Functoriality of Quantum Principal Bundles and Quantum Connections

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Abstract. The purpose of this work is to present a complete categorical point of view of the association between finite dimensional representations of a compact quantum group and quantum vector bundles with quantum linear connections using M. Durdevich’s theory [D1, D2]. This paper is a noncommutative version of the principal result of [SW], which extends the work presented in [D3] by considering connections as well.

MSC 2010: 46L87; 18D35.

Keywords: Quantum principal bundles and quantum principal connections, quantum association functor.

1. Introduction

In differential geometry the study of principal bundles and principal connections is one of the most important subjects [KN], for mathematics and also for physics since Yang-Mills theories have been developed in this context [B]. An special result in the theory is the fact that taking a smooth principal $G$-bundle $GM$ over a manifold $M$ one can associate a fiber bundle over $M$ for every single smooth manifold with a smooth $G$-action. This turns out into a covariant functor $Ass_{GM}$ between the category of manifolds with $G$-actions, $MF_G$, and the category of fiber bundles over $M$, $FB_M$. One of the most important contributions to the study of these kinds of functors was made by M. Nori in [N] but this paper was developed in the framework of algebraic geometry. Another contribution was developed in [SW], which is a generalization of [N] in the framework of differential geometry considering connections as well, that means, it presents a complete characterization of the (covariant) association functor

$$Ass_{GM} : MF_G \rightarrow FB^\nabla_M$$

(this last one is the category of fiber bundles over $M$ with non-linear connections) for a given principal $G$-bundle $GM$ over $M$ with a principal connection $\omega$. In particular, it shows that every covariant functor between $MF_G$ and $FB^\nabla_M$ that satisfies certain properties is naturally isomorphic to $Ass_{GM}$, for an unique (except by isomorphisms) principal bundle $GM$ over $M$ with a principal connection $\omega$. This implies a categorical equivalence between principal bundles over $M$ with principal connections and the category of gauge theory sectors on $M$ with connections, whose objects are essentially these association functors [SW].

In noncommutative differential geometry is common to find several definitions of the same concept, such is the case of quantum vector bundles with quantum linear connections and quantum principal bundles with quantum principal connections. M. Durdevich in [D1, D2] developed a complete theory of quantum principal bundles and quantum principal connections considering the notion of quantum groups $qG$ published by S. L. Woronovicz in [W1].

Date: February 11, 2020.
and [W3], which will play the role of structure groups. The theory developed in [D1] is semi classical in the sense that the base space is the algebra of smooth functions of a compact smooth manifold; while the theory presented in [D2] is completely quantum. This theory was extended later in order to embrace others classical notions of principal bundles, for example characteristic classes [D5]. We have to remark that even when Woronowicz’s quantum groups are used, Durdevich’s theory presents a different differential calculus that the shown in [W2]. On the other hand, the Serre–Swan theorem ([Sw]) gave us a natural way to generalize into noncommutative differential geometry the concept of vector bundle: a finitely generated projective module; although it is possible to use left modules, right modules or even bimodules.

A. Connes, Dubois–Violette and others have studied in a deep way the concept of quantum vector bundles and quantum linear connections [C], [DV].

The aim of this work is to show a categorical result; more specifically the noncommutative version of the categorical equivalence between principal bundles with principal connections and gauge theory sectors with connections shown on [SW] using Durdevich’s theory and Dubois–Violette’s theory (with certain changes), which corresponds with an extension of the paper [D3] when one can find a study of these quantum association functors. The approach presented is important not only because it provides a better support to the general theory, but because talking about categories and functors always involves natural constructions and it promotes a common language. For example the fact that we are able to recreate the classical categorical equivalence could tell us that we are presence of a correct definition of quantum vector bundles with quantum linear connections and principal $qG$–bundles with principal connections, among others concepts.

We going to do some little changes to the notation presented originally in the theories that we will use because we want to highlight similarities and constras with the classical theory shown in [SW]. One of these changes consist in denoting compact quantum spaces as $qX$ and formally represent them as associative unital $*$–algebras over $\mathbb{C}$, $(X, \cdot, 1, *)$ (interpreted like the $*$–algebra of smooth $\mathbb{C}$–valued functions on $qX$). We will identify the quantum space with its algebra, so in general, we going to omit the words compact, associative and unital. Also all our $*$–algebra morphisms will be unital. In some cases we going to work with noncompact quantum spaces, in which case we will point how we going to denote them.

The paper is organized in four sections. The second one is about the notation and basic concepts that we will use and it is split into three parts: the category of quantum representations (or corepresentations) of a compact quantum group $\text{Rep}_{qG}$ in which we will present Woromonicz’s theory but we going to change the traditional definition of morphisms in order to add antilinear maps as well; the category of quantum vector bundles with quantum linear connections $q\text{VB}^\nabla$ (and over a fixed quantum space $qM$, $q\text{VB}^\nabla_{qM}$) in which we will show our definition of these quantum structures using bimodules and our definition of morphisms between them that will include antilinear maps too; and finally the category of quantum principal $qG$–bundles with quantum principal connections, $q\text{PB}^{\omega}$ (and over a fixed quantum space $qM$, $q\text{PB}^{\omega}_{qM}$). In this part we going to present the general theory and after that we will impose several condition on the quantum bundles and on the quantum connections (as the regularity condition [D2]) in order to be able to define quantum association functors $q\text{Ass}^{\omega}$. We have to remark that we shall use the theory presented on [D4] to define quantum principal connections or to be more precisely, covariant derivatives. The third section
is about the quantum association functor

\[ \text{qAss}_{qG}^{q\omega} : \text{Rep}_{qG} \longrightarrow \text{qVB}_{qM}^{q\nabla} \]

for a principal $qG$–bundle over a fixed quantum space $qM$ $q\zeta$ with a quantum principal connection $q\omega$: its construction; its general properties (its relation with some functors defined for $\text{Rep}_{qG}$ and $\text{qVB}_{qM}^{q\nabla}$, its behaviour in monomorphisms and epimorphisms, etc); and finally we going to present the categorical equivalence between $\text{qPB}_{qM}^{q\omega}$ and the category of quantum gauge theory sectors on $qM$ with quantum connections $\text{qGTS}_{qM}^{q\nabla}$, so it will be necessary a way to rebuilt the bundle and the connection for a given contravariant functor between $\text{Rep}_{qG}$ and $\text{qVB}_{qM}^{q\nabla}$ that satisfies certain properties. To do this we will based on the result shown in [D3]. The last section is about some concluding comments.

One of the most important theorems in gauge theory is the Gauge Principle [KMS], [SW], which establishes that given a principal $G$–bundle $GM$ over $M$ with a principal connection and its associated vector bundle with the induced linear connection for a linear representation $\alpha$, there must exist a linear isomorphism between vector bundle–valued $m$–forms on $M$ and the space of all basic $m$–forms of type $\alpha$ on $GM$ for any $m \in \mathbb{N}_0$. This isomorphism commutes with the twisted exterior covariant derivative on vector bundle–valued $m$–forms and the exterior covariant derivative associated to the principal connection on basic $m$–forms of type $\alpha$. For $m = 0$, the Gauge Principle will give us a natural way to define the associated quantum vector bundle for a given quantum principal bundle and a quantum representation $q\alpha$. Also we going to present the noncommutative version of this theorem for $m > 0$ and finally the fact that both covariant derivatives commute will inspire our definition of the induced quantum linear connection given a quantum principal connection.

2. Notation and Basic Concepts

In this section we will present some basic notation and concepts that we going to use in the whole paper; particularly, we will define the necessary categories to fulfill our purpose.

We going to use Sweedler notation and given an arbitrary category $\mathbf{C}$, we will denote by $\text{Obj}(\mathbf{C})$ the class of objects of $\mathbf{C}$ and by $\text{Mor}(\mathbf{C})$ the class of morphisms in $\mathbf{C}$. Furthermore, given $c_1, c_2 \in \text{Obj}(\mathbf{C})$, we going to denote by $\text{Mor}_{\mathbf{C}}(c_1, c_2)$ the class of all morphisms in $\mathbf{C}$ between $c_1$ and $c_2$.

2.1. Representations of Matrix Compact Quantum Groups. We going to use the theory developed in [W3] by S. L. Woronovicz but with a little change of notation. One can also check [MVD].

A compact quantum group (cqg) [W3] will be denote by $qG$; while its dense $*$–Hopf (sub)algebra will be denote by

\[ qG^\infty := (\mathcal{G}, \cdot, 1, \phi, \epsilon, \kappa, *) , \]

where $\phi$ is the comultiplication, $\epsilon$ is the counity and $\kappa$ is the coinverse. It shall treat as the algebra of all smooth $\mathbb{C}$–valued functions definend on $qG$. In other words, $qG^\infty$ defines a smooth structure on $qG$ or a Lie group structure [W2].
Definition 2.1 (Quantum representation). For a given cqg $qG$, a (smooth right) $qG$–representation on a $\mathbb{C}$–vector space $V$ is a linear map

$$q\alpha : V \rightarrow V \otimes \mathcal{G}$$

such that

$$V \xrightarrow{id_V} V \xrightarrow{\otimes G} V \otimes \mathcal{G}$$

(where the horizontal arrow at the bottom of the diagram is the canonical isomorphism $v \mapsto v \otimes 1$) and

$$V \xrightarrow{\otimes G} V \otimes \mathcal{G}$$

$$(1)$$

$$(2)$$

We say that the representation is finite dimensional if $\dim_{\mathbb{C}}(V) < |\mathbb{N}|$. $q\alpha$ usually receives the name of (right) coaction or (right) corepresentation of $qG$ on $V$.

We have to remark that in the general theory presented in [W1], [W3], Diagram (1) is not necessary.

Example 2.2. Given a cqg $qG$ and a $\mathbb{C}$–vector space, one can always take

$$q\alpha_{\text{triv}}^V : V \rightarrow V \otimes \mathcal{G}$$

$$v \mapsto v \otimes 1.$$  

$q\alpha_{\text{triv}}^V$ turns out to be a $qG$–representation on $V$ which is called the trivial quantum representation on $V$.

It is easy to see that a linear map $q\alpha$ that satisfies Diagrams (1), (2) can be thought as

$$q\alpha = \sum_i f_i \otimes g_i \in B(V) \otimes \mathcal{G}$$

(where $B(V) = \{ f : V \rightarrow V \mid f \text{ is linear} \}$) such that

$$\sum_{i,j} f_i \circ f_j \otimes g_i \otimes g_j = (\phi \otimes id_V)(q\alpha) \quad \text{and} \quad id_V \cong \sum_i \epsilon(g_i)f_i.$$

Definition 2.3 (Corepresentation morphism). Let $qG$ be a cqg and $q\alpha_i$ be a $qG$–representation on $V_i$. A corepresentation morphism of degree 0 between them is a linear map

$$f : V_1 \rightarrow V_2$$

such that

$$V_1 \xrightarrow{\otimes \mathcal{G}} V_1 \otimes \mathcal{G}$$

$$(3)$$

$$V_2 \xrightarrow{\otimes \mathcal{G}} V \otimes \mathcal{G}.$$
A correpresentation morphism of degree 1 between these $qG$–representations is an antilinear map

$$f : V_1 \rightarrow V_2$$

such that

$$V_1 \xrightarrow{q\alpha_1} V_1 \otimes G \xrightarrow{f \otimes \ast} V_2 \xrightarrow{q\alpha_2} V \otimes G.$$  

(4)

We will denote by $\text{Mor}_{0}^{\text{Rep}_{qG}}(q\alpha_1, q\alpha_2)$ the set of all degree 0 corepresentation morphisms between two finite dimensional corepresentations $q\alpha_1$, $q\alpha_2$; and by $\text{Mor}_{1}^{\text{Rep}_{qG}}(q\alpha_1, q\alpha_2)$ the set of all degree 1 corepresentation morphisms between these $qG$–representations. Finally we define the set of corepresentation morphisms between $q\alpha_1$ and $q\alpha_2$ as

$$\text{Mor}_{\text{Rep}_{qG}}(q\alpha_1, q\alpha_2) := \text{Mor}_{0}^{\text{Rep}_{qG}}(q\alpha_1, q\alpha_2) \cup \text{Mor}_{1}^{\text{Rep}_{qG}}(q\alpha_1, q\alpha_2).$$

It is easy to show that finite dimensional $qG$–representations turn into a category where composition of morphisms is composition of maps and the identity morphism is just the identity map. With this composition, morphisms have a natural $\mathbb{Z}_2$–grading.

**Definition 2.4 (The category of $qG$–representations).** For a fixed cqg $qG$, let us define $\text{Rep}_{qG}$ as the category whose objects are finite dimensional $qG$–representations and whose morphisms are corepresentation morphisms. The category of $qG$–representations of any dimension will be denote by $\text{Rep}_{\infty}^{qG}$.

A $qG$–representation is unitary if $q\alpha$ is an unitary element of $B(V) \otimes G$ and it can be proven that every finite dimmensional $qG$–representation is isomorphic with a degree 0 morphism to an unitary one [MVD]. In this way we will considerate that every object in $\text{Rep}_{qG}$ is unitary.

Now, for $q\alpha_1$, $q\alpha_2$, $q\beta_1$, $q\beta_2 \in \text{Obj}(\text{Rep}_{qG})$ we define the set of cross corepresentation morphisms between $(q\alpha_1, q\alpha_2)$ and $(q\beta_1, q\beta_2)$ as

$$\text{Mor}_{\text{Rep}_{qG}}^{\mathbb{Z}_2}((q\alpha_1, q\alpha_2), (q\beta_1, q\beta_2)) :=$$

$$\text{Mor}_{0}^{\mathbb{Z}_2} \text{Rep}_{qG}((q\alpha_1, q\alpha_2), (q\beta_1, q\beta_2)) \cup \text{Mor}_{1}^{\mathbb{Z}_2} \text{Rep}_{qG}((q\alpha_1, q\alpha_2), (q\beta_1, q\beta_2)),$$

where elements of

$$\text{Mor}_{0}^{\mathbb{Z}_2} \text{Rep}_{qG}((q\alpha_1, q\alpha_2), (q\beta_1, q\beta_2))$$

are ordered pairs $(f_1, f_2)$ with $f_1 \in \text{Mor}_{\text{Rep}_{qG}}^{0}(q\alpha_1, q\beta_1)$ and $f_2 \in \text{Mor}_{\text{Rep}_{qG}}^{0}(q\alpha_2, q\beta_2)$; and elements of

$$\text{Mor}_{1}^{\mathbb{Z}_2} \text{Rep}_{qG}((q\alpha_1, q\alpha_2), (q\beta_1, q\beta_2))$$

are ordered pairs $(f_1, f_2)$ with $f_1 \in \text{Mor}_{\text{Rep}_{qG}}^{1}(q\alpha_1, q\beta_2)$ and $f_2 \in \text{Mor}_{\text{Rep}_{qG}}^{1}(q\alpha_2, q\beta_1)$.
Definition 2.5 (The cross category of $qG$–representations). For a fixed $cqg$ \( qG \), we define \( \text{Rep}_{\tilde{Z}_2}^{qG} \) as the category whose objects are ordered pairs \( (q\alpha_1, q\alpha_2) \) where \( q\alpha_1, q\alpha_2 \in \text{Obj}(\text{Rep}_{qG}) \) and whose morphisms are cross corepresentation morphisms.

