Approximative solution of the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system

Yoon Seok Choun*

Baruch College, The City University of New York, Natural Science Department, A506, 17 Lexington Avenue, New York, NY 10010

Abstract

In earlier papers[3, 4, 5, 6] Gürsey et al. showed development of a bilocal baryon-meson field from two quark-antiquark fields. The Hamiltonian in the case of vanishing quark masses was shown to have a very good agreement with experiments [5]. The theory for vanishing mass was solved using Confluent Hypergeometric functions [6].

In this paper I construct the normalized wave function for the spin-free Hamiltonian with light quark masses (only up to the first order of the mass of quark). I develop the new kind of special function theory in mathematics that generalize all existing theories of Confluent Hypergeometric types. I call it the Grand Confluent Hypergeometric (GCH) Function. My solution produces previously unknown extra hidden radial quantum numbers relevant for description of supersymmetry and for generating new mass formulas.

This paper is 1st out of 10 in series “Special functions and three term recurrence formula (3TRF)”. See section 6 for all the papers in the series. The next paper in the series describes generalization of three term recurrence relation in linear ordinary differential equations and its applications[8].

Keywords: Supersymmetry, Relativistic quark model, Three term recurrence relation, Orthogonal relations, Generating functions
2000 MSC: 33C47, 33E15, 33E20, 33E30, 34A25, 34A30

1. Introduction

According to Gürsey et al. (1991 [6]), “We derive an effective Hamiltonian of the relativistic quark model. In the limit of zero quark masses, we obtain linear Regge trajectories for mesons. Based on the diquark-antiquark symmetry, we show that the Regge trajectories of baryons and mesons are parallel at high angular moments. We discuss the breaking of the hadronic supersymmetry and obtain a mass relation of mesons and baryons.” Following their analysis, the Hamiltonian in the case of vanishing quark masses was shown to have a very good agreement.

*Correspondence to: Baruch College, The City University of New York, Natural Science Department, A506, 17 Lexington Avenue, New York, NY 10010
Email address: ychoun@gc.cuny.edu; youn.choun@baruh.cuny.edu; ychoun@gmail.com (Yoon Seok Choun)

Preprint submitted to Elsevier
with experiments [5]. Since they neglect mass of quark in a supersymmetric differential equation [6], the power series expansion in closed forms consists of two term recursion relation. The theory for vanishing mass was solved using Confluent Hypergeometric function.

In this paper I include small mass of quark in their supersymmetric differential equation and its power series expansion in closed forms consists of three term recurrence relation. I develop a new kind of special function that generalizes the Confluent Hypergeometric series that I call Grand Confluent Hypergeometric (GCH) function.

GCH ordinary differential equation is of Fuchsian types with the one regular and one irregular singularities in (2.0.26). In contrast, Heun equation of Fuchsian types has the three regular and one irregular singularities [1, 2]. Heun equation has the four kind of confluent forms: (1) Confluent Heun (two regular and one irregular singularities), (2) Doubly confluent Heun (two irregular singularities), (3) Biconfluent Heun (one regular and one irregular singularities), (4) Triconfluent Heun equations (one regular and one irregular singularities). Biconfluent Heun equation is derived from the GCH equation by changing all coefficients $\mu = 1$ and $\varepsilon\omega = -q$. (see section 4 in Ref.[10]) In this paper I will show how Confluent Hypergeometric function is related to Grand Confluent Hypergeometric function analytically.

Due to its complex mathematical calculation in three term recurrence relation of their linear ordinary differential equation, I construct the analytic solution of their supersymmetric differential equation only up to the first order of the extremely small mass of quark. More than second order of the mass of quark is negligible in this paper.

As the mass of quark is negligible in their effective Hamiltonian of the relativistic quark model, its differential equation turns to be Confluent Hypergeometric differential equation. As we all know, there is only one eigenvalue and it has infinite eigennumbers which is called radial quantum number. For another example a hydrogen like atom wave function, only has one eigenvalue and has infinite eigennumbers.

In contrast infinite eigenvalues is arisen in their supersymmetric differential equation as the small mass of quark is included in their Hamiltonian. Each eigenvalue has infinite eigennumbers [15, 16]. The concept of its eigenvalues gives rise to an extra degree of a quantum number I designate as "$i$th kind of hidden radial quantum number" that will be expressed below. This makes it especially applicable to supersymmetric theories having three term recurrence relation in the power series expansion of their differential equations in nature. As we see in Regge trajectory plot of angular momentum vs. square of mass ($J$ vs. $m^2$), there are many linearly increasing lines with same slopes including bunch of eigenvalues corresponding to fermions and bosons. It is not clear what the meaning of many eigenvalues are: more details are explained in section 4.3 in Ref[15].

In section 2, I consider asymptotic behaviors of GCH differential equation including only up to the first order of the mass of quark. In section 3 and 4, I construct the power series expansion in closed forms of GCH function with including the first order of the mass of quark for the polynomial and infinite series. Also I derive the generating function and orthogonal relation of GCH polynomial with including the first order of the mass of quark.
1.1. Analytic solution neglecting the mass of quark in effective Hamiltonian of the relativistic quark model

Following Gürsey et al., there is the spin free Hamiltonian involving only scalar potential for the $q \bar{q}$ system:[3, 4, 5, 6]

$$H^2 = 4 \left[ (m + \frac{1}{2}br)^2 + P^2 + \frac{l(l+1)}{r^2} \right]$$  \hspace{1cm} (1.1.1)

where $P^2 = -\frac{\partial^2}{\partial \theta^2} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \phi^2}$, $m =$ mass of quark, $b =$ real positive, and $l =$ angular momentum quantum number. Gürsey et al. assume that $m$ can be neglected because $m$ is extremely small for $u$ and $d$ quarks.[4, (1.1.1)]

$$H^2 \simeq 4 \left[ \frac{1}{4} b^2 r^2 + P^2 + \frac{l(l+1)}{r^2} \right]$$  \hspace{1cm} (1.1.2)

Its normalized wave function by using orthogonal relation is

$$\psi(r, \theta, \phi) \approx \sqrt{\frac{b^r}{2r-1\Gamma(|\alpha_0| + 1)\Gamma(|\alpha_0| + \gamma)}} \; r^{l+1/2} e^{-br/2} F_{|\alpha_0|} (\gamma = l + 3/2; \; z) \; Y_l^m \; (\theta, \phi)$$  \hspace{1cm} (1.1.3)

where

$$\alpha_0 = 1 - n_0 = \frac{1}{2b} \left( E^2/4 - (l + 3/2) b \right) = 0, -1, -2, -3, \cdots$$  \hspace{1cm} (1.1.4a)

$n_0 = 1, 2, 3, \cdots$ which is primary radial quantum number.

$$F_{|\alpha_0|}(\gamma = l + 3/2; \; z) = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} (-|\alpha_0|)^n z^n$$  \hspace{1cm} (1.1.4b)

is the first kind of the independent solution of Confluent Hypergeometric function. And its eigenvalue is

$$E^2 = 4b (2|\alpha_0| + l + 3/2) = 4b (2n_0 + l + 1/2)$$  \hspace{1cm} (1.1.5)

1.2. Analytic solution including only up to the first order of the mass of quark in effective Hamiltonian of the relativistic quark model

In this paper I consider normalized wave function including the small mass $m$: only up to 1st order of $m$ terms. When wave function $\psi(r) = e^{\pm (r/\hbar)} r^l Y_l^m (\theta, \phi)$ acts on both sides of (1.1.1), it becomes

$$r \frac{\partial^2 \psi}{\partial r^2} + \left( -br^2 - 2mr + 2(l+1) \right) \frac{\partial \psi}{\partial r} + (Qr - 2m(l+1)) \psi = 0$$  \hspace{1cm} (1.2.1)

where

$$Q = E^2/4 - b (l + 3/2)$$  \hspace{1cm} (1.2.2)

By using the function $\gamma(r)$ as Frobinous series in (1.2.1), I obtain two indicial roots which are $\lambda_1 = 0$ and $\lambda_2 = -2l - 1$. Recurrence formula for all $n$ is

$$K_n = A_n + \frac{B_n}{K_{n-1}} \begin{cases} \begin{aligned} K_n & = \frac{C_{n+1}}{C_n} \\ K_{n-1} & = \frac{C_{n-1}}{C_n} \\ A_n & = \frac{D_{n+1}}{D_{n+2}+2(n+2)} \\ B_n & = \frac{E_{n+1}}{F_{n+2}+2(n+2)} \end{aligned} \end{cases}$$  \hspace{1cm} (1.2.3)
where \( C_1 = A_0 C_0 \) and \( n \geq 1 \).

Let’s investigate function \( y(r) \) as \( n \) and \( r \) go to infinity. As \( n \gg 1 \), (1.2.3) is

\[
\lim_{n \gg 1} K_n = \frac{2m}{n} + \frac{b/n}{\lim_{n \gg 1} K_{n-1}}
\]

(1.2.4)

The first term of RHS in (1.2.4) is negligible, since mass \( m \) is extremely small and \( n \) is too large, respectively. Then, (1.2.4) is approximately equal to

\[
\lim_{n \gg 1} K_n \approx \frac{b/n}{\lim_{n \gg 1} K_{n-1}}
\]

(1.2.5)

Classify \( C_n \) to its even and odd parts from (1.2.5) by using

\[
K_n = C_n + \frac{1}{n}C_{n-1}
\]

and

\[
K_n - 1 = C_n - C_{n-1}
\]

(1.2.6)

From (1.2.6) suggesting \( C_0 = 1 \), the function \( y(r) \) approximately is

\[
\lim_{n \gg 1} y(r) \approx \sum_n \frac{2m n!}{(2n)!} \left( \frac{b r^2}{2} \right)^n + m r \sum_n \frac{1}{n!} \left( \frac{b r^2}{2} \right)^n > (1 + mr)e^{\frac{b r^2}{2}}
\]

(1.2.7)

It is unacceptable that wave function \( \psi(r, \theta, \phi) \) is divergent as \( r \) goes to infinity from the quantum mechanical point of view. As \( r \) is extremely large value, the big polynomial of degree \( n \) will take a dominant position. Substitute (1.2.7) into the wave function which gives

\[
\psi(r, \theta, \phi) = Ne^{-\frac{b r^2}{2}} Y^{m}(\theta, \phi)
\]

where \( N \) is normalized constant.

\[
\lim_{r \to \infty} \psi(r, \theta, \phi) > \lim_{r \to \infty} N(1 + mr)r^l e^{\frac{b r^2}{2}} Y^{m+l}(\theta, \phi) \to \infty
\]

(1.2.8)

Even if the mass \( m \) is extremely small, the wave function \( \psi(r, \theta, \phi) \) will blows up as \( r \to \infty \). All wave functions must to go to zero as \( r \) goes to infinity from a quantum mechanical perspective. The first and second term of \( y(r) \) must also be terminated to become a polynomial of degree in this case. As we see in (1.2.7), the first term indicates even term of \( C_n \), and the second term of it has odd term of \( C_n \). Now, let’s try to define the first kind of independent solution as \( \lambda_1 = 0 \).

As \( \lambda_1 = 0 \),

\[
A_{n|\lambda_1=0} = \frac{2(n + l + 1)m}{(n + 1)(n + 2(l + 1))}
\]

(1.2.9a)
\[ B_{n|l=0} = \frac{-Q + b(n-1)}{(n+1)(n+2l+1)} \]  

(1.2.9b)

I define \( B_{i,j,k,l} \) referring to \( B_iB_jB_kB_l \). Classify \( C_n \) to its even and odd parts up to the first order of small mass \( m \) from (1.2.3).

\[ \begin{align*}
C_0 & \quad C_1 = C_0A_0 \\
C_2 & \quad C_2 = C_0B_1 \\
C_3 & \quad C_3 = C_0(A_0B_2 + A_2B_1) \\
C_4 & \quad C_4 = C_0B_{1.3} \\
C_5 & \quad C_5 = C_0(A_0B_{2.4} + A_2B_{1.4} + A_4B_{1.3}) \\
C_6 & \quad C_6 = C_0B_{1.3.5} \\
C_7 & \quad C_7 = C_0A_0B_{2.4.6} + A_2B_{1.4.6} + A_4B_{1.3.6} + A_6B_{1.3.5,7} \\
C_8 & \quad C_8 = C_0B_{1.3.5.7} \\
C_9 & \quad C_9 = C_0(A_0B_{2.4.6.8} + A_2B_{1.4.6.8} + A_4B_{1.3.6.8} + A_8B_{1.3.5.8} + A_8B_{1.3.5.7}) \\
C_{10} & \quad C_{10} = C_0B_{1.3.5.7.9} \\
& \quad \vdots \\
\end{align*} \]

For simplicity plugging \( C_0 = \frac{\Gamma(n_0-1)}{(i+1/2)!} \) in (1.2.10), I obtain

\[ y(r) = \begin{cases} 
QW_{|\alpha_0|,|\alpha_1|}(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, y = l + 3/2; z = \frac{1}{2}br^2) \\
F_{|\alpha_0|}(y; z) + mr \prod_{p=0}^{a_p} (y; z) & \text{only if } |\alpha_0| \leq |\alpha_1| 
\end{cases} \]  

(1.2.12)
where,

\[ F_{|[a_0]|}(\gamma; z) = \frac{\Gamma(|a_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{[a_0]} \frac{(-|a_0|)_n}{n!} z^n \]  

(1.2.13a)

\[ \prod_{[a_0]}^{[a_1]} (\gamma; z) = \frac{\Gamma(|a_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{[a_0]} \frac{(-|a_0|)_n}{n!} \sum_{k=0}^{[a_1]-n} \frac{(n + \frac{1}{2}(\gamma - \frac{1}{2}))\Gamma(n + \frac{1}{2})\Gamma(n + \frac{1}{2})(n - |a_1|)_k}{\Gamma(k + n + \frac{1}{2})\Gamma(k + n + \gamma + \frac{1}{2})} z^k \]

(1.2.13b)

\[ \prod_{[a_0]}^{(1/2)} T(s, t, p, u) = \frac{1}{2\pi i B(|a_1| + 1, \frac{1}{2})} \frac{\Gamma(|a_1| + 1)\Gamma(1/2)}{\Gamma(|a_1| + 3/2)} \left( w_1 \frac{\partial}{\partial w_1} + \frac{1}{2}(\gamma - 1/2) \right) F_{|[a_0]|}(\gamma; w_1) \]  

(1.2.14a)

And,

\[ B(|a_1| + 1, \frac{1}{2}) = \frac{\Gamma(|a_1| + 1)\Gamma(1/2)}{\Gamma(|a_1| + 3/2)} \]  

(1.2.14b)

in the above, \( T(s, t, p, u) \) is the operator which acts on

\[ T(s, t, p, u) = \int_0^\infty \int_{-1}^1 \int_{-1}^1 \int_0^1 ds \, s^{-1/2}(1 + s)^{-(|a_1|+3/2)} dt \, t^{-3/2} dp \, p^{-1} e^{-\frac{1}{2} \frac{s}{u}} \frac{1}{u^{|a_1|+1}(1-u)} \frac{1}{u^{|a_1|+1}(1-u)} \]  

(1.2.15)

We see in (1.2.13a), it is the first kind of confluent hypergeometric polynomial of degree \(|a_0|\). \[ (1.2.13b) \] denoted as \( QW_{|[a_0]||a_1|}(\gamma = n_0 - 1, |a_1| = n_1 - 1, \gamma = l + \frac{3}{2} ; z = \frac{1}{2} br^2) \) is the first kind of Grand Confluent Hypergeometric (GCH) polynomial of degree \(|a_0|\) and \(|a_1|\) with the first order \( m \).

