \) independent variables, which have order not greater than \( p \), (here \( l, m, n \) and \( p \) are fixed integers) can be included within the framework of the formal approach by viewing the left hand sides of the equations themselves as arbitrary elements and taking the empty auxiliary system \( S \), i.e., \( k = l \), \( L = \emptyset \) and \( S \) is the whole set of \( l \)-tuples of functionally independent \( p \)-th order differential functions of \( (x, u) \). Then \( T(L|S) = \{ (\theta, \tilde{\theta}, \varphi) \mid \theta \in S, \tilde{\theta} = \text{pr}_p \varphi F(x, u(p), \theta) \mid \varphi \in G, \partial F / \partial \theta \neq 0 \} \) and \( G^\sim = \{ \Phi = (\varphi(x, u), F(x, u(p), \theta)) \mid \partial \varphi / \partial (x, u) \neq 0, \partial F / \partial \theta |_{\theta=0} \neq 0 \} \). This obviously shows normalization of this class.

The normalization property has been established (in explicit or implicit forms) for a number of different classes of differential equations important for applications. As examples we mention generalized Burgers equations [32], eikonal equations of space dimensions one, two and three [9, 10], (1 + 1)-dimensional general or quasi-linear evolution equations [11, 4, 33, 36, 74] and systems of such equations [61], different multi-dimensional quasi-linear parabolic equations [63], (1 + 1)-dimensional generalized nonlinear wave equations [39], different kinds of nonlinear Schrödinger equations [25, 29, 35, 55, 59, 60, 75].

### 3.3 Normalized classes and group classification problems

The notion of normalized classes naturally arises in group analysis. In an implicit form, it was often used in solving group classification problems for many classes of system of differential equations. Well-known examples include Lie’s classical classifications of second-order ordinary differential equations [41] and of second-order two-dimensional linear partial differential equations [40]. Recently similar classification methods were applied by a number of authors (see, e.g., [4, 9, 25, 37, 39, 59, 74]). In fact all these methods involve the following properties of normalized classes.

**Proposition 6.** Let the class \( L|S \) be normalized and let \( G^i \), \( i = 1, 2 \), be local groups of point transformations in the space of \( (x, u) \), for which \( S^i = \{ \theta \in S \mid G^i \theta = G^i \} \neq \emptyset \). Then \( S^1 \sim S^2 \mod G^\sim \) iff \( G^1 \sim G^2 \mod G^\sim \).

**Proposition 7.** Two systems from a semi-normalized class are transformed into one another by a point transformation iff they are equivalent with respect to the equivalence group of this class.

**Corollary 1.** In a semi-normalized class, the classifications up to the equivalence induced by action of the equivalence group and up to the general point-transformation equivalence coincide.

**Proposition 8.** Each class of systems of differential equations is semi-normalized.

The implementation of group classification in a normalized class always leads to the construction of a tree of subclasses possessing normalization properties. Let us investigate this phenomenon.
Proposition 9. Let the class $L|_S$ be normalized and suppose that a subset $S'$ of $S$ determines a subclass $L|_{S'}$ which is invariant under the action of $G^\sim(L|_S)$. Then the subclass $L|_{S'}$ is normalized (in the same sense). $G^\sim(L|_{S'})$ is a subgroup of $G^\sim(L|_S)$ which generates $T(L|_{S'})$, and which, if $L|_{S'}$ is normalized in the usual sense, coincides with $G^\sim(L|_{S'})$ up to gauge equivalence transformations in $L|_{S'}$.

Proof. $G^\sim(L|_{S'}) \supset G^\sim(L|_S)$, since for any $\Phi \in G^\sim(L|_S)$ and for any $\theta \in S'$ we have $\Phi\theta \in S'$, i.e., $(\theta, \Phi\theta, \Phi|_{(x,u)}) \in T(L|_S)$, which implies $\Phi \in G^\sim(L|_S)$. Since $T(L|_{S'}) \subset T(L|_S)$, for any $(\theta, \varphi, \psi) \in T(L|_{S'})$ there exists $\Phi \in G^\sim(L|_S)$ such that $\Phi = \Phi\theta$ and $\varphi = \Phi|_{(x,u)}$, i.e., the subclass $L|_{S'}$ is normalized. The above part of the proof is easily extended to the generalized case.

Any $\Psi \in G^\sim(L|_{S'})$ and any $\theta \in S'$ give the admissible transformation $(\theta, \Phi\theta, \Psi|_{(x,u)}) \in T(L|_{S'})$. Therefore, there exists $\Phi \in G^\sim(L|_{S'})$ such that $\Psi|_{(x,u)} = \Phi|_{(x,u)}$ and $\Phi\theta = \Phi\theta$. □

Note that under the above assumptions the subclass $L|_{S\setminus S'}$ has similar properties.

Given the class $L|_S$ and a local (connected) group $G$ of point transformations of $(x, u)$ such that $G = G^p_\theta$ for some $\theta \in S$, consider the subsets of $S$

$$S_G = \{ \theta \in S \mid G^p_\theta \supset G \}, \quad S_G' = \{ \theta \in S \mid G^p_\theta = G \}$$

$$\bar{S}_G = \{ \theta \in S \mid G^p_\theta \supset G \text{ mod } G^\sim \}, \quad \bar{S}_G' = \{ \theta \in S \mid G^p_\theta = G \text{ mod } G^\sim \}.$$

Corollary 2. Let the class $L|_S$ be normalized. Then $L|_{S_G}$ and $L|_{S_G'}$ are normalized subclasses of $L|_S$. $G^\sim(L|_S)$ is a subgroup of $G^\sim(L|_{S_G})$ and $G^\sim(L|_{S_G'})$ and generates $T(L|_{S_G})$ and $T(L|_{S_G'})$.

Proposition 10. The subclass $L|_{S_0}$ is invariant with respect to $G^\sim(L|_S)$, where $S_0 = S'_{G^\sim} = G^\sim = G^p_{\theta_0}(L|_S)$.

Proof. Let us fix any $\Phi \in G^\sim(L|_S)$ and any $\theta \in S_0$. We have to show that $\Phi\theta \in S_0$. Now $G^p_{\Phi\theta} = Ad_\Phi G^p_{\theta} = Ad_\Phi G^\sim$, where $Ad_\Phi$ is the action of $\Phi$ on transformation groups: $G \ni \phi \to \phi \circ \psi \circ \phi^{-1} \in Ad_\Phi G$, $\psi := \Phi|_{(x,u)}$. Since $\Phi \in L|_S$, $G^p_{\Phi\theta} \supset G^\sim$. If $G^p_{\Phi\theta} = Ad_\Phi G^\sim \supset G^\sim$ then $G^\sim \supset Ad_{\Phi^{-1}} G^\sim$. But $Ad_{\Phi^{-1}} G^\sim = G_{\Phi^{-1}\theta}^p$, $\Phi^{-1}\theta \in L|_S$ and, therefore, $Ad_{\Phi^{-1}} G^\sim \supset G^\sim$ which implies a contradiction. That is why $G^p_{\Phi\theta} = G^\sim$, i.e., $\Phi\theta \in S_0$. □

Proposition 11. If the class $L|_S$ is normalized in the usual sense, the class $L|_{S_G}$ has the same property. The set $T(L|_{S_G'})$ is generated by the group $G^\sim(L|_{S_G'}) \cap G^\sim(L|_S)$ whose projection onto $(x, u)$ is the normalizer of $G$ in $G^\sim(L|_S)|_{(x,u)}$.

Proof. Let us fix an arbitrary $(\theta, \varphi) \in T(L|_{S_G'})$. Since $T(L|_{S_G'}) \subset T(L|_S)$, there exists $\Phi \in G^\sim(L|_S)$ such that $\Phi\theta = \Phi\theta$ and $\varphi = \Phi|_{(x,u)}$, hence $G = G^p_\theta = \varphi \circ G^p_\theta \circ \varphi^{-1} = \varphi \circ G \circ \varphi^{-1}$, i.e., $\varphi = \Phi|_{(x,u)}$ belongs to the normalizer of $G$ in $G^\sim(L|_S)|_{(x,u)}$.

Consider any transformation $\Phi \in G^\sim(L|_S)$ such that its projection $\varphi = \Phi|_{(x,u)}$ belongs to the normalizer of $G$ in $G^\sim(L|_S)|_{(x,u)}$. Then $(\theta, \Phi\theta, \varphi) \in T(L|_{S_G'})$ for arbitrary $\theta \in S_G'$, since $\Phi\theta \in S_G'$. Indeed, $\Phi \in S$ and $G^p_{\Phi\theta} = \varphi \circ G^p_{\theta} \circ \varphi^{-1} = \varphi \circ G \circ \varphi^{-1} = G$. Therefore, $\Phi \in G^\sim(L|_{S_G'})$. □

Proposition 12. Suppose that $S'_G \neq \emptyset$. Then $G^p_{\theta_0}(L|_{S_G}) = G$, $G^\sim(L|_{S_G}) = G^\sim(L|_{S'_G})$, and if $L|_S$ is normalized in the usual sense, the projections of these equivalence groups onto $(x, u)$ coincide.

Proof. The first statement trivially follows from the definition of $L|_{S_G}$. Then in view of Proposition 10 $L|_{S'_G}$ is invariant with respect to $G^\sim(L|_{S_G})$, i.e., $G^\sim(L|_{S_G}) \subset G^\sim(L|_{S'_G})$ and hence $G^\sim(L|_{S_G}) \cap G^\sim(L|_S) \subset G^\sim(L|_{S'_G}) \cap G^\sim(L|_S)$. At the same time, $G^\sim(L|_{S_G}) \cap G^\sim(L|_S)$ contains all the transformations from $G^\sim(L|_S)$ whose projections onto $(x, u)$ belong to the normalizer of $G$ in $G^\sim(L|_S)|_{(x,u)}$. This implies in view of Proposition 11 that

$$G^\sim(L|_{S_G})|_{(x,u)} \supset G^\sim(L|_{S_G}) \cap G^\sim(L|_S)|_{(x,u)} = G^\sim(L|_{S'_G}) \cap G^\sim(L|_S)|_{(x,u)} = G^\sim(L|_{S'_G})|_{(x,u)}.$$
In particular, $G^\sim(\mathcal{L}|_{S_G}) \cap G^\sim(\mathcal{L}|_S) = G^\sim(\mathcal{L}|_{S'_G}) \cap G^\sim(\mathcal{L}|_S)$.

**Note 4.** In general, the normalization of $\mathcal{L}|_S$ does not imply that the class $\mathcal{L}|_{S_G}$ is normalized.

In view of the above propositions, the group classification problem in any normalized class of differential equations is reduced to the subgroup analysis of the corresponding equivalence group. The property of strong normalization often is an indication that many subgroups will be Lie symmetry groups of systems from the class under consideration. Moreover, under classification a hierarchy of normalized classes corresponding to symmetry extension cases is naturally obtained. The possibility of characterizing certain properties of a class in terms of normalization and the complexity of associated normalized subclasses strongly correlate with the complexity of the corresponding group classification problem.

### 3.4 Normalized subclasses and admissible transformations

An investigation of the normalization of the class $\mathcal{L}|_S$ or its subclasses is necessary for the description of $T(\mathcal{L}|_S)$ and can be included as a step in studying $T(\mathcal{L}|_S)$. Analogously to conditional equivalence groups, only a part of the normalized subclasses is significant for this.

**Definition 11.** A normalized subclass $\mathcal{L}|_{S'}$ of the class $\mathcal{L}|_S$ is called *maximal* if there are no normalized subclasses of $\mathcal{L}|_S$ properly containing $\mathcal{L}|_{S'}$.

The definition of maximality can be formulated in the same way for other kinds of normalization, i.e., for strongly normalized and semi-normalized (in the usual or generalized sense) classes. Generally speaking, a maximal strongly normalized subclass is not necessarily a maximal normalized subclass and a maximal normalized subclass is not necessarily a maximal semi-normalized subclass. Moreover, a maximal normalized or semi-normalized subclass may be non-associated with a maximal conditional equivalence group. At the same time, no proper subclass of a normalized class leads to a maximal conditional equivalence group.

The algorithm for describing sets of admissible transformations can be modified by including as an additional step the investigation of normalization properties of subclasses associated with maximal equivalence groups and the construction of the complete family of maximal normalized subclasses. (The maximal normalized subclasses can by studied up to $G^\sim$-equivalence.) Note that under solving group classification problems, non-maximal normalized subclasses arise as possible classification cases and, therefore, should also be studied.

The problem of classifying admissible transformations can be seen as solved, e.g., in the following cases.

In view of the definition of normalized classes, the set of admissible transformations is known if the class turns out to be normalized and its equivalence group is calculated. Then

$$T(\mathcal{L}|_S) = \{ (\theta, \Phi \theta, \Phi|_{(x,u)} ) \mid \theta \in S, \Phi \in G^\sim \}.$$

This is the case for all classes of nonlinear Schrödinger equations investigated in Section 5. Note that any normalized class has a unique maximal conditional equivalence group which obviously coincides with the equivalence group of this class. A subclass cannot generate a maximal conditional equivalence group of the class if it is strongly contained in a normalized subclass.

Suppose that the class $\mathcal{L}|_S$ is presented as a union of disjoint normalized subclasses, and that there are no admissible transformations between systems from different subclasses. That is, $S = \bigcup_{\gamma \in \Gamma} S_\gamma$, $\mathcal{L}|_S$ is normalized for any $\gamma \in \Gamma$, $S_\gamma \cap S_{\gamma'} = \emptyset$ and $T(\theta, \theta') = \emptyset$, where $\theta \in S_\gamma$, $\theta' \in S_{\gamma'}$, $\gamma \neq \gamma'$. Then obviously $G^\sim(\mathcal{L}|_S) = \cap_{\gamma \in \Gamma} G^\sim(\mathcal{L}|_{S_\gamma})$ and $T(\mathcal{L}|_S)$ is the union of the
subalgebras of the Lie symmetry algebra of the \((1+1)\)-dimensional free Schrödinger equation were
Schrödinger equations. All the equations from the class
the linear case \([44, 46, 47]\). Thereafter, symmetry investigations were extended to nonlinear
arbitrary potential.

