QUASI-POISSON STRUCTURES ON MODULI SPACES OF QUASI-SURFACES

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Abstract. In generalization of the classical Atiyah-Bott Poisson brackets on the moduli spaces of surfaces we define quasi-Poisson brackets on the moduli spaces of quasi-surfaces.

1. Introduction

Moduli spaces of surfaces carry beautiful geometric structures, in particular, Poisson brackets, see [AB]. Our goal is to extend these brackets to the more general setting of quasi-surfaces introduced in [Tu1]. It turns out that the appropriate version of the Jacobi identity involves both a 2-bracket and a 3-bracket. Pairs of brackets satisfying this modified Jacobi identity are said to be quasi-Poisson. Our main result is a construction of quasi-Poisson pairs of brackets on the moduli spaces of quasi-surfaces.

A part of this work concerns an arbitrary algebra $A$. Following [VdB], [Cb], for any integer $n \geq 1$, we consider a trace algebra $A_t^n$ which, under appropriate assumptions on $A$ and the ground ring, is the coordinate algebra of the affine quotient scheme $\text{Rep}_n(A)/\text{GL}_n$. We view this affine scheme as the moduli space of $n$-dimensional representations of $A$ and we view the trace algebra $A_t^n$ as the algebra of functions on this space. In generalization of the work of Crawley-Boevey [Cb], we show how to derive brackets in the trace algebras from so-called braces in $A$. We formulate conditions on the braces ensuring that the induced brackets in the trace algebras form quasi-Poisson pairs.

Quasi-surfaces are topological spaces generalizing surfaces with boundary by allowing singular (non-surface) parts. A number of homotopy-invariant operations on loops in surfaces generalize to quasi-surfaces, see [Tu1] and Sections 5, 6 below. We show that these operations are braces in the group algebra $A$ of the fundamental group. We then show that the induced brackets in the trace algebras $\{A_t^n\}_{n \geq 1}$ form quasi-Poisson pairs.

The first part of the paper (Sections 2, 3) presents our algebraic methods and the second part (Sections 4–6) is devoted to the topological results. In the appendix we discuss a construction of braces from Fox derivatives.

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2. Algebraic preliminaries

Throughout the paper we fix a commutative base ring $R$. By a module we mean an $R$-module and by an algebra we mean an associative $R$-algebra with unit.

2.1. Quasi-Lie algebras. Given an integer $m \geq 1$, an $m$-bracket in a module $M$ is a map $M^m \to M$ which is linear in all $m$ variables. Here $M^m$ is the direct
product of $m$ copies of $M$. An $m$-bracket $\mu$ in $M$ is cyclically symmetric if for all $x_1, \ldots, x_m \in M$, we have
$$\mu(x_1, \ldots, x_{m-1}, x_m) = \mu(x_m, x_1, \ldots, x_{m-1}).$$

A 2-bracket $\mu$ in $M$ is skew-symmetric if $\mu(x_1, x_2) = -\mu(x_2, x_1)$ for all $x_1, x_2 \in M$.

Following [Tv2], a quasi-Lie algebra is a module $M$ carrying a skew-symmetric 2-bracket $[-,-]$ and a cyclically symmetric 3-bracket $[-,-,-]$ such that for any $x, y, z \in M$, we have the following quasi-Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = [x, y, z] - [z, y, x].$$

Note that both sides of (2.1.1) are cyclically symmetric and the left-hand side is the usual Jacobiator of the 2-bracket $[−, −]$ such that for any $x, y, z \in M$, we have the following quasi-Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = [x, y, z] - [z, y, x].$$

A quasi-Lie algebra homomorphism from a quasi-Lie algebra $M$ to a quasi-Lie algebra $N$ is a linear map $f : M \to N$ such that $[f(x), f(y)] = f([x, y])$ and $[f(x), f(y), f(z)] = f([x, y, z])$ for all $x, y, z \in M$.

2.2. Weak derivations. For an algebra $A$, let $A'$ be the submodule of $A$ spanned by the commutators $\{xy - yx \mid x, y \in A\}$. The quotient module $\hat{A} = A/A'$ is the zeroth Hochschild homology of $A$. A derivation of $A$ is a linear map $d : A \to A$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$. The identity

$$d(xy - yx) = d(x)y - yd(x) + xd(y) - d(y)x$$

shows that the derivations of $A$ carry $A'$ into itself and induce linear endomorphisms of the module $\hat{A}$. A linear endomorphism of $\hat{A}$ is a weak derivation if it is induced by a derivation of $A$.

2.3. Braces. An $m$-brace in an algebra $A$ is an $m$-bracket in the module $\hat{A}$ which is a weak derivation in all $m$ variables. Thus, an $m$-brace in $A$ is a map $\mu : (\hat{A})^m \to \hat{A}$ such that for any $1 \leq j \leq m$ and $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m \in \hat{A}$, the map

$$\hat{A} \to \hat{A}, \ x \mapsto \mu(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_m)$$

is a weak derivation. A 1-brace in $A$ is just a weak derivation $A \to A$.

If $A$ is a commutative algebra, then $\hat{A} = A$ and an $m$-brace in $A$ is an $m$-bracket $A^m \to A$ which is a derivation in all $m$ variables. The following lemma fully describes braces in polynomial algebras.

**Lemma 2.1.** Let $X$ be a set and let $A = R[X]$ be the (commutative) algebra of polynomials in the variables $\{x \in X \}$ with coefficients in $R$. For any $m \geq 1$ and any mapping $f : X^m \to A$, there is a unique $m$-brace $F : A^m \to A$ in $A$ such that $F(x_1, \ldots, x_m) = f(x_1, \ldots, x_m)$ for all $x_1, \ldots, x_m \in X$.

**Proof.** The uniqueness of $F$ is clear as the set $X$ generates the algebra $A$. We define $F$ by the following explicit formula: for any $a_1, \ldots, a_m \in A$, set

$$F(a_1, \ldots, a_m) = \sum_{x_1, \ldots, x_m \in X} \left( \prod_{i=1}^{m} \frac{\partial a_i}{\partial x_i} \right) f(x_1, \ldots, x_m).$$

Here the right-hand side has only a finite number of non-zero terms as each $a_i$ is a polynomial in the variables $\{x \mid x \in X\}$ and therefore $\partial a_i/\partial x \neq 0$ for only a finite set of $x \in X$. That $F$ is a derivation in all its variables follows from the Leibnitz formula for the partial derivatives. \(\square\)
2.4. Quasi-Poisson algebras. A quasi-Poisson algebra is an algebra $A$ endowed with a quasi-Lie pair of brackets in the module $\hat{A}$ which both are braces in $A$. Such a pair of braces is called a quasi-Poisson pair. A commutative quasi-Poisson algebra with zero 3-brace is a Poisson algebra in the usual sense.

