Fermion mass hierarchies and mixings
in a $B - L$ model with $D_4 \times Z_4 \times Z_2$ symmetry

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We propose a gauge $B - L$ model with $D_4 \times Z_4 \times Z_2$ symmetry that can explain the charged lepton and quark mass hierarchies as well as the lepton and quark mixing patterns with the CP phase through the type-I seesaw mechanism.

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I. INTRODUCTION

The origin of the fermion mass hierarchy problems and their mixings as well as the CP phases is one of the most exciting issues in particle physics [1, 2]. In the three neutrino framework, the basically experimental results related to flavour problem are: (i) the origin of charged-lepton mass hierarchy $m_e \ll m_\mu \ll m_\tau$ and the origin of the quark mass hierarchies $m_u \ll m_c \ll m_t$ and $m_d \ll m_s \ll m_b$; (ii) the origins of the small values of three quark mixing angles and of the neutrino mixing pattern with the atmospheric and solar neutrino mixing angles are large while the reactor neutrino mixing angle is small, and of the two independent neutrino mass squared splittings $\Delta m_{32}^2 \equiv m_3^2 - m_2^2 \approx 2.5 \times 10^{-3}$ eV$^2$ and $\Delta m_{12}^2 \equiv m_1^2 - m_2^2$; and (iii) why the fermion mass hierarchy reaches a range at least about twelve orders of magnitude from the light neutrino mass (around 0.1 eV [4, 5]) to the top quark mass (around 172 GeV [3]). The charged-lepton mass hierarchy can be explained by the Froggatt-Nielsen mechanism [6]. The neutrino mass squared splittings can be explained through the seesaw mechanism [7-12].

In order to overcome the limitations of the SM, many extensions have been suggested such as symmetry (discrete and/or continuous) extension with larger scalar and/or fermion
sectors. The $B - L$ model [13–29] is appreciated because the simplest way is to add three right-handed neutrinos for generating neutrino masses. Although this model solves many interesting problems such as dark matter [18, 19, 22–25], the muon anomalous magnetic moment [20, 29], leptogenesis [24, 27] and gravitational wave radiation [28], it cannot provide a satisfactory explanation for fermion masses and mixing observables. Non-Abelian discrete symmetries have seem to be the most elegant in explaining the observed lepton and quark mixing patterns (see for instance Res. [30–32] and the references therein). Among the discrete symmetries, $D_4$ has attracted the attention since it provides a very predictive description of the observed patterns of lepton and quark masses and mixing angles [32–46], however, most of these works solved only some of sub-problems of the flavour problem. Thus, it would be wishful to construct a $D_4$ flavor model that can overcome all mentioned sub-problems including the origin of the charged-lepton mass hierarchy, the small values of three quark mixing angles, the neutrino mixing pattern with 2 large and 1 small angles, the two neutrino mass squared splittings, and the fermion mass hierarchy problem. A possible solution to the fermion mass hierarchy problem is to introduce a new family symmetry acting between families [47]. Beside the differences between previous works with $D_4$ symmetry, there is another important difference comes from the fermion mass hierarchy problem which has not been mentioned or has not been achieved in Refs. [32, 46].

In this study, we modify the model in Refs. [32, 45] by additional introduction one $SU(2)_L$ doublet $H'$ assigned in $1'$ under $D_4$ symmetry in the lepton sector [45] and replace one doublet by singlet in the quark sector [32]. The first family of the right handed charged lepton ($l_{1R}$) and the right handed neutrino ($\nu_{1R}$) are assigned in $1'$ under $D_4$ symmetry which is different from those of Refs. [32, 45]. Furthermore, the assignments under $Z_4$ symmetry of right-hand leptons $l_{1R}, l_{\alpha R}, \nu_{1R}, \nu_{\alpha R}$ and singlet scalars $\chi, \varphi, \phi$ in our present work are also different from those of Ref. [32, 45]. As a consequence, the charged lepton mass hierarchy, the neutrino mass hierarchy and the quark mass hierarchy can be naturally achieved. It is not the first time that the $B - L$ model is based on $D_4$ symmetry [38], however, in our previous model and other models with $D_4$ symmetry the mass hierarchy among charged leptons and quarks are usually non-trivial. Oppositely in our present study, the observed fermion mass hierarchies are natural results of our model.

1 In Ref. [40] the charged lepton mass hierarchy is achieved by one $SU(2)_L$ doublet and up to five singlets, and including the Yukawa-like dimension-seven operator.
The rest of this work is layout as follows. In section II we present a $B-L$ model based on $D_4 \times Z_4 \times Z_2$ symmetry as well as introduce the particle content of the model. Sections III and IV are dedicated to lepton and quark masses and mixings, respectively. We make conclusions in Sec. V.

II. THE MODEL

The total symmetry of the model is $\Gamma = SU(3)_C \otimes SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times D_4 \times Z_4 \times Z_2$ where a $Z_2$ symmetry is added compared to those of the Refs. [32, 45]. Furthermore, the assignments under $D_4$, $Z_4$ and $Z_2$ are different from each others. In this study, the first families of the left handed charged lepton, left handed quark, right handed up quark and right handed down quark are assigned in $\mathbf{1}$; the first families of the right handed charged lepton and right handed neutrino are assigned in $\mathbf{1}'$ while the corresponding two others are assigned in $\mathbf{2}$ under $D_4$. In order to explain the hierarchies of quark and lepton masses, the lepton and quark mixing patterns and CP violating phases, three right-handed neutrinos, one $SU(2)_L$ doublet $H'$ with $B-L = 0$ put in $\mathbf{1}'$ under $D_4$ together with two flavons $\chi, \rho$ with $B-L = 0$ respectively put in $\mathbf{1}, \mathbf{2}$ under $D_4$ and two flavons $\varphi, \phi$ with $B-L = 2$ respectively put in $\mathbf{1}$ and $\mathbf{2}$ under $D_4$ are additional introduced. The particle content of the model is shown in Table I.

