WHEN WILL A ONE PARAMETER FAMILY OF UNIMODAL MAPS PRODUCE FINITE LIMIT CYCLES MONOTONICALLY WITH THE PARAMETER?

JOHN TAYLOR

Abstract. In this note we consider a collection $C$ of one parameter families of unimodal maps of $[0,1]$. Each family in the collection has the form $\{\mu f\}$ where $\mu \in [0,1]$. Denoting the kneading sequence of $\mu f$ by $K(\mu f)$, we will prove that for each member of $C$, the map $\mu \mapsto K(\mu f)$ is monotone. It then follows that for each member of $C$ the map $\mu \mapsto h(\mu f)$ is monotone, where $h(\mu f)$ is the topological entropy of $\mu f$. For interest, $\mu f(x) = 4\mu x(1-x)$ and $\mu f(x) = \mu \sin(\pi x)$ are shown to belong to $C$. This extends the work of Masato Tsujii [1].

INTRODUCTION

Metropolis, Stein and Stein were among the first, to my knowledge, to study what are now called finite kneading sequences. These were associated with one parameter families of interval maps which included $\mu f(x) = 4\mu x(1-x)$ and $\mu f(x) = \mu \sin(\pi x)$ [2]. Computer studies strongly suggested a universal topological dynamics for a large class of such families, and many workers were quickly drawn to this emerging field of study. In the 1980’s and 1990’s there was intense interest in the behavior of the logistic map (affinely modified) in the setting of one complex dimension. A central question was (essentially) under what circumstances would finite kneading sequences be monotone with the parameter, as this question was associated with the structure of the boundary of the Mandelbrot set. This question was successfully addressed (in the special case of a real quadratic) in [1,5]. Here we address the question generally for a large class that includes the logistic map, and give a sufficient condition for the solution.

All known proofs of this type apply to the case of a quadratic polynomial only and use complex analytic methods (holomorphic techniques) or depend on complex analysis (Compare [1], [5], [6], [7] [8]). These methods are not used here.

Let $I = [0,1]$. Consider the collection $C$ one parameter families of unimodal maps $\{\mu f\}$ where $x, \mu \in I$, and $\mu f : I \rightarrow I$ with $\mu f$ at least $C^3$ in both $\mu$ and $x$. Notice that since $\mu f(x)$ is linear in $\mu$, it is $C^\infty$ in $\mu$. Denote the single critical point $c \in (0,1)$ and scale the map so that $f(c) = 1$, requiring that $f(0) = 0 = f(1)$. Then $\mu f(c) = \mu$. Denote the $n^{th}$ iterate of $\mu f$ by $f_\mu^n(x) = (\mu f) \circ \cdots \circ (\mu f) \circ (\mu f)\circ (\mu f)(x)$, where the composition is $n$-fold.
For any \( x \in I \), the orbit of \( x \) is the set \( O(x) = \{ f^n_\mu(x) \mid n \geq 0 \} \). We associate with \( O(x) \) the word \( \omega(x) = \omega_0\omega_1\omega_2 \cdots \) with \( \omega_k \in \{ L, C, R \} \) where words are formed as follows:

\[
\omega_k = \begin{cases} 
L, & \text{for } f^k_\mu(x) < c \\
C, & \text{for } f^k_\mu(x) = c \\
R, & \text{for } f^k_\mu(x) > c.
\end{cases}
\]

\( \omega(x) \) is called the itinerary of \( x \) under \( f \). Thus, \( \omega(x) \) will be infinite if and only if \( O(x) \) is aperiodic. We are interested in finite words.

In particular, we will be interested in studying the itinerary associated with \( O(\mu) \). This special itinerary is called the kneading sequence of \( \mu f \), symbolized \( K(\mu f) = \omega(\mu) \).

The following preliminaries are necessary for the statement that the map \( \mu \mapsto K(\mu f) \) is monotone being meaningful; that is, we need a total order on the kneading sequences.

