A REAL VIEWPOINT ON THE INTERSECTION OF COMPLEX QUADRICS AND ITS TOPOLOGY

A. LERARIO

Abstract. We study the relation between a complex projective set $C \subset \mathbb{C}P^n$ and the set $R \subset \mathbb{R}P^{2n+1}$ defined by viewing each equation of $C$ as a pair of real equations. Once $C$ is presented by quadratic equations, we can apply a spectral sequence to efficiently compute the homology of $R$; using the fact that the $\mathbb{Z}_2$-cohomology of $R$ is a free $H^*(C)$-module with two generators we can in principle reconstruct the homology of $C$. Explicit computations for the intersection of two complex quadrics are presented.

1. Introduction

Given a projective algebraic set $C \subset \mathbb{C}P^n$ we are interested in the computation of its $\mathbb{Z}_2$-Betti numbers. The approach we propose is that of studying first the topology of the set $R \subset \mathbb{R}P^{2n+1}$, defined by viewing each equation of $C$ as a pair of real equations, and then recover the Betti numbers of $C$ from those of $R$ using the formula $b_j(C) = \sum_{k=0}^j (-1)^k b_{j-k}(R)$. In the case $C$ is cut by quadrics, the same is true for $R$ and the homology of the former can be computed using the spectral sequence discussed in [1]. If $C$ is the intersection of two quadrics the computations are quite easy and we devote the last section to perform some of them.

In the case $C$ is a complete intersection of two quadrics, its geometric properties are studied in [4]. The classification of pencil of complex quadrics is a well known fact and involves rather complicated algebraic stuff; the reader can see [3] for a classical algebraic treatment or [2] for a more geometric approach. The reduction to a canonical form for a pair of quadrics is studied in [5].

Our interest is only in the rough topology of $C$ and thus our approach is far from the previous analytic ones, but since we do not make regularity assumptions it includes the treatment of very degenerate objects.

2. Remarks on real and complex projective sets

We start by considering the bundle

$$ S^1 \to \mathbb{R}P^{2n+1} \to \mathbb{C}P^n $$

where the map $\pi$ is given by $[x_0, y_0, \ldots, x_n, y_n] \mapsto [x_0 + iy_0, \ldots, x_n + iy_n]$. The fiber of $\pi$ over a point $[v] \in \mathbb{C}P^n$ equals the projectivization of the two dimensional real vector space $\text{span}_\mathbb{R}\{v, iv\} \subset \mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$. Thus $\mathbb{R}P^{2n+1}$ is the total space of the projectivization of the tautological bundle $O(-1) \to \mathbb{C}P^n$ view as a rank two real vector bundle. Applying Leray-Hirsch we get a cohomology class $x \in H^1(\mathbb{R}P^{2n+1}; \mathbb{Z}_2)$, which restricts to a generator of the cohomology of each fiber, such...
that the map \( \alpha \otimes p(x) \mapsto \pi^* \alpha \otimes p(x) \), where \( p \in \mathbb{Z}_2[x]/(x^2) \) and \( \alpha \in H^*(\mathbb{C}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1}) \), gives an isomorphism of \( H^*(\mathbb{C}P^n; \mathbb{Z}_2) \)-modules
\[
H^*(\mathbb{C}P^n; \mathbb{Z}_2) \otimes \{1, x\} \simeq H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}_2).
\]

In particular this tells that \( \pi^* \) is injective with image the even dimensional cohomology (recall that \( |\alpha| = 2 \)).

The following geometric description of the map \( \pi \) also gives an alternative proof of the previous statement. Consider the restriction of \( \pi \) to \( \{[x_0, y_0, x_1, 0, \ldots, 0] \} \cong \mathbb{R}P^2 \) : we see that it maps \( \mathbb{R}P^2 \) to \( \{[z_0, z_1, 0, \ldots, 0] \} \cong \mathbb{C}P^1 \) trough a homeomorphism \( \{x_1 \neq 0\} \cong \{z_1 \neq 0\} \) and by collapsing the line at infinity \( \{x_1 = 0\} \) to the point \( [1, 0, \ldots, 0] \). It follows that the modulo 2 degree of \( \pi|_{\mathbb{R}P^2} \) is one. Using the isomorphism \( H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^{2n+2}), |\beta| = 1 \), we see that
\[
\pi^*: H^*(\mathbb{C}P^n; \mathbb{Z}_2) \to H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}_2)
\]
is given by \( \alpha \mapsto \beta^2 \), where \( \beta|_{\mathbb{R}P^2} \) generates \( H^1(\mathbb{R}P^2; \mathbb{Z}_2) \). If we consider the Gysin sequence with \( \mathbb{Z}_2 \) coefficients for \( \pi \), then the injectivity of \( \pi^* \) implies that for every \( j \) the following portion of the sequence is exact
\[
0 \to H^j(\mathbb{C}P^n; \mathbb{Z}_2) \xrightarrow{\pi^*} H^j(\mathbb{R}P^{2n+1}; \mathbb{Z}_2) \to H^{j-1}(\mathbb{C}P^n; \mathbb{Z}_2) \xrightarrow{e} 0
\]
where \( e = e(\pi) \) is the modulo 2 euler class of \( \pi \), which of course turns out to be zero.

Let now \( I \subset \mathbb{C}[z_0, \ldots, z_n] \) be a homogeneous ideal; we will denote by \( C = C(I) \) its zero locus in \( \mathbb{C}P^n \). If we restrict the bundle \( O(-1) \to \mathbb{C}P^n \) to \( C \) we get a bundle:

\[
\begin{array}{ccc}
E & \to & O(-1) \\
\downarrow & & \downarrow \\
C & \to & \mathbb{C}P^n
\end{array}
\]

and if we consider the previous as rank two real vector bundles and take their projectivization we get:

\[
\begin{array}{ccc}
\mathbb{R}P^1 & \to & R \\
\pi|_R & \downarrow & \pi \\
C & \to & \mathbb{C}P^n
\end{array}
\]

where \( i_R \) and \( i_C \) are the inclusion maps.

