Quasiperiodicity, band topology, and moiré graphene

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A number of moiré graphene systems have nearly flat topological bands where electron motion is strongly correlated. Though microscopically these systems are only quasiperiodic, they can typically be treated as translation invariant to an excellent approximation. Here we reconsider this question for magic angle twisted bilayer graphene that is nearly aligned with a hexagonal boron nitride (h-BN) substrate. We carefully study the effect of the periodic potential induced by h-BN on the low energy physics. The combination of this potential and the moiré lattice produced by the twisted graphene generates a quasi-periodic term that depends on the alignment angle between h-BN and the moiré graphene. We find that the alignment angle has a significant impact on both the band gap near charge neutrality and the behavior of electrical transport. We also introduce and study toy models to illustrate how a quasi-periodic potential can give rise to localization and change in transport properties of topological bands.

I. INTRODUCTION

Following the discovery of correlated insulators and superconductivity in Magic Angle Twisted Bilayer Graphene (MATBG) in 2018 [1, 2] a tremendous amount of attention has been lavished on moiré materials. Other moire systems displaying correlated electron physics include ABC Trilayer Graphene (TLG/hBN) nearly aligned with a hexagonal Boron-Nitride (hBN) substrate[3], twisted double bilayer graphene[4], twisted monolayer-bilayer graphene[5], and twisted transition metal dichalcogenides[6]. Our interest in this paper is on MATBG that is further nearly aligned with a hBN substrate (MATBG/hBN)[7, 8] which alters the observed phenomena.

In MATBG/hBN Ref.7 discovered ferromagnetism and an associated large anomalous Hall effect at $3/4$ filling of the conduction band. Subsequently Ref.8 studied devices of MATBG/hBN which not only showed emergent ferromagnetism at $3/4$ conduction band filling but also observed a quantized anomalous Hall effect with $\sigma_{xy} = \frac{e}{4\pi}$. Theoretically the near alignment with the hBN breaks the $C_3$ symmetry of 180 degree rotation within the graphene plane and opens up a gap between the valence and conduction bands which - in the absence of alignment - touch at Dirac points. The resulting bands within a single valley were found[9, 10] to have Chern number $\pm 1$ (with opposite valleys having opposite Chern number). As discussed in Ref. 11 such nearly flat $\pm$ Chern bands are, in fact, common to a number of moiré graphene materials. Upon including electron-electron interactions, Ref. 11 also proposed these systems to be excellent platforms to show a quantum anomalous Hall effect at total (i.e. including spin and valley) odd integer filling. These ideas were developed further in the specific context[9, 10] of MATBG/hBN, and in ABC TLG/hBN which too displays emergent ferromagnetism and a quantum anomalous Hall effect[3].

In this paper we revisit the theory of single particle states of MATBG/hBN. The presence of h-BN layer has two effects on the nearby graphene. One is that h-BN induces a constant sub-lattice potential difference, which is studied in detail in [9, 10]. The other is that it induces a second periodic moiré potential which may or may not be commensurate with the original moiré potential of the TBLG system. The previous theoretical work ignored the moire potential introduced by the near alignment with the hBN, mostly for simplicity but also on the grounds that its estimated strength is smaller than the TBLG moire potential. In the present paper we go beyond this approximation, and carefully include both moire potentials. We first determine the conditions - which we dub “perfect alignment” - under which the two moire potentials are commensurate. This concept of perfect alignment is distinct from the naive expectation that the perfect situation is when the twist angle between one graphene layer and hBN is zero. When the perfect alignment condition is satisfied , translation invariance is preserved and we can define a crystal momentum and a (reduced) Brillouin zone. Away from perfect alignment, the two moire potentials are incommensurate case, and translational symmetry is completely broken. The low energy physics can be modeled by introducing a quasi-periodic potential to topological bands (in the case of TBLG/hBN, Chern bands with opposite Chern numbers).

Electronic systems with a quasi-periodic potential(QP) has been studied extensively in 1D. (See [12] for a detailed review.) In the 1D Audry-André model, there is a localization transition with the increase of quasi-periodic potential strength.[13] In higher than 1D, an intermediate phase with eigenstates delocalized in both real space and momentum space can exist between an extended phase and a localized phase.[14] The generic existence of such an intermediate phase in a 2D system with quasi-periodic potential has not been settled yet but it is not our focus in this paper. We are particularly interested in the effect of a quasi-periodic potential on topological bands[15]. In momentum space, non-vanishing Chern number can impose non-trivial phase structure on the wave function, which may change the localization properties when a
quasi-periodic potential is added to the system compared to trivial bands.

In the case of perfect alignment, there is a clean separation between valence and conduction bands. Then if - due to interactions - the system is valley and spin polarized at total odd integer filling $\nu_T$, electrons will completely fill a Chern band, and there will be a quantum anomalous Hall effect. Away from perfect alignment, the quasiperiodic potential induces in-gap states - which are not real-space localized - between the valence and conduction bands. Then we show by explicit calculation that even with full valley and spin polarization at odd integer $\nu_T$, there is no quantization of the anomalous Hall conductivity. Thus observation of a quantum anomalous Hall effect at such fillings is aided by studying devices that are tuned close to perfect alignment. We show however that strain can be used to tune the alignment condition, thereby enabling engineering flat well separated Chern bands in TBLG/hBN devices.

Though we do not address many body effects in this paper, we note that the presence of in-gap states are likely to hinder the development of valley/spin polarization in the first place. This is because they can roughly be thought of as increasing the bandwidth of the active valence or conduction band, thereby reducing the ability of interactions to induce ferromagnetism. Thus it is desirable to stay close to perfect alignment. Indeed the two devices studied in Refs. 7 and 8 are nearly perfectly aligned. This condition may be a more stringent requirement for the fractional quantum anomalous Hall states proposed[16–18] for TBLG/hBN.

The periodic modulation induced by the hBN layer is relevant only if the hBN layer is nearly aligned with TBLG since the moiré lattice constants of the superlattices generated by hBN and TBLG is of the same order as the moiré lattice constant of TBLG. For hBN misaligned with TBLG, due to the lattice mismatch, there is no longer any periodic moiré potential induced by hBN so the QP physics are irrelevant in those systems.

In recent years, Anderson localization and many-body localization in the presence of QP have been investigated in cold atom experiments[19–22]. The interplay between quasiperiodicity and interaction near critical points in quantum Ising and related spin models have been the subject of several studies: see, eg, Refs. 23–28 for some representative papers. It is seen that the presence of QP can lead to new interacting critical phases which are different from that found with quenched disorder [28]. The specific moiré graphene system we study here provides an experimental context where strongly interacting quantum phases/phase transitions in the presence of quasiperiodicity may be explored.

The rest of the paper is organized as follows. In section II, we explain how the alignment to hBN induces another moiré pattern on top of the original moiré pattern of TBLG system. We further study two scenarios in section III and in section IV. One is that the two moiré patterns overlap and the other is that they are incommensurate. In section V, we propose a toy model to address the question of the effect of a quasi-periodic potential on a topological band.

## II. TWO MOIRÉ PATTERNS IN HBN/TBLG SYSTEM

Let us consider TBLG with the top graphene layer nearly aligned with hBN. There are two moiré patterns, one formed by the TBLG, the other formed by top graphene layer and h-BN layer. The difference between the two moiré reciprocal lattice vectors is in general not small compared to the reciprocal vectors themselves. Thus, strictly speaking, it is not a valid approximation to define a mini-BZ. Let us first write down the reciprocal vectors explicitly. The reciprocal lattice vectors of the top graphene sheet are $\vec{G}_{t,1} = \frac{\pi}{\sqrt{3}a_{G}}(0,1)$ and $\vec{G}_{t,2} = R_{2\pi/3}\vec{G}_{t,1}$, where $a_G$ is the lattice constant of graphene and $R_{2\pi/3}$ denotes counter-clockwise rotation by $2\pi/3$. Assuming the bottom graphene layer rotates counter-clockwise by an angle $\theta_G \ll 1$, that gives, $\vec{G}_{b,1} = R_{\theta_G}\vec{G}_{t,1}$. The TBLG moiré pattern is determined by the two reciprocal lattice vectors, $\vec{G}_{1} = \vec{G}_{t,1} - \vec{G}_{b,1} = \frac{4\pi}{\sqrt{3}a_{BN}}(\sin \theta_G, 1 - \cos \theta_G)$ and $\vec{G}_{2} = R_{2\pi/3}\vec{G}_{1}$. Now adding h-BN on top, assuming h-BN layer rotates by an angle $\theta_{BN} \ll 1$ with respect to the top layer of TBLG, there is a second moiré pattern, which is generated by the lattice mismatch of the h-BN layer and the top graphene layer. For the reciprocal lattice vectors for this second moiré pattern, we write, $\vec{Q}_{1} = \vec{G}_{t,1} - \vec{G}_{BN,1} = (\frac{4\pi}{\sqrt{3}a_{BN}}\cos \theta_{BN} - \frac{4\pi}{\sqrt{3}a_{BN}}\sin \theta_{BN}, \frac{4\pi}{\sqrt{3}a_{BN}}\cos \theta_{BN} - \frac{4\pi}{\sqrt{3}a_{BN}}\sin \theta_{BN})$ and $\vec{Q}_{2} = R_{2\pi/3}\vec{Q}_{1}$. Now adding h-BN on top, assuming h-BN layer rotates by an angle $\theta_{BN} \ll 1$ with respect to the top layer of TBLG, there is a second moiré pattern, which is generated by the lattice mismatch of the h-BN layer and the top graphene layer.

For special combination of $\theta_{BN}$ and $\theta_G$, the two moiré patterns can be commensurate. For simplicity, we only consider the case where these two patterns overlap, which we call "perfect" alignment. This means that the lattice generated by $\vec{G}_{1,2}$ is the same as the lattice generated by $\vec{Q}_{1,2}$, which can be satisfied as long as $|\vec{G}_{1}| = |\vec{Q}_{1}|$ and the angle between $\vec{Q}_{1}$ and $\vec{G}_{1}$ is $\pi n/3$, where $n$ is an integer. These two conditions can be satisfied either when $\theta_{BN} + \theta_{BN}$ have the same sign or have the opposite sign. See Fig.1 for illustration.

