LOCAL WELL-POSEDNESS FOR THE HALL-MHD SYSTEM IN OPTIMAL SOBOLEV SPACES

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ABSTRACT. We show that the viscous resistive magneto-hydrodynamics system with Hall effect is locally well-posed in $H^s(\mathbb{R}^n) \times H^{s+1-\varepsilon}(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$ and any small enough $\varepsilon > 0$ such that $s + 1 - \varepsilon > \frac{n}{2}$. This space is to date the largest local well-posedness space in the class of Sobolev spaces for the system. It is also optimal according to the predominant scalings of the two equations in the system.

KEY WORDS: Magneto-hydrodynamics; Hall effect; local well-posedness; scaling structure.
CLASSIFICATION CODE: 76D03, 35Q35.

1. INTRODUCTION

Considered in this treatise is the three dimensional incompressible viscous resistive Hall-magneto-hydrodynamics (Hall-MHD) system:

\begin{equation}
\begin{aligned}
    u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p - \nu \Delta u &= 0, \\
    b_t + u \cdot \nabla b - b \cdot \nabla u + \eta \nabla \times ((\nabla \times b) \times b) - \mu \Delta b &= 0, \\
    \nabla \cdot u &= 0,
\end{aligned}
\end{equation}

(1.1)

accompanied with the initial conditions

\begin{equation}
\begin{aligned}
    u(x,0) &= u_0(x), \\
    b(x,0) &= b_0(x), \\
    \nabla \cdot u_0 &= \nabla \cdot b_0 = 0,
\end{aligned}
\end{equation}

(1.2)

for $x \in \mathbb{R}^3$ and $t \geq 0$. In the system, $u$ represents the fluid velocity, $p$ is the fluid pressure and $b$ stands for the magnetic field. The parameters $\nu, \mu$ and $\eta$ denote the fluid viscosity, resistivity (electrical diffusivity) and the Hall effect coefficient, respectively. It is important to observe that, if $\nabla \cdot b_0 = 0$, the divergence free condition for $b$ is propagated by the second equation of (1.1), see [3]. The Hall term $\nabla \times ((\nabla \times b) \times b)$ distinguishes (1.1) from the usual MHD system (system (1.1) with $\eta = 0$). In contrast to the latter one, the Hall-MHD model is more advantageous due to the fact that it can capture the essential characteristics of the magneto-hydrodynamics with strong magnetic reconnection where the Hall effect plays a significant role. Magnetic reconnection is a fundamental dynamical process in highly conductive plasmas in astrophysics, allowing for explosive and efficient magnetic to kinetic energy conversion. For a more comprehensive physical background of the magnetic reconnection phenomena and the Hall-MHD model, we refer the readers to [11, 14, 16] and references therein.

Despite its increasing popularity among the astrophysicists community, the mathematical understanding of the Hall-MHD model is very limited. Conceptually, we can have a peek about the barriers from various perspectives. First, the Hall term launches new physics into the system at small length scales and hence intrinsically
challenging into the mathematical analysis. Second, it is well-known that the main 
problem to understand the turbulent flows governed by the Navier-Stokes equation
(NSE) relies on the nonlinearity such as \((u \cdot \nabla)u\). One can imagine that system
\([1.1]\) is more intricate than the NSE, for the former one contains the NSE and
a magnetic field equation with the Hall term which appears more singular than
\((u \cdot \nabla)u\). Third, the natural scaling structure is a strong motivation in the study
of both the NSE and the MHD system, who share the same scaling. However, the
Hall term destroys such natural scaling. Into more details, for the MHD system, if
\((u(x, t), p(x, t), b(x, t))\) solves \([1.1]\) with \(\eta = 0\) with the initial data \((u_0(x), b_0(x))\),
then the triplet \((u_\lambda(x, t), p_\lambda(x, t), b_\lambda(x, t))\) defined by
\[
(1.3) \quad u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad b_\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t)
\]
solves the same system with the data
\[
u_{0\lambda}(x, t) = \lambda u_0(\lambda x), \quad b_{0\lambda}(x, t) = \lambda b_0(\lambda x).
\]
The scaling \([1.3]\) no longer holds for system \([1.1]\) with \(\eta > 0\). On the other hand,
we can extract the “Hall equation”
\[
b_\lambda + \nabla \times ((\nabla \times b) \times b) = \Delta b
\]
which has the scaling
\[
(1.4) \quad b_\lambda(x, t) = b(\lambda x, \lambda^2 t).
\]
Since the Hall term is the most singular nonlinearity in system \([1.1]\), it suggests
that the predominant scaling for \([1.1]\) could be
\[
(1.5) \quad u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad b_\lambda(x, t) = b(\lambda x, \lambda^2 t).
\]
In fact, based on scaling \([1.5]\), we obtained a regularity criterion for \([1.1]\) in three
dimension which improves various criteria in the literature, see \([9]\).

In this paper our interest is to find the largest possible (optimal) Sobolev space
where system \([1.1]\) is locally well-posed. On this topic, it was first shown in \([7]\)
that system \([1.1]\) in three dimension is locally well-posed in \(H^s(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3)\) with \(s > \frac{5}{2}\).
By taking \([1.3]\) as the dominant scaling, in \([8]\), we obtained the local well-posedness
of \([1.1]\) in \(H^s(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3)\) with \(s > \frac{3}{2}\). Even though the result of \([8]\)
Improves that of \([7]\), it seems that there is still room to have improvement, for the reason
that the NSE is known to be locally well-posed in \(H^s(\mathbb{R}^3)\) with \(s > \frac{1}{2} - 1\). In fact,
motivated by scaling \([1.3]\), one expects that system \([1.1]\) may be locally well-posed
in \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) with \(s > \frac{n}{2} - 1\). In order to justify the conjecture, we need to
treat the energy estimates for \(u\) and \(b\) separately, namely, \(u\ in H^s\) and \(b\ in H^{s+1}\).
In this situation, we encounter the difficulty that no cancelation can be employed to
deal with the two terms \(b \cdot \nabla b\) and \(b \cdot \nabla u\). To overcome this barrier, it comes to our
mind that we need to optimize the estimates of the flux contributed from the two
terms by fully employing the diffusion of both the \(u\) and the \(b\). Techniques based on
the paradifferential calculus enables us to operate such optimizations. Surprisingly,
it turns out that the local well-posedness space we can obtain is slightly larger than the
conjectured one. In deed, we prove the main result below.

