1 Introduction

A graph polynomial is an algebraic object associated with a graph that is usually invariant at least under graph isomorphism. As such, it encodes information about the graph, and enables algebraic methods for extracting this information. This chapter surveys a comprehensive, although not exhaustive, sampling of graph polynomials. It concludes Graph Polynomials and their Applications I: The Tutte Polynomial by continuing the goal of providing a brief overview of a variety of techniques defining a graph polynomial and then for decoding the combinatorial information it contains.

The polynomials we discuss here are not generally specializations of the Tutte polynomial, but they are each in some way related to the Tutte polynomial, and often to one another. We emphasize these interrelations and explore how an understanding of one polynomial can guide research into others. We also discuss multivariable generalizations of some of these polynomials and the theory facilitated by this. We conclude with two examples, one from biology and one from physics, that illustrate the applicability of graph polynomials in other fields.

2 Formulating Graph Polynomials

We have seen two methods for formulating a graph polynomial with the linear recursion (deletion/contraction) and generating function definitions of the Tutte polynomial in the previous chapter. Here we will see several more. We begin with one of the earliest graph polynomials, the edge-difference polynomial, a multivariable polynomial defined as a product and originally studied by Sylvester [Syl78] and Peterson [Pet91] in the late 1800s. More recently, it
has been used to address list coloring questions (see Alon and Tarsi [AT92] and Ellingham and Goddyn [EG96]), where a list coloring of a graph is a proper coloring of the vertices of a graph with the color of each vertex selected from a predetermined list of colors assigned to that vertex.

**Definition 1** The edge-difference polynomial. Let \((v_1,\ldots,v_n)\) be an ordering of the vertices of a graph \(G\). Then \(D(G;x_1,\ldots,x_n) = \prod_{i<j} (x_i - x_j)\), where the product is over all edges \((v_i,v_j)\) of \(G\).

Note that a proper coloring of \(G\) corresponds to finding positive integer values \(N_i\) (not necessarily distinct) for each of the \(x_i\)'s so that \(D(G;N_1,\ldots,N_n) \neq 0\).

There are also several polynomials based on various determinants (or even permanents; see Pathasarthy [Par89] for a survey) involving the adjacency matrix of a graph. Recall that \(A(G)\), the adjacency matrix of a graph, has entries \(a_{ij} = 1\) if \((i,j)\) is an edge of the graph and 0 if it is not. The characteristic polynomial is the classic example of such a graph polynomial, and will be discussed further in Section 3.

**Definition 2** The characteristic polynomial. Let \(A(G)\) be the adjacency matrix of a graph \(G\). Then \(f(G;x) = |xI - A(G)|\).

Other examples of such polynomials are the idiosyncratic polynomial introduced by Tutte, see [Tut79], that is defined by \(\nu(G;x,y) = |A(G) + y(J - I - A) - xI|\), where \(J\) is the matrix having all entries equal to 1. Also \(\mu(G;x,y) = |xI - D(G) + A(G)|\) a polynomial introduced in [Kel65], where \(D(G)\) is the degree matrix of \(G\) that is the diagonal matrix with \(\text{deg}(i)\) in the position \((i,i)\). Note that \(J - I - A\) is the adjacency matrix of the complement of \(G\) and when \(G\) is a simple graph, \(D(G) - A(G)\) is just the Laplacian matrix \(L(G)\) of the graph.

A number of important graph polynomials may be defined by state model formulas. Loosely speaking, a state of a graph is some configuration resulting from making local assignments for substructures (e.g. the edges or vertices) of the graph. These assignments may be, for example, associating an element of a given set to each vertex, or even the result of reconfiguring the edges incident with a vertex. A graph polynomial is formed by associating an expression, often a weighted monomial, to each state of the graph, and then summing over all possible graph states. The language comes from physics, and is also found in knot theory. We will see several state model graph polynomials among those surveyed below, as well as an application of this method in the Potts model of statistical mechanics in Section 5.

An early example of a graph polynomial given by a state model formulation is \(P(G;x)\), the Penrose polynomial. This polynomial graph invariant for planar graphs was defined implicitly by Penrose [Pen69] in the context of tensor diagrams in physics, but an excellent graph theoretical exposition can be found in Aigner [Aig97]. To compute \(P(G;x)\), let \(G\) be a plane graph, and let \(G_m\) be its medial graph, face two-colored with the unbounded face colored white. At each vertex, we consider three possible local reconfigurations, as in Figure 1.
A state $S$ of $G_m$ then results from choosing one of these three reconfigurations at each vertex of $G_m$ and consists of a set of disjoint closed curves (like a knot diagram). Furthermore, to each local reconfiguration at a vertex $v$, we assign a weight $\omega(S,v)$ that is $+1$, $0$, or $-1$ for a white, black, or crossing configuration, respectively.

Figure 1: The three possible local reconfigurations at a vertex $v$, identified, from left to right, as white, black, and crossing. Which strand passes over which in the crossing configuration does not affect the computation.

**Definition 3** The Penrose polynomial. Let $G$ be a planar graph with medial graph $G_m$, and let $St(G_m)$ be the set of states of $G_m$ and let $St'(G_m)$ be the set of states with no black configurations. Then,

$$P(G;x) = \sum_{S \in St(G_m)} \left( \prod_{v \in G_m} \omega(S,v) \right) x^{k(S)} = \sum_{S' \in St'(G_m)} \left( (-1)^{cr(S)} x^{k(S)} \right),$$

where $k(S)$ is the number of components in the graph state $S$, and $cr(S)$ is the number of crossing vertex configurations chosen in the state $S$.

For example, if $G$ is the $\theta$-graph consisting of two vertices joined by three edges in parallel, then $P(G;x) = x^3 - 3x^2 + 2x$, as in Figure 2. The Penrose polynomial may also be computed via a linear recursion relation (see Jaeger [Jae90] for example).

The Penrose polynomial has some surprising properties, particularly with respect to graph coloring. The Four Color Theorem is equivalent to showing that every planar, cubic, connected graph can be properly edge-colored with three colors. The Penrose polynomial, when applied to planar, cubic, connected graphs, encodes exactly this information (see Penrose [Pen69]):

$$P(G; 3) = \left( \frac{-1}{4} \right)^{|V|} P(G; -2) = \text{the number of edge-3-colorings of } G.$$
Computing the Penrose polynomial of a graph $G$ from the states of its medial graph.

Figure 2: Computing the Penrose polynomial of a graph $G$ from the states of its medial graph.

themselves. These relations lead to combinatorial insights as results for any one polynomial then inform those related to it.

### 3.1 Characteristic and Matching Polynomials

The characteristic and matching polynomials are particularly interrelated, so we treat them together here, beginning with the characteristic polynomial $f(G; x)$ already introduced in Definition 2. Note that $f(G; x)$ is a monic polynomial of degree $n$. Furthermore, since the adjacency matrix $A$ is real and symmetric, all its eigenvalues are real, and thus all the zeros of $f(G; x)$ are real.

By using properties of determinants we can find interpretations of the coefficients of $f(G; x)$ in terms of the principal minors of $A$. A principal minor of order $r$ is the determinant of an $r \times r$ submatrix of $A$ obtained by choosing $r$ rows and columns with the same set of indices.

**Proposition 1** Suppose that $f(G; x) = \sum_{i=0}^{n} a_i x^{n-i}$. Then $(-1)^i a_i$ is equal to the sum of the principal minors of $A$ with order $i$.

