Conditional Symmetries and Phase Space Reduction
towards G.C.T. Invariant Wave functions,
for the Class A Bianchi Type VI & VII Vacuum Cosmologies

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Abstract

The quantization of Class A Bianchi Type VI and VII geometries—with all six scale factors, as well as the lapse function and the shift vector present—is considered. A first reduction of the initial 6–dimensional configuration space is achieved by the usage of the information furnished by the quantum form of the linear constraints. Further reduction of the space in which the wave function—obeying the Wheeler-DeWitt equation—lives, is accomplished by revealing a classical integral of motion, tantamount to an extra symmetry of the corresponding classical Hamiltonian. This symmetry generator—member of a larger group—is linear in momenta and corresponds to G.C.T.s through the action of the automorphism group—especially through the action of the outer automorphism subgroup. Thus, a G.C.T. invariant wave function is found, which depends on one combination of the two curvature invariants—which uniquely and irreducibly, characterize the hypersurfaces $t = const$.

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As it is well known, the quantum cosmology approximation consists in freezing out all but a finite number of degrees of freedom of the gravitational field and quantize the rest. This is done by imposing spatial homogeneity. Thus, our –in principle– dynamical variables are the scale factors $\gamma_{\alpha\beta}(t)$, the lapse function $N(t)$ and the shift vector $N^a(t)$, of some spatially homogeneous geometry.

The general Class A Bianchi Type VI and VII cosmologies can be treated on the same footing, because of their similar structure constants.

In this sort communication, we present a complete reduction of the initial configuration space for the above mentioned Bianchi Types; one combination of the two curvature invariants of the corresponding 3-space, is thus revealed as the only true degree of freedom on which the wave function depends.

In this work, we will quantize the known action corresponding to the most general Class A Bianchi Type VI and VII cosmologies, i.e. the action giving Einstein’s field equations derived from the line element

$$ds^2 = (-N^2(t) + N_\alpha(t)N^\alpha(t))dt^2 + 2N_\alpha(t)\sigma^\alpha_i(x)dx^i dt + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^i dx^j, \quad (1)$$

where $\sigma^\alpha_i$ are the invariant basis one-forms of the homogeneous surfaces of simultaneity $\Sigma_t$, satisfying

$$d\sigma^\alpha = C^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma^\alpha_i, j - \sigma^\alpha_j, i = 2C^\alpha_{\beta\gamma} \sigma^\gamma_i \sigma^\beta_j, \quad (2)$$

with $C^\alpha_{\beta\gamma}$ being the structure constants of the corresponding isometry group.

In 3 dimensions, the tensor $C^\alpha_{\beta\gamma}$ admits a unique decomposition in terms of a contravariant symmetric tensor density of weight -1, $m^{\alpha\beta}$, and a covariant vector $\nu_\alpha = \frac{1}{2}C^\rho_{\alpha\rho}$ as follows:

$$C^\alpha_{\beta\gamma} = m^{\alpha\delta} \epsilon_{\beta\delta\gamma} + \nu_\beta \delta^\alpha_\gamma - \nu_\gamma \delta^\alpha_\beta. \quad (3)$$

For the Class A ($\nu_\alpha = 0$) Bianchi Type VI ($\epsilon = 1$) and VII ($\epsilon = -1$), this matrix is

$$m^{\alpha\beta} = \begin{pmatrix}
\epsilon & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (4)$$
resulting in the following non vanishing structure constants (1):

\[ C_{23}^1 = \varepsilon \quad C_{13}^2 = 1. \] (5)

As is well known [2], the Hamiltonian of the above system is \( H = \tilde{N}(t)H_0 + N^\alpha(t)H_\alpha \), where

\[ H_0 = \frac{1}{2}L_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} + \gamma R \] (6)

is the quadratic constraint, with

\[ L_{\alpha\beta\mu\nu} = \gamma_{\alpha\mu}\gamma_{\beta\nu} + \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu} \]

\[ R = C_{\lambda\mu}\gamma_{\alpha\beta}\gamma^{\beta\lambda}\gamma^{\pi\tau} + 2C_{\beta\lambda}\gamma^{\beta\rho}\gamma^{\tau\nu} + 4C_{\mu\nu}\gamma^{\beta\lambda}H_\gamma \] (7)

