Massless Particle Creation in Bianchi I Universes

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Abstract

The Particle Physics and Cosmological motivations of Bianchi I universes are illustrated using Quantum Field Theory in curved space-time. We show that in a non-isotropically expanding universe fermionic and scalar fields acquire mass-like terms which, in the former case, preserve chiral symmetry. We also prove that these do not arise in an isotropic and homogeneous universe and, more generally, any conformally flat manifold. The creation of massless particles in Bianchi I universes is presented as a simple explanation, requiring nothing more than a massless $U(1)$ gauge theory, for the primordial origins of extra-galactic magnetic fields.

I. INTRODUCTION

A. Motivation

In the absence of a consistent quantum theory of gravity, the influence of gravitation on scales far from the Planck mass but where quantum effects become important is described by Quantum Field Theory (QFT) in curved space-time. The gravitational field is classical and treated according to the principles of General Relativity while matter, gauge, and scalar fields are quantized and treated according to the principles of Quantum Field Theory. The quantized fields exist on a curved background space-time, which gives rise to a rich variety of phenomenology not present in flat space-time. Some examples are a modification of the Einstein field equations necessary for renormalizability [11], curvature induced asymptotic freedom, where the coupling constant of a non-abelian gauge theory goes to zero at high curvature [3], and the creation of particles by the background gravitational field for a non-interacting theory (see for example [1] and [2]).

Previous works in this field have focused on space-times with a large degree of symmetry such as the Schwarzschild geometry or the Friedmann-Lemaître-Robertson-Walker (FLRW) universe in Cosmology. In this paper we show that less restrictive geometries offer possible solutions to unsolved problems in both Particle Physics and Cosmology.

In Particle Physics and QFT in general, there is no known theoretical method for predicting an ab-initio value for the masses of the various fields. Also, since a conventional mass term for a spin-$\frac{1}{2}$ field breaks chiral symmetry and there is no experimental evidence to date for right-handed neutrinos, the nature of neutrino masses is an open question. In Cosmology, the origin of extra-galactic magnetic fields is an unsolved problem, although it is broadly accepted that they are primordial. Therefore, they would have to have been formed before the Electroweak Phase Transition, when all fermions and gauge bosons were massless. This represents a theoretical challenge as it was shown in [4] that no massless particles of non-zero spin are created in an FLRW universe and a $U(1)$ gauge theory coupled to massless spinors is conformally invariant if we ignore quantum anomalies. Efforts in addressing this problem have focused mainly on adding conformal invariance breaking interactions to a massless Quantum Electrodynamics (QED)-type theory. However, most of these new interactions are non-renormalizable and hence only viable as Effective Field Theories.

QFT in a less restrictive curved space-time provides a possible solution to both these problems. We will argue that for a non-conformally flat manifold, the spin connection part of the covariant derivative should be interpreted as chiral symmetry preserving mass term for the spinor and that the coupling of a scalar field to the Ricci Scalar should be regarded as a mass term for the former. Both of these are calculable from first principles if the metric is known. We then show that the creation of massless spin-$\frac{1}{2}$ particles is possible in a Bianchi I universe. This offers a much simpler mechanism, which requires nothing more than a QED-type theory, to explain the primordial origins of extra-galactic magnetic fields.

B. Particle Creation by the Background Gravitational Field

To see that a background gravitational field creates particles, consider a non-interacting field of arbitrary spin $\chi(x)$ propagating in curved space-time. The equation of motion is

$$\mathcal{F}\chi = 0$$

(1)

where $\mathcal{F}$ is an operator typically of the form $\nabla^\mu_n \pm m^n$, with $n$ a positive integer denoting the order of the covari-
ant derivative operator $\nabla_\mu$ and $m$ is the mass of the field.

In flat space-time Poincaré symmetry, and in particular invariance under time translation, allows us to write a set of positive and negative frequency plane wave solutions to (1)

$$f^\pm_p(x, t) = \exp \left[ \mp i (\omega t - p \cdot x) \right]$$

where $f^\pm$ denotes the positive and negative frequency solutions respectively.

