DONOGHUE $m$-FUNCTIONS FOR SINGULAR STURM–LIOUVILLE OPERATORS

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Dedicated to Sergey Naboko (1950–2020): Friend and Mathematician Extraordinaire

Abstract. Let $\hat{A}$ be a densely defined, closed, symmetric operator in the complex, separable Hilbert space $\mathcal{H}$ with equal deficiency indices and denote by $\mathcal{N}_i = \ker((\hat{A})^* - iI\mathcal{H})$, $\dim(\mathcal{N}_i) = k \in \mathbb{N} \cup \{\infty\}$, the associated deficiency subspace of $\hat{A}$. If $A$ denotes a self-adjoint extension of $\hat{A}$ in $\mathcal{H}$, the Donoghue $m$-operator $M^{\mathcal{N}_i}_{\mathcal{N}_i}(\cdot)$ in $\mathcal{N}_i$ associated with the pair $(A, \mathcal{N}_i)$ is given by

$$M^{\mathcal{N}_i}_{\mathcal{N}_i}(z) = zI_{\mathcal{N}_i} + (z^2 + 1)P_{\mathcal{N}_i}(A - zI\mathcal{H})^{-1}P_{\mathcal{N}_i} |_{\mathcal{N}_i}, \quad z \in \mathbb{C}\setminus\mathbb{R},$$

with $I_{\mathcal{N}_i}$ the identity operator in $\mathcal{N}_i$, and $P_{\mathcal{N}_i}$ the orthogonal projection in $\mathcal{H}$ onto $\mathcal{N}_i$.

Assuming the standard local integrability hypotheses on the coefficients $p, q, r$, we study all self-adjoint realizations corresponding to the differential expression

$$\tau = \frac{1}{r(x)} \left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x)\right]$$

for a.e. $x \in (a, b) \subseteq \mathbb{R}$, in $L^2((a, b); rdx)$, and, as the principal aim of this paper, systematically construct the associated Donoghue $m$-functions (resp., $2 \times 2$ matrices) in all cases where $\tau$ is in the limit circle case at least at one interval endpoint $a$ or $b$.

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Date: January 13, 2022.

2020 Mathematics Subject Classification. Primary: 34B20, 34B24, 34L05; Secondary: 47A10, 47E05.

Key words and phrases. Singular Sturm–Liouville operators, boundary values, boundary conditions, Donoghue $m$-functions.
1. Introduction

Sergey’s contributions to operator and spectral theory are legendary and will pass the test of time. He had a very keen eye for the interface of complex analysis and operator theory, and was quite interested in all aspects of (operator-valued) \( m \)-functions. We hope our modest contribution would have been something he might have enjoyed.

To set the stage we briefly discuss abstract Donoghue \( m \)-functions following [36] (see also [33], [35]). Given a self-adjoint extension \( A \) of a densely defined, closed, symmetric operator \( \hat{A} \) in \( \mathcal{H} \) (a complex, separable Hilbert space) with equal deficiency indices and the deficiency subspace \( \mathcal{N}_i \) of \( \hat{A} \) in \( \mathcal{H} \), with

\[
\mathcal{N}_i = \ker \left( (\hat{A})^* - iI_\mathcal{H} \right), \quad \dim (\mathcal{N}_i) = k \in \mathbb{N} \cup \{ \infty \},
\]

the Donoghue \( m \)-operator \( M_{A,\mathcal{N}_i}^D (\cdot) \in B(\mathcal{N}_i) \) associated with the pair \((A, \mathcal{N}_i)\) is given by

\[
M_{A,\mathcal{N}_i}^D (z) = P_{\mathcal{N}_i} (zA + I_\mathcal{H}) (A - zI_\mathcal{H})^{-1} P_{\mathcal{N}_i} |_{\mathcal{N}_i}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

with \( I_{\mathcal{N}_i} \) the identity operator in \( \mathcal{N}_i \) and \( P_{\mathcal{N}_i} \) the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{N}_i \). The special case \( k = 1 \), was discussed in detail by Donoghue [26]; for the case \( k \in \mathbb{N} \) we refer to [40].

More generally, given a self-adjoint extension \( A \) of \( \hat{A} \) in \( \mathcal{H} \) and a closed, linear subspace \( \mathcal{N} \) of \( \mathcal{H} \), the Donoghue \( m \)-operator \( M_{A,\mathcal{N}}^D (\cdot) \in B(\mathcal{N}) \) associated with the pair \((A, \mathcal{N})\) is defined by

\[
M_{A,\mathcal{N}}^D (z) = P_{\mathcal{N}} (zA + I_\mathcal{H}) (A - zI_\mathcal{H})^{-1} P_{\mathcal{N}} |_{\mathcal{N}_i}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

with \( I_{\mathcal{N}} \) the identity operator in \( \mathcal{N} \) and \( P_{\mathcal{N}} \) the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{N} \).

Since \( M_{A,\mathcal{N}}^D (z) \) is analytic for \( z \in \mathbb{C} \setminus \mathbb{R} \) and satisfies (see [36, Theorem 5.3])

\[
[\text{Im}(z)]^{-1} \text{Im} \left( M_{A,\mathcal{N}}^D (z) \right) \geq 2 \left[ (|z|^2 + 1) + \left[ (|z|^2 - 1)^2 + 4(\text{Re}(z))^2 \right]^{1/2} \right]^{-1} I_{\mathcal{N}}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

\( M_{A,\mathcal{N}}^D (\cdot) \) is a \( B(\mathcal{N}) \)-valued Nevanlinna–Herglotz function. Thus, \( M_{A,\mathcal{N}}^D (\cdot) \) admits the representation

\[
M_{A,\mathcal{N}}^D (z) = \int_{\mathbb{R}} d\Omega_{A,\mathcal{N}}^D (\lambda) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right], \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

where the \( B(\mathcal{N}) \)-valued measure \( \Omega_{A,\mathcal{N}}^D (\cdot) \) satisfies

\[
\Omega_{A,\mathcal{N}}^D (\lambda) = (\lambda^2 + 1) (P_{\mathcal{N}} E_A (\lambda) P_{\mathcal{N}} |_{\mathcal{N}}),
\]

\[
\int_{\mathbb{R}} d\Omega_{A,\mathcal{N}}^D (\lambda) (1 + \lambda^2)^{-1} = I_{\mathcal{N}},
\]

\[
\int_{\mathbb{R}} d(\xi, \Omega_{A,\mathcal{N}}^D (\lambda) \xi) = \infty \quad \text{for all} \quad \xi \in \mathcal{N} \setminus \{0\},
\]

with \( E_A (\cdot) \) the family of strongly right-continuous spectral projections of \( A \) in \( \mathcal{H} \).
Operators of the type $M_{A,N_i}^\delta(\cdot)$ and some of its variants have attracted considerable attention in the literature. They appear to go back to Krein [52] (see also [53]), Saakjan [75], and independently, Donoghue [26]. The interested reader can find a wealth of additional information in the context of (1.2)–(1.8) in [3], [4], [6]–[9], [10], [11]–[15], [19]–[25], [33]–[40], [41], [45], [54], [55], [57], [59], [60], [62], [63]–[65], [68], [70], [74], and the references therein.

Without going into further details (see [36, Corollary 5.8] for details) we note that the prime reason for the interest in $M_{A,N_i}^\delta(\cdot)$ lies in the fundamental fact that the entire spectral information of $A$ contained in its family of spectral projections $E_A(\cdot)$, is already encoded in the $B(N_i)$-valued measure $\Omega_{A,N_i}^\delta(\cdot)$ (including multiplicity properties of the spectrum of $A$) if and only if $A$ is completely non-self-adjoint in $H$ (that is, if and only if $\hat{A}$ has no invariant subspace on which it is self-adjoint, see [36, Lemma 5.4]).

In the remainder of this paper, we will exclusively focus on the particular case $N = N_i = \ker((\hat{A})^* - iI_H)$ and develop a self-contained approach to constructing Donoghue $m$-functions (resp., $2 \times 2$ matrices) for singular Sturm–Liouville operators on arbitrary intervals $(a,b) \subseteq \mathbb{R}$. More precisely, assuming the standard local integrability hypotheses on the coefficients $p,q,r$ (cf. Hypothesis 2.1) we study all self-adjoint $L^2((a,b); rdx)$-realizations corresponding to the differential expression

$$\tau = \frac{1}{r(x)} \left[ -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] \text{ for a.e. } x \in (a,b) \subseteq \mathbb{R}, \quad (1.9)$$

and systematically determine the underlying Donoghue $m$-functions in all cases where $\tau$ is in the limit circle case at least at one interval endpoint $a$ or $b$.

Turning to the content of each section, we discuss the necessary background in connection with minimal $T_{\min}$ and maximal $T_{\max}$ operators, self-adjoint extensions, etc., corresponding to (1.9) in the underlying Hilbert space $L^2((a,b); rdx)$ in Section 2. In particular, we recall the discussion of boundary values in terms of appropriate Wronskians, especially, in the case where $T_{\min}$ is bounded from below (utilizing principal and nonprincipal solutions). Our strategy for the construction of Donoghue $m$-functions consists of first constructing them for the Friedrichs extension of $T_{\min}$ and then employing Krein-type resolvent formulas to derive Donoghue $m$-functions for the remaining self-adjoint extensions of $T_{\min}$. These Krein-type resolvent formulas use the Friedrichs extension as a reference operator and then explicitly characterize the resolvents of all the remaining self-adjoint extensions of $T_{\min}$ in terms of the Friedrichs extension and the deficiency subspaces for $T_{\min}$. Hence Sections 3 and 4 derive Krein-type resolvent formulas for singular Sturm–Liouville operators in the case where $\tau$ has one, respectively, two, interval endpoints in the limit circle case. Donoghue $m$-functions corresponding to the case where $\tau$ is in the limit circle case in precisely one interval endpoint are derived in Section 5; the case where $\tau$ is in the limit circle case at $a$ and $b$ is treated in detail in Section 6. We conclude this paper with an illustration of a generalized Bessel operator in Section 7 where $a = 0$, $b \in (0, \infty) \cup \{\infty\}$, and $\tau$ takes on the explicit form,

$$\tau_{\delta,\nu,\gamma} = x^{-\delta} \left[ -\frac{d}{dx} x^{\nu} \frac{d}{dx} + \frac{(2 + \delta - \nu)^2 \gamma^2 - (1 - \nu)^2}{4} x^{\nu - 2} \right], \quad (1.10)$$

$$\delta > -1, \ \nu < 1, \ \gamma \geq 0, \ x \in (0, b).$$
Finally, we comment on some of the basic notation used throughout this paper. If $T$ is a linear operator mapping (a subspace of) a Hilbert space into another, then $\text{dom}(T)$ and $\text{ker}(T)$ denote the domain and kernel (i.e., null space) of $T$. The spectrum and resolvent set of a closed linear operator in a Hilbert space will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$, respectively. Moreover, we typically abbreviate $L^2((a,b);rdx)$ as $L^2_2((a,b))$ in various subscripts involving the identity operator $I_{L^2_2((a,b))}$ and the scalar product $(\cdot,\cdot)_{L^2_2((a,b))}$ (linear in the second argument) and associated norm $\|\cdot\|_{L^2_2((a,b))}$ in $L^2((a,b);rdx)$.

2. Some Background

In this section we briefly recall the basics of singular Sturm–Liouville operators. The material is standard and can be found, for instance, in [5, Ch. 6], [18, Chs. 8, 9], [27, Sects. 13.6, 13.9, 13.10], [28], [42, Ch. 4], [46, Ch. III], [66, Ch. V], [67], [69, Ch. 6], [76, Ch. 9], [77, Sect. 8.3], [78, Ch. 13], [80, Chs. 4, 6–8]. Throughout this section we make the following assumptions:

**Hypothesis 2.1.** Let $(a, b) \subseteq \mathbb{R}$ and suppose that $p, q, r$ are (Lebesgue) measurable functions on $(a, b)$ such that the following items (i)–(iii) hold:

(i) $r > 0$ a.e. on $(a, b)$, $r \in L^1_{\text{loc}}((a,b);dx)$.

(ii) $p > 0$ a.e. on $(a, b)$, $1/p \in L^1_{\text{loc}}((a,b);dx)$.

(iii) $q$ is real-valued a.e. on $(a,b)$, $q \in L^1_{\text{loc}}((a,b);dx)$.

Given Hypothesis 2.1, we study Sturm–Liouville operators associated with the general, three-coefficient differential expression $\tau$ of the form

$$\tau = \frac{1}{r(x)} \left[ -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] \text{ for a.e. } x \in (a, b) \subseteq \mathbb{R}. \quad (2.1)$$

If $f \in AC_{\text{loc}}((a, b))$, then the quasi-derivative of $f$ is defined to be $f^{[1]} := pf'$. Moreover, the Wronskian of two functions $f, g \in AC_{\text{loc}}((a, b))$ is defined by

$$W(f, g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x) \text{ for a.e. } x \in (a, b). \quad (2.2)$$

The following result is useful for computing weighted integrals of products of solutions of $(\tau - z)y = 0$: Assume Hypothesis 2.1 and let $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$. If $y(z_j, \cdot)$ is a solution of $(\tau - z_j)y = 0$, $j \in \{1, 2\}$, then for all $\alpha < \alpha < \beta < b$,

$$\int_{\alpha}^{\beta} r(x)dx y(z_1, x)y(z_2, x) = \frac{W(y(z_1, \cdot), y(z_2, \cdot))|^{\beta}_{\alpha}}{z_1 - z_2}. \quad (2.3)$$

**Definition 2.2.** Assume Hypothesis 2.1. Given $\tau$ as in (2.1), the maximal operator $T_{\text{max}}$ in $L^2((a,b);rdx)$ associated with $\tau$ is defined by

$$T_{\text{max}}f = \tau f,$$

$$f \in \text{dom}(T_{\text{max}}) = \{g \in L^2((a, b);rdx) \mid g, g^{[1]} \in AC_{\text{loc}}((a, b)); \tau g \in L^2((a, b);rdx)\}. \quad (2.4)$$

The preminimal operator $\hat{T}_{\text{min}}$ in $L^2((a,b);rdx)$ associated with $\tau$ is defined by

$$\hat{T}_{\text{min}}f = \tau f,$$

$$f \in \text{dom}(\hat{T}_{\text{min}}) = \{g \in L^2((a, b);rdx) \mid g, g^{[1]} \in AC_{\text{loc}}((a, b))\}. \quad (2.5)$$
One can prove that \( \hat{T}_{\min} \) is closable, and one then defines the minimal operator \( T_{\min} \) as the closure of \( \hat{T}_{\min} \).

For \( f, g \in \text{dom}(T_{\max}) \), one can prove that the following limits exist:

\[
W(f, g)(a) = \lim_{x \downarrow a} W(f, g)(x) \quad \text{and} \quad W(f, g)(b) = \lim_{x \uparrow b} W(f, g)(x). \tag{2.6}
\]

In addition, one can prove the following basic fact:

**Theorem 2.3.** Assume Hypothesis 2.1. Then

\[
(\hat{T}_{\min})^* = T_{\max},
\]

and hence \( T_{\max} \) is closed and \( T_{\min} = \hat{T}_{\min} \) is given by

\[
T_{\min}f = \tau f,
\]

for all \( f \in \text{dom}(T_{\min}) \) in the case

\[
g \in L^2((a, b); rdx) \text{ for all } h \in \text{dom}(T_{\max}), W(h, g)(a) = 0 = W(h, g)(b); \tau g \in L^2((a, b); rdx) \}
\]

Moreover, \( \hat{T}_{\min} \) is essentially self-adjoint if and only if \( T_{\max} \) is symmetric, and then \( \hat{T}_{\min} = T_{\min} = T_{\max} \).

Regarding self-adjoint extensions of \( T_{\min} \) one has the following first result.

**Theorem 2.4.** Assume Hypothesis 2.1. An extension \( \hat{T} \) of \( \hat{T}_{\min} \) or of \( T_{\min} = \hat{T}_{\min} \) is self-adjoint if and only if

\[
\hat{T}f = \tau f,
\]

for all \( f \in \text{dom}(\hat{T}) \) if (resp.) \( W(f, g)(a) = W(f, g)(b) \) for all \( f \in \text{dom}(\hat{T}) \).

The celebrated Weyl alternative then can be stated as follows:

**Theorem 2.5** (Weyl’s Alternative).

Assume Hypothesis 2.1. Then the following alternative holds: Either

(i) for every \( z \in \mathbb{C} \), all solutions \( u \) of \( (\tau - z)u = 0 \) are in \( L^2((a, b); rdx) \) near \( b \) (resp., near \( a \)), or,

(ii) for every \( z \in \mathbb{C} \), there exists at least one solution \( u \) of \( (\tau - z)u = 0 \) which is not in \( L^2((a, b); rdx) \) near \( b \) (resp., near \( a \)). In this case, for each \( z \in \mathbb{C} \setminus \mathbb{R} \), there exists precisely one solution \( u_b \) (resp., \( u_a \)) of \( (\tau - z)u = 0 \) (up to constant multiples) which lies in \( L^2((a, b); rdx) \) near \( b \) (resp., near \( a \)).