| Fields | $\psi_{1L}$ | $\psi_{\alpha L}$ | $l_{1R}$ | $l_{\alpha R}$ | $\nu_{1R}$ | $\nu_{\alpha R}$ | $Q_{1L}$ | $Q_{\alpha L}$ | $u_{1R}$ | $u_{\alpha R}$ | $d_{1R}$ | $d_{\alpha R}$ | $H$ | $H'$ | $\chi$ | $\varphi$ | $\phi$ | $\rho$
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| SU(2)$_L$ | 2      | 2      | 1      | 1      | 1      | 1      | 2      | 2      | 1      | 1      | 1      | 1      | 2      | 2      | 1      | 1      | 1      | 1      |
| U(1)$_{B-L}$ | -1    | -1    | -1    | -1    | -1    | -1    | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0      | 0      | 0      | 2      | 2      | 0      |
| $D_4$ | 1      | 2      | $1'$  | 2      | $1'$  | 2      | 1      | 2      | 1      | 2      | 1      | $1'$ | $1'$ | $1'$ | 2      | 2      |
| $Z_4$ | $i$    | $i$    | $i$   | 1      | 1      | $i$   | 1      | $i$   | $-1$  | $i$   | $-1$  | $i$   | 1      | 1      | $i$   | $-1$  | $-i$  |
| $Z_2$ | +      | +      | -      | -      | -      | -      | -      | +      | -      | -      | -      | +      | +      | -      | -      | +      | -      |

With the given particle content, for the charged-lepton sector, under $\Gamma$, $\bar{\psi}_{1L}l_{1R} \sim (1, 2, -\frac{1}{2}, 0, 1', -i, -), \bar{\psi}_{1L}l_{\alpha R} \sim (1, 2, -\frac{1}{2}, 0, 2, 1, -), \bar{\psi}_{\alpha L}l_{1R} \sim (1, 2, -\frac{1}{2}, 0, 2, -i, -), \bar{\psi}_{\alpha L}l_{\alpha R} \sim (1, 2, -\frac{1}{2}, 0, 1 \oplus 1' \oplus 1'' \oplus 1''', 1, -)$. Therefore, $\bar{\psi}_{1L}l_{1R}$ can couple with $H\chi$ which
transforms as $(1,2,\frac{1}{2},0,1',i,-)$ and $\psi_{aLLaR}$ can couple with $H$ and $H'$ to form invariant terms which are responsible for generating the charged-lepton masses. For neutrino sector, $\tilde{\psi}_{1L}\nu_{1R} \sim (1,2,\frac{1}{2},0,1',-i,-)$, $\tilde{\psi}_{1L}\nu_{aR} \sim (1,2,\frac{1}{2},0,2,1,-)$, $\tilde{\psi}_{aLL}\nu_{1R} \sim (1,2,\frac{1}{2},0,2,-i,-)$ and $\tilde{\psi}_{aLL}\nu_{aR} \sim (1,2,\frac{1}{2},0,1\oplus 1'\oplus 1''\oplus 1'''',1,-)$. Thus, $\tilde{\psi}_{1L}\nu_{1R}$ can couple with $\tilde{H}\chi$ which transforms as $(1,2,-\frac{1}{2},0,1',i,-)$ while $\tilde{\psi}_{aLL}\nu_{aR}$ can couple with $\tilde{H}$ and $\tilde{H}'$ to form invariant terms that generate the Dirac neutrino mass matrix. Furthermore, $\tilde{\nu}_{1R}\nu_{1R} \sim (1,1,0,-2,\frac{1}{2},1,+)$, $\tilde{\nu}_{1R}\nu_{aR} \sim (1,1,0,-2,0,1,+)$ and $\tilde{\nu}_{aR}\nu_{aR} \sim (1,1,0,-2,1\oplus 1'\oplus 1''\oplus 1'''',-1,+)$). So, $\tilde{\nu}_{1R}\nu_{1R}$ can couple with $\chi^*\varphi$, $\tilde{\psi}_{1L}\nu_{aR}$ and $\tilde{\psi}_{aLL}\nu_{1R}$ can couple with $\chi\phi$ while $\tilde{\nu}_{aR}\nu_{aR}$ can couple with $\chi\varphi$ to construct the invariant terms that generate the Majorana neutrino mass matrix. For the quark sector, $\tilde{Q}_{1L}\nu_{1R} \sim (1,2,\frac{1}{2},0,1,1,-)$, $\tilde{Q}_{aL}\nu_{1R} \sim (1,2,\frac{1}{2},0,2,1,+)$, $\tilde{Q}_{aL}\nu_{aR} \sim (1,2,\frac{1}{2},0,1\oplus 1'\oplus 1''\oplus 1'''',1,-)$. On the other hand, $\tilde{Q}_{1L}\nu_{1R} \sim (1,2,-\frac{1}{2},0,1,1,-)$, $\tilde{Q}_{aL}\nu_{1R} \sim (1,2,-\frac{1}{2},0,1,1,+)$, $\tilde{Q}_{aL}\nu_{aR} \sim (1,2,-\frac{1}{2},0,1\oplus 1'\oplus 1''\oplus 1'''',1,-)$). Therefore, $\tilde{Q}_{al}\nu_{al}$ can couple with $\tilde{H}$ and $\tilde{H}'$ while $\tilde{Q}_{aL}\nu_{al}$ and $\tilde{Q}_{al}\nu_{al}$ can couple with $\tilde{H}\rho$ and $\tilde{H}'\rho$ which transform as $(1,2,-\frac{1}{2},0,2,-i,+)$ to form invariant terms that generate up-quark masses. Simultaneously, $\tilde{Q}_{al}\nu_{al}$ can couple with $H$ and $H'$ while $\tilde{Q}_{1L}\nu_{1R}$ and $\tilde{Q}_{aL}\nu_{1R}$ can couple with $H\rho$ and $H'\rho$ which transform as $(1,2,\frac{1}{2},0,2,-i,+)$ to form invariant terms that generate down-quark masses.