A quasi-Poisson algebra homomorphism from a quasi-Poisson algebra $A$ to a quasi-Poisson algebra $B$ is a quasi-Lie algebra homomorphism $\hat{A} \to \hat{B}$. The definition of a quasi-Poisson algebra given above differs from parallel definitions in [AKsM], [MT]. The main point of difference is that here we not involve actions of Lie groups or Lie algebras. Our quasi-Poisson algebras generalize $H_0$-Poisson algebras introduced by W. Crawley-Boevey [Cb]. In our terminology, an $H_0$-Poisson structure in an algebra $A$ is a Lie bracket in $\hat{A}$ which is a brace in $A$.

3. Trace algebras

We recall representation schemes and trace algebras following [VdB], [CB]. Then we discuss braces in trace algebras.

3.1. Representation schemes. An algebra $A$ and an integer $n \geq 1$ determine an affine scheme $\text{Rep}_n(A)$, the $n$-th representation scheme of $A$. For each commutative algebra $S$, the set of $S$-valued points of $\text{Rep}_n(A)$ is the set of algebra homomorphisms $A \to \text{Mat}_n(S)$. The coordinate algebra, $A_n$, of the affine scheme $\text{Rep}_n(A)$ is generated over $R$ by the symbols $x_{ij}$ with $x \in A$ and $i,j \in \{1,2,\ldots,n\}$. These generators commute and satisfy the following relations: $1_{ij} = \delta_{ij}$ for all $i,j$, where $\delta_{ij}$ is the Kronecker delta; for all $x,y \in A$, $r \in R$, and $i,j \in \{1,2,\ldots,n\}$,

$$(rx)_{ij} = rx_{ij}, \quad (x + y)_{ij} = x_{ij} + y_{ij} \quad \text{and} \quad (xy)_{ij} = \sum_{l=1}^n x_{il}y_{lj}.$$  

The function on the set of $S$-valued points of $\text{Rep}_n(A)$ determined by $x_{ij}$ assigns to a homomorphism $f : A \to \text{Mat}_n(S)$ the $(i,j)$-entry of the matrix $f(x)$. That these functions satisfy the relations above is straightforward.

The action of the group $G = GL_n(R)$ on $\text{Hom}(A,\text{Mat}_n(S))$ by conjugations induces an action of $G$ on $A_n$ for all $n$. Explicitly, for $g = (g_{kl}) \in G$ and any $x \in A$, $i,j \in \{1,\ldots,n\}$ we have

$$g \cdot x_{ij} = \sum_{k,l=1}^n g_{ik}(g^{-1})_{lj}x_{kl}.$$  

The set of invariant elements $A^G_n = \{a \in A_n \mid Ga = a\}$ is a subalgebra of $A_n$. This is the coordinate algebra of the affine quotient scheme $\text{Rep}_n(A) // G$ which we view as the moduli space of $n$-dimensional representations of $A$.

The linear map $A \to A_n, x \mapsto \sum_{i=1}^n x_{ii}$ is called the trace and denoted $\text{tr}$. The trace annihilates all the commutators in $A$ and so $\text{tr}(A') = 0$. Thus, the trace induces a linear map $A \to A_n$ also denoted $\text{tr}$. The subalgebra of $A_n$ generated by $\text{tr}(A) = \text{tr}(A)$ is denoted $A^n_0$ and is called the $n$-th trace algebra of $A$. A direct computation shows that $\text{tr}(A) \subset A^G_n$ and therefore $A^n_0 \subset A^G_n$. If the ground ring $R$ is an algebraically closed field of characteristic zero and $A$ is a finitely generated algebra, then a theorem of Le Bruyn and Procesi [LBP] implies that $A^n_0 = A^G_n$ so that $A^n_0$ is the coordinate algebra of $\text{Rep}_n(A) // G$. 
3.2. **Braces in trace algebras.** The following lemma - inspired by W. Crawley-Boevey [CB] - will allow us to construct braces in the trace algebras.

**Lemma 3.1.** For any integers \( m, n \geq 1 \) and any \( m \)-brace \( \mu \) in an algebra \( A \), there is a unique \( m \)-brace \( \mu_n \) in the algebra \( A_n^t \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{A}_n^m & \xrightarrow{\mu} & \tilde{A}_n^t \\
\text{tr} \times \cdots \times \text{tr} \downarrow & & \downarrow \text{tr} \\
(A_n^t)^m & \xrightarrow{\mu_n} & A_n^t
\end{array}
\]

If \( \mu \) is cyclically symmetric, then so is \( \mu_n \). If \( m = 2 \) and \( \mu \) is skew-symmetric, then \( \mu_n \) is skew-symmetric.

**Proof.** The uniqueness of \( \mu_n \) is clear as the set \( \text{tr}(A) \) generates the algebra \( A_n^t \). We first prove the existence of \( \mu_n \) for \( m = 1 \). We need to show that for any weak derivation \( \mu : \tilde{A} \to \tilde{A} \), there is a derivation of \( A_n^t \) carrying \( \text{tr}(x) \) to \( \text{tr}(\mu(x)) \) for all \( x \in \tilde{A} \). Pick a derivation \( d : A \to A \) inducing \( \mu \). Then there is a unique derivation \( \tilde{d} : A_n \to A_n \) such that \( \tilde{d}(a_{ij}) = (d(a))_{ij} \) for all \( a \in A \), \( i, j \in \{1, \ldots, n\} \). Indeed, this formula defines \( \tilde{d} \) on the generators of the algebra \( A_n \); the compatibility with the defining relations is straightforward, see [CB] Lemma 4.4. Clearly, \( \tilde{d}(\text{tr}(a)) = \text{tr}(d(a)) \) for all \( a \in A \). Therefore \( \tilde{d}(A_n^t) \subset A_n^t \) and the restriction of \( \tilde{d} \) to \( A_n^t \) is a derivation of the algebra \( A_n^t \) satisfying our requirements.

