The sharp Adams type inequalities in the hyperbolic spaces under the Lorentz-Sobolev norms

Van Hoang Nguyen

January 14, 2020

Abstract

Let $2 \leq m < n$ and $q \in (1, \infty)$, we denote by $W^m L^{\frac{n}{m}, q}(\mathbb{H}^n)$ the Lorentz–Sobolev space of order $m$ in the hyperbolic space $\mathbb{H}^n$. In this paper, we establish the following Adams inequality in the Lorentz–Sobolev space $W^m L^{\frac{n}{m}, q}(\mathbb{H}^n)$

$$\sup_{u \in W^m L^{\frac{n}{m}, q}(\mathbb{H}^n), \|\nabla^m u\|_{\frac{n}{m}, q} \leq 1} \int_{\mathbb{H}^n} \Phi_{\frac{n}{m}, q} \left( \beta_{n,m}^{\frac{q}{q-1}} |u|^{\frac{q}{q-1}} \right) dV_g < \infty$$

for $q \in (1, \infty)$ if $m$ is even, and $q \in (1, n/m)$ if $m$ is odd, where $\beta_{n,m}^{q/(q-1)}$ is the sharp exponent in the Adams inequality under Lorentz–Sobolev norm in the Euclidean space. To our knowledge, much less is known about the Adams inequality under the Lorentz–Sobolev norm in the hyperbolic spaces. We also prove an improved Adams inequality under the Lorentz–Sobolev norm provided that $q \geq 2n/(n-1)$ if $m$ is even and $2n/(n-1) \leq q \leq \frac{n}{m}$ if $m$ is odd,

$$\sup_{u \in W^m L^{\frac{n}{m}, q}(\mathbb{H}^n), \|\nabla^m u\|_{\frac{n}{m}, q} \leq 1} \int_{\mathbb{H}^n} \Phi_{\frac{n}{m}, q} \left( \beta_{n,m}^{\frac{q}{q-1}} |u|^{\frac{q}{q-1}} \right) dV_g < \infty$$

for any $0 < \lambda < C(n, m, n/m)^q$ where $C(n, m, n/m)^q$ is the sharp constant in the Lorentz–Poincaré inequality. Finally, we establish a Hardy–Adams inequality in the unit ball when $m \geq 3$, $n \geq 2m+1$ and $q \geq 2n/(n-1)$ if $m$ is even and $2n/(n-1) \leq q \leq n/m$ if $m$ is odd

$$\sup_{u \in W^m L^{\frac{n}{m}, q}(\mathbb{H}^n), \|\nabla^m u\|_{\frac{n}{m}, q} \leq 1} \int_{\mathbb{B}^n} \exp \left( \beta_{n,m}^{\frac{q}{q-1}} |u|^{\frac{q}{q-1}} \right) dx < \infty.$$
1 Introduction

It is well-known that the Sobolev’s embedding theorems play the important roles in the analysis, geometry, partial differential equations, etc. Let $m \geq 1$, we we traditionally use the notation

\[ \nabla^m = \begin{cases} \Delta^{\frac{m}{2}} & \text{if } m \text{ is even}, \\ \nabla \Delta^{\frac{m-1}{2}} & \text{if } m \text{ is odd} \end{cases} \]

to denote the $m$–th derivatives. For a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and $1 \leq p < \infty$, we denote by $W^{m,p}_0(\Omega)$ the usual Sobolev spaces which is the completion of $C_\infty^0(\Omega)$ under the Dirichlet norm $\|\nabla^m u\|_{L^p(\Omega)} = \left( \int_\Omega |\nabla^m u|^p dx \right)^{\frac{1}{p}}$. The Sobolev inequality asserts that $W^{m,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \leq \frac{np}{n-mp}$ provided $mp < n$. However, in the limits case $mp = n$ the embedding $W^{m,\frac{n}{m}}_0(\Omega) \hookrightarrow L^\infty(\Omega)$ fails. In this situation, the Moser–Trudinger inequality and Adams inequality are perfect replacements. The Moser–Trudinger inequality was proved independently by Yudovic [25], Pohozaev [56] and Trudinger [60]. This inequality was then sharpened by Moser [44] in the following form

\[ \sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_\Omega e^{\alpha |u|^\frac{n}{n-1}} dx < \infty \] (1.1)

for any $\alpha \leq \alpha_n := n\omega_{n-1}^{\frac{1}{n}}$ where $\omega_{n-1}$ denotes the surface area of the unit sphere in $\mathbb{R}^n$. Furthermore, the inequality (1.1) is sharp in the sense that the supremum in (1.1) will be infinite if $\alpha > \alpha_n$. The inequality (1.1) was generalized to higher order Sobolev spaces $W^{m,\frac{n}{m}}_0(\Omega)$ by Adams [2] in the following form

\[ \sup_{u \in W^{m,n}_0(\Omega), \int_\Omega |\nabla^m u|^\frac{n}{m} dx \leq 1} \int_\Omega e^{\alpha |u|^\frac{n}{n-m}} dx < \infty, \] (1.2)

for any

\[ \alpha \leq \alpha_{n,m} := \begin{cases} \left( \frac{1}{\sigma_n} \left( \frac{\pi^{n/2}m! \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)} \right) \frac{n}{n-m} \right) & \text{if } m \text{ is even}, \\ \left( \frac{1}{\sigma_n} \left( \frac{\pi^{n/2}m! \Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{n-m}{2}\right)} \right) \frac{n}{n-m} \right) & \text{if } m \text{ is odd}, \end{cases} \]

where $\sigma_n = \omega_{n-1}/n$ is the volume of the unit ball in $\mathbb{R}^n$. Moreover, if $\alpha > \alpha_{n,m}$ then the supremum in (1.2) becomes infinite though all integrals are still finite.

The Moser-Trudinger inequality (1.1) and Adams inequality (1.2) play the role of the Sobolev embedding theorems in the limiting case $mp = n$. They have many applications to study the problems in analysis, geometry, partial differential equations, etc such as the Yamabe’s equation, the $Q$–curvature equations, especially the problems in partial differential equations with exponential nonlinearity, etc. There have been many generalizations of the Moser–Trudinger inequality and Adams inequality in literature. For examples, the Moser–Trudinger inequality and Adams inequality were established in the Riemannian
manifolds in [5, 11, 20, 26, 35, 41, 42, 63] and were established in the subRiemannian manifolds in [9, 15, 16]. The singular version of the Moser–Trudinger inequality and Adams inequality was proved in [4, 29]. The Moser–Trudinger inequality and Adams inequality were extended to unbounded domains and whole spaces in [1, 6, 27, 28, 30, 33, 57, 58], and to fractional order Sobolev spaces in [22, 23, 43]. The improved version of the Moser–Trudinger inequality and Adams inequality were given in [3, 18, 19, 31–40, 47, 49, 51, 59, 61, 64]. An interesting question concerning to the Moser–Trudinger inequality and Adams inequality is whether or not the extremal functions exist. For this interesting topic, the reader may consult the papers [12, 14, 21, 32–34, 38, 39, 47, 51, 57] and many other papers.

Another generalization of the Moser–Trudinger inequality and Adams inequality is to establish the inequalities of same type in the Lorentz–Sobolev spaces. The Moser–Trudinger inequality and the Adams inequality in the Lorentz spaces was established by Alvino, Ferone and Trombetti [8] and Alberico [7] in the following form

\[
\sup_{u \in W^{m}L^{\frac{n}{m}, q}(\Omega), \|\nabla^{m} u\|_{\frac{n}{m}, q} \leq 1} \int_{\Omega} e^{\alpha|u|^q} \, dx < \infty \tag{1.3}
\]

for any \(\alpha \leq \beta_{n,m}^{\frac{q}{q-1}}\) with

\[
\beta_{n,m} = \begin{cases} 
\frac{\pi n/2^m \Gamma(\frac{m}{2})}{\sigma_{n}^{(m-m)/n} \Gamma(\frac{m}{2}-1)} & \text{if } m \text{ is even}, \\
\frac{\pi n/2^m \Gamma(\frac{m+1}{2})}{\sigma_{n}^{(m-m)/n} \Gamma(\frac{m+1}{2})} & \text{if } m \text{ is odd}.
\end{cases}
\]

The constant \(\beta_{n,m}\) is sharp in (1.3) in the sense that the supremum will become infinite if \(\alpha > \beta_{n,m}^{\frac{q}{q-1}}\). For unbounded domains in \(\mathbb{R}^n\), the Moser–Trudinger inequality was proved by Cassani and Tarsi [13] (see Theorem 1 and Theorem 2 in [13]). In [36], Lu and Tang proved several sharp singular Moser–Trudinger inequalities in the Lorentz–Sobolev spaces which generalize the results in [8, 13] to the singular weights. The singular Adams type inequalities in the Lorentz–Sobolev spaces were studied by the author in [55].

The motivation of this paper is to study the Adams inequalities in the hyperbolic spaces under the Lorentz–Sobolev norm. For \(n \geq 2\), let us denote by \(\mathbb{H}^n\) the hyperbolic space of dimension \(n\), i.e., a complete, simply connected, \(n\)-dimensional Riemmanian manifold having constant sectional curvature \(-1\). The aim in this paper is to generalize the main results obtained by the author in [53] to the higher order Lorentz–Sobolev spaces in \(\mathbb{H}^n\). Before stating our results, let us fix some notation. Let \(V_g, \nabla_g\) and \(\Delta_g\) denote the volume element, the hyperbolic gradient and the Laplace–Beltrami operator in \(\mathbb{H}^n\) with respect to the metric \(g\) respectively. For higher order derivatives, we shall adopt the following convention

\[
\nabla^m_g = \begin{cases} 
\Delta_g^{\frac{m}{2}} & \text{if } m \text{ is even}, \\
\nabla_g(\Delta_g^{\frac{m-1}{2}}) & \text{if } m \text{ is odd}.
\end{cases}
\]

Furthermore, for simplicity, we write \(|\nabla^m_g|\) instead of \(|\nabla^m_g|_g\) when \(m\) is odd if no confusion occurs. For \(1 \leq p, q < \infty\), we denote by \(L^{p,q}(\mathbb{H}^n)\) the Lorentz space in \(\mathbb{H}^n\) and by \(\| \cdot \|_{p,q}\).
the Lorentz quasi-norm in $L^{p,q}(\mathbb{H}^n)$. When $p = q$, $\| \cdot \|_{p,p}$ is replaced by $\| \cdot \|_p$ the Lebesgue $L_p$–norm in $\mathbb{H}^n$, i.e., $\|f\|_p = (\int_{\mathbb{H}^n} |f|^p dV_g)^{\frac{1}{p}}$ for a measurable function $f$ on $\mathbb{H}^n$. The Lorentz–Sobolev space $W^{m,p,q}(\mathbb{H}^n)$ is defined as the completion of $C_0^\infty(\mathbb{H}^n)$ under the Lorentz quasi-norm $\|\nabla_g^m u\|_{p,q} := \|\nabla_g^m u\|_{p,q}$ in $W^{m,p,q}(\mathbb{H}^n)$. In [53, 54], the author proved the following Poincaré inequality in $W^1 L^{p,q}(\mathbb{H}^n)$

$$\|\nabla_g^m u\|_{p,q}^q \geq C(n,m,p)^q \|u\|_{p,q}^q, \quad \forall u \in W^{m,p,q}(\mathbb{H}^n).$$

provided $1 < q \leq p$ if $m$ is odd and for any $1 < q < \infty$ if $m$ is even, where

$$C(n,m,p) = \begin{cases} \left( \frac{(n-1)^2}{pp'} \right) \frac{n}{p} & \text{if } m \text{ is even}, \\ \left( \frac{p-1}{p} \right) \frac{n}{p} \left( \frac{n-1}{pp'} \right) \frac{m-1}{p'} & \text{if } m \text{ is odd}, \end{cases}$$

with $p' = p/(p-1)$. Furthermore, the constant $C(n,m,p)^q$ in (1.4) is the best possible and is never attained. The inequality (1.4) generalizes the result in [46] to the setting of Lorentz–Sobolev space.

