SOME REMARKS OF HOCHSCHILD HOMOLOGY AND SEMI-ORTHOGONAL DECOMPOSITIONS

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Abstract. Given a nontrivial semi-orthogonal decomposition \( \text{Perf}(X) = \langle A, B \rangle \), and assume that the base locus of \( \omega_X \) is a proper closed subset, it was proved by Kotaro Kawatani and Shinnosuke Okawa that all skyscraper sheaves \( k(x) \) with \( x \notin \text{Bs}[\omega_X] \) belong to exactly one and only one of the components. It is natural to ask which one it is, and whether we can determine this by certain linear invariants. In this note we use Hochschild homology of derived category of coherent sheaves with support to provide another proof that if the \(-n^{th}\) Hochschild homology of a component is nonzero, then the skyscraper sheaves we consider above belong to such component, which was originally proved by Dmitrii Pirozhkov [29, Lemma 5.3]. Furthermore, we prove a conjecture proposed by Kuznetsov about classifying \( n \)-Calabi-Yau admissible subcategory of \( \text{Perf}(X) \) (\( \text{dim} \ X = n \)) for certain projective smooth variety \( X \) if we put more assumptions to the Calabi-Yau categories. Finally we remark that the additive invariants of derived category with support could provide more linear obstructions to semi-orthogonal decompositions.

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1. Introduction

Derived category of coherent sheaves play a central role in homological algebra, Mirror symmetry, and representation theory. One way to study the derived category is decomposing itself as a triangulated category. The natural decomposition is orthogonal decomposition. However, it is too restrictive since if the variety is connected, the derived category of coherent sheaves has no nontrivial orthogonal decompositions. The notion of semi-orthogonal decomposition is more flexible and it turns out that there are many examples, and much geometric information are encoded \[23\].

Hochschild homology theory of admissible subcategories of derived category of a projective smooth variety and dg categories were studied by many people. For admissible subcategories, A. Kuznetsov constructed an intrinsic Hochschild homology theory in paper [7]. One of the fantastic properties is that the Hochschild homology is additive for semi-orthogonal decompositions. This provided some interesting obstruction for semi-orthogonal decompositions. For example, if \(\text{Perf}(X)\) admits a full exceptional collection, then \(X\) must be of Hodge-Tate. dg categories are considered as noncommutative counterpart of varieties, they are more flexible than triangulated categories. For instance, many homological theories can be defined at the framework of dg categories. Furthermore, some classical conjectures in algebraic geometry can be generalized to certain dg categories, see [28]. The Hochschild homology (or additive invariants) of dg categories were extensively studied by B. Keller (or G. Tabuada), see [1], [2] (or [22]). Kuznetsov’s additivity of semi-orthogonal decompositions can be obtained in framework of dg categories, even though it is not intrinsic — we use the dg enhancement which would be non-canonical. However, it is expected that Hochschild homology of dg categories should provide more flexible possibilities to the study of semi-orthogonal decompositions or derived categories itself.

Kotaro Kawatani and Shinnosuke Okawa proved that if there is a semi-orthogonal decomposition \(\text{Perf}(X) := \langle A, B \rangle\), then either support of all objects in \(A\) are in \(Z := \text{Bs}|\omega_X|\) (then \(k(x) \in B\) with \(x \in U := X \setminus Z\)) or support of all objects in \(B\) are in \(Z\) (then \(k(x) \in A\) with \(x \in U := X \setminus Z\)) [17]. A natural question is that under the assumption that \(Z\) being proper closed subset, could we determine to which component the skyscraper sheaves belong by certain linear invariants?

In this note, we apply new techniques to study the semi-orthogonal decomposition problems via Hochschild homology of dg categories. The key observation is that \(\text{Perf}_Z(X)\) is a unique enhanced triangulated category [25]. We provide an interesting answer to the question raised above by Hochschild homology of derived categories of coherent sheaves with support. We write \(n\) as the dimension of \(X\). Note that the theorem below was originally proved in [29, Lemma 5.3].

**Theorem 1.1.** (= Theorem 4.5) [29, Lemma 5.3] Let \(X\) be a projective smooth variety of dimension \(n\). Suppose there is a nontrivial semi-orthogonal decomposition \(\text{Perf}(X) = \langle A, B \rangle\) with \(\text{HH}_{-n}(B) \neq 0\). Then the support of any object in \(A\) is contained in \(Z = \text{Bs}|\omega_X|\). The skyscraper
sheaves $k(x)$ with $x \in X \setminus Z$ belongs to $\mathcal{B}$. Furthermore, $\text{HH}_{-n}(\mathcal{A}) = 0$, $\text{HH}_{-n}(\text{Perf}(X)) \cong \text{HH}_{-n}(\mathcal{B})$.

If $\mathcal{B}$ is a $n$ Calabi-Yau category, then it is indecomposable.

Remark 1.2. Note that the statement still holds when changing the role of $\mathcal{A}$ and $\mathcal{B}$. $\mathcal{A}$ should be regarded as the smaller piece in the components of the semi-orthogonal decomposition.

Surprisingly, it is closed related to a conjecture proposed by Kuznetsov that any $n$-Calabi-Yau category of $\text{Perf}(X)$ ($\dim X = n$) is equivalent to a derived category of variety $Y$, and $X$ is blow-up of $Y$ [19]. The essence of the conjecture is that the behaviour of semi-orthogonal decompositions that contain $n$-Calabi-Yau category as a component should be the same with that of the examples constructed from blow-ups. However, Kuznetsov’s conjecture is far from being true, see “Compact Hyper−Kähler Categories” [18, Section 5]. It was shown in the paper that there are infinite many geometric 4-folds containing 4-Calabi–Yau (connected) categories which are not derived category of a projective smooth variety. Hence, we should not expect to recover geometric information by less categorical information. However, we can achieve this if there are more assumptions about the Calabi-Yau subcategory for certain varieties.

Theorem 1.3. (= Theorem 4.21) Let $Y$ be a smooth projective Calabi-Yau variety of dimension $n$, $f : X \to Y$ is the blow-up of $Y$ over points $\{y_i\}$. Define the distinguish objects $D_i = Lf^*k(y_i)$.

Let $\mathcal{B}$ be an $n$ Calabi-Yau admissible subcategory of $\text{Perf}(X)$. If all $D_i \in \mathcal{B}$, then $\mathcal{B} = Lf^*\text{Perf}(Y)$.

Remark 1.4.

(1) The assumption is something like marking points of the category $\mathcal{B}$.

(2) Assume there is an element $\phi$ of $\text{Aut}(\text{Perf}(X))$ which maps the distinguished objects $D_i = Lf^*(k(x_i))$ to the one whose support is disjoint to exceptional divisors (base locus of $K_X$). Then the assumption in the theorem always holds for $\mathcal{B}$, see Theorem 4.19.

Even though the original Kuznetsov conjecture failed, we obtain some interesting theorems which satisfy the original intuition. Thus, we expect that the Kuznetsov conjecture is still true for surfaces because the birational geometry is clearer for surfaces.

Corollary 1.5.

(1) (= Theorem 4.14) Let $X$ be a projective smooth variety of dimension $n$. Suppose $\dim H^0(X, \omega_X) \geq 2$, then any $n$-Calabi–Yau admissible subcategory of $\text{Perf}(X)$ is not a derived category of a smooth projective variety.

(2) (= Theorem 4.17) Let $X$ be a projective smooth variety. Suppose its derived category $\text{Perf}(X)$ admits a nontrivial semi-orthogonal decomposition with component $\mathcal{B}$ being n-CY category. Here $n$ is dimension of $X$. Then, the base locus $Z$ of the canonical bundle is a proper closed subset. Furthermore, any skyscraper sheaves $k(x)$ with $x \in X \setminus Z$ belongs to $\mathcal{B}$. 

One of interesting problem is to find weak Calabi–Yau categories which are non-admissible subcategory of $D^b(Y)$, where $Y$ is a projective smooth variety. It turns out that there are series of examples.

**Theorem 1.6.** (= Theorem 4.1) Let $X$ be a projective smooth variety of dimension $n$. $Z$ is a closed subscheme of $X$.

1. If $X$ is a Calabi-Yau variety, then $\text{Perf}_Z(X)$ is a weak Calabi-Yau category, and it is not an admissible subcategory of derived category of projective smooth varieties.

2. If $Z$ consist of points, then $\text{Perf}_Z(X)$ is a weak Calabi-Yau category, and it is not an admissible subcategory of derived category of projective smooth varieties.

**Remark 1.7.** We must note that probably the easiest way to prove these facts is to prove that $\text{Perf}_Z(X)$ is not smooth, then by Orlov’ result [27], it can not be an admissible subcategory of $\text{Perf}(Y)$ for some smooth projective $Y$.

As for proof of Theorem 1.1 we use recent theorem proved by Kotaro Kawatani and Shinnosuke Okawa (see Proposition 4.3) that if there is a semi-orthogonal decomposition $\text{Perf}(X) = \langle A, B \rangle$, then either support of all objects in $A$ are in $Z := B_0|\omega_X|$ or support of all objects in $B$ are in $Z$. Then we can localize the semi-orthogonal decomposition of $\text{Perf}(X)$ to that of $\text{Perf}_Z(X)$. Applying Hochschild homology theory of $\text{dg}$ categories, we prove Theorem 1.1. We essentially use the fact that $\text{HH}_{-n}(\text{Perf}_Z(X)) = 0$, $\dim X = n$ for any proper closed subset $Z$. Corollary 2.17. Finally, using the fact that Hochschild homology of admissible $n$-Calabi-Yau categories have non-vanishing $-n$ Hochschild homology [24, Corollary 5.4], we have Theorem 1.6.

