Numerical study of the strongly screened vortex glass model in an external field

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The vortex glass model for a disordered high-\(T_c\) superconductor in an external magnetic field is studied in the strong screening limit. With exact ground state (i.e. \(T = 0\)) calculations we show that 1) the ground state of the vortex configuration varies drastically with infinitesimal variations of the strength of the external field, 2) the minimum energy of global excitation loops of length scale \(L\) do not depend on the strength of the external field, however 3) the excitation loops themselves depend sensibly on the field. From 2) we infer the absence of a true superconducting state at any finite temperature independent of the external field.

I. INTRODUCTION

The gauge or vortex glass model has become a paradigm in studying amorphous high-\(T_c\) superconductors or random Josephson-junction arrays (see \(\Box\) for a review). One essential feature of this model is the possible appearance of a glassy state at low enough temperatures, without which true superconductivity (i.e. vanishing resistance) would cease to exist in these disordered materials.

Experimental evidence of such a vortex glass state has been reported for high-\(T_c\) superconductors (see \(\Box\)). From the theoretical side it is now commonly believed that in the absence of screening a true superconducting vortex glass phase occurs at low enough temperatures (see \(\Box\)). If screening is present the original, unscreened 1/r-interaction of the vortex lines is exponentially shielded beyond a particular length scale \(\lambda\) and the situation seems to change, in particular in the limit in which the screening length is zero (i.e. where vortex lines interact only on-site) the low temperature vortex glass phase seems to be destroyed.

In a typical experimental situation the amorphous high-\(T_c\) superconductor is put into a homogeneous magnetic field pointing, say, in the \(z\)-direction. Due to bulk disorder, i.e. inhomogeneities (vacancies, defects, etc.), in the bulk of the sample the vector potential acting on the superconducting phase variables attains a random component (most plausible this mechanism is explained in the context of granular superconductors), however, still there should be a homogeneous back ground field superposed on the random part.

Therefore in this paper we study the question of how is the latter scenario, i.e. the absence of a true superconducting phase in the strongly screened three-dimensional gauge or vortex glass model influenced by the presence of a homogeneous external field in one particular space direction. This is done via the investigation of exact ground states of the vortex glass Hamiltonian and its low energy excitation. First we analyze the sensibility of the minimum energy configuration with respect to the addition of a homogeneous external field, then we study the low energy excitations of length scale \(L\) in the spirit of the usual domain wall renormalization group (DWRG) calculations (see \(\Box\)).

The lattice model describing the phase fluctuations (described by phase variable \(\phi_i \in [0,2\pi]\)) in a strongly disordered superconductor close to a normal-to-superconductor phase transition is the gauge glass model

\[
H = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j - A_{ij} - \lambda^{-1} a_{ij}) \\
+ \frac{1}{2} \sum_i (\nabla \times \mathbf{a})^2, 
\]

(1)

where the first sum runs over all nearest-neighbor pairs \(\langle ij \rangle\) on a \(L \times L \times L\) simple cubic lattice and the second over every elementary plaquette \(\Box\), respectively, and \(a_{ij}\) the fluctuating vector potentials and \(\lambda\) the screening length. \(A_{ij} = A^\text{rand}_{ij} + A^\text{hom}_{ij}\) are the quenched vector potentials consisting of a random component \(A^\text{rand}_{ij} \in [0,2\pi]\) and a homogeneous component \(A^\text{hom}_{ij}\) modeling an external magnetic field in the \(z\)-direction. The parameter \(\lambda\) is the bare screening length. A similar Hamiltonian occurs for ceramic (granular) superconductors including the self-inductance of vortex loops (see \(\Box\)). For simplicity we set \(J = 1\).

After standard manipulations this Hamiltonian can be brought into the vortex representation

\[
H_V = \frac{1}{2} \sum_{i,j} (\mathbf{J}_i - \mathbf{b}_j) G(i - j)(\mathbf{J}_j - \mathbf{b}_j), 
\]

(2)

The \(\mathbf{J}_i\), representing the vortex density of the phase fields \(\phi_i\) on bond \(i\), are three-component integer variables running from \(-\infty\) to \(\infty\), live on the links of the dual lattice and satisfy the divergence constraint \((\nabla \cdot \mathbf{J})_i = 0\) on every site \(i\). The function \(G(r)\) is a lattice Greens function behaving asymptotically like \(G(r) \sim r^{-1} \exp(-r/\lambda)\). The \(\mathbf{b}_j\) are magnetic fields which are constructed from the quenched vector potentials \(A_{ij}\) by a lattice curl, in the present case with the homogeneous external field we have:
Obviously the random part fulfills the divergence free condition \( (\nabla \cdot \mathbf{b}) = 0 \). We specify the boundary conditions for the vortex glass Hamiltonian to be periodic in all space directions (corresponding to fluctuating boundary conditions in the phase variables of the original gauge glass Hamiltonian). Now we choose \( \mathbf{B}^\text{ext} = B \mathbf{e}_z \) i.e. the external field points in the \( z \)-direction (i.e. along \( \mathbf{e}_z \) the unit vector in the \( z \)-direction) and is also divergence-less due to the periodic boundary conditions.

