Operator Product Expansion and Topological States in $c = 1$ Matter Coupled to 2-D Gravity

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Abstract

Factorization of the $N$-tachyon amplitudes in two-dimensional $c = 1$ quantum gravity is studied by means of the operator product expansion of vertex operators after the Liouville zero mode integration. Short-distance singularities between two tachyons with opposite chiralities account for all singularities in the $N$-tachyon amplitudes. Although the factorization is valid, other possible short-distance singularities corresponding to other combinations of vertex operators are absent since the residue vanishes. Apart from the tachyon states, there are infinitely many topological states contributing to the intermediate states. This is a more detailed account of our short communication on the factorization.
§1 Introduction

Recent studies in matrix models\textsuperscript{1,2} have made significant progress in the nonperturbative treatment of two-dimensional quantum gravity and string theory. To make further progress, it is important to clarify these nonperturbative results of matrix models from the viewpoint of the usual continuum approach of the two-dimensional quantum gravity. The most standard way of treating the continuum theory of quantum gravity in two-dimensions is the so-called Liouville theory.\textsuperscript{3–6} Several works have recently appeared to compute the correlation functions on the sphere topology using the Liouville theory and the technique of analytic continuation.\textsuperscript{7–12} These results are consistent with those of matrix models.\textsuperscript{13–15} So far only conformal field theories with central charge $c \leq 1$ have been successfully coupled to quantum gravity.

In a recent publication,\textsuperscript{16} we have reported the result of the factorization analysis of the $c = 1$ quantum gravity. The purpose of this paper is to give a more detailed account of the analysis for understanding the factorization of $c = 1$ quantum gravity in terms of the short-distance singularities arising from the operator product expansion (OPE) of vertex operators.

The $c = 1$ case is the richest and the most interesting. It has been observed that this theory can be regarded effectively as a critical string theory in two dimensions, since the Liouville field zero mode provides an additional “time-like” dimension besides the obvious single spatial dimension given by the zero mode of the $c = 1$ matter.\textsuperscript{17} We have a physical scalar particle corresponding to the center of mass motion of the string. Though it is massless, it is still referred to as a “tachyon” following the usual terminology borrowed from the critical string theory. Since there are no transverse directions, the continuous (field) degrees of freedom are exhausted by the tachyon field. In fact, the partition function for the torus topology was computed in the Liouville theory, and was found to give precisely the same partition function as the tachyon field alone.\textsuperscript{18,19} However, there are indications of the existence of other discrete degrees of freedom in the $c = 1$ quantum
gravity. Firstly, the correlation functions obtained in the matrix model exhibit a characteristic singularity structure.\textsuperscript{14} In the continuum approach of the Liouville theory, Polyakov has observed that special states with discrete momenta and Liouville energies can produce such poles, and has called these operators co-dimension two operators.\textsuperscript{20} Moreover the two-loop partition function has been computed in a matrix model and evidence has been noted for the occurrence of these topological states.\textsuperscript{21} Most recently, the symmetry governing such topological states are beginning to be understood.\textsuperscript{22,23} It is clearly of vital importance to pin down the role played by these topological states as much as possible. In the critical string theory, the particle content of the theory and unitarity have been most clearly revealed through the factorization analysis of scattering amplitudes. On the other hand, the factorization and unitarity of the Liouville theory have not yet been well understood.

Since we are interested in the short distance singularities, we consider correlation functions on a sphere topology only. We find that the singularities of the amplitudes can be understood as short-distance singularities of two vertex operators of tachyons with opposite chiralities. We also find that the other possible short-distance singularities corresponding to other combinations of tachyons are absent since the residue vanishes. It is found that infinitely many discrete states contribute to the intermediate states of the factorized amplitudes, apart from the tachyon states. These are the co-dimension two operators of Polyakov \textsuperscript{20} and presumably are topological in origin. We have also explicitly constructed some of the topological states.

Recently we have received two papers where other groups have discussed the subject partly overlapping with ours.\textsuperscript{24,25} Their results are consistent with ours. In particular, the paper by Di Francesco and Kutasov has shown the decoupling of various topological states in certain kinematical configurations which was only partially demonstrated in our analysis.
§2 Correlation Functions on a Sphere

In this paper we consider \( c = 1 \) conformal matter realized by a single bosonic field (string variable) \( X \) coupled to two-dimensional quantum gravity

\[
S_{\text{matter}}[g, X] = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{g} g^{\alpha\beta} \partial_\alpha X \partial_\beta X,
\]
\[
Z = \int \mathcal{D}g_{\alpha\beta} \mathcal{D}X \frac{1}{V_{\text{gauge}}} e^{-S_{\text{matter}}[g, X]},
\]

(2.1)

where \( \alpha' \) is the “Regge slope parameter” and \( V_{\text{gauge}} \) is the volume of the group of diffeomorphisms. This is the so-called “noncritical string” in one dimension. We use the method of Ref. 3). In this approach the metric is parametrized by a diffeomorphism \( f \) and the Liouville field \( \phi \) representing the freedom of local Weyl rescaling

\[
f^* g_{\alpha\beta} = e^{\phi} \hat{g}_{\alpha\beta}(\tau),
\]

(2.2)

where \( \hat{g}_{\alpha\beta}(\tau) \) is a reference metric that depends on the moduli parameters \( \tau \) of the Riemann surface. The diffeomorphism invariance allows us to choose the conformal gauge. On a sphere there is no moduli and the gauge condition is

\[
g_{\alpha\beta} = e^{\phi} \hat{g}_{\alpha\beta}.
\]

(2.3)

In the following we shall set \( \alpha' = 2 \) for convenience. The matter action in the conformal gauge (2.3) is

\[
S_{\text{matter}}[\hat{g}, X] = \frac{1}{2\pi} \int d^2z \partial X \bar{\partial} X,
\]

(2.4)

where we have used a holomorphic variable \( z = \xi^1 + i\xi^2 \) and an anti-holomorphic variable \( \bar{z} = \xi^1 - i\xi^2 \). If we want to consider \( c < 1 \) case, we need to introduce additional terms with a parameter \( \alpha_0 = -\sqrt{(1-c)/12} \)

\[
S_{\text{matter},\alpha_0}[\hat{g}, X] = \frac{\imath \alpha_0}{4\pi} \int d^2z \sqrt{\hat{g}} \hat{R} X + \frac{\imath \alpha_0}{2\pi} \int ds \hat{k} X,
\]

