Bivariate Extension of the $r$-Dowling Polynomials and the Generalized Spivey’s Formula

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Abstract

In this paper, we extend the $r$-Dowling polynomials to their bivariate forms. Several properties that generalize those of the bivariate Bell and $r$-Bell polynomials are established. Finally, we obtain two forms of generalized Spivey’s formula.

1 Introduction

The Bell numbers $B_n$ are defined by the sum

$$B_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\},$$

(1)

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ denote the Stirling numbers of the second kind, and are known to satisfy the recurrence relation given by

$$B_{n+1} = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) B_k.$$  

(2)

The numbers $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ count the number of ways to partition a set $X$ of $n$ elements into $k$ non-empty subsets. With this, it is obvious that $B_n$ count the total number of partitions of the set $X$. Using the same combinatorial interpretation, Spivey [22] obtained a generalized recurrence for $B_n$ which unifies (1) and (2), viz.

$$B_{\ell+n} = \sum_{k=0}^{\ell} \sum_{i=0}^{n} k^{n-i} \left( \begin{array}{c} n \\ i \end{array} \right) \left\{ \begin{array}{c} \ell \\ k \end{array} \right\} B_i.$$  

(3)

The Bell polynomials, denoted by $B_n(x)$, are defined by

$$B_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k.$$  

(4)
Gould and Quaintance [11] established the polynomial version of (3) as follows

\[ B_{\ell+n}(x) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} k^{n-i} \binom{n}{i} \binom{\ell}{k} B_i(x)x^i \]  

(5)

by means of generating functions. The same identity was also obtained by Belbachir and Mihoubi [3, Theorem 1] using a method that involve decomposition of the \( B_n(x) \) into a certain polynomial basis and by Boyadzhiev [5, Proposition 3.2] using the Mellin derivatives.

Recently, Zheng and Li [23] defined the bivariate Bell polynomials by

\[ B_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} (x)_k y^k, \]

(6)

where \( (x)_k = x(x-1) \cdots (x-k+1) \), \( (x)_0 = 1 \), with the following exponential generating function [23, Theorem 1]:

\[ \sum_{n=0}^{\infty} B_n(x, y) \frac{t^n}{n!} = \left[ 1 + y(e^t - 1) \right]^x. \]

(7)

With this notion, they were able to obtain the bivariate extension of Spivey’s formula [23, Theorem 2] given by

\[ B_{\ell+n}(x, y) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} k^{n-i} \binom{n}{i} \binom{\ell}{k} B_i(x-k, y)(x)_k y^k. \]

(8)

Equation (5) can be recovered from this formula by replacing \( y \) with \( y/x \) and taking the limit as \( x \to \infty \). The \( r \)-Stirling numbers of the second kind, denoted by \( \{n\}_k \), are defined by Broder [6] as the number of partitions of the \( n \)-element set \( X \) into \( k \) non empty disjoint subsets such that the elements 1, 2, \ldots, \( r \) are in distinct subsets. These numbers are known to satisfy the horizontal generating function [6, Theorem 22]

\[ (t + r)^n = \sum_{k=0}^{n} \binom{n + r}{k + r} \binom{\ell}{k} (t)_k \]

(9)

and the exponential generating function [6, Theorem 16]

\[ \sum_{n=0}^{\infty} \binom{n + r}{k + r} \frac{t^n}{n!} = \frac{1}{k!} e^{rt}(e^t - 1)^k. \]

(10)

By defining the bivariate \( r \)-Bell polynomials by

\[ B_{n,r}(x, y) = \sum_{k=0}^{n} \binom{n + r}{k + r} (x)_k y^k, \]

(11)
Zheng and Li [23] were also able to obtain the following generalizations of (8):

\[
B_{\ell+n,r}(x, y) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} k^{n-i} \binom{n}{i} \left\{ \frac{\ell + r}{k + r} \right\} B_{i,r}(x - k, y)(x)k^i y^k
\]  

(12)

and

\[
B_{\ell+n,r}(x, y) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} (k + r)^{n-i} \binom{n}{i} \left\{ \frac{\ell + r}{k + r} \right\} B_{i}(x - k, y)(x)k^i y^k.
\]  

(13)