The references in this paper, related to this subject, represent mainly investigations on classical
integrable models of mathematical physics. At the same time, the physical interpretation of
some known types of nonlinear Schrödinger equations is not completely clear and is a topic of
ongoing research.

A more nontrivial situation concerning admissible transformations is encountered when maximal
normalized subclases have a nonempty intersection. Suppose that \(S', S'' \subset S, S' \cap S'' \neq \emptyset\),
the subclases \(L|_{S'}\) and \(L|_{S''}\) are normalized, \(S' = G^\gamma(L|_{S'}) S' \cap S''\) and \(S'' = G^\gamma(L|_{S''}) S' \cap S''\).
The latter two conditions mean that any equation from \(S'\) (or \(S''\)) is equivalent, with respect
to \(G^\gamma(L|_{S'})\) (or \(G^\gamma(L|_{S''})\)), to an equation from \(S' \cap S''\). Then any admissible transformation
\((\theta', \theta'', \varphi)\) with \(\theta' \in S'\) and \(\theta'' \in S''\), can be represented in the form
\((\theta', \Phi^2(\Phi^1 \theta'), (\Phi^2 \circ \Phi^1)_{(x,u)})\),
where \(\Phi^1 \in G^\gamma(L|_{S'})\), \(\Phi^2 \in G^\gamma(L|_{S''})\) and \(\Phi^1 \theta' \in S' \cap S''\).

A set of admissible transformations of such structure arises, e.g., in the investigation of a
class of variable coefficient diffusion–reaction equations \([69]\).

It is obvious that the above cases comprise only simplest situations of the description of sets
of admissible transformations in terms of normalized subclasses.

4 Lie symmetries of nonlinear Schrödinger equations:
known results

Nonlinear Schrödinger equations (NSchEs) are important objects of investigation due to their
interesting mathematical properties and have found many applications in different fields of physics
and other science. These include optics, nonlinear quantum mechanics, the theory of Bose–
Einstein condensation, plasma physics, computer science, and geophysics among others. NSchEs
also occur in applications as the so-called Madelung fluid equations which are connected with
the standard form via the Madelung transformation \(\psi = \sqrt{R} e^{i\varphi}\), where \(R\) and \(\varphi\) are the new
real-valued unknown functions. See, e.g., the references on Schrödinger equations in this paper
and references therein. The cubic Schrödinger equation is one of the most intensively studied
integrable models of mathematical physics. At the same time, the physical interpretation of
some known types of nonlinear Schrödinger equations is not completely clear and is a topic of
ongoing research.

Schrödinger equations have also been intensively studied by means of symmetry methods.
The references in this paper, related to this subject, represent mainly investigations on classical
Lie symmetries. In fact, the first investigation of Schrödinger equations from the symmetry
point of view was performed by Lie. More precisely, his classification \([40]\) of all the linear
equations with two independent complex variables includes, in an implicit form, the solution
of the classification problem for the linear \((1+1)\)-dimensional Schrödinger equations with an
arbitrary potential.

The specific study of Lie symmetries for Schrödinger equations was begun in the 1970s with
the linear case \([41, 45, 47]\). Thereafter, symmetry investigations were extended to nonlinear
Schrödinger equations. All the equations from the class \(\mathcal{F}\), which are invariant with respect to
subalgebras of the Lie symmetry algebra of the \((1+1)\)-dimensional free Schrödinger equation were
constructed in [11]. Later the more general problem of the description of the equations from the class $\mathcal{F}$, possessing at most three-dimensional Lie invariance algebras, was solved in [75]. It was observed [11, 22] that $(1 + n)$-dimensional NSchEs with nonlinearities of the form $F = f(|\psi|)|\psi|$ are notable for their symmetry properties because any such equation is invariant with respect to a representation of the $(1 + n)$-dimensional Galilean group. Here $n$ is the number of spatial variables. Extensions of the invariance group are possible for logarithmic [20] and power functions [22] and in fact only for these functions [48]. The power $\gamma = 4/n$ is special since the free Schrödinger equation and the NSchE with the nonlinearity $|\psi|^{4/n}\psi$ are distinguished from many similar equations by possessing the complete Galilean group extended with both the scale and conformal transformations [22]. This NSchE also has other exceptional properties, which is why now the value $\gamma = 4/n$ is called the critical power of NSchEs. The complete group classification of constant coefficient NSchEs with nonlinearities of the general form $F = F(\psi, \psi^*)$ was performed in [48]. Lie symmetries of NSchEs with modular nonlinearities and oscillator potential were classified in [28].

Finishing a series of works [23, 24, 25] on group analysis and exact solutions of NSchEs, Gagnon and Winternitz [25] investigated a general class of $(1 + 1)$-dimensional variable coefficient cubic SchEs. Doebner and Goldin applied a symmetry approach to obtain new equations which generalize Schrödinger equations and can be applied in nonlinear quantum mechanics [15]. These equations were investigated in more detail from the symmetry point of view by a number of authors [21, 16, 45]. Different generalizations of NSchEs also arose under the classification of Galilei-invariant nonlinear systems of evolution equations in [18]. Investigations on conditional invariance and related direct reductions led to an extension of the set of NSchEs whose exact solutions were constructed by symmetry or closed methods [14, 19]. Conditional and Lie symmetries of Schrödinger equations were also studied, with mass considered as an additional variable [68]. A number of exact solutions of NSchEs are collected in [51]. Lie symmetries of vector nonlinear Schrödinger systems of the form $i\psi_t + \Delta \psi + S(t, x, \psi, \psi^*)\psi = 0$ were computed in [71, 72, 73] with a symbolic calculation package to demonstrate its effectiveness for solving overdetermined systems of PDEs. Here $x$ and $\psi$ are $n$- and $m$-tuples of the space variables and unknown functions, respectively. $S$ is a real-valued smooth function of its arguments. The determining equations for Lie symmetry operators of certain systems from this class were proposed to be used by other researchers as benchmarks for testing their algorithms and program packages dealing with overdetermined systems of PDEs. See the discussion in Example 3 for details.

5 Nested normalized classes of $(1 + 1)$-dimensional nonlinear Schrödinger equations

We start with the widest (for the present paper) class $\mathcal{F}$ of equations having the general form (11), where $F = F(t, x, \psi, \psi^*, \psi_x, \psi^*_x)$ is an arbitrary smooth complex-valued function of its arguments. Successively prescribing more constraints on the arbitrary element $F$, we construct a series of nested normalized classes of $(1 + 1)$-dimensional nonlinear Schrödinger equations. The final such class still contains Schrödinger equations with modular nonlinearities and potentials and possesses certain properties which allow us to continue with the investigation of Schrödinger equations with modular nonlinearities and potentials. In what follows, subscripts of functions denote differentiation with respect to the corresponding variables.

In the case of the entire class (11), the auxiliary system for arbitrary elements $F$ consists of the equations

$$F_{\psi_{tt}} = F_{\psi_{tx}} = F_{\psi_{xx}} = F_{\psi_{tx}} = F_{\psi_{xx}} = 0, \quad F_{\psi_t} = F_{\psi_x} = 0.$$ 

Observe that we also have to take into account the conjugate equation to equation (11). So, we actually work with a system of two conjugate equations for two conjugate unknown functions.
All invariance and equivalence conditions need only be tested for one of the equations (since they then automatically hold for the conjugate equation as well), yet on the manifold of the whole system. Alternatively we could replace equation (11) by a system of two equations in two real unknown functions but this would lead to more complicated computations.

Any point transformation \( T \) in the space of variables of the class \( \mathcal{F} \) has the form

\[
\tilde{t} = T_t(t, x, \psi, \psi^*), \quad \tilde{x} = T_x(t, x, \psi, \psi^*), \quad \tilde{\psi} = T_\psi(t, x, \psi, \psi^*), \quad \tilde{\psi}^* = T_{\psi^*}(t, x, \psi, \psi^*),
\]

where \( T_{\psi^*} = (T_\psi)^* \) and the Jacobian \( |\partial(T_t, T_x, T_\psi, T_{\psi^*})/\partial(t, x, \psi, \psi^*)| \neq 0 \).

**Lemma 1.** If a point transformation \( T \) connects two equations from the class \( \mathcal{F} \) then

\[
\begin{align*}
T_t^i &= T_\psi^i = T_{\psi^*}^i = 0, \quad (T_x^i)^2 = |T_t^i|, \\
T_\psi^i &= 0 \quad \text{if} \quad T_t^i > 0 \quad \text{and} \quad T_\psi^i = 0 \quad \text{if} \quad T_t^i < 0, \quad \text{i.e.,} \quad T_{\psi^*}^i = 0.
\end{align*}
\]

**Note 5.** Hereafter in case of any complex value \( \beta \) we use the notation

\[
\hat{\beta} = \beta \quad \text{if} \quad T_t^i > 0 \quad \text{and} \quad \hat{\beta} = \beta^* \quad \text{if} \quad T_t^i < 0.
\]

**Proof.** Let us recall that any equation of the form (11) is interpreted as a system of two semi-linear evolution equations in the functions \( \psi \) and \( \psi^* \) (or in the real and imaginary parts of the function \( \psi \)). In [61] more general classes of systems of \((1 + 1)\)-dimensional evolution equations were studied from the viewpoint of admissible transformations and normalization properties. In view of Lemma 4 from [61], the evolutionary character of such systems implies that \( T_t \) is a function only of \( t \). (The above conditions on transformations with respect to \( t \) were earlier proved for single evolution equations [83]). The condition \( T_\psi^i = T_{\psi^*}^i = 0 \) follows, according to Lemma 5 of [61], from the quasi-linearity of the systems. Since all the equations from \( \mathcal{F} \) have the same ratio of the coefficients of \( \psi_t \) and \( \psi_{xx} \) then additionally \( (T_x^i)^2 = |T_t^i| \). The last condition on the transformation \( T \) is derived from the fact that this ratio differs from the ratio of the coefficients of \( \psi_t^i \) and \( \psi_{xx}^i \) in the corresponding conjugate equations.

Any transformation \( T \) satisfying the constraints of Lemma 1 results (after application to an arbitrary equation from the class \( \mathcal{F} \)) in an equation from the same class. The corresponding values of the arbitrary elements are connected in a local way, i.e., the transformation \( T \) belongs to the equivalence group \( G_T^\mathcal{F} \) of the class \( \mathcal{F} \). This allows us to reformulate and to strengthen the results of [73] on the equivalence group \( G_T^\mathcal{F} \).

**Theorem 1.** The class \( \mathcal{F} \) is strongly normalized. The equivalence group \( G_T^\mathcal{F} \) of the class \( \mathcal{F} \) is formed by the transformations

\[
\begin{align*}
\tilde{t} &= T(t), \quad \tilde{x} = \varepsilon x|T_t|^{|1/2} + X(t), \quad \tilde{\psi} = \Phi(t, x, \hat{\psi}), \\
\tilde{F} &= \frac{1}{|T_t|} \left( \Phi \hat{F} - i\varepsilon' \Phi_t + i \left( \frac{T_{tt}}{2|T_t|} x + \frac{\varepsilon \varepsilon'}{|T_t|} X \right) \Phi_x + \Phi \hat{\psi}_x \right) \\
&\quad - \Phi_x x - 2F_{\psi x} \hat{\psi}_x - \Phi \hat{\psi}_x (\hat{\psi}_x)^2,
\end{align*}
\]

where \( T \) and \( X \) are arbitrary smooth real-valued functions of \( t \), \( T_t \neq 0 \), \( \Phi \) is an arbitrary smooth complex-valued function of \( t \), \( x \) and \( \hat{\psi}, \Phi \hat{\psi} \neq 0 \). Hereafter \( \varepsilon = \pm 1, \varepsilon' = \text{sign} \, T \).

**Note 6.** Analogous arguments imply that the wider class of Schrödinger-like equations of the general form \( iv_\psi t + G_{\psi xx} + F = 0 \), where \( F \) and \( G \) are arbitrary smooth complex-valued functions of \( (t, x, \psi, \psi^*, \psi_x, \psi^*_x) \), is normalized. The subclasses with arbitrary \( G = G(t, x, \psi, \psi^* \) or \( G = G(t, x) \) or \( G \) running through real-valued functions of one of the above lists of arguments are also normalized. A number of different constraints on the arbitrary elements can be posed under which the corresponding classes are normalized.
**Note 7.** In fact, the equivalence group $G_2^F$ is generated by the continuous family of transformations of the form (3), where $T_t > 0$ and $\varepsilon = 1$, and two discrete transformations: the space reflection $I_x (\bar{t} = t, \bar{x} = -x, \bar{\psi} = \psi, \bar{F} = F)$ and the Wigner time reflection $I_t (\bar{t} = -t, \bar{x} = x, \bar{\psi} = \psi^*, \bar{F} = F^*)$. Similar statements are true for the equivalence groups of the classes below.

**Note 8.** Strong normalization of the class $F$ is proved easily. In view of the definition of strongly normalized classes and Note 7 it is enough to prove for each of the following transformations $T$ there exists an equation of the form (1) which is invariant under the projection of $T$: the discrete transformations $I_x$ and $I_t$ and the infinitesimal equivalence transformations. The projection of $I_x$ is a point symmetry of equation (1) iff $F$ is an even function in $(x, \psi, \psi^*)$, i.e., $F(t, x, \psi, \psi^* x) = F(t, -x, \psi, -\psi^* x)$. Analogously, equation (1) is invariant with respect to the projection of $I_t$ iff $F(t, x, \psi, \psi^* x) = F^*(-t, x, \psi, \psi^*)$. The projection of any operator generating a one-parameter subgroup of $G_2^F$ has the form

$$Q = \tau(t) \partial_t + \left(\frac{1}{2} \tau x + \chi(t)\right) \partial_x + \eta(t, x, \psi, \psi^*) \partial_{\psi} + \eta^*(t, x, \psi) \partial_{\psi^*},$$

where the coefficients $\tau$, $\chi$ and $\eta$ are arbitrary functions of their arguments. Infinitesimal invariance of equation (1) with respect to the operator $Q$ implies a single first-order partial differential equation in the arbitrary element $F$ which always has a solution.

