Construction of blow-up solution for 5-dimensional critical Fujita-type equation with different blow-up speed

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Abstract. We are concerned with the blow-up solutions of the 5-dimensional energy critical heat equation \(u_t = \Delta u + |u|^{\frac{4}{3}}u\). Our main finding is to show that the existence of type II solutions results in blowing up at any \(k\) points, with arbitrary \(k\) blow-up rates. We have employed the inner–outer gluing method to accomplish this.

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1. Introduction

We consider the equation
\[
u_t = \Delta u + |u|^{p-1}u, \quad \text{in } \Omega \times (0,T).
\] (1)

In 1966, Fujita [11] initiated research on Eq. 1. If we only consider the time variable, 1 becomes an ordinary differential equation \(\dot{g} = |g|^{p-1}g\). The solution
\[g(t) = (p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}},\]
blows up at time \(T\), which is called the type I blow-up. And if a blow-up solution satisfies the following condition
\[
\lim_{t \to T} |u(\cdot, t)(T-t)^{\frac{1}{p-1}}|_\infty = \infty,
\]
then the solution with this blow-up rate is called a type II blow-up solution. It is obvious that type II blow-up is faster than type I blow-up. More results for type I blow-up can be found in [13,14,22,25,29].

Let \(p_S = \frac{n+2}{n-2}\) denote the critical Sobolev exponent. In the subcritical case, Giga and Kohn [12] first proved that only type I blow-up occurs in the
convex domain. In the supercritical case, let $p_{JL}$ denote the Joseph-Lundgreen exponent.

$$p_{JL} = \begin{cases} 
\infty, & 3 \leq n \leq 10, \\
1 + \frac{4}{n - 1 - 2\sqrt{n - 1}}, & n \geq 11.
\end{cases}$$

The first example of a type II solution was given by Herrero and Velázquez [19, 20], they constructed a radial positive solution in the case $p > p_{JL}$. Collot [1] constructed a type II solution with the same profile later. In the case $p_S < p < p_{JL}$, Matano and Merle [21] excluded type II blow-up in the radial case. For the case $p = p_{JL}$, Seki constructed a blow-up solution in [27]. For the case $p = p_S$, Filippas, Herrero and Velázquez [10] proved that the radial positive solution can only be a type I blow-up solution. Using the asymptotic matching method, they have formally obtained the possible blow-up solution with the blow-up rates given by (after slight modification):

$$\|u(\cdot, t)\|_\infty = \begin{cases} 
(T - t)^{-k}, & n = 3, \\
(T - t)^{-k}\ln(T - t)^{\frac{2k}{n - 1}}, & n = 4, \\
(T - t)^{-3k}, & n = 5, \\
(T - t)^{\frac{n}{2}}\ln(T - t)^{\frac{12}{n}}, & n = 6,
\end{cases}$$

where $k = 1, 2, \ldots$. When $n \geq 7$, Collot, Merle, and Raphaël [2] proved that the type II blow-up solution cannot be around the ground state. For $n = 4$, type II solutions were constructed by Schweyer [26] in 2012. In the case $3 \leq n \leq 6$, type II solutions were constructed by Del Pino, Musso, Wei, and Zhou [9]; they constructed a multi-point blow-up in 2019. The case $n = 5$ was first constructed by Del Pino, Musso, Wei [6] in 2018, with the blow-up rate of $k = 1$ in . Junichi Harada [17] finished the construction of a higher blow-up rate for $n = 5$ in 2019. And Junichi Harada [18] also proved the case for $n = 6$. Very recently, Del Pino, Musso, Wei, Zhang, and Zhou [8] finished proof for $n = 3$.

In the above-mentioned construction of the critical case, except for the construction by Schweyer, they all employed the so-called inner-outer gluing method. This technique is not only applied to the construction of finite time blow-up solutions for the critical case but also for the supercritical cases and infinite time blow-up [3, 7]. Apart from the Fujita equation, Dávila, Del Pino, and Wei also constructed a blow-up solution for the harmonic map equation [4].

Our construction is based on the so-called inner-outer gluing method, but the most challenging aspect in multi-point blow-up problems is to derive a solution to the outer problem with the proper vanishing rate at blow-up points. In fact, Del Pino, Musso, and Wei dealt with the multi-point problem in [6], but in their construction, due to the type of solution they selected to match the blow-up profile, which was approximately constant near the blow-up points, they only needed to provide an $L^\infty$ estimate in the outer problem. However, in other cases, we require a function with a vanishing rate of approximately $(T - t)^l$ with different exponents near different blow-up points. Therefore, $L^\infty$ is not sufficient here. To overcome this difficulty, we
introduce a solution to a heat equation to cancel the influence of the other terms. Specifically, we cancel the derivative of other terms at the blow-up points using this function. We achieve our goal by using Taylor’s expansion at the blow-up points. We believe this method can be applied to other problems with multiple blow-up points.

1.1. Main result

For the 5-dimensional energy critical equation, our main result is

\textbf{Theorem 1.} Let $\Omega = \mathbb{R}^n$, $n = 5$, $p = \frac{7}{3}$ be the critical exponent. Given $k$ points $q_1, \ldots, q_k$ in $\mathbb{R}^5$, and $k$ integers $l_1, \ldots, l_k$, then for any sufficiently small $T > 0$, there exists an initial data $u_0$, such that the solution of equation 1 blows up at time $T$, which satisfies $|u(\xi_i, t)| \to \infty$, $i = 1, \ldots, k$, where $\xi_j(t) \to q_j$, and $|u(\xi_i, t)| \sim \lambda_i^{-\frac{3}{2}}$ as $t \to T$. Moreover, the solution has the form

$$u(x, t) = \sum_{i=1}^{k} U_{\lambda_i(t), \xi_i(t)}(x)\tilde{\eta}(x - q_i) - \sum_{i=1}^{k} Z_i \eta_i(x) + \theta(x, t),$$

where

$$U_{\lambda_i(t), \xi_i(t)} = \lambda_i(t)^{-\frac{n-2}{2}} U\left(\frac{x - \xi_i(t)}{\lambda_i(t)}\right),$$

$$U(x) = \alpha_5 \left(\frac{1}{1 + |x|}\right)^{\frac{3}{2}}, \alpha_5 = 15^{\frac{1}{3}}$$

being the radial solution of the steady critical Fujita equation $\Delta U + U^{\frac{7}{3}} = 0$, $Z_i$ is a solution of the heat equation, bear each point $q_i$, we have $Z_i \sim (T - t)^{l_i}$, $\tilde{\eta}$ and $\eta_i$ are cut-off functions satisfies

$\tilde{\eta}(x) = 1$ for $|x| \leq \frac{\min_{i \neq j} \text{dist}(q_i, q_j)}{4}$ and $\tilde{\eta}(x) = 0$ for $|x| \geq \frac{\min_{i \neq j} \text{dist}(q_i, q_j)}{4}$, $\eta_i = \eta\left(\frac{x - q_i}{(T - t)^{\frac{l_i}{2} + b_i}}\right)$, where $b_i = \frac{l_i + \frac{1}{2}}{2l_i + 2}$, when $t \to T$

$$\lambda_i(t) \sim C_i (T - t)^{2l_i + 2}$$

finally,

$$\theta(x, t) \in L^\infty(\mathbb{R}^5 \times (0, T)).$$

Our result extends the existing work of [6] and [17], where the authors respectively studied the case of multi-point blow-up with $n = 5$ and $k = 1$. In [17], the authors considered a radial solution with $n = 5$ and $k \in \mathbb{N}^+$. The novelty of our work lies in the construction of a solution that blows up simultaneously at multiple points, each with a different blow-up rate. Our approach uses new techniques and insights, leading to a significant advance in the understanding of the blow-up behavior of solutions in higher dimensions. Our work opens up new avenues for future research.
2. Basic facts in the construction

As is well known, in the critical case, the steady solution of equation (1) is unique, up to scalings and translations. This unique steady solution is given by the following Talenti-Aubin steady state [28]:

\[ U(x) = \alpha_n \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}}, \]

In the following subsection, we present the construction of the approximated solutions for the problem at hand.

2.1. Construction of the approximated solutions

To simplify notation, we present the proof in the case of \( k = 2 \), with a fixed set of two points, \( q_1 \) and \( q_2 \), in \( \mathbb{R}^5 \). The solution we aim to construct is of the form:

\[ u(x, t) = \sum_{i=1}^{2} \left[ U_{\lambda_i(t),\xi_i(t)}(x - q_i) - Z_i \eta_i \right] + \theta(x, t), \tag{3} \]

where \( \lambda_i(t) \) are the scaling parameters and \( \xi_i(t) \) are the translation parameters. Here, the translation is assumed to be small compared with scaling, that is, \( \xi_i(t) = o(\lambda_i(t)) \). The function \( Z_i := (T - t)^{l_i} e_i^* \), where \( e_i^* = 1 + a_{1,i} |z|^2 + \cdots + a_{i,i} |z|^{2l_i} \) is the \( 2l_i \)-th degree polynomial that solves the eigenvalue problem.

\[ -\Delta e_i^* + z_i \cdot \nabla e_i^* = l_i e_i^*, \]

where \( z_i = \frac{x - q_i}{\sqrt{T - t}} \) is the self-similar variable around \( q_i \). Here \( Z_i \) is actually a solution of the heat equation. Since the solution \( Z_i \) does not decay at infinity, we need to cut it off by multiplying the cut-off function \( \eta_i = \eta\left(\frac{z}{(T - t)^{\frac{l_i}{2}}}\right) \), here we choose \( b_i = \frac{l_i + 1}{2T - t} \), \( \eta(x) = 1 \) if \( x \leq 1 \) and \( \eta(x) = 0 \) if \( x \geq 2 \), it is smooth in the interval \((1, 2)\). We don’t hope \( U_{\lambda_i(t),\xi_i(t)} \) dominate \( Z_2 \) near \( q_2 \) or \( U_{\lambda_2(t),\xi_2(t)} \) dominate \( Z_1 \) near \( q_1 \), hence we need to cut off \( U_{\lambda_i(t),\xi_i(t)} \) by times cut-off function \( \tilde{\eta}_i = \eta\left(\frac{4(x - q_i)}{d}\right) \), where \( d = \min_{i \neq j} \{d(q_i, q_j)\} \).

2.2. Inner–outer gluing

To provide a more detailed explanation of our approach, we write the remainder in the form

\[ \theta(x, t) = \sum_{i=1}^{2} \lambda_i^{-\frac{n-2}{2}} \phi_i(y_i, t) \eta_R(y_i) + \psi(x, t), \]

where \( y_i = \frac{x - \xi_i}{\lambda_i} \), denote \( \eta_{R,i} = \eta\left(\frac{y_i}{2R}\right) \) and \( \eta \) be a cut-off function. We require \( \psi \) to be a continuous function and satisfy the following estimate

\[ |\psi| \leq \begin{cases} \delta_0 (T - t)^{l_i} (1 + |z_i|^{2l_i+2}) & \text{for } |z_i| \leq (T - t)^{-\frac{l_i}{2l_i+2}}, \\ \frac{\delta_0}{1 + |x|^\sigma} & \text{else}, \end{cases} \tag{4} \]
where $\delta_0 = \sup_{i=1,2} \frac{\|Z_i\|_\infty}{10}$. Let us set the error function
\[ S(u) = -u_t + \Delta u + |u|^{p-1} u, \]
and the error of $U_{\lambda_i, \xi_i}$
\[ E_i(y_i, t) = \lambda_i \dot{x}_i \left[ \left( \frac{3}{2} U(y_i) + y_i \cdot \nabla U(y_i) \right) \right] + \lambda_i \dot{\xi}_i \cdot \nabla U(y_i). \] (5)

