A Diagrammer’s Note on Superconducting Fluctuation Transport for Beginners:
II. Hall and Nernst Effects with Perturbational Treatment of Magnetic Field

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Abstract

A diagrammatic approach based on thermal Green function to superconducting fluctuation transport is reviewed focusing on Hall and Nernst effects. The treatment of weak magnetic field is carefully discussed within the linear order perturbation.

1 Introduction

This Note is the second part of the series\(^1\) and I expect that you have already read the first one [I]. In this second Note I shall discuss the effect of the magnetic field perturbationally in the weak field limit. The finite magnetic field shall be discussed non-perturbationally in the third Note. The symbols that have appeared in [I] are used here without explanations. Since the introduction to the series has been given in [I], let us start at once.

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\(^1\) In this Note I shall quote the first Note as [I].

[I] ≡ Narikiyo: arXiv:1112.1513.
2 Boltzmann Transport: Relaxation-Time Approximation

In the relaxation-time approximation of the Boltzmann transport\(^{2}\), the expectation value of the charge current \(J^e\) and the heat current \(J^Q\) in the linear order of the uniform electric field \(E\) and the uniform magnetic field \(H\) are given as\(^3\)

\[
J^e = -2\frac{e^3}{m} \sum_p v_p^2 r^2 \left( -\frac{\partial f(\xi_p)}{\partial \xi_p} \right) (H \times E),
\]

and

\[
J^Q = -2\frac{e^2}{m} \sum_p v_p^2 r^2 \xi_p \left( -\frac{\partial f(\xi_p)}{\partial \xi_p} \right) (H \times E),
\]

where both \(E\) and \(H\) are static. These describe the currents in Hall and Nernst effects. Although formally exact results have been obtained\(^4\), I shall only discuss the results within the relaxation-time approximation in the following.

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\(^{2}\) The Boltzmann transport is formulated in terms of \(E\) and \(H\) so that it is gauge-invariant. See footnote 12 in [I].

\(^{3}\) See (44) and (45) in [I].

\(^{4}\) The exact formula for \(J^e\) via the Boltzmann equation is given by Kotliar, Sengupta and Varma: Phys. Rev. B 53, 3573 (1996) as has been discussed in §6 of [I] and it is also derived from the Fermi-liquid theory in [KY].

The exact formula for \(J^Q\) via the Boltzmann equation is given by Pikulin, Hou and Beenakker: Phys. Rev. B 84, 035133 (2011) and it is also derived from the Fermi-liquid theory in [Kon].

In the Fermi-liquid theory the effect of scatterings beyond the relaxation-time approximation is taken into account as the vertex correction.

[KY] ≡ Kohno and Yamada: Prog. Theor. Phys. 80, 623 (1988).

[Kon] ≡ Kontani: Phys. Rev. B 67, 014408 (2003).
3 Quasi-particle Transport: Relaxation-Time Approximation

We consider the linear response to electric field in the presence of magnetic field. The effect of the magnetic field is treated perturbationally in the weak-field limit.

The expectation value of the charge current $J^e$ is expressed as the linear response to electric field

$$J^e_\mu(k, \omega) = \sum_\nu \sigma_{\mu\nu}(k, \omega) E^\nu(k=0, \omega), \quad (3)$$

where the electric field $E$ is uniform and the $k$-dependence comes from the vector potential $A(x)$ introduced as

$$A(x) = Ae^{i k \cdot x}, \quad (4)$$

with a constant vector $A$. The magnetic field $H$ in the limit of $k \to 0$ is expressed as

$$H = \nabla \times A(x) = i(k \times A). \quad (5)$$

The conductivity tensor is given by the Kubo formula (linear response theory)

$$\sigma_{\mu\nu}(k, \omega) = \frac{1}{i\omega} \left[ \Phi^e_{\mu\nu}(k, \omega + i\delta) - \Phi^e_{\mu\nu}(k, i\delta) \right], \quad (6)$$

where

$$\Phi^e_{\mu\nu}(k, i\omega_\lambda) = \int_0^\beta d\tau e^{i\omega_\lambda\tau} \langle T_\tau \{ J^H_\mu(k; \tau) J^H_\nu(k=0) \} \rangle. \quad (7)$$

Here $J^H_\mu$ is the charge current in the presence of the magnetic field and its Fourier component is given by

$$j^H_\mu(q) = j^e_\mu(q) - \frac{e}{m} \rho(q-k) A_\mu, \quad (8)$$

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5 $E^\nu(k=0, \omega)$ should be used in (68) and (78) of [I].

6 We do not have to consider the $k$-dependence in §8 of [I] where the magnetic field is absent.

7 I follow the derivation of the Hall conductivity by [FEW]. Their derivation for the case of nearly free electrons are extended to the case of interacting electrons by [KY] on the basis of the Fermi-liquid theory. These works are formulated in fully gauge-invariant manner.

[FEW] ≡ Fukuyama, Ebisawa and Wada: Prog. Theor. Phys. 42, 494 (1969).

8 Here the diamagnetic current density is transformed by the integral

$$-\frac{e^2}{m} \sum_\sigma \int d^3 x e^{-i q \cdot x} \psi^\dagger_\sigma(x) \psi_\sigma(x) A(x),$$
where

\[ j^\sigma(q) = e \sum_\sigma \sum_p \frac{1}{2} (v_p + v_{p+q}) c_{p\sigma} c_{p+q\sigma} \]
\[ = \frac{e}{m} \sum_\sigma \sum_p \left( p + \frac{q}{2} \right) c_{p\sigma} c_{p+q\sigma}, \quad (9) \]

and

\[ \rho(q) = e \sum_\sigma \sum_p c_{p\sigma} c_{p+q\sigma}. \quad (10) \]

In this note \( e \) is chosen to be negative \( e < 0 \). It should be noted that in (7) the \( k \)-independence of \( j^H_\nu(k=0) \) results from the coupling to the uniform electric field\(^9\) and the \( k \)-dependence of \( j^H_\mu(k; \tau) \) represents the magnetic-field dependence of the observed current.

The (imaginary) time-dependence is introduced by

\[ j^\mu_H(k; \tau) = e^{K\tau} j^H_\mu(k) e^{-K\tau}, \quad (11) \]

where the coupling to the magnetic field\(^11\)

\[ -j^\sigma(-k) \cdot A, \quad (12) \]

is added to \( K \) in (1) of [I]. By the coupling to the vector potential the momentum of the electron increases by \( k \) as seen from (9). Thus the coupling leads to the electron propagator off-diagonal in the momentum.