We just remark that in the pure case \( (\mathbf{A}^\text{rand} = 0) \) the field strength \( B \) simply plays the role of the usual filling factor \( f \) counting the number of flux units per plaquette giving rise to the uniformly frustrated XY-model (see e.g. and references therein) in the unscreened case \( (\lambda = \infty) \). Here, due to the long range interaction \( G(r) \approx 1/r \), the ground state is indeed non-trivial for irrational filling factors. In the continuum limit the flux lines would actually form a hexagonal lattice, the well known Abrikosov flux line lattice.

For the disordered case one has an interesting interplay between two sorts of frustration: one is also present in the pure case and coming from the external field and the other comes from the quenched disorder. To our knowledge this problem has not been investigated systematically so far. Here, as a first step, we confine ourselves to the strongly screened case \( (\lambda \rightarrow 0) \), for which the vortex Hamiltonian \( (2) \) simplifies to

\[
H^\lambda \rightarrow 0 = \frac{1}{2} \sum_i (\mathbf{J}_i - \mathbf{b}_i)^2. \tag{4}
\]

The problem of finding the ground state, i.e. the minimum energy configuration of this Hamiltonian, is actually a minimum-cost-flow-problem \( \min (\mathbf{J}_i) \sum_i c_i(\mathbf{J}_i) \) subject to the constraint \( (\nabla \cdot \mathbf{J})_i = 0 \), where \( c_i(\mathbf{J}_i) := (\mathbf{J}_i - \mathbf{b}_i)^2/2 \) are the so called (convex) cost functions. This problem can be solved exactly in polynomial time via combinatorial optimization techniques, as described in [11].

II. FIELD CHAOS

First we investigate how the ground state of the vortex glass changes with the external field. Obviously, with increasing field strength \( B \) more and more flux will flow in the \( z \)-direction. In Fig. 1 we depict various quantities reflecting this observation: a) the total flux \( L^{-3} \sum_i |J^z_i| + |J^x_i| + |J^y_i| \), which increases, after a crossover region, linearly with \( B \); b) the fraction of flow variables in the \( x \)- and \( y \)-direction that are zero/nonzero; then the fraction of flow variable in the \( z \)-direction which are c) larger than, d) smaller than, e) equal to zero.

![FIG. 1. Plot of the count of components per site vs. external field for 1000 samples. a) The total flux \( L^{-3} \sum_i |J^z_i| + |J^x_i| + |J^y_i| \); b) \( L^{-3} \sum_i \delta_{J^z_i,0} \) (upper line), \( L^{-3} \sum_i \theta(J^x_i) \) and \( L^{-3} \sum_i \theta(-J^y_i) \) (both lay on the lower line); the \( y \)-values are the same; c) \( L^{-3} \sum_i \theta(J^x_i) \); d) \( L^{-3} \sum_i \theta(-J^y_i) \); e) \( L^{-3} \sum_i \delta_{J^z_i,0} \).

For \( B = 0 \) the vortex variables \( J_i \) are homogeneously distributed: \( L^{-3} \sum_i \delta_{J^z_i,0} \) same values for each direction, where the last two values are equal due to the symmetry of direction. An increasing field \( B > 0 \) only effects the components along the \( z \)-direction (dashed and dotted lines) and let the components perpendicular to it unchanged (solid lines). There is a critical external field \( B_c \) above which all components of the flux lines \( J^z_i \) parallel to the field have a non-vanishing negative amplitude, what is attended by a decreasing of the zero-components, where \( B_c = 3/2 \). The critical fields can be determined by remembering that the random field \( \mathbf{B}^\text{rand} \) is in \([-2,+2]\) and the flux is an integer. Thus the maximal discrepancy between the optimal flux and field is 1/2 per site. The optimal \( J^z_i \) depends on the applied field by a step function and in the worst case the random field the \( \mathbf{b}^\text{rand} = \pm 2 \) so that we have \( \{\mathbf{J}^\text{opt}\} = \{\mathbf{J}_i \text{ integer} \mid \min |\mathbf{J}_i - (\mathbf{B}^\text{ext} \pm 2.0)| \} \). One obtains: \( J^z_i < 0 \forall i \) if \( B > B_{c0} = 5/2 \) and \( J^z_i \leq 0 \forall i \) if \( B > B_{c3} = 3/2 \), in accordance with Fig. 1.

