Optimal and Learning Control for Autonomous Robots

Lecture Notes

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Foreword - ArXiv Edition

Optimal and Learning Control for Autonomous Robots has been taught in the Robotics, Systems and Controls Masters at ETH Zürich with the aim to teach optimal control and reinforcement learning for closed loop control problems from a unified point of view. The starting point is the formulation of an optimal control problem and deriving the different types of solutions and algorithms from there.

These lecture notes aim at supporting this unified view with a unified notation wherever possible, and a bit of a translation help to compare the terminology and notation in the different fields. There are a number of outstanding and complementary text books which inspired some of the material in here. We highly recommend to peruse these textbooks to deepen the knowledge in the specific topics.

Thus, some chapters follow relatively closely some well known text books (see below) with adapted notation, while others section have been originally written for the course.

- Section 1.3 - Robert F Stengel. Stochastic Optimal Control: Theory and Application. J. Wiley & Sons, New York, NY, USA, 1986 [2].

- Chapter 2 - Richard S. Sutton and Andrew G. Barto. Introduction to Reinforcement Learning. MIT Press, Cambridge, MA, USA, 1st edition, 1998 [3].

Slides and additional information on the course can be found at:
http://www.adrl.ethz.ch/doku.php/adrl:education:lecture

The course assumes basic knowledge of Control Theory, Linear Algebra and Stochastic Calculus.

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1 Optimal control

1.1 Principle of optimality

This section describes two different approaches of finding an optimal path through a graph, namely forward and backward search. Backwards search introduces the Principle of Optimality which lies at the core of dynamic programming. This section closely follows [1], pages 18-19.

1.1.1 Graph Search Problem

Consider the directed graph shown in Fig. 1.1. The goal is to find the path with the lowest cost from node A to node E, with the values at the edges corresponding to costs. Two different approaches, namely forward and backward search, are discussed in the following.

![Directed Graph](image)

Figure 1.1: Example of a directed graph with the cost at each edge. The goal is passing from A → E, while accumulating the least costs.

Forward Search

A forward search through this graph consists of computing the cost for all possible paths starting at node A until node E. The 10 possible paths can be seen in Fig. 1.2. The value inside each node corresponds to the accumulated cost from node A up to the specific node. The optimal path is the one with the lowest accumulated cost after reaching node E, namely path A - B₁ - C₁ - D₂ - E (bold path).

Now consider that we reformulate the problem and want to find the shortest path from node C₂ to E, after unintentionally reaching state C₂. We cannot reuse our calculated values of the accumulated costs (node A → E), because we do not care anymore about the cost to reach C₂. To find the
1.1 Principle of optimality

**Optimal control**

The principle of optimality states that an optimal policy has the particular property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

### Forward Search

[Figure 1.2: Forward search tree. The accumulated cost to reach each node from node A is shown inside the nodes. The optimal path with the lowest accumulated cost when reaching node E is shown in bold.]

The optimal path we must rebuild the tree, starting at \( C_2 \), and then again choose the lowest cost path. This rebuilding of the tree for every change of the initial state is computationally expensive. The following approach allows us to store the information in a more reusable way.

### Backward Search

A backward search tree is constructed by starting at the goal node and evaluating all possible paths towards the start node. The resulting tree can be seen in [Fig. 1.3](#). The values in the nodes describe the “cost-to-go” (“value function”) from this specific node to the goal node \( E \). The cost-to-go for each node is equal to the cost-to-go of the node one level above plus the cost of the connecting edge. The highlighted path in [Fig. 1.3](#) starting at node \( A \) is the optimal path from \( A \rightarrow E \) since it has the lowest cost-to-go, with the cost being 5.

[Figure 1.3: Backward search tree. The graph is constructed by starting at the goal node \( E \) and evaluating the possible paths towards the start node \( A \). The values inside each node refer to the cost to go from this node to the goal node \( E \).]
Storing the information about the graph as the cost-to-go, instead of the accumulated cost for each possible path, yields one main advantage: The optimal “control strategy” at each step is the one which leads to the node with the lowest cost-to-go. There is no dependency on how this node has been reached as in the forward search, so there is no need to build a new tree when deviating from our initially planned path. In terms of computational effort, the construction of both trees is comparable (considering one unit of computation for each node of the trees: 28 for the forward search tree, and 30 for the backward search tree). The relative cost for the two methods is only dependent on the structure of the directed graph. This is easily seen by considering what the computational cost of each method would be if the direction of all edges in the example were switched (i.e., reverse the notion of backwards and forwards). The real value in considering the cost-to-go for each node, though, is that we can easily reuse this information when deviating from our original plan.

1.1.2 Principle of optimality

The reason why the backward search tree in the previous example can be constructed as described is due to the Principle of optimality, which states:

If path $ABCDE$ is optimal, then $BCDE$ is optimal for the truncated problem.

In “control” terms: $U_0^*, U_1^*, U_2^*, \ldots, U_N^* \Rightarrow U_n^*, U_{n+1}^*, \ldots, U_N^*$.

This means it is possible to construct an optimal path in a piecemeal fashion: First an optimal path is found for only the last stage of the problem (e.g. $D \rightarrow E$). The problem can then be solved for the next stage, reusing the already solved “tail subproblem”. This iterative process reduces overall complexity and computational time by decomposing the problem into solvable sub-tasks. This principle is the foundation of the dynamic programming (DP) algorithms presented in the following sections.

1.2 Bellman equation

The Bellman equation introduces a functional for the value function (cost-to-go) of a given policy in discrete time systems. It expresses a relationship between the value function of a policy in a specific time and state to its successor states in the next time step. By applying the principle of optimality, the optimal Bellman equation can be derived for any discrete time optimal control problem. This section derives the Bellman equation in four different problem settings. The optimal control problem for discrete time deterministic systems is presented, first over a finite time horizon and then over an infinite time horizon. The optimal control problem for discrete time stochastic systems is then presented, again for both finite and infinite time horizons. For each problem setting, the notions of value function and optimal value function are introduced and the Bellman equation is then derived.

1.2.1 Finite time horizon, deterministic system
**Problem definition**  The problem definition consists of two components. The first is a model that describes the system dynamics by the function \( f \). This corresponds to the nodes and arrows of the graph in Figure 1.1. The second is the cost function \( J \), which captures the cost associated with a certain path taken over the whole time interval considered.

In the discrete time case, time steps \( t = t_n \) are indexed by integer values \( n \in \{0, 1, ..., N\} \). In this section, we consider a finite time interval \( t \in [t_0, t_N] \). For the system state at time \( t_n \), we use the short hand notation \( x_n = x(t_n) \). The discrete time deterministic system is described as

\[
x_{n+1} = f_n(x_n, u_n), \quad n \in \{0, 1, ..., N - 1\}
\]  

(1.1)

where

- \( n \) is the discrete time index,
- \( x_n \) is the state of the system at time \( n \),
- \( u_n \) is the control input at time \( n \) and
- \( f_n \) is the state transition equation.

The cost function \( J \) gives the cost associated with a particular trajectory, i.e. a time sequence of states \( x(t) \), starting from state \( x_0 \) at time step \( n = 0 \) up to the final state \( x_N \) at time step \( n = N \) under a specific control policy \( \mu = \{u_0, u_1, ..., u_{N-1}\} \).

\[
J = \alpha^N \Phi(x_N) + \sum_{k=0}^{N-1} \alpha^k L_k(x_k, u_k)
\]  

(1.2)

\( L(x_n, u_n) \) defines the intermediate cost incurred at each time step, as a function of the state and control input applied at that time step. \( \Phi(x_N) \) defines the terminal cost, which depends only on the state at the final time. \( \alpha \) is a parameter \( 0 \leq \alpha \leq 1 \) called the discount or decay rate, that continuously reduces the effect of costs further away in time (in the finite time horizon case this discount factor is usually set to 1).

The goal of the optimal control problem is to find the optimal control policy \( \mu^* \), that minimizes the cost function, and thus gives to optimal cost \( J^* \). This goal can equivalently be written as

\[
\mu^* = \arg \min_u J
\]  

(1.3)

**Value function**  As motivated in the previous chapter, it is more useful to consider the “cost-to-go” from a specific state to a goal state, than the “accumulated cost” \( J \). The cost-to-go from a specific
state \( x \) at a specific time step \( n \) when following a policy \( \mu \) is described by the value function

\[
V^\mu(n, x) = \alpha^{N-n} \Phi(x_N) + \sum_{k=n}^{N-1} \alpha^{k-n} L_k(x_k, u_k)
\]  

(1.4)

\[
x_n = x
\]

\[
x_{k+1} = f_k(x_k, u_k) \quad k = n, \ldots, N - 1
\]

\[
u_k = \mu(k, x_k)
\]

Note that the value function depends on both time \( n \) and state \( x \), which indicates the initial condition at time \( n \) for integrating the system dynamics. The value function evaluated at the final stage \( N \) corresponds to the terminal cost \( \Phi(\cdot) \) of the cost function.

\[
V^\mu(N, x) = \Phi(x)
\]  

(1.5)

The value function with the lowest value, e.g. the minimum cost, is called the optimal value function and is denoted as

\[
V^*(n, x) = \min_{\mu} V^\mu(n, x)
\]

(1.6)

Notice that in general a control policy, \( \mu \), that minimizes the right-hand side of equation (1.6) is a function of both time and state. Furthermore, due to the Principle of Optimality (Section 1.1.2), the control sequence that minimizes the value function in time \( n \) should be same as the tail sequence of a policy that minimizes the value function for a time step before the time step \( n \).

The corresponding optimal policy for the optimal value function is defined as

\[
\mu^* = \{u^*_n, \ldots, u^*_{N-1}\} = \arg \min_{\mu} V^\mu(n, x) \quad \forall n : 0, \ldots, N - 1
\]

(1.7)

\[
= \arg \min_{u_n, \ldots, u_{N-1}} \left\{ \alpha^{N-n} \Phi(x_N) + \sum_{k=n}^{N-1} \alpha^{k-n} L_k(x_k, u_k) \right\}
\]

Note that the optimal value function \( V^*(\cdot, \cdot) \) at time step 0 corresponds to the optimal accumulated cost \( J^* \)

\[
J^* = V^*(0, x_0)
\]  

(1.8)

**Bellman equation** The Bellman equation is derived starting with the definition of the value function (equation 1.4). Taking the intermediate cost at time \( n \) out of the summation and using the fact
1.2 Bellman equation

Optimal control

that \( x_n = x \) leads to the following equation

\[
V^\mu(n, x) = L_n(x, u_n) + \alpha^{N-n} \Phi(x_N) + \sum_{k=n+1}^{N-1} \alpha^{k-n} L_k(x_k, u_k)
\]  

(1.9)

Factoring \( \alpha \) out of the last terms leads to

\[
V^\mu(n, x) = L_n(x, u_n) + \alpha \left[ \alpha^{N-n-1} \Phi(x_N) + \sum_{k=n+1}^{N-1} \alpha^{k-n-1} L_k(x_k, u_k) \right]
\]

(1.10)

The terms inside the brackets are equal to \( V^\mu(n+1, x_{n+1}) \), where \( x_{n+1} = f(x, u_n) \):

\[
V^\mu(n, x) = L_n(x, u_n) + \alpha V^\mu(n+1, f_n(x, u_n))
\]

(1.11)

with final condition \( V^\mu(N, x) = \Phi(x) \).

As previously discussed, the optimal value function corresponds to the policy that minimizes the right-hand side of (1.11) which is known as the optimal Bellmann equation for deterministic systems

\[
V^*(n, x) = \min_{u_n} [L_n(x, u_n) + \alpha V^*(n+1, f_n(x, u_n))]
\]

(1.12)

and thus the optimal control policy at time \( n \) is computed as

\[
u^*(n, x) = \arg \min_{u_n} [L_n(x, u_n) + \alpha V^*(n+1, f_n(x, u_n))]
\]

(1.13)

To find the optimal value function as defined in equation (1.12), we search for the control input \( u_n \) that minimizes the sum of the instantaneous cost \( L_n \) and the optimal value function at the next time step, \( (n+1) \), considering the state which would be reached by applying \( u_n \). To solve this equation, one should first find the optimal control input for time instance \( n = N - 1 \) and then proceed backwards in time, finding the optimal control input at each step, until the first time instance \( n = 0 \). The optimal state trajectory will be obtained if the calculated optimal policy \( \mu^* = \{u^*_0, u^*_1, ..., u^*_{N-1}\} \) is applied to the system described by equation (1.1).

The advantage of using the optimal Bellman equation compared to solving (1.6) is problem decomposition: whereas in equation (1.6) the entire sequence of control inputs (policy) must be optimized for at once, the Bellman equation allows us to optimize for a single control input \( u_n \) at each time. Therefore the computational effort in the latter case increases only linearly with the number of time steps, as opposed to exponentially for equation (1.6).

1.2.2 Infinite time horizon, deterministic system
Problem definition In the infinite horizon case the cost to minimize resembles the finite horizon case except the accumulation of costs is never terminated, so $N \to \infty$ and (1.2) becomes

$$J = \sum_{k=0}^{\infty} \alpha^k L(x_k, u_k), \quad \alpha \in [0, 1]$$

(1.14)

Since the evaluation of the cost never ends, there exists no terminal cost $\Phi(\cdot)$. In contrast to Section 1.2.1, the discount factor $\alpha$ is usually chosen smaller than 1, since otherwise it is a summation over a infinite numbers which can lead to an unbounded cost. Additionally for the sake of convenience, the system dynamics $f(\cdot)$ and the instantaneous costs $L(\cdot)$ are assumed time-invariant.

Value function The optimal value function for (1.14) at time step $n$ and state $x$ can be defined as

$$V^*(n, x) = \min_{\mu} \left[ \sum_{k=n}^{\infty} \alpha^{k-n} L(x_k, u_k) \right].$$

(1.15)

Calculating the optimal value function for the same state $x$, but at a different time $n + \Delta n$ leads to

$$V^*(n + \Delta n, x) = \min_{\mu} \left[ \sum_{k=n+\Delta n}^{\infty} \alpha^{k-n-\Delta n} L(x_k, u_k) \right]$$

$$= \min_{\mu} \left[ \sum_{k'=n}^{\infty} \alpha^{k'-n} L(x_{k'+\Delta n}, u_{k'+\Delta n}) \right]$$

(1.16)

The only difference between (1.15) and (1.16) is the state trajectory over which the cost is calculated. However, since the system dynamics are time invariant and the initial state for the both paths is the same, these two trajectories are identical except for the shift in time by $\Delta n$. It follows that

$$V^*(n, x) = V^*(n + \Delta n, x) = V^*(x).$$

(1.17)

This shows that the optimal value function for the infinite time horizon is time-invariant and therefore only a function of the state.

Bellman equation Using the results from the finite time horizon Bellman equation and the knowledge that the value function for the infinite time horizon case is time-invariant, we can simplify (1.12) to give

$$V^*(x) = \min_u \{ L(x, u) + \alpha V^*(f(x, u)) \},$$

(1.18)

which is the optimal Bellman equation for the infinite time horizon problem.
1.2 Bellman equation

1.2.3 Finite time horizon, stochastic system

Problem definition We model stochastic systems by adding a random variable $w_n$ to the deterministic system’s dynamics.

$$x_{n+1} = f_n(x_n, u_n) + w_n$$ \hspace{1cm} (1.19)

$w_n$ can take an arbitrary conditional probability distribution given $x_n$ and $u_n$.

$$w_n \sim P_w(\cdot | x_n, u_n)$$ \hspace{1cm} (1.20)

We can now introduce a new random variable $x'$, which incorporates the deterministic and random parts of the system:

$$x_{n+1} = x'$$ \hspace{1cm} (1.21)

where $x'$ is distributed according to a new conditional distribution given $x_n$ and $u_n$.

$$x' \sim P_f(\cdot | x_n, u_n)$$ \hspace{1cm} (1.22)

Although any system described by equations (1.19) and (1.20) can be uniquely formulated by equations (1.21) and (1.22), the opposite is not correct. Therefore, equations (1.21) and (1.22) describe a more general class of discrete systems. We will consider this class of systems in the remainder of this subsection.

Once again, the goal is to find the control policy $\mu^* = \{u^*_0, u^*_1, \ldots, u^*_{N-1}\}$ which results in the path associated with the lowest cost defined by equation (1.23).

$$J = E \left[ \alpha^N \Phi(x_N) + \sum_{k=0}^{N-1} \alpha^{k-n} L_k(x_k, u_k) \right]$$ \hspace{1cm} (1.23)

Value function The value function for a given control policy in a stochastic system is identical to that which was used in the deterministic case, except for the fact that we must take the expectation of the value function to account for the stochastic nature of the system.

$$V^\mu(n, x) = E \left[ \alpha^{N-n} \Phi(x_N) + \sum_{k=n}^{N-1} \alpha^{k-n} L_k(x_k, u_k) \right]$$ \hspace{1cm} (1.24)

The optimal value function and optimal policy similarly follow as

$$V^*(n, x) = \min_{\mu} E \left[ \alpha^{N-n} \Phi(x_N) + \sum_{k=n}^{N-1} \alpha^{k-n} L_k(x_k, u_k) \right]$$ \hspace{1cm} (1.25)
\[ \mu^* = \arg \min_{\mu} E \left[ \alpha^{N-n} \Phi(x_N) + \sum_{k=n}^{N-1} \alpha^{k-n} L_k(x_k, u_k) \right] \] (1.26)

**Bellman equation** Through the same process as in the deterministic case, we can show that the Bellman equation for a given control policy will be as follows

\[ V^\mu(n, x) = L_n(x, u_n) + \alpha E_{x' \sim P_f(x \mid x, u_n)} \left[ V^\mu \left( n + 1, x' \right) \right] \] (1.27)

and the optimal Bellman equation is

\[ V^*(n, x) = \min_{u_n} \left[ L_n(x, u_n) + \alpha E_{x' \sim P_f(x \mid x, u_n)} \left[ V^* \left( n + 1, x' \right) \right] \right] \] (1.28)

and thus the optimal control at time \( n \) is computed as

\[ u^*(n, x) = \arg \min_{u_n} \left[ L_n(x, u_n) + \alpha E_{x' \sim P_f(x \mid x, u_n)} \left[ V^* \left( n + 1, x' \right) \right] \right]. \] (1.29)

Note that it is possible to convert Equation (1.28) to the deterministic case by assuming \( P_f(\cdot \mid x, u) \) as a Dirac delta distribution.

\[ P_f(x' \mid x, u_n) = \delta(x' - f(x, u_n)) \] (1.30)

### 1.2.4 Infinite time horizon, stochastic system

**Problem Definition** The Bellman equation for a stochastic system over an infinite time horizon is a combination of Section 1.2.2 and Section 1.2.3. As in Section 1.2.3 the system dynamics includes the stochastic variable \( w_n \) as

\[ x_{n+1} = f(x_n, u_n) + w_n. \] (1.31)

Since the stochastic cost cannot be minimized directly, we wish to minimize the expectation of the cost in Section 1.2.2 denoted as

\[ J = E \left[ \sum_{k=0}^{\infty} \alpha^k L(x_k, u_k) \right], \quad \alpha \in [0, 1) \] (1.32)

again with a discount factor \( \alpha < 1 \), time invariant systems dynamics \( f(\cdot) \) and instantaneous costs \( L(\cdot) \) as in the deterministic case described in Section 1.2.2.
1.3 Hamilton-Jacobi-Bellman Equation

**Optimal value function**  The optimal value function resembles Eq. (1.15), but is extended by the expectation $E$

$$V^*(x) = \min_{\mu} \mathbb{E} \left[ \sum_{n=0}^{\infty} \alpha^n L(x_n, u_n) \right]$$  \hspace{1cm} (1.33)

**Bellman equation**  The Bellman equation for the infinite time horizon stochastic system also uses the expectation and gives

$$V^*(x) = \min_{u_n} \{ L(x, u) + \alpha \mathbb{E}[V^*(x')] \}$$  \hspace{1cm} (1.34)

### 1.3 Hamilton-Jacobi-Bellman Equation

We now wish to extend the results obtained in the previous section to *continuous time* systems. As you will see shortly, in the continuous time setting, optimal solutions become less straightforward to derive analytically. We will first derive the solution to the optimal control problem for deterministic systems operating over a finite time horizon. This solution is famously known as the Hamilton-Jacobi-Bellman (HJB) equation. We will then extend the results to systems with an infinite time horizon, then to stochastic finite time horizon systems, and finally to stochastic infinite time horizon systems.

#### 1.3.1 Finite time horizon, deterministic system

**Problem Definition**  In this section we will consider a continuous time, non-linear deterministic system of the form,

$$\dot{x}(t) = f_t(x(t), u(t))$$  \hspace{1cm} (1.35)

Its corresponding cost function over a finite time interval, $t \in [t_0, t_f]$, starting from initial condition $x_0$ at $t_0$, is

$$J = e^{-\beta(t_f-t_0)} \Phi(x(t_f)) + \int_{t_0}^{t_f} e^{-\beta(t-t_0)} L(x(t), u(t)) dt,$$  \hspace{1cm} (1.36)

where $L(x(t), u(t))$ and $\Phi(x(t_f))$ are the intermediate cost and the terminal cost respectively. $\beta$ is a parameter $0 \leq \beta$ called the decay or discount rate, that continuously reduces the effect of costs further away in time (in the finite time horizon case this discount factor is usually set to 0). Our goal is to find a control policy which minimizes this cost.

**HJB equation**  In order to obtain an expression for the optimal cost-to-go starting from some $x$ at some time $t \in [t_0, t_f]$, we need to evaluate the cost function over the optimal trajectory, $x^*(t)$,
using the optimal control policy, \( u^*(t) \), during the remaining time interval \([t, t_f]\).

\[
V^*(t, x) = e^{-\beta(t_f - t)} \Phi(x^*(t_f)) + \int_t^{t_f} e^{-\beta(t' - t)} L(x^*(t'), u^*(t')) dt',
\]

(1.37)

We can informally derive the HJB equation from the Bellman equation by discretizing the system into \( N \) time steps as follows

\[
\delta t = \frac{t_f - t_0}{N}, \\
\alpha = e^{-\beta \delta t} \approx 1 - \beta \delta t \\
t_n = t_0 + n \delta t \\
x_{k+1} = x_k + f(x_k, u_k) \cdot \delta t
\]

The value function can now be approximated as a sum of instantaneous costs at each time step.

\[
\tilde{V}(t_n, x) = \alpha^{N-n} \Phi(x_N) + \sum_{k=n}^{N-1} \alpha^{k-n} L(x_k, u_k) \delta t,
\]

(1.38)

where \( n \in \{0, 1, ..., N\} \), and \( x \) is the state at time \( t_n \).

This is similar to the value function of a discrete time system (equation (1.2)). The discrete approximation of the optimal value function is

\[
\tilde{V}^*(t_n, x) = \min_{u \in U} \{ L(x, u) \delta t + \alpha \tilde{V}^*(t_{n+1}, x_{n+1}) \}.
\]

(1.39)

For small \( \delta t \), we can use the Taylor Series Expansion of \( \tilde{V}^* \) to expand the term on the right of the equation above.

\[
\tilde{V}^*(t_{n+1}, x_{n+1}) = \tilde{V}^*(t_n + \delta t, x + f(x, u) \delta t) \\
= \tilde{V}^*(t_n, x) + \Delta \tilde{V}^*(t_n, x) \\
= \tilde{V}^*(t_n, x) + \frac{\partial \tilde{V}^*(t_n, x)}{\partial t} \delta t + \left( \frac{\partial \tilde{V}^*(t_n, x)}{\partial x} \right)^T f(x, u) \delta t
\]

(1.40)

Higher order terms are omitted because they contain \( \delta t \) to the second power or higher, making their impact negligible as \( \delta t \) approaches 0.

Plugging (1.40) into (1.39), subtracting \( \tilde{V}^*(t_n, x) \) from both sides, and rearranging terms we get

\[
-\alpha \frac{\partial \tilde{V}^*(t_n, x)}{\partial t} \delta t + \beta \tilde{V}^*(t_n, x) \delta t = \min_{u \in U} \left\{ L(x, u) \delta t + \alpha \left( \frac{\partial \tilde{V}^*(t_n, x)}{\partial x} \right)^T f(x, u) \delta t \right\}
\]

(1.41)

Here we assumed that \( \delta t \) is small enough to allow us to approximate \( \alpha \) with \((1 - \beta \delta t)\). Letting \( t = t_n \), using \( \delta t \neq 0 \) to remove it from both sides of the equation, and assuming \( \tilde{V}^* \rightarrow V^* \) as \( \delta t \rightarrow 0 \), we
get the HJB equation (notice that $\lim_{\delta t \to 0} \alpha = 1$)

$$\beta V^* - \frac{\partial V^*}{\partial t} = \min_{u \in U} \left\{ L(x, u) + \left( \frac{\partial V^*}{\partial x} \right)^T f(x, u) \right\}$$  \hfill (1.42)

### 1.3.2 Infinite time horizon, deterministic system

**Problem Definition** We will now consider the case where the cost function includes an infinite time horizon. The cost function takes the form:

$$J = \int_{t_0}^{\infty} e^{-\beta(t-t_0)} L(x(t), u(t)) \, dt$$  \hfill (1.43)

where the terminal cost (formerly $\Phi(x(t_f))$) has dropped out. Furthermore, like the discrete case, the system dynamics and the intermediate cost are time-invariant. $\beta$ is a parameter $0 \leq \beta$ called the decay or discount rate, that in the infinite time horizon case is usually set to greater than 0.

**HJB equation** As it was the case for discrete-time systems, if we consider an infinite time horizon problem for a continuous-time system, $V^*$ is not a function of time. This means that $\frac{\partial V^*}{\partial t} = 0$ and the HJB equation simply becomes:

$$\beta V^* = \min_{u \in U} \left\{ L(x, u) + \left( \frac{\partial V^*}{\partial x} \right)^T f(x, u) \right\}$$  \hfill (1.44)

### 1.3.3 Finite time horizon, stochastic system

**Problem Definition** We will now consider a continuous time, stochastic system of the form

$$\dot{x}(t) = f_t(x(t), u(t)) + B(t)w(t), \quad x(0) = x_0$$  \hfill (1.45)

We assume that the stochasticity of the system can be expressed as additive white noise with mean and covariance given by:

$$E[w(t)] = \bar{w} = 0$$  \hfill (1.46)

$$E[w(t)w(\tau)^T] = W(t)\delta(t-\tau)$$  \hfill (1.47)

The Dirac delta in the covariance definition signifies that the noise in the system is uncorrelated over time. Unless $t = \tau$, the covariance will equal zero ($E[w(t)w(\tau)^T] = 0$).