**Notations.** In this note, we assume $X$ to be smooth projective over the base field $k = \mathbb{C}$. Since $D^b(X) \cong \text{Perf}(X)$ for projective smooth variety $X$, we won’t distinguish $D^b(X)$ and $\text{Perf}(X)$. We say a $k$-linear triangulated category to be weak Calabi-Yau if it admits Serre functor of shifting (the triangulated structure). It is not a classical definition of Calabi–Yau category.

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### 2. Hochschild Homology

In this section, we briefly recall the Hochschild homology for algebras, $\text{dg}$ categories and unique enhanced triangulated categories. As for notions of $\text{dg}$ category, there is a great survey by B. Keller, on differential categories [21]. The Hochschild homology theory of $\text{dg}$ algebras and
2.1. **Hochschild homology of algebra.** Given a $k$ algebra $A$, we define the Hochschild complex of $A$ to be

\[ C(A) := A \otimes_{A^e} L_A \]

and define Hochschild homology of $A$ to be $HH_\bullet(A) = H_\bullet(C(A)) = H_\bullet(A \otimes_{A^e} L_A)$. Here $A^e := A \otimes A^{op}$. We can calculate derived tensor by bar resolution of $A$ as module $A^e$: The degree $n$ term of bar complex is $A \otimes A \otimes \cdots \otimes A$, differential of the complex is defined as

\[ d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^n (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_{n+1}. \]

**Remark 2.1.** Using bar resolution, the Hochschild complex $C(A)$ is as follows: the degree $n$ part is $A \otimes A \otimes \cdots \otimes A$. The differential

\[ d_n(a_0 \otimes a_2 \otimes \cdots \otimes a_n) = \oplus_{i=0}^n (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i \cdot a_{i+1} \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}. \]

**Example 2.2.** Consider the polynomial algebra $A = k[x_1, x_2, \cdots, x_n]$. Then

\[ HH_\bullet(A) = \oplus_{i=0}^n \Omega^i(A). \]

For a differential graded algebra over $k$, we still define the Hochschild complex in the same way, but it is a little subtle. The bar resolution is no longer a projective resolution as a differential graded algebra, but the total complex of bar resolution turns out to be a proper one. Thus for differential graded algebra $A$ over $k$

\[ HH_\bullet(A) = H_\bullet(Tot(Bar(A) \otimes A)). \]

See B. Keller’s paper [1, section 1] for a more explicit description.

2.2. **Hochschild homology of differential graded category.** Hochschild homology can be generalized to small (differential graded) categories. We sketch basic notions of the differential graded categories, the reader can refer to the survey, [21, On differential graded categories].

**Definition 2.3.** The $k$-linear category $\mathcal{A}$ is called a dg category if $\text{Mor}(E, F)$ is differential $\mathbb{Z}$-graded $k$-vector spaces for every object $E, F, G \in \mathcal{A}$, and the compositions

\[ \text{Mor}(F, E) \times \text{Mor}(G, F) \to \text{Mor}(G, E) \]

are morphism of complexes and associative. Furthermore, there is a unit $k \to \text{Mor}(E, E)$. Note that $\text{Mor}(E, E)$ is a differential graded algebra because of the composition law.

**Example 2.4.** One of the basic example is $C_{dg}(k)$, objects are complexes of $k$ vector space, the morphism spaces are redefined as follows:

Let $E, F \in C_{dg}(k)$, define degree $n$ piece to be $\text{Mor}(E, F)(n) := \Pi \text{Hom}(E_i, F_{i+n})$. The $n^{th}$ differential is given by $d_n(f) = d_E \circ f - (-1)^n f \circ d_F$, $f \in \text{Mor}(E, F)(n)$. 

\[ dg \] categories is mainly from papers of B. Keller [1] and [2]. The author introduces a Hochschild homology theory of unique enhanced triangulated categories, even though it may be well known for experts.
**Definition 2.5.** We call $F: C \to D$ a dg functor between dg category if it preserves degree of morphisms. We say $F$ is quasi-equivalent if $F$ induces isomorphisms on homologies of morphisms and equivalences of their homotopic categories.

Consider the category of small dg categories, it is well known that there is a model structure with weak equivalence being quasi-equivalence [22] [3]. We denote $\text{Ho}(\text{dg-cat})$ to be the localization of the category of small dg categories to quasi-equivalence. Recall that morphism of $\mathcal{A}$ to $\mathcal{B}$ in $\text{Ho}(\text{dg-cat})$ can be represented by $\mathcal{A} \leftarrow \mathcal{A}_{\text{cof}} \to \mathcal{B}$.

Now back to the definition of Hochschild homology, the reader can refer to Bernhard Keller’s paper [2]. Let $\mathcal{A}$ be a small dg category. Consider the total complex

$$\bigoplus_{E_0, E_1, E_2, \ldots, E_n} \mathcal{A}(E_0, E_1) \otimes \mathcal{A}(E_1, E_2) \otimes \cdots \otimes \mathcal{A}(E_n, E_0)$$

where $E_0, E_1, E_2, \ldots, E_n$ run over the set of objects of $\mathcal{A}$. The tensor product is in the graded sense. The vertical degeneracy differential is defined as follows:

$$d_i(f_0, f_1, \ldots, f_{i-1}, f_i \ldots f_n) = \begin{cases} (f_0, \ldots, f_i f_{i-1}, \ldots, f_n) & \text{if } i < n \\ (-1)^{n+d}(f_n f_0, \ldots, f_{n-1}) & \text{if } i = n \end{cases}$$

The cyclic operator is given by

$$t_n(f_{n-1}, \ldots, f_0) = (-1)^{n+d}(f_0, \ldots, f_{n-1}).$$

$d = \deg(f_0) \cdot (\sum_{i=1}^{n-1} \deg(f_i))$. See [2] 1.3. The Hochschild homology of $\mathcal{A}$ is defined to be homology of the total complex.

**Remark 2.6.** Following paper [2], we can associate a mixed complex whose underline complex is defined above for a small dg category. A mixed complex can be regarded as complex with degree $-1$ operator $B$ such that $B^2 = 0$, $dB + Bd = 0$. It is equivalent to modules of a dg algebra $\Lambda$, $\Lambda := k[\varepsilon]/(\varepsilon^2)$, degree $\varepsilon$ is $-1$ and $d\varepsilon = 0$. Hence, when we talk about Hochschild complexes, we always refer to the mixed complexes in mixed derived category (the derived category of modules over a dg algebra $\Lambda$). The Hochschild homology is the homology of the mixed complex for the original differential $d$.

**Remark 2.7.**

1. A differential graded algebra can be regarded as a differential graded category with one object and the two definitions of Hochschild homology given above coincide.

2. Some people define the Hochschild homology of a mixed complex to be the opposite degree homology of the mixed complex, which is compatible with the Hochschild homology of an algebra. In this paper, we define the Hochschild homology of a mixed complex as the homology of the complex and we regard the Hochschild homology of an algebra concentrates at non-positive degrees.
Definition 2.8. The Hochschild homology can be defined for a localizing pair \( \langle B_0, B \rangle \) of differential graded categories. It is defined as \( \text{Cone}(C(B_0) \to C(B)) \) in the mixed derived category. The cone is taken in the mixed derived category, see Remark 2.6. Here the dg categories are exact (or strong pre-triangulated) dg categories and \( B_0 \) is the exact dg subcategory, see [2, 2.1]. For simplicity, the exact dg category means the shift \([n]\) and the cone exist in degree 0 levels. It is stronger than being a pre-triangulated category since the latter one requires the existence of shifts and cones in the homotopic level. We say the Verdier quotient \([B]/[B_0]\) the associated triangulated category of the localizing pair \( \langle B_0, B \rangle \).

2.3. Hochschild homology of unique enhanced triangulated categories.

Definition 2.9. Let \( T \) be a dg category. Consider Yoneda embedding, \( Y : T \hookrightarrow \text{dg-mod}(T) \). Let \( f : X \to Z \) be of degree zero and closed morphism, \( X, Z \in T \). It would happen that the object \( Y^A[m](A \in T) \) and \( \text{Cone}(Y^X \to Y^Z) \) do not come from the image of \( Y \), i.e. they are not represented. If they are represented in the homotopic category, we call \( T \) pre-triangulated dg category. If they are represented in \( \text{dg-mod}(T) \) and \( T \) has zero object, we call \( T \) strong pre-triangulated (or exact dg category). The typical example of a strong pre-triangulated category is \( C_{dg}(A) \), the dg category of complexes of \( A \) where \( A \) is an abelian category.

Lemma 2.10. Let \( A \) be a dg category. There is a universal exact dg envelop \( \text{Ext}(A) \) of \( A \). That is, there is an embedding \( A \hookrightarrow \text{Ext}(A) \), and for any exact dg category \( B \) and dg functor, \( A \to B \), there is a unique extended dg functor from \( \text{Ext}(A) \) to \( B \).

