Counting Schur Rings over Cyclic Groups of Semi-prime Order

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Abstract

In this paper, we continue the enumeration of Schur rings over cyclic groups. Cyclic groups of semi-prime order \( pq \), where \( p \) and \( q \) are distinct primes, are considered. Additionally, groups of order \( 4p \) are considered. This is accomplished by using counting techniques first developed by the second author from previous work [6, 5] and by counting subgroups of the associated automorphism group.

Keywords: Schur ring, cyclic group, association scheme, lattice of subgroups

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1 Introduction

Let \( G \) be a finite group, and let \( \mathbb{Q}[G] \) denote the rational group algebra. Let \( \mathcal{L}(G) \) denote the lattice of subgroups of \( G \). For any subset \( C \subseteq G \), let \( \overline{C} := \sum_{g \in C} g \in \mathbb{Q}[G] \). Such an element is called a simple quantity. Define \( C^* := \{x^{-1} \mid x \in C\} \) for all \( C \subseteq G \). Let \( \{C_1, C_2, \ldots, C_r\} \) be a partition of \( G \), and let \( \mathcal{S} \) be the

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subspace of $\mathbb{Q}[G]$ spanned by the simple quantities $C_1, C_2, \ldots, C_r$. We say that $\mathcal{S}$ is a Schur ring over $G$ if

1. $C_1 = \{1\}$,
2. For each $i$, there is a $j$ such that $C^*_1 = C_j$,
3. For each $i$ and $j$, $C_i \cdot C_j = \sum_{k=1}^{r} \lambda_{ijk} C_k$, for $\lambda_{ijk} \in \mathbb{N}$.

The sets $C_1, C_2, \ldots, C_r$ are called the $\mathcal{S}$-classes (or primitive sets of $\mathcal{S}$). Note that a Schur ring is uniquely determined by its associated partition of $G$. We will denote this partition as $D(\mathcal{S})$.

For a Schur ring $\mathcal{S}$ over $G$ and a subgroup $H \leq G$, we say $H$ is a subgroup of $\mathcal{S}$ (or an $\mathcal{S}$-subgroup) if $H$ can be partitioned using the primitive sets of $\mathcal{S}$, or, equivalently, $H \in \mathcal{S}$. Then $\mathcal{S}_H := \mathcal{S} \cap \mathbb{Q}[H]$ is a Schur ring over $H$ and is called a Schur subring of $\mathcal{S}$. We say a Schur ring $\mathcal{S}$ is primitive if the only $\mathcal{S}$-subgroups are 1 and $G$.

Schur rings over cyclic groups have been of great interest for the last few decades because of their connection to algebraic graph theory (see [7]). In [6], the second author provides recursive formulas to count the number of Schur rings over a cyclic group of order $p^n$, where $p$ is an arbitrary prime. In [5], the second author extend these formulas to include cyclic groups of order $2p^n$. This paper is a direct continuation of [5] and, as such, we intentionally omit many details found therein for the sake of brevity. We invite the reader to reference [5] for the necessary background and notation presented therein.

Let $\mathbb{Z}_n = \langle z \rangle$ denote the cyclic group of order $n$. Let $\Omega(n)$ denote the number of Schur rings over $\mathbb{Z}_n$. In this paper, we describe general techniques for enumerating Schur rings over cyclic groups, with a focus on orders of the form $pq$ and $4p$, where $p$ and $q$ are distinct primes. We present first the semiprime case $pq$:

**Theorem 1.1.** Let $p$ and $q$ be distinct primes such that $p = \prod_{i=1}^{n} r_i^{k_i} + 1$ and

2
\( q = \prod_{i=1}^{n} r_i^{\ell_i} + 1 \), where \( \{r_1, r_2, \ldots, r_n\} \) is a list of distinct primes. Then

\[
\Omega(pq) = \prod_{i=1}^{n} \sum_{j=0}^{\min(k_i, \ell_i)} \phi(r_i^j)(k_i - j + 1)(\ell_i - j + 1) + 2 \prod_{i=1}^{n} (k_i + 1)(\ell_i + 1) + 1,
\]

where \( \phi \) denotes Euler’s totient function.

Be aware that in the above decompositions of \( p \) and \( q \), the exponents \( k_i \) and \( \ell_i \) may possibly be zero, allowing for a common family of primes \( \{r_1, r_2, \ldots, r_n\} \) between \( p \) and \( q \).

We list next some useful simplifications of Theorem 1.1 when special conditions are placed on the primes \( p \) and \( q \). The proofs of the following corollaries are immediate from Theorem 1.1.

**Corollary 1.2.** Let \( p \) and \( q \) be distinct primes such that \( p = 2^k a + 1 \) and \( q = 2^\ell b + 1 \), where \( a \) and \( b \) are both odd integers and \( \gcd(a, b) = 1 \). Let \( x \) and \( y \) be the number of divisors of \( p - 1 \) and \( q - 1 \), respectively. Then

\[
\Omega(pq) = 3(k + 1)(\ell + 1) + \sum_{j=1}^{\min(k, \ell)} 2^j (k - j + 1)(\ell - j + 1) \left( \frac{xy}{(k + 1)(\ell + 1)} \right) + 1.
\]

**Corollary 1.3.** Let \( p \neq 2 \) be a prime, and let \( x \) be the number of divisors of \( p - 1 \). Then

\[\Omega(2p) = 3x + 1.\]

Corollary 1.2 is particularly useful when \( p \) is a Fermat prime, that is, \( p = 2^k + 1 \). There are only five known Fermat primes: 3, 5, 17, 257, and 65537. It is widely conjectured that these are the only Fermat primes. We illustrate the simplification of Corollary 1.2 for the Fermat primes 3 and 5.

