Three Counterexamples on Semigraphoids

Raymond Hemmecke, Jason Morton, Anne Shiu, Bernd Sturmfels, and Oliver Wienand

Abstract

Semigraphoids are combinatorial structures that arise in statistical learning theory. They are equivalent to convex rank tests and to polyhedral fans that coarsen the reflection arrangement of the symmetric group $S_n$. In this paper we resolve two problems on semigraphoids posed in Studený’s book [18], and we answer a related question by Postnikov, Reiner, and Williams on generalized permutohedra [17]. We also study the semigroup and the toric ideal associated with semigraphoids.

1 Introduction

A conditional independence (CI) statement on a finite set of random variables, indexed by $[n] = \{1, 2, \ldots, n\}$, is a formal symbol $[i \perp \! \! \! \perp j | K]$ where $K \subset [n]$ and $i, j \in [n]\setminus K$. The symbol $[i \perp \! \! \! \perp j | K]$ represents the statement that the random variables $i$ and $j$ are conditionally independent given the joint random variable $K$. For any joint probability distribution on the $n$ random variables, the set $\mathcal{M}$ of all CI statements that are valid for the given distribution satisfies the following axiom:

\[(SG) \text{ If } [i \perp \! \! \! \perp j | K \cup \ell] \text{ and } [i \perp \! \! \! \perp \ell | K] \text{ are in } \mathcal{M} \text{ then so are } [i \perp \! \! \! \perp j | K] \text{ and } [i \perp \! \! \! \perp \ell | K \cup j].\]

A semigraphoid is any set $\mathcal{M}$ of CI statements which satisfies the axiom (SG). Studený’s book [18] gives an introduction to semigraphoids and their role in statistical learning theory. For further details and references see also Matúš [11, 13]. In this paper we construct examples which answer two problems stated by Studený:

(Q1) Is it true that every coatom of the lattice of (disjoint) semigraphoids over $[n]$ is a structural independence model over $[n]$? [18 Question 4, page 194]

(Q2) Is every structural imset over $[n]$ already a combinatorial imset over $[n]$? [18 Question 7, page 207]

Our approach is based on the geometric characterization of semigraphoids which was developed in [15]. Let $\Pi_{n-1}$ denote the $(n-1)$-dimensional permutahedron [12, 19], and let $C_n = [0,1]^n$ denote the standard $n$-dimensional cube. The vertices of $\Pi_{n-1}$ are in bijection with the elements of the symmetric group $S_n$, and with the monotone edge paths from $(0,0,\ldots,0)$ to $(1,1,\ldots,1)$ on the cube $C_n$. The 2-dimensional faces of $C_n$ are in bijection with the CI statements on $[n]$. Namely, $[i \perp \! \! \! \perp j | K] = [j \perp \! \! \! \perp i | K]$ represents the 2-face of $C_n$ with $x_k = 1$ for $k \in K$ and $x_l = 0$
for $l \in [n] \setminus (K \cup \{i, j\})$. The number of these 2-cubes equals $\gamma_n := \binom{n}{2} 2^{n-2}$. There is a natural surjection from the edges of $\Pi_{n-1}$ onto the 2-faces of $C_n$. Namely, an edge of $\Pi_{n-1}$ corresponds to a pair of adjacent monotone edge paths on $C_n$. These adjacent paths differ only along a 2-cube $[i \perp j | K]$. In this manner, we identify any set $\mathcal{M}$ of CI statements on $[n]$ with a set of 2-cubes on the boundary of $C_n$. We also identify $\mathcal{M}$ with a set of edges of the permutohedron $\Pi_{n-1}$, bearing in mind that opposite edges of a square have the same CI statement as their label.

Each 2-face of the permutohedron $\Pi_{n-1}$ is either a square or a hexagon. By [15], the semigraphoid axiom is equivalent to the following geometric condition on $\Pi_{n-1}$:

\((SG')\) If two adjacent edges of a hexagon are in $\mathcal{M}$ then so are their two opposites.

The normal fan of the permutohedron $\Pi_{n-1}$ is the reflection arrangement of $S_n$. Theorem 3 in [15] identifies semigraphoids with fans that coarsen this arrangement. Such fans are called convex rank tests. Namely, $\mathcal{M}$ specifies the set of edges of $\Pi_{n-1}$ whose dual walls in the normal fan are not present in the convex rank test.

A basic question about any semigraphoid $\mathcal{M}$ is whether its corresponding convex rank test is submodular, in other words, whether it is the normal fan of a convex polytope. That polytope would then be a Minkowski summand of $\Pi_{n-1}$. These polytopes are known as generalized permutohedra and they were studied in [16, 17].

Studeny’s first question has the following geometric translations:

\((Q1)\) Is every coarsest convex rank test submodular?
\((Q1)\) Is every fan which maximally coarsens the $S_n$-arrangement the normal fan of a generalized permutohedron?

In the first version of [17], Postnikov, Reiner and Williams asked a similar question:

\((Q3)\) Is every simplicial fan which coarsens the $S_n$-arrangement the normal fan of a simple generalized permutohedron?

This paper answers all three questions. In Section 2 we derive and explain our counterexample for Question (Q3). That example is discussed in [17] Example 3.8]. By Studeny’s classification of the 26424 semigraphoids for $n = 4$, it had been known that the answers to Questions (Q1) and (Q2) are affirmative for $n \leq 4$. In Sections 3 and 4 we construct counterexamples for (Q1) and (Q2) with $n = 5$.

Question (Q2) has the following reformulation in the setting of toric algebra [14 §7]. We represent the semigraphoid axiom as an equation among formal symbols:

\[(SG'')\] $[i \perp j | K \cup \ell] + [i \perp \ell | K] = [i \perp j | K] + [i \perp \ell | K \cup j]$ for all $i, j, l, K$. These relations span the kernel of the linear map

\[A : \mathbb{Z}^{\gamma_n} \to \mathbb{Z}^{2^n}, \quad [i \perp j | K] \mapsto e_{iK} + e_{jK} - e_K - e_{ijK}.\] (1)

A semigraphoid is a solution to the equations \((SG'')\) in the semiring $\{0, +\}$, representing “zero” and “positive”. A semigraphoid is submodular if it is the set of zero coordinates of a solution to \((SG'')\) in the non-negative real numbers. These definitions furnish us with an algebraic representation of a semigraphoid $\mathcal{M}$ and a
systematic method for testing submodularity of $\mathcal{M}$ by linear programming. Studenť’s question (Q2) concerns the $\mathbb{N}$-linear span of the columns of the matrix $\mathcal{A}$:

(Q2) *Is the semigroup $\mathcal{A}(\mathbb{N}^n)$ normal, i.e., does it coincide with $\mathcal{A}(\mathbb{R}_{\geq 0}^n) \cap \mathbb{Z}^n$?*

In Section 5 we study the toric ideal $[\Pi]$ of $\mathcal{A}$ in a polynomial ring in $\gamma_n$ unknowns, and we examine how it differs from the subideal generated by the binomials

\[(SG'') \quad [i \perp j | K \cup \ell] \cdot [i \perp \ell | K] = [i \perp j | K \cup \ell].\]

Proposition 5.4 describes the primary decomposition of this binomial ideal for $n = 4$. We also discuss the problem of deriving the full Markov basis from $(SG'')$.

