SURGERY IN CODIMENSION 3 AND
THE BROWDER–LIVESAY INVARIANTS

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Abstract. The inertia subgroup $I_n(\pi)$ of a surgery obstruction group $L_n(\pi)$ is generated by elements which act trivially on the set of homotopy triangulations $S(X)$ for some closed topological manifold $X^{n-1}$ with $\pi_1(X) = \pi$. This group is a subgroup of the group $C_n(\pi)$ which consists of the elements which can be realized by normal maps of closed manifolds. In all known cases these groups coincide and the computation of them is one of the basic problems of surgery theory. The computation of the group $C_n(\pi)$ is equivalent to the computation the image of the assembly map $A : H_n(B\pi, L_\bullet) \to L_n(\pi)$. Every Browder-Livesay filtration of the manifold $X$ provides a collection of Browder-Livesay invariants which are the forbidden invariants in the closed manifold surgery problem. In the present paper we describe all possible forbidden invariants which can give a Browder-Livesay filtration for computing the inertia subgroup. Our approach is a natural generalization of the approach of Hambleton and Kharshiladze. More precisely, we prove that a Browder-Livesay filtration of a given manifold can give the following forbidden invariants for an element $x \in L_n(\pi_1(X))$ to belong to the subgroup $I_n(\pi)$: the nontrivial Browder-Livesay invariants in codimensions 0, 1, 2 and a nontrivial class of obstructions of a restriction of a normal map to a submanifold in codimension 3.

1. Introduction.

Throughout the paper we consider finitely presented groups $\pi$ equipped with an orientation homomorphism $w: \pi \to \{\pm 1\}$. Let $L_n(\pi)$ be the Wall surgery obstruction groups $L_n^s(\pi, w)$. As usually, $B\pi = K(\pi, 1)$ denotes the classifying space of $\pi$. For a manifold $X$ we suppose that the orientation map $w: \pi = \pi_1(X) \to \{\pm 1\}$ coincides with the Stiefel-Whitney character. We shall work in the category of topological manifolds.

Any element $x \in L_{n+1}(\pi)$ is represented by a normal map of a closed manifold with boundary. The results of [17] provide the following representation. Choose a closed $n$-manifold $X^n$ with $\pi_1(X^n) = \pi$. Then there is a normal map

\[(1.1) \quad (F, B): (W^{n+1}, \partial_0 W, \partial_1 W) \to (X \times I; X \times \{0\}, X \times \{1\})\]

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with the obstruction \( \sigma(F, B) = x \), where \( \partial_0 W = X, F|_{\partial_0 W} = \text{Id}, \partial_1 W = M^n \), and

\[
f = F|M : M \to X
\]
is a simple homotopy equivalence. Here \( B : \nu W \to \nu X \times I \) is a covering \( F \) bundle map of the stable normal bundle of \( W \) in Euclidean space to a bundle over \( X \times I \). In what follows we shall not mention the maps of stable bundles if this does not lead to a confusion.

Let \( X \) be a closed \( n \)-dimensional topological manifold. An orientation preserving simple homotopy equivalence \( f : M^n \to X^n \) of \( n \)-manifolds is called a homotopical triangulation of the manifold \( X \). Two homotopical triangulations \( f_i : M_i \to X \) \((i = 1, 2)\) are said to be equivalent if there exists a homeomorphism \( h : M_1 \to M_2 \) fitting in the following homotopy commutative diagram (see [14], [15], and [17])

\[
\begin{array}{ccc}
M_1 & \xrightarrow{f_1} & X \\
\downarrow{h} & \nearrow{f_2} & \\
M_2 & & \\
\end{array}
\]
The set of equivalence classes \( \mathcal{S}^{TOP}(X) \) fits into the surgery exact sequence (see [14], [15], and [17])

\[
\cdots \to \left[ \Sigma X, G/TOP \right] \xrightarrow{\lambda} L_{n+1}(\pi) \xrightarrow{\sigma} \mathcal{S}^{TOP}(X) \xrightarrow{\sigma} [X, G/TOP] \xrightarrow{\lambda} L_n(\pi).
\]

For a homotopy triangulation \( g : M \to X \), the action \( \lambda \) in (1.3) is defined in a similar way as in the representation (1.1) that gives the action of \( x \in L_{n+1}(\pi) \) on the trivial triangulation \( \text{Id} : X \to X \). By definition, \( [\lambda(x)](\text{Id}) \) is the homotopy triangulation

\[
F|_{\partial_1 W} : \partial_1 W \to X \times \{1\}
\]
of the manifold \( X \) [17, §10]. Let an element \( x \in L_{n+1}(\pi) \) act trivially on a homotopy triangulation of some manifold \( X^n \) with \( \pi_1(X) = \pi \). We denote by \( I_{n+1}(\pi) \) the subgroup generated by such elements.

