Quasiperiodic packings of $G$-clusters and Baake-Moody sets

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Abstract. The diffraction pattern of a quasicrystal admits as symmetry group a finite group $G$, and there exists a $G$-cluster $C$ (a union of orbits of $G$) such that the quasicrystal can be regarded as a quasiperiodic packing of copies of $C$, generally, partially occupied. On the other hand, by starting from the $G$-cluster $C$ we can define in a canonical way a permutation representation of $G$ in a higher dimensional space, decompose this space into the orthogonal sum of two $G$-invariant subspaces and use the strip projection method in order to define a pattern which can also be regarded as a quasiperiodic packing of copies of $C$, generally, partially occupied. This mathematical algorithm is useful in quasicrystal physics, but the dimension of the superspace we have to use in the case of a two or three-shell cluster is rather large. We show that the generalization concerning the strip projection method proposed by Baake and Moody [Proc. Int. Conf. Aperiodic’ 97 (Alpe d’Huez, 27-31 August, 1997) ed M de Boissieu et al. (Singapore: World Scientific, 1999) pp 9-20] allows to reduce this dimension, and present some examples.
1. Introduction

The diffraction pattern of a quasicrystal contains a discrete set of intense Bragg peaks invariant under a finite group $G$, and the high-resolution electron microscopy suggests that the quasicrystal can be regarded as a packing of (partially occupied) copies of a well-defined $G$-invariant atomic cluster $\mathcal{C}$. From a mathematical point of view, the cluster $\mathcal{C}$ can be defined as a finite union of orbits of $G$, and there exists an algorithm [2, 3] which leads from $\mathcal{C}$ directly to a pattern $Q$ which can be regarded as a union of interpenetrating partially occupied translations of $\mathcal{C}$ (the neighbours of each point $x \in Q$ belong to the set $x + \mathcal{C} = \{ x + y \mid y \in \mathcal{C} \}$). This algorithm, based on the strip projection method and group theory, represents an extended version of the model proposed by Katz & Duneau and independently by Elser for the icosahedral quasicrystals.

Unfortunately, the dimension of the superspace used in the case of a two or three-shell icosahedral cluster is rather large. The main purpose of this article is to present a way to reduce this dimension. It is based on the extension of the notion of model set proposed by Baake and Moody [1]. This extension increases the power of the strip projection method, and allows to define a larger class of quasiperiodic patterns, very useful in quasicrystal mathematics. We call them Baake-Moody sets. Some examples are presented in order to illustrate the theoretic considerations.

2. Baake-Moody sets

Let $\mathbb{E}_k = (\mathbb{R}^k, \langle , \rangle)$ be the usual $k$-dimensional Euclidean space, $E$ and $E^\perp$ be two orthogonal subspaces such that $\mathbb{E}_k = E \oplus E^\perp$, and let

$$L = \kappa \mathbb{Z}^k \quad K = [0, \kappa] = \{(x_1, x_2, \ldots, x_k) \mid 0 \leq x_i \leq \kappa \text{ for all } i\} \quad (1)$$

where $\kappa \in (0, \infty)$ is a fixed constant. For each $x \in \mathbb{E}_k$ there exist $x^\parallel \in E$ and $x^\perp \in E^\perp$ uniquely determined such that $x = x^\parallel + x^\perp$. The mappings

$$\pi : \mathbb{E}_k \longrightarrow \mathbb{E}_k : x \mapsto \pi x = x^\parallel \quad \pi^\perp : \mathbb{E}_k \longrightarrow \mathbb{E}_k : x \mapsto \pi^\perp x = x^\perp \quad (2)$$

are the corresponding orthogonal projectors.

By using the bounded set $K = \pi^\perp(K)$ we define in terms of the strip projection method [4] the discrete set

$$Q = \{ \pi x \mid x \in L, \pi^\perp x \in K \} \quad (3)$$

formed by the projection on $E$ of all the points of $L$ lying in the strip $K + E = \{ x + y \mid x \in K, \ y \in E \}$.

It is known [3] that any $\mathbb{Z}$-module $M \subset \mathbb{R}^l$ is the direct sum of a lattice $M_d$ of rank $d$ and a $\mathbb{Z}$-module $M_s$ dense in a vector subspace of dimension $s$, where $d + s$ is the dimension of the subspace generated by $L$ in $\mathbb{R}^l$. In view of this result the $\mathbb{Z}$-module $L^\perp = \pi^\perp(L)$ is the direct sum $L^\perp = L' + D$ of a lattice $D$ of rank $d$ and a $\mathbb{Z}$-module $L'$ dense in a subspace $E' \subset E^\perp$ of dimension $s$, where $d + s = \dim E^\perp$. In this
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Figure 1. The decompositions $E_k = E \oplus E^\perp = E \oplus E' \oplus E'' = E \oplus E''$.

decomposition the space $E'$ is uniquely determined and we denote by $E''$ its orthogonal complement in $E^\perp$

$$E'' = \{ x \in E^\perp \mid \langle x, y \rangle = 0 \text{ for any } y \in E' \}.$$  

We get $E_k = E \oplus E' \oplus E''$. For each $x \in E_k$ there exist $x^\parallel \in E$, $x' \in E'$ and $x'' \in E''$ uniquely determined such that $x = x^\parallel + x' + x''$. The mappings $\pi': E_k \rightarrow E_k$, $\pi' x = x'$ and $\pi'': E_k \rightarrow E_k$, $\pi'' x = x''$ are the orthogonal projectors corresponding to $E'$ and $E''$.

