$K \to \pi\pi$ Decays with Domain Wall Fermions: Lattice Matrix Elements

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We present a lattice calculation of the $K \to \pi\pi$ and $K \to 0$ matrix elements of the $\Delta S = 1$ effective weak Hamiltonian which can be used to determine $\epsilon'/\epsilon$ and the $\Delta I = 1/2$ rule for $K$ decays in the Standard Model. The matrix elements for $K \to \pi\pi$ decays are related to $K \to \pi$ and $K \to 0$ using lowest order chiral perturbation theory. We also present results for the kaon $B$ parameter, $B_K$. Our quenched domain wall fermion simulation was done at $\beta = 6.0$ ($a^{-1} \approx 2$ GeV), lattice size $16^3 \times 32 \times 16$, and domain wall height $M_5 = 1.8$.

1. Introduction

Recent measurements of direct CP violation ($\epsilon'/\epsilon \neq 0$) in $K \to \pi\pi$ decays at FNAL and CERN allow an important test of the Standard Model (in particular, the CKM mixing paradigm). The effective weak Hamiltonian governing strangeness changing $K$ decays has been computed to next-to-leading order in QCD and QED by the Munich and Rome groups; the remaining piece of the puzzle is the hadronic matrix elements of the operators of this effective weak Hamiltonian.

The recent advance of domain wall (and overlap) fermions which maintain chiral symmetry to a high degree of accuracy$^2$ allows for a new attempt at this old problem. Chiral symmetry of domain wall fermions provides a significant advantage when computing light quark QCD observables since the lattice artifacts that arise when this symmetry is explicitly broken are greatly reduced. Mixing and renormalization of operators, which is already complicated in the continuum, is readily handled with domain wall fermions$^2$. However, calculations using improved Wilson fermions were also reported at this meeting$^3$, and it is still unclear which method will prove most advantageous.

In this study, we present preliminary results for the $K \to \pi$ and $K \to 0$ matrix elements of these operators, which when combined with lowest order chiral perturbation theory, yield the desired $K \to \pi\pi$ matrix elements$^4$. The contribution of R. Mawhinney in these proceedings takes up this point$^5$. Here we are concerned with the simpler $K \to \pi$ and $K \to 0$ lattice matrix elements. Also, the CP-PACS collaboration has presented a very similar calculation at this meeting$^6$.

2. Theoretical Framework

The $\Delta S = 1$ effective weak Hamiltonian is generated from the fundamental Standard Model Lagrangian by integrating out the top quark and $W$ boson. The resulting effective Hamiltonian is then evolved to a low scale ($\mu \ll M_W$) appropriate for lattice calculations using the renormalization group equations. The effective Hamiltonian above the charm threshold is:

$$H^{\Delta S = 1} = V_{ud}V_{us}^* \frac{G_F}{\sqrt{2}} \left[ (1 + \frac{V_{td}V_{ts}^*}{V_{ud}V_{us}^*}) \left( C_1(\mu)(Q_1(\mu) - Q_{1c}(\mu)) + C_2(\mu)(Q_{2s}(\mu) - Q_{2c}(\mu)) \right) - \frac{V_{td}V_{ts}^*}{V_{ud}V_{us}^*} \tilde{C}(\mu) \cdot \tilde{Q}(\mu) \right].$$  \hspace{1cm} (1)

where $\mu$ is the renormalization scale, $\tilde{Q}$ are a basis of local four-quark operators which are closed under renormalization, $C_i(\mu)$ the corresponding Wilson coefficients, and $V_{qq'}$ the
Cabibbo-Kobayashi-Maskawa mixing matrix elements which are fundamental parameters of the Standard Model.