(the last term of \( R \) vanishes in Class A Models), \( \gamma \) being the determinant of \( \gamma_{\alpha\beta} \), and

\[ H_\alpha = C_{\alpha\rho}^{\mu} \gamma_{\beta\mu} \pi^{\beta\rho} \] (8)

are the linear constraints. Note that \( \tilde{N} \) appearing in the Hamiltonian, is to be identified with \( N/\sqrt{\gamma} \). The canonical equations of motion, following from (6), are equivalent to Einstein’s equations derived from line element (1) (see [3]).

The quantities \( H_0, H_\alpha \) are weakly vanishing [3], i.e. \( H_0 \approx 0, H_\alpha \approx 0 \). For all Class A Bianchi Types (\( C_{\alpha\beta} = 0 \)), it can be seen –using the basic PBR \{\( \gamma_{\alpha\beta}, \pi^{\mu\nu} \)\} = \( \delta^{\mu\nu} \)– that these constraints are first class, obeying the following algebra

\[ \{H_0, H_\alpha\} = 0 \]

\[ \{H_\alpha, H_\beta\} = -\frac{1}{2}C_{\alpha\beta}^\gamma H_\gamma, \] (9)

which ensures their preservation in time, i.e. \( \dot{H}_0 \approx 0, \dot{H}_\alpha \approx 0 \), and establishes the consistency of the action.

If we follow Dirac’s general proposal [3] for quantizing this action, we have to turn \( H_0, H_\alpha \), into operators annihilating the wave function \( \Psi \).

In the Schrödinger representation

\[ \gamma_{\alpha\beta} \rightarrow \tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} \]

\[ \pi^{\alpha\beta} \rightarrow \tilde{\pi}_{\alpha\beta} = -i\partial_{\gamma_{\alpha\beta}}. \] (10)

with the relevant operators satisfying the basic Canonical Commutation Relations (CCR) –corresponding to the classical ones–:

\[ [\tilde{\gamma}_{\alpha\beta}, \tilde{\pi}^{\mu\nu}] = i\delta_{\alpha\beta}^{\mu\nu} = i \frac{1}{2}(\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu} + \delta_{\beta}^{\mu}\delta_{\alpha}^{\nu}). \] (11)
In general, via the method of characteristics [4], the quantum version of the three independent linear constraints can be used to reduce the dimension of the initial configuration space from 6 ($\gamma_{\alpha\beta}$) to 3 (combinations of $\gamma_{\alpha\beta}$), i.e. $\Psi = \Psi(q^1, q^2, q^3)$ [5], where
\begin{align*}
q^1 &= C^\alpha_{\mu\kappa} C^\beta_{\nu\lambda} \gamma^{\mu\nu} \gamma^\kappa\lambda \gamma_{\alpha\beta} \\
q^2 &= C^\alpha_{\beta\kappa} C^\beta_{\alpha\lambda} \gamma^{\kappa\lambda} \\
q^3 &= \gamma.
\end{align*}
(12)

One can prove [6, 7] that the only G.C.T. (gauge) invariant quantities, which uniquely and irreducibly, characterize a 3-dimensional geometry admitting the Class A Type VI and VII symmetry groups, are the quantities $q^1$ and $q^2$. An outline of the proof, is as follows:

Let two hexads $\gamma^{(1)}_{\alpha\beta}$ and $\gamma^{(2)}_{\alpha\beta}$ be given, such that their corresponding $q^1$, $q^2$ are equal. Then [6], there exists an automorphism matrix $\Lambda$ (i.e. a matrix satisfying $C^\alpha_{\mu\nu} \Lambda^\kappa_\mu = C^\alpha_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu$) connecting them, i.e. $\gamma^{(1)}_{\alpha\beta} = \Lambda^\mu_\alpha \gamma^{(2)}_{\mu\nu} \Lambda^\nu_\beta$ (the inverse is easily seen to be true, as well). But, as it has been shown in the appendix of [7], this kind of changes on $\gamma_{\alpha\beta}$, can be seen to be induced by spatial diffeomorphisms. Thus, 3-dimensional Class A Type VI and VII geometries, are uniquely and irreducibly characterized by some values of these two $q$’s.

