Interaction and Particle–Hole Symmetry of Laughlin Quasiparticles

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The pseudopotentials describing interaction of Laughlin quasielectrons (QE) and quasiholes (QH) in an infinite fractional quantum Hall system are studied. The QE and QH pseudopotentials are similar which suggests the (approximate) particle–hole symmetry recovered in the thermodynamical limit. The problem of the hypothetical symmetry-breaking QE hard-core repulsion is resolved by the estimate that the “forbidden” QE pair state has too high energy and is unstable. Strong oscillations of the QE and QH pseudopotentials persist in an infinite system, and the analogous QE and QH pair states with small relative angular momentum and nearly vanishing interaction energy are predicted.

An important element in our understanding of the incompressible-fluid ground states is formed in a two-dimensional electron gas (2DEG) in high magnetic fields has been the identification of Laughlin correlations 1 in a partially filled lowest Landau level (LL). These correlations can be defined as a tendency to avoid pair eigenstates with the largest repulsion (smallest relative pair angular momentum R) in the low-energy many-body states. The incompressibility results at a series of filling factors (number of particles divided by the number of states) ν = (2p + 1)−1 at which the p leading pair states at R = 1, 3, ..., 2p − 1 are completely avoided in the non-degenerate (Laughlin) ground state, but not in any of the excited states.

Each Laughlin-correlated state can be understood in terms of two types of quasiparticles (QP): quasielectrons (QE) and quasiholes (QH), moving in an underlying Laughlin ground state (“reference” or “vacuum” state). The QP’s are the elementary excitations of the Laughlin fluid and correspond to an excessive (QE) or missing (QE) single-particle state, compared to an exact ν = (2p + 1)−1 filling. They have finite size and (fractional) electric charge of ±(2p + 1)−1e, and thus (in analogy to LL’s of electrons) the single-QP spectrum in a magnetic field is degenerate at a finite energy denoted as εQP. For the QP’s at a complex coordinate z = 0, their wavefunctions are obtained by applying the prefactors ∏k ∂/∂zk (QE) and ∏k zk (QH) to the Laughlin wavefunction Φ2p+1 = ∏k< j (zi − zj)2p+1.

Partially filled lowest LL is not the only many-body system with Laughlin correlations, which generally occur when the single-particle Hilbert space is degenerate and the two-body interaction is repulsive and has short range.) Among other Laughlin-correlated systems are a two-component system of electrons and charged excitons (X−, two electrons bound to a valence hole, formed in an electron–hole plasma in a magnetic field or a system of (bosonic) electron pairs formed near the half-filling of the first excited LL (Moore–Read state at ν = 5 4).

Due to their LL-like macroscopic degeneracy and the Coulomb nature of their interaction, Laughlin correlations can also be expected in a system of Laughlin QP’s.
momenta) exists that should require introduction of a phenomenological hard-core QE–QE repulsion to predict the correct number of many-QE states in numerical energy spectra. Instead, the repulsion energy of the “forbidden” QE pair state is finite but higher than that of a corresponding QH pair state. It even exceeds the Laughlin gap \( \Delta = \varepsilon_{\text{QE}} + \varepsilon_{\text{QH}} \) to create an additional QE–QH pair and makes this QE pair state unstable (and pushes it into the 3QE+QH continuum). This instability explains why Jain’s composite fermion (CF) picture correctly predicts the lowest-energy bands of states, despite the asymmetry of QE and QH LL’s introduced by an (unphysical) effective magnetic field. Also, the similarity of the QE and QH pair states and energies precludes qualitatively different response of a Laughlin-correlated 2DEG to a positively and negatively charged perturbation.

