The quark’s behaviour while influenced by a strong stochastic gluon field is analyzed. An approximate procedure for calculating the effective Hamiltonian is developed and the corresponding ground state within the Hartree-Fock-Bogolyubov approach is found. The comparative analysis of various Hamiltonian models is given and transition to the chiral limit in the Keldysh model is discussed in detail.

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Nowadays we know that the mixing of the zero modes is the microscopic mechanism of the spontaneous breakdown of chiral symmetry in the instanton liquid model [1]. In this approach the quarks are considered in a given gluon background and the spectrum of the respective Dirac operator is calculated in order to be accompanied then by averaging over the gluon ensemble. It is believed that at low energy the zero modes are effectually overlapped and the eigenvalues of Dirac operator spread over some range of virtualities. In other words, studying the behaviour of a single quark in external (stochastic) field one endeavours to guess the corresponding one-particle Green function but for the quark ensemble even now. Being unable to carry out this transition in detail from the first principles one is forced to devise the suitable approximations argued by some general theoretical reasons. Also great care is needed in order to obtain proper thermodynamical limit with nonzero chiral condensate. A lot of that happens to be in striking contrast to the Nambu-Jona-Lasinio (NJL) model [2] which is cognate to the instanton liquid model based actually on the similar multi-fermion interaction. Superficially, the main distinction consists in the appearance of some non-local formfactors instead of corresponding coupling constant. As to the microscopic consideration, the generation of dynamical quark mass in the NJL model is caused by the reconstruction of the Hamiltonian ground state and the quarks manifest themselves already as the quasi-particles [2] although the multi-fermion attractive force should be strong enough, roughly speaking. In this paper we emphasize that an instanton model and several other models which are based on treating the stochastic ensemble of strong gluon field become practically identical in many aspects to the NJL model.

Such an approach is motivated by the conceptual idea of an intricate nature of the QCD vacuum [4] having populated by intensive stochastic gluon fields of nontrivial topological structure. Moreover, studying the corresponding cooled lattice configurations gives evidence of this component presence [5] and using the instantons in the singular gauge to fit the data turns out to be very fruitful [6] and allows one to evaluate the ensemble density (around one topological charge per fm$^4$) and the characteristic size of a saturating configuration (about 1 GeV$^{-1}$). Both estimates are in fairly good agreement with the corresponding results of instanton liquid model [7]. Nevertheless, the keen search of various confining configurations is still going on [8, 9, 10] in parallel with collecting the convincing evidences that the construction of self-consistent ensemble of such configurations is a too complicated problem (see, for example, the estimate for the (anti-)instanton ensemble done in Ref. [11]).

Supposing the high-frequency component of stochastic ensemble of gluon fields as the dominating contribution, we develop, in fact, an effective theory [37] with applying the procedure of simplified (averaged in time) system description which is widely used at studying the dynamical systems. Developing the effective theories which are discussed here has been launched to a considerable extent by studying the behaviour of light quarks in the instanton gas (liquid) [12]. The zero mode approximation has provided for the quantitative picture of spontaneous chiral symmetry breaking [13]. However, an effective Lagrangian of the NJL type was soon received in Ref. [14] by the direct summation of certain leading diagrams and the obtained vertices of multi-quark interactions occurred rather different from those calculated in the zero mode approximation. Analysis of heavy quark systems behaviour affected by the stochastic gluon fields [15] has demonstrated that at constructing the respective effective theory the cluster decomposition of generating functional [16] can be very efficient tool and the specific role of various characteristic correlation times has been clarified to classify the descriptions. These results together with the criticism of zero mode approximation [17] have contributed to widening the cluster decomposition applications. This approach has been used to analyse the light quark behaviour [18] and it is interesting to note the effective Lagrangian has agreed with that obtained

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in \[14\]. In the context of our interest here the cluster decomposition is called upon to describe the correlations in quantum system inspired by an external process. In this situation, as a matter of fact, the description of system behaviour is executed by averaging the generating functional. However, as we show in this paper such a procedure applied to the quantum system could be incomplete and it is more appropriate to base an analysis on the corresponding density matrix. Nonetheless, we argue here that in the ‘white noise’ limit (when the time intervals of stochastic impulses are very short) the procedure of averaging the generating functional occurs quite adequate.

The form of the effective Hamiltonian obtained urges us to search the system ground state as the Bogolyubov trail function. The corresponding dressing transformation will be analysed for various formfactors of effective Hamiltonian. In such an approach the quarks are already treated as the quasi-particles and rather practical way to get beyond the zero mode approximation appears. It grounds on the method of simple iterations of corresponding mean energy functional is demonstrated.

II. THE HARTREE–FOCK–BOGOLYUBOV APPROXIMATION

We consider the quark (anti-quark) ensemble in the background of strong stochastic gluon field and suppose this field is so strong that we could neglect the gluon interchanging processes (quenched approximation). The stochastic gluon field is characterized by a correlation function and its particular form will be discussed and fixed below. The Lagrangian density is the following

\[
\mathcal{L}_E = \bar{q} (i \gamma_\mu D_\mu + im) q ,
\]

here \( q, \bar{q} \) are the quark and anti-quark fields with covariant derivative \( D_\mu = \partial_\mu - ig A^a_\mu t^a \) where \( A^a_\mu \) is the gluon field, \( t^a = \lambda^a/2 \) are the generators of colour gauge group \( SU(N_c) \) and \( m \) is the current quark mass, \( \mu = 1, 2, 3, 4 \).

We work in the context of the Euclidean field theory and \( \gamma_\mu \) mean the Hermitian Dirac matrices \( \{ \gamma_\mu^+, \gamma_\mu^- \} = 2 \delta_{\mu \nu} \) in the chiral representation. Then the corresponding Hamiltonian description results from

\[
\mathcal{H} = \pi \dot{q} - \mathcal{L}_E , \quad \pi = \frac{\partial \mathcal{L}_E}{\partial \dot{q}} = iq^+ ,
\]

and, in particular, for the noninteracting fields we have

\[
\mathcal{H}_0 = -\bar{q} (i \gamma \nabla + im) q .
\]

In the Schrödinger representation the quark field evolution is determined by the equation for the quark probability amplitude \( \Psi \) as

\[
\dot{\Psi} = -H \Psi ,
\]

and the creation and annihilation operators of quarks and anti-quarks \( a^+, a, b^+, b \) have no ‘time’ dependence and consequently look like

\[
q_{\alpha i}(x) = \int \frac{dp}{(2\pi)^3} \frac{1}{(2|p_A|)^{1/2}} \left[ a(p, s, c) \ u_{\alpha i}(p, s, c) \ e^{ipx} + b^+(p, s, c) \ v_{\alpha i}(p, s, c) \ e^{-ipx} \right] ,
\]

here the summation over index \( s \) which stands to describe two quark spin polarizations and index \( c \) which should play the similar role for a colour is implied. Further we make concrete the form of the Dirac conjugated spinor. Fixing a spin polarization as it is known can be done by imposing an additional constraint on spinor (see, below). However, there is no direct analogy with the colour polarization and the particular state should be fixed by the corresponding complete set of diagonal operators which includes the Casimir operators as well. In fact, this complete definition of the spinor colour state is unnecessary for us here. All observables are usually expressed by summing up the polarization states of some bilinear spinor combinations as the singlet and octet states and the singlet component is obviously playing the specific role.

The density of interaction Hamiltonian can be presented as

\[
\mathcal{V}_S = \bar{q}(x) \ t^a \gamma_\mu A^a_\mu(t, x) \ q(x) .
\]
The obvious dependence on ‘time’ in this Hamiltonian is present in the gluon field only. As it is mentioned above we are planning to work with the stochastic gluon field implying the random process for which one may define only a probability of realizing some gluon configuration. Such a nature of gluon field urges (and allows) us to develop the approximate procedure for describing the quark field treating (4) as a probabilistic process. Then the system states are described by the corresponding averages (over a ‘time’ or an ensemble according to the ergodic hypothesis). However, in the quantum theory we face one difficulty in this way because $\Psi$ is a probability amplitude and an immediate averaging of $\langle \Psi \rangle$ can be insignificant. 

Studying a mean probability density $\langle \Psi \bar{\Psi} \rangle$ looks more promising and can be realized by complicating the procedure of continual integration [19]. In Appendix I we analyse convincing quantum mechanical example to illustrate the difference between two approaches. One of those is based on constructing the corresponding density matrix $\langle \Psi \bar{\Psi} \rangle$, and the second approach does use the relevant averaging of the functional $\langle \Psi \rangle$. We argue later the latter could be practical for application in the ‘white noise’ limit with the $\delta$-like time correlation function. Adapting these ideas to the gauge theories we should obviously strive to operate with the gauge invariant quantities which include an ordered exponential, at least. Unfortunately, such a program in what concerns the ensemble consideration is still very far to be realized. However, it is clear that applying the averaging procedure would result in putting in an appearance of a set of corresponding correlation functions $\langle A^2 \rangle$, $\langle A^4 \rangle$ etc. [20].