We compute $S(u)$
\[
S \left( \sum_{i=1,2} U_{\lambda_i, \xi_i, \eta_i}(t) \eta_i - Z_1 \eta_1 - Z_2 \eta_2 + \theta(x, t) \right) \\
= -\theta_t + \Delta \theta + \sum_{i=1,2} pU_{\lambda_i, \xi_i}^{p-1} \eta_i(\theta(x, t) - Z_1 \eta_1 - Z_2 \eta_2) + \sum_{i=1,2} \lambda_i^{-\frac{1}{2}} E_i \eta_i + \mathcal{N} \\
= \sum_{i=1,2} \eta_R, i \lambda_i^{-\frac{1}{2}} \left( -\lambda_i^2 \partial_x \phi_i + \Delta \phi_i + pU_i(y_i)^{p-1} \left[ \phi_i + \lambda_i^2 (-Z_1 \eta_1 - Z_2 \eta_2 + \psi) \right] + E_i \right) \\
- \psi_t + \Delta_x \psi + \sum_{i=1,2} p\lambda_i^{-2}(1 - \eta_R, i)U_i(y_i)^{p-1} \eta_i(-Z_1 \eta_1 - Z_2 \eta_2 + \psi) \\
+ \sum_{i=1,2} A[\phi_i] + \sum_{i=1,2} B[\phi_i] \\
+ \sum_{i=1,2} \lambda_i^{-\frac{1}{2}} E_i((1 - \eta_R, i) \eta_i + \mathcal{N}).
\]

Where
\[
A[\phi_i] := \lambda_i^{-\frac{1}{2}} [\Delta_y, \eta_R, i \phi_i + 2 \nabla y_i \eta_R, i \nabla y_i, \phi_i],
\]
\[
B[\phi_i] := \lambda_i^{-\frac{1}{2}} [\dot{\lambda}_i(y_i) \cdot \nabla y_i, \phi_i + \frac{3}{2} \phi_i) \eta_R, i + \dot{\xi}_i \cdot \nabla y_i, \phi_i \eta_R, i + (\dot{\lambda}_i y_i \cdot \nabla y_i, \eta_R, i \\
+ \dot{\xi}_i \cdot \nabla y_i, \eta_R, i) \phi_i],
\]
\[
\mathcal{N} := \left[ \sum_{i=1,2} U_{\lambda_i, \xi_i, \eta_i} + \theta - \sum_{i=1,2} Z_i \eta_i \right]^{p-1} \left( \sum_{i=1,2} U_{\lambda_i, \xi_i, \eta_i} + \theta - \sum_{i=1,2} Z_j \eta_j \right) \\
- \sum_{i=1,2} \frac{pU_{\lambda_i, \xi_i}^{p-1}}{\lambda_i^{-\frac{1}{2}}} (\theta - Z_1 \eta_1 - Z_2 \eta_2) - pU_{\lambda_i, \xi_i}^{p-1} \eta_i^{p-1} (\theta - Z_1 \eta_1 - Z_2 \eta_2) \\
+ \sum_{i=1,2} \frac{\partial \eta_i}{\partial t} Z_i + \nabla Z_i \nabla \eta_i - \Delta \eta_i Z_i + \sum_{i=1,2} \lambda_i^{-\frac{1}{2}} \nabla y_i, U_i(y_i) \nabla \eta_i \\
+ \lambda_i^{-\frac{1}{2}} U_i(y_i) \Delta_x \eta_i.
\]

We shall find a solution of if we find a pair $(\phi_1(y, t), \phi_2(y, t), \psi(x, t))$ solves the following system of equations
\[
\begin{cases}
\lambda_1^2 \partial_t \phi_1 = \Delta_y \phi_i + pU_i^{p-1}(y_1) \phi_1 + H_1(\psi, \lambda_1, \xi_1) & \text{for } (y_1, t) \in B_{2R}(0) \times (0, T), \\
\phi_1(y_1, 0) = \phi_{1,0} & \text{for } y_1 \in B_{2R}. \\
\lambda_2^2 \partial_t \phi_2 = \Delta_y \phi_2 + pU_i^{p-1}(y_2) \phi_2 + H_2(\psi, \lambda_2, \xi_2) & \text{for } (y_2, t) \in B_{2R}(0) \times (0, T), \\
\phi_2(y_2, 0) = \phi_{2,0} & \text{for } y_2 \in B_{2R}.
\end{cases}
\]
\[
\begin{align*}
\psi_t &= \Delta_x \psi + G(\phi, \psi, \lambda_1, \lambda_2, \xi_1, \xi_2) \quad \text{for } (x, t) \in \mathbb{R}^5 \times (0, T), \\
\psi(\cdot, 0) &= \psi_0 \quad \text{for } x \in \mathbb{R}^5.
\end{align*}
\] (8)

Where for \( i = 1, 2 \)

\[
H_i(\psi, \lambda_i, \xi_i) := \lambda_i^\frac{3}{2} pU(y_i)^{p-1}(-Z_i(\xi_i + \lambda_i y_i, t) + \psi(\xi_i + \lambda_i y_i)) + E_i(y_i, t),
\]

and

\[
G(\phi_1, \phi_2, \psi, \lambda_1, \lambda_2, \xi_1, \xi_2)
:= \sum_{i=1,2} p\lambda_i^{-2}(1 - \eta_{R,i})U(y_i)^{p-1}\tilde{\eta}_i^{p-1}(-Z_1\eta - Z_2\eta_2 + \psi)
+ \sum_{i=1,2} A[\phi_i] + \sum_{i=1,2} B[\phi_i] + \sum_{i=1,2} \lambda_i^{-\frac{7}{2}} E_i(1 - \eta_{R,i})\tilde{\eta}_i + N.
\]

2.3. A result about the linearized equation

To deal with the inner problem, we consider the corresponding linear problem of 6 and 7.

\[
\begin{align*}
\lambda^2 \phi_t &= \Delta_y \phi + pU(y)^{p-1}\phi + h(y, t) \quad \text{for } (y, t) \in B_2 \times [0, T), \\
\phi(y, 0) &= lW_0(y) \quad \text{for } y \in B_2,
\end{align*}
\] (11)

where \( l \) is a constant to be chosen later, \( W_0 \) is the eigenfunction of \( L_0 := \Delta + pU^{p-1} \) with the negative eigenvalue. It is known that

\[
W_0(y) \sim |y|^{-\frac{2}{a}} e^{-\sqrt{|\lambda_0|}|y|}, \quad \text{as } y \to \infty,
\]

where \( \lambda_0 \) is the only negative eigenvalue of \( L_0 \). And \( h(y, t) \) satisfies the orthogonal condition

\[
\int_{B_2} h(y, t)W_i(y)dy = 0 \quad \text{for } i = 1, \ldots, 6, \quad t \in [0, T).\n\] (12)

The kernels of \( L_0 \) are

\[
W_i(x) = \frac{\partial U(x)}{\partial x_i} \quad i = 1, 2, 3, 4, 5,
\]

and

\[
W_6(x) = \frac{\partial U_\lambda(x)}{\partial \lambda} \bigg|_{\lambda=1} = \left( \frac{3}{2} U(x) + x \cdot \nabla U(x) \right).
\] (14)

We define

\[
\|h\|_{a, \nu} := \sup_{y \in B_2, t \in [0, T]} \lambda_0^\nu (1 + |y|^a)|h(y, t)|,
\]

and

\[
\|\phi\|_{a, \nu} := \sup_{y \in B_2, t \in [0, T]} \lambda_0^\nu \left(1 + \frac{|y|^6}{R^{6-a}} \right)|\phi(y, t)|.
\]

We require that the solution \( \phi \) is located in the space where the \( \| \cdot \|_{a, \nu} \) is finite. However, we can only obtain an estimate for this due to the work of Del Pino, Musso, and Wei, who have shown that \( \phi \) belongs to \( \| \cdot \|_{a, \nu} \). The following theorem from [6] states this result.
Theorem 2. There exists constant $C_3 > 0$, for all sufficiently large $R > 0$, if $\|h\|_{2+a,\nu} < +\infty$, $\nu > 0$, and $h$ satisfies orthogonal condition 12, then there exists linear operator $L$ satisfies
\[
\phi = L^{in}[h], \quad l = l[h],
\]
fulfill the Eq. 11, and $l[h]$ satisfies
\[
|l[h]| + \|(1 + y)\nabla_y \phi\|_{*a,\nu} + \|\phi\|_{*a,\nu} \leq C_3 \|h\|_{2+a,\nu}.
\]

2.4. Local properties of the heat equation

In order to solve the outer problem, we first present some known results for the completeness of this paper. We do not restrict ourselves to the case where $n = 5$. Consider the heat equation:
\[
u = \Delta u \quad \text{for} \quad (x, t) \in \mathbb{R}^n \times (0, \infty).
\]

We make use of self-similar variables
\[
w(z, \tau) = u(x, t), \quad z = \frac{x}{\sqrt{T - t}}, \quad T - t = e^{-\tau}.
\]

The function $w(z, \tau)$ solves
\[
w_{\tau} = A_z w \quad \text{for} \quad (z, \tau) \in \mathbb{R}^n \times (0, \infty),
\]
where $A_z = \Delta_z - \frac{z}{2} \cdot \nabla_z$. We define the weighted $L^2$ space
\[
L^2_\rho(\mathbb{R}^n) := \{ f \in L^2_{loc}(\mathbb{R}^n), \|f\|_\rho < \infty \}, \|f\|_\rho^2 = \int_{\mathbb{R}^n} f^2(z) \rho(z)dz,
\]
where $\rho(z) = e^{-\frac{|z|^2}{4}}$. And the inner product is denoted by
\[
(f_1, f_2)_\rho := \int_{\mathbb{R}^n} f_1(z)f_2(z)\rho(z)dz.
\]

Consider the eigenvalue problem of $A_z$ in $L^2_\rho(\mathbb{R}^n)$. Let $e_\alpha$ be the eigenfunction
\[
-A_z e_\alpha = \lambda_\alpha e_\alpha,
\]
where $\alpha$ is multi-index, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, corresponding eigenvalue is
\[
\lambda_\alpha = \frac{|\alpha|}{2} = \alpha_1 + \cdots + \alpha_n.
\]

The eigenfunction
\[
e_\alpha = \prod e_{\alpha_i}, \quad e_{\alpha_i} = H_{\alpha_i},
\]
where $H_{\alpha_i}$ is the $\alpha_i$-th Hermite polynomial. Then $(T - t)^{\lambda_\alpha}e_\alpha$ is a solution of heat equation. We denote $e^{A_z}f_0$ to be the solution with initial data $f_0$, by Duhamel’s principle
\[
e^{A_z}f_0 = \frac{c_n}{(T - e^{-\tau})^\frac{3}{2}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4(T - e^{-\tau})}} f_0(\zeta)d\zeta.
\]

By using this formula, It is easy to obtain the following estimate.
Theorem 3. There exists a constant $C = C_n > 0$, such that
\[
|e^{A_z(\tau-\tau_0)} f_0| \leq C_n \frac{e^{-|z|^2}}{(T - e^{-\tau})^2} \|f_0\|_p \quad \text{for } (z, \tau) \in \mathbb{R}^n \times (0, \infty) \quad (16)
\]
The proof of this theorem can be seen in [16].

Theorem 4. For any $l \in \mathbb{Z}^+$, it has
\[
|e^{A_z(\tau-\tau_0)} |z|^{2l}| \leq C_l (1 + e^{-l(\tau-\tau_0)} |z|^{2l}) \quad \text{for } (z, \tau) \in \mathbb{R}^n \times (0, \infty) \quad (17)
\]
The proof of this theorem can be seen in [15]. The last one is an estimate of the heat equation in different scaling variables.

Theorem 5. Consider nonhomogeneous heat equation
\[
\begin{align*}
\partial_t \Phi - \Delta \Phi &= g(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T), \\
\Phi |_{t=0} &= 0 \quad \text{for } x \in \mathbb{R}^n.
\end{align*}
\]
Suppose
\[
|g(x, t)| \leq \frac{1}{\lambda^2(t)(1 + |y|^{2+a})}, \quad (x, t) \in \mathbb{R}^n \times (0, T)
\]
where $y = \frac{x}{\lambda(t)}$, $0 < a < 1$ be fixed, and $\lambda(t) \sim (T-t)^k$, $\dot{\lambda}(t) \sim -(T-t)^k-1$, for some $k \geq 2, k \in \mathbb{N}^+$, then there exist constants $C_1, \gamma > 0$ such that
\[
|\Phi| \leq C_1 (T^\gamma + \frac{1}{1+|y|^a}).
\]
The proof of this theorem can be seen in theorem 4.2 in [6]. In order to address the multi-rate problem, let us consider the following equation:
\[
\begin{align*}
\partial_t \Phi - \Delta \Phi &= g \quad \text{for } (x, t) \in \mathbb{R}^n \times (0, T), \\
\Phi |_{t=0} &= \Phi_0
\end{align*}
\]
(18)
Our goal is to find a solution of 18, its derivatives of given order equal a given data.

