A MORITA THEOREM FOR DUAL OPERATOR ALGEBRAS

UPASANA KASHYAP

Abstract. We prove that two dual operator algebras are weak∗ Morita equivalent in the sense of [4] if and only if they have equivalent categories of dual operator modules via completely contractive functors which are also weak∗-continuous on appropriate morphism spaces. Moreover, in a fashion similar to the operator algebra case, we characterize such functors as the module normal Haagerup tensor product with an appropriate weak∗ Morita equivalence bimodule. We also develop the theory of the W∗-dilation, which connects the non-selfadjoint dual operator algebra with the W∗-algebraic framework. In the case of weak∗ Morita equivalence, this W∗-dilation is a W∗-module over a von Neumann algebra generated by the non-selfadjoint dual operator algebra. The theory of the W∗-dilation is a key part of the proof of our main theorem.

1. Introduction and Notation

An important and well-known perspective of understanding an algebraic object is to study its category of representations. For example, modules correspond to representations of a ring hence rings are commonly studied in terms of their modules. Once we view an algebraic object in terms of its category of representations, it is natural to compare such categories. This leads to the notion of Morita equivalence. The notion of Morita equivalence of rings arose in pure algebra around 1960. Two rings are defined to be Morita equivalent if and only they have equivalent categories of modules. Morita equivalence is a powerful tool in pure algebra, and it has inspired similar notions in operator algebra theory. In the 1970’s Rieffel introduced and developed the notion of Morita equivalence for C∗-algebras and W∗-algebras. This is a useful and very important tool in modern operator theory. With the advent of operator space theory in the 1990’s, Blecher, Muhly and Paulsen generalized Rieffel’s C∗-algebraic notion of Morita equivalence to non-selfadjoint operator algebras.

Recently we generalized Rieffel’s variant of W∗-algebraic Morita equivalence to dual operator algebras. By a dual operator algebra, we mean a unital weak∗-closed algebra of operators on a Hilbert space which is not necessarily selfadjoint. One can view a dual operator algebra as a non-selfadjoint analogue of a von Neumann algebra. By a non-selfadjoint version of Sakai’s theorem (see e.g. Section 2.7 in [7]), a dual operator algebra is characterized as a unital operator algebra that is also a dual operator space.

In [4] we defined two dual operator algebras M and N to be weak∗ Morita equivalent if there exists a dual operator M-N-bimodule X, and a dual operator N-M-bimodule Y, such that M ∼= X ⊗ Y as dual operator M-bimodules (that

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is, via a completely isometric, weak$^*$-homeomorphism which is also a $M$-bimodule map), and $N \cong Y \otimes^\sigma h M X$ as dual operator $N$-bimodules. Another notion of Morita equivalence for dual operator algebras introduced by Elefttherakis will be briefly discussed later in the introduction.

In the literature of Morita equivalence for rings in pure algebra, there is a popular collection of theorems known as Morita I, II and III. Morita I can be described as the consequences of a pair of bimodules being mutual inverses ($X \otimes_N Y \cong M$ and $Y \otimes_M X \cong N$). For dual operator algebras, most of the appropriate version of Morita I is proved in [1]. Morita II characterizes module category equivalences as tensoring with an invertible bimodule, and our main theorem here is a Morita II theorem for dual operator algebras. The Morita III theorem states that there is a bijection between the set of isomorphism classes of invertible bimodules and the set of equivalence classes of category equivalences; its appropriate version for dual operator algebras follows as in pure algebra and will be presented in [16].

In [4] we proved that two dual operator algebras that are weak$^*$ Morita equivalent in our sense have equivalent categories of dual operator modules. In the present work, we prove the converse, a Morita II theorem: if two dual operator algebras have equivalent categories of dual operator modules then they are weak$^*$-Morita equivalent in the sense of [4]. The functors implementing the categorical equivalences are characterized as the module normal Haagerup tensor product with an appropriate weak$^*$ Morita equivalence bimodule. In Section 2, we develop the theory of the $W^*$-dilation, which connects the non-selfadjoint dual operator algebra with the $W^*$-algebraic framework. In particular, we use the maximal $W^*$-algebra $C$ generated by a dual operator algebra $M$. Every dual operator $M$-module dilates to a dual operator module over $C$ which is called the ‘maximal dilation’. We show that every dual operator module is a weak$^*$-closed submodule of its maximal dilation. Indeed, in the case of weak$^*$ Morita equivalence this maximal dilation turns out to be a $W^*$-module over $C$. The theory of the $W^*$-dilation is a key part of the proof of our main theorem. In Section 3, we discuss some weak$^*$ Morita equivalence and $W^*$-dilation results. In Section 4 and Section 5, we prove our main theorem.

Many of the techniques and ideas in this paper are taken from [1], [2], [3]. We refer the reader to these papers for earlier ideas, proof techniques, and additional details. In some places we just need to modify the arguments in the present setting of weak$^*$-topology, or merely change the tensor product. However, we need to develop new techniques to deal with a number of subtleties that arise in the weak$^*$-topology setting.

Another notion of Morita equivalence for dual operator algebras was considered in [14] and is called $\Delta$-equivalence. In [15] it was shown that the $\Delta$-equivalence implies weak$^*$ Morita equivalence in the sense of [4]. That is, any of the equivalences of [14] is one of our weak$^*$ Morita equivalence. Both the theories have different advantages. For example, the equivalence considered in [14] is equivalent to the very important notion of weak$^*$ stable isomorphism. On the other hand, our theory contains all examples considered up to this point in the literature of Morita-like equivalence in a dual (weak$^*$) setting. There are certain important examples that do not seems to be contained the other theory but are weak$^*$ Morita equivalent in our sense. For example, in the selfadjoint setting the second dual of strongly Morita equivalent $C^*$-algebras are Morita equivalent in Rieffel’s $W^*$-algebraic sense. In the non-selfadjoint case, the second dual of strongly Morita equivalent operator
algebras in the sense of Blecher, Muhly and Paulsen are weak* Morita equivalent in our sense. Also, two ‘similar’ separably acting nest algebras are Morita equivalent in our sense but are not $\Delta$-equivalent.

In [4] we showed that weak* Morita equivalent dual operator algebra have equivalent categories of normal Hilbert space representations (also known as normal Hilbert modules). However, the converse of this is still an open problem and at present we are working on this aspect. The characterization theorem in [14] is in terms of equivalence of categories of normal Hilbert modules, which intertwines not only the representations of the dual operator algebras, but also their restrictions to the diagonals.

We assume that the reader is familiar with the notions from operator space theory. One can refer to [7] and [13] for background and most of the terminology used in this paper. We also assume that all dual operator algebras are unital; that is, they each have an identity of norm 1. We will often abbreviate ‘weak*’ to ‘$w^*$’. We reserve the symbols $M$ and $N$ for dual operator algebras. A normal representation of $M$ is a $w^*$-continuous unital completely contractive homomorphism $\pi : M \to B(H)$. For a dual space $X$, we let $X_*$ denote its predual. We assume that the reader is familiar with the weak* topology and basic duality principles such as the Krein-Smulian theorem (see Theorem A.2.5 in [7]).

A concrete dual operator $M$-$N$-bimodule is a $w^*$-closed subspace $X$ of $B(K,H)$ such that $\theta(M)X\pi(N) \subseteq X$, where $\theta$ and $\pi$ are normal representations of $M$ and $N$ on $H$ and $K$ respectively. An abstract dual operator $M$-$N$-bimodule is defined to be an operator $M$-$N$-bimodule $X$ (by which we mean that $X$ is an operator space and a nondegenerate $M$-$N$-bimodule such that the module actions are completely contractive in the sense of 3.1.3 in [7]), which is also a dual operator space, such that the module actions are separately weak* continuous. Such spaces can be represented completely isometrically as concrete dual operator bimodules (see e.g. Theorem 3.8.3 in [7], [12]). We shall write $M\mathcal{R}$ for the category of left dual operator modules over $M$. The morphisms in $M\mathcal{R}$ are the $w^*$-continuous completely bounded $M$-module maps.

By $M\mathcal{H}$, we mean the category of completely contractive normal Hilbert modules over a dual operator algebra $M$. That is, elements of $M\mathcal{H}$ are pairs $(H,\pi)$, where $H$ is a (column) Hilbert space (see e.g. 1.2.23 in [7]), and $\pi : M \to B(H)$ is a normal representation of $M$. The module action is expressed through the equation $m\cdot \zeta = \pi(m)\zeta$. The morphisms are bounded linear transformations between Hilbert spaces that intertwine the representations; i.e., if $(H_1,\pi_1)$, $i = 1, 2$, are objects of the category $M\mathcal{H}$, then the space of morphisms is defined as:

$$B_M(H_1,H_2) = \{ T \in B(H_1,H_2) : T\pi_1(m) = \pi_2(m)T \text{ for all } m \in M \}.$$ 

Any $H \in M\mathcal{H}$ (with its column Hilbert space structure) is a left dual operator $M$-module. If $E$ and $F$ are sets, then $EF$ denotes the norm closure of the span of products $x y$ for $x \in E$ and $y \in F$.

If $X$ and $Y$ are dual operator spaces, we denote by $CB^\sigma(X,Y)$ the space of completely bounded $w^*$-continuous linear maps from $X$ to $Y$. Similarly if $X$ and $Y$ are left dual operator $M$-modules, then $CB^\sigma_M(X,Y)$ denotes the space of completely bounded $w^*$-continuous left $M$-module maps from $X$ to $Y$.

If $M$ is a dual operator algebra, then a $W^*$-cover of $M$ is a pair $(A,j)$ consisting of a $W^*$-algebra $A$ and a completely isometric $w^*$-continuous homomorphism $j : M \to A$, such that $j(M)$ generates $A$ as a $W^*$-algebra. By the Krein-Smulian
Let $j(M)$ be a $W^*$-closed subalgebra of $A$. The maximal $W^*$-cover $W^*_{\text{max}}(M)$ is a $W^*$-algebra containing $M$ as a $W^*$-closed subalgebra that is generated by $M$ as a $W^*$-algebra, and has the following universal property: any normal representation $\pi : M \rightarrow B(H)$ extends uniquely to a (unital) normal $*$-representation $\tilde{\pi} : W^*_{\text{max}}(M) \rightarrow B(H)$ (see [11]).

We will refer to Rieffel’s $W^*$-algebraic Morita equivalence (see [17]) as ‘weak Morita equivalence’ for $W^*$-algebras, and the associated equivalence bimodules as ‘$W^*$-equivalence-bimodules’ (see e.g. Section 8.5 in [7]).

We use the normal module Haagerup tensor product $\otimes_M^{\sigma_h}$ throughout the paper. We refer to [15] and [4, Section 2] for the universal property and general facts and properties of $\otimes_M^{\sigma_h}$. Loosely speaking, the normal module Haagerup tensor product linearizes completely contractive balanced separately weak$^*$-continuous bilinear maps.

