OPERATOR SPACE GROTHENDIECK INEQUALITIES FOR NONCOMMUTATIVE $L_p$-SPACES

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Abstract. We prove the operator space Grothendieck inequality for bilinear forms on subspaces of noncommutative $L_p$-spaces with $2 < p < \infty$. One of our results states that given a map $u : E \rightarrow F^*$, where $E, F \subset L_p(M)$ ($2 < p < \infty$, $M$ being a von Neumann algebra), $u$ is completely bounded iff $u$ factors through a direct sum of a $p$-column space and a $p$-row space. We also obtain several operator space versions of the classical little Grothendieck inequality for maps defined on a subspace of a noncommutative $L_p$-space ($2 < p < \infty$) with values in a $q$-column space for every $q \in [p', p]$ ($p'$ being the index conjugate to $p$). These results are the $L_p$-space analogues of the recent works on the operator space Grothendieck theorems by Pisier and Shlyakhtenko. The key ingredient of our arguments is some Khintchine type inequalities for Shlyakhtenko’s generalized circular systems. One of our main tools is a Haagerup type tensor norm, which turns out particularly fruitful when applied to subspaces of noncommutative $L_p$-spaces ($2 < p < \infty$). In particular, we show that the norm dual to this tensor norm, when restricted to subspaces of noncommutative $L_p$-spaces, is equal to the factorization norm through a $p$-row space.

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0. Introduction

In the remarkable recent work [PS], Pisier and Shlyakhtenko obtained the operator space version of the famous Grothendieck theorem. This can be stated as follows. Let $E, F \subseteq B(H)$ be operator spaces and $u : E \times F \to \mathbb{C}$ a bilinear form. Assume $E$ and $F$ exact. Then $u$ is jointly completely bounded if there are a constant $K$ and states $f_i, g_i$ ($i = 1, 2$) on $B(H)$ such that

$$|u(a, b)| \leq K \left[ (f_1(a a^*) g_1(b b^*))^{1/2} + (f_2(a^* a) g_2(b b^*))^{1/2} \right], \quad a, b \in E, b \in F.$$

Moreover, if $K$ denotes the least constant in the inequality above, then $K \approx \|u\|_{cb}$, where the relevant equivalence constants depend only on the exactness constants of $E$ and $F$. We refer to the next section for background on operator space theory and all unexplained notions. [PS] also contains several interesting variants of the above statement, especially when both $E$ and $F$ are $C^*$-algebras (then the exactness assumption is needed for only one of them). In this latter case, the corresponding inequality is exactly the version for operator space theory of the noncommutative Grothendieck inequality obtained first by Pisier [PS] with an approximability assumption, and then in the full generality by Haagerup [H2]. Recall that the classical Grothendieck inequality corresponds to the case where both $E$ and $F$ are commutative $C^*$-algebras (in the Banach space theory). [P2] is an excellent reference for the classical and noncommutative Grothendieck inequalities for Banach spaces.

On the other hand, Maurey [M1] extended the classical Grothendieck inequality to bilinear forms on commutative $L_p$-spaces. (In this paper, we use “commutative $L_p$-spaces” to distinguish the usual $L_p$-spaces from the general noncommutative $L_p$-spaces, which are our main objects.) Let $(\Omega, \mu)$ be a measure space and $2 < p, q \leq \infty$. Let $u : L_p(\Omega, \mu) \times L_q(\Omega, \mu) \to \mathbb{C}$ be a bilinear form. Then $u$ is bounded if there are a constant $K$ and positive unit functionals $f \in (L_{p/2}(\Omega, \mu))^*$, $g \in (L_{q/2}(\Omega, \mu))^*$ such that

$$|u(a, b)| \leq K \left[ f(|a|^2) g(|b|^2) \right]^{1/2}, \quad a \in L_p(\Omega, \mu), b \in L_q(\Omega, \mu).$$

Again the best constant $K$ is equivalent to $\|u\|$. Although this statement is not explicitly stated in [M1], it immediately follows from Kwapien’s theorem (cf. [P4] Corollary 3.6) and the little Grothendieck theorem for $L_p$-spaces in [M1]. This latter theorem says that if $u : L_p(\Omega, \mu) \to H$ is a bounded map ($2 < p \leq \infty$; $H$ being a Hilbert space), then there is a positive unit functional $f \in (L_{p/2}(\Omega, \mu))^*$ such that

$$|u(a)| \leq K_0 \|u\| \left[ f(|a|^2) \right]^{1/2}, \quad a \in L_p(\Omega, \mu),$$

where $K_0$ is a universal constant.

It is this last statement which was extended to the noncommutative setting by Lust-Piquard [LP]. Let $M$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace $\tau$. Let $L_p(M)$ be the associated noncommutative $L_p$-space. Let $2 < p \leq \infty$ and $u : L_p(M) \to H$ be a bounded map. Then Lust-Piquard’s theorem claims that there are positive unit functionals $f_1, f_2 \in (L_{p/2}(M))^*$ such that

$$|u(a)| \leq K_0 \|u\| \left[ f_1(a a^*) + f_2(a^* a) \right]^{1/2}, \quad a \in L_p(M).$$

As in the commutative case, using Kwapien’s theorem, we then deduce that for any bounded bilinear form $u : L_p(M) \times L_q(M) \to \mathbb{C}$ ($2 < p, q \leq \infty$) there are positive unit functionals $f_i \in (L_{p/2}(M))^*$ and $g_i \in (L_{q/2}(M))^*$ such that

$$|u(a, b)| \leq K \|u\| \left[ f_1(a a^*) + f_2(a^* a) \right]^{1/2} \left[ g_1(b b^*) + g_2(b b^*) \right]^{1/2}, \quad a, b \in L_p(M).$$

Having all these in mind, one is naturally attempted to find out the operator space versions of (0.1) and (0.2) in the spirit of [PS] for bilinear forms on noncommutative $L_p$-spaces. This is the main concern of the present paper. The noncommutative $L_p$-spaces we use are those constructed by Haagerup [H1]. Thus type III von Neumann algebras are also allowed. The reader is referred to the next section for a brief introduction to noncommutative $L_p$-spaces and their operator space structure. To state our main result, we need some notations (see the next section for more details). Given a Hilbert space $H$ and $1 \leq p \leq \infty$ we denote by $H_p^+$ (resp. $H_p^*$) the Schatten $p$-class $\mathcal{S}_p(\mathbb{C}, H)$ (resp. $\mathcal{S}_p(\mathbb{H}, \mathbb{C})$) equipped with its natural operator space structure. $H_p^+$ (resp. $H_p^*$) is the column
We will need this notably when operator spaces.

Theorem 0.1 above in the case of $p_{0.4}, 0.5$ and Corollary 0.7 there). More precisely, Pisier-Shlyakhtenko’s results correspond to $|E|$, an absolute positive constant $c$.

Recall that the main result of [P6] corresponds again to the case to be either exact or a C*-algebra.

There are two positive functionals for any finite sequences

There is a constant $c$.

Let

Let $E, F \subset L_p(M)$ be two closed subspaces. Let $u : E \times F \rightarrow \mathbb{C}$ be a bilinear form. Then the following assertions are equivalent:

i) $u$ is jointly completely bounded and $\|u\|_{jcb} \leq K_1$.

ii) For any finite sequences $(a_k) \subset E, (b_k) \subset F$ and $(\mu_k) \subset \mathbb{R}_+$

$$\left| \sum_k u(a_k, b_k) \right| \leq K_2 \left[ \left\| \left( \sum_k \mu_k a_k^* a_k \right)^{1/2} \right\|_p + \left\| \left( \sum_k \mu_k^{-1} a_k a_k^* \right)^{1/2} \right\|_p \right].$$

(0.3)

\[ \bullet \left[ \left\| \left( \sum_k \mu_k b_k^* b_k \right)^{1/2} \right\|_p + \left\| \left( \sum_k \mu_k^{-1} b_k b_k^* \right)^{1/2} \right\|_p \right]. \]

iii) There are positive functionals $f_1, f_2$ and $g_1, g_2$ in the unit ball of $(L_{p/2}(M))^*$ such that for any $(a, b) \in E \times F$

$$|u(a, b)| \leq K_3 \left[ \left( f_1(aa^*) g_1(b^*b) \right)^{1/2} + \left( f_2(a^*a) g_2(bb^*) \right)^{1/2} \right].$$

(0.4)

iv) For any finite sequences $(a_k) \subset E, (b_k) \subset F$ and $(\mu_k) \subset \mathbb{R}_+$

$$\left| \sum_k u(a_k, b_k) \right| \leq K_4 \left[ \left\| \left( \sum_k a_k a_k^* \right)^{1/2} \right\|_p + \left\| \left( \sum_k b_k b_k^* \right)^{1/2} \right\|_p \right]$$

(0.5)

$$+ \left\| \left( \sum_k \mu_k a_k^* a_k \right)^{1/2} \right\|_p + \left\| \left( \sum_k \mu_k^{-1} b_k b_k^* \right)^{1/2} \right\|_p \right].$$

v) $u$ admits a decomposition $u = u_1 + u_2$, where $u_1$ and $u_2$ are bilinear forms on $E \times F$ such that the associated linear maps $\tilde{u}_1, \tilde{u}_2 : E \rightarrow F^*$ satisfy

$$\tilde{u}_1 \in \Gamma_{R_p}(E, F^*), \quad \tilde{u}_2 \in \Gamma_{C_p}(E, F^*) \quad \text{and} \quad \max \{ \gamma_{R_p}(\tilde{u}_1), \gamma_{C_p}(\tilde{u}_2) \} \leq K_5.$$

vi) $\tilde{u} \in \Gamma_{R_{p/2}(E)}(E, F^*)$ and $\gamma_{R_{p/2}(E)}(\tilde{u}) \leq K_6$.

Here the $K_i$ are constants; moreover, the best $K_i$ are equivalent uniformly in $p, E$ and $F$, i.e. there is an absolute positive constant $c$ such that $c^{-1}K_j \leq K_i \leq cK_j$ for all $i, j = 1, \ldots, 6$.

This theorem is the $L_p$-space version of the corresponding results in [PS] (see Theorems 0.3, 0.4, 0.5 and Corollary 0.7 there). More precisely, Pisier-Shlyakhtenko’s results correspond to Theorem 0.1 above in the case of $p = \infty$. In this case, one needs the exactness assumption on $E$ and $F$, namely, either both $E$ and $F$ are exact, or both $E$ and $F$ are C*-algebras and one of them is exact.

From Theorem 0.1 we can easily deduce the version of (0.1) for operator spaces, namely, the noncommutative little Grothendieck theorem in the category of operator spaces (see section 4). However, we will prove a more general result in the spirit of [PE]. This is the following theorem. Recall that the main result of [PE] corresponds again to the case $p = \infty$. Then $E$ must be supposed to be either exact or a C*-algebra.

Theorem 0.2. Let $E \subset L_p(M)$ be a subspace with $2 < p < \infty$ and $H$ a Hilbert space. Let $0 \leq \theta \leq 1$ and $\frac{1}{q} = \frac{1}{p} + \frac{\theta}{2}$. Then for any map $u : E \rightarrow H^\oplus_q$ the following assertions are equivalent:

i) $u$ is completely bounded.

ii) There is a constant $K$ such that for all finite sequences $(a_k) \subset E$ and $(\mu_k) \subset \mathbb{R}_+$

$$\sum_k \|u(a_k)\| \leq K^2 \left[ (1 - \theta) \left\| \sum_k \mu_k^\theta a_k^* a_k \right\|_{p/2} + \theta \left\| \sum_k \mu_k^{(1-\theta)} a_k a_k^* \right\|_{p/2} \right].$$

(0.6)

iii) There are two positive unit elements $f, g \in (L_{p/2}(M))^*$ such that

$$\|u(a)\| \leq K \left( f(a^* a) \right)^{(1-\theta)/2} \left( g(aa^*) \right)^{\theta/2}, \quad a \in E.$$

(0.7)
Moreover, if $K$ denotes the best constant in (0.6) and (0.7), then
\[ c_{p,q}^{-1} K \leq \|u\|_{cb} \leq K, \]
where $c_{p,q}$ is a positive constant depending only on $p$ and $q$, which can be controlled by an absolute constant.

As in [PS] and [P6], the key ingredient for the proofs of Theorems 0.1 and 0.2 is some noncommutative Khintchine type inequalities for Shlyakhtenko’s generalized circular systems. Note that the von Neumann algebra generated by a generalized circular system is of type III. This forces us to work with noncommutative $L_p$-spaces based on general von Neumann algebras. These inequalities are presented in section 3. The $L_\infty$ case was already obtained in [PS]. We should emphasize that the consideration of type III von Neumann algebras is inevitable both for [PS] and the present paper.

Besides these Khintchine type inequalities, we will still need two tools. The first one is a new tensor product. This is defined in a way similar to the usual Haagerup tensor product, replacing the row and column spaces $R$ and $C$ by their $L_p$-space counterparts $R_p$ and $C_p$, the $p$-row and $p$-column spaces. This new tensor product shares many properties with the usual Haagerup tensor product. It seems notably interesting when restricted to the subspaces of noncommutative $L_p$-spaces for $p \geq 2$. This is developed in section 2 where the main result is a characterization of maps factorable through $R_p$ (see Theorem 2.6). This is the $L_p$-space analogue of the well known Christensen-Sinclair’s factorization for completely bounded bilinear maps.

The second tool needed is the vector-valued noncommutative $L_p$-space theory developed by Pisier [P2] for injective semifinite von Neumann algebras, and especially, the recent extension by Junge in [J2] and [J3] to QWEP algebras. As said previously, the von Neumann algebra generated by a generalized circular system is of type III. It is non injective. However, it is QWEP. This explains why we really need Junge’s extension of Pisier’s theory. Junge’s work is briefly discussed in section 4.

Sections 5 and 6 are devoted to the proofs of Theorems 0.1 and 0.2, respectively. The last section contains some applications. We mention here two of them. The first one is that any completely bounded map from a subspace of a noncommutative $L_p(M)$ into a quotient of $L_p(M)$ has completely bounded approximation property $(1/p + 1/p' = 1; 2 < p < \infty)$. The second is a characterization of (completely) bounded Schur multipliers from $S_p$ (or a suitable subspace) to its dual $(2 < p < \infty)$. The latter is again the $L_p$-space analogue of the corresponding result in [PS].

1. Preliminaries

In this section we collect some preliminaries necessary to the whole paper. For clarity we divide the section into several subsections.

1.1. Operator spaces:

We will use standard notions and notation from operator space theory. Our references are [ER1] and [P1]. $M_n$ will denote the algebra of all complex $n \times n$ matrices. Given an operator space $E \subset B(H)$ we denote by $M_n(E)$ the space of all $n \times n$ matrices with entries in $E$. $M_n(E)$ is also an operator space equipped with the operator space structure induced by that of $M_n(B(H)) \cong B(\ell_2^n(H))$. Let $u : E \to F$ be a linear map between two operator spaces. $u$ is said to be completely bounded (c.b. in short) if
\[ \|u\|_{cb} = \sup_n \|I_{M_n} \otimes u\|_{M_n(E) \to M_n(F)} < \infty. \]
Let $CB(E,F)$ denote the operator space of all c.b. maps from $E$ to $F$. $u$ is said to be completely isomorphic if $u$ is an isomorphism and both $u$, $u^{-1}$ are c.b. Similarly, we define complete contraction, complete isometry.

Let $E,F,G$ be operator spaces. A bilinear map $u : E \times F \to G$ is said to be jointly completely bounded (j.c.b. in short) if the associated linear map $\tilde{u} : E \to CB(F,G)$ is c.b. Then we set $\|u\|_{jcb} = \|\tilde{u}\|_{cb}$. Let $JCB(E,F;G)$ denote the space of all j.c.b. maps from $E \times F$ to $G$. In particular, a bilinear form $u : E \times F \to \mathbb{C}$ is j.c.b. iff the associated linear map $\tilde{u} : E \to F^*$ is c.b.

Given an operator space $E$, we denote by $\overline{E}$ the complex conjugate of $E$. As a vector space, $\overline{E}$ is the same as $E$ but with the conjugate multiplication by a complex scalar. If $x \in E$, $\bar{x}$
denotes the same vector \( x \) considered as an element of \( E \). The norm of \( M_n(E) \) is defined by \( \| (x_{ij}) \|_{M_n(E)} = \| (x_{ij}) \|_{M_n(E)} \). Thus the map \( x \mapsto \bar{x} \) establishes a complete anti-isometry between \( E \) and \( E \).

1.2. Noncommutative \( L_p \)-spaces and their natural operator space structure:

It is well known by now that there are several equivalent constructions of noncommutative \( L_p \)-spaces associated with a von Neumann algebra. In this paper we will use Haagerup's noncommutative \( L_p \)-spaces (cf. [11]). [11] is our main reference for these spaces. Throughout this paper \( M \) will denote a general von Neumann algebra.

Let \( M \) be a von Neumann algebra. For \( 0 < p < \infty \), the spaces \( L_p(M) \) are constructed as spaces of measurable operators relative not to \( M \) but to a certain semifinite super von Neumann algebra \( \mathcal{M} \), namely, the crossed product of \( M \) by one of its modular automorphism groups. Let \( (\theta_s) \) be the dual automorphism group on \( M \). It is well known that \( M \) is a von Neumann subalgebra of \( \mathcal{M} \) and that the position of \( M \) in \( \mathcal{M} \) is determined by the group \( (\theta_s) \) in the following sense:

\[
\forall x \in \mathcal{M}, \quad x \in M \iff (\forall s \in \mathbb{R}, \theta_s(x) = x).
\]

Moreover, \( \mathcal{M} \) is semifinite and can be canonically equipped with a normal semifinite faithful trace \( \tau \) such that

\[
\tau \circ \theta_s = e^{-s} \tau.
\]

Let \( L_0(\mathcal{M}, \tau) \) be the topological *-algebra of measurable operators associated with \( (\mathcal{M}, \tau) \) (in Nelson's sense [2]). [2] see also [10]). The automorphisms \( \theta_s, s \in \mathbb{R} \), extend to automorphisms of \( L_0(\mathcal{M}, \tau) \). For \( 0 < p \leq \infty \), the space \( L_p(M) \) is defined by

\[
L_p(M) = \{ h \in L_0(\mathcal{M}, \tau) \mid \forall s \in \mathbb{R}, \theta_s(h) = e^{-s/p} h \}.
\]

The space \( L_\infty(M) \) coincides with \( M \) (modulo the inclusions \( M \subset \mathcal{M} \subset L_0(\mathcal{M}, \tau) \)). The spaces \( L_p(M) \) are closed self-adjoint linear subspaces of \( L_0(\mathcal{M}, \tau) \). They are closed under left and right multiplications by elements of \( M \). If \( h = u|h| \) is the polar decomposition of \( h \in L_0(\mathcal{M}, \tau) \), then

\[
h \in L_p(M) \iff u \in M \text{ and } |h| \in L_p(M).
\]

It was shown by Haagerup that there is a linear homeomorphism \( \omega \mapsto h_\omega \) from \( M_\omega \) onto \( L_1(M) \) (equipped with the vector space topology inherited from \( L_0(\mathcal{M}, \tau) \)). This homeomorphism preserves the additional structures (conjugation, positivity, polar decomposition, action of \( M \)). It permits to transfer the norm of \( M_\omega \) into a norm on \( L_1(M) \), denoted by \( \| \|_1 \).

