Treewidth of Erdős-Rényi Random Graphs, Random Intersection Graphs, and Scale-Free Random Graphs

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Abstract

We prove that the treewidth of an Erdős-Rényi random graph $G(n, m)$ is, with high probability, greater than $\beta n$ for some constant $\beta > 0$ if the edge/vertex ratio $m/n$ is greater than 1.073. Our lower bound $m/n > 1.073$ improves the only previously-known lower bound established in [19]. We also study the treewidth of random graphs under two other random models for large-scale complex networks. In particular, our result on the treewidth of random intersection graphs strengthens a previous observation in [16] on the average case behavior of the gate matrix layout problem. For scale-free random graphs based on the Barabási-Albert preferential-attachment model, our result shows that if more than 12 vertices are attached to a new vertex, then the treewidth of the obtained network is linear in the size of the network with high probability.

1 Introduction

Treewidth plays an important role in characterizing the structural properties of a graph and the complexity of a variety of algorithmic problems of practical importance [4, 19]. When restricted to instances with bounded treewidth, many NP-hard problems are polynomially solvable. Dynamic programming algorithms based on the tree-decomposition of graphs have found many applications in research field such as computational biology and artificial intelligence [8, 9].

The theory of random graphs pioneered by the work of Erdős and Rényi [11] deals with the probabilistic behavior of various graph properties such as the connectivity, the colorability, and the size of (connected) components [11, 5, 2, 13]. Random intersection graphs and scale-free random graphs were proposed as more realistic models for large-scale complex networks arising in real-world domains such as communication networks (Internet, WWW, Wireless and P2P networks), computational biology (protein networks), and

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sociology (social networks). It has been hoped that these new models will be able to capture the common features of these networks in a better way and in the mean time, are mathematically approachable and algorithmically tractable [7, 12, 15, 20].

As treewidth is one of the most important structural parameters used to capture the algorithmic tractability of computationally hard problems, it is interesting to see how large the treewidth of a typical graph is in these random models. Of course, studying the probabilistic behavior of the treewidth of these random graphs is itself an interesting combinatorial problem. Except for a result in [19] establishing a lower bound on the threshold of having a linear treewidth of the Erdős-Rényi random graph, we are not aware of any other work in the literature. In the paper, we study the treewidth of random graphs under the following three random models:

1. **The Erdős-Rényi model** [5, 11]. An Erdős-Rényi random graph \( G(n, m) \) is defined on \( n \) vertices and contain \( m \) edges selected from the \( N = \binom{n}{2} \) potential edges uniformly at random and without replacement.

2. **The random intersection model** [16]. A random intersection graph \( G_{\mathcal{I}}(n, m, p) \) on \( n \) vertices is defined as follows. Let \( M = \{1, 2, \ldots, m\} \) be a fixed universe of size \( m \). Each vertex \( v \) is associated with a subset \( S_v \subset M \) that is obtained by including each element in \( M \) independently with probability \( p \). These \( S_v \)'s are determined independently as well. There is an edge between a pair of vertices \( u \) and \( v \) if and only if \( S_u \cap S_v \neq \emptyset \).

3. **The Barabási-Albert scale-free model** [3]. A Barabási-Albert random graph \( G_S(n, m) \) on a set of \( n \) vertices \( \{v_1, \ldots, v_n\} \) is defined by a graph evolution process in which vertices are added to the graph one at a time. In each step, the newly-added vertex is connected to \( m \) existing vertices selected according to the preferential attachment mechanism, i.e. an existing vertex is selected with probability in proportion to its degree.

We establish a lower bound 1.073 on the edge/vertex ratio \( \frac{m}{n} \) above which an Erdős-Rényi random graph \( G(n, m) \) has a treewidth linear to the number of vertices with high probability. Our lower bound improves the previous one \( \frac{m}{n} > 1.18 \) in [19]. We obtain similar results on the behavior of the treewidth for the random intersection graph \( G_{\mathcal{I}}(n, m, p) \) and the Barabási-Albert scale-free random graph \( G_S(n, m) \). Our result on \( G_{\mathcal{I}}(n, m, p) \) complements an observation in [16] on the average case behavior of the gate matrix layout problem. Our result on the scale-free random graph \( G_S(n, m) \) shows that if more than 12 vertices are attached to a new vertex, then the treewidth of the obtained network is linear in the size of the network with high probability. Our results are summarized in the following theorems:

**Theorem 1** Let \( G(n, m) \) be an Erdős-Rényi random graph. For any \( \frac{m}{n} \geq 1.073 \), there is a constant \( \beta > 0 \) such that

\[
\lim_{n \to \infty} \mathbb{P}_{G(n, m)} \{tw(G(n, m)) > \beta n\} = 1.
\]
Theorem 2 Let $G_I(n,m,p)$ be a random intersection graph with the universe $M = \{1, \cdots, m\}$ and $m = n^\alpha$. For any $p \geq \frac{2}{m}$ and $\alpha > 0$, there exists a constant $\beta > 0$ such that
\[ \lim_{n \to \infty} \mathbb{P}_{G_I(n,m,p)} \{ \text{tw}(G_I(n,m,p)) > \beta n \} = 1. \] (1.2)

Theorem 3 Let $G_S(n,m)$ be the Barabási-Albert random graph. For any $m \geq 12$, there is a constant $\beta > 0$ such that
\[ \lim_{n \to \infty} \mathbb{P}_{G_S(n,m)} \{ \text{tw}(G_S(n,m)) > \beta n \} = 1. \] (1.3)

1.1 Technical Contribution

The approach used in [19] is essentially an application of the first-moment method to the random variable that counts the total number of the balanced partitions $(S, A, B)$ where the size of the separator $S$ is at most $\beta n$ (See Section 3 for the formal definition of a balanced partition.) It is further commented in [19] that it was not known whether the 1.18 lower bound can be improved and that the treewidth of the random graph $G(n,m)$ with $\frac{1}{2} < \frac{m}{n} < 1$ is unknown.

Our main contribution in this paper is in the proof of our improved lower bound $\frac{m}{n} > 1.073$. We note that a more refined analytical calculation is able to improve the lower bound 1.18 in [19] to 1.083. The difficulty lies in bringing down the lower bound further from 1.083 to 1.073. To achieve this, we introduce the notion of $d$-rigid and balanced partitions $(S, A, B)$ which are maximally balanced in the sense that no vertex subset of certain size from the larger part, say $B$, can be moved to the smaller one $|A|$ to create a new balanced partition. The motivation is that by considering the expected number of these more restricted partitions, we will be able to get a more accurate estimation when applying Markov’s inequality.

The difficulty we have to overcome in the case of treewidth is the estimation of the expected number of $d$-rigid and balanced partition $(S, A, B)$ in $G(n,m)$. To do this, an exponentially small upper bound is required on the probability that the induced subgraph $G[B]$ of the random graph $G(n,m)$ doesn’t have small-sized tree components.

We managed to obtain such an exponentially small upper bound in a “conditional” probability space, which is equivalent to the Erdő-Rényi random model as far as the size of the treewidth is concerned, by using a Hoeffding-Azuma style inequality. To achieve the best possible Lipschitz constant in our application of the Hoeffding-Azuma inequality, we used a “weighted” count on the number of tree components of size up to a fixed constant $d$. We are not aware of any other application of the Hoeffding-Azuma inequality in the study of random discrete structures where this idea of weighted counts is beneficial.

1.2 Outline of the Paper

The next section fixes our notation and contains preliminaries. Also discussed in this section is a variant of the Erdő-Rényi model for random graphs which we will be using in our proofs.

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1The idea of restricting the kinds of combinatorial objects to be considered have been used in the study of the threshold for the satisfiability of random CNF formulas and the chromatic number of random graphs [14, 11, 10].
Sections 3 - 5 contain the proofs of Theorem 1, Theorem 2, and Theorem 3 respectively. The two appendices contain the proof of some necessary lemmas.

2 Notation and Preliminaries

Throughout this paper, all logarithms are natural logarithms, i.e., to the base $e$. The cardinality of a set $U$ is denoted by $|U|$. All graphs are undirected and standard terminologies in graph theory [23] are used. Given a graph $G(V, E)$ and a vertex $v \in V$, we use $N(v)$ to denote the set of neighbors of $v$, i.e.,

$$N(v) = \{ u \in V \mid u \neq v \text{ and } (u, v) \in E \}.$$

Given a vertex subset $U$, we use $N(U)$ to denote the neighborhood of $U$, i.e.,

$$N(U) = \{ w \in V \setminus U \mid (w, u) \in E \text{ for some } u \in U \}.$$

The induced subgraph on a subset of vertices $U$ is denoted by $G[U]$. By a component of a graph, we mean a maximal connected subgraph.

In the proofs, we will be using the following upper bound on $\binom{n}{\beta n}$ that can be derived from Stirling’s formula:

Lemma 2.1 For any constants $0 < \beta < 1$,

$$\binom{n}{\beta n} \leq \frac{\theta}{\sqrt{\beta(1-\beta)n}} \left( \frac{1}{\beta^\beta (1-\beta)^{1-\beta}} \right)^n,$$

where $\theta > 0$ is a constant.

We also need the following three lemmas on the properties of some useful functions. The proof of these lemmas are included in Appendix 2.

Lemma 2.2 On internal $(0, 1)$, the function $f(t) = t^t (1-t)^{1-t}$ attains its minimum at $t = \frac{1}{2}$ and $\lim_{t \to 0} f(t) = 1$. Furthermore, $f(t)$ is decreasing on the interval $(0, \frac{1}{2}]$ and decreasing on the interval $[\frac{1}{2}, 1)$.

Lemma 2.3 Let $r(t)$ be a function defined as

$$r(t) = \frac{2t^2}{(1+e)^2 c} \left( \frac{1}{e} \right)^{\frac{4ct}{1-2t(1-t)}}$$

where $c > 0$ is a constant. For any $c > 1$ and sufficiently small $\beta > 0$, $r(t)$ is decreasing on the interval $[\frac{1-\beta}{2}, \frac{3}{3}]$.

