Generalized Set of Boussinesq equations for surf zone region
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Abstract

In this report, generalized wave breaking equations are developed using three dimensional fully nonlinear extended Boussinesq equations to encompass rotational dynamics in wave breaking zone. The derivation for vorticity distributions are developed from Reynold based stress equations.

Keywords: Wave breaking, Boussinesq equation, shallow water, surf zone.

1 Introduction

Wave breaking is one of the most complex phenomena that occurs in the nearshore region. During propagation of wave from deep to shallow water, the wave field is transformed due to shoaling and close to the shoreline, they become unstable and break. In the process of breaking, energy is redistributed from fairly organized wave motion to small scale turbulence, large scale currents and waves.

It has been shown by numerous researchers that Boussinesq-type equations for varying water depth can describe nonlinear transformation in the shoaling region quite well. In the last couple of decades, a lot of research effort has gone into improving the predictive capability of these equations in the intermediate water-depth and close to the surf zone (see e.g. Nwogu [1983], Madsen[5], Wei [1995]). It was established that to extend the validity of these equations to the deep water, higher order dispersive terms will have to be retained, and to improve the predictive capability close to breaking, the nonlinear terms will all have to be retained. However, to model wave breaking, these models use additional terms that artificially added to the momentum equation, which would then reproduce the main characteristic of a breaking wave, i.e. the reduction in wave height. For example, wave breaking in FUNWAVE (FUNWAVE is based on the model described by Nwogu [1993]) is modeled by introducing momentum mixing term developed by Kennedy et al [1999].

Most progress have been done for potential flow, starting with the work of Nwogu [1993] and Madsen [1983]. Some work have also been done to address partially rotational flows by Shen [2000]. In the breaking region and in the surf zone, the wave

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breaking introduces vorticity into the fluid. To address this problem, Veeramony & Svendsen [2000] derived breaking terms in Boussinesq equation assuming flow as two-dimensional rotational flow. Here, the breaking process is modeled by assuming that vorticity is generated in the roller region of the breaking wave and solving vorticity transport equation to determine the distribution of the vorticity. This naturally introduces additional terms in the momentum equation which causes wave height reduction as well as changes in the velocity field. However, since this model is based on stream function formulation, it cannot be trivially extended to three-dimensional flow. The phenomena of wave breaking in Boussinesq equations are being modeled using quite few techniques which can preserve the wave shape as well as include energy dissipation mechanism. Shen [2000] developed a generalized form of Boussinesq equation in 3D vortical flow field with arbitrary vorticity distribution up to $O(\mu^2)$. But he did not describe momentum transport equation with full description of rotational flow. Recently, Zou et al [2004] addressed the problem by including the higher order terms in Boussinesq equation in 2D flow. This model solves to vorticity distribution based on the parametric form taken form surface roller data. In this paper, we try to develop a general form for breaking term for fully nonlinear set of Boussinesq equations for three dimensional vortical flow field near surf zone region. Derivation of breaking term from Reynold stress based vorticity transport equation was also developed to describe rotational field as a complete model of Veeramony [2000].

The paper is organized as follows: Section 2 discusses the basic governing equations for continuity and momentum with boundary conditions. Section 3 describes the equation for horizontal and vertical velocity distribution for potential and rotational components. In section 4, the breaking term is derived for velocity transport equation for fully nonlinear case and solved vorticity transport equation analytically from fourier series expansion. In last section, results were discussed with conclusion.