This yields the limit circle/limit point classification of \( \tau \) at an interval endpoint as follows.

**Definition 2.6.** Assume Hypothesis 2.1.

In case (i) in Theorem 2.5, \( \tau \) is said to be in the limit circle case at \( b \) (resp., \( a \)). (Frequently, \( \tau \) is then called quasi-regular at \( b \) (resp., \( a \)).)

In case (ii) in Theorem 2.5, \( \tau \) is said to be in the limit point case at \( b \) (resp., \( a \)).

If \( \tau \) is in the limit circle case at \( a \) and \( b \) then \( \tau \) is also called quasi-regular on \( (a, b) \).
Theorem 2.8. Assume Hypothesis $T$. In particular, at $\alpha_j = 1$ ($\alpha_j = 0$), then the following items hold:

(i) If $\tau$ is in the limit point case at $a$ (resp., $b$), then
\[
W(f,g)(a) = 0 \ (\text{resp.,} \ W(f,g)(b) = 0) \text{ for all } f, g \in \text{dom}(T_{\text{max}}).
\] (2.11)

(ii) Let $T_{\text{min}} = \overline{T}_{\text{min}}$. Then
\[
n_{\pm}(T_{\text{min}}) = \dim\left(\mathcal{N}_{\pm}\right)
\]
= \begin{cases} 
2 & \text{if } \tau \text{ is in the limit circle case at } a \text{ and } b, \\
1 & \text{if } \tau \text{ is in the limit circle case at } a \\
& \text{and in the limit point case at } b, \text{ or vice versa,} \\
0 & \text{if } \tau \text{ is in the limit point case at } a \text{ and } b.
\end{cases} \tag{2.12}

In particular, $T_{\text{min}} = T_{\text{max}}$ is self-adjoint if and only if $\tau$ is in the limit point case at $a$ and $b$.

All self-adjoint extensions of $T_{\text{min}}$ are then described as follows:

Theorem 2.8. Assume Hypothesis 2.1 and that $\tau$ is in the limit circle case at $a$ and $b$ (i.e., $\tau$ is quasi-regular on $(a,b)$). In addition, assume that $v_j \in \text{dom}(T_{\text{max}})$, $j = 1, 2$, satisfy
\[
W(\tilde{v}_1, v_2)(a) = W(\tilde{v}_1, v_2)(b) = 1, \quad W(\tilde{v}_j, v_j)(a) = W(\tilde{v}_j, v_j)(b) = 0, \quad j = 1, 2.
\] (2.13)

(E.g., real-valued solutions $v_j$, $j = 1, 2$, of $(\tau - \lambda)u = 0$ with $\lambda \in \mathbb{R}$, such that $W(v_1, v_2) = 1$.) For $g \in \text{dom}(T_{\text{max}})$ we introduce the generalized boundary values
\[
\tilde{g}_1(a) = -W(v_2, g)(a), \quad \tilde{g}_1(b) = -W(v_2, g)(b), \\
\tilde{g}_2(a) = W(v_1, g)(a), \quad \tilde{g}_2(b) = W(v_1, g)(b).
\] (2.14)

Then the following items (i)–(iii) hold:

(i) All self-adjoint extensions $T_{\alpha, \beta}$ of $T_{\text{min}}$ with separated boundary conditions are of the form
\[
T_{\alpha, \beta}f = \tau f, \quad \alpha, \beta \in [0, \pi), \tag{2.15}
\]
\[
f \in \text{dom}(T_{\alpha, \beta}) = \left\{ g \in \text{dom}(T_{\text{max}}) \mid \begin{array}{c} \cos(\alpha)\tilde{g}_1(a) + \sin(\alpha)\tilde{g}_2(a) = 0, \\
\cos(\beta)\tilde{g}_1(b) + \sin(\beta)\tilde{g}_2(b) = 0 \end{array} \right\}.
\]

(ii) All self-adjoint extensions $T_{\varphi, R}$ of $T_{\text{min}}$ with coupled boundary conditions are of the type
\[
T_{\varphi, R}f = \tau f, \tag{2.16}
\]
\[
f \in \text{dom}(T_{\varphi, R}) = \left\{ g \in \text{dom}(T_{\text{max}}) \mid \begin{array}{c} \tilde{g}_1(b) \overline{\tilde{g}_2(b)} = e^{i\varphi} R \left( \begin{array}{c} \tilde{g}_1(a) \\ \tilde{g}_2(a) \end{array} \right), \end{array} \right\},
\]
where $\varphi \in [0, 2\pi)$, and $R$ is a real $2 \times 2$ matrix with $\det(R) = 1$ (i.e., $R \in SL(2, \mathbb{R})$).

(iii) Every self-adjoint extension of $T_{\text{min}}$ is either of type (i) (i.e., separated) or of type (ii) (i.e., coupled).
Remark 2.9. (i) If $\tau$ is in the limit point case at one endpoint, say, at the endpoint $b$, one omits the corresponding boundary condition involving $\beta \in [0, \pi)$ at $b$ in (2.15) to obtain all self-adjoint extensions $T_{\alpha}$ of $T_{\min}$, indexed by $\alpha \in [0, \pi)$. (In this case item (iii) in Theorem 2.8 is vacuous.) In the case where $\tau$ is in the limit point case at both endpoints, all boundary values and boundary conditions become superfluous as in this case $T_{\min} = T_{\max}$ is self-adjoint.

(ii) In the special case where $\tau$ is regular on the finite interval $[a, b]$, choose $v_j \in \text{dom}(T_{\max})$, $j = 1, 2$, such that

$$v_1(x) = \begin{cases} \theta_0(\lambda, x, a), & \text{for } x \text{ near } a, \\ \theta_0(\lambda, x, b), & \text{for } x \text{ near } b, \end{cases}$$

$$v_2(x) = \begin{cases} \phi_0(\lambda, x, a), & \text{for } x \text{ near } a, \\ \phi_0(\lambda, x, b), & \text{for } x \text{ near } b, \end{cases}$$

(2.17)

where $\phi_0(\lambda, \cdot, d), \theta_0(\lambda, \cdot, d), d \in \{a, b\}$, are real-valued solutions of $(\tau - \lambda)u = 0$, $\lambda \in \mathbb{R}$, satisfying the boundary conditions

$$\phi_0(\lambda, a, a) = \phi_0[1](\lambda, a, a) = 0, \quad \theta_0(\lambda, a, a) = \phi_0[1](\lambda, a, a) = 1,$$

$$\phi_0(\lambda, b, b) = \phi_0[1](\lambda, b, b) = 0, \quad \theta_0(\lambda, b, b) = \phi_0[1](\lambda, b, b) = 1.$$ 

(2.18)

Then one verifies that

$$\tilde{g}_1(a) = g(a), \quad \tilde{g}_1(b) = g(b), \quad \tilde{g}_2(a) = g[1](a), \quad \tilde{g}_2(b) = g[1](b),$$

(2.19)

and hence Theorem 2.8 recovers the well-known special regular case.

(iii) In connection with (2.14), an explicit calculation demonstrates that for $g, h \in \text{dom}(T_{\max}),$

$$\tilde{g}_1(d)\tilde{h}_2(d) - \tilde{g}_2(d)\tilde{h}_1(d) = W(g, h)(d), \quad d \in \{a, b\},$$

(2.20)

interpreted in the sense that either side in (2.20) has a finite limit as $d \downarrow a$ and $d \uparrow b$. Of course, for (2.20) to hold at $d \in \{a, b\}$, it suffices that $g$ and $h$ lie locally in $\text{dom}(T_{\max})$ near $x = d$.

(iv) Clearly, $\tilde{g}_1, \tilde{g}_2$ depend on the choice of $v_j, j = 1, 2$, and a more precise notation would indicate this as $\tilde{g}_{1,v_2}, \tilde{g}_{2,v_1},$ etc.

(v) One can supplement the characterization (2.8) of $\text{dom}(T_{\min})$ by

$$T_{\min}f = \tau f, \quad f \in \text{dom}(T_{\min}) = \{g \in \text{dom}(T_{\max}) \mid \tilde{g}_1(a) = \tilde{g}_2(a) = \tilde{g}_1(b) = \tilde{g}_2(b) = 0\}.$$ 

(2.21)

$\diamond$

Next, we determine when two self-adjoint extensions of $T_{\min}$ are relatively prime with respect to $T_{\min}$.

Definition 2.10. If $T$ and $T'$ are self-adjoint extensions of a symmetric operator $S$, then the maximal common part of $T$ and $T'$ is the operator $C_{T,T'}$ defined by

$$C_{T,T'}u = Tu, \quad u \in \text{dom}(C_{T,T'}) = \{f \in \text{dom}(T) \cap \text{dom}(T') \mid Tf = T'f\}. \quad (2.22)$$

Moreover, $T$ and $T'$ are said to be relatively prime with respect to $S$ if $C_{T,T'} = S$.

Theorem 2.11. Assume Hypothesis 2.1.

(i) If $\alpha, \alpha', \beta, \beta' \in [0, \pi)$ with $\alpha \neq \alpha'$ and $\beta \neq \beta'$, then $T_{\alpha,\beta}$ and $T_{\alpha',\beta'}$ are relatively prime with respect to $T_{\min}$.
(ii) If \( \alpha, \beta, \beta' \in [0, \pi) \) with \( \beta \neq \beta' \), then the maximal common part of \( T_{\alpha, \beta} \) and \( T_{\alpha, \beta'} \) is the restriction of \( T_{\max} \) to the subspace

\[
\{ g \in \text{dom}(T_{\max}) \mid \cos(\alpha)\tilde{g}_1(a) + \sin(\alpha)\tilde{g}_2(a) = 0, \tilde{g}_1(b) = \tilde{g}_2(b) = 0 \}. 
\] (2.23)

(iii) If \( \alpha, \alpha', \beta \in [0, \pi) \) with \( \alpha \neq \alpha' \), then the maximal common part of \( T_{\alpha, \beta} \) and \( T_{\alpha', \beta} \) is the restriction of \( T_{\max} \) to the subspace

\[
\{ g \in \text{dom}(T_{\max}) \mid \tilde{g}_1(a) = \tilde{g}_2(a) = 0, \cos(\beta)\tilde{g}_1(b) + \sin(\beta)\tilde{g}_2(b) = 0 \}. 
\] (2.24)

(iv) Let \( \alpha, \beta \in [0, \pi), \varphi \in [0, 2\pi), R = (R_{j,k})^2_{j,k=1} \in \text{SL}(2, \mathbb{R}) \), and define

\[
d(\alpha, \beta, R) = (\cos(\alpha) \cos(\beta)R_{1,2} + \cos(\alpha) \sin(\beta)R_{2,2} - \sin(\alpha) \cos(\beta)R_{1,1} - \sin(\alpha) \sin(\beta)R_{2,1}).
\] (2.25)

If \( d(\alpha, \beta, R) \neq 0 \), then \( T_{\alpha, \beta} \) and \( T_{\varphi, R} \) are relatively prime with respect to \( T_{\min} \). If \( d(\alpha, \beta, R) = 0 \), then the maximal common part of \( T_{\alpha, \beta} \) and \( T_{\varphi, R} \) is the restriction of \( T_{\max} \) to the subspace

\[
\{ g \in \text{dom}(T_{\varphi, R}) \mid (\tilde{g}_1(b), \tilde{g}_2(b))^T \in V_1 \}. 
\] (2.26)

Proof. To prove (i), it suffices to show that \( f \in \text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\alpha', \beta'}) \) implies \( f \in \text{dom}(T_{\min}) \). If \( f \in \text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\alpha', \beta'}) \), then

\[
\begin{pmatrix}
\cos(\alpha) & \sin(\alpha) \\
\cos(\alpha') & \sin(\alpha')
\end{pmatrix}
\begin{pmatrix}
\tilde{f}_1(a) \\
\tilde{f}_2(a)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\] (2.28)

\[
\begin{pmatrix}
\cos(\beta) & \sin(\beta) \\
\cos(\beta') & \sin(\beta')
\end{pmatrix}
\begin{pmatrix}
\tilde{f}_1(b) \\
\tilde{f}_2(b)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. 
\] (2.29)

The determinants of the \( 2 \times 2 \) coefficient matrices in (2.28) and (2.29) are \( \sin(\alpha - \alpha') \) and \( \sin(\beta - \beta') \), respectively. Since the assumptions on \( \alpha, \alpha', \beta, \beta' \) imply \( \alpha - \alpha', \beta - \beta' \in (-\pi, \pi) \setminus \{0\} \), it follows that the coefficient matrices in (2.28) and (2.29) are invertible. Hence, \( \tilde{f}_1(a) = \tilde{f}_2(a) = \tilde{f}_1(b) = \tilde{f}_2(b) = 0 \), and the characterization of \( \text{dom}(T_{\min}) \) in (2.21) implies \( f \in \text{dom}(T_{\min}) \).

The proofs of (ii) and (iii) are similar, so we only provide the proof of (ii) here. Let \( D \) denote the set in (2.24). To prove (ii), it suffices to show \( \text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\alpha', \beta'}) = D \). If \( f \in \text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\alpha', \beta'}) \), then \( \cos(\alpha)\tilde{f}_1(a) + \sin(\alpha)\tilde{f}_2(a) = 0 \) and (2.29) holds. As in the proof of (i), the determinant of the \( 2 \times 2 \) coefficient matrix in (2.29) is nonzero. Therefore, \( \tilde{f}_1(b) = \tilde{f}_2(b) = 0 \), and it follows that \( f \in D \). Conversely, if \( f \in D \), then it is clear that \( f \) simultaneously belongs to \( \text{dom}(T_{\alpha, \beta}) \) and \( \text{dom}(T_{\alpha', \beta'}) \).

The proof of (iv) begins with a general observation about functions in the intersection \( \text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\varphi, R}) \). If \( f \in \text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\varphi, R}) \), then

\[
\begin{align*}
\cos(\alpha)\tilde{f}_1(a) + \sin(\alpha)\tilde{f}_2(a) &= 0, \\
\cos(\beta)\tilde{f}_1(b) + \sin(\beta)\tilde{f}_2(b) &= 0,
\end{align*}
\] (2.30)
and

\[
\begin{align*}
\bar{f}_1(b) &= e^{i\varphi}R_{1,1}\bar{f}_1(a) + e^{i\varphi}R_{1,2}\bar{f}_2(a), \\
\bar{f}_2(b) &= e^{i\varphi}R_{2,1}\bar{f}_1(a) + e^{i\varphi}R_{2,2}\bar{f}_2(a).
\end{align*}
\]  
(2.31)

Applying (2.31) in (2.30) yields a set of boundary conditions that may be recast in matrix form as

\[
\begin{pmatrix}
\cos(\alpha) & \sin(\alpha) \\
\cos(\beta)R_{1,1} + \sin(\beta)R_{2,1} & \cos(\beta)R_{1,2} + \sin(\beta)R_{2,2}
\end{pmatrix}
\begin{pmatrix}
\bar{f}_1(a) \\
\bar{f}_2(a)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]  
(2.32)

The determinant of the $2 \times 2$ coefficient matrix in (2.32) is $d(\alpha, \beta, R)$.

If $d(\alpha, \beta, R) \neq 0$, then (2.32) implies $\bar{f}_1(a) = \bar{f}_2(a) = 0$. In turn, (2.31) implies $\tilde{f}_1(b) = \tilde{f}_2(b) = 0$. Hence, $f \in \text{dom}(T_{\min})$, and it follows that $T_{\alpha, \beta}$ and $T_{\varphi, R}$ are relatively prime with respect to $T_{\min}$.

To complete the proof of (iv), it remains to show that the set in (2.26), call it $D$, coincides with $\text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\varphi, R})$ when $d(\alpha, \beta, R) = 0$. The containment $\text{dom}(T_{\alpha, \beta}) \cap \text{dom}(T_{\varphi, R}) \subset D$ follows immediately from the definitions of $T_{\alpha, \beta}$, $T_{\varphi, R}$, and $D$. To prove the reverse containment, let $f \in D$, so that $f \in \text{dom}(H_{\varphi, R})$ and $f$ satisfies the boundary condition at $a$ in (2.30). The proof is then reduced to showing $f$ satisfies the boundary condition at $b$ in (2.30). In order to do this, one distinguishes the cases $\alpha \neq 0$ and $\alpha = 0$. If $\alpha \neq 0$, one uses $d(\alpha, \beta, R) = 0$, the conditions in (2.31), and $\sin(\alpha)f_2(a) = -\cos(\alpha)f_1(a)$ to compute

\[
e^{-i\varphi}\sin(\alpha)\left[\cos(\beta)f_1(b) + \sin(\beta)f_2(b)\right]
\]  
(2.33)

\[
= [\cos(\beta)R_{1,1} + \sin(\beta)R_{2,1}]\sin(\alpha)f_1(a)
- [\cos(\beta)R_{1,2} + \sin(\beta)R_{2,2}]\cos(\alpha)f_1(a)
= -d(\alpha, \beta, R)f_1(a)
= 0.
\]

Since $e^{-i\varphi}\sin(\alpha) \neq 0$ when $\alpha \neq 0$, (2.33) implies $f$ satisfies the boundary condition at $b$ in (2.30). If $\alpha = 0$, then $f_1(a) = 0$, and (2.31) simplifies. One then computes

\[
\cos(\beta)f_1(b) + \sin(\beta)f_2(b) = e^{i\varphi}[\cos(\beta)R_{1,2} + \sin(\beta)R_{2,2}]f_2(a)
= e^{i\varphi}d(0, \beta, R)f_2(a)
= 0,
\]  
(2.34)

so $f$ satisfies the boundary condition at $b$ in (2.30).