In the following we focus on the case of nearly free electrons where the electron propagator in the absence of magnetic field is given by

\[ G(p, i\varepsilon_n) = \frac{1}{i\varepsilon_n - \xi_p}, \quad (13) \]

with

\[ \rho(q) = e \sum_\sigma \int d^3x e^{-iq \cdot x} \psi^\dagger_\sigma(x) \psi_\sigma(x). \]

\(^9\) It should be noted that each term in \( j^\sigma(q) \) is proportional to the summation of incoming and outgoing velocities \( v_p + v_{p+q}. \)

\(^10\) The uniform electric field \( E \) is expressed by the uniform vector potential \( A_0 \) as \( E = i\omega A_0 \) where we have put \( \phi = 0 \) with \( \phi \) being the scalar potential. The current \( j^H(k=0) \) couples to \( A_0. \)

\(^11\) It is sufficient to consider the coupling

\[ -\int d^3x j^\sigma(x) \cdot A(x), \]

for the discussion of the observed current linear in \( A. \)
with
\[ \tilde{\varepsilon}_n \equiv \varepsilon_n + \frac{1}{2\tau} \text{sgn}(\varepsilon_n). \] (14)

Here \( \tau \) is the life-time due to quasi-elastic scatterings.

Figure 1: Diagrams with A-linear contributions: (a) The solid line in upward direction represents the electron propagator \(-G(p + k/2, i\varepsilon_n + i\omega_\lambda)\) and the broken line in downward direction represents \(-G(p - k/2, i\varepsilon_n)\).
Throughout the series of three Notes the Zeeman splitting is neglected so that the propagators for up-spin electrons and for down-spin electrons are degenerate. The upper black circle represents a component of \( j_{\mu}(k) \) where the momentum of the electron changes from \( p + k/2 \) to \( p - k/2 \) and the lower black circle represents a component of \( j_{\nu}(0) \) where the momentum does not change.

(b) The broken line in upward direction is \(-G(p + k/2 \leftarrow p - k/2, i\varepsilon_n + i\omega_\lambda)\) and the solid line in downward direction is \(-G(p - k/2, i\varepsilon_n)\). The upper black circle is \( j_{\mu}(k) \) and the lower black circle is \( j_{\nu}(0) \).
(c) The solid line in upward direction is \(-G(p + k/2 \leftarrow p - k/2, i\varepsilon_n + i\omega_\lambda)\) and the solid line in downward direction is \(-G(p - k/2, i\varepsilon_n)\). The upper black circle is \( j_{\mu}(k) \) and the lower gray circle is \(-\frac{e}{m} \rho(\rho)(0)A_{\mu} \) where the momentum changes from \( p - k/2 \) to \( p + k/2 \).
(d) The solid line in upward direction is \(-G(p, i\varepsilon_n + i\omega_\lambda)\) and the solid line in downward direction is \(-G(p, i\varepsilon_n)\). The upper gray circle is \(-\frac{e}{m} \rho(0)A_{\mu} \), where the momentum does not change, and the lower gray circle is \( j_{\nu}(0) \). The integral of this diagram in terms of \( p \) is odd under the variable change \( p_{\nu} \rightarrow -p_{\nu} \) and vanishes so that (d) does not contribute to the conductivity.

The Feynman diagrams for the current-current correlation function\(^{12}\) in

\(^{12}\) The procedure described in the footnote 20 of [I] can be repeated in terms of the off-diagonal propagator (15). Here we put \( F(\tau, \tau) \equiv \langle j^{\tau}(k; \tau) j^{\tau}(k = 0) \rangle \). Neglecting the
(7) are drawn in Fig. I where we only need the first order contributions in $A$ to obtain the conductivity linear in $H$ and the second order one has been neglected. The product of $j^H_\mu(k)$ and $j^H_\nu(0)$ leads to four kinds of terms: (i) $j^e_\mu(k) \cdot j^0_\nu(0)$, (ii) $j^e_\mu(k) \cdot \rho(-k) A_\nu$, (iii) $\rho(0) A_\mu \cdot j^e_\nu(0)$, and (iv) $\rho(0) A_\mu \cdot \rho(-k) A_\nu$. In the case of (i) two current vertices can be connected by two ways as Figs. I(a) and (b) within the linear order of $A$. In the cases of (ii) and (iii) one of the vertices is already proportional to $A$ so that the vertices are connected by the electron propagator diagonal in momentum as Figs. I(c) and (d). The

vertex correction

$$F(\tau, \tau) = 2e \sum_\rho \sum_{\rho'} \frac{v_{p+k/2} + v_{p-k/2}}{2} \langle a^\dagger_{p-k/2}(\tau) a_{p+k/2}(\tau) a^\dagger_{p'}(0) a_{p'}(0) \rangle v_{p'}$$

is factorized as

$$F(\tau, \tau) = 2e \sum_\rho \sum_{\rho'} \left[ \langle a^{\dagger}_{p+k/2}(\tau) a^\dagger_{p+k/2}(0) \rangle \langle a^\dagger_{p-k/2}(\tau) a_{p-k/2}(0) \rangle v_{p+k/2} \right.$$

$$\left. + \langle a^{\dagger}_{p+k/2}(\tau) a^\dagger_{p-k/2}(0) \rangle \langle a^\dagger_{p-k/2}(\tau) a_{p+k/2}(0) \rangle v_{p-k/2} \right]$$

within the linear order of $A$ (See (16)). Since

$$\langle a_p(\tau)a^\dagger_{p'}(0) \rangle = -G(p \leftarrow p', \tau), \quad \langle a^\dagger_{p}(\tau)a_{p}(0) \rangle = G(p, -\tau),$$

for $\tau > 0$,

$$F(\tau, \tau) = -2e \sum_\rho \sum_{\rho'} \left[ \langle a_{p+k/2}(\tau) a^\dagger_{p+k/2}(0) \rangle \langle a^\dagger_{p-k/2}(\tau) a_{p-k/2}(0) \rangle \frac{p+k/2}{m} \right.$$

$$\left. + G(p+k/2 \leftarrow p-k/2, \tau) G(p-k/2 \leftarrow p+k/2, -\tau) \frac{p-k/2}{m} \right].$$

Introducing the Fourier transforms

$$F(\tau, \tau) = \frac{1}{\beta} \sum_m F(\omega_m) e^{-i\omega_m \tau},$$

and

$$G(p, \tau) = \frac{1}{\beta} \sum_n G(p, \epsilon_n) e^{-i\epsilon_n \tau}, \quad G(p \leftarrow p', -\tau) = \frac{1}{\beta} \sum_n G(p \leftarrow p', \epsilon_n) e^{i\epsilon_n \tau},$$

we obtain the contributions in Figs. 1-(a) and (b)

$$F(\omega_m) = -2e \frac{1}{\beta} \sum_n \sum_\rho \frac{p}{m} \left[ G(p+k/2, \omega_m + \epsilon_n) G(p+k/2 \leftarrow p-k/2, \epsilon_n) \frac{p+k/2}{m} \right.$$

$$\left. + G(p+k/2 \leftarrow p-k/2, \omega_m + \epsilon_n) G(p-k/2, \epsilon_n) \frac{p-k/2}{m} \right]$$

6
case (iv) is not considered here, because it is the second order contribution in $A$.