2 Basic Equations

We consider a three-dimensional wave field with free surface $\eta(x, y, t)$ propagating over a variable water depth $h(x, y)$. As we are primarily concerned with wave breaking, we only consider here wave propagation in shallow water. Wave in this region can be characterized by two non-dimensional parameters $\delta = a/h$ and $\mu = h/l$ where $a$ is the characteristic wave amplitude and $l$ the characteristic wave length. The parameter $\mu$ is a measure of frequency dispersion and $\delta$ that of the nonlinearity of the wave. In this study, since we are only considering shallow water waves, we only have to consider weakly dispersive waves (upto $O(\mu^2)$) but have to retain all nonlinear terms.
In this paper, the variables are non-dimensionalized using following scaling:

\[ x = \hat{x}/l, \ y = \hat{y}/l, \ z = \hat{z}/h, \ t = \hat{t}\sqrt{gh}/l, \]
\[ \hat{u} = (\delta\sqrt{gh}) u, \ \hat{v} = (\delta\sqrt{gh}) v, \ \hat{w} = (\delta\mu\sqrt{gh}) w \]

where the \( \hat{\text{\_}} \) represents the dimensional variables, \( g \) is the acceleration due to gravity, \( u \) and \( v \) are the horizontal components of the velocity in the \( x \) and \( y \) directions respectively, \( w \) is the vertical velocity. We start with the Eulerian equations of continuity and momentum in nondimensionalized form for velocity field \( \mathbf{u} = (u, v, w) \) as:

\[
\frac{\partial u}{\partial t} + \delta u \frac{\partial u}{\partial x} + \delta v \frac{\partial u}{\partial y} + \delta w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = 0
\]
\[
\frac{\partial v}{\partial t} + \delta v \frac{\partial v}{\partial x} + \delta v \frac{\partial v}{\partial y} + \delta w \frac{\partial v}{\partial z} + \frac{\partial p}{\partial y} = 0
\]
\[
\delta\mu^2 \frac{\partial w}{\partial t} + \delta^2\mu^2 u \frac{\partial w}{\partial x} + \delta^2\mu^2 v \frac{\partial w}{\partial y} + \delta^2\mu^2 w \frac{\partial w}{\partial z} + \delta \frac{\partial p}{\partial z} + 1 = 0
\]

Since the fluid flow is rotational, we also have three dimensional vorticity field \( \mathbf{s} = (s_x, s_y, s_z) \) in the fluid defined as

\[
\nabla \times \mathbf{u} = \mathbf{s}
\]

where \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \). The continuity equation then becomes,

\[
\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0
\]

Here \( \nabla \cdot \mathbf{u} = (\partial u/\partial x, \partial v/\partial y) \). The above equations satisfy two boundary conditions for velocity at bottom and at free surface. At the free surface \( z = \eta(x, y, t) \), since particles are free to move with fluid velocity, the kinematic boundary condition is

\[
w_\eta = \mathbf{u}_\eta \cdot \nabla \eta + \frac{\partial \eta}{\partial t}
\]

and at bottom \( z = -h(x, y) \)

\[
w_b = -u_b \cdot \nabla h
\]

where \( \mathbf{u}_\eta = (u_\eta, v_\eta) \) is two component horizontal surface velocity. \( \nabla \eta = (\eta_x, \eta_y) \), \( \nabla h = (h_x, h_y) \) refer to horizontal derivative with respect to \( x \) and \( y \) in all subsequent calculations. The horizontal component for vorticity field \( \mathbf{s} = (s_y, -s_x) \) can be described as,

\[
\frac{\partial u}{\partial z} - \mu^2 \nabla w = \mathbf{s}
\]
with \( \mathbf{u} = (u, v) \) as two component horizontal field whereas vertical component of vorticity expressed as

\[
-s_z = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}
\] (10)

This is straightforward calculation from equation (6) and (8) which is the beginning equation in three dimensional vorticity field formulation.

\[
\mu^2 \nabla^2 w + \frac{\partial^2 w}{\partial z^2} = -\nabla \cdot s = S_w
\] (11)

\( w \) represents the vertical velocity of the flow.

In the above equation, once \( w \) solved, horizontal component of velocity \( u, v \) can be solved from vorticity relation. In weakly hydrostatic case ( \( 0 < \mu^2 \ll 1 \) ), solution is typically obtained from iterative perturbation procedure with successive correction term up to \( \mu^2 \).