The knowledge of pseudopotentials defining interactions of Laughlin QP’s is essential in Haldane’s hierarchy of the fractional quantum Hall effect in which they determine those of Laughlin fillings at which the QP’s form (daughter) Laughlin incompressible states of their own. Although they are to a large extent equivalent, Haldane’s hierarchy differs from Jain’s CF picture in the “symmetric” description of the two types of QP’s. Haldane’s elegant argument that both QE and QH excitations are bosons placed “between” the \( N \) (effectively one-dimensional) electrons yields equal numbers of possible QE and QH states, \( g_{\text{QE}} = g_{\text{QH}} = N + 1 \) (tilde mean bosons), which on a sphere correspond to equal single-particle angular momenta, \( l_{\text{QE}} = l_{\text{QH}} = \frac{1}{2} N \) (because \( \tilde{g} = 2\tilde{l} + 1 \); the lowest LL on a Haldane sphere is an angular momentum shell of \( l = 1 \), half the strength of Dirac’s magnetic monopole in the center). In a system of \( n \) QP’s, a mean-field Chern–Simons transformation (MFCST) can further be used to convert such bosonic QP’s to more convenient fermions with \( g = \tilde{g} + (n - 1) \) yielding \( l = l + \frac{1}{2} (n - 1) \). However, for QE’s this value of \( l \) seemed to predict incorrect number of low-energy states in the numerical energy spectra unless the pair state at the maximum angular momentum \( L_{\text{max}} = 2l + 1 \) was forbidden. On a sphere, the relation between \( L = |l_1 + l_2| \) and \( R \) is \( L = 2l - R \), and thus the exclusion of the pair state at \( L_{\text{max}} \) is equivalent to a hypothetical hard-core repulsion, \( V_{\text{QE}}(1) = \infty \).

Such interaction hard-core can be formally removed by an appropriate redefinition of the single-particle Hilbert space. This is accomplished by a fermion-to-fermion MFCST \( g^* \) which replaces \( g \) by \( g^* = g - 2(n - 1) \), and \( \tilde{g} = \tilde{g} - 1 \) by \( \tilde{l}^* = \frac{1}{2} N - \frac{1}{2} (n - 1) \). By “elimination” we mean that the angular momenta \( L_{\text{QE}} \) of states containing \( n \) QE’s can be obtained by simple and unrestricted addition of \( n \) individual angular momentum vectors \( I_{\text{QE}} \) followed by antisymmetrization (QE’s are treated as indistinguishable fermions), just as \( L_{\text{QH}} \) could be obtained by antisymmetric combination of \( n \) vectors \( I_{\text{QH}} \). Although they seem to agree with the “numerical experiments,” no explanation exists for a hard-core in the QE–QE repulsion (and its absence in \( V_{\text{QH}} \)) or the resulting asymmetry between \( l_{\text{QH}} \) and \( l^*_{\text{QH}} \).

This asymmetry is inherent in Jain’s CF picture, in which QE’s and QH’s are converted into particles and vacancies in different CF LL’s whose (different) angular momenta are equal to \( l_{\text{QE}} \) and \( l_{\text{QH}} \), respectively. However, the effective magnetic field leading to the correct values of \( g_{\text{QE}} \) and \( g_{\text{QH}} \) in the CF picture does not physically exist. While for the QH states the effective field is one of possible physical realizations of the MFCST describing Laughlin correlations (the avoidance of most strongly repulsive pair states) in the underlying electron system, no explanation for \( g^*_{\text{QE}} \) being smaller than \( g_{\text{QH}} \) is possible within the CF model itself.

To resolve this puzzle we have examined the QE and QH pseudopotentials calculated for the systems of \( N \leq 12 \) electrons at \( \nu = \frac{1}{3} \) and \( \frac{2}{3} \). In Fig. 1 we compare \( V_{\text{QE}} \) (a) and \( V_{\text{QH}} \) (b) obtained at \( \nu = \frac{1}{3} \) for different values of \( N \). In both frames, \( \mathcal{R} = 2l - L \), with \( l_{\text{QE}} = l_{\text{QH}} = \frac{1}{2} (N + 1) \). To obtain the values of \( V \), the energies of the Laughlin ground state and of the two QP’s are subtracted from the energies of the appropriate QP pair states (such as the QE pair states for \( N = 11 \) and 12 shown in the insets). The energy is measured in the units of \( e^2/\lambda \), and \( \lambda \) is the magnetic length.

In the limit of \( N \rightarrow \infty \), the sphere radius \( R \sim \sqrt{N} \) diverges and the numerical values of \( V(\mathcal{R}) \) converge to those describing an infinite 2DEG on a plane. In this (planar) geometry, \( \mathcal{R} \) is the usual relative pair angular momentum. Remarkably, when \( R_{\text{QE}} \) is defined as

![Fig. 1](image-url)
2l_{QH} − L rather than 2l_{QH}^0 − L, the QE and QH pseudopotentials become quite similar. The main difference is the obvious lack of the \( R_{QH} = 1 \) state and stronger oscillations in the \( V_{QH}(R) \), but the maximum at \( R = 5 \) and the minima at \( R = 3 \) and 7 are common for both \( V_{QH} \) and \( V_{QH} \). The same structure occurs also for the QP’s as in the \( \nu = \frac{1}{3} \) state. Most unexpected in Fig. 1 are the negative signs of \( V_{QH} \). The only positive pseudopotential coefficient is \( V_{QH}(1) \), which might indicate that, despite QP’s being charge excitations, both QE–QE and QH–QH interactions are generally attractive.