**Corollary 1.4.** Let \( p \neq 3 \) be a prime such that \( p = 2^k a + 1 \) where \( a \) is odd, and let \( x \) be the number of divisors of \( p - 1 \). Then

\[\Omega(3p) = \left( \frac{7k + 6}{k + 1} \right) x + 1\]

When \( p \equiv 3 \pmod{4} \), \( \Omega(3p) = \frac{13}{2} x + 1 \).
Corollary 1.5. Let $p \neq 5$ be a prime such that $p = 2^k a + 1$ where $a$ is odd, and let $x$ be the number of divisors of $p - 1$. Then

$$\Omega(5p) = \left(\frac{13k + 7}{k + 1}\right)x + 1$$

When $p \equiv 3 \pmod{4}$, $\Omega(5p) = 10x + 1$.

We also mention that Corollary 1.2 is often applicable when $p$ is a safe prime, that is, $p = 2r + 1$, where $r$ is itself a prime\(^1\). It is widely conjectured that there are infinitely many safe primes, the first few being:\(^2\) 7, 11, 23, 47, 59, 83, and 107. Note that by Corollary 1.2, if $p$ and $q$ are both safe primes, then $\Omega(pq) = 53$. Likewise, if $p$ is a safe prime, then $\Omega(2p) = 13$, $\Omega(3p) = 27$ and $\Omega(5p) = 41$, by Corollaries 1.3, 1.4, and 1.5, respectively.

Using Corollaries 1.3, 1.4, and 1.5 and the above discussion of safe primes, one can easily compute the number of Schur rings over $\mathbb{Z}_{pq}$ for all semi-primes under 100. These are listed in Table 1.1. The one exception here is $n = 91 = 7 \cdot 13 = (2 \cdot 3 + 1)(2^2 \cdot 3 + 1)$. In this case, $\Omega(91)$ can be computed directly using Theorem 1.1.

We next present the counting formula for $n = 4p$:

Theorem 1.6. Let $p$ be an odd prime such that $p = 2^k a + 1$, where $a$ is an odd integer and $x$ the number of divisors of $p - 1$. Then

$$\Omega(4p) = \frac{15k + 14}{k + 1}x + 3.$$ 

In the special case that $p = 2^k + 1$ is a Fermat prime, the above formula simplifies to $\Omega(4p) = 15k + 17$. For safe primes, we always have $\Omega(4p) = 61$. In Table 1.2 we list all integers of the form $4p$ less than 100.

The proof of Theorem 1.1 and Theorem 1.6 (which proofs can be found in Section 3 and Section 4, respectively) can be summarized as following. By the Fundamental Theorem of Schur Rings over Cyclic Groups (due to Leung and

\(^1\)In this case, $r$ is necessarily a Sophie Germain prime.

\(^2\)We have intentionally omitted 5 from the list of safe primes as it is the only safe prime which is Fermat. As a consequence, it is the only safe prime $p$ for which the number of divisors of $p - 1$ is 3 instead of 4.
Table 1.1: Number of Schur Rings over $Z_{pq}$

| $n$ | $\Omega(n)$ | $n$ | $\Omega(n)$ | $n$ | $\Omega(n)$ | $n$ | $\Omega(n)$ | $n$ | $\Omega(n)$ |
|-----|-------------|-----|-------------|-----|-------------|-----|-------------|-----|-------------|
| 6   | 7           | 26  | 19          | 46  | 13          | 65  | 67          | 86  | 25          |
| 10  | 10          | 33  | 27          | 51  | 35          | 69  | 27          | 87  | 41          |
| 14  | 13          | 34  | 16          | 55  | 41          | 74  | 28          | 91  | 97          |
| 15  | 21          | 35  | 41          | 57  | 40          | 77  | 53          | 93  | 53          |
| 21  | 27          | 38  | 19          | 58  | 19          | 82  | 25          | 94  | 13          |
| 22  | 13          | 39  | 41          | 62  | 25          | 85  | 60          | 95  | 61          |

Man [3, 4]), all Schur rings over cyclic groups belong to one of four families, which we call the traditional Schur rings: namely, the indiscrete Schur ring, automorphic Schur rings, direct products of Schur rings, and wedge products of Schur rings (see [5] for definitions and statement of the theorem).

Because these four families often overlap, special care is taken to ensure that an exact count is made. This is accomplished by enumerating the wedge-indecomposable Schur rings over $Z_n$ and its subgroups. Much of this effort will be derived from counting subgroups of the automorphism group $\text{Aut}(Z_n)$ and studying the structure of the lattice of subgroups of this abelian group.

It is well known that $\text{Aut}(Z_n) \cong \prod_{i=1}^k \text{Aut}(Z_{p_i^{e_i}}) \cong \prod_{i=1}^k Z_{(p_i - 1)p_i^{e_i - 1}}$, where $n = \prod_{i=1}^k p_i^{e_i}$ is the prime factorization of $n$. This isomorphism is seen by identifying automorphisms of $Z_n$ with integers coprime to $n$.

In [11] Ziv-Av enumerates all Schur rings over small finite groups up to order 63. There is a large intersection between the Schur rings over cyclic groups counted here and the Schur rings over those groups of small order enumerated.
Table 1.2: Number of Schur Rings over $\mathbb{Z}_{4p}$

| $n$ | $\Omega(n)$ | $n$ | $\Omega(n)$ | $n$ | $\Omega(n)$ | $n$ | $\Omega(n)$ |
|-----|-------------|-----|-------------|-----|-------------|-----|-------------|
| 12  | 32          | 28  | 61          | 52  | 91          | 76  | 90          |
| 20  | 47          | 44  | 61          | 68  | 77          | 92  | 61          |

by Ziv-Av. In all instances, the two enumerations agree.

## 2 Counting Schur Rings

In this section, we remind the reader of important counting techniques introduced in [5].

For any subgroup $\mathcal{H} \leq \text{Aut}(G)$, let $G^\mathcal{H}$ denote the *automorphic Schur ring* associated to $\mathcal{H}$. In the case of $\mathcal{H} = 1$, $G^1 = \mathbb{Q}[G]$, the group algebra itself. For simplicity of notation, this Schur rings, called the *discrete Schur ring* is simply denoted as $G$, as the coefficient ring will provide little consequence. In the case that $G$ is abelian and $\mathcal{H} = \langle \ast \rangle$, we denote $G^\mathcal{H}$ as $G^\pm$.