This property of the characteristic polynomial can be found, for example, in Horn and Johnson [HJ90].

Since the diagonal elements of $A$ are all zero, we have that $a_1 = 0$. The principal minors of order two and three which are not zero are of the form
$|J - I|$, where $J$ is the matrix having all entries +1 and $J - I$ has order 2 or 3. The $2 \times 2$ submatrices $J - I$ of $A(G)$ correspond naturally to the edges of $G$ and the $3 \times 3$ submatrices $J - I$ correspond to the $K_3$ subgraphs of $G$. Thus $c_2 = -|E(G)|$ and $-c_3$ is twice the number of $K_3$ subgraphs of $G$.

A linear subgraph of $G$ is a subgraph whose components are edges or cycles. An expression for the coefficients of $f(G; x)$ in terms of linear subgraphs is given in the following.

**Proposition 2** The coefficients of the characteristic polynomial may be expressed as

$$(-1)^i a_i = \sum_{\Lambda} (-1)^{r(\Lambda)} 2^{|r^*(\Lambda)|},$$

where $r$ is the rank function and the sum is over all linear subgraphs $\Lambda$ of $G$ having $i$ vertices.

Note that because $\Lambda$ is a linear subgraph, $r^*(\Lambda)$ is simply the number of components in $\Lambda$ that are cycles. The proof of Proposition 2 uses Proposition 1 and can be found in Harary [Har62], while a detailed history of this result is given by Cvetković, Doob, and Sachs [CDS80].

As with the Tutte polynomial we also have some reduction formulas and an expression for the derivative of the characteristic polynomial.

**Theorem 1** The characteristic polynomial of a graph satisfies the following identities:

1. $f(G \cup H; x) = f(G; x)f(H; x)$,
2. $f(G; x) = f(G \setminus e, x) - f(G - u - v; x)$ if $e = \{u, v\}$ is a cut-edge of $G$;
3. $\frac{\partial}{\partial x} f(G; x) = \sum_{v \in V(G)} f(G - v; x)$.

A proof of these properties can be found in Godsil [God93]. Item 1 is an easy exercises in matrix theory, as in Horn and Johnson [HJ90]. Item 2 can be proved by using Proposition 2 and considering the linear subgraphs of $G$ that use the edge $e$ and the ones that do not use it. The result follows because $e$ is in no cycle of $G$ if and only if it is a cut-edge. Item 3 can be proved by using Proposition 1 since any principal minor of order $i$ is counted $n - i$ times in the right hand side of the formula in Item 2.

Given a graph $G$, the collection of (unlabeled) subgraphs $G - v$, for $v \in V(G)$, is called the deck of $G$, and the individual subgraphs are called cards. Thus, the deck for a graph on $n$ vertices consists of $n$ graphs, each of which has $n - 1$ vertices. Ulam’s reconstruction conjecture in [Ula60] asserts that any finite graph $G$ with more than two vertices is uniquely determined by its deck, see [Ula60], and we call any graph that satisfies the conjecture reconstructible. Similarly, an invariant of $G$ which can be deduced from the deck is called reconstructible.

Clearly, the number $n$ of vertices of a graph is reconstructible. Also, as every edge is present in exactly $n - 2$ cards, the number of edges is also reconstructible. The following useful example of reconstructibility is due to Kelly [Kel57].
Lemma 1 (Kelly’s lemma) Let $G$ and $H$ be graphs, and let $\nu(H,G)$ denote the number of subgraphs of $G$ isomorphic to $H$. Then

$$\left(|V(G)| - |V(H)|\right)\nu(H,G) = \sum_{v \in V(G)} \nu(H, G - v).$$

The proof is by a double counting argument, and it follows that $\nu(H,G)$ is reconstructible whenever $|V(H)| < |V(G)|$.

Tutte proved in [Tut67] that the Tutte polynomial is reconstructible, and thus the chromatic polynomial, the flow polynomial, the number of spanning trees and any invariant mentioned in the previous chapter are also reconstructible. Tutte also proved that the characteristic polynomial is reconstructible in [Tut79].

Theorem 2 The characteristic polynomial of a graph is reconstructible.

For the proof, note that we have immediately from Theorem 1 that $f'(G;x)$ is reconstructible. It then remains to prove that the constant term of $f(G;x)$ is reconstructible. But by Proposition 1, this is the same as proving that $|A(G)|$ is reconstructible. Then, using Theorem 2 and an extension of Kelly’s lemma (see [Koc81]), the problem is reduced to proving that the number of Hamiltonian cycles is reconstructible. A complete proof of Theorem 2 based on the proof in Kocay [Koc81], can be found in [God93].

Let us turn now to the matching polynomial. An $i$-matching in a graph $G$ is a set of $i$ edges, no two of which have a vertex in common. Let $\Phi_i(G)$ denote the number of $i$-matchings, and set $\Phi_0(G) = 1$. Thus $\Phi_1(G) = m$ is the number of edges of $G$, and if $n$, the number of vertices, is even, then $\Phi_{n/2}(G)$ is the number of perfect matchings of $G$.

Definition 4 Let $G$ be a graph. Then the matching polynomial of $G$ is

$$\mu(G;x) = \sum_{i \geq 0} (-1)^i \Phi_i(G)x^{n-2i}.$$ 

A more natural polynomial might be the matching generating polynomial, given as the generating function of $i$-matchings by

$$g(G;x) = \sum_{i \geq 0} \Phi_i(G)x^i.$$ 

However, the two polynomials are related by the identity

$$\mu(G;x) = x^n g(G;(-x)^2),$$

so there is no essential difference between them.

The matching polynomial is also known as the acyclic polynomial in Gutman and Trinajstić [GT76], matching defect polynomial in Lovász and Plummer [LP86] and reference polynomial in Aihara [Aih76]. It has appeared independently in several different contexts. In combinatorics, it was probably
introduced by Farrell in [Far79a], but since the matching polynomial is essentially the same as the rook polynomial for bipartite graphs (see Farrell [Far88]), then its origin can be traced back at least to Riordan [Rio58]. In statistical physics it appears because of the monomer-dimer problem and was introduced by Heilmann and Lieb in [HL70] and independently by Kunz in [Kun70]. Finally, in theoretical chemistry was introduced by Hosaya in [Hos71] and later in connection with the so-called topological resonance energy by Gutman, Milun and Trinajstić in [GT76,GMT76,GMT77] and independently by Aihara in [Aih76]. For a full account of the history of the matching polynomial see Gutman [Gut91].

As with the Tutte and characteristic polynomials, we have some reduction formulas for the matching polynomial. The proof of the following theorem can be found in Godsil [God93].

**Theorem 3** The matching polynomial satisfies the following identities:

1. \( \mu(G \cup H; x) = \mu(G; x) \mu(H; x) \),
2. \( \mu(G; x) = \mu(G \setminus e; x) - \mu(G - u - v; x) \) if \( e = \{u, v\} \) is an edge of \( G \),
3. \( \mu(G; x) = x \mu(G - u; x) - \sum_{\{u, v\} \in E(G)} \mu(G - v - u; x) \), if \( u \in V(G) \),
4. \( \frac{\partial}{\partial x} \mu(G; x) = \sum_{v \in V(G)} \mu(G - v; x) \).

To choose an \( i \)-matching in \( G \cup H \) you need to choose an \( s \)-matching in \( G \) and a \( t \)-matching in \( H \) such that \( s + t = i \). Item 1 then follows by the fundamental counting principle. For Item 2 notice that the set of \( i \)-matchings can be partitioned into those \( i \)-matchings that use the edge \( e \) and those that do not use it. Item 3 follows similarly. Finally, every \( i \)-matching of \( G \) with \( i < n/2 \) is counted \( n - 2i \) times in the right-hand side of the formula in Item 4 so the result follows.