In fact, Wall [17] constructed an action of an element \( x \in L_{n+1}(\pi) \) on any homotopy triangulation

\[
f : (M, \partial M) \to (X, \partial X)
\]
of compact manifolds relative the boundary. Let \( \pi_1(X^n) = \pi_4(\partial X) = \pi \). Then the results of Wall provide a normal map of 4-ads

\[
F : (W^{n+1}, \partial_0 W, \partial_1 W, V) \to (X \times I; X \times 0, X \times 1, \partial X \times I)
\]
with \( \sigma(F) = x \). In (1.5) we have \( \partial_0 W = X \times 0, V = \partial X \times I, F|_{\partial_0 W \cup V} = \text{Id}, \partial_1 W = M \) with the boundary \( \partial M = \partial X \), and the map

\[
f_1 = F|_{\partial_1 W} : \partial_1 W \to X \times 1
\]
is a simple homotopy equivalence.
The assembly map
\[(1.7) \quad A : H_n(B\pi; L_\bullet) \to L_n(\pi)\]
fits in the following algebraic surgery exact sequence of Ranicki [14] for the space \(B\pi\)
\[(1.8) \quad \cdots \to L_{n+1}(\pi) \to S_{n+1}(B\pi) \to H_n(B\pi; L_\bullet) \xrightarrow{A} L_n(\pi) \to \cdots\]
where \(S_{n+1}(B\pi)\) is the structure set of the topological space \(B\pi\) and \(L_\bullet\) is the
\(\Omega\)-spectrum that is an 1-connected cover of the simply connected surgery \(\Omega\)-spectrum
\(L_\bullet(1)\) with \(\pi_n(L_\bullet(1)) = L_n(1)\)\( (n > 0)\) and \(L_\bullet 0 \simeq G/TOP\) [14]. For a closed
\(n\)-dimensional topological manifold \(X\) the surgery exact sequence (1.3) is isomorphic to the
left part (from the group \(L_n(\pi)\)) of the algebraic surgery exact sequence (1.8) of the space \(X\) with
\[H_n(X; L_\bullet) \cong [X, G/TOP] \quad \text{and} \quad S_{n+1}(X) = S^{TOP}(X)\]
(see [14], [15], and [17]). The image \(C_n(\pi) \subset L_n(\pi)\) of the assembly map consists
of the surgery obstructions \(\sigma(f, b) \in L_n(\pi)\) where \((f, b)\) is a normal map of closed
manifolds with a given orientation map \(w\) [17, §13B]. It was proved in [5] that
\(I_n(\pi) \subset C_n(\pi)\). Additionally, it follows from [7] that images of these groups coincide
in the projective Novikov-Wall groups \(L^p\). The problem of the computation of the
groups \(I_n(\pi)\) and \(C_n(\pi)\) is one of the significant problems of geometrical topology
(see [8]).

The iterated Browder-Livesay invariants provide a collection of the forbidden
invariants in the closed manifold surgery problem for a group \(\pi\) with a subgroup
of index 2 (see [1], [3], [5], [7], [11], and [12]). The natural way to describe iterated
Browder-Livesay invariants for the group \(L_n(\pi)\) is the Browder-Livesay filtration
of the manifold \(X\) with \(\pi_1(X) = \pi\) (see [1], [6], and [13]) (the definition of a
Browder-Livesay filtration and iterated invariants is given in section 2 below). In
the present paper we describe the application of this approach to computing the
inertia subgroup. More precisely, we prove that a Browder-Livesay filtration of a given
manifold can give the following forbidden invariants for an element \(x \in L_n(\pi_1(X))\)
to belong to the subgroup \(I_n(\pi)\): the nontrivial Browder-Livesay invariants in codi-
mensions 0, 1, 2 and a nontrivial class of obstructions of a restriction of a normal
map to a submanifold in codimension 3.

In section 2 we give necessary preliminary material and state the main theorem
and in section 3 we formulate and prove the main results.