It is clear that \( R \) is an algebraic subset of \( \mathbb{R}P^{2n+1} \) whose equations are given by considering each polynomial \( f \in I \) as a pair of polynomials \( f^u = \text{Re}(f), f^v = \text{Im}(f) \in \mathbb{R}[x_0, y_0, \ldots, x_n, y_n] \). Applying Leray-Hirsch to \( \pi|_R \), or the the identity \( e(\pi|_R) = e(\pi)|_C = 0 \), we get the isomorphism of \( H^*(C; \mathbb{Z}_2) \)-modules:
\[
H^*(C; \mathbb{Z}_2) \otimes \{1, x|_R\} \simeq H^*(R, \mathbb{Z}_2).
\]

The previous isomorphism allows us to compute \( \mathbb{Z}_2 \)-Betti numbers of \( C \) once those of \( R \) are known, via the following formula:
\[
b_j(C; \mathbb{Z}_2) = \sum_{k=0}^j (-1)^k b_{j-k}(R; \mathbb{Z}_2).
\]
We have the following equalities for the Stiefel-Whitney classes of $E$, which come from the fact that $E$ is the realification of a complex bundle: $w_{2k}(E) = c_k(E) \mod 2$, where $c_k$ is the $k$-th Chern class of $E$ seen as a complex bundle, and $w_{2k+1}(E) = 0$. Since $E$ has real rank two we have:

$$w_2(E) = i_c^*z \quad \text{and} \quad w_1(E) = 0, \quad i \neq 0, 2,$$

where $z$ is the generator of $H^2(\mathbb{CP}^n, \mathbb{Z}_2)$ and we have used the equalities $w_2(E) = c_1(E) = i_c^*c_1(O(-1)) = i_c^*z$.

The following lemma relates the homomorphisms $i_c^*$ and $i_R^*$.

**Lemma 1.** There exists an odd $r$ such that $(i_R^*)_k : H^k(\mathbb{RP}^{2n+1}; \mathbb{Z}_2) \to H^k(R; \mathbb{Z}_2)$ is injective for $k \leq r$ and zero for $k > r$. Moreover for every $k$ we have

$$rk(i_R^*)_{2k} = rk(i_R^*)_{2k+1}.$$

**Proof.** Let $a$ be such that $(i_R^*)_a \equiv 0$; then using the cup product structure of $H^*(\mathbb{RP}^{2n+1}; \mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^{2n+2})$, we have

$$i_R^*\beta^{a+k} = i_R^*\beta^a \cdot i_R^*\beta^k = 0.$$

For the second part of the statement notice that $R = P(E) \xrightarrow{i_n} \mathbb{RP}^{2n+1}$ is linear on the fibres and thus, letting $y = i_f^*\beta$ we have $y^2 = (w_2(E) + w_1(E)y) = w_2(E)y$ (since $w_1(E)$ is zero), where we interpret $w_i(E)$ as a class on $R$ via $i_n^*$. It follows that

$$y^{2k} = w_2(E)^k \quad \text{and} \quad y^{2k+1} = w_2(E)^k y.$$

On the other hand, since $w_2(E) = i_c^*z$, then the conclusion follows. \qed

### 3. THE QUADRATIC CASE

In this section we study the topology of $R$ in the case $C$ is cut by quadrics, i.e.

$$C = V_{\mathbb{CP}^n}(q_0, \ldots, q_l), \quad q_0, \ldots, q_l \in \mathbb{C}[z_0, \ldots, z_n](2)$$

For a given $q \in \mathbb{C}[z_0, \ldots, z_n](2)$, $q(z) = z^TQz$ with $Q = A - IB$ and $A, B \in \text{Sym}(n+1, \mathbb{R})$ we define the symmetric matrix

$$P = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}.$$

We set $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (it is a $(n+2) \times (n+2)$ matrix) and given $q_0, \ldots, q_l \in \mathbb{C}[z_0, \ldots, z_n](2)$ we define $p : S^{2l+1} \to \text{Sym}(2n+2, \mathbb{R})$ by

$$(a_0, b_0, \ldots, a_l, b_l) \xrightarrow{p} a_0P_0 - b_0JP_0 + \cdots + a_lP_l - b_lJP_l.$$

For every polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ recall that we have defined the polynomials $f^a, f^b \in \mathbb{R}[x_0, y_0, \ldots, x_n, y_n]$ by

$$f^a(x, y) = \text{Re}(f)(x + iy), \quad f^b(x, y) = \text{Im}(f)(x + iy).$$

Thus if $C = V_{\mathbb{CP}^n}(q_0, \ldots, q_l)$, we have

$$R = V_{\mathbb{RP}^{2l+1}}(q_0^b, q_0^a, \ldots, q_l^b, q_l^a).$$

We easily see that $i^+(a_0q_0^a + b_0q_0^b + \cdots + a_lq_l^a + b_lq_l^b) = i^+(p(a_0, b_0, \ldots, b_l, q_l))$; this is simply because $P_j$ and $-JP_j$ are the symmetric matrices associated respectively to the quadratic forms $q^a_j$ and $q^b_j$.

