THE RATIONAL HOMOLOGY RING OF THE BASED LOOP SPACE OF THE GAUGE GROUPS AND THE SPACES OF CONNECTIONS ON A FOUR-MANIFOLD

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ABSTRACT. We provide the rational-homotopic proof that the ranks of the homotopy groups of a simply connected four-manifold depend only on its second Betti number. We also consider the based loop spaces of the gauge groups and the spaces of connections of a simply connected four-manifold and, appealing to [15] and using the models from rational homotopy theory, we obtain the explicit formulas for their rational Pontrjagin homology rings.

1. INTRODUCTION

Rational homotopy theory can be seen as the torsion free part of the homotopy theory. In the 1960’s Sullivan proved that simply connected topological spaces and continuous maps between them can be rationalized to topological spaces $X_\mathbb{Q}$ and maps $f_\mathbb{Q} : X_\mathbb{Q} \to Y_\mathbb{Q}$ such that $H_k(X_\mathbb{Q}) = H_k(X, \mathbb{Q})$ and $\pi_k(X_\mathbb{Q}) = \pi_k(X) \otimes \mathbb{Q}$. The rational homotopy type of $X$ is defined to be the homotopy type of $X_\mathbb{Q}$ and the rational homotopy class of $f : X \to Y$ is defined to be the homotopy class of $f_\mathbb{Q} : X_\mathbb{Q} \to Y_\mathbb{Q}$. Rational homotopy theory studies those properties of spaces and maps between them which depend only on the rational homotopy type of a space and rational homotopy class of a map.

The main advantage of rational homotopy theory is its great computability. This advantage is due to its explicit algebraic modeling obtained by Quillen [11] and Sullivan [13]. As for the topological spaces the result is that the rational homotopy type of a topological space is determined by the isomorphism class of its algebraic model.

In this paper we continue our work on rational homotopy of simply connected four-manifolds, their gauge groups and spaces of connections. In that direction we prove, only by the means of rational homotopy theory,

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that the rational homotopy groups of a simply connected four-manifold can be expressed only in terms of its second Betti number. We also consider the based loop spaces of the gauge groups and the spaces of connection on a simply connected four manifold. Using the results from [15] on the rational cohomology structure of the gauge groups and the spaces of connection, we provide the explicit formulas for the rational Pontrjagin homology rings of the corresponding based loop spaces.

We refer to [6] for very detailed and comprehensive background on rational homotopy theory and its algebraic modeling.

2. ON HOMOTOPY GROUPS OF SIMPLY CONNECTED FOUR MANIFOLDS

2.1. Review of the known results. Let $M$ be a closed simply connected four-manifold. We recall the notion of the intersection form for $M$. The symmetric bilinear form

$$Q_M : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to \mathbb{Z},$$

defined by

$$Q_M(x, y) = \langle x \cup y, [M] \rangle,$$

where $[M]$ is the fundamental class for $M$, is called the intersection form for $M$.

It is well known the result of Pontryagin-Wall [7] that the homotopy type of a simply connected four-manifold is classified by its intersection form.

The intersection form $Q_M$ can be diagonalised over $\mathbb{R}$ with ±1 as the diagonal elements. Following the standard notation, $b_2^+(M)$ denotes the number of $(+1)$ and $b_2^-(M)$ the number of $(-1)$ in the diagonal form for $Q_M$. The numbers

$$b_2(M) = b_2^+(M) + b_2^-(M)$$

and

$$\sigma(M) = b_2^+(M) - b_2^-(M)$$

are known as the rank and the signature of $M$.

The rational homotopy type of a simply connected four-manifold is, by the result of [14], classified by its rank and signature.

The real cohomology algebra of $M$ for $b_2(M) \geq 2$ can be easily described using the intersection form for $M$ as follows:

(1) $$H^*(M) \cong \mathbb{R}[x_1, \ldots, x_{b_2^+}, x_{b_2^++1}, \ldots, x_{b_2}] / I,$$

where $\deg x_i = 2$, and the ideal $I$ is generated by the equalities

$$x_1^2 = \cdots = x_{b_2^+}^2 = -x_{b_2^++1}^2 = \cdots = -x_{b_2}^2,$$

$$x_ix_j = 0, \ i \neq j.$$
and the fourth homotopy group of \( M \) were obtained in [14] by applying Sullivan minimal model theory to the cohomology algebra of \( M \):

\[
\text{rk} \pi_2(M) = b_2(M), \quad \text{rk} \pi_3(M) = \frac{b_2(b_2 + 1)}{2} - 1, \quad \text{rk} \pi_4(M) = \frac{b_2(b_2^2 - 4)}{3}.
\]

The calculation procedure of the ranks of homotopy groups based on the Sullivan minimal theory carried out in [14] suggested that the ranks of all homotopy groups of \( M \) depend only on the second Betti number of \( M \), but we were not able to prove it explicitly using that method.