Also I obtain two eigenvalues which are \( E_0^2 = 4b \, l + 2n_0 - 1/2 \) and \( E_1^2 = 4b \, l + 2n_1 + 1/2 \). The former is the primary radial eigenvalue. And the latter is the first kind of hidden radial eigenvalue. As I let the small mass \( m \) goes to zero in (1.2.13b), its solution is same as the \( 1^{st} \) kind of confluent hypergeometric polynomial. From \( (1.2.12) \), the wave function of it is

\[ \psi(r, \theta, \phi)_{n_0, n_1, l, m} = \hat{N} e^{-\frac{1}{2} \frac{r^2}{b^2}} r^{n_0} QW_{|[a_0]||a_1|}(\gamma = l + \frac{3}{2} ; z = \frac{1}{2} b r^2) Y_{lm}^{(n)}(\theta, \phi) \]  

(1.2.16)

By using orthogonality relation, normalized constant \( \hat{N} \) is

\[ \hat{N} = \frac{2^{l+1} \Gamma(|a_0| + 1)\Gamma(|a_0| + \gamma) - n! \Gamma(|a_0| + \gamma) \Gamma(|a_0| + \gamma + \frac{3}{2})}{b^2 \frac{\Gamma(\gamma)}{\Gamma(\gamma + \frac{3}{2})}} \]  

(1.2.17)

where,

\[ \frac{|a_0| = n_0 - 1}{|a_1| = n_1 - 1} \quad \text{only if} \quad |a_0| \leq |a_1| \]  

(1.2.18)

\( (1.2.16) \) is the \( 1^{st} \) kind of the normalized Grand confluent hypergeometric (GCH) wave function of degree \(|a_0|\) and \(|a_1|\).
As the small mass m goes to zero in (1.2.16) and (1.2.17), it turns out

\[ \lim_{m \to 0} \psi(r, \theta, \phi)_{n_0,n_1,m} = \sqrt{\frac{b^7}{2^{\gamma-1}\Gamma(|\alpha| + 1)\Gamma(|\alpha| + \gamma)}} r^\gamma e^{-\frac{\gamma b}{2} r} F_{|\alpha|} (\gamma; \zeta) Y_1^m (\theta, \phi) \]  

(1.2.19) is equivalent to (1.1.3). There are two eigennumbers \( n_0 \) and \( n_1 \) in (1.2.16). The first eigennumber \( n_0 \), called primary radial quantum number, appears in zeroth order of m term which is \( y(r)_{primary} \). And the second eigennumber \( n_1 \), called the first kind of hidden radial quantum number, appears in the first order of the m term which is \( y(r)_{small} \). As I neglect small mass m, the primary radial quantum number only starts to appear in the wave function. However when I include the small mass m, the second eigennumber which is \( 1^{st} \) kind of hidden radial quantum number is created. As we see in any other special functions such as Laguerre and Associated laguerre functions, Legendre and associated Legendre functions, hypergeometric function, Kummer function, etc, those functions only have one eigennumber.

In this new special function, there are two terms which are \( 0^{th} \) order of m term such as \( y(r)_{primary} \) and \( 1^{st} \) order of m term such as \( y(r)_{small} \). Actually, higher order of m terms do exist. But the mass of quark is extremely small. So I neglect the m terms that are more than \( 2^{nd} \) order. I only count it up to a \( 1^{st} \) order of m term. The full description of function \( y(r) \) include all higher order of mass m in the following way.

\[ y(r) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{n,m} r^n = \sum_{n=0}^{\infty} C_{n,0} r^n + m \sum_{n=0}^{\infty} C_{n,1} r^n + m^2 \sum_{n=0}^{\infty} C_{n,2} r^n + \cdots \] (1.2.20)

If the function \( y(r) \) of (1.2.20) is infinite series, then the wave function \( \psi(r, \theta, \phi) \) will blow up as we see in (1.2.8). So all of each of summation in (1.2.20) must be a polynomial. Then, (1.2.20) become

\[ y(r) = \sum_{m=0}^{\infty} \sum_{n=0}^{N_i} C_{n,m} r^n = \sum_{n=0}^{N_0} C_{n,0} r^n + m \sum_{n=0}^{N_1} C_{n,1} r^n + m^2 \sum_{n=0}^{N_2} C_{n,2} r^n + \cdots \] (1.2.21)

On the above \( N_i \) where \( i = 0, 1, 2, \cdots \) is the eigenvalue in order to make the sub-power series expansion as polynomial. Then, for example, all possible eigenvalues and eigennumbers of \( m^{th} \) order term up to forth order of m are

\[ m^0 \text{ order term:} \quad \begin{cases} E_{0}^2 = 4b(l + 2n_0 - \frac{1}{2}) & n_0 = 1, 2, 3, 4, \cdots \end{cases} \] (1.2.22a)

\[ m^1 \text{ order term:} \quad \begin{cases} E_{0}^1 = 4b(l + 2n_0 - \frac{1}{2}) & n_0 = 1, 2, 3, 4, \cdots \\ E_{1}^1 = 4b(l + 2n_1 + \frac{1}{2}) & n_1 = 1, 2, 3, 4, \cdots \end{cases} \] (1.2.22b)

\[ m^2 \text{ order term:} \quad \begin{cases} E_{0}^2 = 4b(l + 2n_0 - \frac{1}{2}) & n_0 = 1, 2, 3, 4, \cdots \\ E_{1}^2 = 4b(l + 2n_1 + \frac{1}{2}) & n_1 = 1, 2, 3, 4, \cdots \\ E_{2}^2 = 4b(l + 2n_2 + \frac{1}{2}) & n_2 = 1, 2, 3, 4, \cdots \end{cases} \] (1.2.22c)
where

\[
\begin{align*}
E_0^3 & = 4b(l + 2n_0 - \frac{1}{2}) \quad n_0 = 1, 2, 3, 4, \cdots \\
E_1^3 & = 4b(l + 2n_1 + \frac{1}{2}) \quad n_1 = 1, 2, 3, 4, \cdots \\
E_2^3 & = 4b(l + 2n_2 + \frac{3}{2}) \quad n_2 = 1, 2, 3, 4, \cdots \\
E_3^3 & = 4b(l + 2n_3 + \frac{5}{2}) \quad n_3 = 1, 2, 3, 4, \cdots \\
\end{align*}
\]

(1.2.22d)

\[
\begin{align*}
E_0^4 & = 4b(l + 2n_0 - \frac{1}{2}) \quad n_0 = 1, 2, 3, 4, \cdots \\
E_1^4 & = 4b(l + 2n_1 + \frac{1}{2}) \quad n_1 = 1, 2, 3, 4, \cdots \\
E_2^4 & = 4b(l + 2n_2 + \frac{3}{2}) \quad n_2 = 1, 2, 3, 4, \cdots \\
E_3^4 & = 4b(l + 2n_3 + \frac{5}{2}) \quad n_3 = 1, 2, 3, 4, \cdots \\
\end{align*}
\]

(1.2.22e)

m^3 order term:

\[
\begin{align*}
E_0^3 & = 4b(l + 2n_0 - \frac{1}{2}) \quad n_0 = 1, 2, 3, 4, \cdots \\
E_1^3 & = 4b(l + 2n_1 + \frac{1}{2}) \quad n_1 = 1, 2, 3, 4, \cdots \\
E_2^3 & = 4b(l + 2n_2 + \frac{3}{2}) \quad n_2 = 1, 2, 3, 4, \cdots \\
E_3^3 & = 4b(l + 2n_3 + \frac{5}{2}) \quad n_3 = 1, 2, 3, 4, \cdots \\
\end{align*}
\]

m^4 order term:

where

\[
\begin{align*}
n_i & \leq n_j \text{ where } i \leq j \text{ and } i, j = 0, 1, 2, \cdots \\
E_0^i & = \text{primary radial eigenvalue} \\
E_1^i & = \text{ith type hidden radial eigenvalue} \\
n_0 & = \text{primary radial quantum number} \\
n_i & = \text{ith type hidden radial quantum number} \\
\end{align*}
\]

We see \(i^{th}\) term of \(m\) has \((i+1)\) different eigenvalues from (1.2.22d)-(1.2.22e). Actually we don’t have to think about eigenfunctions of all higher order of \(m^{th}\) term, since the mass of quark creates extremely small vibrations and variation because of small mass \(m\). So zeroth order of \(m\) and 1st order of \(m\) terms are sufficient. However, I obtain many eigenvalues whenever the order of \(m^{th}\) term increases. These are reasons why there are many linearly increasing lines with same slopes including bunch of eigenvalues corresponding to fermions and bosons. From (1.2.22d)-(1.2.22e), we can obtain quiet exact mass formulas of many hadronic particles. More detail about Grand Confluent Hypergeometric (GCH) function of all higher order of \(m^{th}\) term and its eigenvalues are explained in Refs. [15, 16].

2. Asymptotic Behavior of Grand Confluent Hypergeometric function

Now let’s generalize this new special function. Suppose that there is a second order differential equation which is

\[
x^n y''(x) + a_0 x y'(x) + (a_1 x^4 + b_1 x^3 + c_1 x^2 + d_1) y(x) = 0 \quad \text{where} \quad a_0, a_1, b_1, c_1, d_1 = \mathbb{R} \quad 0 \leq x \leq \infty
\]

(2.0.24)
is equivalent to (1.1.1). All coefficients in the above are correspondent to the following way.

\[
a_0 \leftrightarrow 2 \\
(a_1 \leftrightarrow \frac{b^2}{4}) \\
b_1 \leftrightarrow -mb \\
c_1 \leftrightarrow \left(\frac{E^2}{4} - m^2\right) \\
d_1 \leftrightarrow -l(l + 1) \\
x \leftrightarrow r
\]

When a function \( y(x) = e^{\frac{i}{\sqrt{a_1}} \sqrt{(a_0 - 1)x}} x^{\frac{1}{2}} (a_0 - 1)^{\frac{3}{2}} (a_0 - 1)^{\frac{3}{2}}) g(x) \) acts on (2.0.24), I have

\[
xg''(x) + \left(\mu x^2 + \varepsilon x + \nu\right) g'(x) + \left(\Omega x + \varepsilon \omega\right) g(x) = 0
\]

where,

\[
\mu = 2i \sqrt{a_1} \\
\varepsilon = \frac{ib_1}{\sqrt{a_1}} \\
\nu = 1 \pm \sqrt{(a_0 - 1)^2 - 4d_1} \\
\Omega = c_1 - \frac{b^2}{4a_1} + i \sqrt{a_1} (2 \pm \sqrt{(a_0 - 1)^2 - 4d_1}) \\
\omega = \frac{1}{2} (1 \pm \sqrt{(a_0 - 1)^2 - 4d_1})
\]

(2.0.26) is equivalent to (1.2.1). In my definition, (2.0.26) is the Grand Confluent Hypergeometric (GCH) differential equation. All coefficients in the above are correspondent to the following:

\[
\mu \leftrightarrow -b \\
\varepsilon \leftrightarrow -2m \\
\nu \leftrightarrow 2(l + 1) \\
\Omega \leftrightarrow Q = \frac{E^2}{4} - b(l + \frac{3}{2}) \\
\omega \leftrightarrow (l + 1) \\
x \leftrightarrow r
\]

First I suggest that \(|\frac{\varepsilon}{\mu}| = \left|\frac{ib_1}{\sqrt{a_1}}\right| \ll 1\). By using the function \(g(x)\) as Frobinous series in (2.0.26), I have two indicial roots, \(\lambda_1 = 0\) and \(\lambda_2 = 1 - \nu\). And, recurrence formula for all \(n\) is

\[
K_n = A_n + \frac{B_n}{K_{n-1}} \quad ; n \geq 1
\]

\[
\begin{align*}
K_n &= C_n \\
K_{n-1} &= C_{n-1} \\
A_n &= -\frac{C_n}{(\nu + l + \frac{3}{2})} \\
B_n &= -\frac{C_{n-1}}{(\nu + l + \frac{3}{2})} \\
C_n &= -\frac{C_{n-1}}{(\nu + l + \frac{3}{2})}
\end{align*}
\]
Let’s test for the convergence of the function \( g(x) \). As \( n \) goes to infinity, recurrence formula is approximately equal to

\[
C_{n+1} \approx -\frac{\mu}{n} C_{n-1} \quad (2.0.30)
\]

Now I can describe \( g(x) \) as power series by substituting \((2.0.30)\) into Frobenius series. For simplicity, I suggest \( C_0 = 1 \)

\[
\lim_{n \to \infty} g(x) = \sum_{n=0}^\infty C_{2n}x^{2n} + \sum_{n=1}^\infty C_{2n+1}x^{2n+1}
\]

\[
= \sum_n \left( \frac{(-\frac{1}{2})!}{(n-\frac{1}{2})!} \right) \left( -\frac{1}{2\mu} \right)^n x^n - \frac{1}{2} e\sqrt{x} e^{-\frac{1}{2}\mu x^2}
\]

\[
= 1 + \sqrt{\pi} e^{-\frac{1}{2}\mu x^2} \sqrt{\frac{1}{2\mu}} \text{Erf} \left( \sqrt{-\frac{1}{2}\mu x^2} \right) - \frac{1}{2} e\sqrt{x} e^{-\frac{1}{2}\mu x^2}
\]

\[
> (1 - \frac{1}{2} e\sqrt{x}) e^{-\frac{1}{2}\mu x^2} \quad (2.0.31)
\]

In the above, \( \text{Erf} \left( \sqrt{-\frac{1}{2}\mu x^2} \right) \) is error function. Substitute \((2.0.31)\) into \( y(x) \).

\[
\lim_{n \to \infty} y(x) \approx x^{1-(a_0-1)} \sqrt{\pi} e^{-\frac{1}{2}\mu x^2} \left( 1 - \frac{ib_1}{2\sqrt{a_1}} x \right)
\]

\[
+ (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} e^{-\frac{1}{2}\mu x^2} \text{Erf} \left( (i)^{3/2} (a_1)^{1/4} x \right)
\]

\[
> e^{-\frac{1}{2}\mu x^2} x^{1-(a_0-1)} \sqrt{\pi} e^{-\frac{1}{2}\mu x^2} \left( 1 - \frac{ib_1}{2\sqrt{a_1}} x \right) \quad (2.0.32)
\]

2.1. As \( a_1 = \text{real positive} \)

(a) If \( \frac{1}{2} \left[ -(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right] < -1 \),
As \( n \gg 1 \) and \( x \to 0 \) in \((2.0.32)\),

\[
\lim_{n \to \infty} y(x) > x^{1-(a_0-1)} \sqrt{\pi} e^{-\frac{1}{2}\mu x^2} \left( 1 - \frac{ib_1}{2\sqrt{a_1}} x \right) \to \infty \quad (2.1.1)
\]

As \( x \to \infty \) and \( a_1 \gg 1 \) in \( \text{Erf} \left( (i)^{3/2} (a_1)^{1/4} x \right) \), the imaginary part of an error function oscillates around zero. And the real part of an error function oscillates at around \(-1\). As \( x \) goes to \( \infty \) in \((2.0.32)\)

\[
\lim_{x \to \infty} y(x) \approx x^{1-(a_0-1)} \sqrt{\pi} e^{-\frac{1}{2}\mu x^2} \left( 1 - \frac{ib_1}{2\sqrt{a_1}} x \right)
\]

\[
+ (i)^{3/2} \sqrt{\pi} (a_1)^{1/4} e^{-\frac{1}{2}\mu x^2} \text{Erf} \left( (i)^{3/2} (a_1)^{1/4} x \right) \to 0 \quad (2.1.2)
\]

