TOWARDS A STRINGY EXTENSION OF SELF–DUAL SUPER YANG–MILLS

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Abstract

Motivated by the search for a space–time supersymmetric extension of the $N=2$ string, we construct a particle model which, upon quantization, describes (abelian) self–dual super Yang–Mills in 2+2 dimensions. The local symmetries of the theory are shown to involve both world–line supersymmetry and kappa symmetry.

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1. Introduction
Recent progress [1, 2] in a stringy description of (2+2)–dimensional self–dual Yang–Mills
and self–dual gravity triggered by investigations of $N=2$ strings (see [3] for reviews con-
taining further references) stimulates interest in the construction of supersymmetric ex-
tensions. The first formulation of such a supersymmetric self–dual system [4] has been
obtained from a constraint system known earlier as the $d=10$ “$ABCD$ superstring” (or
superparticle) [3], by truncating it to a self–dual superspace.

Since the proposed set of constraints fails to form a closed algebra [6], it was suggested
to examine a smaller but closed subset. Two corresponding $N=2$ string models were
constructed [6, 7]. However, their massless states, although consisting of a scalar and
a spinor, do not interact according to self–dual super Yang–Mills (SDSYM) or self–dual
supergravity (SDSG) [6].

It is well known that quantum field theories appear not only as low–energy limits
of string theories but also upon quantization of point–particle models. Guided by the
connection of the $N=1$ superparticle to $N=1$ $d=10$ super Yang–Mills theory, it is tempting
to investigate similar relations in a superspace based on $R^{2,2}$ which we call Kleinian
superspace. The knowledge of the zero mode structure for that case is likely to clarify the
construction of a full superstring.

Yet, the conventional $N=2$ string [10] does not possess space–time supersymmetry
(see, however, Ref. [11]). Its point analogue [12] is the $N=2$ case of a spinning particle
with $N$–extended local world–line supersymmetry [13]. Upon quantization in $d=4$ such a
model describes two massless irreps of the Poincaré group, labelled by helicities $\pm \frac{N}{2}$. For
$N=2$ these are just the self–dual resp. anti–self–dual parts of ordinary Yang–Mills theory.

Even though the $N=1$ spinning string turns into the superstring after the GSO treat-
ment, space–time supersymmetry is not present in spinning particle or $N=2$ string models
and, thus, requires an enlargement of these theories. The incorporation of rigid supersym-
metry in the $d=4$ spinning particle (which provides also a $\kappa$ symmetry) lifts the original
Poincaré irreps to super Poincaré irreps of superhelicity $\pm \frac{N}{2}$ [14, 15]. One immediately
concludes that the only possibility involving the $N=1$ SDSYM multiplet is to choose $N=1$
(see also Ref. [14]).

Our consideration suggests that a stringy extension of SDSYM may have to be doubly
supersymmetric, i.e. possessing both local world–sheet supersymmetry and $\kappa$ symmetry
(for an earlier consideration of doubly supersymmetric models see e.g. Ref. [14]). It is
the purpose of the present letter to construct a doubly supersymmetric particle model in
Kleinian superspace which describes abelian SDSYM when covariantly quantized.

2. Abelian SDSYM
Let us begin by reminding the reader of the spinor notation in Kleinian flat space–time
in order to fix our notation. The covering group of the Lorentz group in $R^{2,2}$ factorizes as

$$Spin(2, 2) = SL(2, R) \times SL(2, R)$$

A successful stringy extension of $N=1$ SDSG was achieved by de Boer and Skenderis [8] who combined
a conventional $N=2$ string in the right–moving sector with a Green–Schwarz–Berkovits type sigma model
[9] in the left–moving sector.
and we distinguish Weyl spinors with respect to the two factors by employing undotted and dotted early Greek indices, respectively. The standard convention is applied when contracting spinor indices, i.e. $\psi^2 = \psi^\alpha \psi_\alpha$ and $\tilde{\phi}^2 = \tilde{\phi}^\alpha \tilde{\phi}^\dot{\alpha}$. Contrary to the case of $SL(2,C)$, the fundamental $SL(2,R)^{(0)}$ spinor representations are Majorana–Weyl and, hence, not related by complex conjugation. We take all spinors to be real whereas sigma matrices are chosen to be imaginary.