To prove (v), let $f \in \text{dom}(T_{\varphi, R}) \cap \text{dom}(T_{\eta, S})$, so that

\[
\begin{pmatrix}
\bar{f}_1(b) \\
\bar{f}_2(b)
\end{pmatrix}
= e^{i\eta}S
\begin{pmatrix}
\bar{f}_1(a) \\
\bar{f}_2(a)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\bar{f}_1(b) \\
\bar{f}_2(b)
\end{pmatrix}
= e^{i\varphi}R
\begin{pmatrix}
\bar{f}_1(a) \\
\bar{f}_2(a)
\end{pmatrix}.
\]  
(2.35)

Using the invertibility of $e^{i\varphi}R$ to solve the second equation in (2.35) for the vector $(\bar{f}_1(a), \bar{f}_2(a))^\tau$ and substituting into the first equation in (2.35) yields

\[
\begin{pmatrix}
e^{i(\eta - \varphi)}SR^{-1} - I_{2}\end{pmatrix}
\begin{pmatrix}
\bar{f}_1(b) \\
\bar{f}_2(b)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]  
(2.36)
If det \((e^{i(\eta-\nu)}SR^{-1} - I_{C_2})\) ≠ 0, then (2.36) implies \(\tilde{f}_1(b) = \tilde{f}_2(b) = 0\). In turn, the invertibility of \(e^{i\varphi}R\) and the second equation in (2.35) yields \(\tilde{f}_1(a) = \tilde{f}_2(a) = 0\). Hence, \(f \in \text{dom}(T_{\min})\), and it follows that \(T_{\varphi,R}\) and \(T_{\eta,S}\) are relatively prime with respect to \(T_{\min}\).

Now, suppose that det \((e^{i(\eta-\nu)}SR^{-1} - I_{C_2})\) = 0, so that 1 is an eigenvalue of \(e^{i(\eta-\nu)}SR^{-1}\) with corresponding eigenspace \(V_1\). Let \(D\) denote the subspace in (2.27). To complete the proof of (v), it suffices to show the subspace \(D\) coincides with \(\text{dom}(T_{\varphi,R}) \cap \text{dom}(T_{\eta,S})\). To this end, let \(f \in \text{dom}(T_{\varphi,R}) \cap \text{dom}(T_{\eta,S})\), so that both equalities in (2.35) hold. In particular, (2.36) holds due to the invertibility of \(e^{i\varphi}R\), and one concludes that \((\tilde{f}_1(b), \tilde{f}_2(b))^\top \in V_1\). Therefore, \(f \in D\). Conversely, if \(f \in D\), then \(f \in \text{dom}(T_{\varphi,R})\), and one only needs to show \(f \in \text{dom}(T_{\eta,S})\) to complete the proof. Using the boundary conditions implied by the inclusion \(f \in \text{dom}(T_{\varphi,R})\) (i.e., the second equality in (2.35)), one computes

\[
e^{i\eta S} \begin{pmatrix} \tilde{f}_1(a) \\ \tilde{f}_2(a) \end{pmatrix} = e^{i(\eta-\nu)}SR^{-1} \begin{pmatrix} \tilde{f}_1(b) \\ \tilde{f}_2(b) \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(b) \\ \tilde{f}_2(b) \end{pmatrix},
\]

where the last equality in (2.37) follows from the fact that \((\tilde{f}_1(b), \tilde{f}_2(b))^\top \in V_1\) by the assumption \(f \in D\). The equality in (2.37) implies \(f \in \text{dom}(T_{\eta,S})\). \(\square\)

Finally, we turn to the characterization of generalized boundary values in the case where \(T_{\min}\) is bounded from below following [34] and [67].

We recall the basics of oscillation theory with particular emphasis on principal and nonprincipal solutions, a notion originally due to Leighton and Morse [56] (see also Rellich [71], [72] and Hartman and Wintner [44, Appendix]). Our outline below follows [16], [42, Sects. 13.6, 13.9, 13.10], [43, Ch. 7], [67], [80, Chs. 4, 6–8].

**Definition 2.12.** Assume Hypothesis 2.1.

(i) Fix \(c \in (a,b)\) and \(\lambda \in \mathbb{R}\). Then \(\tau - \lambda\) is called nonoscillatory at a (resp., b), if every real-valued solution \(u(\lambda, \cdot)\) of \(\tau u = \lambda u\) has finitely many zeros in \((a,c)\) (resp., \((c,b)\)). Otherwise, \(\tau - \lambda\) is called oscillatory at a (resp., b).

(ii) Let \(\lambda_0 \in \mathbb{R}\). Then \(T_{\min}\) is called bounded from below by \(\lambda_0\), and one writes \(T_{\min} \geq \lambda_0 I_{L^2((a,b))}\), if

\[
(u, [T_{\min} - \lambda_0 I_{L^2((a,b))}]u)_{L^2((a,b);rdx)} \geq 0, \quad u \in \text{dom}(T_{\min}).
\]

The following is a key result:

**Theorem 2.13.** Assume Hypothesis 2.1. Then the following items (i)–(iii) are equivalent:

(i) \(T_{\min}\) (and hence any symmetric extension of \(T_{\min}\)) is bounded from below.

(ii) There exists a \(\nu_0 \in \mathbb{R}\) such that for all \(\lambda < \nu_0\), \(\tau - \lambda\) is nonoscillatory at a and b.

(iii) For fixed \(c,d \in (a,b)\), \(c \leq d\), there exists a \(\nu_0 \in \mathbb{R}\) such that for all \(\lambda < \nu_0\), \(\tau u = \lambda u\) has (real-valued) nonvanishing solutions \(u_\alpha(\lambda, \cdot) \neq 0\), \(\tilde{u}_\alpha(\lambda, \cdot) \neq 0\) in the neighborhood \((a,c)\) of a, and (real-valued) nonvanishing solutions \(u_\beta(\lambda, \cdot) \neq 0\), \(\tilde{u}_\beta(\lambda, \cdot) \neq 0\) in the neighborhood \((d,b)\) of b, such that

\[
W(\tilde{u}_\alpha(\lambda, \cdot), u_\alpha(\lambda, \cdot)) = 1, \quad u_\alpha(\lambda, x) = o(\tilde{u}_\alpha(\lambda, x)) \text{ as } x \downarrow a,
\]
In particular, the limits on the right-hand sides in (2.45), (2.46) exist.

**Definition 2.14.** Assume Hypothesis 2.1, suppose that \( T_{\min} \) is bounded from below, and let \( \lambda \in \mathbb{R} \). Then \( u_a(\lambda, \cdot) \) (resp., \( u_b(\lambda, \cdot) \)) in Theorem 2.13(iii) is called a principal (or minimal) solution of \( \tau u = \lambda u \) at \( a \) (resp., \( b \)). A real-valued solution \( \tilde{u}_a(\lambda, \cdot) \) (resp., \( \tilde{u}_b(\lambda, \cdot) \)) of \( \tau u = \lambda u \) linearly independent of \( u_a(\lambda, \cdot) \) (resp., \( u_b(\lambda, \cdot) \)) is called nonprincipal at \( a \) (resp., \( b \)). In particular, \( \tilde{u}_a(\lambda, \cdot) \) (resp., \( \tilde{u}_b(\lambda, \cdot) \)) in (2.39)-(2.42) are nonprincipal solutions at \( a \) (resp., \( b \)).

Next, we revisit in Theorem 2.8 how the generalized boundary values are utilized in the description of all self-adjoint extensions of \( T_{\min} \) in the case where \( T_{\min} \) is bounded from below.

**Theorem 2.15 ( [34, Theorem 4.5] ).** Assume Hypothesis 2.1 and that \( \tau \) is in the limit circle case at \( a \) and \( b \) i.e., \( \tau \) is quasi-regular on \((a,b)\). In addition, assume that \( T_{\min} \geq \lambda_{0}I \) for some \( \lambda_{0} \in \mathbb{R} \), and denote by \( u_a(\lambda_{0}, \cdot) \) and \( \tilde{u}_a(\lambda_{0}, \cdot) \) (resp., \( u_b(\lambda_{0}, \cdot) \)) \( \text{and} \tilde{u}_b(\lambda_{0}, \cdot) \)) principal and nonprincipal solutions of \( \tau u = \lambda_{0} u \) at \( a \) (resp., \( b \)), satisfying

\[
W(\tilde{u}_a(\lambda_{0}, \cdot), u_a(\lambda_{0}, \cdot)) = W(\tilde{u}_b(\lambda_{0}, \cdot), u_b(\lambda_{0}, \cdot)) = 1. \tag{2.43}
\]

Introducing \( v_{j} \in \text{dom}(T_{\max}) \), \( j = 1, 2 \), via

\[
v_1(x) = \begin{cases} 
\tilde{u}_a(\lambda_{0}, x), & \text{for } x \text{ near } a, \\
\tilde{u}_b(\lambda_{0}, x), & \text{for } x \text{ near } b, 
\end{cases} \quad v_2(x) = \begin{cases} 
u_a(\lambda_{0}, x), & \text{for } x \text{ near } a, \\
u_b(\lambda_{0}, x), & \text{for } x \text{ near } b,
\end{cases} \tag{2.44}
\]

one obtains for all \( g \in \text{dom}(T_{\max}) \),

\[
\overline{g}(a) = -W(v_2, g)(a) = \tilde{g}_1(a) = -W(u_a(\lambda_{0}, \cdot), g)(a) = \lim_{x \uparrow a} \frac{g(x)}{u_a(\lambda_{0}, x)}, \tag{2.45}
\]

\[
\overline{g}(b) = -W(v_2, g)(b) = \tilde{g}_1(b) = -W(u_b(\lambda_{0}, \cdot), g)(b) = \lim_{x \uparrow b} \frac{g(x)}{u_b(\lambda_{0}, x)},
\]

\[
\overline{g}'(a) = W(v_1, g)(a) = \tilde{g}_2(a) = W(\tilde{u}_a(\lambda_{0}, \cdot), g)(a) = \lim_{x \downarrow a} \frac{g(x) - \overline{g}(a)\tilde{u}_a(\lambda_{0}, x)}{u_a(\lambda_{0}, x)},
\]

\[
\overline{g}'(b) = W(v_1, g)(b) = \tilde{g}_2(b) = W(\tilde{u}_b(\lambda_{0}, \cdot), g)(b) = \lim_{x \uparrow b} \frac{g(x) - \overline{g}(b)\tilde{u}_b(\lambda_{0}, x)}{u_b(\lambda_{0}, x)} \tag{2.46}
\]

In particular, the limits on the right-hand sides in (2.45), (2.46) exist.

**Remark 2.16.** The notion of “generalized boundary values” in (2.14) and (2.45), (2.46) corresponds to “boundary values for \( \tau \)” in the sense of [27, p. 1297, 1304–1307], see also [31, Sect. 3], [32, p. 57].

The Friedrichs extension \( T_F \) of \( T_{\min} \) now permits a particularly simple characterization in terms of the generalized boundary values \( \overline{g}(a), \overline{g}(b) \) as derived by Niessen and Zettl [67](see also [39], [47], [48], [51], [61], [72], [73], [79]):
Theorem 2.17. Assume Hypothesis 2.1 and that \( \tau \) is in the limit circle case at \( a \) and \( b \) (i.e., \( \tau \) is quasi-regular on \((a,b)\)). In addition, assume that \( T_{\min} \geq \lambda_0 I \) for some \( \lambda_0 \in \mathbb{R} \). Then the Friedrichs extension \( T_F = T_{0,0} \) of \( T_{\min} \) is characterized by

\[
T_F f = \tau f, \quad f \in \text{dom}(T_F) = \{ g \in \text{dom}(T_{\max}) \mid \tilde{g}(a) = \tilde{g}(b) = 0 \}. \tag{2.47}
\]

Remark 2.18. (i) As in (2.20), one readily verifies for \( g, h \in \text{dom}(T_{\max}) \),

\[
\tilde{g}(d)\tilde{h}'(d) - \tilde{g}'(d)\tilde{h}(d) = W(g,h)(d), \quad d \in \{a,b\}, \tag{2.48}
\]

again interpreted in the sense that either side in (2.48) has a finite limit as \( d \downarrow a \) and \( d \uparrow b \).

(ii) As always in this context (cf. Remark 2.9 (i)), if \( \tau \) is in the limit point case at one (or both) interval endpoints, the corresponding boundary conditions at that endpoint are dropped in Theorems 2.15 and 2.17.

3. Krein Resolvent Identities: One Limit Circle Endpoint

Assuming that \( \tau \) is in the limit circle case at \( a \) and in the limit point case at \( b \), we derive in this section the Krein resolvent formulas for all self-adjoint extensions of \( T_{\min} \) using the Friedrichs extension as the reference operator.

Hypothesis 3.1. In addition to Hypothesis 2.1 assume that \( \tau \) is in the limit circle case at \( a \) and in the limit point case at \( b \). Moreover, for \( z \in \rho(T_0) \), let \( \psi(z, \cdot) \) denote the unique solution to \((\tau - z)g = 0\) that satisfies \( \psi(z, \cdot) \in L^2_z((a,b)) \) and \( \tilde{\psi}(z,a) = 1 \).

Assume Hypothesis 3.1. By Theorem 2.8 or Theorem 2.15, the following statements (i) and (ii) hold.

(i) If \( \alpha \in [0, \pi) \), then the operator \( T_{\alpha} \) defined by

\[
T_{\alpha} f = T_{\max} f, \quad f \in \text{dom}(T_{\alpha}) = \{ g \in \text{dom}(T_{\max}) \mid \cos(\alpha)\tilde{g}(a) + \sin(\alpha)\tilde{g}'(a) = 0 \} \tag{3.1}
\]

is a self-adjoint extension of \( T_{\min} \).

(ii) If \( T \) is a self-adjoint extension of \( T_{\min} \), then \( T = T_{\alpha} \) for some \( \alpha \in [0, \pi) \).

Statements analogous to (i) and (ii) hold if \( \tau \) is in the limit point case at \( a \) and in the limit circle case at \( b \); for brevity we omit the details.

Choosing \( \alpha = 0 \) in (3.1) yields the self-adjoint extension \( T_0 \) with a Dirichlet-type boundary condition at \( a \):

\[
\text{dom}(T_0) = \{ g \in \text{dom}(T_{\max}) \mid \tilde{g}(a) = 0 \}. \tag{3.2}
\]

Since the coefficients \( p, q, \) and \( r \) are real-valued, the solution \( \psi(z, \cdot) \) has the following conjugation property:

\[
\overline{\psi(z, \cdot)} = \psi(\overline{z}, \cdot), \quad z \in \rho(T_0). \tag{3.3}
\]

Theorem 3.2. Assume Hypothesis 3.1. If \( \alpha \in (0, \pi) \), then \( T_0 \) and \( T_{\alpha} \) are relatively prime with respect to \( T_{\min} \). Moreover, for each \( z \in \rho(T_0) \cap \rho(T_{\alpha}) \), the scalar

\[
k_{\alpha}(z) = -\cot(\alpha) - \tilde{\psi}'(z,a) \tag{3.4}
\]

is nonzero and

\[
(T_{\alpha} - zI_{L^2_z((a,b))})^{-1} = (T_0 - zI_{L^2_z((a,b))})^{-1} + k_{\alpha}(z)^{-1}(\psi(\overline{z}, \cdot), \cdot)_{L^2_z((a,b))}\psi(z, \cdot). \tag{3.5}
\]
Proof: The claims follow as a direct application of [2, Theorem 3.4] which is stated in terms of boundary conditions bases and the Lagrange bracket. The condition
\[ W(\tilde{u}_a(\lambda_0, \cdot), u_a(\lambda_0, \cdot)) = 1 \]  
implies
\[ \{ u_a(\lambda_0, \cdot), \tilde{u}_a(\lambda_0, \cdot) \} \text{ is a boundary condition basis at } x = a \]
in the sense of [2, Definition 2.15] and [80, Definition 10.4.3]. The generalized implies in terms of boundary conditions bases and the Lagrange bracket. The condition
\[ \text{The claims follow as a direct application of [2, Theorem 3.4] which is stated} \]
Now follow from [2, Theorem 3.4] after a standard reparametrization of the self-adjoint extensions (3.1) to fit the parametrization used in [2, Theorem 2.19].

