HARMONIC CHEEGER-SIMONS CHARACTERS WITH APPLICATIONS

RICHARD GREEN AND VARGHESE MATHAI

Abstract. We initiate the study of harmonic Cheeger-Simons characters, with applications to smooth versions of the Geometric Langlands program in the abelian case.

1. Introduction

Cheeger-Simons characters, also known as differential characters, provide a useful refinement of integral cohomology incorporating differential forms. The original construction and basic properties are summarized in [2], which also details corresponding refined characteristic classes and associated applications. Such refinements of cohomology theories are now known as differential cohomology theories. A recent, elegant treatise on the subject is [10], which includes a construction of differential versions of arbitrary generalized cohomology theories, together with interesting applications, such as to quadratic refinements of the intersection pairing on a smooth manifold. An application to abelian gauge theories, the simplest example being electromagnetism, can be found in [3], where it is shown that in the quantum theory, fluxes may not be simultaneously measurable. There have also been numerous other applications, either of Cheeger-Simons characters or of its equivalent, Deligne cohomology (cf. [1]), both in mathematics and mathematical physics.

For a compact Riemannian manifold $X$, we introduce the notion of harmonic Cheeger-Simons characters in Definition 3.2 and establish the fundamental exact sequences that it satisfies in Lemma 3.3 as well as some of the basic properties such as Poincaré-Pontrjagin duality in subsection 3.1. The group of all harmonic Cheeger-Simons $j$-characters $\mathcal{H}^j(X)$ form a subgroup of the group of all Cheeger-Simons $j$-characters, and a crucial point established here is that $\mathcal{H}^\bullet(X)$ is finite dimensional, and can be viewed as a refinement of the space of harmonic forms on $X$, as it also encodes the torsion subgroup of $H^\bullet(X, \mathbb{Z})$. In fact, we establish an analogue of the Hodge theorem in this context in Proposition 3.4. We also compute these groups for several examples in this section, such as for compact connected Lie groups endowed with a bi-invariant Riemannian metric in Lemma 3.7.

In the last section, we introduce the harmonic Picard variety for compact Riemann surfaces, which is fundamental for several constructions related to the smooth analogue of the Geometric Langlands program in the abelian case, as established in Theorem 4.1. The Geometric Langlands program is currently of central importance in gauge theory and string theory due to the relationship with S-duality in supersymmetric gauge theory in four dimensions. Although still a conjecture in the general nonabelian case, recently several important insights have been given by E. Witten and his collaborators, in a series of long papers starting with [1].

Acknowledgements. R.G. acknowledges the receipt of an Australian Postgraduate Award. V.M. acknowledges support from the Australian Research Council.

2000 Mathematics Subject Classification. 58J28, 81T13, 81T30.
Key words and phrases. harmonic differential characters, harmonic Cheeger-Simons characters, smooth abelian Geometric Langlands Program.
2. Preliminaries

Here we recall for convenience some classical results on line bundles over surfaces and connections on these, whose proofs are included for the convenience of the reader. Also included are sections recalling results on symmetric products of surfaces and characters of the fundamental group, which is mainly used in [1].

2.1. Divisor line bundles on oriented surfaces. Let \( X \) be an oriented surface, and \( p \in X \). Then we can cover \( X \) with two open sets, \( U_1 \cong \mathbb{R}^2 \) a coordinate neighbourhood of \( p \), and \( U_0 = X \setminus \{p\} \). Then \( U_0 \cap U_1 \cong \mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R} \). We define a complex line bundle \( D_p \) over \( X \), called the divisor line bundle associated to \( p \in X \), by taking the transition function \( g_{01} \) on \( U_0 \cap U_1 \) to be the pull-back from \( S^1 \) of an \( S^1 \)-valued smooth map of degree one, whose homotopy class is the generator of \( H^1(S^1, \mathbb{Z}) \). The choice of orientation on \( X \) induces an orientation on \( S^1 \) and hence a choice of generator. Since there are only two open sets in this definition, nothing further needs to be checked. If \( X \) is compact, then the first Chern class, \( c_1(D_p) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z} \) is the generator, and a refinement of this statement will be discussed in the sequel. This construction is a smooth analogue of the well known construction of a holomorphic line bundle associated to a point in a Riemann surface.

2.2. Connections with harmonic curvature on divisor line bundles. Now suppose that \( X \) is a compact oriented surface. Choose a Riemannian metric \( g \) on \( X \) with volume form \( V \) and total volume \( 2\pi \). Then \( V \) is a harmonic 2-form, which represents the image of the generator of \( H^2(X, \mathbb{Z}) \) in \( H^2(X, \mathbb{R}) \). We will now construct a connection, \( \nabla_p \), with curvature \( V \), on the divisor line bundle \( D_p \) associated to a point \( p \in X \).

We shall use the language of currents, or distributional forms. On an \( n \)-manifold \( M \), a smooth \( k \)-form \( \alpha \) defines a linear form on \( \Omega^{n-k}(M) \) by

\[
\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta
\]

and then by Stokes’ theorem

\[
\langle d\alpha, \beta \rangle = (-1)^{k+1} \langle \alpha, d\beta \rangle.
\]

The theory of de Rham currents considers general continuous linear functionals on \( \Omega^{n-k}(M) \), rather than just those coming from \( k \)-forms as above. In particular, an example in the case \( k = n \) is the Dirac delta function \( \delta_p \) associated to a point \( p \in M \), defined as \( \delta_p(\phi) = \phi(p) \). There is also an analogous Hodge theory of currents (cf [2]).

The homology class of a point \( p \in X \) is dual to the de Rham cohomology class of \( V/2\pi \). In terms of currents, the degree 2 current \( V - 2\pi \delta_p \) is null cohomologous. It follows from the Hodge theory of currents that there is a degree 2 current \( H_0 \) such that

\[
\Delta H_0 = V - 2\pi \delta_p.
\]