Proof. Consider Yoneda embedding \( Y : A \hookrightarrow \text{dg-mod}(A) \). Let \( \text{Ext}(A) \) be the full subcategory of \( \text{dg-mod}(A) \) with objects being generated by shifting and cones of \( Y([A]) \) in the homotopic category. \( \text{Ext}(A) \) satisfies the universal property. Clearly, we have equivalent \( Y : [A] \to [\text{Ext}(A)] \) if \( A \) is pre-triangulated. \( \square \)

Definition 2.11. The triangulated category \( T \) has a unique enhancement if it has pre-triangulated enhancement. Moreover, for any two enhancements \( T_1 \) and \( T_2 \), there exists a quasi-functor or a morphism in \( \text{Ho}(\text{dg-cat}) \) (according to Bertrand Toën [3], they are equivalent) which induces an equivalence of homotopic categories.

Lemma 2.12. Let \( X \) be a projective smooth variety and \( Z \) is a closed scheme of \( X \). The triangulated categories \( \text{Perf}(X) \) and \( \text{Perf}_Z(X) \) are unique enhanced triangulated categories [5] [25, Corollary 2]. Here, \( \text{Perf}_Z(X) \) is the full subcategory of \( \text{Perf}(X) \) whose objects support in \( Z \).

Proof. The statement is true for more general \( X \), which is also well known for experts. For the case of projective smooth variety \( X \), there is an easier way to see this. Since \( \text{Perf}(X) \cong \text{D}^b(\text{coh}(X)) \), and \( \text{Perf}_Z(X) \cong \text{D}^b(\text{coh}_Z(X)) \), according to [25, Corollary 2], they are unique enhanced triangulated categories. \( \square \)

Definition 2.13. Let \( T \) be a unique enhanced triangulated category. We define Hochschild homology of \( T \) as Hochschild homology of its pre-triangulated dg enhancement. Note that two
different pre-triangulated dg enhancement are isomorphic in Ho(dg-cat), which implies there is a chain of quasi-equivalences of dg categories connected these two enhancement. Thus, the Hochschild homology of $\mathcal{T}$ is independent of choice of pre-triangulated enhancement because Hochschild homology is a derived Morita invariant.

We calculate the Hochschild homology of $\text{Perf}(X)$ and $\text{Perf}_Z(X)$ via this definition. Before doing these, we need some lemmas.

**Lemma 2.14.** Let $\langle B_0, B \rangle$ be a localizing pair, and denote its Hochschild complex by $C_*$. Suppose $T := [B/B_0] \cong [B]/[B_0]$ is a unique enhanced triangulated categories, then $C(T) \cong C_*$ in mixed derived category.

**Proof.** Since $B$ is a pre-triangulated category, the Drinfeld dg quotient $B/B_0$ (see Drinfeld [4]) is a pre-triangulated dg category [5, Lemma 1.5]. Consider the exact dg envelope $D$, we have a diagram

$$B_0 \to B \to B/B_0 \hookrightarrow D$$

which induces an exact sequence of triangulated categories:

$$[B_0] \to [B] \to [B/B_0] \cong [D].$$

According to Bernhard Keller [2 Thm 4(c)], there is a triangle in the mixed derived category,

$$C(B_0) \to C(B) \to C(D) \to C(B_0[1]).$$

Then, by definition, we have an isomorphism of Hochschild complexes $C_* \cong C(T)$ in the mixed derived category. □

**Proposition 2.15.** ([8 Section 5.7]) Let $X$ be a smooth projective variety, and $Z$ is a closed subset of $X$. Consider the natural exact dg pairs $\text{Perf}(X)$: They are the category of perfect complexes and its full subcategory of acyclic perfect complexes whose associated triangulated categories is $\text{Perf}(X)$. Similar notation $\text{Perf}_Z(X)$ for the natural dg pair of support case. The Hochschild complexes of these dg pairs are isomorphic to $R\Gamma(X, M(O_X))$ and $R\Gamma_Z(X, M(O_X))$ respectively, where $M(O_X)$ is the sheaf of bar complex.

**Theorem 2.16.** Let $X$ be a projective smooth variety, and $Z$ is a closed subset. Then

$$\text{HH}_i(\text{Perf}_Z(X)) \cong \bigoplus_{p-q=i} H^p_X(\Omega^q_X).$$

**Proof.** According to Lemma 2.14 and Proposition 2.15 it suffices to compute $R\Gamma_Z(X, M(O_X))$. If $X$ is smooth, we have a morphism of complexes which is a $k$-quasi-isomorphism

$$\cdots \to O_X \otimes_{O_X} O_X \otimes O_X \bar{\to} O_X \otimes O_X \bar{\to} O_X \otimes O_X \bar{\to} O_X \to 0 \to 0 \to 0$$

$$\cdots \to \Omega^2_X \to \Omega^1_X \to \Omega^0_X \to 0 \to 0.$$
Here $M(O_X)$ is $k$-quasi-isomorphic to sheaf version of the bar complex, $d(f_1 \otimes f_2 \otimes \cdots \otimes f_n) := \frac{1}{n!} f_1(df_2 \wedge \cdots \wedge df_n)$. Note that it is firstly defined via presheaves, and then induces morphism of sheaves. Locally it is a $k$-quasi-isomorphism by usual HKR theorem for regular algebras. Hence the differential induces a $k$-quasi-isomorphism of complexes. Then formula follow easily. \hfill \Box

**Corollary 2.17.** Let $X$ be a projective smooth variety of dimension $n$, and $Z$ is a proper closed subscheme of $X$. Then, $\text{HH}_{-n}(\text{Perf}_Z(X)) = 0$.

**Proof.** According Theorem 2.16

$$\text{HH}_{-n}(\text{Perf}_Z(X)) \cong \bigoplus_{p-q=-n} H^p_Z(X, \Omega^q_X) = H^0_Z(X, \Omega^n_X) = 0.$$

\hfill \Box

### 3. Computation of Hochschild Homology via Fourier-Mukai Kernels

In this section, we will first introduce basic notions of Fourier-Mukai transform in algebraic geometry, the standard reference is [6, D. Huybrechtz’s book]. Then we recompute the geometric formulas of Hochschild homology in Theorem 2.16 by using techniques of Fourier-Mukai kernels.

#### 3.1. Fourier-Mukai Functors.

**Definition 3.1.** Let $X, Y$ be projective smooth varieties, $E \in \text{Perf}(X \times Y)$, the induced functor

$$F: \text{Perf}(X) \to \text{Perf}(Y)$$

given by $F(\bullet) = R\pi_2^*(\pi_1^* \bullet \otimes E)$ is called Fourier-Mukai functor associated to the kernel $E$.

#### Example 3.2. $\text{Id}: X \to X$ is given by kernel $\Delta_* O_X$. This follows from the projection formula

$$R\pi_2^*(\pi_1^* E \otimes \Delta_* O_X) = R\pi_2^*(\Delta^* (\Delta^* p_1^* E \otimes O_X)) = E.$$
The readers can check that derived functors $Rf_* \colon \text{Perf}(X) \to \text{Perf}(Y)$ and $Lf^* \colon \text{Perf}(Y) \to \text{Perf}(X)$ which are induced by morphism $f : X \to Y$ are both Fourier-Mukai functors with some kernels, including Serre functor and the functor tensor with line bundles.

**Example 3.3.** Let $F_1 \in \text{Perf}(X \times Y)$ and $F_2 \in \text{Perf}(Y \times Z)$ be the kernels respect to Furier-Mukai functors $f_1$ and $f_2$. Then the composition $f_2 \circ f_1$ is a Fourier-Mukai functor with kernel $\pi_{XZ}^*(\pi_{XY}^*(F_1) \otimes \pi_{YZ}^*(F_2))$.

It is natural to ask whether the exact functors between derived categories are of Fourier-Mukai functor, it is true for fully faithful functor.

**Proposition 3.4.** (D. Orlov [10, Thm 3.4]) Let $F : \text{Perf}(X) \leftarrow \text{Perf}(Y)$ be an exact fully faithful functor which has adjoint (left or right adjoint). Then it is isomorphic to a Fourier-Mukai functor. In particular, the corresponding Kernel is unique up to isomorphism.

**Remark 3.5.** The assumption that the fully faithful functor obtain adjoint functors can be skipped due to deep results of Bondal, Van den Bergh [11].

**Example 3.6.** The identity functor $\text{id} : \text{Perf}(X) \to \text{Perf}(X)$ is Fourier-Mukai functor with kernel $\mathcal{O}_{\Delta}$. The kernel corresponding to $\text{id}$ is unique up to isomorphism.

The theory of which kinds of functors are Fourier-Mukai functor was generalized to derived categories with support [15, Theorem 1.1]. Assume $Z$ have no components of zero dimension. Clearly the identity functor of $\text{Perf}_Z(X)$ satisfies assumption in [15, Theorem 1.1]. We prove a theorem which will be used to compute Hochschild homology of $\text{Perf}_Z(X)$.

**Theorem 3.7.** Let $X$ be a smooth projective variety, and $Z$ is a closed subscheme with no zero dimensional components. The object $\Delta_* R\Gamma_Z^* (\mathcal{O}_X)$ is a kernel of $\text{id} : \text{Perf}_Z(X) \to \text{Perf}_Z(X)$. If $E \in D_{Z \times Z, \text{qch}}(X \times X)$ such that $\Phi_E \cong \text{id}$, then $E \cong \Delta_* R\Gamma_Z^* (\mathcal{O}_X)$.

Before proving Theorem 3.7, we propose some interesting lemmas about projection formula and support of objects, which will be used in later calculations too.