**Theorem 1.1.** Let \((u_0, b_0) \in H^s(\mathbb{R}^n) \times H^{s+1-\varepsilon}(\mathbb{R}^n)\) with \(s > \frac{n}{2} - 1\) and any small
enough \(\varepsilon > 0\) such that \(s + 1 - \varepsilon > \frac{3}{2} + \frac{n}{2}\). Assume \(\nabla \cdot u_0 = \nabla \cdot b_0 = 0\). There exists a
time $T = T(u, \mu, \|u_0\|_{H^{s}}, \|b_0\|_{H^{s+1-\epsilon}}) > 0$ and a unique solution $(u, b)$ of (1.1) on $[0, T]$ such that

$$(u, b) \in C([0, T); H^{s}(\mathbb{R}^n)) \times C([0, T); H^{s+1-\epsilon}(\mathbb{R}^n)).$$

Regarding the result, the fact that $b$ needs to be in a space with higher regularity is determined by the Hall term. Predicted by the scaling (1.4) of the “Hall equation”, the optimal Sobolev space of well-posedness for $b$ would be $H^{s+1}(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$. However, as stated in Theorem 1.1, the obtained well-posedness space for $b$ is $H^{s+1-\epsilon}(\mathbb{R}^n)$ for any small $\epsilon > 0$. It may be explained by getting a closer look at the term $b \cdot \nabla u$. While estimating $\|b \cdot \nabla u\|_{H^{r}}$ by applying both diffusions of $u$ and $b$, it happens that we need to take $r$ slightly smaller than $s + 1$.

2. Preliminaries

2.1. Notation. In order to avoid confusion, we specify a few notations. We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some absolute constant $C$, and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with absolute constants $C_1, C_2$. For simplification, it is understood that $\| \cdot \|_p = \| \cdot \|_{L^p}$.

2.2. Littlewood-Paley decomposition. As in our previous articles on the local well-posedness of magneto-hydrodynamics systems, the main tool is paradifferential calculus. To be self-contained, we recall the Littlewood-Paley decomposition theory briefly, even though it appears in our earlier work on related topics. For a more detailed description on this theory we refer the readers to [2] and [12].

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively. Define $\lambda_q = 2^q$ for integers $q$. A nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^n)$ is chosen such that

$$\chi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4} \\ 0, & \text{for } |\xi| \geq 1. \end{cases}$$

Let

$$\varphi(\xi) = \chi\left(\frac{\xi}{2}\right) - \chi(\xi)$$

and

$$\varphi_q(\xi) = \begin{cases} \varphi(\lambda_q^{-1}\xi) & \text{for } q \geq 0, \\ \chi(\xi) & \text{for } q = -1. \end{cases}$$

For a tempered distribution vector field $u$ we define the Littlewood-Paley projection

$$h = \mathcal{F}^{-1}\varphi, \quad \tilde{h} = \mathcal{F}^{-1}\chi,$$

$$u_q := \Delta_q u = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1}\xi)\mathcal{F}u) = \lambda_q^n \int h(\lambda_q y)u(x - y)dy, \quad \text{for } q \geq 0,$$

$$u_{-1} = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}u) = \int \tilde{h}(y)u(x - y)dy.$$

By the Littlewood-Paley theory, the identity

$$u = \sum_{q=-1}^{\infty} u_q$$
Lemma 2.4. In principle, the commutators will be used to reveal certain cancellation; and to shift derivative from high modes to low modes. It was shown in [9] they satisfy the following estimates.

\[ \|(\Delta_q u, \nabla v)\|_p \leq C \lambda^q \|u\|_p \]

Definition 2.1. A tempered distribution \(u\) belongs to the Besov space \(B^s_p,\infty\) if and only if

\[ \|u\|_{B^s_p,\infty} = \sup_{q \geq 1} \lambda^q \|u_q\|_p < \infty. \]

We can identify the Sobolev space \(H^s\) by the Besov space \(B^s_{2,2}\), i.e.

\[ \|u\|_{H^s} \sim \left( \sum_{q=-1}^{\infty} \lambda^q u_q^2 \right)^{1/2} \]

for each \(u \in H^s\) and \(s \in \mathbb{R}\).

Lemma 2.2. (Bernstein’s inequality. See [13].) Let \(n\) be the space dimension and \(r \geq s \geq 1\). Then for all tempered distributions \(u\), we have

\[ \|u_q\|_r \lesssim \lambda_q^{(s-\frac{n}{2})} \|u_q\|_s. \]

2.3. Bony’s paraproduct and commutator. Bony’s paraproduct formula

\[ \Delta_q(u \cdot \nabla v) = \sum_{|q-p| \leq 2} \Delta_q(u_{\leq p-2} \cdot \nabla v_p) + \sum_{|q-p| \leq 2} \Delta_q(u_p \cdot \nabla v_{\leq p-2}) \]

will be used constantly to decompose the nonlinear terms in energy estimate. We will also use the notation of the commutator

\[ [\Delta_q, u_{\leq p-2} \cdot \nabla] v_p := \Delta_q(u_{\leq p-2} \cdot \nabla v_p) - u_{\leq p-2} \cdot \nabla \Delta_q v_p. \]

Lemma 2.3. The commutator satisfies the following estimate, for any \(1 < r < \infty\)

\[ \|\Delta_q, u_{\leq p-2} \cdot \nabla\|_{R} \lesssim \|\nabla u_{\leq p-2}\|_{\infty} \|v_p\|_r. \]

2.4. Auxiliary estimates. To handle the Hall term \(\nabla \times ((\nabla \times b) \times b)\), more preparation is needed. We first introduce two more commutators and their estimates. We define that, for vector valued functions \(F\) and \(G\),

\[ [\Delta_q, F \times \nabla \times] G = \Delta_q(F \times (\nabla \times G)) - F \times (\nabla \times G_q), \]

\[ [\Delta_q, \nabla \times F \times] G = \Delta_q(\nabla \times F \times G) - \nabla \times F \times G_q. \]

In principle, the commutators will be used to reveal certain cancellation; and to shift derivative from high modes to low modes. It was shown in [9] they satisfy the following estimates.

Lemma 2.4. Let \(F\) and \(G\) be vector valued functions. Assume \(\nabla \cdot F = 0\) and \(F\), \(G\) vanish at large \(|x| \in \mathbb{R}^3\). For any \(1 < r < \infty\), we have

\[ \|\Delta_q, F \times \nabla \times G\|_r \lesssim \|\nabla F\|_{\infty} \|G\|_r; \]

\[ \|\Delta_q, \nabla \times F \times G\|_r \lesssim \|\nabla F\|_{\infty} \|G\|_r. \]
Lemma 2.5. Let $F$, $G$ and $H$ be vector valued functions. Assume $F$, $G$ and $H$ vanish at large $|x| \in \mathbb{R}^3$. For any $1 < r_1, r_2 < \infty$ with $\frac{1}{r_1} + \frac{1}{r_2} = 1$, we have

$$\left| \int_{\mathbb{R}^3} [\Delta_q, \nabla \times F] G \cdot \nabla \times H \, dx \right| \lesssim \|\nabla^2 F\|_{\infty} \|G\|_{r_1} \|H\|_{r_2}.$$ 

3. A priori estimate

In this section, we establish a priori estimate for smooth solutions in $H^s(\mathbb{R}^n) \times H^r(\mathbb{R}^n)$ with appropriate index $s$ and $r$. Such estimate is the most crucial ingredient in the argument of local well-posedness, which is rather standard for dissipative equations, see [15]. Thus we only present the following theorem and its proof.