In case of breaking waves where vorticity is very strong, so \( (\partial u/\partial z \sim O(1)) \). We assume solution as, \( u = u_o + \mu^2 u_1 + O(\mu^4) \) and \( w = w_0 + \mu^2 w_1 + O(\mu^4) \) for horizontal and vertical velocity component.

Under this assumption, Poisson equation becomes

\[
\frac{\partial^2 w_0}{\partial z^2} = S_w
\] (12)

\[
\frac{\partial^2 w_1}{\partial z^2} = -\left[ \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \right]
\] (13)

\( w_0, w_1 \) can be calculated from bottom boundary conditions using equation (7) separately where the boundary conditions are,

\[
w_{b0} = -u_{b0} \cdot \nabla h
\] (14)

and

\[
w_{b1} + u_{b1} \cdot \nabla h = 0
\] (15)

at bottom boundary \( z = -h \).

Since at any other depth \( \mathbf{z} = z_r \), \( w \) is constrained by continuity equation only, so the equation follows

\[
\frac{\partial w}{\partial z}|_{z_r} = -\nabla \cdot u_m|_{z_r} + \frac{\partial u}{\partial z} \cdot \nabla z_r|_{z_r}
\] (16)

where \( u_m \) is velocity at any arbitrary depth \( z_r \). In Boussinesq type equation, one may take depth average or any intermediate velocity for horizontal velocity between
bottom and free surface as reference velocity. In the wave breaking zone where the vorticity is developed non uniformly, the equations become simpler with the choice of depth average velocity which includes contribution from surface vorticity gradient. We assume solution for velocity comes also from rotational contribution due to vorticity at the wave surface. So the velocity has both potential as well as rotational component, $u = u_p + u_r$, $w = w_p + w_r$. We solve $w_0, w_1$ and $u_0, u_1$ at any depth $z_r$

\[
\frac{\partial w_0}{\partial z}\bigg|_{z_r} = -\nabla \cdot (u_m - z_r s) + z_r \nabla \cdot s \tag{17}
\]

\[
\frac{\partial w_1}{\partial z}\bigg|_{z_r} = [\nabla w_0]_{z_r} \cdot \nabla z_r \tag{18}
\]

\[
\frac{\partial u_0}{\partial z} = s \tag{19}
\]

and

\[
\frac{\partial u_1}{\partial z} = \nabla w_0 \tag{20}
\]

with boundary condition $[u_0]_{z_r} = u_r$ and $[u_1]_{z_r} = 0$. Equations (4) - (16) form basic shallow water Boussinesq equations.

### 3 Equation for horizontal velocity

In the surf zone, vorticity grows very strongly as a non uniform function over depth. Following Shen [2000], we define reference velocity as $\tilde{u} = \bar{u} + \Delta \bar{u} - \eta s_\eta$ in terms of depth average velocity $\bar{u}$ and magnitude of vorticity at free surface $s_\eta$ with the assumption of $\nabla \cdot s \neq 0$. We set here $z_r = \eta$ as linear calibration for $z_r = r(\eta + h) - h$ does not hold here in presence of nonuniform velocity as wave dispersion properties change both spatially and temporally with vorticity. And boundary condition can be set as

\[
\frac{\partial w}{\partial z}\bigg|_{\eta} = \nabla \cdot \tilde{u} + \eta (\nabla \cdot s_\eta) \tag{21}
\]

Integrating equation (9) from bottom to surface and applying boundary condition to (16) we get $w_0$ as,

\[
w_0 = w_{b0} - (-\nabla \cdot \tilde{u} + \eta \nabla \cdot s_\eta) H_z - S_{w0} \tag{22}
\]

where

\[
S_{w0} = \int \int (-\nabla \cdot s) dz \, dz
\]
is the vertical velocity distribution generated by horizontal divergence of vorticity added to surface velocity.