In [W1] there are some functors for $qG$–representations and co–representation morphisms of degree 0 that we will adapt to \( \text{Rep}_{qG} \) and \( \text{Rep}_{\tilde{Z}_2}^{qG} \). First of all, for every \( \mathbb{C} \)–vector space \( V \), we can consider the complex conjugate vector space of \( V \) (this space and \( V \) are equal as additive groups but multiplication by scalars is given by \( \lambda \cdot v = \lambda^* v \)). This vector space will be denote by \( \overline{V} \) and its elements by \( \bar{v} \). There is a canonical antilinear map

\[
\overline{id_V} : V \longrightarrow \overline{V} \quad v \longmapsto \bar{v}
\]

Definition 2.6 (Conjugate functor). Let us define the conjugate functor on \( \text{Rep}_{qG} \) as the graded–preserving covariant functor

\[
- : \text{Rep}_{qG} \longrightarrow \text{Rep}_{qG}
\]

such that on objects is given by

\[
-q\alpha := \overline{q\alpha},
\]

where

\[
\overline{q\alpha} := \sum_i (\overline{id_V} \circ f_i \circ \overline{id_{V^{-1}}}) \otimes g_i^* \text{ if } q\alpha = \sum_i f_i \otimes g_i \in B(V) \otimes G; \text{ and on morphisms is given by }
\]

\[
-(f) := \overline{id_{V_2} \circ f \circ \overline{id_{V_1}}},
\]

if \( q\alpha_i \) coacts on \( V_i \) for \( i = 1, 2 \) and \( f : V_1 \longrightarrow V_2 \). \( \overline{q\alpha} \) receives the name of complex conjugate representation of \( q\alpha \).

Given two linear \( f_1 : V_1 \longrightarrow V_2 \), \( f_2 : W_1 \longrightarrow W_2 \), let us consider the twisted direct sum of \( f_1 \) with \( f_2 \)

\[
f_1 \oplus^T f_2 : V_1 \oplus V_2 \longrightarrow W_2 \oplus W_1 \quad (v_1, v_2) \longmapsto (f_2(v_2), f_1(v_1)).
\]

Definition 2.7 (Direct sum functor). Let us define the direct sum functor on $qG$–representations as the graded–preserving covariant functor

\[
\bigoplus : \text{Rep}_{\tilde{Z}_2}^{qG} \longrightarrow \text{Rep}_{qG}
\]

such that on objects is given by

\[
\bigoplus(q\alpha_1, q\alpha_2) := q\alpha_1 \oplus q\alpha_2,
\]

where

\[
q\alpha_1 \oplus q\alpha_2 := (i_1 \otimes \text{id}_G) \circ q\alpha_1 \circ (\pi_1 \otimes \text{id}_G) + (i_2 \otimes \text{id}_G) \circ q\alpha_2 \circ (\pi_2 \otimes \text{id}_G)
\]

\(^1\)Composition of morphisms is given by \( (h_1 \circ f_1, h_2 \circ f_2) \) when \( (f_1, f_2) \) has degree 0 and \( (h_2 \circ f_1, h_1 \circ f_2) \) when \( (f_1, f_2) \) has degree 1, where \( (f_1, f_2) \in \text{Mor}_{\text{Rep}_{\tilde{Z}_2}^{qG}}((q\alpha_1, q\alpha_2), (q\beta_1, q\beta_2)) \) and \( (h_1, h_2) \in \text{Mor}_{\text{Rep}_{\tilde{Z}_2}^{qG}}((q\beta_1, q\beta_2), (q\gamma_1, q\gamma_2)) \). We must notice that morphisms have a natural \( \mathbb{Z}_2 \)–grading with respect to composition. Finally the identity morphism of any object \( (q\alpha_1, q\alpha_2) \) is \( (\text{id}_{V_1}, \text{id}_{V_2}) \), if \( q\alpha_i \) coacts on \( V_i \), for \( i = 1, 2 \).
with $\iota_i : V_i \rightarrow V_1 \oplus V_2$ and $\pi_i : V_1 \oplus V_2 \rightarrow V_i$ the canonical inclusion and projections (assuming that $q_\alpha_i$ coacts on $V_i$); and on morphisms is given by

$$\bigoplus (f_1, f_2) := f_1 \oplus f_2$$

if $(f_1, f_2)$ has degree 0, and

$$\bigoplus (f_1, f_2) := f_1 \oplus^T f_2$$

if $(f_1, f_2)$ has degree 1. $q_\alpha_1 \oplus q_\alpha_2$ receives the name of direct sum of $q_\alpha_1$ and $q_\alpha_2$.

Given two linear $f_1 : V_1 \rightarrow V_2$, $f_2 : W_1 \rightarrow W_2$, let us consider the twisted tensor product of $f_1$ with $f_2$

$$f_1 \otimes^T f_2 : V_1 \otimes V_2 \rightarrow W_2 \otimes W_1$$

such that

$$f_1 \otimes^T f_2(v_1 \otimes v_2) = f_2(v_2) \otimes f_1(v_1).$$

**Definition 2.8** (Tensor product functor). We define the tensor product functor on $\text{Rep}_{qG}$ as the graded–preserving covariant functor

$$\boxtimes : \text{Rep}_{qG} \rightarrow \text{Rep}_{qG}$$

such that on objects is defined by

$$\boxtimes (q\alpha_1, q\alpha_2) := q\alpha_1 \otimes q\alpha_2,$$

where

$$q\alpha_1 \otimes q\alpha_2 := \sum_{i,j} f_{1i} \otimes f_{2j} \otimes g_{i1}g_{2j}$$

considering $q\alpha_1 = \sum_i f_{1i} \otimes g_{i1}$, $q\alpha_2 = \sum_j f_{2j} \otimes g_{2j}$ (viewed as elements of $B(V_i \otimes G)$; and on morphisms is defined by

$$\boxtimes (f_1, f_2) := f_1 \otimes f_2$$

if $(f_1, f_2)$ has degree 0, and

$$\boxtimes (f_1, f_2) := f_1 \otimes^T f_2$$

if $(f_1, f_2)$ has degree 1. $q\alpha_1 \otimes q\alpha_2$ is usually called the tensor product of $q\alpha_1$ and $q\alpha_2$.

**2.2. Quantum Vector Bundles and Quantum Linear Connections.** This subsection will be based on the general theory [DV] with some changes to adapt it to our purposes. Based on the Serre–Swan theorem [Sw] we have

**Definition 2.9** (Quantum vector bundle). Let $qM = (M, \cdot, 1, \ast)$ be a quantum space. A quantum vector bundle (qvb) over $qM$, is a quantum structure $q\zeta$ formally represented by a $M$–bimodule

$$(\Gamma(qM, qVM), +, \cdot)$$

which is finitely generated and projective as left and as right $M$–module. It represents the space of smooth sections on $q\zeta$ and we will identify $q\zeta$ with $(\Gamma(qM, qVM), +, \cdot)$. 
According to the Serre–Swan theorem, trivial vector bundles are free projective \( C^\infty(M) \)-modules \[Sw\], so in this way we say that a qvb over \( qM \) is trivial if there exist a left and right \( \mathcal{M} \)-basis \( \{x_i\}_{i=1}^{n} \) of \( \Gamma(qM,qVM) \).

For a given \( q\zeta \) over a quantum space \( qM \), a graded differential \(*\)-algebra

\[
(\Omega^*(\mathcal{M}), d, \ast), \quad \Omega^*(\mathcal{M}) := \bigoplus_{k \geq 0} \Omega^k(\mathcal{M})
\]

is an admissible differential \(*\)-calculus if \( \Omega^0(\mathcal{M}) = \mathcal{M} \) and there exists a graded–preserving \( \mathcal{M} \)-bimodule isomorphism

\[
\sigma : \Omega^*(\mathcal{M}) \otimes_\mathcal{M} \Gamma(qM,qVM) \longrightarrow \Gamma(qM,qVM) \otimes_\mathcal{M} \Omega^*(\mathcal{M}).
\]

Whenever we are using an admissible differential \(*\)-calculus for a qvb we will think that the morphism \( \sigma \) is fixed. It is important to notice that we can endow to

\[
\Omega^*(\mathcal{M}) \otimes_\mathcal{M} \Gamma(qM,qVM)
\]

with a graded \( \Omega^*(\mathcal{M}) \)-bimodule structure where the left multiplication \( \cdot : \Omega^*(\mathcal{M}) \otimes (\Omega^*(\mathcal{M}) \otimes_\mathcal{M} \Gamma(qM,qVM)) \longrightarrow \Omega^*(\mathcal{M}) \otimes_\mathcal{M} \Gamma(qM,qVM) \) is just

\[
(6) \quad m_{\Omega^*(\mathcal{M})} \otimes_\mathcal{M} \text{id}_{\Gamma(qM,qVM)}
\]

and the right multiplication \( \cdot : (\Omega^*(\mathcal{M}) \otimes_\mathcal{M} \Gamma(qM,qVM)) \otimes \Omega^*(\mathcal{M}) \longrightarrow \Omega^*(\mathcal{M}) \otimes_\mathcal{M} \Gamma(qM,qVM) \) is given by

\[
(7) \quad \sigma^{-1} \circ (\text{id}_{\Gamma(qM,qVM)} \otimes_\mathcal{M} m_{\Omega^*(\mathcal{M})}) \circ (\sigma \otimes \text{id}_{\Omega^*(\mathcal{M})}),
\]

with \( m_{\Omega^*(\mathcal{M})} \) the product on \( \Omega^*(\mathcal{M}) \). Clearly there is a similar graded \( \Omega^*(\mathcal{M}) \)-bimodule structure for

\[
\Gamma(qM,qVM) \otimes_\mathcal{M} \Omega^*(\mathcal{M})
\]

With this new structure, \( \sigma \) becomes into a graded \( \Omega^*(\mathcal{M}) \)-bimodule isomorphism.

The following definition is clearly a noncommutative version of the classical concept of linear connection \[DV\].

**Definition 2.10** (Quantum linear connection). Let us consider a qvb \( q\zeta = (\Gamma(qM,qVM), +, \cdot) \) and an admissible differential \(*\)-calculus \( (\Omega^*(\mathcal{M}), d, \ast) \) on it. A quantum linear connection (qlc) on \( q\zeta \) is a linear map

\[
q\nabla : \Gamma(qM,qVM) \longrightarrow \Omega^1(\mathcal{M}) \otimes_\mathcal{M} \Gamma(qM,qVM)
\]

that satisfies left and right Leibniz rule: for all \( p \in \mathcal{M} \) and all \( x \in \Gamma(qM,qVM) \)

\[
q\nabla(px) = pq\nabla(x) + dp \otimes_\mathcal{M} x,
\]

\[
q\nabla(xp) = q\nabla(x)p + \sigma^{-1}(x \otimes_\mathcal{M} dp).
\]

A qvb with a qlc will be denote as \((q\zeta, q\nabla)\).

One has to notice that qlcs depend on the choice of the admissible differential \(*\)-calculus \( (\Omega^*(\mathcal{M}), d, \ast) \) (quantum differential forms on \( qM \)).
Example 2.11. Taking a trivial qvb $q \in \mathfrak{t} \mathfrak{v} \mathfrak{b} = (\Gamma(qM,qVM),+,,)$, we say that a qlc $q \nabla$ is trivial if

$$q \nabla(x) = \sum_{i=1}^{n} dp_i \otimes_M x_i,$$

where $x = \sum_{i} p_i x_i$, $p_i \in \mathcal{M}$ and $\{x_i\}_{i=1}^{n}$ is a left and right $\mathcal{M}$–basis of $\Gamma(qM,qVM)$. Every trivial qlc will be denote by $q \nabla^{\text{triv}}$.

Inspiring in the classical case, one can think $\Gamma(qM,qVM)$ and $\Omega^{i}(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM,qVM)$ as qvb–valued 0–forms on $qM$ and qvb–valued 1–forms on $qM$, respectively (and by $\sigma$, $\Gamma(qM,qVM) \otimes_{\mathcal{M}} \Omega^{i}(\mathcal{M})$ is also the space of qvb–valued 1–forms on $qM$). In this way a qlc $q \nabla$ can be extended for qvb–valued forms on $qM$

$$d^q_L : \Omega^{i}(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM,qVM) \rightarrow \Omega^{i}(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM,qVM)$$

(8)

$$d^q_R : \Gamma(qM,qVM) \otimes_{\mathcal{M}} \Omega^{i}(\mathcal{M}) \rightarrow \Gamma(qM,qVM) \otimes_{\mathcal{M}} \Omega^{i}(\mathcal{M})$$

by means of

$$d^q_L(\mu \otimes_M x) = d\mu \otimes_M x + (-1)^{k} \mu q \nabla x$$

if $\mu \in \Omega^{k}(\mathcal{M})$; and

$$d^q_R(x \otimes_M \mu) = ((\sigma \circ q \nabla)(x))\mu + x \otimes_M d\mu.$$  

These maps satisfy

$$d^q_L(\mu \psi) = (-1)^{k} \mu (d^q_L \psi) + (d\mu) \psi$$

for all $\psi \in \Omega^{i}(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM,qVM)$, $\mu \in \Omega^{k}(\mathcal{M})$ and;

$$d^q_R(\hat{\psi} \eta) = (d^q_R \hat{\psi}) \eta + (-1)^{k} \hat{\psi} (d\eta)$$

for all $\hat{\psi} \in \Gamma(qM,qVM) \otimes_{\mathcal{M}} \Omega^{k}(\mathcal{M})$, $\eta \in \Omega^{i}(\mathcal{M})$.

Definition 2.12 (Curvature). Given $(q \zeta,q \nabla)$ a qvb with qlc, we define the curvature of $q \nabla$ as

$$R^{q \nabla} := d^q_L \circ q \nabla : \Gamma(qM,qVM) \rightarrow \Omega^{2}(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM,qVM).$$

Now it should be clear how one can define morphisms between these structures.

Definition 2.13 (Parallel quantum vector bundle morphism). Let $(q \zeta_i,q \nabla_i)$ be a qvb over $qM_i = (\mathcal{M}_i,*,\mathbb{1},\ast)$ with a qlc and admissible differential $*$–calculus $(\Omega^{i}(\mathcal{M}_i),d,**)$, for $i = 1,2$. A parallel qvb morphism of type $\mathcal{II}$ and degree 0 $(pqvb$ morphism $\mathcal{II} - 0)$ or a morphism of qvbs with qlcs of type II and degree 0 is a pair $(F,A)$, where

$$F : \Omega^{i}(\mathcal{M}_1) \rightarrow \Omega^{i}(\mathcal{M}_2)$$

is a graded differential $*$–algebra morphism;

$$A : \Gamma(qM_1,qVM_1) \rightarrow \Gamma(qM_2,qVM_2)$$

is a linear map such that

$$A(pxp') = F(p)A(x)F(p')$$

for all $p, p' \in \mathcal{M}_1$, $x \in \Gamma(qM_1,qVM_1)$;
If \( \Omega^\bullet(M_1), d, * \) = \( \Omega^\bullet(M_2), d, * \), a pqvb morphism of type I and degree 0 or a pqvb morphism of degree 0 is a pqvb morphism \( II \) with \( F = \text{id}_{\Omega^\bullet(M_1)} \). These kinds of morphisms will be denoted just by \( A \).