When the graph \( G \) is a forest, a linear subgraph of \( G \) with \( j \) vertices corresponds to a matching covering \( j \) vertices, with \( j \) even. Thus, Proposition 2 has the following corollary, observed by Hosaya [Hos71] and by Heilmann and Lieb [HL72].

**Corollary 1** If \( G \) is a forest then \( f(G; x) = \mu(G; x) \).

An unexpected property of the matching polynomial, proved by Heilmann and Lieb [HL72], is that all its zeros are real, and furthermore the zeros for any graph \( G \) interlace with the zeros of any of the cards in its deck. The same paper also gives bounds for the zeros.

**Theorem 4** For any graph \( G \), the matching polynomial \( \mu(G; x) \) has only real zeros. Furthermore, if \( u \) is any vertex in \( G \) and if \( a_1, a_2, \ldots, a_n \) are the zeros of \( \mu(G; x) \) while the zeros of \( \mu(G - u; x) \) are \( a'_1, a'_2, \ldots, a'_{n-1} \), then

\[
    a_1 \leq a'_1 \leq a_2 \leq a'_2 \leq \ldots \leq a'_{n-1} \leq a_n,
\]

that is, the zeros of \( \mu(G; x) \) and \( \mu(G - u; x) \) interlace.
Theorem 5 The (real) zeros, $a_1, a_2, \ldots, a_n$, of $\mu(G; x)$, satisfy

$$|a_i| < 2\sqrt{\maxdeg(G) - 1}.$$  

We outline a proof of Theorem 4 from Godsil [God81b] that uses some of the results already mentioned for $\mu(G; x)$ and $f(G; x)$. First, given a graph $G$ and a vertex $u$ in $G$, the path tree $T(G, u)$ is the tree that has as its vertices the paths in $G$ which start at $u$, and where two such vertices are joined by an edge if one represents a maximal proper subpath (i.e. all but the last edge) of the other. We then have the following proposition from [God81b] that leads to a proof of Theorem 4.

Proposition 3 Let $u$ be a vertex in a graph $G$, let $T = T(G, u)$ be the path tree of $G$ with respect to $u$, and let $u'$ be the vertex of $T$ corresponding to the path of length 0 beginning at $u$. Then

$$\frac{\mu(G - u; x)}{\mu(G; x)} = \frac{\mu(T - u'; x)}{\mu(T; x)} = \frac{f(T - u'; x)}{f(T; x)}.$$  

The last equality follows from Corollary 1. Because all the roots of the characteristic polynomial are real, we conclude that all zeros and poles of the rational function $\mu(G - u; x)/\mu(G; x)$ are real. An induction argument on the number of vertices in $G$ then yields the conclusion that all the zeros of $\mu(G; x)$ are real.

An interesting combinatorial consequence of Theorem 4 is the following result of Heilmann and Lieb [HL72], which gives a stark contrast with how little is known about the coefficients of the chromatic polynomial.

Theorem 6 For any graph $G$, the sequence $\Phi_0(G)$, $\Phi_1(G)$, \ldots of coefficients of $g(G; x)$ is log-concave, that is $\Phi_i^2 \geq \Phi_{i-1}\Phi_{i+1}$.

The characteristic polynomial has been well studied, particularly with respect to graphs with the same characteristic polynomial. Godsil [God93] gives a thorough treatment of both the characteristic and the matching polynomials. Another good reference for the characteristic polynomial is Biggs [Big96]. Just as the matching polynomial is a way to study the matchings of a graph, the characteristic polynomial is a way to study the spectra of the adjacency matrix of a graph. Cvetković, Doob, and Sachs have written a book [CDS80] dedicated to the spectra of the adjacency matrix, and Lovász and Plummer [LP86] have a book devoted to the theory of matchings. Furthermore, although the characteristic polynomial is not a complete invariant of graphs, it is conjectured that the characteristic polynomial of a graph $G$ is reconstructible from its polynomial deck, i.e. from the set of characteristic polynomials of the cards of $G$. See Gutman and Cvetković [GC75] for the conjecture, and then Cvetković and Lepović [CL98], where it is proved in the case of trees.

3.2 Ehrhart Polynomial

A convex polytope $P$ is the convex hull of a finite set of points in $\mathbb{R}^m$. We denote the interior of $P$ (in the usual topological sense) by $P^o$. A convex polytope $P$
is said to be a rational or integral polytope if all its vertices have rational or integral coordinates, respectively. We write \( d = \dim P \) and call \( P \) a \( d \)-polytope.

For \( P \subset \mathbb{R}^m \) a rational \( d \)-polytope and \( t \) a nonnegative integer we define the functions 
\[
i(P; t) = |tP \cap \mathbb{Z}^m| \quad \text{and} \quad i(P; t) = |tP^0 \cap \mathbb{Z}^m|,
\]
where \( tP = \{ ta | a \in P \} \) is the \( t \)-fold dilatation of \( P \). Ehrhart proved in [Ehr67a, Ehr67b] that these functions are quasi-polynomials, that is, they are of the form
\[
c_d(t)t^d + c_{d-1}(t)t^{d-1} + \ldots + c_0(t),
\]
where each \( c_i(t) \) is a periodic function with integer period. Since \( i(P; t) \) is a quasi-polynomial, it can be defined for all \( t \in \mathbb{Z} \). In fact, we have the following reciprocity law due to Ehrhart [Ehr67c]:
\[
i(P; -t) = (-1)^d i(P; t).
\]

For more on this beautiful theory see the monograph by Ehrhart [Ehr77].

From Ehrhart [Ehr67a, Ehr67b] (also see Stanley [Sta96]), we have that when \( P \subset \mathbb{R}^m \) is an integral \( d \)-polytope, then \( i(P; t) \) and \( i(P; t) \) are polynomials, which leads to the following definition of the Ehrhart polynomial.

**Definition 5** Let \( P \) be an integral convex \( d \)-polytope. Then the Ehrhart polynomial of \( P \) is
\[
i(P; t) = c_0 + c_1 t + \ldots + c_{d-1} t^{d-1} + c_d t^d.
\]

From the early works of Ehrhart [Ehr67a] and Macdonald [Mac71] it is known that \( c_0 = 1 \) and \( c_d = \text{vol}(P) \), and that \( c_{d-1} \) is half of the surface area of \( P \), normalized with respect to the sub-lattice on each face of \( P \). Specifically, \( c_{d-1} = \frac{1}{d} \sum_F \text{vol}^{d-1}(F) \), where \( F \) ranges over all facets of \( P \) and the volume of a facet is measured intrinsically with respect to the lattice \( \mathbb{Z}^m \cap L_F \), where \( L_F \) is the affine hull of \( F \). The other coefficients were not well understood, until the later work of Betke and Kneser [BK85], Pommersheim [Pom93], Kantor and Khovanskii [KK93] and Diaz and Robins [DR97], but such interpretations go beyond the scope of this chapter. For the complexity of computing these coefficients see Barvinok [Bar94].