2. Browder-Livesay filtration and iterated invariants.

The pair of closed manifolds \((X^n, Y^{n-1})\) is called a Browder-Livesay pair if
\(Y\) is the one-sided submanifold in \(X\) and induced by the natural inclusion map
\(\pi_1(Y) \to \pi_1(X)\) is an isomorphism (see [2], [4], [5], [7], [12], and [17, §11]). Let
\(\rho = \pi_1(X \setminus Y)\) and \(i: \rho \to \pi\) be the natural map induced by the inclusion. Let \(U\)
be a tubular neighborhood of \( Y \) in \( X \) with a boundary \( \partial U \). We obtain a pushout square

\[
F = \begin{pmatrix}
\pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \to & \pi_1(X)
\end{pmatrix} = \begin{pmatrix}
\rho & \to & \rho \\
\downarrow & & \downarrow \\
\pi^{\mp} & \to & \pi^{\pm}
\end{pmatrix}
\]

of fundamental groups with orientation in which horizontal maps are isomorphisms and vertical maps are inclusions of index 2. In (2.1) the upper horizontal map and the vertical maps agree with orientations. The bottom horizontal maps preserve the orientation on the image of the vertical maps and reverse orientations outside these images. We shall denote this fact by superscript "\( + \)" or "\( - \)". We shall omit this superscript if the orientation follows from the context.

By definition (see [15, §7] and [17, §11]), a simple homotopy equivalence \( f : M \to X \) splits along the submanifold \( Y \) if it is homotopic to a map \( g : M \to X \) which is transversal to \( Y \) with \( N = g^{-1}(Y) \), and whose restrictions

\[
(2.2) \\
g|_N : N \to Y \quad \text{and} \quad g|_{(M \setminus N)} : M \setminus N \to X \setminus Y
\]

are simple homotopy equivalences. The splitting obstruction groups

\[
LN_{n-1}(\pi_1(X \setminus Y) \to \pi_1(X)) = LN_{n-1}(\rho \to \pi)
\]

for a Browder-Livesay manifold pair \((X, Y)\) were defined (see [2], [3], [5], [7], [15, §7], and [17]) and are called the Browder–Livesay groups. The algebraic definition of the \( LN_{*} \)-groups is given in [16]. These groups depend functorially on the oriented inclusion \( \rho \to \pi \) and the dimension \( n - q \) mod 4.

These groups fit in the following braid of exact sequences (see [2], [7], [12], [15], and [16])

\[
(2.3) \\
\to L_n(\rho) \xrightarrow{i_*} L_n(\pi) \xrightarrow{\partial} LN_{n-2}(\rho \to \pi) \to L_n(F) \xrightarrow{s} L_n(\pi) \xrightarrow{\partial} LN_{n-2}(\rho \to \pi) \to \Gamma \xrightarrow{\lambda} LN_{n-2}(\rho \to \pi) \to L_n(\rho) \xrightarrow{c} L_{n-1}(\pi) \xrightarrow{\partial} LN_{n-2}(\rho \to \pi) \to L_n(\rho) \xrightarrow{c} L_{n-1}(\pi^-) \xrightarrow{\partial} LN_{n-2}(\rho \to \pi) \to L_n(\rho)
\]

where \( LP_{n-1}(F) \cong L_n(i^-) \) are the surgery obstruction groups for the manifold pair \((X, Y)\) (see [2], [15], and [17]), and \( L_n(i_*) \) are the relative surgery obstruction groups for the inclusion \( i \) (see [2], [10], [15], and [17]). The upper and bottom rows of Diagram (2.3) are chain complexes, and \( \Gamma \) is an isomorphism of the corresponding homology groups. Note that the maps \( s \) and \( q \) are the natural forgetful maps, and the map \( c \) denote passing from surgery problem inside the manifold \( X \) to an abstract surgery problem [17]. The map \( i^- \) is the surgery transfer map, and the map \( \partial \) is the composition

\[
L_n(\pi) \xrightarrow{\lambda} S^{TOP}(X) \to LN_{n-2}(\rho \to \pi)
\]

of the action of an element \( x \) on the trivial triangulation of a closed manifold \( X^{n-1} \) and taking an obstruction to splitting along the submanifold \( Y^{n-2} \subset X \) on the top
boundary of the bordism as in (1.1). For an element \( x \in L_n(\pi) \) which represents a homology class
\[
[x] \in \text{Ker} \partial/ \text{Im} i_*
\]
we have a class
\[
\Gamma([x]) = \{ qs^{-1}(x) | x \in \text{Ker} \partial \} \in \text{Ker} i_*/\text{Im} c
\]
that is represented by an element \( q(y) \) where \( y \in LP_{n-1}(F) \) and \( s(y) = x \).

Let \( X \) be a filtration
\[
X_k \subset X_{k-1} \subset \cdots \subset X_2 \subset X_1 \subset X_0 = X
\]
of a closed manifold \( X^n \) by means of locally-flat closed submanifolds such that every pair of submanifolds is a manifold pair in the sense of Ranicki [15]. A filtration in (2.4), for which every pair of submanifolds \( (X_i, X_{i+1}) \) \( (0 \leq i \leq k-1) \) is a Browder-Livesay pair, is called a Browder-Livesay filtration (see [1], [6], and [13]). In what follows we shall consider only Browder-Livesay filtrations and we shall assume that \( \dim X_k = n - k \geq 5 \). The filtration \( X \) in (2.4) is a stratified manifold in the sense of Browder-Quinn (see [4], [13], and [18]).