**Lemma 3.8.** [14, Tag 01E6] Let $f : X \to Y$ be morphism of ringed space. Suppose $f$ maps $X$ homeomorphic into closed subset, then the general projection formula holds: $E \in D(\mathcal{O}_X)$, $F \in D(\mathcal{O}_Y)$. Then

$$Rf_*(E \otimes^L Lf^* F) \cong Rf_* E \otimes^L F.$$ 

In particular, it is true for closed immersion.

**Remark 3.9.**

(1) Derived Tensor product, pull back and pull forward maps can be defined in generality of $\mathcal{O}_X$ module, namely using $K$-injective and $K$-flat resolution to define. See "resolution of unbounded complex" [9].
Lemma 3.10. Let $X$ be an algebraic variety, $F \in D^b_{qch}(X)$. Take a closed point $x$, consider the embedding $l_x : x \hookrightarrow X$, then $x$ is in the support of $F$ if and only if $\mathbb{L}^*Fx \neq 0$.

Proof. Suppose $x$ is in the support of $F$, but $\mathbb{L}^*Fx = 0$. Let $x \in \text{support of } \mathcal{H}^m(F)$ with maximal integer $m$. Then, there is a canonical truncation: $F_{\leq m-1} \rightarrow F \rightarrow F_{>m-1} \rightarrow F_{\leq m-1}[1]$. Since $\mathbb{L}^*Fx = 0$, we have isomorphism $\mathbb{L}^*F_{>m-1} \cong \mathbb{L}^*F_{\leq m-1}[1]$. Since $l^{-1}_xF_{>m-1} \cong l^{-1}_x\mathcal{H}^m(F)[-m]$, therefore $\mathbb{L}^*F_{>m-1} \cong l^{-1}_x\mathcal{H}^m(F)[-m] \o k(x) \cong \mathbb{L}^*\mathcal{H}^m(F)[-m]$. Take the flat resolution:

$$\cdots \rightarrow l_1 \rightarrow l_0 \rightarrow \mathcal{H}^m(F).$$

Apply the functor (without derived) $\mathbb{L}^*$:

$$\cdots \rightarrow \mathbb{L}^*l_1 \rightarrow \mathbb{L}^*l_0 \rightarrow 0.$$ 

Then the $0^{th}$ homology gives $\mathcal{H}^m(F)_x \o k(x)$. Since by assumption, $x \in \text{support of } \mathcal{H}^m(F)$, then $\mathcal{H}^m(F)_x \neq 0$. Hence by Nakayama Lemma, $\mathcal{H}^m(F)_x \o k(x) \neq 0$. Therefore, $\mathbb{L}^*F_{>m-1} \cong \mathbb{L}^*F_{\leq m-1}[1] \neq 0$ has non zero degree $m$ homology. However, $F_{\leq m-1}[1]$ survives in degree less than $m$, hence after taking a flat resolution, it survives in degree less than $m$. But it means that $\mathbb{L}^*F_{\leq m-1}[1]$ has zero $m$ homology, a contradiction.

Suppose $\mathbb{L}^*Fx \neq 0$, but $x \notin \text{support of } F$. Since $\mathbb{L}^*F = l^{-1}_xF \o \mathbb{L}^*\mathcal{H}^m(F)_x$, and $l^{-1}_xF$ is an acyclic complex, we have $\mathbb{L}^*F \cong 0$, a contradiction. 

Remark 3.11. There is another proof using spectral sequence. Consider the spectral sequence

$$E_2^{p,q} = \text{Tor}_p(\mathcal{H}^q(l^{-1}_xF), k(x)) \Rightarrow \mathcal{H}^{-p-q}(\mathbb{L}^*Fx).$$

Suppose $\mathbb{L}^*Fx \neq 0$, but $x \notin \text{support of } F$. Then $E_2^{p,q} = 0$, a contradiction. Suppose $x \in \text{support of } F$, but $\mathbb{L}^*Fx = 0$. Let $m$ be the maximal integer such that $x \in \text{support of } \mathcal{H}^m(F)$. Then $E^{0,m}_2 \neq 0$ and degenerates to $E_\infty$, a contradiction.

Proof of Theorem 3.7. Let $F \in D_Z(\text{Qch}(X))$. Actually, by [15 Theorem 1.1]), we need to check perfect objects $F$, but here, it is true for more general $F$. Since the diagonal morphism is a closed embedding, the projection formula always holds for $O_X$ module.

$$R\pi_*(R\pi^*F \o \pi^*\Delta_*\text{R}_ZO_X) \cong R\pi_*(\pi^*(\Delta^*R\pi^*F) \o \pi^*\text{R}_ZO_X) \cong F \o \text{R}_ZO_X.$$

Denote $j : X \setminus Z \rightarrow X$, there is a triangle

$$\text{R}_ZO_X \rightarrow O_X \rightarrow Rj_*j^*O_X \rightarrow \text{R}_ZO_X[1].$$

Tensor (derived sense) with $F$, there is a new triangle

$$\text{R}_ZO_X \o F \rightarrow O_X \o F \rightarrow Rj_*j^*O_X \o F \rightarrow \text{R}_ZO_X \o F[1].$$
Clearly, $R_j j^* O_X \otimes^L F \cong R_j j^* F \cong 0$, and hence $R \Gamma_Z O_X \otimes^L F \cong F$. Suppose $E \in D(\text{Qch}_Z(X \times X))$ such that $\Phi_E \cong \text{id}$. Then according to \cite[Lemma 5.3]{15}, $E \in D^b(\text{Qch}_Z(X \times X))$. Thus, according to \cite[Theorem 1.1]{15}, $E \cong \Delta_* R \Gamma_Z O_X$. □

3.2. The computations.

**Theorem 3.12.** Let $X$ be a projective smooth variety and $Z$ is a closed subscheme with no zero dimensional components. Then

$$HH_i(\text{Perf}_Z(X)) \cong \text{Hom}^i(O_{X \times X}, \Delta_* R \Gamma_Z O_X) \otimes^L \Delta_* R \Gamma_Z O_X) \cong \bigoplus_{p-q=i} H^p_Z(X, \Omega^q_X).$$

**Remark 3.13.** We assume $Z$ to be closed subscheme with no zero dimensional components since the kernel for $\text{id}$ is proved to be unique up to isomorphism in this case. We hope that it is true more generally.

Before proving Theorem 3.12 we need some preparations.

**Lemma 3.14.** Let $X$ be a smooth projective variety and $Z$ is a closed subscheme of $X$. Then the categories $D_{\text{qch}}(X)$ and $D_{\text{Z,qch}}(X)$ have a compact generator respectively \cite[Theorem 6.8]{12}. In particular, they are derived equivalent to derived category of some $\text{dg}$ algebras.

**Proof.** For derived category without support, it is well known. The case of support is similar. Let $E$ be a compact generator of $D_{\text{Z,qch}}(X)$. We write $E$ again after resolution to a $K$-injective perfect complex. Define $\Lambda = \text{Hom}_{\text{dg}}(E, E)$. Then

$$L: D(\Lambda) \rightarrow D_{\text{Z,qch}}(X), \quad M \mapsto E \otimes^L_A M.$$

$$R\text{Hom}(E, \cdot): D_{\text{Z,qch}}(X) \rightarrow D(\Lambda), \quad F \mapsto R\text{Hom}(E, F).$$

define equivalence $D(\Lambda) \cong D_{\text{Z,qch}}(X)$. □

**Lemma 3.15.** Let $E$ be a compact generator of $D_{\text{Z,qch}}(X)$. Then $E^\vee$ is also a compact generator too. In particular, $E^\vee \boxtimes E$ is a compact generator of $D_{\text{Z,qch}}(X \times X)$.

**Proof.** Again, this fact is well know for non support case. We provide a proof here which mimic the proof in A. Bondal, Van den Bergh’s paper \cite[Lemma 3.4.1]{11}.

**Definition 3.16.** Let $C$ be a $k$-linear triangulated category. We say $E$ (could be set of objects) generates $C$ if $\text{Hom}(E[n], c) = 0$ for all integer $n$ implies $c = 0$. $E$ classical generates $C$ if the minimal full triangulated subcategory of $C$ containing $E$ must be $C$. The reader can refer to the paper \cite{11} for the definitions.

**Lemma 3.17.** \cite{13} Assume $C$ (triangulated $k$-linear category) is compactly generated, that is $C^c$ generates $C$. Then a set of objects $E \in C^c$ classically generates $C^c$ if and only if it generates $C$.
According to the Lemma 3.17 it suffices to prove $E^\nu$ classical generates $\text{Perf}_Z(X)$. Consider $F \in \text{Perf}_Z(X)$, then $F^\nu \in \text{Perf}_Z(X)$, hence $F^\nu \in (E)$, taking dual, $F \in (E^\nu)$.

Back to object $E^\nu \boxtimes E$, its support $\subseteq Z \times Z$, which can be easily proved by using Lemma 3.10 above. Claim: $E^\nu \boxtimes E$ is a compact generator of $D_{Z \times Z.qch}(X \times X)$. It suffices to prove that $E^\nu \boxtimes E$ classical generates $\text{Perf}_{Z \times Z}(X \times X)$ by Lemma 3.17. But again by Neeman-Ravenel theorem, replace $C$ by $C^c$, we conclude that $E^\nu \boxtimes E$ classical generates $\text{Perf}_{Z \times Z}(X \times X)$ if and only if $E^\nu \boxtimes E$ generates $\text{Perf}_{Z \times Z}(X \times X)$.

Let $W \in \text{Perf}_{Z \times Z}(X \times X)$. Suppose $\text{Hom}(E^\nu \boxtimes E[i], W) = 0$ for any integer $i$.