(b) If \( \frac{1}{2} \left[ -(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right] \equiv -1 \),
As $x$ goes to zero and $x \to 0$ in (2.0.32), turn out to be

$$\lim_{n \to 1} y(x) = \frac{1}{x} e^{-\frac{i}{x} \sqrt{\pi x^2}} - \frac{ib_1}{2 \sqrt{a_1}} e^{-\frac{i}{x} \sqrt{\pi x^2}} + (i)^{3/2} \sqrt{\pi}(a_1)^{1/4} e^{-\frac{i}{x} \sqrt{\pi x^2}} \text{Erf} \left( (i)^{3/2}(a_1)^{1/4} x \right)$$

As $x$ goes to zero, the function $y(x)$ becomes divergent. Since $x \to 0$ in (2.0.32), it then yields

$$\lim_{x \to 0} y(x) > e^{-\frac{i}{x} \sqrt{\pi x^2}} \left( \frac{1}{x} - \frac{ib_1}{2 \sqrt{a_1}} \right) \to \infty$$

As $x$ goes to zero and $x \to 0$ in (2.1.3), it then yields

$$\lim_{n \to 1} y(x) = (i)^{3/2} \sqrt{\pi}(a_1)^{1/4} e^{-\frac{i}{x} \sqrt{\pi x^2}} \text{Erf} \left( (i)^{3/2}(a_1)^{1/4} x \right)$$

(c) If $-1 < \frac{1}{x}(-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) < 0$,

As $x$ goes to zero and $x \to 0$ in (2.0.32),

$$\lim_{n \to 1} y(x) > e^{-\frac{i}{x} \sqrt{\pi x^2}} x^\frac{1}{2} \left( -(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right) \left( 1 - \frac{ib_1}{2 \sqrt{a_1}} x \right) \to \infty$$

(d) If $\frac{1}{x}(-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) = 0$,

(2.0.32) turns out to be

$$\lim_{n \to 1} y(x) = e^{\frac{i}{x} \sqrt{\pi x^2}} - \frac{ib_1}{2 \sqrt{a_1}} e^{\frac{i}{x} \sqrt{\pi x^2}} + (i)^{3/2} \sqrt{\pi}(a_1)^{1/4} xe^{-\frac{i}{x} \sqrt{\pi x^2}} \text{Erf} \left( (i)^{3/2}(a_1)^{1/4} x \right)$$

As $x$ goes zero and $x \to 0$ in (2.1.7),

$$\lim_{n \to 1} y(x) = 1 \quad \lim_{n \to 1} y(x) \to \infty$$

(e) If $\frac{1}{x}(-(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1}) > 0$,

As $x$ goes to $\infty$ and zero in (2.0.32),

$$\lim_{n \to 1} y(x) > e^{\frac{i}{x} \sqrt{\pi x^2}} x^\frac{1}{2} \left( -(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right) \left( 1 - \frac{ib_1}{2 \sqrt{a_1}} x \right) \to \infty$$

$$\lim_{n \to 1} y(x) \Rightarrow x^\frac{1}{2} \left( -(a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} \right) \left( e^{\frac{i}{x} \sqrt{\pi x^2}} - \frac{ib_1}{2 \sqrt{a_1}} xe^{-\frac{i}{x} \sqrt{\pi x^2}} \right)$$

$$+ (i)^{3/2} \sqrt{\pi}(a_1)^{1/4} xe^{-\frac{i}{x} \sqrt{\pi x^2}} \text{Erf} \left( (i)^{3/2}(a_1)^{1/4} x \right) \to 0$$
2.2. As \( a_1 = 0 \)

(2.0.32) simply turns to be

\[
\lim_{x \to \pm \infty} y(x) = x^{\frac{1}{2}} e^{-\frac{(a_0 - 1)\pm \sqrt{(a_0 - 1)^2 - 4d_1}}{2}} \left( 1 - \frac{ib_1}{2 \sqrt{|a_1|}} x \right)
\]  

(2.2.1)

I suggest that \(|\frac{\epsilon}{2}| = |\frac{ib_1}{\sqrt{|a_1|}}| \ll 1\). Then as \( a_1 = 0 \) in the second term of the bracket in (2.2.1), \( |\frac{\epsilon}{2}| = |\frac{ib_1}{\sqrt{|a_1|}}| \to \infty \). The function \( y(x) \) will be divergent no matter what the value of \( x \) is. Therefore, there are no any independent solutions at all in the case of \( a_1 = 0 \).

2.3. As \( a_1 = \text{real negative} \)

Plug \( a_1 = -|a_1| \) into (2.0.32).

\[
\lim_{x \to \pm \infty} y(x) = x^{\frac{1}{2}} e^{-\frac{(a_0 - 1)\pm \sqrt{(a_0 - 1)^2 - 4d_1}}{2}} \left( 1 - \frac{b_1}{2 \sqrt{|a_1|}} x \right)
\]  

(2.3.1)

(a) \( \frac{1}{2}((a_0 - 1)\pm \sqrt{(a_0 - 1)^2 - 4d_1}) < 0 \),

As \( x \) goes to 0 and \( \infty \) in (2.3.1), the function \( y(x) \) is divergent.

\[
\lim_{x \to \pm \infty} y(x) > x^{\frac{1}{2}} e^{-\frac{(a_0 - 1)\pm \sqrt{(a_0 - 1)^2 - 4d_1}}{2}} \left( 1 - \frac{b_1}{2 \sqrt{|a_1|}} x \right) \to \pm \infty
\]  

(2.3.2)

(b) \( \frac{1}{2}((a_0 - 1)\pm \sqrt{(a_0 - 1)^2 - 4d_1}) = 0 \),

As \( x \) goes to 0 and \( \infty \) in (2.3.1).

\[
\lim_{x \to \pm \infty} y(x) \to 1 \quad \lim_{x \to \pm \infty} y(x) \to 1
\]  

(2.3.3)

(c) \( \frac{1}{2}((a_0 - 1)\pm \sqrt{(a_0 - 1)^2 - 4d_1}) > 0 \),

As \( x \) goes to 0 and \( \infty \) in (2.3.1).

\[
\lim_{x \to \pm \infty} y(x) \to 0 \quad \lim_{x \to \pm \infty} y(x) \to 1
\]  

(2.3.4)

2.4. As \( g(x) \) is polynomial for \( a_1 = \text{real negative} \)

I check all possible tests to determine whether an infinite series of function \( y(x) \) converges or diverges on the above. Now let’s test for convergence as the function \( y(x) \) as \( g(x) \) is polynomial for the case of \( a_1 = \text{real negative} \). Substitute \( g(x) = \sum_{n=0}^{N} C_n x^{n+\lambda} \) into \( y(x) \).

\[
y(x) \approx e^{-\frac{1}{2} \sqrt{|a_1|} x^{\lambda} \sqrt{(a_0 - 1)\pm \sqrt{(a_0 - 1)^2 - 4d_1}}} \sum_{n=0}^{N} C_n x^{n+\lambda}
\]  

(2.4.1)
(a) If \( \frac{1}{2}( - (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} ) < 0 \),
As \( x \) goes to 0 and \( \infty \) in (2.4.1),
\[
\lim_{n \gg 1} y(x) \to \infty \quad \lim_{n \gg 1} y(x) \to 0 \quad (2.4.2)
\]
(b) If \( \frac{1}{2}( - (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} ) = 0 \),
As \( x \) goes to 0 and \( \infty \) in (2.4.1),
\[
\lim_{n \gg 1} y(x) \to 1 \quad \lim_{n \gg 1} y(x) \to 0 \quad (2.4.3)
\]
(c) If \( \frac{1}{2}( - (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} ) > 0 \),
As \( x \) goes to 0 and \( \infty \) in (2.4.1),
\[
\lim_{n \gg 1} y(x) \to 0 \quad (2.4.4)
\]
I choose boundary conditions of a function \( g(x) \) for the polynomial in the following:
\[
\begin{cases}
\lim_{x \to 0} g(x) \to \text{convergent} \\
\lim_{x \to \infty} g(x) \to 0 
\end{cases}
(2.4.5)
\]
and the necessary conditions of (2.4.5) is
\[
\begin{cases}
a_1 = \text{real negative} \\
\frac{1}{2}( - (a_0 - 1) \pm \sqrt{(a_0 - 1)^2 - 4d_1} ) \geq 0
\end{cases}
(2.4.6)
\]
After we develop independent solutions for the polynomial, we can expand them as infinite series in simple ways. When we try to find the analytic solution of any differential equations, first we must consider what physical circumstance and mathematical condition make solutions as polynomial or infinite series.

3. Polynomial for \( \nu = \text{non-integer} \)

I consider the power series expansion in closed forms of GCH polynomial only up to the 1st order of \( m \) terms, its integral form, generating function and orthogonal relation. As we know, there are two indicial roots which are \( \lambda_1 = 0 \) and \( \lambda_2 = 1 - \nu \).

3.1. As \( \lambda_1 = 0 \)

3.1.1. Power series expansion in closed forms

From (2.0.29)
\[
A_{n|\lambda=0} = -\frac{\varepsilon(n + \omega)}{(n + 1)(n + \nu)} \quad (3.1.1a)
\]
\[
B_{n|\lambda=0} = -\frac{\Omega + \mu(n - 1)}{(n + 1)(n + \nu)} \quad (3.1.1b)
\]

13
Put \( n = 0 \) in (3.1.1a). I get \( A_0 = -\frac{\omega}{\epsilon} = -\frac{\epsilon}{\epsilon_0} \). From (2.0.29), I have \( K_0 = \frac{\epsilon_0}{\epsilon} \) which is equal to \( A_0 \). Plug \( n = 1 \) in recurrence formula.

\[
K_1 = A_1 + \frac{B_1}{K_0} = A_1 + \frac{B_1}{A_0} = A_1 - \frac{B_1}{(\epsilon/2)}
\]  

(3.1.2)

As we see in (3.1.2), \( A_n \) includes the first order of \( \frac{\epsilon}{\epsilon_0} \) in which has an extremely small value. Then I argue that \( |A_1| \ll \frac{B_1}{(\epsilon/2)} \). (3.1.2) is approximatively equal to \( K_1 \approx \frac{B_1}{A_0} = -\frac{B_1}{(\epsilon/2)} \). By using this process, I can simplify (2.0.29) by giving \( K_n \) in terms of \( A_n \) and \( B_n \) instead of \( K_{n+1} \) up to the first order of \( \frac{\epsilon}{\epsilon_0} \). By using \( K_n = \frac{\epsilon_0}{\epsilon} \), even and odd terms of \( C_n \) are same as (1.2.10). I can describe \( g(x) \) as power series by using (1.2.10).

\[
g(x) = g(x)_{\text{domain}} + g(x)_{\text{small}} = \sum_{n=0}^{\infty} C_{2n} x^{2n} + \sum_{n=0}^{\infty} C_{2n+1} x^{2n+1}
\]

\[
= C_0 \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{B_{2n+1} x^{2n}}{n!} \right) \right] + C_0 \left[ A_0 (x + (A_0 B_2 + A_2 B_1)) x^3 \right]
\]  

(3.1.3)

\[
+ \sum_{n=2}^{\infty} \left[ A_0 \left( \frac{B_{2n+1} x^{2n}}{n!} \right) + A_0 \left( \frac{B_{2n+2} x^{2n+1}}{n!} \right) \right]
\]

By using similar process from the previous case, there are two eigenvalues which are

\[
- \frac{\Omega}{2\mu} = n_0 - 1 = |\alpha_0| \quad \text{where} \quad n_0 = 1, 2, 3, \ldots
\]  

(3.1.4a)

\[
- \frac{\Omega}{2\mu} = n_1 - 1 = |\alpha_1| \quad \text{where} \quad n_1 = 1, 2, 3, \ldots
\]  

(3.1.4b)

(3.1.4a) makes \( B_{2n+1} \) term go to zero at certain value of \( k \) where \( k = 0, 1, 2, \ldots \). And (3.1.4b) makes \( B_{2n+2} \) term go to zero at certain value of \( k \). (3.1.4a) and (3.1.4b) make \( g(x) \) function a polynomial series. The \( g(x)_{\text{small}} \) term is extremely small value relatively compared to \( g(x)_{\text{domain}} \) because it includes \( A_n \) term having \( \epsilon_0 \). First of all, add (3.1.1a) and (3.1.1b) into the first term of \( g(x) \) in (3.1.3), putting \( C_0 = \frac{\Gamma(\nu+\nu_0)}{\Gamma(\nu)} \), \( \gamma = \frac{1}{2} (1 + \nu) \) and \( z = -\frac{\mu}{\epsilon_0} x^2 \).

\[
g(x)_{\text{domain}} = F_{\nu_0}(\gamma ; z) = \frac{\Gamma(\nu_0) + \gamma}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \left( \frac{-|\alpha_0|}{n!} \right)_n z^n
\]  

(3.1.5)

(3.1.5) is same as the first kind of confluent hypergeometric function. Substitute (3.1.1a) and (3.1.1b) into the second term of \( g(x) \) in (3.1.3), plugging \( C_0 = \frac{\Gamma(\nu_0) + \gamma}{\Gamma(\gamma)} \), \( \gamma = \frac{1}{2} (1 + \nu) \) and \( z = -\frac{\mu}{\epsilon_0} x^2 \).

\[
g(x)_{\text{small}} = -\frac{\Gamma(\nu + |\alpha_0|)}{\Gamma(\gamma)} \left\{ \sum_{n=0}^{\infty} \frac{\varepsilon(n + \frac{\gamma}{2})}{2(n + \frac{1}{2})(n + \gamma + \frac{\nu}{2})(n + \gamma + \frac{\nu}{2} - 1)!} |\alpha_0| (|\alpha_0|)! \right\}
\]

(3.1.6)
Putting \(|\alpha_0| = 0, 1, 2, \cdots\) in (3.1.6), I yield

\[
g(x)_{small} = -x e^{\sum_{n=0}^{[\alpha_0]} \frac{(n+\frac{\alpha_0}{2})(n+1/2)!}|\alpha_0|!(|\alpha_0| + 1/2)!}{(n+|\alpha_0|+1/2)!n^{|\alpha_0|}} \prod_{k=0}^{[\alpha_0]} \frac{(-1)^k(n+\frac{\alpha_0}{2})(n+\frac{\alpha_0}{2}-1)!}{(k+\frac{\alpha_0}{2})!(|\alpha_0|-k)!} \frac{1}{(k+\frac{\alpha_0}{2})!} x^k
\]

(3.1.7)

As we see in (3.1.7), maximum value of index \(n\) is \(|\alpha_0|\). The range of index \(k\) is \(n \leq k \leq |\alpha_1|\). In other words, \(0 \leq n \leq |\alpha_0| \leq k \leq |\alpha_1|\). Then I obtain \(|\alpha_0| \leq |\alpha_1|\). If \(|\alpha_0| \geq |\alpha_1|\), the function \(g(x)\) will be infinite series. Then, the function \(g(x)\) will blow up as I plug \(g(x)\) into it no matter what the value of \(x\) is. Such solution does not exist. When we see the second summation of (3.1.7), we can shift index \(k\) to zero at the beginning of summation. Then we can replace the interval of index \(k\) by \(0 \leq k \leq |\alpha_1| - n\).

