MAHLER MEASURES, STABLE PAIRS, AND THE GLOBAL COERCIVE ESTIMATE FOR THE MABUCHI FUNCTIONAL

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ABSTRACT. We show that the Mabuchi energy of any polarized manifold \((X, L)\) is (bounded below) proper on the full space of Kähler metrics in the class \(c_1(L)\) if and only if \((X, L)\) is asymptotically (semi)stable. It now follows from work of Xiuxiong Chen and Jinguri Cheng that \(X\) admits a cscK metric in \(c_1(L)\) iff \((X, L)\) is asymptotically stable, provided that the group \(\text{Aut}(X, L)\) is finite.

1. STATEMENT OF MAIN RESULTS

Let \(G\) be a reductive algebraic group over \(\mathbb{C}\). Let \(W\) be any finite dimensional complex representation of \(G\). Fix \(w \in W \setminus \{0\}\). Define \(O_w := G \cdot w\), the \(G\) orbit of \(w\) in \(W\). Recall that \(w\) is semistable if and only if
\[
0 /\notin \overline{O}_w
\]
(1.1)
where \(\overline{O}_w\) is the Zariski closure of the orbit in \(W\). Next choose any Hermitian norm \(h = || \cdot ||\) on \(W\). We define
\[
dist_h(\overline{O}_w, 0) := \inf\{||\sigma \cdot w|| | \sigma \in G\}.
\]
(1.2)
Then we have the following well known proposition.

**Proposition 1.1.** A point \(w \in W \setminus \{0\}\) is semistable if and only if there is a constant \(C = C(h) \geq 0\) such that
\[
\log \text{dist}_h(\overline{O}_w, 0) \geq -C.
\]
(1.3)

Let \((X, L)\) be a polarized manifold. Fix a large \(k\) embedding of \(X\) into \(\mathbb{P}^N\). Let \(R_X\) and \(\Delta_X\) denote Cayley’s \(X\)-resultant and \(X\)-hyperdiscriminant respectively. Recall that these are irreducible polynomials in the following \(G\) modules:
\[
R_X \in \mathbb{C}_{d(n+1)}[M_{(n+1) \times (N+1)}]^{SL(n+1,\mathbb{C})},
\]
\[
\Delta_X \in \mathbb{C}_{n(n+1)d-d\mu}[M_{n \times (N+1)}]^{SL(n,\mathbb{C})}.
\]
(1.4)

Let \(O_{R\Delta}\) and \(O_R\) denote the projective orbits
\[
O_{R\Delta} := G \cdot \{(R_{X}^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)})\} \subset \mathbb{P}(V \oplus W),
\]
\[
O_{R_X} := G \cdot \{(R_{X}^{\deg(\Delta)}, 0)\} \subset \mathbb{P}(V \oplus \{0\}).
\]
(1.5)

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\(^1\)In this paper we always take \(G = SL(N+1, \mathbb{C})\).
where \( \mathbb{V} \) and \( \mathbb{W} \) are the obvious \( G \) modules. Next choose any Hermitian metric \( h \) on \( L \) with positive curvature. With Proposition 1.1 in mind we make the following definition.

**Definition 1.** A polarized manifold \((X, L)\) is asymptotically semistable if and only if there is a constant \( C = C(h) \geq 0 \) such that for all \( k >> 0 \) we have
\[
\log \tan \text{dist}_0(\mathcal{O}_{R\Delta},\mathcal{O}_R) \geq -C k^{2n}.
\]
(1.6)

As we will explain in the sections that follow, \( \text{dist}_0 \) is simply the distance between the orbit closures measured in the Mahler metric on polynomials. The curious appearance of \( \tan \) in the above definition is due to the fact that the orbits are projective, not affine. Moreover, \( R \) and \( \Delta \) must be scaled to unit length in the Mahler measure. There is a similar but slightly more technical definition of asymptotic stability of \((X, L)\). This is described in detail below. For the moment we remark to the reader that any stable \( X \subset \mathbb{P}^N \) has finite automorphism group.

With this said we can state the main result of this article.

**Theorem 1.1.** Let \((X, L)\) be a polarized manifold. Let \( h \) be a Hermitian metric on \( L \) with positive curvature \( \omega_h \). Then

- \((X, L)\) is asymptotically stable if and only if the Mabuchi energy is proper on \( \mathcal{H}_\omega \).
- \((X, L)\) is asymptotically semistable if and only if the Mabuchi energy is bounded below on \( \mathcal{H}_\omega \).

A variational characterization of the existence of a Kähler Einstein metric on a Fano manifold is provided by the following theorem of Gang Tian [33].

\(\ast\) Let \((X, \omega)\) be a Fano manifold with \( [\omega] = c_1(X) \). Assume that \( \text{Aut}(X) \) is finite. Then \( X \) admits a Kähler Einstein metric if and only if the Mabuchi energy is proper.

Combining \(\ast\) with our main result gives our first corollary.

**Corollary 1.1.** Let \((X, -K_X)\) be an anti-canonically polarized manifold. Assume that \( \text{Aut}(X) \) is finite. Then \((X, -K_X)\) is asymptotically stable if and only if \( X \) admits a Kähler Einstein metric in the class \( c_1(X) \).

This provides another algebraic characterization of the existence of a Kähler Einstein metric on a Fano manifold with finite symmetry group.

An important development in Kähler geometry is the following deep result of Jinguri Cheng and Xiuxiong Chen [11], [12], [13], which generalizes Tian’s properness Theorem to any Kähler class.

\(\ast\ast\) Let \((X, \omega)\) be a compact Kähler manifold. Then the Mabuchi energy is proper (modulo automorphisms of \( X \), if any) on \( \mathcal{H}_\omega \) if and only if there is a metric of constant scalar curvature in the class \([\omega]\).

Combining \(\ast\ast\) with our main result gives our second corollary.

**Corollary 1.2.** Let \((X, L)\) be an arbitrary polarized manifold. Assume that \( \text{Aut}(X, L) \) is finite. Then \((X, L)\) is asymptotically stable if and only if there is a constant scalar curvature metric in \( c_1(L) \).
1.1. **Discussion.** The most important statements in the article are (5.32) and (5.33). These follow at once from Theorem 5.3, the purpose of which is to identify the norms appearing in Theorem A from [24]. In principle the main results of this article were available shortly after the appearance of [24]. The norms that appeared in Theorem A of [24] are conformally equivalent to the standard $L^2$ norms on polynomials. Since the conformal factors are continuous, they are bounded by reasons of compactness. The conclusion was that the Mabuchi energy is almost the distance between the orbits. That is, the distance in the usual Fubini Study metric induced by $L^2$ up to some (unknown) error that depended (somehow) on the degree of the embedding. Based on work in [4], [5], and [6] the author recently found a more sophisticated path to the relationship between the Mabuchi energy restricted to the Bergman metrics and the resultant and hyperdiscriminant of the subvariety which revealed that the error was in fact the difference between the $L^2$ norm and another well known norm, namely the $L^0$ norm, i.e. the Mahler measure (see Theorem (5.3)). The boundedness of the error, initially attributed to compactness, is just an expression of the fact that these norms are comparable. The outcome is that the norm on the space of polynomials which connects the Mabuchi energy to stability of the pair $(R, \Delta)$ is exactly given by the Mahler measure. Now asymptotic stability and global bounds on K-energy maps follow almost at once from Tian’s Thesis [31].

The strategy of restricting to the Bergman metrics is due to Tian and Yau. Tian explained it to the author many years ago. Despite Tian’s many works on the subject, as well as the articles [16] and [15], this strategy was never really developed. Instead, the approach of Tian in [37] as well as Chen-Donaldson-Sun in [10], [9], [8] is to reduce an infinite dimensional estimate to a finite dimensional one. Whereas the approach of this author is to obtain the infinite dimensional estimate from a sequence of finite dimensional estimates.

The finite dimensional estimates are equivalent to the stability of the variety with respect to the given embedding.

As we have mentioned, the precise definition of the asymptotic stability of a polarized manifold appears below. The relevant ideas are contained in definitions (9),(10),(13), and (14). The reader should compare the author’s definition of (semi)stability with the many variations of K-Stability that appear in the literature. First, we consider orbits under all of $G$, not just one parameter subgroups of $G$. Second, from the author’s point of view, stability is not necessarily concerned with a variety in a projective space. Stability is a property that a pair of (non-zero) vectors in a pair of finite dimensional complex representations of an algebraic group may, or may not, possess. As the reader shall see, the stability of a projective variety is a special case of this situation. As we have already mentioned, test configurations do not play a direct part in our definition of stability, they are rather considered as a means to check stability. This is exactly how one parameter subgroups are used in Hilbert and Mumford’s Geometric Invariant Theory. The author is optimistic about the eventual conversion of our stability condition into a (hopefully tractable) combinatorial condition that can be checked in concrete examples. This is due to the fact that a (semi)stable pair is a straightforward extension of Mumford’s stability (see the table at the end of section 2) . In particular the vast array of tools concerning actions of reductive groups on finite dimensional representations can be used. On the other hand, the author does not expect that checking the stability of a given pair will ever be made entirely trivial: indeed, even after so many years, and so

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2That the $L^p$ norms only give norms for $p \in [1, \infty]$ does not matter in this article.
much creative effort, checking the stability of Chow and Hilbert points in dimensions $\geq 3$ still seems to be out of reach.