## 2 A non-submodular simplicial semigraphoid

Let $n = 4$ and consider the 4-dimensional cube $C_4$ and the 3-dimensional permutohedron $\Pi_3$. Each hexagon on $\Pi_3$ corresponds to one of the eight facets of $C_4$. Each facet specifies three semigraphoid axioms, written additively as in $(SG'')$:

\[
\begin{align*}
(*, *, *, 0) & \quad [1 \perp 2|0] + [2 \perp 3|1] = [2 \perp 3|0] + [1 \perp 2|3] \iff \\
& \quad [1 \perp 3|0] + [1 \perp 2|3] = [1 \perp 2|0] + [1 \perp 3|2] \\
& \quad [1 \perp 3|0] + [2 \perp 3|1] = [2 \perp 3|0] + [1 \perp 3|2] \\
(*, *, 0, *) & \quad [1 \perp 2|0] + [2 \perp 4|1] = [2 \perp 4|0] + [1 \perp 2|4] \\
& \quad [1 \perp 2|0] + [1 \perp 4|2] = [1 \perp 4|0] + [1 \perp 2|4] \\
& \quad [1 \perp 4|0] + [2 \perp 4|1] = [2 \perp 4|0] + [1 \perp 4|2] \\
(*, 0, *, *) & \quad [1 \perp 3|0] + [1 \perp 4|3] = [1 \perp 4|0] + [1 \perp 3|4] \\
& \quad [3 \perp 4|0] + [1 \perp 4|3] = [1 \perp 3|0] + [3 \perp 4|1] \\
& \quad [3 \perp 4|0] + [1 \perp 4|3] = [1 \perp 4|0] + [3 \perp 4|1] \\
(0, *, *, *) & \quad [2 \perp 3|0] + [3 \perp 4|2] = [3 \perp 4|0] + [2 \perp 3|4] \\
& \quad [2 \perp 4|0] + [2 \perp 3|4] = [2 \perp 3|0] + [2 \perp 4|3] \\
& \quad [3 \perp 4|0] + [2 \perp 4|3] = [2 \perp 4|0] + [3 \perp 4|2] \\
(*, *, 1) & \quad [3 \perp 4|1] + [2 \perp 3|14] = [2 \perp 3|1] + [3 \perp 4|12] \iff \\
& \quad [2 \perp 4|1] + [2 \perp 3|14] = [2 \perp 3|1] + [2 \perp 4|12] \\
& \quad [2 \perp 4|1] + [3 \perp 4|12] = [3 \perp 4|1] + [2 \perp 4|13] \\
(*, 1, *) & \quad [1 \perp 3|2] + [3 \perp 4|12] = [3 \perp 4|2] + [1 \perp 3|24] \\
& \quad [1 \perp 3|2] + [1 \perp 4|23] = [1 \perp 4|2] + [1 \perp 3|24] \\
& \quad [3 \perp 4|2] + [1 \perp 4|23] = [1 \perp 4|2] + [3 \perp 4|12] \\
(*, 1, *) & \quad [1 \perp 2|3] + [1 \perp 4|23] = [1 \perp 4|3] + [1 \perp 2|34] \iff \\
& \quad [1 \perp 4|3] + [1 \perp 4|23] = [2 \perp 4|3] + [1 \perp 4|23] \\
& \quad [1 \perp 2|3] + [2 \perp 4|13] = [2 \perp 4|3] + [1 \perp 2|34] \\
& \quad [1 \perp 3|4] + [2 \perp 3|14] = [2 \perp 3|4] + [1 \perp 3|24] \\
& \quad [1 \perp 2|4] + [1 \perp 3|24] = [1 \perp 3|4] + [1 \perp 2|34] \\
& \quad [1 \perp 2|4] + [2 \perp 3|14] = [2 \perp 3|4] + [1 \perp 2|34].
\end{align*}
\]
This is a system of 24 equations in $\gamma_4 = 24$ formal symbols $[i \perp j \mid K]$.

A semigraphoid is a solution to these equations over the semiring $\{0, +\}$. More precisely, given such a solution vector in $\{0, +\}^{24}$, the semigraphoid $\mathcal{M}$ consists of all coordinates $[i \perp j \mid K]$ that have the value 0. There are $26424$ such semigraphoids. They form a sublattice of the Boolean lattice $\{0, +\}^{24}$, with $+ < 0$. Question (Q1) concerns the coatoms of this lattice. But let us first resolve Question (Q3).

We consider the following collection of CI statements:

$$\mathcal{M} = \{ [2 \perp 3 \mid 14], [1 \perp 4 \mid 23], [1 \perp 2 \mid \emptyset], [3 \perp 4 \mid \emptyset] \}. \quad (2)$$

These four symbols are highlighted in the 24 equations above by the use of double brackets $[\cdots]$. Each equation (individually) can be solved among the positive reals after these four symbols have been set to zero, or equivalently they can be solved as a system over $\{0, +\}$. This shows that $\mathcal{M}$ is a semigraphoid.

![Figure 1: A simple 3-dimensional polytope with 16 vertices and 10 facets](image)

The semigraphoid $\mathcal{M}$ is represented geometrically by the three-dimensional polytope in Figure 1. This polytope is simple, i.e., each of the 16 vertices is adjacent to three other vertices. The eight vertices whose labels include three bars (such as $4 \mid 1 \mid 23$) correspond to unique permutations in $S_4$ (namely the permutation 4213), while the eight vertices whose labels have two bars (such as $4 \mid 1 \mid 23$) correspond to pairs of permutations in $S_4$ (namely 4123 and 4132). This partition of $S_4$ into eight singletons and eight pairs is the convex rank test of $\mathcal{M}$. The normal fan of the polytope in Figure 1 is a simplicial fan which is combinatorially (but not geometrically) isomorphic to a fan that coarsens the hyperplane arrangement of $S_4$.