For any $n$, there is exactly one *indiscrete Schur ring* over $G$, namely $G^0 := \text{Span}\{1, \overline{G} - 1\}$. As it is primitive, the indiscrete Schur ring will never be wedge-decomposable or be decomposable as a direct product. This is because the wedge products and direct products have a requirements about proper subgroups which are absent for primitive Schur rings, that is, Schur rings without proper, nontrivial subgroups. Again considering subgroups, an automorphic Schur ring over $G$ will contain every characteristic subgroup of $G$ as a subgroup of the Schur ring. As all subgroups of $\mathbb{Z}_n$ are characteristic, this implies that all subgroups of $\mathbb{Z}_n$ are subgroups of every automorphic Schur ring over $\mathbb{Z}_n$. In fact, $(G^\mathcal{H})_H = H^\mathcal{H}$ for all subgroups. As $\mathbb{Z}_n^0$ is primitive, it could only be automorphic over $\mathbb{Z}_p$. In this case, $\mathbb{Z}_n^0 = \mathbb{Z}_p^{\text{Aut}(\mathbb{Z}_p)}$. Excluding this one exception, the indiscrete Schur
ring will contribute a single count to $\Omega(n)$, but it will also be needed recursively as we consider direct and wedge products in these counts.

Counting direct products is the next easiest family to consider. If $G = H \times K$, $\mathcal{S}$ is a Schur ring over $H$, and $\mathcal{T}$ is a Schur ring over $K$, then $\mathcal{S} \times \mathcal{T} = \mathcal{S} \otimes_{\mathbb{Q}} \mathcal{T}$ denotes the direct product of $\mathcal{S}$ and $\mathcal{T}$. Note that both $H$ and $K$ are necessarily subgroups of $\mathcal{S} \times \mathcal{T}$. In fact, $(\mathcal{S} \times \mathcal{T})_H = \mathcal{S}$ and $(\mathcal{S} \times \mathcal{T})_K = \mathcal{T}$. The number of direct product Schur rings over $\mathbb{Z}_{pq}$ will be $\Omega(p) \Omega(q)$, since each such Schur ring has the form $\mathcal{S} \times \mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ are Schur rings over $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively.

Similarly, the number of direct products over $\mathbb{Z}_{4p}$ is $\Omega(4) \Omega(p) = 2\Omega(p)$. Direct products are often automorphic, especially for $n = pq, 4p$. Let $H$ and $K$ be groups and let $H \leq \text{Aut}(H)$ and $K \leq \text{Aut}(K)$. If $G = H \times K$, then we may naturally view $H \times K$ as a subgroup of $\text{Aut}(G)$ by the following rule: if $\sigma \in H$ and $\tau \in K$, then define the map $\sigma \times \tau : H \times K \to H \times K$ as $(h, k) \sigma \times \tau = (h\sigma, k\tau)$.

Using this correspondence, we see the following.

**Lemma 2.1.** Let $G = H \times K$. Then the Schur ring $\mathcal{S} \times \mathcal{T}$ is automorphic if and only if $\mathcal{S}$ and $\mathcal{T}$ are both automorphic.

A section $U$ of $G$ is a pair of subgroups $U = [K, H]$ such that $1 \leq K \leq H \leq G$ and $K \trianglelefteq G$. We say that a section $U = [K, H]$ is proper if $1 < K \leq H < G$. We say that a section is trivial if $K = H$. As all subgroups of $\mathbb{Z}_n$ are normal and uniquely determined by their orders, we shall denote the section $[\mathbb{Z}_d, \mathbb{Z}_e]$ simply as $[d, e]$.

Given any proper section $U = [K, H]$, a Schur ring $\mathcal{S}$ over $H$, and a Schur ring $\mathcal{T}$ over $K$, we form the wedge product $\mathcal{S} \wedge_U \mathcal{T}$ by constructing the common refinement of $D(\mathcal{S})$ and $D(\pi^{-1}(\mathcal{T}))$, where $\pi : G \to G/K$ is the natural map. To guarantee that $(\mathcal{S} \wedge_U \mathcal{T})_H = \mathcal{S}$ and $\pi(\mathcal{S} \wedge_U \mathcal{T}) = \mathcal{T}$, the extra compatibility condition $\pi(\mathcal{S}) = \mathcal{T}_{H/K}$ is required in this construction. When $U = [H, H]$ is trivial, compatibility is automatic. We say a Schur ring is wedge-decomposable when there exists a proper section such that the Schur ring can be expressed as a wedge product of two other Schur rings. In particular, $H$ and $K$ are subgroups of $\mathcal{S} \wedge_U \mathcal{T}$. 

7
Wedge products on the most ubiquitous of all of the traditional Schur rings, at least for cyclic groups. To avoid counting twice wedge-decomposable rings with the other traditional families, we focus on identifying wedge-indecomposable Schur rings. We note that a direct product is wedge-decomposable if either of its factor are decomposable. As such, we will focus only on those direct products of indecomposable factors, which will be indiscrete or automorphic. Of course, automorphic Schur rings can be wedge-decomposable, as is common for cyclic groups of order \( p^n \), as seen in [6]. In the sequel, we will be able to distinguish between these two families using subgroups of the Schur rings.

Lastly, to count automorphic Schur rings over \( \mathbb{Z}_n \), we note that there is a one-to-one correspondence between subgroups of \( \text{Aut}(\mathbb{Z}_n) \) and automorphic Schur rings. So, it suffices to count the number of subgroups of \( \text{Aut}(\mathbb{Z}_n) \), that is, \( |\mathcal{L}(\text{Aut}(\mathbb{Z}_n))| \). The problem of counting the number of subgroups of an abelian group is a well studied problem in literature, for example [8], [9], and [10]. These all derive from a Theorem of Goursat [2]. Our consideration of this problem will follow the method from Călugăreanu [1]. We say that two sections \( U = [K, H] \) and \( U' = [K', H'] \) over \( G \) and \( G' \), respectively, are isomorphic, if \( H/K \cong H'/K' \). If \( G = H \times K \), we say a subgroup \( D \) is diagonal if \( D \in \mathcal{L}(G) \setminus \mathcal{L}(H) \times \mathcal{L}(K) \).