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2. Dual Operator Modules over a Generated $W^*$-Algebra and $W^*$-Dilations

We begin this section with a weak$^*$-topology version of Theorem 3.1 in [1].

**Theorem 2.1.** Let $D$ be a $W^*$-algebra, $B$ a Banach algebra which is also a dual Banach space, and $\theta : D \rightarrow B$ a unital $w^*$-continuous contractive homomorphism. Then the range of $\theta$ is $w^*$-closed, and possesses an involution with respect to which $\theta$ is a $*$-homomorphism and the range of $\theta$ is a $W^*$-algebra.

**Proof.** It is known that (see e.g. Theorem A.5.9 in [7]) the range of a contractive homomorphism between a $C^*$-algebra and a Banach algebra is a $C^*$-algebra and moreover such homomorphisms are $*$-homomorphisms. To see that the range of $\theta$ is $w^*$-closed, consider the quotient map $D/\ker(\theta) \rightarrow B$ which is an isometry, and apply the Krein-Smulian theorem.

Thus if $X$ is a left dual operator module over a $W^*$-algebra $D$, and if we let $\theta : D \rightarrow CB(X)$ be the associated unital $w^*$-continuous contractive (equivalently completely contractive by Proposition 1.2.4 in [7]) homomorphism, then the range of $\theta$ is a $W^*$-algebra.

**Theorem 2.2.** Suppose that $X$ is a dual operator module over a dual operator algebra $M$. Let $\theta : M \rightarrow CB(X)$ be the associated completely contractive homomorphism. Suppose that $D$ is any $W^*$-algebra generated by $M$. Then the $M$-action on $X$ can be extended to a $D$-action with respect to which $X$ is a dual operator $D$-module if and only if $\theta$ is the restriction to $M$ of a $w^*$-continuous contractive (equivalently completely contractive) homomorphism $\phi : D \rightarrow CB(X)$. This extended $D$-action, or equivalently the homomorphism $\phi$, is unique if it exists.

**Proof.** If $\theta$ is the restriction to $M$ of a $w^*$-continuous completely contractive homomorphism $\phi : D \rightarrow CB(X)$ then the $M$-action on $X$ can be extended to a $D$-action via $d \cdot x = \phi(d) \cdot x$. Note that the $D$-module action $x \mapsto dx$ on $X$, for $x \in X$ and $d \in D$, is a multiplier (see e.g. Section 4.5 in [7]), hence it is weak$^*$-continuous by Theorem 4.1 in [6]. The $D$-module action on $X$ is separately $w^*$-continuous and completely contractive. Hence $X$ is a dual operator $D$-module. The converse
is obvious. To see the uniqueness assertion, suppose that $\phi_1$ and $\phi_2$ are two $w^*$-continuous contractive homomorphisms $D \to CB(X)$, extending $\theta$. By Theorem 2.1, the ranges $\mathcal{E}_1$ and $\mathcal{E}_2$, of $\phi_1$ and $\phi_2$ respectively, are each $W^*$-algebras, but with possibly different involutions and weak$^*$-topologies. We will write these involutions as $\ast$ and $\#$ respectively. With respect to these involutions $\phi_1$ and $\phi_2$ are $\ast$-homomorphisms. Note, $CB(X)$ is a unital Banach algebra and $\mathcal{E}_1$ and $\mathcal{E}_2$ may be viewed as unital subalgebras of $CB(X)$, with the same unit. Let $a \in M$ and $f$ be a state on $CB(X)$. Then $f|_{\mathcal{E}_i}$ is a state on $\mathcal{E}_i$ for $i = 1, 2$. Thus $f(\phi_1(a)\ast) = f(\phi_1(a)) \ast f(\phi_2(a)) = f(\phi_2(a)\#)$. Thus $u = \phi_1(a)\ast - \phi_2(a)\#$ is a Hermitian element in $CB(X)$ with numerical radius 0, hence $u = 0$. This implies that $\phi_1(a)\ast = \phi_2(a)\ast$, since $\phi_1$ and $\phi_2$ are $\ast$-homomorphisms. Hence $\phi_1$ equals $\phi_2$ on the $\ast$-subalgebra generated by $M$ in $D$. By weak$^*$-density, it follows that $\phi_1 = \phi_2$ on $D$.

This immediately gives the following:

**Corollary 2.3.** Let $D$ be a $W^*$-algebra generated by a dual operator algebra $M$. If $X_1$ and $X_2$ are two dual operator $D$-modules, and if $T : X_1 \to X_2$ is a $w^*$-continuous completely isometric and surjective $M$-module map, then $T$ is a $D$-module map.

**Corollary 2.4.** Let $D$ be a $W^*$-algebra generated by a dual operator algebra $M$. Then the category $\mathcal{D}R$ of dual operator modules over $D$ is a subcategory of the category $\mathcal{M}R$ of dual operator modules over $M$. Similarly, $\mathcal{D}H$ is a subcategory of $\mathcal{M}H$.

Next we discuss the $W^*$-dilation which we call the ‘$D$-dilation’ of a dual operator $M$-module $X$, where $D$ is a $W^*$-algebra generated by $M$. Strictly speaking, it should be called $W^*$-$D$-dilation, but for brevity we will use the shorter term.

**Definition 2.5.** A pair $(E, i)$ is said to be a $D$-dilation of a left dual operator $M$-module $X$, if the following hold:

1. $E$ is a left dual operator $D$-module and $i : X \to E$ is a $w^*$-continuous completely contractive $M$-module map.
2. For any left dual operator $D$-module $X'$, and any $w^*$-continuous completely bounded $M$-module map $T : X \to X'$, there exists a unique $w^*$-continuous completely bounded $D$-module map $\tilde{T} : E \to X'$ such that $\tilde{T} \circ i = T$, and also $\|T\|_{cb} = \|\tilde{T}\|_{cb}$.

Some authors also use the terminology ‘$D$-adjunct’ for $D$-dilation (see [1]).

The assertion in (2) above implies that $i(X)$ generates $E$ as a dual operator $D$-module. To see this, let $E' = Di(X)^{w^*}$, and consider the quotient map $q : E \to E/E'$. Then $E/E'$ is a left dual operator $D$-module such that $q \circ i = 0$. Hence the assertion in (2) in the above definition implies that the map $q = 0$. Thus $E = E'$.

Up to a complete isometric module isomorphism there is a unique pair $(E, i)$ satisfying (1) and (2) in the above definition. To see this, let $(E', i')$ be any other pair satisfying (1) and (2), then there exists a unique $w^*$-continuous completely contractive $D$-module linear maps $\rho : E \to E'$ and $\phi : E' \to E$ such that $\rho \circ i = i'$ and $\phi \circ i' = i$. One concludes that $\rho \circ \phi$ is the identity map on $i'(X)$ and $\phi \circ \rho$ is the identity map on $i(X)$. Since $i(X)$ and $i'(X)$ generate $E$ as a dual operator $D$-module, and since $\phi$ and $\rho$ are $w^*$-continuous complete contractions, this implies that $\phi$ and $\rho$ are complete isometries.
Remark 2.6. From the above it is clear that the \( D \)-dilation \( (E, i) \) is the unique pair satisfying (1), and such that for all dual operator \( D \)-modules \( X' \), the canonical map \( i^* : CB^2_D(E, X') \rightarrow CB^2_M(X, X') \), given by composition with \( i \), is an isometric isomorphism. Note that by using (1.7) and Corollary 1.6.3 in [2], it is easy to see that \( M_n(CB^*(X, Y)) \cong CB^*(X, M_n(Y)) \) completely isometrically for dual operator spaces \( X \) and \( Y \). If \( X \) is a left dual operator \( M \)-module, then \( M_n(X) \) is also a left dual operator \( M \)-module via \( m \cdot [x_{ij}] = [m \cdot x_{ij}] = I_n \otimes m \cdot [x_{ij}] \), where \( I_n \otimes m \) denotes the diagonal matrix in \( M_n(M) \) with diagonal entries \( m \). Indeed, if \( X \) is a dual operator \( M \)-module, the above module action is completely contractive and by Corollary 1.6.3 in [7], this action is separately \( w^* \)-continuous. This proves that \( M_n(X) \) is a dual operator \( M \)-module. Thus the \( D \)-dilation \( E \) of \( X \) is completely isometrically. Thus the \( D \)-dilation \( E \) of \( X \) satisfies:

\[
(2.1) \quad CB^2_D(E, X') \cong CB^2_M(X, X')
\]

completely isometrically.

By the dual operator module version of Christensen-Effros-Sinclair theorem (see e.g. Theorem 3.3.1 in [7]), \( X' \) in Definition 2.6 can be taken to be \( B(H, K) \), where \( K \) is a normal Hilbert \( D \)-module and \( H \) is a Hilbert space. In fact, by a modification of Theorem 3.8 in [1], we may take \( X' = K \). We are going to prove this important fact in the next theorem but before that we need to recall some tensor products facts.

For operator spaces \( X \) and \( Y \), we denote the Haagerup tensor product of \( X \) and \( Y \) by \( X \otimes_h Y \). If \( Z \) is another operator space, \( CB(X \times Y, Z) \) denotes the space of completely bounded bilinear maps from \( X \times Y \rightarrow Z \) (in the sense of Christensen and Sinclair). It is well known that \( CB(X \times Y, Z) \cong CB(X \otimes_h Y, Z) \) completely isometrically (see e.g. 1.5.4 in [7]).

If \( X \) and \( Y \) are two dual operator spaces, we use \( (X \otimes h Y)^* \) to denote the subspace of \( (X \otimes h Y)^* \) corresponding to the completely bounded bilinear maps from \( X \times Y \rightarrow \mathbb{C} \) which are separately \( w^* \)-continuous. Then we define the normal Haagerup tensor product \( X \otimes^{sh} Y \) to be the operator space dual of \( (X \otimes h Y)^* \). If \( Z \) is another dual operator space, we denote by \( CB^*(X \times Y, Z) \) the space of completely bounded bilinear maps from \( X \times Y \rightarrow Z \) which are separately \( w^* \)-continuous. By the matricial version of (5.22) in [11], \( CB^*(X \times Y, Z) \cong CB^*(X \otimes^{sh} Y, Z) \) completely isometrically.