One can prove \[4\] that the projection $L = (\pi + \pi')(L)$ of the lattice $L$ on the space $E = E \oplus E'$ is a lattice in $E$, $\pi$ restricted to $L$ is injective and $\pi'(L)$ is dense in $E'$. It follows that the collection of spaces and mappings

$$\pi x \leftarrow x : E \leftarrow \pi \quad E \xrightarrow{\pi'} E' : x \rightarrow \pi' x \quad \cup \quad L$$

is a cut and project scheme \[1,7\].

The lattice $L = L \cap E$ is a sublattice of $L$, and necessarily $[L : L]$ is finite. The projection $L'' = \pi''(L)$ of $L$ on $E''$ is a discrete countable set. Let $Z = \{ z_i \mid i \in \mathbb{Z} \}$ be a subset of $L$ such that $L'' = \pi''(Z)$ and $\pi'' u_i \neq \pi'' u_j$ for $i \neq j$. The lattice $L$ is contained in the union of the cosets $E_i = z_i + E = \{ z_i + x \mid x \in E \}$

$$\mathbb{L} \subset \bigcup_{i \in \mathbb{Z}} E_i. \quad (5)$$

Since $\mathbb{L} \cap E_i = z_i + L$ the set

$$L_i = (\pi + \pi')(\mathbb{L} \cap E_i) = (\pi + \pi')z_i + L \quad (6)$$

is a coset of $L$ in $L$ for any $i \in \mathbb{Z}$.

Only for a finite number of cosets $E_i$ the intersection

$$K_i = K \cap E_i = \pi^\perp(\mathbb{K} \cap E_i) \subset \pi'' z_i + E' \quad (7)$$

is non-empty. By changing the indexation of the elements of $Z$ if necessary, we can assume that the subset of $E'$

$$K_i = \pi'(K_i) = \pi'(\mathbb{K} \cap E_i) \subset E' \quad (8)$$
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has a non-empty interior only for $i \in \{1, ..., m\}$. The ‘polyhedral’ set $K_i$ satisfies the conditions:

(a) $K_i \subset E'$ is compact;
(b) $K_i = \overline{\text{int}(K_i)}$;
(c) The boundary of $K_i$ has Lebesgue measure 0 for any $i \in \{1, ..., m\}$.

This allows us to use the Baake-Moody generalization of the notion of model set presented in [1] in order to define the set

$$\Lambda = \bigcup_{i=1}^{m} \{ \pi x \mid x \in L_i, \pi' x \in K_i \}$$

which we call a Baake-Moody set.

One can remark that $\Lambda = \mathbb{Q}$, and hence we have re-defined our pattern $\mathcal{Q}$ as a Baake-Moody set by using the superspace $E$ of dimension, generally, smaller than the dimension $k$ of the initial superspace $E_k$. The main difficulty in this new approach is the determination of the ‘atomic surfaces’ $K_i$.

3. Packings of $G$-clusters obtained by projection

In this section we review our method to obtain packings of clusters by projection. It is a direct generalization of the model proposed by Katz & Duneau [6] and independently by Elser [5] for icosahedral quasicrystals.

Let $\{g : E_n \rightarrow E_n \mid g \in G\}$ be an orthogonal $\mathbb{R}$-irreducible faithful representation of a finite group $G$ in the ‘physical’ space $E_n$ and let $S \subset E_n$ be a finite non-empty set which does not contain the null vector. Any finite union of orbits of $G$ is called a $G$-cluster. Particularly,

$$C = \bigcup_{r \in S} Gr \cup \bigcup_{r \in S} G(-r) = \{e_1, e_2, ..., e_k, -e_1, -e_2, ..., -e_k\}$$

where $Gr = \{gr \mid g \in G\}$, is the $G$-cluster symmetric with respect to the origin generated by $S$.

