Multi-Component Quantum Hall Systems: The Sum of Their Parts and More

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Abstract

The physics of the fractional quantum Hall effect is the physics of interacting electrons confined to a macroscopically degenerate Landau level. In this Chapter we discuss the theory of the quantum Hall effect in systems where the electrons have degrees of freedom in addition to the two-dimensional orbital degree of freedom. We will be primarily interested in the situation where a finite number of states, most-commonly two, are available for each orbital state within a degenerate Landau level and will refer to these systems as multi-component systems. Physical realizations of the additional degree of freedom include the electron spin, the valley index in multi-valley semiconductors, and the layer index in multiple-quantum-well systems. The consideration of multi-component systems expands the taxonomy of incompressible states and fractionally charged excitations, and for example, leads to the appearance of fractions with even denominators. More interestingly, it also leads us to new physics, including novel spontaneously broken symmetries and in some cases, finite temperature phase transitions. We present an introduction to this rich subject.

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# Contents

I  Introduction  
II Multi-Component Wave Functions  
III Chern Simons Effective Field Theory  
IV Fractional Charges in Double-Layer Systems  
V Collective Modes in Double-Layer Quantum Hall Systems  
VI Broken Symmetries  
VII Field Theoretic Approach  
VIII Interlayer Coherence in Double Layer Systems  
A Experimental Indications of Interlayer Phase Coherence  
B Effective action for double layer systems  
C Superfluid Dynamics  
D Merons: Charged Vortex Excitations  
E Kosterlitz-Thouless Phase Transition  
IX Tunneling Between the Layers  
X Parallel Magnetic Field in Double Layer Systems  
XI Summary  
XII Acknowledgments  

References  
Tables  
Figure Captions
I. INTRODUCTION

The fractional quantum Hall effect\textsuperscript{1–4} is a remarkable example of strong correlations in a two-dimensional electron gas (2DEG). In zero magnetic field, a dimensionless measure of the strength of the Coulomb interaction for a system with dielectric constant $\epsilon$, Fermi energy $\epsilon_F$ and Fermi wavevector $k_F$ is

$$\lambda = \frac{e^2 k_F}{\epsilon \epsilon_F},$$

which is small in the limit of high density. In this limit one can frequently treat the effects of the Coulomb interaction perturbatively. Physically this can be visualized as being due to the electrons moving rapidly past each other at the Fermi velocity and thus not scattering so strongly as they would at lower densities. A strong magnetic field completely changes this situation. Semiclassically, the rapid motion of the electrons is converted into circular cyclotron orbits. The particles now scatter strongly from each other and in fact semiclassically do not move except under the $E \times B$ drift induced by their mutual interactions. A full quantum treatment of the motion shows that the kinetic energy is quenched and now occurs only in discrete values $(n + 1/2)\hbar \omega_c$, where $n$ is the Landau level index. Since the kinetic energy within a given Landau level is completely degenerate, the Coulomb interaction inevitably induces highly non-perturbative effects. (The Landau level degeneracy is $N_\phi = BA/\Phi_0$ where $\Phi_0 = h c/e$ is the magnetic flux quantum, and $A$ is the area of the system.)

The essential feature of the fractional quantum Hall effect is a condensation of the electrons into special highly correlated states\textsuperscript{5} which minimize the Coulomb energy by having the electrons avoid each other as much as possible. These states are characterized by an unusual topological order\textsuperscript{6–9} which costs a finite amount of energy to break. Hence the fluid is incompressible and has an excitation gap both for its charged\textsuperscript{5} and neutral\textsuperscript{10,11} excitations. P. W. Anderson has characterized this state as a Mott insulator induced by the magnetic field.\textsuperscript{12}

The essence of this phenomenon is captured in a remarkable class of wave functions first constructed by R. B. Laughlin

$$\psi_m(Z_1, Z_2, \ldots, Z_N) = \prod_{i<j} (Z_i - Z_j)^m \exp \left\{ -\frac{1}{2} \sum_{k} |Z_k|^2 \right\}.$$ 

(2)

Here, since we are in two dimensions, we are using the dimensionless complex number $Z = (x + iy)/\ell$ to represent the position vector $(x, y)$ in units of the magnetic length $\ell = \sqrt{\hbar c/eB}$. This wave function describes spinless fermions in the lowest Landau level (in the symmetric gauge). To satisfy the analyticity requirement placed on the wave function by the constraint of being in the lowest Landau level,\textsuperscript{13} the parameter $m$ must be an integer. To satisfy the antisymmetry requirement for fermions, $m$ must be odd. Laughlin’s plasma analogy\textsuperscript{5} shows that the parameter $m$ fixes the Landau level filling factor to be $\nu \equiv N/N_\phi = 1/m$. Experiment\textsuperscript{14} has indeed observed gapped quantum Hall states at filling factors $\nu = 1, 1/3$, and $1/5$. 

1
It is clear that Laughlin’s wave function builds in good correlations because it vanishes as $|r_i - r_j|^m$ when any two particles $i$ and $j$ approach each other. Thus there is only a small amplitude for the particles to be near each other, and the Coulomb energy is lowered. Note that no pair of particles ever has relative angular momentum less than $m$. Hence the Laughlin function is a zero-energy exact eigenfunction for Hamiltonians with the appropriate finite number of non-zero Haldane pseudopotentials,$^{15}$ $V_n$:

$$H = \sum_{n<m} V_n \sum_{i<j} P_n[i,j],$$

where $P_n[i,j]$ is the lowest Landau level (LLL) projection operator for the relative angular momentum state $n$ of particles $i$ and $j$. In the lowest Landau level, relative angular momentum is proportional to the square of the separation.$^{13}$ Hence the Laughlin wave function is very nearly an exact ground state for any sufficiently short-range repulsive interaction.

In addition to the primary filling fractions $\nu = 1/m$, numerous other fractions have been observed, all of which (for single component systems) have odd denominators (again because of the Pauli principle). These phases have been explained in terms of a hierarchical picture using bosonic,$^{15}$ anyonic,$^{16}$ and fermionic$^{5,17}$ representations. More recently Jain$^{18,19}$ has discussed an appealing composite fermion picture. N. Read has argued that all of these representations are mathematically equivalent$^{20}$ and contain the same physics. Which representation is most convenient depends on circumstances. Jain’s approach has inspired several new and important experiments.$^{19}$

Our purpose here is to consider the nature of the various phases which can occur in multi-component systems. There are several physically different realizations of systems with extra degrees of freedom which require a multi-component representation, and important early work on this problem was done by Halperin.$^{21,22}$

The first and simplest example is that of ordinary electron spin. In free space for electrons with $g$ factor 2 (i.e., neglecting QED corrections) the Zeeman splitting $g\mu_B B$ is precisely equal to the Landau level splitting $\hbar \omega_c$. If this were true in quantum Hall samples, then even a non-interacting system with filling factor $\nu = 1$ would have a large excitation gap for flipping spins and the ground state would be fully polarized at low temperatures. In this case spin excitations are frozen out and we can treat the electrons as being effectively spinless. However in the solid state environment of the 2DEG, two factors conspire to make the effective $g$-factor much smaller in many semiconductors and particularly in GaAs samples in which almost all fractional quantum Hall studies have been done. The first is that the small effective mass ($m^* \approx 0.068$ in GaAs) increases the cyclotron energy by a factor of approximately 15. Secondly, spin-orbit coupling causes the spins to tumble and reduces their coupling to the external magnetic field by roughly a factor of 4. Thus the ratio of the Zeeman splitting to cyclotron splitting is reduced from unity to about 0.02 in GaAs. The spin-orbit contribution is pressure dependent and may also be further reduced as a result of size quantization effects in narrow quantum wells.$^{23}$

For small enough $g$ factor and weak enough magnetic fields, spin fluctuations become an important dynamical degree of freedom and we must use a two-component wave function to describe the system. While the Coulomb force is spin-independent, we shall see below that exchange effects can lead to spontaneous ferromagnetism as well as to gapped ‘local singlet’
spin liquids, which are, loosely speaking, itinerant antiferromagnets. [Unlike antiferromagnets however, they do not break translation symmetry nor do they have gapless Goldstone modes). Ferromagnetism turns out to be important at filling factor $\nu = 1$ where Coulomb exchange effects are much more important than originally realized.\textsuperscript{24–27}

A different class of quantum Hall states with gaps has been observed experimentally\textsuperscript{28–31} at filling fractions with even denominators such as $\nu = 5/2$. These have been argued\textsuperscript{9,32–35} to be either manifestations of spin liquid states or special ‘p-wave pairing’ spin-polarized states known as pfaffian states. One useful signature of spin effects in 2DEG’s is that they are sensitive to tilts of the magnetic field. To a first approximation, the orbital degrees of freedom are sensitive only to the perpendicular component of the field because of the 2D confinement, while the Zeeman splitting is proportional to the total magnetic field. (However, the coupling of orbital degrees of freedom to the parallel component of the magnetic field is not completely negligible in typical quantum wells\textsuperscript{36} and this can considerably complicate the situation.)

A second example of a multi-component system is found in silicon where the conduction band minimum occurs not at the $\Gamma$ point (the zone center) but rather at six symmetry equivalent points lying near the zone boundary along the principal cubic directions. Thus electrons doped into the conduction band of Si must be described by a six-component wave function (if we ignore spin). The presence of the oxide barrier in a Si MOSFET device and the enormous electric field perpendicular to it (which is used to confine the electrons into the inversion layer) breaks the cubic symmetry and lowers the energy of two of the valleys. For typical electron densities, only these two valleys are occupied, thus yielding a system which is effectively two-component and has SU(2) symmetry just like a spin-1/2 system.\textsuperscript{37}

A third example which will be extensively discussed here and in the Chapter in this volume by J. P. Eisenstein\textsuperscript{38} occurs in double quantum well structures.\textsuperscript{39} With modern MBE techniques it is possible to grow GaAs heterostructures containing two 2DEG’s separated by a distance comparable to the spacing between electrons within each layer. Remarkably, it is also possible to make separate electrical contact to each layer. A closely related system is a single wide quantum well in which the two lowest electric subbands are nearly degenerate.\textsuperscript{40} We will make the (only approximately correct) assumption throughout our discussion that the low energy physics of a single wide well can be mapped onto that of a double well with appropriately chosen parameters. In all two component systems it is useful to define a pseudospin representation in which spin up and down refer to the two possible values of the layer index (or subband index) for each electron.\textsuperscript{41} We will frequently frame our discussion in a spin or pseudospin language; the reader should be aware that the discussion applies equally well to double-layer systems. When the distinction between the U(1) and SU(2) symmetries of the interaction term in the Hamiltonian is important, we will say so.

Because the intra- and inter-layer Coulomb matrix elements are different, double well systems do not have full SU(2) symmetry, but rather only U(1) symmetry associated with the conservation of the charge difference between the two layers (assuming there is no interlayer tunneling).\textsuperscript{41–47} These systems exhibit gapped quantum Hall states with both even and odd denominators. In addition, there are gapless XY ordered phases. These phases are destroyed above some critical temperature by a Kosterlitz-Thouless phase transition. This phase transition (not yet observed experimentally) is the first example of a finite temperature phase transition in a quantum Hall system.

This Chapter is organized as follows. In Section [Section number], we introduce the physics of inco-
pressible states in multi-component cases by presenting the appropriate generalizations of Laughlin’s many-particle wavefunctions. Section III briefly directs the reader to references on the Chern Simons effective field theory approach to these problems. Section IV discusses fractional charges in multicomponent systems. Section V discusses collective modes from the point of view of the single-mode approximation. In both these sections we will see that some of the multi-component wavefunctions have correlation functions whose qualitative properties deviate from the norm. These differences lead to changes in the nature of the fractionally charged excitations and in the collective modes. The qualitative difference is associated with a broken symmetry which occurs in some cases as we discuss in Section VI. In our view, it is in the properties of these broken symmetry states that some of the most interesting new physics in multi-component quantum Hall states is revealed. Section VII discusses the field-theoretic gradient expansion approach to the broken symmetry case. In double-layer systems, the broken symmetry state has spontaneous phase coherence between the electrons in different layers even when these layers are isolated apart from inter-layer Coulomb interactions. Section VIII deals with the external symmetry breaking introduced by interlayer tunneling. Spontaneous coherence in double-layer systems leads to remarkable effects upon tilting the magnetic field away from the normal in a double layer system. Some of these effects are discussed in Section X. Finally Section XI presents a summary of the central ideas.

II. MULTI-COMPONENT WAVE FUNCTIONS

In this section we discuss wavefunctions for spin or pseudospin 1/2 particles which can be written in the symmetric gauge in the form

$$\Psi[Z : \chi] = A[\Phi[Z]\alpha_1 \cdots \alpha_{N_\uparrow}, \beta_1 [1] \cdots \beta_{[N_\downarrow]}],$$

(4)

where $A$ is the antisymmetrization operator, $[i] \equiv N_\uparrow + i$, and $\alpha_k$ and $\beta_k$ are the spinors for the $k$th electron aligned respectively parallel and antiparallel to the Zeeman field (which we take to be in the $\hat{z}$ direction). Thus $\Phi[Z]$ is the orbital wave function for the spin configuration in which the first $N_\uparrow$ electrons have spin “up” and the remaining electrons have spin “down”.

The two-component orbital wave functions originally proposed by Halperin\textsuperscript{21,22} have a form analogous to the Laughlin functions

$$\Phi_{m,m',n}[Z] = \prod_{i<j \leq N_\uparrow} (Z_i - Z_j)^m \prod_{k<l \leq N_\downarrow} (Z_{[k]} - Z_{[l]})^{m'} \times \prod_{a=1}^{N_\uparrow} \prod_{b=1}^{N_\downarrow} (Z_a - Z_{[b]})^n \prod_{s=1}^{N} \exp\{-|Z_s|^2/4\},$$

(5)

where $Z_k = (x_k + iy_k)/\ell$ is the 2D layer coordinate of the $k$th electron expressed as a complex number, and $m$ and $m'$ are odd integers. $\Phi_{m,m',n}[Z]$ excludes relative angular momenta less than $m$ between up-spins, less than $m'$ between down-spins, and less than $n$ between an up-spin and a down-spin. Hence the same arguments presented above for the Laughlin
wave function can be used to motivate the idea that the Halperin wave functions are good approximations to the ground state for short-range repulsive potentials.

In the single component case, Laughlin wavefunctions are expected to accurately approximate the ground state for \( m = 3 \) and \( m = 5 \) (\( \nu = 1/3 \) and \( \nu = 1/5 \)); at smaller filling factors the Wigner crystal state takes over. In the two-component case, the realization of a Laughlin state at a particular filling factor may be dependent on other parameters of the system such as the layer separation in double-layer systems. One slight complication is presented by the SU(2) symmetric case. Here the total spin \( S_T \) commutes with the Hamiltonian and one may, without loss of generality, require that the energy eigenstates simultaneously be eigenstates of \( S_T^2 \). This requirement is not satisfied in general by the Halperin wave functions. It is easy to pick out two special cases (among others) that do work however. The choice \( m = m' = n \) yields a fully antisymmetric spatial wave function and hence implies a fully symmetric spin function. This immediately tells us that we have a fully aligned ferromagnetic state with total spin quantum number \( S = N/2 \). On the other hand, the choice \( \{m, m', n\} = \{1, 1, 0\} \) corresponds to a simple Slater determinant with both spin states of the lowest Landau level fully occupied, giving \( \nu = 2 \). Hence it is automatically a spin singlet. We can generalize this to \( \{m, m', n\} = \{m, m, (m - 1)\} \) since this corresponds to simply multiplying the filled Landau level function by a fully symmetric spatial polynomial. Thus these states are also spin singlets.

Using the fact that every extended single-particle orbital in the lowest Landau level involves a polynomial with a fixed (average) density of zeros given by \( B/\Phi_0 \) (where \( \Phi_0 \equiv hc/e \) is the flux quantum) we may derive a pair of equations for the density \( \rho \) of each component

\[
\begin{align*}
\frac{B}{\Phi_0} &= m\rho_\uparrow + n\rho_\downarrow, \\
\frac{B}{\Phi_0} &= m'\rho_\downarrow + n\rho_\uparrow.
\end{align*}
\]

(6)

From this we obtain for the filling factors

\[
\begin{align*}
\nu_\uparrow &= \frac{m' - n}{mm' - n^2}, \\
\nu_\downarrow &= \frac{m - n}{mm' - n^2}.
\end{align*}
\]

(7)

Partial and total filling factors for some of these two-component Laughlin states are listed in Table II. At these partial filling factors \( \Phi_{m,m',n}[Z] \) is unique in the sense that it is the only wave function which excludes its corresponding low relative angular momentum channels and, just as in the case of Laughlin states in one-component systems, we may expect that these wave functions will be nearly exact groundstates for any sufficiently short-ranged repulsive interaction. In the SU(2) invariant case it seems that we should however require that the wave functions be eigenstates of the total spin operator \( S_T \) and that their Zeeman energy be not too unfavorable. There is numerical evidence for instance, that the \( \{3, 3, 2\} \) state which has \( \nu \equiv \nu_\uparrow + \nu_\downarrow = 2/5 \) has a lower energy than the usual hierarchical state if (and for typical field strengths, only if) one ignores the Zeeman energy. For further references and
detailed discussion on this point and other topics related to spin in the FQHE, the reader
is directed to the book by Chakraborty and Pietiläinen.3

We note that Eq.(7) is ill-defined if $m m' - n^2$ vanishes as it does, for example, in the fully
ferromagnetic $\{m, m, m\}$ states. In this case, however, one can compute the filling factor by
simply noting that the fully spin aligned system has an orbital wave function equivalent to the
Laughlin function at total filling factor $\nu = 1/m$. The relative filling factors of the two
components is necessarily ill-defined because of the SU(2) rotational symmetry. There are
$2S + 1 = N + 1$ orthogonal but macroscopically degenerate states differing only by their $S^z$
quadratic number. Thus we have

$$\nu_{\uparrow} + \nu_{\downarrow} = \frac{1}{m},$$

$$\nu_{\uparrow} - \nu_{\downarrow} = \frac{2S^z}{Nm}. \tag{8}$$

The degeneracy of these states leads to a broken symmetry which we discuss in greater detail
in Section VI.

