Subgraph probability of random graphs with specified degrees and applications to chromatic number and connectivity

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Abstract

Given a graphical degree sequence $d = (d_1, \ldots, d_n)$, let $G(n, d)$ denote a uniformly random graph on vertex set $[n]$ where vertex $i$ has degree $d_i$ for every $1 \leq i \leq n$. We give upper and lower bounds on the joint probability of an arbitrary set of edges in $G(n, d)$. These upper and lower bounds are approximately what one would get in the configuration model, and thus the analysis in the configuration model can be translated directly to $G(n, d)$, without conditioning on that the configuration model produces a simple graph. Many existing results of $G(n, d)$ in the literature can be significantly improved with simpler proofs, by applying this new probabilistic tool. One example we give is about the chromatic number of $G(n, d)$.

In another application, we use these joint probabilities to study the connectivity of $G(n, d)$. When $\Delta^2 = o(M)$ where $\Delta$ is the maximum component of $d$, we fully characterise the connectivity phase transition of $G(n, d)$. We also give sufficient conditions for $G(n, d)$ being connected when $\Delta$ is unrestricted.

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1 Introduction

Given a graphical degree sequence \( \mathbf{d} = (d_1, \ldots, d_n) \), let \( \mathcal{G}(n, \mathbf{d}) \) denote a uniformly random graph on vertex set \([n]\) where vertex \( i \) has degree \( d_i \) for every \( 1 \leq i \leq n \). We also use the same notation \( \mathcal{G}(n, \mathbf{d}) \) for the support of \( \mathcal{G}(n, \mathbf{d}) \), i.e. the set of graphs with degree sequence \( \mathbf{d} \). Unless otherwise specified, there is usually no confusion from the context whether \( \mathcal{G}(n, \mathbf{d}) \) refers to a set or a random graph from this set, and we will specify it if there is confusion. We say \( \mathbf{d} \) is graphical if the set \( \mathcal{G}(n, \mathbf{d}) \) is nonempty. In the special case where \( d_i = d \) for every \( 1 \leq i \leq n \), \( \mathcal{G}(n, \mathbf{d}) \) is a uniformly random \( d \)-regular graph. Random graphs with specified degree sequences are among the most studied random graph models. However, unlike the binomial random graph \( \mathcal{G}(n, p) \), edge probabilities in \( \mathcal{G}(n, \mathbf{d}) \) are correlated. In fact, even estimating the probability of a single edge in \( \mathcal{G}(n, \mathbf{d}) \) can be challenging. That makes analysing \( \mathcal{G}(n, \mathbf{d}) \) difficult. A common tool used for analysing \( \mathcal{G}(n, \mathbf{d}) \) is the configuration model introduced by Bollobás [3]. In the configuration model, each vertex is represented as a bin containing \( d_i \) points. Take a uniformly random perfect matching over the total \( M = \sum_{i=1}^{n} d_i \) points, and let \( G^\ast \) be the multigraph obtained by taking each pair in the matching as an edge. That is, if points \( p \) and \( q \) are matched as a pair by the matching then \( uv \) is an edge in \( G^\ast \) where \( u \) contains \( p \) and \( v \) contains \( q \). \( G^\ast \) is a multigraph because there can be more than one edge between two vertices. A simple counting argument shows that every simple graph with degree sequence \( \mathbf{d} \) corresponds to the same number of configurations, and thus, \( G^\ast \), conditioned to the set of simple graphs, has the same distribution as \( \mathcal{G}(n, \mathbf{d}) \). Suppose \( \mathbf{d} \) is such that the probability that \( G^\ast \) is simple is bounded away from 0 for all large \( n \), then any property that holds a.a.s. in the configuration model must hold a.a.s. in \( \mathcal{G}(n, \mathbf{d}) \). Estimating edge probabilities in the configuration model is quite easy, and hence, many properties of \( \mathcal{G}(n, \mathbf{d}) \) are obtained by analysing the configuration model. However, the probability that \( G^\ast \) is simple is bounded away from 0 only for \( \mathbf{d} \) such that \( M = \Theta(n) \) and \( M_2 := \sum_{i=1}^{n} d_i^2 = O(n) \). This condition significantly restricts the type of results one can get by translating from the configuration model.

The purpose of this work is to develop probabilistic tools for translating configuration model analysis to \( \mathcal{G}(n, \mathbf{d}) \) analysis for a rich family of degree sequences. We will illustrate with examples and show how easy it is to improve existing results and prove new properties of \( \mathcal{G}(n, \mathbf{d}) \) using the new tools we develop. Let \( H \) be a graph on \([n]\). Under some mild conditions on \( H \) and \( \mathbf{d} \), one of our main results shows the following.

The probability that \( H \) is a subgraph of \( \mathcal{G}(n, \mathbf{d}) \) is approximately the probability that all edges in \( H \) appear in the configuration model.

Before formally stating the main results, we define a few necessary terms. Given \( \mathbf{d} = (d_1, \ldots, d_n) \), without loss of generality we may assume that \( d_1 \geq d_2 \geq \cdots \geq d_n \). Define

\[
\Delta = d_1, \quad \delta = d_n, \quad M = \sum_{i=1}^{n} d_i, \quad J(\mathbf{d}) = \sum_{i=1}^{\Delta} d_i.
\]

For any \( S \subseteq [n] \) define

\[
d(S) = \sum_{i \in S} d_i, \quad \Delta_S = \max_{i \in S} d_i.
\]

Given a graph \( H \) on vertex set \([n]\), let \( e(H) \) be the number of edges in \( H \), \( \Delta(H) \) be the maximum degree of \( H \), and let \( \mathbf{d}^H = (d_1^H, \ldots, d_n^H) \) denote the degree sequence of \( H \). For graphs on the same vertex set, e.g. on \([n]\), we may treat them as sets of edges. Thus, if two graphs \( H_1 \) and \( H_2 \) are both on vertex set \([n]\), we say \( H_1 \) and \( H_2 \) are disjoint if their edge sets are disjoint. Given a graph \( H \) on \([n]\), let \( H^+ \) denote the event that \( H \subseteq \mathcal{G}(n, \mathbf{d}) \), i.e. \( H \) is a subgraph of \( \mathcal{G}(n, \mathbf{d}) \), and let \( H^− \) denote
Then, we consider the conditional edge probabilities in $G(n, d)$. Let $H_1$ and $H_2$ be two disjoint graphs on $[n]$. Suppose that $d^{H_1} \preceq d$, and $uv \notin H_1 \cup H_2$. Then, $d^{H_2} \preceq d$ and $uv /\notin H_1 \cup H_2$. Suppose further that $J(d) + \Delta \cdot \Delta(H_2) = o(M - 2e(H_1))$, $e(H_2)\Delta^2 = o((M - 2e(H_1))^2)$. Then, $v \in G(n, d) | H_1^+, H_2^-) = (1 + o(1)) \frac{(d_u - d^{H_1}_u)(d_v - d^{H_1}_v)}{M - 2e(H_1) + (d_u - d^{H_1}_u)(d_v - d^{H_1}_v)}$.
Lemmas 2.1 and 2.2 obtained the asymptotic value of such conditional probabilities in a random \(d\)-regular \(r\)-uniform hypergraph. Our corollary above generalises and strengthens all these results above (compared with \[7,\] Lemmas 2.1 and 2.2) for \(r = 2\), our corollary has more relaxed conditions on \(H_1\) and \(H_2\) in addition to relaxed conditions on \(d\). Our result is mostly comparable with McKay’s enumeration results \([20]\), which yield similar probabilities as in Corollary 2 but require stronger conditions on \(\Delta\). These are all for sparse degree sequences. For dense \(d\), such conditional probabilities are estimated in \([1, 14, 22, 12]\).

Repeatedly applying Theorem 1 we obtain the following joint probability bounds for an arbitrary set of edges, under some mild conditions. Given a graph \(H\), let \(\partial(H) = \{v : v \cap x \neq \emptyset\text{ for some } x \in E(H)\}\) be the set of vertices that are incident to some edge in \(H\).

**Theorem 4.** Assume \(H_1\) and \(H_2\) are two disjoint graphs on \([n]\).

(a) If
\[
R(d, H_1, H_2) := \frac{6J(d) + 2\Delta(8 + 2\Delta(H_2))}{M - 2e(H_1)} + \frac{4e(H_2)\Delta^2}{(M - 2e(H_1))^2} \leq 1,
\]
then
\[
P(H_1^+ \mid H_2^-) \leq \prod_{i=1}^{n} (d_i)_{d_i} \cdot \prod_{j=1}^{h} \frac{1 + R(d, H_1, H_2)}{(M - 2j + 2)}.
\]

(b) If
\[
\tau(d, H_1, H_2) := \frac{2J(d) + 6\Delta + 2\Delta(H_2)\Delta + \Delta^2}{M - 2e(H_1)} \leq 1,
\]
then
\[
P(H_1^+ \mid H_2^-) \geq \prod_{i=1}^{n} (d_i)_{d_i} \cdot \prod_{j=1}^{h} \frac{1 - \tau(d, H_1, H_2)}{(M - 2j + 2)}.
\]

By setting \(H_2 = \emptyset\) and proper conditions on \(d\) so that \(R(d, H_1, \emptyset) = o(1)\), we obtain the following useful upper and lower bounds on the joint probability.

**Corollary 5.** Let \(H\) be a graph on \([n]\) and assume that \(d\) is a degree sequence satisfying \(J(d) = o(M - 2e(H))\). Then
\[
P(H^+) \leq \prod_{i=1}^{n} (d_i)_{d_i} \cdot \prod_{i=1}^{e(H)} \frac{1 + o(1)}{M - 2i + 2},
\]
If further we have \(\Delta^2_{\partial(H)} = o((M - 2e(H)))\) then the above holds with equality.

**Remark 6.** Similar bounds on \(P(H^+)\) are given by McKay in \([22,\] Theorem 2.1\) with much more restrictive \(d\) and \(H\).

**Remark 7.** Although probability estimates similar to \([7]\) appeared in the literature for several times, such probabilities were not connected before to the corresponding probabilities in the configuration model, and thus were never used to automatically translate the analysis from the configuration model to \(G(n, d)\). This is one of our main contributions of this work. We compare \([7]\) with the probability in the configuration model. Let \(\sigma^*\) denote a uniformly random perfect matching over the \(M\) points produced by the configuration model, and let \(G^*\) be the multigraph corresponding to \(\sigma^*\). Given a
graph \( H \) on \([n]\) with \( d^H \leq d \), let \( \mathcal{P}(H) \) be the set of matchings \( \sigma \) of size \( e(H) \) over the set of \( M \) points whose corresponding graph is \( H \). Then \( |\mathcal{P}(H)| = \prod_{i=1}^n (d_i)_{d^H} \) and \( \mathbb{P}(\sigma \subseteq \sigma^*) = \prod_{i=1}^n \frac{1}{M - 2i + 1} \).

