An upper bound for the size of s-distance sets in real algebraic sets

Gábor Hegedüs
John von Neumann Faculty of Informatics
Óbuda University
Budapest, Hungary
hegedus.gabor@uni-obuda.hu

Lajos Rónyai*
Institute of Computer Science and Control, Eötvös Loránd Research Network
and Department of Algebra
Budapest University of Technology
Budapest, Hungary
lajos@info.ilab.sztaki.hu

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Abstract

In a recent paper, Petrov and Pohoata developed a new algebraic method which combines the Croot-Lev-Pach Lemma from additive combinatorics and Sylvester’s Law of Inertia for real quadratic forms. As an application, they gave a simple proof of the Bannai-Bannai-Stanton bound on the size of s-distance sets (subsets $A \subseteq \mathbb{R}^n$ which determine at most s different distances). In this paper we extend their work and prove upper bounds for the size of s-distance sets in various real algebraic sets. This way we obtain a novel and short proof for the bound of Delsarte-Goethals-Seidel on spherical s-distance sets and a generalization of a bound by Bannai-Kawasaki-Nitamizu-Sato on s-distance sets on unions of spheres. In our arguments we use the method of Petrov and Pohoata together with some Gröbner basis techniques.

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1 Introduction

Let $\mathcal{A} \subseteq \mathbb{R}^n$ be an arbitrary set. Denote by $d(\mathcal{A})$ the set of non-zero euclidean distances among the points of $\mathcal{A}$:

$$d(\mathcal{A}) := \{d(p_1, p_2); \ p_1, p_2 \in \mathcal{A}, \ p_1 \neq p_2\}.$$  

An $s$-distance set is a subset $\mathcal{A} \subseteq \mathbb{R}^n$ such that $|d(\mathcal{A})| \leq s$. Here we mention just two theorems from the rich area of sets with few distances, more information can be found for example in [14], [3]. Bannai, Bannai and Stanton proved the following upper bound for the size of an $s$-distance set in [4, Theorem 1].

**Theorem 1.** Let $n, s \geq 1$ be integers and suppose that $\mathcal{A} \subseteq \mathbb{R}^n$ is an $s$-distance set. Then

$$|\mathcal{A}| \leq \binom{n + s}{s}.$$ 

Delsarte, Goethals and Seidel investigated $s$-distance sets on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. These are the spherical $s$-distance sets. They proved a general upper bound for the size of a spherical $s$-distance set in [11]. In their proof they used Delsarte’s method (see [3, Subsection 2.2]).

**Theorem 2.** (Delsarte, Goethals, and Seidel) Let $n, s \geq 1$ be integers and suppose that $\mathcal{A} \subseteq S^{n-1}$ is an $s$-distance set. Then

$$|\mathcal{A}| \leq \binom{n + s - 1}{s} + \binom{n + s - 2}{s - 1}.$$ 

Before stating our results, we introduce some notation. Let $\mathbb{F}$ be a field. In the following $S = \mathbb{F}[x_1, \ldots, x_n] = \mathbb{F}[x]$ denotes the ring of polynomials in commuting variables $x_1, \ldots, x_n$ over $\mathbb{F}$. Note that polynomials $f \in S$ can be considered as functions on $\mathbb{F}^n$. For a subset $Y$ of the polynomial ring $S$ and a natural number $s$ we denote by $Y_{\leq s}$ the set of polynomials from $Y$ with degree at most $s$. Let $I$ be an ideal of $S = \mathbb{F}[x]$.

The (affine) Hilbert function of the factor algebra $S/I$ is the sequence of non-negative integers $h_{S/I}(0), h_{S/I}(1), \ldots$, where $h_{S/I}(s)$ is the dimension over $\mathbb{F}$ of the factor space $\mathbb{F}[x_1, \ldots, x_n]_{\leq s}/I_{\leq s}$ (see [8, Section 9.3]). Our main technical result gives an upper bound for the size of an $s$-distance set, which is contained in a given real algebraic set.

**Theorem 3.** Let $I \subseteq \mathbb{R}[x]$ be an ideal in the polynomial ring, and let $\mathcal{A} \subseteq \mathbb{R}^n$ be an $s$-distance set such that every polynomial from $I$ vanishes on $\mathcal{A}$. Then

$$|\mathcal{A}| \leq h_{\mathbb{R}[x]/I}(s).$$ 

The proof is based on Gröbner basis theory and an improved version of the Croot-Pach-Lev Lemma (see [9] Lemma 1) over the reals. Petrov and Pohoata proved this in [20, Theorem 1.2] and used it to give a new proof of Theorem 1. We generalize their result to give a new upper bound for the size of an $s$-distance set, which is contained in a given affine algebraic set in the real affine space $\mathbb{R}^n$.