Without loss of generality, we consider two positive frequency solutions $f_1^+$ and $f_2^+$. The scalar product

$$\left(f_1^+, f_2^+\right) = -i \int_{\partial\Omega} f_1^+ \partial_t f_2^+ - \partial_t \left(f_1^+\right) f_2^{+*}$$

is conserved. $\partial\Omega$ is a spatial volume element at a fixed time $t \ [12]$. If we choose solutions of the form

$$f_p^+(x, t) = (2\omega 2\pi)^{-\frac{n-1}{2}} \exp \left[-i (\omega t - p \cdot x)\right],$$

with $n$ the number of space-time dimensions under consideration, the scalar product [2] provides a set of normalized solutions [11]

$$\left(f^+_p, f^+_p\right) = \delta^{n-1}(p - p')$$

and a complete solution to the equation of motion is given by

$$\chi = \sum_{j,p} a_j f^+_jp + a^*_j f^-_jp$$

where space-time labels were suppressed for clarity. The label $j$ is generic for any quantum numbers characterising the system. After canonical quantization, $a_{jp}$ and its complex conjugate $a_{jp}^*$ are interpreted as annihilation and creation operators respectively for positive frequency particles of quantum number $j$ and momentum $p$. They define a Fock space and an empty vacuum state that is annihilated by $a_{jp}$

$$a_{jp} |0> = 0$$

In curved space-time the notion of positive and negative frequencies is well defined only for stationary manifolds, which possesses a time-like Killing vector. This is certainly not the case for the Bianchi I metric we will consider later or for a general space-time. Several workarounds have been proposed to deal with this complication [8]. In our case, we split the space-time into three causally connected regions. The in and out regions are the asymptotic limits for time $t \rightarrow \mp\infty$ respectively where the metric is stationary (not necessarily flat). In between the two is a region where the time-like Killing vector does not exist.

In the in region, let there be a set of plane wave solutions given by [3] and an empty vacuum $|0_{in}>$. In the out region we assume that the metric is stationary and possesses a vacuum $|0_{out}>$ for another set of solutions given by

$$\chi = \sum_{j,p} \tilde{a}_j \tilde{f}^+_jp + \tilde{a}^*_j \tilde{f}^-_jp$$

Since both sets of solutions are complete, we can write [11]

$$\tilde{f}^\pm_j = \sum_k \alpha_{jk} f^+_k,p + \beta_{jk} f^-_k,p$$

$$\tilde{a}_j = \sum_k \alpha_{jk}^* a_k - \beta_{jk}^* a_k$$

The second equation defines the Bogoliubov transformations for the creation and annihilation operators [7]. The expectation value of the number operator for the out vacuum is

$$\langle 0_{out}|\tilde{a}^*\tilde{a}|0_{out} > = \sum_k |eta_{jk}|^2$$

The out vacuum is not empty, which is interpreted as particle creation by the background gravitational field.

II. MASS FROM QFT IN CURVED SPACE-TIME

A. Tetrad Formalism of General Relativity

The prescription for incorporating gravity into a physical system is to replace all Lorentz tensors with objects that are covariant under general coordinate transformations. Partial derivatives are replaced by covariant derivatives and the Minkowski metric $\eta_{\mu\nu}$ is replaced by a space-time dependent metric $g_{\mu\nu}(x)$. Mathematically, this means that in General Relativity the Poincaré symmetry group of Special Relativity is extended to the $GL(\mathbb{R}, 4)$ group, referred in [5] as the Principle of General Covariance. This approach works for Lorentz vectors and tensors, but not for spinors as there are no representations of $GL(\mathbb{R}, 4)$ that behave like spinors under the Lorentz subgroup. The only way to introduce spinors in curved space-time is through the tetrad formulation of General Relativity, which we introduce briefly here and refer the reader to [8] for further details. In our notation, bars represent Lorentz indices, raised and lowered with $\eta_{\mu\nu}$, unbarred indices represent generally covariant indices, raised and lowered with $g_{\mu\nu}(x)$, and repeated indices of the same kind are summed over.

Invoking the equivalence principle, we erect, at each point $x'$ of a manifold $\mathcal{M}$, a set of locally inertial coordinates $\xi_{\mu}'$. For each specific point $x'$, the line element can
be written as
\[ g_{\mu\nu} (x') \, dx^\mu \, dx'^\nu = e^\hat{\rho}_\mu (x') \, e^\hat{\lambda}_\nu (x') \, \eta_{\hat{\rho}\hat{\lambda}} \, dx^\mu \, dx'^\nu \]
\[ e^\hat{\rho}_\mu (x') = \frac{\partial \xi^\hat{\rho}_\mu (x)}{\partial x^\mu} \bigg|_{x=x'} \]  
(5)

The \( e^\hat{\rho}_\mu (x') \) are called tetrads and can be raised and lowered as follows
\[ e^{\mu\hat{\rho}} (x') = \eta^{\mu\sigma} e^\sigma_\nu (x') \]  
(6)

Using tetrads, we can relate any vector field \( V^\mu (x') \) to the locally inertial coordinate system at \( x' \), \( \xi^\mu (x) \)
\[ V^\mu (x') \equiv e^\mu_\rho (x') \, V^\rho (x') \]  
(7)

where we have used the subscript \( I \) to denote the locally inertial frame.

