Every toroidal graph without triangles adjacent to 5-cycles is DP-4-colorable

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Abstract

DP-coloring, also known as correspondence coloring, is introduced by Dvořák and Postle. It is a generalization of list coloring. In this paper, we show that every connected toroidal graph without triangles adjacent to 5-cycles has minimum degree at most three unless it is a 2-connected 4-regular graph with Euler characteristic ϵ(G) = 0. Consequently, every toroidal graph without triangles adjacent to 5-cycles is DP-4-colorable. In the final, we show that every planar graph without two certain subgraphs is DP-4-colorable. As immediate consequences, (i) every planar graph without 3-cycles adjacent to 4-cycles is DP-4-colorable; (ii) every planar graph without 3-cycles adjacent to 5-cycles is DP-4-colorable; (iii) every planar graph without 4-cycles adjacent to 5-cycles is DP-4-colorable.

1 Introduction

All graphs in this paper are finite, undirected and simple. DP-coloring, also known as correspondence coloring, is introduced by Dvořák and Postle [12].

Definition 1. Let G be a graph. A cover of G is a pair (L, H), consisting of a graph H and a function L : V(G) → Pow(V(H)), satisfying the following requirements:

(C1) the sets \{L(u) : u ∈ V(G)\} form a partition of V(H);
(C2) for every u ∈ V(G), the graph H[L(u)] is complete;
(C3) if E_H(L(u), L(v)) ≠ ∅, then either u = v or uv ∈ E(G);
(C4) if uv ∈ E(G), then E_H(L(u), L(v)) is a matching.

Note that the matching in Definition 1 (C4) is not required to be a perfect matching, and possibly it is empty. A cover (L, H) of G is k-fold if |L(u)| = k for all u ∈ V(G).

Definition 2. Let G be a graph. If (L, H) is a cover of G, then an (L, H)-coloring is an independent set in H of size |V(G)|. The DP-chromatic number χ_D(P)(G) of G is the least integer k such that G has a (L, H)-coloring whenever (L, H) is a k-fold cover of G. A graph G is DP-k-colorable if its DP-chromatic number is at most k.

To see that DP-coloring is a generalization of list coloring, we define H be a graph with vertex set

\{(u, c) : u ∈ V(G) and c ∈ L(u)\},

in which two distinct vertices (u, c) and (v, d) are adjacent if and only if

- either u = v;
- or else uv ∈ E(G) and c = d.

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It is observe that $G$ has an $(L, H)$-coloring if and only if $G$ is $L$-colorable. Dvořák and Postle [12] presented a non-trivial application of DP-coloring to solve a longstanding conjecture by Borodin [8], showing that every planar graph without cycles of lengths 4 to 8 is 3-choosable. Another application of DP-coloring can be found in [3], Bernshteyn and Kostochka extended the Dirac’s theorem on the minimum number of edges in critical graphs to the Dirac’s theorem on the minimum number of edges in DP-critical graphs, yielding a solution to the problem posed by Kostochka and Stiebitz [18].

A cover $(L, H)$ is a degree-cover if $|L(v)| \geq \deg_G(v)$ for all $v \in V(G)$. A graph $G$ is DP-degree-colorable if $G$ is $(L, H)$-colorable whenever $(L, H)$ is a degree-cover. A GDP-tree is a connected graph in which every block is either a cycle or a complete graph.

Bernshteyn, Kostochka, and Pron [7] gave a Brooks’ type result for DP-coloring. More detailed characterization see [15].

**Theorem 1.1** (Bernshteyn, Kostochka, and Pron [7]). Let $G$ be a connected graph. The graph $G$ is not DP-degree-colorable if and only if $G$ is a GDP-tree.

Dvořák and Postle [12] showed that every planar graph is DP-5-colorable, and observe that $\chi_{DP}(G) \leq k + 1$ if $G$ is $k$-degenerate. Thomassen [22] showed that every planar graph is 5-choosable, and Voigt [23] showed that there are planar graphs which are not 4-choosable. Thus it is interesting to give sufficient conditions for planar graphs to be 4-choosable. As a generalization of list coloring, it is also interesting to give sufficient conditions for planar graphs to be DP-4-colorable. Kim and Ozeki [16] showed that each $k \in \{3, 4, 5, 6\}$, every planar graph without $C_k$ is DP-4-colorable. Two cycles are adjacent if they have at least one edge in common; two cycles are normally adjacent if they have exactly one edge in common. Kim and Yu [17] showed that every planar graph without triangles adjacent to 4-cycles is DP-4-colorable. Some other materials on DP-coloring, see [1, 2, 4–6].