A parallel qvb morphism of type II and degree 1 (pqvb morphism \( II - 1 \)) or a morphism of qvbs with qles of type II and degree 1 is a pair \( (F, A) \), with

\[
F : \Omega^\bullet(M_1) \rightarrow \Omega^\bullet(M_2)
\]
a graded differential antilinear and \(*\)-antimultiplicative map;

\[
A : \Gamma(qM_1, qVM_1) \rightarrow \Gamma(qM_2, qVM_2)
\]
an antilinear map such that

\[
A(px^p) = F(p')A(x)F(p)
\]
for all \( p, p' \in M_1, x \in \Gamma(qM_2, qVM_1) \);

\[
\Gamma(qM_1, qVM_1) \xrightarrow{qV_1} \Omega^1(M_1) \otimes_{M_1} \Gamma(qM_1, qVM_1)
\]

\[
\begin{array}{ccc}
A & \circ & \\
\downarrow & & \downarrow F \otimes_{M_1} A \\
\Gamma(qM_2, qVM_2) & \otimes_{qV_2} & \Gamma(qM_2, qVM_2) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\Omega^\bullet(M_1) \otimes_{M_1} \Gamma(qM_1, qVM_1) & \xrightarrow{\sigma_1} & \Gamma(qM_1, qVM_1) \otimes_{M_1} \Omega^\bullet(M_1) \\
\downarrow F \otimes_{M_1} A & & \downarrow A \otimes_{M_1} F \\
\Omega^\bullet(M_2) \otimes_{M_2} \Omega^\bullet(M_2) & \xrightarrow{\sigma_2} & \Omega^\bullet(M_2) \otimes_{M_2} \Omega^\bullet(M_2) \\
\end{array}
\]

where \( F \otimes_{M_1} A \) and \( A \otimes_{M_1} F \) are the twisted tensor product of \( F \) with \( A \) and \( A \) with \( F \), respectively. If \( \Omega^\bullet(M_1), d, * \) = \( \Omega^\bullet(M_2), d, * \), a pqvb morphism of type I and degree 1 or a pqvb morphism of degree 1 is a pqvb morphism \( II - 1 \) with \( F = * \). These kinds of morphisms will be denoted just by \( A \).

We are going to define \( \text{Mor}_{qVB}^0((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) \) as the set of all pqvb morphisms \( II - 0 \) between \( (q\zeta_1, q\nabla_1) \) and \( (q\zeta_2, q\nabla_2) \); and \( \text{Mor}^1_{qVB}((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) \) as the set of all pqvb morphisms \( II - 1 \) between \( (q\zeta_1, q\nabla_1) \) and \( (q\zeta_2, q\nabla_2) \). Finally we define the set of all pqvb morphisms \( II \) between \( (q\zeta_1, q\nabla_1) \) and \( (q\zeta_2, q\nabla_2) \) as

\[
\text{Mor}_{qVB}((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) :=
\]
\[
\text{Mor}^0_{q\text{VB}^\nabla}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) \cup \text{Mor}^1_{q\text{VB}^\nabla}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2))).
\]

In the same way we define
\[
\text{Mor}^0_{q\text{VB}^\nabla_{qM}}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) := \\
\text{Mor}^0_{q\text{VB}^\nabla_{qM}}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) \cup \text{Mor}^1_{q\text{VB}^\nabla_{qM}}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)).
\]

One can show easily that qvbs with qlcs and pqvb morphisms (of any type) form a category where composition of morphisms is \((F_2, A_2) \circ (F_1, A_1) := (F_2 \circ F_1, A_2 \circ A_1)\) and the identity morphism is \((id_{\Omega^*(M)}, id_{\Gamma(qM, qV^{\nabla}_M)})\). With this composition, morphisms have a natural \(Z_2\)-grading.

**Definition 2.14** (The categories of quantum vector bundles with quantum linear connections). We will denote by \(q\text{VB}^{q\nabla}\) the category whose objects are qvbs and whose morphisms are pqvb morphisms of type II. Also for a fixed quantum space \(qM\), we define \(q\text{VB}^{q\nabla}_{qM}\) as the category whose objects are qvbs over a fixed \(qM\) with qlcs and whose morphisms are pqvb morphisms.

Using definition 2.13 it is easy to get relations between the maps \(d_L^{q\nabla}, d_R^{q\nabla}\), for example
\[
\begin{align*}
\Omega^*(M_1) \otimes_{M_1} \Gamma(qM_1, qV M_1) &\xrightarrow{d_L^{q\nabla}} \Omega^*(M_2) \otimes_{M_2} \Gamma(qM_1, qV M_1) \quad (\Omega^*(M_2) \otimes_{M_2} \Gamma(qM_2, qV M_2) \xrightarrow{d_R^{q\nabla}} \Omega^*(M_2) \otimes_{M_2} \Gamma(qM_2, qV M_2)
\end{align*}
\]

for all \((F, A) \in \text{Mor}^0_{q\text{VB}^\nabla}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)).

Now in a similar way that we defined cross corepresentation morphisms we can define cross pqvb morphisms of type II
\[
\text{Mor}_{q\text{VB}^{q\nabla}_{qM}}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2); (q\zeta_3, q\nabla_3), (q\zeta_4, q\nabla_4))
\]

and cross pqvb morphisms
\[
\text{Mor}_{q\text{VB}^{q\nabla}_{qM}}(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2); (q\zeta_3, q\nabla_3), (q\zeta_4, q\nabla_4)).
\]

**Definition 2.15** (The cross categories of quantum vector bundles with quantum linear connections). We define \(q\text{VB}^{q\nabla}_{qM}\) as the category whose objects are ordered pairs \(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2))\) where \((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2) \in \text{Obj}(q\text{VB}^{q\nabla})\) and whose morphisms are cross pqvb morphisms \(\text{II}\). In a similar way we define the category \(q\text{VB}^{q\nabla}_{qM}\).

To finish we going to present a version of the functors defined in last subsection for \(q\text{VB}^{q\nabla}_{qM}\) and \(q\text{VB}^{q\nabla}_{qM}\).

Let \(q\zeta = (\Gamma(qM, qV M), +, \cdot)\) be a qvb. Then the following multiplications
\[
\begin{align*}
\cdot: M \times \Gamma(qM, qV M) &\longrightarrow \Gamma(qM, qV M), \\
(p, x) &\longrightarrow x p^* \\
\n\cdot: \Gamma(qM, qV M) \times M &\longrightarrow \Gamma(qM, qV M) \\
(x, p) &\longrightarrow p^* x
\end{align*}
\]

\(^2\)Composition of morphisms follows the same rules as cross corepresentation morphisms. The identity morphism of any object \(((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2))\) is \((id_{\Omega^*(M_1), id_{\Gamma(qM, qV^{\nabla}_M)})_1), (id_{\Omega^*(M_2), id_{\Gamma(qM, qV^{\nabla}_M)})_2)).\]
endow to \((\Gamma(qM, qVM), +)\) with another \(\mathcal{M}\)–bimodule structure, which will be denote by \((\Gamma(qM, qVM), +, \cdot)\) and it turns out to be a finitely generated projective left–right \(\mathcal{M}\)–module as well. We going to use the notation \(\overline{x}\) for an element of \(\overline{\Gamma(qM, qVM)}\). In this way, taking a qvb over \(qM\), \(q\zeta = (\Gamma(qM, qVM), +, \cdot)\), we define the conjugate qvb of it as the qvb over \(qM\)

\[
q\overline{\zeta} = (\Gamma(qM, qVM), +, \cdot).
\]

Just like we did for \(\mathbf{Rep}_{qG}\), one can always take \(\text{id}_{\overline{\Gamma(qM, qVM)}} : \overline{\Gamma(qM, qVM)} \longrightarrow \overline{\Gamma(qM, qVM)}\)

\[
x \mapsto \overline{x}.
\]

With this, if \((\Omega^\ast(\mathcal{M}), d, \ast)\) is an admissible differential \(\ast\)–calculus for \(q\zeta\), the map

\[
\sigma := (\ast \otimes_{\mathcal{M}} \text{id}_{\overline{\Gamma(qM, qVM)}}) \circ \sigma^{-1} \circ (\ast \otimes_{\mathcal{M}} \text{id}_{\overline{\Gamma(qM, qVM)}})
\]

tells us that \((\Omega^\ast(\mathcal{M}), d, \ast)\) is an admissible differential \(\ast\)–calculus for \(q\overline{\zeta}\) as well, where \(\sigma\) is given in Equation (5). Even more for every qlc \(q\nabla\) on \(q\zeta\), the linear map

\[
q\overline{\nabla} := (\text{id}_{\overline{\Gamma(qM, qVM)}} \otimes_{\mathcal{M}} \ast) \circ q\nabla \circ \text{id}_{\overline{\Gamma(qM, qVM)}}
\]

is a qlc on \(q\overline{\zeta}\) which is usually known as the conjugate qlc of \(q\nabla\).

**Definition 2.16 (Conjugate functor).** Let us define the conjugate functor on \(q\mathbf{VB}_{qM}\) as the graded–preserving covariant endofunctor

\[- : q\mathbf{VB}_{qM}^q\nabla \longrightarrow q\mathbf{VB}_{qM}^q\nabla\]

such that on objects is given by

\[-(q\zeta, q\nabla) := (q\overline{\zeta}, q\overline{\nabla})\]

and on morphisms is given by

\[-(A) := \text{id}_{\overline{\Gamma(qM, qVM_1)}} \circ A \circ \text{id}_{\overline{\Gamma(qM, qVM_2)}}\]

if \(A : \Gamma(qM, qVM_1) \longrightarrow \Gamma(qM, qVM_2)\).

Given two qvbs over \(qM\), \(q\zeta_i = (\Gamma(qM, qVM_i), +, \cdot)\) \((i = 1, 2)\), we define the direct sum (or the Whitney sum) of qvbs as

\[q\zeta_1 \oplus q\zeta_2 = (\Gamma(qM, qVM_1) \oplus \Gamma(qM, qVM_2), +, \cdot, \oplus, \cdot)\].

On the other hand, if \((\Omega^\ast(\mathcal{M}), d, \ast)\) is an admissible differential \(\ast\)–calculus on \(q\zeta_i\) \((i = 1, 2)\), the map (considering the corresponding isomorphism)

\[\sigma^\oplus := \sigma_1 \oplus \sigma_2\]

guarantees us that \((\Omega^\ast(\mathcal{M}), d, \ast)\) is an admissible differential \(\ast\)–calculus on \(q\zeta_1 \oplus q\zeta_2\) as well, where \(\sigma_i\) is the map given in Equation (5) for each qvb. Furthermore, for a qlc \(q\nabla_i\) on \(q\zeta_i\), we define the direct sum of qlcs by means of

\[q\nabla^\oplus := q\nabla_1 \oplus q\nabla_2\]

which is a qlc on \(q\zeta_1 \oplus q\zeta_2\).
**Definition 2.17** (Direct sum functor). The direct sum functor on qvbs with qlcs is the graded–preserving covariant functor

\[ \bigoplus : q\text{VB}_{qM}^{q\nabla Z} \to q\text{VB}_{qM}^{q\nabla} \]

such that on objects is defined by

\[ \bigoplus((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) := (q\zeta_1 \oplus q\zeta_2, q\nabla^{\otimes}) \]

and on morphisms is defined by

\[ \bigoplus(A, A') := A \oplus A' \]

if \((A, A')\) has degree 0, and

\[ \bigoplus(A, A') := A \oplus^T A' \]

if \((A, A')\) has degree 1, where \(A \oplus^T A'\) is the twisted direct sum of \(A\) with \(A'\).

Finally if one takes two qvbs over \(qM\), \(q\zeta_i = (\Gamma(qM, qVM_i), +, \cdot)\) \((i = 1, 2)\), the qvb

\[ q\zeta_1 \otimes q\zeta_2 = (\Gamma(qM, qVM_1) \otimes_{\mathcal{M}} \Gamma(qM, qVM_2), +_{\otimes}, \cdot_{\otimes}) \]

receives the name of the tensor product of qvbs. Also taking an admissible differential \(*\)–calculus on \(q\zeta_i\) \((i = 1, 2)\), \((\Omega^*(\mathcal{M}), d, *)\), we get that it is an admissible differential \(*\)–calculus on \(q\zeta_1 \otimes q\zeta_2\) to by means of

\[ \sigma^{\otimes} := (\text{id}_{\Gamma(qM, qVM_1)} \otimes_{\mathcal{M}} \sigma_2) \circ (\sigma_1 \otimes_{\mathcal{M}} \text{id}_{\Gamma(qM, qVM_2)}). \]

Moreover given a qlc \(q\nabla_i\) on \(q\zeta_i\), the qlc on \(q\zeta_1 \otimes q\zeta_2\)

\[ q\nabla^{\otimes} : \Gamma(qM, qVM_1) \otimes_{\mathcal{M}} \Gamma(qM, qVM_2) \to \Omega^1(\mathcal{M}) \otimes_{\mathcal{M}} (\Gamma(qM, qVM_1) \otimes_{\mathcal{M}} \Gamma(qM, qVM_2)) \]

such that

\[ q\nabla^{\otimes}(x_1 \otimes_{\mathcal{M}} x_2) = q\nabla_1(x_1) \otimes_{\mathcal{M}} x_2 + (\sigma_1^{-1} \otimes_{\mathcal{M}} \text{id}_{\Gamma(qM, qVM_2)})(x_1 \otimes_{\mathcal{M}} q\nabla_2(x_2)) \]

receives the name of the tensor product of qlcs.

**Definition 2.18** (Tensor product functor). The tensor product functor on qvbs with qlcs is defined as the graded–preserving covariant functor

\[ \bigotimes : q\text{VB}_{qM}^{q\nabla Z} \to q\text{VB}_{qM}^{q\nabla} \]

such that on objects is given by

\[ \bigotimes((q\zeta_1, q\nabla_1), (q\zeta_2, q\nabla_2)) := (q\zeta_1 \otimes q\zeta_2, q\nabla^{\otimes}) \]

and on morphisms is defined by

\[ \bigotimes(A, A') := A \otimes_{\mathcal{M}} A' \]

if \((A, A')\) has degree 0, and

\[ \bigotimes(A, A') := A \otimes^T_{\mathcal{M}} A' \]

if \((A, A')\) has degree 1.

\(^3\)It is well–defined by the right Leibniz rule.
It is easy to see that tensor products are associative and distributive over direct sums. Also we have to notice that we could define a qlc as a linear map from $\Gamma(qM,qVM)\otimes M\Omega^\bullet(M)$ and with all necessary changes we could recreate all theory presented here. Furthermore all our functors have an extension to $qV^{\Omega^\bullet}$ and $qVB^{\Omega^\bullet\mathbb{Z}_2}$.