The space \( L_1(M) \) is equipped with a distinguished bounded positive linear functional \( \text{tr} \), the “trace”, defined by

\[
\text{tr} (h_\omega) = \omega(1), \quad \omega \in M_\omega.
\]

Consequently, \( \| h \|_1 = \text{tr} (|h|) \) for every \( h \in L_1(M) \).

For any \( 0 < p < \infty \), the Mazur map \( M_\omega \rightarrow L_0(\mathcal{M}, \tau) \) extends by continuity to a map \( L_0(\mathcal{M}, \tau)_+ \rightarrow L_0(\mathcal{M}, \tau)_+, h \mapsto h^p \) (cf. [4]). Then

\[
\forall h \in L_0(\mathcal{M}, \tau)_+ , \quad h \in L_p(M) \iff h^p \in L_1(M).
\]

For \( h \in L_p(M) \) set \( \| h \|_p = \| |h| \|_1^{1/p} \). Then \( \| \|_p \) is a norm or a \( p \)-norm according to \( 1 \leq p < \infty \), or \( 0 < p \leq 1 \). The associated vector space topology coincides with that inherited from \( L_0(\mathcal{M}, \tau) \).

Another important link between the spaces \( L_p(M) \) is the external product: in fact, the product of \( L_0(\mathcal{M}, \tau) \), \( (h, k) \mapsto h \cdot k \), restricts to a bounded bilinear map \( L_p(M) \times L_q(M) \rightarrow L_r(M) \), where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). This bilinear map has norm one, which amounts to saying that the usual Hölder inequality extends to Haagerup \( L_p \)-spaces (called “noncommutative Hölder inequality”).

Assume that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then the bilinear form \( L_p(M) \times L_q(M) \rightarrow \mathbb{C}, (h, k) \mapsto \text{tr} (h \cdot k) \) defines a duality bracket between \( L_p(M) \) and \( L_q(M) \), for which \( L_p(M) \) is (isometrically) the dual of \( L_p(M) \) (if \( p \neq \infty \)); moreover we have the tracial property:

\[
\text{tr} (hk) = \text{tr} (kh), \quad h \in L_p(M), k \in L_q(M).
\]

In the sequel \( \text{tr} \) will denote this tracial functional associated to any von Neumann algebra. In case of possible ambiguity, we will write \( \text{tr}_M \) to indicate that the von Neumann algebra in consideration is \( M \).
Now we turn to describe the natural operator space structure on $L_p(M)$ as introduced in [P2] and [P1], (see also [J3]). For $p = \infty$, $L_\infty(M) = M$ has its natural operator space structure as a von Neumann algebra. This yields an operator space structure on $M^*$, the standard dual of $M$. Let us consider the case of $p = 1$. Recall that $L_1(M)$ coincides with the predual $M_*$ of $M$ at the Banach space level. Thus one would attempt to define the operator space structure on $L_1(M)$ as the one induced by that of $M^*$ via the natural embedding $M_* \hookrightarrow M^* = (M_*)^{**}$ (again as Banach spaces). However, as explained in [P1] Chapter 7, it is more convenient to consider $L_1(M)$ as the predual of the opposite von Neumann algebra $M^{op}$, which is isometric (but in general not completely isomorphic) to $M$, and to equip $L_1(M)$ with the operator space structure inherited from $M^{op}$. One of the main reasons for this choice is that it insures that the equality $L_1(M_\eta \otimes M) = S_1^{\eta} \otimes L_1(M)$ (operator space projective tensor product) holds true (see [J2]). Finally, the operator space structure of $L_p(M)$ is obtained by complex interpolation, using the well known interpretation of $L_p(M)$ as the interpolation space $(M, L_1(M))_{1/p}$ (see [P2]). If $M$ admits a normal faithful state $\varphi$, we can also use Kosaki’s interpolation [Ko]. Note that in this case all injections $I_q$ of $M$ into $L_1(M)$ ($0 \leq q \leq 1$) considered in [Ko] give completely isometric interpolation spaces, exactly as in the Banach space level.

When $M$ is semifinite, we will always consider $L_p(M)$ as the usual $L_p$-space constructed from a normal semifinite faithful trace. We refer, for instance, to the survey [PX] for semifinite noncommutative $L_p$-spaces and for more references.

1.3. Vector-valued Schatten classes:

One of our main tools is the theory of vector-valued noncommutative $L_p$-spaces. This theory was first introduced and developed by Pisier [P2] for injective semifinite von Neumann algebras. Pisier’s theory can be easily extended to general injective (so not necessarily semifinite) von Neumann algebras. Very recently, Junge [J2] and [J3] has partly extended this theory to QWEP von Neumann algebras. We will discuss Junge’s extension in some more details later in section 5. Here we content ourselves only with a brief description of vector-valued Schatten classes. The reader is referred to [P2] for more information.

The Schatten classes $S_p$ are equipped with their natural operator space structure as described previously. Now let $E$ be an operator space. We define $S_1[E]$ as the operator space projective tensor product $S_1 \otimes E$. Then for any $1 < p < \infty$ we define $S_p[E]$ by interpolation:

$$S_p[E] = (S_\infty[E], S_1[E])_{1/p}.$$ 

Note that in this interpolation formula, $S_\infty[E]$ can be replaced by $B(\ell_2) \otimes_{\min} E$, namely, we have

$$S_p[E] = (B(\ell_2) \otimes_{\min} E, S_1[E])_{1/p}.$$ 

By reiteration, for any $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$

$$(S_{p_0}[E], S_{p_1}[E])_{\theta} = S_p[E],$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. More generally, given a compatible couple $(E_0, E_1)$ of operator spaces we have (completely isometrically)

$$S_{p_0}[E_0], S_{p_1}[E_1])_{\theta} = S_p[(E_0, E_1)_{\theta}].$$

The usual duality for Schatten classes extends to the vector-valued case too. Let $1 \leq p < \infty$.

Then

$$(S_p[E])^* = S_{p'}[E^*]$$

completely isometrically.

Here and throughout this paper, $p'$ always denotes the conjugate index of $p$. The duality is given as follows. Let $x = (x_{ij}) \in S_p[E]$ and $\xi = (\xi_{ij}) \in S_{p'}[E^*]$. Then

$$\langle \xi, x \rangle = \text{Tr}(\xi^* x) = \sum_{ij} \xi_{ij}(x_{ij}),$$

where $\text{Tr}$ is the usual trace on $B(\ell_2)$. We call the reader’s attention to that in the theory of operator spaces the duality between $S_p$ and $S_{p'}$ (in the scalar case) is given by

$$\langle y, x \rangle = \text{Tr}(y^* x) = \sum_{ij} x_{ij}y_{ij}.$$

This is consistent with the natural operator space structure on $S_1$ described previously, which is the predual of $B(\ell_2)^{op}$.

There is a nice description of the norm of $S_p[E]$ in terms of that of $S_\infty[E]$, given by [P2, Theorem 1.5]: for any $x = (x_{ij}) \in S_p[E]$ we have

\[ \|x\|_{S_p[E]} = \inf \left\{ \|\alpha\|_{2p} \|y\|_{S_\infty[E]} \|\beta\|_{2p} : \alpha, \beta \in S_{2p}, y \in S_\infty[E] \right\}, \tag{1.3} \]

where the infimum runs over all factorizations $x = \alpha y \beta$ with $\alpha, \beta \in S_{2p}$ and $y \in S_\infty[E]$. Conversely, the norm of $S_\infty[E]$ can be recovered from that of $S_p[E]$ as follows (see [P2, Lemma 1.7]): for any $x = (x_{ij}) \in S_\infty[E]$

\[ \|x\|_{S_\infty[E]} = \sup \left\{ \|\alpha y \beta\|_{S_\infty[E]} : \alpha, \beta \in S_{2p}, \|\alpha\|_{2p} \leq 1, \|\beta\|_{2p} \leq 1 \right\}. \tag{1.4} \]

From (1.3) and (1.4) we can easily deduce the following more general formula. Let $1 \leq p < q \leq \infty$ and $\frac{1}{p} = \frac{1}{r} - \frac{\theta}{q}$. Then for any $x = (x_{ij}) \in S_q[E]$

\[ \|x\|_{S_q[E]} = \sup \left\{ \|\alpha y \beta\|_{S_q[E]} : \alpha, \beta \in S_{2r}, \|\alpha\|_{2r} \leq 1, \|\beta\|_{2r} \leq 1 \right\}. \tag{1.5} \]

(1.4) also implies the following convenient characterization of c.b. maps, which will be repeatedly used later (cf. [P2, Lemma 1.7]).

**Lemma 1.1.** Let $E$ and $F$ be two operator spaces. Then a linear map $u : E \to F$ is c.b. iff

\[ \sup_n \left\| I_{S_p^n} \otimes u : S_p^n[E] \to S_p^n[F] \right\| < \infty; \]

moreover in this case the supremum above is equal to $\|u\|_{cb}$.

More generally, if $H$ is a Hilbert space, we can analogously define $S_p[H; E]$, the $E$-valued Schatten classes based on $H$. $S_p[E]$ (resp. $S_p^n[E]$) corresponds to the case of infinite dimensional separable $H$ (resp. dim $H = n$). All preceding properties still hold for general $S_p[H; E]$.

We now specialize the discussion above to the case when $E$ is a subspace of a noncommutative $L_p(M)$. In this case, the theory becomes much simpler. Note that there is a natural algebraic identification of $L_p(M_n \otimes M)$ with $M_n(L_p(M))$. Then $S_p^n[L_p(M)]$ is nothing but the linear space $M_n(L_p(M))$ equipped with the norm of $L_p(M_n \otimes M)$. More generally, if $E \subset L_p(M)$ is a closed subspace, the norm on $S_p^n[E]$ is induced by that of $S_p^n[L_p(M)]$. In the infinite dimensional case, $S_p[L_p(M)]$ is completely isometrically identified with $L_p(\mathbb{B}(\ell_2), \mathbb{B}(\ell_2))$ for all $1 \leq p < \infty$. If $E \subset L_p(M)$, then $S_p[E]$ is the closure in $L_p(\mathbb{B}(\ell_2), \mathbb{B}(\ell_2))$ of the algebraic tensor product $S_p \otimes E$.

### 1.4. Column and row spaces :

The column and row spaces, $C$ and $R$, play an important role in the whole theory of operator spaces. Note that $C$ and $R$ are respectively the (first) column and row subspaces of $S_\infty$. The $L_p$-space counterparts of $C$ and $R$ will play an essential role in the present paper. Let $C_p$ (resp. $R_p$) denote the subspace of $S_p$ consisting of matrices whose all entries but those in the first column (resp. row) vanish. So $C_\infty$ and $R_\infty$ are just $C$ and $R$ respectively. It is clear that $C_p$ and $R_p$ are completely 1-complemented subspaces of $S_p$. We have the following completely isometric identities: for any $1 \leq p \leq \infty$

\[ (C_p)^* \cong C_{p'} \cong R_{p'} \quad \text{and} \quad (R_p)^* \cong R_{p'} \cong C_{p'}. \tag{1.6} \]

$C_p$ and $R_p$ can be also defined via interpolation from $C$ and $R$. We view $(C, R)$ as a compatible couple by identifying both of them with $\ell_2$ (in the Banach space level!), i.e. by identifying the canonical bases $(e_{i, k})$ of $C_p$ and $(e_{k, j})$ of $R_p$ with $(e_k)$ of $\ell_2$. Then

\[ C_p = (C, R)_{1/p} = (C_\infty, C_1)_{1/p} \quad \text{and} \quad R_p = (R, C)_{1/p} = (R_\infty, R_1)_{1/p}. \tag{1.7} \]

By reiteration, for any $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$

\[ C_p = (C_{p_0}, C_{p_1})_{\theta} \quad \text{and} \quad R_p = (R_{p_0}, R_{p_1})_{\theta}, \]

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Like $C$ and $R$, $C_p$ and $R_p$ are also 1-homogenous 1-Hilbertian operator spaces. We refer to [P2] for the proofs of all these elementary facts.

More generally, given a Hilbert space $H$ and $1 \leq p \leq \infty$ we denote by $H^p_p$ (resp. $H^r_r$) the Schatten $p$-class $S_p(C, H)$ (resp. $S_p(R, C)$) equipped with its natural operator space structure. When $H$ is separable and infinite dimensional, $H^p_p$ and $H^r_r$ are respectively $C_p$ and $R_p$ above. If
\[ \dim H = n < \infty, \text{ set } H^c = C^p, \text{ and } H^r = R^p. \text{ All properties for } C_p \text{ and } R_p \text{ mentioned above hold for } H^c \text{ and } H^r \text{ too. We will call } H^c \text{ resp. } H^r \text{ a } p}\text{-column (resp. } p\text{-row) space.} \\
\]

Now let \( E \) be an operator space. We denote by \( C_p[E] \) (resp. \( R_p[E] \)) the closure of \( C_p \otimes E \) (resp. \( R_p \otimes E \)) in \( S_p[E] \). Again \( C_p[E] \) and \( R_p[E] \) are completely 1-complemented subspaces of \( S_p[E] \). If \( E \) is a subspace of a noncommutative \( L_p(M) \), the norm of \( C_p[E] \) is easy to determined. For any finite sequence \( (x_k) \subset E \)

\[
\| \sum_k x_k \otimes e_k \|_{C_p[E]} = \left\| \left( \sum_k x_k^* x_k \right)^{1/2} \right\|_{L_p(M)},
\]

where \( (e_k) \) denotes the canonical basis of \( C_p \). More generally, if \( a_k \in C_p \), then

\[
\| \sum_k x_k \otimes a_k \|_{C_p[E]} = \left\| \left( \sum_k \langle a_k, a_j \rangle x_k^* x_j \right)^{1/2} \right\|_{L_p(M)},
\]

where \( \langle \ , \ \rangle \) denotes the scalar product in \( C_p \). (In terms of matrix product, \( \langle a_k, a_j \rangle = a_k^* a_j \).) We also have a similar description for \( R_p[E] \).

1.5. Factorization through \( C_p \) and \( R_p \):

The following definition goes back to [3] (see also [1]). Given operator spaces \( X, E \) and \( F \), we denote by \( \Gamma_X(E, F) \) the family of all maps \( u : E \to F \) which factors through \( X \), namely, all \( u \) which admit a factorization \( E \xrightarrow{\alpha} X \xrightarrow{\beta} F \) with c.b. maps \( \alpha \) and \( \beta \). For \( u \in \Gamma_X(E, F) \) define

\[
\gamma_X(u) = \inf \left\{ \|\alpha\|_{C_p} \|\beta\|_{C_p} : u = \beta \alpha, \alpha \in CB(E, X), \beta \in CB(X, F) \right\}.
\]

We will need this notably when \( X \) is \( R_p, C_p \) or \( R_p \oplus_p C_p \). Then \( X \) can be any one of these spaces associated with an arbitrary Hilbert space. Let us make this more precise. A map \( u : E \to F \) is said to be factorable through \( C_p \) if \( u \) admits a factorization \( E \xrightarrow{\alpha} H^c \xrightarrow{\beta} F \) for some Hilbert space \( H \) and c.b. maps \( \alpha \) and \( \beta \). Let \( \Gamma_{C_p}(E, F) \) denote the space of all maps between \( E \) and \( F \) factorable through \( C_p \). For \( u \in \Gamma_{C_p}(E, F) \) define

\[
\gamma_{C_p}(u) = \inf \left\{ \|\alpha\|_{C_p} \|\beta\|_{C_p} : u = \beta \alpha, \alpha \in CB(E, X), \beta \in CB(X, F) \right\},
\]

where the infimum runs over all factorizations \( u = \beta \alpha \) as above. Then \( (\Gamma_{C_p}(E, F), \gamma_{C_p}) \) is a Banach space. Similarly, we define the factorization through \( R_p \) and \( C_p \oplus R_p \) respectively. The resulting spaces are denoted respectively by \( \Gamma_{R_p}(E, F) \) and \( \Gamma_{C_p \oplus R_p}(E, F) \) equipped with \( \gamma_{R_p} \) and \( \gamma_{C_p \oplus R_p} \).

2. A Haagerup type tensor norm and factorization through \( p\)-row spaces

In this section we introduce a Haagerup type tensor norm. We refer to [1] and [3] for the usual Haagerup tensor product. This new tensor norm enjoys many properties of the usual Haagerup tensor norm. It turns out especially satisfactory when restricted to subspaces of noncommutative \( L_p \)-spaces for \( p \geq 2 \). In this latter case we obtain a description of the linear functionals continuous with respect to this tensor norm. This is the \( L_p \)-space analogue of the well-known factorization theorem due to Christensen and Sinclair.

Let us begin with some conventions. All row and column matrices below contain only finitely many non zero coefficients, so they can be considered as finite row and column matrices. When \( a \) and \( b \) are two row (resp. column) matrices with entries in \( E \), \( (a, b) \) (resp. \( (a, b) \)) is again a row (resp. column) matrix with coefficients in \( E \).