Since the system is not deterministic, the cost function is defined as the expected value of the cost used in (1.36). Once again, $\beta \in [0, +\infty)$ is the decay or discount factor.

$$J = E \left\{ e^{-\beta(t_f-t_0)} \Phi(x(t_f)) + \int_{t_0}^{t_f} e^{-\beta(t'-t_0)} L(x(t'), u(t')) \, dt' \right\}$$  \hfill (1.48)
In order to obtain an expression for the optimal cost-to-go starting from some \( x \) at some time \( t \in [t_0, t_f] \), we need to evaluate the cost function over the optimal trajectory, \( x^*(t) \), using the optimal control policy, \( u^*(t) \), during the remaining time interval \([t, t_f]\).

\[
V^*(t, x) = E \left\{ e^{-\beta(t_f-t)} \Phi(x^*(t_f)) + \int_t^{t_f} e^{-\beta(t'-t)} L(x^*(t'), u^*(t')) dt' \right\}, \tag{1.49}
\]

Taking the total time derivative of \( V^* \) with respect to \( t \) gives the following:

\[
\frac{dV^*(t, x)}{dt} = \beta V^*(t, x) - E \{ L(x, u^*(t)) \} \tag{1.50}
\]

Note that \( \Phi(x^*(t_f)) \) is independent of \( t \), and \( t \) only occurs in the lower limit of the integral in (1.49). Since \( L(x^*(t), u^*(t)) \) is only a function of the optimal trajectory and control input at initial time \( t \), it is known with certainty (i.e. the system noise has no impact on its value at any point in time).

This means that we can remove the expectation to give the following, which will be used again at the end of the derivation.

\[
\frac{dV^*(t, x)}{dt} = \beta V^*(t, x) - L(x, u^*(t)) \tag{1.51}
\]

We can express the incremental change of \( V^* \) over time by its Taylor series expansion, considering that \( V^* \) depends on \( t \) both directly, and indirectly through its dependence on \( x(t) \).

\[
\Delta V^*(t, x) \approx \frac{dV^*(t, x)}{dt} \Delta t
\]

\[
= E \left\{ \frac{\partial V^*(t, x)}{\partial t} \Delta t + \left( \frac{\partial V^*(t, x)}{\partial x} \right)^T \Delta x + \frac{1}{2} x^T \frac{\partial^2 V^*(t, x)}{\partial x^2} \Delta x \Delta t \right\}. \tag{1.52}
\]

Note that the expectation must be taken here because of the appearance of \( \Delta x \), which depends on the system noise. All second-order and higher terms can be dropped from the expansion since the value of \( \Delta t^2 \) is negligible as \( \Delta t \to 0 \). The last term on the right is kept, though, because as you will see shortly, the second partial derivative includes the covariance of our system noise, which has been modeled as a Dirac delta function (equation (1.47)).

We can now plug in the system equation (1.45), using the simplified notation \( f := f_t(x(t), u(t)) \). In addition, \( V_x^* := \frac{\partial V^*}{\partial x} \) is used to simplify notation of the partial derivatives.

\[
\frac{dV_x^*}{dt} \Delta t = E[V_t^* \Delta t + V_x^*T (f + Bw) \Delta t + \frac{1}{2} (f + Bw)^T V_{xx}^* (f + Bw) \Delta t^2] \tag{1.53}
\]

We can pull the first term out of the expectation since \( V_t^* \) only depends on \( x \), which is known with certainty. We can also pull the second term out of the expectation, and use \( E[w(t)] = 0 \) to simplify

\[\text{HJB equation}\]
The third term is replaced by its trace, since the trace of a scalar is equal to itself. This will be useful in the next step. Dividing both sides by $\Delta t$, the time derivative becomes:

$$\frac{dV^*}{dt} = V_t^* + V_x^T f + \frac{1}{2} \text{Tr}\{E[(f + Bw)^T V_{xx}^*(f + Bw)]\Delta t}\}. \quad (1.54)$$

Using the matrix identity $\text{Tr}[AB] = \text{Tr}[BA]$, we rearrange the terms inside of the expectation. Then, since $V_{xx}^*$ only depends on $x$, which is known without uncertainty, we can remove it from the expectation.

$$\frac{dV^*}{dt} = V_t^* + V_x^T f + \frac{1}{2} \text{Tr}\{V_x^* E[(f + Bw)(f + Bw)^T]\Delta t\}. \quad (1.55)$$

Expanding the terms inside of the expectation, and removing all terms known with certainty from the expectation gives:

$$\frac{dV^*}{dt} = V_t^* + V_x^T f + \frac{1}{2} \text{Tr}\left[V_{xx}^* \left(f^T \Delta t + 2fE(w^T)B^T \Delta t + BE(ww^T)B^T \Delta t\right)\right]. \quad (1.56)$$

After plugging in our noise model (equation 1.47), the second term in $\text{Tr}()$ drops out and the last term simplifies to give

$$\frac{dV^*}{dt} = V_t^* + V_x^T f + \frac{1}{2} \text{Tr}\left[V_{xx}^* BWB^T \delta(t)\Delta t\right]. \quad (1.57)$$

Assuming that $\lim_{\Delta t \to 0} \delta(t)\Delta t = 1$, and taking the limit as $\Delta t \to 0$,

$$\frac{dV^*}{dt} = V_t^* + V_x^T f + \frac{1}{2} \text{Tr}\left[V_{xx}^* BWB^T\right]. \quad (1.58)$$

Plugging equation (1.51) into the left side, and indicating that the optimal cost requires minimization over $u(t)$ gives the stochastic principle of optimality

$$\beta V^*(t, x) - V_t^*(t, x) = \min_{u(t)} \left\{ L(x, u(t)) + V_x^T f_t(x, u(t)) + \frac{1}{2} \text{Tr}[V_{xx}^* B(t)W(t)B^T(t)]\right\} \quad (1.59)$$

with the terminal condition,

$$V^*(t_f, x) = \Phi(x). \quad (1.60)$$

### 1.3.4 Infinite time horizon, stochastic system

**Problem Definition** Finally, consider a stochastic time-invariant system similar to the form in equation (1.45) over an infinite time horizon. The cost function is similar to the cost function in equation (1.43), but we now must consider the expected value of the cost due to the stochasticity.
in the system.

\[
J = E \left\{ \int_{t_0}^{\infty} e^{-\beta(t-t_0)} L(x(t), u(t)) dt \right\}, \tag{1.61}
\]

**HJB equation** Once again, over an infinite time horizon, the value function is no longer a function of time. As a result, the second term in the left hand side of equation 1.59 equals zero, and the HJB equation becomes

\[
\beta V^*(x) = \min_{u(t)} \left\{ L(x, u(t)) + V_x^*(x)f_t(x, u(t)) + \frac{1}{2} \text{Tr}[V_{xx}^* B(t) W(t) B^T(t)] \right\} \tag{1.62}
\]
### 1.4 Summary of Results

In the following table, a summary of the results in the two preceding sections is presented.

|                 | Discrete Time                                                   | Continuous Time                                                  |
|-----------------|-----------------------------------------------------------------|------------------------------------------------------------------|
| **Optimization Problem:** | $x_{n+1} = f(x_n, u_n) + w_n$ | $dx = f(x_t, u_t)dt + B(x_t, u_t)dw_t$ |
| $w_n \sim P_w(\cdot | x_n, u_n)$ | $w_t \sim \mathcal{N}(0, \Sigma)$ | $w_t \sim \mathcal{N}(0, 0)$ i.e. $\Sigma = 0$ |
| $\min_{u_0 \rightarrow N-1} E \{ \alpha^N \Phi(N) + \sum_{k=0}^{N-1} \alpha^k L(x_k, u_k) \}$ | $\min_{u_0 \rightarrow t_f} E \{ e^{-\beta t_f} \Phi(t_f) + \int_0^{t_f} e^{-\beta t} L(x_t, u_t) dt \}$ | $\min_{u_t} \{ L(x_t, u_t) + V^*_{xT}(t, x) f(x_t, u_t) + \frac{1}{2} \text{Tr}[V^*_{xx}(t, x) B \Sigma B^T] \}$ |
| $V^*(n, x) = \min_{u_n} \{ L(x, u_n)$ | $\Phi(N) = 0$ $V^*$ is not function of time. | $\Phi(t_f) = 0$ $V^*$ is not function of time. |
| $+ \alpha E[V^*(n+1, x_{n+1})]]}$ | | |
| **Bellman equation:** | $V^*(n, x) = \min_{u_n} \{ L(x, u_n)$ | $\beta V^*(t, x) - V^*_t(t, x) = \min_{u_t} \{ L(x_t, u_t) + V^*_{xT}(t, x) f(x_t, u_t) \}$ |
| $+ \alpha E[V^*(n+1, x_{n+1})]]}$ | $\Phi(N) = 0$ $V^*$ is not function of time. | $\Phi(t_f) = 0$ $V^*$ is not function of time. |
| **Deterministic System:** | | |
| Optimization Problem: | $x_{n+1} = f(x_n, u_n)$ | |
| $\min_{u_0 \rightarrow N-1} \{ \alpha^N \Phi(N) + \sum_{k=0}^{N-1} \alpha^k L(x_k, u_k) \}$ | | |
| $\Phi(N) = 0$ $V^*$ is not function of time. | $\Phi(t_f) = 0$ $V^*$ is not function of time. | |

Section 13.3
1.5 Iterative Algorithms

In general, an optimal control problem with a nonlinear cost function and dynamics does not have an analytical solution. Therefore numerical methods are required to solve it. However, the computational complexity of the optimal control problem scales exponentially with the dimension of the state space. Even though various algorithms have been proposed in the literature to solve the optimal control problem, most of them do not scale to the high dimension problems. In this Section, we introduce a family of methods which approximate this complex problem with a set of tractable sub-problems before solving it.

This section starts by introducing a method for solving optimization problems with static constraints. This is a simpler problem to solve than our optimal control problem, since system dynamics appear there as dynamic constraints. We will then show the extension of this method to an optimal control problem where the constraints are dynamic. Finally we will conclude the section by introducing an algorithm which implements this idea.

1.5.1 Sequential Quadratic Programming: SQP

Assuming that $f(x)$ is a general nonlinear function, a problem of the form

$$\min_x f(x) \quad x \in \mathbb{R}^n$$

s.t.  \quad $f_j(x) \leq 0, \quad j = 1, \ldots, N$

$$h_j(x) = 0, \quad j = 1, \ldots, N$$

is called a nonlinear programming problem with inequality and equality constraints. Setting the gradient of the Lagrangian function equal to zero and solving for $x$ will usually not return a closed form solution (e.g. for $\nabla (x \sin x) = \sin x + x \cos x = 0$). However, there will always exist a closed form solution for the two following special cases:

- Linear Programming (LP): In LP, the function $f$ and all the constraints are linear w.r.t. $x$. An example algorithm for solving an LP is “Dantzig’s Simplex Algorithm”.

- Quadratic Programming (QP): In QP, the function $f$ is quadratic, but all the constraints are linear w.r.t. $x$. Example algorithms for solving a QP are the “Broyden–Fletcher–Goldfarb–Shanno” (BFGS) algorithm and the “Newton-Raphson Method”.

One way to approach a general nonlinear problem of the form (1.63), is to iteratively approximate it by a QP. We start by guessing a solution $\tilde{x}_0$. Then we approximate $f(x)$ by its first and second order Taylor expansion around this initial point. We also approximate all the constraints by their first
order Taylor expansion around this point.

\[
f(x) \approx f(\tilde{x}_0) + (x - \tilde{x}_0)^T \nabla f(\tilde{x}_0) + \frac{1}{2}(x - \tilde{x}_0)^T \nabla^2 f(\tilde{x}_0)(x - \tilde{x}_0)
\]

\[
f_j(x) \approx f_j(\tilde{x}_0) + (x - \tilde{x}_0)^T \nabla f_j(\tilde{x}_0)
\]

\[
h_j(x) \approx h_j(\tilde{x}_0) + (x - \tilde{x}_0)^T \nabla h_j(\tilde{x}_0)
\]

This problem can now be solved by one of the QP solvers mentioned previously to obtain a new solution \(\tilde{x}_1\). We then iteratively approximate the objective function and the constraints around the new solution and solve the QP problem. It can be shown that for a convex problem this algorithm converges to the optimal solution as \(\lim_{i \to \infty} \tilde{x}_i = x^*\).

**Example of a SQP solver - the Newton-Raphson Method:** One method for iteratively solving a SQP, e.g. finding the zeros of the function \(f'(x)\) if no closed form solution exists, is the Newton-Raphson Method. Consider the following update law:

\[
\tilde{x}_1 = \tilde{x}_0 - \frac{f'(|x_0|)}{f''(|x_0|)}
\]

(1.65)

Starting from an initial guess \(\tilde{x}_0\), and knowing the first derivative \(f'(\tilde{x}_0)\) and second derivative \(f''(\tilde{x}_0)\) at that specific point \(\tilde{x}_0\), a better approximation of our optimization problem is given by \(\tilde{x}_1\). This procedure is repeated using the new approximation \(\tilde{x}_1\) until convergence.

![Figure 1.4: Newton-Raphson Method for finding the zeros (x-axis intersection) of a function \(f'(x)\) shown in red, which is equivalent to finding the extreme points of \(f(x)\).](image)

Note that for finding the minimum of a quadratic function \(f(x) = 0.5ax^2 + bx + c\), the Newton-Raphson Method will find the minimizer \(x^*\) of the function in only one iteration step, independent for the sake of simplicity we have omitted all the constraints in this example.
of the initial guess $x_0$:

$$x^* = \tilde{x}_0 - \frac{f'(\tilde{x}_0)}{f''(\tilde{x}_0)} = \tilde{x}_0 - \frac{a\tilde{x}_0 + b}{a} = -\frac{b}{a}$$

### 1.5.2 Sequential Linear Quadratic Programming: SLQ

Sequential Linear Quadratic Programming (SLQ) is a family of algorithms for solving optimal control problem involving a non-linear cost function and non-linear system dynamics. In discrete time this problem is defined as follows

$$\min_{\mu} \left[ \Phi(x(N)) + \sum_{n=0}^{N-1} L_n(x(n), u(n)) \right]$$

s.t. $x(n+1) = f(x(n), u(n))$ $x(0) = x_0$

$$u(n, x) = \mu(n, x)$$

SLQ methods are a class of algorithms which are based on the idea of fitting simplified subproblems over the original problem. Assuming that solving the optimality conditions is easier for these subproblems, the solution can be improved iteratively by optimizing over these subproblems. Since linear-quadratic problems are almost the most difficult problems which have a closed form solution, the subproblems are chosen to be linear-quadratic. One can see the similarity between SLQ and SQP through the way that the original problem is decomposed, and the way subproblems are chosen. In both algorithms the solution of the primary problem is derived by iteratively approximating a linear-quadratic subproblem around the latest update of the solution and optimizing over this subproblem respectively. A general overview of the SLQ algorithm is as follows:

1. Guess an initial (stabilizing) control control policy $\mu^0(n, x)$.

2. “Roll out”: Apply the control policy to the non-linear system \ref{eq:nonlinear_system} (forward integration), which yields the state trajectory $X_k = \{x(0), x(1), \ldots, x(N)\}$ and input trajectory $U_k = \{u(0), u(1), \ldots, u(N-1)\}$.

3. Starting with $n = N - 1$, approximate the value function as a quadratic function around the pair $(x(N-1), u(N-1))$.

4. Having a quadratic value function, the Bellman equation can be solved efficiently. The output of this step is a control policy at time $N - 1$ which minimizes the quadratic value function.

5. “Backward pass”: Repeat steps 3 and 4 for every state-input pair along the trajectory yielding $\delta \mu = \{\bar{u}(0, x), \ldots, \bar{u}(N-1, x)\}$. The updated optimized control inputs are then calculated with an appropriate step-size $\alpha_k$ from

$$\mu^{k+1} = \mu^k + \alpha_k \cdot \delta \mu^k$$

\[1.69\]
6. Iterate through steps 2 → 5 using the updated control policy μ_{k+1} until a termination condition is satisfied, e.g. no more cost improvement or no more control vector changes.

Notice that if our system dynamics is already linear, and the cost function quadratic, then only one iteration step is necessary to find the globally optimal solution, similar to (1.66). In this case the SLQ controller reduces to a LQR controller. In the next section, a member of the SLQ algorithm family, called the Iterative Linear Quadratic Controller (ILQC), will be introduced.

1.6 Iterative Linear Quadratic Controller: ILQC

The ILQC is an iterative Linear-Quadratic method for locally-optimal feedback control of nonlinear, deterministic discrete-time systems. Given an initial, feasible sequence of control inputs, we iteratively obtain a local linear approximation of the system dynamics and a quadratic approximation of the cost function, and then an incremental improvement to the control law, until we reach convergence at a local minimum of our cost function.

1.6.1 ILQC Problem statement

Similar to Section 1.2.1, we consider the discrete-time, finite-horizon, nonlinear dynamic system

\[ x_{n+1} = f_n(x_n, u_n), \quad x(0) = x_0, \quad n \in \{0, 1, ..., N - 1\} \tag{1.70} \]

with state-vector \( x_n \) and control input vector \( u_n \). Let the cost function be

\[ J = \Phi(x_N) + \sum_{n=0}^{N-1} L_n(x_n, u_n) , \tag{1.71} \]

which corresponds to Equation (1.2) with \( \alpha = 1 \). We denote \( L_n(x_n, u_n) \) the intermediate, non-negative cost rate at time-step \( n \) and \( \Phi(x_N) \) the terminal, non-negative cost at time-step \( N \). Therefore, the cost function \( J \) gives the undiscounted cost associated with a particular trajectory starting from state \( x_0 \) at time step \( n = 0 \) up to the final state \( x_N \) at time step \( n = N \) under a deterministic control policy

\[ \mu = \{u_0, u_1, ..., u_{N-1}\} . \tag{1.72} \]

The general goal of optimal control is to find an optimal policy \( \mu^* \) that minimizes the total cost \( J \). Finding the optimal controls in a general, nonlinear setting by solving the Bellman Equation (1.12) is typically infeasible, because there is no analytic solution for the Bellman Equation if the system is nonlinear and the cost-function is non-quadratic. Instead, we aim at finding a locally optimal control law which approximates \( \mu^* \) in the neighborhood of a local minimum.
1.6.2 Local Linear-Quadratic Approximation

The locally-optimal control law is constructed in an iterative way. In each iteration, we begin with a stable control policy $\mu(n, x)$. Starting at initial condition $\bar{x}(0) = x_0$, the corresponding nominal state-trajectory $\{\bar{x}_n\}$ and control input trajectory $\{\bar{u}_n\}$ for the nonlinear system can be obtained by forward integration of Equation (1.70) using the policy $\mu$.

Now, we linearize the system dynamics and quadratize the cost function around every pair $(\bar{x}_n, \bar{u}_n)$. To do so, we introduce the state and control input increments as follows

$$
\delta x_n = x_n - \bar{x}_n \\
\delta u_n = u_n - \bar{u}_n
$$

Since $\{\bar{x}_n\}$ and $\{\bar{u}_n\}$ are satisfying the system dynamics in equation (1.70), we will have $\delta x(0) = 0$.

For linearizing the system dynamics we substitute $x_n$ and $u_n$ by the definitions in equation (1.73) and then approximate $f$ by its first order Taylor expansion

$$
\bar{x}_{n+1} + \delta x_{n+1} = f_n(\bar{x}_n + \delta x_n, \bar{u}_n + \delta u_n) \\
= f_n(\bar{x}_n, \bar{u}_n) + \frac{\partial f(\bar{x}_n, \bar{u}_n)}{\partial x} \delta x_n + \frac{\partial f(\bar{x}_n, \bar{u}_n)}{\partial u} \delta u_n
$$

Using $\bar{x}_{n+1} = f_n(\bar{x}_n, \bar{u}_n)$ to simplify the approximation, we obtain

$$
\delta x_{n+1} \approx A_n \delta x_n + B_n \delta u_n
$$

where $A_n$ and $B_n$ are independent of $\delta x_n$ and $\delta u_n$. Notice that as long as the nominal trajectories $x_n$ and $u_n$ are time dependent, $A_n$ and $B_n$ are time varying. Therefore, the linear approximation transforms a nonlinear (either time-variant or time-invariant) system into a linear time-variant system.

We wish also to quadratize the cost function with respect to the nominal state and control trajectories.

$$
J \approx q_N + \delta x_N^T q_N + \frac{1}{2} \delta x_N^T Q_N \delta x_N \\
+ \sum_{n=0}^{N-1} \{ q_n + \delta x_n^T q_n + \delta u_n^T r_n + \frac{1}{2} \delta x_n^T Q_n \delta x_n + \frac{1}{2} \delta u_n^T R_n \delta u_n + \delta u_n^T P_n \delta x_n \}
$$

(1.77)
1.6 Iterative Linear Quadratic Controller: ILQC

where the cost function elements are defined as

\[ q_n = L_n(\bar{x}_n, \bar{u}_n), \quad q_n = \frac{\partial L(\bar{x}_n, \bar{u}_n)}{\partial x}, \quad Q_n = \frac{\partial^2 L(\bar{x}_n, \bar{u}_n)}{\partial x^2} \]

\[ p_n = \frac{\partial^2 L(\bar{x}_n, \bar{u}_n)}{\partial u \partial x}, \quad r_n = \frac{\partial L(\bar{x}_n, \bar{u}_n)}{\partial u}, \quad R_n = \frac{\partial^2 L(\bar{x}_n, \bar{u}_n)}{\partial u^2} \]

\[ n = N : \quad q_N = \Phi(\bar{x}_N), \quad q_N = \frac{\partial \Phi(\bar{x}_N)}{\partial x}, \quad Q_N = \frac{\partial^2 \Phi(\bar{x}_N)}{\partial x^2} \]

Note that all derivatives w.r.t. \( u \) are zero for the terminal time-step \( N \). Using the above linear-quadratic approximation to the original problem, we can derive an approximately optimal control law.

1.6.3 Computing the Value Function and the Optimal Control Law

In this section we will show that, if the value function (cost-to-go function) is quadratic in \( \delta x_{n+1} \) for a certain time-step \( n+1 \), it will stay in quadratic form during back-propagation in time, given the linear-quadratic approximation presented in Equations (1.75) to (1.78).

Now, suppose that for time-step \( n+1 \), we have the value function of the state deviation \( \delta x_{n+1} \) given as a quadratic function of the form

\[ V^*(n+1, \delta x_{n+1}) = s_{n+1} + \delta x_{n+1}^T s_{n+1} + \frac{1}{2} \delta x_{n+1}^T A_{n+1} s_{n+1} \delta x_{n+1} . \]  

(1.79)

We can write down the Bellman Equation for the value function at the previous time-step \( n \) as

\[ V^*(n, \delta x_n) = \min_{u_n} [L_n(x_n, u_n) + V^*(n+1, \delta x_{n+1})] \]

Assuming that a control input \( \delta u_n \) is given from a policy \( \mu \) and plugging in Equation (1.77) leads to

\[ V^*(n, \delta x_n) = \min_{u_n} \left[ q_n + \delta x_n^T (q_n + \frac{1}{2} Q_n \delta x_n) + \delta u_n^T (r_n + \frac{1}{2} R_n \delta u_n) + \delta u_n^T P_n \delta x_n \right. 

\[ + V^*(n+1, A_n \delta x_n + B_n \delta u_n) \]  

Using equation (1.79) for \( V(n+1, A_n \delta x_n + B_n \delta u_n) \) and re-grouping terms results in

\[ V^*(n, \delta x_n) = \min_{u_n} \left[ q_n + s_{n+1} + \delta x_n^T (q_n + A_n^T s_{n+1}) \right. 

\[ + \frac{1}{2} \delta x_n^T (Q_n + A_n^T S_{n+1} A_n) \delta x_n + \delta u_n^T (g_n + G_n \delta x_n) + \frac{1}{2} \delta u_n^T H_n \delta u_n \]  

(1.80)

| 22 |
where we have defined the shortcuts

\[ g_n \triangleq r_n + B_n^T s_{n+1} \]
\[ G_n \triangleq P_n + B_n^T s_{n+1} A_n \]
\[ H_n \triangleq R_n + B_n^T s_{n+1} B_n \]  

(1.81)

for each time-step \( n \). At this point we notice that the expression for \( V^*(n, \delta x_n) \) is a quadratic function of \( u_n \) (see highlighted box in equation 1.80). In order to minimize \( V^*(n, \delta x_n) \), we set \( u_n \) to the value which makes its gradient w.r.t. \( u_n \) vanish. Therefore, we obtain the optimal control law for this update step as

\[ \delta u_n = -H_n^{-1} g_n - H_n^{-1} G_n \delta x_n \]  

(1.82)

It can be seen that the optimal control input consists of a feed-forward term \( \delta u_n^{ff} = -H_n^{-1} g_n \) and a feedback term \( K_n \delta x_n \) with feedback gain matrix \( K_n := -H_n^{-1} G_n \). Replacing \( \delta u_n \) by \( \delta u_n^{ff} + K_n \delta x_n \), the expression which is highlighted in equation (1.80) becomes

\[ \delta u_n^{ff} g_n + \frac{1}{2} \delta u_n^{ff}^T H_n \delta u_n^{ff} + \delta x^T (G_n^T \delta u_n^{ff} + K_n^T g_n + K_n^T H_n \delta u_n^{ff}) \]

+ \[ \frac{1}{2} \delta x^T (K_n^T H_n K_n + K_n^T G_n + G_n^T K_n) \delta x \]  

(1.83)

where all time-step indices have been omitted for readability. Obviously, Equation (1.83) is quadratic in \( \delta x \) and therefore the whole value-function remains quadratic in \( \delta x \) throughout the backwards step in time. Thus, assuming the terminal cost is also approximated quadratically w.r.t \( \delta x \), the value-function remains quadratic for all \( n \).

By plugging (1.82) back into equation (1.80) and using the quadratic assumption for value function introduced in (1.79) to replace the left hand side of (1.80), we will obtain a quadratic functional equation of \( \delta x \). This functional should always remain identical to zero for any arbitrary \( \delta x \). Thus we can conclude, all the coefficients of this quadratic functional should be identical to zero which will lead to the following recursive equation for \( s_n \), \( s_n \), and \( s_n \).

\[ S_n = Q_n + A_n^T s_{n+1} A_n + K_n^T H_n K_n + K_n^T G_n + G_n^T K_n \]  

(1.84)

\[ s_n = q_n + A_n^T s_{n+1} + K_n^T H_n \delta u_n^{ff} + K_n^T g_n + G_n^T \delta u_n^{ff} \]  

(1.85)

\[ s_n = q_n + s_{n+1} + \frac{1}{2} \delta u_n^{ff}^T H_n \delta u_n^{ff} + \delta u_n^{ff}^T g_n \]  

(1.86)

which is valid for all \( n \in \{0, \cdots, N-1\} \). For the final time-step \( N \) we have the following terminal conditions:

\[ S_N = Q_N, \quad s_N = q_N, \quad s_N = q_N \]  

(1.87)
1.7 Linear Quadratic Regulator: LQR

Equation (1.84) to (1.86) should be solved backward in time starting from the final time-step \( N \) with the terminal conditions (1.87). As we are propagating backward in time, the optimal policy will also be calculated through equation (1.82). However we should notice that this policy is in fact the incremental policy. Therefore for applying this policy to the system we first should add the nominal control trajectory to this policy.

\[
\begin{align*}
\mathbf{u}(n, \mathbf{x}) &= \bar{\mathbf{u}}_n + \delta \mathbf{u}_{\text{ff}} + K_n(\mathbf{x}_n - \bar{\mathbf{x}}_n) \\
\end{align*}
\]  

(1.88)

1.6.4 The ILQC Main Iteration

In this section, we summarize the steps of the ILQC algorithm:

0. **Initialization**: we assume that an initial, feasible policy \( \mu \) and initial state \( x_0 \) is given. Then, for every iteration \( i \):

1. **Roll-Out**: perform a forward-integration of the nonlinear system dynamics (1.70) subject to initial condition \( x_0 \) and the current policy \( \mu \). Thus, obtain the nominal state- and control input trajectories \( \bar{\mathbf{u}}_{\text{ff}}^{(i)}, \bar{\mathbf{x}}_{\text{ff}}^{(i)} \) for \( n = 0, 1, \ldots, N \).

