Approximations of the Ruin Probability in a Discrete Time Risk Model

David J. Santana\textsuperscript{1} and Luis Rincón\textsuperscript{2}

\textsuperscript{1}División Académica de Ciencias Básicas, UJAT, México
\textsuperscript{2}Departamento de Matemáticas, Facultad de Ciencias, UNAM, México

June 3, 2020

Abstract

Based on a discrete version of the Pollaczek-Khinchine formula, a general method to calculate the ultimate ruin probability in the Gerber-Dickson risk model is provided when claims follow a negative binomial mixture distribution. The result is then extended for claims with a mixed Poisson distribution. The formula obtained allows for some approximation procedures. Several examples are provided along with the numerical evidence of the accuracy of the approximations.

1 Introduction

Several models have been proposed for a discrete time risk process \( \{U(t) : t = 0, 1, \ldots \} \). The following model is known as a compound binomial process and was first considered in \cite{6},

\[
U(t) = u + t - \sum_{i=1}^{N(t)} X_i, \tag{1}
\]

where \( U(0) = u \geq 0 \) is an integer representing the initial capital and the counting process \( \{N(t) : t = 0, 1, \ldots \} \) has a Binomial\((t, p)\) distribution, where \( p \) stands for the probability of a claim in each period. The discrete random variables \( X_1, X_2, \ldots \) are i.i.d. with probability function \( f_X(x) = P(X_i = x) \) for \( x = 1, 2, \ldots \) and mean \( \mu_X \) such that \( \mu_X \cdot p < 1 \). This restriction comes from the net profit condition. Each \( X_i \) represents the total amount of claims in the \( i \)-th period where claims existed. In each period, one unit of currency from

\textsuperscript{1}We will reserve the use of letter \( n \) for the approximation procedures proposed later on.
premiums is gained. The top-left plot of Figure 1 shows a realization of this risk process. The ultimate ruin time is defined as

$$\tau = \min \{t \geq 1 : U(t) \leq 0\},$$

as long as the indicated set is not empty, otherwise $\tau := \infty$. Hence, the probability of ultimate ruin is

$$\psi(u) = P(\tau < \infty | U(0) = u).$$

One central problem in the theory of ruin is to find $\psi(u)$. For the above model this probability can be calculated using the following relation known as Gerber’s formula [6],

$$\psi(0) = p \cdot \mu_X,$$

$$\psi(u) = (1 - p)\psi(u + 1) + p \sum_{x=1}^{u} \psi(u + 1 - x) f_X(x) + p \overline{F}_X(u),$$

for $u = 1, 2, \ldots$ where $\overline{F}_X(u) = P(X_i > u) = \sum_{x=u+1}^{\infty} f_X(x)$.

An apparently simpler risk model is defined as follows.

**Definition 1.1.** Let $u \geq 0$ be an integer and let $Y_1, Y_2, \ldots$ be i.i.d. random variables taking values in $\{0, 1, \ldots\}$. The Gerber-Dickson risk process $\{U(t) : t = 0, 1, \ldots\}$ is given by

$$U(t) = u + t - \sum_{i=1}^{t} Y_i.$$  \hspace{1cm} (4)

In this case, at each unit of time there is always a claim of size $Y$. If $\mu_Y$ denotes the expectation of this claim, the net profit condition now reads $\mu_Y < 1$. It can be shown [3, pp. 467] that this condition implies that $\psi(u) < 1$, where the time of ruin $\tau$ and the ultimate ruin probability $\psi(u)$ are defined as before. Under a conditioning argument it is easy to show that the probability of ruin satisfies the recursive relation

$$\psi(0) = \mu_Y,$$

$$\psi(u) = \sum_{y=0}^{u} f_Y(y) \psi(u + 1 - y) + \overline{F}_Y(u), \quad u \geq 1.$$ \hspace{1cm} (6)

Now, given a compound binomial model (1) we can construct a Gerber-Dickson model (4) as follows. Let $R_1, R_2, \ldots$ be i.i.d. Bernoulli($p$) random variables and define $Y_i = R_i \cdot X_i, i \geq 1$. The distribution of these claims is $f_Y(0) = 1 - p$ and $f_Y(y) = p \cdot f_X(y)$ for $y \geq 1$.

Conversely, given model (4) and defining $p = 1 - f_Y(0)$, we can construct a model (1) by letting claims $X_i$ have distribution $f_X(x) = f_Y(x)/p$, for $x \geq 1$. It can be readily checked that $\mu_Y = p \cdot \mu_X$ and that the probability generating function of $U(t)$ in both models coincide.
This shows models (1) and (4) are equivalent in the sense that \( U(t) \) has the same distribution in both models. As expected, the recursive relations (3) and (6) can be easily obtained one from the other.

In this work we will use the notation in the Gerber-Dickson risk model (4) and drop the subindex in the distribution of claims. Also, as time and other auxiliary variables are considered discrete, we will write, for example, \( t \geq 0 \) instead of \( t = 0, 1, \ldots \) Our main objective is to provide some methods to approximate the ultimate ruin probability in the discrete risk model of Gerber and Dickson.

A survey of results and models for discrete time risk models can be found in [11].

2 The Pollaczeck–Khinchine formula

The continuous version of this formula plays a major role in the theory of ruin for the Cramér-Lundberg model. On the contrary, the discrete version is seldom mentioned in the literature on discrete time risk models. In this section we develop this formula and apply it later to find a general method to calculate ultimate ruin probabilities for claims with particular distributions. The construction procedure resembles closely that for the continuous case.

Assuming \( \tau < \infty \), the non-negative random variable \( W = |U(\tau)| \) is known as the severity of ruin. It indicates how large the capital drops below zero at the time of ruin. See the top-right plot of Figure 1. The joint probability of ruin and severity not greater than \( w = 0, 1, \ldots \) is
denoted by
\[ \varphi(u, w) = P(\tau < \infty, W \leq w \mid U(0) = u). \] (7)

In [4] it is shown that, in particular,
\[ \varphi(0, w) = \sum_{x=0}^{w} F(x), \quad w \geq 0. \] (8)

Hence,
\[ P(\tau < \infty, W = w \mid U(0) = 0) = \varphi(0, w) - \varphi(0, w - 1) = F(w). \] (9)

This probability will be useful in finding the distribution of the size of the first drop of the risk process below its initial capital \( u \), see Proposition 2.3 below, which will ultimately lead us to the Pollaczek–Khinchine formula. For every claim distribution, there is an associated distribution which often appears in the calculation of ruin probabilities. This is defined next.