We give several corollaries, where Theorem 3 is applied to specific ideals of the polynomial ring $\mathbb{R}[x]$, the first ones being the principal ideals $I = (F)$, with $F \in \mathbb{R}[x]$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 28(3) (2021), #P3.27 2
Corollary 4. Let $F \in \mathbb{R}[x]$ be a polynomial of degree $d$. Suppose that $s \geq d$. Let $A$ be an $s$-distance set such that $F$ vanishes on $A$. Then

$$|A| \leq \binom{n + s}{n} - \binom{n + s - d}{n}. \leqno{\text{Corollary 4.}}$$

For example, when $n = 2$, then $F$ defines a plane curve of degree $d$. Then for $s \geq d$ we obtain

$$|A| \leq \frac{(2 + s)}{2} - \frac{(2 + s - d)}{2} = ds - \frac{d(d - 3)}{2}. \leqno{\text{Remark 5.}}$$

In particular, when $F(x, y) = y^2 - f(x)$ gives a Weierstrass equation of an elliptic curve, then $|A| \leq 3s$ for $s \geq 3$.

Remark 5. We can now easily derive Theorem 2 for $s > 1$. Indeed, consider the real polynomial

$$F(x_1, \ldots, x_n) = 1 - \sum_{i=1}^{n} x_i^2 \in \mathbb{R}[x_1, \ldots, x_n]$$

of degree 2 which vanishes on $S^{n-1}$. Corollary 4 and the hockey-stick identity gives

$$|A| \leq \binom{n + s}{n} - \binom{n + s - 2}{n} = \binom{n + s - 1}{s} + \binom{n + s - 2}{s - 1}. \leqno{\text{Corollary 6.}}$$

Next, assume that $V = \bigcup_{i=1}^{p} S_i$, where the $S_i$ are spheres in $\mathbb{R}^n$. E. Bannai, K. Kawasaki, Y. Nitamizu, and T. Sato proved the following result in [5, Theorem 1] for the case when the spheres $S_i$ are concentric. We have a much shorter approach to the same bound, in a more general setting, without the assumption on the centers.

Corollary 6. Let $A$ be an $s$-distance set on the union $V$ of $p$ spheres in $\mathbb{R}^n$. Then

$$|A| \leq \binom{n + s - i - 1}{s - i}. \leqno{\text{Corollary 6.}}$$

Let $T_i \subseteq \mathbb{R}$ be given finite sets, where $|T_i| = q \geq 2$ for each $i$ with $1 \leq i \leq n$. A box is a direct product

$$B := \prod_{i=1}^{n} T_i \subseteq \mathbb{R}^n. \leqno{\text{Corollary 7.}}$$

We can easily apply Theorem 3 to obtain an upper bound for the size of $s$-distance sets in boxes.

Corollary 7. Let $B \subseteq \mathbb{R}^n$ be a box as above, and $A \subseteq B$ an $s$-distance set. Then

$$|A| \leq |\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} : 0 \leq \alpha_i \leq q - 1 \text{ for each } i, \text{ and } \sum_i \alpha_i \leq s\}|. \leqno{\text{Corollary 7.}}$$
Remark 8. In the special case $q = 2$ we have
\[
|\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i \leq 1 \text{ for each } i, \text{ and } \sum_i \alpha_i \leq s\}| = \sum_{j=0}^{s} \binom{n}{j},
\]
hence we obtain the upper bound
\[
|A| \leq \sum_{j=0}^{s} \binom{n}{j}. \tag{1}
\]
In the case when $T_i = T$ for $1 \leq i \leq n$ and $|T| = 2$, the Euclidean distance is essentially the same as the Hamming distance. For this case (1) was proved by Delsarte [10], see also [2, Theorem 1].

Remark 9. The bound is sharp, when $q = 2$, $n = 2m$ and $s = m$. Then the 0,1 vectors of even Hamming weight give an extremal family $A \subseteq \mathbb{R}^n$.