Let \( F_i \) \( (0 \leq i \leq k - 1) \) be a square of fundamental groups in the splitting problem for the manifold pair \( (X_i, X_{i+1}) \) of the filtration in (2.4), \( G_i = \pi_1(X_i) \), and \( \rho_i = \pi_1(X_i \setminus X_{i+1}) \). Then \( LN_* (\rho_i \to G_i) \) are the splitting obstruction groups for the manifold pair \( (X_i, X_{i+1}) \).

Every inclusion \( \rho_i \to G_i \) of index 2 gives a commutative braid of exact sequences that is similar to (2.3). Putting together central squares from these diagrams (see [9] and [11]) we obtain the following commutative diagram

\[
\begin{array}{cccc}
\rightarrow & L_n(G_0) & \delta_{0} & LN_{n-2}(\rho_0 \to G_0) \\
LP_{n-1}(F_0) & \Gamma \downarrow & \rightarrow & L_n(\rho_0 \to G_0) \\
& q \nearrow & & \\
& \rightarrow & L_{n-1}(G_1) & \delta_{1} & LN_{n-3}(\rho_1 \to G_1) \\
& \Gamma \downarrow & \rightarrow & L_{n-1}(\rho_1 \to G_1) \\
LP_{n-2}(F_1) & q \nearrow & \rightarrow & L_{n-2}(G_2) & \delta_{2} & LN_{n}(\rho_2 \to G_2) \\
& \Gamma \downarrow & \rightarrow & L_{n-2}(\rho_2 \to G_2) \\
LP_{n-3}(F_2) & q \nearrow & \rightarrow & L_{n-3}(G_3) & \vdots & \\
& \Gamma \downarrow & \rightarrow & L_{n-k+1}(G_{k-1}) & \delta_{k-1} & LN_{n-k-1}(\rho_{k-1} \to G_{k-1}) \\
LP_{n-k}(F_{k-1}) & q \nearrow & \rightarrow & L_{n-k-1}(\rho_{k-1} \to G_{k-1}) \\
& \Gamma \downarrow & \rightarrow & L_{n-k}(G_k) \\
\end{array}
\]
In this diagram we denote by \( s \) and \( q \) the similar maps from different diagrams. However, in what follows, it will be clear from the context which map is under consideration. Note that the groups and the maps in Diagram (2.5) are defined by the subscripts taken mod 4 since \( L_* \)-groups and Diagram (2.3) are four-periodic.

Now we can give an inductive definition of the sets

\[
\Gamma^j(x) \subset L_{n-j}(G_j) \quad (0 \leq j \leq k)
\]

and iterated Browder-Livesay \( j \)-invariants \((1 \leq j \leq k)\) with respect to the filtration (2.4) (see [6], [11], [12], and [13]).

**Definition 2.1.** Let \( x \in L_n(G_0) \). By definition,

\[
\Gamma^0(x) = \{x\} \subset L_n(G_0).
\]

The set \( \Gamma^0(x) \) said to be trivial if \( x \in \text{Image}\{L_n(\rho_0) \to L_n(G_0)\} \). Let a set

\[
\Gamma^j(x) \subset L_{n-j}(G_j) \quad (0 \leq j \leq k-1)
\]

be defined. For \( j \geq 1 \), it is called trivial if \( 0 \in \Gamma^j(x) \).

If \( \Gamma^j(x)(0 \leq j \leq k-1) \) is defined and nontrivial, then the \((j+1)\)-th Browder-Livesay invariant with respect filtration (2.4) is the set

\[
\partial_j(\Gamma^j(x)) \subset LN_{n-j-2}(\rho_{j-1} \to G_{j-1}).
\]

The \((j+1)\)-th invariant is nontrivial if \( 0 \notin \partial_j(\Gamma^j(x)) \).

If the \((j+1)\)-th \((1 \leq j \leq k-1)\) Browder-Livesay invariant is defined and trivial then the set \( \Gamma^{j+1}(x) \) is defined as

\[
\Gamma^{j+1}(x) \overset{def}{=} \Gamma(\Gamma^j(x)) \overset{def}{=} \{qs^{-1}(z)|z \in \Gamma^j(x), \partial_j(z) = 0\} \subset L_{n-j-1}(G_{j+1}).
\]

**Theorem 2.2.** ([11] and [13]) Let \( x \in L_n(G_0) \) be an element with a nontrivial \( j \)-th Browder-Livesay invariant relatively to a Browder-Livesay filtration \( \mathcal{X} \) of the manifold \( X \). Then the element \( x \) cannot be realized by a normal map of closed manifolds.