**Definition 2.1.** Let \( F(y) \) be the distribution function of a discrete random variable with values 0, 1, \ldots and with finite mean \( \mu \neq 0 \). Its equilibrium probability function is defined by
\[ f_e(y) = \frac{F(y)}{\mu}, \quad y \geq 0. \] (10)

The probability function defined by (10) is also known as the integrated-tail distribution, although this name is best suited to continuous distributions. For example, the equilibrium distribution associated to a Geometric(\( p \)) claim distribution with mean \( \mu = 1/(1-p) \) is the same geometric since
\[ f_e(y) = \frac{F(y)}{\mu} = (1-p)^{y+1} p/(1-p) = p \cdot (1-p)^y, \quad y \geq 0. \] (11)

As in the continuous time risk models, let us define the surplus process \( \{Z(t) : t \geq 0\} \) by
\[ Z(t) = u - U(t) = \sum_{i=1}^{t} (Y_i - 1). \] (12)

This is a random walk that starts at zero, it has stationary and independent increments and \( Z(t) \to -\infty \) a.s. as \( t \to \infty \) under the net profit condition \( \mu < 1 \). See bottom-right plot of Figure 1. In terms of this surplus process, ruin occurs when \( Z(t) \) reaches level \( u \) or above. Thus, the ruin probability can be written as
\[ \psi(u) = P(Z(t) \geq u \text{ for some } t \geq 1) = P\left(\max_{t \geq 1} \{Z(t)\} \geq u\right), \quad u \geq 1. \] (13)

As \( u \geq 1 \) and \( Z(0) = 0 \), we can also write
\[ \psi(u) = P\left(\max_{t \geq 0} \{Z(t)\} \geq u\right). \] (14)

We next define the times of records and the severities for the surplus process.
Definition 2.2. Let \( \tau^*_0 := 0 \). For \( i \geq 1 \) the \( i \)-th record time of the surplus process is defined as

\[
\tau^*_i = \min \{ t > \tau^*_{i-1} : Z(t) \geq Z(\tau^*_{i-1}) \},
\]

(15)

when the indicated set is not empty, otherwise \( \tau^*_i := \infty \). The non-negative variable \( Y^*_i = Z(\tau^*_i) - Z(\tau^*_{i-1}) \) is called the severity or size of the \( i \)-th record time, assuming \( \tau^*_i < \infty \).

The random variables \( \tau^*_0 < \tau^*_1 < \cdots \) represent the stopping times when the surplus process \( \{Z(t) : t \geq 0\} \) arrives at a new or the previous maximum, and the severity \( Y^*_i \) is the difference between the maxima at \( \tau^*_i \) and \( \tau^*_{i-1} \). A graphical example of these record times are shown in the bottom-right plot of Figure 1. In particular, observe \( \tau^*_1 \) is the first positive time the risk process is less than or equal to its initial capital \( u \), that is,

\[
\tau^*_1 = \min \{ t > 0 : u - U(t) \geq 0 \},
\]

(16)

and the severity is \( Y^*_1 = Z(\tau^*_1) - u - U(\tau^*_1) \) and this is the size of this first drop below level \( u \). Also, since the surplus process has stationary increments, all severities share the same distribution, that is,

\[
Y^*_i = Z(\tau^*_i) - Z(\tau^*_{i-1}) \sim Z(\tau^*_i) - Z(0) = Y^*_1, \quad i \geq 1,
\]

(17)

assuming \( \tau^*_i < \infty \). We will next find out that distribution.

Proposition 2.3. Let \( k \geq 1 \). Conditioned on the event \( \tau^*_k < \infty \), the severities \( Y^*_1, \ldots, Y^*_k \) are independent and identically distributed according to the equilibrium distribution

\[
P(Y^* = x \mid \tau^*_k < \infty) = F(x)/\mu, \quad x \geq 0.
\]

(18)

Proof. By (17), it is enough to find the distribution of \( Y^*_1 \). Observe that \( \tau^*_1 = \tau \) when \( U(0) = 0 \). By (10) and (11), for \( x \geq 0 \),

\[
P(Y^*_1 = x \mid \tau^*_1 < \infty) = P(\{u - U(\tau^*_1) = x \mid \tau^*_1 < \infty\})
\]

\[
= P(|U(\tau)| = x \mid \tau < \infty, U(0) = 0)
\]

\[
= P(\tau < \infty, Y = x \mid U(0) = 0)/P(\tau < \infty \mid U(0) = 0)
\]

\[
= \frac{F(x)}{\mu}.
\]

The independence property follows from the independence of the claims. Indeed, the severity of the \( i \)-th record time is

\[
Y^*_i = Z(\tau^*_i) - Z(\tau^*_{i-1}) = \sum_{j=\tau^*_{i-1}+1}^{\tau^*_i} \left( Y^*_j - 1 \right), \quad i \geq 1.
\]

Therefore,

\[
P \left( \bigcap_{i=1}^k (Y^*_i = y_i) \right) = P \left( \bigcap_{i=1}^k \left( \sum_{j=\tau^*_{i-1}+1}^{\tau^*_i} \left( Y^*_j - 1 \right) = y_i \right) \right) = \prod_{i=1}^k P(Y^*_i = y_i). \]
Since the surplus process is a Markov process, the following properties hold: For \( i \geq 2 \) and assuming \( \tau_i^* < \infty \), for \( 0 < s < x \),

\[
P(\tau_i^* = x \mid \tau_{i-1}^* = s) = P(\tau_i^* - \tau_{i-1}^* = x - s \mid \tau_{i-1}^* = s) = P(\tau_1^* = x - s).
\] (19)

Also, for \( k \geq 1 \),

\[
P(\tau_k^* < \infty \mid \tau_{k-1}^* < \infty) = P(\tau_1^* < \infty),
\] (20)

\[
P(\tau_k^* = \infty \mid \tau_{k-1}^* < \infty) = P(\tau_1^* = \infty).
\] (21)

The total number of records of the surplus process \( \{Z(t) : t \geq 0\} \) is defined by the non-negative random variable

\[
K = \max \{k \geq 1 : \tau_k^* < \infty\},
\] (22)

when the indicated set is not empty, otherwise \( K := 0 \). Note that \( 0 \leq K < \infty \) a.s. since \( Z(t) \to -\infty \) a.s. under the net profit condition. The distribution of this random variable is established next.

**Proposition 2.4.** The number of records \( K \) has a Geometric\((1 - \mu)\) distribution, that is,

\[
f_K(k) = (1 - \mu)\mu^k, \quad k \geq 0.
\] (23)

**Proof.** The case \( k = 0 \) can be related to the ruin probability with \( u = 0 \) as follows,

\[
f_K(0) = P(\tau_1^* = \infty) = P(\tau = \infty \mid U(0) = 0) = 1 - \psi(0) = 1 - \mu.
\]

Hence, \( P(K > 0) = \psi(0) = \mu \). Let us see the case \( k = 1 \),

\[
f_K(1) = P(\tau_1^* < \infty, \tau_2^* = \infty) = P(\tau_2^* = \infty \mid \tau_1^* < \infty)P(\tau_1^* < \infty).
\]

By (20),

\[
f_K(1) = P(\tau_1^* = \infty)P(\tau_1^* < \infty) = P(K > 0)f_K(0) = \mu(1 - \mu).
\]

Now consider the case \( k \geq 2 \) and let \( A_k = (\tau_k^* < \infty) \). Conditioning on \( A_{k-1} \) and its complement,

\[
P(A_k) = P(\tau_k^* < \infty \mid A_{k-1})P(A_{k-1})
\]

\[
= P(\tau_k^* < \infty \mid \tau_{k-1}^* < \infty)P(A_{k-1})
\]

\[
= P(\tau_1^* < \infty)P(A_{k-1})
\]

\[
= \psi(0)P(A_{k-1}).
\]