(3.1.7) is described as

\[
g(x)_{small} = -x \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{[\alpha_0]} \frac{(-|\alpha_0|)_{n}}{n!} \sum_{k=0}^{[\alpha_1]-n} \frac{(n+\frac{\gamma}{2})!|\alpha_1|!(\gamma + 1/2)!}{(n+|\alpha_1|)!} \frac{1}{(k+\frac{\gamma}{2})!} \frac{1}{|\alpha_1|-k)!} x^k
\]

(3.1.8)

We see the first summation of (3.1.8) is \(\frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{[\alpha_0]} (-|\alpha_0|)_{n} x^{n}\), it is the first kind of confluent hypergeometric polynomial of degree \(|\alpha_0|\) which is denoted as \(F_{|\alpha_0|}(\gamma; z)\). Substitute (3.1.5) and (3.1.8) into (3.1.3).

\[
g(x) = Q^{\gamma \lambda}_{|\alpha_0|, |\alpha_1|} \left(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, \gamma = \frac{1}{2} \right) \frac{1}{(1 + \nu); z = -\frac{1}{2} \mu x^2}
\]

(3.1.9)

where,

\[
F_{|\alpha_0|}(\gamma; z) = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{[\alpha_0]} (-|\alpha_0|)_{n} x^{n}
\]

(3.1.10a)

\[
\prod_{j=0}^{[\alpha_1]} (\gamma; z) = \frac{\Gamma(|\alpha_0| + \gamma)}{\Gamma(\gamma)} \sum_{n=0}^{[\alpha_0]} (-|\alpha_0|)_{n} x^{n} \sum_{k=0}^{[\alpha_1]-n} \frac{(n+\frac{\gamma}{2})!|\alpha_1|!(\gamma + 1/2)!}{(n+|\alpha_1|)!} \frac{1}{(k+\frac{\gamma}{2})!} \frac{1}{|\alpha_1|-k)!} x^k
\]

(3.1.10b)

(3.1.10b) denoted as \(Q^{\gamma \lambda}_{|\alpha_0|, |\alpha_1|} \left(|\alpha_0| = n_0 - 1, |\alpha_1| = n_1 - 1, \gamma = \frac{1}{2} \right) \frac{1}{(1 + \nu); z = -\frac{1}{2} \mu x^2}\) is the first kind of GCH polynomial of degree \(|\alpha_0|\) and \(|\alpha_1|\) with the first order of \(\frac{1}{2}\).

3.1.2. Integral formalism

The solution of the Laguerre differential equation is

\[
L_n(z) = \sum_{k=0}^{n} \frac{(-1)^k \frac{\Gamma(n+k)}{k!}}{n!} e^z \frac{d^n}{dze^n} (e^z e^{-z})
\]

(3.1.11)

And the solution of the associated Laguerre differential equation is

\[
L_n^k(z) = \sum_{j=0}^{n} \frac{(-1)^{(n+k)} (n+j)!}{(n-j)!(k+j)!} \frac{1}{j!} e^z \frac{d^n}{dze^n} (e^z e^{-z})
\]

(3.1.12)
The beta function is

$$\text{Plug } (3.1.16) \text{ into the second term of } (3.1.14) \text{ on RHS.}$$

(3.1.13b)

$$(3.1.10b) \text{ might be described in the following way:}$$

$$\prod_{n=0}^{\lfloor n \rfloor} \Gamma(z) = \sum_{n=0}^{\lfloor n \rfloor} F_n^{\lfloor n \rfloor}(z) \prod_{n}^{\lfloor n \rfloor} (y) = (3.1.13a)$$

where,

$$F_n^{\lfloor n \rfloor}(z) = \frac{\Gamma(\lfloor n \rfloor + \gamma) (-\lfloor n \rfloor) n!}{\Gamma(\gamma)}$$

(3.1.13b)

$$\prod_{n=0}^{\lfloor n \rfloor} \Gamma(z) = \sum_{k=0}^{\lfloor n \rfloor} \frac{(n + \gamma) \Gamma(n + \gamma + \beta) \Gamma(n + \gamma + \delta)(\gamma - \lfloor n \rfloor)}{\Gamma(k + n + \beta) \Gamma(k + n + \gamma + \delta)}$$

(3.1.13c)

Plug $\lfloor n \rfloor = 0$ into (3.1.9), (3.1.10b) and (3.1.13a)–(3.1.13c)

$$QW_{0,\lfloor n \rfloor}(y; z) = 1 - \frac{E}{2} \sum_{k=0}^{\lfloor n \rfloor} \frac{(\gamma + k) \Gamma(n + \gamma + \beta) \Gamma(n + \gamma + \delta)(\gamma - \lfloor n \rfloor)}{\Gamma(k + n + \beta) \Gamma(k + n + \gamma + \delta)}$$

(3.1.14)

The beta function is

$$B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p + q)} = \int_0^1 dt \frac{t^{p-1}}{(1 - t)^{q-1}} = \int_0^\infty dt \frac{t^{p-1}}{(1 + t)^{q+1}} = B(q, p)$$

(3.1.15)

And,

$$\frac{1}{\Gamma(k + 1)} \int_0^1 dp \cdot (1 - p^2)^k = \frac{\Gamma(\frac{1}{2})}{\Gamma(k + \frac{1}{2})}$$

(3.1.16)

Plug (3.1.16) into the second term of (3.1.14) on RHS.

$$\prod_{n=0}^{\lfloor n \rfloor} \Gamma(z) = \frac{\omega}{2} \int_{-1}^{1} dp \int_{0}^{1} dt \frac{t^{n-\gamma} \sum_{k=0}^{\lfloor n \rfloor} (\gamma - \lfloor n \rfloor) \Gamma(\gamma + \beta) \Gamma(\gamma + \delta)}{\Gamma(k + n + \beta) \Gamma(k + n + \gamma + \delta)}$$

(3.1.17)

Replace $p$ and $q$ by $\gamma - \frac{1}{2}$ and $k + 1$ in (3.1.15). Take the new (3.1.15) into (3.1.17).

$$\prod_{n=0}^{\lfloor n \rfloor} \Gamma(z) = \frac{\omega}{2} \int_{-1}^{1} dp \int_{0}^{1} dt \frac{t^{n-\gamma} \sum_{k=0}^{\lfloor n \rfloor} (\gamma - \lfloor n \rfloor) \Gamma(\gamma + \beta) \Gamma(\gamma + \delta)}{\Gamma(k + n + \beta) \Gamma(k + n + \gamma + \delta)}$$

(3.1.18)

Replace $n$ and $z$ by $\lfloor n \rfloor$ and $\lambda = z(1 - t)(1 - p^2)$ in (3.1.11). Take the new (3.1.11) into (3.1.18).

$$\prod_{n=0}^{\lfloor n \rfloor} \Gamma(z) = \frac{\omega}{2} \int_{-1}^{1} dp \int_{0}^{1} dt \frac{t^{n-\gamma} \sum_{k=0}^{\lfloor n \rfloor} (\gamma - \lfloor n \rfloor) \Gamma(\gamma + \beta) \Gamma(\gamma + \delta)}{\Gamma(k + n + \beta) \Gamma(k + n + \gamma + \delta)}$$

(3.1.19)

Plug (3.1.19) into (3.1.14)

$$QW_{0,\lfloor n \rfloor}(y; z) = F_0(y; z) - \frac{E}{2} \left(\frac{\omega}{2}\right) \int_{-1}^{1} dp \int_{0}^{1} dt \frac{t^{n-\gamma} \sum_{k=0}^{\lfloor n \rfloor} (\gamma - \lfloor n \rfloor) \Gamma(\gamma + \beta) \Gamma(\gamma + \delta)}{\Gamma(k + n + \beta) \Gamma(k + n + \gamma + \delta)}$$

(3.1.20)
Plug $|\alpha_0| = 1$ into (3.1.9), (3.1.10a) and (3.1.13a–3.1.13c)

$\mathcal{Q}W_{1,|\alpha_0|}(\gamma;z) = F_1(\gamma;z) - \frac{\epsilon}{2} \left[ F^1_0(\gamma;z) \prod_{n=0}^{[\alpha_0]} (\gamma + z) + F^1_1(\gamma;z) \prod_{n=0}^{[\alpha_0]} (\gamma + z) \right]$  \hspace{1cm} (3.1.21)

Replace $k$ by $k + 1$ in (3.1.16). Take the new (3.1.16) into the second term inside bracket in (3.1.21)

$\prod_{n=0}^{[\alpha_0]} (\gamma + z) = \frac{1}{2} (1 + \omega z) \int_{-1}^{1} dp \left(1 - p^2\right) \sum_{k=0}^{[\alpha_0]-1} \frac{(-1)^k \Gamma(k+\gamma)\Gamma(\gamma + \frac{1}{2})}{\Gamma(k+\gamma)\Gamma(\gamma + \frac{1}{2})} [z(1-p^2)]^k$  \hspace{1cm} (3.1.22)

Replace $p$ and $q$ by $\gamma + \frac{1}{2}$ and $k + 1$ in (3.1.15). Take the new (3.1.15) into (3.1.22).

$\prod_{n=0}^{[\alpha_0]} (\gamma + z) = \frac{1}{2} (1 + \omega z) \int_{-1}^{1} dp \left(1 - p^2\right) \int_{0}^{1} dt \cdot \epsilon^{\frac{1}{2} - \frac{1}{2}} L_{[\alpha_0]-1}^1(\lambda)$  \hspace{1cm} (3.1.23)

Substitute (3.1.19) and (3.1.22) into (3.1.21).

$\mathcal{Q}W_{1,|\alpha_0|}(\gamma;z) = F_1(\gamma;z) - \frac{\epsilon}{2} \left[ F^1_0(\gamma;z) \omega z \int_{-1}^{1} dp \left(1 - p^2\right) \int_{0}^{1} dt \cdot \epsilon^{\frac{1}{2} - \frac{1}{2}} L_{[\alpha_0]-1}^1(\lambda) \right]$  \hspace{1cm} (3.1.24)

As $|\alpha_0| = 2$ in (3.1.9), (3.1.10a) and (3.1.13a–3.1.13c), the result of its solution is similar to the previous case as $|\alpha_0| = 0$ and 1.

$\mathcal{Q}W_{2,|\alpha_0|}(\gamma;z) = F_2(\gamma;z) - \frac{\epsilon}{2} \left[ \sum_{n=0}^{2} \left\{ F^2_0(\gamma;z)(n + \omega z) \frac{\Gamma(n + \frac{1}{2} \Gamma(k+\gamma)\Gamma(\gamma + 1 - n)}{\Gamma(\gamma + \frac{1}{2})\Gamma(\gamma + 1)} \right\} \right] \int_{-1}^{1} dp \left(1 - p^2\right) \int_{0}^{1} dt \cdot \epsilon^{\frac{1}{2} - \frac{1}{2}} L_{[\alpha_0]-1}^1(\lambda)$  \hspace{1cm} (3.1.25)

By repeating this process, I obtain $\mathcal{Q}W_{|\alpha_0|,|\alpha_1|}(\gamma;z)$ where $|\alpha_0| \geq 3$. According to (3.1.21), (3.1.25) and (3.1.26), (3.1.10a) and (3.1.10b) are

$F_{|\alpha_0|}(\gamma;z) = \left( \frac{|\alpha_0|!}{2\pi i} \right) \int dv \frac{e^{-i\alpha_0 v}}{\gamma + \alpha_0 v + (1 - v)^2}$  \hspace{1cm} (3.1.26)

And

$\prod_{n=0}^{[\alpha_0]} (\gamma + z) = \sum_{n=0}^{[\alpha_0]} \left\{ F^|\alpha_0|_n(\gamma;z)(n + \omega z) \frac{\Gamma(n + \frac{1}{2} \Gamma(k+\gamma)\Gamma(\gamma + 1 - n)}{\Gamma(\gamma + \frac{1}{2})\Gamma(\gamma + 1)} \right\} \int_{-1}^{1} dp \left(1 - p^2\right) \int_{0}^{1} dt \cdot \epsilon^{\frac{1}{2} - \frac{1}{2}} L_{[\alpha_0]-1}^1(\lambda)$  \hspace{1cm} (3.1.27)
We know
\[ \frac{\Gamma(n + \frac{1}{2})\Gamma(|a| + 1 - n)}{\Gamma(\frac{1}{2})\Gamma(|a| + 1)} = B(n + \frac{1}{2}, |a| + 1 - n) \quad (3.1.29) \]

Substitute (3.1.13b) and (3.1.29) into (3.1.28).

\[ \prod_{\lambda=0}^{m} \frac{(-1)^n|a_0|!(|a_0| + \gamma - 1)!}{(n - 1)!|a_0| - n!} B(n + \frac{1}{2}, |a_1| + 1 - n) \]
\[ \times \int_{0}^{1} dt \ t^{n-\frac{\gamma}{2}} \int_{-1}^{1} dp \ [zt(1-p^2)]^{n} L_{|a_1|-n}^{\alpha}(d) \]
\[ + \frac{\omega}{2} \sum_{n=0}^{m} \frac{(-1)^n|a_0|!(|a_0| + \gamma - 1)!}{n!|a_0| - n!} B(n + \frac{1}{2}, |a_1| + 1 - n) \]
\[ \times \int_{0}^{1} dt \ t^{n-\frac{\gamma}{2}} \int_{-1}^{1} dp \ [zt(1-p^2)]^{n} L_{|a_1|-n}^{\alpha}(d) \] \quad (3.1.30)

Replace \( p \) and \( q \) by \( n + \frac{1}{2} \) and \( |a_1| + 1 - n \) into (3.1.15). Take the new (3.1.15) into (3.1.30).

\[ \prod_{\lambda=0}^{m} \frac{(-1)^n|a_0|!(|a_0| + \gamma - 1)!}{(n - 1)!|a_0| - n!} B(n + \frac{1}{2}, |a_1| + 1 - n) \]
\[ \times \left\{ \int_{0}^{1} dt \ t^{n-\frac{\gamma}{2}} \int_{-1}^{1} dp \ [zt(1-p^2)]^{n} L_{|a_1|-n}^{\alpha}(d) + \frac{\omega}{2} \sum_{n=0}^{m} \frac{(-1)^n|a_0|!(|a_0| + \gamma - 1)!}{n!|a_0| - n!} L_{|a_1|-n}^{\alpha}(d) \right\} \] \quad (3.1.31)

Integral form of Associated Laguerre polynomial is
\[ L_{\alpha}^{\gamma}(z) = \frac{1}{\sqrt{\pi t}} \int_{u=0}^{\infty} du \frac{e^{-u\gamma t}}{u^{\gamma+1}(1-u)^{\alpha+1}} \quad (3.1.32) \]

Replace \( n, m \) and \( z \) by \( |a_1| - n, n \) and \( \lambda \) in (3.1.32). Take the new (3.1.32) into (3.1.31).

\[ \prod_{\lambda=0}^{m} \frac{(-1)^n|a_0|!(|a_0| + \gamma - 1)!}{(n - 1)!|a_0| - n!} B(n + \frac{1}{2}, |a_1| + 1 - n) \]
\[ \times \left\{ \int_{0}^{1} dt \ t^{n-\frac{\gamma}{2}} \int_{-1}^{1} dp \ e^{-pt} \frac{u^{\gamma t}}{u^{\alpha+1}(1-u)^{\alpha+1}} \right\} \]
\[ \times \left\{ \sum_{n=0}^{m} \frac{(-1)^n|a_0|!(|a_0| + \gamma - 1)!}{(n - 1)!|a_0| - n!} \frac{u^n}{(1-u)^{\alpha+1}} \right\} \]
\[ + \frac{\omega}{2} \sum_{n=0}^{m} \frac{(-1)^n|a_0|!(|a_0| + \gamma - 1)!}{n!|a_0| - n!} \frac{u^n}{(1-u)^{\alpha+1}} \] \quad (3.1.33)
Replace $z$ and $\lambda$ by $w_1 = zts(1 - p^2)\frac{u}{1-u}$ and $z(1-t)(1-p^2)$ into (3.1.33).