Remark 10. The bound of Corollary 7 can be nicely formulated in terms of extended binomial coefficients (see [12, Example 8] or [7, Exercise 16]):
\[
|A| \leq \sum_{j=0}^{s} \binom{n}{j}_q.
\]
Here $\binom{n}{j}_q$ is an extended binomial coefficient giving the number of restricted compositions of $j$ with $n$ terms (summands), where each term is from the set $\{0,1,\ldots,q-1\}$. In particular, we have $\binom{n}{2}_2 = \binom{n}{2}$.

Remark 11. In [16] a weaker, but similar upper bound was given for the size of $s$-distance sets in boxes:
\[
|A| \leq 2|\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i \leq q - 1 \text{ for each } i, \text{ and } \sum_i \alpha_i \leq s\}|.
\]
The bound appearing in Corollary 7 presents an improvement by a factor of 2.

Let $\alpha_1, \ldots, \alpha_n$ be $n$ different elements of $\mathbb{R}$, and $X_n = X_n(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{R}^n$ be the set of permutations of $\alpha_1, \ldots, \alpha_n$, where each permutation is considered as vector of length $n$. It was proved in [17, Section 2] that for $s \geq 0$
\[
h_{X_n}(s) = \sum_{i=0}^{s} I_n(i),
\]
where $I_n(i)$ is the number of permutations of $n$ symbols with precisely $i$ inversions. Using this, Theorem 3 implies the following bound:

**Corollary 12.** Let $A \subseteq X_n$ be an $s$-distance set. Then
\[
|A| \leq \sum_{i=0}^{s} I_n(i).
\]
In [19, Section 5.1.1] Knuth gives a generating function for $I_n(i)$ and some explicit formulae for the values $I_n(i)$, $i \leq n$.

Let $0 \leq d \leq n$ be integers and $Y_{n,d} \subseteq \mathbb{R}^n$ denote the set of 0,1-vectors of length $n$ which have exactly $d$ coordinate values of 1. The following (sharp) bound was obtained by Ray-Chaudhuri and Wilson in [21, Theorem 3], formulated in terms of intersections rather than distances.

**Corollary 13.** Let $0 \leq d \leq n$ and $s$ be integers, with $0 \leq s \leq \min(d, n - d)$. Suppose that $A \subseteq Y_{n,d}$ is an $s$-distance set. Then

$$|A| \leq \binom{n}{s}.$$ 

In some cases data about the complexification of a real affine algebraic set can be used to give a bound. We give next a statement of this type. For a subset $X \subseteq \mathbb{F}^n$ of the affine space we write $I(X)$ for the ideal of all polynomials $f \in \mathbb{F}[x]$ which vanish on $X$.

**Corollary 14.** Let $V \subseteq \mathbb{C}^n$ be an affine variety such that the projective closure $\overline{V}$ of $V$ has dimension $d$ and degree $k$. Suppose also that the ideal $I(V)$ of $V$ is generated by polynomials over $\mathbb{R}$. Let $A \subseteq V \cap \mathbb{R}^n$ be an $s$-distance set. Then we have

$$|A| \leq k \cdot s^d \frac{d!}{d^d} + O(s^{d-1}).$$

For instance, when in Corollary 14 the projective variety $\overline{V}$ is a curve of degree $k$, then the bound is $ks + b$ for large $s$, where $b$ is an integer. More specifically, when $\overline{V}$ is an elliptic curve such that $V \subseteq \mathbb{C}^2$ is the set of zeroes of $y^2 - f(x)$, where $f(x) \in \mathbb{R}[x]$ is a cubic polynomial without multiple roots, then in fact, the preceding bound becomes $|A| \leq 3s + b$ for $s$ large (see also the remark after Corollary 4).

The rest of the paper is organized as follows. Section 2 contains some preliminaries on Gröbner bases, Hilbert functions, and related notions. Section 3 contains the proofs of the main theorem and the proof of the corollaries.

## 2 Preliminaries

A total ordering $\prec$ on the monomials $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ composed from variables $x_1, x_2, \ldots, x_n$ is a **term order**, if 1 is the minimal element of $\prec$, and $uw \prec vw$ holds for any monomials $u, v, w$ with $u \prec v$. Two important term orders are the lexicographic order $\prec_l$ and the deglex order $\prec_{dl}$. We have

$$x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \prec_l x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$$

iff $i_k < j_k$ holds for the smallest index $k$ such that $i_k \neq j_k$. As for the deglex order, we have $u \prec_{dl} v$ iff either $\deg(u) < \deg(v)$, or $\deg(u) = \deg(v)$, and $u \prec_l v$. 