This paper is organized as follows. In section 2 we give an account of the (semi)stability of pairs of rational representations of a reductive algebraic group and provide several elementary examples of such pairs, we also show that the automorphism group of a stable pair is finite. In section 3 we apply the ideas of section 2 to (hyper)discriminants and resultants of a projective variety. In section 4 we discuss the equivalence among the $L^p$ metrics on orbit closures in polynomial representations. Of special importance is the case when $p = 0$. The special metric induced by this measure allows us to define asymptotic (semi)stability for any polarized manifold $(X, L)$. In section 5 we write down the conformal factor that appears in Theorem A of [24]. This allows us to show that the Mabuchi energy is the distance between the orbits where the distance is computed the the Mahler metric. This is enough to establish the equivalence between the global coercive estimate for the Mabuchi energy and the asymptotic stability of the polarized manifold.

2. SEMISTABILITY OF PAIRS

Let $G$ denote any of the classical linear reductive algebraic groups over $\mathbb{C}$. For example, $G$ can be taken to be any one of the following

\begin{equation}
GL(N, \mathbb{C}), SL(N, \mathbb{C}), SO(N, \mathbb{C}), O(N, \mathbb{C}), Sp(2N, \mathbb{C}).
\end{equation}

Primarily we will be interested in the case when $G$ is the special linear group. For any vector space $V$ and any $v \in V \setminus \{0\}$ we let $[v] \in \mathbb{P}(V)$ denote the line through $v$. If $V$ is a $G$ module then we can consider the projective orbit:

\begin{equation}
O_v := G \cdot [v] \subset \mathbb{P}(V).
\end{equation}

We let $\overline{O}_v$ denote the Zariski closure of this orbit.

We consider pairs $(E; e)$ such that $E$ is a finite dimensional complex $G$-module and the linear span of the orbit $G \cdot e$ coincides with $E$.

**Definition 2.** (see [27]) A pair $(U; u)$ dominates $(W; w)$, in which case we write $(U; u) \succeq (W; w)$, if and only if there exists $\pi \in \text{Hom}(U, W)^G$ such that $\pi(u) = w$ and the induced rational map $\pi : \mathbb{P}(U) \dashrightarrow \mathbb{P}(W)$ restricts to a regular finite map $\pi : \overline{O}_u \dashrightarrow \overline{O}_w$ between the Zariski closures of the orbits.

My approach to the Stability Conjectures is based on this definition. In [27], the motivation behind making such a definition seems to the problem of decomposing the symmetric power of an irreducible representation of $GL(n, \mathbb{C})$. It is mysterious that the same definition appears when one seeks to bound (from below) the Mabuchi energy restricted to the space of Bergman metrics.

Observe that the restriction of the map $\pi$ to $\overline{O}_u$ is regular if and only if the following holds

\begin{equation}
\overline{O}_u \cap \mathbb{P}(\ker(\pi)) = \emptyset.
\end{equation}

As the reader can easily check, whenever $(U; u) \succeq (W; w)$ it follows that

\begin{equation}
\pi(U) = W \text{ and } U = \ker(\pi) \oplus W \text{ (G-module splitting)}.
\end{equation}

\footnote{The author was led to the same definition independently. See “semistable pair” below.}
Therefore we may identify $\pi$ with projection onto $W$ and $u$ decomposes as follows
\begin{equation}
(2.5) \quad v = (u_\pi, w), \quad \ker(\pi) \ni u_\pi \neq 0.
\end{equation}
Again the reader can easily check that (2.3) is equivalent to
\begin{equation}
(2.6) \quad \overline{O}_{(u_\pi, w)} \cap \overline{O}_{u_\pi} = \emptyset \quad \text{ (Zariski closure in $\mathbb{P}(\ker(\pi) \oplus W)$)}. \end{equation}

We summarize this discussion in the following way. Given $V$ and $W$ two $G$ representations with (nonzero) points $v$ and $w$ respectively, we consider, as before, the projective orbits
\begin{equation}
(2.7) \quad O_{(v, w)} := G \cdot [(v, w)] \subset \mathbb{P}(V \oplus W), \quad O_v := G \cdot [(v, 0)] \subset \mathbb{P}(V \oplus \{0\}).
\end{equation}
Now we can give the definition of a semistable pair. This definition seems the most appropriate for the Stability Conjectures as it gives precise characterization of the infimum of the Mabuchi energy restricted to the space of Bergman metrics.

**Definition 3.** The pair $(v, w)$ is **semistable** if and only if $O_{(v, w)} \cap O_v = \emptyset$.

The relationship of this with Mumford’s Geometric Invariant Theory is brought out in the following example.

**Example 1.** Let $V \cong \mathbb{C}$ be the trivial one dimensional representation and let $v = 1$. Suppose $W$ is any representation of $G$ and let $w \in W \setminus \{0\}$. Then $([1], [w])$ is a semistable pair if and only if $0 \notin G \cdot w \subset W$.

**Example 2.** Let $V_e$ and $V_d$ be irreducible $SL(2, \mathbb{C})$ modules with highest weights $e \in \mathbb{N}$ and $d \in \mathbb{N}$ respectively. These are well known to be spaces of homogeneous polynomials in two variables. Let $f$ and $g$ be two such polynomials in $V_e \setminus \{0\}$ and $W_d \setminus \{0\}$ respectively. If the pair $(f, g)$ is semistable then we must have that
\begin{equation}
(2.8) \quad e \leq d \quad \text{and for all} \quad p \in \mathbb{P}^1 \quad \text{we have} \quad \ord_p(g) - \ord_p(f) \leq \frac{d - e}{2}.
\end{equation}
In particular when $e = d - 1$ there are no semistable pairs.

Let $E$ be a finite dimensional reducible representation of $G$. Let $u \in E \setminus \{0\}$. Let $O \subset \mathbb{P}(E)$ denote the projective orbit $G \cdot [u]$. We assume that the linear span of $O$ coincides with $\mathbb{P}(E)$. Fix a Borel subgroup $B \leq G$ and a maximal algebraic torus $T \leq B$. Let $\Lambda^+$ denote the dominant integral weights relative to $B$. It is well known that $O$ is a union of orbits at least one of which is closed and each closed orbit corresponds to an irreducible submodule $E_{\mu_*}$ of $E$. We assume that $O$ consists of finitely many orbits. Let $\Lambda^+(O)$ denote the dominant weights corresponding to the closed orbits in $O$. Then we have the decomposition
\[ O = O \cup O_1 \cup \cdots \cup O_k \cup \bigcup_{\mu_* \in \Lambda^+(O)} G \cdot [w_{\mu_*}], \]
where $w_{\mu_*}$ is the corresponding highest weight vector. Now we decompose $E$ according to the orbit $O$
\[ E = V \oplus \bigoplus_{\mu_* \in \Lambda^+(O)} E_{\mu_*}. \]

\[ \text{We do not assume anything about the linear spans of the orbits.} \]
We assume that $\mathbb{V} \neq 0$. Let $\pi_O$ and $\pi_V$ denote the projections onto $\bigoplus_{\mu \in \Lambda^+(\mathcal{O})} E_{\mu}$ and $\mathbb{V}$ respectively. Then we may decompose $u$ as follows

$$u = (v, w) := (\pi_V(u), \pi_O(u)).$$

Then $(v, w)$ is semistable if and only if for every $1 \leq i \leq k$ there exists a $\mu_i \in \Lambda^+(\mathcal{O})$ such that $\pi_{\mu_i}(x_i) \neq 0$ where $O_i = G \cdot [x_i]$ and $\pi_{\mu_i}$ is the projection onto $E_{\mu_i}$.

The simplest situation is the case when $\mathcal{O}$ consists of two orbits (one of which is closed)

$$\mathcal{O} = \mathcal{O} \cup G \cdot [w_{\mu_i}].$$

In this case it is automatic that the pair $(\pi_V(u), \pi_O(u))$ is semistable. Therefore the class of two orbit varieties (or, more generally, quasi-closed orbits) provides many examples of semistable pairs. Such varieties have been completely classified by Stephanie Cupit-Foutou (see [14]) and Alexander Smirnov (see [26]).

**Example 3.** Let $\psi : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}(\wedge^2 \mathbb{C}^3)$ be the rational map $\psi([v], [w]) := [v \wedge w]$. The graph of $\psi$ is

$$\Gamma_\psi := \{(v, w, [v \wedge w]) \mid [v] \neq [w]\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}(\wedge^2 \mathbb{C}^3).$$

Recall that the blow up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the diagonal $\Delta$ is the Zariski closure of $\Gamma_\psi$ inside $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}(\wedge^2 \mathbb{C}^3)$. We will denote the blow up by $B_\Delta (\mathbb{P}^2 \times \mathbb{P}^2)$ and let $E \cong \mathbb{P}(T_{\mathbb{P}^2}^{-1,0})$ denote the exceptional divisor. The situation can be pictured as follows

$$\begin{array}{ccc}
\mathbb{P}^2 \times \mathbb{P}^2 & \xrightarrow{\psi} & \mathbb{P}(\wedge^2 \mathbb{C}^3) \\
\downarrow p_1 & & \downarrow p_3 \\
\mathbb{P}^2 \times \mathbb{P}^2 & \xrightarrow{\psi} & \mathbb{P}(\wedge^2 \mathbb{C}^3)
\end{array}$$

Then we claim that $B = B_\Delta (\mathbb{P}^2 \times \mathbb{P}^2)$ is a two-orbit $G = SL(3, \mathbb{C})$ variety (for the natural $G$ action) with orbit decomposition

$$B = (B \setminus E) \cup E.$$ 

Where $(B \setminus E)$ is necessarily the open orbit. There is a $G$ equivariant identification

$$B \setminus E \cong \mathbb{P}^2 \times \mathbb{P}^2 \setminus \Delta.$$ 

Since $G$ acts transitively on planes in $\mathbb{C}^3$, we easily get that $\mathbb{P}^2 \times \mathbb{P}^2 \setminus \Delta$ is an orbit:

$$G \cdot ([e_1], [e_2]) = \mathbb{P}^2 \times \mathbb{P}^2 \setminus \Delta.$$ 

To see that $E$ is a homogeneous $G$ variety we can proceed as follows. We have the decomposition into irreducible summands

$$\mathbb{C}^3 \times \mathbb{C}^3 \otimes \wedge^2 \mathbb{C}^3 \cong E_{310} \oplus \mathbb{C}^3 \oplus S^2(\wedge^2 \mathbb{C}^3) \oplus \mathbb{C}^3.$$ 