Lemma 2.4 Let $g(t)$ be a function defined as

$$g(t) = \frac{(1 - 2t + 2t^2 + 2\beta t)^c}{t^t (1-t)^{1-t}}$$

where $c > 1$ and $\beta > 0$ are constants. Then for sufficiently small $\beta$, $g(t)$ is increasing on $[\frac{1-\beta}{2}, \frac{2}{3}]$.  

4
2.1 Treewidth and Random Graphs

The notion of treewidth plays an important role in graph theory and in real world computing. Several equivalent definitions of treewidth exist and the one based on $k$-trees is probably the easiest to explain. The graph class of $k$-trees is defined recursively as follows [19]:

1. A clique with $k+1$ vertices is a $k$-tree;
2. Given a $k$-tree $T_n$ with $n$ vertices, a $k$-tree with $n+1$ vertices is constructed by adding to $T_n$ a new vertex and connecting it to a $k$-clique of $T_n$.

A graph is called a partial $k$-tree if it is a subgraph of a $k$-tree. The treewidth $tw(G)$ of a graph $G$ is the minimum value $k$ such that $G$ is a partial $k$-tree.

Since the seminal work of Erdős and Rényi [11], the theory of random graphs has been an active research area in graph theory and combinatorics. The probabilistic behavior of various graph properties such as the connectivity, the colorability, and the size of (connected) components, have been extensively studied. The theory of random graphs has also motivated the study of the probabilistic properties of random instances of other important combinatorial optimization problems, most notably that of the satisfiability of random logic formulas in conjunctive normal form (CNF).

We use $G(n, m)$ to denote an Erdős-Rényi random graph [5] on $n$ vertices with $m$ edges selected from the $N = \binom{n}{2}$ possible edges uniformly at random and without replacement. Throughout this paper by “with high probability”, abbreviated as whp, we mean that the probability of the event under consideration is $1 - o(1)$ as $n$ goes to infinity.

We will be working with a random graph model $\overline{G}(n, m)$ that is slightly different from $G(n, m)$ in that the $m$ edges are selected independently and uniformly at random, but with replacement. There is a one-to-one correspondence between the random graph $\overline{G}(n, m)$ and the product probability space $(\Omega, \mathcal{A}, \mathbb{P}_{\overline{G}(n, m)} \{\cdot\})$ defined as follows:

1. $\Omega = \prod_{i=1}^{m} E_i$ where each $E_i$ is the set of all $\binom{n}{2}$ possible edges. This is a finite and discrete sample space.
2. $\mathcal{A}$ is the $\sigma$-field consisting of all subsets of $\Omega$.
3. The probability measure $\mathbb{P}_{\overline{G}(n, m)} \{\cdot\}$ is

$$\mathbb{P}_{\overline{G}(n, m)} \{\omega\} = \left( \frac{1}{\binom{n}{2}} \right)^m, \forall \omega \in \Omega. \quad (2.6)$$

A sample point $\omega \in \Omega$ is interpreted as an outcome of the random experiment that selects $m$ edges independently, uniformly at random with replacement from the set of all possible edges. Note that the graph corresponding to a sample point $\omega \in \Omega$ is actually a multi-graph, i.e., a graph in which parallel edges are allowed.

It turns out that as far as the property of having a treewidth linear in the number of vertices is concerned, the two random graph models $\overline{G}(n, m)$ and $G(n, m)$ are equivalent. In
fact, the equivalence holds for any monotone increasing combinatorial property in random
discrete structures, as has been observed in \cite{17,1} and formally proved in \cite{18}. For com-
pleteness, we will include in Appendix A an alternative pure measure-theory style proof.

**Proposition 2.1** If there exists a constant $\beta > 0$ such that

$$\lim_{n \to \infty} P \{ tw(G(n,m)) \geq \beta n \} = 1,$$

then

$$\lim_{n \to \infty} P \{ tw(G(n,m)) \geq \beta n \} = 1.$$

Due to Proposition 2.1, we will continue to use the notation $G(n,m)$ instead of $\overline{G}(n,m)$
throughout this paper, but with the understanding that the $m$ edges are selected indepen-
dently and uniformly at random with replacement.

In the rest of the paper, we will always subscript operations such as a probability mea-
sure $P_{G(n,m)} \{ \cdot \}$ and a mathematical expectation $E_{G(n,m)} [ \cdot ]$ to clear indicate the underlying
probability space in which these operations are applied.

In \cite{19}, it proved that the treewidth of an Erdős-Rényi random graph $G(n,m)$ is linear
in the number of vertices whp if the edge/vertex ratio is greater than 1.18. It is mentioned
in \cite{19} that it was unclear whether the lower bound 1.18 can be further improved, and that
the treewidth of a random graph $G(n,m)$ with $\frac{1}{2} < \frac{m}{n} < 1$ is unknown \cite{19}. The main result
of this paper improves the bound to 1.073.

### 2.2 Random Intersection Graphs

The intersection model for random graphs was introduced by Karoński, et al. \cite{16}. As one
of the motivations, Karoński, et al. discussed the application of this model in the average-
case analysis of algorithmic problems in gate matrix circuit design \cite{16}. Other motivations
for the recent interests in random intersection graphs include the possible applications in
modeling large-scale complex networks arising in wireless communications \cite{22} and social
networks.

A random intersection graph $G_T(n,m,p)$ over a vertex set $V$ is defined by a universe
$M$ and three parameters $n$ (the number of vertices), $m = |M|$, and $0 \leq p \leq 1$. Associated
with a vertex $v \in V$ is a random subset $S_v \subset M$ formed by selecting each element in $M$
independently with probability $p$. A pair of vertices $u$ and $v$ is an edge in $G_T(n,m,p)$ if
and only if $S_u \cap S_v \neq \emptyset$.

An alternative view of $G_T(n,p,m)$ is as follows. Let $(C_1, \ldots, C_m)$ be a set of $m$ subsets
of vertices. Each $C_i$ is formed independently by including each vertex independently with
probability $p$. A pair of vertices $u$ and $v$ is an edge in $G_T(n,m,p)$ if and only if some $C_i$
contains both $u$ and $v$. In this sense, a random intersection graph is actually the primal
graph of a random hypergraph consisting of $m$ hyperedges each of which contains a vertex
with probability $p$. 

6
2.3 Barabási-Albert Random Graphs

In recent years, there has been growing interests in random models for large-scale communication networks, biological networks, and social networks. A remarkable observation is that the degree distribution of these large-scale networks follow a power law, i.e., the fraction of vertices of a given degree \( d \) is proportional to \( d^{-\gamma} \) for some constant \( \gamma > 0 \).

The Barabási-Albert model for random graphs is proposed in [3] and has been shown to have a power law degree distribution [6]. In addition to the purpose of modeling, it is also hoped that features such as a power-law degree distribution may be exploited algorithmically and/or mathematically to help solve real-world problems defined on these large-scale networks. See, for example, the work and arguments in [7, 12, 20, 15].

Following the formal definition given in [6], a Barabási-Albert random graph \( G_{S}^{n,m} \) on a set of \( n \) vertices \( V = \{ v_1, \ldots, v_n \} \) is defined by a graph evolution process in which vertices are added to the graph one at a time. In each step, the newly-added vertex is connected to \( m \) existing vertices selected according to the preferential attachment mechanism, i.e. an existing vertex is selected with probability in proportion to its degree. To be more precisely, let \( v_i \) be the vertex to be added and let \( G_{i-1} \) be the graph obtained after vertex \( v_{i-1} \) is added. The \( m \) neighbors of \( v_i \) are selected in \( m \) steps. In step \( 1 \leq j \leq m \), the probability that an existing vertex \( w \) is selected as the neighbor of the new vertex \( v \) is

\[
\frac{d_{G_{i-1}}(w) + d_{w}(j)}{m(i-1) + 2(j-1)},
\]

where

1. \( (i-1)m = \sum_{k \leq i-1} d_{G_{i-1}}(v_k) \) is the total degree of \( G_{i-1} \),
2. \( d_{w}(j) \) is the number of times \( w \) has been picked as the neighbor of \( v \) in the first \( (j-1) \) trials, and
3. the term \( 2(j-1) \) takes into consideration the increase of the total degree as a result of the first \( j-1 \) neighbors.

One also needs to take care of the initial phase, but that won’t have any impact on our analyses.

3 Treewidth of Erdös-Rényi Random Graphs: Proof of Theorem [1]

In this section, we prove Theorem [1] to establish the lower bound \( c^* \) on the edge/vertex ratio \( \frac{m}{n} \) such that whenever \( \frac{m}{n} \geq c^* \), the treewidth of an Erdös-Rényi random graph \( G(n, m) \) is \textit{whp} greater than \( \beta n \) for some constant \( \beta > 0 \). To begin with, we introduce the following concept which will be used as a necessary condition for a graph to have a treewidth of certain size. The following notion of balanced \( l \)-partition was used in [19] to establish the lower bound 1.18.
Definition 3.1 ([19]) Let \( G(V, E) \) be a graph with \(|V| = n\). Let \( W = (S, A, B) \) be a triple of disjoint vertex subsets such that \( V = S \cup A \cup B \) and \(|S| = l + 1\).

We say that \( W \) is balanced if \( \frac{1}{3}(n - l - 1) \leq |A|, |B| \leq \frac{2}{3}(n - l - 1) \). Without lose of generality, we will always assume that \(|B| \geq |A|\).

We say that \( W \) is an \( l \)-partition if \( S \) separates \( A \) and \( B \), i.e., there are no edges between vertices of \( A \) and vertices of \( B \).