2. **Linear-Quadratic Approximation**: build a local, linear-quadratic approximation around every state-input pair \( (\bar{\mathbf{u}}_{\text{ff}}^{(i)}, \bar{\mathbf{x}}_{\text{ff}}^{(i)}) \) as described in Equations (1.75) to (1.78).

3. **Compute the Control Law**: solve equations (1.84) to (1.86) backward in time and design the affine control policy through equation (1.88).

4. Go back to 1. and repeat until the sequences \( \bar{\mathbf{u}}_{\text{ff}}^{(i+1)} \) and \( \bar{\mathbf{u}}_{\text{ff}}^{(i)} \) are sufficiently close.

1.7 Linear Quadratic Regulator: LQR

In this section, we will study the familiar LQR for both discrete and continuous time systems. LQR stands for Linear (as system dynamics are assumed linear) Quadratic (as the cost function is purely quadratic w.r.t states and inputs) Regulator (since the states are regulated to zero). As the naming implies, it is an optimal control problem for a linear system with a pure quadratic cost function to regulate the system’s state to zero. The LQR problem is defined for both discrete and continuous time systems in a deterministic setup.

This section starts by deriving the LQR solution for the discrete time case. To do so, we will use the results obtained in Section 1.6 for the ILQC controller. The similarity between these two problems readily becomes clear by comparing their assumptions for the underlying problem. In Section 1.6 we have seen that ILQC finds the optimal controller by iteratively approximating the nonlinear problem with a subproblem which has linear dynamics and quadratic cost function. Therefore if the original problem has itself linear dynamics and quadratic cost, the algorithm converges to the optimal solution at the fist iteration i.e. the solution to the first subproblem is the solution to the original problem. Furthermore since the LQR problem even has more restricted assumptions, the algorithm can even
be simplified more. In Section 1.7.2 the infinite time horizon LQR solution is obtained by using the result we derive in finite time horizon case.

For the continuous time case, we go back to the original deterministic HJB equation. Using the linear, quadratic nature of the LQR problem, we derive an efficient algorithm to find the global optimal solution of the problem. Finally we extend these results to the infinite time horizon problem. Note that although we have chosen different approach for deriving the discrete-time LQR solution, starting from the original Bellman equation would give an identical solution.

1.7.1 LQR: Finite time horizon, discrete time

From the results derived in the previous section, we can easily derive the familiar discrete-time Linear Quadratic Regulator (LQR). The LQR setting considers very similar assumptions to those used in ILQC, but used in a much stricter sense. It assumes that the system dynamics are linear. Furthermore it assumes that the cost function only consists of pure quadratic terms (no bias and linear terms). As discussed in Section 1.5.2 for the case of linear system with quadratic cost function the SLQ algorithm converges to the global optimal solution in the first iteration. Therefore in order to derive the LQR controller, we just need to solve the first iteration of the iLQC algorithm.

Here we consider a discrete time system with linear (or linearized) dynamics of the form:

\[ x_{n+1} = A_n x_n + B_n u_n \]  \hspace{1cm} (1.89)

This is very similar to the linearized dynamics from equation (1.75), but note that here \( A \) and \( B \) are not seen as local approximations around some nominal trajectories, as they were in Section 1.6.2. We are also no longer representing the state and control input as deviations from nominal trajectories, as in (1.73). This is because in the LQR setting, we assume that we are regulating the system to zero states, which also implies zero control input since the system is linear and has a unique equilibrium point at the origin. Therefore we have

\[
\delta x_n = x_n \\
\delta u_n = u_n
\]

In addition, we assume that we are trying to determine a control policy which minimizes a pure quadratic cost function of the form

\[
J = \frac{1}{2} x_N^T Q_N x_N + \sum_{n=0}^{N-1} \frac{1}{2} x_n^T Q_n x_n + \frac{1}{2} u_n^T R_n u_n + u_n^T P_n x_n. \hspace{1cm} (1.90)
\]

Therefore the first derivative of the cost function with respect to both \( x \) and \( u \) will be zero. From equations (1.78), \( q_n \), \( q_n \), and \( r_n \) must equal 0 for all \( n \). This then implies that in equations (1.81),
1.7 Linear Quadratic Regulator: LQR

Optimal control

\( g_n = r_n + B_n^T s_{n+1} = 0 \)

\( s_n = q_n + A_n^T S_{n+1} + K_n^T H_n \delta u_n^{ff} + K_n^T g_n + G_n^T \delta u_n^{fl} = 0 \)

\( s_n = q_n + s_{n+1} + \frac{1}{2} \delta u_n^{ff}^T H_n \delta u_n^{ff} + \delta u_n^{fl}^T g_n = 0 \)

Substituting these results in the equation (1.79), will result a value function which has a pure quadratic form with respect to \( x \).

\[ V^*(n, x) = \frac{1}{2} x^T S_n x \] (1.91)

where \( S_n \) is calculated from the following final-value recursive equation (derived from equation (1.84))

\[ S_n = Q_n + A_n^T S_{n+1} A_n + K_n^T H_n K_n + K_n^T G_n + G_n^T K_n = Q_n + A_n^T S_{n+1} A_n - G_n^T H_n^{-1} G_n \] (1.92)

This equation is known as the discrete-time Riccati equation. The optimal control policy can be also derived form equation (1.82) as follows

\[ \mu^*(n, x) = -H_n^{-1} G_n x \]

\[ = - (R_n + B_n^T S_{n+1} B_n)^{-1} (P_n + B_n^T S_{n+1} A_n) x \] (1.93)

We can derive the LQR controller by starting from the terminal condition, \( S_N = Q_N \), and then solving for \( S_n \) and \( \mu^*(n, x) \) iteratively, backwards in time.

1.7.2 LQR: Infinite time horizon, discrete time

In this section we will derive the discrete-time LQR controller for the case that the cost function is calculated over an infinite time horizon. Furthermore we have assumed that the system dynamics are time invariant (e.g. a LTI system)

\[ J = \sum_{n=0}^{\infty} \frac{1}{2} x_n^T Q x_n + \frac{1}{2} u_n^T R u_n + u_n^T P x_n. \] (1.94)

Note that the coefficients of the cost function are also assumed to be time invariant. Furthermore the decay factor in the cost function is assumed to be 1. Basically, in order to keep the cost function bounded in an infinite horizon problem, the decay factor should be smaller than one. However for
infinite horizon LQR problem, it can be shown that under mild conditions the optimal quadratic cost function is bounded.

As discussed in Section 1.2.2, the value function in the infinite time horizon problem is not a function of time. This implies that the value function in equation (1.91) changes as follows

\[ V^*(x) = \frac{1}{2} x^T S x \]  

Since \( S \) is not a function of time, the equation (1.92) and the optimal policy are reduced to the following forms

\[ S = Q + A^T SA - (P^T + A^T SB)(R + B^T SB)^{-1}(P + B^T SA) \]  

\[ \mu^*(x) = -(R + B^T SB)^{-1}(P + B^T SA)x \]  

Equation (1.96) is known as the discrete-time algebraic Riccati equation.

1.7.3 LQR: Finite time horizon, continuous time

In this section, we derive the optimal LQR controller for a continuous time system. This time we will start from the basic HJB equation which we derived for deterministic systems in Section 1.3.1.

The system dynamics are now given as a set of linear time varying differential equations.

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t). \]  

and the quadratic cost function the controller tries to optimize is given by

\[ J = \frac{1}{2} x(T)^T Q_T x(T) + \int_0^T \left( \frac{1}{2} x(t)^T Q(t)x(t) + \frac{1}{2} u(t)^T R(t)u(t) + u(t)^T P(t)x(t) \right) dt, \]  

where the matrices \( Q_T \) and \( Q \) are symmetric positive semidefinite, and \( R \) is symmetric positive definite. Starting from the HJB equation Eq. (1.42) and substituting in Eq. (1.98) and Eq. (1.99) we obtain

\[ -\frac{\partial V^*}{\partial t} = \min_{u \in U}\{L(x,u) + \left( \frac{\partial V^*}{\partial x} \right)^T f(x,u) \} \]

\[ = \min_{u \in U}\{\frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + u^T P x + \left( \frac{\partial V^*}{\partial x} \right)^T (A x(t) + B u(t)) \}, \]

\[ V(T,x) = \frac{1}{2} x^T Q_T x. \]

This partial differential equation must hold for the value function to be optimal. Considering the spacial form of this equation and also the quadratic final cost, one sophisticated guess for the solution
is a quadratic function as follows

\[ V^*(t, x) = \frac{1}{2} x^T S(t) x \] (1.102)

where the Matrix \( S \) is symmetric. The time and state derivatives of value function are

\[ \frac{\partial V^*(t, x)}{\partial t} = \frac{1}{2} x^T \dot{S}(t) x \] (1.103)

\[ \frac{\partial V^*(t, x)}{\partial x} = S(t) x \] (1.104)

By substituting equations (1.103) and (1.104) in equation (1.100) we obtain

\[ -x^T \dot{S}(t) x = \min_{u \in U} \{ x^T Q x + u^T R u + 2 x^T P x + 2 x^T S(t) A x + 2 x^T S(t) B u \} \] (1.105)

In order to minimize the right hand side of the equation, we put its derivative with respect to \( u \) equal to zero.

\[ 2 R u + 2 P x + 2 B^T S(t) x = 0 \]
\[ u^*(t, x) = -R^{-1}(P + B^T S(t)) x \] (1.106)

By inserting \( u(t, x) \) into equation (1.105) and few more simplification steps we obtain

\[ x^T [S(t) A(t) + A^T(t) S(t) - (P(t) + B^T(t) S(t))^T R^{-1} (P(t) + B^T(t) S(t)) + Q(t) + \dot{S}(t)] x = 0 \]

Since this equation should hold for all the states, the value inside the brackets should be equal to zero. Therefore we get

\[ \dot{S} = -S A - A^T S + (P + B^T S)^T R^{-1} (P + B^T S) - Q \]

with \( S(T) = Q_T \). (1.107)

This equation is known as the continuous-time Riccati equation. If \( S(t) \) satisfies (1.107) then we found the optimal value function through (1.102). Furthermore the optimal control input can then be computed using (1.106).

### 1.7.4 LQR: Infinite time horizon, continuous time

In this subsection the optimal control input \( u(t)^* \) is derived over an infinitely long time horizon. In contrast to the finite horizon case (1.99), the cost function does not include a terminal cost, since the evaluation is never terminated. The decay factor is also chosen 1. As in the discrete-time case, it can be shown that under mild conditions, the infinite-time LQR cost function will be bounded.

\[ J = \int_0^\infty \left[ \frac{1}{2} x(t)^T Q x(t) + \frac{1}{2} u(t)^T R u(t) + u(t)^T P x(t) \right] dt \] (1.108)
Since the value function in an infinite horizon problem is not a function of time, the time dependency in equation (1.102) is dropped, resulting in

$$V(x) = x^T S x, \quad S : n \times n \text{ symmetric.} \quad (1.109)$$

The evaluation is performed just as in the previous section, by deriving (1.109) w.r.t state $x$ and time $t$ and substituting the values into the HJB equation (1.42). The solution is equal to the continuous time Riccati equation (1.107), apart from the derivative on the left hand side $\dot{S}(t)$. Due to the time independence this derivative is zero resulting in the continuous-time algebraic Riccati equation.

$$SA + A^T S - (P + B^T S)^T R^{-1} (P + B^T S) + Q = 0, \quad (1.110)$$

By solving this equation once for $S$, the optimal control input at every state $x$ is given by

$$u^*(x) = -R^{-1} (P + B^T S)x \quad (1.111)$$

### 1.8 Linear Quadratic Gaussian Regulator: LQG(R)

In this section, we will study the LQG regulator. LQG stands for Linear Quadratic Gaussian regulator. The assumption behind the LQG problem is very similar to the one in LQR problem except one main point. In contrast to the LQR problem, in LQG the system dynamics is corrupted with a Gaussian noise. The introduction of noise in the system dynamics causes the system to demonstrate stochastic behavior e.g. different runs of the system under a similar control policy (or control input trajectory) will generate different state trajectories. This fact implies that a cost function defined for the LQR problem should be stochastic as well. Therefore in the LQG controller, the cost function is defined as an expectation of the LQR cost function.

One important observation about the LQG problem is that the noise is assumed to have a Gaussian distribution, not any arbitrary distribution. Beyond the discussion that the Gaussian noise has some physical interpretations, the main reason for this choice is the nice analytical feature that Gaussian noise is closed under the linear transformation. In other words, if a Gaussian random variable is linearly transformed, it is still a Gaussian random variable.

This section is organized as follows. In the first section we will derive the LQG controller for discrete-time systems with finite-time horizon cost function. We then extend this result to the infinite-time horizon problem. We then derive the LQG controller for continuous-time systems both for the finite-time and infinite-time horizon cost functions.

#### 1.8.1 LQG: Finite time horizon, discrete time

The LQG controller can be seen as a LQR controller with additive Gaussian noise. We can find it by applying the Stochastic Principle of Optimality to the Linear-Quadratic problem. The problem formulation for the discrete time case are as follows:
1.8 Linear Quadratic Gaussian Regulator: LQG(R)

Optimal control

Quadratic discrete cost:

\[
J = \frac{1}{2} E \left\{ \alpha^N x_N^T Q_N x_N + \sum_{n=0}^{N-1} \alpha^n \begin{bmatrix} x_n^T & u_n^T \end{bmatrix} \begin{bmatrix} Q_n & P_n^T \\ P_n & R_n \end{bmatrix} \begin{bmatrix} x_n \\ u_n \end{bmatrix} \right\}
\]

(1.112)

This is identical to the cost function of the discrete time LQR, except for the expectation and the decay factor. As you will see later, the decay factor for the finite horizon problem is usually chosen to be 1. However for the infinite horizon problem, it is absolutely necessary that \( \alpha \) is smaller than 1. Again \( Q \) is a positive semidefinite matrix and \( R \) is a positive definite matrix (which is basically invertible).

Linear system dynamics (or linearized system dynamics):

\[
x_{n+1} = A_n x_n + B_n u_n + C_n w_n \quad x(0) = x_0 \text{ given } \]

(1.113)

Matrices \( Q, P, R, A, B \) and \( C \) can be functions of time. \( w_n \) is a uncorrelated, zero-mean Gaussian process.

\[
E[w_n] = 0
\]

\[
E[w_n w_m^T] = \mathbf{I} \delta(n - m)
\]

(1.114)

In order to solve this optimal control problem, we use the Bellman equation given by equation (1.28). Like in the LQR setting, we assume a quadratic value function as Ansatz but with an additional term to account for the stochasticity. We will call it the Quadratic Gaussian Value Function:

\[
V^*(n, x) = \frac{1}{2} x^T S_n x + \nu_n
\]

(1.115)

Plugging the cost function and that Ansatz into (1.28) yields the following equation for the optimal value function.

\[
V^*(n, x) = \min_{u_n} \frac{1}{2} E \left[ x_n^T Q_n x_n + 2u_n^T P_n x_n + u_n^T R_n u_n + \alpha (A_n x_n + B_n u_n)^T S_{n+1} (A_n x_n + B_n u_n) \right] + 2 \alpha \nu_{n+1}
\]

By the use of the system equation (1.113) and considering \( E\{w\} = 0 \) we can write

\[
V^*(n, x) = \min_{u_n} \frac{1}{2} E \left[ x_n^T Q_n x_n + 2u_n^T P_n x_n + u_n^T R_n u_n + \alpha (A_n x_n + B_n u_n)^T S_{n+1} (A_n x_n + B_n u_n) \right] + \alpha w_n^T C_n^T S_{n+1} C_n w_n + 2 \alpha \nu_{n+1}
\]

Using \( Tr(AB) = Tr(BA) \) and considering \( E\{w_n w_n^T\} = \mathbf{I} \), we get

\[
V^*(n, x) = \min_{u_n} \frac{1}{2} \left\{ x_n^T Q_n x_n + 2u_n^T P_n x_n + u_n^T R_n u_n + \alpha (A_n x_n + B_n u_n)^T S_{n+1} (A_n x_n + B_n u_n) \right\} + \alpha Tr(S_{n+1} C_n C_n^T) + 2 \alpha \nu_{n+1}
\]
In order to minimize the right-hand-side of the equation with respect to \( u \), we set the gradient with respect to \( u \) equal to zero, and then substitute the result back into the previous equation.

\[
u^*(n, x) = -(R_n + \alpha B_n^T S_{n+1} B_n)^{-1}(P_n + \alpha B_n^T S_{n+1} A_n)x
\] (1.116)

\[
V^*(n, x) = \frac{1}{2}x_n^T \left[ Q_n + \alpha A_n^T S_{n+1} A_n - (P_n + \alpha B_n^T S_{n+1} A_n)^T (R_n + \alpha B_n^T S_{n+1} B_n)^{-1} (P_n + \alpha B_n^T S_{n+1} A_n) \right] x_n + \alpha \left[ \frac{1}{2} Tr(S_{n+1} C_n C_n^T) + \nu_{n+1} \right]
\]

Substituting \( V^*(n, x) \) with the Ansatz and bringing all the terms to one side, gives

\[
\frac{1}{2}x_n^T \left[ Q_n + \alpha A_n^T S_{n+1} A_n - (P_n + \alpha B_n^T S_{n+1} A_n)^T (R_n + \alpha B_n^T S_{n+1} B_n)^{-1} (P_n + \alpha B_n^T S_{n+1} A_n) \right] x_n + \left[ \frac{1}{2} Tr(S_{n+1} C_n C_n^T) + \alpha \nu_{n+1} - \nu_n \right] = 0.
\]

In order to satisfy this equality for all \( x \), the terms inside the brackets should equal zero.

\[
S_n = Q_n + \alpha A_n^T S_{n+1} A_n - (P_n + \alpha B_n^T S_{n+1} A_n)^T (R_n + \alpha B_n^T S_{n+1} B_n)^{-1} (P_n + \alpha B_n^T S_{n+1} A_n)
\]

\[
\nu_n = \frac{1}{2} Tr(S_{n+1} C_n C_n^T) + \alpha \nu_{n+1}, \quad S_N = Q_N \quad \nu_N = 0
\] (1.117)

For finite horizon problem, we normally choose \( \alpha \) equal to 1. Therefore the updating equation for \( S \) will reduce to the well known Riccati equation

\[
S_n = Q_n + A_n^T S_{n+1} A_n - (P_n + A_n^T S_{n+1} B_n)(R_n + B_n^T S_{n+1} B_n)^{-1}(P_n + B_n^T S_{n+1} A_n)
\]

\[
\nu_n = \frac{1}{2} Tr(S_{n+1} C_n C_n^T) + \nu_{n+1}, \quad S_N = Q_N \quad \nu_N = 0
\] (1.118)

and the optimal control policy

\[
u^* = -(R_n + B_n^T S_{n+1} B_n)^{-1}(P_n + B_n^T S_{n+1} A_n)x
\] (1.119)

By comparing the optimal control policies for the LQG problem in equation (1.119) and the LQR problem in equation (1.93), we can see that they have the same dependency on \( S \). Furthermore by comparing the recursive formulas for calculating \( S \) in equations (1.118) and (1.92), we see that they are basically the same. In fact, the optimal control policy and the Riccati equation in both LQR and LQG problem are identical.

The only difference between the LQG and LQR problems are their corresponding value functions. It can be shown that the value function in the LQG case is always greater than that in the LQR case. In order to prove this, we need just to prove that \( \nu_n \) is always nonnegative. We will prove it by induction. First we show that the base case \( (n = N) \) is correct. Then we show that if \( \nu_{n+1} \) is nonnegative, \( \nu_n \) should also be nonnegative.
1.8 Linear Quadratic Gaussian Regulator: LQG(R) Optimal control

**Base case:** It is obvious because $v_N$ is equal to zero.

**Induction:** From equation (1.118), we realize that if $v_{n+1}$ is nonnegative, $v_n$ will be nonnegative if and only if $Tr(S_{n+1}C_nC_n^T) \geq 0$. Now we will show this.

\[
Tr(S_{n+1}C_nC_n^T) = Tr(C_n^T S_{n+1} C_n) = \sum_i C_i^T S_{n+1} C_i
\]

where $C_i$ is a the $i$th column of $C_n$. Since $S_n$ is always positive semidefinite, $C_i^T S_{n+1} C_i \geq 0$ holds for all $i$. Therefore $Tr(S_{n+1}C_nC_n^T) \geq 0$, and $v_n$ is always nonnegative.

**1.8.2 LQG: Infinite time horizon, discrete time**

In this section the LQG optimal control problem with an infinite-time horizon cost function is introduced. The quadratic cost function is defined as follows

\[
J = \frac{1}{2} E \left\{ \sum_{n=0}^{\infty} \alpha^n \left[ \begin{array}{c} x_n^T \\ u_n^T \end{array} \right] \left[ \begin{array}{cc} Q & P^T \\ P & R \end{array} \right] \left[ \begin{array}{c} x_n \\ u_n \end{array} \right] \right\}
\]

(1.120)

and the system dynamics

\[
x_{n+1} = Ax_n + Bu_n + Cw_n \quad x(0) = x_0 \text{ given}
\]

(1.121)

where $w_n$ is a Gaussian process with the characteristic described at (1.114). All of the matrices are time independent. An interesting difference between the infinite-time LQR problem and the LQG problem is that the decay factor must be always smaller than 1 ($0 < \alpha < 1$), otherwise the quadratic cost function will not be bounded. As you will see shortly, although $\alpha$ can approach in limit to 1, it is absolutely necessary that $\alpha \neq 1$.

In order to solve this problem we will use the results we have obtained from the previous section. The only difference between these two problems is that in the infinite time case the value function is only a function of state, not time. Therefore the value function can be expressed as

\[
V^*(x) = \frac{1}{2} x^T S x + v
\]

(1.122)

Using equation (1.117), we get

\[
S = Q + \alpha A^T SA - (P + \alpha B^T SA)^T (R + \alpha B^T SB)^{-1} (P + \alpha B^T SA)
\]

(1.123)

\[
v = \frac{\alpha}{2(1-\alpha)} Tr(SCC^T)
\]

(1.124)

and the optimal control policy

\[
u^*(x) = -(R + \alpha B^T SB)^{-1} (P + \alpha B^T SA)x
\]

(1.125)
As equation (1.124) illustrates, if $\alpha$ approaches 1, $\nu$ grows to infinity. Therefore $\alpha$ should always be smaller than 1. However, it could approach in limit to 1, in which case equations (1.123) and (1.125) simplify as

$$
S = Q + A^T S A - (P + B^T S A)^T (R + B^T S B)^{-1} (P + B^T S A) \\
u^*(x) = -(R + B^T S B)^{-1} (P + B^T S A) x
$$

Equation (1.126) is the discrete-time algebraic Riccati equation similar to the LQR one in equation (1.96).

### 1.8.3 LQG: Finite time horizon, continuous time

In this section, we solve the LQG problem for continuous-time systems. The problem formulation is as follows:

**Quadratic cost function:**

$$
J = \frac{1}{2} E \left\{ e^{-\beta T} x^T(T) Q_T x(T) + \int_0^T e^{-\beta t} \left[ x^T(t) \quad u^T(t) \right] \begin{bmatrix} Q(t) & P(t) \\ P^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \right\}
$$

(1.128)

$Q$ is a positive semidefinite matrix and $R$ is a positive definite matrix (which is basically invertible). As you will see later, the decay factor ($\beta$) for the finite horizon problem is usually chosen 0. However for the infinite horizon problem, it is absolutely necessary that $\beta$ be greater than 0.

**Stochastic linear system dynamics:**

$$
\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)w(t), \quad x(0) = x_0
$$

(1.129)

Matrices $Q$, $P$, $R$, $A$, $B$, and $C$ can be functions of the time. $w(t)$ is an uncorrelated, zero-mean Gaussian process.

$$
E[w(t)] = 0 \\
E[w(t)w(\tau)^T] = I \delta(t-\tau)
$$

(1.130)

The solution procedure is like in the deterministic case, except that now we are accounting for the effect of noise. We make an Ansatz like in equation (1.115): A quadratic value function with stochastic value function increment.

$$
V^*(t, x) = \frac{1}{2} x^T(t) S(t)x(t) + \nu(t)
$$

(1.131)

where $S(t)$ is a symmetric matrix and $\nu(t)$ is the stochastic value function increment.
From the Ansatz (1.131) we can derive the first partial derivative with respect to \( t \) and the first and second partial derivatives with respect to \( x \) as:

\[
V_t^*(t, x) = \frac{1}{2} x^T(t) \dot{S}(t)x(t) + \dot{\nu}(t)
\]

\[
V_x^*(t, x) = S(t)x(t)
\]

\[
V_{xx}^*(t, x) = S(t)
\]  

(1.132)

Plugging the Ansatz and its derivatives into the Stochastic HJB equation and following a similar simplification strategy as in the discrete case, we get:

\[
\beta V^* - V_t^* = \min_u \frac{1}{2} \left\{ x^T Q x + 2u^T P x + u^T R u + 2 x^T S(Ax + Bu) + Tr(SCC^T) \right\}
\]

The optimal control is found following the same procedure as in the LQR case, namely differentiating the right side of the equation with respect to \( u \) and setting the result to 0:

\[
u^*(t, x) = -R(t)^{-1} (P(t) + B^T(t)S(t)) x
\]  

(1.133)

In order to find the unknown \( S(t) \), we substitute \( u \) and \( V_t^* \) into the previous equation

\[
\frac{1}{2} x^T \left[ S(t)A(t) + A^T(t)S(t) - (P(t) + B^T(t)S(t))^T R^{-1} (P(t) + B^T(t)S(t)) + Q(t) + \dot{S}(t) - \beta S \right] x + \left[ \dot{\nu}(t) - \beta \nu(t) + \frac{1}{2} Tr(SCC^T) \right] = 0
\]

In order to satisfy this equality for all \( x \), the terms inside the brackets should equal zero. Therefore we have

\[
\dot{S} = \beta S - SA - A^T S + (P + B^T S)^T R^{-1} (P + B^T S) - Q, \quad \text{with } S(T) = Q_T. \quad (1.134)
\]

\[
\dot{\nu} = \beta \nu(t) - \frac{1}{2} Tr(SCC^T) \quad \text{with } \nu(T) = 0. \quad (1.135)
\]

For finite horizon problems, we normally choose \( \beta \) equal to 0. Therefore the updating equation for \( S \) will reduce to the following:

\[
\dot{S} = -SA - A^T S + (P + B^T S)^T R^{-1} (P + B^T S) - Q, \quad \text{with } S(T) = Q_T. \quad (1.136)
\]

\[
\dot{\nu} = -\frac{1}{2} Tr(SCC^T) \quad \text{with } \nu(T) = 0. \quad (1.137)
\]

Equation (1.135) is known as the continuous time Riccati equation. By comparing this equation and the one in (1.107), we realize that the Riccati equation is the same for both LQR and LQG problems. Furthermore the control policies are identical in both cases. However the value functions are different. One can easily prove that the value function in the LQG problem is greater and equal to the LQR value function.
1.8.4 LQG: Infinite time horizon, continuous time

In this section, the continuous time LQG problem with an infinite time horizon cost function is studied. The quadratic cost function is defined as follows

\[
J = \frac{1}{2} E \left\{ \int_0^\infty e^{-\beta t} \left[ x^T(t) \begin{bmatrix} Q & P^T \\ P & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right] dt \right\}
\]

(1.138)

and the system dynamics

\[
\dot{x}(t) = Ax(t) + Bu(t) + Cw(t) \quad x(0) = x_0 \quad \text{given}
\]

(1.139)

Matrices \(Q, P, R, A, B\) and \(C\) are all time invariant. \(w(t)\) is an uncorrelated, zero-mean Gaussian process with the characteristic described at (1.130). The defined cost function is comparable to the one introduced in (1.108) except the expectation and the decay factor. The expectation is introduced because of the stochasticity of the problem. The decay rate is chosen to be non-zero since the pure quadratic cost function will never be bounded in the LQG case though it is bounded for LQR. As you will see shortly, although \(\beta\) can approach in limit to 0, it is absolutely necessary that \(\beta > 0\).