Let $\prec$ be a fixed term order. The leading monomial $\text{lm}(f)$ of a nonzero polynomial $f$ from the ring $S = \mathbb{F}[x]$ is the largest (with respect to $\prec$) monomial which occurs with nonzero coefficient in the standard form of $f$.

Let $I$ be an ideal of $S$. A finite subset $G \subseteq I$ is a Gröbner basis of $I$ if for every $f \in I$ there exists a $g \in G$ such that $\text{lm}(g)$ divides $\text{lm}(f)$. It can be shown that $G$ is in fact a basis of $I$. A fundamental result is (cf. [6, Chapter 1, Corollary 3.12] or [1, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal $I$ of $S$ has a Gröbner basis with respect to $\prec$.

A monomial $w \in S$ is a standard monomial for $I$ if it is not a leading monomial of any $f \in I$. Let $\text{Sm}(\prec, I, \mathbb{F})$ denote the set of all standard monomials of $I$ with respect to the term-order $\prec$ over $\mathbb{F}$. It is known (see [6, Chapter 1, Section 4]) that for a nonzero ideal $I$ the set $\text{Sm}(\prec, I, \mathbb{F})$ is a basis of the factor space $S/I$ over $\mathbb{F}$. Hence every $g \in S$ can be written uniquely as $g = h + f$ where $f \in I$ and $h$ is a unique $\mathbb{F}$-linear combination of monomials from $\text{Sm}(\prec, I, \mathbb{F})$.

If $X \subseteq \mathbb{F}^n$ is a finite set, then an interpolation argument gives that every function from $X$ to $\mathbb{F}$ is a polynomial function. The latter two facts imply that

$$|\text{Sm}(\prec, I(X), \mathbb{F})| = |X|,$$

where $I(X)$ is the ideal of all polynomials from $S$ which vanish on $X$, and $\prec$ is an arbitrary term order.

The initial ideal $\text{in}(I)$ of $I$ is the ideal in $S$ generated by the set of monomials $\{\text{lm}(f) : f \in I\}$.

It is easy to see [8, Propositions 9.3.3 and 9.3.4] that the value at $s$ of the Hilbert function $h_{S/I}$ is the number of standard monomials of degree at most $s$, where the ordering $\prec$ is deglex:

$$h_{S/I}(s) = |\text{Sm}(\prec_{\text{deglex}}, I, \mathbb{F}) \cap \mathbb{F}[x]_{\leq s}|.$$  \hspace{1cm} (3)

In the case when $I = I(X)$ for some $X \subseteq \mathbb{F}^n$, then $h_X(s) := h_{S/I}(s)$ is the dimension of the space of functions from $X$ to $\mathbb{F}$ which are polynomials of degree at most $s$.

Next we recall a known fact about the Hilbert function. It concerns the change of the coefficient field. Let $\mathbb{F} \subseteq \mathbb{K}$ be fields and let $I \subseteq \mathbb{F}[x]$ be an ideal, and consider the corresponding ideal $J = I \cdot \mathbb{K}[x]$ generated by $I$ in $\mathbb{K}[x]$.

**Lemma 15.** For the respective affine Hilbert functions for $s \geq 0$ we have

$$h_{\mathbb{F}[x]/I}(s) = h_{\mathbb{K}[x]/J}(s).$$

For the convenience of the reader we outline a simple proof.

**Proof.** It follows from Buchberger’s criterion [8, Theorem 2.6.6] that a deglex Gröbner basis of $I$ in $\mathbb{F}[x]$ will be a deglex Gröbner basis of $J$ in $\mathbb{K}[x]$, implying that the initial ideals $\text{in}(I)$ and $\text{in}(J)$ contain exactly the same set of monomials, hence their respective
factors have the same Hilbert function $h_{\mathbb{F}[x]/\text{in}(I)}(s) = h_{\mathbb{K}[x]/\text{in}(J)}(s)$, see [8, Proposition 9.3.3]. Then by [8, Proposition 9.3.4] we have

$$h_{\mathbb{F}[x]/I}(s) = h_{\mathbb{F}[x]/\text{in}(I)}(s) = h_{\mathbb{K}[x]/\text{in}(J)}(s) = h_{\mathbb{K}[x]/J}(s),$$

for every integer $s \geq 0$.