The use of tetrads has the effect of replacing a single four-vector \( V^\mu (x') \) with four scalars \( V_I^\mu (x') \). With this technique, any vector or tensor can be expressed as a set of scalars. However, the choice of the local inertial frame defined by \( \xi^\mu (x) \) is not unique: invoking the Equivalence Principle once again, the laws of Special Relativity must be applicable irrespective of the local inertial frame chosen. In other words, the \( V_I^\mu (x') \) must transform covariantly under the Lorentz group
\[ \psi_n (x) \rightarrow D[\Lambda (x)]_{mn} \psi^m (x) \]  
(8)

where \( \psi_n (x) \) is an arbitrary field and the use of roman indices \( mn \) reflects the fact that these are not necessarily Lorentz indices. They could, for example, be the Dirac indices of spinors. \( D[\Lambda (x)] \) is a matrix representation of the Lorentz group.

The prescription for constructing an action in curved space-time is then simple. Starting from the Minkowski action, we use tetrads to write all vectors and tensors as scalars and replace the partial derivative, which transforms like a generally covariant vector, with an operator that transforms covariantly under local Lorentz transformations
\[ \nabla_\mu \psi_I (x) \rightarrow \Lambda^\mu_\nu \nabla_\nu \psi_I (x) \]  
(9)

From [8], this is given by
\[ \omega^{\hat{\alpha}\hat{\beta}} = e^{\mu\hat{\rho}} \Gamma_{\mu\nu}^\rho \, e^\alpha_\nu + e^{\mu\hat{\rho}} \partial_{\mu} e^{\hat{\rho} \hat{\lambda}} \]  
(10)

where and \( \omega^{\hat{\alpha}\hat{\beta}} \) are the spin connection and \( \Gamma_{\mu\nu}^\rho \) are the Christoffel symbols. \( \sigma^{\hat{\alpha}\hat{\beta}} \) are the anti-symmetric matrices that furnish a representation of the Lorentz group. The spin connection can also be written explicitly in terms of the tetrads as [8]:
\[ \omega^{\hat{\alpha}\hat{\beta}} = \frac{1}{2} \epsilon^{\rho\sigma} \epsilon^{\lambda\lambda} (C_{\alpha\beta\mu} - C_{\beta\alpha\mu} - C_{\mu\alpha\beta}) \]
\[ C_{\alpha\beta\mu} = e^\sigma_\alpha (\partial_\beta e^\mu_{\kappa} - \partial_\mu e^\kappa_{\beta}) \]  
(11)

B. Conformal Flatness and Particle Creation

It was shown in [4] that no massless particles of non-zero spin are created in an FLRW universe. For the latter space-time, in the absence of a mass term the free field equations of motion are conformally invariant and can be recast into their flat space-time form using a conformal transformation. The creation and annihilation operators are then space-time independent and the background gravitational field does not create any massless particles. Such a conformal transformation is possible because the FLRW metric is conformally flat. In this section we will generalise the result in [4] and prove that no massless particles of non-zero spin are created in any conformally flat geometry, with some comments about scalar fields at the end.

A conformal transformation is a local rescaling of the line element defined by
\[ g_{\mu\nu} (x) \, dx^\mu \, dx'^\nu \rightarrow \exp \left[ -2 \Omega (x) \right] g_{\mu\nu} (x) \, dx^\mu \, dx'^\nu \]  
(12)

with \( \Omega (x) \) a continuous, non-vanishing, finite, real function [8] [11].

A manifold \( M \) with a metric \( h_{\mu\nu} (x) \) is conformally flat if there exists a space-time transformation which allows the line element to be recast in the form
\[ h_{\mu\nu} (x) \, dx^\mu \, dx'^\nu = H^2 (x) \, \eta_{\mu\nu} dx^\mu \, dx'^\nu \]  
(13)

where \( H (x) \) is a space-time dependent function. Under the conformal transformation
\[ \Omega (x) = \log [H (x)] \]  
(14)

\[ h_{\mu\nu} (x) \, dx^\mu \, dx'^\nu \rightarrow \eta_{\mu\nu} dx^\mu \, dx'^\nu \]  

hence the name conformally flat.