It is worthy to note that, to obtain the desired mass matrices, additional symmetries $U(1)_{B-L}, D_4, Z_4$ and $Z_2$ play crucial roles. Namely, $(\tilde{\psi}_{1L}\nu_{1R})(H\varphi)$, $(\tilde{\psi}_{1L}\nu_{1R})(H\varphi)$, $(\tilde{\nu}_{1R}\nu_{1R})(\chi\varphi^*)$, $(\tilde{\nu}_{1R}\nu_{aR} + \tilde{\nu}_{aR}\nu_{1R})(\varphi\phi^*)$, $(\tilde{\nu}_{1R}\nu_{aR} + \tilde{\nu}_{aR}\nu_{1R})(\varphi\phi^*)$, $(\tilde{\nu}_{aR}\nu_{aR})(\chi^*\varphi^*)$ and $(\tilde{\nu}_{aR}\nu_{aR})(\chi^*\varphi^*)$ are prevented by $U(1)_{B-L}$ symmetry. Furthermore, $(\tilde{\psi}_{1L}\nu_{1R})(H'\chi)$, $(\tilde{\psi}_{aL}\nu_{1R})(H'\chi)$ and $(\tilde{\psi}_{1L}\nu_{1R})(H'\chi)$ are forbidden by $D_4$ symmetry. Moreover, $(\tilde{\psi}_{1L}\nu_{1R})(H'\chi^*)$, $(\tilde{\psi}_{aL}\nu_{1R})(H'\chi^*)$, $(\tilde{\psi}_{aL}\nu_{1R})(H'\chi^*)$ and $(\tilde{\psi}_{1L}\nu_{1R})(\tilde{H}\chi^*)$, $(\tilde{\psi}_{1L}\nu_{1R})(\tilde{H}\chi^*)$, $(\tilde{\psi}_{1L}\nu_{1R})(\tilde{H}\chi^*)$, $(\tilde{\psi}_{aL}\nu_{1R})(\tilde{H}\chi^*)$, $(\tilde{\psi}_{aL}\nu_{1R})(\tilde{H}\chi^*)$ and $(\tilde{\psi}_{aL}\nu_{1R})(\tilde{H}\chi^*)$ are disallowed by $Z_4$ symmetry. In addition, $(\tilde{\psi}_{aL}\nu_{1R})(H\rho^*)$ and $(\tilde{\psi}_{aL}\nu_{1R})(H'\rho^*)$ are prevented by $Z_2$ symmetry.
From the above analysis, the following charged lepton, neutrino and quark Yukawa terms, which are invariant under $\Gamma$, read:

$$-\mathcal{L}_Y = \frac{h_1}{\Lambda} (\bar{\psi}_1 L'_{1R}) \langle H \chi \rangle + h_2 (\bar{\psi}_2 L'_{2R}) H + h_3 (\bar{\psi}_3 L'_{3R}) \tilde{H} H'$$

$$+ \frac{x_1}{\Lambda} (\bar{\psi}_1 \nu_{1R}) \langle \tilde{H} \chi \rangle + x_2 (\bar{\psi}_2 \nu_{2R}) \tilde{H} + x_3 (\bar{\psi}_3 \nu_{3R}) \tilde{H} H'$$

$$+ \frac{y_1}{2\Lambda} (\bar{\nu}_{1R} \nu_{1R}) \langle \chi' \varphi \rangle + \frac{y_2}{2\Lambda} (\bar{\nu}_{2R} \nu_{2R}) \langle \chi \varphi \rangle$$

$$+ \frac{y_3}{2\Lambda} (\bar{\nu}_{1R} \nu_{1R})_{2R} (\phi \chi) + \text{H.c.},$$

(1)

$$-\mathcal{L}_V = x_1 (\bar{Q}_{aL} u_{aR}) \tilde{H} + x_2 (\bar{Q}_{aL} u_{aR}) \tilde{H}'$$

$$+ \frac{x_3}{\Lambda} (\bar{Q}_{aL} u_{aR} + \bar{Q}_{aL} u_{1R})_2 \tilde{H} \rho_2 + \frac{x_4}{\Lambda} (\bar{Q}_{aL} u_{aR} + \bar{Q}_{aL} u_{1R})_2 \tilde{H}' \rho_2$$

$$+ x_5 (\bar{Q}_{aL} d_{aR}) \tilde{H} + x_6 (\bar{Q}_{aL} d_{aR}) \tilde{H}' + \frac{x_7}{\Lambda} (\bar{Q}_{aL} d_{aR} + \bar{Q}_{aL} d_{1R})_2 \tilde{H} \rho_2$$

$$+ \frac{x_8}{\Lambda} (\bar{Q}_{aL} d_{aR} + \bar{Q}_{aL} d_{1R})_2 \tilde{H}' \rho_2 + \text{H.c.},$$

(2)

where $h_{1,2,3}; x_{1,2,3}, y_{1,2,3}$ and $x_{1,2,3,4}^{u,d}$ are the Yukawa-like dimensionless couplings and $\Lambda$ is the cut-off scale.

We now determine the vacuum expectation value (VEV) alignments of the scalar fields. We will show that the VEVs of scalar fields satisfying the minimum condition of scalar potential take the following form:

$$\langle H \rangle = (0 \ v)^T, \quad \langle H' \rangle = (0 \ v')^T, \quad \langle \chi \rangle = v_\chi, \quad \langle \varphi \rangle = v_\varphi, \quad \langle \phi \rangle = ((\phi_1), \ (\phi_2)),$$

$$\langle \rho \rangle = ((\rho_1), \ 0), \quad \langle \rho_1 \rangle = v_\rho.$$

(3)

For this purpose we use the assumption that the VEVs of scalars are real, i.e., $v_k^* = v_a \ (v_k = v, v', v_\chi, v_\varphi, v_\phi, v_\rho)$, the minimization condition reduces to

$$\frac{\partial V_{\text{scalar}}}{\partial v_k} = 0, \quad \frac{\partial^2 V_{\text{scalar}}}{\partial v_k^2} > 0,$$

(4)

where $V_{\text{scalar}}$ is the scalar potential with the explicit expression given in Appendix $\Lambda$.