Let us narrow the class under consideration to the subclass $F'$ with the assumption that the arbitrary element $F$ does not depend on the derivatives $\psi_x$ and $\psi^*_x$, i.e., we supplement the auxiliary system on $F$ with the constraints $F_{\psi_x} = F_{\psi^*_x} = 0$. The additional constraints on $F$ in the subclass $F'$ imply more conditions on the components of admissible transformations with respect to $\psi$ and $\psi^*$. Namely,

$$T_{\psi} = 0, \quad T_{\psi^*} = \frac{i \varepsilon^\prime T_{\xi}^2}{2|T_t|^{1/2}} T_{\psi^*},$$

Analogously to Theorem 1 we arrive at the following statement on the class $F'$. (Strong normalization of $F'$ is proved in a way similar to Note 8)

**Theorem 2.** The subclass $F'$ of $F$ satisfying the condition that the arbitrary element $F$ depends only on $t$, $x$, $\psi$ and $\psi^*$ is strongly normalized. Its equivalence group $G_2^{F'}$ is a subgroup of $G_2^F$ and is formed by the transformations (3) where

$$\Phi = \hat{\psi} \exp \left(\frac{i}{8} T_t \left(\frac{1}{2} |T_t| \right)^2 x + \frac{i \varepsilon^\prime}{2} X_t \left(\frac{1}{2} |T_t| \right)^{1/2} x + \Theta(t) + i \Psi(t)\right) + \Phi^0(t, x),$$

and $T$, $X$, $\Theta$ and $\Psi$ are arbitrary smooth real-valued functions of $t$, $T_t \neq 0$, and $\Phi^0$ is an arbitrary smooth complex-valued function of $t$ and $x$.

We next narrow the class further, in a more specific way: Consider the class $S$ of equations

$$i \psi_t + \psi_{xx} + S(t, x, |\psi|) \psi = 0, \quad S_{|\psi|} \neq 0$$

which encompasses the class of $(1+1)$-dimensional Schrödinger equations with potentials and modular nonlinearity and is more convenient, in some sense, for a preliminary group classification. Here $S$ is an arbitrary complex-valued function depending on $t$, $x$ and $|\psi|$, and we additionally assume $S_{\rho} \neq 0$. The latter condition is invariant under any point transformation which transforms a fixed equation of the form (4) to an equation of the same form. The converse condition $S_{\rho} = 0$ corresponds to the linear case which should be investigated separately because of its singularity. Hence the imposed inequality is natural.
The class $S$ is singled out from the class $\mathcal{F}$ by the representation $F = S(t, x, |\psi|)\psi$, i.e., the arbitrary element $F$ satisfies, additionally to the above auxiliary equations, the conditions

$$(\psi \partial_\psi - \psi^* \partial_{\psi^*})(F/\psi) = 0, \quad (\psi \partial_\psi + \psi^* \partial_{\psi^*})(F/\psi) \neq 0.$$ 

For convenience we will consider $S = F/\psi$ instead of $F$ as an arbitrary element depending on no derivatives and also satisfying the conditions

$$\psi S_\psi - \psi^* S_{\psi^*} = 0, \quad \psi S_\psi + \psi^* S_{\psi^*} \neq 0.$$ 

**Theorem 3.** The class $S$ is strongly normalized. The equivalence group $G_S^\sim$ of the class $S$ is the subgroup of $G_{\mathcal{F}}^\sim$ defined by the condition $\Phi^0 = 0$, i.e., it is formed, in terms of the arbitrary element $S$, by the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = \varepsilon x |T_t|^{1/2} + X(t), \quad \tilde{\psi} = \tilde{\psi} \exp \left( \frac{i}{8} \frac{T_{tt}}{|T_t|} x^2 + \frac{i \varepsilon^2}{2 |T_t|^{1/2}} \right) x + \Theta(t) + i \Psi(t),$$

$$\tilde{S} = \frac{\hat{S}}{|T_t|} + \frac{2T_{tt}T_t - 3T_t^2}{16\varepsilon^2 T_t^3} x^2 + \frac{-\varepsilon^4}{2 |T_t|^{1/2}} \left( \frac{X_t}{T_t} \right) x + \frac{\Psi_t - i \Theta_t}{T_t} - \frac{X_t^2 + i T_{tt}}{4T_t^2}.$$ 

Here $T, X, \Phi$ and $\Psi$ are arbitrary smooth real-valued functions of $t$, $\varepsilon = \pm 1$, $\varepsilon' = \text{sign} T$.

See Note 10 for a discussion of the proof that the class $S$ is strongly normalized.

**Corollary 3.** For any equation from the class $S$ the value $\rho S_{\rho}/S_\rho$ is preserved under any transformation which maps this equation to an equation from the same class, excluding $I_t$. In particular, if $\rho S_{\rho}/S_\rho$ is a real-valued function then it is an invariant of the admissible transformations in the class $S$.

**Note 9.** It follows from Theorem 3 that equivalence transformations from the equivalence group of any subclass of $S$ have the form (6). This statement is also true for Lie symmetry transformations of any equation from the class $S$ if we assume in (6) that $S$ is an invariant function.

In particular, Theorem 3 jointly with the infinitesimal Lie method results in the following statement on Lie symmetry operators of equations from the class $S$.

**Theorem 4.** Any operator $Q$ from the maximal Lie invariance algebra $A(S)$ of equation (4) with an arbitrary function $S$ ($S_\rho \neq 0$) can be represented in the form $Q = D(\tau) + G(\chi) + \lambda M + \zeta I$, where

$$D(\tau) = \tau \partial_\tau + \frac{1}{2} \tau_t x \partial_x + \frac{1}{8} \tau_{tt} x^2 M, \quad G(\chi) = \chi \partial_x + \frac{1}{2} \chi_t x M,$$

$$M = i (\psi \partial_\psi - \psi^* \partial_{\psi^*}), \quad I = \psi \partial_\psi + \psi^* \partial_{\psi^*},$$

where $\chi = \chi(t)$, $\tau = \tau(t)$, $\lambda = \lambda(t)$ and $\zeta = \zeta(t)$ are arbitrary smooth real-valued functions of $t$. Moreover, the coefficients of $Q$ have to satisfy the classifying condition

$$\tau S_t + \left( \frac{1}{2} \tau_t x + \chi \right) S_x + \zeta \rho S_\rho + \tau_\tau S = \frac{1}{8} \tau_{tt} x^2 + \frac{1}{2} \chi_t x + \lambda_t - i \zeta_t - \frac{1}{4} \tau_{tt}.$$ 

Theorem 4 can also be proved by direct application of the infinitesimal Lie method. Namely, consider an operator from $A(S)$ in the most general form $Q = \xi^t \partial_t + \xi^x \partial_x + \eta \partial_\psi + \eta^* \partial_{\psi^*}$, where $\xi^t$, $\xi^x$ and $\eta$ are smooth functions of $t, x, \psi$ and $\psi^*$. The infinitesimal invariance condition [49, 50] of
Theorem 5. \( \hat{\tau} \), where \( \hat{\tau} \) is an arbitrary complex-valued potential depending on \( t, x, \psi \).

Note 10. Under the proof of strong normalization of the class \( S \) in a way similar to Note 8, the operator projections have the form \( Q = D(\tau) + G(\chi) + \lambda M + \zeta I \) as operators in Theorem 4. In view of the classifying condition \( S \), \( Q \) is a Lie invariance operator of an equation of form (4) iff \((\tau, \chi, \zeta) \neq (0, 0, 0)\) or \( \lambda = 0 \). It is enough to deduce the statement on strong normalization of the class \( S \).

Assuming \( S \) to be arbitrary and splitting \( S \) with respect to \( S_t, S_x \) and \( S_\rho \), we obtain that the Lie algebra of the kernel \( G_{\mathbb{C}} \) of maximal Lie invariance groups of equations from the class \( S \) is \( A_{\mathbb{C}} = (M) \). The complete group \( G_{\mathbb{C}} \) coincides with the projection, to \((t, x, \psi)\), of the normal subgroup \( \hat{G}_{\mathbb{C}} \) of \( G_{\mathbb{C}} \), which include the transformations \( \hat{G} \) acting on the arbitrary element \( S \) identically (i.e., \( T = t \) and \( X = \Theta = \Psi = \Theta = 0 \), see Section 2.4).

Theorem 5. \( \hat{G}_{\mathbb{C}} \) is formed by the transformations \( \tilde{t} = t, \tilde{x} = x, \tilde{\psi} = \psi e^{i\Phi}, \tilde{S} = S \), where \( \Phi \) is an arbitrary constant.

Note 11. The operators \( D(\tau), G(\chi), \lambda M \) and \( \zeta I \), where \( \tau, \chi, \lambda \) and \( \zeta \) run through the whole set of smooth functions of \( t \), generate an infinite-dimensional Lie algebra \( A_{\mathbb{C}} \) under the usual Lie bracket of vector fields. The non-zero commutation relations between the basis operators of \( A_{\mathbb{C}} \) are the following ones:

\[
[D(\tau^1), D(\tau^2)] = D(\tau^1 \tau^2 - \tau^2 \tau^1), \quad [D(\tau), G(\chi)] = G(\tau \chi - \frac{1}{2} \tau^2 \chi),
\]

\[
[D(\tau), \lambda M] = \tau \lambda t M, \quad [D(\tau), \zeta I] = \tau \zeta t I, \quad [G(\chi^1), G(\chi^2)] = \frac{1}{2} (\chi^1 \chi^2 - \chi^2 \chi^1) M.
\]

Note 12. Sometimes (e.g. for reduction and construction of solutions) it is convenient to use the amplitude \( \rho \) and the phase \( \varphi \) instead of the wave function \( \psi = \rho e^{i\varphi} \). Then equation (4) is replaced by the following system for the two real-valued functions \( \rho \) and \( \varphi \):

\[
\rho_t + 2\rho_x \varphi_x + \rho \varphi_{xx} + \rho \Im S = 0, \quad -\rho \varphi_t - \rho (\varphi_x)^2 + \rho \varphi_{xx} + \rho \Re S = 0,
\]

where \( S = S(t, x, \rho) \). The constraining system for \( S \) takes the form \( S_{\varphi} = 0, S_\rho \neq 0 \). In the variables \( (\rho, \varphi) \) the operators \( D(\tau) \) and \( G(\chi) \) have the same form (7), and \( M = \partial_{\varphi}, I = \rho \partial_{\rho} \).

Below we use the variables \( (\rho, \varphi) \) and \( (\psi, \psi^*) \) simultaneously.

6  Group classification of \((1 + 1)\)-dimensional nonlinear Schrödinger equations with potentials and modular nonlinearities

Let us pass to the subclass \( V \) of the class \( S \) which consists of the equations of the general form

\[
i \psi_t + \psi_{xx} + f(|\psi|) \psi + V \psi = 0,
\]

where \( f \) is an arbitrary complex-valued nonlinearity depending only on \( \rho = |\psi| \), \( f_\rho \neq 0 \), and \( V \) is an arbitrary complex-valued potential depending on \( t \) and \( x \). The arbitrary element \( S \) is
represented as \( S = f(\rho) + V(t, x) \), where \( f_\rho \neq 0 \). Therefore, this subclass is derived from the class \( S \) by the condition \( S_{\rho t} = S_{\rho x} = 0 \) and \( S_\rho \neq 0 \) or, in terms of \( \psi \) and \( \psi^* \),

\[
\psi S_{\psi t} + \psi^* S_{\psi^* t} = \psi S_{\psi x} + \psi^* S_{\psi^* x} = 0, \quad \psi S_\psi + \psi^* S_{\psi^*} \neq 0.
\]

The group classification problem for the class \( \mathcal{V} \) is solved in this section. Although the class \( \mathcal{V} \) is not normalized, the approach based on normalization is still applicable to this class due to its representation in the form of a union of normalized classes.

### 6.1 Equivalence groups and admissible transformations

To find the equivalence group \( G_\gamma \) of the class \( \mathcal{V} \) in the framework of the direct method, we look for all point transformations in the space of the variables \( t, x, \psi, \psi^* \), \( S \) and \( S^* \) which preserve the system formed by equations \( \{1\}, \{3\} \) and \( \{10\} \). Moreover, in the same way we can classify all admissible point transformations in the class \( \mathcal{V} \).

**Theorem 6.** \( G_\gamma \) is formed by the transformations \( \{1\} \) where \( T_{tt} = 0 \) and \( \Psi_t = 0 \). The class \( \mathcal{V} \) is not normalized. The subclass \( \mathcal{V}' \) of \( \mathcal{V} \) under the additional condition that the value \( \rho f_{pp}/f_\rho \) (\( \equiv \rho S_{pp}/S_\rho \)) is not a real constant has the same equivalence group and is normalized. There exist two different cases for additional (conditional) equivalence transformations in the class \( \mathcal{V} \) (\( \sigma \) is a complex constant):

1. \( \rho f_{pp}/f_\rho = -1 \), i.e., \( f = \sigma \ln \rho \).
2. \( \rho f_{pp}/f_\rho = \gamma - 1 \in \mathbb{R} \) and \( \gamma \neq 0 \), i.e., \( f = \sigma \rho^\gamma \).

For any real constant \( \gamma \) the subclass \( \mathcal{P}_\gamma \) consisting of equations \( \{9\} \), where \( \rho f_{pp}/f_\rho = \gamma - 1 \), is normalized. There are no point transformations between equations from different subclasses taken from the set \( \{\mathcal{V}', \mathcal{P}_\gamma, \gamma \in \mathbb{R}\} \).

**Note 13.** It is possible to find equivalence transformations in another way, considering \( f \) and \( V \) as arbitrary elements instead of \( S \). Then we have to look for all point transformations in the space of the variables \( t, x, \psi, \psi^* \), \( f \), \( f^* \), \( V \) and \( V^* \) which preserve the system formed by the equations

\[
i\psi_t + \psi_{xx} + (f + V)\psi = 0, \quad f_t = f_x = 0, \quad \psi f_\psi - \psi^* f_{\psi^*} = 0, \quad V_\psi = V_{\psi^*} = 0,
\]

and additionally \( \psi f_{\psi} + \psi^* f_{\psi^*} \neq 0 \). Due to the representation \( S = f + V \) we additionally obtain only gauge equivalence transformations of the form \( \tilde{f} = f + \beta \), \( \tilde{V} = V - \beta \), where \( \beta \) is an arbitrary complex number and \( t, x \) and \( \psi \) are not changed. We neglect these transformations, choosing \( f \) in the most suitable form. For example, this is the reason why we can assume \( f = \sigma \ln \rho \) in the case \( \rho f_{pp}/f_\rho = -1 \). Analogously, we put \( f = \sigma \rho^\gamma \) up to gauge equivalence transformations if \( \rho f_{pp}/f_\rho = \gamma - 1 \in \mathbb{R} \) and \( \gamma \neq 0 \).