In order to solve this problem we will use the results we have obtained from the previous section. The only difference between these two problems is that in the infinite time case the value function is only a function of state, not time. Therefore the value function can be expressed as

\[
V^*(x) = \frac{1}{2} x^T S x + \nu
\]

(1.140)

Using equations (1.134) and (1.135), we get

\[
-\beta S + SA + A^T S - (P + B^T S)^T R^{-1} (P + B^T S) + Q = 0
\]

(1.141)

\[
\nu(t) = \frac{1}{2\beta} Tr(SCC^T)
\]

(1.142)

and the optimal control policy at every state \(x\) is given by

\[
u^*(x) = -R^{-1}(P + B^T S) x
\]

(1.143)

As equation (1.142) illustrates, if \(\beta\) approaches 0, \(\nu\) grows to infinity. Therefore \(\beta\) should always be greater than 0. However it could be arbitrary small. Thus if \(\beta\) approaches to 0, equation (1.141) simplifies to

\[
SA + A^T S - (P + B^T S)^T R^{-1} (P + B^T S) + Q = 0
\]

(1.144)

\[
u^*(x) = -R^{-1}(P + B^T S) x
\]

(1.145)

Equation (1.144) is the continuous-time algebraic Riccati equation similar to the LQR one in equation (1.110).
2 Classical Reinforcement Learning

In the field of Reinforcement Learning (RL) different notations are used even though the problem setting is the same as in the optimal control case. In order to keep consistency with the previous chapters of the script we will use the notation of optimal control also in this chapter. The following table summarizes the relationship between important terms used in RL and optimal control. It should help the reader link the material here to the external literature.

| RL                        | Optimal Control                  |
|----------------------------|----------------------------------|
| environment                | system                           |
| agent                      | controller                       |
| state: \( s \)             | state: \( x \)                   |
| control action: \( a \)    | control input: \( u \)           |
| reward: \( r \)            | intermediate cost: \( L \)       |
| discount factor: \( \gamma \) | discount factor \( \alpha \)    |
| stochastic policy: \( \pi \) | deterministic policy \( \mu \)  |

One notation that we will adopt from RL is the reward. In order to evaluate a specific control policy RL does not punish elements that lead to unwanted behavior but it rewards behavior that is desired. Therefore instead of defining a cost function like in optimal control, RL computes a reward function, which has to be maximized. Conceptually, these concepts are trivially equivalent though, since \( \text{cost} = -\text{reward} \).

Note that this chapter follows in parts very closely Sutton and Barto’s classical and excellent RL textbook. Following the aim of the course to give a unified view on the two fields of RL and Optimal Control, we have reproduced here some figures and developments from the book using the unified notation.

*Richard S. Sutton and Andrew G. Barto. Introduction to Reinforcement Learning. MIT Press, Cambridge, MA, USA, 1st edition, 1998*
2.1 Markov Decision Process

A stochastic process satisfies the Markov property if the following conditional probability distribution holds.

\[ Pr\{x_{n+1} \mid x_n, x_{n-1}, \ldots, x_0\} = Pr\{x_{n+1} \mid x_n\} \]

This means that the behavior of the process (system) at each time step is based only on information from the previous time step. In this sense, there is no memory in the system and its behavior in each state does not depend on how it got there, but only on the action taken at that point in time. According to this definition, all of the dynamical systems we have defined so far in Chapter 1 are Markov processes.

In RL, a system that satisfies the Markov property is called Markov Decision Process (MDP). It consists of a finite set of states and actions (control inputs) and a reward function. Loosely speaking, states are defined based on information we gather from the environment and the actions (e.g. a motor command of a robot) are the way we affect the surrounding environment. The reward function is a memoryless stochastic process that assesses the value of the current state and the subsequent action of the agent. The term “agent” in RL refers to the entity which interacts with the environment in order to optimize some criteria of its performance.

In most cases, states and control inputs are regarded as finite sets

\[ x \in X = \{x_1, \ldots, x_{\|X\|}\}, \quad u \in U = \{u_1, \ldots, u_{\|U\|}\} \]

(2.1)

where \( \|X\| \) and \( \|U\| \) are the number of possible states and actions at each time step respectively (for simplicity we have assumed that the number of the possible actions in each state is the same). The process’s dynamics are defined by what is called the transition probability distribution. This defines the probability of moving from state \( x_n = x \) to state \( x_{n+1} = x' \) after applying the control input \( u_n = u \):

\[ \mathcal{P}_x^{uu} = Pr\{x_{n+1} = x' \mid x_n = x, u_n = u\} \]

(2.2)

The process’s rewards are also stochastic. The expected intermediate reward received at time \( n \) by performing action \( u \) in state \( x \) and transiting to state \( x' \) is defined as:

\[ \mathcal{R}_x^{uu} = E\{r_n \mid x_{n+1} = x', u_n = u, x_n = x\} \]

(2.3)

Three points worth noticing here are:

- The reward function is stochastic in contrast to the optimal control cost function which has always been defined as a deterministic function.
• We have only defined the first moment of the reward process. As we will see later, this is the only piece of information needed to find the optimal policy.

• Last but not the least, in contrast to the cost function that is defined based on the current state and current control input, the reward function in this chapter is defined based on the triple of the current state, the current control input, and the next state. However, one can derive a reward function only based on the current state and control input from this reward function as follows

\[ R_u^x = \sum_{x'} P_{ux'} R_{ux'} \] (2.4)

Actually, as we will see later, whenever \( R_{ux'} \) appears, it is always marginalized with respect to \( x' \). Therefore, the actual probability distribution that we are interested in is \( R_u^x \). Using \( R_{ux'} \) is a design choice which sometimes makes the definition of the reward function more intuitive.

In the problems of previous chapters we have mostly considered deterministic policies that we denoted by \( \mu(x) \). On the contrary, MDPs assume more general policies, which include the stochastic policy class. In the following sections, stochastic policies will be expressed as \( \pi(x, u) \), or more precisely \( \pi(u|x) \). \( \pi \) is a probability distribution over the action (control input) set and conditioned over the current state.

Except for the restricting assumption that the state and action are discrete, the modeling assumptions behind the MDP problem are more general than in the optimal control problem of the previous chapter. We should note, however, that the discrete state and action assumptions are often impractical when working on real-world robotic platforms.

2.2 The RL Problem

In the optimal control problem, our objective was to find a policy (controller) that minimizes the expected total cost. Similarly, in the RL problem we want to find a policy, \( \pi \), which maximizes the expected accumulated reward. The optimal control counterpart of the RL problem is the infinite-horizon discrete time optimal control problem. However in RL the state and the actions are discrete (or discretized). Furthermore the reward function is not a deterministic function.

**Problem Statement:** Assuming an MDP problem, we seek a stochastic policy that maximizes the expected accumulated reward. The accumulated reward is defined as

\[ R_0 = r_0 + \alpha r_1 + \alpha^2 r_2 + \cdots + \alpha^n r_n + \cdots = \sum_{k=0}^{\infty} \alpha^k r_k \] (2.5)

and the optimal policy as

\[ \pi^* = \arg \max_{\pi} E[R_0] \] (2.6)
In equation (2.5), $\alpha \in [0, 1]$ is the decay or discount factor. In an episodic task (i.e., a task that has some terminal states or finishes after a finite number of time steps) $\alpha$ can be 1. Otherwise, it should be chosen in such a way to ensure that the infinite summation of intermediate rewards exists. Furthermore, we should notice that $R_0$ is a stochastic variable. Its stochasticity originates from three sources. First, the reward function is a stochastic process. Therefore, $R_0$ as a summation of random variables is also a random variable. Second, the state transition is governed by a stochastic process. Therefore, the state sequence, which the reward function depends on, is stochastic. Finally, the policy can also be stochastic, which affects both state and control sequences.

In equation (2.6), the expectation is taken with respect to all sources of stochasticity in the accumulated reward. As you can see, since the rewards at each time step are summed (which is a linear operation), we can easily pull the expectation inside of the sum, resulting in a sum of expected intermediate rewards. This means that, in the interest of finding the optimal policy, the only relevant characteristic of the reward function’s distribution is its first moment.

For the sake of the notion simplicity, we introduce a more general definition for the accumulated reward. In this definition, the accumulated reward at time step $n$ is defined as

$$ R_n = \sum_{k=0}^{\infty} \alpha^k r_{n+k} $$

(2.7)

One can verify that $R_n$ at time zero is the originally defined accumulated reward. In Figure 2.1, a flow chart of state, control input and reward in RL is given.

![Flow chart of state, control input and reward in RL](image)

Figure 2.1: Flow chart of state, control input and reward in the RL notation. Reproduced from [3].

### 2.3 State Value Function and the Bellman Equation

As was the case with the optimal control problem, in order to find a policy which maximizes a process’s expected reward, we first need to define a value function which quantifies how effective a given policy is. In this chapter, in order to clearly distinguish between the value function and the action value function (which will be introduced in the next section), we will refer to the value function as the state value function. Similar to the optimal control setting, the state value function $V^\pi(x)$ captures the accumulated reward expected when starting at $x$ and following the given policy.
\begin{equation}
V^\pi(x) = E\{R_n \mid x_n = x\} = E \left\{ \sum_{k=0}^{\infty} \alpha^k r_{n+k} \mid x_n = x \right\}
\tag{2.8}
\end{equation}

From equation (2.8) the Bellman equation can be derived by relating the state value function of state \( x \) with the state value function of its successor states \( x' \). Here we are using the fact that in the infinite-horizon problem the value function is not a function of time, but just a function of states. For a more detailed discussion, refer to Section 1.2.2.

Extracting the first reward term from the summation in equation (2.8) and then factoring out an \( \alpha \) gives

\begin{equation}
V^\pi(x) = E \left\{ r_n + \alpha \sum_{k=0}^{\infty} \alpha^k r_{n+k+1} \mid x_n = x \right\}
\end{equation}

Since the expectation is a linear operator, we can change its order with the first summation

\begin{equation}
V^\pi(x) = E \{ r_n \mid x_n = x \} + E \left\{ \alpha \sum_{k=0}^{\infty} \alpha^k r_{n+k+1} \mid x_n = x \right\}
\end{equation}

Using the law of total expectation (\( E\{\ldots \mid y\} = E\{E\{\ldots \mid z, y\} \mid y\}\)), we can additionally condition the term on the right by the control input applied and the subsequent value of the state, adding another expectation over these additional conditional arguments.

\begin{equation}
V^\pi(x) = E \{ r_n \mid x_n = x \}
+ E_{u_n} \left\{ E_{x_{n+1}} \left\{ \alpha \sum_{k=0}^{\infty} \alpha^k r_{n+k+1} \mid x_{n+1} = x', u_n = u, x_n = x \right\} \mid u_n = u, x_n = x \right\} \mid x_n = x
\end{equation}

Here we explicitly include the variables over which the expectation is taken for clarity.

By means of the problem’s Markov property (cf. the definition of reward in equation 2.3), we can verify that for all \( k \)s greater and equal to zero

\begin{equation}
Pr \{ r_{n+k+1} \mid x_{n+1} = x' \} = Pr \{ r_{n+k+1} \mid x_{n+1} = x', u_n = u, x_n = x \}, \quad \forall k \geq 0.
\end{equation}
Therefore the inner expectation can be simplified to the following

\[
V_\pi(x) = E \{ r_n \mid x_n = x \} + E_u \left\{ E_{x_{n+1}} \left\{ E_\pi \left\{ \sum_{k=0}^{\infty} \alpha^k r_{n+k+1} \mid x_{n+1} = x' \right\} \mid u_n = u, x_n = x \right\} \mid x_n = x \right\}
\]

\[
= E \{ r_n \mid x_n = x \} + E_u \left\{ \alpha V_\pi(x') \mid u_n = u, x_n = x \right\} \mid x_n = x \right\}
\]

In the last equality, we have substituted the most inner expectation by the value function at state \(x'\). By rolling back the expanded expectations, we obtain

\[
V_\pi(x) = E \{ r_n \mid x_n = x \} + E \{ \alpha V_\pi(x') \mid x_n = x \}
\]

And then pulling the expectation outside of the sum:

\[
V_\pi(x) = E_{u_n, r_n, x_{n+1}} \{ r_n + \alpha V_\pi(x') \mid x_n = x \} \quad (2.9)
\]

Equation (2.9) is the Bellman equation for a given policy \(\pi\). This equation is essentially the same as the Bellman equation for the optimal control problem in stochastic systems (Equation 1.34). The only difference is that Equation (2.9) evaluates the expectation with respect to the reward function \(r_n\) and policy, as well as the state transition, since the reward function and policy may now be stochastic as well.

We can solve the Bellman equation in equation (2.9) using the state transition probability, the policy, and the reward function.

\[
V_\pi(x) = E_{u_n, r_n, x_{n+1}} \{ r_n + \alpha V_\pi(x') \mid x_n = x \}
\]

\[
= E_u \left\{ E_{r_n, x_{n+1}} \left\{ r_n + \alpha V_\pi(x') \mid u_n = u, x_n = x \right\} \mid x_n = x \right\}
\]

Substituting the appropriate probability distributions for each of these expectations gives

\[
V_\pi(x) = \sum_u \pi(x, u) \sum_{x'} P^u_{xx'} [R^u_{xx'} + \alpha V_\pi(x')] \quad (2.10)
\]

Equation (2.10) has a few interesting features. First, it shows that the value of the state value function at state \(x\) is related to the value of other states' value function. Therefore a change in one state's value function will affect the value of the other states' value functions as well. Second it shows that the relationship between the value function of different states is linear. Therefore if we collect the value functions associating to all the states in a vector of the size \(\|X\|\) (number of states
in MDP) and call it $V$, we will have

$$V = AV + B$$

$$[A_{i,j}] = \alpha \sum_u \pi(x_i, u) P^u_{x_i x_j}$$

$$[B_i] = \sum_u \pi(x_i, u) \sum_{x'} P^u_{x_i x'} R^u_{x_i x'}$$

### 2.4 Action Value Function and the Bellman Equation

We introduce the action value function as the adversary to the state value function: It is the expected accumulated reward starting at $x$ and choosing control input $u$, then following the policy $\pi$. It is defined as follows. Notice the additional conditioning with respect to $u$ inside the brackets of the expectation.

$$Q^\pi(x, u) = E_{\pi}\{R_n \mid x_n = x, u_n = u\} = E\left\{\sum_{k=0}^{\infty} \alpha^k r_{n+k} \mid x_n = x, u_n = u\right\}$$

The difference between the action value function and the state value function is that in the state value function the control input sequence is totally generated according to the given policy, $\pi$, while in the action value function, the first control input is fixed and is not extracted from the policy $\pi$ (note that the rest of the control input sequence will be extracted form $\pi$). Based on this definition, we can write the relationship between state value function and action value function as follows

$$V^\pi(x) = \sum_u \pi(x, u)Q^\pi(x, u)$$

In some literature, the action value function is also referred as state-value function. It is also common to call it as Q-table. The reason for calling it a table is that we can represent a set of action value functions as a table, associating rows to the states and columns to the control inputs. If the number of the states is $\|X\|$ and the number of control inputs is $\|U\|$, then the system’s complete Q-table will be of the size $\|X\|$-by-$\|U\|$.

Similar to equation (2.9), we can derive the Bellman equation for the action value function. Due to the similarity between this derivation and the one for the value function, we will skip some steps.

$$Q^\pi(x, u) = E\{R_n \mid x_n = x, u_n = u\}$$

$$= E\left\{r_n + \alpha \sum_{k=0}^{\infty} \alpha^k r_{n+k+1} \mid x_n = x, u_n = u\right\}$$

$$= E_{r_n, x_{n+1}}\left\{E_{r_{n+1}}\left\{r_n + \alpha E_{r_{n+k+1}}\left\{r_{n+k+1} \mid x_{n+1} = x'\right\} \mid x_n = x, u_n = u\right\} \mid x_{n+1} = x', u_{n+1} = u\right\}$$

$$= E_{r_n, x_{n+1}}\left\{r_n + \alpha V^\pi(x') \mid x_n = x, u_n = u\right\}$$
By plugging in equation (2.12)

\[
Q^\pi(x, u) = E_{r_n, x_{n+1}} \left\{ r_n + \alpha \sum_{u'} \pi(x', u')Q^\pi(x', u') \mid x_n = x, u_n = u \right\}
\]  
(2.13)

Equation (2.13) is the Bellman equation for the action value function. Again we can solve this equation using the state transition probability, the policy and the reward function.

\[
Q^\pi(x, u) = \sum_{x'} P_{xx'}^u \left[ R_{xx'}^u + \alpha \sum_{u'} \pi(x', u')Q^\pi(x', u') \right]
\]  
(2.14)

Since the control input is fixed in the action value function, the outer summation here is over the finite set \(X\) of future states. All the statements we made in the previous section for equation (2.10) hold for this equation as well.

### 2.5 Optimal Policy

Solving the RL problem means finding a policy that achieves the maximum accumulated reward over an infinite-time horizon. This means we want to find a policy whose state value function satisfies the following for all \(x \in X\) and for any possible policy \(\pi\).

\[
V^*(x) \geq V^\pi(x)
\]  
(2.15)

In other words, we can write

\[
V^*(x) = \max_\pi V^\pi(x)
\]  
(2.16)

All the optimal policies which result in \(V^*\) are denoted as \(\pi^*\). Equivalently, the optimal action value function is defined for all \(x \in X\) and \(u \in U\) as,

\[
Q^*(x, u) = \max_\pi Q^\pi(x, u)
\]  
(2.17)

The relationship between \(V\) and \(Q\) is given by equation (2.12), therefore we can write the same for the optimal policy

\[
V^*(x) = \sum_a \pi^*(x, u)Q^*(x, u)
\]  
(2.18)

Since \(\pi^*(x, u)\) is always between 0 an 1 for all the control inputs, the following inequality holds

\[
V^*(x) = \sum_u \pi^*(x, u)Q^*(x, u) \leq \max_u Q^*(x, u)
\]  
(2.19)
Therefore, if we have a policy which always chooses a control input that has the maximum action value function, its value function will be higher or equal to the optimal value function. However, this can only be correct if the policy is actually the optimal policy. Therefore in the previous equation, the inequality should be replaced by equality.

\[ V^*(x) = \max_u Q^*(x, u) \]  \hspace{1cm} (2.20)

Equation (2.20) is the relationship between the optimal state value function and the optimal action value function.

Applying the Bellman equation (2.13) for \( Q^* \) leads to

\[ Q^*(x, u) = \mathbb{E} \left\{ r_n + \alpha \sum_{u'} \pi^*(x', u') Q^*(x', u') \mid x_n = x, u_n = u \right\} \]

By using the results from equations (2.19) and (2.20), we will have

\[ Q^*(x, u) = \mathbb{E} \left\{ r_n + \alpha \max_u Q^*(x', u') \mid x_n = x, u_n = u \right\} \]  \hspace{1cm} (2.21)

This is the optimal Bellamn equation for the action value function. We can again solve this equation to give,

\[ Q^*(x, u) = \sum_{x'} P^{u}_{xx'} \left[ R^{u}_{xx'} + \gamma \max_u Q^*(x', u') \right]. \]  \hspace{1cm} (2.22)

In order to derive the Bellman equation for the optimal state value function, we use the equation (2.9) for the optimal policy

\[ V^*(x) = \max_u \mathbb{E} \left\{ r_n + \alpha V^*(x') \mid x_n = x \right\} \]  \hspace{1cm} (2.23)

This is the optimal Bellman equation for the optimal value function. We can also solve this equation to give,

\[ V^*(x) = \max_{u \in U} \sum_{x'} P^u_{xx'} \left[ R^u_{xx'} + \alpha V^*(x') \right]. \]  \hspace{1cm} (2.24)

## 2.6 Policy Evaluation

We would like to be able to compute the optimal policy, given our state transition probability distribution (equation 2.2) and the expected reward (equation 2.3). In order to do this, we first need to be able to calculate the value function \( V^\pi \), given some arbitrary policy, \( \pi \). This is referred to as Policy Evaluation.
From equation (2.11), we can see that performing Policy Evaluation involves solving a system of \( \|X\| \) (the number of possible states) linear equations with \( \|X\| \) unknowns (each \( V^\pi(x) \)). While finding a closed-form solution to this system of equations is often not feasible due to the size of the state space, a solution can be found iteratively. If we start with some arbitrary approximation of the value function, \( V_0 \), we can eventually obtain the true value function by iteratively pushing our value function towards the true value function.

Iterative improvement of the approximated value functions requires an update rule which guarantees that the subsequent value function is always a better approximation. Here, we will use the Bellman equation which was derived in the previous section (equation 2.10).

\[
V_{k+1}(x) = \sum_u \pi(x, u) \sum_{x'} P_{xx'}^{u}[R_{xx'} + \gamma V_k(x')]
\]

You can see that \( V_k = V^\pi \) is a fixed point of this equation, since substituting it in gives the form of the Bellman equation. Though it will not be proven here, this method is guaranteed to converge to the true value function as long as either \( \alpha < 1 \) or "eventual termination is guaranteed from all states under policy \( \pi^* \)" (Sutton p.88). Furthermore since it is not feasible in practice to iterate infinitely, evaluation is typically terminated once \( \max_{x \in X} |V_{k+1}(x) - V_k(x)| \) is sufficiently small.

**Algorithm 1** Iterative Policy Evaluation Algorithm

**Input:** \( \pi \), the policy to be evaluated

**Initialize** \( V(x) = 0 \), for all \( x \in X^+ \)

**repeat**

\( \Delta \leftarrow 0 \)

**for** each \( x \in X \)

\( v \leftarrow V(x) \)

\( V(x) \leftarrow \sum_u \pi(x, u) \sum_{x'} P_{xx'}^{u}[R_{xx'} + \gamma V(x')] \)

\( \Delta \leftarrow \max(\Delta, |v - V(x)|) \)

**until** \( \Delta < \theta \) (a small positive number)

**Return:** \( V \approx V^\pi \)

This algorithm can similarly be used to find the action Value Function. Once again, we simply set some initial guess \( Q_0(x, u) \) for \( \forall x \in X, \forall u \in U \), and use the Bellman equation as the update rule, and iterate until convergence.

\[
Q_{k+1}(x, u) = \sum_{x'} P_{xx'}^{u} \left[ R_{xx'}^{u} + \alpha \sum_{u'} \pi^*(x', u')Q_k(x', u') \right]
\]
2.7 Policy Improvement

Now that we can evaluate a policy, the next step is to determine how to improve a policy. We will first define what makes one policy “better” than another. A policy \( \pi' \) is better than policy \( \pi \) if the following two conditions hold:

\[
\begin{align*}
\forall x \in X & : \quad V^{\pi'}(x) \geq V^\pi(x) \\
\exists x \in X & : \quad V^{\pi'}(x) > V^\pi(x)
\end{align*}
\] (2.27)

This means that a superior policy must perform better than an inferior policy in at least one state, and cannot perform worse in any state.

Practically speaking, what we would like to be able to do is the following: Given some arbitrary policy, \( \pi \), whose corresponding value function \( V^\pi \) has already been found using the Iterative Policy Evaluation Algorithm, we want to be able to decide if modifying this policy at some state yields a better policy. We can evaluate this decision by considering how our reward would change if we took action \( u \), which doesn’t follow out existing policy, at \( x \), then continued following the existing policy \( \pi \) afterwards. Evaluating this decision requires comparing the original policy’s value function \( V^\pi(x) \) to the action value function of the altered policy, which can conveniently be expressed as \( Q^\pi(x, u) \). In order to compare these quantities, we will use the Policy Improvement Theorem, which is explained below.

To better illustrate the Policy Improvement Theorem, we will temporarily consider deterministic policies, even though the results derived also apply to stochastic policies. To simplify the notation, we define the function \( \mu(x) \) to be:

\[
\mu(x) = a, \quad \text{where: } \pi(u = a \mid x) = 1.
\] (2.28)

Assume that we have two policies, \( \pi \) and \( \pi' \). These policies are identical at all states except \( x \) (i.e. \( \pi'(x) \neq \pi(x) \)). If we assume that the value of taking the action of policy \( \pi' \) at \( x \), and then following policy \( \pi \) afterwards will yield more reward than always following policy \( \pi \), we can say,

\[
Q^\pi(x, \mu'(x)) \geq V^\pi(x)
\] (2.29)

where \( \mu'(x) \) is defined as equation 2.28 for the deterministic policy \( \pi' \).

We can not directly, however, perform this action update at every state and expect this result to hold. It would be impossible to follow the old policy after the first time step because new policy would completely override the old policy. Because of this, the question becomes, if we greedily choose the action with higher action value function in each state, will the new policy \( \pi' \) always be a better policy than \( \pi \).
Following the same procedure, we can expand the inequality to the end. 

\[ V^\pi(x) \leq \frac{Q^\pi(x, \mu'(x))}{E_\pi \left\{ r_n + \alpha V^\pi(x_{n+1}) | u_n = \mu'(x), x_n = x \right\} } \]

note that the highlighted terms are equivalent.