Theorem 6. Suppose $g(x, t)$ is smooth and bounded function in $\mathbb{R}^n \times (0, T)$, given $k$ points $p_1, \ldots, p_k$ in $\mathbb{R}^n$, $s_{i,j}$, $t_{i,j}$ for $1 \leq i \leq k$, $1 \leq j \leq L_i$, $1 \leq r \leq n$, and $\beta_{i,j} \in \mathbb{R}, i = 1, \ldots, k, 1 \leq j \leq L_i$, there exist initial data $\Phi_0$, such that the solution of nonhomogeneous heat equation 18, satisfies
\[
\partial_{s_{1,j}}^{i_{1,j}} \cdots \partial_{s_{n,j}}^{i_{n,j}} \partial_{t}^{i_{1,j}} \Phi(p_i, T) = \beta_{i,j}, \quad 1 \leq i \leq k, \quad 1 \leq j \leq L_i.
\]
(19)

Proof: Denote $I := \{(i, j), i = 1, \ldots, k, 1 \leq j \leq L_i\}$, and $m = L_1 + \cdots L_k$, choose $m$ smooth function supported in a bounded set $g_{i',j'}, (i',j') \in I$, such that
\[
\det|\partial_{s_{1,j}}^{i_{1,j}} \cdots \partial_{s_{n,j}}^{i_{n,j}} \partial_{t}^{i_{1,j}} g_i(p_i, T)| \neq 0,
\]
(20)
where $[\partial_{s_{1,j}}^{i_{1,j}} \cdots \partial_{s_{n,j}}^{i_{n,j}} \partial_{t}^{i_{1,j}} g_i(p_i, T)]$ is a $m \times m$ matrix, then, let us solving the following linear equations about $\mu_{i,j}$,
\[
\sum_{i=1}^{m} \mu_{i,j} \partial_{s_{1,j}}^{i_{1,j}} \cdots \partial_{s_{n,j}}^{i_{n,j}} \partial_{t}^{i_{1,j}} g_{i',j'}(p_i, T)
\]
Therefore it is reasonable for us to ignore the term $\lambda$. We formally derive an approximate equation of $\lambda$.

In this section, we shall solve the scaling parameter $\lambda_i$. The parameters $\lambda_i$ and $\xi_i$ essentially determined the blow-up rates.

3. The parameters $\lambda_i$ and $\xi_i$

In this section, we shall solve the scaling parameter $\lambda_i$ and translation parameter $\xi_i$. They essentially determined the blow-up rates.

3.1. Formal derivation of $\lambda_i$ and $\xi_i$

We formally derive an approximate equation of $\lambda_i$ and $\xi_i$ and determine the first order of the parameter. Recall the inner problem

$$
\lambda_i^2 \partial_t \phi_i = \Delta y_i \phi_i + p U^{p-1}(y_i) + H_i(\psi, \lambda_i, \xi_i) \quad \text{for} \quad (y_i, t) \in B_{2R} \times (0, T),
$$

and the notation $y_i = \frac{x-\xi_i}{\lambda_i}$, where $i = 1, 2$ and

$$
H_i(\psi, \lambda, \xi) := \lambda_i^3 p U(y_i)^{p-1}(-Z_i(\xi_i + \lambda_i y_i, t) + \psi(\xi_i + \lambda y_i)) + E_i(y_i, t).
$$

Because we We aim to find a type II blow-up solution, which is equivalent to

$$
|\lambda_i| << \sqrt{T-t}.
$$

Therefore it is reasonable for us to ignore the term $\lambda_i^2 \partial_t \phi_i$, which reduces the equation to an elliptic equation as follows

$$
\Delta y_i \phi_i + p U^{p-1}(y_i) + H_i(\psi, \lambda_i, \xi_i) = 0. \quad (22)
$$

According to the Fredholm theorem, the solvability condition for this equation is that the non-homogeneous term be orthogonal to the operator’s kernel. This orthogonal condition can be approximated as

$$
\int_{R^5} H_i(y_i, t) W_j(y_i) dy_i = 0 \quad j = 1, \ldots, 6, \quad i = 1, 2. \quad (23)
$$

We require that the remainder term $\psi$ has a negligible influence and can be omitted. Notice that in the region $y_i \in B_{2R}$, $Z_i = (T-t)^{l_i} e_i, i = 1, 2$ is approximately equal to $(T-t)^l$. And from 5

$$
E_i(y_i, t) = \lambda_i \dot{\lambda}_i \left[ \left( \frac{3}{2} U(y_i) + y_i \cdot \nabla U(y_i) \right) \right] + \lambda_i \dot{\xi}_i \cdot \nabla U(y_i).
$$

Also

$$
W_i(y_i) = \left( \frac{3}{2} U(y_i) + y_i \cdot \nabla U(y_i) \right)
$$

is an even function, while $W_j(y_i) = \frac{\partial U(y_i)}{\partial y_j}$ is odd in variable $y_i$. Then, the condition

$$
\int_{R^5} H_i(y_i, t) W_i(y_i) dy_i = 0, \quad i = 1, 2,
$$


can approximately be written as
\[ \dot{\lambda}_i \lambda_i \int_{R^5} W_6^2(y_i) dy_i - (T - t) \lambda_i^3 \int_{R^5} pU(y_i)^{p-1} W_6(y_i) dy_i = 0. \]

Let
\[ \kappa_i := -\frac{\int_{R^5} pU(y_i)^{p-1} W_6(y_i) dy_i}{\int_{R^5} W_6^2(y_i) dy_i} = \frac{3}{2} \frac{\int_{R^5} U(y_i)^2 dy_i}{\int_{R^5} W_6^2(y_i) dy_i}. \]

Then the main order of \( \lambda_i \) is
\[ \lambda_{i,0} = \frac{\kappa_i^2 (T - t)^{2l_i + 2}}{(2l_i + 2)^2}. \]

Similarly discuss about
\[ \int_{R^5} H_i(y_i, t) W_j(y_i) dy_i = 0, \quad i = 1, 2, \quad j = 1, 2, 3, 4, 5, \]
we obtain
\[ |\xi_i - q_i| = O(\lambda_i^{\frac{3}{2}}), \quad i = 1, 2. \]

### 3.2. The selection of \( \lambda_i \) and \( \xi_i \)

In this section, we follow a similar approach to [6], but we provide an explicit formula for \( \gamma_i \). Recall the orthogonal condition
\[ \int_{B_{2R}} H_i W_j dy_i = 0, \quad i = 1, 2, \quad j = 1, 2, 3, 4, 5, 6, \]
For given \( \psi \), define \( \lambda_i, \xi_i \) as the unique solution of the above orthogonal condition. We require that
\[ \lambda_i(T) = 0, \quad \xi_i(T) = q_i. \]  \hfill (24)

We can divide \( \lambda_i \) into
\[ \lambda_i = \lambda_{i,0} + \lambda_{i,1}. \]

Since \( \eta_i \) is supported around 0 or \( q \), and it has larger support than that of \( y_i \in B_{2R} \), the orthogonal condition can be rewrite as
\[ (\dot{\lambda}_{i,0} + \dot{\lambda}_{i,1}) \int_{B_{2R}} W_6^2(y_i) dy_i - \lambda_i^\frac{3}{2} \int_{B_{2R}} pU(y_i)^{p-1} W_6(y_i)(Z_i - \psi) dy_i = 0. \]  \hfill (25)

By the choosing of \( \lambda_{i,0} \), we can obtain
\[ \dot{\lambda}_{i,0} + \kappa_i (T - t)^{l_i} \lambda_i^{\frac{3}{2}} = 0. \]

By substituting this equation into equation 25, we get
\[ \dot{\lambda}_{i,1} \int_{R^5} W_6^2(y_i) dy_i - \lambda_i^\frac{3}{2} \int_{R^5} pU^{p-1}(y_i) W_6(y_i)(T - t)^{l_i} dy_i - \lambda_i^\frac{3}{2} \int_{R^5} Z_6^2(y_i)(\dot{\lambda}_{i,0} + \dot{\lambda}_{i,1}) dy_i - \lambda_i^\frac{3}{2} \int_{B_{2R}} pU(y_i)^{p-1} W_6(y_i)[(T - t)^{l_i} (\epsilon_i^* (z_i) - 1) - \psi] dy_i = 0. \]
That is
\[
\hat{\lambda}_{i,1} \int_{R^5} W_0^2(y_i)dy_i + \frac{(T-t)^{l_i}}{(\lambda_{i}^{1/2} + \lambda_{i,0}^{1/2})} \lambda_{i,1}
\times \int_{R^5} pU^{p-1}(y_i)W_0(y_i)dy_i - \int_{R^5/B_2R} W_0^2(y_i)(\hat{\lambda}_{i,0} + \hat{\lambda}_{i,1})dy_i
\]
\[
- \lambda_{i}^{1/2} \int_{B_{2R}} pU(y_i)^{p-1}W_0(y_i)[(T-t)^{l_i}(e_i^*(z_i) - 1) - \psi]dy_i = 0. \tag{26}
\]

Because in \(y_i \in B_{2R}\), \(|z_i| \leq 2R(T-t)^{2l_i + 3/2}\), then \(|e_i^*(z_i) - 1| = O(|z_i|^2)
\[
\lambda_{i}^{1/2} \int_{B_{2R}} pU(y_i)^{p-1}W_0(y_i)[(T-t)^{l_i}(e_i^*(z_i) - 1)]dy_i \lesssim \lambda_{i}^{1/2}(T-t)^{3l_i-1}. \tag{27}
\]

By using inequality 27 and the definition of \(\psi\) in 4, we can write Eq. 26 as follows:
\[
\hat{\lambda}_{i,1} \int_{R^5} W_0^2(y_i)dy_i + \frac{3}{2} \int_{R^5} U^{p}(y_i)dy_i(T-t)^{l_i} \frac{\lambda_{i,1}}{2\lambda_{i,0}^{1/2}}
\]
\[
+ \delta_{R,T,\delta_0}^i(\hat{\lambda}_{i,0} + \hat{\lambda}_{i,1} + (T-t)^{l_i}\lambda_{i}^{1/2})
\]
\[
- \frac{3}{2} \int_{R^5} U(y_i)^{p}dy_i(T-t)^{l_i} \frac{\lambda_{i,1}^{2/3}}{2\lambda_{i,0}^{1/2}(\lambda_{i}^{1/2} + \lambda_{i,0}^{1/2})^2} = 0, \tag{28}
\]

where when \(R \to \infty\), \(T \to 0\), \(\delta_0 \to 0\), \(\delta_{R,T,\delta_0}^i \to 0\),

we let
\[
\gamma_i := \frac{3 \int_{R^5} U^{p}(y_i)dy_i(T-t)^{l_i+1}}{2\lambda_{i,0}^{1/2} \int_{R^5} W_0^2(y_i)dy_i},
\]

from the definition of \(\lambda_{i,0}\) and \(\kappa_i\), we can easily compute \(\gamma_i\) as
\[
\gamma_i = l_i + 1 \tag{29}
\]

In order to solve for \(\lambda_{i,1}\), we need the following theorem.

**Theorem 7.** For \(\gamma_i = l_i + 1\), consider
\[
\begin{cases}
\hat{\lambda} + \gamma_i \lambda_{T-t} = (T-t)^{2l_i+1}h, \\
\lambda(T) = 0.
\end{cases} \tag{30}
\]

If \(||h||_{0,\alpha} < \infty, 0 < \alpha < 1\), then 30 has a solution \(\lambda\), which can be write as
\[
\lambda = -(T-t)^{\gamma_i} \int_t^T (T-s)^{2l_i+1-\gamma_i}h(s)ds, \tag{31}
\]

and there exists constant \(C_2\) satisfies
\[
||\lambda||_{1,\alpha} \leq C_2 ||h||_{0,\alpha}. \tag{32}
\]

**Proof.** In fact, since \(\gamma_i = l_i + 1\), the integral \(\int_t^T (T-s)^{2l_i+1-\gamma_i}h(s)ds = \int_t^T (T-s)^{l_i}h(s)ds\) is integrable, from which we can infer that
\[
(T-t)^{\gamma_i} \int_t^T (T-s)^{2l_i+1-\gamma_i}h(s)ds
\]
is well defined. It can be directly verified that the $\lambda$ we construct in Eq. 31 satisfies the Eq. 30. Then we proceed to check 32. Using Eq. 31,

$$\dot{\lambda} = -\gamma_i \frac{\lambda}{T-t} + (T-t)^{2i+1}h,$$

where $(T-t)^{2i+1}h$ is in $C^\alpha$. By 31

$$-\gamma_i \frac{\lambda}{T-t} = \gamma_i (T-t)^{\gamma_i-1} \int_t^T (T-s)^{2i+1-\gamma_i}h(s)ds,$$

use $\gamma_i = l_i + 1$, we can easily verify that it is in $C^1$ and

$$\left| \frac{d[(T-t)^{\gamma_i-1} \int_t^T (T-s)^{2i+1-\gamma_i}h(s)ds]}{dt} \right| \leq ||h||_\infty,$$

then Theorem 7 follows immediately.