A graph $G$ is a minimal non-DP-$k$-colorable graph if it is not DP-$k$-colorable, but every proper subgraph of $G$ is DP-$k$-colorable. A toroidal graph is a graph that can be embedded in a torus. Any graph which can be embedded in a plane can also be embedded in a torus, thus every planar graph is also a toroidal graph.

In section 2, we give some structural results on the minimal non-DP-$k$-colorable graphs. In section 3, we show that every connected toroidal graph without triangles adjacent to 5-cycles has minimum degree at most three unless it is a 2-connected 4-regular graph with Euler characteristic $\epsilon(G) = 0$. Consequently, every toroidal graph without triangles adjacent to 5-cycles is DP-4-colorable. In section 4, we show that every planar graph without subgraphs isomorphic to the configurations in Fig. 2 is DP-4-colorable.

**Theorem 1.2.** Every connected toroidal graph without triangles adjacent to 5-cycles has minimum degree at most three unless it is a 2-connected 4-regular graph with Euler characteristic $\epsilon(G) = 0$.

**Theorem 1.3.** Every toroidal graph without triangles adjacent to 5-cycles is DP-4-colorable.

**Theorem 1.4.** Every planar graph without subgraphs isomorphic to the configurations in Fig. 2 is DP-4-colorable.

## 2 Structural results on minimal non-DP-$k$-colorable graphs

In this section, we may assume that $G$ is a minimal non-DP-$k$-colorable graph. Let $(L, H)$ be a $k$-fold cover of $G$ having no $(L, H)$-coloring. Let $V_k$ be the set of all the vertices having degree $k$.

**Lemma 1.** The graph $G$ is connected and the minimum degree is at least $k$.

**Proof.** It is observe that $G$ is connected. Suppose that $G$ has a vertex $w$ of degree at most $k − 1$. By the minimality of $G$, the graph $G' = G − w$ has an $(L', H')$-coloring $I'$, where $L'(v) = L(v)$ for each $v \neq w$, and $H' = H − L(w)$. Note that $L(w) − N_H(I')$ is nonempty, so we can choose a color $c$ in $L(w) − N_H(I')$ such that $I = I' \cup \{c\}$ is an $(L, H)$-coloring of $G$. □

**Lemma 2.** If $F$ is a 2-connected subgraph of $G$ and $V(F) \subseteq V_k$, then the subgraph induced by $V(F)$ is a cycle or a complete graph.
Proof. By the minimality of \( G \), the graph \( G' = G - V(F) \) has an \((L', H')\)-coloring \( I' \), where \( L'(v) = L(v) \) for each \( v \in V(G) - V(F) \), and

\[
H' = H - \bigcup_{v \in V(F)} L(v).
\]

Let \( L^*(v) = L(v) - N_H(I') \) for each \( v \in V(F) \) and

\[
H^* = H - \bigcup_{v \notin V(F)} L(v).
\]

Note that \( |L^*(v)| \geq |L(v)| - \deg_{G'}(v) = k - \deg_{G'}(v) = \deg_{G[V(F)]}(v) \), thus \((L^*, H^*)\) is a degree-cover of \( G[V(F)] \). If \( G[V(F)] \) has an \((L^*, H^*)\)-coloring \( I' \), then \( I' \cup I^* \) is an \((L, H)\)-coloring of \( G \), a contradiction. Thus, \( G[V(F)] \) has no \((L^*, H^*)\)-coloring. The graph \( F \) is 2-connected, thus \( G[V(F)] \) is also 2-connected. By Theorem 1.1, \( G[V(F)] \) must be a cycle or a complete graph. \( \square \)

3 Toroidal graph without triangles adjacent to 5-cycles

We recall our structural result on toroidal graphs without triangles adjacent to 5-cycles.

Theorem 1.2. Every connected toroidal graph without triangles adjacent to 5-cycles has minimum degree at most three unless it is a 2-connected 4-regular graph with Euler characteristic \( e(G) = 0 \).

Proof. Suppose that \( G \) is a connected toroidal graph without triangles adjacent to 5-cycles and the minimum degree is at least four. We may assume that \( G \) has been 2-cell embedded in the plane or torus.

