In this paper, the inconsistent LR fuzzy matrix equation $A\tilde{X} = \tilde{B}$ is proposed and discussed. Firstly, the LR fuzzy matrix equation is transformed into two crisp matrix equations in which one determines the mean value and the other determines the left and right extends of fuzzy approximate solution. Secondly, the approximate solution of the LR fuzzy matrix equation is obtained by solving two crisp matrix equations according to the generalized inverse of crisp matrix theory. Then, sufficient conditions for the existence of strong LR fuzzy approximate solution are given. Finally, some numerical examples are given to illustrate our proposed method.

1. Introduction

People often encounter vague concepts in production practice, scientific experiments, and daily life. With the development of science and technology, quantitative analysis is often needed for some vague practical problems in various disciplines, which makes fuzzy mathematics flourishing and attracts some scholars’ attention [1–5].

The concept of fuzzy numbers and fuzzy arithmetic operations were introduced and investigated by Zadeh [6]. After that, Dubois and Prade [7], S. Kandel [8], Puri and Ralescu [9], Goetschel and Voxman [10], and Wu and Ma [11, 12] gave some different approaches to fuzzy numbers and structure of fuzzy number space. In 1998, Friedman et al. [13] proposed an approach to solve fuzzy linear equations by the embedding method. Later, a lot of research works have been made by some scholars to solve numerical fuzzy linear systems, see [14–27]. For examples, Allahviranloo et al. [16–23] have completed a series of attempts about how to compute the fuzzy linear system and pointed out that the weak fuzzy solution was not existed sometimes [15] based on triangle fuzzy numbers. Asady et al. [21] considered the $m \times n$ fuzzy linear system with the full row rank in 2005. Later, Zheng and Wang [28] discussed the $m \times n$ general fuzzy linear system and the inconsistent fuzzy linear systems in which we know the coefficient matrix of the model equation is singular or rectangular. New theory and method for fuzzy linear system is emerging in endlessly recently.

In 2009, Allahviranloo et al. [21] discussed firstly the fuzzy linear matrix equations (FLMEs) of the form $A\tilde{X} = \tilde{B}$. By means of the parametric form of the fuzzy number, they derived necessary and sufficient conditions for the existence condition of fuzzy solutions and designed a numerical procedure for calculating the solutions of the original system. In the past decade, we have made systematic investigation on fuzzy matrix equations. In 2011, Gong and Guo [29] investigated a class of fuzzy matrix equations $A\tilde{X} = \tilde{B}$ by the same way. In 2012, Guo et al. [24, 30] proposed a computing method of fuzzy symmetric solutions to fuzzy matrix equations $A\tilde{X} + \tilde{X}B = \tilde{C}$ with LR fuzzy numbers in the next year. In 2014, Gong and Guo et al. [31] studied the general dual fuzzy matrix systems $A\tilde{X} + \tilde{B} = C\tilde{X} + D$ according to arithmetic operations of LR fuzzy numbers. In 2017, Guo et al. [32, 33] studied the fuzzy matrix equation with the form of $\tilde{X}A = \tilde{B}$ by a matrix method and made a further investigation to dual fuzzy matrix equation $A\tilde{X} + \tilde{B} = C\tilde{X} + D$. In 2018, Guo and Shang [34] introduced a class of complex fuzzy matrix equation $\tilde{Z}C = \tilde{W}$ and...
proposed a general model to deal with it. Recently, Guo et al. [35] proposed a new method for solving linear fuzzy matrix equations $AXB = C$ based on LR fuzzy numbers.

There are two reasons that make us to consider the inconsistent LR fuzzy matrix equation. They are as follows:

1. When the uncertain elements of fuzzy systems were denoted by the parametric form of fuzzy numbers, it may lead to two defects in dealing with fuzzy linear systems. One is that the extended linear equations always contains parameter $r$, $0 \leq r \leq 1$, which makes their computation inconvenient. The other is that sometimes the weak fuzzy solution of fuzzy linear systems does not exist [15] based on the triangle fuzzy number.