In the special case that \( P \) is a zonotope there is a combinatorial interpretation for the coefficients of the Ehrhart polynomial. First recall that if \( A \) is an \( r \times m \) real matrix written in the form \( A = [a_1, \ldots, a_r] \), then it defines a zonotope \( Z(A) \) which consists of those points \( p \) of \( \mathbb{R}^m \) which can be expressed in the form
\[
p = \sum_{i=1}^{m} \lambda_i a_i, \quad 0 \leq \lambda_i \leq 1.
\]
In other words, \( Z(A) \) is the Minkowski sum of the line segments \([0, a_i]\) for \( 1 \leq i \leq n \). For more on zonotopes, see McMullen [McM71].

When \( A \) has integer entries, Stanley [Sta80], using techniques from Shephard [She74], proved that \( i(P; t) = \sum_X f(X) t^{|X|} \), where \( X \) ranges over all linearly independent subsets of columns of \( A \) and where \( f(X) \) denotes the greatest common divisor of all minors of sizes \(|X|\) of the matrix \( A \).
When \( A \) is a \textit{totally unimodular matrix}, that is, the determinant of every square submatrix is 0 or \( \pm 1 \), then \( Z(A) \) is described as a \textit{unimodular zonotope}. For these polytopes the previous result shows that

\[
i(Z(A); t) = \sum_{k=0}^{r} f_k t^k,
\]

where \( f_k \) is the number of subsets of columns of the matrix \( A \) which are linearly independent and have cardinality \( k \). In other words, the Ehrhart polynomial \( i(Z(A); t) \) is the generating function of the number of independent sets in the regular matroid \( M(A) \).

The incidence matrix \( D(G) \) of a graph \( G \) is totally unimodular, a long-standing result due to Poincaré [Poi01] with a modern treatment given by Biggs [Big96]. A linearly independent subset of columns in \( D \) corresponds to a subset of edges with no cycle. Thus, the coefficient \( f_k \) in this case is the number of spanning forests of \( G \) with exactly \( k \) edges. From the previous chapter we know that \( T(G; x + 1, y) = \sum_{k=0}^{r} f_k x^{r-k} \), where \( r \) is the rank of the graph \( G \). With these ingredients we get the following relations with the Tutte polynomial from Welsh [Wel97].

**Theorem 7** If \( G \) is a graph and \( D \) is its incident matrix then the Ehrhart polynomial of the unimodular zonotope \( Z(D) \) is given by

\[
i(Z(D); t) = t^r T(G; 1 + \frac{1}{t}, 1),
\]

where \( r \) is the rank of \( G \).

In this case, the zonotope \( Z(D) \) is a \( r \)-polytope in \( \mathbb{R}^n \), where \( n \) is the number of vertices of \( G \).

The reciprocity law of Theorem 7 leads to the following geometric result, also from Welsh [Wel97].

**Corollary 2** If \( D \) is the incidence matrix of a rank \( r \) graph \( G \) with \( n \) vertices then for any positive integer \( \lambda \) the number of lattice points of \( \mathbb{R}^n \) lying strictly inside the zonotope \( tZ(D) \) is given by

\[
i(Z(D); t) = (-t)^r T(G; 1 - \frac{1}{t}, 1).
\]

In particular we have that the number of lattice points strictly inside \( Z(D) \) is \((-1)^r T(G; 0, 1)\).

### 3.3 The Topological Tutte Polynomial of Bollobás and Riordan

The classical Tutte polynomial discussed in the previous chapter is an invariant of abstract graphs, so it encodes no information specific to graphs embedded
in surfaces. In [BR01, BR02], Bollobás and Riordan generalize the classical Tutte polynomial to topological graphs, that is, graphs embedded in surfaces. In [BR01], Bollobás and Riordan define the cyclic graph polynomial, a three variable deletion/contraction invariant for graphs embedded in oriented surfaces. They extend this work in [BR02], using a different approach, with the four variable ribbon graph polynomial. Both of these polynomials extend the classical Tutte polynomial, but in such a way that topological information about the embedding is encoded. The version for oriented surfaces is subsumed by the version for arbitrary surfaces, so we focus on the latter here. The ribbon graph polynomial is also sometimes called the Bollobás-Riordan polynomial after the authors or the topological Tutte polynomial to emphasize that it simultaneously encodes topological information while generalizing the classical Tutte polynomial.

First recall that a cellular embedding of a graph in an orientable or unorientable surface can be specified by providing a sign for each edge and a rotation scheme for the set of half edges at each vertex, where a rotation scheme is simply a cyclic ordering of the half edges about a vertex. This is equivalent to a ribbon (or fat) graph, which is a surface with boundary where the vertices are represented by a set of disks and the edges by ribbons, with the ribbon of an edge with a negative sign having a half-twist. This can also be thought of as taking a slight ‘fattening’ of the edges of the graph as it is embedded in the surface, or equivalently as ‘cutting out’ the graph together with a small neighborhood of it from the surface. Figure 3 shows a graph with two vertices and two parallel edges, one positive and one negative. It is embedded on a Klein bottle, and the ribbon graph is a Möbius band with boundary.

![Figure 3: A ribbon graph which is a Möbius band with boundary](image)

In addition to the usual graphic characteristics such as number of vertices, connected components, rank, and nullity for a ribbon graph $G$ we also consider $bc(G)$, the number of boundary components of the surface, and $t(G)$, an index of the orientability of the surface. The value of $t(G)$ is 0 if the surface is orientable, and 1 if it is not. Thus, $t(G)$ is 1 if and only if for some cycle in $G$, the product of the signs of the edges is negative.

**Definition 6** Let $G$ be a ribbon graph, that is, a graph embedded in a surface.
The topological Tutte polynomial of Bollobás and Riordan is given by

$$R(G; x, y, z, w) = \sum_{A \subseteq E(G)} (x - 1)^{r(G) - r(A)} y^{n(A)} z^{\kappa(A) - bc(A) + n(A)} w^{t(A)} \in \mathbb{Z}[x, y, z, w]/\langle w^2 - w \rangle.$$ 

As previously, $r(A)$, $\kappa(A)$, $n(A)$, and now also $bc(A)$ and $t(A)$, refer to the spanning subgraph of $G$ with edge set $A$, here with its embedding inherited from $G$.

Clearly, by comparing with the rank-nullity generating function definition of the classical Tutte polynomial given in the previous chapter, this generalizes the classical Tutte polynomial. Like the classical Tutte polynomial, $R(G; x, y, z, w)$ is multiplicative on disjoint unions and one point joins of ribbon graphs. More importantly, it retains the essential property of obeying a deletion/contraction reduction relation.

We must first define deletion and contraction in the context of embedded graphs. The ribbon graph resulting from deleting an edge is clear, but contraction requires some care. Let $e$ be a non-loop edge. First assume the sign of $e$ is positive, by flipping one endpoint if necessary to remove the half twist (this reverses the cyclic order of the half edges at that vertex and toggles their signs). Then $G/e$ is formed by deleting $e$ and identifying its endpoints into a single vertex $v$. The cyclic order of edges at $v$ comes from the original cyclic order at one endpoint, beginning where $e$ had been, and continuing with the cyclic order at the other endpoint, again beginning where $e$ had been.

**Theorem 8** If $G$ is a ribbon graph, then

$$R(G; x, y, z, w) = R(G/e; x, y, z, w) + R(G - e; x, y, z, w)$$

if $e$ is an ordinary edge and $R(G; x, y, z, w) = xR(G/e; x, y, z, w)$ if $e$ is a bridge.

The proof depends on a careful analysis of how each of the relevant parameters $r(A)$, $\kappa(A)$, $n(A)$, $bc(A)$ and $t(A)$ changes with the deletion or contraction of an edge.