Suppose \( X \) is a right dual operator \( M \)-module and \( Y \) is a left dual operator \( M \)-module. A bilinear map \( u : X \times Y \rightarrow Z \) is \( M \)-balanced if \( u(xm, y) = u(x, my) \) for \( m \in A \). We let \( (X \otimes_{hM} Y)^* \) denote the subspace of \( (X \otimes_{hM} Y)^* \) corresponding to the completely bounded balanced bilinear maps from \( X \times Y \rightarrow \mathbb{C} \) which are separately \( w^* \)-continuous, where \( \otimes_{hM} \) denotes the module Haagerup tensor product (see e.g. 3.4.2, 3.4.3 in [7]). By Proposition 2.1 in [15], the module normal Haagerup tensor product \( X \otimes_{hM}^n Y \) may be defined to be the operator space dual of \( (X \otimes_{hM} Y)^* \). If \( Z \) is another dual operator space, we denote by \( CB^{M*}(X \times Y, Z) \) the space of completely bounded balanced separately \( w^* \)-continuous bilinear maps. By Proposition 2.2 in [15], \( CB^{M*}(X \times Y, Z) \cong CB^*(X \otimes_{hM}^n Y, Z) \) completely isometrically.
In order to prove the next lemma, we will introduce some notation. Let $CB^{S\sigma}(X \otimes Y, Z)$ denote the subspace of $CB(X \otimes Y, Z)$ consisting of completely bounded maps from $X \otimes Y$ to $Z$ that are induced by the jointly completely bounded bilinear maps from $X \times Y \to Z$ which are separately $w^*$-continuous, where $\otimes$ denotes the operator space projective tensor product (see e.g. 1.5.11 in [1]). In the case, when $Z = \mathbb{C}$, we denote $CB^{S\sigma}(X \otimes Y, \mathbb{C})$ by $(X \otimes Y)_c^\sigma$.

**Lemma 2.7.** For any Hilbert spaces $H$ and $K$ and dual operator space $X$, $CB^\sigma(X, B(H, K)) \cong CB^\sigma(X \otimes^h H^c, K^c) \cong (\overline{K} \otimes^h X \otimes^h H^c)_c^\sigma$, completely isometrically.

**Proof.** For any dual operator space $X$, we have the following isometries:

$CB^\sigma(X \otimes^h H^c, K^c) \cong CB^\sigma(X \times H^c, K^c)$

$\cong CB^{S\sigma}(X \otimes H^c, K^c)$

$\cong CB^\sigma(X, CB(H^c, K^c))$

$\cong CB^\sigma(X, B(H, K))$

using Proposition 1.5.14 (1) and (1.50) from [1]. Consider

$CB^\sigma(X \otimes^h H^c, K^c) \cong (\overline{K} \otimes^h (X \otimes^h H^c))_c^\sigma$

$\cong (\overline{K} \otimes^h (X \otimes^h H^c))_c^\sigma$

$\cong (\overline{K} \otimes^h_h (X \otimes^h H^c))_c^\sigma$

$\cong (\overline{K} \otimes^h (X \otimes^h H^c)_c^\sigma$

using (1.51) and Proposition 1.5.14 (1) in [1], and associativity of the normal Haagerup tensor product. \qed

Similarly we have the module version of the above lemma:

**Lemma 2.8.** Let $X$ be a left dual operator $M$-module and $K$ be a normal Hilbert $M$-module. Then for any Hilbert space $H$, $CB^\sigma_M(X, B(H, K)) \cong CB^\sigma_M(X \otimes^h H^c, K^c) \cong (\overline{K} \otimes^h M X \otimes^h H^c)_c^\sigma$, completely isometrically.

**Proof.** The first isomorphism follows as above with completely bounded maps replaced with module completely bounded maps. Consider

$CB^\sigma_M(X \otimes^h H^c, K^c) \cong (\overline{K} \otimes^h_M (X \otimes^h H^c))_c^\sigma$

$\cong (\overline{K} \otimes^h_M (X \otimes^h H^c))_c^\sigma$

$\cong (\overline{K} \otimes^h (X \otimes^h H^c))_c^\sigma$

$\cong (\overline{K} \otimes^h_M X \otimes^h H^c)_c^\sigma$,

using Corollary 3.5.10 in [1], $K^r \otimes^h_M = K^r \otimes^h_M$ and a variant of Proposition 2.9 in [1]. \qed

We would like to thank David Blecher for the proof of the following lemma.

**Lemma 2.9.** Let $S : X \to Y$ be a $w^*$-continuous linear map between dual operator spaces. The following are equivalent:

(i) $S$ is a complete isometry and surjective.

(ii) For some Hilbert space $H$, $S \otimes I_H : X \otimes^h H^c \to Y \otimes^h H^c$ is a complete isometry and surjective.
Proof. Firstly, suppose \( S \) is a completely isometric and \( w^\ast \)-homeomorphic map. Then, by the functoriality of the normal Haagerup tensor product \( S \otimes_{1} I_H \) and \( S^{-1} \otimes I_H \) are completely contractive \( w^\ast \)-continuous maps, where \( I_H \) denotes the identity map on \( H \). Also \( (S^{-1} \otimes I_H) \circ (S \otimes I_H) = Id \) on a \( w^\ast \)-dense subset \( X \otimes H \). By \( w^\ast \)-density, \( (S^{-1} \otimes I_H) \circ (S \otimes I_H) = Id \) on \( X \otimes H^c \). Similarly, \( (S \otimes I_H) \circ (S^{-1} \otimes I_H) = Id \). Thus \( S \otimes I_H \) is a completely isometric and \( w^\ast \)-homeomorphic map.

Conversely, suppose \((ii)\) holds. Fix a \( \eta \in H \) with \(|\|\eta\|| = 1 \). Let \( v : X \to X \otimes H^c \eta : x \mapsto x \otimes \eta \). Since \( X \subseteq X \otimes H^c \) completely isometrically via \( v \), and \( X \otimes H^c \subseteq X \otimes H^c \) completely isometrically, this implies that \( v \) is a complete isometry. If \( S \otimes I_H \) is a complete isometry, then \( S \otimes I_H \) restricted to \( X \otimes H^c \eta \) is a complete isometry. Similarly, let \( u : Y \to Y \otimes H^c \eta : y \mapsto y \otimes \eta \). Then, \( S = u^{-1} \circ (S \otimes I_H) \circ v \) is a complete isometry. To see \( S \) is onto, suppose for the sake of contradiction that it is not. Then by the Krein-Smulian theorem \( G = \text{Ran}(S) \) is a weak*-closed proper subspace of \( Y \). Let \( \varphi \in G^\perp \) and \( \varphi \neq 0 \). Consider a map \( r : Y \otimes H^c \to C \otimes H^c : y \otimes \zeta \mapsto \varphi(y) \otimes \zeta \). Then, \( r \circ (S \otimes I_H) = 0 \), hence \( r \) vanishes on a \( w^\ast \)-dense subset \( Y \otimes H^c \). So \( r = 0 \). Hence \( \varphi(y) \otimes \zeta = 0 \) for all \( \zeta \in H \) and \( y \in Y \). This implies \( \varphi = 0 \), which is a contradiction. \( \square \)

**Theorem 2.10.** Suppose \( E \) is a left dual operator \( D \)-module and \( i : X \to E \) is a \( w^\ast \)-continuously completely contractive \( M \)-module map. Then \( (E, i) \) is the \( D \)-dilation of \( X \) if and only if the canonical map \( i^* : CB^D_1(E, K) \to CB^D_1(X, K) \) as defined above is a complete isometric isomorphism, for all normal Hilbert \( D \)-modules \( K \).

It is sufficient to take \( K = B \) to be the normal universal representation of \( D \) or any normal generator for \( D \) in the sense of [10], [17].

**Proof.** Consider the following sequence of complete contractions:

\[
\mathcal{K} \otimes^h_M X \xrightarrow{id \otimes i} \mathcal{K} \otimes^h_M E \cong \mathcal{K} \otimes^h_D D \otimes^h_M E \to \mathcal{K} \otimes^h_D E.
\]

where the last map in the sequence comes from the multiplication \( D \times E \to E \).

Taking the composition of the above maps, we get a complete contraction \( S : \mathcal{K} \otimes^h M X \to \mathcal{K} \otimes^h_M E \). Tensoring \( S \) with the identity map on \( H \), we get a \( w^\ast \)-continuous, completely contractive linear map \( S_1 = S \otimes id_H : \mathcal{K} \otimes^h_M X \otimes H^c \otimes^h_D E \otimes H^c \to \mathcal{K} \otimes^h_M E \otimes H^c \otimes^h_D E \) by Corollary 2.4 in [4]. From a well known weak*-topology fact, \( S_1 = T^* \) for some \( T : (\mathcal{K} \otimes^h_M E \otimes H^c)_\ast \to (\mathcal{K} \otimes^h M X \otimes H^c)_\ast \). From Lemma 2.8 and standard weak*-density arguments, it follows that \( T \) equals \( i^\ast \), as defined earlier. Indeed, we use the duality pairing, namely, \( \langle \psi \otimes x \otimes \eta, T \rangle = \langle T(x)(\eta), \psi \rangle \), for \( T \in CB^D_1(X, B(H, K)) \), \( x \in X \), \( \eta \in H \), \( \psi \in K^\ast \), to check that \( (i^\ast)^\ast = S_1 \) on the weak*-dense subset \( \mathcal{K} \otimes X \otimes H^c \). Then by weak*-density, it follows that \( (i^\ast)^\ast = S_1 = T^* \), so \( i^\ast = T \). Hence, \( i^\ast \) is an isometric isomorphism if and only if \( S_1 \) is an isometric isomorphism if and only if \( S \) is an isometric isomorphism by Lemma 2.9. Note that with \( H = C \) in Lemma 2.8 \( CB^D_1(X, K^\ast) = (\mathcal{K} \otimes^h_M X)_\ast \).

From Lemma 2.8 it is clear that \( CB^D_1(E, K^\ast) \cong CB^D_1(X, K^\ast) \) if and only if \( CB^D_1(E, H^c, K^\ast) \cong CB^D_1(X, H^c, K^\ast) \) for all normal Hilbert \( D \)-modules \( K \). For the last assertion, note that every nondegenerate normal Hilbert \( D \)-module \( K \) is a complemented submodule of a direct sum of \( I \) copies of the normal universal representation or normal generator, for some cardinal \( I \) (see e.g. [10]). Therefore we need to show that if \( CB^D_1(E, K) \cong CB^D_1(X, K) \) completely isometrically then \( CB^D_1(E, K^I) \cong CB^D_1(X, K^I) \) completely isometrically as well, where \( K^I \) denotes...
the Hilbert space direct sum of $I$-copies of $K$. This follows from the operator space fact that $\text{CB}_M(X,Y') \cong M_{1,1}(\text{CB}_M(X,Y))$ completely isometrically for any dual operator spaces $X$ and $Y$ which are also $M$-modules (see page 156 in [12]). Here $M_{1,1}(X)$ denotes the operator space of columns of length $I$ with entries in $X$, whose finite subcolumns have uniformly bounded norm. □

The following lemma shows the existence of the $D$-dilation. The normal module Haagerup tensor product $D \otimes^h_M X$ (which is a dual operator $D$-module by Lemma 2.3 in [4]) acts as the $D$-dilation of $X$. We note that, since by Lemma 2.10 in [4] $M \otimes^h_M X \cong X$, there is a canonical $w^*$-continuous completely contractive $M$-module map $i : X \to D \otimes^h_M X$ taking $x \mapsto 1 \otimes^h_M x$.

**Lemma 2.11.** For any left dual operator module $X$ over $M$, the dual operator $D$-module $E = D \otimes^h_M X$ is the $D$-dilation of $X$.