The summand $E_{310}$ appears as follows

$$0 \to E_{310} \cong \text{Ker}(\pi) \to S^2(\mathbb{C}^3) \otimes \wedge^2(\mathbb{C}^3) \xrightarrow{\pi} \mathbb{C}^3 \to 0,$$

where the map $\pi$ is defined by

$$\pi(v \cdot w \otimes \alpha) = \alpha(v)w + \alpha(w)v.$$
Note that 

\[ e_1^2 \otimes (e_1 \wedge e_2) \in \text{Ker}(\pi). \]

Since \( e_1^2 \otimes (e_1 \wedge e_2) \) is a highest weight \((310)\) vector we see that \( E_{310} \) is a summand of \( \text{Ker}(\pi) \). Since these spaces have the same dimension (which is 15 by the Weyl dimension formula) they coincide. Next we observe that

\[ ([e_1 + te_2], [e_1], [e_1 \wedge e_2]) \in \Gamma, \quad \text{for all } t \in \mathbb{C}^*. \]

As \( t \to 0 \) we have

\[ ([e_1 + te_2], [e_1], [e_1 \wedge e_2]) \to ([e_1], [e_1], [e_1 \wedge e_2]) \in E. \]

Let \( S : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}(\wedge^2 \mathbb{C}^3) \to \mathbb{P}(E_{310} \oplus \mathbb{C}^3 \oplus S^2(\wedge^2 \mathbb{C}^3) \oplus \mathbb{C}^3) \) denote the Segre map. Then we have that

\[ S([e_1], [e_1], [e_1 \wedge e_2]) = [e_1^2 \otimes (e_1 \wedge e_2)]. \]

Therefore

\[ S(E) = G \cdot [e_1^2 \otimes (e_1 \wedge e_2)]. \]

Since \( S \) is an embedding \( E \) is a closed orbit with stabilizer

\[
\begin{pmatrix}
* & * & *
\end{pmatrix}
\begin{pmatrix}
0 & * & *
\end{pmatrix}
\begin{pmatrix}
0 & 0 & *
\end{pmatrix}
\]

therefore we identify \( E \) with \( F(1, 2, \mathbb{C}^3) \) the space of complete flags in \( \mathbb{C}^3 \). The projection

\[ F(1, 2, \mathbb{C}^3) \xrightarrow{p_1} \mathbb{P}^2 \]

exhibits \( F(1, 2, \mathbb{C}^3) \) as a projective bundle with fiber

\[ p_1^{-1}([v]) = \mathbb{P}(\mathbb{C}^3/\mathbb{C}v). \]

Therefore if \( Q \) denotes the quotient bundle over \( \mathbb{P}^2 \) then we have the \( G \) equivariant identifications

\[ F(1, 2, \mathbb{C}^3) \cong \mathbb{P}(Q) \cong \mathbb{P}(\mathcal{O}(1) \otimes Q) = \mathbb{P}(T_{\mathbb{P}^2}^{1,0}) \]

as expected. \( S \) maps the point \([v \otimes w \otimes (v \wedge w)]\) in \( X \setminus E \) to

\[ v \cdot w \otimes (v \wedge w) + (v \wedge w)^2 \in E_{310} \oplus S^2(\wedge^2 \mathbb{C}^3) \cong E_{310} \oplus E_{220}. \]

We conclude that the pair

\[ (e, f) := ((e_1 \wedge e_2)^2, e_1 \cdot e_2 \otimes (e_1 \wedge e_2)) \in E_{220} \oplus E_{310} \]

is semistable.

\[ \square \]

**Remark 1.** The semistability of the pair \((v, w)\) depends only on \(([v], [w])\). The reader should also observe that the definition is not symmetric in \( v \) and \( w \). In virtually all examples where the pair \((v, w)\) is semistable \((w, v)\) is not semistable.
2.1. **Numerical Semistability.** If the pair \((v, w)\) is semistable then obviously we have

\[
\mathcal{T} \cdot [(v, w)] \cap \mathcal{T} \cdot [(v, 0)] = \emptyset
\]

for all algebraic tori \(T\) of \(G\). We may as well assume that \(T\) is maximal. In this section we relate semistability to lattice polytopes. To begin we let \(M_{\mathbb{Z}}\) be the character lattice of \(T\)

\[
M_{\mathbb{Z}} := \text{Hom}_{\mathbb{Z}}(T, \mathbb{C}^*)
\]

As usual, the dual lattice is denoted by \(N_{\mathbb{Z}}\). It is well known that \(u \in N_{\mathbb{Z}}\) corresponds to an algebraic one parameter subgroup \(\lambda\) of \(T\). These are algebraic homomorphisms

\[
\lambda : \mathbb{C}^* \rightarrow T
\]

The correspondence is given by

\[
(\cdot, \cdot) : N_{\mathbb{Z}} \times M_{\mathbb{Z}} \rightarrow \mathbb{Z}, \quad m(\lambda(\alpha)) = \alpha^{(u, m)} \quad m \in M_{\mathbb{Z}}.
\]

We introduce associated real vector spaces by extending scalars

\[
M_{\mathbb{R}} := M_{\mathbb{Z}} \otimes \mathbb{R}, \quad N_{\mathbb{R}} := N_{\mathbb{Z}} \otimes \mathbb{R}.
\]

Then the one parameter subgroups \(\lambda\) of \(T\) may be viewed as integral linear functionals

\[
l_{\lambda} : M_{\mathbb{R}} \rightarrow \mathbb{R}.
\]

Any rational representation \(\mathbb{E}\) decomposes under the action of \(T\) into weight spaces

\[
\mathbb{E} = \bigoplus_{a \in \mathcal{A}} \mathbb{E}_a, \quad \mathbb{E}_a := \{ e \in \mathbb{E} \mid \tau \cdot e = a(\tau) e, \quad \tau \in T \}
\]

\(\mathcal{A}\) denotes the \(T\)-support of \(\mathbb{E}\)

\[
\mathcal{A} := \{ a \in M_{\mathbb{Z}} \mid \mathbb{E}_a \neq 0 \}.
\]

Observe that \(\mathcal{A}\) is a finite subset of \(M_{\mathbb{Z}}\).

Given \(e \in \mathbb{E} \setminus \{0\}\) the projection of \(e\) into \(\mathbb{E}_a\) is denoted by \(e_a\). The support of any (nonzero) vector \(e\) is then defined by

\[
\mathcal{A}(e) := \{ a \in \mathcal{A} \mid e_a \neq 0 \}.
\]

**Definition 4.** Let \(T\) be any maximal torus in \(G\). Let \(e \in \mathbb{E} \setminus \{0\}\). The **weight polytope** of \(e\) is the compact convex lattice polytope \(\mathcal{N}(e)\) given by

\[
\mathcal{N}(e) := \text{conv} \mathcal{A}(e) \subset M_{\mathbb{R}}
\]

where \(\text{conv} \mathcal{A}(e)\) denotes the convex hull of the finite set \(\mathcal{A}(e)\).

**Definition 5.** Let \(\mathbb{E}\) be a rational representation of \(G\). Let \(\lambda\) be any 1psg of \(T\). The **weight** \(w_{\lambda}(e)\) of \(\lambda\) on \(e \in \mathbb{E} \setminus \{0\}\) is the integer

\[
w_{\lambda}(e) := \min_{x \in \mathcal{N}(e)} l_{\lambda}(x) = \min\{(a, \lambda) \mid a \in \mathcal{A}(e)\}.
\]

Alternatively, \(w_{\lambda}(e)\) is the unique integer such that

\[
\lim_{|t| \rightarrow 0} t^{-w_{\lambda}(e)} \lambda(t)e \quad \text{exists in} \quad \mathbb{E} \quad \text{and is not zero}.
\]

Next, given \(d \in \mathbb{N}\) and \(a \in \mathcal{A}\) recall that the \(T\) semi-invariants \(P \in \mathbb{C}_d[\mathbb{E}]\) of degree \(d\) are characterized by

\[
P(\tau \cdot e) = a(\tau) P(e) \quad \text{for all} \quad \tau \in T.
\]
Proposition 2.1. Let $T$ be a maximal algebraic torus of $G$, and let $V$ and $W$ two finite dimensional rational $G$-modules. Then the following are equivalent:

1) $T \cdot [(v, w)] \cap T \cdot [(v, 0)] = \emptyset$
2) $\mathcal{N}(v) \subset \mathcal{N}(w)$
3) $w_\lambda(w) \leq w_\lambda(v)$ for all 1psg's $\lambda : \mathbb{C}^* \to T$
4) For every $\chi \in \mathcal{A}(v)$ there is an $f \in \mathbb{C}[\mathbb{V} \oplus \mathbb{W}]_{d\chi}^T$ such that $f((v, w)) \neq 0$ and $f|_\mathbb{V} \equiv 0$.

Proof. The equivalence of 1) and 2) follows by a simple modification of the argument in [3], the equivalence of 1) and 4) follows from the Nullstellensatz, the remaining equivalences are left to the reader. □

There should be an analogue of the Hilbert Mumford numerical criterion in our situation.