**Note 14.** Theorem \( \{4\} \) gives an exhaustive description of the set of admissible transformations of the class \( \mathcal{V} \). Indeed, the class \( \mathcal{V} \) is not normalized but it is presented as the union of disjoint normalized subclasses \( \mathcal{W} \) and \( \mathcal{P}_\gamma \), \( \gamma \in \mathbb{R} \), and there are no equations from different subclasses which are equivalent with respect to point transformations. Therefore, the set of admissible transformations of the class \( \mathcal{V} \) is the union of the sets of admissible transformations in the subclasses, which are generated by the corresponding conditional equivalence groups. This gives an example on the second simplest structure of sets of admissible transformations, described in Subsection 3.4.

The equivalence groups \( G_\gamma^\mathcal{V} \) and \( G_\gamma^\mathcal{P} \), \( \gamma \in \mathbb{R} \), exhaust the set of the maximal conditional equivalence groups of the class \( \mathcal{V} \). (See the next sections for exact formulas.) Since they are different, the subclasses \( \mathcal{V}' \) and \( \mathcal{P}_\gamma \), \( \gamma \in \mathbb{R} \), are all the maximal normalized subclasses of the class \( \mathcal{V} \).
Admitting generalized equivalence groups, we can unite the subclasses $\mathcal{P}_\gamma$, $\gamma \neq 0$, into the total subclass $\hat{\mathcal{P}}$ of equations (9) with power nonlinearities. The subclass $\hat{\mathcal{P}}$ is normalized in the generalized sense with respect to its generalized equivalence group $G_\gamma^S$ which is an interlacement of the groups $G_\gamma^\mathcal{P}$, $\gamma \neq 0$, by the parameter $\gamma$. The equivalence group $G_\gamma^S$ is generalized since transformations with respect to $\psi$ depend on the arbitrary element $\gamma$ of the class $\hat{\mathcal{P}}$. Therefore, the class $\mathcal{V}$ possesses only three maximal conditional generalized equivalence groups $G_\mathcal{V}^\gamma$, $G_\mathcal{P}^\gamma$ and $G_\mathcal{P}_0^\gamma$ corresponding to the maximal normalized (in the generalized sense) subclasses $\mathcal{V}'$, $\mathcal{P}$ and $\mathcal{P}_0$. There are no other maximal subclasses of $\mathcal{V}$, which are normalized in the generalized sense.

Since the class $\mathcal{V}$ is not normalized and possesses nontrivial conditional equivalence groups, the entire set of admissible transformations is much wider than its subset associated with the equivalence group $G_\mathcal{V}^\gamma$. Therefore, under group classification with standard techniques a number of $G_\mathcal{V}^\gamma$-inequivalent but point-transformation equivalent cases should be included in the final classification list of equations with Lie symmetry extensions and then a number of additional equivalence transformations should be found between such cases for the list to be simplified.

We use a more efficient way. Namely, we separately carry out the group classification in each maximal normalized subclass with respect to the corresponding conditional equivalence group, using the approach to group classification in normalized classes of differential equations developed above. The proposed algorithm is implemented in the different subclasses in a maximally unified way to demonstrate its capacities. The classification list for the whole class $\mathcal{V}$ is constructed as the union of the lists obtained for the subclasses. Due to the special structure of the set of maximal normalized subclasses of the class $\mathcal{V}$, this list is formed by all the extension cases inequivalent with respect to point transformations. It can be expanded to the list of $G_\mathcal{V}^\gamma$-inequivalent extension cases by the conditional equivalence transformations factorized with respect to $G_\mathcal{V}^\gamma$.

**Note 15.** Theorem 3 and the results of Sections 6.2–6.4 imply that $G_\mathcal{V}^\gamma$ is generated by the transformational parts of admissible transformations of class (9). More precisely, for any fixed $\gamma \neq 0$ each transformation from $G_\mathcal{V}^\gamma$ is presented as a composition of transformations from the maximal conditional equivalence groups $G_\mathcal{P}_0^\gamma$ and $G_\mathcal{P}^\gamma$. At the same time, the class $\mathcal{S}$ is not the minimal normalized superclass of the class (9). It is obvious that such a class is given by the subclass of $\mathcal{S}$ which is determined by the condition $S = R'(t)f(R(t)|\psi|) + V(t, x)$, where $R$ and $R'$ are arbitrary smooth real-valued functions of $t$.

### 6.2 General case of nonlinearity

In this section we adduce results only for the general case $\rho f_{\rho \rho}/f_\rho \neq \text{const} \in \mathbb{R}$ (the subclass $\mathcal{V}'$). The cases $f = \sigma \ln \rho$ and $f = \sigma \rho^\gamma$ (the subclass $\mathcal{P}_0$ and $\mathcal{P}_\gamma$, $\gamma \neq 0$) which admit extensions of conditional equivalence groups are considered in the next sections in detail. The classifications of the subclasses are maximally unified to demonstrate the universality of our approach to group classification problems.

In spite of the absence of equivalence extensions, equations with nonlinearities of the general case can possess sufficiently large Lie invariance algebras. Moreover, special nonlinearities of this kind arise in applications and mathematical investigations [23, 51].

**Corollary 4.** A potential $V$ in (9) with nonlinearity of the general form can be made to vanish by means of point transformations iff it is a function which is real-valued up to gauge equivalence transformations and is linear with respect to $x$.

**Note 16.** The action of $G_\mathcal{V}^\gamma$ on $f$ is only multiplication with non-zero real constants and/or complex conjugation. That is why the general case can be split into an infinite number of subclasses, and each subclass is formed by equations with nonlinearities which are proportional.
to an arbitrary fixed function or its conjugation with real constant coefficients and is normalized. Moreover, we can restrict our consideration to the class of equations with an arbitrary fixed nonlinearity $f(\rho)$, assuming $f$ as determined up to a real multiplier or/and complex conjugation and only $V$ as an arbitrary element. The equivalence group of such a restricted class $\mathcal{V}^f$, where $\rho f_{\rho\rho}/f_\rho$ is not a real constant, will be denoted by $G^\gamma_f$ and is formed by the transformations (6) with $T_i = 1$ ($T_i = \pm 1$ if $f$ is a real-valued function) and $\Psi = 0$.

Substitution of $S = f(\rho) + V(t, x)$ into the classifying condition (3) and subsequent splitting under the condition $\rho f_{\rho\rho}/f_\rho \neq \text{const} \in \mathbb{R}$ imply the equations $\tau_i = 0$, $\zeta = 0$. In view of Theorem 4 the following statement is true.

**Lemma 2.** Any operator $Q$ from the maximal Lie invariance algebra $A_f(V)$ of equation (2) in the case where $\rho f_{\rho\rho}/f_\rho$ is not a real constant can be presented in the form $Q = c_0 \partial_t + G(\chi) + \lambda M$, where $\chi = \chi(t)$ and $\lambda = \lambda(t)$ are arbitrary smooth real-valued functions of $t$, $c_0 = \text{const}$. Moreover, the coefficients of $Q$ should satisfy the classifying condition

$$c_0V_t + \chi V_x = \frac{1}{2} \chi Hu + \lambda t. \quad (11)$$

The kernel of the maximal Lie invariance groups of equations from the class $\mathcal{V}^f$ is $G^\gamma_f = G^0_f = G^0_f$, and its Lie algebra is $A^0_f = \langle M \rangle$.

Let us briefly sketch a chain of statements which yields, in view of the results of Section 3, a complete group classification of the class $\mathcal{V}^f$.

The set $A^0_f = \{Q = c_0 \partial_t + G(\chi) + \lambda M\}$ is an (infinite-dimensional) Lie algebra under the usual Lie bracket of vector fields. For any $Q \in A^0_f$ where $(c_0, \chi) \neq (0, 0)$ we can find $V$ satisfying condition (11), i.e., $A^0_f = \langle \cup_i A_f(V) \rangle$. Moreover, the space reflection belongs to the point symmetry groups of certain equations from $\mathcal{V}^f$. The same statement is true for the Wigner time reflection if $f$ is a real-valued function. Therefore, $\mathcal{V}^f$ is a strongly normalized class of differential equations. (This was a reason for introducing the class $\mathcal{V}^f$ since the class $\mathcal{V}$ is not strongly normalized.)

The group $G^\gamma_f$ acts on $A^0_f$ and on the set of equations of the form (11) and, therefore, generates equivalence relations in these sets. The automorphism group generated on $A^0_f$ and the equivalence group generated on the set of equations (11) are isomorphic to $G^\gamma_f/\hat{G}^\gamma_f$. (The transformations from $\hat{G}^\gamma_f$ act on (11) as gauge equivalence transformations and can be neglected.) $A^0_f = \langle M \rangle$ coincides with the center of the algebra $A^0_f$ and is invariant with respect to $G^\gamma_f$.

Let $A^1$ and $A^2$ be the maximal Lie invariance algebras of some equations from the class $\mathcal{V}^f$, and $\mathcal{V}_i = \{V \mid A_f(V) = A^i\}$, $i = 1, 2$. Since the class $\mathcal{V}^f$ is normalized then $\mathcal{V}_1 \sim \mathcal{V}_2 \text{ mod } G^\gamma_f$ iff $A^1 \sim A^2 \text{ mod } G^\gamma_f$.

A complete list of $G^\gamma_f$-inequivalent one-dimensional subalgebras of $A^0_f$ is exhausted by the algebras $\{\partial_t\}$, $\langle G(\chi) \rangle$, $\langle \lambda M \rangle$, where $\chi$ and $\lambda$ are arbitrary fixed functions of $t$. (There exist additional equivalences in $\{(G(\chi))\}$ and $\{(\lambda M)\}$, which are generated by translations with respect to $t$ and, if $f$ is real-valued, by the Wigner time reflection $I_t$.)

**Note 17.** For convenience we use below the double numeration $T.N$ of classification cases where $T$ is a table number and $N$ is a row number. We mean that the invariance algebras for Cases 1.0, 1.1 and the corresponding ones from the next tables are maximal if these cases are inequivalent under the corresponding equivalence group to the other, more specialized, cases from the same table.

**Theorem 7.** A complete set of inequivalent potentials admitting extensions of the maximal Lie invariance algebra of equation (9) in the case where $\rho f_{\rho\rho}/f_\rho$ is not a real constant is exhausted by the potentials given in Table 1.
Moreover, in the same way we can easily classify all possible point transformations in the class $S$ formed by equations (5) and (13) under the condition that the class $P_{t, x, \psi, \psi}$ does not depend on derivatives. The functions $\chi^1 = \chi^1(t)$ and $\chi^2 = \chi^2(t)$ form a fundamental system of solutions for the ordinary differential equation $\chi_{tt} = 4v\chi$.

Proof. Suppose that equation (9) has an extension of the Lie invariance algebra for a potential $V$, i.e., $A_f(V) \neq A_f^0$. Then there exists an operator $Q = c_0 \partial_t + G(\chi) + \lambda M \in A_f(V)$ such that $(c_0, \chi) \neq (0, 0)$.

If $c_0 \neq 0$ then $\langle Q \rangle \sim \langle \partial_t \rangle \bmod G_\gamma^1$, i.e., we obtain Case 1. If $c_0 = 0$ then $\langle Q \rangle \sim \langle G(\chi) \rangle \bmod G_\gamma^1$. It follows from (11) that the potential $V$ has the form $V = v(t)x^2 + iw(t) + \bar{w}(t)$, and $\bar{w} = 0 \bmod G_\gamma^1$. For $(v_t, w_t) \neq (0, 0)$ we have Case 1. The condition $v, w = \text{const}$ results in Cases 1.3, 1.4 and 1.5 depending on the sign of $v$. If $v = 0$ and $w = \text{const}$ we can reduce $w$ by means of equivalence transformations to either 0 or 1.

### 6.3 Logarithmic modular nonlinearity

Consider the first subclass $\mathcal{P}_0$ of class $\mathcal{V}$, which is introduced in Theorem 5 and admits extensions of the equivalence and Lie symmetry groups in comparison with the whole class $\mathcal{V}$. It is formed by the equations

$$i\psi_t + \psi_{xx} + \sigma \psi \ln |\psi| + V(t, x)\psi = 0.$$  \hfill (12)

Here $\sigma$ is an arbitrary non-zero complex number and $V$ is an arbitrary complex-valued function of $t$ and $x$. This subclass is distinguished from the larger class $S$ of equations (11) by the condition $S_{pt} = S_{px} = 0$ and $(\rho S_\rho)_\rho = 0$, i.e.,

$$\psi S_{\psi t} + \psi^* S_{\psi^* t} = \psi S_{\psi x} + \psi^* S_{\psi^* x} = 0, \quad (\psi \partial_\psi + \psi^* \partial_{\psi^*})^2 S = 0.$$  \hfill (13)

Schrödinger equations with logarithmic nonlinearity were first proposed by Bialynicki-Birula and Mycielski [6] as possible models of nonlinear quantum mechanics. Although the possibility was later called in question, due to their nice mathematical properties these equations are applied for the description of nonlinear phenomena in many other fields of physics (the theory of dissipative systems, nuclear physics, optics and geophysics), see, e.g., references in [7]. The maximal Lie invariance algebras of such equations with the zero potential were found in [20], their exact solutions were already constructed in [6]. Solutions for other potentials are also known [7].