2.3. Quantum Principal Bundles and Quantum Principal Connections. This subsection will be based on the theory developed by M. Durdevich in the text [SZ] written by S. Sonz (especially because we going to use the notation of this book with little changes). Also one can check this theory in the original work [D1], [D2].

Definition 2.19 (Quantum principal $qG$–bundle). Let $qM = (\mathcal{M}, \cdot, 1, *)$ be a quantum space and let $qG$ be a cqg. A quantum principal $qG$–bundle (qpqgb) over $qM$ is a quantum structure formally represented by the triplet 

$$q\zeta = (qGM, qM, gM\Phi),$$

where $qGM = (\mathcal{G}M, \cdot, 1, *)$ is a quantum space called the total quantum space, with $qM$ as quantum subspace which receives the name of base quantum space, and 

$$gM\Phi : \mathcal{G}M \rightarrow \mathcal{G}M \otimes \mathcal{G}$$

is a $*$–algebra morphism that satisfies

1. $gM\Phi$ is a qG–representation.
2. $gM\Phi(x) = x \otimes 1$ if and only if $x \in \mathcal{M}$.
3. The linear map $\beta : \mathcal{G}M \otimes \mathcal{G}M \rightarrow \mathcal{G}M \otimes \mathcal{G}$ given by

$$\beta(x \otimes y) := x \cdot gM\Phi(y) = (x \otimes 1) \cdot gM\Phi(y)$$

is surjective.

One has to notice that in this situation $qM$ appears as a secondary object: right invariant elements. There are a lot of extra structure that we have to add in order to get a noncommutative version of the concept of principal connections. First of all

Definition 2.20 (Differential calculus). Given a qpqgb over $qM$, $q\zeta$, a graded differential calculus on it is

1. A graded differential $*$–algebra over $\mathcal{G}M$ ($\Omega^\bullet(\mathcal{G}M), d, *$), such that it is generated as graded differential $*$–algebra by $\Omega^0(\mathcal{G}M) = \mathcal{G}M$ (quantum differential forms on $qGM$).
2. A bicovariant $*$–FODC over $\mathcal{G}$ ($\Gamma, d$).
3. The map $gM\Phi$ is extendible to a graded differential $*$–algebra morphism

$$\Omega \Psi : \Omega^\bullet(\mathcal{G}M) \rightarrow \Omega^\bullet(\mathcal{G}M) \otimes \Gamma^\wedge,$$

where $(\Gamma^\wedge, d)$ is the universal differential envelope $*$–calculus of the $*$–FODC $(\Gamma, d)$ (which is just called universal differential calculus in [SZ]).

Second

Definition 2.21 (Horizontal forms). Let $q\zeta$ be a qpqgb over $qM$ with a graded differential calculus. We define the space of horizontal forms as

$$\text{Hor}^\bullet \mathcal{G}M := \{\varphi \in \Omega^\bullet(\mathcal{G}M) \mid \Omega \Psi(\varphi) \in \Omega^\bullet(\mathcal{G}M) \otimes \mathcal{G}\}.$$
It can be proven that Hor$^\cdot \mathcal{G} \mathcal{M}$ is a graded $*$-subalgebra of $\Omega^\cdot(\mathcal{G} \mathcal{M})$ and
\[ \Omega \Psi(\text{Hor}^\cdot \mathcal{G} \mathcal{M}) \subseteq \text{Hor}^\cdot \mathcal{G} \mathcal{M} \otimes \mathcal{G}, \]
so $H^\Phi := \Omega \Psi|_{\text{Hor}^\cdot \mathcal{G} \mathcal{M}}$ turns into a $q\mathcal{G}$-representation [SZ]. Also we have

**Definition 2.22 (Base forms).** Let $q\zeta$ be a qpqgb over $qM$ with a graded differential calculus. We define the space of base forms as
\[ \Omega^\cdot(\mathcal{M}) := \{ \mu \in \Omega^\cdot(\mathcal{G} \mathcal{M}) \mid \Omega \Psi(\mu) = \mu \otimes \text{id} \}. \]
\[ \Omega^\cdot(\mathcal{M}) \] is a graded differential $*$-subalgebra of $\Omega^\cdot(\mathcal{G} \mathcal{M})$ [SZ]. It is important to mention that in general, $\Omega^\cdot(\mathcal{M})$ is not generated as graded differential $*$-algebras by $\Omega^0(\mathcal{M}) = \mathcal{M}$. Furthermore, it turns out that the graded $*$-algebra Hor$^\cdot \mathcal{G} \mathcal{M}$ is generally not generated by $\mathcal{G} \mathcal{M}$ and Hor$^1 \mathcal{G} \mathcal{M}$ [SZ].

**Definition 2.23 (Vertical forms).** Let $q\zeta$ be a qpqgb over $qM$ with a graded differential calculus. We define the space of vertical forms as
\[ \text{Ver}^\cdot \mathcal{G} \mathcal{M} := \mathcal{G} \mathcal{M} \otimes \text{inv} \Gamma^\wedge, \]
where
\[ \text{inv} \Gamma^\wedge := \{ \theta \in \Gamma^\wedge \mid \Phi_{\text{inv}}(\theta) = 1 \otimes \theta \}, \]
with $\Phi_{\text{inv}}$ the extension of the canonical corepresentation of $\mathcal{G}$ on $\Gamma$. Even more, since $\text{inv} \Gamma^\wedge$ is a graded differential $*$-subalgebra of $\Gamma^\wedge$, $\text{Ver}^\cdot \mathcal{G} \mathcal{M}$ has a natural structure of graded vector space and defining the operations
\[ (x \otimes \theta)(y \otimes \hat{\theta}) := xy^{(0)} \otimes (\theta \circ y^{(1)}\hat{\theta}), \]
\[ (x \otimes \theta)^* := x^{(0)*} \otimes (\theta^* \circ x^{(1)*}) \]
and
\[ d_v(x \otimes \theta) = x \otimes d\theta + x^{(0)} \otimes \pi(x^{(1)})\hat{\theta}, \]
(Ver$^\cdot \mathcal{G} \mathcal{M}, d_v, *)$ is a graded differential $*$-algebra generated by Ver$^0 \mathcal{G} \mathcal{M} = \mathcal{G} \mathcal{M} \otimes \mathbb{C} = \mathcal{G} \mathcal{M}$, where $\pi : \mathcal{G} \longrightarrow \text{inv} \Gamma := \text{inv} \Gamma^\wedge$ is the quantum germs map,
\[ \pi(g) \circ g' = \pi(gg' - \epsilon(g)g') \]
and $\mathcal{G} \mathcal{M} \Phi(x) = x^{(0)} \otimes x^{(1)}$ (we are using Sweedler’s notation) [SZ].

It is really important to emphasize that unlike the classical case, here in the noncommutative case there are not canonical calculus over the spaces. This gives us a richer theory.

**Definition 2.24 (Quantum principal connection).** Let $q\zeta$ be a qpqgb over $qM$ with a graded differential calculus. A linear map
\[ q\omega : \text{inv} \Gamma \longrightarrow \Omega^1(\mathcal{G} \mathcal{M}) \]
is a quantum principal connection (qpc) if it satisfies
\[ (1) \quad q\omega(\theta^*) = q\omega(\theta)^* \]
\[ (2) \quad \Omega \Psi(q\omega(\theta)) = (q\omega \otimes \text{id}_\mathcal{G}) \text{ad}(\theta) + 1 \otimes \theta, \]
where $\text{ad} : \text{inv} \Gamma \longrightarrow \text{inv} \Gamma \otimes \mathcal{G}$ is the right adjoin $q\mathcal{G}$-representation.
A qpqgb with a qpc will be denote by \((q\zeta, q\omega)\).

A qpc is called regular if for all \(\varphi \in \text{Hor}^k \mathcal{G}\mathcal{M}\) and \(\theta \in \text{inv} \Gamma\), we have

\[
q\omega(\theta) \varphi = (-1)^k \varphi^{(0)} q\omega(\theta \circ \varphi^{(1)}),
\]

where \(H\Phi(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)}\).

A qpc is called multiplicative if \(q\omega(\pi(g^{(1)})) q\omega(\pi(g^{(2)})) = 0 \) for all \(g \in \mathcal{R}\) with \(\phi(g) = g^{(1)} \otimes g^{(2)}\), where we are considering that \(\mathcal{R} \subseteq \text{Ker}(\epsilon)\) is the right ideal of \(\mathcal{G}\) associated to the bicovariant \(*-\text{FODC}\) \([\text{SZ}]\).

A really useful characteristic of regular qpc is that any homogeneous element of \(\Omega^\bullet(\mathcal{M})^\bullet\text{graded-commutes with all elements of Im}(q\omega)\). In the whole paper we will assume that all our qpcs are regular, so we will omit the word regular.

**Definition 2.25 (Curvature).** Taking a qpqgb with a qpc \((q\zeta, q\omega)\), we define the curvature of \(q\omega\) as

\[
R^{q\omega} := d \circ q\omega - \langle \omega, \omega \rangle
\]

with

\[
\langle \omega, \omega \rangle : \text{inv} \Gamma \longrightarrow \Omega^2(\mathcal{G}\mathcal{M})
\]

given by

\[
\langle \omega, \omega \rangle(\theta) = m_{\Omega^\bullet(\mathcal{G}\mathcal{M})} \circ (q\omega \otimes q\omega) \circ \delta,
\]

where \(\delta : \text{inv} \Gamma \longrightarrow \text{inv} \Gamma \otimes \text{inv} \Gamma\) is an embedded differential \([\text{D2}]\) and \(m_{\Omega^\bullet(\mathcal{G}\mathcal{M})}\) is the multiplication map of \(\Omega^\bullet(\mathcal{G}\mathcal{M})\).

A really important property of multiplicative qpcs is the fact that for these connections the curvature does not depend on the map \(\delta\) \([\text{D2}]\).

**Definition 2.26 (Covariant derivative).** For a given qpqgb with a qpc \((q\zeta, q\omega)\), the first-order linear map

\[
D^{q\omega} : \text{Hor}^\bullet \mathcal{G}\mathcal{M} \longrightarrow \text{Hor}^\bullet \mathcal{G}\mathcal{M}
\]

such that for every \(\varphi \in \text{Hor}^k \mathcal{G}\mathcal{M}\)

\[
D^{q\omega}(\varphi) = d\varphi - (-1)^k \varphi^{(0)} q\omega(\pi(\varphi^{(1)}))
\]

with \(H\Phi(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)}\) is called the covariant derivative of \(q\omega\).

We have to remark that the last definition is not the most general way to define the covariant derivative \([\text{D2}]\), but it will be enough for our purposes. Furthermore one can prove that \(D^{q\omega}\) satisfies

\[
D^{q\omega} \in \text{MOR}^0_{\text{Rep}^\infty_{\mathcal{G}}}(H\Phi, H\Phi).
\]

and

\[
D^{q\omega}(\varphi \psi) = D^{q\omega}(\varphi)\psi + (-1)^k \varphi D^{q\omega}(\psi), \quad D^{q\omega} \circ * = * \circ D^{q\omega}, \quad D^{q\omega}|_{\Omega^\bullet(\mathcal{M})} = d|_{\Omega^\bullet(\mathcal{M})}
\]

where \(\varphi \in \text{Hor}^k \mathcal{G}\mathcal{M}\) and \(\psi \in \text{Hor}^\bullet \mathcal{G}\mathcal{M}\) \([\text{SZ}]\).
Remark 2.27. Let $T$ a complete set of mutually nonequivalent irreducible finite dimensional $qG$–representations. From this point until the end of this paper we going to consider for a given $qζ = (qGM, qM, g_M Φ)$ and each $qα ∈ T$ that there exists

$$\{T_k^L\}_{k=1}^d ⊆ \text{Mor}^0_{\text{Rep}^∞_q}(qα, g_M Φ)$$

such that

$$\sum_{k=1}^d x_{ki}^{qα} x_{kj}^{qα} = δ_{ij} 1,$$

with $x_{ki}^{qα} := T_k L(e_i)$, where $\{e_i\}_{i=1}^n$ is some fixed basis of the vector space $V^{qα}$ on which $qα$ coacts; and the following relation holds

$$(Z^{qα} X^{qα})^T X^{qα} = \text{Id}_n,$$

where $X^{qα} = (x_{ij}^{qα}) ∈ M_{d×n}(G_M)$, $\text{Id}_n$ is the identity element of $M_n(G_M)$ and $Z^{qα} = (z_{ij}^{qα}) ∈ M_d(\mathbb{C})$ is a strictly positive element [D2].

Given a pqpqb, $qζ = (qGM, qM, g_M Φ)$, let us consider a graded $*$–algebra $(Ω^•_H, 1, *)$ such that $Ω^0_H = G_M$ with a graded $*$–subalgebra $Ω^•_M$ with structure of graded differential $*$–algebra generated by its degree 0 elements $Ω^0_M = M$ and a $qG$–representation $Φ$ coacting on $Ω^•_M$ such that $Ω^•_M$ is exactly the set of all $Φ$–invariant elements, where $Φ$ is also a graded $*$–algebra morphism which extends $g_M Φ$. Now we going to denote by

$$\text{Der}$$

the space of all first–order linear maps $D$ that satisfies Equations (14), (15) with respect to

$$\{Φ, Ω^•_H, (Ω^•_M, d, *)\}.$$ 

It can be proven that there exists a bicovariant $*$–FODC over $G$, $(Γ, d)$ such that together

$$\{Φ, Ω^•_H, (Ω^•_M, d, *)\}$$

one can get a differential calculus on $qζ$ [D1]. Even more, $Ω^•_H$ is the space of horizontal forms of this calculus, $Ω^•_M$ corresponds to the space of base forms and qpcs (relative to this differential calculus) on $qζ$ are in bijection with elements of $\text{Der}$ in a natural way: for every qpc $qω$, $D^{qω} ∈ \text{Der}$ and for every $D ∈ \text{Der}$ there exists a unique qpc $qω$ such that $D^{qω} = D$ [D1]. In other words, we just need $\text{Der}$ to get all the previous structures that we have just presented in this subsection. Another important result of this particular way to get the differential calculus is that every qpc is multiplicative as well.

Remark 2.28. From this point until the end of this paper we shall assume that every qpc is given by the above conditions.