The Moser–Trudinger inequality in the hyperbolic spaces was firstly proved by Mancini and Sandeep [41] in the dimension $n = 2$ (another proof of this result was given by Adimurthi and Tintarev [5]) and by Mancini, Sandeep and Tintarev [42] in higher dimension $n \geq 3$ (see [24] for an alternative proof)

$$\sup_{u \in W^{1, n}(\mathbb{H}^n), \int_{\mathbb{H}^n} |\nabla_g u|_2^2 dV_g \leq 1} \int_{\mathbb{H}^n} \Phi(\alpha_n |u|^{n-1}) dV_g < \infty,$$

where $\Phi(t) = e^t - \frac{n-2}{2} t^2$. Lu and Tang [35] also established the sharp singular Moser–Trudinger inequality under the conditions $\|\nabla u\|_{L^n(\mathbb{H}^n)} + \tau \|u\|_{L^n(\mathbb{H}^n)} \leq 1$ for any $\tau > 0$ (see Theorem 1.4 in [35]). In [48], the author improves the inequality (1.5) by proving the following inequality

$$\sup_{u \in W^{1, n}(\mathbb{H}^n), \int_{\mathbb{H}^n} |\nabla_g u|_2^2 dV_g - \lambda \int_{\mathbb{H}^n} |u|_2^2 dV_g \leq 1} \int_{\mathbb{H}^n} \Phi(\alpha_n |u|^{n-1}) dV_g < \infty,$$

for any $\lambda < \left( \frac{n-1}{n} \right)^n$. The Adams inequality in the hyperbolic spaces were proved by Karmakar and Sandeep [26] in the following form

$$\sup_{u \in C_0^\infty(\mathbb{H}^{2n}), \int_{\mathbb{H}^{2n}} P_n u u dV_g \leq 1} \int_{\mathbb{H}^{2n}} \left( e^{\alpha_{2n} u^2} - 1 \right) dV_g < \infty,$$

where $P_k$ is the GJMS operator on the hyperbolic spaces $\mathbb{H}^{2n}$, i.e., $P_1 = -\Delta_g - n(n-1)$ and

$$P_k = P_1(P_1 + 2) \cdots (P_1 + k(k - 1)), \quad k \geq 2.$$

In recent paper, Fontana and Morpurgo [23] established the following Adams inequality in the hyperbolic spaces $\mathbb{H}^n$,

$$\sup_{u \in W^{m,n}(\mathbb{H}^n), \int_{\mathbb{H}^n} |\nabla_g^m u|_2^2 dV_g \leq 1} \int_{\mathbb{H}^n} \Phi_{\frac{m}{2}}(\alpha_{m,n} |u|^{\frac{n}{m-2}}) dV_g < \infty$$

(1.7)
where
\[
\Phi_n(t) = e^t - \sum_{j=0}^{\infty} \frac{t^j}{j!}, \quad \text{and} \quad j_n = \min\{j : j \geq \frac{n}{m}\} \geq \frac{n}{m}.
\]

In [45], Ngo and the author proved several Adams type inequalities in the hyperbolic spaces.

To our knowledge, much less is known about the Trudinger–Moser inequality and Adams inequality under the Lorentz–Sobolev norm on complete noncompact Riemannian manifolds except Euclidean spaces. Recently, Yang and Li [62] proves a sharp Moser–Trudinger inequality in the Lorentz–Sobolev spaces defined in the hyperbolic spaces. More precisely, their result ([62, Theorem 1.6]) states that for \(1 < q < \infty\) it holds
\[
\sup_{u \in W^{1, L}((\mathbb{H}^n), \|\nabla u\|_{L^q})} \int_{\mathbb{H}^n} \Phi_n(q) |u|^{q-1} dV_g < \infty,
\]
where
\[
\Phi_n(q) = e^t - \sum_{j=0}^{\infty} \frac{t^j}{j!}, \quad \text{where} \quad j_n = \min\{j : j > 1 + a(q-1)/q\},
\]
with \(a > 1\).

The first aim in this paper is to establish the sharp Adams inequality in the hyperbolic spaces under the Lorentz–Sobolev norm which generalize the result of Yang and Li to higher order derivatives. Our fist result in this paper reads as follows.

**Theorem 1.1.** Let \(n > m \geq 2\) and \(q \in (1, \infty)\). Then it holds
\[
\sup_{u \in W^{m, L}((\mathbb{H}^n), \|\nabla^m u\|_{L^q})} \int_{\mathbb{H}^n} \Phi_{n,m} \left(\beta_{n,m}^q |u|^{\frac{q}{q-1}}\right) dV_g < \infty,
\]
for any \(q \in (1, \infty)\) if \(m\) is even, or \(1 < q \leq \frac{n}{m}\) if \(m\) is odd. Futhermore, the constant \(\beta_{n,m}^q\) is sharp in the sense that the supremum in (1.8) will become infinite if \(\beta_{n,m}^q\) is replaced by any larger constant.

Let us make some comments on Theorem 1.1. When \(q = \frac{n}{m}\), we obtain the inequality (1.7) of Fontana and Morpurgo from Theorem 1.1. However, our approach is completely different with the one of Fontana and Morpurgo. Notice that in the case that \(m\) is odd, we need an extra assumption Notice \(q \leq \frac{n}{m}\) comparing with case that \(m\) is even. This extra condition is a technical condition in our approach for which we can apply the Pólya–Szegö principle in the hyperbolic space (see Theorem 2.2 below). This principle was proved by the author in [53] which generalizes the classical Pólya–Szegö principle in Euclidean space to the hyperbolic space. Note that when \(m = 1\), the extra condition is not need by the result of Yang and Li [62]. The approach of Yang and Li is based on an representation formula for function via Green’s function of the Laplace-Beltrami \(-\Delta_g\) (similar with the
one of Fontana and Morpurgo [23]). Hence, we believe that the extra condition \( q \leq \frac{n}{m} \) is superfluous when \( m > 1 \) is odd. One reasonable approach is to follow the one of Fontana and Morpurgo by using the representation formulas and estimates in [23, Section 5]. This problem is left for interesting reader.

Next, we aim to improve the Lorentz–Adams inequality in Theorem 1.1 in spirit of (1.6). In the case \( m = 1 \), an analogue of (1.6) under Lorentz–Sobolev norm was obtained by the author in [53, Theorem 1.3]. The result for \( m > 1 \) is given in the following theorem.

**Theorem 1.2.** Let \( n > m \geq 2 \) and \( q \geq \frac{2n}{n-1} \). Suppose in addition that \( q \leq \frac{n}{m} \) if \( m \) is odd. Then we have

\[
\sup_{u \in W^{1,m}L^{\frac{n}{m},q}(\mathbb{B}^n), \|\nabla u\|_{\frac{n}{m},q} \leq \lambda \|u\|_{n,m}^{\frac{n}{m}}} \int_{\mathbb{B}^n} \Phi_{\frac{n}{m},q}(\beta_{n,m}^{\frac{n}{m}} |u|^{\frac{n}{m-1}}) \, dV_g < \infty. \tag{1.9}
\]

for any \( \lambda < C(n,m,\frac{n}{m})^q \).

Obviously, Theorem 1.2 is stronger than Theorem 1.1. The extra condition \( q \geq \frac{2n}{n-1} \) in Theorem 1.2 is to apply a crucial point-wise estimate in \( 1.1 \). Let \( n > m \geq 2 \), we have \( m = 1 \) and Morpurgo by using the representation formulas and estimates in [54] which we will recall in Section 8 below.

The Hardy–Moser–Trudinger inequality was proved by Wang and Ye (see [61]) in dimension 2

\[
\sup_{u \in W^{1,2}B^2, \int_{B^2} |\nabla u|^2 \, dx \leq 1} \int_{B^2} e^{4\pi u^2} \, dx < \infty. \tag{1.10}
\]

The inequality (1.10) is stronger than the classical Moser–Trudinger inequality in \( \mathbb{B}^2 \). It connects both the sharp Moser–Trudinger inequality in \( \mathbb{B}^2 \) and the sharp Hardy inequality in \( \mathbb{B}^2 \)

\[
\int_{B^2} |\nabla u|^2 \, dx \geq \int_{B^2} \frac{u^2}{(1-|x|^2)^2} \, dx, \quad u \in W^{1,2}_0(\mathbb{B}^2).
\]

The higher dimensional version of (1.10) was recently established by the author [52]

\[
\sup_{u \in W^{1,n}B^n, \int_{B^n} |\nabla u|^n \, dx \leq 1} \int_{B^n} e^{\alpha_n |u|^{\frac{n}{n-1}}} \, dx < \infty.
\]

For higher order derivatives, the sharp Hardy–Adams inequality was proved by Lu and Yang [37] in dimension 4 and by Li, Lu and Yang [31] in any even dimension. The approach in [31,37] relies heavily on the Hilbertian structure of the space \( W^{m,2}_0(B^n) \) with \( n \) even for which the Fourier analysis in the hyperbolic spaces can be applied. Our next motivation in this paper is to establish the sharp Hardy–Adams inequality in any dimension. Our next result reads as follows.

**Theorem 1.3.** Let \( m \geq 3 \), \( n \geq 2m+1 \) and \( q \geq \frac{2n}{n-1} \). Suppose in addition that \( q \leq \frac{n}{m} \) if \( m \) is odd. Then it holds

\[
\sup_{u \in W^{m,L^{\frac{n}{m},q}}(\mathbb{B}^n), \|\nabla u\|_{\frac{n}{m},q} \leq C(n,m,\frac{n}{m})^q \|u\|_{\frac{n}{m},q}^{\frac{n}{m}}} \int_{\mathbb{B}^n} \exp \left( \beta_{n,m}^{\frac{n}{m}} |u|^{\frac{n}{m-1}} \right) \, dx < \infty. \tag{1.11}
\]
Notice that the condition $m \geq 3$ is crucial in our approach. Indeed, under this condition we can make some estimates for $\|\nabla_g^m u\|_{L^q(q,n)} - C(n,m,\frac{n}{m})^q \|u\|_{L^q(q,n)}$ for which we can apply the results from Theorem 1.1 and Theorem 1.2. We do not know an analogue of (1.11) when $m = 2$. When $q = \frac{n}{m}$, we obtain the following Hardy–Adams inequality

$$\sup_{u \in W^{m,\frac{n}{m}}_0(\mathbb{H}^n), \int_{\mathbb{H}^n} |\nabla_g^n u|^\frac{n}{m} dV_g - C(n,m,\frac{n}{m})^\frac{n}{m} \int_{\mathbb{H}^n} |u|^\frac{n}{m} dV_g \leq 1} \int_{\mathbb{B}^n} \exp(\alpha_{n,m} |u|^\frac{n}{n-m}) dx < \infty.$$ 

The rest of this paper is organized as follows. In Section §2, we recall some facts on the hyperbolic spaces, the non-increasing rearrangement argument in the hyperbolic spaces and some important results from [54] which are used in the proof of Theorem 1.2 and Theorem 1.3. The proof of Theorem 1.1 is given in Section §3. Section §4 is devoted to prove Theorem 1.2. Finally, in Section §5 we provide the proof of Theorem 1.3.