As shown in [1], diagonal subgroups of \( G = H \times K \) correspond exactly to automorphisms between isomorphic, non-trivial sections of the lattices \( \mathcal{L}(H) \) and \( \mathcal{L}(K) \). We illustrate Călugăreanu’s technique below.

**Example 2.2.** In \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) there are 6 diagonal subgroups. To see this, note that the sections in \( \mathbb{Z}_4 \) are \([1, 2], [2, 4], \) and \([1, 4] \). The first two sections are isomorphic to \( \mathbb{Z}_2 \). Since there is only one automorphism over \( \mathbb{Z}_2 \), \( 1 \cdot 2 \cdot 2 = 4 \) of the diagonal subgroups arise from the combinations of \([1, 2] \) and \([2, 4] \). The last \( 2 \cdot 1 \cdot 1 = 2 \) come from there being two automorphisms over \( \mathbb{Z}_4 \) and the single combination of \([1, 4] \) and itself. Given that \( |\mathcal{L}(\mathbb{Z}_4)| = 3 \), this shows that \( |\mathcal{L}(\mathbb{Z}_4 \times \mathbb{Z}_4)| = 3^2 + 6 = 15. \)  

\[ \square \]
Lemma 2.3. The number of subgroups of $\mathbb{Z}_{p^k} \times \mathbb{Z}_{p^\ell}$ is given as

$$\left| \mathcal{L}(\mathbb{Z}_{p^k} \times \mathbb{Z}_{p^\ell}) \right| = \sum_{j=0}^{\min(k,\ell)} \phi(p^j)(k-j+1)(\ell-j+1),$$

where $\phi$ denotes Euler’s totient function.

Proof. To begin, we note that $\mathcal{L}(\mathbb{Z}_{p^k})$ is a linear chain, that is, $1 \leq \mathbb{Z}_{p} \leq \mathbb{Z}_{2p} \leq \cdots \leq \mathbb{Z}_{p^k}$. Hence, $|\mathcal{L}(\mathbb{Z}_{p^k})| = k+1$. Likewise, $|\mathcal{L}(\mathbb{Z}_{p^\ell})| = \ell+1$. Thus,

$$|\mathcal{L}(\mathbb{Z}_{p^k}) \times \mathcal{L}(\mathbb{Z}_{p^\ell})| = |\mathcal{L}(\mathbb{Z}_{p^k})||\mathcal{L}(\mathbb{Z}_{p^\ell})| = (k+1)(\ell+1).$$

To count the diagonal subgroups, we first consider all the possible sections on $\mathbb{Z}_{p^k}$. Since $\mathcal{L}(\mathbb{Z}_{p^k})$ is a linear chain, all the sections are of the form $[a, b]$ where $1 \leq a < b \leq k$. Organizing the sections by isomorphism, this gives $k-j+1$ sections isomorphic to $\mathbb{Z}_{p^j}$ for $1 \leq j \leq k$. An isomorphism between nontrivial sections of $\mathcal{L}(\mathbb{Z}_{p^k})$ and $\mathcal{L}(\mathbb{Z}_{p^\ell})$ is then determined by a matching between isomorphic sections and an automorphism on their common quotient.

For $1 \leq j \leq \min(k,\ell)$, the number of matchings is $(k-j+1)(\ell-j+1)$. For $j > \min(k,\ell)$, the number of matchings is 0. As $|\text{Aut}(\mathbb{Z}_{p^j})| = \phi(p^j)$, there are $\phi(p^j)(k-j+1)(\ell-j+1)$ many isomorphism between sections of $\mathcal{L}(\mathbb{Z}_{p^k})$ and $\mathcal{L}(\mathbb{Z}_{p^\ell})$ whose quotients are isomorphic to $\mathbb{Z}_{p^j}$. Thus, there exists $\sum_{j=1}^{\min(k,\ell)} \phi(p^j)(k-j+1)(\ell-j+1)$ diagonal subgroups in $\mathbb{Z}_{p^k} \times \mathbb{Z}_{p^\ell}$. Therefore, combining these two observations with the final observation that $\phi(p^0) = 1$, we have

$$|\mathcal{L}(\mathbb{Z}_{p^k} \times \mathbb{Z}_{p^\ell})| = (k+1)(\ell+1) + \sum_{j=1}^{\min(k,\ell)} \phi(p^j)(k-j+1)(\ell-j+1)$$

$$= \sum_{j=0}^{\min(k,\ell)} \phi(p^j)(k-j+1)(\ell-j+1).$$

Note that for $\mathbb{Z}_4 \times \mathbb{Z}_4$, we see that

$$|\mathcal{L}(\mathbb{Z}_4 \times \mathbb{Z}_4)| = 1(3)(3) + 1(2)(2) + 2(1)(1) = 15,$$

which agrees with Example 2.2.
3 Proof of Theorem 1.1

We proceed to the proof of Theorem 1.1. Consider $Z_{pq}$, where $p = \prod_{i=1}^{n} r_i^{k_i} + 1$ and $q = \prod_{i=1}^{n} s_i^{l_i} + 1$ are primes. By the Fundamental Theorem, all the Schur rings over this group are traditional, that is, all Schur rings fall into one of four families: indiscrete, automorphic, direct products, or wedge products. We now consider each of these families.

As mentioned above, the indiscrete case only overlaps with the other families when the order is prime. Thus, this family is disjoint from the other three and contains exactly one ring.

Before continuing, let us consider the Schur rings over the subgroups of $Z_{pq}$.

Lemma 3.1. All Schur rings over $Z_p$ are indecomposable and automorphic, where $p$ is prime. Furthermore, $\Omega(p) = x$, where $x$ is the number of divisors of $p - 1$.