**Proof.** If $T : X \to X'$ is as in Definition 2.5, then by the functoriality of the normal module Haagerup tensor product, $1_D \otimes T : D \otimes^h_M X \to D \otimes^h_M X'$ is $w^*$-continuous completely bounded. Composing this with the $w^*$-continuous module action $D \otimes^h_M X' \to X'$ gives the required map $\tilde{T}$. It is routine to check that $\tilde{T}$ has the required properties. □

**Lemma 2.12.** If $X$ is a left dual operator $M$-module, and if $D$ is a $W^*$-algebra generated by $M$, then the following are equivalent:

1. There exists a dual operator $D$-module $X'$ and a completely isometric $w^*$-continuous $M$-module map $j : X \to X'$.

2. The canonical $w^*$-continuous $M$-module map $i : X \to D \otimes^h_M X$, is a complete isometry.

**Proof.** The one direction (2) implies (1) is obvious. For the difficult direction, suppose that $m$ is the module action on $X'$. Then we have the following sequence of canonical $w^*$-continuous completely contractive $M$-module maps:

$$X \xrightarrow{i} D \otimes^h_M X \xrightarrow{1_{D \otimes^h_M X}} D \otimes^h_M X' \xrightarrow{m} X'.$$

The composition of these maps equals $j$, which is a complete isometry. This forces $i$ to be a complete isometry which proves the assertion. □

In the case that $D = \mathcal{C} = W^*_{\text{max}}(M)$, we call $\mathcal{C} \otimes^h_M X$ the ‘maximal $W^*$-dilation’ or ‘maximal dilation’. This is the key point in proving our main theorem (Section 4). The reason we work mostly with maximal dilation instead of any arbitrary dilation is the following result.

**Corollary 2.13.** For any left dual operator $M$-module $X$, the canonical $M$-module map $i : X \to \mathcal{C} \otimes^h_M X$ is a $w^*$-continuous complete isometry.

**Proof.** This follows from the previous result, the Christensen-Effros-Sinclair-representation theorem for dual operator modules, and the fact that every normal Hilbert $M$-module is a normal Hilbert $\mathcal{C}$-module for the maximal $W^*$-algebra generated by $M$ (i.e. the universal property of $\mathcal{C}$). □

Hence, we may regard $X$ as a $w^*$-closed $M$-submodule of $\mathcal{C} \otimes^h_M X$. There is a similar notion of $W^*$-dilation for right dual operator modules or dual operator bimodules. The results in this section carry through analogously to these cases.
3. Morita equivalence of dual operator algebras

In this section, $M$ and $N$ are again dual operator algebras. We reserve the symbols $\mathcal{C}$ and $\mathcal{D}$ for the maximal $\ast$-algebras $W_{\text{max}}(M)$ and $W_{\text{max}}(N)$ generated by $M$ and $N$ respectively. We refer the reader to [4] if further background for this section is needed.

We begin with the following normal Hilbert module characterization of $\ast$-algebras which is proved in Proposition 7.2.12 in [7].

**Proposition 3.1.** Let $M$ be a dual operator algebra. Then $M$ is a $\ast$-algebra if and only if for every completely contractive normal representation $\pi : M \to B(H)$, the commutant $\pi(M)'$ is selfadjoint.

**Corollary 3.2.** Suppose $M$ and $N$ are dual operator algebras such that the categories $\mathcal{M}_H$ and $\mathcal{N}_H$ are completely isometrically equivalent; i.e., there exist completely contractive functors $F : \mathcal{M}_H \to \mathcal{N}_H$ and $G : \mathcal{N}_H \to \mathcal{M}_H$, such that $FG \cong \text{Id}$ and $GF \cong \text{Id}$ completely isometrically, then:

1. If $M$ is a $\ast$-algebra then so is $N$.
2. Also $\mathcal{C}_H$ and $\mathcal{D}_H$ are completely isometrically equivalent.

**Proof.** Suppose $F : \mathcal{M}_H \to \mathcal{N}_H$ and $G : \mathcal{N}_H \to \mathcal{M}_H$, are functors as in the statement of the corollary. If $M$ is a $\ast$-algebra, then for $H \in \mathcal{M}_H$, $B_M(H)$ is a $\ast$-algebra by Proposition 3.1. The map $T \mapsto F(T)$ from $B_M(H)$ to $B_N(F(H))$ is a surjective isometric homomorphism (see Lemma 2.2 in [3] or Lemma 4.4 below). Hence by Theorem A.5.9 in [7], this is a $\ast$-homomorphism if $M$ is a $\ast$-algebra, and consequently its range $B_N(F(H))$ is a $\ast$-algebra. Thus, if $M$ is a $\ast$-algebra, then $B_N(H)$ is a $\ast$-algebra for all normal Hilbert $N$-modules $H$. From Proposition 3.1 it follows that $N$ is a $\ast$-algebra. For $H \in \mathcal{M}_H$, we have $B_C(H)$ is a subalgebra of $B_M(H)$. The proof that $F$ restricts to a functor from $\mathcal{C}_H$ to $\mathcal{D}_H$ and similar assertion for $G$, follows identically to the $C^*$-algebra case (see e.g. Proposition 5.1 in [1]).

**Definition 3.3.** (1) Suppose that $\mathcal{E}$ and $\mathcal{F}$ are weakly Morita equivalent $\ast$-algebras in the sense of Rieffel [17], and that $Z$ is a $\ast$-equivalence $\mathcal{F}$-$\mathcal{E}$-bimodule (see 8.5.12 in [7]), and that $W = \overline{Z}$ is the conjugate $\mathcal{E}$-$\mathcal{F}$-bimodule of $Z$. Then we say that $(\mathcal{E}, \mathcal{F}, W, Z)$ is a $\ast$-Morita context (or $\ast$-context for short).

(2) Suppose that $M$ and $N$ are dual operator algebras, and suppose that $\mathcal{E}$ and $\mathcal{F}$ are $\ast$-algebras generated by $M$ and $N$ respectively. Suppose that $(\mathcal{E}, \mathcal{F}, W, Z)$ is a $\ast$-Morita context, $X$ is a $\ast$-closed $M$-$N$-submodule of $W$, and $Y$ is a $\ast$-closed $N$-$M$-submodule of $Z$. Suppose that the natural pairings $Z \times W \to \mathcal{F}$ and $W \times Z \to \mathcal{E}$ restrict to maps $Y \times X \to N$, and $X \times Y \to M$ respectively, both with $\ast$-dense range. Then we say $(M, N, X, Y)$ is a subcontext of $(\mathcal{E}, \mathcal{F}, W, Z)$. If further, $\mathcal{E}$ and $\mathcal{F}$ are maximal $\ast$-covers (as defined in the introduction) of $M$ and $N$ respectively, then we say that $(M, N, X, Y)$ is a maximal subcontext.

(3) A subcontext $(M, N, X, Y)$ of a $\ast$-Morita context $(\mathcal{E}, \mathcal{F}, W, Z)$ is left dilatable if $W$ is the left $\mathcal{E}$-dilation of $X$, and $Z$ is the left $\mathcal{F}$-dilation of
Y. In this case we say that M and N are left weakly subequivalent and 
(M,N,X,Y) is a left subequivalence context.

There is a similar definition and symmetric theory where we replace the words
‘left’ by ‘right’ or ‘two-sided’.

Remark 3.4. Note that (2) in the above definition implies that X and Y are non-
deregenerate dual operator modules over M and N.

Write $L^w$ for the set of $2 \times 2$ matrices

$$L^w = \left\{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} : a \in M, b \in N, x \in X, y \in Y \right\}.$$ 

Write $L'$ for the same set, but with entries from the $W^*$-context $(E,F,W,Z)$. It is
well known that $L'$ is canonically a $W^*$-algebra, called the ‘linking $W^*$-algebra’ of
the $W^*$-context $(E,F,W,Z)$ (see e.g. 8.5.10 in [7]). Saying that $(M,N,X,Y)$ is a
subcontext of $(E,F,W,Z)$ implies that $L^w$ is a $w^*$-closed subalgebra of $L'$. Thus
a subcontext gives a linking dual operator algebra $L^w$. Clearly $L^w$ has a unit. We
shall see that $L^w$ generates $L'$ as a $W^*$-algebra.

The proof of the following theorem is similar to the proof of Theorem 5.2 in [4]
with an arbitrary $W^*$-dilation in place of $W^*_{\text{max}}(M)$ and hence we omit it.

Theorem 3.5. Suppose that dual operator algebras M and N are linked by a weak$^*$
Morita context $(M,N,X,Y)$ in the sense of [4]. Suppose that M is rep-
resented normally and completely isometrically as a subalgebra of $B(H)$ nondegen-
erately, for some Hilbert space $H$, and let $E$ be the $W^*$-algebra generated by M in
$B(H)$. Then $Y \otimes_{\pi^h M} E$ is a right $W^*$-module over $E$. Also (as in the proof of Theo-
rem 5.2 in [4]) $Y \otimes_{\pi^h M} E \cong \overline{YE}^{w^*}$ completely isometrically and $w^*$-homeomorphically
and hence $Y \otimes_{\pi^h M} E$ contains $Y$ as a $w^*$-closed $M$-submodule completely isometrically.
Also, via this module, $E$ is weakly Morita equivalent (in the sense of Rieffel) to the
$W^*$-algebra $F$ generated by the completely isometric induced normal representation
of N on $Y \otimes_{\pi^h M} H$.

If $C$ is a $W^*$-algebra generated by M, then we shall write $F(C)$ for $Y \otimes_{\pi^h M} C \otimes_{\pi^h M} X$.
From an obvious modification of Theorem 5.2 in [4], we have that $F(C)$ is a $W^*$
algebra containing a copy of $N$, which is $*$-isomorphic and $w^*$-homeomorphic to
$(YCX)^{-w^*}$. The copy of $N$ may be identified with $(YMX)^{-w^*}$. Thus, Theorem
5.5 tells us that C is weakly Morita equivalent to $F(C)$ as $W^*$-algebras.

Similarly, if D is a $W^*$-algebra generated by N, then we write $G(D)$ for $X \otimes_{\pi^h N} D \otimes_{\pi^h N} Y$.
Again $G(D) \cong (XDY)^{-w^*}$ $*$-isomorphically and $w^*$-homeomorphically.
By associativity of the module normal Haagerup tensor product and Lemma 2.10
in [4], $G(F(C)) \cong C$, and $F(G(D)) \cong D$ $*$-isomorphically. One can think of $F$ as a
mapping between $W^*$-covers of M and N. There is a natural ordering of $W^*$-covers
of a dual operator algebra. If $(A,j)$ and $(A',j')$ are $W^*$-covers of $M$, we then define
$(A,j) \leq (A',j')$ if and only if there is a $w^*$-continuous $*$-homomorphism $\pi : A' \to A$
such that $\pi \circ j' = j$. It is an easy exercise (using that the range of $\pi$ is $w^*$-closed)
to check that $\pi$ is surjective.