Irrespective of the space–time signature, abelian SYM is described by a single Majorana spinor superfield ($W_\alpha, \tilde{W}^\dot{\alpha}$). A convenient way to incorporate the self–duality condition in Kleinian superspace is to drop the anti–chiral spinor superfield strength and work with $W_\alpha$ alone [17]. Thus, our starting point is

$$\tilde{D}^\dot{\alpha} W_\beta(x,\theta,\tilde{\theta}) = 0 \quad \text{and} \quad D^\alpha W_\alpha(x,\theta,\tilde{\theta}) = 0 ,$$

(2)

where $D_\alpha = \partial_\alpha + \tilde{\theta}^\dot{\alpha} \partial_{a\dot{\alpha}}$ and $\tilde{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - \theta^\alpha \partial_{a\dot{\alpha}}$ are the usual covariant derivatives. For $R^{2,2}$ this is consistent because $W_\alpha$ is real and automatically fulfills the Majorana–Weyl condition.

The first of equations (2) is just the chirality condition which specifies the field content of the problem and reduces the component expansion of $W_\alpha$ to

$$W_\alpha(x,\theta,\tilde{\theta}) = \lambda_\alpha(x) + \theta^\beta F_{\beta\alpha}(x) + \theta^2 \phi_\alpha(x) + \theta^\beta \tilde{\theta}^\dot{\beta} \partial_{\dot{\beta}} \lambda_\alpha(x) + \frac{1}{2} \theta^2 \tilde{\theta}^2 \Box \lambda_\alpha(x) .$$

(3)

The second condition in (2) is the equation of motion. Decomposing $F_{\alpha\beta} = F_{(\alpha\beta)} + \epsilon_{\alpha\beta} F$ one finds

$$F(x) = 0 , \quad \phi_\alpha(x) = 0 ,$$

(4)

$$\partial^\alpha \lambda_\alpha(x) = 0 .$$

(5)

The first of Eqs. (2) implies that $F_{\alpha\beta}$ are the components of a self–dual field strength; the first of Eqs. (3) is nothing but the corresponding Bianchi identity. The latter guarantees (locally) the existence of a self–dual vector potential (we symmetrize with weights),

$$F_{(\alpha\beta)} = \partial_\gamma \hat{A}_{\beta\gamma} \quad \text{with} \quad 0 = \partial_{(\dot{\gamma}} \hat{A}_{\beta)\dot{\gamma}} .$$

(6)

All equations above are invariant under abelian gauge transformations (acting only on $A$) as well as under (1,1) supersymmetry:

$$\delta \lambda_\alpha = \epsilon^\beta F_{(\alpha\beta)} \quad \text{and} \quad \delta A_{a\dot{\alpha}} = 2 \lambda_\alpha \tilde{\epsilon}_{\dot{\alpha}} \rightarrow \delta F_{(\alpha\beta)} = 2 \partial_\gamma \hat{A}_{(\beta\gamma)\dot{\gamma}} \tilde{\epsilon}_{\dot{\alpha}}$$

(7)

where $\epsilon^\alpha$ and $\tilde{\epsilon}^{\dot{\alpha}}$ are two independent (real) Majorana–Weyl spinors of opposite chirality. It is worth mentioning that the supersymmetry algebra closes (modulo gauge transformations) only on–shell, i.e. with the use of the self–duality condition.

We shall go on to construct a spinning (super) particle model which, when covariantly quantized, yields Eqs. (2) (in their complexified form). It is known that the quantum wave

\[3\text{The integrability conditions for (2) imply that } W_\alpha \text{ obeys the Weyl equation in the external index.} \]
function of a conventional spinning particle is realized in terms of four component Dirac (and not Weyl) spinors \[18\]. It is therefore convenient to rewrite our starting point (2) as

\[(1 + \gamma_5)_{AB} W_B = 0 , \quad \gamma^n_{AB} \partial_n W_B = 0 ,\]  

(8)

\[\tilde{D}_\alpha W_A = 0 , \quad \partial^\alpha \tilde{D}_\alpha W_A = 0 ,\]  

(9)

with \(W_A = \left(\begin{array}{c} W^\alpha \\ \tilde{W}^\dot{\alpha} \end{array}\right)\) a Majorana spinor and \(\gamma_n = \left(\begin{array}{cc} 0 & \sigma_n \\ \sigma_n & 0 \end{array}\right)\) the Dirac matrices in Kleinian flat space–time. The first two equations define an on–shell chiral fermion, while the remaining ones govern supersymmetry.