The current \( H_0 \) is unique up to the addition of a constant multiple of \( V \), and by elliptic regularity \( H_0 \) is a 2-form which is smooth except at \( p \) where the function \( \phi = *H_0 \) has a singularity of the form \( \phi = \text{constant} \cdot \log(r) + \ldots \). Choose a local smooth 2-form \( H_1 \) on the coordinate neighborhood \( U_1 \) such that \( \Delta H_1 = V \) and define the 1-forms \( F_0 = d^* H_0 \) on \( U_0 = X \setminus \{p\} \) and \( F_1 = d^* H_1 \) on \( U_1 \). Note that \( F_0 \) is now independent of the choice of \( H_0 \). Then \( dF_0 = dd^* H_0 = \Delta H_0 = V \) on \( U_0 \) and \( dF_1 = dd^* H_1 = V \) on \( U_1 \), therefore \( d(F_1 - F_0) = 0 \) on \( U_0 \cap U_1 \). Hence, the locally defined 1-forms \( \{F_1, F_0\} \) will patch together to give a connection on the line bundle \( D_p \) with curvature \( V \), provided \( (F_1 - F_0)/2\pi = d \log(g_{01}) \) on the overlap \( U_0 \cap U_1 \cong \mathbb{R}^2 \setminus \{0\} \) determines a cohomology class which is the generator of \( H^1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z} \). Consider a closed ball \( B \) centred at \( p \) in the coordinate
neighborhood $U_1$ and $\varphi$ a smooth function of compact support in $U_1$ which is identically 1 in a neighborhood of $B$. By the definition of $H_0$ and $H_1$,

$$\langle dd^* H_0, \varphi \rangle = \int_{U_1} \varphi V - 2\pi \varphi(p) \quad \text{and} \quad \int_{U_1} \varphi dd^* H_1 = \int_{U_1} \varphi V,$$

so subtracting,

$$\langle d(d^* H_0 - d^* H_1), \varphi \rangle = -2\pi$$

since $\varphi \equiv 1$ near $p$. Since $d^* H_0 - d^* H_1$ is a smooth closed 1-form outside $p$, so we can equivalently consider its restriction to $B$, and since $\varphi = 1$ on $B$, we see by Stokes theorem that,

$$-2\pi = \langle d(d^* H_0 - d^* H_1), \varphi \rangle = \int_B d(d^* H_0 - d^* H_1) = \int_{\partial B} (d^* H_0 - d^* H_1).$$

Thus $(F_1 - F_0)/2\pi = d^*(H_1 - H)/2\pi$ determines an integral cohomology class, which is a generator of the cohomology group as asserted. Thus we have produced a connection on the divisor line bundle $D_p$, represented by the pair of local 1-forms $\{F_1, F_0\} = \nabla_p$, whose curvature is equal to the harmonic 2-form $V$.

2.3. Symmetric products of spaces. Let $M$ be smooth manifold. On the $n$-fold Cartesian product $M^n$, one can define an action of the symmetric group $S_n$ as follows. Recall that $S_n$ consists of bijections $\theta : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ (also called permutations of $n$ letters) and has order equal to $n!$, where $S_2 = \mathbb{Z}_2$ and $S_n$ is nonabelian when $n > 2$. The action of $S_n$ on $M^n$ is given by

$$M^n \times S_n \rightarrow M^n$$

$$(x_1, \ldots, x_n, \theta) \rightarrow (x_{\theta(1)}, \ldots, x_{\theta(n)})$$

The quotient $M^n/S_n$ is an orbifold denoted by $\text{Sym}^{(n)}(M)$ and is called the $n$-th symmetric product of $M$. It consists of all the unordered $n$-tuples of points in $M$.

In the special case when $\dim(M) = 2$, the symmetric product $\text{Sym}^{(n)}(M)$ is not just an orbifold, but is a manifold; details can be found in [6], page 236. The key point is the observation that $\text{Sym}^{(n)}(\mathbb{C})$ is diffeomorphic to $\mathbb{C}^n$ via the diffeomorphism $\phi$ given by $\phi(z_1, \ldots, z_n) = (\sigma_1, \ldots, \sigma_n)$, where $\sigma_j, j = 1, \ldots, n$ are the elementary symmetric functions in $z_1, \ldots, z_n$, that is, the coefficients of the polynomial $p(z) = \prod_{j=1}^n (z - z_j)$.

For example, $\text{Sym}^{(n)}(\mathbb{C}P^1) \cong \mathbb{C}P^n$, which can be deduced using the Schubert cell decomposition of $\mathbb{C}P^1$ and $\mathbb{C}P^n$. Also, $\text{Sym}^{(n)}(\mathbb{T}^2)$ is a fiber bundle over $\mathbb{T}^2$ with fiber diffeomorphic to $\mathbb{C}P^{n-1}$, cf. [12]. This description enables one to determine many useful facts about the topology of $\text{Sym}^{(n)}(\mathbb{T}^2)$.

The product map $M^n \times M^m \rightarrow M^{n+m}$ descends to a continuous map of the quotient spaces

$$h_{m,n} : \text{Sym}^{(m)}(M) \times \text{Sym}^{(n)}(M) \rightarrow \text{Sym}^{(m+n)}(M), \quad (2.1)$$

and is given by $([x_1, \ldots, x_m], [y_1, \ldots, y_n]) \rightarrow [x_1, \ldots, x_m, y_1, \ldots, y_n]$.

Choosing a point $p \in M$ determines an embedding $M^n \hookrightarrow M^{n+1}$ given by $(x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_n, p)$, and the embedding $\text{Sym}^{(n)}(M) \hookrightarrow \text{Sym}^{(n+1)}(M)$ given by $[x_1, \ldots, x_n] \rightarrow [x_1, \ldots, x_n, p]$. Define

$$\text{Sym}^{(\infty)}(M) = \bigcup_{n \geq 1} \text{Sym}^{(n)}(M)$$
in the weak topology. It is a commutative, associative H-space with a strict identity element. The Dold-Thom theorem (cf [3]) asserts that there is a homotopy equivalence

$$\text{Sym}^{(\infty)}(M) \sim \prod_{n \geq 1} \text{K}(H_n(M, \mathbb{Z}), n)$$

where the right hand side denotes the cartesian product of Eilenberg-Maclance spaces. In particular, when \( M \) is a compact Riemann surface of genus equal to \( g \), one has the homotopy equivalence

$$\text{Sym}^{(\infty)}(M) \sim \mathbb{T}^{2g} \times \mathbb{C}^{\infty}$$

If \( f : M \to N \) is a continuous map, then it induces in a natural way a map on the cartesian products \( f^n : M^n \to N^n \), which descends to a continuous map \( \text{Sym}^{(n)}(f) : \text{Sym}^{(n)}(M) \to \text{Sym}^{(n)}(N) \). If \( f, g : M \to N \) are continuous maps that are homotopic, then so are the continuous maps \( \text{Sym}^{(n)}(f) \) and \( \text{Sym}^{(n)}(g) \). In particular, if there is a homotopy equivalence \( M \sim N \), then there is a homotopy equivalence \( \text{Sym}^{(n)}(M) \sim \text{Sym}^{(n)}(N) \).

2.4. Characters of the fundamental group and of the first homology group. Suppose \( M \) is path connected. Then the Hurewicz theorem gives a canonical homomorphism \( \phi : \pi_1(M) \to H_1(M, \mathbb{Z}) \), which is surjective, and has kernel the commutator subgroup \( [\pi_1(M), \pi_1(M)] \) of \( \pi_1(M) \). As such, this homomorphism descends to an isomorphism between the abelianization of the fundamental group, \( \pi_1(M)_{ab} \cong H_1(M, \mathbb{Z}) \). Since every character of \( \pi_1(M) \) factors through its abelianization, it induces a character of \( H_1(M, \mathbb{Z}) \). Conversely, every character of \( H_1(M, \mathbb{Z}) \) pulls back via \( \phi \) to a character of \( \pi_1(M) \). This sets up a bijective correspondence between characters of \( \pi_1(M) \) and characters of \( H_1(M, \mathbb{Z}) \). The following proposition, which will be used in §4, is due to [7] and asserts in particular the surprising fact that the fundamental group of symmetric products of a Riemann surface, is abelian.