For arbitrary integer $m$, we obtain

$$\text{Hom}(E^\nu[i + m], R\text{Hom}_{Z \times Z}(p_2^* E, W[m])) = 0.$$ 

$R\text{Hom}_{Z \times Z}(p_2^* E, W[m])$ supports in $Z$, the proof is as follows: Firstly, the support of $R\text{Hom}_{Z \times Z}(p_2^* E, W[m])$ is in $Z \times Z$, which follows from Lemma 3.10. We claim that $R\text{Hom}(F)$ supports in $Z$ if $F$ supports in $Z \times Z$. There is a commutative diagram, $U = X \setminus Z$.

$$
\begin{array}{ccc}
U & \xrightarrow{j} & X \\
\downarrow{p_1} & & \downarrow{p_1} \\
X & \xrightarrow{\text{Id}} & X \\
\end{array}
$$

According to flat base change theorem, $Lj^* R\text{Hom}_{Z \times Z}(p_2^* E, W[m]) \cong 0$, which means the support of $R\text{Hom}_{Z \times Z}(p_2^* E, W[m])$ supports in $Z$.

Since $E^\nu$ is a compact generator of $D_{Z.qch}(X)$,

$$R\text{Hom}_{Z \times Z}(p_2^* E, W[m + n]) = 0.$$ 

Take any affine open sub-scheme $U$ of $X$, we have $\text{Hom}_{U \times X}(p_2^* E \mid_U \times X, W \mid_U \times X[m + n]) \cong 0$, and then $\text{Hom}_{X}(E, (R\text{Hom}_{U \times X}[p_2^* E, W][m])[n]) = 0$. Since $W \mid_U \times X$ supports in $(Z \cap U) \times Z$, $R\text{Hom}_{U \times X}[n]$ supports in $Z$ because of the same reason above. We obtain

$$R\text{Hom}_{U \times X}[n] = 0$$

for any integer $n$. Again taking any affine open sub-scheme of $V$ of $X$, we have

$$\Gamma(U \times V, W \mid_{U \times V}) = 0$$

for any open affine $U$, $V$. Therefore, $W \cong 0$.

**Proposition 3.18.** Choose a compact generator $E$ of $D_{Z.qch}(X)$. Let $\Lambda = R\text{Hom}(E, E)$. There is a commutative diagram
The morphism in the upper row is Fourier-Mukai functor associated with \( M \otimes_{\Lambda^{\text{op}} \otimes \Lambda} E^\vee \boxtimes E \).

**Proof.** We need to prove:

\[
\text{Rp}_2(p_1^* F \otimes^L E^\vee \boxtimes E \otimes_{\Lambda^{\text{op}} \otimes \Lambda} M) \cong \text{RHom}(E, F) \otimes_{\Lambda^{\text{op}}} M \otimes^L_{\Lambda} E.
\]

Firstly, we replace \( M \) with \( \Lambda^{\text{op}} \otimes \Lambda \), then by projection formula

\[
\text{Left} = (\text{RHom}(E, F) \boxtimes E) \otimes^L_{\Lambda^{\text{op}} \otimes \Lambda} \Lambda^{\text{op}} \otimes \Lambda = \text{Right}.
\]

For general \( M \), we have semi-free resolution, without loss of generality, assume \( M \) semi-free. That is, there is a filtration: \( 0 \subset \phi_1 \subset \phi_2 \subset \cdots \subset \phi_n \subset \cdots \subset M \), with quotient being direct sum (maybe infinite) of \( \Lambda^{\text{op}} \otimes \Lambda[n] \). Then since the formula holds for \( \Lambda^{\text{op}} \otimes \Lambda[n] \), and hypercohomology and tensor product commute with direct sum (infinite), it holds for \( \phi_i \). According to [14, Tag 09KL], there is a triangle

\[
\oplus \phi_i \rightarrow \oplus \phi_i \rightarrow M.
\]

Thus, it holds for \( M \). \( \square \)

**Corollary 3.19.** \( \Lambda \otimes^L_{\Lambda^{\text{op}} \otimes \Lambda} E^\vee \boxtimes E \) is the Fourier-Mukai kernel corresponding to identity functor of \( D_{Z,qch}(X) \) and hence of \( \text{Perf}_Z(X) \). In particular, \( \Lambda \otimes^L_{\Lambda^{\text{op}} \otimes \Lambda} E^\vee \boxtimes E \cong \Delta_\ast R\Gamma_Z(O_X) \) if \( Z \) has no zero dimensional components.

**Proof.**

\[
\text{Rp}_2(p_1^* F \otimes^L (p_1^* E^\vee \otimes_{O_X}^L p_2^* E) \otimes_{\Lambda^{\text{op}} \otimes \Lambda} \Lambda) \cong (\text{RHom}(E, F) \boxtimes E) \otimes^L_{\Lambda^{\text{op}} \otimes \Lambda} \Lambda
\]

\[
\cong \text{RHom}(E, F) \otimes^L_{\Lambda} E
\]

\[
\cong F.
\]

By Theorem 3.7, \( \Lambda \otimes^L_{\Lambda^{\text{op}} \otimes \Lambda} E^\vee \boxtimes E \cong \Delta_\ast R\Gamma_Z(O_X) \).

\( \square \)

**Proof of Theorem 3.12.** The method is that we compute \( \text{Perf}_Z(X) \) via special dg enhancement.

Choose a compact generator \( E \) of \( D_{Z,qch}(X) \). According to Lemma 3.14, \( \text{Perf}(\Lambda) \cong \text{Perf}_Z(X) \) where \( \Lambda = \text{RHom}(E, E) \). Therefore, the dg category \( \text{Per}_{\text{dg}}(\Lambda) \) is a dg enhancement of \( \text{Perf}_Z(X) \). Thus,

\[
\text{HH}_\ast(\text{Perf}_Z(X)) \cong \text{HH}_\ast(\text{Per}_{\text{dg}}(\Lambda)) \cong \text{HH}_\ast(\Lambda).
\]

The second isomorphism is because the natural Yoneda embedding \( \Lambda \rightarrow \text{Per}_{\text{dg}}(\Lambda) \) is a derived Morita equivalence.
Therefore:

Since for any object \( \text{H} \), by induction (resolution of semi-free module), we have \( M \otimes L N \cong \text{R} \Gamma(X, (M \otimes L E) \otimes L (E^\vee \otimes L N)) \) for right \( \text{H} \) module \( M \) and left \( \text{H} \) module \( N \). Similarly, \( \text{R} \Gamma(X \times X, (E^\vee \otimes E) \otimes L (E^\vee \otimes E)) \cong \Lambda^{op} \otimes \Lambda \). Hence in general, for \( \Lambda^{op} \otimes \Lambda \)-module \( M \) and \( N \), we have:

\[
M \otimes_{\Lambda^{op} \otimes \Lambda} N \cong \text{R} \Gamma(X \times X, (M \otimes E^\vee \otimes E) \otimes (E^\vee \otimes E \otimes N)).
\]

In particular, \( \Lambda \otimes_{\Lambda^{op} \otimes \Lambda} \Lambda \cong \text{R} \Gamma(X \times X, (\Lambda \otimes E^\vee \otimes E) \otimes (E^\vee \otimes E \otimes \Lambda)) \).

According to Corollary 3.19, we have \( \text{HH}_n(\Lambda) \cong \text{Hom}^*(O_{X \times X}, \Delta, \text{R} \Gamma_Z(O_X) \otimes L \Delta, \text{R} \Gamma_Z(O_X)[i]) \). The remaining thing is to compute \( \text{Hom}^*(O_{X \times X}, \Delta, \text{R} \Gamma_Z(O_X) \otimes L \Delta, \text{R} \Gamma_Z(O_X)) \).

There is a triangle:

\[
\text{R} \Gamma_Z O_X \rightarrow O_X \rightarrow \text{Rj}_j^* O_X \rightarrow \text{R} \Gamma_Z O_X[1].
\]

Apply functor \( \Delta^* \), we have:

\[
\Delta^* \text{R} \Gamma_Z O_X \rightarrow \Delta^* O_X \rightarrow \Delta^* \text{Rj}_j^* O_X \rightarrow \Delta^* \text{R} \Gamma_Z O_X[1].
\]

Tensor (derived) with the object \( \Delta^* \text{R} \Gamma_Z O_X \) we get the triangle:

\[
\Delta^* \text{R} \Gamma_Z O_X \otimes L \Delta^* \text{R} \Gamma_Z O_X \rightarrow \Delta^* O_X \otimes L \Delta^* \text{R} \Gamma_Z O_X \rightarrow \Delta^* \text{Rj}_j^* O_X \otimes L \Delta^* \text{R} \Gamma_Z O_X \rightarrow +.
\]

By projection formula for closed immersion:

\[
\Delta^* \text{Rj}_j^* O_X \otimes L \Delta^* \text{R} \Gamma_Z O_X \cong \Delta^*(\text{Rj}_j^* O_X \otimes L \Delta^* \Delta^* \text{R} \Gamma_Z O_X).
\]

Since for any object \( F \in D(X) \):

\[
(L\Delta^* \Delta^* O_X) \otimes L F \cong L\Delta^*(\Delta^* O_X \otimes L p^* F).
\]

\[
\Delta^* O_X \otimes L p^* F \cong \Delta^*(O_X \otimes L \Delta^* p^* F) \cong \Delta^* F.
\]

Therefore

\[
(L\Delta^* \Delta^* O_X) \otimes L F \cong L\Delta^* \Delta^* F.
\]