Let us consider an an infinite diagram \( \mathcal{D}_\infty \) of groups with orientations

\[
\begin{array}{cccccc}
(2.6) & \cdots & \rho_2 & \rho_1 & \rho_0 \\
G_3 & \cong & G_2 & \cong & G_1 & \cong & G_0,
\end{array}
\]

which is commutative as the diagram of groups. The maps \( \rho_i \to G_i \) and \( \rho_i \to G_{i+1} \) in (2.6) are index 2 inclusions of groups with orientations, and the horizontal maps preserve the orientations on the images of the groups \( \rho_i \) and reverse the orientations outside these images. Each commutative triangle from (2.6) defines an algebraic version of diagram (2.3) for the inclusion \( \rho_i \to G_i \) [16]. Putting together central squares of these diagrams we obtain an infinite in bottom direction diagram which is similar to diagram (2.5) (see [1], [6], [9], and [13]). Thus we can define the iterated Browder–Livesay invariants of an element \( x \in L_n(G_0) \) relatively to a diagram in (2.6) similarly to the case of the filtration \( \mathcal{X} \). A result similar to Theorem 2.2 is true for Browder-Livesay invariants of an element \( x \) relatively to a diagram in (2.6) [13]. Note that the finite subdiagram (from \( G_k \) until \( G_0 \)) of diagram (2.6) provides a commutative diagram (2.5). Denote this diagram by \( \mathcal{D}_k \).
**Definition 2.3.** ([1], [6], and [12]) Let \( x \in L_n(G_0) \). The element \( x \) is the element of the second type with respect to \( D_\infty \) if all the sets \( \Gamma^j(x) \) \((j \geq 0)\) are defined and nontrivial and all Browder-Livesay invariants with respect to \( D_\infty \) are defined and trivial.

**Theorem 2.4.** ([11] and [12]) Let \( x \in L_n(G_0) \) be an element of the second type with respect to \( D_\infty \). Then \( x \) cannot be realized by a normal map of closed manifolds.

Now we state the main results of the paper.

**Theorem 2.5.** Let \( x \in L_n(G_0) \) \((n \geq 5)\) be an element for which the set \( \Gamma^3(x) \) is defined and nontrivial with respect to the subdiagram \( D_3 \) of diagram (2.6). Then the element \( x \) does not belong to the subgroup \( I_n(\pi) \).

**Remark.** To define \( j \)-th Browder-Livesay invariant of \( x \) for \( j \geq 4 \) relatively to diagram \( D_\infty \) we must have a nontrivial set \( \Gamma^3(x) \) with respect to subdiagram \( D_3 \) of \( D_\infty \). Thus, the nontriviality of a the \( j \)-th Browder-Livesay invariant for \( j \geq 4 \) automatically implies nontriviality of the invariant \( \Gamma^3(x) \). Also, if the element \( x \) has the second type, then by Definition 2, the invariant \( \Gamma^3(x) \) is nontrivial.

3. Proof of Theorem 2.5.

**Lemma 3.1.** Let \( X^n = X^n_0 \) be a manifold with the fundamental group \( G_0 \) and \( D_k \) be a finite subdiagram of diagram (2.6) such that \( n - k \geq 5 \). Then there exists a Browder-Livesay filtration as in (2.4) \( X \) of \( X = X_0 \) which corresponds to the diagram \( D_k \).

**Proof.** Consider a map
\[
\phi : X^n \to \mathbb{R}P^N
\]
to a real projective space of high dimension which induces an epimorphism of fundamental groups \( G_0 \to \mathbb{Z}/2 \) that has the kernel \( \rho_0 \). Using the standard arguments (see [7], [9], [12], and [17, §11, 12C]), we can suppose that the map \( \phi \) is transversal to \( \mathbb{R}P^{N-1} \subset \mathbb{R}P^N \) with \( Y^{n-1} = (\phi_0)^{-1}(\mathbb{R}P^{N-1}) \) such that the induced map \( \pi_1(Y) \to \pi_1(X) \) is an isomorphism. The pair \( (X,Y) \) is the Browder-Livesay pair that gives the filtration with \( D_1 \). Iterating this process we obtain the desired result.