Question. In addition to the requirement that $\mathcal{N}(v) \subset \mathcal{N}(w)$ for all maximal algebraic tori $T \leq G$ are further combinatorial conditions required to insure that the pair $(v, w)$ is semistable?

In order to define a strictly stable (henceforth stable) pair we need a large (but fixed) integer $m$ and the auxiliary left regular representation of $G$

$$G \times \mathcal{G}\mathcal{L}(N + 1, \mathbb{C}) \ni (\sigma, A) \to \sigma \cdot A.$$ (2.22)

Recall that $\mathcal{G}\mathcal{L}(N + 1, \mathbb{C})$ is the vector space of square matrices of size $N + 1$. The action is matrix multiplication. The standard $N$-simplex, denoted by $Q_N$, is defined to be the weight polytope of the identity operator

$I \in \mathcal{G}\mathcal{L}(N + 1, \mathbb{C})$. (2.23)

$Q_N$ is full-dimensional and contains the origin in its strict interior

$$0 \in Q_N := \mathcal{N}(I) \subset M_\mathbb{R}.$$ (2.24)

Let $\mathbb{V}$ be a $G$ module. We define the degree of $\mathbb{V}$ as follows

$$\deg(\mathbb{V}) := \min \left\{ k \in \mathbb{Z}_{>0} \mid \mathcal{N}(v) \subseteq kQ_N \text{ for all } 0 \neq v \in \mathbb{V} \right\}. $$ (2.25)

For example, if $G = SL(N + 1, \mathbb{C})$ and $\mathbb{V} = \text{Sym}^d(\mathbb{C}^{N+1})^\vee$ then the degree of $\mathbb{V}$ is $d$.

Let $v \in \mathbb{V}$, $w \in \mathbb{W}$, and $m \in \mathbb{N}$ we define

$$v^m := v^\otimes m \in \mathbb{V}^\otimes m, \quad w^{m+1} := w^\otimes (m+1) \in \mathbb{W}^\otimes (m+1)$$ (2.26)

$$I^q := I^\otimes q \in \mathcal{G}\mathcal{L}(N + 1, \mathbb{C})^\otimes q.$$ Finally we can give the definition of a stable pair.

Definition 6. The pair $(v, w)$ is stable if and only if there is a positive integer $m$ such that $(I^q \otimes v^m, \ w^{m+1})$ is semistable where $q$ denotes the degree of $\mathbb{V}$.

We define the automorphism group of the pair $(v, w)$ as

$$\text{Aut}(v, w) := G_{[v]} \cap G_{[w]}.$$ (2.27)

We have developed enough of the theory of (semi)stable pairs in this section to state the following proposition.
Proposition 2.2. The automorphism group of a stable pair is finite.

Proof. Stability of \((v, w)\) is equivalent to the inequality
\[
(2.28) \quad m (\log ||\sigma \cdot w|| - \log ||\sigma \cdot v||) \geq \deg(V) \log ||\sigma|| - \log ||\sigma \cdot v||
\]
for all \(\sigma \in G\) where \(m\) is a positive integer. Decompose \(\text{Aut}(v, w)\) into its reductive \((S)\) and unipotent \((U)\) parts
\[
(2.29) \quad \text{Aut}(v, w) = S \cdot U.
\]
Since \(U\) has no non-trivial characters stability implies that there is a constant \(C\) such that
\[
(2.30) \quad C \geq \log ||u|| \quad \text{for all} \quad u \in U.
\]
Since a (euclidean) bounded affine algebraic variety is a finite collection of points, we see that \(U\) must be finite. Since stability implies semistability the weights of any \(\lambda : \mathbb{C}^* \to \text{Aut}(v, w)\) must coincide. Precisely
\[
(2.31) \quad w_\lambda(w) = w_\lambda(v) \quad \text{for all 1psg's} \quad \lambda \text{ of } S.
\]
Once more stability shows that for all such \(\lambda\) we have
\[
(2.32) \quad \deg(V) w_\lambda(I) - w_\lambda(v) \geq 0.
\]
On the other hand, by definition of the degree of a representation we have
\[
(2.33) \quad \mathcal{N}(v) \subset \deg(V) \mathcal{N}(I)
\]
which implies equality (remember that bigger polytopes have smaller weights)
\[
(2.34) \quad \deg(V) w_\lambda(I) = w_\lambda(v).
\]
Since \(\lambda\) lies in \(\text{Aut}(v, w)\) we have
\[
(2.35) \quad w_\lambda(v) = -w_{\lambda^{-1}}(v).
\]
Therefore we see that for all \(\lambda\) in \(S\) we have
\[
(2.36) \quad w_\lambda(I) = -w_{\lambda^{-1}}(I).
\]
Observe that we may diagonalize \(\lambda\)
\[
(2.37) \quad \lambda = (a_0 \geq a_1 \geq \cdots \geq a_N) \quad a_i \in \mathbb{Z} \quad \sum_{0 \leq i \leq N} a_i = 0.
\]
(2.36) implies that \(a_0 = a_N\) and therefore that \(\lambda\) is trivial. This completes the proof. \(\square\)

Choose Hermitian inner products on \(V\) and \(W\). If we give \(V \oplus W\) the orthogonal sum metric then we may define the usual Fubini Study Riemannian metric \(g_{FS}\) on \(\mathbb{P}(V \oplus W)\). Choose any \(\sigma, \tau \in G\). The well known formula for the distance between two points in the Fubini Study metric gives the inequality
\[
(2.38) \quad \cos \text{dist}_{g_{FS}}(\sigma \cdot [(v, w)], \tau \cdot [(v, 0)]) \leq \cos \text{dist}_{g_{FS}}(\sigma \cdot [(v, w)], \sigma \cdot [(v, 0)]).
\]
In particular we see that
\[
(2.39) \quad \text{dist}_{g_{FS}}(\sigma \cdot [(v, w)], \sigma \cdot [(v, 0)]) \leq \text{dist}_{g_{FS}}(\sigma \cdot [(v, w)], \tau \cdot [(v, 0)]).
\]
The fact that the group elements are the same on the left hand side of this inequality implies the following.
Proposition 2.3. We have the identity
\[
\inf_{\sigma \in G} \text{dist}_{gFS}(\sigma \cdot [v, w], \sigma \cdot [(v, 0)]) = \text{dist}_{gFS}(\overline{O(v, w)}), \overline{O(v)}).
\]

Another direct application of the distance formula gives
\[
\log \tan^2 \text{dist}_{gFS}(\sigma \cdot [v, w], \sigma \cdot [(v, 0)]) = \log \|\sigma \cdot w\|^2 - \log \|\sigma \cdot v\|^2.
\]

Therefore we have the following proposition.

Proposition 2.4. We have the identity
\[
\inf_{\sigma \in G} \left(\log \|\sigma \cdot w\|^2 - \log \|\sigma \cdot v\|^2\right) = \log \tan^2 \text{dist}_{gFS}(\overline{O(v, w)}), \overline{O(v)}).
\]

We end this section with a direct comparison of Mumford’s stability and the author’s stability of pairs. Observe that the left hand column of Table 2.1 below arises from the right when we take \(V = \mathbb{C}\) (the trivial one dimensional representation) and \(v = 1\). Recall that \(q\) denotes the degree of \(V\).

| Table 2.1. Hilbert Mumford Semistability vs. Semistable Pairs |
|---------------------------------------------------------------|
| For all \(T \leq G\) \(\exists d \in \mathbb{Z}_{>0}\) and \(f \in \mathbb{C}\) such that \(f(w) \neq 0\) and \(f(0) = 0\) | For all \(T \leq G\) and \(\chi \in \mathcal{A}(v)\) \(\exists d \in \mathbb{Z}_{>0}\) and \(f \in \mathbb{C}_{d}[\mathbb{V} \oplus \mathbb{W}]_{d\chi}\) such that \(f((v, w)) \neq 0\) and \(f|_{V} \equiv 0\) |
| \(0 \notin G \cdot w\) | \(\overline{O}(v, w) \cap \overline{O}_{v} = \emptyset\) |
| \(w_{\lambda}(w) \leq 0\) for all \(1\)psg’s \(\lambda\) of \(G\) | \(w_{\lambda}(w) - w_{\lambda}(v) \leq 0\) for all \(1\)psg \(\lambda\) of \(G\) |
| \(0 \in \mathcal{N}(w)\) all \(T \leq G\) | \(\mathcal{N}(v) \subset \mathcal{N}(w)\) all \(T \leq G\) |
| \(\exists C \geq 0\) such that \(\log \|\sigma \cdot w\|^2 \geq -C\) all \(\sigma \in G\) | \(\exists C \geq 0\) such that \(\log \|\sigma \cdot w\|^2 - \log \|\sigma \cdot v\|^2 \geq -C\) all \(\sigma \in G\) |
| \(G \cdot w\) closed and \(G_w\) finite | \(\exists m \in \mathbb{N}\) such that \((I^q \otimes v^m, w^{m+1})\) is semistable |

3. Stability of Projective Varieties

Fix \(L \subset \mathbb{C}^{N+1}\), \(\dim(L) = n+1 < N+1\). Choose \(l \in \mathbb{N}\) satisfying \(0 \leq l \leq n\). Consider the Zariski open subset \(\mathcal{U}_{L}\) of the Grassmannian defined by
\[
\mathcal{U}_{L} := \{E \in G(N-l, \mathbb{C}^{N+1})\mid H^{\bullet}(0 \rightarrow E \cap L \rightarrow E, \pi_{L} \rightarrow \mathbb{C}^{N+1}/L) = 0\}.
\]

Observe that \(E \in \mathcal{U}_{L}\) if and only if
\[
\dim(\pi_{L}(E)) = N - n.
\]

Consider the subvariety \(Z_{L}\) defined by
\[
Z_{L} := G(N-l, \mathbb{C}^{N+1}) \setminus \mathcal{U}_{L}.
\]

Then \(E \in Z_{L}\) if and only if \(-\dim(\pi_{L}(E)) > n - N\).
The rank plus nullity theorem implies that
\[
\dim(E \cap L) + \dim(\pi L(E)) = N - l
\]
for any \( E \in G(N - l, \mathbb{C}^{N+1}) \).