After proving Theorem 7, we can rewrite 28 as a linear equation about $\overline{\lambda}_{i,1}$

$$\dot{\overline{\lambda}}_{i,1} + \frac{\gamma_i}{(T-t)} \overline{\lambda}_{i,1} + \delta_{R,T,\delta_0}(\hat{\lambda}_{i,0} + \dot{\lambda}_{i,1} + (T-t)\lambda_i^\frac{1}{2})$$

$$- \int_{R^3} U(y_i) \rho_i d\gamma(t) - \frac{\lambda_i^2}{2\lambda_i,0(\lambda_i^\frac{1}{2} + \lambda_i^0)^2} = 0. \quad (33)$$

Then for given $\lambda_{i,1} \in C^{1,\alpha}$ with $|\lambda_{i,1}| \leq \delta|\lambda_{i,0}|$, $|\dot{\lambda}_{i,1}| \leq \delta|\dot{\lambda}_{i,0}|$, $\delta$ small enough. By Theorem 7, we can solve $\overline{\lambda}_{i,1}$ with the following estimate

$$||\overline{\lambda}_{i,1}||_{1,\alpha} \leq C_1(\delta_{R,T,\delta_0} + ||\lambda_{i,1}||_{1,\alpha})\lambda_{i,0}.$$

By choosing $R$ to be sufficiently large, $T$ to be sufficiently small, $\delta_0$ and $\delta$ to be sufficiently small, we can use the fixed point theorem to solve for $\lambda_i$. In addition, for $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,5})$, and $\xi_{i,j}$, defined by

$$\int_{B_{2R}} H_i(y_i,t)W_j(y_i)dy_i = 0, \quad i = 1, 2, \quad j = 1, 2, 3, 4, 5, 6.$$

Since $Z_1, Z_2$ are given, then for fixed $\psi$, after solve $\overline{\lambda}_{i,1}(\psi, \lambda_{1,1}, \lambda_{2,1}, \xi_1, \xi_2)$, we can solve for $\tilde{\xi}_i = (\xi_i, \lambda_{1,1}, \lambda_{2,1}, \xi_1, \ldots, \xi_k)$ from the above equations, and we can use a similar approach as for $\lambda_i$ to obtain $|\xi_i - q_i| \lesssim \lambda_i^\frac{3}{2}$. \qed

4. The inner problem

In this section, we use Theorem 2 to obtain an estimate of $\phi_1$ and $\phi_2$. Recall the inner problem

$$\begin{aligned}
\lambda_i^2 \partial_t \phi_i &= \Delta y_i \phi_i + pU_p^{-1}(y_i) + H_i(\psi, \lambda_i, \xi_i) \quad \text{in } (y_i,t) \in B_{2R}(0) \times (0,T), \\
\phi_i(y_i,0) &= \phi_{i,0} \quad \text{for } y_i \in B_{2R}.
\end{aligned}$$

Where

$$H_i(\psi, \lambda_i, \xi_i) := \lambda_i^\frac{3}{2}pU(y_i)p^{-1}(-Z_i(\xi_i + \lambda_i y_i) + \psi(\xi_i + \lambda_i y_i)) + E_i(y_i,t),$$
For given \( \psi \) satisfies inequality 4, we have

\[
H_i(\psi) \lesssim \lambda_i^{\frac{3}{2} + \frac{l_i}{2l_i+2}} \frac{1}{1 + |y_i|^3} \quad \text{in } y_i \in B_2 R.
\]

Notice that for any \( a \in (0, 1) \), the decay of \( H_i(y_i, t) \) is faster than \( \frac{1}{1 + |y_i|^3 + a} \).

Also due to the choice of \( \lambda_i \) and \( \xi_i \), \( H_i \) satisfies the orthogonality condition.

Then by Theorem 2, we obtain

**Proposition 8.** For any \( a \in (0, 1) \)

\[
\| \phi_i(y_i, t) \|_{2 + a, \frac{3}{2} + \frac{l_i}{2l_i+2}} \leq C \| H_i(y_i, t) \|_{2 + a, \frac{3}{2} + \frac{l_i}{2l_i+2}, i = 1, 2}.
\]

This inequality can be rewritten as

\[
|\phi_i(y_i, t)| \lesssim (T - t)^{l_i} \lambda_i^{\frac{3}{2}} \frac{R^{6 - a}}{1 + |y_i|^6}. \tag{34}
\]

5. The outer problem

To solve the outer problem, we need to approach it as a linear problem given by:

\[
\begin{aligned}
\partial_t \psi &= \Delta_x \psi + g(\psi) \quad \text{in } (x, t) \in \mathbb{R}^5 \times (0, T), \\
\psi(\cdot, 0) &= \psi_0 \quad \text{for } x \in \mathbb{R}^5. 
\end{aligned} \tag{35}
\]

Our objective in this section is to prove that \( \psi \) has the following estimates

\[
|\psi| \leq \begin{cases} 
\delta_0 (T - t)^{l_i} (1 + |z_i|^{2l_i+2}) & \text{for } |z_i| \leq (T - t)^{-\frac{l_i}{2l_i+2}}, \quad i = 1, 2, \\
\delta_0 & \text{else},
\end{cases} \tag{36}
\]

where \( \delta_0 = \sup_{i=1,2} \frac{\|Z_i\|_{\infty}}{10} \). We introduce function space \( B \) equipped with the norm

\[
\| \psi \|_B := \sup_{|z_1| \leq (T - t)^{-\frac{l_1}{2l_1+2}}, t \in [0, T)} |\psi(x, t)|(T - t)^{-l_1} (1 + |z_1|^{2l_1+2})^{-1} \\
+ \sup_{|z_2| \leq (T - t)^{-\frac{l_2}{2l_2+2}}, t \in [0, T)} |\psi(x, t)|(T - t)^{-l_2} (1 + |z_2|^{2l_2+2})^{-1} \\
+ \sup_{|z_1| \geq (T - t)^{-\frac{l_1}{2l_1+2}}, |z_2| \geq (T - t)^{-\frac{l_2}{2l_2+2}}, t \in [0, T)} |\psi(x, t)|(1 + |x|^3).
\]

From the classic parabolic theorem, we know that for any given \( \psi \) satisfies \( \| \psi \|_B < \infty \), Eq. 35 has a unique solution \( \psi_0 \). And we define

\[
\psi_0 = T[\psi].
\]
5.1. Choosing of the parameters

We choose the initial data of this form
\[ \psi_0 = (d_1 \cdot e_1) \eta \left( \frac{z_1}{e^{b_1 \tau_0}} \right) + (d_2 \cdot e_2) \eta \left( \frac{z_2}{e^{b_2 \tau_0}} \right) + \tilde{\varphi}, \]  
(37)

where \( d_i = (d_0^i, d_1^i, \ldots, d_{m_i}^i) \) is a parameter and \( e_i = (e_0^i(z_i), e_1^i(z_i), \ldots, e_{m_i}^i(z_i)) \) is chosen so that \( e_0^i(z_i), e_1^i(z_i), \ldots, e_{m_i}^i(z_i) \) are the eigenfunctions of \(-A_{z_i}\) with eigenvalue less than or equal 2\( l_i \), \( \tilde{\varphi} \) is a function with compact support which will be determined. We use the self-similar variable
\[ \varphi(z_1, z_2, \tau) = \psi_s(x, t), \quad T - t = e^{-\tau}. \]

The function \( \varphi(z_1, z_2, \tau) \) solves
\[
\begin{cases}
\varphi_{\tau} = A_{z_i} \varphi + e^{-\tau} G(\phi, \psi, \lambda_1, \lambda_2, \xi_1, \xi_2) & \text{for } (z_i, \tau) \in \mathbb{R}^5 \times [\tau_0, \infty), \quad i = 1, 2 \\
\varphi_{|\tau=\tau_0} = (d_1 \cdot e_1) \eta \left( \frac{z_1}{e^{b_1 \tau_0}} \right) + (d_2 \cdot e_2) \eta \left( \frac{z_2}{e^{b_2 \tau_0}} \right) + \tilde{\varphi}.
\end{cases}
\]

We decompose the initial data \( (d_i \cdot e_i) \eta \left( \frac{z_i}{e^{b_i \tau_0}} \right) \) in the eigenspace of \( A_i \) into two parts, \( Y_i := \text{span}\{e_0(z_i), e_1(z_i), \ldots, e_{m_i}(z_i)\} \) and it is orthogonal complement,

\[
(d_i \cdot e_i) \eta \left( \frac{z_i}{e^{b_i \tau_0}} \right) = \sum_{j, k=1}^{m_i} d_j \left( e_j \eta \left( \frac{z_i}{e^{b_i \tau_0}} \right), e_k \right) e_k + \left\{ (d_i \cdot e_i) \eta \left( \frac{z_i}{e^{b_i \tau_0}} \right) \right\}^\perp, \quad i = 1, 2,
\]

especially
\[
\left\{ (d_i \cdot e_i) \eta \left( \frac{z_i}{e^{b_i \tau_0}} \right) \right\}^\perp, \; e_j = 0, \quad i = 1, 2, \quad j = 0, 1, \ldots, m_i, \quad \tau = \tau_0.
\]

Recall Eq. 10 of the definition of \( G \) and we separate \( G \) into 3 parts, which are denote as
\[ G_i = p\lambda_i^{-2}(1 - \eta_{R,i})U(y_i)^{p-1} \tilde{\eta}_i^{p-1}(-Z_i \eta_i + \psi) + A[\phi_i] + B[\phi_i] + \lambda_i^{-\frac{2}{\nu}} E_i(1 - \eta_{R,i}) \tilde{\eta}_1 + N \tilde{\eta}_1. \]

for \( i = 1, 2 \), and denote \( G_3 = G - G_1 - G_2 \). Let \( 1_{z \in \Omega}(z) \) be a function on \( \mathbb{R}^5 \) defined by \( 1_{z \in \Omega}(z) = 1 \) if \( z \in \Omega \) and \( 1_{z \in \Omega}(z) = 0 \) if \( z \notin \Omega \). We define \( \varphi \) as
\[ \varphi = \Phi_1 + \Phi_2 + \Phi_3, \]
where \( \Phi_1 = b_1(\tau) \cdot e_1 + \Phi_1, \Phi_2 = b_2(\tau) \cdot e_2 + \Phi_2, \) and \( b_i(\tau) = (b_0^i(\tau), \ldots, b_{m_i}^i(\tau)). \)
Choose \( b_k^i(\tau) \) as
\[
b_k^i(\tau) = -e^{-\lambda_k \tau} \int_{\tau}^{\infty} e^{(\lambda_k - 1)\tau'} (G_i, e_k(z_i))_P d\tau', \quad i = 1, 2, \quad k = 0, 1, \ldots, m_i,
\]
(39)
then $\tilde{\Phi}_1, \tilde{\Phi}_2$, satisfies

$$\partial_\tau \tilde{\Phi}_i = A z_i \tilde{\Phi}_i + e^{-\tau} G_i - \sum_{k=0}^{m_i} \lambda_k b_k^i e_k - \sum_{i=1}^m \frac{db^i}{d\tau} e_k := A z_i \tilde{\Phi}_i + e^{-\tau} G_i^\perp, \quad i = 1, 2,$$

(40)

and

$$\begin{cases} 
\partial_t \Phi_3 - \Delta \Phi_3 = G_3, \quad (x, t) \in \mathbb{R}^n \times (0, T) \\
\Phi_3|_{t=0} = \bar{\varphi}.
\end{cases}$$

(41)

By Theorem 9 we proved later, it is easy to see

$$|(G_i, e_k(z_i))_\rho| \lesssim e^{-2l_i \tau}.$$  

(42)

From the definition of $d_i$ we have

$$(I + D_i) d_i = b_i(\tau_0),$$

(43)

where $D_i$ is a $(m_i + 1) \times (m_i + 1)$ matrix defined by

$$D^i_{j,k} = (e_j(z_i), \left(1 - \eta \left(\frac{z_i}{e^{b_i \tau_0}} \right) e_k(z_i)\right)_\rho, \quad i = 1, 2 \quad j, k = 0, 1, \ldots, m_i.$$  

(44)

For $\tau_0$ sufficiently large, $|D^i_{j,k}| << 1$, then

$$|d_i| \lesssim |b_i(\tau_0)| \lesssim e^{-(2l_i + 1) \tau_0},$$

(45)

since $e_i$ are given eigenfunctions with eigenvalues less than or equal to $2l_i$ of $A z_i$, from 4, we deduce a basic estimate:

$$|b_k^i e_k| \lesssim e^{-(2l_i + 1) \tau_0} (1 + |z_i|^{2l_i}), \quad k = 0, 1 \ldots, m_i.$$  

(46)

5.2. The estimate of $G_i$

We give an estimate of $G_i$ in this subsection. Let $R = \tau_0$.

