CMB temperature polarization correlation and primordial gravitational waves

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ABSTRACT

We examine the use of the TE cross-correlation power spectrum of the cosmic microwave background (CMB) as a complementary test to detect primordial gravitational waves (PGWs). The first method used is based on the determination of the lowest multipole, ℓ0, where the TE power spectrum, CTEℓ, first changes sign. The second method uses Wiener filtering on the CMB TE data to remove the density perturbations contribution to the TE power spectrum. In principle this leaves only the contribution of PGWs. We examine two toy experiments (one ideal and another more realistic) to see their ability to constrain PGWs using the TE power spectrum alone. We found that an ideal experiment, one limited only by cosmic variance, can detect PGWs with a ratio of tensor to scalar metric perturbation power spectra r = 0.3 at 99.9 per cent confidence level using only the TE correlation. This value is comparable with current constraints obtained by the Wilkinson Microwave Anisotropy Probe based on the 2σ upper limits to the B-mode amplitude. We demonstrate that to measure PGWs by their contribution to the TE cross-correlation power spectrum in a realistic ground-based experiment when real instrumental noise is taken into account, the tensor-to-scalar ratio, r, should be approximately three times larger.

Key words: gravitational waves – polarization – cosmic microwave background – cosmological parameters.

1 INTRODUCTION

Primordial gravitational waves (PGWs) polarize the cosmic microwave background (CMB) (see e.g. Basko & Polnarev 1980; Polnarev 1985; Crittenden, Davis & Steinhardt 1993; Frewin, Polnarev & Coles 1994; Coles, Frewin & Polnarev 1995; Kamionkowski, Kosowsky & Stebbins 1997; Seljak 1997; Seljak & Zaldarriaga 1997; Kamionkowski & Kosowsky 1998; Baskaran, Grishchuk & Polnarev 2006; Keating et al. 2006). Current experiments are using the polarization of the CMB to search for this PGW background (Bowden et al. 2004; Taylor et al. 2004; Yoon et al. 2006). This polarization can be used as a direct test of inflation. An alternative probe of the inflationary epoch which does not use the PGW background was studied by Spergel & Zaldarriaga (1997). This probe was used in recent analyses by the Wilkinson Microwave Anisotropy Probe (WMAP) team (Peiris et al. 2003) to provide plausibility for the inflationary paradigm. This paper presents a test similar in spirit to that of Spergel & Zaldarriaga (1997).

CMB polarization can be separated into two independent components: E-mode (grad) polarization and B-mode (curl) polarization. B-mode polarization can only be generated by PGWs (see e.g. Seljak 1997; Seljak & Zaldarriaga 1997; Kamionkowski & Kosowsky 1998), therefore most CMB polarization experiments which are searching for evidence of PGWs focus on measuring the BB power spectrum. However, the TE cross-correlation power spectrum offers another method to detect PGWs (Crittenden, Coulson & Turok 1995). The TE power spectrum is two orders of magnitude larger than the BB power spectrum and it was suggested that it may therefore be easier to detect gravitational waves in the TE power spectrum (Bowden et al. 2004; Grishchuk 2007).

In this paper we first discuss the method of detection of PGWs by measuring the TE power spectrum for low ℓ. This method, originally proposed in Baskaran et al. (2006), is based on a measurement of ℓ0, the multipole where the TE power spectrum first changes sign. Hereafter we will call this method ‘the zero-multipole method’. The TE power spectrum due to density perturbations is positive on large scales, corresponding to ℓ < ℓ0, changes sign at ℓ = ℓ0, and then oscillates for ℓ > ℓ0, while for PGWs the TE power spectrum must be negative for small ℓ and then also oscillates for larger ℓ. The current best set of cosmological parameters, obtained in Spergel et al. (2007), gives, in the absence of PGWs, ℓ0 = 53. Therefore, the measurement of the difference between the multipole number, ℓ0, where the TE power spectrum changes sign, and ℓ = 53 is the way to detect PGWs. We will then consider an alternative method...
based on Wiener filtering, removing the contribution to the TE power spectrum due to density perturbations. Since the TE power spectrum due to PGWs is negative on large scales a test of negativity of the resulting TE power spectrum is a test of PGWs. In this paper, we present an analysis of both of these methods, based on Monte Carlo simulations.

At the present time, the main priority and the main challenge in CMB polarization observations is the detection of the PGW background via the BB power spectrum. In connection with BB experiments, the methods based on the TE cross-correlation can be considered as very useful auxiliary measurements of PGWs because systematic effects in TE measurements are not degenerate with those in BB measurements. For example, T/B leakage or even E/B leakage could swamp a detection of BB, whereas T/E leakage would be small and well controlled (see Shimon et al. 2007). These BB systematics could falsely indicate a detection of PGWs, but measurements of the TE power spectrum provide insurance against such a spurious detection. Additionally, galactic foreground contamination affects BB and TE in different ways, which enables us to perform powerful cross-checks and subtraction of foregrounds in BB measurements.

Another advantage of TE measurements for experiments, which measure a small fraction of the sky, is related to the fact that a significant contaminant to the B modes is caused by E/B mixing. This limits the power spectrum of PGWs that can be detected (Challinor & Chon 2005). The E modes are practically unaffected by E/B mixing so, in contrast to the BB measurements, the TE power spectrum should be nearly the same for both full and partial sky measurements.

The plan of this paper is the following. In Section 2, we introduce the primordial power spectra of scalar (density) and tensor (PGW) perturbations (Section 2.1). Then following Crittenden et al. (1995) and Baskaran et al. (2006), we explain why the sign of the TE power spectra for scalar and tensor perturbations is opposite for large scales (Section 2.2). In Section 3, we describe in more detail the zero-multipole method for the detection of PGWs. In Section 4, we describe the method for detection of PGWs based on Wiener filtering along with the statistical tests used and a comparison of the tests. In Section 5, we present results of numerical Monte Carlo simulations for two toy experiments. In the first toy experiment we neglect instrumental noise and the uncertainties are limited only by cosmic variance (Section 5.1). In the second toy experiment, along with cosmic variance, we take into account instrumental noise which is comparable to real noise in current ground experiments (Section 5.2). For comparison, we also present results of simulations for the two satellite experiments, WMAP (Section 5.3) and Planck (Section 5.4). In Section 6, we compare the signal-to-noise ratio (S/N) of the TE measurements with those of BB measurements.

### 2 TE CROSS-CORRELATION

The power spectrum of TE correlations is determined by primordial power spectra of scalar and tensor perturbations and time evolution of these perturbations during the epoch of recombination.