Hence,

\[
\mathbb{P}(H \subseteq G^*) \leq \sum_{\sigma \in \mathcal{P}(H)} \mathbb{P}(\sigma \subseteq \sigma^*) = \prod_{i=1}^n (d_i)_{d^H} \prod_{i=1}^n \frac{1}{M - 2i + 1}.
\tag{2}
\]

Thus, under the condition \( J(d) = o(M - 2e(H)) \), our upper bound in (1) differs from the corresponding probability bound \( (3) \) in the configuration model by a relative \( 1 + o(1) \) factor in each term of the product. Such an approximation is enough to translate a lot of configuration model analysis to \( G(n, d) \). See examples in Sections 1.1.2.

Another advantage of using the configuration model is that, it is very easy to bound the probability of having a certain number of edges joining two sets of vertices. Given two subsets of vertices \( S_1, S_2 \subseteq [n] \), let \( e(S_1, S_2) \) be the number of edges with one end in \( S_1 \) and the other end in \( S_2 \). When \( S_1 = S_2 \), \( e(S_1, S_2) \) is simply the number of edges induced by \( S_1 \). In the configuration model, the probability that \( e(S_1, S_2) \geq \ell \) is at most

\[
\left( \frac{d(S_1)}{\ell} \right) \left( \frac{d(S_2)}{\ell} \right) \prod_{i=1}^{\ell} \frac{1 + o(1)}{M - 2i + 2} \leq \left( \frac{d(S_1)}{\ell} \right) \left( \frac{d(S_2)}{\ell} \right) \frac{(M/2)\ell(2 + o(1))}{\ell}.
\tag{3}
\]

In the following corollary we show a similar bound for this probability in \( G(n, d) \).

**Corollary 8.** Suppose \( S_1, S_2 \subseteq [n] \). Let \( 1 \leq \ell < M/2 \) be an integer. Assume that \( d \) satisfies \( J(d) = o(M - 2\ell) \). Then,

\[
\mathbb{P}(e(S_1, S_2) \geq \ell) \leq \left( \frac{d(S_1)}{\ell} \right) \left( \frac{d(S_2)}{\ell} \right) \prod_{i=1}^{\ell} \frac{1 + o(1)}{M - 2i + 2} \leq \left( \frac{d(S_1)}{\ell} \right) \left( \frac{d(S_2)}{\ell} \right) \frac{(M/2)\ell(2 + o(1))}{\ell}.
\tag{4}
\]

**Remark.** Since \( J(d) \leq \Delta^2 \), both Corollaries 5 and 8 hold when \( J(d) \) is replaced by \( \Delta^2 \).

**Proof.** Let \( \mathcal{H}_\ell \) denote the set of graphs on \([n]\) having exactly \( \ell \) edges, all of which have exactly one end in \( S_1 \) and the other end in \( S_2 \). Let \( \{i_1, \ldots, i_k\} \) denote the set of vertices in \( S_1 \cup S_2 \). By the union bound and Corollary 5

\[
\mathbb{P}(e(S_1, S_2) \geq \ell) \leq \sum_{H \in \mathcal{H}_\ell} \mathbb{P}(H \subseteq G(n, d)) \leq \left( \prod_{i=1}^{\ell} \frac{1 + o(1)}{M - 2i + 2} \right) \sum_{H \in \mathcal{H}_\ell} \prod_{j=1}^{k} (d_{i_j})_{d^H_{i_j}}.
\]

Next we give a combinatorial interpretation of \( \sum_{H \in \mathcal{H}_\ell} \prod_{j=1}^{k} (d_{i_j})_{d^H_{i_j}} \) above. Represent each vertex \( i_j \) in \( S_1 \cup S_2 \) by a bin containing \( d_{i_j} \) points. Given \( H \in \mathcal{H}_\ell \), how many matchings of size \( \ell \) out of the total \( d(S_1 \cup S_2) \) points are there so that if \( uv \) is an edge in \( H \) then there is a point \( p \) in bin \( u \) and a point \( q \) in bin \( v \) such that \( p \) and \( q \) are matched by the matching? It is easy to see that there are \( \prod_{j=1}^{k} (d_{i_j})_{d^H_{i_j}} \) such matchings. Hence, \( \sum_{H \in \mathcal{H}_\ell} \prod_{j=1}^{k} (d_{i_j})_{d^H_{i_j}} \) is bounded above by the total number of size-\( \ell \) matchings when every pair \( (p, q) \) in the matching is of the form that \( p \) is in some bin in \( S_1 \) and \( q \) is in some bin in \( S_2 \). There are \( \left( \frac{d(S_1)}{\ell} \right) \) ways to choose the \( \ell \) ends that are in bins in \( S_1 \) and there are \( \left( \frac{d(S_2)}{\ell} \right) \) ways to choose the other ends from bins in \( S_2 \) and match them to the \( \ell \) ends chosen before. Hence,

\[
\sum_{H \in \mathcal{H}_\ell} \prod_{j=1}^{k} (d_{i_j})_{d^H_{i_j}} \leq \left( \frac{d(S_1)}{\ell} \right) \left( \frac{d(S_2)}{\ell} \right).
\]
Theorem 9. If \( d \) satisfies condition (A1) and \( \Delta = o(n) \) then a.a.s. \( \chi(G(n, d)) = O(d/\ln d) \). If \( J(d) = o(M) \) then a.a.s. \( \chi(G(n, d)) = \Omega(d/\ln d) \).

The proof of Theorem 9 will be given in Section 2, which is obtained by translating the existing analysis of the configuration model to \( G(n, d) \). We believe that many other results of \( G(n, d) \) can be obtained or improved in a similar manner. For instance, the order of the largest component of \( G(n, d) \) was determined by Molloy and Reed [23] for the so-called “well-behaved” degree sequences. Their proof relies on an analysis in the configuration model. We believe that most of the analysis can be immediately translated to \( G(n, d) \) by using the conditional edge probabilities in Theorem 1. We also believe that the new probabilistic tools developed in Section 1.1.1 will be useful in studying other properties of \( G(n, d) \). We give another example in Section 1.1.3.

1.1.2 Chromatic number of \( G(n, d) \)

Let \( \chi(G) \) denote the chromatic number of graph \( G \), i.e., the minimum number of colours required to colour vertices of \( G \) so that all pairs of adjacent vertices receive distinct colours. It is known that a.a.s. the chromatic number of a random \( d \)-regular graph is asymptotically \( d/2 \ln d \), for \( d = \omega(1) \) and \( d = o(n) \); see [11, 5, 18]. In the paper [10] Frieze, Krivelevich and Smyth asked under what conditions on \( d \) would we have a.a.s. \( \chi(G(n, d)) = \Theta(d/\ln d) \), where \( d = M/n \) is the average degree of graphs in \( G(n, d) \). Let \( D_k = \sum_{i=1}^{k} d_i \) and \( M_2 = \sum_{i=1}^{n} d_i^2 \). It was shown in [10] that if

1. there exist constants \( 1/2 < \alpha < 1, K_0 > 0 \) such that \( D_k \leq K_0dn(k/n)^\alpha \) for all \( 1 \leq k \leq \epsilon n \);
2. \( \Delta^5 = o(M_2) \),

then a.a.s. \( \chi(G(n, d)) = O(d/\ln d) \). On the other hand, if

3. \( \Delta^4 = o(M_1) \),

then a.a.s. \( \chi(G(n, d)) = \Omega(d/\ln d) \).

We significantly relax the conditions (A2) and (A3) and obtain the following result for \( \chi(G(n, d)) \).

Theorem 9. If \( d \) satisfies condition (A1) and \( \Delta = o(n) \) then a.a.s. \( \chi(G(n, d)) = O(d/\ln d) \). If \( J(d) = o(M) \) then a.a.s. \( \chi(G(n, d)) = \Omega(d/\ln d) \).

1.1.3 Connectivity transition of \( G(n, d) \)

The connectivity is one of the best studied graph properties for random graphs. Erdős and Rényi [6] determined the threshold of the connectedness for \( G(n, p) \). Indeed, for every fixed integer \( k \geq 1 \), Erdős and Rényi [6] determined when \( G(n, p) \) becomes \( k \)-connected. Random graph \( G(n, p) \) becomes connected when isolated vertices disappear, which happens when \( p \approx \log n/n \). For \( p \) in this range, the average degree of the random graphs is around \( \log n \) and there are very few vertices of degree one or two, and these vertices are pair-wise far away in graph distance. Consequently the vertices of degree one or two do not affect the connectedness of \( G(n, p) \). The most natural sparser random graph model for the study of connectivity would be \( G(n, d) \), and it is natural to ask when are such random graphs connected. As we will see, the vertices of degree one or two play crucial roles for the connectedness of \( G(n, d) \).
The first work about the connectivity of $G(n, d)$ was by Wormald [24] in 1981. In this pioneering work, the author studied the connectivity of $G(n, d)$ where $\delta \geq 3$ and $\Delta$ is bounded by some absolute constant $R$ (i.e. $R$ does not depend on $n$), and proved that a.a.s. the connectivity of $G(n, d)$ is equal to $\delta$ for such degree sequences. Frieze [9] studied the connectivity of random $\delta$-regular graphs where $3 \leq \delta = o(n^{0.2})$, and proved that a.a.s. a random $\delta$-regular graph is $\delta$-connected, for $\delta$ in the aforementioned range. Later, Cooper, Frieze and Reed [4] extended this result to $3 \leq d \leq cn$ where $c > 0$ is a sufficiently small constant. Luczak [19] extended their work to non-regular degree sequences, and considered also degree sequences permitting $\delta = 2$. Luczak showed that, for any $d$ where $\delta \geq 3$ and $\Delta \leq n^{0.01}$ then a.a.s. $G(n, d)$ is $\delta$-connected. When $\delta = 2$ and $\Delta \leq n^{0.01}$ he characterised the structure of $G(n, d)$ and determined when is $G(n, d)$ a.a.s. 2-connected. In a more recent work, Federico and Van der Hofstad [8] considered degree sequences permitting $\delta = 1$ and fully characterized the connectivity transition of $G(n, d)$ for $d \in \mathcal{D}$, where $\mathcal{D} = \{d : M = \Theta(n), \sum_{i=1}^{n} d_i^2 = O(n)\}$. Let $n_1$ be the number of components in $d$ with value 1, and $n_2$ the number of components in $d$ with value 2. Federico and Van der Hofstad showed that for $d \in \mathcal{D}$ that satisfies some additional “smoothness” condition, $G(n, d)$ is a.a.s. connected if $n_1 = o(\sqrt{n})$ and $n_2 = o(n)$, and $G(n, d)$ is disconnected if $n_1 = \omega(\sqrt{n})$. All the work that we have discussed so far are for $d$ where either the maximum degree is not large (at most $n^{0.01}$), or $d$ corresponds to a regular degree sequence, and the degree is nearly sublinear (at most $cn$ for some sufficiently small $c$). For $d$ linear in $n$, Krivelevich, Sudakov, Vu and Wormald [18] proved several properties of random $d$-regular graphs, including the connectivity. Recently, Isaev, McKay and the first author [12] proved several properties including the connectivity of $G(n, d)$ for near-regular $d$ where $d = \omega(\log n)$ and $d_i \sim d$ for every $i$.

In this work, we characterise the connectivity transition of $G(n, d)$ for a much larger family of degree sequences. For the family of $d$ where $J(d) = o(M)$ (in particular, when $\Delta^2 = o(M)$) we fully characterise the phase transition of the connectedness of $G(n, d)$. When $\Delta$ is unrestricted we give sufficient conditions under which $G(n, d)$ is a.a.s. connected.