Following [1] for every $j \in \mathbb{N}$ we define

$$\Omega_j = \{ \alpha \in S^{2l+1} | i^+(p(\alpha)) \geq j \}$$
and if we let $B$ be the unit ball in $\mathbb{R}^{2l+2}$, $\partial B = S^{2l+1}$ we recall the existence of a first quadrant spectral sequence $(E_r, d_r)_{r \geq 0}$ such that:

$$(E_r, d_r) \Rightarrow H_{2n+1-\ast}(R; \mathbb{Z}_2), \quad E_2^{ij} = H^i(B, \Omega^{j+1}; \mathbb{Z}_2).$$

For $j \in \mathbb{N}$ if we let $P^j \subset S^{2l+1} \subset \mathbb{C}^{l+1}$ be defined by

$$P^j = \{(a_0, \ldots, a_l) \in S^{2l+1} \mid \text{rk}_\mathbb{C}(a_0q_0 + \ldots + a_lq_l) \geq j\}$$

we can rewrite Theorem A of [1] in the following more natural way.

**Theorem 2.** There exists a cohomology spectral sequence of the first quadrant $(E_r, d_r)$, converging to $H_{2n+1-\ast}(R; \mathbb{Z}_2)$ such that $E_2^{ij} = H^i(B, P^{j+1}; \mathbb{Z}_2)$.

**Proof.** We will prove that for every $j$ the two sets $P^{j+1}$ and $\Omega^{j+1}$ are homeomorphic, and in fact if $\tau : \mathbb{C}^{l+1} \to \mathbb{C}^{l+1}$ denotes complex conjugation that we have

$$\tau(P^{j+1}) = \Omega^{j+1}.$$

If we use the matrix notation for each $q_j$ we have $q_j(z) = z^TQ_jz$ for $Q_j \in \text{Sym}(n+1, \mathbb{C})$ and writing $Q_j = A_j - iB_j$ with $A_j, B_j \in \text{Sym}(n+1, \mathbb{R})$

$$q_j^a(x, y) = \langle (\bar{z}, (A_j B_j - A_j B_j)(\bar{z})) \rangle, \quad q_j^b(x, y) = \langle (\bar{z}, (A_j - A_j)(\bar{z})) \rangle.$$

In particular notice that the matrix associated to the real quadratic form $a_0q_0^a + b_0q_0^b + \cdots + a_lq_l^a + b_lq_l^b$ is of the form

$$M = \left( \begin{array}{cc} A & B \\ B & -A \end{array} \right)$$

for $A, B \in \text{Sym}(n+1, \mathbb{R})$. If $\lambda$ is an eigenvalue of $M$ and $V_\lambda$ is the corresponding eigenspace, then the map $(u, v) \mapsto (-v, u)$ gives an isomorphism $V_\lambda \simeq V_{-\lambda}$. This implies

$$2i^+(M) = \text{rk}_\mathbb{R}(M).$$

On the other side it is easy to show that

$$\text{rk}_\mathbb{R} \left( \begin{array}{cc} A & B \\ B & -A \end{array} \right) = 2\text{rk}_\mathbb{C}(A - iB)$$

in fact the map $(u, v) \mapsto u + iv$ gives an isomorphism of real vector space $\ker(M) \simeq \ker(A - iB)$. Comparing now the matrices associated to $a_0q_0^a + b_0q_0^b + \cdots + a_lq_l^a + b_lq_l^b$ and to $(a_0 - ib_0)q_0 + \ldots + (a_l - ib_l)q_l$ we get the result.

□

**Remark 1.** Even more natural than the sets $\{P^j\}_{j \in \mathbb{N}}$ are the sets

$$Y^j = \{[\alpha] \in \mathbb{CP}^l, \alpha \in S^{2l+1} \mid \text{rk}(p(\alpha)) \geq j\}.$$ 

If we consider the hopf bundle $S^1 \to S^{2l+1} \xrightarrow{h} \mathbb{CP}^l$ we see that $h(P^j) = Y^j$ and thus

$$P^j \neq S^{2l+1} \Rightarrow H^*(P^j) = H^*(Y^j) \otimes H^*(S^1).$$

In this way we see that it is possible to express all the data for $E_2$ of the previous spectral sequence only in terms of the linear system $P(\text{span}(q_0, \ldots, q_l)) \subset P(\mathbb{C}[z_0, \ldots, z_n]_{(2)}).$

We recall now from [1] the following description for the second differential of $(E_r, d_r)_{r \geq 0}$

For each $P \in \text{Sym}(2n + 2, \mathbb{R})$ we order the eigenvalues of $P$ in increasing way:

$$\lambda_1(P) \geq \cdots \geq \lambda_{2n+2}(P).$$
and we define

\[ D_j = \{ \alpha \in S^{2l+1} \mid \lambda_j(p(\alpha)) \neq \lambda_{j+1}(p(\alpha)) \}. \]

Then there is a naturally defined bundle \( \mathbb{R}^j \to L_j \to D_j \) whose fiber over a point \( \alpha \in D_j \) equals \( (L_j)_{\alpha} = \text{span}\{v \in \mathbb{R}^{2n+2} \mid p(\alpha)v = \lambda_i v, i = 1, \ldots, j\} \) and whose vector bundle structure is given by the inclusion \( L_j \hookrightarrow D_j \times \mathbb{R}^{2n+2} \). We define \( w_{1,j} \in H^1(D_j) \) to be the first Stiefel-Whitney class of \( L_j \) and

\[ \gamma_{1,j} = \partial^* w_{1,j} \in \mathbb{H}^2(B, D_j) \]

where \( \partial^* : H^1(D_j) \to \mathbb{H}^2(B, D_j) \) is the connecting isomorphism. In [1] it is proved that \( d_2 : \mathbb{H}^1(B, \Omega^{j+1}) \to \mathbb{H}^{j+2}(B, \Omega^j) \) is given by

\[ d_2(x) = (x - \gamma_{1,j})(B, \Omega^j) \]

(notice that \( \Omega^j \subset \Omega^{j+1} \cup D_j \)).