Next we study the sensibility of the ground state (or optimal flow configuration) with respect to small changes in the external field \( B \). To this end we compare the ground state configurations of samples with the same quenched disorder and slightly different external field \( B \). Denoting with \( \mathbf{J}_1 \) the zero field \( (B = 0) \) ground state and with \( \mathbf{J}_1(B) \) the ground state of the same sample in non-vanishing external field \( B \) we define the Hamming distance of the two configuration \( \mathbf{J}_1 \) and \( \mathbf{J}_1(B) \) by

\[
b_i = \frac{1}{2\pi} (\nabla \times \mathbf{A}^\text{rand})_i + \mathbf{B}_i^\text{ext}.
\]
FIG. 2. Scaling plot of the Hamming distance $D_B(L)$ vs. $LB^{1/\zeta}$ for $L \leq 32$: 5000 samples for $L \leq 16$, 2000 for $L = 24$ and 500 for $L = 32$. The chosen values for $B$ are $B = 0.0001$, 0.0010, 0.0100 and 0.1000. The best data collapse is achieved by a chaos exponent $\zeta = 3.8 \pm 0.2$. The error bars are less than the size of the symbols and thus omitted.

$$D_B(L) = \sum_i (J_i(B) - \langle J_i \rangle)^2,$$

so that a small value of $D_B(L)$ means a strong correlation of the ground states.

In [3] it has been found that an infinitesimal random perturbation of the vector potential $A_{ij}^{\text{rand}}$ leads to a chaotic rearrangement of the ground state configuration. There it was demonstrated that, like in spin glasses, beyond a particular length scale, the so-called overlap length, the two ground states (perturbed and unperturbed) decorrelate. For the non-random magnetic field perturbation we study here we take over this concept and demonstrate the existence of an overlap length $l^*$ scaling with the strength of the external field $B$ like $l^* \propto B^{1/\zeta}$, where $\zeta$ is the chaos exponent. For this length scale to exist the finite size scaling form $D_B(L) = d(L/l^*) = d(L/B^{1/\zeta})$ should hold, which is indeed satisfied, as is shown in Fig. 2. We obtain a relatively large chaos exponent $\zeta = 3.8 \pm 0.2$. Remarkably this exponent coincides (within the error bars) with the chaos exponent for a random perturbation which has been reported to be $\zeta^{\text{rand}} = 3.9 \pm 0.2$ [3].

III. DEFECT ENERGY (DWRG)

In this section we study the scaling behavior of low energy excitations $\Delta E(L)$ of length scale $L$ (to be defined below) in the presence of an external field, which provides the essential evidence about the stability of the ground state with respect to thermal fluctuations. If $\Delta E(L)$ decreases with increasing length $L$ it implies that it costs less energy to turn over larger domains thus indicating the absence of a true ordered (glass) state at any non-vanishing temperature. Usually one studies such excitation of length scale $L$ by manipulating the boundary conditions (b.c.) for the phase variables of the original Hamiltonian $H$, see [4]. One induces a so called domain wall of length scale $L$ into the system by changing the b.c. of a particular sample from periodic to anti-periodic (or vice versa) in one space direction and measures the energy of such an excitation by comparing the energy of these two ground state configurations. This is the common procedure for a DWRG analysis, which, however, bears some technical complications and some conceptual ambiguities in it.

Here we follow the basic idea of DWRG, we will, however, avoid the complications and the ambiguities that appear by manipulating the b.c. and try to induce the low energy excitation in a different way, as it has first been done by one of us in [4] for the zero-field case.

First we clarify what a low energy excitation of length scale $L$ is: in the model under consideration here it is certainly a global vortex loop encircling the 3d torus (i.e. the $L \times L \times L$ lattice with periodic b.c.) once (or several times) with minimum energy cost. For the pure case the global minimum energy loop is simply a straight line that costs energy $\Delta E(L) = JL$, which is exactly also what one would expect for a domain wall of length $L$ in a three-dimensional XY-model, and which is also obtained from the energy difference between ground states with periodic and anti-periodic b.c.