A formula for all homotopy groups of a simply connected four manifold is obtained later in [5] by proving that any simply connected four-manifold admits \( S^1 \)-covering of the form \( \#b_2(M) - 1 S^2 \times S^3 \). It implies that

\[
\pi_k(M) = \begin{cases} 
\bigoplus b_2(M) \mathbb{Z} & \text{for } k = 2, \\
\pi_k(\#b_2(M) - 1 S^2 \times S^3) & \text{for } k > 2.
\end{cases}
\]

This formula in particular proves that the ranks of the homotopy groups of \( M \) depend only on the second Betti number \( b_2(M) \). The recent results obtained in this direction are much stronger. Denote by \( \Omega M \) the space of based loops on \( M \). It is proved in [3] that for a simply connected four-manifolds \( M \) and \( N \) it holds:

\[
\Omega M \cong \Omega N \iff H^2(M) \cong H^2(N).
\]

This is done via homotopy theoretic methods by decomposing \( \Omega M \) into a product of spaces, up to homotopy:

\[
\Omega M \cong S^1 \times \Omega(S^2 \times S^3) \times \Omega(J \lor (J \land \Omega(S^2 \times S^3))),
\]

where \( J = \lor_{i=1}^{k-2} (S^2 \lor S^3) \) if \( k > 2 \) and \( J = \ast \) if \( k = 2 \). Since \( H^2(M) = \pi_2(M) = \bigoplus b_2(M) \mathbb{Z} \) we have that the based loop spaces \( \Omega M \) and \( \Omega N \) are homotopy equivalent if and only if \( b_2(M) = b_2(N) \). On the other hand, the application of the Sullivan minimal model theory to the algebra [14] gives that, in general, the ranks of homotopy groups of a simply connected four-manifold \( M \) can be expressed only in terms of the rank and possible signature of the corresponding intersection form. This together proves that the ranks of homotopy groups for \( M \) depend only on \( b_2(M) \).

We would like to mention here the results from [1] which give the explicit formulas on the ranks of homotopy groups of the suspension of a finite simplicial complex:

\[
\text{rk} \pi_{j+1}(\sum X) = \frac{(-1)^j}{j} \sum_{d|j} (-1)^d \mu\left(\frac{i}{d}\right) S_d(2 - P(z)),
\]
where $P(z)$ is the Poincaré polynomial for $X$, by $\mu$ is denoted the Möbius function and $S_d$ is the Newton polynomial of the roots of the polynomial inverse to the polynomial $2 - P(z)$.

In the case of a simply connected four-manifold $M$, the following explicit formula for the rational homotopy groups has been obtained recently in [2]:

$$\text{rk}(\pi_{n+1}(M)) = \sum_{d \mid n} (-1)^{n+\frac{n}{d}} \frac{\mu(d)}{d} \sum_{a+2b = \frac{n}{d}} (-1)^b \binom{a+b}{b} \frac{b_2^a}{a+b},$$

where $b_2$ denotes the second Betti number of $M$.

2.2. The background from rational homotopy theory. We provide here the simple proof, based only on the methods of rational homotopy theory, that the rational homotopy groups of a simply connected four-manifold $M$ can be expressed only in terms of $b_2(M)$. In this way we complete the work [14] by the means of the rational homotopy theory.

Let us recall some notions from rational homotopy theory, for the background we refer to [6]. Let $TV$ denotes the tensor algebra on a graded vector spaces $V$. It is a graded Lie algebra with the commutator bracket. The sub Lie algebra generated by $V$ is called the free graded Lie algebra on $V$ and it is denoted by $L_V$. An element in $L_V$ is said to have the bracket length $i$ if it is linear combination of the elements of the form $[v_1, \ldots, [v_{k-1}, v_k], \ldots]$.