Replacing \( y \) with \( 1/x \) and taking the limit as \( x \to \infty \) in these formulas give [23, Corollaries 9 and 10]

\[
B_{\ell+n,r} = \sum_{k=0}^{\ell} \sum_{i=0}^{n} k^{n-i} \binom{n}{i} \left\{ \frac{\ell + r}{k + r} \right\} B_{i,r}
\]  

(14)

and

\[
B_{\ell+n,r} = \sum_{k=0}^{\ell} \sum_{i=0}^{n} (k + r)^{n-i} \binom{n}{i} \left\{ \frac{\ell + r}{k + r} \right\} B_{i}.
\]  

(15)

Equation (15) is a generalization of (3) proved by Mező [20, Theorem 2] using the combinatorial interpretation of \( \binom{n}{k} \) for the \( r \)-Bell numbers [19, Equation 2]

\[
B_{n,r} = \sum_{k=0}^{n} \binom{n + r}{k + r}.
\]  

(16)

Notice that we used the notation \( \frac{\ell + r}{k + r} \) in the above equations instead of just \( \frac{\ell}{k} \) for consistency. On the other hand, equation (14) and its polynomial version (12) appear in the paper of Mangontarum and Dibagulun [15, Corollary 3] as particular case of the formula [15, Theorem 2]

\[
D_{m,r}(\ell + n; x) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} (mk)^{n-i} W_{m,r}(n, k) \binom{n}{i} D_{m,r}(i; x)x^k,
\]  

(17)

where \( D_{m,r}(\ell + n; x) \) denote the \( r \)-Dowling polynomials [7] defined by

\[
D_{m,r}(n; x) = \sum_{k=0}^{n} W_{m,r}(n, k)x^k
\]  

(18)

and \( W_{m,r}(n, k) \) denote the \( r \)-Whitney numbers of the second kind [18]. Equation (17) was proved using the classical operators \( X \) and \( D \) satisfying the commutation relation

\[
[D, x] := DX - XD = 1.
\]

Inspecting equations (14) and (15), we see that generalizing Spivey’s formula yields two forms. In the first form, as seen in the right-hand side of (14), the \( r \)-Bell numbers \( B_{\ell+n,r} \) are expressed recursively in terms of the \( r \)-Bell numbers \( B_{i,r} \). In the second form, as seen in (15), the right-hand side involves the usual Bell numbers \( B_{i} \) instead of \( B_{i,r} \). We also notice the presence of \( (k + r)^{n-i} \) instead of \( k^{n-i} \).

In this paper, we will extend the \( r \)-Dowling polynomials to the bivariate case and investigate generalizations of Spivey’s formula that are analogous to the two forms mentioned above.
2 Bivariate $r$-Dowling polynomials

The $r$-Whitney numbers of the second kind are defined as coefficients in the expansion of the horizontal generating function \[ (mt + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n, k)(t)_k \tag{19} \]
and have the exponential generating function
\[
\sum_{n=k}^{\infty} W_{m,r}(n, k) \frac{t^n}{n!} = \frac{e^{rt} - 1}{m} \left( \frac{e^{mt} - 1}{m} \right)^k \tag{20}
\]
and the explicit formula
\[
W_{m,r}(n, k) = \frac{1}{m^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (mj + r)^n. \tag{21}
\]

Apparently, other mathematicians worked on numbers which are equivalent to $W_{m,r}(n, k)$. More precisely, the $(r, \beta)$-Stirling numbers \[ \langle n \rangle_{k, r, \beta} \] defined by \[
t^n = \sum_{k=0}^{n} \binom{\frac{t-r}{\beta}}{k} \beta^k k! \langle n \rangle_{k, r, \beta}, \]
the Ruciński-Voigt numbers \[ S^n_k(a) \] defined by \[
t^n = \sum_{k=0}^{n} S^n_k(a) P^a_k(x), \]
where $a = (a, a + r, a + 2r, a + 3r, \ldots)$ and $P^a_k(x) = \prod_{i=0}^{k-1} (t - a + ir)$, and the noncentral Whitney numbers of the second kind \[ \widetilde{W}^n_m(a, k) \] defined by \[
\langle \frac{n}{k} \rangle_{r, \beta} = W_{r, x}(n, k), S^n_k(a) = W_{r, a}(n, k), \widetilde{W}^n_m(a, k) = W_{m, a}(n, k). \]

Furthermore, aside from the classical Stirling numbers of the second kind which are given by $W_{1,0}(n, k) = \{n\}_k$, the numbers considered by previous authors in \[ [6, 13, 12, 14, 17] \] can also be obtained from $W_{m,r}(n, k)$ by assigning suitable values to the parameters $m$ and $r$.