The following notion of a \( d \)-rigid partition plays an important role in establishing our improved lower bound:

Definition 3.2 Let \( d > 0 \) be an integer. A triple \( W = (S, A, B) \) with \(|B| > |A| + d\) is said to be \( d \)-rigid if there is no subset of vertices \( U \subset B \) with \(|U| \leq d\) that induces a connected component of \( G[B] \).

A \( d \)-rigid and balanced \( l \)-partition generalizes Kloks’s balanced \( l \)-partition by requiring that any vertex set of size at most \( d \) in the larger subset of a partition cannot be moved to the other subset of the partition, and hence the word “rigid”. As we will have to consider all the vertex sets of size at most \( d \) to get the best possible estimation, the requirement of connectivity is a kind of “maximality” condition to avoid repeated counting of vertex sets of different sizes. For the case of \( d = 1 \), being \( d \)-rigid means that \( G[B] \) has no isolated vertices.

We note that the idea of imposing various restrictions on the combinatorial objects under consideration has been used in recent years to increase the power of the first moment method when dealing with combinatorial problems in discrete random structures such as the satisfiability of random CNF formulas [17, 10] and the colorability of random graphs [1]. Our result is a further illustration of the power of this idea in the context of treewidth of random graphs.

Lemma 3.1 Let \( d \geq 1 \) be an integer. Any graph with a treewidth at most \( l > 4 \) must have a balanced \( l \)-partition \( W = (S, A, B) \) such that either \(|B| \leq |A| + d\) or \( W \) is \( d \)-rigid.

Proof. From [19], any graph with treewidth at most \( l > 4 \) must have a balanced \( l \)-partition \( W = (S, A, B) \). If \(|B| \leq |A| + d\), we are done. Otherwise, if the triple \( W \) is not \( d \)-rigid, then there must be a vertex subset \( U \subset B \) that induces a component of \( G[B] \) and consequently

\[ N(U) \cap (B \setminus U) = \emptyset. \]

Therefore, we can move \( U \) from \( B \) to \( A \) and create a new balanced \( l \)-partition with the size of \( B \) decreased by \(|U|\). Continuing this process until either \(|B| \leq |A| + d\) or the partition becomes \( d \)-rigid. \( \blacksquare \)

3.1 Conditional Probability of a \( d \)-rigid and balanced \( l \)-partition

We now bound the conditional probability that a balanced triple \( W = (S, A, B) \) with \(|S| = l + 1\) and \(|B| \geq |A| + d\) is \( d \)-rigid given that it is an \( l \)-partition of \( G(n, m) \). To facilitate
the presentation, we define the following function

\[ x(t, c) = \frac{2ct}{2t^2 - 2t + 1}, \]

\[ g(t, c) = \sum_{i=2}^{d} \frac{i^i - 2}{i!} \left( x(t, c) e^{-x(t, c)} \right)^{i-1}, \]

\[ r(t, c) = \frac{2t^2}{(1 + \epsilon)^2c} e^{-2x(t, c)} \]  \hspace{1cm} (3.8)

where \( \epsilon = \frac{1}{d-1} \).

**Theorem 4** Let \( G(n, m), c = \frac{m}{n} \), be a random graph and let \( \mathbf{W} = (S, A, B) \) be a balanced triple such that \(|S| = l + 1, |A| = a, \) and \(|B| = b = tn\). Let \( d > 0 \) be a constant integer less than \( l + 1 \). Then for \( n \) sufficiently large,

\[ \mathbb{P}_{G(n,m)} \{ \mathbf{W} \text{ is d-rigid} \mid \mathbf{W} \text{ is an l-partition} \} \leq \left( \frac{1}{e} \right)^{r(1+g)^2n} \]  \hspace{1cm} (3.9)

where \( \epsilon = \frac{1}{d-1} \),

\[ r = r(t) = \frac{2t^2}{(1 + \epsilon)^2c} \left( \frac{1}{e} \right)^{\frac{4ct}{2(1-\epsilon)}} \]

and

\[ g = g(t) = \sum_{i=2}^{d} \frac{i^i - 2}{i!} \frac{c^{i-1}}{e^{\frac{2(1-\epsilon)ct}{2t^2 + 2t - 1}}}. \]

**Proof.** Conditional on that \( \mathbf{W} \) is an \( l \)-partition of \( G(n, m) \), each of the \( m \) edges can only be selected from the set of edges

\[ E_W = V^2 \setminus \{(u, v) : u \in A, v \in B\}, \]

where \( V^2 \) denotes the set of unordered pair of vertices. Let \( s \) be the size of \( E_W \), we have

\[ s = |E_W| = \frac{n(n-1)}{2} - ba = \frac{n(n-1)}{2} - tn(n - tn - (l + 1)). \]

In the rest of the proof, we will work on the conditional probability space \( \mathcal{P} = (\Omega, \mathbb{P}_{\mathcal{P}} \{\cdot\}) \) where \( \Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m \) and \( \Omega_i = E_W \) for each \( 1 \leq i \leq m \). A sample point \( \omega = (\omega_1, \cdots, \omega_m) \in \Omega \) corresponds to an outcome of selecting \( m \) edges from \( E_W \) uniformly at random and with replacement such that \( \mathbf{W} \) is a balanced \( l \)-partition of the graph determined by \( \omega \). The probability measure \( \mathbb{P}_{\mathcal{P}} \{\cdot\} \) is

\[ \mathbb{P}_{\mathcal{P}} \{\omega\} = \left( \frac{1}{s} \right)^m. \]

The following lemma guarantees that we can obtain Equation (3.9) by studying the probability \( \mathbb{P}_{\mathcal{P}} \{\mathbf{W} \text{ is d-rigid} \} \):
Lemma 3.2
\[ \mathbb{P}_{G(n,m)} \{ W \text{ is } d\text{-rigid} \mid W \text{ is an } l\text{-partition} \} = \mathbb{P}_P \{ W \text{ is } d\text{-rigid} \} . \]

Proof. Recall that \( \mathbb{P}_{G(n,m)} \{ \cdot \} \) is the probability measure for the probability space \( (\Omega, \mathbb{P}_{G(n,m)} \{ \cdot \}) \) and \( \mathbb{P}_P \{ \cdot \} \) is the probability measure for the probability space \( \mathcal{P} = (\Omega, \mathbb{P}_P \{ \cdot \}) \). Note that \( \Omega \) is the set of sample points \( \omega \) in \( \overline{\Omega} \) such that \( W \) is an \( l\)-partition in the graph determined by \( \omega \). Let \( Q \subset \overline{\Omega} \) be the set of sample points \( \omega \) such that \( W \) is \( d\)-rigid in the graph determined by \( \omega \). We have
\[
\mathbb{P}_{G(n,m)} \{ W \text{ is } d\text{-rigid} \mid W \text{ is an } l\text{-partition} \} = \frac{\sum_{\omega \in Q \cap \Omega} \mathbb{P}_{G(n,m)} \{ \omega \}}{\sum_{\omega \in \Omega} \mathbb{P}_{G(n,m)} \{ \omega \}} \quad \text{(definition of conditional probability)}
\]
\[
= \frac{|Q \cap \Omega|}{|\Omega|} = \frac{|Q \cap \Omega|}{s^m}
\]
\[
= \frac{\sum_{\omega \in Q \cap \Omega} \mathbb{P}_P \{ \omega \}}{\sum_{\omega \in \Omega} \mathbb{P}_P \{ \omega \}} \quad \text{(definition of the two probability spaces)}
\]
\[
= \mathbb{P}_P \{ W \text{ is } d\text{-rigid} \} .
\]
This proves the lemma. ■

Continuing the proof of Theorem 4, we need to bound \( \mathbb{P}_P \{ W \text{ is } d\text{-rigid} \} \). To make thing simpler, we will bound the probability that there exist tree components, instead of general connected components, of size at most \( d \) in the subgraph of \( G(n,m) \) induced on the vertex set \( B \). We use the following variate of Hoeffding-Azuma inequality:

Lemma 3.3 (Lemma 1.2 [21] and Theorem 1.19 [5]) Let \( \Omega = \prod_{i=1}^{m} \Omega_i \) be a independent product probability space where each \( \Omega_i \) is a finite set, and \( f : \Omega \to R \) be a random variable satisfying the following Lipschitz condition
\[
|f(\omega) - f(\omega')| \leq c_f \tag{3.10}
\]
if \( \omega, \omega' \in \Omega \) differs only in one coordinate. Then, for any \( t > 0 \),
\[
\mathbb{P} \{ f(\omega) \leq E \{ f(\omega) \} - t \} \leq e^{-\frac{2t^2}{c_f^2 m}}.
\]

In our case, the probability space is \( \mathcal{P} = (\Omega, \mathbb{P}_P \{ \cdot \}) \) and we may use any the function \( f : \Omega \to R \) such that the total number of tree components of size at most \( d \) is larger than zero whenever \( f > 0 \). To achieve the best possible Lipschitz constant \( c_f \) in Equation (3.10), we consider a weighted sum \( I \) of all tree components of size at most \( d \) defined as follows.

For any \( 1 \leq i \leq d \), let \( \mathcal{U}_i = \{ U \subset B : |U| = i \} \) be the collection of size-\( i \) vertex sets in \( B \) and let
\[
\mathcal{U} = \bigcup_{i=1}^{d} \mathcal{U}_i.
\]
For a vertex set $U \in \mathcal{U}$, we use $I_U$ to denote the indicator function of the event that $G[U]$ is a tree component of $G[B]$, i.e., $G[U]$ is a tree and $N(U) \cap (B \setminus U) = \emptyset$. Define

$$I = \sum_{U \in \mathcal{U}} (1 - (|U| - 1)\epsilon) I_U = \sum_{i=1}^d \sum_{U \in \mathcal{U}_i} (1 - (i - 1)\epsilon) I_U$$

where $\epsilon = \frac{1}{d-1}$. The idea is that instead of counting the total number of tree components of size at most $d$, we use the random variable $I$ as a “weighted count” to which the contribution of a tree component on a vertex set of size $i$ is $(1 - (i - 1)\epsilon)$. Note that the constant $\epsilon$ can be made arbitrarily small by taking an arbitrarily large (but constant) $d$. The purpose is to make $|I(\omega) - I(\omega')|$ as close to 1 as possible for every pair $\omega$ and $\omega'$ that differs only on one coordinate.

