AN ELEMENTARY CONSTRUCTION OF NONLINEAR SEMIGROPS
FOR DELAY EQUATIONS IN HILBERT SPACES

Mikhail Anikushin

Abstract. We study the well-posedness of non-autonomous delay equations in Hilbert spaces. We present a construction of solving operators (non-autonomous case) or nonlinear semigroups (autonomous case) for a large class of delay equations, arising in practice. Our approach has lesser limitations and much more elementary than some previously known constructions of such semigroups and solving operators based on the theory of accretive operators. We also discuss the differentiability properties of such semigroups that allows to apply various dimension estimates using the Hilbert space geometry.

1. Introduction

In this paper we consider the following class of nonlinear non-autonomous delay differential equations in $\mathbb{R}^n$:
\begin{equation}
\dot{x}(t) = \bar{A}x(t) + \bar{B}F(t,\bar{C}x(t)),
\end{equation}
where $x(t) := x(t \circ \theta), \theta \in [-\tau,0]$, denotes the history segment; $\tau > 0$ is a constant; $\bar{A}: C([−\tau,0];\mathbb{R}^n) \to \mathbb{R}^n$, $\bar{B}: \mathbb{R}^m \to \mathbb{R}^n$ and $\bar{C}: C([−\tau,0];\mathbb{R}^n) \to \mathbb{R}^m$ are bounded linear operators and $F: \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^m$ is a nonlinear continuous function such that for some constant $\Lambda = \Lambda(t) > 0$, which is bounded in $t$ from compact intervals, we have
\begin{equation}
|F(t,y_1) - F(t,y_2)| \leq \Lambda(t)|y_1 - y_2| \mbox{ for all } y_1, y_2 \in \mathbb{R}^r, t \in \mathbb{R}.
\end{equation}
For $t \geq s$ we put $\Lambda_s^t := \sup_{s \leq \theta \leq t} \Lambda(\theta)$.

From the classical theory (that is the application of the Banach fixed point theorem), it follows that for any $\phi_0 \in C([−\tau,0];\mathbb{R}^n)$ and $t_0 \in \mathbb{R}$ there exists a unique classical solution $x(\cdot) = x(\cdot, t_0, \phi_0): [t_0 − \tau, +\infty) \to \mathbb{R}^n$ such that $x_{t_0} \equiv \phi_0$, $x(\cdot) \in C^1([t_0, +\infty);\mathbb{R}^n)$ and $x(\cdot)$ satisfies $\Lambda_s^t$ for $t \geq t_0$. We define the family of solving operators $\bar{U}(t,s): C([−\tau,0];\mathbb{R}^n) \to C([−\tau,0];\mathbb{R}^n)$, where $t \geq s$, given by $\bar{U}(t,s)\phi_0 := x_{t}(\cdot, s, \phi_0)$, where $x_{t}(\theta, s, \phi_0) = x(t + \theta, s, \phi_0)$ for $\theta \in [−\tau,0]$.

Consider the Hilbert space $\mathbb{H} := \mathbb{R}^n \times L_2(−\tau,0;\mathbb{R}^n)$ and the operator $A: D(A) \subset \mathbb{H} \to \mathbb{H}$ given by
\begin{equation}
(x, \phi) \mapsto \left( \bar{A}\phi, \frac{d}{d\theta}\phi \right),
\end{equation}
where $(x, \phi) \in D(A) := \{(x, \phi) \in \mathbb{H} \mid \phi(0) = x, \phi \in W^{1,2}(-\tau,0;\mathbb{R}^n)\}$. The bounded linear operator $B: \mathbb{R}^m \to \mathbb{H}$ is defined as $B\xi := (\xi,0)$ and we define the unbounded linear operator

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\( C : \mathbb{H} \to \mathbb{R}^r \) as \( C(x, \phi) := \tilde{C}\phi. \) Now (1.1) can be written as an abstract evolution equation in \( \mathbb{H} : \)
\[
\dot{v}(t) = Av(t) + BF(t, Cv(t)).
\]
(1.4)

We put \( E := C([\tau, 0]; \mathbb{R}^n) \) and consider the embedding \( E \subset \mathbb{H} \) given by \( \phi \mapsto (\phi(0), \phi). \) We identify the elements of \( E \) and \( \mathbb{H} \) under such an embedding. The only assumption we require for the measurement operator \( C, \) is the following.