Suppose now that \( m \geq 2 \) and consider the (commutative) polynomial algebra \( B = R[\tilde{A}] \) in the variables \( \{x\}_{x \in \tilde{A}} \). By Lemma 2.1 there is a unique \( m \)-brace \( F: B^m \rightarrow B \) such that

\[
F(x_1, \ldots, x_m) = \mu(x_1, \ldots, x_m) \in \tilde{A} \subset B \quad \text{for all} \quad x_1, \ldots, x_m \in \tilde{A}.
\]

Let \( \tau : B \rightarrow A_n^t \) be the algebra homomorphism carrying each \( x \in \tilde{A} \subset B \) to \( \text{tr}(x) \in A_n^t \). Claim: if \( y_l \in \text{Ker} \tau \), then \( \tau F(y_1, y_2, \ldots, y_m) = 0 \) for all \( y_2, \ldots, y_m \in B \). Since \( F \) is a derivation in \( y_2, \ldots, y_m \), it suffices to prove this claim when \( y_2, \ldots, y_m \in \tilde{A} \subset B \). Consider the linear endomorphism \( x \mapsto \mu(x, y_2, \ldots, y_m) \) of \( \tilde{A} \). Since \( \mu \) is a brace in \( \tilde{A} \), this endomorphism is a weak derivation. By the case \( m = 1 \) discussed above, there is a derivation \( D : A_n^t \rightarrow A_n^t \) such that

\[
D(\text{tr}(x)) = \text{tr}(\mu(x, y_2, \ldots, y_m))
\]

for all \( x \in \tilde{A} \). Then for any finite sequence \( x_1, \ldots, x_N \in \tilde{A} \), we have

\[
\tau F(\prod_{i=1}^{N} x_i) = \tau \left( \prod_{k=1}^{N} F(x_k, y_2, \ldots, y_m) \prod_{l \neq k} x_l \right)
\]

\[
= \sum_{k=1}^{N} \tau F(x_k, y_2, \ldots, y_m) \prod_{l \neq k} \text{tr}(x_l) = \sum_{k=1}^{N} \text{tr}(\mu(x_k, y_2, \ldots, y_m)) \prod_{l \neq k} \text{tr}(x_l)
\]

\[
= \sum_{k=1}^{N} \text{tr}(D(x_k)) \prod_{l \neq k} \text{tr}(x_l) = D(\prod_{i=1}^{N} \text{tr}(x_i)) = D(\text{tr}(\prod_{i=1}^{N} x_i)).
\]

Since each \( y \in B \) is a linear combination of monomials in the generators \( \{x\}_{x \in \tilde{A}} \) we have \( \tau F(y_1, y_2, \ldots, y_m) = D(y) \). For \( y_l \in \text{Ker} \tau \), we get \( \tau F(y_1, y_2, \ldots, y_m) = 0 \). Similar arguments show that if \( y_l \in \text{Ker} \tau \) for some \( i = 1, \ldots, m \) then \( \tau F(y_1, \ldots, y_m) = 0 \).
Adding these three expansions and using the skew-symmetry of \(-Z\) elements of a generating set of such that the following diagram commutes:

\[
\begin{array}{c}
B^m \\
\tau \times \cdots \times \tau
\end{array} \xrightarrow{F} B
\]

\[
\begin{array}{c}
(A_n^t)^m \\
\mu_n
\end{array} \xrightarrow{} A_n^t.
\]

Since \(F\) is a derivation in all variables and \(\tau\) is an algebra epimorphism, \(\mu_n\) is a derivation in all variables, i.e., a brace. Restricting the latter diagram to \(A \subset B\) we obtain the diagram \([3.2.1]\). The last two claims of the lemma are straightforward. \(\square\)

**Lemma 3.2.** Let \(Z\) be a commutative algebra carrying a skew-symmetric 2-brace \([-,-]\) and a cyclically symmetric 3-brace \([-,-,-]\). If the relation \((2.1.1)\) holds for elements of a generating set of \(Z\), then it holds for all elements of \(Z\).

**Proof.** Let \(L(x, y, z)\) and \(R(x, y, z)\) be respectively the left and the right hand-sides of \((2.1.1)\). Since both sides are linear in \(x, y, z\) and cyclically symmetric, it suffices to verify the following: if \((2.1.1)\) holds for triples \(x, y, z \in Z\) and \(x, y, t \in Z\), then it holds for the triple \(x, y, zt\). Since \([-,-]\) is a brace in a commutative algebra,

\[
[[x, y], zt] = z[[x, y], t] + [[x, y], z]t,
\]

\[
[y, zt], x = [z[y, t], x] + [[y, z], x]t,
\]

\[
= z[[y, t], x] + [z, x][y, t] + [[y, z], x]t.
\]

Similarly,

\[
[[zt, x], y] = [z[t, x], y] + [[z, x]t, y]
\]

\[
= z[[t, x], y] + [z, y][t, x] + [[z, x]t, y] + [[z, x], y]t.
\]

Adding these three expansions and using the skew-symmetry of \([-,-]\), we get

\[
L(x, y, zt) = zL(x, y, t) + L(x, y, z)t.
\]

Thus, \(L\) satisfies the Leibnitz rule in the last variable. Since the bracket \([-,-,-]\) also satisfies this rule, so does \(R(x, y, z) = [x, y, z] = [z, y, x]\). Consequently, if \((2.1.1)\) holds for the triples \(x, y, z\) and \(x, y, t\), then it holds for the triple \(x, y, zt\). \(\square\)

**Theorem 3.3.** For any quasi-Poisson algebra \(A\) and integer \(n \geq 1\), there is a unique structure of a quasi-Poisson algebra in \(A_n^t\) such that the trace \(\text{tr} : A \rightarrow A_n^t\) is a quasi-Poisson algebra homomorphism.

**Proof.** Let \([-,-],[-,-,-]\) be the given braces in \(A\) forming a quasi-Lie pair. By Lemma \([3.1]\) there are unique braces \([-,-],[-,-,-]\) in \(A_n^t\) such that

\[
|\text{tr}(x), \text{tr}(y)| = \text{tr}([x, y]) \quad \text{and} \quad |\text{tr}(x), \text{tr}(y), \text{tr}(z)| = \text{tr}([x, y, z])
\]

for all \(x, y, z \in A\). Thus, the quasi-Jacobi relation \((2.1.1)\) holds for all elements of the set \(\text{tr}(A) \subset A_n^t\) generating \(A_n^t\). Lemma \([3.2]\) implies that \((2.1.1)\) holds for all elements of \(A_n^t\). Since the brace \([-,-]\) in \(A\) is skew-symmetric and the brace \([-,-,-]\) in \(A\) is cyclically symmetric, so are the induced braces in \(A_n^t\). Therefore these braces form a quasi-Poisson pair. They turn \(A_n^t\) into a quasi-Poisson algebra satisfying the conditions of the theorem. \(\square\)
When the quasi-Poisson algebra \( A \) in Theorem 3.3 has zero 3-bracket, the induced 3-bracket in \( A_n^t \) also is zero. Then the 2-bracket in \( \hat{A} \) is a Lie bracket and so is the induced 2-bracket in \( A_n^t \). So, \( A_n^t \) is a Poisson algebra in the usual sense. This case of Theorem 3.3 is due to W. Crawley-Boevey [CB].