The Euler's formula \(|V| - |E| + |F| = e(G)\) for \( G \) can be rewritten as the following:

\[
\sum_{v \in V(G)} (2 \deg(v) - 6) + \sum_{f \in F(G)} (\deg(f) - 6) = -6e(G) \leq 0. \tag{1}
\]

Initially, we give every vertex \( v \) an initial charge \( \mu(v) = 2 \deg(v) - 6 \), and give every face \( f \) an initial charge \( \mu(f) = \deg(f) - 6 \). Note that every vertex has a positive initial charge and every face has a nonnegative initial charge unless it is a \( 5^- \)-face. Next, we redistribute the charges between the vertices and the \( 5^- \)-faces, preserving their sum, such that the final charge \( \mu'(x) \) of every element \( x \in V(G) \cup F(G) \) is nonnegative.

R1 Each 3-face receives 1 from each incident vertex;

R2 Each 4-face receives \( \frac{1}{2} \) from each incident vertex;

R3 Each 5-face receives \( \frac{1}{2} \) from each incident vertex.

The final charges. Note that no triangle is adjacent to 5-cycle, it is easy to obtain the following claim.

Claim 1. (i) There is no three consecutive 3-faces. (ii) If \( w w_1 \) is incident with a 3-face and a \( 4^+ \)-face \( f \), then \( f \) must be a \( 6^- \)-face. Consequently, if \( v \) is incident with at least one 3-face, then \( v \) is incident with at least two \( 6^- \)-faces.

Each 3-face has final charge \( \mu'(v) = 3 - 6 + 3 \cdot 1 = 0 \) by R1. Each 4-face has final charge \( \mu'(v) = 4 - 6 + 4 \cdot \frac{1}{2} = 0 \) by R2. Each 5-face has final charge \( \mu'(v) = 5 - 6 + 5 \cdot \frac{1}{2} = \frac{3}{2} > 0 \) by R3. Each 6-face has final charge zero and each \( 7^+ \)-face has positive final charge.

If \( v \) is incident with at least one 3-face, then \( v \) is incident with at least two \( 6^- \)-faces due to Claim 1, thus it has final charge \( \mu'(v) \geq 2 \deg(v) - 6 - (\deg(v) - 2) \cdot 1 = \deg(v) - 4 \geq 0 \), and the equality holds only if \( v \) is a 4-vertex. If \( v \) is not incident with any 3-face, then \( v \) has final charge \( \mu'(v) \geq 2 \deg(v) - 6 - \deg(v) \cdot \frac{1}{2} = \frac{3}{2} \deg(v) - 6 \geq 0 \), and the equality holds only if \( v \) is a 4-vertex.

Hence, every element in \( V(G) \cup F(G) \) has a nonnegative final charge. By (1), every element in \( V(G) \cup F(G) \) has final charge zero, thus \( G \) is 4-regular, \( e(G) = 0 \) and \( F(G) \) has only 3-, 4- and 6-faces. If \( w \) is incident with a cut-edge \( w w_1 \), then \( w w_1 \) is incident with an \( 8^+ \)-face, a contradiction.

Suppose that \( w \) is a cut-vertex but it is not incident with any cut-edge. Note that \( G \) is 4-regular, it follows that \( G - w \) has exactly two components \( C_1 \) and \( C_2 \), and \( w \) has exactly two neighbors in each of \( C_1 \) and \( C_2 \). We may assume that
Fig. 1: The $F_{3,5}$-subgraph

\[ \begin{array}{c}
Fig. 2: Forbidden configurations
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\[ \begin{array}{c}
w_1, w_2, w_3, w_4 \text{ are four incident edges in the cyclic order and } w_1, w_2 \in V(C_1) \text{ and } w_3, w_4 \in V(C_2). \text{ It is observe that the face incident with } w_2 \text{ and } w_3 \text{ is also incident with } w_1 \text{ and } w_4. \text{ Thus it must be a 6-face with boundary } w_1w_2w_3w_4w. \text{ Since } G \text{ is a simple 4-regular graph, } w \text{ cannot incident with two 3-faces, this implies that the final charge of } w \text{ is positive, a contradiction. Hence, } G \text{ has no cut-vertex and it is a 2-connected 4-regular graph.} \quad \Box
\end{array} \]

Corollary 1 (Cai, Wang and Zhu [10]). Every connected toroidal graph without 5-cycles has minimum degree at most three unless it is a 4-regular graph.

The following corollary is a direct consequence of Theorem 1.2, which is stronger than that every planar graph without 5-cycles is 3-degenerate [24].