(MES) There is a constant \( M_C > 0 \) such that for all \( T > 0 \)
\[
\int_0^T |Cv(t)|dt \leq M_C \left( |v(0)|_E + ||v(\cdot)||_{L_1(0, T; \mathbb{H})} \right).
\]
(1.5)

for all \( v(\cdot) \in L_1(0, T; \mathbb{H}) \cap C([0, T]; E). \)

Besides the case, when \( C : \mathbb{H} \to \mathbb{H} \) is bounded, the property in (MES) is satisfied for \( \delta \)-like measurements. For example, let \( r = 1, v(t) = (x(t), x_t) \) and \( \tilde{C}\phi := \phi(\tau). \) Then we have
\[
\int_0^T |Cv(t)|dt = \int_0^T |x(t - \tau)|dt = \int_{-\tau}^0 |x(t)|dt + \int_0^{T-\tau} |x(t)|dt \leq |v(0)|_E + ||v(\cdot)||_{L_1(0, T; \mathbb{H})}.
\]
(1.6)

Thus, (MES) covers all the cases arising in practise.

The main theorem of this paper is the following.

**Theorem 1.** Suppose that for equation (1.1) we have (1.2) and (MES) satisfied. Then for any \( t_0 \in \mathbb{R} \) and \( v_0 \in \mathbb{H} \) there is a unique generalized solution \( v(t) = v(t, t_0, v_0) \) to (1.4), which is a continuous function \( [t_0, +\infty) \to \mathbb{H}. \) This solution is uniquely determined by the property that the family of solving operators \( U(t, s)v_0 := v(t, s, v_0), \ t \geq s, \ in \ \mathbb{H} \) agrees with the family \( \tilde{U}(t, s) \) on \( E. \) Moreover, the following properties are satisfied

(LIP) For any \( T > 0 \) and \( s \in \mathbb{R} \) there are constants \( M_1 = M_1(T, \Lambda^T_\tau) > 0 \) and \( \varkappa = \varkappa(T, \Lambda^T_\tau) > 0 \) such that for all \( t \in [s, s + T] \) and \( v_1, v_2 \in \mathbb{H} \) we have
\[
|U(t, s)v_1 - U(t, s)v_2|_E \leq M_1e^{\varkappa(t-s)}|v_1 - v_2|_E.
\]
(1.7)

(EMB) For all \( t \geq s + \tau \) we have \( U(t, s)E \subset E \) and for any \( T \geq \tau, s \in \mathbb{R} \) and \( t \in [s + \tau, s + T] \) we have
\[
||U(t, s)v_1 - U(t, s)v_2||_E \leq M_1e^{\varkappa(t-s)}|v_1 - v_2|_E.
\]
(1.8)

(COM) The map \( U(t, s) : \mathbb{H} \to \mathbb{H} \) is compact for \( t \geq s + 2\tau. \)

(REG) For \( v_0 \in D(A) \) the generalized solution \( v(t) = v(t, t_0, v_0), \ t \geq t_0, \) is a classical solution to (1.4), i.e. we have \( v(\cdot) \in C^1([0, +\infty); \mathbb{H}), \ v(t) \in D(A) \) and \( v(t) \) satisfies (1.4) for all \( t \geq t_0. \)

The proof of Theorem 1 is given in the next section.

Our approach is very simple and it is based on three steps. The first one is the well-posedness of the linear problem in \( \mathbb{H}, \) including exponential estimates and the variation of constants formula. Such results for (partial) delay equations are given in [7]. The second step is the existence of classical solutions (in the space of continuous functions \( E \)) for the nonlinear problem, which give rise to classical solutions of the non-linear problem in \( \mathbb{H}. \) This is well-known for ordinary delay equations [12] and for partial delay equations [10, 11]. The third step is the derivation of elementary a priori estimates for the norm of classical solutions in \( \mathbb{H} \) with the use of the variation of constants formula, estimates for the linear problem, (1.2) and (MES). This allows us to obtain generalized solutions by the continuity.
Thus, the conclusion of Theorem 1 can be easily extended to some partial delay equations (see Remark 1 below for a discussion).