An iterative argument shows that \( P(A_k) = (\psi(0))^k \), \( k \geq 2 \). Therefore,

\[
f_K(k) = P(\tau_{k+1}^* = \infty, A_k) = P(\tau_{k+1}^* = \infty \mid A_k)P(A_k) = P(\tau_1^* = \infty)(\psi(0))^k = (1 - \mu)\mu^k.
\]

In the following proposition it is established that the ultimate maximum of the surplus process has a compound geometric distribution. This will allow us to write the ruin probability as the tail of this distribution.
Proposition 2.5. For a surplus process \( \{ Z(t) : t \geq 0 \} \) with total number of records \( K \geq 0 \) and record severities \( Y_1^*, Y_2^*, \ldots, Y_K^* \),

\[
\max_{t \geq 0} \{ Z(t) \} = \sum_{i=1}^{K} Y_i^*.
\]

Hence,

\[
\psi(u) = P \left( \sum_{i=1}^{K} Y_i^* \geq u \right), \quad u \geq 1.
\]

Proof.

\[
\sum_{i=1}^{K} Y_i^* = \sum_{i=1}^{K} \left( Z(\tau_i^*) - Z(\tau_{i-1}^*) \right) = Z(\tau_K^*) = \max_{t \geq 0} \{ Z(t) \} \quad \text{a.s.}
\]

Thus, for \( u \geq 1 \),

\[
\psi(u) = P \left( \max_{t \geq 0} \{ Z(t) \} \geq u \right) = P \left( \sum_{i=1}^{K} Y_i^* \geq u \right).
\]

Proposition 2.6. (Pollaczek–Khinchine formula, discrete version) The probability of ruin for a Gerber-Dickson risk process can be written as

\[
\psi(u) = (1 - \mu) \sum_{k=1}^{\infty} P(S_k^* \geq u) \mu^k, \quad u \geq 0,
\]

where \( S_k^* = \sum_{i=1}^{k} Y_i^* \).

Proof. For \( u = 0 \), the sum in (27) reduces to \( \mu \) which we know is \( \psi(0) \). For \( u \geq 1 \), by (23) and (25),

\[
\psi(u) = P \left( \sum_{i=1}^{K} Y_i^* \geq u \right) = \sum_{k=0}^{\infty} P \left( \sum_{i=1}^{K} Y_i^* \geq u \mid K = k \right) f_K(k) = (1 - \mu) \sum_{k=1}^{\infty} P(S_k^* \geq u) \mu^k.
\]

For example, suppose claims have a Geometric(\( p \)) distribution with mean \( \mu = (1 - p)/p \). The net profit condition \( \mu < 1 \) implies \( p > 1/2 \). We have seen that the associated equilibrium distribution is again Geometric(\( p \)), and hence the \( k \)-th convolution is Negative Binomial(\( k, p \)), \( k \geq 0 \). Straightforward calculations show that the Pollaczek–Khinchine formula gives the known solution for the probability of ruin,

\[
\psi(u) = \left( \frac{1 - p}{p} \right)^{u+1}, \quad u \geq 0.
\]

This includes in the same formula the case \( u = 0 \). In the following section we will consider claims that have a mixture of some distributions.
3 Negative binomial mixture

Negative binomial mixture (NBM) distributions will be used to approximate the ruin probability when claims have a mixed Poisson (MP) distribution. Although NBM distributions are the analogue of Erlang mixture distributions, they cannot be used to approximate any discrete distribution with non-negative support. However, it turns out that they can approximate mixed Poisson distributions. This is stated in [16, Theorem 1], where the authors define NBM distributions those with probability generating function

\[ G(z) = \lim_{m \to \infty} \sum_{k=1}^{m} q_{k,m} \left( \frac{1 - p_{k,m}}{1 - p_{k,m} z} \right)^{r_{k,m}}, \quad z < 1, \]

where \( q_{k,m} \) are positive numbers and sum 1 over index \( k \). This is a rather general definition for a NBM distribution. In this work we will consider a particular case of it.

We will denote by \( nb(k,p)(x) \) the probability function of a negative binomial distribution with parameters \( k \) and \( p \), and by \( NB(k,p)(x) \) its distribution function, namely, for \( x \geq 0 \),

\[ nb(k,p)(x) = \binom{k + x - 1}{x} p^k (1 - p)^x, \quad \text{and} \quad NB(k,p)(x) = 1 - \sum_{i=0}^{k-1} nb(x + 1, 1 - p)(i). \]

**Definition 3.1.** Let \( q_1, q_2, \ldots \) be a sequence of numbers such that \( q_k \geq 0 \) and \( \sum_{k=1}^{\infty} q_k = 1 \). A negative binomial mixture distribution with parameters \( \pi = (q_1, q_2, \ldots) \) and \( p \in (0,1) \), denoted by \( \text{NBM}(\pi, p) \), is a discrete distribution with probability function

\[ f(x) = \sum_{k=1}^{\infty} q_k \cdot nb(k,p)(x), \quad x \geq 0. \]

It is useful to observe that any NBM distribution can be written as a compound sum of geometric random variables. Indeed, let \( N \) be a discrete random variable with probability function \( q_k = f_N(k), k \geq 1 \), and define \( S_N = \sum_{i=1}^{N} X_i \), where \( X_1, X_2, \ldots \) are i.i.d. r.v.s Geometric\((p)\) distributed and independent of \( N \). Then

\[ \sum_{k=1}^{\infty} q_k \cdot nb(k,p)(x) = \sum_{k=1}^{\infty} q_k \cdot P \left( \sum_{i=1}^{k} X_i = x \right) = P(S_N = x), \quad x \geq 0. \]

Thus, given any \( \text{NBM}(\pi, p) \) distribution with \( \pi = (f_N(1), f_N(2), \ldots) \), we have the representation

\[ S_N = \sum_{i=1}^{N} X_i \sim \text{NBM}(\pi, p). \quad (29) \]
In particular,
\[
E(S_N) = E(N) \left( \frac{1-p}{p} \right),
\]
(30)
\[
F_{SN}(x) = \sum_{k=1}^{\infty} f_N(k) \cdot NB(k, p)(x), \quad x \geq 0,
\]
(31)
and the p.g.f. has the form \( G_{SN}(r) = G_N(G_X(r)) \). The following is a particular way to write the distribution function of a NBM distribution.

**Proposition 3.2.** Let \( S_N \sim \text{NBM}(\pi, p) \), where \( \pi = (f_N(1), f_N(2), \ldots) \) for some discrete r.v. \( N \). For each \( x \geq 0 \), let \( Z \sim \text{NegBin}(x + 1, 1 - p) \). Then
\[
F_{SN}(x) = E(F_N(Z)), \quad x \geq 0.
\]
(32)

**Proof.**
\[
F_{SN}(x) = \sum_{k=1}^{\infty} f_N(k) \cdot NB(k, p)(x)
\]
\[
= \sum_{k=1}^{\infty} f_N(k) \left[ 1 - \sum_{i=0}^{k-1} nb(x + 1, 1 - p)(i) \right]
\]
\[
= \sum_{i=0}^{\infty} \left[ \sum_{k=1}^{i} f_N(k) \right] nb(x + 1, 1 - p)(i)
\]
\[
= E(F_N(Z)).
\]

We will show next that the equilibrium distribution associated to a NBM distribution is again NBM. For a distribution function \( F(x) \), \( \overline{F}(x) \) denotes \( 1 - F(x) \).

