A STUDY OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH JUMP DIFFUSION COEFFICIENTS

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Abstract. As a simplified model for subsurface flows elliptic equations may be utilized. Insufficient measurements or uncertainty in those are commonly modeled by a random coefficient, which then accounts for the uncertain permeability of a given medium. As an extension of this methodology to flows in heterogeneous fractured porous media, we incorporate jumps in the diffusion coefficient. These discontinuities then represent transitions in the media. More precisely, we consider a second order elliptic problem where the random coefficient is given by the sum of a (continuous) Gaussian random field and a (discontinuous) jump part. To estimate moments of the solution to the resulting random partial differential equation, we use a pathwise numerical approximation combined with multilevel Monte Carlo sampling. In order to account for the discontinuities and improve the convergence of the pathwise approximation, the spatial domain is decomposed with respect to the jump positions in each sample, leading to path-dependent grids. Hence, it is not possible to create a nested sequence of grids which is suitable for each sample path a-priori. We address this issue by an adaptive multilevel algorithm, where the discretization on each level is sample-dependent and fulfills given refinement conditions.

1. Introduction

Uncertainty quantification plays an increasingly important role in a wide range of problems in the Engineering Sciences and Physics. Examples of sources of uncertainty are imprecise or insufficient measurements and noisy data. In the underlying dynamical system this is modeled via a stochastic operator, stochastic boundary conditions and/or stochastic data. As an example, to model subsurface flow more realistically the coefficient of an (essentially) elliptic equation is assumed to be stochastic. A common approach in the literature is to use (spatially) correlated random fields that are built from uniform distributions or colored log-normal fields. The resulting marginal distributions of the field are (shifted) normally, resp. log-normally distributed. Neither choice is universal enough to accommodate all possible types of permeability, especially not if fractures are incorporated (see [39]), the medium is very heterogeneous or porous.

The last decade has been an active research period on elliptic equations with random data. A non-exhaustive list of publications in this field includes [1, 8, 9, 12, 19, 20, 25, 31, 33, 37, 38]. One can find various ways to approximate the distribution or moments of the solution to the elliptic equation. Next to classical Monte Carlo methods, their multilevel variants and other variance reduction techniques have been applied. The concept of multilevel Monte Carlo simulation has been developed in [29] to calculate parametric integrals and has been rediscovered in [26] to estimate the expected value of functionals of stochastic differential equations. Ever since, multilevel Monte Carlo techniques have been successfully applied to various problems, for instance in the context of elliptic random PDEs in [12, 19, 31, 38] to just name a few. These sampling algorithms are fundamentally different from approaches using Polynomial Chaos. The latter suffer from the fact that one is rather restricted when it comes to possibilities to model the stochastic coefficient. While in the case of fields built from uniform distributions or colored log-normal fields these algorithms can outperform sampling strategies, approaches – like stochastic Galerkin methods –
are less promising in our discontinuous setting due to the rather involved structure of the coefficient. In fact, it is even an open problem to define them in the case that a Lévy-field is used.

Our main objective in this paper is to show existence and uniqueness of the solution to the elliptic equation when the coefficient is modeled as a jump-diffusion. By that we mean a field which consists of a deterministic, a Gaussian and non-continuous part. As we show in the numerical examples, this jump-diffusion coefficient can be used to model a wide array of scenarios. This generalizes the work in [31] and uses partly [30]. To approximate the expectation of the solution we develop and test, further, variants of the multilevel Monte Carlo method which are tailored to our problem: namely adaptive and bootstrapping multilevel Monte Carlo methods. Adaptivity is needed in the jump-diffusion setting, since the coefficient is not continuous. Our analysis shows that the non-adaptivity in a multilevel setting with non-continuous coefficients entails a larger error than when an adaptive algorithm is used. This result is not surprising, since the essence of the multilevel algorithm is that many samples are calculated on coarse grids, where the distributional error from misjudging jump-locations is high. Adaptivity, however, comes to the price that in certain scenarios solving the underlying system of equation becomes computationally expensive. In these settings the advantageous time-to-error performance of adaptive methods may be worse.

Bootstrapping, as a simplified version of Multifidelity Monte Carlo sampling (see [31]), reuses samples across levels and is preferred when sampling from a certain distribution is computationally expensive. The bootstrapping algorithms outperform, in general, algorithms with a standard sampling strategy of multilevel Monte Carlo, as it actually reduces the mean square error.

In Section 2 we introduce the model problem, define \textit{pathwise weak solutions} of random partial differential equations (PDEs) and show almost surely existence and uniqueness under relatively weak assumptions on the model parameters. The main contribution of this section is the existence and uniqueness result in Theorem 2.5, which is then readily transferred to the special case of a jump-diffusion problem in the subsequent sections of this article. In Section 3 we define the jump-diffusion coefficient and construct suitable approximations. Both stochastic parts of the jump-diffusion coefficient are approximated: The Gaussian one by a standard truncation of its basis representation, and the jump part (if direct sampling from the jump distribution is not possible) by a technique based on Fourier Inversion. We show \( L^p \)-type convergence for all existing moments of the approximation. From this result follows immediately convergence of the approximated solution. The approximated solution has still to be discretized to actually estimate moments of the solution. The approximation error from misjudging jump-locations is high. Adaptivity, however, comes to the price that in certain scenarios solving the underlying system of equation becomes computationally expensive. The bootstrapping algorithms outperform, in general, algorithms with a standard sampling strategy of multilevel Monte Carlo, as it actually reduces the mean square error.

2. Elliptic boundary value problems and existence of solutions

We consider the following random elliptic equation in a general setting before we specify in Section 2 our precise choice of coefficient function. Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a complete probability space and \(\mathcal{D} \subset \mathbb{R}^d\), for some \(d \in \mathbb{N}\), be a bounded and connected Lipschitz domain. In this paper we consider the linear, random elliptic problem

\[
-\nabla \cdot (a(\omega, x) \nabla u(\omega, x)) = f(\omega, x) \quad \text{in } \Omega \times \mathcal{D},
\]

where \(a : \Omega \times \mathcal{D} \to \mathbb{R}\) is a stochastic jump-diffusion coefficient and \(f : \Omega \times \mathcal{D} \to \mathbb{R}\) is a random source function. The Lipschitz boundary \(\partial \mathcal{D}\) consists of open \((d - 1)\)-dimensional manifolds which are grouped into two disjoint subsets \(\Gamma_1\) and \(\Gamma_2\) such that \(|\Gamma_1| > 0\) and \(\partial \mathcal{D} = \Gamma_1 \cup \Gamma_2\). We impose mixed Dirichlet-Neumann boundary conditions

\[
\begin{align*}
&u(\omega, x) = 0 \quad \text{on } \Omega \times \Gamma_1, \\
&a(\omega, x) \vec{n} \cdot \nabla u(\omega, x) = g(\omega, x) \quad \text{on } \Omega \times \Gamma_2,
\end{align*}
\]

on Eq. (2.1), where \(\vec{n}\) is the outward unit normal vector to \(\Gamma_2\) and \(g : \Omega \times \Gamma_2 \to \mathbb{R}\), assuming that the exterior normal derivative \(\vec{n} \cdot \nabla u\) on \(\Gamma_2\) is well-defined for any \(u \in C^1(\overline{\mathcal{D}})\). To obtain a pathwise
variational formulation of this problem, we use the standard Sobolev space \( H^1(\mathcal{D}) \) equipped with the norm
\[
\|v\|_{H^1(\mathcal{D})} = \left( \int_{\mathcal{D}} |v|^2 + |\nabla v|^2 \, dx \right)^{1/2}
\] for \( v \in H^1(\mathcal{D}) \).
On the Lipschitz domain \( \mathcal{D} \), the existence of a bounded, linear operator
\[
T : H^1(\mathcal{D}) \to H^{1/2}(\partial \mathcal{D})
\]
with
\[
T : H^1(\mathcal{D}) \cap C^\infty(\overline{\mathcal{D}}) \to H^{1/2}(\partial \mathcal{D}), \quad v \mapsto v|_{\partial \mathcal{D}}
\]
and
\[
(2.3) \quad \|Tv\|_{H^{1/2}(\partial \mathcal{D})} \leq C_D \|v\|_{H^1(\mathcal{D})}
\]
for \( v \in H^1(\mathcal{D}) \) and some constant \( C_D > 0 \), depends only on \( \mathcal{D} \), is ensured by the trace theorem, see for example [22]. At this point, one might argue that the trace operator \( T \) needs to be defined pathwise for any \( \omega \in \Omega \), since the Neumann part of the boundary conditions in Eq. (2.2) may contain a random function \( g : \Omega \times \Gamma_2 \to \mathbb{R} \). This is true if one works with \( T \) on \( \Gamma_2 \subset \partial \mathcal{D} \), as the trace \( Tv \) then has to match the boundary condition given by \( g(\omega, \cdot) \) on \( \Gamma_2 \) for \( \mathbb{P} \)-almost all \( \omega \in \Omega \) and \( v \in H^1(\mathcal{D}) \). In our case, for simplicity, we may treat \( T \) independently of \( \omega \in \Omega \), since we only consider the trace operator on the homogeneous boundary part \( \Gamma_1 \) to define \( V \) as follows: The subspace of \( H^1(\mathcal{D}) \) with zero trace on \( \Gamma_1 \) is then
\[
V := \{ v \in H^1(\mathcal{D}) | Tv|_{\Gamma_1} = 0 \},
\]
with norm
\[
\|v\|_V := \left( \int_{\mathcal{D}} |v|^2 + |\nabla v|^2 \, dx \right)^{1/2}.
\]

**Remark 2.1.** The condition \( |\Gamma_1| > 0 \) implies that \( V \) is a closed linear subspace of \( H^1(\mathcal{D}) \). We may as well work with non-homogeneous boundary conditions on the Dirichlet part, i.e. \( u(\omega, x) = g_1(\omega, x) \) for \( g_1 : \Omega \times \Gamma_1 \to \mathbb{R} \). The corresponding trace operator \( T \) is still well defined if, for a \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), \( g_1(\omega, \cdot) \) can be extended to a function \( \overline{g}_1(\omega, \cdot) \in H^1(\mathcal{D}) \). Then, we consider for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) the problem
\[
-\nabla : (a(\omega, x) \nabla ((u - \overline{g}_1)(\omega, x))) = f + \nabla : (a(\omega, x) \nabla \overline{g}_1(\omega, x)) \quad \text{on } \Omega \times \mathcal{D},
\]
\[
(u - \overline{g}_1)(\omega, x) = 0 \quad \text{on } \Gamma_1 \times \mathcal{D} \text{ and}
\]
\[
a(\omega, x) \overline{g}_1 : \nabla ((u - \overline{g}_1)(\omega, x)) = g(\omega, x) - a(\omega, x) \overline{g}_1 : \nabla \overline{g}_1(\omega, x) \quad \text{on } \Gamma_2 \times \mathcal{D}.
\]
But this is in fact a version of Problem [21] equipped with Eqs. (2.2) where the source term and Neumann-data have been changed (see also [24] p. 317).

As the coefficient and the boundary conditions are given by random functions, the solution \( u \) is also a random function. Besides pathwise properties, \( u \) may also have certain integrability properties with respect to the underlying probability measure. To this end, we introduce the space of *Bochner integrable* random variables resp. random functions (see [21] for an overview).

**Definition 2.2.** Let \( (B, \| \cdot \|_B) \) be a Banach space and define the norm \( \| \cdot \|_{L^p(\Omega; B)} \) for a \( B \)-valued random variable \( v : \Omega \to B \) as
\[
\|v\|_{L^p(\Omega; B)} := \begin{cases} \mathbb{E}(\|v\|_B^p)^{1/p} & \text{for } 1 \leq p < +\infty \\ \text{esssup}_{\omega \in \Omega} \|v\|_B & \text{for } p = +\infty \end{cases}.
\]

The corresponding space of Bochner-integrable random variables is then given by
\[
L^p(\Omega; B) := \{ w : \Omega \to B \text{ is strongly measurable and } \|w\|_{L^p(\Omega; B)} < +\infty \}.
\]

The following set of assumptions on \( a, f, \) and \( g \) allows us to show existence and uniqueness of the solution to Eq. (2.3). Consequently, we denote by \( \mathcal{V}' \) the topological dual of a vector space \( \mathcal{V} \).

**Assumption 2.3.** Let \( H := L^2(\mathcal{D}) \). For \( \mathbb{P} \)-almost all \( \omega \in \Omega \) it holds that:
- \( a_-(\omega) := \inf_{x \in \mathcal{D}} a(\omega, x) > 0 \) and \( a_+(\omega) := \sup_{x \in \mathcal{D}} a(\omega, x) < +\infty \).
If Assumption 2.3 holds, then there exists a unique pathwise weak solution to Problem (2.1).

\[ (2.5) \]

On the other hand, choose \( u \) and \( v \) such that

\[ \int_D a(\omega, x) \nabla u(\omega, x) \cdot \nabla v(x) \, dx = \int_D f(\omega, x) v(x) \, dx + \int_{\Gamma_2} g(\omega, x) [Tv](x) \, dx. \]

Consider the bilinear form \( B_{\alpha}(\omega) \):

\[ B_{\alpha}(\omega) : V \times V \to \mathbb{R}, \quad (u, v) \mapsto \int_D a(\omega, x) \nabla u(\omega, x) \cdot \nabla v(x) \, dx \]

and

\[ F_\omega : V \to \mathbb{R}, \quad v \mapsto \int_D f(\omega, x) v(x) \, dx + \int_{\Gamma_2} g(\omega, x) [Tv](x) \, dx, \]

where the integrals in \( F_\omega \) are understood as the duality pairings

\[ \int_D f(\omega, x) v(x) \, dx = \langle f(\omega, \cdot), v \rangle_V \]

and

\[ \int_{\Gamma_2} g(\omega, x) [Tv](x) \, dx = \langle g(\omega, \cdot), Tv \rangle_{H^{-1/2}(\Gamma_2)}. \]

Equation (2.4) then leads to the pathwise variational formulation of Problem (2.1): For \( \mathbb{P} \)-almost all \( \omega \in \Omega \), given \( f(\omega, \cdot) \in V' \) and \( g(\omega, \cdot) \in H^{-1/2}(\Gamma_2) \), find \( u(\omega, \cdot) \in V \) such that

\[ (2.5) \]

\[ B_{\alpha}(\omega)(u(\omega, \cdot), v) = F_\omega(v) \]

for all \( v \in V \). A function \( u(\omega, \cdot) \in V \) that fulfills the pathwise variational formulation is then called pathwise weak solution to Problem (2.1).