The projective (homogenized) version of the next statement is discussed in [13, Example 6.10].

**Proposition 16.** Let $F \in \mathbb{F}[x]$ be a polynomial of degree $d$. Then for $s \geq d$ we have

$$h_{\mathbb{F}[x]/(F)}(s) = \binom{n+s}{n} - \binom{n+s-d}{n}.$$

If $0 \leq s < d$, then

$$h_{\mathbb{F}[x]/(F)}(s) = \binom{n+s}{n}.$$

**Proof.** By definition

$$h_{\mathbb{F}[x]/(F)}(s) = \dim \mathbb{F}[x]_{\leq s}/(F)_{\leq s} = \dim \mathbb{F}[x]_{\leq s} - \dim (F)_{\leq s}.$$ Clearly

$$\dim \mathbb{F}[x]_{\leq s} = \binom{n+s}{n}.$$

Moreover

$$(F)_{\leq s} = \{ G \in \mathbb{F}[x]_{\leq s} : \text{ there exists an } H \in \mathbb{F}[x] \text{ such that } FH = G \}.$$ Using the fact that $\mathbb{F}[x]$ is a domain, we see that the dimension of the latter subspace is

$$\dim \{ H \in \mathbb{R}[x] : \deg(H) \leq s - d \} = \dim \mathbb{F}[x]_{\leq(s-d)}.$$ The statement now follows from the fact that if $s \geq d$, then

$$\dim \mathbb{F}[x]_{\leq(s-d)} = \binom{n+s-d}{n},$$

while for $s < d$ we have

$$\dim \mathbb{F}[x]_{\leq(s-d)} = 0.$$

\qed
3 Proofs

3.1 Proof of the main result

Petrov and Pohoata proved the following result [20, Theorem 1.2]. They used it to give a short proof of Theorem 1. This improved version of the Croot-Lev-Pach Lemma has a crucial role in the proof of our results.

**Theorem 17.** Let $W$ be an $n$-dimensional vector space over a field $\mathbb{F}$ and let $A \subseteq W$ be a finite set. Let $s \geq 0$ be an integer an let $p(x,y) \in \mathbb{F}[x,y]$ be a $2n$-variate polynomial of degree at most $2s + 1$. Consider the matrix $M(A,p)_{a,b \in A}$, where

$$M(A,p)(a,b) = p(a,b).$$

This matrix corresponds to a bilinear form on $\mathbb{F}^A$ by the formula

$$\Phi_{A,p}(f,g) = \sum_{a,b \in A} p(a,b)f(a)g(b),$$

for each $f,g : A \to \mathbb{F}$. This $\Phi_{A,p}$ defines a quadratic form $\Phi_{A,p}(f,f)$. In the case $\mathbb{F} = \mathbb{R}$ denote by $r_+(A,p)$ and $r_-(A,p)$ the inertia indices of the quadratic form $\Phi_{A,p}(f,f)$. Then

(i) $\text{rank}(M(A,p)) \leq 2h_A(s)$,

(ii) if $\mathbb{F} = \mathbb{R}$, then $\max(r_+(A,p), r_-(A,p)) \leq h_A(s)$.

By combining Theorem 17 with facts about standard monomials, we have the following simple and elegant upper bound for the degree of deglex standard monomials of an $s$-distance set.

**Theorem 18.** Let $A \subseteq \mathbb{R}^n$ be an $s$-distance set. Then

$$\text{Sm}(\langle_{\text{deg}}, I(A), \mathbb{F} \rangle) \subseteq \mathbb{R}[x]_{\leq s}.$$ 

**Proof.** We follow the argument of [20, Theorem 1.1]. Let $A \subseteq \mathbb{R}^n$ denote an $s$-distance set. Recall that $d(A)$ denotes the set of (non-zero) distances among points of $A$. Define the $2n$–variate polynomial by:

$$p(x,y) = \prod_{t \in d(A)} \left(t^2 - \|x - y\|^2\right) \in \mathbb{R}[x,y].$$

Then we can apply Theorem 17 for $p(x,y)$ whose degree is $2s$. The matrix $M(A,p)$ is a positive diagonal matrix, giving that

$$r_+(A,p) = |A|.$$

It follows from Theorem 17 (ii) that

$$|A| = r_+(A,p) \leq h_A(s).$$
But equations (3), (2) and the finiteness of \( A \) imply that
\[
|A| \leq h_{\mathcal{A}}(s) = |\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R}) \cap \mathbb{R}[x]_{\leq s}| \leq |\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R})| = |A|.
\]
We infer that
\[
|\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R}) \cap \mathbb{R}[x]_{\leq s}| = |\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R})|,
\]
and hence
\[
\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R}) \subseteq \mathbb{R}[x]_{\leq s}.
\]