2. The definition of a right shape function $R(\cdot)$ is similar to that of $L(\cdot)$. A LR fuzzy number $\bar{M}$ is symbolically shown as $\bar{M} = (m, \alpha, \beta)_{LR}$.

Clearly, $\bar{M}$ is positive (negative) if and only if $m - \alpha > 0(m + \beta < 0)$. Noticing that $\alpha > 0, \beta > 0$, in Definition 1, which limits its applications, we extend the definition of LR fuzzy numbers as follows.

**Definition 2.** (generalized LR fuzzy numbers). Let $\bar{M} = (m, \alpha, \beta)_{LR}$, and we define

$$
\mu_{\bar{M}}^{-1}(x) = \begin{cases} 
0, & x \leq m, \\
\frac{x - m}{\max[-\alpha, \beta]}, & x \geq m.
\end{cases}
$$

(2) If $\alpha > 0$ and $\beta < 0$, the $\bar{M} = (m, \max[\alpha, -\beta], 0)_{LR}$, and

$$
\mu_{\bar{M}}^{-1}(x) = \begin{cases} 
\frac{m - x}{\max[\alpha, -\beta]}, & x \leq m, \\
0, & x \geq m.
\end{cases}
$$

(3) If $\alpha < 0$ and $\beta < 0$, the $\bar{M} = (m, -\alpha, -\beta)_{LR}$, and

$$
\mu_{\bar{M}}^{-1}(x) = \begin{cases} 
\frac{m - x}{-\beta}, & x \leq m, \\
\frac{x - m}{-\alpha}, & x \geq m.
\end{cases}
$$

**Definition 3.** For arbitrary LR fuzzy numbers $\bar{M} = (m, \alpha, \beta)_{LR}$ and $\bar{N} = (n, \gamma, \delta)_{LR}$, we have

1. Addition:

$$
\bar{M} + \bar{N} = (m, \alpha, \beta)_{LR} + (n, \gamma, \delta)_{LR} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}.
$$

(5)

2. Subtraction:

$$
\bar{M} - \bar{N} = (m, \alpha, \beta)_{LR} - (n, \gamma, \delta)_{LR} = (m - n, \alpha - \gamma, \beta - \delta)_{LR}.
$$

(6)

3. Scalar multiplication:

$$
\lambda \bar{M} = \lambda(m, \alpha, \beta)_{LR} \equiv \begin{cases} 
\lambda m, & \lambda \geq 0, \\
\lambda m - \lambda \beta, & \lambda < 0.
\end{cases}
$$

(7)
Definition 4. The matrix equation:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1m} \\
    a_{21} & a_{22} & \cdots & a_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nm}
\end{pmatrix} \begin{pmatrix}
    x_{11} & x_{12} & \cdots & x_{1n} \\
    x_{21} & x_{22} & \cdots & x_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix} =
\begin{pmatrix}
    b_{11} & b_{12} & \cdots & b_{1m} \\
    b_{21} & b_{22} & \cdots & b_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{m1} & b_{m2} & \cdots & b_{mn}
\end{pmatrix}
\] (8)

where the coefficient matrix \( A = a_{ij} \) is \( m \times n \) crisp matrix and \( \tilde{b}_{ij}, \ 1 \leq i, j \leq m \) are LR fuzzy numbers, which are called LR fuzzy matrix equations (LRFME).

3. Solving Inconsistent LR Fuzzy Matrix Equation

Theorem 1. The fuzzy matrix equation \( AX = B \) can be extended into the following matrix equations:

\[
\begin{cases}
(A^+ + A^-)X = B, \\
(A^+ - A^-)(X^l) = (B^l), \\
(-A^- A^+)X = (B^r),
\end{cases}
\] (9)

where \( \tilde{X} = (X, X^l, X^r) \). And the elements \( a_{ij}^+ \) of matrix \( A^+ \) and \( a_{ij}^- \) of matrix \( A^- \) are determined by the following way.