2 Preliminaries

We start this section by briefly recalling some basis facts on the hyperbolic spaces and the Lorentz–Sobolev space defined in the hyperbolic spaces. Let $n \geq 2$, a hyperbolic space of dimension $n$ (denoted by $\mathbb{H}^n$) is a complete, simply connected Riemannian manifold having constant sectional curvature $-1$. There are several models for the hyperbolic space $\mathbb{H}^n$ such as the half-space model, the hyperboloid (or Lorentz) model and the Poincaré ball model. Notice that all these models are Riemannian isometry. In this paper, we are interested in the Poincaré ball model of the hyperbolic space since this model is very useful for questions involving rotational symmetry. In the Poincaré ball model, the hyperbolic space $\mathbb{H}^n$ is the open unit ball $B_n \subset \mathbb{R}^n$ equipped with the Riemannian metric

$$g(x) = \left(\frac{2}{1-|x|^2}\right)^2 dx \otimes dx.$$

The volume element of $\mathbb{H}^n$ with respect to the metric $g$ is given by

$$dV_g(x) = \left(\frac{2}{1-|x|^2}\right)^n dx,$$

where $dx$ is the usual Lebesgue measure in $\mathbb{R}^n$. For $x \in B_n$, let $d(0,x)$ denote the geodesic distance between $x$ and the origin, then we have $d(0,x) = \ln(1+|x|)/(1-|x|)$. For $\rho > 0$, $B(0,\rho)$ denote the geodesic ball with center at origin and radius $\rho$. If we denote by $\nabla$ and $\Delta$ the Euclidean gradient and Euclidean Laplacian, respectively as well as $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{R}^n$, then the hyperbolic gradient $\nabla_g$ and the Laplace–Beltrami operator $\Delta_g$ in $\mathbb{H}^n$ with respect to metric $g$ are given by

$$\nabla_g = \left(\frac{1-|x|^2}{2}\right)^2 \nabla, \quad \Delta_g = \left(\frac{1-|x|^2}{2}\right)^2 \Delta + (n-2)\left(\frac{1-|x|^2}{2}\right)\langle x, \nabla \rangle,$$
respectively. For a function $u$, we shall denote $\sqrt{g(\nabla_g u, \nabla_g u)}$ by $|\nabla_g u|_g$ for simplifying the notation. Finally, for a radial function $u$ (i.e., the function depends only on $d(0, x)$) we have the following polar coordinate formula

$$
\int_{\mathbb{H}^n} u(x) dx = n \sigma_n \int_0^\infty u(\rho) \sinh^{n-1}(\rho) d\rho.
$$

It is now known that the symmetrization argument works well in the setting of the hyperbolic. It is the key tool in the proof of several important inequalities such as the Poincaré inequality, the Sobolev inequality, the Moser–Trudinger inequality in $\mathbb{H}^n$. We shall see that this argument is also the key tool to establish the main results in the present paper. Let us recall some facts about the rearrangement argument in the hyperbolic space $\mathbb{H}^n$. A measurable function $u : \mathbb{H}^n \to \mathbb{R}$ is called vanishing at the infinity if for any $t > 0$ the set $\{|u| > t\}$ has finite $V_g$-measure, i.e.,

$$
V_g(\{|u| > t\}) = \int_{\{|u| > t\}} dV_g < \infty.
$$

For such a function $u$, its distribution function is defined by

$$
\mu_u(t) = V_g(\{|u| > t\}).
$$

Notice that $t \to \mu_u(t)$ is non-increasing and right-continuous. The non-increasing rearrangement function $u^*$ of $u$ is defined by

$$
u^*(t) = \sup\{s > 0 : \mu_u(s) > t\}.
$$

The non-increasing, spherical symmetry, rearrangement function $u^\sharp$ of $u$ is defined by

$$
u^\sharp(x) = u^*(V_g(B(0, d(0, x))))), \quad x \in \mathbb{H}^n.
$$

It is well-known that $u$ and $u^\sharp$ have the same non-increasing rearrangement function (which is $u^*$). Finally, the maximal function $u^{**}$ of $u^*$ is defined by

$$
u^{**}(t) = \frac{1}{t} \int_0^t u^*(s) ds.
$$

Evidently, $u^*(t) \leq u^{**}(t)$.

For $1 \leq p, q < \infty$, the Lorentz space $L^{p,q}(\mathbb{H}^n)$ is defined as the set of all measurable function $u : \mathbb{H}^n \to \mathbb{R}$ satisfying

$$
\|u\|_{L^{p,q}(\mathbb{H}^n)} := \left( \int_0^\infty \left( \frac{\int_0^t u^*(s) ds}{t} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.
$$

It is clear that $L^{p,p}(\mathbb{H}^n) = L^p(\mathbb{H}^n)$. Moreover, the Lorentz spaces are monotone with respect to second exponent, namely

$$
L^{p,q_1}(\mathbb{H}^n) \subset L^{p,q_2}(\mathbb{H}^n), \quad 1 \leq q_1 < q_2 < \infty.
$$
The functional \( u \to \| u \|_{L^p,q(\mathbb{H}^n)} \) is not a norm in \( L^p,q(\mathbb{H}^n) \) except the case \( q \leq p \) (see [10, Chapter 4, Theorem 4.3]). In general, it is a quasi-norm which turns out to be equivalent to the norm obtained replacing \( u^* \) by its maximal function \( u^{**} \) in the definition of \( \| \cdot \|_{L^p,q(\mathbb{H}^n)} \). Moreover, as a consequence of Hardy inequality, we have

**Proposition 2.1.** Given \( p \in (1, \infty) \) and \( q \in [1, \infty) \). Then for any function \( u \in L^p,q(\mathbb{H}^n) \) it holds

\[
\left( \int_0^\infty \left( t^{\frac{1}{p}} u^{**}(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq \frac{p}{p-1} \left( \int_0^\infty \left( t^{\frac{1}{p}} u^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \frac{p}{p-1} \| u \|_{L^p,q(\mathbb{H}^n)}. \tag{2.1}
\]

For \( 1 \leq p, q < \infty \) and an integer \( m \geq 1 \), we define the \( m \)-th order Lorentz–Sobolev space \( W^m L^p,q(\mathbb{H}^n) \) by taking the completion of \( C_0^\infty(\mathbb{H}^n) \) under the quasi-norm

\[ \| \nabla_g^m u \|_{p,q} := \| |\nabla_g^m u| \|_{p,q}. \]

It is obvious that \( W^m L^p,p(\mathbb{H}^n) = W^m(\mathbb{H}^n) \) the \( m \)-th order Sobolev space in \( \mathbb{H}^n \). In [53], the author established the following Pólya–Szegö principle in the first order Lorenz–Sobolev spaces \( W^1 L^p,q(\mathbb{H}^n) \) which generalizes the classical Pólya–Szegö principle in the hyperbolic space.

**Theorem 2.2.** Let \( n \geq 2, \ 1 \leq q \leq p < \infty \) and \( u \in W^1 L^p,q(\mathbb{H}^n) \). Then \( u^\# \in W^1 L^p,q(\mathbb{H}^n) \) and

\[ \| \nabla_g^1 u^\# \|_{p,q} \leq \| \nabla_g u \|_{p,q}. \]

For \( r \geq 0 \), define

\[ \Phi(r) = n \int_0^r \sinh^{n-1}(s)ds, \quad r \geq 0, \]

and let \( F \) be the function such that

\[ r = n\sigma_n \int_0^{F(r)} \sinh^{n-1}(s)ds, \quad r \geq 0, \]

i.e., \( F(r) = \Phi^{-1}(r/\sigma_n) \).

The following results was proved in [54] (see the Section §2).

**Proposition 2.3.** Let \( n \geq 2 \). Then it holds

\[ \sinh^n(F(t)) > \frac{t}{\sigma_n}, \quad t > 0. \tag{2.2} \]

Furthermore, the function

\[ \varphi(t) = \frac{t}{\sinh^{n-1}(F(t))} \]

is strictly increasing on \((0, \infty)\), and

\[ \lim_{t \to \infty} \varphi(t) = \frac{n\sigma_n}{n-1} > \frac{t}{\sinh^{n-1}(F(t))}, \quad t > 0. \tag{2.3} \]
It should be remark that under an extra condition $q \geq \frac{2n}{n-1}$, a stronger estimate which combines both (2.2) and (2.3) was established by the author in [53, Lemma 2.1] that

$$\sinh^{q(n-1)}(F(t)) \geq \left(\frac{t}{\sigma_n}\right)^{\frac{q-1}{n}} + \left(\frac{n-1}{n}\right)^q \left(\frac{t}{\sigma_n}\right)^q, \quad t > 0.$$ 

Let $u \in C_0^\infty(\mathbb{H}^n)$ and $f = -\Delta_g u$. It was proved by Ngo and the author (see [46, Proposition 2.2]) that

$$u^*(t) \leq v(t) := \int_0^\infty \frac{s f^{**}(s)}{(n\sigma_n \sinh^{n-1}(F(s)))^2} ds, \quad t > 0.$$ 

(2.4)

The following results which were proved in [53, 54] play the important role in the proof of our main results,

**Proposition 2.4.** Let $p \in (1, n)$ and $\frac{2n}{n-1} \leq q \leq p$. Then we have

$$\|\nabla_g u\|_{p,q}^q - \left(\frac{n-1}{p}\right)^q \|u\|_{p,q}^q \geq \left(\frac{n-2p}{pp'}\right)^q \sigma_n^{-\frac{2}{p'}} \|u\|_{p'^*,q}^q, \quad u \in C_0^\infty(\mathbb{H}^n).$$

(2.5)

where $p' = p/(p-1)$,

and

**Proposition 2.5.** Let $n \geq 2$, $p \in (1, n)$ and $q \in (1, \infty)$. If $p \in (1, \frac{n}{2})$ then it holds

$$\|\Delta_g u\|_{p,q}^q \geq \left(\frac{n(n-2p)}{pp'}\right)^q \sigma_n^{-\frac{2}{p'}} \|u\|_{p'^*,q}^q.$$ 

(2.6)

If $p \in (1, n)$ and $q \geq \frac{2n}{n-1}$ then we have

$$\|\Delta_g u\|_{p,q}^q - C(n, 2, p)^q \|u\|_{p,q}^q \geq \left(n^2 \sigma_n^{-\frac{2}{p'}}\right)^q \int_0^\infty \left|v'(t)\right|^q q^{q+\frac{n-2}{p}-1} dt.$$ 

(2.7)

Furthermore, if $p \in (1, \frac{n}{2})$ and $q \geq \frac{2n}{n-1}$ and $\frac{2n}{n-1} \leq q \leq p$ then we have

$$\|\Delta_g u\|_{p,q}^q - C(n, 2, p)^q \|u\|_{p,q}^q \geq \left(n(n-2p)\right)^q \sigma_n^{-\frac{2}{p'}} \|u\|_{p'^*,q}^q, \quad u \in C_0^\infty(\mathbb{H}^n).$$

(2.8)

Proposition 2.4 follows from [53, Theorem 1.2] while Proposition 2.5 follows from Theorem 2.8 in [54].
3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The main point is the proof of the case $m = 2$. For the case $m \geq 3$, the proof is based on the iteration argument by using the inequalities (2.6) and (3.10) below.