Proof. If $Z_p$ had a direct or wedge product Schur ring, it would imply that $Z_p$ has nontrivial, proper subgroups, which it does not. Also, $\mathcal{Z}_p = \mathcal{Z}_p^{\text{Aut}(Z_p)}$. Thus, all Schur rings are automorphic over $Z_p$. Now, these automorphic Schur rings are in one-to-one correspondence with subgroups of $\text{Aut}(Z_p) \cong Z_{p-1}$. As $Z_{p-1}$ is cyclic, each automorphic Schur ring corresponds to a divisor of $p - 1$. Thus, $\Omega(p) = x$. \qed

Combining Lemma 3.1 with Lemma 2.1 shows that all direct product Schur rings over $Z_{pq}$ are automorphic. Therefore, it suffices to only count the automorphic, indiscrete, and wedge product Schur rings over $Z_{pq}$.

We consider next the wedge-decomposable Schur rings over $Z_{pq}$. Over $Z_{pq}$, the only possible proper sections are $[p, p]$ and $[q, q]$. In the first case, any possible Schur ring over $Z_p$ could be wedged with any possible Schur ring over $Z_q$. This produces $\Omega(p)\Omega(q)$ distinct Schur rings. The subgroups of each of these Schur rings will be exactly 1, $Z_p$, and $Z_{pq}$. Note that $Z_q$ is missing since elements of order $q$ and $pq$ are fused together in cosets of $Z_p$. Thus, these Schur rings are not automorphic. The second case is similar and produces $\Omega(q)\Omega(p)$ distinct Schur rings over $Z_{pq}$. \qed
distinct Schur rings by wedging a Schur ring from $\mathbb{Z}_q$ with a Schur ring from $\mathbb{Z}_p$. Likewise, these Schur rings will have only the subgroups 1, $\mathbb{Z}_q$, and $\mathbb{Z}_{pq}$. Thus, these Schur rings are distinct from the automorphic Schur rings as well as the wedge products already mentioned. Therefore, there are $2\Omega(p)\Omega(q)$ wedge products over $\mathbb{Z}_{pq}$.

Note that by the above decomposition of $p$, $p - 1$ has $\prod_{i=1}^{n} (k_i + 1)$ divisors. Likewise, $q - 1$ has $\prod_{i=1}^{n} (\ell_i + 1)$ divisors. Considering the wedge-decomposable and indiscrete families, we have already accounted for $2 \prod_{i=1}^{n} (k_i + 1) (\ell_i + 1) + 1$ distinct, non-automorphic Schur rings. Thus, to prove Theorem 1.1 it suffices to count the number of distinct automorphic Schur rings over $\mathbb{Z}_{pq}$. As these Schur rings are in direct one-to-one correspondence with the subgroups of $\text{Aut}(\mathbb{Z}_{pq})$, we see that

$$\Omega(pq) = |\mathcal{L}(\text{Aut}(\mathbb{Z}_{pq}))| + 2 \prod_{i=1}^{n} (k_i + 1) (\ell_i + 1) + 1. \quad (3.1)$$

For a finite group $G$, let $G = \prod_{i=1}^{k} P_i$ be its primary decomposition. Then it is well-known that $\mathcal{L}(G) \cong \prod_{i=1}^{k} \mathcal{L}(P_i)$. In the case of $\mathbb{Z}_{pq}$, we see that $\text{Aut}(\mathbb{Z}_{pq}) \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$. Hence, the primary decomposition for $\text{Aut}(\mathbb{Z}_{pq})$ is given as

$$\text{Aut}(\mathbb{Z}_{pq}) \cong \prod_{i=1}^{n} (\mathbb{Z}_{r_{ki}^i} \times \mathbb{Z}_{r_{\ell_i}^i}).$$

By Lemma 2.3,

$$|\mathcal{L}(\mathbb{Z}_{r_{ki}^i} \times \mathbb{Z}_{r_{\ell_i}^i})| = \sum_{j=0}^{\min(k_i, \ell_i)} \phi(r_i^j)(k_i - j + 1)(\ell_i - j + 1).$$

Therefore,

$$|\mathcal{L}(\text{Aut}(\mathbb{Z}_{pq}))| = \prod_{i=1}^{n} \sum_{j=0}^{\min(k_i, \ell_i)} \phi(r_i^j)(k_i - j + 1)(\ell_i - j + 1). \quad (3.2)$$

Finally, Theorem 1.1 follows immediately from (3.1) and (3.2), which finishes the proof.

**Example 3.2.** We present a complete enumeration of the Schur rings over $\mathbb{Z}_{21}$ as an example to illustrate the previous proof. There are $\Omega(3) = 2$ Schur
rings over $\mathbb{Z}_3$, namely $\mathbb{Z}_3^0$ and $\mathbb{Z}_3$. There are $\Omega(7) = 4$ Schur rings over $\mathbb{Z}_7$, namely, $\mathbb{Z}_7^0$, $\mathbb{Z}_7^{(2)}$, $\mathbb{Z}_7^\pm$, and $\mathbb{Z}_7$.

Below we list the $\Omega(21) = 27$ Schur rings over $\mathbb{Z}_{21}$:

- $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^0$, $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^{(2)}$, $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^\pm$, $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^0$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7^0$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7^{(2)}$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7^\pm$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7$,
- $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^0$, $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^{(2)}$, $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^\pm$, $\mathbb{Z}_3^0 \wedge \mathbb{Z}_7^0$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7^0$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7^{(2)}$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7^\pm$, $\mathbb{Z}_3 \wedge \mathbb{Z}_7$,
- $\mathbb{Z}_3^0 \times \mathbb{Z}_7^0$, $\mathbb{Z}_3^0 \times \mathbb{Z}_7^{(2)}$, $\mathbb{Z}_3^0 \times \mathbb{Z}_7^\pm$, $\mathbb{Z}_3^0 \times \mathbb{Z}_7^0$, $\mathbb{Z}_3 \times \mathbb{Z}_7^0$, $\mathbb{Z}_3 \times \mathbb{Z}_7^{(2)}$, $\mathbb{Z}_3 \times \mathbb{Z}_7^\pm$, $\mathbb{Z}_3 \times \mathbb{Z}_7$, $(\cong \mathbb{Z}_{21})$,
- $\mathbb{Z}_{21}^0$, $\mathbb{Z}_{21}^{(5)}$, $\mathbb{Z}_{21}^\pm$.