Therefore \( E \in Z_L \) if and only if
\[
\dim(E \cap L) > N - l + n - N = n - l.
\]

Therefore
\[
Z_L = \{E \in G(N - l, \mathbb{C}^{N+1}) \mid \dim(E \cap L) \geq n - l + 1\}.
\]

Now we apply the previous linear algebra to a projective variety \( X^n \subset \mathbb{P}^N \). Recall that for any \( p \in X \) that the embedded tangent space to \( X \) at \( p \) is the \( n \) dimensional projective linear subspace
\[
\mathbb{T}_p(X) \in G(n, \mathbb{P}^N)
\]
obtained (for example) by projectivizing the tangent space the the cone over \( X \) at any point \( v \in \mathbb{C}^{N+1} \setminus \{0\} \) lying over \( p \).

Given any \( 0 \leq l \leq n \) we define the following subvariety \( Z_l(X) \) of the Grassmannian by
\[
Z_l(X) := \{E \in G(N - (l + 1), \mathbb{P}^N) \mid \exists p \in X \cap E \text{ and } \dim(E \cap \mathbb{T}_p(X)) \geq n - l\}.
\]

Generally \( Z_l(X) \) has codimension one in \( G(N - (l + 1), \mathbb{P}^N) \).

To make the defining polynomial of \( Z_l(X) \) concrete we view the Grassmannian in Steifel coordinates \([29]\) by observing that there is a dominant map
\[
M_{(l+1) \times (N+1)}^o \ni A \longrightarrow \pi(\ker(A)) \in G(N - (l + 1), \mathbb{P}^N).
\]

We may then consider the divisor (also denoted by \( Z_l(X) \))
\[
\pi^{-1}(Z_l(X)) \subset M_{(l+1) \times (N+1)}.
\]

Our “new” \( Z_l(X) \) is now an irreducible algebraic hypersurface in an affine space and hence is cut out by a single polynomial.

3.1. Resultants. Let \( X^n \subset \mathbb{P}^N \) be an irreducible, \( n \)-dimensional, linearly normal, complex projective variety of degree \( d \).

**Definition 7.** (Cayley 1840’s) The **associated hypersurface** to \( X^n \subset \mathbb{P}^N \) is given by
\[
Z_n(X) = \{L \in G(N - n - 1, N) \mid L \cap X \neq \emptyset\}.
\]

As we have remarked, it is known that \( Z_n(X) \) enjoys the following properties

i) \( Z_n(X) \) is a divisor in \( G(N - n - 1, N) \) (and hence \( M_{(n+1) \times (N+1)} \)).

ii) \( Z_n(X) \) is irreducible.

iii) \( \deg(Z_n(X)) = d \) (\( = d(n + 1) \) in Steifel coordinates).

---

5The superscript \( o \) denotes matrices of maximal rank.
Therefore there exists \( R_X \in H^0(G(N - n - 1, N), \mathcal{O}(d)) \) such that
\[
(3.12) \quad \{ R_X = 0 \} = Z_n(X).
\]

\( R_X \) is the Cayley-Chow form of \( X \). Modulo scaling, \( R_X \) is unique. Following the terminology of Gelfan’d [18] we call \( R_X \) the \textit{X-resultant}. We will always view \( R_X \) as a polynomial \(^6\) in the matrix entries
\[
(3.13) \quad R_X \in \mathbb{C}_{d(n+1)}[M_{(n+1)\times(N+1)}]^{SL(n+1, \mathbb{C})}.
\]

### 3.2. Hyperdiscriminants

Assume that \( X \subset \mathbb{P}^N \) has degree \( d \geq 2 \). Let \( X^{sm} \) denote the smooth points of \( X \). For \( p \in X^{sm} \) let \( \mathbb{T}_p(X) \) be the embedded tangent space to \( X \) at \( p \).

**Definition 8.** The \textit{dual variety} of \( X \), denoted by \( X^\vee \), is the Zariski closure of the set of \textit{tangent hyperplanes} to \( X \) at its smooth points
\[
(3.14) \quad X^\vee = \{ f \in \mathbb{P}^{N^\vee} \mid \mathbb{T}_p(X) \subset \ker(f), \ p \in X^{sm} \}.
\]

Generally \( X^\vee \) is codimension one in \( \mathbb{P}^{N^\vee} \). This holds, for example, whenever \( X \) is a (nonlinear) projective curve or surface. Observe that we have the identity
\[
X^\vee = Z_1(X).
\]

For the purposes of understanding the Mabuchi energy, what is important is not the dual variety \( X^\vee \) but the variety \( Z_{n-1}(X) \). This divisor also has a simple geometric description.
\[
Z_{n-1}(X) = \{ L \in G(N - n, \mathbb{P}^N) \mid # (L \cap X) \neq \deg(X) \}
\]

It is known that \( Z_n(X) \) enjoys the following properties

\[
i) Z_{n-1}(X) \text{ is a divisor in } G(N - n, N) \quad (\text{and hence } M_{n \times (N+1)}) .
\]

\[
ii) Z_{n-1}(X) \text{ is irreducible}.
\]

\[
iii) \deg(Z_{n-1}(X)) = n(n + 1)d - d\mu \text{ in Steifel coordinates}.
\]

Therefore there exists \( \Delta_X \in H^0(G(N - n, N), \mathcal{O}((n + 1)d - d\mu)) \) such that
\[
(3.15) \quad \{ \Delta_X = 0 \} = Z_{n-1}(X)
\]

Modulo scaling, \( \Delta_X \) is unique. Inspired by the terminology of Gelfan’d we call \( \Delta_X \) the \textit{X-hyperdiscriminant}. We will always consider \( \Delta_X \) as a polynomial \(^7\) in the appropriate matrix entries
\[
(3.16) \quad \Delta_X \in \mathbb{C}_{n(n+1)d-d\mu}[M_{n \times (N+1)}]^{SL(n, \mathbb{C})}.
\]

A word on notation is appropriate here. In [18] the symbol \( \Delta_X \) is used to denote the \textit{X-discriminant}. That is, the defining polynomial (when it exists) of the \textit{dual variety} \( Z_0(X) \) of \( X \subset \mathbb{P}^N \), whereas in this article \( \Delta_X \) is used to denote the defining polynomial of \( Z_{n-1}(X) \). In [29] the defining polynomial of \( Z_{n-1}(X) \) is denoted by \( Hu_X \) and is called the \textit{Hurwitz form} of \( X \subset \mathbb{P}^N \). The hyperdiscriminant and the Hurwitz form are the same polynomial.

\(^6\)It is necessarily invariant under the natural action of \( SL(n+1, \mathbb{C}) \).

\(^7\)It is necessarily invariant under the natural action of \( SL(n, \mathbb{C}) \).
We summarize our constructions: Let $X^n \subset \mathbb{P}^N$ be a smooth, linearly normal complex projective variety. We may associate two divisors $Z_n(X)$ and $Z_{n-1}(X)$ cut out by irreducible polynomials $R_X$ and $\Delta_X$ respectively

\begin{align}
R_X &\in \mathbb{C}d(n+1)[M_{(n+1)\times(N+1)}]^{SL(n+1, \mathbb{C})} \\
\Delta_X &\cong \mathbb{C}n(n+1)d-\mu[M_{n\times(N+1)}]^{SL(n, \mathbb{C})}.
\end{align}

For our purpose we must normalize the degrees of these polynomials. From this point on we are interested in the pair

$$(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)}) .$$

Now we are prepared to make the following definitions. $X$ will always denote a smooth, linearly normal subvariety of $\mathbb{P}^N$.

**Definition 9.** $X$ is semistable if and only if the pair $(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)})$ is semistable for the action of $G$. Explicitly, the orbit closures are disjoint

$$\overline{O}_{R\Delta} \cap \overline{O}_R = \emptyset .$$

In (3.19) we have defined

$$O_{R\Delta} := G \cdot [(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)})] , \quad O_R := G \cdot [(R_X^{\deg(\Delta_X)}, 0)].$$

**Definition 10.** $X$ is stable if and only if the pair $(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)})$ is semistable for the action of $G$. Explicitly, there is an integer $m \geq 2$ such that the pair

$$(I^q \otimes R_X^{(m-1)\deg(\Delta_X)}, \Delta_X^{m\deg(R_X)})$$

is semistable for the action of $G$ and $q = \deg(R_X) \deg(\Delta_X)$.

Proposition 2.2 immediately implies the following corollary

**Corollary 3.1.** The automorphism group of a stable variety is finite.

The reader should note that (semi)stability is independent of which lifts of $R$ or $\Delta$ are chosen. Recall that we can only construct the divisors, there will always be a scalar ambiguity in the choice of defining polynomial.

### 4. Asymptotic (Semi)Stability of Polarized Varieties

We begin this section with a brief discussion of the well known equivalence among the various of $L^p$ norms on spaces of homogeneous polynomials.