We only consider degree sequences where $\delta \geq 1$ since otherwise $G(n, d)$ is disconnected trivially. Given the degree sequence $d$ where $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta \geq 1$, define

$$n_1 = \sum_{i=1}^{n} 1\{d_i = 1\}, \quad n_2 = \sum_{i=1}^{n} 1\{d_i = 2\}.$$

**Theorem 10.** Assume $d$ is a graphical degree sequence with $J(d) = o(M)$. Let $c > 0$ be a fixed constant.

(a) If $n_1 = o(\sqrt{M})$ and $n_2 = o(M)$ then a.a.s. $G(n, d)$ is connected.

(b) If $n_1 = \omega(\sqrt{M})$ then a.a.s. $G(n, d)$ is disconnected.

(c) If $n_1 \geq c\sqrt{M}$ or $n_2 \geq c\sqrt{M}$ then there exists $\delta = \delta(c) > 0$ such that for all sufficiently large $n$, $P(G(n, d) \text{ disconnected}) \geq \delta$.

Since $J(d) \leq \Delta^2$, we immediately have the following corollary.

**Corollary 11.** Theorem 10 holds if $J(d) = o(M)$ is replaced by $\Delta^2 = o(M)$.

Next, we deal with degree sequences where $\Delta$ is rather large. Define

$$\mathcal{H} = \{i : d_i \geq \sqrt{M}/\log M\}.$$

Our next result gives sufficient conditions for the connectedness of $G(n, d)$. 

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Theorem 12. Assume $M - 2d(H) = \Omega(M)$. If $n_1 = o(\sqrt{M})$ and $n_2 = o(M)$ then a.a.s. $\mathcal{G}(n, d)$ is connected.

Conditions in Theorem 12 are not necessary. We can easily make up $d$ where $d_i = n - 1$, $M = \Theta(n)$ and a linear number of vertices have degree 1 (or of degree 2). Conditions in Theorem 12 are not satisfied but $\mathcal{G}(n, d)$ is always connected for such degree sequences.

1.2 Degree sequences satisfying $J(d) = o(M)$

A rich family of degree sequences satisfies the condition $J(d) = o(M)$ for which we may apply the probabilities in Theorems 4 and 4. For instance, it covers all regular sublinear degree sequences, i.e. $d_i = d$ for all $i$ and $d = o(n)$. We give two additional examples below that might be of interest in applications.

- $\Delta = o(n)$ and a linear (in $n$) number of vertices have degree $\Omega(\Delta)$.
- $d$ is composed of i.i.d. power-law variables of exponent $\tau > 2$, conditioned to even sum.

The reader may wonder what types of degree sequences do not satisfy $J(d) = o(M)$. Certainly, regular degree sequences with linear degrees do not satisfy this condition. More generally, if there is a linear (in $n$) number of vertices with degree $\Theta(n)$, then that degree sequence does not satisfy $J(d) = o(M)$.

We will prove Theorems 4 and 4 in Section 3. The proof of Theorem 9 will be given in Section 2 and the proofs for Theorems 10 and 12 will be presented in Section 4.

2 Proof of Theorem 9: chromatic number

We first briefly sketch the proof in [10]. For the upper bound, the authors first obtained an $O(d/\ln d)$ upper bound for the multigraph $G^*$ from the configuration model. A multigraph is properly coloured if every pair of adjacent and distinct vertices are coloured differently. Only Condition (A1) is needed for this part of the proof. Then they applied a sequence of switching operations which repeatedly switch away the loops and multiple edges in $G^*$. Then they proved that (a), every simple graph is obtained with asymptotically the same probability after applying the switchings; (b) if $H$ is the graph induced the by set of edges added during the switchings, then a.a.s. $\chi(H) = O(1)$. It follows immediately that a.a.s. the chromatic number of $\mathcal{G}(n, d)$ is at most $\chi(G^*) \cdot \chi(H) = O(d/\ln d)$. Condition (A2) is needed to guarantee (a) and (b).

For the lower bound, they proved that for any partition $\sigma$ of vertices in to $t = b \cdot d/\ln d$ parts, where $b > 0$ is a sufficiently small constant, the probability that $\sigma$ specifies a proper $t$-colouring of $G^*$ is at most $t^{-2n}$. Condition (A3) was applied to obtain a lower bound on the probability that $G^*$ is a simple graph. When $\Delta^2 = o(n)$, the probability $t^{-2n}$ is small enough to beat the union bound over all such partitions $\sigma$, and the inverse of the probability that $G^*$ is simple. Note that (A3) implies $\Delta^2 = o(n)$.

To prove Theorem 9, we carry all analysis from [10] for $G^*$ to $\mathcal{G}(n, d)$. Both the upper and lower bound proofs for $\chi(G^*)$ in [10] use upper bounds on the probability of $G^*$ containing some set of edges $H$ where $M - 2e(H) = \Omega(M)$. The upper bound of $\chi(G^*)$ follows by [10] Lemmas 1–3]. These lemmas hold for $\mathcal{G}(n, d)$ with exactly the same proofs, by applying inequalities [1] and [4] instead of [2] and [3]. The additional $1 + o(1)$ factors in [1] and [4], compared with (2) and (3), do not affect the proof (in fact, any constant factor would do). As no switching analysis is required any more, we do not need condition (A2). Instead, we need $J(d) = o(M)$ in order to
apply (1) and (4). This is guaranteed by our assumption (A1) and $\Delta = o(n)$ as follows: by (A1), $J(d)/M = O((\Delta/n)^{\alpha})$ which is $o(1)$ if $\Delta = o(n)$.

The same translation of analysis holds for the lower bound proof. As we are working on $G(n,d)$ instead of $G^*$, it is sufficient if the probability $t^{-2n}$ beats the union bound over the total number of such partitions. This is always the case as there can be at most $t^n$ partitions. Hence, for the lower bound we do not need conditions (A3) or $\Delta^2 = o(n)$ any more. Instead, we impose $J(d) = o(M)$ which validates the application of the probability bounds (1) and (4).

3 Proofs of Theorems 1 and 4

Proof of Theorem 4. Let $e_1 = u_1v_1, \ldots, e_h = u_hv_h$ be an enumeration of the set of edges in $H_1$ where $h = e(H_1)$. Let $G_0 = \emptyset$ and $G_j = G_{j-1} \cup \{e_j\}$ for every $1 \leq j \leq h$. Then,

$$
P(H_1^+ | H_2^-) = \prod_{j=1}^{h} \mathbb{P}(e_j \mid G_{j-1}^+, H_2^-).$$

By Theorem 1

$$
\prod_{j=1}^{h} \mathbb{P}(e_j \mid G_{j-1}^+, H_2^-) \leq \prod_{j=1}^{h} (d_{u_j} - d_{u_j}^{G_{j-1}})(d_{v_j} - d_{v_j}^{G_{j-1}}) \frac{f(d, G_{j-1}, H_2)}{M - 2j + 2}
$$

$$
= \prod_{i=1}^{n} (d_{i})_{d_{i}}^{\mu_{1}} \prod_{j=1}^{h} \frac{f(d, G_{j-1}, H_2)}{(M - 2j + 2)},
$$

where

$$
f(d, G_{j-1}, H_2) \leq \left(1 - \frac{3J(d) + \Delta(8 + 2\Delta(H_2))}{M - 2j + 2} - \frac{2e(H_2)\Delta^2}{(M - 2j + 2)^2}\right)^{-1}
$$

$$
\leq 1 + \frac{6J(d) + 2\Delta(8 + 2\Delta(H_2))}{M - 2j + 2} + \frac{4e(H_2)\Delta^2}{(M - 2j + 2)^2} = 1 + \mathfrak{R}(d, H_1, H_2),
$$

for every $1 \leq j \leq h$. The second inequality above holds by the fact that $(1 - x)^{-1} \leq 1 + 2x$ for all $x \in [0, 1/2]$ and the assumption that $\mathfrak{R}(d, H_1, H_2) \leq 1$. This yields our upper bound on $P(H_1^+ | H_2^-)$. Again by the lower bound in Theorem 1

$$
\prod_{j=1}^{h} \mathbb{P}(e_j \mid G_{j-1}^+, H_2^-) \geq \prod_{j=1}^{h} (d_{u_j} - d_{u_j}^{G_{j-1}})(d_{v_j} - d_{v_j}^{G_{j-1}}) \frac{g(d, G_{j-1}, H_2)}{M - 2j + 2}
$$

$$
= \prod_{i=1}^{n} (d_{i})_{d_{i}}^{\mu_{1}} \prod_{j=1}^{h} \frac{g(d, G_{j-1}, H_2)}{(M - 2j + 2)},
$$

where

$$
g(d, G_{j-1}, H_2) = \left(1 - \frac{2J(d) + 6\Delta + 2\Delta(H_2)\Delta}{M - 2j + 2}\right) \left(1 + \frac{(d_{v_j} - d_{v_j}^{G_{j-1}})(d_{u_j} - d_{u_j}^{G_{j-1}})}{M - 2j + 2}\right)^{-1}
$$

$$
\geq 1 - \frac{2J(d) + 6\Delta + 2\Delta(H_2)\Delta + d_{v_j}d_{u_j}}{M - 2j + 2} \geq 1 - \mathfrak{r}(d, H_1, H_2),
$$
for every $1 \leq j \leq h$. The second inequality above holds by the fact that $(1 + x)^{-1} \geq 1 - x$ for all $x \geq 0$. This yields our lower bound on $\mathbb{P}(H^+_1 \mid H^-_2)$. $\blacksquare$

Proof of Theorem 1. Let $\mathcal{G}$ denote the set of graphs $G$ on $[n]$ with degree sequence $d_1, \ldots, d_n$, such that $H_1 \subseteq G$, $G \cap H_2 = \emptyset$. Let $S$ denote the set of graphs in $\mathcal{G}$ that contain $uv$ as an edge, and let $S = \mathcal{G} \setminus S$. Then,

$$\mathbb{P}(uv \in \mathcal{G}(n, d) \mid H^+_1, H^-_2) = \frac{|S|}{|S| + |\overline{S}|} = \frac{1}{1 + |\overline{S}|/|S|}$$

We will obtain upper and lower bounds on the ratio $|\overline{S}|/|S|$ by analysing switchings that relate graphs in $S$ to graphs in $\overline{S}$. We first define the switching.

Given $G \in S$, a forward switching specifies an ordered 4-tuple $(x, a, y, b) \in [n]^4$ satisfying the following conditions:

1. $u, v, x, y, a, b$ are all distinct, except $x = y$ is permitted.
2. $xa$ and $yb$ are edges in $G \setminus H_1$.
3. None of $xu$, $yv$, and $ab$ are edges in $G \cup H_2$.

Then the forward switching converts $G$ to a graph $G' \in \overline{S}$ by deleting the edges $uv$, $xa$ and $yb$ from $G$ and adding the edges $xu$, $yv$ and $ab$. See Figure 1 for an illustration, where solid lines denote edges in the graph and dashed lines denote non-edges.

Let $f(G)$ denote the number of forward switchings that can be applied to $G$. We will show the following upper and lower bounds on $f(G)$:

Claim 13.