**Proof of Theorem 2.5.** Let the element \( x \in L_n(G_0) \) act trivially on a manifold \( X^{n-1} \). Taking the product with \( \mathbb{R}P^4 \), we can suppose that dimension \( n - 1 \geq 8 \) (see [17]). Consider a Browder-Livesay filtration
\[
X_3 \subset X_2 \subset X_1 \subset X_0 = X^{n-1}
\]
which gives the diagram \( D_3 \) by Lemma 1. Let \( U \) be a tubular neighborhood of \( X_3 \) in \( X_0 \). Note, that \( \pi_1(X_0 \setminus X_3) = \pi_1(X_0) = G_0 \). In accordance with [17], we can construct the action of the element \( x \) on the manifold \( X_0 \) "outside the tubular neighborhood \( U \). The proof is identically with the proof of Theorem 5.8 (respectively Theorem 6.5) in [17] because \( \pi_1(X \setminus U) \cong \pi_1(X) \).

This means that we can represent \( X_0 \times I \) as
\[
X_0 \times I = (X_0 \setminus U) \times I \bigcup_{\partial U \times I} \overline{U} \times I
\]
such that the normal map

\[(3.1) \quad F: (W^n; \partial_0W, \partial_1W) \rightarrow (X \times I; X \times \{0\}, X \times \{1\})\]

has the following properties.

i) The manifold \(W^n\) is a union

\[W^n = V^n \bigcup_{\partial U \times I} \overline{U} \times I\]

where

\[\partial V = \partial_0V \bigcup_{\partial U \times \{0\}} \partial U \times I \bigcup_{\partial U \times \{1\}} \partial_1V,\]

\[\partial_0V = (X \setminus U) \times \{0\},\]

the boundary of \(\partial_0V\) is equal to

\[\partial U \times \{0\},\]

the boundary of \(\partial_1V\) is equal to

\[\partial U \times \{1\},\]

\[\partial_0W = X \times \{0\} = (X \setminus U) \times \{0\} \bigcup_{\partial U \times \{0\}} \overline{U} \times \{0\},\]

\[\partial_1W = \partial_1V \bigcup_{\partial U \times \{1\}} \overline{U} \times \{1\}.\]

ii) The restriction

\[F|_{(X \times \{0\}) \cup \overline{U} \times I}: X \times \{0\} \bigcup_{\overline{U} \times \{0\}} \overline{U} \times I \rightarrow X \times \{0\} \bigcup_{\overline{U} \times \{0\}} \overline{U} \times I\]

is the identity map, the restriction

\[F|_{\partial_0V}: \partial_0V \rightarrow (X \setminus U) \times \{1\}\]

is a homeomorphism which is identity map on the boundary \(\partial U \times \{1\}\), and the restriction

\[(3.2) \quad F|_{\partial_1W}: \partial_1W \rightarrow X \times \{1\}\]

is a homeomorphism.

Changing

\[F|_V: V \rightarrow (X \setminus U) \times I\]

relatively boundary \(\partial V\) in the class of normal bordisms, we can suppose that it is transversal to

\[(X_1 \setminus (X_1 \cap U)) \times I \subset (X \setminus U) \times I\]
with a transversal preimage \((V_1, \partial V_1)\), where

\[
\partial V_1 = \partial_b V_1 \bigcup_{(X_1 \cap \partial U) \times \{0\}} (X_1 \cap \partial U) \times I \bigcup_{(X_1 \cap \partial U) \times \{1\}} \partial_1 V_1 = [X_1 \setminus (X_1 \cap U)] \times \{0\} \bigcup_{(X_1 \cap \partial U) \times \{0\}} (X_1 \cap \partial U) \times I \bigcup_{(X_1 \cap \partial U) \times \{1\}} \partial_1 V_1,
\]

the restriction of \(F|_V\) to

\[
\partial_b V_1 \bigcup_{(X_1 \cap \partial U) \times \{0\}} (X_1 \cap \partial U) \times I
\]

is the identity map, and the restriction of \(F|_V\) to \(\partial_1 V_1\) is a homeomorphism

\[
F|_{\partial_1 V_1}: \partial_1 V_1 \to [X_1 \setminus (X_1 \cap U)] \times I.
\]

It follows that the map \(F\) in (3.1) is transversal to \(X_1 \times I\) with a transversal preimage \((W_1, \partial W_1)\) where \(\partial W_1 = \partial_b W_1 \cup \partial_1 W_1\). Let \(U_1 = U \cap X_1\) be the tubular neighborhood of \(X_3 \subset X_1\). The restriction \(F_1 = F|_{W_1}\) is a normal map

\[
(3.3) \quad F_1: (W_1^{n-1}; \partial W_1, \partial_1 W_1) \to (X_1 \times I; X_1 \times \{0\}, X_1 \times \{1\})
\]

with the following properties:

i) the manifold \(W_1^{n-1}\) is a union

\[
W_1 = V_1 \bigcup_{\partial V_1 \times I} \overline{U_1} \times I
\]