Here $G(p + k/2 \leftarrow p - k/2, i\varepsilon_n)$ is the Fourier transform of the off-


diagonal propagator in momentum variable

$$G(p + k/2 \leftarrow p - k/2, \tau) = -\langle T_\tau \{ \epsilon_{p+k/2,\sigma} \tau \} \epsilon_{p-k/2,\sigma} \rangle, \quad (15)$$

and evaluated by

$$G(p + k/2 \leftarrow p - k/2, i\varepsilon_n) \approx G(p + k/2, i\varepsilon_n) \cdot (-e v_p \cdot A) \cdot G(p - k/2, i\varepsilon_n)$$

$$\approx G(p, i\varepsilon_n) \cdot (-e v_p \cdot A) \cdot G(p, i\varepsilon_n), \quad (16)$$

which is the first order perturbation in terms of (12) in the limit of $k \to 0$.

In order to obtain the conductivity proportional to the magnetic field defined in (5), which is linear in both $A$ and $k$, we have to extract the $k$-linear contribution from the processes shown in Figs. 1-(a), (b), (c). A $k$-linear contribution comes from the propagators diagonal in momentum variable $p \pm k/2$ represented by the solid line. It also comes from $j_e(0)$ but does not from $j_e(\mu)$. 16

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13 If we write down the rule of Feynman diagram faithfully:

$$- G(p + k/2 \leftarrow p - k/2, i\varepsilon_n) \approx \left[ - G(p, i\varepsilon_n) \right] \cdot \left[ -(-e v_p \cdot A) \right] \cdot \left[ -G(p, i\varepsilon_n) \right].$$

I prefer the textbook, Lifshitz and Pitaevskii: *Statistical Physics Part 2* (Pergaman Press, Oxford, 1980), because such a faithful description is given concisely.

14 The same relation is obtained via the gauge transformation, $p \to p - eA$. For example, in the case of free electron propagator

$$G_0(p, i\varepsilon_n) = \left[ i\varepsilon_n - \left( \frac{p^2}{2m} - \mu \right) \right]^{-1},$$

is transformed into

$$G_0(p - eA, i\varepsilon_n) \approx G_0(p, i\varepsilon_n) + G_0(p, i\varepsilon_n) \cdot \left( -\frac{e}{m} p \cdot A \right) \cdot G_0(p, i\varepsilon_n).$$

15 In the case of free electrons the $k$-linear contribution of the free propagator is easily extracted as

$$G_0(p \pm k/2, i\varepsilon_n) \approx G_0(p, i\varepsilon_n) + G_0(p, i\varepsilon_n) \cdot \left( \pm \frac{p \cdot k}{2m} \right) \cdot G_0(p, i\varepsilon_n).$$

16 In this footnote we use diagonal propagators which appear in the right-hand side of (10). From $j_\mu(0)$ vertex we obtain the contribution as

$$G(p \pm k/2, i\varepsilon_n + i\omega_\lambda) \cdot \frac{e}{2m} \left[ \left( p_\nu + \frac{k_\nu}{2} \right) + \left( p_\nu \pm \frac{k_\nu}{2} \right) \right] \cdot G(p \pm k/2, i\varepsilon_n),$$
The $k$-linear part $\Phi^{(1)}_{(a)}(i\omega)\Phi^{(1)}_{(b)}(k,i\omega)$ is the summation of the $k$-linear contributions, $\Phi^{(a)}$, $\Phi^{(b)}$, $\Phi^{(c)}$, which are extracted from the processes shown in Fig. 1(a), (b), (c). Since $j^a_\mu(k)$ leads to the factor $(e/m)p_x$, $j^a_\nu(0)$ to $(e/2m)k_y$, and $G(p+k/2\leftrightarrow p-k/2,i\varepsilon_n)$ to $G(p,i\varepsilon_n)[-(e/m)p\cdot A]\cdot G(p,i\varepsilon_n)$,

$$\Phi^{(a)} = -2T\sum_n\sum_p \frac{1}{2} \left( \frac{e}{m} \right)^3 p^2_x G(p,i\varepsilon_n + i\omega) G(p,i\varepsilon_n)^2 k_y A_x, \quad (17)$$

where the product of the fermion-loop factor and the spin-degeneracy factor $2$, has been included. Since $j^a_\mu(0)$ leads to the factor $-(e/m)k_y$,

$$\Phi^{(b)} = -2T\sum_n\sum_p \frac{1}{2} \left( \frac{e}{m} \right)^3 p^2_x G(p,i\varepsilon_n + i\omega) G(p,i\varepsilon_n)^2 k_y A_x. \quad (18)$$

Since $\rho(-k)$ leads to the factor $-(e^2/m_A)y$, and $G(p+k/2,i\varepsilon_n + i\omega)$ to $G(p,i\varepsilon_n + i\omega) \cdot [(1/2m)p \cdot k] \cdot G(p,i\varepsilon_n + i\omega)$, and $G(p-k/2,i\varepsilon_n)$ to $G(p,i\varepsilon_n) \cdot [(1/2m)p \cdot k] \cdot G(p,i\varepsilon_n)$,

$$\Phi^{(c)} = -2T\sum_n\sum_p \left[ \frac{1}{2} \left( \frac{e}{m} \right)^3 p^2_x G(p,i\varepsilon_n + i\omega) G(p,i\varepsilon_n)^2 k_x A_y 
+ \frac{1}{2} \left( \frac{e}{m} \right)^3 p^2_x G(p,i\varepsilon_n + i\omega) G(p,i\varepsilon_n)^2 k_y A_x \right]. \quad (19)$$

which leads to a $k$-linear contribution

$$G(p,i\varepsilon_n + i\omega) \cdot \left( \pm \frac{e}{2m}k_y \right) \cdot G(p,i\varepsilon_n).$$

From $j^\mu_\nu(k)$ vertex we obtain

$$G(p+k/2,i\varepsilon_n + i\omega) \cdot \frac{e}{2m} \left[ \left( p_\mu + \frac{k_\mu}{2} \right) + \left( p_\mu - \frac{k_\mu}{2} \right) \right] \cdot G(p-k/2,i\varepsilon_n),$$

which reduces to a $k$-independent contribution

$$G(p,i\varepsilon_n + i\omega) \cdot \left( \frac{e}{m}p_\mu \right) \cdot G(p,i\varepsilon_n).$$

in the limit of $k \to 0$.

See the footnote for the current-current correlation function in 17.