3. Chiral fermion model

A technique for obtaining the massless Dirac equation from first–quantized particle mechanics has been known for a long time. It suffices to start with the action \[18\]

\[S = \int d\tau \left\{ \frac{1}{2} e (\dot{x}^n + i\chi \psi^n)^2 + i\dot{\psi}_n \psi^n \right\} ,\]  

(10)

where \(\psi^n\) and \(\chi\) are anticommuting real vector and scalar variables, respectively. Quantization modifies the Grassmann algebra to a Clifford algebra, so that the \(\psi^n\) will be represented by gamma matrices. Incorporation of the chirality condition \(1 + \gamma_5 = 0\) into the model \[11\] turns out to be more involved. As was shown in Ref. \[19\], naively implementing the classically analogous constraint \[12\] by adding a Lagrange multiplier term

\[\delta S = \int d\tau \, \phi \left( 1 + \epsilon^{abcd} \psi_a \psi_b \psi_c \psi_d \right) \]  

(11)

is classically inconsistent since taking the body of the constraint equation implies \(0 = 1\). Interestingly, the contradiction seems to be avoided when one imposes only the time derivative of the classical constraint. On the quantum level, however, one has to be more inventive. An elegant solution is provided through an extension of the system by a conventional fermionic oscillator.\[16\] So, let us consider the following action,

\[S = \int_{\tau_1}^{\tau_2} d\tau \left\{ \frac{1}{2e} (\dot{x}^n + \omega e^{nmkl} \psi_m \psi_k \psi_l + \chi \psi^n)^2 + \psi_n \dot{\psi}^n + \xi \ddot{\xi} + \phi (i\dot{\xi} \xi - \frac{2}{3} e^{nmkl} \psi_n \psi_m \psi_k \psi_l) \right\} .\]  

(12)

The variables \(e, x^n, \phi\) are even, the remaining ones are odd. We regard all variables to be real except for \(\bar{\xi} = \xi^*\). Reality of the kinetic term\[17\] implies the boundary condition \(\xi|_{\tau_2} = 0\), which restricts the trajectories entering the variational problem. The equations of motion in the \((\xi, \bar{\xi})\) sector,

\[\ddot{\xi} + i\phi \xi = 0 \quad \text{and} \quad \ddot{\bar{\xi}} - i\phi \bar{\xi} = 0 ,\]  

(13)

\[4\] Note that \(\gamma_n^T = C^{-1} \gamma_n C\) with \(C = \gamma_0 \gamma_1\) in Kleinian space–time.

\[5\] The approach followed in Ref. \[20\] suffers from the same problem \[13\].

\[6\] A similar extension for the massive Dirac particle has been employed in Ref. \[21\].

\[7\] Alternatively, we could start with a conventional quantum description in terms of creation/annihilation operators.
can easily be integrated to yield
\[ \xi = \xi_0 e^{-i\int_0^\tau d\tau' \phi(\tau')} \quad \text{and} \quad \bar{\xi} = \bar{\xi}_0 e^{i\int_0^\tau d\tau' \bar{\phi}(\tau')} , \] (14)

with a constant \( \xi_0 = (\bar{\xi}_0)^\ast \) conforming to the boundary condition. From Eq. (14) one observes that only the constant modes \( \xi_0 \) and \( \bar{\xi}_0 \) enter the \( \phi \) equation of motion,
\[ i\bar{\xi}\xi - \frac{2}{3} \epsilon_{nmkl} \dot{\psi}_n \psi_m \psi_k \psi_l = 0 . \] (15)

Since \( (\xi, \bar{\xi}) \) do not appear elsewhere, this is consistent with other equations of motion, in particular with
\[ \epsilon_{nmkl} \dot{\psi}_n \psi_m \psi_k \psi_l = 0 . \] (16)