Corollary 2. Every planar graph without triangles adjacent to 5-cycles is 3-degenerate.

Remark 1. Note that not every toroidal graph without triangles adjacent to 5-cycles is 3-degenerate. For example, the Cartesian product of an $m$-cycle and an $n$-cycle is a 2-connected 4-regular graph with Euler characteristic $\epsilon(G) = 0$.

We recall our first main result on DP-coloring.

Theorem 1.3. Every toroidal graph without triangles adjacent to 5-cycles is DP-4-colorable.

Proof. Suppose that $G$ is a minimal counterexample to Theorem 1.3. It is observe that $G$ is a minimal non-DP-4-colorable graph. By Lemma 1, $G$ is connected and the minimum degree is at least four. By Theorem 1.2, $G$ is a 2-connected 4-regular graph. By Lemma 2, the graph $G$ can only be a complete graph on five vertices, but this contradicts the fact that $G$ has no triangles adjacent to 5-cycles. \quad \Box

Corollary 3 (Cai, Wang and Zhu [10]). (i) Every toroidal graph without 3-cycles is 4-choosable. (ii) Every toroidal graph without 5-cycles is 4-choosable.

4 Certain planar graphs

An $F_{3,5}$-subgraph of a graph $G$ is a subgraph isomorphic to a 6-cycle $v_1v_2v_3v_4v_5v_6$ with a chord $v_1v_5$ and all the vertices having degree four in $G$, see Fig. 1. Lam, Xu and Liu [19] showed that every planar graph without four cycles has minimum degree at most three unless it contains an $F_{3,5}$-subgraph. Borodin and Ivanova [9] further improved this
to that every planar graph without triangles adjacent to 4-cycles has minimum degree at most three unless it contains an $F_{3,5}$-subgraph. Kim and Yu [17] recovered this structure and showed that every planar graph without triangles adjacent to 4-cycles is DP-4-colorable.

Borodin and Ivanova [9] (independently, Cheng-Chen-Wang [11]) showed that every planar graph without triangles adjacent to 4-cycle is 4-choosable. Xu and Wu [25] showed that a planar graph without 5-cycles simultaneously adjacent to 3-cycles and 4-cycles is 4-choosable. Actually, they gave the following stronger structural result.

**Theorem 4.1** (Xu and Wu [25]). If $G$ is a planar graph without subgraphs isomorphic to the configurations in Fig. 2, then it has minimum degree at most three unless it contains an $F_{3,5}$-subgraph.

We recall our second main result on DP-coloring.

**Theorem 1.4.** Every planar graph without subgraphs isomorphic to the configurations in Fig. 2 is DP-4-colorable.

**Proof.** Suppose to the contrary that $G$ is a minimal counterexample to Theorem 1.4. It is observe that $G$ is a minimal non-DP-4-colorable graph. By Theorem 4.1, the minimum degree of $G$ is at most three or $G$ contains an $F_{3,5}$-subgraph, but this contradicts Lemma 1 and Lemma 2. □

**Remark 2.** Each of the graph in Fig. 2 contains a 3-cycle, a 4-cycle and a 5-cycle, and these three short cycles are mutually adjacent. Thus,

(i) every planar graph without 3-cycles adjacent to 4-cycles is DP-4-colorable;

(ii) every planar graph without 3-cycles adjacent to 5-cycles is DP-4-colorable;

(iii) every planar graph without 4-cycles adjacent to 5-cycles is DP-4-colorable.

**Remark 3.** Theorem 4.1 cannot be extended to toroidal graphs; once again, the Cartesian product of an $m$-cycle and an $n$-cycle is a counterexample. Thus, it is interesting to extend Theorem 1.4 to toroidal graphs.

Kim and Ozeki [16] pointed out that DP-coloring is also a generalization of signed (list) coloring of a signed graph $(G, \sigma)$, thus Theorem 1.4 implies the following result, which partly extends that in [14, Theorem 3.5]. For details on signed (list) coloring of signed graph, we refer the reader to [13, 14, 20, 21].

**Theorem 4.2.** If $(G, \sigma)$ be a signed planar graph and $G$ has no subgraphs isomorphic to the configurations in Fig. 2, then $(G, \sigma)$ is signed 4-choosable.

**Acknowledgments.** This work was supported by the National Natural Science Foundation of China (xxxxxxxxx) and partially supported by the Fundamental Research Funds for Universities in Henan (YQPY20140051).

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