**Proposition 2.1.** The simplicial semigraphoid $\mathcal{M}$ is not submodular.
Proof. Suppose that \( \mathcal{M} \) were submodular. Then the above equations have a solution in \( (\mathbb{R}_{\geq 0})^{24} \) whose coordinates in \( \mathcal{M} \) are zero and whose other 20 coordinates are positive. The four equations marked by an “\( \Leftarrow \)” give the following four equations:

\[
\begin{align*}
[2 \perp 3 | 1] & = [2 \perp 3 | \emptyset] + [1 \perp 2 | 3] \\
[1 \perp 4 | 3] & = [1 \perp 4 | \emptyset] + [3 \perp 4 | 1] \\
[3 \perp 4 | 1] & = [2 \perp 3 | 1] + [3 \perp 4 | 12] \\
[1 \perp 2 | 3] & = [1 \perp 4 | 3] + [1 \perp 2 | 34].
\end{align*}
\]

Adding the left hand sides and the right hand sides of the four equations yields

\[
[2 \perp 3 | \emptyset] + [1 \perp 4 | \emptyset] + [3 \perp 4 | 12] + [1 \perp 2 | 34] = 0.
\]

This contradicts the assumption that these four values are strictly positive.

The set of all non-negative solutions to the 24 equations is an 11-dimensional cone in \( (\mathbb{R}_{\geq 0})^{24} \). This cone is isomorphic to the 16-dimensional cone of submodular functions on \( 2^4 \), modulo its 5-dimensional lineality space. Its 22108 faces are in bijection with the submodular semigraphoids, or, equivalently, with the generalized permutohedra for \( n = 4 \). In addition to these, there are 4316 semigraphoids that are not submodular. Each of the latter can be represented by a polytope of dimension \( \leq 3 \) as in Figure 1. These polytopes have the combinatorial properties of generalized permutohedra, but they cannot be realized as Minkowski summands of \( \Pi_3 \). For example, see [10, Figure 5] for a polytope that depicts Studený’s example of a semigraphoid that is not submodular (see [15] and [18, Section 2.2.4]).

We now give a classification of non-submodular semigraphoids for \( n = 4 \) and \( |\mathcal{M}| \) small. All simplicial examples are coarsenings (up to relabeling) of the particular semigraphoid \( \mathcal{M} \) in Proposition 2.1. The following table lists the number of semigraphoids classified by number of CI statements, their type, and whether they are simplicial. Here, the type of a semigraphoid is the triple \( (m_0, m_1, m_2) \) where \( m_t \) is the number of CI statements \( [i \perp j | K] \) in \( \mathcal{M} \) such that \( |K| = m_t \).

| \(|\mathcal{M}|\) | type  | non-simplicial | simplicial | total |
|----------|-------|----------------|------------|-------|
| 3        | (0, 3, 0) | 8              | 0          | 8     |
| 4        | (0, 4, 0) | 78             | 0          | 78    |
| 4        | (1, 2, 1) | 30             | 0          | 30    |
| 4        | (2, 0, 2) | 0              | 6          | 6     |
| 5        | (0, 5, 0) | 300            | 0          | 300   |
| 5        | (1, 2, 2) | 30             | 0          | 30    |
| 5        | (1, 3, 1) | 84             | 0          | 84    |
| 5        | (2, 0, 3) | 12             | 12         | 24    |
| 5        | (2, 2, 1) | 30             | 0          | 30    |
| 5        | (3, 0, 2) | 24             | 0          | 24    |
| 6        | (0, 6, 0) | 604            | 0          | 604   |
| 6        | (1, 3, 2) | 84             | 0          | 84    |
| 6        | (1, 4, 1) | 78             | 0          | 78    |
| 6        | (2, 0, 4) | 30             | 3          | 33    |
### A non-submodular coarsest semigraphoid

We now consider the case $n = 5$. There are $\gamma_5 = 80$ CI statements, one for each two-dimensional face of the 5-cube $C_5$. There are 120 semigraphoid axioms ($\text{SG}''$), three for each of the 40 three-dimensional faces of $C_5$, listed as additive equations in the Appendix. The semigraphoids are the solutions of these equations over $\{0, +\}^{80}$. These solutions include the all-zero vector $0$ which represents the semigraphoid that consists of all 80 CI statements, and which is the maximal element in the lattice of semigraphoids. A semigraphoid is said to be coarsest if it is maximal among non-

| $|\mathcal{M}|$ | type   | non-simplicial | simplicial | total |
|---------|--------|---------------|------------|-------|
| 6       | (2, 2, 2) | 30            | 0          | 30    |
| 6       | (2, 3, 1) | 84            | 0          | 84    |
| 6       | (3, 0, 3) | 74            | 12         | 96    |
| 6       | (4, 0, 2) | 30            | 3          | 33    |
| 7       | (0, 7, 0) | 684           | 0          | 684   |
| 7       | (1, 4, 2) | 78            | 0          | 78    |
| 7       | (1, 5, 1) | 24            | 0          | 24    |
| 7       | (2, 0, 5) | 18            | 0          | 18    |
| 7       | (2, 3, 2) | 84            | 0          | 84    |
| 7       | (2, 4, 1) | 78            | 0          | 78    |
| 7       | (3, 0, 4) | 132           | 0          | 132   |
| 7       | (4, 0, 3) | 132           | 0          | 132   |
| 7       | (5, 0, 2) | 18            | 0          | 18    |
| 8       | (0, 8, 0) | 450           | 0          | 450   |
| 8       | (1, 5, 2) | 24            | 0          | 24    |
| 8       | (2, 0, 6) | 3             | 0          | 3     |
| 8       | (2, 4, 2) | 48            | 0          | 48    |
| 8       | (2, 5, 1) | 24            | 0          | 24    |
| 8       | (3, 0, 5) | 72            | 0          | 72    |
| 8       | (4, 0, 4) | 174           | 0          | 174   |
| 8       | (5, 0, 3) | 72            | 0          | 72    |
| 8       | (6, 0, 2) | 3             | 0          | 3     |
| 9       | (0, 9, 0) | 212           | 0          | 212   |
| 9       | (3, 0, 6) | 12            | 0          | 12    |
| 9       | (4, 0, 5) | 84            | 0          | 84    |
| 9       | (5, 0, 4) | 84            | 0          | 84    |
| 9       | (6, 0, 3) | 12            | 0          | 12    |
| 10      | (0, 10, 0)| 60            | 0          | 60    |
| 10      | (4, 0, 6) | 15            | 0          | 15    |
| 10      | (5, 0, 5) | 24            | 0          | 24    |
| 10      | (6, 0, 4) | 15            | 0          | 15    |
| 11      | (0, 11, 0)| 12            | 0          | 12    |
| 11      | (5, 0, 6) | 6             | 0          | 6     |
| 11      | (6, 0, 5) | 6             | 0          | 6     |
semigraphoids. Geometrically, such a semigraphoid corresponds to a fan which coarsens the $S_5$-arrangement but cannot be coarsened to a non-trivial fan.