For case 1, $\theta_{BN} > 0$ and $\theta_{BN} > 0$. The angle between $\vec{G}_{1}$ and $\vec{Q}_{1}$ is $\pi/3$. For case 2, $\theta_{BN} < 0$ and $\theta_{BN} > 0$. The angle between $\vec{G}_{1}$ and $\vec{Q}_{1}$ is $2\pi n/3$. We only consider $a_{BN} > a_G$. Using the law of sines one can get,

$$\cot |\theta_{BN}| = \left(2\sin \left(\frac{\theta_{BN}}{4}\right)\sin \left(\frac{\theta_{BN}}{4} + \frac{\pi}{n}\right)\right)^{-1} - \cot \left(\frac{\theta_{BN}}{4} + \frac{\pi}{n}\right)$$

$$a_{BN} = 2\sin \left(\frac{\theta_{BN}}{4}\right)\sin \left(\frac{\theta_{BN}}{4} + \frac{\pi}{n}\right)$$

where we take "-" for case 1 and "+" for case 2.

From the perfect alignment conditions Eq.1, there are two free parameters in $a_G, a_{BN}, \theta_G$ and $\theta_{BN}$. If one fix $a_{BN}$ to be $2.504\text{Å}$ and $a_G$ to be $2.46\text{Å}$, the corresponding $\theta_{G, BN}$ are $\theta_G \approx 1.16^\circ$, $\theta_{BN} \approx 0.58^\circ$ for case 1 and
\[ \theta_G \approx 1.17^\circ, \theta_{BN} \approx -0.61^\circ \] for case 2. If the graphene sheets are under strain, \( a_G \) can be slightly changed. From Fig. 2, we find that \( \theta_{BN} \) and \( \theta_G \) are highly sensitive to the lattice constants and \( \theta_G \) can be tuned to magic angle with 0.2% change to \( a_G/a_{BN} \). We also plot the experimental value of \( \theta_G \) and \( \theta_{BN} [7, 8] \) in Fig. 2(b) and compare them to the perfect alignment case. Note that the sample studied in Ref.[8] is closer to the perfect alignment. This provides an explanation to the better quantization of Hall conductivity in Ref.[8] than in Ref.[7].

Next, we continue to discuss the two kinds of perfect alignments in details in section III.

### III. PERFECT ALIGNMENT

As an idealized limit, in this section, we consider perfect alignment between h-BN and TBLG. We can still define mini Brillouin zone and momentum is a good quantum number in this limit. We ignore the hopping between h-BN layer and the bottom layer of the TBLG system. Hopping between h-BN and top layer of TBLG induces two kinds of terms in momentum space of the graphene. One is hopping terms between \( k \) and \( k + \mathbf{Q}_i \)'s, where \( \mathbf{Q}_i \) are the reciprocal vectors of the moiré pattern generated by h-BN and top graphene layer. The other one is a constant AB sublattice potential due to the lattice relaxation in h-BN and in graphene and electron-electron interaction (If the lattice is rigid, the sublattice potential vanishes due to the lattice mismatch between h-BN and graphene.).[29] In momentum space, the Hamiltonian of the h-BN and TBLG system for one valley and one spin can be written as,

\[ H = H_{TBLG} + H_V, \]  
where \( H_V \) contains two terms,

\[ H_V = \sum_{\mathbf{k}} f_k^{\dagger} m z \sigma_z f_k + \sum_{\mathbf{k}, i} (f_k^{\dagger} V(\mathbf{Q}_i)f_{k+i\mathbf{Q}_i} + h.c.), \]

where \( f_k = (f_{kA}, f_{kB})^T \) denotes the electron annihilation operators for sublattice A and B. \( \sigma_z \) acts on sublattice degree of freedom. Index \( i = 1, \ldots, 6 \) labels different reciprocal vectors. \( \mathbf{Q}_1 \) is defined in section II and all the other \( \mathbf{Q}_i \)'s are generated by performing \( C_6 \) rotation of \( \mathbf{Q}_1 \) consecutively.

\[ V(\mathbf{Q}_i)'s \] can be parametrized in the following way[30],

\[ V(\mathbf{Q}_i) = \begin{pmatrix} H_0(\mathbf{Q}_i) + H_z(\mathbf{Q}_i) & H_{AB}(\mathbf{Q}_i) \\ H_{BA}(\mathbf{Q}_i) & H_0(\mathbf{Q}_i) - H_z(\mathbf{Q}_i) \end{pmatrix}, \]

where \( H_{0,z}(\mathbf{Q}_1) = H_{0,z}(\mathbf{Q}_4) = H_{0,z}(\mathbf{Q}_5) = C_0 e^{i \phi_{0,z}}, H_{0,z}(\mathbf{Q}_2) = H_{0,z}(\mathbf{Q}_3) = C_0 e^{-i \phi_{0,z}} \) and \( H_{AB}(\mathbf{Q}_1) = H_{AB}(\mathbf{Q}_4) = C_{AB} e^{i \phi_{AB}}, H_{BA}(\mathbf{Q}_3) = H_{AB}(\mathbf{Q}_5) = C_{AB} e^{-i \phi_{AB}} \), \( H_{AB}(\mathbf{Q}_2) = C_{AB} e^{i(\pi - \phi_{AB})}, H_{BA}(\mathbf{Q}_3) = H_{AB}(\mathbf{Q}_5) = C_{AB} e^{i(\pi - \phi_{AB})} \) from Hermiticity.

From ab initio study[30], at \( \theta_{BN} = 0^\circ \), taking the lattice relaxation into account, the parameters for the periodic terms are \( C_0 = -9.07, \phi_0 = 97.99^\circ, C_z = -5.64, \phi_z = -3.66^\circ \). All
From the numerical calculation, we demonstrate that two bands near charge neutrality has Chern number with TBLG. Further calculation shows that in case 1, the Berry curvature is also similar to Chern numbers for case 1 and case 2. The distribution of $V$ terms is the contribution to the gap of the momentum dependent Fig.3 along a path in mini BZ for both case 1 and case 2. The distribution of Berry curvature of valence band for various cases is plotted in Fig.4. For $\theta_G = 1.15^\circ$, we get the same Chern numbers for case 1 and case 2. The distribution of Berry curvature is also similar to $\theta_G = 1.2^\circ$ (see appendix A). From the numerical calculation, we demonstrate that the alignment with hBN has a significant effect on the low energy physics of the TBLG system. In particular, the periodic potential induced by hBN cannot be ignored.

**IV. INCOMMENSURATE ALIGNMENT**

In general, the alignment between hBN and TBLG is not commensurate. The periodic potential term $\hat{V}(\tilde{Q}_i)$ induces a quasi-periodic potential relative to the TBLG superlattice. The spectrum of the TBLG will get broadened but since the coupling strength between hBN and graphene is much smaller than the band gaps from the flat bands to the other bands in TBLG system, we can ignore the other bands and only consider an effective description for the flat bands.

The extra hBN layer breaks the $C_2T$ symmetry of the TBLG system such that there is no obstruction of constructing localized Wannier orbitals using the two bands near charge neutrality in one valley. The low energy effective tight binding model is obtained in two steps. First, take $m_z$ term in $H_V$ together with $H_{TBBLG}$ and construct localized Wannier orbitals for the two bands in one valley. Second, project the $\hat{V}(\tilde{Q}_i)$ terms to the active bands and transform to Wannier basis.

We use projection method to obtain Wannier functions.[34] The relationship between Bloch function and Wannier function can be written as,

$$\psi^\dagger_{n,\tilde{x}_0} = \frac{1}{\sqrt{N}} \sum_{\tilde{k},m} e^{-i\tilde{k}\cdot\tilde{x}_0} \psi^\dagger_{m,\tilde{k}}(U_{\tilde{k}})_{mn},$$

(5)

where $\psi^\dagger_{n,\tilde{x}_0}$ is the creation operator for Wannier orbital labeled by n at position $\tilde{x}_0$ and $\psi^\dagger_{m,\tilde{k}}$ is the creation operator for Bloch state and $m \in \{c,v\}$ labels the conduction and valence band in one valley. $U_{\tilde{k}}$ is a unitary matrix, defined as $U_{\tilde{k}} = A_{\tilde{k}} (A_{\tilde{k}}^\dagger A_{\tilde{k}})^{-1/2}$, where $(A_{\tilde{k}})_{mn} = \langle \mu_m(\tilde{k}) | g_n(\tilde{k}) \rangle$ is the overlap matrix between the Bloch wave function $|\mu_m(\tilde{k})\rangle$ and k-space representation of a localized wave function ansatz $|g_n(\tilde{k})\rangle$. In the numerical calculation below, we take $|g_n(\tilde{k})\rangle = e^{-\tilde{E}^2/2} e^{-i\tilde{k}\cdot\tilde{x}_0} |\varphi_n\rangle$ so after inverse Fourier transform, $|g_n(\tilde{x})\rangle$ is localized near $\tilde{x}_0$. $|\varphi_n\rangle$ is a constant vector in $\tilde{k}$ space and it is chosen to maximize the singular values of $A_{\tilde{k}}$. [35]

The projected hopping terms and quasi-periodic potential terms can be written as,

$$t_{mn}(\tilde{x}_{ij}) = \frac{1}{N} \sum_{\tilde{k}} e^{i\tilde{k}\cdot\tilde{x}_{ij}} (U_{\tilde{k}}^\dagger U_{\tilde{k}+\tilde{q}})_{mn}$$

$$V_{mn}(\tilde{x}_{ij}, \tilde{x}_j) = \frac{1}{N} \sum_{\tilde{q}} e^{-i\tilde{q}\cdot\tilde{x}_j} \sum_{\tilde{k}} e^{i\tilde{k}\cdot\tilde{x}_{ij}} (U_{\tilde{k}}^\dagger f^{\tilde{q}}(\tilde{k}) U_{\tilde{k}+\tilde{q}})_{mn}$$

(6)

where $\tilde{x}_{ij} = \tilde{x}_i - \tilde{x}_j$ is the displacement of the two lattice points, $\epsilon_{\tilde{k}} = diag(\epsilon_c(\tilde{k}), \epsilon_v(\tilde{k}))$, $\epsilon_c(\tilde{k})$, $\epsilon_v(\tilde{k})$ being the dispersion of the conduction/valence bands. $\tilde{Q}_i$ is defined in the first BZ of the TBLG system and is related to $\hat{Q}_i$ by addition of integer multiples of $\hat{G}_1$ and $\hat{G}_2$. $f^{\tilde{q}}(\tilde{k})$ is the
form factor, whose matrix element is,

\[
f_{n_1,n_2}(\vec{k}) = \sum_{m,n} \psi_{n_1,\vec{k}+m\vec{G}_1}^* \psi_{n_2,\vec{k}+n\vec{G}_2} (V_q)_{n_1,n_2} \psi_{n_2,\vec{k}+n\vec{G}_2+q\vec{G}_1+n\vec{G}_2},
\]

where \(\psi_{n_1,\vec{k}}\) denotes the wave function in \(\vec{k}\)-space of the \(n_1\) sublattice of the top layer graphene and \(V_q\) is the coupling matrix of the quasi-periodic potential term, in the form of Eq.4.