Let $e_i = (e_{i1}, e_{i2}, ..., e_{in})$, and let $\varepsilon_1 = (1, 0, ..., 0)$, $\varepsilon_2 = (0, 1, 0, ..., 0)$, ..., $\varepsilon_k = (0, ..., 0, 1)$ be the canonical basis of $E_k$. For each $g \in G$, there exist the numbers $s_{g1}^i$, $s_{g2}^i$, ..., $s_{gm}^i \in \{-1; 1\}$ and a permutation of the set $\{1, 2, ..., k\}$ denoted also by $g$ such that,

$$ge_j = s_{g(j)}^i e_{g(j)} \quad \text{for all } j \in \{1, 2, ..., k\}.$$  

**Theorem 1.** [2, 3] The group $G$ can be identified with the group of permutations $\{C \rightarrow C : r \mapsto gr \mid g \in G\}$ and the formula

$$g\varepsilon_j = s_{g(j)}^i \varepsilon_{g(j)} \quad \text{for all } j \in \{1, 2, ..., k\}.$$  

defines the orthogonal representation

$$g(x_1, x_2, ..., x_k) = (s_{g1}^{x_1} x_{g^{-1}(1)}, s_{g2}^{x_2} x_{g^{-1}(2)}, ..., s_{gm}^{x_m} x_{g^{-1}(m)})$$

of $G$ in $E_k$. 

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**Theorem 2.** \[2, 3\] The subspaces

$$E = \{ (r, e_1), (r, e_2), ..., (r, e_k) \mid r \in \mathbb{E}_n \}$$

$$E^\perp = \left\{ (x_1, x_2, ..., x_k) \in \mathbb{E}_k \mid \sum_{i=1}^k x_i e_i = 0 \right\}$$

(14)

of $\mathbb{E}_k$ are $G$-invariant, orthogonal, and $\mathbb{E}_k = E \oplus E^\perp$.

**Theorem 3.** \[2, 3\] The vectors $v_1 = \rho(1, 1, e_1, ..., e_k), v_n = \rho(1, n, e_2, ..., e_k)$, where $\rho = 1/\sqrt{(e_1)^2 + (e_2)^2 + ... + (e_k)^2}$ form an orthonormal basis of $E$.

**Theorem 4.** \[2, 3\] The subduced representation of $G$ in $E$ is equivalent with the representation of $G$ in $\mathbb{E}_n$, and the isomorphism

$$\varphi : \mathbb{E}_n \rightarrow E \quad \varphi(r) = (\rho r, e_1, \rho r, e_2, ..., \rho r, e_k)$$

(15)

with the property $\varphi(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$ allows us to identify the ‘physical’ space $\mathbb{E}_n$ with the subspace $E$ of $\mathbb{E}_k$.

**Theorem 5.** \[2, 3\] The matrix of the orthogonal projector $\pi : \mathbb{E}_k \rightarrow \mathbb{E}_k$ corresponding to $E$ in the basis $\{ \varepsilon_1, \varepsilon_2, ..., \varepsilon_k \}$ is

$$\pi = \rho^2 \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & ... & \langle e_1, e_k \rangle \\ \langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle & ... & \langle e_2, e_k \rangle \\ ... & ... & ... & ... \\ \langle e_k, e_1 \rangle & \langle e_k, e_2 \rangle & ... & \langle e_k, e_k \rangle \end{pmatrix}.$$  

(16)

Let $\kappa = 1/\rho$, $\mathbb{L} = \kappa \mathbb{Z}^k$, $\mathbb{K} = [0, \kappa]^k$, and let $K = \pi^\perp(\mathbb{K})$, where $\pi^\perp : \mathbb{E}_k \rightarrow \mathbb{E}_k$, $\pi^\perp x = x - \pi x$ is the orthogonal projector corresponding to $E^\perp$.

**Theorem 6.** \[2, 3\] The $\mathbb{Z}$-module $\mathbb{L} \subset \mathbb{E}_k$ is $G$-invariant, $\pi(\kappa \varepsilon_i) = \varphi(\varepsilon_i)$, that is, $\pi(\kappa \varepsilon_i) = e_i$ if we take into consideration the identification $\varphi : \mathbb{E}_n \rightarrow E$, and

$$\pi(\mathbb{L}) = \mathbb{Z} e_1 + \mathbb{Z} e_2 + ... + \mathbb{Z} e_k.$$  

(17)

The pattern defined by using the strip projection method \[6\]

$$Q = \{ \pi x \mid x \in \mathbb{L}, \pi^\perp x \in K \}$$

(18)

can be regarded as a union of interpenetrating partially occupied copies of $C$. For each point $\pi x \in Q$ the set of all the arithmetic neighbours of $\pi x$

$$\{ \pi y \mid y \in \{ x + \kappa \varepsilon_1, ..., x + \kappa \varepsilon_k, x - \kappa \varepsilon_1, ..., x - \kappa \varepsilon_k \}, \pi^\perp y \in K \}$$

is contained in the translated copy

$$\{ \pi x + e_1, ..., \pi x + e_k, \pi x - e_1, ..., \pi x - e_k \} = \pi x + C$$

of the $G$-cluster $C$. A fragment of $Q$ can be obtained by using, for example, the algorithm presented in \[9\].