**Proposition 2.1.** If \( M \) is a Riemann surface and \( n > 1 \), then

$$\pi_1(\text{Sym}^{(n)}(M)) \cong H_1(M, \mathbb{Z}),$$

**Corollary 2.2.** The characters of \( \pi_1(M) \) are in bijective correspondence with the characters of \( \pi_1(\text{Sym}^{(n)}(M)) \).

3. Harmonic Cheeger-Simons characters

We begin by reviewing the definition and some basic properties of Cheeger-Simons characters on a smooth manifold \( X \), cf. [2, 4, 10]. If \( X \) is a compact Riemannian manifold, then we introduce the notion of harmonic Cheeger-Simons characters in Definition 3.2 and establish the basic properties in Lemma 3.3 and Proposition 3.4.

**Definition 3.1.** A *Cheeger-Simons j-character* is an element \( \chi \in \text{Hom}(Z_{j-1}(X), \mathbb{R}/\mathbb{Z}) \) such that there is a \( j \)-form \( F_\chi \) on \( X \) with the property that if \( \Sigma \in B_{j-1}(X) \), i.e \( \Sigma = \partial Q \), then

$$\chi(\Sigma) = \int_Q F_\chi \mod \mathbb{Z}.$$

This automatically implies that \( F_\chi \) is a closed \( j \)-form, with integral periods, which we shall write as \( F_\chi \in \Omega^j_0(X) \). The form \( F_\chi \) is uniquely determined by the character \( \chi \), and referred to as its fieldstrength or curvature. The group of Cheeger-Simons \( j \)-characters is denoted by \( \tilde{H}^j(X) \), where the group law is given by pointwise sum of Cheeger-Simons characters. We often also denote
elements in $\tilde{H}^j(X)$ by $[A]$, where $A$ is a locally defined $(j-1)$-form on $X$ sometimes called a vector potential, such that $dA$ is the globally defined form given by the fieldstrength.

In addition to the mapping $F : H^j(X) \rightarrow \Omega^j_Z$ given by the fieldstrength, there is a mapping $c : \tilde{H}^j(X) \rightarrow H^j(X, \mathbb{Z})$, called the characteristic class map, for which the following diagram commutes

$$
\begin{array}{ccc}
\tilde{H}^j(X) & \xrightarrow{F} & \Omega^j_Z(X) \\
\downarrow c & & \downarrow c \\
H^j(X, \mathbb{Z}) & \xrightarrow{c} & H^j(X, \mathbb{R}).
\end{array}
$$

Here the rightmost arrow is given by taking de Rham cohomology classes, and the bottom arrow is induced by the inclusion $\mathbb{Z} \rightarrow \mathbb{R}$.

There are two fundamental exact sequences associated to $\tilde{H}^j(X)$. The first is related to the fieldstrength and given by

$$
0 \rightarrow H^{j-1}(X, \mathbb{R}/\mathbb{Z}) \rightarrow \tilde{H}^j(X) \xrightarrow{F} \Omega^j_Z(X) \rightarrow 0. 
$$

(3.1)

The other is related to the characteristic class and given by

$$
0 \rightarrow \Omega^{j-1}(X)/\Omega^{j-1}_Z(X) \rightarrow \tilde{H}^j(X) \xrightarrow{c} H^j(X, \mathbb{Z}) \rightarrow 0.
$$

(3.2)

These exact sequences give a picture of the structure of the Cheeger-Simons cohomology groups, viz that they are infinite dimensional, with components labelled by $H^j(X, \mathbb{Z})$ and each component is a torus bundle with typical fiber $H^{j-1}(X, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$, over the infinite dimensional vector space $\Omega^{j-1}(X)/\Omega^{j-1}_Z(X)$, where $\Omega^{j-1}_Z(X)$ denotes closed forms.

In particular, the exact sequences (3.1) and (3.2) show that the Cheeger-Simons groups $\tilde{H}^\bullet(X)$ can be viewed as a refinement of the de Rham cohomology of $X$, as it also encodes the torsion subgroup of $H^\bullet(X, \mathbb{Z})$.

If $X = (X, g)$ is a compact Riemannian manifold, then the space of harmonic $j$-forms $\mathcal{H}^j(X)$ can be defined and is, by the Hodge theorem, isomorphic to $H^j(M, \mathbb{R})$, while the harmonic $j$-forms with integral periods $\mathcal{H}^j_Z(X)$ form a lattice isomorphic to the image of $H^j(M, \mathbb{Z})$ in $H^j(M, \mathbb{R})$. In this situation we will define as follows, a finite dimensional subgroup of the full Cheeger-Simons character group which encodes the same topological information.

For the remainder of this section, let $X$ be a compact Riemannian manifold. The following is the main new definition that we introduce in the paper.

**Definition 3.2.** A harmonic Cheeger-Simons $j$-character is a character $\chi \in \tilde{H}^j(X)$ whose fieldstrength $F_\chi$ is a harmonic differential form.

**Remark.** The harmonic Cheeger-Simons $j$-characters form a subgroup of $\tilde{H}^j(X)$ which we denote by $\mathcal{H}^j(X)$. The crucial point established in the following Lemma is that $\mathcal{H}^\bullet(X)$ is finite dimensional, and can be viewed as a refinement of the space of harmonic forms on $X$, as it also encodes the torsion subgroup of $H^\bullet(X, \mathbb{Z})$. Moreover, it has many properties analogous to the full Cheeger-Simons character group.

**Lemma 3.3.** There are two fundamental short exact sequences associated to $\mathcal{H}^j(X)$, namely a fieldstrength sequence,

$$
0 \rightarrow H^{j-1}(X, \mathbb{R}/\mathbb{Z}) \rightarrow \mathcal{H}^j(X) \xrightarrow{F} \mathcal{H}^j_Z(X) \rightarrow 0.
$$

(3.3)
and a characteristic class sequence,

\[ 0 \to \mathcal{H}^{j-1}(X)/\mathcal{H}^{j-1}_\mathbb{Z}(X) \to \bar{\mathcal{H}}^j(X) \xrightarrow{\varphi} H^j(X, \mathbb{Z}) \to 0. \] (3.4)

**Proof.** The restriction of \( F : \bar{\mathcal{H}}^j(X) \to \Omega^j_\mathbb{Z}(X) \) to \( \bar{\mathcal{H}}^j(X) \), \( F_{\text{res}} : \bar{\mathcal{H}}^j(X) \to \mathcal{H}^j_\mathbb{Z}(X) \), is by definition surjective. Since \( \ker(F) \subseteq \bar{\mathcal{H}}^j(X) \) it follows that \( \ker(F_{\text{res}}) = \ker(F) \simeq H^{j-1}(X, \mathbb{R}/\mathbb{Z}) \). From which we deduce the exact sequence \( (3.3) \) immediately.