In terms of the three $q$’s, one can define the following –according to [5]– induced “physical” metric, given by the relation
\begin{align*}
g^{ij} &= L_{\alpha\beta\mu\nu} \frac{\partial q^i}{\partial \gamma_{\alpha\beta}} \frac{\partial q^j}{\partial \gamma_{\mu\nu}} = \begin{pmatrix} 5q^1q^1 - 16q^2q^2 & q^1q^2 & q^1q^3 \\
q^2q^2 & q^2q^2 & q^2q^3 \\
q^3q^2 & q^3q^3 & -3q^3q^3 \end{pmatrix}. \tag{13}
\end{align*}

Note that the first-class algebra satisfied by $H_0, H_\alpha$, ensures that indeed, all components of $g^{ij}$ are functions of the $q^i$.

According to Kuchar’s and Hajicek’s [5] prescription, the “kinetic” part of $H_0$ is to be realized as the conformal Laplacian, based on the “physical” metric (13). However, in the case under discussion here, there is a further reduction.

For full pure gravity, Kuchar [8] has shown that there are no other first-class functions, homogeneous and linear in the momenta, except the linear constraints. If however, we impose extra symmetries (e.g. the Class A Type VI and VII –here considered), such quantities may emerge –as it will be shown. We are therefore –according to Dirac [3]– justified to seek the generators of these extra symmetries, whose quantum-operator form will be imposed as additional conditions on the wave function. Thus, these symmetries
are expected to lead us to the final reduction, by revealing the true degrees of freedom. Such quantities are, generally, called in the literature “Conditional Symmetries” \[8\].

The automorphism group for the Class A Type VI and VII, is described by the following 4 generators –in matrix notation and collective form:

\[
\lambda_{(I)\beta}^\alpha = \begin{pmatrix}
  a & \varepsilon & c & b \\
  c & a & d \\
  0 & 0 & 0
\end{pmatrix},
\]

(14)

with the defining property

\[
C_{\mu\nu}^\alpha \lambda_\alpha^\kappa = C_{\mu\sigma}^\kappa \lambda_\nu^\sigma + C_{\sigma\nu}^\kappa \lambda_\mu^\sigma.
\]

(15)

From these matrices, we can construct the linear –in momenta– quantities

\[
A_{(I)} = \lambda_{(I)\beta}^\alpha \gamma_{\alpha\rho} \pi^{\rho\beta}.
\]

(16)

Three of these are the \(H_\alpha\)'s, since \(C_{(\rho)\beta}^\alpha\) correspond to the inner automorphism subgroup –designated by the \(c, b\) and \(d\) parameters in \(\lambda_{(I)\beta}^\alpha\). The remaining is the generator of the outer automorphisms and is given by the essentially unique matrix

\[
\epsilon_\beta^\alpha = \begin{pmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & 0
\end{pmatrix}.
\]

(17)

The corresponding –linear in momenta– quantity is

\[
E = \epsilon_\beta^\alpha \gamma_{\alpha\rho} \pi^{\rho\beta}.
\]

(18)

It is straightforward to calculate the Poisson Brackets of \(E\) with \(H_0\), \(H_\alpha\):

\[
\{E, H_\alpha\} = -\frac{1}{2} \lambda_\alpha^\beta H_\beta \\
\{E, H_0\} = -2a \gamma R = -2a \gamma (q^1 + 2q^2).
\]

(19)