By continuing to choose a control input which has a higher action value function (being greedy) on the next time step \( n + 1 \), we should substitute \( V^\pi(x_{n+1}) \) by \( Q^\pi(x_{n+1}, \mu'(x_{n+1})) \).

\[ V^\pi(x) \leq E_{\pi} \left\{ r_n + \alpha Q^\pi(x_{n+1}, \mu'(x_{n+1})) | u_n = \mu'(x), x_n = x \right\} \]

\[ V^\pi(x) \leq E_{u_{n+1}} \left\{ r_{n+1} + \alpha V^\pi(x_{n+2}) | u_{n+1} = \mu'(x'), x_{n+1} = x' \right\} | u_n = \mu'(x), x_n = x \}

\[ V^\pi(x) \leq E_{\pi} \left\{ r_n + \alpha r_{n+1} + \alpha^2 V^\pi(x_{n+2}) | u_{n+1} = \mu'(x'), x_{n+1} = x', u_n = \mu'(x), x_n = x \} \right\} \]

where the last equation indicates that we choose the first two control inputs according to the policy \( \pi' \), then we follow the policy \( \pi \) afterwards. To simplify the notation, we show this equation as

\[ V^\pi(x) \leq E_{u_{n+1}} \left\{ r_{n+1} + \alpha^2 V^\pi(x_{n+2}) | x_n = x \right\} \]

Following the same procedure, we can expand the inequality to the end.

\[ V^\pi(x) \leq E_{u_{n+1,n+2,...}} \left\{ r_n + \alpha r_{n+1} + \alpha^2 r_{n+2} + \alpha^3 r_{n+3} + ... | x_n = x \right\} \]

Since we are following the policy \( \pi' \) from time \( n \) to the end, we can simply omit the subscription \( \pi \) from the expectation.

\[ V^\pi(x) \leq E_{\pi'} \left\{ r_n + \alpha r_{n+1} + \alpha^2 r_{n+2} + \alpha^3 r_{n+3} + ... | x_n = x \right\} = V^{\pi'}(x) \]

\[ V^\pi(x) \leq V^{\pi'}(x) \quad (2.30) \]

Here we have shown that the altered policy, \( \pi' \) is indeed better than the original policy. Let us now consider greedy improvement of our policy. Consider a policy update rule of the form

\[ \pi'(x) = \arg \max_u Q^\pi(x, u) = \arg \max_u E \left\{ r_n + \alpha V^\pi(x_{n+1}) | x_n = x, u_n = u \right\} \quad (2.31) \]

The original policy is modified to take the action which maximizes the reward in the current time step, according to \( Q^\pi \). From the Policy Improvement Theorem, we know that this greedy policy update will **always** yield a policy which is greater than or equal to the original one. This greedy approach is called **Policy Improvement**.
Now, suppose that the new greedy policy is strictly equal to the original policy \( V^{\pi'} = V^\pi \). From equation (2.31),

\[
V^{\pi'}(x) = \max_u \mathbb{E}\left\{ r_n + \alpha V^{\pi'}(x_{n+1}) | x_n = x, u_n = u \right\} \\
= \max_u \sum_{x'} P_{xx'}^u [R_{xx'}^u + \alpha V^{\pi'}(x')] 
\]

which is the same as the optimal Bellman equation (equation 2.24)! This means that Policy Improvement will always give us a better policy, unless the policy is already optimal. Once again, this section only considered deterministic policies, even though the methods and results described here do in general extend to stochastic policies, as we will show later for a class of stochastic policies known as \( \varepsilon \)-greedy policies (See [3], p. 94 for more details).

### 2.8 Model-Based RL: Generalized Policy Iteration

In this section, we will introduce three algorithms for solving the RL optimization problem. These algorithms are based on the idea of iteratively evaluating a policy and then improving it (Figure 2.2). The granularity of the interaction between the Policy Evaluation and the Policy Improvement can be at any level from completing one exhaustively before starting the other to alternating between them after every single step of both processes.

This section starts with introducing the Policy Iteration algorithm, in which the Policy Evaluation and the Policy Improvement process are completely separated. Then the Value Iteration algorithm is introduced. The Value Iteration algorithm has a finer level of interaction between Policy Evaluation and the Policy Improvement. Finally, we introduce a more general algorithm referred to as Generalized Policy Improvement (GPI) which essentially allows to use any level of granularity between the Policy Evaluation and the Policy Improvement.

![Figure 2.2: Generalized Policy Iteration. Reproduced from [3].](image-url)
2.8 Model-Based RL: Generalized Policy Iteration

### 2.8.1 Policy Iteration

Now armed with the techniques of **Policy Evaluation** and **Policy Improvement**, finding an optimal policy is quite straightforward. Starting from some arbitrary policy, $\pi_0$, we can iteratively find the corresponding value function, $V^{\pi_0}$. We can then greedily improve our policy with respect to that value function to obtain a new policy $\pi_1$. We then use the previous value function as the initialization for the following **Policy Evaluation**. We can repeat this process until our policy converges. Each subsequent policy is guaranteed to be a strict improvement to the previous one unless it is already optimal. Also, since a finite MDP has only a finite number of possible actions, and therefore a finite number of possible policies, we know our solution will converge in a finite number of iterations. This process is called **Policy Iteration**. The complete algorithm is given in Algorithm 2.

---

**Algorithm 2 Policy Iteration**

1. **Initialization**
   
   select $V(x) \in \mathbb{R}$ and $\pi(x) \in \mathcal{U}$ arbitrarily for all $x \in \mathcal{X}$

2. **Policy evaluation**

   repeat
   
   $\Delta \leftarrow 0$

   for each: $x \in \mathcal{X}$

   $v \leftarrow V(x)$

   $V(x) \leftarrow \sum_u \pi(x, u) \sum_{x'} \mathcal{P}_{xx'}^{u} [R_{xx'}^{u} + \alpha V(x')]$

   $\Delta \leftarrow \max(\Delta, |v - V(x)|)$

   until $\Delta < \theta$ (a small positive number)

3. **Policy Improvement**

   $policyIsStable \leftarrow true$

   for $x \in \mathcal{X}$ do

   $b \leftarrow \pi(x)$

   $\pi(x) \leftarrow \arg\max_u \sum_{x'} \mathcal{P}_{xx'}^{u} [R_{xx'}^{u} + \alpha V(x')]$

   if $b \neq \pi(x)$ then

   $policyIsStable \leftarrow false$

   end if

   end for

   if $policyIsStable$ then

   stop

   else

   go to 2

   end if

Return: a policy, $\pi$, such that: $\pi(x) = \arg\max_u \sum_{x'} \mathcal{P}_{xx'}^{u} [R_{xx'}^{u} + \alpha V(x')]$
2.8.2 Value Iteration

One of the drawbacks of the Policy Iteration is that each iteration of the algorithm involves a Policy Evaluation, which may require several sweeps of the whole state space. Since Policy Evaluation typically converges to the optimal policy in the limit, we normally need to exit after a finite number of sweeps. This means that we need to truncate the Policy Evaluation part after some iterations. One extreme of this approach would be to exit after the first sweep of the Policy Evaluation and then perform a Policy Improvement process. In this case the policy Evaluation and Policy Improvement can be merged in a single update rule as

$$V_{k+1}(x) = \max_u \sum_{x'} P_{xx'}^u [R_{xx'}^u + \alpha V_k(x')]$$  \hspace{1cm} (2.33)

The algorithm which is based on this update rule is called Value Iteration. It is proven that Value Iteration converges to the optimal state value function in limit. Also, this update rule has another interesting interpretation. By comparing it to the optimal Bellman equation for value function (Eq. (2.24)), you can see that it is an iterative backup rule for solving this equation. The complete algorithm is given as Algorithm 3.

**Algorithm 3 Value Iteration**

Initialization: $V(x) \in \mathbb{R}$ and $\pi(x) \in \mathbb{U}$ arbitrarily for all $x \in X$

repeat

$\Delta \leftarrow 0$

for $x \in X$ do

$v \leftarrow V(x)$

$V(x) \leftarrow \max_u \sum_{x'} P_{xx'}^u [R_{xx'}^u + \alpha V(x')]$

$\Delta \leftarrow \max(\Delta, |v - V(x)|)$

end for

until $\Delta < \theta$ (a small positive number)

Return: a policy, $\pi$, such that: $\pi(x) = \arg\max_u \sum_{x'} P_{xx'}^u [R_{xx'}^u + \alpha V(x')]$

2.8.3 Generalized Policy Iteration

Generalized Policy Iteration (GPI) refers to the class of algorithms which are based on the general idea of synchronously performing Policy Evaluation and Policy Improvement, independent of the granularity of the two processes. For example in the Policy Iteration algorithm these two process are performed one after the other, each completing before the other starts. In value Iteration algorithm, only after a single sweep of the Policy Evaluation, the Policy Improvement is performed. In general, there two processes can be interleaved in any fine level. Almost all of the algorithms in this chapter conform to this GPI idea. The convergence to the optimal value is typically guaranteed as long as both processes continue to update one after the other.
2.9 Sample-Based RL: Monte Carlo Method

Monte Carlo methods propose a sample-based approach to solve the RL problem, previously introduced in Section 2.2. Unlike the model-based approaches, they are based on samples generated from the agent’s interaction with its environment instead of requiring the reward and the transition probability distributions. Since the information for the Monte Carlo Algorithm is provided through samples, there should be a practical way to extract data and save it. This simply implies that tasks with infinite execution time will not be considered in these approaches. Therefore in this section, we restrict our discussion to tasks with limited durations which we refer to as episodic tasks.

The main idea of the Monte Carlo approach carries over from the GPI algorithm (section 2.8.3). However, looking at a policy iteration algorithm (see e.g. Algorithm 2), we realize that model-knowledge is required in the form of the transition probabilities and the reward function, in both the policy evaluation and policy improvement steps. Furthermore, in order to obtain the optimal policy from the calculated optimal value function, we need to use the following equation, which is based on the state transition model

$$
\pi^*(x) = \arg\max_\pi E \left\{ r_n + \alpha V^*(x') \mid x_n = x \right\}
= \arg\max_\pi \sum_{x'} p_{x|x'}^u [R_{x|x'}^u + \alpha V^*(x')]
$$

However, if the action value function is calculated directly, there is no need for a transition model, since

$$
\pi^*(x) = \arg\max_\pi Q^*(x, u)
$$

As we shall see, if a model is not available, it is particularly useful to estimate action value functions $Q^*(x, u)$ instead of state value functions $V^*(x)$.

2.9.1 Sample-Based GPI with Action Value Function

Algorithm 4 states a GPI that utilizes action value functions instead of state value functions (note that the policy evaluation stopping criterion is left open as design choice). In the following sections, we will examine how the fundamental elements of GPI (namely Policy Evaluation and Policy Improvement) can be carried over to the sample-based case.

Model-free Policy Improvement  First, we consider the Policy Improvement step of the GPI as given in Algorithm 4. Given an action value function $Q^*(x, u)$, policy improvement is done by making the policy greedy w.r.t. the current action value function. In other words, for every state $x$ we evaluate all possible control inputs $u$ and choose the one which leads to the the best possible
Algorithm 4 Generalized Policy Iteration (GPI) using the action value function $Q^\pi(x, u)$

1. Initialization
   
   $Q^\pi(x, u) \in \mathbb{R}$

2. Policy Evaluation (PE)
   
   repeat
      for select a pair $(x, u)$ with $x \in X$, $u \in U$ do
         $\nu \leftarrow Q^\pi(x, u)$
         $Q^\pi(x, u) \leftarrow \sum_{x'} P_{xx'}^u [R_{xx'}^u + \alpha \sum_u \pi(x', u) Q^\pi(x', u)]$
      end for
   until "individual PE criterion satisfied"

3. Policy Improvement
   
   policy-stable $\leftarrow$ true
   for $x \in X$ do
      $b \leftarrow \pi(x)$
      $\pi(x) \leftarrow \arg \max_u \sum_{x'} P_{xx'}^u [R_{xx'}^u + \alpha V(x')]$
      if $b \neq \pi(x)$ then
         policy-stable $\leftarrow$ false
      end if
   end for
   if (policy-stable == true) then
      stop;
   else
      go to 2.
   end if

combination of reward and action-value of the successor state. This can be equivalently written as

$$\pi(x) = \arg \max_u Q^\pi(x, u)$$

which does not require a model to construct and is compliant with the Policy Improvement Theorem (2.30).

Model-free Policy Evaluation Consider the Policy Evaluation step of GPI as given in Algorithm 4. Recall that the action value is the expected (discounted) accumulative reward starting from a certain state $x$ by choosing control $u$ and then following a policy $\pi$, thus

$$Q^\pi(x, u) = E_\pi \{ R_n \mid x_n = x, u_n = u \} .$$

Taking an episodic point of view, an obvious way to estimate $Q^\pi(x, u)$ is simply by averaging the returns observed after visits to that state-action pair. After a total of $N$ visits (each observation is
indexed with \(i\), we write the averaged discounted return \(\tilde{Q}_N(x, u)\) as

\[
\tilde{Q}_N(x, u) \approx \frac{1}{N} \sum_{i=1}^{N} R_i(x, u) = \frac{1}{N} \sum_{i=1}^{N} \left( r_i^n + \alpha r_{n+1}^i + \alpha^2 r_{n+2}^i + \ldots \right) .
\] (2.34)

Equation (2.34) implies that for calculating the action value function for each state-control pair, we need to store all the retrieved samples as they are received at runtime and then calculate the average for each. This approach is usually called batch computation. However as you may guess, it is an inefficient approach both in terms of memory usage and computation requirements. On the other hand, we can perform this averaging in a recursive way. To do so, assume that a new sample indexed as \(N+1\) is acquired. We decompose \(\tilde{Q}_{N+1}(x, u)\) as

\[
\tilde{Q}_{N+1}(x, u) = \frac{1}{N+1} \sum_{i=1}^{N+1} R_i(x, u)
\]

\[
= \frac{1}{N+1} \left( \sum_{i=1}^{N} R_i(x, u) + R_{N+1}(x, u) \right)
\]

\[
= \frac{N}{N+1} \cdot \tilde{Q}_N(x, u) + \frac{1}{N+1} \cdot R_{N+1}(x, u) \quad \text{by Equation (2.34)}
\]

\[
= \tilde{Q}_N(x, u) + \frac{1}{N+1} \left( R_{N+1} - \tilde{Q}_N(x, u) \right)
\] (2.35)

which yields a recursive update equation for the estimated action-value function with "learning rate" \(\omega_{N+1} = 1/(N+1)\). In the stationary case, by taking the limit \(N \to \infty\), expression (2.35) converges towards the true value of the action value function

\[
\lim_{N \to \infty} \tilde{Q}_N(x, u) = Q^*(x, u) .
\]

However it can be shown, for any arbitrary learning rate sequence \(\{\omega_N\}\) that meets the Robbins-Monro conditions (equation (2.36) to (2.38)) the recursive update equation converges to the true value in the stationary case (i.e. the case that the stochastic property of the problem does not change over time).

\[
\lim_{N \to \infty} \omega_N = 0 \quad \text{(2.36)}
\]

\[
\sum_{N=1}^{\infty} \omega_N = \infty \quad \text{(2.37)}
\]

\[
\sum_{N=1}^{\infty} \omega_N^2 < \infty \quad \text{(2.38)}
\]

The first condition (2.36) ensures that the successive corrections decrease in magnitude so that the process can converge to a limit. The second condition (2.37) is required to ensure that the algorithm
does not converge prematurely and the third condition \((2.38)\) is needed in order to ensure that the accumulated noise has finite variance and hence does not spoil convergence. However, the above conditions are tailored to the stationary case. Note that in sample-based methods, the involved distributions are typically non-stationary. This is because the return depends on the policy, which is changing and improving over time. It can be shown that, in order to reach a good approximation of the true expected value by sampling in the non-stationary case, conditions \((2.36)\) and \((2.38)\) are not required to hold. Therefore we normally choose a constant learning rate.

To summarize, we define the update-rule for the \((N+1)\)-th action value after receiving a new sample reward \(R_{n+1}\) for a particular action-value pair \((x, u)\) as

\[
\tilde{Q}^{\pi}_{N+1}(x, u) = \tilde{Q}^{\pi}_{N}(x, u) + \omega_{N+1} \cdot (R_{N+1} - \tilde{Q}^{\pi}_{N}(x, u)) \tag{2.39}
\]

which can be initialized with any arbitrary values (e.g. zero or a random number).

### 2.9.2 Naive Monte Carlo Control with “Exploring Starts”

As shown in Section 2.9.1, the GPI algorithm can be used in a purely model-free way. The key elements for moving from a model-based to a sample-based algorithm were: 1) estimating the action value function instead of the state value function, 2) replacing the Bellman updating rule in the Policy Evaluation by a numerical averaging, 3) using the estimated action value function in the Policy Improvement process. This implementation of the GPI algorithm which is based on the raw samples rather than the model is referred to as the Monte Carlo method.

Equipped with the Monte Carlo method, we can solve the RL problem by only using samples. A sample in this context is defined as a sequence of state-action-rewards which is acquired by initializing the agent in an arbitrary state-action pair and then interacting with the environment according to the policy at hand. As indicated by Algorithm 4, we need to estimate the action value function for every possible state-action pair. This is ensured by initializing the agent in every one of these pairs.

Furthermore, in order to increase the sampling efficiency, we can reuse the already extracted samples by considering each of the sub-sequences as a new sample which has a different starting state-action pair. Although this reusing scheme helps to acquire samples from state-action pairs that the agent is not initialized from, it cannot guarantee a perfect coverage of all the state-action pairs. This is the general problem of maintaining exploration. Since the current version of the algorithm uses a greedy policy (greedy with respect to latest approximation of the action value function), in order to
2.9 Sample-Based RL: Monte Carlo Method

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maintain the exploration, we should start from each possible state-input pair.

\[ x_n, u_n, r_n, x_{n+1}, u_{n+1}, r_{n+1}, x_{n+2}, u_{n+2}, r_{n+2}, \ldots, x_{T-1}, u_{T-1}, r_{T-1}, x_T, u_T, r_T \]

\[ x_{n+1}, u_{n+1}, r_{n+1}, x_{n+2}, u_{n+2}, r_{n+2}, \ldots, x_{T-1}, u_{T-1}, r_{T-1}, x_T, u_T, r_T \]

\[ x_{n+2}, u_{n+2}, r_{n+2}, \ldots, x_{T-1}, u_{T-1}, r_{T-1}, x_T, u_T, r_T \]

\[ \vdots \]

\[ x_{T-1}, u_{T-1}, r_{T-1}, x_T, u_T, r_T \]

Relying only on samples to solve the RL problem makes it possible to use the algorithm on real world scenarios where the samples are directly drawn from the interaction between the agent and the world. Therefore, the agent can improve its performance while it interacts with the environment for fulfilling its task. This process is referred to in the literature as “Learning while Living”.

However, as noted before, in order to guarantee that the algorithm actually converges to the optimal action value function (and the optimal corresponding policy), we need to initialize it from every possible state-input pair. This is contrary to the idea of “Learning while Living” since we need to artificially place the agent in different initial states. This approach is called “exploring starts” and it actually is a naive implementation of Monte Carlo methods - hence we also call the corresponding algorithm “Naive Monte Carlo”. A pseudo-code of the algorithm is shown in Algorithm 5. In the next section, we will introduce another implementation of the Monte Carlo method which overcomes this limitation by introducing a non-greedy exploration policy.

Algorithm 5 Naive Monte Carlo Algorithm Assuming Exploring Starts

Initialize, for all \( x \in X, u \in U \)

\( Q(x, u) \leftarrow \) arbitrary

\( \pi \leftarrow \) an arbitrary deterministic policy

Repeat forever:

(a) Exploring start: select random pair \( (x, u) \)

(b) Select a policy \( \pi \) and generate an episode: \( x_0, u_0, r_0, x_1, u_1, r_1, \ldots, x_N, u_N, r_N \).

(c) Sample-based Policy Evaluation:

for each pair \( x, u \) appearing in the episode:

\( R \leftarrow \) return following the first occurrence of \( x, u \)

\( Q(x, u) \leftarrow Q + \omega \cdot (R - Q(x, u)) \)

(d) Policy improvement:

\( \pi(x) \leftarrow \arg \max_u Q(x, u) \)

2.9.3 On-Policy Monte Carlo Control with \( \varepsilon \)-soft Policy

In this section we introduce an on-policy method, that is, a method that attempts to improve the policy that is used to make the decisions. In on-policy Monte Carlo control methods, the policies are generally soft, meaning that \( \pi(x, u) > 0 \) for all control inputs in each state.
While in the previous algorithm, we should initialize the agent in each state-action pair in order to balance exploration-exploitation, this implementation removes this restriction by introducing a soft policy (called $\varepsilon$-greedy policy) instead of the greedy one.

Therefore in the Policy Improvement process of the algorithm, we use an $\varepsilon$-greedy policy improvement instead of the greedy policy improvement. That means, with higher probability we choose an action that has maximum estimated action value, but with some non-zero probability $\varepsilon$, we select other actions randomly. The $\varepsilon$-greedy policy is defined as follows

$$
\pi(u \mid x) = \begin{cases} 
\frac{\varepsilon}{\|U\|} & \text{for the non-greedy action} \\
1 - \varepsilon \left(1 - \frac{1}{\|U\|}\right) & \text{for the greedy action}
\end{cases}
$$

(2.40)

for some $\varepsilon > 0$ and $\|U\|$ being the number of the control inputs.

The overall idea of on-policy Monte Carlo is still based on GPI, however, with the assumption of the $\varepsilon$-greedy policy, we cannot improve the policy by making it greedy w.r.t. the current action value function. In the following, we show that the Policy Improvement theorem still holds for $\varepsilon$-greedy policies. In particular, we show that any $\varepsilon$-greedy policy with respect to the latest estimation of $Q^\pi$ is an improvement over the current $\varepsilon$-greedy policy. This is similar to what we showed in equation (2.29) for the greedy Policy Improvement (see Section 2.7), but now for the $\varepsilon$-greedy case. Let $\pi'$ be the $\varepsilon$-greedy policy w.r.t. the $Q^\pi$ (the action value function associated to the policy $\pi$).

The proof steps are very similar to the one for the greedy policy. First, we show that if at an arbitrary time step e.g. $n$, we perform w.r.t. the new policy $\pi'$, then we follow the old policy afterwards, the expected accumulated reward is greater or equal to the value function of the old policy, $V^\pi$.

$$
E_{u \sim \pi'} \{ R_n \mid x_n = x \} = \sum_u \pi'(x, u)Q^\pi(x, u) = \frac{\varepsilon}{\|U\|} \sum_u Q^\pi(x, u) + (1 - \varepsilon) \max_u Q^\pi(x, u)
$$

(2.41)

Now the goal is to find the relation between the left hand side of the equation and the value function of the policy $\pi$. To do so, we use the following trick which implies that the weighted average of a set of numbers is always less or equal to their maximum. Therefore, we can write

$$
\max_u Q^\pi(x, u) \geq \sum_u \frac{\pi(x, u) - \varepsilon}{1 - \varepsilon} Q^\pi(x, u).
$$

(2.42)
Note that \( \left\{ \frac{\pi(x,u) - U(x,u)}{1 - \varepsilon} \right\} \) are the nonnegative averaging weights which actually sum up to one.

\[
\begin{align*}
\sum_u \frac{\pi(x,u) - \varepsilon}{1 - \varepsilon} &= \sum_u \frac{\pi(x,u)}{1 - \varepsilon} - \sum_u \frac{\varepsilon}{1 - \varepsilon} \\
&= \frac{1}{1 - \varepsilon} \sum_u \pi(x,u) - \frac{\varepsilon}{1 - \varepsilon} \sum_u 1 \\
&= \frac{1}{1 - \varepsilon} - \frac{\varepsilon}{1 - \varepsilon} \|U\| = 1
\end{align*}
\]

By substituting \( \max_u Q^\pi(x,u) \) in the equation (2.41) using the inequality (2.42), we get

\[
E_{u \sim \pi'} \{ R_n | x_n = x \} \geq \varepsilon \|U(x)\| \sum_u Q^\pi(x,u) + (1 - \varepsilon) \sum_u \frac{\pi(x,u) - U(x,u)}{1 - \varepsilon} Q^\pi(x,u)
\]

The whole right-hand side expression can be rewritten as

\[
\varepsilon \|U(x)\| \sum_u Q^\pi(x,u) - \varepsilon \|U(x)\| \sum_u Q^\pi(x,u) + \sum_u \pi(x,u)Q^\pi(x,u) = \sum_u \pi(x,u)Q^\pi(x,u)
\]

Therefore we have

\[
E_{u \sim \pi'} \{ R_n | x_n = x \} \geq \sum_u \pi(x,u)Q^\pi(x,u)
\]

which clearly resolves to be equal to \( V^\pi(x) \), hence

\[
E_{u \sim \pi'} \{ R_n | x_n = x \} \geq V^\pi(x)
\] (2.43)

In the second step of the proof, we show that the expected accumulated reward in case we perform the first two steps according to the policy \( \pi' \) and then follow the policy \( \pi \) is greater or equal to

\[
E_{u \sim \pi'} \{ R_n | x_n = x \} \geq V^\pi(x)
\]

Since this proof is very similar to the previous it will be skipped but the results will be used:

\[
E_{u \sim \pi'} \{ R_n | x_n = x \} \geq E_{u \sim \pi'} \{ R_n | x_n = x \}
\]

Following the same procedure, we can replace the policy \( \pi' \) until the end. Therefore

\[
E_{u \sim \pi'} \{ R_n | x_n = x \} \geq V^\pi(x)
\] (2.44)
The left-hand side of this inequality is actually the value function for the policy $\pi'$. So we have

$$V^{\pi'}(x) \geq V^\pi(x) \tag{2.45}$$

which implies that the policy $\pi'$ is better than the policy $\pi$. Therefore, we have proven the Policy Improvement theorem for the $\varepsilon$-greedy policy case. It can be shown that the Policy Improvement for $\varepsilon$-greedy policies converges when the policy is optimal amongst the $\varepsilon$-greedy policies (for more details refer to Sutton section 5.4).

A major difference between this method and the Policy Iteration from the previous sections is that it does not find the best overall policy — it just converges to the best $\varepsilon$-greedy policy, which contains occasional exploration moves (which are suboptimal). But at the same time, it eliminates the need for exploring starts. To tackle this issue, we can eventually decrease the $\varepsilon$ to zero. Therefore, in the end the algorithm converges to the optimal greedy policy. However it is important that the decrease has a reasonable rate since a fast decrease can cause a premature convergence. The reason is that by decreasing the $\varepsilon$, we favor more exploitation against exploration which can cause some favorable states to never be evaluated. The $\varepsilon$-soft, On-Policy Monte Carlo Control is given as Algorithm 6.

**Algorithm 6 $\varepsilon$-soft, On-Policy Monte Carlo Algorithm**

1. choose a constant learning rate, $\omega$
2. choose a positive $\varepsilon \in (0, 1]$
3. $Q^\pi(x, u) \leftarrow$ arbitrary
4. $\pi \leftarrow$ an arbitrary $\varepsilon$-soft policy
5. **Repeat forever:**
   6. (a) generate an episode using $\pi$
   7. (b) Policy Evaluation
      - for each pair $(x, u)$ appearing in the episode
        - $R \leftarrow$ return following the first occurrence of $(x, u)$
        - $Q^\pi(x, u) \leftarrow Q^\pi(x, u) + \omega (R - Q^\pi(x, u))$
   8. (c) Policy Improvement
      - for each: $x$ in the episode:
        - $u^* \leftarrow \arg \max_u Q^\pi(x, u)$
        - For all $a \in \mathcal{U}(x)$:
          - $\pi(x, u) \leftarrow \begin{cases} 
            \frac{\varepsilon}{|\mathcal{U}(x)|} & \text{if } u \neq u^* \\
            1 - \varepsilon \left(1 - \frac{1}{|\mathcal{U}(x)|}\right) & \text{if } u = u^*
          \end{cases}$
   9. (d) (optional) decrease $\varepsilon$.  

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2.10 Sample-Based RL: Q-Learning

One of the most important breakthroughs in reinforcement learning was the development of Q-learning. Roughly speaking, it is a combination of Monte Carlo ideas and Dynamic Programming ideas. Like Monte Carlo methods, Q-learning learns directly from raw experience without requiring model information. However like Dynamic Programming, it updates its estimates based partly on the neighboring state estimates, without waiting for the episode end.