If \( a_{ij} \geq 0, a_{ij}^+ = a_{ij} \), else \( a_{ij}^+ = 0, 1 \leq i \leq n, 1 \leq j \leq m \); if \( a_{ij} < 0, a_{ij}^- = a_{ij} \), else \( a_{ij}^- = 0, 1 \leq i \leq n, 1 \leq j \leq m \).

Thus, we

\[
\begin{align*}
A^+(X, X^l, X^r) + A^-(X, X^l, X^r) &= (A^+X, A^+X^l, A^+X^r) + (A^-X, -A^-X^r, -A^-X^l) \\
&= A^+X + A^-X, A^+X^l - A^-X^r, A^+X^r - A^-X^l = (B, B^l, B^r).
\end{align*}
\] (14)

Proof. We denote the right fuzzy matrix \( \tilde{B} \) with \( \tilde{B} = (B, B^l, B^r) = (b_{ij}, \tilde{b}_{ij}, \tilde{b}_{ij})_{m \times n} \) and the unknown fuzzy matrix \( \tilde{X} \) by \( \tilde{X} = (X, X^l, X^r) = (x_{ij}, x_{ij}^l, x_{ij}^r)_{m \times n} \). We also suppose

\[
\begin{bmatrix}
A &= (A^+ + A^-), \\
S &= (A^+ - A^-) = (E \ F).
\end{bmatrix}
\] (10)

in which the elements \( a_{ij}^+ \) of matrix \( A^+ \) and \( a_{ij}^- \) of matrix \( A^- \) are determined by the following way.

If \( a_{ij} \geq 0, a_{ij}^+ = a_{ij} \), else \( a_{ij}^+ = 0, 1 \leq i \leq n, 1 \leq j \leq m \); if \( a_{ij} < 0, a_{ij}^- = a_{ij} \), else \( a_{ij}^- = 0, 1 \leq i \leq n, 1 \leq j \leq m \).

For fuzzy matrix equation \( A\tilde{X} = \tilde{B} \), we can express it as

\[
(A^+ + A^-)(X, X^l, X^r) = (B, B^l, B^r).
\] (11)

Since

\[
k\tilde{x}_{ij} = \begin{cases}
(kx_{ij}, x_{ij}^l, kx_{ij}^r), & k \geq 0, \\
(kx_{ij}^l, -kx_{ij}^r, -kx_{ij}^l), & k < 0,
\end{cases}
\] (12)

we have

\[
A\tilde{X} = \begin{cases}
(AX, AX^l, AX^r), & A \geq 0, \\
(AX, -AX^l, -AX^r), & A < 0.
\end{cases}
\] (13)

So, equation (10) can be rewritten as

\[
\begin{align*}
A^+(X, X^l, X^r) + A^-(X, X^l, X^r) &= (A^+X, A^+X^l, A^+X^r) + (A^-X, -A^-X^r, -A^-X^l) \\
&= A^+X + A^-X, A^+X^l - A^-X^r, A^+X^r - A^-X^l = (B, B^l, B^r).
\end{align*}
\] (14)

Thus, we

\[
\begin{align*}
A^+X + A^-X &= B, \\
A^+X^l - A^-X^r &= B^l, \\
A^+X^r - A^-X^l &= B^r,
\end{align*}
\] (15)

It concludes the proof.

We know that for \( n \times n \) matrix equation, when \( A \) is nonsingular and \( S \) maybe singular. However, when \( S \) is nonsingular and \( A \) must be nonsingular. We could conclude it from the following result.

\[ \square \]

Theorem 2. The matrix \( S \) is nonsingular if and only if both matrices \( A = E + F \) and \( E - F \) are nonsingular.