#### 2.1 Primordial power spectra

The primordial power spectra describing the initial scalar (density) perturbations (denoted by $s$) and tensor (PGW) perturbations (denoted by $t$) are (see e.g. Spergel et al. 2007)

$$P_s(k) = A_s \left( \frac{k}{k_0} \right)^{n_s},$$

$$P_t(k) = A_t \left( \frac{k}{k_0} \right)^{n_t},$$

where $k_0 = 0.002 \text{ Mpc}^{-1}$, this value of $k_0$ is obtained by fitting of CMB data (Smith, Kamionkowski & Cooray 2006). The variables $n_s$ and $n_t$ are the scalar and tensor spectral indices, respectively. The variable $\alpha_s$ is the running of the scalar spectral index. In terms of $A_s$ and $A_t$, the tensor-to-scalar ratio, $r$, is

$$r \equiv \frac{A_t}{A_s} = \frac{P_t(k_0)}{P_s(k_0)}.$$  \hspace{1cm} (2)

The location of $k_0$ is determined by the parameters $n_s$ and $r$. In this paper, we do not specify particular cosmological models considering the generation of primordial spectra, $P_s(k)$ and $P_t(k)$, which means that for our purposes we consider $n_s$, $n_t$, and $r$ as independent parameters. This is not true if we use some particular cosmological model. For example, in standard inflation models, the parameters $n_s$ and $r$ are related by the consistency relation, $n_t = -r/8$ (see e.g. Peiris et al. 2003). In other words, we consider all parameters $n_s$, $n_t$, and $r$ as independent except in Sections 5.1 and 5.2, where along with model-independent we give also model-dependent constraints on $r$.

#### 2.2 Opposite signs of scalar and tensor perturbations to TE correlation

Taking into account that scalar and tensor perturbations are not correlated, the TE power spectrum is simply a sum of two TE power spectra for scalar and tensor perturbations correspondingly.

First, the physical motivation for the difference in the cross-correlation contributions produced by scalar and tensor perturbations for small $\ell$ was demonstrated and physically interpreted for the cross-correlation of the Stokes parameters $T$ and $Q$ in Crittenden et al. (1995). For scalar perturbations the Stokes parameter $Q$ contains only E modes, hence the TE correlation is identical with the TQ correlation and is positive for small $\ell$. As was then emphasized in Baskaran et al. (2006), the sign of the TE correlation for tensor perturbations is negative for small $\ell$. The simple qualitative physical interpretation of the fact that the contributions of the TE correlation are different for scalar and tensor perturbations is the following. For both scalar and tensor perturbations, the temperature fluctuations, $T(\ell)$, for small $\ell$ [when oscillations of $T(\ell)$ are absent] are proportional to the metric perturbations $h$ at the moment of recombination, while the E-mode fluctuations, $E(\ell)$, are proportional to $h$ at the moment of recombination. Hence, the TE correlation is proportional to $h^2 \propto d(h^2)/dh$. Taking into account the growth of scalar perturbations and tensor perturbations decay, one can see that the contributions to the TE correlation for scalar and tensor perturbations are opposite.

To understand this in more detail, following Baskaran et al. (2006), we consider the multipole expansion of the TE cross-correlation with coefficients $C_{\ell}^{TE}$. These coefficients are related to the spherical harmonic expansion coefficients of the temperature anisotropy and polarization by

$$C_{\ell}^{TE} = \left\langle a_{T,\ell m} a^{*}_{E,\ell m} \right\rangle,$$  \hspace{1cm} (3)

where the brackets denote averaging over all possible statistical realizations. The statistical properties of the CMB field in general, and the TE cross-correlation specifically, follow from the statistical
properties of the underlying scalar or tensor metric field. Assuming Gaussianity together with statistical isotropy and homogeneity, the TE cross-correlation takes the form

\[
C^\text{TE}_\ell = \int \frac{dk}{k} a_{\ell,\ell}(k) a_{0,0}(k),
\]

where \( a_{\ell,\ell}(k) \) is the contribution from temperature perturbation while \( a_{0,0}(k) \) is the contribution from E polarization. The integration over \( k \) takes into account the contribution from all the possible wavenumbers.

It was shown in Baskaran et al. (2006)

\[
\begin{align*}
\alpha_{\text{TE}}(n) &\sim h_n(\eta)\bigg|_{\eta=\eta_{\text{rec}}} , \\
\alpha_{\text{TE}}(n) &\sim \frac{dh_n(\eta)}{d\eta}\bigg|_{\eta=\eta_{\text{rec}}},
\end{align*}
\]

where \( h_n(\eta) \) is the mode function of the metric perturbation, and \( \eta_{\text{rec}} \) is the conformal time at recombination. It follows that the TE correlation is approximately

\[
C^\text{TE}_\ell \propto \int d\ell F(\ell, k) \left( \frac{dh_n^2(\eta)}{d\eta} \right) \bigg|_{\eta=\eta_{\text{rec}}},
\]

where \( F(\ell, k) \) is a strictly positive function which peaks at \( \ell \approx k(\eta_{\text{today}} - \eta_{\text{rec}}) \). Heuristically, the function \( F(\ell, k) \) projects the space on to the \( \ell \) space. Therefore the sign of the integral in the right-hand side (RHS) of equation (7) evaluated at around \( \ell \approx k(\eta_{\text{today}} - \eta_{\text{rec}}) \) determines the sign of \( C^\text{TE} \) on large scales.

The adiabatic decrease of the gravitational wave amplitude upon entering the Hubble radius is preceded by the monotonic decrease of the gravitational wave mode function \( h_n(\eta) \) as a function of \( \eta \). Since \( h_n(\eta) \) is decreasing the integral on the RHS of equation (7) is negative. The RHS of equation (7) is negative for \( k(\eta_{\text{today}} - \eta_{\text{rec}}) < 90 \) since \( h_n \) is decreasing over that range. Therefore, for \( \ell < 90 \) the correlation \( C^\text{TE} \) must be negative. For larger \( \ell \), the \( F(\ell, k) \) in equation (7) and, hence, the TE cross-correlation power spectrum changes sign as a function of \( \ell \).

Thus the TE cross-correlation, due to density perturbations, must be positive at lower \( \ell \) (as mentioned above, the TE cross-correlation in absence of PGWs changes sign at \( \ell_0 \approx 53 \)). If we were able to separate them we could use this signature for detection of PGWs. However, even without such separation the presence of PGWs manifests itself in the value of \( \ell_0 \), which is the smallest \( \ell \) where the total TE correlation power spectrum (scalar plus tensor) changes its sign. Thus, the sign of the TE correlation is a very prominent signature of PGWs. For this reason, in the next section, we investigate the dependence of \( \ell_0 \) on \( r, n_s, A_s \), and \( n_t \).

3 DEPENDENCE OF \( \ell_0 \) ON PARAMETERS OF PGW POWER SPECTRUM

The method of detecting PGWs which implies a calculation of \( \ell_0 \), where the TE power spectrum first goes to zero, will be called hereafter as the zero-multipoles method. We take into account uncertainties in determination of \( C_\ell \) values which are unavoidable in any experiment:

\[
\begin{align*}
\Delta C^\text{TE}_\ell &= \frac{1}{(2\ell + 1)f_{\text{sky}}} \left[ (C^\text{TE}_\ell)^2 \\
&+ (C^\text{TT}_\ell + N^\text{TT}_\ell)(C^\text{EE}_\ell + N^\text{EE}_\ell) \right],
\end{align*}
\]

(see e.g. Dodelson 2003). Even in an ideal experiment, when we neglect instrumental noise (\( N_t = 0 \)) and measure the full sky (\( f_{\text{sky}} = 1 \)), we still have uncertainties related with cosmic variance (which arises from the fact that we have only one realization of the sky in CMB measurements) (see e.g. Dodelson 2003). For a more realistic experiment, we take into account noise and partial sky coverage (see Section 5). Over small multipole bands it is reasonable to approximate the power spectrum as linear. In the range \( 70 \leq \ell \leq 70 \), it seems reasonable to use a linear approximation for \( (\ell + 1)C^\text{TE}_\ell/2\pi \). It seems unlikely that in this range any deviations from a linear approximation can be larger than mentioned above uncertainties.