It is possible to construct a total order \( \prec \) on the set of all kneading sequences, and more generally, on the set of all words made from the alphabet \( \{ L, C, R \} \), in such a way that it reflects the order of the real line, in the sense that \( x < y \) implies \( \omega(x) \preceq \omega(y) \), is defined as follows: First define \( L \prec C \prec R \). Then, if \( A = \{ a_k \} \neq B = \{ b_k \} \), let \( N \) be the smallest index for which \( a_N \neq b_N \) and let \( \rho_N-1 \) be the number of \( R \)'s in the word \( a_1 \cdots a_{N-1} \). Then define \( A \prec B \) if \( a_N < b_N \) and \( \rho_N-1 \) is even or if \( a_N > b_N \) and \( \rho_N-1 \) is odd.

This order it sometimes referred to as the parity-lexicographic order. The intuition derives from the fact that for \( x \in [c, 1] \), \( \mu f \) is orientation reversing, that is, \( x < y \) implies that \( f(x) > f(y) \). In order that the ordering on the words be consistent with the order in the real numbers, this reversal of orientation is accounted for in the manner just described.

A word \( \omega \) is called maximal (or shift-maximal) provided it is greater (in the parity lexicographic order) than all of its shifts, where, as usual, the shift operator \( \sigma \) is defined by the action \( \sigma(\omega) = \omega_2\omega_3\cdots \) on the word \( \omega = \omega_0\omega_1\omega_2\cdots \). Shift maximal words correspond to periodic orbits.

In kneading theory there are several versions of an “intermediate value theorem”. This type of theorem is fundamental in that it relates abstract words to the behavior of dynamical systems. That is, it connects the set of kneading sequences ordered by the relation \( \prec \) and the parameter space (an interval in the real line) with the usual order. The following version is essentially that found in [3]:

**Theorem A** Let \( \{ \mu f \} \) be any one parameter family of \( C^1 \) unimodal maps. If \( \mu_1 < \mu_2 \) are two parameter values with corresponding kneading sequences \( K(\mu_1 f) \prec K(\mu_2 f) \), and if \( \omega \) is any shift-maximal sequence with the property that \( K(\mu_1 f) \prec \omega \prec K(\mu_2 f) \), then there exists a \( \mu \) such that \( \mu_1 < \mu < \mu_2 \) and \( \omega = K(\mu f) \).

Since \( \mu f \) double covers \( [0, \mu] \) in such a way that \( \mu f([0, c]) = [0, \mu] = \mu f([c, 1]) \), the functions \( (\mu f)^{-1}_L \), \( \omega \in \{ L, R \} \) have the action \( (\mu f)^{-1}_L([0, \mu]) = [0, c] \) and \( (\mu f)^{-1}_R([0, \mu]) = [c, 1] \).
For all $n \geq 0$, let $G_n(\mu)$ denote the graph of $f^n_\mu$.

By the chain rule, $\frac{d}{dx} f^n_\mu(x) = \prod_{k=0}^{n-1} \mu f^k [f^k_\mu(x)]$. Therefore, if $x$ is an extreme point of $G_n(\mu)$ then there exists $k, 0 \leq k \leq n$ such that $f^k(x) = c$.

**Definition** For $\mu$ fixed, define $x^0(\mu) = c$, and for $1 \leq k \leq n - 1$, denote by the symbol $x^k_\omega$ any $k^{th}$ preimage of $c$, specifically

$$f^{-k}_\mu(c) = \{ (\mu f)_j^{-1} \circ \cdots \circ (\mu f)_0^{-1}(c) \mid P = \omega_1 \omega_2 \cdots \omega_k \omega_l \in \{L,R\}\}.$$

Fixed points of the functions $x^k_\omega(\mu)$, which, given $f$, are functions of $\mu$ alone, will be central in what follows.

Denote the graph $\Gamma f^n_\mu$ by $G_k(\mu)$.

For any $C^3$ function $\psi$, let $S(\psi) = \frac{\psi''' - 3}{2 \psi'}$. A simple computation reveals that if $\phi$ is also a $C^3$ function with $S(\phi) > 0$ and $S(\psi) > 0$, then $S(\psi \circ \phi) > 0$.