Generalizations of the Halperin states can be made to the case of an arbitrary number
of components.51–53 This would have application for example to a superlattice of closely
spaced quantum wells, should these become technologically feasible to produce at some
point in the future. It is known experimentally28–31 that there exists an incompressible (but
unusually delicate) Hall state at filling factor $\nu = 5/2$. This state has also been observed
numerically for various artificially chosen interaction models, but not however, for a pure
Coulomb interaction.3,33,34,54 It has been argued that the ground state at this filling factor
is not fully spin polarized because the Hall state is easily destroyed by tilting the magnetic
field at constant filling fraction.31 In principle, the same state should be observed at filling
$\nu = 1/2$ because $\nu = 5/2 = 2 + 1/2$ has both spin states of the LLL filled and these electrons
are essentially inert, leaving an effective filling factor $\nu_{\text{eff}} = 1/2$ in the next Landau level.
However the analogous state is not observed at $\nu = 1/2$ because, to reach this lower filling
factor (in the same GaAs sample), it is necessary to increase the perpendicular component of
the magnetic field by a factor of 5. This increases the Zeeman energy advantage of the fully
spin polarized state (which is gapless and does not exhibit a Hall plateau) and makes it the
ground state. Thus non-observation of a quantum Hall state at $\nu = 1/2$ provides additional
evidence that the $\nu = 5/2$ state is not fully spin polarized. As described in the Chapter
in this volume by Eisenstein, tilted field experiments can also help identify spin-reversed
quasi-particle excitations above polarized ground states.3,55

Haldane and Rezayi32 have proposed a two-component spin-singlet wave function to
explain the existence of the $\nu = 5/2$ quantum Hall plateau. This wave function is an exact
ground state for the so-called ‘hollow-core’ model32 and may be written in two different ways
by modifying two different Halperin states, each of which has filling factor one half.56 The
first uses the Halperin fermionic $\{3, 3, 1\}$ function

$$\Phi_{\text{HC}} = \Phi_{331 \text{ per}|M|}, \tag{9}$$

where $M$ is an $N/2 \times N/2$ matrix whose $ij$ element is given by

$$M_{ij} = (Z_i - Z_{ij})^{-1}. \tag{10}$$
The permanent of the matrix, \( \text{per}|M| \), is by definition, just like the determinant except that there are no minus signs for odd permutations. The subtle effect of this permanent on the wave function is to cause it to be an eigenfunction of total spin (with \( S = 0 \)) without changing the density (at least in the thermodynamic limit).

Note that the permanent causes some up and down spin particles to have a finite probability of having relative angular momentum zero. It turns out that no particles ever have relative angular momentum 1 in this state. Hence this wave function is an exact zero energy singlet ground state for the ‘hollow core’ potential model

\[
V_m = V\delta_{m,1}.
\]

Despite the unphysical appearance of this model, it has been argued that it might capture the correct physics when the form of the effective pseudopotentials in the second Landau level is taken into account.\(^\text{32}\)

A second way to write the same state uses the \((222)\) bosonic Halperin wave function

\[
\Phi_{HC} = \Phi_{222} \text{Det}|\tilde{M}|,
\]

where \( M \) is an \( N/2 \times N/2 \) matrix whose \( ij \) element is given by

\[
M_{ij} = \left(Z_i - Z_{ij}\right)^{-2}.
\]

A curious mathematical identity\(^\text{32}\) allows one to show that these two representations are precisely equivalent.

It is possible that orbital effects\(^\text{36}\) confuse the tilted field test for spin unpolarized states and the \( \nu = 5/2 \) is actually spin-polarized. A competing spin-polarized ‘pfaffian’ state developed by Read\(^\text{9}\) and also studied by Greiter et al.\(^\text{35}\) is a kind of ‘p-wave’ paired state. It is not known for certain at this point what the true nature of the very delicate 5/2 state is.\(^\text{38, 54}\) It may be one of the proposed states\(^\text{9, 32–34}\) or something completely unknown.

Jason Ho has recently considered interesting connections between the internal order in these types of wave functions and analogous order in superfluid \(^3\)He.\(^\text{57}\) In particular he has discussed the ”incompressible” deformation of the various states into each other, connecting for example the \( \{3, 3, 1\} \) state and the pfaffian state.\(^\text{35, 57, 58}\)

### III. CHERN SIMONS EFFECTIVE FIELD THEORY

One interesting approach to the quantum Hall effect in general, and multi-component systems in particular, is the Chern Simons effective field theory. Unfortunately space limitations prevent us from discussing this approach in any detail. The reader is directed to the Chapter in this volume by B. I. Halperin,\(^\text{59}\) the references therein and to the many references in what is now a vast literature.\(^\text{24, 42–44, 46, 60–73}\) A succinct and introductory summary of the bosonic representation for double layer systems is given in Ref.\([46]\). We present a brief discussion of the collective mode predictions of the Chern Simons approach for double-layer systems in Section \(\nabla\) of this Chapter.
IV. FRACTIONAL CHARGES IN DOUBLE-LAYER SYSTEMS

The fractional quantum Hall effect occurs because, at particular filling factors, electrons in a partially filled Landau level are able to organize themselves into such strongly correlated states that the energy cost of making decoupled particles and holes remains finite, even in the thermodynamic limit, i.e., there is a charge gap. It is unusual to have a charge gap due entirely to electron-electron interactions (i.e., in a translationally invariant continuum system) although the example of superconductivity is familiar. In the fractional quantum Hall effect, not only do interactions produce a charge gap, but the free-charges responsible for the thermally activated dissipation measured experimentally contain only a fraction of the charge of an electron. The fact that sharply-defined fractional charges occur in the fractional quantum Hall effect can be understood as a necessary consequence of the quantization of the Hall conductance.\(^5,74\) (See the related discussion in Section VII.) It can also be understood in terms of the variational wavefunctions introduced by Laughlin in his pioneering work on the theory of the fractional quantum Hall effect.\(^5\) In this section we discuss the fractionally charged excitations of the \(\Psi_{m,m',n}\) states by generalizing the plasma arguments made by Laughlin for single-layer systems.

The \(\Psi_{m,m',n}\) wavefunctions have the property that pairs of electrons are excluded from certain relative angular momentum states. A low-energy charged state must have a localized excess or deficiency of charge without destroying the energetically favorable correlations associated with the relative angular momentum state exclusions. For a single-component system, Laughlin suggested that an accurate approximation to the many-body wavefunction for a state with a charged hole at the origin could be obtained simply by multiplying his wavefunctions for incompressible states by the factor \(\prod_i Z_i\). For two-component systems this argument has an obvious generalization. We can produce two, in general different, charged excitations centered on the origin, by multiplying the orbital many-particle wavefunction by the product of \(Z_i\) for all electrons in one pseudospin state. This operation is the variational wavefunction equivalent of introducing an unattached flux-tube in Chern-Simons theories and the conclusions we reach below can equally well be obtained by using an algebraically equivalent argument in that language. The plasma analogy results from writing the quantum distribution function, the square of the many-body wavefunction, as a classical statistical mechanics distribution function for interacting particles in an external potential

\[
|\psi|^2 = e^{-U}. \tag{14}
\]

The classical systems that result are generalized two-dimensional Coulomb plasmas\(^75\) and it is convenient in discussing them to adopt the convention of using Roman indices for one component of the plasma and Greek indices for the other component. With this notation, the trial wavefunctions we consider for charged excitations are

\[
\Psi^A_{m,m',n} \equiv \prod_{i=1}^{N_R} Z_i \Psi_{m,m',n} \tag{15}
\]

and

\[
\Psi^B_{m,m',n} \equiv \prod_{\alpha=1}^{N_G} Z_\alpha \Psi_{m,m',n}. \tag{16}
\]
These trial wavefunctions clearly reduce the density near the origin without ruining the good correlations in $\Psi_{m,m',n}^A$. At first sight it might appear that only the Roman particle density is reduced in $\Psi_{m,m',n}^A$ and only the Greek particle density is reduced in $\Psi_{m,m',n}^B$, but this is not the case in general because of the correlation factors.

The classical potential energy corresponding to $\Psi_{m,m',n}^A$ is

$$U_{m,m',n}^A = m \sum_{i<j} (-2 \ln |Z_i - Z_j|) + m' \sum_{\alpha<\beta} (-2 \ln |Z_{\alpha} - Z_{\beta}|) + n \sum_{i,\alpha} (-2 \ln |Z_i - Z_{\alpha}|)$$

$$+ \sum_i \frac{|Z_i|^2}{2} + \sum_{\alpha} \frac{|Z_{\alpha}|^2}{2} + \sum_i -2 \ln |Z_i|.$$ 

This potential is a generalized two-dimensional Coulomb plasma in the sense that the coupling constants outside the sums in the interaction terms are not constrained to be the products of charges for the two species, i.e., we allow $n \neq (mm')^{1/2}$. This difference results in long-range interactions in the plasma which depend on the density of each species separately, rather than just on the total ‘charge’. $U_{m,m',n}^A$ is the potential energy function for a system consisting of Roman and Greek particles. All particles have repulsive mutual two-dimensional Coulomb interactions with coupling constant $m$ between two Roman particles, coupling constant $m'$ between two Greek particles, and coupling constant $n$ between a Roman particle and a Greek particle. All particles are attracted to a neutralizing background which can be considered to have resulted from interaction with unit coupling constant with non-responding particles of uniform charge density $(2\pi \ell^2)^{-1}$. For $U^A$, only Roman particles interact with unit coupling constant with an impurity particle located at the origin.

The charge densities induced in each species of particles by the impurity can be calculated using the perfect screening properties which result from the long-range interactions of the plasma. Far enough from the impurity the direct long-range interaction must vanish for each species of particle; i.e., the sum of the impurity charge times its coupling constant plus the induced charges in each plasma component times the coupling strength for that plasma component must vanish so that

$$\left( \begin{array}{cc} m & n \\ n & m' \end{array} \right) \left( \begin{array}{c} e_R^A \\ e_G^A \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

In Eq.(17) $e_R^A$ is the contribution, in units of the magnitude of the electron charge, to the quasiparticle charge from Roman particles and $e_G^A$ is the contribution from Greek particles. Eq.(17) can be solved for $e_R^A$ and $e_G^A$ and the total quasiparticle charge $e_T^A \equiv e_R^A + e_G^A$:

$$e_R^A = \frac{m'}{mm' - n^2}, \quad e_G^A = \frac{-n}{mm' - n^2}, \quad e_T^A = \frac{m' - n}{mm' - n^2}.$$ (18)

The fractional charges for $\Psi_{m,m',n}^B$ differ only through the interchange of $m$ and $m'$:

$$e_R^B = \frac{-n}{mm' - n^2}, \quad e_G^B = \frac{m}{mm' - n^2}, \quad e_T^B = \frac{m - n}{mm' - n^2}.$$ (19)

Fractional charges calculated from these expressions are listed in Table II.
We note that the total charge of what is presumably the lowest energy charged excitation at each filling factor, has a value \( e/q \) where \( q \) is the denominator of the fractional total filling factor. When the two-components of the incompressible Hall fluid are correlated, a reduction of charge density in one-component leads to an increase of charge density in the other component. The total charge thus tends to consist of partially canceling contributions from the two-layers. This cancellation reaches its extreme limit for the case where \( m = m' = n \), for which the total charge of the excitation is well defined but its separation into contributions from separate components cannot be fixed by the perfect screening requirement on the plasma. We will see later that what is behind this behavior is the existence of long-range-order in the \( \Psi_{m,m,m} \) wavefunction. This long range order has delivered a bonanza of new physics in two-component systems, which will be the focus of much of this Chapter.

V. COLLECTIVE MODES IN DOUBLE-LAYER QUANTUM HALL SYSTEMS

In this section we will discuss both intra-Landau-level and inter-Landau-level (cylotron) collective (neutral) excitations of the incompressible ground states whose origin we have explained in previous sections. Our discussion is based on the projected single-mode-approximation, which has proved extremely useful\(^{10,76,77}\) in understanding the nature of the collective mode structure in single-layer systems. The projected single-mode-approximation can be appropriate in the strong magnetic field limit where there is little Landau level mixing in either the ground state or the low-lying excited states. Many-body eigenstates of the system can then be distinguished by the quantized integer number of units of \( \hbar \omega_c \) by which the kinetic energy exceeds the minimum value \( N\hbar \omega_c/2 \) (where \( N \) is the number of particles). The single-mode-approximation for the collective energy spectrum follows from the assumption that there is a unique many-body state, \( |\Psi_k^n\rangle \), with energy \( E_k^n \) within each quantized kinetic energy manifold, which is coupled to the ground state by the one-body density operator:

\[
\rho_k \equiv \sum_i \exp(i k \cdot r_i). \tag{20}
\]

[We will employ complex number notation \((k \equiv k_x + ik_y)\) for two-dimensional vectors when convenient.]

To use the single-mode approximation it is necessary to separately evaluate contributions to moments of the dynamic structure factor from transitions involving different numbers of quantized kinetic energy units. We write for the dynamic structure factor

\[
s(k, \epsilon) = \sum_n s_n(k, \epsilon), \tag{21}
\]

where

\[
s_n(k, \epsilon) = \frac{1}{A} \sum_i |\langle \Psi_{i,n} | \rho_k | \Psi_0 \rangle|^2 \delta(\epsilon - E_{i,n} + E_0), \tag{22}
\]

and \( A \) is the area of the system. Here \( |\Psi_{i,n}\rangle \) is an exact eigenstate of the Hamiltonian involving \( n \) excess quantized kinetic energy units. In the single-mode approximation it is assumed that only a single eigenstate contributes to the sum in Eq.(22),
\[ s_n(k, \epsilon) = \frac{1}{A} |\langle \Psi_{k,n} | \rho_k | \Psi_0 \rangle|^2 \delta(\epsilon - E_k^n + E_0). \] (23)

Both the matrix element and the energy which appear in this expression have physical significance. The matrix element determines how strongly one-body external probes (for example, far infrared or microwave radiation) couple to the collective excitation. The energies of the collective modes can be measured in transmission experiments or in inelastic light-scattering experiments.\textsuperscript{78} The thermodynamics and linear response functions of the system depend qualitatively on the presence or absence of collective modes whose energies vanish in the limit of long wavelengths.

Two moments of \( S_n(k, \epsilon) \) are relatively easy to evaluate and we will use these two moments to determine both the matrix element and the collective mode energy:

\[ s_n(k) \equiv \int_0^\infty d\epsilon s_n(k, \epsilon), \] (24)

and

\[ f_n(k) \equiv \int_0^\infty d\epsilon \epsilon s_n(k, \epsilon). \] (25)

Given \( s_n(k) \) and \( f_n(k) \), we have for the matrix element

\[ |\langle \Psi_k^n | \rho_k | \Psi_0 \rangle|^2 = As_n(k), \] (26)

and for the excitation energy

\[ \Delta_k^n \equiv E_k^n - E_0 = f_n(k)/s_n(k). \] (27)

Since \( \rho_{k=0} \) is a constant, and the ground and excited states must be orthogonal, it follows that at long-wavelength \( s_n(k) \) must vanish at least as fast as \( k^2 \). Long wavelength probes, like far infrared radiation, produce\textsuperscript{78} observable coupling to a long-wavelength collective mode only if \( s_n(k) \propto k^2 \) at long-wavelengths. We will refer to modes for which \( s_n(k) \propto k^2 \) as dipole active and those for which \( s_n(k) \) vanishes with a higher power of \( k \) as dipole inactive. The quantity \( f_n(k) \) is proportional to the product of the square of the matrix-element and the excitation energy and in analogy with atomic physics we refer to this quantity as the projected oscillator strength. We will refer to \( s_n(k) \) as the projected static structure factor.

The usual ‘f-sum rule’, valid with or without a magnetic field, states that for parabolic bands

\[ \sum_n f_n(k) = \frac{N}{A} \frac{\hbar^2 k^2}{2m^*}. \] (28)

To evaluate projected ‘f-sum rules’ we exploit the property that the single-particle Hilbert space of a charged particle in a magnetic field can\textsuperscript{74} be considered as the product space of a factor space in which states are distinguished by the number of quantized kinetic energy units in the cyclotron orbit and a factor space for the cyclotron-orbit-center degree of freedom which exists within each Landau level and is responsible for the macroscopic Landau level degeneracy. To separate the dynamic structure factor into contributions associated with different quantized kinetic energies we write the kinetic energy operator in the form
\[ \rho_k = \sum_{n',n} \rho_{k,n',n}^{n',n}, \]  

(29)

where

\[ \rho_{k,n',n}^{n',n} = \sum_i |n'\rangle_i \langle n| G_{n',n}(k) B_i(k). \]  

(30)

The \( G_{n',n}(k) \) are related to Laguerre polynomials, the sum over \( i \) is over particle labels, and \( B_i(k) \) is a factor coming from the projection of \( \rho_k \) onto a single Landau level and operates on the intra-Landau-level degree of freedom of particle \( i \). The commutators which appear below are evaluated by using Eq. (30) and the identity,

\[ B_i(k_1)B_i(k_2) = \exp(k_1^*k_2/2)B_i(k_1 + k_2). \]  

(31)

At strong magnetic fields, only \( \rho_{k,0}^{n,0} \) contributes to \( s_n(k) \) and \( f_n(k) \). We will restrict our attention to the cases of intra-Landau-level (\( n = 0 \)) and magnetoplasmon (\( n = 1 \)) collective excitations. (Excitation modes out of the LLL with \( n > 1 \) are never dipole active.)

Using completeness relations of the many-body eigenstates it follows from the above definitions that

\[ s_0(k) = \frac{1}{A} \langle \Psi_0| \rho_{-k,0}^{0,0} \rho_{k,0}^{0,0} |\Psi_0 \rangle \]  

(32)

and that

\[ f_0(k) = \frac{1}{2A} \langle \Psi_0|[\rho_{-k,0}^{0,0}, \hat{V}], \rho_{k,0}^{0,0}]|\Psi_0 \rangle \]  

(33)

where \( \hat{V} \) is the electron-electron interaction term in the Hamiltonian. From general properties of (translation invariant) many-body eigenstates within the lowest Landau levels it can be shown that for small \( k \), \( s_0(k) \propto k^4 \) so that the intra-Landau level collective mode is dipole inactive. After a somewhat laborious calculation, involving repeated application of Eq.(31), the \( n = 0 \) projected oscillator strength can be expressed in terms of \( s_0(k) \) and one finds the result that \( f_0(k) \propto k^4 \), independent of any details of the electron-electron interaction or the ground state wavefunction, so that the magnetoroton intra-Landau-level collective modes have a gap at long-wavelengths:

\[ \lim_{k \to 0} \Delta_0^k \neq 0. \]  

(34)

On the other hand an elementary calculation based on Eq.(31) and the strong-field-limit assumption that the ground state lies entirely within the lowest-Landau-level subspace of the full Hilbert space, implies that

\[ s_1(k) = \frac{N k^2 \ell^2}{A} \exp(-|k\ell|^2/2). \]  

(35)

Similarly

\[ f_1(k) = \hbar \omega_c s_1(k) + \frac{1}{2A} \langle \Psi_0|[\rho_{-k,1}^{0,1}, \hat{V}], \rho_{k,0}^{1,0}]|\Psi_0 \rangle, \]  

(36)
and the second term on the right-hand-side of Eq. (36) can be shown to vanish as $k^3$ for small $k$. The magnetoplasmon mode is dipole active and, in the long-wavelength limit, completely exhausts the full f-sum rule. For a single-layer, the magnetoplasmon mode is the only dipole-active mode and its energy is not shifted from $\hbar \omega_c$ by electron-electron interactions. These behaviors result from the conservation of particle number and invariance under translation: a long wavelength electromagnetic field couples only to the cyclotron motion of the center of mass of the system.\textsuperscript{79}

We are now prepared to discuss how these results are altered in double layer systems. We restrict our attention to the case where the two 2DEG’s are identical and tunneling between them may be neglected. In this case the number of electrons in each layer is a good quantum number and collective modes corresponding to the sum and difference of the density oscillations in the two layers decouple. To generalize the projected single-mode-approximation to the double-layer case we evaluate separately projected oscillator strengths for both sum (in phase) and difference (out of phase) modes. We will find that the behavior of the sum modes for double-layer systems is similar to the behavior of the modes of a single-layer system while the behavior of the difference modes departs from this pattern.