We obtain the decomposition $L_V = \bigoplus L_V^{(i)}$, where $L_V^{(i)}$ consists of elements having the bracket length $i$. Then it holds $TV \cong U_L V$, where $U_L V$ is the universal enveloping algebra for $L_V$. Let $(L_V, d)$ be a differential graded free Lie algebra. The linear part of $d$ is the differential $d_V : V \rightarrow V$ defined by $dv - d_V v \in \bigoplus_{k \geq 2} L_V^{(k)}$. An algebra $(L_V, d)$ is said to be minimal if the linear part $d_V$ of its differential vanishes.

The Quillen functor $C_*$ is a functor from the connected chain Lie algebras $\{(L; d_L)\}$ to one-connected cocommutative chain coalgebras $\{C_*(L; d_L)\}$. By dualizing Quillen’s construction one obtains commutative differential graded algebras $\{(C_*(L; d_L))\}$. In the case when $(L; d_L)$ is a chain algebra, $C_*(L; d_L)$ is a cochain algebra.

Let $X$ be a simply connected space with rational homology of finite type and $AP_L(X)$ its commutative cochain algebra. A Lie model for $X$ is a connected chain Lie algebra $(L; d_L)$ of finite type with a differential graded algebras quasi-isomorphism

$$m : C_*(L; d_L) \xrightarrow{\cong} AP_L(X).$$

Any such space $X$ has a minimal free Lie model which is unique, up to isomorphism. For a minimal free model $(L_V, d)$ for $X$, it is known that
there exists a Lie algebra isomorphism

$$\pi_\ast(\Omega X) \otimes \mathbb{Z} \mathbb{Q},$$

where the Lie algebra structure on $\pi_\ast(\Omega X) \otimes \mathbb{Z} \mathbb{Q}$ is given by the Whitehead product. This isomorphism is dual to the isomorphism $H(\wedge V_X) \cong H^4(X, \mathbb{Q})$ in the context of Sullivan algebras, where $\wedge V_X$ denotes the minimal model for $X$. The isomorphism (2) suspends to

$$sH(\mathbb{L}_V, d) \cong \pi_\ast(X) \otimes \mathbb{Z} \mathbb{Q}.$$

Assume now that $X$ is an adjunction space, meaning that

$$X = Y \cup_f (\cup_a D^{n_a+1}),$$

where:

- $Y$ is simply-connected space with rational homology of finite type,
- $f = \{f_\alpha : (S^{n_\alpha}, *) \to (Y, y_0)\}$,
- the cells $D^{n_\alpha+1}$ are all of dimension $\geq 2$, with finitely many of them in any dimension.

Let $\tau : sH(\mathbb{L}_V, d) \cong \pi_\ast(X) \otimes \mathbb{Q}$ be an isomorphism, where $\mathbb{L}_V$ is a minimal free Lee model for $X$. Using this isomorphism the classes $[f_\alpha] \in \pi_{n_\alpha}(X)$ determine the classes $s[z_\alpha] = \tau^{-1}([f_\alpha])$ represented by the cycles $z_\alpha \in \mathbb{L}_V$. Let $W$ be a graded vector space with basis $\{w_\alpha\}$ such that $\deg w_\alpha = n_\alpha$. One can extend $\mathbb{L}_V$ to $\mathbb{L}_V \oplus W$ by putting $dw_\alpha = z_\alpha$. It is proved in [6] that the chain algebra $(\mathbb{L}_V \oplus W, d)$ is a Lie model for $X$.

2.3. On rational homotopy groups of simply connected four-manifolds.

In this section, appealing to the methods of rational homotopy theory, we prove:

Proposition 1. The rational homotopy groups of a simply-connected four manifold depend only on its second Betti number.

Proof. It is the classical result, see [8] and [16], that a simply-connected four-manifold $M$ is homotopically an adjunction space:

$$M \cong (\vee b_2(M)S^2) \cup_f D^4.$$

A Lie model for the space $X = \vee_\alpha S^{n_\alpha+1} = pt \cup_f (\cup_a D^{n_a+1})$ is (see [6]) given by $(\mathbb{L}_V, 0)$, where $V = \{V_i\}_{i \geq 1}$ is a vector space of finite type with basis $v_\alpha$ such that $\deg v_\alpha = n_\alpha$. It implies that a Lie model for $\vee b_2(M)S^2$ is given by $(\mathbb{L}_V, 0)$, where $V$ has a basis $v_1, \ldots, v_{b_2(M)}$ such that $\deg v_\alpha = 1$ for all $1 \leq \alpha \leq b_2(M)$. For $[f] \in \pi_3(\vee b_2(M)S^2, x_0)$ denote by $z$, according to the previous section, a cycle in $\mathbb{L}_V$ such that $s[z] = \tau^{-1}([f]) \in sH(\mathbb{L}_V)$. We have that $\deg z = 2$. Let $W = \mathcal{L}(w)$ be a vector space with the basis $w$ such that $\deg w = 3$. 
Then the Lie model for $M$ is given by

$$(L_V \oplus W = L_{L^{(v_1, \ldots, v_{b_2}, w)}, d}, \text{ where } dv_i = 0 \text{ } dw = z \in L_V).$$

We note that the ranks of the cohomology groups in $H(L_V \oplus W, d)$ depend only on $b_2(M)$. Together with (2) this proves the statement. 