(2.5)

where \( \hat{R} \) is the curvature of the two-dimensional surface with the metric \( \hat{g}_{\alpha\beta} \) and \( \hat{k} \) is the geodesic curvature along the boundary of the surface parametrized by \( s \).
After converting the Liouville field path integral measure into the translationally invariant measure for a usual scalar field,\(^3\) we can treat the Liouville field almost as a free field except for the nontrivial dynamics of the zero mode due to the cosmological term with the cosmological constant \(\mu\).\(^4\),\(^5\) The Liouville action is given by the following action involving parameters \(Q\), \(\alpha\) and \(\mu\)\(^3\)–\(^6\):

\[
S_{L}[\hat{g}, \phi] = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \left( \hat{g}^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi - Q \hat{R} \phi + 8\mu e^{\alpha\phi} \right) - \frac{Q}{4\pi} \int ds \hat{k} \phi
\]

\[
= \frac{1}{2\pi} \int d^2z \left( \partial_{\phi} \bar{\partial}_{\phi} \phi - \frac{Q}{4} \hat{R} \phi + 2\mu e^{\alpha\phi} \right) - \frac{Q}{4\pi} \int ds \hat{k} \phi. \tag{2.6}
\]

The correlation functions of \(X\) and \(\phi\) are given by

\[
\langle X(z, \bar{z})X(w, \bar{w}) \rangle = \langle \phi(z, \bar{z})\phi(w, \bar{w}) \rangle = -\log|z - w|^2. \tag{2.7}
\]

The energy-momentum tensor is given by

\[
T(z) = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}Q \partial^2 \phi. \tag{2.8}
\]

If we want to consider \(c < 1\) case with \(\alpha_0 \neq 0\) in Eq. (2.5), we should add \(i\alpha_0 \partial^2 X\) to \(T(z)\).

The parameter \(Q\) is fixed by requiring\(^3\)–\(^6\) that the theory does not depend on a choice of the reference metric \(\hat{g}_{\alpha\beta}\). One finds in the present case of \(c = 1\)

\[
Q = 2\sqrt{2}. \tag{2.9}
\]

If we want to consider the \(c < 1\) case with \(\alpha_0 \neq 0\) in Eq. (2.5), we should use \(Q = \sqrt{(25 - c)/3} = 2\sqrt{2 + \alpha_0^2}\) instead. The other parameter \(\alpha\) is determined by requiring that the naive cosmological term operator \(e^{\alpha\phi}\) is a primary field of
In the present case of a one-dimensional string \((c = 1)\), one finds

\[
\alpha = -\sqrt{2}.
\]  

(2.11)

If we consider the \(c < 1\) case, we should use \(\alpha_+ = -Q/2 + |\alpha_0|\) instead.

We have introduced a naive cosmological term \(8\mu e^{\alpha\phi}\). Although the naive cosmological term operator \(e^{\alpha\phi}\) has conformal weight \((1, 1)\), it has been argued that the operator does not correspond to a genuine local operator in the case of \(c = 1\) because of the Liouville zero mode path integral.\(^4\) The correct cosmological term operator in the \(c = 1\) case is given by \(\phi^e^{\alpha\phi}\) rather than \(e^{\alpha\phi}\). We can obtain the correlation functions with the correct cosmological term operator \(\mu \phi^e^{\alpha\phi}\) from those with the naive cosmological term operator \(\mu e^{\alpha\phi}\) by the following limiting procedure: we replace \(\alpha\) by \((1 - \epsilon)\alpha\) and \(\mu\) by \(\mu_r/(2\epsilon)\) and take the \(\epsilon \to 0\) limit.\(^8,10\)

This procedure is guaranteed to give the correct result, since the naive cosmological terms decouple from the correlation functions as we will explain later.

The gravitationally dressed tachyon vertex operator \(O_p\) with momentum \(p\) is given by

\[
O_p = \int d^2 z \sqrt{g} e^{ipX} e^{\beta(p)\phi}.
\]  

(2.12)

We see that the Liouville zero mode can be regarded as an “imaginary time” and the exponent \(\beta(p)\) as “energy”. The gravitational dressing of the tachyon vertex operator is determined by requiring that the dressed operator has conformal weight \((1, 1)\):

\[
\frac{1}{2}p^2 - \frac{1}{2} \beta(\beta + Q) = 1.
\]  

(2.13)

There are two solutions for the Liouville energy \(\beta(p) = -Q/2 \pm |p|\). It has been argued\(^4,5\) that the Liouville zero mode path integral is well-defined only if \(\beta >
\(-Q/2\). Hence we choose in the case of \(c = 1\)

\[
\beta(p) = -\sqrt{2} + |p|. \tag{2.14}
\]

For \(c < 1\), \(p^2/2\) in Eq. (2.13) should be replaced by \(p(p - 2\alpha_0)/2\) and the solution becomes \(\beta(p) = -\frac{Q}{2} \pm |p - \alpha_0|\).

The \(N\)-point correlation function of the tachyon vertex operators (2.12) on the sphere without boundary is given by the path integral

\[
\langle O_{p_1} \cdots O_{p_N} \rangle = \int \frac{\mathcal{D}X \mathcal{D}\phi}{V_{SL(2,\mathbb{C})}} \ e^{-S_{\text{matter}}[\hat{g},X] - S_L[\hat{g},\phi]} \ O_{p_1} \cdots O_{p_N}
\]

\[
= \int \prod_{i=1}^{N} \left[ d^2z_i \sqrt{\hat{g}} \right] \frac{1}{V_{SL(2,\mathbb{C})}} \left\langle e^{ip_1X(z_1)} \cdots e^{ip_NX(z_N)} \right\rangle_X
\]

\[
\times \left\langle e^{\beta_1\phi(z_1)} \cdots e^{\beta_N\phi(z_N)} \right\rangle_{\phi}, \tag{2.15}
\]

where \(V_{SL(2,\mathbb{C})}\) is the volume of the \(SL(2,\mathbb{C})\) group which is generated by the conformal Killing vectors on the sphere

\[
V_{SL(2,\mathbb{C})} = \int \frac{d^2z_a d^2z_b d^2z_c}{|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_a|^2}. \tag{2.16}
\]

In Eq. (2.15), we have omitted powers of the string coupling constant \(g_{\text{st}}^{-2}\) for the sphere topology, since we will deal with correlation functions on the sphere only. It is convenient to separate the zero modes in the path integral: the zero mode \(\phi_0\) of the Liouville field \((\phi = \phi_0 + \tilde{\phi})\) and the zero mode \(X_0\) of the matter field \((X = X_0 + \tilde{X})\) and perform the zero mode integrations first. The integration over the zero mode \(\phi_0\) of the Liouville field gives

\[
\left\langle \prod_{i=1}^{N} e^{\beta_i\phi} \right\rangle_{\phi} = \int \mathcal{D}\phi e^{-S_L[\hat{g},\phi]} e^{\beta_1\phi} \cdots e^{\beta_N\phi}
\]

\[
= \frac{1}{\alpha} \Gamma(-s) \int \mathcal{D}\tilde{\phi} e^{-S_L,0[\hat{g},\tilde{\phi}]} \left( \frac{\mu}{\alpha} \int d^2w \sqrt{\hat{g}} e^{\alpha \phi} \right)^s \prod_{i=1}^{N} e^{\beta_i\tilde{\phi}}, \tag{2.17}
\]
where $s$ is given for the sphere as

$$ s = -\frac{1}{\alpha} \left( Q + \sum_{i=1}^{N} \beta_i \right). \quad (2.18) $$

