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Inverse problem of potential theory

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ABSTRACT

P. Novikov in 1938 has proved that if \( u_1(x) = u_2(x) \) for \( |x| > R \), where \( R > 0 \) is a large number,

\[
u_j(x) := \int_{D_j} g_0(x, y)dy, \quad g_0(x, y) := \frac{1}{4\pi|x - y|},
\]

and \( D_j \subset \mathbb{R}^3, j = 1, 2, D_j \subset B_R \), are bounded, connected, smooth domains, star-shaped with respect to a common point, then \( D_1 = D_2 \). Here \( B_R := \{ x : |x| \leq R \} \).

Our basic results are:
(a) the removal of the assumption about star-shapeness of \( D_j \),
(b) a new approach to the problem,
(c) the construction of counter-examples for a similar problem in which \( g_0 \) is replaced by \( g = \frac{e^{ik|x-y|}}{4\pi|x-y|} \), where \( k > 0 \) is a fixed constant.

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1. Introduction

Suppose there are two bodies \( D_j, j = 1, 2 \), uniformly charged with charge density 1. Let the corresponding potentials be \( u_j(x) = \int_{D_j} \frac{dy}{4\pi|x - y|} \). Assume that \( u_1(x) = u_2(x) \) for \( |x| > R \), where \( R > 0 \) is a large number. The classical question (inverse problem of potential theory) is: does this imply that \( D_1 = D_2 \)?

P. Novikov in 1938 (see [1]) has proved a uniqueness theorem for the solution of inverse problem (IP) of potential theory under a special assumption, see Proposition 1.

Let

\[
u(x) = \int_{\overline{D}} g_0(x, y)dy, \quad g_0(x, y) := \frac{1}{4\pi|x - y|}, \tag{1}\]

where \( D \subset \mathbb{R}^3 \) is a bounded, connected, \( C^2 \)-smooth domain.

We use the following notations: \( D_j, j = 1, 2 \), are two different domains \( D \), \( S_j \) is the boundary of \( D_j \), \( D'_j = \mathbb{R}^3 \setminus D_j \), \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), \( B_R \) is the ball of radius \( R \), centered at the origin, \( B'_R = \mathbb{R}^3 \setminus B_R \).

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\[D_j \subset B_R, D_{12} := D_1 \cup D_2, D'_{12} := \mathbb{R}^3 \setminus D_{12}, D := D_{12} = D_1 \cap D_2, D' = \mathbb{R}^3 \setminus D, S\] is the boundary of \(D\), and let \(u_j(x) = \int_{D_j} g_0(x,y)dy, j = 1, 2\).

P. Novikov has proved the following result, see [1]:

**Proposition 1.** If \(u_1(x) = u_2(x)\) for \(|x| > R\), then \(D_1 = D_2\) provided that \(D_j, j = 1, 2\), are star-shaped with respect to a common point.

In [2] this result is generalized: the existence of the common point with respect to which \(D_1\) and \(D_2\) are star-shaped is not assumed, but \(D_j\) are still assumed star-shaped.

In [3], p. 334, a new proof of Proposition 1 was given. In [4–11], some inverse problems and symmetry problems are studied.

The goal of this paper is to give a new method for a proof of a generalization of Proposition 1. In this generalization
(a) The assumptions about star-shapeness of \(D_j, j = 1, 2\), are discarded (Theorem 1);
(b) A new approach to the IP is developed;
(c) A similar inverse problem is studied in the case when \(g_0(x,y)\) is replaced by the Green’s function of the Helmholtz operator, \(g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, k = \text{const} > 0\) is fixed, and \(u_j\) is replaced by

\[U_j(x) = \int_{D_j} g(x,y)dy.\]

The result is formulated in Theorem 2.

The idea of our proof does not use the basic idea of [1,2], or [3]. Our proof is based on some lemmas.

**Lemma 1.** If \(R(\phi)\) is the element of the rotation group in \(\mathbb{R}^3\) then \(\frac{\partial R(\phi)x}{\partial \phi}|_{\phi=0} = [\alpha, x]\), where \([x,y]\) is the cross product of two vectors, \(\alpha\) is the unit vector around which the rotation by the angle \(\phi\) takes place, and \(x\) is an arbitrary vector.

This lemma is proved in [3], p. 416.

**Lemma 2.** The set of restrictions on \(S\) of all harmonic in \(B_R\) functions is dense in \(L^2(S)\), where \(S\) is the boundary of \(D\).

**Proof of Lemma 2.** Let us assume the contrary and derive a contradiction. Without loss of generality one may assume \(f\) to be real-valued. Suppose \(f \not\equiv 0\) is orthogonal in \(L^2(S)\) to any harmonic function \(h\), that is,

\[\int_S fhds = 0\] (2)

for all harmonic functions in \(B_R\).