Before going over to the Hamiltonian analysis we list the Lagrangian local world–line symmetries of the model:
\[
\begin{aligned}
\delta x^n &= \alpha \dot{x}^n + \psi^n \epsilon - \lambda \epsilon_{nmkl} \psi_m \psi_k \psi_l , \\
\delta \psi^n &= \alpha \psi^n + \frac{1}{2} \Pi^n \epsilon + \frac{4}{3} \beta \epsilon_{nmkl} \psi_m \psi_k \psi_l - \frac{3}{2} \epsilon_{nmkl} \Pi_m \psi_k \psi_l , \\
\delta \xi &= \alpha \dot{\xi} - i\beta \bar{\xi} , \quad \delta \bar{\xi} = \alpha \dot{\bar{\xi}} + i\beta \xi , \\
\delta e &= (\alpha \epsilon)^\cdot + \chi \epsilon , \quad \delta \phi = (\alpha \phi)^\cdot + \beta , \\
\delta \chi &= (\alpha \chi)^\cdot + \dot{\epsilon} , \quad \delta \omega = (\alpha \omega)^\cdot + \frac{4}{3} \dot{\epsilon} + \frac{4}{3} \chi \beta + \lambda ,
\end{aligned}
\] (17)

where we denote \( \Pi^n \equiv \dot{z}^n + \omega \epsilon_{nmkl} \psi_m \psi_k \psi_l + \chi \psi^n \). In addition to world–line reparametrization invariance and local supersymmetry of the massless spinning particle (11), there appears a couple of new local symmetries with bosonic \( (\beta) \) and fermionic \( (\lambda) \) parameters and their corresponding gauge fields \( \phi \) and \( \omega \), respectively. It is straightforward to check that the symmetry algebra closes only modulo the equations of motion.

Let us now analyze the system in the Hamiltonian formalism. Evaluating the set of primary constraints\(^8\)
\[
\begin{aligned}
p_e &= 0 , \quad p_\phi = 0 , \quad p_\omega = 0 , \quad p_\chi = 0 , \\
p_\psi_n - \psi_n &= 0 , \quad p_\xi - \bar{\xi} = 0 , \quad p_{\bar{\xi}} = 0 ,
\end{aligned}
\] (18)

one finds the canonical Hamiltonian to be
\[
H = p_e \lambda_e + p_\phi \lambda_\phi + p_\omega \lambda_\omega + (p_\psi_n - \psi_n) \lambda_\psi^n + p_\chi \lambda_\chi + (p_{\bar{\xi}} - \bar{\xi}) \lambda_{\bar{\xi}} + p_{\bar{\xi}} \lambda_{\bar{\xi}}
+ \frac{1}{2} e p^2 - \omega \epsilon_{nmkl} p_n \psi_m \psi_k \psi_l - \chi p^n \psi_n - \phi \left( i\bar{\xi}\xi - \frac{2}{3} \epsilon_{nmkl} \psi_n \psi_m \psi_k \psi_l \right) ,
\] (19)

where the \( \lambda \)'s are the Lagrange multipliers associated to the primary constraints. Consistency conditions for the primary constraints imply the secondary ones,
\[
\begin{aligned}
p^n \psi_n &= 0 , \quad i\bar{\xi}\xi - \frac{2}{3} \epsilon_{nmkl} \psi_n \psi_m \psi_k \psi_l = 0 , \\
p^2 &= 0 , \quad \epsilon_{nmkl} p_n \psi_m \psi_k \psi_l = 0 ,
\end{aligned}
\] (20)

\(^8\)We define momenta conjugate to anticommuting variables to be right derivatives of the Lagrangian with respect to their velocities.
and fix some of the Lagrange multipliers,

\[
\lambda_{\psi}^n = \frac{1}{2} \chi \xrightarrow{n} - \frac{3}{2} \omega \varepsilon_{nmkl} p_m \psi_k \psi_l + \frac{1}{3} \phi \varepsilon_{nmkl} \psi_m \psi_k \psi_l ,
\]
\[
\lambda_{\xi} = -i \phi \xi \quad \text{and} \quad \lambda_{\bar{\xi}} = i \phi \bar{\xi} .
\]

To show that no tertiary constraints arise at the next stage of the Dirac procedure, one needs the well–known identity

\[
\varepsilon^{abcd} \varepsilon_{efgd} = 6 \delta^{[a} \delta_{b} \delta^{c]}_{d} = 6 \delta^{a} \delta_{[b} \delta_{c]}_{d} .
\]