We now present the counterexample which answers question (Q1). Our constructions make use of the identification of semigraphoids with convex rank tests that was derived in [15]. Let $\Gamma$ denote the partition of the symmetric group $S_5$ into fourteen classes as follows. There are eight classes containing 12 permutations each:

$15|234$ $234|15$ $123|45$ $235|14$
$124|35$ $245|13$ $134|25$ $345|12$.

And there are six classes containing four permutations each:

$12|5|34$ $25|1|34$ $13|5|24$
$35|1|24$ $14|5|23$ $45|1|23$.

Here $15|234$ denotes the class of all permutations $ijklm$ with $\{i, j\} = \{1, 5\}$ and $\{k, l, m\} = \{2, 3, 4\}$. Similarly, $45|1|23$ denotes the class of all permutations $ijklm$ with $\{i, j\} = \{4, 5\}$, $k = 1$, and $\{l, m\} = \{2, 3\}$. Clearly, $\Gamma$ is a pre-convex rank test, as each of the 14 classes is the set of all linear extensions of a poset on $[5] = \{1, 2, 3, 4, 5\}$. Note that the stabilizer of the pre-convex rank test $\Gamma$ in $S_5$ has order 12, because $\Gamma$ is fixed under permutations of $\{1, 5\}$ and permutations of $\{2, 3, 4\}$.

The 14 classes of $\Gamma$ are represented by the 14 vertices of the polytope in Figure 2.

Each pair of adjacent permutations in a given class of $\Gamma$ specifies a CI statement. For instance, the four-element class $45|1|23$ specifies the two CI statements $[ [4 \perp \perp 5|\emptyset] ]$ and $[ [2 \perp \perp 3|145] ]$, while the 12-element class $15|234$ specifies the seven CI statements $[ [1 \perp \perp 5|\emptyset] ]$, $[ [2 \perp \perp 3|145] ]$, $[ [2 \perp \perp 4|135] ]$, $[ [3 \perp \perp 4|15] ]$, $[ [3 \perp \perp 4|125] ]$.

Altogether, we obtain 44 CI statements $[ \cdot | \cdot ]$ from the 14 classes, and we identify the pre-convex rank test $\Gamma$ with this set of 44 CI statements. We now prove:

**Theorem 3.1.** $\Gamma$ is a coarsest convex rank test which is not submodular.

**Proof.** To establish this theorem, we must prove the following three claims:

- $\Gamma$ is a convex rank test, i.e. it satisfies the semigraphoid axioms (SG).
- There is no proper convex rank test which is coarser than $\Gamma$.
- The convex rank test $\Gamma$ is not submodular.

We shall prove all three statements at once, by examining the semigraphoid equations (SG$^\cdot$). As in Section 2, the 44 symbols in $\Gamma$ are denoted with double brackets $[\cdot|\cdot]$, while the 36 symbols not in $\Gamma$ are denoted with brackets $\cdot|\cdot$. With this distinction between brackets, there are four symmetry types of semigraphoid equations that involve the 36 positive unknowns $\cdot|\cdot$. The full list is given in the Appendix:

- **Type I** $[3\perp\perp5|12] + [3\perp\perp4|125] = [3\perp\perp4|12] + [3\perp\perp5|124]$
- **Type II** $[1\perp\perp5|2] + [1\perp\perp3|25] = [1\perp\perp3|2] + [1\perp\perp5|23]$
- **Type III** $[4\perp\perp5|1] + [2\perp\perp5|14] = [2\perp\perp5|1] + [4\perp\perp5|12]$
- **Type IV** $[1\perp\perp2|5] + [2\perp\perp3|15] = [2\perp\perp3|5] + [1\perp\perp2|35]$

Here, $\perp$ indicates a constraint that the variables do not appear together in a given equation. Each of these equations is a semigraphoid equation that holds in $\Gamma$ and does not hold in any convex rank test that is coarser than $\Gamma$. This proves that $\Gamma$ is a coarsest convex rank test which is not submodular.
Figure 2: Schlegel diagram of a 4-dimensional polytope with 10 facets

After setting the 44 unknowns \([\cdot | \cdot]\) to zero, we are left with 120 equations in the 36 strictly positive unknowns. For instance, the first three types give

- **Type I** \([3 \bot 5 | 12] = [3 \bot 4 | 12]\)
- **Type II** \([1 \bot 5 | 2] + [1 \bot 3 | 25] = [1 \bot 5 | 23]\)
- **Type III** \([4 \bot 5 | 1] + [2 \bot 5 | 14] = [2 \bot 5 | 1] + [4 \bot 5 | 12]\)

The axiom \((SG'')\) merely requires that each of these equations is individually solvable. This is obviously the case. Hence \(\Gamma\) is a semigraphoid.

The 78 equations of Type I listed in the Appendix imply that all 36 positive unknowns must be equal. So, if another CI statement is added to the semigraphoid \(\Gamma\), then all others must be added in order for \((SG)\) to remain valid. This proves our second claim that \(\Gamma\) is a coarsest convex rank test.

Given that the 36 unknowns \([\cdot | \cdot]\) must be equal, the 12 Type II equations imply that their common value is zero, contradicting the requirement that they be positive. Hence the 120 orginal equations altogether have no non-negative real solution that is consistent with \(\Gamma\). This proves our third claim that \(\Gamma\) is not submodular.

Every semigraphoid for \(n = 5\) corresponds to a 4-dimensional fan. Intersecting this fan with a sphere around the origin, we obtain a polyhedral cell decomposition of the 3-dimensional sphere. We do not know whether each of these 3-spheres can
be realized as the boundary of a 4-dimensional polytope. However, using [19, §5], every semigraphoid can be represented by a 3-dimensional diagram as in Figure 2.

For the specific semigraphoid \( \Gamma \) of Theorem [3.1] the diagram in Figure 2 is indeed the boundary of a 4-polytope with f-vector \((14, 36, 32, 10)\). The following coordinates for this polytope were found by a direct calculation, using the techniques described in [2]. Each of the following ten row vectors represents a facet of our polytope:

**POINTS**

\[
\begin{array}{ccccc}
1 & 1/4 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & -1/4 & 1/4 & 1/4 & 5/4 \\
1 & 280/893 & -280/893 & 25/893 & 0 \\
1 & 1/57 & 1/57 & -1/57 & 17/19 \\
1 & 1 & 1 & 0 & -5 \\
1 & 2/37 & 20/37 & 1/37 & 10/37 \\
1 & 37 & 1/37 & 10/37 & -2/37
\end{array}
\]

For instance, the last row represents the facet-defining inequality

\[
\frac{2}{37} \cdot x_1 + \frac{20}{37} \cdot x_2 + \frac{1}{37} \cdot x_3 + \frac{10}{37} \cdot x_4 - \frac{2}{37} \cdot x_5 \leq 1.
\]