It is obvious that $I > 0$ if and only if the total number of tree components of size at most $d$ is greater than zero. By the definition of a $d$-rigid triple, we have

$$\mathbb{P}_P \{ W \text{ is } d\text{-rigid} \} \leq \mathbb{P}_P \{ I = 0 \} .$$

By Lemma 3.2 and Lemma 3.3, we have

$$\mathbb{P}_{G(n,m)} \{ W \text{ is } d\text{-rigid} \mid W \text{ is an l-partition} \}
= \mathbb{P}_P \{ I = 0 \mid W \text{ is an l-partition} \}
\leq \mathbb{P}_P \{ I - \mathbb{E}_P [I] \leq -\mathbb{E}_P [I] \}
\leq \left( \frac{1}{e} \right)^{\frac{2\epsilon^2|U|}{c_f^2en}}$$

where $c_f = \max |I(\omega) - I(\omega')|$ with the maximum taken over all pairs of $\omega$ and $\omega'$ in $\Omega$ that differ only on one coordinate. The following lemma bounds $\max |I(\omega) - I(\omega')|$. (Note that if we had used the unweighted sum $I = \sum_{U \in \mathcal{U}} I_U$, the best we can have is max $|I(\omega) - I(\omega')| \leq 2$.)

**Lemma 3.4** For any $\omega, \omega' \in \Omega$ that differ only in one coordinate,

$$|I(\omega) - I(\omega')| \leq 1 + \epsilon.$$

**Proof.** Note that $\omega$ and $\omega'$ represent two possible outcomes of the $m$ independent random experiments that select the $m$ edges of a random graph. If $\omega, \omega' \in \Omega$ differ only in one coordinate, say the $i$-th coordinate, then the edge sets of the corresponding graphs $G_\omega$ and $G_{\omega'}$ only differ in the $i$-th edge.

Let us consider the change of the value of $I$ when we modify $G_\omega$ to $G_{\omega'}$ by removing the $i$-th edge of $G_\omega$ and adding the $i$-th edge of $G_{\omega'}$. First, removing the $i$-th edge can only increase $I$ by $\delta^+(I)$. The maximum increase occurs situations where a tree component $T$ is broken up into two smaller tree components $T_1$ and $T_2$. Suppose that there are $i$ vertices in $T_1$ and $j$ vertices in $T_2$, we have

$$\delta^+_i = (1 - (i - 1)\epsilon) + (1 - (j - 1)\epsilon) - (1 - (i + j - 1)\epsilon) I_{i+j \leq d}.$$
where \( i + j \leq d \) if \( i + j \leq d \) and \( I_{i+j} = 0 \) otherwise. If \( i + j \leq d \), we have
\[
\delta^+_i = (1 - (i - 1)\epsilon) + (1 - (j - 1)\epsilon) - (1 - (i + j - 1)\epsilon) = (1 + \epsilon).
\]
If \( i + j > d \), we have (since \( \epsilon = \frac{1}{d - 1} \))
\[
\delta^+_i = 2 - (i + j - 2)\epsilon = 2 - (i + j - 1)\epsilon + \epsilon < 1 + \epsilon.
\]
Secondly, adding the \( i \)-th edge can only decrease \( I \) by \( \delta^+_i - I \). The maximum decrease occurs in situations where two tree components are merged into a larger one, and \( \delta^+_i - I \leq 1 + \epsilon \) as well.

Therefore, the maximum net change of \( I \) is \( (1 + \epsilon) \) and is achieved when \( \delta^+_i = 1 + \epsilon \) and \( \delta^-_i = 0 \), or \( \delta^+_i = 0 \) and \( \delta^-_i = -(1 + \epsilon) \). Consequently,
\[
|I(\omega) - I(\omega')| \leq 1 + \epsilon.
\]
The proves the lemma.

To complete the proof of Theorem 4, we estimate in the following lemma the expected number of tree components \( E_P[I] \).

**Lemma 3.5** Let \( I = I(\omega) \) be the number of tree components on at most \( d \) vertices in \( G[B] \). We have
\[
E_P[I] \geq te^{-x(t,c)} \left( 1 + \sum_{i=2}^{d} \frac{i^{-2}}{i!} (x(t,c)e^{-x(t,c)})^{i-1} \right) n \quad (3.13)
\]

**Proof.** Let \( U, |U| = i \), be a vertex set in \( \mathcal{U}_i \) and recall that in \( G(n, m) \), the \( m = cn \) edges are selected uniformly at random and with replacement. Conditional on the event that \( W = (S, A, B) \) is a balanced \( l \)-partition, the \( m \) edges are selected from the set \( E_W \) uniformly at random with replacement. Therefore for \( i \geq 2 \), the probability that \( G[U] \) is an induced tree component in \( G[B] \) is
\[
\mathbb{P}_P \{ I_U = 1 \} = \binom{cn}{i-1} i^{-2} \left( \frac{i-1}{s} \cdots \frac{1}{s} \right) \left( 1 - \frac{i(tn-i) + \binom{i}{2}}{s} \right)^{cn-i+1} = e^{-1} n^{-1} i^{-2} \left( \frac{1}{s} \right)^{i-1} \left( 1 - \frac{i(tn-i) + \binom{i}{2}}{s} \right)^{cn-i+1}
\]
For the case of \( |U| = 1 \), the probability \( \mathbb{P}_P \{ I_U = 1 \} \) is the probability that the single vertex in \( U \) is isolated in \( G[B] \), and thus
\[
\mathbb{P}_P \{ I_U = 1 \} = \left( 1 - \frac{(tn-1)}{s} \right)^{cn}.
\]
Since there are \( \binom{tn}{i} \) vertex subsets of size \( i \) in \( B \), the expected number of tree components in \( G[B] \) on at most \( d \) vertices is

\[
\mathbb{E}_P[I] = \sum_{U \in \mathcal{U}} P\{I_U = 1\} = tn \left( 1 - \frac{(tn - 1)}{s} \right)^{cn} + \sum_{i=2}^{d} \binom{tn}{i} c^{i-1} n^{i-1} i^{i-2} \left( \frac{1}{s} \right)^{i-1} \left( 1 - i(tn - i) + \frac{(i)}{s^2} \right)^{cn-i+1}
\]

Since \( s = \frac{n(n-1)}{2} - tn(n - tn - (l+1)) = \frac{(1-2t(1-t))n^2 + tn(l+1)-n}{2} \), we have that for sufficiently large \( n \)

\[
\mathbb{E}_P[I] \geq tn \left( e^{-\frac{2t}{1-2t(1-t)}} + \sum_{i=2}^{d} \frac{t^{i-1}i^{i-2}2^{i-1}}{(2t^2 - 2t + 1)^{i-1}i!} c^{i-1} e^{-\frac{2ict}{2t^2 - 2t + 1}} \right) = te^{-x(t,c)} \left( 1 + \sum_{i=2}^{d} \frac{i^{i-2}}{i!} \left(x(t,c)e^{-x(t,c)}\right)^{i-1} \right) n.
\]

This proves Lemma 3.5.

To complete the proof of Theorem 4, we see that Equation (3.9) follows from Lemma 3.4, Lemma 3.5 and Equation (3.12).

3.2 Proof of Theorem 1

We prove Theorem 1 by applying Markov’s inequality and the upper bound obtained in Section 3.1 on the conditional probability of a \( d \)-rigid and balanced \( l \)-partition.

Let \( l + 1 = \beta n \) where \( \beta > 0 \) is a sufficiently small number to be determined at the end of the proof. Let \( J_1 \) be the total number of balanced \( \beta n \)-partition \( W = (S, A, B) \) such that \( |A| \leq |B| \leq |A| + d \), and let \( J_2 \) be the total number of balanced \( \beta n \)-partition \( W = (S, A, B) \) such that \( |B| > |A| + d \) and \( W \) is \( d \)-rigid.

By Lemma 3.1 if the treewidth of \( G(n, m) \) is at most \( \beta n \), then either \( J_1 > 0 \) or \( J_2 > 0 \). It follows that

\[
\mathbb{P}_{G(n,m)}\{tw(G(n,m)) \leq \beta n\} \leq \mathbb{P}_{G(n,m)}\{J_1 + J_2 > 0\}.
\]

If we can show that \( \mathbb{E}_{G(n,m)}[J_1 + J_2] \) tends to zero as \( n \) goes to infinity, Theorem 1 follows from Markov’s inequality.

Define

\[
\phi_1(t) = (1 - 2t + 2t^2 + 2t\beta + O(1/n))^c,
\]

\[
\phi_2(t) = \left(e^{-\frac{1}{2}t(1+g(t,c))} \right)^c,
\]

\[
\phi(t) = \phi_1(t)\phi_2(t)
\]

For the expectation of \( J_1 \), we have
Lemma 3.6 For any $c > 1$, there is a constant $\beta^*_1 > 0$ such that for any $\beta < \beta^*_1$, 
\[ \lim_{n \to \infty} E_{G(n,m)} [J_1] = 0. \]

Proof. Consider a partition $W = (S, A, B)$ of the vertices of $G(n, m)$ such that $|B| \geq |A|$. Since $|A| + |B| = (1 - \beta)n$, we see that $|B| \leq |A| + d$ if and only if $|B| \leq \frac{(1 - \beta)n + d}{2}$.