Plots of \( C^\text{TE}_\ell \) for different values of \( r \) are shown in Fig. 1 plotted for \( n_t = 0 \). It can be seen that a linear fit to the TE power spectrum does well approximate \( (\ell + 1)C^\text{TE}_\ell/2\pi \) near \( \ell_0 \).

Thus near \( \ell_0 \), \( (\ell + 1)C^\text{TE}_\ell/2\pi \) can be approximated as a line with negative slope \( a - bt \), where \( a \) and \( b \) are positive real numbers. For any set of experimental data, we can find \( a \) and \( b \) by applying a least-squares fit. The values \( a \) and \( b \) correspond to the best fit obviously can be used for prediction of \( \ell_0 = a/b \). This value, \( \ell_0 \), can then be used to constrain the parameter \( \tau \) under some assumptions about spectral indices \( n_s \) and \( n_t \).

We need to investigate how \( \ell_0 \) depends on the cosmological parameters \( r, n_s, n_t, A_s, P_0(k_0) \), and the optical depth to reionization, \( \tau \). The value of \( \ell_0 \) for a standard \( \Lambda \)CDM cosmology described in Spergel et al. (2007) as a function of \( n_s \) and \( r \) is shown in Figs 2 and 3. All power spectra were generated with the code camb1 (Lewis, Challinor & Lasenby 2000). If \( r = 0, \ell_0 = 53 \), while if \( r = 0.3 \) (the WMAP3 upper limit on the tensor-to-scalar ratio) and \( n_s = 0 \), we find that \( \ell_0 = 49 \).

From Figs 2 and 3, one can see that \( \ell_0 \) decreases with increase of \( r \). This effect is more pronounced for smaller \( n_t \). For example, if \( r = 0.3 \), then \( \ell_0 = 52 \) for \( n_t = +0.5 \), \( \ell_0 = 49 \) for \( n_t = 0 \), and \( \ell_0 = 38 \) for \( n_t = -0.5 \). The fact that \( \ell \) is discrete (the plots are composed of a set of step functions) puts limitations on using this method for

1 See http://camb.info on web.
Harrison–Zel’dovich scale-free spectrum), \( \delta \ell \) is in the range where the TE power spectrum for scalar and tensor perturbations depend on \( \tau \) in the same way for instantaneous reionization histories to cause a change in \( \ell_0 \) as shown in Kaplinghat et al. (2003); however, we will assume instantaneous reionization for the purpose of this paper.

Thus, even if we cannot separate the contributions of scalar and tensor perturbations to the TE power spectra, PGWs still leave their imprint on the value of \( \ell_0 \). In the next section, we will consider the possibility of such separation with the help of Wiener filtering.

### 4 Wiener Filtering of the TE Cross-Correlation Power Spectrum

Wiener filtering has been used often in the case of CMB data analysis. For example, it was used to combine multifrequency data in order to remove foregrounds and extract the CMB signal from the observed data (Tegmark & Efstathiou 1996; Bouchet, Prunet & Sethi 1999). Here we examine the use of the Wiener filter to subtract the PGW signal from the total TE correlation signal. This is done because the Wiener filter reduces the contribution of noise in a total signal by comparison with an estimation of the desired noiseless signal (Vaseghi 2006). In our case, the signal is the one due to PGWs only, and the signal contributed by density perturbations is considered to be ‘noise’.

The observed signal can be written as

\[
C_{\ell}^{\text{TE}} = C_{\ell}^{\text{TE}_{\text{PGW}}} + C_{\ell}^{\text{TE}_{\text{PGW}}},
\]

where ‘s’ and ‘t’ refer to the contributions to the power spectrum due to scalar and tensor perturbations, respectively. The values \( C_{\ell}^{\text{TE}_{\text{PGW}}} \) and \( C_{\ell}^{\text{TE}_{\text{PGW}}} \) refer to the spherical harmonic coefficients of the temperature and polarization maps. In our application to TE correlation, we consider the Wiener filter, \( W_{\ell}^{\text{TE}} \):

\[
W_{\ell}^{\text{TE}} = \frac{C_{\ell}^{\text{TE}_{\text{PGW}}}}{C_{\ell}^{\text{TE}_{\text{PGW}}}},
\]

The filtered signal, \( \alpha'_{\ell} \) (for \( X = T \) and \( E \)), is obtained from the measured signal, \( \alpha_{\ell} \), as

\[
\alpha'_{\ell} = \alpha_{\ell} W_{\ell}^{1/2}
\]

In this paper, we assume the Wiener filter is perfect, in the sense that it leaves the signal due to PGWs only. We then get, for the filtered multipoles \( C_{\ell}^{\text{TE}_{\text{filtered}}} \):

\[
C_{\ell}^{\text{TE}_{\text{filtered}}} = \frac{\langle \alpha'_{\ell a}\alpha'_{\ell m} \rangle}{C_{\ell}^{\text{TE}}}
\]

\[
W_{\ell}^{\text{TE}} C_{\ell}^{\text{TE}} = C_{\ell}^{\text{TE}_{\text{filtered}}}
\]

In practice this is not true, because we are trying to determine \( C_{\ell}^{\text{TE}_{\text{filtered}}} \), which is not known in advance. Nevertheless, the assumption that the Wiener filter is perfect is good as a first approximation and illustrates the detectability of PGWs with the help of TE correlation measurements.

The filtering can reduce the measured signal to the desired signal, but, since we are trying to remove the density perturbations and not the actual noise, we can not reduce the measurement uncertainties. These uncertainties in \( C_{\ell}^{\text{TE}_{\text{filtered}}} \) are then entirely determined by the noise in the original signal.

We have shown that the TE power spectrum due to PGWs is negative on large scales, hence a test determining whether the Wiener filtered power spectrum is negative or not is a probe of PGWs.

There are three different statistical tests we use to see if we can measure a negative TE power spectrum. The first test is a Monte Carlo simulation to determine S/N (Section 4.1). The other...
two tests are standard non-parametric statistical tests: the sign test (Section 4.2) and the Wilcoxon rank sum test (Wilcoxon 1945) (Section 4.3).