\[
\prod_{\gamma}^{[n]}(y; z) = \frac{1}{2\pi i B(\frac{1}{2}, |\alpha| + 1)} \int_0^\infty ds \ s^{\frac{3}{4}(1 + s)}\ e^{-i(\gamma + \frac{1}{2})s} \int_0^1 dt \ t^{-1/2} \int_{-1}^1 dp \ \prod_{\gamma}^{[n]} e^{-i\omega t}\left[w_1 \partial_{\alpha} + \frac{\omega}{2} \right]F_{\gamma}(y; w_1 = zts(1 - p^2)\frac{u}{1-u})
\]

(3.1.34) can be described as various integral forms of several different special function in the following way:

\[
\prod_{\gamma}^{[n]}(y; z) = \frac{(\alpha_n)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha| + 1)} \int_0^\infty ds \ s^{\frac{3}{4}(1 + s)}\ e^{-i(\gamma + \frac{1}{2})s} \int_0^1 dt \ t^{-1/2} \int_{-1}^1 dp \ \prod_{\gamma}^{[n]} e^{-i\omega t}\left[w_2 = z(1 - p^2)\left[(1 - t) + \frac{tsv}{(1 - t)}\right]\right] \frac{dv}{u^{\alpha_1+1}(1-v)}
\]

\[
\prod_{\gamma}^{[n]}(y; z) = \frac{(\alpha_n)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha| + 1)} \int_0^\infty ds \ s^{\frac{3}{4}(1 + s)}\ e^{-i(\gamma + \frac{1}{2})s} \int_0^1 dt \ t^{-1/2} \int_{-1}^1 dp \ \prod_{\gamma}^{[n]} e^{-i\omega t}\left[w_2 = z(1 - p^2)\left[(1 - t) + \frac{tsv}{(1 - t)}\right]\right] \frac{dv}{u^{\alpha_1+1}(1-v)}
\]

\[
\prod_{\gamma}^{[n]}(y; z) = \frac{(\alpha_n)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha| + 1)} \int_0^\infty ds \ s^{\frac{3}{4}(1 + s)}\ e^{-i(\gamma + \frac{1}{2})s} \int_0^1 dt \ t^{-1/2} \int_{-1}^1 dp \ \prod_{\gamma}^{[n]} e^{-i\omega t}\left[w_2 = z(1 - p^2)\left[(1 - t) + \frac{tsv}{(1 - t)}\right]\right] \frac{dv}{u^{\alpha_1+1}(1-v)}
\]

\[
\prod_{\gamma}^{[n]}(y; z) = \frac{(\alpha_n)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha| + 1)} \int_0^\infty ds \ s^{\frac{3}{4}(1 + s)}\ e^{-i(\gamma + \frac{1}{2})s} \int_0^1 dt \ t^{-1/2} \int_{-1}^1 dp \ \prod_{\gamma}^{[n]} e^{-i\omega t}\left[w_2 = z(1 - p^2)\left[(1 - t) + \frac{tsv}{(1 - t)}\right]\right] \frac{dv}{u^{\alpha_1+1}(1-v)}
\]

In the above, $M(a, b, z)$ is the first kind of Kummer function which is

\[
M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n = e^z M(b - a, b, -z)
\]

\[
= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 du \ e^{\alpha u^{a-1}}(1 - u)^{b-a-1}
\]

where, $\text{Re}(b) > \text{Re}(a) > 0$
And \( U(a, b, z) \) is the second kind of Kummer function which is

\[
U(a, b, z) = \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} M(a, b, z) + \frac{\Gamma(b - 1)}{\Gamma(a)} \zeta^{1-b} M(a - b + 1, 2 - b, z)
\]

\[
= \zeta^{1-b} U(1 + a - b, 2 - b, z)
\]

\[
= \frac{1}{\Gamma(a)} \int_0^\infty dt \ e^{-t \zeta} (1 + t)^{b-a-1} \quad \text{where, } \text{Re}(a) > 0
\]

Substitute integral form of the first kind of confluent hypergeometric function into (3.1.41).

\[
\int \_{\frac{|v|}{|\alpha|}} \left( y; z \right) = \frac{(\alpha_0)!}{(2\pi i)^2 B(\frac{1}{2}, |\alpha| + 1)} \int_0^\infty ds \ s^{-\frac{1}{2}} (1 + s)^{-|\alpha|+\frac{1}{2}}
\]

\[
\times \int_0^1 dt \ t^{-\frac{1}{2}} \int_{-1}^1 dp \ \frac{dv}{\sqrt{1 - v^2} (1 - v)^{\gamma}}
\]

\[
\times \int_0^1 du \ u^{\alpha_0 + 1} \left( 1 - u \right) \left( -\frac{z\alpha_0 (1 - v^2)}{1 - u (1 - v)} + \frac{\alpha_0}{2} \right) e^{-\frac{v}{2} (1 - v^2)}
\]

\[
\quad \text{where} \quad |v_0| < 1, \quad |\alpha_0| \quad \text{is integral formalism is exactly equivalent (3.1.39).}
\]

Description I gave in (3.1.35)- (3.1.39) as integral formalism is exactly equivalent (3.1.39). And I obtain the integral representation of the first kind of GCH function according to (3.1.27) and (3.1.39).

### 3.1.3 Generating function

Now, let’s try to get the generating function of the 1st kind of GCH function. First I multiply

\[
\sum_{|\alpha|=|\alpha_0|} B(|\alpha| + 1, \frac{1}{2}) \ y_0^{|\alpha|} W_{\gamma_0, |\alpha|} \left( y = \frac{1}{2} (1 + v); z = -\frac{1}{2} i \alpha \right) = I - \frac{K}{2} \chi
\]

\[
\text{where,}
\]

\[
I = \sum_{|\alpha|=|\alpha_0|} B(|\alpha| + 1, \frac{1}{2}) y_0^{|\alpha|} F_{\gamma_0}(z)
\]

\[
\text{II} = \sum_{|\alpha|=|\alpha_0|} B(|\alpha| + 1, \frac{1}{2}) y_0^{|\alpha|} \prod_{|\alpha|} (y; z)
\]

The 1st kind of hypergeometric function is

\[
_2F_1(a, b, c, z) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k k!} t^k = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 dt \ t^{b-1} (1 - t)^{c-b-1} (1 - z)^{-a}
\]

only if \(|z| < 1, \text{Re}(c) > \text{Re}(b) > 0\)

Substitute (3.1.42) into (3.1.41a).

\[
I = \frac{\Gamma(|\alpha_0| + 1) \Gamma(\frac{1}{2})}{\Gamma(|\alpha_0| + \frac{1}{2})} y_0^{|\alpha_0|} _2F_1(1, |\alpha_0| + 1, |\alpha_0| + \frac{3}{2}, y_0) = \int_0^1 dt \ (v_0 t)^{|\alpha_0|} (1 - t)^{-\frac{1}{2}} (1 - v_0 t)^{-1}
\]

(3.1.43)
Substitute (3.1.39) into (3.1.41b).

\[
\mathbb{II} = \frac{(i\alpha_0)!}{2\pi i} \int_0^\infty ds \int_0^1 dt \int_0^1 dp \int_0^1 dv \sum_{\nu=0}^\infty \frac{d\nu}{v^{\nu+1}(1+v)}
\]

\[
\times \left\{ \frac{1 + s}{1 + s - v_0} - \frac{\nu \pi t}{1 + s - v_0} + \frac{\gamma}{2} \right\} e^{-\frac{\nu \pi t}{1 + s - v_0}} \left( 1 - \frac{\gamma}{2} \right) (1 + s - v_0)^{\nu+1}
\]

(3.1.44)

Multiply \( \sum_{|\nu|=0}^{\infty} \frac{v_i^{\nu}}{|(\alpha_i)|!} \) with both sides of (3.1.40) where \(|\nu_i| < 1\).

\[
\sum_{|\nu_i|=0}^{\infty} \sum_{|\nu|=|\nu_i|} B(|\alpha_i| + 1, \frac{1}{2}) v_0^{\nu_i} F_{|\nu_i|} I (\nu = \frac{1}{2}(1 + y); z = -\frac{1}{2}y^2)
\]

\[
= A - \frac{e}{2} B
\]

(3.1.45)

where,

\[
A = \sum_{|\nu|=0}^{\infty} \frac{v_i^{\nu}}{|(\alpha_i)|!} F_{|\nu|} I
\]

(3.1.46a)

\[
B = \sum_{|\nu|=0}^{\infty} \frac{v_i^{\nu}}{|(\alpha_i)|!} \mathbb{II}
\]

(3.1.46b)

Plug (3.1.43) into (3.1.46a).

\[
A = \int_0^1 dt (1 - t)^{-\frac{1}{2}} (1 - v_0 t)^{-1} \sum_{|\nu|=0}^{\infty} \frac{(v_0 v_i t)^{|\nu|}}{|(\nu_i)|!} F_{|\nu_i|} I
\]

\[
= \int_0^1 dt (1 - t)^{-\frac{1}{2}} (1 - v_0 v_i t)^{-1} e^{-\frac{v_0 v_i t}{1 + v_i t}}
\]

\[
= \sum_{n,m=0}^{\infty} \frac{(\gamma + n + m - 1)!((-v_0 v_i)^n (v_0 v_i)^m)!}{n! m! (\gamma + n - 1)!} \int_0^1 dt t^{\gamma + m}(1 - t)^{-\frac{1}{2}} (1 - v_0 t)^{-1}
\]

(3.1.47)

Replace a,b,c and z by 1, n + m + 1, n + m + \frac{1}{2} and v_0 into (3.1.42). And substitute the new (3.1.42) into (3.1.47).

\[
A = \sum_{n,m=0}^{\infty} B(n + m + 1, \frac{1}{2} (\gamma + n m) (-v_0 v_i)^n (v_0 v_i)^m)\binom{2F(1, n + m + 1, n + m + \frac{3}{2}, v_0)}{n! m!}
\]

\[
= \sum_{n,m,j=0}^{\infty} B(n + m + j + 1, \frac{1}{2} (\gamma + n m) (-v_0 v_i)^n (v_0 v_i)^m)\binom{2F(1, n + m + j, n + m + j + m)}{n! m!}
\]

(3.1.48)

I can describe (3.1.48) in a different way. First there is the 1st kind of Appell hypergeometric function which is

\[
F_1(\alpha; \beta; \gamma; x, y) = \sum_{j,k=0}^{\infty} \frac{(\alpha)_j (\beta)_j (\gamma)_k}{j! k! (\gamma)_r} x^j y^k
\]

for \(|x|, |y| < 1\)

(3.1.49)
The function $F_1$ can be expressed by simple integral

$$
\frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F_1(\alpha; \beta; \gamma; x, y) = \int_0^1 du \ u^{\alpha-1} (1-ux)^{-\beta}(1-uy)^{-\beta}
$$

where $\text{Re}(\alpha) > 0$, $\text{Re}(\gamma - \alpha) > 0$ (3.1.50)

Put $\alpha = n + 1$, $\gamma = n + \frac{3}{2}$, $\beta = 1$, $\beta' = y + n$, $x = v_0$ and $y = v_0 v_1$ in (3.1.50). And substitute the new (3.1.51) into (3.1.47).

$$
A = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2})(-zv_0v_1)^n}{\Gamma(n + \frac{1}{2})} F_1(n + 1; 1, \gamma + n; n + \frac{3}{2}; v_0, v_0 v_1)
$$

$$
= \sum_{n, j, k=0}^{\infty} \frac{B(n + k + j + 1, \frac{1}{2})(y + n)k(1-zv_0)^n}{n! \ k!} \int_0^{v_0 v_1} \int_0^1 dt \ t^{n+k+j}
$$

(3.1.51) is exactly equivalent to the second line of (3.1.48). Substitute (3.1.44) into (3.1.46b).

$$
B = \Gamma_1(s, p, t) + \Gamma_2(s, p, t)
$$

where

$$
\Gamma_1(s, p, t) = \frac{-zv_0v_1}{(1 - v_1)^{1+s}} \int_0^{\infty} ds \ s^\frac{1}{2}(1+s)^{-\frac{3}{2}}(1+s-v_0)^{-1} \int_0^1 dt \ t^{s-\frac{3}{2}}
$$

$$
\times \int_0^1 dp \ p(1-p)^{-\frac{3}{2}} e^{-\frac{v_0}{v_1} \left(1-t + \frac{t s v_1}{1-v_1}\right)}
$$

(3.1.53)

$$
\Gamma_2(s, p, t) = \frac{v_0}{(1 - v_1)^{1+s}} \int_0^{\infty} ds \ s^\frac{1}{2}(1+s)^{-\frac{3}{2}}(1+s-v_0)^{-1} \int_0^1 dt \ t^{s-\frac{3}{2}}
$$

$$
\times \int_0^1 dp \ (1-p)^{-\frac{3}{2}} e^{-\frac{v_0}{v_1} \left(1-t + \frac{t s v_1}{1-v_1}\right)}
$$

(3.1.54)

By using the first kind of Kummer function,

$$
\int_0^1 dp \ p(1-p)^{-\frac{3}{2}} e^{-\alpha p} = e^{-\alpha} \left\{ 2 M(\frac{1}{2}, \frac{3}{2}, \alpha) - \frac{2}{3} \ M(\frac{3}{2}, \frac{5}{2}, \alpha) \right\}
$$

where $\alpha = \frac{zv_0}{1+s-v_0} \left(1-t + \frac{t s v_1}{1-v_1}\right)$ (3.1.55)

Plug (3.1.55) into (3.1.53).

$$
\Gamma_1(s, p, t) = \gamma_1(s, t) + \gamma_2(s, t)
$$

(3.1.56)

$$
\gamma_1(s, t) = \frac{-2zv_0v_1}{(1 - v_1)^{1+s}} \int_0^{\infty} ds \ s^\frac{1}{2}(1+s)^{-\frac{3}{2}}(1+s-v_0)^{-1} e^{-\frac{v_0}{v_1} s v_0}
$$

(3.1.57)

$$
\times \int_0^1 dt \ t^{s-\frac{3}{2}} e^{-\frac{v_0}{v_1} \left(1-t + \frac{t s v_1}{1-v_1}\right)} M(\frac{1}{2}, \frac{3}{2}, \frac{zv_0}{1+s-v_0} \left(1-t + \frac{t s v_1}{1-v_1}\right))
$$

22
\[ \gamma_2(s, t) = \frac{2zv_0y}{3(1-v_1)^{y+1}} \int_0^\infty ds \ s^\frac{1}{2}(1+s)^{-\frac{1}{2}}(1+s-v_0)^{-2} e^{-\frac{sv_0}{1-v_1}} \times \int_0^t dt \ t^{-\frac{1}{2}} e^{\frac{sv_0}{1-v_1}(1-t)} M\left(\frac{3}{2}, \frac{5}{2}; \frac{zv_0}{1-s-v_0}(1-t) + \frac{tsv_1}{1-v_1}\right) \] (3.1.58)

There are addition theorem for \( M(a, b, x + y) \).

\[ M(a, b, x + y) = \sum_{n=0}^\infty \frac{(a)_n x^n}{(b)_n n!} M(a + n, b + n, x) \] (3.1.59a)

\[ M(a, b, x + y) = e^y \sum_{n=0}^\infty \frac{(b-a)_n (-y)^n}{(b)_n n!} M(a + n, b + n, x) \] (3.1.59b)