The equation of motion for a free massless field \( \chi (x) \) is
\[ \nabla_\chi (x) = 0 \]  
(15)

where \( \nabla_\chi (x) \) represents the covariant derivative acting on the fields appropriately contracted using tetrads and with tensor indices suppressed. This equation is first order for spin-1 fields and second order for spin-0 and spin-1 fields. To see that for a conformally flat metric [15] can be recast into its flat space-time form by the conformal transformation [14] we consider the spin connection for a diagonal metric
\[ \omega^{\hat{\alpha}\hat{\beta}} = \frac{1}{2} \epsilon^{\rho\sigma} \epsilon^{\lambda\lambda} \left[ \partial_\lambda e_{\hat{\rho}} e_{\hat{\rho}} (e^{\rho}_{\hat{\rho}} + e^{\rho}_{\mu}) \delta_{\hat{\rho}\hat{\beta}} - \partial_\rho e_{\hat{\rho}} e_{\hat{\rho}} \left( e^{\hat{\alpha}}_{\hat{\alpha}} + e^{\hat{\mu}}_{\mu} \right) \delta_{\hat{\mu}\hat{\beta}} \right] \]  
(16)

where for an arbitrary index \( \hat{\kappa} = \kappa \) (the bar only serves to distinguish Lorentz indicies from generally covariant
indices). Under a conformal transformation
\begin{equation}
\tilde{\omega}^\lambda_\mu \rightarrow \frac{1}{2} e^{\tilde{\tilde{\mu}}_\rho} e^{\lambda_\sigma} \left[ \left( \partial_\lambda e_{\tilde{\tilde{\mu}}_\rho} - \left( \partial_\lambda \Omega \right) e_{\tilde{\tilde{\mu}}_\rho} \right) \left( e^{\tilde{\tilde{\mu}}_\rho} + e^{\tilde{\tilde{\mu}}_\rho} \right) \delta_{\tilde{\tilde{\mu}}_\rho} + \left( \partial_\rho e^\lambda_\mu \right) \left( e^{\tilde{\tilde{\mu}}_\rho} + e^{\tilde{\tilde{\mu}}_\rho} \right) \delta_{\tilde{\tilde{\mu}}_\rho} \right]
\end{equation}

For a conformally flat metric $e_{\tilde{\tilde{\mu}}_\rho} = \eta_{\tilde{\tilde{\mu}}_\rho} H (x)$. Substituting the latter and (14) into (17) we find that $\tilde{\omega}^\lambda_\mu$ is contained in the matrix representation of the $R$ possible to couple the Ricci Scalar, derivative is reduced to a partial derivative irrespective of a conformally flat metric to zero by (14), the covariant derivative is reduced to a partial derivative irrespective of the spin of the field.

In the tetrad formalism all the information about spin is contained in the matrix representation of the Lorentz group $\sigma^{\alpha\beta}$. By setting the spin connection for a conformally flat metric to zero by (14), the covariant derivative is reduced to a partial derivative irrespective of the spin of the field.

For scalar fields, as we will see in the next section, it is possible to couple the Ricci Scalar, $R (x)$, and the fields in renormalizable and gauge invariant units. The resulting term is quadratic in the fields and has been interpreted as an effective mass term $\frac{1}{2} L_{\phi}$ [11, 12]. The equation of motion for a scalar field in an arbitrary gravitational field with such a coupling is
\begin{equation}
\left[ g^{\mu\nu} (x) \partial_\mu \partial_\nu + m^2 + \xi R (x) \right] \phi (x) = 0
\end{equation}
where $\xi$ is a dimensionless factor. It was shown in [11] that for $m = 0$ (18) is conformally invariant when $\xi = \frac{n-2}{4(n-4)}$, where $n$ is the number of space-time dimensions under consideration. In such a conformally coupled case with $m = 0$ no particle creation occurs, irrespective of the gravitational field. In the next section we will take $\xi = 1$ and argue that this coupling should be regarded as an ab-initio mass term for the scalar field. However, there is no experimental evidence to date for such a coupling - even less so for the conformally coupled case. Since for $m = 0$ and $\xi = 0$ (18) is not conformally invariant, massless scalar fields can be created by a gravitational field.

C. Chiral Symmetry and Mass

Conformal flatness implies that in an idealized perfectly isotropic and homogeneous universe gravity does not significantly influence Particle Physics in the pre Electroweak Phase Transition era. In this section we motivate considerations for the effects of anisotropies and inhomogeneities by showing that a less restrictive background gravitational field possibly provides ab-initio mass terms for spinor and scalar fields.