Theorem 3.1. Let $(u_0, b_0) \in H^s(\mathbb{R}^n) \times H^r(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$ and $\frac{n}{2} < r \leq s + 1 - \varepsilon$ for small enough $\varepsilon > 0$. There exists a time $T = T(\nu, \mu, \|u_0\|_{H^s}, \|b_0\|_{H^r}) > 0$ such that the Hall-MHD system (1.1) has a solution $(u, b)$ satisfying

$$u \in L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^n)),$$

$$b \in L^\infty(0, T; H^r(\mathbb{R}^n)) \cap L^2(0, T; H^{r+1}(\mathbb{R}^n)).$$

The proof involves certain amount of computations and estimates which will be divided into several lemmas, each carrying an estimate for a flux term. To start, multiplying the first equation of (1.1) by $\lambda_q^2 \nabla_q u_q$ and the second one by $\lambda_q^2 \nabla_q b_q$, and adding up for all $q \geq -1$, we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|^2_2 + \nu \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|^2_2 \leq -I_1 - I_2,$$

$$\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda_q^{2r} \|b_q\|^2_2 + \mu \sum_{q \geq -1} \lambda_q^{2r+2} \|b_q\|^2_2 \leq -I_3 - I_4 - I_5,$$

with

$$I_1 = \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \cdot u_q \, dx,$$

$$I_2 = - \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla b) \cdot u_q \, dx,$$

$$I_3 = \sum_{q \geq -1} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla b) \cdot b_q \, dx,$$

$$I_4 = - \sum_{q \geq -1} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla u) \cdot b_q \, dx,$$

$$I_5 = - \sum_{q \geq -1} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q ((\nabla \times b) \cdot b) \cdot \nabla \times b_q \, dx.$$

To fully exploit cancelations in the flux terms $I_1$, $I_3$ and $I_5$, we will apply commutator estimates along with Bony’s paraproduct and some fundamental inequalities. While $r \neq s$, there is no cancelation in $I_2 + I_4$, and hence $I_2$ and $I_4$ will be treated in slightly different ways.

Lemma 3.2. Let $s > \frac{n}{2} - 1$. We have that, for some absolute constants $\gamma_1, \gamma_2 > 0$,

$$|I_1| \leq \frac{\nu}{8} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|^2_2 + C\nu \|u\|^{2+\gamma_1}_{H^s} + C\nu \|u\|^{2+\gamma_2}_{H^s}.$$
Proof: Using Bony’s paraproduct (2.7) followed by the commutator notation (2.8), $I_1$ is decomposed as

$$I_1 = \sum_{q \geq 1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_{\leq p-2} \cdot \nabla u_p)u_q \, dx$$

$$+ \sum_{q \geq 1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla u_{\leq p-2})u_q \, dx$$

$$+ \sum_{q \geq 1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_p \cdot \nabla u)u_q \, dx$$

$$= I_{11} + I_{12} + I_{13},$$

with

$$I_{11} = \sum_{q \geq 1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, u_{\leq p-2} \cdot \nabla]u_p u_q \, dx$$

$$+ \sum_{q \geq 1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} u_{\leq q-2} \cdot \nabla \Delta_q u_p u_q \, dx$$

$$+ \sum_{q \geq 1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla \Delta_q u_p u_q \, dx$$

$$= I_{111} + I_{112} + I_{113}.$$
such that for some $\theta_1 > 0$, $\theta_2 > 0$

$$ |I_{111}| \leq \frac{\nu}{64} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_\nu \sum_{q \geq -1} (\lambda_q^{2s} \|u_q\|_2^{(2-\theta)r_2}) $$

$$ + \frac{\nu}{64} \sum_{q \geq -1} \sum_{p \leq q} \lambda_p^{2s+2} \|u_p\|_2 \lambda_{p-q}^{\theta_1} + C_\nu \sum_{q \geq -1} \sum_{p \leq q} (\lambda_p^{2s} \|u_p\|_2^{(2-\theta)r_2}) $$

$$ \leq \frac{\nu}{32} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_\nu \left( \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2 \right) \left( \sum_{q \geq -1} \lambda_q^{\theta_2} \|u_q\|_2^2 \right) $$

Notice that (3.13) and (3.14) imply that $s > \frac{\theta}{2} - 1$.

To estimate $I_{113}$, it follows from Hölder, Bernstein and Young's inequalities that

$$ |I_{113}| \leq \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2s} \|u_{p-q}\|_2 \|\nabla u_p\|_\infty \|u_q\|_2 $$

$$ \leq \sum_{q \geq -1} \lambda_q^{2s+\frac{\theta}{2}+1} \|u_q\|_2^3 $$

$$ \leq \sum_{q \geq -1} \lambda_q^{(s+1)\theta} \|u_q\|_2^{\theta} \lambda_q^{(3-\theta)} \|u_q\|_2^{3-\theta} \lambda_q^{\frac{\theta}{2}+1-s-\theta} $$

$$ \leq \frac{\nu}{32} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_\nu \left( \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^2 \right) $$

for $s \geq \frac{\theta}{2} + 1 - \theta$ and $0 < \theta < 2$. Thus

$$ I_1 \leq \frac{\nu}{8} \|\nabla u\|_{H^s}^2 + C_\nu \|u\|_{H^s}^{2+\gamma_1} + C_\nu \|u\|_{H^s}^{2+\gamma_2} $$

for $s > \frac{\theta}{2} - 1$ and some $\gamma_1, \gamma_2 > 0$.

**Lemma 3.3.** Let $\frac{\theta}{2} + s - 2r \leq 0$ and $s < r$. The following estimate holds

$$ |I_2| \leq \frac{\nu}{8} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_\nu \|b\|_{H^r}^4 $$

**Proof:** We first decompose $I_2$ by using Bony's paraproduct,

$$ I_2 = - \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_{\leq p-2} \cdot \nabla b_p) u_q \, dx $$

$$ - \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla b_{\leq p-2}) u_q \, dx $$

$$ - \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla \tilde{b}_p) u_q \, dx $$

$$ = I_{21} + I_{22} + I_{23}.$$
Due to the lack of cancelation, $I_{21}$ is the worst term which can be estimated as

$$|I_{21}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+1} \|b_{q-p-2}\|_\infty \|b_p\|_2 \|u_q\|_2$$

$$\leq \sum_{q \geq -1} \lambda_q^{2s+1} \|b_q\|_2 \|u_q\|_2 \sum_{p \leq q} \lambda_p^2 \|b_p\|_2$$

$$\leq \sum_{q \geq -1} \lambda_q^{s+1} \|u_q\|_2 \lambda_q^{r} \|b_q\|_2 \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{s-r} \lambda_p^{\frac{s}{2}+2r}$$

$$\leq \sum_{q \geq -1} \lambda_q^{s+1} \|u_q\|_2 \lambda_q^{r} \|b_q\|_2 \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{s-r}$$

for $\frac{n}{2} + s - 2r \leq 0$. As a result, Young’s inequality gives rise to

$$|I_{21}| \leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_r \sum_{q \geq -1} \left( \lambda_q^{r} \|b_q\|_2 \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{s-r} \right)^2$$.

