GREEN FUNCTION’S PROPERTIES AND EXISTENCE THEOREMS FOR NONLINEAR SINGULAR-DELAY-FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we are dealing with singular fractional differential equations (DEs) having delay and $p$-Laplacian operator. In our problem, we contemplate two fractional order differential operators that is Riemann–Liouville and Caputo’s with fractional integral and fractional differential initial boundary conditions. The SFDE is given by

$$
\begin{align*}
&D^\gamma [u_0^{\gamma} (D^\kappa x(t))] + \varrho(t) \zeta_1(t, x(t - \varrho^*)) = 0, \\
&D^\delta x(0) = 0, \quad x(1) = x'(0), \quad x^{(k)}(0) = 0 \quad \text{for} \quad k = 2, 3, \ldots, n - 1,
\end{align*}
$$

$\zeta_1$ is a continuous function and singular at $t$ and $x(t)$ for some values of $t \in [0, 1]$. The operator $D^\gamma$, is Riemann–Liouville fractional derivative while $D^\delta$, $D^\kappa$ stand for Caputo fractional derivatives and $\delta, \gamma \in (1, 2]$, $n - 1 < \kappa \leq n$, where $n \geq 3$. For the study of the EUS, fixed point approach is followed in this paper and an application is given to explain the findings.

1. Introduction. The use of fractional calculus in different scientific fields has attracted the attention of scientists in last two decades to large extent. Mathematical models via FDEs were studied fluid dynamics, hydrodynamics, signals, control theory, biology, viscoelastic theory, image processing, computer networking and many others [6–8,22,26,32,33].

Recently, Ahmad and Luca [10], Ahmad et al. [1] contemplated EU of solution for sequential fractional integro DEs with the help of fixed point approach and provided applications of their results. Zhang et al. [40] studied some necessary conditions EU of solutions for singular fractional DEs and provided applications. Srivastava et al. [35] contemplated a fractional order DE for the EU of solutions using fixed approach and provided applications. Lopez et al. [30] established necessary

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conditions required for the EU of positive solutions of fractional DEs using classical results of fixed points approach in Hölder spaces. Luca [31] used Guo-Krasnosel’skii theorem for the existence of positive solutions and multiplicity results for a nonlinear class of singular fractional DEs and applications were given. Khan et al. [29] studied singular fractional DE with $0^p$-operator for the EU of solutions and given applications of their results. Xie and Xie [36] studied high order nonlinear fractional DEs with singularity for the EU of solutions and multiplicity results with the help of topological degree approach. Yan et al. [37] used upper and lower solution technique for the existence and uniqueness of solutions for a fractional DE with $0^p$-operator.

Guo et al. [18] studied EU solution for fractional DEs with $p$-Laplacian and several parameters. Ghanmia et al. [19] discussed necessary conditions for the EU of solutions for fractional DEs with singularity using the Riemann-Liouville definition of fractional derivative. Karimov and Sadarangani [25] studied the EU of solution for singular fractional DEs with initial and boundary conditions using fixed point technique. Chang and Ponce [15] studied integro-differential system for exponential stability and applications. Ji [24] studied singular fractional DEs with $p$-Laplacian using fixed point technique. Saoudi [34] produced the necessary conditions required for the study of multiplicity results for a class of singular fractional DEs. Henderson and Luca [17] studied singular and non-singular fractional DEs with multiple boundary conditions for existence of positive solution and multiplicity results.

Zhang and Zhang [39] studied high order fractional DEs with singularities with the help of Krasnosel’skii and Leggett-Williams fixed point results and presented applications. For some more related results, we refer the readers to [9,12–14].

Inspired from the recently produced results of the scientists in the fractional calculus and applications, we use fixed point theorems to prove EPS and HUS of the below nonlinear SFDE with $0^p$-operator

\[
\begin{align*}
\mathcal{D}^\gamma \left[ 0^p \mathcal{D}^\kappa x(t) \right] + \mathcal{Q}(t)\zeta_1(t, x(t - g^s)) &= 0, \\
\mathcal{I}_0^{1-\gamma} (0^p \mathcal{D}^\kappa x(t))|_{t=0} &= 0 = \mathcal{I}_0^{2-\gamma} (0^p \mathcal{D}^\kappa x(t))|_{t=0}, \\
x'(0) &= 0, \quad x'(1) = 0 = \mathcal{D}^\delta x(1) = 0, \quad x^{(k)}(0) = 0, \quad \text{for} \quad k = 3, \ldots, n - 1,
\end{align*}
\]

where $\zeta_1$ is continuous functions involving singularity with respect to $t$ and $s$. The fractional orders $n-1 < \kappa, \gamma \leq n$, where $n \geq 3$, $\zeta_1 \in \mathcal{L}[0, 1]$ and $\mathcal{D}^\kappa, \mathcal{D}^\gamma$ are Caputo’s fractional derivative, $0^p \mathcal{D}^\kappa x = |x|^{p-2}x$ is the well-known $p$-Laplacian operator.

