Dilaton-driven confinement

Steven S. Gubser
Lyman Laboratory of Physics
Cambridge, MA 02138
Tel: (617) 496-8362
Fax: (617) 496-8396
ssgubser@born.harvard.edu

Abstract

We derive a solution of type IIB supergravity which is asymptotic to $\text{AdS}_5 \times S^5$, has $SO(6)$ symmetry, and exhibits some of the features expected of geometries dual to confining gauge theories. At the linearized level, the solution differs from pure $\text{AdS}_5 \times S^5$ only by a dilaton profile. It has a naked singularity in the interior. Wilson loops follow area law behavior, and there is a mass gap. We suggest a field theory interpretation in which all matter fields of $\mathcal{N} = 4$ gauge theory acquire a mass and the infrared theory is confining.

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1 Introduction

One of the interesting aspects of the AdS/CFT conjecture [1, 2, 3] is that energy plays the role of an extra dimension: a fifth dimension if we are working with four-dimensional quantum field theories. The expectation is that the gravitational equations which dictate the bulk geometry to be equivalent in some sense to the renormalization group (RG) equations for the quantum field theory (QFT). It is difficult from a mathematical point of view to see how this can be, since the RG equations are first order while the supergravity equation are second order. It is also difficult to gain intuition regarding the problem from examining the conformal case, since there the anti-de Sitter is self-similar with respect to motions in the fifth dimension, and the evolution under RG is trivial.

It is desirable, then, to come up with examples where the quantum field theory is not conformal. The obvious approach is to start with $\mathcal{N} = 4$ super-Yang-Mills theory and add some relevant operators, corresponding to tachyon fields ($m^2 < 0$) in $AdS_5$. It turns out to be very difficult to find a supergravity solution with such fields excited because they all have $SO(6)$ quantum numbers: that is, they come from ten-dimensional fields with some non-trivial variation on $S^5$. Recently, progress has been made in this direction by considering the truncation of the Kaluza-Klein (KK) reduction of type IIB supergravity on $S^5$ to five-dimensional gauged supergravity [4, 5]. This amounts to keeping only the first few spherical harmonics for each ten-dimensional field in the five-dimensional theory. In this setup, RG flows in a boundary QFT are supposed to be reflected in a dependence on the fifth dimension of the scalars in the gauged supergravity theory. It has been speculated [6] that any solution of five-dimensional maximally supersymmetric gauged supergravity can be lifted to an exact solution of the ten-dimensional theory, in analogy with the embedding of maximally supersymmetric four-dimensional gauged supergravity in eleven-dimensional supergravity [7]. So far, the only examples in which the flow equations are known explicitly break all supersymmetry.

In this paper, we wish to take an approach which is much simpler from the point of view of supergravity calculations: we will study geometries which preserve the full $SO(6)$ invariance as well as the 3+1-dimensional Poincaré symmetry of the boundary quantum field theory. It would seem to follow from the no-hair theorems that any such geometry other than the full D3-brane metric must have a naked singularity, and this indeed is the case for the geometry which we will exhibit in section 3. Unlike the full D3-brane metric, our geometry is asymptotic to $AdS_5 \times S^5$ far from the singularity. It is tempting to guess that its field theory “image” (in the holographic sense of AdS/CFT) is $\mathcal{N} = 4$ super-Yang-Mills theory deformed by a relevant operator. The difficulty is that at strong coupling, AdS/CFT itself predicts that there are no relevant operators
which are $SO(6)$ singlets. In section 5 we will suggest that a solution to this dilemma is to add to the lagrangian an $SO(6)$-invariant mass term for scalars. This operator corresponds to an excited string state in $AdS_5 \times S^5$.

In section 4 we show that our geometry satisfies the usual criterion for confinement, namely the area law for Wilson loops and a mass gap. This is hardly unexpected from the perturbative field theory point of view, since confinement is the generic behavior for gauge theory coupled to massive matter. Finally, in section 6, we discuss the global structure of the geometry and obtain the conditions under which supergravity is a valid approximation.

2 The equations

In ten-dimensional type IIB supergravity, the most general ansatz with $SO(6)$ symmetry, 3+1-dimensional Poincaré invariance, and $N$ units of five-form flux through an $S^5$ is

$$
\begin{align*}
\hat{g}_{MN}dx^Mdx^N &= e^{-\frac{20}{3}\chi+2\sigma}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2) + L^2 e^{2\chi}d\Omega_5^2 \\
F_5 &= \frac{N\sqrt{\pi}}{2\text{Vol} S^5} (\text{vol}_{S^5} + \ast \text{vol}_{S^5}) ,
\end{align*}
$$

where $\chi$, $\sigma$, and also the dilaton $\phi$ and the axion $C$, are allowed to depend only on the radial coordinate $z$. The dilaton and the axion combine to form the complex coupling $\tau = C + ie^{-\phi}$, which in the AdS/CFT correspondence is identified with $\theta/2\pi + 4\pi i/g_{YM}^2$. We will always work in Einstein frame unless otherwise noted explicitly.