By result 16, Thm 4.1], \( L\Delta^* \Delta^* O_X \cong \bigoplus_{1 \leq i \leq n} \Omega[i] \). Hence \( L\Delta^* \Delta^* \text{R} \Gamma_Z O_X \) supports in \( Z \). Again by the similar reason,

\[
\text{Rj}_j^* O_X \otimes L \Delta^* \Delta^* \text{R} \Gamma_Z O_X \cong 0.
\]

Therefore:

\[
\Delta^* \text{Rj}_j^* O_X \otimes L \Delta^* \text{R} \Gamma_Z O_X \cong 0.
\]
Hence
\[ \Delta_!R\Gamma_Z O_X \otimes^L \Delta_!R\Gamma_Z O_X \cong \Delta_!O_X \otimes^L \Delta_!R\Gamma_Z O_X. \]

Then
\[ \text{Hom}^\bullet(X \times X, \Delta_!R\Gamma_Z O_X \otimes^L \Delta_!R\Gamma_Z O_X) \cong \text{Hom}^\bullet(X \times X, \Delta_!R\Gamma_Z O_X \otimes^L \Delta_!O_X). \]

According to projection formula for closed immersion again:
\[ \Delta_!R\Gamma_Z O_X \otimes^L \Delta_!O_X \cong \Delta_!(R\Gamma_Z O_X \otimes^L L\Delta^* \Delta_!O_X). \]

Hence
\[ \text{Hom}^\bullet(X \times X, \Delta_!R\Gamma_Z O_X \otimes^L \Delta_!R\Gamma_Z O_X) \cong \text{Hom}^\bullet(X, R\Gamma_Z O_X \otimes \bigoplus_{1 \leq i \leq n} \Omega^i[i]). \]

But derived tensor \( \bigoplus_{1 \leq i \leq n} \Omega^i[i] \) with the triangle:
\[ R\Gamma_Z O_X \to O_X \to Rj_*j^* O_X \to R\Gamma_Z O_X[1]. \]

We have
\[ R\Gamma_Z O_X \otimes^L \bigoplus_{1 \leq i \leq n} \Omega^i[i] \to O_X \otimes^L \bigoplus_{1 \leq i \leq n} \Omega^i[i] \to Rj_*j^* O_X \otimes^L \bigoplus_{1 \leq i \leq n} \Omega^i[i] \to R\Gamma_Z O_X \otimes^L \bigoplus_{1 \leq i \leq n} \Omega^i[i][1]. \]

Since
\[ Rj_*j^* O_X \otimes^L \bigoplus_{1 \leq i \leq n} \Omega^i[i] \cong Rj_*j^* \bigoplus_{1 \leq i \leq n} \Omega^i[i], \]

which is compatible with the triangle, we have
\[ R\Gamma_Z O_X \otimes \bigoplus_{1 \leq i \leq n} \Omega^i[i] \cong R\Gamma_Z \bigoplus_{1 \leq i \leq n} \Omega^i[i]. \]

Finally
\[ \text{Hom}^\bullet(X \times X, \Delta_!R\Gamma_Z O_X \otimes^L \Delta_!R\Gamma_Z O_X) \cong R\Gamma_Z(X, \bigoplus_{1 \leq i \leq n} \Omega^i[i]) \cong \bigoplus_{t \geq 0} \bigoplus_{p-q=t} H^p(X, \Omega^q_X)[t]. \]

\[ \square \]

4. Applications

In this section, we apply the Hochschild homology theory of derived categories with support to certain problems of derived categories of variety and problems of semi-orthogonal decompositions.
4.1. **weak CY categories which are not admissible.** It is interesting to find some examples of weak Calabi-Yau categories which are not admissible subcategories of Perf($Y$), here $Y$ is assumed to be projective smooth. Note that we say weak Calabi–Yau category if the Serre functor is a shifting. We obtain a series of examples as follows.

**Theorem 4.1.** Let $X$ be a projective smooth variety of dimension $n$. $Z$ is a closed subscheme of $X$.

1. If $X$ is a Calabi-Yau variety, then Perf$_Z(X)$ is a weak $n$-Calabi-Yau category, and it is not an admissible subcategory of derived category of projective smooth varieties.
2. If $Z$ consist of points, then Perf$_Z(X)$ is a weak $n$-Calabi-Yau category, and it is not an admissible subcategory of derived category of projective smooth varieties.

**Proof.** the Serre functor of Perf$_Z(X)$ is $\otimes \omega_X[n]$. For (1), since $\omega_X \cong O_X$, therefore the Serre functor of Perf$_Z(X)$ is $[n]$. For (2), since $\omega_X$ is locally trivial around the points $Z$, $E \otimes \omega_X \cong E$ for any object $E \in$ Perf$_Z(X)$. Thus, the Serre functor of Perf$_Z(X)$ is $[n]$. For both (1) and (2), we have that Perf$_Z(X)$ admits Serre functor $[n]$. To prove that Perf$_Z(X)$ are not admissible subcategory of some derived category of projective smooth varieties, we need a lemma.

**Lemma 4.2.** ([24, Corollary 5.4]) Let $B$ be an admissible subcategory of some derived category of a projective smooth variety. Suppose it is a $m$-Calabi–Yau category, then $\text{HH}^{-m}(B) \neq 0$.

Back to the proof. Suppose Perf$_Z(X) \cong B$ is an admissible subcategory of Perf($Y$) where $Y$ is a smooth projective variety. Since the Hochschild homology of $B$ in paper [24] is compatible with some special dg enhancement of $B$, let $A = \text{RHom}(F,F)$ which is a dg algebra. Then, $\text{HH}_n(A) \cong \text{HH}_n(B)$. Thus, $\text{HH}_n(\text{Perf}_Z(X)) \cong \text{HH}_n(A) \cong \text{HH}_n(B) \neq 0$. But according to Corollary 2.17, $\text{HH}_n(\text{Perf}_Z(X)) = 0$, a contradiction. □

4.2. **Applications to semi-orthogonal decomposition.** Let $X$ be a projective smooth variety. It an interesting question that whether Perf($X$) have no nontrivial semi-orthogonal decompositions. Kotaro Kawatani and Shinnosuke Okawa constructed an obstruction for the existence of nontrivial semi-orthogonal decompositions.

**Proposition 4.3.** ([17, Kotaro Kawatani and Shinnosuke Okawa, Thm 3.1]) Let $X$ be a proper smooth variety, suppose there is a nontrivial semi-orthogonal decomposition, Perf($X$) = $\langle A, B \rangle$. Then only one of the following cases happen:

1. The support of object in $A$ is contained in $Z = Bs|\omega_X|$.
2. The support of object in $B$ is contained in $Z = Bs|\omega_X|$.

Furthermore, if (1) (or (2)) holds, then for $x \in X \setminus Z$, $k(x) \in B$ (or $A$).

**Remark 4.4.** This fact is generalized to para-canonical base locus $Z = \text{PBs}|\omega_X|$ [30].

It is natural to ask which component in Proposition 4.3 support in the base locus of $\omega_X$. We provide a criteria via Hochshcild homology.
Theorem 4.5. [29, Lemma 5.3] Let $X$ be a projective smooth variety of dimension $n$. Suppose there is a nontrivial semi-orthogonal decomposition $\text{Perf}(X) = \langle A, B \rangle$ with $\text{HH}_{-n}(B) \neq 0$. Then the support of any object in $A$ is contained in $Z = Bs|\omega_X|$. The skyscraper sheaves $k(x)$ with $x \in X \setminus Z$ belongs to $B$. Furthermore, $\text{HH}_{-n}(A) = 0$, $\text{HH}_{-n}(\text{Perf}(X)) \cong \text{HH}_{-n}(B)$. If $B$ is a $n$ Calabi-Yau category, then it is indecomposable.

Remark 4.6. $A$ should be regarded as the smaller piece in the components of the semi-orthogonal decomposition.

Before proving Theorem 4.5, we need some preparations that Hochschild homology is additive with respect to semi-orthogonal decompositions. Again, it is well known for experts, we provide proof here for completion.