At this point, it is crucial to observe that we can construct a classical integral of motion, i.e. an extra gauge symmetry of the corresponding classical action: notice that the trace of the canonical momenta, \(\gamma_{\mu\nu} \pi^{\mu\nu}\), has vanishing PB with \(H_\alpha\), \(E\), and a –similar to \(E\)– PB with \(H_0\) equal to \(2\gamma R\); thus, if we define

\[
T = E - a\gamma \gamma_{\mu\nu} \pi^{\mu\nu},
\]

(20)
we can easily derive the following PB of $T$ with $H_0, H_α$:

\[
\{T, H_0\} = 0 \\
\{T, H_1\} = -\frac{a}{2} H_1 \approx 0 \\
\{T, H_2\} = -\frac{a}{2} H_2 \approx 0 \\
\{T, H_3\} = 0.
\]  

(21)

The quantity $T$, is thus revealed to be first-class, and therefore an integral of motion (since the Hamiltonian, is a linear combination of the constraints):

\[
\dot{T} = \{T, H\} \approx 0 \Rightarrow T = \text{const} = C_T.
\]  

(22)

The quantum version of $T$, is taken to be [5]:

\[
\hat{T} = \epsilon^α_β γ_ρ \frac{∂}{∂γ_ρ} - aγ_α β \frac{∂}{∂γ_α β}
\]  

(23)

(without any loss of generality –see (17)– we can safely suppress $a$, whenever convenient, by setting it to 1).

Following the spirit of Dirac, we require:

\[
\hat{T} \Psi = \epsilon^α_β γ_ρ \frac{∂Ψ}{∂γ_ρ} - aγ_α β \frac{∂Ψ}{∂γ_α β} = (q^1 \frac{∂Ψ}{∂q^1} + q^2 \frac{∂Ψ}{∂q^2} - γ \frac{∂Ψ}{∂γ}) = C_T \Psi.
\]  

(24)

The general solution [4] to the above equation has the form

\[
Ψ = γ^{-C_T} Θ(q^1, q^2, γq^2),
\]  

(25)

$Θ$ being an arbitrary function in its arguments.

Now, the number of our dynamical variables, is reduced from 3 ($q^i, i = 1, 2, 3$) to 2, namely the combinations $w^1 = q^1/q^2$ and $w^2 = q^2/q^3$. So, we have a further reduction of the 3-dimensional configuration space spanned by the 3 $q$’s. Again, in terms of the $w$’s, the finally reduced “physical” –although singular– metric, is given by the following relation

\[
s^{kl} = g^{ij} \frac{∂w^k}{∂q^i} \frac{∂w^l}{∂q^j} = \begin{pmatrix} -16 + 4(w^1)^2 & 0 \\ 0 & 0 \end{pmatrix}.
\]  

(26)

The singular character of this metric is not unexpected; its origin lies in the fact that $L^{αβγδ} T_{αβ} T_{γδ} = 0$, where $L^{αβγδ} = \frac{1}{4} (γ^{αγ}γ^{βδ} + γ^{αδ}γ^{βγ} - 2γ^{αβ}γ^{γδ})$ is the covariant supermetric (inverse to $L_{αβγδ}$) and $T_{αβ}$ are the components of $T$, seen as vector field in the initial
superspace spanned by \( \gamma_{\alpha\beta} \). Indeed, it is known that reducing to null surfaces entails all sort of peculiarities.

So far, the degrees of freedom are two \((w^1, w^2)\). The vanishing of the \( s^{22} \) component indicates that \( w^2 \) is not dynamical at the quantum level. This fact has its analogue at the classical level; indeed, consider the quantity:

\[
\Omega = \left( \frac{\gamma q^2}{\gamma q^2} \right) = L_{\alpha\beta\mu\nu} \gamma_{\mu\nu} \pi_{\alpha\beta} - L_{\kappa\lambda\rho\sigma} \pi_{\rho\sigma} C_{\beta\mu} C_{\alpha\nu} \gamma_{\mu\kappa} \gamma_{\nu\lambda} \frac{q^2}{q^2}.
\]

In (27), the transition from velocity phase space to momentum phase space has been made using the Hamiltonian (6). It is straightforward to verify that:

\[
\Omega = \frac{2}{a} T - \frac{4\varepsilon \gamma^{23}}{q^2} H_1 - \frac{4\varepsilon \gamma^{13}}{q^2} H_2.
\]

Taking into account the weak vanishing of the linear constraints, it is deduced that

\[
\Omega = \frac{2}{a} T = \frac{2C_T}{a} = 2C_T,
\]

if we set \( a = 1 \).