17 The integrand is proportional to $(p_x A_x + p_y A_y + p_z A_z)\cdot p_x$ but the terms proportional to $A_x$ and $A_z$ are odd in $p_x$ and vanish by the integration over $p_x$. In the same manner the integrand in 19 is proportional to $(p_x k_x + p_y k_y + p_z k_z)\cdot p_x$ but the terms proportional to $k_y$ and $k_z$ vanish.

18 Throughout the series of three Notes the Zeeman splitting is neglected so that the spin degrees of freedom only appears as the degeneracy factor 2.

19 This factor is already proportional to $k_y$ so that we can put $k = 0$ for all the propagators in the diagrams (a) and (b), because we only need the contribution linear in $k_y$. 

8
Thus we obtain
\[
\Phi(a) + \Phi(b) + \Phi(c) = -\frac{1}{2} \left(\frac{e}{m}\right)^3 (k_x A_y - k_y A_x) T \sum_n \sum_p p_x^2 
\times \left[ G(p, i\varepsilon_n + i\omega)G(p, i\varepsilon_n)^2 - G(p, i\varepsilon_n + i\omega)^2 G(p, i\varepsilon_n) \right].
\]

If we set \( \mathbf{H} = (0, 0, H) \) so that \( H = i(k_x A_y - k_y A_x) \), (20) leads to
\[
\Phi^{(1)}(i\omega) = -\frac{H}{i} \left(\frac{e}{m}\right)^3 T \sum_n \sum_p p_x^2 
\times \left[ G(p, i\varepsilon_n + i\omega)G(p, i\varepsilon_n)^2 - G(p, i\varepsilon_n + i\omega)^2 G(p, i\varepsilon_n) \right].
\]

Figure 2: Diagrams for a fixed gauge \( \mathbf{A} = (A_x, 0, 0) \): The left diagram corresponds to the one in Fig. 1-(a) and the right to Fig. 1-(b). The broken line represents the coupling to the magnetic field (12). (left) The solid line in upward direction is \(-G(p, i\varepsilon_n + i\omega)\). The downward process is the product \([-G(p, i\varepsilon_n)] \cdot [-(-J_x A_x)] \cdot [-G(p, i\varepsilon_n)]\). The upper black circle is \( J_x \) and the lower black circle is \((\partial J_y / \partial p_y)k_y/2\). (right) The solid line in downward direction is \(-G(p, i\varepsilon_n)\). The upward process is the product \([-G(p, i\varepsilon_n + i\omega)] \cdot [-(-J_x A_x)] \cdot [-G(p, i\varepsilon_n + i\omega)]\). The upper black circle is \( J_x \) and the lower black circle is \(-(\partial J_y / \partial p_y)k_y/2\). Here \( J_x \equiv (e/m)p_x \) and \( \partial J_y / \partial p_y \equiv e/m \).

If we choose the gauge \( \mathbf{A} = (A_x, 0, 0) \), we do not have to consider the contribution \( \Phi(c) \) of the diamagnetic current. Such a choice make the calculation simpler\(^{21}\) The relevant processes leading to \( \mathbf{H} \)-linear contribution are

\(^{20}\) Eq. (2.19) in [FEW] obtained for general dispersion reduces to \(^{21}\) for isotropic dispersion \( \nu_p = p/m \).

\(^{21}\) The approach by Altshuler, Khmel’nitzkii, Larkin and Lee: Phys. Rev. B 22, 5142 (1980) considers only paramagnetic contributions \( \Phi(a) \) and \( \Phi(b) \) by fixing the gauge.
summarized in Fig. [2]. The expressions (17) and (18) are rewritten as

\[-\frac{1}{2}\Phi_c(a) = T \sum_n \sum_p J_x \frac{\partial J_y}{\partial p_y} \frac{k_y}{2} G(i\epsilon_n)( - J_x A_x)G(i\epsilon_n)G(i\epsilon_n + i\omega_\lambda), \tag{22}\]

\[-\frac{1}{2}\Phi_c(b) = T \sum_n \sum_p J_x \frac{\partial J_y}{\partial p_y} \left( - \frac{k_y}{2} \right) G(i\epsilon_n)G(i\epsilon_n + i\omega_\lambda)( - J_x A_x)G(i\epsilon_n + i\omega_\lambda), \tag{23}\]

where \(J_\mu \equiv (e/m)p_\mu\) and \(G(i\epsilon_n) \equiv G(p, i\epsilon_n).\) In this case (21) is rewritten as

\[-\frac{1}{2}\Phi_e(i\omega_\lambda) = H \sum_n \sum_p (J_x)^2 \frac{\partial J_y}{\partial p_y} \times \left[G(i\epsilon_n + i\omega_\lambda)G(i\epsilon_n)^2 - G(i\epsilon_n + i\omega_\lambda)^2 G(i\epsilon_n)\right], \tag{24}\]

where the fermion-loop factor and the spin-degeneracy factor are moved to the left-hand side. Here \(H = -ik_y A_x.\)

In the following I adopt a shortcut calculation along [FEW] for

\[I_e(\omega + i\delta) - I_e(i\delta) \approx -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left( - \frac{\partial f(\epsilon)}{\partial \epsilon} \right) G^A(\epsilon)G^R(\epsilon),\]

and (167) as

\[I^Q(\omega + i\delta) - I^Q(i\delta) \approx -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left( - \frac{\partial f(\epsilon)}{\partial \epsilon} \right) \epsilon G^A(\epsilon)G^R(\epsilon).\]

The integral

\[-\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} G^A(\epsilon)G^R(\epsilon) = i\omega \tau,\]

by the residue readily leads to (165) and (171) in [I].

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22 Using the shortcut formula (27) the summation (156) in [I] is evaluated as

\[I^e(\omega + i\delta) - I^e(i\delta) \approx -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left( - \frac{\partial f(\epsilon)}{\partial \epsilon} \right) G^A(\epsilon)G^R(\epsilon),\]

and (167) as

\[I^Q(\omega + i\delta) - I^Q(i\delta) \approx -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left( - \frac{\partial f(\epsilon)}{\partial \epsilon} \right) \epsilon G^A(\epsilon)G^R(\epsilon).\]
and neglecting the contributions from $C_1$ and $C_4$ we obtain

$$I^e(i\omega) \equiv \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \left[ -X^A(\epsilon) Y^R(\epsilon + i\omega) + X^A(\epsilon - i\omega) Y^R(\epsilon) \right],$$  \hspace{1cm} (26)

where the first integrand is the contribution along $C_2$ and the second along $C_3$. The contours of the integral, $C_1$, $C_2$, $C_3$ and $C_4$, are defined in Fig. 5 of [I]. To calculate the DC conductivity we only need the $\omega$-linear contribution

$$I^e(\omega + i\delta) - I^e(i\delta) \equiv -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \left[ X^A(\epsilon) \frac{\partial Y^R(\epsilon)}{\partial \epsilon} + \frac{\partial X^A(\epsilon)}{\partial \epsilon} Y^R(\epsilon) \right]$$