For simplicity we work with the following benchmark point:

$$\lambda_1^{H' \chi} = \lambda_2^{H' \chi} = \lambda_1^{H' \chi} = \lambda_2^{H' \chi} = \lambda_1^{H' \rho} = \lambda_2^{H' \rho} = \lambda_1^{H' \phi} = \lambda_2^{H' \phi} = \lambda^{H' \rho},$$

$$\lambda_1^{H' \varphi} = \lambda_2^{H' \varphi} = \lambda_1^{H' \varphi} = \lambda_2^{H' \varphi} = \lambda_1^{H' \phi} = \lambda_2^{H' \phi} = \lambda_1^{H' \phi} = \lambda_2^{H' \phi} = \lambda^{H' \phi},$$

$$\lambda_1^{\phi \rho} = \lambda_2^{\phi \rho} = \lambda_1^{\phi \rho} = \lambda_2^{\phi \rho} = \lambda_1^{\phi \phi} = \lambda_2^{\phi \phi} = \lambda_1^{\phi \phi} = \lambda_2^{\phi \phi} = \lambda_1^{\phi \rho} = \lambda_2^{\phi \rho} = \lambda^{\phi \rho},$$

$$\lambda_1^{H H'} = \lambda_2^{H H'} = \lambda_1^{H H'} = \lambda_2^{H H'} = \lambda_1^{H H'} = \lambda_2^{H H'} = \lambda_1^{H H'} = \lambda_2^{H H'} = \lambda_{1,2}^{H H' \rho} = \lambda_{1,2}^{H H' \phi} = \lambda_{1,2}^{H H' \rho} = \lambda_{1,2}^{H H' \phi} = \lambda_{1,2}^{H H' \rho} = \lambda_{1,2}^{H H' \phi}.$$
As a result, the condition (4) is equivalent to the following expressions

\[
\begin{align*}
\mu_H^2 + 2 \left[ \lambda^H v^2 + \lambda^{H'H'} v'^2 + \lambda^{H'H'} v v'_\rho + \lambda^H \left( v^2 + 2 v^2 + v^2 + v^2 \right) \right] &= 0, \\
\mu_{H'}^2 + 2 \left[ \lambda^{H'H'} v^2 + \lambda^{H'H'} v v'_\rho + \lambda^H \left( v^2 + 2 v^2 + v^2 + v^2 \right) \right] &= 0, \\
\mu_\chi^2 + 2 \left[ \lambda^\chi v^2 + \lambda^{H'} \left( v^2 + v^2 \right) + \lambda^\rho \left( 2 v^2 + v^2 + v^2 \right) \right] &= 0, \\
\mu_\rho^2 + 2 \left[ \lambda^\rho \left( v^2 + v^2 \right) \right] &= 0, \\
\mu_\omega^2 + 2 \lambda^\omega \left( v^2 + v^2 \right) + 8 \lambda^\rho v^2 + \lambda^\rho \left( 2 v^2 + v^2 + v^2 \right) &= 0,
\end{align*}
\]

The system of Eqs. (6)-(11) always own the following solution:

\[
\begin{align*}
\lambda^H &= -\frac{\mu_H^2}{2 v^2} - \frac{\lambda^H \left( v^2 + 2 v^2 + v^2 + v^2 \right)}{v^2} - \lambda^{H'H'} v^2 - \frac{\lambda^{H'H'} v v'_\rho}{v^3}, \\
\lambda^{H'} &= -\frac{\mu_{H'}^2}{2 v'^2} - \frac{\lambda^{H'} \left( v^2 + 2 v^2 + v^2 + v^2 \right)}{v'^2} - \lambda^{H'H'} v^2 - \frac{\lambda^{H'H'} v v'_\rho}{v'^3}, \\
\lambda^\chi &= -\frac{\mu_\chi^2}{2 v^2} - \frac{\lambda^\rho \left( 2 v^2 + v^2 + v^2 \right)}{v^2} - \lambda^{H} \left( v^2 + v^2 \right), \\
\lambda^\rho &= -\frac{\mu_\rho^2}{2 v^2} - \frac{\lambda^\rho \left( 2 v^2 + v^2 + v^2 \right)}{v^2} - \lambda^{H} \left( v^2 + v^2 \right), \\
\lambda^\omega &= -\frac{\mu_\omega^2}{8 v^2} - \frac{\lambda^\rho \left( 2 v^2 + v^2 + v^2 \right)}{8 v^2} - \lambda^{H} \left( v^2 + v^2 \right), \\
\lambda^\rho &= -\frac{\mu_\rho^2}{2 v^2} - \frac{\lambda^\rho \left( 2 v^2 + v^2 + v^2 \right)}{2 v^2} - \frac{\lambda^{H} \left( v^2 + v^2 \right)}{2 v^2}.
\end{align*}
\]

Assuming that \( \mu_x^2 (x = H, H', \chi, \varphi, \phi, \rho) \) are negative and of the same order of magnitude and the same as that of the SM \( 3 \),

\[
\mu_x^2 = -10^4 \text{ GeV}.
\]
The fact that the electroweak symmetry breaking scale is\[^2\] (14) (15)
\[ v \sim v' \sim 10^2 \text{ GeV}. \] (20)
Furthermore, the scale of $B - L$ symmetry breaking is unknown, ranging from TeV to much higher scales [18]. For instance, it may be at $(10^3 - 10^4)$ GeV [49, 50] or $(10^{14} - 10^{16})$ GeV [51] or $(0.7 - 1.6)10^{15}$ GeV [52] or $(10^{13} - 10^{15})$ GeV [53]. In this study, we use the cut-off scale $\Lambda \sim 10^{14}$ GeV [54] and assume that the $B - L$ symmetry breaking scale at TeV and the VEV of singlets are at a very high energy scale, i.e.,

\[ v_\phi \sim v_\varphi = 10^3 \text{ GeV}, \quad v_\chi \sim v_\rho \sim 10^{11} \text{ GeV}. \] (21)

With the aid of Eqs. (18)-(21), $\delta_{H'}^2 = \delta_H^2$ depends on three parameters $\lambda_{H'H}$, $\lambda_{HH'\rho}$ and $\lambda H\varphi$, $\delta_\rho^2$ depends on three parameters $\lambda_{H'H\rho}$, $\lambda H\phi$ and $\lambda\varphi\rho$, and $\delta_\lambda^2$, $\delta_\varphi^2$ and $\delta_\rho^2$ depend on two parameters $\lambda H\phi$ and $\lambda\varphi\rho$ which are respectively plotted in Figs. 1-3. Figures 1-3 imply that the inequalities (12)-(17) are always satisfied by the VEV alignments in Eq. (3).