We start the discussion by recalling the Bellman Equation for the optimal action value functions which is given in Equation (2.21) as

$$Q^*(x_n, u_n) = E \left\{ r_n + \alpha \max_{u'} Q^*(x', u') \mid x_n = x, u_n = u \right\}$$

where we denote the control action that follows $u$ as $u'$. Since we would like to learn the action value $Q^*(x, u)$ of a certain state-action pair from experience without using a model, we sample and average action values over $N$ occurrences of $(x, u)$. This leads us to conventional averaging as in equation (2.34). However, this time, we are not considering the whole tail of episode until its termination – instead, we only consider the following state $x'_i$ and append the current reward ($r^i_n$) to the best action value function that can be gained by selecting a particular $u'$. Therefore, figuratively speaking, we average over instances of the right-hand side of the Bellman optimality equation as we are sampling.

$$\tilde{Q}(x, u) = \frac{1}{N} \sum_{i=1}^{N} \left( r^i_n + \alpha \max_{u} Q(x'_i, u) \right)$$

which can also be formulated recursively similar to what we did in Equation (2.35). That leads us to the following update equation for a sample that appears in episode $i + 1$ at time $n$:

$$\tilde{Q}^{i+1}(x_n, u_n) = \tilde{Q}^i(x_n, u_n) + \omega_{i+1} \left[ r^i_n + \alpha \max_{u'} \tilde{Q}^i(x'_n, u'_n) - \tilde{Q}^i(x_n, u_n) \right]$$

Note that so far we have not talked about the policy $\pi$. The learned action-value function, $\tilde{Q}(x, u)$, directly approximates $Q^*(x, u)$, the optimal action-value function, independent of the policy being followed. Of course, the policy has an effect in that it determines which state-action pairs are visited and updated. However, all that is required for correct convergence is that all pairs continue to be updated. Therefore, Q-learning is an off-policy control method. It learns about the greedy policy, independent of what policy is actually being followed by the agent (as long as it tries all state-action pairs). Thus, it is learning about one policy while following another, which is the defining characteristic of off-policy methods.

Q-learning learns its estimates in part on the basis of other estimates, i.e. it looks ahead one timestep, pre-selects an optimal following action and action value and appends it to the immediate reward. This is what is commonly called “bootstrapping”. Q-learning is shown in procedural form in Algorithm 7.
Algorithm 7 Q-Learning

Initialize $Q(x, u)$ arbitrarily

Repeat for each episode:

Initialize $x$

repeat (for each step of episode):

Choose $u$ from $x$ using policy derived from $Q$

(e.g., $\varepsilon$-greedy)

Take action $u$, observe $r, x'$

$Q(x, u) \leftarrow Q(x, u) + \omega[r + \gamma \max_{u'} Q(x', u') - Q(x, u)]$

$x \leftarrow x'$

until $x$ is terminal
3 Path Integrals

3.1 The Brownian Motion

Brownian motion, or random walk, is defined as a stochastic process of some variable, \( w \), with a Gaussian probability distribution as follows

\[
P_w(t, w) = \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp\left( -\frac{(w - \mu t)^2}{2\sigma^2 t} \right)
\] (3.1)

At each instance in time, this process has a Gaussian distribution with the following mean and variance:

\[
\mathbb{E}\{w(t)\} = \mu t
\]
\[
\text{Var}\{w(t)\} = \sigma^2 t
\] (3.2)

To define the probability distribution of a stochastic process, we need to define not only \( P_w(t, w) \) but also the joint distribution of the stochastic process for an arbitrary subset of time steps i.e. \( P_w(t_1, w_1, \ldots, t_N, w_N) \).

Instead of defining the joint probability distribution directly, we will define the Brownian motion’s increment process, \( dw(t) = \lim_{\Delta t \to 0} w(t + \Delta t) - w(t) \) by the two following characteristics:

1. The increment process, \( dw(t) \), has a Gaussian distribution with the mean and the variance, \( \mu \Delta t \) and \( \sigma^2 \Delta t \) respectively.

2. The increment process, \( dw(t) \), is statistically independent of \( w(s) \) for all \( s \leq t \).

With these characteristics of the increment process and the probability distribution introduced in equation (3.1), we can derive the joint distribution of any arbitrary set of time instances. For example, we can show that the joint distribution for \( w(t) \) and \( w(s) \) is a Gaussian distribution with the cross covariance

\[
\mathbb{E}\{(w(t) - \mu t)(w(s) - \mu s)\} = \sigma^2 \min(t, s)
\]

Figure 3.1 illustrates 15 samples extracted from a Brownian motion with \( \mu = 5 \) and \( \sigma^2 = 4 \) within a time period \([0, 2]\). An interesting aspect of Brownian motion is that even though it is continuous over time, it is not differentiable.

In order to simulate the Brownian motion, either we should find the joint probability distribution of \( \{w(0), w(\Delta t), w(2\Delta), \ldots, w(N\Delta t)\} \) and then extract samples from it or we should forward integrate...
the following equation (which is in fact a discretized stochastic differential equation)

\[ w(t + \Delta t) = w(t) + \mu \Delta t + \sqrt{\Delta t \sigma^2} \varepsilon, \quad w(0) = 0 \]

where \( \varepsilon \) is extracted from a normal distribution with zero mean and variance one. In this equation, the terms \( \mu \Delta t + \sqrt{\Delta t \sigma^2} \varepsilon \) are used to approximate the increment process, \( dw \). In the limit when \( \Delta t \) approaches to zero, this term will be equal to the increment process. Throughout this chapter we will mostly use the increment process instead of the Brownian motion itself.

Finally, the Brownian motion can be easily extended to n-dimension vector form as

\[ P_{w(t,w)} = \frac{1}{(2\pi)^{n/2}(\Sigma t)^{1/2}} \exp \left( -\frac{1}{2}(w - \mu t)^T(\Sigma t)^{-1}(w - \mu t) \right) \]

### 3.2 Stochastic Differential Equations

The following equation describes a Stochastic Differential Equation (SDE)

\[ dx = f(t,x)dt + g(t,x)dw \quad \text{(3.3)} \]

where \( x(t) \) is a stochastic vector process of dimension \( n \) and \( w(t) \) is a Brownian motion with zero mean and identity covariance matrix (\( dw \) is the increment process). \( f(t,x) \) and \( g(t,x) \) are called respectively the drift and the diffusion coefficients. Note that for \( g(t,x) = 0 \) the SDE reduces to the well-known Ordinary Differential Equation (ODE). Furthermore defining the mean and covariance of the Brownian motion to zero and the identity matrix, respectively, is not a restrictive assumption. Since any arbitrary Brownian motion can be put in this form by adding the mean to the drift coefficient and multiplying the diffusion coefficient by the square root of the covariance matrix.

One simple approach for simulating the SDE in equation (3.3) is to discretize it over time by some sufficiently small time step \( \Delta t \) and replace \( dw \) by \( \sqrt{\Delta t} \varepsilon \). In doing this, we assume that during the time step \( \Delta t \) the Brownian motion has a constant increment with mean zero and covariance \( \Delta t I \).
The discretized SDE is then expressed as follows:

\[ x(t_{n+1}) = x(t_n) + f(t_n, x(t_n)) \Delta t + g(t_n, x(t_n)) \sqrt{\Delta t} \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, I) \] (3.4)

Note that when equation (3.3) is non-linear, as is typically the case, \( x \) will have a non-Gaussian distribution. However the following conditioned probability distribution for an infinitely small time step is always Gaussian.

\[ P_x(t + \Delta t, x | t, y) = \mathcal{N}(y + f(t, y) \Delta t, g(t, y) g^T(t, y) \Delta t) \] (3.5)

### 3.3 The Fokker Planck Equation

Consider the SDE in equation (3.3). Despite the fact that the Brownian motion has a Gaussian distribution, generally the resulting process does not have a Gaussian distribution (except in the linear case). However the probability distribution associated to the SDE in equation (3.3) can be derived as a solution to an initial value Partial Differential Equation (PDE) known as Fokker Plank equation.

We define \( P_x(t, x | s, y) \) as the conditioned probability distribution of \( x(t) \) knowing that the process at time \( s \) has value \( y \) (\( x(s) = y \)). The governing PDE on this probability distribution is defined by the following Fokker Planck equation

\[ \partial_t P = -\nabla_x^T (fP) + \frac{1}{2} \text{Tr} \left[ \nabla_{xx} (gg^T P) \right] \] (3.6)

with initial condition \( P_x(t = s, x | s, y) = \delta(x - y) \). The operators \( \nabla_x() \) and \( \nabla_{xx}() \) are defined as:

\[
\nabla_x() = \begin{bmatrix}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2} \\
\vdots \\
\frac{\partial}{\partial x_n}
\end{bmatrix}, \quad \nabla_{xx}() = \begin{bmatrix}
\frac{\partial}{\partial x_{11}} & \cdots & \frac{\partial}{\partial x_{1n}} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial x_{n1}} & \cdots & \frac{\partial}{\partial x_{nn}}
\end{bmatrix}
\]

where \( x_i \) is the \( i \)th element of the \( n \)-dimension stochastic process vector \( x \).

### 3.4 Linearly-Solvable Markov Decision Process

The Linearly-Solvable Markov Decision Process (LMDP) is a class of optimal control problems in which the optimality equation (characterized by the Bellman equation or the HJB equation) can be transformed into a linear form. In this section we will consider a special case of this class with stochastic dynamics defined as follows

\[ dx = f(t, x) dt + g(t, x) (udt + dw), \quad dw \sim \mathcal{N}(0, \Sigma dt) \] (3.7)
where \( x \) is the state vector of size \( n \), \( u \) is the control input of size \( m \), and \( w \) is the \( m \)-dimension Brownian motion with zero mean and covariance \( \Sigma dt \). In the control literature, the system described by equation (3.7) is called a control affine system since the control input appears linearly in the differential equations. Note that if we divide both sides of the equation by \( dt \), and then substitute \( \frac{dw}{dt} \) for \( \varepsilon \) (stationary white noise), we get a system equation in the class of systems introduced in Section 1.3.3 (equation 1.45).

\[
\dot{x} = f(t, x) + g(t, x) (u + \varepsilon), \quad \varepsilon \sim \mathcal{N}(0, \Sigma)
\]

The cost function for this optimal control problem is defined as

\[
J = E \left\{ \Phi(x(t_f)) + \int_{t_0}^{t_f} q(t, x) + \frac{1}{2} u^T R u \, dt \right\}
\] (3.8)

The expectation is over the trajectories extracted from the stochastic process described by equation (3.7). The only difference between this cost function and the general one introduced by equation (1.48) is that the control input cost is quadratic.

In order to make this optimal control problem a LMDP, we need to add another condition which is

\[
R \Sigma = \lambda I
\] (3.9)

where \( \lambda \) is an arbitrary positive real number. This condition has the following intuitive interpretation:

Let’s assume for simplicity that both \( \Sigma \) and \( R \) are diagonal matrices. Then we get

\[
R_{ii} \Sigma_{ii} = \lambda, \quad \text{for all } i \in \{1, \ldots, m\}
\]

This means that if the covariance of the noise affecting control input \( i \) is relatively high, the cost for that control input should be lower. Higher control effort should be tolerated to counteract the noise (note that in equation 3.7 the process noise is added to the control input).

Now we will show how the optimal control problem can be transformed to an LMDP. We will start by writing the HJB equation for this problem. We will use the HJB equation given in equation (1.59) with the following substitutions

\[
\begin{align*}
\beta &\leftarrow 0 \\
L(x, u(t)) &\leftarrow q(t, x) + \frac{1}{2} u^T R u \\
f_i(x, u(t)) &\leftarrow f(t, x) + g(t, x) u \\
B(t) &\leftarrow g(t, x) \\
W(t) &\leftarrow \Sigma
\end{align*}
\]

Then we get

\[
-\partial_t V^*(t, x) = \min_u \left\{ q(t, x) + \frac{1}{2} u^T R u + \nabla_x V^*(t, x) (f(t, x) + g(t, x) u) + \frac{1}{2} \text{Tr}[\nabla_{xx} V^*(t, x) g(t, x) \Sigma g^T(t, x)] \right\}
\] (3.10)
where $V^*$ is the optimal value function. For simplicity, we will omit all the dependencies with respect to time and state. In equation (3.10) we have also adopted the notation introduced in Section 3.3.

If we minimize the left hand side with respect to $u$, we get

$$-\partial_t V^* = q - \frac{1}{2} \nabla_x^T V^* g R^{-1} g^T \nabla_x V^* + \nabla_x^T V^* f + \frac{1}{2} \text{Tr} [\nabla_{xx} V^* g \Sigma g^T]$$

(3.11)

and the optimal control

$$u^*(t, x) = -R^{-1} g^T(t, x) \nabla_x V(t, x)$$

(3.12)

Equation (3.11) is a nonlinear PDE, since $\nabla_x V^*$ appears in a quadratic form. In order to shorten the notion, we substitute $g R^{-1} g^T$ by $\Xi$. Using the condition 3.9, we can also write $g \Sigma g^T = \lambda \Xi$.

Therefore

$$-\partial_t V^* = q - \frac{1}{2} \nabla_x^T V^* \Xi \nabla_x V^* + \nabla_x^T V^* f + \frac{1}{2} \text{Tr} [\nabla_{xx} V^* \Xi]$$

(3.13)

In spite of the nonlinearity of the resulting equation, it can be shown that under a log transformation we get a linear PDE. Assume the following transformation $\Psi(t, x)$ is called the desirability function

$$V^*(t, x) = -\lambda \log \Psi(t, x),$$

(3.14)

we can write the derivatives as

$$\partial_t V^*(t, x) = -\lambda \frac{\partial \Psi}{\Psi}$$

$$\nabla_x V^*(t, x) = -\lambda \frac{\nabla_x \Psi}{\Psi}$$

$$\nabla_{xx} V^*(t, x) = \frac{1}{\lambda} \nabla_x V^* \nabla_x^T V^* - \lambda \frac{\nabla_{xx} \Psi}{\Psi}.$$  

(3.15)

Furthermore, for the scalar value $-\frac{1}{2} \nabla_x^T V^* \Xi \nabla_x V^*$ in equation (3.13) we can write

$$-\frac{1}{2} \nabla_x^T V^* \Xi \nabla_x V^* = -\frac{1}{2} \text{Tr} [\nabla_x V^* \Xi \nabla_x V^*] = -\frac{1}{2} \text{Tr} [\nabla_x V^* \nabla_x^T V^* \Xi].$$

(3.16)

By substituting equations (3.15) and (3.16) into equation (3.13), we get

$$\lambda \frac{\partial \Psi}{\Psi} = q - \frac{1}{2} \text{Tr} [\nabla_x V^* \nabla_x^T V^* \Xi] - \lambda f^T \frac{\nabla_x \Psi}{\Psi} + \frac{1}{2} \text{Tr} [\nabla_x V^* \nabla_x^T V^* \Xi] - \frac{\lambda^2}{2} \text{Tr} [\nabla_{xx} \Psi \Xi] .$$

(3.17)

Finally, we multiply both sides by $-\Psi/\lambda$

$$-\partial_t \Psi = -\frac{1}{\lambda} q \Psi + f^T \nabla_x \Psi + \frac{\lambda}{2} \text{Tr} [\Xi \nabla_{xx} \Psi].$$

(3.17)
3.5 Path Integral Optimal Control

Note that we used the matrix identity $\text{Tr}[AB] = \text{Tr}[BA]$ to write $\text{Tr}[\nabla_{xx}\Psi \Xi] = \text{Tr}[\Xi \nabla_{xx}\Psi]$. Equation (3.17) is linear with respect to $\Psi$ and its derivatives. This equation is normally written in the form of the linear operator $H$ as follows

$$-\partial_t \Psi = H[\Psi]$$

(3.18)

where $H$ is defined as

$$H = -\frac{1}{\lambda}q + f^T \nabla_x + \frac{\lambda}{2} \text{Tr}[\Xi \nabla_{xx}]$$

$$= -\frac{1}{\lambda}q + \sum_i f_i \frac{\partial}{\partial x_i} + \frac{\lambda}{2} \sum_{i,j} \Xi_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

(3.19)

From the principle of optimality, we know that the original optimal control problem is a final value problem, meaning that only the terminal condition is known prior to solving. Equation (3.17) is therefore also a final value problem, which has the following terminal value.

$$\Psi(t_f, x) = \exp \left(-\frac{1}{\lambda}V^*(t_f, x)\right)$$

$$= \exp \left(-\frac{1}{\lambda}\Phi(x)\right)$$

(3.20)

The equation (3.18) is consider to be an LMDP. In the next section, we will introduce the Path Integral method to solve this PDE.

### 3.5 Path Integral Optimal Control

The Path Integral framework introduces a method for solving the linear backward PDE introduced in equation (3.18) through a forward integration approach. We start the proof by introducing a new function over time and state, $\rho(t, x)$. We also define the following inner product in the function space which is in fact a function of time.

$$< \rho | \Psi >= \int \rho(t, x) \Psi(t, x) dx$$

(3.21)

We also assume that

$$\lim_{\|x\| \to \infty} \rho(t, x) = 0$$

(3.22)

The $H^\dagger$ operator is called the Hermitian conjugate of the operator $H$ (equation 3.19) if it satisfies the following equality.

$$< \rho | H[\Psi] >= < H^\dagger[\rho] | \Psi >$$

(3.23)
The $H^\dagger$ formulation can be derived as follows by starting from the definitions of the inner product and the $H$ operator.

$$\langle \rho | H[\Psi] \rangle = \int \rho(t, x)H[\Psi(t, x)]dx$$

$$= \int \rho(t, x) \left(-\frac{1}{\lambda} \frac{q}{\Psi} + \sum_i f_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} \Xi_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right) \Psi(t, x) dx \quad (3.24)$$

Using integration by parts and the assumption in 3.22 we can easily show

$$\langle \rho | H[\Psi] \rangle = \int \Psi \left(-\frac{1}{\lambda} q \rho - \sum_i \frac{\partial (f_i \rho)}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 (\Xi_{ij} \rho)}{\partial x_i \partial x_j} \right) dx$$

$$= \int \Psi(t, x)H^\dagger[\rho(t, x)]dx$$

Therefore we can write $H^\dagger$ as

$$H^\dagger = -\frac{1}{\lambda} q - \sum_i \frac{\partial f_i}{\partial x_i} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \Xi_{ij}}{\partial x_i \partial x_j}$$

$$= -\frac{1}{\lambda} q - \nabla^T_x f + \frac{1}{2} \text{Tr}[\nabla_{xx} \Xi] \quad (3.25)$$

So far the only restriction that we have posed over $\rho(t, x)$ is the condition 3.22. We will now pose another restriction on $\rho(t, x)$ that $\langle \rho | \Psi \rangle$ is time independent. Therefore we have

$$\frac{d}{dt} \langle \rho | \Psi \rangle = 0 \quad (3.26)$$

Then we can write

$$0 = \frac{d}{dt} \langle \rho | \Psi \rangle$$

$$= \int \partial_t \left(\rho(t, x)\Psi(t, x)\right) dx$$

$$= \int \partial_t \rho(t, x) \Psi(t, x) + \rho(t, x) \partial_t \Psi(t, x) dx$$

$$= \langle \partial_t \rho | \Psi \rangle + \langle \rho | \partial_t \Psi \rangle$$

By using the equation (3.18), we get

$$0 = \langle \partial_t \rho | \Psi \rangle - \langle \rho | H[\Psi] \rangle$$

Now we use the Hermitian conjugate operator $H^\dagger$

$$0 = \langle \partial_t \rho | \Psi \rangle - \langle H^\dagger[\rho] | \Psi \rangle$$
Therefore we can write
\[
< \partial_t \rho - H^\dagger[\rho] \mid \Psi > = 0
\]

A trivial solution to this equation is
\[
\partial_t \rho = H^\dagger[\rho] = -\frac{1}{\lambda} q \rho - \nabla^T_x (f \rho) + \frac{\lambda}{2} \nabla^T_x [\nabla_xx(\Xi \rho)]
\]
(3.27)

Since equation (3.18) is a final value problem, equation (3.27) should be an initial value problem, otherwise the terminal value for \( \rho \) cannot be defined freely, as the inner product should be time independent.

Furthermore it can be shown that if \( \rho(t, x) \) satisfies the equation (3.27), it will always satisfy the condition in (3.22). Therefore by only determining the initial value, we can uniquely define a \( \rho(t, x) \) function which satisfies all of the conditions. In the literature this equation is sometimes referred to as a forward diffusion process.

We will now define an appropriate initial condition for this forward diffusion process. Equation (3.27) resembles the Fokker Planck equation (equation 3.6). The only difference is the term \(-\frac{1}{\lambda} q \rho\). The effect of this extra term is that the probability distribution decays over time. Based on this similarity, we can show that the following process has the same probability distribution as the solution of equation (3.27) (note that we are misusing the term “probability distribution”. As mentioned the term \(-\frac{1}{\lambda} q \rho\) attenuates the probability distribution over time, therefore its integral eventually becomes less than one).

\[
\begin{align*}
dx(t_{i+1}) &= f(t_i, x(t_i))dt + g(t_i, x(t_i))d\mathbf{w}, \quad x(t_0 = s) = y \\
\begin{cases}
x(t_{i+1}) = x(t_i) + dx(t_i) & \text{with probability } \exp \left(-\frac{1}{\lambda} q dt\right) \\
x(t_{i+1}) : \text{annihilation} & \text{with probability } 1 - \exp \left(-\frac{1}{\lambda} q dt\right)
\end{cases}
\end{align*}
\]
(3.28)

where \( \mathbf{w} \) is a Brownian motion with zero mean and the covariance \( \Sigma dt \) and \( t_{i+1} = t_i + dt \). Here “annihilation” means that we discard the sample with some probability. Through the similarity to the Fokker Planck equation, we also can determine an appropriate initial condition. If the state vector at time \( s \) is \( y \) then the probability distribution of the state at the initial time should be a delta function at \( y \).

\[
\rho(t = s, x) = \delta(x - y)
\]
(3.29)

In order to emphasize that the \( \rho(t, x) \) is a solution to the diffusion process with the initial condition \( x(s) = y \), sometimes the initial condition is explicitly written in the condition i.e. we write \( \rho(t, x \mid s, y) \).
Path Integrals

3.5 Path Integral Optimal Control

Since the process described by equation (3.28) is a Markov Process we can write the joint probability distribution of a trajectory $\tau = \{x(t_0), x(t_1), \ldots x(t_N)\}$, with $t_0 = s$, as

$$
\rho(\tau \mid s, y) = \prod_{i=0}^{N-1} \rho(t_{i+1}, x(t_{i+1}) \mid t_i, x(t_i)) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}q(t_i,x(t_i))dt \right) \mathcal{N} \left( x(t_i) + f(t_i, x(t_i))dt, \Xi(t_i, x(t_i))dt \right)
$$

(3.30)

where $N$ is the number of the time steps and the initial state at time $s$ is $y$. Furthermore the conditioned probability distribution will have a Gaussian like distribution.

$$
\rho(t_{i+1}, x(t_{i+1}) \mid t_i, x(t_i)) = e^{-\frac{1}{2}q(t_i,x(t_i))dt} \mathcal{N} \left( x(t_i) + f(t_i, x(t_i))dt, \Xi(t_i, x(t_i))dt \right)
$$

(3.31)

Here $\Xi(t_i, x(t_i))$ is defined as Section 3.4. The effect of the annihilation process is incorporated by multiplying the distribution by $\exp (-\frac{1}{4}qdt)$. Therefore the probability distribution of the trajectory $\tau$ can be written as

$$
\rho(\tau \mid s, y) = \prod_{i=0}^{N-1} e^{-\frac{1}{4}q(t_i,x(t_i))dt} \mathcal{N} \left( x(t_i) + f(t_i, x(t_i))dt, \Xi(t_i, x(t_i))dt \right)
$$

(3.32)

where $\mathbb{P}_{uc}(\tau \mid s, y)$ is introduced in equation (3.33) and it is actually the probability distribution of the trajectory $\tau$ generated by the system in equation (3.28) without the annihilation when initialized at $x(s) = y$. Furthermore by omitting the annihilation term from equation (3.28), it will reduce to the same system as LMDP (equation 3.7) when the $u$ is set to zero. In control literature such a system is normally referred to as the uncontrolled system (hence the subscript “uc”).

$$
\mathbb{P}_{uc}(\tau \mid s, y) = \prod_{i=0}^{N-1} \mathcal{N} \left( x(t_i) + f(t_i, x(t_i))dt, \Xi(t_i, x(t_i))dt \right)
$$

(3.33)

Now we go back to the characteristic that the inner product of $\Psi(t, x)$ and $\rho(t, x)$ is time independent. Since the inner product is time independent, its value at any time will be always the same. Therefore for the initial time and the final time we can write

$$
< \rho \mid \Psi \mid (t = s) = < \rho \mid \Psi \mid (t = t_f)
$$

\[
\int \rho(s, x_s)\Psi(s, x_s)d\mathbf{x}_s = \int \rho(t_f, x_{tf})\Psi(t_f, x_{tf})d\mathbf{x}_{tf}
\]

(3.34)
By substituting the initial condition given in equation (3.29) and integrating it we will get

\[ \int \delta(x_s - y)\Psi(s, x_s)dx_s = \int \rho(t_f, x_{t_f})\Psi(t_f, x_{t_f})dx_{t_f} \]

\[ \Psi(s, y) = \int \rho(t_f, x_{t_f})\Psi(t_f, x_{t_f})dx_{t_f} \]

Now we will use the terminal condition of \(\Psi\) from equation (3.20).

\[ \Psi(s, y) = \int \rho(t_f, x_{t_f})e^{-\frac{1}{\lambda}\Phi(x_{t_f})}dx_{t_f} \]

To calculate the RHS integral we need to define the probability distribution at the last time step \(t_f\). We can derive \(\rho(t_f, x_{t_f})\) by marginalizing the joint probability distribution introduced in equation (3.32) with respect to \(\{x(t_1), \ldots, x(t_{N-1})\}\). Note that the time index starts from \(t_1\) and goes up to \(t_{N-1}\).

\[ \rho(t_f, x_{t_f}) = \int \rho_{uc}(\tau | s, y) e^{-\frac{1}{2}\sum_{i=0}^{N-1} q(t_i, x(t_i))dt} dx(t_1) \ldots dx(t_{N-1}) \]

By substituting this in the previous equation, we get (we have changed the name of \(x_{t_f}\) to \(x(t_N)\))

\[ \Psi(s, y) = \int \rho_{uc}(\tau | s, y) e^{-\frac{1}{2}\left(\Phi(x(t_N)) + \sum_{i=0}^{N-1} q(t_i, x(t_i))dt\right)} dx(t_1) \ldots dx(t_{N-1})dx(t_N) \quad (3.35) \]

This equation is equal to the following expectation

\[ \Psi(s, y) = E_{\tau_{uc}} \left\{ e^{-\frac{1}{2}\left(\Phi(x(t_N)) + \sum_{i=0}^{N-1} q(t_i, x(t_i))dt\right)} \right\} \]

\[ = E_{\tau_{uc}} \left\{ e^{-\frac{1}{2}\left(\Phi(x(t_f)) + \int_s^{t_f} q(t, x(t)) dt\right)} \right\} \quad (3.36) \]

where the first equation is the discretized equivalent of the latest one. We can estimate this expectation through a Monte Carlo method by extracting samples and averaging them. The samples of this expectation are generated through the uncontrolled dynamics system

\[ dx = f(t, x)dt + g(t, x)d\mathbf{w}, \quad d\mathbf{w} \sim \mathcal{N}(0, \Sigma dt), \quad x(t = s) = y \quad (3.37) \]

Therefore in order to estimate the desirability function at time \(s\) and state \(y\), we should forward simulate the uncontrolled dynamics several times with the initial condition \(x(s) = y\). Then we can estimate the expectation by averaging the value of \(-\frac{1}{2}\left(\Phi(x(t_f)) + \int_s^{t_f} q(t, x(t)) dt\right)\) for each sample.
After calculating the desirability function for all times and states, we can derive the optimal control input as

\[ u^*(s, y) = - R^{-1} g^T(s, y) \nabla_y V^*(s, y) \]

\[ = \lambda R^{-1} g^T(s, y) \frac{\nabla_y \Psi(s, y)}{\Psi(s, y)} \]  
(3.38)

However we can also calculate the optimal control input directly by a similar path integral approach

\[ u^*(s, y) = \lim_{\Delta s \to 0} \frac{E_{\tau_{uc}} \left\{ \int_s^{s+\Delta s} d\mathbf{w} e^{-\frac{1}{2} \left( \Phi(x(t)) + \int_t^{t_f} q(t, x) dt \right) } \right\}}{\Delta s \cdot E_{\tau_{uc}} \left\{ e^{-\frac{1}{2} \left( \Phi(x(t)) + \int_t^{t_f} q(t, x) dt \right) } \right\}} \]  
(3.39)

again if we use the white noise notation where \( d\mathbf{w} dt = \varepsilon \), we can write

\[ u^*(s, y) = \frac{E_{\tau_{uc}} \left\{ \varepsilon e^{-\frac{1}{2} \left( \Phi(x(t)) + \int_t^{t_f} q(t, x) dt \right) } \right\}}{E_{\tau_{uc}} \left\{ e^{-\frac{1}{2} \left( \Phi(x(t)) + \int_t^{t_f} q(t, x) dt \right) } \right\}} \]  
(3.40)

or equivalently

\[ u^*(s, y) = E_{\tau_{uc}} \left\{ \varepsilon \frac{e^{-\frac{1}{2} \left( \Phi(x(t)) + \int_t^{t_f} q(t, x) dt \right) }}{e^{-\frac{1}{2} \left( \Phi(x(t)) + \int_t^{t_f} q(t, x) dt \right) }} \right\} \]  
(3.41)

The samples are generated by

\[ \dot{x} = f(t, x) + g(t, x) \varepsilon, \quad \varepsilon \sim N(0, \Sigma), \quad x(t = s) = y \]  
(3.42)

Before we move on to the next section, we would like to give few insights into the path integral formulation for optimal control. In this discussion we will mainly use equations (3.40) and (3.42), however the same argument holds for the formulation in equations (3.39) and (3.37).