The well-posedness of nonlinear autonomous and non-autonomous (partial) delay equations in Banach spaces was studied in several papers, for example, [11, 21, 22]. The main approach in these papers is based on applications of the theory of accretive operators. Besides the fact that the theory itself is non-elementary and its applications require pages and pages of various estimates, the given applications have huge restrictions. For example, in [21, 22] the number of “independent” discrete delays, roughly speaking, cannot exceed \( n \) that is unnatural. In [22] there is also assumed some smoothness of the right-hand side in \( t \), which is linked with the construction of a family of equivalent norms to obtain the accretiveness condition. In [11] only the case of non-autonomous delay differential equations in \( \mathbb{R}^n \) is considered and the main restriction is posed on the nonlinear part that must be everywhere defined in \( \mathbb{H} \) (i.e., no discrete delays can appear in the nonlinear term) and some smoothness in \( t \) is also assumed. However, the authors of [11] considered a non-autonomous linear part and showed the well-posedness for the linear problem, but their assumptions on the linear part in our context allows to consider it as a nonlinear part of (1.1), i.e., (1.2) and (MES) are satisfied. Thus, our Theorem 1 covers in most and largely extends the final result of [11] as well as results from [21, 22]. Moreover, if we put \( \tilde{A} = 0 \) in (1.1), then the generation of a \( C_0 \)-semigroup for the operator \( A \) is obvious. Now, if we make \( F(t, \tilde{C}x_t) \) to be a linear function of \( x_t \) for each \( t \), then Theorem 1 can be considered as a well-posedness theorem for linear non-autonomous delay equations in \( \mathbb{H} \).

The consideration of delay equations in Hilbert spaces is rarely seen in the works on dynamical systems. Recent monographs on infinite-dimensional dynamical systems [8, 10] treat the delay equations in the classical context, i.e., in the space of continuous functions. However, posing delay equations in a proper Hilbert space setting sometimes has advantages. The first advantage is dimension estimates, which uses the Hilbert space geometry. There are few works [16, 19], where this is done for delay equations. In this direction we prove the quasi-differentiability property for the semigroup in \( \mathbb{H} \) given by (1.1), which allows to apply well-known dimension estimates [9, 13, 20] (see Section 3). Some recent advantages are given by the present author. In [2] it is proved a version of the frequency theorem (a theorem on the solution of special operator inequalities [14, 15, 1], which covers equations with discrete delays and leads to an extension of the well-known circle criterion for the global asymptotic stability of delay equations or existence of almost periodic solutions (see also [5, 6] for applications of the frequency theorem to study the existence and dimensional-like properties of almost periodic solutions to almost periodic ODEs). Essential parts of the proof are the Hilbert space geometry and a similar to (MES) assumption. In [4] the present author unified and generalized the results of R. A. Smith [17, 18] concerning the Poincaré-Bendixson theory, convergence theorems and construction of inertial manifolds for infinite-dimensional dynamical systems. For (partial) delay equations the application of these results is possible if we consider them in a proper Hilbert space setting and apply the frequency theorem to obtain a special operator.

This paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we prove a theorem on quasi-differentiability of the semigroup given by (1.4) in the autonomous case (Theorem 2).

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1However, the pioneering paper [16] has the lack of explanation as to how one should obtain semigroups in \( \mathbb{H} \) from delay equations and why they should be differentiable. It seems that the result from [16] concerning delay equations should be clarified, using the theory from the present paper.
2. Construction of delay semigroups

The following lemma is Theorem 3.23 from [7].

**Lemma 1.** The operator $A$ given by (1.3) is the generator of a $C_0$-semigroup $G(t)$, $t \geq 0$, in $\mathbb{H}$. In particular, there are constants $M_0, \varkappa > 0$ such that

$$|G(t)v_0|_{\mathbb{H}} \leq M_0 e^{\varkappa t} |v_0|_{\mathbb{H}} \text{ for all } t \geq 0 \text{ and } v_0 \in \mathbb{H}. \quad (2.1)$$

The following lemma is a particular case of Lemma 3.6 from [7].

**Lemma 2.** Let $x : [−\tau, \infty) \to \mathbb{R}^n$ be a function such that $x \in W^{1,2}_{loc}([−\tau, \infty); \mathbb{R}^n)$. Define the history function $\phi(t) := x_t$ for $t \geq 0$. Then $\phi \in C^1([0, +\infty); L_2(−\tau, 0; \mathbb{R}^n))$ and for all $t \geq 0$ we have $\phi(t) \in W^{1,2}(−\tau, 0; \mathbb{R}^n)$ and

$$\dot{\phi}(t) = \frac{d}{dt} \phi(t) \in L_2(−\tau, 0; \mathbb{R}^n). \quad (2.2)$$

An immediate corollary of Lemma 2 is the following.