4 Proof of Theorem 1.6

For any positive integer $n$, any divisor $d \mid n$, and any Schur ring $\mathcal{S}$ over $\mathbb{Z}_d$, let $\Omega(n, \mathcal{S})$ count the number of all Schur rings $\mathcal{T}$ over $\mathbb{Z}_n$ such that $\mathcal{T} \wedge \mathbb{Z}_d = \mathcal{S}$.

Note that if $U = [K, H]$ then the number of Schur rings of the form $\mathcal{S} \wedge U \mathcal{T}$ for a fixed $\mathcal{S}$ is $\Omega(n/\lvert K \rvert, \pi(\mathcal{S}))$, where $\pi : \mathbb{Z}_n \to \mathbb{Z}_n/K$ is the natural map.

**Lemma 4.1.** Let $p$ be a prime, and let $\mathcal{S}$ be a Schur ring over $\mathbb{Z}_p$. Then $\Omega(2p, \mathcal{S}) = 2$ and $\Omega(2p, \mathbb{Z}_2) = 2\Omega(p)$.

**Proof.** Revisiting the proof of Theorem 1.1 with the simplification $q = 2$, shows that there are exactly $\Omega(p)$ many Schur rings of the form $\mathbb{Z}_2 \wedge \mathcal{T}$, $\Omega(p)$ many Schur rings of the form $\mathcal{T} \wedge \mathbb{Z}_2$, $\Omega(p)$ many Schur rings of the form $\mathbb{Z}_2 \times \mathcal{T}$, and one ring of the form $\mathbb{Z}_{2p}^0$, where $\mathcal{T}$ is a generic Schur ring over $\mathbb{Z}_p$. Then $\mathcal{S} \wedge \mathbb{Z}_2$ and $\mathcal{S} \times \mathbb{Z}_2$ are the only Schur rings that contain the subring $\mathcal{S}$, and $\mathbb{Z}_2$ is a subring of all the Schur rings of the form $\mathbb{Z}_2 \wedge \mathcal{T}$ and $\mathbb{Z}_2 \times \mathcal{T}$.

The following observation is immediate from the previous proof.

**Lemma 4.2.** There are $\Omega(p) + 1$ indecomposable Schur rings over $\mathbb{Z}_{2p}$, for which $\Omega(p)$ have the form $\mathbb{Z}_2 \times \mathcal{T}$ and one has the form $\mathbb{Z}_{2p}^0$.

With the above lemmas established, we proceed to prove Theorem 1.6. Let $p = 2^k a + 1$, where $a$ is odd. Let $x$ denote the number of divisors of $p - 1$. Then $x/(k + 1)$ is the number of divisors of $a$. 

12
The construction of wedge products will be more complicated in this case as the compatibility condition will have to be checked. There is also the concern that different sections can potentially create the same wedge product, e.g. if $U = [2, 4]$ and $U' = [4, 4]$, then $Z_4 \wedge_U Z_6 = Z_4 \wedge_{U'} Z_3$. This is a consequence of the fact that all cosets of $Z_4$ are unions of cosets of $Z_2$. To avoid this issue, we will restrict our attention to sections $[K, H]$ where $K$ is a minimal subgroup. Additionally, $(Z_2 \wedge Z_2) \wedge Z_3 = Z_2 \wedge (Z_2 \wedge Z_3)$. To avoid these multiple occurrences of the same Schur ring as a consequence of multiple wedges, we restrict to wedge products of the form $\mathcal{S} \wedge_U \mathcal{T}$, where $\mathcal{S}$ is wedge-indecomposable. We note that all primitive Schur rings are necessarily indecomposable, such as indiscrete Schur rings and all Schur rings of prime order. Likewise, the direct product of indecomposable rings is indecomposable.

We will organize Schur rings over $\mathbb{Z}_{4^p}$ according to the order of its left wedge-factor. That is, we will consider Schur rings of the form $\mathcal{S} \wedge_U \mathcal{T}$ where $\mathcal{S}$ is an indecomposable Schur ring over $\mathbb{Z}_m$ for $m = 2, p, 4, 2p, 4p$ ($m = 1$ is omitted as it offers no proper sections). We note that if $\mathcal{S}$ is primitive, the only possible section would be trivial, that is, $U = [m, m]$. As such, there is no restriction on the Schur ring $\mathcal{T}$, which implies there would be $\Omega(4^p/m)$ Schur rings of this type.

We begin with $m = 2$. There is only one Schur ring over $\mathbb{Z}_2$, $\mathbb{Z}_2$ itself, that is, $\Omega(2) = 1$. As it is primitive, there are $\Omega(2p)$ Schur rings over the form $\mathbb{Z}_2 \wedge \mathcal{T}$. Likewise, if $m = p$, there are $\Omega(p)\Omega(4)$ Schur rings of the type $\mathcal{S} \wedge \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are Schur rings over $\mathbb{Z}_p$ and $\mathbb{Z}_4$, respectively. This again follows from the fact that all Schur rings of prime order are primitive.

Consider next $m = 4$. We mention that there are three Schur rings over $\mathbb{Z}_4$, namely, $Z_4^0, Z_2 \wedge Z_2$, and $Z_4$, that is, $\Omega(4) = 3$. Thus, two of them are indecomposable, namely, $Z_4^0$ and $Z_4$. If $\mathcal{S} = Z_4^0$, then $\mathcal{S}$ is primitive and, hence, there are $\Omega(p)$ options for $\mathcal{T}$. On the other hand, if $\mathcal{S} = Z_4$, then we select $U = [2, 4]$ (as wedge products using the section $[4, 4]$ are a subset of the products we are considering presently). Then there are $\Omega(2p, Z_2)$ options for $\mathcal{T}$.