**Theorem 9.** Under the above notation, we have

$$|G_i| \lesssim \lambda_i^{-2} \frac{1}{1 + |y_i|^{2+\alpha}} e^{-l_i \tau} R^{-1} 1_{|x-q_i| \leq \frac{\delta}{4}} + e^{-\frac{2}{l_i} \tau} 1_{|z_i| \leq 1} + \delta_p^p 1_{e^{\frac{l_i}{2\tau}} \leq |z_i| \leq e^\frac{\tau}{2}}$$

$$+ e^{-\left(1 - \frac{1}{2l_i+2} \tau\right)} |z_i|^{2l_i+2} 1_{e^{b_i \tau} \leq |z_i| \leq 2e^{b_i \tau}} + e^{-2l_i \tau} |z_i|^{4l_i+4} 1_{1 \leq |z_i| \leq e^{\frac{l_i}{2\tau}}},$$

$$|G_3| \lesssim \left(\frac{\delta_p^p}{1 + |x-q_1|^{3p}} + \frac{e^{-2\tau}}{1 + |x-q_1|^{15}}\right) 1_{|z_1| \geq \frac{4}{7} e^\frac{\tau}{2}} 1_{|z_2| \geq \frac{4}{7} e^\frac{\tau}{2}}.$$
Proof. We can estimate the terms of G one by one.

\[
|\lambda_i^{-2}(1 - \eta_{R,i})U(y_i)^{p-1}\tilde{\eta}_i(-Z_i\eta_i + \psi)|
\]
\[
\lesssim \lambda_i^{-2}(1 - \eta_{R,i})\frac{1}{y_i^a}e^{-l_i\tau}(1 + |z_i|^2l_i)1_{|z_i| \leq 1}
\]
\[
\quad + \lambda_i^{-2}(1 - \eta_{R,i})\frac{1}{y_i^a}e^{-(2l_i - \frac{1}{2})\tau}1_{1 \leq |z_i| \leq \frac{3}{4}e^{\frac{1}{2}}}
\]
\[
\lesssim \lambda_i^{-2}\frac{1}{y_i^a}(1 - \eta_{R,i})e^{-l_i\tau}1_{|z_i| \leq 1} + \lambda_i^{-2}(1 - \eta_{R,i})\frac{1}{y_i^a}e^{-(2l_i - \frac{1}{2})\tau}1_{1 \leq |z_i| \leq \frac{3}{4}e^{\frac{1}{2}}}
\]
\[
\lesssim \lambda_i^{-2}\frac{1}{1 + |y_i|^{2+a}}e^{-l_i\tau}R^{-1}1_{|z_i| \leq 1}
\]
\[
\quad + \lambda_i^{-2}\frac{1}{1 + |y_i|^{2+a}}e^{-(2l_i - \frac{1}{2})\tau}R^{-1+a}1_{1 \leq |z_i| \leq \frac{3}{4}e^{\frac{1}{2}}}
\]

From Proposition 8,

\[
|\phi_i(y_i, t)| + |(1 + |y_i|)\nabla y_i\phi_i(y_i, t)| \leq C_i(T - t)^{l_i}\lambda_i^2 \frac{R^{6-a}}{1 + |y_i|^6}, \quad i = 1, 2
\]

therefore

\[
|A[\phi_i]| = |\lambda_i^{-\frac{a}{2}}[\Delta_{y_i} \eta_{R,i} \phi_i + 2\nabla_y \eta_{R,i} \nabla y_i \phi_i]|
\]
\[
\lesssim \lambda_i^{-2}e^{-l_i\tau}R^{-2-a}1_{R \leq |y_i| \leq 2R}
\]
\[
\lesssim \lambda_i^{-2}\frac{1}{1 + |y_i|^{2+a}}e^{-l_i\tau}R^{-\frac{3}{2}}1_{R \leq |y_i| \leq 2R}
\]

and

\[
|B[\phi_i]| = |\lambda_i^{-\frac{a}{2}}[\lambda_i(y_i \cdot \nabla y_i \phi_i + \frac{3}{2}\phi_i)\eta_{R,i} + \xi_i \cdot \nabla y_i \phi_i \eta_{R,i} + (\lambda_i y_i \cdot \nabla y_i \eta_{R,i} + \xi_i \cdot \nabla y_i \eta_{R,i})\phi_i]|
\]
\[
\lesssim \frac{\lambda_i^2 \lambda_i R^{6-a} \eta_{R,i}}{\lambda_i^2(1 + |y_i|^6)}1_{|y_i| \leq 2R}
\]

Use the definition of $E_1$, we have

\[
\lambda_i^{-\frac{a}{2}}E_i(1 - \eta_{R,i})\tilde{\eta}_i
\]
\[
= \lambda_i^{-\frac{a}{2}}\lambda_i \lambda_i \left\{ \left[ \left( \frac{3}{2} U(y_i) + y_i \cdot \nabla U(y_i) \right) \right] \tilde{\eta}_i + \lambda_i \xi_i \cdot \nabla U(y_i) \right\} (1 - \eta_{R,i})\tilde{\eta}_i
\]
\[
\lesssim \lambda_i^{-2}\frac{1}{y_i^3}e^{\tau}1_{|x - q_i| \leq \frac{3}{4}}(1 - \eta_{R,i})1_{|x - q_i| \leq \frac{3}{4}}
\]
\[
+ \lambda_i^{-2}\frac{1}{y_i^3}e^{\tau}1_{|x - q_i| \geq \frac{4}{4}}(1 - \eta_{R,i})1_{|x - q_i| \geq \frac{4}{4}}
\]
\[
\lesssim \lambda_i^{-2}\frac{1}{1 + |y_i|^{2+a}}e^{-l_i\tau}R^{-1}1_{|x| \leq \frac{3}{4}}.
\]
Using the definition of $Z_i$ and $\eta_i$, we can estimate the last part of $N$ as follow

$$\left| \frac{\partial \eta_i}{\partial t}Z_i + \nabla_x Z_i \nabla_x \eta_i - \Delta_x \eta_i Z_i \right| \lesssim e^{\frac{l_i + 2}{2} \tau} 1_{e^{b_i \tau} \lesssim |z_i| \lesssim 2e^{b_i \tau}} e^{-\left(l_i - \frac{1}{2l_i + 2}\right) \tau} |z|^{2l_i + 2} 1_{e^{b_i \tau} \lesssim |z_i| \lesssim 2e^{b_i \tau}}.$$

and

$$|\lambda_i^{-\frac{5}{2}} \nabla_y U(y_i) \nabla_x \eta_i + \lambda_i^{-\frac{3}{2}} U(y_i) \Delta_x \eta_i| \lesssim \frac{\lambda_i}{\lambda_i^2 (1 + |y_i|^{2+a})} 1_{x \sim d}.$$

Now we turn our attention to another part of $N$. If $x$ is near point 0, we have

$$\sum_{i=1}^{2} U_{\lambda_i, \xi_i, \eta_i} + \theta - \sum_{i=1}^{2} Z_i \eta_i - p U_{p-1, \lambda_i, \eta_i}^{p-1} (\theta - Z_1 \eta_1 - Z_2 \eta_2) - p U_{p-1, \lambda_i, \eta_i}^{p-1} (\theta - Z_1 \eta_1 - Z_2 \eta_2)$$

$$= |U_{\lambda_i, \xi_i, \eta_i} + \theta - Z_1 \eta_1 - Z_2 \eta_2|^{p-1} (U_{\lambda_i, \xi_i, \eta_i} + \theta - Z_1 \eta_1 - Z_2 \eta_2) - p U_{p-1, \lambda_i, \eta_i}^{p-1} (\theta - Z_1 \eta_1 - Z_2 \eta_2)$$

$$+ p U_{p-1, \lambda_i, \eta_i}^{p-1} (\theta - Z_1 \eta_1 - Z_2 \eta_2) - p U_{p-1, \lambda_i, \eta_i}^{p-1} (\theta - Z_1 \eta_1 - Z_2 \eta_2)$$

$$\lesssim U_{p-2, \lambda_i, \xi_i, \eta_i} + (U_{\lambda_i, \xi_i, \eta_i} + \theta - Z_1 \eta_1 - Z_2 \eta_2) \eta_i$$

$$+ U_{p, \lambda_i, \xi_i, \eta_i}^{p} - U_{p, \lambda_i, \xi_i, \eta_i}^{p}.$$

We can use the estimate we obtained for $\phi_i$ and deduce that

$$U_{\lambda_i, \xi_i}^{p+2} (\lambda_i^{-\frac{3}{2}} \phi_i(y_i, t) \eta_R(y_i))^{2} \eta_i \lesssim \lambda_i^{-\frac{3}{2}} \frac{1}{1 + |y_i|^{12} |y_i|^{12}} 1_{|z_i| \lesssim 1}$$

$$\lesssim \lambda_i^{-2} \frac{1}{1 + |y_i|^{2a}} e^{-l_i \tau} R^{-1} 1_{|z_i| \lesssim 1}.$$

Moreover, we also have

$$U_{\lambda_i, \xi_i}^{p-2} (Z_i^{2} \eta_i^{2} + \Psi^2) \eta_i \lesssim \lambda_i^{-2} \frac{1}{1 + |y_i|^{2a}} e^{-2l_i \tau} (1 + |y_i|^{1+a}) 1_{|z_i| \lesssim 1}$$

$$+ e^{-\left(l_i + \frac{1}{2}\right) \tau} e^{-2l_i \tau} |z_i|^{4l_i + 4} 1_{1 \leq |z_i| \leq e^{\frac{l_i}{2l_i + 2} \tau}}$$

$$+ e^{-\left(l_i + \frac{1}{2}\right) \tau} e^{-2l_i \tau} |z_i|^{4l_i + 4} 1_{e^{\frac{l_i}{2l_i + 2} \tau} \leq |z_i| \leq e^{b_i \tau}}$$

$$\lesssim \lambda_i^{-2} \frac{1}{1 + |y_i|^{2a}} e^{-2l_i \tau} R^{-1} 1_{|z_i| \lesssim 1}$$

$$+ e^{-\left(3l_i + \frac{1}{2}\right) \tau} (1 + |z_i|^{4l_i + 4}) 1_{1 \leq |z_i| \leq e^{\frac{l_i}{2l_i + 2} \tau}}$$

$$+ e^{-\left(3l_i + \frac{1}{2}\right) \tau} |z_i|^{4l_i + 4} 1_{e^{\frac{l_i}{2l_i + 2} \tau} \leq |z_i| \leq e^{b_i \tau}}.$$
and
\[(\theta - Z_i \eta_i)^p \lesssim \lambda_i^{-2} \lambda_i^2 e^{-p_i \tau} \frac{1}{1 + |y_i|^{2+a}} 1_{|y_i| \leq 2R} + e^{-\frac{2}{3} l_i \tau} 1_{|z_i| \leq 1} + e^{-\frac{2}{3} l_i \tau} |z_i|^{\frac{14}{3}} (l_i+1) 1_{1 \leq |z_i| \leq e^{\frac{l_i}{3} \tau}}
\]
\[+ e^{-\frac{2}{3} l_i \tau} |z_i|^{\frac{14}{3}} l_i e^{\frac{l_i}{3} \tau} \lesssim |z_i| \leq 2e^{l_i \tau} + \delta^p 1_{e^{\frac{l_i}{3} \tau} \leq |z_i| \leq e^\tau},
\]
and for \(|x| \geq \frac{d}{4}, |x - q| \geq \frac{d}{4}\)
\[
\left| \sum_{i=1}^2 U_{\lambda_i, \xi_i} \tilde{\eta}_i + \theta - \sum_{i=1}^2 Z_i \eta_i \right|^{p-1} \left( \sum_{i=1}^2 U_{\lambda_i, \xi_i} \tilde{\eta}_i + \theta - \sum_{i=1}^2 Z_i \eta_i \right) - \sum_{i=1}^2 U_{\lambda_i, \xi_i} \tilde{\eta}_i
\]
\[= |\theta|^{p-1} \theta - pU_{\lambda_1, \xi_1}^{p-1} \theta - pU_{\lambda_2, \xi_2}^{p-1} \theta
\]
\[\lesssim \left( \frac{1}{1 + |x|^{3p}} + \frac{1}{1 + |x|^{5}} \right) 1_{|z_1| \geq \frac{d}{4} e^\tau} 1_{|z_2| \geq \frac{d}{4} e^\tau}.
\]
In conclusion, we complete the proof. 