In Fig. 2, we plot a few leading pseudopotential coefficients (those at the smallest values of \( R \)) \( V_{QH} \) and \( V_{QH} \) for Laughlin correlants at \( \nu = \frac{1}{3} \) and \( \frac{1}{5} \) as a function of \( N^{-1} \). Clearly, the corresponding coefficients of all four pseudopotentials behave similarly which confirms the correct use of \( l_{QH} \) rather than \( l_{QH}^0 \) in the definition of \( R_{QH} \). It is also clear that all coefficients \( V \) increase with increasing \( N \) (although at a different rate for \( QE \)’s and \( QH \)’s) and it seems that none of them will remain negative in the \( N \to \infty \) limit. In attempt to estimate the magnitude of \( V \) in this limit we have drawn straight lines that approximately extrapolate our data for some of the coefficients. Most noteworthy values are: \( V_{QH, \nu = 1/3}(1) \approx 0.03 e^2/\lambda \) being about three times larger than \( V_{QH, \nu = 1/5}(1) \) as expected from the comparison of interacting charges (\( \frac{1}{3} e \) and \( \frac{1}{5} e \), respectively), the \( V(3) \) coefficients (seemingly) vanishing in all four plots, and \( V_{QH, \nu = 1/3}(5) \approx 0.005 e^2/\lambda \) being about twice smaller than \( V_{QH, \nu = 1/3}(5) \).

The predicted small value of \( V(3) \) and of some other leading coefficients is by itself quite interesting, although it can be understood from the fact that QP’s are more complicated objects than electrons, and the oscillations in \( V_{QH}(R) \) reflect the oscillations in their more complicated charge density profile (similar oscillations occur in the electron pseudopotentials in higher LL’s). The consequences of this fact are even more important.

First, from a general criterion \( |V_{QH}(5)| \) for Laughlin correlations at \( \nu \approx (2p + 1)^{-1} \) (defined as the avoiding of pair states with \( R < 2p + 1 \) in the low-energy many-body states) in a system interacting through a pseudopotential \( V(R) \) we find that the QP’s of the parent Laughlin state of electrons form Laughlin states of their own only at \( \nu_{QP} = \frac{1}{3} \). These states and their \( \nu_{QE} = \frac{1}{3} \) daughters exhaust Jain’s \( \nu = n(2pn + 1)^{-1} \) sequence. No other incompressible daughter states occur in the hierarchy, including the (ruled out earlier) \( \nu = \frac{1}{7} \) or \( \frac{1}{11} \) states. Despite all the differences between Haldane’s hierarchy and Jain’s CF model, our conclusion makes their predictions of the incompressibility at a given \( \nu \) completely equivalent.

Second, the (near) vanishing of \( V_{QH}(3) \) explains the stability of the \( hQ_{E}(3) \) complex \( \nu \) in the 2DEG interacting with an (optically injected) valence hole. Being the most strongly bound and the only radiative state of all “fractionally charged excitons” \( hQ_{E} \), the \( hQ_{E} \) is most likely the complex observed \( \nu \) in the PL spectra of the 2DEG at \( \nu > \frac{1}{3} \).