Proof of Theorem 1.1. We divide the proof of (1.8) into three following cases:

Case 1: $m = 2$. It is enough to consider $u \in C_0^\infty(\mathbb{H}^n)$ with $\|\Delta_g u\|_{p,q} \leq 1$. Denote $f = -\Delta_g u$ and define $v$ by (2.4), then we have $u^* \leq v$. By [54, Theorem 1.1], we have $\|u\|_{p,q} \leq C$. Here and in the sequel, we denote by $C$ a generic constant which does not depend on $u$ and whose value maybe changes on each line. For any $t > 0$, we have

$$
\frac{n}{2q} u^*(t)^{\frac{q}{2}} \leq \int_0^t u^*(s)^{\frac{q}{2}} s^{2-n} ds \leq \|u\|_{p,q} \leq C,
$$

which yields $u^*(t) \leq Ct^{-\frac{2}{n}}$, $t > 0$. Therefore, it is not hard to see that

$$
\Phi_{p,q}(\beta_{\frac{q}{2},2} u^*(t)^{\frac{q}{2}}) \leq Cu^*(t)^{\frac{q}{2}} v(t)^{\frac{q}{2}} \leq Ct^{-\frac{q}{n} + (\frac{2}{n} - 1)} \leq Ct^{-\frac{q}{n} + (\frac{2}{n} - 1)}, \quad \forall t \geq 1.
$$

By the choice of $j_{\frac{q}{2},q}$, we then have

$$
\int_1^\infty \Phi_{p,q}(\beta_{\frac{q}{2},2} u^*(t)^{\frac{q}{2}}) dt \leq C. \quad (3.1)
$$

On the other hand, we have

$$
\int_0^1 \Phi_{p,q}(\beta_{\frac{q}{2},2} u^*(t)^{\frac{q}{2}}) dt \leq \int_0^1 \exp \left( \beta_{\frac{q}{2},2} u^*(t)^{\frac{q}{2}} \right) dt
\leq \int_0^1 \exp \left( \beta_{\frac{q}{2},2} v(t)^{\frac{q}{2}} \right) dt
= \int_0^\infty \exp \left( -t + \beta_{\frac{q}{2},2} v(e^{-t})^{\frac{q}{2}} \right) dt. \quad (3.2)
$$

Notice that

$$
v(e^{-t}) = \int_{e^{-t}}^{\infty} \frac{r}{(n\sigma_n \sinh^{n-1}(F(r)))^2} f^{**}(r) dr = \int_{-\infty}^t \frac{e^{-2(1-\frac{\lambda}{2})s}}{(n\sigma_n \sinh^{n-1}(F(e^{-s})))^2} e^{-\frac{2}{n} s} f^{**}(e^{-s}) ds.
$$

Denote

$$
\phi(s) = \frac{n-2}{n} e^{-\frac{2}{n} s} f^{**}(e^{-s}),
$$

we then have

$$
\int_\mathbb{R} \phi(s)^q ds = \left( \frac{n-2}{n} \right)^q \int_0^\infty (f^{**}(t)^{\frac{q}{2}})^q \frac{dt}{t} \leq 1, \quad (3.3)
$$

11
here we used the Hardy inequality (2.1) and $\|\Delta u\|_{L^{\frac{p}{2}}(\mathbb{H}^n)} \leq 1$. Define the function

$$a(s, t) = \begin{cases} \beta_{n, 2} \frac{n}{n - 2} \frac{e^{-2(1 - \frac{1}{q}) s}}{(n \sigma_n \sinh^{n-1}(F(e^{-s})))^2} & \text{if } s \leq t, \\ 0 & \text{if } s > t. \end{cases}$$

Using the inequality $\sigma_n \sinh^n(F(r)) \geq r$, we have for $0 \leq s \leq t$

$$a(s, t) \leq \beta_{n, 2} \frac{1}{n(n - 2) \sigma_n^n} = 1. \quad (3.4)$$

Moreover, for $t > 0$ we have

$$\int_{-\infty}^{0} a(s, t) q' ds + \int_{t}^{\infty} a(s, t) q' ds = \beta_{n, 2}^q \left( \frac{n}{n - 2} \right)^q \int_{-\infty}^{0} e^{-(1 - \frac{1}{n}) s} \left( n \sigma_n \sinh^{n-1}(F(e^{-s})) \right)^2 ds$$

$$\leq \beta_{n, 2}^q \left( \frac{n}{n - 2} \right)^q (n - 1)^{-2q} \int_{-\infty}^{0} e^{\frac{2}{q'} q'} ds$$

$$= \beta_{n, 2}^q \left( \frac{n}{n - 2} \right)^q (n - 1)^{-2q} \frac{n}{2q'}$$

here we used $n \sigma_n \sinh^{n-1}(F(r)) \geq (n - 1)r$. Hence

$$\sup_{t > 0} \left( \int_{-\infty}^{0} a(s, t) q' ds + \int_{t}^{\infty} a(s, t) q' ds \right)^{\frac{1}{q'}} \leq \left( \beta_{n, 2}^q \left( \frac{n}{n - 2} \right)^q (n - 1)^{-2q} \frac{n}{2q'} \right)^{\frac{1}{q'}}. \quad (3.5)$$

Notice that

$$\beta_{n, 2} v(e^{-t}) \leq \int_{\mathbb{R}} a(s, t) \phi(s) ds. \quad (3.6)$$

With (3.2), (3.3), (3.4), (3.5) and (3.6) at hand, we can apply Adams’ Lemma [2] to obtain

$$\int_{0}^{1} \Phi_{\frac{n}{q}}(\beta_{n, 2}^q u(t) \frac{e^{-t}}{q}) dt \leq \int_{0}^{\infty} e^{-t + \beta_{n, 2}^q v(t) q'} dt \leq C. \quad (3.7)$$

Combining (3.1) and (3.7) together, we arrive

$$\int_{\mathbb{R}^n} \Phi_{\frac{n}{q}}(\beta_{n, 2}^q |u|^q) dx = \int_{0}^{\infty} \Phi_{\frac{n}{q}}(\beta_{n, 2}^q (u^*)(t)) dt \leq C,$$

for any $u \in W^2L^{\frac{p}{2}}(\mathbb{H}^n)$ with $\|\Delta u\|_{L^{\frac{p}{2}}(\mathbb{H}^n)} \leq 1$. This proves (1.8) for $m = 2$.

Case 2: $m = 2k$, $k \geq 2$. To obtain the result in this case, we apply the iteration argument. Firstly, by iterating the inequality (2.6), we have that for $k \geq 1$, $q \in (1, \infty)$ and $p \in (1, \frac{4}{3})$

$$\|\Delta_g^k u\|_{p, q}^q \geq S(n, 2k, p)^q \|u\|_{p_{2k}, q}^q.$$
Hence, if \( u \in W^{2k}L^{\frac{qn}{2k}}(\mathbb{H}^n) \) with \( \|\Delta_{g}^{k}u\|_{\frac{qn}{2k}} \leq 1 \), then we have
\[
S(n,2(k-1),\frac{n}{2k})\|\Delta_{g}u\|_{\frac{n}{2k}} \leq 1.
\]
Define \( w = S(n,2(k-1),\frac{n}{2k})u \), then \( \|w\|_{\frac{n}{2k}} \leq 1 \). Using the result in the Case 1 with remark that
\[\beta_{n,2k} = \beta_{n,2}S(n,2(k-1),\frac{n}{2k}),\]
we obtain
\[
\int_{\mathbb{H}^n} \Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u|t|^{q})dV_{g} \leq C. \tag{3.8}
\]
By the Lorentz–Poincaré inequality (1.4), we have \( \|u\|_{\frac{qn}{2k}} \leq 1 \). Similarly in the Case 1, we get \( u^{*}(t) \leq Ct^{-\frac{2k}{q}}, \ t > 0 \). Hence, for \( t \geq 1 \), it holds
\[
\Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u^{*}(t)|^{q}) \leq C(u^{*}(t))^{q(j^{\frac{n}{2k}}-1)} \leq Ct^{-\frac{2k}{q}q(j^{\frac{n}{2k}}-1)},
\]
which implies
\[
\int_{1}^{\infty} \Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u^{*}(t)|^{q})dt \leq C \tag{3.9}
\]
by the choice of \( j_{\frac{n}{2k},q} \). Since
\[
\lim_{t \to \infty} \frac{\Phi_{\frac{n}{2k}}(t)}{\Phi_{\frac{n}{2k}}(t)} = 1,
\]
then there exists \( A \) such that \( \Phi_{\frac{n}{2k}}(t) \leq 2\Phi_{\frac{n}{2k}}(t) \) for \( t \geq A \). Hence, we have
\[
\int_{0}^{1} \Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u^{*}(t)|^{q})dt = \int_{\{t \in (0,1):u^{*}(t) < A^{j^{\frac{n}{2k}}-1}\beta_{n,2k}^{-1}\}} \Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u^{*}(t)|^{q})dt
\]
\[
\quad + \int_{\{t \in (0,1):u^{*}(t) \geq A^{j^{\frac{n}{2k}}-1}\beta_{n,2k}^{-1}\}} \Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u^{*}(t)|^{q})dt
\]
\[
\leq C + 2 \int_{\{t \in (0,1):u^{*}(t) \geq A^{j^{\frac{n}{2k}}-1}\beta_{n,2k}^{-1}\}} \Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u^{*}(t)|^{q})dt
\]
\[
\leq C + \int_{0}^{1} \Phi_{\frac{n}{2k}}(\beta_{n,2k}^{j}u^{*}(t)|^{q})dt
\]
here we have used (3.8). Combining the previous inequality together with (3.9) proves the result in this case.