Given a homogeneous degree $d$ polynomial $P$ on $\mathbb{C}^{n+1}$ we identify it with a section of $O(d)$ over $\mathbb{P}^N$. If we fix a Hermitian metric on $\mathbb{C}^{n+1}$ recall that the pointwise norm $|P|_{h_{FS}^d}([z])$ is given by

$$|P|_{h_{FS}^d}^2([z]) := \frac{|P(z_0, \ldots, z_n)|^2}{(|z_0|^2 + \ldots |z_n|^2)^{d}} .$$

(4.1)
For any $p \in [0, \infty]$ we define the $L^p$ norms by
\[
\|P\|_0 := \exp \left( \int_{\mathbb{P}^N} \log |P|_{h_{FS}^d} \omega_{FS}^N \right), \quad \|P\|_{\infty} := \sup_{[z] \in \mathbb{P}^N} |P|_{h_{FS}^d}([z]) \quad (4.2)
\]
\[
\|P\|_p := \left( \int_{\mathbb{P}^N} |P|_{h_{FS}^d}^p \omega_{FS}^N \right)^{\frac{1}{p}}, \quad p \in (0, \infty).
\]

**Remark 2.** These satisfy the triangle inequality only for $p \in [1, \infty]$.

Observe that $\log \|P\|_0$ is the **logarithmic Mahler measure** of $P$.

The following proposition is well known, we provide a simple proof below.

**Proposition 4.1.** (see [17], [19], [7]) For any homogeneous polynomial $P$ of degree $d$ on $\mathbb{P}^N$ we have
\[
-\frac{d}{2} \left( \sum_{j=1}^{N-1} \frac{1}{j} \right) + \log \|P\|_{\infty} \leq \log \|P\|_0 \leq \log \|P\|_{\infty}. \quad (4.3)
\]

**Proof.** The content of the inequality (4.3) is the left hand side since the sup norm clearly dominates any $L^p$ norm. Recall that the mean zero Green’s function for the scalar Fubini-Study Laplacian on $\mathbb{P}^N$ is given by (where $\rho$ denotes the geodesic distance between two points)
\[
G_{FS}(\rho) = \frac{1}{2N} \left( \sum_{j=1}^{N-1} \frac{1}{(N-j) \sin^{2N-2j}(\rho)} - 2 \log(\sin(\rho)) + \frac{1}{N} - 2 \sum_{j=1}^{N} \frac{1}{j} \right). \quad (4.4)
\]

In particular
\[
G_{FS}(\rho) \geq -\frac{1}{N} \sum_{j=1}^{N} \frac{1}{j}.
\]

For any (homogeneous) polynomial $P$ the Green’s representation formula gives
\[
\log |P|_{h_{FS}^d}^2([w]) = \int_{\mathbb{P}^N} \log |P|_{h_{FS}^d}^2 \omega_{FS}^N - \int_{\mathbb{P}^N} G_{FS} \Delta \log |P|_{h_{FS}^d}^2 \omega_{FS}^N \quad \text{for } [w] \notin Z(P). \quad (4.5)
\]

Since $G_{FS}$ has mean zero we have
\[
\int_{\mathbb{P}^N} G_{FS} \Delta \log |P|_{h_{FS}^d}^2 \omega_{FS}^N = \int_{\mathbb{P}^N} G_{FS}(dN + \Delta \log |P|_{h_{FS}^d}^2) \omega_{FS}^N. \quad (4.6)
\]

Plurisubharmonicity of $\log |P|_{h_{FS}^d}^2$ gives
\[
dN + \Delta \log |P|_{h_{FS}^d}^2 \geq 0. \quad (4.7)
\]

Therefore
\[
G_{FS}(dN + \Delta \log |P|_{h_{FS}^d}^2) \geq -\frac{1}{N} \sum_{j=1}^{N} \frac{1}{j} (dN + \Delta \log |P|_{h_{FS}^d}^2). \quad (4.8)
\]

Integrating this inequality gives the result. \qed

As a consequence of Proposition 4.1 we deduce the following.
Corollary 4.1. For all $p \in (0, \infty)$ we have

$$-\frac{d}{2} \left(\sum_{j=1}^{N} \frac{1}{j}\right) + \log \|P\|_p \leq \log \|P\|_0 \leq \log \|P\|_p.$$ \hfill (4.9)

The right hand side of (4.9) comes from Jensen’s inequality. In particular we get that the $L^2$ norm and the Mahler measure are equivalent, which is all that we need. Also observe that the inequality in Proposition (4.1) becomes an equality for $P(z_0, \ldots, z_N) = z_0^d$.

Now we return to the situation of $X \subset \mathbb{P}^N$. We assume that $X$ is smooth and linearly normal. Let $R_X$ and $\Delta_X$ denote the resultant and (hyper)discriminant. We remind the reader that these polynomials are only given up to scale. Propositions 4.1 and 2.4 justify the following definitions.

Definition 11. Let $X \subset \mathbb{P}^N$. Let $p \in [0, \infty]$. Choose any $L^p$ normalized $R$ and $\Delta$. Then the $L^p$ distance between the points

$$\sigma \cdot [(R_X^{\deg(\Delta_X)}, \Delta_X^{\deg(R_X)})] \text{ and } \sigma \cdot [(R_X^{\deg(\Delta_X)}, 0)]$$

is defined by

$$\log \tan \text{dist}_p(\sigma) := \log \|\sigma \cdot \Delta_X^{\deg(R_X)}\|_p - \log \|\sigma \cdot R_X^{\deg(\Delta_X)}\|_p.$$ \hfill (4.10)

Definition 12. The $L^p$ distance between the orbit closures is defined to be

$$\log \tan \text{dist}_p(\overline{O}_{R\Delta}, \overline{O}_R) := \inf_{\sigma \in G} \log \tan \text{dist}_p(\sigma).$$ \hfill (4.11)

The point is that all of the $L^p$ distances measure the same thing: any one of them detects the semistability of $X \subset \mathbb{P}^N$. What is extraordinary is that the infimum of the Mabuchi energy restricted to the Bergman metrics at level $k$ is exactly the distance between the orbit closures in the $L^0$ distance.

Now we are prepared to the introduce asymptotic (semi)stability of a polarized manifold $(X, L)$. We require an auxiliary Hermitian metric $h$ on $L$ with positive curvature $\omega_h$. The definition of asymptotic (semi)stability is independent of which $h$ is chosen. We must scale $R_X$ and $\Delta_X$ to have unit length in the norm $\|\cdot\|_0$.

Definition 13. A polarized manifold $(X, L)$ is asymptotically semistable if and only if there is a uniform constant $C = C(h) \geq 0$ such that

$$\text{dist}_0(\overline{O}_{R\Delta}, \overline{O}_R) \gtrsim \exp(-Cd^2)$$

for all sufficiently large $L^k$-embeddings of degree $d = k^n$.

The author’s previous work [24] shows that the orbit closures must be disjoint for all powers of $L$, otherwise the Mabuchi energy is unbounded from below and no canonical metric exists. Asymptotic semistability not only requires orbit closure separation for each embedding, but also that the orbit closures are not allowed to approach one another too quickly in the Mahler metric as the degree of the embedding increases.

Definition 14. A polarized manifold $(X, L)$ is asymptotically stable if and only if there are uniform constants $m \in \mathbb{Z}_{>0}$ and $C = C(h, m)$ such that

$$\text{dist}_0(\overline{O}_{(v, w)}, \overline{O}_v) \gtrsim \exp(-Ck^{2n+1})$$

for all sufficiently large $L^k$-embeddings of degree $d = k^n$.
for all sufficiently large \( k \): the power of the embedding). 

\[
(v, w) := (I^q \otimes R_X^{(km-1) \text{deg}(\Delta X)}, \Delta_X^{km \text{deg}(R_X)})
\]

As in the definition of asymptotic semistability, both \( R_X \) and \( \Delta_X \) have been scaled to have length one in the norm \( \| \cdot \|_0 \). The reader should observe that the speeds of approach of the orbit closures in the definitions and asymptotic stability and semistability differ by a single factor of \( k \).

5. Asymptotic Stability and Properness of the Mabuchi Energy

We recall some definitions surrounding Mabuchi’s K-energy map. Let

\[
(X^n, \omega), \ n = \dim_C(X)
\]

be a compact Kähler manifold. Recall that the Kähler form \( \omega \) is given locally by a Hermitian positive definite matrix of functions

\[
\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.
\]

The Ricci form of \( \omega \) is the smooth \((1,1)\) form on \( X \) given by

\[
\text{Ric}(\omega) := -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \det(g_{i\bar{j}}) = \sum_{i,j} -\frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz_i \wedge d\bar{z}_j.
\]

The scalar curvature is by definition the contraction of the Ricci curvature

\[
\text{Scal}(\omega) := \sum_{i,j} g^{i\bar{j}} R_{i\bar{j}} \in C^\infty(X).
\]

The volume \( V \) and the average of the scalar curvature \( \mu \) depend only on \([\omega]\) and are given by

\[
V = \int_X \omega^n, \quad \mu = \frac{1}{V} \int_X \text{Scal}(\omega) \omega^n.
\]

The space of Kähler metrics in the class \([\omega]\) is defined by

\[
\mathcal{H}_\omega := \{ \varphi \in C^\infty(X) \mid \omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi > 0 \}.
\]

**Definition 15.** (Mabuchi [22]) The **K-energy map** \( \nu_\omega : \mathcal{H}_\omega \rightarrow \mathbb{R} \) is given by

\[
\nu_\omega(\varphi) := -\frac{1}{V} \int_0^1 \int_X \dot{\varphi}_t (\text{Scal}(\omega_{\varphi_t}) - \mu) \omega_{\varphi_t}^n dt
\]

\( \varphi_t \) is a \( C^1 \) path in \( \mathcal{H}_\omega \) satisfying \( \varphi_0 = 0, \varphi_1 = \varphi \).