Now, once \( w_0 \) is calculated, \( u_1 \) can be calculated from eqn (15) with surface boundary condition \([u_0]_\eta = u_m \) and \([u_1]_\eta = 0\)

Finally, we calculate horizontal velocity as

\[
\begin{align*}
  \bar{u} &= u_\eta - \Delta \bar{u} + \mu^2 \frac{2}{3} H_\eta^2 \bigtriangledown \big( \eta \cdot \bar{u} - \eta \bigtriangledown \cdot s_\eta \big) \\
  &\quad - \mu^2 \frac{2}{3} H_\eta \big[ \bigtriangledown (\eta s_\eta) \cdot \bigtriangledown h - (\bigtriangledown \cdot \eta s_\eta) \bigtriangledown h \big] + O(\mu^4)
\end{align*}
\]

which on averaging over depth yields,

\[
\begin{align*}
  \bar{u} &= u_\eta - \Delta \bar{u} + \mu^2 \frac{2}{3} H_\eta^2 \bigtriangledown \big( \eta \cdot \bar{u} - \eta \bigtriangledown \cdot s_\eta \big) \\
  &\quad - \mu^2 \frac{2}{3} H_\eta \big[ \bigtriangledown (\eta s_\eta) \cdot \bigtriangledown h - (\bigtriangledown \cdot \eta s_\eta) \bigtriangledown h \big] + O(\mu^4)
\end{align*}
\]

\( \Delta \bar{u} = \frac{1}{H_\eta} \int_{-h}^{\eta} \Delta u(z) dz \) is the average surface velocity contribution due to vorticity and it is significant for suspended sediment particles in the flow. The term \( \Delta u(z) = \int_{-h}^{\eta} s dz \) is the change due to depth variation of vorticity \( S \). The total water depth \( H_z \) and surface elevation \( H_\eta \) are taken as \( H_z = z + h \) and \( H_\eta = \eta + h \)

The contribution for velocity has and rotational component apart from potential due to vorticity generation. After we redefine \( H_\eta = d \) and \( z = H_z/H_\eta \), we express potential and rotational component up to order \( O(\mu^2) \) as

\[
\begin{align*}
  u_p(z) &= \bar{u}_p + \mu^2 \frac{2}{3} (1 - z^2) d^2 \bigtriangledown \big( \bigtriangledown \cdot \bar{u}_p \big) \\
  &\quad + \mu^2 \big[ \frac{1}{2} - z \big] d \big[ \bigtriangledown (\bar{u}_p \cdot \bigtriangledown h) + (\bigtriangledown \cdot \bar{u}_p) \bigtriangledown h \big]
\end{align*}
\]

\[
\begin{align*}
  u_r(z) &= \bar{u}_r - \Delta u(z) + \eta s_\eta + \mu^2 (S_{wl} - \bar{S}_{wl}) \\
  &\quad - \mu^2 \big[ \frac{1}{3} - z^2 \big] d^2 \eta \bigtriangledown \big( \bigtriangledown \cdot s_\eta \big) \\
  &\quad - \mu^2 \big[ \frac{1}{2} - z \big] d \eta \big[ (\bigtriangledown \cdot s_\eta) \bigtriangledown h - \bigtriangledown (s_\eta \cdot \bigtriangledown h) \big]
\end{align*}
\]
Similar expressions for vertical velocity are
\[
\begin{align*}
    w_p(z) &= -(h + z) \triangle \cdot u_p(z) \\
    &= - \triangle \cdot [(h + z) \tilde{u}_p] - \frac{\mu^2}{2} \left( \frac{1}{3} - z^2 \right) \triangle \cdot [d^2(h + z)(\triangle \cdot \tilde{u}_p)] \\
    & \quad - \mu^2 \left( \frac{1}{2} - z \right) \left[ \triangle \cdot (h + z)(\triangle(\nabla \cdot \tilde{u}_p)) + (\triangle \cdot u_p) \nabla h \right] \\
    w_r(z) &= - \nabla \cdot (h + z) \tilde{u}_r - \nabla \cdot [(h + z) \eta s_\eta] \\
    & \quad - \frac{\mu^2}{2} \left[ \nabla \cdot (h + z)(\frac{1}{3} - z^2)d^2(\eta \nabla \cdot s_\eta) \right] \\
    & \quad + \mu^2 \nabla \cdot [(\frac{1}{2} - z)d[\nabla(\eta s_\eta \cdot \nabla h) - (\nabla \cdot s_\eta \cdot \nabla h)] \\
\end{align*}
\]