**Case 3:** \( m = 2k + 1, \ k \geq 1 \). Let \( f = -\Delta_{g}^{k}u \). Since \( q \leq \frac{n}{2k+1} \), then it was proved in [53] (the formula after (2.8) with \( u \) replaced by \( f \) ) that
\[
\|\nabla_{g}^{m}u\|_{\frac{n}{2k+1}}^{\frac{n}{2k+1}} = \|\nabla_{g}f\|_{\frac{n}{2k+1}}^{\frac{n}{2k+1}} \geq \int_{0}^{\infty} |(f^{*})'(t)|^{q(n\sigma_{n}\sinh^{n-1}(F(t)))^{(2k+1)n-1}dt.
\]
Using (2.2), we have

$$
\|\nabla_g^m u\|_{2k+q}^q \geq n^q \sigma_n^q \int_0^\infty |f^*(t)|^q t^{\frac{n+q-1}{n}} dt.
$$

Applying the one-dimensional Hardy inequality, it holds

$$
\|\nabla_g^m u\|_{2k+q}^q \geq (2k)^q \sigma_n^q \int_0^\infty |f^*(t)|^q t^{\frac{n+q-1}{n}} dt = (2k)^q \sigma_n^q \|\Delta_g u\|_{2k+q}^q.
$$

(3.10)

For any \( u \in W^{2k+1} L^{\frac{n}{2k+q}}(\mathbb{H}^n) \) with \( \|\nabla_g^m u\|_{2k+q} \leq 1 \), define \( w = 2k \sigma_n^\frac{1}{q} u \). By (3.10), we have \( \|w\|_{\frac{n}{2k+q}} \leq 1 \). Using the result in the Case 2 with remark that

$$
\beta_{n,2k+1} = 2k \sigma_n^\frac{1}{q} \beta_{n,2k},
$$

we obtain

$$
\int_{\mathbb{H}^n} \Phi_{\frac{n}{2k+q}}(\beta_{n,2k+1}^q |u|^q) dV_g \leq C.
$$

(3.11)

Using (3.11) together with the last arguments in the proof of the Case 2 proves the result in this case.

It remains to check the sharpness of constant \( \beta_{n,m}^{\frac{n}{q}} \). To do this, we construct a sequence of test functions as follows

$$
v_j(x) = \begin{cases} 
\frac{(\ln j)^{1/q}}{\beta_{n,m}} + \frac{n \beta_{n,m}}{2(\ln j)^{1/q}} \sum_{i=1}^{m-1} \frac{(1-j^\frac{1}{q} |x|^2)^i}{i!} & \text{if } 0 \leq |x| \leq j^{-\frac{1}{q}}, \\
-j^\frac{1}{q} \ln |x| & \text{if } j^{-\frac{1}{q}} < |x| \leq 1, \quad j \geq 2 \\
\xi_j(x) & \text{if } 1 < |x| < 2,
\end{cases}
$$

where \( \xi \in C_0^\infty(2\mathbb{B}^n) \) are radial function chosen such that \( \xi_j = 0 \) on \( \partial \mathbb{B}^n \) and for \( i = 1, \ldots, m-1 \)

$$
\frac{\partial^i \xi_j}{\partial r^i} |_{\partial \mathbb{B}^n} = (-1)^i (i-1)! n \beta_{n,m}^{-1} (\ln j)^{-1/q},
$$

and \( \xi_j, |\nabla^i \xi_j| \) and \( |\nabla^m \xi_j| \) are all \( O((\ln j)^{-1/q}) \) as \( j \to \infty \). For \( \epsilon \in (0,1/3) \) let us define \( u_{\epsilon,j}(x) = v_j(x/\epsilon) \). Then \( u_{\epsilon,j} \in W^m L^{\frac{n}{2k+q}}(\mathbb{H}^n) \) has support contained in \( \{|x| \leq 2\epsilon\} \). It is easy to check that

$$
|\nabla_g^m u_{\epsilon,j}(x)| \leq \left( 1 - \frac{|x|^2}{2} \right)^m C(\epsilon^{-1} j^{\frac{1}{q}})^m (\ln j)^{-1/q} \leq C2^{-m} (\epsilon^{-1} j^{\frac{1}{q}})^m (\ln j)^{-1/q}
$$

for \( |x| \leq \epsilon j^{-\frac{1}{q}} \), and

$$
|\nabla_g^m u_{\epsilon,j}(x)| \leq C \epsilon^{-m} (\ln j)^{-\frac{1}{q}} \left( 1 - \frac{|x|^2}{2} \right)^m \leq C2^{-m} \epsilon^{-m} (\ln j)^{-\frac{1}{q}}
$$

for \( |x| \geq \epsilon j^{-\frac{1}{q}} \).
for $|x| \in (\epsilon, 2\epsilon)$ with a positive constant $C$ independent of $\epsilon < \frac{1}{3}$ and $j$. Furthermore, we can check that

$$|
abla g^m u_{\epsilon,j}(x)| \leq \left(\frac{1 - |x|^2}{2}\right)^m \left(||x|^n \sigma_n \right)^{-\frac{m}{n}} + C|x|^{-m+1}) (\ln j)^{-\frac{1}{q}}$$

$$\leq 2^{-m} (\ln j)^{-\frac{1}{q}} \left(||x|^n \sigma_n \right)^{-\frac{m}{n}} + C|x|^{-m+1})$$

and

$$|
abla g^m u_{\epsilon,j}(x)| \geq \left(\frac{1 - |x|^2}{2}\right)^m \left(||x|^n \sigma_n \right)^{-\frac{m}{n}} - C|x|^{-m+1}) (\ln j)^{-\frac{1}{q}}$$

$$\geq \left(\frac{1 - \epsilon^2}{2}\right)^m (\ln j)^{-\frac{1}{q}} \left(||x|^n \sigma_n \right)^{-\frac{m}{n}} - C|x|^{-m+1})$$

for $|x| \in (\epsilon^{-\frac{1}{q}}, \epsilon)$ with $\epsilon > 0$ small enough where $C$ is a positive constant independent of $\epsilon$ and $j$. Define

$$h_1(x) = \begin{cases} C2^{-m}(\epsilon^{-1}j^{\frac{1}{q}})^m (\ln j)^{-\frac{1}{q}} & \text{if } |x| \leq \epsilon^{-\frac{1}{q}} \\ 2^{-m}(\ln j)^{-\frac{1}{q}} \left(||x|^n \sigma_n \right)^{-\frac{m}{n}} + C|x|^{-m+1}) & \text{if } |x| \in (\epsilon^{-\frac{1}{q}}, \epsilon) \\ C2^{-m}e^{-m}(\ln j)^{-\frac{1}{q}} & \text{if } |x| \in (\epsilon, 2\epsilon) \\ 0 & \text{if } |x| \in (2\epsilon, 1), \end{cases}$$

Then we have $0 \leq |
abla g^m u| \leq h_1$. Consequently, we get $0 \leq |
abla g^m u|^* \leq h_1^*$. Let us denote by $h_1^{*,e}$ the rearrangement function of $h_1$ with respect to Lebesgue measure. Since the support of $h_1$ is contained in $\epsilon\{|x| \leq \epsilon\}$, then we can easy check that

$$h_1^*(t) \leq h_1^{*,e}\left(\left(\frac{1 - \epsilon^2}{2}\right)^n t\right).$$

Consequently, we have

$$\|\nabla g^m u_{\epsilon,j}\|_q^q \geq \left(\frac{2}{1 - \epsilon^2}\right)^m \int_0^\infty h_1^{*,e}(t)^q t^{\frac{m}{n} - 1} dt$$

Notice that by enlarging the constant $C$ (which is still independent of $\epsilon$ and $j$), we can assume that

$$C2^{-m}e^{-m}(\ln j)^{-\frac{1}{q}} \geq h_1\Big|_{|x| = \epsilon} = 2^{-m}(\ln j)^{-\frac{1}{q}} \epsilon^{-m} \left(\sigma_n^{-\frac{m}{n}} + C\epsilon\right)$$

for $\epsilon > 0$ small enough. For $j$ larger enough, we can chose $x_0$ with $\epsilon j^{-\frac{1}{q}} < |x_0| \leq \epsilon$ such that $C2^{-m}e^{-m}(\ln j)^{-\frac{1}{q}} = h_1(x_0)$. It is easy to see that $\epsilon \sigma \leq |x_0| \leq C\epsilon$ for constant $C, c > 0$ independent of $\epsilon$ and $j$. We have

$$h_1(x) \leq g(x) := \begin{cases} h_1(x) & \text{if } |x| \leq |x_0| \\ C2^{-m}e^{-m}(\ln j)^{-\frac{1}{q}} & \text{if } |x| \in (|x_0|, 2\epsilon) \\ 0 & \text{if } |x| \geq 2\epsilon. \end{cases}$$
Notice that $g$ is non-increasing radially symmetric function in $\mathbb{B}^n$, hence $g^{t,e} = g$. Using the function $g$, we can prove that

$$\int_0^\infty h_1^{t,e}(t)q \frac{m}{m-1}dt \leq 2^{-mq}(1 + C(ln j)^{-1}).$$

Therefore, we have

$$\|\nabla_g^m u_{e,j}\|_{q,m,q}^q \leq \left(\frac{1}{1 - \epsilon^2}\right)^{mq}(1 + C(ln j)^{-1})$$

Set $w_{e,j} = u_{e,j}/\|\nabla_g^m u_{e,j}\|_{m,q}$. For any $\beta > \beta_{n,m}^{q'}$, we choose $\epsilon > 0$ small enough such that $\gamma := \beta(1 - \epsilon^2)^{\frac{m}{m-1}} > \beta_{n,m}^{q'}$. Then we have

$$\int_{\mathbb{H}^n} \Phi_m q(\beta |w_{e,j}|^{q'})dV_g \geq \int_{\{|x| \leq \epsilon j^{-\frac{1}{m}}\}} \Phi_m q\left(\frac{\gamma}{(1 + C(ln j)^{-1})^{\frac{q'}{m-1}}} |u_{e,j}|^{q'}\right)dV_g$$

$$\geq 2^n \int_{\{|x| \leq \epsilon j^{-\frac{1}{m}}\}} \Phi_m q\left(\frac{\gamma}{(1 + C(ln j)^{-1})^{\frac{q'}{m-1}}} |u_{e,j}|^{q'}\right)dx$$

$$= 2^n \epsilon^n \int_{\{|x| \leq \epsilon j^{-\frac{1}{m}}\}} \Phi_m q\left(\frac{\gamma}{(1 + C(ln j)^{-1})^{\frac{q'}{m-1}}} |v_{e,j}|^{q'}\right)dx$$

$$\geq 2^n \epsilon^n \int_{\{|x| \leq \epsilon j^{-\frac{1}{m}}\}} \Phi_m q\left(\frac{\gamma}{(\beta_{n,m}^{q'} (1 + C(ln j)^{-1})^{\frac{q'}{m-1}})\ln j} \right)dx$$

$$= 2^n \epsilon^n \sigma_n \Phi_m q\left(\frac{\gamma}{(\beta_{n,m}^{q'} (1 + C(ln j)^{-1})^{\frac{q'}{m-1}})\ln j} \right)e^{-ln j}.$$  

Since

$$\lim_{j \to \infty} \frac{\gamma}{\beta_{n,m}^{q'} (1 + C(ln j)^{-1})^{\frac{q'}{m-1}}} = \infty,$$

then

$$\Phi_m q\left(\frac{\gamma}{\beta_{n,m}^{q'} (1 + C(ln j)^{-1})^{\frac{q'}{m-1}}} \right) \geq C e^{\frac{\gamma}{\beta_{n,m}^{q'} (1 + C(ln j)^{-1})^{\frac{q'}{m-1}}}\ln j}$$

for $j$ larger enough. Consequently, we get

$$\int_{\mathbb{H}^n} \Phi_m q(\beta |w_{e,j}|^{q'})dV_g \geq 2^n \epsilon^n \sigma_n C e^{\frac{\gamma}{\beta_{n,m}^{q'} (1 + C(ln j)^{-1})^{\frac{q'}{m-1}}}\ln j} \rightarrow \infty$$

as $j \to \infty$ since $\gamma > \beta_{n,m}^{q'}$. This proves the sharpness of $\beta_{n,m}^{q'}$.