Similarly to the previous section, we have two ways of finding the equivalence group $G_{\mathcal{P}_0}$ of the class $\mathcal{P}_0$ in the framework of the direct method. In the first approach we look for all point transformations in the space of the variables $t, x, \psi, \psi^*, S$ and $S^*$ which preserve the system formed by equations (5) and (13) under the condition that $S$ does not depend on derivatives. Moreover, in the same way we can easily classify all possible point transformations in the class $\mathcal{P}_0$.
Moreover, the coefficients of $Q$ with an arbitrary potential $\rho$ corresponding restricted class $P_\sigma$ and $T$ complex conjugation. That is why we can fix an arbitrary value remembering that all objects marked in this way are parametrized by transformations with respect to $\sigma$ of the whole class $G$.

Theorem 8. $G^{\sim}_{\mathcal{P}_0}$ is formed by the transformations (6) where $T_{it} = 0$. The corresponding transformations with respect to $\sigma$ and $V$ have the form

$$\tilde{\sigma} = \frac{\sigma}{|T_i|}, \quad \tilde{V} = \frac{V}{|T_i|} + \frac{\varepsilon}{2|T_i|^{3/2}} X - \tilde{\sigma} \frac{\Theta}{|T_i|} - \frac{1}{4} \frac{X^2}{T_i} + \frac{\Psi}{T_i} T_i.$$

Moreover, the class $\mathcal{P}_0$ is normalized.

Corollary 5. A potential $V$ can be made to vanish in (12) by means of point transformations iff $V_{xx} = 0$, and the coefficient of $x$ in $V$ is a real-valued function of $t$.

Note 18. The action of $G^{\sim}_{\mathcal{P}_0}$ on $\sigma$ is simply multiplication with non-zero real constants and/or complex conjugation. That is why we can fix an arbitrary value $\sigma$, supposing e.g. $|\sigma| = 1$ and $\sigma_2 \geq 0$, and assuming only $V$ as an arbitrary element. The equivalence group $G^{\sim}_{\mathcal{P}_0}$ is formed by the transformations (6) where $T_i = 1$ if $\sigma_2 > 0$ and $T_i = \pm 1$ if $\sigma_2 = 0$. In what follows we use the notation $\sigma_1 = \text{Re} \sigma, \sigma_2 = \text{Im} \sigma$.

Substituting $S = \sigma \ln \rho + V(t, x)$ into equation (5) and subsequently splitting with respect to $\rho$ implies the additional equation $\tau_i = 0$. As a result, we obtain the following statement in view of Theorem 4.

Lemma 3. Any operator $Q$ from the maximal Lie invariance algebra $A_{\ln}(V)$ of equation (12) with an arbitrary potential $V$ can be represented in the form $Q = c_0 \partial_t + G(\chi) + \lambda M + \zeta I$. Moreover, the coefficients of $Q$ have to satisfy the classifying condition

$$c_0 V_t + \chi V_x = \frac{1}{2} \chi \mu x + \lambda t - i \zeta t - \sigma \zeta.$$

The kernel $G^{\cap}_{\ln}$ of the maximal Lie invariance groups of equations from the class $\mathcal{P}_0^\sigma$ is formed by the transformations (6), where $T = t, X = 0, \Theta = \Theta^0 \sigma_2 e^{-\sigma_2 t}, \Psi = \Psi^0 - \Theta^0 \sigma_1 e^{-\sigma_2 t}$ if $\sigma_2 \neq 0$ and $\Theta = \Theta^0, \Psi = \Psi^0 + \Theta^0 \sigma_1 t$ if $\sigma_2 = 0$. (Here $\Theta^0$ and $\Psi^0$ are arbitrary constants.) Its Lie algebra $A_{\ln}^\cap$ is $(\mathcal{M}, I')$, where $I' = e^{-\sigma_2 t}(\sigma_2 I - \sigma_1 M)$ if $\sigma_2 \neq 0$ and $I' = I + \sigma_1 t M$ if $\sigma_2 = 0$.

Hereafter we use the subscript ‘ln’ instead of ‘$\mathcal{P}_0^\sigma$’ for simplicity. It is necessary, though, to remember that all objects marked in this way are parametrized by $\sigma$.

The following chain of statements results in a complete group classification of (12).

The set $A_{\ln}^\cup = \{Q = c_0 \partial_t + G(\chi) + \lambda M + \zeta I\}$ is an (infinite-dimensional) Lie algebra under the usual Lie bracket of vector fields. For any $Q \in A_{\ln}$ where $(c_0, \chi) \neq (0, 0)$ we can find $V$ satisfying condition (14), i.e., $A_{\ln}^\cap = \langle U \cap A_{\ln}(V) \rangle$. Moreover, the space reflection belongs to the point symmetry groups of certain equations from $\mathcal{P}_0^\sigma$. The same statement is true for the Wigner time reflection if $\sigma$ is real. Therefore, $\mathcal{P}_0^\sigma$ is a strongly normalized class of differential equations. (As in the previous section, this was a reason for introducing the class $\mathcal{P}_0^\sigma$ since the whole class $\mathcal{P}_0$ is not strongly normalized.)

The group $G_{\ln}^\cap$ acts on $A_{\ln}^\cup$ and on the set of equations of the form (14) and, therefore, generates equivalence relations in these sets. The automorphism group generated on $A_{\ln}^\cup$ is isomorphic to $G_{\ln}^\cap/G_{\ln}^\cup$. The non-trivial equivalence group generated on the set of equations (14) is isomorphic to $G_{\ln}^\cap/G_{\ln}^\cap$ where $G_{\ln}^\cap$ is the normal subgroup of $G_{\ln}^\cap$ corresponding to $G_{\ln}^\cup$. The transformations from $G_{\ln}^\cap$ are gauge equivalence transformations of (14). $A_{\ln}^\cap = \langle M, I' \rangle$ is an ideal of the algebra $A_{\ln}^\cup$ and is invariant with respect to $G_{\ln}^\cap$. 

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Let $A^1$ and $A^2$ be the maximal Lie invariance algebras of some equations from the class $\mathcal{P}_0$, and $\gamma^i = \{ V \mid A_{\text{in}}(V) = A^i \}, i = 1, 2$. Then $V^1 \sim V^2 \text{ mod } G_{\text{in}}^0 \iff A^1 \sim A^2 \text{ mod } G_{\text{in}}^\sim$.

A complete list of $G_{\text{in}}^\sim$-inequivalent one-dimensional subalgebras of $A_{\text{in}}^i$ is exhausted by the algebras $\langle \partial_t \rangle$, $\langle G(\chi) + \zeta I \rangle$, $\langle \lambda M + \zeta | \rangle$. (There exist additional equivalences in $\{ \langle G(\chi) + \zeta I \rangle \}$ and $\{ \langle \lambda M + \zeta | \rangle \}$, which are generated by translations with respect to $t$ and, if $\sigma_2 = 0$, by the Wigner time reflection $I_t$.)

**Theorem 9.** A complete set of inequivalent cases of $V$ admitting extensions of the maximal Lie

invariance algebra of equation of the form (12) is exhausted by the potentials given in Table 2.

| N | $V$ | Basis of $A_{\text{in}}(V)$ |
|---|---|---|
| 0 | $V(t, x)$ | $M$, $I'$ |
| 1 | $V(x)$ | $M$, $I'$, $\partial_t$ |
| 2 | $v(t)x^2 + iv(t)x$ | $M$, $I'$, $G'(\chi^1)$, $G'(\chi^2)$ |
| 3 | $\nu x$ | $M$, $I'$, $\partial_t$, $G'(\chi)$, $G'(t)$ |
| 4 | $\mu^2 x^2 + ivx$ | $M$, $I'$, $\partial_t$, $G'(e^{-2\mu t})$, $G'(e^{2\mu t})$ |
| 5 | $-\mu^2 x^2 + ivx$ | $M$, $I'$, $\partial_t$, $G'(\cos 2\mu t)$, $G'(\sin 2\mu t)$ |

Here $v(t), w(t), \mu, \nu \in \mathbb{R}$, $(v_t, w_t) \neq (0, 0), \mu > 0, \nu \geq 0$. The functions $\chi^1 = \chi^1(t)$ and $\chi^2 = \chi^2(t)$ form a fundamental system of solutions to the ordinary differential equation $\chi_{\mu} = 4\chi_{t}$. $G'(\chi) = G(\chi) - \chi I - \sigma_1 \int e^{-\sigma_2 t} \chi dt M$, where $\chi = \int e^{\sigma_2 t} w \chi dt$ and $w = \nu = \text{const}$ in Cases 3, 4 and 5.

**Proof.** Let $A_{\text{in}}(V) \neq A^1_{\text{in}}$, i.e., equation (12) has a Lie symmetry extension for the potential $V$. This means that $A_{\text{in}}(V)$ contains an operator $Q = c_0 \partial_t + G(\chi) + \lambda M + \zeta I$ with $(c_0, \chi) \neq (0, 0)$.

If $c_0 \neq 0$ then $\langle Q \rangle \sim \langle \partial_t \rangle \text{ mod } G_{\text{in}}^\sim$, i.e., we obtain Case 2[1]. The investigation of additional extensions is reduced to the next case.

If $c_0 = 0$ then $\langle Q \rangle \sim \langle G(\chi) + \zeta I \rangle \text{ mod } G_{\text{in}}^\sim$. It follows from (14) that the potential $V$ has the form $V = v(t)x^2 + (\tilde{w}(t) + iv(t))x + \tilde{u}(t) + iv(t)$, and $\tilde{u}, \tilde{w} = 0 \text{ mod } G_{\text{in}}^\sim$. For $(v_t, w_t) \neq (0, 0)$ we have Case 2[2]. The condition $v, w = \text{const}$ results in Cases 2[3], 2[4] and 2[5] depending on the sign of $v$, and $w = \nu$.

### 6.4 Power nonlinearity

The most interesting (and, at the same time, most difficult) subclass of class (9) from the point

of view of group analysis is formed by the equations with power modular nonlinearities

$$i\psi_t + \psi_{xx} + \sigma |\psi|^\gamma \psi + V(t, x)\psi = 0.$$  \hspace{1cm} (15)

Here $\sigma$ and $\gamma$ are arbitrary non-zero complex and real constants, respectively, $\gamma \neq 0$ and $V$ is an arbitrary complex-valued potential depending on $t$ and $x$. In view of Theorem 8 this subclass admits extensions of the equivalence and Lie symmetry groups of its equations in comparison with the whole class (9).

It is possible to consider the whole class $\mathcal{P}$ of equations having the form (15), where $\gamma$ runs through $\mathbb{R}\setminus\{0\}$. A drawback of the above approach is the need to consider the extended equivalence group (equivalence transformations of $x$ will depend on $\gamma$, see Theorem 10), since $\mathcal{P}$ is a normalized class in the extended sense only.

In view of Corollary 8 $\gamma$ is an invariant of all admissible transformations in the superclass (11), i.e., equations with different values of $\gamma$ do not transform into one another. Therefore, it is more natural to interpret (15) as a family of classes parametrized with $\gamma$ and then to fix an arbitrary...
value of \( \gamma \). We assume below that \( \gamma \) is fixed and denote the class of equations (15) corresponding to the fixed value \( \gamma \) as \( P_\gamma \). (See also Section 6.2 where the class \( P_\gamma \) is first introduced.) It is derived from the superclass (11) by imposing the conditions \( S_{\rho t} = S_{\rho x} = 0, (\rho S_\rho)_\rho = \gamma S_\rho, \) i.e.,

\[
\psi S_{\psi t} + \psi^* S_{\psi^* t} = \psi S_{\psi x} + \psi^* S_{\psi^* x} = 0, \quad (\psi \partial_\psi + \psi^* \partial_{\psi^*})^2 S = \gamma (\psi \partial_\psi + \psi^* \partial_{\psi^*}) S.
\]

(16)

Similarly to the previous sections, we have two ways of finding the equivalence group \( G^\gamma_\gamma \) of the class \( P_\gamma \) in the framework of the direct method. In the first approach we look for all point transformations in the space of the variables \( t, x, \psi, \psi^* \) and \( S \) and \( S^* \), under which the system formed by equations (5) and (16) is invariant. Moreover, in the same way we can easily classify all possible point transformations in \( P_\gamma \) using Note 9. The second, completely equivalent, way of proceeding is to consider \( \sigma \) and \( V \) as arbitrary elements instead of \( S \) and to find all point symmetry transformations for the system

\[
i \psi_t + \psi_{xx} + \sigma |\psi|^2 \psi + V \psi = 0, \quad V_\psi = V_{\psi^*} = 0, \quad \sigma_t = \sigma_x = \sigma_\psi = \sigma_{\psi^*} = 0.
\]

(Under the prolongation procedure for equivalence transformations, we suppose \( \psi \) is a function of \( t \) and \( x \) as well as that \( \sigma \) and \( V \) are functions of \( t, x, \psi \) and \( \psi^* \).)

Theorem 10. The class \( P_\gamma \), where \( \gamma \neq 0 \), is normalized. The equivalence group \( G^\gamma_\gamma \) is formed by the transformations (10), where \( e^\Theta = \kappa |T_t|^{-1/\gamma}, \kappa > 0 \). The corresponding transformations with respect to \( \sigma \) and \( V \) have the form

\[
\tilde{V} = \frac{V}{|T_t|} + \frac{2 T_{tt} T_t - 3 T_{tt}}{16 \varepsilon T_t^3} x^2 + \frac{\varepsilon \varepsilon'}{2 |T_t|^{1/2}} \left( \frac{X_t}{T_t} \right)_t + \frac{\Psi_t}{T_t} - \frac{X_t^2}{4 T_t^2} + i \gamma T_t \frac{\gamma}{T_t^2}, \quad \tilde{\sigma} = \frac{\sigma}{\kappa^\gamma}
\]

with \( \gamma' = \frac{1}{\gamma} - \frac{1}{4} \).

Corollary 6. A potential \( V \) in (15) can be transformed to an \( x \)-free one if

\[
V = v(t)x^2 + u(t)x + \tilde{w}(t) + i w(t),
\]

(17)

where \( v, u, \tilde{w} \) and \( w \) are real-valued functions of \( t \). In particular, if \( \gamma = 4 \) any real-valued potential quadratic in \( x \) can be made to vanish. In the case \( \gamma \neq 4 \) the potential (17) is equivalent to zero iff \( 16(\gamma')^2 v = 2 \varepsilon \varepsilon' w_t + w^2 \).