For all of our tests, we calculate a random variable. If the data satisfy the hypothesis that \( r = 0 \), we can calculate the mean and uncertainty in the variables. If we make one realization of data, the random variable is determined from its distribution. Because we are not using any real observational data, we must run a Monte Carlo simulation to reduce the risk of randomly getting a value for the variable taken from the outlying area of its distribution. To do this, the filtered multipoles, \( C_{\ell,i}^{\text{TE}} \), are randomly chosen from a Gaussian distribution with mean \( C_{\ell,i}^{\text{TE}} \) and standard deviation \( \Delta C_{\ell,i}^{\text{TE}} \), where

\[
\left( \Delta C_{\ell,i}^{\text{TE}} \right)^2 = \frac{1}{(2\ell + 1) f_{\text{sky}}} \left( C_{\ell,i}^{\text{TE}} \right)^2 + \left( C_{\ell,i}^{\text{TE}} + N_{\ell,i}^{\text{TE}} \right) \left( C_{\ell,i}^{\text{TE}} + N_{\ell,i}^{\text{EE}} \right)
\]

(13)

(see e.g. Dodelson 2003), the variable \( f_{\text{sky}} \) refers to the fraction of the sky covered by observations and \( N_{\ell,i} \) is the effective power spectrum of the instrumental noise (see Dodelson 2003 for details on how \( N_{\ell,i} \) is related to actual instrumental noise).

Our determination of \( C_{\ell,i}^{\text{TE}} \) is dependent on \( \ell \). However, for two of our tests we ignore the value of \( \ell \) in the calculation of the random variable. We assume that the calculated random variable is Gaussian. In order for this to work, the random variable must be calculated from Gaussian variables. The errors on the multipoles for the ‘ideal’ toy experiment are large enough so that we can assume the multipoles are taken from a single distribution and not from a distribution that depends on \( \ell \).

### 4.1 Monte Carlo S/N test

For this test, the random variable we calculate, \( S/N \), is defined as

\[
S/N = \sum_{\ell = 2}^{53} \frac{C_{\ell,i}^{\text{TE}}}{\Delta C_{\ell,i}^{\text{TE}}}
\]

(14)

The reason why the sum in this equation is taken in the range \( 2 < \ell < 53 \) is because only in this range \( \text{sgn} (C_{\ell}^{\text{TE}} \text{ (scalar)}) = -\text{sgn}C_{\ell}^{\text{EE}} \text{ (tensor)}) \). In other words, if we include higher multipoles we confront with a danger of a false detection, because the total TE power spectrum is negative for \( \ell > 53 \).

The value of \( S/N \) is Gaussian distributed because it is a sum of many modes of squares of Gaussian distributed values, \( C_{\ell,i} = a_{\ell,i}^{g,m} \). We approximate each \( C_{\ell,i}^{\text{TE}} \) as being Gaussian distributed for the purpose of this paper. For each set of parameters we run this simulation one million times to determine the mean, \( \langle S/N \rangle \), and standard deviation, \( \sigma_{S/N} \). The mean of this distribution is determined by the pre-assumed value of \( r \), while the standard deviation is determined by parameters of the experiment and gives the confidence level of detection. We run such Monte Carlo simulations for different values of \( r \) to determine in what range of \( r \) we can detect PGWs. When then using real observational data, we can compare the actual value of \( S/N \) with the results of Monte Carlo simulations to infer the likelihood, as function of \( r \), which determines the probability that \( r \neq 0 \), or that PGWs exist at detectable levels.

### 4.2 Sign test

The sign test is a test of compatibility of observational data with the hypothesis that \( r = 0 \). If we do have \( r = 0 \), then \( C_{\ell,i}^{\text{TE}} \) will be equally distributed around zero. Application of this test to the filtered data is very simple. In practice, all observational data are distributed between several bins and the averaging of the signal is produced in each bin separately. Let \( N_{\text{bin}} \) be the number of such bins. The sign test actually gives the probability that in \( N_{\text{bin}} \) bins the average is negative and in \( N_{\text{bin}} - N_{\text{pos}} \) it is positive, if \( r = 0 \). This probability, \( P \), is given by the binomial distribution

\[
P(N_{\text{pos}}) = \left( \begin{array}{c} N_{\text{bin}} \\ N_{\text{pos}} \end{array} \right) 0.5^{N_{\text{bin}}} = \frac{N_{\text{bin}}!}{N_{\text{pos}}! N_{\text{num}}!}
\]

(15)

The probability that the hypothesis \( r = 0 \) is wrong is

\[
P(r \neq 0) \approx 1 - 2 \sum_{i=0}^{N_{\text{bin}}} \frac{N_{\text{pos}}!}{N_{\text{pos}}! N_{\text{num}}!}
\]

(16)

The value \( \sum_{i=0}^{N_{\text{bin}}} P(i) \) is the probability that we would get \( \leq N_{\text{pos}} \) positive values given \( r = 0 \). This is the same as the probability of getting \( \leq N_{\text{num}} \) negative values given \( r = 0 \). Therefore our confidence that \( r \neq 0 \) is just 100 per cent minus the sum of the probabilities describe above (the probability that the \( N_{\text{pos}} \) is closer to the mean, \( N_{\text{bin}}/2 \), if \( r = 0 \)). This equation only makes sense if \( N_{\text{pos}} \leq N_{\text{bin}}/2 \), since that is required for \( r > 0 \). If \( N_{\text{pos}} > N_{\text{bin}}/2 \), that would imply \( r < 0 \), which is not physical. We would have to interpret the result as a random realization of \( r \geq 0 \), with the most likely result of \( r = 0 \). Therefore we would not be able to say \( r \neq 0 \) with any confidence.

Let us consider the following example: we put all measurements of \( C_{\ell,i}^{\text{TE}} \) into 11 bins and in three of them the average is positive. In this example, the probability that the hypothesis \( r = 0 \) is wrong is equal to 89 per cent.

One possible drawback of this method is that it does not take into account any measure of the S/N of individual measurements. As we show in Section 4.4, it is possible to have two completely different sets of data with the same probability of having \( r = 0 \). This test is also unable to make any prediction as to the value of \( r \), only that it differs from zero.

### 4.3 Wilcoxon rank sum test

This statistical test deals with two sets of data. The first set of data is taken from a real experiment which measures \( C_{\ell,i}^{\text{TE}} \) with some unknown \( r \). The second set of data is generated by Monte Carlo simulations (see Section 4.1) with \( r = 0 \). The objective of the Wilcoxon rank sum test is to give the probability that the hypothesis \( r = 0 \) is wrong (Wilcoxon 1945).

First, we choose some random variable \( U \), whose probability distribution is known if \( r = 0 \). For that, let us combine all data from first set with \( n_1 \) multipoles and second set with \( n_2 \) multipoles into one large data set, which obviously contains \( n_1 + n_2 \) multipoles. Then, we rank all multipoles in the large data set from 1 to \( n_1 + n_2 \) according to their amplitude (rank 1 for the smallest and rank \( n_1 + n_2 \) for the largest). Now, the variables \( R_1 \) and \( R_2 \) are defined as the sum of the ranks for the first original data set and the second original data set, correspondingly. Finally, the variable \( U \) is

\[
U = \min(U_1, U_2), \quad \text{where} \quad U_i = R_i - n_i(n_i + 1)/2, \quad i = 1, 2.
\]

(17)

If all multipoles of the first data set are larger than all multipoles of the second data set, then \( U_1 = n_1 n_2 \) and \( U_2 = 0 \). It is not difficult to show that \( U_1 + U_2 = n_1 n_2 \). If both sets of measurements have no evidence for PGWs, \( \langle U_1 \rangle = \langle U_2 \rangle \). It is also simple to see that \( U_1 + U_2 = n_1 n_2 \).
It is important to emphasize that the ranks of multipoles are random variables because all multipoles themselves are random variables, hence $U_1$, $U_2$, and $U$ are random variables. If $n_1 + n_2$ is large, the distribution of $U$ can be approximated as a Gaussian with a known mean and standard deviation. In this approximation we have

$$m_U = n_1 n_2 / 2,$$

(18)

$$\sigma_U = \sqrt{n_1 n_2 (n_1 + n_2 + 1) / 12}.$$  

(19)

In some cases, instead of $U$, the variables $R_1$ or $R_2$ are used. The reason $U$ is used here is because $m_U$ is symmetric in the data sets. If $r = 0$ in both sets of data, then the distributions of $U_1$ and $U_2$ are the same, no matter what $n_1$ and $n_2$ are. The distributions of $R_1$ and $R_2$ would be the same only if $n_1 = n_2$. The probability that the first data set corresponds to $r \neq 0$ obtained from the test in which $R_1$ or $R_2$ is used is the same as if $U$ is used.

