A SIMPLE PROOF OF THE BORSUK-ULAM THEOREM FOR \( \mathbb{Z}_p \)-ACTIONS

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Abstract. In this note, we give a simple proof of the Borsuk-Ulam theorem for \( \mathbb{Z}_p \)-actions. We prove that, if \( S^n \) and \( S^m \) are equipped with free \( \mathbb{Z}_p \)-actions (\( p \) prime) and \( f : S^n \to S^m \) is a \( \mathbb{Z}_p \)-equivariant map, then \( n \leq m \).

Introduction

Let \( S^n \) be the unit \( n \)-sphere in \( \mathbb{R}^{n+1} \). There is a natural involution on \( S^n \), called the antipodal involution and given by \( x \mapsto -x \). The well known Borsuk-Ulam theorem states that: If there is a map \( f : S^n \to S^m \) taking a pair of antipodal points to a pair of antipodal points, then \( n \leq m \).

Over the years, there have been several generalizations of the theorem in many directions. We refer the reader to an interesting article \cite{steinlein} by Steinlein, which lists 457 publications concerned with various generalizations of the Borsuk-Ulam theorem. Also the recent book \cite{matousek} by Matoušek contains a detailed account of various generalizations and applications of the Borsuk-Ulam theorem. There are several proofs of this theorem in literature, in fact, most algebraic topology texts contains a proof.

The purpose of this note is to give a simple proof of a generalization of this theorem in the setting of group actions. Let \( G \) be a group acting on a space \( X \) with the action \( \times \) \( X \to X \) denoted by \( (g,x) \mapsto gx \). There is associated with the group action, the orbit space \( X/G \), obtained by identifying all the points in the orbit of \( x \) (denoted by \( x \)) for each \( x \in X \). The orbit map \( X \to X/G \) is given by \( x \mapsto \overline{x} \).

If spaces \( X \) and \( Y \) carry \( G \)-actions, then a map \( f : X \to Y \) is called \( G \)-equivariant if \( f(gx) = g(f(x)) \) for all \( x \in X \) and \( g \in G \). An equivariant map \( f : X \to Y \) induces a map \( \overline{f} : X/G \to Y/G \) given by \( \overline{f}([x]) = \overline{f(x)} \). Recall that a \( G \)-action is said to be free if \( gx = x \) implies \( g = e \), the identity of \( G \).

In 1983, Liulevicius \cite{liulevicius} published the following generalization of the Borsuk-Ulam theorem:

If a map \( f : S^n \to S^m \) commutes with some free actions of a non-trivial compact Lie group \( G \) on the spheres \( S^n \) and \( S^m \), then \( n \leq m \).

An alternative, but relatively simple, proof of the later theorem was also given by Dold \cite{dold} in 1983. There are also some other generalizations of the result, see for example \cite{liulevicius}. In this note, we give a simple proof of the above result for free actions of the cyclic group \( \mathbb{Z}_p \) of prime order \( p \) involving only elementary algebraic topology. More precisely, we prove the following theorem.

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Borsuk-Ulam Theorem. Let $S^n$ and $S^m$ be equipped with free $\mathbb{Z}_p$-actions. If there is a $\mathbb{Z}_p$-equivariant map $f : S^n \to S^m$, then $n \leq m$.

Before proceeding to prove the theorem, we recall the universal coefficient theorem for singular cohomology.

Universal Coefficient Theorem. [6, p.243] There is a natural short exact sequence

$$0 \to \text{Ext}(H_{k-1}(X;\mathbb{Z}),\mathbb{Z}_p) \to H^k(X;\mathbb{Z}_p) \to \text{Hom}(H_k(X;\mathbb{Z}),\mathbb{Z}_p) \to 0$$

for each $k \geq 0$.

Proof of the Borsuk-Ulam Theorem

Suppose that $n > m$. Let the $\mathbb{Z}_p$-actions on $S^n$ and $S^m$ be generated by $T$ and $S$ respectively. Note that the map $f : S^n \to S^m$ is $\mathbb{Z}_p$-equivariant if $f(T(x)) = S(f(x))$ for all $x \in X$. Let $q_1 : S^n \to S^n/T$ and $q_2 : S^m \to S^m/S$ be the orbit maps which are also $p$-sheeted covering projections. We claim that $f^* : \pi_1(S^n/T) \to \pi_1(S^m/S)$ is zero. This will give a lift $\tilde{f}$ of $f$, that is, the following diagram commutes

$$\begin{array}{ccc}
S^n & \xrightarrow{f} & S^m \\
\downarrow{q_1} & & \downarrow{q_2} \\
S^n/T & \xrightarrow{T} & S^m/S.
\end{array}$$

Since, $\text{Ext}(H_0(S^n/T;\mathbb{Z}),\mathbb{Z}_p) = 0$, taking $k = 1$ in the universal coefficient theorem, we have $H^1(S^n/T;\mathbb{Z}_p) \cong \text{Hom}(H_1(S^n/T;\mathbb{Z}),\mathbb{Z}_p)$. The same holds for $S^m/S$ also. By naturality of the universal coefficient formula, the map $T : S^n/T \to S^m/S$ gives the following commutative diagram

$$\begin{array}{ccc}
H^1(S^n/T;\mathbb{Z}_p) & \xrightarrow{\cong} & \text{Hom}(H_1(S^n/T;\mathbb{Z}),\mathbb{Z}_p) \\
\downarrow{T^*} & & \downarrow{\alpha \mapsto \alpha T_*} \\
H^1(S^m/T;\mathbb{Z}_p) & \xrightarrow{\cong} & \text{Hom}(H_1(S^m/T;\mathbb{Z}),\mathbb{Z}_p).
\end{array}$$