Proof. By adding the \((n + i)\)th row of \( S \) to its \( i \)th row for \( 1 \leq i \leq n \), we obtain

\[
S = \begin{bmatrix} E & F \\ F & E \end{bmatrix} \longrightarrow \begin{bmatrix} E + F & E + F \\ F & E \end{bmatrix} = S_1.
\] (16)

Next, we subtract the \( j \)th column of \( S \), from its \((n + j)\)th column for \( i \leq j \leq n \) and obtain

\[
S_1 = \begin{bmatrix} E + F & E + F \\ F & E \end{bmatrix} \longrightarrow \begin{bmatrix} E + F & 0 \\ F & E - F \end{bmatrix} = S_2.
\] (17)

Clearly,

\[
|S| = |S_1| = |S_2| = |E + F||E - F| = |A||E - F|.
\] (18)

Therefore, \(|S| \neq 0\) if and only if \(|A| \neq 0\) and \(|E + F| \neq 0\). It concludes the proof.

In order to solve the original fuzzy matrix equation (8), there are some main results for solvability of model equation
(9). For convenience, we suppose \( T = \begin{pmatrix} X^1 \\ X^r \end{pmatrix} \) and 
\( Y = \begin{pmatrix} B^1 \\ B^r \end{pmatrix} \).

\[ \sqrt[3]{\text{Theorem 3 (see [32])}. \text{ The } 2m \times 2n \text{ crisp matrix equation exists solution if and only if the rank of matrix } S \text{ equals to that of matrix } (S, Y), \text{i.e.,}} \]
\[ \text{Rank}(S) = \text{Rank}(S, Y). \quad (19) \]

When \( \text{Rank}(S) < \text{Rank}(S, Y) \), the equation does not have any solution, when \( \text{Rank}(S) = \text{Rank}(S, Y) = 2n \), the equation has a unique solution, and when \( \text{Rank}(S) = \text{Rank}(S, Y) < 2n \), the equation has infinite many solutions.

\[ \text{Definition 5.} \]
\[ \text{If } \text{Rank}(A) \neq \text{Rank}(A, X), \text{Rank}(S) \neq \text{Rank}(S, Y), \quad (20) \]
in model matrix equations
\[ \begin{cases} 
(A^* + A^-)X = B, \\
(A^* - A^-)(X^1) \begin{pmatrix} X^1 \\ X^r \end{pmatrix} = \begin{pmatrix} B^1 \\ B^r \end{pmatrix},
\end{cases} \quad (21) \]
to solve LR fuzzy matrix equation (8), the LR fuzzy matrix equation (8) is called a inconsistent LR fuzzy matrix equations (ILRFMEs).

\[ \text{Lemma 1 (see [28])}. \text{ Vector } \bar{X} \text{ is a LR least squares solution of the equation (9) if and only if}} \]
\[ \begin{cases} 
AX = AA^{(1,3)}B, \\
SC = SS^{(1,3)}Y.
\end{cases} \quad (22) \]

Thus, the general LR least squares solution is
\[ \begin{cases} 
X = A^{(1,3)}B + (I_n - A^{(1,3)}A)Z, \\
C = S^{(1,3)}Y + (I_{2n} - S^{(1,3)}S)Z,
\end{cases} \quad (23) \]

where \( A^{(1,3)} \) is a 1,3-inverse of \( A \), \( S^{(1,3)} \) is a 1,3-inverse of \( S \), and \( Z \) is an arbitrary vector.

It will be noted that the LR least squares solution is unique only when \( A \) and \( S \) are of full column rank; otherwise, (23) is an infinite set of such solutions.

\[ \text{Lemma 2 (see [28])}. \text{ Among the LR least squares solution of (9), } A' B \text{ and } S' Y \text{ are the one of minimum norm LR least squares solution, where } A' \text{ and } S' \text{ is the Moore–Penrose inverse of } A \text{ and } S. \]

It is well known that Moore–Penrose inverse is unique [36], and the minimum norm LR least squares solution of (9) is unique. To illustrate the LR fuzzy least squares solution to a LR fuzzy matrix equation, we now discussed the generalized inverse of the matrix in a special structure.