Consider next $m = 2p$. If $\mathcal{S} = Z_{2p}^0$, then $\mathcal{S}$ is primitive and there are $\Omega(2) = \ldots$
1 options for \( T \), that is, \( Z_{2p} \wedge Z_2 \) is the only option. By Lemma 4.2, the only other indecomposable Schur rings over \( Z_{2p} \) are \( \mathcal{G} = Z_2 \times \mathcal{G}' \), where \( \mathcal{G}' \) is a Schur ring over \( Z_p \). Unlike all the previous cases we have seen already, \( \mathcal{G} \) has two distinct minimal subgroups, namely, \( Z_2 \) and \( Z_p \). As such, two proper sections need to be considered, namely, \([2, 2p]\) and \([p, 2p]\). For fixed \( \mathcal{G}' \), there are \( \Omega(2p, \mathcal{G}') \) Schur rings of the form \((Z_2 \times \mathcal{G}' \wedge [2, 2p]) T\) and \( \Omega(4, Z_2) \) many Schur rings of the form \((Z_2 \times \mathcal{G}') \wedge [p, 2p] T\). Note that \( \Omega(4, Z_2) = 2 \) since \((Z_2 \wedge Z_2) Z_2 = (Z_4) Z_2 = Z_2\).

Now, as cosets of \( Z_{2p} \) are unions of cosets of \( Z_2 \) and unions of cosets of \( Z_p \), an inclusion-exclusion argument is necessary here, that is, we need to remove wedge products associated to the trivial section \([2p, 2p]\) which were counted twice. Therefore, for a fixed \( \mathcal{G}' \), there are \( \Omega(2p, \mathcal{G}') + \Omega(4, Z_2) - \Omega(2) \) of the type \((Z_2 \times \mathcal{G}') \wedge U T\). Allowing \( \mathcal{G}' \) to vary, there are \( \Omega(p) (\Omega(2p, \mathcal{G}') + \Omega(4, Z_2) - \Omega(2)) \) options for \((Z_2 \times \mathcal{G}') \wedge U T\). This completes \( m = 2p \).

It remains to consider the case \( m = 4p \). In this case, as there is no proper section possible here, we mean to only consider those indecomposable Schur rings over \( Z_{4p} \). There is, of course, the indiscrete Schur ring \( Z_{4p}^0 \), as well as the indecomposable automorphic Schur rings and those direct products of the form \( Z_{4p}^0 \times \mathcal{G} \), for some Schur rings \( \mathcal{G} \) of order \( p \). In regard to the automorphic Schur rings, we know the total count is equal to \( |\mathcal{L} \text{Aut}(Z_{4p})|\). Since \( \text{Aut}(Z_{4p}) \cong Z_2 \times Z_{2p-1} \cong (Z_2 \times Z_2^*) \times Z_a \), \( \mathcal{L}(Z_2) \times \mathcal{L}(Z_{p-1}) \) will consist of two lattice-isomorphic copies of \( \mathcal{L}(Z_{p-1}) \). The full lattice \( \mathcal{L}(Z_2 \times Z_{p-1}) \) contains these two layers and all the diagonal entries that sit “between” the top and bottom layers. Those automorphic Schur rings that correspond to the top layer have the form \( Z_4 \times \mathcal{G} \), for some Schur ring \( \mathcal{G} \) of order \( p \), and are indecomposable. Those automorphic Schur rings that correspond to the bottom layer have the form \((Z_2 \wedge Z_2) \times \mathcal{G}\) and are wedge-decomposable. Notice that these decomposable automorphic Schur rings are in one-to-one correspondence with those indecomposable Schur rings of the form \( Z_{4p}^0 \times \mathcal{G} \). Thus, if every diagonal automorphic Schur ring over \( Z_{4p} \) is indecomposable, which we claim, then the number of indecomposable Schur
rings over $\mathbb{Z}_{4p}$ is $1 + |\mathcal{L}(\text{Aut}(\mathbb{Z}_{4p}))|$. By Lemma 2.3,

$$|\mathcal{L}(\text{Aut}(\mathbb{Z}_{4p}))| = |\mathcal{L}(\mathbb{Z}_2 \times \mathbb{Z}_2^*)||\mathcal{L}(\mathbb{Z}_p)| = (2(k + 1) + k) \left( \frac{x}{k+1} \right) = \frac{3k+2}{k+1}x.$$ 

To prove the claim, we introduce the representation $\omega : \mathbb{Q}[\mathbb{Z}_n] \rightarrow \mathbb{Q}(\zeta_n)$ which maps a generator of $\mathbb{Z}_n$ to $\zeta_n := e^{2\pi i/n}$. We remind the reader that in [5] we saw that an automorphic Schur ring $\mathcal{R}$ is wedge-decomposable if and only if $\omega(S) \leq \mathbb{Q}(\zeta_d)$ for some proper divisor $d$ of $n$ (excluding, of course, the case when $n$ is prime). By definition, the diagonal automorphic Schur rings are not contained in $\mathcal{L}(\text{Aut}(\mathbb{Z}_4)) \times \mathcal{L}(\text{Aut}(\mathbb{Z}_p))$, which implies that there image is not contained in $\mathcal{L}(\mathbb{Q}(\zeta_4)) \times \mathcal{L}(\mathbb{Q}(\zeta_p))$. This implies they are all indecomposable, as claimed.

As we have now exhausted all possibilities, we see that

$$\begin{align*}
\Omega(4p) &= \Omega(2)\Omega(2p) + \Omega(p)\Omega(4) + \left( \Omega(p) + \Omega(2p, \mathbb{Z}_2) \right) + \left( \Omega(2) + \Omega(p)(\Omega(2p, \mathcal{R}) + \Omega(4, \mathbb{Z}_2) - \Omega(2)) \right) + \left( 1 + |\mathcal{L}(\text{Aut}(\mathbb{Z}_{4p}))| \right) \\
&= (3x + 1) + (3x) + (x + 2x) + (1 + x(2 + 2 - 1)) + \left( 1 + \frac{3k+2}{k+1}x \right) \\
&= 12x + 3 + \frac{3k+2}{k+1}x = \frac{15k+14}{k+1}x + 3,
\end{align*}$$

which finishes the proof of Theorem 1.6.