The definition below is a generalization of the Haagerup tensor product, just replacing \( R_{\infty} \) and \( C_{\infty} \) in the usual Haagerup tensor product by \( R_p \) and \( C_p \). Recall that if \( a \in R_p[E] \) is a row matrix and \( b \in C_q[F] \) a column one, \( a \odot b \) denotes the element in \( E \otimes F \) given by

\[
a \odot b = \sum_k a_k \otimes b_k.
\]

**Definition 2.1.** Let \( E \) and \( F \) be operator spaces and \( 1 \leq p, q \leq \infty \). Given \( x \in E \otimes F \) we define

\[
\| x \|_{h_{p,q}} = \inf \left\{ \|a\|_{R_p[E]} \|b\|_{C_q[F]} : x = a \odot b, \ a \in R_p[E], \ b \in C_q[F] \right\}.
\]

If \( p = q \), \( \| x \|_{h_{p,q}} \) is simply denoted by \( \| x \|_{h_p} \).

In general, \( \| \cdot \|_{h_{p,q}} \) is not a norm, but only a quasi-norm (see the proposition below). However, when \( E \) and \( F \) verify a certain 2-convexity, then \( \| \cdot \|_{h_{p,q}} \) is indeed a norm.
Definition 2.2. Let $E$ be an operator space and $1 \leq p \leq \infty$. $E$ is said to be $R_p$-2-convex (resp. $C_p$-2-convex) if for any $a, b \in R_p[E]$ (resp. $C_p[E]$)

$$\|(a, b)\|_{R_p[E]} \leq \left(\|a\|_{R_p[E]}^2 + \|b\|_{R_p[E]}^2\right)^{1/2}$$

(resp.

$$\|t(a, b)\|_{C_p[E]} \leq \left(\|a\|_{C_p[E]}^2 + \|b\|_{C_p[E]}^2\right)^{1/2}$$

where $\sum_{k} W(k)$ is the weight of non-commutative $L_p$-spaces ($p \geq 2$) are $R_p$-2-convex and $C_p$-2-convex. Any operator space is $R_\infty$-2-convex and $C_\infty$-2-convex.

Remark. Let $E$ (resp. $F$) be an $R_p$-2-convex (resp. $C_p$-2-convex) operator space. Then the functional $\|\|_{h_{p,q}}$ defined by Definition 2.1 is a $\gamma$-norm in the sense of [13]. Indeed, given a positive element $u = \sum a_k \otimes \bar{a}_k$ in $E \otimes \overline{E}$, define

$$r_p(u) = \|(a_1, a_2, \cdots)\|_{R_p[E]}^2.$$

Then the $R_p$-2-convexity of $E$ guarantees that $r_p$ is a weight on $(E \otimes \overline{E})$ in Pisier’s sense. Similarly, we have a weight $c_q$ on $(F \otimes \overline{F})$ corresponding to the norm of $C_q[F]$. Then $\|\|_{h_{p,q}}$ is exactly the $\gamma$-norm associated to $r_p$ and $c_q$ defined in [13, section 6].

The following is elementary. As usual, an element $x \in E \otimes F$ is also regarded as a map from $E^*$ to $F$. Its adjoint $^tx$ is from $F^*$ to $E$. Then the norm $\|t^t x\|_{h_{p,q}}$ is the norm of $x$ in $F \otimes h_{p,q} E$.

Proposition 2.3. Let $E$ and $F$ be operator spaces and $1 \leq p, q \leq \infty$.

i) $\|\|_{h_{p,q}}$ is a quasi-norm on $E \otimes F$. If in addition $E$ and $F$ are $R_p$-2-convex and $C_q$-2-convex, respectively, then $\|\|_{h_{p,q}}$ is a norm.

ii) For any $x \in E \otimes F$ there are $a = (a_1, \ldots, a_n) \in R_p[E]$ and $b = (b_1, \ldots, b_n) \in C_q[F]$ such that both $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are linearly independent and such that

$$\|x\|_{h_{p,q}} = \|\|_{R_p[E]} \|b\|_{C_q[F]}.$$

Moreover, $a$ and $b$ can be chosen to further satisfy

$$\|t^t x\|_{h_{p,q}} = \|t^t a\|_{C_q[E]} \Delta^{-1} \|\|_{R_p[E]} \|\|_{C_q[E]}.$$

where $\Delta$ is a positive definite diagonal matrix.

Proof. i) We have the following quasi-triangle inequality:

$$\|x + y\|_{h_{p,q}} \leq 2(\|x\|_{h_{p,q}} + \|y\|_{h_{p,q}}).$$

This is proved by a standard argument that is left to the reader. The 2-convex assumption implies the validity of the triangle inequality. Thus it remains to check that $\|x\|_{h_{p,q}} = 0$ implies $x = 0$. To this end we show $\|x\|_\varepsilon \leq \|x\|_{h_{p,q}}$, where $\|\|_\varepsilon$ denotes the Banach space injective tensor norm. Let $\xi \in E^*$ and $\eta \in F^*$ be unit vectors. Then for any factorization $x = a \otimes b$

$$|\langle \xi \otimes \eta, x \rangle| = \left|\sum_k \xi(a_k) \eta(b_k)\right| \leq \left(\sum_k |\xi(a_k)|^2\right)^{1/2} \left(\sum_k |\eta(b_k)|^2\right)^{1/2}.$$

Let $(\alpha_1, \alpha_2, \cdots) \in \ell_2$ be a unit vector such that

$$\left(\sum_k |\xi(a_k)|^2\right)^{1/2} = \sum_k \alpha_k \xi(a_k) \leq \left(\sum_k \alpha_k a_k\right).$$

Writing

$$\sum_k \alpha_k a_k = (a_1, a_2, \cdots) \cdot (\alpha_1, \alpha_2, \cdots),$$

we get

$$\|\sum_k \alpha_k a_k\| \leq \|a\|_{R_p[E]} \cdot \|t^t (\alpha_1, \alpha_2, \cdots)\|_{B(\ell_2)} \leq \|a\|_{R_p[E]}.$$

Thus

$$\left(\sum_k |\xi(a_k)|^2\right)^{1/2} \leq \|a\|_{R_p[E]}.$$
Similarly, 
\[
\left( \sum_k |\eta(b_k)|^2 \right)^{1/2} \leq \|b\|_{C_q[E]}. 
\]

Hence it follows that \( \|x\|_e \leq \|x\|_{h_{p,q}}. \)

ii) Recall that the \( R_p \) and \( C_q \) norms satisfy the following elementary property: for any scalar matrix \( \alpha \in B(\ell_2) \)
\[
\|a\alpha\|_{R_p[E]} \leq \|a\|_{R_p[E]} \|\alpha\| \quad \text{and} \quad \|\alpha b\|_{C_q[F]} \leq \|\alpha\| \|b\|_{C_q[F]}.
\]

Using this property one can easily prove the first assertion of ii) exactly as in the case of the usual Haagerup tensor product. See [ER1, section 9.2] for more details. Similarly, the second part can be proved as [PS, Proposition 1.7].

We denote by \( E \otimes_{h_{p,q}} F \) the completion of \( (E \otimes F, \|\cdot\|_{h_{p,q}}) \). Again \( E \otimes_{h_{p,q}} F \) is denoted by \( E \otimes_{h_{p,q}} F \) in the case of \( p = q \). By definition, the tensor product \( E \otimes_{h_{p,q}} F \) is projective. Proposition [2.3 ii) implies that it is also injective.

From now on we consider only the case where \( p = q \geq 2 \), and specialize the above tensor product to subspaces of noncommutative \( L_p \)-spaces. (Recall that all these subspaces are both \( R_p \)-2-convex and \( C_p \)-2-convex.) For these spaces we will have a satisfactory description of the dual space of \( E \otimes_{h_{p,q}} F \). We first need to characterize the c.b. maps from a subspace of a noncommutative \( L_p \) to \( R_p \). The following is the \( L_p \)-space analogue of a well-known result on maps with values in \( R \) due to Effros and Ruan (cf. [ER2]). The main point here is the implication iv) \( \Rightarrow i \). We should emphasize that this result (without the assertion ii) below) is a special case of Theorem [1.2] corresponding to the endpoint cases \( \theta = 0 \) and \( \theta = 1 \). In this special case, the arguments are elementary and much simpler than that for the general case as in Theorem [1.2].

Proposition 2.4. Let \( E \subset L_p(M) \) be a closed subspace \( (2 < p < \infty) \) and \( H \) a Hilbert space. Let \( u : E \to H_p^* \) be a linear map. Then the following assertions are equivalent

i) \( u \) is c.b.

ii) \( I_{R_p} \otimes u \) extends to a bounded map from \( R_p[E] \) to \( R_p[H_p^*] \).

iii) There is a constant \( c \) such that for any finite sequence \( (a_k) \subset E \)
\[
\left( \sum_k \|u(a_k)\|^2 \right)^{1/2} \leq c \left( \sum_k \|a_k a_k^*\| \right)^{1/2} \|p\|
\]
(2.1)

iv) There is a positive unit element \( f \in (L_{p/2}(M))^* = L_{(p/2)'}(M) \) such that
\[
\|u(a)\| \leq c \left( f(a a^*) \right)^{1/2}, \quad a \in E.
\]

Moreover, if one of these assertions holds, \( \|u\|_{cb}, \|I_{R_p} \otimes u\| \) and the smallest constants \( c \) in [2.1] and [2.4] are all equal, and \( u \) admits a c.b. extension to \( L_p(M) \) with the same c.b. norm. We have a similar result for maps with values in \( H_p^* \) (with necessary changes in ii)-iv) above).

Proof. i) \( \Rightarrow ii \). By Lemma 1.1, \( u \) is c.b. iff \( I_{S_p} \otimes u \) extends to a bounded map from \( S_p[E] \) to \( S_p[H_p^*] \). In particular, i) implies ii).

ii) \( \Rightarrow iii \). Assume ii). Then for any finite row \( a = (a_1, a_2, \ldots) \in R_p[E] \)
\[
\|I_{R_p} \otimes u(a)\|_{R_p[H_p^*]} \leq c \|a\|_{R_p[E]}.
\]

However,
\[
\|I_{R_p} \otimes u(a)\|_{R_p[H_p^*]} = \left( \sum_k \|u(a_k)\|^2 \right)^{1/2} \quad \text{and} \quad \|a\|_{R_p[E]} = \left( \sum_k \|a_k a_k^*\| \right)^{1/2} \|p\|.
\]

Thus (2.1) follows.

iii) \( \Rightarrow iv \). This can be done by a standard application of the Hahn-Banach theorem. We give a sketch of the proof for completeness. Let \( S \) denote the positive cone of the unit ball of \( (L_{p/2}(M))^* = L_{(p/2)'}(M) \). Then \( S \) is a compact space when equipped with the \( w^* \)-topology. Given a finite sequence \( (a_k) \subset E \) set
\[
\varphi(a_k)(s) = c^2 s(\sum_k a_k a_k^*) - \sum_k \|u(a_k)\|^2, \quad s \in S.
\]
Then \( \varphi(a_k) \) is a continuous function on \( S \) and (2.1) implies that its maximum is positive. Let \( A \) denote the closure of the family of all such functions \( \varphi(a_k) \). Then \( A \) is a closed convex cone and disjoint from \( A_\ast \), where \( A_\ast \) is the open convex cone of all negative continuous functions on \( S \). Therefore, by the Hahn-Banach theorem there is a probability measure \( \mu \) on \( S \) such that

\[
\int_S \left( c^2 s(aa^*) - \|u(a)\|^2 \right) d\mu(s) \geq 0.
\]

Define

\[
f(a) = \int_S s(a) d\mu(s), \quad a \in L_{p/2}(M).
\]

Then \( f \) is a positive unit element in \( L_{[p/2]}(M) \) satisfying (2.2). iv) \( \Rightarrow \) i). Note that \( \langle b, a \rangle = f(ab^*) \) defines a semi-scalar product on \( E \). Quotiented by its kernel, \( E \) becomes a pre-Hilbert space whose completion is denoted by \( K \). It is clear that the identity on \( E \) induces a contractive inclusion of \( E \) into \( K \), denoted by \( i_E \). On the other hand, (2.2) implies that there is a bounded operator \( \tilde{u} : K \to H \) such that \( \|\tilde{u}\| \leq c \) and \( u = \tilde{u} \circ i_E \). We now equip \( K \) with the operator space structure of \( K_p^{r} \), i.e. we consider \( K \) as \( K_p^{r} \) in the category of operator spaces. Then by the homogeneity of \( p \)-row spaces, \( \tilde{u} : K_p^r \to H_p^r \) is automatically c.b. and \( \|\tilde{u}\|_{cb} = \|\tilde{u}\| \). Therefore, it remains to show that \( i_E : E \to K_p^r \) is completely contractive. By Lemma 1.1 it then suffices to prove

\[
\|I_{S_p^\alpha} \otimes i_E : S_p^{\alpha}[E] \to S_p^{\alpha}[K_p^r] \| \leq 1, \quad \forall n \geq 1.
\]

To this end we first need to identify the action of an operator in \( S_p^{\alpha}[K_p^r] \). Recall that \( K_p^r = S_p(\bar{K}, \mathbb{C}) \), and so \( S_p^{\alpha}[K_p^r] = S_p(\ell_2^p(\bar{K}), \ell_{p^\alpha}^2) \). Let \( y = (y_{ij}) \in S_p^{\alpha}[K_p^r] \). Considered as an operator in \( S_p(\ell_2^p(\bar{K}), \ell_{p^\alpha}^2) \), \( y \) and \( y^* \) act as follows: for any \( \beta = (\beta_k) \in \ell_2^p(\bar{K}) \) and \( \alpha = (\alpha_k) \in \ell_{p^\alpha}^2 \)

\[
y(\beta) = \left( \sum_j \langle \beta_j, y_{ij} \rangle \right)_{1 \leq i \leq n} \quad \text{and} \quad y^*(\alpha) = \left( \sum_i \alpha_i \bar{y}_{ij} \right)_{1 \leq j \leq n}.
\]

Then it is easy to deduce that the \( n \times n \) complex matrix \( yy^* \) is given by

\[
yy^* = \left( \sum_k \langle y_{jk}, y_{ik} \rangle \right)_{1 \leq i, j \leq n}.
\]

(Here \( \langle \cdot, \cdot \rangle \) stands for the scalar product in \( K \).)

Now let \( x = (x_{ij}) \in S_p^{\alpha}[E] \) and \( y = I_{S_p^\alpha} \otimes i_E(x) \). Then the discussion above yields

\[
yy^* = \left( \sum_k f(x_{ik}x_{jk}^*) \right)_{1 \leq i, j \leq n}.
\]

Let \( \alpha \in S_p^{\alpha}[E] \) of norm 1. Then

\[
\text{Tr}(\alpha yy^*) = \text{Tr} \otimes \text{tr}(\alpha \otimes f)(xx^*) \leq \|\alpha \otimes f\|_{(p/2)'} \|xx^*\|_{p/2} \leq \|xx^*\|_{p/2}.
\]

Taking the supremum over all such \( \alpha \), we obtain

\[
\|yy^*\|_{p/2} \leq \|xx^*\|_{p/2},
\]

and so

\[
\|I_{S_p^\alpha} \otimes i_E(x)\|_{S_p^{\alpha}[K_p^r]} \leq \|x\|_{S_p^{\alpha}[E]}.
\]

Therefore, \( i_E \) is completely contractive. Thus we have proved the equivalence between i)–iv).

Finally, suppose one of i)–iv) holds. Then from the previous arguments we see that all the relevant constants are equal. Moreover, from the proof of iv) \( \Rightarrow \) i), \( u \) factors as \( E \xrightarrow{i_E} K_p^r \xrightarrow{\tilde{u}} H_p^r \). From this one easily deduces that \( u \) admits a c.b. extension to the whole \( L_p(M) \). Indeed, let \( \bar{K} \) be the Hilbert space constructed from \( L_p(M) \) with respect to the semi-scalar product \( \langle b, a \rangle = f(ab^*) \) as previously. Then \( \bar{K} \) contains \( K \) as an isometric subspace. Let \( i_{L_p(M)} : L_p(M) \to \bar{K}_p^r \) be the natural inclusion. Then \( i_{L_p(M)} \) is completely contractive. Let \( P_K : \bar{K} \to K \) be the orthogonal projection. By the homogeneity of \( \bar{K}_p^r \), \( P_K \) is completely contractive. Then \( U = \tilde{u} P_K i_{L_p(M)} \) is the desired extension of \( u \).

**Remarks.** i) The equivalence in Proposition 2.4 does not hold for general \( R_p^q \)-2-convex spaces. In fact, a simple interpolation argument shows that if \( 2 \leq q < p \), any noncommutative \( L_q \) is \( R_p^q \)-2-convex. Consequently, \( R_q \) is \( R_p^q \)-2-convex for such \( p, q \). However, one can prove that the
complete boundedness of a map \( u : R_q \to R_p \) is not equivalent to the boundedness of \( I_{R_q} \otimes u : R_q[R_q] \to R_p[R_p] \). On the other hand, \( R_q \) is a quotient of a subspace of \( R_p \oplus_p C_p \) (see \( \text{[X3]} \)). Thus this example also shows that Proposition 2.4 does not hold for quotients of subspaces of noncommutative \( L_p \)-spaces too.

ii) Using the weight \( r_p \) introduced in the remark following Definition 2.2 part iii) of Proposition 2.4, we have that \( \pi_2, r_p(u) \leq c \), where \( \pi_2, r_p \) is the \((2, r_p)\)-summing norm defined in \( \text{section 5} \). Thus for any \( u \) as in Proposition 2.4, we have \( \| u \|_{cb} = \pi_2, r_p(u) \).

The proof of the implication iii) \( \Rightarrow \) i) of Theorem 1.2 in section 5 gives an alternative proof of the implication iv) \( \Rightarrow \) i) above (corresponding to \( \theta = 1 \) in Theorem 1.2). The proof given above has an advantage that it yields the natural factorization of \( u \) by a kind of change of density as in the commutative case. The natural inclusion \( i_E : E \to K^*_p \) constructed previously will be used several times in the sequel. For later reference let us explicitly record this as follows.

**Remark 2.5.** Let \( E \subset L_p(M) \) be a closed subspace with \( p > 2 \) and \( f \) a positive unit functional on \( L_{p/2}(M) \). Let \( K \) be the Hilbert space obtained from \( E \) relative to the semi-scalar product \( \langle b, a \rangle = f(ab^*) \). Then the natural inclusion \( i_E : E \to K^*_p \) is completely contractive. A similar statement holds for the semi-scalar product \( \langle b, a \rangle = f(b^*a) \) (the resulting Hilbertian operator space is then a \( p \)-column space).