Now, looking at the defining relations in equations \[ [6] \text{ and } [11] \], it is natural to define the bivariate $r$-Dowling polynomials by \[
D_{m,r}(n; x, y) = \sum_{k=0}^{n} W_{m,r}(n, k)(x)_k y^k. \tag{22}
\]

In the following theorems, we will present some combinatorial properties of $D_{m,r}(n; x, y)$:
Theorem 1. The bivariate $r$-Dowling polynomials satisfy the following exponential generating function:
\[
\sum_{n=0}^{\infty} D_{m,r}(n; x, y) \frac{t^n}{n!} = e^{rt} \left[ 1 + \frac{y(e^{mt} - 1)}{m} \right]^x .
\] (23)

Proof. Making use of the exponential generating function in (20) and the binomial theorem, we get
\[
\sum_{n=0}^{\infty} D_{m,r}(n; x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} W_{m,r}(n, k) x^k y^k \right] \frac{t^n}{n!}
= \sum_{k=0}^{\infty} (x)_{k} y^k \frac{e^{rt} (e^{mt} - 1)}{k!}
= e^{rt} \sum_{k=0}^{x} \left( \begin{array}{c} x \\ k \end{array} \right) \left[ \frac{y(e^{mt} - 1)}{m} \right]^k
= e^{rt} \left[ 1 + \frac{y(e^{mt} - 1)}{m} \right]^x
\]

as desired. \qed

The results in (7) and in [23, Theorem 3] are special cases of this theorem, i.e. when $m = 1$ and $r = 1$, and $m = 1$, respectively.

Theorem 2. The bivariate $r$-Dowling polynomials satisfy the following explicit formula:
\[
D_{m,r}(n; x, y) = \sum_{i=0}^{x} \binom{x}{i} (mi + r)^n \left( \frac{y}{m} \right)^i \left( 1 - \frac{y}{m} \right)^{x-i} .
\] (24)

Proof. By applying the explicit formula in (21),
\[
D_{m,r}(n; x, y) = \sum_{k=0}^{n} \left[ \frac{1}{m^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (m(k-j) + r)^n \right] x^k y^k
= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{(-1)^j (m(k-j) + r)^n (x)_{k} y^k}{m^k j! (k-j)!}
\]
Letting $i = k - j$ and since $(x)_{i+j} = (x)_i (x-i)_j$,
\[
D_{m,r}(n; x, y) = \sum_{i=0}^{\infty} \frac{(mi + r)^n (x)_i}{i!} \left( \frac{y}{m} \right)^i \sum_{j=0}^{\infty} \frac{(-y)^j (x-i)_j}{m^j j!}
= \sum_{i=0}^{\infty} \binom{x}{i} \left( \frac{y}{m} \right)^i (mi + r)^n \sum_{j=0}^{\infty} \binom{x-i}{j} \left( -\frac{y}{m} \right)^j
\]
which simplifies into (24). \qed
Similar formulas for the bivariate Bell and \( r \)-Bell polynomials can be obtained directly from this theorem.

**Corollary 3.** The bivariate Bell and \( r \)-Bell polynomials satisfy the following explicit formulas:

\[
D_{1,r}(n; x, y) := B_{n,r}(x, y) = \sum_{i=0}^{x} \binom{x}{i} (i + r)^n y^i (1 - y)^{x-i} \tag{25}
\]

\[
D_{1,0}(n; x, y) := B_n(x, y) = \sum_{i=0}^{x} \binom{x}{i} i^n y^i (1 - y)^{x-i}. \tag{26}
\]

Before proceeding, we first cite the binomial inversion formula given by

\[
f_n = \sum_{j=0}^{n} \binom{n}{j} g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f_j.
\]

This identity will be used in the proof of the next theorem.