(27)

(28)

4 Breaking Model [fully nonlinear case]

Conventional time dependent Boussinesq equations for surface wave height and consequent breaking term calculation are very straightforward and published previously in case of irrotational waves. Here we take up fully nonlinear calculation as vorticity becomes a large fraction of water depth in the surf zone or shoaling waves. So, while developing Boussinesq equations for horizontal momentum, we retain up to order \(O(\delta^2)\) and \(O(\delta \mu^2)\) in our fully nonlinear calculation. Fully nonlinear Boussinesq equations for long wave have been derived by Mei [1983] for flat bottom and by Wei et al. [1995] for variable bottom surface in case of irrotational wave. Shen [2000] addressed problems in developing generalized three dimensional irrotational propagating wave field to include rotational motion in general did not describe the vorticity breaking terms. For horizontal propagation of waves, the three dimensional problem can be reduced in terms of two horizontal velocity by integrating over depth and retaining up to order \(O(\delta^2)\) and \(O(\delta \mu^2)\) As horizontal velocity is governed by momentum equation at the surface \(\eta\) by,

\[
\frac{Du}{Dt}|_{\eta} = \left( \frac{Dw}{Dt} \right)|_{\eta} + \nabla \eta \tag{29}
\]

In the surf zone region of sloping beach, waves break due to high vorticity and the breaking of wave later being converted to turbulence. So horizontal variation of water depth \(h(x, y)\) must be considered in this case. We express surface propagation equation in terms of average velocity description and total time derivative of horizontal momentum can be written as,

\[
\frac{D\tilde{u}}{Dt}|_{\eta} = \frac{\partial u}{\partial t}|_{\eta} + u_\eta \cdot (\nabla u)|_{\eta} \tag{30}
\]

7
where surface velocity is given by,

\[ u_\eta = \bar{u} + \eta S_\eta - \frac{\mu^2}{3} d^2 (\nabla \cdot \tilde{u} - \eta \nabla \cdot s_\eta) \]

\[ + \frac{\mu^2}{2} d [\nabla (\tilde{u} - \Delta \tilde{u})_{\eta} + \eta S_\eta] \cdot \nabla h - (\nabla \cdot \tilde{u} - \eta \nabla \cdot s_\eta) \nabla h \]  

(31)

We consider \( \nabla H_\eta = \nabla \eta + \nabla h \) for wavy bottom

\[
\frac{D u_\eta}{D t} = \frac{\partial \tilde{u}}{\partial t} + \eta \frac{\partial S_\eta}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} - \frac{\mu^2}{3} d^2 [\nabla (\nabla \cdot \tilde{u}) - \eta \nabla \cdot \frac{\partial S_\eta}{\partial t}] \\
+ \tilde{u} \cdot \nabla (\nabla \cdot \tilde{u} - \eta \nabla \cdot S_\eta) + \frac{\mu^2}{2} d [\nabla (\frac{\partial \tilde{u}}{\partial t} + \eta \frac{\partial S_\eta}{\partial t}) \cdot \nabla h \\
- (\nabla \cdot \frac{\partial \tilde{u}}{\partial t} - \eta \nabla \cdot \frac{\partial S_\eta}{\partial t}) \nabla h \\
+ \tilde{u} \cdot \nabla ] + O(\mu^4)
\]

(32)