\[ \text{Theorem 4. Let } S \text{ be in form (9); then, the matrix} \]
\[ S^{(1,3)} = \frac{1}{2} \begin{pmatrix} (E + F)^{(1,3)} + (E - F)^{(1,3)} & (E + F)^{(1,3)} - (E - F)^{(1,3)} \\ (E + F)^{(1,3)} - (E - F)^{(1,3)} & (E + F)^{(1,3)} + (E - F)^{(1,3)} \end{pmatrix} \quad (24) \]
is a \((1, 3)\)-inverse of the matrix \( S \), where \((E + F)^{(1,3)} \text{ and } (E - F)^{(1,3)} \) are \((1, 3)\)-inverse of matrices \((E + F) \text{ and } (E - F)\), respectively; in particular, the Moor–Penrose inverse of the matrix \( A \) is
\[ S' = \frac{1}{2} \begin{pmatrix} (E + F)^T + (E - F)^T & (E + F)^T - (E - F)^T \\ (E + F)^T - (E - F)^T & (E + F)^T + (E - F)^T \end{pmatrix}, \quad (25) \]

\[ \text{Proof}. \text{ By the theory of generalized inverse, it is sufficient to show that} \]
\[ SS^{(1,3)} = S, (SS^{(1,3)})^T = SS^{(1,3)}, \quad (26) \]
\[ S'S = S, S'S’ = S’, (SS')^T = SS^{(1,3)} = S' S, \]
where \((\cdot)^T \) denotes the transpose of a matrix \((\cdot)\).

From (10) and (15), we have
\[ SS^{(1,3)} = \begin{pmatrix} E & F \\ F & E \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} (E + F)^{(1,3)} + (E - F)^{(1,3)} & (E + F)^{(1,3)} - (E - F)^{(1,3)} \\ (E + F)^{(1,3)} - (E - F)^{(1,3)} & (E + F)^{(1,3)} + (E - F)^{(1,3)} \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & E \end{pmatrix} \]

\[ = \frac{1}{2} \begin{pmatrix} 2E & 2F \\ 2F & 2E \end{pmatrix} = \begin{pmatrix} E & F \\ F & E \end{pmatrix} = S. \]

Similarly, it is easy to verify \((SS^{(1,3)})^T = SS^{(1,3)}\) and (29). It concludes the proof.

From the above analysis, we know that the LR fuzzy matrix equation (8) is inconsistent when \( \text{Rank}(S) \neq \text{Rank}(S, Y) \) in its extended crisp matrix equation (10). If a LR fuzzy matrix equation (8) is inconsistent, we can consider its LR least squares solutions. However, the LR least squares solution matrix may still not be an appropriate LR fuzzy matrix. According to the theory of generalized inverse, we have the following result about the LR least squares solutions to the matrix equation (8).
Definition 6. Let $\bar{X} = (X, X^l, X^r)$, if $X, X^l, X^r$ is the minimal solution of the (9) such that $X^l \geq 0, X^r \geq 0$. Then, we say $\bar{X} = (X, X^l, X^r)$ is a strong LR fuzzy minimal solution of (9).

Remark 1. From Definition 6, we know that the Moore–Penrose inverse $S^r$ and $A^r$ being a special 1, 3-inverse, if $S^r$ and $A^r$ are nonnegative, then the system has a strong LR fuzzy least squares solution, by Lemma 2, which is the LR minimum norm fuzzy least squares solution.

The following result are given for $S^{(1,3)}$ and $S^r$ being nonnegative, as usual, $(\cdot)^r$ denotes the transpose of a matrix $(\cdot)$.

Theorem 5 (see [36]). The matrix $S$ of rank $r$ with no zero row or zero column, admits a nonnegative $S^{(1,3)}$-inverse if and only if there exists some permutation matrices, $Q$ such that

$$PSQ = [R, *],$$

where $R$ is a direct sum of $r$ positive and rank-one matrices.

