Erratum to

The braided Thompson’s groups are of type $F_{\infty}$

(J. reine angew. Math. 718 (2016), 59–101)

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There is a mistake in Lemma 3.9 of [1], which has no consequences for the rest of the article. The assumption that the link of every $k$-simplex in $X$ be $(m - 2k - 2)$-connected is insufficient to get the induction step to work, and needs to be replaced by $(m - k - 2)$-connected. The correct statement therefore reads:

**Lemma 3.9 (Reduction to the simplexwise injective case).** Let $Y$ be a compact $m$-dimensional combinatorial manifold. Let $X$ be a simplicial complex and assume that the link of every $k$-simplex in $X$ is $(m - k - 2)$-connected. Let $\psi: Y \to X$ be a simplicial map whose restriction to $\partial Y$ is simplexwise injective. Then after possibly subdividing the simplicial structure of $Y$, $\psi$ is homotopic relative $\partial Y$ to a simplexwise injective map.

We became aware of this mistake through discussions with Søren Galatius involving an equivalent result with the correct bound [2, Theorem 2.4].

In the new formulation the old proof applies verbatim, but we extend the presentation to confirm that the induction step, which breaks down when using the old bound, now works.

**Proof.** The proof is by induction on $m$ and the statement is trivial for $m = 0$.

If $\psi$ is not simplexwise injective, there exists a simplex whose vertices do not map to pairwise distinct points. In particular we can choose a simplex $\sigma \subseteq Y$ of maximal dimension $k > 0$ such that for every vertex $x$ of $\sigma$ there is another vertex $y$ of $\sigma$ with $\psi(x) = \psi(y)$. By assumption, $\sigma$ is not contained in $\partial Y$. Maximality of the dimension of $\sigma$ implies that the restriction of $\psi$ to the $(m - k - 1)$-sphere $\text{lk}_Y(\sigma)$ is simplexwise injective. It also implies that $\psi(\text{lk}_Y(\sigma)) \subseteq \text{lk}_X(\psi(\sigma))$. Note further that $\psi(\sigma)$ has dimension at most $\frac{1}{2}(k - 1) \leq k - 1$. Therefore its link in $X$ is $(m - k - 1)$-connected by assumption. Hence there is an $(m - k)$-disk $B$ with $\partial B = \text{lk}_Y(\sigma)$ and a map $\varphi: B \to \text{lk}_X(\psi(\sigma))$ such that $\varphi|_{\partial B}$ coincides with $\psi|_{\text{lk}_Y(\sigma)}$. Inductively applying the lemma, we may assume that $\varphi$ is simplexwise injective.

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This inductive step is indeed possible because if $\tau$ is a $d$-simplex in $\text{lk}_X(\psi(\sigma))$, then $\psi(\sigma) \vee \tau$ is a simplex of dimension at most $\frac{1}{2}(k - 1) + d + 1 \leq k + d$ such that

$$\text{lk}_X(\psi(\sigma))(\tau) = \text{lk}_X(\psi(\sigma) \vee \tau).$$

By assumption that complex is $((m - k) - d - 2)$-connected.

We now replace $Y$ by $Y'$, the space obtained by replacing the closed star of $\sigma$ by $B \ast \partial \sigma$. The map $\psi': Y' \to X$ is the map that coincides with $\psi$ outside the open star of $\sigma$, coincides with $\varphi$ on $B$ and is affine on simplices. It is clearly homotopic to $\psi$, since the image of $B$ under $\psi$ is contained in $\text{lk}_X(\psi(\sigma))$. Since the restriction of $\psi'$ to $B$ is simplexwise injective, the restriction to any $k$-simplex of $B \ast \partial \sigma$ is injective. Since $Y$ is compact, by repeating this procedure finitely many times we eventually obtain a map that is simplexwise injective. □

This change is inconsequential for the rest of the article, because in both applications of Lemma 3.9 the new bound is still met. Indeed in the proof of Theorem 3.10 the estimate reads

$$\eta(n - 3k - 3) - 1 = \left[\frac{n - 3k - 4}{4}\right] - 1$$

$$\geq \eta(n) - \left[\frac{3}{4}k\right] - 2$$

$$> m - k - 2,$$

because $m < \eta(n) - 1$, and in the alternate proof of Theorem 3.8 it reads

$$\nu(n - 2k - 2) - 1 = \left[\frac{n - 2k - 1}{3}\right] - 1$$

$$\geq \nu(n) - \left[\frac{2}{3}k\right] - 2$$

$$> m - k - 2,$$

because $m < \nu(n) - 1$. (Recall that $\nu(n) = \left[\frac{n+1}{3}\right] - 1$ and $\eta(n) = \left[\frac{n-1}{4}\right].$)

The following example shows that the original formulation of the lemma is incorrect, and not just its proof.

**Example.** Let $Y$ consist of two triangles with vertices $a, b, c$ and $b, c, d$, respectively. Then $Y$ is a $2$-dimensional combinatorial manifold whose boundary is the circle $[a, b] \cup [b, d] \cup [d, c] \cup [c, a]$.

Let $X$ be a single edge $[v, w]$. The link of each vertex of $X$ is a single point, hence contractible, while $\text{lk}[v, w] = \emptyset$ is $(2 - 1 - 2)$-connected (which is an empty condition) but not $(2 - 1 - 2)$-connected (which would mean non-empty). Consider the simplicial map that takes $a$ and $d$ to $v$ and $b$ and $c$ to $w$. Its restriction to the boundary is simplexwise injective. However, invariance of domain forbids that an interior point of $Y$ could have a neighborhood that is mapped injectively to $X$.

The proof of the lemma would introduce an additional vertex $x$ in the interior of $[v, w]$ and then map $[b, c]$ to $[x, w]$. However, then the induction fails since the non-empty link of $[b, c]$ cannot be mapped to the empty link of $[x, w]$. 


References

[1] K.-U. Bux, M. G. Fluch, M. Marschler, S. Witzel and M. C. B. Zaremsky, The braided Thompson’s groups are of type $F_\infty$. With an appendix by Zaremsky, J. reine angew. Math. 718 (2016), 59–101.

[2] S. Galatius and O. Randal-Williams, Homological stability for moduli spaces of high dimensional manifolds. I, J. Amer. Math. Soc. 31 (2018), no. 1, 215–264.

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