This long wave momentum equation upon simplification over flat bottom case can be compared to the one derived by Shen [2000]. The vertical velocity can be obtained similarly,

\[
\frac{D w}{D t} \bigg|_\eta = \frac{\partial w}{\partial t} \bigg|_\eta + u_\eta \cdot \nabla w + \frac{\partial w}{\partial z} \bigg|_\eta
\]

(33)

So, we can write the horizontal momentum equation as,

\[
\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \eta = \frac{\mu^2}{3} d^2 \{\nabla (\nabla \cdot \tilde{u}) \cdot \nabla \tilde{u} + \nabla (\nabla \cdot \frac{\partial \tilde{u}}{\partial t}) \}
+ \tilde{u} \cdot \nabla (\nabla \cdot \tilde{u}) \right) - \mu^2 d [\nabla \cdot \frac{\partial \tilde{u}}{\partial t} - d^2 \nabla (\nabla \cdot \frac{\partial \tilde{u}}{\partial t}) + (\nabla \cdot u_\eta)^2
- \tilde{u} \cdot \nabla (\nabla \cdot \tilde{u}) \right) \nabla \eta
\]

(34)

\( \tilde{u} \) is defined in previous section. In contrast to the result by Shen [2000], additional contribution factor here arises from vorticity variation which is significant for surf zone wave. Wei et al [1995] also breaking term for irrotational long wave momentum equation over a variable bottom wave. The intermediate depth velocity \( z_\alpha \) is being used there proportional to \( h \) instead of depth average velocity used here which may not be valid inside the fluid. The use of \( z_\alpha \) in our approach avoids this difficulty. Finally we try to generalize equation by solving vorticity from vorticity transport equation in next section.
5 Vorticity transport equation in breaking zone

Madsen and Svendsen [1983] used a cubic vertical distribution of rotational velocity based on roller jump data which cannot be considered in three dimension case as it is not guaranteed to bring accuracy in the simulation. So we try to solve vorticity function from Reynolds stress based equation.

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla p \tag{35}
\]

Taking the curl on both sides and use vorticity function \( s = \nabla \times u \) we get,

\[
\frac{\partial s}{\partial t} - (s \cdot \nabla)s + (u \cdot \nabla)s = \nu \nabla^2 s \tag{36}
\]

\( (s \cdot \nabla)s \) is "vorticity stretching" factor due to change in gradient in vorticity. This term leads to change of rotation of material particles present in the flow to the beach. Contribution of this term cannot be incorporated from two dimension roller jump data.

We generalize the equation in three dimension as

\[
\frac{\partial s}{\partial t} + \delta u \frac{\partial s}{\partial x} + \delta v \frac{\partial s}{\partial y} + \delta w \frac{\partial s}{\partial z} - \delta s \frac{\partial u}{\partial x} - \delta s \frac{\partial v}{\partial y} - \delta s \frac{\partial w}{\partial z} = \nu [\mu^2 \frac{\partial^2 s}{\partial x^2} + \mu^2 \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}] \tag{37}
\]

After changing the variable from \((x,y,z,t)\) to wave following coordinates \((x,y,\sigma,t)\), we write the vorticity equation as

\[
\frac{\partial s}{\partial t} - \frac{\delta \sigma}{(h + \delta \eta)} \frac{\partial \sigma}{\partial \sigma} \frac{\partial s}{\partial \sigma} + \delta u (\nabla \cdot s) - \delta s (\nabla \cdot u) - \frac{\delta}{(h + \delta \eta)} [s \frac{\partial w}{\partial \sigma} - w \frac{\partial s}{\partial \sigma}] \\
- \delta^2 \frac{\sigma u}{(h + \delta \eta)} (\nabla \cdot \eta) \frac{\partial s}{\partial \sigma} + \delta^2 \frac{\sigma s}{(h + \delta \eta)} (\nabla \cdot \eta) \frac{\partial u}{\partial \sigma} \\
= \nu [\mu^2 \nabla^2 s + \frac{1}{(h + \delta \eta)} \frac{\partial^2 s}{\partial \sigma^2}] + O(\mu^2) + O(h_x) + O(h_y) \tag{38}
\]