The probability that $W$ is a balanced $l$-partition is
\[ P_{G(n,m)} \{ W \text{ is a } \beta n\text{-partition} \} = \left( 1 - \frac{tn(n - tn - \beta n)}{n(n - 1)/2} \right)^{cn} = (1 - 2t + 2t^2 + 2t\beta + O(1/n))^{cn} = \phi_1(t). \] 
For a fixed vertex set $S$, there are $\binom{n-\beta m}{b}$ ways ($\frac{1}{2}n \leq b = |B| \leq \frac{2}{3}n$) to choose the pair $(A, B)$ such that one of them has the size $b$. It follows that
\[ E_{G(n,m)} [J_1] = \binom{n}{\beta n} \sum_{\frac{1}{2}n \leq b \leq \frac{1}{2}(1 - \beta)n + d} \binom{n - \beta n}{b} \left( \phi_1 \left( \frac{b}{n} \right) \right)^n \leq \binom{n}{\beta n} \sum_{\frac{1}{2}n \leq b \leq \frac{1}{2}(1 - \beta)n + d} \binom{n}{b} \left( \phi_1 \left( \frac{b}{n} \right) \right)^n. \]
Since $\binom{n}{b}$ attains its maximum at $b = \frac{n}{2}$ and the function $\phi_1(t)$ is increasing in the interval $[\frac{1 - \beta}{2}, 1]$, we have by Stirling’s formula (Lemma 2.1) that
\[ E_{G(n,m)} [J_1] \leq d \binom{n}{\beta n} \frac{n}{2} \left( \phi_1 \left( \frac{1}{2} \right) \right)^n \leq d \binom{n}{\beta n} 2^n \left( \frac{1}{2} + \beta \right)^cn \leq d \left( \frac{1}{\beta^*(1 - \beta)^{1-\beta}} \right)^n \left( 2 \left( \frac{1}{2} + \beta \right)^c \right)^n. \]
For any $c > 1$, there is some $\beta_1 > 0$ such that $2 \left( \frac{1}{2} + \beta \right)^c < 1$ for any $\beta < \beta_1$. Since $\lim_{\beta \to 0} \frac{1}{\beta^*(1 - \beta)^{1-\beta}} = 1$, there exists some $\beta_2 > 0$ such that $\frac{1}{\beta^*(1 - \beta)^{1-\beta}} \leq (2 \left( \frac{1}{2} + \beta_1 \right)^c)^{-1}$.

Taking $\beta^* = \min(\beta_1, \beta_2)$, we see that for any $\beta < \beta^*$,
\[ E_{G(n,m)} [J_1] \leq d \left( \frac{1}{\beta^*(1 - \beta)^{1-\beta}} \right)^n \left( 2 \left( \frac{1}{2} + \beta_1 \right)^c \right)^n \leq d\gamma^n \]
where $0 < \gamma < 1$. Lemma 3.6 follows.

For the expectation of $J_2$, we need to take into consideration the requirement of being $d$-rigid in order to get a better bound.

Lemma 3.7 For $c = 1.073$, there is a constant $\beta^*_2 > 0$ such that for any $\beta < \beta^*_2$, 
\[ \lim_{n \to \infty} E_{G(n,m)} [J_2] = 0. \]
Proof. Consider a partition $W = (S, A, B)$ of the vertices of $G(n, m)$ such that $|S| = l + 1 = \beta n$, $|B| \geq |A| + d$, $|B| = b = tn$, with $\frac{1-\beta}{3} \leq t \leq \frac{2(1-\beta)}{3}$. Let $I_W$ be the indicator function of the event that $W$ is a $d$-rigid and balanced $l$-partition. We have

$$
E_{G(n,m)}[I_W] = P_{G(n,m)} \{ W \text{ is a $d$-rigid and balanced $\beta n$--partition} \}
= P_{G(n,m)} \{ W \text{ is a balanced $\beta n$--partition} \} \times
P_{G(n,m)} \{ W \text{ is $d$-rigid } | \text{ W is a balanced $\beta n$--partition} \}. \tag{3.16}
$$

From Theorem 4, we know that

$$
P_{G(n,m)} \{ W \text{ is $d$-rigid } | \text{ W is a balanced $\beta n$--partition} \} \leq e^{-r(1+g)^2n}.
$$

By the definition of a balanced partition,

$$
P_{G(n,m)} \{ W \text{ is a balanced $\beta n$--partition} \} = \left(1 - \frac{tn(n - tn - \beta n)}{n(n - 1)/2}\right)^{cn}
= \phi_1(t). \tag{3.17}
$$

For a fixed vertex set $S$ with $|S| = \beta n$, there are $\binom{n-\beta n}{b}$ ways ($\frac{1}{2} n \leq b \leq \frac{2}{3} n$) to choose the pair $(A, B)$ such that $|B| = b$. Therefore,

$$
E_{G(n,m)}[J_2] = \sum_W E_{G(n,m)}[I_W]
\leq \binom{n}{\beta n} \sum_{\frac{1}{2} n \leq b \leq \frac{2}{3} n} \binom{n-\beta n}{b} \left(\frac{\phi(b)}{n}\right)^n
\leq \binom{n}{\beta n} \sum_{\frac{1}{2} n \leq b \leq \frac{2}{3} n} \binom{n}{b} \left(\frac{\phi(b)}{n}\right)^n.
$$

By Lemma 2.1, we have for $n$ large enough

$$
E_{G(n,m)}[J_2] \leq \left(\frac{1}{(\beta \beta (1-\beta)^{1-\beta})}\right)^n \sum_{\frac{1}{2} n \leq b \leq \frac{2}{3} n} \left(\phi_1(\frac{b}{n})\phi_2(\frac{b}{n})\right)^n.
$$

Recall that

$$
\phi_2(t) = \left(e^{1/2r(t,c)(1+g(t,c))2}c\right),
$$

and see Equation (3.8) for the definition of $r(t,c)$ and $g(t,c)$. By Lemma 2.3, $r(t)$ and $g(t)$ are decreasing on $[\frac{1-\beta}{2}, \frac{2}{3}]$. Consequently $\phi_2(t)$ is increasing on $[\frac{1-\beta}{2}, \frac{2}{3}]$. It follows that

$$
\phi_2\left(\frac{b}{n}\right) \leq \phi_2\left(\frac{2}{3}\right).
$$

By Lemma 2.4

$$
\frac{\phi_1\left(\frac{b}{n}\right)}{\frac{b}{n}^{\frac{b}{n} (1-\frac{b}{n})^{1-\frac{b}{n}}}} \leq \phi_1\left(\frac{2}{3}\right) \left(\frac{\phi\left(\frac{2}{3}\right)}{(\frac{3}{5})^\frac{2}{3} (\frac{1}{5})^\frac{1}{3}}\right)^{\frac{2}{3} (\frac{1}{4})^\frac{1}{3}}.
$$
Therefore,  
\[ E_{G(n,m)} [J_2] \leq O(n) \left( \frac{1}{\beta \beta (1 - \beta)^{1-\beta}} \right)^n \left( \frac{(\frac{5}{3} + \frac{4}{3} \beta) \phi_2(\frac{2}{\beta})}{(\frac{2}{3})^\frac{2}{3} (\frac{1}{3})^\frac{1}{3}} \right)^n. \]  
(3.18)

Consider the function 
\[ z(\beta, \epsilon, c) = \frac{(\frac{5}{3} + \frac{4}{3} \beta) \phi_2(\frac{2}{\beta})}{(\frac{2}{3})^\frac{2}{3} (\frac{1}{3})^\frac{1}{3}}. \]

Numerical calculations using MATLAB shows that for \( c = 1.073, \beta = 0, \) and \( \epsilon = 0, \) we have 
\[ z(0, 0, 1.073) < 1. \]

Since \( z(\beta, \epsilon, 1.073) \) is continuous in \( \beta \) and \( \epsilon \) on \([0, 1]\), there exist constants \( \beta_1 > 0 \) and \( \epsilon_1 > 0 \) such that 
\[ z(\beta_1, \epsilon, 1.073) < 1, \forall \epsilon < \epsilon_1. \]

By Lemma 2.4, there exits a constant \( \beta_2 > 0 \) such that 
\[ \frac{1}{\beta^2 (1 - \beta)^{1-\beta}} < \frac{1}{z(\beta_1, \epsilon_1, 1.073)}, \forall \beta \leq \beta_2. \]

Let \( \beta^* = \min(\beta_1, \beta_2). \) It follows that for any \( \beta < \beta^* \) and \( \epsilon < \epsilon_2, \)
\begin{align*}
E_{G(n,m)} [J_2] &\leq O(n) \left( \frac{1}{\beta^* \beta (1 - \beta)^{1-\beta}} \right)^n \frac{z(\beta^*, \epsilon^*, 1.073)}{z(\beta_1, \epsilon, 1.073)} \leq \frac{1}{\beta^*_2 (1 - \beta_2)^{1-\beta_2}} \frac{z(\beta_1, \epsilon, 1.073)}{z(\beta_1, \epsilon, 1.073)} \leq O(n) \gamma^n. \quad (3.19)
\end{align*}

for some constant \( 0 < \gamma < 1. \) This proves Lemma 3.7. \( \blacksquare \)

It follows from Equation (3.14) that for any \( \beta \leq \beta^*, \)
\[ \lim_n P_{G(n,m)} \{ tw(G(n,m)) \leq \beta n \} = 0, \text{ if } \frac{m}{n} = 1.073. \]

Since the property that the treewidth of a graph is greater \( \beta n \) is a monotone increasing graph property, we have that for any \( c \geq 1.073, \)
\[ \lim_n P_{G(n,m)} \{ tw(G(n, cn)) \leq \beta n \} = 0. \]

Theorem 1 follows. \( \blacksquare \)