In this article, our goal is to study two important aspects of SFDE with $0^p$ (1.1) they are EPS and HUS. For these, we will convert the problem (1.1) into an integral equation form by the use of Green function. And will study the nature of the Green function whether decreasing or increasing and positive or negative. Then, some fixed point theorems will be utilized for the analysis of EPS and HUS will be examined by the same techniques as studied by Khan et al. in [28] for a class of Hybrid FDEs. Researchers can study the SFDE (1.1) with nonlinear $0^p$ for multiplicity results. For the recent related development in the fractional differential equations, we suggest the readers to [2–5,16,20,21,23,38,41].

**Definition 1.1.** Fractional order integral of $\zeta : (0, +\infty) \rightarrow \mathbb{R}$ for order $\kappa > 0$ is

\[
\mathcal{I}_0^\kappa \zeta(t) = \frac{1}{\Gamma(\kappa)} \int_0^t (t-s)^{\kappa-1} \zeta(s) \, ds,
\]
such that the integral is defined on \((0, +\infty)\), where
\[
\Gamma(\kappa) = \int_{0}^{+\infty} e^{-s^\kappa} ds.
\]

**Definition 1.2.** For a fractional order \(\kappa > 0\), Caputo’s derivative for \(\zeta(t) : (0, +\infty) \to \mathbb{R}\) is given by
\[
\mathcal{D}^\kappa \zeta(t) = \frac{1}{\Gamma(k - \kappa)} \int_{0}^{t} (t - s)^{k-\kappa-1} \zeta^{(k)}(s) ds,
\]
for \(k = [\kappa] + 1\), where \([\kappa]\) is used for the integer part of \(\kappa\).

**Lemma 1.1.** For \(\zeta \in C^{n-1}\) and \(\kappa \in (n-1, n]\), and we have
\[
\mathcal{I}_{0}^\kappa \mathcal{D}^\kappa \zeta(t) = \zeta(t) + k_{0} + k_{1}t + k_{2}t^{2} + \ldots + k_{m-1}t^{m-1},
\]
for the \(k_{i} \in \mathbb{R}\) for \(i = 0, 1, 2, \ldots, m - 1\).

**Theorem 1.2.** [29] Consider a Banach’s space \(\mathcal{S}\) and a cone \(\mathcal{B}^* \in \mathcal{S}\) be a cone. Let \(\mathcal{P}^*_1, \mathcal{P}^*_2\) be two bounded subsets of \(\mathcal{S}\) such that \(0 \in \mathcal{P}^*_1\), \(\overline{\mathcal{P}^*_1} \subset \mathcal{P}^*_2\). Then, a completely continuous operator \(T_0 : \mathcal{B}^* \cap (\overline{\mathcal{P}^*_2}\setminus \mathcal{P}^*_1) \to \mathcal{B}^*\), satisfying
\[
\begin{align*}
(A_1) \quad &\|T_0x\| \leq \|x\| \text{ if } x \in \mathcal{B}^* \cap \partial \mathcal{P}^*_1 \text{ and } \|T_0x\| \geq \|x\| \text{ if } x \in \mathcal{B}^* \cap \partial \mathcal{P}^*_2, \text{ or} \\
(A_2) \quad &\|T_0x\| \geq \|x\| \text{ if } x \in \mathcal{B}^* \cap \partial \mathcal{P}^*_1 \text{ and } \|T_0x\| \leq \|x\| \text{ if } x \in \mathcal{B}^* \cap \partial \mathcal{P}^*_2,
\end{align*}
\]
has a fixed point in \(\mathcal{B}^* \cap (\overline{\mathcal{P}^*_2}\setminus \mathcal{P}^*_1)\).

**Lemma 1.3.** [29] For a \(p\)-Laplacian operator \(\mathcal{U}^*_p\), we have
(1) If \(|\theta_1|, |\theta_2| \geq \rho > 0\), \(1 < p \leq 2, \theta_1\theta_2 > 0\), then
\[
|\mathcal{U}^*_p(\theta_1) - \mathcal{U}^*_p(\theta_2)| \leq (p - 1)p^{\rho - 2}|\theta_1 - \theta_2|.
\]
(2) If \(p > 2\), and \(|\theta_1|, |\theta_2| \leq \rho^*\), then
\[
|\mathcal{U}^*_p(\theta_1) - \mathcal{U}^*_p(\theta_2)| \leq (p - 1)p^{\rho^* - 2}|\theta_1 - \theta_2|.
\]

**Organization.** We have given five sections in the paper. The first section “Introduction” contains most relevant research including the literature, definitions and basic results. The second section is dedicated to the study of the properties Green’s function and its application to the existence and uniqueness of solution of the singular fractional DE with \(\mathcal{U}_p(1.1)\). The HU-stability is given in the third section. An application is presented for the explanation of the findings in the previous sections. Finally, the paper is summarized in the conclusion, section five.