The restricted map \( c_{\text{res}} : \bar{\mathcal{H}}^j(X) \to H^j(X, \mathbb{Z}) \) can be seen to be onto as follows. Consider a class \([u] \in H^j(X, \mathbb{Z})\). Then by the Hodge theorem, there is a harmonic form \( h \in \mathcal{H}^j(X) \) such that \([u] = [h]\) as real cohomology classes, and hence \( h - u = \delta T \) for some cochain \( T \in C^{k-1}(M, \mathbb{R}) \). Restricting \( T \) to cycles \( Z_{k-1}(X) \) and reducing modulo \( \mathbb{Z} \) gives the required harmonic character \( \chi = T|_{Z_{k-1}} \) with characteristic class \( c(\chi) = [u] \).

The kernel of \( c_{\text{res}} \) can be computed by considering the map

\[ \varphi : \Omega^{j-1}(X)/\Omega^{j-1}_\mathbb{Z}(X) \to \bar{\mathcal{H}}^j(X) \]

appearing in \( (3.2) \), which maps onto \( \ker(c) \). This map \( \varphi \) takes \( \alpha \in \Omega^{j-1}(X) \) (viewed as a cochain via integration over chains) to the differential character \( f_\alpha = \delta|_{Z_{j-1}} \), which has curvature \( d\alpha \). The Hodge decomposition implies that closed forms split as \( \Omega^j_{\text{cl}}(X) = B^j(X) \oplus \mathcal{H}^j(X) \), where \( B^j(X) \) denotes exact forms. It then follows that \( \varphi^{-1}(\ker(c) \cap \bar{\mathcal{H}}^j(X)) = \Omega^j_{\text{cl}}(X)/\Omega^j_{\mathbb{Z}}(X) \). Finally, the inclusion \( \mathcal{H}^j(X) \to \Omega^j_{\text{cl}}(X) \) induces, via the Hodge theorem, an isomorphism \( \mathcal{H}^j(X)/\mathcal{H}^j_{\mathbb{Z}}(X) \to \Omega^j_{\text{cl}}(X)/\Omega^j_{\mathbb{Z}}(X) \), so that the exactness of \( (3.4) \) follows.

The analogue of the Hodge theorem in this context is the following.

**Proposition 3.4.** The inclusion \( i : \bar{\mathcal{H}}^j(X) \hookrightarrow \bar{\mathcal{H}}^j(X) \) is a homotopy equivalence (and is in fact a deformation retraction).

**Proof.** From the short exact sequences \( (3.1) \) and \( (3.3) \), we see that the quotient group \( \bar{\mathcal{H}}^j(X)/\bar{\mathcal{H}}^j(X) \) is isomorphic to the quotient \( \Omega^j_{\mathbb{Z}}(X)/\mathcal{H}^j_{\mathbb{Z}}(X) \). Since \( B^j(X) \) is a subgroup of \( \Omega^j_{\mathbb{Z}}(X) \), the Hodge decomposition implies \( \Omega^j_{\mathbb{Z}}(X) = B^j(X) \oplus \mathcal{H}^j_{\mathbb{Z}}(X) \). Hence the quotient \( \Omega^j_{\mathbb{Z}}(X)/\mathcal{H}^j_{\mathbb{Z}}(X) \) is isomorphic to \( B^j(X) \), which is contractible, and the assertion follows.

We now give some examples of the group of harmonic Cheeger-Simons characters.

1. When \( X = pt \), the only nontrivial groups of harmonic Cheeger-Simons characters are \( \bar{\mathcal{H}}^0(pt) = \mathbb{Z} \) and \( \bar{\mathcal{H}}^1(pt) = \mathbb{R}/\mathbb{Z} \).
2. If \( X \) is a compact, connected oriented Riemannian manifold of dimension equal to \( N \), then \( \bar{\mathcal{H}}^0(X) = \mathbb{Z} \) and \( \bar{\mathcal{H}}^{N+1}(X) \cong \mathbb{R}/\mathbb{Z} \). More precisely, the isomorphism is defined as follows. Any class in \( \mathcal{H}^{N+1}(X) \) is topologically trivial, hence any vector potential is a globally defined form \( A \in \Omega^N(X) \), and the integration homomorphism \( \int_X^H : \mathcal{H}^{N+1}(X) \to \mathbb{R}/\mathbb{Z} \) is given by

\[ \int_X^H [A] = \int_X A \mod \mathbb{Z}. \]

It is just the restriction to \( \mathcal{H}^{N+1}(X) \) of the integration map on the group of Cheeger-Simons characters \( \bar{\mathcal{H}}^{N+1}(X) \).

3. Using the fundamental exact sequence of Lemma \( (3.3) \) we see that \( \mathcal{H}^1(X) \cong \mathbb{R}/\mathbb{Z} \times \mathcal{H}^0_\mathbb{Z}(X) \).
It is well known (and easy to show) that $H^1(X, \mathbb{Z}) \cong [X, \mathbb{R}/\mathbb{Z}]$ and that $\tilde{H}^1(X) \cong C^\infty(X, \mathbb{R}/\mathbb{Z})$. There is an analogous interpretation of the group of degree 1 harmonic Cheeger-Simons characters, given in terms of harmonic maps, defined as follows.

**Definition 3.5.** A smooth map $f : X \rightarrow \mathbb{R}/\mathbb{Z}$ is said to be harmonic if $\log(f) : \tilde{X} \rightarrow \mathbb{R}$ is a harmonic function, where $\tilde{X}$ denotes the universal cover of $X$, and where we observe that $\gamma^*\log(f) = \log(f) + c_\gamma$, $\forall \gamma \in \pi_1(X)$, where $c_\gamma \in 2\pi\mathbb{Z}$. Let $\text{Harm}(X, \mathbb{R}/\mathbb{Z})$ denote the space of all harmonic functions from $X$ to $\mathbb{R}/\mathbb{Z}$.

The interpretation is then given as follows.

**Lemma 3.6.** Let $X$ be a compact, connected oriented Riemannian manifold. Then

$$\tilde{H}^1(X) \cong \text{Harm}(X, \mathbb{R}/\mathbb{Z}).$$

**Proof.** We show that via the isomorphism $C^\infty(X, \mathbb{R}/\mathbb{Z}) \cong \tilde{H}^1(X)$, the subgroup $\text{Harm}(X, \mathbb{R}/\mathbb{Z})$ of $C^\infty(X, \mathbb{R}/\mathbb{Z})$ corresponds to the subgroup $\tilde{\mathcal{H}}^1(X)$ of $H^1(X)$. If $f \in C^\infty(X, \mathbb{R}/\mathbb{Z})$ then consider the closed 1-form on $\tilde{X}$, $\theta = d(\log(f))$. Now $\gamma^*\theta = d\gamma^*(\log(f)) = d(\log(f)) = \theta$, therefore $\theta$ descends to a 1-form $\tilde{\theta}$ on $X$, which coincides with the fieldstrength of $f$, viewed as a differential character.