Theorem 6 (see [37]). $S^r \geq 0$ if and only if

$$S^r = \begin{pmatrix} GE^r & GF^r \\ GF^r & GE^r \end{pmatrix},$$

for some positive diagonal matrix $G$; in this case,

$$(E + F)^r = G(E + F)^s, (E - F)^r = G(E - F)^s.$$  

Here, we give an algorithm for solving inconsistent fuzzy matrix equation as follows. (Algorithm 1)

4. Numerical Examples

Example 1. Consider the following fuzzy matrix equation:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} \\ \bar{x}_{21} & \bar{x}_{22} \end{pmatrix} = \begin{pmatrix} (1,2,2)_{LR} & (3,2,1)_{LR} \\ (2,1,2)_{LR} & (2,1,1)_{LR} \end{pmatrix}.$$  

The extended $4 \times 4$ matrix $S$ is

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

and the augmented matrix is

$$\bar{x}_{ij} = \begin{cases} (x_{ij}, x^l_{ij}, x^r_{ij}), \\ (x_{ij}, 0, \max(-x^r_{ij}, x^l_{ij})), \\ (x_{ij}, \max(x^l_{ij}, -x^r_{ij}), 0), \\ (x_{ij}, -x^r_{ij}, -x^l_{ij}), \end{cases}$$

otherwise, the $\bar{X} = (X, X^l, X^r)$ is said to a weak LR fuzzy minimal solution of fuzzy matrix equation (9) given by

$$S^r \geq 0, (1 \leq i, j \leq n) \text{ and } x^r_{ij} > 0, x^l_{ij} > 0.$$

Otherwise, the $\bar{X} = (X, X^l, X^r)$ is said to a weak LR fuzzy minimal solution of fuzzy matrix equation (9) given by

$$S^r \geq 0, (1 \leq i, j \leq n) \text{ and } x^r_{ij} > 0, x^l_{ij} < 0.$$  

Therefore, the original fuzzy matrix equation has a strong LR fuzzy solution, which is the LR minimum norm fuzzy least squares solution.
and the augmented matrix is
\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]

**Algorithm 1:** The steps for solving inconsistent fuzzy matrix equation is as follows.

(i) Step 1. Decomposing the matrix \( A \) with \( A = A^+ + A^- \).
(ii) Step 2. Setting up the model
\[
\begin{pmatrix}
(A^+ + A^-)X = B, \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
A^T - A^- \\
-A^+ + A^- \\
\end{pmatrix}
\begin{pmatrix}
X^T \\
X^- \\
\end{pmatrix}
= \begin{pmatrix}
B^T \\
B^- \\
\end{pmatrix}.
\]
(iii) Step 3. Solving the model
\[
\begin{pmatrix}
X = A^{(1,3)}B + (I_n - A^{(1,3)})Z, \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
X^T, X^- \\
\end{pmatrix}
= S^{(1,3)}Y + (I_{2n} - S^{(1,3)})Z.
\]
In generally,
\[
\begin{pmatrix}
X = (A^+ + A^-)^TB, \\
\end{pmatrix}
\]
\[
\begin{pmatrix}
X^T = (A^+ - A^-)^TB^T, \\
\end{pmatrix}
\]
(iv) Step 4. Judging and giving strong LR fuzzy minimal solution
\[
\bar{x}_{ij} = \begin{cases}
(x_{ij}, x^+_{ij}, x^-_{ij}), & x^+_{ij} > 0, x^-_{ij} > 0, \\
(x_{ij}, 0, \max(-x^-_{ij}, x^+_{ij})), & x^+_{ij} < 0, x^-_{ij} > 0, \\
(x_{ij}, \max(x^+_{ij}, -x^-_{ij})), & 0, x^+_{ij} > 0, x^-_{ij} < 0, \\
(x_{ij}, -x^+_{ij}, -x^-_{ij}), & x^+_{ij} < 0, x^-_{ij} < 0.
\end{cases}
\]

by Definition 6.