2. Green’s function.

**Theorem 2.1.** Assume an integrable function \(\zeta_1 \in C[0,1]\) satisfying the singular fractional DE with \(\mathcal{U}_p(1.1)\). Then for \(\gamma \in (1,2]\), \(\kappa \in (n - 1, n]\) for \(n \geq 4\), the solution of SFDE with \(p\)-Laplacian operator and delay \(\rho^* > 0\):
\[
\begin{align*}
\mathcal{D}^\gamma [\mathcal{U}^*_p(\mathcal{D}^\kappa x(t))] + \zeta_1(t, x(t - \rho^*)) &= 0, \\
\mathcal{X}^{(1 - \gamma)} [\mathcal{U}^*_p(\mathcal{D}^\kappa x(t))] |_{t=0} &= 0 = \mathcal{X}^{2 - \gamma} [\mathcal{U}^*_p(\mathcal{D}^\kappa x(t))] |_{t=0}, \\
x'(0) = 0, x'(1) = 0 = \mathcal{D}^\kappa x(1) = 0, \quad x^{(j)}(0) = 0, \quad \text{for } j = 3, \ldots, n - 1,
\end{align*}
\]
is
\[
x(t) = \int_{0}^{1} \mathcal{M}^\kappa(t, s) \mathcal{U}^*_q \left( \frac{1}{\Gamma(\gamma)} \int_{0}^{s} (s - \zeta)^{\gamma - 1} Q(\zeta) \zeta_1(\zeta, x(x - \rho^*)) d\zeta \right) ds, \quad (2.2)
\]
where the multi valued Green's function $M^\kappa(t,s)$ is

$$
M^\kappa(t,s) = \begin{cases} 
\frac{-(t-s)^{\kappa-1}}{\Gamma(\kappa)} + t\frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + t^2 \frac{\Gamma(3-\delta)}{\Gamma(1+\delta)}(1-s)^{\kappa-\delta-1}, & s \leq t \leq 1, \\
\frac{-(t-s)^{\kappa-2}}{\Gamma(\kappa-1)} + t\frac{(1-s)^{\kappa-3}}{\Gamma(\kappa-2)} + t^2 \frac{\Gamma(3-\delta)}{\Gamma(1+\delta)}(1-s)^{\kappa-\delta-1}, & t \leq s \leq 1.
\end{cases}
$$

(2.3)

**Proof.** With the help of integral operator of fractional order $I_0^\gamma$ on (2.1) and Lemma 1.1, the singular fractional DE (2.1) converts to:

$$
U_p^* [D^\kappa x(t)] = -I_0^\gamma [\zeta_1(t,x(t\theta^*))] + c_1 t^{\gamma-1} + c_2 t^{\gamma-2}.
$$

(2.4)

By $I_0^\gamma (U_p^* [D^\kappa x(t)])|_{t=0} = 0 = I_0^\gamma (U_p^* [D^\kappa x(t)])|_{t=0}$, we have $c_1 = c_2 = 0$. Using values of $c_i$ for $i = 1, 2$, (2.4) becomes

$$
U_p^* [D^\kappa x(t)] = -I_0^\gamma [\zeta_1(t,x(t\theta^*))].
$$

(2.5)

From (2.5), we have

$$
D^\kappa x(t) = -U_p^* (I_0^\gamma [\zeta_1(t,x(t\theta^*))] dt).
$$

(2.6)

Again, applying $I_0^\gamma$ on (2.6) and using Lemma 1.1, we get

$$
x(t) = -I_0^\gamma (U_p^* (I_0^\gamma [\zeta_1(t,x(t\theta^*))])) + k_1 + k_2 t + k_3 t^2 + \ldots + k_n t^{n-1}.
$$

(2.7)

By the help of $x^{(j)}(0) = 0$ where $j = 3, \ldots, n - 1$ in (2.7), we evaluate $k_4 = k_5 = \ldots = k_n = 0$. Now using $x(0) = 0$, we find $k_1 = 0$. By condition $x'(1) = 0$, we devise $k_2 = I_0^{\gamma-1} (U_p^* (I_0^\gamma [\zeta_1(t,x(t\theta^*))]) dt)|_{t=1}$, and $D^\kappa x(1) = 0$ confer that $k_3 = \frac{\Gamma(3-\delta)}{\Gamma(3-\delta-1)}I_0^{\gamma-\delta} (I_0^\gamma [\zeta_1(t,x(t\theta^*))] dt)|_{t=1}$. Putting the calculated values of the constants in (2.7), we get

$$
x(t) = -I_0^\gamma (U_p^* (I_0^\gamma [Q(t)\zeta_1(t,x(t\theta^*))])) + tI_0^{\gamma-1} (U_p^* (I_0^\gamma [\zeta_1(t,x(t\theta^*))]))|_{t=1}
$$

$$
+ \frac{t^2 \Gamma(3-\delta)}{2} I_0^{\gamma-\delta} (U_p^* (I_0^\gamma [Q(t)\zeta_1(t,x(t\theta^*))]))|_{t=1}
$$

$$
= -\int_0^t (t-s)^{\kappa-1} \frac{1}{\Gamma(\kappa)} - \int_0^s (s-\zeta)^{\gamma-1} [Q(\zeta)\zeta_1(\zeta, x(\zeta, \theta^*))] d\zeta * ds
$$

$$
+ \int_0^1 (1-s)^{\kappa-2} \frac{1}{\Gamma(\kappa-1)} - \int_0^s (s-\zeta)^{\gamma-1} [Q(\zeta)\zeta_1(\zeta, x(\zeta, \theta^*))] d\zeta * ds
$$

$$
+ \frac{t^2 \Gamma(3-\delta)}{2} \int_0^1 (1-s)^{\kappa-\delta-1} \frac{1}{\Gamma(\kappa-\delta)} - \int_0^s (s-\zeta)^{\gamma-1} \frac{1}{\Gamma(\gamma)} - \int_0^s (s-\zeta)^{\gamma-1} [Q(t)\zeta_1(\zeta, x(\zeta, \theta^*))] d\zeta * ds,
$$