The non-zero mode part of the Liouville action $S_{L,0}$ is given by the free field action. Integration over the matter zero mode $X_0$ gives the momentum conservation

$$ \sum_{j=1}^{N} p_j = 0. \quad (2.19) $$

It is convenient to introduce two-momenta for tachyons $p_j = (p_j, -i\beta_j)$, for the naive cosmological terms $q = (0, -i\alpha)$ and for the source $Q = (0, -iQ)$ respectively. The definition of $s$ together with the momentum conservation allow us to write down “energy-momentum conservation” using two-momenta

$$ \sum_{j=1}^{N} p_j + sq + Q = 0. \quad (2.20) $$

Let us note that, in the case of $c < 1$, the source two-momentum becomes $Q = (-2\alpha_0, -iQ)$ and the momentum conservation is modified to $\sum_{j=1}^{N} p_j = 2\alpha_0$. Hence the two-momentum conservation (2.20) is unchanged. Therefore we obtain the $N$-point correlation function after the zero mode integration

$$ \left\langle \prod_{j=1}^{N} O_{p_j} \right\rangle 
= 2\pi\delta \left( \sum_{j=1}^{N} p_j \right) \frac{\Gamma(-s)}{-\alpha} \int \frac{D\tilde{X}D\tilde{\phi}}{V_{SL(2,\mathbb{C})}} \prod_{j=1}^{N} O_{p_j} \left( \frac{\mu}{\pi} \int d^2w \sqrt{\hat{g}} e^{\alpha\tilde{\phi}(w)} \right)^s e^{-S_0} 
= 2\pi\delta \left( \sum_{j=1}^{N} p_j \right) \frac{\Gamma(-s)}{-\alpha} \int \prod_{i=1}^{N} \left[ d^2z_i \sqrt{\hat{g}} \right] \frac{1}{V_{SL(2,\mathbb{C})}} \left\langle \prod_{j=1}^{N} e^{ip_j\tilde{X}(z_j)} \right\rangle \left\langle \left( \frac{\mu}{\pi} \int d^2w \sqrt{\hat{g}} e^{\alpha\tilde{\phi}(w)} \right)^s \prod_{j=1}^{N} e^{\beta_j\tilde{\phi}(z_j)} \right\rangle_{\tilde{\phi}}, \quad (2.21) $$

The expectation values with $\tilde{\phi}, \tilde{X}$ denote the path integral over the non-zero modes.
\[ \phi, \bar{\phi} \] with the free action

\[ S_0 = \frac{1}{2\pi} \int d^2z \left( \partial \bar{\phi} \partial \phi + \partial \bar{X} \partial X \right). \quad (2.22) \]

For a non-negative integer \( s \), we can evaluate the non-zero mode \( \bar{\phi} \) integral by regarding the amplitude as a scattering amplitude of \( N \)-tachyons and \( s \) naive cosmological terms.

\[ \left\langle N \prod_{i=1}^{N} e^{\beta_i \phi} \right\rangle_{\phi} = \frac{1}{\Gamma(-s)} \left( \frac{\mu}{\pi} \right)^s \int \prod_{j=1}^{s} d^2 w_j \prod_{1 \leq j < k \leq s} |w_j - w_k|^{-2\alpha^2} \]

\[ \times \prod_{1 \leq i < j \leq N} |z_i - z_j|^{-2\beta_i \beta_j} \prod_{i=1}^{N} \prod_{j=1}^{s} |z_i - w_j|^{-2\beta_i \alpha}. \quad (2.23) \]

After performing the path integral over \( X \) and fixing the \( SL(2, C) \) gauge invariance \( (z_1 = 0, z_2 = 1, z_3 = \infty) \), we obtain an integral representation for the \( N \)-tachyon amplitude. It is convenient to factor out the momentum conservation and to define the normalized amplitude \( \tilde{A} \)

\[ \left\langle N \prod_{j=1}^{N} O_{p_j} \right\rangle = 2\pi \delta \left( \sum_{j=1}^{N} p_j \right) \frac{1}{\Gamma(-s)} \tilde{A}(p_1, \ldots, p_N). \quad (2.24) \]

The normalized \( N \)-tachyon amplitude is given by

\[ \tilde{A} = \left( \frac{\mu}{\pi} \right)^s \int \prod_{i=4}^{N} d^2 z_i \prod_{j=1}^{s} d^2 w_j \prod_{i=4}^{N} (|z_i|^{2p_1 \cdot p_i} |1 - z_i|^{2p_2 \cdot p_i}) \prod_{4 \leq i < j \leq N} |z_i - z_j|^{2p_i \cdot p_j} \]

\[ \times \prod_{i=4}^{N} \prod_{j=1}^{s} |z_i - w_j|^{2p_1 \cdot q} \prod_{j=1}^{s} (|w_j|^{2p_1 \cdot q} |1 - w_j|^{2p_2 \cdot q}) \prod_{1 \leq j < k \leq s} |w_j - w_k|^{2q \cdot q}. \quad (2.25) \]

In spite of the non-analytic relation (2.14) between energy \( \beta \) and momentum \( p \), we need to continue analytically the formula into general complex values of
momenta in order to explore the singularity structure. Therefore it is convenient to
define the chirality of tachyons: the tachyon has positive (negative) chirality if the
tachyon energy-momentum satisfies \((\beta + \sqrt{2})/p = 1(-1)\) irrespective of the actual
values of momentum.\(^{20}\) It seems to us that the operators with \(\beta < -\sqrt{2}\) in Eq.
\((2.15)\) are free from the trouble noted in Refs. 4), 5) since the Liouville zero mode
\(\phi_0\) has already been integrated out. The physical values of momenta are reached
by analytic continuation in \(s\), since \(s\) is related to other momenta through (2.18).
For generic physical values of momenta \((s \neq 0)\), one finds a finite result for the
\(N\)-tachyon amplitudes (both \(\langle \prod_{j=1}^{N} O_{p_j} \rangle\) and \(\tilde{A}(p_1, \cdots, p_N)\) are finite). However,
the result is different in different chirality configurations, since the amplitude is
non-analytic in momenta.