**MAIN SECTION**

Here we prove that for each member of $C$, the map $\mu \mapsto K(\mu f)$ is monotone.

All families in $C$ have the following properties:

1) For each $\mu$ there exists a unique fixed point for $\mu f$ in $(0,1)$.

2) For each fixed $\mu$ and for all $n \geq 1$, $f^n_\mu$ has at most one attracting periodic orbit, and $O(\mu)$ is asymptotic to this attracting periodic orbit.

3) $S[(\mu f)^{-1}_\omega] > 0$ for all $\mu$, where $\omega \in \{L,R\}$.

**Remarks**

(i) Concave maps, for example, have property 1.

(ii) It is known that if $S(f) < 0$ for all $x$, then property 2 holds. [4]

**Lemma** Assume that $\omega = \omega_1 \omega_2 \cdots \omega_{k-1} = K(\mu f)$ and that $f^k_\mu(c) = c$ has primitive period $k$, so that by continuity there is an open set of parameter values for which the composition

$$(\mu f)^{-1}_{\omega_1} \circ \cdots \circ (\mu f)^{-1}_{\omega_{k-1}}(c)$$

is defined. If $\mu$ is such that $f^k_\mu(c) = c$, then

$$x^{k-1}_\omega(\mu) = \mu \Rightarrow f^{k-1}_\mu(c) = c.$$

**Proof**

$$x^{k-1}_\omega(\mu) = (\mu f)^{-1}_{\omega_1} \circ \cdots \circ (\mu f)^{-1}_{\omega_{k-1}}(c) = \mu \Rightarrow c = f^{-k-1}_\mu \left[ x^{k-1}_\omega(\mu) \right] = f^{k-1}_\mu \left[ (\mu f)^{-1}_{\omega_1} \circ \cdots \circ (\mu f)^{-1}_{\omega_{k-1}}(c) \right] = f^{k-1}_\mu(\mu) = f^{k-1}_\mu(\mu f(c)) = f^{k}_\mu(c).$$

$\Box$
Remarks

(i) A super stable point of primitive period \( n \) occurs in association with the equation \( x_\omega^{n-1}(\mu) = \mu \), where \( K(\mu f) = \omega \).

(ii) The trajectories of distinct preimages can never intersect.

It follows from the implicit function theorem that, for all \( n \geq 1 \), \( 0 \leq k \leq n - 1 \), level functions of order \( k \) exist so long as the intersection of \( G_n(\mu) \) with the line \( y = c \) exists.

But this intersection exists for all \( \mu > \mu^* \), where \( \mu^* \) is the parameter value with the property that, for \( 1 \leq k \leq n \), \( x_\omega^{k-1}(\mu^*) = \mu^* \); for then, \( f_\mu^{k}(c) = c \) in \( G_n(\mu) \) by the lemma. Therefore, so long as \( \mu^* \) is unique with the above property, we see that for all \( \mu > \mu^* \), the intersection of \( G_n(\mu) \) and the line \( y = c \) persists, and so, the level functions \( x_\omega^{n-1}(\mu) \) exist on a connected domain. Further, a certain number of these \( x_\omega^{n-1}(\mu) \) will have fixed points. The number is known to be

\[
\frac{1}{2^n} \sum_{\mu(d)2^{n/d}}
\]

where the sum is taken over all odd square free divisors of \( n \). [5]

**Theorem** For each member of \( \mathcal{C} \), the map \( \mu \mapsto K(\mu f) \) is monotone.

**Proof** The proof is by strong induction. First, notice that \( x_\omega^1(\mu) \) exists on the connected domain \([c, 1]\). Since we assume that \( S(x_\omega^1(\mu)) > 0 \), \( \frac{d}{d\mu} x_\omega^1(\mu) \) cannot have a positive local maximum. Therefore,

\[
\exists \mu^{*} [x_\omega^1(\mu^*) = \mu^*] \Rightarrow \exists \mu^{*} [f_{\mu^*}^2(c) = c] \quad \text{for} \quad K(\mu^* f) = RC \quad \text{by the lemma.}
\]

In other words, \( \exists \mu_\omega [f_{\mu_\omega}^1(c) = c] \) with \( \omega = K(\mu_\omega f) \Rightarrow \exists \mu_{\tau} [f_{\mu_{\tau}}^2(c) = c] \) with \( \tau = K(\mu_{\tau} f) \). Here \( \omega = C \) and \( \tau = RC \).