Next we have to clarify how we induce the above mentioned global vortex loop, if not by manipulating the b.c. The solution is to manipulate instead the costs for flow in one particular space direction. Suppose we have the exact ground state of one particular sample, which specifies also the current cost for increasing the the flow variables $J^x,y,z$ with respect to $J^1$ by one unit, e.g.: $\Delta c_i^x = c_i(J^x_i + 1) - c_i(J^x_i) = j_i^x - b_i^x + 1/2$. If we decrease smoothly these variables $\Delta c_i^x$ and apply our min-cost-flow algorithm to this modified problem at some point a configuration $J^1_i$ that is the original ground state plus a global loop in the $x$-direction appears as the new optimal flow configuration for the modified problem. This extra loop, which can be easily identified by comparing the new optimum with the original ground state, is the low energy excitation we are looking for. Its energy $\Delta E(L)$ is simply the difference of this state (ground state with loop), $H(J^1)$ minus the energy of the ground state $H(J^0)$. Note that this is always positive, since it is definitely an excitation (in contrast to the usual DWRG procedure where the b.c. is modified).

Four remarks are in order: 1) small, topologically simply connected loops are not generated by this procedure, since all what can be gained in energy is lost again on the return. 2) In the pure case this procedure would not work, since at some point spontaneously all links in the $z$ direction would increase their flow value by one. It is only for the disordered case with a continuous distribution for the random variables $b_i$ that a unique loop can
be expected. 3) Sometimes (in ca. 5% of the samples) the global flux changes discontinuously by more than one unit, a typical example for such an elementary excitation loop is shown in Fig. 3, however we define these still to be elementary excitations of length scale L. 4) In the presence of a homogeneous external field one has to discriminate between different excitation loops: those parallel and those perpendicular to the external field need not to have the same energy (however, it turns out that the disorder averaged defect energy is identical in all directions, see below).

Schematically the numerical procedure is the following:
1. Calculate the exact ground state configuration \( \{ J^0 \} \) of the vortex Hamiltonian(2).
2. Determine the resulting global flux along, say, the \( x \)-axis \( f_x = \frac{1}{L} \sum_i J^0_{ix} \);
3. Study a minimum-cost-flow problem in the actual cost for increasing the flow in the \( x \)-direction \( \Delta c_x = c_x(J^0_{ix} + 1) - c_x(J^0_{ix}) - b_x/2 + 1/2 \) is smoothly modified letting the cost of a topologically simple connected loop unchanged and only affecting global loops.
4. Reduce the \( \Delta c_x \) until the optimal flow configuration \( \{ J^1 \} \) for this min-cost-flow problem has the global flux \( f_x + 1 \), corresponding to the so called elementary low energy excitation on the length scale \( L \);
5. Finally, the defect energy is \( \Delta E = H(\{ J^1 \}) - H(\{ J^0 \}) \).

It turns out that the computation time grows linearly with \( B \) and it took about 150 seconds per samples for \( L = 24 \) with \( B = 1 \) on a SUN workstation (167MHz).

As for the zero-field case one expects for the disorder averaged excitation energy (or defect energy)

\[
[\Delta E(B, L)]_{av} \sim L^\theta ,
\]

where \( B \) is fixed, \( [\cdots]_{av} \) denotes the disorder average and \( \theta \) is the stiffness exponent and its sign determines whether there is a finite temperature phase transition or not, as explained above. If \( \theta < 0 \), i.e. the transition to a true superconducting vortex state appears only at zero temperature \( T = 0 \), scaling predicts that the thermal correlation length \( \xi \) diverges at \( T = 0 \) as \( \xi \propto T^{-\nu} \) with \( \nu = 1/|\theta| \), c.f. (6).

In the non-random case \( A_{ij}^{\text{rand}} = 0 \) an elementary excitation of the kind we described, i.e. a global loop of length \( L \) around the torus costs an energy of \( \Delta E \sim L^{\frac{2}{\nu}} \) without any dependence of the external field as well parallel as well as perpendicular to the external field (i.e. the \( z \)-direction). Note, that this is the situation for of the pure classical 3d XY-model with strong screening.