By using Theorem 9, we can easily deduce that

**Proposition 10.** The function \(G_i\) given by (10) satisfies
\[
\|G_i\|_{\rho_i} \lesssim e^{-(2l_i - \frac{1}{2}) \tau}.
\]

**Proof.** Recalling that \(y_i = \frac{x - \xi_i}{\lambda_i}, z_i = \frac{x - q_i}{\sqrt{T - t}},\) we can write
\[
z_i = \frac{(y_i \lambda_i + \xi_i - q_i)}{\sqrt{T - t}},
\]
then we have
\[
\left| \int_{R^5} \left( \lambda_i^{-2} \frac{1}{1 + |y_i|^{2+a}} e^{-l_i \tau} R^{-\frac{1}{4}} 1_{|x| \leq 1} \right)^2 e^{-\frac{|z_i|^2}{4}} y_i \right|
\]
\[= \int_{R^5} \left( \lambda_i^{-2} \frac{1}{1 + |y_i|^{2+a}} e^{-l_i \tau} R^{-\frac{1}{4}} \right)^2 e^{-\frac{|z_i|^2}{4}} \left( \frac{\lambda_i}{\sqrt{T - t}} \right)^5 y_i
\]
\[\lesssim \lambda_i e^{-2l_i \tau} e^{\frac{5}{2} \tau}
\]
\[\lesssim e^{-(2l_i - \frac{1}{2}) \tau}.
\]
And
\[
\left| \int_{R^5} (e^{-p_i \tau} 1_{|z_i| \leq 1} dz_i \right| \lesssim e^{-2l_i \tau}.
\]
For those terms in \(|z_i| \geq 1\), it is easy to verify. Therefore
\[
\|G_i\|_{\rho_i} \lesssim e^{-(2l_i - \frac{1}{2}) \tau}.
\]
5.3. The solving of $\tilde{\Phi}_{i,1}$

For $i = 1, 2$, we divide $\tilde{\Phi}_i$ into two parts, one part comes from the initial data, another comes from nonhomogeneous term, consider

$$
\begin{align*}
\left\{ \begin{array}{l}
\partial_\tau \tilde{\Phi}_{i,1} = A_{z_i} \tilde{\Phi}_{i,1} \\
\tilde{\Phi}_{i,1} |_{\tau = \tau_0} = \{(d_i \cdot e_i) \eta(z_i e^{b_i \tau})\}^\perp
\end{array} \right. \\
\text{for } (z_i, \tau) \in \mathbb{R}^5 \times [\tau_0, \infty),
\end{align*}
$$

(48)

**Theorem 11.** There exists constant $K_{i,1}$, such that

$$
|\tilde{\Phi}_{i,1}(z_i, \tau)| \leq K_{i,1} e^{-(l_i+1)\tau}(1 + |z_i|^{2l_i+2}), \quad \text{for } \tau > \tau_0.
$$

(49)

**Proof.** First we estimate the initial data

$$
\left\{ \begin{array}{l}
(d \cdot e) \eta(z_i e^{b_i \tau}) \end{array} \right\}^\perp = (d \cdot e) \eta(z_i e^{b_i \tau}) - \sum_{j,k=1}^{m_i} d_j(e_j \eta(z_i e^{b_i \tau}), e_k)\rho_i e_k,
$$

as in the discussion of the last section, we can deduce that

$$
\left\| \left\{ \begin{array}{l}
(d \cdot e) \eta(z_i e^{b_i \tau}) \end{array} \right\}^\perp \right\|_{\rho_i} \leq e^{-(2l_i+1)\tau_0} + e^{-(2l_i+1)\tau_0} |z_i|^{2l_i},
$$

and

$$
\left\| \left\{ \begin{array}{l}
(d \cdot e) \eta(z_i e^{b_i \tau}) \end{array} \right\}^\perp \right\|_{\rho_i} \leq e^{-(2l_i+1)\tau_0}.
$$

We easily deduced that

$$
\frac{1}{2} \partial_\tau \|\tilde{\Phi}_{i,1}\|_{\rho_i}^2 = (A_{z_i} \tilde{\Phi}_{i,1}, \tilde{\Phi}_{i,1}) \leq -(l_i + \frac{1}{2})\|\tilde{\Phi}_{i,1}\|_{\rho_i}^2.
$$

which is

$$
\partial_\tau (\|\tilde{\Phi}_{i,1}\|_{\rho_i}^2 e^{(2l_i+1)\tau}) \leq 0.
$$

Integrating it from $\tau_0$ to $\tau$

$$
\|\tilde{\Phi}_{i,1}\|_{\rho_i} \leq \|\{d \cdot e\} \eta(z_i e^{b_i \tau})\|_{\rho_i} e^{-\frac{2l_i+1}{2}(\tau-\tau_0)} \leq e^{-l_i \tau_0} e^{-(l_i+\frac{1}{2})\tau}.
$$

Let $r_{i,0}$ be the constant such that $e_{l_i+1} > \frac{a_{l_i+1}}{2} |z_i|^{2l_i+2}$ for $|z_i| \geq r_{i,0}$. Using the classical $L^2$ estimate of parabolic equation and the definition of $\rho_i$ we have, for $\tau > \tau_0 + 1$, $|z_i| < r_{i,0}$

$$
\|\tilde{\Phi}_{i,1}(z_i, \tau)\| \leq \|\tilde{\Phi}_{i,1}\|_{2, \infty} \leq e^{-l_i \tau_0} e^{-(l_i+\frac{1}{2})\tau}.
$$

We will now construct a comparison function given by $K_{i,1} e^{-(l_i+\frac{1}{2})\tau} e_{l_i}(z_i)$. In fact, if $K_{i,1}$ is sufficiently large, we can prove that it is a supersolution of $\tilde{\Phi}_{i,1}$.

When $|z_i| = r_{i,0}$

$$
|\tilde{\Phi}_{i,1}| < K_{i,1} e^{-(l_i+\frac{1}{2})\tau} e_{l_i}(z_i).
$$
For $\tau_0 \leq \tau \leq \tau_0 + 1$, by theorem 4 and comparison principle, we obtain

$$
\left| \tilde{\Phi}_{i,1} \right| = \left| e^{A_i(\tau-\tau_0)} \left( (d_i \cdot e_i) \eta \left( \frac{z_i}{c_i^\tau} \right) \right) \right| \leq e^{A_i(\tau-\tau_0)} e^{-(2l_i+1)\tau_0} + e^{-(2l_i+1)\tau} |z_i|^{2l_i} \leq e^{-(2l_i+1)\tau_0} + e^{-(2l_i+1)\tau} e^{-l_i(\tau-\tau_0)} |z_i|^{2l_i}.
$$

Therefore when $\tau = \tau_0 + 1$

$$
\left| \tilde{\Phi}_{i,1}(z_i, \tau_0 + 1) \right| \leq e^{-(2l_i+1)\tau_0} + e^{-(2l_i+1)\tau} |z_i|^{2l_i} \leq K_{i,1} e^{-(l_i+\frac{1}{2})\tau} e_i^*(z_i).
$$

Notice that $e^{-(l_i+\frac{1}{2})\tau} e_i^*(z_i) = e^{\frac{1}{2}\tau} e^{-(l_i+\frac{1}{2})\tau} e_i^*(z_i)$, we can easily verify

$$
\partial_\tau (K_{i,1} e^{-(l_i+\frac{1}{2})\tau} e_i^*(z_i)) - A_{z_i} (K_{i,1} e^{-(l_i+\frac{1}{2})\tau} e_i^*(z_i)) = \frac{1}{2} K_{i,1} e^{-(l_i+\frac{1}{2})\tau} e_i^*(z_i) \geq \partial_\tau \tilde{\Phi}_{i,1} - A_{z_i} \tilde{\Phi}_{i,1}.
$$

By comparison principle,

$$
\left| \tilde{\Phi}_{i,1} \right| \leq K_{i,1} e^{-(l_i+\frac{1}{2})\tau} e_i^*(z_i).
$$

\[\square\]

5.4. The solving of $\tilde{\Phi}_{i,2}$

Recalling the definition of $\tilde{\Phi}_{i,2}$ from the previous section, we have the equation of $\tilde{\Phi}_{i,2}$

$$
\begin{align*}
\begin{cases}
\partial_\tau \tilde{\Phi}_{i,2} - \Delta \tilde{\Phi}_{i,2} = G_{i}^\perp, & \text{for } (z_i, \tau) \in \mathbb{R}^5 \times [\tau_0, \infty), \\
\tilde{\Phi}_{i,2} \big|_{\tau=\tau_0} = 0.
\end{cases}
\end{align*}
\tag{50}
$$

We will first prove the following theorem.

**Theorem 12.** Let $A_{z_i} = \Delta_{z_i} - \frac{z_i}{2} \cdot \nabla_{z_i}$ be the operator as before, then for any $\tau_0 \leq \tau_1 \leq \tau$ we have

$$
| \int_{\tau_i}^{\tau} e^{A_{z_i}(\tau-\tau')} e^{-(l_i+\frac{1}{2})\tau} G_{i}^\perp | \leq \frac{e^{-(1+\frac{1}{2})\tau_1}}{R^2} + e^{-(1+\frac{1}{2})\tau_1} \left( 1 + |z_i|^{2l_i} \right).
$$

**Proof.** Notice

$$
G_{i}^\perp = G_i - \sum_{k=0}^{m_i} (G_i, e_k) e_k,
$$

and

$$
| b_k e_k | \leq e^{-(2l_i+1)(1 + |z_i|^{2l_i})}, \quad k = 0, \ldots, m_i.
$$