3.3. Remark. A smooth vector field \( v \) on a smooth manifold \( N \) induces a derivation \( d_v \) of the commutative algebra \( C^\infty(N) \) of smooth \( \mathbb{R} \)-valued functions on \( N \). By definition, \( d_v(f) = df(v) \) for \( f \in C^\infty(N) \). The map \( v \mapsto d_v \) defines a Lie algebra isomorphism from the Lie algebra of smooth vector fields on \( N \) (with the Jacobi-Lie bracket) onto the Lie algebra of derivations of \( C^\infty(N) \) with the Lie bracket \( [d_1, d_2] = d_1d_2 - d_2d_1 \). For any algebra \( A \) and any \( n \geq 1 \), this suggests to view derivations of the trace algebra \( A_n^t \) as vector fields on (the smooth part) of the affine scheme \( \text{Rep}_n(A)//G \). More generally, we can view \( m \)-braces in \( A_n^t \) as \( m \)-tensor fields on \( \text{Rep}_n(A)//G \) for all \( m \geq 1 \).

4. The main theorem

We define braces in the modules of loops, recall quasi-surfaces from [Tu1], and state our main results.

4.1. Braces in the module of loops. A loop in a topological space \( X \) is a continuous map \( a : S^1 \to X \) where the circle \( S^1 = \{ p \in \mathbb{C} \mid |p| = 1 \} \) is oriented counterclockwise. Two loops \( a, b : S^1 \to X \) are freely homotopic if there is a continuous map \( F : S^1 \times [0, 1] \to X \) such that \( F(p, 0) = a(p) \) and \( F(p, 1) = b(p) \) for all \( p \in S^1 \). The set of free homotopy classes of loops in \( X \) is denoted by \( L(X) \). The free module with basis \( L(X) \) is denoted by \( M(X) \).

For path-connected \( X \), we define braces in \( M(X) \) as follows. Pick a point \( * \in X \) and set \( \pi = \pi_1(X, *) \). For the group algebra \( A = R[\pi] \), the module \( A' \subset A \) is generated by the set \( \{ uv - vu \mid u, v \in \pi \} \). Since \( uv = u(vu)u^{-1} \) for \( u, v \in \pi \), the module \( A' \) is generated by the set \( \{ wu - u w \mid w \in \pi \} \). Thus, \( \hat{A} = A/A' = R\hat{\pi} \) is the module freely generated by the set \( \hat{\pi} \) of conjugacy classes of elements of \( \pi \).

Note that the map \( \pi \to L(X) \) carrying the homotopy classes of loops to their free homotopy classes induces a bijection \( \pi \to L(X) \). Thus \( \hat{A} = R\hat{\pi} = M(X) \). A bracket in the module \( M(X) \) is a brace if it is a brace in the algebra \( A \). It is straightforward to see that this definition does not depend on the choice of the base point \( * \).

4.2. Quasi-surfaces. By a segment we mean a closed segment and by a surface we mean a smooth oriented 2-manifold with boundary. A quasi-surface is a path-connected topological space obtained by gluing a surface \( \Sigma \) to a topological space \( Y \) along a continuous mapping from a union of several disjoint segments in \( \partial \Sigma \) to \( Y \). We call \( \Sigma \) the surface core and \( Y \) the singular part of \( X \). For example, taking a path-connected surface \( \Gamma \) and collapsing several disjoint subsegments of \( \partial \Gamma \) into a single point we obtain a quasi-surface with 1-point singular part. When only one segment in \( \partial \Gamma \) is collapsed, the resulting quasi-surface is homeomorphic to \( \Gamma \). Another way to turn \( \Gamma \) into a quasi-surface is as follows: pick a surface \( \Sigma \subset \Gamma \) meeting \( \Gamma \setminus \Sigma \) at a finite non-empty set of disjoint segments with endpoints in \( \partial \Gamma \). Taking \( \Sigma \) as the surface core and \( \Gamma \setminus \Sigma \) as the singular part, we turn \( \Gamma \) into a quasi-surface. In particular, given a finite tree \( T \subset \Gamma \) meeting \( \partial \Gamma \) at the vertices of degree 1, we can take a closed regular neighborhood of \( T \) in \( \Gamma \) as the surface core.
and take the closure of the rest of $\Gamma$ as the singular part. In this way, $T$ gives rise to a structure of a quasi-surface on $\Gamma$.

4.3. Main results. For any quasi-surface $X$, the author constructed in [Tu1] a skew-symmetric 2-bracket $[-,-]_X$ and cyclically symmetric $m$-brackets $\{\mu^m\}_{m \geq 1}$ in the module $M(X)$. We now state our main theorem.

Theorem 4.1. The brackets $[-,-]_X$ and $\{\mu^m\}_{m \geq 1}$ in $M(X)$ are braces.

By Theorem 4.2 of [Tu2], the brackets $[-,-]_X$ and $\mu^3$ form a quasi-Lie pair. Combining with Theorem 4.1 we obtain the following claim.

Corollary 4.2. The group algebra $R[\pi_1(X)]$ endowed with the braces $[-,-]_X$ and $\mu^3$ is a quasi-Lie algebra.

Corollary 4.2 and Theorem 4.3 imply the following claim.

Corollary 4.3. For the algebra $A = R[\pi_1(X)]$ and any integer $n \geq 1$, there is a unique structure of a quasi-Poisson algebra in $A^i_n$ such that the trace $\text{tr} : \hat{A} \to A^i_n$ is a quasi-Poisson algebra homomorphism.

In the next two sections we recall the definitions of the brackets $\{\mu^m\}_{m \geq 1}, [-,-]_X$ and prove Theorem 4.1.

5. Gates

We define gates in an arbitrary topological space $X$ and show how gates give rise to braces.