In the interaction representation, where $\Psi = e^{H_{\text{tot}}t} \Phi$, Eq. (4) can be rewritten as

$$\dot{\Phi} = -V \Phi, \quad V = e^{H_{\text{tot}}t} V S C e^{-H_{\text{tot}}t}. \quad (7)$$

Now the ‘time’ dependence appears in quark operators as well. Now we remind some features of the averaging description as formulated in Ref. [16]. Presenting Eq. (7) in the integral form as

$$\Phi(t) = \Phi(0) - \int_0^t d\tau V(\tau) \Phi(\tau), \quad (8)$$

where $\Phi(0)$ is an arbitrary initial state of ensemble and performing another iteration one receives

$$\Phi(t) = \Phi(0) - \int_0^t d\tau V(\tau) \Phi(0) + \int_0^t d\tau V(\tau) \int_0^\tau d\tau' V(\tau') \Phi(\tau').$$

By averaging the fast-changing component and uncoupling the correlators one approximately approaches the long wavelength component $\langle \Phi \rangle$ in the highest order (also taking into account that $\langle V \rangle = 0$) as follows

$$\langle \Phi(t) \rangle \approx \Phi(0) + \int_0^t d\tau \int_0^\tau d\tau' \langle V(\tau) V(\tau') \rangle \langle \Phi(\tau') \rangle. \quad (9)$$

Certainly, it is assumed the characteristic correlation time of stochastic process is smaller than the time characteristic for the process $\langle \Phi \rangle$. By differentiating Eq. (9) it is easy to get rid of the initial condition $\Phi(0)$ and to have

$$\langle \Phi(t) \rangle = + \int_0^t d\tau' \langle V(t) V(\tau') \rangle \langle \Phi(\tau') \rangle.$$ 

Actually this equation should describe a steady-state process and at reversing a time the solution, in general case, will not return to the initial magnitude $\Phi(0)$. Changing the integration variable as $\tau' = t - \tau$ one comes to

$$\langle \Phi \rangle = + \int_0^t d\tau \langle V(t) V(t - \tau) \rangle \langle \Phi(t - \tau) \rangle. \quad (10)$$

It is usually supposed the correlations are quickly decaying then the upper limit of integration might be changed for $\infty$ and in order to deal with the local process it is well justified (without a precision loss) to change the argument of function $\langle \Phi \rangle$ for $t$. Eventually, as a result we have

$$\langle \Phi(t) \rangle = + \int_0^\infty d\tau \langle V(t) V(t - \tau) \rangle \langle \Phi(t) \rangle. \quad (11)$$

(The requirements to validate the factorization of the long-wavelength component are discussed, for example, in [16].) Implementing the approximation (11) in the quantum field theory models, we run into the trouble at trying to get the most general form of correlation function if the characteristic quark and gluon correlation times are comparable. Fortunately, if the quark fields are considered to be practically constant on the gluon background the problem receives essential simplification. The gluon field contribution may be factorized as a corresponding correlation function $\langle A^2_\mu(x) A^2_\nu(y) \rangle$ [20]. Recent lattice measurements of this correlation function provide us with a reasonable arguments to interpret the result as gluon ‘mass’ generation ($\sim 300$ – $400$ MeV) in the momentum region of order 200 MeV [21].

It is curious to notice that the averaging over ensemble (‘time’) in the right hand side of Eq. (11) is performed in both the correlator and $\langle \Phi(t) \rangle$. It means that by resuming and averaging a certain class of diagrams in the quantum field theory models, one may take into account high order correlator contributions in different ways if the form of function $\langle \Phi(t) \rangle$ is specified. Besides, the correlation functions in models interesting to us should be translation invariant and it implies that the correlator in Eq. (11) has the following form

$$\langle V(t) V(t - \tau) \rangle = F(\tau),$$

i.e., for example, an one-dimensional process after having done the integration in Eq. (11) will be described by a constant which characterizes the slow process. In quantum field theory for the problem we are interested in, the correlator connecting two space points
\[
\langle \Phi(t) \rangle = \int dx \, \bar{q}(x, t) \, t^a \gamma_\mu \, q(x, t) \int_0^\infty d\tau \int dy \, \bar{q}(y, t - \tau) \, t^b \gamma_\nu \, q(y, t - \tau) \, g^2 \langle A^a_\mu (t, x) A^b_\nu (t - \tau, y) \rangle \langle \Phi(t) \rangle
\]

appears instead of a constant. Assuming the correlation function is rapidly decreasing with time we change the 'time' \( t - \tau \) dependence in the quark fields for \( t \) and per-

form the inverse transformation to the Schrödinger representation. Then introducing the function \( \chi = e^{-H_{\text{ind}}(\Phi)} \) we have \[39\] the following equation

\[
\dot{\chi} = -H_{\text{ind}} \chi,
\]

\[
\mathcal{H}_{\text{ind}} = -\bar{q} \left( i\gamma \nabla + im \right) q - \bar{q} \, t^a \gamma_\mu \, q \int dy \, \bar{q}' \, t^b \gamma_\nu \, q' \int_0^\infty d\tau \, g^2 \langle A^a_\mu A^b_\nu \rangle,
\]

where \( q = q(x) \), \( \bar{q} = \bar{q}(x) \), \( q' = q(y) \), \( \bar{q}' = \bar{q}(y) \), \( A^a_\mu = A^a_\mu (t, x) \) and \( A^b_\nu = A^b_\nu (t - \tau, y) \).

In order to receive the final result we should fix the form of correlation function. In this paper we rely on the stochastic ensemble of (anti-)instantons in the singular gauge

\[
A^a_\mu (x) = \sum_{i=1}^N A^a_\mu (x; \gamma_i),
\]

and instanton solution reads as

\[
A^a_\mu (x) = \frac{2}{g} N \omega_{ab} \eta_{nu} \int \frac{dq}{(2\pi)^4} q_n \, \phi(q) \, e^{i q(x - z)},
\]

\[
\phi(q) = \frac{1}{q^2} \left( K_2(q \rho) - \frac{2}{q^2 \rho^2} \right),
\]

where \( K_2 \) is the modified Bessel function of imaginary argument, \( \rho \) is the instanton size, the matrix \( \omega \) appoints the pseudo-particle orientation in colour space, \( z \) is the coordinate of instanton center and \( \eta \) stands for the 't Hooft symbol. The distribution of the pseudo-particle orientation in colour space is supposed to be homogeneous \( \sim d\omega \) as well as the probability to find a pseudo-particle in the volume element is proportional \( \sim dV \) where \( V \) is the volume of the system under consideration. Apparently, specifying the saturating configuration in the form of Eq. \[13\] is, in a direct way, the gauge fixing procedure. Calculating the quantum corrections for every single pseudo-particle in one-loop approximation (what corresponds to the zeroth order of the \( N/V \)-expansion), and exploiting the variation principle \[7\], \[11\] allows one to ascertain the size distribution of pseudo-particles. In this way it is possible to attach clear meaning to the functional and to construct in the thermodynamical limit \( \lim_{V \to \infty} N/V \to n \) the state possessing a negative energy density and developing a non-zero gluon condensate. (Uncertain interrelation of perturbative and non-perturbative contributions into the path integral \[22\] makes the computability of generating functional highly nontrivial as for now.) In Eq. \[12\] we imply the correlation function integrated over the 'time' for which we receive in the highest order in the density \( n \) of (anti-)instanton ensemble

\[
\int_0^\infty dx_4 \, \langle A^a_\mu(x) A^b_\nu(y) \rangle = \frac{1}{2} \int_{-\infty}^\infty dx_4 \, \langle A^a_\mu(x) A^b_\nu(y) \rangle = \frac{4(4\pi^2)^2}{g^2} \frac{\delta_{ab}}{N_c^2 - 1} \left( \delta_{\mu\nu} \delta_{\alpha\beta} - \delta_{\mu\alpha} \delta_{\nu\beta} \right) \times \int \frac{dp}{(2\pi)^4} \, p_{\alpha\beta} \, e^{i p(x - y)} \, \phi(-p) \phi(p) \frac{1}{2} \delta(p_4).
\]

The first equality is valid due to the symmetry properties of instanton solution. Then the correlation function can be presented as

\[
\langle \tilde{A}^a_\mu \tilde{A}^b_\nu (P) \rangle = \frac{(4\pi^2)^2}{g^2} \frac{\delta_{ab}}{N_c^2 - 1} \left( I(p) \delta_{\mu\nu} - J_{\mu\nu}(p) \right)
\]

\[
I(p) = p^2 \phi(-p) \phi(p), \quad J_{ij}(p) = p_i p_j \phi(-p) \phi(p),
\]

\[
J_{44} = J_{41} = J_{44} = 0.
\]

We suppose in what follows the various stochastic ensembles of gluon fields are characterized by their profile functions \( I(p) \), \( J_{\mu\nu}(p) \) and analyze the contribution of quadratic correlator only. However, this deficiency of fixing the gauge implicitly for the truncated system is compensated, in a sense, by our investigation of full spectrum of reasonable correlation functions (including an opposite limiting correlators when they are extrapolated even into the perturbative region). Recent considerable progress in studying the confining configurations of lattice gauge theories, in particular, revealing the monopole clusters and their role in confinement (see, for review \[23\]) as well as detecting the specific features of quark behaviour in the uncooled configurations and
the indications that low-lying Dirac eigenmodes are localized on the objects of dimension inherent in monopoles and vortices looks entirely urging. But these results are also bringing the perilous tendencies because reveal some features of lattice gauge theories common with the compact electrodynamics. This fact makes almost inevitable to draw in the singular (in the continual limit) objects and to give them an underlying physical meaning. Searching the formfactors (the corresponding ensembles of saturating configurations) interesting for applications one should compare to the reasonable results for the four nonets of light mesons obtained in the NJL model. Apparently, the constants of effective four-quark Hamiltonian (scalar, pseudo-scalar, vector and axial-vector channels) and the parameters of integral saturation (cut-off) should be comparable. Seems, it might be carefully supposed that those singular objects (at still an unknown scale) have to reproduce the major features of successful effective NJL Hamiltonian after the corresponding averaging. In any case, the problems to find the specific features of such singular objects which admit their experimental identification and to analyse the quark behaviour in the ensembles of monopoles or vortices are of really great interest.