If we let \( |a_0, \ldots, a_l| = |a_0 + ib_0, \ldots, a_l + ib_l| \in \mathbb{C}P^l \) such that \( \alpha = (a_0, b_0, \ldots, a_l, b_l) \in S^{2l+1} \) then \( p|_{H^{-1}(a_0, \ldots, a_l)} : S^1 \to \text{Sym}(2n + 2, \mathbb{R}) \) equals

\[ \theta \mapsto \left( \begin{smallmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{smallmatrix} \right) p(\alpha) \]

as one can easily check. The following lemma is the main ingredient for the explicit computations of \( d_2 \).

**Lemma 3.** Let \( A, B, I \in \text{Sym}(n + 1, \mathbb{R}) \), with \( I \) the identity matrix, \( R(\theta) = \left( \begin{smallmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{smallmatrix} \right) \) and \( M = \left( \begin{smallmatrix} A & B \\ -B & -A \end{smallmatrix} \right) \). Let \( c : S^1 \to \text{Sym}(2n + 2, \mathbb{R}) \) be defined by

\[ \theta \mapsto R(\theta)M. \]

Consider the bundle \( c^*L \) over \( S^1 \) whose fibre at the point \( \theta \in S^1 \) is

\[ (c^*L)_{\theta} = \text{span}\{w \in \mathbb{R}^{2n+2} \mid \exists \lambda > 0 : c(\theta)w = \lambda w \} \]

and whose vector bundle structure is given by its inclusion in \( S^1 \times \mathbb{R}^{2n+2} \). Then the following holds for the first Stiefel-Whitney class of \( c^*L \):

\[ w_1(c^*L) = \text{rk}_C(A - iB) \text{ mod } 2. \]

**Proof.** First notice that if \( w = (\begin{smallmatrix} u \\ v \end{smallmatrix}) \) is an eigenvector of \( \left( \begin{smallmatrix} A & B \\ -B & -A \end{smallmatrix} \right) \) for the eigenvalue \( \lambda \), then \( Jw = (\begin{smallmatrix} v \\ -u \end{smallmatrix}) \) is an eigenvector for the eigenvalue \( -\lambda \). It follows that there exists a basis \( \{w_1, Jw_1, \ldots, w_{n+1}, Jw_{n+1}\} \) of \( \mathbb{R}^{2n+2} \) of eigenvectors of \( \left( \begin{smallmatrix} A & B \\ -B & -A \end{smallmatrix} \right) \) such that \( \left( \begin{smallmatrix} A & B \\ -B & -A \end{smallmatrix} \right) w_j = \lambda_j w_j \) with \( \lambda_j \geq 0 \). Let now \( W_j = \text{span}\{w_j, Jw_j\} \). Then \( W_j \) is \( R(\theta) \)-invariant: \( R(\theta)w_j = \cos \theta w_j - \sin \theta Jw_j \) and \( R(\theta)Jw_j = \sin \theta w_j + \cos \theta Jw_j \). Thus, using the above basis, we see that \( R(\theta) \) is congruent to

\[ M^T R(\theta)M = \text{diag}(D_1(\theta), \ldots, D_{n+1}(\theta)), \quad D_j(\theta) = \lambda_j \left( \begin{smallmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{smallmatrix} \right) \]

If \( c_j : \theta \mapsto D_j(\theta) \), then clearly we have the splitting \( c^*L = c_1^*L \oplus \cdots \oplus c_{n+1}^*L \). Since \( w_j(c^*L) = 0 \) if and only if \( \lambda_j = 0 \), then

\[ w_1(c^*L) = \frac{1}{2} \text{rk}_C\left( \begin{smallmatrix} A & B \\ -B & -A \end{smallmatrix} \right) = \text{rk}_C(A - iB) \]

where the last equality comes from the proof of Theorem [2].

**Corollary 4** (The cohomology of one single quadric). Let \( q \in \mathbb{C}[z_0, \ldots, z_n]_{(2)} \) be a quadratic form with \( \text{rk}(q) = \rho > 0 \) and

\[ C = V(q) \subset \mathbb{C}P^n \]
Then the Betti numbers of $C$ are:

- **$\rho$ even:** $b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ 1 & \text{if } j \text{ is even, } 0 \leq j \leq 2n - 2, j \neq 2n - \rho \\ 2 & \text{if } j = 2n - \rho \end{cases}$

- **$\rho$ odd:** $b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ 1 & \text{if } j \text{ is even, } 0 \leq j \leq 2n - 2 \end{cases}$

**Proof.** We first compute $H^*(R)$ using Theorem 2: in this case if $Q = A - iB$, then $R$ is the intersection of the two quadrics defined by the symmetric matrices $P = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$ and $-JP = \begin{pmatrix} B & A \\ A & -B \end{pmatrix}$ and $p : S^1 \to \text{Sym}(2n + 2, \mathbb{R})$ equals $\theta \mapsto R_\theta \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$. The function $i^+$ has constant value $\rho$ and thus the $E_2$ table for $R$ has the following picture:

$$
\begin{array}{c|c|c|c|c}
2n + 1 & \mathbb{Z}_2 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\rho & \mathbb{Z}_2 & 0 & 0 \\
\rho & \mathbb{Z}_2 & 0 & 0 \\
0 & 0 & \mathbb{Z}_2 & \\
\vdots & \vdots & \vdots & \\
0 & 0 & \mathbb{Z}_2 & 0 \\
\end{array}
$$

The only (possibly) nonzero differential is

$$d_2 : E_2^{0,\rho} \to E_2^{2,\rho - 1}$$

which by the previous discussion equals $1 \mapsto \partial^*w_1(p^*L)$. Lemma 3 implies now

$$d_2(1) = \rho \mod 2$$

Thus if $\rho$ is even $E_2 = E_\infty$ and if $\rho$ is odd the $(\rho - 1)$-th and the $\rho$-th row of $E_3 = E_\infty$ are zero. Applying the formula $b_j(C; \mathbb{Z}_2) = \sum_{k=0}^j (-1)^k b_{j-k}(R; \mathbb{Z}_2)$ gives the result.