Note 19. It follows from Theorem 10 that any point transformation in \( P_\gamma \) acts on \( \sigma \) only as multiplication with non-zero real constants and/or complex conjugation. Therefore, we can assume that \( \sigma \) is fixed in our consideration below under the assumption that \( |\sigma| = 1 \) and \( \sigma_2 \geq 0 \) and consider only \( V \) as an arbitrary element. The equivalence group \( G^\gamma_\gamma \) of the corresponding restricted class \( P^\gamma_\gamma \) is formed by the transformations (11) where \( T_t > 0 \) if \( \sigma_2 > 0 \) and \( \kappa = 1 \). Hereafter \( \sigma_1 = \Re \sigma, \sigma_2 = \Im \sigma \).

Below we use the subscript ‘\( \gamma \)’ instead of ‘\( P^\gamma_\gamma \)’ for simplicity. Recall, however, that all objects marked in such a way may be parametrized by \( \sigma \).

The classifying condition (8) with \( S = \sigma \rho^\gamma + V(t, x) \) implies under splitting with respect to \( \rho \) that \( t_\gamma + \gamma \zeta = 0 \). Therefore, we have the following statement as a consequence of Theorem 4

Lemma 4. Any operator \( Q \) from the maximal Lie invariance algebra \( A_\gamma(V) \) of equation (15) with an arbitrary potential \( V \) can be represented in the form \( Q = D^\gamma(\tau) + G(\chi) + \lambda M, \) where \( D^\gamma(\tau) = D(\tau) - \gamma^{-1} \tau^1 I \). Moreover, the coefficients of \( Q \) have to satisfy the classifying condition

\[
\tau V_t + \left( \frac{1}{2} \tau t x + \chi \right) V_x + \tau V = \frac{1}{8} \tau t x^2 + \frac{1}{2} \chi_t x + \lambda t + i \gamma' \tau u.
\]

(18)

The kernel of maximal Lie invariance groups of equations from class (11) is \( G^\gamma_\gamma = \hat{G}^\gamma_\gamma \), and its Lie algebra is \( A^\gamma_\gamma = \langle M \rangle \).
Let us apply the framework of normalized classes for obtaining the complete group classification of the class $\mathcal{P}_\gamma$.

The set $A^\mathcal{L}_\gamma = \{ Q = D^\gamma(\tau) + G(\chi) + \lambda M \}$ is an (infinite-dimensional) Lie algebra under the usual Lie bracket of vector fields. For any $Q \in A^\mathcal{L}_\gamma$ where $(\tau, \chi) \neq (0, 0)$ we can find $V$ satisfying condition (18), i.e., $A^\mathcal{L}_\gamma = \langle \bigcup V, A_\gamma(V) \rangle$. Moreover, the space reflection belongs to the point symmetry groups of certain equations from $\mathcal{P}_\gamma$. The same statement is true for the Wigner time reflection if $\sigma$ is real. Therefore, $\mathcal{P}_\gamma^\perp$ is a strongly normalized class of differential equations. (Again this was a reason for introducing the class $\mathcal{P}_\gamma^\perp$ since the whole class $\mathcal{P}_\gamma$ is not strongly normalized.)

The group $G^\gamma_\sim$ acts on $A^\mathcal{L}_\gamma$ and on the set of equations of the form (18) and, therefore, generates equivalence relations in these sets. The automorphism group generated on $A^\mathcal{L}_\gamma$ and the non-trivial equivalence group generated on the set of equations (18) are isomorphic to $G^\gamma_\sim / \hat{G}^\gamma_\parallel$. $A^\mathcal{L}_\gamma = \langle M \rangle$ coincides with the center of the algebra $A^\mathcal{L}_\gamma$ and is invariant with respect to $G^\gamma_\sim$.

Let $A^1$ and $A^2$ be the maximal Lie invariance algebras of some equations from $\mathcal{P}_\gamma^\perp$, and $V_i = \{ V \mid A_\gamma(V) = A^i \}, i = 1, 2$. Then $V^1 \sim V^2 \mod G^\gamma_\sim$ iff $A^1 \sim A^2 \mod G^\gamma_\sim$.

A complete list of $G^\gamma_\sim$-inequivalent one-dimensional subalgebras of $A^\mathcal{L}_\gamma$ is exhausted by the algebras $\langle \partial_t \rangle$, $\langle \partial_x \rangle$, $\langle tM \rangle$, $\langle M \rangle$.

**Corollary 7.** If $A_\gamma(V) \neq A^\mathcal{L}_\gamma$ then $V_x V_x = 0 \mod G^\gamma_\sim$.

**Proof.** Under these assumptions there exists an operator $Q = D^\gamma(\tau) + G(\chi) + \lambda M \in A(V)$ which does not belong to $\langle M \rangle$. Condition (18) implies $(\tau, \chi) \neq (0, 0)$. Therefore, $\langle Q \rangle \sim \langle \partial_t \rangle$ or $\langle \partial_x \rangle \mod G^\gamma_\sim$, i.e., $V_x V_x = 0 \mod G^\gamma_\sim$.

**Theorem 11.** A complete set of inequivalent classes of $V$ admitting extensions of the maximal Lie invariance algebra of equations from $\mathcal{P}_\gamma$ is exhausted by the potentials given in Table 3.

| $\gamma$ | $\text{Basis of } A_\gamma(V)$ |
|----------|-------------------------------|
| $\gamma = 0$ | $M$ |
| $\gamma = 1$ | $iW(t)$, $M$, $\partial_x$, $G(t)$ |
| $\gamma = 2$ | $\frac{2\gamma t + \nu}{t^2 + 1}$, $M$, $\partial_x$, $G(t)$, $D^\gamma(t^2 + 1)$ |
| $\gamma = 3$ | $D^\gamma(t)$, $\partial_t$, $G(t)$ |
| $\gamma = 4$ | $\partial_t$, $G(t)$, $D^\gamma(t^2)$ |
| $\gamma = 5$ | $\partial_t$, $G(t)$, $D^\gamma(t)$ |
| $\gamma = 6$ | $\partial_t$, $G(t)$, $D^\gamma(t)$ |
| $\gamma = 7$ | $\partial_t$, $G(t)$, $D^\gamma(t^2)$ |

**Proof.** In view of Corollary 7 for proving Theorem 11 it is sufficient to study two cases: $V_x = 0$ and $V_t = 0$.

The subclass of equations from $\mathcal{P}_\gamma^\perp$ with potentials satisfying the additional assumption $V_x = 0$, i.e., $V = V(t)$, is also strongly normalized. Its equivalence group is generated by transformations of the form (19), where $T = (a_t t + a_0)/(b_1 t + b_0)$, $X = c_1 T + c_0$, $e^{i\Omega} = |T|^{-1/\gamma}$, and $\Psi$ is an arbitrary smooth function of $t$. $a_t$, $b_1$ and $c_1$ are arbitrary constants such that $a_1 b_0 - b_1 a_0 \neq 0$. We can make the real part of any $x$-free potential vanish with the above equivalence transformations. The more restricted subclass $W$ of equations from $\mathcal{P}_\gamma^\perp$ with purely imaginary $x$-free potentials is also strongly normalized, and moreover, its equivalence group $G_{\mathcal{W}}^\gamma$
Theorem 12. The class $\mathcal{C}$ is strongly normalized. The equivalence group $G_{\mathcal{C}}^\cap$ of this class is formed by the transformations

$$
\tilde{t} = T, \quad \tilde{x} = |T_t|^{1/2} O x + X,
$$

$$
\tilde{\psi} = |T_t|^{-1/2} \exp \left( i \frac{T_t}{8 |T_t|} x_a x_a + \frac{i}{2 |T_t|^{1/2}} O^{ba} x_a + i \Psi \right) \tilde{\psi},
$$

$$
\tilde{V} = \frac{\tilde{V}}{|T_t|} + \frac{2 T_{tt} T_t - 3 T_{tt}^2}{16 T_t} x_a x_a + \frac{\varepsilon_T}{2 |T_t|^{1/2}} \left( \frac{X_t^{b}}{|T_t|} O^{ba} x_a + \frac{\Psi_t}{T_t} - \frac{X_t^{a} X_t^{a}}{4 T_t^2} \right).
$$

Here $T, X = (X^1, X^2)$ and $\Psi$ are arbitrary smooth real-valued functions of $t,$ $T_t \neq 0,$ $\varepsilon_T = \text{sign } T$ and $O = (O^{ab})$ is an arbitrary constant two-dimensional orthogonal matrix.
Note 22. The equivalence group $G^\circ_C$ is generated by the continuous family of transformations of the form (19), where $T_t > 0$ and $\det O = 1$, and two discrete transformations: the space reflection $I_a$ for a fixed $a$ ($t = t$, $\tilde{x}_a = -x_a$, $\tilde{x}_b = -x_b$, $b \neq a$, $\tilde{\psi} = \psi$, $\tilde{\Phi} = \Phi$) and the Wigner time reflection $I_t$ ($\tilde{t} = -t$, $\tilde{x} = x$, $\tilde{\psi} = \psi^*$, $\tilde{\Phi} = \Phi^*$).

Any operator $Q$ from the maximal Lie invariance algebra $A(V)$ of the equation (2) with an arbitrary element $V$ can be represented in the form $Q = D(\tau) + \kappa J + G(\bar{\sigma}) + \chi M$, where

$$D(\tau) = \tau \partial_t + \frac{1}{2} \tau_t x_a \partial_a + \frac{1}{8} \tau_{tt} x_a x_a M - \frac{1}{2} \tau_I I,$$

$$G(\bar{\sigma}) = \sigma^a \partial_a + \frac{1}{2} \bar{\sigma}_t^a x_a M, \quad J = x_1 \partial_2 - x_2 \partial_1,$$

$\kappa$ is a constant, $\tau$, $\bar{\sigma} = (\sigma^1, \sigma^2)$ and $\chi$ are smooth real-valued functions of $t$ and $M = i(\psi \partial_\psi - \psi^* \partial_{\psi^*})$, $I = \psi \partial_\psi + \psi^* \partial_{\psi^*}$.

Moreover, the coefficients of $Q$ have to satisfy the classifying condition

$$\tau_t V + \left( \frac{1}{2} \tau_t x_a + \kappa \varepsilon_{ab} x_b + \sigma^a \right) V_a + \tau_t V = \frac{1}{8} \tau_{tt} x_a x_a + \frac{1}{2} \sigma^a \tau_t x_a + \chi_t,$$  \hspace{1cm} (20)

where $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{21} = -\varepsilon_{12} = 1$.

The operators $D(\tau)$, $J$, $G(\bar{\sigma})$ and $\chi M$, where $\tau$, $\bar{\sigma} = (\sigma^1, \sigma^2)$ and $\chi$ run through the whole set of smooth functions of $t$, generate an infinite-dimensional Lie algebra $A^\circ_C$ under the usual Lie bracket of vector fields. The non-zero commutation relations between the basis operators of $A^\circ_C$ are the following ones:

$$[D(\tau^1), D(\tau^2)] = D(\tau^1 \tau^2_t - \tau^2 \tau^1_t), \quad [D(\tau), G(\bar{\sigma})] = G(\tau \bar{\sigma}_t - \frac{1}{2} \tau_I \bar{\sigma}),$$

$$[D(\tau), \chi M] = \tau \chi_t M, \quad [J, G(\bar{\sigma})] = G(\sigma^2, -\sigma^1),$$

$$[G(\bar{\sigma}^1), G(\bar{\sigma}^2)] = \frac{1}{2} (\sigma^1 a \sigma^2 a - \sigma^2 a \sigma^1 a) M.$$  

Therefore, the subspaces $\langle \chi M \rangle$, $\langle G(\bar{\sigma}), \chi M \rangle$, $\langle J, G(\bar{\sigma}), \chi M \rangle$ and $\langle D(\tau), G(\bar{\sigma}), \chi M \rangle$ are ideals of $A^\circ_C$. The subspace $\langle D(\tau) \rangle$ is a subalgebra of $A^\circ_C$.

Assuming $V$ to be arbitrary and splitting (20) with respect to $V$, $V_t$ and $V_a$, we obtain that the Lie algebra of the kernel $G^\circ_C$ of maximal Lie invariance groups of equations from the class $C$ is $A^\circ_C = \langle M \rangle$. The complete group $G^\circ_C$ coincides with the projection, to $(t, x, \psi)$, of the normal subgroup $\tilde{G}^\circ_C$ of $G^\circ_C$, which include the transformations (19) acting as identity on the arbitrary element $V$ (i.e., $T = t$ and $X = \Psi_t = 0$).

For any fixed value of the arbitrary element $V$, the classifying condition (20) implies, in particular, a linear system of ordinary differential equations in the coefficients $\tau$, $\sigma^a$ and $\chi$ of the general form

$$\tau_{tt} = g^{00} \tau_t + g^{01} \tau + g^{0, a+1} \sigma^a + g^{04} \chi,$$

$$\sigma^a_{tt} = g^{a0} \tau_t + g^{a1} \tau + g^{a, a+1} \sigma^a + g^{a4} \chi,$$

$$\chi_t = g^{40} \tau_t + g^{41} \tau + g^{4, a+1} \sigma^a + g^{44} \chi,$$

where $g^{pq}$, $p, q = 1, \ldots, 4$, are functions of $t$ which are determined by $V$. Therefore, for any $V$ we obviously have

$$\dim A(V) \leq 9, \quad A(V) \cap \langle \chi M \rangle = \langle M \rangle, \quad \dim A(V) \cap \langle G(\bar{\sigma}), \chi M \rangle \leq 5,$$

$$\dim \text{pr}_{\langle D(\tau) \rangle} A(V) \cap \langle D(\tau), G(\bar{\sigma}), \chi M \rangle \leq 3.$$
We first list all possible inequivalent cases of Lie symmetry extensions for the class $C$, then briefly describe how to derive the presented classification list, and finally provide additional explanations on the extension cases. A detailed proof will be the subject of a forthcoming paper.