**Theorem 4.** The bivariate \( r \)-Dowling polynomials satisfy the following recurrence relations:

\[
D_{m,r+1}(n; x, y) = \sum_{j=0}^{n} \binom{n}{j} D_{m,r}(j; x, y) \tag{27}
\]

\[
D_{m,r}(n; x, y) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} D_{m,r+1}(j; x, y). \tag{28}
\]

**Proof.** From [7, Corollary 3.5], the \( r \)-Whitney numbers of the second kind satisfy the vertical recurrence relation

\[
W_{m,r+1}(n, k) = \sum_{j=k}^{n} \binom{n}{j} W_{m,r}(j, k).
\]

Multiplying both sides by \((x)_k y^k\) and summing over \(k\) yields

\[
\sum_{k=0}^{n} W_{m,r+1}(n, k)(x)_k y^k = \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} W_{m,r}(j, k)(x)_k y^k.
\]

Thus, by (22), we get (27). Moreover, with \(f_n = D_{m,r+1}(n; x, y)\) and \(g_j = D_{m,r}(j; x, y)\), (28) is obtained by using the binomial inversion formula. This completes the proof. \(\square\)

The next corollary is obvious.

**Corollary 5.** The bivariate \( r \)-Bell polynomials satisfy the following recurrence relations:

\[
B_{n,r+1}(x, y) = \sum_{j=0}^{n} \binom{n}{j} B_{j,r}(x, y) \tag{29}
\]

\[
B_{n,r}(x, y) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{j,r+1}(x, y). \tag{30}
\]
Mezó [19, Theorem 3.2] established the ordinary generating function of the $r$-Bell polynomials as

$$
\sum_{n=0}^{\infty} B_{n,r}(x)t^n = \frac{-1}{rt-1} \cdot \frac{1}{e^x} \cdot {}_1F_1 \left( \frac{rt-1}{rt+t-1} \middle| x \right),
$$

(31)

where

$$
_pF_q \left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \middle| t \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k} \frac{t^k}{k!},
$$

(32)

is the hypergeometric function. This formula was then generalized in the papers of Corcino and Corcino [10, Theorem 4.1] and Mangontarum et al. [16, Theorem 39]. Since the generating function in [7, pp. 2339] can be written as

$$
\sum_{n=0}^{\infty} W_{m,r}(n,k)t^n = \frac{1}{m^k(1-rt)} \cdot \frac{(-1)^k}{\langle (m+r)t-1 \rangle_k},
$$

then

$$
\sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} W_{m,r}(n,k)t^n \right) (x)ky^k = \frac{1}{1-rt} \sum_{k=0}^{\infty} \frac{(-x)_k\langle 1 \rangle_k}{\langle (m+r)t-1 \rangle_k} \frac{y^k}{k!}.
$$

By (32),

$$
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} W_{m,r}(n,k)(x)ky^k \right) t^n = \frac{1}{1-rt} \cdot {}_2F_1 \left( \begin{array}{c} -x, 1 \\ \frac{y}{m} \end{array} \middle| \frac{y}{m} \right)
$$

and the next theorem follows by applying the formula [11, pp. 559]

$$
(1-t)^{-b} {}_2F_1 \left( \begin{array}{c} b, c-a \\ \frac{t}{t-1} \end{array} \middle| \frac{t}{t-1} \right) = {}_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| t \right)
$$

with $c = \frac{(m+r)t-1}{mt}$, $a = \frac{rt-1}{mt}$ and $b = -x$.

**Theorem 6.** The bivariate $r$-Dowling polynomials have the following ordinary generating function:

$$
\sum_{n=0}^{\infty} D_{m,r}(n;x,y)t^n = \frac{1}{1-rt} \left( \frac{m-y}{m} \right)^x {}_2F_1 \left( \begin{array}{c} \frac{rt-1}{mt}, -x \\ \frac{y}{y-m} \end{array} \middle| \frac{y}{y-m} \right).
$$

(33)

This yields similar generating functions for $B_n(x,y)$ and $B_{n,r}(x,y)$.