Mabuchi shows that \( \nu_\omega \) is independent of the path chosen. It is clear that \( \varphi \) is a critical point for \( \nu_\omega \) if and only if

\[
\text{Scal}(\omega_\varphi) \equiv \mu.
\]

What is relevant for the present article is the following theorem, first established by Bando and Mabuchi in the case \( L = -K_X \), and then generalized some years later by Donaldson and Li.
Theorem 5.1. (see [2], [15], [13], [20]) Let \((X, L)\) be a polarized manifold, and assume that there is a constant scalar curvature metric in the class \(c_1(L)\). Then the Mabuchi energy is bounded below on \(H_\omega\) where \(h\) is any Hermitian metric on \(L\) with positive curvature \(\omega\).

We recall the Aubin \(J_\omega\) functional (see [1]) and the associated energy \(F_\omega^0\)

\[
J_\omega(\varphi) := \frac{1}{V} \int_X \frac{\sqrt{-1}}{2\pi} \sum_{i=0}^{n-1} \varphi \wedge \overline{\varphi} \wedge \omega^{n-i-1} \\
F_\omega^0(\varphi) := J_\omega(\varphi) - \frac{1}{V} \int_X \varphi \omega^n.
\]

Definition 16. (Tian [33]) Let \((X, \omega)\) be a Kähler manifold. The Mabuchi energy is proper provided there exists constants \(a > 0\) and \(b\) such that for all \(\varphi \in H_\omega\) we have

\[
\nu(\varphi) \geq aJ_\omega(\varphi) + b.
\]

Let \((X, L)\) be a polarized manifold. Let \(h\) be a smooth Hermitian metric on \(L\) with positive curvature \(\omega\). Choose \(k\) large enough so that there is an embedding \(i_k : X \to \mathbb{P}(H^0(X, L^k)^*)\).

We will always assume that the embedding is given by a unitary basis of sections \(\{S_i\}\). Similarly we outfit \(H^0(X, L_k)\) with the Hodge \(L^2\) inner product. We let \(\omega_{FS}\) denote the corresponding Fubini-Study Kähler metric on the (dual) projective space of sections. Then

\[
i_k^*\omega_{FS}|_{i_k(X)} = k\omega_h + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log \left( \sum_{i=0}^{N_k} |S_i|^2 \right).
\]

Let \(G = SL(H^0(X, L^k))\), then \(\sigma \in G\) acts on the sections by

\[
\sigma \cdot S_i = \sum_{0 \leq j \leq N_k} \sigma_{ij} S_j.
\]

Define

\[
\Psi_\sigma := \log \sum_{0 \leq i \leq N_k+1} |\sigma \cdot S_i|^2.
\]

The Bergman metrics of level \(k\) are given by

\[
\mathcal{B}_{N_k} := \{ \frac{1}{k} \Psi_\sigma \mid \sigma \in SL(N_k + 1, \mathbb{C}) \} \subset H_\omega.
\]

In the discussion below we need to distinguish between the potential \(\Psi_\sigma\) which is a function on \(X\), and the closely related potential \(\varphi_\sigma\), which is a function on \(\mathbb{P}^{N_k}\). Note that \(\varphi_\sigma\) is a Kähler potential on \(i_k(X)\) relative to the restriction \(\omega_{FS}|_{i_k(X)}\). These two potentials are related as follows.

\[
\sigma^* \omega_{FS} = \omega_{FS} + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \varphi_\sigma, \quad \varphi_\sigma([z]) = \log \frac{|\sigma \cdot z|^2}{|z|^2}
\]

\[
i_k^*(\sigma^* \omega_{FS}|_{i_k(X)}) = k\omega_h + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \Psi_\sigma
\]

\[
\Psi_\sigma = \varphi_\sigma \circ i_k + \log \sum_{0 \leq i \leq N_k+1} |S_i|^2.
\]

A key ingredient in this paper is the following result of Tian [31].
Theorem 5.2. (Tian’s Thesis) The spaces $\mathcal{B}_{N_k}$ are dense in the $C^2$ topology
\begin{equation}
\bigcup_k \mathcal{B}_{N_k} = \mathcal{H}_\omega.
\end{equation}

Now we are prepared to establish that the asymptotic (semi)stability of $(X, L)$ is equivalent to the (lower bound) global coercive estimate for the Mabuchi energy $\nu_{\omega}$ for any $\omega \in c_1(L)$. We need to compare the Mabuchi and Aubin energies of the reference metric $\omega := \omega_h$ with the restrictions of the Fubini-Study metrics coming from the large projective embeddings. It is easy to see that $\nu_{\omega}$ does not scale but $F_{\omega}$ does scale as we pass between $\omega$ and $\omega_{FS}|_{\iota_k(X)}$. We collect the precise comparisons below, where $o(1)$ denotes any quantity that converges to 0 as $k \to \infty$. The $o(1)$’s below have the form $O\left(\log\left(\frac{k}{k}\right)\right)$.

\begin{equation}
\nu_{\omega}\left(\frac{\Psi_\sigma}{k}\right) = \nu_{\omega_{FS}|_{\iota_k(X)}}(\varphi_\sigma) + o(1), \quad J_{\omega}\left(\frac{\Psi_\sigma}{k}\right) = \frac{1}{k}J_{\omega_{FS}|_{\iota_k(X)}}(\varphi_\sigma) + o(1)
\end{equation}

\begin{equation}
\int_X \frac{\Psi_\sigma \omega^n}{k \nu_o} = \frac{1}{V} \int_{\iota_k(X)} \frac{\varphi_\sigma \omega^n_{FS}}{k} + o(1).
\end{equation}

All of the results in this article depend on the following theorem which completely describes the Mabuchi energy restricted to the space of Bergman metrics associated to the embedding $X^n \subset \mathbb{P}^N$.

Theorem A. (\cite{24}) There is a norm $\| \cdot \|$ on the space of polynomials such that
\begin{equation}
d^2(n + 1)\nu_{\omega_{FS}|_X}(\varphi_\sigma) = \deg(R_X) \log \frac{||\sigma \cdot \Delta_X||^2}{||\Delta_X||^2} - \deg(\Delta_X) \log \frac{||\sigma \cdot R_X||^2}{||R_X||^2}.
\end{equation}

The norm appearing in (5.18) was first considered by Gang Tian in his early works on CM stability \cite{35}, \cite{32}, \cite{33}, \cite{34}, \cite{36}. This norm is conformally equivalent to the $L^2$ norm with a continuous potential $\theta$
\begin{equation}
\| \cdot \| := e^\theta \| \cdot \|_{L^2}.
\end{equation}

In the situation considered by Tian in \cite{33} it seems there is little one could say about $\theta$ beyond it’s (Hölder) continuity. However, for families of divisors, the situation considered here, $\theta$ can be described explicitly which allows us to significantly improve Theorem A.

Theorem 5.3. Let $X \subset \mathbb{P}^N$ be a smooth, linearly normal complex projective variety then the following holds for all $\sigma \in G$
\begin{equation}
d^2(n + 1)\nu_{\omega_{FS}|_X}(\varphi_\sigma) = \deg(R_X) \log \frac{||\sigma \cdot \Delta_X||^2}{||\Delta_X||^2} - \deg(\Delta_X) \log \frac{||\sigma \cdot R_X||^2}{||R_X||^2}.
\end{equation}

Proof. We identify the conformal factor $\theta$. Consider $\mathbb{C}^{n+1}$ equipped with it’s standard metric. Let $d$ be any positive integer. We identify the space of homogeneous polynomials of degree $d$ on $\mathbb{C}^{n+2}$ with $H^0(\mathbb{P}^n, \mathcal{O}(d))$. We let $B$ denote the corresponding complete linear system and $\mathcal{X}_d$ the universal family of hypersurfaces over $B$
\begin{equation}
B := \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d))) \quad \mathcal{X}_d := \{([S], [w]) \in B \times \mathbb{P}(W) | S(w) = 0\}.
\end{equation}
Using the projections

\[
\mathcal{X}_d \xrightarrow{p_2} \mathbb{P}^n \\
\downarrow p_1 \\
B
\]

we define a closed positive \((1, 1)\) current \(u\) on \(B\) by

\[
u := p_1^* p_2^*(\omega_{FS}^n) .
\]

Explicitly, for any smooth form \(\alpha\) on \(B\) of correct type we define

\[
\int_B u \wedge \alpha := \int_{\mathcal{X}_d} p_2^*(\omega^n) \wedge p_1^*(\alpha) .
\]

Since DeRham and current cohomology on \(B\) coincide we see at once that \([u] = [\omega_B]\) where \(\omega_B\) is a smooth \((1, 1)\) form on \(B\).