**Proposition 3.3.** Let \( S_N \sim \text{NBM}(\pi, p) \), with \( \pi = (f_N(1), f_N(2), \ldots) \) and \( E(N) < \infty \). The equilibrium distribution of \( S_N \) is \( \text{NBM}(\pi_e, p) \), where \( \pi_e = (f_{Ne}(1), f_{Ne}(2), \ldots) \) and
\[
f_{Ne}(j) = \frac{\overline{F}_N(j - 1)}{E(N)}, \quad j \geq 0.
\]
(33)

**Proof.**
\[
f_e(x) = \frac{\overline{F}_{SN}(x)}{E(S_N)} = \frac{p \sum_{i=0}^{\infty} F_N(i) \binom{x+i}{i} p^i (1-p)^{x+1}}{(1-p)E(N)} = \sum_{i=0}^{\infty} \frac{F_N(i)}{E(N)} \binom{x+i}{i} p^{i+1}(1-p)^x.
\]

Naming \( j = i + 1 \),
\[
f_e(x) = \sum_{j=1}^{\infty} \frac{\overline{F}_N(j - 1)}{E(N)} \binom{j+x-1}{x} p^j (1-p)^x = \sum_{j=1}^{\infty} f_{Ne}(j) \cdot nb(j, p)(x).
\]
It can be checked that (33) is a probability function. It is the equilibrium distribution associated to \( N \). In what follows, a truncated geometric distribution will be used. This is denoted by TGeometric(\( \rho \)), where \( 0 < \rho < 1 \), and defined by the probability function \( f(k) = \rho (1 - \rho)^{k-1} \), for \( k \geq 1 \).

The following proposition states that a compound geometric NBM distribution is again NBM. This result is essential to calculate the ruin probability when claims have NBM distribution.

**Proposition 3.4.** Let \( M \sim \text{TGeometric}(\rho) \) and let \( N_1, N_2, \ldots \) be a sequence of independent random variables with identical distribution \( \pi = (f_N(1), f_N(2), \ldots) \). Let \( S_{N_1}, S_{N_2}, \ldots \) be random variables with NBM(\( \pi, p \)) distribution. Then

\[
S := \sum_{j=1}^{M} S_{N_j} \sim \text{NBM}(\pi^*, p),
\]

where \( \pi^* = (f_{N^*}(1), f_{N^*}(2), \ldots) \) is the distribution of \( N^* = \sum_{j=1}^{M} N_j \) and is given by

\[
\begin{align*}
  f_{N^*}(1) &= \rho f_N(1), \\
  f_{N^*}(k) &= (1 - \rho) \sum_{i=1}^{k-1} f_N(i) f_{N^*}(k-i) + \rho f_N(k), \quad k \geq 2.
\end{align*}
\]

**Proof.** For \( x \geq 1 \) and \( m \geq 1 \),

\[
P(S = x \mid M = m) = P\left( \sum_{j=1}^{m} S_{N_j} = x \right) = P\left( \sum_{j=1}^{m} \sum_{i=1}^{N_j} X_{i,j} = x \right) = P\left( \sum_{\ell=1}^{N_m} X_{\ell} = x \right),
\]

where \( N_m = \sum_{i=1}^{m} N_i \) and \( X_{\ell} \sim \text{Geometric}(p) \) for \( \ell \geq 1 \). Therefore,

\[
P(S = x) = \sum_{m=1}^{\infty} P(S = x \mid M = m) f_M(m) = \sum_{m=1}^{\infty} P\left( \sum_{\ell=1}^{N_m} X_{\ell} = x \right) f_M(m) = P\left( \sum_{\ell=1}^{N^*} X_{\ell} = x \right),
\]

where \( N^* = \sum_{j=1}^{M} N_j \). Using Panjer’s formula it can be shown that \( N^* \) has distribution \( \pi^* \) given by (35) and (36). Since \( X_{\ell} \sim \text{Geometric}(p) \), \( \sum_{\ell=1}^{N^*} X_{\ell} \sim \text{NBM}(\pi^*, p) \). Lastly, we consider the probability of the event \( (S = 0) \).

\[
P(S = 0) = \sum_{k=1}^{\infty} f_{N^*}(k) \text{nb}(k, p)(0) = \sum_{k=1}^{\infty} f_{N^*}(k) p^{k} = f_{N^*}(1) p + \sum_{k=2}^{\infty} f_{N^*}(k) p^{k}.
\]

Substituting \( f_{N^*}(k) \) from (35) and (36), one obtains

\[
P(S = 0) = \rho G_N(p) + (1 - \rho) G_N(p) P(S = 0).
\]
Therefore,

\[ P(S = 0) = \frac{\rho G_N(p)}{1 - (1 - \rho)G_N(p)} = G_M(G_N(p)) = G_M(G_N(G_{X,j}(0))). \tag{38} \]

The last term is the p.g.f. of a \( \text{NBM}(\pi^*, p) \) distribution evaluated at zero. \( \square \)

From (35) and (36), it is not difficult to derive a recursive formula for \( F_{N^*}(k) \), namely,

\[ F_{N^*}(k) = (1 - \rho) \sum_{j=1}^{k} f_{N}(j) F_{N^*}(k - j) + F_{N}(k), \quad k \geq 1. \tag{39} \]

The following result establishes a formula to calculate the ruin probability when claims have a \( \text{NBM}(\pi, p) \) distribution.

**Theorem 3.5.** Consider the Gerber-Dickson model with claims having a \( \text{NBM}(\pi, p) \) distribution, where \( \pi = (f_N(1), f_N(2), \ldots) \) and \( E(N) < \infty \). For \( u \geq 1 \) define \( Z_u \sim \text{NegBin}(u, 1 - p) \).

Then the ruin probability can be written as

\[ \psi(u) = \sum_{k=0}^{\infty} \overline{C}_k \cdot P(Z_u = k) = E(\overline{C}_Z_u), \quad u \geq 1, \tag{40} \]

where the sequence \( \{\overline{C}_k\}_{k=0}^{\infty} \) is given by

\[ \overline{C}_0 = E(N)(1 - p)/p, \tag{41} \]

\[ \overline{C}_k = \overline{C}_0 \left[ \sum_{i=1}^{k} f_{N^e}(i) \overline{C}_{k-i} + \overline{F}_{N^e}(k) \right], \quad k \geq 1, \tag{42} \]

\[ f_{N^e}(i) = \frac{F_N(i - 1)}{E(N)}, \quad i \geq 1. \tag{43} \]