The method presented in the previous section allows to re-define $Q$ as a Baake-Moody set by using, generally, a smaller dimensional superspace. Some exemples are presented in sections 4-7.
4. A 2D Penrose pattern

The relations
\[ \begin{align*}
a(x, y) &= (cx - sy, sx + cy) \\
b(x, y) &= (x, -y)
\end{align*} \] (19)
where \( c = \cos(\pi/5) = (1 + \sqrt{5})/4, s = \sin(\pi/5) = \sqrt{10 - 2\sqrt{5}}/4 \) define the usual two-dimensional representation of the dihedral group
\[ D_{10} = \langle a, b \mid a^{10} = b^2 = (ab)^2 = e \rangle. \]

Let \( \varepsilon_1 = (1, 0, \ldots, 0), \varepsilon_2 = (0, 1, 0, \ldots, 0), \ldots, \varepsilon_5 = (0, \ldots, 0, 1) \) be the canonical basis of \( \mathbb{E}_5 \),
and let \( c' = \cos(2\pi/5) = (\sqrt{5} - 1)/4, s' = \sin(2\pi/5) = \sqrt{10 + 2\sqrt{5}}/4 \). The \( D_{10} \)-cluster (containing only one orbit) generated by the set \( \mathcal{S} = \{(1, 0)\} \) is
\[ \mathcal{C} = D_{10}(1, 0) = \{e_1, e_2, e_3, e_4, e_5, -e_1, -e_2, -e_3, -e_4, -e_5\} \]
where \( e_1 = (1, 0), e_2 = (c', s'), e_3 = (-c, s), e_4 = (-c, -s), e_5 = (c', -s') \). It is formed by the vertices of a regular decagon.

The action of \( a \) and \( b \) on \( \mathcal{C} \) is described by the signed permutations
\[ a = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\
-e_4 & -e_5 & -e_1 & -e_2 & -e_3 \end{pmatrix} \quad b = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\
e_1 & e_5 & e_4 & e_3 & e_2 \end{pmatrix} \] (20)
and the corresponding transformations \( a, b : \mathbb{E}_6 \rightarrow \mathbb{E}_6 \)
\[ a(x_1, x_2, x_3, x_4, x_5) = (-x_3, -x_4, -x_5, -x_1, -x_2) \]
\[ b(x_1, x_2, x_3, x_4, x_5) = (x_1, x_5, x_4, x_3, x_2). \] (22)

The vectors \( v_1 = \varrho(1, c', -c, -c, c') \), \( v_2 = \varrho(0, s', s, -s, -s') \), where \( \varrho = \sqrt{2/\sqrt{5}} \), form an orthonormal basis of the \( D_{10} \)-invariant subspace
\[ E = \{(r, e_1), (r, e_2), \ldots, (r, e_5) \mid r \in \mathbb{E}_2\} \] (23)
and the isometry (which is an isomorphism of representations)
\[ \varphi : \mathbb{E}_2 \rightarrow E : r \mapsto (\varrho < r, e_1>, \varrho < r, e_2>, \ldots, \varrho < r, e_5>) \] (24)
with the property \( \varphi(\alpha, \beta) = \alpha v_1 + \beta v_2 \) allows us to identify the physical space \( \mathbb{E}_2 \) with the subspace \( E \) of \( \mathbb{E}_5 \). The matrices of the orthogonal projectors \( \pi, \pi^\perp : \mathbb{E}_5 \rightarrow \mathbb{E}_5 \)
corresponding to \( E \) and
\[ E^\perp = \{x \in \mathbb{E}_5 \mid \langle x, y \rangle = 0 \text{ for all } y \in E\} \] (25)
in the basis \( \{\varepsilon_1, \ldots, \varepsilon_5\} \) are
\[ \pi = \mathcal{M}(2/5, -\tau'/5, -\tau/5) \quad \pi^\perp = \mathcal{M}(3/5, \tau'/5, \tau/5) \] (26)
where $\tau = (1 + \sqrt{5})/2$, $\tau' = (1 - \sqrt{5})/2$ and

$$
\mathcal{M}(\alpha, \beta, \gamma) = \begin{pmatrix}
\alpha & \beta & \gamma & \gamma & \beta \\
\beta & \alpha & \beta & \gamma & \\
\gamma & \beta & \alpha & \beta & \gamma \\
\beta & \gamma & \beta & \alpha & \\
\end{pmatrix}
$$

(27)

Let $\kappa = 1/\varrho = \sqrt{5/2}$, $L = \kappa \mathbb{Z}^5$, $K = [0, \kappa]^5$, and let $K = \pi^\perp(K)$. The pattern defined in terms of the strip projection method

$$
Q = \{ \pi x \mid x \in L, \pi^\perp x \in K \}
$$

(28)

is the set of vertices of a 2D Penrose (singular) pattern $\mathbb{Q}$. For each $i \in \{1, 2, 3, 4, 5\}$ we have $\pi(\kappa \varepsilon_i) = \varphi(e_i)$, that is, $\pi(\kappa \varepsilon_i) = e_i$ if we use the identification of $E_2$ with $E$. Since the arithmetic neighbours of each point $\pi x \in Q$ belong to the set $\pi x + C$ we can regard $Q$ as a quasiperiodic packing of partially occupied copies of the $D_{10}$-cluster $C$, that is, a quasiperiodic packing of decagons.