**Theorem 2.6.** Let \( M \) be a von Neumann algebra and \( E, F \subset L_p(M) \) be closed subspaces \((2 < p < \infty)\). Let \( u : E \otimes F \to \mathbb{C} \) be a linear functional and \( c > 0 \) a constant. The following assertions are equivalent

i) \( u \) defines a continuous functional on \( E \otimes_{h_p} F \) of norm \( \leq c \).

ii) For all finite sequences \((a_k) \subset E\) and \((b_k) \subset F\)

\[
\left| \sum_k u(a_k \otimes b_k) \right| \leq c \left\| \left( \sum_k a_k a_k^* \right)^{1/2} \right\|_p \left\| \left( \sum_k b_k b_k^* \right)^{1/2} \right\|_p.
\]

iii) There are positive unit functionals \( f, g \in (L_{p/2}(M))^* \) such that

\[
|u(a \otimes b)| \leq c (f(aa^*))^{1/2} g(b^*b)^{1/2}, \quad a \in E, \ b \in F.
\]

iv) The associated linear map \( \bar{u} : E \to F^* \) belongs to \( \Gamma_{R_p}(E, F^*) \) and \( \gamma_{R_p}(\bar{u}) \leq c \).

Moreover, if one of these assertions holds, \( u \) has an extension to \( L_p(M) \otimes_{h_p} L_p(M) \) with the same norm.

We have a similar result for maps belonging to \( \Gamma_{C_p}(E, F^*) \).

**Proof.** Going back to the definition of the norm of \( E \otimes_{h_p} F \), we see that ii) is just a reformulation of i). The implication ii) \( \Rightarrow \) iii) is shown by a standard argument using the Hahn-Banach theorem as in the proof of iii) \( \Rightarrow \) iv) of Proposition 2.4. Conversely, iii) \( \Rightarrow \) ii) is a simple consequence of the Hölder inequality. It remains to show the equivalence iii) \( \Leftrightarrow \) iv).

First assume iii). Let \( H \) be the Hilbert space obtained from \( E \) relative the semi-scalar product \( \langle a, \ a' \rangle = f(a^*a') \) (see Remark 2.3). Similarly, let \( K \) be the Hilbert space associated with \( F \) and the semi-scalar product \( \langle b, \ b' \rangle = g(b^*b') \). Let \( i_E \) and \( i_F \) be the natural inclusions of \( E \) into \( H \), respectively, of \( F \) into \( K \). Then ii) implies that there is a bounded operator \( \bar{u} : H \to K \) with \( \| \bar{u} \| \leq c \) such that

\[
u(a, b) = \langle \bar{u} i_E(a), \ i_F(b) \rangle, \quad a \in E, \ b \in F.
\]

Thus we deduce

\[
u = i_F \bar{u} i_E.
\]

We now equip \( H \) (resp. \( K \)) with the operator space structure of \( H^p_v \) (resp. \( K^*_p \)). Then by Remark 2.4 \( i_E \) : \( E \to H^p_v \) and \( i_F \) : \( F \to K^*_p \) are completely contractive. Thus \( i_F^* : K^*_p \to F^* \) is also completely contractive. On the other hand, \( \bar{u} : H^p_v \to K^*_p \) is c.b. and has \( \| \bar{u} \| \) as its cb-norm. Set \( \alpha = i_E \bar{u} \) and \( \beta = i_F^* \bar{u} \). Then \( \alpha \in CB(E, H^p_v), \ \beta \in CB(H^p_v, F^*), \ \bar{u} = \beta \alpha \) and \( \| \alpha \|_{cb} \| \beta \|_{cb} \leq c \). Thus \( \bar{u} \in \Gamma_{R_p}(E, F^*) \) and \( \gamma_{R_p}(\bar{u}) \leq c \).

Conversely, assume iv). By Proposition 2.4 it is not hard to see that \( u \) satisfies ii). Therefore, the equivalence between i) \( - \) iv) has been proved.

Moreover, if iv) is verified, then by Proposition 2.4 \( u \) extends to \( L_p(M) \otimes_{h_p} L_p(N) \) with the same norm.
Theorem 2.5 has the following extension with almost the same proof.

**Remark 2.7.** Let $2 \leq p, q \leq \infty$ and $p \neq q$. Let $E \subset L_p(M)$ and $F \subset L_q(M)$ be closed subspaces. Let $u : E \otimes F \rightarrow \mathbb{C}$ be a linear functional and $c > 0$ a constant. The following assertions are equivalent

i) $u$ defines a continuous functional on $E \otimes_{b_{p,q}} F$ of norm $\leq c$.

ii) For all finite sequences $(a_k) \subset E$ and $(b_k) \subset F$

$$\left| \sum_k u(a_k \otimes b_k) \right| \leq c \left\| \left( \sum_k a_k a_k^* \right)^{1/2} \right\|_p \left\| \left( \sum_k b_k b_k^* \right)^{1/2} \right\|_q .$$

iii) There are positive unit functionals $f \in (L_{p/2}(M))^*$ and $g \in (L_{q/2}(M))^*$ such that

$$|u(a \otimes b)| \leq c \left( f(aa^*) g(b^*b) \right)^{1/2}, \quad a \in E, \ b \in F.$$

iv) There are positive unit functionals $f \in (L_{p/2}(M))^*$ and $g \in (L_{q/2}(M))^*$ such that the associated linear map $\tilde{u} : E \rightarrow F^*$ admits the following factorization

$$\begin{array}{ccc}
E & \xrightarrow{\tilde{u}} & F^* \\
\phi & \downarrow & \downarrow \phi \\
H_p^r & \xrightarrow{\tilde{u}} & K_q^r ,
\end{array}$$

where $i_E$ and $i_F$ are the natural inclusions associated to $f$ and $g$, respectively, given by Remark 2.6 and where $\tilde{u}$ is a bounded map.

In general, the map $\tilde{u}$ in iv) above cannot be chosen to be c.b.. Indeed, let $E = R_p$ and $F = C_q$ with $2 \leq p, q \leq \infty$. Then both $R_p[R_p]$ and $C_q[C_q]$ are isometrically identified with $S_2$. It follows that $u : R_p \otimes C_q \rightarrow \mathbb{C}$ satisfies ii) above iff $\tilde{u} : R_p \rightarrow C_q$ is bounded. However, by using the fact that $C_q \cong R_q$, it is not hard to prove that $B(R_p, C_q^*) \neq CB(R_p, C_q^*)$ for $p \neq q$.

**Remak 2.8.** Let $E, F$ be two operator spaces and $u : E \times F \rightarrow \mathbb{C}$ a bilinear form. Let $p, q \geq 2$. Mimicking the notion of completely bounded bilinear forms in Christensen - Sinclair’s sense (cf. [CS1], [CS2]), we say that $u$ is $(p, q)$-multiplicatively bounded if there is a constant $c$ such that for all $n \geq 1$ and all $a = (a_{ij}) \in S^n_p[E], \ b = (b_{ij}) \in S^n_q[F]$

$$\left\| \left( \sum_k u(a_{ik}, b_{kj}) \right)_{1 \leq i, j \leq n} \right\|_{S^n_p} \leq c \left\| a \right\|_{S^n_p[E]} \left\| b \right\|_{S^n_q[F]} ,$$

where $1/r = 1/p + 1/q$. Let $\|u\|_{(p,q)\text{-}mb}$ denote the smallest of such constants $c$. If $p = q$, $(p, q)$-multiplicatively bounded forms are simply called $p$-multiplicatively bounded forms. Now assume $E \subset L_p(M)$ and $F \subset L_q(M)$. Then $u$ is $(p, q)$-multiplicatively bounded iff one of the assertions in Remark 2.7 holds.

**Proof.** If $u$ is $(p, q)$-multiplicatively bounded, considering only row and column matrices in the definition above, we see that the assertion ii) of Remark 2.7 is verified. Conversely, assume that iv) of Remark 2.7 holds. Let $\tilde{u}$ have the factorization (2.6). Then for any $a \in E$ and $b \in F$, $u(a, b)$ can be written as a product of three operators:

$$u(a, b) = i_E(a) \circ \tilde{u}^* \circ i_F(b).$$

Recall that $i_F(b) \in K_q^r = S_q(\mathbb{C}, K)$ and $i_E(a) \in H_p^r = S_p(\overline{T}, \mathbb{C})$. Therefore, for any $a = (a_{ij}) \in S^n_p[E], \ b = (b_{ij}) \in S^n_q[F]$

$$\left( \sum_k u(a_{ik}, b_{kj}) \right)_{ij} = [I_{S^n_p} \otimes i_E(a)] \circ [I_{\ell_2^n} \otimes \tilde{u}^*] \circ [I_{S^n_q} \otimes i_F(b)].$$

Here $I_{S^n_p} \otimes i_F(b) \in S^n_q[K_q^r] = S_q(\ell_2^n, \ell_2^n(K)), I_{S^n_p} \otimes i_E(a) \in S^n_p[H_p^r] = S_p(\ell_2^n(\overline{T}), \ell_2^n)$ and $I_{\ell_2^n} \otimes \tilde{u}^* \in B(\ell_2^n(K), \ell_2^n(\overline{T}))$. Thus by the Hölder inequality

$$\left\| \left( \sum_k u(a_{ik}, b_{kj}) \right)_{ij} \right\|_{S^n_p} = \left\| [I_{S^n_p} \otimes i_E(a)] \circ [I_{\ell_2^n} \otimes \tilde{u}^*] \circ [I_{S^n_q} \otimes i_F(b)] \right\|_{S^n_p} \leq \left\| I_{S^n_p} \otimes i_E(a) \right\|_{S^n_p[H_p^r]} \left\| I_{\ell_2^n} \otimes \tilde{u}^* \right\| \left\| I_{S^n_q} \otimes i_F(b) \right\|_{S^n_q[K_q^r]} \leq \left\| \tilde{u} \right\| \left\| a \right\|_{S^n_p[E]} \left\| b \right\|_{S^n_q[F]} .$$
Therefore, \( u \) is \((p,q)\)-multiplicatively bounded, and so we have proved the announced result. \( \square \)

In particular, in the situation of Theorem 2.4 i.e. when \( p = q \) in Remark 2.5 \( u : E \times F \to \mathbb{C} \) is \( p \)-multiplicatively bounded if the associated linear map \( \bar{u} : E \to F^* \) belongs to \( \Gamma_{R_p}(E,F^*) \). Consequently, \( p \)-multiplicatively bounded forms are j.c.b.. Conversely, Theorem 0.1 implies that any j.c.b. form \( u : E \times F \to \mathbb{C} \) (still with \( E,F \subset L_p(M) \) and \( p \geq 2 \)) is the sum of a \( p \)-multiplicatively bounded form and the adjoint of a \( p \)-multiplicatively bounded form. However, if \( p \neq q \), \((p,q)\)-multiplicatively bounded forms are in general not j.c.b..

3. Noncommutative Khintchine inequalities

In this section we give the main ingredient of the proofs of Theorems 0.1 and 0.2. This is the noncommutative Khintchine type inequalities for generalized circular systems. In the sequel, \( \mathcal{H} \) will be a fixed infinite dimensional separable Hilbert space with an orthonormal basis \( \{e_{\pm k}\}_{k \geq 1} \). \( \mathcal{F}(\mathcal{H}) \) stands for the associated free Fock space:

\[
\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n,
\]

where \( \mathcal{H}^\otimes 0 = \mathbb{C} \Omega \) with \( \Omega \) a distinguished unit vector. Let \( \ell(e) \) (resp. \( \ell^*(e) \)) denote the left creation (resp. annihilation) operator associated with a vector \( e \in \mathcal{H} \). Recall that \( \ell^*(e) = (\ell(e))^* \).

Let \( 0 \leq \theta \leq 1 \) and \( \{\lambda_k\}_{k \geq 1} \) be a sequence of positive numbers. Let

\[
(3.1) \quad s_k = \ell(e_k) + \lambda_k^{-1}\ell^*(e_{-k}) \quad \text{and} \quad g_k = \lambda_k^\theta s_k, \quad k \geq 1.
\]

The \( s_k \) are generalised circular variables studied by Shlyakhtenko \( \mathbf{S} \). We will also call \( (g_k)_{k \geq 1} \) a generalized circular system (with parameters \( \theta \) and \( \lambda_k \)). Let \( \Gamma \) be the von Neumann algebra on \( \mathcal{F}(\mathcal{H}) \) generated by the \( s_k \) (or equivalently by \( g_k \)). Let \( \rho \) be the vector state on \( \Gamma \) determined by the vacuum \( \Omega \). By \( \mathbf{S} \), \( \rho \) is faithful on \( \Gamma \). Thus the Haagerup \( L_p \)-space \( L_p(\Gamma) \) can be constructed from \( \rho \). Let \( D \) denote the density of \( \rho \) in \( L_1(\Gamma) \). Recall that \( \rho \) can be recovered from \( D \) as follows:

\[ \rho(x) = \text{tr}(Dx), \quad x \in \Gamma. \]

Also recall that the modular group \( \sigma^\rho_t \) is given by

\[
\sigma^\rho_t(x) = D^{-t}xD^t, \quad x \in \Gamma, \quad t \in \mathbb{R}.
\]

The \( s_k \)'s are eigenvectors of the modular group \( \sigma^\rho_t \). More precisely, we have the following formulas from \( \mathbf{S} \)

\[
(3.2) \quad \sigma_t(s_k) = \lambda_k^{-it}s_k, \quad \sigma_t(g_k) = \lambda_k^{-it}g_k, \quad k \geq 1, \quad t \in \mathbb{R}
\]

(see \( \mathbf{S} \) pp.342-343); note that the minor difference on parameters \( \lambda \) between our definition of \( s_k \) above and that of \( y \) in \( \mathbf{S} \)). The sequence \( \{g_k\} \) satisfies the following orthogonality with respect to the state \( \rho \): For any \( 0 \leq \eta \leq 1 \)

\[
(3.3) \quad \text{tr}(g_j^*D^n g_k D^{1-\eta}) = \delta_{j,k} \lambda_k^{2(\theta-\eta)}.
\]

Indeed, by \( \mathbf{S} \), the left hand side of \( \mathbf{S} \) is equal to

\[
\text{tr}(g_j^*\sigma_{-\eta}(g_k)D) = \text{tr}(g_j^*\lambda_k^{-2\eta} g_k D) = \lambda_k^{-2\eta}(g_j \Omega, g_k \Omega) = \delta_{j,k} \lambda_k^{2(\theta-\eta)}.
\]

The following is the noncommutative Khintchine type inequalities for generalized circular systems. The case \( p = \infty \) was already obtained in \( \mathbf{PS} \).

**Theorem 3.1.** Let \( 1 \leq p \leq \infty \) and \( \theta = 1/p \). Let \( \{\lambda_k\} \) be a positive sequence. Set

\[
(3.4) \quad g_{k,p} = D_{\theta}^\frac{1}{p} g_k D_{\frac{1}{p}}^{-\theta}.
\]

where \( \{g_k\} \) is defined by \( \mathbf{S} \). Let \( M \) be a von Neumann algebra and \( (x_n) \) a finite sequence in \( L_p(M) \).
i) If \( p \geq 2 \),
\[
\max \left\{ \left\| \left( \sum_k \lambda_k^{2(1-\frac{1}{p})} x_k^* x_k \right)^{\frac{1}{2}} \right\|_p, \left\| \left( \sum_k \lambda_k^{-2(1-\theta)(1-\frac{1}{p})} x_k^* x_k \right)^{\frac{1}{2}} \right\|_p \right\} 
\leq \left\| \sum_k x_k \otimes g_{k,p} \right\|_p \leq B_p \max \left\{ \left\| \left( \sum_k \lambda_k^{2(1-\frac{1}{p})} x_k^* x_k \right)^{\frac{1}{2}} \right\|_p, \left\| \left( \sum_k \lambda_k^{-2(1-\theta)(1-\frac{1}{p})} x_k^* x_k \right)^{\frac{1}{2}} \right\|_p \right\}. 
\]
(3.5)

ii) If \( p < 2 \),
\[
A_p^{-1} \inf \left\{ \left\| \left( \sum_k \lambda_k^{2(1-\frac{1}{p})} a_k^* a_k \right)^{\frac{1}{2}} \right\|_p + \left\| \left( \sum_k \lambda_k^{-2(1-\theta)(1-\frac{1}{p})} b_k b_k^* \right)^{\frac{1}{2}} \right\|_p \right\} \leq \left\| \sum_k x_k \otimes g_{k,p} \right\|_p \leq \inf \left\{ \left\| \left( \sum_k \lambda_k^{2(1-\frac{1}{p})} a_k^* a_k \right)^{\frac{1}{2}} \right\|_p + \left\| \left( \sum_k \lambda_k^{-2(1-\theta)(1-\frac{1}{p})} b_k b_k^* \right)^{\frac{1}{2}} \right\|_p \right\},
\]
(3.6)

where the infimum runs over all decompositions \( x_k = a_k + b_k \) in \( L_p(M) \). The two positive constants \( A_p \) and \( B_p \) depend only on \( p \) and can be controlled by a universal constant.

iii) Let \( G_p \) be the closed subspace of \( L_p(\Gamma) \) generated by \( \{g_{k,p}\}_{k \geq 1} \). Then there is a completely bounded projection \( P_p : L_p(\Gamma) \to G_p \) such that
\[
\| P_p \|_b \leq 2^{1-\frac{1}{p}}.
\]

**Remarks.** i) The Khintchine inequalities above also play a crucial role in [X2] on the embedding of Pisier’s OH spaces, and more generally, the \( q \)-column spaces \( C_q \) into noncommutative \( L_p \)-spaces (\( 1 \leq p < q \leq 2 \)).

ii) [821] is a particular case of a more general inequality for free random series in [X2].

**Proof of Theorem [X2]:** i) The state \( \rho \) on \( \Gamma \) extends to a contractive functional on \( L_p(\Gamma) \) for all \( 1 \leq p \leq \infty \). More generally, let us consider the normal faithful conditional expectation \( \Phi \) defined by \( I_M \otimes \rho : M \otimes \Gamma \to M \). By [X2] Lemma 2.2, it extends to a contractive projection from \( L_p(M \otimes \Gamma) \) onto \( L_p(M) \) for all \( p \geq 1 \), still denoted by \( \Phi \) in the following.