With such a form of induced four-fermion interaction we are going to search the ground state as the Bogolyubov probe function with vacuum quantum numbers

\[
|\sigma\rangle = T\ |0\rangle, \\
T = \Pi_{p,s,c} \exp\{ \varphi \left[ a^+(p, s, c) b^+(-p, s, c) + a(p, s, c) b(-p, s, c) \right] \},
\]

which is defined by minimizing mean energy

\[
E = \langle \sigma | H | \sigma \rangle,
\]

here \( \varphi = \varphi(p) \) and \(|0\rangle \) is the vacuum of free Hamiltonian, i.e. \( a(p, s, c) \ |0\rangle = 0, b(p, s, c) \ |0\rangle = 0 \). Introducing with the dressing \( T \) transformation the creation and annihilation operators of quasi-particles \( (T^{-1} = T^\dagger \) for fermions)

\[
A = T \ a \ T^{-1}, \quad B^+ = T \ b^+ T^{-1},
\]

we present the operator Eq. (5) as, with the Dirac conjugate spinor

\[
\begin{align*}
q(x) &= \int \frac{dp}{\sqrt{2\pi^2}} \frac{1}{|p_4|^{1/2}} \left[ A(p, s, c) U(p, s, c) e^{ipx} + B^+(p, s, c) V(p, s, c) e^{-ipx} \right], \\
\bar{q}(x) &= \int \frac{dp}{\sqrt{2\pi^2}} \frac{1}{|p_4|^{1/2}} \left[ A^+(p, s, c) \bar{U}(p, s, c) e^{-ipx} + B(p, s, c) \bar{V}(p, s, c) e^{ipx} \right],
\end{align*}
\]

where the spinors \( U \) and \( V \) are defined as

\[
U(p, s, c) = \cos(\varphi) u(p, s, c) - \sin(\varphi) v(-p, s, c), \\
V(p, s, c) = \sin(\varphi) u(-p, s, c) + \cos(\varphi) v(p, s, c),
\]

with \( U(p, s, c) = U^+(p, s, c) \gamma_4 \) and \( V(p, s, c) = V^+(p, s, c) \gamma_4 \). Now we have to specify the choice of spinors in the Euclidean variables. They obey the Dirac equations

\[
(p - im) u(p, s) = 0, \quad (p + im) v(p, s) = 0,
\]

with \( p = p_\gamma \gamma_4 + p_\gamma \) and additional constraint which fixes the spinor polarization

\[
i\gamma_5 \, \hat{s} \, u(p, s) = u(p, s), \quad i\gamma_5 \, \hat{s} \, v(p, s) = v(p, s),
\]

where \( \gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4 \), and the four-vector \( s \) is normalized to unit and orthogonal to the four-vector \( p \), i.e. \( s^2 = 1, \ (ps) = 0 \). It could be, for example,

\[
s_4 = \frac{(pn)}{im}, \quad s = n + \frac{(pn)}{im} p / (p_4 - im),
\]

where \( n \) is an arbitrary unit vector. If the covariant normalization conditions are satisfied

\[
\bar{u} u = 2im, \quad \bar{v} v = -2im,
\]

the spinors are defined with the precision up to phase factor. All these conditions allow us to formulate the following matrix representation

\[
\begin{align*}
(u(p, s), \bar{u}(p, s)) &= \frac{1}{2} (1 + i\gamma_5 \, \hat{s}), \\
(v(p, s), \bar{v}(p, s)) &= \frac{1}{2} (1 - i\gamma_5 \, \hat{s}).
\end{align*}
\]
Calculating the mean energy Eq.(17) we meet spinors with opposite moments. We introduce the four-vector \( q = (p_1, -p) \) in order to simplify notations. Using the projection operator we can express the spinor \( v(q, s) \) through the spinor \( u(p, s) \) (see [28])

\[
v(q, s) = \alpha \frac{\hat{q} - im}{-2im} + \frac{i\gamma_5 \hat{s}}{2} u(p, s). \tag{24}
\]

The coefficient \( \alpha \) is fixed by the covariant normalization Eq.(22) up to the phase factor as

\[
\alpha = -\frac{2m^2}{(pq + m^2)} = \frac{m^2}{|p|}. \nonumber
\]

Then summing up over the spinor states results in

\[
\sum_s u(q, s) \bar{v}(p, s) = \alpha \frac{\hat{q} + im}{2im} (\hat{p} - im), \quad \sum_s v(p, s) \bar{u}(q, s) = *\alpha (\hat{p} - im) \frac{\hat{q} + im}{2im},
\]

\[
\sum_s u(p, s) \bar{v}(q, s) = *\alpha (\hat{p} + im) \frac{\hat{q} - im}{2im}, \quad \sum_s v(q, s) \bar{u}(p, s) = \alpha \frac{\hat{q} - im}{2im} (\hat{p} + im). \tag{25}
\]

The polarization in which the momentum \( p \) and unit polarization vector \( n \) are orthogonal \( (pn) = 0 \) turns out to be the most convenient for handling. In such a situation both operators \( \hat{p} \) and \( \hat{q} \) commute with \( \gamma_5 \hat{s} \) and the polarization directions of quark and anti-quark could be taken identical (although in general case they should be two different directions). Then the summation over polarization of quarks and anti-quarks is performed separately in the final equations. It allows us not to control the obligatory constraint to have the vacuum quantum numbers of the pairs present in the intermediate calculations.

When calculating the mean energy Eq.(17) nontrivial contribution bilinear in quark operators comes from the terms of type \( \langle \sigma | B B^+ | \sigma \rangle \) (remember \( B |\sigma \rangle = 0, \ A |\sigma \rangle = 0 \)). The contribution from the terms like \( \langle \sigma | A A^+ | \sigma \rangle \) is absent because of the particular representation of bilocal operator we are using as \( \hat{pq} \) (then quadratic terms are expressed by the spinors \( V, \bar{V} \). Due to the similar reasons the four-quark operators develop only two nonzero contributions \( \langle \sigma | B B^+ B^+ B^+ | \sigma \rangle \) and \( \langle \sigma | B A A^+ B^+ B^+ | \sigma \rangle \). The first combination corresponds to the contribution of so-called tadpole diagrams and the latter is related to the asterisk [41] diagrams. As a result the four-fermion interaction contribution can be presented in the following form

\[
\tilde{\langle A_\mu^a A_\nu^b(0) \rangle} = \int \frac{dp}{(2\pi)^3} \frac{dp'}{(2\pi)^3} \frac{1}{|p_4||p'_4|} \tilde{V}_{\alpha i}(p, s, c) t_{ij}^\a \gamma_\mu \tilde{V}_{bj}(p, s, c) \tilde{V}_{\gamma k}(p', s', c') t_{kl}^b \gamma_\delta \tilde{V}_{\delta l}(p', s', c') + \tilde{V}_{\alpha i}(p, s, c) t_{ij}^\a \gamma_\mu \tilde{V}_{bj}(p, s, c) \tilde{U}_{\gamma k}(p', s', c') t_{kl}^b \gamma_\delta \tilde{U}_{\delta l}(p', s', c') \langle A_\mu^a A_\nu^b(p + p') \rangle.
\]

Here \( \tilde{\langle A_\mu^a A_\nu^b \rangle} \) is the Fourier transform of the gluon correlator and the summation over spinor and colour indices is implied. The contribution of the first tadpole diagram is an identical zero due to completeness of the spinor basis in color space, giving a unit color matrix (in particular it is valid for colour singlet quark configuration). In electrodynamics the considered term provides a dominant contribution. But it is interesting to remark that the singular character of photon propagator in the infrared region makes this abelian theory even more complicated to research than in the nonabelian one. In the compact \( U(1) \) electrodynamics (on a lattice) the infrared behaviour of correlation function is formed by the monopole contributions but nowadays it is still difficult to define a scale where these effects show up themselves. In the octet channel of nonabelian theory we obtain the quark repulsion \( \sim -1/(4N_c) \) and therefore this regime might be omitted when searching the minimum of mean energy Eq.(17). Then for the spinors with polarizations summed up we have
\[ \mathcal{V} = p_4 \gamma_4 + \cos(\theta) (p \gamma - im) - \frac{\alpha + \alpha}{2im} \sin(\theta) (p^2 - im p \gamma) , \]
\[ \mathcal{U} = p_4 \gamma_4 + \cos(\theta) (p \gamma + im) + \frac{\alpha + \alpha}{2im} \sin(\theta) (p^2 + im p \gamma) , \]

where angle \( \theta = 2\varphi \). In the formulae above the phase inherent in the sum \( \alpha + \alpha \) (a spinor is defined up to such a phase) is still indefinite. The direct analysis of the mean energy functional demonstrates that the most preferable value of the phase factor (responsible for the colour interaction of quarks) is the value when the coefficient \( \alpha \) appears to be a real number. For definiteness we put \( \alpha = +|m|/p \). The curious fact is that the results of summation are not equal \( (\mathcal{V} \mathcal{U}(m) = \mathcal{U} \mathcal{V}(-m)) \) and they coincide in the chiral limit \( m = 0 \) only, i.e. particles and antiparticles formally generate the different contributions.