Using the previous spectral sequence we can easily compute the rank of the map induced on the $\mathbb{Z}_2$-cohomology by the inclusion

$$i_C : C \hookrightarrow \mathbb{C}P^n.$$

We recall that from Theorem C of [1] we have $\dim(E_\infty^{0, 2n+1-k}) = \text{rk}(i_C^* \mathbb{Z}_2)$; thus applying Lemma 4 we get the following.

**Theorem 5.** For every $k$ we have

$$\text{rk}(i_C^* \mathbb{Z}_2)_k = \dim E_\infty^{0, 2n+1-k}$$

and the zeroth column of $E_\infty$ must be the following:

$$E_\infty^{0,*} = \begin{pmatrix} \mathbb{Z}_2 \\
\vdots \\\n\mathbb{Z}_2 \\
0 \\
\vdots \\
0 \end{pmatrix}$$
where the number of \( \mathbb{Z}_2 \) summand is an even number \( r + 1 \), and \( r \) is that given by Lemma 7.

Notice in particular that \( E_\infty^{0,2a} = \mathbb{Z}_2 \) iff \( E_\infty^{0,2a+1} = \mathbb{Z}_2 \).

4. The intersection of two complex quadrics

We apply here the previous result to compute the cohomology of the intersection of two complex quadrics:

\[
C = V(q_0, q_1) \subset \mathbb{C}P^n.
\]

We define \( \Sigma_j = \{[\alpha] \in \mathbb{C}P^1 \mid \text{rk}(\alpha q_0 + \alpha q_1) \leq j - 1\} \); for \( j \leq \mu = \max (i) \), we see that \( \Sigma_j \) consists of a finite number of points (it is a proper algebraic subset) \( \Sigma_j = \{[\alpha_1], \ldots, [\alpha_{\sigma_j}]\} \), where we have set

\[
\sigma_j = \text{card}(\Sigma_j), \quad j \leq \mu.
\]

The discussion of the previous sections implies that for every \([\alpha] \in \mathbb{C}P^1\) the function \(i^+_{h^{-1}[\alpha]}\) is constant on the circle \( h^{-1}[\alpha] \subset S^3\) with value

\[
i^+_{h^{-1}[\alpha]} = \text{rk}(\alpha q_0 + \alpha q_1) = \rho([\alpha]).
\]

Thus it is defined the bundle \( R_{\rho([\alpha])} \to L_{[\alpha]} \to h^{-1}[\alpha] \) of positive eigenspace of \( p|_{h^{-1}[\alpha]} \) and Lemma 3 implies

\[
w_1(L_{[\alpha]}) = \rho([\alpha]) \mod 2.
\]

For every \([\alpha] \in \mathbb{C}P^1\) we let \( m_{[\alpha]} \) be the multiplicity of \([\alpha]\) as a solution of \( \det(\alpha_0 Q_0 + \alpha q_1) = 0 \); notice that in general \( n + 1 - \rho([\alpha]) \neq m_{[\alpha]}\).

For every \( j \in \mathbb{N} \) we see that

\[
\Omega^j+1 = S^3 \setminus h^{-1}(\Sigma_j+1)
\]

If we let \( \nu \) be the minimum of \( i^+ \) over \( S^3 \), we see that for \( i > 0 \) and \( \nu + 1 \leq j + 1 \leq \mu \)

\[
E_2^{i,j} = H^i(B, S^3 \setminus h^{-1}(\Sigma_j+1)) \simeq \tilde{H}_{3-i}(h^{-1}(\Sigma_j+1)) = \begin{cases} 0 & \text{if } i \neq 2, 3; \\ \mathbb{Z}_2^{\sigma_j+1} & \text{if } i = 2 \\ \mathbb{Z}_2^{\sigma_j+1-1} & \text{if } i = 3 \end{cases}
\]

This gives the following picture for the table of ranks of \( E_2 \):

\[
\begin{array}{cccc|cc}
2n + 1 & 1 & 0 & \vdots & \vdots & \vdots \\
\mu & 1 & 0 & \sigma_\mu & \sigma_\mu - 1 & 0 \\
\mu - 1 & 0 & 0 & \vdots & \vdots & \vdots \\
\nu & 0 & 0 & \sigma_{\nu+1} & \sigma_{\nu+1} - 1 & 0 \\
0 & 0 & 0 & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}
\]

We proceed now with the computation of the second differential; the only two possibly nonzero differential are \( d_2^{\mu,\mu} \) and \( d_2^{\nu,\nu} \), for which the following theorem holds; for an integer \( m \) we let \( m \in \mathbb{Z}_2 \) be its residue modulo 2.
We start with

\[ d_2^{2,ν}(x_1, \ldots, x_{σ_{ν+1}}) = \sum_{k=1}^{σ_{ν+1}} x_k. \]

Moreover in the case \( μ = n + 1 \), we also have the following explicit expression for \( d_2^{0,n+1} : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^{σ_{n+1}} \)

\[ d_2^{0,n+1}(x) = (\overline{m_1}, \ldots, \overline{m}_ν) \]

where \( m_k = m_{[α_k]} \).