If $f \in \text{Harm}(X, \mathbb{R}/\mathbb{Z})$, $\log(f)$ is a harmonic function on $\tilde{X}$, and $d^*\theta = d^*d(\log(f)) = 0$. Therefore $\theta$ is a harmonic 1-form on $\tilde{X}$ and hence $\tilde{\theta}$, the fieldstrength of $f$, is a harmonic 1-form on $X$.

Conversely if $f$ has fieldstrength $\tilde{\theta} \in \tilde{\mathcal{H}}^1(X)$, then $d(\log(f)) = \pi^*\tilde{\theta}$, where $\pi : \tilde{X} \rightarrow X$ is the projection map. Since $\tilde{\theta}$ is harmonic, we have $d^*d(\log(f)) = d^*\pi^*\tilde{\theta} = 0$. So $f \in \text{Harm}(X, \mathbb{R}/\mathbb{Z})$. □

### 3.1. Poincaré-Pontrjagin duality

For a compact, connected, oriented manifold $X$, Poincaré-Pontrjagin duality asserts that there is a perfect pairing defined by,

$$\langle \cdot, \cdot \rangle : \tilde{\mathcal{H}}^j(X) \times \tilde{\mathcal{H}}^{N+1-j}(X) \rightarrow \mathbb{R}/\mathbb{Z},$$

$$\langle \chi_1, \chi_2 \rangle = \int_X \chi_1 \ast \chi_2 \quad (3.5)$$

where $\ast$ is the star product on Cheeger-Simons characters. We caution the reader that the product of harmonic Cheeger-Simons characters is a Cheeger-Simons character, but which may not be harmonic. This is because the field strength $F_{\chi_1 \ast \chi_2} = F_{\chi_1} \wedge F_{\chi_2}$ is not necessarily a harmonic form even though $F_{\chi_1}$ and $F_{\chi_2}$ are harmonic forms. What is meant by a perfect pairing is that every homomorphism $\tilde{\mathcal{H}}^j(X) \rightarrow \mathbb{R}/\mathbb{Z}$ is given by pairing with an element of $\tilde{\mathcal{H}}^{N+1-j}(X)$.

In order to justify Poincaré-Pontrjagin duality, we consider two exact sequences. The first is the fieldstrength sequence in degree $j$

$$0 \rightarrow H^{j-1}(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\cdot} \tilde{\mathcal{H}}^j(X) \xrightarrow{F} \tilde{\mathcal{H}}^j_0(X) \rightarrow 0 \quad (3.6)$$

together with the characteristic class sequence in degree $N + 1 - j$,

$$0 \rightarrow \mathcal{H}^{N-j}(X)/\mathcal{H}^N_0(X) \xrightarrow{\cdot} \mathcal{H}^{N+1-j}(X) \xrightarrow{c} H^{N+1-j}(X, \mathbb{Z}) \rightarrow 0 \quad (3.7)$$

Recall that there is a standard perfect pairing

$$H^{j-1}(X; \mathbb{R}/\mathbb{Z}) \times H^{N+1-j}(X; \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$$
given by multiplication and integration, and also the perfect pairing,
\[ \mathcal{H}_Z^j(X) \times \left( \mathcal{H}^{N-j}(X)/\mathcal{H}_Z^{N-j}(X) \right) \longrightarrow \mathbb{R}/\mathbb{Z} \]
\[ (F, A_D) \longrightarrow \int_X F \wedge A_D \mod \mathbb{Z}. \]

These pairings induce isomorphisms which we denote by
\[ \alpha : H^{j-1}(X, \mathbb{R}/\mathbb{Z}) \rightarrow \text{Hom}(H^{N+1-j}(X, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \]
and
\[ \beta : \mathcal{H}_Z^j(X) \rightarrow \text{Hom}(\mathcal{H}^{N-j}(X)/\mathcal{H}_Z^{N-j}(X), \mathbb{R}/\mathbb{Z}) \]
respectively. The pairing given by (3.5) induces a mapping \( \varphi \) respectively. The pairing given by (3.5) induces a mapping \( \varphi \). We deduce that \( \varphi \) is an isomorphism as follows. The functor \( \text{Hom}(\cdot, \mathbb{R}/\mathbb{Z}) \) is exact, so when applied to (3.7) we obtain an exact sequence which can be combined with the exact sequence (3.6) into the following diagram,
\[ \begin{array}{cccccc}
0 & \longrightarrow & H^{j-1}(X, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \mathcal{H}_Z^j(X) & \longrightarrow & 0 \\
& & \downarrow{\alpha} & & \downarrow{\varphi} & & \\
0 & \longrightarrow & D(H^{N+1-j}(X, \mathbb{Z})) & \longrightarrow & D(\mathcal{H}^{N+1-j}(X)) & \longrightarrow & 0,
\end{array} \]
where we have abbreviated \( \text{Hom}(\cdot, \mathbb{R}/\mathbb{Z}) \) to \( D(\cdot) \). An elementary fact from homological algebra, known as the ‘short five lemma’, states that for any such diagram with exact rows which is commuting and where \( \alpha \) and \( \beta \) are isomorphisms, \( \varphi \) must also be an isomorphism. Since here \( \alpha \) and \( \beta \) are known to be isomorphisms, it only remains to show that the diagram commutes to apply this lemma.

To see that the right square commutes, consider \( \chi \in \mathcal{H}_Z^j(X) \) and \( h \in \mathcal{H}^{N-j}(X) \). Then it is sufficient to show that \( \int_X \chi \ast I(h) = \int_X F \chi \wedge h \mod \mathbb{Z} \). However the product on \( \mathcal{H}^\bullet(X) \) satisfies \( \chi \ast I(h) = I(F \chi \wedge h) \) so
\[ \int_X \chi \ast I(h) = \int_X F \chi \wedge h = \int_X F \chi \wedge h \mod \mathbb{Z}. \]

Now consider \( a \in H^{j-1}(X, \mathbb{R}/\mathbb{Z}) \) and \( \chi \in \mathcal{H}^{N+1-j}(X) \). Then since \( \iota(a) \ast \chi \in \mathcal{H}^{N+1}(X) \) there is a form \( \tau \in \Omega^N(X) \) such that \( \iota(a) \ast \chi = I(\tau) \). However \( \iota(a) \ast \chi = \iota(a \cup c(\chi)) \), so it must have vanishing fieldstrength, \( d\tau = 0 \), and \( \tau \mod \mathbb{Z} \) is a representative cocycle for \( a \cup c(\chi) \). Hence
\[ \int_X \iota(a) \ast \chi = \int_X \tau \mod \mathbb{Z} = \tau(X) \mod \mathbb{Z} = (a \cup c(\chi))[X], \]
and it follows that the left square in the diagram commutes.