For the difference modes, both $n = 0$ and $n = 1$ modes are dipole active. The $n = 1$ mode is shifted from $\hbar \omega_c$ and the $n = 0$ mode usually has a finite energy as $k \rightarrow 0$. An exception occurs for those ground state wavefunctions which have a type of long range order which we have not yet made explicit. This long range order is associated with a broken symmetry ground state. These broken symmetry ground states are, arguably, responsible for the most surprising and appealing new physics which is introduced going from one-component to two-component fractional quantum Hall systems and we will have much more to say about them later in this Chapter. The difference in behavior between sum and difference modes is due to the fact that the Hamiltonian is not invariant under relative translations of the two layers.

The Hamiltonian of the double-layer system in the absence of interlayer tunneling may be written in the following form which is convenient for calculations:

$$H = \hbar \omega_c \sum_i [a_i^\dagger(L)a_i(L) + a_i^\dagger(R)a_i(R)] +$$

$$\frac{1}{2} \sum_q [V_{qLL}^\dagger \rho_q(L)\rho_{-q}(L) + V_{qRR}^\dagger \rho_q(R)\rho_{-q}(R) + 2V_{LR}^\dagger \rho_q(L)\rho_{-q}(R)],$$

(37)

where $a_i(L)$ is the Landau level lowering\textsuperscript{74} operator for particle $i$ in the left (L) layer, $V_{qLL}^R$ is the intra-layer Coulomb interaction, $V_{LR}^R$ is the inter-layer Coulomb interaction, and $\rho_q(X)$ is the density operator for layer X. [For explicit calculation we ignore the finite thickness of the 2D layers so that $V_{qLL}^R = 2\pi e^2/(\epsilon q)$ and $V_{LR}^R = \exp(-qd)V_{LL}^R$.] In Eq. (37) the Hamiltonian includes infinite constant terms corresponding to the self-interaction of each electron in the system. It is convenient to retain these terms so that the interaction terms in the Hamiltonian can be expressed in terms of density operators. Since, in all subsequent calculations, the Hamiltonian enters only in commutators, these non-physical constant terms never contribute. The operators which generate the sum and difference collective modes are $\Omega_k^{n\pm} = [\rho_k^{n\pm}(L) \pm \rho_k^{n\pm}(R)]/\sqrt{2}$. With the Hamiltonian expressed in terms of density operators, a laborious but direct calculation makes it possible to express $f_k^{n\pm}$ and $s_k^{n\pm}$ in terms of $s_k^{n\pm}$ using Eq. (31). Explicit expressions for the collective mode energies and the coupling matrix...

13
elements are given elsewhere.\textsuperscript{65,80} Here we only comment on some of the physically interesting
conclusions of these calculations.

As mentioned above the energy of the \( n = 1 \) sum mode approaches the cyclotron energy \( \hbar \omega_c \) in the \( k \to 0 \) limit, in agreement with Kohn’s theorem. (Since interactions are invariant
under simultaneous translations in both layers, the proof for a single-layer system\textsuperscript{79} trivially
generalizes to the case of the two-layer in-phase mode.) However the energy of the \( n = 1 \) difference mode is shifted from the cyclotron energy in the long wavelength limit where it is
given by the following expression:

\[
\Delta^1-(k = 0) = \hbar \omega_c - \int \frac{d^2q}{(2\pi)^2} q^2 V_q^{LR} h^{LR}(q).
\]  

(38)

Here \( h^{LR}(q) \) is the Fourier transform of the inter-layer pair correlation:

\[
h^{LR}(q) = \frac{1}{N} \langle \Psi_0 | \rho_q(L) \rho_{-q}(R) | \Psi_0 \rangle,
\]

(39)

and

\[
h^{LL}(q) = \frac{1}{N} \langle \Psi_0 | \rho_q(L) \rho_{-q}(L) | \Psi_0 \rangle - 1,
\]

(40)

where \( N \) is the number of particles per layer. If the layers are uncorrelated, \( h_{LR}(q) = h_{LL}(q) = 0 \) and \( \Delta^1-(k = 0) = \hbar \omega_c \). For correlated layers \( h_{LR}(q) \) tends to be negative, at
least at small \( q \) since the density in the left layer will tend to be reduced when the density in
the right layer is increased, and we can expect that \( \Delta^1-(k = 0) > \hbar \omega_c \). In fact, it is possible
to prove that interactions (of either sign!) always increase the frequency of this mode as we
mention below.

By expanding the expression for \( f_{0\pm} \) at small \( k \) it can be shown that \( f_{0+} \sim k^4 \) whereas

\[
f_{0+} = -\frac{N}{A} (k^2/2) \sum_q q^2 V_q^{LR} h^{LR}(q) + \mathcal{O}(k^4).
\]

(41)

The \( n = 0 \) difference mode is dipole active. It is interesting that for the difference mode the
interaction contributions to the dipole (\( \propto |k|^2 \)) portions of the \( n = 0 \) and \( n = 1 \) oscillator
strengths are identical. By definition \( f_{0-} \) is positive definite so that \( \Delta^-(k = 0) = \hbar \omega_c \) must also be positive. In the single-mode approximation, the \( n = 1 \) (cyclotron) difference
mode is always shifted to higher energy by electron-electron interactions. The situation
is similar to that for the effect of disorder on the vibration modes of the Wigner crystal
at strong magnetic fields where pinning of the crystal shifts both intra-Landau-level and
inter-Landau-level modes upward by the same amount.\textsuperscript{81}

To determine whether or not the \( n = 0 \) mode is gapped, it is necessary to determine
how \( s_{0\pm} \) behaves at small \( k \). From general properties of wavefunctions in the lowest Landau
level it is possible\textsuperscript{65} to conclude that\textsuperscript{82} \( s_{0+} \sim k^4 \), whereas the behavior of the \( n = 0 \)
difference mode structure factor depends on the difference between inter-layer and intra-
layer correlation functions. It is possible to prove\textsuperscript{65} that \( s_{0-} \sim k^2 \) at small \( k \) provided that
intra-layer and inter-layer correlation functions separately vanish at large spatial separations.
In general it is easy to show from the plasma arguments outlined in Section IV that this
is a property of the $\Psi_{m,m,n}$ wavefunctions. However an exception occurs for $n = m$. In this case there is no distinction between the effective plasma interactions of particles in the same layer and particles in different layers. The weighting of particle configurations depends only on the total charge densities of the two-layers and only correlations in the total charge densities of the two-layers go to zero at large distances. For the broken symmetry state $\Psi_{m,m,m}$ it is possible\(^6\) to show explicitly that $s_k^{0-} = \exp(-|k\ell|^2/2)$ which goes to a constant for $k \to 0$. The SMA collective mode then goes like $k^2$ for this special case if the ground state is approximated by $\Psi_{m,m,m}$. (The italics above are pregnant as we will see later.) For the $\{m, m, m\}$ states, the $n = 0$ difference mode (intra-Landau level) is gapless at long-wavelengths. It is often the case that gapless collective modes can be identified as Goldstone modes associated with a broken symmetry in the ground state, and that is indeed the case here, as we shall see.

We emphasize that the results discussed above follow from general sum rules and are independent of the approximate many-body wavefunction ($\Psi_{m,m',n}$) in terms of which we have framed our discussion so far. Since $f_k^{0+} \sim k^4$ and $f_k^{0-} \sim k^2$ independent of the long wavelength behavior of $s_k^{0\pm}$, it follows quite generally that the $n = 0$ sum mode has a gap and that the $n = 0$ difference mode has a gap except in the case where long-range order is present which results in correlation functions which do not vanish at large distances. In Fig. we show results obtained for the collective mode energies of a double-layer system with a total Landau level filling factor $\nu_T = 1/2$ and a layer separation $d = 1.5\ell$, close to the effective layer separation value for which novel double-layer fractional Hall effects have recently been observed.\(^8\) Numerical calculations\(^8\) have established that the ground state at this value of $d/\ell$ is accurately approximated by the $\{m, m, n\} = \{3, 3, 1\}$ Halperin\(^2\) wavefunction and we have used the correlation functions\(^8\) of that wavefunction to evaluate the oscillator strengths and structure factors. For $k \to 0$ the $n = 1$ sum mode (the Kohn mode) is unshifted by interactions while the $n = 1$ difference mode is shifted to higher energies as discussed above. The shift, which is a direct measure of inter-layer correlations, should\(^8\) be observable in cyclotron resonance experiments in double-layer systems. Note also that both sum and difference $n = 0$ modes have a finite gap as expected from the above discussion.

We are now able to compare our results for the collective modes with the Chern-Simons Landau-Ginzburg (CSLG) theory of the double-layer system.\(^4\) In the CSLG random-phase-approximation, the sum and difference density response functions are given by:

$$\rho_+(\omega, q) = \frac{N q^2/A m^*}{\omega^2 - \omega_+^2}, \quad \rho_- (\omega, q) = \frac{N q^2/A m^*}{\omega^2 - \omega_-^2},$$

where the collective mode frequencies are given by $\omega_+ = \omega_c = eB/m^*c$, $\omega_- = \omega_c (m - n)/(m + n)$ and $m^*$ is the effective band mass of the electrons. For the single-layer case the CSLG random-phase-approximation predictions are correct for the dipole active mode and we might have expected the same to be true in double-layer systems. The sum mode for double-layer systems can be clearly identified with the Kohn mode.\(^6\) However there are difficulties in identifying the density difference modes in this theory. From Eq. we see that the $n = 0$ difference mode is dipole active and should have a dipole oscillator strength proportional to $V_{LR}^2 \hbar^{LR}$. One might therefore be tempted to identify $\omega_-$ with the $n = 0$ difference mode. Then, for the case of the $\{m, m, m\}$ state random-phase-approximation, $\omega_- = 0$, and it would then be tempting to identify this mode with the gapless $n = 0$
Goldstone mode. Unfortunately, the single density-difference mode calculated within the double-layer CSLG theory saturates the full dipole oscillator strength $Nq^2/m^*A$. This is not acceptable since an excitation within the lowest Landau level can not contain explicit dependence on the band mass $m^*$. The second possibility is to interpret the difference mode obtained in Eq. (42) as the $n = 1$ difference mode. In this case one is faced with the difficulty that the mode energy is shifted downwards from the cyclotron energy by an amount proportional to $\omega_c$, whereas the SMA calculations show that it should be shifted upwards by an amount proportional to the interlayer Coulomb energy. It has been suggested$^{89}$ that these difficulties can be resolved by including the mixing of the vortex excitations with the gaussian fluctuations in the CSLG theory. Similar difficulties arise in the fermion-Chern-Simons theory of the single-component $\nu = 1/2$ state and can be resolved in that case by taking sufficient care with the Landau parameters of the composite-fermion Fermi liquid.$^{59}$

VI. BROKEN SYMMETRIES

Ferromagnetic states break spin rotation symmetry since they are defined by an order parameter $\langle \mathbf{S} \rangle$ which gives the magnitude and orientation of the magnetization. For the SU(2) symmetric case with no Zeeman term, this orientation is arbitrary. As we will see below, the case of a double layer system is described by a pseudospin with easy-plane anisotropy [U(1) or XY symmetry]. Here the magnetization vector is forced to lie in the XY plane in the ground state. The origin of ferromagnetism in all these systems is the Coulomb interaction just as it is for itinerant ferromagnets like iron. Exchange effects are particularly crucial in a 2DEG in a large magnetic field because the kinetic energy is quenched into highly degenerate Landau levels. It is advantageous to follow Hund’s rule and maximize the spin in order to make the spatial wave function fully antisymmetric, thereby lowering the Coulomb energy. Since the Landau level is degenerate, this spin alignment can in some cases be complete since it costs no kinetic energy as it does in iron.$^{90}$

Before considering the physical consequences of this broken symmetry, let us return to Table I to consider how the total spin quantum number $S$ for a state can be determined. A portion of our discussion here follows that of Ref.$^{[22]}$. As already mentioned, states of the form $\{m,m,m\}$ are fully ferromagnetically aligned and have total spin $S = N/2$. To derive the spin quantum numbers for the other states in Table I we write

$$S^2_{\uparrow} \Psi[Z : \chi] = A[\Phi'[Z] \alpha_1 \cdots \alpha_N, \beta_{[1]} \cdots \beta_{[N]}],$$  \hspace{1cm} (43)

and use the fact that

$$S^2_{\uparrow} = (S^z)^2 + \frac{1}{2} (S^+ S^- + S^- S^+)$$

$$= (S^z)^2 + \frac{N}{2} + \frac{1}{2} \sum'_{k,l} (S^+_k S^-_l + S^-_k S^+_l).$$  \hspace{1cm} (44)

We see that

$$\Phi'[Z] = \left[ \left( \frac{N_+ - N_-}{2} \right)^2 + \left( \frac{N}{2} \right) \right] \Phi[Z]$$

16
prime on the sum in Eq. (44) indicates that $k$ is no longer perfectly antisymmetric. This energy gap is quite large ($\sim$ quantum number. 

Moreover

$S$ states where $m$ = 0 and $S$ are eigenstates of determinant these orbital wave functions have the up-spin Landau level full (see table I). They are eigenstates of $S^z$ with eigenvalue

$$\frac{(N_\uparrow - N_\downarrow)}{2} = N_\downarrow (m' - 1)/2 = N (m' - 1)/2 (m' + 1) \equiv S_{m'}.$$ (46)

Moreover $S^+\Psi_{1,m',0} \equiv 0$ since the up-spin Landau level is already full and there are no wave functions with a larger value of $N_\uparrow$. It follows that $\Phi_{1,m,0}[Z]$ satisfies Eq. (45) with $S = (N_\uparrow - N_\downarrow)/2$ and hence that

$$\sum_{i=1}^{N_\uparrow} \sum_{j=1}^{N_\downarrow} e(i, [j]) \Phi_{1,m',0}[Z] = N_\downarrow \Phi_{1,m',0}[Z].$$ (47)

(This result can also be established by an explicit algebraic proof). Since $e(i, [j])Q[Z]\Phi[Z] = Q[Z]e(i, [j])\Phi[Z]$ for any symmetric polynomial $Q[Z]$, we have from Eqs. (47) and (45) that

$$S^z \Phi_{1+p,m'+2p,2p}[Z : \chi] = S_{m'}(S_{m'} + 1)\Phi_{1+p,m'+2p,2p}[Z : \chi].$$ (48)

In addition, it follows from Eq. (13) that $S^z \Psi[Z : \chi] = N(N/2 + 1)/2 \Phi[Z : \chi]$ for any completely antisymmetric function $\Phi[Z]$, and in particular for generalized Laughlin states with $m = m' = n$. These states are merely the $S^z = 0$ members of the set of the $(N + 1)$ fully polarized Laughlin states which are degenerate in the absence of the Zeeman term.

We return now to the question of the physical consequences of the spontaneously broken symmetry of ferromagnetic states. We will focus initially on the SU(2) invariant case of ‘real’ spins with zero $g$ factor. Consider a state with $\nu = 1$ and all spins up. Because the up Landau level is maximally filled, the Pauli principle forces us to flip a spin if we are to move an electron to create a pair of charged excitations. This is illustrated in Fig.[2 a] and shows that spin and charge are intimately connected in this case.

Transport dissipation measures the thermal activation of charged excitations. In the absence of interactions, the energy cost of charged excitations is zero and there will be no $\nu = 1$ quantum Hall plateau because there is no gap. In the presence of Coulomb interactions a flipped spin particle-hole pair causes a loss of exchange energy of

$$E_0 = \sqrt{\frac{\pi}{2} \epsilon^2 \ell},$$ (49)

where $\epsilon$ is the bulk dielectric constant. This occurs because the electron spatial wave function is no longer perfectly antisymmetric. This energy gap is quite large ($\sim 150 K$ at $B = 10T$).
and is vastly larger than the bare Zeeman splitting. Hence Coulomb interactions and the associated ferromagnetism play a dramatic role in producing the charge gap at \( \nu = 1 \) (and because of the spontaneous magnetic ordering, will continue to do so, even if \( g \) is strictly zero).

The exchange energy cost of particle-hole pair excitations is so large that it is worth searching for some modified form of the excitation which is less costly. A prescient analysis of smooth spin textures by Sondhi, et al.\(^{24}\) yielded the exciting idea that ‘skyrmion’\(^{60}\) like spin textures (shown in cross section in Fig.[2 b]) can have relatively low energy and carry fermion number proportional to their topological charge (Pontryagin index)\(^{24,46}\)

\[
\Delta N = -\frac{\nu}{8\pi} \int d^2r \epsilon_{\mu\nu} \mathbf{m}(\mathbf{r}) \cdot [\partial_\mu \mathbf{m}(\mathbf{r}) \times \partial_\nu \mathbf{m}(\mathbf{r})],
\]

where \( \mathbf{m}(\mathbf{r}) \) is the unit vector field representing the local spin orientation. The fermion number \( \Delta N \) is an integer multiple of \( \nu \) because it is the number of times the unit sphere is wrapped around by the order parameter. That is, it is the winding number of the spin texture.\(^{91}\) For the Laughlin parent states \( \nu = 1/m \), elementary spin-textures carry the same fractional charge as the quasiparticles discovered by Laughlin\(^5\) for spinless electrons. As we discuss below, the fact that the charges are the same follows from very general considerations.

Actually, the spin texture states we have defined must contain precisely the same number of particles as \( |\psi_0\rangle \) since the spin-rotation operator does not change the total electron number. However the spin-density may contain a number of well-separated textures with well-defined non-zero topological charge densities and hence well localized charges; only the net charge in the spin-texture states defined above will be zero. The system clearly has states with locally non-zero net charge in the spin textures.