**Proof.** We start with

\[ d_2 : E_2^{2,ν} \cong \tilde{H}^1(h^{-1}(Σ_{ν+1})) \rightarrow E_2^{4,ν-1} = \mathbb{Z}_2 \]

which is given by \( x \mapsto (x - γ_{ν})|_{(B, Ω^{ν})}. \) In order to do that we choose a small neighborhood \( U(ν) \) of \( Σ_{ν+1} = \{ [β_1], \ldots, [β_{σ_{ν+1}}] \} \) and we define \( C(ν) = h^{-1}(U(ν)). \)

If we set \( γ_{ν,ν} = γ_{ν,ν}|_{(B, C(ν))} \), then since \( C(ν) \cup Ω^{ν+1} = Ω^{ν} = S^3 \),

\[ d_2^{2,ν} = (x - γ_{ν,ν})|_{(B, C(ν))}. \]

We let \( \partial^*c_{1}, \ldots, \partial^*c_{σ_{ν+1}} \) be the generators of \( H^2(B, C(ν)) \) \( \cong H^1(C(ν)) \), where \( c_k \) is the dual of \( h^{-1}[β_k], k = 1, \ldots, σ_{ν+1} \). Lemma 3 implies now that \( w_1(L[β_i]) = ν \mod 2 \) because \( ν = \min i^* = \text{rk}(p(β_k)) \) for every \( k = 1, \ldots, σ_{ν+1} \). It follows that \( γ_{ν,ν} = \sum_{k=1}^{σ_{ν+1}} \partial^*c_k. \)

If we let now \( \partial^*g_1, \ldots, \partial^*g_{σ_{ν+1}} \) be the generators of \( H^2(B, Ω^{ν+1}) \) \( \cong H^1(Ω^{ν+1}) \), where \( g_k = \text{lk}(\cdot, h^{-1}[β_k]), k = 1, \ldots, σ_{ν+1} \), we have the following formula

\[ d_2^{2,ν}(x) = \partial^*c_k \sum_{k=1}^{σ_{ν+1}} x^k = \sum_{k=1}^{σ_{ν+1}} x^k \partial^*g_k. \]

We assume now that \( μ = n + 1 \) and we compute

\[ d_2 : E_2^{0,n+1} \cong \mathbb{Z}_2 \rightarrow E_2^{2,n} \cong \tilde{H}_1(h^{-1}(Σ_{n+1})). \]

Consider thus \( Σ_{n+1} = \{ [α_1], \ldots, [α_{σ_{n+1}}] \} \) and let \( f_1, \ldots, f_{σ_{n+1}} \) be the generators of \( \tilde{H}_1(S^3 \backslash h^{-1}(Σ_{n+1})): \)

\[ f_k(c) = \text{lk}(c, h^{-1}[α_k]) \ \forall c \in \tilde{H}_1(S^3 \backslash h^{-1}(Σ_{n+1})). \]

In this way we have

\[ H^2(Σ_{n+1}) = \langle \partial^*f_1, \ldots, \partial^*f_{σ_{n+1}} \rangle. \]

It is shown in [I] that

\[ w_{1,n+1} = p^*\text{lk}(\cdot, \{ λ_{n+1} = λ_n \}) \]

In our case \( p^{-1}\{ λ_{n+1} = λ_{n+2} = h^{-1}(Σ_{n+1}) \} : \) if \( \alpha \notin h^{-1}(Σ_{n+1}) \), then \( \text{rk}(p(α)) = n + 1 \) and thus \( i^*(p(α)) = n + 1 \) and \( λ_{n+1}(p(α)) > λ_{n+2}(p(α)) \); on the contrary if \( α \in h^{-1}(Σ_{n+1}) \), then \( \text{rk}(p(α)) \leq n \) and \( λ_{n+1}(p(α)) = λ_{n+2}(p(α)) = 0 \). Since \( γ_{1,n+1} = \partial^*w_{1,n+1,1} \), then we have

\[ d_2^{0,n+1}(1) = γ_{1,n+1} = \sum_{k=1}^{σ_{n+1}} \overline{m}_k \partial^*f_k \]
where \( m_k = m_{[\alpha_k]} \) comes from the fact that we are taking the pull-back of the class 
\( \text{lk}(\cdot; \{\lambda_{n+1} = \lambda_{n+2}\}) \) through \( p \) and multiplicities have to be taken into account.

\[ m\kappa = m[\alpha_k] \]

**Remark 2.** Notice that if \( \mu = n + 1 \) and \( \nu = n \), then
\[
d_{2}^{n} \circ d_{2}^{0,n+1}(1) = \pi \sum m_{i} = n(n + 1) = 0.
\]

**Remark 3.** Consider the bundle \( \mathbb{R}^\mu \to L_{\mu} \to D_{\mu} \) as defined in the second section and its projectivization \( \mathbb{P}^{\mu-1} \to P_{\mu} \to D_{\mu} \). Since \( L_{\mu} \subset D_{\mu} \times \mathbb{R}^{2n+2} \), then \( P_{\mu} \subset D_{\mu} \times \mathbb{P}^{2n+1} \) and the restriction of the projection on the second factor
\[
l : P_{\mu} \to \mathbb{P}^{2n+1}
\]
is a map which is a linear embedding on the fibres. It is not difficult to prove that for this map we have \( \text{rk}(l^*k) \leq 1 - \text{rk}(iR^*)c_{2n+1-k} \) (see [1]). Thus by Theorem C of [1] we have the following implication:
\[
\text{rk}(l^*k) = 1 \Rightarrow E_{0,k}^{0,\infty} = 0.
\]

Applying the same reasoning and computing \( y_k \) for \( k \geq \mu + 1 \) we get similar conditions for the vanishing of \( E_{0,k}^{0,\mu} = 0 \). Such considerations suggest that higher differential \( d_{r}^* \) for \((E_{r}, d_{r})\) are closely related to higher characteristic classes.

We get as a corollary of the previous theorem the following well known fact from plane geometry.

**Corollary 7.** The intersection of two quadrics in \( \mathbb{C}P^2 \) consists of four points if and only if the associated pencil has exactly three singular elements.