Define

\[v(x) := \int_S g_0(x,s)f ds.\]

Assumption (2) implies \(v(x) = 0\) in \(\mathbb{R}^3\). Indeed, there exists a unique solution to the problem

\[\Delta h = 0 \text{ in } D, \ h|_S = f.\]

For this \(h\) Eq. (2) implies that \(f = 0\), so \(v = 0\) in \(\mathbb{R}^3\). \(\square\)
Remark 1. The proof of Lemma 2 is valid for closed surfaces \( S \) which are not necessarily connected. For example, it is valid for \( S \) which is a union of two surfaces.

It is known (see, for example, [3]) that

\[
g_0(x, y) = g_0(|x|) \sum_{\ell \geq 0} \frac{|y|^\ell}{|x|} Y_\ell(y^0) Y_\ell(x^0), \quad |x| > |y|,
\]

where \( y^0 := x/|x| \), \( Y_\ell \) are the spherical harmonics, normalized in \( L^2(S^2) \), and \( |y|^\ell Y_\ell(y^0) \) are harmonic functions. The set of these functions for all \( \ell \geq 0 \) is dense in the set of all harmonic functions in \( B_R \).

Lemma 3. If \( S \) is a smooth closed surface and \([s, N] = 0 \) on \( S \), then \( S \) is a sphere.

Proof of Lemma 3. Let \( s = s(p, q) \) be a parametric equation of \( S \). Then \( N \) is proportional to \([s_p, s_q] \), where \( s_p \) is the partial derivative \( \frac{\partial s}{\partial p} \). If \([s, N] = 0 \), then \([s, [s_p, s_q]] = 0 \). Thus, \( s_p s_q - s_q s_p = 0 \), where \( s \cdot s_q \) is the dot product of two vectors. Since \( S \) is smooth, vectors \( s_p \) and \( s_q \) are linearly independent on \( S \). Therefore, \( \frac{\partial s q}{\partial p} = 0 \), and \( \frac{\partial s p}{\partial q} = 0 \). Consequently, \( s \cdot s = \text{const} \). This means that \( S \) is a sphere.

Lemma 3 is proved. \( \square \)

Lemma 3 is Lemma 11.2.2 in [3], see also Theorem 2 in [7]. Its short proof is included for convenience of the reader.

It follows from our proof that if \( S \) has finitely many points of non-smoothness, then the parts of \( S \), joining these points, are spherical segments.

Our new results are the following theorems.

Theorem 1. If \( u_1(x) = u_2(x) \) for \(|x| > R \), then \( D_1 = D_2 \).

Theorem 2. There may exist, in general, countably many different \( D_j \) such that the corresponding potentials \( U_j \) are equal in \( B'_R \) for sufficiently large \( R > 0 \).

Remark 2. If

\[
V_j := \int_{S_j} g(x, t) \, dt, \quad j = 1, 2,
\]

then there exist \( S_1 \neq S_2 \) for which \( V_1(x) = V_2(x) = 0 \) \( \forall x \in B'_R \).

An example can be constructed similarly to the one given in the proof of Theorem 2.

In Section 2 proofs are given.

2. Proofs

Proof of Theorem 1. If \( u_1 = u_2 \) for all \( x \in B'_R \) then it follows from the asymptotic of \( u_j \) as \(|x| \to \infty \) that \(|D_1| = |D_2| \). Thus, the case \( D_1 \subset D_2 \) is not possible if \( u_1 = u_2 \) for all \( x \in B'_R \).

The functions \( u_j \) are harmonic functions in \( D'_j \), that is, \( \Delta u_j = 0 \) in \( D'_j \). If \( u_1 = u_2 \) in \( B'_R \) and \( D_1 \neq D_2 \), then \( u_1 = u_2 \) in \( D'_1 \cap D'_2 \) by the unique continuation property for harmonic functions.

Let \( w := u_1 - u_2 \). Then \( w = 0 \) in \( D'_{12} \),

\[
\Delta w = \chi_2 - \chi_1,
\]

where \( \chi_j \) is the characteristic function of \( D_j \).
Let \( h \) be an arbitrary harmonic function in \( B_R \). Then
\[
\int_{D_2} h(x)dx - \int_{D_1} h(x)dx = 0,
\]
as one gets multiplying (4) by \( h \), integrating by parts and taking into account that \( w \) vanishes outside \( D_{12} \).

If \( h(x) \) is harmonic, so is \( h(R(\phi)x) \). Thus,
\[
\int_{D_2} h(R(\phi)x)dx - \int_{D_1} h(R(\phi)x)dx = 0. \tag{5}
\]
Differentiate (5) with respect to \( \phi \) and then set \( \phi = 0 \). Using Lemma 1, one gets:
\[
\int_{S_2} \nabla h \cdot [\alpha, x]ds - \int_{S_1} \nabla h \cdot [\alpha, x]ds = 0, \tag{6}
\]
where \( \alpha \in S^2 \) is arbitrary, and \( h \) is an arbitrary harmonic function in \( B_R \). Since
\[
\nabla h \cdot [\alpha, x] = \nabla \cdot (h[\alpha, x]),
\]
it follows from (6) and the divergence theorem that
\[
\int_{S_2} Nh[\alpha, s]ds - \int_{S_1} Nh[\alpha, s]ds = 0, \tag{7}
\]
for all \( \alpha \in S^2 \) and all harmonic \( h \) in \( B_R \). Here \( N \) is the unit normal to the boundary pointing out of \( D_j \).