Then we apply Jensen’s inequality, if $s < r$,

$$|I_{21}| \leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_r \sum_{q \geq -1} \lambda_q^{r} \|b_q\|_2^2 \sum_{p \leq q} \lambda_p^r \|b_p\|_2^2 \lambda_q^{s-r}$$

$$\leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_r \left( \sum_{q \geq -1} \lambda_q^{r} \|b_q\|_2^2 \right)^2.$$

We claim that $I_{22}$ shares the same estimate as $I_{21}$. Indeed, the following inequality holds

$$|I_{22}| \lesssim \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2 \|u_q\|_2 \sum_{p \leq q} \lambda_p^{s+1} \|b_p\|_2 \lesssim |I_{21}|.$$ 

To move the derivative from high modes to low modes in $I_{23}$, we apply integration by parts

$$|I_{23}| = \left| \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \otimes \hat{b}_p) \cdot \nabla u_q \, dx \right|.$$ 

It then follows from Hölder’s and Bernstein’s inequalities

$$|I_{23}| \lesssim \sum_{q \geq -1} \lambda_q^{2s+1} \|u_q\|_2 \|b_q\|_2 \|b_p\|_\infty$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2s+1} \|u_q\|_2 \sum_{p \geq q-4} \lambda_p^2 \|b_p\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \|u_q\|_2 \sum_{p \geq q-4} \lambda_p^r \|b_p\|_2 \lambda_q^{s-r} \lambda_p^{\frac{s}{2}+s-2r}$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \|u_q\|_2 \sum_{p \geq q-4} \lambda_p^r \|b_p\|_2 \lambda_q^{s-r}.$$
for $\frac{n}{2} + s - 2r \leq 0$. Applying Young’s inequality, Jensen’s inequality and changing order of the summations yields

$$|I_{33}| \leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_{\nu} \sum_{q \geq -1} \left( \sum_{p \geq q-4} \lambda_p^{2r} \|b_p\|_2^2 \lambda_{q-p}^s \right)^2$$

$$\leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_{\nu} \sum_{q \geq -1} \sum_{p \geq q-4} \lambda_p^{4r} \|b_p\|_2^2 \lambda_{q-p}^s$$

$$\leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_{\nu} \sum_{q \geq -1} \sum_{p \geq q-4} \lambda_p^{4r} \|b_p\|_2^2$$

It completes the proof.

\[ \Box \]

**Lemma 3.4.** Let $s > \frac{n}{4} - 1$ and $\frac{n}{4} + \frac{s}{2} < r < s + 2 - \varepsilon$ with small enough $\varepsilon > 0$. We have the estimate

$$|I_3| \leq \frac{\nu}{8} \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 + \frac{H}{8} \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2 + C_{\nu,\mu} \|u\|_H^{2s+\gamma_3} + C_{\nu,\mu} \|b\|_H^{2s+\gamma_4}$$

for some constants $\gamma_3, \gamma_4 > 0$.

**Proof:** As for $I_1$, we first decompose $I_3$ by Bony’s paraproduct

$$I_3 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(u_{p \leq p-2} \cdot \nabla b_p) b_q \, dx$$

$$+ \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(u_{p \cdot \nabla b_{p \leq p-2}}) b_q \, dx$$

$$+ \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(u_{p \cdot \nabla b_p}) b_q \, dx$$

$$= I_{31} + I_{32} + I_{33},$$

and further decompose $I_{31}$ by using the commutator to

$$I_{31} = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} [\Delta_q, u_{p \leq p-2} \cdot \nabla] b_p b_q \, dx$$

$$- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} u_{p \leq p-2} \cdot \nabla \Delta_q b_p b_q \, dx$$

$$- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} (u_{p \leq p-2} - u_{q \leq q-2}) \cdot \nabla \Delta_q b_p b_q \, dx$$

$$= I_{311} + I_{312} + I_{313}.$$
It is not hard to see that $I_{312} = 0$. By the commutator estimate in Lemma 2.3, we infer

$$\left| I_{311} \right| \leq \sum_{q \geq 1} \sum_{p \leq q} \lambda_q^{2r} \left\| \nabla u_{p-2} \right\|_\infty \left\| b_p \right\|_2 \left\| b_q \right\|_2$$

for parameters $\theta$ and $\delta$ satisfying $0 < \theta < 2$, $0 < \delta < 1$ and

$$s \geq \frac{n}{2} + 1 - \theta - \delta. \tag{3.15}$$

It then follows from Young’s inequality with $(r_1, r_2, r_3, r_4) \in (1, \infty)^4$ satisfying

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1, \quad r_1 = \frac{2}{\theta}, \quad r_3 = \frac{2}{\delta} \tag{3.16}$$

such that

$$\left| I_{311} \right| \leq \sum_{q \geq 1} \lambda_q^{2s+2} \left\| u_q \right\|_2^2 + \sum_{q \geq 1} \lambda_q^{2r+2} \left\| b_q \right\|_2^2$$

$$+ C_{\nu, \mu} \left( \sum_{q \geq 1} \lambda_q^{2s} \left\| u_q \right\|_2^2 \right)^{1+\gamma_3} + C_{\nu, \mu} \left( \sum_{q \geq 1} \lambda_q^{2r} \left\| b_q \right\|_2^2 \right)^{1+\gamma_4}$$

for some constants $\gamma_3, \gamma_4 > 0$. Notice that (3.15) and (3.16) imply for large enough $r_2$ and $r_1$, and $\delta, \theta$ close enough to 1, there exists a small $\varepsilon > 0$ such that

$$s \geq \frac{n}{2} - \theta + \varepsilon > \frac{n}{2} - 1.$$ 

We observe that $|I_{313}| \lesssim |I_{311}|$, and hence $I_{313}$ enjoys the same estimate of $I_{311}$. Following similar strategy as for $I_{311}$, we estimate $I_{32}$ as follows,

$$\left| I_{32} \right| \leq \sum_{q \geq 1} \sum_{p \leq q} \lambda_q^{2r} \left\| u_p \right\|_2 \left\| \nabla b_{p-2} \right\|_\infty \left\| b_q \right\|_2$$

for $0 < \theta < 1$ and

$$s \geq \frac{n}{2} - \theta. \tag{3.17}$$
It then follows from Young’s inequality and Jensen’s inequality, with the triplet $(2, \frac{2}{1-\theta}, \frac{2}{1-\theta})$ satisfying

\begin{equation}
 r - s - 1 - \theta < 0
\end{equation}

such that

\[
|I_{32}| \leq \frac\nu{32} \sum_{q \geq -1} \lambda_q^{2s+2}||u_q||_2^2 + \mu \frac{\mu}{32} \sum_{q \geq -1} \lambda_q^{2s+2}||b_q||_2^2 + C_{\nu,\mu} \left( \sum_{q \geq -1} \lambda_{p}^{2r}||b_p||_2^2 \right)^{\frac{1}{1-\theta}}.
\]