Since this test requires Monte Carlo simulations for the second set of data, we ran this test many times for many different data sets to get an accurate mean value for $U$.

To reject the hypothesis $r = 0$ means to detect PGWs. Using the Wilcoxon rank sum test the allowable value of $r$ is determined, if instead of comparing with simulated data with $r = 0$, we compare with simulated data with $r = r_0 \neq 0$. In order to get a range of allowable values for $r$, we need to run multiple Monte Carlo simulations with multiple values for $r_0$. This is where the assumption that the $C_{\ell,i}$ are from a random distribution that is independent of $\ell$ is used. This implies that the ranks are random variables. If the errors on the $C_{\ell,i}$ are small enough, then the ranks will be predetermined. Therefore, our assumption about the distribution of $U$ will not be true and the test would have to be modified. Fortunately, this is not the case for even an experiment only limited by cosmic variance.

To illustrate how this test works, let us consider the following example. Assume there are four multipoles in the first set of data and consider that $r = 0.3$ is the correct value. There are also four multipoles in the second set of data (which for sure corresponds to $r = 0$). All quantities below are expressed in $\mu$ K$^2$. The value for the first data set are $C_{10}^{\text{TE}} = -0.005$, $C_{20}^{\text{TE}} = 0.02$, $C_{30}^{\text{TE}} = -0.015$ and $C_{40}^{\text{TE}} = -0.01$. The values for the second data set are $C_{10}^{\text{TE}} = 0.03$, $C_{20}^{\text{TE}} = 0.003$, $C_{30}^{\text{TE}} = -0.002$ and $C_{40}^{\text{TE}} = -0.003$. A ranking of multipoles gives the ordering from lowest to highest, with 1 referring to the first data set and 2 referring to the second data set, as $21112212$. This results in $R_1 = 2 + 3 + 4 + 7 = 16$, $U_1 = 16 - 10 = 6$ and $U_2 = 16 - 6 = 10$. Therefore $U = \min \{10, 6\} = 6$. For $n_1 = n_2 = 4$, to reject the hypothesis that $r = 0$ at 95 per cent confidence level, $U_1$ should be less than 1 (see e.g. Lehmann 1975). In this example, since $U_1 = 6 > 1$, the first set of data cannot be considered as a detection of PGWs.

### 4.4 Comparison of tests

The $S/N$ test is greatly affected by outlying measurements. A measurement of one large negative multipole could falsely imply a detection. Both the sign test and the Wilcoxon rank sum test are not affected by individual outlying measurements. In the sign test, the value of individual measurements is irrelevant, because the test is sensitive only to the sign of individual measurements. The Wilcoxon rank sum test is affected by outliers, but considerably less than the $S/N$ test. If the outlier is larger (or smaller) than every other multipole, its rank does not depend on its particular value.

If we have two completely different sets of data, the main disadvantage of the sign test, as mentioned in Section 4.2, is that it could give the same result, while for the two other tests the chance to obtain the same value of $r$ is negligible. For example, one set of data, consisting of four small negative multipoles and four large positive multipoles, gives the same result as another set of data, consisting of four large negative multipoles and four small positive multipoles. The $S/N$ test gives two very different values of $S/N$ for these two sets of data. We can also use the Wilcoxon rank sum test to compare these two sets of data. In this case $U = 16 = (1/2)m_U$, which corresponds to a confidence level of hypothesis that $r = 0$ of less than 10 per cent.

With observational data, the sign test can be applied and does not require any Monte Carlo simulations (which could be considered as an advantage of this test). The $S/N$ test requires Monte Carlo simulations, but only for the distribution of the random variable $S/N$. The Wilcoxon rank sum test requires large Monte Carlo simulations and combines the data sets generated by these simulations with observational data. In other words, Monte Carlo simulations are absolutely necessary after obtaining observational data, which may be considered a disadvantage of this test. Thus, each of the three tests has advantages and disadvantages, suggesting that the best way to work out observational data is to apply all these three tests.

### 5 Discussion and Results

Baskaran et al. (2006) used equal amplitudes of scalar and tensor perturbations to sharpen the discussion in their plots. They defined the tensor-to-scalar ratio, $R$, as the ratio of the temperature quadrupoles, $R = C_{TT}^{TE}/C_{TT}^{TE}$ and set $R = 1$. Using standard WMAP3 cosmological parameters (Spergel et al. 2007), the definition of tensor-to-scalar ratio, $r$, used in this paper is approximately twice as large as their definition of $R$. The exact relationship between $r$ and $R$ will depend on the cosmological parameters used. This means that $R = 1$ is equivalent to $r \approx 2$, which has currently been strongly ruled out by WMAP in combination with previous experiments (Spergel et al. 2007). We need to see if this method can detect a value of $r$ that is currently within the limits. We assume that there is no foreground contamination. In reality foregrounds affect the measured location of $\ell_0$ (we will consider the effects of foregrounds on $\ell_0$ elsewhere). For the experiments that do not observe the full sky, correlations between multipoles must be taken into account. The multipoles are binned together of such width that the correlations between the bins are sufficiently small.

Two different toy experiments, along with the two satellite experiments WMAP and Planck, are considered to constrain $r$. The first toy experiment is a full sky experiment. It is idealized in two aspects. The first idealization is that we can take measurements over the full sky while the second idealization is that we assume there is no detector noise. The only uncertainty is due to cosmic variance. Such experiment represents the best limit to which the gravitational waves can be detected by the CMB TE correlation. This toy experiment is close to a space-based experiment with access to the full sky. It is similar to what the Beyond Einstein inflation probe would be able to detect. This toy experiment will be hereafter referred to as the ideal experiment. The second toy experiment is a more realistic one. In this experiment, measurements are on 3 per cent of the sky, the frequency is 100 GHz, and the duration of the experiment is 3 yr. The noise in each detector of the 50 polarization sensitive bolometer pairs can be described by their noise equivalent temperature of 450 $\mu$K$/\sqrt{\ell}$. The detectors’ beam profile is assumed to be Gaussian and it is described by the full width at half of the maximum sensitivity, abbreviated as FWHM of 0.85.