**Theorem 2.5.** If Assumption 2.3 holds, then there exists a unique pathwise weak solution \( u(\omega, \cdot) \in V \) to Problem (2.5) for \( \mathbb{P} \)-almost all \( \omega \in \Omega \). Furthermore, \( u \in L^r(\Omega; V) \) and bounded by

\[ \|u\|_{L^r(\Omega; V)} \leq C(a_{-\cdot}, D, p)(\|f\|_{L^q(\Omega; H)} + \|g\|_{L^q(\Omega; L^{2}(\Gamma_2))}), \]

where \( C(a_{-\cdot}, D, p) > 0 \) is a constant depending only on the indicated parameters.

**Proof.** Choose \( \omega \in \Omega \) such that Assumption 2.3 is fulfilled. For all \( u, v \in V \), we obtain by the Cauchy-Schwarz inequality

\[ |B_{\alpha}(\omega)(u, v)| \leq \left( \int_D (a(\omega, x))^2 |\nabla u(x)|^2 \, dx \int_D |\nabla v(x)|^2 \, dx \right)^{1/2} \leq a(\omega) \|u\|_V \|v\|_V. \]

On the other hand,

\[ B_{\alpha}(\omega)(u, u) \geq a_{\alpha}(\omega) \int_D |\nabla u(x)|^2 \, dx \]

\[ = \frac{a_{\alpha}(\omega)}{2} (\|\nabla u\|_{L^2(D)}^2 + \|\nabla u\|_{L^2(D)}^2) \]

\[ \geq \frac{a_{\alpha}(\omega)}{2} ((\|u\|_{L^2(D)}^2 + C_{D}^{-2}\|u\|_{L^2(D)}^2) \]

\[ \geq \frac{a_{\alpha}(\omega)}{2} \min(1, C_{D}^{-2}) \|u\|_V^2, \]
where the constant $C^2_{|D|} > 0$ stems from the constant in Poincaré’s inequality, $C_{|D|}$, and only depends on $|D|$. Hence the bilinear form $B_{a(\omega)} : V \times V \to \mathbb{R}$ is continuous and coercive. We use that $H = H' \subset V'$ and the trace theorem (Equation (2.3)) to bound $F_\omega$ by

$$F_\omega(v) \leq \|f(\omega,\cdot)\|_{V'} \|v\|_{V} + \|g(\omega,\cdot)\|_{H^{1/2}(\Gamma_D)} \|T_v\|_{H^{1/2}(\Gamma_D)}$$

$$\leq \|f(\omega,\cdot)\|_{H} + C_D \|g(\omega,\cdot)\|_{L^2(\Gamma_D)} \|v\|_{V}.$$ 

This shows that $F_\omega$ is a bounded linear functional on $V$ (and therefore continuous) for almost all $\omega \in \Omega$. The existence of a unique pathwise weak solution $u(\omega,\cdot)$ is then guaranteed by the Lax-Milgram lemma $\mathbb{P}$-almost surely. If $u(\omega,\cdot)$ is a solution of Eq. (2.5) for given $f(\omega,\cdot) \in H$ and $g(\omega,\cdot) \in L^2(\Gamma_D)$, then

$$\frac{a(\omega)}{2} \min(1, C_{|D|}^2) \|u(\omega,\cdot)\|_{V}^2 \leq B_{a(\omega)}(u(\omega,\cdot), u(\omega,\cdot))$$

$$= F_\omega(u(\omega,\cdot)) \leq \|f(\omega,\cdot)\|_{H} + C_D \|g(\omega,\cdot)\|_{L^2(\Gamma_D)} \|u(\omega,\cdot)\|_{V}$$

Using Hölder’s and Minkowski’s inequality together with $r = (1/p + 1/q)^{-1} \geq 1$ yields:

$$\|u\|_{L^r(\Omega;V)} \leq \frac{2 \max(1, C_D)}{\min(1, C_{|D|}^2)} \mathbb{E} \left[ \frac{a^{-p}}{2} \mathbb{E} \left( \left( \|f\|_H + \|g\|_{L^2(\Gamma_D)} \right)^r 0$$

Using Hölder’s and Minkowski’s inequality together with $r = (1/p + 1/q)^{-1} \geq 1$ yields:

$$\|u\|_{L^r(\Omega;V)} \leq \frac{2 \max(1, C_D)}{\min(1, C_{|D|}^2)} \|1/a\|_{L^r(\Omega;\mathbb{R})} \|f\|_{L^r(\Omega;H)} + \|g\|_{L^r(\Omega;L^2(\Gamma_D))} < +\infty.$$ 

Using Hölder’s and Minkowski’s inequality together with $r = (1/p + 1/q)^{-1} \geq 1$ yields:

$$\|u\|_{L^r(\Omega;V)} \leq \frac{2 \max(1, C_D)}{\min(1, C_{|D|}^2)} \|1/a\|_{L^r(\Omega;\mathbb{R})} \|f\|_{L^r(\Omega;H)} + \|g\|_{L^r(\Omega;L^2(\Gamma_D))} < +\infty.$$ 

In the next section, we introduce the diffusion coefficient $a$, which allows us to incorporate discontinuities at random points or areas in $D$. We show the existence and uniqueness of a weak solution to the discontinuous diffusion problem by choosing $a$ such that Assumption 2.3 is fulfilled and Theorem 2.4 may be applied.

### 3. DISCONTINUOUS RANDOM ELLIPTIC PROBLEMS

The stochastic coefficient $a$ in a jump-diffusion model should incorporate random discontinuities as well as a Gaussian component. We achieve this characteristic form of $a$ by defining the coefficient as a Gaussian random field with additive discontinuities on random areas of $D$. Since this usually involves infinite series expansions in the Gaussian component or sampling errors in the jump measure, we also describe how to obtain tractable approximations of $a$. Subsequently, existence and uniqueness results for weak solutions of the unapproximated resp. approximated jump-diffusion problem based on the results in Section 2 are proved. We conclude this section by showing pathwise and $L^2$-convergence of the approximated solution to the solution $u : \Omega \to V$ to the (unapproximated) discontinuous diffusion problem.

#### 3.1. Jump-diffusion coefficients and their approximations.

**Definition 3.1.** The jump-diffusion coefficient $a$ is defined as

$$a : \Omega \times D \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \overline{a}(x) + \Phi(W(\omega, x)) + P(\omega, x),$$

where

- $\overline{a} \in C^1(D; \mathbb{R}_{\geq 0})$ is non-negative, continuous and bounded.
- $\Phi \in C^1(D, \mathbb{R}_{\geq 0})$ is a continuously differentiable, positive mapping.
- $W \in L^2(\Omega; H)$ is a (zero-mean) Gaussian random field associated to a non-negative, symmetric trace class operator $Q : H \to H$.
- $\lambda$ is a finite measure on $(D, B(D))$ and $\mathcal{T} : \Omega \to B(D)$, $\omega \mapsto \{T_1, \ldots, T_n\}$ is a random partition of $D$ with respect to $\lambda$. The number $\tau$ of elements in $\mathcal{T}$ is a random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(\tau) = \lambda(D)$. 

• \((P_i, i \in \mathbb{N})\) is a sequence of non-negative random variables on \((\Omega, A, \mathbb{P})\) and

\[
P : \Omega \times \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} 1_{(\tau_i)}(x) P_i(\omega).
\]

The sequence \((P_i, i \in \mathbb{N})\) is independent of \(\tau\) (but not necessarily i.i.d.).

**Remark 3.2.** The definition of the measure \(\lambda\) on \((\mathcal{D}, \mathcal{B}(\mathcal{D}))\) in Def. 3.1 relates not only to the average number of partition elements \(E(\tau)\), but may further be utilized to concentrate discontinuities of the jump-diffusion coefficient \(a\) to certain areas of \(\mathcal{D}\). Choosing, for instance, \(\lambda\) as the Lebesgue measure on \(\mathcal{D}\) corresponds to uniformly distributed jumps and on average equally sized partition elements \(\mathcal{J}_i\). In contrast, if \(\lambda\) is a Gaussian measure on \(\mathcal{D}\) around some center point \(x_C \in \mathcal{D}\), the number of discontinuities (resp. size of partition elements) will decrease (resp. increase) as one moves away from \(x_C\). We refer to the numerical experiments in Section 6 where we give interpretations of \(\lambda\) to model certain characteristics of different jump-diffusion coefficients.

On a further note, we do not require stochastic independence of \(W\) and \(P\).

In general, the structure of \(a\) as in Def. 3.1 does not allow us to draw samples from the exact distribution of this random function. For an approximation of the Gaussian field, one usually uses truncated Karhunen-Loève expansions: Let \((\eta_i, e_i), i \in \mathbb{N}\) denote the sequence of eigenpairs of \(Q\), where the eigenvalues are given in decaying order \(\eta_1 \geq \eta_2 \geq \cdots \geq 0\). Since \(Q\) is trace class the Gaussian random field \(W\) admits the representation

\[
W = \sum_{i=1}^{\infty} \sqrt{\eta_i} e_i Z_i,
\]

where \((Z_i, i \in \mathbb{N})\) is a sequence of independent and standard normally distributed random variables. The series above converges in \(L^2(\Omega; H)\) and \(\mathbb{P}\)-almost surely (see i.e. [3]). The truncated Karhunen-Loève expansion \(W_N\) of \(W\) is then given by

\[
W_N := \sum_{i=1}^{N} \sqrt{\eta_i} e_i Z_i,
\]

where we call \(N \in \mathbb{N}\) the *cut-off index* of \(W_N\). In addition, it may be possible that the sequence of jumps \((P_i, i \in \mathbb{N})\) cannot be sampled exactly but only with an intrinsic bias (see also Remark 3.4). The biased samples are denoted by \((\tilde{P}_i, i \in \mathbb{N})\) and the error which is induced by this approximation is represented by the parameter \(\varepsilon > 0\) as in Assumption 3.3. To approximate \(P\) using the biased sequence \((\tilde{P}_i, i \in \mathbb{N})\) instead of \((P_i, i \in \mathbb{N})\) we define the jump part approximation

\[
P_{\varepsilon} : \Omega \times \mathcal{D} \rightarrow \mathbb{R}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} 1_{(\tau_i)}(x) \tilde{P}_i(\omega).
\]

The *approximated jump-diffusion coefficient* \(a_{N,\varepsilon}\) is then given by

\[
a_{N,\varepsilon}(\omega, x) = \mathcal{P}(x) + \Phi(W_N(\omega, x)) + P_{\varepsilon}(\omega, x),
\]

and the corresponding stochastic PDE with approximated jump-diffusion coefficient reads

\[
-\nabla \cdot (a_{N,\varepsilon}(\omega, x) \nabla u_{N,\varepsilon}(\omega, x)) = f(\omega, x) \quad \text{in } \Omega \times \mathcal{D},
\]

\[
u_{N,\varepsilon}(\omega, x) = 0 \quad \text{on } \Omega \times \Gamma_1,
\]

\[
a_{N,\varepsilon}(\omega, x) \mathcal{N} \cdot \nabla u_{N,\varepsilon}(\omega, x) = g(\omega, x) \quad \text{on } \Omega \times \Gamma_2.
\]

For \(\omega \in \Omega\) and given samples \(a_{N,\varepsilon}(\omega, \cdot), f(\omega, \cdot)\) and \(g(\omega, \cdot)\), we consider the pathwise weak solution \(u_{N,\varepsilon}(\omega, \cdot) \in V\) to Problem 3.3 for fixed approximation parameters \(N \in \mathbb{N}\) and \(\varepsilon > 0\). The variational formulation of Eq. 3.3 is then analogous to Eq. 2.5 given by: For almost all \(\omega \in \Omega\) with given \(f(\omega, \cdot), g(\omega, \cdot)\), find \(u_{N,\varepsilon}(\omega, \cdot) \in V\) such that

\[
B_{a_{N,\varepsilon}}(u_{N,\varepsilon}(\omega, \cdot), v) := \int_{\mathcal{D}} a_{N,\varepsilon}(\omega, x) \nabla u_{N,\varepsilon}(\omega, x) \nabla v(x) dx
\]

\[
= \int_{\mathcal{D}} f(\omega, x) v(x) dx + \int_{\Gamma_2} g(\omega, x) [Tv](x) dx = F_\omega(v)
\]

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for all \( v \in V \). The following assumptions guarantee that we can apply Theorem 2.3 also in the jump-diffusion setting and that therefore pathwise solutions \( u \) and \( u_{N,\varepsilon} \) exists.

**Assumption 3.3.**

(i) The eigenfunctions \( e_i \) of \( Q \) are continuously differentiable on \( D \) and there exist constants \( \alpha, \beta, C_\alpha, C_\beta > 0 \) such that for any \( i \in \mathbb{N} \)

\[
\|e_i\|_{L^\infty(D)} \leq 1, \quad \|\nabla e_i\|_{L^\infty(D)} \leq C_\alpha i^\alpha \quad \text{and} \quad \sum_{i=1}^{\infty} \eta_i i^\beta \leq C_\beta < +\infty.
\]

(ii) Furthermore, the mapping \( \Phi \) as in Definition 3.1 and its derivative are bounded by

\[
\Phi(w) \geq \phi_1 \exp(-\psi_1 w^2), \quad |\Phi'(w)| \leq \phi_2 \exp(\psi_2 |w|), \quad w \in \mathbb{R},
\]

where \( 0 < \psi_1 < (2 \operatorname{tr}(Q))^{-1} \) with \( \operatorname{tr}(Q) := \sum_{i\in\mathbb{N}} \eta_i \) and \( \phi_1, \phi_2, \psi_2 > 0 \) are arbitrary constants.

(iii) Given \( \psi_1 \) and \( \operatorname{tr}(Q) \), there exists a \( q > (1 - 2 \operatorname{tr}(Q) \psi_1)^{-1} = (1 - \eta^*)_1 \geq 1 \) such that

\[
f \in L^q(\Omega; H) \quad \text{and} \quad g \in L^q(\Omega; \Gamma_2).
\]

(iv) Finally, for some \( \bar{s} \in [1, \infty) \), \( (P_i, i \in \mathbb{N}) \) consists of \( s \)-integrable random variables, i.e. \( P_i \in L^s(\Omega; \mathbb{R}) \) for all \( i \in \mathbb{N} \) and \( s \in [1, \bar{s}] \), further there exists a sequence of approximations \( (\tilde{P}_i, i \in \mathbb{N}) \) so that the sampling error is bounded, for \( \varepsilon > 0 \), by

\[
\mathbb{E}(|\tilde{P}_i - P_i|^s) \leq \varepsilon, \quad i \in \mathbb{N}, \quad s \in [1, \bar{s}].
\]

**Remark 3.4.** Assumption 3.3 (i) on the eigenpairs of \((\eta_i, e_i), i \in \mathbb{N}\) is natural. For instance, the case that \( W \) is a Brownian-motion-like random field or that \( Q \) is a Matérn covariance operator are included. The bounds on \( \Phi \) and the regularity assumptions on \( f \) and \( g \) (Assumption 3.3 (ii), (iii)) are necessary to ensure that the solution \( u \) has at least finite expectation. The sampling error \( \mathbb{E}(|\tilde{P}_i - P_i|^s) \) in Assumption 3.3 (iv) may be interpreted in several ways: For instance, it may account for uncertainties in the distribution of \( P_i \), like parameters for which only confidence intervals are available. Another possibility is, that realizations of \( P_i \) may not be simulated directly or only at relatively high computational costs, for example by Acceptance Rejection algorithms, see [5, Chapter II]. In this case, it may be favorable to generate the approximation \( \tilde{P}_i \) by a more efficient numerical algorithm (i.e. Fourier inversion techniques, see [13] instead and to control for the error. The resulting sampling error can then be equilibrated with the truncation error from the Gaussian field to achieve a desired overall accuracy.