**Proof of Theorem 3.** Theorem 18 gives that
\[
\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R}) \subseteq \mathbb{R}[x]_{\leq s}.
\]
Since \( I \) vanishes on \( A \), we have \( I \subseteq I(\mathcal{A}) \), hence
\[
\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R}) \subseteq \text{Sm}(\prec_{dt}, I, \mathbb{R}).
\]
The preceding two relations imply that
\[
\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R}) \subseteq \text{Sm}(\prec_{dt}, I, \mathbb{R}) \cap \mathbb{R}[x]_{\leq s}.
\]
Now it follows from (3) and (2) that
\[
|A| = |\text{Sm}(\prec_{dt}, I(\mathcal{A}), \mathbb{R})| \leq |\text{Sm}(\prec_{dt}, I, \mathbb{R}) \cap \mathbb{R}[x]_{\leq s}| = h_{\mathbb{R}[x]/I}(s).
\]

### 3.2 Proofs of the Corollaries

**Proof of Corollary 4.** From Theorem 3 we obtain the bound \( |A| \leq h_{\mathbb{R}[x]/(F)}(s) \), therefore for \( s \geq d \) we have
\[
|A| \leq h_{\mathbb{R}[x]/(F)}(s) = \binom{n+s}{n} - \binom{n+s-d}{n},
\]
by Proposition 16.

**Proof of Corollary 6.** It is easy to verify that
\[
\sum_{i=0}^{2p-1} \binom{n+s-i-1}{s-i} = \binom{n+s}{s} - \binom{n+s-2p}{n}.
\]
Let \( V = \bigcup_{i=1}^{p} S_i \), and assume, that the center of the sphere \( S_i \) is the point \((a_{1,i}, \ldots, a_{n,i}) \in \mathbb{R}^n\), and the radius of \( S_i \) is \( r_i \in \mathbb{R} \) for \( i = 1, \ldots, p \). Next consider the polynomials
\[
F_i(x_1, \ldots, x_n) = \left( \sum_{m=1}^{n} (x_m - a_{m,i})^2 \right) - r_i^2 \in \mathbb{R}[x_1, \ldots, x_n]
\]
for each \( i \) and put \( F := \prod_i F_i \). Then \( \deg(F) = 2p \) and \( F \) vanishes on \( V \). We may apply Corollary 4 for the polynomial \( F \). Then for \( s \geq 2p \) we obtain the desired bound
\[
|A| \leq \binom{n+s}{n} - \binom{n+s-2p}{n}.
\]
When \( s < 2p \), the bound follows from the Bannai-Bannai-Stanton theorem.
Proof of Corollary 7. It is well-known and easily proved that the following set of polynomials is a (reduced) Gröbner basis of the ideal $I(\mathcal{B})$ (with respect to any term order):

$$\left\{ \prod_{t \in T_i} (x_i - t) \mid 1 \leq i \leq n \right\}.$$  

This readily gives the (deglex) standard monomials for $I(\mathcal{B})$:

$$\text{Sm}(\prec_{\text{dl}}, I(\mathcal{B}), \mathbb{R}) = \{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i \leq q - 1 \text{ for each } i \}.$$  

It follows from Theorem 3 and equation (3) that

$$|\mathcal{A}| \leq h_{\mathcal{B}}(s) = |\text{Sm}(\prec_{\text{dl}}, I(\mathcal{B}), \mathbb{R}) \cap \mathbb{R}[x]_{\leq s}| = |\{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i \leq q - 1 \text{ for each } i, \text{ and } \sum_i \alpha_i \leq s \}|. \quad \Box$$

Proof of Corollary 13. The statement follows at once from the result

$$h_{V_{n,d}}(s) = \binom{n}{s}$$

proved by Wilson in [22] (formulated there in the language of inclusion matrices, see also [18, Corollary 3.1]), and Theorem 3. \quad \Box