5.1. Gates. A cylinder neighborhood of a set $C \subset X$ is a pair consisting of a closed set $U \subset X$ with $C \subset \text{Int}(U)$ and a homeomorphism $U \approx C \times [-1,1]$ carrying $\text{Int}(U)$ onto $C \times (-1,1)$ and carrying each point $c \in C$ to $(c,0)$. A gate in $X$ is a closed path-connected subspace $C$ of $X$ endowed with a cylinder neighborhood and such that all loops in $C$ are contractible in $X$. We will identify the cylinder neighborhood in question with $C \times [-1,1]$ via the given homeomorphism.

For a gate $C \subset X$, consider the map $H : X \to S^1$ carrying the complement of $C \times (-1,1)$ in $X$ to $-1 \in S^1$ and carrying $C \times \{t\}$ to $\exp(\pi it) \in S^1$ for all $t \in [-1,1]$. We say that a loop $a : S^1 \to X$ is transversal to $C$ if the map $Ha : S^1 \to S^1$ is transversal to $1 \in S^1$. Then the set $a^{-1}(C) = (Ha)^{-1}(1)$ is finite. For each $p \in a^{-1}(C)$, we define the crossing sign $\varepsilon_p(a)$: if at $p$ the loop $a$ goes from $C \times (-1,0)$ to $C \times (0,1)$ then $\varepsilon_p(a) = +1$, otherwise, $\varepsilon_p(a) = -1$.

5.2. Gate brackets. We start with notation. For any loop $a : S^1 \to X$ we call the point $a(1) \in X$ the base point of $a$. For $p \in S^1$, we let $a_p : S^1 \to X$ be the loop obtained as the composition of $a$ with the rotation $S^1 \to S^1$ carrying $1 \in S^1$ to $p$. This loop is based at $a(p)$ and is called a reparametrization of $a$. Set $L = L(X)$ and $M = M(X)$, see Section 4.1 for notation. For any loop $a$ in $X$, we let $\langle a \rangle \in L \subset M$ be the free homotopy class of $a$.

Each gate $C \subset X$ determines brackets $\{\mu^m_C\}_{m \geq 1}$ in the module $M$ as follows. Pick a point $\ast \in C$ and, for each $c \in C$, pick a path $\gamma_c$ in $C$ from $\ast$ to $c$. Given a loop $a$ in $X$ with $a(1) \in C$ we let $a^\gamma = \gamma_{a(1)} a \gamma_{a(1)}^{-1}$ be the loop based at $\ast$ and obtained from $a$ by conjugation along the path $\gamma_{a(1)}$. Since $C$ is a gate, the homotopy class of $a^\gamma$ in $\pi_1(X,\ast)$ does not depend on the choice of $\gamma_{a(1)}$. 
Consider $m \geq 1$ loops $a_1, \ldots, a_m$ in $X$ transversal to $C$. For any $i = 1, \ldots, m$ and $p_i \in a_i^{-1}(C) \subset S^1$, we have the reparametrization $(a_i)_{p_i}$ of $a_i$ based at $a_i(p_i) \in C$ and the loop $(a_i)_{p_i}^\gamma = ((a_i)_{p_i})^\gamma$ based at $*$. Set

$$
\mu_C^m(a_1, \ldots, a_m) = \sum_{p_1 \in a_1^{-1}(C), \ldots, p_m \in a_m^{-1}(C)} \prod_{i=1}^m \varepsilon_{p_i}(a_i) \prod_{i=1}^m (a_i)_{p_i}^\gamma \in M.
$$

The following claim is proved in [Tu2, Section 3].

**Lemma 5.1.** $\mu_C^m(a_1, \ldots, a_m) \in M$ depends only on the free homotopy classes of the loops $a_1, \ldots, a_m$ and, in particular, does not depend on the choice of the point $*$. 

Using the product structure in the cylinder neighborhood of $C$, we easily observe that each loop in $X$ is freely homotopic to a loop transversal to $C$. Therefore Lemma 5.1 yields a map

$$
L^m \to M, \, ((a_1), \ldots, (a_m)) \mapsto \mu_C^m(a_1, \ldots, a_m).
$$

This map extends by linearity to an $m$-bracket $\mu_C^m$ in $M$. Since $\langle uv \rangle = \langle vu \rangle$ for any loops $u, v$ in $X$ based at $*$, the bracket $\mu_C^m$ is cyclically symmetric.

**Lemma 5.2.** The bracket $\mu_C^m$ in $M$ is a brace.

**Proof.** Pick a point $* \in X \setminus C$ and consider the group algebra $A = R[\pi_1(X, *)]$. We need to prove that the bracket $\mu_C^m$ in $M = \hat{A}$ is a brace in $A$. Since $\mu_C^m$ is cyclically symmetric, it suffices to show that it is a weak derivation in the first variable. To this end, we refine the definition of $\mu_C^m(a_1, \ldots, a_m)$ whenever the loop $a_1$ is based at $*$. For any points $p_1 \in a_1^{-1}(C), \ldots, p_m \in a_m^{-1}(C)$ consider the product path

$$
(a_1, \ldots, a_m)
$$

where $a_p^-$ is the path in $X$ obtained as the restriction of $a_1$ to the arc in $S^1$ leading from $1 \in S^1$ to $p_1$ and $a_p^+$ is the path in $X$ obtained as the restriction of $a_1$ to the arc in $S^1$ leading from $p_1$ to $1$. The path (5.2.1) is a loop based at $*$. Let $[p_1, \ldots, p_m] \in \pi_1(X, *)$ be the homotopy class of this loop. Set

$$
\mu_C^m(a_1, \ldots, a_m) = \sum_{p_1 \in a_1^{-1}(C), \ldots, p_m \in a_m^{-1}(C)} \prod_{i=1}^m \varepsilon_{p_i}(a_i) [p_1, \ldots, p_m] \in A.
$$