First we consider a free spinor $\psi (x, t)$ of mass $m$. The action is
\begin{equation}
S = \int \sqrt{-g} d^4 x \left( i e^\rho_\mu \gamma^\rho \nabla_\mu \psi - m \right) \psi (x, t)
\end{equation}
where $g$ is the metric determinant and $\nabla_\mu$ is the covariant derivative for spinor fields, with $\sigma^{\rho\lambda}_\mu$ furnishing the spinor representation of the Lorentz group
\begin{equation}
\sigma^{\rho\lambda}_\mu = \frac{i}{2} [\gamma_\rho, \gamma_\lambda]_-
\end{equation}
The $4 \times 4$ gamma matrices satisfy the Clifford Algebra
\begin{equation}
[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu}
\end{equation}
Defining $\tilde{\gamma}^\mu \equiv e_\rho^\mu \gamma^\rho$, we can write
\begin{equation}
[\gamma^\mu, \tilde{\gamma}^\nu]_+ = 2 g^{\mu\nu}
\end{equation}
It is instructive to split (19) into the chiral fields $\psi_L (x, t)$ and $\psi_R (x, t)$ defined by
\begin{align}
\psi_L (x, t) &= \frac{1}{2} (1 - \gamma_5) \psi (x, t) \\
\psi_R (x, t) &= \frac{1}{2} (1 + \gamma_5) \psi (x, t)
\end{align}
where $\gamma_5 = \mp \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$. In this form
\begin{equation}
S_F = \int \sqrt{-g} d^4 x \left( \psi_L \gamma^\rho \partial_\mu \psi_L - \tilde{\psi}_R \gamma^\rho \partial_\mu \psi_R \\
+ \psi_L \omega_\mu^{\rho\lambda} \gamma^\rho \sigma_{\lambda\rho} \psi_L + \tilde{\psi}_R \omega_\mu^{\rho\lambda} \gamma^\rho \sigma_{\lambda\rho} \psi_R \\
+ m \left( \psi_L \psi_R + \tilde{\psi}_R \psi_L \right) \right)
\end{equation}
where the subscript $F$ stands for fermions.

The second line of (24) preserves chiral symmetry and is quadratic in the chiral fields. We argue that within the framework of QFT in curved space-time, the spin connection, which has natural units of mass and consists of derivatives of the metric $g_{\mu\nu} (x)$, should be interpreted as a chiral mass term. Fermions would then have two contributions to their mass: a gravitationally induced mass due to the spin connection that respects chiral symmetry and the usual chiral symmetry breaking mass term acquired by the spontaneous symmetry breaking of the scalar boson $n$.

It should be noted that this mechanism, which generates a mass through the kinetic term of a field in curved space-time, only works for fermions. For a free scalar field $\phi (x)$, the most general renormalizable action is
\begin{equation}
S_S = \int \sqrt{-g} d^4 x \left[ g^{\mu\nu} (x) \partial_\mu \partial_\nu \phi - m^2 \phi^2 - R (x) \phi^2 \right]
\end{equation}
where the subscript $S$ stands for scalar fields and we have suppressed the space-time dependence of $\phi (x)$ for clarity. This action clearly has no quadratic term in the field arising from the covariant derivative. The last term, which is a coupling between the curvature scalar and the field $\phi (x)$, can be considered as a potential ab-initio
mass term for the latter, with \( m = \sqrt{\mathcal{R}(x)} \). \( \mathcal{R}(x) \),

having canonical mass dimension 2, cannot be coupled to spinors, which have dimension \( \frac{3}{2} \), in a renormalizable and gauge invariant way.

For vector fields \( A_\mu (x) \), which have canonical mass dimension 1, we consider, without loss of generality, an abelian gauge theory. The free action is

\[
S_G = \int \sqrt{-g} dt^4 \left[ g^{\mu \nu} F_{\mu \nu} F_{\alpha \beta} \right]
\]

\[
F_{\mu \nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + (\Gamma^\rho_\nu_\mu - \Gamma^\rho_\mu_\nu) A_\rho
\]

\[
= \partial_\mu A_\nu - \partial_\nu A_\mu
\]

(26)

where the subscript \( G \) stands for gauge fields and third line of (26) is obtained from the second line because the manifold in General Relativity is torsion free. The vector field \( A_\mu (x) \) does not acquire a mass through its kinetic term in a background gravitational field. A term of the form \( \mathcal{R}(x) A_\mu A^\mu \), despite being renormalizable because it has canonical mass dimension 4, is forbidden as it breaks gauge invariance. Spontaneous symmetry breaking therefore remains the only mechanism to generate a mass term for gauge fields \([11]\), even in curved space-time.

III. PARTICLE CREATION IN BIANCHI I UNIVERSES

We now consider a Bianchi I universe, which is spatially homogeneous but not isotropic, and restrict ourselves to a diagonal metric for simplicity

\[
ds^2 = dt^2 - \sum_{j=1}^{3} r_j(t) dx_j^2,
\]

(27)

where \( r_j(t) \) \( (j = 1,2,3) \) are arbitrary functions of time assumed to be increasing.

From \([19]\), the action is

\[
S = \int \sqrt{-g} dt^4 \bar{\psi}(x,t) \left[ i\gamma_0 \partial_0 + i\gamma_j(t) \partial_j \right] \psi(x,t)
\]

(28)

\[
\bar{\psi}(x,t) [i\gamma_0 R(t) - m] \psi(x,t)
\]

where \( \gamma_j(t) \equiv \frac{\gamma_j(t)}{r_j(t)} \) and there is an implied sum over \( j \) and \( R(t) = \frac{1}{2} \sum_{j=1}^{3} \dot{r}_j(t) \) is the spin connection.