4 Treewidth of Random Intersection Graphs: Proof of Theorems 2

Let \( p = \frac{c}{m}. \) Consider a balanced triple \( W = (S, A, B) \) with \( |S| = \beta n \) and \( |A| = tn. \) We upper bound the probability that \( W \) is a balanced \( \beta n \)-partition and then use Markov’s
inequality. By the definition of random intersection graphs, there is no edge between the two vertex sets $A \setminus S$ and $B \setminus S$ if and only if

$$ e \notin \left( \bigcup_{v \in A \setminus S} S_v \right) \cap \left( \bigcup_{v \in B \setminus S} S_v \right), \forall e \in M, \tag{4.20} $$

which in turn is equivalent to the following: for every $e \in M$,

$$ e \notin S_v, \forall v \in A \setminus S, \text{ or } e \notin S_v, \forall v \in B \setminus S. \tag{4.21} $$

Since $S_v$’s are formed independently and since $P\{e \in S_v\} = p$ for any $e \in M$ and $v \in V$, the probability for the event in Equation (4.21) to occur is

$$ \left( (1-p)^{an} + (1-p)^{bn} - (1-p)^{(1-\beta)n}\right)^m. $$

It follows that

$$ P\{W \text{ is a balanced } \beta n\text{-partition} \} = \left( (1-p)^{an} + (1-p)^{bn} - (1-p)^{(1-\beta)n} \right)^m. \tag{4.22} $$

There are $\binom{n}{\beta n}$ ways to choose $S$ and for each fixed $S$, there are $\binom{n-\beta n}{tn}$ ways to choose $A$ with $|A| = tn$. Since the treewidth of $G_I(n, m, p)$ is at most $\beta n$ implies that there is a balanced $\beta n$-partition, we have by Markov’s inequality that for $p \geq \frac{c}{m}, c > 2$,

$$ P\{tw(G_I(n, m, p)) \leq \beta n \} \leq P\{\text{There exists a balanced } \beta n\text{-partition} \} \leq O(1)n\left(\frac{n}{\beta n}\right) \sum_{\frac{1}{2}n \leq a \leq \frac{1}{2}n} \left( \frac{n}{a} \right)^{an} \left(1 + (1-p)^{(b-a)} - (1-p)^{(1-\beta)n-a-b}\right)^m \leq O(1)n\left(\frac{n}{\beta n}\right) \left( \frac{1}{\frac{a}{n}} \right)^{\frac{2}{c}} \left( \frac{1}{1-\frac{a}{n}} \right)^{\frac{1}{1-\frac{a}{n}}} \leq O(1)n\left(\frac{n}{\beta n}\right) \left( \frac{\frac{2}{c}}{\frac{2}{3}} \right)^{\frac{1}{3}} \left( \frac{\frac{1}{2}}{\frac{1}{2}} \right)^{\frac{1}{2}} $$

where last inequality is because the function $\frac{\left(\frac{1}{2}\right)^{\frac{2}{c}}}{\left(\frac{2}{3}\right)^{\frac{1}{3}}} \left( \frac{1}{1-\frac{a}{n}} \right)^{\frac{1}{1-\frac{a}{n}}}$ is decreasing on $[\frac{1}{3}, \frac{1}{2}]$ for any $c > 2$.

Note that $\frac{\left(\frac{1}{2}\right)^{\frac{2}{c}}}{\left(\frac{2}{3}\right)^{\frac{1}{3}}} \left( \frac{1}{1-\frac{a}{n}} \right)^{\frac{1}{1-\frac{a}{n}}} < 1$. Therefore, for sufficiently small $\beta$, we have

$$ \lim_{n \to \infty} P\{tw(G_I(n, m, p)) \leq \beta n \} = 0. $$

This proves Theorem 2. ■
5 The Barabási-Albert Model: Proof of Theorem 3

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices in $G_S(n, m)$ and $V_i = \{v_1, \ldots, v_i\}$. Without loss of generality, assume that the vertices are added to $G_S(n, m)$ in this order in the iterative construction of $G_S(n, m)$. Let $I_1$ be the first half of the vertices, i.e., $I_1 = \{v_1, v_2, \ldots, v_{\frac{s}{2}}\}$, and $I_2$ be the second half $\{v_{\frac{n+1}{2}}, \ldots, v_n\}$.

Let $W = (S, A, B)$ be a balanced triple of disjoint vertex subsets such that $|S| = \beta n$. (See Definition 3.1 for the details). Write $|A| = an$ and $|B| = bn$. Assume, without loss of generality, that $|A| \leq |B|$ so that $\frac{1-\beta}{3} \leq a \leq \frac{1-\beta}{2}$. Considering the way in which $A$ and $B$ intersect with $I_1$ and $I_2$, let us write

$$|I_1 \cap A| = sn, \quad |I_2 \cap A| = (a-s)n;$$
$$|I_1 \cap B| = tn, \quad |I_2 \cap B| = (b-t)n;$$

(5.23)

where $s$ and $t$ shall satisfy

$$0 \leq s \leq \frac{1-\beta}{2}, \quad s + t = \frac{1-\beta}{2}.$$ 

We upper bound the probability $\mathbb{P}_{G_S(n,m)} \{W \text{ is a balanced } \beta n\text{-partition} \}$. Let $E$ be the event that $W$ is a balanced $\beta n$-partition, and focus on what happens when the second half of the vertices, i.e., those in $I_2$, are added to $G_S(n, m)$. Define the following events

$$E_i = \begin{cases} 
\{N(v_i) \cap (I_1 \cap B) = \emptyset\}, & \text{if } v_i \in I_2 \cap A \\
\{N(v_i) \cap (I_1 \cap A) = \emptyset\}, & \text{if } v_i \in I_2 \cap B 
\end{cases} \quad (5.24)$$

We have

$$E \subset E_{\frac{n}{4}+1} \cap \cdots \cap E_n.$$ 

Therefore,

$$\mathbb{P}_{G_S(n,m)} \{E\} \leq \mathbb{P}_{G_S(n,m)} \{E_{n/2+1} \cap \cdots \cap E_n\}. \quad (5.25)$$

The following lemma bounds the conditional probability of $E_i$ given $G_S(n, m)[V_{i-1}]$.

Lemma 5.1

$$\mathbb{P}_{G_S(n,m)} \{E_i \mid G_S(n,m)[V_{i-1}]\} \leq \begin{cases} 
(1 - \frac{s}{2})^m, & \text{if } v_i \in I_2 \cap B \\
(1 - \frac{t}{2})^m, & \text{if } v_i \in I_2 \cap A 
\end{cases} \quad (5.26)$$

Proof. Consider a vertex $v_i \in I_2 \cap B$ (The case that $v_i \in I_2 \cap A$ is similar). The total vertex degree of $G_S(n, m)[V_{i-1}]$ is $2(i-1)m \leq 2nm$. The total vertex degree of the vertices in $I_1 \cap A$ is at least $snm$. Note that the event $E_i$ occurs implies that none of the vertices in $I_1 \cap A$ is selected as the neighbor of $v_i$ in the $m$-step procedure to pick $v_i$'s neighbors.
By the definition of preferential attachment mechanism in the Barabási-Albert model, Equation (2.7), we have that

\[
P_{G_S(n,m)} \{ E_i \mid G_S(n,m)[V_{i-1}] \} \leq (1 - \frac{s \cdot 2nm}{2(i-1)m})^m = (1 - \frac{s}{2})^m.
\]

This proves the lemma. ■

Continue the proof of Theorem 3. From Lemma, we have

\[
P_{G_S(n,m)} \{ E \} \leq \prod_{i=n/2+1}^n P_{G_S(n,m)} \{ E_i \mid G_S(n,m)[V_{i-1}] \}
\]

\[
\leq (1 - \frac{s}{2})^{2n} \prod_{i=2}^{n} \left( 1 - \frac{s}{2} \right) = (1 - \frac{s}{2})^{2n}.
\]

Taking into consideration that \( a + b = (1 - \beta)n \), we see that

\[
P_{G_S(n,m)} \{ E \} \leq \left( 1 - \frac{s}{2} \right)^{2n} \prod_{i=2}^{n} \left( 1 - \frac{s}{2} \right) = \left( 1 - \frac{s}{2} \right)^{2n} \prod_{i=2}^{n} \left( 1 - \frac{s}{2} \right).
\]

(5.27)

Consider the behavior of the function

\[
f(s, \beta) = (1 - \frac{s}{2})^{s-a+\frac{1-\beta}{2}} \left( \frac{3}{4} + \frac{s}{2} \right)^{a-s}
\]

(5.29)

for \( 0 \leq s \leq \frac{1}{2} \) and \( \frac{1-\beta}{3} \leq a \leq \frac{1-\beta}{2} \). We have

**Lemma 5.2** There is a constant \( \beta^* > 0 \) such that for any \( \beta < \beta^* \),

\[
f_{\max} = \max \{ f(s, \beta) : s \in [0, 1/2], a \in [(1 - \beta)/3, (1 - \beta)/2] \} < 0.9425.
\]

(5.30)

**Proof.** Note that the last term \( (1 - s/2)^{-\beta/2} \) of \( f(s, \beta) \) can be made arbitrarily to 1 by requiring that \( \beta \) is less than a sufficiently small number, say \( \beta_0 \). We, therefore, only need to consider the function

\[
f(s) = \left( 1 - \frac{s}{2} \right)^{s-a} \left( 1 - \frac{s}{2} \right)^{1/2}.
\]
First, we claim that $f(s) \leq \left(\frac{7}{5}\right)^{\frac{3}{2}}$ for any $s \in \left[\frac{1}{4}, \frac{1}{2}\right]$. To see this, we take the logarithm on both sides of Equation (5.29) to obtain

$$\log f(s) = (s-a-\frac{1}{2})\log(1-\frac{s}{2}) + (a-s)\log(\frac{3}{4} + \frac{s}{2}).$$

Taking derivative on both sides in the above, we get

$$\frac{1}{f(s)} f'(s) = \log \left(\frac{s}{4} + \frac{s}{2}\right) - \frac{1}{2} \left(\frac{2s - \frac{7}{4}a + \frac{3}{8}}{(1 - \frac{s}{2})(\frac{3}{4} + \frac{s}{2})}\right).$$

Since for any $s \geq \frac{1}{4}$ and $\frac{1-\beta}{3} \leq a \leq \frac{1-\beta}{2}$, $\frac{1-\frac{s}{2}}{4 + \frac{s}{2}} \leq 1$ and $2s - \frac{7}{4}a + \frac{3}{8} > 0$, we see that $f'(s) < 0$. The claim holds since $f(\frac{1}{4}) = \left(\frac{7}{5}\right)^{\frac{3}{2}} = 0.9354$.