$$= -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left( -\frac{\partial f(\epsilon)}{\partial \epsilon} \right) X^A(\epsilon) Y^R(\epsilon),$$  \hspace{1cm} (27)

where we have employed the integration by parts. By the same approximation leading to (164) in [I], the application of (27) to (21) results in

$$\Phi^{e(1)}(\omega + i\delta) - \Phi^{e(1)}(i\delta)$$

$$\sim -i\omega \frac{e^3}{m} \sum_p i_x^2 \left( -\frac{\partial f(\xi_p)}{\partial \xi_p} \right) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left[ G^A(p,\epsilon) G^R(p,\epsilon)^2 - G^A(p,\epsilon)^2 G^R(p,\epsilon) \right],$$  \hspace{1cm} (28)

where

$$G^R(p,\epsilon) = \frac{1}{\epsilon - \xi_p + i/2\tau}, \quad G^A(p,\epsilon) = \frac{1}{\epsilon - \xi_p - i/2\tau},$$  \hspace{1cm} (29)

which is the analytic continuation of the thermal propagator (13). The integral over $\epsilon$ is performed by evaluating the residue so that we finally obtain

$$\sigma_{xy} \equiv \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{i\omega} \left[ \Phi^{e(1)}_{xy}(k,\omega + i\delta) - \Phi^{e(1)}_{xy}(k,\delta) \right]$$

$$\sim 2\frac{e^3}{m} H \sum_p \left( -\frac{\partial f(\xi_p)}{\partial \xi_p} \right) i_x^2 \tau^2,$$  \hspace{1cm} (30)

23 Here we employ the propagator and the location of its pole is explicitly known. The integral on $C_1$ with $G^R$ only vanishes by choosing a closed contour in which there is no pole. The integral on $C_4$ with $G^A$ only also vanishes. On the other hand, the integrals on $C_2$ and $C_3$, which contain a pair of $G^R \cdot G^A$ at least, are determined by the pole of either $G^R$ or $G^A$.

$$\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left[ G^A(p,\epsilon) G^R(p,\epsilon)^2 - G^A(p,\epsilon)^2 G^R(p,\epsilon) \right] = -2\tau^2.$$
which coincides with the result of the Boltzmann transport (1) taking into account that $H = (0, 0, H)$ and $E = (0, E, 0)$.

The expectation value of the heat current $J^Q$ is expressed as the linear response to electric field

$$J^Q_\mu(\mathbf{k}, \omega) = \sum_\nu \tilde{\alpha}_{\mu\nu}(\mathbf{k}, \omega) E_\nu(\mathbf{k} = 0, \omega), \quad (31)$$

with

$$\tilde{\alpha}_{\mu\nu}(\mathbf{k}, \omega) = \frac{1}{i\omega} [\Phi^Q_{\mu\nu}(\mathbf{k}, \omega + i\delta) - \Phi^Q_{\mu\nu}(\mathbf{k}, i\delta)]. \quad (32)$$

As has been discussed in the above, in order to extract the contribution proportional to $H$, we only need to take the heat current $j^Q_\mu(\mathbf{k})$ in the absence of the magnetic field so that only the difference between $\Phi^Q_{\mu\nu}(\mathbf{k}, \omega + i\delta)$ and $\Phi^Q_{\mu\nu}(\mathbf{k}, i\delta)$ is the factor $\frac{1}{i\epsilon_n + (\epsilon_n + \omega/2)}$ as in the case of §11 in [I].

By repeating the above shortcut calculation for

$$I^Q(i\omega_\lambda) \equiv -\frac{1}{\beta} \sum_n \left( i\epsilon_n + \frac{i\omega_\lambda}{2} \right) X(i\epsilon_n) Y(i\epsilon_n + i\omega_\lambda), \quad (33)$$

we obtain

$$I^Q(\omega + i\delta) \equiv \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \left[-\left(\epsilon + \frac{\omega}{2}\right) X^A(\epsilon) Y^R(\epsilon + \omega) + \left(\epsilon - \frac{\omega}{2}\right) X^A(\epsilon - \omega) Y^R(\epsilon)\right]. \quad (34)$$

The $\omega$-linear contribution becomes

$$I^Q(\omega + i\delta) - I^Q(i\delta) \equiv -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} f(\epsilon) \left\{ \epsilon \left[ X^A(\epsilon) \frac{\partial Y^R(\epsilon)}{\partial \epsilon} + \frac{\partial X^A(\epsilon)}{\partial \epsilon} Y^R(\epsilon) \right] + X^A(\epsilon) Y^R(\epsilon) \right\}$$

$$= -\omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left(-\frac{\partial f(\epsilon)}{\partial \epsilon}\right) \epsilon X^A(\epsilon) Y^R(\epsilon), \quad (35)$$

by the integration by parts.

Thus the $\mathbf{k}$-linear part $\Phi^Q_{(1)}(i\omega_\lambda)$ of $\Phi^Q_{xy}(\mathbf{k}, i\omega_\lambda)$ is obtained as

$$\Phi^Q_{(1)}(\omega + i\delta) - \Phi^Q_{(1)}(i\delta)$$

$$\sim -i\omega \sum_p \frac{e^2}{m} \sum_p v_p^2 \left(-\frac{\partial f(\xi_p)}{\partial \xi_p}\right) \xi_p \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i} \left[ G^A(p, \epsilon) G^R(p, \epsilon)^2 - G^A(p, \epsilon)^2 G^R(p, \epsilon) \right]. \quad (36)$$

---

25 We only need the charge current $j^Q_\mu(\mathbf{k})$ in the absence of the magnetic field to obtain $\Phi^Q_{\mu\nu}(\mathbf{k}, \omega + i\delta)$ proportional to $H$.

26 It should be noted that the factor is proportional to the summation of incoming and outgoing frequencies $\epsilon_n + (\epsilon_n + \omega_\lambda)$. 

12
where we have pulled out the factor \(- (\partial f(\epsilon)/\partial \epsilon) \cdot \epsilon\) from the integral over \(\epsilon\) as in the case of §11 in [I]. Finally we obtain

\[
\tilde{\alpha}_{xy} \equiv \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{i\omega} \left[ \Phi^Q_{xy}(k, \omega + i\delta) - \Phi^Q_{xy}(k, i\delta) \right]
\]

\[
\sim 2 \frac{e^2}{m} H \sum_{p} \left( - \frac{\partial f(\xi_p)}{\partial \xi_p} \right) v^2 \tau x^2 \xi_p,
\]

(37)

which coincides with the result of the Boltzmann transport (2).