Note that any two $*$-based loops in $X$ transversal to $C$ and homotopic in the class of $*$-based loops may be related by a finite sequence of homotopies of the following two types (and inverse homotopies): (i) deformations in the class of $*$-based loops transversal to $C$ and (ii) deformations pushing a branch of the loop in $X \setminus C$ across $C$ and creating two new crossings with $C$. It is clear that homotopies of $a_1$ of type (i) preserve $\mu_C^m(a_1, \ldots, a_m)$. A homotopy of $a_1$ of type (ii) creates two new points $p, p'$ in $a_1^{-1}(C)$ such that $\varepsilon_p(a_1) = -\varepsilon_{p'}(a_1)$ and $[p, p_2, \ldots, p_m] = [p', p_2, \ldots, p_m]$ for any $\{p_i \in a_i^{-1}(C)\}_{i=2}^m$. Therefore the expression $\mu_C^m(a_1, \ldots, a_m)$ is preserved under deformations of $a_1$ in the class of $*$-based loops. So, the formula $a_1 \mapsto \mu_C^m(a_1, \ldots, a_m)$ defines a mapping $\pi_1(X, *) \to A$. This mapping extends by linearity to a linear map $A \to A$ which is easily seen to be a derivation and a lift of the linear map $\hat{A} \to \hat{A}, (a_1) \mapsto \mu_C^m(a_1, \ldots, a_m)$. Thus, $\mu_C^m$ is a weak derivation in the first variable. This proves the lemma. □
5.3. Remark. The arguments in the proof of Lemma 5.2 may be used to show that the sum (5.2.2) is preserved under free homotopies of the loops $a_2, \ldots, a_m$. Therefore (5.2.2) defines a map $A \times M^{m-1} \to A$ which is a derivation in the first variable and is linear in the other $m - 1$ variables.

6. Quasi-surfaces and proof of Theorem 4.1

6.1. More on quasi-surfaces. Consider a quasi-surface $X$ with surface core $\Sigma$ and singular part $Y$. We suppose that $X$ is obtained by gluing $\Sigma$ to $Y$ along a continuous map $\alpha \to Y$ where $\alpha \subset \partial \Sigma$ is a union of a finite number ($\geq 1$) of disjoint closed segments in $\partial \Sigma$. Note that $Y \subset X$ and $X \setminus Y = \Sigma \setminus \alpha$. We fix a closed neighborhood of $\alpha$ in $\Sigma$ and identify it with $\alpha \times [-1, 1]$ so that

$$\alpha = \alpha \times \{-1\} \quad \text{and} \quad \partial \Sigma \cap (\alpha \times [-1, 1]) = \alpha \cup (\partial \alpha \times [-1, 1]).$$

The surface

$$\Sigma' = \Sigma \setminus (\alpha \times [-1, 0)) \subset \Sigma \setminus \alpha \subset X$$

is a copy of $\Sigma$ embedded in $X$ and disjoint from $Y$. We provide $\Sigma'$ with the orientation induced from that of $\Sigma$.

Set $\pi_0 = \pi_0(\alpha)$. For $k \in \pi_0$, we let $\alpha_k$ be the corresponding segment component of $\alpha \times \{0\} \subset \partial \Sigma' \subset X$. We call the segments $\{\alpha_k\}_{k \in \pi_0}$ the gates of $X$. It is clear that these segments are gates of $X$ in the sense of Section 6.1. They separate $\Sigma' \subset X$ from the rest of $X$. By Section 5.2 every gate $\alpha_k$ determines a sequence of brackets $\{\mu^m_{\alpha_k}\}_{m \geq 1}$ in the module $M = M(X)$. For each $m \geq 1$, we define the total gate $m$-bracket of $X$ to be the sum

$$\mu^m = \sum_{k \in \pi_0} \mu^m_{\alpha_k} : M^m \to M.$$

We keep the objects $X, \Sigma, \Sigma', \alpha, \pi_0$ till the end of Section 6.

6.2. Loops in $X$. For any loop $a : S^1 \to X$ and a gate $\alpha_k$, we set $a \cap \alpha_k = a(S^1) \cap \alpha_k$. We say that an (ordered) pair of loops $a, b$ in $X$ is admissible if these loops do not meet at the gates and for any gate $\alpha_k$ of $X$ and any points $p \in a \cap \alpha_k$, $q \in b \cap \alpha_k$, the pair (a vector tangent to $\alpha_k$ and directed from $p$ to $q$, a vector at $p$ directed inside $\Sigma'$) is positively oriented in $\Sigma'$.

We say that a loop $a : S^1 \to X$ is generic if (i) all branches of $a$ in $\Sigma'$ are smooth immersions meeting $\partial \Sigma'$ transversely at a finite set of points which all lie in the interior of the gates, and (ii) all self-intersections of $a$ in $\Sigma'$ are double transversal intersections in $\text{Int}(\Sigma') = \Sigma' \setminus \partial \Sigma'$. A generic loop $a$ traverses any point of a gate $\alpha_k$ at most once so that the restriction of the map $a : S^1 \to X$ to $a^{-1}(\alpha_k) \subset S^1$ is a bijection onto the set $a \cap \alpha_k$. In this context, we adjust notation of Section 5 and use the letter $p$ for elements of the set $a \cap \alpha_k$ rather than for their preimages under $a$. Accordingly, the crossing sign $\varepsilon_p(a)$ at $p \in a \cap \alpha_k$ is +1 if $a$ goes at $p$ from $X \setminus \Sigma'$ to $\text{Int}(\Sigma')$ and is −1 otherwise.

A pair of loops in $X$ is generic if both loops are generic and all their intersections in $\Sigma'$ are double transversal intersections in $\text{Int}(\Sigma')$. In particular, such loops do not meet at the gates. It is easy to see that each ordered pair of loops in $X$ is freely homotopic to an admissible generic pair of loops.
6.3. The bracket $[-,-]_X$. We recall the 2-bracket $[-,-]_X$ in the module $M = M(X)$, see [Tu1]. For a loop $a : S^1 \to X$ and a point $r \in X$ traversed by $a$ exactly once, we let $a_r$ be the loop which starts at $r$ and goes along $a$ until coming back to $r$. For any loops $a, b$ in $X$ set
\[ a \cap b = a(S^1) \cap b(S^1) \cap \Sigma'. \]

If $a, b$ is a generic pair of loops then the set $a \cap b \subset \mathrm{Int}(\Sigma')$ is finite and each point $r \in a \cap b$ is traversed by $a$ and $b$ exactly once so that we can consider the loops $a_r, b_r$ based at $r$, their product $a_rb_r$, and the free homotopy class $\langle a_r b_r \rangle \in L(X)$. Set $\varepsilon_r(a, b) = 1$ if the tangent vectors of $a$ and $b$ at $r$ form a positive basis in the tangent space of $\Sigma'$ at $r$ and set $\varepsilon_r(a, b) = -1$ otherwise. Also, a generic pair of loops $a, b$ determines a finite set of triples
\[ T(a, b) = \{ (k, p, q) \mid k \in \pi_0, p \in a \cap \alpha_k, q \in b \cap \alpha_k \}. \]

For any triple $(k, p, q) \in T(a, b)$, we can multiply the loops $a_p, b_q$ based at $p, q$ using an arbitrary path connecting $p, q$ in $\alpha_k$. The product loop determines a well-defined element of $L(X)$ denoted $\langle a_p b_q \rangle$.