In what follows, $U$ is an arbitrary real-valued function of its arguments or an arbitrary complex constant, the other functions and constants take real values,

$$ |x| = \sqrt{x_1^2 + x_2^2}, \quad \phi = \arctan \frac{x_2}{x_1}, \quad \omega = x_1 \cos t + x_2 \sin t, \quad \theta = -x_1 \sin t + x_2 \cos t. $$

Each item of the classification list consists of a value of the arbitrary element $V$ and a basis of the maximal Lie invariance algebra $A(V)$ of the corresponding equation $E$. More precisely, the presented algebras are maximal for the values of $V$ which are $G_C$-inequivalent to that possessing additional extensions of the Lie symmetry algebra. In those cases where the associated inequivalence conditions admit a simple formulation, they are explicitly indicated after the classification list.

0. $V = V(t, x): \quad M$

1. $V = U(x_1, x_2): \quad M, D(1)$

2. $V = U(\omega, \theta): \quad M, D(1) + J$

3. $V = |x|^{-2}U(\zeta), \quad \zeta = \phi - 2\beta \ln |x|, \beta > 0: \quad M, D(1), D(t) + \beta J$

4. $V = |x|^{-2}U(\phi): \quad M, D(1), D(t), D(t^2)$

5. $V = U(t, |x|) + \epsilon \phi, \epsilon \in \{0, 1\}: \quad M, J + \epsilon t M$

6. $V = U(|x|) + \epsilon \phi, \epsilon \in \{0, 1\}: \quad M, J + \epsilon t M, D(1)$

7. $V = |x|^{-2}U, U \neq 0: \quad M, J, D(1), D(t), D(t^2)$

8. $V = U(t, x_1): \quad M, G(0, 1), G(0, t)$

9. $V = U(\zeta), \quad \zeta = x_1: \quad M, G(0, 1), G(0, t), D(1)$

10. $V = t^{-1}U(\zeta), \quad \zeta = |t|^{-1/2}x_1: \quad M, G(0, 1), G(0, t), D(t)$

11. $V = (t^2 + 1)^{-1}U(\zeta), \quad \zeta = (t^2 + 1)^{-1/2}x_1: \quad M, G(0, 1), G(0, t), D(t^2 + 1)$

12. $V = x_1^{-2}U, U \neq 0: \quad M, G(0, 1), G(0, t), D(1), D(t), D(t^2)$

13. $V = U(t, \omega) + \frac{1}{2}(h_t - h)h^{-1}\theta^2 + h_t h^{-1}\omega \theta, h = h(t) \neq 0: \quad M, G(h \cos t, h \sin t)$

14. $V = U(\omega) + \frac{1}{3}(\alpha^2 - 1)\theta^2 + \alpha \omega \theta, \alpha \neq 0: \quad M, D(1) + J, G(e^{\alpha t} \cos t, e^{\alpha t} \sin t)$

15. $V = U(\omega) - \frac{1}{2}\theta^2 + \beta \theta: \quad M, D(1) + J, G(\cos t, \sin t) + \beta t M$

16. $V = h^{ab}(t)x_a x_b + ih^{00}(t), h^{12} = h^{21}, (h^{12} \neq 0 \lor h^{11} \neq h^{22}): \quad M, G(\bar{\sigma}^p), \quad p = 1, \ldots, 4,$

where $\{\bar{\sigma}^p = (\sigma^{p1}(t), \sigma^{p2}(t)), \quad p = 1, \ldots, 4\}$ is a fundamental set of solutions of the system $\sigma_{tt} = H \sigma, \quad H = (h^{ab})$.

17. $V = \frac{1}{4}\alpha x_1^2 + \frac{1}{2}\beta x_2^2 + i\gamma, \alpha \neq \beta: \quad M, G(\sigma^{11}), G(\sigma^{21}), G(0, \sigma^{12}), G(0, \sigma^{22}), D(1)$,

where the functions $\sigma^{11}$ and $\sigma^{21}$ (resp. $\sigma^{12}$ and $\sigma^{22}$) of $t$ form a fundamental set of solutions of the equation $\sigma_{tt} = \alpha \sigma$ (resp. $\sigma_{tt} = \beta \sigma$).
18. \( V = \frac{1}{4} \omega^2 + \frac{1}{4} \beta \theta^2 + i \gamma, \alpha \neq \beta: \)
\[ M, G(\sigma^{p_1} \cos t + \sigma^{p_2} \sin t, -\sigma^{p_1} \sin t + \sigma^{p_2} \cos t), p = 1, \ldots, 4, D(1) + J, \]
where \( \{\sigma^p = (\sigma^{p_1}(t), \sigma^{p_2}(t)), p = 1, \ldots, 4\} \) is a fundamental set of solutions of the system
\[ \sigma^2 - 2\sigma^2 = (\alpha + 1)\sigma^1, \sigma^2 \sigma^2 = (\beta + 1)\sigma^2. \]
19. \( V = iW(t): M, J, G(1, 0), G(t, 0), G(0, 1), G(0, t). \)
20. \( V = i: M, J, G(1, 0), G(t, 0), G(0, 1), G(0, t), D(1). \)
21. \( V = i\nu t^{-1}, \nu > 0: M, J, G(1, 0), G(t, 0), G(0, 1), G(0, t), D(t). \)
22. \( V = 2i\nu (t^2 + 1)^{-1}, \nu > 0: M, J, G(1, 0), G(t, 0), G(0, 1), G(0, t), D(t^2 + 1). \)
23. \( V = 0: M, J, G(1, 0), G(t, 0), G(0, 1), G(0, t), D(1), D(t), D(t^2). \)

The general case (Case 0) is included in the list for completeness.

To single out the main cases of the classification, we introduce the \( G_{c-}^\sim \)-invariant values
\[
\begin{align*}
  r_\sigma &= \text{rank}\{\sigma(t) | \exists \chi(t): G(\sigma) + \chi M \in A(V)\} \in \{0, 1, 2\}, \\
  r_J &= \text{dim}(\text{pr}_{(J)} A(V) \cap \langle J, G(\sigma), \chi M \rangle) \in \{0, 1\}
\end{align*}
\]
depending on the arbitrary element \( V \).

Let \( r_\sigma = r_J = 0. \) The set \( S^\tau(V) = \{\tau(t) | \exists \kappa, \sigma(t), \chi(t): D(\tau) + \kappa J + G(\sigma) + \chi M \in A(V)\} \) is a linear space which is closed with respect to the usual Poisson bracket of functions, i.e., \( \tau^1, \tau^2 \in S^\tau(V) \) if \( \tau^1, \tau^2 \in S^\tau(V). \) \( \dim S^\tau(V) \leq 3 < \infty. \) Lie’s classification of realizations of finite-dimensional Lie algebras by vector fields on a line implies that, up to local diffeomorphisms of \( t, S^\tau(V) \in \{1, (1, t), (1, t, t^2)\}. \) Hence, up to \( G_{c-}^\sim \)-equivalence, a basis of the algebra \( A(V) \) may include, additionally to the common operator \( M, \) one of the sets of operators \( \{D(1), \} \), \( \{D(1) + J, \} \), \( \{D(1), D(t) + J, \} \), \( \{D(1), D(t), D(t^2)\} \) which correspond to Cases 1, 2, 3 and 4. The algebra whose basis is presented in Case 3 (resp. Case 4) is the maximal Lie invariance algebra if and only if \( \beta \neq 0 \) and \( U_\zeta \neq 0 \) (resp. \( U_\phi \neq 0 \)).

Suppose that \( r_\sigma = 0 \) and \( r_J = 1. \) Then the algebra \( A(V) \) contains the operator \( J + G(\sigma) + \chi M \), where \( \sigma = 0 \) mod \( G_{c-}^\sim. \) The general form of \( V \) for which the associated equation from the class \( \mathcal{C} \) admits the operator \( J + \chi M \) is \( V = U(t, |x|) + h(t)\phi. \) We also have \( (|x|^{-1}U_{|x|} |x| \neq 0 \) or \( h \neq 0 \) if otherwise \( r_\sigma > 0. \) The subclasses of \( \mathcal{C} \) corresponding to the sets \( \{V = U(t, |x|) + h(t)\phi \} \) \( (|x|^{-1}U_{|x|} |x| \neq 0 \) and \( \{V = U(t, |x|) + h(t)\phi \} \) \( h \neq 0 \) as well as their union and their intersection are normalized. The equivalence groups of these subclasses are isomorphic and induced by the transformations of the form \( \text{[19]} \) with \( X = 0. \) The possible inequivalent extensions are exhausted by Cases 5–7.

Analogously, Cases 8–15 are singled out by the conditions \( r_\sigma = 1 \) and \( r_J = 0. \) In Cases 8–12 the following \( G_{c-}^\sim \)-invariant condition is additionally satisfied: If an operator \( G(\sigma) + \chi M \) belongs to \( A(V) \) then \( \sigma^2 = \sigma^1 \sigma^2, \) i.e., \( \sigma \) is proportional to a tuple of constants. The further consideration in this specific case is simplified due to the fact that any fixed operator with this property is \( G_{c-}^\sim \)-equivalent to the operator \( G(0, 1). \) If \( G(0, 1) \in A(V) \) then the potential \( V \) has the form \( V = U(t, x_1) \) and also \( G(0, t) \in A(V). \) The subclass of \( \mathcal{C} \) corresponding to the set \( \{V = U(t, x_1) | U_{111} \neq 0\} \) is normalized. Its equivalence group is induced by the transformations of the form \( \text{[19]} \) with \( T \) fractional linear in \( t \) and \( X = c_1 T + c_0, \) where \( c_1 \) and \( c_0 \) are arbitrary constants. Therefore, the inequivalent extensions in this subclass are exhausted by the inequivalent subalgebras of the algebra \( \langle D(1), D(t), D(t^2) \rangle, \) namely, by the subalgebras \( \langle D(1) \rangle, \langle D(t) \rangle, \langle D(t^2 + 1) \rangle, \langle D(1), D(t) \rangle \) and \( \langle D(1), D(t), D(t^2) \rangle \). The basis elements presented in Cases 9–11 give the maximal Lie invariance algebras for the corresponding values of the arbitrary element \( V \) if and only if \( U_{111} \neq 0 \) and \( \langle \zeta^2 U \rangle \zeta \neq 0. \) Cases 13–15 arise under the condition that
for an operator \( G(\sigma) + \chi M \) from \( A(V) \). Then \( \dim A(V) \cap \langle G(\sigma), \chi M \rangle = 2 \) since \( r_\sigma = 1 \). Moreover, each operator from the complement of \( \langle G(\sigma), \chi M \rangle \) in \( A(V) \) has to nontrivially involve both an operator \( D(\tau) \) and the operator \( J \), i.e., only one-dimensional extensions are possible.

The values of \((r_\sigma, r_J)\) not considered so far are \((2, 0)\), \((1, 1)\) and \((2, 1)\). In view of the commutation relation \([J, G(\sigma)] = G(\sigma^2, -\sigma^3)\), the case \( r_\sigma = r_J = 1 \) is impossible and we necessarily have \( r_\sigma = 2 \). The condition \( r_\sigma = 2 \) is equivalent to the potential \( V \) being a quadratic polynomial in \( x_1 \) and \( x_2 \) with coefficients depending on \( t \). More precisely,

\[
V = h^{ab}(t)x_ax_b + h^{0b}(t)x_b + i\hbar^{00}(t) + \hbar^{00}(t),
\]

where all the functions \( h \) are real-valued, \( h^{12} = h^{21} \). Denote by \( \mathcal{C}_q \) the subclass of equations from \( \mathcal{C} \) with these values of the arbitrary element \( V \). The subclass \( \mathcal{C}_q \) is normalized, its equivalence group coincides with the equivalence group \( G'_C \) of the whole class \( \mathcal{C} \) and it is partitioned by the conditions \( r_J = 0 \) and \( r_J = 1 \) into two normalized subclasses \( \mathcal{C}^0_q \) and \( \mathcal{C}^1_q \) with the same equivalence group as that admitted by \( \mathcal{C} \) and \( \mathcal{C}_q \). In terms of the functions \( h \), the subclass \( \mathcal{C}^1_q \) (resp. \( \mathcal{C}^0_q \)) is singled out by the condition \( h^{12} = h^{21} = 0 \) or \( h^{11} = h^{22} \) (resp. its negation).

The regular classification case for the class \( \mathcal{C}^0_q \) is Case 16. Only one-dimensional extensions necessarily involving operators of the form \( D(\tau) \) are possible in this class (Cases 17 and 18).

Any equation from the class \( \mathcal{C}^1_q \) is \( G'_C \)-equivalent to an equation from the same class whose potential has the form \( V = iW(t) \) with a real-valued function \( W = W(t) \). The subclass \( \mathcal{C}^1_q \) corresponding to the set \( \{ V = iW(t) \} \) is normalized. The regular classification case for this subclass is Case 19. The equivalence group of \( \mathcal{C}^1_q \) is induced by the transformations of the form \( \text{Case } 19 \) with \( T \) fractional linear in \( t \), \( X^a = c^a_1 T + c^a_0 \) and \( \Psi = \frac{1}{4} c^a_1 c^a_2 + d_0 \), where \( c^a_1, c^a_0 \) and \( d_0 \) are arbitrary constants. Therefore, the inequivalent extensions in \( \mathcal{C}^1_q \) are exhausted by the inequivalent subalgebras of the algebra \( \langle D(1), D(t), D(t^2) \rangle \), namely, by the subalgebras \( \langle D(1), \langle D(t) \rangle, \langle D(1), D(t), D(t^2) \rangle \rangle \) (Cases 20, 21, 22 and 23, respectively). The algebra \( \langle D(1), D(t) \rangle \) does not give a proper extension since the invariance of an equation from \( \mathcal{C}^1_q \) with respect to \( D(1) \) and \( D(t) \) implies vanishing \( V \) and, therefore, the invariance with respect to \( D(t^2) \). There exists a discrete equivalence transformation \( T \) for the set of potentials \( ivt^{-1}, \nu \in \mathbb{R}, \) which has the form \( \text{Case } 19 \) with \( T = -t^{-1}, X^a = 0, \Psi = 0 \). Its action on \( \nu \) is equivalent to alternating the sign of \( \nu \). Analogously, the Wigner time reflection \( I \) induces a change of sign of \( \nu \) in the potential \( iv(t^2 + 1)^{-1}. \) Therefore, up to \( G'_C \)-equivalence we can set \( \nu > 0 \) in Cases 21 and 22.