Definition 2.29 (Quantum principal bundle morphism). Let $qζ_i = (qGM_i, qM_i, g_M Φ_i)$ be a quantum principal $qG_i$–bundle with a qpc $qω_i$ ($i = 1, 2$). A parallel quantum principal bundle morphism of type II (pqpb morphism II) is a pair

$$(h, F),$$

where

$$h : G_1 → G_2$$

is a $*$–hopf algebra morphism and

$$F : \text{Hor}^• G_M_1 → \text{Hor}^• G_M_2$$
is a graded $*$-algebra morphism such that

$$\begin{align*}
\text{Hor}^*\mathcal{G}M_1 & \xrightarrow{\text{Hor}^*\mathcal{G}M_1 \otimes \mathcal{G}_1} \text{Hor}^*\mathcal{G}M_1 \otimes \mathcal{G}_1 \\
\downarrow F & \quad \circ \quad \downarrow F \otimes h \\
\text{Hor}^*\mathcal{G}M_2 & \xrightarrow{\text{Hor}^*\mathcal{G}M_2 \otimes \mathcal{G}_2} \text{Hor}^*\mathcal{G}M_2 \otimes \mathcal{G}_2.
\end{align*}$$

with

$$F \circ D^{q\omega_1} = D^{q\omega_2} \circ F.$$ 

If $(\Omega^*(\mathcal{M}_1), d, *) = (\Omega^*(\mathcal{M}_2), d,*)$, a pqpb morphism of type I or a pqpb morphism is a pqpb morphism II with $F|_{\Omega^*(\mathcal{M}_1)} = \text{id}_{\Omega^*(\mathcal{M}_1)}$.

It is clear that qpbs with qpcs and pqpb morphisms become into a category where composition of morphisms is composition of maps and the identity morphism is $\text{id}_{\text{Hor}^*\mathcal{G}M}$. In this way

**Definition 2.30** (The category of quantum principal $qG$–bundles with quantum principal connections). We define $\text{qPB}^{q\omega}$ as the category whose objects are triplets $(qG, q\zeta, q\omega)$, where $qG$ is cqg and $(q\zeta, q\omega)$ is a qpqgb with a qpc; and whose morphisms are pqpb morphisms II. We will denote by $\text{qPB}^{q\omega}_{qM}$ the category whose objects are the same as before but $q\zeta$ is a qpqgb over a fixed quantum space $qM$ and whose morphisms are pqpb morphisms.

### 3. The functor $\text{qAss}^{q\omega}_{q\zeta}$

In this section we going to define the association functor in the framework of noncommutative differential geometry using the theory presented in last section and we going to give a characterization of it.

**3.1. Construction.** Let $q\alpha \in \mathcal{T}$ coacting on the vector space $V^{q\alpha}$ and let $q\zeta = (qGM, qM, \mathcal{G}_M \Phi)$ be a qpqgb. Let us define

$$\Gamma(qM, qV^{q\alpha}M) := \text{Mor}_{qG}^0(q\alpha, \mathcal{G}_M \Phi).$$

Notice that $\Gamma(qM, qV^{q\alpha}M)$ is a $\mathcal{M}$–bimodule by means of

$$p \otimes T \mapsto pT \quad \quad \quad T \otimes p \mapsto Tp.$$ 

Furthermore, for every $T \in \Gamma(qM, qV^{q\alpha}M)$,

$$T = \sum_{k=1}^d p_k^T T_k^L = \sum_{k=1}^d T_k^R \hat{p}_k^T,$$

where $T_k^R = \sum_{i=1}^d z_{ki} T_i^L$ and

$$p_k^T = \sum_{i=1}^n T(e_i) x_{ki}^{q\alpha} \ast, \quad \hat{p}_k^T = \sum_{i,j=1}^{n,d} y_{ik}^{q\alpha} w_{ij}^{q\alpha} \ast T(e_j) \in \mathcal{M}.$$
with \( (w_{ij}^{q\alpha}) = Z^{q\alpha}X^{q\alpha} \) and \( Y^{q\alpha} = (y_{ij}^{q\alpha}) \in M_d(\mathbb{C}) \) the inverse of \( Z^{q\alpha} \) (see Remark 2.27). These sets of \( \mathcal{M} \)-generators help us to show that \( \Gamma(qM, qV^{q\alpha}M) \) is actually a finitely generated projective left–right \( \mathcal{M} \)-module; so

\[
q\zeta = (\Gamma(qM, qV^{q\alpha}M), +, \cdot)
\]

is a qvb over \( qM \).

Now let us fix a qpc \( q\omega \) on \( q\zeta \). The maps

\[
\Upsilon_{q\alpha}^{-1}: \Omega^\bullet(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\alpha}M) \rightarrow \text{MOR}_0^\alpha_{\text{Rep}_{q\omega}^G}(q\alpha, \mathbb{H}\Phi)
\]

\[
\tilde{\Upsilon}_{q\alpha}^{-1}: \Gamma(qM, qV^{q\alpha}M) \otimes_{\mathcal{M}} \Omega^\bullet(\mathcal{M}) \rightarrow \text{MOR}_0^\alpha_{\text{Rep}_{q\omega}^G}(q\alpha, \mathbb{H}\Phi)
\]

such that

\[
\Upsilon_{q\alpha}^{-1}(\mu \otimes_{\mathcal{M}} T) = \mu T \quad \text{and} \quad \tilde{\Upsilon}_{q\alpha}^{-1}(T \otimes_{\mathcal{M}} \mu) = T\mu
\]

are \( \mathcal{M} \)-bimodule isomorphisms, where \( \text{MOR}_0^\alpha_{\text{Rep}_{q\omega}^G}(q\alpha, \mathbb{H}\Phi) \) has the \( \mathcal{M} \)-bimodule structure similar to the one of \( \Gamma(qM, qV^{q\alpha}M) \). Specifically their inverses are given by

\[
\Upsilon_{q\alpha}(\tau) = \sum_{k=1}^{d} \mu^r_k \otimes_{\mathcal{M}} T^L_k \quad \text{and} \quad \tilde{\Upsilon}_{q\alpha}(\tau) = \sum_{k=1}^{d} T^R_k \otimes_{\mathcal{M}} \hat{\mu}^r_k,
\]

with

\[
\mu^r_k = \sum_{i=1}^{n} \tau(\epsilon_i) x^{q\alpha}_{ki}, \quad \hat{\mu}^r_k = \sum_{i,j=1}^{d,n} y^{q\alpha}_{ik} w^{q\alpha \ast}_{ij} \tau(\epsilon_j) \in \Omega(\mathcal{M}).
\]

In this way, taking

\[
\sigma_{q\alpha} := \tilde{\Upsilon}_{q\alpha} \circ \Upsilon_{q\alpha}^{-1}
\]

we obtain that the space of base forms \( \Omega^\bullet(\mathcal{M}), d, * \) is an admissible differential \(*\)-calculus for \( q\zeta_{q\alpha} \) and the linear map

\[
\nabla^{q\omega}_{q\alpha}: \Gamma(qM, qV^{q\alpha}M) \rightarrow \Omega^1(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\alpha}M)
\]

\[
T \mapsto \nabla_{q\alpha} \circ D^{q\omega} \circ T,
\]

is a qlc on \( q\zeta_{q\alpha} \). In summary \( (q\zeta_{q\alpha}, \nabla^{q\omega}_{q\alpha}) \in \text{OBJ}(q\text{VB}^{q\alpha}_{qM}) \) provided that \( q\alpha \in \mathcal{T} \).

**Proposition 3.1.** If \( q\alpha_i \in \mathcal{T} \) coacts on \( V_i \) (for \( i = 1, 2 \)) and \( f \in \text{MOR}^0_{\text{Rep}_{q\omega}^G}(q\alpha_1, q\alpha_2) \), then the map

\[
A_f : \Gamma(qM, qV_2 M) \rightarrow \Gamma(qM, qV_1 M)
\]

\[
T \mapsto T \circ f
\]

is an element of \( \text{MOR}^0_{q\text{VB}^{q\omega}_{qM}}((q\zeta_{q\alpha_2}, \nabla^{q\omega}_{q\alpha_2}), (q\zeta_{q\alpha_1}, \nabla^{q\omega}_{q\alpha_1})) \). Also if \( f \in \text{MOR}^1_{\text{Rep}_{q\omega}^G}(q\alpha_1, q\alpha_2) \), the map

\[
A^*_f : \Gamma(qM, qV_2 M) \rightarrow \Gamma(qM, qV_1 M)
\]

\[
T \mapsto T^* \circ f
\]

is an element of \( \text{MOR}^1_{q\text{VB}^{q\omega}_{qM}}((q\zeta_{q\alpha_2}, \nabla^{q\omega}_{q\alpha_2}), (q\zeta_{q\alpha_1}, \nabla^{q\omega}_{q\alpha_1})) \), where \( T^* : V_2 \rightarrow \hat{\mathcal{G}}M \) is given by \( T^*(v) = T(v)^* \).
Proof. It is clear that $A_f$ is a $\mathcal{M}$–bimodule morphism. For every $T \in \Gamma(qM, qV_2M)$

$$\Upsilon_{q\alpha_2}(D^{q\omega} \circ T) = \nabla^{q\omega}_{q\alpha_2}(T) = \sum_{k=1}^{d_2} \mu_k^2 \otimes_{\mathcal{M}} T_k^{1,2},$$

for some $\mu_k^2 \in \Omega(\mathcal{M})$ such that $D^{q\omega} \circ T = \sum_{k=1}^{d_2} \mu_k^2 T_k^{1,2}$ with $\{T_k^{1,2}\}_{k=1}^{d_2} \subseteq \Gamma(qM, qV_2M)$ the set of left generators; so evaluating on any basis it is easy to see that they are the same element and therefore

$$(\text{id}_{\Omega^1(\mathcal{M})} \otimes_{\mathcal{M}} A_f)\nabla^{q\omega}_{q\alpha_2}(T) = \sum_{k=1}^{d_2} \mu_k^2 \otimes_{\mathcal{M}} T_k^{1,2} \circ f.$$ 

On the other hand

$$\Upsilon_{q\alpha_1}(D^{q\omega} \circ A_f(T)) = \nabla^{q\omega}_{q\alpha_1}(A_f(T)) = \sum_{k=1}^{d_1} \mu_k^1 \otimes_{\mathcal{M}} T_k^{1,1},$$

with $\mu_k^1 \in \Omega(\mathcal{M})$ such that $D^{q\omega} \circ A_f(T) = D^{q\omega} \circ T \circ f = \sum_{k=1}^{d_1} \mu_k^1 T_k^{1,1}$ where $\{T_k^{1,1}\}_{k=1}^{d_1} \subseteq \Gamma(qM, qV_1M)$ is the set of left generators. Applying $\Upsilon^{-1}_{q\alpha_1}$ in both last equalities it is easy to show that they are the same element, so $A_f$ satisfies Diagram (10) for $F = \text{id}_{\Omega^1(\mathcal{M})}$. In a similar way it can be proven that $A_f$ satisfies Diagram (10) for $F = \text{id}_{\Omega^0(\mathcal{M})}$ and hence $A_f \in \text{MOR}^0_{q\mathcal{VB}_{qM}^{qV}}((q\zeta_{q\alpha_2}, \nabla^{q\omega}_{q\alpha_2}), (q\zeta_{q\alpha_1}, \nabla^{q\omega}_{q\alpha_1})).$

Analogously one can prove that $A_f^* \in \text{MOR}^1_{q\mathcal{VB}_{qM}^{qV}}((q\zeta_{q\alpha_2}, \nabla^{q\omega}_{q\alpha_2}), (q\zeta_{q\alpha_1}, \nabla^{q\omega}_{q\alpha_1})).$ For example, for all $T \in \Gamma(qM, qV_2M)$

$$\hat{\Upsilon}^{-1}_{q\alpha_1}((\ast \otimes_{\mathcal{M}} A_f^*) \nabla^{q\omega}_{q\alpha_2}(T)) = \sum_{k=1}^{d_2} (T_k^{1,2*} \circ f) \mu_k^{2*},$$

while

$$(\hat{\Upsilon}^{-1}_{q\alpha_1} \circ \sigma_{q\alpha_1} \circ \nabla^{q\omega}_{q\alpha_1})(A_f^*(T)) = D^{q\omega}(T^* \circ f);$$

so evaluating on any basis it is easy to see that they are the same element and therefore $A_f^*$ satisfies Diagram (11). 

Due to the fact that every $qG$–representation is (isomorphic with degree 0 morphisms to) a direct sum of irreducible ones $\mathbb{M}[V]$, we can extend in a really natural way all our previous results using the functor $\bigoplus$ (see Definitions 2.7, 2.17). Thus taking $(q\zeta, q\omega) \in \text{Obj}(q\mathcal{PB}_{qG, qM}^{qV})$, we define the association quantum vector bundle of $q\zeta$ with respect to any $qG$–representation $q\alpha$ as the qvb over $qM$, $q\zeta_{q\alpha}$, formally represented by its space of smooth sections

$$(\Gamma(qM, qV^{q\alpha}M), +, \cdot),$$

where $\Gamma(qM, qV^{q\alpha}M) := \text{MOR}^0_{qG}(q\alpha, qM\Phi)$; and we define the induced quantum linear connection of $q\omega$ on $q\zeta_{q\alpha}$ as the linear map

$$\nabla^{q\omega}_{q\alpha} : \Gamma(qM, qV^{q\alpha}M) \longrightarrow \Omega^1(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\alpha}M)$$

given by Equation (21) (using $\bigoplus$).
Remark 3.2. Definition and interpretation of \( q \zeta_{q \alpha} = (\Gamma(qM, qV^q_{q \alpha}M), +, \cdot) \) is in accordance with Gauge Principle [KMS], [SW]. The Gauge Principle also gives us a characterization of vector–bundle differential forms which we can observe with the isomorphisms \( \Upsilon_{q \alpha}, \hat{\Upsilon}_{q \alpha} \). Finally it also tells us that induced connections on associated bundles are given by covariant derivatives of principal connections, which is in clear resonance with our definition.

Proposition 3.1 also can be extended using \( \bigoplus \), so finally we have

Definition 3.3 (Quantum association functor). Let \( (q \zeta, q \omega) \) be a qpqgb over \( qM \) with a qpc. Then we define the quantum association functor as the graded–preserving contravariant functor

\[
q\text{Ass}_{q \zeta}^{q \omega} : \text{Rep}_{qG} \to q\text{VB}_{qM}^{q \nabla}
\]

such that on objects is given by

\[
q\text{Ass}_{q \zeta}^{q \omega} q\alpha = (q \zeta_{q \alpha}, \nabla_{q \omega}^{q \nabla})
\]

and on morphisms is given by

\[
q\text{Ass}_{q \zeta}^{q \omega}(f) = A_f
\]

if \( f \) has degree 0, and

\[
q\text{Ass}_{q \zeta}^{q \omega}(f) = A^*_f
\]

if \( f \) has degree 1.

3.2. Properties. Now that we have our desired functor, we are going to prove some properties of it before the proof of the categorical equivalence. First of all notice that \( q\text{Ass}_{q \zeta}^{q \omega} \) is linear in morphisms no matters the grade. Second

Proposition 3.4. \( q\text{Ass}_{q \zeta}^{q \omega} q\alpha_{\text{triv}} = (q \zeta_{q \alpha_{\text{triv}}}^q, \nabla_{q \alpha_{\text{triv}}}^{q \omega}) \) is a trivial qvb with a trivial qlc (see Examples 2.2, 2.11)

Proof. Let \( \{e_i\}_{i=1}^n \) be a basis of \( V \) and \( \{f_i\}_{i=1}^n \) its dual basis. Then the set \( \{T_{f_i}\}_{i=1}^n \), where

\[
T_{f_i} : V \to M
\]

\[
v \to f_i(v)1,
\]

is a left and right \( M \)–basis of \( \Gamma(qM, qV^{q \alpha}M) \), so \( q \zeta_{q \alpha_{\text{triv}}}^q \) is a trivial qvb. With this basis it is easy to show that \( \nabla_{q \alpha_{\text{triv}}}^{q \omega} \) is trivial. \( \blacksquare \)

Corollary 3.5. \( q \zeta_{q \alpha_{\text{triv}}}^q \cong (M, +, \cdot) \) and under this isomorphism the induced qlc is just \( D^{q \omega}|_M = d|_M \). Moreover, \( \sigma_{q \zeta_{\text{triv}}} = \text{id}_{q \nabla_{\text{triv}}^q}(M) \).