(2.8)

where $M^\kappa(t,s)$ is given in (2.3). \qed

**Lemma 2.2.** For the $M^\kappa(t,s)$ given by (2.3), the subsequent are satisfied:

\begin{enumerate}
    
    \item[(N_1)] $M^\kappa(t,s) > 0$ for all $0 < s, t < 1$;
    
    \item[(N_2)] $M^\kappa(t,s)$ is an increasing function and $\max_{t \in [0,1]} M^\kappa(t,s) = M^\kappa(1,s)$;
    
    \item[(N_3)] $t^{\kappa-1} \max_{t \in [0,1]} M^\kappa(t,s) \leq M^\kappa(1,s)$ for $0 < s, t < 1$.
\end{enumerate}
Proof. For \((N_1)\), we assume that

**Case 1.** When \(0 < s \leq t < 1\). Since \(\kappa \geq 4\) this confer \(\kappa - 1 \geq 3\). This confer

\[
M^\kappa(t,s) = \frac{-(t-s)^{\kappa-1}}{\Gamma(\kappa)} + \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t^2 \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{2 \Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} \\
= -t^{\kappa-1} \frac{(1 - \frac{s}{t})^{\kappa-1}}{\Gamma(\kappa)} + \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t^2 \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{2 \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} \\
\geq -t^{\kappa-1} \frac{(1-s)^{\kappa-1}}{\Gamma(\kappa)} + t^{\kappa-1} \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t^2 \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{2 \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} \\
> 0.
\]

**Case 2.** For \(t \leq s\), we devise

\[
M^\kappa(t,s) = t \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + t^2 \frac{\Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{\Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} > 0. \tag{2.10}
\]

With \((2.9), (2.10)\), we get \(0 < M^\kappa(t,s) \forall s, t \in (0,1)\).

For \((N_2)\), we contemplate the subsequent two cases:

**Case 1.** For \(s \leq t\), we evaluate

\[
\frac{\partial}{\partial t} M^\kappa(t,s) = -\frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{\Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} \\
= t \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{\Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} \\
\geq 0.
\]

**Case 2.** Consider \(t \leq s\), then \(\forall s, t \in (0,1)\)

\[
\frac{\partial}{\partial t} M^\kappa(t,s) = \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{\Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} > 0. \tag{2.12}
\]

With \((2.11), (2.12)\), we have \(\frac{\partial^2}{\partial t^2} M^\kappa(t,s) > 0 \forall s, t \in (0,1)\). Correspondingly, we have \(\frac{\partial^2}{\partial t^2} M^\kappa(t,s) > 0 \forall s, t \in (0,1)\). This confer the increasing nature of \(M^\kappa(t,s)\) with respect to \(t\), which shows that

\[
\max_{t \in [0,1]} M^\kappa(t,s) = -\frac{(1-s)^{\kappa-1}}{\Gamma(\kappa)} + \frac{1}{2} \frac{(1-s)^{\kappa-3}}{\Gamma(\kappa-2)} = M^\kappa(1,s). \tag{2.13}
\]

For \((N_3)\), assume two cases.

**Case 1.** For \(s \leq t\), where \(\kappa \geq 4\) this confer \(\kappa - 1 \geq 3\), which implies

\[
M^\kappa(t,s) = -\frac{(t-s)^{\kappa-1}}{\Gamma(\kappa)} + \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t^2 \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{2 \Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} \\
= -t^{\kappa-1} \frac{(1 - \frac{s}{t})^{\kappa-1}}{\Gamma(\kappa)} + \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t^2 \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{2 \Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1} \\
\geq -t^{\kappa-1} \frac{(1-s)^{\kappa-1}}{\Gamma(\kappa)} + t^{\kappa-1} \frac{(1-s)^{\kappa-2}}{\Gamma(\kappa-1)} + \frac{t^2 \Gamma(3 - \delta^*) (1-s)^{\kappa-\delta^*-1}}{2 \Gamma(1+\delta^*) \Gamma(\kappa-\delta^*)} (1-s)^{\kappa-\delta^*-1}.
\]

**Case 2.**
Assume by (2.14) and (2.15), the result is satisfied. Let \( Q \in \mathcal{S} \) be a Banach space in which we define \( \|v\| = \max_{t \in [0,1]} |v(t)| : x \in \mathcal{S} \). Let \( Q \) be a cone of positive functions in \( \mathcal{S} \) of the kind \( Q = \{v \in \mathcal{S} : v(t) \geq t^r \|v\|, \ t \in [0,1] \} \).