Here, we are considering the vectors \((x_1, x_2, x_3, x_4, x_5)\) to be elements in the quotient of \(\mathbb{R}^5\) modulo the one-dimensional linear subspace spanned by \((4, 1, 1, 1, 1)\). Our format is that of the software Polymake [7]. If the above eleven lines are put in a file named `mypolytope` then the following command in Polymake will verify that this polytope does indeed have the combinatorial structure displayed in Figure 2:

```
polymake mypolytope F_VECTOR VERTICES_IN_FACETS
```

The 10 facets of our 4-polytope correspond to the facets of the 5-cube, and they comprise all classes of permutations in \(S_5\) in which the first or last coordinate is fixed. The facets corresponding to permutations with 1 or 5 in the first coordinate have seven vertices, twelve edges, and eight 2-faces. The facets corresponding to permutations with 2, 3 or 4 first have seven vertices, 13 edges, and eight 2-faces. The facets for 1 or 5 last are tetrahedra. The facets for 2, 3 or 4 last are cubes in which one edge has been contracted; they have seven vertices and 11 edges.

### 4 The semigraphoid semigroup is not normal

Continuing to assume \(n = 5\), we now consider the linear map \(A\) in the Introduction. It maps the free abelian group \(\mathbb{Z}^{80}\) spanned by the CI statements to the free abelian group \(\mathbb{Z}^{32}\) with basis \(\{e_K : K \subseteq [5]\}\) as specified in [11]. The matrix representing \(A\) has 32 rows and 80 columns; each column has four non-zero entries: two \(+1\)'s and two \(-1\)'s. The rank of \(A\) is 26. The *semigraphoid semigroup* is \(A(\mathbb{N}^{80})\), the non-negative integer span of the columns of this \(32 \times 80\)-matrix. This is a subsemigroup
of \( \mathbb{Z}^{32} \). Equivalently, the semigraphoid semigroup is the affine semigroup with 80 generators and 120 relations (given in the Appendix). Note that the polyhedral cone dual to the semigraphoid semigroup is the cone of submodular functions.

In the language of [13], the vectors in \( \mathbb{Z}^{32} \) are called imsets, the columns of \( A \) are elementary imsets, and the elements of \( \mathcal{A}(\mathbb{N}^{(80)}) \) are combinatorial imsets. A structural imset is a lattice point which lies in the polyhedral cone spanned by the elementary imsets. Studený’s question (Q2) whether each structural imset is combinatorial translates into the question whether the semigroup \( \mathcal{A}(\mathbb{N}^{(80)}) \) is normal.

**Theorem 4.1.** The semigraphoid semigroup is not normal for \( n = 5 \).

**Proof.** Consider the following element in the free abelian group \( \mathbb{Z}^{80} \):

\[
[1\perp 5\mid 2] + [1\perp 4\mid 3] + [2\perp 3\mid 4] + [2\perp 3\mid 5] + [3\perp 4\mid 12] + [2\perp 5\mid 13] + [1\perp 2\mid 45] + [1\perp 3\mid 45] + [4\perp 5\mid 23] - [2\perp 3\mid 45].
\]

(3)

The image of this element under the map \( \mathcal{A} : \mathbb{Z}^{80} \to \mathbb{Z}^{32} \) is the imset

\[
b := -e_2 - e_3 - e_4 - e_5 - e_{23} + e_{24} + 2e_{25} + 2e_{34} + e_{35} - e_{45} + 2e_{123} + e_{124} - e_{125} - e_{134} + e_{135} + 2e_{145} - e_{1234} - e_{1235} - e_{1245} - e_{1345}.
\]

(4)

The imset \( b \) is structural because \( 2 \cdot b \) is a combinatorial imset. It is the image of

\[
[4\perp 5\mid 2] + [4\perp 5\mid 3] + [1\perp 3\mid 4] + [1\perp 2\mid 5] + [2\perp 5\mid 14] + [3\perp 4\mid 15] + [1\perp 4\mid 23] + [1\perp 5\mid 23] + [1\perp 5\mid 2] + [1\perp 4\mid 3] + [2\perp 3\mid 4] + [2\perp 3\mid 5] + [3\perp 4\mid 12] + [2\perp 5\mid 13] + [1\perp 2\mid 45] + [1\perp 3\mid 45] \in \mathbb{N}^{80}
\]

under the linear map \( \mathcal{A} \).

Suppose that \( b \) were a combinatorial imset. Then there exists \( x \in \mathbb{N}^{80} \) such that \( \mathcal{A} \cdot x = b \). We write \( x = \sum [a_i \perp b_i | K_i] \), where we allow repetition in the sum. In any elementary imset, the basis vector \( e_\emptyset \) occurs with coefficient \(-1\) or 0, and the basis vector \( e_{12345} \) occurs with coefficient \(-1\) or 0. However, neither \( e_\emptyset \) nor \( e_{12345} \) appears in the imset \( b \), so we conclude that \( |K_i| = 1 \) or \( |K_i| = 2 \) for all terms \([a_i \perp b_i | K_i]\) in the representation of \( x \). The first four terms \(-e_2 - e_3 - e_4 - e_5 \) in \( b \) imply that \( x \) has precisely four terms \([a_i \perp b_i | K_i]\) with \( |K_i| = 1 \), and the terms \(-e_{1234} - e_{1235} - e_{1245} - e_{1345} \) imply that \( x \) has precisely four terms with \( |K_i| = 2 \).

Each of the eight terms in \( x \) evaluates to an alternating sum of 4 terms under the map \( \mathcal{A} \). Some cancellation occurs among the resulting 32 terms. Prior to that cancellation, our imset had been written as the sum of two subsums, \( b = \mathcal{A} \cdot x = -e_2 - e_3 - e_4 - e_5 + e_{24} + 2e_{25} + 2e_{34} + e_{35} + e_{A_1} + e_{A_2} - e_{125} - e_{134} - e_{B_1} - e_{B_2} - e_{23} - e_{45} - e_{A_1} - e_{A_2} + 2e_{123} + e_{124} + e_{135} + 2e_{145} + e_{B_1} + e_{B_2} - e_{1234} - e_{1235} - e_{1245} - e_{1345}, \)

where \( |A_1| = |A_2| = 2 \) and \( |B_1| = |B_2| = 3 \). The first line is the sum of the four elementary imsets \( \mathcal{A}(\{a_i \perp b_i | K_i\}) \) with \( |K_i| = 1 \), and the second line is the sum of the four elementary imsets with \( |K_i| = 2 \). A contradiction will arise when we try to determine the unknown pairs \( A_1 \) and \( A_2 \). The term \(-e_{125} \) in the first line must come from \( K_i = \{2\} \) or \( K_i = \{5\} \). This implies that either \( \{1,2\} \) or \( \{1,5\} \) is in
A_* = \{A_1, A_2\}. Similarly, the term \(-e_{134}\) shows that either \{1, 3\} or \{1, 4\} is in \(A_*\). Now consider the second line. The presence of the term \(2e_{123}\) implies that \{1, 2\} or \{1, 3\} is in \(A_*\), and the term \(2e_{145}\) implies that \{1, 4\} or \{1, 5\} is in \(A_*\). The term \(e_{124}\) shows that \{1, 2\}, \{1, 4\}, or \{2, 4\} is in \(A_*\), and, finally, the term \(e_{135}\) shows that \{1, 3\}, \{1, 5\}, or \{3, 5\} is in \(A_*\). However, no such pair of pairs \(A_*\) satisfies these six restrictions. This proves that \(b\) is not a combinatorial imset.