By construction, it should be clear that \( q\text{Ass}_{q \zeta}^{q \omega} \) commutes with \( \bigoplus \), but also

Proposition 3.6. For every \( q \alpha \in \text{Obj}(\text{Rep}_{qG}) \), there is a natural isomorphism between \( q\text{Ass}_{q \zeta}^{q \omega} \) and \( -q\text{Ass}_{q \zeta}^{q \omega} \) (see Definitions 2.6, 2.16). Moreover, the isomorphism given has degree 0.

Proof. Notice that it is enough to prove the proposition for elements of \( T \); so let \( q \alpha \in T \) and

\[
q \zeta_{q \alpha_{\text{triv}}} = (\Gamma(qM, qV^{q \alpha}M), +, \cdot), \quad q \zeta_{q \alpha} = (\Gamma(qM, qV^{q \alpha}M), +, \cdot).
\]
Actually we have
\[ \Gamma(qM, q\nabla M) = \text{Mor}^1_{\text{Rep}_{qG}}(q\alpha, qG\Phi). \]
Let us consider the \( M \)-bimodule isomorphism
\[ A_{q\alpha} : \Gamma(qM, q\nabla M) \rightarrow \Gamma(qM, q\nabla q\alpha M) \]
\[ T \mapsto T^*. \]
If \( \nabla^{q\omega}_{q\alpha} \) is the induced connection on \( q\zeta_{q\alpha} \) and \( \nabla^{q\omega}_{q\alpha} \) is the induced connection on \( q\zeta_{q\alpha} \), then for all \( T \in \Gamma(qM, q\nabla q\alpha M) \)
\[ (\nabla^{q\omega}_{q\alpha}(A_{q\alpha}(T))) = \nabla^{q\omega}_{q\alpha}(T^*) = \sum_{k=1}^d (\text{id}_{\Gamma(qM, q\nabla q\alpha M)} \otimes \sigma_{q\alpha} (\mu_k^{D^{q\omega}oT^*} \otimes_M T_k^L)) \]
\[ = \sum_{k=1}^d (\text{id}_{\Gamma(qM, q\nabla q\alpha M)} \otimes T_k^R \otimes \dot{\mu}_k^{D^{q\omega}oT^*}) \]
\[ = \sum_{k=1}^d (\dot{\mu}_k^{D^{q\omega}oT^*})^* \otimes_M T_k^R = \sum_{k=1}^d (\dot{\mu}_k^{D^{q\omega}oT^*})^* \otimes_M T_k^R \]
where in the ultimate equality we have used Equation (15) and the Equation (19) for \( q\alpha \) taking into account that \( \text{Mor}^0_{\text{Rep}_{qG}}(q\alpha, H\Phi) = \text{Mor}^1_{\text{Rep}_{qG}}(q\alpha, H\Phi). \) Also we get
\[ ((\text{id}_{\Omega^*(M)} \otimes_M A_{q\alpha}) \circ \nabla^{q\omega}_{q\alpha})(T) = \sum_{k=1}^d (\text{id}_{\Omega^*(M)} \otimes_M A_{q\alpha}) [(\dot{\mu}_k^{D^{q\omega}oT^*})^* \otimes_M T_k^R] \]
\[ = \sum_{k=1}^d (\dot{\mu}_k^{D^{q\omega}oT^*})^* \otimes_M T_k^R \]
and hence \( A_{q\alpha} \) satisfies Diagram (9) for \( F = \text{id}_{\Omega^*(M)}. \) A similar calculation shows that
\[ ((A_{q\alpha} \otimes_M \text{id}_{\Omega^*(M)}) \circ \sigma_{q\alpha})(\mu \otimes_M T) = \sum_{k=1}^d T_k^L \otimes_M (\mu^T \mu^*) = (\sigma_{q\alpha} \circ (\text{id}_{\Omega^*(M)} \otimes_M A_{q\alpha}))(\mu \otimes_M T) \]
and by linearity we conclude that \( A_{q\alpha} \) fulfills Diagram (10); so
\[ q\text{Ass}_{q\alpha} - q\alpha \cong -q\text{Ass}_{q\alpha}^\alpha \]
Finally, a quick calculation shows that for any \( f \in \text{Mor}_{\text{Rep}_{qG}}(q\alpha_1, q\alpha_2) \)
\[ \Gamma(qM, q\nabla M) \xrightarrow{q\text{Ass}^\alpha_{q\alpha}(f)} \Gamma(qM, q\nabla M) \]
(22)
\[ \overline{A}_{q\alpha_1} \quad \circ \quad \overline{A}_{q\alpha_2} \]
and proposition follows. \[ \blacksquare \]
Proposition 3.7. For every \((q\alpha_1, q\alpha_2) \in \text{Obj}(\text{Rep}_{q\mathbb{G}}^Z)\), there is a natural isomorphism between \(q\text{Ass}^{q\omega}_{q\mathbb{C}} \otimes \otimes(q\text{Ass}^{q\omega}_{q\mathbb{C}}, q\text{Ass}^{q\omega}_{q\mathbb{C}})\) (see Definitions 2.8, 2.18), defining
\[
(q\text{Ass}^{q\omega}_{q\mathbb{C}}, q\text{Ass}^{q\omega}_{q\mathbb{C}}) : \text{Rep}_{q\mathbb{G}}^Z \longrightarrow \text{qVB}_{qM}
\]
in the obviously way. In addition, the isomorphism given has degree 0.

Proof. As before, it is enough to prove the proposition for elements of \(\mathcal{T}\); so let us take \(q\alpha_i \in \mathcal{T}\) coacting on \(V_i\) for \(i = 1, 2\). Using the sets of \(\mathcal{M}\)-generators shown in Remark 2.27 one can prove that the map
\[
A^{-1}_{q\alpha_1,q\alpha_2} : \Gamma(qM, qV_1M) \otimes \mathcal{M} \longrightarrow \Gamma(qM, qV_2M)
\]
such that
\[
A^{-1}_{q\alpha_1,q\alpha_2}(T^1 \otimes \mathcal{M} T^2) : V_1 \otimes V_2 \longrightarrow \mathcal{G}M
\]
is given by
\[
A^{-1}_{q\alpha_1,q\alpha_2}(T^1 \otimes \mathcal{M} T^2)(v_1 \otimes v_2) = T^1(v_1)T^2(v_2)
\]
is a \(\mathcal{M}\)-bimodule isomorphism, where \(\Gamma(qM, qV_iM) := \text{Mor}_q^{\mathcal{G}M}(q\alpha_i, g, \mathcal{G}M\Phi)\) with \(i = 1, 2\) and \(\Gamma(qM, q(V_1 \otimes V_2)M) := \text{Mor}_q^{\mathcal{G}M}(q\alpha_1 \otimes q\alpha_2, g, \mathcal{G}M\Phi)\). Using \(A_{q\alpha_1,q\alpha_2}, \Upsilon_{q\alpha_1 \otimes q\alpha_2}\) and \(\Upsilon_{q\alpha_1 \otimes q\alpha_2}\) we can induce \(\mathcal{M}\)-bimodule isomorphisms
\[
\text{Mor}_q^{\mathcal{G}M}(q\alpha_1 \otimes q\alpha_2, H\Phi) \cong \Omega^*(\mathcal{M}) \otimes \mathcal{M} \Gamma(qM, qV^1M) \otimes \mathcal{M} \Gamma(qM, qV^2M)
\]
and
\[
\text{Mor}_q^{\mathcal{G}M}(q\alpha_1 \otimes q\alpha_2, H\Phi) \cong \Gamma(qM, qV^1M) \otimes \mathcal{M} \Gamma(qM, qV^2M) \otimes \mathcal{M} \Omega^*(\mathcal{M})
\]
which we going to denote by \(\Lambda\) and \(\hat{\Lambda}\), respectively. With this a direct calculation like in the last proposition proves that
\[
\Lambda^{-1} \circ (\text{id}_{\Omega^*(\mathcal{M})} \otimes \mathcal{M} \ A_{q\alpha_1,q\alpha_2}) \circ \nabla^q_{q\alpha_1 \otimes q\alpha_2} = \Lambda^{-1} \circ \nabla^q \circ A_{q\alpha_1,q\alpha_2},
\]
\[
\hat{\Lambda}^{-1} \circ (A_{q\alpha_1,q\alpha_2} \otimes \mathcal{M} \text{id}_{\Omega^*(\mathcal{M})}) \circ \sigma_{q\alpha_1 \otimes q\alpha_2} = \hat{\Lambda}^{-1} \circ \sigma^q \circ (\text{id}_{\Omega^*(\mathcal{M})} \otimes \mathcal{M} A_{q\alpha_1,q\alpha_2}).
\]
Thus \(A_{q\alpha_1,q\alpha_2}\) is a pqvb isomorphism of degree 0 between
\[
(q\text{Ass}^{q\omega}_{q\mathbb{C}} \otimes (q\alpha_1, q\alpha_2)) \quad \text{and} \quad \otimes(q\text{Ass}^{q\omega}_{q\mathbb{C}} q\alpha_1, q\text{Ass}^{q\omega}_{q\mathbb{C}} q\alpha_2);
\]
and also one can prove
\[
\Gamma(qM, q(V_1 \otimes V'_1)M) \xrightarrow{\text{qAss}^{q\omega}_{q\mathbb{C}} \otimes (f,f')} \Gamma(qM, q(V_2 \otimes V'_2)M)
\]
for any \((f, f') \in \text{Mor}^{\mathcal{G}M}_{q\mathbb{G}} ((q\alpha_1, q\alpha_1'), (q\alpha_2, q\alpha_2'))\).

It is easy to show that the set of isomorphisms \(\{A_{q\alpha_1,q\alpha_2}\}\) satisfies

Proposition 3.8. In the context of last proposition
\[
(A_{q\alpha_1,q\alpha_2} \otimes \mathcal{M} \text{id}_{V_3M}) \circ A_{q\alpha_1,q\alpha_2 \otimes q\alpha_3} = (\text{id}_{\Gamma(qM,qV_1M)} \otimes \mathcal{M} A_{q\alpha_2,q\alpha_3}) \circ A_{q\alpha_1,q\alpha_2 \otimes q\alpha_3}.
\]
In other words, \(q\text{Ass}^{q\omega}_{q\mathbb{C}}\) preserves associativity of \(\otimes\).
A well-known result in the theory of $qG$–representations is that for any two irreducible finite dimensional corepresentations $q\alpha$, $q\beta$ coacting on $V$ and $W$ respectively,

$$\text{Mor}_0^{\text{Rep}_{qG}}(q\alpha, q\beta) = \{0\} \quad \text{or} \quad \text{Mor}_0^{\text{Rep}_{qG}}(q\alpha, q\beta) = \{\lambda f \mid \lambda \in \mathbb{C}\},$$

with $f : V \rightarrow W$ a linear isomorphism [MVD]. Also we have the same result for degree 1 morphisms:

$$\text{Mor}_1^{\text{Rep}_{qG}}(q\alpha, q\beta) = \{0\} \quad \text{or} \quad \text{Mor}_1^{\text{Rep}_{qG}}(q\alpha, q\beta) = \{\lambda f \mid \lambda \in \mathbb{C}\},$$

with $f : V \rightarrow W$ an antilinear isomorphism.

**Proposition 3.9.** The functor $q\text{Ass}_{q\zeta}$ sends degree $k$ monomorphisms into degree $k$ epimorphisms and vice versa, for $k = 0$, 1.

**Proof.** Let $f \in \text{Mor}_{\text{Rep}_{qG}}(q\alpha, q\beta)$. Of course if $f$ is surjective, $q\text{Ass}_{q\zeta}(f)$ is injective. Let us take a degree $k$ injective morphism $f$. We know that there exists irreducible $qG$–representations $q\alpha_i$, $q\beta_i$ coacting on $V_i$ and $W_i$ respectively such that $[\text{MVD}]$

$$q\alpha = \bigoplus_{i=1}^n q\alpha_i \quad \text{and} \quad q\beta = \bigoplus_{j=1}^m q\beta_j.$$

If $\pi_j : W \rightarrow W_j$ is the canonical projection and $\iota_i : V_i \rightarrow V$ is the canonical embedding, then $\pi_j \circ f \circ \iota_i \in \text{Mor}_k^{\text{Rep}_{qG}}(q\alpha_i, q\beta_i)$, so $(f \circ \iota_i)(V_i) = W_j$ or $\text{Im}(f \circ \iota_i)(V_i) \cap W_j = \{0\}$. Without loss of generality, we shall assume $(f \circ \iota_i)(V_i) = W_i$ for $i = 1, \ldots, n$. Fixing $T \in \Gamma(qM, qVM)$ we can choose $T_j \in \Gamma(qM, qW_jM)$ for $j > n$ and in this way, for $k = 0$ the linear map

$$T^{\text{ext}} : W = W_1 \times \cdots \times W_n \times W_{n+1} \times \cdots \times W_m \rightarrow GM$$

$$(w_1, \ldots, w_n, w_{n+1}, \ldots, w_n) \mapsto \sum_{j=1}^n (T \circ f^{-1})(w_j) + \sum_{j=n+1}^m T_j(w_j)$$

is an element of $\Gamma(qM, qWM)$ which satisfies

$$A_f(T^{\text{ext}}) = T;$$

and for $k = 1$

$$T^{\text{ext}} : W = W_1 \times \cdots \times W_n \times W_{n+1} \times \cdots \times W_m \rightarrow GM$$

$$(w_1, \ldots, w_n, w_{n+1}, \ldots, w_n) \mapsto \sum_{j=1}^n (T^* \circ f^{-1})(w_j) + \sum_{j=n+1}^m T_j(w_j)$$

is also an element of $\Gamma(qM, qWM)$ that fulfills

$$A^*_f(T^{\text{ext}}) = T.$$

Hence $q\text{Ass}_{q\zeta}(f)$ is surjective. $\blacksquare$

**Corollary 3.10.** The functor $q\text{Ass}_{q\zeta}$ is exact.
For any qpggb \( q = (qGM, qM, qGM\Phi) \) and \( q\alpha \in T \) that coacts on \( V^{q\alpha} \), let \( GM^{q\alpha} \subseteq GM \) be the multiple irreducible subspace corresponding to this \( qG \)-representation. Each \( GM^{q\alpha} \) is a \( M \)-bimodule and

\[
GM \cong \bigoplus_{q\alpha \in T} GM^{q\alpha}.
\]

as \( M \)-bimodules. Furthermore,

\[
GM^{q\alpha} \cong \Gamma(qM, qV^{q\alpha}M) \otimes V^{q\alpha}
\]

as \( M \)-bimodules via the map

\[
T(v) \longleftrightarrow T \otimes v,
\]

with \( q\zeta_{q\alpha} = (\Gamma(qM, qV^{q\alpha}M), +, \cdot) \). This identification is a degree 0 corepresentation isomorphism between \([D2]\]

\[
gM\Phi|GM^{q\alpha} \quad \text{and} \quad \text{id}_{\Gamma(qGM, qV^{q\alpha}M) \otimes q\alpha}.
\]

According to Corollary 3.5, there is a canonical inclusion of \( M \) in the right side of Equation (24) since \( q\alpha_{\text{triv}} \in T \). Even more, using Proposition 3.7 we can get a \(*\)-algebra structure on the right side of Equation (24) by means of

\[
T^1 \cdot T^2 := A_{q\alpha_1, q\alpha_2}^{-1}(T^1 \otimes_M T^2);
\]

and the \(*\) operation is defined such that

\[
[qAss_q\zeta(f)(T)]^* = T \circ f
\]

for any degree 1 corepresentation morphism \( f \). Since \( qG \) coacts on the right side of Equation (24) with the direct sum of \( \text{id}_{\Gamma(qGM, qV^{q\alpha}M) \otimes q\alpha} \), Equation (24) holds as qpbs \([D3]\). In other words, given \( qAss_q\zeta \), we can recreate \( q\zeta \).