We can re-define $Q$ as a Baake-Moody set by using the general method presented in section 2. In this case, we have to use the decomposition $E^\perp = E' \oplus E''$ corresponding to the orthogonal projectors

$$
\pi' = \mathcal{M}(2/5, -\tau/5, -\tau'/5) \quad \pi'' = \mathcal{M}(1/5, 1/5, 1/5).
$$

We get

$$
\mathcal{E} = E \oplus E' = \{(x_1, x_2, x_3, x_4, x_5) \in E_5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0 \}
$$

(29)

$$
E'' = \{(x_1, x_2, x_3, x_4, x_5) \in E_5 \mid x_1 = x_2 = x_3 = x_4 = x_5 \}
$$

(30)

$$
L = (\pi + \pi')(L) = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3 + \mathbb{Z}w_4
$$

(31)

$$
L = L \cap \mathcal{E} = 5L = \{(x_1, x_2, x_3, x_4, x_5) \in L \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0 \}
$$

(32)

where

$$
w_1 = \frac{1}{\sqrt{10}}(4, -1, -1, -1, -1) \quad w_2 = \frac{1}{\sqrt{10}}(-1, 4, -1, -1, -1) \quad w_3 = \frac{1}{\sqrt{10}}(-1, -1, 4, -1, -1) \quad w_4 = \frac{1}{\sqrt{10}}(-1, -1, -1, 4, -1).
$$

(33)

We can choose $z_j = (\kappa j, 0, 0, 0, 0)$ since $\pi'' z_i \neq \pi'' z_j$ for $i \neq j$,

$$
\mathcal{E}_j = z_j + \mathcal{E} = \{(x_1, x_2, x_3, x_4, x_5) \in E_5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = \kappa j \}
$$

and

$$
L \subset \bigcup_{j \in \mathbb{Z}} \mathcal{E}_j.
$$
The set $K \cap E_i$ is non-empty only for $i \in \{0, 1, 2, 3, 4, 5\}$, but $K_i = \pi'(K \cap E_i)$ has non-empty interior only for $i \in \{1, 2, 3, 4\}$. The set $K_i$ is the regular pentagon with the vertices

$$
\begin{align*}
\pi'(\kappa, 0, 0, 0, 0) &= \frac{1}{\sqrt{10}} (2, -\tau, -\tau', -\tau', -\tau) \\
\pi'(0, \kappa, 0, 0, 0) &= \frac{1}{\sqrt{10}} (-\tau, 2, -\tau, -\tau', -\tau') \\
\pi'(0, 0, \kappa, 0, 0) &= \frac{1}{\sqrt{10}} (-\tau', -\tau, 2, -\tau, -\tau') \\
\pi'(0, 0, 0, \kappa, 0) &= \frac{1}{\sqrt{10}} (-\tau', -\tau', -\tau, 2, -\tau) \\
\pi'(0, 0, 0, 0, \kappa) &= \frac{1}{\sqrt{10}} (-\tau, -\tau', -\tau', -\tau', 2)
\end{align*}
$$

$K_2 = -\tau K_1$, $K_3 = \tau K_1$, $K_4 = -K_1$, and we can re-define the Penrose pattern $Q$ as the Baake-Moody set

$$
Q = \bigcup_{i=1}^{4} \{ \pi x \mid x \in L_i, \pi' x \in K_i \}
$$

where $L_j = (\pi + \pi') z_j + L = j w_1 + L$. This definition is directly related to de Bruijn’s definition [7].

5. A 3D Penrose pattern

The icosahedral group $Y = 235 = \langle a, b \mid a^5 = b^2 = (ab)^3 = I \rangle$ has five irreducible non-equivalent representations. Its character table is

$$
\begin{array}{ccccccc}
& 1 & e & 12 a & 15 b & 20 ab & 12 a^2 \\
\Gamma_1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\Gamma_2 & 3 & \tau & -1 & 0 & \tau' & 0 \\
\Gamma_3 & 3 & \tau' & -1 & 0 & \tau & 0 \\
\Gamma_4 & 4 & -1 & 0 & 1 & -1 & 1 \\
\Gamma_5 & 5 & 0 & 1 & -1 & 0 & 0
\end{array}
$$

(36)