The direct calculations lead to the following result for the mean energy \[ 17 \]

\[ \langle \sigma | H_{\text{ind}} | \sigma \rangle = - \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{2N_c p_4^2}{|p_4|} (1 - \cos \theta) - \frac{G}{p_4} \int \frac{d\mathbf{p} d\mathbf{q}}{(2\pi)^6} \left\{ (3\mathbf{i} - \mathbf{J}) \frac{p_4 q_4}{|p_4||q_4|} + (4\mathbf{i} - \mathbf{J}) \frac{p q}{|p_4||q_4|} \left( \sin \theta - \frac{m}{p} \cos \theta \right) \left( \sin \theta' - \frac{m}{q} \cos \theta' \right) - (2\mathbf{i} \delta_{ij} + 2\mathbf{J}_{ij} - \mathbf{J}_{ij}) \frac{p_i q_j}{|p_4||q_4|} \left( \cos \theta + \frac{m}{p} \sin \theta \right) \left( \cos \theta' + \frac{m}{q} \sin \theta' \right) \right\} , \]

here we designated \( p = |\mathbf{p}|, q = |\mathbf{q}|, \mathbf{i} = \mathbf{i}(\mathbf{p} + \mathbf{q}), \mathbf{J} = \sum_{i=1}^3 \mathbf{J}_i, \mathbf{p}_4^2 + p^2 = q_4^2 + q^2 = -m^2, \theta = \theta(q), G = (4\pi^2)^2 n \rho^4 \) as and as a matter of convenience we singled out the colour factor \( G = \frac{2}{N_c} \mathcal{G} \). To obtain this result we performed the regularization (subtracting the free Hamiltonian \( H_0 \)). It results in the presence of a unit (together with \( -\cos \theta \)) in the parentheses of the first integral. Let us also remind that in the Euclidean space \( p_4^2 \) is a negative magnitude. Then with Eq. \[ 26 \] available one can find the most advantageous value of the angle \( \theta \) from the condition

\[ \frac{d\langle \sigma | H_{\text{ind}} | \sigma \rangle}{d\theta} = 0 . \]

Henceforth we characterize the different stochastic ensembles of the gluon fields by their profile functions \( I(p) \), \[ J_{\mu\nu}(p) \).

**II. NAMBU-JONA-LASINIO MODEL**

Now let us consider the example in which the correlation function behaves in the coordinate space as the \( \delta \)-function (simply we assume \( J_{\mu\nu}(p) = 0 \)). Actually, it corresponds to the Nambu-Jona-Lasinio (NJL) model \[ 2 \]. As well known the regularization is required to obtain an intelligent result in this model. We adjust the NJL model with the parameter set given by Ref. \[ 29 \], and limit the integration interval over momentum in Eq. \[ 26 \] with the quantity \( |p| < \Lambda (\Lambda = 631 \text{ MeV}) \). Then the functional \[ 26 \] is written in the following form (unessential terms contributing the constant values are omitted)

\[ W = \int^{\Lambda} \frac{d\mathbf{p}}{(2\pi)^3} \left[ p_4 (1 - \cos \theta) - G \frac{p}{|p_4|} \left( \sin \theta - \frac{m}{p} \cos \theta \right) \right] \int^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^3} \frac{q}{|q_4|} \left( \sin \theta' - \frac{m}{q} \cos \theta' \right) , \]

where \( m = 5.5 \text{ MeV} \). The equation to calculate the optimal angle \( \theta \) \[ 27 \] reads as

\[ (p^2 + m^2) \sin \theta - M (p \cos \theta + m \sin \theta) = 0 , \]

where

\[ M = 2G \int^{\Lambda} \frac{d\mathbf{p}}{(2\pi)^3} \frac{p}{|p_4|} \left( \sin \theta - \frac{m}{p} \cos \theta \right) . \]
The constant of four-fermion interaction is \( G = \frac{1}{\Lambda^2} \tilde{G} \)
while expressed in the same units as the mean energy functional in Eq. (26). For the NJL model, Eq. (29) makes it possible to contract a functional space in which the minimum of mean energy functional can be realized. This equation parameterizes the function \( \theta(p) \) on the whole interval \( p \in [0, \Lambda] \) of searching the solution. Moreover, Eq. (29) itself does not impose any restrictions on the parameter \( M \) which may be any real number. Then the functional (28) simply becomes the function of parameter \( W(M) \). Now if one expresses the trigonometrical functions via parameter \( M \) it is possible to make the representation of minimizing function and the result of its integration (30) agree. As a result we receive three external points, two of them correspond the minimal points with negative and positive values of \( M \) and the negative value conforms to the state of more stability. The point of unstable equilibrium is located in the vicinity of co-ordinate origin \( \sim m \). The induced quark mass for the parameter magnitudes fixed is \( M = -335 \text{ MeV} \) and the quark condensate

\[
\langle \sigma|\bar{q}q|\sigma \rangle = \frac{i}{N_c} \frac{\pi}{\Lambda^2} \int_0^\Lambda dp \frac{p^2}{|p_4|} \left( p \sin \theta - m \cos \theta \right), \quad (31)
\]
develops the magnitude of \( \langle \sigma|\bar{q}q|\sigma \rangle = -i \; (245 \text{ MeV})^3 \).

The situation, if the correlator \( J_{\mu\nu}(p) \) is not equal to zero and has the same form of the \( \delta \)-function in coordinate space, can be similarly analyzed. The numerical analysis done teaches that its influence can be essential but we do not show these results due to the lack of any phenomenological estimates of the correlation function magnitude. The non-local version of the NJL model in which the correlator has the separable form \( I(p,q) = K(p)K(q) \) can be similarly analyzed. In fact, it again displays the above mentioned property which replaces the functional analysis for the analysis of function dependence on some parameter. Although one important difference does exist and it shows that the procedure of integral cutting off is unnecessary for the functions \( K(p) \). The regularization is naturally performed by the \( K(p) \) kernel and so strong regularization is caused by the separable form interaction kernel. Certainly, such a property can manifest itself in much weaker form for more realistic correlators.

### III. THE KELDYSH MODEL

Here we are going to analyse the limit in which the correlation function has a \( \delta \)-function form in the momentum space

\[
I(p) = (2\pi)^3 \tilde{G} \delta(p).
\]

This limit is an analogue of the Keldysh model which is well known in the physics of condensed matter (30) and the mean energy functional (26) develops the following form in this case (42)

\[
W(m) = \int \frac{dp}{(2\pi)^3} \left[ |p_4| (1 - \cos \theta) - G \frac{p^2}{|p_4|^2} \left( \sin \theta - \frac{m}{p} \cos \theta \right)^2 \right]. \quad (32)
\]

The optimal values of angle \( \theta \) are determined by the solutions of the following equation

\[
|p_4|^3 \sin \theta - 2G (p \cos \theta + m \sin \theta) (p \sin \theta - m \cos \theta) = 0 \quad (33)
\]
and we start analyzing these solutions in the chiral limit \( m = 0 \). One of the solutions corresponds to the zero angle \( \theta = 0 \) but the non-trivial one takes the form

\[
\cos \theta = \frac{p}{2G}. \quad (34)
\]
Both the positive and negative angles \( \theta \) are suitable as the solutions because of the parity (positive) property of the functional (32) and these (real) solutions (additional to the trivial one) exist on the limited momentum interval \( p < 2G \). There are one real solution for the trivial angle and two imaginary (complex-conjugate) solutions beyond this interval. Analyzing the NJL model above we noticed its very convenient property when the solution \( \theta(p) \) is defined on the whole interval and, in fact, the functional is parametrized by a single number which is the integral \( M \). In the Keldysh model the situation is much more sophisticated and the phase portrait of its solutions in the chiral limit (for example, \( \sin \theta \) as a function of momentum \( p \)) consists of two arches (with positive and negative \( \sin \theta \), see Fig.1) and straight line corresponding to the trivial solution. Thus, the semi-axis \( p \in [0, \infty) \) can be divided into two parts. There are three branches (solutions) at the interval \( p \in [0, 2G] \), two of those correspond to the positive and negative angles \( \theta = \pm \arccos(p/2G) \) and the trivial one where \( \theta = 0 \). At the interval \( p \in [2G, \infty) \) only one trivial solution \( \theta = 0 \) exists and in order to construct the solution on the whole semi-axis \( p \in [0, \infty) \) one has to add the trivial solution on the interval \( p \in [2G, \infty) \) to any detached branch of solutions on the interval \( p \in [0, 2G] \). It is easy to see that
functions will lead to the magnitudes of functional $W$ which fill up the interval $[0, 2G]$ densely, a similar pattern takes place for the chiral condensate. With the natural parametrization

$$\sin \theta = \frac{M_\theta}{(p^2 + M_\theta)^{1/2}} ,$$

we obtain for the mass $M_\theta$ which characterize the angle at the top arch the following result

$$M_\theta = \left(4G^2 - p^2\right)^{1/2} .$$

It is interesting to notice that then the respective energy of nontrivial solutions $E(p) = \sqrt{p^2 + M_\theta^2}$ becomes constant $E(p) = 2G$.

After having done the analysis in the chiral limit which is shown by the dotted lines in Fig. [I] we would like to comment on the situation beyond this limit, i.e. where $m \neq 0$. The evolution of corresponding branches is available on the same plot [I] where the behaviour of $\theta(p)$ as the function of momentum $p$ in MeV is shown for the solution of Eq. (33). The semi-axis $p \in [0, \infty)$ where we are searching the solution can be subdivided into two sectors which are demonstrated by the vertical dashed line on the plot. Three solutions denoted by $a$, $b$ and $c$ are developing at the first sector denoted in Fig. [I] by I. Besides, there are three solutions at the second sector denoted by II, one real solution designated as $A$ for the negative pairing angle and two complex-conjugate roots with the positive real parts. The imaginary parts of solutions are plotted in Fig. [II] by the dashed lines. The solution $A$ in the domain II develops the behaviour of $\theta \sim -\frac{2Gm}{p^2}$ with increasing momentum. As in the chiral limit the minimum of the mean energy functional $W(m)$ can be obtained with the piecewise continuous functions which are properly represented by the trajectories $aA$, $bA$, $cA$ (for real solutions). The first symbol of this complicated designation implies the branches $a$, $b$ or $c$ at the first sector, the second symbol corresponds to the branch at the sector II. Thus, at low momenta we start with the solution of branches $a$, $b$ or $c$, then relevant solution passes to the branches interchanging its position in any subinterval. But in any case there is only one way to continue the real solution when momenta goes to infinity and it is related with the branch $A$ when the angle is going to the zero value. Moreover, if the angle $\theta$ could take strictly zero value in the sector II then the second term of Eq. (33) leads to the singular contribution coming from the term $\frac{m^2}{p^2} \cos^2 \theta$ with the linear divergency at large momentum. Besides, the other terms develop the logarithmic divergencies as well. It is an amusing fact the mean energy functional out of chiral limit goes to an infinity at any nonzero value of current quark mass $m$ although in the chiral limit $W(0)$ is well defined. (It is worthwhile to remember here the current mass singularity of zero mode approximation which was discovered in Ref. [17]).