Theorem 3.6. The correspondence $C \mapsto F(C)$ is bijective and order preserving.
Theorem 3.9. A W*-continuous quotient +-homomorphism between two W*-algebras generated by \( M \), such that \( \phi|_M = \text{Id}_M \). Then by Corollary 2.4 in [3] \( \phi = \text{Id}_Y \otimes \phi \otimes \text{Id}_X : Y \otimes^h \sigmah c_1 \otimes \sigmah c_2 \otimes \sigmah c_3 X \to Y \otimes^h \sigmah c_1 \otimes \sigmah c_2 \otimes \sigmah c_3 X \) is a W*-continuous completely contractive map with w*-dense range, which equals the identity when restricted to the copy of N.

It is easy to check that \( \phi \) is a homomorphism on the w*-dense subset \( Y \otimes \sigmah c_1 \otimes X \). Therefore by w*-density, \( \phi \) is a homomorphism. Hence by Proposition A.5.8 in [7], \( \phi \) is a +-homomorphism and is onto. Hence, \( \phi \) is order preserving.

**Corollary 3.7.** If \( \mathcal{L}^w \) is the linking dual operator algebra for a weak* Morita equivalence of dual operator algebras \( M \) and \( N \), and if \( \mathcal{L}' \) is the corresponding linking W*-algebra of the weak Morita equivalence of W*-algebras \( W_{\text{max}}(M) \) and \( W_{\text{max}}(N) \), then \( W_{\text{max}}(\mathcal{L}^w) = \mathcal{L}' \).

**Proof.** Suppose \( W_{\text{max}}(M) \) is normally and faithfully represented on \( B(H) \) for some Hilbert space \( H \). Then, by Lemma 1.1 in [3], \( H \) is a normal universal Hilbert \( M \)-module. Also \( M \) is weak* Morita equivalent to \( \mathcal{L}^w \), via the dual bimodule \( M \oplus c Y \) (see Corollary 4.1 in [3]). By Theorem 3.10 in [3], this induces a normal representation of \( \mathcal{L}^w \) on the Hilbert space \( (M \oplus c Y) \otimes \sigmah c H^c \). By Proposition 4.2 in [3] we have that

\[
(M \oplus c Y) \otimes \sigmah c M^c H^c \cong (H \oplus K)^c
\]

unitarily, where \( K = Y \otimes \sigmah c M^c H^c \) and \( K \) is also a normal universal Hilbert \( N \)-module (see e.g. remark on page 6 in [10]). As in the proof of Theorem 5.2 in [3], \( W_{\text{max}}(\mathcal{L}^w) \) may be taken to be the W*-algebra generated by \( \mathcal{L}^w \) in \( B(H \oplus K) \), which is \( \mathcal{L}' \). □

The above corollary should have a variant valid for arbitrary W*-covers which we hope to include in [10]. That is, if \( \mathcal{L}' \) is the corresponding linking W*-algebra of the weak Morita equivalence of arbitrary W*-covers then \( \mathcal{L}' \) is a W*-cover of \( \mathcal{L}^w \).

**Proposition 3.8.** If \( (M, N, X, Y) \) is a subcontext of a W*-Morita context \( (\mathcal{E}, \mathcal{F}, W, Z) \), then

1. \( X \) and \( Y \) generate \( W \) and \( Z \) respectively as left dual operator modules; i.e., \( W \) is the smallest w*-closed left \( \mathcal{E} \)-submodule of \( W \) containing \( X \). Similar assertions hold as right dual operator modules, by symmetry.
2. The linking algebra \( \mathcal{L} \) of \( (M, N, X, Y) \) generates the linking W*-algebra \( \mathcal{L}' \) of \( (\mathcal{E}, \mathcal{F}, W, Z) \).
3. If \( M \) or \( N \) is a W*-algebra, then \( (M, N, X, Y) = (\mathcal{E}, \mathcal{F}, W, Z) \).

**Proof.** Since the pairing \( [\cdot, \cdot] : Y \times X \to N \) has w*-dense range, we can pick a net \( e_t \) in \( N \) which is a sum of terms of the form \( [y, x] \), for \( y \in Y \), \( x \in X \), such that \( e_t \overset{w^*}{\to} 1_N \). Hence \( w e_t \overset{w^*}{\to} w \) for all \( w \in W \). Thus, sums of terms of the form \( w[y, x] \), for \( w \in W \), \( x \in X \), \( y \in Y \) are w*-dense in \( W \). However, \( w[y, x] = (w, y)x \in \epsilon X \) which shows that \( \epsilon X \) is w*-dense in \( W \). Thus, \( X \) generates \( W \) as a left dual operator \( \epsilon \)-module. Assertions (2) and (3) follow from (1). For example, if \( M \) is a W*-algebra, then clearly \( X = W \). Since \( Y \) generates \( Z \) as a right dual operator module, we have \( Z = Y Y^* = Y \epsilon Y^* = Y \). Since the ranges of the natural pairings \( Z \times W \to \mathcal{F} \) and \( Y \times X \to N \) are weak*-dense, this implies that \( \mathcal{F} = N \). □

**Theorem 3.9.** If \( (M, N, X, Y) \) is a weak* Morita context which is a subcontext of a W*-Morita context \( (\mathcal{E}, \mathcal{F}, W, Z) \), then it is a dilatable subcontext.
Proof. By Proposition 3.8, $w$ operator modules. Hence we have a $w^*$-continuous complete contraction $E \otimes_{\sigma h}^N X \to W$ with $w^*$-dense range. On the other hand,

$$W \cong W \otimes_{\sigma h}^N N \cong W \otimes_{\sigma h}^N Y \otimes_{\sigma h}^M X \cong (W \otimes_{\sigma h}^N Y) \otimes_{\sigma h}^M X$$

completely isometrically and $w^*$-homeomorphically. However, the pairing $(\cdot, \cdot): W \times Y \to E$ determines a $w^*$-continuous complete contraction $W \otimes_{\sigma h}^N Y \to E$, and so we obtain a $w^*$-continuous complete contraction $W \to E \otimes_{\sigma h}^M X$. Recall from [4] that $N$ has an ‘approximate identity’ of the form $\sum_{i=1}^{n_i} [y_i^t, x_i^t]$. Under the above identifications,

$$w \mapsto w \otimes_N 1_N \mapsto w \otimes_N w^*\lim_{t} \sum_{i=1}^{n_i} y_i^t \otimes_M x_i^t \mapsto w^*\lim_{t} \sum_{i=1}^{n_i} (w \otimes_N y_i^t) \otimes_M x_i^t$$

$$\mapsto w^*\lim_{t} \sum_{i=1}^{n_i} (w, y_i^t)x_i^t \mapsto w^*\lim_{t} \sum_{i=1}^{n_i} w[y_i^t, x_i^t] = w.$$ 

Hence, the composition of these maps

$$E \otimes_{\sigma h}^M X \to W \to E \otimes_{\sigma h}^M X$$

is the identity map, from which it follows that $W \cong E \otimes_{\sigma h}^M X$. Similarly $Z$ is the dilation of $Y$. \[\square\]

Theorem 3.10. If $(M, N, X, Y)$ is a left dilatable maximal subcontext of a $W^*$-context, then $M$ and $N$ are weak* Morita equivalent dual operator algebras. Indeed, it also follows that $(M, N, X, Y)$ is a weak* Morita context. Conversely, every weak* Morita equivalence of dual operator algebras occurs in this way.

That is, every weak* Morita context is a left dilatable maximal subcontext of a $W^*$-Morita context.

Proof. Every weak* Morita context is a left dilatable maximal subcontext of a $W^*$-Morita context is proved in Theorem 5.2 in [4]. For the converse, let $C$ and $D$ be the usual maximal $W^*$-algebras of $M$ and $N$ respectively, and let $(M, N, X, Y)$ be a left dilatable subcontext of $(C, D, W, Z)$. Using Lemmas 2.13 and 2.11 we have

$$Y \otimes_{\sigma h}^N X \subset (D \otimes_{\sigma h}^N Y) \otimes_{\sigma h}^N X \cong Z \otimes_{\sigma h}^N X \cong (Z \otimes_{\sigma h}^N C) \otimes_{\sigma h}^M X \cong Z \otimes_{\sigma h}^N W \cong D,$$

complete isometrically and $w^*$-homeomorphically. On the other hand, we have the canonical $w^*$-continuous complete contraction

$$Y \otimes_{\sigma h}^N X \to N \subset D$$

coming from the restricted pairing in Definition 4.20 (2). It is easy to check that the composition of maps in these two sequences agree. Thus the canonical map $Y \otimes_{\sigma h}^N X \to N$ is a $w^*$-continuous completely isometric isomorphism. Similarly, $X \otimes_{\sigma h}^N Y \to M$ is a $w^*$-continuous completely isometric isomorphism. Hence by the Krein-Smuelian theorem, $X \otimes_{\sigma h}^N Y \cong M$ and $Y \otimes_{\sigma h}^M X \cong N$ completely isometrically and $w^*$-homeomorphically. Thus $M$ and $N$ are weak* Morita equivalent dual operator algebras. \[\square\]
4. The Main Theorem

**Definition 4.1.** Two dual operator algebras $M$ and $N$ are (left) dual operator Morita equivalent if there exist completely contractive functors $F: M \mathcal{R} \to N \mathcal{R}$ and $G: N \mathcal{R} \to M \mathcal{R}$ which are weak* -continuous on morphism spaces, such that $FG \cong Id$ and $GF \cong Id$ completely isometrically. Such $F$ and $G$ will be called dual operator equivalence functors.

Note that by Corollary 3.5.10 in [7], $CB_M(V,W)$ for $V,W \in M \mathcal{R}$ is a dual operator space, but $CB_M^*(V,W)$ is not a $w^*$-closed subspace of $CB_M(V,W)$. In the above definition, by the functor $F$ being $w^*$-continuous on morphism spaces, we mean that if $(f_t) \subseteq CB_M^*(V,W)$, $f_t \longrightarrow f$ in $CB_M(V,W)$, and if $f$ also lies in $CB_M^*(V,W)$, then $F(f_t) \longrightarrow F(f)$ in $CB_N(F(V),F(W))$. Similarly for the functor $G$. We also assume that the natural transformations coming from $GF \cong Id$ and $FG \cong Id$ are weak* -continuous in the sense that for all $V \in M \mathcal{R}$, the natural transformation $\omega_V : GF(V) \to V$ is a weak* -continuous map. Similarly for $FG \cong Id$.

There is an obvious analogue to ‘right dual operator Morita equivalence’, where we are concerned with right dual operator modules. Throughout, we write $\mathcal{C}$ and $\mathcal{D}$ for $W^*_{\max}(M)$ and $W^*_{\max}(N)$ respectively.