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Figure 1. $\delta_{H'}^2 \times 10^{-38} = \delta_H^2 \times 10^{-38}$ versus $\lambda_{H'H}$, $\lambda_{HH'\rho}$ and $\lambda_{H\phi}$ with $\lambda_{H\phi} \in (-10^{-2}, -10^{-4})$, $\lambda_{HH'\rho} \in (-5 \times 10^{-3}, -5 \times 10^{-4})$ and $\lambda_{H'H} \in (-10^{-2}, -10^{-4})$ (left side), $\delta_\rho^2 \times 10^{-38}$ versus $\lambda_{\varphi\rho}$, $\lambda_{HH'\rho}$ and $\lambda_{H\phi}$ with $\lambda_{\varphi\rho} \in (-10^{-2}, -10^{-4})$, $\lambda_{HH'\rho} \in (-5 \times 10^{-3}, -5 \times 10^{-4})$ and $\lambda_{H\phi} \in (-10^{-2}, -10^{-4})$ (right side).

\[^2\] In the SM, the Higgs VEV is 246 GeV, fixed by the W boson mass and the gauge coupling, $m_W^2 = \frac{g^2}{4} v_{\text{weak}}^2$. In our model, $M_W^2 = \frac{g^2}{2} (v^2 + v'^2)$; thus, we can identify $v_{\text{weak}}^2 = 2 (v^2 + v'^2) = (246 \text{ GeV})^2$ and then $v' \sim v = 123 \text{ GeV}$. 
III. LEPTON MASS AND MIXING

Using the Clebsch-Gordan coefficients of $D_4$ symmetry [55], after symmetry breaking, from Eq. (1), we obtain the charged lepton masses as follows:

$$m_e = \frac{v\chi}{\Lambda} h_1 v, \quad m_{\mu,\tau} = h_2 v \pm h_3 v'.$$  \hspace{1cm} (22)

The left-and right charged-lepton diagonalization matrices $U_{IL,R}$ get the diagonal form, $U_{IL} = U_{IR} = \mathbb{I}_{3 \times 3}$, i.e., the lepton mixing matrix is that of the neutrino, $U_{\text{Lep}} \equiv U_\nu$. Equation (22) implies that $m_\mu$ and $m_\tau$ are separated by $v'$ and this is reason why $H'$ is required in comparison to $H$. Furthermore, from Eq. (22), the hierarchy of the charged-lepton masses are naturally explained by a factor of $\frac{v\chi}{\Lambda} \sim 10^{-3}$.

At the best-fit values of $m_{e,\mu,\tau}$ taken from Ref. [3], $m_e \simeq 0.51099$ MeV, $m_\mu \simeq 105.65837$ MeV,
\[ m_r \simeq 1776.86 \text{ MeV}, \text{ together with } v \sim v' = 123 \text{ GeV and the cut-off scale } \Lambda = 10^{14} \text{ GeV, we get} \]
\[ h_1 = 4.15 \times 10^{-3}, \quad h_2 = 7.65 \times 10^{-3}, \quad h_3 = -6.79 \times 10^{-3}, \tag{23} \]
which are all of the same order of magnitude.

We now turn to the neutrino sector. With the help of the Clebsch-Gordan coefficients of \( D_4 \) group, the neutrino Yukawa Lagrangian reads:
\[ -\mathcal{L}_\nu = \frac{x_1}{\Lambda} (\bar{\psi}_1 \nu_{1R}) (\bar{H} \chi) + x_2 (\bar{\psi}_2 \nu_{2R} + \bar{\psi}_3 \nu_{3R}) \bar{H} + x_3 (\bar{\psi}_2 \nu_{2R} - \bar{\psi}_3 \nu_{3R}) \bar{H}' \]
\[ + \frac{y_1}{2\Lambda} (\bar{\nu}_{1R} \nu_{1R}) (\chi^* \varphi) + \frac{y_2}{2\Lambda} (\bar{\nu}_{2R} \nu_{2R} + \bar{\nu}_{3R} \nu_{3R}) (\chi \varphi) \]
\[ + \frac{y_3}{2\Lambda} [(\bar{\nu}_{1R} \nu_{2R} + \bar{\nu}_{2R} \nu_{1R}) (\phi_1 \chi) + (\bar{\nu}_{1R} \nu_{3R} + \bar{\nu}_{3R} \nu_{1R}) (\phi_2 \chi)] + \text{H.c.} \tag{24} \]

After symmetry breaking, the Dirac and Majorana neutrino mass matrices get the following forms
\[ M_D = \begin{pmatrix} x_D & 0 & 0 \\ 0 & y_D + z_D & 0 \\ 0 & 0 & y_D - z_D \end{pmatrix}, \quad M_R = \begin{pmatrix} x_R & z_R & z_R \\ z_R & y_R & 0 \\ z_R & 0 & y_R \end{pmatrix}, \tag{25} \]
where
\[ x_D = \frac{x_1 v v_\chi}{\Lambda}, \quad y_D = x_2 v, \quad z_D = x_3 v', \]
\[ x_R = \frac{y_1 v_\chi^* v_\varphi}{\Lambda}, \quad y_R = \frac{y_2 v_\chi v_\varphi}{\Lambda}, \quad z_R = \frac{y_3 v_\chi v_\varphi}{\Lambda}. \tag{26} \]

Using the type-I seesaw mechanism, the effective neutrino mass matrix gets the form
\[ M_{\text{eff}} = -M_D^T M_R^{-1} M_D = M_0 + \delta M, \tag{27} \]
where
\[ M_0 = \begin{pmatrix} a_0 & b_0 & b_0 \\ b_0 & c_0 - \frac{b_0^2}{a_0} & \frac{b_0^2}{a_0} \\ b_0 & \frac{b_0^2}{a_0} & c_0 - \frac{b_0^2}{a_0} \end{pmatrix}, \quad \delta M = \begin{pmatrix} 0 & \epsilon & -\epsilon \\ \epsilon & \delta & 0 \\ -\epsilon & 0 & -\delta \end{pmatrix}, \tag{28} \]
with
\[ a_0 = \frac{x_D^2 y_R}{2z_R^2 - x_R y_R}, \quad b_0 = \frac{x_D y_D z_R}{x_R y_R - 2z_R^2}, \quad c_0 = \frac{x_R y_D^2}{2z_R^2 - x_R y_R}, \tag{29} \]
\[ \epsilon = \frac{x_D z_D^2 z_R}{x_R y_R - 2z_R^2}, \quad \delta = y_D z_D \left( \frac{x_R}{2z_R^2 - x_R y_R} - \frac{1}{y_R} \right). \tag{30} \]