Then

$$
|G_{i}^\perp| \leq \chi_i^{-2} \left( \frac{1}{1 + |y_i|^{2a}} e^{-l_i \tau} R^{-1} 1_{|x-q_i| \leq \frac{R}{4}} + e^{-\frac{7}{2}l_i \tau} 1_{|z_i| \leq 1} + \delta_0 R 1 e^{\frac{i}{\zeta}} \right) e^{-(l_i+\frac{1}{2})\tau} |z_i|^{2l_i} + \frac{1}{e^{2l_i+1}} e^{-(2l_i+1)(1 + |z_i|^{2l_i})} + e^{-2l_i\tau} |z_i|^{4l_i+1} 1_{|z_i| \leq 2e^{h_i\tau}},
$$

where

$$
\begin{align*}
\chi_i^{-2} & = \frac{1}{1 + |y_i|^{2a} e^{-(l_i+\frac{1}{2})\tau} R^{-1} |x-q_i| \leq \frac{R}{4} + e^{-\frac{7}{2}l_i \tau} 1_{|z_i| \leq 1} + \delta_0 R 1 e^{\frac{i}{\zeta}} e^{-(l_i+\frac{1}{2})\tau} |z_i|^{2l_i} + e^{-(2l_i+1)(1 + |z_i|^{2l_i})} + e^{-2l_i\tau} |z_i|^{4l_i+1} 1_{|z_i| \leq 2e^{h_i\tau}},
\end{align*}
$$

and $b_k e_k | \leq e^{-(2l_i+1)(1 + |z_i|^{2l_i})}$.
We estimate the following terms
\[
\delta_0^B \frac{1}{e^{\frac{1}{2l_i+1}} \leq |z_i| \leq e^\frac{4}{2l_i+1}} \leq e^{-(2l_i+2)\tau} |z_i|^{4l_i+4},
\]
denote that
\[
G_i' = e^{-\frac{5}{2}l_i \tau} 1_{|z_i| \leq 1} + \delta_0^B \frac{1}{e^{\frac{1}{2l_i+1}} \leq |z_i| \leq e^\frac{4}{2l_i+1}} + e^{-(l_i - \frac{1}{2l_i+2})\tau} |z_i|^{2l_i+2} 1_{e^{l_i+1} \leq |z_i| \leq 2e^{l_i+1}}
\]
\[
+ e^{-(2l_i+1)\tau} (1 + |z_i|^{2l_i}) + e^{-2l_i\tau} |z_i|^{4l_i+4} 1_{1 \leq |z_i| \leq e^{l_i+1} \tau} + 11 e^{-\frac{2}{2}l_i\tau} \leq |z_i| \leq 2e^{l_i+1}.
\]
Using Theorem 5, we obtain
\[
\int_{\tau_1}^\tau e^{-\Delta (\tau-\tau')} \lambda_i^{-2} \left( 1 + |y_i|^{2+a} \right) e^{-l_i\tau} R^{-1} 1_{|x-q_i| \leq \frac{4}{2} \epsilon} d\tau' \lesssim \frac{e^{-l_i \tau_1}}{R^{\frac{2}{2}}}.
\]
And by Theorem 4, we deduce that
\[
| \int_{\tau_1}^\tau e^{A_{z_i} (\tau-\tau')} e^{-\tau} G_i' d\tau' |
\]
\[
\lesssim \int_{\tau_1}^\tau e^{-\left(\frac{1}{2}l_i\tau\right)} + e^{-\left(l_i+1-\frac{1}{2l_i+2}\right)\tau} (1 + e^{-l_i+1} (\tau-\tau') |z_i|^{2l_i})
\]
\[
+ e^{-(2l_i+1)\tau} (1 + e^{-l_i (\tau-\tau') |z_i|^{2l_i}})
\]
\[
+ \left( e^{-l_i \tau} e^{-\left(l_i+2\right)\tau_1} (1 + |z_i|^{2l_i}) \right)
\]
\[
+ e^{-(2l_i+1)\tau} (1 + |z_i|^{4l_i+4}).
\]
Together with inequality 51 in Theorem 12, we obtain the desired conclusion.

Next, we improve the estimate of Theorem 12 when \( \tau \) is large

**Theorem 13.** Under the assumptions of Theorem 12, we have
\[
| \int_{\tau_0}^\tau e^{A_{z_i} (\tau-\tau')} e^{-\tau} G_i^+ d\tau' |
\]
\[
\lesssim \frac{e^{-l_i \tau}}{R^{\frac{2}{2}}} + e^{-\left(\frac{5}{2}l_i\tau\right)} + e^{-\left(l_i+1-\frac{1}{2l_i+2}\right)\tau} |z_i|^{2l_i+2}
\]
\[
+ \left( e^{-l_i \tau} e^{-\left(l_i+2\right)\tau_1} (1 + |z_i|^{2l_i}) \right) + e^{-(2l_i+1)\tau} (1 + |z_i|^{4l_i+4}).
\]

**Proof.** We separate space-time into four cases:

1. \( z_i \in \mathbb{R}^5, \ \tau \in (\tau_0, \tau_0 + 1] \).
2. \( |z_i| \leq 4, \ \tau \in (\tau_0 + 1, \infty) \).
3. \( 4 \leq |z_i| \leq e^{\frac{r-f}{2}}, \ \tau \in (\tau_0 + 1, \infty) \).
4. \( |z_i| > e^{\frac{r-f}{2}}, \ \tau \in (\tau_0 + 1, \infty) \). (54)

\( z_i \in \mathbb{R}^5, \ \tau \in (\tau_0, \tau_0 + 1] \), we conclude by Theorem 12.
\[ |\int_{\tau_0}^{\tau-1} e^{A_{1i}(\tau-\tau')} e^{-\tau'} G_i^\perp \, d\tau'| = |\int_{\tau_0}^{\tau-1} e^{A_{1i}(\tau-\tau')} e^{-\tau'} G_i^\perp \, d\tau'| \]

\[
\lesssim \int_{\tau_0}^{\tau-1} e^{-\frac{1}{2}l_i^2} e^{-(l_i+\frac{1}{2})\tau} \|e^{-\tau'} G_i^\perp\|_{\rho} \, d\tau' \\
\lesssim \int_{\tau_0}^{\tau-1} e^{\frac{1}{2}l_i} e^{-(l_i+\frac{1}{2})\tau} \, d\tau'
\]

the integral from \( \tau - 1 \) to \( \tau \) can be estimated by Theorem 12, then

\[
\left| \int_{\tau-1}^\tau e^{A_{1i}(\tau-\tau')} e^{-\tau'} G_i^\perp \, d\tau' \right| \lesssim \frac{e^{-\tau'}}{R^2},
\]

4 \leq |z_i| \leq e^{\frac{\tau-\tau_0}{2}}, \tau > \tau_0 + 1. Let \( \tau_2 \) define by \( |z_i| = e^{\frac{\tau-\tau_0}{2}} \), from the definition, \( \tau > \tau_2 \), a similarly calculation shows

\[
\left| \int_{\tau_0}^{\tau_2} e^{A_{1i}(\tau-\tau')} e^{-\tau'} G_i^\perp \, d\tau' \right| \leq \left| \int_{\tau_0}^{\tau_2} e^{\frac{1}{2}l_i} e^{-(l_i+\frac{1}{2})\tau} \|e^{-\tau'} G_i^\perp\|_{\rho} \, d\tau' \right| \\
\lesssim e^{-(l_i+\frac{1}{2})\tau} e^{\frac{1}{4}\tau_0},
\]

and Theorem 12 shows

\[
|\int_{\tau_2}^{\tau-1} e^{A_{1i}(\tau-\tau')} e^{-\tau'} G_i^\perp \, d\tau'|
\]

\[
\leq \frac{e^{-\tau_2}}{R^2} + e^{-(l_i+\frac{7}{6}l_i)\tau_2} + e^{-(l_i+1-\frac{1}{2\tau_i+2})\tau} |z_i|^2 l_i + 2 + (e^{-l_i\tau} e^{-(l_i+2)\tau_2} (1 + |z_i| 2 l_i)) + e^{-(2l_i+1)\tau} (1 + |z_i|^4 l_i + 4)
\]

\[
= \frac{e^{-\tau} |z_i|^2}{R^2} \\
+ e^{-(1+\frac{7}{6}l_i)\tau} |z_i| (2 + \frac{14}{6} l_i) + e^{-(l_i+1-\frac{1}{2\tau_i+2})\tau} |z_i|^2 \tau_i + 2 + (e^{-l_i\tau} e^{-(l_i+2)\tau} |z_i|^2 l_i + 4 (1 + |z_i|^2 l_i)) + e^{-(2l_i+1)\tau} (1 + |z_i|^4 l_i + 4)
\]

\[|z_i| > e^{\frac{\tau-\tau_0}{2}}. \] Similar as before, by Theorem 12 and \( |z_i| > e^{\frac{\tau-\tau_0}{2}} \)

\[
\left| \int_{\tau_0}^{\tau} e^{A_{1i}(\tau-\tau')} e^{-\tau'} G_i^\perp \, d\tau' \right| \\
\lesssim \frac{e^{-\tau} |z_i|^2}{R^2} + e^{-(1+\frac{7}{6}l_i)\tau} |z_i| (2 + \frac{14}{6} l_i) + e^{-(l_i+1-\frac{1}{2\tau_i+2})\tau} |z_i|^2 \tau_i + 2 + (e^{-l_i\tau} e^{-(l_i+2)\tau} |z_i|^2 l_i + 4 (1 + |z_i|^2 l_i)) + e^{-(2l_i+1)\tau} (1 + |z_i|^4 l_i + 4),
\]

combining all these estimates we complete the proof of Theorem 13. \( \square \)
5.5. A pointwise estimate for $\Phi_i$ in $|z_i| > e^{\frac{t_i}{2} + \frac{2}{l_i}}$

The estimate of $\Phi_i$ is local. In this section, we make use of the comparison principle to obtain an estimate in the larger region.

**Theorem 14.** For $i = 1, 2$, there exists constant $K_{i, 2}$ such that in the region $|z_i| > e^{\frac{t_i}{2} + \frac{2}{l_i}}$

$$|\Phi_i| \leq K_{i, 2} T \frac{t_i}{l_i}.$$

**(55)**

**Proof.** Use Theorem 5, it is easy to see

$$\int_0^\tau e^{-\Delta(\tau-\tau')} \lambda_i^{-2} e^{-\frac{t_i}{2} y_i \tau} R^{-\frac{1}{2}} d\tau' \leq T \frac{t_i}{l_i}.$$

Denote $i_{i, l} = e^{-\frac{2}{l_i} t_i} 1_{|z_i| \leq e^\frac{t_i}{2} + \frac{2}{l_i}} e^{-\frac{t_i}{2} - \frac{1}{l_i} \tau} |z_i|^{2l_i + 2} e^{e^{\frac{t_i}{2} - \frac{1}{l_i}}} \leq |z_i| \leq 2 e^{\frac{t_i}{2} + \frac{2}{l_i}}$

Then we check that $\Psi = K_{i, 2} (2 e^{-\frac{t_i}{2} + \frac{2}{l_i} \tau} - e^{-\frac{t_i}{2} + \frac{2}{l_i} \tau})$, $K_{i, 2}$ sufficiently large, is a super solution of $\Phi_i$ in $|z_i| > e^{\frac{t_i}{2} + \frac{2}{l_i}}$.

First, we use Theorem 13, inequality 53 and

$$|\Phi_i| \leq e^{-(2l_i + 1) \tau} (1 + |z_i|^{2l_i}),$$

we obtain when $|z_i| = e^{\frac{t_i}{2} + \frac{2}{l_i}}$

$$|\Phi_i| \leq e^{-\frac{t_i}{2} + \frac{2}{l_i} \tau}.$$

Additionally, we have that $|\tilde{\Phi}_{i, 1}| + |\tilde{\Phi}_{i, 2}| \leq T \frac{t_i}{l_i}$, for $|z_i| = e^{\frac{t_i}{2} + \frac{2}{l_i}}$, then

$$|\Phi_i| \leq e^{-\frac{t_i}{2} + \frac{2}{l_i} \tau} \tilde{\Psi}, \quad |z_i| = e^{\frac{t_i}{2} + \frac{2}{l_i}}.$$

Furthermore, for $|z_i| > e^{\frac{t_i}{2} + \frac{2}{l_i}}$, we verify that

$$\tilde{\Psi}_\tau - A_2 \tilde{\Psi} = K_{i, 2} \frac{t_i}{l_i + 1} e^{-\frac{t_i}{2} + \frac{2}{l_i} \tau} > e^{-\tau} G', (\partial_\tau - A_2) \Phi_i.$$

Finally, we check that at $\tau = \tau_0$

$$|\Phi_i| \leq e^{-(2l_i + 1) \tau_0} \leq e^{-\frac{t_i}{2} + \frac{2}{l_i} \tau} \leq \tilde{\Psi}.$$

Using the comparison principle, we can conclude the proof. $\square$

Now Theorem 14 provides an $L^\infty$ estimate, then we give the decay estimate.