The same conclusion is valid for the chiral condensate (see Eq. (31)) in which the first and second terms are developing the linear and quadratic divergencies, respectively. We could conclude here that if the cut-off factor is not used in the integrals when dealing with the solutions on the whole axis the functional $W(m)$ and quark condensate $\langle \sigma | \bar{q} q | \sigma \rangle (m)$ are ill-defined.

Let us remember here that by definition the approximation (12) should describe the quark behaviour in the background of stochastic gluon field (which is averaged).
at low energies. Then it looks quite natural to introduce an effective cut-off (in momentum) parameter $\tilde{\Lambda}$. The condition for factorization of gluon and quark field contributions gets broken at the momenta above $\tilde{\Lambda}$. In such a situation the dependence of mean energy and quark condensate on the current quark mass is defined not only by the form of integrand but by the value of parameter $\tilde{\Lambda}$ as well. And if this value is pretty large $\tilde{\Lambda} \gg M$ (where $M$ is the dynamical quark mass) the dependence on $m$ of all the observables is mainly defined by the magnitude of cut-off parameter $\tilde{\Lambda}$ because of the singular character of integrals (for example, for the NJL model this magnitude could be estimated as $\sim 1 \text{GeV}$). Obviously, it means in order to get the dependence of observables on the current quark mass we need to draw essential additional information.

As to the possible interpretation of the singularities available in the mean energy functional we could assume, for example, the mechanism similar to the Cooper pairing which takes place at every scale of the increasing momenta $\tilde{\Lambda}, \tilde{\Lambda}_1, \ldots$. Certainly, we should correct the existing results about four-quark interaction potential to put the pairing effect on realistic ground. For example, the contribution of the stochastic configurations like the small size instantons which is exponentially suppressed is hardly relevant to provide an efficient pairing mechanism for the momenta above $\tilde{\Lambda}$. Apparently, the hard gluon exchange looks like a more adequate mechanism at small distances. Then the gluon correlation function in Eq. (12) should be transformed in the corresponding gluon propagator. The effective four-quark interaction we are interested in can be derived by the quasi-average formalism which approximates smoothly the infrared and ultraviolet momentum regions although an alternative scenario could also be quite meaningful (see, for instance, the discussion in [31]). The fact that the Cooper attraction is still large enough despite the coupling constant weakening could signal the dominance of more fundamental fields at very small distances. (Here it is easy to see all the different models might be classified by the convergence of the integral over momentum with the constant of four-fermion interaction

$$I_G = \int dp \, G(p) ,$$

as the integrand. The model falls under the category of a singular one if this integral diverges.)

In Fig. 2 we compare the equilibrium angles $\theta$ for the NJL model (solid line) and the $aA$ solution of the Keldysh model (dashed curve) as the functions of momentum $p$ in MeV with the current quark mass $m = 5.5 \text{ MeV}$. It is interesting to notice that out of the chiral limit the solution (which has a spherical symmetry) passed over zero at $p = 0$ (see, Eq. (37)). Besides, Fig 2 demonstrates that out of the chiral limit the pairing process becomes essential not at zero momentum value (as it takes place in the chiral limit) but it is shifted to the magnitude about $p_0 \approx 40 \text{ MeV}$ for the fitting parameters used. For example in the NJL model it can be obtained

$$p_0 = [m \ |M - m|]^{1/2} .$$

The quantity $r_0 = 1/p_0$ determines the characteristic size of the region which is efficient for the pairing process. In the chiral limit this region is formally extending to infinity. The curves shown in the plot correspond to the opposite, in a sense, limiting regimes and it is interesting to evaluate where the model with more realistic correlator could be found out.

One of the important motivations to study the Keldysh model was the question of a natural regularization which presents for the interaction with separable kernel. We have seen that in the chiral limit for the kernel with most extensively expressed regularizing property as, for example, the momentum $\delta$-function, both the mean energy and chiral condensate are well defined. Out of the chiral limit the unexpected singularity appears. In Ref. [32] it was discussed the possibility of continuing the mean energy functional and the quark condensate by performing the respective regularization. As well known the meson masses in the NJL model can be presented by the quark condensate what hints the corresponding expressions in the Keldysh model could be singular as well and one needs to perform another regularization to provide them with clear physical meaning. However, despite the present singularity of chiral condensate the meson observables are finite and are well matched with the experimental mass scale (see [33]). The reason to have these meson observables as the smooth functions of current quark mass is in the regularizing role of additional vertex formfactors which enter the meson mass formulae. Then we may summarize that it does not make sense to debate about an absolute value of quark condensate (in vacuum) for considered mechanism of spontaneous chiral symmetry breakdown because its magnitude de-

**FIG. 2:** The equilibrium angle $\theta$ (in degrees) as a function of momentum $p$ in MeV. The solid line shows the result of the NJL model and the dashed line corresponds to the most stable branch of the Keldysh model, the current quark mass is taken as $m = 5.5 \text{ MeV}$.
The function of the constant $G$ solid line is calculated with the current quark mass $m$ MeV and the dashed line is calculated in the chiral limit.

The kernels of integral equations look like the equation to derive the optimal angle in the following form

$$|p_4|^2 \sin \theta - \frac{4G}{\pi} \left( \cos \theta + \frac{m}{p} \sin \theta \right) \int_0^\infty dqdx \frac{q}{|q_4|} (q \sin \theta' - m \cos \theta') I(x) \sin(px) \sin(qx) = 0.$$  

(37)

Considering the solution behaviour at high momentum values $p$ we are interested in analyzing solutions in which the angle $\theta$ is going to zero magnitude. Assuming the $\theta$ value to be rather small we expand (37) up to the terms of the $\theta$ order and have

$$p^2 \theta - \frac{4G}{\pi} \int_0^\infty dq(q \theta' - m) I(p,q) = 0.$$ 

If the function $\theta$ is decreasing faster than $1/q$ the most essential contribution to the integral comes from the term proportional to $m$ and if the kernel $I(p,q)$ is integrable the asymptotic behaviour has the following form

$$\theta = -\frac{4G m}{\pi p^2} \int_0^\infty dq I(p,q).$$

Let us consider now two concrete examples, with exponential behaviour of the correlator $I(x) = \exp(-a |x|)$, and with the Gaussian behaviour $I(x) = \exp(-a^2 x^2)$. The integration over $x$ can be performed exactly for both cases and the kernels of integral equations look like

$$\int_0^\infty dx e^{-a x} \sin(px) \sin(qx) =$$

$$= \frac{a}{2} \left( \frac{1}{a^2 + (p-q)^2} - \frac{1}{a^2 + (p+q)^2} \right),$$

for the exponential correlator and as

$$\int_0^\infty dx e^{-a^2 x^2} \sin(px) \sin(qx) =$$

$$= \frac{\sqrt{\pi}}{4a} \left( e^{-\frac{(p-q)^2}{4a^2}} - e^{-\frac{(p+q)^2}{4a^2}} \right),$$

for the Gaussian one. Now let us hold the contribution of the first term only at large momentum values $p$ for both examples. Then as a result the corresponding asymptotic behaviours are expressed by the constants which are defined by the integrals with the kernels $I(p,q)$. It allows us to conclude that we have again the singular functional for the mean energy out of the chiral limit. The parameter $M_\theta$ (see Eq. (37)) and the quark condensate as functions of the constant $G$ for the Gaussian correlator (both obtained by the numerical computation of Eq. (37)) are depicted in Fig. 3. The solid line demonstrates the solution with the current quark mass $m = 5.5$ MeV and the dashed line is calculated in the chiral limit as the quark condensate presented by the points. The intrinsic change of the parameter $M_\theta$ generation out of the chiral limit is easily seen. The similar features are observed for the exponential correlator as well.

Unfortunately, it is a very serious problem to get all the solutions of the nonlinear integral equation (37) and here we are working with only one of its (the most stable) branches. As it was demonstrated above such a situation generates a lot of difficulties for extracting a reliable information on the observables out of the chiral limit.