The constraints (3.17) and (3.18) implies that for $\theta = 1 - \varepsilon$

\[
s > r - 1 - \theta, \quad s \geq \frac{n}{2} - 1 + \varepsilon > \frac{n}{2} - 1.
\]

The term $I_{33}$ can be estimated in an analogous way as for $I_{23}$. To not overload the analysis with computations, we omit the details and claim

\[
|I_{33}| \leq \frac\nu{32} \sum_{q \geq -1} \lambda_q^{2s+2}||u_q||_2^2 + \mu \frac{\mu}{32} \sum_{q \geq -1} \lambda_q^{2s+2}||b_q||_2^2 + C_{\nu,\mu} \left( \sum_{q \geq -1} \lambda_{p}^{2r}||b_p||_2^2 \right)^{1+\gamma_4/2}
\]

for some constant $\gamma_4 > 0$.

\[\Box\]

**Lemma 3.5.** Let the index $r$ and $s$ satisfy conditions in Lemma 3.4. In addition, assume $r \leq s + 1 - \varepsilon$ for a small enough constant $\varepsilon > 0$. We have

\[
|I_4| \leq \frac\nu{32} \sum_{q \geq -1} \lambda_q^{2s+2}||u_q||_2^2 + \mu \frac{\mu}{32} \sum_{q \geq -1} \lambda_q^{2s+2}||b_q||_2^2 + C_{\nu,\mu} ||u||_{H^{2+\gamma_5}} + C_{\nu,\mu} ||b||_{H^{2+\gamma_6}} + C_{\nu,\mu} ||b||_{H^{2+\gamma_7}}
\]

for various constants $C_{\nu,\mu}$ depending on $\nu, \mu$, and some constants $\gamma_5, \gamma_6, \gamma_7 > 0$.

**Proof:** As usual, using Bony’s paraproduct, $I_4$ can be written as

\[
I_4 = -\sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(b_{\leq p-2} \cdot \nabla u_p)b_q \, dx
\]

\[
-\sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(b_{p} \cdot \nabla b_{\leq p-2})b_q \, dx
\]

\[
-\sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q(b_{p} \cdot \nabla u_p)b_q \, dx
\]

\[=I_{41} + I_{42} + I_{43}.
\]

First we notice that $I_{42}$ and $I_{43}$ can be estimated as $I_{311}$ and $I_{33}$, respectively. While $I_{41}$ needs to be treated in a different way, since cancellation is not available
here. Applying Hölder’s inequality and Bernstein’s inequality first, we get

\[
|I_{11}| \leq \sum_{q \geq -1} \sum_{|r| \leq 2} \lambda_q^{2r} \|b_{-p}^r\|_2 \|\nabla u_p\|_2 \|b_q\|_2
\]

\[
\lesssim \sum_{q \geq -1} \lambda_q^{2q+1} \|b_q\|_2 \|u_q\|_2 \sum_{p \leq q} \|b_p\|_\infty
\]

\[
\lesssim \sum_{q \geq -1} \lambda_q^{2q+1} \|b_q\|_2 \|u_q\|_2 \sum_{p \leq q} \lambda_p^2 \|b_p\|_2
\]

\[
\lesssim \sum_{q \geq -1} \left( \lambda_q^{(r+1)\delta} \|b_q\|_2^{\delta} \right) \left( \lambda_q^{r(1-\delta)} \|b_q\|_2^{1-\delta} \right) \left( \lambda_q^{s+1}\eta \|u_q\|_2^{\eta} \right) \left( \lambda_q^{s(1-\eta)} \|u_q\|_2^{1-\eta} \right)
\]

\[
\cdot \left( \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{r+1-s-\delta-\eta} \lambda_p^{1-s-\delta-\eta} \right)
\]

\[
\lesssim \sum_{q \geq -1} \left( \lambda_q^{(r+1)\delta} \|b_q\|_2^{\delta} \right) \left( \lambda_q^{r(1-\delta)} \|b_q\|_2^{1-\delta} \right) \left( \lambda_q^{s+1}\eta \|u_q\|_2^{\eta} \right) \left( \lambda_q^{s(1-\eta)} \|u_q\|_2^{1-\eta} \right)
\]

\[
\cdot \left( \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{r+1-s-\delta-\eta} \right)
\]

provided that \( \frac{r}{2} + 1 - s - \delta - \eta \leq 0 \). We apply Young’s inequality with parameters

\[1 \leq r_1, r_2, r_3, r_4, r_5 \leq \infty\]

satisfying

\[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5} = 1, \quad r_1 = \frac{2}{\delta}, \quad r_3 = \frac{2}{\eta},\]

for some \( \delta, \eta \in (0, 1) \). It yields that

\[
|I_{11}| \leq \frac{\nu}{64} \sum_{q \geq -1} \lambda_q^{2q+2} \|u_q\|_2^2 + \frac{\mu}{64} \sum_{q \geq -1} \lambda_q^{2r+2} \|b_q\|_2^2 + C_{\nu, \mu} \sum_{q \geq -1} \lambda_q^{(1-\delta)r_2} \|b_q\|_2^{(1-\delta)r_2}
\]

\[+ C_{\nu, \mu} \sum_{q \geq -1} \lambda_q^{(1-\eta)r_4} \|u_q\|_2^{(1-\eta)r_4} + C_{\nu, \mu} \sum_{q \geq -1} \left( \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{r+1-s-\delta-\eta} \right) \]