A simple variational wave function for a skyrmion of size \( \lambda \) centered on the origin and carrying \( p \) units of topological charge is given by

\[
\psi_\lambda = \prod_j \left( \frac{Z_j^p}{\lambda^p} \right) \Phi_{mmm},
\]

where \( \Phi_{mmm} \) is defined in Eq.(5), \( \{\}_j \) refers to the spinor for the \( j \)th particle, and the variational parameter \( \lambda \) is a fixed length scale. This is a skyrmion because the \( \hat{x} - \hat{y} \) component of the spin has a vortex centered on the origin and the \( \hat{z} \)-component is purely down at the origin (where \( Z_j = 0 \)) and purely up at infinity (where \( |Z_j| \gg \lambda \)) as shown in Fig.[2]. The parameter \( \lambda \) is simply the size scale of the skyrmion.\(^{24,92}\) Notice that in the limit \( \lambda \rightarrow 0 \) [where the continuum effective field theory is invalid (see Section [VII]), but this microscopic wave function is still sensible] we recover a fully spin polarized filled Landau level with \( p \) Laughlin quasiholes at the origin. Hence the number of flipped spins associated with the presence of the skyrmion interpolates continuously from zero to infinity as \( \lambda \) increases.

In order to analyze the skyrmion wave function in Eq.(51), we use the Laughlin plasma analogy.\(^5\) In this analogy the norm of \( \psi_\lambda \), \( \text{Tr}_\{\sigma\} \int D[z] |\Psi[z]|^2 \) is viewed as the partition function of a Coulomb gas. In order to compute the density distribution we simply need to take a trace over the spin (we specialize here to the case \( p = 1 \) for simplicity)

\[
Z = \int D[z] e^{\frac{i}{\hbar} \sum_{i>j} \log |Z_i-Z_j| + \frac{\pi}{\Phi} \sum_k \log(|Z_k|^2+\lambda^2) - \frac{\pi}{\Phi} \sum_k |Z_k|^2 },
\]

(52)
This partition function describes the usual logarithmically interacting charge \( m \) Coulomb gas with uniform background charge plus a spatially varying impurity background charge \( \Delta \rho_b(r) \),

\[
\Delta \rho_b(r) \equiv -\frac{1}{2\pi} \nabla^2 V(r) = -\frac{\lambda^2}{\pi(r^2 + \lambda^2)^2},
\]

and

\[
V(r) = \frac{1}{2} \log(r^2 + \lambda^2).
\]

For large enough scale size \( \lambda \gg \ell \), local neutrality of the plasma\(^6\) implies that the excess electron number density is precisely \( \frac{1}{m}\Delta \rho_b(r) \), so that Eq.(54) is in agreement with the standard result for the topological density,\(^9^2\) and the skyrmion carries electron number \( \frac{1}{m} \) (for \( p = 1 \)) and \( \frac{p}{m} \) in general. These objects are roughly analogous to the Laughlin quasi-hole. Explicit wave functions for the the corresponding quasi-electron objects (‘anti-skyrmions’) are more difficult to write down just as they are for the Laughlin quasi-electron due to the analyticity constraint.\(^5\)

Sondhi, et al.\(^{2^4}\) have shown that for the case of pure Coulomb interactions (i.e., with no finite inversion layer thickness corrections) the optimal skyrmion configuration costs precisely half the energy of the simple spin flip.\(^9^3\) The reason for this is simply that the skyrmion keeps the orientation of spins close to that of their neighbors and so loses less exchange energy. This will be discussed in more detail from a field theoretic point of view in Section VII.

Optical\(^9^4\) and standard transport\(^9^5\) experiments show that the charge excitation gap is indeed much larger than would be expected if interactions were neglected and has approximately the correct Coulomb scale, although there is not yet precise agreement between the observed gap and the best estimates including the finite \( g \) factor.\(^{2^4, 2^5}\) The main source of error is probably neglect of finite thickness effects. Calculations including finite thickness corrections do not exist at present. Quantum fluctuation effects (i.e., corrections to Hartree-Fock) may also be important. It should be noted that the uniform ground state of the \( \nu = 1 \) ferromagnet does not have quantum fluctuations (Hartree-Fock is exact here). However for finite \( g \) factor, the length scales associated with the skyrmion are small and quantum corrections could well become important.

The idea that skyrmions are the lowest energy excitations has received very strong and unequivocal support from numerical simulations which show that quite remarkably, adding a single electron to a \( \nu = 1 \) system (with \( g = 0 \)) suddenly changes it from a fully aligned ferromagnet \((S = N/2)\) to a spin singlet \((S = 0)\) due to the formation of a skyrmion.\(^{2^4, 9^6}\) (It should be noted that this occurs in the spherical geometry. Things are slightly more complicated on the torus.\(^4^6\))

The notion of charges being carried by skyrmion textures has received additional dramatic experimental confirmation in recent optically pumped NMR measurements by Barrett et al.\(^{2^6, 2^7}\) Their Knight shift measurements (see Fig.\(\[3\]) indicate that the electron gas spin polarization has a maximum at filling factor \( \nu = 1 \) and falls off sharply on each side. The rate of fall off indicates that each charge added or removed from the Landau level turns over about 7 spins. This is consistent with the charge being carried by skyrmions of finite size. The size of the skyrmion is determined by a competition between the Zeeman coupling which
wants to minimize the number of flipped spins and the Coulomb self-energy which wants to
expand the skyrmion to spread out the excess charge density over the largest possible area.
As mentioned above, Rezayi found numerically that a single charge entirely destroys the
spin polarization of a system (if \( g = 0 \)). Using an effective field-theory approach, Sondhi et
al.\(^{24}\) have estimated the skyrmion size in the regime of small \( g \) factor. Microscopic Hartree-
Fock calculations expected to be more accurate in the physically accessible Zeeman energy
regime have been performed by Fertig et al.\(^{25}\) These estimates are roughly consistent with
the experimental value of 7 spins per unit of charge. It should be noted that for finite \( g \)
factor, the energetic advantage of the skyrmion over the simple flipped spin is considerably
reduced.\(^{25}\) Wu and Sondhi have shown that in higher Landau levels, skyrmions have higher
energy than other charged excitations.\(^{97}\)

For the SU(2) symmetric case discussed in this section, the existence of a vector order
parameter \( \langle \mathbf{S} \rangle \) is in some sense trivial because the magnetization commutes with the Hamiltonian.
For the case of double layer systems we will see that the the pseudospin Hamiltonian
has only U(1) symmetry and the fact that a non-zero expectation value of \( \langle \mathbf{S} \rangle \) appears is
highly non-trivial. We will discuss this in more detail in Section \( \text{VII} \).

\section*{VII. FIELD THEORETIC APPROACH}

In this section we study the ferromagnetic broken symmetry ground state and its ex-
citations from the point of view of (quantum) Ginzburg-Landau effective field theory. We
will begin with the SU(2) invariant case of ‘real’ spin and then move on to the pseudospin
analogy in double-layer systems. We will give here an introductory qualitative discussion of
the physics. Part of our discussion follows fairly closely the presentation in Ref.[46]. The
technical details of the calculations can be found there and elsewhere.\(^{24,25,47}\)

The standard first step in this procedure is always to identify the slowly fluctuating order
parameter field, which in this case we have reason to believe is the local magnetization. We
believe that on long length scales the (coarse-grained) magnetization fluctuates very slowly,
because we know that the zero-wavevector component of the spin operator (i.e., the total
spin) commutes with the Hamiltonian and so is a constant of the motion. We will focus on
slow tilts of the spin orientation ignoring variations in the magnitude of the coarse-grained
magnetization, and so define the order in terms of a local unit vector field \( \mathbf{m}(\mathbf{r}) \).

General symmetry arguments can now be used to deduce the form of the Lagrangian. We
cannot have any terms that break spin rotational symmetry and thus the leading term
which is an allowed scalar is

\[ E = \frac{1}{2} \rho_s \int d^2 r \left[ \nabla m^\mu(\mathbf{r}) \right] \cdot [\nabla m^\mu(\mathbf{r})], \]

where \( \rho_s \) is a phenomenological spin stiffness coefficient and the energy is relative to the
ground state energy. It expresses the cost due to loss of Coulomb exchange energy when the
spin orientation varies with position. For the SU(2) invariant \( \nu = 1 \) case, this stiffness may
be computed exactly.\(^{46}\) For Coulomb interactions (with no finite thickness corrections) this
calculation yields

\[ \rho_s = \frac{e^2/\epsilon \ell}{16 \sqrt{2\pi}}. \]

20
In two dimensions the stiffness has units of energy and is approximately 4 K at a field of 10 T. Numerical estimates for $\nu = 1/3$ yield a value which is about 25 times smaller.\cite{46}

As usual, there is a linear time derivative term in the Lagrangian which can be deduced from the fact that each spin precesses under the influence of its local exchange field. Equivalently we may note that when the orientation of a spin is moved around a closed loop, the quantum system picks up a Berry’s phase\cite{98} proportional to the solid angle $\Omega$ enclosed by the path $\omega$ of the tip of the spin on the unit sphere as shown in Fig.[4]. Noting that a charged particle moving on the surface of a unit sphere with a magnetic monopole at the origin also picks up a Berry’s phase proportional to the solid angle subtended by the path\cite{98} we may express the Berry’s phase for a spin $S$ as

$$e^{i\gamma} = e^{i\Omega_S} = e^{i \oint_{\omega} \mathbf{m} \cdot \mathbf{A}(\mathbf{m})},$$

or equivalently

$$e^{i\gamma} = e^{i \int dt \partial \mathbf{m} / \partial t \cdot \mathbf{A}(\mathbf{m})},$$

where $\mathbf{A}(\mathbf{m})$ is the vector potential of a unit monopole\cite{24,46,91} at the center of the sphere evaluated at the point $\mathbf{m}$. That is, $\nabla \mathbf{m} \times \mathbf{A} = \mathbf{m}$. This phase is correctly reproduced in the quantum action by adding the following total derivative term to the Lagrangian for the spin

$$L_1 = S \partial \mathbf{m} / \partial t \cdot \mathbf{A}(\mathbf{m}).$$

Using the fact that the electronic density is $\nu / 2\pi \ell^2$, the analog for the present problem of a large collection of electrons with $S = 1/2$ at filling factor $\nu$ may be simply written

$$L_1 = \frac{\nu}{4\pi \ell^2} \int d^2 r \partial \mathbf{m} / \partial t \cdot \mathbf{A}[\mathbf{m}],$$

which yields the Lagrangian

$$L = \frac{\nu}{4\pi \ell^2} \int d^2 r \partial \mathbf{m} / \partial t \cdot \mathbf{A}[\mathbf{m}] - \frac{1}{2} \rho_s \int d^2 r [\nabla m^\mu(\mathbf{r})] \cdot [\nabla m^\mu(\mathbf{r})].$$

We shall see shortly that higher gradient terms can be unexpectedly significant, but this Lagrangian is adequate to recover the correct spin wave collective mode. Taking the spins to be aligned in the $\hat{z}$ direction and looking at small transverse oscillations at wave vector $\mathbf{q}$ we obtain from this Lagrangian the following equation of motion

$$\frac{d \mathbf{m}_q}{dt} = \frac{4\pi \rho_s q^2}{h \nu} \hat{z} \times \mathbf{m}_q.$$  

This yields the dispersion relation

$$\hbar \omega = \frac{4\pi \rho_s q^2}{\nu},$$

which agrees with the long-wavelength limit of exact results obtained by a variety of means.\cite{99,100}
At this point we have expanded the Lagrangian to lowest order in gradients and we have correctly found the neutral collective spin wave modes. Their dispersion is quadratic in wave vector just as it is for the Heisenberg ferromagnet on a lattice. However here we have an itinerant magnet and we have so far seen no sign of the charge degrees of freedom. It turns that we have to go to higher-order gradient terms in the action to see charged objects.

We have already seen in the discussion of Fig.[2] in Section VI, that for a filled Landau level, the Pauli principle forces there to be a connection between charge excitations and flipped spins. It turns out that the existence of a finite Hall conductivity in this itinerant magnet causes smooth spin textures to carry charge proportional to their topological density. One can derive this result from a Chern-Simons effective field theory, or from microscopic considerations involving the fact that the spin density and charge density operators do not commute when projected onto the lowest Landau level, or from macroscopic considerations connecting the Berry phase term to the Hall conductivity. The latter is the least technical and the most instructive so we shall pursue it here.

Imagine that the order parameter of the ferromagnetic system is distorted into a smooth texture as illustrated in Fig.[5]. As an electron travels around in real space along a path \( \partial \Gamma \) which is the boundary of the region \( \Gamma \), the spin is assumed to follow the orientation of the local exchange field \( b(\mathbf{r}) \) and hence traces out a path in spin space labeled \( \omega \) in the (schematic) illustration in Fig.[4]. That is, given any sufficiently smooth spin texture, we can write a Hartree-Fock like Hamiltonian for the electrons which will reproduce this texture self-consistently

\[
H = \sum_{j=1}^{N} b(\mathbf{r}) \cdot \mathbf{S}_j.
\]  

(64)

If we drag one electron around in real space along \( \partial \Gamma \) and its spin follows the local \( b(\mathbf{r}) \) adiabatically (which we expect since the exchange energy is so large) then the electron will acquire a Berry’s phase \( \Omega/2 \) where \( \Omega \) is the solid angle subtended by the region \( \omega \) shown in Fig.[4].

In addition to the Berry’s phase from the spin, the electron will acquire a Bohm-Aharonov phase from the magnetic flux enclosed in the region \( \Gamma \). At least in the adiabatic limit, the electron can not distinguish the different sources of the two phases. The electron would acquire the same total phase in the absence of the spin texture if instead an additional amount of flux

\[
\Delta \Phi = \frac{\Omega}{4\pi} \Phi_0,
\]  

(65)

(where \( \Phi_0 \) is the flux quantum) were added to the region \( \Gamma \). We know however that adding flux to a region in a system with a finite Hall conductivity changes the total charge in that region. To see this, let \( \Phi(t) \) be the time-dependent flux inside \( \Gamma \). Then the electric field along the perimeter obeys (from Faraday’s Law)

\[
\oint_{\partial \Gamma} \mathbf{E} \cdot d\mathbf{r} = -\frac{1}{c} \frac{d\Phi}{dt}.
\]  

(66)

Because of the Hall conductivity (and the fact that \( \sigma_{xx} = 0 \)), the field at the perimeter induces a current obeying
\[ \hat{z} \cdot \mathbf{J} \times \mathbf{dr} = \sigma_{xy} \mathbf{E} \cdot \mathbf{dr}. \] (67)

Integrating this expression along the boundary and using the continuity equation, we have that the total charge inside \( \Gamma \) obeys

\[ \frac{dQ}{dt} = +\frac{\sigma_{xy} d\Phi}{c dt}, \] (68)

or

\[ \Delta Q = -e\nu \frac{\Delta \Phi}{\Phi_0}, \] (69)

where we have used the fact that the Hall conductivity is quantized (and is negative for \( B = |B| \hat{z} \)):

\[ \sigma_{xy} = -\nu e^2/h. \] (70)

Thus \( \nu \) electrons flow into the region for each quantum of flux added to the region. This makes sense when we recall that there is one state in each Landau level per quantum of flux penetrating the sample.

From Eq.(65) we see that the spin texture thus induces an extra charge of

\[ \Delta Q = -e\nu \frac{\Omega}{4\pi}. \] (71)

The solid angle \( \Omega \) is of course a functional of the spin texture in the region \( \Gamma \). For simplicity of analysis of this functional let us consider making up \( \Gamma \) out of a set of infinitesimal square loop circuits in real space of the form

\[ (x, y) \rightarrow (x + dx, y) \rightarrow (x + dx, y + dy) \rightarrow (x, y + dy) \rightarrow (x, y). \] (72)

The corresponding circuit in spin space illustrated in Fig.[6] is,

\[ \mathbf{m}(x, y) \rightarrow \mathbf{m}(x + dx, y) \rightarrow \mathbf{m}(x + dx, y + dy) \rightarrow \mathbf{m}(x, y + dy) \rightarrow \mathbf{m}(x, y). \] (73)

Approximating this circuit as a parallelogram as shown in Fig.[7], the solid angle subtended is (to a sufficient approximation)

\[ d\omega = [\mathbf{m}(x + dx, y) - \mathbf{m}(x, y)] \times [\mathbf{m}(x, y + dy) - \mathbf{m}(x, y)] \cdot \mathbf{m}(x, y). \] (74)

This may be rewritten in a suggestive form which tells us the curl of the Berry ‘connection’: \(^{98}\)

\[ d\omega = \frac{1}{2} \epsilon_{\mu\nu} \mathbf{m} \cdot \partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m} \, dx \, dy. \] (75)

We can now add up all the infinitesimal contributions to obtain

\[ \Omega = \int_{\Gamma} dx \, dy \, \frac{1}{2} \epsilon_{\mu\nu} \mathbf{m} \cdot \partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m}, \] (76)
which yields a total charge of

$$\Delta Q = -\frac{e\nu}{8\pi} \int \Gamma \, dx \, dy \, \epsilon_{\mu\nu} \mathbf{m} \cdot \partial_{\mu} \mathbf{m} \times \partial_{\nu} \mathbf{m}, \quad (77)$$

or a local charge density deviation of

$$\delta \rho = -\frac{e\nu}{8\pi} \epsilon_{\mu\nu} \mathbf{m} \cdot \partial_{\mu} \mathbf{m} \times \partial_{\nu} \mathbf{m}. \quad (78)$$

The expression on the right hand side of Eq.(78) is simply the Pontryagin topological charge density of the spin texture. Its integral over all space is an integer and is a topologically invariant winding number known as the Pontryagin index. The spin textures which have non-zero Pontryagin index are the ‘skyrmion’ configurations illustrated in Fig.2b. A microscopic variational wave function for these spin textures was discussed in Section VI.

The charge density in Eq.(78) can be viewed as the time-like component of a conserved (divergenceless) topological ‘three-current’ which results in the following beautiful formula

$$j^\alpha = -\frac{\nu}{8\pi} \epsilon^{\alpha\beta\gamma} \epsilon_{abc} m^a(r) \partial_\beta m^b(r) \partial_\gamma m^c(r). \quad (79)$$

Using the fact that \( \mathbf{m} \) is a unit vector, it is straightforward to verify that \( \partial_\mu j^\mu = 0 \). We note that the fact that the expression for the topological current is not parity invariant is a direct reflection of the lack of parity symmetry in the presence of the external magnetic field.

The mechanism we have seen here that associates charge with flux is the reason that quantum Hall fluids are described by Chern-Simons theories, and is the same mechanism which causes Laughlin quasiparticles (which are topological vortices) to carry quantized fractional charge proportional to the quantized value of \( \sigma_{xy} \).