\(^3\) Here, the second order parameters \( \gamma_1 = \frac{x_D^2 y_R^2}{x_R y_R^2 - 2y_R z_R^2} \) and \( \gamma_2 = \frac{x_R y_D^2}{x_R y_R^2 - 2y_R z_R^2} \) have been omitted.
The expressions (27) and (28) show that the structures of $M_D$ and $M_R$ are the same as those of our previous work [45], however, the expression of the parameters $x_D, y_D, z_D, x_R, y_R$ and $z_R$ are completely different from each others. Therefore we will not consider in detail the neutrino sector in this work (for a similar analysis, the reader is referred to Ref. [45]). Using the obtained values of neutrino masses in Ref. [45], $m_1 \sim 1\,\text{meV}, m_2 \sim 8.7\,\text{meV}, m_3 \sim 50.3\,\text{meV}$, we can estimate the order of magnitude of the Yukawa couplings in the neutrino sector:

$$|x_1| \sim |x_2| \sim 1, \quad |x_3| \sim 10^{-3}, \quad |y_1| \sim |y_3| \sim 10^{-2}, \quad |y_2| \sim 10^{-3},$$

which are about only three orders of magnitude.

IV. QUARK MASS AND MIXING

Using the Clebsch-Gordan coefficients of $D_4$ symmetry [55], from Eq. (2), the Yukawa interactions in the quark sector can be rewritten in the following form:

$$-\mathcal{L}_Y = x_1^u (Q_{2L} u_{2R} + Q_{3L} u_{3R}) \tilde{H} + x_2^u (Q_{2L} u_{2R} - Q_{3L} u_{3R}) \tilde{H}'$$

$$+ \frac{x_3^u}{\Lambda} \left[ (Q_{1L} u_{2R} + Q_{2L} u_{1R})(\tilde{H} \rho_1) + (Q_{1L} u_{3R} + Q_{3L} u_{1R})(\tilde{H} \rho_2) \right]$$

$$+ \frac{x_4^u}{\Lambda} \left[ (Q_{1L} u_{2R} + Q_{2L} u_{1R})(\tilde{H}' \rho_1) - (Q_{1L} u_{3R} + Q_{3L} u_{1R})(\tilde{H}' \rho_2) \right]$$

$$+ x_1^d (Q_{2L} d_{2R} + Q_{3L} d_{3R}) H + x_2^d (Q_{2L} d_{2R} - Q_{3L} d_{3R}) H'$$

$$+ \frac{x_3^d}{\Lambda} \left[ (Q_{1L} d_{2R} + Q_{2L} d_{1R})(H \rho_1) + (Q_{1L} d_{3R} + Q_{3L} d_{1R})(H \rho_2) \right]$$

$$+ \frac{x_4^d}{\Lambda} \left[ (Q_{1L} d_{2R} + Q_{2L} d_{1R})(H' \rho_1) - (Q_{1L} d_{3R} + Q_{3L} d_{1R})(H' \rho_2) \right] + \text{H.c.}$$

(32)

After symmetry breaking, i.e., the scalar fields get VEVs as shown in Eq. (3), we find the up-and down-quark mass matrices as follows:

$$m_q = \begin{pmatrix}
0 & a_{3q} + a_{4q} & 0 \\
 a_{3q} + a_{4q} & a_{1q} + a_{2q} & 0 \\
0 & 0 & a_{1q} - a_{2q}
\end{pmatrix} \quad (q = u, d),$$

(33)

where

$$a_{1q} = x_1^q v, \quad a_{2q} = x_2^q v', \quad a_{3q} = x_3^q \frac{v_\rho}{\Lambda}, \quad a_{4q} = x_4^q \frac{v_\rho}{\Lambda}. \quad (34)$$
Next, let us construct the following hermitian matrices $M_q^2 = m_q m_q^\dagger$ which can obtained from Eqs. (33) and (34) as follows:

$$M_q^2 = \begin{pmatrix} A_{0q} & D_{0q} e^{-i\omega_q} & 0 \\ D_{0q} e^{i\omega_q} & B_{0q} & 0 \\ 0 & 0 & C_{0q} \end{pmatrix},$$  \hspace{1cm} (35)$$

where $A_{0q}, B_{0q}, C_{0q}, D_{0q}$ are real and positive parameters,

$$A_{0q} = |a_{3q}|^2 + |a_{4q}|^2 + 2 |a_{3q}| |a_{4q}| \cos \alpha_{34}^q,$$

$$B_{0q} = |a_{1q}|^2 + |a_{2q}|^2 + |a_{3q}|^2 + |a_{4q}|^2 + 2 |a_{1q}| |a_{2q}| \cos \alpha_{12}^q + 2 |a_{3q}| |a_{4q}| \cos \alpha_{34}^q,$$

$$C_{0q} = |a_{1q}|^2 + |a_{2q}|^2 - 2 |a_{1q}| |a_{2q}| \cos \alpha_{12}^q,$$

$$D_{0q} e^{-i\omega_q} = (|a_{1q}| e^{-i\alpha_{1q}} + |a_{2q}| e^{-i\alpha_{2q}}) (|a_{3q}| e^{i\alpha_{3q}} + |a_{4q}| e^{i\alpha_{4q}}),$$

with $q = u, d$ and $\alpha_{12}^q = \alpha_{1q} - \alpha_{2q}, \alpha_{34}^q = \alpha_{3q} - \alpha_{4q}, \alpha_{iq} = \text{arg}(a_{iq}) \ (i = 1 \div 4)$.