For a single configuration we find that a change of the external magnetic field \( B \) drastically affects the defect energy \( \Delta E \) (Fig. 3). \( \Delta E \) is a piecewise linear function that behaves in particular intervals \([B_a, B_b] \) as

\[
\frac{\partial \Delta E}{\partial B} = 2L n^2 ,
\]

\( n \) is in a range, where the defect energy \( \Delta E \) varies linear with respect to the field (see inset of Fig. 3). Note that the loop has also winding number \( n^z = 1 \) in the direction parallel to the external field. Hence \( \partial \Delta E / \partial B = 2L \). b) (middle) The same sample as in (a) with \( B \in [0.0070, 0.0075] \). In this interval the defect energy is constant, no loop along the direction of the applied field occurs. c) (right) The same sample as in (a,b) with \( B \in [0.0076, 0.0081] \). The system is very sensitive to the variation of applied field \( \Delta B \). Even for a small change by \( \Delta B = 0.0001 \) the form of the excitation loop changes drastically (compare with a and b).
which can be understood as follows: the external field varies continuously and the integer valued flow changes only in discrete steps, thus the minimum energy excitation loop may not change in a whole interval say \([B_a, B_b]\). In this interval \(\Delta E\) changes linearly with \(B\) since \(H(J^z) - H(J^y)\) is simply proportional to length of the excitation loop in the \(z\)-direction, which is \(n^z \cdot L\), with \(n^z\) the winding number of the loop in the \(z\)-direction \((n^z \in \{\ldots, -2, -1, 0, +1, +2, \ldots\})\).

Furthermore not only the ground state itself is extremely sensible to the small variations of the external field strength (as we have seen further above), but also the excitation loops themselves change their form dramatically, as it is exemplified in Fig. 4. Only small parts of the loop seem to persist over a significant range of the field strength, see for instance the in the vicinity of the plane \(z = 20\) in Fig. 3.

Now we study the behavior of the disorder averaged energy of excitation loops perpendicular to the applied magnetic field along, say, the \(z\)-direction (full diamonds in Fig. 4). Note that it is only necessary to study the situation \(B \in [0, 1]\), since all physical properties of the vortex glass Hamiltonian (3) are periodic in the strength of the external field \(B\), i.e. the filling factor. As can easily be seen in Fig. 4 the defect energy \([\Delta E(L, B)]_{av}\) is independent of the value of \(B\). For any fixed value of \(B\) the finite size scaling relation (1) is confirmed and gives \(\theta = -0.95 \pm 0.04\), c.f. (1).

We want to note that this behavior of excitations perpendicular to the applied field depends neither on the length of the system in the \(z\)-direction nor on the topology in this direction: we also studied the situation in which the vortex Hamiltonian (3) lives on a lattice with free instead of periodic b.c. in the \(z\)-direction. In this case the external field has an appropriate source and sink outside the system. We find here the same result as before: the disorder averaged defect energy is independent of the strength of the external field \(B\).

Finally, we also observe a field-independent domain wall energy for elementary excitation loops (open boxes in Fig. 4) parallel to the external field. Thus we conclude that the disorder averaged defect energy is also independent of the direction of the homogeneous external field.

### IV. SUMMARY

We have studied the three-dimensional vortex glass model in the strong screening limit in the presence of a homogeneous external field. The ground state is extremely sensible to small external field variations. Ground state configurations at different field values \(B\) and \(B + \Delta B\) decorrelate beyond the overlap correlation length \(l^* \sim \Delta B^{\zeta}\), where \(\zeta\) is the so called chaos exponent which we estimate to be \(\zeta = 3.8 \pm 0.2\). This value agrees within the error-bars with the chaos exponent for random perturbations of the quenched disorder that have been investigated in (1).

For individual disorder configurations the change of the defect energy \(\Delta E_L(B)\) with respect to the applied field \(B\) is piecewise linear, analytic and accompanied by a drastic deformation of the minimum energy global excitation loop. On the other hand the disorder averaged value of the defect energy \([\Delta E_L(B)]_{av}\) is independent of the strength of the external field \(B\). Moreover it turned...
out that also the excitation loops parallel as well as perpendicular to the external field yield the same disorder averaged value $|\Delta E_L(B)|_{av}$. Thus the scaling behavior of the defect energy is independent of $B$, i.e., identical to the case $B = 0$ already studied in \cite{10}. Therefore, as in the $B = 0$ case, we infer the absence of a true superconducting low temperature phase in the strongly screened vortex glass model in an external field. The stiffness exponent is again, as in the $B = 0$ case, $\theta = -0.95 \pm 0.04$, giving an estimate $\nu = 1/|\theta| = 1.05 \pm 0.05$ for the thermal correlation length exponent.

Concluding we would like to note that it would be interesting to perform the same analysis for non-vanishing screening length and for the unscreened case, where due to the long range repulsion of the vortex lines important new physics might appear. In particular it is an open whether a homogeneous external field has a significant effect on the existence of the low temperature vortex glass phase in the unscreened case.

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