For $p$ odd, both $n$ and $m$ are odd. It is known that for a free action of $\mathbb{Z}_p$ on a sphere $S^{2k-1}$, there are integers $n_1, \ldots, n_k$ such that $S^{2k-1}/\mathbb{Z}_p$ is homotopy equivalent to the lens space $L^{2k-1}(p; n_1, \ldots, n_k)$. Thus both $S^n/T$ and $S^m/S$ are homotopy equivalent to lens spaces and have the following cohomology algebras [8, p. 251]

$$H^*(S^n/T;\mathbb{Z}_p) \cong \mathbb{Z}_p[s, t]/\langle s^2, t^{n+1} \rangle,$$

$$H^*(S^m/S;\mathbb{Z}_p) \cong \mathbb{Z}_p[s_1, t_1]/\langle s_1^2, t_1^{m+1} \rangle.$$
with \( t = \beta(s) \) and \( t_1 = \beta(s_1) \), where \( \beta \) is the mod-p Bockstein homomorphism. Naturality of the Bockstein homomorphism gives the commutative diagram

\[
\begin{array}{ccc}
H^1(S^n/S; \mathbb{Z}_p) & \xrightarrow{\beta} & H^2(S^n/S; \mathbb{Z}_p) \\
\downarrow \phi & & \downarrow \phi \\
H^1(S^n/T; \mathbb{Z}_p) & \xrightarrow{\beta} & H^2(S^n/T; \mathbb{Z}_p).
\end{array}
\]

If \( \phi \) is non zero, then \( \phi(s) = s \). From the diagram we have \( \phi(t_1) = t \). But \( 0 = \phi(t_1) = \phi(t_1) \frac{m+1}{2} = t \frac{m+1}{2} \), a contradiction as \( n > m \). Hence \( \phi \) is zero in this case.

For \( p = 2 \), both \( S^n/T \) and \( S^m/S \) have the homotopy type of real projective spaces and hence have the cohomology algebras \([3, p. 250] \)

\[
\begin{align*}
H^*(S^n/T; \mathbb{Z}_2) &\cong \mathbb{Z}_2[s]/(s^{n+1}), \\
H^*(S^m/S; \mathbb{Z}_2) &\cong \mathbb{Z}_2[s_1]/(s_1^{m+1}),
\end{align*}
\]

where \( s \) and \( s_1 \) are homogeneous elements of degree one each.

If \( \phi \) is non zero, then \( \phi(s_1) = s_1 \). But \( 0 = \phi(s_1) = \phi(s_1) \frac{m+1}{2} = s_1 \frac{m+1}{2} = 3m+1 \), a contradiction as \( n > m \). Hence, \( \phi \) must be zero and by the commutativity of the second diagram, the map \( \alpha \mapsto \alpha \phi \) is zero. From this we get \( \phi_s : H_1(S^n/T; \mathbb{Z}) \to H_1(S^m/S; \mathbb{Z}) \) is zero. Now by naturality of the Hurewicz homomorphism

\( h : \pi_1(S^n/T) \to H_1(S^n/T; \mathbb{Z}) \)

(which is an isomorphism in our case), we have the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(S^n/T) & \xrightarrow{\pi_1} & \pi_1(S^m/S) \\
\downarrow h & & \downarrow h \\
H_1(S^n/T; \mathbb{Z}) & \xrightarrow{\phi} & H_1(S^m/S; \mathbb{Z}),
\end{array}
\]

which shows that \( \phi : \pi_1(S^n/T) \to \pi_1(S^m/S) \) is zero and hence the lift exists.

The commutativity of the first diagram shows that both \( f \) and \( \tilde{f}q_1 \) are lifts of \( f \).

Let \( x_0 \in S^n \), then by definition of \( q_2 \),

\[
q_2(f(x_0)) = q_2(Sf(x_0)) = q_2(S^2f(x_0)) = \ldots = q_2(S^{p-1}f(x_0)),
\]

that is, the fiber over \( q_2(f(x_0)) \) is the set

\[
\{ f(x_0), Sf(x_0), \ldots, S^{p-1}f(x_0) \}.
\]

Also \( q_2(\tilde{f}q_1(x_0)) = \tilde{f}q_1(x_0) = q_2f(x_0) \). Therefore, \( \tilde{f}q_1(x_0) = f(x_0) \) or \( \tilde{f}q_1(x_0) = S^if(x_0) \) for some \( 1 \leq i \leq p-1 \). Note that in the later case we have \( \tilde{f}q_1(T^i(x_0)) = \tilde{f}q_1(x_0) = S^if(x_0) = fT^i(x_0) \). Hence in either case, the lifts \( f \) and \( \tilde{f}q_1 \) agree at a point and therefore by uniqueness of lifting, we have \( f = \tilde{f}q_1 \). Now for any \( x \in S^n \), \( q_1(x) = q_1T(x) \). But \( \tilde{f}q_1(x) = \tilde{f}q_1T(x) = fT(x) = Sf(x) \neq f(x) \), a contradiction. Hence \( n \leq m \).

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