The relevant equations of type IIB supergravity, truncated to the fields we are interested in, read [8]

$$
\begin{align*}
\Box \phi &= e^{2\phi}(\partial_M C)^2 \\
\Box C &= -2(\partial_M \phi)(\partial^M C) \\
\hat{R}_{MN} &= \frac{1}{2}\partial_M \phi \partial_N \phi + \frac{1}{2}e^{2\phi} \partial_M C \partial_N C + \frac{\kappa^2}{6} F_{MP_1...P_4} F_{N P_1...P_4} .
\end{align*}
$$

The Einstein equations in the $S^5$ directions are satisfied if

$$
L^4 = \frac{\kappa N}{2\pi^{5/2}} .
$$

The remaining equations can be expressed in purely five-dimensional terms:

$$
\begin{align*}
\Box \phi &= e^{2\phi}(\partial_\mu C)^2 \\
\Box C &= -2(\partial_\mu \phi)(\partial^\mu C) \\
\Box \chi &= \frac{4}{L^2} \left( e^{-\frac{16}{3}\chi} - e^{-\frac{40}{3}\chi} \right) \\
R_{\mu\nu} &= \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{2\phi} \partial_\mu C \partial_\nu C + \frac{40}{3} \partial_\mu \chi \partial_\nu \chi - \frac{g_{\mu\nu}}{L^2} \left( \frac{20}{3} e^{-\frac{16}{3}\chi} - \frac{8}{3} e^{-\frac{40}{3}\chi} \right) ,
\end{align*}
$$

2
where now the metric
\[ ds_5^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{2\sigma}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2) \] (5)
is used to compute \( R_{\mu\nu} \) and to contract indices. The equations (4) follow from the five-dimensional action
\[ S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{g} \left[ R - \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} e^{2\phi} (\partial_\mu C)^2 - \frac{40}{3} (\partial_\mu \chi)^2 + \frac{1}{L^2} \left( 20e^{-\frac{4\phi}{3}} - 8e^{-\frac{4\phi}{3}} \right) \right], \] (6)
as was first noted in [9]. The gravitational couplings \( \kappa_5 \) and \( \kappa \) in five and ten dimensions are related by
\[ \frac{1}{2\kappa^2} = \frac{\pi^3}{2\kappa_5^2} \frac{L^5}{N^2} \frac{1}{8\pi^2 L^3}. \] (7)
The Weyl factor on the non-compact part of the metric in (1) was chosen expressly so that the wave equation \( \Box \eta = 0 \) for a minimal scalar in the s-wave would reduce to the five-dimensional wave equation \( \Box \eta = 0 \). This is the same requirement that makes \( ds_5^2 \) the Einstein frame metric.

It is possible to reduce the equations of motion (6) to a set of coupled non-linear second order ordinary differential equations in \( \phi, C, \chi, \) and \( \sigma \). These equations seem too complicated to deal with in general, but there is an obvious simplification: \( C = 0 \) and \( \chi = 0 \). Then the equations become much simpler:
\[ \partial_z^2 \sigma + 3(\partial_z \sigma)^2 = \frac{4}{L^2} e^{2\sigma} \]
\[ 4\partial_z^2 \sigma = -\frac{1}{2} (\partial_z \phi)^2 + \frac{4}{L^2} e^{2\sigma} \]
\[ e^{-5\sigma} \partial_z e^{3\sigma} \partial_z \phi = 0. \] (8)
In future sections we will usually deal with the full ten-dimensional Einstein metric. Because \( \chi = 0 \) there is no distinction between the five-dimensional Einstein metric and the ten-dimensional Einstein metric restricted to the five-dimensional non-compact space. So we will drop the hats from ten-dimensional quantities.

3 The solution

The last equation in (8) can be integrated directly to give
\[ \phi(z) = \phi_\infty + \frac{B}{L} \int_0^z d\tilde{z} e^{-3\sigma(\tilde{z})}, \] (9)
where \( \phi_\infty \) is the value of the dilaton at the boundary of the asymptotically \( AdS_5 \) geometry, and \( B \) is another arbitrary constant. We can take \( B > 0 \) since \( S \)-duality
sends $\phi \rightarrow -\phi$ while preserving the Einstein metric. Substituting (9) back into (8), defining $u = z/L$, and rearranging slightly, one obtains

$$\partial_u^2 \sigma = e^{2\sigma} - \frac{B^2}{8} e^{-6\sigma}$$

$$\left(\partial_u \sigma\right)^2 = e^{2\sigma} + \frac{B^2}{24} e^{-6\sigma}.$$  \hspace{1cm} (10)

The first equation follows from differentiating the second, so we see that (8) is a consistent system of equations despite being overdetermined. Physically, (10) describes the zero-energy trajectory of a classical particle with unit mass moving in the potential $V(\sigma) = -\frac{1}{2} e^{2\sigma} - \frac{B^2}{48} e^{-6\sigma}$, which is depicted in figure 1. We regard $z = uL$ as a radial variable which goes to 0 at the boundary of the geometry, where $\sigma$ becomes large. Thus in our mechanical analog, we are starting the particle at $\sigma = \infty$ at “time” $u = 0$. In finite time the particle gets to $\sigma = 0$. If $B = 0$, then it continues to move to the left in figure 1 ever more slowly as $u \rightarrow \infty$. But for $B \neq 0$, the particle reaches a minimum velocity at some $\sigma < 0$, and then falls back down the other side of the potential after a finite time $u_0$. The $B = 0$ solution is pure $AdS_5$ with constant dilaton. Not unexpectedly, the $B \neq 0$ geometry is geodesically incomplete and singular at the point $u = u_0$. To find $u_0$ explicitly, we use the standard trick for integrating the motion in a one-dimensional potential well:

$$u = \int_{\sigma}^{\infty} \frac{d\tilde{\sigma}}{\sqrt{-2V(\tilde{\sigma})}} = \int_{\sigma}^{\infty} \frac{d\tilde{\sigma}}{\sqrt{e^{2\tilde{\sigma}} + \frac{B^2}{24} e^{-6\tilde{\sigma}}}}$$

$$= \frac{3^{1/8} \Gamma(3/8) \Gamma(1/8)}{8^{7/8} \sqrt{\pi} B^{1/4}} - \sqrt{\frac{8}{3} B} F\left(\frac{3}{8}, \frac{1}{2}; \frac{11}{8}; -\frac{24 e^{8\sigma}}{B^2}\right).$$  \hspace{1cm} (11)
where $F(\alpha, \beta; \gamma; z)$ is the usual hypergeometric function. The second term vanishes as $\sigma \to -\infty$, so we find $u_0 = \frac{3^{1/4}\Gamma(3/8)\Gamma(1/8)}{8^{7/8}\sqrt{\pi}B^{1/4}}$.

We can find the dilaton explicitly in terms of $\sigma$ by performing the integral in (9):

$$\phi = \phi_\infty + \sqrt{\frac{3}{2}} \text{arcoth} \sqrt{1 + \frac{24}{B^2} e^{8\sigma}}. \quad (12)$$

Note that $\phi$ increases as $\sigma$ decreases, and $\phi \to \infty$ as $\sigma \to -\infty$, which is what we expect since the far interior of the geometry ($\sigma \to -\infty$) corresponds to the infrared in the gauge theory. We can also write the ten-dimensional Einstein metric explicitly if we use $\sigma$ rather than $z$ as the radial variable:

$$ds_{10}^2 = e^{2\sigma}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{L^2 d\sigma^2}{1 + \frac{B^2}{24} e^{-8\sigma}} + L^2 d\Omega_5^2. \quad (13)$$

From (12) and (13) it is evident that we can cancel the factors of $\frac{B^2}{24}$ by sending $\sigma \to \sigma + \frac{1}{8} \log \frac{B^2}{24}$. In order not to reintroduce factors of $\frac{B^2}{24}$ in the world-volume components of the metric, we should also rescale $t \to \left(\frac{B^2}{24}\right)^{-1/8} t$ and $x_i \to \left(\frac{B^2}{24}\right)^{-1/8} x_i$ for $i = 1, 2, 3$. If we also rescale $z \to \left(\frac{B^2}{24}\right)^{-1/8} z$ then the net result is the same as if we had set $B^2 = 24$ throughout this section. Choice of the radial coordinate in $AdS_5$ corresponds to choice of one out of a given class of conformally equivalent boundary metrics. Thus we see that the freedom to change $B$ in the solution (12), (13) corresponds merely to the asymptotic scale invariance of the boundary theory.

### 4 Confinement

The string coupling gets strong as $\sigma \to -\infty$, but this does not necessarily mean confinement, as illustrated in [10], where the far interior of a D3-brane geometry in a type 0 string theory was found to be $AdS_5 \times S^5$ geometry with formally infinite dilaton.* The most straightforward test of confinement in the context of holography [12] is the area law for Wilson loops. The metric felt by strings is not the Einstein metric (13), but rather the string metric

$$ds_{\text{string}}^2 = G_{MN} dx^M dx^N = e^{(\phi - \phi_\infty)/2} ds_{\text{Einstein}}^2 = e^{2\sigma + (\phi - \phi_\infty)/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2) + L^2 e^{(\phi - \phi_\infty)/2} d\Omega_5^2. \quad (14)$$

*In [11], more generic interior geometries were explored and shown to exhibit properties of confinement. These solutions are similar in spirit and in global structure to ours. We thank J. Minahan for bringing this work to our attention.
Figure 2: $\log n = 2\sigma + (\phi - \phi_{\infty})/2$ as a function of $\sigma$ for $B^2 = 24$ (solid line) and $B = 0$ (dashed line).

Following [13, 14, 12], we consider a fundamental string following a trajectory ending on specified points of the boundary of the asymptotically $AdS_5$ geometry which minimizes the Nambu-Goto action,

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{\det \left(G_{MN} \frac{\partial x^M}{\partial \xi^\alpha} \frac{\partial x^N}{\partial \xi^\beta} \right)}$$

$$= \frac{1}{2\pi\alpha'} \int dt dx e^{2\sigma + (\phi - \phi_{\infty})/2} \sqrt{1 + \left(\frac{dz}{dx}\right)^2}.$$  \hspace{1cm} (15)

In the second line we have assumed that the string is at a constant angular position in $S^5$, and we have set $\xi^0 = t$, $\xi^1 = x_1 = x$, and $x_2 = x_3 = 0$. The time integral is trivial, so we see that the integral whose minimum determines the string trajectory is

$$V = \frac{1}{2\pi\alpha'} \int dx n(z) \sqrt{1 + \left(\frac{dz}{dx}\right)^2},$$  \hspace{1cm} (16)

where we have defined $n(z) = e^{(\phi(z) - \phi_{\infty})/2 + 2\sigma(z)}$. The form (16) is suggestive of Fermat’s Principle for the path of light rays: if $n(z)$ is regarded as the “refractive index,” then $V$ is the total “time” (or more properly, the total distance in the flat metric $ds^2 = dx^2 + dz^2$) it takes to traverse the trajectory followed by the “light ray” (more properly, the string). As one can see from figure 2, $n(z)$ has a global minimum when $B \neq 0$, namely $n_s = (\sqrt{2} + \sqrt{3})\sqrt{3/8} \left(\frac{B^2}{48}\right)^{1/4}$ at $\sigma_s = \frac{1}{8} \log \frac{B^2}{48}$. In the light ray analogy, confinement is realized as total internal reflection: for large separation of endpoints, the trajectory which minimizes $V$ locates itself very nearly at the minimum of $n(z)$ for
most of its length. This is similar to the way fiber optics work: the light ray is guided

to the minimum of $n(z)$. When the separation $r$ of the two endpoints is much larger
than $L$, we have

$$V(r) \approx \frac{n_\ast}{2\pi \alpha'} r.$$  \hfill (17)