A. Canonical Quantization in Bianchi I Universes

The equation of motion obtained from (28) is

\[
\left[ i\gamma_0 \left( \frac{\partial}{\partial t} + R(t) \right) + i \sum_{j=1}^{3} r_j^{-1}(t) \gamma_j \frac{\partial}{\partial x_j} - m \right] \psi(x,t) = 0.
\]

(29)

We use the Dirac representation in which the \( \gamma \) matrices are given by

\[
\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}
\]

(30)

where \( j = 1,2,3 \). \( I_2 \) is the 2×2 identity matrix and the \( \sigma_j \)'s are the Pauli matrices.

Plane wave solutions to (28) are given by

\[
\psi(x,t) = \sum_{\alpha,s} \int \sqrt{-g} dt \, a_{(\alpha,s)}(p,t) \exp(-i\omega(p,t) x)
\]

(31)

where \( p \cdot x = \int_{t_0}^{t} dt' [\omega(p,t') - p \cdot x] \). The energy \( \omega(p,t) \equiv p^0 \) is defined by the relation

\[
p^2 = g_{\mu \nu}(x) p^\mu p^\nu = m^2
\]

(32)

We have condensed the notation for \( u^{(\alpha,s)}(p,t) \) and \( a_{(\alpha,s)}(p,t) \): \( \alpha = \pm 1 \) refers to a fermion and an anti-fermion respectively (commonly called \( u \) and \( v \) for spinors and \( a \) and \( b \) for the creation/annihilation operators in Minkowski QFT) and \( s = \pm 1 \) refers to spin up or spin down.

We make an ansatz for the spinors that reduces to the usual flat space-time form when taking \( r_j(t) \rightarrow 1 \forall j \) (see Appendix \( A \) for details). The orthogonality relations are

\[
\bar{u}_{(\alpha,s)}(p,t) u^{(\alpha',s')}(p,t) = \alpha \delta_{\alpha \alpha'} \delta_{ss'} \frac{R^2(t) + 2m(m + \omega(p,t))}{(\omega(p,t) + m)^2 + R^2(t)}
\]

(33)

We assume that we are given, at an initial time \( t_0 \), creation and annihilation operators

\[
a_{(\alpha,s)}(p,t_0) = A_{(\alpha,s)}(p)
\]

(34)

and impose the anti-commutation relation

\[
\left[ A_{(\alpha,s)}(p,t), A^\dagger_{(\alpha',s')}(p') \right]_+ = \delta_{pp'} \delta_{\alpha \alpha'} \delta_{ss'}
\]

(35)

For their time evolution, we generalize the ansatz made in \([8]\) and define the Bogoliubov relation \([8]\)

\[
a_{(\alpha,s)}(p,t) = D_{\alpha \alpha'}^{\beta s}(p,t) A_{(\beta,r)}(p)
\]

(36)

The \( D_{\alpha \alpha'}^{\beta s}(p,t) \) are elements of the Bogoliubov matrix that depend on the particle momentum and time. Their components are denoted by \( (\alpha s) \), where \( \alpha = \pm 1 \) denotes the rows/columns 1 and 2 and 3 and 4 respectively and \( s = \pm 1 \) denotes the rows/columns 1 and 3 and 2 and 4 respectively. Thus, \( D_{11}^{-1}(p,t) \) is the time-dependent coefficient of \( A_{(1,-1)}(p,t) \) for \( a_{(1,1)}(p,t) \). Relation (36) implies, for example, that the creation operator of a fermion of spin projection \( s \) at a time \( t \) is a linear combination of the creation operators of two fermions.
with spin projection $\pm$ and the annihilation operators of two anti-fermions with spin projections $\pm$ at time $t_0$.

We now proceed to obtain expressions for $D_{\alpha s}^{\beta r}(p, t)$. Substituting $36$ into $29$ and multiplying from the right by $u^{(\alpha', s')}_{(\alpha', s')} (p, t)$,

$$
\bar{u}_{(\alpha', s')} u_{(\alpha, s)} \frac{d D_{\alpha s}^{\beta r}}{dt} = D_{\alpha s}^{\beta r} \times 
\left[ -\bar{u}_{(\alpha', s')} \frac{d}{dt} u_{(\alpha, s)} + i u_{(\alpha', s')}^{\dagger} O_{\alpha s} u_{(\alpha, s)} \right]
$$

(37)

where $O_{\alpha s}(p, t) \equiv \alpha \omega(p, t) \gamma_0 + i R(t) \gamma_0 - i \gamma_3 (t) p^j - m$.