The first line in Eqs. (18) contains a set of first class constraints. Imposing the gauge

\[
e = 1 , \quad \phi = 0 , \quad \omega = 0 , \quad \chi = 0
\]

allows us to omit four canonical pairs, the fixed Lagrange multipliers being

\[
\lambda_{e} = 0 , \quad \lambda_{\phi} = 0 , \quad \lambda_{\omega} = 0 , \quad \lambda_{\chi} = 0 .
\]

Rewriting the second of Eqs. (20) in the equivalent form

\[
i p_{\xi} \xi + i p_{\bar{\xi}} \bar{\xi} - \frac{2}{3} \varepsilon_{nmkl} \psi_n \psi_m \psi_k \psi_l = 0 ,
\]

one decouples the (second class) constraints in the sector \((\xi, p_{\xi})\) from the others. In this fashion we may drop those variables, after having introduced the associated Dirac bracket. The brackets for the remaining variables prove to be canonical. Finally, by making use of the shift

\[
\psi_n \rightarrow \psi_n' = \psi_n + \frac{1}{2} (p_{\psi_n} - \psi_n) \quad \text{so that} \quad \{\psi_n', \psi_m'\} = \frac{1}{2} \eta_{nm} ,
\]

one can also isolate the second class constraint \(p_{\psi_n} - \psi_n = 0\). Regarding it further as a strong equation, one finally arrives at the following constraint set,

\[
p_{n}^{\psi} \psi_{n} = 0 , \quad i p_{\xi} \xi - \frac{2}{3} \varepsilon_{nmkl} \psi_{n} \psi_{m} \psi_{k} \psi_{l} = 0 ,
\]
\[
p_{2} = 0 , \quad \varepsilon_{nmkl} p_{n} \psi_{m} \psi_{k} \psi_{l} = 0 ,
\]

which forms a closed algebra under the Dirac bracket.

Quantization is now straightforward. One promotes classical variables to quantum operators (here in position representation),

\[
\hat{x}^n = x^n , \quad \hat{p}_n = -i \frac{\partial}{\partial x^n} , \quad \hat{\psi}_n = \frac{1}{2} e^{\frac{4m}{\hbar}} \gamma_n , \quad \hat{\xi} = \xi , \quad \hat{p}_{\xi} = i \frac{\partial}{\partial \xi} ,
\]

which act on the corresponding quantum state represented by a wave function

\[
\Psi_{A}(x, \xi) = U_{A}(x) + \xi W_{A}(x) .
\]

\[^9\] Eqs. (23) applied to Eqs. (21) also imply \(\lambda_{\xi} = 0 = \lambda_{\bar{\xi}}\) which puts the Hamiltonian (13) into its real form. It was originally complex due to our choice for the \(\xi\) kinetic term.
The latter is a Dirac spinor since it is acted upon by $R_{\gamma\delta}^2$ Dirac matrices $\gamma^n_{AB}$. In what follows, we assume the $qp$-ordering in the sector $(\xi, p\xi)$. In agreement with general principles we assign positive Grassmann parity to the $x$-dependent functions in the expansion (29). Hence, the complete state has no definite parity.

Physical states are characterized by the fact that the first class constraint operators annihilate them. Eqs. (27) then yield

$$U(x) = 0, \quad (1 + \gamma_5)W(x) = 0, \quad \gamma^n\partial_nW(x) = 0. \quad (30)$$

At the quantum level, the second line in Eqs. (27) follows from the first one. Thus, after quantization, the model (12) describes a single on-shell chiral fermion.