By [S], the \( g_k \)'s are free in \( (\Gamma, \rho) \). Thus by [X2], given a finite sequence \( (x_k) \subset L_p(M) \) (\( 2 \leq p \leq \infty \)) we have
\[
S \leq \left\| \sum_k x_k \otimes g_{k,p} \right\|_p \leq B_p S,
\]
where
\[
S = \max \left\{ 2^{1-\frac{1}{p}} \left( \sum_k \left\| x_k \otimes g_{k,p} \right\|_p^{\frac{1}{p}} \right)^{\frac{1}{2}}, \left\| \left( \sum_k \Phi(x_k^* x_k \otimes g_{k,p}^* g_{k,p}) \right)^{\frac{1}{2}} \right\|_p, \left\| \left( \sum_k \Phi(x_k^* x_k \otimes g_{k,p}^* g_{k,p}) \right)^{\frac{1}{2}} \right\|_p \right\}.
\]

By [822],
\[
g_{k,p} = \sigma^{-\frac{2(1-\theta)}{p}}(g_k) D^{\frac{1}{2}} = \lambda_k^{-\frac{2(1-\theta)}{p}} g_k D^{\frac{1}{2}}.
\]
Thus,
\[
\Phi(x_k^* x_k \otimes g_{k,p}^* g_{k,p}) = x_k^* x_k \otimes \left[ \lambda_k^{-\frac{2(1-\theta)}{p}}(g_k^* g_k) D^{\frac{1}{2}} \right] = \lambda_k^{2(1-\frac{1}{p})} x_k^* x_k \otimes D^{\frac{1}{2}}.
\]
Therefore,
\[
\left\| \left( \sum_k \Phi(x_k^* x_k \otimes g_{k,p}^* g_{k,p}) \right)^{\frac{1}{2}} \right\|_p = \left\| \left( \sum_k \lambda_k^{2(1-\frac{1}{p})} x_k^* x_k \right)^{\frac{1}{2}} \right\|_p.
\]
Similarly,
\[
\left\| \left( \sum_k \Phi(x_k^* x_k \otimes g_{k,p}^* g_{k,p}) \right)^{\frac{1}{2}} \right\|_p = \left\| \left( \sum_k \lambda_k^{-2(1-\theta)(1-\frac{1}{p})} x_k^* x_k \right)^{\frac{1}{2}} \right\|_p.
\]
Combining the previous inequalities, we get the lower estimate in [821].
To prove the upper estimate we only need to show that the term \((\sum_k \|x_k \otimes g_k,p\|_p)^{1/p}\) is controlled by the two others. To this end we first observe that
\[
\|g_k,p\|_p \leq \|g_k\|_2 \|g_k,2\|_2 \leq (\lambda_k^\theta + \lambda_k^{-1+\theta})^{-1/\theta};
\]
whence
\[
\|x_k \otimes g_k,p\|_p \leq (\lambda_k^\theta + \lambda_k^{-1+\theta})^{-1/\theta} \|x_k\|_p .
\]
Thus
\[
(\sum_k \|x_k \otimes g_k,p\|_p)^{1/\theta} \leq (\sum_k \lambda_k^{(1-\theta)/2} \|x_k\|_p)^{1/\theta} + (\sum_k \lambda_k^{-(1-\theta)/(1-\theta)} \|x_k\|_p)^{1/\theta}.
\]
However, for any \((y_k) \subset L_p(M)\) we have
\[
(\sum_k \|y_k\|_p)^{1/\theta} \leq \min \left\{ \| (\sum_k y_k^* y_k)^{1/2} \|_p, \| (\sum_k y_k y_k^*)^{1/2} \|_p \right\}.
\]
Therefore,
\[
(\sum_k \|x_k \otimes g_k,p\|_p)^{1/\theta} \leq \left( \sum_k \lambda_k^{2\theta (1-\theta)/2} x_k^* x_k \right)^{1/\theta} + \left( \sum_k \lambda_k^{-2(1-\theta)/(1-\theta)} x_k^* x_k \right)^{1/\theta} .
\]
whence the upper estimate in (3.5).

ii) The minoration here follows from the majoration in i) by a simple duality argument. Indeed, let \((y_k) \subset L_{p'}(M)\) be a finite sequence such that
\[
\max \left\{ \left\| \left( \sum_k \lambda_k^{2\theta (1-\theta)/2} y_k^* y_k \right)^{1/2} \right\|_{p'}, \left\| \left( \sum_k \lambda_k^{-2(1-\theta)/(1-\theta)} y_k y_k^* \right)^{1/2} \right\|_{p'} \right\} \leq 1 .
\]
Let
\[
x = \sum_k x_k \otimes g_k,p \quad \text{and} \quad y = \sum_k y_k \otimes g_k,p' .
\]
Then by (3.3)
\[
\sum_k \text{tr}(y_k^* x_k) = \text{tr} \otimes \text{tr}(y^* x)
\]
and by i)
\[
\|y\|_{p'} \leq B_{p'} \max \left\{ \left\| \left( \sum_k \lambda_k^{2\theta (1-\theta)/2} y_k^* y_k \right)^{1/2} \right\|_{p'}, \left\| \left( \sum_k \lambda_k^{-2(1-\theta)/(1-\theta)} y_k y_k^* \right)^{1/2} \right\|_{p'} \right\} \leq B_{p'}.
\]
Hence
\[
\left| \sum_k \text{tr}(y_k^* x_k) \right| \leq \|x\|_p \|y\|_{p'} \leq B_{p'} \|x\|_p .
\]
Taking the supremum over all \((y_k)\) as above yields the lower estimate in (3.5) with \(A_p = B_{p'}\). The majoration is a consequence of the following elementary inequality (with \(1 \leq p \leq 2\))
\[
\|x\|_p \leq \left\| \left( \Phi(x^* x) \right)^{1/2} \right\|_p , \quad \forall x \in L_p(M \otimes \Gamma).
\]
By duality, this immediately follows from
\[
\|\Phi(y^* y)\|_{p'/2} \leq \|y^* y\|_{p'/2} , \quad \forall y \in L_{p'}(M \otimes \Gamma).
\]

iii) The following type of arguments is rather standard today (cf. [HP2] in the case of free groups).
Recall that \(\Omega\) is a separating vector for \(\Gamma\) (cf. [S]). Thus any operator \(a \in \Gamma\) is uniquely determined by \(a\Omega\). Set \(\Gamma_2 = \{a\Omega : a \in \Gamma\} \). Then \(\Gamma_2\) is a vector subspace of \(\mathcal{H}(\Gamma)\), which is isometric to \(L_2(\Gamma)\). For any \(\xi \in \Gamma_2\) we denote by \(W(\xi)\) the unique operator in \(\Gamma\) such that \(W(\xi)\Omega = \xi\). It is well known (and easy to check) that all tensors from \(\mathcal{H}(\Omega)^n\) belong to \(\Gamma_2\) \((n \in \mathbb{N})\). Now we use multi-index notation. Recall that \(\{e_{i\pm k}\}_{k \geq 1}\) is an orthonormal basis of \(\mathcal{H}\). For any \(i_1, \ldots, i_n \in \{0, 1, \ldots, n\}\) we put \(i = (i_1, \ldots, i_n)\) and \(e_i = e_{i_1} \otimes \cdots \otimes e_{i_n}\). If \(i = 0\), we set \(e_i = \Omega\). Then \(\{e_i\}_{i \in \mathbb{N}}\) is an orthonormal basis of \(\mathcal{F}(\mathcal{H})\). By the discussion above, every \(e_i\) belongs to \(\Gamma_2\) and \(W(e_i)\) is the unique operator in \(\Gamma\) such that \(W(e_i)\Omega = e_i\). By the definition of \(g_k\) in (3.3), we have
\[
W(e_k) = \lambda_k^{-\theta} g_k \quad \text{and} \quad W(e_{-k}) = \lambda_k^{1-\theta} g_k , \quad k \geq 1.
\]
Given $a \in \Gamma$, developing $a\Omega$ in the orthonormal basis $\{e_i\}$ we can (symbolically) write

$$a = \sum_{i} c_i(a) W(e_i),$$

where $c_i = \langle e_i, a\Omega \rangle$. Let $\Gamma_0$ be the *-subalgebra of $\Gamma$ of all operators which admit a finite development as above. Note that $\Gamma_0$ is w*-dense in $\Gamma$. Consequently, $D^{1/2p} \Gamma_0 D^{1/2p}$ is dense in $L_p(\Gamma)$ for any $1 \leq p < \infty$ (cf. [JX1, Lemma 1.1]; see also [J1]).

Now for any $a \in \Gamma_0$ we define

$$P_\infty(a) = \sum_{k \geq 1} c_k(a) W(e_k)$$

and for $1 \leq p < \infty$

$$P_p : D_{\frac{1}{p}} \Gamma_0 D_{\frac{1}{p}} \rightarrow D_{\frac{1}{p}} \Gamma_0 D_{\frac{1}{p}} \text{ by } P_p(D_{\frac{1}{p}} a D_{\frac{1}{p}}) = D_{\frac{1}{p}} P_\infty(a) D_{\frac{1}{p}}.$$

We are going to show that $P_p$ extends to a completely bounded projection from $L_p(\Gamma)$ onto $G_p$. In fact, what will be shown is that $I \otimes P_p$ extends to a bounded projection from $L_p(M \otimes \Gamma)$ onto $L_p(M) \otimes G_p$ for any von Neumann algebra $M$, where $L_p(M) \otimes G_p$ is the closure of $L_p(M) \otimes G_p$ in $L_p(M \otimes \Gamma)$ (relative to the w*-topology for $p = \infty$). This is clear for $p = 2$; moreover, the extension of $I \otimes P_2$ is the orthogonal projection from $L_2(M \otimes \Gamma)$ onto $L_2(M) \otimes G_2$.

Then consider the case $p = \infty$. Let $\{e_i\} \subseteq M$ be a finite family in $M$. Let $x = \sum_{i} x_i \otimes W(e_i) \in M \otimes \Gamma$.

Then

$$I \otimes P_\infty(x) = \sum_{k \geq 1} x_k \otimes W(e_k) = \sum_{k} \lambda_k^{-\theta} x_k \otimes g_k.$$

Therefore, by (38)

$$\|I \otimes P_\infty(x)\|_\infty \leq 2 \max \left\{\left\|\left(\sum_k x_k^* x_k\right)^{1/2}\right\|_\infty, \left\|\left(\sum_k \lambda_k^{-2\theta} x_k x_k^*\right)^{1/2}\right\|_\infty\right\}.$$

Let $\xi \in H$ be a unit vector ($H$ being the Hilbert space at which $M$ acts). Then by the orthornormality of $\{e_i\}$

$$\|x\|_\infty^2 \geq \langle x(\xi \otimes \Omega), x(\xi \otimes \Omega) \rangle = \sum_{i} \|x_i(\xi)\|^2 = \langle \xi, \sum_{i} x_i^* x_i(\xi) \rangle \geq \langle \xi, \sum_{k} x_k^* x_k(\xi) \rangle;$$

whence

$$\|\sum_{k} x_k^* x_k\|_\infty \leq \|x\|_\infty^2.$$

On the other hand,

$$\langle x^*(\xi \otimes \Omega), x^*(\xi \otimes \Omega) \rangle = \sum_{i} \langle x_i^* x_i(\xi), x_i^* x_i(\xi) \rangle \langle W(e_i)^*(\Omega), W(e_i)^*(\Omega) \rangle$$

$$= \sum_{i} \langle \xi, x_i x_i^*(\xi) \rangle \rho[W(e_i)W(e_i)^*]$$

$$= \sum_{i} \langle \xi, x_i x_i^*(\xi) \rangle \rho[W(e_i)^*\sigma_{-i}(W(e_i))].$$

However, by [S] one can easily show

$$\sigma_{-i}(W(e_i)) = \beta_i^{-\theta} W(e_i),$$

where $\beta_i$ is a finite product involving $\lambda_k^{\pm \theta}$ and $\lambda_k^{(1-\theta)}$. Then it follows that

$$\rho[W(e_i)^*\sigma_{-i}(W(e_i))] = (W(e_i)\Omega, \sigma_{-i}(W(e_i))\Omega) = \delta_{i,j} \beta_i^2.$$
Hence
\[
\langle x^* (\xi \otimes \Omega), x^* (\xi \otimes \Omega) \rangle \geq \sum_{k \geq 1} \langle \xi, x_k x_k^* (\xi) \rangle \rho [W(e_k)^* \sigma_{-1}(W(e_k))] = \sum_{k \geq 1} \langle \xi, x_k x_k^* (\xi) \rangle \lambda_k^{-2} = \langle \xi, \sum_{k \geq 1} \lambda_k^{-2} x_k x_k^* (\xi) \rangle.
\]
Therefore
\[
\left\| \sum_{k \geq 1} \lambda_k^{-2} x_k x_k^* \right\|_{\infty} \leq \|x\|_{\infty}^2.
\]
Combining the preceding inequalities, we obtain
\[
\|I \otimes P_{\infty}(x)\|_{\infty} \leq 2\|x\|_{\infty}.
\]
This is the key inequality of this part of the proof. From this we get the extension property for \(p = 1\) by virtue of the following easily checked duality equality: for any \(a, b \in \Gamma_0\) and \(1 \leq p \leq \infty\)
\[
\langle P_p(D_{\Gamma} b D_{\Gamma}), D_{\Gamma} a D_{\Gamma} \rangle = \langle D_{\Gamma} b D_{\Gamma}, P_{p'}(D_{\Gamma} a D_{\Gamma}) \rangle.
\]
Indeed, by this equality combined with the preceding boundedness of \(P_p\) on \(\Gamma_0\), we deduce that \(I \otimes P_1\) is bounded on \(L_1(M) \otimes [D_{\Gamma} \Gamma_0 D_{\Gamma}]\) with respect to the \(L_1\)-norm and is of norm \(\leq 2\). Thus by the density of \(D_{\Gamma} \Gamma_0 D_{\Gamma}\) in \(L_1(\Gamma)\), we deduce that \(I \otimes P_1\) extends to a bounded map on \(L_1(M \otimes \Gamma)\). By duality once more, we see that the adjoint of this extension of \(I \otimes P_1\) yields the desired (normal) extension of \(I \otimes P_{\infty}\) on \(M \otimes \Gamma\). The remaining case for \(1 < p < 2\) or \(2 < p < \infty\) is proved by Kosaki’s interpolation theorem \([Ko]\).
Therefore the proof of Theorem 3.1 is complete.

4. VECTOR-VALUED NONCOMMUTATIVE \(L_p\)

This section contains the second main tool of the proofs of Theorems 3.1 and 3.2, i.e., the vector-valued noncommutative \(L_p\)-spaces for QWEP von Neumann algebras. The theory of vector-valued noncommutative \(L_p\)-spaces was first developed by Pisier \([P2]\) for injective semifinite von Neumann algebras. Very recently, Junge \([J2], [J3]\) partly extended this theory to QWEP von Neumann algebras. Junge’s idea is to represent QWEP von Neumann algebras as images of normal conditional expectations on ultraproducts of injective von Neumann algebras, and then apply Pisier’s theory. His approach relies heavily upon the theory of ultraproducts of noncommutative \(L_p\)-spaces developed recently by Raynaud \([R]\).

We first recall some known results on the ultraproducts of von Neumann algebras and noncommutative \(L_p\)-spaces. Let \(U\) be a free ultrafilter on some index set \(I\). If \(X\) is an operator space, we use the notation \(X^{\mathcal{U}}\) to denote the ultrapower of \(X\). \(X^{\mathcal{U}}\) is equipped with its natural operator space structure as introduced by Pisier (cf. \([P1]\)). Now let \(M\) be a von Neumann algebra. Groh proved that the ultrapower \((M_e)^{\mathcal{U}}\) of the predual \(M_e\) is again a predual of von Neumann algebra (cf. \([G]\); see also \([K]\)). In fact, assuming \(M\) acts standardly on some Hilbert space \(H\), the ultrapower \(M^{\mathcal{U}}\) is a C*-algebra, which can be naturally represented on the Hilbert space ultrapower \(H^{\mathcal{U}}\). Then the von Neumann algebra \(((M_e)^{\mathcal{U}})^*\) is the w*-closure of \(M^{\mathcal{U}}\) in \(B(H^{\mathcal{U}})\). In the sequel we will denote \(((M_e)^{\mathcal{U}})^*\) by \(M^{\mathcal{U}}\). On the other hand, Raynaud developed the theory of ultraproducts of noncommutative \(L_p\)-spaces. In particular, he proved that the ultrapower \((L_p(M))^{\mathcal{U}}\) can be identified with \(L_p(M^{\mathcal{U}})\); moreover, this identification is natural in the sense that it preserves all algebraic operations such as product, involution, positivity ...

Recall that a C*-algebra is called WEP (for weak expectation property) in Lance’s sense if the natural inclusion \(A \hookrightarrow A^{**}\) can be factorized completely contractively through some \(B(H)\). \(A\) is called QWEP if \(A\) is a quotient of a WEP C*-algebra (cf. \([K]\)).

The following characterization of QWEP due to Junge \([J2]\) will play an important role later.

**Proposition 4.1.** A von Neumann algebra \(M\) is QWEP iff there are a Hilbert space \(H\) and a free ultrafilter \(U\) on some index set \(I\) such that \(M\) is the image of a normal conditional expectation on \(B(H)^{\mathcal{U}}\).

Let \(M\) and \(B(H)^{\mathcal{U}}\) be as in the proposition above. Let \(\Phi : B(H)^{\mathcal{U}} \rightarrow M\) be the corresponding normal conditional expectation (\(\Phi\) is, in general, not faithful). In this case, \(L_p(M)\) can be naturally identified as a subspace of \(L_p(B(H)^{\mathcal{U}})\). It is also known that \(\Phi\) defines a contractive projection \(\Phi_p\).
from $L_p(B(H)_{\mathcal{U}})$ onto $L_p(M)$ for any $1 \leq p \leq \infty$ (cf. [JX1 Proposition 2.3]). Let $N$ be any von Neumann algebra. Then $I_{L_p(N)} \otimes \Phi_p$ is also a contractive projection from $L_p(N \overline{\otimes} B(H)_{\mathcal{U}})$ onto $L_p(N \overline{\otimes} M)$. Indeed, it is obvious that $I_N \otimes \Phi$ is a normal conditional expectation from $N \overline{\otimes} B(H)_{\mathcal{U}}$ onto $N \overline{\otimes} M$. Applying the previous result to $I_N \otimes \Phi$, we get the announced one. In particular, $\Phi_p$ is completely contractive.