Repeated application of this theorem reduces a ribbon graph to a disjoint union of embedded blossom graphs, that is, graphs each consisting of a single vertex with some number of loops. Because of the embedding, the loops are signed, and there is a rotation system of half-edges about the single vertex. Not surprisingly, the topological information is distilled into these minors of the original graph, and to complete a deletion/contraction linear recursion computation, it is necessary to specify an evaluation of these terminal forms.

Signed chord diagrams provide a useful device for determining the relevant parameters of an embedded blossom graph. Recall that a chord diagram consists of a circle with $n$ symbols on its perimeter, with each symbol appearing twice and a chord drawn between each pair of like symbols. A signed chord diagram simply has a sign on each chord. A signed chord diagram $D$ corresponds to an embedded blossom graph $G$ by assigning a symbol to each loop and arranging
them on the perimeter of the circle in the chord diagram in the same order as the cyclic order of the half-edges about the vertex. A chord receives the same
sign as the loop it represents. If we ‘fatten’ the chords as in Figure 4 with
a negative chord receiving a half-twist, then \( bc(G) \) is equal to the number
of components in the resulting diagram, which is denoted \( bc(D) \). Similarly,
since \( G \) has only one vertex, \( n(G) \) is the number of edges of \( G \), which is the
number of chords of \( D \), so we denote this by \( n(D) \). We also set \( t(D) = t(G) \), and note
that \( t(D) = 0 \) if all chords of \( D \) have a positive sign, and \( t(D) = 1 \) otherwise.
This, combined with the definition of \( R(G; x, y, z, w) \) above, gives the following
evaluation for these terminal forms.

\[
R(G; x, y, z, w) = \sum_{D' \subseteq D} y^{n(D')} z^{1 - bc(D')} w^{t(D')},
\]

where the sum is over all subdiagrams \( D' \) of \( D \).

Theorems 8 and 9 taken together give a linear recursion definition for \( R(G; x, y, z, w) \). There are a number of technical considerations, similar to the care
that must be taken in contracting edges, but nevertheless many other properties
analogous to those of the classical Tutte polynomial hold. For example, \( R(G; x, y, z, w) \) has a spanning tree expansion, a universality property, and
duality relation (in addition to the Bollobás and Riordan’s originating work
in [BR01,BR02], see also recent work by Chmutov [Chm] and Moffatt [Mof08]).
Furthermore, Chmutov and Pak [CP07], and Moffatt [Mof,Mof08] have shown
that \( R(G; x, y, z, w) \) also extends the relation between the classical Tutte poly-
nomial and the Kauffman bracket and Jones polynomial of knot theory due to
Thistlethwaite [Thi87] and Kauffman [Kau89].

3.4 Martin, or Circuit Partition, Polynomials
In his 1977 thesis, Martin [Mar77] recursively defined polynomials \( M(G, x) \)
and \( m(\vec{G}; x) \) that encode, respectively, information about the families of cir-

Figure 4: A signed blossom graph, its signed chord diagram, and the boundary
components of the signed chord diagram

\[
\begin{align*}
\text{a} & \quad \text{b} \\
\text{a} & \quad \text{b}
\end{align*}
\]

\[
\begin{align*}
\text{a} & \quad \text{b} \\
\text{a} & \quad \text{b}
\end{align*}
\]

\[
\begin{align*}
\text{a} & \quad \text{b} \\
\text{a} & \quad \text{b}
\end{align*}
\]

\[
\begin{align*}
\text{a} & \quad \text{b} \\
\text{a} & \quad \text{b}
\end{align*}
\]
cuits in 4-regular Eulerian graphs and digraphs. Las Vergnas subsequently found a state model expression for these polynomials, extended their properties to general Eulerian graphs and digraphs, and further developed their theory (see [Las79,Las83,Las88]). Both Martin [Mar78] and Las Vergnas [Las88] found combinatorial interpretations for some small integer evaluations of the polynomials, while combinatorial interpretations for all integer values as well as some derivatives were given in [E-M00,E-M04a,E-M04b], and by Bollobás [Bol02].

Transforms of the Martin polynomials, \(J(G; x)\) and \(j(\vec{G}; x)\), given in [E-M98], and then aptly named circuit partition polynomials in [ABS00], facilitate these computations, and for this reason we give the definitions below in terms of \(J\) and \(j\). Like many of the polynomials surveyed here, the circuit partition polynomials have several definitions, including linear recursion formulations, generating function formulations, and state model formulations. We give the state model definition, and refer the reader to [E-M04a,E-M04b] for the others.

As with other state model formulations, we must first specify what we mean by a state of a graph (or digraph) in this context. Here an Eulerian graph must have vertices all of even degree, but it need not be connected. An Eulerian digraph must have the indegree equal to the outdegree at each vertex, and again need not be connected.

**Definition 7** An Eulerian graph state of an Eulerian graph \(G\) is the result of replacing each \(2n\)-valent vertex \(v\) of \(G\) with \(n\) 2-valent vertices joining pairs of half edges originally adjacent to \(v\). An Eulerian graph state of an Eulerian digraph \(\vec{G}\) is defined similarly, except here each incoming half edge must be paired with an outgoing edge.

Note that a Eulerian graph state is a disjoint union of cycles, each consistently oriented in the case of a digraph.

**Definition 8** The circuit partition polynomial. Let \(G\) be an Eulerian graph, let \(St(G)\) be the set of states of \(G\), and let \(c(S)\) be the number of components in a state \(S \in St(G)\). Then the circuit partition polynomial has a state model formulation given by

\[
J(G; x) = \sum_{S \in St(G)} x^{c(S)}.
\]

The circuit partition polynomial is defined similarly for Eulerian digraphs as

\[
j(\vec{G}; x) = \sum_{S \in St(\vec{G})} x^{c(S)}.
\]

The transforms between the circuit partition polynomials and the original Martin polynomial, as extended to general Eulerian graphs and digraphs by Las Vergnas, are:

\[
J(G; x) = xM(G; x + 2), \text{ for } G \text{ an Eulerian graph, and}
\]

\[
j(\vec{G}; x) = xm(\vec{G}; x + 1), \text{ for } \vec{G} \text{ an Eulerian digraph.}
\]
The circuit partition polynomials have ‘splitting’ formulas, analogous to Tutte’s identity for the chromatic polynomial given in the previous chapter, proofs for which may be found in [E-M98,E-M04b]. These formulas derive from the Hopf algebra structures of the generalized transition polynomial discussed in Subsection 4.3, but may also be proved combinatorially, as by Bollobás [Bol02] and [E-M04b].

**Theorem 10** Let $G$ be an Eulerian graph and $\vec{G}$ be an Eulerian digraph. Then

$$J(G; x + y) = \sum J(A; x) J(A^c; y),$$

where the sum is over all subsets $A \subseteq E(G)$ such that $G$ restricted to both $A$ and $A^c = E(G) - A$ is Eulerian. Also,

$$j(\vec{G}; x + y) = \sum j(\vec{A}; x) j(\vec{A}^c; y),$$

where the sum is over all subsets $\vec{A} \subseteq E(\vec{G})$ such that $\vec{G}$ restricted to both $\vec{A}$ and $\vec{A}^c$ is an Eulerian digraph.

The connection between the circuit partition polynomial of a digraph and the Tutte polynomial of a planar graph $G$ is through the oriented medial graph $\vec{G}_m$ described in the previous chapter. Martin [Mar77] proved the following, which we extend to the circuit partition polynomial via (3.2).