The proof of Theorem 1.1 is then completely finished. \[\square\]
4 Proof of Theorem 1.2

This section is devoted to prove Theorem 1.2. The proof is based on the inequalities (2.5) and (2.8), the iteration argument and Theorem 1.2 for \( m \geq 3 \). The case \( m = 2 \) is proved by using inequality (2.7) and the Moser–Trudinger inequality involving the fractional dimension in Lemma 4.1 below. Let \( \theta > 1 \), we denote by \( \lambda_\theta \) the measure on \( [0, \infty) \) of density

\[
d\lambda_\theta = \theta \sigma_\theta x^{\theta - 1} dx, \quad \sigma_\theta = \frac{\pi^{\frac{\theta}{2}}}{\Gamma(\frac{\theta}{2} + 1)}.
\]

For \( 0 < R \leq \infty \) and \( 1 \leq p < \infty \), we denote by \( L^p_\theta(0, R) \) the weighted Lebesgue space of all measurable functions \( u : (0, R) \to \mathbb{R} \) for which

\[
\| u \|_{L^p_\theta(0, R)} = \left( \int_0^R |u|^p d\lambda_\theta \right)^{\frac{1}{p}} < \infty.
\]

Besides, we define

\[
W^{1, p}_\alpha, \theta(0, R) = \left\{ u \in L^p_\theta(0, R), \| u' \|_{L^p_\alpha(0, R)}, \lim_{x \to R^-} u(x) = 0 \right\}, \quad \alpha, \theta > 1.
\]

In [17], de Oliveira and do Ó prove the following sharp Moser–Trudinger inequality involving the measure \( \lambda_\theta \): suppose \( 0 < R < \infty \) and \( \alpha, \theta \geq 2 \), then

\[
D_{\alpha, \theta}(R) := \sup_{u \in W^{1,\alpha}_\alpha(0, R), \| u' \|_{L^\alpha_\theta(0, R)} \leq 1} \int_0^R e^{\mu_{\alpha, \theta}|u|^\frac{\alpha}{\alpha - 1}} d\lambda_\theta < \infty
\]

where \( \mu_{\alpha, \theta} = \theta \alpha^{\frac{1}{\alpha - 1}} \sigma_\alpha^{-\frac{1}{\alpha - 1}} \). Denote \( D_{\alpha, \theta} = D_{\alpha, \theta}(1) \). It is easy to see that \( D_{\alpha, \theta}(R) = D_{\alpha, \theta} R^\theta \).

Lemma 4.1. Let \( \alpha > 1 \) and \( q \geq 2 \). There exists a constant \( C_{\alpha, q} > 0 \) such that for any \( u \in W^{1, q}_{\alpha, q}(0, \infty) \), \( u' \leq 0 \) and \( \| u \|_{L^q_\alpha(0, \infty)} + \| u' \|_{L^q_\alpha(0, \infty)} \leq 1 \), it holds

\[
\int_0^\infty \Phi_{\alpha, q, \theta}(\mu_{\alpha, q}, 1)|u|^\frac{\alpha}{\alpha - 1} d\lambda_\theta \leq C_{\alpha, q}.
\]

Proof. We follows the argument in [57]. Since \( u' \leq 0 \) then \( u \) is a non-increasing function. Hence, for any \( t > 0 \), it holds

\[
u(r)^q \leq \frac{1}{\sigma_\alpha r^\alpha} \int_0^r \nu(s)^q d\lambda_\alpha \leq \frac{\int_0^\infty \nu(s)^q d\lambda_\alpha}{\sigma_\alpha r^\alpha} \leq \frac{\| u \|_{L^q_\alpha(0, \infty)}^q}{\sigma_\alpha r^\alpha}.
\]

For \( R > 0 \), define \( w(r) = u(r) - u(R) \) for \( r \leq R \) and \( w(r) = 0 \) for \( r > R \). Then \( w \in W^{1, \frac{q}{\alpha}}(0, R) \) and

\[
\| w \|_{L^q_\alpha(0, R)}^q = \int_0^R |w(s)|^q d\lambda_\theta \leq 1 - \| u \|_{L^q_\alpha(0, \infty)}^q.
\]
For $r \leq R$, we have $u(r) = w(r) + u(R)$. Since $q \geq 2$, then there exists $C > 0$ depending only on $q$ such that

$$u(r)^q \leq w(r)^q + C w(r)^{q-1} u(R) + u(R)^q.$$

Applying Young’s inequality and (4.3), we get

$$u(r)^q \leq w(r)^q \left( 1 + \frac{C}{q} u(R)^q \right) + \frac{q-1}{q} + u(R)^q$$

$$\leq w(r)^q \left( 1 + \frac{C}{q\sigma R^\alpha} \right) + \frac{q-1}{q} + \left( \frac{1}{\sigma R^\alpha} \right)^{q-1}. \quad (4.5)$$

Fix a $R \geq 1$ large enough such that $\frac{C}{q\sigma R^\alpha} \leq 1$, and set

$$v(r) = w(r) \left( 1 + \frac{C}{q\sigma R^\alpha} \right)^{\frac{q}{q-1}}.$$

Using (4.4) and the choice of $R$, we can easily verify that $\|v\|_{L^q(0, R)} \leq 1$. Hence, applying (4.1), we get

$$\int_0^R e^{\mu_1 |u|^q} d\lambda_1 \leq D_{q,1} R. \quad (4.6)$$

For $r \geq R$, we have $u(r) \leq \sigma R^{-\frac{q}{q-1}}$, hence it holds

$$\Phi_{q,\alpha}(\mu q, 1 |u(r)|^q) \leq C |u(r)|^q \sigma^{-\frac{q}{q-1}} \leq Q R^{-\frac{q}{q-1}}.$$

By the choice of $j_{\alpha,q}$, we have

$$\int_R^\infty \Phi_{q,\alpha}(\mu q, 1 |u(r)|^q) d\lambda_1 \leq C. \quad (4.7)$$

Putting (4.5), (4.6), (4.7) together and using $R \geq 1$, we get

$$\int_0^R \Phi_{q,\alpha}(\mu q, 1 |u|^q) d\lambda_1 \leq \int_0^R \Phi_{q,\alpha}(\mu q, 1 |u|^q) d\lambda_1 + \int_R^\infty \Phi_{q,\alpha}(\mu q, 1 |u|^q) d\lambda_1$$

$$\leq \int_0^R \exp \left( \mu q, 1 |u|^q \right) d\lambda_1 + C$$

$$\leq \int_0^R \exp \left( \mu q, 1 |u|^q + \mu q, 1 \left( \frac{q-1}{q} + \sigma R^{-\frac{q}{q-1}} \right) \right) d\lambda_1 + C$$

$$\leq \exp \left( \mu q, 1 \left( \frac{q-1}{q} + \sigma R^{-\frac{q}{q-1}} \right) \right) D_{q,1} R + C$$

$$\leq C.$$
For any $\tau > 0$ and $u \in W^{1,q}_{q,q}(0,\infty)$, such that $u' \leq 0$ and $\tau \|u\|_{L^q(0,\infty)} + \|u'\|_{L^q(0,\infty)} \leq 1$. Applying (4.2) for function $u_\tau(x) = u(\tau^{-\frac{1}{q}}x)$ and making the change of variables, we obtain

$$\int_0^\infty \Phi_{q,q}(\mu_{q,1}|u'|)d\lambda_1 \leq C\tau^{-\frac{1}{q}}. \tag{4.8}$$

We are now ready to give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We divide the proof into the following cases.

**Case 1:** $m = 2$. Let $u \in C_0^\infty(\mathbb{H}^n)$ with $\|\Delta_g u\|_{L^q}^q - \lambda \|u\|_{L^q}^q \leq 1$. Define $v$ by (2.4) and $\tilde{v}(x) = v(V_q(B(0,d(0,x))))$, then $u^* \leq v$, $\|\Delta_g u\|_{L^q}^q = \|\Delta_g \tilde{v}\|_{L^q}^q$ and $\|u\|_{L^q}^q \leq \|\tilde{v}\|_{L^q}$. So, we have

$$\|\Delta_g \tilde{v}\|_{L^q}^q - \lambda \|\tilde{v}\|_{L^q}^q \leq 1.$$  

We show that $\int_{\mathbb{H}^n} \Phi_{q,q}(\beta_{n,2}|\tilde{v}'|)dV_g \leq C$. Set $\kappa = C(n,2,n/2)^q - \lambda > 0$. Applying the inequality (2.7) for $\tilde{v}$, we get

$$\left( n(n-2)\sigma_n^{\frac{1}{q}} \right)^q \int_0^\infty |v'(t)|^q t^{q-1}dt + \kappa \int_0^\infty v(t)^q t^{2\frac{q}{n}-1}dt \leq 1.$$  

Define

$$w = \frac{n(n-2)\sigma_n^{\frac{1}{q}}}{q\sigma_q^{\frac{1}{q}}}, \quad \tau = \frac{q\sigma_q}{(n(n-2)\sigma_n^{\frac{1}{q}})^{2q} n^{2q} \sigma_n^{2q}},$$

then, we have

$$\int_0^\infty |w'|^q d\lambda_q + \tau \int_0^\infty |w|^q d\lambda_{2q} \leq 1.$$  

Applying the inequality (4.8), we obtain

$$\int_0^\infty \Phi_{q,q}(\mu_{q,1}w)\frac{w^{\frac{q}{n}-1}}{\tau}d\lambda_1 \leq C_{2q,q} \tau^{-\frac{q}{n}}.$$  

Notice that

$$\int_{\mathbb{H}^n} \Phi_{q,q}(\beta_{n,2}|\tilde{v}'|)dV_g = \frac{1}{2} \int_0^\infty \Phi_{q,q}(\beta_{n,2}|\tilde{v}'|)d\lambda_1 = \frac{1}{2} \int_0^\infty \Phi_{q,q}(\mu_{q,1}w)\frac{w^{\frac{q}{n}-1}}{\tau}d\lambda_1.$$  

Hence, it holds

$$\int_{\mathbb{H}^n} \Phi_{q,q}(\beta_{n,2}|\tilde{v}'|)dV_g \leq \frac{1}{2} C_{2q,q} \tau^{-\frac{q}{n}}.$$  

This completes the proof of this case.