The term in brackets on the RHS of (37) is clearly not zero since the first term involves time derivatives of the spinor but the second one does not. Therefore, only $\alpha = \alpha'$ and $s = s'$ yield non-trivial solutions for $D_{\alpha s}^{\beta r} (p, t)$. An explicit calculation with spinors given in Appendix A shows that

$$
\alpha \delta_{\alpha \alpha'} \delta_{s s'} \left[ R^2(t) + 2m (m + \omega(p, t)) \right] \frac{d D_{\alpha s}^{\beta r}}{dt} (p, t) = D_{\alpha s}^{\beta r} (p, t) \left[ -f_\alpha (p, t) - ig_\alpha (p, t) \right]
$$

(38)

where

$$
g_\alpha (p, t) = R(t) (\omega(p, t) + i \alpha R(t) + m) (R(t) - i 2 \omega(p, t))
$$

$$
f_\alpha (p, t) = \alpha \left[ \sum_{j=1}^{3} \frac{\hat{r}_j (t) p_j^2}{j} - i \frac{sp_1 p_2}{r_1 (t) r_2 (t)} \left( \hat{r}_1 (t) - \hat{r}_2 (t) \right) \right] + \left( \omega^2 (p, t) - m^2 \right) \frac{W_{\alpha} (p, t)}{\omega (p, t) + m + i \alpha R(t)}
$$

$$
W_{\alpha} (p, t) = -\frac{1}{\omega (p, t)} \sum_{j=1}^{3} \frac{\hat{r}_j (t) p_j^2}{j} + i \alpha \hat{R}(t)
$$

An integral solution is given by

$$
D_{\alpha s}^{\beta r}(t_0, t) = D_{\alpha s}^{\beta r}(t_0, t_0) - \alpha \int_{t_0}^{t} dt' \left[ f_\alpha (p, t) \times D_{\alpha s}^{\beta r}(t_0, t') \right]
$$

(39)

$$
F_{\alpha} (p, t) \equiv \int_{t_0}^{t} dt' \left[ f_\alpha (p, t) \times D_{\alpha s}^{\beta r}(t_0, t') \right]
$$

where we have dropped momentum labels for clarity and used the notation $D_{\alpha s}^{\beta r}(t_0, t)$ to denote the time evolution of the creation and annihilation operators from an initial time $t_0$ to a final time $t$. (39) can be solved iteratively and put in time-ordered form to give

$$
D_{\alpha s}^{\beta r}(t_0, t) = \hat{T} \exp \left[ -\alpha \int_{t_0}^{t} dt' F_{\alpha} (t') \times D_{\alpha s}^{\beta r}(t_0, t) \right]
$$

$$
D_{\alpha s}^{\beta r}(t_0, t_0) = \delta_{\alpha \beta} \delta_{s r}
$$

(40)
to $\langle N_{\alpha s} (p, t) \rangle$ equally. Explicitly

$$\langle N_{\alpha s} (t) \rangle = 4 \exp \left[ 2 \int_{t_0}^{t} \sum_{j=1}^{3} \frac{\tilde{p}_j(t')}{R^2(t') + (\omega(t') + m)^2} \right] \frac{R(t')}{R^2(t') + 2m(\omega(t') + m)} \left( \tilde{R}(t') (\omega(t') - m^2) + R^2(t') - 2\omega(t') (\omega(t') + m) \right)$$

where $\tilde{p}_j(t) = \frac{p_j(t)}{v_{\alpha s}(t)}$ and momentum dependence labels were once again suppressed for clarity. Since $\langle N_{\alpha s} (t) \rangle$ is independent of both $\alpha$ and $s$, the number of particles of spin projection $s$ created is the same as the number of anti-particles created of the same spin projection.

IV. RESULTS & CONCLUSIONS

Gravitationally Induced Mass

The second line of [28] written in terms of the chiral fields is

$$iR(t) \psi^\dagger_L \psi_L + iR(t) \psi^\dagger_R \psi_R - m\psi^\dagger_L \psi_L - m\psi^\dagger_R \psi_R$$

where we dropped space-time labels for clarity and used the Weyl basis.

$R(t)$ is therefore the chiral symmetry preserving gravitationally induced mass while $m$ is the chiral symmetry breaking scalar boson mass. For a purely left or right handed fermion, $m = 0$ but non-trivial functions $R(t)$ require it to be massive. Massless fermions are therefore a special feature of flat or conformally flat space-times. This mechanism for generating chiral mass terms may be of interest in Neutrino Physics.