The matrices $M_q^2$ in Eq. (35) are diagonalized by the following biunitary matrices $V_{q(L,R)}$:

$$V_{uL}^\dagger M_u^2 V_{uR} = \text{diag} \left( m_u^2, m_c^2, m_t^2 \right),$$  \hspace{1cm} (40)$$

$$V_{dL}^\dagger M_d^2 V_{dR} = \text{diag} \left( m_d^2, m_s^2, m_b^2 \right),$$  \hspace{1cm} (41)$$

where

$$V_L^q = V_R^q = \begin{pmatrix} \cos \theta_q & -\sin \theta_q \cdot e^{-i\omega_q} & 0 \\ \sin \theta_q \cdot e^{i\omega_q} & \cos \theta_q & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (42)$$

with

$$\theta_q = \arctan \left( K_q^{-1} \right),$$  \hspace{1cm} (43)$$

$$K_u = \frac{m_u^2 - B_{0u}}{D_{0u}} = \frac{A_{0u} - m_c^2}{D_{0u}},$$  \hspace{1cm} (44)$$

$$K_d = \frac{m_d^2 - B_{0d}}{D_{0d}} = \frac{A_{0d} - m_s^2}{D_{0d}},$$  \hspace{1cm} (45)$$

and

$$m_{u,c}^2 = \frac{1}{2} \left( A_{0u} + B_{0u} \pm \sqrt{(A_{0u} - B_{0u})^2 + 4D_{0u}^2} \right), \hspace{1cm} m_t^2 = C_{0u},$$  \hspace{1cm} (46)$$

$$m_{d,s}^2 = \frac{1}{2} \left( A_{0d} + B_{0d} \pm \sqrt{(A_{0d} - B_{0d})^2 + 4D_{0d}^2} \right), \hspace{1cm} m_b^2 = C_{0d}.$$

\hspace{1cm} (47)
Combining Eqs. (34), (36)-(39) and (43)-(47) yields:

\[
\begin{aligned}
&\left\{ \frac{v^2}{X^2} (v^2|x_{3u}|^2 + v^2|x_{4u}|^2 + 2vv'|x_{3u}||x_{4u}| \cos \alpha_{34}^u) = m_u^2 \cos^2 \theta_u + m_c^2 \sin^2 \theta_u, \\
&\frac{v^2}{X^2} (v^2|x_{3u}|^2 + v^2|x_{4u}|^2 + 2vv'|x_{3u}||x_{4u}| \cos \alpha_{34}^d) + v^2|x_{1u}|^2 + v^2|x_{2u}|^2 \\
&\quad + 2vv'|x_{1u}||x_{2u}| \cos \alpha_{12}^u = m_c^2 \cos^2 \theta_u + m_u^2 \sin^2 \theta_u, \\
&v^2|x_{1u}|^2 + v^2|x_{2u}|^2 - 2vv'|x_{1u}||x_{2u}| \cos \alpha_{12}^u = m_t^2, \\
&\frac{v^2}{X^2} \{(v^2|x_{1u}|^2 + v^2|x_{2u}|^2 + 2vv'|x_{1u}||x_{2u}| \cos \alpha_{12}^u) (v^2|x_{3u}|^2 + v^2|x_{4u}|^2 \\
&\quad + 2vv'|x_{3u}||x_{4u}| \cos \alpha_{34}^u) \}^{\frac{1}{2}} = (m_c^2 - m_u^2) \sin \theta_u \cos \theta_u, \\
&\frac{v^2}{X^2} (v^2|x_{3d}|^2 + v^2|x_{4d}|^2 + 2vv'|x_{3d}||x_{4d}| \cos \alpha_{34}^d) = m_d^2 \cos^2 \theta_d + m_s^2 \sin^2 \theta_d, \\
&\frac{v^2}{X^2} (v^2|x_{3d}|^2 + v^2|x_{4d}|^2 + 2vv'|x_{3d}||x_{4d}| \cos \alpha_{34}^d) + v^2|x_{1d}|^2 + v^2|x_{2d}|^2 \\
&\quad + 2vv'|x_{1d}||x_{2d}| \cos \alpha_{12}^d = m_s^2 \cos^2 \theta_d + m_d^2 \sin^2 \theta_d, \\
&v^2|x_{1d}|^2 + v^2|x_{2d}|^2 - 2vv'|x_{1d}||x_{2d}| \cos \alpha_{12}^d = m_b^2, \\
&\frac{v^2}{X^2} \{(v^2|x_{1d}|^2 + v^2|x_{2d}|^2 + 2vv'|x_{1d}||x_{2d}| \cos \alpha_{12}^d) (v^2|x_{3d}|^2 + v^2|x_{4d}|^2 \\
&\quad + 2vv'|x_{3d}||x_{4d}| \cos \alpha_{34}^d) \}^{\frac{1}{2}} = (m_d^2 - m_s^2) \sin \theta_d \cos \theta_d.
\end{aligned}
\]

For simplicity, we consider the following benchmark point:

\[
\begin{aligned}
&\cos \alpha_{12}^u = \cos \alpha_{12}^d = -1, \quad \cos \alpha_{34}^u = \cos \alpha_{34}^d = 1, \\
&|x_{3u}| = |x_{4u}| = |x_u|, \quad |x_{3d}| = |x_{4d}| = |x_d|, \\
&|x_{1u}| = 0.697, \quad |x_{2u}| = 0.707, \quad |x_u| = 0.213, \\
&|x_{1d}| = 0.0166, \quad |x_{2d}| = 0.0176, \quad |x_d| = 0.0847,
\end{aligned}
\]

(50), (51)

which are about only two orders of magnitude.
The quark mixing matrix is defined from Eq. (42) as

\[ V_{\text{CKM}} = V_L^d V_L^u = \begin{pmatrix} \cos \theta_u \cos \theta_d + e^{i(\omega_u - \omega_d)} \sin \theta_u \sin \theta_d & e^{-i\omega_d} \cos \theta_u \sin \theta_d - e^{-i\omega_u} \sin \theta_u \cos \theta_d & 0 \\ e^{i\omega_u} \sin \theta_u \cos \theta_d - e^{i\omega_d} \cos \theta_u \sin \theta_d & \cos \theta_u \cos \theta_d + e^{-i(\omega_u - \omega_d)} \sin \theta_u \sin \theta_d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (56)

This is a good approximation for the realistic quark mixing matrix, which implies that the mixings among the quarks are very small \cite{3},

\[ V_{\text{CKM}} \sim \begin{pmatrix} 0.9740 & 0.2265 & 0 \\ 0.2265 & 0.9730 & 0 \\ 0 & 0 & 0.999 \end{pmatrix}. \] (57)

By using the best-fit points of \((V_{\text{CKM}})_{11}\) taken from Ref. \cite{3}, \((V_{\text{CKM}})_{11} = 0.97401\) and with the help of Eq. (53), we obtain:

\[ \cos(\omega_d - \omega_u) = 0.111 \ (\omega_d - \omega_u = 83.60^\circ), \] (58)

and the quark mixing matrix in Eq. (56) becomes

\[ |V_{\text{CKM}}| = \begin{pmatrix} 0.9740 & 0.2265 & 0 \\ 0.2265 & 0.9740 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \] (59)

which is consistent with the approximate pattern of the quark mixing matrix in Eq. (57).