IV. THE EXPONENTIAL AND GAUSSIAN CORRELATORS

Here we turn to a more realistic situation and analyze the solutions possessing a spherical symmetry in the regime where the correlation function $I(x)$ is rather quickly decreasing with the distance increasing. Performing the integration over the azimuthal angles we can get the equation to derive the optimal angle in the following form

$$\int_0^\infty dqdx e^{\frac{q^2}{4a^2}} \sin(px) \sin(qx) =$$

$$= \frac{\sqrt{\pi}}{4a} \left( e^{-\frac{(p-q)^2}{4a^2}} - e^{-\frac{(p+q)^2}{4a^2}} \right),$$

FIG. 3: The parameter $M_\theta$ (solid and dashed lines) and quark condensate (without an imaginary unit, in power 1/3) in MeV (shown by points calculated in the chiral limit $m = 0$) as a function of the constant $G$ for the Gaussian correlator. The solid line is calculated with the current quark mass $m = 5.5$ MeV and the dashed line is calculated in the chiral limit.
limit. Due to this reason we calculate here the dynamical quark mass \( M = -335 \text{ MeV} \) and chiral condensate \(|\langle \sigma | \bar{q} q | \sigma \rangle \sim (245 \text{ MeV})^3\) in the chiral limit collating the dynamical quark mass with \( M_\theta \) and fitting the parameters \( a \) and \( G \). The parameter \( a \) for the exponential and Gaussian correlators reads as

\[
a_{ex} = 0.15 \text{ GeV}, \quad a_{gs} = 0.16 \text{ GeV}.
\]

The most suitable values of \( G \) are equal to

\[
G_{ex} = 0.35, \quad G_{gs} = 0.31
\]

and \( |M_{ex}^{ex}| = 338 \text{ MeV}, \quad |\langle \sigma | \bar{q} q | \sigma \rangle_{ex}^{ex}| = (228 \text{ MeV})^3, \quad |M_{gs}^{gs}| = 340 \text{ MeV}, \quad |\langle \sigma | \bar{q} q | \sigma \rangle_{gs}^{gs}| = (245 \text{ MeV})^3 \). Actually we can collate the gotten value of the four-fermion interaction constant with the packing fraction parameter of which the basic one for the instanton vacuum model is \( G = (4\pi^2)^2 n \rho^4 \). The result of this exercise \( n \rho^4 \sim 10^{-3} \) is quite realistic. However, there is a pretty serious discrepancy in the estimates of characteristic configuration size (we should keep in mind the calculations are done in the chiral limit).

The parameter \( M_\theta \) as a function of momentum \( p \) in GeV calculated with the parameters corresponding the best fit of ‘experimental data’ is depicted in Fig. 4. The solid curve is obtained for the Gaussian correlator with \( m = 5.5 \text{ MeV} \) and the dashed line is calculated for the same correlator but in the chiral limit. Here we do not mention the results obtained for the behaving exponentially correlator because they practically coincide with the results for the Gaussian correlator. The parameter \( p_\theta \) is estimated at the current quark mass \( m = 5.5 \text{ MeV} \) to be as \( p_\theta \sim 150 \text{ MeV} \), i.e. of the \( \pi \)-meson mass order. The treatment of the correlator with instanton profile together with the detailed analysis of exponential and Gaussian correlators is worthy of special paper and will be studied in the next paper.

V. CONCLUSION

In the present paper we undertake the efforts to systematically study the quark’s behaviour in various ensembles of stochastic gluon fields developing simple ensemble approximation which is grounded on the circumstantial analysis of two-particle correlation function. An approximate procedure developed enables to calculate the effective functional for the mean energy and to estimate the ground state parameters within the Hartree-Fock-Bogolyubov approach. The models with the exponential and Gaussian behaviours of correlators are analyzed in the chiral limit and their parameters are fitted. The results obtained are used to estimate the characteristic region size \( r_\theta \) in which the possible processes of quark–anti-quark pairing might become significant. This size was estimated to be \( r_\theta \sim 1/150 \text{ MeV}^{-1} \) for the parameters inherent in the NJL model. For the exponential and Gaussian correlators this estimate looks like \( r_\theta \sim 1/150 \text{ MeV}^{-1} \).

Besides, we clearly demonstrate the presence of singularity in the mean energy functional outside the chiral limit. Finally, let us emphasize the quark ensemble characteristics discussed in the paper are not physically observable and in order to make the intelligent conclusion about the model effectiveness one should explore, for example, the meson correlation function. In fact, it has been done for the Keldysh model in \( \int_{ph} \) and result turned out to be quite encouraging. Despite the singular character of the mean energy of the system the meson observables are finite, quite recognizable and comparable with the energy scale coming from an experiment.

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VI. APPENDIX I

Here dealing with the simple quantum mechanical example we demonstrate the difference between the description based on the averaging of density matrix and the approach in which the averaging of the generating functional is used (considering the model system in a real time).

1. Let us suppose the particle described by the stationary Hamiltonian \( H_0 \) is also affected by the time dependent force \( f(t) \). Then the particle state is circumscribed by the Schrödinger equation

\[
i\dot{\Psi} = H \Psi, \quad H = H_0 + V, \quad V = f x,
\]
where $\Psi$ is the corresponding wave function and we search the perturbative solution as
\[ \Psi = \Psi^{(0)} + \Psi^{(1)} + \ldots \]
We expand the wave function of the zeroth order in the eigenfunctions of Hamiltonian $H_0$ as
\[ \Psi^{(0)} = c_n \ e^{-i\lambda_n \ t \ \psi_n} , \ H_0 \ \psi_n = \lambda_n \ \psi_n , \]
the constants $c_n$ are defined by the initial condition here, and the next perturbative orders are calculated in the following form
\[ \Psi^{(j)} = d^{(i)}_n \ e^{-i\lambda_n \ t \ \psi_n} , \]
where the coefficient $d^{(i)}_n$ is determined by the iterations as
\[ d^{(j)}_n(t) = -i \ \langle \psi^*_n x \psi_m \rangle \ \int_0^t d\tau \ f(\tau) \ e^{i(\lambda_n-\lambda_m)\tau} \ d^{(j-1)}_m(\tau) , \]
and $(\psi^*_n x \psi_m)$ stands here for the matrix element over the eigenfunctions of $H_0$, $j = 1, 2, \ldots$, $d^{(0)}_n = c_n$. The energy operator after averaging over the final state $\Psi$ can be presented in the form of trace $\text{Tr} \ {H\rho}$ with the pseudo-density matrix $\rho = \Psi \ \Psi^*$. In general, an energy being averaged over such a pseudo-matrix will be time dependent but at analysing the quasi-stationary processes, for example, it might be useful to study its averages in time as reads
\[ \text{Tr} \ {H\rho} = \int_0^T dt \ \text{Tr} \ {H\rho}/T . \]
For the sake of clarity we suppose for the force mean value that $\bar{f} = 0$. Then a nontrivial contribution into the interaction mean energy comes from the cross terms of the zeroth and first orders of perturbation expansion $\langle \psi^*_0 \ \Psi^{(1)} \rangle \ \Psi^{(0)} \ \Psi^{(1)}$ and we have
\[
\text{Tr} \ {V\rho} = i \ c_k \ c_m \ \langle \psi^*_k x \psi^*_m \rangle \langle \psi_n x \psi_m \rangle \times \ e^{i(\lambda_n-\lambda_k)t} \ \int_0^t d\tau \ f(\tau) \ f(t) \ e^{i(\lambda_k-\lambda_m)\tau} + \text{c.c} . \tag{38}
\]
At estimating the impact of the stochastic force its contribution may be factorized with a help of the corresponding correlation function $\overline{f(\tau) f(t)} \sim \overline{F(\tau-t)}$ if the characteristic frequencies $\lambda_n$ are smaller then the stochastic ones. In the particular case of the ‘white noise’ (when the profile function $F$ has the $\delta$-function shape) the time dependence in the intermediate states $\psi^*_n, \psi_n$ in Eq. (38) disappears (see the corresponding exponentials depending on $\lambda_n$). Due to the assumed completeness of eigenvalues basis of $H_0$, i.e. $\sum_n |\psi^*_n \rangle \langle \psi_n| = 1$, Eq. (38) may be presented as
\[ \text{Tr} \ {V\rho^{(2)}} = i c_k \ c_m \ \langle \psi^*_k x^2 \psi^*_m \rangle \ e^{i(\lambda_n-\lambda_k)t} + \text{c.c} . \]
It allows us to conclude that the resulting averaged final state density matrix is weighed with the effective ‘potential’ of form $\overline{F^2} \ x^2$. The similar results can be received in the next perturbative orders. The cluster decomposition of stochastic exponential $\ e^{i f x} \ $ is practical to demonstrate that the same results for the effective ‘potential’ of interaction take place for a ‘white noise’ at averaging the generating functional $\langle \Psi \rangle$ (as it is claimed in the first section of this paper). In general consideration there appears a certain nonlocal ‘potential’ and its properties are dependent of the system state.

2. Now let us turn to the description in terms of a density matrix only. It is defined by the following equations
\[ i \ \dot{\rho} = H \ \rho - \rho \ H' , \tag{39} \]
and the density matrix is dependent on the coordinates and time $\rho(x,y,t)$. The operator $H$ is acting on the coordinate $x$ and the operator $H'$ is acting on the coordinate $y$. In the zeroth order of perturbative expansion we have
\[ \rho^{(0)} = c_{nm} \ e^{i\lambda_n t} \ \psi^*_n (y) \ e^{-i\lambda_m t} \psi_m (x) , \]
where $c_{nm}$ is a hermitian matrix which is defined by the initial data. In the first order of the perturbation series we present the solution in the following form
\[ \rho^{(1)} = d^{(1)}_{nm} \ e^{i\lambda_n t} \ \psi^*_n (y) \ e^{-i\lambda_m t} \psi_m (x) . \]
It is possible to have for the matrix $d^{(1)}$ the representation as
\[
\begin{align*}
\text{d}^{(1)}_{nm} \ &= \ -i \ c_{nk} \ \langle \psi^*_n x \psi_k \rangle \ \int_0^t d\tau \ f(\tau) \ e^{i(\lambda_n-\lambda_k)\tau} + \\
&+ i \ c_{km} \ \langle \psi^*_k y \psi_n \rangle \ \int_0^t d\tau \ f(\tau) \ e^{i(\lambda_k-\lambda_m)\tau}.
\end{align*}
\]
Then for the density matrix $\rho^{(1)}$ the following form appears (in order to get it we have to interchange the indices $m$ and $n$ in the second term)
\[
\rho^{(1)} = -ic_{nk} \left( \psi_m \bar{\psi}_k \right) e^{i(\lambda_n - \lambda_k)t} \bar{\psi}_k \left( y \right) \psi_m \left( x \right) \int_0^t d\tau f \left( \tau \right) e^{i(\lambda_n - \lambda_k)\tau} + \\
+ ic_{kn} \left( \bar{\psi}_k \psi_m \right) e^{i(\lambda_m - \lambda_k)t} \psi_m \left( y \right) \psi_n \left( x \right) \int_0^t d\tau f \left( \tau \right) e^{i(\lambda_m - \lambda_k)\tau}.
\]

Due to the hermitian property of density matrix we have \( c_{kn} = c_{nk} \) the second term is complex conjugate with the first one at the coinciding arguments \( x = y \). Then calculating the mean interaction energy \( \text{Tr}\{V\rho^{(1)}(x, y; t)\}_{y \rightarrow x} \), we are convinced that the result is identical to what we found out at the beginning of this section.