4. Adding rigid supersymmetry

To obtain the remaining equations (19) from the quantization of our particle mechanics it suffices to extend the original configuration space $\{x^n\}$ to a superspace $\{(x^n, \theta^\alpha, \tilde{\theta}^{\dot{\alpha}})\}$, by adjoining a pair $(\theta^\alpha, \tilde{\theta}^{\dot{\alpha}})$ of Kleinian Majorana–Weyl spinors, and to provide a couple of first class constraints

$$p_{\bar{\theta}^{\dot{\alpha}}} - \theta^\alpha p_{\alpha\dot{\alpha}} = 0 \quad \text{and} \quad p_{\theta^\alpha} p^{\alpha\dot{\alpha}} = 0 \quad (31)$$

when passing to the Hamiltonian formalism. This is achieved by extending the action (12) just like in the construction of Siegel’s superparticle [5] (see also related works [22, 23]),

$$S = \int_{\tau_1}^{\tau_2} d\tau \left\{ \frac{1}{2\epsilon}(\dot{x}^n - i\dot{\theta}^\alpha \sigma^n \tilde{\theta} + i\theta^\alpha \sigma^n \dot{\tilde{\theta}} + \omega \epsilon^{nmkl} \psi_m \psi_k \psi_l + \chi \psi^n + i\rho \sigma^n \bar{\mu})^2 + \psi^n \psi_n + \bar{\xi} \bar{\xi} - \rho \dot{\bar{\theta}} + \phi (i\bar{\xi} \dot{\bar{\xi}} - \frac{2}{3} \epsilon^{nmkl} \psi_n \psi_m \psi_k \psi_l) \right\}. \quad (32)$$

Here, $\rho^\alpha$ and $\bar{\mu}^{\dot{\alpha}}$ are auxiliary (real) odd variables. We switch back to spinor notation. The model is invariant under rigid space–time supersymmetry transformations

$$\delta x^{\alpha\dot{\alpha}} = 2(\theta^\alpha \epsilon^{\dot{\alpha}} - \epsilon^\alpha \tilde{\theta}^{\dot{\alpha}}), \quad \delta \theta^\alpha = \epsilon^\alpha, \quad \delta \tilde{\theta}^{\dot{\alpha}} = \tilde{\epsilon}^{\dot{\alpha}}. \quad (33)$$

The set of local symmetries is extended by a pair of new symmetries, including kappa symmetry,

$$\delta \theta^\alpha = -\epsilon^{-1} \Pi^{\alpha\dot{\alpha}} \tilde{\kappa}^{\dot{\alpha}}, \quad \delta \tilde{\theta}^{\dot{\alpha}} = 0, \quad \delta x^{\alpha\dot{\alpha}} = 2(\delta \theta^\alpha \tilde{\theta}^{\dot{\alpha}} - \rho^\alpha \tilde{\kappa}^{\dot{\alpha}}), \quad (34)$$

and

$$\delta \rho^\alpha = 0, \quad \delta \tilde{\mu}^{\dot{\alpha}} = \tilde{\kappa}^{\dot{\alpha}}, \quad \delta e = -4 \tilde{\kappa}^{\dot{\alpha}} \theta^\alpha \quad (35)$$

where

$$\Pi^{\alpha\dot{\alpha}} = \dot{x}^{\alpha\dot{\alpha}} - 2 \dot{\theta}^\alpha \tilde{\theta}^{\dot{\alpha}} + 2 \tilde{\theta}^{\dot{\alpha}} \dot{\theta}^\alpha + \omega \psi^{\alpha\dot{\beta}} \psi_{\beta\dot{\beta}} \psi^{\dot{\beta} \dot{\alpha}} + \chi \psi^{\alpha\dot{\alpha}} + 2 \rho^\alpha \tilde{\mu}^{\dot{\alpha}}. \quad (36)$$
As is seen from the transformations above, the $\tilde{\nu}$ symmetry allows us to remove the $\tilde{\theta}$ variable, which is compatible with the first of the first class constraints (31) arising in the Hamiltonian formalism.

The Hamiltonian analysis for the extended model proceeds along the lines of the previous section. In addition to the seven earlier primary constraints (18), one finds four new ones,

$$p_{\rho\alpha} = 0 \ , \ p_{\tilde{\theta}\alpha} - \theta^{\alpha} p_{\alpha\tilde{\theta}} = 0 \ , \ p_{\tilde{\mu}\tilde{\alpha}} = 0 \ , \ p_{\theta\alpha} - \tilde{\theta}^{\alpha} p_{\alpha\tilde{\theta}} - \rho = 0 \ , \quad (37)$$