The effective Hamiltonian can therefore be written as,

\[
H_{tb} = \sum_{ij} (t_{mn} (\vec{x}_i,\vec{x}_j)c^\dagger_{in}c_{jn} + h.c.) + \sum_{ij} (V_{mn}(\vec{x}_i,\vec{x}_j)c^\dagger_{in}c_{jn} + h.c.),
\]

where \(c_i\) lives on the moiré lattice formed by the TBLG and we can write \(\vec{r}_i = n_i\vec{a}_M + m_i\vec{a}_M\), where \(a_M = \frac{a}{\sin(\theta_G/2)}\) and \(a_M = a_M(0,1)\). \(a_M = \frac{2a}{\sin(\theta_G/2)}\) being the moiré lattice constant.

After \(U_J\) is obtained, we get \(t_{mn}\)’s and \(V_{mn}\)’s from Eq.6. Let us consider \(t_{mn}\)’s first. We find that in order to reproduce the band gap and band structure well, we need to keep the hopping terms up to the third nearest unit cell. (See Table I and Table II.) \(t_{0AA}\) and \(t_{0BB}\) are on-site potentials for site A and B. The meaning of the other labels is explained in Fig.5.

Without the quasi-periodic terms, the dispersion of \(H_{tb}\) is plotted in Fig.6. The valence and conduction bands have Chern number \(\pm 1\) respectively.

Let us consider the \(V_{mn}\) terms. There are two effects of the twist angle between hBN and graphene \(\theta_{BN}\) on the tight-binding Hamiltonian: one is the change of the \(Q\)’s and the other is the change of projected amplitude of the quasi-periodic potential terms. We study two different \(\theta_{BN}\)’s numerically: \(\theta_{BN} = 0^\circ\) and \(0.8^\circ\). We project the quasi-periodic terms to the Wannier orbitals and calculate \(V_{mn}(\vec{x}_i,\vec{x}_j)\). We find that although the amplitude of the quasi-periodic terms decays with \(|\vec{x}_i - \vec{x}_j|\) but within

The black line is the boundary of the first BZ and the black dot is the \(\Gamma\) point.

| \(\theta_G\) | \(1.2^\circ\) |
|---|---|
| \(t_{0AA}\) | 4.375 |
| \(t_{0BB}\) | -1.270 |
| \(t_{1AA}\) | 1.547e\((0.197)\) |
| \(t_{1BB}\) | -1.613e\((-0.188)\) |
| \(t_{2AA}\) | 0.482e\((-0.349)\) |
| \(t_{2BB}\) | -0.452e\((0.316)\) |
| \(t_{3AA}\) | 0.506e\((-0.134)\) |
| \(t_{3BB}\) | -0.521e\((-0.13)\) |

TABLE I: Hopping between the same type of lattice sites

| \(\theta_G\) | \(1.2^\circ\) |
|---|---|
| \(t_{1AB}\) | 2.249e\((0.082)\) |
| \(t_{2AB}\) | -1.544e\((0.333)\) |
| \(t_{3AB1}\) | -0.398e\((0.123)\) |
| \(t_{3AB2}\) | 0.668e\((-0.304)\) |
| \(t_{4AB1}\) | -0.412e\((0.131)\) |
| \(t_{4AB2}\) | -0.509e\((0.133)\) |
| \(t_{5AB}\) | -0.270e\((0.314)\) |
| \(t_{6AB1}\) | -0.320e\((-0.142)\) |
| \(t_{6AB2}\) | -0.165e\((-0.289)\) |

TABLE II: Hopping between AB lattice sites
FIG. 5: Labels of the hopping terms from site A (orange circle) to other nearby sites. For example, hopping from A to an orange site labelled '2' corresponds to $t_{2AA}$ and from A to an green site labelled $3_1$ corresponds to $t_{3AB}$.  

FIG. 6: Dispersion of the effective tight binding model. 

We introduce PR (participation ratio) in $\vec{k}$-space, $PR = \frac{\sum_{\vec{k}} |\psi_{\vec{k}}|^2}{\sum_{\vec{k}} |\psi_{\vec{k}}|^4}$. States that are localized in real space are extended in momentum space and so we expect $|\psi_{\vec{k}}| \sim 1/N$, where $N \times N$ is the system size in $\vec{k}$-space. Thus PR $\sim N^2$ for localized states and PR $\sim$ constant for extended states. From Fig. 8, we find that the PR share similar feature as the density of states, which means that near the dips of density of states (DOS), there are more extended states while near the peaks of DOS, there are more localized states. Localized states in $\vec{k}$-space are extended in real space. Thus, we indeed get metallic behavior near charge neutrality for $\theta_{BN} = 0^\circ$ and mobility edges exist. The density of states indicates that the alignment of hBN has a strong effect on the low energy physics especially near charge neutrality. We then calculate the Hall conductivity $\sigma_{xy}$ using Kubo formula for one valley and one spin species to further address the difference in electrical transport. In the full many body system, this is the Hall conductivity obtained (within Hartree-Fock) if the system is spontaneously fully spin and valley polarized at the filling considered. Thus at $3/4$ filling of the conduc-
FIG. 8: PR at (a) $\theta_{BN} = 0^\circ$ and (b) $\theta_{BN} = 0.8^\circ$ for different states throughout entire energy spectrum. The dotted line indicates the energy of the middle state in the spectrum. Different color indicates different system size. From bottom to top: $N = 30, 50, 70$ and the system size is $N \times N$ in $\vec{k}$-space.

FIG. 9: $\sigma_{xy}$ vs. chemical potential at (a) $\theta_{BN} = 0^\circ$ and (b) $\theta_{BN} = 0.8^\circ$ for a $50 \times 50$ grid in $\vec{k}$-space. The dotted line indicates the energy of the middle state in the spectrum.

The natural occurrence of topological bands and a quasi-periodic potential in TBLG/hBN discussed in previous sections leads to a number of interesting theoretical questions. For ordinary non-topological bands, the question of how different the effects of a quasiperiodic potential are as compared to a random potential on the electronic wave functions has begun to be addressed in recent years. [14, 36, 37] Here we are interested instead in similar questions when, in addition, the bands are topological. Within a free fermion theory, what is the behavior of the conductivity as a function of chemical potential? As part of addressing this question, it is important to understand in the first place how to couple in an external vector potential to the electrons in the topological band which is itself a subtle question, as we shall see.

Here we address these questions within a simple context. Let us consider a system with a Chern band and add a quasi-periodic potential to it. If the strength of the quasi-periodic potential is much smaller than the band gaps between the topological band considered and all the other bands, the minimal approach is to project the Hamiltonian to the low energy Chern band. Since there is...
Wannier obstruction, a tight-binding model in real space is not possible. Thus, we write the effective Hamiltonian in momentum space. For simplicity, assume that we have a flat band to begin with. In momentum space, the Hamiltonian can be written as,

$$H = \sum_{\vec{k}} \sum_{\vec{Q}_i} c_{\vec{k}}^\dagger V(\vec{Q}_i) c_{\vec{k}+\vec{Q}_i} \lambda(\vec{k},\vec{k}+\vec{Q}_i) + h.c., \quad (9)$$

where $\lambda(\vec{k},\vec{k}+\vec{Q}_i)$ is the form factor and $\vec{Q}_i$'s are the reciprocal vectors for the quasi-periodic potential. For each $\vec{k}$, the Hamiltonian can be viewed as a tight-binding model in momentum space with a lattice generated by $\vec{Q}_i$'s. For $\vec{Q}_i$'s that are incommensurate with the original reciprocal lattice vectors $\vec{G}_{1,2}$ that generate the Brillouin zone, we expect $\vec{k} + n_1 \vec{Q}_1 + n_2 \vec{Q}_2$ mod $(m_1 \vec{G}_1 + m_2 \vec{G}_2)$ ($n_{1,2}$ and $m_{1,2}$ are integers) to be dense in the first Brillouin zone. In this case, we only need to consider one lattice that is generated by $\vec{k} + n_1 \vec{Q}_1 + n_2 \vec{Q}_2$ with a fixed $\vec{k}$. To keep contact with moiré graphene, we will let $Q_{1,2}$ generate a triangular lattice but similar discussion can be carried out on any lattice.

For trivial bands, one can take $\lambda(\vec{k},\vec{k}+\vec{Q}_i) = 1$. In this case, the eigenvectors are plane waves in $\vec{k}$ space and therefore they are localized in real space. Thus at large quasi-periodic potential strength for trivial bands, there is always localization.

For a Chern band, on the other hand, the form factor is non-trivial. For small $|\vec{Q}|$, it can be written as $\lambda(\vec{k},\vec{k}+\vec{Q}_i) = F(\vec{k},\vec{k}+\vec{Q}_i)e^{-i \int_{\vec{k}}^{\vec{k}+\vec{Q}_i} A(\vec{q}) \cdot d\vec{q}}$, where $F(\vec{k},\vec{k}+\vec{Q}_i)$ is real and positive and the path of the integral is taken to be a straight line from $\vec{k}$ to $\vec{k}+\vec{Q}_i$.

To simplify the problem, we assume homogeneous Berry curvature and further let $F(\vec{k},\vec{k}+\vec{Q}_i) = 1$ for now. Then the Hamiltonian is equivalent to a tight-binding model in a uniform perpendicular magnetic field. We choose Landau gauge such that $A(\vec{k}) = (−Bk_y, 0, 0)$. $B$ is proportional to Chern number and the magnetic flux is in general not rational. For this choice of gauge, $\lambda(\vec{k},\vec{k}') = e^{i \frac{\pi}{4}(k_x'-k_x)(k_y'+k_y)}$. The Hamiltonian can be written as,

$$H = \sum_{\vec{k}} \left( V_1 c_{\vec{k},k_y}^\dagger c_{\vec{k}+1,k_y}e^{iBk_y} + V_2 c_{\vec{k},k_y}^\dagger c_{\vec{k}-1/2,k_y+\sqrt{3}/2}e^{-i\frac{\pi}{4}(k_y+\sqrt{3})} + V_3 c_{\vec{k},k_y}^\dagger c_{\vec{k}-1/2,k_y-\sqrt{3}/2}e^{i\frac{\pi}{4}(k_y-\sqrt{3})} + h.c. \right) \quad (10)$$

where the lattice spacing is set to be 1 and $V_{1,2,3}$ are taken to be real.