**Example 4.3.** We present a complete enumeration of the Schur rings over $\mathbb{Z}_{12}$ as an example to illustrate the previous proof. Note $12 = 4(3)$ and $3 = 2^1 \cdot 1 + 1$. There are $\Omega(3) = 2$ Schur rings over $\mathbb{Z}_3$, namely $\mathbb{Z}_3^0$ and $\mathbb{Z}_3$. There are $\Omega(6) = 7$ Schur rings over $\mathbb{Z}_6$, namely

$$\mathbb{Z}_2 \wedge \mathbb{Z}_3^0, \mathbb{Z}_2 \wedge \mathbb{Z}_3, \mathbb{Z}_2^0 \wedge \mathbb{Z}_3, \mathbb{Z}_3 \wedge \mathbb{Z}_2, \mathbb{Z}_2 \wedge \mathbb{Z}_2^0, \mathbb{Z}_2 \times \mathbb{Z}_3^0 (= \mathbb{Z}_6^+), \mathbb{Z}_2 \times \mathbb{Z}_3 (= \mathbb{Z}_6).$$

Below we list the $\Omega(12) = 32$ Schur rings over $\mathbb{Z}_{12}$:

$$\begin{align*}
\mathbb{Z}_2 \wedge \mathbb{Z}_3^0, \mathbb{Z}_2 \wedge \mathbb{Z}_3, \mathbb{Z}_2^0 \wedge \mathbb{Z}_3, \mathbb{Z}_2 \wedge \mathbb{Z}_2^0 \wedge \mathbb{Z}_2, \mathbb{Z}_2 \wedge \mathbb{Z}_2 \wedge \mathbb{Z}_2 \wedge \mathbb{Z}_2^0, \mathbb{Z}_2 \wedge \mathbb{Z}_6^+, \mathbb{Z}_2 \wedge \mathbb{Z}_6, \\
\mathbb{Z}_3^0 \wedge \mathbb{Z}_2^0, \mathbb{Z}_3^0 \wedge \mathbb{Z}_2, \mathbb{Z}_3^0 \wedge \mathbb{Z}_2, \mathbb{Z}_3 \wedge \mathbb{Z}_2, \mathbb{Z}_3 \wedge \mathbb{Z}_2^0, \mathbb{Z}_3 \wedge \mathbb{Z}_3^0, \mathbb{Z}_3 \wedge \mathbb{Z}_6^+, \mathbb{Z}_3 \wedge \mathbb{Z}_6, \\
\mathbb{Z}_2^0 \wedge \mathbb{Z}_3^0, \mathbb{Z}_2^0 \wedge \mathbb{Z}_3, \mathbb{Z}_4 \wedge \mathbb{Z}_2^0 (\mathbb{Z}_2 \wedge \mathbb{Z}_3^0), \mathbb{Z}_4 \wedge \mathbb{Z}_2^0 (\mathbb{Z}_2 \wedge \mathbb{Z}_3^0), \mathbb{Z}_4 \wedge \mathbb{Z}_2^0 \wedge \mathbb{Z}_6^+, \mathbb{Z}_4 \wedge \mathbb{Z}_2^0 \wedge \mathbb{Z}_6.
\end{align*}$$
\[ \mathbb{Z}_6 \wedge \mathbb{Z}_2, \mathbb{Z}_6^\pm \wedge \mathbb{Z}_2, \mathbb{Z}_6^\pm \wedge [2,6], \mathbb{Z}_6^\pm \wedge [3,6], \mathbb{Z}_4, \mathbb{Z}_6 \wedge \mathbb{Z}_2, \mathbb{Z}_6 \wedge [2,6], \mathbb{Z}_6 \wedge [3,6], \mathbb{Z}_4, \]

\[ \mathbb{Z}_4 \wedge \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 = \mathbb{Z}_{12}, \mathbb{Z}_{12}^\pm. \]

5 Conclusion

We can extrapolate from the proofs of Theorems 1.1 and 1.6, as well as those proofs from [6, 5] a general strategy for counting Schur rings over cyclic groups. We begin by identifying the indecomposable Schur rings for all divisors of the order \( n \). Once this is complete, we proceed to enumerate all wedge products choosing as the left factor only indecomposable Schur rings and choosing as sections only those of the form \([K, H]\), where \( K \) is the minimal subgroup of the left wedge-factor. This is trivial for primitive Schur rings and the Principle of Inclusion-Exclusion is necessary when distinct minimal subgroups are present. For every left factor in \( \mathcal{S} \wedge [d,e] \mathcal{T} \), there are \( \Omega(\pi(\mathcal{S})) \) options for \( T \). The sum of these mutually exclusive cases gives \( \Omega(n) \).

This general strategy comes with two major obstacles. First, it requires a strong understanding of the recursive nature of the function \( \Omega(n, \mathcal{S}) \). While ad hoc arguments are used here to handle the cases considered herein, the potential complexity of \( \Omega(n, \mathcal{S}) \) is seen in [6, 5]. Second, it requires we be able to effectively enumerate the indecomposable Schur rings. While primitive Schur rings are easy to identify for cyclic groups (only indiscrete rings or prime order) and direct products are indecomposable only if their direct factors are (a fact that can be established recursively), the indecomposable automorphic Schur rings are a greater challenge. As we saw in the proof of Theorem 1.6, as well as in [6], we can identify the indecomposable automorphic rings using the lattice \( \mathcal{L}(\text{Aut}(\mathbb{Z}_n)) \), but this lattice becomes increasingly more difficult as the rank of \( \text{Aut}(\mathbb{Z}_n) \) increases. Călugăreanu’s technique provides an effective method of counting subgroups of an abelian group of rank 2, but it becomes far less effective for rank 3 and beyond. An explicit formula for enumerating subgroups of an abelian groups of rank 3 is fairly recent (see the aforementioned references
and, at the time of writing, any explicit formula beyond rank 3 has yet to be discovered. As such, any explicit formula for enumerating Schur rings over cyclic groups is unlikely without an explicit formula for counting subgroups of abelian groups.

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