Let us consider the solution for density matrix in the operator form (and without specifying the basis functions) as

\[
\rho = e^{i(H_0 - H_0)t} \bar{\rho},
\]

where the matrix \( \bar{\rho} \) is determined by the solution of following integral equation

\[
\bar{\rho}(t) = -i \int_0^t d\tau e^{i(H_0 - H_0)\tau} f(\tau) (x-y) e^{i(H_0 - H_0)\tau} \bar{\rho}(\tau) + \bar{\rho}(0),
\]

and effective interaction Hamiltonian is given by the operator expression as

\[
\text{Tr}\{H\rho\} = \text{Tr} \left\{ [H_0 + f(t) x] ( -i ) e^{i(H_0 - H_0)t} x \right\} \left( \int_0^t d\tau e^{i(H_0 - H_0)\tau} f(\tau) (x-y) \rho(x, y; \tau) \right|_{y \rightarrow x} \right\}.
\]

3. Further we analyse some details of the particular exercise which admits of receiving the overt expressions and consider the forced oscillations defined by the Hamiltonian

\[
H_0 = -\frac{1}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2.
\]

The continual integral is exactly calculated for this exercise [19] and the presentation of pseudo-density matrix which we are interested in looks like

\[
\psi \left( y_2, t_2 \right) = \int_{-\infty}^{\infty} dx_1 dy_1 \ K \left( y_2, t_2; y_1, t_1 \right) \times \\
\times K \left( x_2, t_2; x_1, t_1 \right) \varphi \left( y_1, t_1 \right) \varphi \left( x_1, t_1 \right),
\]

where \( \varphi \left( x_1, t_1 \right) \) is an initial state. The transformation kernel is expressed by the overt formula like

\[
K \left( x_2, t_2; x_1, t_1 \right) = \left( \frac{m\omega}{2\pi i \sin \omega T} \right)^{1/2} e^{iS},
\]

where the action is given as

\[
S = \frac{m\omega}{2\sin \omega T} [ \cos \omega T (x_2^2 + x_1^2) - 2x_2 x_1 + 2x_2 \phi_2 + 2x_1 \phi_1 - F],
\]

and the phase factor have the following form

\[
\phi_1 = \frac{1}{m\omega} \int_{t_1}^{t_2} d\tau f (\tau) \sin \omega (t_2 - \tau),
\]

\[
\phi_2 = \frac{1}{m\omega} \int_{t_1}^{t_2} d\tau f (\tau) \sin \omega (\tau - t_1),
\]

\( T = t_2 - t_1 \). The term \( F \) depends on the time parameters only and is immaterial because it is cancelled in the exponential exponent of pseudo-density matrix. We introduce the new variable \( x_2 = \bar{x}_2 + \phi_1 \) and transform the exponential exponent in the \( K \) kernel as

\[
\cos \omega T (x_2^2 + x_1^2) - 2x_2 x_1 + 2x_2 \phi_2 + 2x_1 \phi_1 = \\
= \cos \omega T (\bar{x}_2^2 + x_1^2) - 2\bar{x}_2 x_1 + 2(\cos \omega \phi_1 + \phi_2) \bar{x}_2 + \\
+ \cos \omega T \phi_1^2 + 2\phi_1 \phi_2.
\]

The similar transformations should be done in the kernel \( K \) with the variable \( y_2 = \bar{y}_2 + \phi_1 \). The formulae take more convenient form if we introduce the auxiliary factor \( \bar{\phi}_1 = (\cos \omega \phi_1 + \phi_2) / \sin \omega T \) which can be written down by the simple transformations in the following form

\[
\bar{\phi}_1 = \frac{1}{m\omega} \int_{t_1}^{t_2} d\tau f (\tau) \cos \omega (t_2 - \tau).
\]

Making use the well known representation of the kernel \( K \) for non-perturbated oscillator \(( f = 0) \), see [19],

\[
K_0 (x_2, t_2; x_1, t_1) = \sum_{n=0}^{\infty} e^{-i\lambda_n T} \psi_n^* (x_2) \psi_n (x_1).
\]

It is easy to understand that the important terms of pseudo-density matrix kernel \( \tilde{K} \) at \( y_2 \rightarrow x_2 \) are represented in the similar form

\[
\tilde{K} = \sum_{n=0}^{\infty} e^{i\lambda_n T} \psi_n (y_2) \psi_n^* (y_1) e^{-im\omega \phi_1 \bar{y}_2} \times \\
\times e^{im\omega \phi_1 \bar{x}_2} \sum_{m=0}^{\infty} e^{-i\lambda_m T} \psi_m (x_2) \psi_m (x_1) \mid_{y_2 = x_2}.
\]

The following matrix element

\[
\int_{-\infty}^{\infty} dx_2 \psi_n (\bar{y}_2) e^{-im\omega \phi_1 \bar{y}_2} H (x_2) e^{im\omega \phi_1 \bar{x}_2} \psi_m (x_2) \mid_{y_2 = x_2}
\]

will be faced at calculating the mean energy. Now moving the exponential \( e^{im\omega \phi_1 \bar{x}_2} \) to the left and changing the
variable \( x_2 = \bar{x}_2 + \phi_1 \) in the Hamiltonian \( H(x_2) \) we obtain the representation
\[
e^{-i m \omega \phi_1 \bar{t}} H(x_2) e^{i m \omega \phi_1 \bar{t}} x_2 = H_0(\bar{x}_2) - \omega \bar{t} + \frac{d}{d \bar{t}^2} + (m \omega^2 \phi_1 + f) \bar{x}_2 + \frac{m \omega^2}{2}(\phi_1^2 + \bar{\phi}_1^2) + f \phi_1.
\]
which allows us to see that the mean energy calculated over the final state is expressed by the diagonal elements of coordinate and momentum as well as
\[
\int_{-\infty}^{\infty} dx_2 \, K(\bar{x}_2) H(x_2) K = \sum_{n=0}^{\infty} H_{n,n} \psi_n(x_1) \psi_n^*(y_1) + \sum_{n=0}^{\infty} H_{n,n-1} e^{i \omega T} \psi_n(x_1) \psi_{n-1}^*(y_1) + \sum_{n=0}^{\infty} H_{n-1,n} e^{-i \omega T} \psi_{n-1}(x_1) \psi_n^*(y_1),
\]
where
\[
H_{n,n} = \left( n + \frac{1}{2} \right) \omega + \frac{m \omega^2}{2}(\phi_1^2 + \bar{\phi}_1^2) + f \phi_1,
\]
\[
H_{n,n-1} H_{n-1,n} = \left[ m \omega^2 (i \phi_1 + \phi_1) + f \right] \left( \frac{n}{2 m \omega} \right)^{1/2}.
\]

4. Now we would like to analyse the example of oscillations initiated by a periodic perturbation defined as
\[
f(t) = F \sin \Omega t.
\]
Then the phase factors develop the following form
\[
m \omega^2 \phi_1 = F \frac{\omega}{\Omega^2 - \omega^2}(\Omega \sin \omega T - \omega \sin \Omega T),
\]
\[
m \omega^2 \bar{\phi}_1 = F \frac{\Omega \omega}{\Omega^2 - \omega^2} (\cos \Omega T - \cos \omega T),
\]
(for the sake of simplicity we take the parameter as \( t_1 = 0 \)). As in the limit of classical mechanics these expressions include a resonance behaviour at coinciding the external frequency \( \Omega \) and oscillator frequency \( \omega \), and in the resonance vicinity \( \Omega = \omega + \varepsilon \) with the small deviation \( \varepsilon \) from the oscillator frequency) the motion behaves as the beats, i.e. the small oscillations with the frequency \( \omega \) and large amplitude. Now we are going to resolve the corresponding classical equation
\[
\ddot{x}_c + \omega^2 x_c = -f / m,
\]
with the initial conditions as \( x_c(0) = 0, \dot{x}_c(0) = 0 \).
\[
x_c = \frac{F}{m \omega} \frac{\Omega \sin \omega T - \omega \sin \Omega T}{\Omega^2 - \omega^2}.
\]
Comparing Eqs. (43) \( x_c, \dot{x}_c \) we are able to express the phase factors \( \phi_1, \bar{\phi}_1 \) as the classical coordinates \( \phi_1 = x_c \) and velocity \( \dot{\phi}_1 = \dot{x}_c / \omega \). In particular, the correction to the diagonal element of effective Hamiltonian (42) can be presented in the following form
\[
\frac{m \omega^2}{2}(\phi_1^2 + \bar{\phi}_1^2) + f \phi_1 = \frac{m}{2} \dot{x}_c^2 + \frac{m \omega^2}{2} x_c^2 + f x_c.
\]
Averaging the mean energy with pseudo-density matrix we get the quadratic form as
\[
\text{Tr}(H \rho)_{y_2=x_2} = \sum_{n,m=0}^{\infty} c_n H_{n,m} c_m,
\]
with the coefficients \( c_n \) defined by the initial state and normalized as \( \sum |c_n|^2 = 1 \). In the considered situation of periodic force acting for very long (unlimited) time it becomes clear the value of mean energy received is time dependent (analogously to the classical description) and it means a certain asymptotic value for mean energy (as for other observables) simply does not exist. Physical meaning of this fact appears quite transparent. The quantum system is carrying out the repeated transitions to the upper levels of excited state and back (these transitions are controlled by pseudo-density matrix) eventually resulting in some quasi-stationary process which can be pithily characterized by some observable values averaged in time. Thus, the averaged magnitude of diagonal element of pseudo-density matrix \( \overline{H_{n,n}} = \int_0^T dt \, H_{n,n} / T \) takes the form
\[
\overline{H_{n,n}} = \left( n + \frac{1}{2} \right) \omega + \frac{F^2}{m} \frac{\Omega^2 + 3 \omega^2}{4(\Omega^2 - \omega^2)^2} + \frac{3 F^2}{8 m} \frac{1}{\Omega^2 - \omega^2} \sin 2 \Omega T + \frac{\Omega \omega}{2 m \omega^2} \frac{\Omega - 2 \omega \sin(\Omega - \omega) T}{\Omega - \omega} - \frac{\Omega + 2 \omega \sin(\Omega + \omega) T}{\Omega + \omega}.
\]
Then it is not difficult to see that asymptotically a quasi-stationary regime of quantum ensemble oscillations as the whole can be set in, indeed, and now the question of interest is to determine the minimum of functional (44).
which corresponds to some effective ground state of the system while under the external influence. The effective Hamiltonian \[ H_{\text{eff}}(x) = e^{-im\omega\hat{y}} H(x) e^{im\omega\hat{y}} \]
can be presented by using the classical variables \( x_c, \dot{x}_c \) in the following form
\[
H_{\text{eff}}(x) = \frac{(\hat{p} + p_c)^2}{2} + \frac{m\omega^2}{2} (\dot{x} + x_c)^2 + f(\dot{x} + x_c),
\]
where \( p_c = m\dot{x}_c \). This quantity (at the certain conditions) may be treated in such a way that it is practical to search the ground state with the biased coordinate \( x_c \) and momentum \( p_c \). Certainly, the treatment of excited states turns out the nontrivial problem in this situation.