**Corollary 1.** Let $x(t) = x(t, t_0, \phi_0)$ be the classical solution to (1.1) such that $\phi_0 \in W^{1,2}(−\tau, 0; \mathbb{R}^n)$. Then the function $v(t) = (x(t), x_t)$, $t \geq t_0$, is a classical solution to (1.4) with $v_0 := v(t_0) = \phi_0$ (or, more precisely, $v(t_0) = (\phi_0(0), \phi_0)$). Moreover, if we put $\xi(t) := F(Cv(t))$, then $v(\cdot)$ also satisfies the inhomogeneous equation

$$\dot{v}(t) = Av(t) + B\xi(t) \quad (2.3)$$

and, in particular, the variation of constants formula for $t \geq t_0$

$$v(t) = G(t − t_0)v_0 + \int_{t_0}^{t} G(t − s)BF(Cv(s))ds \quad (2.4)$$

is valid.

**Remark 1.** An analog of Lemma 2 for parabolic or hyperbolic equations can also be proved (see, for example, Theorem 3.29 in [7]). To proceed from Lemma 2 to Corollary 1 we have to establish the well-posedness (=existence of classical solutions) in the space of continuous functions. This is also well-studied for partial delay equations. See, for example, [8] [10].

Now we can give a proof of Theorem 1.

**Proof.** Let $x_j(t) = x_j(t, t_0, \phi_{0,j})$, where $t \geq t_0$ and $j = 1, 2$, be two classical solutions to (1.1) and put $v_j(t) := (x_j(t), x_t)$. If we suppose that $\phi_{0,j} \in W^{1,2}(−\tau, 0; \mathbb{R}^n)$, then $v_j(\cdot)$ satisfies (2.4). From this, (2.4), (1.2) and (MES) we get

$$|v_1(t) − v_2(t)|_{\mathbb{H}} \leq \leq M_0 e^{\varkappa(t−t_0)}|v_1(t_0) − v_2(t_0)|_{\mathbb{H}} + M_0 \Lambda_0^T \|B\| \int_{t_0}^{t} e^{\varkappa(s−t)}|C(v_1(s) − v_2(s))|ds \leq \leq (M_0 + M_0 M_C \Lambda_0^T \|B\|) e^{\varkappa(t−t_0)}|v_1(t_0) − v_2(t_0)|_{\mathbb{H}} + + M_0 M_C \Lambda_0^T \|B\| e^{\varkappa(t−t_0)} \int_{t_0}^{t} |v_1(s) − v_2(s)|_{\mathbb{H}}ds. \quad (2.5)$$

Putting $M_1 := (M_0 + M_0 M_C \Lambda_0^T \|B\|), \xi := (M_0 M_C \Lambda_0^T \|B\| + 1)e^{\varkappa T}$ and applying the Gronwall lemma, we get

$$|v_1(t) − v_2(t)|_{\mathbb{H}} \leq M_1 e^{\varkappa(t−t_0)}|v_1(t_0) − v_2(t_0)|_{\mathbb{H}} \text{ for all } t \in [t_0, t_0 + T]. \quad (2.6)$$

\footnote{Here, as in the previous section, $x_t(\theta) := x(t + \theta)$ for $\theta \in [−\tau, 0]$.}
From (2.6) for any \( t_0 \in \mathbb{R}, v_0 \in \mathbb{H} \) we can define a generalized solution \( v(t, t_0, v_0), t \geq t_0 \), by the continuity and density of \( W^{1,2}(-\tau, 0; \mathbb{R}^n) \) in \( \mathbb{E} \) and \( \mathbb{H} \). Indeed, let \( v_{0,k} \in W^{1,2}(-\tau, 0; \mathbb{R}^n) \) tend to \( v_0 \) in \( \mathbb{H} \). Then (2.6) shows that \( v_k(t) = v_k(t, t_0, v_{0,k}), t \in [t_0, t_0 + T], \) is fundamental in \( \mathbb{H} \) for any \( T > 0 \). Its limit is the generalized solution \( v(t, t_0, v_0), t \geq t_0 \), which is independent on the choice of \( v_{0,k} \). This proves the initial statement of Theorem 1 and (LIP). Moreover for \( T \geq t \geq t_0 + \tau \) we have
\[
\|v_1(t) - v_2(t)\|_{\mathbb{H}} \leq \sup_{\theta \in [-\tau, 0]} |v_1(t + \theta) - v_2(t + \theta)|_{\mathbb{H}} \leq M_1 e^{\kappa (t - t_0)} |v_1(t_0) - v_2(t_0)|_{\mathbb{H}}.
\]
(2.7) This proves that, in fact, the sequence \( v_k(t), t \in [t_0, t_0 + T] \), defined above is fundamental in \( \mathbb{E} \) and therefore \( v(t, t_0, v_0) \in \mathbb{E} \) for \( t \geq t_0 + \tau \). This proves the property stated in (EMB). In particular, the map \( U(t, s) \) for \( t \geq s + \tau \) takes bounded sets in \( \mathbb{H} \) into bounded sets in \( \mathbb{E} \), where \( U(t, s) \) coincides with \( \bar{U}(t, s) \). From the Arzelá-Ascoli theorem, the map \( \bar{U}(t, s) \) for \( t \geq s + \tau \) takes bounded sets in \( \mathbb{E} \) into precompact sets in \( \mathbb{E} \). Consequently, \( U(t, s); \mathbb{H} \to \mathbb{E} \) is compact for \( t \geq s + 2\tau \). This shows (COM). Summarizing the above, it is clear that (REG) is also satisfied. The proof is finished. 