We now state our main theorem:

**Theorem 4.2.** Two dual operator algebras are weak* Morita equivalent if and only if they are left dual operator Morita equivalent, and if and only if they are right dual operator Morita equivalent. Suppose that $F$ and $G$ are the left dual operator equivalence functors, and set $Y = F(M)$ and $X = G(N)$. Then $X$ is a weak* Morita equivalence $M$-$N$-bimodule. Similarly $Y$ is a weak* Morita equivalence $N$-$M$-bimodule; that is, $(M,N,X,Y)$ is a weak* Morita context. Moreover, $F(V) \cong Y \otimes_M^{\sigma h} V$ completely isometrically and weak*-homeomorphically (as dual operator $N$-modules) for all $V \in M \mathcal{R}$. Thus, $F \cong Y \otimes_M^{\sigma h} \quad$ and $G \cong X \otimes_N^{\sigma h} \quad$ completely isometrically. Also $F$ and $G$ restrict to equivalences of the subcategory $M \mathcal{H}$ with $N \mathcal{H}$, the subcategory $c \mathcal{H}$ with $d \mathcal{R}$, and the subcategory $c \mathcal{D}$ with $d \mathcal{R}$.

We will use techniques similar to those of [2] and [3] to prove our main theorem. Mostly this involves the change of tensor product and modification of arguments in the present setting of weak*-topology.

The following lemmas will be very useful to us. Their proofs are almost identical to analogous results in [2] and therefore are omitted.

**Lemma 4.3.** Let $V \in M \mathcal{R}$. Then $v \mapsto r_v$ is a $w^*$-continuous complete isometry of $V$ onto $CB_M(M,V)$. In this case, $CB_M(M,V) = CB_M^*(M,V)$ i.e. $V \cong CB_M^*(M,V)$ completely isometrically and $w^*$-homeomorphically.

**Lemma 4.4.** If $V,V' \in M \mathcal{R}$ then the map $T \mapsto F(T)$ gives a completely isometric surjective linear isomorphism $CB_M^*(V,V') \cong CB_N^*(F(V),F(V'))$. If $V = V'$, then this map is a completely isometric surjective homomorphism.

**Lemma 4.5.** For any $V \in M \mathcal{R}$, we have $F(R_m(V)) \cong R_m(F(V))$ and $F(C_m(V)) \cong C_m(F(V))$ completely isometrically.

**Lemma 4.6.** The functors $F$ and $G$ restrict to a completely isometric functorial equivalence of the subcategories $M \mathcal{H}$ and $N \mathcal{H}$. 
Proof. Let $H \in M\mathcal{H}$. Recall that $H$ with its column Hilbert space structure $H^c$ is a left dual operator $M$-module. We need to show that $K = F(H^c) \in N\mathcal{H}$ or equivalently $F(H^c)$ is a column Hilbert space. For any dual operator space $X$ and $m \in \mathbb{N}$, we have $X \otimes_h C_m = X \otimes^{\sigma_h} C_m$. Hence by Proposition 2.4 in [2], it suffices to show that the identity map $K \otimes_{min} C_m \to K \otimes^{\sigma_h} C_m$ is a complete contraction for all $m \in \mathbb{N}$. Since all operator space tensor products coincide for Hilbert column spaces, we have $C_m(H^c) \cong H^c \otimes_{min} C_m \cong H^c \otimes_h C_m \cong H^c \otimes^{\sigma_h} C_m$. Thus

$$K \otimes_{min} C_m \cong C_m(F(H^c))$$

using Lemma 4.6 and $G(K) \cong H^c$. Also, using Lemma 4.3 and Lemma 4.4 we have

$$G(K) \cong CB_M(M,G(K)) \cong CB_N^*(Y,F(G(K))) \cong CB_N^*(Y,K).$$

By Lemma 2.3 in [4] we get a complete contraction $G(K) \otimes^{\sigma_h} C_m \to CB_N^*(Y,K) \otimes^{\sigma_h} C_m$. Now $CB_N^*(Y,K) \otimes^{\sigma_h} C_m \to CB_N^*(Y,K \otimes^{\sigma_h} C_m) : T \otimes z \mapsto y \mapsto T(y) \otimes z$ for $T \in CB_N^*(Y,K)$ and $z \in C_m$, is a complete contraction. Again using Lemma 4.3 and Lemma 4.4 we have $CB_N^*(Y,K \otimes^{\sigma_h} C_m) \cong CB_N^*(M,G(K \otimes^{\sigma_h} C_m)) \cong G(K \otimes^{\sigma_h} C_m)$. Taking the composition of above maps gives a complete contraction $G(K) \otimes^{\sigma_h} C_m \to G(K \otimes^{\sigma_h} C_m)$. Applying $F$ to this map, we get a complete contraction $F(G(K) \otimes^{\sigma_h} C_m) \to K \otimes^{\sigma_h} C_m$. This together with $K \otimes_{min} C_m \cong F(G(K) \otimes^{\sigma_h} C_m)$ gives the required complete contraction $K \otimes_{min} C_m \to K \otimes^{\sigma_h} C_m$. \hfill $\square$

Corollary 4.7. The functors $F$ and $G$ restrict to a completely isometric equivalence of $\mathcal{C}\mathcal{H}$ and $\mathcal{D}\mathcal{H}$.

Proof. This is Corollary 3.2 proved earlier. \hfill $\square$

Also, this restricted equivalence is a normal $*$-equivalence in the sense of Rieffel [17], and so $\mathcal{C}$ and $\mathcal{D}$ are weak Morita equivalent in the sense of Definition 7.4 in [17].

Lemma 4.8. For a dual operator $M$-module $V$, the canonical map $\tau_V : Y \otimes V \to F(V)$ given by $y \otimes v \mapsto F(r_v)(y)$ is separately $w^*$-continuous and extends uniquely to a completely contractive map on $Y \otimes^{\sigma_h}_M V$. Moreover, this map has $w^*$-dense range.

Proof. Since the functor $F$ is $w^*$-continuous on morphism spaces, it is easy to check that $\tau_V : Y \times V \to F(V)$ is a separately $w^*$-continuous bilinear map. To see that $\tau_V$ has $w^*$-dense range, suppose the contrary. Let $Z = F(V)/\text{Range}(\tau_V)$ and let $Q : F(V) \to Z$ be the nonzero $w^*$-continuous quotient map. Then $G(Q) : G(F(V)) \to G(Z)$ is nonzero. Thus there exists $v \in V$ such that $G(Q)w_v^{-1}r_v \neq 0$ as a map on $M$, where $w_v$ is the $w^*$-continuous completely isometric natural transformation $GF(V) \to V$ coming from $GF \cong Id$. Hence $FG(Q)F(w_v^{-1}r_v) \neq 0$, and thus $QTF(r_v) \neq 0$ for some $w^*$-continuous module map $T : F(V) \to F(V)$ since $w_v^{-1}$ is $w^*$-continuous by the Krein-Smulian theorem.
By Lemma 1.3, \( T = F(S) \) for some \( w^* \)-continuous module map \( S : V \rightarrow V \), so that \( QF(r_{v'}) \neq 0 \) for \( v' = S(v) \in V \). Hence \( Q \circ \tau_V \neq 0 \), which is a contradiction. Again as in the proof of Lemma 2.6 in \([3]\), \( \tau_V \) is a complete contraction. Thus, \( \tau_V \) is a separately \( w^* \)-continuous completely contractive bilinear map. The result follows from the universal property of \( Y \otimes_{\mathcal{M}} V \).

□

Let \((M, N, C, D, F, G, X, Y)\) be as above. We let \( H \in \mathcal{M} \mathcal{H} \) be the Hilbert space of the normal universal representation of \( \mathcal{C} \) and let \( K = F(H) \). By Lemma 4.3 and Corollary 4.7, \( F \) and \( G \) restrict to equivalences of \( \mathcal{M} \mathcal{H} \) with \( \mathcal{N} \mathcal{H} \), and restrict further to normal \( \ast \)-equivalences of \( \mathcal{C} \mathcal{H} \) with \( \mathcal{P} \mathcal{H} \). By Proposition 1.3 in \([17]\), \( D \) acts faithfully on \( K \). Hence, we can regard \( D \) as a subalgebra of \( B(K) \). Define \( Z = F(C) \) and \( W = G(D) \).

From Lemma 1.3 with \( V = M \), it follows that \( Y \) is a right dual operator \( M \)-module with module action \( y \cdot m = F(r_m)(y) \), for \( y \in Y, m \in M \) and \( r_m : M \rightarrow M : c \mapsto cm \) is simply right multiplication by \( m \). Similarly, \( X \) is a right dual operator \( N \)-module, and \( Z \) and \( W \) are dual operator \( N \)-\( \mathcal{C} \)- and \( M \)-\( \mathcal{D} \)-bimodules respectively.

The inclusion \( i \) of \( M \) in \( C \) induces a completely contractive \( w^* \)-continuous inclusion \( F(i) \) of \( Y \) in \( Z \). One can check that \( F(i) \) is a \( N \)-\( M \)-module map. By Lemma 4.3 below and its proof, it is easy to see that \( F(i) \) is a complete isometry. Hence we may regard \( Y \) as a \( w^* \)-closed \( N \)-\( M \)-submodule of \( Z \) and similarly \( X \) may be regarded as a \( w^* \)-closed \( M \)-\( N \)-submodule of \( W \).

With \( V = X \) in Lemma 1.3, there is a left \( N \)-module map \( Y \otimes X \rightarrow F(X) \) defined by \( y \otimes x \mapsto F(r_x)(y) \). Since \( F(X) = FG(N) \cong N \), we get a left \( N \)-module map \([ \cdot ] : Y \otimes X \rightarrow N \). In a similar way we get a module map \(( \cdot ) : X \otimes Y \rightarrow M \). In what follows we may use the same notation for the unlinearized bilinear maps, so for example we may use the symbol \([ y, x ]\) for \([ y \otimes x ]\). These maps \(( \cdot )\) and \([ \cdot ]\) have natural extensions to \( Y \otimes W \rightarrow \mathcal{D} \) and \( X \otimes Z \rightarrow \mathcal{C} \) respectively, which we denote by the same symbols. Namely, \([ y, w ]\) is defined via \( \tau_W \) for \( y \in Y \) and \( w \in W \). By Lemma 1.3 these maps have weak*-dense ranges.

**Lemma 4.9.** The canonical maps \( X \rightarrow CB^*_N(Y, N) \) and \( Y \rightarrow CB^*_M(X, M) \), induced by \([ \cdot ]\) and \(( \cdot )\) respectively, are completely isometrically isomorphic. Similarly, the extended maps \( W \rightarrow CB^*_N(Y, \mathcal{D}) \) and \( Z \rightarrow CB^*_M(X, \mathcal{C}) \) are complete isometries.