Assume \( r < s - 1 + \delta + \eta \). Using Jensen’s inequality to the last term and exchanging the order of summation gives rise to

\[
\sum_{q \geq -1} \left( \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{r+1-s-\delta-\eta} \right) \]

\[
\lesssim \sum_{q \geq -1} \sum_{p \leq q} \lambda_p^{r+1-s-\delta-\eta} \lambda_q^r \|b_p\|^r \|b_q\|^r
\]

\[
\lesssim \sum_{p \leq -1} \lambda_p^{r} \|b_p\|_2^{r} \sum_{q \geq -1} \lambda_q^{r+1-s-\delta-\eta}
\]

\[
\lesssim \left( \sum_{p \leq -1} \lambda_p^{2r} \|b_p\|_2^{2r} \right)^{\frac{r}{2r}}
\]

Thus one can choose \( \delta \) and \( \eta \) close enough to 1 and \( r_2, r_4, r_5 \) large enough such that \( (1 - \delta)r_2 = 2 + \gamma_5 \), \( (1 - \eta)r_4 = 2 + \gamma_6 \) and \( r_5/2 = 1 + \gamma_7/2 \) with \( \gamma_5, \gamma_6, \gamma_7 > 0 \). It
then follows that
\[
|I_{41}| \lesssim \frac{\mu}{64} \sum_{q \geq -1} \lambda_q^{2r+2} \|u_q\|_2^2 + \frac{\mu}{64} \sum_{q \geq -1} \lambda_q^{2r+2} \|b_q\|_2^2 \\
+ C_{\nu, \mu} \|x\|^{2+\gamma_6} + C_{\nu, \mu} \|b\|_H^{2+\gamma_6} + C_{\nu, \mu} \|b\|_H^{2+\gamma_6}
\]

Indeed, one can choose \( \delta + \eta = 2 - \varepsilon \) with \( \varepsilon = \frac{1}{2} \left[ s - \left( \frac{d}{2} - 1 \right) \right] \).

**Lemma 3.6.** Let \( r > \frac{d}{2} \). Then \( I_5 \) satisfies
\[
|I_5| \lesssim \frac{\mu}{16} \sum_{q \geq -1} \lambda_q^{2r+2} \|b_q\|_2^2 + C_{\nu} \|b\|_H^{2+\gamma_8} + C_{\mu} \|b\|_H^{2+\gamma_9}
\]
for some constants \( \gamma_8, \gamma_9 > 0 \).

**Proof:** Applying Bony’s paraproduct first, we decompose \( I_5 \) to
\[
I_5 = \sum_{q \geq -1} \sum_{[q-p] \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q (b_{\leq p-2} \times (\nabla \times b_p)) \cdot \nabla \times b_q \, dx \\
+ \sum_{q \geq -1} \sum_{[q-p] \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q (b_p \times (\nabla \times b_{\leq p-2})) \cdot \nabla \times b_q \, dx \\
+ \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q (b_p \times (\nabla \times b_{p})) \cdot \nabla \times b_q \, dx
\]

where we used the fact \( \sum_{q-2 \leq p \leq q+2} \Delta_q b_p = b_q \). It is clear that \( I_{512} = 0 \) due to the cross product property. By the commutator estimate in Lemma 2.5, we infer
\[
|I_{511}| \lesssim \sum_{q \geq -1} \sum_{[p-q] \leq 2} \lambda_q^{2r+1} \|\nabla b_{\leq p-2}\|_\infty \|b_p\|_2 \|b_q\|_2 \\
\lesssim \sum_{q \geq -1} \lambda_q^{2r+1} \|b_q\|_2^2 \sum_{p \leq q} \lambda_p \|b_p\|_\infty \\
\lesssim \sum_{q \geq -1} \lambda_q^{2r+1} \|b_q\|_2^2 \sum_{p \leq q} \lambda_p^{\frac{1+\theta}{2}} \|b_p\|_2 \\
\lesssim \sum_{q \geq -1} \lambda_q^{(r+1)\theta} \|b_q\|_2^2 \sum_{p \leq q} \lambda_p^{(r+1)\delta} \|b_p\|_2^2 \lambda_q^{(1-\delta)} \|b_p\|_2 \lambda_q^{1-\delta} \lambda_q^{1-\theta} \\
\lesssim \sum_{q \geq -1} \lambda_q^{(r+1)\theta} \|b_q\|_2^2 \sum_{p \leq q} \lambda_p^{(r+1)\delta} \|b_p\|_2^2 \lambda_q^{(1-\delta)} \|b_p\|_2 \lambda_q^{1-\delta} \lambda_q^{1-\theta}
\]
for $0 < \theta < 2$, $0 < \delta < 1$ and
\begin{equation}
(3.19) \quad r \geq \frac{n}{2} + 2 - (\theta + \delta), \quad 1 - \theta < 0.
\end{equation}
It then follows from Young’s inequality with $(r_1, r_2, r_3, r_4) \in (1, \infty)^4$ satisfying
\begin{equation}
(3.20) \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1, \quad r_1 = \frac{2}{\theta}, \quad r_3 = \frac{2}{\delta}
\end{equation}
such that
\begin{equation}
|I_{511}| \leq \frac{\mu}{16} \sum_{q \geq -1} \lambda_q^{2r} \|b_q\|^2_2 + C_\mu \left( \sum_{q \geq -1} \lambda_q^{2r} \|b_p\|^2_2 \right)^{1+\tilde{\gamma}_1} + C_\mu \left( \sum_{q \geq -1} \lambda_q^{2r} \|b_p\|^2_2 \right)^{1+\tilde{\gamma}_2}
\end{equation}
for some constants $\tilde{\gamma}_1, \tilde{\gamma}_2 > 0$. The conditions (3.19) and (3.20) imply that
\begin{equation}
(3.21) \quad r \geq \frac{n}{2} + 2 - \theta > \frac{n}{2}, \quad \alpha > \frac{1}{\theta} = \frac{1}{2 - \varepsilon} > \frac{1}{2}
\end{equation}
provided $\theta$ close enough to 2 and $\delta$ close enough to 0.

The term $I_{513}$ is estimated as follows,
\begin{equation}
|I_{513}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} |(b_{\leq p-2} - b_{q-2}) \times (\nabla \times (b_p)_q) \cdot \nabla \times b_q| \ dx
\end{equation}
\begin{equation}
\leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \|\nabla b_q\|_\infty \|b_{\leq p-2} - b_{q-2}\|_2 \|\nabla b_p\|_2
\end{equation}
\begin{equation}
\leq \sum_{q \geq -1} \lambda_q^{2r + \frac{3}{2} - \theta} \|b_q\|_2^3
\end{equation}
\begin{equation}
\leq \sum_{q \geq -1} \lambda_q^{(r+1)\theta} \|b_q\|_2^\theta \lambda_q^{(3-\theta)\|b_q\|_2^3} \lambda_q^{2r + 2 - \theta}
\end{equation}
\begin{equation}
\leq \sum_{q \geq -1} \lambda_q^{(r+1)\theta} \|b_q\|_2^\theta \lambda_q^{(3-\theta)\|b_q\|_2^3}
\end{equation}
for $0 < \theta < 2$ and
\begin{equation}
(3.22) \quad r \geq \frac{n}{2} + 2 - \theta = \frac{n}{2} + 2 - \varepsilon > \frac{n}{2}
\end{equation}
provided $\theta = 2 - \varepsilon$ with small enough $\varepsilon$. Thus, we have by Young’s inequality that
\begin{equation}
|I_{513}| \leq \frac{\mu}{16} \sum_{q \geq -1} \lambda_q^{2r+2} \|b_q\|^2_2 + C_\mu \left( \sum_{q \geq -1} \lambda_q^{2r} \|b_p\|^2_2 \right)^{1+\tilde{\gamma}_3}
\end{equation}
for some constant $\tilde{\gamma}_3 > 0$.