Proof of Corollary 14. Write $I = I(V) \cap \mathbb{R}[x]$ and $J = I(V) \subseteq \mathbb{C}[x]$. It follows from Theorem 3 and Proposition 15 that

$$|\mathcal{A}| \leq h_{\mathbb{R}[x]/I}(s) = h_{\mathbb{C}[x]/J}(s).$$

From [8, Theorem 9.3.12] we obtain that the affine Hilbert function $h_{\mathbb{C}[x]/J}(s)$ is the same as the projective Hilbert function $h_{\mathbb{P}^d}(s)$ of the projective variety $\overline{V}$. Now [15, Proposition 13.2] and the subsequent remark imply that for $s$ large the Hilbert function will be the same as the Hilbert polynomial: $h_{\mathbb{P}^d}(s) = p_{\mathbb{P}^d}(s)$, moreover

$$p_{\mathbb{P}^d}(s) = \frac{k}{d!} \cdot s^d + \text{ terms of degree at most } d - 1 \text{ in } s.$$  

This proves the statement. \quad \Box

References

[1] W. W. Adams, and P. Loustaunau. *An Introduction to Gröbner bases.* AMS, Providence, 1994.

[2] L. Babai, H. Snevily, and R. M. Wilson. A new proof of several inequalities on codes and sets. *Journal of Combinatorial Theory, Series A*, 71(1), 146-153 (1995).
[3] E. Bannai, and E. Bannai. A survey on spherical designs and algebraic combinatorics on spheres. *European Journal of Combinatorics*, **30**, 1392-1425 (2009).

[4] E. Bannai, E. Bannai, and D. Stanton. An upper bound for the cardinality of an $s$-distance subset in real Euclidean space II. *Combinatorica*, **3(2)**, 147-152 (1983).

[5] E. Bannai, K. Kawasaki, Y. Nitamizu, and T. Sato. An upper bound for the cardinality of an $s$-distance set in Euclidean space. *Combinatorica*, **23(4)**, 535-557 (2003).

[6] A. M. Cohen, H. Cuypers, and H. Sterk (eds.). *Some tapas of computer algebra*. Springer-Verlag, Berlin, Heidelberg, 1999.

[7] L. Comtet. *Advanced Combinatorics*. D. Reidel Publishing Company, Dordrecht, 1974.

[8] D. Cox, J. Little, and D. O’Shea. *Ideals, varieties, and algorithms*. Springer-Verlag, Berlin, Heidelberg, 1992.

[9] E. Croot, V. Lev, and P. Pach. Progression-free sets in $\mathbb{Z}^n_4$. *Annals of Mathematics*, **185**, 331-337 (2017).

[10] P. Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Research Reports Supplements*, **10**, 1-97 (1973).

[11] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. *Geometriae Dedicata*, **6(3)**, 363-388 (1977).

[12] S. Egger. Restricted weighted integer compositions and extended binomial coefficients. *Journal of Integer Sequences*, **16**, Article 13.1.3 (2013).

[13] D. Eisenbud, and J. Harris. *3264 and all that: A second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[14] A. Glazyrin, and W. H. Yu. Upper bounds for $s$-distance sets and equiangular lines. *Advances in Mathematics*, **330**, 810-833 (2018).

[15] J. Harris. *Algebraic geometry: a first course*. Springer Science and Business Media, New York, 2013.

[16] G. Hegedűs. A new upper bound for the size of $s$-distance sets in boxes. *Acta Mathematica Hungarica*, **160**, 168–174 (2020).

[17] G. Hegedűs, A. Nagy and L. Rónyai. Gröbner bases for permutations and oriented trees. *Annales Univ. Sci. Budapest., Sectio Computatorica*, **23**, 137-148 (2004).

[18] G. Hegedűs, and L. Rónyai. Gröbner bases for complete uniform families. *Journal of Algebraic Combinatorics*, **17(2)**, 171-180 (2003).

[19] D. E. Knuth. *The art of computer programming, Volume 3, Sorting and searching*. Second. ed., Addison-Wesley, Upper Saddle River, 1998.

[20] F. Petrov, and C. Pohoata. A remark on sets with few distances in $\mathbb{R}^d$. *Proceedings of the American Mathematical Society*, **149(2)**, 569-571 (2021).

[21] D. K. Ray-Chaudhuri, and R. M. Wilson. On $t$-designs. *Osaka Journal of Mathematics*, **12**, 737-744 (1975).

[22] R. M. Wilson. A diagonal form for the incidence matrices of $t$-subsets vs. $k$-subsets. *European Journal of Combinatorics*, **11**, 609–615 (1990).