**Lemma 6.1.** There is a unique 2-bracket $[-,-]_X$ in $M = M(X)$ such that for any admissible generic pair of loops $a, b$ in $X$, we have
\[ (\langle a \rangle, \langle b \rangle)_X = 2 \sum_{r \in a \cap b} \varepsilon_r(a, b) \langle a_r b_r \rangle - \sum_{(k, p, q) \in T(a, b)} \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle. \]

The uniqueness of such a bracket follows from the last observation in Section 6.2.

The existence follows from [Tu1], Section 4.4.

The bracket $[-,-]_X$ generalizes Goldman’s bracket [Go1], [Go2]: its value on any pair of free homotopy classes of loops in $\Sigma' \subset X$ is twice their Goldman bracket.

6.4. Proof of Theorem 4.1.1 That the total gate brackets $\{ \mu^m \}_{m \geq 1}$ in $M = M(X)$ are braces follows from Lemma 6.2 and the obvious fact that a sum of braces is a brace. We need to prove that the 2-bracket $[-,-]_X$ is a brace. Since this bracket is skew-symmetric, it suffices to prove that for any $y \in L = L(X)$ the linear map $x \mapsto [x, y]_X : M \to M$ is a weak derivation.

Pick an arbitrary base point $* \in Y$: set $\pi = \pi_1(X, *)$ and $A = R[\pi]$. For a loop $a$ based at $*$, we let $[a] \in \pi \subset A$ be the homotopy class of $a$. We construct a derivation $d : A \to A$ as follows. First, we represent $y$ by a generic loop $b$ in $X$. Since $b$ meets the gates only in their interior points, every element of the group $\pi = \pi_1(X, *)$ can be represented by a loop $a$ based at $*$ such that the pair $a, b$ is generic and admissible. For any point $r \in a \cap b$ we define a loop $a \circ_r b$: it starts at $*$ and goes along $a$ till $r$, then it goes along the whole loop $b_r$ back to $r$, then it continues along the remaining part of $a$ to $*$. For any triple $(k, p, q) \in T(a, b)$, we define a loop $a \circ_{p,q} b$: it starts at $*$ and goes along $a$ till $p$, then it goes along $\alpha_k$ to $q$, then it goes along the whole loop $b_q$ back to $q$, then it follows along $\alpha_k$ back to $p$, then it continues along the remaining part of $a$ to $*$. Set
\[ d([a]) = 2 \sum_{r \in a \cap b} \varepsilon_r(a, b) [a \circ_r b] - \sum_{(k, p, q) \in T(a, b)} \varepsilon_p(a) \varepsilon_q(b) [a \circ_{p,q} b] \in A. \]

We claim that (i) $d([a]) \in A$ depends only on $[a] \in \pi$ and does not depend on the choice of the loop $a$ in its homotopy class; (ii) the linear extension $A \to A$ of $d$ is a derivation; (iii) this derivation is a lift of the linear map $x \mapsto [x, y]_X : M \to M$. 

The proof of (i) uses the same arguments as the proof of [Tu1], Lemma 3.1. The key
observation is that for any homotopic loops \(a_0, a_1\) such that the pairs \(a_0, b\) and \(a_1, b\)
are admissible, there is a homotopy \((a_t)_{t \in [0,1]}\) of \(a_0\) into \(a_1\) such that the pair \(a_t, b\)
is admissible for all \(t\). The proof of (ii) is straightforward: if \(a, a'\) are loops based
at \(*\) and such that both pairs \(a, b\) and \(a', b\) are admissible and generic, then the pair
\(aa', b\) is admissible. Deforming slightly \(a'\) we can ensure that this pair is generic.
Then the set \((aa', b)\) is a disjoint union of the sets \(a \cap b\) and \(a' \cap b\). Similarly, the
set \(T(aa', b)\) is a disjoint union of the sets \(T(a, b)\) and \(T(a', b)\). Using these facts
and computing \(d([aa'])\) via (6.4.1) we get \(d([aa']) = d([a])|a'| + |a|d([a'])\). Finally,
Claim (iii) is obtained by direct comparison of Formulas (6.3.1) and (6.4.1).

6.5. Remark. Similar arguments show that the expression (6.4.1) is preserved under
homotopies of the loop \(b\). As a result, we obtain a well-defined bilinear form
\(A \times M(X) \to A\). In the case of surfaces, this form was first constructed by N.
Kawazumi and Y. Kuno [KK1], [KK2] who proved that it defines a right action of
the Goldman-Lie algebra of loops on the group algebra of the fundamental group.

Appendix A. Fox derivatives

We define Fox derivatives in group algebras and explain how they give rise to
braces and how they arise from gates in topological spaces.

A.1. Fox derivatives. Let \(\pi\) be a group. A (left) Fox derivative in the group
algebra \(A = \mathbb{R}[\pi]\) is a linear map \(\partial : A \to A\) such that
\(\partial(xy) = \partial(x) + x\partial(y)\) for all \(x, y \in \pi \subset A\). For arbitrary \(x, y \in A\), we have then
\(\partial(xy) = \partial(x)\) \(\text{aug}(y) + x\partial(y)\) where \(\text{aug} : A \to \mathbb{R}\) is the linear map
carrying all elements of \(\pi\) to 1. For \(x \in \pi\), we can uniquely expand \(\partial(x) = \sum_{a \in x} (x/a)a\) where \((x/a)a \in \mathbb{R}\) is non-zero only for
a finite set of \(a\). We define a linear map \(\Delta_\partial : A \to A\) by
\[
\Delta_\partial(x) = \sum_{a \in \pi} (x/a)a^{-1}xa \quad \text{for all} \quad x \in \pi.
\]

Lemma A.1. \(\Delta_\partial(A') = 0\).

Proof. It suffices to prove that \(\Delta_\partial(xy - yx) = 0\) for any \(x, y \in \pi\). We have
\[
\partial(xy) = \partial(x) + x\partial(y) = \sum_{a \in \pi} ((x/a)a + (y/a)b xa).
\]

Therefore, by the definition of \(\Delta_\partial\),
\[
\Delta_\partial(xy) = \sum_{a \in \pi} ((x/a)a^{-1}xya + (y/a)b (xa)^{-1}xy(xa))
\]
\[
= \sum_{a \in \pi} ((x/a)a^{-1}xya + (y/a)b a^{-1}yx) = 0.
\]
The latter expression is invariant under the permutation \(x \leftrightarrow y\). So, \(\Delta_\partial(xy) =
\Delta_\partial(yx)\) and \(\Delta_\partial(xy - yx) = 0\).