4 GL Transport

The linearized GL transport theory gives

\[
\sigma_{xy} = \frac{e^2 \tau_2}{48d \tau_1} \frac{h}{\epsilon^2},
\]

(38)

and

\[
\alpha_{xy} = \frac{|e|}{4\pi d} \frac{h}{\epsilon},
\]

(39)

where \(h \equiv 2|e|H\xi_0^2\). The derivation of these results shall be discussed by the non-perturbational treatment of the magnetic field in the third Note.

\[27\] The remark in the footnote 40 of [I] also applies to (37).

\[28\] See the footnote 46 in [I]. Since \(\tau_2 < 0\) in 3D as discussed in the footnote 61 of [I], \((38)\) for Cooper pairs has the same sign as \((30)\) for electrons. It is natural, because \(\sigma_{xy}\) is related to the cyclotron motion of the charged object and both electrons and Cooper pairs carry negative charge. By the same reason \((37)\) for free electrons in 3D and \((39)\) for Cooper pairs have the same sign. The same discussion also applies to \(\alpha_{xx}\) so that \((171)\) in [I] for free electrons in 3D and \((199)\) in [I] for Cooper pairs have the same sign.

\[29\] The conductivity tensor \((38)\) and the thermo-electric tensor \((39)\) are given in (3.52) and (4.36) of [3] in [I]. The contribution of the magnetization current modifies (4.36) into (4.38) that is identical to the result in TABLE I of [5] in [I].
5 Cooper-Pair Transport: AL Process

The calculations for electrons in §3 are translated into those for Cooper pairs straightforwardly.\(^\text{30}\) The contribution of the left diagram in Fig. 3 is

\[
\Phi_{(a)} = T \sum_m \sum_q \tilde{\Delta}_x^e \frac{\partial \tilde{\Delta}_y^e}{\partial q_y} \frac{k_y}{2} L(i\omega_m)(-\tilde{\Delta}_x^e A_x) L(i\omega_m) L(i\omega_m + i\omega_\lambda), \tag{40}
\]

which corresponds to (22) and that of the right diagram is

\[
\Phi_{(b)} = T \sum_m \sum_q \tilde{\Delta}_x^e \frac{\partial \tilde{\Delta}_y^e}{\partial q_y} \left(-\frac{k_y}{2}\right) L(i\omega_m) L(i\omega_m + i\omega_\lambda)(-\tilde{\Delta}_x^e A_x) L(i\omega_m + i\omega_\lambda), \tag{41}
\]

which corresponds to (23) where \(\tilde{\Delta}_x^e \equiv 4eN(0)\xi_0 q_x\) and \(L(i\omega_m) \equiv L(q, i\omega_m)\).

Thus (24) is translated into

\[
\Phi^{(1)}_{e}(i\omega_\lambda) = \frac{H}{2i} T \sum_m \sum_q \left(\tilde{\Delta}_x^e \right)^2 \frac{\partial \tilde{\Delta}_y^e}{\partial q_y}
\times \left[ L(i\omega_m + i\omega_\lambda) L(i\omega_m)^2 - L(i\omega_m + i\omega_\lambda)^2 L(i\omega_m) \right]. \tag{42}
\]

This result is rewritten as

\[
\Phi^{(1)}_{e}(i\omega_\lambda) = 32 \frac{H}{i} (e\xi_0^2)^3 \sum_q q_x^2 \tilde{I}^e(i\omega_\lambda), \tag{43}
\]

with

\[
\tilde{I}^e(i\omega_\lambda) \equiv T \sum_m \left[ \tilde{L}(i\omega_m + i\omega_\lambda)^2 \tilde{L}(i\omega_m) - \tilde{L}(i\omega_m + i\omega_\lambda) \tilde{L}(i\omega_m)^2 \right]. \tag{44}
\]

\(^{30}\) As has been discussed in the footnote 7 of [I], \(L\) can be identified with the propagator \(D_\Delta\) near \(T_c\). The order-parameter field \(\Psi\), (177) in [I], is related to the gap function \(\Delta\) as

\[
\Psi = \sqrt{z\Delta} \text{ where } z = \frac{7\zeta(3)n}{8\pi^2T_c^2} \text{ in 3D.}
\]

\(^{31}\) The perturbational calculation in terms of electron propagators shall be given in the Supplement noticed in the footnote 63 of [I].
Figure 3: AL process for a fixed gauge $A = (A_x, 0, 0)$: These diagrams correspond to those in Fig. 2. The broken line represents the coupling to the magnetic field (12). (left) The wavy line in right-side is $-L(q, i\omega + i\lambda)$. The upper black circle is $\Delta^e_x$ and the lower black circle is $(\partial \Delta^e_y / \partial q_y) k_y / 2$. (right) The wavy line in left-side is $-L(q, i\omega_m)$. The right-side process is the product $[-L(q, i\omega_m + i\omega_L)] \cdot [-(-\Delta^e_x A_x)] \cdot [-L(q, i\omega_m)]$. The upper black circle is $\Delta^e_x$ and the lower black circle is $-(\partial \Delta^e_y / \partial q_y) k_y / 2$. Here $\Delta^e_x \equiv 4eN(0)\xi^2_0 q_x$ and $\partial \Delta^e_y / \partial q_y \equiv 4eN(0)\xi^2_0$.

where $L(i\omega_m) \equiv -\tilde{L}(i\omega_m)/N(0)$. Analytical continuation of this discrete summation becomes

$$\tilde{I}^e(\omega + i\delta) = \int_{-\infty}^{\infty} dx \frac{1}{\pi} n(x) \left\{ [R(+)^2 - A(-)^2] \text{Im} R - [R(+) - A(-)] \text{Im} R^2 \right\},$$

where

$$R = \frac{1}{\eta - i\tau_0 x}, \quad A = \frac{1}{\eta + i\tau_0^* x},$$

$$R(+) = \frac{1}{\eta - i\tau_0(x + \omega)}, \quad A(-) = \frac{1}{\eta + i\tau_0^*(x - \omega)).$$

Here $\tau_0$ is complex, $\tau_0 \equiv \tau_1 + i\tau_2$, and $\tau_1 \gg |\tau_2|$. The $\omega$-linear contribution is evaluated as

$$\tilde{I}^e(\omega + i\delta) - \tilde{I}^e(i\delta) \equiv \omega \int_{-\infty}^{\infty} dx \frac{1}{\pi} n(x) \left\{ [2R \frac{\partial R}{\partial x} + 2A \frac{\partial A}{\partial x}] \text{Im} R - \left[ \frac{\partial R}{\partial x} + \frac{\partial A}{\partial x} \right] \text{Im} R^2 \right\}$$

$$= \omega \int_{-\infty}^{\infty} dx \frac{1}{2\pi i} n(x) (R - A) \left\{ R \frac{\partial R}{\partial x} + A \frac{\partial A}{\partial x} - R \frac{\partial A}{\partial x} - A \frac{\partial R}{\partial x} \right\}.$$
where \( \text{Im} R = (R - A)/2i \) and \( \text{Im} R^2 = (R + A)(R - A)/2i \). Employing the high-temperature expansion
\[ n(x) \approx T/x, \]