**Proof.** Notice that a for a pencil of quadrics in \( \mathbb{C}P^2 \) generated by \( Q_0, Q_1 \) the following four possibilities can happen for
\[
\{[\alpha_0, \alpha_1] \in \mathbb{C}P^1 | \det(\alpha_0Q_0 + \alpha_1Q_1) = 0\} = \begin{cases} \mathbb{C}P^1 & (\infty) \\ \text{one point} & (1) \\ \text{two points} & (2) \\ \text{three points} & (3) \end{cases}
\]

The general table for the ranks of \( E_2(R) \) has the following picture:

|   | 1 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | c | c' | 0 |   |
| b | 0 | d | d' | f |   |
| 0 | 0 | e | e' | g |   |
Now \( b_0(C) = b_0(R) \leq 1 + \epsilon' + f \) and
\[
(\infty) : a = 1, \ c = \epsilon = f = 0 \text{ and } b_0(C) = 1.
\]
\[
(1), (2) : a = b = 0, \ c' = c - 1 \leq 1, \ f \leq 1 \text{ and } b_0(C) \leq 3.
\]
\[
(3) : a = b = 0, \ c = 3, \ c' = 2, \ f = 1 \text{ and by Theorem } E_{0,5}^{3,4} \text{ is identically zero}
\]
\[
(\nu = 2 \text{ is even); also, since } E_{0,5}^{3,4} = \mathbb{Z}_2, \text{ Theorem } \text{ implies } E_{0,4}^{3,4} = \mathbb{Z}_2 \text{ (the number of } \mathbb{Z}_2 \text{ summands in } E_{0,4}^{3,4} \text{ is even); thus } d_2^{3,4} = d_3^{3,4} = d_4^{3,4} \equiv 0 \text{ and } b_0(C) = 4.
\]

It follows that
\[
b_0(C) = 4 \iff (3).
\]

□

\textbf{Example 1 (The complete intersection of two quadrics).} We recall from [3] that the condition for \( C = V(q_0, q_1) \) to be a complete intersection is equivalent to have \( \mu = n + 1, \sigma_\mu = n + 1 \text{ and } \nu = n. \) In other words the equation \( \det(\alpha_0 Q_0 + \alpha_1 Q_1) = 0 \) must have \( n + 1 \) distinct roots and at each root \( [\alpha_0, \alpha_1] \) the pencil must by simply degenerate, i.e. the rank of \( \alpha_0 Q_0 + \alpha_1 Q_1 \) must be \( n \) (notice in particular that for the case \( n = 2 \) we have the above result).

Thus the table for the rank of \( E_2 \) is the following:

| \( \text{rk}(E_2) = \frac{n + 1}{n} \) | 2n + 1 | 1 | 0 |
|-----------------------------|--------|---|---|
| \( n + 1 \) | \( n + 1 \) | \( 0 \) | \( n \) |
| \( 0 \) | \( 0 \) | \( n + 1 \) | \( 0 \) |
| \( 0 \) | \( 0 \) | \( 0 \) | \( 1 \) |
| \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

We distinguish the two cases \( n \) even and \( n \) odd.

\textbf{(n even)}: In this case, by Theorem \( d_2^{n+1} \) is injective and \( d_2^{n+1} \) is zero. Hence the table for the rank of \( E_3 \) is the following:

| \( \text{rk}(E_3) = \frac{n + 1}{n} \) | 2n + 1 | 1 | 0 |
|-----------------------------|--------|---|---|
| \( n + 1 \) | \( n + 1 \) | \( 0 \) | \( 0 \) |
| \( 0 \) | \( 0 \) | \( n \) | \( n \) |
| \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( 0 \) | \( 0 \) | \( 0 \) | \( 1 \) |

Since \( d_3^{n+3} = d_4^{n+3} = 0 \) then \( E_{0,3}^{n+3} = \mathbb{Z}_2; \text{ since } n \text{ is even, then by Theorem } \text{ we have } E_{0,3}^{n+2} = E_{0,4}^{n+3} = \mathbb{Z}_2 \text{ and thus } d_3^{n+2} = d_4^{n+2} = 0. \) This implies
\[
E_3 = E_\infty.
\]
Thus the $\mathbb{Z}_2$-Betti numbers of $R$ are:

$$b_j(R) = \begin{cases} 1 & \text{if } j \neq n - 2, n - 1, 0 \leq j \leq 2n - 1; \\ n + 2 & \text{if } j = n - 2, n - 1. \end{cases}$$

Consequently the $\mathbb{Z}_2$-Betti numbers of $C$ are:

$$b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ 1 & \text{if } j \text{ is even, } j \neq n - 2 \text{ and } 0 \leq j \leq 2n - 2 \\ n + 2 & \text{if } j = n - 2. \end{cases}$$

($n$ odd) : in this case, by Theorem 6 $d_2^{0,n+1}$ is injective and $d_2^{2,n}$ is surjective. Thus the table for the rank of $E_3$ is the following:

$$\begin{array}{c|cccc} 2n + 1 & 1 & 0 & & \\ \vdots & \vdots & & & \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{array}$$

$$\text{rk}(E_3) = \begin{array}{c|cccc} n + 1 & & & & \\ \hline n & 0 & 0 & n - 1 & n - 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ \end{array}$$

Since $E_0^{0,n+1} = 0$ and $n$ is odd, then by Theorem 5 we have $E_0^{0,n+2} = 0$, thus $d_3^{0,n+2}$ must be injective and the table of rank of $E_4 = E_\infty$ must be the following:

$$\begin{array}{c|cccc} 2n + 1 & 1 & 0 & & \\ \vdots & \vdots & & & \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{array}$$

$$\text{rk}(E_4) = \text{rk}(E_\infty) = \begin{array}{c|cccc} n + 1 & & & & \\ \hline n & 0 & 0 & n - 1 & n - 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ \end{array}$$