**Example 2.** Consider the following fuzzy matrix equation:
\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
\bar{x}_{11} \\
\bar{x}_{21} \\
\bar{x}_{31}
\end{pmatrix}
= \begin{pmatrix}
(1,2,3)_{LR} \\
(2,1,1)_{LR} \\
(2,3,1)_{LR}
\end{pmatrix}.
\]

The extended \( 6 \times 6 \) matrix \( S \) is
\[
S = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix},
\]
and the augmented matrix is
\[
SY = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\
1 & 0 & 0 & 0 & 1 & 1 & 3 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 & 2 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 2 & 3
\end{pmatrix}.
\]

where \( A \) is nonsingular, while \( S \) is singular, which implies the original is inconsistent.

One \((1,3)\)-inverse of \( A \) and \( S \) is
\[
A^{(1,3)} = \begin{pmatrix}
0.50 & 0.00 & 0.50 \\
0.00 & 0.50 & -0.50 \\
0.50 & -0.50 & 0.00
\end{pmatrix},
\]
\[
S^{(1,3)} = \begin{pmatrix}
0.62 & -0.20 & 0.24 & 0.12 & -2.00 & -0.26 \\
-0.01 & 0.62 & -0.45 & -0.01 & 0.12 & 0.05 \\
0.05 & -0.26 & 0.37 & -0.45 & 0.24 & 0.37 \\
0.12 & -2.00 & -0.26 & 0.62 & -0.20 & 0.24 \\
-0.01 & 0.12 & 0.05 & -0.01 & 0.62 & -0.45 \\
-0.45 & 0.24 & 0.37 & 0.05 & -0.26 & 0.37
\end{pmatrix},
\]
i.e., one solution of the equations is
\[
\begin{pmatrix}
\bar{x}_{11} \bar{x}_{12} \bar{x}_{13} \\
\bar{x}_{21} \bar{x}_{22} \bar{x}_{23} \\
\bar{x}_{31} \bar{x}_{32} \bar{x}_{33}
\end{pmatrix} = \begin{pmatrix}
(1.50, 1.67, 1.17) \\
(2.50, 0.43, 0.93) \\
(3.00, -0.39, 0.61)
\end{pmatrix},
\]
\[
\begin{pmatrix}
0.00, -0.60, 0.41 \\
-1.00, -0.07, -0.07 \\
0.00, 1.06, -0.44
\end{pmatrix},
\]
\[
\begin{pmatrix}
-0.50, 0.24, 0.74 \\
0.50, 0.63, 1.13 \\
0.00, 0.81, 1.31
\end{pmatrix}.
\]

Since \( \bar{x}_{13}, \bar{x}_{21}, \bar{x}_{22} \), and \( \bar{x}_{23} \) are not LR fuzzy numbers, the corresponding solution is a weak LR fuzzy least squares solution given by
Example 3. Consider the following fuzzy matrix equation:

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
\bar{x}_{11} & \bar{x}_{12} \\
\bar{x}_{21} & \bar{x}_{22} \\
\bar{x}_{31} & \bar{x}_{32}
\end{pmatrix} =
\begin{pmatrix}
(1,2,3)_{LR} & (2,1,2)_{LR} \\
(2,1,1)_{LR} & (1,1,1)_{LR}
\end{pmatrix}.
\]

The extended $4 \times 6$ matrix $S$ is

\[
S = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix},
\]