Now consider the interval $[0, \frac{1}{4}]$. Let $\beta_1$ be a constant such that for any $\beta < \beta_1$, $\frac{1}{4} - \frac{1-\beta}{3} < 0$. Split $[0, \frac{1}{4}]$ into $d+1$ segments and consider the $(d+1)$ intervals $[s_i, s_{i+1}]$ where $s_i = \frac{i}{2d}$, $0 \leq i \leq d$. Since $g(s) = \left(\frac{1-s/2}{4 + s/2}\right)^{s-a}$ is decreasing in $[0,1/2]$, $s-a < s - \frac{1}{3} < 0$ for any $s \in [0,1/4]$ and $a \in [(1-\beta)/3, (1-\beta)/2]$, and $h(s) = (1-s/2)^{\frac{3}{2}}$ is decreasing in $[0,1/4]$, we have

$$\max_{s \in [0,1/4]} f(s) = \max_{0 \leq i \leq d} \left\{ \max_{s \in [s_i,s_{i+1}]} f(s) \right\}$$

\[
\leq \max_{0 \leq i \leq d} (g(s_{i+1}) h(s_i)).
\]

Numerical calculations\(^2\) using $d = 10$ gives us $\max_{0 \leq i \leq d} (g(s_{i+1}) h(s_i)) < 0.9425$. Take $\beta^{\ast} = \min\{\beta_0, \beta_1\}$, we get Equation (5.30).

To complete the proof of Theorem 3, we see from Markov inequality that the expected number of balanced $\beta$-partition is at most

$$\left(\frac{n}{\beta n}\right)^n \left(\frac{n}{a}\right)^n \left(1 - \frac{s}{2}\right)^{s-a+\frac{1}{2}} \left(\frac{3}{4} + \frac{s}{2}\right)^{a-s} \leq \left(\frac{n}{\beta n}\right)^n \left(\frac{n}{an}\right)^n 0.9425^{mn}.$$

Numerical calculation shows that $0.9425^{12} < 1$. Since $an \leq \frac{1}{2} an$, we have by Lemma 2.1 and Lemma 2.4 that there is a constant $\beta_2$ such that for any $\beta < \beta_2$,

$$\lim_{n \to \infty} \left(\frac{n}{\beta n}\right)^n \left(\frac{n}{an}\right)^n 0.9425^{mn} = 0.$$

Let $\beta = \min\{\beta^{\ast}, \beta_2\}$ where $\beta^{\ast}$ is the constant required in Lemma 5.30. It follows that for any $m \geq 12$, the expected number of balanced $\beta n$-partitions in $G_S(n,m)$ tends to zero, and consequently

$$\lim_{n \to \infty} \mathbb{P}_{G_S(n,m)} \{tw(G_S(n,m)) > \beta n\} = 1.$$

This completes the proof of Theorem 3.

\(^2\)We also tried $d$ up to 10, 000, and found out that the value seems to converge to 0.9424
A Proof of Proposition 2.1

The result actually holds for any monotone increasing combinatorial property in random discrete structures, as has been observed in [17, 1] and formally proved in [18]. For completeness, we give an alternative pure measure-theory style proof here.

Recall that a random graph can be identified with a properly-defined probability space. The Erdős-Rényi random graph $G(n, m)$ corresponds to the probability space $(\Omega_m, \mathbb{P}_{G(n,m)} \{\cdot\})$ where $\Omega$ is the collection of the $\binom{N}{m}$ subsets of $m$ edges ($N = \binom{n}{2}$), and $\mathbb{P}_{G(n,m)} \{\cdot\}$ is

$$\mathbb{P}_{G(n,m)} \{\omega\} = \frac{1}{\binom{N}{m}}, \forall \omega \in \Omega.$$  \hspace{1cm} (A.31)

Each sample point $\omega \in \Omega_m$ corresponds to a set of $m$ edges selected uniformly at random without replacement from the $N$ potential edges.

The random graph $G(n, m)$, where the $m$ are selected uniformly at random, but with replacement, can be identified with the following probability space $(\overline{\Omega}, \mathbb{P}_{G(n,m)} \{\cdot\})$ where

1. $\overline{\Omega}_m = \prod_{i=1}^{m} \mathcal{E}_i$ where each $\mathcal{E}_i$ is the set of all $\binom{n}{2}$ possible edges. A sample point $\overline{\omega} = \{\omega_i, 1 \leq i \leq m\} \in \overline{\Omega}_m$ corresponds to a multi-graph with $m$ edges.

2. The probability measure $\mathbb{P}_{G(n,m)} \{\cdot\}$ is

$$\mathbb{P}_{G(n,m)} \{\omega\} = \left(\frac{1}{\binom{n}{2}}\right)^m.$$

Each sample point $\omega \in \overline{\Omega}_m$ is an outcome of the random experiment of selecting $m$ edges independently and uniformly at random with replacement from the set of all possible edges. Also note that the graph represented by a sample point in $\overline{\Omega}_m$ is actually a multi-graph, i.e., there are may be more than one edges between a pair of vertices.

Let $\beta > 0$ be a fixed constant. Let $\overline{Q}_m \subset \overline{\Omega}_m$ be the set of sample points $\overline{\omega}$ such that the treewidth of the multi-graph determined by $\overline{\omega}$ is greater than $\beta n$, and let $Q_m \subset \Omega_m$ be the set of sample points $\omega$ such that the treewidth of the simple graph determined by $\omega$ is greater than $\beta n$.

For each $\overline{\omega} \subset \overline{\Omega}_m$, let $r(\overline{\omega}) \in \Omega_{|r(\overline{\omega})|}$ be the set of distinct edges that $\overline{\omega}$ has, and let $E_i = \{\overline{\omega} \in \overline{\Omega}_m : |r(\overline{\omega})| = i\}$ be the set of sample points in $\overline{\Omega}$ that have exactly $i$ distinct edges. For each sample point $\omega \in \Omega_i$, define

$$T_i(\omega) = \{\overline{\omega} \in \overline{\Omega}_m : r(\overline{\omega}) = \omega\}.$$

We claim that $\{T_\omega : \omega \in \Omega_i\}$ satisfies the following

$$\bigcup_{\omega \in \Omega_i} T_i(\omega) = E_i; \hspace{1cm} (A.33)$$
\( T_i(\omega_1) \cap T_i(\omega_2) = \emptyset, \forall \omega_1, \omega_2 \in \Omega_i; \) \hspace{1cm} (A.34)

\( |T_i(\omega_1)| = |T_i(\omega_2)|, \forall \omega_1, \omega_2 \in \Omega_i. \) \hspace{1cm} (A.35)

If there is an \( \omega \) that belongs to both \( T_i(\omega_1) \) and \( T_i(\omega_2) \), then it must be the case that \( \omega_1 = \omega_2 \). Therefore, \( T_i(\omega_1) \cap T_i(\omega_2) = \emptyset, \forall \omega_1, \omega_2 \in \Omega_i \). To see that \( |T_i(\omega_1)| = |T_i(\omega_2)| \), note that any one-to-one mapping \( \text{map}(\cdot) \) between the two sets of edges \( \omega_1 \) and \( \omega_2 \) defines a one-to-one mapping between \( T_i(\omega_1) \) and \( T_i(\omega_2) \).

From Equations (A.33) through (A.35), the additive property of a probability measure, and the fact that \( |\Omega_i| = \binom{N}{i} \), we see that for any \( T_i(\omega) \),

\[ P_{G(n,m)} \{ T_i(\omega) \} = \frac{P_{G(n,m)} \{ E_i \}}{\binom{N}{i}}. \] \hspace{1cm} (A.36)

Since parallel edges have no impact on treewidth, we have

either \( T_i(\omega) \cap \overline{Q}_m = \emptyset \) or \( T_i(\omega) \subset \overline{Q}_m \), \hspace{1cm} (A.37)

and consequently

\[ \bigcup_{\omega \in Q_i} T_i(\omega) = \overline{Q}_m \cap E_i. \] \hspace{1cm} (A.38)

We have

\[ P_{G(n,m)} \{ \overline{Q}_m \} = \sum_{i=1}^{m} P_{G(n,m)} \{ \overline{Q}_m \cap E_i \} \]

\[ = \sum_{i=1}^{m} \sum_{\omega \in Q_i} P_{G(n,m)} \{ T_i(\omega) \} \quad \text{(due to (A.34), (A.37), and (A.38))} \]

\[ = \sum_{i=1}^{m} \frac{|Q_i|}{\binom{N}{i}} \frac{P_{G(n,m)} \{ E_i \}}{\binom{N}{i}} \quad \text{(due to (A.36))} \]

\[ = \sum_{i=1}^{m} P_{G(n,i)} \{ Q_i \} P_{G(n,m)} \{ E_i \} \quad \text{(due to (A.31))} \]

\[ \leq P_{G(n,m)} \{ Q_m \} \sum_{i=1}^{m} P_{G(n,m)} \{ E_i \} \]

\[ = P_{G(n,m)} \{ Q_m \} \quad \text{(A.39)} \]

where the second last inequality is due to the fact that the graph property represented by the set of sample points \( Q_i \) is monotone increasing and Theorem 2.1 in [5] on the probability of monotone increasing properties in the Erdős-Rényi random graph \( G(n, m) \). This completes the proof of the proposition. \[ \blacksquare \]