On the other hand, we can always define a contravariant functor between \( \text{Rep}_{qG} \) and the category of graded \( \Omega^*(\mathcal{M})\)-bimodules (with their corresponding graded morphisms of degree 1)

\[
qAss_q^H : \text{Rep}_{qG} \rightarrow \Omega^*(\mathcal{M}) - \text{GradBimod}
\]

such that in objects is

\[
qAss_q^H q\alpha = \text{MOR}^0_{\text{Rep}_{qG}}(q\alpha, H\Phi)
\]

and in morphisms is

\[
qAss_q^H (f) = A^H_f
\]

if \( f \) has degree 0 and

\[
qAss_q^H (f) = A^*_f
\]

if \( f \) has degree 1, where \( A^H_f \) and \( A^*_f \) are defined in a similar way that \( A_f \) and \( A^*_f \). Due to the fact that

\[
(25) \quad \text{Hor}^*GM \cong \bigoplus_{q\alpha \in T} \text{Hor}^*GM^{q\alpha},
\]

as graded \( \Omega^*(\mathcal{M})\)-bimodules with

\[
\text{Hor}^*GM^{q\alpha} \cong \text{MOR}^0_{\text{Rep}_{qG}}(q\alpha, H\Phi) \otimes V^{q\alpha}
\]

one can use \( qAss_q^H \) to rebuilt the graded \(*\)-algebra \( \text{Hor}^*GM \) and the coaction \( H\Phi \) like we have just done for \( GM \) and \( gM\Phi \). According to \([D4]\), using \( \{GM\Phi, \text{Hor}^*GM, (\Omega^*(\mathcal{M}), d, \ast)\} \)
one can recreate the whole differential calculus on \( q \zeta \) (see Definition 2.20). Furthermore, in this context qpcs are in bijection with elements of \( \mathcal{O} \mathfrak{t} \) (see Equation (16)).

Considering that the graded \( \Omega^\ast(M) \)-bimodule structure on \( \Omega^\ast(M) \otimes_M \Gamma(qM, qV^{q_\alpha} M) \) is given by Equations (6), (7); \( \Upsilon_{q_\alpha} \) becomes into a graded–preserving \( \Omega^\ast(M) \)-bimodule isomorphism (clearly we have an analogous result for \( \Gamma(qM, qV^{q_\alpha} M) \otimes_M \Omega^\ast(M) \) and \( \Upsilon_{q_\alpha} \)).

With this in mind, for every \( q_\alpha, q_\beta \) ∈ \( \mathcal{T} \), the Equivalence.

Theorem 3.11. Let \( q^\ast : \text{Rep}_G \to q \mathcal{B} \) be a graded–preserving contravariant functor. For every \( q_\alpha \in \text{Rep}_G \) that coacts on \( V^{q_\alpha} \), let

\[
\hat{q} \tilde{\zeta} : \text{Rep}_G \to q \mathcal{B} \]

be a graded–preserving contravariant functor. For every \( q_\alpha \in \text{Rep}_G \) that coacts on \( V^{q_\alpha} \), let

\[
\hat{q} \tilde{\zeta} (q_\alpha) := \hat{\zeta} (q_\alpha, q_\zeta, q_\omega)
\]

with

\[
\hat{\zeta} (q_\alpha) := (\Gamma(qM, qV^{q_\alpha} M), +, \cdot)
\]

and

\[
\hat{q} \tilde{\zeta} (f) := \hat{A}_f
\]
if \( f \) has degree 0 and
\[
q{\mathfrak{F}}(f) =: \hat{A}_f^*
\]
if \( f \) has degree 1. The existence of \( q{\hat{\nabla}}_{q\alpha} \) implies the existence of an \( \mathcal{M} \)-bimodule isomorphism which we will denote by
\[
\hat{\sigma}_{q\alpha} : \Omega^\bullet(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\alpha}M) \rightarrow \Gamma(qM, qV^{q\alpha}M) \otimes_{\mathcal{M}} \Omega^\bullet(\mathcal{M}).
\]
Let us assume that \( q{\mathfrak{F}} \) satisfies

1. \( q{\mathfrak{F}} q\alpha_{\text{triv}} \cong ((\mathcal{M}, +, \cdot), d|_{\mathcal{M}}) \).
2. For every \((q\alpha_1, q\alpha_2) \in \text{Obj}(\text{Rep}_{qG}^Z)\) with \( q\alpha_1, q\alpha_2 \in \mathcal{T} \), there is a natural isomorphism between \( q{\mathfrak{F}} \otimes \otimes (q{\mathfrak{F}}, q{\mathfrak{F}}) \), defining
\[
(q{\mathfrak{F}}, q{\mathfrak{F}})_G : \text{Rep}_{qG}^Z \rightarrow q\mathcal{V}B_{qM}^{q\bar{\nabla}Z}
\]
in the obviously way. All isomorphisms given have degree 0.
3. If \( \{v_{q\alpha, q\beta}\} \) is the set of all previous isomorphisms, then
\[
(u_{q\alpha_1, q\alpha_2} \otimes_{\mathcal{M}} \text{id}_{V^{q\alpha_2}M}) \circ v_{q\alpha_1, q\alpha_2 \otimes q\alpha_3} = (\text{id}_{\Gamma(qM, qV^{q\alpha_1}M)} \otimes_{\mathcal{M}} u_{q\alpha_2, q\alpha_3}) \circ v_{q\alpha_1, q\alpha_2 \otimes q\alpha_3}.
\]
4. For every \( q\alpha \in \mathcal{T} \) there exists a basis \( \{e_k\} \) of \( V^{q\alpha} \) and a set \( \{x_k\} \subseteq \Gamma(qM, qV^{q\alpha}M) \) such that
\[
\sum_{k=1}^d u_{q\alpha, q\beta}^{-1}(\hat{A}_{\text{id}_{V^{q\alpha}}}^R (x_k^L) \otimes_{\mathcal{M}} x_k^L, 1) \otimes (e_i \otimes e_j) = \delta_{ij} 1 \otimes 1 \in \mathcal{M} \otimes \mathbb{C},
\]
and there exists a strictly positive matrix \( Z^{q\alpha} = (Z_{ij}^{q\alpha}) \in M_d(\mathbb{C}) \) such that
\[
\sum_{k,l,t} u_{q\alpha, q\beta}^{-1}(\hat{A}_{\text{id}_{V^{q\alpha}}}^R (x_k^L) \otimes_{\mathcal{M}} x_k^L, 1) \otimes (e_i \otimes e_j) = \delta_{ij} 1 \otimes 1 \in \mathcal{M} \otimes \mathbb{C},
\]
where \( C = (c_{ij}) \in M_n(\mathbb{C}) \) is the canonical degree 0 corepresentation morphism between \( q\alpha \) and its second contragredient representation.
5. For every \( q\alpha \in \mathcal{T} \)
\[
\hat{\sigma}_{q\alpha}(1 \otimes_{\mathcal{M}} x) = x \otimes_{\mathcal{M}} 1 \quad \text{and} \quad \hat{d}_{q\alpha}^R = \hat{\sigma}_{q\alpha} \circ \hat{d}_{q\alpha}^L = \hat{\sigma}_{q\alpha}^{-1}.
\]

Then there exists a quantum principal \( qG \)-bundle over \( qM, q\zeta \) such that \( q{\mathfrak{F}} \) is naturally isomorphic to \( q\text{Ass}_{qG} \).

Proof. Define a graded–preserving contravariant functor
\[
q{\mathfrak{F}}^H : \text{Rep}_{qG} \rightarrow \Omega^\bullet(\mathcal{M}) \otimes \text{GradBimod}
\]
such that in objects is
\[
q{\mathfrak{F}}^H q\alpha = \Omega^\bullet(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\alpha}M)
\]
(considering the \( \Omega^\bullet(\mathcal{M}) \)-bimodule structure on \( \Omega^\bullet(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\alpha}M) \) given by Equations (9), (17)); and in morphisms is
\[
q{\mathfrak{F}}^H(f) = \hat{A}_f^H := \text{id}_{\Omega^\bullet(\mathcal{M})} \otimes_{\mathcal{M}} \hat{A}_f, \quad q{\mathfrak{F}}^H(f) = \hat{A}_f^H := \hat{\sigma}_{q\alpha_1}^{-1} \circ (\ast \otimes_{\mathcal{M}} \hat{A}_f^L),
\]
depending on the degree of \( f \in \text{Mor}_{\text{Rep}_{qG}}(q\alpha_1, q\alpha_2) \). Notice that
\[
q{\mathfrak{F}}^H q\alpha_{\text{triv}} \cong \Omega^\bullet(\mathcal{M})
\]
(as graded \(\ast\)-algebras), \(\hat{\sigma}_{qG_{\text{triv}}} = \text{id}_{\Omega^\bullet(M)}\) and \(d^d_{L,M} = d|_{\Omega^\bullet(M)}\).

On the other hand it is easy to prove that for every \((q\alpha_1, q\alpha_2) \in \text{Obj}(\text{Rep}_{qG}^{Z_2})\) with \(q\alpha_1, q\alpha_2 \in \mathcal{T}\), there exists a natural isomorphism between \(q\mathfrak{g}^H \otimes \mathfrak{g}^H\) and \(\mathfrak{g}^H \otimes (q\mathfrak{g}^H, q\mathfrak{g}^H)\) and these isomorphisms are given by

\[
\{\text{id}_{\Omega^\bullet(M)} \otimes_M \upsilon_{q\alpha, q\beta}\},
\]

which have degree 0. It follows that \(q\mathfrak{g}^H\) fulfills the hypothesis (3) of this theorem.

Let us define the \(\mathcal{M}\)-bimodule

\[
\mathcal{G}\mathcal{M} := \bigoplus_{q\alpha \in \mathcal{T}} \Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M) \otimes V^{q\alpha}
\]

and the graded \(\Omega^\bullet(M)\)-bimodule

\[
\text{Hor}^\bullet \mathcal{G}\mathcal{M} := \bigoplus_{q\alpha \in \mathcal{T}} \Omega^\bullet(M) \otimes_M \Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M) \otimes V^{q\alpha},
\]

where now we are assuming that \(q\alpha\) coacts on \(V^{q\alpha}\). \(qG\) naturally coacts on \(\mathcal{G}\mathcal{M}\) and \(\text{Hor}^\bullet \mathcal{G}\mathcal{M}\) via

\[
\mathcal{G}\mathcal{M}\Phi := \bigoplus_{q\alpha \in \mathcal{T}} \text{id}_{\Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M)} \otimes q\alpha \quad \text{and} \quad \text{Hor}^\bullet \mathcal{G}\mathcal{M}\Phi := \bigoplus_{q\alpha \in \mathcal{T}} \text{id}_{\Omega^\bullet(M)} \otimes_M \text{id}_{\Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M)} \otimes q\alpha.
\]

Notice that \(\text{Hor}^0 \mathcal{G}\mathcal{M} = \mathcal{G}\mathcal{M}\) and \(\text{Hor}^\bullet \mathcal{G}\mathcal{M}\Phi\) extends \(\mathcal{G}\mathcal{M}\Phi\). Using the method presented before (\[D3\]), one can equip to \(\mathcal{G}\mathcal{M}\) (\(\text{Hor}^\bullet \mathcal{G}\mathcal{M}\)) with a (graded) \(\ast\)-algebra structure such that \(\mathcal{G}\mathcal{M}\Phi\) is a (graded) \(\ast\)-algebra morphism and in such way that \(\Omega^\bullet(M)\) is a graded \(\ast\)-algebra. This implies that

\[
q\zeta := (qGM, qM, \mathcal{G}\mathcal{M}\Phi) := ((\mathcal{G}\mathcal{M}, \cdot, 1, \ast), (M, \cdot, 1, \ast, \mathcal{G}\mathcal{M}\Phi))
\]

is a quantum principal \(qG\)-bundle and the set \(\{(\ast \Phi, \text{Hor}^\bullet \mathcal{G}\mathcal{M}, (\Omega^\bullet(M), d, \ast))\}\) provides us a differential calculus on \(q\zeta\) (see Definition \[2.20\] \[D4\]). Hypothesis (4) guarantees us that \(q\zeta\) satisfies the written in Remark \[2.27\]. Under these conditions we have

\[
\Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M) \cong \text{Mor}^0_{\text{Rep}_{qG}^{Z_2}}(q\alpha, \mathcal{G}\mathcal{M}\Phi)
\]

as \(\mathcal{M}\)-bimodules \[D3\]; and

\[
\Omega^\bullet(M) \otimes_M \Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M) \cong \text{Mor}^0_{\text{Rep}_{qG}^{Z_2}}(q\alpha, \text{Hor}^\bullet \mathcal{G}\mathcal{M}\Phi) \cong \Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M) \otimes_M \Omega^\bullet(M)
\]

as graded \(\Omega^\bullet(M)\)-bimodules\(^4\). The first isomorphism agrees with the map \(\mu \otimes_M T \mapsto \mu T\) (taking in consideration the corresponding structures). Denoting this map by \(\Upsilon^{-1}_{q\alpha}\), the isomorphism between \(\Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M) \otimes_M \Omega^\bullet(M)\) and \(\text{Mor}^0_{\text{Rep}_{qG}^{Z_2}}(q\alpha, \text{Hor}^\bullet \mathcal{G}\mathcal{M}\Phi)\) is given by \(\Upsilon^{-1}_{q\alpha} := \Upsilon^{-1}_{q\alpha} \circ \hat{\sigma}_{q\alpha}^{-1}\), so Equation (20) is satisfied. Using hypothesis (5) one can show that \(\hat{\Upsilon}_{q\alpha}^{-1}\) agrees with the map \(T \otimes_M \mu \mapsto T\mu\). Moreover, under these identifications \[D3\]

\[
\hat{A}_f = A_f \quad \text{and} \quad \hat{A}_f^H = A_f^H,
\]

\(^4\)We are assuming that the graded \(\Omega^\bullet(M)\)-bimodule structure on \(\Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M) \otimes_M \Omega^\bullet(M)\) is similar to the one of \(\Omega^\bullet(M) \otimes_M \Gamma(q\mathcal{M}, q\mathfrak{V}^{q\alpha} M)\).
for degree 0 corepresentation morphisms; and
\[ \hat{A}_f^g = A_f^g \quad \text{and} \quad \hat{A}_f^g = A_f^g, \]
for degree 1 corepresentation morphisms.