**Proposition 5.1.** ([33] Lemma 8.7 pg. 32) There is a continuous function \(\theta\) on \(B\) such that, in the sense of currents we have

\[
u = \omega_B + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta [\omega_B] = [c_1(\mathcal{O}_B(1))] .
\]

With this said the conformal factor that appears on the right hand side of (5.18) is \(e^\theta\) and for any section \(S\) of \(\mathcal{O}_{\mathbb{P}^n+1}(d)\) it’s norm is defined to be

\[
||S|| := e^{\theta([S])} ||S||_{L^2} .
\]

This introduces a bounded “error” on the right hand side of (5.18) when we relate the Mabuchi energy to the \(L^2\) norm. An explicit description of \(\theta\) is obtained by noting that \(\mathcal{X}_d\) is a divisor in \(B \times \mathbb{P}^{n+1}\) cut out by a section \(\Psi\) of \(p_1^* \mathcal{O}_B(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^n}(d)\)

\[
\xymatrix{ \mathcal{X}_d \ar[r]^\iota & B \times \mathbb{P}^n \\
\Psi([S], [w]) := 1_S \otimes S([w])}
\]

Observe that, in the natural Hermitian metric on \(p_1^* \mathcal{O}_B(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^n}(d)\) the log of the length of \(\Psi\) is

\[
\log |\Psi([S], [w])|_h^2 = \log \frac{|S([w])|_{h_{FS}^d}^2}{||S||_{L^2}^2} .
\]
Next observe that the Poincaré-Lelong formula gives

\[
\int_{\mathcal{X}} p^*_d(\omega^{n+1}) \wedge p^*_1(\alpha) = \int_{B \times \mathbb{P}^n} \left( dp^*_1(\omega_{FS}) + p^*_2(\omega_B) + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |\Psi|^2_h \right) \wedge p^*_2(\omega_{FS}^n) \wedge p^*_1(\alpha) = \int_{B} \left( \omega_B + \frac{\sqrt{-1}}{2\pi} \partial_B \bar{\partial}_B \int_{\mathbb{P}^n} \log |\Psi|^2_h \omega_{FS}^n \right) \wedge \alpha.
\]

(5.28)

Therefore we get

\[
\theta([S]) = \int_{\mathbb{P}^n} \log \frac{|S([w])|^2_h \omega_{FS}^n}{||S||^2_{L^2}}.
\]

(5.29)

Inserting (5.29) into the right hand side of (5.25) gives that the norm of the discriminant and resultant are given explicitly by

\[
\log ||\sigma \cdot \Delta||^2_h = \int_{\mathcal{P}(M_{n+1}(N+1))} \log |\sigma \cdot \Delta|^2_{h_{FS}} - \int_{\mathcal{P}(M_{n+1}(N+1))} \log |\Delta|^2_{h_{FS}}
\]

(5.30)

\[
\log ||\sigma \cdot R||^2_h = \int_{\mathcal{P}(M_{n+1}(N+1))} \log |\sigma \cdot R|^2_{h_{FS}} - \int_{\mathcal{P}(M_{n+1}(N+1))} \log |R|^2_{h_{FS}}.
\]

In other words the norm ||·|| in (5.25) is given by

\[
||S|| = ||S||_0 \text{ for any } S \in H^0(\mathbb{P}^n, \mathcal{O}(d)).
\]

(5.31)

\[\square\]

The second part of Theorem 1.1, namely equivalence between asymptotic semistability and a global lower bound for the Mabuchi energy, follows from (5.20), [31] and the corollary below.

**Corollary 5.1.** For any polarized manifold \((X, L)\) and any large \(k\) embedding \(X \subset \mathbb{P}^N\) the infimum of the Mabuchi energy restricted to \(G = SL(N_k + 1, \mathbb{C})\) is given by

\[
\inf_{\sigma \in G} d^2(n + 1) \nu_{FS}|_{X}(\varphi_\sigma) = \log \tan \text{dist}_0(\mathcal{O}_{R\Delta}, \mathcal{O}_R).
\]

(5.32)

**Proof.** This follows at once from the definition of the distance in the \(L^p\) metrics. \[\square\]

**Remark 3.** The reader should compare (5.32) with the corollary on pg. 257 of [24].

Now we can show the first part of Theorem 1.1, namely equivalence between asymptotic stability and the global coercive estimate for the Mabuchi energy.

**Proposition 5.2.** Let \(m\) be a positive integer. For any polarized manifold \((X, L)\) and any large \(k\) embedding we have

\[
\inf_{m \in \mathbb{Z}_{N_k}} \left( m \nu_{h} \left( \frac{\Psi_\sigma}{k} \right) - \frac{\deg(\Delta_X)}{d} J_{h} \left( \frac{\Psi_\sigma}{k} \right) \right) = \frac{h^{-2(n+1)}}{(n+1)} \log \tan \text{dist}_0(\mathcal{O}_{R}, \mathcal{O}_R) + O(1),
\]

(5.33)
where we have defined the pair \((v, w)\) and \(q\) as follows
\[
(v, w) := \left( I^q \otimes R_X^{(km-1) \deg(\Delta_X)}, \Delta_X^{km \deg(R_X)} \right), \quad q := \deg(R_X) \deg(\Delta_X)
\]
and the distance \(d_{\text{dist}}\) in \((5.33)\) has been extended to \(I\) by simply using the Hilbert-Schmidt norm on matrices.

**Proof.** We begin with the following crucial observation, which was shown to the author by Gang Tian.

**Lemma 5.1.** There is a uniform constant \(C\) such that for all sufficiently large \(k \in \mathbb{N}\) we have
\[
C + \frac{1}{k} \log \left( \frac{\|\sigma\|^2}{N_k + 1} \right) \leq \int_X \frac{\Psi_{\sigma} \omega^n}{k V_o}.
\]

**Proof.** If \(\|\sigma\|^2 := \text{Trace}(\sigma^* \sigma)\) then we observe that the unitarity of the basis gives
\[
\sum_{0 \leq i \leq N_k} \frac{||\sigma \cdot S_i||^2}{||\sigma||^2} = 1.
\]
Therefore there is an index \(j\) such that
\[
\log ||\sigma \cdot S_j||^2 \geq \log \frac{||\sigma||^2}{N_k + 1}.
\]
Define
\[
T^\sigma_j := \frac{\sigma \cdot S_j}{||\sigma \cdot S_j||}.
\]
Let \(\alpha(L)\) be Tian’s alpha invariant [30], and choose any \(0 < \beta < \alpha(L)\) then there exists a uniform constant \(C(\beta) > 0\) such that
\[
\int_X \left( \frac{1}{|T^\sigma_j|^2} \right)^{\frac{\beta}{2}} \omega^n V \leq C(\beta).
\]
Jensen’s inequality gives
\[
-\frac{\beta}{k} \int_X \left( \log |\sigma \cdot S_j|^2 - \log ||\sigma \cdot S_j||^2 \right) \frac{\omega^n V}{V} \leq \log C(\beta).
\]
Equivalently
\[
\frac{\beta}{k} \log ||\sigma \cdot S_j||^2 \leq \frac{\beta}{k} \int_X \log |\sigma \cdot S_j|^2 + \log C(\beta).
\]
Applying inequality \((5.36)\) we see that
\[
-\frac{1}{\beta} \log C(\beta) + \frac{1}{k} \log \left( \frac{||\sigma||^2}{N_k + 1} \right) \leq \frac{1}{k} \int_X \log |\sigma \cdot S_j|^2 \frac{\omega^n V}{V_o} \leq \int_X \frac{\Psi_{\sigma} \omega^n}{k V_o}.
\]

The comparison formulas \((5.17)\) and the preceding lemma imply that
\[
J_{\text{wh}} \left( \frac{\Psi_{\sigma}}{k} \right) = \frac{1}{k} F^\omega_{PSL_1(k)(\mathcal{X})}(\varphi_{\sigma}) + \frac{1}{k} \log ||\sigma||^2 + O(1).
\]
Recall the well known proposition \[8\]

**Proposition 5.3.** ([25], [28]) For any linearly normal projective variety \( X \subset \mathbb{P}^N \) we have

\[
- \deg(R_X) F_{\omega_{FS}|_{\kappa(X)}}^{\omega} (\phi_\sigma) = \log ||\sigma \cdot R_X||_0 .
\] (5.42)

In the above Proposition we have chosen \( R_X \) to have length one in the Mahler norm. Inserting (5.41) into Proposition 5.3 allows us to express \( J_{\omega_h|_{\kappa_N}} \) as a distance function

\[
\frac{\deg(\Delta_X)}{d} J_{\omega_h} \left( \frac{\Psi_\sigma}{k} \right) = \frac{1}{k^{2n+1}(n+1)} \left( - \deg(\Delta_X) \log ||\sigma \cdot R_X||_0^2 + q \log ||\sigma||^2 \right) + O(1) .
\] (5.43)

Theorem 5.3 and the comparison formulas (5.17) give

\[
m_{\nu_{\omega_h}} \left( \frac{\Psi_\sigma}{k} \right) = \frac{1}{k^{2n+1}(n+1)} \left( km \deg(R_X) \log ||\sigma \cdot \Delta_X||_0^2 - km \deg(\Delta_X) \log ||\sigma \cdot R_X||_0^2 \right) + o(1) .
\] (5.44)

As usual, we have chosen representatives satisfying \( ||R_X||_0 = ||\Delta_X||_0 = 1 \). Now subtract (5.43) from (5.44) and use the definition of the \( L^0 \) distance to get

\[
m_{\nu_{\omega_h}} \left( \frac{\Psi_\sigma}{k} \right) - \frac{\deg(\Delta_X)}{d} J_{\omega_h} \left( \frac{\Psi_\sigma}{k} \right) = k^{-2n+1} \frac{(n+1)}{d} \log \tan \text{dist}_0(\boldsymbol{w}, \sigma) + O(1) .
\] (5.45)

Recall that the pair \((v, w)\) is given by

\[
(v, w) := (I^q \otimes R_X^{km-1}) (\Delta_X^{km} \deg(R_X)) .
\] (5.46)

Taking the \( \inf \) over \( G \) on both sides of (5.45) completes the proof of Theorem 1.1. \( \square \)

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This work was carried out during two visits to the Mathematics Department of the Massachusetts Institute of Technology. The first visit was in the spring of 2013 and the second in the spring of 2021, where this work was finally completed. The author thanks his host, \[8\] This amounts to expressing the Faltings height of \( X \) in terms of the Mahler measure of the Cayley (Chow) form of \( X \). This was first shown by P. Phillipon and independently by C. Soulé around 1991. See also [38], [21], [23].
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