**Theorem 11** Let $G$ be a connected planar graph, and let $\vec{G}_m$ be its oriented medial graph. Then relationships among the Martin polynomial, circuit partition polynomial, and Tutte polynomial are:

$$j(\vec{G}; x) = xm(\vec{G}_m; x + 1) = xt(G; x + 1, x + 1).$$

The proof of this theorem depends on a fundamental observation relating deletion/contraction in $G$ with choices of configurations at a vertex in an Eulerian graph state of $\vec{G}_m$, as illustrated in Figure 5. Theorems 10 and 11 combine to give the basis for many of the combinatorial interpretation of the Tutte polynomial along the line $y = x$ described in the previous chapter. For more details, see Martin [Mar77,Mar78], Las Vergnas [Las79,Las83,Las88], Bollobás [Bol02], and also [E-M98,E-M00,E-M04a,E-M04b].

Evolving from the relation between the Tutte and Martin polynomials is the theory of isotropic systems, which unifies essential properties of 4-regular graphs and pairs of dual binary matroids. A series of papers throughout the 1980’s and 1990’s, including work by Bouchet [Bou87a], [Bou87b], [Bou88], [Bou89], [Bou91], [Bou93], as well as Bouchet and Ghier [BG96], and Jackson [Jac91], significantly extends the relationship between the Tutte polynomial of a planar graph and the Martin polynomial of its medial graph via the theory of isotropic systems.
An edge $e$ in a planar graph $G$, with the corresponding vertex $v$ in the oriented medial graph $\vec{G}_m$ (dotted edges). Deleting $e$ corresponds to one possible configuration at $v$ in an Eulerian graph state of $\vec{G}_m$, while contracting $e$ corresponds to the other.

Figure 5: An edge $e$ in a planar graph $G$, with the corresponding vertex $v$ in the oriented medial graph $\vec{G}_m$ (dotted edges). Deleting $e$ corresponds to one possible configuration at $v$ in an Eulerian graph state of $\vec{G}_m$, while contracting $e$ corresponds to the other.

3.5 Interlace Polynomial

In [ABS00], Arratia, Bollobás and Sorkin defined a one-variable graph polynomial motivated by questions arising from DNA sequencing by hybridization addressed by Arratia, Bollobás, Coppersmith and Sorkin in [ABC01], an application we will return to in Section 5. In [ABS04b], Arratia, Bollobás, and Sorkin defined a two-variable interlace polynomial, and showed that the original polynomial of [ABS00] is a specialization of it, renaming the original one-variable polynomial as the vertex-nullity interlace polynomial due to its relationship with the two-variable generalization.

Remarkably, despite very different terminologies, motivations and approaches, the original vertex-nullity interlace polynomial of a graph may be realized as the Tutte-Martin polynomial of an associated isotropic system (see Bouchet [Bou09]). For exploration of this relationship, see the works mentioned in Subsection 3.4 as well as Aigner [Aig00], Aigner and Mielke [AM00], Aigner and van der Holst [AvdH04], Allys [All94], and also Bouchet’s series on multimatroids [Bou97, Bou98a, Bou98b, Bou01].

Both the vertex-nullity interlace polynomial of a graph and the two-variable interlace polynomial may be defined recursively via a pivot operation. This pivot is defined as follows. Let $vw$ be an edge of a graph $G$, and let $A_v$, $A_w$ and $A_{vw}$ be the sets of vertices in $V(G) \setminus \{v, w\}$ adjacent to $v$ only, $w$ only, and to both $v$ and $w$, respectively. The pivot operation “toggles” the edges among $A_v$, $A_w$, and $A_{vw}$, by deleting existing edges and inserting edges between previously non-adjacent vertices. The result of this operation is denoted $G^{vw}$. More formally, $G^{vw}$ has the same vertex set as $G$, and edge set equal to the symmetric difference $E(G) \Delta S$, where $S$ is the complete tripartite graph with vertex classes $A_v$, $A_w$ and $A_{vw}$. See Figure 6.
Figure 1: Pivoting on the edge \(vw\). \(A_v, A_w\) and \(A_{vw}\) are the sets of vertices of \(G\) adjacent to \(v\) only, \(w\) only, and to both \(v\) and \(w\), respectively. These sets are constant in all the diagrams. Vertices of \(G\) adjacent to neither \(v\) nor \(w\) are omitted. Heavy lines indicate that all edges are present, and dotted lines represent non-edges. Note the interchange of edges and non-edges among \(A_v, A_w\) and \(A_{vw}\).

![Diagram](image)

Figure 6: Pivoting on the edge \(vw\). \(A_v, A_w\) and \(A_{vw}\) are the sets of vertices of \(G\) adjacent to \(v\) only, \(w\) only, and to both \(v\) and \(w\), respectively. These sets are constant in all the diagrams. Vertices of \(G\) adjacent to neither \(v\) nor \(w\) are omitted. Heavy lines indicate that all edges are present, and dotted lines represent non-edges. Note the interchange of edges and non-edges among \(A_v, A_w\) and \(A_{vw}\).

Also, \(G^a\) is the local complementation of \(G\), defined as follows. Let \(N(a)\) be the neighbors of \(a\), that is, the set \(\{w \in V : a\) and \(w\) are joined by an edge\}\}. The graph \(G^a\) is equal to \(G\) except that we “toggle” the edges among the neighbors of \(a\), switching edges to non-edges and vice-versa.

**Definition 9** Let \(G\) be a graph of order \(n\), which may have loops, but no multiple loops or multiple edges. The two-variable interlace polynomial may be given recursively by \(q(E_n) = y^n\) for \(E_n\), the edgeless graph on \(n \geq 0\) vertices, with

\[
q(G) = q(G - a) + q(G_{ab} - b) + ((x - 1)^2 - 1)q(G_{ab} - a - b),
\]

for any edge \(ab\) where neither \(a\) nor \(b\) has a loop, and

\[
q(G) = q(G - a) + (x - 1)q(G^a - a),
\]

for any looped vertex \(a\).

Alternatively, the interlace polynomial has the following generating function representation.

**Definition 10** Let \(G\) be a graph of order \(n\), which may have loops, but no multiple loops or multiple edges. Then the two-variable interlace polynomial may be given by

\[
q(G; x, y) = \sum_{S \subseteq V(G)} (x - 1)^r(G|_S)(y - 1)^n(G|_S),
\]

where \(r(G|_S)\) and \(n(G|_S) = |S| - r(G|_S)\) are, respectively, the \(\mathbb{F}_2\)-rank and nullity of the adjacency matrix of \(G|_S\), the subgraph of \(G\) restricted to \(S\).
**Definition 11** The vertex-nullity interlace polynomial is defined recursively as:

\[ q_N(G; x) = \begin{cases} 
  x^n & \text{if } G = E_n, \text{ the edgeless graph on } n \text{ vertices} \\
  q_N(G - v; x) + q_N(G^{vw} - w; x) & \text{if } vw \in E(G).
\] 

This polynomial was shown to be well defined by Arratia, Bollobás, and Sorkin for all simple graphs in [ABS00], and then was shown in [ABS04b] to be a specialization of the two-variable interlace polynomial as follows.

\[ q_N(G; y) = q(G; 2, y) = \sum_{W \subseteq V(G)} (y - 1)^{n(G|_W)}. \]

An equivalent formulation for \( q_N(G; x) \) for simple graphs is given by Aigner and van der Holst in [AvdH04].