while the Hamiltonian (19) acquires the contribution

$$H_{\text{add}} = p_{\rho\alpha} \lambda_{\rho}^{\alpha} + p_{\tilde{\mu}\tilde{\alpha}} \lambda_{\tilde{\mu}}^{\tilde{\alpha}} + (p_{\theta\alpha} - \tilde{\theta}^{\alpha} p_{\alpha\tilde{\theta}} - \rho) \lambda_{\theta}^{\alpha} + (p_{\tilde{\theta}\alpha} - \theta^{\alpha} p_{\alpha\theta}) \lambda_{\tilde{\theta}}^{\tilde{\alpha}} - \rho^{\alpha} \tilde{\mu}^{\tilde{\alpha}} p_{\alpha\tilde{\theta}} . \quad (38)$$

Time translation invariance of the new constraints (37) yields a new secondary one as well,

$$\rho^{\alpha} p_{\alpha\tilde{\theta}} = 0 \ , \quad (39)$$

and specifies some of the Lagrange multipliers, i.e.

$$\lambda_{\theta\alpha} = \tilde{\mu}^{\tilde{\alpha}} p_{\alpha\tilde{\theta}} \ , \quad \lambda_{\rho\alpha} = -2 \lambda_{\tilde{\theta}}^{\tilde{\alpha}} p_{\alpha\tilde{\theta}} . \quad (40)$$

No tertiary constraints appear at the next stage of the Dirac procedure. In the sector of the additional variables $(\theta, \tilde{\theta}, \rho, \tilde{\mu})$ the full set of constraints may then be written as

$$p_{\theta\alpha} - \tilde{\theta}^{\alpha} p_{\alpha\tilde{\theta}} - \rho = 0 \ , \quad p_{\rho\alpha} = 0 \ , \quad (41)$$

$$\quad (p_{\theta\alpha} - \tilde{\theta}^{\alpha} p_{\alpha\tilde{\theta}}) p_{\alpha\tilde{\theta}} + 2 p_{\rho}^{\alpha} p_{\alpha\tilde{\theta}} = 0 \ , \quad p_{\tilde{\mu}\tilde{\alpha}} = 0 \ , \quad (42)$$

The constraints (42) are first class. Imposing the gauge (implying $\lambda_{\tilde{\mu}} = 0$)

$$\tilde{\mu}^{\tilde{\alpha}} = 0 \ , \quad (43)$$

one can omit the variables $(\tilde{\mu}, p_{\tilde{\mu}})$. The pair (11) is second class and can be taken as strong equations after introducing the associated Dirac bracket. The Dirac brackets for the remaining variables turn out to coincide with the Poisson ones. Upon quantization, the remaining constraints will append two additional restrictions to equations (30), namely

$$\tilde{D}_{\alpha} W_{A}(x, \theta, \tilde{\theta}) = 0 \quad \text{and} \quad \partial^{\alpha\dot{\alpha}} D_{\alpha} W_{A}(x, \theta, \tilde{\theta}) = 0 \ , \quad (44)$$

with $W_{A}$ from (30) acquiring now $(\theta, \tilde{\theta})$ dependence. Beautifully enough, these are precisely the equations describing abelian self–dual super Yang–Mills in their complexified form.10

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10 In $R^{1,3}$, time reversal is often used to impose a reality condition on the wave function [12], which is, of course, only admissible for energy eigenfunctions. For $R^{2,2}$ however, this is not an option because there is no notion of orthochronicity.
5. Concluding remarks
To summarize, in this letter we have constructed a particle model which describes (abelian) SDSYM upon quantization. The theory is essentially doubly supersymmetric, i.e. combines the features of the Green–Schwarz and Neveu–Schwarz–Ramond formalisms. In order to attempt a superstring generalization, it seems natural to start with a string analog of the action (12), followed by the implementation of kappa symmetry. We envisage that the fermionic oscillator degree of freedom, being inessential in the particle case, may exhibit a nontrivial influence on the string spectrum. Another (heterotic) possibility is provided by combining the standard NSR set of constraints in the left-moving sector with the constraints of Ref. [6] in the right-moving sector. Such an approach is likely to be related to that of Ref. [8].

As has been discussed in Ref. [15], the global symmetry structure underlying a wide class of doubly supersymmetric particles is that of the superconformal group. It would be interesting to perform a similar analysis for the formulation at hand.

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