Now we introduce the vector-valued noncommutative $L_p$-spaces for QWEP von Neumann algebras. Let $M$ and $H$ be as in Proposition 4.1. Given an operator space $X$, the $X$-valued Schatten class $S_p[H; X]$ is defined in section 1. Now let $E$ be a finite dimensional operator space. Following Junge, we define $L_p(M; E)$ ($1 \leq p \leq \infty$) simply as the ultrapower $(S_p[H; E])^{\mathcal{U}}$. Then for any operator space $X$, $L_p[M; X]$ is defined as the closure in $(S_p[H; X])^{\mathcal{U}}$ of $L_p[M; E]$ when $E$ runs over all finite dimensional spaces of $X$.

We will need the following from [J3].

**Proposition 4.2.** Let $M$ be QWEP and $1 \leq p < \infty$.

i) Let $u : E \to F$ be a c.b. map between operator spaces. Then $I_{L_p(M)} \otimes u$ extends to a c.b. map from $L_p[M; E]$ into $L_p[M; F]$ and $\|I_{L_p(M)} \otimes u\|_{cb} \leq \|u\|_{cb}$. Moreover, if $u$ is a complete isometry, then so is $I_{L_p(M)} \otimes u$.

ii) Let $N$ be another von Neumann algebra. Then

$$L_p[M; L_p(N)] = L_p(M \overline{\otimes} N)$$

completely isometrically.

Note that part i) above easily follows from the definition and the corresponding results on the vector-valued Schatten classes in [P2]. Part ii) is more substantial. It is a consequence of Junge’s noncommutative Fubini theorem (cf. [J2]). We refer to [P2] and [J3] for more details.

**Proposition 4.3.** Let $E$ be an operator space, $1 \leq p < q \leq \infty$ and $1/r = 1/p - 1/q$.

i) If $q < \infty$, then for any $x \in L_q(B(H)_{\mathcal{U}}) \otimes E$

$$\|x\|_{L_q[B(H)_{\mathcal{U}}; E]} = \sup \{\|axb\|_{L_p[B(H)_{\mathcal{U}}; E]} : a, b \in S_p[H; E], \|a\|_2 \leq 1, \|b\|_2 \leq 1\},$$

where the supremum runs over all $a$ and $b$ in the unit ball of $L_2p(B(H)_{\mathcal{U}})$.

ii) Assume $q = \infty$ (so $r = p$). If $E$ is exact, then for any $x \in B(H)_{\mathcal{U}} \otimes E$

$$\|x\|_{B(H)^{\mu \otimes \min E}} \leq \sup \{\|axb\|_{L_p[B(H)_{\mathcal{U}}; E]} : a, b \in S_p[H; E], \|a\|_2 \leq 1, \|b\|_2 \leq 1\} \leq \lambda \|x\|_{B(H)^{\mu \otimes \min E}},$$

where the supremum runs over all $a$ and $b$ in the unit ball of $L_2p(B(H)_{\mathcal{U}})$, and where $\lambda = ex(E)$ is the exactness constant of $E$. Conversely, if (4.2) holds for some infinite dimensional Hilbert space $H$ and some constant $\lambda$, then $E$ is exact and $ex(E) \leq \lambda$.

**Proof.** i) Let $x \in L_q(B(H)_{\mathcal{U}}) \otimes E$. Passing to a finite dimensional subspace if necessary, we may assume $E$ finite dimensional. Then by definition

$$L_q[B(H)_{\mathcal{U}}; E] = (S_q[H; E])^{\mathcal{U}}.$$

By (1.5), for any $x_i \in S_q[H; E]$ we have

$$\|x_i\|_{S_q[H; E]} = \sup \{\|a_i x_i b_i\|_{S_p[H; E]} : a_i, b_i \in S_{2r}(H), \|a_i\|_{2r} \leq 1, \|b_i\|_{2r} \leq 1\}.$$

On the other hand, by [Ra] for any $t < \infty$

$$L_t(B(H)_{\mathcal{U}}) = (S_t(H))^{\mathcal{U}}.$$

Combining these, we easily deduce (1.1).

ii) Again, we can assume $E$ finite dimensional. By (1.4), for any $x_i \in B(H) \otimes E$

$$\|x_i\|_{B(H)^{\otimes \min E}} = \sup \{\|a_i x_i b_i\|_{S_p[H; E]} : a_i, b_i \in S_{2p}(H), \|a_i\|_{2p} \leq 1, \|b_i\|_{2p} \leq 1\}.$$

Thus as before, we get

$$\|x\|_{(B(H)^{\otimes \min E})^{\mathcal{U}}} = \sup \{\|axb\|_{L_p[B(H)_{\mathcal{U}}; E]} : a, b \in L_2p(B(H)_{\mathcal{U}}), \|a\|_{2p} \leq 1, \|b\|_{2p} \leq 1\}.$$

Therefore (4.2) can be rewritten as

$$\|x\|_{B(H)^{\mu \otimes \min E}} \leq \|x\|_{(B(H)^{\otimes \min E})^{\mathcal{U}}} \leq \lambda \|x\|_{B(H)^{\mu \otimes \min E}}.$$
It is known that the first inequality above is always true, while the validity of the second is equivalent to the exactness of $E$; moreover, the least constant $\lambda$ is then equal to the exactness constant of $E$. We omit the details and refer to [P1] Chapter 17.

\textbf{Corollary 4.4.} Let $p, q, r$ be as in Proposition 4.3. Given $a, b \in L_2(B(H)_{\mathcal{U}t})$ we define $M_{a,b}(x) = axb$.

i) If $q < \infty$, $M_{a,b}$ defines a c.b. map from $L_q[B(H)_{\mathcal{U}t}; E]$ into $L_p[B(H)_{\mathcal{U}t}; E]$ and $\|M_{a,b}\|_{cb} \leq \|a\|_{2r}\|b\|_{2r}$.

ii) If $q = \infty$ and $E$ is exact, $M_{a,b}$ defines a c.b. map from $(B(H))^t \otimes_{\min} E$ into $L_p[B(H)_{\mathcal{U}t}; E]$ and $\|M_{a,b}\|_{cb} \leq ex(E)\|a\|_{2r}\|b\|_{2r}$.

\textbf{Proof.} We only prove i). The proof for ii) is similar. It is immediate from Proposition 4.3 that $M_{a,b}$ is bounded and $\|M_{a,b}\| \leq \|a\|_{2r}\|b\|_{2r}$. (Only this boundedness will be needed later.) To prove the complete boundedness we use Lemma 11 so we have to show

$$\left\|S_q^n \otimes_{\min} M_{a,b} : \left[ L_q[B(H)_{\mathcal{U}t}; E]\rightarrow S_q^n [L_p[B(H)_{\mathcal{U}t}; E]] \right]\right\| \leq \|a\|_{2r}\|b\|_{2r}, \quad \forall n \in \mathbb{N}.$$ 

However,

$$S_q^n [L_q[B(H)_{\mathcal{U}t}; E]] = L_q[B(\ell_q^2(H)_{\mathcal{U}t}); E].$$

Thus for any $\alpha, \beta \in S_q^2$, $M_{\alpha,\beta}$ is bounded from $L_q[B(\ell_q^2(H)_{\mathcal{U}t}); E]$ to $L_p[B(\ell_q^2(H)_{\mathcal{U}t}); E]$ and of norm $\leq \|\alpha \otimes \beta \|_{2r} = \|\alpha\|_{2r}\|\beta\|_{2r}$. Taking the supremum over all $\alpha$ and $\beta$ such that $\|\alpha\|_{2r} \leq 1$ and $\|\beta\|_{2r} \leq 1$, and using (4.3), we deduce the announced result. \hfill $\square$

\section{Proof of Theorem 4.1}

This section and the next are devoted to the proofs of Theorems 4.1 and 4.2 respectively. The common key ingredient of both proofs is Theorem 3.1. The patterns of our proofs are similar to those of the corresponding results for $L_{\infty}$ in [PS] and [P7].

\textbf{Proof of Theorem 4.1 i) $\Rightarrow$ ii).} Let $(\mu_k)_k$ be a sequence of positive numbers. Set $\lambda_k = \mu_k^{p/(p-2)}$. Let $\{g_k\}_k$ be the generalized circular system with parameters $(\lambda_k)_k$ and $\theta = 1/2$ defined by (3.1). We will use the notations introduced in section 3: $\Gamma$ is the von Neumann algebra generated by the $g_k, \rho$ the normal faithful state on $\Gamma$ given by the vacuum and $D$ the density of $\rho$ in $L_1(\Gamma)$. It was proved in [PS] that $\Gamma$ is QWEP. Thus by Proposition 4.1, $\Gamma$ is the image of a normal conditional expectation $\Phi$ on some $B(H)_{\mathcal{U}t}$. Consequently, $L_p(\Gamma)$ can be naturally identified as a subspace of $L_p(B(H)_{\mathcal{U}t})$.

To prove ii) we can clearly assume $E$ and $F$ finite dimensional. Then by definition, $L_p(\Gamma; E)$ is a subspace of $L_p[B(H)_{\mathcal{U}t}; E]$. By Proposition 4.2

$$w \overset{\text{def}}{=} I_{L_p[B(H)_{\mathcal{U}t}]} \otimes \bar{u} : L_p[B(H)_{\mathcal{U}t}; E] \rightarrow L_p[B(H)_{\mathcal{U}t}; F^*] \quad \text{is bounded.}$$

On the other hand, letting $\frac{1}{r} = \frac{1}{p} - \frac{1}{2} = 1 - \frac{2}{p}$ (so $r$ is the conjugate index of $p/2$) and by Corollary 4.3 (noting that $\|D^{1/2r}\|_{2r} = 1$),

$$w \overset{\text{def}}{=} M_{D^{1/2r},D^{1/2r}} : L_p[B(H)_{\mathcal{U}t}; F^*] \rightarrow L_{p'}[B(H)_{\mathcal{U}t}; F^*] \quad \text{is contractive.}$$

Therefore,

$$\left\|wv : L_p[B(H)_{\mathcal{U}t}; E] \rightarrow L_{p'}[B(H)_{\mathcal{U}t}; F^*] \right\| \leq \|u\|_{jcb}.$$ 

Now let $(a_k) \subset E$ be a finite sequence. Then (recalling that $g_k, p$ is defined by (3.1) with $\theta = \frac{1}{2}$)

$$wv\left(\sum_k a_k \otimes g_{k,p}\right) = \sum_k \bar{u}(a_k) \otimes g_{k,p}.$$ 

Thus

$$\left\|\sum_k \bar{u}(a_k) \otimes g_{k,p}\right\| \leq \|wv\|_{jcb} \left\|\sum_k a_k \otimes g_{k,p}\right\|_{L_p[B(H)_{\mathcal{U}t}; E]}.$$ 

Let $F^\perp \subset L_{p'}(M)$ be the orthogonal complement of $F$. Then

$$L_{p'}[B(H)_{\mathcal{U}t}; F^*] = (S_{p'}[H; F^*])^{\mathcal{U}t} = \left(\frac{S_{p'}[H; L_{p'}(M)]^{\mathcal{U}t}}{S_{p'}[H; F^\perp]}\right)^{\mathcal{U}t}.$$
On the other hand, by Proposition 2.2 we have the following isometric inclusions
\[ L_1[B(H)_{lt}; X] \subset L_1[B(H)_{lt}; L_1(M)] = L_1(B(H)_{lt} \overline{\otimes} M) \subset (S_1[H; L_1(M)])^\mathcal{U} \]
for any \(1 \leq t < \infty\) and any subspace \(X \subset L_t(M)\). It is clear that
\[ (S_\mathcal{U}^{\mathcal{U}}[H; F^{\perp}])^\mathcal{U} \subset (L_\mathcal{U}^{\mathcal{U}}[B(H)_{lt}; F])^{\perp}. \]

Given a von Neumann algebra \(N\) we use the following duality bracket between \(L_p(N)\) and \(L_p^{\mathcal{U}}(N)\) in the category of operator spaces
\[ (y, x) = \text{tr}(\overline{y}^* x), \quad x \in L_p(N), \ y \in L_p^{\mathcal{U}}(N). \]
This duality is consistent with the operator space structure on \(L_p^{\mathcal{U}}(N)\) (recalling that \(L_1^{\mathcal{U}}(N)\) is the predual of \(N^{\mathcal{U}}\) and \(N^{\mathcal{U}} \cong \overline{N}\)). With this duality, the dual of \(L_p(N)\) is \(L_p^{\mathcal{U}}(N)\) completely isometrically.

Note that by 3.3 we have
\[ (g_j, p, g_k, p) = \text{tr}(g_j^{\mathcal{U}} D_{j}^{\mathcal{U}} g_k D_{k}^{\mathcal{U}}) = \delta_{j,k}. \]
Then for any finite sequence \((b_k) \subset F\) we deduce that
\[
\left| \sum_k u(a_k, b_k) \right| = \left| \sum_k \bar{u}(a_k)(b_k) \right|
= \left| \left( \sum_k \bar{u}(a_k) \otimes g_k, p, \sum_k b_k \otimes g_k, p \right) \right|
\leq \left\| \sum_k b_k \otimes g_k, p \right\|_{L_p[B(H)_{lt}; F]} \left\| \sum_k \bar{u}(a_k) \otimes g_k, p \right\|_{L_p^{\mathcal{U}}[B(H)_{lt}; F^{\perp}]}. \]
Therefore, combining the preceding inequalities with Theorem 3.1 we get
\[
\left| \sum_k u(a_k, b_k) \right| \leq \|u\|_{jcb} \left( \left\| \sum_k a_k \otimes g_k, p \right\|_{L_p[B(H)_{lt}; E]} \left\| \sum_k b_k \otimes g_k, p \right\|_{L_p[B(H)_{lt}; F]} \right)
\leq \|u\|_{jcb} \left( \left\| \sum_k a_k \otimes g_k, p \right\|_{L_p(B(1 \overline{\otimes} M))} \left\| \sum_k b_k \otimes g_k, p \right\|_{L_p(1 \overline{\otimes} M)} \right)
\leq B_p\|u\|_{jcb} \left[ \left\| \left( \sum_k \lambda_k^{\frac{1}{2}} a_k^* a_k \right)^{\frac{1}{2}} \right\|_p + \left\| \left( \sum_k \lambda_k^{\frac{1}{2}} b_k^* b_k \right)^{\frac{1}{2}} \right\|_p \right]. \]
This is 3.3 by the relation between \(\lambda_k\) and \(\mu_k\).

ii) \(\Rightarrow\) iii). This is done by a standard Hahn-Banach separation argument as in [PS]. For completeness, we include the main lines. Assume all the \(\mu_k\) are equal, say, to \(s\). Then by 3.3
\[
\left| \sum_k u(a_k, b_k) \right| \leq 2K_2 \left[ \left\| \sum_k s a_k^* a_k \right\|_{\mathcal{F}}^{\frac{1}{2}} + \left\| \sum_k s^{-1} a_k a_k^* \right\|_{\mathcal{F}}^{\frac{1}{2}} \right]^{\frac{1}{2}}
\leq K_2 \left[ \left\| \sum_k s a_k^* a_k \right\|_{\mathcal{F}}^{\frac{1}{2}} + \left\| \sum_k s^{-1} a_k a_k^* \right\|_{\mathcal{F}}^{\frac{1}{2}} \right]^{\frac{1}{2}}
\leq K_2 \left[ \left\| \sum_k s a_k^* a_k \right\|_{\mathcal{F}}^{\frac{1}{2}} + \left\| \sum_k s^{-1} a_k a_k^* \right\|_{\mathcal{F}}^{\frac{1}{2}} \right]^{\frac{1}{2}}
+ \left\| \sum_k s b_k^* b_k \right\|_{\mathcal{F}}^{\frac{1}{2}} + \left\| \sum_k s^{-1} b_k b_k^* \right\|_{\mathcal{F}}^{\frac{1}{2}}. \]
Then by a Hahn-Banach argument as in the proof of Proposition 2.4 we get positive operators \(f_1, f_2, g_1, g_2 \in E_r(M)\) (\(r\) being the conjugate index of \(p/2\)), all of them with norms \(\leq 1\), such that for any \(a \in E, b \in F\)
\[ |u(a, b)| \leq K_2 [f_1(s^{-1} a a^*) + f_2(s a^* a) + g_1(s b^* b) + g_2(s^{-1} b b^*)]. \]
Replacing $a$ and $b$ respectively by $ta$ and $t^{-1}b$ in the above inequality and then taking the infimum over all $t > 0$, we deduce that
\[
|u(a, b)| \leq 2K_2 \left[ f_1(s^{-1}aa^*) + f_2(sa^*a) \right]^{1/2} [g_1(sb^*b) + g_2(s^{-1}bb^*)]^{1/2}
\]
\[
= 2K_2 \left[ f_1(aa^*)g_1(b^*b) + f_2(a^*a)g_2(bb^*) \right] + s^2 f_2(a^*a)g_1(b^*b) + s^{-2} f_1(aa^*)g_2(bb^*) ]^{1/2}.
\]
Now taking the infimum over all $s > 0$, we finally get (1.4) with $K_3 \leq 2K_2$.

This is a special case of the following result due to Pisier, which has independent interest.

We end this section with an alternate direct proof of the implication iii) $\Rightarrow$ iv). This is a successuse use of the Cauchy- Schwarz and Hölder inequalities. Indeed, by (0.4) we have
\[
\left| \sum_k u(a_k, b_k) \right| \leq K_3 \left[ \sum_k \left( f_1(a_k a_k^*)g_1(b_k^* b_k) \right)^{1/2} + \sum_k \left( f_2(\mu_k a_k^* a_k)g_2(\mu_k^{-1} b_k^* b_k) \right)^{1/2} \right]
\]
This is (0.5) with $K_4 \leq K_3$. It is easy to see that iv) $\Rightarrow$ i) with $K_2 \leq K_4$ (although we will not need this). Therefore, ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv).