A somewhat circuitous route through the circuit partition polynomial relates the vertex-nullity interlace polynomial to the Tutte polynomial. First recall that a circle graph on \( n \) vertices is a graph \( G \) derived from a chord diagram. Two vertices \( v \) and \( w \) in \( G \) share an edge if and only if their corresponding chords intersect in the chord diagram. Note that \( G \) is necessarily simple.

For circle graphs, the vertex-nullity interlace polynomial and the circuit partition polynomial are related by the following theorem, noting that although \( \vec{G} \) may be a multigraph, \( H \) is necessarily simple.

**Theorem 12** (Arratia, Bollobás and Sorkin [ABS00], Theorem 6.1). If \( \vec{G} \) is a 4-regular Eulerian digraph, \( C \) is any Eulerian circuit of \( \vec{G} \), and \( H \) is the circle graph of the chord diagram determined by \( C \), then \( j(\vec{G}; x) = xq_N(H; x + 1) \).

This now allows us to relate the vertex-nullity interlace polynomial to the Tutte polynomial, a relation proved in [E-MS07] and also observed by Arratia, Bollobás and Sorkin at the end of Section 7 in [ABS04b].

**Theorem 13** If \( G \) is a planar graph, and \( H \) is the circle graph of some Eulerian circuit of \( \vec{G}_m \), then \( q_N(H; x) = t(G; x, x) \).

**Proof.** By Theorem 12, \( j(\vec{G}_m; x) = xq_N(H; x + 1) \), but recalling that the circuit partition and Martin polynomials are simple translations of each other, we have from Theorem 11 that \( j(\vec{G}_m; x) = xm(\vec{G}_m; x + 1) \), and hence \( q_N(H; x) = m(\vec{G}_m; x) = t(G; x, x) \).

The interlace polynomial has generated further interest and other applications in Balister, Bollobás, Cutler, and Pebody [BBCP02], and Balister, Bollobás, Riordan, and Scott [BBRS01], Glantz and Pelillo [GP06], Ellis-Monaghan and Sarmiento [E-MS07].

### 4 Multivariable Extensions

Multivariable extensions have proved valuable theoretical tools for many of the polynomials we have seen since they capture information not encoded by the
original polynomial. More critically, powerful algebraic tools not applicable to
the original polynomial may be available to the multivariable version, providing
new means of extracting combinatorial information from the polynomial.
While the multivariable indexing may make the defining notation somewhat
bulky, these generalizations are natural extensions of classical versions, computed in exactly the same ways, only now also keeping track of some additional
parameters in the computation processes.

4.1 Generalized coloring polynomials

and the U-polynomial

The evaluation of the chromatic polynomial at \( \lambda \) can be written as

\[
\chi_G(\lambda) = \sum_{\phi: V \to \{1, \ldots, \lambda\} \text{ proper}} 1.
\]

(4.1)

This was generalized to a symmetric function over (commuting) indeterminate
s \( x_1, x_2, \ldots \) by Stanley [Sta95] in the following way.

**Definition 12** Let \( G = (V, E) \) be a graph, let \( \phi : V \to \mathbb{P} = \{1, 2, \ldots\} \), and denote the product \( \prod_{v \in V} x_{\phi(v)} \) by \( x^{\phi} \). Then the symmetric function generalization of the chromatic polynomial is

\[
X_G(x) = X(G; x_1, x_2, \ldots) = \sum_{\phi: V \to \mathbb{P} \text{ proper}} x^{\phi}.
\]

That this is a generalization of the chromatic polynomial can be seen by
setting \( x_i = 1 \) for \( 1 \leq i \leq \lambda \) and \( x_j = 0 \) for \( j > \lambda \) and noting that the expression
in (4.1) for the chromatic polynomial evaluated at \( \lambda \) results.

Generalizing polynomial graph invariants is not a theoretical exercise. The
original invariant encodes combinatorial information, and the multivariable
generalization will encode not only the same information but also more refined
information. For example, the chromatic polynomial of any tree with \( n \)
vertices has chromatic polynomial \( x(x - 1)^{n-1} \). But not all trees have the same \( X_G(x) \).
For example, if \( K_{1,3} \) is the 4-star graph and \( P^4 \) is the path of order 4, \( X_{K_{1,3}}(x) \)
has a term \( x_i x_j^3 \) for all \( i \neq j \), but such a term is not present in \( X_{P^4}(x) \). In fact,
it is still an open question if \( X \) distinguishes trees, that is if \( X_{T_1}(x) \neq X_{T_2}(x) \),
whenever \( T_1 \) and \( T_2 \) are not isomorphic trees.

A similar multivariable extension of the bad coloring polynomial is also natural,
especially given the importance of the latter because of its being equivalent
to the Tutte polynomial. The following generalization of the bad coloring
polynomial is also due to Stanley [Sta98].

**Definition 13** Let \( G = (V, E) \) be a graph, let \( \phi : V \to \mathbb{P} = \{1, 2, \ldots\} \), and let
\( b(\phi) \) be the set of monochromatic edges in the coloring given by \( \phi \). Then the
symmetric function generalization of the bad coloring polynomial over indeterminates $x_1, x_2, \ldots$ and $t$ is

$$X_G(x, t) = \sum_{\phi : V \to \mathbb{P}} (1 + t)^{|b(\phi)|} x^{\phi},$$

where the sum is over all possible colorings $\phi$ of the graph $G$.

Again, by setting $x_i = 1$ for $1 \leq i \leq \lambda$ and $x_j = 0$ for $j > \lambda$ we get the bad-coloring polynomial, and hence the Tutte polynomial. Therefore, $X_G(x, t)$ is a multivariable generalization of the Tutte polynomial.

There is another multivariable generalization of the Tutte polynomial that was developed independently and for very different reasons. This generalization is called the U-polynomial and is due to Noble and Welsh in [NW99].

**Definition 14** Let $G = (V, E)$ be a graph. Then the U-polynomial of $G$ is

$$U_G(x, y) = \sum_{A \subseteq E} x_{n_1} \cdots x_{n_k} (y - 1)^{|A| - r(A)},$$

where $n_1, \ldots, n_k$ are the numbers of vertices in the $k$ different components of $G$ restricted to $A$.

Clearly, this is a generalization of the Tutte polynomial, as by setting $x_i = (x - 1)$ for all $i$ in $U_G(x, y)$ we get $(x - 1)^{k(G)} T_G(x, y)$. Note that the factor $x_{n_1} \cdots x_{n_k}$ in every term keeps track of the number of vertices in the different components in $A$. This is a refinement of the rank-nullity generating-function definition of the Tutte polynomial where the factors $x^{r(G) - r(A)} = x^{n(A) - n(G)}$ in each term keep track of the number of components in $A$.

That $U_G$ captures more combinatorial information from $G$ than the Tutte polynomial can be seen by noting that $U_G$ contains the matching generating polynomial, and thus the matching polynomial, as a specialization as well.

**Theorem 14** For any graph $G$,

$$g(G; x) = U_G(1, t, 0, \ldots, 0, \ldots, y = 1).$$

The U-polynomial has a deletion/contraction reduction relationship not in the class of graphs but in the class of weighted graphs. To see this, we turn to the W-polynomial also due to Noble and Welsh in [NW99]. A weighted graph consists of a graph $G = (V, E)$, together with a weight function $\omega : V \to \mathbb{Z}^+$. If $e$ is an edge of $(G, \omega)$ then $(G \setminus e, \omega)$ is the weighted graph obtained from $(G, \omega)$ by deleting $e$ and leaving $\omega$ unchanged. If $e$ is not a loop, $(G/e, \omega/e)$ is the weighted graph obtained from $(G, \omega)$ by contracting $e$, that is deleting $e$ and identifying its endpoints $v, v'$ into a single vertex $v''$. The weight function $\omega/e$ is defined as $\omega/e(u) = \omega(u)$ for all $u \in V \setminus \{v, v'\}$ and $\omega/e(v'') = \omega(v) + \omega(v')$. 