The boundary conditions in new coordinate system are,

\[
s(\sigma = 1, t) = s(x,y,t); s(\sigma = 0, t) = 0; s(\sigma, t = 0) = 0 \tag{39a}
\]

After we redefine \( s = \Omega + \sigma \omega_s \), which transforms the equation to which is easier
to solve:
\[
\frac{\partial \Omega}{\partial t} + \sigma \frac{\partial \omega_s}{\partial t} - \delta \frac{\sigma}{(h + \delta \eta)} (\sigma \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial t} \frac{\partial \Omega}{\partial \sigma}) \\
+ \delta u (\nabla \cdot \Omega) - \frac{\delta \sigma}{(h + \delta \eta)} (\nabla \cdot \eta) \frac{\partial \Omega}{\partial \sigma} - \frac{\delta \sigma^2}{(h + \delta \eta)} (\nabla \cdot \eta) \\
+ \frac{\delta u}{(h + \delta \eta)} \frac{\partial \Omega}{\partial \sigma} - \frac{\delta \sigma}{(h + \delta \eta)} \frac{\partial \omega_s}{\partial \sigma} \\
- \delta (\nabla \cdot u) - \delta \sigma \omega_s (\nabla \cdot u) + \frac{\delta^2 \sigma \Omega}{(h + \delta \eta)} (\nabla \cdot \eta) \frac{\partial u}{\partial \sigma} + \frac{\delta^2 \sigma \Omega}{(h + \delta \eta)} (\nabla \cdot \Omega) \frac{\partial u}{\partial \sigma} \\
+ \frac{\delta^2 \sigma \omega_s}{(h + \delta \eta)} (\nabla \cdot \eta) \frac{\partial u}{\partial \sigma}
\] (40)

with new boundary,
\[
\Omega(\sigma = 1, t) = 0, \Omega(\sigma = 0, t) = 0
\] (41)

with initial condition \(\Omega(\sigma, t = 0) = 0\). This additional equation can be solved numerically as done by Briganti et al [2004] for the two dimensional case or an analytical solution can be formulated as shown by Veeramony & Svendsen [2000]. The analytical solution can be calculated by assuming \(\Omega = \omega^{(1)} + \delta \omega^{(2)}\) which gives first and second solution as \(O(1)\) Problem
\[
\frac{\partial \omega^{(1)}}{\partial t} + \sigma \frac{\partial \omega_s}{\partial t} = \frac{\nu}{h^2} \frac{\partial^2 \omega^{(1)}}{\partial t^2}
\] (42)

where the solution is
\[
F_n^{(1)} = (-1)^n \frac{2}{n\pi} \frac{\partial \omega_s}{\partial t}
\] (43)

assuming \(-\sigma \frac{\partial \omega_s}{\partial t} = \sum_{n=1}^{\infty} F_n^{(1)} \sin n\pi \sigma\) And to solve \(\omega^{(1)}\), assume \(\omega^{(1)} = \Sigma_n G_n^{(1)} \sin n\pi \sigma\) which gives zeroth order solution as
\[
G_n^{(1)} = (-1)^n \frac{2}{n\pi} \int_0^t \frac{\partial \omega_s}{\partial \tau} e^{n^2 \pi^2 \kappa (t-\tau)} d\tau
\] (44)

To consider \(O(\delta)\)Problem
\[
\frac{\partial \omega^{(2)}}{\partial \sigma} - \frac{\nu}{h} \frac{\partial^2 \omega^{(2)}}{\partial \sigma^2} = F^{(2)}
\] (45)
where

\[ F^{(2)} = \frac{\sigma}{h} \frac{\partial \eta}{\partial \tau} \frac{\partial \omega^{(1)}}{\partial \sigma} - \frac{\sigma^2}{h} \frac{\partial \eta}{\partial \tau} - \frac{\sigma}{h} (\nabla \cdot \eta) \frac{\partial \omega^{(1)}}{\partial \sigma} - \frac{\sigma^2}{h} (\nabla \cdot \eta) + u(\nabla \cdot \omega^{(1)}) \]  