(a) $f(G) \leq (M - 2e(H_1))^2$

(b) $f(G) \geq (M - 2e(H_1))^2 \left(1 - \frac{3J(d) + \Delta(8 + d_{H_2} + d_H) - 2e(H_2)\Delta^2}{M - 2e(H_1)}\right)$.

Next, given $G' \in \overline{S}$, we count the number of forward switchings that can produce $G'$. In order to do so, we define a backward switching on $G'$ as an ordered 4-tuple $(x, a, y, b) \in [n]^4$ satisfying the following:

1. $u, v, x, y, a, b$ are all distinct, except $x = y$ is permitted.
(2') \( xa \) and \( yb \) are not edges in \( G' \cup H_2 \).

(3') \( xu, yv, \) and \( ab \) are edges in \( G' \setminus H_1 \).

Then the backward switching deletes the edges \( xu, yv, \) and \( ab, \) and adds the edges \( uv, xa \) and \( yb. \)

Obviously, a backward switching on \( G' \) is exactly the inverse of a forward switching which produces \( G' \). Let \( b(G') \) be the number of backward switchings that can be applied to \( G' \). We will show the following.

Claim 14.

\begin{align*}
(a) & \quad b(G') \leq (d_u - d_{u_1})(d_v - d_{v_1})(M - 2e(H_1)) \\
(b) & \quad b(G') \geq (d_u - d_{u_1})(d_v - d_{v_1})(M - 2e(H_1)) \left( 1 - \frac{2J(d) + 6\Delta + 2\Delta(H_2)\Delta}{M - 2e(H_1)} \right).
\end{align*}

Let \( T \) be the total number of forward switchings from \( S \) to \( S^\prime \). By definition,

\[ T = \sum_{G \in S} f(G) = \sum_{G' \in S} b(G'). \]

By Claim \( 13(a) \) and Claim \( 14(b) \),

\[ |S|(d_u - d_{u_1})(d_v - d_{v_1})(M - 2e(H_1)) \left( 1 - \frac{2J(d) + 6\Delta + 2\Delta(H_2)\Delta}{M - 2e(H_1)} \right) \leq T \leq |S|(M - 2e(H_1))^2. \]

Thus,

\[ \frac{|S|}{|S| + |S^\prime|} \geq \frac{(d_u - d_{u_1})(d_v - d_{v_1})}{M - 2e(H_1)} \left( 1 - \frac{2J(d) + 6\Delta + 2\Delta(H_2)\Delta}{M - 2e(H_1)} \right) \left( 1 + \frac{(d_u - d_{u_1})(d_v - d_{v_1})}{M - 2e(H_1)} \right)^{-1} \]

\[ = \frac{(d_u - d_{u_1})(d_v - d_{v_1})}{M - 2e(H_1)} \cdot g(d, H_1, H_2). \]

Similarly, Claim \( 13(b) \) and Claim \( 14(a) \),

\[ \frac{|S|}{|S| + |S^\prime|} \leq \frac{(d_u - d_{u_1})(d_v - d_{v_1})}{M - 2e(H_1)} \cdot f(d, H_1, H_2). \]

Hence, we have shown the upper and lower bounds of \( P(uv \in G(n, d) \mid H_1^+, H_2^-) \) as desired.

It only remains to prove the two claims. They follow from simple inclusion-exclusion counting arguments as follows.

Proof of Claim \( 13 \). The upper bound is obvious as there are at most \( M - 2e(H_1) \) ways to choose vertices \( x \) and \( a, \) and then at most \( M - 2e(H_1) \) ways to choose vertices \( y \) and \( b. \) To get the required lower bound, we subtract from the above upper bound the number of choices where one of the conditions in (1)–(3) is violated. If condition (1) is violated, then \( \{x, y, a, b\} \cap \{u, v\} = \emptyset, \) or \( a \in \{y, b\}, \) or \( x = b. \) There are at most \( 2(M - 2e(H_1)) \cdot 2 \cdot 2 + 2\Delta(M - 2e(H_1)) + \Delta(M - 2e(H_1)) = 11\Delta(M - 2e(H_1)) \) ways to choose such 4-tuples. In our upper bound, we only considered choices where condition (2) is satisfied. Thus, it only remains to subtract the number of choices where condition (3) is violated. That means either (a), \( xu, yv, \) or \( ab \) is an edge in \( G; \) or (b), \( xu, yv, \) or \( ab \) is an edge in \( H_2. \) We call an ordered triple of vertices \( (v_1, v_2, v_3) \) a directed 2-path at \( v_1, \) if both \( v_1v_2 \) and \( v_2v_3 \) are edges in the graph. Note that for any graph \( G \) with degree sequence \( d, \) and any
v \in [n]$, the number of directed 2-paths at $v$ in $G$ is always at most $\sum_{i=1}^{\Delta} (d_i - 1) = J(d) - \Delta$. Hence, the number of choices for (a) is at most $3(J(d) - \Delta)(M - 2e(H^+))$. The number of choices for (b) is at most $(d_u^H \Delta + d_v^H \Delta)(M - 2e(H_1)) + 2e(H_2)\Delta^2$. These give the lower bound for $f(G)$ as desired.

**Proof of Claim 14.** The upper bound is obvious. There are at most $d_u - d_u^{H_1}$ ways to choose $x$, at most $d_v - d_v^{H_1}$ ways to choose $y$, and at most $M - 2e(H_1)$ ways to choose $a$ and $b$. From this upper bound, we need to subtract the number of choices where condition (1') or (2') is violated (note that our choices in the upper bound guarantee condition (3') already). If condition (1') is violated then $\{a, b\} \cap \{u, v, x, y\} \neq \emptyset$. There are at most $4 \cdot 2 \cdot (d_u - d_u^{H_1})(d_v - d_v^{H_1})\Delta$ such choices. If condition (2') is violated then either (a'), $xa$ or $yb$ is an edge in $G'$; or (b'), $xa$ or $yb$ is an edge in $H_2$. The number of choices for (a') is at most $2(d_u - d_u^{H_1})(d_v - d_v^{H_1})(J(d) - \Delta)$, and the number of choice for (b') is at most $2(d_u - d_u^{H_1})(d_v - d_v^{H_1})\Delta(H_2)\Delta$. Subtracting these upper bounds on the number of invalid choices from the upper bound on $b(G')$ yields the lower bound on $b(G')$ as desired.

4 Proof of Theorems 10 and 12: connectivity

4.1 Proof techniques: the old and the new

Approximately four proof techniques or a hybrid of them have been used for proving the connectivity of $G(n, d)$ and for analysing properties of $G(n, d)$ in general, when $\Delta$ is not too large. The first, and perhaps the most well known method uses the configuration model. Recall that all a.a.s. results can be translated from the configuration model to $G(n, d)$ if $d \in \mathcal{D}$ where $\mathcal{D} = \{d : M = \Theta(n), \sum_{i=1}^{n} d_i^2 = O(n)\}$. Federico and Hofstad’s work [3] is an example of this proof method. Due to the ease in handling with the configuration model, they managed to prove more accurate result including a critical window analysis during the connectivity phase transition, but such distributional results cannot be directly translated to $G(n, d)$.

Another proof method uses graph enumeration. Assume we want to bound the probability that $e(S, \overline{S}) = 0$. This probability is simply

$$\frac{|G(S, d|_S) \cdot |G(S, d|_{\overline{S}})|}{|G(n, d)|},$$

where $d|_S$ denotes the degree sequence obtained by restricted to vertices in $S$, and $G(S, d|_S)$ denote the set of graphs on vertex set $S$ and with degree sequence $d|_S$. Applying known asymptotic enumeration results on $|G(n, d)|$ one can get asymptotic probability for the event that $e(S, \overline{S})$, which can further be used to bound the probability that $G(n, d)$ is disconnected. This approach was taken by Wormald [21]. It may be interesting to note that the enumeration results on which Wormald’s proof was based are by Bender and Canfield [2], which requires $\Delta$ to be absolutely bounded. Then Bollobás introduced the configuration model and deduced a probabilistic proof of [2]. Afterwards the configuration model became popularised. In that sense, Wormald’s proof can be viewed as a “detour” of the first method aforementioned.

The third method combines the configuration model with the switching method introduced by McKay [20, 21]. As mentioned before, a.a.s. results can be translated from the configuration model to $G(n, d)$ if $d \in \mathcal{D}$. What can we do for $d \notin \mathcal{D}$, e.g. when the average degree of $d$ is growing with $n$? McKay’s switching method starts with $G^*$, the multigraph produced by the configuration model, and repeatedly switches away multiple edges in the multigraph from the configuration model. Using simple counting argument one can show that when $\Delta$ is below $M^{1/4}$ then the distribution
of the final simple graph obtained is very close to the uniform distribution. Then, we can deduce properties of \( G(n, d) \) by analysing the configuration model and the switching algorithm. There are many results of \( G(n, d) \) obtained this way, e.g. \( \chi(G(n, d)) \) in [10] discussed in Section 2. See more examples in [15]. In terms of the connectivity, proofs in [9, 19] followed this path.

The last method applies the switching technique directly to random graph \( G(n, d) \). Partition the set of graphs \( G(n, d) \) into two parts \( S \) and \( T \) where graphs in \( S \) have a certain property \( P \) and graphs in \( T \) do not. Then defining switchings that relate graphs in \( S \) to graphs in \( T \). By counting the number of ways to perform switchings one can estimate the ratio \( |S|/|T| \) and the probability of property \( P \). This approach was used in [4] for the connectivity of random \( d \)-regular graphs for \( d \) up to \( cn \) where \( c \) is sufficiently small.

In this work, we use the new tool in Corollary [5] to characterise the connectivity phase transition for the family of degree sequences where \( J(d) = o(M) \) (Theorem 10). This result is a generalisation of [8] but works for a much larger family of degree sequences. For Theorem 12 we will use switchings to prove that the set of edges incident with \( H \) spans a subgraph with \( O(1) \) components. Then, we expose the set of edges incident with \( H \), and then analyse the subgraph induced by \([n] \setminus H\). As the degrees of vertices in \([n] \setminus H\) are not too large, we can apply Corollary [5] again.

### 4.2 Proof of Theorem 10(b,c)

Let \( Y \) denote the number of isolated edges, i.e. edges whose ends are both of degree 1. Let \( Z \) be the number of isolated triangles, i.e., triples of vertices \( \{x, y, z\} \) which induce a \( K_3 \) and all of the three vertices are of degree 2. With standard first and second moment calculations using the asymptotic probabilities in Corollary 5 we immediately have the following lemma, whose proof we omit.

**Lemma 15.**
- If \( n_1 = \Omega(\sqrt{M}) \) then \( \mathbb{E}Y \sim n_1^2/2M \) and \( \mathbb{E}Y(Y - 1) \sim n_1^4/4M^2 \).
- If \( n_2 = \Omega(M) \) then \( \mathbb{E}Z \sim 4n_2^3/3M^3 \) and \( \mathbb{E}Z(Z - 1) \sim 16n_2^6/9M^6 \).

Now Theorem 10(b) follows by Chebyshev’s inequality, and Theorem 10(c) follows by the Paley-Zygmund inequality.