**Corollary 7.** The bivariate Bell and $r$-bell polynomials have the following ordinary generating functions:

$$
\sum_{n=0}^{\infty} B_{n,r}(x,y)t^n = \frac{(1-y)^x}{1-rt} {}_2F_1 \left( \begin{array}{c} \frac{rt-1}{1+r}, -x \\ \frac{y}{y-1} \end{array} \middle| \frac{y}{y-1} \right)
$$

(34)

$$
\sum_{n=0}^{\infty} B_n(x,y)t^n = (1-y)^x {}_2F_1 \left( \begin{array}{c} \frac{1}{t-1}, -x \\ \frac{y}{y-1} \end{array} \middle| \frac{y}{y-1} \right)
$$

(35)
Let \( v_k, k = 0, 1, 2, \ldots \), be a sequence of real numbers. \( v_k \) is convex \([8, pp. 268]\) on an interval \([a, b]\), where \([a, b]\) contains at least three consecutive integers, if

\[
v_k \leq \frac{1}{2} (v_{k-1} + v_{k+1}), \quad k \in [a + 1, b - 1].
\]

This is called convexity property. Corcino and Corcino \([10, Theorem 2.2]\) showed that Hsu and Shiue’s \([12]\) generalized exponential polynomials obey the convexity property. Special cases can also be seen in \([16, Theorem 42]\) and \([17, Theorem 9]\).

**Theorem 8.** The bivariate \( r \)-Dowling polynomials satisfy the convexity property.

**Proof.** Let \( m + r \geq 0 \) so that \((m + r)^2 \geq 0\) and

\[
0 \leq 1 - 2(m + r) + (m + r)^2.
\]

Rewrite this as

\[
m + r \leq \frac{1}{2} [1 + (m + r)^2]
\]

and multiply \((m + r)^n\) to both sides to get

\[
(m + r)^{n+1} \leq \frac{1}{2} [(m + r)^n + (m + r)^{n+2}].
\]

Multiplying both sides of this inequality by \( \binom{x}{i} \left( \frac{y}{m} \right)^i (1 - \frac{y}{m})^{x-i} \), summing over \( i \) and using \([24]\) gives

\[
D_{m,r}(n + 1; x, y) \leq \frac{1}{2} [D_{m,r}(n; x, y) + D_{m,r}(n + 2; x, y)]
\]

which is the desired result. \( \square \)

**Remark 9.** By assigning suitable values to \( m \) and \( r \), it can be shown that convexity property is preserved for the cases of both the bivariate Bell and \( r \)-Bell polynomials.

### 3 Generalized Spivey’s formula

Let \( f(x) \) be the exponential generating function of the sequence \( \{A_n\} \) given by

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} A_n = f(x).
\]

The exponential generating function of the sequence \( \{A_{i+j}\} \) is given by

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i! j!} A_{i+j} = f(x + y).
\]

Zheng and Li \([23, Equations 7 and 8]\) used this identity in the derivation of their main results. Adopting the same method they employed in their paper, we present the following theorems:
Theorem 10 (Generalized Spivey’s formula, first form). The following formulas hold:

\[ D_{m,r}(\ell + n; x, y) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} (mk)^{n-i} \binom{n}{i} W_{m,r}(\ell, k) D_{m,r}(i; x - k, y)(x) ky^k \]  \hspace{1cm} (37)

\[ D_{m,r}(\ell + n) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} (mk)^{n-i} \binom{n}{i} W_{m,r}(\ell, k) D_{m,r}(i). \]  \hspace{1cm} (38)

Proof. According equations (36) and (23), we may write

\[ \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} D_{m,r}(\ell + n; x, y) \frac{y^\ell v^n}{\ell! n!} = e^{r(u+v)} \left[ 1 + \frac{y(e^{m(u+v)} - 1)}{m} \right]^x \]  \hspace{1cm} (39)

In the right-hand side,

\[ \left[ 1 + \frac{y(e^{m(u+v)} - 1)}{m} \right]^x = \left[ 1 + \frac{y(e^{mv} - 1)}{m} + ye^{mv}(e^{mu} - 1) \right]^x. \]

Hence, by the binomial theorem,

\[ e^{r(u+v)} \left[ 1 + \frac{y(e^{m(u+v)} - 1)}{m} \right]^x = e^{r(u+v)} \sum_{k=0}^{\infty} \binom{x}{k} \left[ 1 + \frac{y(e^{mv} - 1)}{m} \right]^{x-k} \left[ ye^{mv}(e^{mu} - 1) \right]^k. \]