3. ON GAUGE GROUPS AND SPACES OF CONNECTIONS

Let $\pi : P \to M$ be a $G$-principal bundle, where $G$ is a compact semisimple simply connected Lie group and $M$ is a compact simply connected four-manifold. The gauge group $G$ of this principal bundle is the group of $G$-equivariant automorphisms of $P$ which induce identity on the base. Let $A$ denote the space of all connections and $A^*$ the space of all irreducible connections on the bundle $P$. It is assumed that these spaces are equipped with certain Sobolev topologies $L^2_p$ for $A$ and $L^2_p$ for $G$, where $p$ is large enough.

The action of $G$ on $A$ and $A^*$ is not free in general. Instead one should consider the group $G_0$ of the gauge transformations for $P$ which fix one fiber. The group $G_0$ acts freely on the spaces of connections and irreducible connections. The group $\tilde{G} = G/Z(G)$ also acts freely on $A^*$, where $Z(G)$ is the center of the group $G$, which gives a fibration

$$(5) \quad \tilde{B} = A/G_0, \quad \tilde{B}^* = A^*/G_0, \quad B^* = A^*/\tilde{G}.$$ 

It is known, see [4] that $A$ is contractible, then $A^*$ is weakly homotopy equivalent to $A$ and $B^*$ is weakly equivalent to $\tilde{B}$.

The well known result of Singer [12] states that the weak homotopy type of $G_0$ is independent of $P$:

$$(6) \quad G_0 \approx Map_*(M, BG).$$

This means that $G_0$ is weakly homotopy equivalent to the pointed mapping space $Map_*(M, BG)$.

Let $G^e$ be the identity component of the gauge group $G$. The rational cohomology algebras of the spaces $G^e$, $\tilde{B}$ and $B^*$ are computed in [15] and they are as follows:

- $H^*(G^e)$ is an exterior algebra in $(b_2(M) + 2) \text{rk } G - 1$ generators of odd degree, where the number of generators in degree $j$ is equal to $b_2(M) \text{rk } \pi_{j+2}(G) + \text{rk } \pi_j(G) + \text{rk } \pi_{j+4}(G)$.
- $H^*(\tilde{B})$ is a polynomial algebra in $(b_2(M) + 1) \text{rk } G - 1$ generators of even degree, where the number of generators of degree $j$ is equal to $b_2(M) \text{rk } \pi_{j+1}(G) + \text{rk } \pi_{j+3}(G)$. 

\( H^*(B^*) \) is a polynomial algebra in \( (b_2(M) + 2) \text{rk } G - 1 \) generators of even degree, where the number of generators in degree \( j \) is equal to \( b_2(M) \text{rk } \pi_{j+1}(G) + \text{rk } \pi_{j-3}(G) + \text{rk } \pi_{j+3}(G) \).

Remark 1. Since for all these spaces their rational cohomology algebras are free, they are all formal in the sense of rational homotopy theory as remarked in [15]. Therefore the minimal models, in the sense of rational homotopy theory, for the spaces \( G^e, \tilde{B} \) and \( B^* \) coincide with the minimal models for their rational cohomology algebras. Furthermore, being free, their rational cohomology algebras coincide with their minimal models.

3.1. The rational Pontrjagin homology ring of a based loop space. For a topological spaces \( X \) its based loop space \( \Omega X \) is an \( H \)-space where one of possible multiplication of the loops is given by loop concatenation. The ring structure induced in \( H_*(\Omega X) \) is called Pontrjagin homology ring. We are interested in Pontrjagin homology ring of the spaces \( G, \tilde{B} \) and \( B^* \).