By the help of Theorem 2.1, (1.1) is equivalent to

\[
x(t) = \int_0^1 M^\kappa(t,s)\mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s-\varsigma)^{\gamma-1} [Q(\varsigma^*)\zeta_1(\varsigma^*,x(\varsigma^*-\varrho^*))]d\varsigma^* ds \right). \tag{2.16}
\]

Assume \( \mathcal{T}_0 : P \setminus \{0\} \to \mathcal{S} \) for \( (i = 1, 2) \) such that

\[
\mathcal{T}_0 x(t) = \int_0^1 M^\kappa(t,s)\mathcal{U}_q \left( \int_0^s M^\gamma(s,\varsigma^*)[Q(\varsigma^*)\zeta_1(\varsigma^*,x(\varsigma^*-\varrho^*))]d\varsigma^* ds \right). \tag{2.17}
\]

With the help of Theorem 2.1, solution of the SFDE with \( \mathcal{U}_p \) given by (1.1) is a fixed point \( x(t) \) of \( \mathcal{F} \), where

\[
x(t) = \mathcal{T}_0 x(t). \tag{2.18}
\]

We assume that:

- (P1) \( \zeta_1 : ((0,1) \times (0, +\infty)) \to [0, +\infty) \) is continuous;
- (P2) \( Q : (0,1) \to [0, +\infty) \) is non vanishing and continuous on \((0,1)\) with \( \|Q\| = \max_{t \in [0,1]} |Q(t)| < +\infty \);
- (P3) With \( a_1, M_{\kappa_1}^* > 0 \) and \( k_1 \in [0,1] \), function \( \zeta_1 \) satisfying

\[
|\zeta_1(t,x(t-\varrho^*))| \leq \mathcal{U}_p^*(a_1|x(t)|^{k_1} + M_{\kappa_1}^*); \tag{2.19}
\]

- (P4) There are constants \( \lambda_{\zeta_1} \) such that for all \( u, v \in \mathcal{S} \), the subsequent is satisfied

\[
|\zeta_1(t,x(t-\varrho^*)) - \zeta_1(t,v(t-\varrho^*))| \leq \lambda_{\zeta_1} |x(t) - v(t)|.
\]

**Theorem 2.3.** Let (P1) – (P3) are hold true. Then \( \mathcal{T}_0 \) is completely continuous operator.

**Proof.** For any \( x \in \mathcal{L}^\kappa(\tau_2) \setminus \mathcal{L}^\kappa(\tau_1) \), with the help of Lemma 2.2 and (2.17), we evaluate

\[
\mathcal{T}_0 x(t) = \int_0^1 M^\kappa(t,s)\mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s-\varsigma)^{\gamma-1} [Q(\varsigma)\zeta_1(\varsigma^*,x(\varsigma^*-\varrho^*))]d\varsigma^* ds \right). \tag{2.19}
\]
\[ T_0 x(t) = \int_0^1 \mathcal{M}^*(t, s) \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \tag{2.20} \]

\[ \geq t^{\gamma - 1} \int_0^1 \mathcal{M}^*(1, s) \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds. \]

With help of (2.19) and (2.20), we proceed

\[ T_0 x(t) \geq t^{\gamma - 1} \| T_0 x \|, \quad t \in [0, 1]. \tag{2.21} \]

This confer \( T_0 : L^\infty(\tau_2) \setminus L^\infty(\tau_1) \rightarrow P \). At the moment, for continuity of \( T_0 \), we show that \( \| T_0(x_n) - T_0(x) \| \rightarrow 0 \) as \( n \rightarrow \infty \), let us contemplate

\[
|T_0 x_n(t) - T_0 x(t)| = \left| \int_0^1 \mathcal{M}^*(t, s) \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \right|
\]

\[
- \int_0^s \mathcal{M}^*(t, s) \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \right| \quad (2.22)
\]

\[
\leq \int_0^1 |\mathcal{M}^*(t, s)| \left| \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \right|
\]

\[
- \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds. \]

By continuity of the function \( \zeta_1 \) and (2.22), we contrive \( |T_0 x_n(t) - T_0 x(t)| \rightarrow 0 \) as \( n \) goes to \( +\infty \). This affirms that \( T_0 \) is continuous. At the moment, for uniformly boundedness of \( T_0 \), by (Q1) and (2.17), we conspire

\[
|T_0 x(t)| = \left| \int_0^1 \mathcal{M}^*(t, s) \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \right|
\]

\[
= \int_0^1 |\mathcal{M}^*(t, s)| \left| \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \right|
\]

\[
\leq \int_0^1 |\mathcal{M}^*(1, s)| \left| \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \mathcal{Q} \left\| \overline{U}_q^* \right\| \left( a_1 \left\| x \right\| k_1 + M_{\zeta_1} \right) d\varsigma* \right) ds \right|
\]

\[
\leq \left( \frac{1}{\Gamma(\kappa + 1)} + \frac{1}{\Gamma(\kappa)} + \Gamma(\gamma) + \frac{1}{\Gamma(1 + \delta*)} \Gamma(\kappa - \delta* + 1) \right) \left( \frac{1}{\Gamma(\gamma + 1)} \right) ^{\gamma - 1} \mathcal{Q} \left\| \mathcal{Q} \right\| ^{\gamma - 1} \quad (2.23)
\]

\[
\times \left( a_1 \left\| x \right\| k_1 + M_{\zeta_1} \right),
\]

where \( \Delta_1 = \left( \frac{1}{\Gamma(\kappa + 1)} + \frac{1}{\Gamma(\kappa)} + \Gamma(3 - \delta*) \right) \Gamma(\gamma + 1) ^{\gamma - 1} \) By (2.23), the operator \( T_0 : L^\infty(\tau_2) \setminus L^\infty(\tau_1) \) is uniformly bounded. Now for the equiuniformity of the operator \( T_0 \), by (P3), (2.17) and Theorem 2.1, for \( t_1, t_2 \in [0, 1] \), we devise