Theorem 4.7. Let $D_{dg}$ be a small $dg$ category with homotopic category $[D_{dg}] = D$. Suppose there is a semi-orthogonal decomposition of $D := \langle A, B \rangle$. Let $A_{dg}$ and $B_{dg}$ be the full sub $dg$ category of $D_{dg}$ such that objects belong to $A$ and $B$ respectively. Then,

$$\text{HH}_\bullet(D_{dg}) \cong \text{HH}_\bullet(A_{dg}) \oplus \text{HH}_\bullet(B_{dg}).$$

Proof. A part of proof here is taken from G. Tabuada’s book [26, Proposition 2.2]. Consider the $dg$ bi-module

$$M: B_{dg}^{op} \otimes A_{dg} \to C_{dg}(k) \quad (b, a) \mapsto D_{dg}(X)(a, b)$$

Define the gluing of $dg$ categories with respect to $M$ as

$$\mathcal{T}(A_{dg}, B_{dg}; M)(x, y) = \begin{cases} A_{dg}(x, y), & \text{if } x, y \in A_{dg} \\ B_{dg}(x, y), & \text{if } x, y \in B_{dg} \\ M(x, y), & \text{if } x \in A_{dg}, y \in B_{dg} \\ 0, & \text{otherwise, } x \in B_{dg}, y \in A_{dg}. \end{cases}$$

The natural morphism of $dg$ categories

$$\mathcal{T} \to D_{dg}$$

is a derived Morita equivalence. We have natural isomorphism of Hochschild homology

$$\text{HH}_\bullet(\mathcal{T}) \cong \text{HH}_\bullet(D_{dg}).$$

There is a diagram of $dg$ categories

$$\begin{array}{ccc} B_{dg} & \xleftarrow{i_2} & \mathcal{T} & \xrightarrow{i_1} & A_{dg} \end{array}$$
It induces a commutative diagram of triangles of Hochschild complex

\[
\begin{array}{cccc}
C(B_{dg}) & \xrightarrow{i_2} & C(T) & \xrightarrow{L} & C(A_{dg}) & \xrightarrow{=} & C(B_{dg})[1] \\
\downarrow{id} & & \downarrow{R+L} & & \downarrow{id} & & \\
C(B_{dg}) & \xrightarrow{} & C(B_{dg}) & \oplus & C(A_{dg}) & \xrightarrow{} & C(B_{dg})[1])
\end{array}
\]

It induces an isomorphism of Hochschild complex \(C(T) \cong C(A_{dg}) \oplus C(B_{dg})\). Thus

\[
HH_\ast(D_{dg}) \cong HH_\ast(T) \cong HH_\ast(A_{dg}) \oplus HH_\ast(B_{dg}).
\]

\[\square\]

**Proof of Theorem 4.7.** By HKR and the additive theory of Hochschild homology Theorem 7, \(H^0(X, \omega_X) \neq 0\). Hence if the nontrivial semi-orthogonal decomposition exists, then the canonical bundle is not trivial, and the base locus of the canonical bundle is a proper closed subset of \(X\).

We write \(\text{Per}_{dg}(X)\) as the natural \(dg\) enhancement of \(\text{Perf}(X)\). The objects are \(K\)-injective perfect complexes. Define \(\text{Per}_{Z,dg}(X)\) as the full \(dg\) subcategory of \(\text{Per}_{dg}(X)\) whose objects support in \(Z\). We write \(A_{dg}\) and \(B_{dg}\) as the natural \(dg\) enhancement corresponding to \(\text{Per}_{dg}(X)\).

According to Proposition 4.3 either all objects of \(A\) support in \(Z\), or all objects of \(B\) support in \(Z\). Suppose all objects of \(B\) support in \(Z\), then the original semi-orthogonal decomposition induces a semi-orthogonal decomposition

\[
\text{Perf}_Z(X) := \langle A_Z, B \rangle
\]

where \(A_Z\) is the full triangulated subcategory of \(A\) whose objects support in \(Z\). First, given any object \(E \in \text{Perf}_Z(X)\), there is a triangle with respect to the original semi-orthogonal decomposition

\[
E_1 \to E \to E_2 \to E_1[1].
\]

That is, \(E_2 \in A\) and \(E_1 \in B\). Furthermore, support of \(E\) and \(E_2\) are in \(Z\). Thus, using Lemma 3.10 \(E_1\) also supports in \(Z\). Hence \(E_1 \in A_Z\). Second, \(\text{Hom}(B, A_Z) = 0\) by the original semi-orthogonal decomposition.

According to Theorem 4.7, we have isomorphism of Hochschild homology

\[
HH_\ast(\text{Per}_{Z,dg}(X)) \cong HH_\ast(A_{Z,dg}) \oplus HH_\ast(B_{dg}).
\]

Now, with the same technique in proof of Theorem 4.1 the Hochschild homology \(HH_\ast(B)\) is isomorphic to Hochschild homology of a special \(dg\) enhancement of \(B\): Choosing a strong compact generator \(F\) of \(B\) (assume it is \(K\)-injective), define \(dg\) algebra \(\Lambda = \text{Hom}_{dg}(F, F)\). Then \(HH_\ast(\Lambda) \cong HH_\ast(B)\). We regard the \(dg\) algebra \(\Lambda\) as a \(dg\) category with one object \(\circ\). There is a natural morphism of \(dg\) categories

\[
\Lambda \to B_{dg} \quad \circ \mapsto F.
\]

It is a derived Morita morphism since this natural map induce natural equivalence \(D(\Lambda) \cong D(B_{dg}) \cong B\). Thus, \(HH_\ast(\Lambda) \cong HH_\ast(B_{dg})\). Finally, \(HH_{n}(B_{dg}) \cong HH_{-n}(B) \neq 0\) should be a
subspace of $\text{HH}_{-n}(\text{Perf}_{Z}(X)) = 0$, a contradiction. So, we prove that every objects of $A$ support in $Z$.

We prove that in this case, $\text{HH}_{-n}(A) = 0$. Otherwise, the same method shows that every objects of $B$ should support in $Z$, a contradiction. It is easy to see that we can exchange the role of $A$ and $B$.

At last, we prove $B$ is indecomposable if it is $n$ Calabi-Yau category. If not, let $B = \langle B_1, B_2 \rangle$. Clearly, $B_1$ and $B_2$ are both $n$-Calabi-Yau categories. There is a semi-orthogonal decomposition

$$\text{Perf}(X) = \langle A, B_1, B_2 \rangle$$

The same argument above shows that $k(x) \in B_2$ and $k(x) \in \langle A, B_1 \rangle$ for any closed points $x \in X \setminus Z$, a contradiction. □

**Example 4.8.** Let $Y$ be a Calabi–Yau variety of dimension $n$, and $X$ be the blow-up of one ponit of $Y$. According to Orlov blow-up formula, there is a semi-orthogonal decomposition,

$$\text{Perf}(X) = \langle \mathcal{O}_{E}(E), \text{Perf}(Y) \rangle.$$  

$E$ is the exceptional locus of the blow-up $f: X \to Y$. The base locus of $\omega_X$ is $E$, and clearly support of $\langle \mathcal{O}_{E}(E) \rangle \subseteq E$. Point sheaf $k(x) \in \text{Perf}(Y)$ for $x \in X \setminus E \cong Y \setminus \text{pt}$. Furthermore, $\text{HH}_{-n}(\langle \mathcal{O}_{E}(E) \rangle) \cong \text{HH}_{-n}(\text{D}^b(\text{pt})) = 0$.

**Remark 4.9.** From the above’s proof, the $-n$ Hochschild homology of $A$ vanishes. It coincides with our intuition that $A$ should be smaller. In addition, the $-n$ Hochschild homology of $A$ and $B$ can’t be nonzero at the same time. The situation $\text{Perf}(X) = \langle \text{D}^b(Y_1), \text{D}^b(Y_2), \cdots \rangle$ can’t happen that $\dim X = \dim Y_1 = \dim Y_2$ and both $Y_1, Y_2$ are varieties with canonical bundles having non zero global sections.

Actually, we can also only use Theorem 3.12 to prove Theorem 4.5 since the following theorem.

**Theorem 4.10.** Assume there is a non trivial semi-orthogonal decomposition, $\text{Perf}(X) = \langle A, B \rangle$. Denote $Z = \text{Bs} |_{\omega_X}$. It could happen that $Z$ have zero dimensional components. Deleting zero component of $Z$, denote it $Z'$, then the support of $A$ is in $Z'$ or the support of $B$ is in $Z'$.

**Proof.** According to the Proposition 4.3, the support of $A$ or the support of $B$ is in $Z$. Without loss of generality, assume the support of $A$ is in $Z$. We prove that it is actually in $Z'$. Suppose $\exists \ E \in A$ whose support contains some point components, then it is easy to show that $A_{Z \setminus Z}'$ is nontrivial. Then we have an orthogonal decomposition $A = \langle A_{Z \setminus Z}', A_{Z} \rangle$. Therefore we have a nontrivial semi-orthogonal decomposition, $\text{Perf}(X) = \langle A_{Z \setminus Z}', C \rangle$ where $C: = \langle A_{Z \setminus Z}', B \rangle$. However, $A_{Z \setminus Z}' \otimes \omega_X = A_{Z \setminus Z}'$, hence it is a nontrivial orthogonal decomposition, contradicts that $X$ is connected. □
4.3. **Applications to a Kuznetsov’s conjecture.** Kuznetsov problem [19]: Let \( X \) be a projective smooth variety, \( B \) be a \( n \)-Calabi–Yau category such that \( n > \dim X \). Then \( B \) can not be an admissible subcategory of \( \text{Perf}(X) \) since otherwise \( 0 = \HH_{-n}(X) \geq \HH_{-n}(B) \neq 0 \). Hence in order to find \( n \)-Calabi–Yau subcategory inside \( \text{Perf}(X) \), we must have \( n \leq \dim X \). It was conjectured that the case that \( n = \dim X \) should be a certain boundary condition, and that the semi-orthogonal decomposition only comes from Orlov blow-up formula.

**Conjecture 4.11.** [24] Suppose we have a proper fully faithful embedding \( \text{Perf}(Y) \hookrightarrow \text{Perf}(X) \), \( \dim X = \dim Y \), \( Y \) is a Calabi–Yau variety. Then, one can find a Calabi–Yau variety \( Y' \) such that \( X \) is a certain blow-up of \( Y' \). Furthermore, \( \text{Perf}(Y') \cong \text{Perf}(Y) \).