**Proof.** By Lemma 1.3 and Lemma 1.4, we have \( X \cong CB^*_M(M, X) \cong CB^*_N(Y, F(X)) \cong CB^*_N(Y, N) \) completely isometrically. Taking the composition of these maps shows that \( x \in X \) corresponds to the map \( y \mapsto [ y, x ] \) in \( CB^*_N(Y, N) \). Similarly for the other maps. □

Next consider maps \( \phi : Z \rightarrow B(H, K) \), and \( \rho : W \rightarrow B(K, H) \) defined by \( \phi(\zeta)(\xi) = F(r_{\zeta})(\xi) \), and \( \rho(\zeta, \eta)(w) = \omega_HG(r_{\eta})(w) \), for \( \zeta \in H \) and \( \eta \in K \) where \( \omega_H : GF(H) \rightarrow H \) is the \( w^* \)-continuous \( M \)-module map coming from the natural transformation \( GF \cong Id \). Again \( r_{\zeta} : \mathcal{C} \rightarrow H \) and \( r_{\eta} : \mathcal{D} \rightarrow K \) are the obvious right multiplications. As \( \omega_H \) is an isometric onto map between Hilbert spaces, \( \omega_H \) is unitary and hence also a \( \mathcal{C} \)-module map by Corollary 2.8. One can check that:

\[
(1.1) \quad \rho(x)\phi(z) = (x, z) \quad \text{and} \quad \phi(y)\rho(w) = [y, w]V
\]

for all \( x \in X, y \in Y, z \in Z, w \in W \) and where \( V \in B(K) \) is a unitary operator in \( \mathcal{D}' \) composed of two natural transformations. A similar calculation as in Lemma 4.3 in \([3]\), shows that the unitary \( V \) is in the center of \( \mathcal{D} \), hence \( \phi(y)\rho(w) \in \mathcal{D} \) for all \( y \in Y \) and \( w \in W \).
Lemma 4.10. The map \( \phi \) (respectively \( \rho \)) is a completely isometric \( w^* \)-continuous \( N\)-\( C \)-module map (respectively \( M \)-\( D \)-module map). Moreover, \( \phi(z_1)^*\phi(z_2) \in C \) for all \( z_1, z_2 \in Z \), and \( \rho(w_1)^*\rho(w_2) \in D \), for all \( w_1, w_2 \in W \).

Proof. We will prove that the maps \( \phi \) and \( \rho \) are \( w^* \)-continuous. The rest of the assertions follow as in Lemma 4.2 in [3] and by von Neumann’s double commutant theorem. To see that \( \phi \) is \( w^* \)-continuous, let \( (z_i) \) be a bounded net in \( Z \) such that \( z_i \xrightarrow{w^*} z \) in \( Z \). For \( \zeta \in H \), we have \( F(r_\zeta) \in CB_Z^w(Z,K) \). Hence \( F(r_\zeta)(z) \) weakly. That is, \( \phi(z_i) \rightarrow \phi(z) \) in the WOT and it follows that \( \phi \) is weak\(^*\)-continuous. A similar argument works for \( \rho \). \( \square \)

We will follow the approach of [2] to prove the selfadjoint analogue of our main theorem, which involves a change of the tensor product. Nonetheless, for completeness we will give the proof.

Theorem 4.11. Two \( W\)-algebras \( A \) and \( B \) are weakly Morita equivalent in the sense of Rieffel if and only if they are dual operator Morita equivalent in the sense of Definition 4.1. Suppose that \( F \) and \( G \) are the dual operator equivalence functors, and set \( Z = F(A) \) and \( W = G(B) \). Then, \( W \) is a \( W^* \)-equivalence \( A-B \)-bimodule, \( Z \) is a \( W^* \)-equivalence \( B-A \)-bimodule, and \( Z \) is unitarily and \( w^* \)-homeomorphically isomorphic to the conjugate \( W^* \)-bimodule \( \overline{W} \) of \( W \). Moreover, \( F(V) \cong Z \otimes^h_A V \) completely isometrically and \( \rho \)-homeomorphically (as dual operator \( B \)-modules) for all \( V \in A \mathcal{R} \). Thus \( F \cong Z \otimes^h_A \) and \( G \cong W \otimes^h_B \) completely isometrically. Also \( F \) and \( G \) restrict to equivalences of the subcategory \( A \mathcal{H} \) with \( B \mathcal{H} \).

Proof. In [4] we saw that the weakly Morita equivalent \( W^* \)-algebras (in the sense of Rieffel) are weak\(^* \) Morita equivalent. Hence by Theorem 3.5 in [4], they have equivalent categories of dual operator modules and the assertion about the form of the functors also holds.

For the other direction, observe that by Corollary 4.10 the functors \( F \) and \( G \) restrict to a completely isometric equivalence of \( A \mathcal{H} \) and \( B \mathcal{H} \). Hence, by Definition 7.4 in [17], \( A \) and \( B \) are weakly Morita equivalent in the sense of Rieffel. We will follow [2] to prove rest of the assertions.

By the polarization identity and Lemma 4.10, \( W \) is a right \( C^* \)-module over \( B \) with inner product \( \langle w_1, w_2 \rangle_B = \rho(w_1)^*\rho(w_2) \), for \( w_1, w_2 \in W \). Similarly, \( W \) is a left \( C^* \)-module over \( A \) by setting \( A(w_1, w_2) = \rho(w_1)\rho(w_2)^* \). To see that this inner product lies in \( A \), note that, since the range of (.) is \( w^* \)-dense in \( A \), we can choose a net in \( A \) of the form \( e_\alpha = \sum_{k=1}^{n(\alpha)} (w_k, z_k) = \sum_{k=1}^{n(\alpha)} \rho(w_k)\phi(z_k) \) where \( z_k \in Z \) and \( w_k \in W \), such that \( e_\alpha \xrightarrow{w^*} 1_A \). Then, \( e_\alpha^* = \xrightarrow{w^*} 1_A \). Since \( \rho \) is a weak\(^* \)-continuous \( A \)-module map, \( \rho(w)^* = \xrightarrow{w^*} \lim_{\alpha} \rho(e_\alpha)^* = \xrightarrow{w^*} \lim_{\alpha} \rho(w)^*e_\alpha \), it follows that \( \rho(w)^*e_\alpha \) is a weak\(^* \) limit of finite sums of terms of the form \( \rho(w)(\rho(w)^*\rho(w_k))\phi(z_k) = \rho(w)\phi(bz_k) = (w, bz_k) \in A \), where \( b = \rho(w)^*\rho(w_k) \in B \). Thus \( \rho(w)^*e_\alpha \in A \). By the polarization identity \( \rho(w_1)\rho(w_2)^* \in A \). Similarly, \( Z \) is both a left and a right \( C^* \)-module. To see that \( Z \) is a \( w^* \)-full right \( C^* \)-module over \( A \), rechoose a net in \( A \) of the form \( e_\alpha = \sum_{k=1}^{n(\alpha)} \rho(w_k)\phi(z_k) \) such that \( e_\alpha \rightarrow I_H \) strongly, so that \( e_\alpha^*e_\alpha \rightarrow I_H \) weak\(^* \) as done in Theorem 4.4 in [4]. However \( e_\alpha^*e_\alpha = \sum_{k,j} \phi(z_k)^*b_{kj}\phi(z_j) \) where \( b_{kj} = \rho(w_k)^*\rho(w_j) \in B \). Since \( P = |b_{kj}| \) is a positive matrix, it has a square root \( R = \{r_{ij}\} \), with \( r_{ij} \in B \). Thus \( e_\alpha^*e_\alpha = \sum_{k,j} \phi(z_k)^*z_j^2 \) where \( z_j^2 = \sum_{k} r_{kj}z_j \). From this one can easily deduce that the \( A \)-valued inner product on \( Z \) has \( w^* \)-dense range. Similarly \( Z \) is a weak\(^* \)-full left \( C^* \)-module over \( B \). Similarly for \( W \). Since \( \rho \) and
Theorem 4.12. The $W^*$-algebras $C$ and $D$ are weakly Morita equivalent. In fact, $Z$, which is a dual operator $N$-$C$-bimodule, is a $W^*$-equivalence $D$-$C$-bimodule. Similarly, $W$ is a $W^*$-equivalence $C$-$D$-bimodule, and $W$ is unitarily and $w^*$-homeomorphically isomorphic to the conjugate $W^*$-bimodule $Z$ of $Z$ (and as dual operator bimodules).

Proof. By Lemma 4.10 it follows that $\rho(W)$ is a $w^*$-closed TRO (a closed subspace $Z \subseteq B(K, H)$ with $Z^* Z \subseteq Z$). Hence, by 8.5.11 in [7] and Lemma 4.10 $W$ (or equivalently $\rho(W)$) is a right $W^*$-module over $D$ with inner product $\langle w_1, w_2 \rangle_D = \rho(w_1)^* \rho(w_2)$. Since $\rho$ is a complete isometry, the induced norm on $W$ coming from the inner product coincides with the usual norm. Similarly $Z$ is a right $W^*$-module over $C$. Also, $W$ (or equivalently $\rho(W)$) is a $w^*$-full left $W^*$-module over $E = \text{weak}^*$ closure of $\rho(W)\rho(W)^*$, with the obvious inner product $\epsilon \langle w_1, w_2 \rangle = \rho(w_1)\rho(w_2)^*$. We will show that $E = C$. Analogous statements hold for $D$ and $\phi$. It will be understood that whatever a property is proved for $W$, by symmetry, the matching assertions for $Z$ hold.

Let $\mathcal{L}^w$ be the linking $W^*$-algebra for the right $W^*$-module $W$, viewed as a weak* closed subalgebra of $B(H \oplus K)$. We let $\mathcal{A} = \text{weak}^*$ closure of $\rho(W)\phi(Y)$. It is easy to check, using the fact that $\phi(Y)\rho(W) \in D$ (see above Lemma 4.10) and Lemma 4.10 that $\mathcal{A}$ is a dual operator algebra. By the last assertion of Lemma 4.8 and (4.11), $M = \rho(X)\phi(Y)^{w^*} \subseteq \mathcal{A}$ and the identity of $M$ is an identity of $\mathcal{A}$. We let $\mathcal{U}$ be the weak* closure of $D\phi(Y)$, and we define $\mathcal{L}$ to be the following subset of $B(H \oplus K)$:

\[
\begin{bmatrix}
\mathcal{A} & \rho(W) \\
\mathcal{U} & D
\end{bmatrix}
\]

Using (4.11) and Lemma 4.10 it is easy to check that $\mathcal{L}$ is a subalgebra of $B(H \oplus K)$. By explicit computation and Cohen’s factorization theorem, $\mathcal{L}^w \mathcal{L} = \mathcal{L}$ and $\mathcal{L}^w \mathcal{L} = \mathcal{L}^w$. Indeed, by Lemma 4.10 and the fact that $\rho(W)$ is a TRO, it follows that $\mathcal{L}^w \mathcal{L} \subseteq \mathcal{L}$. Again by using (4.11), Lemma 4.10 and the fact that $\rho(W)^*$ is a left $W^*$-module over $D$, it follows that $\mathcal{L}^w \mathcal{L} \subseteq \mathcal{L}^w$. As $\rho(W)$ is a right $W^*$-module over $D$ so $\rho(W)$ is a nondegenerate $D$-module (see e.g. 8.1.3 in [7]), hence $\rho(W) = \rho(W)D$ by Cohen’s factorization theorem (A.6.2 in [7]). For the same reason, $\rho(W) = \rho(W)\rho(W)^*\rho(W)$. Now one can easily check that $\mathcal{L} \subseteq \mathcal{L}^w \mathcal{L}$ and similarly $\mathcal{L}^w \subseteq \mathcal{L} \mathcal{L}^w$. Hence $\mathcal{L} \mathcal{L}^w \mathcal{L} = \mathcal{L}$ and $\mathcal{L}^w \mathcal{L} = \mathcal{L}$. Therefore, we conclude that $\mathcal{L}^w = \mathcal{L}$. Comparing corners of these algebras gives $E = \mathcal{A}$ and $\mathcal{U} = \rho(W)^*$. Thus, $M \subseteq \mathcal{E}$, from which it follows that $\mathcal{C} \subseteq \mathcal{E}$, since $\mathcal{C}$ is the $W^*$-algebra generated by $M$ in $B(H)$. Thus $\rho(W)$ is a left $\mathcal{C}$-module, so $W$ can be made into a left $\mathcal{C}$-module by $w^*$ continuity, the inner products are separately $w^*$-continuous. Hence, by Lemma 8.5.4 in [7], $W$ and $Z$ are $W^*$-equivalence bimodules, implementing the weak Morita equivalence of $A$ and $B$. Note that by Corollary 8.5.8 in [7], $CB_A(W, A) = CB_A^2(W, A)$. Thus by (8.18) in [7] and Lemma 4.10, $Z \cong \mathcal{W}$ completely isometrically.