Having established that the electron charge density is proportional to the topological density of the spin order parameter field, we must now return to our Lagrangian to see what modifications this implies. We have already taken into account the long-range Coulomb force but it led only to the local spin gradient term whose coefficient is the spin wave stiffness. There are however additional effects of the charge (topological) density fluctuations which we must take into account

$$\mathcal{L} = \frac{\nu}{4\pi\ell^2} \int d^2 r \frac{\partial \mathbf{m}}{\partial t} \cdot \mathbf{A}[\mathbf{m}] - \frac{1}{2} \rho_s \int d^2 r [\nabla m^\mu(r)] \cdot [\nabla m^\mu(r)]$$

$$+ \sum_q V(-q)\delta \rho_q + \frac{1}{2} \sum_q \frac{2\pi}{\epsilon q} \delta \rho_{-q}\delta \rho_q. \quad (80)$$

Here \( \delta \rho_q \) is the Fourier transform of the charge density in Eq.(78). Note that it is second order in spin gradients. The first of the new terms in Eq.(80) represents the coupling of the charge fluctuations to the external and random disorder potentials \( V(-q) \) and is second order in spin gradients. The second new term represents the mutual interaction of the charge fluctuations via the Coulomb potential. Note that this is fourth-order in gradients (and so is not a duplicate of the \( \rho_s \) term which also comes from the Coulomb interaction). In general there will be additional fourth-order terms allowed by symmetry, but we do not bother to
write them down since they will not have the divergent Coulomb interaction coefficient \(2\pi/\epsilon q\) which makes the term we have kept effectively third order in \(q\).

We can immediately conclude several interesting things from the rather peculiar nature of our itinerant ferromagnet. First, unlike the case of a regular ferromagnet, a scalar potential can induce the formation of charged skyrmions in the ground state. Thus sufficiently strong disorder would have the effect of greatly reducing the net spin polarization of the ground state, something which should be directly observable experimentally.

Secondly, we note that (at the classical level) the energy of a skyrmion due to the gradient term is scale invariant

\[
E = \frac{1}{2} \rho_s \int d^2r [\nabla m^\mu(r)] \cdot [\nabla m^\mu(r)] = \frac{E_0}{4},
\]

because we have two spatial integrations and two derivatives. [The quantity \(E_0\) is defined in Eq.(49).] Now however the Coulomb self-energy will want to expand the size of the skyrmion. In real spin systems this effect competes against the small but (usually) non-zero Zeeman coupling which wants to minimize the number of flipped spins. This competition has been studied in some detail and appears to be essential to explain the experiments of Barrett et al.\textsuperscript{24–27}

\section*{VIII. INTERLAYER COHERENCE IN DOUBLE LAYER SYSTEMS}

The details of the double layer experiments of Murphy et al.\textsuperscript{39} are described in the Chapter by J. P. Eisenstein. Here we briefly introduce the main ideas. Double layer quantum Hall systems (and wide single well systems\textsuperscript{40}) exhibit a variety of non-trivial collective states at different filling factors. Here we will focus on the case of total filling factor \(\nu = 1\) (that is, 1/2 in each layer) which is most closely analogous to the fully ferromagnetic broken symmetry state for \(\nu = 1\) with ‘real’ spins that we have discussed in the previous sections. There are many other interesting states which we do not have room to discuss here. One example, is the state at total filling factor \(\nu = 1/2\) (that is, 1/4 in each layer) which is believed to be described by Halperin’s \(\{3, 3, 1\}\) wave function.\textsuperscript{3, 21, 22, 84, 85} This state is more nearly like a gapped spin liquid state, although, as we have already seen, it does not satisfy the Fock cyclic condition and so is not a true spin singlet.

The schematic energy level diagram for the growth direction degree-of-freedom in the double-layer system is shown in Fig.[7]. For simplicity we assume that electrons can only occupy the lowest electric subband in each quantum well. If the barrier between the wells is not too strong, tunneling from one side to the other is allowed. The lowest energy eigenstates split into symmetric and antisymmetric combinations separated by an energy gap \(\Delta_{\text{SAS}}\) which can, depending on the sample, vary from essentially zero to hundreds of Kelvins. The splitting can therefore be much less than or even greater than the interlayer interaction energy scale, \(E_c \equiv e^2/(\epsilon d)\).

The analogy with the spin systems studied in the previous sections is that within the approximations just mentioned, the electrons in a two-layer system have a double-valued internal quantum number—namely the layer index. The tunnel splitting in the double well plays the role of the Zeeman splitting for spins.
In addition to double quantum wells, there are wide single quantum wells in which the two lowest electric subband states are strongly mixed by Coulomb interactions. These systems exhibit very similar physics, and can also be approximately modeled as a double layer system.

Throughout our discussion we will assume that the ‘real’ spins are aligned and their dynamics frozen out by the small Zeeman energy. This is not necessarily a good approximation in experimentally relevant cases but greatly simplifies matters. Dynamics of ‘real’ spins in double layers is currently a topic of investigation.

A. Experimental Indications of Interlayer Phase Coherence

Here we will very briefly review the main experimental indications that double well and wide single well systems at \( \nu = 1 \) can show coherent pseudospin phase order over long length scales and exhibit excitations which are highly collective in nature.

When the layers are widely separated, there will be no correlations between them and we expect no dissipationless quantum Hall state since each layer has \( \nu = 1/2 \). For smaller separations, it was predicted theoretically and subsequently observed experimentally that there is an excitation gap and a quantized Hall plateau. The resulting phase diagram is shown in Fig. and discussed in more detail in the Chapter in this volume by J. P. Eisenstein. The existence of a gap has either a trivial or a highly non-trivial explanation, depending on the ratio \( \Delta_{\text{SAS}}/E_c \). For large \( \Delta_{\text{SAS}} \) the electrons tunnel back and forth so rapidly that it is as if there is only a single quantum well. The tunnel splitting \( \Delta_{\text{SAS}} \) is then analogous to the electric subband splitting in a (wide) single well. All symmetric states are occupied and all antisymmetric states are empty and we simply have the ordinary integer Hall effect. Correlations are irrelevant in this limit and the excitation gap is close to the single-particle gap \( \Delta_{\text{SAS}} \) (or \( \hbar \omega_c \), whichever is smaller). What is highly non-trivial about this system is the fact that the \( \nu = 1 \) quantum Hall plateau survives even when \( \Delta_{\text{SAS}} \ll E_c \) (See Fig.). In this limit the excitation gap has clearly changed to become highly collective in nature since the observed gap can be on the scale of 10-20 K even when \( \Delta_{\text{SAS}} \sim 1 \text{K} \). Because of the spontaneous broken symmetry, the excitation gap actually survives the limit \( \Delta_{\text{SAS}} \rightarrow 0 \)!

This cross-over from single-particle to purely collective gap is quite analogous to the result we discussed earlier that for spin polarized single layers, the excitation gap survives the limit of zero Zeeman splitting. Hence, to borrow a delightful phrase from Sondhi et al., ‘\( \nu = 1 \) is a fraction too.’

A second indication of the highly collective nature of the excitations can be seen in the Arrhenius plots showing thermally activated dissipation. The low temperature activation energy \( \Delta \) is, as already noted, much larger than \( \Delta_{\text{SAS}} \). If \( \Delta \) were nevertheless somehow a single-particle gap, one would expect the Arrhenius law to be valid up to temperatures of order \( \Delta \). Instead one observes a fairly sharp leveling off in the dissipation as the temperature increases past values as low as \( \sim 0.1\Delta \). This is consistent with the notion of a thermally induced collapse of the order that had been producing the collective gap. This behavior is very similar to that seen in ‘real’ spins.

The third significant feature of the experimental data pointing to a highly-ordered collective state is the strong response of the system to relatively weak magnetic fields \( B_{||} \)
applied in the plane of the 2D electron gases. Within a model that neglects higher electric subbands, we can treat the electron gases as strictly two-dimensional. This is important since \( B_\parallel \) can then affect the system only if there are processes that carry electrons around closed loops containing flux. A prototypical such process is illustrated in Fig. 9. An electron tunnels from one layer to the other at point A, and travels to point B. Then it (or another indistinguishable electron) tunnels back and returns to the starting point. The parallel field contributes to the quantum amplitude for this process (in the 2D gas limit) a gauge-invariant Aharonov-Bohm phase factor \( \exp \left( \frac{2 \pi i \Phi}{\Phi_0} \right) \) where \( \Phi \) is the enclosed flux and \( \Phi_0 \) is the quantum of flux.

Such loop paths evidently contribute significantly to correlations in the system since the activation energy gap is observed to decrease very rapidly with \( B_\parallel \), falling by factors of order two or more until a critical field, \( B_\parallel^* \sim 0.8 \text{T} \), is reached at which the gap essentially ceases changing.\(^{39}\) To understand how remarkably small \( B_\parallel^* \) is, consider the following. We can define a length \( L_\parallel \) from the size of the loop needed to enclose one quantum of flux: \( L_\parallel B_\parallel^* d = \Phi_0 \). \((L_\parallel [\text{Å}] = 4.137 \times 10^5 / d [\text{Å}] B_\parallel^* [\text{T}]\). For \( B_\parallel^* = 0.8 \text{T} \) and \( d = 210 \text{Å} \), \( L_\parallel = 2460 \text{Å} \) which is approximately six times the spacing between electrons in a given layer and more than twenty times larger than the quantized cyclotron orbit radius \( \ell \equiv (\hbar c/eB_\perp)^{1/2} \) within an individual layer. Significant drops in the excitation gap are already seen at fields of 0.1T implying enormous phase coherent correlation lengths must exist. Again this shows the highly collective nature of the ordering in this system.

The fourth indication about the nature of the coherent ordering is the fact that the gapped quantum Hall state at \( \nu = 1 \) can survive a finite amount of imbalance in the layer charge densities.\(^{38}\) Charge imbalance can be controlled by applying a gate voltage which increases the equilibrium charge density in one layer and decreases it in the other. In some cases, such as the gapped state at \( \nu = 1/2 \), layer imbalance immediately destroys the quantum Hall state. For \( \nu = 1 \) it does not. This is a strong hint about the different natures of the ordering in the two cases. We defer discussion of what this hint means to subsection VIII C.

### B. Effective action for double layer systems

Having established the correct form of the effective low-energy Lagrangian for the SU(2) invariant case, let us now turn to the U(1) symmetric case in double layer systems. The spin analogy is relatively straightforward, but can be confusing until one gets used to it. The idea is simply that each electron can be in either the upper layer or the lower layer and we refer to these two states as pseudospin up and down respectively. For the moment we shall assume the layers are identical and also neglect the possibility of tunneling between the two layers. The confusing point is that quantum mechanics nevertheless forces us to consider the possibility that an electron can be in a coherent superposition of the two pseudospin states so that its layer index is uncertain. Portions of our discussion here and in the following subsections follows that of Ref.\(^{46}\). Further technical details can be found therein. The formal mapping that we use to define the pseudospin density operators is the following. The \( z \) component of the pseudospin density represents the local charge density difference between the layers.
\[ S^z(r) = \frac{1}{2} \left[ \psi^\dagger(r) \psi^\dagger(r) - \psi^\dagger(r) \psi^\dagger(r) \right]. \]  

(82)

The \( x \) and \( y \) components of the pseudospin density are off-diagonal and can be combined to form tunneling operators

\[ S^+(r) = \left[ S^-(r) \right]^\dagger = \frac{1}{2} \psi^\dagger(r) \psi^\dagger(r). \]  

(83)

If for instance, tunneling is present then the Hamiltonian contains a term

\[ T = -t \int d^2r \left[ S^+(r) + S^-(r) \right] = -2t \int d^2r S^z(r). \]  

(84)

We know that ferromagnetism in real spin systems is not the result of truly spin-dependent forces but rather a byproduct of Coulomb exchange forces. Hence the effective spin Hamiltonian must contain only spin scalars such as \( S_i \cdot S_j \). Here however the Coulomb forces are explicitly pseudospin dependent since the intra- and inter-layer Coulomb interactions are not identical. Following our previous discussion\textsuperscript{41} we define

\[ V^0_k \equiv \frac{1}{2} (V^A_k + V^E_k), \]  

(85)

\[ V^z_k \equiv \frac{1}{2} (V^A_k - V^E_k), \]  

(86)

where \( V^A_k \) is the Fourier transform with respect to the planar coordinate of the interaction potential between a pair of electrons in the same layer and \( V^E_k \) is the Fourier transform of the interaction potential between a pair of electrons in opposite layers. If we neglect the finite thickness\textsuperscript{41} of the layers, \( V^A_q = 2\pi e^2/(eq) \) and \( V^E_q = \exp(-qd)V^A_q \), where \( d \) is the layer separation. The interaction Hamiltonian can then be separated into a pseudospin-independent part with interaction \( V^0 \) and a pseudospin-dependent part. The pseudospin dependent term in the Hamiltonian is

\[ V_{sb} = 2 \sum_k V^z_k \bar{S}^z_k \bar{S}^z_{-k}. \]  

(87)

Here \( \bar{S}_k \) is the Fourier transform of the spin density at wave vector \( k \) and the overbar indicates projection onto the lowest Landau level.\textsuperscript{46} Since \( V^A_k > V^E_k \), \( V^z_k \) is positive and this term produces an easy-plane, as opposed to Ising, pseudospin anisotropy. That is, this term prefers for the spin to lie in the XY plane. If the spin orientation moves out of the XY plane so that \( \langle S^z \rangle \neq 0 \), then the energy increases. We can view this energy cost as simply the charging energy of the capacitor formed by the two layers since the pseudospin component \( S^z \) measures the charge difference between the two layers.

The pseudospin symmetry of the Hamiltonian is reduced from \( SU(2) \) to \( U(1) \) by this term. In addition, this term increases the quantum fluctuations in the system since it does not commute with the order parameter

\[ [V_{sb}, S^\mu] \neq 0, \]  

(88)
where \( \mu = x, y \). Thus total spin is no longer a sharp quantum number. However for small layer separations, we expect the quantum fluctuations to be small. At very large layer separations we expect the quantum fluctuations to become dominant and produce the disordering phase transition which uncouples the two layers.

In the absence of the symmetry breaking term \((d = 0)\), we know the exact quantum ground state for \( \nu = 1 \) since it is simply a fully occupied pseudospin-polarized Landau level. This state is \( 2S + 1 = N + 1 \) fold degenerate (since there is no charging energy in this unphysical limit). As a first approximation for finite \( d \), we assume that the eigenstates remain exactly the same, but the charging energy lifts the degeneracy among these states by favoring the ones with \( S^z \sim 0 \). That is, we assume that the spin vector is still fully polarized and non-fluctuating, but it now lies in the XY plane.

We argue that the form of the energy-functional we derive must remain valid even when quantum fluctuations due to the pseudospin-dependent terms in the Hamiltonian are present. However, the coefficients which appear in the energy-functional will be altered by quantum fluctuations and the explicit expressions we derive below are accurate only when the pseudospin-dependent interactions are weak, i.e., only when the layers are close together. Estimates have been obtained for the quantum fluctuation corrections to these coefficients from finite-size exact diagonalization and many-body perturbation theory calculations.\(^{46, 47}\)

To better understand what we are assuming, consider the specific example of a state which has \( S = N/2 \) and has its spin oriented in the \( \hat{x} \) direction so that it is an eigenstate of total \( S^x \)

\[
|\Psi\rangle = \prod_X \frac{1}{\sqrt{2}} \left( c_{X\uparrow}^\dagger + c_{X\downarrow}^\dagger \right) |0\rangle.
\]

This state has zero net charge on the capacitor on the average \((\langle S^z\rangle = 0)\) and so first-order perturbation theory in \( V_{SB} \) favors it for the ground state. The confusion that many people have at this point is over the fact that this is a coherent state with uncertain \( S^z \). It makes perfect sense if the Hamiltonian includes tunneling that the charge in each layer will be uncertain. However we are arguing that this state is a good approximation to the ground state even in the absence of tunneling because it has favorable Coulomb energy. That is, we are assuming that the system spontaneously breaks the \( U(1) \) symmetry associated with conservation of layer charge difference\(^{42–44, 106}\) and is acting much like a superfluid\(^{41}\) which breaks \( U(1) \) symmetry. We refer to this state as having spontaneous interlayer phase coherence. It is quite analogous to a BCS state which has an uncertain total number of Cooper pairs. In the absence of tunneling, the electron energy cannot be sensitive to the actual relative phase between the two layers and so any coherent state of the form

\[
|\Psi\rangle = \prod_X \frac{1}{\sqrt{2}} \left( c_{X\uparrow}^\dagger + e^{i\varphi} c_{X\downarrow}^\dagger \right) |0\rangle
\]

29
will have equally good energy. The global U(1) phase \( \varphi \) simply determines the orientation of the total spin in the XY plane

\[
\langle \Psi | S^+ | \Psi \rangle \equiv \langle \Psi | S^x + iS^y | \Psi \rangle = \frac{N}{2} e^{i\varphi}.
\] (91)

This is exactly analogous to the situation in a BCS superconductor where the energy is independent of the phase \( \varphi \) which determines the coherence between states of different numbers of Cooper pairs

\[
\langle \psi_1^\dagger \psi_2^\dagger \rangle \sim e^{i\varphi}.
\] (92)

The complex order parameter in a double layer system given by Eq.(91) has an amplitude and a phase like that of a superconductor, but in many ways is more reminiscent of an excitonic insulator since it is charge neutral (i.e., it contains \( \psi_1^\dagger \psi_1 \) not \( \psi_1^\dagger \psi_2^\dagger \)) and physically represents a particle in one layer bound to a hole in the other layer (but we do not know which layer contains the particle and which contains the hole). Some years ago, Datta\textsuperscript{110} considered similar states in double layer systems in zero B field.

At first, it may seem counter-intuitive that, in the absence of tunneling, the system energy could depend on the relative phase \( \varphi \) for finding the particle in the upper or lower layer. Indeed the energy is unchanged when a constant is added to \( \varphi \). However the energy does depend on gradients of \( \varphi \). The ‘spin stiffness’ \( \rho_s \) is non-zero because of the loss of exchange energy which occurs when \( \varphi \) varies with position. Imagine that two particles approach each other. They are in a linear superposition of states in each of the layers (even though there is no tunneling!). If these superpositions are characterized by the same phase, then the wave function is symmetric under pseudospin exchange and so the spatial wave function is antisymmetric and must vanish as the particles approach each other. This lowers the Coulomb energy. If a phase gradient exists, then there is a greater amplitude for the particles to be near each other and hence the energy is higher. This loss of exchange energy is the source of the finite spin stiffness and is what causes the system to spontaneously ‘magnetize’. We may describe this by a term in the effective action of the form

\[
E = \frac{1}{2} \rho_s \int d^2r |\nabla \varphi|^2.
\] (93)

We see immediately that at finite temperatures this system will be described by a classical XY model and will undergo a Kosterlitz-Thouless phase transition. We will cover this in more detail further below.