However, Kuznetsov’s conjecture is far from being true, see “Compact Hyper–Kähler Categories” [18, Section 5]. It was shown in the paper that there are infinite many geometric 4-folds containing 4-Calabi–Yau (connected) categories which are not derived category of a projective smooth variety. The problem still make sense when we firstly assume the Calabi–Yau category to be derived category of variety, that is, Kuznetsov problem can be refined as follows:

**Conjecture 4.12.** Suppose we have a proper fully faithful embedding \( \text{Perf}(Y) \hookrightarrow \text{Perf}(X) \), \( \dim X = \dim Y \), \( Y \) is a Calabi–Yau variety. Then, one can find a Calabi–Yau variety \( Y' \) such that \( X \) is a certain blow-up of \( Y' \). Furthermore, \( \text{Perf}(Y') \cong \text{Perf}(Y) \).

**Remark 4.13.** We can not expect that \( Y' \cong Y \), since there may be nonbirational Fourier-Mukai partners of some Calabi–Yau varieties, for example, the general \( K_3 \) surfaces.

**Theorem 4.14.** Let \( X \) be a projective smooth variety of dimension \( n \). Suppose \( \dim H^0(X, \omega_X) \geq 2 \), then any \( n \)-Calabi–Yau admissible subcategory of \( \text{Perf}(X) \) is not equivalent to a derived category of a smooth projective variety.

**Proof.** Suppose there is a \( n \)-Calabi–Yau subcategory \( B \subset \text{Perf}(X) \). According to Theorem 4.5, \( \dim \HH_{-n}(B) = \dim \HH_{-n}(\text{D}^b(X)) \geq 2 \). Thus \( B \) is not connected (An admissible subcategory is connected if \( \HH^0(\bullet) \cong k \)). Suppose \( B \) is geometric, that is \( B \cong \text{Perf}(Y) \) for some Calabi–Yau variety \( Y \). Since \( B \) is not connected, \( Y \) is not connected too, a contradiction. \( \square \)

**Conjecture 4.15.** Let \( B \) be an admissible \( m \)-CY subcategory of \( \text{D}^b(Y) \), where \( Y \) is a projective smooth variety, and \( m \geq 0 \). Then, \( B \) is indecomposable if and only if \( \HH^0(B) \cong k \).

**Remark 4.16.** If the Conjecture 4.15 is true, then Theorem 4.14 implies that there is no \( n \)-Calabi-Yau subcategory in \( \text{Perf}(X) \) (\( \dim X = n \)) if \( \dim H^0(X, \omega_X) \geq 2 \), which also satisfies Kuznetsov’s original intuition.

The following theorem shows some intuition about this A. Kuznetsov’s problem:

**Theorem 4.17.** Let \( X \) be a projective smooth variety. Suppose its derived category \( \text{Perf}(X) \) admits a nontrivial semi-orthogonal decomposition with component \( B \) being \( n \)-CY category. Here \( n \) is dimension of \( X \). Then, the base locus \( Z \) of the canonical bundle is a proper closed subset. Furthermore, any point sheaves \( k(x) \) with \( x \in X \setminus Z \) belongs to \( B \).
Proof. This follows from Theorem 4.5 since $\text{HH}_{-n}(B) \neq 0$ for Calabi-Yau category $B$. \qed

Since the good knowledge of the birational geometry of the surfaces, we expect that the Kuznetsov’s problem can be achieved for the cases of surfaces. Since the Theorem 4.14, we always assume the 2-Calabi-Yau category to be connected.

1. $k(X) = -\infty$. If $X$ is a rational surface, then by Hochschild homology techniques, there is no CY-2 category in $\text{Perf}(X)$. If $X$ is ruled surface, write $X = \mathbb{P}^1 \times C$ for some curves $C$. Then again by Hochschild homology theory, $\text{Perf}(X)$ has no admissible CY-2 category.

2. $k(X) = 0$, it is well known that the Enrique surfaces (and their blow-ups) or bi-elliptic surfaces (and their blow-ups) is of geometric genus 0, hence there is no admissible CY-2 in $D^b(X)$. According to classical classification of projective smooth surfaces, it remains to the cases: blow-up of abelian surfaces, $K_3$ surfaces, elliptic surfaces, and minimal surfaces of general type.

Finally, we propose an interesting strategy to refine this Kuznetsov’s problem. We need a lemma which is interesting in itself.

Lemma 4.18. Let $T$ be a triangulated category with a semi-orthogonal decomposition $T = \langle A, B \rangle$. Suppose there is a fully faithful embedding of triangulated categories $i: T_\infty \hookrightarrow T$ with right adjoint $R$, and $T_\infty$ admits Serre functor. Let $\{E_i[m]\}$ be a set of objects that is a spanning class of $T_\infty$. Assume $i$ maps this set of objects into $B$, then $i$ maps $T_\infty$ into $B$. Note that the statement is true if replacing $B$ with $A$, and right adjoint with left adjoint.

Proof. Write $\{E_i[m]\}_{i \in I}$ as a set of spanning class of $T_\infty$. Since the category $T_\infty$ has Serre functor, we can assume the spanning class as the left spanning class. Let $E \in T_\infty$, and any $a \in A$. By assumption, $\text{Hom}(i(E_i[m]), A) = 0$ for any integers $m$ and $j$, then $\text{Hom}(i(E_i[m]), a) \cong \text{Hom}(E_i[m], R(a)) = 0$. Since $\{E_i[m]\}_{i \in I}$ is a spanning class, we have $R(a) \cong 0$. But $\text{Hom}(i(E), a) \cong \text{Hom}(E, R(a)) \cong 0$, hence $i(E) \in B$. \qed

Let $X$ be the blow-up of $Y$ over points $\{y_i\}_i$. We write the blow-up morphism as $f: X \to Y$. Let $\{C_i\}$ be the exceptional divisors which are contracted to the points $y_i$. We get an embedding $Lf^*: \text{Perf}(Y) \hookrightarrow \text{Perf}(X)$. Define the distinguished objects $D_i$ attached to the contraction $f$ as $Lf^*k(y_i)$. Note that the distinguished object $D_i$ supports at the exceptional divisor $C_i$.

Theorem 4.19. Let $X$ be the blow-up of dimension $n$ Calabi-Yau variety $Y$. Assume there is an element $\phi$ of $\text{Aut}(\text{Perf}(X))$ which maps all distinguished objects $D_i$ to the one whose support is disjoint to all exceptional divisors (base locus of $K_X$). Then, let $B$ be any admissible CY-$n$ of $\text{Perf}(X)$, we have $D_i \in B$.

Proof. Suppose there is a CY-$n$ $B$ in $\text{Perf}(X)$. Consider the semi-orthogonal decomposition $\text{Perf}(X) = \langle B^-, B \rangle$ and the embedding $\text{Perf}(Y) \hookrightarrow \text{Perf}(X)$. After an auto-equivalence of the derived category $\text{Perf}(X)$, we get a new semi-orthogonal decomposition with a CY-$n$ component $B' \cong B$. We see that the embedding $Lf^*$ composing with the auto-equivalence $\phi$ maps the
distinguish objects to $B'$. The reason is as follows: $\phi D_i$ belongs to $B'$ or $B'\perp$. According to Theorem 4.5, any objects in $B'\perp$ must support in the exceptional divisors $\cup C_i$, hence we have $\phi D_i \in B'$. Thus, there exist $D'_i \in B'$ such that $\phi D_i \cong D'_i$, and then $D_i = \phi^{-1} D'_i$ which belongs to $B$.

Remark 4.20.

(1) Unfortunately, the usually automorphism of varieties, twisted by line bundles, and shifting do not satisfy our assumption in Theorem 4.19. It is interesting to the author that if there is a nontrivial auto-equivalence satisfying assumption in Theorem 4.19.

(2) The assumption can’t not happen for surfaces. This follows easily from Proposition 12.15 [6] which is originally proved by Kawamata: There is a correspondence $\Gamma$ which induces automorphism of $X$. The support of objects $\phi(D_i)$ under the auto-equivalence $\phi$ must contain exceptional curve $C_i$. It will still be interesting if the assumption could happen for higher dimensional varieties.

Theorem 4.21. Let $Y$ be a smooth projective Calabi-Yau variety of dimension $n$, $f : X \to Y$ is the blow-up of $Y$ over points $\{y_i\}$. Define the distinguish objects $D_i = Lf^* k(y_i)$. Let $B$ be an $n$ Calabi-Yau admissible subcategory of $\text{Perf}(X)$. If all $D_i \in B$, then $B = Lf^* \text{Perf}(Y)$.

Proof.

Since the skyscraper sheaves of all the closed points (and shifting) form a spanning class, according to Theorem 4.19 and Theorem 4.5, $Lf^* k(y_i) \in B$, and then by Lemma 4.18 we get an embedding $\text{Perf}(Y) \hookrightarrow B$. The embedding $\text{Perf}(Y) \hookrightarrow B$ must induce an orthogonal decomposition because $B$ is a Calabi-Yau category.

$B = \langle \text{Perf}(Y)^\perp, \text{Perf}(Y) \rangle$.

Applying additive theory of Hochschild cohomology for orthogonal decompositon, see [7] Proposition 5.5], we have $\text{HH}^0(B) \cong \text{HH}^0(\text{Perf}(Y)) \oplus \text{HH}^0(\text{Perf}(Y)^\perp)$. If we assume $\text{Perf}(Y)^\perp$ to be nontrivial, then $\text{HH}^0(\text{Perf}(Y)^\perp) \neq 0$, and obviously $\text{HH}^0(\text{Perf}(Y)) \neq 0$. However, by Theorem 4.5 we have $\text{HH}^0(B) = \text{HH}^0(Y) = k$, a contradiction. Thus, the embedding $\text{Perf}(Y) \hookrightarrow B$ is an equivalence.

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