Again, we apply (23) to get

\[ e^{r(u+v)} \left[ 1 + \frac{y(e^{m(u+v)} - 1)}{m} \right]^x = \sum_{k=0}^{\infty} \binom{x}{k} y^k \frac{e^{mu}(e^{mu} - 1)^k}{k! m^k} \sum_{i=0}^{\infty} D_{m,r}(i; x - k, y) \frac{v^i}{i!} \sum_{j=0}^{\infty} (mv)^j \frac{1}{j!}. \]

From (20),

\[ e^{r(u+v)} \left[ 1 + \frac{y(e^{m(u+v)} - 1)}{m} \right]^x = \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{x}{k} y^k W_{m,r}(n, k) D_{m,r}(i; x - k, y) \frac{u^\ell v^{i+j}(mk)^j}{\ell! i! j!}. \]

Reindexing the sums with \( i + j = n \), and after a few simplifications

\[ e^{r(u+v)} \left[ 1 + \frac{y(e^{m(u+v)} - 1)}{m} \right]^x = \sum_{\ell=0}^{\infty} \left\{ \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\ell} \sum_{i=0}^{n} \binom{x}{k} y^k W_{m,r}(n, k) \right. \right. \left. \right. \times D_{m,r}(i; x - k, y)(mk)^{n-i} \binom{n}{i} \right\} \frac{v^n}{n!} u^\ell \frac{1}{\ell!}. \]

We arrive at the desired result in (37) by combining the last equation with (39) and comparing the coefficients of \( \frac{v^n}{n!} \cdot \frac{u^\ell}{\ell!} \). For (38), we simply replace \( y \) with \( 1/x \) and then take the limit as \( x \to \infty \). \( \square \)
Now, if we make use of the exponential generating function of $B_n(x, y)$ in (7) instead on (23), then after applying (20), (39) becomes
\[
e^r(u+v) \left[ 1 + \frac{y(e^m(u+v) - 1)}{m} \right]^x = \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (x)_{k} y^k W_{m,r}(n, k) \times B_i \left( x - k, \frac{y}{m} \right) \frac{u^\ell m^j v^i (mk + r)^j}{\ell! i! j!}.
\]
Therefore, we can directly deduce the following from the previous theorem:

**Theorem 11** (Generalized Spivey’s formula, second form). The following formulas hold:

\[
D_{m,r}(\ell + n; x, y) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} (mk + r)^{n-i} \binom{n}{i} W_{m,r}(\ell, k) B_i \left( x - k, \frac{y}{m} \right) (x)_{k} y^k \quad (40)
\]

\[
D_{m,r}(\ell + n) = \sum_{k=0}^{\ell} \sum_{i=0}^{n} (mk + r)^{n-i} \binom{n}{i} W_{m,r}(\ell, k) B_i \left( \frac{1}{m} \right). \quad (41)
\]

### 4 Conclusion

It is easy to see that when $m = 1$, we recover from equations (37) and (40) Zheng and Li’s [23] identities in (12) and (13), respectively. On the other hand, equations (38) and (41) are both generalizations of Spivey’s formula since the two equations reduce to (3) when $m = 1$ and $r = 0$. Finally, observe that when $n = 0$ in (38), we get

\[
D_{m,r}(\ell; x, y) = \sum_{k=0}^{\ell} W_{m,r}(n, k) (x)_{k} y^k,
\]

exactly the defining relation in (22); and when $\ell = 1$ in the same equation, we have the recurrence relation

\[
D_{m,r}(n + 1; x, y) = \sum_{i=0}^{n} m^{n-i} \binom{n}{i} D_{m,r}(i; x - 1, y) xy. \quad (42)
\]

This scenario is very similar to how Spivey’s formula generalizes both equations (1) and (2) as mentioned earlier in this paper. When $m = 1$ and $r = 0$, this results to

\[
B_{n+1}(x, y) = \sum_{i=0}^{n} \binom{n}{i} B_i(x - 1) xy, \quad (43)
\]
the bivariate extension of (2).
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