According to equation (3.41) in order to derive the optimal control at time \( s \) and state \( y \), we should repeatedly forward simulate the noise-driven system from that moment and point until the end of the time horizon \( t_f \). Then we should weight the first noise element of each sample by \( \alpha(\tau_{uc}; s, y) \) which is defined as

\[ \alpha(\tau_{uc}; s, y) = e^{-\frac{1}{2} \left( \Phi(x(t_f)) + \int_t^{t_f} q(t, x) dt \right) } \]  
(3.43)
3.6 Path Integral with Importance Sampling

Since in $\alpha(\tau_{uc}; s, y)$ the accumulated cost is in the exponential, this weighting will be almost zero for a costly trajectory and only a very small portion of the samples with near optimal or optimal cost will have non-negligible weights. However, the probability of generating optimal trajectories through the noise-driven system (a.k.a. random walk) is very low. Therefore in order to be able to estimate the optimal input, we need abundantly many samples. This issue becomes even more severe in high dimensional problems, quickly to the point that calculating the above exponential becomes intractable. In the next section, an importance sampling scheme will be introduced which can boost the sampling efficiency of the Path Integral.

3.6 Path Integral with Importance Sampling

In order to make path integral sampling more efficient, we will use importance sampling. Before introducing the method, we will briefly describe the idea behind importance sampling.

Assume the following expectation problem where $x$ is a random variable with probability distribution $p(x)$ and $f(x)$ is an arbitrary deterministic function.

$$E_p[f(x)] = \int_{-\infty}^{\infty} f(x) \ p(x) \ dx$$

We also assume that we have another random variable named $y$ with the probability distribution $q(y)$. Let's assume that calculating the expectation of an arbitrary function for this random variable is less costly than the previous one. Is there a way to calculate $E_p[f(x)]$, while we are only sampling from $y$?

The answer is yes, as long as we can guarantee that $q(\cdot)$ is non-zero everywhere $p(\cdot)$ is. In other words if the probability of extracting a particular value of $x$ is non-zero, $y$ should also be able to
have that value with some non-zero probability. If this condition satisfies we can write

$$E_p[f(x)] = E_q[w(y)f(y)], \quad w(y) = \frac{p(y)}{q(y)}$$

$w$ is called the importance sampling weighting. Proving this relation is very simple, we just need to write the definition of the expectation. To do such, we will start from the right hand side

$$E_q[w(y)f(y)] = \int_{-\infty}^{\infty} w(y)f(y)q(y) \, dy$$

$$= \int_{-\infty}^{\infty} \frac{p(y)}{q(y)} f(y)g(y) \, dy$$

$$= \int_{-\infty}^{\infty} p(y)f(y) \, dy = E_p[f(x)]$$

Note that changing $y$ to $x$ is eligible since they are just dummy variables.

Now that we have introduced the idea of importance sampling, we will go back to the path integral problem. First we will introduce importance sampling for path integrals, then we will discuss how it can help to improve the sample efficiency.

Lets assume that, for calculating the path integral expectations in equation $3.40$, we use the following controlled system dynamics instead of the uncontrolled dynamics (equation 3.42).

$$\dot{x} = f(t, x) + g(t, x) (u + \varepsilon), \quad \varepsilon \sim \mathcal{N}(0, \Sigma), \quad x(t = s) = y$$ \hspace{1cm} (3.44)

Now according to the importance sampling method, we need to incorporate the importance sampling weighting into the expectation. This weighting should be defined as

$$\mathbb{P}_{uc}(\tau | s, y) = \frac{\prod_{i=0}^{N-1} \mathcal{N}(x_{t_i} + f(t_i, x_{t_i})dt, \Xi(t_i, x_{t_i})dt)}{\prod_{i=0}^{N-1} \mathcal{N}(x_{t_i} + f(t_i, x_{t_i})dt + g(t_i, x_{t_i})u(t_i)dt, \Xi(t_i, x_{t_i})dt)}$$

where $\mathbb{P}_c$ is the probability distribution of the trajectory generated by the controlled system in equation (3.44). For the sake of simplicity, we will temporarily drop the time and state dependency of the functions. Therefore we will have

$$\frac{\mathbb{P}_{uc}(\tau | s, y)}{\mathbb{P}_c(\tau | s, y)} = \frac{\prod_{i=0}^{N-1} \mathcal{N}(x_i + f_i dt, \Xi_i dt)}{\prod_{i=0}^{N-1} \mathcal{N}(x_i + f_i dt + g_i u_i dt, \Xi_i dt)}$$

$$= \prod_{i=0}^{N-1} \frac{\exp \left(-\frac{1}{2} \|x_{i+1} - x_i - f_i dt\|^2_{\Xi_i dt}\right)}{\exp \left(-\frac{1}{2} \|x_{i+1} - x_i - f_i dt - g_i u_i dt\|^2_{\Xi_i dt}\right)}$$
3.6 Path Integral with Importance Sampling

Path Integrals

Since the trajectory is generated through the controlled dynamics, we know $dx_{i+1} = f_i dt + g_i u_i dt + g_i dw_i$. Hence we get

$$\begin{align*}
P_{uc}(\tau | s, y) &= \prod_{i=0}^{N-1} \exp \left( -\frac{1}{2} \|g_i u_i dt + g_i dw_i\|^2_{\mathbb{E}_i dt} \right) \\
&= \prod_{i=0}^{N-1} \exp \left( -\frac{1}{2} \|g_i u_i dt + g_i dw_i\|^2_{\mathbb{E}_i dt} + \frac{1}{2} \|g_i dw_i\|^2_{\mathbb{E}_i dt} \right) \\
&= \prod_{i=0}^{N-1} \exp \left( -\frac{1}{2} u_i^T g_i^T \Xi_i^{-1} g_i u_i dt - \frac{1}{2} u_i^T g_i^T \Xi_i^{-1} g_i dw_i \right) \\
&= \prod_{i=0}^{N-1} \exp \left( -\frac{1}{2 \lambda} u_i^T R u_i dt - \frac{1}{\lambda} u_i^T R dw_i \right) \\
&= \exp \left( -\frac{1}{\lambda} \sum_{i=0}^{N-1} \frac{1}{2} u_i^T R u_i dt + \frac{1}{2} u^T R dt \right) \\
&= \exp \left( -\frac{1}{\lambda} \int_{t_0}^{t_f} \frac{1}{2} u^T R dt + u^T R dt \right)
\end{align*}$$

Now if we multiply this importance sampling correction term to the expectations in equation (3.40), we get

$$\begin{align*}
u^*(s, y) &= \frac{\mathbb{E}_{\tau_c} \left\{ (u + \varepsilon) e^{-\frac{1}{2} \left( \Phi(x(t_f)) + \int_{t_0}^{t_f} \Phi(q(t, x) + \frac{1}{2} u^T R u \ dt + u^T R dw) \right) } \right\}}{\mathbb{E}_{\tau_c} \left\{ e^{-\frac{1}{2} \left( \Phi(x(t_f)) + \int_{t_0}^{t_f} \Phi(q(t, x) + \frac{1}{2} u^T R u \ dt + u^T R dw) \right) } \right\} } \\
&= \frac{\mathbb{E}_{\tau_c} \left\{ \mathbb{E}_{\tau_c} \left\{ (u + \varepsilon) e^{-\frac{1}{2} \int_{t_0}^{t_f} \Phi(q(t, x) + u^T R u \ dt + u^T R dw) } \right\} \right\}}{\mathbb{E}_{\tau_c} \left\{ e^{-\frac{1}{2} \int_{t_0}^{t_f} \Phi(q(t, x) + u^T R u \ dt + u^T R dw) } \right\} } \tag{3.45}
\end{align*}$$

We have changed the $\tau_{uc}$ subscript to $\tau_c$ and substituted $\varepsilon$ by $u + \varepsilon$. We can introduce the return of a trajectory as

$$R(\tau; s, y) = \Phi(x(t_f)) + \int_{t_0}^{t_f} \left( q(t, x) + \frac{1}{2} u^T R u \right) dt + \int_{t_0}^{t_f} u^T Rdw \tag{3.46}$$

Note that since the Brownian motion has a zero mean, the expectation of the return (at initial time $t_0$ and initial state $x_0$) is equal to the cost function of the policy $u$, i.e. $J = \mathbb{E}[R(\tau; t_0, x_0)]$. Using the return notation, the optimal control equation reduces to the following form

$$\begin{align*}
u^*(s, y) &= \frac{\mathbb{E}_{\tau_c} \left\{ \varepsilon(s) e^{-\frac{1}{2} R(\tau; s, y)} \right\}}{\mathbb{E}_{\tau_c} \left\{ e^{-\frac{1}{2} R(\tau; s, y)} \right\} } \\
&= \frac{\mathbb{E}_{\tau_c} \left\{ \varepsilon(s) e^{-\frac{1}{2} R(\tau; s, y)} \right\}}{e^{-\frac{1}{2} R(\tau; s, y)}} \tag{3.47}
\end{align*}$$
3.7 Path Integral with Function Approximation

The equation (3.47) indicates that in order to calculate the optimal control at a specific time and state we must calculate two expectations. In practice for a nonlinear system we estimate these expectations through a Monte Carlo method. This means that we extract a sufficiently big set of samples by forward simulating the controlled dynamics from a given time and state. Then we estimate the expectations by a numerical average over the samples. However in order to find the optimal policy, we need to repeat this process for all the time steps in the period \([t_0, t_f]\) and at least a subset of the state space. As you can imagine, this is typically quite a costly process. On the other hand, we know that the final optimal control manifold is relatively smooth (or at least we prefer to have a smooth approximation of it), which means the estimated optimal control for a specific time and state could possibly be used for its neighboring time steps and states. This idea can be included in the path integral approach by using function approximation on the optimal control. Let’s assume that we want to approximate the optimal control input through the following linear model (linear w.r.t. the parameter vector)

\[
    u_i^*(s, y) = \mathbf{\Upsilon}_i^T(s, y)\theta_i^* + \text{error}
\]  

(3.48)

where \(u_i\) refers to the \(i\)th control input. \(\mathbf{\Upsilon}_i(s, y)\) is a nonlinear basis function vector for the control input \(i\) which can have any arbitrary functionality of time and state. \(\theta_i^*\) is the optimal parameter vector. Calculating the optimal parameter vector for this linear model introduces a linear regression problem. In order to solve the regression problem, we must define an appropriate objective function. Here we will consider the mean squared error criterion. The optimal parameter vector for each of the \(m\) control inputs can be calculated through the following optimization.

\[
    \theta_i^* = \arg\max_{\theta_i} L(\theta_i)
\]

\[
    = \arg\max_{\theta_i} \int_{t_0}^{t_f} \int p(s, y) \left\| u_i^*(s, y) - \mathbf{\Upsilon}_i^T(s, y)\theta_i^* \right\|^2 \, dy \, ds
\]

(3.49)

where \(p(s, y)\) is an arbitrary weighting function that has a unit integral (therefore it can be considered as a probability distribution function). Since the gradient of the optimal parameter vector should be zero, we can write

\[
    \frac{\partial L(\theta_i^*)}{\partial \theta_i} = \int_{t_0}^{t_f} \int \left( u_i^*(s, y) - \mathbf{\Upsilon}_i^T(s, y)\theta_i^* \right) \mathbf{\Upsilon}_i(s, y) \, p(s, y) \, dy \, ds = 0
\]

(3.50)

Through multiplying and dividing the integrand by \(E_{\tau_i} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \right\} \), we get

\[
    \int_{t_0}^{t_f} \int E_{\tau_i} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \right\} \left( u_i^*(s, y) - \mathbf{\Upsilon}_i^T(s, y)\theta_i^* \right) \mathbf{\Upsilon}_i(s, y) \, p(s, y) \, dy \, ds = 0
\]

(3.51)
We then push all of the terms inside of the nominator's $E_{\tau c} \{ \cdot \}$. Notice that this operation is eligible because the initial state $y$ is not a part of the expectation and is assumed to be known.

$$\int_{t_0}^{t_f} \int_{\Omega} E_{\tau c} \left\{ \frac{e^{-\frac{i}{\lambda} R(\tau; s, y)}}{E_{\tau c} \{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \}} \left( u_i^*(s, y) - \mathbf{Y}_i^T(s, y) \theta_i^* \right) \mathbf{Y}_i(s, y) p(s, y) \right\} dy ds = 0 \quad (3.52)$$

In order to shorten notation, we will show all of the expectations in integral form and only use one integral sign, instead of three.

$$\int e^{-\frac{i}{\lambda} R(\tau; s, y)} E_{\tau c} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \left( u_i^*(s, y) - \mathbf{Y}_i^T(s, y) \theta_i^* \right) \mathbf{Y}_i(s, y) \right\} P_{\tau c}(\tau \mid s, y) p(s, y) d\tau dy ds = 0 \quad (3.53)$$

In the next step, we will add and subtract the terms $u_i(s, y) + \varepsilon(s)$.

$$\int e^{-\frac{i}{\lambda} R(\tau; s, y)} E_{\tau c} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \left( u_i^* - u_i - \varepsilon + u_i + \varepsilon - \mathbf{Y}_i^T \theta_i^* \right) \mathbf{Y}_i(s, y) \right\} P_{\tau c}(\tau \mid s, y) p(s, y) d\tau dy ds = 0 \quad (3.54)$$

We break this integral into two integrals as follows

$$\int e^{-\frac{i}{\lambda} R(\tau; s, y)} E_{\tau c} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \left( u_i^* - u_i - \varepsilon \right) \mathbf{Y}_i(s, y) \right\} P_{\tau c}(\tau \mid s, y) p(s, y) d\tau dy ds +$$

$$\int e^{-\frac{i}{\lambda} R(\tau; s, y)} E_{\tau c} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \left( u_i(s, y) + \varepsilon - \mathbf{Y}_i^T(s, y) \theta_i^* \right) \mathbf{Y}_i(s, y) \right\} P_{\tau c}(\tau \mid s, y) p(s, y) d\tau dy ds = 0 \quad (3.55)$$

For the first integral, we will first integrate it with respect to trajectory $\tau$ and then use equation (3.47).

$$\int e^{-\frac{i}{\lambda} R(\tau; s, y)} E_{\tau c} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \left( u_i^* - u_i - \varepsilon \right) \mathbf{Y}_i(s, y) \right\} P_{\tau c}(\tau \mid s, y) p(s, y) d\tau dy ds =$$

$$\int \left( u_i^*(s, y) - u_i(s, y) - \varepsilon \right) \mathbf{Y}_i(s, y) (\tau \mid s, y) p(s, y) dy ds = 0$$

By substituting this result back into the previous equation we get

$$\int e^{-\frac{i}{\lambda} R(\tau; s, y)} E_{\tau c} \left\{ e^{-\frac{1}{\lambda} R(\tau; s, y)} \left( u_i(s, y) + \varepsilon - \mathbf{Y}_i^T(s, y) \theta_i^* \right) \mathbf{Y}_i(s, y) \right\} P_{\tau c}(\tau \mid s, y) p(s, y) d\tau dy ds = 0 \quad (3.56)$$
which is equivalent to the following optimization problem

$$\theta^*_i = \arg\min_{\theta_i} \int \frac{e^{-\frac{1}{2} R(\tau; s, y)}}{E_{\tau_c} \left\{ e^{-\frac{1}{2} R(\tau; s, y)} \right\}} \left\| \Upsilon^T_i (s, y) \theta_i - u_i (s, y) - \varepsilon \right\|_2^2 P_{\tau_c}(\tau | s, y)p(s, y)d\tau dy ds \quad (3.57)$$

If we use the same function approximation model for $u_i(s, y)$ and $u_i(s, y) \approx \Upsilon^T_i (s, y) \theta_{i,c}$, we get

$$\theta^*_i = \theta_{i,c} + \arg\min_{\Delta \theta_i} \int \frac{e^{-\frac{1}{2} R(\tau; s, y)}}{E_{\tau_c} \left\{ e^{-\frac{1}{2} R(\tau; s, y)} \right\}} \left\| \Upsilon^T_i (s, y) \Delta \theta_i - \varepsilon \right\|_2^2 P_{\tau_c}(\tau | s, y)p(s, y)d\tau dy ds \quad (3.58)$$

Equation (3.57) or (3.58) introduce a method to blend the function approximation and optimal control problems into a single optimization problem. In the next section, we will introduce the general path integral algorithm which uses importance sampling and function approximation.

### 3.8 General Path Integral Algorithm

In this section, we will present the path integral general algorithm. In this algorithm we will use all the components that we already have introduced in the previous sections. Before putting things all together, we will describe the reason behind each of the component separately.

**Importance Sampling** As discussed previously, the original path integral algorithm requires that the samples are extracted from the uncontrolled system. Therefore, estimating the optimal control through a Monte Carlo method will be extremely inefficient. However by using the Importance Sampling scheme introduced in Section 3.6, we can implement an incremental method to estimate the optimal control which is more sample efficient.

In this approach, we can start the sampling with a sophisticated guess of the optimal control. Note that this is optional, though, and we could also start from the uncontrolled system. In each iteration of the algorithm, we use our latest guess of the optimal control to extract a batch of samples. Using this batch of samples, we can improve our estimation of the optimal control. Our sampling efficiently inherently improves at the same time, since, as our estimation improves, we sample at the area with lower accumulated cost and therefore the sample weightings, $\alpha(\tau; s, y)$ (equation 3.43), will have more significant values. It can be shown that if we are in the $\epsilon$-vicinity of the optimal solution the sample efficiency will improve linearly with $\epsilon$.

**Function Approximation** The original path integral theorem introduces a sampling method for estimating the optimal control at a specific time and state. Therefore for estimating the optimal control in a given time period and some subspace of the state space, we should repeat the estimation individually for every point. In Section 3.7 we introduced a function approximation scheme in which the estimated optimal control for a single point generalizes to its neighboring area. Another advantage of function approximation is the introduction of $p(s, y)$ (the weighting function of MSE).
Algorithm 8 General Path Integral Algorithm

given
The cost function:
\( J = \Phi(x(t_f)) + \int_{t_s}^{t_f} \left( q(t, x) + \frac{1}{2} u^T R u \right) dt \)

A PDF defining the quality of approximation of optimal control at each time-state pair: \( p(t, x) \)

An initial policy and a Linear Model: \( u(t, x) = \left[ u_i(t, x) \right] = \left[ Y_i^T(t, x) \theta_i \right] \)

repeat
(a) Randomly choose a time-state pair from \( p(t, x) \): \( (s, y) \)
(b) Forward simulate the controlled system for \( K \) different rollouts: \( \{ \tau^k \}_{k=1}^K \)
\[ \dot{x} = f(t, x) + g(t, x) (u + \varepsilon) \]
\[ u(t, x) = [Y_i^T(t, x) \theta_i], \varepsilon \sim \mathcal{N}(0, \Sigma), x(t = s) = y \]
(c) Calculate the return for each rollout: \( \{ R^k \}_{k=1}^K \)
\[ R(\tau; s, y) = \Phi(x(t_f)) + \int_{t_s}^{t_f} \left( q(t, x) + \frac{1}{2} u^T R u \right) dt + \int_{t_s}^{t_f} u^T R d w \]
(d) Calculate \( \{ \alpha^k \}_{k=1}^K \)
\[ \alpha^k(s, y) = \exp\left(-\frac{1}{\lambda} R^k\right) \frac{1}{K} \sum_{j=1}^{K} \exp\left(-\frac{1}{\lambda} R^j\right) \]
(e) Solve the following linear regression problem for each control input \( i \):
\[ \Delta \theta_i = \text{argmin} \sum_{k=1}^{K} \alpha^k \| Y_i^T(s, y) \Delta \theta_i - \varepsilon_i^k(s) \|^2_2 \]
(f) Update the parameter vector for each control input \( i \):
\[ \theta_i \leftarrow \theta_i + \omega \Delta \theta_i \]
until convergence

For example, assume we have chosen \( p(s, y) \) as the probability distribution of the state under the latest estimation of the optimal control. In this case, the function approximation will be more precise in the area more likely to be visited using the optimal policy.

Algorithm 8 shows the General Path Integral Algorithm which uses importance sampling and function approximation.

3.9 PI2 Algorithm: Time-dependent Policy

The Policy Improvement Path Integral algorithm (PI2) is a variant of General Path Integral algorithm. PI2 is an instance of a sample-based reinforcement learning algorithm that can retrieve trajectory samples directly from the real environment i.e. the real system. In this section, we will derive the PI2 algorithm from the General Path Integral algorithm.

Originally the PI2 algorithm was developed for optimizing parameterized, time-dependent policies. The time-dependent policy can be either a parameterized dynamical system like Dynamic Movement Primitives (DMPs) or simply a function approximation of the control input like the one in equation (3.59). The essential requirements for the algorithm are linearity w.r.t the parameter vector, and the usage of time-dependent basis functions. In this section, we will use function-approximation policies since it will later give us the opportunity to extend the PI2 algorithm to more general polices.
that are functions of time and state.

\[ u_i(t) = \mathbf{\Upsilon}^T(t)\theta_i \]

\[ \mathbf{\Upsilon}(t) = [\mathbf{\Upsilon}_n(t)]_{N \times 1} = \begin{bmatrix} e^{-\frac{1}{2}(t-\mu_n)^2}{\sigma_n^2} \end{bmatrix}_{N \times 1} \quad (3.59) \]

In this equation, \( u_i \) refers to the \( i \)th control input and \( \mathbf{\Upsilon} \) is a time-dependent basis function. Each element of \( \mathbf{\Upsilon} \) (denoted by \( \mathbf{\Upsilon}_n \)) is a bell-shape function with a mean and variance, \( \mu_n \) and \( \sigma_n^2 \) respectively. The dimension of the parameter vector as well as the basis function vector is \( N \).

In order to derive PI2 from the General Path Integral algorithm, we should make the following assumptions:

1. The weighting function of MSE, \( p(t, x) \), is assumed to be the probability distribution of the state under the latest estimation of the optimal control. Therefore the rollout trajectory’s states are considered to be extracted from \( p(t, x) \).

2. The return function, \( R(\tau; s, y) \), won’t be a function of state if the system has been initialized from a similar initial condition. An immediate result of this assumption is that \( \alpha_k(s, y) \) is only a function of time i.e. \( \alpha_k(s) \). Therefore for a batch of trajectories extracted from a similar initial condition, \( \alpha_k(s) \) can be estimated for each time step by the following.

\[ \alpha_k(s) = \frac{\exp(-\frac{1}{\lambda}R_k(s))}{\frac{1}{K}\sum_{j=1}^{K}\exp(-\frac{1}{\lambda}R_j(s))} \quad (3.60) \]

Notice that in contrast to the general path integral algorithm, where we are required to estimate \( \alpha_k(s, y) \) by extracting trajectories starting from \( (s, y) \), we can use trajectories with different state values but the same time indexes to estimate \( \alpha \). Therefore, if we have a batch of trajectories that are generated from similar initial conditions, we can estimate \( \alpha \) for all of the states and over the given time horizon, \([t_0, t_f]\).

3. The basis function vector for all of the control inputs is the same. Hence we will drop the \( i \) subscript of the basis function as \([u_i(t)] = [\mathbf{\Upsilon}^T(t)\theta_i] \).

4. Instead of adding noise to the control input, the noise is added directly to the parameter vector. Therefore, the input noise can be expressed as \( \varepsilon_i = \mathbf{\Upsilon}(t)\epsilon_i \), where \( \epsilon_i \) is the noise that is added to the parameter vector of the \( i \)th control input.

5. The PI2 regression problem should be modified as follows

\[ \Delta \theta_i = \arg\min_{\theta_i} \sum_{s=t_0}^{t_f} \sum_{k=1}^{K} \alpha_k(s)\|\mathbf{\Upsilon}_k^T(s)\Delta \theta_i - \varepsilon_k(s)\|_2^2 \quad (3.61) \]
In contrast to the General Path Integral Algorithm, PI2 assumes that samples are extracted over the entire time horizon, not only for a single time step. Therefore, the regression problem should be over all of the sampled trajectories as well as all of the sub-trajectories of each individual trajectory, where a sub-trajectory is a portion of a trajectory starting from some time \( s \) until the end of the time horizon (as was illustrated in Section 2.9.2).