For every \( q\alpha \in \mathcal{T} \) we define a first–order linear map
\[ \hat{D}_{q\alpha} : \text{MOR}^0_{\text{Rep}_{q\alpha}}(q\alpha, H\Phi) \to \text{MOR}^0_{\text{Rep}_{q\alpha}}(q\alpha, H\Phi) \]

exactly like in Equation (26) (using \( d^q_{\nabla} \) instead of \( d^q_{\nabla} \)) and one can prove that \( \hat{D}_{q\alpha} \) satisfies Equation (27) as well. For example, since (see Equation (8))
\[ d^q_{\nabla} (\psi) = \hat{\sigma}_{q\alpha} \circ d^q_{\nabla} \circ \hat{\sigma}^{-1} \]
for any \( \psi \in \Omega^k(M) \otimes_M \Gamma(qM, qV \omega M) \), it is possible to show that (see Definition 2.18)
\[ d^q_{\nabla} (\psi) = \hat{\sigma}_{q\alpha} \circ d^q_{\nabla} \circ \hat{\sigma}^{-1} \]
for any \( \psi \in \Omega^k(M) \otimes_M \Gamma(qM, qV \omega M) \), \( \psi_1 \) in \( \Omega^k(M) \otimes_M \Gamma(qM, qV \omega M) \), \( \psi_2 \) in \( \Omega^k(M) \otimes_M \Gamma(qM, qV \omega M) \), and then
\[ \hat{D}_{q\alpha} \otimes \hat{D}_{q\alpha} (\tau_1 \cdot \tau_2) = \hat{D}_{q\alpha} (\tau_1) \tau_2 + (-1)^{k} \tau_1 \hat{D}_{q\alpha} (\tau_2). \]

In this way one can induce a first–order antiderivation on \( \text{Hor}^*GM \) that satisfies Equations (14), (15). This implies the existence of a covariant derivative (see Definition 2.26)
\[ D^q_{\omega} : \text{Hor}^*GM \to \text{Hor}^*GM, \]
for a unique \( q\omega \) (\( [D4] \)). Now it should be clear that by construction, \( q\tilde{\mathcal{G}} \) and \( q\text{Ass}^q_{\mathcal{G}} \) are natural isomorphic.

Objects in the category \( \mathcal{qGTS}_{qM} \) of quantum gauge theory sectors on \( qM \) with quantum connections are tuples \( (qG, q\tilde{\mathcal{G}}) \) formed by a \( cqG \) and a contraviariant functor \( q\tilde{\mathcal{G}} \) between \( \text{Rep}_{qG} \) and \( \mathcal{qVB}^\mathcal{G}_{qM} \) that satisfy hypothesis of Theorem 3.11. In \( \mathcal{qGTS}_{qM} \) morphisms between two objects \( (qG_1, q\tilde{\mathcal{G}}_1), (qG_2, q\tilde{\mathcal{G}}_2) \) are pairs \( (h, NT) \) where \( h \) is a \(*\)-Hopf algebra morphism and
\[ NT : q\tilde{\mathcal{G}}_1 \to q\tilde{\mathcal{G}}_2 \hat{h} \]
is a natural transformation with
\[ \hat{h} : \text{Rep}_{qG_1} \to \text{Rep}_{qG_2} \]
the graded–preserving covariant functor given by
\[ \hat{h} q\alpha = (\text{id}_V \otimes \hat{h}) \circ q\alpha \]
if \( q\alpha \) coacts on \( V \) (notice that \( \hat{h} q\alpha \) coacts on \( V \) as well) and
\[ \hat{h}(f) = f \]
for morphisms such that
(1) \( NT q\alpha \) is always a degree 0 morphism.
(2) \( NT \) commutes with \( - \) and \( \otimes \).
Now it is possible to interpret the quantum association functor $q\text{Ass}^\omega_{q\zeta}$ as another functor (see Definition 2.30)

$$q\text{Ass} : q\text{PB}_{qM}^\omega \longrightarrow q\text{GTS}_{qM}^\omega$$

such that on objects is defined by

$$(qG, q\zeta, q\omega) \mapsto (qG, q\text{Ass}^\omega_{q\zeta})$$

and morphisms by means of

$$q\text{Ass}(h, F) := (h, \hat{F}),$$

where

$$\hat{F} : q\text{Ass}^\omega_{q\zeta_1} \longrightarrow q\text{Ass}^\omega_{q\zeta_2} \hat{h}$$

is the natural transformation given by

$$\hat{F} q\alpha : q\text{Ass}^\omega_{q\zeta_1} q\alpha \longrightarrow q\text{Ass}^\omega_{q\zeta_2} \hat{h} q\alpha \quad T \mapsto F \circ T.$$

The next theorem is the propose of this paper and finally we have the tools to prove it. One can check [SW] to compare this theorem with its classical version.

**Theorem 3.12.** For every quantum space $qM$, the functor $q\text{Ass}$ provides us with an equivalence of categories from $q\text{PB}_{qM}^\omega$ to $q\text{GTS}_{qM}^\omega$.

**Proof.** By Theorem 3.11 every $(qG, q\hat{\mathcal{F}}) \in \text{Obj}(q\text{GTS}_{qM}^\omega)$ is isomorphic in $q\text{GTS}_{qM}^\omega$ to $(qG, q\text{Ass}^\omega_{q\zeta})$ for a suitable principal $qG$-bundle over $qM$ with a quantum principal connection $q\omega$; so in order to prove the statement we just need to show $q\text{Ass}$ induces a bijection between

$$\text{Mor}_{q\text{PB}_{qM}^\omega}((qG_1, q\zeta_1, q\omega_1), (qG_2, q\zeta_2, q\omega_2))$$

and

$$\text{Mor}_{q\text{GTS}_{qM}^\omega}((qG_1, q\text{Ass}^\omega_{q\zeta_1}), (qG_2, q\text{Ass}^\omega_{q\zeta_2}))$$

for arbitrary $(qG_1, q\zeta_1, q\omega_1), (qG_2, q\zeta_2, q\omega_2) \in \text{Obj}(q\text{PB}_{qM}^\omega)$. According to our previous results we know that

$$\mathcal{G}M_i \cong \bigoplus_{q\alpha_i \in \mathcal{T}_i} \Gamma(qM, qV^{q\alpha_i}M) \otimes V^{q\alpha_i},$$

$$\text{Hor}^*\mathcal{G}M_i \cong \bigoplus_{q\alpha_i \in \mathcal{T}_i} \Omega^*(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\alpha_i}M) \otimes V^{q\alpha_i},$$

and

$$\mathcal{G}M_i \Phi \cong \bigoplus_{q\alpha_i \in \mathcal{T}_i} \text{id}_{\Gamma(qM, qV^{q\alpha_i}M)} \otimes q\alpha_i, \quad h^\Phi \cong \bigoplus_{q\alpha_i \in \mathcal{T}_i} \text{id}_{\Omega^*(\mathcal{M})} \otimes_{\mathcal{M}} \text{id}_{\Gamma(qM, qV^{q\alpha_i}M)} \otimes q\alpha_i,$$

where we are assuming that $q\alpha_i$ coacts on $V^{q\alpha_i}$ and $\mathcal{T}_i$ is a complete set of mutually nonequivalent irreducible finite dimensional $qG_i$-representations for $i = 1, 2$. Also the maps

$$D_{q\alpha_i} = \Upsilon_{q\alpha_i}^{-1} \circ d^\omega_{q\alpha_i} \circ \Upsilon_{q\alpha_i}$$
consistently recreate the covariant derivative $D^{q\omega_i}$ and hence the qpc $q\omega_i$ \cite{[D4]}. Taking $(h,\text{nt}) \in \text{Mor}_{\mathbf{qGTS}^\text{ev}_\Gamma}((qG_1, q\text{Ass}_{q\zeta_1}^{q\omega_1}), (qG_2, q\text{Ass}_{q\zeta_2}^{q\omega_2}))$, let us define

$$F: \bigoplus_{q\omega_1 \in T} \Omega^*(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\omega_1}M) \otimes V^{q\omega_1} \rightarrow \bigoplus_{q\omega_2 \in \mathcal{T}_2} \Omega^*(\mathcal{M}) \otimes_{\mathcal{M}} \Gamma(qM, qV^{q\omega_2}M) \otimes V^{q\omega_2}$$

such that

$$F(\mu \otimes_{\mathcal{M}} T \otimes v) := \bigoplus_{q\omega_2 \in \mathcal{T}_2} \mu \otimes_{\mathcal{M}} T_{q\omega_2} \otimes v_{q\omega_2},$$

if $T \in \Gamma(qM, qV^{q\omega_1}M)$ and

$$\text{nt } q\omega_1(T) = \bigoplus_{q\omega_2 \in \mathcal{T}_2} T_{q\omega_2} \subseteq \Gamma(qM, qV^{q\omega_2}M),$$

$$v = \bigoplus_{q\omega_2 \in \mathcal{T}_2} v_{q\omega_2} \in V^{q\omega_1} = \bigoplus_{q\omega_2 \in \mathcal{T}_2} V^{q\omega_2},$$

where in the last three expressions we have used the same finite number of corepresentations \{q\omega_2\} $\in \mathcal{T}_2$. Now a direct calculation using the defined $\ast$–algebra structure and properties of nt proves that $(h, F) \in \text{Mor}_{\mathbf{qPB}^\text{ev}_{q\Lambda}}((qG_1, q\zeta_1, q\omega_1), (qG_2, q\zeta_2, q\omega_2))$ and by construction $\hat{F} = \text{nt}$. This implies that $\text{qAss}(h, F) = (h, \text{nt})$. On the other hand if $\text{qAss}(h_1, F_1) = (h_1, \hat{F}_1) = (h_2, \hat{F}_2) = \text{qAss}(h_2, F_2)$ it is clear that $h_1 = h_2$ and since $\hat{F}_1 = \hat{F}_2$ we get that $F_1 \circ T = F_2 \circ T$ for all $T \in \Gamma(qM, qV^{q\omega_1}M)$ and all $q\omega_1 \in \mathcal{T}_1$. Considering the decomposition of Hor$^\ast \mathcal{G} M_1$ into the direct sum it follows that $F_1 = F_2$, so $(h_1, F_1) = (h_2, F_2)$. This completes the proof. \hfill \blacksquare

4. Concluding Comments

First of all we going to talk about the differential calculus that we have used on quantum groups. In general, it is enough to consider a differential calculus (covering the $\ast$–FODC $(\mathcal{G}, d)$) that allows us to extend the comultiplication map $\phi: \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$. These kinds of differential calculus are bicovariant. For example, in \cite{[W2]} we can see the definition of another differential calculus called the braided exterior calculus which leaves to extend $\phi$ as well. It can be proven that the universal differential $\ast$–calculus is maximal in this sense \cite{[D1], [SZ]}, and the braided exterior calculus is minimal \cite{[W2]}. The maps getting by this universal properties are such that they reducing to the identity on $\Gamma$ and $\mathcal{G}$. Also we have to highlight that definition of the universal differential $\ast$–calculus is independent of the quantum group structure on $q\mathcal{G}$.

Notice we have changed the traditional definitions of corepresentation morphism and qvb morphism in order to define the $\ast$ operation in the right side of Equation \cite{[24]}. If we forget the $\ast$ structure in the whole work, i.e., if we just work with (associative unital) algebras, this change in morphism would not be necessary. This is a recent point of view in noncommutative geometry in which one can consider that the $\ast$ structure is not an essential initial condition \cite{[D7]}. Of course, this gives us a richer and more general theory.

All our conditions presented in Remark \cite{[227]} are not just technical, specially the first one. For example, in \cite{[D5]} one can check that there is a bijective relation between the existence of \{T^L_k\}_k=1 and $d$–classifying maps $(\rho, \gamma)$. Reference \cite{[D6]} shows a noncommutative–geometric
generalization of classical Weil Theory of characteristic classes for qpb considering regular qpc and in the general case; in particular, one can find the noncommutative counterpart of the Chern character for structures admitting regular connections. In addition, one also uses this particular set of $M$–generators in the theory presented in [D4], in which we based as well; to say nothing of using the both set of $M$–generators it can be proven that $\text{Mor}_{\text{qG}}^0(\mathcal{G}_\alpha, \varphi_M \Phi)$ is a finitely generated projective left–right $M$–module [D2]. Even more, Remark 2.27 presents sufficiente condition for built the quantum association bundle and get the categorical equivalence; in clearly difference with the classical case in which for each principal bundle with a principal connection the association functor always exists [SW].

Now we have to talk about the regularity condition that we have imposed in quantum principal connections. Regularity condition leaves us to prove that $\nabla_{\mathcal{G}_\alpha}^{\varphi_\alpha} = T_{\mathcal{G}_\alpha} \circ D_{\mathcal{G}_\alpha}^\varphi \circ T$ satisfies the right Leibniz rule as we defined it in Definition 2.10; also it is crucial in the theory developed in [D4], which allows us to built quantum connections using just a few structures, not the complet differential calculus. Our definition of qpb morphisms is based on this fact (see Definition 2.29) and we used it strongly in Theorem 3.11 so it intervenes implicitly in Theorem 3.12. If we remove this condition and if we still want to have a quantum association functor, we would have to consider qlc as linear maps that just satisfy the left Leibniz rule and the morphism $\sigma$ would not be necessary, at least for this definition. In addition there would be certain properties that would not be fulfilled. For example, the quantum association functor would not satisfies Theorem 3.7 and we could not get a qpc in Theorem 3.11 since this reconstruction is based on the regularity condition [D4]. Nevertheless putting away all qpcs that are not regular would be a mistake that would separate us from many interesting examples to study, even if we do not have a categorical equivalence for these qpcs. It is important to emphasize that every (classical) principal connection is regular and multiplicative.

Finally and as we have mentioned before, the theory developed in this paper is an extention of the one presented in [D3]: we have considered qpcs with the same importance that qpbs. In [D3] we can see some examples that show us this theory is not trivial. Clearly these examples also work if we consider regular qpcs on these quantum bundles. We must remark that all our imposed conditions (or restrictions) are just properties that guarantees us noncommutative generalizations of classical conditions [SW]. The fact that we were able to recreate the classical categorical equivalence could tell us that our definitions of qvb, qlc, qpb, qpc and associated qvb are the correct ones.

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