\[
|T_0 x(t_1) - T_0 x(t_2)|
\]

\[
= \left| \int_0^1 \mathcal{M}^*(t_1, s) \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \right|
\]

\[
- \int_0^1 \mathcal{M}^*(t_2, s) \overline{U}_q^*(s) \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma*)^{\gamma - 1} \left[ \mathcal{Q}(\varsigma*) \zeta_1(\varsigma*, x(\varsigma* - \rho*)) \right] d\varsigma* \right) ds \right|.
\]
\begin{equation}
\begin{aligned}
&\leq \int_0^1 \left| \mathcal{M}^\kappa(t_1,s) - \mathcal{M}^\kappa(t_2,s) \right| \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} \|Q\| \right. \\
&\quad \times \mathcal{U}_q^* \left( a_1 \|x\|^k_1 + \mathcal{M}^\kappa_1 \right) \mathcal{A} \mathcal{U}^* \left. \right) ds \\
&\leq \left( \frac{|t_1^* - t_2^*|}{\Gamma(\kappa + 1)} + \frac{|t_1 - t_2|}{\Gamma(\kappa)} + |t_1^* - t_2^*| \frac{\Gamma(3 - \delta^*)}{\Gamma(\kappa - \delta^* + 1) \Gamma(1 + \delta^*)} \right) \frac{1}{\Gamma(\gamma + 1)} \|Q\|^{q-1} \\
&\quad \|Q\|^{q-1} (a_1 \|x\|^k_1 + \mathcal{M}^\kappa_1).
\end{aligned}
\end{equation}

It is clear from (2.24) that \( t_1 \to t_2 \) confer that \( |T_0^x(t_1) - T_0^x(t_2)| \to 0 \). Consequently, \( T_0(\mathcal{L}^*(\tau_2) \setminus \mathcal{L}^*(\tau_1)) \) is equicontinuous. By Arzela-Ascoli theorem \( T_0(\mathcal{L}^*(\tau_2) \setminus \mathcal{L}^*(\tau_1)) \) is compact which confer \( T_0 \) is compact in \( \mathcal{L}^*(\tau_2) \setminus \mathcal{L}^*(\tau_1) \). Subsequently, we conspire completely continuity of \( T_0 : \mathcal{L}^*(\tau_2) \setminus \mathcal{L}^*(\tau_1) \to P \).

Let us define the subsequent height functions as growth for function \( \zeta_1(t, x(t - \varphi^*)) \), \( \forall t \in (0, 1) \):

\[
\begin{align*}
\mathcal{U}^*_1(t, \varphi^*) &= \max\{\zeta_1(t, x(t - \varphi^*)) : t^{\kappa-1} \varphi^* \leq x \leq \varphi^*\}, \quad (t \in (0, 1), r > 0), \\
\mathcal{U}^*_0(t, \varphi^*) &= \min\{\zeta_1(t, x(t - \varphi^*)) : t^{\kappa-1} \varphi^* \leq x \leq \varphi^*\}, \quad (t \in (0, 1), \varphi^* > 0). 
\end{align*}
\]

(2.25)

**Theorem 2.4.** Assume that \((P_1) - (P_3)\) hold true and there are \( a, b \in \mathbb{R}^+ \) such that any of the subsequent conditions is satisfied:

\((B_1)\) \( a \leq \int_0^1 \mathcal{M}^\kappa(1,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} \mathcal{U}^*_0(\varsigma, a) d\varsigma \right) ds < +\infty \) and

\( \int_0^1 \mathcal{M}^\kappa(1,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} \mathcal{Q}(\varsigma) \mathcal{U}^*_0(\varsigma, b) d\varsigma \right) ds \leq b; \)

\((B_2)\) \( b \leq \int_0^1 \mathcal{M}^\kappa(1,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} [\mathcal{Q}(\varsigma) \mathcal{U}^*_0(\varsigma, a)] d\varsigma \right) ds < a \) and

\( \frac{1}{\mathcal{M}^\kappa(1,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} [\mathcal{Q}(\varsigma) \mathcal{U}^*_0(\varsigma, b)] d\varsigma \right) ds} < +\infty. \)

Then the problem (1.1) has an increasing positive solution \( x^* \in P \) such that \( a \leq \|x^*\| \leq b. \)

**Proof.** With no loss of generality, we can contemplate \((B_1)\). If \( x \in \partial \mathcal{L}^*(a) \), then we devise \( \|x\| = a \) and \( \forall t \in (0, 1) \), \( t^{\kappa-1} a \leq x(t) \leq a. \) By (2.25), we devise \( \mathcal{U}^*_0(t,u) \leq \zeta_1(t,u - \varphi^*) \), we get

\[
\|T_0^x(t)\| = \max_{t \in [0,1]} \int_0^1 \mathcal{M}^\kappa(t,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} [\mathcal{Q}(\varsigma) \zeta_1(\varsigma, x(\varsigma - \varphi^*))] d\varsigma \right) ds
\]

\[
\geq t^{\kappa-1} \int_0^1 \mathcal{M}^\kappa(t,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} [\mathcal{Q}(\varsigma) \zeta_1(\varsigma, x(\varsigma - \varphi^*))] d\varsigma \right) ds
\]

\[
\geq \int_0^1 \mathcal{M}^\kappa(1,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} [\mathcal{Q}(\varsigma) \zeta_1(\varsigma, x(\varsigma - \varphi^*))] d\varsigma \right) ds \geq a = \|x\|.
\]