The linear map \(\dot{A} = A/A' \to A\) induced by \(\Delta_\partial : A \to A\) is denoted by \(\dot{\Delta}_\partial\).

Theorem A.2. Let \(p : A \to \dot{A}\) be the projection. For any \(m \geq 1\) Fox derivatives
\(\partial_1, \ldots, \partial_m\) in \(A\), the map \(\mu^m : \dot{A}^m \to \dot{A}\) defined by
\[
\mu^m(x_1, \ldots, x_m) = p(\dot{\Delta}_{\partial_1}(x_1) \cdots \dot{\Delta}_{\partial_m}(x_m))
\]
for \(x_1, \ldots, x_m \in \dot{A}\) is an \(m\)-brace in \(A\).
Proof. We need to prove that \( \mu^m \) is a weak derivation in all variables, i.e., that for any \( i = 1, \ldots, m \) and \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m \in \mathbb{A} \), the map

\[
(A.1.2) \quad \mathbb{A} \to \mathbb{A}, \ x \mapsto \mu^m(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m)
\]
is induced by a derivation in \( \mathbb{A} \). Set

\[ G = \Delta_{\partial_1}(x_1) \cdots \Delta_{\partial_{i-1}}(x_{i-1}) \in \mathbb{A} \quad \text{and} \quad H = \Delta_{\partial_{i+1}}(x_{i+1}) \cdots \Delta_{\partial_m}(x_{im}) \in \mathbb{A}. \]

For \( x \in \pi \), we expand \( \partial_i(x) = \sum_{a \in \pi} (x/a)a \) with \( (x/a) = (x/a)\partial_i \). Then

\[
\mu^m(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m) = p(G\Delta_{\partial_i}(x)H) = p\left(\sum_{a \in \pi} (x/a)Ga^{-1}xaH\right) = p\left(\sum_{a \in \pi} (x/a)aHGa^{-1}x\right)
\]

where we use that \( p(Ga^{-1}xaH) = p(aHGa^{-1}x) \). Thus, the map \( (A.1.2) \) is induced by the linear map \( \mathbb{A} \to \mathbb{A} \) carrying any \( x \in \pi \) to \( \sum_{a \in \pi} (x/a)Fa^{-1}x \) is a derivation in \( \mathbb{A} \). Indeed, for \( x, y \in \pi \), we have

\[
d(x) = \sum_{a \in \pi} (x/a)aFa^{-1}x \quad \text{and} \quad d(y) = \sum_{a \in \pi} (y/a)aFa^{-1}y.
\]

Also,

\[
\partial_i(xy) = \partial_i(x) + x\partial_i(y) = \sum_{a \in \pi} ((x/a)a + (y/a)xa)
\]

and so

\[
d(xy) = \sum_{a \in \pi} ((x/a)aFa^{-1}xy + (y/a)xaF(xa)^{-1}xy) = d(x)y + xd(y).
\]

This completes the proof of the theorem. \( \square \)

Combining Theorem A.2 with Lemma 3.1 we obtain the following.

Corollary A.3. For any integers \( m, n \geq 1 \) and any Fox derivatives \( \partial_1, \ldots, \partial_m \) in \( \mathbb{A} \), there is a unique \( m \)-brace \( \mu_n^m \) in \( \mathbb{A}^n \) such that

\[
\mu_n^m(\text{tr}(x_1), \ldots, \text{tr}(x_m)) = \text{tr}(\Delta_{\partial_1}(x_1) \cdots \Delta_{\partial_m}(x_m))
\]

for all \( x_1, \ldots, x_m \in \mathbb{A} \).

A.2. Gate derivatives. Consider a topological space \( X \) with base point \( * \). A based gate in \( X \) is a gate \( C \subset X \setminus \{*\} \) endowed with a path \( \gamma : [0,1] \to X \) such that \( \gamma(0) = * \) and \( \gamma(1) \in C \). We show that such a pair \((C, \gamma)\) gives rise to a Fox derivative in the group algebra \( R[\pi] \) where \( \pi = \pi_1(X, *) \).

Pick any loop \( a : S^1 \to X \) based at \( * \) and transversal to \( C \). For \( p \in a^{-1}(C) \subset S^1 \), we let \( a^p \) be the product of the following three paths in \( X \): the restriction of \( a \) to the arc \((1,p) \subset S^1 \); any path \( \beta \) in \( C \) from \( a(p) \) to \( \gamma(1) \); the path inverse to \( \gamma \).

Clearly, \( a^p \) a loop in \( X \) based at \(*\). Since all loops in \( C \) are contractible in \( X \), the element \([a^p]\) of \( \pi \) represented by this loop does not depend on the choice of \( \beta \). Set

\[
(A.2.1) \quad \partial(a) = \sum_{p \in a^{-1}(C)} \varepsilon_p(a) [a^p] \in R[\pi]
\]

where \( \varepsilon_p(a) = \pm 1 \) is the crossing sign defined in Section 5.1.

Lemma A.4. Formula \( (A.2.1) \) determines a well-defined map \( \partial : \pi \to R[\pi] \). Its linear extension \( R[\pi] \to R[\pi] \) is a Fox derivative.
Proof. It is clear that all elements of $\pi$ can be represented by loops based at $*$ and transversal to $C$. The arguments in the proof of Lemma 5.2 can be used to show that if two loops $a, a'$ based at $*$ and transversal to $C$ represent the same element of $\pi$, then $\partial(a) = \partial(a')$. If $a, b$ are loops based at $*$ and transversal to $C$, then so is their product, and it follows from the definitions that $\partial(ab) = \partial(a) + a\partial(b)$. This implies the second claim of the lemma.

Combining Lemma A.4 with the results of Section A.1 we conclude that each sequence of $m \geq 1$ based gates in $X$ (not necessarily disjoint or distinct) gives rise to an $m$-brace in $A = R[\pi]$ and to $m$-braces in the algebras $\{A_n^t\}$. In particular, a sequence of $m$ copies of the same based gate $(C, \gamma)$ determines a cyclically symmetric $m$-brace in $A$. We leave it to the reader to verify that this brace does not depend on $\gamma$ and coincides with the $m$-brace $\mu^m_C$ defined in Section 5.2.

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