The relations

$$L^4 = \frac{N \kappa}{2\pi^{5/2}} g_{YM}^2 = 4\pi g_s \quad e^{\phi_\infty} = g_s$$  \hfill (18)

result in the simple formula $L^2/\alpha' = \sqrt{g_{YM}^2 N}$. Using this in (17) we obtain the QCD
string tension as

$$T_{QCD} = \frac{n_\ast}{2\pi \alpha'} = \frac{n_\ast}{2\pi L^2} \sqrt{g_{YM}^2 N} = \left( \frac{B^2}{48} \right)^{1/4} \left( \sqrt{2} + \sqrt{3} \right) \sqrt{3/8} \frac{g_{YM}^2 N}{2\pi L^2}.$$  \hfill (19)

Note that in (18) we have defined $g_s$ as the string coupling at the boundary of the
asymptotically AdS$_5$ geometry.

The second test of confinement used in [12] is the presence of a mass gap. This is
straightforward to check for perturbations of the dilaton in our geometry. Because the
dilaton equation of motion $\Box \phi = 0$ is linear, we can ignore the background dilaton
profile and compute the spectrum of $m^2$ in the differential equation

$$\left[ \sqrt{e^{-8\sigma} + \frac{B^2}{24} e^{-16\sigma} \partial_\sigma \sqrt{e^{8\sigma} + \frac{B^2}{24} \partial_\sigma + e^{-2\sigma} m^2 L^2 - \ell(\ell + 4) \right]} R(\sigma) = 0.$$  \hfill (20)

This radial equation follows from plugging the ansatz $\delta \phi = e^{ik \cdot x} R(\sigma) Y_\ell$ into the
equation $\Box \delta \phi = 0$, where $k^2 = -m^2$ and $Y_\ell$ is a spherical harmonic on $S^5$ with
$\Box_{S^5} Y_\ell = \ell(\ell + 4) Y_\ell$. Note that the Einstein metric, not the string metric, enters into
these computations. A convenient change of variables is

$$y = \sqrt{\frac{2}{3}} (\phi - \phi_\infty) = \text{arccoth} \left[ 1 + \frac{24}{B^2} e^{8\sigma} \right] \quad E = \sqrt{24 m^2 L^2 \sqrt{2}}.$$  \hfill (21)

In these variables, the equation (20) reads

$$\left[ 16 \partial_y^2 - \frac{E}{(\sinh y)^{3/2} \sinh^2 y} - \frac{\ell(\ell + 4)}{ \sinh^2 y} \right] R(y) = 0.$$  \hfill (22)

The small $y$ region is near the boundary of the asymptotically AdS$_5$ geometry. Replacing $\sinh y$ by $y$ in (22) results in a form of Bessel’s equation. In the large $y$ region the
potential terms vanish, so $R(y)$ is linear. Altogether we have the asymptotics

$$R(y) \rightarrow \begin{cases} c_1 \sqrt{y} J_{\ell+2}(4\sqrt{E} y^{1/4}) + c_2 \sqrt{y} N_{\ell+2}(4\sqrt{E} y^{1/4}) \\ c_3 + c_4 y \end{cases} \quad \text{for } y \to 0$$  \hfill (23)

for $y \to \infty$.  \quad \text{for } y \to \infty.$
For $\delta \phi$ to be everywhere small, we must choose $c_2 = c_4 = 0$. These conditions determine a boundary-value problem with a discrete spectrum for $E$. It has been proposed [15] that the corresponding massive states in the gauge theory have an interpretation of glueballs. Let us label the eigen-energies as $E_{n\ell}$, where $n = 1, 2, 3, \ldots$ labels the “glueball” excitation level and $\ell = 0, 1, 2, 3, \ldots$ is its $SO(6)$ principle quantum number. From (21) we extract the masses:

$$m_{n\ell}^2 = \sqrt{\frac{B}{24}} \frac{E_{n\ell} L^2}{\sqrt{E_{n\ell} L^2}}. \quad (24)$$

We tabulate numerical values for the first few glueball states in table 1. One sees immediately that the same problem that vexed early efforts [15, 16] using the near-extremal D4-brane approach to QCD also occurs here: namely, the “glueballs” with $SO(6)$ charge have masses on the same order as the $SO(6)$-neutral glueballs.