4. Krein Resolvent Identities: Two Limit Circle Endpoints

Assuming that \( \tau \) is in the limit circle case at \( a \) and \( b \), we now derive the Krein resolvent formulas for all self-adjoint extensions of \( T_{\min} \) using once more the Friedrichs extension as the reference operator (in this context we also refer to [17]).

**Hypothesis 4.1.** In addition to Hypothesis 2.1 assume that \( \tau \) is in the limit circle case at \( a \) and \( b \). Moreover, for \( z \in \rho(T_{0,0}) \), let \( \{ u_j(z, \cdot) \}_{j=1,2} \) denote solutions to \( \tau u = zu \) which satisfy the boundary conditions
\[ \tilde{u}_1(z, a) = 0, \quad \tilde{u}_1(z, b) = 1, \]
\[ \tilde{u}_2(z, a) = 1, \quad \tilde{u}_2(z, b) = 0. \]  

Assume Hypotheses 4.1. By Theorem 2.8 or Theorem 2.15, the following statements (i)–(iii) hold.

(i) If \( \alpha, \beta \in [0, \pi) \), then the operator \( T_{\alpha,\beta} \) defined by
\[ T_{\alpha,\beta}f = T_{\max}f, \]
\[ f \in \text{dom}(T_{\alpha,\beta}) = \left\{ g \in \text{dom}(T_{\max}) \mid \cos(\alpha)\tilde{g}(a) + \sin(\alpha)\tilde{g}'(a) = 0, \cos(\beta)\tilde{g}(b) + \sin(\beta)\tilde{g}'(b) = 0 \right\}, \]
is a self-adjoint extension of \( T_{\min} \).

(ii) If \( \varphi \in [0, 2\pi) \) and \( R \in \text{SL}(2, \mathbb{R}) \), then the operator \( T_{\varphi, R} \) defined by
\[ T_{\varphi, R}f = T_{\max}f, \]
\[ f \in \text{dom}(T_{\varphi, R}) = \left\{ g \in \text{dom}(T_{\max}) \mid \left( \begin{array}{c} \tilde{g}(b) \\ \tilde{g}'(b) \end{array} \right) = e^{i\varphi}R \left( \begin{array}{c} \tilde{g}(a) \\ \tilde{g}'(a) \end{array} \right) \right\}, \]
is a self-adjoint extension of \( T_{\min} \).

(iii) If \( T \) is a self-adjoint extension of \( T_{\min} \), then \( T = T_{\alpha,\beta} \) for some \( \alpha, \beta \in [0, \pi) \) or \( T = T_{\varphi, R} \) for some \( \varphi \in [0, 2\pi) \) and some \( R \in \text{SL}(2, \mathbb{R}) \).

**Notational Convention.** To describe all possible self-adjoint boundary conditions associated with self-adjoint extensions of \( T_{\min} \) effectively, we will frequently employ
the notation $T_{A,B}$, $M^{P\rho}_{A,B}(\cdot)$, etc., where $A,B$ represents $\alpha,\beta$ in the case of separated boundary conditions and $\varphi,R$ in the context of coupled boundary conditions.

Choosing $\alpha = \beta = 0$ in (4.2) yields the self-adjoint extension with Dirichlet-type boundary conditions at $a$ and $b$:

$$\text{dom}(T_{0,0}) = \{ g \in \text{dom}(T_{\max}) \mid \tilde{g}(a) = \tilde{g}(b) = 0 \}. \quad (4.4)$$

Since the coefficients of the Sturm–Liouville differential expression are real, the following conjugation property holds:

$$u_j(\beta, \cdot) = u_j(\alpha, \cdot), \quad \beta \in \rho(T_{0,0}), \quad j \in \{1, 2\}. \quad (4.5)$$

Applying (4.1), one computes

$$W(u_1(\alpha, \cdot), u_2(\alpha, \cdot))(a) = -\tilde{u}_1'(\alpha, a),$$

$$W(u_1(\alpha, \cdot), u_2(\alpha, \cdot))(b) = \tilde{u}_2'(\alpha, b), \quad \beta \in \rho(T_{0,0}). \quad (4.6)$$

In particular, since the Wronskian of two solutions is constant,

$$\tilde{u}_2'(\alpha, b) = -\tilde{u}_1'(\alpha, a), \quad \beta \in \rho(T_{0,0}). \quad (4.7)$$

**Theorem 4.2.** Assume Hypothesis 4.1. Then the following statements (i)–(v) hold.

(i) If $\alpha,\beta \in (0,\pi)$, then $T_{0,0}$ and $T_{\alpha,\beta}$ are relatively prime with respect to $T_{\min}$. Moreover, for each $\beta \in \rho(T_{0,0}) \cap \rho(T_{\alpha,\beta})$ the matrix

$$K_{\alpha,\beta}(z) = \begin{pmatrix} \cot(\beta) + \tilde{u}_1'(z, b) & -\tilde{u}_1'(z, a) \\ \tilde{u}_2'(z, b) & -\cot(\alpha) - \tilde{u}_2'(z, a) \end{pmatrix} \quad (4.8)$$

is invertible and

$$(T_{\alpha,\beta} - z I_{L^2_\ell((a,b))})^{-1} = (T_{0,0} - z I_{L^2_\ell((a,b))})^{-1}$$

$$+ \sum_{j,k=1}^{2} [K_{\alpha,\beta}(z)^{-1}]_{j,k} (u_j(\beta, \cdot), \cdot)_{L^2_\ell((a,b))} u_k(\beta, \cdot). \quad (4.9)$$

(ii) If $\beta \in (0,\pi)$, then the maximal common part of $T_{0,0}$ and $T_{0,\beta}$ is the restriction of $T_{\max}$ to the set

$$S_1 = \{ y \in \text{dom}(T_{\max}) \mid \tilde{y}(a) = \tilde{y}(b) = \tilde{y}'(b) = 0 \}. \quad (4.10)$$

Moreover, for each $\beta \in \rho(T_{0,0}) \cap \rho(T_{0,\beta})$ the scalar

$$K_{0,\beta}(z) = \cot(\beta) + \tilde{u}_1'(z, b) \quad (4.11)$$

is nonzero and

$$(T_{0,\beta} - z I_{L^2_\ell((a,b))})^{-1}$$

$$= (T_{0,0} - z I_{L^2_\ell((a,b))})^{-1} + K_{0,\beta}(z)^{-1} (u_1(\beta, \cdot), \cdot)_{L^2_\ell((a,b))} u_1(\beta, \cdot). \quad (4.12)$$

(iii) If $\alpha \in (0,\pi)$, then the maximal common part of $T_{0,0}$ and $T_{\alpha,0}$ is the restriction of $T_{\max}$ to the set

$$S_2 = \{ y \in \text{dom}(T_{\max}) \mid \tilde{y}(a) = \tilde{y}(b) = \tilde{y}'(a) = 0 \}. \quad (4.13)$$

Moreover, for each $\beta \in \rho(T_{0,0}) \cap \rho(T_{\alpha,0})$ the scalar

$$K_{\alpha,0}(z) = -\cot(\alpha) - \tilde{u}_2'(z, a) \quad (4.14)$$

is nonzero and

$$(T_{\alpha,0} - z I_{L^2_\ell((a,b))})^{-1} \quad (4.15)$$
is nonzero, and
\[
K_{\varphi,R}(z) = \left( \begin{array}{cc} -\frac{R_{2,2}}{R_{1,2}} + \bar{u}'_1(z, b) & \frac{e^{-i\varphi}}{R_{1,2}} - \bar{u}'_1(z, a) \\ \frac{e^{i\varphi}}{R_{1,2}} + \bar{u}'_2(z, b) & -\frac{R_{1,1}}{R_{1,2}} - \bar{u}'_2(z, a) \end{array} \right) \tag{4.16}
\]
is invertible and
\[
(T_{\varphi,R} - zI_{L_2^2((a,b))})^{-1} = (T_{0,0} - zI_{L_2^2((a,b))})^{-1} + \sum_{j,k=1}^{2} [K_{\varphi,R}(z)^{-1}]_{j,k}(u_j(\tau, \cdot), \cdot)L_2^2((a,b))u_k(z, \cdot). \tag{4.17}
\]

(v) If \( R_{1,2} = 0 \), then the maximal common part of \( T_{\varphi,R} \) and \( T_{0,0} \) is the restriction of \( T_{\max} \) to the set
\[
S_{\varphi,R} = \{ y \in \text{dom}(T_{\max}) \mid \bar{y}(a) = \bar{y}(b) = 0, \bar{y}'(b) = e^{i\varphi}R_{2,2}\bar{y}'(a) \}. \tag{4.18}
\]
Moreover, for each \( z \in \rho(T_{0,0}) \cap \rho(T_{\varphi,R}) \), the scalar
\[
k_{\varphi,R}(z) = -R_{2,1}R_{2,2} - e^{i\varphi}R_{2,2}\bar{u}'_{\varphi,R}(z, a) + \bar{u}'_{\varphi,R}(z, b) \tag{4.19}
\]
is nonzero, and
\[
(T_{\varphi,R} - zI_{L_2^2((a,b))})^{-1} = (T_{0,0} - zI_{L_2^2((a,b))})^{-1} + k_{\varphi,R}(z)^{-1}(u_{\varphi,R}(\tau, \cdot), \cdot)L_2^2((a,b))u_{\varphi,R}(z, \cdot), \tag{4.20}
\]
where
\[
u_{\varphi,R}(\zeta, \cdot) = e^{-i\varphi}R_{2,2}u_2(\zeta, \cdot) + u_1(\zeta, \cdot), \quad \zeta \in \rho(T_{0,0}). \tag{4.21}
\]

Proof. Statements (i)–(v) are direct applications of the Krein identities for singular Sturm–Liouville operators obtained in [2] which are stated in terms of boundary conditions bases and the Lagrange bracket. The conditions
\[
W(\tilde{u}_a(\lambda_0, \cdot), u_a(\lambda_0, \cdot)) = W(\tilde{u}_b(\lambda_0, \cdot), u_b(\lambda_0, \cdot)) = 1 \tag{4.22}
\]
imply that
\[
\{ u_c(\lambda_0, \cdot), \tilde{u}_c(\lambda_0, \cdot) \} \text{ is a boundary condition basis at } x = c \text{ for } c \in \{ a, b \} \tag{4.23}
\]
in the sense of [2, Definition 2.15] and [80, Definition 10.4.3]. The generalized boundary values take the form
\[
[g, u_a(\lambda_0, \cdot)](a) = \bar{g}(a), \quad [g, u_b(\lambda_0, \cdot)](b) = \bar{g}(b),
\]
\[
[g, \bar{u}_a(\lambda_0, \cdot)](a) = -\bar{g}'(a), \quad [g, \bar{u}_b(\lambda_0, \cdot)](b) = -\bar{g}'(b), \tag{4.24}
\]
where \([\cdot, \cdot]\) denotes the Lagrange bracket (see (3.9)). Using the boundary condition bases in (4.23) and the identities in (4.24), statements (i)–(v) now follow from [2, Theorems 4.4, 4.5, 4.6, and 4.7] after a standard reparametrization of the self-adjoint extensions (4.2) and (4.3) to fit the parametrization used in [2, Theorem 2.20].
Remark 4.3. As an illustration of Theorem 4.2, we consider the Krein extension, 
\( T_{0,RK} \), under the additional assumption that \( T_{\min} \geq \epsilon L_{L^2((a,b))} \) for some \( \epsilon > 0 \). Then applying [30, Thm. 3.5 (ii)] and Theorem 4.2 (iv), one computes for the matrix \( K_{0,RK} \) in (4.16),

\[
K_{0,RK}(z) = \begin{pmatrix}
\bar{u}_1(z,b) - \bar{u}_1(0,b) & \bar{u}_1(0,a) - \bar{u}_1(z,a) \\
\bar{u}_2(z,b) - \bar{u}_2(0,b) & \bar{u}_2(0,a) - \bar{u}_2(z,a)
\end{pmatrix}, \quad z \in \rho(T_{0,0}) \cap \rho(T_{0,RK}),
\]

where we note that \( 0 \in \sigma(T_{0,RK}) \).

5. DONOGHUE \( m \)-FUNCTIONS: ONE LIMIT CIRCLE ENDPOINT

In this section we construct the Donoghue \( m \)-functions in the case where \( \tau \) is in the limit circle case at precisely one endpoint (which we choose to be \( a \) without loss of generality). We first focus on the Friedrichs extension of \( T_{\min} \) and then use the Krein resolvent formulas from Section 3 to treat all remaining self-adjoint extensions of \( T_{\min} \).

Throughout this section we shall assume that Hypothesis 3.1 holds so that \( \tau \) is in the limit circle case at \( a \) and in the limit point case at \( b \). We begin by obtaining a general expression for the Donoghue \( m \)-function of an arbitrary self-adjoint extension \( T_\alpha \) of \( T_{\min} \) in terms of a unit vector \( \phi(i, \cdot) \in \mathcal{N}_i \). This general expression will then be made more explicit in terms of \( \psi(i, \cdot) \) (cf. Hypothesis 3.1) in the analysis below. The Donoghue \( m \)-function for \( T_\alpha \) is given by (see, e.g., [36, Eq. (5.5)])

\[
M_{T_\alpha}^{D_0}(i)(z) = P_{\mathcal{N}_i}(zT_\alpha + I_{L^2((a,b))})(T_\alpha - zI_{L^2((a,b))})^{-1}P_{\mathcal{N}_i}|_{\mathcal{N}_i}, \quad z \in \mathbb{C}\setminus \mathbb{R},
\]

where \( P_{\mathcal{N}_i} \) denotes the orthogonal projection onto \( \mathcal{N}_i \). According to (5.1),

\[
M_{T_\alpha}^{D_0}(i)(z) \in \mathcal{B}(\mathcal{N}_i), \quad z \in \mathbb{C}\setminus \mathbb{R}, \quad \text{and} \quad M_{T_\alpha}^{D_0}(i)(\pm i) = \pm iP_{\mathcal{N}_i}.
\]

The unit vector \( \phi(i, \cdot) \) spans the one-dimensional subspace \( \mathcal{N}_i \), so the orthogonal projection onto \( \mathcal{N}_i \) is

\[
P_{\mathcal{N}_i} = (\phi(i, \cdot), \cdot)L^2((a,b))\phi(i, \cdot).
\]

Thus, the action of \( M_{T_\alpha}^{D_0}(i, \cdot) \) may be computed directly in terms of \( \phi(i, \cdot) \) as follows:

\[
M_{T_\alpha}^{D_0}(i, \cdot)f = [zI_{\mathcal{N}_i} + (z^2 + 1)P_{\mathcal{N}_i}(T_\alpha - zI_{L^2((a,b))})^{-1}P_{\mathcal{N}_i}|_{\mathcal{N}_i}]f
\]

\[
= zf + (z^2 + 1)P_{\mathcal{N}_i}(T_\alpha - zI_{L^2((a,b))})^{-1}f
\]

\[
= (\phi(i, \cdot), [zI_{\mathcal{N}_i} + (z^2 + 1)(T_\alpha - zI_{L^2((a,b))})^{-1}]L^2((a,b))\phi(i, \cdot))
\]

\[
= (\phi(i, \cdot), [zI_{\mathcal{N}_i} + (z^2 + 1)(T_\alpha - zI_{L^2((a,b))})^{-1}]\phi(i, \cdot))L^2((a,b))
\]

\[
\times (\phi(i, \cdot), f)L^2((a,b))\phi(i, \cdot). \quad f \in \mathcal{N}_i, \quad z \in \mathbb{C}\setminus \mathbb{R},
\]
where one uses \( f = (\phi(i, \cdot), f)_{L^2_t((a,b))} \phi(i, \cdot) \) to obtain the fourth equality in (5.4). Hence,
\[
M_{T_{a,N}N_0}^D(z) = \left[ z + (z^2 + 1) \frac{1}{(\phi(i, \cdot), (T_0 - zI_{L^2_t((a,b))})^{-1}\phi(i, \cdot))_{L^2_t((a,b))}} \right] (5.5)
\]
\[
\times (\phi(i, \cdot), (T_0 - zI_{L^2_t((a,b))})^{-1}\phi(i, \cdot))_{N_0^c}, \quad z \in \mathbb{C}\setminus\mathbb{R}.
\]
In order to determine \( M_{T_{a,N}N_0}^D(\cdot) \) in terms of \( \psi(i, \cdot) \), one must compute the fixed inner product in (5.5). That is, one must compute
\[
(\phi(i, \cdot), (T_0 - zI_{L^2_t((a,b))})^{-1}\phi(i, \cdot))_{L^2_t((a,b))}, \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{5.6}
\]
In light of (5.2), it suffices to compute (5.6) under the additional assumption that \( z \neq \pm i \). We will first do this for the Dirichlet-type extension \( T_0 \) (cf. (3.2)).