\[ \square \]

3. Differentiability of delay semigroups

In this section we suppose that (1.1) is autonomous, i.e. \( F \) is independent of \( t \). We suppose also that \( F \in C^1(\mathbb{R}^r; \mathbb{R}^m) \). Note that from (1.2) it immediately follows that \( F' \) is bounded. Let the assumptions of Theorem 1 hold. Then there is a semi-flow \( \varphi^t: \mathbb{H} \to \mathbb{H} \) given by equation (1.4). For \( v_0 \in \mathbb{H} \) one can formally write from (1.4) the linearized along the trajectory \( \varphi^t(v_0) \) equation
\[
\dot{V}(t) = [A + BF'(C\varphi^t(v_0))C]V(t)
\]
(3.1)
Putting \( \bar{F}(t, \sigma) := F'(C\varphi^t(v_0))\sigma \), we see that (3.1) is of the form (1.3) with \( F \) changed to \( \bar{F} \). Thus, we have the following well-posedness result for (3.1), which immediately follows from Theorem 1.

Lemma 3. For any \( v_0 \in \mathbb{H} \) and \( \xi_0 \in \mathbb{H} \) equation (3.1) has a unique generalized solution \( V(t) = V(t; \xi_0; v_0) \) such that \( V(0) = \xi_0 \). For \( \xi_0 \in \mathcal{D}(A) \) the solution \( V(t) = V(t; \xi_0; v_0) \) is a classical solution, i.e. \( V(\cdot) \in C^1([0, +\infty); \mathbb{H}) \), \( V(t) \in \mathcal{D}(A) \) and \( V(t) \) satisfies (3.1) for all \( t \geq 0 \).

Let \( K \subset \mathbb{H} \) be an invariant compact set, i.e. \( \varphi^t(K) = K \) for all \( t \geq 0 \). From the smoothing property (EMB) and the fact that a compact invariant set consists of complete trajectories we have the inclusion \( K \subset \mathcal{D}(A) \).

(MES*) There is a constant \( M^* > 0 \) such that for all \( T > 0 \)
\[
\int_0^T |Cv(t)|^2 dt \leq MC \left( |v(0)|^2_{\mathbb{H}} + \|v(\cdot)\|^2_{L_2(0,T; \mathbb{H})} \right).
\]
(3.2)
for all \( v(\cdot) \in L_2(0, T; \mathbb{H}) \cap C([0, T]; \mathbb{H}) \).

Although (MES*) is slightly different than (MES), it is also satisfied for discrete delays (\( \delta \)-like measurements).

Lemma 4. Let \( F \in C^2(\mathbb{R}^r; \mathbb{R}^m) \) and (MES*) be satisfied. Then for any \( T > 0 \) there exists \( M_d > 0 \) such that
\[
|\varphi^t(v_2) - \varphi^t(v_1) - V(t; v_1; v_2 - v_1)|_{\mathbb{H}} \leq M_d |v_1 - v_2|_{\mathbb{H}}.
\]
(3.3)
for all \( t \in [0, T] \) and \( v_1, v_2 \in K \).
From this, (3.8) and the Gronwall inequality, we get the existence of

\[ \|v(t)\|_\mathcal{K} \leq M \int_0^t e^{\alpha(t-s)} |C(\varphi^s(v_2) - \varphi^s(v_1))|^2 ds + M \int_0^t e^{\alpha(t-s)} \|\delta(s)\|_\mathcal{H} ds. \]  

From (MES*) and (LIP) we get that there is a constant \( M' = M'(T) > 0 \) such that

\[ \int_0^t e^{\alpha(t-s)} |C(\varphi^s(v_2) - \varphi^s(v_1))|^2 ds \leq M' |v_1 - v_2|^2. \]  

From this, and the Gronwall inequality, we get the existence of \( M_d = M_d(T) > 0 \)

\[ \|\delta(t)\|_\mathcal{H} \leq M_d |v_1 - v_2|^2 \]  

for all \( t \in [0, T] \). The proof is finished.