Let us consider the kinematical configuration where all tachyons except one
have the same chirality. If \(p_1\) has negative chirality and the rest \(p_2, \cdots, p_N\) positive
chirality, momentum conservation reads
\[
p_1 + p_2 + \cdots + p_N = 0 \tag{2.26}
\]
and energy conservation dictates that \((\alpha = -\sqrt{2})\)
\[
-p_1 + p_2 + \cdots + p_N = \sqrt{2}(N + s - 2). \tag{2.27}
\]
Thus one obtains the kinematical constraints
\[
p_1 = -\frac{N + s - 2}{\sqrt{2}}, \quad \beta_1 = \frac{N + s - 4}{\sqrt{2}}. \tag{2.28}
\]
It has been shown that the \(N\)-tachyon amplitude is given in this kinematical config-
configuration by\(^{7)-11}\)
\[
\tilde{A}(p_1, \cdots, p_N) = \frac{\pi^{N-3}[\mu \Delta(-\rho)]^s}{\Gamma(N + s - 2)} \prod_{j=2}^{N} \Delta(1 - \sqrt{2}\rho_j), \tag{2.29}
\]
where \(\Delta(x) = \Gamma(x)/\Gamma(1 - x)\). The regularization parameter \(\rho\) is given by \(\rho = -\alpha^2/2\) and is eventually set equal to \(-1\) after analytic continuation (in the central
charge $c$). We see immediately that the insertion of the naive cosmological term operator always gives vanishing correlation functions, since $\Delta(-\rho) = \Gamma(-\rho)/\Gamma(1 + \rho)$ vanishes at $\rho = -1$. This decoupling of the naive cosmological term operator guarantees the validity of our procedure in computing the correlation function with the correct cosmological term operators $\phi e^{\alpha\bar{\phi}}$: we replace $\alpha$ by $(1 - \epsilon)\alpha$ and $\mu$ by $\mu/(2\epsilon)$ and take the $\epsilon \to 0$ limit. In effect, we should just replace the combination $\mu\Delta(-\rho)$ by the correct (renormalized) cosmological constant $\mu_r$. For the case of negative chirality for $p_1$ and positive chirality for the remaining $p_2, \ldots, p_N$, the momentum of $p_1$ is fixed because of the kinematical constraints (2.28). As a function of the other momenta $p_2, \ldots, p_N$, the $N$-tachyon amplitude exhibits singularities at

$$p_j = \frac{n + 1}{\sqrt{2}}, \quad j = 2, \ldots, N; \quad n = 0, 1, 2, \ldots,$$

(2.30)

but has no singularities in other combinations of momenta contrary to the dual amplitudes in the critical string theory. The first pole will be shown to correspond to tachyon as an intermediate state in the next section. Other higher level poles for $n = 1, 2, \ldots$ will be shown to correspond to topological states as argued by several people.\(^{14,20}\)

Let us examine other kinematical configurations. The amplitudes with one tachyon of positive chirality and the rest negative are given by changing the sign of $p_j$ in Eq. (2.29). On the other hand, if each chirality has two or more tachyons, the normalized amplitude $\tilde{A}$ is finite for generic momenta but has the factor $1/\Gamma(-s)$. Hence $\tilde{A}$ vanishes for more than two tachyons in each chirality, when we consider non-negative integer $s$ in the following. This property has been explicitly demonstrated for the four- and five-tachyon amplitudes in the Liouville theory,\(^{8,20}\) and has been argued to be a general property using the matrix model.\(^{14}\) Therefore we take it for granted that the tachyon scattering amplitudes $\tilde{A}$ vanish for non-negative integer $s$, unless there is only one tachyon in either one of the chiralities.

Let us note that our assertion is consistent with the argument for vanishing S-matrix by Gross and Klebanov\(^{14}\): they absorb the $\Delta(1 \pm \sqrt{2}p)$ factor for the
individual momenta into a renormalization factor of tachyon vertex operators. This momentum dependent renormalization factor is harmless if the momenta are at some generic values \( p \neq (n + 1)/\sqrt{2} \). However, the two-dimensional kinematics forces one of the renormalization factors to be infinite, if there is only one tachyon in either one of the chiralities\(^{8-10}\) since the kinematical constraints fix the momentum of the tachyon to be at one of the poles (2.30). In the case of negative chirality for \( p_1 \) and positive chirality for the rest \( p_2, \cdots, p_N \), the momentum \( p_1 \) is fixed to be (2.28). Because of this infinite renormalization, the renormalized amplitudes of Gross and Klebanov vanish even if there is only one tachyon in either one of the chiralities.

Let us suppose that we are interested in the case of vanishingly small values of the cosmological constant \( \mu \). We note that, even if the cosmological constant \( \mu \) is infinitesimal, the cosmological term \( \mu e^{\alpha \phi} \) can become arbitrarily large for sufficiently large negative values of the Liouville field \( \phi \) (\( \alpha < 0 \) in our convention). Therefore the Liouville field in the path integral is suppressed for large negative values. Since the large positive values of the Liouville field should be cut off as an ultraviolet or short-distance cut-off, the tachyon field space is restricted to a large but finite volume (proportional to \( \ln \mu \))\(^4,5\) It is important to remember that we cannot neglect the cosmological constant completely even if it is infinitesimally small. On the other hand, the contribution to the amplitudes proportional to \( \ln \mu \) is given by the tachyon interaction in the bulk and hence is insensitive to the details of the cut-off of the Liouville field space. The nonlinearity of the Liouville interactions remains only in the form of the restricted field space. We are precisely interested in this bulk interaction of tachyons. The finite values of the normalized scattering amplitudes \( \tilde{A} \) correspond to a divergent correlation function at \( s = \) positive integers. Since the correlation functions are given by \( \tilde{A} \) multiplied by \( (\mu)^s \Gamma(-s) \), the divergent correlation functions are more properly interpreted as the logarithmically divergent contribution as we let \( \mu \to 0 \). Therefore the finite tachyon scattering amplitudes \( \tilde{A} \) at \( s = \) non-negative integers represent the tachyon interaction proportional to the volume of the Liouville field space\(^{14,20,8}\) Hence they
are often called the bulk or resonant amplitudes.

§3 Higher Level Operators

Since the short-distance singularities should come from terms in the OPE, we first examine the operators responsible for these singularities. We will consider the OPE of vertex operators after the zero mode integration of \( \phi \) and \( X \) in Eq. (2.21). Therefore the operators we consider in this section consist of only non-zero mode \( X = (\tilde{\phi}, \tilde{X}) \), which have the free action (2.22). The simplest operator is \( e^{i p \cdot X} \), which is the tachyon operator (2.12) with zero modes omitted. For higher levels, it has been pointed out that there are only null states at generic values of momenta.\(^{20,14}\) However, there are exceptional values of momenta where the null states degenerate and new primary states emerge as a result. These new primary states are called co-dimension two states by Polyakov,\(^{20}\) and special states, topological states or discrete states by other people.\(^{14,19}\) We can construct vertex operators for these topological states in the following way.