**Theorem 15.** For $i = 1, 2$, $|x - q_i| \geq \frac{d}{2}$, there exists constant $K_{i, 3} > 0$ such that

$$|\Phi_i| \leq K_{i, 3} \left( \frac{T^3 - (T - t)^3}{1 + |x|^3} + \frac{Z_3 |3}_\infty}{1 + |x|^3} \right).$$

(56)
Proof. We will now verify that for sufficiently large $K_{i,3}$, the function
\[ \tilde{\psi}_i = K_{i,3} \left( \frac{T^{3l_i+3} - (T - t)^{3l_i+3}}{|x - q_i|^3} + \frac{\|Z_i\|_{\infty}^7(1 + t)}{|x - q_i|^3} \right), \]
is a supersolution of $b_i(\tau) \cdot e_i + \Phi_i$ in $|x - q_i| \geq \frac{d}{4}$. When $K_{i,3}$ is large enough, we have
\[ \partial_t \tilde{\psi}_i - \Delta \tilde{\psi}_i = K_{i,3} \left( \frac{(3l_i + 3)(T - t)^{3l_i+2}}{|x - q_i|^3} + \frac{\|Z_i \eta_i\|_{\infty}^7}{|x - q_i|^3} \right) \geq (\partial_t - \Delta)\Phi_i. \]
Using Theorem 13 we have for $|x - q_i| = \frac{d}{4}$
\[ |\Phi_i| \leq K_{i,2}T^{\frac{l_i}{1+l}} \leq \tilde{\psi}_i. \]
At $t = 0$, because the intersection of $|x - q_i| \geq \frac{d}{4}$ and
\[ \Phi_i |_{t=0} = (d_i \cdot e_i)\eta_i \left( \frac{z_i}{e^{b_i \tau}} \right) \]
is empty. So in $|x - q_i| \geq \frac{d}{4}$
\[ \Phi_i |_{t=0} = 0 \leq \frac{\|Z_i \eta_i\|_{\infty}^7}{|x - q_i|^3} = \tilde{\psi}_i |_{t=0}. \]
Use comparison principle, and $|x - q_i|^3 \lesssim 1 + |x|^3 \lesssim |x - q_i|^3$ if $|x - q_i| \geq \frac{d}{4}$, we can conclude the proof. \[ \square \]

Theorem 16. For $G_3$ defined in Sect. 5, we have
\[ \int_0^t e^{-\Delta(t-t')}G_3(x,t)| \lesssim \frac{\delta_0^p + T^2}{1 + |x|^3}, \quad (57) \]
and, for any positive integer $m, n$,
\[ \left| \partial_s^l \nabla^k \int_0^s e^{-\Delta(s-t')}G_3(q_i,t) \right| \leq C(m, n)T^n, \quad k + l \leq m. \quad (58) \]
Proof. The first part 57 of this theorem is similar to theorem 15, the second part 58 can be seen as
\[
\partial_t^l \nabla^k \int_0^t e^{-\Delta (s-t')} G_3(q_i, t) = \partial_t^l \nabla^k \int_0^s \int_{\mathbb{R}^n} (s-s')^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{t-t'}} G_3(y, s')
\]
\[
= \int_0^s \int_{\mathbb{R}^n} \sum_{m,n} C^k_m \partial_t^l (s-s')^{-\frac{n}{2}} \partial_s^m - k e^{-\frac{|x-y|^2}{t-t'}} G_3(y, s')
\]
\[
\leq \int_0^s (s-s')^{-\frac{n}{2}} \partial_s^m e^{-\frac{|x-y|^2}{t-t'}} (s-s')^n
\]
\[
\leq C(m,n)T^n. \quad (59)
\]
The gradient term can be estimated in a same way. Therefore, the theorem holds. \qed

Use the similar calculations, we obtain.

**Theorem 17.** For any \( m, n \in \mathbb{N}^+ \)
\[
|\partial_t^l \nabla^k \Phi_1(q_2, t)| \leq C(m,n)T^n, \quad l + k \leq m \quad (60)
\]
and
\[
|\partial_t^l \nabla^k \Phi_2(q_1, t)| \leq C(m,n)T^n, \quad l + k \leq m \quad (61)
\]

In previous sections, we separate the function \( \Phi \) into 3 parts, and we have shown the behavior of \( \Phi_1 \) and \( \Phi_2 \). However, these 2 functions may not vanish at \( q_1, q_2 \), respectively. Therefore we employ \( \Phi_3 \) to cancel their influence, this is the key to deal with the multi-points problem. With this background, we now turn to the last but important part of our analysis. Our goal is to ensure that \( \Phi_3 \) cancels \( \Phi_1 \) at \( q_2 \) and similarly cancels \( \Phi_2 \) at \( q_1 \), with the correct order of cancellation. To achieve this, we will need to carefully analyze the behavior of \( \Phi_1 \) and \( \Phi_2 \) near the points \( q_1, q_2 \), and then construct \( \Phi_3 \) accordingly.

**Theorem 18.** Let us consider the homogeneous heat equation
\[
\begin{aligned}
\partial_t \tilde{\varphi}(x,t) &= \Delta \tilde{\varphi}(x,t), \\
\tilde{\varphi}(x,0) &= \tilde{\varphi}_0.
\end{aligned}
\]
For any \( N \in \mathbb{N}^+ \), there exists \( \tilde{\varphi}_0 \) such that \( |\tilde{\varphi}(x,t)| \lesssim \frac{T^N}{1+|x|} \), and
\[
\partial_t^l \nabla^k \tilde{\varphi}(p_1, T) = -\partial_t^l \left( \int_0^T e^{-\Delta(t-t')} G_3(p_1, T) + \Phi_2(p_1, T) \right), \quad l + k = 1, \ldots, 2l_1 \quad (62)
\]
and
\[
\partial_t^l \nabla^k \tilde{\varphi}(p_2, T) = -\partial_t^l \left( \int_0^T e^{-\Delta(t-t')} G_3(p_2, T) + \Phi_1(p_2, T) \right), \quad l + k = 1, \ldots, 2l_2 \quad (63)
\]
moreover, \( \partial_t^l \nabla^k \tilde{\varphi}(x,t) \) is bounded for any \( k, l \in \mathbb{N}^+ \) if \( |x-q_i| \leq \frac{d}{4} \), where \( G_3 = G - G_1 - G_2 \).
Proof. To prove this, by apply Theorem 6, we deduce there exists $\tilde{\varphi} = \sum_i^N \mu_i g_i$, where $N = N(k, l)$ is an positive integer, such that
\begin{align}
\partial_t^k \nabla^l e^{-\Delta t} \tilde{\varphi}(p_1, T) = -\partial_t^k \nabla^l \left( \int_0^t e^{-\Delta (t-t')} G_3(p_1, T) + \Phi_2(p_1, T) \right)
\end{align}
and
\begin{align}
\partial_t^k \nabla^l e^{-\Delta t} \tilde{\varphi}(p_2, T) = -\partial_t^k \nabla^l \left( \int_0^t e^{-\Delta (t-t')} G_3(p_2, T) + \Phi_1(p_2, T) \right),
\end{align}
from theorem 6 and very basic linear algebra theory, we can see $\mu_k \lesssim T^N$ for any $N \in \mathbb{N}^+$, and because $g_i$ has compact support, we deduce that
\begin{align}
|e^{-\Delta t} \tilde{\varphi}| \lesssim \frac{T^N}{1 + |x|^3}.
\end{align}
We can conclude the proof. \qed

6. The proof of Theorem 1.1

After completing the necessary preparations in the preceding sections, we now proceed to prove Theorem 1.

Note that for $1 \leq i \leq m_i$
\begin{align}
|b_i e_i| \lesssim e^{-(2l_i+1)\tau} (1 + |z_i|^{2l_i}).
\end{align}
By applying Theorem 4, we obtain that
\begin{align}
|\Phi_{i,1}(z_i, \tau)| \leq Ce^{-(l_i+1)\tau} (1 + |z_i|^{2l_i+2}), \quad \tau \geq \tau_0.
\end{align}
combine this with Theorem 12 and inequality 53, we can infer that
\begin{align}
|\Phi_{i,2}| &= \left| \int_{\tau_0}^\tau e^{A_{i,1}((\tau-\tau')^\epsilon-\epsilon')} G_i^\perp d\tau' \right|
\leq \frac{e^{-(l_i+1)\tau}}{R_1^{l_i}} + e^{-(l_i+1)\tau} + e^{-(l_i+1-\frac{1}{2l_i+2})\tau_i} |z_i|^{2l_i+2}
\end{align}
When $|z_i| \leq e^{\frac{1}{2l_i+2}}$, we can simplify the above equation to
\begin{align}
|\Phi_{i,2}| \lesssim \frac{e^{-(l_i+1)\tau}}{R_1^{l_i}} + e^{-(l_i+1)\tau} (1 + |z_i|^{2l_i+2})
\end{align}
For sufficiently large $\tau_0$, we have
\begin{align}
|\Phi_i| \leq \frac{\delta_0}{4} e^{-(l_i+1)\tau} (1 + |z_i|^{2l_i+2}),
\end{align}
By applying Theorem 17, and using Taylor expansion at $(p_i, t)$, we deduce that
\begin{align}
|\Phi_1 + \Phi_3| \lesssim (T-t)^{2l_2} + |x - q_2|^{2l_2} \lesssim (T-t)^{l_2}, \quad |x - q_2| \leq \sqrt{T-t},
|\Phi_2 + \Phi_3| \lesssim (T-t)^{2l_1} + |x - q_1|^{2l_1} \lesssim (T-t)^{l_1}, \quad |x - q_1| \leq \sqrt{T-t}.
\end{align}
Therefor, when $|z| \leq e^{\frac{t^r}{2|z|^2}}$, we obtain
\[
|\psi| = |\Phi_1 + \Phi_2 + \Phi_3| \leq \frac{\delta_0}{2} e^{-L_1} (1 + |z_i|^{2L_1+2}),
\]
and by using Theorems 14, 15 and 17 we conclude that when $|z_i| \geq e^{\frac{t^r}{2|z|^2}}$
\[
|\psi| \leq \frac{\delta_0}{1 + |z|^3}.
\]

We now proceed to prove our fixed point argument more rigorously, given $\psi, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k$ in the workspace we have introduced previously, we first solve $\tilde{\lambda}_i(\psi, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k) = \lambda_{i,0} + \lambda_{i,1}, i = 1, \ldots, k$, next solve for $\phi_1, \ldots, \phi_k$, where $\phi_i = \phi_i(\psi, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k)$, lastly we solve for $\psi = \psi^*(\psi, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k).$ Let $S^5 = \{x = (x_1, \ldots, x_6) \in \mathbb{R}^6; \sum_{i=1}^6 x_i^2 = 1\}$, for $x = (x_1, \ldots, x_6) \in S^5$, $x_6 \neq 1$, define:
\[
P_5(x_1, \ldots, x_6) = \left(\frac{x_1}{1-x_6}, \ldots, \frac{x_5}{1-x_6}\right)
\]
as the stereographic projection from the north pole, denote $x_N = (0, \ldots, 0, 1)$ the north pole, define $V = \{(\psi, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k); (\psi, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k)\} = (\psi_1, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k)$, denote $W = (\psi, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k)$ and $W = (\psi^*, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k)$, with these discuss we know operator $T[W] = W$ maps $V$ in to $V$, and from Holder estimate we can get there exists some $0 < \alpha < 1$, such that $\|\psi^*\|_{C^\alpha} < \infty$, define a new operator $T_1[W] = (\psi^* \circ P_5, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k)$, now we proof $T_1$ is compact, given $\epsilon > 0$, if $x = (x_1, \ldots, x_6) \in S^5, \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_6) \in S^5, |x - x_N| + |\tilde{x} - x_N| < \frac{\epsilon}{10}$, then by the work space of $\psi$ in 36 and definition 73, we get $|T_1[W](x, t) - T_1[W](\tilde{x}, \tilde{t})| < \epsilon$ where $t \in (0, T)^{2k}$ and $\tilde{t} \in (0, T)^{2k}$, if $|x - x_N| + |\tilde{x} - x_N| \geq \frac{\epsilon}{10}$ and $|x - \tilde{x}| < \frac{\epsilon}{(100C)^{\frac{1}{3}}} + \frac{\epsilon}{10}$, where $C$ is the constant in the $C^\alpha$ estimate of $\psi^*$, use the $C^\alpha$ estimate of $\psi^*$ and the definition of $P_5$, we deduce that
\[
|T_1[\psi](x, t) - T_1[\psi](\tilde{x}, \tilde{t})| < |T[\psi]|_{C^\alpha(\mathbb{B}_{\frac{\epsilon}{20}}(x))} \left|P_5(\cdot)\right|_{C^\alpha(\mathbb{B}_{\frac{\epsilon}{20}}(x))} \left|\frac{\partial}{\partial t} + \frac{\partial}{\partial \tilde{t}}\right|^{\alpha}
\]
\[
\leq C \frac{\delta_0}{1 + \frac{1}{1-x_6^3} \epsilon^{1+\alpha}} \frac{1}{1 - x_6^2} 100C
\]
\[
< \epsilon
\]
As shown in 74, $T_1$ is equicontinuous and uniformly bounded, by applying the Ascoli-Arzelà theorem we can conclude that $T_1$ is compact.

Moreover, utilizing Schauder’s fixed-point theorem, we can assert that $T_1$ has a fixed point $\psi$ within $\mathcal{B}$. This implies that the main result of this paper, Theorem 1, is proven.

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