Notice that
\begin{equation}
|I_{52}| = \left| \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q (\nabla \times b_{\leq p-2} \times b_p) \cdot \nabla \times b_q \ dx \right|
\end{equation}
\begin{equation}
\leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2r+1} \|b_p\|_2 \|\nabla b_{\leq p-2}\|_\infty \|b_q\|_2,
\end{equation}
thus $I_{52}$ enjoys the same estimate as for $I_{511}$. 
To estimate $I_{53}$, we proceed as, by using Hölder's inequality and Bernstein's inequality
\[
|I_{53}| \leq \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^2 \int_{\mathbb{R}^3} |\Delta_q(b_p \times \nabla \times \hat{b}_p) \cdot \nabla \times \hat{b}_q| \, dx
\]
\[
\lesssim \sum_{q \geq -1} \lambda_q^{2r} \|\nabla b_q\|_{\infty} \sum_{p \geq q-3} \|b_p\|_2 \|\nabla b_p\|_2
\]
\[
\lesssim \sum_{q \geq -1} \lambda_q^{2r+1+2} \|b_q\|_2 \sum_{p \geq q-3} \lambda_p \|b_p\|_2^2
\]
\[
\lesssim \sum_{q \geq -1} \lambda_q^{(r+1)\theta} \|b_q\|_2^\theta \sum_{p \geq q-3} \lambda_p^{(r+1)\delta} \|b_p\|_2^\delta \sum_{p \geq q-3} \lambda_p^{2(2-\delta)} \|b_p\|_2^{2-2\delta} - \lambda_q\delta_{\theta-2} \delta_{\theta-2}^2
\]
\[
\lesssim \sum_{q \geq -1} \lambda_q^{(r+1)\theta} \|b_q\|_2^\theta \sum_{p \geq q-3} \lambda_p^{(r+1)\delta} \|b_p\|_2^\delta \sum_{p \geq q-3} \lambda_p^{2(2-\delta)} \|b_p\|_2^{2-2\delta} - \lambda_q\delta_{\theta-2} \delta_{\theta-2}^2
\]
for $0 < \theta < 1$, $0 < \delta < 2$ and
\[
\tag{3.23}
r \geq \frac{n}{2} + 2 - (\theta + \delta), \quad 1 - 2r - \delta < 0.
\]
Then by Young's inequality with $(r_1, r_2, r_3, r_4) \in (1, \infty)^4$ satisfying
\[
\tag{3.24}
\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1, \quad r_1 = \frac{2}{\theta}, \quad r_3 = \frac{2}{\delta}
\]
and Jensen's inequality, we have
\[
|I_{53}| \leq \frac{\mu}{16} \sum_{q \geq -1} \lambda^2_{q+2} \|b_q\|_2^2 + C_\mu \left( \sum_{q \geq -1} \lambda^2_{q} \|b_p\|_2^2 \right)^{1+\gamma_4}
\]
\[
+ C_\mu \left( \sum_{q \geq -1} \lambda^2_{q} \|b_p\|_2^2 \right)^{1+\gamma_5}
\]
for some constants $\gamma_4, \gamma_5 > 0$. Again, (3.23) and (3.24) imply
\[
r > \frac{n}{2}
\]
provided $r_2, r_4$ are large enough. To summarize, we have for $r > \frac{n}{2}$
\[
|I_{53}| \leq \frac{\mu}{16} \sum_{q \geq -1} \lambda^2_{q+2} \|b_q\|_2^2 + C_\mu \left( \sum_{q \geq -1} \lambda^2_{q} \|b_p\|_2^2 \right)^{1+\gamma_6}
\]
\[
+ C_\mu \left( \sum_{q \geq -1} \lambda^2_{q} \|b_p\|_2^2 \right)^{1+\gamma_9}
\]
for some constants $\gamma_8, \gamma_9 > 0$. In fact, we can take $\gamma_8/2$ as the smallest number of $\gamma_1, \ldots, \gamma_5$ and $\gamma_9/2$ as the largest one of these constants.

We are ready to show the uniform estimate for $\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2$ on a short time interval.

**Lemma 3.7.** Assume $r$ and $s$ satisfy
\[
s > \frac{n}{2} - 1, \quad r > \frac{n}{2}, \quad \frac{n}{4} + \frac{s}{2} < r \leq s + 1 - \varepsilon
\]
for a small enough constant $\varepsilon > 0$. There exists a time $T = T(\nu, \mu, \|u_0\|_{H^s}, \|b_0\|_{H^r})$ and a constant $C_{\nu, \mu}$ depending on $\nu$ and $\mu$ such that
\[
\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2 \leq C_{\nu, \mu} \left( \|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2 \right), \quad \forall t \in [0, T].
\]
Proof: Combining (3.11), (3.12), and the estimates in Lemma 3.2 to Lemma 3.6, there exist various constants $C_{\nu, \mu}$ depending on $\nu$ and $\mu$ such that
\[
\frac{d}{dt} \left( \|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right) + \nu \|\nabla u\|_{H^s}^2 + \mu \|\nabla b\|_{H^r}^2 \leq C_{\nu, \mu} \left( \|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right)^{1+\gamma} + C_{\nu, \mu} \left( \|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right)^{1+\tau}
\]
with constants $\gamma = \min\{\gamma_1, ..., \gamma_9\}$ and $\tau = \max\{\gamma_1, ..., \gamma_9\}$. Denote $\psi(t) = \|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2$. Let
\[
T = \frac{1}{2} \min \left\{ \frac{1}{C_{\nu, \mu} \gamma_2(0)}, \frac{1}{C_{\nu, \mu} \tau(0)} \right\}.
\]
It follows from the energy inequality above that for $t \in [0, T],
\[
\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2 \leq \frac{\|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2}{\left[ 1 - \gamma C_{\nu, \mu} \left( \|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2 \right)^2 \right]^{1/2}} + \frac{\|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2}{\left[ 1 - \tau C_{\nu, \mu} \left( \|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2 \right)^2 \right]^{1/2}}.
\]
It completes the proof of the lemma and concludes the proof of Theorem 3.1

4. Uniqueness and Continuity

In this section, we establish the uniqueness of solutions stated in Theorem 1.1. The continuity in time can be obtained through a rather standard procedure, see [15]; hence we omit the proof.