The main point of the above proof was to show that the linear system \(A \cdot x = b\) has no solution with non-negative integer coordinates. This can also be verified automatically using integer programming software. In fact, using such software we found that \(A \cdot x = b\) has only one solution with non-negative real coordinates, namely, that unique solution \(x \in (\mathbb{R}_{\geq 0})^{80}\) is the expression in (5) scaled by 1/2.

The reader might now inquire how the imset \(b\) was found. There are several algorithms that test whether a given affine semigroup is normal, including one recently proposed by Takemura, Yoshida and the first author [9], and the method of Bruns and Koch [3] which is implemented in their software normaliz.

Our original attempts to apply these methods directly to the 32 \times 80-matrix \(A\) were unsuccessful. Instead we succeeded by partially computing a Markov basis for the matrix \(A\) using the software 4ti2 [5]. The imset \(b\) was found by inspecting the partial results produced by 4ti2. We explain the details in the next section.

5 Computation in toric algebra

Let \(\mathbb{Q}[\text{Cl}_n]\) denote the polynomial ring over the field of rational numbers \(\mathbb{Q}\) generated by the symbols \([i \perp j | K]\). Thus \(\mathbb{Q}[\text{Cl}_n]\) is a polynomial ring in \(\gamma_n\) unknowns, one for each 2-face of the \(n\)-cube \(C_n\). We write \(\prod \text{Cl}_n\) for the product of all the unknowns. We define the semigraphoid ideal to be the ideal \(I_{SG}\) generated by the binomials in \((SG'')\). Thus the generators of \(I_{SG}\) represent the semigraphoid axioms. Following [14] §7, we introduce the toric ideal \(I_A\) which is obtained from \(I_{SG}\) by saturation:

\[
I_A := (I_{SG} : (\prod \text{Cl}_n)^\infty).
\] (6)

The binomials in \(I_A\) represent the vectors in the kernel of the linear map \(A : \mathbb{Z}^{\gamma_n} \rightarrow \mathbb{Z}^{2^n}\). A minimal set of binomials which generates \(I_A\) is said to be a Markov basis for the matrix \(A\). See [4] for a discussion of Markov bases in the context of statistics.

Let us illustrate these concepts for \(n = 3\). The polynomial ring \(\mathbb{Q}[\text{Cl}_3]\) has six unknowns, one for each facet of the 3-cube. They are the entries of the 2 \times 3-matrix

\[
\begin{pmatrix}
\{1 \perp 2 | 0\} & \{1 \perp 3 | 0\} & \{2 \perp 3 | 0\} \\
\{1 \perp 2 | 3\} & \{1 \perp 3 | 2\} & \{2 \perp 3 | 1\}
\end{pmatrix}.
\] (7)

The semigraphoid ideal \(I_{SG}\) is generated by the three 2 \times 2-minors of the matrix (7). This is a prime ideal of codimension 2 and degree 3, and hence we have \(I_{SG} = I_A\). Here the Markov basis for \(A\) consists precisely of the three semigraphoid axioms.

We next consider the case \(n = 4\). The polynomial ring \(\mathbb{Q}[\text{Cl}_4]\) has 24 unknowns, one for each 2-face of the 4-cube. They are the entries of eight 2 \times 3-matrices as in (7), one for each of the eight facets of the 4-cube. Thus the semigraphoid ideal \(I_{SG}\)
is generated by 24 quadrics, one for each of the 24 axioms \((\text{SG}'')\) in the list given in Section 2. For instance, the last axiom in that list translates into the quadratic binomial \([1 \sqcap 2]4 \cdot [2 \sqcap 3]14 - [2 \sqcap 3]4 \cdot [1 \sqcap 2]34\), which is one of the 24 generators of \(I_{\text{SG}}\). Using the software \text{Macaulay}2 we derived the following result:

**Proposition 5.1.** The semigraphoid ideal \(I_{\text{SG}}\) is a radical ideal which is the intersection of the toric ideal \(I_{\text{A}}\) and 17 additional associated monomial prime ideals.

Before discussing this prime decomposition in detail, let us make a few general remarks. We wish to argue that toric algebra and algebraic geometry provide useful algorithmic tools for the research directions presented in [5]. For any ideal \(I\) of \(\mathbb{Q}[\mathbb{C}_n]\) and any subset \(\Omega\) of the complex affine space \(\mathbb{C}^n\), the variety \(V(I)\) is defined as the set of all vectors in \(\Omega\) which are common zeros of all the polynomials in \(I\). Then \(V_{\mathbb{C}}(I_{\text{SG}})\) is a set of CI statements, reducible for \(n \geq 4\), one of whose irreducible components is the complex toric variety \(V_{\mathbb{C}}(I_{\text{A}})\). Inside this toric variety are the real toric variety \(V_{\mathbb{R}}(I_{\text{A}})\). Its non-negative part \(V_{\mathbb{R}_{\geq 0}}(I_{\text{A}})\) is homeomorphic to the cone spanned by the elementary imsets. Our next result shows that the semigraphoids are precisely the points on these varieties whose coordinates are 0 or 1.

**Theorem 5.2.** The semigraphoids on \([n]\) are in bijection with the points in \(V_{\{0,1\}}(I_{\text{SG}})\). The submodular semigraphoids on \([n]\) are in bijection with the points in \(V_{\{0,1\}}(I_{\text{A}})\).

**Proof.** We replace the additive semiring \(\{0, +\}\) with the multiplicative semiring \(\{1, \cdot\}\). This translates from the additive notation \((\text{SG}'')\) to the multiplicative notation \((\text{SG}''')\). With this translation, the first statement in Theorem 5.2 is obvious.