(46)

To solve above equation, assume \( \omega^{(2)} = \Sigma_n^{(2)} sinn\pi \sigma \) where solution becomes

\[ G_n^{(2)} = 2 \int_0^1 F_n^{(2)} e^{n^2 \pi^2 \kappa (\tau - t)} d\tau \]  

(47)

with

\[ F_n^{(2)} = 2 \int_0^1 F^{(2)} sinn\pi \sigma d\sigma \]  

(48)

The solution for vorticity \( s \) becomes,

\[ s = \sigma \omega_s + \Sigma_1 G_n^{(1)} sinn\pi \sigma + \Sigma_1 G_n^{(2)} sinn\pi \sigma \]  

(49)

To solve breaking term, we need value of \( \omega_s \) for boundary and eddy viscosity value as input data.

6 Conclusion

Finally we conclude here by developing a most generalized form of fully nonlinear Boussinesq equations for wave propagation in surf zone region with variable bathymetry with vorticity distribution from Vorticity Transport Equation (VTE). In this wave breaking zone, vorticity generated by the shear stress of current is very strong, so contribution to the surface velocity due to vorticity variation has significant contribution in fluid flow. These extra terms in generalized equation complicate the numerical technique as these terms are present in the equation in multiple form of equations for vorticity components which has to be solved in coupled solution technique. Veeramony [2000] used simplified the formulation by taking constant eddy viscosity value but this oversimplified case may bring inaccuracy in calculation. Briganti et.al [2004] formulated a numerical technique scheme to solve VTE using generalized depth variable eddy viscosity \( \nu = \nu(x, y) \) in two dimension case. In three dimensional formulation, the nonlinear terms in the vorticity transport equation (VTE) will complicate the calculation and so proper numerical technique have to be developed. This work is under way.
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References

[1] Nwogu, O. (1993), An alternative form of Boussinesq equations for nearshore wave propagation. ASCE J. Waterway Port, Coastal and Ocean Engineering, 119, 618- 638.

[2] Veeramony, J. and I.A. Svendsen (2000), The flow in surf zone waves, Coastal Engineering, 39, 93-122.

[3] Kennedy, A.B., Q. Chen, J.T. Kirby and R.A. Dalrymple (1999), Boussinesq modeling of wave transformation, breaking and run-up I: One dimension, J. Waterway, Port, Coastal and Ocean Engineering, 126, 206-214, 2000.

[4] Shen C. (2000), Constituent Boussinesq Equations for Waves and Currents, J. Physical Oceanography, 31, 850-859.

[5] Zhou,Z., J. Kirby and F. Shi (2004), 2D Higher order Boussinesq Equations for Waves in Flows with Vorticity, Proc 29th Int. Conf Coastal Eng., Lisbon, September; in press.

[6] Mei, C.C. (1983), The Applied Dynamics of Ocean Surface Waves, J. Wiley and sons, 740 pp.

[7] Wei,G., J.T. Kirby, S.T. Grilli and R. Subramanya (1995), A fully nonlinear Boussinesq model for surface waves., Part 1,Highly nonlinear unsteady waves, J. Fluid Mechanics, 294, 71-92.

[8] Madsen,P.A. and I.A. Svendsen (1983), Turbulent bores and hydraulic jump, J.Fluid Mech, 129, 1-25.

[9] Briganti R. and R.E. Musumeci (2004), Boussinesq modeling of breaking waves: Description of turbulence, J. Geophys. Research, 109 , C07015.

[10] Veeramony J., Modeling the flow in surf zone , Ph.D Thesis, Univ. of Delaware, 1999.