### 4.3 Proof of Theorem 10(a)

**Proposition 16.** For any \( d \) with even sum,

\[
|G(n, d)| \leq \frac{M!}{2^{M/2}(M/2)!\prod_{i \in [n] d_i!}}.
\]

**Proof.** Represent vertex \( i \) by a bin containing exactly \( d_i \) points. A perfect matching over the total \( M \) points in the \( n \) bins is called a *pairing*. A pairing produces a multigraph with degree sequence \( d \) by representing each \( \{v(p), v(q)\} \) as an edge where \( p \) and \( q \) are points matched by the pairing and \( v(p) \) and \( v(q) \) are the bins/vertices that contain points \( p \) and \( q \) respectively. It is easy to see that every graph in \( G(n, d) \) corresponds to exactly \( \prod_{i \in [n]} d_i! \) pairings. On the other hand, there are exactly \( M!/2^{M/2}(M/2)! \) perfect matchings over \( M \) points. The assertion follows.

Given \( S \subseteq [n] \) let \( X_S \) denote the indicator variable for the event that \( e(S, \overline{S}) = 0 \). Recall that \( d(S) = \sum_{i \in S} d_i \) and \( d|S = (d_i)_{i \in S} \).

**Lemma 17.** Assume \( J(d) = o(M) \). Suppose \( S \subseteq [n] \) where \( M - d(S) = \Omega(M) \). Then,

\[
\mathbb{E}X_S \leq (\sqrt{2} + o(1)) \left( \frac{1 + o(1)d(S)}{M - d(S)} \right)^{d(S)/2} \left( 1 - \frac{d(S)}{M} \right)^{M/2}.
\]
Lemma 18. A.a.s. there are no nonempty sets \( S \subseteq [n] \) where \( d(S) \leq M/2 \) and \( e(S, \overline{S}) = 0 \).

**Proof.** Let \( \epsilon = 0.5 \). Suppose \( S \) is a set of vertices with \( d(S) = h \leq M/2 \) and \( d(S) \geq (2 + \epsilon)|S| \). Then, \(|S| \leq h/(2 + \epsilon)\). Hence, by Lemma 17, for all sufficiently large \( n \),

\[
\sum_{S \subseteq [n]: \ d(S)=h \ \text{and} \ h \geq (2+\epsilon)|S|} \mathbb{E} X_S \leq 2 \left( (\frac{n}{h/(2 + \epsilon)}) \left( \frac{1_{\{h/2h < \frac{\epsilon}{2}\}} + 2^{n1_{\{h/2h \geq \frac{\epsilon}{2}\}}} \right) \left( \frac{1 + o(1)}{1 - \rho} \right) \right)^{h/2} (1 - \rho)^{M/2},
\]

where \( \rho = h/M < 1/2 \). By the assumption that \( n_1 = o(\sqrt{M}) \) and \( n_2 = o(M) \), we must have \( M \geq (3 - o(1))n \). Thus the above is at most

\[
2 \left( 2e(1 + \epsilon)/3 \right)^{2/(2+\epsilon)} \frac{(\epsilon/2+\epsilon)}{1 - \rho} (1 - \rho)^{M/2} + 2(2^{2/2-\epsilon/2}\rho/(1 - \rho))^{h/2}(1 - \rho)^{M/2}.
\]

We prove that

\[
\sum_{2 \leq h \leq M/2} \left( 2e(1 + \epsilon)/3 \right)^{2/(2+\epsilon)} \frac{(\epsilon/2+\epsilon)}{1 - \rho} (1 - \rho)^{M/2} = o(1),
\]

and

\[
\sum_{2 \leq h \leq M/2} (2^{2-\epsilon/2}\rho/(1 - \rho))^{h/2}(1 - \rho)^{M/2} = o(1),
\]

which will complete the proof of the lemma. Note that

\[
\sum_{2 \leq h \leq M/2} \left( 2e(1 + \epsilon)/3 \right)^{2/(2+\epsilon)} \frac{(\epsilon/2+\epsilon)}{1 - \rho} (1 - \rho)^{M/2}
= \sum_{2 \leq h \leq \ln n} \left( O(1)(\ln n/M)^{\epsilon/(2+\epsilon)} \right)^{h/2} + \sum_{\ln n \leq h < 0.01M} 0.9^{h/2} + \sum_{0.01M \leq h \leq M/2} \exp \left( (\rho)M/2 \right),
\]

Proof. By Proposition 16, the number of graphs on \( S \) with degree sequence \( d|_S \) is at most

\[
\frac{d(S)!}{2^{d(S)/2}(d(S)/2)!} \prod_{i \in S} d_i!
\]

By (1),

\[
\mathbb{E} X_S \leq \frac{d(S)!}{2^{d(S)/2}(d(S)/2)!} \prod_{i \in S} d_i! \prod_{i = 1}^{d(S)/2} 1 + o(1)
\]

\[
\leq (\sqrt{2} + o(1)) \frac{d(S)!}{2^{d(S)/2}(d(S)/2)!} \cdot (2 + o(1))^{-d(S)/2} ((M - d(S))/2)! (M/2)!
\]

\[
\leq (\sqrt{2} + o(1)) \left( \frac{1 + o(1)d(S)}{M - d(S)} \right)^{d(S)/2} \left( 1 - \frac{d(S)}{M} \right)^{M/2}.
\]

If \( G(n, d) \) is disconnected then there is a component of \( G(n, d) \) with total degree at most \( M/2 \). Thus, it is sufficient to show that a.a.s. there is no \( S \subseteq [n] \) where \( d(S) \leq M/2 \) and \( e(S, \overline{S}) = 0 \). In the next lemma, we first bound the expected number of such \( S \) where \( d(S) \geq 2.5|S| \).

Lemma 18. A.a.s. there are no nonempty sets \( S \subseteq [n] \) where \( 2.5|S| \leq d(S) \leq M/2 \) and \( e(S, \overline{S}) = 0 \).
Lemma 17, the probability that $e$ take different values at different places where, the actual values of the constants do not matter. By $C$ which implies that $Hence, the expected number of sets $S$ follows immediately that $\sum_{2h \leq M/2} \left(2e(1 + \epsilon)/3\right)^{2/(2+\epsilon)} \rho^h/\left(2+\epsilon\right) h^{h/2} \left(1 - \rho\right)^{M/2} = o(1)$.

Bounding $\sum_{2h \leq M/2}(2^{2-\epsilon/2} \rho/(1 - \rho))^{h/2}(1 - \rho)^{M/2}$ by $o(1)$ can be done in a similar manner.

Proof of Theorem 10(a). By Lemma 18 it only remains to show that a.a.s. there are no $1/\epsilon \leq 2 \leq 3 \leq 2M/\rho$ vertices of degree at least 3, $d(S) \leq M/2$ and $d(S) \leq 2.5|S|$. The number of ways to choose such a set $S$ is at most $(\binom{n}{\ell_1})^2 (\binom{n}{\ell_2})^{\ell_2}$. Given such an $S$, $d(S) \geq \ell_1 + 2\ell_2 + 3\ell_3$. It follows immediately that

$$\ell_1 + 2\ell_2 + 3\ell_3 \leq 2.5(\ell_1 + \ell_2 + \ell_3),$$

which implies that

$$\ell_3 \leq 3\ell_1 + \ell_2, \quad \ell_1 + 2\ell_2 + 3\ell_3 \leq d(S) \leq 2.5(\ell_1 + \ell_2 + \ell_3) \leq 10\ell_1 + 5\ell_2. \quad (5)$$

In the rest of the proof, for simplicity we use $C$ to denote an absolute positive constant, which may take different values at different places where, the actual values of the constants do not matter. By Lemma 17 the probability that $e(S, S) = 0$ is at most

$$\left(\frac{Cd(S)}{M}\right)^{d(S)/2} \leq \left(\frac{C(\ell_1 + \ell_2)}{M}\right)^{\ell_1/2 + \ell_2 + 3\ell_3}. \frac{\ell_1}{2+\ell_2+3\ell_3/2}$$

Hence, the expected number of sets $S$ where $d(S) \leq 2.5|S|$, $d(S) \leq M/2$ and $e(S, S) = 0$ is at most

$$\sum_{\ell_1, \ell_2, \ell_3 \geq 3} \left(\frac{Cn_1}{\ell_1} \frac{\sqrt{\ell_1 + \ell_2}}{\ell_1 M}\right)^{\ell_1} \left(\frac{Cn_2(\ell_1 + \ell_2)}{\ell_2 M}\right)^{\ell_2} \left(\frac{Cn(\ell_1 + \ell_2)^{3/2}}{\ell_3^{3/2} M^{3/2}}\right)^{\ell_3} \leq \sum_{\ell_1, \ell_2, \ell_3 \geq 3} \left(\frac{C\xi}{\ell_1} \frac{\sqrt{\ell_1 + \ell_2}}{\ell_1 M}\right)^{\ell_1} \left(\frac{C\xi}{\ell_2} \frac{\sqrt{\ell_1 + \ell_2}}{\ell_2 M}\right)^{\ell_2} \left(\frac{C(\ell_1 + \ell_2)^{3/2}}{\ell_3^{3/2} M^{3/2}}\right)^{\ell_3}.$$

We split the above sum into two parts, one restricted to $\ell_1 \geq \ell_2$ and the other restricted to $\ell_1 < \ell_2$, and we show that each sum is $o(1)$. Suppose $\ell_1 \geq \ell_2$. Then $\ell_1 + \ell_2 \leq 2\ell_1$. Hence,

$$\sum_{\ell_1, \ell_2, \ell_3 \geq 3} \left(\frac{C\xi}{\ell_1} \frac{\sqrt{\ell_1 + \ell_2}}{\ell_1 M}\right)^{\ell_1} \left(\frac{C\xi}{\ell_2} \frac{\sqrt{\ell_1 + \ell_2}}{\ell_2 M}\right)^{\ell_2} \left(\frac{C(\ell_1 + \ell_2)^{3/2}}{\ell_3^{3/2} M^{3/2}}\right)^{\ell_3} \leq \sum_{\ell_1, \ell_2, \ell_3 \geq 3} \left(\frac{C\xi}{\ell_1} \frac{\sqrt{\ell_1 + \ell_2}}{\ell_1 M}\right)^{\ell_1} \left(\frac{C\xi}{\ell_2} \frac{\sqrt{\ell_1 + \ell_2}}{\ell_2 M}\right)^{\ell_2} \left(\frac{C\xi}{\ell_3^{3/2} M^{3/2}}\right)^{\ell_3}.$$
Let \( g(x) = (K/x)^x \) on \( x \geq 0 \) where \( K > 0 \). By considering the derivative of \( \ln(g(x)) \), it is easy to see that \( g(x) \) is maximised at \( x = K/e \). Thus,

\[
\left( C \frac{\ell_1}{\ell_2} \right)^{\ell_2} \leq \exp \left( \frac{C \xi \ell_1}{e} \right), \quad \text{and} \quad \left( \frac{C \ell_1^{3/2}}{\ell_{\geq 3} \sqrt{M}} \right)^{\ell_{\geq 3}} \leq \exp \left( \frac{C \ell_1^{3/2}}{e \sqrt{M}} \right).
\]