**Proof.** Let \( R_0 = \sum_{j=1}^{M_0} Y_{e,j} \), where \( M_0 \sim \text{Geometric}(\rho) \) with \( \rho = 1 - \psi(0) \), and let \( Y_{e,1}, Y_{e,2}, \ldots \) be r.v.s distributed according to the equilibrium distribution associated to \( \text{NBM}(\pi, p) \) claims. By Proposition \[ \text{3.3} \] we know this equilibrium distribution is \( \text{NBM}(\pi_e, p) \), where \( \pi_e \) is given by equations (35) and (36). Now define

\[ \overline{C}_k = (1 - \rho)\overline{F}_{N^*}(k), \quad k \geq 0. \tag{44} \]
Therefore, using (32),

\[ \psi(u) = (1 - \rho)P(R > u) = (1 - \rho)E\left(F_{N^*}(Z_u)\right) = \sum_{k=0}^{\infty} \overline{C}_k P(Z_u = k). \]

Finally, we calculate the coefficients \( \overline{C}_k \) where \( \rho = 1 - \psi(0) = 1 - E(N)(1 - p)/p. \) First,

\[ \overline{C}_0 = (1 - \rho)F_{N^*}(0) = 1 - \rho = E(N)(1 - p)/p, \]

and by (39),

\[ \overline{C}_k = (1 - \rho)F_{N^*}(k) = \overline{C}_0 \left[ \sum_{i=1}^{k} f_{N_e}(i)\overline{C}_{k-i} + F_{N_e}(k) \right], \quad k \geq 1. \]

As an example consider claims with a geometric distribution. This is a NBM distribution with \( \pi = (1, 0, 0, \ldots) \). Equations (41–43) yield

\[ \overline{C}_k = \left((1 - p)/p\right)^{k+1}, \quad k \geq 0. \]

Substituting in (40) together with \( \psi(0) = (1 - p)/p, \) we recover the known solution \( \psi(u) = ((1 - p)/p)^{u+1}, \quad u \geq 0. \)

4 Mixed Poisson

This section contains the definition of a mixed Poisson distribution and some of its relations with NBM distributions.

**Definition 4.1.** Let \( X \) and \( \Lambda \) two non-negative random variables. If \( X \mid (\Lambda = \lambda) \sim \text{Poisson}(\lambda) \), then we say that \( X \) has a mixed Poisson distribution with mixing distribution \( F_\Lambda \). In this case, we write \( X \sim MP(F_\Lambda) \).

Observe the distribution of \( X \mid (\Lambda = \lambda) \) is required to be Poisson, but the unconditional distribution of \( X \), although discrete, is not necessarily Poisson. A large number of examples of these distributions can be found in [8] and a study of their general properties is given in [7]. In particular, it is not difficult to see that \( E(X) = E(\Lambda) \) and the p.g.f. of \( X \) can be written as

\[ G_X(r) = \int_0^\infty e^{-\lambda(1-r)}dF_\Lambda(\lambda), \quad r < 1. \]  

(45)

The following proposition establishes a relationship between the Erlang mixture distribution and the negative binomial distribution. The former will be denoted by ErlangM(\( \pi, \beta \)), with similar meaning for the parameters as in the notation NBM(\( \pi, p \)) used before. In the ensuing calculations the probability function of a Poisson(\( \lambda \)) distribution will be denoted by poisson(\( \lambda \))(\( x \)).
Proposition 4.2. Let \( \Lambda \) be a random variable with distribution \( \text{ErlangM}(\pi, \beta) \). The distributions \( \text{MP}(F_\Lambda) \) and \( \text{NBM}(\pi, \beta/(\beta + 1)) \) are the same.

Proof. Let \( X \sim \text{MP}(F_\Lambda) \). For \( x \geq 0 \),
\[
P(X = x) = \int_0^\infty \text{poisson}(\lambda)(x) \cdot \sum_{k=1}^{\infty} q_k \cdot \text{erl}(k, \beta)(\lambda) \, d\lambda
\]
\[
= \sum_{k=1}^{\infty} q_k \cdot \left( \frac{\beta}{\beta + 1} \right)^k \left( \frac{1}{\beta + 1} \right)^x \frac{(k + x - 1)!}{(k - 1)! \, x!}
\]
\[
= \sum_{k=1}^{\infty} q_k \cdot \text{nb}(k, \beta/(\beta + 1))(x).
\]
As a simple example consider the case \( \Lambda \sim \text{Exp}(\beta) \) and \( \pi = (1, 0, 0, \ldots) \). By Proposition 4.2, \( P(X = x) = \text{nb}(1, \beta/(\beta + 1))(x) \) for \( x \geq 0 \). That is, \( X \sim \text{Geometric}(p) \) with \( p = \beta/(\beta + 1) \).

Next proposition will be useful to show that a \( \text{MP} \) distribution can be approximated by \( \text{NBM} \) distributions. Its proof can be found in [7].

Proposition 4.3. Let \( \Lambda_1, \Lambda_2, \ldots \) be positive random variables with distribution functions \( F_1, F_2, \ldots \) and let \( X_1, X_2, \ldots \) be random variables such that \( X_i \sim \text{MP}(F_i), i \geq 1 \). Then \( X_n \xrightarrow{D} X \), if and only if, \( \Lambda_n \xrightarrow{D} \Lambda \), where \( X \sim \text{MP}(F_\Lambda) \).

Finally we establish how to approximate an \( \text{MP} \) distribution.

Proposition 4.4. Let \( X \sim \text{MP}(F_\Lambda) \), and let \( X_n \) be a random variable with distribution \( \text{NBM}(\pi_n, p_n) \) for \( n \geq 1 \), where \( p_n = n/(n + 1) \), \( \pi_n = (q(1, n), q(2, n), \ldots) \) and \( q(k, n) = F_\Lambda(k/n) - F_\Lambda((k - 1)/n) \). Then \( X_n \xrightarrow{D} X \).

Proof. First, suppose that \( F_\Lambda \) is continuous. Let \( \Lambda_1, \Lambda_2, \ldots \) be random variables, where \( \Lambda_n \) has distribution given by the following Erlangs mixture (see [14]),
\[
F_n(x) = \sum_{k=1}^{\infty} q(k, n) \cdot \text{Erl}(k, n)(x), \quad x > 0
\]
with \( q(k, n) = F_\Lambda(k/n) - F_\Lambda((k - 1)/n) \). It is known [14] that
\[
\lim_{n \to \infty} F_n(x) = F_\Lambda(x), \quad x > 0.
\]
Then, by Proposition 4.3, \( X_n \xrightarrow{D} X \), where \( X_n \sim \text{MP}(F_n) \). This is an \( \text{NBM}(\pi, p_n) \) by Proposition 4.2 where \( \pi = (q(1, n), q(2, n), \ldots) \) and \( p_n = n/(n + 1) \).
Now suppose $F_\Lambda$ is discrete. Let $Y_n \sim \text{NegBin}(\lambda_n, n/(n + 1))$, where $\lambda$ and $n$ are positive integers and let $Z \sim \text{Poisson}(\lambda)$. The probability generating functions of these random variables satisfy

$$\lim_{n \to \infty} G_{Y_n}(r) = \lim_{n \to \infty} \left( 1 + \frac{1 - r}{n} \right)^{-\lambda n} = \exp\{-\lambda(1 - r)\} = G_Z(r).$$