Since the hypotheses of the ‘short five lemma’ hold, we deduce that \( \varphi \) is an isomorphism, giving the Poincaré-Pontryagin duality (3.5).

3.2. Harmonic Cheeger-Simons characters on compact Lie groups. Here we will identify harmonic Cheeger-Simons characters on a compact connected Lie group \( G \). A Cheeger-Simons character \( \chi \) on \( G \) is said to be left invariant if \( L_\gamma^* (\chi) = \chi \) for all \( \gamma \in G \), where \( L_\gamma \) denotes left translation by \( \gamma \in G \). \( \chi \) is said to be right invariant if \( R_\gamma^* (\chi) = \chi \) for all \( \gamma \in G \), where \( R_\gamma \) denotes right translation by \( \gamma \in G \). \( \chi \) is said to be bi-invariant if it is both left invariant and right invariant. Clearly the fieldstrength of a bi-invariant Cheeger-Simons character is a bi-invariant
Lemma 3.7. Let $G$ be a compact connected Lie group endowed with a bi-invariant Riemannian metric. Then the harmonic Cheeger-Simons characters on $G$ are precisely the bi-invariant Cheeger-Simons characters on it.

Proof. Since $G$ is connected, it acts trivially on $\tilde{H}^j(G, \mathbb{R}/\mathbb{Z})$, and we can choose an equivariant set theoretic splitting of the fieldstrength exact sequence. This splitting then induces an isomorphism of $G$-sets $\phi : \tilde{H}^j(G) \simeq H^{j-1}(G, \mathbb{R}/\mathbb{Z}) \times \Omega^j_Z(G)$, which is compatible with the fieldstrength map, $F \circ \phi = p_2$, where $p_2$ is projection onto the second factor. From this we deduce that a character is bi-invariant if and only if it has bi-invariant curvature. By a theorem of Hodge, cf. Theorem 7.8 in [9], the harmonic forms on $G$ coincide with the bi-invariant differential forms on it, and so the bi-invariant characters are precisely the harmonic characters.

The star product $\chi_1 \star \chi_2$ of harmonic Cheeger-Simons characters $\chi_j$, $j = 1, 2$, is not in general a harmonic Cheeger-Simons character, since the fieldstrength satisfies $F_{\chi_1 \star \chi_2} = F_{\chi_1} \wedge F_{\chi_2}$, and because it is well known that the wedge product of harmonic forms is not in general a harmonic form. However, in the case of compact connected Lie groups, we can use the Lemma 3.7 above and the fact that the wedge product of bi-invariant forms is bi-invariant, to deduce the following.

Corollary 3.8. Let $G$ be a compact connected Lie group endowed with a bi-invariant Riemannian metric. Then the star product $\chi_1 \star \chi_2$ of harmonic Cheeger-Simons characters $\chi_j$, $j = 1, 2$, is a harmonic Cheeger-Simons character.

4. The harmonic Picard variety and the smooth geometric Hecke correspondence

In this section, we introduce the harmonic Picard variety, which is fundamental for several constructions. We also establish the geometric Hecke correspondence for the abelian group $U(1)$ in the smooth context. This correspondence in the holomorphic context was established by Deligne, cf. the nice lecture notes [5].

4.1. Geometric Hecke operators. Suppose that $X$ is a compact oriented surface with a Riemannian metric $g$ on $X$ whose volume form is $V$ and whose total volume is equal to $2\pi$. Then define the harmonic Picard variety,

$$\tilde{\text{Pic}}(X, g) = \{(\mathcal{L}, \nabla)|F_{\mathcal{L}} = kV \text{ for some } k \in \mathbb{Z}\} / \sim,$$

where $\mathcal{L}$ is a complex line bundle over $X$, $\nabla$ a connection on $\mathcal{L}$ whose curvature 2-form is equal to $F_{\mathcal{L}}$, and $\sim$ denotes isomorphism of line bundles with connection. It is well known, cf [1] [10] [4] and references therein, that the 2nd Cheeger-Simons cohomology group $\tilde{H}^2(X)$ is isomorphic to,

$$\{(\mathcal{L}, \nabla)\} / \sim,$$

where $\mathcal{L}$ is a complex line bundle over $X$, $\nabla$ a connection on $\mathcal{L}$, and $\sim$ denotes isomorphism of line bundles with connection. Under this isomorphism $\tilde{H}^2(X) \subseteq \tilde{H}^2(X)$ corresponds to (isomorphism classes of) line bundles with connection with harmonic curvature. Since $X$ is compact and connected, $H^2(X, \mathbb{R}) \simeq \mathbb{R}$, which with the Hodge decomposition implies,

$$\tilde{\text{Pic}}(X, g) \simeq \tilde{H}^2(X).$$

Therefore $\tilde{\text{Pic}}(X, g)$ is a finite dimensional Lie group and has components labelled by the integers, with $\tilde{\text{Pic}}(X, g)^{(0)}$ being canonically isomorphic to the Jacobian variety $\tilde{\text{Jac}}(X)$ of flat line bundles
modulus gauge equivalence. \( \text{Jac}(X) \) is equivalently described as the space of all flat connections \( \text{Jac}(X) = \{(L, \nabla) | F_{\nabla} = 0 \} / \sim \), since the holonomy of a flat connection on a line bundle determines a character of the fundamental group, and vice-versa. Since the de Rham operator \( d \) is a flat connection, and since any two connections differ by a closed 1-form, and gauge transformations can be identified with closed 1-forms with integral periods, then we see that \( \text{Jac}(X) = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) \), so in particular \( \pi^*\text{Jac}(X) = H^1(X, \mathbb{Z}) \). Moreover, by Abelian Yang-Mills theory, one also sees that all of the other components \( \text{Pic}(X, g)^{(n)} \), \( n \in \mathbb{Z} \setminus \{0\} \) of \( \text{Pic}(X, g) \) are (non-canonically) isomorphic to \( \text{Jac}(X) \).

Consider the geometric Hecke operators,

\[
\mathbb{H}_p : \text{Pic}(X, g)^{(n)} \rightarrow \text{Pic}(X, g)^{(n+1)}
\]

as follows. A flat line bundle \( L \) over \( X \) is equivalent to a character \( \rho_L \) of the fundamental group \( \pi_1(X) \). Also, a character \( \rho_L \) of \( \pi_1(X) \) factors through a unique character \( \rho'_L \) of the Abelianization \( \pi_1(X)^{ab} = H_1(X, \mathbb{Z}) \) of the fundamental group \( \pi_1(X) \). The Riemann surface is oriented, Poincaré duality gives a natural isomorphism \( H_1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) = \pi_1(\text{Jac}(X)) \). Then the definition of \( \mathbb{K}_L \) is the flat line bundle over \( \text{Jac}(X) = \text{Pic}(X, g)^{(0)} \) determined by the character \( \rho'_L \) of the fundamental group \( \pi_1(\text{Jac}(X)) \).