The lowest glueball mass associated with the dilaton may not be the minimum energy excitation $M_{\text{gap}}$ of the theory, since there are other excitations of comparable energy associated with the other supergravity fields. However $M_{\text{gap}}$ should at least be on the order of the lowest glueball mass in table 1. From (24) and (19) we see that $T_{\text{QCD}}/M_{\text{gap}}^2 \sim \sqrt{g_Y^2 M N}$. This relation also obtains in other supergravity models of confinement, but to our knowledge has not been understood simply in field theory.

| $m_{n\ell}^2 L^2$ | $n = 1$ | $n = 2$ | $n = 3$ |
|-------------------|---------|---------|---------|
| $\ell = 0$       | 9.30178 | 30.8883 | 64.8142 |
| $\ell = 1$       | 15.5574 | 43.4014 | 83.5198 |
| $\ell = 2$       | 25.0644 | 59.1776 | 105.503 |

Table 1: Numerical values for glueball masses with $B^2 = 24$.

5 Field theory interpretation

In the previous section we saw that the geometry (13) exhibits the basic features of confinement: the area law for Wilson loops and a mass gap. Given the ultraviolet behavior of the dilaton, $\phi \sim \phi_\infty + \frac{B z^4}{4 L^4}$, an obvious interpretation in view of [17] is that we have described a state with nonzero $\langle \text{tr } F^2 \rangle$. In this section we would like to investigate an alternative interpretation suggested by the confining properties of the infrared geometry: namely, that the matter fields of $\mathcal{N} = 4$ super-Yang-Mills theory have been made massive in an $SO(6)$-invariant fashion, breaking all supersymmetry as well as conformal invariance, and leaving us with only the gauge fields below the mass scale.
We can give the scalars an $SO(6)$-invariant mass by adding $m_X^2 \text{tr} \sum_{I=1}^6 X_I^2$ to the lagrangian. Also we can give an $SO(6)$ symmetric Majorana mass to the fermions, since they transform in a real representation of the gauge group.† In perturbation theory, either type of mass would induce the other via loops, and the theory in the infrared would be a confining gauge theory, perhaps coupled to massive matter if some of it is no heavier than the confinement scale. Let us focus on the scalar mass, and write $O_K = \text{tr} \sum_I X_I^2$.

Thinking of $\mathcal{N} = 4$ gauge theory in $\mathcal{N} = 1$ language, the operator $O_K$ is the first component of the Konishi superfield, $\text{tr} \sum_{i=1}^3 \Phi_i^\dagger e^\nu \Phi_i$, where $\Phi_i$ are the three chiral matter multiplets. $O_K$ is neither a chiral primary nor the descendent of a chiral primary, so its dimension is not protected by the superconformal algebra.‡ In [2, 3] it was conjectured that all such operators are dual to excited string states in $AdS_5 \times S^5$, and that they acquire dimensions on the order $(g_{YM}^2 N)^{1/4}$ at large ’t Hooft coupling. In the absence of a completely satisfactory formalism for quantizing strings in $AdS_5 \times S^5$, what we mean by excited string states is simply all fundamental string states except the ones which participate in the ten-dimensional type IIB supergravity multiplet. But the geometry described in section 3 involved only the supergravity fields, so how can $O_K$ have anything to do with it? Precisely because $\Delta_K$ is large, the influence of the “field” $\phi_K$ dual to $O_K$ on the geometry should fall off very quickly as one moves toward the interior of $AdS_5$.§ The only effect that $\phi_K$ could have on the geometry which would persist into the interior is to source a low-dimension $SO(6)$-invariant field. The only candidates (see the tables in [18]) are the dilaton, the axion, and the overall volume of $S^5$—that is, the fields involved in the ansatz (1). The corresponding operators in the gauge theory are, respectively, $\text{tr} F^2$, $\text{tr} F \tilde{F}$, and \[ \text{Str} \left[ F_{\mu_1 \nu_2} F_{\mu_2 \nu_3} F_{\mu_3 \nu_4} F_{\mu_4 \nu_1} - \frac{1}{4} (F_{\mu_1 \nu_2} F_{\mu_2 \nu_1})^2 \right], \] plus contributions from the gauginos and scalars, and their dimensions are, respectively, four, four, and eight. $\text{Str}$ is the symmetrized trace. Let us call these operators $O_4$, $\tilde{O}_4$, and $O_8$. We can exclude $\tilde{O}_4$ from consideration by insisting on CP invariance. The operator $O_8$ itself is irrelevant, and so is also expected to have only a brief influence on the geometry as one flows inward from the boundary. If we exclude this operator from consideration, then the conclusion is that the only way in which a perturbation of $\phi_K$ near the boundary $AdS_5$ can influence the geometry far from the boundary is to produce a gradient for the dilaton. It seems guaranteed that $\phi_K$ will source the

†We thank A. Grant for pointing this out.
‡The discussion that follows is simplified if $O_K$ has definite dimension. This is true of $O_K = \text{tr} \sum_I X_I^2$ at zero gauge coupling. As we turn on the gauge coupling, $\text{tr} \sum_I X_I^2$ may mix with other operators. If so, then we think of $O_K$ as the resulting operator of definite dimension, and we expect that $O_K$ still has some component of $\text{tr} \sum_I X_I^2$.
§We put the word “field” in quotes because it is not clear that a field theory of excited string states exists in any meaningful form. Nevertheless we will use the notion of the field $\phi_K$ as an intuitive guide.
dilaton, because the integrated three-point function
\[
\langle \mathcal{O}_K(x) \mathcal{O}_K(0) \int \mathcal{O}_4 \rangle \sim g_{YM} \frac{\partial}{\partial g_{YM}} \langle \mathcal{O}_K(x) \mathcal{O}_K(0) \rangle \sim g_{YM} \frac{\partial}{\partial g_{YM}} \frac{N^2}{|x|^{2\Delta_K}},
\]
does not vanish when $\Delta_K$ depends on the gauge coupling. And it does: $\Delta_K \sim (g_{YM}^2 N)^{1/4}$.