The effective operators renormalized at the scale μ are

\begin{align*}
Q_1 &= \bar{s}_\alpha \gamma_\mu P_L d_\alpha \bar{u}_\beta \gamma_\mu P_L u_\beta \\
Q_2 &= \bar{s}_\alpha \gamma_\mu P_L d_\beta \bar{u}_\beta \gamma_\mu P_L u_\alpha \\
Q_{3,5} &= \bar{s}_\alpha \gamma_\mu P_L d_\alpha \sum_{u,d,s,c...} \bar{q}_\beta \gamma_\mu \gamma_\nu P_{(L,R)} q_\beta \\
Q_{4,6} &= \bar{s}_\alpha \gamma_\mu P_L d_\beta \sum_{u,d,s,c...} \bar{q}_\beta \gamma_\mu \gamma_\nu P_{(L,R)} q_\alpha \\
Q_{7,9} &= \frac{3}{2} \bar{s}_\alpha \gamma_\mu P_L d_\alpha \sum_{u,d,s,c...} e_\nu \bar{q}_\beta \gamma_\mu \gamma_\nu P_{(L,R)} q_\beta \\
Q_{8,10} &= \frac{3}{2} \bar{s}_\alpha \gamma_\mu P_L d_\beta \sum_{u,d,s,c...} e_\nu \bar{q}_\beta \gamma_\mu \gamma_\nu P_{(L,R)} q_\alpha \\
Q_{1c} &= \bar{s}_\alpha \gamma_\mu P_L d_\alpha \bar{c}_\beta \gamma_\mu P_L c_\beta \\
Q_{2c} &= \bar{s}_\alpha \gamma_\mu P_L d_\beta \bar{c}_\beta \gamma_\mu P_L c_\beta,
\end{align*}

with color indices α and β, \( P_{(L,R)} \equiv 1 \mp \gamma_5 \), the sums are taken over active quark flavors at the scale μ, and summation over ν is implied. \( Q_{1,2,1c,2c} \) are often referred to as current-current operators, \( Q_{3-6} \) QCD penguin operators, and \( Q_{7-10} \) electroweak penguin operators. It is useful to split the above operators according to their isospin, with \( Q_i \equiv Q_i^{1(2)} + Q_i^{3(2)} \).

As mentioned earlier, lowest order chiral perturbation theory relates \( K \to \pi \pi \) matrix elements to a linear combination of \( K \to \pi \) and \( K \to 0 \). For all \( Q_i \) except the electroweak penguins \( Q_{7,8} \) and all particles at rest, we have \[^4\]

\begin{equation}
\langle \pi^+ \pi^- | Q | K^0 \rangle = \frac{4i(m_K^2 - m_\pi^2)\alpha_1}{f^3} \\
\langle \pi^+ | Q | K^+ \rangle = \frac{4m_M^2(\alpha_1 - \alpha_2)}{f^4} \\
\langle 0 | Q | K^0 \rangle = \frac{4i(m_K^2 - m_\pi^2)\alpha_2}{f},
\end{equation}

where \( m_M \) is the meson mass for unphysical pseudoscalar states with \( m_s = m_d \). For \( Q_i \) which transform in a (27,1) chiral multiplet \( \alpha_2 = 0 \). Note that each matrix element vanishes linearly with the meson mass squared. This is an important prediction of chiral perturbation theory, and therefore QCD, and provides a solid test of the chiral symmetry properties of domain wall fermions. The strength of the above approach is that it allows less computationally demanding \( K \to \pi \) and \( K \to 0 \) matrix elements to be calculated on the lattice. A significant drawback to this approach is that it manifestly does not contain information on the final state interactions of the pions (for calculation of \( \epsilon' \) the final state s-wave scattering phases from experiment can be put in by hand, however).

In the case of the electroweak penguins \( Q_{7,8} \), the contribution in lowest order chiral perturbation theory to the \( K \to \pi \) matrix elements is constant in the chiral limit. The \( K \to 0 \) matrix element, however, still vanishes.

Since on the lattice \( \alpha_2 \) is quadratically divergent, the process of combining the second and third lines in Eq. 4 to obtain an expression for \( K \to \pi \pi \) requires a delicate cancellation of this quadratic divergence. See R. Mawhinney’s contribution for details.

Finally, lattice counterparts of the operators in Eq. \[^2\] must be matched to the continuum and renormalized since they are logarithmically divergent after power divergences have been subtracted. In addition, operators in the same symmetry multiplets mix through renormalization group running from \( M_W \) down to the low scale μ. This poses a serious challenge for lattice calculations. In our calculation this problem is handled remarkably well with the nonperturbative renormalization method of the Rome-Southampton group which was explained in the talk by C. Dawson \[^2\].