### 3.2. Existence of \( u \) and \( u_{N,\varepsilon} \)

We first show the existence of a weak solution for both, the jump-diffusion problem (2.1) with \( a \) as in Definition 3.1 and the approximated problem (3.3).

**Lemma 3.5.** Let \( a \) be a jump-diffusion coefficient (as defined in Def. 3.1). If \( a, f \) and \( g \) fulfill Assumptions 3.2, then the elliptic problem (2.1) has a unique weak solution \( u \in L^r(\Omega; V) \), where \( r \in [1, (1/q + \eta^*)^{-1}) \) for \( \eta^* := 2 \operatorname{tr}(Q) \psi_1 \).

**Proof.** By Theorem 2.3 it is sufficient to show that \( a, f \) and \( g \) fulfill Assumption 2.3. Clearly, \( 0 < a_-(\omega) \leq a_+(\omega) < \infty \) for almost all \( \omega \in \Omega \), as \( \tilde{f} \) and \( P \) are non-negative and we have the lower bound on \( \Phi \) in Assumption 3.3 (ii) by definition. Consequently, it is sufficient to bound the expectation of \((\mathbb{E}_{x \in D} \Phi(W(x))^p \), for \( 1 \leq p < (\eta^*)^{-1} \), from below (see also [10] Section 2.3).

The random variable \( W(x) - W(y) \) follows a centered normal distribution for any \( x, y \in D \). To see this, consider the finite sum

\[
\overline{W}_M(x, y) := \sum_{i=1}^{M} \sqrt{\eta_i}(e_i(x) - e_i(y))Z_i
\]

for \( M \in \mathbb{N} \) and a sequence \( (Z_i, i = 1, \ldots, M) \) of i.i.d. \( \mathcal{N}(0,1) \)-distributed random variables. Clearly, \( \overline{W}_M(x, y) \) is normally distributed with zero mean and characteristic function

\[
\phi_M(t) := \mathbb{E}(\exp(it\overline{W}_M(x, y))) = \exp(-t^2 \sum_{i=1}^{M} \eta_i(e_i(x) - e_i(y))^2), \quad t \in \mathbb{R}.
\]
Using that $Q$ is trace class with $|e_i(z)| \leq 1$ for all $i \in \mathbb{N}$ and $z \in D$, it follows
\[
\sigma^2(x, y) := \sum_{i=1}^{\infty} \eta_i(e_i(x) - e_i(y))^2 \leq 2tr(Q) < +\infty
\]
and hence
\[
\lim_{M \to \infty} \phi_M(t) = \exp(-\frac{t}{2} \sum_{i=1}^{\infty} \eta_i(e_i(x) - e_i(y))^2), \quad t \in \mathbb{R},
\]
where the right hand side is the characteristic function of a normal distribution with zero mean and variance $\sigma^2(x, y)$. By the Lévy continuity theorem this implies that
\[
W(x) - W(y) \sim \mathcal{N}(0, \sigma^2(x, y))
\]
for all $x, y \in D$. Next we show that $W$ has $\mathbb{P}$-almost surely Hölder continuous paths (see also [16, Proposition 3.4]): Let $0 < b \leq \min(1, \frac{\alpha}{2\beta})$ (where $\alpha, \beta$ are defined in Ass. 3.3 (i)) and $Z \sim \mathcal{N}(0,1)$. For $x, y \in D \subset \mathbb{R}^d$ and any $k \in \mathbb{N}$ we have
\[
\mathbb{E}(|W(x) - W(y)|^{2k}) = \mathbb{E}((|\sqrt{\sigma^2_N(x, y)}Z|)^{2k})
\]
\[
= \frac{(2k)!}{2k!} \left(\sum_{i \in N} \eta_i(e_i(x) - e_i(y))^2\right)^k
\]
\[
\leq \frac{(2k)!}{2k!} 2^{2(1-k)\alpha b}\sigma^2_c \left(\sum_{i \in N} \eta_i e_i^2\right)^k \|x - y\|^{2bk},
\]
where $0 < b < \beta/(2\alpha)$, the constant $C_c$ stems from Ass. 3.3 (i) and $\| \cdot \|_2$ denotes the Euclidean norm on $\mathbb{R}^d$. The second equality results from the fact that $\mathbb{E}(Z^{2k}) = (2k)!/(2k!)$ for all $k \in \mathbb{N}$ and the sum in the inequality is finite because $2\alpha b < \beta$. For any dimension $d \in \mathbb{N}$, we may choose $k > d/(2b)$ and obtain by the Kolmogorov-Chentsov theorem ([21, Theorem 3.5]) that $W$ has a Hölder continuous modification with Hölder exponent $\gamma \in (0,(2bk - d)/2k)$. Hence, $W$ is a centered Gaussian process and almost surely bounded on $D$. By [2] Theorem 2.1.1 this implies $\mathbb{E} := \mathbb{E}(\sup_{x \in D} W(x)) < \infty$ and
\[
P(\sup_{x \in D} W(x) - E \geq c) \leq \exp(-\frac{c^2}{2\sigma^2})
\]
for all $c > 0$ and $\sigma^2 := \sup_{x \in D} \mathbb{E}(W(x)^2)$. With Assumption 3.3 (ii) and since $\|\mathbb{E}(W)||_{L^\infty(D)} \leq \exp(||W||_{L^\infty(D)})$ we further obtain
\[
\mathbb{E}(1/a^2) \leq \mathbb{E}(\mathbb{E}(W(x)^2)^{-p}) = \mathbb{E}(\sup_{x \in D} W(x)^{-p})
\]
\[
\leq \frac{1}{\phi_1^2} \mathbb{E}(\mathbb{E}(p\psi_1 W(x)^2))
\]
\[
\leq \frac{1}{\phi_1^2} \mathbb{E}(\mathbb{E}(p\psi_1 ||W||_{L^\infty(D)}^2)).
\]
By Fubini's Theorem and integration by parts we may bound
\[
\mathbb{E}(\mathbb{E}(p\psi_1 ||W||_{L^\infty(D)}^2)) = \int_0^\infty \mathbb{E}(\mathbb{E}(p\psi_1 c^2)\mathbb{P}(||W||_{L^\infty(D)} > c) dc
\]
\[
\leq \mathbb{E}(p\psi_1 \mathbb{E}(p\psi_1 E^2)) + \int_0^\infty \mathbb{E}(\mathbb{E}(p\psi_1 c^2)\mathbb{P}(||W||_{L^\infty(D)} > c) dc
\]
\[
\leq \mathbb{E}(p\psi_1 \mathbb{E}(p\psi_1 E^2)) + 2 \int_0^\infty \mathbb{E}(\mathbb{E}(p\psi_1 (c^2 - \frac{1}{2})) dc)
\]
where we have used $\mathbb{P}(||W||_{L^\infty(D)} > c) \leq 2\mathbb{P}(\sup_{x \in D} W(x) > c)$, by the symmetry of $W$, and Ineq. (3.5) in the last step. The last expectation is finite if and only if $p\psi_1 < \frac{1}{2\sigma^2}$. For this to hold, $p < (\eta^*)^{-1}$ is sufficient, since $W(x) \sim \mathcal{N}(0, \sum_{i=1}^{\infty} \eta_i e_i(x))$ and thus $\sigma^2 \leq tr(Q)$. \qed
Remark 3.6. From Lemma 5.3 it follows immediately that one cannot expect finite second moments of the solution $u$ for $\eta^* \geq 1/2$ or $q \leq 2$. If we assume that $q = 3$ we need $\eta^* < 1/6$ to have finite second moments (in the case of, for instance, a log-normal Gaussian field). Note that, for all covariance kernels and functionals we use (e.g. log-Gaussian fields with Matérn class covariance or Brownian-motion like covariance kernels), $\psi$ and $\eta^*$ are much smaller than $1/2$. If one assumes that $f$, $g$ and $a$ are stochastically independent, then the regularity of the solution (in $\Omega$) is at least the same as the lowest regularity of the data, i.e. $f$, $g$ or $a$.

**Lemma 3.7.** Let $a_{N, \varepsilon}$ be the approximated jump-diffusion coefficient (as in Eq. (3.2)) and define the random variables

$$a_{N, \varepsilon, \omega} : \Omega \rightarrow \mathbb{R}, \quad \omega \mapsto \inf_{x \in \mathcal{D}} a_{N, \varepsilon}(\omega, x) \quad \text{and} \quad a_{N, \varepsilon, \omega} : \Omega \rightarrow \mathbb{R}, \quad \omega \mapsto \sup_{x \in \mathcal{D}} a_{N, \varepsilon}(\omega, x)$$

on $(\Omega, \mathcal{A}, \mathbb{P})$. If Assumption 5.3 holds, then, for any $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists a unique weak solution $u_{N, \varepsilon} \in L^r(\Omega; V)$ to Problem (5.3), where $r \in [1, (1/q + \eta^*)^{-1}]$, for $\eta^* := 2 \text{tr}(Q)\psi_1$. Furthermore, for $p \in [1, (\eta^*)^{-1}]$ the norm $\|1/a_{N, \varepsilon, \omega}\|_{L^p(\Omega; \mathbb{R})}$ is bounded uniformly with respect to $\varepsilon$ and $N$.

**Proof.** The proof is carried out identically to Lemma 5.3 where we replace $a_\varepsilon$ by $a_{N, \varepsilon, \omega}$, $\sigma(x, y)^2$ by $\sum_{i=1}^N \eta_i (e_i(x) - e_i(y))^2$ and $\text{tr}(Q)$ by $\sum_{i=1}^N \eta_i$. Again, by Eq. (5.2), $a_{N, \varepsilon, \omega} < +\infty$ $\mathbb{P}$-almost surely. In the case that $\eta_1 = 0$, the random field $W$ is degenerated and equal to zero. Hence, $a_{N, \varepsilon}(\omega, x) \geq \phi_1 > 0$ for all $(\omega, x) \in \Omega \times \mathcal{D}$ and the claim follows immediately. Otherwise, if $\eta_1 > 0$, we obtain in the fashion of Lemma 5.3

$$\mathbb{E}(1/a_{N, \varepsilon, \omega}^p) \leq \frac{1}{\phi_1^p} \mathbb{E}(\exp(p\psi_1\|W_N\|^2_{L^\infty(\mathcal{D})})) \leq \frac{1}{\phi_1^p} \mathbb{E}(\exp(p\psi_1\|W\|^2_{L^\infty(\mathcal{D})}))$$

which is again finite if $p < (\eta^*)^{-1}$. The proof is concluded by noting that the last estimate is independent of $N$ and $\varepsilon$. \hfill $\square$

### 3.3. Convergence of the approximated diffusion coefficient

The convergence of the approximated solution $u_{N, \varepsilon}$ to $u$ depends on the convergence of the approximated jump-diffusion coefficient $a_{N, \varepsilon}$ to $a$. We investigate this convergence by deriving separately convergence rates of the truncated Karhunen-Loève series and the approximation of the jump part.

**Theorem 3.8.** If Assumption 5.3 holds, then for any $p \geq 1$ and $N \in \mathbb{N}$ we have that

$$\|W - W_N\|_{L^p(\Omega; L^\infty(\mathcal{D}))} \leq C_p \Xi_N^{1/2},$$

where $C_p > 0$ is independent of $N$ and $\Xi_N := \sum_{i=1}^N \eta_i < \infty$.

In order to prove the bound for the truncation error, we need the following Fernique-type result.

**Theorem 3.9.** [32, Theorem 2.9] Let $(W(x), x \in \mathcal{D})$ with $\mathcal{D} \subset \mathbb{R}^d$ be a centered Gaussian field. For $\varepsilon, \delta > 0$ let

$$\varrho(\varepsilon) := \sup_{x \in \mathcal{D}, |x - y| < \varepsilon} \mathbb{E}(W(x) - W(y))^2)\delta^{1/2}$$

and $\Theta(\delta) := \int_0^\infty \varrho(\delta \exp(-y^2))dy$.

Then for all $c > 0$

$$\mathbb{P}(\sup_{x \in \mathcal{D}} W(x) > c) \leq C(\Theta^{-1}(1/c))^{-d} \exp(-\frac{\delta^2}{c^2}),$$

where $\Theta^{-1}$ is the inverse function of $\Theta$, $C > 0$ is an absolute constant and $\delta := \sup_{x \in \mathcal{D}} \mathbb{E}(W(x)^2)^{1/2}$.

**Proof of Theorem 3.8.** With a similar continuity argument as in Lemma 5.3 we obtain that

$$(W - W_N)(x) - (W - W_N)(y) \sim N(0, \sigma_N^2(x, y))$$

where $\sigma_N^2(x, y) := \sum_{i=1}^N \eta_i (e_i(x) - e_i(y))^2$ for all $x, y \in \mathcal{D}$. Hence

$$\bar{\sigma}_N^2 := \sup_{x \in \mathcal{D}} \mathbb{E}((W - W_N)(x)^2) = \sum_{i=1}^N \eta_i (e_i(x))^2 \leq \sum_{i=1}^N \eta_i = \Xi_N$$
and we see by the proof of Lemma 3.8 that for $g$ as in Theorem 3.9

$$g(\varepsilon) \leq 2^{1-b}c_e \left( \sum_{i \geq N} \eta_i^{2ab} \right)^{1/2} \leq 2^{1-b}c_e \left( \sum_{i \geq N} \eta_i^{2ab} \right)^{1/2} \varepsilon^b = : C_{e,b} \varepsilon^b$$

for any $b \in (0, \min(\beta/(2\alpha), 1))$ and each $\varepsilon > 0$. We use the estimate on $g$ to bound $\Theta$ for $\delta > 0$ by

$$\Theta(\delta) \leq C_{e,\alpha} \int_0^\infty \delta^b \exp(-y^2b)dy = \frac{C_{e,b}\sqrt{\pi}}{4\sqrt{2b}}.$$

Since $Q$ and $Q^{-1}$ are monotone increasing this yields with Theorem 3.9 for any $c > 0$

$$\mathbb{P}(\sup_{x \in D}(W - W_N)(x) > c) \leq C \left( \frac{4\sqrt{2b^2}}{C_{e,b} \sqrt{\pi}} \right)^{-d/b} \mathbb{E}^{d/b} \exp(-\frac{c^2}{2\sigma_N^2}) := C_{e,\alpha,b} \mathbb{E}^{d/b} \exp(-\frac{c^2}{2\sigma_N^2}).$$

Using again that $\mathbb{P}(\sup_{x \in D}|(W - W_N)(x)| > c) \leq 2\mathbb{P}(\sup_{x \in D}|W - W_N)(x)| > c)$ and Fubini’s Theorem, we have for any $p \geq 1$

$$\mathbb{E}(||W - W_N||_{L^p(D)}) = \int_0^\infty p\mathbb{P}^{p-1}(\sup_{x \in D}|(W - W_N)(x)| > c)dc$$

\[\leq 2C_{e,b} \mathbb{E} \int_0^\infty d/b \exp(-\frac{c^2}{2\sigma_N^2})dc\]

\[= 2^{1+p/2+d/(2b)}C_{e,\alpha,b} \mathbb{E} \Gamma((p + d/b)/2)\mathbb{E}^{d/b}\]

\[\leq 2^{1+p/2+d/(2b)}C_{e,\alpha,b} \mathbb{E} \Gamma((p + d/b)/2)\mathbb{E}^{d/b}\mathbb{E}^{p/2}2,\]

where $\Gamma(\cdot)$ is the Gamma function and we have used the substitution $z = c^2/(2\sigma_N^2)$ in the second equality. This proves the claim because the above estimate holds for any $b \in (0, \min(\beta/(2\alpha), 1))$.