Third, since \( V_{QH}(5) \) is about twice larger than \( V_{QH}(5) \), it is also plausible that \( V_{QH}(1) \) could be much larger than \( V_{QH}(1) \), so that the \( R_{QH} = 1 \) state would fall in the continuum and could not be identified in the energy spectra. In Fig. 2(c), on top of the 11-electron spectrum at Dirac’s monopole strength (the number of magnetic flux quanta piercing the Haldane sphere) \( 2S = 28 \), marked with full dots, in which the lowest-energy states contain two QE’s at \( \nu = \frac{1}{3} \), with open circles we have marked another spectrum calculated for the same \( N = 11 \) but at \( 2S = 32 \), whose lowest-energy band contains two QH’s. The second spectrum is vertically shifted so that the energies of the QH and QE pair states coincide at \( L = 1 \) (i.e. at \( R = 11 \)) at which \( V_{QH} \) and \( V_{QH} \) are both negligible, but the energy units \( (e^2/\lambda) \) are the same. Since the Laughlin gap \( \Delta \) to the continuum of states with additional QE–QH pairs involves the sum of QE and QH energies, it is roughly the same in both spectra. However, the minima and maxima in \( V_{QH}(R) \) are stronger than those in \( V_{QH}(R) \), and the difference \( |V_{QH} - V_{QH}| \) increases at larger \( L \). While it is hardly possible to rescale \( V_{QH} \) so as to reproduce \( V_{QH} \) at \( L \leq 9 \) and convincingly predict its value at \( L = 11 (R_{QH} = 11) \), it seems likely that \( V_{QH}(11) \) is indeed larger than \( \Delta \), which would explain the absence of the \( R_{QH} = 11 \) state below the continuum. An example of such “rescaling” procedure is shown in Fig. 2(c) with the line obtained by stretching \( V_{QH} \) so that it crosses \( V_{QH}(5) \) and \( V_{QH}(3) \) at \( L = 7 \) and 9, respectively. Similar lines are shown in Fig. 2(d) for the 12-electron spectrum corresponding to two QE’s in the lowest band (\( 2S = 31 \)). Certainly, this procedure, based
on the assumption that $V_{\text{QE}}(3)$ and $V_{\text{QH}}(3)$ are small and that $V(1)$ is proportional to $V(5)$, is not accurate. Nevertheless, having in mind the similarities of $V_{\text{QE}}$ and $V_{\text{QH}}$ in Figs. 1 and 2 and in the absence of any physical reason why the $\mathcal{R}_{\text{QE}} = 1$ state might not exist while the $\mathcal{R}_{\text{QH}} = 1$ state does, we believe that it is more reasonable to assume that $V_{\text{QE}}(1)$ is finite, although larger than $\Delta$. The fact that the $\mathcal{R}_{\text{QE}} = 1$ state is pushed into the 3QE+QH continuum simply means that it is unstable toward spontaneous creation of a low-energy QE–QH pair with finite angular momentum (magneto-rotor).

The assumption that $\Delta < V_{\text{QE}}(1) < \infty$ restores the elegant symmetry of Haldane’s picture of QP’s “placed” between electrons. It replaces the problem of explaining the QE hard-core by a question of why $V_{\text{QE}}$ is larger than $V_{\text{QH}}$ at short distance (e.g., at $\mathcal{R} = 1$ and 5; see Fig. 4). But the fact that $V_{\text{QE}}$ and $V_{\text{QH}}$ are not equal at short distance is by no means surprising since the QE and QH have different wavefunctions.

In conclusion, we have calculated the pseudopotentials $V_{\text{QP}, \nu}(\mathcal{R})$ describing interaction of QE’s and QH’s in Laughlin $\nu = (2p + 1)^{-1}$ states of an infinite 2DEG. These pseudopotentials are all similar, showing strong repulsion at $\mathcal{R} = 1$ and 5, and virtually no interaction at $\mathcal{R} = 3$. The unexpected QE–QE and QH–QH attraction which results in few-electron calculations disappears in the limit of an infinite system. Because the QP charge at $\nu = (2p + 1)^{-1}$ decreases with increasing $p$, the QP interaction at $\nu = \frac{1}{3}$ is stronger than at $\nu = \frac{1}{5}$. Because of different QE and QH wavefunctions, $V_{\text{QE}}$ is larger than $V_{\text{QH}}$ at small $\mathcal{R}$ (short distance). The coefficient $V_{\text{QE}}(1)$ exceeds the Laughlin gap $\Delta$ to create an additional QE–QH pair, which makes the QE pair state at $\mathcal{R} = 1$ unstable. This instability, rather than a mysterious QE hard-core or an inherent asymmetry between the QE and QH angular momenta, is the reason for the overcounting of few-QE states when, following Haldane, QE’s are treated as bosons with $l = \frac{1}{2} N$. In particular, it explains the absence of the $L = N$ multiplet in the low-energy band of states in the $N$-electron numerical spectra at the values of $2S = (2p + 1)(N - 1) - 2$, corresponding to two QE’s in the Laughlin $\nu = (2p + 1)^{-1}$ state. The (near) vanishing of $V_{\text{QP}}(3)$ is the reason why no hierarchy states other than those from Jain’s $\nu = n(2pm \pm 1)^{-1}$ sequence are stable. It is also the reason for the strong binding of the fractionally charged exciton $h\text{QE}_2$.

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