Let us recall the theorem of Milnor and Moore and some constructions from the rational homotopy theory which will be directly applied to describe the Pontrjagin homology rings of the spaces we consider. The theorem of Milnor-Moore [9] states that the rational homology algebra of the based loop space \( \Omega X \) of a simply connected space \( X \) is given by

\[
H_*(\Omega X) \cong UL_X \cong T(L_X)/\langle xy - (-1)^{\deg x \deg y}y x - [x, y] \rangle,
\]

where \( L_X \) is the rational homotopy Lie algebra of \( X \) and \( UL_X \) is the universal enveloping algebra for \( L_X \). The algebra \( L_X \) is a graded Lie algebra defined by \( L_X = (\pi_*(\Omega X) \otimes \mathbb{Q}) \) and the commutator \([ , ]\) is given by the Samelson product.

It is the result of rational homotopy theory [6] that there is an isomorphism between the rational homotopy Lie algebra \( L_X \) and the homotopy Lie algebra \( L \) of the minimal model for \( X \). The homotopy Lie algebra \( L \) is defined as follows. Let \( (\wedge V, d) \) be a minimal model for \( X \). Then \( sL = \text{Hom}(V, \mathbb{Q}) \), where the suspension \( sL \) is defined in the standard way by \( (sL)_i = (L)_{i-1} \). The Lie brackets \( L \) are defined by

\[
\langle v; s[x, y] \rangle = (-1)^{\deg y + 1} \langle d_1 v; s x, s y \rangle \quad \text{for} \quad x, y \in L, v \in V.
\]

Here \( d_1 \) is quadratic part in the differential \( d \) for the minimal model \( (\wedge V, d) \) and it is defined by \( d - d_1 \in \wedge^{k \geq 3} V \). Thus \( d_1 v = v_1 \wedge v_2 \) for some \( v_1, v_2 \in V \) and the expression \( \langle d_1 v; s x, s y \rangle \) is defined by

\[
\langle v_1 \wedge v_2; s x, s y \rangle = \langle v_1; s x \rangle \langle v_2; s y \rangle - \langle v_2; s x \rangle \langle v_1; s y \rangle
\]

In order to apply Milnor-Moore theorem we will assume that the spaces \( \tilde{B} \) and \( B^* \) are simply connected. Using (5) we see that that this condition is
equivalent to the condition that the groups $\mathcal{G}_0$ and $\tilde{\mathcal{G}}$ are connected, which is further equivalent to the condition that the gauge group $\mathcal{G}$ is connected.

**Theorem 1.** Assume that the gauge group $\mathcal{G}$ is connected. Then the rational Pontrjagin homology rings of the based loop spaces $\Omega \tilde{B}$ and $\Omega B^*$ are as follows:

- $H_*(\Omega \tilde{B})$ is an exterior algebra in $(b_2(M) + 1) \text{rk} G - 1$ generators of odd degree, where the number of generators of degree $j$ is equal to $b_2(M) \text{rk} \pi_{j+2}(G) + \text{rk} \pi_{j+4}(G)$.
- $H_*(\Omega B^*)$ is an exterior algebra in $(b_2(M) + 2) \text{rk} G - 1$ generators of odd degree, where the number of generators of degree $j$ is equal to $b_2(M) \text{rk} \pi_{j+2}(G) + \text{rk} \pi_j(G) + \text{rk} \pi_{j+4}(G)$.

**Proof.** By Remark 1 we have that the differentials in the minimal models for the spaces $\tilde{B}$ and $B^*$ are trivial implying that their quadratic parts $d_1$ are trivial as well. The definition of the brackets given by (8) implies that the corresponding homotopy Lie algebras are commutative. It implies, by (7) that $UL$ is commutative and, therefore the loop space homology algebra for these spaces coincide with their rational cohomology algebras with the degrees of generators shifted by $-1$. 

Let us comment on the assumption that the gauge group $\mathcal{G}$ is connected. The group $\mathcal{G}$ is not connected in general. The result of Singer (6) together with the homotopy cofibration (4) gives the exact homotopy sequence, see (15):

$$\pi_3(\mathcal{G}) \leftarrow \oplus b_2(M) \pi_2(\mathcal{G}) \leftarrow [M; BG] \leftarrow \pi_4(\mathcal{G}) \leftarrow \ldots.$$  

It implies that $\mathcal{G}_0$ is connected if $\pi_2(\mathcal{G}) = \pi_4(\mathcal{G}) = 0$. This condition is satisfied when $G = SU(n)$ for $n \geq 3$ and $G = Spin(n)$ for $n \geq 6$ (this follows from the Bott periodicity).