The Hamiltonian in Eq.10 is a special case of that considered in [38]. Following the arguments in [38], we can perform a Fourier transform along $k_y$ direction. The 2D model is then equivalent to a 1D lattice model with quasi-periodic(QP) potential. One can write $k_y = k_y^0 + n \sqrt{3}/2$, where $n$ is an integer. Note that we assume $\vec{k}$-lattice is dense in the original BZ. We can take $k_y^0$ to be 0. The flux quanta is $\Phi = \sqrt{3}B/4\pi$. Thus the 1D lattice model with QP potential is,

$$E\phi_n = 2V_1 \cos(2\pi n\Phi + \nu)\phi_n + A_{n+,n+1}\phi_{n+1} + A_{n+,n-1}\phi_{n-1}, \quad (11)$$

where $A_{n+,n+1} = V_2 e^{-i[\pi\Phi(n+1/2)]-i\frac{\pi}{2}} + V_3 e^{i[\pi\Phi(n+1/2)]+i\frac{\pi}{2}}$ and $A_{n-,n-1} = A_{n-,n-1}^{\ast}$. $\psi(k_x,k_y)$ is the eigenfunction for the Hamiltonian 10 at energy $E$, $\psi(k_x,k_y) = e^{ik_x\nu}\phi(k_y)$ and $\phi_n \equiv (k_y^0 + n \sqrt{3}/2)$.

Depending on the relative strength of $V_{1,2,3}$, $\phi(k_y)$ can be either localized or extended in $k_y$ space. If there is $C_3$ rotational symmetry, which corresponds to $V_1 = V_2 = V_3$, the 1D system is at the critical point of the localization transition and thus the eigenstates are not localized in real space, which is different from the trivial band case.

If the $C_3$ symmetry is broken, we get the Lyapunov exponent (inverse of the localization length) by considering three different gauge choices, i.e., along the three axes of the triangular lattice[38, 39] and the Lyapunov exponent $\lambda(E;V_1, V_2, V_3)$ for $\phi_n$ in Eq.11 is,

$$\lambda(E;V_1, V_2, V_3) = \ln \left( \frac{|V_1|}{|V_3|} \right), \quad (12)$$

if $|V_1| \geq |V_3| \geq |V_2|$. And

$$\lambda(E;V_1, V_2, V_3) = \ln \left( \frac{|V_1|}{|V_2|} \right), \quad (13)$$

if $|V_1| \geq |V_2| \geq |V_3|$. $\lambda(E;V_1, V_2, V_3) = 0$ otherwise.

Even though we get localized or extended $\psi(k_x,k_y)$, depending on the choices of $V_{1,2,3}$, we still need to address the question of what effect it will have on the physical observables. Thus, we study the DC transport of the system in the following.

In the trivial case, all the states are localized so we expect the conductance to vanish. In the topological case, we need to couple the tight-binding Hamiltonian in $k$-space to external electric field. First we need to obtain current operators in the presence of external electric field. The strategy is to apply a probe vector potential $A'$ and $J_\mu = -\frac{\partial H[\vec{E},\vec{A}]}{\partial A'_\mu} \big|_{A'_\mu \to 0}$. The vector potential $A'$ will "shift" the momenta $\vec{k}$. We have to be careful about what we mean by "shift". In comparison to the trivial band, there is a gauge structure in $k$-space. If we change $c_{\vec{k}} \to c_{\vec{k}} e^{i\theta(\vec{k})}$ and $A(\vec{k})$ to $A(\vec{k}) + \partial_{\vec{k}} \theta(\vec{k})$, where $\theta(\vec{k})$ is a differentiable function in $\vec{k}$, $c_{\vec{k}} \to c_{\vec{k}+\vec{Q}} A(\vec{k},\vec{k}+\vec{Q})$ is invariant. In order to keep the gauge invariance of the theory, we cannot simply replace $\lambda(\vec{k},\vec{k}')$ by $\lambda(\vec{k}+\vec{A}',\vec{k}+\vec{A}')$. The only gauge-invariant deformation of the form factor $\lambda(\vec{k},\vec{k}')$ is to attach a small plaquette with Berry curvature as flux. This generalizes the idea of Peierls substitution. Again, let us only consider the phase factor in the form factor for now. The gauge invariant change in the form factor...
Thus the electric potential can be written as,

$$\lambda(\vec{k}, \vec{k} + \vec{Q}_i)$$

by shifting the momentum by $\vec{A}$ is,

$$\lambda_{\vec{A}}(\vec{k}, \vec{k} + \vec{Q}_i) = \lambda(\vec{k}, \vec{k} + \vec{Q}_i) e^{iB(\vec{q} \times \vec{A}) \cdot \hat{e}_z},$$

(14)

where $\lambda_{\vec{A}}(\vec{k}, \vec{k} + \vec{Q}_i)$ denotes the shifted form factor and it is defined as the Wilson loop of the Berry connection along the green curve in Fig. 10. $\hat{e}_z$ is the directional vector along $z$-axis.

The way that we construct the coupling to the external vector potential in Eq. 14 seems to rely on the specific form of the form factor $\lambda(\vec{k}, \vec{k} + \vec{Q})$. For a generic form factor, we also construct the gauge coupling (see Appendix B for details), which agrees with Eq. 14.

The tight-binding Hamiltonian coupled to the probe vector potential and external electric field can be written as,

$$H[\vec{E}, \vec{A}] = \sum_{\vec{k}} (V_1 c_{\vec{k}, x, y}^\dagger c_{\vec{k} + \vec{A} \cdot \hat{e}_x} e^{iBk_x} e^{iBA_y} + V_2 c_{\vec{k}, x, y}^\dagger c_{\vec{k}, y, x} e^{-iB(k_y + \sqrt{3}/2x) e^{iBA_y}} + V_3 c_{\vec{k}, x, y}^\dagger c_{\vec{k}, -y, x} e^{-iB(k_y - \sqrt{3}/2x) e^{iBA_y}} + h.c.) + V(\vec{E}, \vec{A})$$

(15)

$$V(\vec{E}, \vec{A}) = \int d^2x \vec{E} \cdot \vec{x} \rho(\vec{x}, \vec{A})$$

is the electric potential, where $\rho(\vec{x}, \vec{A})$ is the density operator. In $k$-space, we define the Fourier transformation of $\rho(\vec{x}, \vec{A})$ as $\rho(\vec{q}, \vec{A})$ and

$$\rho(\vec{q}, \vec{A}) = \sum_{\vec{k}} c_{\vec{k} + \vec{q}}^\dagger c_{\vec{k}} \lambda(\vec{k}, \vec{k} + \vec{q}) e^{iB(\vec{q} \times \vec{A}) \cdot \hat{e}_z} = \rho(\vec{q}, \vec{A})$$

(16)

Thus the electric potential can be written as,

$$V(\vec{E}, \vec{A}) = \int d^2q d^2x \vec{E} \cdot \vec{x} \rho(\vec{q}, \vec{A}) = V(\vec{E}, \vec{A} = 0) + B\vec{E} \cdot (\vec{A} \times \hat{e}_z)$$

(17)

to linear order in $B$, and the probe vector field $\vec{A}$ only couples to the second term above.

We can do a sanity check of the above expression. Suppose there’s no quasi-periodic potential, we can get $J_{x,0} = -1/S \sum_k c_k^\dagger c_k B E_y$ and $J_{y,0} = 1/S \sum_k c_k^\dagger c_k B E_x$, where $S$ is the total area of the system. For a fully-filled Chern band with Chern number $C$, $B = 2\pi C \equiv C_{A_{\text{cell}}}/2\pi$, where $A_{\text{BZ}}(A_{\text{cell}})$ is the area of the Brillouin zone(unit cell). Thus the Hall conductivity $\sigma_{xy} = \frac{B}{\frac{C_{A_{\text{cell}}}}{2\pi}}$ is quantized.

Now that we have some confidence in the Peierls substitution, let us take the quasi-periodic potential into account. The current density operators can be expressed as,

$$J_x = \frac{1}{S} \sum_k (i\sqrt{3}/2 V_2 B c_{\vec{k}, x, y}^\dagger c_{\vec{k}, -2/k_y + \sqrt{3}/2x} e^{-i\frac{B}{4}(k_y + \sqrt{3}/2x)} - i\sqrt{3}/2 V_3 B c_{\vec{k}, x, y}^\dagger c_{\vec{k}, -2/k_y - \sqrt{3}/2x} e^{-i\frac{B}{4}(k_y - \sqrt{3}/2x)} + h.c.) + \frac{1}{S} \sum_k c_k^\dagger c_k B E_y$$

$$J_y = -\frac{1}{S} \sum_k (iB V_1 c_{\vec{k}, x, y}^\dagger c_{\vec{k}, y, x} - iB V_2 c_{\vec{k}, x, y}^\dagger c_{\vec{k}, -2/k_y + \sqrt{3}/2x} e^{-i\frac{B}{4}(k_y + \sqrt{3}/2x)} - iB V_3 c_{\vec{k}, x, y}^\dagger c_{\vec{k}, -2/k_y - \sqrt{3}/2x} e^{-i\frac{B}{4}(k_y - \sqrt{3}/2x)} + h.c.) + \frac{1}{S} \sum_k c_k^\dagger c_k B E_x$$

(18)

Let us consider $E_x = E$ and $E_y = 0$. The transport properties are the same as a tight-binding model in real space if we view $k_x$ as $y$-coordinate, $k_y$ as $x$-coordinate. By making this mapping, we transform the problem of non-trivial Berry curvature in $k$-space to the problem of a real-space tight binding model under perpendicular magnetic field. In the original model, $\vec{E}$ is along $x$-direction while $\vec{E}$ is along $y$-direction in the real-space model. This is the same as what happens in lowest Landau level(LLL). Indeed, if we view the wave functions for the LLL as wave functions for the flat Chern band at $C = 1$ and calculate the form factors, by comparing with [40], we get the drift current exactly in the form of Eq.18.