Let \( \mathcal{K} \) be a compact invariant set. We say that the family \( \varphi^t \) is quasi-differentiable on \( \mathcal{K} \) w. r. t. the family of quasi-differentials \( L(t; v) \in \mathcal{L}(\mathcal{H}), t \geq 0 \) and \( v \in \mathcal{K} \), if for all \( t \geq 0 \)

\[ \sup_{v_1, v_2 \in \mathcal{K}, |v_1 - v_2|_{\mathcal{H}} \leq \varepsilon} \frac{|\varphi^t(v_2) - \varphi^t(v_1) - L(t; v_1)(v_2 - v_1)|_{\mathcal{H}}}{|v_1 - v_2|_{\mathcal{H}}} \to 0 \text{ as } \varepsilon \to 0^+ \]  

and the following properties are satisfied

\begin{align*}
\text{(QD1)} & \quad \sup_{t \in [0,1]} \|L(t; v)\| < \infty; \\
\text{(QD2)} & \quad L(t + s; v) = L(t; \varphi^s(v))L(s; v) \text{ for all } t, s \geq 0 \text{ and } v \in \mathcal{K}.
\end{align*}
Using the Gronwall lemma argument as above, one can show that $v$ is a bounded linear operator and $\delta \in K \ni (\varphi^t, t \geq 0)$, given by equation (3.11). Then the family $\varphi^t$ is quasi-differentiable w. r. t. the family of quasi-differentials $L(t; v)$ given by

$$L(t; v_0) := V(t; v_0; \xi),$$

where $V(t) = V(t; v_0; \xi)$ is the solution to (3.1) with $V(0) = \xi$. Moreover, the map $v_0 \mapsto L(t, v_0)$ is continuous as a map from $K$ to $L(H)$ for all $t \geq 0$.

**Proof.** If the linear operator $L(t; v): H \to H$ is defined as in (3.12) from Lemma 4, we have that (3.11) is satisfied for all $t \geq 0$. From (LIP) of Theorem 4 it follows that $L(t; v)$ is a bounded linear operator and (QD1) is satisfied. The property in (QD2) is equivalent to

$$V(t + s; v; \xi) = V(t; \varphi^s(v); V(s; v; \xi)) \text{ for all } t, s \geq 0, v \in K, \xi \in H. \quad (3.13)$$

but this is just the uniqueness of solutions to (3.1) given by Lemma 5. To show that the map $K \ni v_0 \mapsto L(t; v_0) \in L(H)$ is continuous for every $t \geq 0$ we define $\delta(t) := L(t; v_2)\xi - L(t; v_1)\xi = V(t; v_2; \xi) - V(t; v_1; \xi)$. Suppose that $\xi \in D(A)$. Then $\delta(\cdot)$ satisfies the equation

$$\delta(t) = \int_0^t G(t - s)B(F'(C\varphi^s(v_2))CV(t; v_2; \xi) - F'(C\varphi^s(v_1))CV(t; v_1; \xi))ds =$$

$$= \int_0^t G(t - s)B(F'(C\varphi^s(v_2)) - F'(C\varphi^s(v_1)))CV(t; v_1; \xi)ds +$$

$$+ \int_0^t G(t - s)BF'(C\varphi^s(v_2))C\delta(s)ds. \quad (3.14)$$

Using the Gronwall lemma argument as above, one can show that $v_2 \to v_1$ implies that $\delta(t) \to 0$ uniformly in $\xi \in D(A)$ with $|\xi| \leq 1$. But this gives the desired continuity. The proof is finished. \qed

The continuity of the map $v_0 \to L(t; v_0)$ plays an important role in dimension estimates. It is shown in [9] that this property is the only missing ingredient that makes the Hausdorff dimension estimate obtained by Constantin, Foias and Temam [20] hold for the fractal dimension also.

Using Theorem 2 one can proceed in a standard way [20, 19] to obtain an estimate for the function of singular values, using the linearized equation (3.1), and then an estimate for the fractal dimension [9] of invariant compacta.

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