Equations (23), (31), (54) and (55) implies that the Yukawa couplings in our model can reach a range only three orders of magnitude.

**V. CONCLUSIONS**

A gauge \(B - L\) model with \(D_4 \times Z_4 \times Z_2\) symmetry has been proposed that can accommodate the charged lepton and quark mass hierarchies as well as the lepton and quark mixing patterns with the CP phase through the type-I seesaw mechanism. The obtained Yukawa couplings in our model can reach a range only three orders of magnitude in both lepton and quark sectors.
Appendix A: Scalar potential

The potential invariant under $\Gamma$ gets the following form\[^{4}\]

\[
V_{\text{scalar}} = V(H) + V(H') + V(\chi) + V(\varphi) + V(\phi) + V(\rho) + V(H, H')
\]
\[
+ V(H, \chi) + V(H, \varphi) + V(H, \phi) + V(H, \rho) + V(H', \chi) + V(H', \varphi)
\]
\[
+ V(H', \phi) + V(H', \rho) + V(\chi, \varphi) + V(\chi, \phi) + V(\chi, \rho) + V(\varphi, \phi)
\]
\[
+ V(\varphi, \rho) + V(\phi, \rho) + V(H, H', \rho),
\]

where

\[
V(H) = \mu_1^2 H^\dagger H + \lambda^H (H^\dagger H)^2, \quad V(H') = V(H \rightarrow H'), \quad V(\chi) = V(H \rightarrow \chi),
\]
\[
V(\varphi) = V(H \rightarrow \varphi), \quad V(\phi) = \mu_2^2 \phi^* \phi + \lambda_1^\phi (\phi^* \phi)_1(\phi^* \phi)_1 + \lambda_2^\phi (\phi^* \phi)_1(\phi^* \phi)_1
\]
\[
+ \lambda_3^\phi (\phi^* \phi)_1(\phi^* \phi)_1 + \lambda_4^\phi (\phi^* \phi)_1(\phi^* \phi)_1, \quad V(\rho) = V(\phi \rightarrow \rho),
\]
\[
V(H, H') = \lambda_1^{HH'} (H^\dagger H)_1(H^\dagger H)_1 + \lambda_2^{HH'} (H^\dagger H)_1(H^\dagger H)_1,
\]
\[
V(H, \chi) = (H, H' \rightarrow \chi), \quad V(H, \varphi) = (H, H' \rightarrow \varphi),
\]
\[
V(H, \phi) = \lambda_1^\phi (H^\dagger H)_1(\phi^* \phi)_1 + \lambda_2^\phi (H^\dagger H)_1(\phi^* \phi)_1, \quad V(H, \rho) = V(H, H' \rightarrow \rho),
\]
\[
V(H', \chi) = \lambda_1^{H'H} (H^\dagger H)_1(\chi^* \chi)_1 + \lambda_2^{H'H} (H^\dagger H)_1(\chi^* \chi)_1, \quad V(H', \varphi) = V(H', \chi \rightarrow \varphi),
\]
\[
V(H', \phi) = V(H \rightarrow H', \phi), \quad V(H', \rho) = V(H \rightarrow H', \rho), \quad V(\chi, \varphi) = V(H' \rightarrow \phi, \chi),
\]
\[
V(\chi, \phi) = V(H \rightarrow \chi, \phi), \quad V(\chi, \rho) = V(\chi, \phi \rightarrow \rho), \quad V(\varphi, \phi) = V(\chi \rightarrow \varphi, \phi),
\]
\[
V(\varphi, \rho) = V(\chi \rightarrow \varphi, \rho), \quad V(\phi, \rho) = \lambda_1^\phi (\phi^* \phi)_1(\rho^* \rho)_1 + \lambda_2^\phi (\phi^* \phi)_1(\rho^* \rho)_1
\]
\[
+ \lambda_3^\phi (\phi^* \phi)_1(\rho^* \rho)_1 + \lambda_4^\phi (\phi^* \phi)_1(\rho^* \rho)_1,
\]

\[
V(H, H', \rho) = \lambda_1^{HH'} (H^\dagger H)_1(\rho^* \rho)_1 + \lambda_2^{HH'} (H^\dagger H)_1(\rho^* \rho)_1
\]
\[
+ \lambda_3^{HH'} (H^\dagger H)_1(\rho^* \rho)_1 + \lambda_4^{HH'} (H^\dagger H)_1(\rho^* \rho)_1.
\]

\[^{4}\] Other Yukawa terms with three and four different scalars including $V(H, H', \chi), V(H, H', \varphi), V(H, \chi, \varphi), V(H, \chi, \phi), V(H, \varphi, \phi), V(H, \rho, \rho), V(H', \chi, \varphi), V(H', \chi, \phi), V(H', \varphi, \phi), V(H', \chi, \rho), V(H', \varphi, \rho), V(H', \chi, \rho), V(H', \varphi, \rho), V(H', \chi, \rho), V(H', \chi, \rho)$ and $V(\chi, \varphi, \rho)$ are all not invariant under one or some of symmetries of $G$; thus, they were not included in the expressions of $V_{\text{scalar}}$. Furthermore, $V(H, H', \phi) = \lambda_1^{HH'} (H^\dagger H)_1(\phi^* \phi)_1 + \lambda_2^{HH'} (H^\dagger H)_1(\phi^* \phi)_1 + \lambda_3^{HH'} (H^\dagger H)_1(\phi^* \phi)_1 + \lambda_4^{HH'} (H^\dagger H)_1(\phi^* \phi)_1 = 0$ due to the VEV alignment of $\phi$ in Eq. \^[3].
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