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contribution of PGWs with shows the contribution to the TE mode of density perturbations, in bins of width $\Delta \ell / \ell$ with an uncertainty of $r_0$, then we measure $A$ plot of TE for $r = 0.3$ and $n_t = 0$, with an error bars for the ideal experiment binned in intervals of $\Delta \ell = 10$.

This second toy experiment is similar to current ground-based experiments and the constraints from this experiment represent those that can and will be obtained in the next several years. This will be referred to as the realistic experiment.

The predicted errors for Planck are based on using the 100-, 143- and 217-GHz channel in the High Frequency Instrument. The numbers are gotten from the Planck science case, the 'bluebook'.

The WMAP noise was obtained by 3 yr of observations of the $Q, V$ and $W$-band detectors.

5.1 Ideal experiment

A plot of TE for $r = 0.3$ and $n_t = 0$ with error bars for $\ell$ binned in bins of width $\Delta \ell = 10$ is shown in Fig. 4. This figure separately shows the contribution to the TE mode of density perturbations, contribution of PGWs with $r = 0.3$, when the TE power spectrum due to density perturbations is approximately five times larger than the power spectrum due to PGWs at $\ell < \ell_0$.

The Monte Carlo simulation for the calculation of $\ell_0$, with an input model of $r = 0.3$ and $n_t = 0$, results in the value of $\ell_0 \approx 49$ with an uncertainty of $\Delta \ell_0 \approx 1.3$. A contour plot of the limits on the resulting measurement of $r$ is shown in Fig. 5. The white is the allowed region for $r$ and $n_t$ that falls within the $1\sigma$ errors of $\ell_0$. The black is the region forbidden with 68 per cent confidence. If $n_t = 0$, then we measure $r \approx 0.3 \pm 0.1$. If we consider the inflationary

$\text{Figure 4.}$ The black line is the total TE mode with a $r = 0.3$. The red line is the contribution from PGWs only while the light blue line is the contribution from the density perturbations. Blue is the error bars for the ideal experiment binned in intervals of $\Delta \ell = 10$.

$\text{Figure 5.}$ This is a plot of the allowed $r$ and $n_t$ for the $1\sigma$ region of $\ell_0$ for the ideal experiment. The white is the $1\sigma$ region while the black is the forbidden region.

$\text{Figure 6.}$ The S/N for the zero-multipole method are shown as the solid black, for ideal experiment, and dashed blue, for realistic experiment, lines. The S/N for realistic measurements of the BB power spectrum is shown as the dash–dotted red curve. For all curves $n_t = 0$.

$\text{Figure 7.}$ This is a plot of the distribution of the number of positive multipoles for the Monte Carlo simulation for the ideal experiment (upper left-hand panel), the realistic experiment (upper right-hand panel), Planck (lower left-hand panel) and WMAP (lower right-hand panel). The dotted red line shows where $N_+ = (1/2)N_{\text{bins}}$.

consistency relation, $n_t = -r/8$ (Peiris et al. 2003), we then get the constraint $r = 0.3 \pm 0.09$. The uncertainty is smaller, but not by much. We predict a $3\sigma$ detection of PGWs by the zero-multipole method.

The detectability of $\ell_0$ using the ideal experiment is shown in Fig. 6. For $n_t = 0$ the effective number of $\sigma$ detection is $\sigma \approx 10r$. We make this approximation by determining the detectability for several values of $r$ and then approximating a line. For comparison the results are also shown for the zero-multipole method with the realistic experiment and for measurements of the BB power spectrum with the realistic experiment described above. We assume we can make measurements over a range of 60 multipoles for BB measurements.

The Monte Carlo simulation for the Wiener filtering gives an average of 19 measured TE power spectrum multipoles greater than zero out of a total of 52 independent multipoles. If the null hypothesis was true, the sign test would indicate there is a 3.5 per cent chance of measuring $\leq 19$ positive multipoles. This is equivalent to a $\approx 1.8\sigma$ detection. A plot of the distribution of the number of positive multipoles is shown in the upper panel plot of Fig. 7. There is an 81 per cent chance for the observed $N_+$ to give a $1\sigma$ detection of PGWs.

The S/N test gives a mean value of $S/N = -17.1$ and standard deviation of $7.21$. The upper left-hand panel in Fig. 8 shows the
The distribution of $S$ shows where $S/N = 0$.

Figure 8. The $S/N$ statistic distribution for the ideal experiment (upper left-hand panel), realistic experiment (upper right-hand panel), Planck (lower left-hand panel) and WMAP (lower right-hand panel). The dotted red line shows where $S/N = 0$.

Figure 9. This is a plot of $S/N$ and $\sigma_{S/N}$ as a function of $r$ for the ideal experiment. The black line is $S/N$ and the red line is $\sigma_{S/N}$.

distribution of the $S/N$ values for the Monte Carlo simulation with $r = 0.3$. If $r = 0.3$ we would have a 0.8 per cent probability of the measured $S/N > 0$. This negative value signifies that a non-zero tensor-to-scalar ratio produced an anticorrelation. We can assume that the standard deviation would be the same if the mean of $S/N$ was 0 (equivalent to $r = 0$), because it is equivalent to adding a constant value to every measured value (and hence adding a constant to $S/N$ which would not change the error). Therefore, if $r = 0$, the probability of getting $S/N < -17.4$ is 0.8 per cent, and hence we have a 99 per cent chance that $r \neq 0$. A plot of $\langle S/N \rangle$ and $\sigma_{S/N}$ as a function of $r$ is shown in Fig. 9. As can be seen from the plot, we can predict a value of $r$ for any value of $S/N$. The value of $\sigma_{S/N}$ is a relatively constant function of $r$ and so our prediction about the distribution of $S/N$ for different values of $r$ is a good approximation to the true distribution.

The Wilcoxon rank sum test gives $U_{\text{avg}} - m_U = -1.23\sigma_U$. The variable $U_{\text{avg}}$ is the mean value for $U$ in the Monte Carlo simulations described earlier. The values $m_U$ and $\sigma_U$ are given in Section 4.3. The distribution of $U$ for the Monte Carlo simulations with $r = 0.3$ is shown in Fig. 10. The standard deviation of the distribution of measured $U$ is the same as the standard deviation of the distribution of $U$ assuming the hypothesis that $r = 0$. The only difference between the distributions is that $m_U$ is shifted by a constant value. Therefore, there is a 22 per cent chance that $U - m_U < -2\sigma_U$. There is also a 40 per cent chance that we measure $U - m_U < -1\sigma_U$, and are not even able to make a 1$\sigma$ detection of PGWs.

Figure 10. This is the plot of the distribution of $U$ for the ideal experiment (upper left-hand panel), realistic experiment (upper right-hand panel), Planck (lower left-hand panel) and WMAP (lower right-hand panel). The red dotted line is the value for $m_U$ and the light blue dashed lines enclose the 1$\sigma$ region for $U$ assuming the hypothesis that $r = 0$.

Figure 11. This is the plot of the $S/N$ (number of $\sigma$) for different values of $r$ for the three different tests. The black line is the $S/N$ test, the dashed dark blue line is the sign test, and the dot–dashed light blue line is the Wilcoxon rank sum test.