The energy-momentum tensor for \( X \) is given by

\[
T(z) = -\frac{1}{2} \partial X \cdot \partial X - \frac{1}{2} i Q \cdot \partial^2 X \\
= -\frac{1}{2} \partial \tilde{X} \partial \tilde{X} - \frac{1}{2} \partial \tilde{\phi} \partial \tilde{\phi} - \frac{1}{2} Q \partial^2 \tilde{\phi}
\]

(3.1)

and the anti-holomorphic component given by (3.1) with \( \partial \) replaced by \( \bar{\partial} \). They satisfy the Virasoro algebra of the central charge 26. We should construct the field of conformal weight \((1, 1)\) with respect to this energy-momentum tensor by taking linear combinations of monomials of derivatives of \( X \) multiplied by \( e^{i p \cdot X} \). The operator at level \( n \) has \( n \) \( \partial \)'s and \( n \) \( \bar{\partial} \)'s for each monomial. The condition for the conformal weight to be \((1, 1)\) at level \( n \) is

\[
\frac{1}{2} p \cdot (p + Q) + n = 1.
\]

(3.2)
We should note that there are two branches of the solution for the condition (3.2)

$$\beta = -\sqrt{2} \pm \sqrt{p^2 + 2n}. \quad (3.3)$$

Seiberg has noted that the vertex operator with $\beta > -\sqrt{2}$ gives an ill-defined integration over the Liouville zero mode.\(^\text{(4,5)}\) Hence the lower sign in Eq. (3.3) is forbidden by this condition. However, we consider both cases here, since we are considering vertex operators consisting of non-zero mode $\tilde{\phi}$ only. In fact, the naive cosmological terms are a result of the Liouville zero mode $\phi_0$ integration. We shall call the upper sign solution S- (Seiberg) type and the lower sign A- (anti-Seiberg) type.

At level $n = 1$ the general form of the vertex operator with one $\partial$ and one $\bar{\partial}$ is

$$V = \zeta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{i p \cdot X}. \quad (3.4)$$

For this operator to be a primary field of the unit conformal weight, the OPE with the energy-momentum tensor (3.1) must be

$$T(z) V(w) \sim \frac{1}{(z-w)^2} V(w) + \frac{1}{z-w} \partial V(w), \quad (3.5)$$

which gives the conditions on the polarization tensor

$$(p + Q)^\mu \zeta_{\mu\nu} = 0 = \zeta_{\mu\nu}(p + Q)^\nu \quad (3.6)$$

and the on-shell condition (3.2) with $n = 1$. A similar condition should also be satisfied for the anti-holomorphic part in order for the operator to be a $(1,1)$ primary. Solving these conditions we find only one primary field with weight $(1,1)$ at generic values of momentum, i.e. $p \neq 0$

$$V^{(1)} = p \cdot \partial X p \cdot \bar{\partial} X e^{i p \cdot X} = -L_{-1} \bar{L}_{-1} e^{i p \cdot X}. \quad (3.7)$$

The above state is clearly null. However, the situation changes at $p = 0$. For the S-type, the operator vanishes at $p = 0$. Therefore we can construct a new operator
by a limit
\[ V_{(1,1)} = \lim_{p \to 0} \frac{V^{(1)}}{p^2} = \partial X \bar{\partial} X. \] (3.8)

We easily find that this field is primary and not null. This kind of a peculiar operator exists only at a discrete momentum. Since there are two kinematical constraints to specify the state, one for the energy and the other for the momentum, the state is called co-dimension two.\(^{20}\)

As for the A-type at \( p = 0 \), we find that the \((1, 1)\) operator condition does not constrain the polarization tensor multiplying the operator \( \partial X \bar{\partial} X e^{i p \cdot X} \). Hence we again obtain a new primary field
\[ V'_{(1,1)} = \partial X \bar{\partial} X e^{-2\sqrt{2} \phi}. \] (3.9)

At level two, the general form of operators is given by (we present only the holomorphic part)
\[ V = (\zeta_{\mu} \partial^2 X^\mu + \zeta_{\mu\nu} \partial X^\mu \partial X^\nu) e^{i p \cdot X}. \] (3.10)

The OPE (3.5) gives the conditions
\[ \zeta_{\mu \mu} + i(3Q + 2p)^\mu \zeta_{\mu} = 0, \]
\[ \zeta_{\mu} - i(p + Q)^\nu \zeta_{\nu \mu} = 0 \] (3.11)
and Eq. (3.2) with \( n = 2 \). We find two independent solutions to these conditions for the holomorphic part
\[ V^{(2)} = \left( L_{-2} + \frac{3}{2} L_{-1}^2 \right) e^{i p \cdot X}, \]
\[ V^{(3)} = L_{-1} \left( \frac{1}{4} i [(8 - p \cdot Q)p - 2Q] \cdot \partial X e^{i p \cdot X} \right). \] (3.12)

Both fields are null. At a special value of the momentum these two operators are linearly dependent and we obtain a topological state. For instance, the \((2,1)\)
topological state of S-type is given by

\[
V_{(2,1)} = \lim_{p \to \sqrt{2}} \frac{6\sqrt{2}}{p} (V^{(2)} - V^{(3)}) \\
= \left(13 \partial X \partial X - \partial \phi \partial \phi - 6 i \partial X \partial \phi - \sqrt{2} i \partial^2 X - \sqrt{2} \partial^2 \phi \right) e^{\frac{1}{\sqrt{2}} i (X - i\phi)}.
\]

(3.13)

We find exactly the same situation for the anti-holomorphic part. We can continue to explore (1, 1) operators at higher levels similarly. We expect that these (1, 1) operators are null fields for generic values of momenta, and that, at special values of momenta, these null states are not linearly independent, namely they degenerate. Then we obtain a new primary state from a limit of an appropriate linear combination of these null states. We expect to have both S-type and A-type topological states.

There are other procedures to obtain topological states. These states were found to originate from the gravitational dressing of the primary states in the \(c = 1\) conformal field theory which create null descendants at level \(n\). The momentum \(p\) of the initial primary state and the level \(n\) are specified by two positive integers \((r, t)\)

\[
p = \frac{r - t}{\sqrt{2}}, \quad n = rt.
\]

(3.14)

Since the level \(n\) corresponds to the \(n\)-th derivatives, we find that the energy \(\beta\) of the topological state is determined by the (1, 1) conformal weight condition (3.3) and is given by

\[
\beta = \frac{-2 \pm (r + t)}{\sqrt{2}}.
\]

(3.15)

The upper (lower) sign corresponds to the S- (A-) type solution. For example, \((r, t) = (1, 1)\) and \((2, 1)\) operators of S-type are

\[
V_{(1,1)} = \partial X \bar{\partial} X, \\
V_{(2,1)} = \left( \partial X \partial X + \frac{1}{\sqrt{2}} i \partial^2 X \right) \left( \bar{\partial} X \bar{\partial} X + \frac{1}{\sqrt{2}} i \bar{\partial}^2 X \right) e^{\frac{1}{\sqrt{2}} i (X + \phi)}.
\]

(3.16)

We find that these operators coincide with our operators (3.8) and (3.13) respec-
tively up to only a certain amount of null operators.