5.1. The Donoghue \( m \)-function \( M_{T_{a,N}N_0}^D(\cdot) \) for \( T_0 \). Here we shall consider the Dirichlet-type self-adjoint extension \( T_0 \) of \( T_{\min} \). Assuming Hypothesis 3.1 and taking
\[
T_\alpha = T_0 \quad \text{and} \quad \phi(i, \cdot) := \|\psi(i, \cdot)\|_{L^2_t((a,b))}^{-1}\psi(i, \cdot), \tag{5.7}
\]
we shall compute the inner product (5.6) and use (5.5) to obtain an explicit expression for the Donoghue \( m \)-function \( M_{T_{a,N}N_0}^D(\cdot) \) for \( T_0 \) in terms of \( \psi(i, \cdot) \).

For the purposes of evaluating the inner product (5.6), we introduce the generalized Cayley transform of \( T_0 \),
\[
U_{0,z,z'} = (T_0 - zI_{L^2_t((a,b))})(T_0 - zI_{L^2_t((a,b))})^{-1}
\]
\[
= I_{L^2_t((a,b))} + (z - z')(T_0 - zI_{L^2_t((a,b))})^{-1}, \quad z, z' \in \rho(T_0), \tag{5.8}
\]
which forms a bijection from \( N_{\gamma_t} \) to \( N_{\gamma} \). One verifies that
\[
U_{0,z,z'}\psi(z', \cdot) = \psi(z, \cdot), \quad z, z' \in \rho(T_0). \tag{5.9}
\]
In fact, for fixed \( z, z' \in \rho(T_0) \), one uses the fact that \( U_{0,z,z'} \) maps into \( N_{\gamma} \) to write
\[
U_{0,z,z'}\psi(z', \cdot) = c_0 \psi(z, \cdot) \tag{5.10}
\]
for some scalar \( c_0 \in \mathbb{C} \). The second equality in (5.8) then implies
\[
U_{0,z,z'}\psi(z', \cdot) = \psi(z', \cdot) + (z - z')(T_0 - zI_{L^2_t((a,b))})^{-1}\psi(z', \cdot), \tag{5.11}
\]
so that
\[
[U_{0,z,z'}\psi(z', \cdot)]^{-1}(a) = \frac{1}{\psi(z', a)} = 1. \tag{5.12}
\]
Taking the generalized boundary value at \( a \) throughout (5.10) and using (5.12) yields \( c_0 = 1 \) in (5.10), and (5.9) follows.

Let \( z \in \mathbb{C}\setminus\mathbb{R} \) with \( z \neq \pm i \) be fixed. Applying (5.8), one computes:
\[
(\phi(i, \cdot), (T_0 - zI_{L^2_t((a,b))})^{-1}\phi(i, \cdot))_{L^2_t((a,b))} \tag{5.13}
\]
\[
= \frac{(\psi(i, \cdot), (T_0 - zI_{L^2_t((a,b))})^{-1}\psi(i, \cdot))_{L^2_t((a,b))}}{\|\psi(i, \cdot)\|_{L^2_t((a,b))}}
\]
\[
= \frac{(\psi(i, \cdot), [U_{0,z,i} - I_{L^2_t((a,b))}]\psi(i, \cdot))_{L^2_t((a,b))}}{(z - i)\|\psi(i, \cdot)\|_{L^2_t((a,b))}}
\]
\[
= \frac{1}{i - z} \frac{(\psi(i, \cdot), \psi(z, \cdot))_{L^2_t((a,b))}}{(z - i)\|\psi(i, \cdot)\|_{L^2_t((a,b))}}.
\]
Furthermore, by (2.3) and Theorem 2.7 (i),

\[
(\psi(i, \cdot), \psi(z, \cdot))_{L^2((a, b), \bar{\mathbb{C}})} = \int_a^b r(x) dx \psi(-i, x)\psi(z, x) \quad (5.14)
\]

\[
= \frac{W(\psi(-i, \cdot), \psi(z, \cdot))}{z + i} = \frac{\psi'(z, \alpha) - \psi'(-i, \alpha)}{z + i},
\]

where we have used that since \( \gamma \) is in the limit point case at \( b \) and \( \psi(-i, \cdot), \psi(z, \cdot) \in \text{dom}(T_{max}) \), an application of Theorem 2.7 (i) yields

\[
W(\psi(-i, \cdot), \psi(z, \cdot))(b) = 0,
\]

and by Hypothesis 3.1, \( \tilde{\psi}(-i, a) = \tilde{\psi}(z, a) = 1, \) so that

\[
W(\psi(-i, \cdot), \psi(z, \cdot))(a) = \psi'(z, a) - \psi'(-i, a). \quad (5.16)
\]

Therefore, (5.13)–(5.16) yield

\[
(\phi(i, \cdot), (T_0 - zI_{L^2((a, b), \bar{\mathbb{C}})})^{-1}\phi(i, \cdot))_{L^2((a, b), \bar{\mathbb{C}})} = \frac{1}{i} + \frac{\psi'(z, a) - \psi'(-i, a)}{(z^2 + 1)\|\phi(i, \cdot)\|_{L^2((a, b), \bar{\mathbb{C}})}}. \quad (5.17)
\]

By (2.3), Hypothesis 3.1, and the limit point assumption at \( b, \)

\[
\|\psi(i, \cdot)\|_{L^2((a, b), \bar{\mathbb{C}})}^2 = \int_a^b r(x) dx \psi(-i, x)\psi(i, x) = -\frac{W(\psi(-i, \cdot), \psi(i, \cdot))}{2i}
\]

\[
= \frac{1}{2i} \Im(\psi'(i, a) - \psi'(-i, a)) = \frac{1}{2i} \Im(\psi'(i, a)). \quad (5.18)
\]

Applying (5.17) in (5.4) and taking simplifications and (5.18) into account, one obtains the following fact.

**Theorem 5.1.** Assume Hypothesis 3.1. The Donoghue \( m \)-function \( M_{T_0, N_i}^D(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{B}(N_i) \) for \( T_0 \) satisfies

\[
M_{T_0, N_i}^D(\pm i) = \pm iN_i, \quad M_{T_0, N_i}^D(\cdot) - i\psi'(z, \alpha) - \psi'(-i, \alpha) \begin{bmatrix} \psi'(i, \cdot) \\ -i \end{bmatrix}_{N_i} \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad z \neq \pm i. \quad (5.19)
\]

5.2. The Donoghue \( m \)-function for Self-Adjoint Extensions Other Than \( T_0 \). The Donoghue \( m \)-function for \( T_0 \) was computed explicitly in Theorem 5.1. If \( T_\alpha, \alpha \in (0, \pi) \), is any other self-adjoint extension of \( T_{min} \), then the resolvent identity in Theorem 3.2 may be used to obtain an explicit representation of the Donoghue \( m \)-function \( M_{T_\alpha, N_i}^D(\cdot) \) for \( T_\alpha \).

**Theorem 5.2.** Assume Hypothesis 3.1 and let \( \alpha \in (0, \pi) \). The Donoghue \( m \)-function \( M_{T_\alpha, N_i}^D(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{B}(N_i) \) for \( T_\alpha \) satisfies

\[
M_{T_\alpha, N_i}^D(\pm i) = \pm iN_i, \quad M_{T_\alpha, N_i}^D(\cdot) = \begin{bmatrix} \psi'(z, \alpha) - \psi'(-i, \alpha) \\ -i \end{bmatrix}_{L^2((a, b), \bar{\mathbb{C}})} \Im(\psi'(i, \cdot))_{N_i}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad z \neq \pm i. \quad (5.20)
\]
Proof. Let $\alpha \in (0, \pi)$ be fixed. By (5.2), $M_{F_{\alpha}, N_{i}}(\pm i) = \pm iN_{i}$. In order to establish (5.20), let $z \in \mathbb{C}\setminus \mathbb{R}$, $z \neq \pm i$, be fixed. Considering (5.1) and invoking (3.5), one obtains

\begin{equation}
M_{F_{\alpha}, N_{i}}^{(a)}(z) = M_{F_{\alpha}, N_{i}}^{(a)}(z) + (z^{2} + 1)k_{\alpha}(z)^{-1}(\psi(z, \cdot))L_{2}((a, b))P_{N_{i}}\psi(z, \cdot)\big|_{N_{i}}.
\end{equation}

(5.21)

Using (5.14) in (5.21), one obtains

\begin{equation}
M_{F_{\alpha}, N_{i}}^{(a)}(z) = M_{F_{\alpha}, N_{i}}^{(a)}(z) + (z^{2} + 1)k_{\alpha}(z)^{-1}(\psi(i, \cdot), \psi(z, \cdot))L_{2}((a, b))P_{N_{i}}(\psi(i, \cdot))_{N_{i}}.
\end{equation}

Using (5.14) in (5.21), one obtains

\begin{equation}
M_{F_{\alpha}, N_{i}}^{(a)}(z) = M_{F_{\alpha}, N_{i}}^{(a)}(z) + (z - i)\frac{\psi'(z, a) - \psi'(-i, a)}{k_{a}(z)(\psi(z, \cdot), \cdot)L_{2}((a, b))}\psi(i, \cdot)\big|_{N_{i}}.
\end{equation}

Finally, (5.20) follows from (5.22) after using the precise form for $k_{a}(z)$ in (3.4). □

6. Donoghue $m$-functions: Two Limit Circle Endpoints

The construction of Donoghue $m$-functions in the case where $\tau$ is in the limit circle case at $a$ and $b$ is the primary aim of this section. Once more we first focus on the Friedrichs extension of $T_{\min}$ and then use the Krein resolvent formulas from Section 4 to treat all remaining self-adjoint extensions of $T_{\min}$.

Throughout this section, we shall assume that Hypothesis 4.1 holds so that $\tau$ is in the limit circle case at $a$ and $b$. We begin by obtaining a general expression for the Donoghue $m$-function of an arbitrary self-adjoint extension $T_{A, B}$ of $T_{\min}$ in terms of an orthonormal basis for $N_{i}$. Recall that the Donoghue $m$-function for $T_{A, B}$ is given by (see, e.g., [36, Eq. (5.5)])

\begin{equation}
M_{F_{\alpha}, N_{i}}^{(a)}(z) = P_{N_{i}}(zT_{A, B} + IL_{2}((a, b)) - zL_{2}((a, b)))^{-1}P_{N_{i}}\big|_{N_{i}}
\end{equation}

(6.1)

where $P_{N_{i}}$ denotes the orthogonal projection onto $N_{i}$ with $M_{F_{\alpha}, N_{i}}^{(a)}(z) \in B(N_{i})$, $z \in \mathbb{C}\setminus \mathbb{R}$, and

\begin{equation}
M_{F_{\alpha}, N_{i}}^{(a)}(\pm i) = \pm iN_{i}.
\end{equation}

(6.2)

Let $\{v_{j}\}_{j=1,2}$ be an orthonormal basis for the subspace $N_{i}$. The orthogonal projection onto $N_{i}$ is

\begin{equation}
P_{N_{i}} = \sum_{k=1}^{2}(v_{k}, \cdot)L_{2}((a, b))v_{k}.
\end{equation}

(6.3)

Therefore, the action of $M_{F_{\alpha}, N_{i}}^{(a)}(\cdot)$ may be computed directly in terms of $\{v_{j}\}_{j=1,2}$ as follows:

\begin{equation}
M_{F_{\alpha}, N_{i}}^{(a)}(z)f = [zI_{N_{i}} + (z^{2} + 1)P_{N_{i}}(T_{A, B} - zL_{2}((a, b)))^{-1}P_{N_{i}}\big|_{N_{i}}]f
\end{equation}

(6.4)

\begin{equation}
= zf + (z^{2} + 1)P_{N_{i}}(T_{A, B} - zL_{2}((a, b)))^{-1}f
\end{equation}

\begin{equation}
= \sum_{j=1}^{2}\left(zI_{N_{i}} + (z^{2} + 1)(T_{A, B} - zL_{2}((a, b)))^{-1}\right)f_{j}L_{2}((a, b))v_{j}
\end{equation}
must compute the fixed inner products in (6.5). That is, one must compute

\[ \langle v_j, (T_{A,B} - zI_L^2((a,b)))^{-1}v_k \rangle_{L^2_t((a,b))} \]

In particular, (6.7) implies

\[ \langle v_j, (T_{A,B} - zI_L^2((a,b)))^{-1}v_k \rangle_{L^2_t((a,b))} = f \in \mathcal{N}_i, z \in \mathbb{C} \setminus \mathbb{R}, \]

where one uses \( f = \sum_{j=1}^{2} (v_j, f)_{L^2_t((a,b))} v_j \) to obtain the fourth equality in (6.4). Hence,

\[ M^D_{T_{A,B}, \mathcal{N}_i}(z) = \sum_{j,k=1} \left[ z\delta_{j,k} + (z^2 + 1)(v_j, (T_{A,B} - zI_L^2((a,b)))^{-1}v_k) \right] (v_j, f)_{L^2_t((a,b))} v_j, \]

where \( z \in \mathbb{C} \setminus \mathbb{R} \).

In order to determine \( M^D_{T_{A,B}, \mathcal{N}_i}(\cdot) \) in terms of the orthonormal basis \( \{v_j\}_{j=1}^{2} \), one must compute the fixed inner products in (6.5). That is, one must compute

\[ \langle v_j, (T_{A,B} - zI_L^2((a,b)))^{-1}v_k \rangle_{L^2_t((a,b))}, \quad j, k \in \{1, 2\}, z \in \mathbb{C} \setminus \mathbb{R}. \]

In light of (6.2), it suffices to compute (6.6) under the additional assumption that \( z \neq \pm i \). We will first do this for the Dirichlet-type extension \( T_{0,0} \) (cf. (4.4)).

### 6.1. The Donoghue m-function \( M^D_{T_{0,0}, \mathcal{N}_i}(\cdot) \) for \( T_{0,0} \)

Here we shall consider the Dirichlet-type self-adjoint extension \( T_{0,0} \) of \( T_{min} \). Assuming Hypothesis 4.1 and taking the orthonormal basis for \( \mathcal{N}_i \) obtained by applying the Gram–Schmidt process to \( \{u_j(\pm i, \cdot)\}_{j=1}^{2} \), we shall compute the inner products (6.6) and use (6.5) to obtain an explicit expression for the Donoghue m-function \( M^D_{T_{0,0}, \mathcal{N}_i}(\cdot) \) for \( T_{0,0} \).

In the analysis below, it will be convenient to also introduce an orthonormal basis for \( \mathcal{N}_{-i} \). To set the stage for applying Gram–Schmidt to \( \{u_j(\pm i, \cdot)\}_{j=1}^{2} \), one applies (2.3) and (4.1), to compute

\[ (u_j(\pm i, \cdot), u_k(\pm i, \cdot))_{L^2_t((a,b))} = \int_a^b r(x)dx \overline{u_j(\pm i, x)}u_k(\pm i, x) \]

\[ = \int_a^b r(x)dx u_j(\mp i, x)u_k(\pm i, x) = W(\overline{u_j(\mp i, \cdot)}, u_k(\pm i, \cdot)) \bigg|_{a}^{b} \mp i - (\pm i) \]

\[ = \mp \frac{1}{2i} W(\bar{u}_j(\mp i, \cdot), u_k(\pm i, \cdot)) \bigg|_{a}^{b} \]

\[ = \mp \frac{1}{2i} \left\{ \bar{u}_j(\mp i, b)\bar{u}_k(\pm i, b) - \bar{u}_j(\mp i, b)\bar{u}_k(\pm i, b) - \bar{u}_j(\mp i, a)\bar{u}_k(\mp i, a) \right\} \]

\[ = \mp \frac{1}{2i} \left\{ \bar{u}_k(\pm i, b)\delta_{j,1} - \bar{u}_j(\mp i, b)\delta_{k,1} - \bar{u}_k(\pm i, a)\delta_{j,2} - \bar{u}_j(\mp i, a)\delta_{k,2} \right\}, \quad j, k \in \{1, 2\}. \]

In particular, (6.7) implies

\[ (u_1(\pm i, \cdot), u_2(\pm i, \cdot))_{L^2_t((a,b))} = \mp \frac{1}{2i} \left[ \bar{u}_2(\pm i, b) + \bar{u}_1(\mp i, a) \right] \]
Applying the Gram–Schmidt process to \( \{v_j(±i, \cdot)\}_{j=1,2} \) then yields an orthonormal basis \( \{v_j(±i, \cdot)\}_{j=1,2} \) for \( \mathcal{N}_{±i} \) as follows:

\[
v_1(±i, \cdot) = c_1(±i)u_1(±i, \cdot),
\]

\[
v_2(±i, \cdot) = c_2(±i) \left[ \frac{u_2(±i, \cdot) - \langle u_1(±i, \cdot), u_2(±i, \cdot) \rangle_{L^2_{±i}(a,b)} u_1(±i, \cdot)}{\| u_1(±i, \cdot) \|_{L^2_{±i}(a,b)}^2} \right], \tag{6.12}
\]

where

\[
c_1(±i) = \| u_1(±i, \cdot) \|_{L^2_{±i}(a,b)}^2 = \left[ ± \text{Im}(\bar{u}_1'(±i, b)) \right]^{-1/2}, \tag{6.13}
\]

\[
c_2(±i) = \left[ u_2(±i, \cdot) - \frac{\text{Im}(\bar{u}_2'(±i, b))}{\text{Im}(\bar{u}_1'(±i, b))} u_1(±i, \cdot) \right]_{L^2_{±i}(a,b)} = \left[ ± \text{Im}(\bar{u}_2'(±i, a)) \pm \left[ \frac{\text{Im}(\bar{u}_2'(±i, b))}{\text{Im}(\bar{u}_1'(±i, b))} \right]^2 \right]^{-1/2}, \tag{6.14}
\]

and the equality \( \text{Im}(\bar{u}_2'(±i, b)) / \text{Im}(\bar{u}_1'(±i, b)) = \text{Im}(\bar{u}_2'(±i, b)) / \text{Im}(\bar{u}_1'(±i, b)) \) has been applied. Based on (4.5), one infers that

\[
c_j(i) = c_j(-i), \quad j \in \{1, 2\}. \tag{6.15}
\]