Thus the $\mathbb{Z}_2$-Betti numbers of $R$ are:

$$b_j(R) = \begin{cases} 1 & \text{if } j \neq n - 2, n - 1, 0 \leq j \leq 2n - 1; \\ n & \text{if } j = n - 2, n - 1. \end{cases}$$

Consequently the $\mathbb{Z}_2$-Betti numbers of $C$ are:

$$b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd and } j \neq n - 2; \\ 1 & \text{if } j \text{ is even, } 0 \leq j \leq 2n - 2 \\ n + 1 & \text{if } j = n - 2. \end{cases}$$

Thus the complete intersection of two quadrics $C$ in $\mathbb{CP}^n$ has complex dimension $m = n - 2$ and its $m$-th Betti number is $m + 4$ if $m$ is even and $m + 1$ if $m$ is odd.
Example 2. Consider the two quadrics

\[ q_0(z_0, z_1, z_2) = z_0z_2 - z_1^2 \quad \text{and} \quad q_1(z_0, z_1, z_2, z_3) = z_0z_3 - z_1z_2. \]

Then \( \det(\alpha_0 Q_0 + \alpha_1 Q_1) = \alpha_1^4 \) and \( \text{rk}(\alpha_0 Q_0 + \alpha_1 Q_1) \equiv 4 \) except at the point \([1, 0] \) where we have \( \text{rk}(Q_0) = 3 \). Notice in this case that \( \text{rk}(p([1, 0])) \neq n + 1 - m_{[\alpha]} = 4 - m_{[\alpha]} = 0 \).

The table for the rank of \( E_2 \) has the following picture:

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{array}
\]

Since \( \mu = 4 = n + 1 \), then we can use the previous formula for \( d_{2, \mu}^0 \) and since we have \( m_{[1, 0]} = 4 \) it follows \( d_{2, 4}^0 \equiv 0 \). On the other hand \( d_{2, 4}^1 \) is multiplication by \( \nu = 3 \mod 2 \), hence it is an isomorphism. Hence the table for the rank of \( E_3 \) has the following picture:

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{array}
\]

Since \( d_{3, 5}^0 = d_{4, 5}^0 \equiv 0 \), then \( E_{\infty}^{0, 4} = Z_2 \) and Theorem B implies also \( E_{\infty}^{5, 4} = Z_2 \). Thus \( E_3 = E_4 = E_{\infty} \). Hence, for the only possible nonzero Betti numbers of \( R \) we have \( b_0(R) = b_1(R) = 1, b_2(R) = b_3(R) = 2 \). This implies the following for the Betti numbers of \( C \):

\[ b_0(C) = 1, b_2(C) = 2 \quad \text{and} \quad b_i(C) = 0, i \neq 0, 2. \]

Using Theorem 5 we see that \( (i_C)_0^* \) and \( (i_C)_2^* \) are injective.

Looking directly at the equations for \( C \) we see that it equals the union of the skew-cubic and a (complex projective) line meeting at one point; thus topologically \( C \sim S^2 \lor S^2 \).

Example 3. Consider the two quadrics

\[ q_0(z_0, z_1, z_2) = z_0^2 - z_1^2 \quad \text{and} \quad q_1(z_0, z_1, z_2) = 2z_0(z_1 + z_2). \]

We have \( \det(\alpha_0 Q_0 + \alpha_1 Q_1) \equiv 0 \) and \( \text{rk}(\alpha_0 Q_0 + \alpha_1 Q_1) = 2 \) for every \([\alpha_0, \alpha_1] \in \mathbb{C}P^1 \).

Thus the table for the rank of \( E_2 \) is:
By dimensional reasons, the only possibly nonzero differential is \( d_4 \). Since \( d_4^0 = 0 \), then \( E_{\infty}^0,2 = \mathbb{Z}_2 \) and by Theorem 5 also \( E_{\infty}^0,3 = \mathbb{Z}_2 \). Since \( E_{\infty}^0,3 = \mathbb{Z}_2 \), then \( d_4^0 = 0 \); hence \( E_{\infty}^0,0 = \mathbb{Z}_2 \). On the other side we have \( d_4^0 = 0 \) and hence \( E_{\infty}^0,5 = \mathbb{Z}_2 \); Theorem 5 implies \( E_{\infty}^0,4 = \mathbb{Z}_2 \). Since \( E_{\infty}^0,4 = \mathbb{Z}_2 \), then \( d_4^0 = 0 \) and \( E_{\infty}^1,1 = \mathbb{Z}_2 \).

All this tells us that \( E_{\infty} = E_2 \). The only possible nonzero Betti numbers of \( R \) are \( b_0(R) = b_1(R) = 2, b_2(R) = b_3(R) = 1 \). This implies the following for the Betti numbers of \( C \):

\[
b_0(C) = 2, \ b_2(C) = 1 \quad \text{and} \quad b_i(C) = 0, \ i \neq 0, 2.
\]

Looking directly at the equations of \( C \) we see that it equals the union of the point \([1, 1, 0]\) and the complex projective line \( \{z_0 + z_1 = 0\} \).

**References**

[1] A. A. Agrachev, A. Lerario: *Systems of quadratic inequalities*, arXiv:1012.5731v2.
[2] A. Dimca: *A geometric approach to the classification of pencils of quadrics*, Geometriae Dedicata Volume 14, Number 2, 105-111
[3] W. V. D. Hodge, D. Pedoe: *Methods of algebraic geometry, volume 2*, Cambridge University Press, 1952.
[4] M. Reid: *The complete intersection of two or more quadrics*, 1972.
[5] R. C. Thompson: *Pencils of complex and real symmetric and skew matrices*, Linear Algebra and its Applications, Volume 147, March 1991, 323-371.