The second statement is less obvious and is based on the geometry of toric varieties. Specifically, we shall use the characterization of *facial* index sets which is developed in [5]. If we consider our specific \(2^n \times \gamma_n\)-matrix \(\mathcal{A}\) then the role of the set \(\{1, \ldots, m\}\) in [5] is played by the set of CI statements, and a subset of CI statements is facial for \(\mathcal{A}\) if and only if it is submodular semigraphoid. With this observation, our second assertion follows from Lemma A.2 in the Appendix of [5]. \(\square\)

Using Theorem 5.2, we can study semigraphoids by studying the zero-dimensional ideals obtained by adding \(\langle x^2 - x : x \in \mathbb{C}_n \rangle\) to the ideal \(I_{\text{SG}}\) or \(I_{\text{A}}\). For instance, with the command `degree` in \text{Macaulay}2, it takes only a few seconds to compute

\[
\#V_{\{0,1\}}(I_{\text{SG}}) = 26424 \quad \text{and} \quad \#V_{\{0,1\}}(I_{\text{A}}) = 22108. \quad (8)
\]

The difference between these numbers is explained geometrically by the prime decomposition in Proposition 5.1, which we shall now describe in explicit terms.

The 17 associated monomial primes of \(I_{\text{SG}}\) come in three symmetry classes. First there are two primes of codimension 12. A representative is the ideal

\[
\langle [1 \sqcap 2]0, [1 \sqcap 3]0, [1 \sqcap 4]0, [2 \sqcap 3]0, [2 \sqcap 4]0, [3 \sqcap 4]0, [3 \sqcap 4]12, [2 \sqcap 4]13, [2 \sqcap 3]14, [2 \sqcap 3]23, [1 \sqcap 3]24, [1 \sqcap 2]34 \rangle.
\]

The semigraphoid ideal \(I_{\text{SG}}\) has 12 associated primes of codimension 15, such as

\[
\langle [1 \sqcap 2]0, [1 \sqcap 3]0, [1 \sqcap 4]0, [3 \sqcap 4]0, [1 \sqcap 3]2, [1 \sqcap 4]2, [3 \sqcap 4]2, [1 \sqcap 2]3, [2 \sqcap 4]3, [1 \sqcap 2]4, [2 \sqcap 3]4, [3 \sqcap 4]12, [2 \sqcap 4]13, [2 \sqcap 3]14, [1 \sqcap 2]34 \rangle.
\]
Next, $I_{SG}$ has three associated primes of codimension 16. A representative is

\[
\begin{bmatrix}
1 \parallel 2 \mid 0, & 1 \parallel 3 \mid 0, & 2 \parallel 4 \mid 0, & 3 \parallel 4 \mid 0, & 2 \parallel 4 \mid 1, & 3 \parallel 4 \mid 1, & 1 \parallel 3 \mid 2, & 3 \parallel 4 \mid 2, \\
1 \parallel 2 \mid 3, & 2 \parallel 4 \mid 3, & 1 \parallel 2 \mid 4, & 1 \parallel 3 \mid 4, & 3 \parallel 4 \mid 12, & 2 \parallel 4 \mid 13, & 1 \parallel 3 \mid 24, & 1 \parallel 2 \mid 34
\end{bmatrix}.
\]

Each of the 4316 non-submodular semigraphoids is a \{0,1\}-valued point not in $V(I_A)$ but in one of the 17 coordinate subspaces corresponding to these primes.

Finally, the last associated prime of $I_{SG}$ is the toric ideal $I_A$. This ideal has codimension 13 and degree 396. Its minimal generating set consists of 52 binomials. Besides the 24 quadrics (the axioms), the Markov basis of $A$ contains four cubics

\[
\begin{align*}
2 \parallel 3 \mid 1 \cdot & 3 \parallel 4 \mid 2 \cdot 1 \parallel 3 \mid 4 - 3 \parallel 4 \mid 1 \cdot 1 \parallel 3 \mid 2 - 2 \parallel 3 \mid 4, \\
2 \parallel 3 \mid 1 \cdot & 2 \parallel 4 \mid 3 \cdot 1 \parallel 2 \mid 4 - 2 \parallel 4 \mid 1 \cdot 1 \parallel 2 \mid 3 - 2 \parallel 3 \mid 4, \\
1 \parallel 3 \mid 2 \cdot & 1 \parallel 4 \mid 3 \cdot 1 \parallel 2 \mid 4 - 1 \parallel 4 \mid 2 \cdot 1 \parallel 2 \mid 3 - 1 \parallel 3 \mid 4, \\
2 \parallel 4 \mid 1 \cdot & 3 \parallel 4 \mid 2 \cdot 1 \parallel 4 \mid 3 - 3 \parallel 4 \mid 1 \cdot 1 \parallel 4 \mid 2 - 2 \parallel 4 \mid 3,
\end{align*}
\]

and 24 quartics such as

\[
1 \parallel 2 \mid 0 \cdot 3 \parallel 4 \mid 0 \cdot 2 \parallel 4 \mid 13 \cdot 1 \parallel 3 \mid 24 - 1 \parallel 3 \mid 0 \cdot 2 \parallel 4 \mid 0 \cdot 3 \parallel 4 \mid 12 \cdot 1 \parallel 2 \mid 34.
\]

We now come to case $n = 5$. It will be a challenge for future commutative algebra software to compute a primary decomposition of the semigraphoid ideal $I_{SG}$ for $n = 5$. At present we do not know even whether $I_{SG}$ is radical. Let us therefore focus on the main component of this ideal, namely, the toric ideal $I_A$. Here our main goal is to compute its minimal generators, that is, the Markov basis of $A$. We attacked this problem using the software 4ti2, and we now discuss the results.

First, we started a Markov basis computation for the toric ideal $I_A$ using the function markov of 4ti2, but this computation turned out to be non-trivial. In the hope that a counterexample would not involve all 80 variables, we set several variables to 0 and tried to compute the Markov basis of smaller ideals that are contained in $I_A$. For the one-day computation that finally produced a counterexample, we set the first 18 formal symbols to zero and found the Markov basis move

\[
g := (\alpha + 2 \cdot [2 \parallel 3 \mid 45]) - (\beta + 2 \cdot [4 \parallel 5 \mid 23]) \in \mathbb{N}^{80},
\]

where

\[
\begin{align*}
\alpha &= [4 \parallel 5 \mid 2] + [4 \parallel 5 \mid 3] + [1 \parallel 3 \mid 4] + [1 \parallel 2 \mid 5] + [2 \parallel 5 \mid 14] + [3 \parallel 4 \mid 15] + [1 \parallel 4 \mid 23] + [1 \parallel 5 \mid 23], \\
\beta &= [1 \parallel 5 \mid 2] + [1 \parallel 4 \mid 3] + [2 \parallel 3 \mid 4] + [2 \parallel 5 \mid 3] + [3 \parallel 4 \mid 12] + [2 \parallel 5 \mid 13] + [1 \parallel 2 \mid 45] + [1 \parallel 3 \mid 45].
\end{align*}
\]