where

\[
\partial V_1 = \partial_b V_1 \bigcup_{\partial U_1 \times \{0\}} \partial U_1 \times I \bigcup_{\partial U_1 \times \{1\}} \partial_1 V_1, \\
\partial_b V_1 = (X_1 \setminus U_1) \times \{0\},
\]

the boundary of \(\partial_b V_1\) is equal to

\[
\partial U_1 \times \{0\},
\]

the boundary of \(\partial_1 V_1\) is equal to

\[
\partial U_1 \times \{1\},
\]

\[
\partial_b W_1 = X_1 \times \{0\} = (X_1 \setminus U_1) \times \{0\} \bigcup_{\partial U_1 \times \{0\}} \overline{U_1} \times \{0\} = \partial_b V_1 \bigcup_{\partial U_1 \times \{0\}} \overline{U_1} \times \{0\}, \\
\partial_1 W_1 = \partial_1 V_1 \bigcup_{\partial U_1 \times \{1\}} \overline{U_1} \times \{1\};
\]

ii) the restriction

\[
F_1|_{X_1 \times \{0\} \bigcup \overline{U_1} \times I}: X_1 \times \{0\} \bigcup_{\overline{U_1} \times \{0\}} \overline{U_1} \times I \to X_1 \times \{0\} \bigcup_{\overline{U_1} \times \{0\}} \overline{U_1} \times I
\]
is the identity map, the restriction
\[ F_1|_{\partial_0 V_1} : \partial_1 V_1 \to (X_1 \setminus U_1) \times \{1\} \]
is a homeomorphism which is identity map on the boundary \( \partial U_1 \times \{1\} \), and hence the restriction

\[ (3.4) \quad F_1|_{\partial_1 W_1} : \partial_1 W_1 \to X_1 \times \{1\} \]
is a homeomorphism.

Now consider the diagram (2.5) for which \( x \in L_n(G_0) \). Since \( \sigma(F) = x \) and the map in (3.2) is a homeomorphism. It follows from the geometrical definition of the map \( \partial_0 : L_n(G_0) \to LN_{n-2}(\rho_0 \to G_0) \) (see [7], [12], and [17] that \( \partial_0(x) = 0 \). It follows from geometrical definition of the map \( \Gamma \) in diagram (2.5) ([11], [12] and [17]) that the normal map \( F_1 \) in (3.3) has a surgery obstruction \( \sigma(F_1) = x_1 \in L_{n-1}(G_1) \) which lies in the class \( \Gamma(x) \). As above, since the restriction \( F_1|_{\partial_1 W_1} \) in (3.4) is a homeomorphism, we obtain \( \partial_1(x_1) = 0 \in L_{n-3}(\rho_1 \to G_1) \). For passing from the map \( F_1 \) to a map \( F_2 \), which is restriction of \( F_1 \) to the a transversal preimage of \( X_2 \times I \), we can use the same line of arguments as for the passing from the map \( F \) to the map \( F_1 \). Let \( U_2 = U \cap X_2 \) be the tubular neighborhood of \( X_3 \subset X_2 \). Thus we obtain a normal map

\[ (3.5) \quad F_2 : (W_2^{n-2}, \partial_0 W_2, \partial_1 W_2) \to (X_2 \times I; X_2 \times \{0\}, X_2 \times \{1\}) \]

with a decomposition
\[ W_2 = V_2 \bigcup_{\partial V_2 \times I} U_2 \times I \]
which is similar to the decomposition above. In particular,
\[ \partial V_2 = \partial_0 V_2 \bigcup_{\partial U_2 \times \{0\}} \partial U_2 \times I \bigcup_{\partial U_2 \times \{1\}} \partial_1 V_2, \]
\[ \partial_0 V_2 = (X_2 \setminus U_2) \times \{0\}, \]
the boundary of \( \partial_0 V_2 \) is equal to
\[ \partial U_2 \times \{0\}, \]
the boundary of \( \partial_1 V_2 \) is equal to
\[ \partial U_2 \times \{1\}, \]
\[ \partial_0 W_2 = X_2 \times \{0\}, \quad \partial_1 W_2 = \partial_1 V_2 \bigcup_{\partial U_2 \times \{1\}} U_2 \times \{1\}. \]
The map \( F_2 \) is the identity on
\[ \partial_0 W_2 \bigcup_{U_2 \times \{0\}} U_2 \times I = X_2 \times \{0\} \bigcup_{U_2 \times \{0\}} U_2 \times I, \]
and the restriction of $F_2$ to $\partial_1 W_2$ is a homeomorphism