Put $X = (E \otimes_{h_p} F) \oplus_1 (F \otimes_{h_p} E)$ and $Y = \{ (x, t^* x) : x \in E \otimes F \}$.

Then by Proposition 2.5 ii), (0.5) implies that $u$ defines a continuous linear functional on $Y$ with norm $\leq K_4$. Hence $u$ extends to a continuous functional $\varphi$ on $X$ with the same norm. Then $\varphi$ can be decomposed as $\varphi = \varphi_1 + \varphi_2$ such that
\[
\varphi_1 \in (E \otimes_{h_p} F)^*, \varphi_2 \in (F \otimes_{h_p} E)^* \quad \text{and} \quad \max\{ \| \varphi_1 \|, \| \varphi_2 \| \} = \| \varphi \| \leq K_4.
\]

Going back to bilinear forms on $E \times F$ and using Theorem 2.6, we see that $\varphi_1$ and $\varphi_2$ define respectively bilinear forms $u_1$ and $u_2$ such that
\[
\gamma_{R_p}(\tilde{u}_1) = \| \varphi_1 \|, \gamma_{R_p}(\tilde{u}_2) = \| \varphi_2 \|.
\]

Then clearly, $u = u_1 + u_2$ yields the desired decomposition.

vi) $\Rightarrow$ i). Suppose $u_1$ and $u_2$ are as in vi). Then there are a Hilbert space $H$ and c.b. factorizations $E \overset{\alpha_1}{\rightarrow} H_p^{\beta_1} \overset{\beta_2}{\rightarrow} F^*$ for $\tilde{u}_1$ and $E \overset{\alpha_2}{\rightarrow} H_p^{\beta_2} \overset{\beta_1}{\rightarrow} F^*$ for $\tilde{u}_2$. Define $\alpha : E \rightarrow H_p^{\beta_1} \oplus_p H_p^{\beta_2}$ by $\alpha(a) = (\alpha_1(a),\alpha_2(a))$ and $\beta : H_p^{\beta_1} \oplus_p H_p^{\beta_2} \rightarrow F^*$ by $\beta(\xi,\eta) = \beta_1(\xi) + \beta_2(\eta)$. Then $\tilde{u} = \beta\alpha$ and
\[
\| \alpha \|_c \leq (\| \alpha_1 \|_c + \| \alpha_2 \|_c)^{1/2}, \| \beta \|_c \leq (\| \beta_1 \|_c + \| \beta_2 \|_c)^{1/2}.
\]

It then follows that $\tilde{u} \in \Gamma_{R_p \oplus_p C_p}$ and
\[
\gamma_{R_p \oplus_p C_p}(\tilde{u}) \leq \max \{ \gamma_{R_p}(\tilde{u}_1), \gamma_{C_p}(\tilde{u}_2) \}.
\]

vi) $\Rightarrow$ i). This is evident.

Therefore, we have proved that all assertions in Theorem 0.1 are equivalent. The last part of the theorem is clear from the preceding proof.

Remark. The previous proof also works for the case of $p = \infty$ with the additional assumption that both $E$ and $F$ are exact. The place where we need this assumption is only the implication i)$\Rightarrow$ii), for which we have to use Corollary 4.4 ii).

We end this section with an alternate direct proof of the implication iii)$\Rightarrow$vi) in Theorem 0.1. This is a special case of the following result due to Pisier, which has independent interest.
**Proposition 5.1.** Let $E, F$ be vector spaces and $H_i, K_i$ Hilbert spaces $(i = 1, 2)$. Let $I_i : E \to H_i$ and $J_i : F \to K_i$ be linear maps. Assume a bilinear form $u : E \times F \to \mathbb{C}$ satisfies

\[(5.1) \quad |u(a, b)| \leq \|I_1(a)\|_{H_1}\|J_1(b)\|_{K_1} + \|I_2(a)\|_{H_2}\|J_2(b)\|_{K_2}, \quad (a, b) \in E \times F.\]

Then $u$ can be used as a sum of two bilinear forms $u = u_1 + u_2$ such that

\[|u_1(a, b)| \leq \|I_1(a)\|_{H_1}\|J_1(b)\|_{K_1} \quad \text{and} \quad |u_2(a, b)| \leq \|I_2(a)\|_{H_2}\|J_2(b)\|_{K_2}, \quad (a, b) \in E \times F.\]

**Proof.** On the vector space $E \otimes F$ we introduce the following semi-norm. For $x \in E \otimes F$ define

\[
\|x\|_1 = \inf \left\{ \left( \sum_k \|I_1(a_k)\|^2_{H_1} \right)^{1/2} \left( \sum_k \|J_1(b_k)\|^2_{K_1} \right)^{1/2} \right\},
\]

where the infimum runs over all decompositions of $x$ as $x = \sum_k a_k \otimes b_k$. Similarly, we define a semi-norm $\|\|_2$ by using $H_2$ and $K_2$. Setting

\[a = (a_1, ..., a_n) \quad \text{and} \quad b = (b_1, ..., b_n), \]

we can rewrite $x = \sum_k a_k \otimes b_k$ as $x = a \otimes b$. Then

\[
\|x\|_1 = \inf \left\{ \|I_{\ell_2} \otimes I_1(a)\|_{\ell_2(H_1)} \|I_{\ell_2} \otimes J_1(b)\|_{\ell_2(K_1)} : x = a \otimes b \right\}.
\]

Since the semi-norm $\|I_{\ell_2} \otimes I_1(a)\|_{\ell_2(H_1)}$ is given by a quadratic form, for any operator $\alpha \in B(\ell_2)$

\[
\|I_{\ell_2} \otimes I_1(a)\|_{\ell_2(H_1)} \leq \|I_{\ell_2} \otimes I_1(a)\|_{\ell_2(H_1)} \|\alpha\|,
\]

\[
\|I_{\ell_2} \otimes J_1(b)\|_{\ell_2(K_1)} \leq \|\alpha\| \|I_{\ell_2} \otimes J_1(b)\|_{\ell_2(K_1)}.
\]

From this observation and using the same argument as in the proof of [PS, Proposition 1.7], we can deduce the following lemma.

**Lemma 5.2.** For any $x \in E \otimes F$ there is a decomposition $x = \sum_{k=1}^n a_k \otimes b_k$ and positive numbers $\lambda_1, ..., \lambda_k$ such that $a_1, ..., a_n$ (resp. $b_1, ..., b_n$) are linearly independent and such that

\[
\|x\|_1 = \left( \sum_k \|I_1(a_k)\|^2_{H_1} \right)^{1/2} \left( \sum_k \|J_1(b_k)\|^2_{K_1} \right)^{1/2},
\]

\[
\|x\|_2 = \left( \sum_k \lambda_k \|I_2(a_k)\|^2_{H_2} \right)^{1/2} \left( \sum_k \lambda_k^{-1} \|J_2(b_k)\|^2_{K_2} \right)^{1/2}.
\]

This lemma allows us to finish the proof of the proposition. Indeed, let

\[X = (E \otimes F, \|\|_1) + 1 (E \otimes F, \|\|_2) \quad \text{and} \quad Y = \{(x, x) : x \in E \otimes F\} \subset X.\]

By (5.1) and Lemma 5.2 considered as a linear functional on $E \otimes F$ (and so on $Y$ too), $u$ is continuous and of norm $\leq 1$ with respect to the semi-norm of $Y$. Therefore, by the Hahn-Banach theorem, $u$ extends to a contractive functional $\tilde{u}$ on $X$. Write

\[\tilde{u}(x, y) = \tilde{u}(x, 0) + \tilde{u}(0, y) \overset{\text{def}}{=} u_1(x) + u_2(y), \quad x, y \in E \otimes F.\]

Considered back to bilinear forms on $E \times F$, $u_1$ and $u_2$ give the required decomposition of $u$. □

**Remark.** As the reader can see, the above proof is similar to that of the implication iv) $\Rightarrow$ v) of Theorem 11. However, it has an advantage that the original functionals $f_i$ and $g_i$ in (11.1) can be used for $u_1$ and $u_2$ respectively in Theorem 11.11 v); see Theorem 24 for the existence of such functionals for $u_1$ and $u_2$. (Concerning this point see a remark in [PS, p.189].)

6. **Proof of Theorem 12**

Now we pass to the proof of Theorem 12.2. We will need the following result, which is a generalization of [23, Theorem 8.4].

**Proposition 6.1.** Let $2 < p < \infty, 0 \leq \theta \leq 1$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}$. Let $(e_k)$ denote the canonical basis of $C_q$. 

\[\text{Proposition 6.1.} \quad \text{Let } 2 < p < \infty, 0 \leq \theta \leq 1 \text{ and } \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}. \quad \text{Let } (e_k) \text{ denote the canonical basis of } C_q.\]
i) For any finite sequence \((x_k) \subset S_p\) we have

\[
\|
\sum_k x_k \otimes e_k
\|_{S_p[C_q]} = \sup \left\{ \left( \sum_k \|\alpha x_k \beta\|_2^2 \right)^{1/2} \right\},
\]

where the supremum runs over all \(\alpha\) and \(\beta\) respectively in the unit balls of \(S_{2^q-1}\) and \(S_{2^{q-1}}\), \(r\) being the conjugate index of \(p/2\). Moreover, the supremum can be restricted to all \(\alpha\) and \(\beta\) in the positive parts of these unit balls.

ii) Let \(H\) be a Hilbert space and \(U\) a free ultrafilter. Let \(B(H)_U\) be the associated ultrapower von Neumann algebra. Then for any finite sequence \((x_k) \subset L_p(B(H)_U)\)

\[
\|
\sum_k x_k \otimes e_k
\|_{L_p[B(H)_U], C_q} = \sup \left\{ \left( \sum_k \|\alpha x_k \beta\|_2^2 \right)^{1/2} \right\},
\]

where the supremum runs over all \(\alpha\) and \(\beta\) respectively in the unit balls of \(L_{2^q-1}(B(H)_U)\) and \(L_{2^{q-1}}(B(H)_U)\); again the supremum can be restricted to all \(\alpha\) and \(\beta\) in the positive parts of these unit balls.

**Proof.** i) Let \(x = \sum_k x_k \otimes e_k \in S_p[C_q]\). By (1.5),

\[
\|x\|_{S_p[C_q]} = \sup \left\{ \|axb\|_{S_q[C_q]} : a, b \in S_{2t}, \|a\|_{2t} \leq 1, \|b\|_{2t} \leq 1 \right\},
\]

where \(s\) is determined by \(\frac{1}{q} = \frac{1}{p} + \frac{1}{t}\). However, \(S_q[C_q] = C_q[S_q]\). Thus by (1.8),

\[
\|axb\|^2_{S_q[C_q]} = \|\sum_k b^* x_k^* a^* ax_k b\|_\theta.
\]

Assume \(q \geq 2\). It then follows that

\[
\|x\|^2_{S_p[C_q]} = \sup \left\{ \sum_k \text{Tr}(e^* b^* x_k^* a^* ax_k b) : a, b \in S_{2t}, c \in S_{2t}, \|a\|_{2t} \leq 1, \|b\|_{2t} \leq 1, \|c\|_{2t} \leq 1 \right\},
\]

where \(t\) is the index conjugate to \(q/2\). Set \(\alpha = a\) and \(\beta = bc\). Note that \(\alpha \in S_{2^q-1}\) and \(\beta \in S_{2^{q-1}}\). We then deduce (6.1) in the case of \(q \geq 2\). The case of \(q \leq 2\) can be done similarly by using the identification \(C_q \cong R_p\) (see (1.6)).

ii) It suffices to prove (6.2) for any \(C^n_q\) instead of \(C_q\). Then by definition, \(L_p[B(H)_U; C^n_q]\) is the ultrapower \((S_p[H]; C^n_q)_U\). On the other hand, \(L_p(B(H)_U)\) is also the ultrapower \((S_p(H))_U\) (Raynaud’s theorem). Recall that again by [Ra], ultraproduct preserves all algebraic structures on noncommutative \(L_p\)-spaces, in particular, the product. With the help of all these, we can easily deduce (6.2) from (6.1). \(\square\)

**Remarks.** i) We are grateful to the referee for the short proof of part i) above, which is much simpler than our original one.

ii) One can show that Proposition 6.1 ii) holds for any QWEP von Neumann algebra \(M\) in place of \(B(H)_U\).

iii) Proposition 6.1 yields a simple description of the norm in the complex interpolation space \((C_p[L_p(M)], R_p[L_p(M)])_\theta\) when \(M\) is \(B(H)\) or an ultrapower of \(B(H)\) \((p \geq 2)\). More generally, one can describe the norm of \((C_p[L_p(M)], R_p[L_p(M)])_\theta\) for any von Neumann algebra \(M\) by a formula like (6.1) in the case \(p \geq 2\), and by a similar dual formula in the case \(p < 2\). This will be pursued elsewhere.

**Proof of Theorem 0.2.** Without loss of generality, we can assume \(H\) separable and infinite-dimensional. Thus \(H_c = C_q\).

i) \(\Rightarrow\) ii). Assume \(\|u\|_{cb} \leq 1\). As for the proof of Theorem 0.1, the noncommutative Khintchine inequality in section 3 will be the key ingredient for the present proof too. Fix a positive sequence \((\lambda_k)\). We maintain the notations introduced at the beginning of the proof of Theorem 0.1 but with \(g_k\) being now defined by

\[
g_k = \lambda_k^\theta \ell(e_k) + \lambda_k^{-(1-\theta)} \ell^*(e_{-k}).
\]

The \(g_{k,p}\) are defined by (3.3) with the \(g_k\) above. Then by Proposition 4.2,

\[
\|I_{L_p(\Gamma)} \otimes u\| \leq 1.
\]
Thus by Theorem 6.1 for any finite sequence \((a_k) \subset E\)
\[
\left\| \sum_k u(a_k) \otimes g_{k,p} \right\|_{L_v[G:C_q]} \leq \left\| \sum_k a_k \otimes g_{k,p} \right\|_{L_v[G:E]} 
\leq B_p \max \left\{ \left\| \left( \sum_k \lambda_k^{2(1-\theta)} a_k^* a_k \right)^{1/2} \right\|_p, \left\| \left( \sum_k \lambda_k^{-2(1-\theta)(1-\theta)} a_k a_k^* \right)^{1/2} \right\|_p \right\}
\]

Therefore, to prove (6.6) we must show that for any finite sequence \((z_k) \subset C_q\)
\[
(\sum_k \|z_k\|^2)^{1/2} \leq \left\| \sum_k z_k \otimes g_{k,p} \right\|_{L_v[G:C_q]}.
\]

To this end we can clearly assume all \(z_k\) are finitely supported, and so \(C_q\) can be replaced by a \(C_q^n\). Write \(z_k\) in the canonical basis of \(C_q^n\):
\[
z_k = \sum_{j=1}^n z_{k,j} e_j.
\]

Then
\[
\sum_k z_k \otimes g_{k,p} = \sum_j e_j \otimes x_j,
\]
where
\[
x_j = \sum_k z_{k,j} g_{k,p} \in L_p(\Gamma).
\]

Now since \(\Gamma\) is QWEP, \(\Gamma\) is the image of a normal conditional expectation \(\Phi\) on some ultrapower von Neumann algebra \(B(H)_\mathcal{U}\). Then by definition, \(L_p[\Gamma; C_q^n]\) is a subspace of \(L_p[B(H)_\mathcal{U}; C_q^n]\). By Proposition 6.1 ii), for any \(\alpha\) and \(\beta\) in the unit balls of \(L_{2r\theta^{-1}}(B(H)_\mathcal{U})\) and \(L_{2r(1-\theta)^{-1}}(B(H)_\mathcal{U})\), respectively, we have
\[
(\sum_j \|\alpha x_j \beta\|_2^2)^{1/2} \leq \left\| \sum_j e_j \otimes x_j \right\|_{L_p[\Gamma; C_q^n]}.
\]

In particular, this is true for \(\alpha = D^{\theta/2r}\) and \(\beta = D^{(1-\theta)/2r}\). For this choice of \(\alpha\) and \(\beta\), we have
\[
\|\alpha x_j \beta\|_2^2 = \|\alpha x_j \beta\|_{L_2(\Gamma)}^2 = \text{tr}(\beta^* x_j^* \alpha^* \alpha x_j \beta)
= \sum_{k,k'} z_{k,j}^* z_{k',j} \text{tr}[g_k D^\theta g_{k'} D^{1-\theta}]
= \sum_k |z_{k,j}|^2 \text{ by (6.3)}.
\]

Summing up over all \(j\) and using (6.3), we get (6.4). Therefore, (6.6) is proved.

ii) \(\Rightarrow\) iii). This is a standard application of the Hahn-Banach theorem as in the proof of ii) \(\Rightarrow\) iii) in Theorem 6.1. Conversely, it is trivial that iii) \(\Rightarrow\) ii).

iii) \(\Rightarrow\) i). Let \(a = (a_{i,j}) \in S_p^n[E]\) be a unit element. We have to show
\[
\|I_{S_p^n} \otimes u(a)\|_{S_p[K]} \leq K.
\]

To this end we use again Proposition 6.1. Let \(u_k\) be the \(k\)-th component of \(u\) relative to the canonical basis of \(C_q\). Set \(x_k = (u_k(a_{i,j}))_{1 \leq i,j \leq n} \in S_p^n\). Then
\[
I_{S_p^n} \otimes u(a) = \sum_k x_k \otimes e_k.
\]

Let \(\alpha\) (resp. \(\beta\)) be a positive matrix in the unit ball of \(S_{2r\theta^{-1}}\) (resp. \(S_{2r(1-\theta)^{-1}}\)). We are going to estimate \(\sum_k \|\alpha x_k \beta\|_2^2\). In virtue of the invariance of the norm of \(S_p^n[E]\) by multiplication from left and right by unitary matrices, and changing the matrix \(a\) if necessary, we can assume both \(\alpha\) and
\( \beta \) are diagonal. Let \( \alpha_i \) and \( \beta_i \) be respectively their diagonal entries. Then by (3.7) and the H\ölder inequalities

\[
\sum_k \|\alpha_k \beta_k\|_2^2 = \sum_{i,j} \alpha_i^2 \beta_j^2 |x_k(i,j)|^2 = \sum_{i,j} \alpha_i^2 \beta_j^2 \|u(a_{ij})\|^2 \\
\leq K^2 \sum_{i,j} \alpha_i^2 \beta_j^2 [f(a_{ij}^* a_{ij})]^{1-\theta} \|g(a_{ij} a_{ij}^*)\|^\theta \\
\leq K^2 \left( \sum_{i,j} \beta_j^{2(1-\theta)-1} f(a_{ij}^* a_{ij}) \right)^{1-\theta} \left( \sum_{i,j} \alpha_i^{2\theta-1} \|g(a_{ij} a_{ij}^*)\|^\theta \right) \\
\leq K^2 \left( \sum_{i,j} \beta_j^{2(1-\theta)-1} \sum_j a_{ij} a_{ij}^* \|L_{L_p(M)}\|^{2(1-\theta)} \right)^\frac{2(1-\theta)}{\theta} \left( \sum_{i,j} \alpha_i^{2\theta-1} \|a_{ij} a_{ij}^* \|_{L_p(M)}^{2\theta} \right) \\
\leq K^2 \|a\|^2 \|S_{p,E}\|_\theta,
\]

where for the last inequality we have used the following elementary fact that for any von Neumann algebra \( M \) and any \( a \in S_{p,L_p(M)} \) with \( 2 \leq p \leq \infty \)

\[
\left( \sum_j \left\| \left( \sum_i a_{ij}^* a_{ij} \right)^{1/2} \right\|_{L_p(M)}^p \right)^{1/p} \leq \|a\|_{S_{p,L_p(M)}},
\]

\[
\left( \sum_i \left\| \left( \sum_j a_{ij} a_{ij}^* \right)^{1/2} \right\|_{L_p(M)}^p \right)^{1/p} \leq \|a\|_{S_{p,L_p(M)}}.
\]

Therefore, \( \|I_{S_{p,E}} \otimes u(a)\|_{S_{p,E}} \leq K \) for any \( n \geq 1 \). Then by Lemma 4.1 \( \|u\|_{cb} \leq K \). Therefore, we have proved Theorem 0.2. \qed

7. Applications

In this section we present some applications of Theorems 0.1 and 0.2. We first give a factorization for maps satisfying one of the conditions in Theorem 0.2 through a real interpolation space of \( M \). Therefore, we have proved Theorem 0.2.