**C. Superfluid Dynamics**

We consider now the dynamics of a two layer system with easy-plane pseudospin anisotropy but no tunneling. Let us back up and note that implicit in Eq.(93) is the assumption that the spin lies exactly in the XY plane. To be more complete we should derive the effective Lagrangian in the presence of the easy plane anisotropy. On general symmetry grounds this must be of the form (neglecting higher order gradients needed to describe charge fluctuations)
\[ \mathcal{L} = \frac{\nu}{4\pi\ell^2} \int d^2r \frac{\partial m}{\partial t} \cdot \mathbf{A}[m] - \int d^2r \left\{ \beta (m^z)^2 + \frac{\rho_A}{2} |\nabla m^z|^2 + \frac{\rho_E}{2} [|\nabla m^x|^2 + |\nabla m^y|^2] \right\}. \quad (94) \]

Within the Hartree-Fock approximation which is presumably valid only for small layer separations, the coefficients \(\beta, \rho_A,\) and \(\rho_E\) can be evaluated in terms of the density-density correlation function of the ground state.\(^{46}\) (The gradient expansion in \(m^z\) is not strictly valid since it turns out\(^{46}\) that the long range of the Coulomb interaction leads to a non-local \(m^z m^z\) interaction term not included above. This term vanishes in the SU(2) invariant case but here is more important than the \(|\nabla m^z|^2\) term and less important than the \(\beta |m^z|^2\) term at long wavelengths. We retain the \(|\nabla m^z|^2\) term in the following expressions only to remind us of its importance in the SU(2) invariant limit.)

Since \(m^z\) is massive, the equations of motion derived from this Lagrangian lead to a linear rather than quadratic collective mode dispersion\(^{41-44, 46, 69, 106}\) like that of the Goldstone mode in a superfluid or an antiferromagnet. We take the pseudospin of the system to be polarized in the \(\hat{x}\) direction and consider the linear response to a time– and space–dependent pseudospin ‘Zeeman’ field in the \(\hat{y} - \hat{z}\) plane. Using the equations of motion determined from the Lagrangian in Eq.(94) and Fourier transforming with respect to both time and space we find that

\[
\begin{pmatrix}
-i\omega \\
\frac{4\pi}{\nu} (q^2 \rho_s) \\
\frac{4\pi}{\nu} (2\beta + q^2 \rho_A)
\end{pmatrix}
\begin{pmatrix}
m_y \\
m_z \\
h_y
\end{pmatrix}
= 
\begin{pmatrix}
-h_z \\
-\omega
\end{pmatrix}, \quad (95)
\]

where \(h_y\) and \(h_z\) are the Fourier coefficients of the pseudospin magnetic field at frequency \(\omega\) and wavevector \(q\). Physically \(h_z\) corresponds to a time– and space–dependent bias potential between the two wells, while \(h_y\) could arise from a space– and time–dependent interlayer tunneling amplitude. We see immediately that the response has a singularity at the collective mode frequency

\[ \omega_q^2 = \left( \frac{4\pi}{\nu} \right)^2 \left[ 2\beta + q^2 \rho_A \right] q^2 \rho_s. \quad (96) \]

For the \(d/\ell = 0\) case, \(\beta = 0, \rho_A = \rho_s = \rho_s^0\), and the collective mode frequency reduces to the result obtained previously for the spin-wave collective mode of isotropic ferromagnets \([\omega_q = 4\pi q^2 \rho_s^0 / \nu]\). The collective mode corresponds to a spin-precession whose ellipticity increases as the long-wavelength limit is approached. The presence of the mass term \((\beta \neq 0)\) changes the collective mode dispersion at long wavelengths from quadratic to linear.\(^{41}\) In the limit of small \(q\)

\[ \omega_q = \frac{4\pi}{\nu} \sqrt{2\beta \rho_s} q. \quad (97) \]

Thus we see that the system is acting very much like a bosonic superfluid of weakly repulsively interacting particles which has a Goldstone mode (if and only if there is non-zero repulsion).\(^{41-46, 62, 69}\) However we must again emphasize that we have a charge neutral order parameter. The singular response occurs for electric fields of opposite signs in the two layers. Another way to say this is that the ‘charge’ conjugate to the U(1) phase field \(\varphi\) is \(S^z\). The current associated with phase gradients,
\[ J_{zz} = \frac{2\rho_s}{\hbar} \nabla \varphi, \]  

(98) is the flow of pseudospin density which is the difference between the electrical currents in the two layers.\(^{43,44,46,69}\) This simply reflects the fact that the object which condenses is a neutral excitonic composite of a particle and hole.

These analogies with superfluidity naturally raise the question of analogs of other phenomenon known to occur in superfluids and superconductors. It is believed that there is no analog of the Meissner effect in these systems.\(^{44-46}\) However, it has been suggested that Josephson or ‘Josephson like’ effects should occur in the tunneling transport between the two layers.\(^{42-44}\) In this picture one views each layer as analogous to one side of a Josephson junction. This appealing analogy appears to imply measureable consequences only in the case where superconducting contacts are made to each two-dimensional electron layer; the distinction between the system and an ordinary Josephson junction system then seems to be artificial. More interesting, in our view, is the recent ‘sideways tunneling’ proposal of Wen and Zee\(^{111}\) which appears to provide a quite precise analog of the Josephson effect, but in this case for pseudospin superfluidity. Here one views the pair of layers as constituting a single superfluid on one side of a Josephson junction. A separate pair of layers is imagined to be coupled to the first by weak ‘sideways’ tunneling. In each layer, there is a phase variable $\varphi$ and a pseudospin supercurrent given by Eq.\((\ref{98})\). When this current reaches the junction, it can coherently jump across the weak link and continue onwards. Physically this coherent tunneling of pseudospin involves an electron tunneling one way between the upper layers and another electron tunneling in the opposite direction between the lower layers. It should be possible in principle to see AC Josephson oscillations of the current at a frequency of $2eV/\hbar$ where $V$ an appropriately applied bias voltage (of opposite sign in the two layers). It is unlikely however that a sample with no tunneling within each pair of layers, but finite tunneling across the junction, can ever be produced. One could imagine using a parallel B field to effectively shut off the tunneling within each pair, but even weak disorder will pin the discommensurations and probably ruin the effect.

As an aside to the question of spin channel superfluidity, we point out that double layer systems are ideal for probing electron-electron interactions via mutual drag effects in transport at zero field.\(^{112}\) More details on some of the mutual transport effects associated with spontaneous interlayer phase coherence are presented in Ref.\([46]\). In addition, it has been predicted that for double-layer fractional quantum Hall states without broken symmetries, but with inter-layer correlations of the type described by the Halperin $\{m,m',n\}$ wavefunctions, anomalously large (and quantized!) mutual drag effects will occur.\(^{61,62}\)

We are now in a position to return to a discussion of the fourth experimental indication of interlayer phase coherence that was briefly alluded to at the end of subsection \[\text{VIII A}\], namely the fact that unlike the $\nu = 1/2$ state, the $\nu = 1$ state survives a finite amount of charge imbalance in the two layers. The rigidity of the $\nu = 1/2$ state can be understood macroscopically from the Chern Simons effective field theory point of view.\(^{58,113}\) or microscopically from the fact that the Halperin $\{3,3,1\}$ state is believed to give a good description of the state. The plasma analogy shows that this state has a sharply defined total filling $\nu = 1/2$ and that the density in each layer must be exactly equal (see Section \[\text{IV}\]). Another way to say this is that the $\{3,3,1\}$ state is almost a singlet spin liquid which must have
$S^z = 0$ (although strictly speaking, it is not an eigenstate of total spin). That is to say, it is pseudospin incompressible.

In contrast to this, the $\nu = 1$ state is pseudospin compressible. This is clear from the existence of pseudospin superfluidity with its gapless collective Goldstone mode. The state is ferromagnetically ordered and there is a whole family of states differing only in $S^z$ which would be degenerate were it not for the charging energy. The charging energy lifts this degeneracy and picks out the $S^z = 0$ state. Application of a bias voltage to unbalance the layers, simply picks a different state having non-zero $S^z$ out of the manifold.

We can supplement this picture by thinking macroscopically of the XY model describing the ordering. Layer imbalance corresponds to tilting the order parameter slightly up out of the XY plane. This changes no symmetries in the problem, and simply renormalizes the spin stiffness slightly because the projection of the spin onto the XY plane is reduced. Thus the ordering which produces the charge gap is weakened but not (immediately) destroyed by layer imbalance. One can also describe this microscopically by modifying the variational wave function in Eq. (90) to the form

$$|\Psi \rangle = \prod_x \frac{1}{\sqrt{1 + \gamma^2}} \left( c^\dagger_{x\uparrow} + \gamma e^{i\phi} c^\dagger_{x\downarrow} \right) |0\rangle,$$

which obeys

$$\langle \Psi | S^z | \Psi \rangle = \frac{N}{2} \left( 1 - \gamma^2 \right).$$

We close by noting that a class of experiments that would be very useful in probing the pseudospin superfluidity would be inelastic (Raman) light scattering to try to detect the gapless collective excitation mode. To see this at finite frequency requires finite wavevectors which in turn would require some sort of grating coupler. These are difficult to fabricate on sufficiently short length scales. Here however, it turns out that it is possible to have the system act as its own grating by tilting the applied magnetic field which will induce a twisting of the order parameter as we will discuss in Section X.

D. Merons: Charged Vortex Excitations

Continuing the superfluid analogy, we study vortex excitations in this section and discuss the Kosterlitz-Thouless phase transition induced by the unbinding of these topological defects. The order parameter of the system in the presence of a vortex at the origin has the following approximate form

$$m = \left\{ \pm \sqrt{1 - [m^z(r)]^2} \cos \theta, \sqrt{1 - [m^z(r)]^2} \sin \theta, m^z(r) \right\}.$$

Here the $\pm$ refers to right and left handed vortices respectively, and $\theta$ is the azimuthal angle made by the position vector $\mathbf{r}$. At asymptotically large radii, $m^z$ vanishes to minimize the charging energy. However in the vortex core we must have $m^z \rightarrow \pm 1$ (and $m^x, m^y \rightarrow 0$) to prevent a singularity in the gradient energy. Thus there are four flavors of topologically
stable objects which we refer to as ‘merons’ since it turns out that they are essentially half skyrmions. These are illustrated in Fig. [10].

The local topological charge density calculated from \( \delta \rho = -\frac{1}{8\pi} \epsilon_{ij}(\partial_i \mathbf{m} \times \partial_j \mathbf{m}) \cdot \mathbf{m} \) can be expressed in the form

\[
\delta \rho(r) = \frac{1}{4\pi r} \frac{dm_z}{dr},
\]

and the total charge is

\[
Q = \int d^2r \delta \rho(r) = \frac{1}{2}[m_z(\infty) - m_z(0)].
\]

For a meron, the spin points up or down at the core center and tilts away from the \( \hat{z} \) direction as the distance from the core center increases. At asymptotically large distances from the origin, the spins point purely radially in the \( \hat{x} - \hat{y} \) plane. Thus the topological charge is \( \pm \frac{1}{2} \) depending on the polarity of core spin. The general result for the topological charge of the four meron flavors may be summarized by the following formula:

\[
Q = \frac{1}{2}[m_z(\infty) - m_z(0)] n_v,
\]

where \( n_v \) is the vortex winding number. The formulae derived above for the meron charge do not rely for their validity on the variational ansatz assumed in Eq. (101). They are quite general and follow from the fact that a meron topologically has half the spin winding of a skyrmion. The meron charge of \( \pm 1/2 \) is a topological invariant and implies that the electrical charge is \( \pm ne/2 \).

The fact that merons carry fractional charge \( \pm ne/2 \) can be deduced from a Berry phase argument similar to the one used to find the skyrmion charge. We simply note that an electron moving at a large distance around a meron will have its spin rotated through \( 2\pi \) in the \( \hat{x} - \hat{y} \) plane due to the vorticity. We know that the Berry’s phase for rotating a spin one-half object in such a way is \( \exp \left( i \frac{\pi}{2} S \right) = -1 \). Thus the meron produces the same Berry’s phase as half a flux quantum. From Eq. (69) we then obtain \( \Delta Q = \pm ne/2 \). [The ambiguity of the sign of the charge associated with half a flux quantum can only be resolved by examining the behavior of \( m_z \) in the meron core. It depends on whether the mid-gap state induced by the topological defect (discussed below) is empty or occupied.]

It is instructive to write down explicit microscopic variational wave functions for vortices (merons). We start with the simplest example: a meron with vorticity +1 and charge \( -\frac{1}{2} \) that has the smallest possible core size:

\[
|\Psi_{+1,-\frac{1}{2}}\rangle = \prod_{m=0}^{M} \left( \frac{1}{\sqrt{2}} c_{m+\uparrow}^\dagger + \frac{1}{\sqrt{2}} c_{m+\downarrow}^\dagger \right) |0\rangle.
\]

Here \( |0\rangle \) is the fermion vacuum, \( c_{m\uparrow(\downarrow)}^\dagger \) creates an electron in the upper (lower) layer in the angular momentum \( m \) state in the LLL, and \( M \) is the angular momentum quantum number corresponding to the edge. The vorticity is +1 because far away the spin wave function is essentially
\[ \chi(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta} \\ 1 \end{pmatrix}, \]

where \( \theta \) is the polar angle. The charge is \(-\frac{1}{2}\) because we have created a hole in the center of the lower layer (every state has occupancy 1/2 except \( m = 0 \downarrow \) which is unoccupied). Since the spin is pointing up at the center, this agrees with the spin-charge relation derived earlier. From the spin-charge relation we know we can change the sign of the charge of a meron by changing the direction of spins in the core region without changing the vorticity. This can be seen explicitly from the wave function:

\[ |\Psi_{\uparrow+\downarrow/2}\rangle = c_0 \prod_{m=0}^{M} \left( \frac{1}{\sqrt{2}} c_m^\dagger + \frac{1}{\sqrt{2}} c_{m+1}^\dagger \right) |0\rangle. \]

This state has charge \(+\frac{1}{2}\) because we have put in an electron in the \( m = 0 \) state in the lower layer. Obviously what we did (in terms of the spin texture language) is to flip the spins in the core region to the down direction without changing the vorticity of the meron at long distances. In this construction, one sees that in a sense, the merons are like fractionally charged ‘mid-gap states’ in polyacetylene which can be empty or occupied.\(^{115}\)

A meron with vorticity -1 is readily obtained by simply interchanging the labels \( m \) and \( m + 1 \) in the subscripts in Eq.(105) and Eq.(107). Invariance under pseudospin reversal guarantees the equality of the energies of corresponding \( \pm \) vorticity merons. However the two charge states for a given vorticity are not necessarily degenerate, just as Laughlin quasiholes and quasiparticles are in general non-degenerate. [For the special case \( \nu = 1 \) particle-hole symmetry guarantees degeneracy.]

It is instructive to attempt to find a variational wave function for a pair of merons to show that a meron is half a skyrmion. Consider the situation where we have a pair of merons of opposite vorticity but the same charge, located at points \( \bar{Z}_1 \) and \( \bar{Z}_2 \). To achieve this one meron must have \( m^z = +1 \) in its core and the other must have \( m^z = -1 \) in its core. The following wave function seems to do the job:

\[ \psi_{\lambda} = \prod_j \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi}(Z_j - \bar{Z}_1) \\ (Z_j - \bar{Z}_2) \end{pmatrix} \Phi_{\text{mmm}}, \]

where \( \Phi_{\text{mmm}} \) is defined in Eq.(5), \( \varphi \) is an arbitrary constant phase angle, and \(()\) refers to the spinor for the \( j \)th particle. Note that at very large distances away from \( \bar{Z}_1 \) and \( \bar{Z}_2 \) the spinor for each particle asymptotically approaches

\[ Z_j \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi} \\ 1 \end{pmatrix}, \]

and so corresponds to a fixed spin orientation in the xy plane at an angle \( \varphi \) from the x axis. Hence the net vortex content of the pair of excitations is zero. It is clear from the construction that the spin orientation is purely up for an electron located at \( \bar{Z}_2 \) and purely down at \( \bar{Z}_1 \). Furthermore the net charge must be \(+\nu e\) since asymptotically the factor of \( Z_j \) is the same one as for the Laughlin quasihole in a spin polarized state. By symmetry, it seems that there must be half that amount, \(+\nu e/2\) associated with each of the two objects.
located at $Z_1$ and $Z_2$. (It turns out however, that the charge is not split in half and localized near $\tilde{Z}_1$ and $\tilde{Z}_2$ as we would really want for a meron pair.)

We can easily show that this particular wave function actually is just another way to represent a skyrmion. Let $Z_1 = \lambda$ and $Z_2 = -\lambda$ and, for simplicity, assume that the spins are asymptotically oriented in the $\hat{x}$ direction so that $\varphi = 0$. Now perform a global rotation of all the spins about the $\hat{y}$ axis by an angle of $-\pi/2$. Using

$$\exp \left\{ \frac{i\pi}{4} \sigma^y \right\} \frac{1}{\sqrt{2}} \left( \begin{pmatrix} Z_j - \lambda \\ Z_j + \lambda \end{pmatrix} \right) = \left( \begin{pmatrix} Z_j \\ \lambda \end{pmatrix} \right),$$

we see that we have recovered the variational skyrmion wave function studied previously in Eq. (72). The form in Eq. (108) is the appropriate one in the U(1) symmetric case since it keeps the spins primarily in the XY plane. We expect that a pair of merons can be deformed into a skyrmion and each meron can be properly viewed as carrying half the topological and electrical charge of a skyrmion. However there must exist variational wave functions which are better for the U(1) case than the one discussed here in the sense that the present one has a smooth charge distribution centered in between $\tilde{Z}_1$ and $\tilde{Z}_2$ rather than being more closely associated with the two points defining the meron positions.

Finally we note that merons carry fractional statistics $1/4$; they are ‘quarterons’.\textsuperscript{46} This can be seen from the Chern Simons theory\textsuperscript{46} or from the fact that two of them together make a skyrmion which is a fermion.

E. Kosterlitz-Thouless Phase Transition

It is well known that the presence of vortex topological defects can disorder the XY phase of the ground state. This will occur even at zero temperature due to quantum fluctuations if the layer separation exceeds a critical value $d^*$. One could also conveniently tune through this transition in a single sample with fixed $d$ by varying a gate voltage to induce layer imbalance which will renormalize the spin stiffness. We focus here however on the effect of finite temperature and thermally induced vortices in samples which are otherwise ordered at zero temperature (because $d < d^*$).