$$F_2|_{\partial_1 W_2} : \partial_1 W_2 \to X_2 \times \{1\}$$

which is identity on

$$\cup_2 \times \{1\} \subset \partial_1 W_2.$$  

As above, the normal map $F_2$ in (3.5) has a surgery obstruction $\sigma(F_2) = x_2 \in L_{n-2}(G_2)$ which lies in the class $\Gamma^2(x)$ and $\partial_2(x_2) = 0$. By our construction,

$$F_2^{-1}(U_2 \times \{I\}) = U_2 \times \{I\} \subset W_3$$

and $F_2|_{U_2 \times \{I\}}$ is identity. Since $U_2$ is a tubular neighborhood of $X_3$ in $X_2$ we obtain that a restriction of $F_2$ to the transversal preimage of $X_3 \times I$

$$F_3 = F_2|_{F_2^{-1}(X_3 \times I)}$$

is the identity. Thus, the surgery obstruction $\sigma(F_3) \in L_{n-3}(G_3)$ is trivial. This obstruction lies in the class $\Gamma^3(x)$, and hence $0 \in \Gamma^3(x)$ and $\Gamma^3(x)$ is trivial. The theorem is thus proved.

There are many examples of nontrivial first and second Browder-Livesay invariants and notrivial classes $\Gamma^3$ (see for example [12]). We do not know examples with nontrivial third Browder-Livesay invariant. Also, in all known cases nontriviality of the class $\Gamma^2(x)$ for some element $x \in L_n(\pi)$ implies that element $x$ does not lie in $C_n(\pi)$. It would be very interesting to understand this problem.

**References**

1. A. Bak – Yu. V. Muranov, *Splitting a simple homotopy equivalence along a submanifold with filtration*, Sbornik : Mathematics **199** (2008), no. 6, 123.
2. A. Bak – Yu. V. Muranov, *Splitting along submanifolds, and L-spectra (Russian)*, Sovrem. Mat. Prilozh., Topol., Anal. Smezh. Vopr. (2003), no. 1, 3–18; English translation in J. Math. Sci. (N. Y.) 123 (2004), no. 4, 4169–4184.
3. W. Browder – G. R. Livesay, *Fixed point free involutions on homotopy spheres*, Bull. Amer. Math. Soc. **73** (1967), 242–245.
4. W. Browder – F. Quinn, *A surgery theory for G-manifolds and stratified sets*, in Manifolds–Tokyo 1973 (1975), Univ. of Tokyo Press, 27–36.
5. S. E. Cappell – J. L. Shaneson, *Pseudo-free actions. I*, Lecture Notes in Math. **763** (1979), 395–447.
6. A. Cavicchioli – Yu. V. Muranov – F. Spaggiari, *On the elements of the second type in surgery groups*, Preprint MPI (2006).
7. I. Hambleton, *Projective surgery obstructions on closed manifolds*, Lecture Notes in Math. **967** (1982), 101–131.
8. I. Hambleton – J. Milgram – L. Taylor – B. Williams, *Surgery with finite fundamental group*, Proc. London Mat. Soc. **56** (1988), 349–379.
9. I. Hambleton – A. F. Kharshiladze, *A spectral sequence in surgery theory*, Mat. Sbornik **183** (1992), 3–14; English transl. in Russian Acad. Sci. Sb. Math. **77** (1994), 1–9.
10. I. Hambleton – A. Ranicki – L. Taylor, *Round L-theory*, J. Pure Appl. Algebra **47** (1987), 131–154.
11. A. F. Kharshiladze, *Iterated Browder-Livesay invariants and oozing problem*, Mat. Zametki **41** (1987), 557–563; English transl. in Math. Notes **41** (1987).
12. A. F. Kharshiladze, *Surgery on manifolds with finite fundamental groups*, Uspechi Mat. Nauk **42** (1987), 55–85; English transl. in Russian Math. Surveys **42** (1987).
13. Yu. V. Muranov – D. Repovš – R. Jimenez, Surgery spectral sequence and manifolds with filtrations 67 (2006), Trudy MMO (in Russian), 294–325.
14. A. A. Ranicki, The total surgery obstruction, Lecture Notes in Math. 763 (1979), 275–316.
15. A. A. Ranicki, Exact Sequences in the Algebraic Theory of Surgery, Math. Notes 26, Princeton Univ. Press, Princeton, N. J., 1981.
16. A. A. Ranicki, The L-theory of twisted quadratic extensions, Canad. J. Math. 39 (1987), 245–364.
17. C. T. C. Wall, Surgery on Compact Manifolds, Academic Press, London – New York, 1970; Second Edition (A. A. Ranicki, ed.), Amer. Math. Soc., Providence, R.I., 1999.
18. S. Weinberger, The Topological Classification of Stratified Spaces, The University of Chicago Press, Chicago – London, 1994.

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