In addition, by taking conjugates throughout (6.11)–(6.14) and applying (4.5), one obtains

\[
v_j(±i, \cdot) = v_j(±i, \cdot), \quad j \in \{1, 2\}. \tag{6.16}
\]

Taking the orthonormal basis \( \{v_j(i, \cdot)\}_{j=1,2} \) for \( \mathcal{N}_i \) in (6.5) then yields the following expression for the Donoghue \( m \)-function \( M^D_{T_{0,0},R,N_i}(z) \) for \( T_{0,0} \):

\[
M^D_{T_{0,0},R,N_i}(z) = \sum_{j,k=1}^2 \left[ z \delta_{j,k} + (z^2 + 1) \langle v_j(i, \cdot), (T_{0,0} - zI_{L^2_{±i}(a,b)})^{-1} v_k(i, \cdot) \rangle_{L^2_{±i}(a,b)} \right] \times (v_k(i, \cdot), v_j(i, \cdot))_{L^2_{±i}(a,b)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{6.17}
\]

In the special cases \( z = ±i \), one obtains (cf. (6.2))

\[
M^D_{T_{0,0},R,N_i}(±i) = ±iI_{N_i}. \tag{6.18}
\]
Thus, to obtain an explicit representation for $\mathcal{D}_{a,b,H}^\alpha (\cdot )$, it remains to evaluate the inner products
\[
  (v_j(i, \cdot ), (T_{0,0} - z I_{L_2^2((a,b))})^{-1} v_k(i, \cdot ))_{L_2^2((a,b))}, \quad j, k \in \{1, 2\}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad z \neq \pm i.
\]  
(6.19)

For the purposes of evaluating the inner products (6.19), we introduce the generalized Cayley transform of $T_{0,0}$,
\[
  U_{0,0,z,z'} = (T_{0,0} - z' I_{L_2^2((a,b))}) (T_{0,0} - z I_{L_2^2((a,b))})^{-1}
\]  
(6.20)

which forms a bijection from $\mathcal{N}_z$ to $\mathcal{N}_{z'}$. One verifies that
\[
  U_{0,0,z,z'} u_j(z', \cdot ) = u_j(z, \cdot ), \quad j \in \{1, 2\}, \quad z, z' \in \rho(T_{0,0}).
\]  
(6.21)

In fact, for fixed $z, z' \in \rho(T_{0,0})$, one uses the fact that $U_{0,0,z,z'}$ maps into $\mathcal{N}_z$ to write
\[
  U_{0,0,z,z'} u_j(z', \cdot ) = \alpha_{j,1} u_1(z, \cdot ) + \alpha_{j,2} u_2(z, \cdot ), \quad j \in \{1, 2\},
\]  
(6.22)

for some scalars $\alpha_{j,k} \in \mathbb{C}, j, k \in \{1, 2\}$. The second equality in (6.20) then implies
\[
  U_{0,0,z,z'} u_j(z', \cdot ) = u_j(z', \cdot ) + (z - z')(T_{0,0} - z I_{L_2^2((a,b))})^{-1} u_j(z', \cdot ),
\]  
(6.23)

so that
\[
  [U_{0,0,z,z'} u_j(z', \cdot )]^{-1}(x) = \tilde{u}_j(z', x), \quad x \in \{a, b\}, \quad j \in \{1, 2\}.
\]  
(6.24)

Evaluating (6.22) and (6.24) at $a$ yields $\alpha_{1,2} = 0$ and $\alpha_{2,2} = 1$. Similarly, evaluating (6.22) and (6.24) at $b$ yields $\alpha_{1,1} = 1$ and $\alpha_{2,1} = 0$. Hence, (6.21) follows.

We will now calculate the inner products (6.19). Let
\[
  z \in \mathbb{C} \setminus \mathbb{R} \text{ be fixed with } z \neq \pm i.
\]  
(6.25)

The system $\{v_j(z, \cdot )\}_{j=1,2}$ defined by
\[
  v_j(z, \cdot ) = U_{0,0,z,i} v_j(i, \cdot ), \quad j \in \{1, 2\}
\]  
(6.26)

is a basis for the subspace $\mathcal{N}_z$. Applying (6.11)–(6.12) and (6.21) in (6.26), one obtains
\[
  v_1(z, \cdot ) = c_1(i) u_1(z, \cdot ),
\]
\[
  v_2(z, \cdot ) = c_2(i) \left[ u_2(z, \cdot ) - \frac{\text{Im}(\tilde{u}_2'(i, b))}{\text{Im}(\tilde{u}_1'(i, b))} u_1(z, \cdot ) \right].
\]  
(6.27)

The inner products (6.19) can be recast in terms of $\{v_j(z, \cdot )\}_{j=1,2}$ as follows:
\[
  (v_j(i, \cdot ), (T_{0,0} - z I_{L_2^2((a,b))})^{-1} v_k(i, \cdot ))_{L_2^2((a,b))}
\]
\[
  = \frac{1}{z - i} (v_j(i, \cdot ), [U_{0,0,z,i} - I_{L_2^2((a,b))}] v_k(i, \cdot ))_{L_2^2((a,b))}
\]
\[
  = \frac{1}{z - i} \delta_{j,k} + \frac{1}{z - \frac{i}{2}} (v_j(i, \cdot ), v_k(z, \cdot ))_{L_2^2((a,b))}, \quad j, k \in \{1, 2\}.
\]  
(6.28)

In turn, by (2.3) and (6.16), one obtains
\[
  (v_j(i, \cdot ), v_k(z, \cdot ))_{L_2^2((a,b))} = \int_a^b r(x) dx v_j(-i, x) v_k(z, x)
\]
for $z$ that appear in (6.31) can be computed by applying (4.1) and (6.27). One obtains $m$

The relations (6.18) and (6.31)–(6.36) now yield an explicit representation for the

After substituting (6.30) in (6.17) and taking cancellations into account, one obtains $M_{D_{0,0},N_{i}}(z)$

$$M_{D_{0,0},N_{i}}(z) = \sum_{j,k=1}^{2} \left[ -i\delta_{j,k} - W(v_{j}(-i, \cdot), v_{k}(z, \cdot)) \right]_{\delta_{j,k}}.$$ (6.32)

The Wronskians

$$W_{j,k}(z) := W(v_{j}(-i, \cdot), v_{k}(z, \cdot)),$$ (6.33)

that appear in (6.31) can be computed by applying (4.1) and (6.27). One obtains for $z \in \mathbb{C} \setminus \mathbb{R}, z \neq \pm i$:

$$W_{1,1}(z) = [c_{1}(i)]^{2} \left[ \bar{u}_{1}(z, b) - \bar{u}_{1}(-i, b) \right],$$ (6.34)

$$W_{1,2}(z) = c_{1}(i)c_{2}(i) \left\{ \frac{\text{Im}(\bar{u}_{2}(i, b))}{\text{Im}(\bar{u}_{1}(i, b))} \left[ \bar{u}_{1}(-i, b) - \bar{u}_{1}(z, b) \right] \right\},$$ (6.35)

$$W_{2,2}(z) = [c_{2}(i)]^{2} \left\{ \bar{u}_{2}(-i, b) - \bar{u}_{2}(z, b) \right\}.$$ (6.36)

The relations (6.18) and (6.31)–(6.36) now yield an explicit representation for the Donoghue $m$-function $M_{D_{0,0},N_{i}}(\cdot)$ for $T_{0,0}$.

**Theorem 6.1.** Assume Hypothesis 4.1 and let $\{v_{j}(i, \cdot)\}_{j=1,2}$ be the orthonormal basis for $N_{i}$ defined in (6.11)–(6.14). The Donoghue $m$-function $M_{D_{0,0},N_{i}}(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(N_{i})$ for $T_{0,0}$ satisfies $M_{D_{0,0},N_{i}}(\pm i) = \pm iI_{N_{i}}$,.
where the matrix $\left(W_{j,k}(\cdot)\right)_{j,k=1}^2$ is given by (6.33)–(6.36).

6.2. The Donoghue $m$-function for Self-Adjoint Extensions Other Than $T_{0,0}$. The Donoghue $m$-function $M_{T_{0,0},\mathcal{N}}^{D_{0}}(\cdot)$ for $T_{0,0}$ was computed explicitly in Theorem 6.1. If $T_{A,B}$ is any other self-adjoint extension of $T_{\text{min}}$, then the resolvent identities in Theorem 4.2 may be used to obtain an explicit representation of the Donoghue $m$-function for $T_{A,B}$.

We begin with the case when either $T_{A,B} = T_{\alpha,\beta}$ for $\alpha, \beta \in (0, \pi)$ or $T_{A,B} = T_{\varphi,R}$ for some $\varphi \in [0, 2\pi)$, $R \in \text{SL}(2, \mathbb{R})$, with $R_{1,2} \neq 0$. In this case, items (i) and (iv) in Theorem 4.2 imply

\[
(T_{A,B} - zI_{L^2((a,b))})^{-1} = (T_{0,0} - zI_{L^2((a,b))})^{-1} + \sum_{j,k=1}^{2} [K_{A,B}(z)^{-1}]_{j,k}(u_j(\varpi, \cdot), \cdot)L_{2}((a,b))u_k(z, \cdot),
\]

\[
z \in \rho(T_{0,0}) \cap \rho(T_{A,B}),
\]

where $K_{A,B}(\cdot) = K_{\alpha,\beta}(\cdot)$ or $K_{A,B}(\cdot) = K_{\varphi,R}(\cdot)$ (cf. (4.8) and (4.16)) according to whether $T_{A,B} = T_{\alpha,\beta}$ or $T_{A,B} = T_{\varphi,R}$, respectively. Employing (6.38) in (6.1), one obtains the following representation for the Donoghue $m$-function $M_{T_{A,B},\mathcal{N}}^{D_{0}}(\cdot)$ of $T_{A,B}$:

\[
M_{T_{A,B},\mathcal{N}}^{D_{0}}(z) = zI_{\mathcal{N}} + (z^2 + 1)P_{\mathcal{N}}(T_{0,0} - zI_{L^2((a,b))})^{-1}P_{\mathcal{N}} \big|_{\mathcal{N}} \bigg) \sum_{j,k=1}^{2} [K_{A,B}(z)^{-1}]_{j,k}(u_j(\varpi, \cdot), \cdot)L_{2}((a,b))P_{\mathcal{N}}u_k(z, \cdot),
\]

\[
z \in \mathbb{C} \setminus \mathbb{R}.
\]

In light of (6.2), to obtain a final expression for $M_{T_{A,B},\mathcal{N}}^{D_{0}}(\cdot)$, one must compute $P_{\mathcal{N}}u_k(z, \cdot)$, $k \in \{1, 2\}$, for $z \in \mathbb{C} \setminus \mathbb{R}$, $z \neq \pm i$. Let $z \in \mathbb{C} \setminus \mathbb{R}$, $z \neq \pm i$. Invoking the orthonormal basis \{\(\upsilon_i(i, \cdot)\)\}_{i=1,2} for $\mathcal{N}$ defined in (6.11)–(6.14), one obtains

\[
P_{\mathcal{N}}u_k(z, \cdot) = \sum_{i=1}^{2} (\upsilon_i(i, \cdot), u_k(z, \cdot))_{L^2((a,b))}\upsilon_i(i, \cdot), \quad k \in \{1, 2\}.
\]

By (2.3),

\[
(\upsilon_i(i, \cdot), u_k(z, \cdot))_{L^2((a,b))} = \int_{a}^{b} r(x)dx \upsilon_i(-i, x)u_k(z, x)
\]
The Wronskians
\[
W_{\ell,k}^{Kr}(z) := W(v_{\ell}(i, \cdot), u_k(z, \cdot))|_a^b, \quad \ell, k \in \{1, 2\},
\]
that appear in (6.41) can be computed by applying (4.1) and (6.11)–(6.12). One obtains:
\[
\begin{align*}
W_{1,1}^{Kr}(z) &= c_1(i) \left[ \tilde{u}_1'(z, b) - \tilde{u}_1'(-i, b) \right], \quad (6.43) \\
W_{1,2}^{Kr}(z) &= c_1(i) \left[ \tilde{u}_2'(z, b) + \tilde{u}_1'(-i, a) \right], \quad (6.44) \\
W_{2,1}^{Kr}(z) &= \bar{v}_2(-i, b) \tilde{u}_1'(z, b) - \bar{v}_2'(-i, b) - \bar{v}_2(-i, a) \tilde{u}_1'(z, a) \\
&= -c_2(i) \left\{ \frac{\text{Im}(\tilde{u}_2'(i, b))}{\text{Im}(\tilde{u}_1'(i, b))} \left[ \tilde{u}_1'(z, b) - \tilde{u}_1'(-i, b) \right] + \tilde{u}_2'(-i, b) + \tilde{u}_1'(z, a) \right\}, \quad (6.45) \\
W_{2,2}^{Kr}(z) &= \bar{v}_2(-i, b) \tilde{u}_2'(z, b) - \bar{v}_2'(-i, a) \tilde{u}_2'(z, a) + \bar{v}_2'(-i, a) \\
&= -c_2(i) \left\{ \frac{\text{Im}(\tilde{u}_2'(i, b))}{\text{Im}(\tilde{u}_1'(i, b))} \left[ \tilde{u}_2'(z, b) + \tilde{u}_2'(-i, a) \right] + \tilde{u}_2'(z, a) - \tilde{u}_2'(-i, a) \right\}.
\end{align*}
\]

Therefore, (6.40) may be recast as
\[
P_{\mathcal{N}_i} u_k(z, \cdot) = -\frac{1}{z + i} \sum_{\ell=1}^2 W_{\ell,k}^{Kr}(z) v_{\ell}(i, \cdot), \quad k \in \{1, 2\}. \tag{6.47}
\]

By combining (6.39) and (6.47), one obtains
\[
M_{T_{\alpha,\beta},\mathcal{N}_i}^{D_0}(z) = M_{T_{0,0},\mathcal{N}_i}^{D_0}(z) \tag{6.48}
\]
\[
+ (i - z) \sum_{j,k,\ell=1}^2 \left[ K_{\alpha,\beta}(z)^{-1} \right]_{j,k} W_{\ell,k}^{Kr}(z) (u_j(\mathcal{N}_i, \cdot, \cdot) L_{\mathcal{Z}((a,b))} v_{\ell}(i, \cdot) |_{\mathcal{N}_i}.
\]

These considerations are summarized next.

**Theorem 6.2.** Assume Hypothesis 4.1 and let \( \{v_j(i, \cdot)\}_{j=1,2} \) be the orthonormal basis for \( \mathcal{N}_i \) defined in (6.11)–(6.14). The following items (i) and (ii) hold.

(i) If \( \alpha, \beta \in (0, \pi) \), then the Donoghue m-function \( M_{T_{\alpha,\beta},\mathcal{N}_i}^{D_0}(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{N}_i) \) for \( T_{\alpha,\beta} \) satisfies
\[
M_{T_{\alpha,\beta},\mathcal{N}_i}^{D_0}(\pm i) = \pm i I_{\mathcal{N}_i}, \quad (6.49)
\]
\[
M_{T_{\alpha,\beta},\mathcal{N}_i}^{D_0}(z) = M_{T_{0,0},\mathcal{N}_i}^{D_0}(z) + (i - z) \sum_{j,k,\ell=1}^2 \left[ K_{\alpha,\beta}(z)^{-1} \right]_{j,k} W_{\ell,k}^{Kr}(z) (u_j(\mathcal{N}_i, \cdot, \cdot) L_{\mathcal{Z}((a,b))} v_{\ell}(i, \cdot) |_{\mathcal{N}_i},
\]

where the matrices \( K_{\alpha,\beta}(\cdot) \) and \( (W_{\ell,k}^{Kr}(\cdot))_{\ell,k=1}^2 \) are given by (4.8) and (6.43)–(6.46), respectively.