**Note 23.** Different potentials corresponding to the same case of the classification may be equivalent. This kind of equivalence can be used to additionally constrain parameters. In particular, we can set \( h^{11} + h^{22} = 0 \) in Case 16. In Cases 17 and 18 the parameters \( \alpha \) and \( \beta \) can be scaled up to permutation, i.e., we can assume that either \( \alpha = 1 \) and \( \beta \in [-1; 1) \) or \( \alpha = -1 \) and \( \beta \in (-1; 1). \)

### 8 Possible applications

Classical Lie symmetries and point equivalent transformations are traditionally used for the construction of exact solutions by different methods \([3, 59, 50]\), finding conservation laws \([39]\) etc. The implementation of this machinery for nonlinear Schrödinger equations with potentials and modular nonlinearities will be the subject of a forthcoming paper \([35]\). Here we only give a few examples on applications of equivalence transformations with the aim of establishing connections between known results and the derivation of new ones. (Note that we use scaling of variables and notations which may differ from those employed in the cited papers.)
Example 1. Consider cubic Schrödinger equations with stationary potentials, i.e., equations (15) with \( \gamma = 2 \), \( V = V(x) \) and \( \sigma \) real, so that it can be put equal to 1. After the Madelung transformation \( \psi = \sqrt{R} e^{i\varphi} \) in these equations, up to inessential signs we obtain the so-called Madelung fluid equations

\[
i\varphi_t + (\varphi_x)^2 + \frac{1}{\sqrt{R}} (\sqrt{R})_{xx} = R + V(x), \quad R_t + (R \varphi_x)_x = 0.
\]

Here \( R \) and \( \varphi \) are the new real-valued unknown functions of \( t \) and \( x \). In [5] (see also [8, p. 214]) the Lie symmetry properties of the Madelung fluid equations were investigated for the case \( V = c_1 x \). It was found that the equation with \( V = c_1 x \) possesses a five-dimensional maximal Lie invariance algebra. Then the Lie symmetry operations were used for the standard Lie reduction of the equations under consideration and for constructing their exact solutions. At the same time, it is evident from Theorem [11] and Corollary [6] that the equation \( i\psi_t + \psi_{xx} + |\psi|^2 \psi + c_1 x \psi = 0 \) is reduced to the usual cubic Schrödinger equation with \( V = 0 \) by means of the transformation (6) with \( T = t \), \( X = -c_1 t^2 \), \( \varepsilon = 1 \) and \( \Psi = c_1^2 t^3 / 3 \), i.e., the transformation

\[
\tilde{t} = t, \quad \tilde{x} = x - c_1 t^2, \quad \tilde{\psi} = \psi \exp \left( i c_1 t x + \frac{i}{3} c_1^2 t^3 \right).
\]

In the term of the probability density \( R \) and the phase function \( \varphi \) the transformation is \( \tilde{t} = t \), \( \tilde{x} = x - c_1 t^2 \), \( \tilde{R} = R \), \( \tilde{\varphi} = \varphi + c_1 t x + c_1^2 t^3 / 3 \). Therefore, all results on Lie symmetry and exact solutions of the Madelung equations with \( V = c_1 x \) can be derived from the analogous results for the usual cubic Schrödinger equation.

Example 2. The cubic Schrödinger equations with heterogeneity and dielectric loss have the general form

\[
i\psi_t + \psi_{xx} + R(t,x)|\psi|^2 \psi + V(t,x) \psi = 0,
\]

where \( V \) and \( R \) are complex- and real-valued functions of \( t \) and \( x \). The heterogeneity is implemented via the dependence of \( V \) and \( R \) on \( t \) and \( x \) (\( t \) is a spatial variable in some applications). The imaginary part of the potential \( V \) represents the dielectric loss. In [12] all the equations (21) possessing third-order generalized symmetries were found. They are exhausted by the equations with \( R \) depending only on \( t \) and

\[
V = \frac{(\chi^2)_{tt}}{4\chi^2} x^2 + i \left( \frac{\chi t}{\chi} + \frac{1}{2} \frac{R_t}{R} \right) + \zeta x + \varsigma,
\]

where \( \chi, \zeta \) and \( \varsigma \) are arbitrary functions of \( t \). It was proved in [12] that any such equation is reduced by point transformations to the standard cubic Schrödinger equations \( (R = \pm 1, V = 0) \). This evidently follows from results of our paper. Indeed, class (21) is a subclass of the class \( S \) from Section 5. For equations from this subclass, \( S = R(t,x)|\psi|^2 + V(t,x) \). Since the class \( S \) is normalized, any admissible transformation in class (21) is generated by a transformation from the equivalence group \( G_S \) of the class \( S \). Moreover, the transformations from \( G_S \), rewritten in terms of \( R \) and \( V \) instead of \( S \), form the equivalence group of class (21). Therefore, this class itself is normalized as well. It is obvious that any equation (21) possessing third-order generalized symmetries is reduced to the standard cubic Schrödinger equations by the transformation (6) with the functions \( T, X, \Psi \) and \( \Theta \) of \( t \), determined by the equations

\[
T_t = \chi^{-4}, \quad (\chi^4 X)_t = -2\chi^2 \zeta, \quad \Psi_t - \chi^4 X_t^2 / 4 = -\varsigma, \quad e^\Theta = \chi^2 |R|^{1/2}.
\]

Another strategy is to carry out the transformation step by step. At first, the parameter-function \( R \) is rendered equal to \( \pm 1 \) by the transformation \( \psi = \psi |R|^{1/2} \). The resulting equation belongs to the class \( P_2 \) and can be further simplified according to Theorem [10] and Corollary [6].
Example 3. As mentioned in the introduction, in the works \[71, 72, 73\] Wittkopf and Reid compute Lie symmetries of vector nonlinear Schrödinger systems of the form

\[ i\psi_t + \Delta \psi + S(t, x, \psi \cdot \psi^*) \psi = 0 \quad (22) \]

to test their programs for the symbolic solution of overdetermined systems of PDEs. Here \( S \) is a real-valued smooth function of its arguments, and its derivative with respect to \( \psi \cdot \psi^* \) does not vanish. The tuples \( x \) and \( \psi \) consist of \( n \) and \( m \) components, respectively.

For \( m = n = 1 \) the systems become equations and form the (normalized) subclass \( S' \) of the class \( S \) defined in Section 5. The additional auxiliary condition on the arbitrary element \( S \) is \( \text{Im} S = 0 \). It was proved in \[71, 73\] that the dimensions of the Lie invariance algebras of equations from the class \( S' \) do not exceed six; and the equations possessing six-dimensional Lie invariance algebras correspond to the arbitrary elements of the form \( S = \sigma |\psi|^4 + v(t)x^2 + u(t)x + w(t) \) (and, therefore, belong to the class \( \mathcal{P}_4 \)). Hereafter \( v, u \) and \( w \) are arbitrary (real-valued) functions of \( t \) and \( \sigma \) is an arbitrary (real) nonzero constant. In view of Corollary 6 of our paper, any such arbitrary element is point-transformation equivalent to \( S = \pm |\psi|^4 \). Since the classes \( S' \) and \( \mathcal{P}_4 \) are normalized, the point-transformation equivalence coincides with the equivalences generated by the corresponding equivalence groups. Therefore, the first statement of Theorem 1 from \[73\] can be reformulated in the following way. \textit{Any equation from class \( S' \), possessing a Lie invariance algebra of the maximal dimension for this class (i.e., dimension 6), is equivalent to the quintic Schrödinger equation.} The above theorem contains also the statement that the equations associated with the arbitrary elements of the form \( S = \sigma |\psi|^4 + u(t)x + w(t) \), where \( \gamma \neq 0, 4 \), admit five-dimensional Lie invariance algebras. According to Corollary 6 they are reduced to the potential-free equation with \( S = \pm |\psi|^4 \).

For general values of \( m \) and \( n \), the arbitrary elements of the form

\[ S = \sigma |\psi|^\gamma + v(t)x^2 + u(t) \cdot x + w(t) \]

were considered in \[71, 72, 73\]. The problem of finding Lie symmetries of the corresponding systems was proposed as a benchmark due to the evident dependence of its computational complexity on \( m \) and \( n \). In calculations for low values of \( n \) and \( m \), in particular, the equations with \( \gamma = 4/n \) were singled out as the ones having Lie invariance algebras of the maximal possible dimension in this class. It was conjectured that this dimension equals \( m^2 + n(n + 3)/2 + 3 \).

Note that in the case \( \gamma = 4/n \) the parameter-functions \( v, u \) and \( w \) can be made to vanish by means of an obvious generalization of the transformations from Theorem 10 of our paper. The system \( i\psi_t + \Delta \psi + |\psi|^{4/n} \psi = 0 \) admits a Lie invariance algebra \( g \) isomorphic to \( \text{sch}_n \oplus \text{su}_m \), where \( \text{su}_m \) is the special unitary algebra of order \( m \) and \( \text{sch}_n \) is the Lie invariance algebra of the corresponding single equation. (The particular case \( n = 2 \) is presented in \[67\].) The algebra \( g \) is generated by the operators

\[
\begin{align*}
\partial_t, & \quad t\partial_t + \frac{1}{2} x_j \partial_{x_j} - \frac{n}{4} I, \quad t^2 \partial_t + tx_j \partial_{x_j} - \frac{n}{2} tI + \frac{1}{4} x_j x_j M, \\
\partial_{x_j}, & \quad t\partial_{x_j} + \frac{1}{2} x_j M, \quad x_k \partial_{x_j} - x_j \partial_{x_k}, \ k < j, \\
\psi^a \partial_{\psi^b} - \psi^b \partial_{\psi^a} & + \psi^a \partial_{\psi^*} - \psi^b \partial_{\psi^*}, \ a < b, \\
i\psi^a \partial_{\psi^b} + i\psi^b \partial_{\psi^a} & - i\psi^a \partial_{\psi^*} - i\psi^b \partial_{\psi^*}, \ a \leq b.
\end{align*}
\]

where \( I := \psi^a \partial_{\psi^a} + \psi^{a*} \partial_{\psi^{a*}} \), \( M := i\psi^a \partial_{\psi^a} - i\psi^{a*} \partial_{\psi^{a*}} \). The indices \( j \) and \( k \) run from 1 to \( n \). The indices \( a \) and \( b \) run from 1 to \( m \) and summation over repeated indices is understood.

This implies that the above conjecture on the dimension is true.
Example 4. In [65] the matrix cubic Schrödinger equations with the potentials \( V = \frac{1}{2}(t + b)^{-1} \), i.e., the equations of the form
\[
i\Psi_t + \Psi_{xx} \pm \Psi\Psi^*\Psi + \frac{i}{2(t + b)}\Psi = 0 \tag{23}
\]
were investigated. Here \( \Psi = \Psi(t, x) \) is an \( m_1 \times m_2 \) matrix-function and \( \Psi^* \) is the corresponding conjugate (i.e., complex conjugate and transpose) matrix. The constant \( b \) is complex. The zero curvature representations of equations (23) were found and families of exact solutions were constructed by extending a version of the Bäcklund–Darboux transformation first introduced in [64]. In the scalar case \( (m_1 = m_2 = 1) \), equations (23) belong to the class \( P_2 \) (i.e., the class (15) with \( \gamma = 2 \)). Moreover, in this case for real values of \( b \) the potential \( V \) satisfies the conditions of Corollary 6 and, therefore, can be made to vanish by the transformation described in Note 20 up to translations. The explicit form of this transformation is
\[
\tilde{t} = -\frac{1}{t + b}, \quad \tilde{x} = \frac{x}{t + b}, \quad \tilde{\Psi} = (t + b) \exp\left(-\frac{i}{4(t + b)}x^2\right)\Psi.
\]
This transformation can be formally treated if \( \text{Im} \, b \neq 0 \). Also, it extends to the general matrix case without changing its form. Therefore, the transformation of equations (23) to the standard matrix cubic equation can be considered as a way of deriving results of [65] from analogous results of [64]. See also [66] for a generalization of this example.

9 Conclusion

The approach to group classification problems, proposed in [55, 54] and developed in this paper, seems to be quite universal. It is based on the notion of normalized classes of differential equations, which can be considered as a core for the further enhancement of group classification methods.

Depending on normalization properties of classes of differential equations, different strategies of group classification can be implemented. For a normalized class, the group classification problem is reduced to subgroup analysis of its equivalence group (or to subalgebra analysis of the corresponding algebra in the infinitesimal approach). No modifications of the classical formulation of group classification problems are necessary. A non-normalized class can be embedded into a normalized class which, possibly, is not minimal among the normalized superclasses. Another way is to partition the non-normalized class into a family of normalized subclasses and then to classify each subclass separately. In fact, both strategies are simultaneously applied in our paper to the class \( \mathcal{V} \) of \((1+1)\)-dimensional nonlinear Schrödinger equations with modular nonlinearities and potentials. The normalized superclass is the class \( \mathcal{S} \). The partition considered is formed by the subclasses of equations with logarithmical, power and general nonlinearities. If a partition into normalized subclasses is difficult to construct due to the complicated structure of the set of admissible transformations, conditional equivalence groups and additional equivalence transformations may be involved in the group classification.

Employing the machinery of normalized classes, we are enabled to effectively pose and solve new kinds of classification problems in classes of differential equations, e.g., to classify conditional equivalence groups or to describe the corresponding sets of admissible transformations.

The investigation of admissible transformations and normalization properties of classes of multidimensional partial differential equations is much more complicated than in the case of two independent variables. Nevertheless such investigations are possible and constitute an effective tool for solving group classification problems in the multidimensional case. The group classification of \((1+2)\)-dimensional cubic Schrödinger equations with potentials has been presented in
this paper as an example for the applicability of classification technique based on normalization properties in the case of more than two independent variables.

An interesting subject for further study is to derive results on existence and uniqueness of solutions of some boundary or initial value problems from known ones by means of equivalence transformations \([13]\) or to prove existence and uniqueness for similarity solutions \([31]\).

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