A comparison of the three tests is shown in Fig. 11. This is obtained by simulated with several values of $r$ and then interpolating between them. A 2$\sigma$ detection is obtained for $r = 0.26$ ($S/N$ test), $r = 0.3$ (sign test), and $r = 0.5$ (Wilcoxon rank sum test), highlighting its intended use as a monitor of a false positive detection for large $r$.

5.2 Realistic ground-based experiment

A plot of the error bars for the realistic experiment is shown in Fig. 12 with $r = 0.9$. Observations on an incomplete sky require the multipoles to be binned in sizes of $\Delta \ell = 10$. This experiment has much larger error bars than the ideal experiment and it is not able to detect low values of $r$ with the TE cross-correlation only. Plots of the TE power spectrum due to density perturbations and PGWs are shown in Fig. 12 along with the combined TE power spectrum.

For this experiment, the constraints on measuring $\ell_0$ are significantly larger than those for the ideal experiment. The 1$\sigma$ uncertainty on $\ell_0$ is $\Delta \ell_0 \approx 10$. This corresponds to a limit of $r < 0.9$ with 68 per cent confidence. If we want a 2$\sigma$ limit, then the constraint expands to $r < 1.5$. If we assume the inflationary consistency relation, then this error on $\ell_0$ would correspond to a $1\sigma$ upper limit of about $r \lesssim 0.7$. Fig. 13 shows the region of $r$ and $n_s$ allowed with 68 per cent confidence of $\ell_0$.
result in terms of the three tests for the Wiener filtered data. The realistic experiment will not be able to constrain \( r < 0.3 \) using the TE cross-correlation power spectrum. Its limit is closer to \( r < 0.7 \) at only 68 per cent confidence depending on the test used. For a higher confidence in a detection of PGWs, the value of \( r \) would need to be much higher. Since the observed distribution of \( U \) corresponds almost exactly to the simulated distribution of \( U \) under the assumption that \( r = 0 \), therefore we have a 16 per cent chance of measuring \( U - m_U < -1\sigma_U \).

5.3 WMAP

A constraint on \( r \) using a measurement of \( \ell_0 \) for WMAP is almost impossible. Using error bars consistent with WMAP noise, we get \( \Delta \ell_0 \approx 15 \) for an input of \( r = 0.3 \) and \( n_t = 0 \). The published results of WMAP give limits of \( r < 0.3 \) so adding this method to the WMAP results would not change constraints significantly. In fact, using the real WMAP data\(^3\) we get \( \ell_0 \approx 48 \). With an uncertainty of \( \Delta \ell_0 \approx 15 \), the probability of getting a value farther away from \( \ell_0 = 53 \) is larger than 50 per cent, so we cannot detect PGWs in the published WMAP data using the zero-multipoles method.

The results of the Wiener filtering showed that the WMAP cannot make a detection of gravitational waves using the TE cross-correlation power spectrum alone. As with the two toy experiments, the result of the scalar and tensor separation was similar. The Monte Carlo simulation gave on average gave 13 positive multipoles out of a total of 26 uncorrelated multipoles. We would get the same result if the input data had \( r = 0.0 \) so we cannot detect PGWs with WMAP using only the TE power spectrum. A plot of the distribution of the number of positive multipoles is shown in the lower right-hand panel of Fig. 7. As can be seen, this distribution of \( N_+ \) for WMAP noise and \( r = 0.3 \) is simply the distribution for \( r = 0 \).

For WMAP, the S/N test gives the value of \( S/N = -0.02 \) with a standard deviation of 5.09. The distribution is shown in the lower right-hand panel of Fig. 8. The distribution is centred around \( S/N = 0 \) so there is no chance of using this test to detect PGWs in WMAP’s TE power spectrum. The probability of getting a \( 1\sigma \) or \( 2\sigma \) detection is the same probability that we would randomly get a detection if there are no PGWs.

The rank sum test gives a value of \( U_{avg} - m_U = -0.004\sigma_U \), which is implies no ability to distinguish WMAP’s observed TE data from a data set with no PGWs. A plot of the distribution of \( U \) for WMAP error bars is shown in the lower right-hand panel of Fig. 10. We reach the same conclusion for WMAP noise as for the realistic experiment. There is only a 16 per cent chance that we can measure \( U - m_U < -1\sigma_U \), and make a \( 1\sigma \) detection of \( r = 0.3 \).

The published WMAP results show an anticorrelation of TE power spectrum at large scales. Unfortunately this is not a detection of PGWs as theorized in Baskaran et al. (2006). The contribution to the TE power spectrum due to PGWs only changes sign once for \( \ell \approx 60 \). If a claimed evidence for gravitational waves is to be believed, then the TE power spectrum would have to change sign three times for \( \ell \approx 60 \). In fact, other than the two anticorrelations at low \( \ell \), the rest of the multipoles, up to \( \ell = 53 \), are consistent with \( r = 0 \). None of the described tests applied to the current WMAP data will give any detection of PGWs.

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\(^3\)http://lambda.gsfc.nasa.gov/.
5.4 Planck

The uncertainty in $\ell_0$ is much better for Planck than for the realistic experiment and about twice as large for the ideal experiment. The Monte Carlo simulations resulted in $\Delta \ell_0 \approx 3.75$ for an input TE power spectrum with $r = 0.3$ and $n_t = 0$. This results in $\approx 68$ per cent confidence that $r \neq 0$, under the assumption that $n_t = 0$.

The sign test gives on average 10 positive measurements of the TE power spectrum out of a total of 26 uncorrelated multipoles. There is a 16 per cent chance of getting $\leq 10$ positive multipoles if $r = 0$. A plot of the distribution of the number of positive multipoles for Planck is shown in the lower left-hand panel of Fig. 7. There is a 50 per cent chance that we will measure $N_+ < 10$ and hence have a 1σ detection of $r = 0.3$.

The S/N test gives a value of $S/N = -6.24$ with a standard deviation of 5.09. There is only a 10 per cent chance that the $S/N$ test results in a value of $S/N$ larger than $N_0 = 0.3$, and a 10 per cent chance getting $S/N < -3.12$ if $r = 0$. This is close to a 90 per cent probability of detection. The distribution of the S/N variable is shown in lower left-hand panel of Fig. 8.

Again, the rank sum test gives the lowest confidence result with a value of $U_{\text{avg}} - m_U = -0.66 \sigma_U$. A plot of the distribution of $U$ is shown in the lower left-hand panel of Fig. 10. There is a 37 per cent probability that we will measure $U - m_U < -1\sigma_U$ and a 9 per cent probability that we measure $U - m_U < -2\sigma_U$ for Planck.