Put \( a = \frac{3}{2}, b = \frac{5}{2}, x = \frac{zv_0}{1-s-v_0} \), and \( y = -\frac{zv_0}{1-v_1} \) into (3.1.59b), and take the new (3.1.59b) into (3.1.58) and replace \( s \) by \( \frac{1}{2} - 1 \) in it.

\[ \gamma_1(s, t) = 2 \sum_{n=0}^\infty \frac{(1-v_1)^{(-n+1)}(zv_0)^{n+1}(-v_1)^{n+1}}{(y+n+\frac{1}{2})(\frac{3}{2})_n} \int_0^t dt \ (1-t)^{\frac{1}{2}}(1-v_0t)^{-(n+1)}(1 - \frac{t}{v_1})^n \times M\left(n+1, n + \frac{3}{2}; \frac{zv_0}{1-v_0} \right) \] (3.1.60)

Put \( a = \frac{3}{2}, b = \frac{5}{2}, x = \frac{zv_0}{1-s-v_0} \), and \( y = -\frac{zv_0}{1-v_1} \) into (3.1.59b), and take the new (3.1.59b) into (3.1.58) and replace \( s \) by \( \frac{1}{2} - 1 \) in it.

\[ \gamma_2(s, t) = -3 \sum_{n=0}^\infty \frac{(1-v_1)^{(-n+1)}(zv_0)^{n+1}(-v_1)^{n+1}}{(y+n+\frac{1}{2})(\frac{3}{2})_n} \int_0^t dt \ (1-t)^{\frac{1}{2}}(1-v_0t)^{-(n+1)}(1 - \frac{t}{v_1})^n \times M\left(n+1, n + \frac{5}{2}; \frac{zv_0}{1-v_0} \right) \] (3.1.61)

We know that

\[ M(n+1, n + \frac{3}{2}; \frac{-zv_0}{1-v_0}) = \sum_{m=0}^\infty \frac{(n+1)_m (-zv_0)^m}{(n + \frac{3}{2})_m m!} t^m (1-v_0t)^{-m} \] (3.1.62a)

\[ M(n+1, n + \frac{5}{2}; \frac{-zv_0}{1-v_0}) = \sum_{m=0}^\infty \frac{(n+1)_m (-zv_0)^m}{(n + \frac{5}{2})_m m!} t^m (1-v_0t)^{-m} \] (3.1.62b)

Plug (3.1.62a) and (3.1.62b) into (3.1.60) and (3.1.61). Put \( \alpha = m + 1, \beta = n + m + 2, \beta' = -n, \gamma = m + \frac{5}{2}, x = v_0 \) and \( y = \frac{1}{v_1} \) in (3.1.50). And substitute the new (3.1.50) into the new (3.1.60) and (3.1.61).

\[ \gamma_1(s, t) = 2 \sum_{n,m=0}^\infty \frac{(-1)^{n+m+1}(1-v_1)^{(-n+y+1)}(zv_0)^{n+m+1}(v_1)^{n+1}}{(y+n+\frac{1}{2})(\frac{3}{2})_m (n + \frac{3}{2})_m \Gamma(m + \frac{3}{2})} \times F_1\left(m+1; n + m + 2; -n; m + \frac{5}{2}; v_0, \frac{1}{v_1} \right) \] (3.1.63)
\[ \gamma_2(s, t) = -\frac{2}{3} \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1}(1 - v_1)^{-(\alpha+\gamma+1)}(zv_0)^{n+m+1}(v_1)^{n+1} \Gamma(\frac{1}{2})}{(\gamma + n + \frac{1}{2}) \Gamma(m + \frac{3}{2})} \times F_1\left(m + 1; n + m + 2, -n; m + \frac{5}{2}; v_0, \frac{1}{v_1}\right) \] (3.1.64)

Substitute (3.1.63) and (3.1.64) into (3.1.56).

\[ \Gamma_1(s, p, t) = \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1}(n + m + 1) \Gamma(\frac{1}{2}) \Gamma(\frac{3}{2})(1 - v_1)^{-(\alpha+\gamma+1)}(zv_0)^{n+m+1}(v_1)^{n+1} \Gamma(m + \frac{5}{2})}{(\gamma + n + \frac{1}{2}) \Gamma(m + \frac{3}{2})} \times F_1\left(m + 1; n + m + 2, -n; m + \frac{5}{2}; v_0, \frac{1}{v_1}\right) \] (3.1.65)

Replace \( s \) and \( p \) by \( \frac{1}{v} - 1 \) and \( 1 - p^2 \) in (3.1.54).

\[ \Gamma_2(s, p, t) = \frac{\omega}{(1 - v_1)^{1/2}} \int_0^1 dv (1 - v)^{-1/2} (1 - v_0 v)^{-1} \int_0^1 dt \ t^{v_1} e^{-\frac{zv_0 v}{2}} \left(1 + \frac{v_1(t - v)}{v_1(v_1)}\right) \times \frac{\omega}{(1 - v_1)^{1/2}} \int_0^1 dv (1 - v)^{-1/2} (1 - v_0 v)^{-1} \int_0^1 dt \ t^{v_1} \times M\left(1, \frac{3}{2}, \frac{zv_0 v}{v(1 - v_1)}; 1 - t + \frac{v_1(1 - v)}{v_1}\right) \] (3.1.66)

Put \( a = 1, b = \frac{3}{2}, x = -\frac{zv_0 v}{v(1 - v_1)} \) into (3.1.59a), and take the new (3.1.59a) into (3.1.66).

\[ \Gamma_2(s, p, t) = \frac{\omega}{(1 - v_1)^{1/2}} \sum_{n,m=0}^{\infty} \frac{v_1^n(1 - v_1)^{-(\alpha+\gamma+1)}(zv_0)^{n+m+1}(v_1)^{n+1} \Gamma(m + \frac{5}{2})}{(\gamma + n + \frac{1}{2}) \Gamma(m + \frac{3}{2})} \times \int_0^1 dv v_1^n(1 - v)^{-1/2} (1 - v_0 v)^{-1} \left(1 - \frac{v_1}{v_1}\right)^n \] (3.1.67)

Put \( \alpha = m + 1, \gamma = m + \frac{3}{2}, \beta = -n, \beta = n + m + 1, x = v_0 \) and \( y = \frac{1}{v_1} \) in (3.1.50). Take the new (3.1.50) into (3.1.67).

\[ \Gamma_2(s, p, t) = \omega \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1} \Gamma(\frac{1}{2})(n + 1) \Gamma(\frac{3}{2})(1 - v_1)^{-(\alpha+\gamma+1)}(zv_0)^{n+m}(v_1)^{n+1} \Gamma(m + \frac{5}{2})}{(\gamma + n + \frac{1}{2}) \Gamma(m + \frac{3}{2})} \times F_1\left(m + 1; n + m + 2, -n; m + \frac{5}{2}; v_0, \frac{1}{v_1}\right) \] (3.1.68)
Substitute (3.1.65) and (3.1.68) into (3.1.52).

\[
B = \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1}(n + m + 1)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)(1 - v_1)^{-(n+\gamma+1)}(zv_0)^{n+m+1}(v_1)^{n}}{(\gamma + n + \frac{1}{2})\Gamma(m + \frac{1}{2})\Gamma(n + m + \frac{1}{2})} \\
\times F_1\left(m + 1; n + m + 2, -n; m + \frac{3}{2}; v_0, \frac{1}{v_1}\right) \\
+\omega \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}\Gamma\left(\frac{1}{2}\right)(n + 1)_m(1 - v_1)^{-(n+\gamma)}(zv_0)^{n+m}(v_1)^{n}}{(\gamma + n - \frac{1}{2})(\gamma + n + \frac{1}{2})_m\Gamma(m + \frac{1}{2})\Gamma(n + m + \frac{1}{2})} \\
\times F_1\left(m + 1; n + m + 1, -n; m + \frac{3}{2}; v_0, \frac{1}{v_1}\right)
\]

Plug (3.1.51) and (3.1.69) into (3.1.45). Then I obtain the generating function for 1st kind of GCH function in the following way.

\[
\sum_{|\alpha|,|\beta|=0}^{\infty} \sum_{|\beta|=0}^{\infty} \frac{B(|\alpha| + 1, \frac{1}{2})}{(|\alpha|)!} v_0^{|\beta|} v_1^{|\alpha|} W_{|\alpha|,|\beta|} (y = \frac{1}{2}(1 + y); z = -\frac{1}{2} \mu^2)
\]

\[
= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}\right)(-zv_0v_1)^n}{\Gamma(n + \frac{1}{2})} F_1\left(n + 1; 1, \gamma + n; n + \frac{3}{2}; v_0, v_0v_1\right)
\]

\[
-\frac{\omega}{2^3} \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m+1}(n + m + 1)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)(1 - v_1)^{-(n+\gamma+1)}(zv_0)^{n+m+1}(v_1)^{n}}{(\gamma + n + \frac{1}{2})\Gamma(m + \frac{1}{2})\Gamma(n + m + \frac{1}{2})} \\
\times F_1\left(m + 1; n + m + 2, -n; m + \frac{5}{2}; v_0, \frac{1}{v_1}\right) \\
+\omega \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}\Gamma\left(\frac{1}{2}\right)(n + 1)_m(1 - v_1)^{-(n+\gamma)}(zv_0)^{n+m}(v_1)^{n}}{(\gamma + n - \frac{1}{2})(\gamma + n + \frac{1}{2})_m\Gamma(m + \frac{1}{2})\Gamma(n + m + \frac{1}{2})} \\
\times F_1\left(m + 1; n + m + 1, -n; m + \frac{3}{2}; v_0, \frac{1}{v_1}\right)
\]

\[
= \sum_{n,k=0}^{\infty} \frac{B(n + k + j + 1, \frac{1}{2})(y + n)(-z)^n}{n! k!} v_0^{n+kj} v_1^{n} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2} \sum_{n,m,k,l=0}^{\infty} \left\{ \begin{array}{l}
(1)^{n+m+1}(n + m + 1)(n + \gamma + 1)_s(n + m + 1)_{j,k}(n + m + 2)\frac{(n)_k}{v_0^{n+m+1}} v_1^{n+m+j+k}
\times F_1\left(m + 1; n + m + 2, -n; m + \frac{5}{2}; v_0, \frac{1}{v_1}\right)
\end{array} \right\}
\]

Also, I can describe the generating function of it as the integral formalism by using (3.1.47), (3.1.53) and (3.1.54).
\[
\sum_{|\alpha_1|=0}^{\infty} \sum_{|\alpha| = |\beta|}^{\infty} \frac{B(|\alpha_1|+1, \frac{1}{2})}{(1+|\alpha_1|)!} v_0^{(|\alpha_1|)} v_1^{(|\alpha|)} QW_{\varphi_{\alpha_1}\beta_\gamma}(\gamma = \frac{1}{2}(1+\nu); z = -\frac{1}{2} \mu x^2)
\]

\[
= \int_0^1 dt (1-t)^{-\nu}(1-v_0 t)^{-1}(1-v_0 v_1 t)^{-\nu} e^{-\frac{v_0 v_1}{1-v_1 t}} \times \frac{e}{2 (1-v_1 t)^{\nu}} \int_0^1 du (1-u)^{-\nu}(1-v_0 u)^{-1} \int_0^1 dt v^\nu \int_0^1 dp (1-p)^{-\nu} \\
\times \left\{ \frac{\omega}{2} - \frac{zp t(1-u) v_0}{(1-v_1 t)(1-uv_1)} \right\} e^{-\frac{v_0 v_1}{1-v_1 t}} \frac{\nu}{1-v_1 t} \right\} \right) (3.1.71)
\]

### 3.1.4 Orthogonal relation

From the differential equations satisfied by \( QW_{\varphi_{\alpha_1}\beta_\gamma}(z) \) and \( QW_{\varphi_{\alpha_1}\beta_\gamma}(z) \), namely,

\[
xQW_{\varphi_{\alpha_1}\beta_\gamma}(z) + (\mu x^2 + \epsilon x + \nu)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) + (\Omega_{\varphi_{\alpha_1}\beta_\gamma} x + \epsilon \omega)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) = 0 \quad (3.1.72a)
\]

\[
xQW_{\varphi_{\alpha_1}\beta_\gamma}(z) + (\mu x^2 + \epsilon x + \nu)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) + (\Omega_{\varphi_{\alpha_1}\beta_\gamma} x + \epsilon \omega)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) = 0 \quad (3.1.72b)
\]

where,

\[
\Omega_{\varphi_{\alpha_1}\beta_\gamma} = \left\{ \begin{array}{ll} -2\mu|\alpha_1| & -2\mu|\alpha_1| + \frac{1}{2} \\ -2\mu|\beta_\gamma| & -2\mu|\beta_\gamma| + \frac{1}{2} \end{array} \right\} (3.1.73)
\]

Multiplying (3.1.72a) and (3.1.72b) by \( QW_{\varphi_{\alpha_1}\beta_\gamma}(z) \) and \( QW_{\varphi_{\alpha_1}\beta_\gamma}(z) \) respectively and subtracting, I have

\[
x^dM(z) + (\mu x^2 + \epsilon x + \nu)M(z) = (\Omega_{\varphi_{\alpha_1}\beta_\gamma} x - \Omega_{\varphi_{\alpha_1}\beta_\gamma} x)QW_{\varphi_{\alpha_1}\beta_\gamma}(z)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) = 0 \quad (3.1.74)
\]

where

\[
M(z) = QW_{\varphi_{\alpha_1}\beta_\gamma}(z)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) - QW_{\varphi_{\alpha_1}\beta_\gamma}(z)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) \quad (3.1.75)
\]

Multiply \( x^{\nu-1}e^{\frac{\mu x^2 + \epsilon x}{2}} \) on both side of (3.1.74) Then integrate it with respect to \( x \) from 0 to \( \infty \).

\[
\left( \Omega_{\varphi_{\alpha_1}\beta_\gamma} - \Omega_{\varphi_{\alpha_1}\beta_\gamma} \right) \int_0^\infty dx x^\nu e^{\frac{\mu x^2 + \epsilon x}{2}} QW_{\varphi_{\alpha_1}\beta_\gamma}(z)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) = \left[ x^\nu e^{\frac{\mu x^2 + \epsilon x}{2}}(QW_{\varphi_{\alpha_1}\beta_\gamma}(z)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) - QW_{\varphi_{\alpha_1}\beta_\gamma}(z)QW_{\varphi_{\alpha_1}\beta_\gamma}(z)) \right]_0^\infty \quad (3.1.76)
\]

Therefore, only if \( |\alpha_1| \neq |\beta_\gamma| \) and \( |\alpha_1| \neq |\beta_\gamma| \), then

\[
\int_0^\infty dx x^\nu e^{\frac{\mu x^2 + \epsilon x}{2}} QW_{\varphi_{\alpha_1}\beta_\gamma}(z)QW_{\varphi_{\alpha_1}\beta_\gamma}(z) = 0 \quad (3.1.77)
\]

Let’s think about the case in which are \( |\alpha_1| = |\beta_\gamma| \) and \( |\alpha_1| = |\beta_\gamma| \). First of all, from (3.1.9)

\[
\left[ QW_{\varphi_{\alpha_1}\beta_\gamma}(z) \right]^2 = \left[ F_{\varphi_{\alpha_1}}(z) \right]^2 - \epsilon x F_{\varphi_{\alpha_1}}(z) \int \frac{|\alpha_1|}{|\alpha_1|} (z) + \frac{\nu^2}{4} \left[ \int \frac{|\alpha_1|}{|\alpha_1|} (z) \right]^2 \]

\[
\simeq \left[ F_{\varphi_{\alpha_1}}(z) \right]^2 - \epsilon x F_{\varphi_{\alpha_1}}(z) \int \frac{|\alpha_1|}{|\alpha_1|} (z) \quad 26 \quad (3.1.78)
\]
$\epsilon$ is extremely small. So anything beyond the second order of $\epsilon$ is negligible in \((3.1.78)\). Multiply $x^\epsilon e^{\mu x^2+\epsilon x}$ in \((3.1.78)\) and integrate them with respect to $x = [0, \infty]$.