**Remark 3.10.** In [10], the author proves a similar error bound for the truncation error in the Gaussian field, namely

$$\mathbb{E}(||W - W_N||_{L^p(\Omega;\mathbb{C}^{\mathbb{N}^*}(D))}) \leq C_{b,\gamma,p} \max \left( \sum_{i \geq N} \eta_i, C_e^2 \sum_{i \geq N} \eta_i^{2b} \right)^{1/2}.$$ 

In the jump-diffusion setting the error bound in Theorem 3.9 is advantageous for several reasons:

- In our setting, $a$ will in general not have Hölder continuous paths, but involves discontinuities, hence only the error in $L^p(\Omega; L^\infty(D))$ is of interest.
- In general it is rather easy to calculate the sum $\Xi_N$ if the first $N$ eigenvalues are known (can be approximated), whereas this is not necessarily the case for $\sum_{i \geq N} \eta_i^{2b}$.

**Theorem 3.11.** Under Assumption 3.3 the sampling error is, for all $s \in [1, \overline{s}]$, bounded by

$$\|P - P_\varepsilon\|_{L^s(\Omega; L^\infty(D))} \leq (\lambda(D)\varepsilon)^{1/s}.$$ 

**Proof.** For any $\omega \in \Omega$, we have

$$\|P(\cdot, \cdot) - P_\varepsilon(\cdot, \cdot)\|_{L^\infty(D)} = \sup_{s=1,\ldots,T(\omega)}|P_s(\omega) - \overline{P}_s(\omega)|.$$

By Fubini’s Theorem and integration by parts this yields

$$\|P - P_\varepsilon\|_{L^s}(\Omega; L^\infty(D)) = \int_0^\infty s^{e-1} \mathbb{P}\left( \max_{s=1,\ldots,T(\omega)}|P_s - \overline{P}_s| \geq \varepsilon \right)dc,$$

since $\lim_{c \to 0^+} \mathbb{P}(\max_{s=1,\ldots,T}|P_s - \overline{P}_s| \geq c) = 0$. For fixed $c > 0$ and $i \in \mathbb{N}$, we define the sets

$$T_i := \{\omega \in \Omega | T(\omega) = i\} \quad \text{and} \quad A_i := \{\omega \in \Omega | P_s(\omega) - \overline{P}_s(\omega) \geq c\}$$

to obtain the identity

$$\mathbb{P}\left( \max_{s=1,\ldots,T(\omega)}|P_s - \overline{P}_s| \geq c \right) = \mathbb{P}\left( \bigcup_{i \in \mathbb{N}} T_i \cap \bigcup_{j=1}^{\infty} A_j \right).$$
By the independence of $|P_i - \bar{P}_i|$ and $\tau$ this yields

$$\mathbb{P}\left(\max_{i=1,\ldots,\tau} |P_i - \bar{P}_i| \geq c \right) \leq \sum_{i \in \mathbb{N}} \mathbb{P}\left(\bigcap_{j=1}^i (\bigcup_{j=1}^{\infty} A_j) \right) = \sum_{i \in \mathbb{N}} \mathbb{P}(T_i) \mathbb{P}(\bigcup_{j=1}^i A_j) \leq \sum_{i \in \mathbb{N}} \mathbb{P}(T_i) \sum_{j=1}^i \mathbb{P}(A_j)$$

and thus by Fubini’s theorem

$$\|P - P\|_{L^r(\Omega; L^\infty(D))} = \int_0^\infty \int_0^y s^{c-1} d\mathbb{P}(\|P - P\|_{L^\infty(D)} \geq c) \, dy$$

$$\leq \int_0^\infty \mathbb{P}(\tau = i) \sum_{j=1}^i s^{c-1} \mathbb{P}(|P_j - \bar{P}_j| \geq c) \, dy$$

$$= \sum_{i \in \mathbb{N}} \mathbb{P}(\tau = i) \int_0^\infty s^{c-1} \mathbb{P}(|P_j - \bar{P}_j| \geq c) \, dy = \sum_{i \in \mathbb{N}} \mathbb{P}(\tau = i) \int_0^\infty \mathbb{E}(|P_j - \bar{P}_j|^s) \, dy$$

$$\leq \sum_{i \in \mathbb{N}} \mathbb{P}(\tau = i) \varepsilon i^c.$$ 

The claim then follows since $\mathbb{E}(\tau) = \lambda(D) < +\infty$ by Definition 3.11.

With Theorems 5.8 and 5.11 follows convergence of the approximated diffusion coefficient:

**Theorem 3.12.** Let Assumption 5.5 hold, then there exists a constant $C(s, \Phi, Q, \lambda, D) > 0$, depending only on the indicated parameters, such that for any $N \in \mathbb{N}$ and $\varepsilon > 0$

$$\|a - a_{N, \varepsilon}\|_{L^r(\Omega; L^\infty(D))} \leq C(s, \Phi, Q, \lambda, D) \left( \frac{1}{2N} + \varepsilon \frac{1}{p_1} \right).$$

Hence, $a_{N, \varepsilon}$ converges to $a$ in $L^r(\Omega; L^\infty(D))$, as $N \to \infty$ and $\varepsilon \to 0$.

**Proof.** Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $(\omega, x) \in \Omega \times D$ be fixed. By the mean-value theorem

$$a(\omega, x) - a_{N, \varepsilon}(\omega, x) = \Phi'(\xi_N(\omega, x))(W(\omega, x) - W_N(\omega, x)) + P(\omega, x) - P(\omega, x),$$

where $\xi_N(\omega, x) \in (W(\omega, x), W_N(\omega, x))$. With Assumption 5.3 (ii) on $\Phi'$ and by the triangle inequality, we obtain the estimate

$$|\Phi'(\xi_N(\omega, x))| \leq \phi_2 \exp(\psi_2(|W_N(\omega, x)| + |W(\omega, x) - W_N(\omega, x)|)).$$

The random fields $W_N$ and $W - W_N$ are independent, so for any $s \in [1, \infty)$ it holds that

$$\|a - a_{N, \varepsilon}\|_{L^r(\Omega; L^\infty(D))} \leq C(s, \Phi, Q, \lambda, D) \left( \frac{1}{2N} + \varepsilon \frac{1}{p_1} \right).$$

We proceed as in Lemma 5.5 and conclude by Fubini’s theorem that $C_2(p_1, \Phi, Q) < \infty$ and note that $C_2(p_1, \Phi, Q)$ is independent of $N$. Similarly,

$$\|a - a_{N, \varepsilon}\|_{L^r(\Omega; L^\infty(D))} \leq \phi_2 C_2(\varepsilon, |W_N(\cdot, x)|^{2s} \hat{t}(Q)) \exp\left(\frac{\|W - W_N\|_{L^\infty(D)}}{4 \hat{t}(Q)}\right) \leq C_1(s, \phi_2, Q) \leq \infty$$

for any $s$ and $C_1(s, \phi_2, Q)$ is independent of $N$. Altogether, this yields

$$\|a - a_{N, \varepsilon}\|_{L^r(\Omega; L^\infty(D))} \leq \phi_2 C_2(\varepsilon, |W_N(\cdot, x)|^{2s} \hat{t}(Q)) \exp\left(\frac{\|W - W_N\|_{L^\infty(D)}}{4 \hat{t}(Q)}\right) + \|P - \bar{P}\|_{L^r(\Omega; L^\infty(D))}$$

and the claim follows by Theorems 5.8 and 5.11.

\(\square\)
Corollary 3.13. Let Assumptions 3.3 hold, then there exists a sequence of approximation parameters \((N_i, \varepsilon_i), i \in \mathbb{N}\) in \(\mathbb{N}^N \times (0, \infty)^N\), depending only on \(\beta\) and \(\omega\), such that the error \(\|u - a_{N_i, \varepsilon_i}\|_{L^\infty(D)}\) converges to zero \(\mathbb{P}\)-almost surely as \(i \to \infty\).

Proof. For any \(\delta > 0\) we get by Markov’s inequality

\[
\mathbb{P}(\|a_{N_i, \varepsilon_i} - a\|_{L^\infty(D)} \geq \delta) \leq \frac{\|a_{N_i, \varepsilon_i} - a\|_{L^\infty(D)}^p}{\delta^p}.
\]

Using Theorem 3.12 and the inequality \((a + b)^s \leq 2^{s-1}(a^s + b^s)\) for \(a, b > 0, s \geq 1\) this leads to

\[
\sum_{i \in \mathbb{N}} \mathbb{P}(\|a_{N_i, \varepsilon_i} - a\|_{L^\infty(D)} \geq \delta) \leq \frac{2^{s-1}C(s, \psi_2, Q, \lambda, D)^s}{\delta^s} \sum_{i \in \mathbb{N}} \varepsilon_i^{2s/2} + \varepsilon_i.
\]

By Assumption 3.3 there exists \(\beta > 0\) such that

\[
\Xi_i = \sum_{j \in N_i} \eta_j j^\beta \leq N_i^{-\beta} \sum_{j \in N_i} \eta_j j^\beta \leq N_i^{-\beta} \sum_{j \in \mathbb{N}} \eta_j j^\beta \leq C_i N_i^{-\beta}.
\]

Now, choosing \(\delta > 2, N_i := [2^{i/\beta}]\) and \(\varepsilon_i := i^{-\delta/2}\) for \(i \in \mathbb{N}\) yields the estimate

\[
\sum_{i \in \mathbb{N}} \mathbb{P}(\|a_{N_i, \varepsilon_i} - a\|_{L^\infty(D)} \geq \delta) \leq \frac{2^{s-1}C(s, \psi_2, Q, \lambda, D)^s}{\delta^s} (C_i^{\delta/2} + 1) \sum_{i \in \mathbb{N}} i^{-\delta/2} \leq \infty,
\]

and the sequence \((a_{N_i, \varepsilon_i}, i \in \mathbb{N})\) converges almost surely by the Borel-Cantelli lemma. \(\square\)

3.4. Convergence of the approximated solution. We conclude this chapter by showing the convergence of \(u_{N, \varepsilon}\) towards \(u\), given that \(a_{N, \varepsilon} \to a\) in \(L^p(\Omega; L^{\infty}(D))\).

Theorem 3.14. Let Assumptions 3.3 hold such that \(t := (2/p + 1/q + 1/s)^{-1} \geq 1\), where \(p \in [1, (\eta^*)^{-1}]\) for \(\eta^*\) as in Lemma 3.3. Then \(u_{N, \varepsilon}\) converges to \(u\) in \(L^t(\Omega; V)\) as \(N \to +\infty\) and \(\varepsilon \to 0\).

Proof. Existence and uniqueness of weak solutions \(u \in L^t(\Omega; V)\) resp. \(u_{N, \varepsilon} \in L^t(\Omega; V)\), for \(r \in [1, (1/q + \eta^*)^{-1}]\), is guaranteed by Lemma 3.3 resp. Lemma 3.7 \(\mathbb{P}\)-almost surely. For notational convenience, we omit the argument \(\omega \in \Omega\) in the following pathwise estimates with respect to the norm \(|\cdot|_V\). With Poincaré’s inequality and \(a_{N, \varepsilon} - (\omega) > 0\) \(\mathbb{P}\)-a.s., we obtain the (pathwise) estimate

\[
\|u - u_{N, \varepsilon}\|_V^2 \leq \frac{1}{a_{N, \varepsilon} - a} \int_D a_{N, \varepsilon} ((u - u_{N, \varepsilon})^2 + (\nabla u - \nabla u_{N, \varepsilon})^2) dx
\]

\[
\leq \frac{1 + C_{[D]}^2}{a_{N, \varepsilon} - a} \int_D a_{N, \varepsilon} (\nabla u - \nabla u_{N, \varepsilon})^2 dx,
\]

where \(C_{[D]} > 0\) denotes the Poincaré constant. Since \(u\) and \(u_{N, \varepsilon}\) are weak solutions of Problem 2.1 with \(\eta = \omega\) resp. Eq. 3.3, we have

\[
\int_D a_{N, \varepsilon} \nabla u \cdot \nabla u_{N, \varepsilon} dx = \int_D a_\nu \nabla u \cdot \nabla u_{N, \varepsilon} dx \quad \text{and} \quad \int_D a_{N, \varepsilon} (\nabla u_{N, \varepsilon})^2 dx = \int_D a \nabla u \cdot \nabla u_{N, \varepsilon} dx,
\]

almost surely, and hence

\[
\int_D a_{N, \varepsilon} (\nabla u - \nabla u_{N, \varepsilon})^2 dx = \int_D (a_{N, \varepsilon} - a) \nabla u \cdot (\nabla u - \nabla u_{N, \varepsilon}) dx.
\]

By Hölder’s inequality, \(V \subset L^2(D)\), Eq. 2.3 and Theorem 2.5 we have

\[
\|u - u_{N, \varepsilon}\|_V^2 \leq \frac{1 + C_{[D]}^2}{a_{N, \varepsilon} - a} \|a_{N, \varepsilon} - a\|_{L^\infty(D)} \|\nabla u\|_{L^2(D)} \|\nabla (u - u_{N, \varepsilon})\|_{L^2(D)}
\]

\[
\leq \frac{1 + C_{[D]}^2}{a_{N, \varepsilon} - a} \|a_{N, \varepsilon} - a\|_{L^\infty(D)} \|u\|_{L^2(D)} \|u - u_{N, \varepsilon}\|_V
\]

\[
\leq 2(1 + C_{[D]}^2) \max(1, C_{[D]}^2) (\|u\|_{L^2(D)} + \|\nabla (u)\|_{L^2(D)}) \min(1, C_{[D]}^2) a_{N, \varepsilon} - a \|a_{N, \varepsilon} - a\|_{L^\infty(D)} \|u - u_{N, \varepsilon}\|_V.
\]
Using $t = (2/p + 1/q + 1/s)^{-1} \geq 1$, the $t$-th moment of the pathwise error is bounded by
\begin{equation}
\|u - u_{N,\varepsilon}\|_{L^t(\Omega; V)} \leq (1 + C^2_{|D|}) C(a, D, p) \left[ \|a_{N,\varepsilon}\|_{L^p(\Omega, \mathbb{R})} \cdot (\|a\|_{L^1(\Omega)} + \|a\|_{L^1(\Omega, L^2(\Gamma_2))}) \right] a_{N,\varepsilon} - a \|_{L^t(\Omega, L^\infty(D))},
\end{equation}
where $C(a, D, p) > 0$ is as in Theorem 2.5. The convergence now follows by Theorem 3.12 since $\|a_{N,\varepsilon}\|_{L^p(\Omega, \mathbb{R})}$ is bounded uniformly in $N$ and $\varepsilon$ by Lemma 3.7.