For commensurate flux $\Phi = 2\pi q$, translational symmetry is restored and the energy spectrum is divided into $q$ magnetic sub-bands. For simplicity, we only consider zero temperature. $\sigma_{xz} = 0$ if the chemical potential is within band gaps. $\sigma_{xy}$ can be obtained through TKNN formula[41],

$$\sigma_{xy} = \sum_m \int \frac{d^2\nu}{(2\pi)^2} f_m(\nu) \left[ F_m(\nu) + \frac{C}{2\pi} \right],$$

(19)

where $m \in \{1, \ldots, q\}$ is the index of magnetic bands, $f_m(\nu)$ is the Fermi-Dirac distribution. $\nu$ takes value
with the magnetic Brillouin zone \( \nu_x \in (-\pi, \pi] \) and \( \nu_y \in (-\pi/q, \pi/q] \). \( F_m(\nu) + \frac{mC}{\nu} \) is the total Berry curvature of the states at \( \nu \) of the \( m \)-th band, where the first term takes care of the contribution of the QP potential and the second term comes from the background Berry curvature.

Note that \( \sigma_{xy} \) is quantized although it is not obvious from the expression in Eq.19. Following Ref.[41, 42], Eq. 19 is reduced to \( \sigma_{xy} = \frac{mC}{2\pi} \), where \( m \) is an integer that satisfies the Diophantine equation \(-p\Delta n + qm = 1\), where \( \Delta n \) is an integers.

For incommensurate flux, by mapping to the real-space model, \( \sigma_{xx} = 0 \) if the filled states are localized along \( k_y \)-direction, which is the case for \( |V_1| > max(|V_2|, |V_3|) \). Otherwise, if \( |V_1| < max(|V_2|, |V_3|) \), all states are extended.

Let us consider \( \sigma_{xy} \) next. If the chemical potential is within the gap and the gap is not closed if we continuously tune the value of \( \Phi \) from the expression in Eq.19. Following Ref.[41, 42], Eq. 19 is set to be

\[
\frac{H_{\pm}}{2} = \sum_{\vec{k}} \left( \epsilon_{\vec{k}} c_{\vec{k}\uparrow} c_{\vec{k}\downarrow} - \epsilon_{\vec{k}} c_{\vec{k}\downarrow}^\dagger c_{\vec{k}\uparrow}^\dagger \right) + V_0 \sum_{\vec{k},\vec{Q}} \epsilon_{\vec{k}+\vec{Q}} \lambda_{\pm \pm}(\vec{k}, \vec{k} + \vec{Q}) + V_1 \sum_{\vec{k}} \epsilon_{\vec{k}} c_{\vec{k}\uparrow} c_{\vec{k}\downarrow}^\dagger \lambda^0_{\pm \pm}(\vec{k}) + h.c.,
\]

where the subscripts \( \pm \) label the different bands. The \( V_0 \) terms are projected quasi-periodic potential terms and the \( V_1 \) terms are inter-band hopping between the two \( \pm \) Chern bands. The form factor \( \lambda_{\pm \pm}(\vec{k}, \vec{k} + \vec{Q}) \) is \( \langle \psi_{\vec{k} \pm \uparrow} | e^{-i \vec{Q} \cdot \vec{x}} | \psi_{\vec{k} \pm \downarrow} \rangle \), where \( |\psi_{\vec{k} \pm \alpha} \rangle \) are Bloch states and similarly, \( \lambda^0_{\pm \pm}(\vec{k}) = \langle \psi_{\vec{k} \pm \uparrow} | \psi_{\vec{k} \pm \downarrow} \rangle \). \( \Delta \) is set to be positive. For the purpose of illustration, we take \( |\psi_{\vec{k} \alpha} \rangle \) to be the same as in the LLL and choose a Landau gauge \( \vec{A}_\pm = (\mp By, 0) \) such that \( \langle \vec{x} | \psi_{\vec{k} \pm \alpha} \rangle = \sum_m e^{i(zmk_y + k_x + mBx + zmk_x/B)} \Phi_0(y \pm \frac{k_y + mB}{2}) \) (see Appendix C for details), where \( \Phi_0(y) = (\frac{B}{\pi})^\frac{1}{2} e^{-\frac{y^2}{2B}} \). We further let \( \tilde{Q}_1 = Q(2\pi, 0) \) and \( \tilde{Q}_2 = Q(0, 2\pi) \), where \( Q \) is an irrational number (Note that in LLL, we can always define the magnetic Brillouin zone \( \tilde{Q}_{1,2} \) are aligned with hBN system by considering two flat topological bands with Chern number \( \pm 1 \). For illustration purpose, we only consider a square lattice; the quasi-periodic potential only contains the lowest harmonics and the system has \( C_4 \) rotational symmetry. The quasi-periodic potential also only acts within the same band and there is an inter-band mixing term. The Hamiltonian thus can be written as,

\[
\frac{H_{\pm}}{2} = \sum_{\vec{k}} \left( \epsilon_{\vec{k}} c_{\vec{k}\uparrow} c_{\vec{k}\downarrow} - \epsilon_{\vec{k}} c_{\vec{k}\downarrow}^\dagger c_{\vec{k}\uparrow}^\dagger \right) + V_0 \sum_{\vec{k},\vec{Q}} \epsilon_{\vec{k}+\vec{Q}} \lambda_{\pm \pm}(\vec{k}, \vec{k} + \vec{Q}) + V_1 \sum_{\vec{k}} \epsilon_{\vec{k}} c_{\vec{k}\uparrow} c_{\vec{k}\downarrow}^\dagger \lambda^0_{\pm \pm}(\vec{k}) + h.c.,
\]

where the subscripts \( \pm \) label the different bands. The \( V_0 \) terms are projected quasi-periodic potential terms and the \( V_1 \) terms are inter-band hopping between the two \( \pm \) Chern bands. The form factor \( \lambda_{\pm \pm}(\vec{k}, \vec{k} + \vec{Q}) \) is \( \langle \psi_{\vec{k} \pm \uparrow} | e^{-i \vec{Q} \cdot \vec{x}} | \psi_{\vec{k} \pm \downarrow} \rangle \), where \( |\psi_{\vec{k} \pm \alpha} \rangle \) are Bloch states and similarly, \( \lambda^0_{\pm \pm}(\vec{k}) = \langle \psi_{\vec{k} \pm \uparrow} | \psi_{\vec{k} \pm \downarrow} \rangle \). \( \Delta \) is set to be positive. For the purpose of illustration, we take \( |\psi_{\vec{k} \alpha} \rangle \) to be the same as in the LLL and choose a Landau gauge \( \vec{A}_\pm = (\mp By, 0) \) such that \( \langle \vec{x} | \psi_{\vec{k} \pm \alpha} \rangle = \sum_m e^{i(zmk_y + k_x + mBx + zmk_x/B)} \Phi_0(y \pm \frac{k_y + mB}{2}) \) (see Appendix C for details), where \( \Phi_0(y) = (\frac{B}{\pi})^\frac{1}{2} e^{-\frac{y^2}{2B}} \). We further let \( \tilde{Q}_1 = Q(2\pi, 0) \) and \( \tilde{Q}_2 = Q(0, 2\pi) \), where \( Q \) is an irrational number (Note that in LLL, we can always define the magnetic Brillouin zone \( \tilde{Q}_{1,2} \) are aligned with the mBZ reciprocal lattice vectors \( \tilde{G}_{1,2} \), which are set to be \( (2\pi, 0) \) and \( (0, 2\pi) \) here). As elaborated in Appendix C, the form factors are,

\[
\lambda_{\pm \pm}(\vec{k}, \vec{k} + \vec{Q}) = e^{\mp i(2k_x + 2Q_x + 2Q_y - 4k_x^2 - 4k_y^2) + \frac{4\pi^2}{B^2} \Phi_0(y - \frac{k_y + mB}{2})} \sum_{m=-\infty}^{\infty} e^{\mp i(2k_x + 2Q_x + 2Q_y - 4k_x^2 - 4k_y^2) + \frac{4\pi^2}{B^2} \Phi_0(y - \frac{k_y + mB}{2})},
\]

Plugging in the definition of \( \tilde{Q}_1 \) and \( \tilde{Q}_2 \), we have, \( \lambda_{\pm \pm}(\vec{k}, \vec{k} + \tilde{Q}_1) = e^{-\frac{2\pi^2}{k_B^2} \Phi_0(y)} \) and \( \lambda_{\pm \pm}(\vec{k}, \vec{k} + \tilde{Q}_2) = e^{-\frac{2\pi^2}{k_B^2} \Phi_0(y)} \).

One can check that the \( c_{\vec{k} \pm \alpha} \) bands are topological bands with Chern number \( \pm 1 \) in two ways. First, by taking derivatives of \( \tilde{Q} \) in Eq.21 around \( \tilde{Q} = 0 \), one gets uniform Berry curvature of \( \pm \frac{1}{B} \) for \( c_{\vec{k} \pm \alpha} \) bands respectively.

Second, since the phases of the Bloch wave func-
tical and the “+” bands are shifted with an energy

\[ \theta \]

The eigenvectors are,

\[ \mathbf{A}_{\pi, \pm}(k_x, \pm) = \mathbf{A}_{\pi, \mp}(k_x, \mp) = \mathbf{A}_{\pi, \mp}(k_x, \pm) \mp \mathbf{e}_x, \]

where \( \mathbf{A}_{\pi, \pm}(k_x, \pm) \) is the Berry connection and \( \mathbf{e}_x \) is the unit vector along \( k_x \). Thus the contour integrals of \( \mathbf{A}_{\pi, \pm}(k_x, \pm) \) along the boundary of the BZ yield Chern number \( \pm 1 \).