If \( x(t) \in \partial \mathcal{L}^*(b) \), then \( \forall t \in [0,1], \|x\| = b \) and \( t^{\kappa-1} b \leq x(t) \leq b \). With the help of (2.25), we devise \( \mathcal{U}^*_0(t,u) \leq \zeta_1(t,x(t - \varphi^*)) \) for \( t \in (0, 1) \), which implies

\[
\|T_0^x(t)\| = \max_{t \in [0,1]} \int_0^1 \mathcal{M}^\kappa(t,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} [\mathcal{Q}(\varsigma) \zeta_1(\varsigma, x(\varsigma - \varphi^*))] d\varsigma \right) ds
\]

\[
\leq t^{\kappa-1} \int_0^1 \mathcal{M}^\kappa(1,s) \mathcal{U}_q^* \left( \frac{1}{\Gamma(\gamma)} \int_0^s (s - \varsigma)^{\gamma-1} [\mathcal{Q}(\varsigma) \zeta_1(\varsigma, x(\varsigma - \varphi^*))] d\varsigma \right) ds
\]
Thus $x^\ast \in \mathcal{L}^2(b) \setminus \mathcal{L}^1(a)$ is a fixed of $T_0$. Consequently, $a \leq \|x^\ast\| \leq b$. Further, by Lemma 2.2 and Theorem 2.1, the $x^\ast(t) \geq t^\alpha \|x\ast\| \geq at^\alpha - 1 > 0$, $\forall t \in (0, 1)$, which confer that $x^\ast$ is positive. We further have

$$\frac{\partial}{\partial t} x^\ast(t) = \frac{\partial}{\partial t} (T_0 x^\ast(t))$$

(2.28)

which confer the increase of solution $x^\ast$.

## 3. HU-stability

In this section, we discuss HU-stability of singular fractional DE with $\mathcal{U}_p$ given by (1.1) with the help of work in [27, 28].

**Definition 3.1.** The fractional order integral system (2.16) is HU-stability provided that there is a constant $D^* > 0$ sustaining:

For every $\lambda > 0$, if

$$|x(t) - \int_0^1 \mathcal{M}^\ast(t, s) \mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \right) \int_0^s (s - \zeta^\ast)^{\gamma - 1} [Q(\zeta^\ast) \xi_1(\zeta^\ast, x(\zeta^\ast - \varrho^\ast))] d\zeta^\ast ds \right| \leq \lambda,$$

(3.1)

there exist a pair say $x^\ast(t)$ sustaining

$$x^\ast(t) = \int_0^1 \mathcal{M}^\ast(t, s) \mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \right) \int_0^s (s - \zeta^\ast)^{\gamma - 1} [Q(\zeta^\ast) \xi_1(\zeta^\ast, x^\ast(\zeta^\ast - \varrho^\ast))] d\zeta^\ast ds,$$

such that

$$|x(t) - x^\ast(t)| \leq D^* \lambda.$$

(3.2)

**Theorem 3.1.** Assume that ($P_1$), ($P_2$) and ($P_3$) hold true. Then, the SFDE with nonlinear $p$-Laplacian operator (1.1) is HUS.

**Proof.** By Definition 3.1 and Theorem 2.4, assume that $x(t)$ be exact solution of (2.16) and $x^\ast(t)$ an approximation sustaining (3.2), then

$$|x(t) - x^\ast(t)|$$

$$= \left| \int_0^1 \mathcal{M}^\ast(t, s) \mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \right) \int_0^s (s - \zeta^\ast)^{\gamma - 1} [Q(\zeta^\ast) \xi_1(\zeta^\ast, x(\zeta^\ast - \varrho^\ast))] d\zeta^\ast ds \right|

- \int_0^1 \mathcal{M}^\ast(t, s) \mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \right) \int_0^s (s - \zeta^\ast)^{\gamma - 1} [Q(\zeta^\ast) \xi_1(\zeta^\ast, x^\ast(\zeta^\ast - \varrho^\ast))] d\zeta^\ast ds$$

(3.3)

$$\leq (p - 1)\rho^{p - 2} \|Q\|^{q - 1} \left( \int_0^1 |\mathcal{M}^\ast(t, s)| \right)$$

$$\mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \right) \int_0^s (s - \zeta^\ast)^{\gamma - 1} [Q(\zeta^\ast) \xi_1(\zeta^\ast, x(\zeta^\ast - \varrho^\ast))] d\zeta^\ast ds$$

$$- \mathcal{U}_q \left( \frac{1}{\Gamma(\gamma)} \right) \int_0^s (s - \zeta^\ast)^{\gamma - 1} [Q(\zeta^\ast) \xi_1(\zeta^\ast, x^\ast(\zeta^\ast - \varrho^\ast))] d\zeta^\ast ds$$