Employing the high-temperature expansion,
\[ 33 \eta(x) \approx \frac{T}{x}, \]

Employing the high-temperature expansion,
\[ \tilde{I}^e(i \delta) - \tilde{I}^e(i \delta) \approx \omega T \cdot J, \]

with
\[ J \equiv \int_{-\infty}^{\infty} \frac{dx}{2\pi i} \left\{ \frac{R - A}{i x} \right\}, \]

Here we have picked up
\[ R - A \]

in the integrand, because \( \text{Im} L_R(x)/x \) is proportional to the Lorentz function in \( x \) and basic quantity. Using
\[ \frac{\partial R}{\partial x} = i \tau_0 R^2, \quad \frac{\partial A}{\partial x} = -i \tau_0^* A^2, \]

we obtain
\[ J = \int_{-\infty}^{\infty} \frac{dx}{2\pi i} (\tau_0^* + \tau_0) RA \{ -\tau_0 R^3 + \tau_0^* A^3 \}. \]

The integral is evaluated by the residue so that
\[ J = -i \frac{(\tau_0^* \tau_0 (\tau_0 - \tau_0^*))}{(\tau_0^* + \tau_0)^2} \frac{1}{\eta^4} \approx \frac{\tau_2}{2} \frac{1}{\eta^4}. \]

---

33 The statement between (210) and (211) in [I] is insufficient. The cut-off frequency \( \omega_c \) of the fluctuation propagator, (35) in [I], is determined by the condition \( \epsilon = \omega_c \tau_0 \). Since \( \tau_0 = \pi/8T, \omega_c/T = 8\epsilon/\pi \) so that we can use the high-temperature expansion for the integrand with \( x < \omega_c \) in the limit of \( \epsilon \to 0 \).

34 \( \text{Im} L^R(x) = (R - A)/2i \).

35 We have used
\[ \int_{-\infty}^{\infty} dx \left( \frac{R A}{\eta} \right) RA = 0. \]

This is derived by the integration by parts noting that
\[ \frac{\partial}{\partial x} (RA)^2 = 2RA \left( \frac{\partial A}{\partial x} + A \frac{\partial R}{\partial x} \right). \]

36
\[ \int_{-\infty}^{\infty} dx R^4 A = \frac{2\pi}{\tau_0} \frac{1}{(1 + \tau_0^*)^4} \frac{1}{\eta^4}, \]
\[ \int_{-\infty}^{\infty} dx R A^4 = \frac{2\pi}{\tau_0} \frac{1}{(1 + \tau_0^*)^4} \frac{1}{\eta^4}. \]
Therefore
\[ \Phi^e(\omega + i\delta) - \Phi^e(i\delta) = 16\frac{H_i e^3}{t} \omega T \tau_2 \sum_q \frac{\xi_0^6 q_x^2}{(\epsilon + \xi_0^2 q^2)^4}, \]
(55)
and this leads to
\[ \sigma_{xy} = -16\frac{H_i e}{t} \omega T \tau_2 \sum_q \frac{\xi_0^6 q_x^2}{(\epsilon + \xi_0^2 q^2)^4}. \]
(56)
In 2D the \(q\)-summation is performed as
\[ \sum_q \frac{\xi_0^4 q_x^2}{(\epsilon + \xi_0^2 q^2)^4} = \frac{1}{8\pi d} \int_0^\infty \frac{dx}{(x + \epsilon)^4} = \frac{1}{48\pi d \epsilon^2}. \]
(57)
Finally we obtain (58) by using \(\tau_1 = \pi/8T\).

Only the difference between \(\Phi^{e,\mu}(k, i\omega_\lambda)\) and \(\Phi^{Q,\mu}(k, i\omega_\lambda)\) is the factor \((i\omega_m + i\omega_\lambda/2)/2\epsilon\) as has been discussed in §3 so that we readily obtain
\[ \Phi^Q(1)(i\omega_\lambda) = 32\frac{H_i}{t} (e\xi_0^2) \sum_q q_x^2 \frac{1}{2\epsilon} J^Q(i\omega_\lambda), \]
(58)
with
\[ J^Q(i\omega_\lambda) \equiv T \sum_m \left(i\omega_m + \frac{i\omega_\lambda}{2}\right) \left[ \hat{L}(i\omega_m + i\omega_\lambda)^2 \hat{L}(i\omega_m) - \hat{L}(i\omega_m + i\omega_\lambda) \hat{L}(i\omega_m)^2 \right]. \]
(59)
Analytical continuation of this discrete summation becomes
\[ J^Q(\omega + i\delta) = \int_{-\infty}^{\infty} \frac{dx}{\pi} n(x) \left(x + \frac{\omega}{2}\right) \left\{ R(+) \cdot \text{Im} R - R(+) \cdot \text{Im} R^2 \right\} 
+ \int_{-\infty}^{\infty} \frac{dx}{\pi} n(x) \left(x - \frac{\omega}{2}\right) \left\{ \text{Im} R^2 \cdot A(-) - \text{Im} R \cdot A(-)^2 \right\} 
\equiv \tilde{J}^Q(\omega + i\delta) + \tilde{J}^Q_2(\omega + i\delta), \]
(60)
\[ 37 \int_0^\infty dx \frac{x}{(x + \epsilon)^4} = \int_0^\infty dx \frac{1}{(x + \epsilon)^4} - \epsilon \int_0^\infty dx \frac{1}{(x + \epsilon)^3} = \frac{1}{6 \epsilon^2}. \]
38 Previously published formulae, (35) in [Uss], the formula between (10.35) and (10.36) in [2] of [I] and (7) in [LNV], differ significantly from ours [37].

[Uss] ≡ Ussishkin: Phys. Rev. B 68, 024517 (2003).
[LNV] ≡ Levchenko, Norman and Varlamov: Phys. Rev. B 83, 020506 (2011).
where
\[ \tilde{I}^Q_1(\omega + i\delta) = \int_{-\infty}^{\infty} \frac{dx}{\pi} n(x) x \left\{ [R(+) + A(-)]^2 \text{Im}R - [R(+) - A(-)]^2 \text{Im}R^2 \right\}. \]  
(61)

\[ \tilde{I}^Q_2(\omega + i\delta) = \frac{\omega}{2} \int_{-\infty}^{\infty} \frac{dx}{\pi} n(x) \left\{ [R(+) + A(-)]^2 \text{Im}R - [R(+) - A(-)]^2 \text{Im}R^2 \right\}. \]
(62)

In the following the imaginary part of \( \tau_0 \) is neglected: \( \tau_0^* = \tau_0 \) or \( \tau_2 = 0 \).