We begin by defining \( \mathcal{K}_L \) over \( \text{Pic}(X, g)^{(n)} \) for all \( n > 0 \). Let \( p_j : X^n \rightarrow X \) denote the projection to the \( j \)-th factor for \( j = 1, \ldots , n \). Recall that the external tensor product \( L_{\otimes n} \) is defined as the tensor product \( \bigotimes_{j=1}^n p_j^* L \) of flat line bundles \( p_j^* L \), \( j = 1, \ldots , n \) over \( X^n \), and so is also a flat line bundle over \( X^n \). Also, there is an action of the symmetric group \( S_n \) on \( L_{\otimes n} \) given as follows.

\[
L_{\otimes n} \times S_n \rightarrow L_{\otimes n}
\]

\[
((v_1 \otimes \ldots \otimes v_n), \theta) \rightarrow (v_{\theta(1)} \otimes \ldots \otimes v_{\theta(n)})
\]

**Theorem 4.1.** Given a flat line bundle \( L \) over \( X \), there exists a unique flat, line bundle \( \mathcal{K}_L \) over \( \text{Pic}(X, g) \), satisfying the Hecke eigenvalue property: for \( (L, \nabla) \in \text{Pic}(X, g)^{(n)} \) and \( p \in X \),

\[
\mathcal{K}_L \big|_{\mathbb{H}_p(L, \nabla)} = L_p \otimes \mathcal{K}_L \big|_{(L, \nabla)}.
\]

Globally, the Hecke eigenvalue property can be rephrased as,

\[
\mathbb{H}^* \mathcal{K}_L = L \otimes \mathcal{K}_L.
\]

The Hecke correspondence is the bijection,

\[
\text{Jac}(X) \leftrightarrow \text{Jac}_{\text{Hecke}}(\text{Pic}(X, g))
\]

\[
L \leftrightarrow \mathcal{K}_L.
\]

Here \( \text{Jac}_{\text{Hecke}}(\text{Pic}(X, g)) \) is the subspace of \( \text{Jac}(\text{Pic}(X, g)) \) consisting of all those flat line bundles on \( \text{Pic}(X, g) \) that satisfy the Hecke eigenvalue property (4.1).

**Proof.** It is easy to construct \( \mathcal{K}_L^{(0)} \in \text{Jac}(\text{Jac}(X)) \), as follows. A flat line bundle \( L \) over \( X \) is equivalent to a character \( \rho_L \) of the fundamental group \( \pi_1(X) \). The Riemann surface is oriented, Poincaré duality gives a natural isomorphism \( H_1(X, \mathbb{Z}) \cong H^1(X, \mathbb{Z}) = \pi_1(\text{Jac}(X)) \). Then the definition of \( \mathcal{K}_L^{(0)} \) is the flat line bundle over \( \text{Jac}(X) = \text{Pic}(X, g)^{(0)} \) determined by the character \( \rho'_L \) of the fundamental group \( \pi_1(\text{Jac}(X)) \).
The invariant vectors $\text{Sym}^{(n)}(L) := (L^\otimes n)^{S_n}$ is a flat orbifold line bundle over the orbifold $\text{Sym}^{(n)}(M)$, defined by the character $\text{Sym}^{(n)}(\rho^L)$ of the fundamental group of $\text{Sym}^{(n)}(X)$, or equivalently by a character $\text{Sym}^{(n)}(\rho^L)$ of its first homology group. Consider the canonical map $h : \text{Sym}^{(n)}(X) \times X \to \text{Sym}^{(n+1)}(X)$ given by $([x_1, \ldots, x_n], x_{n+1}) \mapsto [x_1, \ldots, x_{n+1}]$. Then clearly

$$h^*(\text{Sym}^{(n+1)}(L)) = L \boxtimes \text{Sym}^{(n)}(L).$$

Therefore $\text{Sym}^{(n)}(L)$ satisfies the analogue of the Hecke eigenvalue property. More generally,

$$h_{m,n}^*(\text{Sym}^{(m+n)}(L)) = \text{Sym}^{(m)}(L) \boxtimes \text{Sym}^{(n)}(L),$$

where $h_{m,n}$ is defined in (2.1).

The smooth Abel-Jacobi map $A : X \to \text{Pic}(X, g)^{(1)}$ defined by $p \mapsto [(D_p, \nabla_p)]$, extends to $A_n : \text{Sym}^{(n)}(X) \to \text{Pic}(X, g)^{(n)}$ given by $[p_1, \ldots, p_n] \mapsto [(\bigotimes_{j=1}^n D_{p_j}, \nabla_{p_1} \otimes 1 + \ldots 1 \otimes \nabla_{p_n})]$ (notice that the last map is well defined since $\text{Pic}(X, g)$ is an Abelian group). The following diagram is commutative, showing the compatibility of the various maps

$$
\begin{array}{ccc}
\text{Sym}^{(m)}(X) \times \text{Sym}^{(n)}(X) & \xrightarrow{h_{m,n}} & \text{Sym}^{(m+n)}(X) \\
A_m \times A_n & \downarrow & A_{m+n} \\
\text{Pic}(X, g)^{(m)} \times \text{Pic}(X, g)^{(n)} & \xrightarrow{\otimes} & \text{Pic}(X, g)^{(m+n)}
\end{array}
$$

The Abel-Jacobi theorem asserts that $A_n$ induces an isomorphism,

$$H_1(\text{Sym}^{(n)}(X), \mathbb{Z}) \cong H_1(\text{Pic}(X, g)^{(n)}, \mathbb{Z}),$$

for all $n > 0$, which is closely related to Proposition 2.1 since by that proposition, $\pi_1(\text{Sym}^{(n)}(X))$ is an abelian group, and so by the Hurewicz theorem, it is naturally isomorphic to $H_1(\text{Sym}^{(n)}(X), \mathbb{Z})$. Via this isomorphism, the character $\text{Sym}^{(n)}(\rho^L)$ also defines a flat line bundle $K_L^{(n)}$ over $\text{Pic}(X, g)^{(n)}$. Since the flat line bundle $\text{Sym}^{(n)}(L)$ over $\text{Sym}^{(n)}(X)$ has the analogue Hecke eigenvalue property of equation (4.2), it follows that for all $n > 0$, the Hecke eigenvalue property also holds for the flat line bundle $K_L^{(n)}$ over $\text{Pic}(X, g)^{(n)}$ as claimed.

Note that, if we can define the flat line bundle $K_L^{(n)}$ over $\text{Pic}(X, g)^{(n)}$ then we can uniquely extend $K_L$ to a flat line bundle $K_L^{(n-1)}$ over $\text{Pic}(X, g)^{(n-1)}$ via the Hecke eigenvalue property as follows.