It follows now that

\[
\sum_{\ell_1, \ell_2, \ell_{\geq 3}} \left( \frac{C \xi}{\sqrt{\ell_1}} \right)^{\ell_1} \left( \frac{C \xi}{\ell_2} \right)^{\ell_2} \left( \frac{C \ell_1^{3/2}}{\ell_{\geq 3} \sqrt{M}} \right)^{\ell_{\geq 3}} = \sum_{\ell_1, \ell_2, \ell_{\geq 3}} \left( \frac{C \xi}{\sqrt{\ell_1}} \right)^{\ell_1} \left( \frac{C \xi}{\ell_2} \right)^{\ell_2} \left( \frac{C \ell_1^{3/2}}{\ell_{\geq 3} \sqrt{M}} \right)^{\ell_{\geq 3}}
\]

\[
+ \sum_{M^{1/3}/\log M < \ell_1 \leq n_1} \left( \frac{C \xi}{\sqrt{\ell_1}} \right)^{\ell_1} \cdot \ell_1 \exp \left( \frac{C \xi \ell_1}{e} \right) \cdot \sum_{\ell_{\geq 3}} \left( \frac{1}{\ell_{\geq 3} \log M} \right)^{\ell_{\geq 3}}
\]

\[
+ \sum_{M^{1/3}/\log M < \ell_1 \leq n_1} \left( \frac{C \xi}{\sqrt{\ell_1}} \right)^{\ell_1} \cdot \ell_1 \exp \left( \frac{C \xi \ell_1}{e} \right) \cdot n \exp \left( \frac{C \ell_1^{3/2}}{e \sqrt{M}} \right)
\]

\[
\leq O \left( \frac{1}{\log M} \right) \sum_{\ell_1 \leq M^{1/3}/\log M} \ell_1 \left( \frac{C \xi}{\sqrt{\ell_1}} \right)^{\ell_1} e^{C \xi/e + C \sqrt{\ell_1/M}} + \sum_{M^{1/3}/\log M < \ell_1 \leq n_1} \ell_1 n \left( \frac{C \xi}{\sqrt{\ell_1}} \right)^{\ell_1}
\]

\[
= o(1),
\]

as \( n_1 = o(\sqrt{M}) \).

Similarly, we have

\[
\sum_{\ell_1, \ell_2, \ell_{\geq 3}} \left( \frac{C \xi \sqrt{\ell_1} + \ell_2}{\ell_1} \right)^{\ell_1} \left( \frac{C \xi}{\ell_2} \right)^{\ell_2} \left( \frac{C (\ell_1 + \ell_2)^{3/2}}{\ell_{\geq 3} \sqrt{M}} \right)^{\ell_{\geq 3}}
\]

\[
\leq \sum_{\ell_1, \ell_2, \ell_{\geq 3}} \left( \frac{C \xi \sqrt{\ell_2}}{\ell_1} \right)^{\ell_1} \left( \frac{C \xi}{\ell_2} \right)^{\ell_2} \left( \frac{C \ell_2^{3/2}}{\ell_{\geq 3} \sqrt{M}} \right)^{\ell_{\geq 3}}
\]

\[
\leq \sum_{\ell_2 \leq M^{1/3}/\log M} \ell_2 (C \xi e^{C \xi/\sqrt{\ell_2}})^{\ell_2} \sum_{\ell_{\geq 3}} \left( \frac{1}{\ell_{\geq 3} \log M} \right)^{\ell_{\geq 3}} + \sum_{M^{1/3}/\log M < \ell_2 \leq n_2} \ell_2 n (C \xi e^{C \xi/\sqrt{\ell_2} + C \sqrt{\ell_2/M}})^{\ell_2}
\]

\[
= o(1),
\]

as \( n_2 = o(M) \). By Markov’s inequality, a.a.s. there are no sets \( S \subseteq [n] \) where \( d(S) \leq 2.5|S| \), \( d(S) \leq M/2 \) and \( e(S, S') = 0 \). This, together with Lemma \( \text{18} \) completes the proof for Theorem \( \text{10(a)} \). ■

5 Proof of Theorem \[12\]

We start by some structural result involving vertices in \( \mathcal{H} \).
Lemma 19. Suppose $d$ is a degree sequence satisfying either of the following two conditions.

- $|H| = \omega(1)$.
- $\mathcal{H} = O(1)$ and $d(\mathcal{H}) \geq M^{2/3}$.

A.a.s. all vertices in $\mathcal{H}$ are contained in the same component of $\mathcal{G}(n, d)$.

**Remark.** This lemma does not assume $M - 2d(\mathcal{H}) = \Omega(M)$. It may be useful for studying the connectivity of $\mathcal{G}(n, d)$ where $d(\mathcal{H}) \geq (1/2 - o(1))M$. However we do not attempt that in this paper.

**Proof of Lemma 19.** The case $|H| \geq \log^7 M$ follows by [16] Lemma 28. Assume $|H| < \log^7 M$ and $|H| = \omega(1)$. Our proof considers two cases depending on if $d(H)$ is at most $M/16$. When $d(\mathcal{H}) \leq M/16$ we can apply the following claim.

**Claim 20.** Suppose $M - 2d(\mathcal{H}) = \Omega(M)$. Let $u, v \in \mathcal{H}$. Then, for any two disjoint graphs $H_1$ and $H_2$ on $\mathcal{H}$, and $uv \notin H_1 \cup H_2$,

$$\mathbb{P}(uv \in \mathcal{G}(n, d) \mid H_1^+, H_2^+) = \Omega\left(\frac{1}{\log^2 M}\right).$$

Let $\mathcal{G}(\mathcal{H}, p)$ be the binomial random graph on vertex set $\mathcal{H}$ where each pair of vertices are adjacent independently with probability $p$, and let $G[\mathcal{H}]$ be the subgraph of $\mathcal{G}(n, d)$ induced by $\mathcal{H}$. By Claim 20, we can couple $G[\mathcal{H}]$ with $\mathcal{G}(\mathcal{H}, c/\log^2 M)$ such that $G[\mathcal{H}]$ contains $\mathcal{G}(\mathcal{H}, c/\log^2 M)$ as a subgraph for some constant $c > 0$. As $|\mathcal{H}| = \omega(1)$ and $|\mathcal{H}| < \log^7 M$, we know a.a.s. $\mathcal{G}(\mathcal{H}, c/\log^2 M)$ is connected (the connectivity threshold for $\mathcal{G}(n, d)$ is at $p = \ln(|\mathcal{H}|)/|\mathcal{H}|$). Hence, a.a.s. $G[\mathcal{H}]$ is connected.

Next consider the case where $|\mathcal{H}| = \omega(1)$ and $d(\mathcal{H}) > M/16$. Let $\mathcal{H}^+ = \{i : d_i \geq M^{2/3}/\log M\}$. Obviously $\mathcal{H}^+$ is nonempty as $|\mathcal{H}| < \log^7 M$ and $d(\mathcal{H}) > M/16$ but all vertices in $\mathcal{H} \setminus \mathcal{H}^+$ have degree less than $M^{2/3}/\log M$. The assertion of the lemma in this case follows from the following claim.

**Claim 21.** Suppose $\mathcal{H}^+ \neq \emptyset$. Then a.a.s. $v$ is adjacent to every vertex in $\mathcal{H}$, for every $v \in \mathcal{H}^+$.

Finally, if $|\mathcal{H}| = O(1)$ and $d(\mathcal{H}) \geq M^{2/3}$ then $\mathcal{H}^+$ is nonempty and thus our assertion in this case also follows by Claim 21. This completes the proof of Lemma 19.

It only remains to prove the two claims above.

**Proof of Claim 20.** Let $W$ be the set of graphs $G$ in $\mathcal{G}(n, d)$ where $H_1 \subseteq G$, $H_2 \cap G = \emptyset$ and $uv \notin G$. Let $W'$ be the set of graphs $G$ on $\mathcal{G}(n, d)$ where $H_1 \subseteq G$, $H_2 \cap G = \emptyset$ and $uv \in G$. Then,

$$\mathbb{P}(uv \in \mathcal{G}[\mathcal{H}] \mid H_1^+, H_2^+) = \frac{|W'|}{|W| + |W'|}.$$

We estimate the ratio $|W'|/|W|$ using switchings as follows. Given $G \in W$, a forward switching identifies an ordered 4-tuple of vertices $(x_1, x_2, x_3, x_4)$, all of which from $[n] \setminus \mathcal{H}$ satisfying the following conditions: (a), all six vertices $u, v$ and $x_i, 1 \leq i \leq 4$ are distinct expect that $x_4 = x_1$ is permitted; (b), $ux_1, x_2x_3$ and $vx_4$ are edges in $G$; (c) $x_1x_2$ and $x_3x_4$ are non-edges in $G$. The switching then replaces $ux_1, vx_4$ and $x_2x_3$ by $uv, x_1x_2$ and $x_3x_4$. The resulting graph $G'$ is in $W'$. Given $G$, there are at least $d_4 - |\mathcal{H}|$ and $d_4 - |\mathcal{H}|$ ways to choose $x_1$ and $x_4$ respectively. Then, there are at least $M - 2d(\mathcal{H}) - 2(\sqrt{M}/\log M)^2$ ways to choose $x_2$ and $x_3$ so that both vertices are
in \([n] \setminus \mathcal{H}\) and none of \(x_1x_2\) and \(x_3x_4\) are edges in \(G\). Thus, the total number of forward switchings that can be applied to \(G\) is at least

\[
(d_u - |\mathcal{H}|)(d_v - |\mathcal{H}|)(M - 2d(\mathcal{H}) - 2M/\log^2 M) = \Omega(M^2/\log^2 M),
\]
as \(d_u, d_v \geq \sqrt{M}/\log M\), \(|\mathcal{H}| < \log^7 M\) and \(M - 2d(\mathcal{H}) = \Omega(M)\). On the other hand, the number of ways to perform a backward switching to any \(G' \in W'\) is at most \(M^2\) (at most \(M\) ways to choose \(x_1\) and \(x_2\) and then at most \(M\) ways to choose \(x_3\) and \(x_4\)). Thus, \(|W'|/|W| \geq \Omega(M^2/\log^2 M)/M^2 = \Omega(1/\log^2 M)\). It follows now that \(\mathbb{P}(uv \in G[\mathcal{H}] \mid H_1^+, H_2^-) = \Omega(1/\log^2 M)\). \(\blacksquare\)

Proof of Claim 21. Let \(v \in \mathcal{H}^+\) and \(u \in \mathcal{H}\). We will show that \(\mathbb{P}(uv \notin \mathcal{G}(n, \mathbf{d})) = o(\log^{-14} M)\) and our assertion of the claim follows by taking the union bound over all pairs of \((v, u)\) where \(v \in \mathcal{H}^+\) and \(u \in \mathcal{H}\).

Let \(W\) be the set of graphs in \(\mathcal{G}(n, \mathbf{d})\) where \(u\) is not adjacent to \(v\) and \(W' = \mathcal{G}(n, \mathbf{d}) \setminus W\). Given \(G \in W\), a forward switching on \(G\) specifies a pair of vertices \((x, y)\) satisfying the following conditions: (a) \(x\) and \(y\) are both in \([n] \setminus \mathcal{H}\); (b) \(ux_1\) and \(vx_2\) are edges in \(G\); (c) \(xy\) is not an edge in \(G\). The switching replaces edges \(ux\) and \(vy\) by \(uv\) and \(xy\). The resulting graph \(G'\) is in \(W'\).