Corollary 3.15. With the Assumptions of Theorem 3.14, there exists a sequence of approximation parameters $((N_i, \varepsilon_i), i \in \mathbb{N})$ in $\mathbb{N}^3 \times (0, \infty)^N$, depending only on $\beta$ and $t$, such that $\|u_{N_i,\varepsilon_i} - u\|_{V}$ converges to zero $\mathbb{P}$-almost surely as $i \to +\infty$.

Proof. The proof is analogous to the one of Corollary 3.13 with Markov’s inequality applied to the mapping $x \mapsto x^t$ and Ineq. (3.7). The sequence $(N_i, \varepsilon_i, i \in \mathbb{N})$ may then also be constructed in the same way as in Corollary 3.13 where we simply replace $s$ by $t$.

Knowing that $u_{N,\varepsilon}$ converges to $u$ pathwise and in $L^t(\Omega; V)$, we aim to estimate moments of $u$ by drawing samples from the distribution of $u_{N,\varepsilon}$. In general, the distribution of $u_{N,\varepsilon}$ is not known. Further, each pathwise solution is an element of the infinite dimensional Hilbert space $V$, which in turn means that we are only able to simulate pathwise approximations of the functions $u_{N,\varepsilon}(\omega, \cdot)$ in a finite-dimensional subspace of $V$. Next, we show how to construct these approximations in some appropriate subspaces $V_\ell$ of $V$ and how the discretization error may be controlled.

4. Adaptive pathwise discretization

The variational problem (3.4) to Eq. (3.3) is to find for almost all $\omega \in \Omega$ and given $f(\omega, \cdot)$, $g(\omega, \cdot)$, $N$ and $\varepsilon$ a function $u_{N,\varepsilon}(\omega, \cdot) \in V$ such that
\begin{equation}
B_{a_{N,\varepsilon}}(u_{N,\varepsilon}(\omega, \cdot), v) := \int_D a_{N,\varepsilon}(\omega, x) \nabla u_{N,\varepsilon}(\omega, x) \cdot \nabla v(x) dx
= \int_D f(\omega, x) v(x) dx + \int_{\Gamma_2} g(\omega, x) [Tv](x) dx = F_v(v)
\end{equation}
for all $v \in V$. To find suitable approximations of $u_{N,\varepsilon}(\omega, \cdot)$, we use a standard Galerkin approach with a nested sequence $V = (V_\ell, \ell \in \mathbb{N})$ of finite dimensional subspaces $V_\ell \subset V$. The corresponding family of refinement sizes is given by a sequence $(h_\ell, \ell \in \mathbb{N})$, which decreases monotonically to zero as $\ell \to \infty$. For any $\ell \in \mathbb{N}$, let $d_\ell \in \mathbb{N}$ and $\{v_1, \ldots, v_{d_\ell}\}$ be a basis of $V_\ell$. The discrete version of the variational formulation (3.3) is then: Find $u_{N,\varepsilon, \ell}(\omega, \cdot) \in V_\ell$ such that
\begin{equation}
B_{a_{N,\varepsilon}}(u_{N,\varepsilon, \ell}(\omega, \cdot), v_\ell) = F_v(v_\ell)
\end{equation}
for all $v_\ell \in V_\ell$. The function $u_{N,\varepsilon, \ell}(\omega, \cdot)$ may be expanded with respect to the basis $\{v_1, \ldots, v_{d_\ell}\}$ as
\begin{equation}
u_{N,\varepsilon, \ell}(\omega, x) = \sum_{i=1}^{d_\ell} c_i v_i(x),
\end{equation}
where $c_1, \ldots, c_{d_\ell} \in \mathbb{R}$ and $c := (c_1, \ldots, c_{d_\ell})^T$ is the respective coefficient (column-)vector. With this, the discrete variational problem in the finite dimensional space $V_\ell$ is equivalent to solving the linear system of equations
\begin{equation}
A(\omega)c = F(\omega),
\end{equation}
with stochastic stiffness matrix $(A(\omega))_{ij} = B_{a_{N,\varepsilon}}(\omega)(v_i, v_j)$ and load vector $(F(\omega))_i = F_v(v_i)$ for $i, j \in \{1, \ldots, d_\ell\}$. Since the jump-diffusion coefficient is not continuous, in general one would not expect the full convergence rate of the Galerkin approximation.

Example 4.1. For a polygonal domain $D \subset \mathbb{R}^d$, we define by $K = (K_\ell, \ell \in \mathbb{N})$ a nested sequence of triangulations on $D$. The maximum diameter of each triangulation is defined by $h_\ell := \max_{K \in K_\ell} \text{diam}(K)$, $\ell \in \mathbb{N}$
and the finite-dimensional subspaces are given by
\begin{equation}V_\ell := \{v \in V | v|_K \in \mathcal{P}_1, K \in K_\ell\},
\end{equation}
where $\mathcal{P}_1$ is the space of all polynomials up to degree one. This yields a sequence $\mathcal{V} = (V_\ell, \ell \in \mathbb{N}_0)$ of nested subspaces in $V$ with refinement parameters $(h_\ell, \ell \in \mathbb{N}_0)$. For fixed $\ell \in \mathbb{N}_0$, let $\{v_1, \ldots, v_d_\ell\}$ be a basis of piecewise linear functions of $V_\ell$. Given that $u_{N, \varepsilon} \in L^2(\Omega; H^{1+\kappa_a}(D))$ for some $\kappa_a > 0$, the pathwise discretization error is bounded by

$$
\|u_{N, \varepsilon}(\omega, \cdot) - u_{N, \varepsilon}(\omega, \cdot)\|_V \leq C\|u_{N, \varepsilon}(\omega, \cdot)\|_{H^{1+\kappa_0}(D)} h_\ell^{\min(\kappa_a, 1)}
$$

almost surely, see e.g. [28 Theorem 8.4.8 f.], where the constant $C > 0$ only depends on $\mathcal{D}$ and $K$. If $\mathcal{K}$ and thus $(h_\ell, \ell \in \mathbb{N}_0)$ is fixed for any $\omega$ and there exists an uniform bound $\|u_{N, \varepsilon}\|_{L^2(\Omega; H^{1+\kappa_0}(D))} \leq \overline{C}_u$ in $N$ and $\varepsilon$, we readily obtain

$$
\|u_{N, \varepsilon} - u_{N, \varepsilon, \ell}\|_{L^2(\Omega; V)} \leq C_u h_\ell^{\min(\kappa_a, 1)}
$$

where $C_u := C\overline{C}_u$. We note that a uniform a-priori bound $\overline{C}_u$ may require higher moments of $1/a_{N, \varepsilon}, f$ and $g$ or even essential bounds on $a_{N, \varepsilon}$ which are not ensured by Assumption 3.3.

**Remark 4.2.** For jump-diffusion problems, we obtain, in general, a discretization error of order $\kappa_a \in (1/2, 1)$. We cannot expect the pathwise "full" order of convergence $\kappa_a = 1$ of the finite-dimensional discretization error, since the diffusion coefficient $a_{N, \varepsilon}$ is almost surely discontinuous. Most results that ensure $H^2(D)$-regularity of the pathwise weak solution $u_{N, \varepsilon}(\omega, \cdot)$ need that $a_{N, \varepsilon}(\omega, \cdot)$ is continuously differentiable or that $a_{N, \varepsilon}(\omega, \cdot) \in W^{1, \infty}(D)$, see for instance [24]. The latter would imply that $a_{N, \varepsilon}(\omega, \cdot)$ is continuous by the Sobolev embedding theorem, which contradicts our setting. As we consider pathwise regularity, we can rely on results for deterministic elliptic problems with discontinuous coefficients. We refer to [35] and the references therein, where the author emphasizes that the regularity of the solution depends on the shape and magnitude of the discontinuities. For several examples $H^{1+\kappa_a}$-regularity with $\kappa_a < 1$ is shown. In [7] the author states that without special treatment of the interfaces with respect to the triangulation one, in general, cannot expect a better pathwise convergence rate than $\kappa_a = 1/2$.

### 4.1. Adaptive triangulations.

In view of the previous remark, we aim to increase the order of convergence $\kappa_a$ with respect to $h_\ell$. For this, we employ path-dependent triangulations to match the interfaces generated by the samples of the jump-diffusion coefficient. Hence, we need to reformulate the discrete problem with respect to $\omega$, since the triangulation and matching basis functions may be sample-dependent. Given a fixed $\omega$ and $\ell$, we consider a finite dimensional subspace $\tilde{V}_\ell(\omega) \subset V$ with sample-dependent basis $\{\tilde{v}_1(\omega), \ldots, \tilde{v}_{d_\ell}(\omega)\}$ and stochastic dimension $d_\ell(\omega) \in \mathbb{N}$. As before, we assume the subspaces $(\tilde{V}_\ell(\omega), \ell \in \mathbb{N}_0)$ to be nested and denote the corresponding sequence of (random) refinement parameters by $(\tilde{h}_\ell(\omega), \ell \in \mathbb{N}_0)$, i.e. for a given random partition $\mathcal{T}(\omega) = (\mathcal{T}_i, i = 1, \ldots, \tau(\omega))$ of $D$, we choose a triangulation $\mathcal{K}_\ell(\omega)$ of $D$ with respect to $\mathcal{T}(\omega)$ such that

$$
\mathcal{T}(\omega) \subset \mathcal{K}_\ell(\omega) \quad \text{and} \quad \tilde{h}_\ell(\omega) := \max_{K \subset \mathcal{K}_\ell(\omega)} \text{diam}(K) \leq \overline{\tilde{h}}_\ell \quad \text{for} \quad \ell \in \mathbb{N}_0,
$$

where $(\mathcal{T}_i, \ell \in \mathbb{N}_0)$ is a positive sequence of deterministic refinement thresholds, decreasing monotonically to zero. This guarantees that $\tilde{h}_\ell(\omega) \to 0$ almost surely, although the absolute speed of convergence may vary for each $\omega$.

The pathwise discrete variational problem in the sample-adaptive subspace $\tilde{V}_\ell(\omega)$ now reads: Find $\tilde{u}_{N, \varepsilon, \ell}(\omega, \cdot) \in \tilde{V}_\ell(\omega)$ such that

$$
B_{u_{N, \varepsilon}, i}(\omega, \cdot, \tilde{u}_{N, \varepsilon, \ell}(\omega, \cdot), \tilde{v}_i(\omega)) = F_{\omega}(\tilde{v}_i(\omega)) \quad \text{for all} \quad \tilde{v}_i(\omega) \in \tilde{V}_\ell(\omega) \subset V.
$$

Since the triangulation is aligned with the discontinuities of $a_{N, \varepsilon}$, this approximation admits an increase of the (pathwise) order of convergence $\kappa_a$ compared to the non-adaptive, deterministic Galerkin approximations. For piecewise linear basis functions $\tilde{v}_i$, we can expect

$$
\|u_{N, \varepsilon}(\omega, \cdot) - \tilde{u}_{N, \varepsilon, \ell}(\omega, \cdot)\|_V \leq \overline{C}_1(\omega) \left( \left\| u_{N, \varepsilon}(\omega, \cdot) \right\|_V + \sum_{i=1}^{\tau(\omega)} \| u_{N, \varepsilon}(\omega, \cdot) \|_{L^2(\mathcal{T}_i(\omega))} \right) \tilde{h}_\ell(\omega)
$$

$$
\Rightarrow \overline{C}_u(\omega)
$$
\(\mathbb{P}\)-almost surely by results from domain decomposition methods for deterministic elliptic problems, see i.e. \([14, 15]\). This estimate holds provided the random partitions are polygonal and \(u_{N,\varepsilon}(\omega, \cdot)\) is almost surely piecewise in \(H^2(T_i)\). In this case, the adaptive triangulations perfectly fit the random subdomains in \(D\) and the order of convergence is the same as if \(a_{N,\varepsilon} \in H^2(D)\). If the random partitions are not polygonal but form \(C^2\)-interfaces within \(D\), we obtain log-linear rates:

\[
\|u_{N,\varepsilon}(\omega, \cdot) - \tilde{u}_{N,\varepsilon}(\omega, \cdot)\|_V \leq C_2(\omega)\tilde{C}_u(\omega) \log(\tilde{h}_\ell(\omega))^{0.5}
\]

with \(\tilde{C}_u(\omega)\) as in Ineq. (4.3), see \([18]\).

**Remark 4.3.** Based on Ass. 4.4 and the pathwise estimates in Eqs. (4.3), (4.4), the question arises whether it is possible to derive estimates in a mean-square sense

\[
\|u_{N,\varepsilon} - \tilde{u}_{N,\varepsilon}\|_{L^2(\Omega; V)} \leq \tilde{C}_u \mathbb{E}(\tilde{h}_\ell^2)^{1/2},
\]

where the constant \(\tilde{C}_u\) is independent of \(N\) and \(\varepsilon\). As the independence of \(\varepsilon\) is not an issue, an uniform estimate with respect to \(N\) requires further summability conditions (i.e. \(\beta \geq 2\alpha\)) on the eigenvalues in Ass. 3.3 and would result in a piecewise differentiable diffusion coefficient \(a\). It is in this more restrictive setting straightforward to derive the corresponding \(L^2(\Omega; V)\)-estimates, however, this would exclude the important cases where the covariance operator of \(W\) is of Brownian-motion-type or exponential. Practically, as we show in Section 5 the pathwise adaptive convergence rates may also be recovered for the \(L^2(\Omega, V)\)-error if \(a\) has only piecewise Hölder-continuous trajectories.