$$\int_0^\infty dx ~x^\epsilon e^{\mu x^2+\epsilon x}
\left[QW_{[\alpha_0],[\alpha_1]}(z)\right]^2 \approx E_1 - \epsilon E_2$$ \hspace{1cm} (3.1.79)

where

$$E_1 = \int_0^\infty dx ~x^\epsilon e^{\mu x^2+\epsilon x}
\left[F_{[\alpha_0]}(z)\right]^2$$ \hspace{1cm} (3.1.80a)

$$E_2 = \int_0^\infty dx ~x^{\epsilon+1} e^{\mu x^2+\epsilon x}
F_{[\alpha_0]}(z) \prod_{[\nu_0]} (z)$$ \hspace{1cm} (3.1.80b)

Apply the generating function of the first kind of confluent hypergeometric function into \((3.1.80a)\) and \((3.1.80b)\).

$$E_1 \approx \frac{2^{\gamma-1}}{|\mu|^{\gamma+\frac{3}{2}}} \Gamma(\alpha_0 + 1) \Gamma(\alpha_0 + \gamma) + \epsilon
\left(-\frac{1}{\mu}\right)^{\gamma/2} \frac{\Gamma(\alpha_0 + \gamma - \frac{1}{2}) \Gamma(\alpha_0 + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})}$$ \hspace{1cm} (3.1.81)

$$E_2 \approx \frac{2^{\gamma-1}}{|\mu|^{\gamma+\frac{3}{2}}} \frac{\Gamma(\alpha_0 + \gamma) \sum_{n=0}^{\infty} \frac{(-1)^n (n + \frac{1}{2}) \Gamma(n + \gamma - \frac{1}{2}) \Gamma(n + \gamma + \frac{1}{2})}{n!} \Gamma(n + k - |\alpha_0| + \frac{1}{2}) \Gamma(n + k + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})}$$ \hspace{1cm} (3.1.82)

I neglect any terms which include more than second order of $\epsilon$, extremely small, in \((3.1.81)\) and \((3.1.82)\). Substitute \((3.1.81)\) and \((3.1.82)\) into \((3.1.79)\) with neglecting any terms larger than second order of $\epsilon$. Therefore, the orthogonal relation of it is

$$\int_0^\infty dx ~x^\epsilon e^{\mu x^2+\epsilon x}
\left[QW_{[\alpha_0],[\alpha_1]}(z)\right]QW_{[\nu_0],[\nu_1]}(\tilde{z})$$

$$\approx \frac{2^{\gamma-1}}{|\mu|^{\gamma+\frac{3}{2}}} \frac{\Gamma(\alpha_0 + \gamma - \frac{1}{2}) \Gamma(\alpha_0 + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})} \left\{ \frac{\Gamma(\alpha_0 + \gamma - \frac{1}{2}) \Gamma(\alpha_0 + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})} \right\} \prod_{[\nu_0]} \delta_{[\nu_0],[\nu_1]} \delta_{|\nu_0|,|\nu_1|}$$ \hspace{1cm} (3.1.83)

Also, if there is an analytic function $\Psi(x)$ having first and second continuous derivatives in \([0, \infty]\) and approaching zero when $x \to \infty$, they can be expanded in terms of $QW_{[\alpha_0],[\alpha_1]}(\gamma; z)$ by using \((3.1.83)\):

$$\Psi(x) = \sum_{[\alpha_0]=0}^{\infty} \sum_{[\alpha_1]=|\alpha_0|}^{\infty} A_{[\alpha_0],[\alpha_1]} QW_{[\alpha_0],[\alpha_1]}(\gamma = \frac{1}{2} (1 + \nu); z = -\frac{1}{2} \mu x^2)$$ \hspace{1cm} (3.1.84)
where,

\[ A_{\nu_1,\nu_1} \approx \int_0^{\infty} dx \ x^\frac{1}{2} e^{\frac{1}{2} x + x} Q e^{1/2} \nu_0 e_1^2 \mu x^2 + e x Q W_{\nu_1,\nu_1}(y; z) \Psi(x) \left[ \frac{2^{\nu-1} \Gamma(\nu_0 + 1) \Gamma(\nu + \gamma) + e^{(-1)^{\nu_0}} 2^{\nu-1}}{\mu^{\nu+1}} \right] \]

\times \left\{ \frac{\Gamma(\nu_0 + \gamma - \frac{1}{2}) \Gamma(\nu_0 + \gamma + \frac{1}{2})}{\Gamma(\gamma - \frac{1}{2})} \right\}^{-1} \quad (3.1.85)

\[ \frac{1}{2} \left\{ \sum_{n=0}^{\nu_0} \left( -\nu_0 \right)_n \sum_{k=0}^{\nu_1-n} \frac{(n + \frac{\nu_0}{2}) \Gamma(n + \gamma - \frac{1}{2}) \Gamma(n + \gamma + \frac{1}{2}) (n + 1)}{2 \Gamma(n + 1)} \right\}^{-1} \]

3.2. As \( \lambda_2 = 1 - \nu \)

### 3.2.1. Power series expansion in closed forms and integral formalism

In the previous case, I obtain the first kind of independent solution of GCH function, putting \( \lambda_1 = 0 \).

Now put \( \lambda_2 = 1 - \nu \) into (2.0.29) to get the second solution of GCH function.

\[ A_{\nu_1,\nu_1} = -\frac{e(n + \omega - \nu + 1)}{(n + 1)(n + 2 - \nu)} \quad (3.2.1a) \]

\[ B_{\nu_1,\nu_1} = -\frac{\Omega + \mu(n - \nu)}{(n + 1)(n + 2 - \nu)} \quad (3.2.1b) \]

And there are eigenvalues where \( \gamma = \frac{1}{2}(1 + \nu) \) which are

\[ -\frac{\Omega}{2\mu} + \gamma - 1 = \psi_0 \quad \text{where } \psi_0 = 1, 2, 3, \cdots \quad (3.2.2a) \]

\[ -\frac{\Omega}{2\mu} + \gamma - \frac{3}{2} = \psi_1 \quad \text{where } \psi_1 = 1, 2, 3, \cdots \quad (3.2.2b) \]

Substitute (3.2.1a) and (3.2.1b) into (3.1.3). Then I obtain the second independent solution of GCH function by using similar process as I did before.

\[ R W_{\psi_0,\psi_1}(y = \frac{1}{2}(1 + \nu); z = -\frac{1}{2} \mu x^2) = e^{1-\gamma} \left\{ A_{\psi_0}(y; z) - \frac{e^{-1}}{2} \int_{\psi_0}^{\psi_1} \right\} \]

where,

\[ A_{\psi_0}(y; z) = \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_0} (-\psi_0)_n \left( \frac{1}{n!} \right)^n \psi_0 \quad (3.2.3) \]

(3.2.4)
\[
\psi_{\gamma}^{\nu}(\gamma; z) = \Gamma(\gamma_0 + 2 - \gamma) \sum_{n=0}^{\nu_0} (\gamma_0)_n \frac{(-\gamma_0)_n}{\Gamma(2 - \gamma)} z^n \\
\times \sum_{k=0}^{\nu - n} \left( \frac{\gamma_0 + n + 1 - \gamma}{\Gamma(k + n + \frac{1}{2})} \right) \left( \frac{\gamma + \frac{3}{2} - \gamma}{\Gamma(k + n + \frac{3}{2} - \gamma)} \right) z^k
\]

(3.2.5)

\[
= \frac{1}{2\pi i B(\frac{1}{2}, \nu_0 + 1)} \int_0^\infty ds \int_0^\infty dt \int_0^1 dp \int du e^{-\frac{\pi}{2\nu_0 + 1}(1 - p^2)} \\
\times \left( w_1 \partial_{w_1} + \left( \frac{\nu}{2} + 1 - \gamma \right) \nu_{\nu_0}(\gamma; w_1 = \frac{z_2}{1 - p^2}) \frac{u}{(1 - u)} \right)
\]

(3.2.6)

And its integral representation is

\[
\Re W_{\nu_0, \gamma}(\gamma = \frac{1}{2}(1 + v); z = \frac{1}{2} x) = z^{-\gamma} \left( \frac{\psi_0!}{2\pi i} \int_{\nu_{\nu_0} + 1} e^{\frac{\gamma + 1}{2}(1 - p^2)} \right) z^{\nu_0} \\
\times \left\{ \frac{z_1}{2(1 - v)} U\left( \frac{3}{2} - \gamma_0 + 1, w_5 \right) + \left( \frac{\nu}{2} + 1 - \gamma \right) U\left( \frac{1}{2}, -\gamma_0, w_5 \right) \right\}
\]

(3.2.7)
where,
\[
\int_{V_0}^{v_1} (y; z) = \frac{\psi_0^2}{(2\pi i)^2 B(\frac{1}{2}, \psi + \frac{1}{2})} \int_0^\infty ds \frac{s^{-2} (1 + s)^{-1}}{\Gamma(1 + s)} \gamma \int_0^1 dt t^{\gamma} \int_1^1 dp \times \int_0^{\psi_0} dw \psi_0^{\psi_0+1} (1 - v)^{2 - \gamma} \int_0^{\psi_0+1} (1 - u) \left[ -z stuv (1 - p^2) \right] \frac{1}{(1 - v)(1 - v)} + \left( \frac{\omega}{2} + 1 - \gamma \right) e^{-\frac{(2 - \gamma)(1 - p^2)}{(1 - v)(1 - v)}} \right] \right)
\]

Due to space restrictions of (3.2.3)-(3.2.8) are not included in the paper, but feel free to contact me for the proofs.

3.2.2. Generating function

Let's try to construct the generating function of the 2nd kind of GCH function. First, multiply
\[
\sum_{\psi_0 = 0}^\infty \sum_{\psi_1 = 0}^\infty \frac{B(\psi_1 + 1, \frac{1}{2})}{\psi_1!} v_0^{\psi_0} v_1^{\psi_1} \text{ on both sides of integral form of 2nd kind of GCH function in (3.2.3)-(3.2.5)}
\]

where \(|v_0| < 1\) and \(|v_1| < 1\). Then, its solution is the following way.

\[
\sum_{\psi_0 = 0}^\infty \sum_{\psi_1 = 0}^\infty \frac{B(\psi_1 + 1, \frac{1}{2})}{\psi_1!} v_0^{\psi_0} v_1^{\psi_1} \text{ } \Re W_{\psi_0, \psi_1} \left( \gamma = \frac{1}{2} (1 + \gamma); z = -\frac{1}{2} \mu x^2 \right)
\]

\[
= z^{-1 - \gamma} \sum_{m=0}^\infty \frac{\Gamma(\frac{1}{2}) (-z v_0 v_1)^m}{\Gamma(m + \frac{1}{2})} F_1 \left( m + 1, 2 - \gamma + m; m + 3 \right) \frac{v_0 v_1}{v_0} \nu_1
\]

\[
= e^{-\frac{\omega}{2} \psi_0} \sum_{m,n=0}^\infty \frac{\Gamma(\frac{1}{2}) \gamma (n + m + 1) (n + m + 1) (1 - v_1)^{-(n+\gamma)} (-z v_0)^{m+n+1} (v_1)^{n+1}}{(n + \frac{3}{2} - \gamma) \Gamma(n + m + \frac{3}{2}) \Gamma(n + m + \frac{\gamma}{2})} \nu_1
\]

\[
+ \left( \frac{\omega}{2} + 1 - \gamma \right) \sum_{n,m=0}^\infty \frac{\Gamma(\frac{1}{2}) \gamma (n + m + 1) (n + m + 1) (1 - v_1)^{-(n+\gamma)} (-z v_0)^{m+n+1} (v_1)^{n+1}}{(n + \frac{3}{2} - \gamma) \Gamma(n + m + \frac{3}{2}) \Gamma(n + m + \frac{\gamma}{2})} \nu_1
\]

\[
= z^{-1 - \gamma} \sum_{m,n,k=0}^\infty \frac{B(m + k + j + 1, \frac{1}{2}) (m + 2 - \gamma) \gamma (z)^m (\gamma)^m \nu_1}{v_0^{m+k+j+1} v_1^{m+k+1} - \frac{\Gamma(\frac{1}{2}) \gamma}{2} \Gamma(\frac{1}{2}) \gamma} \nu_1
\]

\[
\times \sum_{m,n,j,k=0}^\infty \left\{ \begin{array}{c}
\Gamma(n + s + 3 - \gamma) \Gamma(n + m + j + 1) (n + m + 1) (1 - v_0)^{m+n+1} (v_0)^{n+1} (n + m + j + 1) (n + m + 1) (1 - v_0)^{m+n+1} (v_0)^{n+1} \\
+ \frac{\Gamma(\frac{1}{2}) (\omega + \gamma) \gamma (n + m + j + 1) (n + m + 1) (1 - v_0)^{m+n+1} (v_0)^{n+1}}{n + \frac{3}{2} - \gamma) \Gamma(n + m + j + 1) (n + m + 1) (1 - v_0)^{m+n+1} (v_0)^{n+1}} \nu_1
\end{array} \right\}
\]

(3.2.9)
And its integral representation of generating function is

\[
\begin{align*}
&\sum_{\phi_0=0}^{\infty} \sum_{\psi_1=0}^{\infty} \frac{B(\psi_1 + 1 + \frac{1}{\mu})}{\phi_0!} v_0^{\phi_1} v_1^{\phi_0} R W_{\psi_1,\psi_0}(y = \frac{1}{2}(1 + \nu); z = -\frac{1}{2} \mu x^2) \\
&= z^{-1/2} \left\{ \int_0^1 dt (1-t)^{-1/2}(1-v_0 t)^{-1} e^{-\int_0^1 \frac{e}{2(1-v_0 t)} dt} \right\}
\end{align*}
\]


\[
\int_0^1 dy (1-y)^{1/2}(1-v_0 y)^{-1}
\]

Due to space restrictions, proofs of \((3.2.9)\) and \((3.2.10)\) are not included in the paper, but feel free to contact me for the proofs.