This lattice vector corresponds to a binomial $x^g^+ - x^g^-$ which is in the toric ideal $I_A$ and has the property that both of its monomials are not square-free. We then verified that $x^g^+ - x^g^-$ is not only indispensable for the smaller ideal (with 18 variables set to zero) but also indispensable for $I_A$. Recall (e.g. from [1]) that a binomial $x^g^+ - x^g^-$ in the toric ideal $I_A$ is called indispensable if

\[
\{z \in \mathbb{N}^{80} : A \cdot z = A \cdot g^+ \} = \{g^+, g^-\}.
\]

This means that the Markov move $g$ corresponds to a 2-element fiber given by the right-hand side and consequently, $g$ must belong to every Markov basis of $I_A$. In
order to check this condition for our given move \( g \), we computed the minimal Hilbert basis (that is, the \( \leq \)-minimal integer solutions) of the cone

\[
\{(z, u) \in \mathbb{R}^{81} : A \cdot z - (A \cdot g^+) \cdot u = 0, (z, u) \geq 0\}.
\]

This was done using the function \texttt{hilbert} of 4ti2 which produced precisely the two expected elements \((g^+, 1)\) and \((g^-, 1)\) within a few seconds.

From our special Markov move \( g = (\alpha + 2 \cdot \{2 \parallel 3|45\}) - (\beta + 2 \cdot \{4 \parallel 5|23\}) \), we then constructed the imset \( b \) presented in Section 4. We first checked that \( b \) was not a combinatorial imset by showing that \( Ax = b \) has no solutions with non-negative integer coordinates. Using the functions \texttt{hilbert} and \texttt{rays} of the program 4ti2, we computed the Hilbert basis and the extreme rays of the cone

\[
\{(z, u) \in \mathbb{R}^{81} : A \cdot z = b \cdot u \text{ and } (z, u) \geq 0\}.
\]

Both computations quickly finished. They showed that this cone has dimension one and is generated by the single vector \((\alpha + \beta, 2)\). Consequently, the only non-negative real solution to \( A \cdot x = b \) is \((\alpha + \beta)/2\), which is not an integer solution.

We are currently in the process of computing the complete minimal Markov basis of the toric ideal for semigraphoids with \( n = 5 \). That Markov basis has well over a million elements. Yet, we are convinced that 4ti2 will succeed. The completion of that Markov basis will represent a computational milestone in toric algebra.

Acknowledgments

Jason Morton and Bernd Sturmfels were supported by the DARPA \textit{Fundamental Laws of Biology} program, and Bernd Sturmfels was also supported by the NSF. Anne Shiu was supported by a Lucent Technologies Bell Labs Graduate Research Fellowship. Oliver Wienand was supported by the Wipprecht foundation.

References

[1] S Aoki, A Takemura, and R Yoshida. Indispensable monomials of toric ideals and Markov bases. Preprint, \url{math.ST/0511290}.

[2] J Bokowski and B Sturmfels. Polytopal and non-polytopal spheres—An algorithmic approach. \textit{Israel Journal of Mathematics} 57 (1987) 257–271.

[3] W Bruns and R Koch. Computing the integral closure of an affine semigroup. Effective methods in algebraic and analytic geometry, 2000 (Kraków). \textit{Univ. Iagel. Acta Math.} 39 (2001) 59–70.

[4] P Diaconis and B Sturmfels. Algebraic algorithms for sampling from conditional distributions. \textit{Annals of Statistics} 26 (1998) 363-397.

[5] D Geiger, C Meek and B Sturmfels. On the toric algebra of graphical models. \textit{Annals of Statistics} 34 (2006) 1463–1492.
[6] D Grayson and M Stillman. Macaulay 2, a software system for research in algebraic geometry. Available from [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).

[7] E Gawrilow and M Joswig: Polymake: a framework for analyzing convex polytopes, in Polytopes — Combinatorics and Computation, eds. G. Kalai and G.M. Ziegler, Birkhäuser, 2000, pp. 43–74.

[8] R Hemmecke, R Hemmecke, and P Malkin: 4ti2 Version 1.2—Computation of Hilbert bases, Graver bases, toric Gröbner bases, and more. Available from [www.4ti2.de](http://www.4ti2.de), September 2005.

[9] R Hemmecke, A Takemura, and R Yoshida. Computing holes in semi-groups. Preprint, [math.CO/0607599](http://arxiv.org/abs/math.CO/0607599).

[10] H Hirai. Sequences of stellar subdivisions. Preprint, 2006.

[11] F Matúš. Conditional independences among four random variables. III. Final conclusion. *Combin. Probab. Comput.* 8 (1999) 269–276.

[12] F Matúš. Conditional probabilities and permutohedron. *Ann. Inst. H. Poincaré Probab. Statist.* 39 (2003) 687–701.

[13] F Matúš. Towards classification of semigraphoids. *Discrete Mathematics* 277 (2004) 115–145.

[14] E Miller and B Sturmfels. *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics, Springer Verlag, New York, 2004.

[15] J Morton, L Pachter, A Shiu, B Sturmfels, and O Wienand. Geometry of rank tests. *Probabilistic Graphical Models (PGM 3)*, Prague, Czech Republic, September 2006, [math.ST/0605173](http://arxiv.org/abs/math.ST/0605173).

[16] A Postnikov. Permutohedra, associahedra, and beyond. 2005. Preprint, [math.CO/0507163](http://arxiv.org/abs/math.CO/0507163).

[17] A Postnikov, V Reiner, and L Williams. Faces of simple generalized permutohedra. Preprint, [math.CO/0609184](http://arxiv.org/abs/math.CO/0609184).

[18] M Studený. *Probabilistic Conditional Independence Structures*. Springer Series in Information Science and Statistics, Springer-Verlag, London, 2005.

[19] G Ziegler. *Lectures on Polytopes*. Graduate Texts in Mathematics, Springer-Verlag, New York, 1998.

Authors’ addresses:

Raymond Hemmecke, Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg, 39106 Magdeburg, Germany, raymond@hemmecke.de

Jason Morton, Anne Shiu and Bernd Sturmfels, Department of Mathematics, University of California, Berkeley, CA 94720, USA, [mortonj, annejls, berndj]@math.berkeley.edu

Oliver Wienand, Fachbereich Mathematik, Universität Kaiserslautern, 67653 Kaiserslautern, Germany, wienand@rhrk.uni-kl.de
6 Appendix: The 120 semigraphoid axioms

Here is the list of all 120 semigraphoid axiom for n = 5, grouped into triples according to which 3-face of the 5-cube they come from. The two types of brackets specify the non-submodule coresest semigraphoid Π which was discussed in Section 6.