Now we equip \( K_f \) (resp. \( K_g \)) with the operator space structure of \( K_{f,p}^r \) (resp. \( K_{g,p}^r \)), and consider real interpolation space \( (K_{f,p}^r, K_{g,p}^r)_{\theta, 1} \) (see [11] for the real interpolation theory in the category of operator spaces). By Remark 2.3 \( i_f : E \to K_{f,p}^r \) and \( i_g : E \to K_{g,p}^r \) are completely contractive, so by interpolation they induce a completely contractive map \( i_{f,g} : E \to (K_{f,p}^r, K_{g,p}^r)_{\theta, 1} \).

**Corollary 7.1.** Let \( E, p, q, \theta \) be as in Theorem 0.2. Then a map \( u : E \to H_q^c \) is c.b. iff there are two positive unit functionals \( f \) and \( g \) on \( L_{p/2}(M) \) such that \( u \) admits a factorization of the following form

\[
\begin{array}{cccc}
E & \overset{u}{\longrightarrow} & H_q^c \\
\downarrow i_{f,g} & & \downarrow \tilde{u} \\
(K_{f,p}^r, K_{g,p}^r)_{\theta, 1} & \overset{u}{\longrightarrow} & H_q^c
\end{array}
\]
where \( \tilde{u} \) is a bounded map and \( \| \tilde{u} \| \leq K \). Moreover, the smallest of such constants \( K \) is universally equivalent to \( \| u \|_{cb} \).

**Proof.** Let \( u : E \to H_q^\circ \) be c.b.. Then by Theorem 0.2, \( u \) satisfies 0.7. By the previous discussion, we clearly have the required factorization. Conversely, if \( u \) admits such a factorization, then we have 1.1, and so \( u \) is c.b.. \( \square \)

**Remarks.** i) Using [P6], we can get a factorization similar to that in Corollary 7.1 in the case of \( p = \infty \) with the additional assumption that either \( E \subset M \) is exact or \( E = M \).

ii) We do not know whether \( \tilde{u} \) in Corollary 7.1 can be chosen to be c.b..

Let us isolate out the special case of \( \theta = 1/2 \) in Theorem 0.2 and Corollary 7.1 because of the particular importance of \( OH \).

**Corollary 7.2.** Let \( E \subset L_p(M) \) be a subspace with \( 2 < p \leq \infty \). In the case of \( p = \infty \) we assume in addition that \( E \) is either exact with \( \text{ex}(E) = 1 \) or \( E = M \). Then for any map \( u : E \to OH(I) \) (with \( I \) an index set) the following assertions are equivalent:

i) \( u \) is c.b..

ii) There is a constant \( K \) such that for all finite sequences \( (a_k) \subset E \) and \( (\mu_k) \subset \mathbb{R}_+ \)

\[
\sum_k \| u(a_k) \|^2 \leq \frac{K^2}{2} \left[ \sum_k \| \mu_k a_k^* a_k \|_{p/2}^2 + \sum_k \| \lambda_k^{-1} a_k a_k^* \|_{p/2}^2 \right].
\]

iii) There are two positive unit functionals \( f, g \) on \( L_{p/2}(M) \) such that

\[
\| u(a) \| \leq K \left( f(a^* a)^{1/4} (g(aa^*))^{1/4} \right), \quad a \in E.
\]

iv) There are two positive unit functionals \( f, g \) on \( L_{p/2}(M) \) such that \( u \) admits a factorization of the form (7.1) with \( \| \tilde{u} \| \leq K' \).

Moreover, the best constants \( K \) and \( K' \) are universally equivalent to \( \| u \|_{cb} \).

This is the little Grothendieck theorem for noncommutative \( L_p \)-spaces in the category of operator spaces. The case of \( p = \infty \) goes back to [PS]. Compare 7.3 with (0.1): the arithmetic mean in (0.1) is replaced by the geometric mean in (7.3).

**Corollary 7.3.** In the situation of Theorem 0.2, if \( u : E \times F \to \mathbb{C} \) is j.c.b., then \( u \) admits an extension \( U : L_p(M) \times L_p(M) \to \mathbb{C} \) with \( \| U \|_{cb} \leq c \| u \|_{jcb} \), where \( c \) is a universal constant.

**Proof.** We use the decomposition \( u = u_1 + u_2 \) in Theorem 0.2 v). Regarding \( u_1, u_2 \) as linear functionals on \( E \otimes F \), we see that \( u_1 \) and \( u_2 \) satisfy iv) of Theorem 2.6. Therefore, \( u_1 \) (resp. \( u_2 \)) admits an extension \( U_1 \) (resp. \( U_2 \)) on \( L_p(M) \otimes_{h_p} L_p(M) \) (resp. \( L_p(M) \otimes_{h_p} L_p(M) \)). Then \( U = U_1 + U_2 \) is the required extension of \( u \). \( \square \)

We say that a c.b. map \( T : E \to F \) between two operator spaces has the completely bounded approximation property (CBAP in short) if there are a constant \( \lambda \) and a net \( (T_i) \) of finite rank maps from \( E \) to \( F \) such that \( T_i \) converges to \( T \) in the point-norm topology and \( \sup_i \| T_i \|_{cb} \leq \lambda \| T \|_{cb} \).

In this case, we also say that \( T \) has the \( \lambda \)-CBAP if we want to emphasize the constant \( \lambda \). Note that \( E \) has the CBAP iff the identity of \( E \) does. We recall the open problem in [PS] whether any c.b. map from a \( C^* \)-algebra to the dual of a \( C^* \)-algebra has automatically the CBAP. However, the corresponding problem in the \( L_p \)-space case is easily solved by virtue of Theorem 0.1.

**Corollary 7.4.** Let \( E, F \) be as in Theorem 0.1. Then any map \( T \in CB(E, F^*) \) has the \( \lambda \)-CBAP with \( \lambda \) a universal constant.

**Proof.** By Theorem 0.1, \( T \) belongs to \( \Gamma_{R_p \otimes C_p}(E, F^*) \). It remains to note that \( R_p \oplus C_p \) has the 1-CBAP. \( \square \)

**Corollary 7.5.** i) Let \( E \) be an operator space. If both \( E \) and \( E^* \) are completely isomorphic to subspaces of a noncommutative \( L_p(M) \) with \( 1 < p < 2 \), then \( E \) is completely isomorphic to a quotient of a subspace of \( H_p^\circ \oplus_p K_p^\circ \) for some Hilbert spaces \( H \) and \( K \).

ii) If we assume in addition that the completely isomorphic copies of \( E \) and \( E^* \) are completely complemented in \( L_p(M) \), then \( E \) is completely isomorphic to \( H_p^\circ \oplus_p K_p^\circ \).
Proof. Based on Theorem 0.1, the proof of this corollary is the same as that of Corollaries 3.1 and 3.3 in [PS] (which corresponds to the case \( p = 1 \) with an additional assumption on \( E \)), so we omit it. \( \square \)

**Remarks.**

i) Let \( H \) be a Hilbert space and \( 1 < p < 2 \). It is proved in [X2] that a quotient of a subspace of \( H_p^q \oplus H_p^q \) is completely isomorphic to a subspace of a noncommutative \( L_p \).

ii) Moreover, for any \( p < q \leq 2 \), \( H_q^p \) is a quotient of a subspace of \( H_p^q \oplus H_p^q \). Consequently, any quotient of a subspace of \( H_p^q \oplus H_p^q \) is also completely isomorphic to a subspace of a noncommutative \( L_p \) (\( 1 \leq p \leq q \leq 2 \)). See [X2] for more details.

We end this section with an application to Schur multipliers. Let \( \varphi \) be a function on \( \mathbb{N} \times \mathbb{N} \). We recall that \( \varphi \) is a Schur multiplier from \( S_p \) to \( S_q \) if the map \( M_\varphi : x \mapsto (\varphi(i,j)x(i,j)) \) defined for finite matrices \( x \) extends to a bounded map from \( S_p \) to \( S_q \) (which is still denoted by \( M_\varphi \)). (Note that we change slightly the matrix notation by regarding a matrix as a function on \( \mathbb{N} \times \mathbb{N} \).)

**Corollary 7.6.**

i) Let \( 1 \leq q \leq 2 \leq p \leq \infty \) and \( r = \frac{pq}{p-q} \). Then \( \varphi \) is a Schur multiplier from \( S_p \) to \( S_q \) if and only if \( \varphi \in \ell_r(\ell_\infty) \), i.e. \( \ell_r(\ell_\infty) \) is the space of all complex functions \( \varphi \) on \( \mathbb{N} \times \mathbb{N} \) such that

\[
\| \varphi \|_{\ell_r(\ell_\infty)} = \left( \sum_{i,j} |\varphi(i,j)|^r \right)^{1/r} < \infty.
\]

Set \( \ell_r(\ell_\infty) = \{ \varphi : \varphi \in \ell_r(\ell_\infty) \} \), where \( \varphi(i,j) = \varphi(j,i) \).

The following, except i'), is again the \( L_p \)-space analogue of the corresponding results in [PS], which correspond to the case where \( q = 1 \) and \( p = \infty \).

The proof is similar to those of Theorems 4.1 and 4.2 in [PS]. Thus we will be very brief.

i) The case \( p = \infty \) and \( q = 1 \) corresponds to Theorem 4.1. Let \( \varphi \) be a Schur multiplier from \( S_p \) to \( S_q \) with \( \| M_\varphi \| \leq 1 \). Let \( u : S_p \times S_q' \rightarrow \mathbb{C} \) be the bilinear form defined by \( M_\varphi \):

\[
u(x,y) = \sum_{i,j} \varphi(i,j)x(i,j)y(i,j), \quad x \in S_p, \ y \in S_q'. \]

Then by [X2], \( u \) can be decomposed as \( u = u_1 + u_2 + u_3 + u_4 \) with

\[
|u_1(x,y)| \leq K(1xx^*)g_1(y^*y)^{1/2}, \quad |u_2(x,y)| \leq K(f_2(x^*x)g_2(yy^*))^{1/2},
\]

\[
|u_3(x,y)| \leq K(f_1(xx^*)g_2(yy^*))^{1/2}, \quad |u_4(x,y)| \leq K(f_2(x^*x)g_1(y^*y)^{1/2},
\]

Moreover,

\[
\| M_\varphi \|_{cb} \approx \inf \{ \| \varphi_1 \|_{\ell_r(\ell_\infty)} + \| \varphi_2 \|_{\ell_r(\ell_\infty)} : \varphi = \varphi_1 + \varphi_2, \varphi_1, \varphi_2 \in \ell_r(\ell_\infty) \}.
\]

ii') With the same proof as in ii), let \( \Lambda \subset \mathbb{N} \times \mathbb{N} \). Then every c.b. Schur multiplier from \( S_p^\Lambda \) to \( S_q^\Lambda \) extends to a c.b. Schur multiplier from \( S_p \) to \( S_q \).

Proof. The proof is similar to those of Theorems 4.1 and 4.2 in [PS]. Thus we will be very brief.

i) The case \( p = \infty \) and \( q = 1 \) corresponds to Theorem 4.1. Let \( \varphi \) be a Schur multiplier from \( S_p \) to \( S_q \) with \( \| M_\varphi \| \leq 1 \). Let \( u : S_p \times S_q' \rightarrow \mathbb{C} \) be the bilinear form defined by \( M_\varphi \):

\[
\nu(x,y) = \sum_{i,j} \varphi(i,j)x(i,j)y(i,j), \quad x \in S_p, \ y \in S_q'. \]

Then by [X2], \( u \) can be decomposed as \( u = u_1 + u_2 + u_3 + u_4 \) with
where \( f_1, f_2 \) (resp. \( g_1, g_2 \)) are positive unit functionals on \( S_{p/2} \) (resp. \( S_{q'/2} \)). By an elementary average argument as in [PS], we can assume that each \( u_k \) is given by a Schur multiplier as \( u \), say \( \varphi_k \), and the functionals \( f_i, g_i \) are diagonal matrices. Then we have
\[
|\varphi_1(i,j)| \leq K \left( f_1(i,j)g_1(j,i) \right)^{1/2},
|\varphi_2(i,j)| \leq K \left( f_2(j,i)g_2(i,j) \right)^{1/2},
|\varphi_3(i,j)| \leq K \left( f_3(i,j)g_3(j,i) \right)^{1/2},
|\varphi_4(i,j)| \leq K \left( f_4(j,i)g_4(i,j) \right)^{1/2}.
\]
Thus \( \varphi_3 \) and \( \varphi_4 \) are in \( \ell_r(\ell_\infty) \). On the other hand, \( \varphi_1 \) and \( \varphi_2 \) can be decomposed into sums of two such elements, i.e. \( \varphi_1, \varphi_2 \in \ell_r(\ell_\infty) + \ell_r(\ell_\infty) \). Hence \( \varphi \in \ell_r(\ell_\infty) + \ell_r(\ell_\infty) \).

Conversely, suppose \( \varphi \in \ell_r(\ell_\infty) \). Then for any \( x \in S_p \) and \( y \in S_{q'} \)
\[
\left| \sum_{i,j} \varphi(i,j)x(i,j)y(i,j) \right| \leq \sum_i \sup_j |\varphi(i,j)| \sum_j |x(i,j)y(i,j)| \\
\leq \sum_i \sup_j |\varphi(i,j)| \left( \sum_j |x(i,j)|^2 \right)^{1/2} \left( \sum_j |y(i,j)|^2 \right)^{1/2} \\
\leq \|\varphi\|_{\ell_r(\ell_\infty)} \|x\|_{S_p} \|y\|_{S_{q'}},
\]
where we have used the following elementary inequality
\[
\left( \sum_i \left( \sum_j |x(i,j)|^2 \right)^{p/2} \right)^{1/p} \leq \|x\|_{S_p}.
\]
Therefore \( \varphi \) is a Schur multiplier from \( S_p \) to \( S_q \). Similarly, every matrix in \( \ell_r(\ell_\infty) \) is also a Schur multiplier from \( S_p \) to \( S_q \). This proves part i).

i') Let \( \varphi \) be a Schur multiplier from \( S^\Lambda_p \) to \( S_q \). Since \( S_p \) and \( S_q \) have respectively type 2 and cotype 2, by Kwapien’s theorem (cf. [P1 Corollary 3.6]), \( M_\varphi : S_p^\Lambda \to S_q \) factors through a Hilbert space. Then Maurey’s extension theorem [M2] implies that \( M_\varphi \) admits an extension \( T : S_p \to S_q \). Averaging \( T \) over the group of all unitary diagonal matrices in \( B(\ell_2) \), we deduce an extension of \( M_\varphi \) which is again a Schur multiplier.

ii) This part is proved in a way similar to that of i); the only difference is that this time instead of [0.2], we use Theorem 0.1. We omit the details.

ii') Assume \( M_\varphi : S^\Lambda_p \to S^\Lambda_{p'} \) is c.b. Then \( M_{\varphi'} \) has a c.b. extension from \( S_p \to S_{p'} \). This follows from Corollary 0.6 of [PS] in the case \( p = \infty \), and from Corollary 7.3 for \( p < \infty \). Then as above for i'), we get a Schur extension of \( \varphi \).

The second part of Corollary 7.6 gives a characterization of c.b. Schur multipliers from \( S_p \) to \( S_{p'} \), i.e. from \( S_p \) to its dual. We do not know how to characterize the c.b. Schur multipliers from \( S_p \) into \( S_q \) for any \( 1 \leq q \leq 2 \leq p \leq \infty \), as in the first part at the Banach space level.

Problem 7.7. Let \( 1 \leq q \leq 2 \leq p \leq \infty \) with \( p \neq q \). Characterize c.b. Schur multipliers from \( S_p \) into \( S_q \), in a way similar to that in Corollary 7.6.

This problem might be related to the following

Problem 7.8. Let \( M \) be a von Neumann algebra and \( p, q \geq 2 \) with \( p \neq q \). Let \( E \subset L_p(M) \) and \( F \subset L_q(M) \). Find a Grothendieck type inequality for j.c.b. forms \( u : E \times F \to \mathbb{C} \).

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