**Example 1.** Let us consider the $SU(3)$-principal bundles over a simply connected four-manifold $M$. The gauge group of any such bundle is connected. It is well known that $\text{rk} \pi_3(SU(3)) = \text{rk} \pi_5(SU(3)) = 1$ and $\text{rk} \pi_j(SU(3)) = 0$ for $j \neq 3, 5$. Therefore, Theorem 1 implies:

$$H_*(\Omega \tilde{B}) \cong \wedge(x_1, \ldots, x_{b_2(M)+1}, y_1, \ldots, y_{b_2(M)}), \quad \text{deg } x_i = 1, \quad \text{deg } y_j = 3.$$  

$$H_*(\Omega B^*) \cong \wedge(x_1, \ldots, x_{b_2(M)+1}, y_1, \ldots, y_{b_2(M)+1}, z),$$  

$$\text{deg } x_i = 1, \quad \text{deg } y_j = 3, \quad \text{deg } z = 5.$$  

**Example 2.** Let us now consider the $SU(2)$-principal bundles over a simply connected four-manifold $M$. It is well known that $\pi_4(SU(2)) = \mathbb{Z}_2$. In this case the fundamental group $\pi_1(B^*)$ depends on the $SU(2)$-principal bundle

$$H_*(\Omega \tilde{B}) \cong \wedge(x_1, \ldots, x_{b_2(M)+1}, y_1, \ldots, y_{b_2(M)}), \quad \text{deg } x_i = 1, \quad \text{deg } y_j = 3.$$  

$$H_*(\Omega B^*) \cong \wedge(x_1, \ldots, x_{b_2(M)+1}, y_1, \ldots, y_{b_2(M)+1}, z),$$  

$$\text{deg } x_i = 1, \quad \text{deg } y_j = 3, \quad \text{deg } z = 5.$$
$P \to M$. This fundamental group is computed in [10] and we recall these results. For $G = SU(2)$ we have that $\pi_0(\mathcal{G}) = [M, S^3]$ and due to Steenrod theorem it follows:

$$\pi_0(\mathcal{G}) = \begin{cases} 0, & \text{if the intersection form for } M \text{ is odd} \\ \mathbb{Z}_2, & \text{if the intersection form for } M \text{ is even.} \end{cases}$$

It implies, using (5) that

$$\pi_1(\tilde{\mathcal{G}}) = \begin{cases} 0, & \text{if the intersection form for } M \text{ is odd} \\ \mathbb{Z}_2, & \text{if the intersection form for } M \text{ is even.} \end{cases}$$

Assume that the intersection form for $M$ is odd. Theorem 1 gives $H_*(\Omega \tilde{\mathcal{G}}) \cong \wedge(x_1, \ldots, x_{b_2(M)}), \deg x_i = 1$.

Also the fibration $Z(G) \to \mathcal{G} \to \tilde{\mathcal{G}}$ gives the exact sequence

$$(10) \quad \ldots \to \mathbb{Z}_2 \xrightarrow{j_\ast} \pi_0(\mathcal{G}) \to \pi_0(\tilde{\mathcal{G}}) \to 0,$$

which implies that $\tilde{\mathcal{G}}$ is connected meaning that $B^*$ is simply connected. Therefore in this case Theorem 1 gives $H_*(\Omega \tilde{\mathcal{G}}) \cong \wedge(x_1, \ldots, x_{b_2(M)}, y), \deg x_i = 1, \deg y = 3$.

Assume that the intersection form for $M$ is even. It is proved in [10] that the map $j_\ast$ from (10) depends on the second Chern number of the principal $SU(2)$-bundle $P$ over $M$. More precisely, it is proved that $j_\ast$ is 0-map when $c_2(P)$ is even and $j_\ast$ is surjective when $c_2(P)$ is odd. It implies that $\pi_0(\tilde{\mathcal{G}}) = \mathbb{Z}_2$ if $c_2(P)$ is even and $\pi_0(\tilde{\mathcal{G}}) = 0$ if $c_2(P)$ is odd. Thus

$$\pi_1(B^*) = \begin{cases} \mathbb{Z}_2, & c_2(P) \text{ is even} \\ 0, & c_2(P) \text{ is odd} \end{cases}$$

Therefore when the intersection form for $M$ is even and the second Chern number of the $SU(2)$-principal bundle $P \to M$ is odd, Theorem 1 gives that:

$$H_*(\Omega B^*) \cong \wedge(x_1, \ldots, x_{b_2(M)}, y), \deg x_i = 1, \deg y = 3.$$

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