Spatially coupled serially concatenated codes via parity re-encoding

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The recently introduced spatially coupled turbo-like codes are attractive in trading-off performances between the error-floor region and the water-fall region. This letter is concerned with the construction of spatially coupled serially concatenated codes. Particularly, a new construction of spatially coupled serially concatenated codes is presented, in which a portion of the encoding output of the previous inner encoder is taken as the input of the current inner encoder. The authors show that by proper selection of the component codes and the re-encoding ratio, performance gain about 0.25 dB is obtained with the proposed construction.

Introduction: Spatially coupled codes are constructed by interconnecting a series of underlying uncoupled codes into a single code. Spatially coupled low-density parity-check (SC-LDPC) codes [1–3] is the first class of spatially coupled codes. One of the most promising property of SC-LDPC codes is that their iterative decoding threshold achieves the maximum a posteriori (MAP) threshold of the underlying uncoupled LDPC block ensemble. As a result, we may use spatial coupling to approach the capacity by increasing the density of the parity-check matrix of the uncoupled LDPC ensemble. This phenomenon is termed as threshold saturation, which was first observed numerically in [2] and rigorously proved in [4]. Optimised construction of SC-LDPC codes was studied in [5].

Spatial coupling can also be applied to other block coding schemes to construct high performance codes. Braided convolutional codes and braided block codes were proposed in [6]. In [7, 8], the authors proposed the block Markov superposition transmission codes, in which spatial coupling is achieved with block-oriented superposition. When applied to turbo-like codes, we obtain the spatially coupled turbo-like (SC-TL) codes [9]. Partially information re-encoding was proposed in [10] to achieve spatial coupling, resulting in the partially information-re-encoding coupled codes.

As a class of spatially coupled codes, SC-TL codes include spatially coupled serially concatenated codes (SC-SCC), spatially coupled parallel concatenated codes (SC-PCC), and braided convolutional codes as subclasses. As a result, SC-TL codes are very attractive in trading off performances between the water-fall region and the error-floor region. Particularly, SC-PCC are superior in the water-fall region, while the SC-SCC are superior in the error-floor region [11]. However, as a class of promising coding schemes, the construction of high-performance SC-TL codes has not received enough attention.

In the SC-SCC of [9], spatial coupling is achieved by taking a portion of the encoding output of the previous outer encoder as the encoding input of the current inner encoder. Hence, this construction results in a class of non-recoverable codes. In this letter, we present a new construction of spatially coupled serially concatenated codes. In the proposed construction, we take a portion of the parity-check bits of the previous inner encoder as the encoding input of the current inner encoder. As a result, the proposed construction results in a class of recoverable codes. As a class of spatially coupled codes, the proposed SC-SCC can be efficiently decoded with the sliding-windowed iterative decoding algorithm. We show by simulation that the proposed SC-SCC performs better than the SC-SCC constructed in [9] in the water-fall region.

Spatially Coupled Serially Concatenated Codes: In the proposed SC-SCC, spatial coupling is achieved by taking a portion of the parity-check of the previous inner encoder as the input of the current inner encoder. Hence, the proposed codes are termed as partially coupled SCC (PPC-SCC). Let \( C_1 \) and \( C_2 \) be two binary systematic linear block codes with efficient encoding and decoding algorithms. We assume that \( k_2 = n_2 + d, d = \sum_{i=1}^{m_2} d_i, \) and \( m_2 \leq n_2 - k_2 \), where \( m \) and \( d \), \( 1 \leq m \leq n \), are positive integers. The encoding algorithm of the proposed PPC-SCC with encoding memory \( m \) is presented in the following. Let \( u \) be a binary sequence of length \( L \). We partition \( u \) into \( L \) sub-blocks, denoted by \( u^{(1)}, u^{(2)}, \ldots, u^{(L)} \), where \( u^{(l)} \in \mathbb{F}_2^m \). The size of each block is \( k \) bits. The \( L \) sub-blocks are encoded in Algorithm 1 (see Figure 1 for reference), where \( \Pi_1 \) and \( \Pi_2 \) are two random interleavers of size \( n_2 + d \) and \( n_2 - k_2 \), respectively. For termination, \( T \) length-\( k \) all-zero sequences are taken to be the final input of the encoder.