(ii) If \( \varphi \in [0, 2\pi) \) and \( R \in \text{SL}(2, \mathbb{R}) \) with \( R_{1,2} \neq 0 \), then the Donoghue m-function \( M_{T_{\varphi,R},\mathcal{N}_i}^{D_0}(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{N}_i) \) for \( T_{\varphi,R} \) satisfies
\[
M_{T_{\varphi,R},\mathcal{N}_i}^{D_0}(\pm i) = \pm i I_{\mathcal{N}_i},
\]

\[ M_{T_{\varphi},R,N_i}(z) = M_{T_{0,0},N_i}(z) \]

\[ + (i - z) \sum_{j,k,l=1}^{2} [K_{\varphi,R}(z)^{-1}]_{j,b} W_{\ell,k}^{K_r}(z)(u_{j}(z,a), \cdot) L_{2}^2((a,b)) v_{l}(i, \cdot) \big|_{N_i}, \]

\[ z \in \mathbb{C} \setminus \mathbb{R}, z \neq \pm i, \]

where the matrices \( K_{\varphi,R}(\cdot) \) and \( (W_{\ell,k}^{K_r}(\cdot))_{\ell,k=1}^{2} \) are given by (6.16) and (6.43)–(6.46), respectively.

It remains to compute the Donoghue \( m \)-functions for \( T_{0,\beta} \) and \( T_{\alpha,0} \) with \( \alpha, \beta \in (0, \pi) \) and \( T_{\varphi,R} \) for \( \varphi \in [0,2\pi) \) and \( R \in \text{SL}(2, \mathbb{R}) \) with \( R_{1,2} = 0 \).

**Theorem 6.3.** Assume Hypothesis 4.1 and let \( \{v_{j}(i, \cdot)\}_{j=1,2} \) be the orthonormal basis for \( N_i \) defined in (6.11)–(6.14). The following items (i) and (ii) hold.

(i) If \( \alpha \in (0, \pi) \), then the Donoghue \( m \)-function \( M_{T_{0,0},N_i}(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(N_i) \) for \( T_{0,0} \) satisfies

\[ M_{T_{0,0},N_i}(\pm i) = \pm i I_{N_i}, \]

\[ M_{T_{0,0},N_i}(z) = M_{T_{0,0},N_i}(z) \]

\[ + \frac{z - i}{\cot(\alpha) + u_{1}(z,a)} (u_{2}(z,a), \cdot) L_{2}^2((a,b)) \sum_{\ell=1}^{2} W_{\ell,1}^{K_r}(z) v_{1}(i, \cdot) \big|_{N_i}, \]

\[ z \in \mathbb{C} \setminus \mathbb{R}, z \neq \pm i, \]

where the scalars \( \{W_{\ell,2}^{K_r}(\cdot)\}_{\ell=1,2} \) are given by (6.44) and (6.46).

(ii) If \( \beta \in (0, \pi) \), then the Donoghue \( m \)-function \( M_{T_{0,\alpha},N_i}(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(N_i) \) for \( T_{0,\alpha} \) satisfies

\[ M_{T_{0,\alpha},N_i}(\pm i) = \pm i I_{N_i}, \]

\[ M_{T_{0,\alpha},N_i}(z) = M_{T_{0,\alpha},N_i}(z) \]

\[ - \frac{z - i}{\cot(\beta) + u_{1}(z,b)} (u_{1}(z,b), \cdot) L_{2}^2((a,b)) \sum_{\ell=1}^{2} W_{\ell,1}^{K_r}(z) v_{1}(i, \cdot) \big|_{N_i}, \]

\[ z \in \mathbb{C} \setminus \mathbb{R}, z \neq \pm i, \]

where the scalars \( \{W_{\ell,1}^{K_r}(\cdot)\}_{\ell=1,2} \) are given by (6.43) and (6.45).

(iii) If \( \varphi \in [0, 2\pi) \) and \( R \in \text{SL}(2, \mathbb{R}) \) with \( R_{1,2} = 0 \), then the Donoghue \( m \)-function \( M_{T_{\varphi,R},N_i}(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(N_i) \) for \( T_{\varphi,R} \) satisfies

\[ M_{T_{\varphi,R},N_i}(\pm i) = \pm i I_{N_i}, \]

\[ M_{T_{\varphi,R},N_i}(z) = M_{T_{0,0},N_i}(z) \]

\[ - \frac{z - i}{k_{\varphi,R}(z)} (u_{\varphi,R}(z,a), \cdot) L_{2}^2((a,b)) \sum_{\ell=1}^{2} \left[ e^{-i\varphi} R_{2,2} W_{\ell,2}^{K_r}(z) + W_{\ell,1}^{K_r}(z) \right] v_{1}(i, \cdot) \big|_{N_i}, \]

\[ z \in \mathbb{C} \setminus \mathbb{R}, z \neq \pm i, \]

where the scalar \( k_{\varphi,R}(\cdot) \) and the matrix \( (W_{\ell,k}^{K_r}(\cdot))_{\ell,k=1}^{2} \) are given by (4.19) and (6.43)–(6.46), respectively.
Finally, (6.53) follows by combining (6.58) and (6.59).

Using (6.47) with $k=2$ in (6.54), one obtains

$$M_{T_{0,0},N}(z) = M_{T_{0,0},N}(z) + (z^2 + 1)K_{0,0}(z)^{-1}(u_0(z, \cdot))_{L^2((a,b))}P_{N_i}u_2(z, \cdot) |_{N_i}.$$  

(6.54)

Hence, (6.51) follows from (6.55) by applying the precise form for $K_{0,0}(z)$ given in (4.14). This completes the proof of item (i).

To prove item (ii), let $\beta \in (0, \pi)$. By (6.2), $M_{T_{0,0},N}(\pm i) = \pm iN_i$. In order to establish (6.52), let $z \in \mathbb{C} \setminus \mathbb{R}$, $z \neq \pm i$, be fixed. Taking $T_{A,B} = T_{0,0}$ in (6.1) and invoking (4.12), one obtains

$$M_{T_{0,0},N}(z) = M_{T_{0,0},N}(z) + (z^2 + 1)K_{0,0}(z)^{-1}(u_1(z, \cdot))_{L^2((a,b))}P_{N_i}u_1(z, \cdot) |_{N_i}.$$  

(6.56)

Using (6.47) with $k=1$ in (6.56), one obtains

$$M_{T_{0,0},N}(z) = M_{T_{0,0},N}(z) + (i - z)K_{0,0}(z)^{-1}(u_1(z, \cdot))_{L^2((a,b))}W_{N_i}^z v_1(i, \cdot) |_{N_i}.$$  

(6.57)

Hence, (6.52) follows from (6.57) by applying the precise form for $K_{0,0}(z)$ given in (4.11). This completes the proof of item (ii).

To prove item (iii), let $\phi \in [0, 2\pi)$ and $R \in \text{SL}(2, \mathbb{R})$ with $R_{1,2} = 0$. By (6.2), $M_{T_{0,0},N}(\pm i) = \pm iN_i$. In order to establish (6.53), let $z \in \mathbb{C} \setminus \mathbb{R}$, $z \neq \pm i$, be fixed. Taking $T_{A,B} = T_{\phi,R}$ in (6.1) and invoking (4.20), one obtains

$$M_{T_{\phi,R},N}(z) = M_{T_{0,0},N}(z) + (z^2 + 1)K_{\phi,R}(z)^{-1}(u_{\phi,R}(z, \cdot))_{L^2((a,b))}P_{N_i}u_{\phi,R}(z, \cdot) |_{N_i}.$$  

(6.58)

By (6.41) and (6.42),

$$P_{N_i}u_{\phi,R}(z, \cdot) = \sum_{\ell=1}^{2} (v_\ell(i, \cdot), e^{-i\phi}R_{2,2}u_2(z, \cdot) + u_1(z, \cdot))_{L^2((a,b))}v_\ell(i, \cdot)$$  

$$= -\frac{1}{z+i} \sum_{\ell=1}^{2} [e^{-i\phi}R_{2,2} W_{N_i}^z v_\ell(i, \cdot) + W_{N_i}^z v_\ell(i, \cdot)].$$  

(6.59)

Finally, (6.53) follows by combining (6.58) and (6.59).  

7. A Generalized Bessel-Type Operator Example

As an illustration of these results, we consider the following explicitly solvable generalized Bessel-type equation following the analysis in [37] (see also [30]). Let
Then is singular at the endpoint \( x = 0 \), regular at \( x = b \) when \( b \in (0, \infty) \), and in the limit point case at \( x = b \) when \( b = \infty \). Furthermore, \( \tau_{\delta, \nu, \gamma} \) is in the limit circle case at \( x = 0 \) if \( 0 \leq \gamma < 1 \) and in the limit point case at \( x = 0 \) when \( \gamma \geq 1 \).

Solutions to \( \tau_{\delta, \nu, \gamma} u = zu \) are given by (cf. [49], [50, No. 2.162, p. 440])

\[
y_1,\delta,\nu,\gamma(z, x) = x^{(1-\nu)/2} J_\gamma \left( 2^{1/2} z (2 + \delta - \nu) / (2 + \delta - \nu) \right), \quad \gamma \geq 0, \\
y_2,\delta,\nu,\gamma(z, x) = \begin{cases} 
\left( (1-\nu)^{-1} x^{(1-\nu+(2+\delta-\nu)\gamma)/2} \right), & \gamma \notin \mathbb{N}_0, \\
\left( (1-\nu) x^{(1-\nu)/2} \ln(1/x) \right), & \gamma \in \mathbb{N}_0, \quad \gamma \geq 0,
\end{cases}
\]

where \( J_\mu(\cdot), Y_\mu(\cdot) \) are the standard Bessel functions of order \( \mu \in \mathbb{R} \) (cf. [1, Ch. 9]).

In the following we assume that

\[
\gamma \in [0, 1)
\]

to ensure the limit circle case at \( x = 0 \). In this case it suffices to focus on the generalized boundary values at the singular endpoint \( x = 0 \) following [34]. For this purpose we introduce principal and nonprincipal solutions \( u_{0,\delta,\nu,\gamma}(0, \cdot) \) and \( \tilde{u}_{0,\delta,\nu,\gamma}(0, \cdot) \) of \( \tau_{\delta, \nu, \gamma} u = 0 \) at \( x = 0 \) by

\[
u_{0,\delta,\nu,\gamma}(0, x) = (1 - \nu)^{-1} x^{(1-\nu+(2+\delta-\nu)\gamma)/2}, \quad \gamma \in [0, 1), \\
\tilde{u}_{0,\delta,\nu,\gamma}(0, x) = \begin{cases} 
(1-\nu)[(2 + \delta - \nu)\gamma]^{-1} x^{(1-\nu-(2+\delta-\nu)\gamma)/2}, & \gamma \in (0, 1), \\
(1-\nu)x^{(1-\nu)/2}\ln(1/x), & \gamma = 0,
\end{cases}
\]

\[
\delta > -1, \quad \nu < 1, \quad x \in (0, 1).
\]

Remark 7.1. Since the singularity of \( q \) at \( x = 0 \) renders \( \tau_{\delta, \nu, \gamma} \) singular at \( x = 0 \) (unless, of course, \( \gamma = (1 - \nu)/(2 + \delta - \nu) \)), in which case \( \tau_{\delta, \nu, (1-\nu)/(2+\delta-\nu)} \) is regular at \( x = 0 \), there is a certain freedom in the choice of the multiplicative constant in the principal solution \( u_{0,\delta,\nu,\gamma} \) of \( \tau_{\delta, \nu, \gamma} u = 0 \) at \( x = 0 \). Our choice of \( (1 - \nu)^{-1} \) in (7.6) reflects continuity in the parameters when comparing to boundary conditions in the regular case (cf. [34, Remark 3.12 (ii)]), that is, in the case \( \delta > -1, \nu < 1, \) and \( \gamma = (1 - \nu)/(2 + \delta - \nu) \) treated in [29].
The generalized boundary values for \( g \in \text{dom}(T_{\max,\delta,\nu,\gamma}) \) are then of the form

\[
\tilde{g}(0) = -W(u_{0,\delta,\nu,\gamma}(0, \cdot), g)(0)
\]

\[
= \begin{cases} 
\lim_{x \downarrow 0} g(x)/[(1 - \nu)[(2 + \delta - \nu)\gamma]^{-1}x^{[(1-\nu)-(2+\delta-\nu)\gamma]/2}], & \gamma \in (0, 1), \\
\lim_{x \downarrow 0} g(x)/[(1 - \nu)x^{(1-\nu)/2\ln(1/x)}], & \gamma = 0,
\end{cases}
\]

(7.7)

\[
\tilde{g}'(0) = W(\tilde{u}_{0,\delta,\nu,\gamma}(0, \cdot), g)(0)
\]

\[
= \begin{cases} 
\lim_{x \downarrow 0} [g(x) - \tilde{g}(0)(1 - \nu)[(2 + \delta - \nu)\gamma]^{-1}x^{[(1-\nu)-(2+\delta-\nu)\gamma]/2}] \\
/[(1 - \nu)^{-1}x^{(1-\nu)/2}], & \gamma \in (0, 1), \\
\lim_{x \downarrow 0} g(x)/[(1 - \nu)x^{(1-\nu)/2\ln(1/x)}], & \gamma = 0.
\end{cases}
\]

(7.8)

Next, introducing the standard normalized (at \( x = 0 \)) fundamental system of solutions \( \phi_{\delta,\nu,\gamma}(z, \cdot, 0) \), \( \theta_{\delta,\nu,\gamma}(z, \cdot, 0) \) of \( \tau_{\delta,\nu,\gamma}u = zu \), \( z \in \mathbb{C} \), that is real-valued for \( z \in \mathbb{R} \) and entire with respect to \( z \in \mathbb{C} \) by

\[
\tilde{\phi}_{\delta,\nu,\gamma}(z, 0, 0) = 0, \quad \tilde{\phi}'_{\delta,\nu,\gamma}(z, 0, 0) = 1,
\]

\[
\tilde{\theta}_{\delta,\nu,\gamma}(z, 0, 0) = 1, \quad \tilde{\theta}'_{\delta,\nu,\gamma}(z, 0, 0) = 0, \quad z \in \mathbb{C},
\]

(7.9)

one obtains explicitly,

\[
\phi_{\delta,\nu,\gamma}(z, x, 0) = (1 - \nu)^{-1}(2 + \delta - \nu)^\gamma\Gamma(1 + \gamma)z^{-(\gamma/2)y_1,\delta,\nu,\gamma}(z, x),
\]

\( \delta > -1, \nu < 1, \gamma \in [0, 1), z \in \mathbb{C}, x \in (0, b) \),

(7.10)

\[
\theta_{\delta,\nu,\gamma}(z, x, 0) = \begin{cases} 
(1 - \nu)(2 + \delta - \nu)^{-\gamma-1}\gamma^{-1}\Gamma(1 + \gamma)z^{\gamma/2}y_{2,\delta,\nu,\gamma}(z, x), & \gamma \in (0, 1), \\
(1 - \nu)(2 + \delta - \nu)^{-1}[\pi y_{2,\delta,\nu,\gamma}(z, x) \\
+ (\ln(z) - 2\ln(2 + \delta - \nu) + 2\gamma_E)\gamma_{1,\delta,\nu,\gamma}(z, x)], & \gamma = 0,
\end{cases}
\]

\( \delta > -1, \nu < 1, z \in \mathbb{C}, x \in (0, b) \),

(7.11)

\[
W(\theta_{\delta,\nu,\gamma}(z, \cdot, 0), \phi_{\delta,\nu,\gamma}(z, \cdot, 0)) = 1, \quad z \in \mathbb{C},
\]

(7.12)

where \( \Gamma(\cdot) \) denotes the Gamma function, and \( \gamma_E = 0.57721\ldots \) represents Euler’s constant.

We now turn to the cases of computing Donoghue \( m \)-functions for the generalized Bessel operator in general on the infinite interval and for the Krein–von Neumann extension on the finite interval.

**Example 7.2 (Infinite Interval).** Let \( b = \infty \). We begin by finding \( \psi_{0,\delta,\nu,\gamma}(z, \cdot) \) described in Hypothesis 3.1 for this example.