If \( V_0 = 0 \), the Hamiltonian in Eq. 20 is block-diagonal in \( k_x \) space. By solving the 2 \( \times 2 \) block, we have the two eigenvalues \( \pm \epsilon_k = \pm \sqrt{\frac{A^2}{4} + V_1^2|\lambda^0_{\pi-(\mathbf{k})}|^2} \) so the system is always gapped if \( \Delta \neq 0 \) and has a gap that is \( \geq \Delta \). The eigenvectors are,

\[
\begin{align*}
    d_{\mathbf{p},+} & = \frac{1}{\sqrt{2\pi}} \frac{e^{i\phi} + e^{i\theta(\mathbf{p})}}{\sqrt{-\Delta^2}} c_{\mathbf{p},+} + \frac{e^{i\phi} + e^{i\theta(\mathbf{p})}}{\sqrt{2\pi}} c_{\mathbf{p},-} \\
    d_{\mathbf{p},-} & = -e^{-i\phi} \frac{1}{\sqrt{2\pi}} \frac{e^{i\phi} + e^{i\theta(\mathbf{p})}}{\sqrt{-\Delta^2}} c_{\mathbf{p},+} + \frac{e^{i\phi} + e^{i\theta(\mathbf{p})}}{\sqrt{2\pi}} c_{\mathbf{p},-},
\end{align*}
\] (22)

where \( d_{\mathbf{p},\pm} \) are the annihilation operators for eigenstates in \( \pm \epsilon_k \) energy bands and \( \theta(\mathbf{p}) = \text{Arg}[\lambda^0_{\pi-(\mathbf{k})}] \). We choose the phase factors such that in the limit of \( V_1 \to 0 \), \( d_{\mathbf{p},\pm} \to c_{\mathbf{p},\pm} \). Since the \( V_1 \) term does not close the gap, we expect the \( d_{\mathbf{k},\pm} \) bands to have the same Chern number as the \( c_{\mathbf{k},\pm} \) bands. Note that \( \lambda^0_{\pi-(\mathbf{k})} = 0 \) at \( (k_x, k_y) = (\pm \pi, \pm \pi) \) and \( \theta(\mathbf{k}) \) is not well-defined at these singular points. However, the factors associated with \( \theta(\mathbf{k}) \) in Eq. 22 vanish at \( (k_x, k_y) = (\pm \pi, \pm \pi) \) so the \( d_{\mathbf{k},\pm} \) fields can be continuously defined in the whole BZ.

If \( V_1 = 0 \), \( c_{\mathbf{k},+} \) and \( c_{\mathbf{k},-} \) bands are decoupled and each one of the bands is a flat band with QP. As we discussed before, the spectrum of each band has fractal structure and the width of the spectrum is of the order of \( V_0 \). Moreover, from Eq. 20, after a partial Fourier transformation along \( k_x \), we find that the \( V_0 \) terms are the same for \( c_{\mathbf{k},+} \)'s and \( c_{\mathbf{k},-} \)'s. Thus the energy spectra of \( \pm \epsilon_k \) bands are identical and the \( "+" \) bands are shifted with an energy \( \Delta \) from the \( "-" \) bands, with the same corresponding energy eigenstates. If \( V_0 \ll \Delta \), there is a band gap between the two “fractal” bands that consist of \( c_{\mathbf{k},\pm} \) degrees of freedom respectively, and we get a Chern insulator at half filling. If \( V_0 \sim \Delta \), the gap at half filling will close. If \( V_0 \gg \Delta \), the fractal bands contributing positive Hall conductivity and negative Hall conductivity almost overlap, resulting in nearly zero Hall conductivity.

Now we take both \( V_0 \) and \( V_1 \) into account. If we fix \( V_1 \) and \( \Delta \) and increase \( V_0 \), the band gap decreases and eventually vanishes. We further calculate the IPR (see Appendix D for details). We find that there are extended states (in real space) near band edges when the band gap is not closed. Upon increasing \( V_0 \), after the band gap closes, there are localized states near zero energy. Similar “leviation” and “pair annihilation” behavior of the extend states is also observed in disordered topological insulators [44, 45]. If the strength of QP potential further increases, the states near zero energy get de-localized since in the \( V_0 \gg V_1 \) limit, the model reduces to two decoupled AA models at critical points.

VI. CONCLUSION

In this paper we showed that when magic-angle twisted bilayer graphene is nearly aligned with h-BN, the single particle physics is sensitive to the quasi-periodic potential produced by the interference between two moiré potentials: one produced by the relative twist of the two graphene layers, and the other produced by the h-BN substrate. The periodic modulation induced by h-BN cannot be treated as a small perturbation due to the narrow bandwidth of the valence and conduction bands. By exact diagonalization, we find that for TBLG twist angle 1.2°, for alignment angle \( \theta(BN) = 0° \) and \( \theta(BN) = 0.8° \), localized states and extend states are both present and there is no clear mobility edge. For \( \theta(BN) = 0° \), the charge gap near neutrality is closed. In the presence of valley polarization (due to interactions), the Hall conductivity \( \sigma_{xy} \) is not quantized when \( \theta(BN) = 0° \) while for \( \theta(BN) = 0.8° \), the charge gap is reduced and \( \sigma_{xy} \) is quantized.

In order to study the electron properties of topological bands in the presence of quasi-periodic potential, it is more straightforward to begin with a model in momentum space since the non-triviality is manifest in the form factor. In the limit of flat band and uniform Berry curvature, we find that quasi-periodic potential induces hopping between different momentum, which can be mapped to a tight-binding model coupled to magnetic field. We discussed localization properties and transport in such toy models. The next step will be to introduce dispersion and electron-electron interaction, which we leave for future studies.

VII. ACKNOWLEDGEMENT

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Appendix A: Numerical results for $\theta_G = 1.15^\circ$

1. Perfect Alignment

We plot the dispersion in Fig.11 and berry curvature distribution of the valence band in Fig.12.
2. Incommensurate Alignment

We study the density of states and PR for $\theta_G = 1.15^\circ$ and $\theta_{BN} = -0.6^\circ$, which is close to perfect alignment. Indeed we find a clear gap near charge neutrality and there is no indication of localization from PR. (See Fig.13)

Appendix B: Minimal coupling in a topological band

The gauge transformation operator can be written as,

$$U = e^{i \int_q \theta(q, \tau) \rho(q)}.$$  (B1)
where $\rho(\vec{q})$ is the projected density operator in momentum space,

$$
\rho(\vec{q}) = \int \frac{d^2k}{(2\pi)^2} \hat{c}_k^\dagger \hat{c}_k \lambda(\vec{k}, \vec{q}).
$$

(B2)

$\lambda(\vec{k}, \vec{q}) = \langle u_k | u_{-\vec{k} - \vec{q}} \rangle$ is the form factor (note that the definition is different from the main text). And $\theta(\vec{q}, \tau)$ is the Fourier transform of a real function $\theta(\tau, \tau)$, so $\theta(\vec{q}, \tau)^* = \theta(-\vec{q}, \tau)$.

For an infinitesimal gauge transformation, $U \approx 1 + i \int_{\vec{q}} \theta(\vec{q}, \tau) \rho(\vec{q})$ and an operator $\hat{O} \rightarrow U^\dagger \hat{O} U$ under gauge transformation. Thus we have,

$$
U^\dagger \hat{c}_k U \approx \hat{c}_k + i \int_{\vec{q}} \theta(\vec{q}, \tau) \lambda(\vec{k} + \vec{q}, \vec{q}) c_{k+q}^\dagger
$$

$$
U^\dagger \hat{c}_k U \approx \hat{c}_k - i \int_{\vec{q}} \theta(\vec{q}, \tau) \lambda(\vec{k}, \vec{q}) c_{k-q}^\dagger
$$

(B3)

Consider an action in Euclidean signature that contains three terms, $S_0 = \int d^2k \int d\tau c_k^\dagger \partial_\tau c_k - \epsilon_k c_k^\dagger c_k - \sum_{\vec{Q}} V_{\vec{Q}} \rho_{\vec{Q}}$, where $\vec{Q}$ does not need to be commensurate with the reciprocal lattice vector. Performing a gauge transformation to these terms, we have,

$$
\delta(\int c_k^\dagger \partial_\tau c_k) \approx -i \int_{\vec{k}, \vec{q}} \theta(\vec{q}, \tau) \lambda(\vec{k}, \vec{q}) c_{k-q}^\dagger \partial_\tau c_k + i \int_{\vec{k}, \vec{q}} c_k^\dagger \partial_\tau [\theta(\vec{q}, \tau) \lambda(\vec{k} + \vec{q}, \vec{q}) c_{k+q}]
$$

$$
\delta(\int \epsilon_k c_k^\dagger c_k) \approx -i \int_{\vec{k}, \vec{q}} \theta(\vec{q}, \tau) \lambda(\vec{k}, \vec{q}) c_{k-q}^\dagger \epsilon_k c_k + i \int_{\vec{k}, \vec{q}} c_k^\dagger \epsilon_k \theta(\vec{q}, \tau) \lambda(\vec{k} + \vec{q}, \vec{q}) c_{k+q}
$$

$$
\delta(V_{\vec{Q}} \rho_{\vec{Q}}) \approx -i V_{\vec{Q}} \int_{\vec{k}, \vec{q}} \lambda(\vec{k}, \vec{Q}) \theta(\vec{q}, \tau) \lambda(\vec{k} - \vec{Q}, \vec{q}) c_{k-q}^\dagger c_k + i V_{\vec{Q}} \int_{\vec{k}, \vec{q}} \lambda(\vec{k}, \vec{Q}) c_{k-q}^\dagger \theta(\vec{q}, \tau) \lambda(\vec{k} + \vec{q}, \vec{q}) c_{k+q}
$$

$$
\approx i V_{\vec{Q}} \int_{\vec{k}, \vec{q}} \theta(\vec{q}, \tau) c_{k-q}^\dagger [\lambda(\vec{k} - \vec{Q}, \vec{Q}) \lambda(\vec{k}, \vec{q}) - \lambda(\vec{k}, \vec{Q}) \lambda(\vec{k} - \vec{Q}, \vec{q})]
$$

(B4)

If we further consider long wave-length gauge transformation, we only need to take small $\vec{q}$ in $\theta(\vec{q}, \tau)$ into account. Note that $\partial_k \epsilon_k = \bar{v}_k$ and

$$
\lambda(\vec{k} - \vec{q}, \vec{Q}) \lambda(\vec{k}, \vec{q}) - \lambda(\vec{k}, \vec{Q}) \lambda(\vec{k} - \vec{q}, \vec{q}) = -\frac{\bar{v}_k}{\epsilon_k} \lambda(\vec{k}, \vec{Q}) + i \lambda(\vec{k}, \vec{Q}) (\bar{A}_k - \bar{A}_{k-q}) + o(q^2),
$$

(B5)

where $\bar{A}_k = -i \langle u_{\vec{k}} | \partial_{\vec{k}} u_{\vec{k}} \rangle$ is the Berry connection in momentum space.