Employing the high-temperature expansion, \( n(x) \equiv T/x \), we obtain
\[ \tilde{I}^Q_1(\omega + i\delta) - \tilde{I}^Q_1(i\delta) \approx \frac{T}{\pi} \omega \int_{-\infty}^{\infty} dx \left\{ \left[ 2R \frac{\partial R}{\partial x} + 2A \frac{\partial A}{\partial x} \right] \text{Im}R - \left[ \frac{\partial R}{\partial x} + \frac{\partial A}{\partial x} \right] \text{Im}R^2 \right\} \]
\[ = \frac{T}{\pi} \omega \int_{-\infty}^{\infty} dx \frac{\tau_0}{2}(R + A)(R - A)^3 = 0, \]  
(63)

\[ \tilde{I}^Q_2(\omega + i\delta) - \tilde{I}^Q_2(i\delta) \approx \frac{T}{2\pi} \omega \int_{-\infty}^{\infty} dx \{ \left( R^2 + A^2 \right) \text{Im}R - \left( R + A \right) \text{Im}R^2 \} \]
\[ = \frac{T}{2\pi} \omega \int_{-\infty}^{\infty} dx (-2\tau_0)R^2A^2 \]
\[ = -\frac{T}{2} \omega \frac{1}{\eta^3}. \]  
(64)

Therefore
\[ \Phi^{Q(1)}(\omega + i\delta) - \Phi^{Q(1)}(i\delta) \approx 8He^2i\omega T \sum_q \frac{\xi_0^2 q^2}{(\epsilon + \xi_0^2 q^2)^3}. \]  
(65)

In 2D the \( q \)-summation is performed as
\[ \sum_q \frac{\xi_0^2 q^2}{(\epsilon + \xi_0^2 q^2)^3} = \frac{1}{8\pi d} \int_0^{\infty} dx \frac{x}{(x + \epsilon)^3} = \frac{1}{16\pi d} \frac{1}{\epsilon}. \]  
(66)

\[ \text{If we put } R \equiv u(x) + iv(x) \text{ and } A \equiv u(x) - iv(x), u(x) \text{ is even: } u(-x) = u(x) \text{ and } \]
\[ v(x) \text{ is odd: } v(-x) = -v(x) \text{ in } x. \]  
Namely \( R + A \) is even and \( R - A \) is odd. Therefore the integrand \((R + A)(R - A)^3\) is odd so that the integral in 63 vanishes.

\[ \int_{-\infty}^{\infty} dx R^2A^2 = \frac{\pi}{2} \frac{1}{\tau_0 \eta^3}. \]

\[ \int_0^{\infty} dx \frac{x}{(x + \epsilon)^3} = \int_0^{\infty} dx \frac{1}{(x + \epsilon)^2} - \epsilon \int_0^{\infty} dx \frac{1}{(x + \epsilon)^3} = \frac{1}{2} \frac{1}{\epsilon}. \]
Finally we obtain (39) via
\[ \tilde{\alpha}_{xy} = T \frac{e^2}{2\pi d} \frac{H \xi_0^2}{\epsilon}. \] (67)

The calculations in the absence of the magnetic field are also performed in the same manner. The integral (207) in [I] with real \( \tau_0 \) is evaluated as

\[ I^e(\omega + i\delta) - I^e(i\delta) = 41 \int_0^\infty \frac{T}{N(0)^2} \frac{T \omega}{\pi} \int_{-\infty}^{\infty} dx \left[ \frac{\partial R}{\partial x} - \frac{\partial A}{\partial x} \right] R - A \]
\[ = \frac{1}{N(0)^2} \int_{-\infty}^{\infty} dx \left( R^2 + A^2 \right) RA \]
\[ = \frac{T}{2 N(0)^2} i \omega \tau_0 \frac{1}{\eta^3}. \] (68)

The integral (216) in [I] with complex \( \tau_0 \) is evaluated as

\[ I^Q(\omega + i\delta) - I^Q(i\delta) = 42 \int_0^\infty \frac{T}{N(0)^2} \frac{T \omega}{\pi} \int_{-\infty}^{\infty} dx \left[ \frac{\partial R}{\partial x} - \frac{\partial A}{\partial x} \right] R - A \]
\[ + \frac{1}{N(0)^2} \int_{-\infty}^{\infty} dx \left( R - A \right) \frac{R - A}{2i} \]
\[ = \frac{T}{2 N(0)^2} \frac{\omega \tau_0}{\eta^2}. \] (69)

to give (220) in [I].

6 Remarks

The sections of Exercise and Acknowledgements are common to [I] so that I do not repeat here. Some typographic errors in [I] are listed below.

\[ \int_{-\infty}^{\infty} dx R^2 A = \int_{-\infty}^{\infty} dx RA = \frac{\pi}{4} \frac{1}{\tau_0 \eta^2}. \]

\[ \int_{-\infty}^{\infty} R^2 A \]

41 Noting that
\[ \frac{\partial}{\partial x} (R - A)^2 = 2(R - A) \left( \frac{\partial R}{\partial x} - \frac{\partial A}{\partial x} \right), \]
the integral in the first line of (69) is shown to vanish by integration by parts. The integral in the second line is evaluated using (51) and

\[ \int_{-\infty}^{\infty} dx R^2 A = 2 \frac{\pi}{\tau_0} \frac{1}{(1 + \frac{\tau_0}{\tau})^2} \frac{1}{\eta^2}; \]
\[ \int_{-\infty}^{\infty} dx RA^2 = 2 \frac{\pi}{\tau_0} \frac{1}{(1 + \frac{\tau_0}{\tau})^2} \frac{1}{\eta^2}. \]
• The thermodynamic relation in p. 11 should be \( dQ = dU - \mu dN \).

• The electric field in (68) and (78) should be \( E_\nu(k=0, \omega) \).

• The factor \( e \) in the first line of (73) should be \( e/m \).

• In Fig. 5 the subscript \( \lambda \) for the lower cut \( C_\lambda \) has dropped by the font-error at arXiv.

• In the footnote 47 the first term in the right-hand side of the complex GL equation should be \( (1 + i\epsilon_0)\bar{\Psi} \).

References

[1] In the following I only list the references that you must read. Neither originality nor priority is considered here. Other references are cited in the footnotes.

[2] Fukuyama, Ebisawa and Tsuzuki: Prog. Theoer. Phys. 46, 1028 (1971).\[43\]

\[43\] The overall minus sign in the right-hand side of the formula (2.21) should be removed. We should take care that \( e > 0 \) in this reference. On the other hand, \( -1 \) should be multiplied to the right-hand side of (2.27) and (2.28). Consequently these two errors cancel so that the final result (2.30) is correct in their notation. These two signs are corrected in Nishio and Ebisawa: Physica C 290, 43 (1997).