B Proof of Lemmas 2.2, 2.3 and 2.4

B.1 Proof of Lemma 2.2

Taking derivative on both sides of

\[ \log f(t) = t \log t + (1 - t) \log(1 - t), \]
we see that \( f(t) \) is increasing on \((0, \frac{1}{2})\) and decreasing on \([\frac{1}{2}, 1]\). The lemma follows.

\[\square\]

**B.2 Proof of Lemma 2.3**

To show that the function
\[
r(t) = \frac{2t^2}{(1 + \epsilon)(1 - \epsilon)} \left(1 \right) \frac{4c t}{t - 1}\]

is decreasing in \( t \) on the interval \([\frac{1-\beta}{2}, \frac{2}{3}]\), we show that its derivative \( r'(t) < 0, \forall t \in \left(\frac{1-\beta}{2}, \frac{2}{3}\right)\). To this end, we take the derivative of the logarithm of \( r(t) \)

\[
\log(r(t)) = 2\log(t) - \frac{4ct}{1 - 2t + 2t^2} - \log((1 + \epsilon)c)
\]
to get

\[
\frac{1}{r(t)} r'(t) = \frac{2(1 - 2t + 2t^2)^2 - 4c(t - 2t^2 + 2t^3) - 4c(-2t^2 + 4t^3)}{t(1 - 2t + 2t^2)^2}.
\]

Since \( r(t) > 0 \) and \( t(1 - 2t + 2t^2)^2 > 0 \), we only need to show that the numerator of the right-hand side in the above, i.e., the function

\[
h(t) = 2(1 - 2t + 2t^2)^2 - 4c(t - 2t^2 + 2t^3) - 4c(-2t^2 + 4t^3).
\]

is less than zero.

Note \( h\left(\frac{1}{2}\right) = \frac{1}{2} - c < 0 \) and \( h\left(\frac{2}{3}\right) = \frac{50}{81} - \frac{144}{81}c < 0 \). As \( h(t) \) is continuous, we have that for sufficiently small \( \beta > 0 \), \( h\left(\frac{1-\beta}{2}\right) < 0 \) as well. It is thus sufficient to show that \( h(t) \) itself is monotone. The first and second derivatives of the function \( h(t) \) are respectively

\[
h'(t) = 4(-2 + 8t - 12t^2 + 8t^3) - 4c(1 - 8t + 18t^2)
\]

and

\[
h''(t) = 4[(8 - 24t + 24t^2) - c(-8 + 36t)].
\]

Note that as a quadratic polynomial, \( h''(t) = 4(24t^2 - (24 + 36c)t + 8(1 + c)) \) can be shown to be always less than 0 for any \( t \in \left[\frac{1}{2}, \frac{2}{3}\right] \). As \( h'(t) \) is continuous and \( h'(\frac{1}{2}) = -4c\left(1 + \frac{1}{2}\right) < 0 \), we see that for sufficiently small \( \beta > 0 \), \( h\left(\frac{1-\beta}{2}\right) < 0 \) as well. It follows that \( h'(t) < 0, \forall t \in \left[\frac{1-\beta}{2}, \frac{2}{3}\right] \). Therefore \( h(t) \) is monotone as required.

\[\square\]

**B.3 Proof of Lemma 2.4**

First, since both \( 1 - 2t + 2t^2 + 2\beta t \) and \( \frac{1}{r(t(1-t))} \) are increasing on the interval \([\frac{1-\beta}{2}, \frac{1}{2}]\), we have that

\[
g(t) = \frac{(1 - 2t + 2t^2 + 2\beta t)c}{t^t(1-t)^{1-t}}
\]
is increasing on the interval \([\frac{1-\beta}{2}, \frac{1}{2}]\).

Focusing now on the interval \([\frac{1}{2}, \frac{2}{3}]\), let us consider the logarithm of the function \( g(t) \),

\[
h(t) = \log g(t) = c\log(1 - 2t + 2t^2 + 2\beta t) - t \log t - (1 - t) \log(1 - t).
\]

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The derivative of $h(t)$ is
\[
h'(t) = c \frac{-2 + 4t + 2\beta}{1 - 2t + 2t^2 + 2\beta t} - \log t + \log(1 - t)
\]
and $h'(\frac{1}{2}) \geq 0$. The second-order derivative of $h(t)$ is
\[
h''(t) = \frac{c}{(1 - 2t + 2t^2 + 2\delta t)^2} \times z(t, \delta)
\]
where
\[
z(t, \beta) = 4(1 - 2t + 2t^2 + 2\beta t)(1 - t)t - (4t - 2 + 2\beta)^2(1 - t)t - (1 - 2t + 2t^2 + 2\beta t)^2.
\]
First, assume that $\beta = 0$. On the interval $[\frac{1}{2}, \frac{5}{3}]$, we have
\[
(4t - 2 + 2\beta)^2 \leq (4 \times \frac{2}{3} - 2)^2 = \frac{4}{9},
\]
\[
\frac{2}{9} \leq t(1 - t) \leq \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}
\]
and
\[
\frac{1}{2} \leq (1 - 2t + 2t^2 + 2\beta t)^2 \leq (1 - 2 \times \frac{2}{3} + 2 \times (\frac{2}{3})^2)^2 = \frac{5}{9}.
\]
It follows that
\[
z(t, \beta = 0) \geq 4 \times \frac{1}{2} \times \frac{2}{9} - \frac{1}{9} - (\frac{5}{9})^2 = \frac{2}{81} > 0.
\]
Since the family of functions $z(t, \beta), \beta > 0$ are uniformly continuous on $[\frac{1}{2}, \frac{5}{3}]$, we have that for small enough $\beta$, $z(t, \beta) > 0, \forall t \in [\frac{1}{2}, \frac{5}{3}]$. It follows that the second-order derivative $h''(t)$ is always greater than zero. Since $h''(\frac{1}{2}) > 0$, we have that $h'(t) > 0, \forall t \in [\frac{1}{2}, \frac{5}{3}]$. It follows that $h(t)$ is increasing. Consequently, $g(t)$ is also increasing since $g(t) > 1, \forall t \in [\frac{1}{2}, \frac{5}{3}]$. 

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References

[1] D. Achlioptas. Threshold Phenomena in Random Graph Colouring and Satisfiability. PhD thesis, Department of Computer Science, University of Toronto, Toronto, Canada, 1999.

[2] D. Achlioptas and E. Friedgut. A sharp threshold for k-colorability. Random Structures and Algorithms, 14(1):63–70, 1999.
[3] R. Albert and A. Barabási. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74(47), 2002.

[4] H. L. Bodlaender. A tourist guide through treewidth. Technical report, Technical Report RUU-CS-92-12, Department of Computer Science, Utrecht University, 1992.

[5] B. Bollobas. *Random Graphs*. Cambridge University Press, 2001.

[6] B. Bollobas, O. Riordan, J. Spencer, and G. Tusnady. The degree sequence of a scale-free random graph process. *Random Structures and Algorithms*, 18:279–290, 2001.

[7] C. Cooper, R. Klasing, and M. Zito. Lower bounds and algorithms for dominating sets in web graphs. *Internet Mathematics*, 2(3):275–300, 2005.

[8] V. Dalmau, P. Kolaitis, and M. Y. Vardi. Constraint satisfaction, bounded treewidth, and finite-variable logics. In *Proceedings of Principles and Practices of Constraint Programming (CP-2002)*, pages 310–326. Springer, 2002.

[9] R. Dechter and Y. Fattah. Topological parameters for time-space tradeoff. *Artificial Intelligence*, 125(1-2):93–118, 2001.

[10] J. Díaz, L. Kirousis, D. Mitsche, and X. Pérez-Giménez. On the satisfiability threshold of formulas with three literals per clause. *Theoretical Computer Science*, 410:2920–2934, 2009.

[11] P. Erdős and A. Renyi. On the evolution of random graphs. *Publ. Math. Inst. Hungar. Acad. Sci.*, 5:17–61, 1960.

[12] A. Ferrante, G. Pandurangan, and K. Park. On the hardness of optimization in power-law graphs. *Theoretical Computer Science*, 393(1-3):220–230, 2008.

[13] E. Friedgut. Sharp thresholds of graph properties, and the k-SAT problem. *J. Amer. Math. Soc.*, 12:1017–1054, 1999.

[14] Y. Gao. On the threshold of having a linear treewidth in random graphs. In *Proceedings of 12th Annual International Conference on Computing and Combinatorics (COCOON’06)*, pages 226–234, 2006.

[15] Y. Gao. The degree distribution of random k-trees. *Theoretical Computer Science*, 410(8-10):688–695, 2009.

[16] M. Karoński, E. Scheinerman, and K. Singer-Cohen. On random intersection graphs: The subgraph problem. *Combinatorics, Probability, and Computing*, pages 131–159, 1999.

[17] L. Kirousis, P. Kranakis, D. Krizanc, and Y. Stamatiou. Approximating the unsatisfiability threshold of random formulas. *Random Structures and Algorithms*, 12(3):253–269, 1994.

[18] L. Kirousis and Y. Stamatiou. An inequality for reducible, increasing properties of randomly generated words. Technical Report TR-96.10.34, Computer Technology Institute, University of Patras, Patras, Greece, 1996.
[19] T. Kloks. *Treewidth: Computations and Approximations*. Springer-Verlag, 1994.

[20] S. Lattanzi and D. Sivakumar. Affiliation networks. In *Proceedings of the 41st ACM Annual Symposium on Theory of Computing*, pages 427–434, 2009.

[21] C. McDiarmid. On the method of bounded differences. In *Surveys in Combinatorics*, London Mathematical Society Lecture Note Series, vol. 141, pages 148–188. Cambridge Univ. Press, 1989.

[22] S. Nikoletseas, C. Raptopoulos, and P. Spirakis. Large independent sets in general random intersection graphs. *Theoretical Computer Science*, pages 215–224, 2008.

[23] D. West. *Introduction to Graph Theory*. Prentice Hall, 2001.