In order to solve this regression problem, PI2 breaks it into two separate optimizations. In doing so, it is assumed that the regression error is zero mean over the samples. In this method, the optimization is first solved for each time step separately. Therefore the first optimization will find a time-dependent parameter vector increment that has the minimum error over the rollouts at each time step. In the second optimization, we find a parameter vector increment that approximates the time-dependent one.

The first optimization is defined as follows for each time step \( s \)

\[
\Delta \theta^*_i(s) = \arg\min \sum_{k=1}^{K} \alpha^k(s) \| \mathbf{Y}_i^T(s) \Delta \theta_i - \mathbf{e}^k_i(s) \|_2^2
\]

(3.62)

In this optimization, all of the regressors are the same. Therefore in order to find the solution to this problem, we should use the right inverse which will give a solution with the minimum-norm \( \Delta \theta^*_i(s) \)

\[
\Delta \theta^*_i(s) = \sum_{k=1}^{K} \alpha^k(s) \frac{\mathbf{Y}_i(s)}{\mathbf{Y}_i^T(s) \mathbf{Y}_i(s)} \mathbf{e}^k_i(s)
\]

(3.63)

If we use the fourth assumption that the noise is directly added to the parameter vector, we get

\[
\Delta \theta^*_i(s) = \sum_{k=1}^{K} \alpha^k(s) \frac{\mathbf{Y}_i(s) \mathbf{Y}_i^T(s)}{\mathbf{Y}_i^T(s) \mathbf{Y}_i(s)} \mathbf{e}^k_i(s)
\]

(3.64)

The second optimization for finding the optimal \( \Delta \theta^*_i \) is defined in equation (3.65). The index \( n \) refers to the \( n \)th element of the vector \( \Delta \theta^*_i = [\Delta \theta^*_{i,n}] \)

\[
\Delta \theta^*_{i,n} = \arg\min_{\Delta \theta^*_{i,n}} \sum_{s=f_0}^{t_f} (\Delta \theta_{i,n}(s) - \Delta \theta^*_{i,n}(s))^2 \mathbf{Y}_n(s)
\]

(3.65)
\( \Upsilon_n(t) \) is the \( n \)th element of the basis function vector \( \Upsilon(t) \). The solution to this optimization can be easily calculated as

\[
\Delta \theta^*_i = \frac{\sum_{t=t_0}^{t_f} \Delta \theta^*_i(s) \Upsilon_n(s)}{\sum_{t=t_0}^{t_f} \Upsilon_n(s)}
\]

(3.66)

By using element-wise multiplication and division and replacing the summation with an integral, we can also write

\[
\Delta \theta^*_i = \left( \int_{t_0}^{t_f} \Delta \theta^*_i(s) \circ \Upsilon(s) ds \right) \div \int_{t_0}^{t_f} \Upsilon(s) ds
\]

(3.67)

where \( \circ \) and \( \div \) are element-wise multiplication and division.

Using these assumptions, we can derive PI2 from the General Path Integral algorithm. The complete algorithm is given as Algorithm 9. Notice that in PI2, we do not need to artificially initialize the system at random points \((s, y)\) according to \( p(s, y) \). Therefore this algorithm is a sample-based reinforcement learning algorithm and can be used to learn the optimal policy through samples that are generated directly from the real physical system.
Algorithm 9 PI2 Algorithm for time-dependent policy

given
The cost function:
\[ J = \Phi(x(t_f)) + \int_{t_0}^{t_f} \left( q(t, x) + \frac{1}{2} u^T R u \right) dt \]
A linear model for function approximation: \[ u(t) = \left[ u_i(t) \right] = \left[ \gamma(t) \theta_i \right] \]
Initialize \[ \theta_i \] with a sophisticated guess
Initialize exploration noise standard deviation: \[ c \]
repeat
Create \( K \) rollouts of the system with the perturbed parameter \( \theta_i + [\epsilon_i], \quad \epsilon_i \sim \mathcal{N}(0, c^2 I) \)
for the \( i \)th control input do
for each time, \( s \) do
Calculate the Return from starting time \( s \) for the \( k \)th rollout:
\[ R(\tau^k(s)) = \Phi(x(t_f)) + \int_{s}^{t_f} \left( q(t, x) + \frac{1}{2} u^T R u \right) dt \]
Calculate \( \alpha \) from starting time \( s \) for the \( k \)th rollout:
\[ \alpha^k(s) = \exp\left(-\frac{1}{\lambda} R(\tau^k(s))\right) / \sum_{i=1}^{K} \exp\left(-\frac{1}{\lambda} R(\tau^i(s))\right) \]
Calculate the time varying parameter increment \( \Delta \theta_i(s) \):
\[ \Delta \theta_i(s) = \sum_{k=1}^{K} \alpha^k(s) \gamma(s)^T \theta_i \gamma(s) \epsilon^T_i(s) \]
end for
Time-averaging the parameter vector
\[ \Delta \theta_i = \left( \int_{t_0}^{t_f} \Delta \theta_i(s) \cdot \gamma(s)ds \right) / \int_{t_0}^{t_f} \gamma(s)ds \]
Update parameter vector for control input \( i \), \( \theta_i \):
\[ \theta_i \leftarrow \theta_i + \Delta \theta_i \]
end for
Decrease \( c \) for noise annealing
until maximum number of iterations

3.10 PI2 Algorithm: Feedback Gain Learning

In a first attempt to tailor the PI2 algorithm to more general policies, we will introduce a method to learn a linear time-varying feedback gain. Consider the following tracking problem
\[
\begin{align*}
\dot{x} &= f(t, x) + g(t, x)u \\
u &= K^T(t)(x - x_{ref})
\end{align*}
\] (3.68)

For now, we will assume that the control input dimension is one. If we substitute the linear feedback controller back into the system dynamics, we will get
\[
\dot{x} = f(t, x) + g(t, x)(x - x_{ref})^T K(t)
\] (3.69)
According to this equation, we can assume a new system with $K(t)$ as its control input. Considering this new system, we can use Algorithm 9 to learn the time varying feedback gains. This method can be easily extended to a system with multiple control inputs. The learning algorithm in this case will assume a system that has as many inputs as the total number of gains. In other words if the dimensions of the state and control input vectors are $\dim[x]$ and $\dim[u]$ respectively, the dimension of control input from the perspective of PI2 will be $\dim[x] \times \dim[u]$.

In the next section, we will introduce a more general algorithm which will learn both time-dependent policies and the feedback gains simultaneously.

### 3.11 PI2 Algorithm: General Algorithm

In this section, the General PI2 algorithm will be modified to allow policies with linear state dependency and a nonlinear time dependency to be learned. An instance of such a policy is as follows

$$u_i(t, x) = \text{grand sum} \left[ \Upsilon(t, x) \circ \theta_i \right]$$

$$\Upsilon(t, x) = \Upsilon(t) \begin{bmatrix} 1 \\ x^T \end{bmatrix} = \begin{bmatrix} e^{-\frac{1}{2} \frac{(t-\mu_n)^2}{\sigma_n^2}} \\ 1 \end{bmatrix} x^T \right]_{N \times (1+\dim[x])}$$

(3.70)

The grand sum operator calculates the sum of all the elements in a matrix and $\circ$ is the element-wise multiplication. $\theta_i$ is a $N \times (1+\dim[x])$ parameter matrix for $i$th control input approximation. $\Upsilon(t, x)$ is the basis function matrix of the same size as $\theta_i$. Finally $\Upsilon(t)$ is defined as in equation (3.59).

By setting the parameters which are multiplied by the states to zero, this policy reduces to the time varying policy primarily introduced in the equation (3.59). Using the same argument from the previous section we can assume that the state dependent segment of the policy is part of the system dynamics and PI2 will be used to learn the time-varying gains. Based on the policy in equation (3.70), we can assume that these gains are sufficiently approximated by the using time-dependent basis function, $\Upsilon(t)$. The complete algorithm is given in Algorithm 10.
Algorithm 10 General PI2 Algorithm

given
The cost function:
\[ J = \Phi(x(t_f)) + \int_{t_0}^{t_f} \left( q(t, x) + \frac{1}{2} u^T R u \right) dt \]

A linear model for function approximation: \( u(t, x) = [u_i(t, x)] = \text{grand sum} [\Upsilon(t, x) \circ \theta_i] \)

Initialize \{\theta_i\} with a sophisticated guess

Initialize exploration noise standard deviation: \( c \)

repeat
Create \( K \) rollouts of the system with the perturbed parameter \{\theta_i + \{\epsilon_i\}, \{\epsilon_{i,j}\} \sim \mathcal{N}(0, c^2 \mathbf{I})\)

for the \( i \)th control input do
  
  for each time, \( s \) do
    
    Calculate the Return from starting time \( s \) for the \( k \)th rollout:
    \[ R(\tau^k(s)) = \Phi(x(t_f)) + \int_{t_0}^{t_f} q(t, x) + \frac{1}{2} u^T R u dt \]
    
    Calculate \( \alpha \) from starting time \( s \) for the \( k \)th rollout:
    \[ \alpha^k(s) = \exp(-\frac{1}{\lambda} R(\tau^k(s))) / \sum_{k=1}^{K} \exp(-\frac{1}{\lambda} R(\tau^k(s))) \]
    
    Calculate the time varying parameter increment \( \Delta \theta_i(s) \):
    \[ \Delta \theta_i(s) = \sum_{k=1}^{K} \alpha^k(s) \frac{\Upsilon(s) \Upsilon^T(s)}{\Upsilon^T(s) \Upsilon(s)} \epsilon^k_i(s) \]

  end for

  for the \( j \)th column of \( \Delta \theta_i \) matrix, \( \Delta \theta_{i,j} \) do
    
    Time-averaging the parameter vector
    \[ \Delta \theta_{i,j} = \frac{\int_{t_0}^{t_f} \Delta \theta_{i,j}(s) \circ \Upsilon(s) ds}{\int_{t_0}^{t_f} \Upsilon(s) ds} \]

  end for

  Update parameter vector for control input \( i \), \( \theta_i \):
  \[ \theta_i \leftarrow \theta_i + \Delta \theta_i \]

end for

- Decrease \( c \) for noise annealing

until maximum number of iterations
4 Policy Gradient

Assume the following optimal control problem with the cost function, $J$ defined as follows

$$ J = E \left[ \Phi(x(N)) + \sum_{k=0}^{N-1} L_k (x(k), u(k)) \right] $$

(4.1)

and the system dynamics as

$$ x(n + 1) = f_n(x(n), u) + w(n) $$

(4.2)

where $w(n)$ is an arbitrary random process. Finding the optimal controller for this problem is nearly impossible, except in very special cases. Let's assume that, instead of finding the actual optimal control, we want to find a parameterized policy that has the lowest cost among a given class of parametrized policies. If we express the parameterized policy as $\mu(n, x; \theta)$, the optimal control problem will be defined as follows

$$ \theta^* = \arg \min_{\theta} J(\theta) = \arg \min_{\theta} E \left[ \Phi(x(N)) + \sum_{k=0}^{N-1} L_k (x(k), u(k)) \right] $$

$$ \begin{cases} x(n + 1) = f_n(x(n), u) + w(n) \\ u(n, x) = \mu(n, x; \theta) \end{cases} $$

(4.3)

From another perspective, we can look at this problem as a function approximation problem for the optimal control. As opposed to the functional approximation method used in the PI2 algorithm, in which the policies had to be linear with respect to the parameters, here we can consider a more general class of policies which are non-linear with respect to $\theta$.

The proposed optimal control problem in equation (4.3) can be directly solved if we can calculate the gradient of the cost function with respect to the parameter vector, $\theta$. Then we can simply use an optimization algorithm to find the optimal parameter vector, $\theta^*$. Algorithm 11 introduces an instance of this optimization algorithm which uses the gradient descent method. In this algorithm, $\omega$ is the learning rate.

However in order to find a closed form formula for the cost function gradient we first need to calculate the cost function as a function of the parameter vector. This requires that the states are computed as a function of the parameter vector by solving the difference equation of the system. The solution to this problem does not exist, though, for a general nonlinear system.

In the absence of the closed form solution, we need to use a numerical method to directly estimate the gradient. In the reinforcement learning and optimal control literature, numerous methods have
Algorithm 11 Gradient Descent Algorithm

\begin{algorithm}
\setstretch{1.1}
\textbf{given}\n\begin{itemize}
  \item A method to compute $\nabla_\theta J(\theta)$ for all $\theta$
  \item An initial value for the parameter vector: $\theta \leftarrow \theta_0$
\end{itemize}
\textbf{repeat}\n\begin{itemize}
  \item Compute the cost function gradient at $\theta$ \\
    $g = \nabla_\theta J(\theta)$
  \item Update the parameter vector \\
    $\theta \leftarrow \theta - \omega g$
\end{itemize}
\textbf{until} convergence
\end{algorithm}

been proposed to estimate the gradient (as well as Hessian matrix if we want to use an optimization method that requires the Hessian matrix). In this chapter we will introduce a very basic, yet effective, method to estimate the cost function gradient. This technique is known as the Finite Difference method.

4.1 Finite Difference

The Finite Difference (FD) method estimates the cost function gradient with respect to the optimal control parameter vector numerically. Before introducing the algorithm in its complete form, we will start with a slightly simpler problem. First consider a policy that has a single parameter and assume that there is no stochasticity in the problem i.e. $w(n) = 0$. The gradient of the cost function will be a scalar which is the derivative of the cost with respect to the parameter. Based on the definitions, the derivative of a function can be calculated through each of the two following formulas.

\begin{align}
\frac{dJ(\theta)}{d\theta} &= \lim_{d\theta \to 0} \frac{J(\theta + d\theta) - J(\theta)}{d\theta} \quad (4.4) \\
\frac{dJ(\theta)}{d\theta} &= \lim_{d\theta \to 0} \frac{J(\theta + d\theta/2) - J(\theta - d\theta/2)}{d\theta} \quad (4.5)
\end{align}

The method in equation (4.4) is called the single-sided derivative and the one in equation (4.5) is called the double-sided derivative. For an infinitely small perturbation, $d\theta$, these methods calculate the derivative of the cost function at $\theta$. However with a sufficiently small perturbation $\Delta \theta$, we can approximate the derivative through the followings methods

\begin{align}
\frac{dJ(\theta)}{d\theta} &\approx \frac{J(\theta + \Delta \theta) - J(\theta)}{\Delta \theta} \quad (4.6) \\
\frac{dJ(\theta)}{d\theta} &\approx \frac{J(\theta + \Delta \theta/2) - J(\theta - \Delta \theta/2)}{\Delta \theta} \quad (4.7)
\end{align}

Therefore by only calculating the value of the cost function at two different points ($\theta$ and $\theta + \Delta \theta$ for single-sided approximation and $\theta - \frac{1}{2}\Delta \theta$ and $\theta + \frac{1}{2}\Delta \theta$ for the double-sided approximation), we can approximate the derivative at $\theta$. To compute the cost function, we need to simulate the system dynamics with the given parameter value and then calculate the cost function for the rollout. Since
there is no stochasticity in this simple example, we are guaranteed to get the true value of the cost function from this single rollout.

A natural extension to this approach is to assume that we have a vector of parameters instead of a single parameter. In this case we need to calculate the gradient of the cost. By definition, in order to calculate one element of the gradient vector, we should only perturb the parameter associated with that element (Figure 4.1). In this case if we use the single-sided approximation, we need to calculate the cost function $\dim[\theta] + 1$ times. One for the cost at the non-perturbed parameter $\theta$ and $\dim[\theta]$ times for each element of the parameter vector. If we use the double-sided method, we need $2 \times \dim[\theta]$ times of the cost function’s evaluations, with 2 perturbations per parameter element.

In the next section, we will examine a case where the perturbations are not evaluated at each parameter element separately, but instead are applied in a random directions.

### 4.1.1 FD with Random Perturbations

Before introducing the FD method with random perturbations, we will use two simple examples as an introduction to the main idea. Both of the examples are considered to be deterministic. In the first example, we will assume that we have just two parameters. In this example we will use the single-sided method. Thus we need three evaluations of the cost function. Let’s assume that these measurements are $J(\theta)$, $J(\theta + \Delta \theta_1)$, $J(\theta + \Delta \theta_2)$ which are the values of the cost function for the parameter vector without perturbation and two vectors with random perturbations, $\Delta \theta_1$ and $\Delta \theta_2$ (Figure 4.2). In order to estimate the gradient at $\theta$, we will use the first order Taylor expansion of the cost function around $\theta$

$$J(\theta + \Delta \theta_1) \approx J(\theta) + \Delta \theta_1^T \nabla J(\theta)$$

$$J(\theta + \Delta \theta_2) \approx J(\theta) + \Delta \theta_2^T \nabla J(\theta)$$

(4.8)
Using matrix notation, these two equalities can be written in the following form

\[
\begin{bmatrix}
\Delta \theta_1^T \\
\Delta \theta_2^T
\end{bmatrix}
\nabla J(\theta) =
\begin{bmatrix}
J(\theta + \Delta \theta_1) - J(\theta) \\
J(\theta + \Delta \theta_2) - J(\theta)
\end{bmatrix}
\tag{4.9}
\]

In order to find the gradient of the cost function, we need to solve this equality. If the perturbations are not parallel (linearly independent), \([\Delta \theta_1^T \Delta \theta_2^T]\) is a 2-by-2 invertible matrix. Therefore the gradient can be calculated as

\[
\nabla J(\theta) = \left[ \Delta \theta_1^T \Delta \theta_2^T \right]^{-1}
\begin{bmatrix}
J(\theta + \Delta \theta_1) - J(\theta) \\
J(\theta + \Delta \theta_2) - J(\theta)
\end{bmatrix}
\tag{4.10}
\]

In the second example, we will consider the same problem except this time, instead of computing the cost function twice perturbed and once not perturbed, we will compute the cost function using three different perturbed values of the parameter vector (Figure 4.3). Therefore we have the following: \(J(\theta + \Delta \theta_1), J(\theta + \Delta \theta_2), J(\theta + \Delta \theta_3)\).
In contrast to the previous example, in this case we don’t have the cost function’s value at the nominal value of the parameters. We can still estimate both the gradient of the cost function and the value of the cost function for this point, however. Again we start with the first order Taylor expansion of the cost function

\[
\begin{align*}
J(\theta + \Delta \theta_1) &= J(\theta) + \Delta \theta_1^T \nabla J(\theta) \\
J(\theta + \Delta \theta_2) &= J(\theta) + \Delta \theta_2^T \nabla J(\theta) \\
J(\theta + \Delta \theta_3) &= J(\theta) + \Delta \theta_3^T \nabla J(\theta)
\end{align*}
\] (4.11)

In this case, the unknown entities are \( J(\theta) \) and \( \nabla J(\theta) \). We can reformulate these equalities in the following matrix form.

\[
\begin{bmatrix}
\Delta \theta_1^T \\
\Delta \theta_2^T \\
\Delta \theta_3^T
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
J(\theta + \Delta \theta_1) \\
J(\theta + \Delta \theta_2) \\
J(\theta + \Delta \theta_3)
\end{bmatrix}
\] (4.12)

If the perturbations are pairwise independent (i.e., each pair of vectors are independent from each other), the left matrix is invertible and we can solve for the desired quantities. Notice that since the dimension of the parameter space is two, we can have a maximum of two independent vectors. Thus the set of three perturbations will never be completely independent.

\[
\begin{bmatrix}
\nabla J(\theta) \\
J(\theta)
\end{bmatrix}
= 
\begin{bmatrix}
\Delta \theta_1^T \\
\Delta \theta_2^T \\
\Delta \theta_3^T
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
^{-1}
\begin{bmatrix}
J(\theta + \Delta \theta_1) \\
J(\theta + \Delta \theta_2) \\
J(\theta + \Delta \theta_3)
\end{bmatrix}
\] (4.13)

This equation estimates the cost function and the gradient of the cost function simultaneously. In the general case, for a parameter vector of the length \( p \) with \( (p+1) \) perturbations such that each \( p \)-member subset of perturbations is independent, we can write the following

\[
\begin{align*}
J(\theta + \Delta \theta_1) &= J(\theta) + \Delta \theta_1^T \nabla J(\theta) \\
J(\theta + \Delta \theta_2) &= J(\theta) + \Delta \theta_2^T \nabla J(\theta) \\
\vdots \\
J(\theta + \Delta \theta_p) &= J(\theta) + \Delta \theta_p^T \nabla J(\theta) \\
J(\theta + \Delta \theta_{p+1}) &= J(\theta) + \Delta \theta_{p+1}^T \nabla J(\theta)
\end{align*}
\]

Then we can estimate the cost function’s value and gradient at \( \theta \) by the following

\[
\begin{bmatrix}
\nabla J(\theta) \\
J(\theta)
\end{bmatrix}
= \Delta \Theta^{-1} J
\] (4.14)
In the next section, we will consider the case that we have more than $p + 1$ perturbations. Extra perturbations can be helpful for two main reasons: compensating for the numerical error and dealing with the stochasticity of the problem.

### 4.1.2 FD for Stochastic Problems

In this section, we will consider the original problem in equation (4.3) in which the system dynamics are stochastic. A similar idea introduced in the previous section can be used here as well. We should notice that the cost function in this case is defined as an expectation. Thus for estimating this expectation through a numerical method, we need to compute the value of the function for many executions of a given policy. Here we will define the return function, $R$, as the value of the cost function for a single execution of the policy. Notice that $J = E[R]$.

$$R = \Phi(x(N)) + \sum_{k=0}^{N-1} L_k (x(k), u(k))$$

If we assume that we need $K$ samples to evaluate the cost function for a single policy, we will need $K \times (p + 1)$ evaluations of the cost function to estimate the cost function’s gradient. We can re-write the previous equations as follows

$$\begin{bmatrix}
\frac{1}{K} \sum_{k=1}^{K} R^k(\theta + \Delta \theta_1) \\
\frac{1}{K} \sum_{k=1}^{K} R^k(\theta + \Delta \theta_2) \\
\vdots \\
\frac{1}{K} \sum_{k=1}^{K} R^k(\theta + \Delta \theta_{p+1})
\end{bmatrix}
\begin{bmatrix}
\Delta \Theta^T_1 & 1 \\
\Delta \Theta^T_2 & 1 \\
\vdots & \vdots \\
\Delta \Theta^T_{p+1} & 1
\end{bmatrix}
J
$$

and the cost function’s value and gradient can be estimated as

$$\begin{bmatrix}
\nabla J(\theta) \\
J(\theta)
\end{bmatrix}
= \Delta \Theta^{-1} J \tag{4.15}$$

By extending the idea of random perturbations, we can assume that instead of $K$ evaluations of the cost function for each fixed perturbation, we can evaluate the cost function for $N \leq K \times (p + 1)$ different perturbations. Therefore we can write

$$\begin{bmatrix}
R(\theta + \Delta \theta_1) \\
R(\theta + \Delta \theta_2) \\
\vdots \\
R(\theta + \Delta \theta_N)
\end{bmatrix}
\begin{bmatrix}
\Delta \Theta^T_1 & 1 \\
\Delta \Theta^T_2 & 1 \\
\vdots & \vdots \\
\Delta \Theta^T_N & 1
\end{bmatrix}
\begin{bmatrix}
\nabla J(\theta) \\
J(\theta)
\end{bmatrix}$$
Notice that $\Delta \Theta$ is a $N \times (p + 1)$ matrix. If we assume that $N \geq p + 1$, $\Delta \Theta$ will have rank $p + 1$. Therefore, for estimating the cost function’s value and gradient, we should use the left pseudo-inverse of $\Delta \Theta$.

$$
\begin{bmatrix}
\nabla J(\theta) \\
J(\theta)
\end{bmatrix} = \Delta \Theta^\dagger R = (\Delta \Theta^T \Delta \Theta)^{-1} \Delta \Theta^T R
$$

(4.16)

**4.1.3 Finite Difference: The General Method**

In this section, we will introduce the general FD method which unifies the previous sections. Let’s assume the general problem introduced in equation (4.3). The goal is to estimate the value of the cost function and its gradient. First we generate $N$ ($N \geq p + 1$) random perturbations using an isometric distribution. Then we evaluate the return for each of these perturbations. The cost function’s value and gradient can be estimated as follows

$$
\begin{bmatrix}
\nabla J(\theta) \\
J(\theta)
\end{bmatrix} = \Delta \Theta^\dagger R = (\Delta \Theta^T \Delta \Theta + \lambda I)^{-1} \Delta \Theta^T R
$$

(4.17)

where $R$ and $\Delta \Theta$ are defined as

$$
R = 
\begin{bmatrix}
R(\theta + \Delta \theta_1) \\
R(\theta + \Delta \theta_2) \\
\vdots \\
R(\theta + \Delta \theta_N)
\end{bmatrix},
\Delta \Theta = 
\begin{bmatrix}
\Delta \theta_1^T \\
\Delta \theta_2^T \\
\vdots \\
\Delta \theta_N^T
\end{bmatrix}
$$

The term $\lambda I$ in equation (4.17) is a regularization term. Although it adds a bias to the estimation of the gradient, it is used to reduce the covariance of the estimation. Notice that even if we have a deterministic problem, we still might want to use more perturbations than $p + 1$ in order to increase the numerical accuracy of the estimation. This is actually the case if we want to use the double-sided method for approximating the gradient. Note also that in deterministic problems, $R$ is equal to $J$.

Algorithm 12 illustrates an implementation of the gradient descend algorithm which uses the generalized Finite Difference method for estimating the gradient of the cost function. Even though the algorithm is written for a finite-horizon discrete optimal control problem, it can be easily modified to the continuous time problem by changing the computation of the return function to an appropriate integral form. For the infinite-time horizon case, we also need to change the return function to incorporate the decay factor. However we should notice that generating rollouts for an infinite-horizon problem is impractical unless the task is episodic (ie the task ends by reaching a particular set of states after a certain maximum time horizon).
4.1 Finite Difference Policy Gradient

Algorithm 12: Gradient Descend Algorithm with Finite Difference Method

**given**
- The cost function:
  \[ J = E \left[ \Phi(x(N)) + \sum_{k=0}^{N-1} L_k(x(k), u(k)) \right] \]
- A policy (function approximation) for the control input: \( u(n, x) = \mu(n, x; \theta) \)
- An initial value for the parameter vector: \( \theta \leftarrow \theta_0 \)
- The parameter exploration standard deviation: \( c \)
- The regularization coefficient: \( \lambda \)
- The learning rate: \( \omega \)

**repeat**
- Create \( N \) rollouts of the system with the perturbed parameters \( \theta + \Delta \theta, \Delta \theta \sim N(0, c^2 I) \)
- Calculate the return from the initial time and state for the \( n \)th rollout:
  \[ R(\theta + \Delta \theta_n) = \Phi(x^n(N)) + \sum_{k=0}^{N-1} L_k(x^n(k), u^n(k)) \]
- Construct \( R \) and \( \Theta \) matrices as:
  \[ R_{N \times 1} = [R(\theta + \Delta \theta_n)], \Theta_{N \times (p+1)} = [\Delta \theta_n^T \ 1] \]
- Calculate the value and gradient of the cost function at \( \theta \)
  \[ \begin{bmatrix} \nabla J(\theta) \\ J(\theta) \end{bmatrix} = (\Delta \Theta^T \Delta \Theta + \lambda I)^{-1} \Delta \Theta^T R \]
- Update the parameter vector:
  \[ \theta \leftarrow \theta - \omega \nabla J(\theta) \]

**until** convergence
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