Massless Particles

In the FLRW limit

$$\frac{dD_{\alpha s}^{\beta r}(p, t)}{dt} = -m \delta_{\alpha \alpha'} \delta_{s s'} \frac{\dot{R}(t)}{R(t)} D_{\alpha s}^{\beta r}(p, t)$$

This is consistent with equations (31) - (33) of [28], although not exactly the same since we define our spinors differently. However, the main conclusion that massless particles cannot be created by an FLRW background gravitational field is clear from [35]: the Bogoliubov coefficients become time-independent in the massless limit. In contrast, the massless limit of [35] is

$$\alpha \delta_{\alpha \alpha'} \delta_{s s'} R^2(t) \frac{dD_{\alpha s}^{\beta r}}{dt} = D_{\alpha s}^{\beta r} [-f_{\alpha} (p, t) - ig_{\alpha} (p, t)] \bigg|_{m=0}$$

where

$$\lim_{m \to 0} g_{\alpha} (t) = R(t) (\omega(t) + i\alpha R(t)) (R(t) - i2\alpha \omega(t))$$

$$\lim_{m \to 0} f_{\alpha} (t) = \alpha \left[ \sum_{j=1}^{3} \frac{\dot{r}_j(t) p_j^2}{r_j(t)} - i \frac{s p_1 p_2}{r_1(t) r_2(t)} (\dot{r}_1(t) - \dot{r}_2(t)) \right] + \omega^2(t) \frac{W_{\alpha} (t)}{\omega(t) + i\alpha R(t)}$$

From [15], in the massless limit the Bogoliubov coefficients are still time-dependent; massless spin-$\frac{1}{2}$ particles can be created in Bianchi I universes.

Anisotropies make it possible for an expanding universe to create massless fermions, anti-fermions, and, from the discussion in [14] and [15] gauge bosons such as the photon. This represents a promising approach in primordial magnetogenesis that requires no interactions beyond the renormalizable ones in QFT. We believe that a QFT in a Bianchi I background gravitational field, which for generality would consist of interacting fermionic, scalar, and gauge fields locally invariant under a gauge group that contains a $U(1)$ sub-group (not necessarily that of the Standard Model of Particle Physics) offers the simplest explanation for the origins of extra-galactic magnetic fields in the universe. Inflation would certainly be useful but is not a requirement. For a rigorous treatment, the effect of the conformal anomaly due to the non-invariance of the path integral measure under conformal transformations should be accounted for by using an effective lagrangian. This is interesting subject for future investigations.

Appendix A: Bianchi I Spinors

In this appendix we give the explicit form of the spinors used in this paper. They were constructed such that they reduce to their flat space-time form in the limit $r_j(t) \to 1 \forall j$. This is achieved with the following substitutions: $p_j \to \frac{p_j}{r_j(t)}$ and $\omega(p, t) \to \omega(p, t) + iR(t) \equiv \tilde{\omega}(p, t)$ for fermions and $\tilde{\omega}(p, t) = \omega(p, t) - iR(t)$ for anti-fermions.

Choosing eigenvectors of the $\sigma_3$ matrix, $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as basis we obtain

$$\begin{pmatrix} \tilde{\psi}_L(p, t) \\ \tilde{\psi}_R(p, t) \end{pmatrix}$$
\[
\begin{align*}
    u_{(1,1)}(p,t) &= \begin{pmatrix}
        1 & 0 \\
        0 & \frac{1}{\omega(p,t)+m} p_3 \\
        \frac{1}{\omega(p,t)+m} p_1 r_1(t) + \frac{1}{\omega(p,t)+m} p_2 r_2(t) \\
    \end{pmatrix} \\

    u_{(1,-1)}(p,t) &= \begin{pmatrix}
        0 & 1 \\
        1 & \frac{1}{\omega(p,t)+m} p_3 \\
        \frac{1}{\omega(p,t)+m} p_1 r_1(t) - \frac{1}{\omega(p,t)+m} p_2 r_2(t) \\
    \end{pmatrix} \\

    u_{(-1,1)}(p,t) &= \begin{pmatrix}
        \frac{1}{\omega^*(p,t)+m} p_3 \\
        \frac{1}{\omega^*(p,t)+m} p_1 r_1(t) + \frac{1}{\omega^*(p,t)+m} p_2 r_2(t) \\
        \frac{1}{\omega^*(p,t)+m} p_3 r_3(t) \\
    \end{pmatrix} \\

    u_{(-1,-1)}(p,t) &= \begin{pmatrix}
        \frac{1}{\omega^*(p,t)+m} p_1 r_1(t) - \frac{1}{\omega^*(p,t)+m} p_2 r_2(t) \\
        \frac{1}{\omega^*(p,t)+m} p_3 r_3(t) \\
        0 \\
    \end{pmatrix}
\end{align*}
\]

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