It can be seen that the rate of proposed PPC-SCC is

\[
R = \frac{Lk}{Ln + T(n - k)} = \frac{L}{L/k + L/n}
\]

Note that the \( T \) all-zero sub-blocks for termination are not transmitted.

We present in Figure 2(b) the encoder of the SC-SCC proposed in [9] with encoding memory \( m = 1 \). For comparison, we also present in Figure 2(a) the encoder of the proposed PPC-SCC with encoding memory \( m = 1 \). Note that the encoding outputs of the outer encoders of Figure 2 are codewords of the outer codes, as opposed to parity-check bits. It can be seen that these two classes of spatially coupled codes are quite different. In the proposed construction, previous encoding outputs influence the current encoding output. Hence, the proposed codes are recursive in nature. Due to the recursive characteristic of the proposed codes, the last \( T \) parity-check sub-blocks are usually non-zero. This is different from the spatially coupled SCC proposed in [9].

Numerical Evaluation: In this section, we present numerical results to show the performance advantages of the proposed SC-SCC by comparing with the codes in [9].

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**Algorithm 1 Encoding of a PPC-SCC**

1. **Initialisation:** For \( t < 0 \), set \( w^{(t)} = (w^{(t)}_0, w^{(t)}_1, \ldots, w^{(t)}_{m-1}) = \emptyset \in \mathbb{F}_2^m \), where \( w^{(t)}_i \in \mathbb{F}_2^m (0 \leq i \leq m) \) and \( d_0 = n_1 \).
2. **Recursion:** For \( t = 0, 1, \ldots, L-1 \),
   - **Outer Encoding:** Encode \( u^{(t)} \) by an encoder of \( C_1 \), resulting in \( w^{(t)}_i \in \mathbb{F}_2^m \).
   - **Combining:** Combine \( w^{(t)}_i \) (0 \( \leq i \leq m \)) to obtain \( v^{(t)}_i \), where \( v^{(t)}_i = (w^{(t)}_0, w^{(t)}_1, \ldots, w^{(t)}_{m-1}) \).
   - **Inner Encoding:** Encode \( \Pi_1(v^{(t)}) \) by a systematic encoder of \( C_2 \), resulting in \( v^{(t)}_i \) is the interleaved version of \( v^{(t)}_i \) with the interleaver \( \Pi_1 \). The resulting parity-check sequence is denoted as \( d^{(t)} = v^{(t)}_{n_1} \).
   - **Interleaving and Transmission:** Interleave \( v^{(t)}_i \) with \( \Pi_2 \), resulting in \( \Pi_2(v^{(t)}_i) \) with \( \Pi_2(v^{(t)}_i) \).
3. **Termination:** For \( t = L, L+1, \ldots, L+T-1 \), set \( u^{(t)} = 0 \in \mathbb{F}_2^m \) and compute \( d^{(t)} \) following Step (2).
Consider the rate-half recursive systematic convolutional codes \( C_1 \) and \( C_2 \), specified by the generator matrix

\[
G_1(D) = \left[ \frac{1}{1 + D} \right]
\]

and

\[
G_2(D) = \left[ \frac{1 + D + D^2}{1 + D + D^2 + D^3} \right],
\]

respectively. We construct a PPC-SCC with encoding memory one. The outer code is the terminated convolutional code \( C_1 \), \( [2050, 1024] \). Two inner codes, which are the terminated convolutional codes \( C_2 [5096, 2546] \) and \( C_2 [5304, 2650] \), are considered. The corresponding re-encoding lengths are \( d_1 = 496 \) and \( d_1 = 600 \). We present in Figure 3 the bit error rates (BER) of these two PPC-SCCs. The windowed iterative decoding algorithm with decoding window size 4 is implemented for decoding. For comparison, we also present in Figure 3 the BER of a comparable algorithm with decoding window size 4 is implemented for decoding.

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