In view of the preceding remarks and Example 4.1 we make the following assumption on the mean-square discretization error:

**Assumption 4.4.** There exists constants \(C_{u,a}, C_{u,n}, \kappa_a, \kappa_n > 0\), such that for any \(N, \ell \in \mathbb{N}_0\) and \(\varepsilon > 0\), the finite-dimensional approximation errors of \(u_{N,\varepsilon}\) in the subspaces \(\tilde{V}_\ell\) resp. \(V_\ell\) are bounded by

\[
\|u_{N,\varepsilon} - \tilde{u}_{N,\varepsilon}\|_{L^2(\Omega; V)} \leq C_{u,n} \mathbb{E}(\tilde{h}_\ell^{2\alpha_n})^{1/2} \quad \text{resp.} \quad \|u_{N,\varepsilon} - \tilde{u}_{N,\varepsilon}\|_{L^2(\Omega; V)} \leq C_{u,n} h_\ell^{\alpha_n}.
\]

The constants \(C_{u,a}, C_{u,n}\) may depend on \(a, f\) and \(g\), but are independent of \(\tilde{h}_\ell, h_\ell, \kappa_a\) and \(\kappa_n\).

Note that in general \(1 \geq \kappa_a > \kappa_n > 0\) by the previous observations. We consider \(\mathbb{E}(\tilde{h}_\ell^{2\alpha_n})^{1/2}\) instead of \(\mathbb{E}(\tilde{h}_\ell^{2\alpha_n})\) as both quantities depend to a great extend on the geometry introduced by \(a\). The parameter \(\tilde{h}_\ell\) has been merely introduced to ensure that \(\lim_{\ell \to \infty} \mathbb{E}(\tilde{h}_\ell^2) = 0\).

5. Estimation of expectations by Monte Carlo methods

As we are able to generate samples from \(u_{N,\varepsilon}\) and control for the discretization error in each sample, we may estimate the expected value \(\mathbb{E}(u)\) of the weak solution to Eq. (2.1). We focus on multilevel Monte Carlo estimators as they are easily implemented and do not require much regularity of \(u\). Monte Carlo estimators introduce an additional statistical bias besides the error contributions of \(a_{N,\varepsilon}\) and the pathwise discretization error from Section 4. However, this error can be controlled under natural assumptions and it may be equilibrated according to the other error terms. In this section, we first recall briefly standard Monte Carlo and multilevel Monte Carlo methods to estimate \(\mathbb{E}(u)\) and then control for the mean-squared error in both algorithms. We also suggest a modification of the multilevel Monte Carlo estimation to increases computational efficiency before we verify our results on several numerical examples in Section 6.

5.1. Monte Carlo and multilevel Monte Carlo estimators. Consider a sequence \((u^{(i)}, i \in \mathbb{N})\) of i.i.d. copies of the \(V\)-valued random variable \(u\). For \(M \in \mathbb{N}\) independent samples, the Monte Carlo estimator of \(\mathbb{E}(u)\) is defined as

\[
E_M(u) := \frac{1}{M} \sum_{i=1}^{M} u^{(i)}.
\]
While this is straightforward for $M$ equilibrated to obtain an RMSE of order $\mathcal{O}(\sqrt{N})$. The estimate in Theorem 5.2 suggests that all four error contributions should be bounded uniformly with respect to these parameters by Lemma 3.7. Finally, Assumption 4.4 and Remark 5.3 yield for the third term

\[
\frac{1}{\sqrt{M}} + \mathcal{O}(\sqrt{N}) + \mathcal{O}(\eta^2) + \mathcal{O}(\kappa^2),
\]

where $C_{MC} > 0$ is a constant independent of $M, \Xi_N$, and $\tilde{h}_k$.

**Proof.** We apply the triangle inequality to obtain

\[
\|\hat{u} - u\|_{L^2(V)} \leq \|\hat{u} - u\|_{L^2(V)} + \|u - u_N\|_{L^2(V)} + \|u_N - u\|_{L^2(V)}.
\]

By the first part of Lemma 5.1 and Theorem 2.2 we bound the first term by

\[
\|\hat{u} - u\|_{L^2(V)} \leq C(a, g, D, p)(\|f\|_{L^2(\Omega; H)} + \|g\|_{L^2(\Omega; \mathbb{R})})\frac{1}{\sqrt{M}}.
\]

The second part of Lemma 5.1 yields with the estimate of Theorem 3.14

\[
\|u_N - u\|_{L^2(V)} \leq \bar{C}(a, f, g, D)\|u_N\|_{L^2(\Omega; \mathbb{R})} + \mathcal{O}(\sqrt{N}) + \mathcal{O}(\eta^2) + \mathcal{O}(\kappa^2),
\]

where $\bar{C}(a, f, g, D), C(a, f, g, D) > 0$ are independent of $N$ and $\varepsilon$, because $\|u_N - u\|_{L^2(\Omega; \mathbb{R})}$ is bounded uniformly with respect to these parameters by Lemma 5.7. Finally, Assumption 4.3 and Lemma 5.1 yield for the third term

\[
\|u_N - u\|_{L^2(V)} \leq C_{u,a}\mathbb{E}(\tilde{h}_k^2)\sqrt{N},
\]

where $C_{u,a}$ is also independent of $N$ and $\varepsilon$ by assumption.

**Remark 5.3.** The estimate in Theorem 5.2 suggests that all four error contributions should be equilibrated to obtain an RMSE of order $\mathcal{O}(\sqrt{N})$. For this, we may choose $M, N$ and $\varepsilon$ such that

\[
M^{-1} \simeq \mathbb{E}_N \simeq \varepsilon^{2/\beta} \simeq \mathbb{E}(\tilde{h}_k^2).
\]

While this is straightforward for $M$ and usually also for $\varepsilon$, the choice of the truncation index $N$ involves a few difficulties. In general, the eigenvalues $(\lambda_i, i \in N)$ of the covariance operator $Q : H \to H$ will not be available in closed form. The decay parameter $\beta > 0$ may be unknown and only be bounded from below, which would result in an overestimation of the term $\Xi_N$. One possibility to find $N$ is applicable if $Q$ is an integral operator of the form

\[
(Q\varphi)(x) = v \int_D k_Q(x, y)\varphi(y)dy, \quad \varphi \in H,
\]

with some nonnegative, symmetric and bounded kernel function $k_Q : D^2 \to \mathbb{R}$ and $v > 0$. In this case, the eigenvalues of $Q$ fulfill the identity $v \int_D dx = \sum_{i \in N} \xi_i$ (see [23, 40]). Operators with this property are widely used in practice and include for instance Matérn class, $\gamma$-exponential and...
rational quadratic covariance functions (see [36]). The first eigenvalues of $Q$ have to be, in any case, determined (numerically) to approximate the Gaussian field $W$, so we select $N$ such that

$$\Xi_N = u \int_{\mathcal{D}} dx - \sum_{i=1}^{N} \eta_i \Xi(\hat{N}_t^{2\epsilon}).$$

In most cases, the sampling of $a_{N,\epsilon}$ and $u_{N,\epsilon,\ell}$ for given boundary data will be computationally expensive: If the eigenvalues of $Q$ decay slowly, it is necessary to include a large number of terms in the Karhunen-Loève expansion to achieve $\Xi_N \simeq \Xi(\hat{N}_t^{2\epsilon})$. In addition, sampling of the sequence $(\hat{P}_i, i \in \mathbb{N})$ might also be time-consuming if a small error $\mathbb{E}(|\hat{P}_i - P_i|) \leq \epsilon$ is desired. Given a sample of $u_{N,\epsilon}$, one has then to rely on numerical integration schemes to calculate the entries of the stiffness matrix $A(\omega)$ and the load vector $F(\omega)$, and solve a possibly large system of linear equations. This motivates the use of advanced Monte Carlo techniques, such as multilevel Monte Carlo, to achieve essentially the same accuracy with reduced computational effort. We briefly recall the idea of the multilevel Monte Carlo sampling in the following.

For $L \in \mathbb{N}$ we consider nested, finite-dimensional subspaces $\mathcal{V}_0 \subset \cdots \subset \mathcal{V}_L$ of $\mathcal{V}$ with refinement sizes $\hat{h}_0 > \cdots > \hat{h}_L > 0$. We define $u_{N,\epsilon,\ell,-1} = 0$ and expand the “finest level approximation” $u_{N,\epsilon,\ell,L}$ into a telescopic sum to obtain

$$\mathbb{E}(u_{N,\epsilon,\ell,L}) = \sum_{\ell=0}^{L} \mathbb{E}(\hat{g}_{N,\epsilon,\ell,L} - \hat{u}_{N,\epsilon,\ell-1,L}).$$

Instead of estimating the left hand side by the ordinary Monte Carlo method, we estimate the expected corrections $\mathbb{E}(\hat{g}_{N,\epsilon,\ell,L} - \hat{u}_{N,\epsilon,\ell-1,L})$ by generating $M_\ell$ independent realizations $\hat{g}_{N,\epsilon,\ell} = \hat{u}_{N,\epsilon,\ell-1}$ on each level and calculating the Monte Carlo estimator $E_{M_\ell}(\hat{g}_{N,\epsilon,\ell,L} - \hat{u}_{N,\epsilon,\ell-1,L})$.

The multilevel Monte Carlo estimator of $u_{N,\epsilon,\ell,L}$ is then defined as

$$(5.1) \quad E^L(u_{N,\epsilon,\ell,L}) := \sum_{\ell=0}^{L} E_{M_\ell}(\hat{g}_{N,\epsilon,\ell,L} - \hat{u}_{N,\epsilon,\ell-1,L}) = \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \hat{g}_{N,\epsilon,\ell} - \hat{u}_{N,\epsilon,\ell-1}.$$}

To achieve a desired target RMSE of $\epsilon_{RMS} > 0$, this estimator requires less computational effort than the standard Monte Carlo approach under certain assumptions. This, by now, classical result was proven in [25] Theorem 3.1. The proof is rather general and may readily be transferred to the problem of estimating moments of random PDEs (see [12]).

**Theorem 5.4.** Let Assumptions 3.3 and 4.4 hold such that $t := (2/p + 1)\in (p)^{-1} \geq 2$, where $p \in (1, (\gamma^*)^{-1})$ for $\gamma^*$ as in Lemma 4.3, and for $L \in \mathbb{N}$, let $h_\ell > 0$, $M_\ell, N_\ell \in \mathbb{N}$ and $\epsilon_{\ell} > 0$ be the level dependent approximation parameters for any $\ell = 0, \ldots, L$. The RMSE of the multilevel Monte Carlo estimator is then bounded by

$$\|E(u) - E^L(u_{N,\epsilon,\ell,L})\|_{L^2(\Omega; V)} \leq C\left(\xi_{N_{L}}^{1/2} + \epsilon_{L}^{1/2} + \mathbb{E}(\hat{N}^{2\epsilon}_{L})^{1/2} + \frac{1}{\sqrt{M_0}} \sum_{\ell=0}^{L} \frac{\xi_{N_{\ell}}^{1/2} + \epsilon_{\ell}^{1/2} + \mathbb{E}(\hat{N}^{2\epsilon}_{\ell})^{1/2}}{\sqrt{M_\ell}}\right).$$

where $C > 0$ is a constant independent of $L$ and the level-dependent approximation parameters.

**Proof.** Using the triangle inequality and Jensen’s inequality for expectations, we observe that

$$\|E(u) - E^L(u_{N,\epsilon,\ell,L})\|_{L^2(\Omega; V)} \leq \|E(u) - E(\hat{u}_{N,\epsilon,\ell,L})\|_{L^2(\Omega; V)} + \|E(\hat{u}_{N,\epsilon,\ell,L}) - E^L(\hat{U}_{N,\epsilon,\ell,L})\|_{L^2(\Omega; V)}.$$

Theorem 3.14 and Assumption 4.4 give a bound for the first term by

$$I \leq C(a, f, g, D)(\xi_{N_{L}}^{1/2} + \epsilon_{L}^{1/2}) + C_{a, a} \mathbb{E}(\hat{N}^{2\epsilon}_{L})^{1/2}.$$
where \(C(a, f, g, D) > 0\) is an independent constant. For the second error term, the definition of \(E^L\) in Eq. (5.1) together with Lemma 5.1 yield

\[
II \leq \sum_{\ell=0}^{L} \|\mathbb{E}(\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell} - \tilde{u}_{N_{\ell-1}, \varepsilon_{\ell-1}, \ell-1})\|_{L^2(\Omega; V)}
\]

\[
\leq \sum_{\ell=0}^{L} \frac{1}{M_\ell} \|\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell} - \tilde{u}_{N_{\ell-1}, \varepsilon_{\ell-1}, \ell-1}\|_{L^2(\Omega; V)}
\]

\[
\leq \sum_{\ell=0}^{L} \frac{1}{M_\ell} \|\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell} - u\|_{L^2(\Omega; V)} + \frac{1}{M_\ell} \|u - \tilde{u}_{N_{\ell-1}, \varepsilon_{\ell-1}, \ell-1}\|_{L^2(\Omega; V)}.
\]

Now, the terms \(\|\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell} - u\|_{L^2(\Omega; V)}\) may be treated analogously to \(I\):

\[
\|\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell} - u\|_{L^2(\Omega; V)} \leq C(a, f, g, D)(\|u\|_{L^2(\Omega; H)} + \|u\|_{L^2(\Omega; V)})
\]

Finally, we arrive at the estimate

\[
I + II \leq C(a, f, g, D)(\|u\|_{L^2(\Omega; H)} + \|u\|_{L^2(\Omega; V)})
\]

and set \(C := \max(2C(a, f, g, D), 2C_{\varepsilon, \ell}, C(a, f, g, D)(\|f\|_{L^2(\Omega; H)} + \|g\|_{L^2(\Omega; V)})). \)

Regarding a single realization of \(\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell}\), we want the pathwise error \(\|\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell}(\omega, \cdot) - u(\omega, \cdot)\|_V\) to decrease as \(\ell\) increases, meaning we need to select the parameters \(\varepsilon_{\ell}\) and \(\mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell}})^{1/2}\) decreasing in \(\ell\) and \(N_\ell\) increasing in \(\ell\). For example, the refinement parameter \(\mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell}})^{1/2}\) may be divided by a factor of two in each level, i.e. \(2\mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell}})^{1/2} = \mathbb{E}(\tilde{h}_{\ell-1}^{2\varepsilon_{\ell}})^{1/2}\) for any \(\ell \in \mathbb{N}\). Similar refining factors may be imposed for the sum of the remaining eigenvalues \(\sum N_\ell\) and the sampling errors \(\varepsilon_{\ell}\). One advantage of the multilevel Monte Carlo estimator is that we are now able to even out the error contributions of the sampling bias \(\|\tilde{u}_{N_\ell, \varepsilon_{\ell}, \ell} - u\|_V\) and the statistical error (with respect to \(M_\ell\)) on each level. This is achieved by generating relatively few of the accurate, but expensive, samples for large \(\ell\) and generating more of the cheap, but less accurate, samples on the lower levels.