Thus,

$$Y_n \overset{D}{\to} Z. \quad (47)$$

On the other hand, suppose that $X$ is a mixed Poisson random variable with probability function $f_X(x)$ for $x \geq 0$ and mixing distribution $F_\Lambda(\lambda)$ for $\lambda \geq 1$. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables with distribution

$$f_n(x) = \sum_{k=1}^{\infty} q(k, n) \cdot \text{nb}\left( k, \frac{n}{n+1} \right)(x), \quad n \geq 1, x \geq 0, \quad (48)$$

where $q(k, n) = F_\Lambda(k/n) - F_\Lambda((k - 1)/n)$. Note that for any natural number $n$, if $k$ is not a multiple of $n$, then $q(k, n) = 0$. Let $k = \lambda n$ with $\lambda \geq 1$. Then $q(k, n) = F_\Lambda(\lambda) - F_\Lambda(\lambda-1/n) = f_\Lambda(\lambda)$. Therefore, for $x \geq 0$,

$$f_{X_n}(x) = \sum_{\lambda=1}^{\infty} q(\lambda n, n) \cdot \text{nb}(\lambda n, n/(n + 1))(x) = \sum_{\lambda=1}^{\infty} f_\Lambda(\lambda) \cdot \text{nb}(\lambda n, n/(n + 1))(x).$$

Therefore,

$$\lim_{n \to \infty} f_{X_n}(x) = \sum_{\lambda=1}^{\infty} f_\Lambda(\lambda) \cdot \lim_{n \to \infty} \text{nb}(\lambda n, n/(n + 1))(x) = \sum_{\lambda=1}^{\infty} f_\Lambda(\lambda) \cdot \text{poisson}(\lambda)(x).$$

For $X_n \sim \text{NBM}(\pi_n, p_n)$ as in the previous statement, it easy to see that

$$E(X_n) < 1. \quad (49)$$

As a consequence of Proposition 4.4, for $X \sim \text{MP}(F_\Lambda)$, its probability function can be approximated by NBM distributions with suitable parameters. That is, for sufficiently large values of $n$,

$$P(X = x) \approx \sum_{k=1}^{\infty} q(k, n) \cdot \text{nb}(k, p_n)(x), \quad (50)$$

where $q(k, n) = F_\Lambda(k/n) - F_\Lambda((k - 1)/n)$ and $p_n = n/(n + 1)$.
5 Ruin probability approximations

We here consider the case when claims in the Gerber-Dickson risk model have distribution function $F \sim \text{MP}(F, \Lambda)$. Let $\psi_n(u)$ denote the ruin probability when claims have distribution $F_n(x)$ as defined in Proposition 4.1. If $n$ is large enough, $F_n(x)$ is close to $F(x)$, and is expected that $\psi_n(u)$ will be close to $\psi(u)$, the unknown ruin probability. This procedure is formalized in the following theorem.

**Theorem 5.1.** If claims in the Gerber-Dickson model have a MP$(F, \Lambda)$ distribution, then

$$\psi(u) = \lim_{n \to \infty} \psi_n(u), \quad u \geq 0,$$

where

$$\psi_n(u) = \sum_{k=0}^{\infty} C_{k,n} P(Z = k) = E(C_{Z,n}), \quad (51)$$

with $Z \sim \text{NegBin}(u, 1/(1+n))$. The sequence $\{C_{k,n}\}_{k=0}^{\infty}$ is determined by

$$C_{0,n} = \sum_{j=0}^{\infty} F_{\lambda}(j/n)/n, \quad (52)$$

$$C_{k,n} = C_{0,n} \left[ \sum_{i=1}^{k} f_{Ne}(i) C_{k-i,n} + F_{Ne}(k) \right], \quad k \geq 1, \quad (53)$$

$$f_{Ne}(i) = \frac{F_{\lambda}((i-1)/n)}{\sum_{j=0}^{\infty} F_{\lambda}(j/n)}, \quad i \geq 1. \quad (54)$$

**Proof.** Suppose $X \sim \text{MP}(F, \Lambda)$ with $E(X) < 1$ and equilibrium probability function $f_e(x)$. Let $X_1, X_2, \ldots$ be an approximating sequence of NBM($\pi_n, p_n$) random variables to $X$, where $\pi_n = (q(1,n), q(2,n), \ldots)$, with $q(k,n) = F_{\lambda}(k/n) - F_{\lambda}((k-1)/n)$ and $p_n = n/(n+1)$. That is,

$$f_{X_n}(x) = \sum_{k=1}^{\infty} q(k,n) \cdot \text{nb}(k,n/(n+1))(x), \quad x \geq 0. \quad (55)$$

By (30),

$$E(X_n) = \sum_{k=1}^{\infty} k q(k,n) \cdot \frac{1/(n+1)}{n/(n+1)} = \sum_{k=1}^{\infty} (k/n) \cdot [F_{\lambda}(k/n) - F_{\lambda}((k-1)/n)].$$

Taking the limit,

$$\lim_{n \to \infty} E(X_n) = \int_0^{\infty} x \, dF_{\lambda}(x) = E(\Lambda) = E(X). \quad (56)$$

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Now, by Proposition 4.4, since $X_n \xrightarrow{D} X$, we have
$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x), \quad x \geq 0.$$ Combining the above with (56),
$$\lim_{n \to \infty} \frac{F_{X_n}(x)}{E(X_n)} = \frac{F_X(x)}{E(X)}.$$
This means the equilibrium probability function $f_{e,n}(x)$ associated to $f_{X_n}(x)$ satisfies
$$\lim_{n \to \infty} f_{e,n}(x) = f_e(x), \quad x \geq 0. \quad (57)$$
Using probability generating functions and (57), it is also easy to show that for any $k \geq 1$,
$$\lim_{n \to \infty} F_{e,k,n}(x) = F_{e,k}(x), \quad x \geq 0. \quad (58)$$
Now, let $X_{n1}, X_{n2}, \ldots$ be i.i.d. random variables with probability function $f_{e,n}(x)$ and set $S_{k,n} := \sum_{i=1}^{k} X_{ni}$. By the Pollaczek-Khinchine formula, for $u \geq 0$,
$$\psi_n(u) = \sum_{k=1}^{\infty} P(S_{k,n} \geq u) (1 - E(X_n)) E^k(X_n) = \sum_{k=1}^{\infty} (1 - F_{e,k}(u - 1)) (1 - E(X_n)) E^k(X_n).$$
Taking the $n \to \infty$ limit and using (56) and (58),
$$\lim_{n \to \infty} \psi_n(u) = \sum_{k=1}^{\infty} (1 - F_{e,k}(u - 1)) (1 - E(X)) E^k(X) = \psi(u), \quad u \geq 1.$$
On the other hand, since claims $X_n$ have a NBM($\pi_n, p_n$) distribution, with $\pi_n = (q(1, n), q(2, n), \ldots)$, $q(k, n) = F\Lambda(k/n) - F\Lambda((k - 1)/n)$ and $p_n = n/(n + 1)$, by Theorem 3.5,
$$\psi_n(u) = \sum_{k=0}^{\infty} C_{k,n} \cdot P(Z = k) = E(C_{Z,n}), \quad u \geq 1,$$
where $Z \sim \text{NegBin}(u, 1/(n + 1))$ and the sequence $\{C_{k,n}\}_{k=0}^{\infty}$ is given by
\[
\begin{align*}
C_{0,n} &= E(N_n)/n, \\
C_{k,n} &= C_{0,n} \left[ \sum_{i=1}^{k} f_{Ne}(i) C_{k-i,n} + F_{Ne}(k) \right], \quad k \geq 1, \\
f_{Ne}(i) &= \frac{F_{Ne}(i - 1)}{E(N_n)}, \quad i \geq 1.
\end{align*}
\]
where $N_n$ is the r.v. related to probabilities $q(k,n)$. Thus, it only remains to calculate the form of $E(N_n)$ and $F_{N_n}(i-1)$.