Thus, the change in action $S_0$ can be written as,

$$
\delta S_0 = \int d\tau \int_{\vec{k}, \vec{q}} (i \partial_\tau \theta(\vec{q}, \tau) + i \bar{v}_k \theta(\vec{q}, \tau)) \lambda(\vec{k}, \vec{q}) c_{k-q}^\dagger c_k
$$

$$
+ \sum_{\vec{Q}} i V_{\vec{Q}} \int_{\vec{k}, \vec{q}} \theta(\vec{q}, \tau) c_{k-q-Q}^\dagger c_k [\lambda(\vec{k}, \vec{Q} + \vec{q}) + i \lambda(\vec{k}, \vec{Q} + \vec{q}) (\bar{A}_k - \bar{A}_{k-q-Q})] + o(q^2)
$$

(B6)

Now let us consider the electromagnetic potential $A(\vec{q}, \tau) = (A_0, \bar{A})$. Note that the projection to the topological band should only affect the gauge transformation of the projected degrees of freedom and the gauge transformation of the electromagnetic potential should remain the same as before the projection. Thus under a gauge transformation, we have,

$$
A_0(\vec{q}, \tau) \rightarrow A_0(\vec{q}, \tau) - \partial_\tau \theta(\vec{q}, \tau)
$$

$$
\bar{A}(\vec{q}, \tau) \rightarrow \bar{A}(\vec{q}, \tau) + i \bar{v}_k \theta(\vec{q}, \tau).
$$

(B7)
The goal is to construct terms involving the electromagnetic potential $A(q', \tau)$ such that $\delta S_0$ can be cancelled by the gauge transformation of the $A(q', \tau)$ field. As a first attempt, we consider the following action,

\[
S_1 = \int d\tau \int \left( i A_0(q', \tau) - \tilde{A}(q', \tau) \cdot \vec{v}_k \right) \lambda(\vec{k}, q) c_{k,q}^\dagger c_{k,q}^+ + \int d\tau \int \tilde{A}(q', \tau) \cdot \sum_q V_q c_{k,q}^\dagger c_{k,q} \left[ \delta_{k,q} \lambda(\vec{k}, q) + i \lambda(\vec{k}, q + \vec{q})(A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}}) \right]. \tag{B8}
\]

Note that $\delta S_0$ is cancelled by terms in $\delta S_1$ but there are other terms in $\delta S_1$ so we have,

\[
\delta S_1 = -\delta S_0 + \delta S'_1 \tag{B9}
\]

and

\[
- i \delta S'_1 = \int d\tau \int \left( i A_0(q', \tau) - \tilde{A}(q', \tau) \cdot \vec{v}_k \right) \lambda(\vec{k}, q) [-c_{k,q}^\dagger c_{k,q} \theta(q', \tau) \lambda(\vec{k} - \vec{q}, q') + c_{k,q}^\dagger c_{k,q}^+ \theta(q', \tau) \lambda(\vec{k} + \vec{q}, q')] - \int d\tau \int \tilde{A}(q', \tau) \cdot \sum_q V_q \left[ \delta_{k,q} \lambda(\vec{k}, q) + q + \vec{q})(A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}}) \right] \times [-c_{k,q}^\dagger c_{k,q} \theta(q', \tau) \lambda(\vec{k} - \vec{q} - \vec{q}) + c_{k,q}^\dagger c_{k,q}^+ \theta(q', \tau) \lambda(\vec{k} + \vec{q}, q')] = \int d\tau \int \left( i A_0(q', \tau) c_{k,q}^\dagger c_{k,q} \theta(q', \tau) \lambda(\vec{k}, q) \right) + e^{i \int \tilde{A}(q', \tau) \cdot \sum_q V_q c_{k,q}^\dagger c_{k,q} \theta(q', \tau) \lambda(\vec{k}, q) + \frac{1}{2} \lambda(\vec{k} - \vec{q}, \vec{q}) \left( A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}} \right) \lambda(\vec{k}, q)]
\]

\[
\approx - \int d\tau \int \left( i A_0(q', \tau) - \tilde{A}(q', \tau) \cdot \vec{v}_k c_{k,q}^\dagger c_{k,q} \theta(q', \tau) \right) \lambda(\vec{k} - \vec{q}, q') + \tilde{A}(q', \tau) \lambda(\vec{k}, q) + i \lambda(\vec{k}, q)(A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}})]
\]

\[
+ \int d\tau \int \tilde{A}(q', \tau) \cdot \sum_q V_q c_{k,q}^\dagger c_{k,q} \theta(q', \tau) \lambda(\vec{k} - \vec{q}, q') - \delta_{k,q} \lambda(\vec{k}, q + \vec{q})(A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}})] + \epsilon((q')^2)
\]

\[
\approx i \int d\tau \int \left( i A_0(q', \tau) - \tilde{A}(q', \tau) \cdot \vec{v}_k c_{k,q}^\dagger c_{k,q} \lambda(\vec{k} - \vec{q}, q') + \frac{1}{2} \lambda(\vec{k}, q + \vec{q})(A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}})] \right) \times (q' \cdot \frac{\partial}{\partial \vec{k}})
\]

\[
+ \int d\tau \int \tilde{A}(q', \tau) \cdot \sum_q V_q c_{k,q}^\dagger c_{k,q} \theta(q', \tau) \lambda(\vec{k}, q + \vec{q}) (A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}})]
\]

\[
\times \left[ \tilde{A}(q', \tau) \left( -i \frac{\partial}{\partial \vec{k}} + \frac{1}{2} \lambda(\vec{k}, q + \vec{q}) (A_{\vec{k}} - \tilde{A}_{\vec{k} - \vec{q} - \vec{q}})] \right) \right]
\]

where $B_k = \partial_{k_x} A_{k,y} - \partial_{k_y} A_{k,x}$ is the Berry curvature.
Now, let us further consider the possible terms that cancel $\delta S_1^i$. Suppose there is $S_2$, and

$$-iS_2 = - \int d\tau \int_{\vec{k}, \vec{q}, \vec{q}'} iA_0(\vec{q}, \tau)(\vec{A}(\vec{q}, \tau) \times \vec{q}) \cdot \hat{z}c_{\vec{k}-\vec{q}}^\dagger c_{\vec{k}}\lambda(\vec{k}, \vec{q} + \vec{q}')B_{\vec{k}}^i$$

$$+ \int d\tau \int_{\vec{k}, \vec{q}, \vec{q}'} (1/2(\partial_\tau \vec{A}(\vec{q}, \tau) \times \vec{A}(\vec{q}', \tau))) \cdot \hat{z}c_{\vec{k}-\vec{q}}^\dagger c_{\vec{k}}\lambda(\vec{k}, \vec{q} + \vec{q}')B_{\vec{k}}^i$$

$$+ \int d\tau \int_{\vec{k}, \vec{q}, \vec{q}'} A_\mu(\vec{q}, \tau)c^{\dagger}_{\vec{k}-\vec{q}}c_{\vec{k}}\lambda(\vec{k}, \vec{q} + \vec{q}')/2 + (\vec{A}(\vec{q}, \tau) \cdot \vec{v}_k)(\vec{A}(\vec{q}', \tau) \times \vec{q}) \cdot \hat{z}c_{\vec{k}-\vec{q}}^\dagger c_{\vec{k}}\lambda(\vec{k}, \vec{q} + \vec{q}')B_{\vec{k}}^i$$

$$+ \frac{i}{2} \int d\tau \int_{\vec{k}, \vec{q}, \vec{q}'} V_{\vec{q}}c_{\vec{k}-\vec{q}}^\dagger c_{\vec{k}}\lambda(\vec{q}, \tau) \cdot [-i\partial_k + \vec{A}_k - \vec{A}_{\vec{k}+\vec{q}}] \lambda(\vec{k}, \vec{q} + \vec{q}')$$

(B11)

One can verify that $\delta S_2$ cancels $\delta S_1^i$ for small $\vec{q}$ and $\vec{q}'$. The first two terms in Eq.B11 can be combined to a Chern-Simons(CS) term. To see this, define $\phi_B(\vec{q}) = \int_{\vec{k}} c_{\vec{k}-\vec{q}}^\dagger c_{\vec{k}}\lambda(\vec{k}, \vec{q})B_{\vec{k}}^i$ and in real space, the first two terms in Eq.B11 reduce to,

$$\frac{1}{2} \int d\tau \int d^2x \phi_B(-\vec{x})A(\vec{x}, \tau)dA(\vec{x}, \tau),$$

(B12)

where $\phi_B(\vec{x})$ is the Fourier transformation of $\phi_B(\vec{q})$.

As a sanity check, consider a Chern insulator. The ground state expectation value of $\phi_B(\vec{x})$ is $\langle \phi_B(\vec{x}) \rangle_{G.S.} = \frac{e^2}{\pi}$, where $C$ is the Chern number. This gives the correct quantized coefficient for the CS term.

Thus, at long wavelength, the action for a topological band that is minimally coupled to gauge field is,

$$S[A] = S_0 + S_1[A] + S_2[A]$$

(B13)

The current density operator is $J^\mu = \frac{\delta S_2}{\delta A^\mu}$. Let us consider applying an external static electric field and choose a gauge such that $\vec{E} = -\nabla A_0(\vec{x}, t)$ and $\vec{A}(\vec{x}, t) = 0$. The current density operator around $\vec{q} = 0$ is thus,

$$J^\mu (\vec{q} = 0) \approx - \int_{\vec{k}} \epsilon^{\mu \nu} E^\nu c_{\vec{k}}^\dagger c_{\vec{k}}\frac{\delta}{\delta A^\mu} + \vec{A}_k - \vec{A}_{\vec{k}+\vec{q}}) \lambda(\vec{k}, \vec{q} + \vec{q})$$

(B14)

For a flat topological band, $\vec{v}_k = 0$ so the total current $\vec{J}$ is,

$$\vec{J} = J^\mu (\vec{q} = 0) = -\epsilon^{\mu \nu} \sum_{\vec{k}} c_{\vec{k}}^\dagger c_{\vec{k}}B_{\vec{k}}E^\nu - i \sum_{\vec{q}} V_{\vec{q}} \sum_{\vec{k}} c_{\vec{k}-\vec{q}}^\dagger c_{\vec{k}}[-i\partial_k + \vec{A}_k - \vec{A}_{\vec{k}+\vec{q}}) \lambda(\vec{k}, \vec{Q})]$$

(B15)

In the main text, we take $\lambda(\vec{k}, \vec{Q}) = e^{i\int_{\vec{k}}^{\vec{k}+\vec{Q}} \vec{A}}$ so we have,

$$[-i\partial_k + \vec{A}_k - \vec{A}_{\vec{k}+\vec{q}}) \lambda(\vec{k}, \vec{Q}) = \lambda(\vec{k}, \vec{Q}) \int_{\vec{k}}^{\vec{k}+\vec{Q}} B(\vec{k})e^{i\mu k}d\vec{k}.$$  

(B16)

If the Berry curvature is uniform, the above expression will reduce to $-\epsilon^{\mu \nu} Q_\nu \lambda(\vec{k}, \vec{Q})B$ so we find that Eq.B15 agrees with Eq.18 in the main text.