If \( D_1 = D_2 \) then (7) is an identity. Suppose that \( D_1 \neq D_2 \). Since \( \alpha \) is arbitrary, it follows from (7) that
\[
\int_{S_2} h[N, s]ds - \int_{S_1} h[N, s]ds = 0, \tag{8}
\]
for all harmonic in \( B_R \) functions \( h \).

By Lemma 2 and Remark 1 it follows from (8) that \( [N, s] = 0 \) on \( S_2 \) and on \( S_1 \).

By Lemma 3 it follows that \( S_1 \) and \( S_2 \) are spheres, so \( D_1 \) and \( D_2 \) are balls. These balls must be of the same radius, as was mentioned earlier.

Now we have a contradiction unless \( D_1 = D_2 \), because two uniformly charged balls with the same total charge but with different centers cannot have the same potential in \( B'_R \). This follows from the explicit formula for their potentials. Theorem 1 is proved. □

**Proof of Theorem 2.** Let \( D_j = B_{a_j} \), where \( a_j > 0 \) are some numbers which are chosen below. Then
\[
U_j(x) := \int_{|y| \leq a_j} g(x,y)dy = 0 \text{ in the region } B'_{a_j} \text{ if and only if}
\]
\[
\int_0^{a_j} r^2 j_0(kr)dr = 0,
\]
where \( j_0(r) \) is the spherical Bessel function,
\[
j_0(kr) := \left( \frac{\pi}{2kr} \right)^{1/2} J_{1/2}(kr) = \frac{\sin(kr)}{kr}.
\]
This follows from the formula \( U_j(x) = \int_{B_{a_j}} g(x,y)dy \), and from the known formula for \( g(x,y) \) (see, for example, [3]):
\[
g(x,y) = \sum_{\ell \geq 0} ikj_\ell(k|y|)h_\ell(k|x|)Y_\ell(y^0)Y_\ell(x^0), \quad |y| < |x|.
\]
Here \( Y_\ell \) are the normalized spherical harmonics (see [3], p. 261), \( j_\ell \) and \( h_\ell \) are the spherical Bessel and Hankel functions (see [3], p. 262), and the known formula
\[
\int_{S^2} Y_\ell(y^0)dy^0 = 0, \quad \ell > 0
\]
was used.
One has
\[ \int_0^{a_j} r^2 j_0(kr) dr = \frac{\sin(ka_j)}{k^2} - \frac{a_j \cos(ka_j)}{k^2} = 0 \]
if and only if
\[ \tan(ka_j) = a_j. \]
This equation has countably many positive solutions. To each of these solutions there corresponds a ball \( B_{a_j} \) such that \( U_j = 0 \) in \( B'_{a_j} \). Thus, there are many different balls for which \( U_j \) are the same in \( B'_{R} \), namely \( U_j = 0 \) in \( B'_{R} \) for \( R > a_j \). Theorem 2 is proved. \( \square \)

References

[1] P. Novikov, On the uniqueness of inverse problem of potential theory, Dokl. Acad. Sci. USSR 18 (1938) 165–168.
[2] A. Margulis, Equivalence and uniqueness in the inverse problem of potential theory for homogeneous star-shaped bodies, Dokl. Acad. Sci. USSR 312 (1990) 577–580.
[3] A.G. Ramm, Inverse Problems, Springer, New York, 2005.
[4] A.G. Ramm, A symmetry problem, Ann. Polon. Math. 92 (2007) 49–54.
[5] A.G. Ramm, Scattering by Obstacles, D. Reidel, Dordrecht, 1986.
[6] A.G. Ramm, Scattering by Obstacles and Potentials, World Sci, Singapore, 2017 (in press).
[7] A.G. Ramm, The Pompeiu problem, Global J. Math. Anal. (GJMA) 1 (N1) (2013) 1–10 Open access Journal: http://www.sciencepubco.com/index.php/GJMA.
[8] A.G. Ramm, Symmetry problem, Proc. Amer. Math. Soc. 141 (N2) (2013) 515–521.
[9] A.G. Ramm, A symmetry result for strictly convex domains, Analysis 35 (1) (2015) 29–32.
[10] A.G. Ramm, Uniqueness of the solution to inverse obstacle scattering with non-over-determined data, Appl. Math. Lett. 58 (2016) 81–86.
[11] A.G. Ramm, Solution to the Pompeiu problem and the related symmetry problem, Appl. Math. Lett. 63 (2017) 28–33.