Theorem 4.1. Let $\varepsilon > 0$ be small enough. Assume $(u_1, b_1, p_1)$ and $(u_2, b_2, p_2)$ are solutions of (1.1) in $H^s(\mathbb{R}^n) \times H^{s+1-\varepsilon}(\mathbb{R}^n)$ satisfying the estimates in Theorem 3.7. Then $(u_1, b_1) = (u_2, b_2)$.

Proof: The difference $(U, B, \pi) = (u_1 - u_2, b_1 - b_2, p_1 - p_2)$ satisfies the equations
\[
U_t + u_2 \cdot \nabla U - b_2 \cdot \nabla B + U \cdot \nabla u_1 - B \cdot \nabla b_1 + \nabla \pi = \nu \Delta U,
\]
\[
B_t + u_2 \cdot \nabla B - b_2 \cdot \nabla U + U \cdot \nabla b_1 - B \cdot \nabla u_1 - \nabla \times ((\nabla \times b_2) \times B) + \nabla \times ((\nabla \times B) \times b_1) = \mu \Delta B.
\]

The goal is to obtain a Grönwall type of inequality for the $L^2$ energy of $(U, B)$. Thus, we take inner product of the equations of $U$ and $B$ in (4.25) with $U$ and $B$, respectively, to arrive at
\[
\frac{d}{dt} \left( \frac{1}{2} \|U\|_2^2 + \frac{1}{2} \|B\|_2^2 \right) + \nu \|\nabla U\|_2^2 + \mu \|\nabla B\|_2^2
\]
\[
= \int_{\mathbb{R}^n} (b_2 \cdot \nabla) B \cdot U \, dx + \int_{\mathbb{R}^n} (B \cdot \nabla)b_1 \cdot U \, dx - \int_{\mathbb{R}^n} (u_2 \cdot \nabla) U \cdot U \, dx
\]
\[
- \int_{\mathbb{R}^n} (U \cdot \nabla) u_1 \cdot U \, dx + \int_{\mathbb{R}^n} (b_2 \cdot \nabla) U \cdot B \, dx - \int_{\mathbb{R}^n} (B \cdot \nabla) u_1 \cdot B \, dx
\]
\[
- \int_{\mathbb{R}^n} (u_2 \cdot \nabla) B \cdot B \, dx - \int_{\mathbb{R}^n} (U \cdot \nabla)b_1 \cdot B \, dx
\]
\[
+ \int_{\mathbb{R}^n} \nabla \times ((\nabla \times b_2) \times B) \cdot B \, dx - \int_{\mathbb{R}^n} \nabla \times ((\nabla \times B) \times b_1) \cdot B \, dx.
\]
We are left to estimate the five non-zero flux terms. The first one is estimated as

\[
\int_{\mathbb{R}^n} (u_2 \cdot \nabla) U \cdot U \, dx = 0, \quad \int_{\mathbb{R}^n} (u_2 \cdot \nabla) B \cdot B \, dx = 0,
\]

\[
\int_{\mathbb{R}^n} \nabla \times ((\nabla \times B) \cdot b_1) \cdot B \, dx = 0
\]

\[
\int_{\mathbb{R}^n} (b_2 \cdot \nabla) B \cdot U \, dx + \int_{\mathbb{R}^n} (b_2 \cdot \nabla) U \cdot B \, dx = 0.
\]

We are left to estimate the five non-zero flux terms. The first one is estimated as

\[
\left| \int_{\mathbb{R}^n} (B \cdot \nabla) b_1 \cdot U \, dx \right| = \left| \int_{\mathbb{R}^n} (B \cdot \nabla) U \cdot b_1 \, dx \right|
\leq \|B\|_2 \|\nabla U\|_2 \|b_1\|_{\infty},
\]

\[
\leq \frac{\mu}{8} \|\nabla U\|_2^2 + C_\nu \|B\|_2 \|\nabla b_1\|_{H^{s+1}}^2,
\]

\[
\int_{\mathbb{R}^n} (U \cdot \nabla) b_1 \cdot U \, dx \leq \frac{\mu}{8} \|\nabla b_1\|_2^2 + C_\mu \|U\|_2 \|\nabla b_1\|_{H^{s+1}}^2,
\]

We estimate the Hall term as follows

\[
\left| \int_{\mathbb{R}^n} \nabla \times ((\nabla \times b_2) \times B) \cdot B \, dx \right| = \left| \int_{\mathbb{R}^n} ((\nabla \times b_2) \times B) \cdot \nabla \times B \, dx \right|
\leq \|\nabla \times B\|_2 \|\nabla \times b_2\|_\infty \|B\|_2
\leq \frac{\mu}{8} \|\nabla b_2\|_2^2 + C_\mu \|\nabla \times b_2\|_{H^{s+1-\varepsilon}}^2 \|B\|_2^2
\leq \frac{\mu}{8} \|\nabla b_2\|_2^2 + C_\mu \|\nabla \times b_2\|_{H^{s+1-\varepsilon}}^2 \|B\|_2^2.
\]

The estimates above along with (1.26) give us

\[
\frac{d}{dt} (\|U\|_2^2 + \|B\|_2^2) + \nu \|\nabla U\|_2^2 + \mu \|\nabla B\|_2^2
\leq C_\nu,\mu \left( \|u_1\|_{H^{s+1}}^2 + \|\nabla b_2\|_{H^{s+1-\varepsilon}}^2 \right) \left( \|U\|_2^2 + \|B\|_2^2 \right)
\leq C_\nu,\mu \left( \|u_1\|_{H^{s+1}}^2 + \|\nabla b_2\|_{H^{s+1-\varepsilon}}^2 + C \right) \left( \|U\|_2^2 + \|B\|_2^2 \right).
\]

It follows from Grönwall's inequality that

\[
\|U(t)\|_2^2 + \|B(t)\|_2^2
\leq (\|U(0)\|_2^2 + \|B(0)\|_2^2) e^{CC_\nu,\mu t} \exp \left\{ C_\nu,\mu \int_0^t \|u_1(\tau)\|_{H^{s+1}}^2 + \|\nabla b_2(\tau)\|_{H^{s+1-\varepsilon}}^2 \, d\tau \right\}.
\]
Since $U(0) = B(0) = 0$, $u_1 \in L^2(0, T; H^{s+1})$ and $b_2 \in L^2(0, T; H^{s+2-\varepsilon})$, we infer
\[
\|U(t)\|_2^2 + \|B(t)\|_2^2 = 0, \quad \forall t \in [0, T].
\]

□

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