\[
E(N_n) = \sum_{j=1}^{\infty} P(N_n > j-1) = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} q(i, n) = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (F_{\Lambda}(i/n) - F_{\Lambda}((i-1)/n)) = \sum_{j=0}^{\infty} F_{\Lambda}(j/n).
\]

Thus,

\[
C_{0,n} = \sum_{j=0}^{\infty} F_{\Lambda}(j/n).
\]

Also,

\[
F_{N_n}(i-1) = P(N_n > i - 1) = \sum_{k=i}^{\infty} q(k, n) = \sum_{k=i}^{\infty} (F_{\Lambda}(k/n) - F_{\Lambda}((k-1)/n)) = F_{\Lambda}((i-1)/n).
\]

Then,

\[
f_{Ne}(i) = \frac{F_{\Lambda}((i-1)/n)}{\sum_{j=0}^{\infty} F_{\Lambda}(j/n)}, \quad i \geq 1.
\]

### 5.1 First approximation method

Our first proposal of approximation to $\psi(u)$ is presented as a corollary of Theorem 5.1. Note that $C_{0,n} = \sum_{j=0}^{\infty} F_{\Lambda}(j/n)$ is an upper sum of the integral of $F_{\Lambda}$. Thus, $C_{0,n} \to E(\Lambda)$ as $n \to \infty$. For the approximation methods we propose, we will take $C_{0,n} = E(\Lambda)$, for any value of $n$.

**Corollary 5.2.** Suppose a Gerber-Dickson model with MP($F_{\Lambda}$) claims is given. For large $n$,

\[
\psi(u) \approx \sum_{k=0}^{\infty} \overline{C}_{k,n} \cdot nb(u, 1/(1 + n))(k),
\]  

where

\[
\overline{C}_{0,n} = E(\Lambda),
\]

\[
\overline{C}_{k,n} = E(\Lambda) \left[ \sum_{i=1}^{k} f_{Ne}(i) \overline{C}_{k-i,n} + F_{Ne}(k) \right], \quad k \geq 1,
\]

\[
f_{Ne}(i) = \frac{F_{\Lambda}((i-1)/n)}{\sum_{j=0}^{\infty} F_{\Lambda}(j/n)}, \quad i \geq 1.
\]
For the examples shown in the next section, we have numerically found that the sum in (59) quickly converge to its value. This will allow us to truncate the infinite sum without much loss of accuracy.

For example, suppose claims have a $MP(F_{\Lambda})$ distribution, where $\Lambda \sim \text{Exp}(\beta)$. In this case, claims have $\text{Geo}(\beta/(1 + \beta))$ distribution and by (28),

$$\psi(u) = \left(\frac{1/(1 + \beta)}{\beta/(1 + \beta)}\right)^{u+1} = \frac{1}{\beta^{u+1}}.$$  

We will check that our approximation (59) converges to this solution as $n \to \infty$. First, the following is easily calculated: $f_{Ne}(i) = e^{-i\beta/n}(e^{\beta/n} - 1)$ and $F_{Ne}(k) = e^{-\beta k/n}$. After some more calculations, one can obtain

$$C_{k,n} = \frac{1}{\beta} \left[ \frac{1}{\beta} (1 - e^{-\beta/n}) + e^{-\beta/n} \right]^k.$$  

Substituting (63) into (59) and simplifying,

$$\psi_n(u) = \frac{1}{\beta} \left( 1 - \frac{1}{\beta} (1 - e^{-\beta/n})/\beta + n (1 - e^{-\beta/n}) \right)^{-u} \to 1/\beta^{u+1} \quad \text{as} \quad n \to \infty.$$

### 5.2 Second approximation method

Our second method to approximate the ruin probability is a direct application of the Law of Large Numbers.

**Corollary 5.3.** Suppose a Gerber-Dickson model with $MP(F_{\Lambda})$ claims is given. Let $z_1, \ldots, z_m$ be a random sample of a $\text{NegBin}(u, 1/(1 + n))$ distribution. For large $n$ and $m$,

$$\psi(u) \approx \frac{1}{m} \sum_{i=1}^{m} C_{z_i,n},$$

where $\{C_{k,n}\}_{k=0}^{\infty}$ is given by (60), (61) and (62).

### 6 Numerical examples

In this section we apply the proposed approximation methods in the case when the mixing distribution is Erlang, Pareto and Lognormal. The results obtained show that the approximated ruin probabilities are extremely close to the exact probabilities. The latter were calculated recursively using formulas (5) and (6), or by numerical integration. In all cases the approximations were calculated for $u = 0, 1, 2, \ldots, 10$ and using the software R. For the first proposed
approximation method, \( n = 500 \) was used and for the second method, \( m = 1000 \) values were generated from a \( \text{NegBin}(u, 1/(n + 1)) \) distribution and again \( n = 500 \). The sum (59) was truncated up to

\[
k^* = \max\{x > un : \text{nb}(u, 1/(1 + n))(x) > 0.00001\}. \tag{65}
\]

**Erlang distribution**

In this example we assume claims have a \( \text{MP}(F_\Lambda) \) distribution with \( \Lambda \sim \text{Erlang}(2, 3) \). In this case \( E(\Lambda) = 2/3 \). Table 1 below shows the results of the approximations. Columns \( E, N_1 \) and \( N_2 \) show for each value of \( u \), the exact value of \( \psi(u) \), the approximation with the first method and the approximation with the second method, respectively. Relative errors \( (\hat{\psi} - \psi)/\psi \) are also shown. The left-hand side plot of Figure 2 shows the values of \( u \) against \( E, N_1 \) and \( N_2 \). The right-hand side plot shows the values of \( u \) against the relative errors.

**Pareto distribution**

In this example claims have a \( \text{MP}(F_\Lambda) \) distribution with \( \Lambda \sim \text{Pareto}(3, 1) \). For this distribution, \( E(\Lambda) = 1/2 \). Table 2 shows the approximations results in the same terms as in Table 1. Figure 3 shows the results graphically.

**Lognormal distribution**

In this example we suppose claims have a \( \text{MP}(F_\Lambda) \) distribution with \( \Lambda \sim \text{Lognormal}(-1, 1) \). For this distribution \( E(\Lambda) = e^{-1/2} \). Table 3 shows the approximations results and Figure 4 shows the related graphics.

As can be seen from the tables and graphs shown, the two approximating methods yield ruin probabilities close to the exact probabilities for the examples considered.
\( E \approx N_1 \approx N_2 \approx PK \)

Figure 2: Approximation when claims are MP(\( \Lambda \)) and \( \Lambda \sim \text{Erl}(2, 3) \).