By the global Hecke eigenvalue property, $\mathbb{H}^* K_L^{(n)} = L \boxtimes K_L^{(n-1)}$. Then

$$(L^* \boxtimes 1) \otimes \mathbb{H}^* K_L^{(n)} \cong (L^* \boxtimes 1) \otimes (L \boxtimes K_L^{(n-1)}) \cong 1 \boxtimes K_L^{(n-1)}$$

determines the flat line bundle $K_L^{(n-1)}$ over $\text{Pic}(X, g)^{(n-1)}$. By induction, it follows that we can uniquely extend $K_L^{(j)}$ over $\text{Pic}(X, g)^{(j)}$ for all $j \leq n$ and having the Hecke eigenvalue property.

This completes the construction of the flat line bundle $K_L$ over $\text{Pic}(X, g)$ having the Hecke eigenvalue property.

Suppose that $K_L$ is the trivial bundle. Then by the Hecke eigenvalue property, it follows that $L$ is trivial, and therefore the Hecke correspondence is injective. On the other hand, suppose that $K \in \text{Jac}^{\text{Hecke}}(\text{Pic}(X, g))$. Then define the flat line bundle $L := K|_X$ over $X$, where $X$ is identified with its image under the (injective) Abel-Jacobi map $A : X \to \text{Pic}(X, g)^{(1)} \subset \text{Pic}(X, g)$. By injectivity of the Hecke correspondence, $K$ is equivalent to $K_L$. \qed
4.2. **Primitive property of flat line bundles.** The harmonic Picard variety $\tilde{\text{Pic}}(X, g)$ is an Abelian Lie group under tensor product,

$$
\mu : \tilde{\text{Pic}}(X, g) \times \tilde{\text{Pic}}(X, g) \rightarrow \tilde{\text{Pic}}(X, g);
$$

$$
((\mathcal{L}, \nabla), (\mathcal{L}', \nabla')) \rightarrow [(\mathcal{L} \otimes \mathcal{L}', \nabla \otimes 1 + 1 \otimes \nabla')]
$$

The main result in this section is that any flat line bundle $L$ on $\tilde{\text{Pic}}(X, g)$ is a primitive line bundle in the sense that it satisfies,

$$
\mu^* L \cong L \boxtimes L.
$$

Equivalently, for all elements $\mathcal{L}, \mathcal{L}' \in \tilde{\text{Pic}}(X, g),$ we must have:

$$
L_{\mathcal{L} \otimes \mathcal{L}'} \cong L_{\mathcal{L}} \otimes L_{\mathcal{L}'}.
$$

Recall that this is equivalent to the requirement that the total space of the flat principal circle bundle associated to $L$, is a $U(1)$-central extension of $\text{Pic}(X, g)$.

This is but a special case of a more general result that we will now prove.

**Lemma 4.2.** Let $G$ be a Lie group with fundamental group $\pi_1(G)$. Let $\chi : \pi_1(G) \rightarrow U(1)$ be a character of the fundamental group, and $P_{\chi} = \tilde{G} \times_{\chi} U(1)$ be the associated flat principal $U(1)$-bundle, where $\tilde{G}$ denotes the universal covering space of $G$, which is always also a Lie group. Then $P_{\chi}$ is a Lie group, which is a $U(1)$-central extension of $G$.

**Proof.** To see this, recall that $P_{\chi}$ consists of equivalence classes of pairs $(g, z) \in \tilde{G} \times U(1)$, where $(g, z) \sim (gn, \chi(n)^{-1}z)$. Consider the product $(g, z)(g', z') = (gg', zz').$ We need to show that this is well defined. But $(gn, \chi(n)^{-1}z)(g'n', \chi(n')^{-1}z') = (gn \chi(n)^{-1}g' \chi(n')^{-1}zz')$, where we use the fact that $\pi_1(G)$ is a subgroup of the center of $G$. This shows that the product is well defined, and that $\{1\} \times U(1)$ is a central subgroup of $P_{\chi}$, and therefore that $P_{\chi}$ is a $U(1)$-central extension of $G$ as claimed. \hfill $\square$

**Corollary 4.3.** For any flat line bundle $L$ over $X$, the associated flat line bundle $K_L$ is a primitive line bundle over the Abelian Lie group $\tilde{\text{Pic}}(X, g)$, i.e. it satisfies

$$
\mu^* K_L \cong K_L \boxtimes K_L.
$$

REFERENCES

[1] J.-L. Brylinski, Loop spaces, characteristic classes and geometric quantization. Progress in Mathematics, 107, Birkhuser Boston, Inc., Boston, MA, 1993.

[2] J. Cheeger and J. Simons, Differential characters and geometric invariants. Geometry and topology, 50–80, Lecture Notes in Math., 1167, Springer-Verlag, Berlin, 1985.

[3] G. de Rham, Differentiable manifolds. Forms, currents, harmonic forms. Grundlehren der Mathematischen Wissenschaften, 266. Springer-Verlag, Berlin, 1984.

[4] D. S. Freed, G. W. Moore, G. Segal, Heisenberg Groups and Noncommutative Fluxes. Annals Phys. 322 (2007) 236-285. [hep-th/0605200]. *ibid*, The uncertainty of fluxes. Comm. Math. Phys. 271 (2007), no. 1, 247–274. [hep-th/0605198].

[5] E. Frenkel, Lectures on the Langlands Program and Conformal Field Theory, in “Frontiers in Number Theory, Physics, and Geometry II”, editors, P. Cartier, P. Moussa, B. Julia and P. Vanhove, pages 387–533, Springer Berlin Heidelberg, 2007. [hep-th/0512172].

[6] P. Griffiths and J. Harris, Principles of algebraic geometry. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.

[7] A. Grothendieck, Revetements etales et groupe fondamental, Seminaire de Geometrie Algebrique du Bois-Marie (SGA 1), Lecture Notes in Math. 224, Springer, Berlin, 1971.

[8] A. Hatcher, Algebraic topology. Cambridge University Press, Cambridge, 2002.
[9] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces. Providence, R.I.: American Mathematical Society, 2001.

[10] M. J. Hopkins, I. M. Singer, Quadratic functions in geometry, topology, and M-theory. J. Differential Geom. 70 (2005), no. 3, 329–452.

[11] A. Kapustin and E. Witten, Electric-Magnetic Duality And The Geometric Langlands Program. Commun. in Number Theory and Physics 1 (2007) 1–236. [hep-th/0604151v3]

[12] A. Mattuck, Symmetric products and Jacobians. Amer. J. Math. 83 (1961) 189–206.

(Richard Green) Department of Pure Mathematics, University of Adelaide, Adelaide 5005, Australia
E-mail address: ric.green@adelaide.edu.au

(Varghese Mathai) Department of Pure Mathematics, University of Adelaide, Adelaide 5005, Australia
E-mail address: mathai.varghese@adelaide.edu.au