Assume that for \( 1 \leq k \leq n \), and for all \( \omega = K(\mu f) \) (with the length of \( \omega \) not exceeding \( n \)), \( \exists \mu_\omega [x_\omega^{k-1}(\mu_\omega) = \mu_\omega] \), that is \( \exists \mu_\omega [f_{\mu_\omega}^k(c) = c] \) with \( \omega = K(\mu f) \).

Since \( \exists \mu_\omega [x_\omega^{k-1}(\mu_\omega) = \mu_\omega] \), that is, \( \exists \mu_\omega [f_{\mu_\omega}^k(c) = c] \), \( \text{dom}(x_\omega^k(\mu)) \) is connected.

If \( x_\omega^k(\mu) \) has a fixed point, that is, if \( P = K(\mu f) \) for some \( \mu \), then \( f_{\mu}^{k+1}(c) = c \) when \( x_\omega^k(\mu) = \mu \).

But \( S(x_\omega^k(\mu)) > 0 \Rightarrow \exists \mu [x_\omega^k(\mu) = \mu] \), that is, \( \exists \mu [f_{\mu}^{k+1}(c) = c] \) with \( \omega = K(\mu f) \).

In particular, \( \exists \mu_\omega [f_{\mu_\omega}^{k+1}(c) = c] \) with \( \omega = K(\mu_\omega f) \Rightarrow \exists \mu_{\tau} [f_{\mu_{\tau}}^{k+1}(c) = c] \) with \( \tau = K(\mu_{\tau} f) \). \( \Box \)

**Remarks** (1) One computes that \( S((\mu f)^{-1}_\omega) > 0 \), \( \omega \in \{L, R\} \) when \( \mu f(x) = 4\mu x(1 - x) \) and \( \mu f(x) = \mu \sin(\pi x) \).

(2) The topological entropy of maps in the class \( \mathcal{C} \) is evidently monotone with the parameter. This is because orbit production for these never decreases.
References

[1] Masato Tsujii, A simple proof of monotonicity of entropy in the quadratic family, Ergodic Theory & Dynamical Systems 20(2000)925-933
[2] N. Metropolis, M.L. Stein and Paul Stein, On Finite Limit Sets for Transformations of the Unit Interval, J. Comb. Theory 15(1973), 25-44.
[3] W.A. Beyer, R.D. Mauldin, P.R. Stein, ol Shift Maximal Sequences in Function Iteration: Existence, Uniqueness, and Multiplicity, J. Math. Anal. Appl. 115 (1986), 305-362
[4] D Singer, on Stable Orbits and Bifurcations of Maps of the Interval SIAM J. Appl. Math 35 (1978), no. 2, 260-267.
[5] J. Milnor, W. Thurston, ol On Iterated Maps of the Interval Lecture Notes in Mathematics, no. 1342, Springer 1988
[6] A. Douday, Topological entropy of unimodal maps: Monotonicity for quadratic polynomials, pp. 65-87 of Real and Complex Dynamical Systems, (B. Branner and P. Hjorth Eds.) (Kluwer, Dordrecht, 1995).
[7] A. Douday and J. H. Hubbard, Etude dynamique des polynômes quadratiques complexes, I (1984) & (1985), Publ. Mat. d’Orsay
[8] W. de Melo and van Strien, One Dimensional Dynamics, (Springer Verlag, Berlin, 1993)

Department of Mathematical Sciences, United Arab Emirates University, Al Ain, UAE
E-mail address: john.taylor@uaeu.ac.ae