§4 OPE and Factorization

To understand the poles of the amplitudes in terms of short-distance singularities in the OPE, we shall consider the case of $s = \text{non-negative integers}$ by choosing the momentum configuration appropriately. As we explained before, these amplitudes at non-negative integer $s$ represent the so-called bulk or resonant interactions.\textsuperscript{14,20} The only nonvanishing $N$-tachyon amplitudes $\tilde{A}$ at $s = \text{non-negative integers}$ are for the kinematical configuration where one of the chiralities have only a single tachyon and the rest opposite chirality. Here we shall take the case of $s = 0$ for the $N$-tachyon amplitude with only one negative chirality tachyon ($p_1$), and examine the $s = \text{positive integers}$ case at the end.

First we shall illustrate the origin of short-distance singularities in the simplest context of the four tachyon scattering amplitude with $s = 0$. As in Eq. (2.25) we fix $z_2 = 0$, $z_3 = 1$, $z_4 = \infty$ and set $z_1 = z$ to find

$$\tilde{A}(p_1, \cdots, p_4) = \int d^2 z \left| z \right|^{2p_1 \cdot p_2} \left| 1 - z \right|^{2p_1 \cdot p_3}. \quad (4.1)$$

The short-distance singularities corresponding to $z_1 \sim z_2$ ($z \to 0$) can be exhibited by expanding the integrand around $z = 0$

$$\tilde{A}(p_1, \cdots, p_4) \approx \int d^2 z \left| z \right|^{2p_1 \cdot p_2} \left| \sum_{n=0}^{\infty} \left( \frac{\Gamma(1 + p_1 \cdot p_3)}{n! \Gamma(p_1 \cdot p_3 - n + 1)} \right) (-z)^n \right|^2 \quad (4.2)$$

$$\approx \sum_{n=0}^{\infty} \frac{\pi}{n + 1 + p_1 \cdot p_2} \left( \frac{\Gamma(1 + p_1 \cdot p_3)}{n! \Gamma(p_1 \cdot p_3 - n + 1)} \right)^2.$$

We see that the so-called noncritical string of the $c = 1$ quantum gravity exhibits exactly the same type of short-distance singularities as the familiar critical string
theory. The kinematics at the pole reflects the peculiarities of two-dimensional physics

\[ p_1 = (\sqrt{2}, 0), \quad p_2 = (n + 1, -n - 1), \quad p_j = (p_j, -i(-\sqrt{2} + p_j)) \quad j = 3, 4. \]  

(4.3)

We can express the short-distance singularities in terms of these momenta to find

\[ \tilde{A}(p_1, \cdots, p_4) \approx \sum_{n=0}^{\infty} \frac{(-1)^n \pi}{(n!)^2} \prod_{j=3}^{4} \Delta(1 - \sqrt{2} p_j). \]  

(4.4)

This shows that all the singularities in \( p_2 \) in the full amplitude are correctly accounted for by these short-distance singularities near \( z_1 \sim z_2 \). Similarly, the singularities in \( p_3 \) and \( p_4 \) are accounted for by the short-distance singularities for \( z_1 \sim z_3 \) and \( z_1 \sim z_4 \) respectively. Hence we see that all the singularities in the amplitude are nothing but the short-distance singularities associated with the opposite chirality tachyons approaching each other.

We can understand these short-distance singularities by means of the OPE of two vertex operators

\[ : e^{i p_1 \cdot X(z_1)} : \cdot : e^{i p_2 \cdot X(z_2)} : \sim \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{|z_1 - z_2|^2} |2p_1 \cdot p_2 + 2n| V_n(z_2) \right). \]  

(4.5)

The operators on the right hand side are given by

\[ V_n = : e^{i p_2 \cdot X} \partial^n \bar{\partial}^n e^{i p_1 \cdot X} : 
\quad = : (-p_1 \cdot \partial^n X p_1 \cdot \bar{\partial}^n X + \cdots) e^{i p \cdot X} :, \]  

(4.6)

where \( p = p_1 + p_2 \). It can be shown that \( V_n \) is a primary field with a conformal weight \((1, 1)\) when \( p \) satisfies Eq. (3.2). Integration of Eq. (4.5) by \( z_1 \) over the region \( |z_1 - z_2| \leq \epsilon \) gives singularities in the momentum \( p \)

\[ \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^2 \frac{\pi}{\|p - (p + Q) + n|} V_n(z_2). \]  

(4.7)

To discuss the OPE in a general context, we now consider the \( N \)-tachyon amplitude where \( p_1 \) has negative chirality and \( p_2, \cdots, p_N \) have positive chirality. From
the energy-momentum conservation (2.26) and (2.27) for this kinematical configuration, the two-momentum of the tachyon 1 takes a fixed value

\[ \mathbf{p}_1 = \left( -\frac{N - 2}{\sqrt{2}}, -i\frac{N - 4}{\sqrt{2}} \right). \]  

(4.8)

This kinematical constraint implies for the intermediate state momentum \( \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 \)

\[ \frac{1}{2} \mathbf{p} \cdot (\mathbf{p} + \mathbf{Q}) + n - 1 = (N - 3)(1 - \sqrt{2}p_2) + n. \]  

(4.9)

At the pole \( p_2 = 1/\sqrt{2} \) \((n = 0)\), the intermediate state momentum becomes

\[ \mathbf{p} = \left( -\frac{N - 3}{\sqrt{2}}, -i \left( -\sqrt{2} + \frac{N - 3}{\sqrt{2}} \right) \right). \]  

(4.10)

Hence we find that the pole at \( p_2 = 1/\sqrt{2} \) \((n = 0)\) in Eq. (4.7) is due to the tachyon intermediate state \( V_0 \) of negative chirality. The higher level poles \((n \geq 1)\) are due to the topological states in \( V_n \) \((n \geq 1)\) that we have discussed in the previous section. These topological state operators can take various appearances depending on the amount of null states. If we let

\[ N - 3 = t, \quad n = rt, \]  

(4.11)

then the singular factor in Eq. (4.7) for \( n \geq 1 \) becomes

\[ \frac{1}{2} \mathbf{p} \cdot (\mathbf{p} + \mathbf{Q}) + n - 1 = \frac{1}{t(r + 1 - \sqrt{2}p_2)}. \]  

(4.12)

Hence we obtain the pole

\[ p_2 \rightarrow \frac{r + 1}{\sqrt{2}}. \]  

(4.13)

In this case the intermediate state momentum becomes

\[ \mathbf{p} = \left( \frac{r - t}{\sqrt{2}}, -i \left( -\sqrt{2} + \frac{r + t}{\sqrt{2}} \right) \right) \]  

\[ = \left( \frac{r - t}{\sqrt{2}}, -i \left( -\sqrt{2} + \sqrt{\left( \frac{r - t}{\sqrt{2}} \right)^2 + rt} \right) \right). \]  

(4.14)

This two-momentum is precisely the two-momentum (3.14) and (3.15) of the topo-
logical state at level \( n = rt \) of the S-type. Therefore we find that the \((r, t)\) topological state is contained in \( V_n \) if the number of tachyons \( N \) and the level of the intermediate state \( n \) are specified by Eq. (4.11) and that the operator must be of S-type. Thus we find that each \((r, t)\) primary topological state appears as an intermediate state of the \( N \)-tachyon amplitude at level \( n \) exactly once. This together with the above tachyon singularity explains all singularities in Eq. (2.30).