**Corollary 5.5.** Let the Assumptions of Theorem 5.4 hold. For \(L \in \mathbb{N}\) and given refinement parameters \(\mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell}})^{1/2} > \cdots > \mathbb{E}(\tilde{h}_{L-1}^{2\varepsilon_{\ell}})^{1/2} > 0\) choose \(N_\ell \in \mathbb{N}\) and \(\varepsilon_{\ell} > 0\) such that

\[
\Xi_{N_\ell} \equiv \varepsilon_{\ell}^{2\varepsilon_{\ell}} \equiv \mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell}}) \quad \text{for} \quad \ell = 0, \ldots, L,
\]

and \(M_\ell \in \mathbb{N}\) such that for some arbitrary \(\varepsilon > 0\)

\[
M_0^{-1} = \mathbb{E}(\tilde{h}_L^{2\varepsilon}) \quad \text{and} \quad M_\ell^{-1} = \mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell-1}}) \mathbb{E}(\tilde{h}_{\ell-1}^{2\varepsilon_{\ell}}) (\ell + 1)^{-2(1+\varepsilon)} \quad \text{for} \quad \ell = 1, \ldots, L.
\]

The RMSE of the corresponding multilevel Monte Carlo estimator is then of order \(\mathbb{E}(\tilde{h}_L^{2\varepsilon})^{1/2}\):

\[
\|\mathbb{E}(u) - E^L(\tilde{u}_{N_{\varepsilon_{\ell}, \ell}, L})\|_{L^2(\Omega; V)} = O(\mathbb{E}(\tilde{h}_L^{2\varepsilon})^{1/2}).
\]

**Proof.** With the above choice of the approximation parameters \(N_\ell, \varepsilon_{\ell}\) and Theorem 5.4 we obtain

\[
\|\mathbb{E}(u) - E^L(\tilde{u}_{N_{\varepsilon_{\ell}, \ell}, L})\|_{L^2(\Omega; V)} \leq C \left(4\mathbb{E}(\tilde{h}_L^{2\varepsilon})^{1/2} + \sum_{\ell=0}^{L} \frac{\mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell}})^{1/2}}{\sqrt{M_\ell}}\right)
\]

\[
\leq C \left(4\mathbb{E}(\tilde{h}_L^{2\varepsilon})^{1/2} + \mathbb{E}(\tilde{h}_0^{2\varepsilon})^{1/2}\mathbb{E}(\tilde{h}_L^{2\varepsilon})^{1/2} + \sum_{\ell=0}^{L} \mathbb{E}(\tilde{h}_L^{2\varepsilon_{\ell}})^{1/2}(\ell + 1)^{-1-\varepsilon}\right)
\]

\[
\leq C (4 + \mathbb{E}(\tilde{h}_0^{2\varepsilon})^{1/2} + \zeta(1+\varepsilon))\mathbb{E}(\tilde{h}_L^{2\varepsilon})^{1/2},
\]

where \(\zeta(\cdot)\) is the Riemann zeta function. \(\square\)
For the level-dependent choice of $N_{L}$ and $\varepsilon_{L}$ we refer to Remark 5.3. In the remainder of this section, we introduce a modification of the multilevel Monte Carlo method to further reduce computational complexity.

5.2. Bootstrapping multilevel Monte Carlo. Recall the multilevel Monte Carlo estimator

$$E^{L}(\hat{u}_{N_{L},\varepsilon_{L},L}) = \sum_{\ell=0}^{L} \frac{1}{M_{\ell}} \sum_{i=1}^{M_{\ell}} (\hat{u}_{N_{L},\varepsilon_{L},\ell}^{(i)} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell-1}^{(i)})$$

of $E(u)$ as in Eq. (5.1), where the terms in the second sum are independent copies of the corrections $\hat{u}_{N_{L},\varepsilon_{L},\ell} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell-1}$. In total, one has to generate $M_{L} + M_{L+1}$ samples of $\hat{u}_{N_{L},\varepsilon_{L}}$ for each $\ell = 0, \ldots, L$ (where we have set $M_{L+1} = 0$). This could be expensive even in low dimensions $d$. We can reduce this effort if we “recycle” the already available samples and generate the differences $\hat{u}_{N_{L},\varepsilon_{L},\ell}^{(i)} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell-1}^{(i)}$ based on the same realization $\hat{u}_{N_{L},\varepsilon_{L}}^{(i)}$ for $\ell = 0, \ldots, L$. That is, we drop the second superscript $\ell$ in $\hat{u}_{N_{L},\varepsilon_{L},\ell}^{(i)}$ and arrive at the bootstrap multilevel Monte Carlo estimator

$$E_{BS}^{L}(\hat{u}_{N_{L},\varepsilon_{L},L}) := \sum_{\ell=0}^{L} \frac{1}{M_{\ell}} \sum_{i=1}^{M_{\ell}} (\hat{u}_{N_{L},\varepsilon_{L}}^{(i)} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell-1}^{(i)})$$

This entails generating $M_{\ell}$ realizations of the random variable $\hat{u}_{N_{L},\varepsilon_{L}}$ instead of $M_{L} + M_{L+1}$. The samples $\hat{u}_{N_{L},\varepsilon_{L}}^{(i)}$ are then independent in $i$, but not anymore across all levels $\ell$ for a fixed index $i$. The bootstrap estimator is unbiased in the sense that

$$\lim_{L \to +\infty} E(E_{BS}^{L}(\hat{u}_{N_{L},\varepsilon_{L},L})) = \lim_{L \to +\infty} E(E^{L}(\hat{u}_{N_{L},\varepsilon_{L},L})) = E(u).$$

The modified calculator is a simplified version of the Multifidelity Monte Carlo estimator (see 34). Under suitable assumptions on the variance of $u$, it is shown in [34] that the Multifidelity Monte Carlo approach achieves the same rate of convergence as the standard multilevel Monte Carlo method with reduced computational effort. The bootstrap estimator corresponds to a Multifidelity Monte Carlo estimator where the weighting coefficients for all level corrections $\hat{u}_{N_{L},\varepsilon_{L},\ell} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell}$ are set equal to one.

We emphasize that the error bounds derived in Thm. 5.3 and Cor. 5.3 do not require independence of the sampled differences $\hat{u}_{N_{L},\varepsilon_{L},\ell} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell}$ across the levels $\ell$. Thus, the asymptotic order of convergence also holds for the bootstrapping estimator. However, we have now introduced additional correlation across the levels, which may entail higher variance, an thus slower convergence of the bootstrapping method. The variance of the standard multilevel Monte Carlo estimator is easily calculated as

$$\text{Var}(E^{L}(\hat{u}_{N_{L},\varepsilon_{L},L})) = \frac{L}{M_{L}} \sum_{\ell=0}^{L} \text{Var}(\hat{u}_{N_{L},\varepsilon_{L},\ell} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell-1}).$$

wheras

$$\text{Var}(E_{BS}^{L}(\hat{u}_{N_{L},\varepsilon_{L},L})) = \text{Var} \left( \sum_{\ell=0}^{L} \frac{1}{M_{\ell}} \sum_{i=1}^{M_{\ell}} (\hat{u}_{N_{L},\varepsilon_{L}}^{(i)} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell}^{(i)}) \right)$$

$$= \text{Var} \left( \sum_{i=1}^{M_{0}} \frac{\hat{u}_{N_{0},\varepsilon_{0},0}^{(i)} - \hat{u}_{N_{0},\varepsilon_{0},0}^{(i)}}{M_{0}} + \sum_{i=M_{0}+1}^{M_{1}+M_{2}} \frac{\hat{u}_{N_{1},\varepsilon_{1},1}^{(i)} - \hat{u}_{N_{0},\varepsilon_{0},0}^{(i)}}{M_{1}} + \frac{\hat{u}_{N_{0},\varepsilon_{0},0}^{(i)}}{M_{0}} \right)$$

$$+ \cdots + \sum_{i=M_{L-1}+1}^{M_{L}} \sum_{\ell=0}^{L} \frac{\hat{u}_{N_{L},\varepsilon_{L},\ell}^{(i)} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell}^{(i)}}{M_{L}}$$

$$= \sum_{\ell_{1}=0}^{L} (M_{\ell_{1}} - M_{\ell_{1}+1}) \text{Var} \left( \sum_{\ell=0}^{\ell_{1}} (\hat{u}_{N_{L},\varepsilon_{L},\ell} - \hat{u}_{N_{L-1},\varepsilon_{L-1},\ell-1}) \right).$$
In case that the differences \( \tilde{u}_{N,\epsilon,\ell} - \tilde{u}_{N,\epsilon,\ell-1} \) are positively correlated across the levels, we trade in simulation time for a possibly higher RMSE, where the ratio of this trade is problem-dependent and hard to access beforehand. Nevertheless, we will compare this modified estimator with the standard multilevel Monte Carlo estimator in different scenarios to show its advantages.

6. Numerical examples

Throughout this section, we plot the error rates against the smallest estimated root-mean-square refinement size \( E(\tilde{h}_L^2)^{1/2} \). For the standard, non-adaptive algorithms this corresponds to the preset deterministic refinement size \( h_L \) and all error contributions may be equilibrated a-priori as in Cor. 5.5. In the adaptive algorithm, to align the discretization error with the error contributions of \( \sum_{\ell>N} \eta_\ell \), we sample a few realizations of the diffusion coefficient before the start of the Monte Carlo loop. This allows us to estimate of the values of \( E(\tilde{h}_L^2)^{1/2} \) for each \( \ell = 0, \ldots , L \) and choose \( N, \epsilon \) and the number of samples on each level accordingly. All numerical examples are implemented with MATLAB and calculated on a workstation with 16 GB Memory and Intel quadcore processor with 3.4 GHz.

6.1. Numerical examples in 1D. For all test scenarios in this subsection, we consider the diffusion problem (2.1) in the one dimensional domain \( \mathcal{D} = (0,1) \) with homogeneous Dirichlet boundary conditions, i.e. \( \Gamma_1 = \partial \mathcal{D} \), and source term \( f \equiv 1 \). The deterministic part of the diffusion coefficient is \( \pi \equiv 0 \) and we consider a log-Gaussian component, i.e. \( \Phi(w) = \exp(w) \), where the Gaussian field \( W \) is characterized by either the Brownian motion covariance operator \( Q_{BM} : H \rightarrow H \) with

\[
[Q_{BM} \varphi](y) := \int_D \min(x,y) \varphi(x) dx, \quad \varphi \in H
\]

or the squared exponential covariance operator

\[
[Q_{SE} \varphi](y) := v \int_D \exp \left( -\frac{|x-y|^2}{2\sigma^2} \right) \varphi(x) dx, \quad \varphi \in H
\]

with variance parameter \( v > 0 \) and correlation length \( r > 0 \). The eigenbasis of \( Q_{BM} \) is given by

\[
\eta_i = \left( \frac{8}{(2i+1)^2} \right)^{1/2}, \quad c_i(x) = \sin \left( \frac{(2i+1)\pi x}{2} \right), \quad i \in \mathbb{N}_0,
\]

see for example [4] p. 46 ff., where the spectral basis of \( Q_{SE} \) may be efficiently approximated by Nyström’s method, see [36]. The number of partition elements is given by \( \tau = \mathcal{P} + 2 \), where \( \mathcal{P} \) is Poisson-distributed random variable with intensity parameter 10. On average, this splits the domain in 12 disjoint intervals and the diffusion coefficient has almost surely at least one discontinuity. The random positions of the \( \tau - 1 \) jumps \( x_1, \ldots , x_{\tau-1} \subset \mathcal{D} \) in the interior of \( \mathcal{D} \) are uniformly distributed over \( \mathcal{D} \), generating the random partition \( \mathcal{T} = \{(0,x_1),(x_1,x_2),\ldots,(x_{\tau-1},1)\} \) for each realization of \( \tau \) and the of jump positions \( x_1, \ldots , x_{\tau-1} \). This fits into our framework of the jump-diffusion coefficient by setting \( \lambda = 12\Lambda \), where \( \Lambda \) denotes the Lebesgue measure on \((0,1),\mathcal{B}(0,1))\): The uniform distribution of the discontinuities on \( \mathcal{D} \) corresponds to a distribution with respect to \( \Lambda \) on \( \mathcal{D} \) and the multiplication with 12 ensures that \( E(\tau) \) is as desired. In the subsequent examples we vary the distribution of the jump heights \( P_\ell \).

To obtain pathwise approximations of the samples \( u_{N,\epsilon}(\omega,\cdot) \), we use non-adaptive and adaptive piecewise linear elements and compare both approaches. In addition, we combine each discretization method with regular and bootstrapping multilevel Monte Carlo sampling, so in total we compare four different algorithms. The adaptive triangulation is based on each sampled partition \( \mathcal{T}(\omega) \) as described in Section [4] see Fig. [1] and [2]. In the graphs below, we plot the RMSE of the adaptive algorithms against the inverse estimated refinement size \( E(\tilde{h}_L^2)^{-1/2} \), for the non-adaptive algorithms this corresponds to the (deterministic) parameters \( h_L^{-1} \). The entries of the stiffness matrix are approximated by the midpoint rule, which ensures a pathwise error of order \( h_\ell^2 \) on each simplex \( K \). To ensure that this is sufficiently precise, we repeated our experiments with a five-point Gauss-Legendre quadrature, which did not entail significant changes. In the multilevel Monte Carlo algorithm, the non-adaptive triangulations are generated with refinement \( h_\ell = 2^{-k-1} \), whereas we set the same threshold as maximum refinement size \( \tilde{h}_\ell = h_\ell = 2^{-k-1} \) in the adaptive.
algorithm. As realized and maximal values of \( \bar{h}_\ell \) may differ significantly, we set \( \Xi_N = \varepsilon_{\ell} = E(\bar{h}_\ell^2) \) for \( \ell = 0, \ldots, L \) and choose the number of samples as \( M_\ell = \lceil E(\bar{h}_\ell^2) - 1 \rceil \) with \( \varepsilon = 0.01 \) for \( \ell = 1, \ldots, L \). The same error equilibration is used for the non-adaptive method, only with \( E(\bar{h}_\ell^2) \) replaced by \( h_\ell^2 \). We consider the test cases with \( L = 0, \ldots, 7 \) and use \( E_L(\bar{u}_{N_L,\varepsilon_L,L}) \) with \( L = 9 \), as a reference estimate for \( E(u) \). The RMSE \( ||E^L(\bar{u}_{N_L,\varepsilon_L,L}) - E(u)||_L^2(\Omega;V) \) is then estimated by averaging 20 samples of the error \( ||E^L(\bar{u}_{N_L,\varepsilon_L,L}) - E^9(u_{N_9,\varepsilon_9,L})||^2_V \) for \( L = 0, \ldots, 7 \).