$$\leq (p - 1)\rho^{p - 2} \lambda_{\xi_1} \left( \frac{1}{\Gamma(\kappa + 1)} + \frac{1}{\Gamma(\kappa)} + \frac{1}{\Gamma(\delta^* + 1)} \right) \left[ \frac{1}{\Gamma(\gamma + 1)} \right]^{q - 1}$$

$$\times \|Q\|^{q - 1} \|x - x^\ast\|$$

\(\square\)
where $\mathcal{D}^* = \|Q\|^{-1}(p - 1)\rho^{p-2}\xi\frac{1}{\Gamma(\kappa+1)} + \frac{1}{\Gamma(\kappa)} + \frac{\Gamma(3-\delta^*)}{\Gamma(1+\delta^*)\Gamma(\kappa-\delta^*)+1})\frac{1}{\Gamma(\kappa+1)}\|Q\|^{-1}$. Thus, with the use of (3.3), the equation (2.16) is HU-stable. Ultimately, the singular fractional DE with $\mathcal{U}_p$ operator (1.1) is HU-stable.

4. An application. In this section, we provide an application of our theorems.

Example 4.1. For $p = 3$, $\delta^* = 1.5$, $\gamma = 1.5$, $\kappa = 3.5$, $\rho = 0.5$, $t \in [0, 1]$, $Q = \frac{1}{\sqrt{t+1}}$, $\zeta_1(t, x(t - \rho^*)) = x^3(t) + \frac{1-x^*}{3\sqrt{x(t)}}$, we contemplate an example given by

\[
\begin{aligned}
\mathcal{D}^*[U_p^*(\mathcal{D}^*x(t))] + Q(t)\zeta_1(t, x(t - \rho^*)) &= 0, \\
(U_p^*[\mathcal{D}^*x(t)])^{(i)}|_{t=0} = 0 = x^{(i)}(0) = x^{(i)}(1), & \text{for } k = 0, 1, 2, i = 0, 1, 2.
\end{aligned}
\]

(4.1)

Clearly $Q \in C((0, 1), [0, +\infty))$, $\zeta_1 \in C((0, 1) \times (0, +\infty), [0, +\infty))$. Consider

\[
\begin{aligned}
\mathcal{U}^*_{\max}(t, r) &= \max\{u^3 + \frac{0.5}{3u^2} : t^2r \leq x \leq r^3 + \frac{0.5}{3t^2r^2}, \\
\mathcal{U}^*_{\min}(t, r) &= \min\{u^3 + \frac{0.5}{3u^2} : t^2r \leq x \leq r^3 + \frac{0.5}{3t^2r^2},
\end{aligned}
\]

\[
\int_0^1 \mathcal{M}(1, s)\mathcal{U}^*_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s - \zeta^*)^{-1} [Q(\zeta^*)\mathcal{U}^*_\max(\zeta^*, b)] d\zeta^* \right) ds
\]

\[
= \int_0^1 \mathcal{M}(1, s)\mathcal{U}^*_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s - \zeta^*)^{-1} [Q(\zeta^*)\mathcal{U}^*_\max(\zeta^*, 1)] d\zeta^* \right) ds
\]

(4.2)

\[
\int_0^1 \frac{1}{\Gamma(\gamma)} \int_0^s (s - \zeta^*)^{-1} \left[\frac{1}{\sqrt{1-\zeta^*} \left(1 + \frac{1}{3\sqrt{\zeta^*}}\right)}\right] d\zeta^* ds
\]

\[
= 0.0185946 < 1.
\]

Correspondingly, we have

\[
\begin{aligned}
\int_0^1 \mathcal{M}(t, s)\mathcal{U}^*_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s - \zeta^*)^{-1} [Q(\zeta^*)\mathcal{U}^*_\min(\zeta^*, a)] d\zeta^* \right) ds
\end{aligned}
\]

\[
= \int_0^1 \mathcal{M}(1, s)\mathcal{U}^*_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s - \zeta^*)^{-1} [Q(\zeta^*)\mathcal{U}^*_\min(\zeta^*, 1)] d\zeta^* \right) ds
\]

(4.3)

\[
\int_0^1 \frac{1}{\Gamma(\gamma)} \int_0^s (s - \zeta^*)^{-1} \left[\frac{1}{\sqrt{1-\zeta^*} \left(1 + \frac{1}{3\sqrt{\zeta^*}}\right)}\right] d\zeta^* ds
\]

\[
= 0.00389515 > \frac{1}{1000}.
\]

Thus, with the help of Theorem 2.4, the singular fractional DE with $\mathcal{U}_p$ (4.1) has a non trivial solution $u^*$ which satisfies $\frac{1}{1000} \leq \|x^*\| \leq 1$. 

5. **Conclusion.** In this paper we are considered stability and existence criteria for a SFDE with $\dot{U}_p(1.1)$. For these aims, the research problem was transferred into its equivalent integral equation form by the use of Green function. The Green function was studied for its nature in the interval $(0, 1)$ and it was proved that the function is an increasing and positive. After these, with the help of fixed point approach, existence and uniqueness results were established. Then HU-stability of $(1.1)$ was also determined. At the end, for the application, an application was given in which the integral was evaluated for the numerical values by *Mathematica*. For future research, we suggest reconsideration of the singular fractional DE with $\dot{U}_p(1.1)$ for multiplicity results.

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