INTERSECTION COHOMOLOGY OF $S^1$-ACTIONS ON PSEUDOMANIFOLDS

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To my wife with love.

Abstract. For any smooth free action of the unit circle $S^1$ in a manifold $M$; the Gysin sequence of $M$ is a long exact sequence relating the DeRham cohomologies of $M$ and its orbit space $M/S^1$. If the action is not free then $M/S^1$ is not a manifold but a stratified pseudomanifold and there is a Gysin sequence relating the DeRham cohomology of $M$ with the intersection cohomology of $M/S^1$. In this work we extend the above statements for any stratified pseudomanifold $X$ of length 1, whenever the action of $S^1$ preserves the local structure. We give a Gysin sequence relating the intersection cohomologies of $X$ and $X/S^1$ with a third term $G$, the Gysin term; whose cohomology depends on basic cohomological data of two flavors: global data concerns the Euler class induced by the action, local data relates the Gysin term and the cohomology of the fixed strata with values on a locally trivial presheaf.

Foreword

For an entire version of this article the reader should go better to Indag. Math. Vol. 15 (3), 383–412.

A pseudomanifold is a topological space $X$ with two features. First, there is a closed $\Sigma \subset X$ called the singular part, which is the disjoint union of smooth manifolds. The $X - \Sigma$ is a dense smooth manifold. We call strata the connected components of $\Sigma$ and $X - \Sigma$; they constitute a locally finite partition of $X$. The second feature is the local conical behavior of $X$, the model being a product $U \times c(L)$ of a smooth manifold $U$ with the open cone of a compact smooth manifold $L$ called the link of $U$. A careful reader will notice that stratified pseudomanifolds with arbitrary length have a richer and more complicated topological structure; in this article we deal with stratified pseudomanifolds of length $\leq 1$, which we call just pseudomanifolds.

Between the various ways for defining the intersection (co)homology; the reader can see [6], [8] for a definition in pl-stratified pseudomanifolds; [11], [17], [13] for a definition with sheaves; [15] for an approach with $L^2$-cohomology; [4] for an exposition in Thom-Mather spaces.

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In this article, we use the DeRham-like definition exposed in [22] where the reader will find a beautiful proof of the DeRham theorem for stratified spaces. We work with differential forms in $X - \Sigma$ and measure their behavior when approaching to $\Sigma$, through an auxiliary construction called an unfolding of $X$. Although $X$ may have many different unfoldings, its intersection cohomology does not depend on any particular choice. This point of view is the dual of the intersection homology defined by King [12], who works with a broader family of perversities. When $S^1$ acts on $X$ preserving the local structure then the orbit space $X/S^1$ is again a pseudomanifold with an unfolding.

The well known Gysin sequence of a smooth manifold $M$ with a principal action of $S^1$ is the long exact sequence

$$\cdots \to H^i(M) \xrightarrow{\bar{f}} H^{i-1}(M/S^1) \xrightarrow{\varepsilon} H^{i+1}(M/S^1) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots$$

where $\pi^*$ is induced by the orbit map $\pi : M \to M/S^1$, which is a smooth $S^1$-principal bundle. The map $\bar{f}$ is induced by the integration along the fibers and the connecting homomorphism $\varepsilon$ is the multiplication by the Euler class $\varepsilon \in H^2(M/S^1)$.

When the action of $S^1$ on $M$ is not free then the base space is not anymore a smooth manifold, but a stratified pseudomanifold $M/S^1$ whose length depends on the number of orbit types. There is a Gysin sequence of $M$ relating the DeRham cohomology of $M$ with the intersection cohomology of $M/S^1$

$$\cdots \to H^i(M) \xrightarrow{\bar{f}} H^{i-1}_{\bar{q}, \bar{z}}(M/S^1) \xrightarrow{\varepsilon} H^{i+1}_{\bar{q}}(M/S^1) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots$$

where $\bar{q}$, $\bar{z}$ are perversities in $M/S^1$. The connecting homomorphism is again the multiplication by the Euler class $\varepsilon \in H^2(M/S^1)$. The fixed points’ subspace $M^{S^1}$ is naturally contained in $M/S^1$. The link of a fixed stratum $S \subset M/S^1$ is always a cohomological complex projective space $[10], [13]$.

In this article we extend the above situation for any pseudomanifold $X$ and any action of $S^1$ on $X$ preserving the local structure. The orbit map $\pi : X \to X/S^1$ induces a long exact sequence

$$\cdots \to H^i_{\bar{\chi}}(X) \to H^i(\mathcal{G}_{\bar{\chi}}(X)/S^1)) \xrightarrow{\partial} H^{i+1}_{\bar{\chi}}(X/S^1) \xrightarrow{\Sigma} H^{i+1}_{\bar{\chi}}(X) \to \cdots$$

relating the intersection cohomologies of $X$ and $X/S^1$ with a third term $H^*(\mathcal{G}_{\bar{\chi}}(X)/S^1))$ whose cohomology can be given in terms of local and global basic cohomological data; we call it the Gysin term. The above long exact sequence is the Gysin sequence.

Global data concerns the Euler class $\varepsilon \in H^2_\bar{\chi}(X/S^1)$. For instance, if $\varepsilon = 0$ then $H^*(\mathcal{G}_{\bar{\chi}}(X)/S^1)) = H^*_{\bar{\chi}, \bar{\tau}}(X/S^1)$ where $\bar{\tau}$ is the perversity defined by

$$\bar{\tau}(S) = \begin{cases} 1 & S \text{ a fixed stratum} \\ 0 & \text{else} \end{cases}$$
The connecting homomorphism $\partial$ of the Gysin sequence depends on the Euler class, though it’s not the multiplication. The Euler class vanishes if and only if there is a foliation on $X - \Sigma$ transverse to the orbits of the action $[18, 23]$.

Local data relates the Gysin term with the fixed strata. In general, there is a second long exact sequence

$$
\cdots \to H^i_{\tau - \chi}(X/\mathbb{S}^1) \to H^i(\mathcal{U}pp_{\tau}(X/\mathbb{S}^1)) \xrightarrow{\partial^i} H^{i+1}(G^\tau_{\tau}(X/\mathbb{S}^1)) \xrightarrow{\iota^i} H^{i+1}_{\tau - \chi}(X/\mathbb{S}^1) \to \cdots
$$

the residual term satisfying

$$
H^*(\mathcal{U}pp_{\tau}(X/\mathbb{S}^1)) = \prod_S H^*(S, \text{Im}(\varepsilon_L))
$$

where $S$ runs over the fixed strata and $H^*(S, \text{Im}(\varepsilon))$ is the cohomology of $S$ with values on a locally trivial constructible presheaf $[8, \text{Im}(\varepsilon_L)]$ with stalk

$$
\mathcal{F} = \text{Im}\{\varepsilon_L : H^{(S)-1}(L/\mathbb{S}^1) \to H^{(S)+1}(L/\mathbb{S}^1)\}
$$

the image of the multiplication by the Euler class $\varepsilon_L \in H^2(L/\mathbb{S}^1)$ of the action on the Link $L$ of $S$. Since $L$ may not be a sphere, this term could not vanish.

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