Given \(G\), there are at least \(d_u - |\mathcal{H}|\) and \(d_v - |\mathcal{H}|\) ways to choose vertices \(x\) and \(y\) respectively. Among such choices, at most \(d_u\sqrt{M}/\log M\) are such that \(xy\) is an edge (if \(xy\) is an edge then \(uxy\) is a 2-path and there are at most \(d_u\sqrt{M}/\log M\) such 2-paths where \(x \in [n] \setminus \mathcal{H}\)). Thus, the number of forward switchings that can be applied to \(G\) is at least

\[
(d_u - |\mathcal{H}|)(d_v - |\mathcal{H}|) - d_u\sqrt{M}/\log M = (1 + o(1))d_u d_v - d_u\sqrt{M}/\log M = (1 + o(1))d_u d_v,
\]
as \(d_v \geq M^{2/3}/\log M\), \(d_u \geq \sqrt{M}/\log M\) and \(|\mathcal{H}| < \log^7 M\).

On the other hand, the number of ways a graph \(G'\) can be created by a forward switching (that is, the number of backward switchings that can be applied to \(G'\)) is at most \(M\). Thus, \(|W'|/|W| \leq (1 + o(1))M/d_u d_v\), and therefore,

\[
\mathbb{P}(uv \notin \mathcal{G}(n, \mathbf{d})) = \frac{|W|}{|W| + |W'|} = (1 + o(1))M/d_u d_v = O(\log^2 M/M^{1/6}),
\]
as \(d_v \geq M^{2/3}/\log M\) and \(d_u \geq \sqrt{M}/\log M\). This completes the proof of the claim. \(\blacksquare\)

Now we are ready to prove the second main theorem about the connectivity of \(\mathcal{G}(n, \mathbf{d})\).

Proof of Theorem 13. Expose the set of edges incident with \(\mathcal{H}\) and let \(\mathbf{d}' = (d_i)_{i \in [n] \setminus \mathcal{H}}\) denote the remaining degree sequence for vertices in \([n] \setminus \mathcal{H}\). That is, \(d'_i = d_i - x_i\) where \(x_i\) is the number of edges between \(i\) and \(\mathcal{H}\). In this proof we will focus on the subgraph of \(\mathcal{G}(n, \mathbf{d})\) induced by \([n] \setminus \mathcal{H}\). Conditioning on \(\mathbf{d}'\), this subgraph is distributed as \(\mathcal{G}([n] \setminus \mathcal{H}, \mathbf{d}')\), a uniformly random graph on \([n] \setminus \mathcal{H}\) with degree sequence \(\mathbf{d}'\). Note that some vertices in \([n] \setminus \mathcal{H}\) may have degree 0 with respect to \(\mathbf{d}'\). They are not of interest for study as they are just isolated vertices in \(\mathcal{G}([n] \setminus \mathcal{H}, \mathbf{d}')\), and they are known to be adjacent to some vertex in \(\mathcal{H}\). Hence, let \(V'\) be the set of vertices \(v \in [n] \setminus \mathcal{H}\) where \(d'_v \geq 1\). Let \(M' = \sum_{i \in V'} d'_i\). Conditioning on \(V'\) and \(\mathbf{d}'\), the subgraph of \(\mathcal{G}(n, \mathbf{d})\) induced by \(V'\) is distributed as \(\mathcal{G}(V', \mathbf{d}'|_{V'})\). By the theorem hypothesis that \(M - 2d(\mathcal{H}) = \Omega(M)\), it follows that \(M' \geq M - 2d(\mathcal{H}) = \Omega(M)\). Hence, by the definition of \(\mathcal{H}\) we have

\[
d_i' \leq \sqrt{M}/\log M = O(\sqrt{M'/\log M'}), \quad \text{for all} \ i \in V'.
\]

Claim 22. A.a.s. there exists \(v \in [n] \setminus \mathcal{H}\) such that \(d_v \geq 2\) and \(v\) is adjacent to \(\mathcal{H}\).
Given set $S \subseteq [n]$, we say $v$ is adjacent to $S$ if there exists $u \in S$ which is adjacent to $v$. Next, we colour the vertices in $[n] \setminus \mathcal{H}$ that are adjacent to $\mathcal{H}$ as follows. If $|\mathcal{H}| = \omega(1)$, or if $|\mathcal{H}| = O(1)$ and $d(\mathcal{H}) \geq M^{2/3}$ then let $U$ be the set of vertices $v \in [n] \setminus \mathcal{H}$ where $v$ is adjacent to some vertex in $\mathcal{H}$. Colour all vertices in $U$ with colour 1 and let $V_1 = U \cap V'$. I.e. $V_1$ is the set of vertices in $V_1$ with degree at least 1 with respect to $d'$. If $|\mathcal{H}| = O(1)$ and $d(\mathcal{H}) < M^{2/3}$, then the subgraph $H$ induced by the set of edges incident with $\mathcal{H}$ has $O(1)$ components. Let $C_1, \ldots, C_k$ be an enumeration of these components. Colour all vertices in $V(C_1) \setminus \mathcal{H}$ with colour $i$ and let $V_i = V(C_1) \cap V'$ for $i \in [k]$. I.e. $V_i$ is the set of vertices in $V(C_1) \setminus \mathcal{H}$ with $d'_i \geq 1$. Combining both cases and by Claim 22 we have some $1 \leq k = O(1)$ where $V'$ is partitioned to at most $k + 1$ parts. The vertices in the first $k$ parts $V_1, \ldots, V_k$ are coloured with colours 1, 2, \ldots, $k$ respectively, and the vertices in the last part are uncoloured. By Lemma 19, we may assume that all monochromatic vertices are contained in the same component of $G(n, d)$. Suppose $V_i \neq \emptyset$ for every $i \in [k]$. Then, the connectivity of $G(n, d)$ is implied if we can prove that there is no partition of $V'$ into $S$ and $T$ where $e(S, T) = 0$, and no colour $i$ such that both $S \cap V_i \neq \emptyset$ and $T \cap V_i \neq \emptyset$. However this implication is not true if there exists $i$ where $V_i = \emptyset$, as the set of vertices in $[n] \setminus \mathcal{H}$ coloured $i$ (they are all isolated vertices with respect to $d'$) together with their neighbours in $\mathcal{H}$ may lie in a distinct component from the vertices of other colours, and the uncoloured vertices. The next claim excludes such a possibility.

**Claim 23.** A.a.s. for every $i \in [k]$, if there is some vertex $u$ coloured $i$ and $d'_u = 0$ then $V_i \neq \emptyset$.

Therefore we may assume that $V_i \neq \emptyset$ for all $i \in [k]$. In the rest of the proof, we will focus on $G(V', d'_i|_{V'})$, and we call $d'_i$ the degree of $v$ for $v \in V'$. When we use graph notation such as $e(U, V)$ and $d(U)$, the graph referred to is $G(V', d'_i|_{V'})$ unless otherwise specified. By construction, all vertices in $V'$ has degree at least 1. Moreover, by the theorem hypothesis on $n_1$ and $n_2$ and by the facts that $n'_1 \leq n_1$, $n'_2 \leq n_2$ and $M' = \Omega(M)$, where $n'_1$ is the number of uncoloured vertices of degree 1 in $V'$, and $n'_2$ is the number of uncoloured vertices of degree 2 in $V'$, it follows that

$$n'_1 = o(\sqrt{M'}), \quad n'_2 = o(M').$$

Now, as argued above, the connectivity of $G(n, d)$ immediately follows from the following two claims.

**Claim 24.** A.a.s. there is no $S \subset V' \setminus (\cup_{i \in [k]} V_i)$ where $d(S) \leq M'/2$ and $e(S, V' \setminus S) = 0$.

**Claim 25.** A.a.s. for every $I \subseteq [k]$, there exists no $T \subseteq [n] \setminus \mathcal{H}$ where $\cup_{i \in I} V_i \subseteq T$, $T \cap (\cup_{i \in [k] \setminus I} V_i) = \emptyset$, $d(T) \leq M'/2$ and $e(T, V' \setminus T) = 0$.

Now Theorem 12 follows.

The proofs of Claims 24 and 25 are analogous to the proof of Theorem 10. We briefly sketch the arguments.

**Proof of Claim 24.** By (1), $n'_1 \leq \xi \sqrt{M'}$, and $n'_2 \leq \xi M'$, for some $\xi = o(1)$. Moreover, by (6), joint probabilities in (1) can be applied to $G(V', d'_i|_{V'})$. The rest of the proof is identical to that of Lemma 18 and Theorem 10 noting that $S$ contains only uncoloured vertices.

**Proof of Claim 25.** We fix $I$, which fixes $V_i := \cup_{i \in I} V_i$. Let $D$ denote the total degree of $V_i$, i.e. $D = \sum_{u \in V_i} d'_u$. Next, given a vector $\ell_1, \ell_2, \ldots$, the number of ways to choose $T$ where $\cup_{i \in I} V_i \subseteq T$, $T \cap (\cup_{i \in [k] \setminus I} V_i) = \emptyset$, and there are $\ell_i$ uncoloured vertices of degree $i$ in $T$ is at most

$$\left(\frac{\xi \sqrt{M'}}{\ell_1}\right) \left(\frac{\xi M'}{\ell_2}\right) \left(\frac{n'_i}{\ell_{\geq 3}}\right),$$
where \( \ell_{\geq 3} = \sum_{i \geq 3} \ell_i \). By Lemma 18 we may assume that

\[
d(T) = D + \sum_{i \geq 1} i\ell_i < 2.5 \left( \sum_{i \geq 1} \ell_i + D \right).
\]

The probability that \( e(T, V' \setminus T) = 0 \) is at most

\[
\left( \frac{d(T)}{M' - d(T)} \right)^{d(T)/2} \left( 1 - \frac{d(T)}{M'} \right)^{(M' - d(T))/2}
\]

\[
= \left( \frac{D + \sum_{i \geq 1} i\ell_i}{M' - (D + \sum_{i \geq 1} i\ell_i)} \right)^{(D + \sum_{i \geq 1} i\ell_i)/2} \left( 1 - \frac{D + \sum_{i \geq 1} i\ell_i}{M'} \right)^{(M' - (D + \sum_{i \geq 1} i\ell_i))/2}.
\]

Given \( \ell_1, \ell_2, \ldots \), the above function is monotonically decreasing on \( D \) on the domain where \( (D + \sum_{i \geq 1} i\ell_i)/M' \leq 1/2 \). Hence, the above probability is maximised at \( D = 0 \). Hence, the probability of existing such a set \( T \), given \( I \), and \( \ell_1, \ell_2, \ldots \) is at most

\[
\left( \frac{\xi \sqrt{M'}}{\ell_1} \right) \left( \frac{\xi M'}{\ell_2} \right) \left( \frac{n'}{\ell_{\geq 3}} \right) \left( \frac{\sum_{i \geq 1} i\ell_i}{M' - \sum_{i \geq 1} i\ell_i} \right)^{\sum_{i \geq 1} i\ell_i/2} \left( 1 - \frac{\sum_{i \geq 1} i\ell_i}{M'} \right)^{(M' - \sum_{i \geq 1} i\ell_i)/2}
\]

where \( \sum_{i \geq 1} i\ell_i < 2.5 \sum_{i \geq 1} \ell_i \) implying

\[
\ell_{\geq 3} \leq 3\ell_1 + \ell_2, \quad \ell_1 + 2\ell_2 + 3\ell_{\geq 3} \leq 2.5(\ell_1 + \ell_2 + \ell_{\geq 3}) \leq 10\ell_1 + 5\ell_2.
\]

The rest of the analysis is the same as in Theorem 10.