Integrating out the massive $m^2$ fluctuations and going to finite temperature, we are led to a classical XY model of the form shown in Eq. (83). Hartree-Fock estimates of the spin stiffness at finite layer separation give values in the range 0.1-0.5 K for typical experimental sample parameters. The Kosterlitz-Thouless phase transition is an unbinding of vortex defects in the XY model and occurs at a temperature given approximately by the value of the bare spin stiffness. These unbound vortices will cause the long wavelength effective spin stiffness in Eq. (83) to renormalize discontinuously to zero at the Kosterlitz-Thouless critical point.

In the usual way, the scale-invariant classical action of Eq. (83) leads to a logarithmic interaction among vortices. A gas consisting of M merons will have an energy of the form

$$E = M E_{\text{core}} - \sum_{i<j}^M n_i n_j 2\pi \rho_s \ln \left( \frac{R}{R_{\text{core}}} \right) + \sum_{i<j}^M q_i q_j \frac{e^2}{4\epsilon R_{ij}}. \quad (111)$$

36
Here $E_{\text{core}}$ is the meron core energy,$^{116}$ $R_{\text{core}}$ is the meron core size, $R_{ij}$ is the separation between the $i$th and $j$th merons, $n_i$ is the vortex charge (winding number, $\pm 1$) of the $i$th meron. The last term is new and is unique to the present problem. It represents the Coulomb interaction among the fractional charges bound to the merons. $q_i = \pm 1$ is the sign of the electrical charge ($\pm e/2$) of the $i$th meron.

The origin of the logarithmic interaction in a superconducting or superfluid film is the kinetic energy stored in the supercurrents circulating around the vortices due to order parameter phase gradients. Here the logarithmic interaction arises from the loss of Coulomb exchange potential energy in the presence of phase gradients produced by the vortices.

The Coulomb interaction in Eq. (111) falls off more rapidly than the log interaction (it is effectively one order higher in derivatives than the log interaction) and so is perturbatively irrelevant at the Kosterlitz-Thouless critical point. That is, the Kosterlitz-Thouless temperature may be shifted somewhat but the phase transition itself is unaffected. However in the limit of strong Coulomb effects (due to small $\epsilon$ or large vortex fugacity) the global phase diagram becomes extremely rich. Among other things there is a phase transition to a chiral state with non-zero order parameter $\langle n_i q_i \rangle$ in which vortex charge and electrical charge are no longer independent. The rich physics and the novel phase diagram of this model has been elucidated in an interesting paper by Tupitsyn, Wallin, and Rosengren.$^{117}$

If found experimentally, this KT transition would be the first finite temperature phase transition in a quantum Hall system. All other transitions between plateaus, etc. are zero temperature transitions because the vortices there (Laughlin quasiparticles) do not interact logarithmically and are unconfined by an analog of the Anderson-Higgs mechanism.$^6,7,63,64$ As noted above the characteristic energy scale for $\rho_s$ is 0.1-0.5 K. This is not a problem from the point of view of experimentally achievable temperatures. However, in order to have the needed U(1) symmetry it is essential (as we discuss in Section [X]) that the tunneling amplitude between the layers be much smaller than this scale. Such samples have not yet been constructed, but could be in principle because the tunneling amplitude falls off exponentially with layer separation while the Coulomb interactions which control the stiffness $\rho_s$ fall off only as a power law. Nevertheless it will not be easy since present samples are already close to the critical separation $d^*/\ell$. The latter could perhaps be circumvented by going to lower density samples and correspondingly lower magnetic fields.

The standard experimental signatures of the KT transition are zero linear response dissipation below $T_{\text{KT}}$, but zero critical current, and a characteristic jump in the exponent associated with the non-linear response

$$V \sim I^p,$$

from $p = 1$ above the transition to $p = 3$ just below the transition.$^{118}$ Dissipation is caused by phase slips associated with the motion of unconfined vortices. A transport current exerts a magnus force whose direction depends on the sign of the vorticity and which moves the vortices at right angles to the current. Below the transition the vortices are confined into (vorticity) neutral pairs by their logarithmic attraction and no longer couple to the supercurrent.

In a superconductor or superfluid at finite temperature there is always some sort of ‘normal’ fluid present (thermally excited Bogoljubov quasiparticles, for example) which can
produce dissipation. However there are no electric fields to couple to these particles since the supercurrent shorts them out. Thus there is no dissipation. Analogous normal fluid excitations exist in the quantum Hall systems we are studying here. If we define normal fluid as anything which is charged but has no net vorticity then we see that a pair of opposite vorticity, but like-charged merons constitutes normal fluid that can be thermally activated. The validity of the meron pair picture has been confirmed by numerical calculations. In the pseudospin superfluid channel (i.e., opposite electrical currents in each layer) this normal fluid will not couple to the supercurrent since it is vortex neutral. The linear response dissipation will therefore drop to zero below $T_{KT}$ and all the other usual signatures of the Kosterlitz-Thouless transition will be present.

Doing transport experiments in the pseudospin channel requires separately contacting the two layers and having extremely high tunneling resistance between the layers without moving them so far apart that they decouple. It would be easier to do the experiment in the ordinary charge channel in which both layers are contacted simultaneously and current flows in the same direction in both layers. Unfortunately in this channel there is a net electric field induced by the $\nu = 1$ quantized Hall resistivity and the normal fluid will couple to this producing thermally activated dissipation. While we would expect to see the dissipation decrease rapidly towards zero below $T_{KT}$, it would not show any discontinuity as it would in the pseudospin channel. This is the same mechanism that causes the small but non-zero thermally activated dissipation in ordinary quantum Hall plateaus.

The energy for an excitation of ‘normal fluid’ consisting of a pair of like-charged but opposite vorticity merons given by Eq.(111) is minimized at a separation of

$$R_0 = \frac{e^2}{8\pi \epsilon \rho_s}.$$  \hspace{1cm} (113)

For typical values of $\rho_s$, $R_0 \geq 10\ell$, and so is large enough to justify the field-theoretic continuum approximations used in deriving Eq.(111).

**IX. TUNNELING BETWEEN THE LAYERS**

A finite tunneling amplitude $t$ between the layers breaks the $U(1)$ symmetry

$$H_{\text{eff}} = \int d^2 r \left[ \frac{1}{2} \rho_s |\nabla \varphi|^2 - \frac{t}{2\pi \ell^2} \cos \varphi \right]$$  \hspace{1cm} (114)

by giving a preference to symmetric tunneling states. This can be seen from the tunneling Hamiltonian

$$H_T = -t \int d^2 r \left\{ \psi^\dagger (\mathbf{r}) \psi (\mathbf{r}) + \psi^\dagger (\mathbf{r}) \psi (\mathbf{r}) \right\},$$  \hspace{1cm} (115)

which can be written in the pseudospin representation as

$$H_T = -2t \int d^2 r S^x (\mathbf{r}).$$  \hspace{1cm} (116)

(Recall that the eigenstates of $S^x$ are symmetric and antisymmetric combinations of up and down.)
We can shed some more light on the spontaneous symmetry breaking by considering the tunneling Hamiltonian $H_T$ in Eq. (115) as a weak perturbation. Naively, since particle number is separately conserved in each layer for $t = 0$, one might expect

$$\lim_{t \to 0} \frac{1}{t} \langle \psi | H_T | \psi \rangle = 0.$$  \hspace{1cm} (117)

That is, one might expect that the first-order term in the perturbation series for the energy shift due to $t$ to vanish. Instead however, we find that the energy shifts linearly in $t$

$$\lim_{t \to 0} \lim_{A \to \infty} \frac{1}{tA} \langle \psi | H_T | \psi \rangle = \lim_{t \to 0} \lim_{A \to \infty} -\frac{1}{tA} \left( \psi \left| \int d^2 r \ 2S^x(r) \right| \psi \right),$$  

where $A$ is the system area, and $m^x$ is, by definition, the magnetization which is the system’s order parameter.\(^{120}\) If the interlayer spacing $d$ is taken to be zero, one can readily show\(^{46}\) that the variational wave function in Eq. (89) is exact, hence $\lim_{t \to 0} m^x = 1$, and $t = \Delta_{SAS}/2$. For finite $d$, Eq. (89) is no longer exact and quantum fluctuations will\(^{46}\) reduce the magnitude of $m^x$ and we must renormalize the hopping parameter $t$ appropriately.

As the layer separation $d$ increases, a critical point $d^*$ will be reached at which the magnetization vanishes and the ordered phase is destroyed by quantum fluctuations.\(^{41,45,46}\) This is illustrated in Fig. [8] and discussed in more detail from the experimental point of view in the Chapter by J. P. Eisenstein.\(^{38}\) For finite tunneling $t$, the collective mode becomes massive and quantum fluctuations will be less severe.\(^{47}\) Hence the phase boundary in Fig. [8] curves upward with increasing $\Delta_{SAS}$. For $\Delta_{SAS} = 0$ the destruction of long-range order and the charge excitation gap are intimately related and occur simultaneously at $d^*$ and zero temperature. For finite $\Delta_{SAS}$ the system always has non-zero $m^x$ even in the phase with zero charge gap.

We have already seen that finite layer separation reduces the pseudospin symmetry from SU(2) to U(1). The introduction of finite tunneling amplitude destroys the U(1) symmetry and makes the simple vortex-pair configuration extremely expensive, thereby destroying the KT transition. To lower the energy, the system distorts the spin deviations into a domain wall or ‘string’ connecting the vortex cores as shown in Fig. [11]. The spins are oriented in the $\hat{x}$ direction everywhere except in the shaded domain line region where they tumble rapidly through $2\pi$. The domain line has a fixed energy per unit length and so the vortices are now confined by a linear ‘string tension’ rather than logarithmically. We can estimate the string tension by examining the energy of a domain line of infinite length. The optimal form for a domain line lying along the $y$ axis is given by

$$\varphi(r) = 2\arcsin [\tanh (x/\xi)],$$  

where the characteristic width of the string is

$$\xi = \left[ \frac{2\pi \ell^2 \rho_s}{t} \right]^{\frac{1}{2}}.$$  \hspace{1cm} (120)

The resulting string tension is\(^{121}\)
\[
T_0 = 8 \left[ \frac{t \rho_s}{2 \pi \ell^2} \right]^{\frac{1}{2}} = \frac{8 \rho_s}{\xi}.
\]  
(121)

Provided the string is long enough \((R \gg \xi)\), the total energy of a segment of length \(R\) will be well-approximated by the expression

\[
E'_\text{pair} = 2E'_\text{mc} + \frac{e^2}{4\epsilon R} + T_0 R.
\]  
(122)

The prime on \(E_{mc}\) in Eq. (122) indicates that the meron core energy can depend \(\Delta_{SAS}\). \(E'_\text{pair}\) is minimized at \(R = R'_0 \equiv \sqrt{e^2/(4\epsilon T_0)}\). Note that apart from the core energies, the charge gap at fixed layer separation (and hence fixed \(\rho_s\)) is proportional to \(T_0^{1/2} \sim \Delta_{SAS}^{1/4}\) which contrasts with the case of free electrons, for which the charge gap is proportional to \(\Delta_{SAS}\). Note that because the exponent 1/4 is so small, there is an extremely rapid initial increase in the charge gap when tunneling first becomes important. (See below.) Note that the present classical analysis ignores the stabilizing effect of the tunneling-induced gap on the quantum fluctuations of the pseudospin magnetization.

The crossover between the meron-pair pseudospin texture which holds for \(t \equiv 0\) and the domain line string pseudospin texture described above occurs at a finite value of \(t\) which we can estimate by the following argument. Recall from Eq. (113), that the equilibrium separation of a meron pair in a system with no tunneling is defined to be \(R_0\). For \(R'_0 > R_0\) the vortices are already bound by the logarithmic attraction due to the gradient energy before the linear attraction due to the hopping becomes important at larger separations. In this regime tunneling does not play an important role in determining the nature of the lowest energy charged pseudospin texture. As \(t\) increases \(R'_0 \propto t^{-1/4}\) decreases and will eventually reach \(R_0\) which is, of course, independent of \(t\). Since

\[
\frac{R'_0}{R_0} = \left( \frac{2\pi \rho_s}{e^2/(\epsilon \xi)} \right)^{1/2} = \frac{\pi \xi}{4R'_0},
\]  
(123)

the characteristic width of the domain line becomes comparable to \(R'_0\) in the same range of \(t\) values where \(R'_0\) and \(R_0\) become comparable. We may conclude that the nature of the charged pseudospin texture crosses over directly from the meron pair form to the finite length domain line string form for \(\rho_s/[e^2/(\epsilon \xi)] \sim 1/25\), or equivalently for \(t \sim t_{cr}\) where

\[
t_{cr} \approx 3.9 \times 10^3 \left[ \frac{\rho_s}{e^2/(\epsilon \ell)} \right]^3.
\]  
(124)

The crossover tunneling amplitude is thus typically smaller than \(5 \times 10^{-4}[e^2/(\epsilon \ell)]\). Typical tunneling amplitudes in double-layer systems are smaller than \(\sim 10^{-1}[e^2/(\epsilon \ell)]\) and can be made quite small by adjusting the barrier material or or making the barrier wider. Nevertheless, it seems likely that \(t\) will be larger than \(t_{cr}\) except for samples which are carefully prepared to make \(t\) as small as possible. As \(t\) increases beyond \(t_{cr}\), \(R'_0\) will continue to decrease. When \(R'_0\) becomes comparable to the microscopic length, \(\ell\), the description given here will become invalid and the lowest energy charged excitations will have single-particle character. However, the domain-wall string picture of the charged pseudospin texture has a
very large range of validity since \( R_0' \propto t^{-1/4} \) decreases very slowly with increasing \( t \) at large \( t \). Writing \( R_0' \sim \left[ e^2/(\epsilon \pi \rho_s) \right] (t_{cr}/t)^{1/4} \) we find that \( R_0' \sim \ell \) only for \( t \sim 10^{-2}[(e^2/\epsilon \ell)^2/\rho_s] \). Using typical values of \( \rho_s \) we see that the charged excitation crosses over to single particle character only when the hopping energy \( t \) becomes comparable to the microscopic interaction energy scale. The various regimes for the charge excitations of double-layer systems are summarized in Table III. Almost all typical double-layer systems lie within the regime of the domain-wall-string pseudospin texture charge excitation, and hence will not exhibit a true Kosterlitz-Thouless phase transition.

**X. PARALLEL MAGNETIC FIELD IN DOUBLE LAYER SYSTEMS**

Murphy et al.\(^{39}\) have shown that the charge gap in double layer systems is remarkably sensitive to the application of relatively weak magnetic fields \( B_\parallel \), oriented in the plane of the 2D electron gas. Experimentally this field component is generated by slightly tilting the sample relative to the magnetic field orientation. Tilting the field (or sample) has traditionally been an effective method for identifying effects due to (real) spins because orbital motion in a single-layer 2DEG system is primarily\(^{36}\) sensitive to \( B_\perp \), while the (real) spin Zeeman splitting is proportional to the full magnitude of \( B \). Adding a parallel field component will tend to favor more strongly spin-polarized states. For the case of the double layer \( \nu = 1 \) systems studied by Murphy et al.,\(^{39}\) the ground state is known to already be an isotropic ferromagnetic state of the *true spins* and the addition of a parallel field would not, at first glance, be expected to influence the low energy states since they are already fully spin-polarized. (At a fixed Landau level filling factor, \( B_\perp \) is fixed and so the total \( B \) and the corresponding Zeeman energy increase with tilt). Nevertheless experiments\(^{39}\) have shown that these systems are very sensitive to \( B_\parallel \). The activation energy drops rapidly (by factors varying from two up to an order-of-magnitude in different samples) with increasing \( B_\parallel \). At \( B_\parallel = B^*_\parallel \) there appears to be a phase transition to a new state whose activation gap is approximately independent of further increases in \( B_\parallel \).

The effect of \( B_\parallel \) on the *pseudospin* system can be visualized in two different pictures, one microscopic, the other macroscopic. We will present the latter here. The technical details of the former are described elsewhere.\(^{45,47}\) Recent work presents a discussion of higher Landau levels.\(^{122}\)

We use a gauge in which \( B_\parallel = \nabla \times A_\parallel \) where \( A_\parallel = B_\parallel (0,0,x) \). In this gauge the vector potential points in the \( \hat{z} \) direction (perpendicular to the layers) and varies with position \( x \) as one moves parallel to the layers. In this gauge, the only change in the Hamiltonian caused by the parallel field is in the term which describes tunneling between layers. As an electron tunnels from one layer to the other it moves along the direction in which the vector potential points and so the tunneling matrix element acquires a position-dependent phase \( t \rightarrow t e^{iQx} \) where \( Q = 2\pi/L_\parallel \) and \( L_\parallel = \Phi_0/B_\parallel d \) is the length associated with one flux quantum \( \Phi_0 \) between the layers (defined in Fig. 3). This modifies the tunneling Hamiltonian to \( H_T = -\int d^2r \ h(r) \cdot S(r) \) where \( h(r) \) ‘tumbles’: i.e., \( h(r) = 2t \ (\cos Qx, \sin Qx, 0) \). The effective XY model now becomes

\[
H = \int d^2r \left\{ \frac{1}{2} \rho_s |\nabla \varphi|^2 - \frac{t}{2\pi \ell} \cos [\varphi(r) - Qx] \right\};
\]

(125)
which is precisely the Pokrovsky-Talapov (P-T) model\textsuperscript{123} and has a very rich phase diagram. For small $Q$ and/or small $\rho_s$ the phase obeys (at low temperatures) $\varphi(r) \equiv Qx$; the order parameter rotates commensurately with the pseudospin Zeeman field. However, as $B_\parallel$ is increased, the local field tumbles too rapidly and a continuous phase transition to an incommensurate state with broken translation symmetry occurs. This is because at large $B_\parallel$, it costs too much exchange energy to remain commensurate and the system rapidly gives up the tunneling energy in order to return to a uniform state $\nabla \varphi \approx 0$ which becomes independent of $B_\parallel$. As explained in further detail below we find that the phase transition occurs at zero temperature for\textsuperscript{45, 47}

\[ B^*_\parallel = B_\perp (2\ell/\pi d)(2t/\pi \rho_s)^{1/2}. \]

(126)

Using the parameters for the sample of Murphy et al.\textsuperscript{39} with weakest tunneling\textsuperscript{124} ($\Delta_{SAS} = 0.45$ K) and neglecting quantum fluctuation renormalizations of both $t$ and $\rho_s$ (i.e., using Hartree Fock) we find that the critical field for the transition is $\approx 1.3$T which is slightly larger than the observed value\textsuperscript{39, 124} of 0.8 T, but still correctly corresponds to a very large length $L_\parallel$. A graphical comparison showing the qualitative agreement between the predicted and observed values of the critical tilt angle for several samples is shown in Fig.(15) of the Chapter by Eisenstein in this volume.