6 COMPARISON OF MEASUREMENTS OF THE TE POWER SPECTRUM WITH THE BB POWER SPECTRUM

As mentioned earlier, it was originally suggested that it might be easier to detect PGWs using the TE power spectrum instead of the BB power spectrum. For both methods, this turned out not to be true. The reason for this is because we are trying to measure the TE power spectrum at the place where the signal is lowest ($C_{\ell\ell}^{\text{TE}} = 0$). In measurements of the BB power spectrum, if we neglect instrumental noise, the signal decreases with a decrease in $r$ and so does the cosmic variance limited uncertainty. This is not the case for the TE power spectrum. The uncertainty in the measurement of the TE power spectrum due to PGWs is determined by the total TE, TT and EE power spectra. When the TE power spectrum goes to zero, the TT and EE power spectrum do not approach zero (in fact, they increase as we approach $\ell_0$). We therefore have a low S/N around $\ell_0$ making it very hard to detect PGWs using the zero-multipole method. Below we give simple summarizing arguments why the same is true for the Wiener filtering of the TE power spectrum if $N_t < C_{\ell\ell}^{\text{BB}}$, the S/N for the BB power spectrum is

$$\frac{(S/N)_{\text{BB}}}{\Delta C_{\ell\ell}^{\text{BB}}} = \gamma\frac{C_{\ell\ell}^{\text{BB}}}{C_{\ell\ell}^{\text{BB}} + N_t} \approx \gamma.$$  (20)

where

$$\gamma = \sqrt{\frac{(2\ell + 1)f_{\text{sky}}}{2}}.$$  (21)

If $N_t > C_{\ell\ell}^{\text{BB}}$ then we will not be able to detect PGWs and a comparison with the TE power spectrum is not worthwhile.

If $N_t \ll C_{\ell\ell}^{\text{EE}}$ and $r < 1$, for the TE power spectrum, the S/N is

$$\frac{(S/N)_{\text{TE}}}{\Delta C_{\ell\ell}^{\text{TE}}} = \frac{C_{\ell\ell}^{\text{TE}}}{\Delta C_{\ell\ell}^{\text{TE}}} \approx \frac{\sqrt{2\gamma}}{\sqrt{\left(C_{\ell\ell}^{\text{TE}}\right)^2 + \left(C_{\ell\ell}^{\text{BB}} + N_t\right)\left(C_{\ell\ell}^{\text{EE}} + N_t\right)}}.$$  (17)

$$\approx \frac{\sqrt{2\gamma}}{\left(C_{\ell\ell}^{\text{TE}}\right)^{1/2} + \left(C_{\ell\ell}^{\text{BB}} + C_{\ell\ell}^{\text{EE}}\right)^{1/2}},$$  (22)

where $\alpha$ and $\beta$ are

$$\alpha = \sqrt{\frac{C_{\ell\ell}^{\text{TE}}}{D_{\ell\ell}^{\text{TE}}}},$$

$$\beta = \frac{2C_{\ell\ell}^{\text{TE}}D_{\ell\ell}^{\text{TE}} + D_{\ell\ell}^{\text{TE}}C_{\ell\ell}^{\text{EE}} + C_{\ell\ell}^{\text{BB}}D_{\ell\ell}^{\text{EE}}}{2D_{\ell\ell}^{\text{TE}}}.$$  (23)

where

$$D_{\ell\ell}^{\text{XY}} = C_{\ell\ell}^{\text{XY}}/r.$$  (24)

One can see that $\alpha$ and $\beta$ are of the order of unity. Therefore, the S/N is approximated as

$$\frac{(S/N)_{\text{TE}}}{\Delta C_{\ell\ell}^{\text{TE}}} \approx \frac{\sqrt{2\gamma}}{\alpha + \beta r} \approx \frac{\sqrt{2\gamma}}{\alpha} r.$$  (25)

In other words if $r < \alpha / \beta \sim 1$, BB measurements have the obvious advantage in comparison with the Wiener filtering of the TE power spectrum. Indeed if $r \lesssim 1$, $(S/N)_{\text{BB}} \sim \gamma^2$, while $(S/N)_{\text{TE}} \sim \gamma r < \gamma$. This is because in BB measurements, applying proper data analysis, we can entirely eliminate contributions of scalar perturbations to CMB polarization signal as well as to the uncertainties. For the perfect Wiener filtering of the TE power spectrum, we can eliminate the contribution of scalar perturbations to the signal only, but cannot eliminate their contribution to the uncertainties.

7 CONCLUSION

The measurement of where the TE cross-correlation first changes sign can be used to detect or put constraints on PGWs. Such constraints are not as strong as the ones given by measurements of the BB power spectrum; however, it is useful to have a supplementary method to detect PGWs. We have shown how well the TE mode can constrain the amount of PGWs from just a measurement of the angular scale where it first changes sign for two different toy experiments and two real satellite experiments. The absolute best limit with which we can measure $\ell_0$ only gives us less than a $3\sigma$ detection of the PGW component if $r = 0.3$. The current confidence limits give us $r < 0.3$ at 95 per cent confidence level. Current and future experiments are optimized to measure the BB power spectrum if $r \lesssim 0.1$ even in the presence of foregrounds, which are not taken into account in this paper. Future satellite experiments should be able to detect $r < 0.01$ which is 10 times better than the sensitivity to $r$ than the result of the ideal experiment. If one neglects even cosmic variance, the discreteness of $\ell$ limits the calculation of $\ell_0$, and the sensitivity to $r$, to values considerably larger than 0.01. The cosmic variance is largest at low $\ell$ and is proportional to the total power spectrum. Since the TE cross-correlation has contributions from density perturbations the errors in the measured TE power spectrum make detecting deviations of $\ell_0$ from 53
difficult, though they also provide insurance against a false detection or imperfect subtraction of instrumental and foreground systematic effects.

The other method described in this paper is one in which we filter out the signal due to density perturbations, leaving only the contribution to the TE power spectrum due to PGWs. We then test the resulting TE power spectrum to see if it is negative. Three different statistical tests were used to see if there was a significant detection of PGWs. The S/N test can give a value for r using a comparison with Monte Carlo simulations, while the Wilcoxon rank sum test can only give an allowable range for r. The sign test will only tell us if r ≠ 0.

Using the Wiener filtering method, we are unable to make as significant of a detection as using the zero-multipole method. The best result was for the S/N test which would give a 2.3σ detection of r = 0.3. To detect PGWs on the level of 3σ, the tensor-to-scalar ratio r should be r ≳ 0.4. The sign test would give 2σ detection for r = 0.3 and a 3σ detection for r = 0.45. The Wilcoxon ranked sum test gives only a 1.2σ detection for r = 0.3 and a 3σ detection for r = 0.7. Similar results were gotten for the other three experiments tested. Thus in the sense of potential to detect PGWs, the zero-multipole method is the best, next best is the S/N test, then the sign test, and the worst is the Wilcoxon ranked sum test.

Baskaran et al. (2006) present illustrative examples in which high r is consistent with measured TT, EE and TE correlations. The value of r is so high in these examples that if PGWs with such r really existed, current BB experiments would already detect PGWs. All models predict that the TE cross-correlation power spectrum change sign only once for ℓ < 100. The fact WMAP cannot exclude several multipoles with C_{TE}^ℓ > 0 in between multipoles of C_{TE}^ℓ < 0 means that the TE cross-correlation power spectrum either changes sign several times for ℓ < 100 or there is some instrumental noise which causes some anticorrelation measurements. Using instrumental noise consistent with WMAP, our Monte Carlo simulations give ∆ℓ_0 ≈ 16 and ℓ_0 > 40, which means that there is no evidence of PGWs in the TE correlation power spectrum.

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