Let $V \in \mathcal{A}$. By Lemma 4.3 and Lemma 4.4 above, Theorem 2.8 in [5], and the fact that $Z \cong \mathcal{W}$, we have the following sequence of isomorphisms:

$$F(V) \cong CB_B^2(B, F(V)) \cong CB_A^2(W, V) \cong Z \otimes_{\text{cb}} V$$

as left dual operator $B$-modules. Thus the conclusions of the theorem all hold. □
Theorem 5.3. Suppose that the dual operator equivalence functors are weakly Morita equivalent and let $(\mathcal{M}, \mathcal{N}, \mathcal{X}, \mathcal{Y})$ be a weak* Morita context. Then from Theorem 5.2 in \cite{4} we know that $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent $W^*$-algebras, with $W^*$-equivalence $\mathcal{D}$-$\mathcal{C}$-bimodule $Z = Y \otimes_{\mathcal{M}}^h \mathcal{C}$. From Theorem 3.5 in \cite{4}, $F(V) = Y \otimes_{\mathcal{M}}^h V$, for $V$ a dual operator $M$-module. However, if $V$ is a dual operator $\mathcal{C}$-module, $Y \otimes_{\mathcal{M}}^h V \cong Y \otimes_{\mathcal{M}}^h \mathcal{C} \otimes_{\mathcal{C}}^h V \cong \mathcal{Z} \otimes_{\mathcal{C}}^h V$. Hence, $F$ restricted to $\mathcal{C}$ is equivalent to $Y \otimes_{\mathcal{C}}^h -$, and thus is $W^*$-restrictable.

5. $W^*$-restrictable equivalences

**Definition 5.1.** We say that a dual operator equivalence functor $F$ is $W^*$-restrictable, if $F$ restricts to a functor from $\mathcal{C}$ into $\mathcal{D}$. We prove our main theorem under the assumption that the functors $F$ and $G$ are $W^*$-restrictable. Later we will prove that this condition is automatic; i.e., the functors $F$ and $G$ are automatically $W^*$-restrictable.

**Remark 5.2.** The canonical equivalence functors coming from a given weak* Morita equivalence are $W^*$-restrictable. Suppose that $M$ and $N$ are weak* Morita equivalent and let $(\mathcal{M}, \mathcal{N}, \mathcal{X}, \mathcal{Y})$ be a weak* Morita context. Then from Theorem 5.2 in \cite{4} we know that $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent $W^*$-algebras, with $W^*$-equivalence $\mathcal{D}$-$\mathcal{C}$-bimodule $Z = Y \otimes_{\mathcal{M}}^h \mathcal{C}$. From Theorem 3.5 in \cite{4}, $F(V) = Y \otimes_{\mathcal{M}}^h V$, for $V$ a dual operator $M$-module. However, if $V$ is a dual operator $\mathcal{C}$-module, $Y \otimes_{\mathcal{M}}^h V \cong Y \otimes_{\mathcal{M}}^h \mathcal{C} \otimes_{\mathcal{C}}^h V \cong \mathcal{Z} \otimes_{\mathcal{C}}^h V$. Hence, $F$ restricted to $\mathcal{C}$ is equivalent to $Y \otimes_{\mathcal{C}}^h -$, and thus is $W^*$-restrictable.

**Theorem 5.3.** Suppose that the dual operator equivalence functors $F$ and $G$ are $W^*$-restrictable. Then the conclusions of the Theorem 4.2 all hold.

**Proof.** Clearly, $F$ and $G$ gives a dual operator Morita equivalence of $\mathcal{C}$ and $\mathcal{D}$ when restricted to these subcategories. Set $Y = F(\mathcal{M})$, $Z = F(\mathcal{C})$, $X = G(\mathcal{N})$, and $W = G(\mathcal{D})$ as before. By Theorem 1.11 $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent von Neumann algebras with $Z$ and $W$ as $W^*$-equivalence bimodules. From the discussion above Lemma 4.2, $Y$ is a right dual operator $M$-module and $X$ is a right dual operator $N$-module. Also $Y$ is a $w^*$-closed $N$-$M$-submodule of $Z$ and $X$ is a $w^*$-closed $M$-$N$-submodule of $W$.

For any left dual operator $\mathcal{C}$-module $X'$, we have the following sequence of canonical complete isometries by Lemma 4.3 and Lemma 4.4:

\[
\begin{align*}
CB^*_M(X, X') &\cong CB^*_N(N, F(X')) \\
&\cong F(X') \\
&\cong CB^*_D(D, F(X')) \\
&\cong CB^*_W(W, X').
\end{align*}
\]

Hence, by the discussion following Definition 2.5 and by Lemma 2.11 we have $W \cong D \otimes_{\mathcal{M}}^h Y$.

For any dual operator $M$-module $V$, we have, $V \otimes_{\mathcal{M}}^h Y \subset (D \otimes_{\mathcal{M}}^h Y) \otimes_{\mathcal{C}}^h V \cong Z \otimes_{\mathcal{C}}^h V$ completely isometrically, since any dual operator module is contained in its maximal dilation. On the other hand, using Lemma 4.8, Lemma 4.4 and
Theorem 4.11 respectively, we have the following sequence of canonical completely contractive $N$-module maps:

$$Y \otimes_{M}^{h} V \to F(V) \to F(C \otimes_{M}^{h} V) \cong Z \otimes_{C}^{h}(C \otimes_{M}^{h} V) \cong Z \otimes_{C}^{h} V.$$ 

The composition of the maps in this sequence coincides with the composition of complete isometries in the last sequence. Hence, the canonical map $Y \otimes_{M}^{h} V \to F(V)$ is a $w^{*}$-continuous complete isometry. Since this map has $w^{*}$-dense range, by the Krein-Smulian theorem, it is a complete isometric isomorphism. Thus $F(V) \cong Y \otimes_{M}^{h} V$, and similarly $G(U) \cong X \otimes_{N}^{h} U$. Finally, $M \cong GF(M) \cong X \otimes_{N}^{h} Y$, using Lemma 2.10 in [4] and similarly $N \cong Y \otimes_{M}^{h} X$ completely isometrically and $w^{*}$-homeomorphically. □

**Corollary 5.4.** Dual operator equivalence functors are automatically $W^{*}$-restrictable.

**Proof.** Firstly, we will show that $W$ is the maximal dilation of $X$, and $Z$ is the maximal dilation of $Y$. In Theorem 4.12, we saw that the set $U$ equals $Z$. This implies that $Y$ generates $Z$ as a left dual operator $D$-module. Similarly, $X$ generates $W$ as a left dual operator $C$-module.

By Lemma 4.3 and Lemma 4.4 we have the following sequence of maps

$$CB^{*}_{M}(X,H) \cong CB^{*}_{N}(N,K) \cong K \cong CB^{*}_{D}(D,K) \to CB^{*}_{M}(W,H).$$

One can check that $\eta \in K$ corresponds under the last two maps in the sequence to the map $w \mapsto \rho(w)(\eta)$, which lies in $CB^{*}_{D}(W,H)$, since $\rho$ is a left $C$-module map. Thus, the composition $R$ of the maps in the above sequence has range contained in $CB^{*}_{D}(W,H)$. Also, $R$ is an inverse to the restriction map $CB^{*}_{D}(W,H) \to CB^{*}_{M}(X,H)$. Thus $CB^{*}_{D}(W,H) \cong CB^{*}_{M}(X,H)$. Since $H$ is a normal universal representation of $C$ (see the paragraph below Lemma 4.8), it follows from Theorem 2.10 that $W$ is the maximal dilation of $X$. Similarly $Z$ is the maximal dilation of $Y$.

Let $V \in c\mathcal{R}$. By Lemma 4.3, Lemma 4.4, Definition 2.5, Theorem 2.8 in [5], and Theorem 4.12 we have the following sequence of isomorphisms

$$F(V) \cong CB^{*}_{N}(N,F(V)) \cong CB^{*}_{M}(X,V) \cong CB^{*}_{C}(W,V) \cong Z \otimes_{C}^{h} V,$$

as left dual operator $N$-modules. Since $Z \otimes_{C}^{h} V$ is a left dual operator $D$-module, we see that $F(V)$ is a left dual operator $D$-module and by Theorem 2.2 this $D$-module action is unique. Also by Corollary 2.6 the map $Z \otimes_{C}^{h} V \to F(V)$ coming from the composition of the above isomorphisms is a $D$-module map. This map $Z \otimes_{C}^{h} V \to F(V)$ is defined analogously to the map $\tau_{V}$ defined in Lemma 4.8. One can check that if $T : V_{1} \to V_{2}$ is a morphism in $c\mathcal{R}$, then the following diagram commutes:

$$
\begin{array}{ccc}
Z \otimes_{C}^{h} V_{1} & \longrightarrow & F(V_{1}) \\
\downarrow I_{Z} \otimes T & & \downarrow F(T) \\
Z \otimes_{C}^{h} V_{2} & \longrightarrow & F(V_{2})
\end{array}
$$

By Corollary 2.4 in [4], $I_{Z} \otimes T$ is a $w^{*}$-continuous $D$-module map and both the horizontal arrows above are $w^{*}$-continuous $D$-module maps. Hence, $F(T)$ is
a $w^*$-continuous $D$-module map; that is, $F(T)$ is a morphism in $\mathcal{D}$. Thus $F$ is $W^*$-restrictable. By Theorem 5.3, our main theorem is proved. □

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Department of Mathematics, University of Houston, Houston, TX 77204-3008
E-mail address: upasana@math.uh.edu