The skeptic may now wonder how it is possible to add a highly irrelevant operator to $\mathcal{N} = 4$ gauge theory at some high energy scale and see dramatic effects in the infrared. The optimist would reply that it is an extreme example of a dangerous irrelevant operator—by which we mean an operator which, as a perturbation of the ultraviolet theory, has large dimension, but which becomes marginal or even relevant along the RG flow. (Usually in discussions of dangerous irrelevant operators one has in mind perturbing a theory which is gaussian in the UV, but here we are thinking of deforming strong-coupling $\mathcal{N} = 4$ gauge theory, which is never gaussian). In fact, the supergravity geometry deviates significantly from $AdS_5$ only for $z \gtrsim L/\sqrt{B}$, so if the $\phi_K$ perturbation is located at a much smaller $z$, then there is a large “scaling region” (in the sense of critical phenomena: many orders of magnitude in energy) in which the theory is nearly conformal. We should add that an operator $\mathcal{O}_K$ with definite dimension and a finite $\text{tr} \sum_I X_I^2$ component is not the only operator one might consider adding. In the language of the optimist, any dangerous irrelevant operator will do, provided it is an $SO(6)$ singlet, and provided one can argue (along the lines of (25), for example) that its dual “field” in string theory triggers a dilaton flow. Our field theory intuition is that only operators which are in some sense strong coupling analogs of soft mass breakings or relevant Yukawa couplings are candidates for dangerous irrelevant operators. Confinement is generic behavior once superconformal invariance is lost. As we flow toward the infrared past the point where the heavy string field has damped out, the only relevant information for determining the subsequent flow is the coefficient $B$ in (12). In short, $B$ parametrizes our ignorance of how the superconformal invariance is broken in the ultraviolet.

If we are willing to make some crude estimates, we can see how the mass gap depends on the strength of the $\mathcal{O}_K$ perturbation. Suppose we cut off the geometry at $z = \epsilon$ rather than allowing it to go all the way out to the natural boundary at $z = 0$. Let us define the theory at $z = \epsilon$ by setting the value of the dilaton, $\phi = \phi_\infty$, and also a finite value for the excited string “field,” $\phi_K = \mu^2$. We regard $\phi_K$ as a dimensionless field, so $\mu$ is a dimensionless mass parameter: $\phi_K = \mu^2$ on the boundary corresponds to having a term in the lagrangian of the form $m_X^2 \text{tr} \sum_I X_I^2$ where $m_X = \mu/\epsilon$. This is to be compared with the near-extremal D4-brane approach to QCD$_4$ [12], where $m_{\text{fermions}} = \pi T$ and $T$ is the temperature of the 4+1-dimensional theory. In both cases the terms in the bare 3+1-dimensional lagrangian which give matter fields masses have a coefficient which is an energy squared.
If we regard $\phi_K$ as a linear perturbation of $AdS_5 \times S^5$ with $\phi = \phi_\infty$ everywhere, then

$$
(\Box + m^2_K)\phi_K = 0
$$

$$
\phi_K = \mu^2 \left( \frac{z}{\epsilon} \right)^{4-\Delta_K}.
$$

(26)

This would be our starting point for computing a two-point function of $O_K$. Beyond the linearized approximation there is a coupling of $\phi_K$ to the dilaton. Again crudely, we take as the five-dimensional lagrangian

$$
L = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (\partial \phi_K)^2 - \frac{1}{2} m^2_K \phi_K^2 + \frac{\lambda}{L^2} \phi_K^2 (\phi - \phi_\infty).
$$

(27)

In order for the relation (25) between $\langle O_K O_K \rangle_k$ and $\langle O_K O_K \rangle$ to hold, we should pick $\lambda \sim \Delta_K$. If we set the same boundary conditions $\phi = \phi_\infty$ and $\partial_z \phi = 0$ at $z = \epsilon$ as obtained in the pure $AdS_5 \times S^5$ geometry, then we can solve approximately for the dilaton in the region where the geometry is still nearly $AdS_5$. We have

$$
\Box \phi = \frac{\lambda}{L^2} \phi_K^2
$$

$$
\frac{1}{z^3} \partial_z \phi = \int_{\epsilon}^{z} \frac{d\tilde{z}}{\tilde{z}^5} \lambda \phi_K^2,
$$

$$
\approx \lambda \mu^4 \int_{\epsilon}^{\infty} \frac{d\tilde{z}}{\tilde{z}^5} \left( \frac{\tilde{z}}{\epsilon} \right)^{8-2\Delta_K} = \frac{\lambda}{2\Delta_K - 4} \frac{\mu^4}{\epsilon^4} \quad \text{for} \ z \gg \epsilon.
$$

(28)

Comparing this with the form which follows from (9) in a nearly $AdS_5$ geometry, namely

$$
\phi \approx \phi_\infty + \frac{B z^4}{4 L^4},
$$

(29)

we find $\frac{B}{L^4} \sim \frac{1}{\Delta_K} \frac{\mu^4}{\epsilon^4} \sim \frac{\mu^4}{\epsilon^4}$ provided $\lambda \sim \Delta_K$. The mass gap as computed in section 4 is

$$
M_{\text{gap}} \sim \frac{B^{1/4}}{L} \sim \frac{\mu}{\epsilon} = m_X.
$$

(30)