Another way to show that \( \Omega \) is constant, is the following:

\[
\{\Omega, H\} = \left\{ \frac{\gamma q^2}{\gamma q^2}, H \right\} = \frac{4\gamma}{(\gamma q^2)^2} (4\gamma_{11} H_1^2 + 4\gamma_{22} H_2^2 - 8\varepsilon \gamma_{12} H_1 H_2).
\]

Again, the weak vanishing of the linear constraints, ensures that \( \Omega \) is constant.

Using (27), (29) and the action, we have:

\[
\gamma q^2 = C_1 \exp\{2C_T \int \tilde{N}(t) dt\}.
\]

Now, returning to the quantum domain, we observe that out of the three arguments of the wave function given in (25), only \( q^1/q^2 \) is G.C.T. invariant –in the sense previously explained. This suggests that we must somehow eliminate \( \gamma, \gamma q^2 \). To this end, we adopt the value zero for the classical constant \( C_T \). This amounts to restricting to a 2-parameter subspace of the classical space of solutions, spanned by the 3 essential constants [9]. This means that we base our quantum theory on this subspace and decree the wave function, to be applicable to all configurations (classical or not). The benefit of such an action, is twofold: \( \gamma^{-C_T} \) drops out, while at the same time \( w^2 \equiv \gamma q^2 \) is set equal to the constant \( C_1 \) –see (31). These facts, along with the obligation that no derivatives with respect to \( w^2 \),
are to enter the Wheeler-DeWitt equation –see (26)–, allow us to arrive at the following form for the wave function (25):

$$\Psi = \Theta \left( \frac{q^1}{q^2}, C_1 \right),$$

(32)

and of course

$$w^2 = q^2 q^3 = \text{const} = C_1.$$  
(33)

Now, the final reduction of the configuration space is achieved. Our dynamic variable is the ratio $q^1/q^2$, which is a combination of the only curvature invariants existing in Class A Bianchi Type VI and VII, and emerges as the only true quantum degree of freedom.

Consequently, following the spirit of [3], we have to construct the quantum analogue of $H_0$ as the conformal Laplacian, based on the non-singular part of the “physical” metric (26), i.e.:

$$\hat{H}_0 = -\frac{1}{2} \nabla_c^2 + w^2 (2 + w^4),$$

(34)

where

$$\nabla_c^2 = \nabla^2 = \frac{1}{\sqrt{s_{11}}} \partial_{w^1} \{ \sqrt{s_{11}} s^{11} \partial_{w^1} \}$$

(35)

is the 1-dimensional Laplacian based on $s_{11}$ ($s^{11} s_{11} = 1$). Note that in 1-dimension the conformal group is totally contained in the G.C.T. group, in the sense that any conformal transformation of the metric can not produce any change in the –trivial– geometry and is thus reachable by some G.C.T. Therefore, no extra term in needed in (35), as it can also formally be seen by taking the limit $d = 1, \ R = 0$ in the general definition:

$$\nabla_c^2 \equiv \nabla^2 + \frac{(d-2)}{4(d-1)} R = \nabla^2.$$

Thus, the Wheeler-DeWitt equation, reads:

$$\hat{H}_0 \Psi = \sqrt{(w^2 - 4)} \partial_w \{ \sqrt{(w^2 - 4)} \partial_w \} \Psi - \frac{C_1 (2 + w)}{2} \Psi = 0,$$

(36)

where, for simplicity, we set $w^1 \equiv w$ and $w^2 = C_1$.

Using the transformation $w = 2 \cosh(z)$, the previous equation takes the form

$$\frac{\partial^2 \Psi}{\partial z^2} - (C_1 + \cosh(z)) \Psi = 0.$$  
(37)
The solutions to this family of equations are the Mathieu Modified Functions—see [10] and references therein, for an extended treatise, in various cases.