Let us discuss the residues of these short-distance singularities. These poles can be associated with the intermediate tachyon or topological states appearing in the OPE. As illustrated in Fig. 1 the residues of these poles are given by a product of two parts. One of them, the right side blob in Fig. 1, has \( N - 2 \) tachyons with incoming momenta \( p_3, \cdots, p_N \), and an intermediate particle with incoming momentum \( p = p_1 + p_2 \), and is given by a kind of dual amplitude. The other part has two tachyons with incoming momenta \( p_1, p_2 \) and the intermediate particle as shown in the left side blob in Fig. 1. First we examine the pole at \( p_2 = 1/\sqrt{2} \), namely at the lowest level \((n = 0)\). This pole is due to the tachyon intermediate state with negative chirality. In fact we find that the residue of the pole \( p_2 = 1/\sqrt{2} \) is precisely given by a product of the tachyon three point function and the \( N - 1 \) tachyon amplitude with a single (intermediate state) tachyon \( p \) having negative chirality and the rest \( p_3, \cdots, p_N \) having positive chirality

\[
\hat{A}(p_1, p_2, p_3, \cdots, p_N) \approx \frac{\pi}{(N-3)(1-\sqrt{2}p_2)} \hat{A}(p, p_3, \cdots, p_N) \\
= \hat{A}(p_1, p_2, -p) \frac{2\pi}{p \cdot (p + Q)} - 2 \hat{A}(p, p_3, \cdots, p_N). \tag{4.15}
\]

This shows that the factorization is valid similarly to critical string theory. By symmetry, we can explain the lowest poles in each individual momentum \( p_j = 1/\sqrt{2} \) as the tachyon intermediate state in the OPE of \( p_1 \) and \( p_j \).

For higher level poles, we explicitly evaluate the residue of the short-distance singularities up to \( p = 3/\sqrt{2} \) and up to \( N = 5 \). For instance, the five tachyon
amplitude has short-distance singularities at

\[ p_2 = \frac{2}{\sqrt{2}} \text{ and } \frac{3}{\sqrt{2}} \]

\[
\tilde{A}(p_1, \cdots, p_5) \approx \left[ -\frac{\pi^2}{2(2 - \sqrt{2} p_2)} + \frac{\pi^2}{8(3 - \sqrt{2} p_2)} \right] \prod_{j=3}^{5} \Delta(1 - \sqrt{2} p_j). \quad (4.16)
\]

The residues of these poles correctly reproduce the residues of the poles in the full amplitude. Each term of Eq. (4.16) can also be written as

\[
\tilde{A}(p_1, p_2; -p) \frac{2\pi}{\mathbf{p} \cdot (\mathbf{p} + \mathbf{Q}) + 2(n-1)} \tilde{A}(p; p_3, \cdots, p_N), \quad (4.17)
\]

where \( n = 1 \) and \( 2 \) for the first and the second terms respectively and \( -p \) and \( p \) are the momenta of the intermediate particles. We have explicitly verified that these amplitudes appearing in the residue agree with those obtained from the three and \( N-1 \) point amplitudes with one topological state of momentum \( -p \) and \( p \). It is rather difficult to compute the short-distance singularities explicitly to an arbitrary level except for the four-point amplitude that we have already worked out in Eq. (4.2). Therefore we content ourselves with the computation of lower level singularities in explicitly demonstrating that the singularities of the amplitudes all come from the short-distance singularities of \( p_1 \) and \( p_j \).

The OPE suggests that there may be other short-distance singularities in other combinations of momenta if one considers corresponding combinations of vertex operators approaching to the same point. For instance, short-distance singularities corresponding to \( k \) vertex operators approaching each other, say \( z_1, \cdots, z_k \), should give poles in \( p_2 + \cdots + p_k \). It is most convenient to fix reduced variables \( u_j = (z_j - z_2)/(z_1 - z_2) \) \( (j = 1, \cdots, k) \) in taking the short-distance limit \( z_1 \to z_2 \). The amplitude exhibits short-distance singularities whose residues are given by a
product of two dual amplitudes (Fig. 2)

\[ \tilde{A}(p_1, \ldots, p_N) \approx \frac{1}{V_{SL(2,\mathbb{C})}} \int d^2 z_1 d^2 z_2 \prod_{i=3}^{k} d^2 u_i \prod_{j=k+1}^{N} d^2 z_j \left| z_1 - z_2 \right|^p \cdot p + Q \cdot p - 4 \]

\[ \times \prod_{1 \leq i < j \leq k} \left| u_i - u_j \right|^2 p_i \cdot p_j \prod_{i=1}^{k} \prod_{j=k+1}^{N} \left| 1 + \frac{z_1 - z_2}{z_2 - z_j} u_i \right|^2 p_i \cdot p_j \]

\[ \times \prod_{i=k+1}^{N} \left| z_2 - z_i \right|^2 p_i \cdot p_j \prod_{k+1 \leq i < j \leq N} \left| z_i - z_j \right|^2 p_i \cdot p_j. \]

(4.18)

The dual amplitude with the original variables \( z_i \) \((i = 2, k + 1, \ldots, N)\) has \( N - k \) positive chirality tachyons \( p_{k+1}, \ldots, p_N \) and the intermediate state particle \( p \) (right side blob in Fig. 2), whereas the dual amplitude with the reduced variables \( u_j \) \((j = 3, \ldots, k)\) has the intermediate state particle \( -p - Q \) and \( k \) tachyons \( p_1, \ldots, p_k \) whose chiralities are positive except \( p_1 \) (left side blob in Fig. 2).