3. Simulation details

We have calculated matrix elements on 200 quenched gauge configurations at \( \beta = 6.0 \), with lattice four volume \( 16^3 \times 32 \), domain wall fermion extra dimension size \( L_s = 16 \), and domain wall height \( M_5 = 1.8 \). We have calculated with light quark masses \( m_\ell = 0.01 - 0.05 \), and charm quark masses \( m_c = 0.1 - 0.4 \). The physical kaon state made from degenerate quarks corresponds to \( m_\ell \approx 0.02 \), and \( m_c \approx 0.5 \) for the physical charm quark.
We extract matrix elements from three-point correlation functions. The external pseudoscalar states are interpolated from wall sources near the time direction boundaries, \( t = 5 \) and 27, and the operator is inserted between them. When the operator is far from either boundary, the desired lowest mass states dominate the correlation function. The forward and backward (in time) quark propagators used to interpolate the \( K \) and \( \pi \) states are linear combinations of propagators computed with periodic and anti-periodic boundary conditions which amounts to doubling the gauge field configuration in the time direction.

The closed fermion loops necessary for operators that have self contractions are computed from a complex Gaussian random source spread over time slices 14-17. All results are given as averages over these four time slices.

We have performed several important checks of our computer code. Most importantly, a completely independent check code was written to compare with our two production versions (the check code and one of the production codes run on the QCDSP supercomputer and the other production code, based on the MILC code, runs on the NERSC T3E). Output from each code generated on the same configuration agreed up to machine precision.

As a final useful check, the left-left operators in Eq. 2 go into themselves under a Fierz transformation. Thus color-mixed contractions can be compared to corresponding color-diagonal ones. We find perfect agreement in all cases.

The following results were obtained on the RIKEN BNL and Columbia University QCDSP supercomputers.

4. Results

Fig. 1 shows \( \langle \pi | Q_2^{(1/2)} | K \rangle \) as a function of quark mass, \( m_f = m_s = m_d \). An uncorrelated linear extrapolation yields a zero intercept, within statistical errors, which is in agreement with chiral perturbation theory. For strictly low energy QCD observables, we expect quantities to vanish at \( m_f = -m_{res} \). Since the \( \Delta f = 1/2 \) operators have contributions from physics scales near the (high energy) lattice cut-off, this is no longer true. Thus, for this matrix element, the statistical errors are not small enough to resolve these systematic effects. However, in R. Mawhinney’s contribution, we see that such effects are visible in the subtracted operator. Presumably this is due to the strong statistical correlations between the \( K \to \pi \) matrix element and the subtraction term. In Fig. 2 we show a similar plot for \( Q_6 \).

Figure 1. The \( K \to \pi \) matrix element of the bare operator \( Q_2^{(1/2)} \).

Here, an uncorrelated linear extrapolation has a non-zero intercept of roughly three standard deviations. Note that it vanishes for \( m_f > 0 \). Thus explicit chiral symmetry breaking effects are visible, though small. In Fig. 3 we show \( \langle \pi | Q_8^{(3/2)} | K \rangle \) which exhibits noticeable nonlinearity and does not vanish in the chiral limit, as expected.

Finally, we show an example of a \( K \to 0 \) matrix element in Fig. 4. Note that for the \( K \to 0 \) matrix elements we use \( m_s \neq m_d \) and fit to the form \( \langle 0 | Q_i | K \rangle / \langle 0 | \bar{s} \gamma_5 d | K \rangle = \text{const} + (m_s - m_d) \eta_i \) where chiral perturbation theory predicts \( \text{const} = 0 \). This ratio is useful since it is exactly the coefficient of the subtraction operator used to remove the quadratic divergence in \( \langle \pi | Q_i | K \rangle \). Since the quark masses enter as a difference, we expect ex-
plicit chiral symmetry breaking effects which do not depend on the quark mass to cancel. From the fit depicted in Fig. 4 we find that the constant term is zero within errors. Note that this ratio is extremely well resolved, and quite linear.

We take this opportunity to quote our value for the kaon $B$ parameter, $B_K^{\overline{MS}}(2\text{ GeV}) = 0.538(8)$, with $Z_L/Z_A = 0.928(6)$ computed nonperturbatively in the RI scheme and matched to the $\overline{MS}$-NDR scheme\cite{7}. The errors are statistical only, and the error on $B_K$ is obtained by adding the errors on the matrix element and the renormalization factor in quadrature. Our value is lower than the one quoted by Taniguchi at this meeting\cite{8}, probably due to the fact that our renormalization constant is lower than the perturbative one used in that study. Note our result is for $\beta = 6.0$ with a single lattice size.

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