A realization of $\Gamma_2$ in the Euclidean 3D space $\mathbb{E}_3$ is the representation generated by the rotations $a$, $b : \mathbb{E}_3 \rightarrow \mathbb{E}_3$

$$
\begin{align*}
a(\alpha, \beta, \gamma) &= \left( \frac{\tau}{\sqrt{2}} \alpha - \frac{\tau'}{\sqrt{2}} \beta + \frac{1}{\sqrt{2}} \gamma, \frac{\tau}{\sqrt{2}} \alpha + \frac{\tau}{\sqrt{2}} \beta + \frac{1}{\sqrt{2}} \gamma, -\frac{1}{\sqrt{2}} \alpha + \frac{\tau}{\sqrt{2}} \beta + \frac{\tau'}{\sqrt{2}} \gamma \right) \\
b(\alpha, \beta, \gamma) &= (-\alpha, -\beta, \gamma).
\end{align*}
$$

(37)

If in relation (37) we replace $\tau$ by $\tau'$ then we obtain a realization of $\Gamma_3$.

Let $\varepsilon_1 = (1, 0, 0, 0, 0, 0, 0)$, $\varepsilon_2 = (0, 1, 0, 0, 0, 0, 0)$, ..., $\varepsilon_6 = (0, ..., 0, 1)$ be the canonical basis of $\mathbb{E}_6$. The points of the one-shell $Y$-cluster

$$
C = Y(1, \tau, 0) = \{ e_1, e_2, ..., e_6, -e_1, -e_2, ..., -e_6 \}
$$

where

$$
\begin{align*}
e_1 &= (1, \tau, 0) & e_3 &= (-\tau, 0, 1) & e_5 &= (\tau, 0, 1) \\
e_2 &= (-1, \tau, 0) & e_4 &= (0, -1, \tau) & e_6 &= (0, 1, \tau)
\end{align*}
$$

(38)
are the vertices of a regular icosahedron. The action of \( a \) and \( b \) on the set \( C \) is described by the signed permutations

\[
a = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ e_2 & e_3 & e_4 & e_5 & e_1 & e_6 \end{pmatrix} \quad b = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ -e_1 & -e_2 & -e_5 & -e_6 & e_3 & e_4 \end{pmatrix}
\]

(39)

and the corresponding transformations \( a, b : \mathbb{E}_6 \rightarrow \mathbb{E}_6 \)

\[
a = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \\ \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_1 & \varepsilon_6 \end{pmatrix} \quad b = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_5 & -\varepsilon_6 & \varepsilon_3 & \varepsilon_4 \end{pmatrix}
\]

(40)

generate the orthogonal representation of \( Y \) in \( \mathbb{E}_6 \)

\[
a(x_1, x_2, x_3, x_4, x_5, x_6) = (x_5, x_1, x_2, x_3, x_4, x_6) \\
b(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_1, -x_2, x_5, x_6, x_3, x_4).
\]

(41)

The vectors

\[
v_1 = \varrho(1, -1, -\tau, 0, \tau, 0) \quad v_2 = \varrho(\tau, \tau, 0, -1, 0, 1) \quad v_3 = \varrho(0, 0, 1, \tau, 1, \tau)
\]

(42)

where \( \varrho = 1/\sqrt{4 + 2\tau} \) form an orthonormal basis of the \( Y \)-invariant subspace

\[
E = \{(r, e_1), (r, e_2), ..., (r, e_6) \mid r \in \mathbb{E}_3\}.
\]

(43)

The isometry

\[
\varphi : \mathbb{E}_3 \rightarrow E : r \mapsto (\varrho(r, e_1), \varrho(r, e_2), ..., \varrho(r, e_6))
\]

(44)

which is an isomorphism of representations \( [2, 3] \) of \( Y \) with the property \( \varphi(\alpha, \beta, \gamma) = \alpha v_1 + \beta v_2 + \gamma v_3 \) allows us to identify the ‘physical’ space \( \mathbb{E}_3 \) with \( E \).

The matrices of the orthogonal projectors \( \pi, \pi^\perp : \mathbb{E}_6 \rightarrow \mathbb{E}_6 \) corresponding to \( E \) and \( E^\perp = \{ x \in \mathbb{E}_6 \mid \langle x, y \rangle = 0 \text{ for all } y \in E \} \) in the basis \( \{ \varepsilon_1, ..., \varepsilon_6 \} \) are

\[
\pi = \mathcal{M}(1/2, \sqrt{5}/10) \quad \pi^\perp = \mathcal{M}(1/2, -\sqrt{5}/10)
\]

(45)

where

\[
\mathcal{M}(\xi, \beta) = \begin{pmatrix}
\alpha & \beta & \beta & \beta & \beta & \beta \\
\beta & \alpha & \beta & -\beta & -\beta & \beta \\
\beta & \beta & \alpha & \beta & -\beta & -\beta \\
\beta & -\beta & \beta & \alpha & \beta & -\beta \\
\beta & -\beta & -\beta & \beta & \alpha & \beta \\
\beta & \beta & -\beta & -\beta & \beta & \alpha \\
\end{pmatrix}.
\]

(46)

They can be obtained one from the other by using the transformation \( \sqrt{5} \mapsto -\sqrt{5} \).