In addition to the Hartree-Fock calculations described here, we have made numerical exact diagonalization studies on small systems to find the critical value of the parallel field. Although it is difficult to extrapolate the results to the thermodynamic limit, they do confirm that at finite layer separation, quantum fluctuations can reduce the predicted critical parallel field.\textsuperscript{47}

As previously mentioned, the observed value $B^*_\parallel = 0.8$T corresponds in these samples to a large value for $L_\parallel$: $L_\parallel/\ell \sim 20$ indicating that the transition is highly collective in nature. We emphasize again that these very large length scales are possible in a magnetic field only because of the interlayer phase coherence in the system associated with condensation of a neutral object.

Having argued for the existence of the commensurate-incommensurate transition, we must now connect it to the experimentally observed transport properties. In the commensurate phase, the order parameter tumbles more and more rapidly as $B_\parallel$ increases. As we shall see below, it is this tumbling which causes the charge gap to drop rapidly. In the incommensurate phase, the state of the system is approximately independent of $B_\parallel$ and this causes the charge excitation gap to saturate at a fixed value.

Recall that in the presence of tunneling, the cheapest charged excitation was found to be a pair of vortices of opposite vorticity and like charge (each having charge $\pm 1/2$) connected by a domain line with a constant string tension. In the absence of $B_\parallel$ the energy is independent of the orientation of the string. The effect of $B_\parallel$ is most easily studied by changing variables to

\[ \theta(r) \equiv \varphi(r) - Qx. \]

(127)

This variable is a constant in the commensurate phase but not in the incommensurate phase. In terms of this new variable, the P-T model energy is
\[ H = \int d^2 r \left\{ \frac{1}{2} \rho_s \left[ (\partial_x \theta + Q)^2 + (\partial_y)^2 \right] - \frac{t}{2\pi \ell^2} \cos(\theta) \right\}. \]  

(128)

We see that \( B_\parallel \) defines a preferred direction in the problem. Domain walls will want to line up in the \( y \) direction and contain a phase slip of a preferred sign \((-2\pi \text{ for } Q > 0)\) in terms of the field \( \theta \). Since the extra term induced by \( Q \) represents a total derivative, the optimal form of the soliton solution is unchanged. However the energy per unit length of the soliton, which is the domain line string tension, decreases linearly with \( Q \) and hence \( B_\parallel \):

\[ T = T_0 \left[ 1 - \frac{B_\parallel}{B^*_\parallel} \right], \]  

(129)

where \( T_0 \) is the tension in the absence of parallel B field given by Eq.(121), and \( B^*_\parallel \) is the critical parallel field at which the string tension goes to zero.\(^{125}\) We thus see that by tuning \( B_\parallel \) one can conveniently control the ‘chemical potential’ of the domain lines. The domain lines condense and the phase transition occurs (in mean field theory) when the string tension becomes negative.

Recall that the charge excitation gap is given by the energy of a vortex pair separated by the optimal distance \( R_0 = \sqrt{e^2/(4\epsilon T)} \). From Eq.[122] we have that the energy gap far on the commensurate side of the phase transition is given by

\[ \Delta = 2E'_{mc} + \frac{\sqrt{e^2T}}{\epsilon} \left[ 1 - \left( \frac{B_\parallel}{B^*_\parallel} \right)^{1/2} \right]. \]  

(130)

As \( B_\parallel \) increases the reduced string tension allows the Coulomb repulsion of the two vortices to stretch the string and lower the energy. Far on the incommensurate side of the phase transition, the possibility of interlayer tunneling becomes irrelevant. From the discussion of the previous section one can argue that the ratio of the charge gap at \( B_\parallel = 0 \) to the charge gap at \( B_\parallel \to \infty \) should be given (very roughly) by

\[ \frac{\Delta_0}{\Delta_\infty} = (t/t_{cr})^{1/4} \approx \frac{(e^2/\epsilon \ell)^{1/2} t^{1/4}}{8 \rho_s^{3/4}}. \]  

(131)

Putting in typical values of \( t \) and \( \rho_s \) gives gap ratios in the range \( \sim 1.5 - 7 \) in qualitative agreement with experiment. According to the discussion of the previous section, gap ratios as large as \( \sim (t_{max}/t_{cr})^{1/4} \sim 0.07(e^2/\epsilon \ell)/\rho_s \), can be expected in the regime where the pseudospin texture picture applies. Here \( t_{max} \) is the hopping parameter at which the crossover to single-particle excitations occurs. Thus gap ratios as large as an order of magnitude are easily possible. Of course, all the discussion here neglects orbital effects (electric subband mixing) within each of the electron gas layers, and these will always become important at sufficiently strong parallel fields. [Note also that for \( d \) near \( d^* \), the system will have enhanced sensitivity to renormalization of parameters by electric subband mixing in tilted fields.]
It should be emphasized that only this highly collective picture involving large length scale distortions of topological defects can possibly explain the extreme sensitivity of the charge gap to small tilts of the B field. Recall that at $B^*_\parallel$, the tumbling length $L^{\parallel}$ is much larger than the particle spacing and the magnetic length. Simple estimates of the cost to make a local one-body type excitation (a spin-flip pair for example) shows that the energy decrease due to $B^{\parallel}$ is extremely small since $\ell/L^{\parallel}$ is so small. Numerical exact-diagonalization calculations on small systems confirm the existence of this phase transition and show that the fermionic excitation gap drops to a much smaller value in the incommensurate phase.\textsuperscript{47}

The collective excitation modes of the system in the commensurate and incommensurate phases appear to be quite interesting.\textsuperscript{114} As mentioned previously, the tumbling of the order parameter may produce a self-grating effect which will allow one to tune the wavevector which couples to optical probes.

All of our discussion of the phase transition in a parallel field has been based on mean-field theory. Close to the phase transition, thermal fluctuations will be important. At finite temperatures there is no strict phase transition at $B^*_\parallel$ in the the P-T model. However there is a finite temperature KT phase transition at a nearby $B^{\parallel} > B^*_\parallel$. At finite temperatures translation symmetry is restored\textsuperscript{123} in the incommensurate phase by means of dislocations in the domain string structure. Thus there are two separate KT transitions in this system, one for $t = 0$, the other for $t \neq 0$ and $B^{\parallel} > B^*_\parallel$. Recently Read\textsuperscript{126} has studied this model at finite temperatures in some detail and has shown that just at the critical value of $B^{\parallel}$ there should be a square-root singularity in the charge gap. The existing data does not have the resolution to show this however. M. P. A. Fisher has pointed out\textsuperscript{127} that at zero-temperature the commensurate-incommensurate phase transition must be treated quantum mechanically. It is necessary to take account of the world sheets traced out by the time evolution of the strings which fluctuate into existence due to quantum zero-point motion. He has also pointed out that the inevitable random variations in the tunneling amplitude with position, which we have not considered at all here, cause a relevant perturbation.

\textbf{XI. SUMMARY}

We have discussed the origin of spontaneous ordering in multicomponent fractional quantum Hall effect systems. For ‘real’ spin at filling factors $\nu = 1$ this spontaneous ferromagnetism induces a very large charge excitation gap even in the absence of a Zeeman gap. The charged excitations are interesting topological ‘skyrmion’-like objects. Magnetic resonance experiments\textsuperscript{26,27} have confirmed this remarkable picture which was developed analytically by Sondhi et al.\textsuperscript{24} to explain the numerical results of Rezayi.\textsuperscript{96}

We have discussed in detail a pseudospin analogy which shows how spontaneous interlayer coherence in double layer quantum Hall systems arises. This coherent XY phase order occurring over long length scales is essential to explain the experimental observations of Murphy et al.\textsuperscript{39} described in the Chapter in this volume by J. P. Eisenstein. The essential physics is condensation of a charge-neutral bosonic order parameter field (pseudospin magnetization). This condensation controls the charge excitation gap and is very sensitive to interlayer tunneling and parallel magnetic field.

We summarize a portion of this rich set of phenomena in the schematic zero-temperature
phase diagram of double layer systems shown in Fig.[12]. First consider the plane with \( \Delta_{\text{SAS}} = 0 \) (zero tunneling). We have argued that the system develops spontaneous interlayer phase coherence despite the fact that the tunneling amplitude is zero. However if the layer spacing \( d \) exceeds a critical value \( d^* \), the system is unable to support a state with strong interlayer correlations and the spontaneous \( U(1) \) symmetry breaking is destroyed by quantum fluctuations.\(^{41}\) At this same point we expect the fermionic gap \( \Delta \mu \) to collapse. Little is known about the nature of this quantum transition which can be viewed as arising from proliferation of quantum induced vortices (merons). This is similar to the quantum XY model, but in the present case, the merons are fractional-statistics anyons which will presumably change the universality class of the transition.\(^{46}\)

At finite tunneling, the \( U(1) \) symmetry is destroyed and the quantum fluctuations are gapped and hence stabilized. This causes the critical layer spacing to increase as shown in Fig.[12].

The third axis in the figure is the tilt of the magnetic field. Magnetic flux between the layers causes the order parameter to want to ‘tumble’. For small tilts, the system is in a commensurate phase with the order parameter tumbling smoothly. However above a critical value of the parallel field, this tumbling costs too much exchange energy, and the system goes into an incommensurate phase which spontaneously breaks translation symmetry (in the absence of disorder). This phase transition has been observed by Murphy et al.\(^{38,39}\) through the rapid drop in the charge gap as the field is tilted. We have presented arguments that the charge gap is determined by the cost of creating a highly collective object: a pair of fractionally charged vortices connected by a string. It is the decrease of the string tension with tilt which causes the extreme sensitivity to small tilts.

In addition to all this, there is (for zero tunneling) a finite temperature Kosterlitz-Thouless phase transition. If observed experimentally, this would represent the first finite temperature phase transition in a quantum Hall system.

We have tried to keep the present discussion as qualitative as possible. The reader is directed to the many references for more detailed discussion of technical points. We close by noting that there are still many open questions in this very rich field concerning such things as edge states in multicomponent systems, proper treatment of quantum fluctuations for the highly collective excitations which appear to exist in these systems, and the nature of the phase transition at the critical value of the layer separation.

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TABLE I. Generalized Laughlin states for two component systems ($S$ is the total spin quantum and $\ast$ denotes a state which is not an eigenstate of $\mathbf{S}^2$). The nominal filling factors $\nu_\uparrow$ and $\nu_\downarrow$ are shown in parentheses for the ferromagnetic $\{m, m, m\}$ states because these are not unique (only their sum $\nu$ is fixed). (After Ref.[22]).

| $m$ | $m'$ | $n$ | $\nu_\uparrow$ | $\nu_\downarrow$ | $\nu$ | $S$ |
|-----|------|-----|---------------|---------------|------|-----|
| 1   | 1    | 0   | 1             | 1             | 2    | 0   |
| 1   | 1    | 1   | (1/2)         | (1/2)         | 1    | $N/2$ |
| 1   | 3    | 0   | 1             | 1/3           | 4/3  | $N/4$ |
| 1   | 5    | 0   | 1             | 1/5           | 6/5  | $N/3$ |
| 3   | 3    | 0   | 1/3           | 1/3           | 2/3  | $\ast$ |
| 3   | 3    | 1   | 1/4           | 1/4           | 1/2  | $\ast$ |
| 3   | 3    | 2   | 1/5           | 1/5           | 2/5  | 0   |
| 3   | 3    | 3   | (1/6)         | (1/6)         | 1/3  | $N/2$ |
| 3   | 5    | 0   | 1/3           | 1/5           | 8/15 | $\ast$ |
| 3   | 5    | 1   | 2/7           | 1/7           | 3/7  | $\ast$ |
| 3   | 5    | 2   | 3/11          | 1/11          | 4/11 | $N/4$ |
| 5   | 5    | 0   | 1/5           | 1/5           | 2/5  | $\ast$ |
| 5   | 5    | 1   | 1/6           | 1/6           | 1/3  | $\ast$ |
| 5   | 5    | 2   | 1/7           | 1/7           | 2/7  | $\ast$ |
| 5   | 5    | 3   | 1/8           | 1/8           | 1/4  | $\ast$ |
| 5   | 5    | 4   | 1/9           | 1/9           | 2/9  | 0   |
| 5   | 5    | 5   | (1/10)        | (1/10)        | 1/5  | $N/2$ |
### TABLE II. Fractional charges for some two-component fractional quantum Hall effect states.

$e_{X'}^X$ gives the contribution to the charge from the $X'$ component in quasihole state $X$.

| $m$ | $m'$ | $n$ | $e_R^A$ | $e_G^A$ | $e_T^A$ | $e_R^B$ | $e_G^B$ | $e_T^B$ |
|-----|------|-----|--------|--------|--------|--------|--------|--------|
| 1   | 1    | 0   | 1      | 0      | 1      | 0      | 1      | 1      |
| 1   | 1    | 1   | ?      | ?      | 1      | ?      | ?      | 1      |
| 1   | 3    | 0   | 1      | 0      | 1      | 0      | 1/3    | 1/3    |
| 1   | 5    | 0   | 1      | 0      | 1      | 0      | 1/5    | 1/5    |
| 3   | 3    | 0   | 1/3    | 0      | 1/3    | 0      | 1/3    | 1/3    |
| 3   | 3    | 1   | 3/8    | −1/8   | 1/4    | −1/8   | 3/8    | 1/4    |
| 3   | 3    | 2   | 3/5    | −2/5   | 1/5    | −2/5   | 3/5    | 1/5    |
| 3   | 3    | 3   | ?      | ?      | 1/3    | ?      | ?      | 1/3    |
| 3   | 5    | 1   | 5/14   | −1/14  | 2/7    | −1/14  | 3/14   | 1/7    |
| 3   | 5    | 2   | 5/11   | −2/11  | 3/11   | −2/11  | 3/11   | 1/11   |
| 5   | 5    | 0   | 1/5    | 0      | 1/5    | 0      | 1/5    | 1/5    |
| 5   | 5    | 1   | 5/24   | −1/24  | 1/6    | −1/24  | 5/24   | 1/6    |
| 5   | 5    | 2   | 5/21   | −2/21  | 1/7    | −2/21  | 5/21   | 1/7    |
| 5   | 5    | 3   | 5/16   | −3/16  | 1/8    | −3/16  | 5/16   | 1/8    |
| 5   | 5    | 4   | 5/9    | −4/9   | 1/9    | −4/9   | 5/9    | 1/9    |
| 5   | 5    | 5   | ?      | ?      | 1/10   | ?      | ?      | 1/10   |

### TABLE III. Charged Spin Texture Energies at $\nu_T = 1$ for Double Layers Systems with Tunneling.

$\tilde{\rho}_s \equiv \rho_s/(e^2/\ell)$ and $\tilde{t} \equiv t/(e^2/\ell)$ where $\rho_s$ is the pseudospin stiffness, $t$ is the renormalized tunneling amplitude, $\ell$ is the magnetic length, $T_0 = 8\rho_s/\xi$ is the soliton string tension and $\xi = \left(\frac{2\pi e\rho_s}{t}\right)^{1/2}$ is the domain wall width.

| Regime | $\tilde{t} \leq 4 \times 10^3 \tilde{\rho}_s^3$ | $4 \times 10^3 \tilde{\rho}_s^3 \leq \tilde{t} \leq 10^{-2} / \tilde{\rho}_s$ | $10^{-2} / \tilde{\rho}_s \leq \tilde{t}$ |
|--------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| Nature of Charged Excitations | Meron Pairs | Finite Length | Single Particle |
| Excitation Size | $\sim \frac{e^2}{\xi \rho_s}$ | $\sim \sqrt[4]{e^2 / 4\ell} \propto t^{-1/4}$ | $\ell$ |
| Excitation Energy | $\sim 2\pi \rho_s$ | $\sim \sqrt{e^2 T_0} \propto t^{1/4}$ | $t$ |
FIGURES

FIG. 1. Collective mode dispersion for a double-layer system at $\nu_T = 1/2$ and $d/\ell = 1.5$. The energies of the inter-Landau-level modes are measured from $\hbar \omega_c$. The ground state is approximated by the $(3, 3, 1)$ Halperin state. The plotting symbols refer to the following modes: triangles ($n = 1$ sum mode); circle ($n = 1$ difference mode); square ($n = 0$ sum mode); ; diamond ($n = 0$ difference mode).

FIG. 2. (a) Illustration of a simple spin-flip excitation which creates a widely separated particle hole pair. (b) A skyrmion spin configuration (shown in cross section). The spins gradually and smoothly rotate from up at the perimeter to down at the origin in a circularly symmetric spin textural defect. For the case of Coulomb interactions, this object costs only $1/2$ the energy of the simple spin flip.

FIG. 3. Knight shift measurements of the electron spin polarization of a 2DEG in the vicinity of filling factor $\nu = 1$. (After Barrett et al., Ref.[26]).

FIG. 4. Illustration of the path $\omega$ of a spin $S$ on a unit sphere. When viewed from the origin of spin space, the path subtends a solid angle $\Omega$ and so the path contributes a Berry’s phase $S\Omega$.

FIG. 5. A smooth spin texture. An electron moving along the boundary of the region $\Gamma$ in real space has its spin follow the path in spin space along the boundary of the region labeled $\omega$ in Fig.[4].

FIG. 6. Infinitesimal circuit in spin space associated with an infinitesimal circuit in real space.

FIG. 7. Schematic conduction band edge profile for a double-layer two-dimensional electron gas system. Typical widths and separations are $W \sim d \sim 100\,\text{Å}$ and are comparable to the spacing between electrons within each inversion layer.

FIG. 8. Phase diagram for the double layer QHE system (after Murphy et al.39). Only samples whose parameters lie below the dashed line exhibit a quantized Hall plateau and excitation gap.

FIG. 9. A process in double-layer two-dimensional electron gas systems which encloses flux from the parallel component of the magnetic field. The quantum amplitude for such paths is sensitive to the parallel component of the field.

FIG. 10. The four flavors of merons. These are vortices which are right or left handed and have topological charge $\pm 1/2$. 
FIG. 11. Meron pair connected by a domain wall. Each meron carries a charge $e/2$ which tries to repel the other one.

FIG. 12. Schematic zero-temperature phase diagram (with $d/\ell$ increasing downwards). The lower surface is $d^*$ below which $d > d^*$ and the interlayer correlations are too weak to support a fermionic gap, $\Delta \mu$. The upper surface gives $B_{\parallel}^*$, the commensurate-incommensurate phase boundary. As $d$ approaches $d^*$, quantum fluctuations soften the spin stiffness and therefore increase $B_{\parallel}^*$. 