Since \( \tau_{\delta,\nu,\gamma} \) is in the limit point case at \( \infty \) (actually, it is in the strong limit point case at infinity since \( q \) is bounded on any interval of the form \( [R, \infty) \), \( R > 0 \), and the strong limit point property of \( \tau_{\delta,\nu,\gamma} = (1 - \nu)/(2 + \delta - \nu) \) has been shown in [29]), to find the Weyl–Titchmarsh solution and \( m \)-function corresponding to the Friedrichs (resp., Dirichlet) boundary condition at \( x = 0 \), one considers the requirement

\[
\psi_{0,\delta,\nu,\gamma}(z, \cdot) = \theta_{\delta,\nu,\gamma}(z, \cdot, 0) + m_{0,\delta,\nu,\gamma}(z)\phi_{\delta,\nu,\gamma}(z, \cdot, 0) \in L^2((0, \infty); x^\delta dx),
\]

\( z \in \mathbb{C}\setminus[0, \infty). \)
This implies

$$
\psi_{0,\delta,\nu,\gamma}(z, x) = \begin{cases}
  i(1 - \nu)(2 + \delta - \nu)^{-\gamma - 1}\Gamma(1 - \gamma)\sin(\pi\gamma)z^{\gamma/2} \\
  \times x^{(1-\nu)/2}H^1_{\nu}(2z^{1/2}x(2+\delta-\nu)/2/(2 + \delta - \nu)), & \gamma \in (0, 1), \\
  i\pi(1 - \nu)/(2 + \delta - \nu)x^{(1-\nu)/2} \\
  \times H^1_{\nu}(2z^{1/2}x(2+\delta-\nu)/2/(2 + \delta - \nu)), & \gamma = 0, \\
  \delta > -1, \nu < 1, z \in \mathbb{C}\setminus[0, \infty), x \in (0, \infty),
\end{cases}
$$

(7.14)

$$
m_{0,\delta,\nu,\gamma}(z) = \begin{cases}
  -e^{-i\pi\gamma}(1 - \nu)^2(2 + \delta - \nu)^{-2\gamma - 1}\Gamma(1 - \gamma)/\Gamma(1 + \gamma)z^{\gamma}, & \gamma \in (0, 1), \\
  (1 - \nu)^2/(2 + \delta - \nu) \\
  \times [\pi - \ln(z) + 2\ln(2 + \delta - \nu) - 2\gamma e], & \gamma = 0, \\
  \delta > -1, \nu < 1, z \in \mathbb{C}\setminus[0, \infty),
\end{cases}
$$

(7.15)

where $H^1_{\nu}(\cdot)$ is the Hankel function of the first kind and of order $\mu \in \mathbb{R}$ (cf. [1, Ch. 9]). In particular, it is immediate from (7.13) and (7.9) that $\psi_{0,\delta,\nu,\gamma}(z, 0) = 1$. We mention that the results (7.14) and (7.15) coincide with the ones obtained in [34] when $\delta = \nu = 0$ and [29] when $\gamma = (1 - \nu)/(2 + \delta - \nu)$.

Substituting the explicit form of $\psi_{0,\delta,\nu,\gamma}(z, \cdot)$ given in (7.14) into Theorems 5.1 and 5.2 yields the Friedrichs extension Donoghue $m$-function, $M^D_{0,\delta,\nu,\gamma, N}(z)$, and the Donoghue $m$-function for all other self-adjoint extensions, $M^D_{\alpha,\delta,\nu,\gamma, N}(z) = \psi_{0,\delta,\nu,\gamma}(z, 0)$ one finds from Theorem 5.1 and (7.15),

$$
M^D_{0,\delta,\nu,\gamma, N}(z) = \left[ -i + \frac{m_{0,\delta,\nu,\gamma}(z) - m_{0,\delta,\nu,\gamma}(-i)}{\text{Im}(m_{0,\delta,\nu,\gamma}(i))} \right] I_{N},
$$

$$
\quad = \left\{ \left\{ -i + [\sin(\pi\gamma/2)]^{-1}e^{-i\pi\gamma}(z^{\gamma}) - e^{3i\pi/2}\right\} \right\} I_{N}, \quad \gamma \in (0, 1),
$$

$$
\quad \left\{ [-i + (2/\pi)](3i\pi/2) - \ln(z)]\right\} I_{N}, \quad \gamma = 0, \\
\quad \delta > -1, \nu < 1, z \in \mathbb{C}\setminus[0, \infty),
$$

(7.16)

where the branch of the logarithm is chosen so that $\ln(-i) = 3i\pi/2$. Thus, by Theorem 5.2 with $\alpha \in (0, \pi)$,

$$
M^D_{\alpha,\delta,\nu,\gamma, N}(z) = M^D_{0,\delta,\nu,\gamma, N}(z) + (i - z)\frac{m_{0,\delta,\nu,\gamma}(z) - m_{0,\delta,\nu,\gamma}(-i)}{\cot(\alpha) + m_{0,\delta,\nu,\gamma}(z)}
$$

$$
\times \psi_{0,\delta,\nu,\gamma}(z, \cdot) = \left. \psi_{0,\delta,\nu,\gamma}(i, \cdot) \right|_{N}, \quad \delta > -1, \nu < 1, z \in \mathbb{C}\setminus\mathbb{R}.
$$

(7.17)

**Example 7.3 (Finite Interval).** Let $b \in (0, \infty)$. It is well known that $T_{\min,\delta,\nu,\gamma} \geq \varepsilon I_{L^2((a,b))}$ for some $\varepsilon > 0$ (see, e.g., the simpler case $\delta = \nu = 0$ treated in [38, Thm. 5.1]). Thus, the Krein–von Neumann extension $T_{0,\delta,\nu,\gamma}$ of $T_{\min,\delta,\nu,\gamma}$ is of the form (see [30, Example 4.1])

$$
T_{0,\delta,\nu,\gamma}f = \tau_{\delta,\nu,\gamma}f,
$$

(7.18)

$$
f \in \text{dom}(T_{0,\delta,\nu,\gamma}) = \left\{ g \in \text{dom}(T_{\max,\delta,\nu,\gamma}) \mid \left( \begin{array}{c}
  g(b) \\
  g'([1](b))
\end{array} \right) = R_{\delta,\nu,\gamma}\left( \begin{array}{c}
  \tilde{g}(0) \\
  \tilde{g}'(0)
\end{array} \right) \right\},
$$

where
\[
R_{K,\delta,\nu,\gamma} = \begin{cases} 
\frac{1 - \nu}{(2 + \delta - \nu)\gamma} b^{1-\nu-\frac{(2+\delta-\nu)\gamma}{2}} 
\times \left( \frac{1 - \nu}{2(2 + \delta - \nu)\gamma} - \frac{1}{2} + \frac{1}{2(1 - \nu)} \right) \frac{b^{1-\nu+(2+\delta-\nu)\gamma}}{b^{2+\delta-\nu} \gamma} , & \gamma \in (0, 1), \\
\frac{(1 - \nu) \ln(1/b) b^{(1-\nu)/2} - 2(1 - \nu) b^{(\nu-1)/2}}{2^{2}} , & \gamma = 0, \\
\frac{1}{2} b^{(1-\nu)/2} b^{(\nu-1)/2} , & \delta > -1, \nu < 1. \end{cases}
\]

One now explicitly finds the solutions in (4.1) for this example by choosing
\[
u_{1,\delta,\nu,\gamma}(z, x) = \phi_{\delta,\nu,\gamma}(z, x, 0)/\phi_{\delta,\nu,\gamma}(z, b, 0),
\]
\[
u_{2,\delta,\nu,\gamma}(z, x) = \theta_{\delta,\nu,\gamma}(z, x, 0) - [\theta_{\delta,\nu,\gamma}(z, b, 0)/\phi_{\delta,\nu,\gamma}(z, b, 0)] \phi_{\delta,\nu,\gamma}(z, x, 0),
\]
\[
\delta > -1, \nu < 1, \gamma \in [0, 1), x \in (0, b),
\]
from which substituting (7.20) into (6.27) yields the expressions for \(\nu_{1,\delta,\nu,\gamma}(z, \cdot),\) \(j = 1, 2,\) and the explicit form of \(K_{0,RK,\delta,\nu,\gamma}(z)\) given in (4.25) (utilizing (7.6)) into Theorems 6.1 and 6.2 yields the Friedrichs extension Donoghue \(m\)-function, \(M_{D_{0,0,\delta,\nu,\gamma}}^{\delta,\nu,\gamma}(z),\) and the Krein–von Neumann extension Donoghue \(m\)-function, \(M_{D_{0,RK,\delta,\nu,\gamma}}^{\delta,\nu,\gamma}(z),\) respectively.

Acknowledgments. R. N. would like to thank the U.S. National Science Foundation for summer support received under Grant DMS-1852288 in connection with REU Site: Research Training for Undergraduates in Mathematical Analysis with Applications in Allied Fields. M. P. was supported by the Austrian Science Fund under Grant W1245.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, 9th printing, Dover, New York, 1972.
[2] S. B. Allan, J. H. Kim, G. Michajlyszyn, R. Nichols, and D. Rung, Explicit Krein resolvent identities for singular Sturm–Liouville operators with applications to Bessel operators, Oper. Matrices 14, No. 4, 1043–1099 (2020).
[3] D. Alpay and J. Behrndt, Generalized Q-functions and Dirichlet-to-Neumann maps for elliptic differential operators, J. Funct. Anal. 257, 1666–1694 (2009).
[4] W. O. Amrein and D. B. Pearson, M operators: a generalization of Weyl–Titchmarsh theory, J. Comp. Appl. Math. 171, 1–26 (2004).
[5] J. Behrndt, S. Hassi, and H. De Snoo, Boundary Value Problems, Weyl Functions, and Differential Operators, Monographs in Math., Vol. 108, Birkhäuser, Springer, 2020.
[6] J. Behrndt and M. Langer, Boundary value problems for elliptic partial differential operators on bounded domains, J. Funct. Anal. 243, 536–565 (2007).
[7] J. Behrndt and T. Micheler, Elliptic differential operators on Lipschitz domains and abstract boundary value problems, J. Funct. Anal. 267, 3657–3709 (2014).
[8] J. Behrndt and J. Rohleder, Spectral analysis of selfadjoint elliptic differential operators, Dirichlet-to-Neumann maps, and abstract Weyl functions, Adv. Math. 285, 1301–1338 (2015).
[9] J. Behrndt and J. Rohleder, *Titchmarsh–Weyl theory for Schrödinger operators on unbounded domains*, J. Spectral Theory 6, 67–87 (2016).

[10] J. F. Brasche, M. Malamud, and H. Neidhardt, *Weyl function and spectral properties of self-adjoint extensions*, Integr. Equ. Oper. Th. 43, 264–289 (2002).

[11] B. M. Brown, G. Grubb, and I. G. Wood, *M-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems*, Math. Nachr. 282, 314–347 (2009).

[12] B. M. Brown, J. Hinchcliffe, M. Marletta, S. Naboko, and I. Wood, *The abstract Titchmarsh–Weyl M-function for adjoint operator pairs and its relation to the spectrum*, Integral Equ. Operator Theory 63, 297–320 (2009).

[13] B. M. Brown, M. Marletta, S. Naboko, and I. Wood, *Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices*, J. London Math. Soc. (2) 77, 700–718 (2008).

[14] B. M. Brown, M. Marletta, S. Naboko, and I. Wood, *Inverse problems for boundary triples with applications*, Studia Math. 237, 241–275 (2017).

[15] J. Brüning, V. Geyler, and K. Pankrashkin, *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*, Rev. Math. Phys. 20, 1–70 (2008).

[16] S. Clark, F. Gesztesy, and R. Nichols, *Principal solutions revisited*, in *Stochastic and Infinite Dimensional Analysis*, C. C. Bernido, M. V. Carpio-Bernido, M. Grothaus, T. Kuna, M. J. Oliveira, and J. L. da Silva (eds.), Trends in Mathematics, Birkhäuser, Springer, 2016, pp. 85–117.

[17] S. Clark, F. Gesztesy, R. Nichols, and M. Zinchenko, *Boundary data maps and Krein’s resolvent formula for Sturm–Liouville operators on a finite interval*, Oper. Matrices 8, 1–71 (2014).

[18] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Krieger Publ., Malabar, FL, 1985.

[19] V. Derkach, S. Hassi, M. Malamud, and H. de Snoo, *Boundary relations and generalized resolvents of symmetric operators*, Russian J. Math. Phys. 16, 1–95 (2009).

[20] V. A. Derkach and M. M. Malamud, *On the Weyl function and Hermitian operators with gaps*, Sov. Math. Dokl. 35, 393–398 (1987).

[21] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. 95, 1–95 (1991).

[22] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, J. Math. Sci. 73, 141–242 (1995).

[23] V. A. Derkach and M. M. Malamud, *On some classes of holomorphic operator functions with nonnegative imaginary part*, in *Operator Algebras and Related Topics*, A. Gheondea, R. N. Gologan, and T. Timotin (eds.), The Theta Foundation, Bucharest, 1997, pp. 113–147.

[24] V. A. Derkach and M. M. Malamud, *Weyl function of a Hermitian operator and its connection with characteristic function*, arXiv:1503.08956.

[25] V. A. Derkach, M. M. Malamud, and E. R. Tsekanovskii, *Sectorial extensions of a positive operator, and the characteristic function*, Sov. Math. Dokl. 37, 106–110 (1988).

[26] W. F. Donoghue, *On the perturbation of spectra*, Commun. Pure Appl. Math. 18, 559–579 (1965).

[27] N. Dunford and J. T. Schwartz, *Linear Operators. Part II: Spectral Theory*, Wiley, Interscience, New York, 1988.

[28] J. Eckhardt, F. Gesztesy, R. Nichols, and G. Teschl, *Weyl–Titchmarsh theory for Sturm–Liouville operators with distributional potentials*, Opuscula Math. 33, 467–563 (2013).

[29] W. N. Everitt and A. Zettl, *On a class of integral inequalities*, J. London Math. Soc. (2) 17, 291–303 (1978).

[30] G. Fucci, F. Gesztesy, K. Kirsten, L. L. Littlejohn, R. Nichols, and J. Stanfill, *The Krein–von Neumann extension revisited*, Applicable Anal., 25p. (2021). DOI: 10.1080/00036811.2021.1938905

[31] C. T. Fulton, *Parametrizations of Titchmarsh’s ‘m(λ)’-Functions in the Limit Circle Case*, Ph.D. Thesis, Technical University of Aachen, Germany, 1973.

[32] C. T. Fulton, *Parametrizations of Titchmarsh’s m(λ)-functions in the limit circle case*, Trans. Amer. Math. Soc. 229, 51–63 (1977).
[60] M. Marletta, *Eigenvalue problems on exterior domains and Dirichlet to Neumann maps*, J. Comp. Appl. Math. **171**, 367–391 (2004).
[61] M. Marletta and A. Zettl, *The Friedrichs extension of singular differential operators*, J. Diff. Eq. **160**, 404–421 (2000).
[62] V. Mogilevskii, *Boundary triplets and Titchmarsh–Weyl functions of differential operators with arbitrary deficiency indices*, Meth. Funct. Anal. Topology **15**, 280–300 (2009).
[63] S. N. Naboko, *Boundary values of analytic operator functions with a positive imaginary part*, J. Soviet Math. **44**, 786–795 (1990).
[64] S. N. Naboko, *Nontangential boundary values of operator-valued R-functions in a half-plane*, Leningrad Math. J. **1**, 1255–1278 (1990).
[65] S. N. Naboko, *The boundary behavior of $S_p$-valued functions analytic in the half-plane with nonnegative imaginary part*, Functional Analysis and Operator Theory, Banach Center Publications, Vol. **30**, Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1994, pp. 277–285.
[66] M. A. Naimark, *Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space*, Transl. by E. R. Dawson, Engl. translation edited by W. N. Everitt, Ungar Publishing, New York, 1968.
[67] H.-D. Niessen and A. Zettl, *Singular Sturm–Liouville problems: the Friedrichs extension and comparison of eigenvalues*, Proc. London Math. Soc. (3) **64**, 545–578 (1992).
[68] K. Pankrashkin, *An example of unitary equivalence between self-adjoint extensions and their parameters*, J. Funct. Anal. **265**, 2910–2936 (2013).
[69] D. B. Pearson, *Quantum Scattering and Spectral Theory*, Academic Press, London, 1988.
[70] A. Posilicano, *Boundary triplets and Weyl functions for singular perturbations of self-adjoint operators*, Meth. Funct. Anal. Topology **10**, 57–63 (2004).
[71] F. Rellich, *Die zulässigen Randbedingungen bei den singulären Eigenwertproblemen der mathematischen Physik. (Gewöhnliche Differentialgleichungen zweiter Ordnung.)*, Math. Z. **49**, 702–723 (1943/44).
[72] F. Rellich, *Halbbeschrankte gewöhnliche Differentialoperatoren zweiter Ordnung*, Math. Ann. **122**, 343–368 (1951). (German.)
[73] R. Rosenberger, *A new characterization of the Friedrichs extension of semibounded Sturm–Liouville operators*, J. London Math. Soc. (2) **31**, 501–510 (1985).
[74] V. Ryshov, *A general boundary value problem and its Weyl function*, Opuscula Math. **27**, 305–331 (2007).
[75] Sh. N. Saakjan, *Theory of resolvents of a symmetric operator with infinite defect numbers*, Akad. Nauk. Armjan. SSR Dokl., **41**, 193–198 (1965). (Russian.)
[76] G. Teschl, *Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators*, 2nd ed., Graduate Studies in Math., Vol. 157, Amer. Math. Soc., RI, 2014.
[77] J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Vol. 68, Springer, New York, 1980.
[78] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil II: Anwendungen*, Teubner, Stuttgart, 2003.
[79] S. Yao, J. Sun, and A. Zettl, *The Sturm–Liouville Friedrichs extension*, Appl. Math. **60**, 299–320 (2015).
[80] A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, Vol. 121, Amer. Math. Soc., Providence, RI, 2005.