As our first numerical example, we use the Brownian motion covariance operator \( Q_{BM} \) and i.i.d uniformly distributed jump heights \( P_i \sim U([0,10]) \), hence the sampling error \( \varepsilon_\ell \) is equal to zero on every level and may be omitted for this scenario. A sample of the corresponding diffusion coefficient with illustrated adaptive and non-adaptive FEM-basis is given in Fig. 1. Fig. 2 indicates that the adaptive algorithm converges considerably faster than the estimator with non-adaptive FEM basis. Asymptotically, we see that both adaptive RMSE curves decay with rate nearly one, whereas the non-adaptive methods only show a rate of \( \kappa \approx 0.75 \). One sees that the adaptive multilevel Monte Carlo estimator also has a better time-to-error ratio, so it is possible to reduce the RMSE significantly using a little more computational effort to adjust the FEM basis in each sample. Surprisingly there is little difference in the convergence speed whether or not we use a bootstrapping algorithm combined with adaptive resp. non-adaptive FEM. Here one would expect at least a slightly higher RMSE of the bootstrapping algorithms, but in this example, the error of both bootstrapping estimators is even lower compared to their non-bootstrapping alternatives. Naturally, bootstrapping decreases computational time (see Fig. 2).
In the next example, we consider the squared exponential covariance operator and a more involved distribution of jump heights, where sampling is rather expensive and may not be realized in a straightforward manner. The jump heights $P_i$ now follow a continuous generalized inverse Gaussian (GIG) distribution with density

$$f_{\text{GIG}}(x) = \frac{\sqrt{\psi \chi}}{2 \lambda^2} K_{\lambda}(\sqrt{\psi \chi} x) x^{-1} \exp(-\frac{1}{2}(\psi x + \chi x^{-1})), \quad x > 0$$

and parameters $\chi, \psi > 0$, $\lambda \in \mathbb{R}$, where $K_{\lambda}$ is the modified Bessel function of the second kind with $\lambda$ degrees of freedom, see [10, 11]. As shown in [6], sampling this distribution by Acceptance-Rejection is possible, but expensive when $\lambda < 0$, since the vast majority of outcomes has to be rejected. We rather generate approximations $\tilde{P}_i$ of $P_i$ by a Fourier inversion technique such that $\mathbb{E}(\tilde{P}_i - P_i)^2 \leq \varepsilon \ell$ for a given $\varepsilon \ell > 0$. For details on the Fourier inversion algorithm, the sampling of GIG distributions and the corresponding error bounds we refer to [13]. The GIG parameters are set as $\psi = 0.25, \chi = 9$ and $\lambda = -1$, the resulting density $f_{\text{GIG}}$ and a sample of the diffusion coefficient are given in Fig. 3. The error curves show a similar behavior compared to the first example, with asymptotic error rates of 1 resp. 0.75 for the adaptive resp. non-adaptive algorithms, see Fig. 4. Again, bootstrapping tends to produce a slightly lower RMSE. The expensive sampling from the GIG distribution causes increased computational times, which entails that the bootstrapping is even more favorable in a setting with a rather challenging jump height distributions.

![Generalized inverse Gaussian density](image)

**Figure 3** Left: GIG density, right: sample of the diffusion coefficient with log-squared exponential covariance and GIG distributed jumps.

![Residual vs. FEM size](image)

**Figure 4** Left: RMSE of the example with GIG-distributed jumps, Right: Time-to-error plot.
The Gaussian random field $W : \Omega \times \mathcal{D} \to \mathbb{R}$ with the Brownian motion covariance operator is not differentiable in $\mathbb{P}$-a.e. $x \in \mathcal{D}$, but only Hölder continuous. In addition, the covariance of the random variables $W(\cdot, x_1)$ and $W(\cdot, x_2)$, where $x_1, x_2 \in \mathcal{D}$ is given by the kernel function $\min(x_1, x_2)$. For a fixed distance between $x_1$ and $x_2$, this implies that the correlation in the random field increases as one moves to the right boundary of the domain. In some applications, however, one might instead want a random field with a constant correlation structure and/or more spatial regularity. This can be achieved with the introduced jump-diffusion coefficient by using, for instance, $Q_{SE}$ or another Matérn class covariance operator. These covariance operators give constant correlation throughout the field and also increase the regularity of $W \in \mathcal{D}$. It is further possible to vary the position and magnitude of the discontinuities of $a$ to model certain desirable characteristics of the diffusion. For example, one could enforce only one jump per sample which is concentrated in some small neighborhood located around a single point in $\mathcal{D}$. The corresponding jump heights on each partition may then also be chosen concentrated around certain values to model, for instance, transitions in heterogeneous or fractured media.

6.2. **Numerical results in 2D.** In the two-dimensional setting, we work on the domain $\mathcal{D} = (0, 1)^2$, with homogeneous Dirichlet or mixed Neumann-Dirichlet boundary conditions and we assume that the deterministic part of the drift coefficient is zero ($\bar{a} \equiv 0$). The Gaussian part of $a$ is given by the Karhunen-Loève expansion with spectral basis given by

$$\eta_i := v \exp(-\pi^2 r^2), \quad e_i(x) := \sin(\pi i x) \sin(\pi i x),$$

with correlation length $r > 0$ and total variance $v > 0$. This basis is related to the two-dimensional heat kernel

$$G(t, x, y) : [0, \infty) \times \mathcal{D}^2 \to \mathbb{R}^*, \quad (t, x, y) \mapsto \frac{1}{4\pi t} \exp(-\frac{|x - y|^2}{4t})$$

in the sense that it solves the associated integral equation for $t = r^2/2$:

$$v \int_\mathcal{D} \exp(-\frac{|x - y|^2}{2r^2}) e_i(y) dy = \eta_i e_i(x), \quad i \in \mathbb{N}$$

with the boundary condition $e_i = 0$ on $\partial \mathcal{D}$, see [27]. Compared with a Gaussian field generated by a squared exponential covariance operator, this field shows a very similar behavior, except that it is zero on the boundary. It, further, has the advantage, that all expressions are available in closed form and we forgo the numerical approximation of the eigenbasis. For all experiments in this section we use the parameters $v = 0.25$ and $r = 0.02$. As before, we consider a log-Gaussian random field, meaning $\Phi(w) = \exp(w)$. To illustrate the flexibility of a jump-diffusion coefficient $a$ as in Def. 3.1, we vary the random partitioning of $\mathcal{D}$ for each example and give a detailed description below. Again, we approximate the entries of the stiffness by the midpoint rule on each simplex $K$. To ensure that this is sufficient, we also tested a four-point Gauss-Legendre quadrature rule for triangular domains, which did not change the outcomes significantly. The multilevel approximation parameters are identical to the 1D example, except that $h_\ell = \frac{\sqrt{2}}{2^{\ell-\ell}}$ and we now calculate the reference solution on level 7 to estimate the RMSE (by averaging 10 independent multilevel Monte Carlo estimations) up to level $L = 4$ or $L = 5$, depending on the example. All RMSE curves are plotted against the inverse estimated refinement $E(\hat{h}^2)^{-0.5}$ on the abscissa.

In the first 2D example, the random partitions $\mathcal{T}$ of $\mathcal{D}$ are generated by random lines through the domain. More precisely, we sample independent Poisson random variables $\mathcal{P}_x, \mathcal{P}_y \sim \text{Poi}(1)$ and a total of $2(\mathcal{P}_x + \mathcal{P}_y + 2)$ independent $U([0, 1])$-distributed random variables $U_1, \ldots, U_{2(\mathcal{P}_x + \mathcal{P}_y + 2)}$. The first $\mathcal{P}_x + 1$ uniform random variables represent the jump positions on $(0, 1) \times \{0\}$, the second set $U_{\mathcal{P}_x+2}, \ldots, U_{2\mathcal{P}_x+2}$ are the positions of the discontinuities on the opposing axis $(0, 1) \times \{1\}$ in $\partial \mathcal{D}$. We connect to opposing points in ascending order by straight lines to obtain a vertical random partition of $\mathcal{D}$. Analogously, the horizontal splitting is achieved by distributing and connecting the remaining $2\mathcal{P}_y + 2$ uniform random variables on the sets $\{0\} \times (0, 1)$ and $\{1\} \times (0, 1)$. As we obtain an average of $E((\mathcal{P}_x + 2)(\mathcal{P}_y + 2)) = 9$ partition elements of uniformly distributed size and location, $\lambda$ may be set as $\lambda = 9\Lambda$ in this example, where $\Lambda$ is the Lebesgue measure on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$. 

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To each of the \((P_x + 2)(P_y + 2)\) random quadrangles we assign a jump height \(P_i\), where the sequence \((P_i, i \in \mathbb{N})\) is i.i.d. \(U([0,5])\)-distributed. This specific structure of \(a\) may be used, for example, to model stationary flows through heterogeneous media, where the hydraulic conductivity varies on sub-domains of the medium. We assume homogeneous Dirichlet-boundary conditions on \(\Gamma_1 = \partial D\) and \(f \equiv 1\) as source term. A sample of the diffusion coefficient and the FEM approximation is given in Fig. 5. Compared to a solution with constant coefficient, the influence of the discontinuous diffusion is clearly visible in the contour of the approximated solution.

Fig. 6 shows that the adaptive multilevel Monte Carlo and non-adaptive multilevel Monte Carlo perform quite similar, where the asymptotic error rate of the nonadaptive methods is slightly lower with 0.85. Compared to that, we recover a convergence rate of 0.9 and lower absolute errors for the adaptive method. In both cases, bootstrapping and normal multilevel Monte Carlo sampling produces almost identical errors which results in the best time-to-error ratio of the adaptive bootstrap multilevel Monte Carlo estimator, see Fig. 6.

In the second 2D example, we aim to mimic the structure of a fractured porous medium. To this end, we set \(\lambda = 5\Lambda\) and sample accordingly \(\tau = 5\) uniformly distributed random points \(x_1, \ldots, x_5\) on the domain \(D\). Then, for each point \(x_i\), a random length \(l_i\) with distribution \(U([0.5,1.5])\) is generated. We sample, further, five random angles \(\alpha_1, \ldots, \alpha_5\) with uniform distribution on the set \([-\pi/9, -\pi/36] \cup [\pi/36, \pi/9]\). We define \(x_i\) as the center of a line with length \(l_i\) rotated by \(\alpha_i\), where three of the five lines are orientated horizontally and the remaining two lines are vertical. Finally, the line segments outside of \(D\) are removed and each random line is assigned a positive preset width of...
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0.04, which results in a “trench structure” of the diffusion coefficient as depicted in Fig. 7. On the trenches, we set the jump height to \( P_1 = 100 \), while the jump heights on the remaining quadrangles of the partition is set to 1. Fixing the jump heights captures increasing permeability in the cracks of a certain medium, the Gaussian field still accounts for some uncertainty within each partition element of the domain. The source function is given as \( f \equiv 5 \). We split \( \partial D \) by \( \Gamma_1 := \{0,1\} \times [0,1] \) and \( \Gamma_2 := (0,1) \times \{0,1\} \) and impose the (pathwise) mixed Dirichlet-Neumann boundary conditions

\[
(6.1) \quad u(\omega, \cdot) = \begin{cases} 
0.1 & \text{on } \{0\} \times [0,1], \\
0.3 & \text{on } \{1\} \times [0,1] 
\end{cases} \quad \text{and } \quad a(\omega, \cdot) \vec{n} \cdot \nabla u(\omega, \cdot) = 0 \quad \text{on } \Gamma_2
\]

for each \( \omega \in \Omega \). A sample of the approximated solution is displayed in Fig. 7.

![Figure 7](image_url)

**Figure 7** Left: Sample of the fractured porous medium diffusion coefficient with adaptive triangulation. Right: Approximated FEM solution to the sample of \( a \).

Compared to the first 2D example, there is now a larger gap between the RMSE curves of the adaptive and non-adaptive estimators, see Fig. 8. The asymptotic rate for the error is again close to order one for both adaptive methods, while we obtain 0.6 for the non-adaptive algorithms. This is possibly due to the higher magnitude of the discontinuities in the diffusion coefficient compared to the first example. Bootstrapping now leads to a higher RMSE in each algorithm and this effect is more pronounced in the adaptive setting. Asymptotically, the rates of both bootstrapping estimators remain comparable to standard multilevel Monte Carlo. An adaptive triangulation for samples of this particular diffusion coefficient often entails very fine meshes, even if the desired maximum diameter of each triangle is comparably high, as Fig. 7 and Fig. 8 illustrate. Due to this increase in complexity when using adaptive FEM, the simulation times for the respective estimators are now considerably longer as in the previous scenario, see Fig. 8. Nevertheless, the adaptive methods still have significantly better time-to-error ratios, but now the standard adaptive method outperforms and the corresponding bootstrapping estimator.

As a last 2D example, we discuss a medium with inclusions. To this end, we sample a discrete uniformly-distributed random variable \( \tau \) where \( \tau \in \{1,2,3,4\} \) and define \( \lambda := \frac{\mathbb{E}(\tau)}{0.01} \cdot \Lambda_{(0.1,0.9)^2} \). Scaling and restricting the Lebesgue measure on \((0.1,0.9)^2\) means that we now draw \( \tau \)-many uniformly distributed points within the the sub-domain \((0.1,0.9)^2 \subset D \). To each of this points we assign a circle of random radius. The radii are \( U([0.075,0.1]) \)-distributed. On each circle we assign a jump height of 1, while this parameter is set to 20 outside of the circles. We assume the same Neumann–Dirichlet boundary conditions as in our second 2D-example (see Eq. (6.1)) and as a source term we set \( f \equiv 10 \). A sample of this jump-diffusion coefficient with corresponding FEM solution is shown in Fig. 9.

As Fig. 10 indicates, we obtain a similar behavior of convergence as in the previous example: The RMSE of the adaptive estimators is again significantly lower on all levels and the non-adaptive has again an asymptotic RMSE of order 0.6. For the adaptive methods, we obtain log-linear error decay of order \( O(h \| \log(h) \|^{\frac{1}{2}}) \). This corresponds to the expected pathwise rate for an adaptive
FEM solution of this diffusion problem, as the discontinuities have $C^2$-boundaries. Bootstrapping slightly reduces the RMSE of the non-adaptive method, but has little effect on the adaptive multilevel Monte Carlo estimator. The computational complexity of this scenario is comparable to the heterogeneous media example and thus significantly lower as in case of the fractured porous medium. Finally, the adaptive algorithms again attain better time-to-error with the bootstrapping estimator slightly outperforming the standard adaptive method, see Fig. 10.

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Figure 10 Left: RMSE of the example for a medium with inclusions, Right: Time-to-error plot.

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