We have assumed in this analysis that there is a scaling region, where $z \gg \epsilon$ but the geometry is still nearly $AdS_5$. Significant deviations from $AdS_5$ come at $z = L/B^{1/4} \sim \epsilon/\mu \gg \epsilon$ provided $\mu \ll 1$, so the story is consistent provided that the scalar mass $m_X = \mu/\epsilon$ which we put into the theory is much lower than the scale $1/\epsilon$ at which we define it. In fact, the explicit cutoff $\epsilon$ can be taken as small as one likes, and the same bulk physics will result if one keeps $m_X = \mu/\epsilon$ constant, provided the dilaton is assumed to be flat at $z = \epsilon$. Thus, somewhat surprisingly, the operator $O_K \sim \sum_I \text{tr} X^2_I$ when added to the Wilsonian action as a finite perturbation is effectively dimension two.
The result $M_{\text{gap}} \sim m_X$ is similar to what is found in the near-extremal D4-brane approach to QCD: there $M_{\text{gap}} \sim m_{\text{fermions}}$. It is rather different from the weak coupling result, where at one loop one expects $M_{\text{gap}} \sim \exp \left( -\frac{8\pi^2}{b_0 g_Y^2} \right) m_{\text{soft}}$. Here $b_0$ is the leading coefficient in the beta function below the soft breaking scale: if there are no matter fields below this scale, then $b_0 = \frac{11}{3} N$. $M_{\text{gap}} \sim m_X$ renders comprehensible the presence of “glueballs” with $SO(6)$ charge with mass comparable to the lowest neutral glueball mass $M_{\text{gap}}$: the masses of the $SO(6)$-charged matter fields are comparable to the QCD scale, so we should indeed expect to see $SO(6)$-charged color singlet states at that same scale, just as we see strange hadrons in the real world with masses on the order of $\Lambda_{QCD}$.

It may seem that we have dropped the operator $\mathcal{O}_8$ from consideration prematurely: \textit{a priori} it seems plausible that the dual field $\chi$ might be the natural candidate for triggering a dilaton profile. Inspection of the equations of motion (4) reveals that this is not the case. In five-dimensional Einstein frame (and still with $C = 0$), the dilaton equation is $\Box \phi = 0$ regardless of what $\chi$ is doing. Thus constant $\phi$ is always a solution. As one can verify from the absence of a $\chi \chi \phi$ term in (6), the three point function $\langle \mathcal{O}_8 \mathcal{O}_8 \mathcal{O}_4 \rangle = 0$, which is certainly consistent with the fact that $\langle \mathcal{O}_8 \mathcal{O}_8 \rangle$ is independent of $g_Y$ if $\mathcal{O}_8$ is properly normalized. With $\phi$ constant and $\chi$ non-constant, one could for example recover the full D3-brane metric. In [19] it was suggested that in a particular large $g_s N$, small $\alpha'$ double scaling limit [20], the world-volume gauge theory corresponding to the full D3-brane metric is

$$\mathcal{L} = \mathcal{O}_4 + L^4 \mathcal{O}_8,$$

(31)

defined in a Wilsonian sense at a cutoff scale $1/L$. As usual $L$ is the radius of the $S^5$ far down the throat of the D3-brane. The second term in (31) was thought to characterize the deviations of the geometry from $AdS_5 \times S^5$ for radii $r \gg L$. In particular it seemed to match the scaling form of corrections to the absorption cross-section of minimally coupled scalars such as the dilaton [21].

6 Discussion

To summarize our results, it is useful to draw a Carter-Penrose diagram of the spacetime (13) (see figure 3). The conformal structure of the solution (13) is most obvious in the original radial variable $z$. A light ray sent in from the boundary can get to the time-like naked singularity and back in finite coordinate time $\Delta t \sim L/B^{1/4} \sim 1/\Lambda_{QCD}$.

The geometry deviates substantially from anti-de Sitter space for $\sigma \leq \sigma_*$, where $\frac{B^2}{24} e^{-8\sigma_*} = 2$. The location $\sigma = \sigma_*$ in the bulk is where long Wilson loops prefer to

\footnote{We thank O. Aharony for pointing out to us this comparison with the one-loop analysis.}
Figure 3: A Carter-Penrose diagram for the solution (13). Each point represents a copy of $\mathbb{R}^3 \times S^5$. 
run, and the “refractive index” \( n_* = n(\sigma_*) \) defined in section 4 is what determines the QCD string tension (that is, the coefficient on the area law for the Wilson loops). It is straightforward to check that the glueball wavefunctions quickly become flat to the right of this radius in figure 3.