and the augmented matrix is

\[
SY = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 & 2 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}.
\]
which implies that the original equation is inconsistent since
\( \text{Rank } (S) = 3 \) and \( \text{Rank } (S, Y) = 4 \).
One \((1,3)\)-inverse of \( A \) and \( S \) is
\[
A^{(1,3)} = \begin{pmatrix}
0.25 & 0.25 \\
0.25 & 0.25 \\
0.50 & -0.50
\end{pmatrix},
\]
\[
S^{(1,3)} = \begin{pmatrix}
0.25 & 0.25 \\
0.25 & 0.25 \\
0.50 & -0.50
\end{pmatrix},
\]
i.e., one solution of the equation is
\[
\begin{pmatrix}
\bar{x}_{11} \\
\bar{x}_{21} \\
\bar{x}_{31}
\end{pmatrix} = \begin{pmatrix}
0.75, -0.01, 1.08 \\
0.75, 1.66, 1.08 \\
-0.50, -0.10, 0.40
\end{pmatrix},
\]
\[
\begin{pmatrix}
\bar{x}_{12} \\
\bar{x}_{22} \\
\bar{x}_{32}
\end{pmatrix} = \begin{pmatrix}
0.75, 0.18, 0.70 \\
0.75, 0.73, 0.70 \\
0.50, 0.06, 0.56
\end{pmatrix}.
\]
Since \( \bar{x}_{11} \) and \( \bar{x}_{31} \) are not LR fuzzy numbers, the corresponding solution is a weak LR fuzzy least squares solution given by
\[
\begin{pmatrix}
\bar{u}_{11} \\
\bar{u}_{21} \\
\bar{u}_{31}
\end{pmatrix} = \begin{pmatrix}
0.75, 1.08, 0.00 \\
0.75, 1.66, 1.08 \\
-0.50, 0.40, 0.00
\end{pmatrix},
\]
\[
\begin{pmatrix}
\bar{u}_{12} \\
\bar{u}_{22} \\
\bar{u}_{32}
\end{pmatrix} = \begin{pmatrix}
0.75, 0.18, 0.70 \\
0.75, 0.73, 0.70 \\
0.50, 0.06, 0.56
\end{pmatrix}.
\]
The corresponding solution is a weak LR fuzzy least squares solution.
The Moore–Penrose inverse of \( A \) and \( S \) is
\[
A^+ = \begin{pmatrix}
0.25 & 0.25 \\
0.25 & 0.25 \\
0.50 & -0.50
\end{pmatrix},
\]
\[
S^+ = \begin{pmatrix}
0.21 & -0.04 & -0.04 \\
0.21 & -0.04 & 0.21 \\
0.33 & -0.17 & 0.21 \\
-0.04 & 0.21 & 0.21 \\
0.33 & 0.33 & 0.33 \\
-0.17 & 0.33 & 0.33
\end{pmatrix},
\]
Then, the solution is
\[
\begin{pmatrix}
\bar{x}_{11} \\
\bar{x}_{21} \\
\bar{x}_{31}
\end{pmatrix} = \begin{pmatrix}
0.75, 0.46, 0.71 \\
0.75, 0.46, 0.71 \\
-0.50, 0.33, 0.83
\end{pmatrix},
\]
\[
\begin{pmatrix}
\bar{x}_{12} \\
\bar{x}_{22} \\
\bar{x}_{32}
\end{pmatrix} = \begin{pmatrix}
0.75, 0.29, 0.54 \\
0.75, 0.29, 0.54 \\
0.50, 0.17, 0.67
\end{pmatrix}.
\]
Therefore, the original equation has a strong LR fuzzy solution, which is the minimum norm fuzzy least squares solution.

5. Conclusion
In this work, we proposed a general model to solve a class of LR inconsistent fuzzy matrix equation \( A\bar{x} = \bar{b} \), in which \( A \) is a \( m \times n \) crisp matrix. By the embedding method, the original system was converted two crisp system, and we analyzed the solvability to the LR general fuzzy matrix equation and obtained the LR fuzzy least squares solution to the inconsistent fuzzy equation system by using generalized inverses of the matrix \( S \). Finally, we provided a sufficient condition for the LR least squares solution being a strong fuzzy solution. Our results enriched the fuzzy linear systems theory.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Acknowledgments
The work was supported by the Natural Scientific Funds of PR China (nos. 61967014 and 11861059) and Scientific Research Project of Gansu Province Colleges and Universities (no. 2019A-004).

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