As for the measure, it is commonly accepted that there is not a unique solution. A natural choice, is to adopt the measure that makes the operator in (36) hermitian, that is \( \mu(w) = \frac{1}{\sqrt{w^2 - 4}} \), or in the variable \( z \), \( \mu(z) = 1 \). However, the solutions to (37) can be seen to violently diverge for various values of \( z \in [0, \infty) \), which is the classically allowed region. If we wish to avoid this difficulty, we can abandon hermiticity, especially in view of the fact that we are interested in the zero eigenvalues of the operator, and thus does not make any harm to lose realness of the eigenvalues. If we adopt this attitude, we can find suitable measures, e.g. \( \mu(z) = e^{-z^2} \). The probability density \( \rho(z) = \mu(z)|\Psi(z)|^2 \) is now finite, enabling one to assign a number between 0 and 1 to each Class A Type VI and VII geometry.

Another feature of the reduction to (37) is that the final dynamical argument of \( \Psi \) is the ratio \( q_1/q_2 \), which is of degree zero in the scale factors, as seen from equation (12). This consists a kind of build-in regularization with respect to the volume of the 3-space. Moreover, the solutions to this equation exhibit an increasingly oscillatory behaviour, as \( C_1 \) increases. This is most welcomed and expected in view of \( C_1 \) being some kind of measure of the 3-volume, since it contains \( \gamma \)—see (33).

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In this work, we were able to express the wave function for the Class A Bianchi Type VI and VII Vacuum cosmologies, in terms of a unique quantum degree of freedom, i.e. the ratio of the two curvature invariants \( (q^1, q^2) \), which completely and irreducibly characterize the 3-geometries under discussion. At a first stage, this was done by imposing the quantum versions of the first-class linear constraints, as conditions on the wave function \( \Psi \). At a second stage, the indirect usage—through a classical integral of motion and its quantum version—of the outer automorphisms reduced the dimension of the configuration space—spanned by the \( q \)'s—from 3 to 2. The final reduction came through the assumption/condition that one ought to end up with a G.C.T. invariant wave function. Thus, the classically conserved value of \( T \)—which was shown to be related to the other degree of freedom, i.e. the product \( q_2^3 \)—was set to zero. Anyway, this is not such a blunder, since the ultimate goal of finding a \( \Psi \), is to give weight to all states—being
classical ones, or not. So, in conclusion, we see that not only the true degree of freedom is isolated, but also the time problem has been solved—in the sense that a square integrable wave function $\Psi$ is found. This is accomplished by revealing all the—hidden—gauge symmetries of the system. This wave function is G.C.T. invariant and well defined on any spatially homogeneous 3-geometry.

A similar situation holds for Class A Type II; objects analogous to $T$ also exist, and thus, a reduction to the variable $\gamma q^1$ is possible, with the help of this $T$ plus another one—the $\Omega$—(see [11]) and the 2 independent $H_\alpha$’s ($q^1 = q$ is the unique independent curvature invariant for these homogeneous 3-geometries). The situation concerning Type VIII and IX is more difficult: there are no outer automorphisms, and consequently, no object analogous to $T$, exists. The $H_\alpha$’s suffice to reduce the configuration space to $q^1$, $q^2$ and $q^3 = Det[m]/\sqrt{\gamma}$, now needed to specify the 3-geometry. The reduced supermetric is still Lorentzian, leading to a hyperbolic Wheeler-DeWitt equation. This fact is generally considered as a drawback, since it prohibits the ensuing wave function from being square integrable.

A very interesting case is that of Bianchi Type I; there, the automorphism group is the whole $GL(3, \mathbb{R})$. This peculiarity may prove extremely helpful in reducing the initial configuration space through the many integrals of motion. A work of ours on this issue, is in progress and it will be available as soon as possible.

**Acknowledgments**
The authors acknowledge financial support by the University of Athens’ Special Account for the Research.
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