Let \( \kappa = 1/\varrho \), \( \mathcal{L} = \kappa \mathbb{Z}^6 \), \( \mathbb{K} = [0, \kappa]^6 \), and let \( K = \pi^\perp(\mathbb{K}) \). The pattern defined in terms of the strip projection method

\[
\mathcal{Q} = \{ \pi x \mid x \in \mathcal{L}, \pi' x \in K \}
\]

(47)

is the set of vertices of a 3D Penrose (singular) pattern \( [3] \). It can be regarded as a quasiperiodic packing of interpenetrating icosahedra. Only a very small part of the icosahedra occurring in this pattern are fully occupied \( [4] \).
Quasiperiodic packings of $G$-clusters and Baake-Moody sets

In this case, $E'' = \{0\}, \mathcal{E} = \mathbb{E}_6, \mathcal{L} = \mathbb{L}$,

$$
\pi x \leftarrow x : E \leftarrow \mathbb{E}_6 \xrightarrow{\pi'} \mathbb{E}_6' : x \rightarrow \pi' x
$$

is a cut and project scheme, and $\mathcal{Q}$ is a model set [7], that is a particular case of Baake-Moody set.

6. A quasiperiodic packing of dodecahedra

The points of the one-shell $Y$-cluster

$$
\mathcal{C} = Y(1,1,1) = \{e_1, e_2, ..., e_{10}, -e_1, -e_2, ..., -e_{10}\}
$$

where

$$
e_1 = (1, 1, 1) \quad e_4 = (1 - \tau, 0, \tau) \quad e_7 = (\tau, \tau - 1, 0)$$
$$e_2 = (0, \tau, \tau - 1) \quad e_5 = (\tau - 1, 0, \tau) \quad e_8 = (0, \tau, 1 - \tau)$$
$$e_3 = (-1, 1, 1) \quad e_6 = (1, -1, 1) \quad e_9 = (-\tau, \tau - 1, 0)$$
$$e_{10} = (-1, -1, 1)
$$

are the vertices of a regular dodecahedron. By using the method from the previous section and the canonical basis $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_{10}\}$ of $\mathbb{E}_{10}$ we define the permutation representation

$$
a = \begin{pmatrix}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 & \varepsilon_9 & \varepsilon_{10} \\
\varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 & \varepsilon_9 & \varepsilon_{10} & \varepsilon_6
\end{pmatrix}
$$
$$
b = \begin{pmatrix}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 & \varepsilon_9 & \varepsilon_{10} \\
\varepsilon_1 & -\varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_8 & \varepsilon_9 & \varepsilon_{10} & -\varepsilon_2
\end{pmatrix}
$$

of $Y$ in $\mathbb{E}_{10}$.

The vectors

$$
v_1 = \varrho(1, 0, -1, 1 - \tau, \tau - 1, 1, \tau, 0, -\tau, -1)
$$
$$v_2 = \varrho(1, \tau, 1, 0, 0, -1, \tau - 1, \tau, \tau - 1, -1)
$$
$$v_3 = \varrho(1, \tau - 1, 1, \tau, \tau, 1, 0, 1 - \tau, 0, 1)
$$

where $\varrho = 1/\sqrt{10}$, form an orthonormal basis of the $Y$-invariant subspace

$$
E = \{(r, e_1), (r, e_2), ..., (r, e_{10}) \mid r \in \mathbb{E}_3\}
$$

and the isometry

$$
\varphi : \mathbb{E}_3 \longrightarrow E : r \mapsto (\varrho(r, e_1), \varrho(r, e_2), ..., \varrho(r, e_{10}))
$$

which is an isomorphism of representations [3] of $Y$ with the property $\varphi(\alpha, \beta, \gamma) = \alpha v_1 + \beta v_2 + \gamma v_3$ allows us to identify the ‘physical’ space $\mathbb{E}_3$ with $E$.