**Case 2:** $m = 2k$, $k \geq 2$. Denote $\tau = C(n,2k,\frac{n}{2k})^q - \lambda > 0$. We have

$$1 \geq \|\Delta_g^k u\|_{L^q}^q - \lambda \|u\|_{L^q}^q \geq \tau \|u\|_{L^q}^q.$$
which yields
\[ \|u\|_{\frac{n}{2k+1},q}^q \leq \tau^{-1}. \] (4.9)

On the other hand, by the Lorentz–Poincaré inequality (1.4) and the Poincaré–Sobolev
inequality under Lorentz–Sobolev norm (2.8), we have
\[
\|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q - \lambda \|u\|_{\frac{n}{2k+1},q}^q \geq \|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q - C(n, 2k, \frac{n}{2k}) \|u\|_{\frac{n}{2k+1},q}^q + \tau \|u\|_{\frac{n}{2k+1},q}^q
\]
\[
\geq \|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q - C(n, 2, \frac{n}{2k}) \|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q + \tau \|u\|_{\frac{n}{2k+1},q}^q
\]
\[
\geq (2(k-1))(n-2k)\sigma^2_n \beta_n^k u \|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q + \tau \|u\|_{\frac{n}{2k+1},q}^q.
\]

Set \( w = 2(k-1)(n-2k)\beta_n^k u \) we have \( \|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q \leq 1 \). Applying the Adams inequality (1.8), we obtain
\[
\int_{\mathbb{H}^n} \Phi_{n, 2(k-1), q}(\beta_n^k u \|u\|_{\frac{n}{2k+1},q}^q) dV_g = \int_{\mathbb{H}^n} \Phi_{n, 2(k-1), q}(\beta_n^k u \|u\|_{\frac{n}{2k+1},q}^q) \leq C,
\]
here we use
\[ \beta_{n, 2k} = 2(k-1)(n-2k)\sigma_n^2 \beta_{n, 2(k-1)}. \]

Using (4.9) and repeating the last argument in the proof of Case 2 in the proof of Theorem 1.1, we obtain (1.9) in this case.

Case 3: \( m = 2k + 1, k \geq 1 \). Denote \( \tau = C(n, 2k + 1, \frac{n}{2k+1}) - \tau > 0 \). Since \( \frac{2n}{n-1} \leq q \leq \frac{n}{2k+1} \), then using the Lorentz–Poincaré inequality (1.4) and the Poincaré–Sobolev inequality under Lorentz–Sobolev norm (2.5), we get
\[
1 \geq \|\nabla_g \Delta_g^k u\|_{\frac{n}{2k+1},q}^q - \lambda \|u\|_{\frac{n}{2k+1},q}^q
\]
\[
\geq \|\nabla_g \Delta_g^k u\|_{\frac{n}{2k+1},q}^q - C(n, 2k + 1, \frac{n}{2k+1}) \|u\|_{\frac{n}{2k+1},q}^q + \tau \|u\|_{\frac{n}{2k+1},q}^q
\]
\[
\geq \|\nabla_g \Delta_g^k u\|_{\frac{n}{2k+1},q}^q - \left( \frac{2(k+1)(n-1)}{n} \right) \|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q + \tau \|u\|_{\frac{n}{2k+1},q}^q
\]
\[
\geq (2k\sigma_n^2 \beta_n^k) \|\Delta_g^k u\|_{\frac{n}{2k+1},q}^q + \tau \|u\|_{\frac{n}{2k+1},q}^q.
\]

We now can use the argument in the proof of Case 2 to obtain the result in this case. The proof of Theorem 1.1 is then completely finished. \( \square \)

5 Proof of Theorem 1.3

In this section, we provide the proof of Theorem 1.3. The proof uses the Lorentz–Poincaré
inequality (1.4), the Poincaré–Sobolev inequality under Lorentz–Sobolev norm (2.5) and
(2.8), and the Adams type inequality (1.8).
Proof of Theorem 1.3. We divide the proof in two cases according to the facts that \( m \) is even or odd.

Case 1: \( m = 2k, \ k \geq 2 \). Using the Lorentz–Poincaré inequality (1.4) and the inequality (2.8), we have

\[
1 \geq \| \Delta_g^k u \|_{\frac{q}{2k}, q}^q - C(n, 2k, \frac{n}{2k}) \| u \|_{\frac{q}{2k}, q}^q \geq \| \Delta_g^k u \|_{\frac{q}{2k}, q}^q - C(n, 2, \frac{n}{2k}) \| \Delta_g^{k-1} u \|_{\frac{q}{2k}, q}^q \\
\geq (2k - 1)(n - 2k)\sigma^2_n \| \Delta_g^{k-1} u \|_{\frac{q}{2(k-1)}, q}^q.
\]

Let us define the function \( w = 2(k-1)(n-2k)\sigma^2_n u \). Then we have \( \| \Delta_g^{k-1} u \|_{\frac{q}{2(k-1)}, q}^q \leq 1 \). Applying the Adams type inequality (1.8), we obtain

\[
\int_{\mathbb{R}^n} \Phi_{n, 2(k-1), q}(\beta_{n, 2k}^{q'}|u|^{q'}) dV_g = \int_{\mathbb{R}^n} \Phi_{n, 2(k-1), q}(\beta_{n, 2(k-1)}^{q'}|u|^{q'}) dV_g \leq C,
\]

here we use

\[
\beta_{n, 2k} = 2(k - 1)(n - 2k)\sigma^2_n \beta_{n, 2(k-1)}.
\]

It follows from (5.1) and the fact \( \Phi_{n, 2(k-1), q}(t) \geq Ct^{-\frac{n}{2(k-1)-q}} \) that

\[
\int_0^\infty (u^*(t))^{q'(j\frac{n}{2(k-1)-q} - 1)} dt = \int_{\mathbb{R}^n} |u|^{q'(j\frac{n}{2(k-1)-q} - 1)} dV_g \leq C.
\]

Using the non-increasing of \( u^* \), we can easily verify that

\[
u^*(t) \leq Ct^{-1/(q'(j\frac{n}{2(k-1)-q} - 1))}
\]

for any \( t > 0 \). Let \( x_0 \in \mathbb{B}^n \) such that \( V_g(B(0, d(0, x_0))) = 1 \). Since the function \( h(x) = (1 - |x|^2)^n \) is decreasing with respect to \( d(0, |x|) \), then \( h^\sharp = h \). Using Hardy–Littlewood inequality, we have

\[
\int_{\mathbb{B}^n} e^{\beta_{n, 2k}^{q'}|u|^{q'}} dx = 2^{-n} \int_{\mathbb{B}^n} e^{\beta_{n, 2k}^{q'}|u|^{q'}} h(x) dV_g \leq 2^{-n} \int_{\mathbb{B}^n} e^{\beta_{n, 2k}^{q'}|u|^{q'}} h(x) dV_g \\
= 2^{-n} \int_0^\infty e^{\beta_{n, 2k}^{q'}(t^{*})^{q'}} h(t) dt.
\]

For \( t \geq 1 \) we have \( u^*(t) \leq C \), hence it holds

\[
2^{-n} \int_1^\infty e^{\beta_{n, 2k}^{q'}(t^{*})^{q'}} h(t) dt \leq C 2^{-n} \int_1^\infty h(t) dt = C \int_{\{|x| \geq |x_0|\}} dx \leq C \sigma_n.
\]

Notice that

\[
e^t = \Phi_{n, 2(k-1), q}(t) + \sum_{j=0}^{\frac{n}{2(k-1)-q} - 2} \frac{t^j}{j!}.
\]
Using Young’s inequality, we get
\[ e^t \leq \Phi \frac{n}{2(\frac{n}{2} - 1)} q(t) + C(1 + t^{\frac{n}{4(\frac{n}{2} - 1)} q - 2}). \]

Consequently, by using the previous inequality and the inequality (5.1) and the fact \( h \leq 1 \), we obtain
\[
\int_0^1 e^{\beta' q_{2k}|u^*(t)|q'} h(t) dt \leq \int_0^1 \Phi \frac{n}{2(\frac{n}{2} - 1)} q(\beta_{2k}^q |u^*(t)|q') dt + C \int_0^1 \left( 1 + (u^*(t))^q(j^{\frac{n}{4(\frac{n}{2} - 1)} q - 2}) \right) dt
\]
\[
\leq \int_0^\infty \Phi \frac{n}{2(\frac{n}{2} - 1)} q(\beta_{2k}^q |u^*(t)|q') dt + C \int_0^1 (u^*(t))^q(j^{\frac{n}{4(\frac{n}{2} - 1)} q - 2}) dt
\]
\[
\leq \int_{\mathbb{H}^n} \Phi_{n,2(k-1),q}(\beta_{2k}^q |u|^q) dV_g + C \int_0^1 t^{\frac{n}{4(\frac{n}{2} - 1)} q} dt \leq C.
\]
Combining (5.2), (5.3) and (5.4) we obtain the desired estimate.

**Case 2:** \( m = 2k + 1 \), \( k \geq 1 \). Since \( \frac{2n}{n-1} \leq q \leq \frac{n}{2k+1} \), then by using the Lorentz–Poincaré inequality (1.4) and the Poincaré–Sobolev inequality under Lorentz–Sobolev norm (2.5), we get
\[
1 \geq \|\nabla g \Delta^k_g u\|^q_{\frac{n}{2k+1}, q} - C(n, 2k + 1, \frac{n}{2k+1}) \|u\|^q_{\frac{n}{2k+1}, q}
\]
\[
\geq \|\nabla g \Delta^k_g u\|^q_{\frac{n}{2k+1}, q} - \left(\frac{(2k+1)(n-1)}{n}\right)^q \|\Delta^k_g u\|^q_{\frac{n}{2k+1}, q}
\]
\[
\geq (2k\sigma_n^\frac{1}{n})^q \|\Delta^k_g u\|^q_{\frac{n}{2k+1}, q}.
\]

Setting \( w = 2k \sigma_n^\frac{1}{n} u \), we have \( \|\Delta^k_g w\|^q_{\frac{n}{2k+1}, q} \leq 1 \). Applying the Adams type inequality (1.8), we obtain
\[
\int_{\mathbb{H}^n} \Phi_{n,2k,q}(\beta_{2k}^q |u|^q) dV_g = \int_{\mathbb{H}^n} \Phi_{n,2k,q}(\beta_{2k}^q |w|^q) dV_g \leq C,
\]
here we use
\[ \beta_{2k+1} = 2k \sigma_n^\frac{1}{n} \beta_{2k}. \]

Similarly in the Case 1, the inequality (5.5) yields
\[
\int_0^\infty (u^*(t))^{q(j^{\frac{n}{4(\frac{n}{2} - 1)} q - 1}) dt = \int_{\mathbb{H}^n} |u|^{q(j^{\frac{n}{4(\frac{n}{2} - 1)} q - 1}) dV_g \leq C,
\]
which implies
\[ u^*(t) \leq Ct^{-\frac{1}{q(j^{\frac{n}{4(\frac{n}{2} - 1)} q - 1})}}, \quad t > 0. \]
Repeating the last arguments in the proof of Case 1, we obtain the result in this case.

The proof of Theorem 1.3 is then completed.
References

[1] S. Adachi and K. Tanaka. Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents. *Proc. Amer. Math. Soc.*, 128(7):2051–2057, 2000.

[2] D. R. Adams. A sharp inequality of J. Moser for higher order derivatives. *Ann. of Math. (2)*, 128(2):385–398, 1988.