It is important to clarify the kinematical constraints on the intermediate states when the amplitudes are factorized. In the left side blob in Fig. 2, the incoming momenta of tachyons \( p_1, \ldots, p_k \) are balanced by the incoming momentum \( -p = -p_1 - \cdots - p_k \). However, if we want to interpret the left side blob as a dual amplitude coming from a path integral with our action (2.4) and (2.6), we need to assign the incoming two-momentum of the intermediate particle to be \( -p - Q \), since the action dictates that an external source two-momentum \( Q \) should be present. Therefore if \( p \) is the intermediate state momentum flowing into the right side blob in Fig. 2, the corresponding momentum for the dual amplitude of the left side blob should be regarded as \( -p - Q \). Consequently, if the intermediate state is a tachyon, the chirality of the tachyon for the left side blob is the same as that of the tachyon for the right side blob

\[ p = (p, -i(-\sqrt{2} \pm p)), \quad -p - Q = (-p, -i(-\sqrt{2} \pm (-p))). \]

(4.19)

Similarly, if the intermediate state is a topological state, the type (S or A) of the intermediate state for the left side blob is the opposite to that of the intermediate
state for the right side blob

\[ p = (p, -i(-\sqrt{2} \pm \sqrt{p^2 + 2n})), \quad -p - Q = (-p, -i(-\sqrt{2} \mp \sqrt{p^2 + 2n})). \] (4.20)

In the present case, we have only a single tachyon with the negative chirality whose two-momentum is determined by the two-dimensional kinematics (4.8). Since we pinch together the single negative chirality tachyon with the positive chirality tachyons \( p_1, \ldots, p_k \), we find the intermediate particle to have the two-momentum

\[
p = \left( \frac{-N + 2}{\sqrt{2}} + \sum_{j=2}^{k} p_j, -i \left( \frac{N - 2 - 2k}{\sqrt{2}} + \sum_{j=2}^{k} p_j \right) \right). \] (4.21)

Since \( k \leq N - 2 \), it is tachyon if

\[
\sum_{j=2}^{k} p_j = \frac{k - 1}{\sqrt{2}}
\] (4.22)

and its chirality is always negative. In this case, the left side blob has more than two tachyons for each chirality. On the other hand, the tachyon amplitudes are non-vanishing only if a single tachyon has one of the chiralities and the rest have opposite chirality. Therefore the dual amplitude with the reduced variables (left side blob in Fig. 2) vanishes except when it is the three-point function \((k = 2)\). This is precisely the case we have evaluated already in Eq. (4.15).

As for the intermediate topological states, kinematics dictates that it is of S-type for the dual amplitude with the original variables corresponding to the right side blob in Fig. 2

\[
p = \left( -\frac{N - k - 1}{\sqrt{2}} + \frac{n}{2(N - k - 1)}, \right.
\]

\[
- i \left( -\sqrt{2} + \frac{N - k - 1}{\sqrt{2}} + \frac{n}{\sqrt{2}(N - k - 1)} \right).
\] (4.23)

for the level \( n \) topological state. As we have already explained, the intermediate state \(-p - Q\) for the right side blob is of A-type. The three-point function with
the A-type topological state \((k = 2)\) is nothing but the OPE coefficient \((4.5)\) that we have seen non-vanishing. Four- and more-point functions with the A-type topological state \((k \geq 3)\) are more difficult to compute. The topological state of level \(n\) consists of a linear combination of monomials of derivatives of \(X\) multiplied by a vertex operator \(e^{ip \cdot X}\). Both the number of \(\partial\) and the number of \(\bar{\partial}\) should be \(n\) for each monomial. The two momentum \(p\) is given by \((r - t)/\sqrt{2}, -i(-2 - r - t)/\sqrt{2}\) for the \((r, t)\) topological state of type A. If we do not specify the coefficients of the monomials, we obtain an operator containing the \((r, t)\) topological state together with a certain amount of null states. We have taken such an operator as a substitute for the \((r, t)\) topological state of A-type at the level \(n = rt\), and have explicitly evaluated the dual amplitude with the topological state for the case of four-point function. We have found it to vanish. This amplitude arises as the left side blob in Fig. 2 contributing to the pole of level \(n = rt\) in the case of \(k = N - r - 1 = 3\). We conjecture in general that the A-type topological state gives vanishing dual amplitude except for the three-point function \((k = 2)\). Only in the three-point dual amplitude \((k = 2)\), we can simply regard the factor for the blob of particles pinched together (left side blob in Fig. 2) as the coefficient of the OPE rather than the dual amplitude. We should emphasize that this decoupling of the topological states of the A-type is very crucial in explaining the simple singularity structure of the \(N\)-tachyon scattering amplitudes. Let us note that the decoupling of these states has been shown by Di Francesco and Kutasov in a recent preprint.\(^{25}\) This property is presumably related to Seiberg’s finding that only the S-type is physical. These decoupling properties of both tachyon and topological states originate partly from a peculiarity of the two-dimensional kinematics (one dimension from the matter \(X\) zero mode and the other from the Liouville zero mode), but they also seem to be a manifestation of the large symmetry characteristic to the \(c = 1\) matter coupled to quantum gravity. Hence they are worth studying further.

Other possibilities are short-distance singularities from the pinching of \(k\) tachyons—all with positive chirality, say \(p_2, \ldots, p_{k+1}\). Actually the short-distance singular-
ities due to the pinching of tachyons all with positive chirality can be regarded as the same short-distance singularities as the pinching of the other tachyons instead. Namely these limits are equivalent to configurations in which the tachyons $p_1$ and $p_{k+2}, \ldots, p_N$ are pinched rather than $p_2, \ldots, p_{k+1}$ by the $SL(2,\mathbb{C})$ invariance. Since one of the other tachyons has the opposite chirality, the present case is actually the same situation as the previously analyzed case: pinching together the single negative chirality tachyon with the opposite chirality tachyons. Such configurations have already been discussed above and we need not to consider them.

These observations explain why there are only singularities in the individual $p_j$, and none in any combinations of momenta, although the factorization of the $N$-tachyon amplitudes is valid through the OPE as we have seen.

Let us finally discuss the case of $s = $ positive integer. The amplitudes with $s = $ positive integer can be obtained from the $s = 0$ case as follows: we consider the $N + s$ tachyon scattering amplitude and take a limit of vanishing momenta for $s$ tachyons and multiply by $(\mu/\pi)^s$. There is one subtlety: at the vanishing momenta, the chirality is ill-defined. We define the vanishing momenta taking the limit from the positive chirality tachyon. In the limit, we obtain an $s$-th power of a singular factor $\mu \Delta(0)$, which should be replaced by the correct (renormalized) cosmological constant $\mu_r$. This procedure gives the insertion of the correct cosmological term operator $\phi e^{\alpha\phi}$, as we explained earlier. In this way we find that the short-distance singularities of the amplitudes with non-vanishing $s$ can be obtained correctly once the short-distance singularities in the $s = 0$ amplitude are correctly obtained. Using the previous argument, we find that the only non-vanishing short-distance singularities come from the OPE of two vertex operators for tachyons. Short-distance singularities from one or more naive cosmological term operators approaching the tachyon vertex operators give a vanishing value for the residue.
Acknowledgements

One of the authors (NS) thanks Y. Kitazawa and D. Gross for a discussion of the Liouville theory. We would like to thank Patrick Crehan for a careful reading of the manuscript. This work is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.01541237).

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**Figure Captions**

1) The factorization of the $N$-tachyon amplitude by the OPE of the operators 1 and 2. The signs $+$ and $-$ denote the chirality of the tachyons.

2) The factorization of the $N$-tachyon amplitude by the OPE of the operators 1, $\cdots$, $k$. The signs $+$ and $-$ denote the chirality of the tachyons.