The projectors corresponding to $E$ and $E^\perp = \{x \in \mathbb{E}_{10} \mid \langle x, y \rangle = 0 \text{ for all } y \in E\}$ are

$$
\pi = \mathcal{M} \begin{pmatrix} 3/10 & \sqrt{5}/10 & 1/10 \end{pmatrix} \quad \pi^\perp = \mathcal{M} \begin{pmatrix} 7/10 & -\sqrt{5}/10 & -1/10 \end{pmatrix}
$$
where

\[ M(\alpha, \beta, \gamma) = \begin{pmatrix}
\alpha & \beta & \gamma & \beta & \gamma & -\gamma & -\gamma \\
\beta & \alpha & \beta & \gamma & -\gamma & \gamma & -\gamma \\
\gamma & \beta & \alpha & \beta & -\gamma & -\gamma & \gamma \\
\gamma & \gamma & \beta & \alpha & -\gamma & -\gamma & -\gamma \\
\beta & \gamma & \gamma & \beta & \alpha & -\gamma & -\gamma \\
\gamma & -\gamma & -\gamma & -\gamma & -\gamma & -\gamma & -\gamma \\
\beta & \gamma & \gamma & -\gamma & -\gamma & -\gamma & -\gamma \\
-\gamma & -\gamma & \gamma & -\gamma & -\gamma & -\gamma & -\gamma \\
-\gamma & -\gamma & -\gamma & -\gamma & -\gamma & -\gamma & -\gamma \\
\end{pmatrix}. \quad (56) \]

Let \( \kappa = 1/\varrho, \ L = \kappa \mathbb{Z}^4, \ K = [0, \kappa]^4, \) and \( K = \pi^+(\mathbb{K}). \) Following the analogy with the Penrose case we define the icosahedral pattern

\[ Q = \{ \pi x \mid x \in L, \ \pi^+ x \in K \}. \quad (57) \]

Since the arithmetic neighbours of a point \( \pi x \in Q \) are distributed on the vertices of the regular dodecahedron \( \pi x + \mathcal{C}, \) the pattern \( Q \) can be regarded as a quasiperiodic packing of interpenetrating dodecahedra. Evidently, only a very small part of the dodecahedra occurring in this pattern can be fully occupied.

The pattern \( Q \) can be re-defined as a Baake-Moody set by using the decomposition \( E^\perp = E' \oplus E'' \), where \( E' \) and \( E'' \) are the subspaces corresponding to the orthogonal projectors

\[ \pi' = M \left( \frac{3}{10}, -\frac{\sqrt{5}}{10}, \frac{1}{10} \right), \quad \pi'' = M \left( \frac{2}{5}, 0, -\frac{1}{5} \right). \quad (58) \]

The projectors \( \pi \) and \( \pi' \) can be obtained one from the other by using the transformation \( \sqrt{5} \mapsto -\sqrt{5} \). The superspace \( E = E \oplus E' \) is six-dimensional, but in order to use it we have to determine the 'atomic surfaces' \( K_i \).

7. Concluding remarks

If in the previous example we replace the starting cluster \( Y(1,1,1) \) by the icosidodecahedron \( Y(1,0,0) \) then we get a quasiperiodic packing of icosidodecahedra. If we replace it by the two-shell \( Y \)-cluster

\[ \mathcal{C} = Y\{\alpha(1, \tau, 0), \beta(1,1,1)\} = Y(\alpha, \alpha \tau, 0) \cup Y(\beta, \beta, \beta) = \{e_1, ..., e_{16}, -e_1, ..., -e_{16}\} \]

where \( \alpha, \beta \) are rational positive numbers, then we get a pattern \( Q \) such that the arithmetic neighbours of each point \( \pi x \in Q \) are distributed on two shells, namely, on the vertices of a regular icosahedron of radius \( \alpha \sqrt{\tau + 2} \) and on the vertices of a regular dodecahedron of radius \( \beta \sqrt{3} \). The pattern \( Q \) can be regarded as a quasiperiodic packing of interpenetrating copies of \( \mathcal{C} \). Only a very small part of the copies of the cluster \( \mathcal{C} \) occurring in \( Q \) can be fully occupied. We think that the frequency of occurrence of
the fully occupied icosahedra may be much greater in this pattern than in the 3D Penrose pattern. The pattern $Q$ can be re-defined as a Baake-Moody set by using a 6D superspace, but we have to determine some rather complicated ‘atomic surfaces’ $K_i$.

From our general theory it follows that the permutation representation defined in $E_k$ by any $Y$-cluster $C$ contains the irreducible representation $\Gamma_2$, and the corresponding $Y$-invariant subspace $E$ can be determined explicitly. If we change the sign of $\sqrt{5}$ in the expression of the orthogonal projector $\pi$ corresponding to $E$ we get the orthogonal projector $\pi'$ corresponding to another 3D $Y$-invariant subspace $E'$. The subduced representation of $Y$ in $E'$ belongs to $\Gamma_3$, and the orthogonal projector $\pi + \pi'$ corresponding to $E = E \oplus E'$ has rational entries. A quasiperiodic packing $Q$ of copies of $C$ can be defined in a natural way in terms of the strip projection method by using the decomposition $E_k = E \oplus E^\perp$. The same pattern can be re-defined as a Baake-Moody set in the 6D superspace $E = E \oplus E'$ by using some rather complicated ‘atomic surfaces’ $K_i$.

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