[3] Adimurthi and O. Druet. Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. *Comm. Partial Differential Equations*, 29(1-2):295–322, 2004.

[4] Adimurthi and K. Sandeep. A singular Moser-Trudinger embedding and its applications. *NoDEA Nonlinear Differential Equations Appl.*, 13(5-6):585–603, 2007.

[5] Adimurthi and K. Tintarev. On a version of Trudinger-Moser inequality with Möbius shift invariance. *Calc. Var. Partial Differential Equations*, 39(1-2):203–212, 2010.

[6] Adimurthi and Y. Yang. An interpolation of Hardy inequality and Trudinger-Moser inequality in $\mathbb{R}^N$ and its applications. *Int. Math. Res. Not. IMRN*, (13):2394–2426, 2010.

[7] A. Alberico. Moser type inequalities for higher-order derivatives in Lorentz spaces. *Potential Anal.*, 28(4):389–400, 2008.

[8] A. Alvino, V. Ferone, and G. Trombetti. Moser-type inequalities in Lorentz spaces. *Potential Anal.*, 5(3):273–299, 1996.

[9] Z. M. Balogh, J. J. Manfredi, and J. T. Tyson. Fundamental solution for the $Q$-Laplacian and sharp Moser-Trudinger inequality in Carnot groups. *J. Funct. Anal.*, 204(1):35–49, 2003.

[10] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[11] J. Bertrand and K. Sandeep. Adams inequality on pinched hadamard manifolds. *preprint, arXiv:1809.00879*, 2019.

[12] L. Carleson and S.-Y. A. Chang. On the existence of an extremal function for an inequality of J. Moser. *Bull. Sci. Math. (2)*, 110(2):113–127, 1986.

[13] D. Cassani and C. Tarsi. A Moser-type inequality in Lorentz-Sobolev spaces for unbounded domains in $\mathbb{R}^N$. *Asymptot. Anal.*, 64(1-2):29–51, 2009.

[14] L. Chen, G. Lu, and M. Zhu. Existence and nonexistence of extremals for critical adams inequalities in $\nabla^4$ and trudinger–moser inequalities in $\nabla^2$. *preprint, arXiv:1812.00413*, 2018.
[15] W. S. Cohn and G. Lu. Best constants for Moser-Trudinger inequalities on the Heisenberg group. *Indiana Univ. Math. J.*, 50(4):1567–1591, 2001.

[16] W. S. Cohn and G. Z. Lu. Best constants for Moser-Trudinger inequalities, fundamental solutions and one-parameter representation formulas on groups of Heisenberg type. *Acta Math. Sin. (Engl. Ser.)*, 18(2):375–390, 2002.

[17] J. F. de Oliveira and J. a. M. do Ó. Trudinger-Moser type inequalities for weighted Sobolev spaces involving fractional dimensions. *Proc. Amer. Math. Soc.*, 142(8):2813–2828, 2014.

[18] A. DelaTorre and G. Mancini. Improved adams–type inequalities and their extremals in dimension $2m$. preprint, *arXiv:1711.00892*, 2017.

[19] J. a. M. do Ó and M. de Souza. A sharp inequality of Trudinger-Moser type and extremal functions in $H^{1,n}(\mathbb{R}^n)$. *J. Differential Equations*, 258(11):4062–4101, 2015.

[20] Y. Q. Dong and Q. H. Yang. An interpolation of Hardy inequality and Moser-Trudinger inequality on Riemannian manifolds with negative curvature. *Acta Math. Sin. (Engl. Ser.)*, 32(7):856–866, 2016.

[21] M. Flucher. Extremal functions for the Trudinger-Moser inequality in 2 dimensions. *Comment. Math. Helv.*, 67(3):471–497, 1992.

[22] L. Fontana and C. Morpurgo. Sharp exponential integrability for critical Riesz potentials and fractional Laplacians on $\mathbb{R}^n$. *Nonlinear Anal.*, 167:85–122, 2018.

[23] L. Fontana and C. Morpurgo. Adams inequalities for Riesz subcritical potentials. *Nonlinear Anal.*, 192:111662, 32, 2020.

[24] L. Fontana and C. Morpurgo. Adams inequalities for Riesz subcritical potentials. *Nonlinear Anal.*, 192:111662, 32, 2020.

[25] V. I. Judović. Some estimates connected with integral operators and with solutions of elliptic equations. *Dokl. Akad. Nauk SSSR*, 138:805–808, 1961.

[26] D. Karmakar and K. Sandeep. Adams inequality on the hyperbolic space. *J. Funct. Anal.*, 270(5):1792–1817, 2016.

[27] N. Lam and G. Lu. Sharp Adams type inequalities in Sobolev spaces $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for arbitrary integer $m$. *J. Differential Equations*, 253(4):1143–1171, 2012.

[28] N. Lam and G. Lu. Sharp Moser-Trudinger inequality on the Heisenberg group at the critical case and applications. *Adv. Math.*, 231(6):3259–3287, 2012.

[29] N. Lam and G. Lu. Sharp singular Adams inequalities in high order Sobolev spaces. *Methods Appl. Anal.*, 19(3):243–266, 2012.
[30] N. Lam and G. Lu. A new approach to sharp Moser-Trudinger and Adams type inequalities: a rearrangement-free argument. *J. Differential Equations*, 255(3):298–325, 2013.

[31] J. Li, G. Lu, and Q. Yang. Fourier analysis and optimal Hardy-Adams inequalities on hyperbolic spaces of any even dimension. *Adv. Math.*, 333:350–385, 2018.

[32] X. Li and Y. Yang. Extremal functions for singular Trudinger-Moser inequalities in the entire Euclidean space. *J. Differential Equations*, 264(8):4901–4943, 2018.

[33] Y. Li and B. Ruf. A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$. *Indiana Univ. Math. J.*, 57(1):451–480, 2008.

[34] K.-C. Lin. Extremal functions for Moser’s inequality. *Trans. Amer. Math. Soc.*, 348(7):2663–2671, 1996.

[35] G. Lu and H. Tang. Best constants for Moser-Trudinger inequalities on high dimensional hyperbolic spaces. *Adv. Nonlinear Stud.*, 13(4):1035–1052, 2013.

[36] G. Lu and H. Tang. Sharp singular Trudinger-Moser inequalities in Lorentz-Sobolev spaces. *Adv. Nonlinear Stud.*, 16(3):581–601, 2016.

[37] G. Lu and Q. Yang. Sharp Hardy-Adams inequalities for bi-Laplacian on hyperbolic space of dimension four. *Adv. Math.*, 319:567–598, 2017.

[38] G. Lu and Y. Yang. Adams’ inequalities for bi-Laplacian and extremal functions in dimension four. *Adv. Math.*, 220(4):1135–1170, 2009.

[39] G. Lu and M. Zhu. A sharp Trudinger-Moser type inequality involving $L^n$ norm in the entire space $\mathbb{R}^n$. *J. Differential Equations*, 267(5):3046–3082, 2019.

[40] G. Mancini and L. Martinazzi. Extremals for fractional moser–trudinger inequalities in dimension 1 via harmonic extensions and commutator estimates. *preprint, arXiv:1904.10267*, 2019.

[41] G. Mancini and K. Sandeep. Moser-Trudinger inequality on conformal discs. *Commun. Contemp. Math.*, 12(6):1055–1068, 2010.

[42] G. Mancini, K. Sandeep, and C. Tintarev. Trudinger-Moser inequality in the hyperbolic space $\mathbb{H}^N$. *Adv. Nonlinear Anal.*, 2(3):309–324, 2013.

[43] L. Martinazzi. Fractional Adams-Moser-Trudinger type inequalities. *Nonlinear Anal.*, 127:263–278, 2015.

[44] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20:1077–1092, 1970/71.
[45] Q. A. Ngô and V. H. Nguyễn. Sharp Adams–Moser–Trudinger type inequalities in the hyperbolic space. *to appear in Revista Matemática Iberoamericana*, 2016.

[46] Q. A. Ngô and V. H. Nguyễn. Sharp constant for Poincaré-type inequalities in the hyperbolic space. *Acta Math. Vietnam.*, 44(3):781–795, 2019.

[47] V. H. Nguyễn. A sharp Adams inequality in dimension four and its extremal functions. *preprint, arXiv:1701.08249*, 2017.

[48] V. H. Nguyễn. Improved Moser-Trudinger type inequalities in the hyperbolic space \( \mathbb{H}^n \). *Nonlinear Anal.*, 168:67–80, 2018.

[49] V. H. Nguyễn. Improved singular Moser-Trudinger and their extremal functions. *Potential Analysis, to appear.*, 2018.

[50] V. H. Nguyễn. The sharp Poincaré-Sobolev type inequalities in the hyperbolic spaces \( \mathbb{H}^n \). *J. Math. Anal. Appl.*, 462(2):1570–1584, 2018.

[51] V. H. Nguyễn. Extremal functions for the Moser-Trudinger inequality of Adimurthi-Druet type in \( W^{1,N}(\mathbb{R}^N) \). *Commun. Contemp. Math.*, 21(4):1850023, 37, 2019.

[52] V. H. Nguyễn. The sharp hardy-moser-trudinger inequality in dimension \( n \). *preprint, arXiv:1909.12587*, 2019.

[53] V. H. Nguyễn. The sharp Sobolev type inequalities in the Lorentz–Sobolev spaces in the hyperbolic spaces. *preprint*, 2019.

[54] V. H. Nguyen. The sharp higher order Lorentz-Poincaré and Lorentz-Sobolev inequalities in the hyperbolic spaces. *preprint*, 2020.

[55] V. H. Nguyễn. Singular adams inequalities in Lorentz–Sobolev spaces. *in preparation*, 2020.

[56] S. I. Pohožaev. On the eigenfunctions of the equation \( \Delta u + \lambda f(u) = 0 \). *Dokl. Akad. Nauk SSSR*, 165:36–39, 1965.

[57] B. Ruf. A sharp Trudinger-Moser type inequality for unbounded domains in \( \mathbb{R}^2 \). *J. Funct. Anal.*, 219(2):340–367, 2005.

[58] B. Ruf and F. Sani. Sharp Adams-type inequalities in \( \mathbb{R}^n \). *Trans. Amer. Math. Soc.*, 365(2):645–670, 2013.

[59] C. Tintarev. Trudinger-Moser inequality with remainder terms. *J. Funct. Anal.*, 266(1):55–66, 2014.

[60] N. S. Trudinger. On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.*, 17:473–483, 1967.
[61] G. Wang and D. Ye. A Hardy-Moser-Trudinger inequality. *Adv. Math.*, 230(1):294–320, 2012.

[62] Q. Yang and Y. Li. Trudinger-Moser inequalities on hyperbolic spaces under Lorentz norms. *J. Math. Anal. Appl.*, 472(1):1236–1252, 2019.

[63] Q. Yang, D. Su, and Y. Kong. Sharp Moser-Trudinger inequalities on Riemannian manifolds with negative curvature. *Ann. Mat. Pura Appl. (4)*, 195(2):459–471, 2016.

[64] Y. Yang. A sharp form of Moser-Trudinger inequality in high dimension. *J. Funct. Anal.*, 239(1):100–126, 2006.