3.2.3. Orthogonal relation

From the differential equations satisfied by \(\mathcal{R}W_{\psi_0,\psi_1}(z)\) and \(\mathcal{R}W_{\psi_0,\phi_1}(c)\), namely,

\[
x \mathcal{R}W_{\psi_0,\psi_1}(x) + (\mu x^2 + e x + v) \mathcal{R}W_{\psi_0,\psi_1}(z) + (\Omega_{\psi_0,\phi_1} x + e \omega) \mathcal{R}W_{\psi_0,\phi_1}(z) = 0
\]

\[(3.2.11a)\]

\[
x \mathcal{R}W_{\psi_0,\phi_1}(x) + (\mu x^2 + e x + v) \mathcal{R}W_{\psi_0,\phi_1}(z) + (\Omega_{\psi_0,\phi_1} x + e \omega) \mathcal{R}W_{\psi_0,\phi_1}(z) = 0
\]

\[(3.2.11b)\]

where,

\[
\Omega_{\psi_0,\psi_1} = \left\{ \begin{array}{ll} -2\mu(\psi_0 + 1 - \gamma) \\
-2\mu(\psi_1 + \frac{1}{2} - \gamma) \end{array} \right. \quad \Omega_{\psi_0,\phi_1} = \left\{ \begin{array}{ll} -2\mu(\psi_0 + 1 - \gamma) \\
-2\mu(\phi_1 + \frac{1}{2} - \gamma) \end{array} \right. \]

Multiplying \((3.2.11a)\) and \((3.2.11b)\) by \(\mathcal{R}W_{\psi_0,\phi_1}(z)\) and \(\mathcal{R}W_{\psi_0,\phi_1}(z)\) respectively and subtracting, I have

\[
x \frac{dN(z)}{dx} + (\mu x^2 + e x + v) N(z) = (\Omega_{\psi_0,\phi_1} - \Omega_{\psi_0,\phi_1}) x \mathcal{R}W_{\psi_0,\phi_1}(z) \mathcal{R}W_{\psi_0,\phi_1}(z)
\]

\[(3.2.13)\]

where

\[
N(z) = \mathcal{R}W_{\psi_0,\phi_1}(z) \mathcal{R}W_{\psi_0,\phi_1}(z) - \mathcal{R}W_{\psi_0,\phi_1}(z) \mathcal{R}W_{\psi_0,\phi_1}(z)
\]

\[(3.2.14)\]

Multiply \(x^{1/2} e^{\mu x^2 + ex}\) on both side of \((3.2.13)\) And integrate it with respect to \(x\) from 0 to \(\infty\),

\[
(\Omega_{\psi_0,\phi_1} - \Omega_{\psi_0,\phi_1}) \int_0^\infty dx x^e e^{\mu x^2 + ex} \mathcal{R}W_{\psi_0,\phi_1}(z) \mathcal{R}W_{\psi_0,\phi_1}(z)
\]

\[
= \left[ x^e e^{\mu x^2 + ex} (\mathcal{R}W_{\psi_0,\phi_1}(z) \mathcal{R}W_{\psi_0,\phi_1}(z) - \mathcal{R}W_{\psi_0,\phi_1}(z) \mathcal{R}W_{\psi_0,\phi_1}(z)) \right]_0^\infty
\]

\[(3.2.15)\]

By using similar process as I did it in the case of the first independent solution of GCH function, I obtain

\[
\int_0^\infty dx x^e e^{\mu x^2 + ex} \mathcal{R}W_{\psi_0,\phi_1}(z) \mathcal{R}W_{\psi_0,\phi_1}(z)
\]

\[
= \left\{ \begin{array}{ll} \frac{\Gamma(2 - \gamma) B(3 - \gamma, \psi_0)}{2^{1-\gamma}} + e^{(-1)^{\psi_0} \Gamma(\psi_0 + \frac{3}{2} - \gamma)} \frac{1}{\Gamma(\frac{1}{2} - \gamma)} \\
\phi_k \Gamma(\gamma + 1) \Gamma(n + 1) \Gamma(n + \frac{3}{2} - \gamma) \Gamma(n - \psi_1) \end{array} \right. \}
\]

\[(3.2.16)\]
If there is an analytic function $\Phi(x)$ having first and second continuous derivatives in $[0, \infty]$ and approaching zero when $x \to \infty$, it can be expanded in terms of $R^W_{\rho_0,\rho_1}(y;z)$.

$$\Phi(x) = \sum_{\psi_0=0}^{\infty} \sum_{\rho_1=0}^{\infty} B_{\psi_0,\rho_1} R^W_{\psi_0,\rho_1}(y;z) = \frac{1}{2}(1 + \gamma); z = -\frac{1}{2} \mu x^2) \quad (3.2.17)$$

where,

$$B_{\rho_0,\rho_1} \approx \int_0^{\infty} dx \, x^2 e^{\pm \mu x^2} R^W_{\rho_0,\rho_1}(y;z) \Phi(x) \left[ \frac{\rho_0 \Gamma(2 - \gamma) B(3 - \gamma, \rho_0)}{2^{1-\gamma} \Gamma(6 - 3\gamma, \rho_0) \left( \Gamma(2 - \gamma) \sum_{n=0}^{\infty} \frac{\psi^{n+1} \Gamma(n+1)\Gamma(n+\frac{3}{2} - \gamma)(n-\rho_0)}{n!(2-\gamma)n^!} \right)^{-1} \right]$$

$$+ e\left( -1 \right)^{\rho_0} x^{\frac{3}{2}} \left( \frac{\Gamma(\psi_0 + \frac{3}{2} - \gamma) \Gamma(\psi_0 + \frac{1}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)} \right)^{-1} \left( \frac{\psi_0 + \frac{3}{2} - \gamma) \sum_{n=0}^{\infty} \frac{\psi^{n+1} \Gamma(n+1)\Gamma(n+\frac{3}{2} - \gamma)(n-\rho_0)}{n!(2-\gamma)n^!} \right)^{-1}$$

Due to space restriction proofs of (3.2.16)-(3.2.18) are not included in the paper, but feel free to contact me for the proofs.

4. Infinite series for $\nu=\text{non-integer}$

In the previous section, I discussed the solutions of polynomial case.

The $1^{st}$ independent solution has two eigenvalues which are

$$\begin{cases}
|\alpha| = -\frac{\Omega}{\mu} \Rightarrow 0, 1, 2, 3, \cdots \\
|\beta| = -\frac{\Omega}{\mu} - \frac{1}{2} \Rightarrow 0, 1, 2, 3, \cdots
\end{cases} \quad \text{; only if } |\alpha| \geq |\alpha_0| \quad (4.0.19)$$

Also, the $2^{nd}$ independent solution has two eigenvalues which are

$$\begin{cases}
\psi_0 = -\frac{\Omega}{\mu} + \gamma - 1 \Rightarrow 0, 1, 2, 3, \cdots \\
\psi_1 = -\frac{\Omega}{\mu} + \gamma - \frac{5}{2} \Rightarrow 0, 1, 2, 3, \cdots
\end{cases} \quad \text{; only if } \psi_1 \geq \psi_0 \quad (4.0.20)$$

These conditions make the solutions as polynomial.

However, if

$$-\frac{\Omega}{\mu} \neq \left\{ \begin{array}{l}
0, 1, 2, 3, \cdots \\
\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots
\end{array}, -\frac{\Omega}{\mu} \neq \left\{ \begin{array}{l}
1 - \gamma, 2 - \gamma, 3 - \gamma, \cdots \\
\frac{3}{2} - \gamma, \frac{5}{2} - \gamma, \cdots
\end{array} \right\} \quad (4.0.21)$$

Then analytic solutions of GCH function turn to be infinite series.

From the $1^{st}$ kind of GCH polynomial, let $|\alpha_0| = -\frac{\Omega}{\mu}$ and $|\beta_1| = -\frac{\Omega}{\mu} - \frac{1}{2}$ in (3.1.59), (3.1.109) and (3.1.10b) where $|\alpha_0|, |\beta_1| \neq 0, 1, 2, 3, \cdots$. The series turns to the infinite series for the case of $\lambda_1 = 0$. I denote the $1^{st}$ independent solution of GCH function for the infinite series as $QW(y;1 + \gamma); z = -\frac{1}{2} \mu x^2)$. From the $2^{nd}$ kind of GCH polynomial, let $\psi_0 = -\frac{\Omega}{\mu} + 1 + \gamma$ and $\psi_1 = -\frac{\Omega}{\mu} - \frac{1}{2} + \gamma$ in (3.2.3)–(3.2.6) where $\psi_1 \neq 0, 1, 2, 3, \cdots$. The series turns to the infinite series for the case of $\lambda_2 = 1 - \nu$. I denote the $2^{nd}$ independent solution of GCH function for the infinite series as
\( RW(γ = \frac{1}{2}(1 + ν); z = -\frac{1}{2}μx^2) \). Due to space restriction power series expansions in closed forms of GCH function for the infinite series are not included in the paper, but feel free to contact me for more details.

When \( ν \) is integer, one of two solution of the GCH equation does not have any meaning, because \( RW(γ = \frac{1}{2}(1 + ν); z = -\frac{1}{2}μx^2) \) and \( RW(γ = \frac{1}{2}(1 + ν); z = -\frac{1}{2}μx^2) \) can be described as \( QW_{\text{nor}}(γ = \frac{1}{2}(1 + ν); z = -\frac{1}{2}μx^2) \) and \( RW(γ = \frac{1}{2}(1 + ν); z = -\frac{1}{2}μx^2) \) as long as \( |λ_1 - λ_2| = |ν - 1| \) = integer. It is required that \( γ \neq 0, -1, -2, \ldots \) for the first kind of independent solution of GCH function for the polynomial and infinite series. Also \( γ \neq 2, 3, 4, \ldots \) is required for the second kind of independent solution of GCH function for both the polynomial and infinite series. Due to space restriction I skip the polynomial and infinite series of GCH function for \( r \in Z \). If the time is permitted, I will publish the power series expansion in closed forms and its integral forms of GCH function for \( r \in Z \).

5. Conclusion

By using Frobenius method and putting the power series expansion into GCH ordinary differential equation, three recursive relation of coefficients starts to appear. Currently the analytic solution of three term recurrence relation is unknown. Due to its complexity more than three term case has been neglected. In this paper I claim that \( ε \), one of coefficients, is extremely small quantity because of its physical meaning: \( ε \) is correspondent to the small mass of quark. Therefore I construct the analytic solution of GCH function including only up to the first order of \( ε \). More than second order of \( ε \) is negligible.

From the above all, I show asymptotic expansions of GCH function for the infinite series and polynomial. I construct the power series expansions in closed forms of GCH function including only up to the first order of \( ε \) (infinite series and polynomials). As we see all solutions of power series expansions in GCH function, denominators and numerators in \( g(λ)_{\text{dom}} \) and \( g(λ)_{\text{smal}} \) terms arise with Pochhammer symbol: the meaning of this is that the analytic solutions of GCH ordinary differential equation can be described as Hypergeometric function in a strict mathematical way.

I construct representations in closed form integrals of GCH functions in an easy way since I have power series expansions with Pochhammer symbols in numerators and denominators. I show that the first and second kind of Confluent hypergeometric functions appear in the sub-integral forms of GCH function in \( \text{(3.1.34)} \) and \( \text{(3.2.6)} \). As we see \( \text{(3.1.35), (3.1.35)} \) and \( \text{(3.2.6)} \), the Confluent hypergeometric function in the sub-integral forms of GCH function is able to be transformed to other well-known special functions analytically such as the first and second Kummer and Laguerre functions. Understanding the connection between other special functions is important in the mathematical and physical points of views as we all know.

Analytic integral forms including only up to the first order of \( ε \) of GCH functions are derived from power series expansion in closed forms of GCH differential equation. I construct the generating functions of GCH polynomial from the its integral forms and I derive orthogonal relations of GCH polynomial including only up to the first order of \( ε \). The orthogonal relation of GCH polynomial is important in the physical point of view because we can derive recurrence relations and the expectation values of physical quantities. For the case of hydrogen-like atoms, the normalized wave function is derived from the generating function of associated Laguerre polynomial. The expectation value of physical quantities such as position and momentum is constructed by applying the recursive relation of associated Laguerre polynomial.
Furthermore, I will generalize three term recurrence relation in linear differential equation. I will derive the analytic solution of the three term recurrence for polynomials and infinite series \[8\]. I will show how to solve mathematical equations having three term recursion relation and go on producing the analytic solutions of some of the well known special function theories that include Heun \[9, 10\], Mathieu \[11\], Lame \[12, 13, 14\] and the GCH Functions \[15, 16\]. I hope these new functions and their solutions will produce remarkable new range of applications not only in supersymmetric field theories as is shown here, but in the areas of all different classes of mathematical physics, applied mathematics and in engineering applications.

6. Series “Special functions and three term recurrence formula (3TRF)”

This paper is 1st out of 10.

1. “Approximative solution of the spin free Hamiltonian involving only scalar potential for the \(q - \bar{q}\) system” \[7\] - In order to solve the spin-free Hamiltonian with light quark masses we are led to develop a totally new kind of special function theory in mathematics that generalize all existing theories of confluent hypergeometric types. We call it the Grand Confluent Hypergeometric Function. our new solution produces previously unknown extra hidden quantum numbers relevant for description of supersymmetry and for generating new mass formulas.

2. “Generalization of the three-term recurrence formula and its applications” \[8\] - Generalize three term recurrence formula in linear differential equation. Obtain the exact solution of the three term recurrence for polynomials and infinite series.

3. “The analytic solution for the power series expansion of Heun function” \[9\] - Apply three term recurrence formula to the power series expansion in closed forms of Heun function (infinite series and polynomials) including all higher terms of \(A_n\)’s.

4. “Asymptotic behavior of Heun function and its integral formalism”, \[10\] - Apply three term recurrence formula, derive the integral formalism, and analyze the asymptotic behavior of Heun function (including all higher terms of \(A_n\)’s).

5. “The power series expansion of Mathieu function and its integral formalism”, \[11\] - Apply three term recurrence formula, analyze the power series expansion of Mathieu function and its integral forms.

6. “Lame equation in the algebraic form” \[12\] - Applying three term recurrence formula, analyze the power series expansion of Lame function in the algebraic form and its integral forms.

7. “Power series and integral forms of Lame equation in the Weierstrass’s form and its asymptotic behaviors” \[13\] - Applying three term recurrence formula, derive the power series expansion of Lame function in the Weierstrass’s form and its integral forms.

8. “The generating functions of Lame equation in the Weierstrass’s form” \[14\] - Derive the generating functions of Lame function in the Weierstrass’s form (including all higher terms of \(A_n\)’s). Apply integral forms of Lame functions in the Weierstrass’s form.
9. “Analytic solution for Grand Confluent Hypergeometric function” - Apply three term recurrence formula, and formulate the exact analytic solution of Grand Confluent Hypergeometric function (including all higher terms of $A_n$’s). Replacing $\mu$ and $\sigma\omega$ by 1 and $-q$, transforms the grand confluent hypergeometric function into Biconfluent Heun function.

10. “The integral formalism and the generating function of Grand Confluent Hypergeometric function” - Apply three term recurrence formula, and construct an integral formalism and a generating function of Grand Confluent Hypergeometric function (including all higher terms of $A_n$’s).

Acknowledgment

I thank Bogdan Nicolescu. The discussions I had with him on number theory was of great joy.

References

[1] Heun, K., Zur Theorie der Riemann’schen Functionen zweiter Ordnung mit vier Verzweigungspunkten, Mathematische Annalen 33, (1889)161.
[2] Ronveaux, A., Heuns Differential Equations, Oxford University Press, (1995).
[3] Catto, S. and Gürsey, F., Algebraic treatment of effective supersymmetry, Nuovo Cim. 86A, (1985)201.
[4] Catto, S. and Gürsey, F., New realizations of hadronic supersymmetry, Nuovo Cim. 99A, (1985)685.
[5] Gürsey, F., Comments on hardonic mass formulae, in A. Das., ed., From Symmetries to Strings: Forty Years of Rochester Conferences. World Scientific, Singapore, (1990).
[6] Catto, S., Cheung, H. Y., Gursey, F., Effective Hamiltonian of the relativistic Quark model Mod. Phys. Lett. A 38, (1991)3485.
[7] Link to arXiv:1302.7309
[8] Link to arXiv:1303.0806
[9] Link to arXiv:1303.0830
[10] Link to arXiv:1303.0876
[11] Link to arXiv:1303.0820
[12] Link to arXiv:1303.0873
[13] Link to arXiv:1303.0878
[14] Link to arXiv:1303.0879
[15] Link to arXiv:1303.0815
[16] Link to arXiv:1303.0819