It is interesting to note that the supergravity approximation continues to be valid far to the right of \( \sigma_* \) in figure 3. Supergravity is valid if the string coupling \( e^\phi \) is small and the string frame curvature is much less than the string scale. The first condition can be arranged to hold as close to the singularity at \( z = z_0 \) as desired, simply by choosing \( g_s = e^{\phi_\infty} \) sufficiently small. The second condition means that curvature invariants such as \( R, R_{MNPQ}R^{MNPQ}, \) etc., computed in string frame, are much smaller than the appropriate power of \( 1/\alpha' \). This will be true if

\[
\left( \frac{B^2}{24} e^{-8\sigma} \right)^{1-\sqrt{3/32}} \ll \frac{L^2}{\alpha'} = \sqrt{\frac{g_{YM}^2 N}{\Lambda^2}}.
\] (32)

The line labeled “string scale curvatures” in figure 3 indicates the radius at which the \( \ll \) in (32) becomes approximate equality. The important point is not the peculiar exponent in (32), but rather the fact that for \( N \) and \( g_{YM}^2 N \) sufficiently large but \( g_{YM}^2 \) small, all the physics of glueballs and Wilson loops takes place in a region of the bulk spacetime where the supergravity approximation is good.

In section 5, we proposed to define the theory at a cutoff \( z = \epsilon \), corresponding to an energy \( 1/\epsilon \), as \( N = 4 \) deformed by a \( SO(6) \)-invariant scalar mass term \( m_X^2 \text{tr} \sum_i X_i^2 \). The cutoff served to give the massive bulk scalar field \( \phi_K \) dual to this operator a definite value, \( m_X \epsilon \), at \( z = \epsilon \). This cutoff can be removed, without altering the infrared physics, in the following way: hold \( m_X \) fixed, send \( \epsilon \) gradually to zero, and at each successively smaller value of \( \epsilon \) use the boundary conditions \( \phi = \phi_\infty \) and \( \partial_z \phi = 0 \) at \( z = \epsilon \).

The infrared physics we found in section 4 was confinement at a scale \( \Lambda \sim m_X \). The spectrum of massive excitations did not appear to be that of pure QCD, nor should it be expected to, since the adjoint scalars have a mass on the order of the QCD scale. The adjoint fermions also get a mass from radiative corrections which is probably also comparable to \( \Lambda \). Direct comparisons of this supergravity model or of the others that have been proposed with pure Yang-Mills on the lattice seem doomed by the presence of “glueballs” with charge under a global symmetry of the massive matter fields. It would be more relevant, and more interesting, to compare the supergravity results with some alternative analysis of a confining gauge theory coupled to adjoint matter fields with masses on the order of the confinement scale and some global flavor symmetry. The vexing Kaluza-Klein glueballs then have a sensible interpretation as flavored hadrons.
Note Added

When this work was near completion, we received [22], in which the solution described in section 3 was independently obtained and some properties of Wilson loops and the running of the gauge coupling were discussed in the nearly anti-de Sitter region.

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References

[1] J. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* 2 (1998) 231, hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” *Phys. Lett.* B428 (1998) 105, hep-th/9802109.

[3] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* 2 (1998) 253, hep-th/9802150.

[4] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, “Novel local CFT and exact results on perturbations of N=4 superYang Mills from AdS dynamics,” hep-th/9810126.

[5] J. Distler and F. Zamora, “Nonsupersymmetric conformal field theories from stable anti-de Sitter spaces,” hep-th/9810206.

[6] A. Khavaev, K. Pilch, and N. P. Warner, “New vacua of gauged N=8 supergravity in five-dimensions,” hep-th/9812035.

[7] B. de Wit and H. Nicolai, “The consistency of the $S^7$ truncation in $d = 11$ supergravity,” *Nucl. Phys.* B281 (1987) 211.

[8] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*. Cambridge University Press, Cambridge, 1987.

[9] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, “Coupling constant dependence in the thermodynamics of N=4 supersymmetric Yang-Mills theory,” *Nucl. Phys.* B534 (1998) 202, hep-th/9805156.

[10] I. R. Klebanov and A. A. Tseytlin, “Asymptotic freedom and infrared behavior in the type 0 string approach to gauge theory,” hep-th/9812089.

[11] J. A. Minahan, “Asymptotic freedom and confinement from type 0 string theory,” hep-th/9902074.

[12] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* 2 (1998) 505, hep-th/9803131.

[13] J. Maldacena, “Wilson loops in large N field theories,” *Phys. Rev. Lett.* 80 (1998) 4859, hep-th/9803002.

[14] S.-J. Rey and J. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” hep-th/9803001.
[15] C. Csaki, H. Ooguri, Y. Oz, and J. Terning, “Glueball mass spectrum from supergravity,” hep-th/9806021.

[16] H. Ooguri, H. Robins, and J. Tannenhauser, “Glueballs and their Kaluza-Klein cousins,” Phys. Lett. B437 (1998) 77, hep-th/9806171.

[17] V. Balasubramanian, P. Kraus, and A. Lawrence, “Bulk vs. boundary dynamics in anti-de Sitter space-time,” Phys. Rev. D59 (1999) 046003, hep-th/9805171.

[18] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, “The mass spectrum of chiral $N = 2$ $d = 10$ supergravity on $S^5$,” Phys. Rev. D32 (1985) 389.

[19] S. S. Gubser and A. Hashimoto, “Exact absorption probabilities for the D3-brane,” hep-th/9805140.

[20] I. R. Klebanov, “World volume approach to absorption by nondilatonic branes,” Nucl. Phys. B496 (1997) 231, hep-th/9702076.

[21] S. S. Gubser, A. Hashimoto, I. R. Klebanov, and M. Krasnitz, “Scalar absorption and the breaking of the world volume conformal invariance,” Nucl. Phys. B526 (1998) 393, hep-th/9803023.

[22] A. Kehagias and K. Sfetsos, “On running couplings in gauge theories from type IIB supergravity,” hep-th/9902125.