Now we prove Claims 22 and 23. Both claims concern events related to edges incident with vertices in \( \mathcal{H} \). We will use switchings to bound probabilities of such events.

**Proof of Claim 22** Let \( \mathcal{E} \) denote the event that \( e(\mathcal{H}, [n] \setminus \mathcal{H}) > 0 \) and all edges between \( \mathcal{H} \) and \( [n] \setminus \mathcal{H} \) has one end whose degree equals 1 in \( G(n, d) \). If there is no vertex \( v \in [n] \setminus \mathcal{H} \) where \( d_v \geq 2 \) and \( v \) is adjacent to \( \mathcal{H} \), then we must have either \( \mathcal{E} \) or \( e(\mathcal{H}, [n] \setminus \mathcal{H}) = 0 \). It is then sufficient to show that \( \mathbb{P}(\mathcal{E}) = o(1) \) and \( \mathbb{P}(e(\mathcal{H}, [n] \setminus \mathcal{H}) = 0) = o(1) \). Let \( \mathcal{G} \) be the class of graphs in \( G(n, d) \) where \( e(\mathcal{H}, [n] \setminus \mathcal{H}) = 0 \) and let \( \mathcal{G}' \) be the class of graphs in \( G(n, d) \) where \( e(\mathcal{H}, [n] \setminus \mathcal{H}) = 2 \). If \( \mathcal{G} = \emptyset \) then \( \mathbb{P}(e(\mathcal{H}, [n] \setminus \mathcal{H}) = 0) = 0 \). Otherwise,

\[
\mathbb{P}(e(\mathcal{H}, [n] \setminus \mathcal{H}) = 0) \leq \frac{|\mathcal{G}|}{|\mathcal{G}'|}.
\]

We define a switching from \( G \in \mathcal{G} \) by choosing an edge \( xy \) in \( G|_{[\mathcal{H}]} \) and another edge \( uv \) in \( G|_{[n] \setminus \mathcal{H}]}. \) Replace these two edges by \( xu \) and \( yv \). The resulting graph \( G' \) is in \( \mathcal{G}' \). There are \( d(\mathcal{H})/2 \) = \( \Omega(\sqrt{M}/ \log M) \) ways to choose \( xy \) and \( d([n] \setminus \mathcal{H}) = \Omega(M) \) ways to choose \( uv \). So the number of switchings applicable on \( G \) is at least \( \Omega(M^{3/2}/ \log M) \). On the other hand, for every \( \mathcal{G}' \), it can be produced by at most 1 way, as there are exactly 2 edges between \( \mathcal{H} \) and \( [n] \setminus \mathcal{H} \). It follows then that

\[
\mathbb{P}(e(\mathcal{H}, [n] \setminus \mathcal{H}) = 0) \leq \frac{|\mathcal{G}|}{|\mathcal{G}'|} = O(\log M/M^{3/2})
\]
as desired.
Next, let $\mathcal{G}$ be the class of graphs in $G(n,d) \cap \mathcal{E}$, and let $\mathcal{G}'$ be the class of graphs in $G(n,d)$ where there is exactly one neighbour of $\mathcal{H}$ in $[n] \setminus \mathcal{H}$ with degree at least 2, and all the other neighbours of $\mathcal{H}$ in $[n] \setminus \mathcal{H}$ have degree equal to 1. Define a switching from $\mathcal{G}$ to $\mathcal{G}'$ as follows. Given $G \in \mathcal{G}$, choose 4 vertices $(u,v,x,y)$ such that $u \in \mathcal{H}$, $v,x,y \notin \mathcal{H}$, $uw$ and $xy$ are edges, and $d_x \geq 2$. Since $d_x > 1$ and $d_v = 1$ it follows immediately that $ux$ and $vy$ are not edges. The switching replaces $uv$ and $xy$ by $ux$ and $vy$. The resulting graph $G'$ is in $\mathcal{G}'$ since $x$ becomes a neighbour of $\mathcal{H}$ and $d_x \geq 2$. There is at least one way to choose $v$, since $G \in \mathcal{E}$. The total degree of $[n] \setminus \mathcal{H}$ is $\Omega(M)$ by the theorem assumption, and there are at most $o(\sqrt{M})$ vertices in the set whose degree is 1. Moreover, all vertices in $[n] \setminus \mathcal{H}$ has degree at most $\sqrt{M}/\log M$. Hence, there are at least $\Omega(M)/(\sqrt{M}/\log M) = \Omega(\sqrt{M} \log M)$ vertices in $[n] \setminus \mathcal{H}$ whose degree is at least 2. Hence, the number of choices for $x$ and $y$ is $\Omega(\sqrt{M} \log M)$. Thus, the number of switchings that can be applied to $G$ is $\Omega(\sqrt{M} \log M)$. On the other hand, given $G' \in \mathcal{G}$, there is a unique way to choose $u$ and $x$, and at most $n_1$ ways to choose $v$ and $y$ so that an inverse switching can be applied. Thus, $$\mathbb{P}(\mathcal{E}) = O \left( \frac{n_1}{\sqrt{M} \log M} \right) = o(1),$$ and our assertion follows.

Proof of Claim 23. We consider two cases. In the first case, we assume $|\mathcal{H}| = O(1)$ and $d(\mathcal{H}) \leq M^{2/3}$. We prove that a.a.s. for every vertex $u \in \mathcal{H}$, $u$ is adjacent to some vertex $v \in V'$. That will imply the assertion in the claim. Fix $u \in \mathcal{H}$. Let $\mathcal{G}$ be the set of graphs in $G(n,d)$ where for each neighbour $v \in [n] \setminus \mathcal{H}$ of $u$, $d'_v = 0$. Let $\mathcal{G}'$ be the set of graphs in $G(n,d)$ where for all but exactly one neighbours $v \in [n] \setminus \mathcal{H}$ of $u$, $d'_v = 0$. Define a switching from $\mathcal{G}$ to $\mathcal{G}'$ as follows. Let $G \in \mathcal{G}$. Choose a neighbour $v \in [n] \setminus \mathcal{H}$ of $u$, and then choose two vertices $(u',v')$ in $[n] \setminus \mathcal{H}$ such that $d'_v \geq 2$ and $u'v'$ is an edge. Since $d'_v = 0$ and $d'_v > 0$ we know that both $v'v$ and $u'v'$ are not edges. Then the switching replaces edges $uv$ and $vy$ by $v'y'$ and $u'v'$. The resulting graph $G'$ is in $\mathcal{G}'$ because $d'_u \geq 1$ in $G'$ and $v'$ is adjacent to $u$. Given $G$, the number of choices for $v$ is at least $\sqrt{M}/\log M$ since $d_u \geq \sqrt{M}/\log M$. By definition of $\mathcal{G}$, we know $\sum_x d'_x = 0$ where the summation is over all $x \in [n] \setminus \mathcal{H}$ that are neighbours of $u$. However, since $d(\mathcal{H}) < M^{2/3}$, $M = \sum_y [n] \setminus \mathcal{H} d'_y \geq M - M^{2/3}$. Let $n'_1$ be the number of vertices in $V'$ with degree (with respect to $d'$) one. Since every edge incident with $\mathcal{H}$ can create at most one new vertex $v$ with $d'_v = 1$, we must have $n'_1 \leq n_1 + d(\mathcal{H}) < 2M^{2/3}$. It follows that $$\sum_{x \in [n] \setminus \mathcal{H}} d'_x \geq (M - 2M^{2/3}) - n'_1 \geq (1 - o(1))M.$$ Since $d'_y \leq \sqrt{M}/\log M$ for every $y \in [n] \setminus \mathcal{H}$, the number of choices for $v'$ is at least $((1 - o(1))M)/(\sqrt{M}/\log M) = \Omega(\sqrt{M} \log M)$. Consequently, the number of ways to perform a switching to $G \in \mathcal{G}$ is at least $\Omega(\sqrt{M} \log M)$. On the other hand, for every $G' \in \mathcal{G}$, $G'$ can be created by at most $O(M^{2/3})$ different switchings, since given $G'$ there is at most one way to choose $v'$, and at most $n_1 + M^{2/3} \leq 2M^{2/3}$ ways to choose $v$, who must satisfy $d'_v = 1$. Hence, the probability that $u$ is not adjacent to any vertex in $V'$ is at most $$\frac{|\mathcal{G}|}{|\mathcal{G}'|} = O(M^{2/3}/M) = O(M^{-1/3}).$$ Our claim in this case follows by taking the union bound over the $O(1)$ vertices $u \in \mathcal{H}$.

In the second case, we assume $\mathcal{H} = \omega(1)$ or $d(\mathcal{H}) \geq M^{2/3}$. In this case, all vertices adjacent to $\mathcal{H}$ are coloured with 1. Let $\mathcal{P}$ denote the set of graphs in $G(n,d)$ with the property in Claim 22.
i.e. there exists \( v \in [n] \setminus \mathcal{H} \) where \( d_v \geq 2 \) and \( v \) is adjacent to \( \mathcal{H} \). Let \( \mathcal{G} \) denote the set of graphs in \( \mathcal{P} \) where for all \( v \in [n] \setminus \mathcal{H} \) adjacent to \( \mathcal{H} \), \( d'_v = 0 \). Let \( \mathcal{G}' \) be the set of graphs in \( \mathcal{P} \) where there are exactly two vertices \( v_1, v_2 \in [n] \setminus \mathcal{H}' \) such that \( d'_{v_1} = 1 \), \( d'_{v_2} \geq 1 \), and \( d'_v = 0 \) for all \( v \in ([n] \setminus \mathcal{H}) \setminus \{v_1, v_2\} \). Define a switching from \( \mathcal{G} \) to \( \mathcal{G}' \) as follows. Given \( G \in \mathcal{G} \), the switching chooses 4 vertices \( (u, v, x, y) \) such that \( u \in \mathcal{H} \), \( v, x, y \in [n] \setminus \mathcal{H} \), \( uv \) and \( xy \) are edges, \( d_v \geq 2 \) and \( d'_x \geq 2 \). Since \( d'_x > 0 \) and \( d'_v = 0 \), we know that \( ux \) and \( vy \) are not edges. The switching replaces edges \( uv \) and \( xy \) by \( ux \) and \( vy \). Let \( G' \) denote the resulting graph. In \( G' \), both \( x \) and \( v \) are adjacent to \( \mathcal{H} \), since \( x \) is adjacent to \( u \in \mathcal{H} \), and \( v \) is adjacent to some vertex \( u' \in \mathcal{H} \) where \( u' \neq u \), as \( d_v \geq 2 \) and \( d'_x \geq 2 \).\( \square \)

This completes the proof of the claim. \( \Box \)

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