Table 1: Ruin probability approximation for MP(\( F_\Lambda \)) claims with \( \Lambda \sim \text{Erlang}(2, 3) \).

| \( u \) | \( E \) | \( N_1 \) | \( \frac{\hat{\psi} - \psi}{\psi} \) | \( N_2 \) | \( \frac{\hat{\psi} - \psi}{\psi} \) | \( PK \) | \( \frac{\hat{\psi} - \psi}{\psi} \) |
|-----|-----|-----|----------------|-----|----------------|-----|----------------|
| 0   | 0.66667 | 0.66667 | 0.00000 | 0.66667 | 0.00000 | 0.66667 | 0.00000 |
| 1   | 0.40741 | 0.40775 | 0.00084 | 0.40326 | -0.01019 | 0.4089 | 0.00366 |
| 2   | 0.24280 | 0.24328 | 0.00196 | 0.24551 | 0.01115 | 0.2397 | -0.01276 |
| 3   | 0.14358 | 0.14401 | 0.00306 | 0.14317 | -0.00282 | 0.1456 | 0.01410 |
| 4   | 0.08469 | 0.08504 | 0.00414 | 0.08647 | 0.02096 | 0.084 | -0.00818 |
| 5   | 0.04992 | 0.05018 | 0.00521 | 0.05063 | 0.01419 | 0.0512 | 0.02566 |
| 6   | 0.02942 | 0.02960 | 0.00628 | 0.02989 | 0.01607 | 0.0311 | 0.05726 |
| 7   | 0.01733 | 0.01746 | 0.00735 | 0.01732 | -0.00079 | 0.0172 | -0.00763 |
| 8   | 0.01021 | 0.01030 | 0.00842 | 0.01009 | -0.01208 | 0.0105 | 0.02818 |
| 9   | 0.00602 | 0.00607 | 0.00949 | 0.00586 | -0.02682 | 0.0061 | 0.01379 |
| 10  | 0.00355 | 0.00358 | 0.01056 | 0.00335 | -0.05468 | 0.0031 | -0.12559 |
\[ E \approx N_1 \approx N_2 \approx PK \]

Figure 3: Approximation when claims are MP(Λ) and Λ ∼ Pareto(3, 1).

Table 2: Ruin probability approximation for MP(\(F_\Lambda\)) claims with Λ ∼ Pareto(3, 1).

| \(u\) | \(E\) | \(N_1\) | \(\frac{\psi - \psi}{\psi}\) | \(N_2\) | \(\frac{\psi - \psi}{\psi}\) | \(PK\) | \(\frac{\psi - \psi}{\psi}\) |
|------|------|------|----------------|------|----------------|------|----------------|
| 0    | 0.50000 | 0.50000 | 0.00000 | 0.50000 | 0.00000 | 0.50000 | 0.00000 |
| 1    | 0.28757 | 0.28751 | -0.00023 | 0.28484 | -0.00950 | 0.29170 | 0.01435 |
| 2    | 0.18050 | 0.18046 | -0.00022 | 0.18216 | 0.00921 | 0.17690 | -0.01995 |
| 3    | 0.12014 | 0.12010 | -0.00034 | 0.11960 | -0.00448 | 0.12040 | 0.00215 |
| 4    | 0.08348 | 0.08344 | -0.00053 | 0.08445 | 0.01159 | 0.08170 | -0.02135 |
| 5    | 0.06001 | 0.05996 | -0.00076 | 0.06034 | 0.00547 | 0.06080 | 0.01317 |
| 6    | 0.04437 | 0.04432 | -0.00100 | 0.04450 | 0.00301 | 0.04600 | 0.03681 |
| 7    | 0.03360 | 0.03356 | -0.00127 | 0.03343 | -0.00528 | 0.03270 | -0.02686 |
| 8    | 0.02599 | 0.02595 | -0.00154 | 0.02577 | -0.00865 | 0.02280 | -0.12288 |
| 9    | 0.02049 | 0.02045 | -0.00181 | 0.02019 | -0.01467 | 0.02020 | -0.01419 |
| 10   | 0.01643 | 0.01639 | -0.00209 | 0.01612 | -0.01858 | 0.01750 | 0.06527 |
\[ \psi(u) \]

\[ E \approx N_1 \approx N_2 \]

Figure 4: Approximation when claims are MP(\(\Lambda\)) and \(\Lambda \sim \text{Lognormal}(-1,1)\).

Table 3: Approximations for MP(\(F_\Lambda\)) claims with \(\Lambda \sim \text{Lognormal}(-1,1)\).

| \(u\) | \(E\) | \(N_1\) | \(\frac{\psi - \psi}{\psi}\) | \(N_2\) | \(\frac{\psi - \psi}{\psi}\) | \(PK\) | \(\frac{\psi - \psi}{\psi}\) |
|------|------|------|----------------|------|----------------|------|----------------|
| 0    | 0.60653 | 0.60653 | 0.00000 | 0.60653 | 0.00000 | 0.60653 | 0.00000 |
| 1    | 0.38126 | 0.38124 | -0.00005 | 0.37816 | -0.00813 | 0.37960 | -0.00436 |
| 2    | 0.25231 | 0.25238 | 0.00025 | 0.25426 | 0.00772 | 0.25340 | 0.00431 |
| 3    | 0.17287 | 0.17294 | 0.00042 | 0.17198 | -0.00515 | 0.17520 | 0.01349 |
| 4    | 0.12128 | 0.12135 | 0.00053 | 0.12282 | 0.01264 | 0.12010 | -0.00976 |
| 5    | 0.08661 | 0.08666 | 0.00060 | 0.08715 | 0.00624 | 0.08960 | 0.03456 |
| 6    | 0.06272 | 0.06276 | 0.00064 | 0.06297 | 0.00397 | 0.06280 | 0.00124 |
| 7    | 0.04597 | 0.04600 | 0.00067 | 0.04574 | -0.00487 | 0.04390 | -0.04498 |
| 8    | 0.03404 | 0.03406 | 0.00067 | 0.03373 | -0.00914 | 0.03420 | 0.00466 |
| 9    | 0.02545 | 0.02546 | 0.00066 | 0.02502 | -0.01686 | 0.02420 | -0.04902 |
| 10   | 0.01919 | 0.01920 | 0.00063 | 0.01874 | -0.02346 | 0.02030 | 0.05791 |
7 Conclusions

We have first provided a general formula for the ultimate ruin probability in the Gerber-Dickson risk model when claims follow a negative binomial mixture (NBM) distribution. The ruin probability is expressed as the expected value of a deterministic sequence \( \{C_k\} \), where index \( k \) is the value of a negative binomial distribution. The sequence is not given explicitly but can be calculated recursively. We then extended the formula for claims with a mixed Poisson (MP) distribution. The extension was possible due to the fact that MP distributions can be approximated by NBM distributions. The formulas obtained yielded two immediate approximation methods. These were tested using particular examples. The numerical results showed high accuracy when compared to the exact ruin probabilities. The general results obtained in this work bring about some other questions that we have set aside for further work: error bounds for our estimates, detailed study of some other particular cases of the NBM and MP distributions, properties and bounds for the sequence \( \{C_k\} \), and the possible extension of the ruin probability formula to more general claim distributions.

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