At every time moment the pseudo-density matrix is a pure magnitude because the equality \( \rho^2 = \rho \) is identically valid. However, it is possible to estimate the purity degree of trail quasi-stationary state \( \gamma = \text{Tr} \{ \rho^2 \} \), with the time averaged density matrix and to find such states which allow us to develop a description close to one in the terms of the Schrödinger equation.

The density matrix formalism is very practical in more general situations, for example, at studying the influence of other quantum ensembles on a particle. It is very actively discussed and developing (being often quite far from our concerns) \[54\] but our purpose here was to illustrate the difference in describing a system with averaging a density matrix and averaging a generating functional.

**VII. APPENDIX II**

The standard way to formulate an effective theory is to use the path integral formalism. In order to transit to such a description we should construct the corresponding Lagrangian action density from the effective Hamiltonian \[42\]
\[
\mathcal{L} = \bar{q}(i\gamma_\mu \partial_\mu + im)q - G' \bar{q} t^a \gamma_\mu \bar{q} \int dy I_{\mu\nu}(x - y) \bar{q} t^a \gamma_\nu \bar{q},
\]
where \( q = q(x, t), \bar{q} = \bar{q}(x, t), q' = q(y, t), \bar{q}' = \bar{q}(y, t) \).

For the highest order in \( N_c \) the sum of colour group generators looks like \( \sum_{a=1}^{N_c^2-1} t^a_{ij} t^a_{jk} \approx \frac{1}{2} \delta_{ij}\delta_{kj} \). For the sake of simplicity we consider the correlator of the following form only \( I_{\mu\nu}(x - y) = \delta_{\mu\nu} \bar{I}(x - y) \). Using the Fierz transformation \( \gamma_\mu \gamma_\nu = 1 \otimes 1 + i\gamma_5 \otimes i\gamma_5 - \frac{1}{2} \gamma_\mu \gamma_5 \otimes \gamma_\nu \gamma_5 \), and holding only the scalar contribution we receive in the mean field approximation the following effective Lagrangian density
\[
\mathcal{L} = \bar{q} \left( i\gamma_\mu \partial_\mu + im \right) q - G' \int dy \bar{I}(x - y) \langle \bar{q} q' \rangle \bar{q}' q.
\]
The brackets in this expression imply the calculation of the corresponding averages. The self-consistency condition of approximation which may be formulated as the following integral equation
\[
-i M(p) = \int \frac{dq}{(2\pi)^4} G'(p - q) Tr \frac{1}{q + im + iM(q)},
\]
allows us to calculate the quark mass. Integrating over the fourth component of momentum
\[
\int_{-\infty}^{\infty} \frac{dq_4}{2\pi} \frac{1}{q^2 + (m + M(q))^2} = \frac{1}{2} \frac{1}{[q^2 + (m + M(q))^2]^{1/2}}
\]
we have
\[
M(p) = 2G'N_c \int \frac{dq}{(2\pi)^3} I(p - q) \frac{m + M(q)}{|q^2 + (m + M(q))^2|^{1/2}}.
\]

With the correlator corresponding to the NJL model we obtain the well known gap equation
\[
M = 2G'N_c \int \frac{dq}{(2\pi)^3} \frac{m + M(p)}{|p^2 + (m + M(p))^2|^{1/2}},
\]
and remember that \( I(p) = (2\pi)^3 \delta(p) \). Then it is easy to understand that the solution can be presented as a function of \( p(M) \) for convenient handling.

In the Hartree-Fock-Bogolyubov approach the following sum over the colour matrices is used \( \sum_{a=1}^{N_c^2-1} t^a_{ij} t^a_{jk} = \frac{N_c^2-1}{2N_c} \delta_{ik} \) and then we have for the quark mass
\[
M_q(p) = 2G' \frac{N_c^2 - 1}{N_c} \int \frac{dq}{(2\pi)^3} I(p - q) \frac{|q|}{|q_4|} \sin \theta(q).
\]
Comparing this expression to Eq.\[43\] it becomes clear that the four-fermion interaction constant acquires the small correction \( \sim 1/N_c \) which is rooted in the mean field approximation while the higher order terms in \( N_c \) are held. The patent formula for the Keldysh model can be simply received in the chiral limit. In the mean field approximation we have
\[
M(p) = \left[ (2G'N_c)^2 - p^2 \right]^{1/2}
\]
and in the Hartree-Fock-Bogolyubov we receive
\[
M(p) = \left[ \left( 2G' \frac{N_c^2 - 1}{N_c} \right)^2 - p^2 \right]^{1/2},
\]
see also Eq.\[36\]. At \( m \neq 0 \) the momentum dependencies of masses are quite different. For the Keldysh model in
the mean field approximation at zero momentum, for example, we have $M(0) = 2G'N_c$ whereas in the Hartree-Fock approach $M_b(0) = 0$. At large momenta the mass in the mean field approximation behaves as $|M(p)| \rightarrow 2G'N_c \frac{m}{p^2}$ and in the Hartree-Fock-Bogolyubov approximation it is the following $|M(p)| \rightarrow \left(2G'\frac{N^2-1}{N_c}\right)^2 \frac{m}{p^2}$, see also Fig. 3. However, generally, if one takes an orientation to the analysis of integral characteristics $M(p)$ the results are not so different. The similar relations could be obtained for the NJL model as well. Apparently it is reasonable to notice here that our analysis of the Hamiltonian Eq. (12) (Lagrangians Eqs. (45), (46)) is also valid for the Lorentz-invariant formulation when the $\tau$ (‘time’) integration is performed for the quark fields as well (for the Lagrangians (45) and (46) the integration over ‘time’ is retained and the formfactor becomes a function of four-vector $I(x-y)$).

$$L = \bar{q} (i\gamma_{\mu} \partial_{\mu} + im) q - G \int dy \, I(x-y) \langle \bar{q} \, q \rangle \bar{q} \, q \, . \quad (50)$$

The selfconsistency condition Eq. (47) acquires the covariant form. In particular, in the Keldysh model in four-dimensional formulation when $I(p) = (2\pi)^4 \delta(p)$ the mass gap equation reads as

$$M = 4G N_c \frac{m + M}{p^2 + (m + M)^2}.$$  

It allows us to conclude that a quark never comes on the mass shell because

$$p^2 + (m + M)^2 = 4N_c \frac{m + M}{M} > 0.$$  

This feature has already been noticed in Ref. [33]. The similar behaviour has been also observed in the analytic models of confinement [34]. Meanwhile, an absence of bound states in the four-dimensional Keldysh model (unlike the Keldysh model with three-dimensional formfactors) is its shortage. There appears the additional integration over the fourth component of auxiliary four-momentum $l$ in Eq. (20)

$$\int \frac{dq}{(2\pi)^3} \rightarrow \int \frac{dl_4}{2\pi} \int \frac{dq}{(2\pi)^3} \frac{1}{I(l_4)} \frac{1}{|p_4| + |q_4| - i l_4},$$

where $I(l_4)$ is the respective part of the formfactor. In particular, for the four-dimensional Keldysh model with $I(l_4) = 2\pi \delta(l_4)$ the mean energy functional can be presented in the following way

$$W(m) = \int \frac{dp}{(2\pi)^3} \left[ |p_4| \, (1 - \cos \theta) - G \frac{p^2}{|p_4|^2} \frac{1}{2|p_4|} \left( \sin \theta - \frac{m}{p} \cos \theta \right)^2 \right]. \quad (51)$$

The singularity revealed in three-dimensional Keldysh model manifests itself as weaker (logarithmic only) one in the four-dimensional consideration.

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