To conclude, in the derivation of Eq.B15 we don’t assume any specific form of $\lambda(\vec{k}, \vec{Q})$ so the expression of the current operator can be used in any topological band with non-trivial Berry curvature.

Appendix C: Bloch wave function in LLL and form factors

In a Landau gauge $\vec{A} = (-By, 0)$, the magnetic translation operator are,

$$\begin{align*}
T_x &= e^{iP_x} \\
T_y &= e^{i(P_y + Bz)}.
\end{align*}$$

(C1)
One can verify that \([T_x, T_y] = 0\) and \(T_x, T_y\) commute with the kinetic momenta \(\vec{p} - \vec{A}\), since the magnetic flux \(\Phi = B = 2\pi\), where we set the lattice constant to 1.

The eigenfunction \(\psi(\vec{x})\) of \(T_{x,y}\) can be labeled by the momenta \((k_x, k_y)\) such that the eigenvalues are \(e^{ik_x y}\). In order to construct such eigenfunctions, we first examine how \(T_{x,y}\) act on a wave function in LLL, that is \(\phi_{k_x}(\vec{x}) = e^{ik_x y} \Psi_0(0 + k_x/B)\), where \(\Psi_0(y) = (\frac{B}{\pi})^{\frac{1}{2}} e^{-\frac{B y^2}{2}}\). It is readily seen that \(\phi_{k_x}(\vec{x})\) is an eigenfunction of \(T_x\) and \(T_y\phi_{k_x}(\vec{x}) = \phi_{k_x+B}(\vec{x})\). Thus the eigenfunction of \(T_{x,y}\) can be written as,

\[\psi_{k}(\vec{x}) = \sum_{m=-\infty}^{\infty} e^{-imk_y - ik_x k_x/B} \phi_{k_x+mB}(\vec{x}) = \sum_{m=-\infty}^{\infty} e^{ik_x y + mB} \phi_{k_x+mB}(\vec{x}) \Psi_0(y + \frac{k_x}{B} + m),\]  

(C2)

where \(m\) is an integer. Note that we choose a gauge in \(\vec{k}\) space such that \(\psi(k_y, k_x+2\pi)(\vec{x}) = \psi(\vec{x})\).

For an opposite magnetic field, we can choose a Landau gauge such that \(\vec{A} = (By, 0)\). Then the wavefunction in the LLL can be written as \(\phi_{k_x}(\vec{x}) = e^{ik_x y} \Psi_0(0 - \frac{k_x}{B})\). And the magnetic translation operators are \(T_x = T_x, T_y = e^{i(P_y - B)}\) so the corresponding eigenfunction can be written as,

\[\psi_{k}(\vec{x}) = \sum_{m=-\infty}^{\infty} e^{imk_y + ik_x k_x/B} \phi_{k_x+mB}(\vec{x}) = \sum_{m=-\infty}^{\infty} e^{ik_x y + mB} \phi_{k_x+mB}(\vec{x}) \Psi_0(y - \frac{k_x}{B} - m).\]  

(C3)

Now let us consider various form factors,

\[\lambda_{\pm,\pm}(\vec{k}, \vec{Q}) = \sum_{m, m'} \int d\gamma e^{\mp(iQ_m + k_y m_y + k_x m_x) / B - i(mB - m'B)x} e^{i\gamma} \Psi_0(0, \vec{x} + \vec{Q}) \Psi_0(0, \vec{x} - \vec{Q})\]

\[\times \Psi_0(0, \vec{x} + \vec{Q}) \Psi_0(0, \vec{x} - \vec{Q}) \Psi_0(0, \vec{x} + \vec{Q}) \Psi_0(0, \vec{x} - \vec{Q})\]

\[= \sum_{m} \int d\gamma e^{\mp(iQ_m + k_y m_y + k_x m_x) / B - i(mB - m'B)x} e^{i\gamma} \Psi_0(0, \vec{x} + \vec{Q}) \Psi_0(0, \vec{x} - \vec{Q})\]

\[= \sum_{m} \int d\gamma e^{\mp(iQ_m + k_y m_y + k_x m_x) / B - i(mB - m'B)x} e^{i\gamma} \Psi_0(0, \vec{x} + \vec{Q}) \Psi_0(0, \vec{x} - \vec{Q})\]

\[= e^{\pm \frac{iQ_x k_x + Q_y k_y}{2B} - \frac{Q_x^2 + Q_y^2}{4m}}.\]

(C4)

Now let us consider \(\lambda_{0, -}(\vec{k}) = \langle \psi_{k_x, -} | \psi_{k_x, -} \rangle\).

\[\lambda_{0, -}(\vec{k}) = \sum_{m} \int d\gamma e^{i2k_x m_x + 2k_y m_y / B} \Psi_0(0, \vec{x} + \frac{k_x + mB}{B}) \Psi_0(0, \vec{x} - \frac{k_x}{B} + mB)\]

\[= \sum_{m} \int d\gamma e^{i2k_x m_x + 2k_y m_y / B} \Psi_0(0, \vec{x} + \frac{k_x + mB}{B}) \Psi_0(0, \vec{x} - \frac{k_x}{B} + mB)\]

\[= \sum_{m} e^{i2k_y m_y / B} \Psi_0(0, \vec{x} + \frac{k_x + mB}{B}) \Psi_0(0, \vec{x} - \frac{k_x}{B} + mB)\]

(C5)

Appendix D: Inverse participation ratio of various systems

1. IPR of single band models

We consider the Hamiltonian in Eq.10 with \(V_1 = V_2 = V_3 = V\) and further include a dispersion term \(H_{dis} = \sum_k c_k^\dagger \epsilon_k c_k\), where we take \(\epsilon_k = t \sum_{j=1}^{3} \cos (k \cdot \vec{b}_j)\). \(\vec{b}_j\)'s are the lattice vectors, taken to be \(\vec{b}_1 = \frac{4\pi Q}{\sqrt{3}} (0, 1)\) and \(\vec{b}_{2,3} = \frac{4\pi Q}{\sqrt{3}} (\pm \sqrt{3}, 1/2)\). We take the QP lattice to be parallel to the original lattice and \(Q\) is an irrational number. We calculate the IPR for various QP potential strength \(V\) and system sizes (Fig. 14) give \(t = 1\) and \(Q = \sqrt{5} - 1\). We compare the system with vanishing Berry curvature (trivial) and uniform Berry curvature given by a \(C = 1\) band
(non-trivial). We find that in both cases, IPR has a stronger system size dependence with increasing QP potential, which indicates that there are more extended states in $\vec{k}$-space. Moreover, we fit the dependence of $\log(\text{IPR})$ to the logarithm of linear system size $L$ and find a linear dependence, $\log(\text{IPR}) = z \log(L) + b$. And the trivial model has a slope $z \approx -1.88$, while for the non-trivial model, $z \approx -1.37$. The slope of the trivial model is closer to the ideal scaling $z = 2$, which indicates that there are more localized states in real space in the trivial model than in the non-trivial model with large QP potential.

We further consider a non-uniform Berry curvature in $\vec{k}$-space with $B(\vec{k}) = B_0 + B_1 \sum_{j=1}^{3} \cos (\vec{k} \cdot \vec{b}_j)$, where $B_0$ is the uniform part as considered before. The $B_1$ term acts as a QP hopping term. We calculate the IPR with trivial and non-trivial $B_0$ and increasing $B_1$ (Fig.14). For the trivial case, we find $z \approx -1.92$. For the non-trivial case, at small value of $B_1$, $z \approx -1.77$ and $z$ decreases with increasing $B_1$, $z \approx 1.12$ within $2 \leq B_1 \leq 4$. The above results indicate that in the presence of non-uniform Berry curvature, there are more non-localized states (extended or critical) in real space in the non-trivial model compared to the trivial model.

![Image of IPR plots](image_url)

**FIG. 14:** IPR for single band model with dispersion. (a),(b): trivial case. (c),(d): non-trivial case. In (a)(b), $\log(\text{IPR})$ is plotted, x-axis labels different eigenstates in ascending order of their energies and y-axis labels different QP potential $V$ from 0 to 4. The system size is $60 \times 60$. In (b)(d), green dots with different brightness connected by the same line label different QP potential $V$, from 0 to 4 (top to bottom), with 0.2 interval. The black line is the linear fitting for $V = 4$. The linear system sizes are from 20 to 60.
FIG. 15: IPR for single band model with non-uniform Berry curvature. (a): trivial($B_0 = 0$) and (b): non-trivial($B_0 \approx 7.26$). Green dots with different brightness connected by the same line label different $B_1$, from 0 to 4 (top to bottom), with 0.2 interval. The black line in (a) is the linear fitting of the average IPR. The black line in (b) is the linear fitting of the average IPR for $B_1 \geq 2$ and the blue line in (b) is the linear fitting for $B_1 = 0.2$.

2. IPR of the two band model

We consider the Hamiltonian in Eq.20. We let $V_1 = \Delta = 1$ and change $V_0$. The IPR’s for various $V_0$ are shown in Fig.16. We find that the slope of Log(IPR)-Log(L) curve goes from $z = 0$ to $z \approx -1.58$ when $V_0$ is increased from 0 to 1. At small $V_0$, there is a band gap in the middle of the spectrum. When $V_0/\Delta \gtrapprox 0.2$, the gap closes and the IPR values in the middle of the spectrum get larger after the gap closes, which indicates that there are extended states in the middle of the spectrum and they get localized with increasing $V_0$. We study the IPR of the states that are in the middle of the spectrum (Fig.17) and find that there is an intermediate regime of $V_0$ where the IPRs of the states can be fit to a power law dependence to the linear system size with a power $z \approx -1.97$. Thus, the states in the middle of the spectrum are indeed localized in real space.

FIG. 16: IPR for the two band model. (a) Color plot of Log(IPR). X-axis labels different eigenvalues. (b) Dependence of Log(IPR) with linear system size. The green lines from bright to dark labels different $V_0$ from 0 to 1. The black line is the linear fitting to the Log(IPR) at $V_0 = 1$. 
FIG. 17: IPR of the states that is in the middle of the spectrum. The x-axis is \( \log(L) \) and the y-axis is \( \log(\text{IPR}) \) for (a) small \( V_0 \), (b) intermediate \( V_0 \) and (c) large \( V_0 \).