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Definition 15 Let \((G, \omega)\) be a weighted graph. The \(W\)-polynomial may be given recursively by the following rules. If \(e\) is an ordinary edge or a bridge, then

\[
W(G, \omega)(x, y) = W(G/e, \omega)(x, y) + W(G/\omega/e)(x, y).
\]

If \(e\) is a loop, then \(W(G, \omega)(x, y) = yW(G/e, \omega)(x, y)\). Otherwise, \((G, \omega)\) is \(E_n\), the edgeless graph on \(n \geq 0\) vertices, with weights \(a_1, \ldots, a_n\) and \(W(E_n, \omega)(x, y) = x^{a_1} \cdots x^{a_n}\).

That the resulting multivariate polynomial \(W\) is independent of the order in which the edges are deleted and contracted is proved in [NW99]. This can easily be done by induction on the number of edges once it is proved that the order in which you contract or delete edges in \((G, \omega)\) does not affect the weighted graph which you obtain.

The \(U\)-polynomial is obtained from the \(W\)-polynomial by setting all weights equal to 1 and a proof that this definition is equivalent to Definition 14 can be found in [NW99]. Actually in [NW99] it is proved that \(W\) has a representation of the form

\[
W(G, \omega)(x, y) = \sum_{A \subseteq E} x^{c_1} \cdots x^{c_k} (y - 1)^{|A| - \tau(A)},
\]

where \(c_i, 1 \leq i \leq k\), is the total weight of the \(i\)th component of the weighted subgraph \((A, \omega)\).

Noble and Welsh [NW99] show that the symmetric function generalization of the bad coloring polynomial and the \(U\)-polynomial are equivalent in the following sense.

Theorem 15 For any graph \(G\), the polynomials \(U_G\) and \(X_G\) determine each other in that if \(p_0 = 1\) and \(p_r = \sum_i x_i^r\), then

\[
X_G(x, t) = t^{|V|} U_G(x_j = \frac{p_j}{t}, y = t + 1).
\]

There is yet another polynomial, the polychromate, introduced originally by Brylawski in [Bry81], that is as general as \(U_G\) or \(X_G\). Given a graph \(G\) and a partition \(\pi\) of its vertices into non-empty blocks, let \(e(\pi)\) be the number of edges with both ends in the same block of the partition. If \(\tau(\pi) = (n_1, \ldots, n_k)\) is the type of partition \(\pi\), we denote by \(x_{\tau(\pi)}\) the monomial \(\prod_{i=1}^k x_i^{n_i}\).

Definition 16 Let \(G\) be a graph. Then the polychromate \(\chi_G(x, y)\) is

\[
\chi_G(x, y) = \sum_{\pi} y^{e(\pi)} x_{\tau(\pi)},
\]

where the sum is over all partitions of \(V(G)\).

We have the following theorem due to Sarmiento in [Sar00] but see [MN] for a different proof.

Theorem 16 The polynomials \(U_G(x, y)\) and \(\chi_G(x, y)\) are equivalent.
The story doesn’t end here. All three polynomials $U_G(x,y)$, $X_G(x,t)$ and $\chi_G(x,y)$ have natural extensions. For example the extension of the $X_G(x,t)$ replaces the $t$ variable by countably infinitely many variables $t_1, t_2, \ldots$, enumerating not just the total number of monochromatic edges but the number of monochromatic edges of each color. It is defined as follows.

$$X_G(x,t) = \sum_{\phi: V \to P} \left( \prod_{i=1}^{\infty} (1 + t_i)^{|b_i(\phi)|} \right) x^\phi,$$

where the sum is over all colorings $\phi$ of $G$ and $b_i(\phi)$ is the set of monochromatic edges for which both end points have color $i$. By setting $t_i = t$ for all $i \geq 1$ we regain $X_G(x,t)$.

For the other extension the reader is referred to [MN]. There it is also proved that all these extensions are equivalent.

### 4.2 The Parametrized Tutte Polynomial

The basic idea of a parametrized Tutte polynomial is to allow each edge of a graph to have four parameters (four ring values specific to that edge), which apply as the Tutte polynomial is computed via a deletion/contraction recursion. Which parameter is applied in a linear recursion reduction depends on whether the edge is deleted or contracted as an ordinary edge, or whether it is contracted as an isthmus or deleted as a loop. The difficulty lies in assuring that a well-defined function, that is, one independent of the order of deletion/contraction, results. This requires a set of relations, coming from three very small graphs, to be satisfied. Interestingly, additional constraints are necessary for there to be a corank-nullity expansion or even for the function to be multiplicative or a graph invariant, that is, equal on isomorphic graphs.

The motivation for allowing edge-specific values for the deletion/contraction recursion comes from a number of applications where it is natural. This includes graphs with signed edges coming from knot theory, graphs with edge-specific failure probability in network reliability, and graphs whose edges represent various interaction energies within a molecular lattice in statistical mechanics. While there is compelling motivation for allowing various edge parameters, the technical details of a general theory are challenging. The two major works in this area are Zaslavsky [Zas92] and Bollobás and Riordan [BR99]. However, these two works take different approaches, which were subsequently reconciled with a mild generalization in [E-MT06], and for this reason we adopt the formalism of [E-MT06]. Bollobás and Riordan [BR99] also give a succinct historical overview of the development of these multivariable extensions.

For the purposes of the following, we consider a class of graphs minor-closed if it is closed under the deletion of loops, the contraction of bridges, and the contraction and deletion of ordinary edges; however we do not require closure under the deletion of bridges. Some formalism is necessary to handle the parameters.

**Definition 17** Let $U$ be a class, and let $R$ be a commutative ring. Then an $R$-parametrization of $U$ consists of four parameter functions $x, y, X, Y : U \to R$,
denoted $e \rightarrow x_e, y_e, X_e, Y_e$.

**Definition 18** Let $U$ be an $R$-parametrized class, and let $\Gamma$ be a minor-closed class of graphs with $E(G) \subseteq U$ for all $G \in \Gamma$. Then a parametrized Tutte polynomial on $\Gamma$ is a function $T : \Gamma \rightarrow R$ which satisfies the following: $T(G) = X_e T(G/e)$ for any bridge $e$ of $G \in \Gamma$, and $T(G) = Y_e T(G - e)$ for any loop $e$ of $G \in \Gamma$, and $T(G) = y_e T(G - e) + x_e T(G/e)$ for any ordinary edge $e$.

The following theorem gives the central result. Identity in Item 1 comes from requiring to be equal the two ways of carrying out deletion/contraction reductions on a graph on two vertices with two parallel edges $e_1$ and $e_2$ having parameters $\{x_e, y_e, X_e, Y_e\}$. Similarly, identities in Items 2 and 3 come from considering the $\theta$-graph and $K_3$. Here again $E_n$ is the edgeless graph on $n$ vertices.

**Theorem 17 (The generalized Zaslavsky-Bollobás-Riordan theorem)**

Let $R$ be a commutative ring, let $\Gamma$ be a minor-closed class of graphs whose edge-sets are contained in an $R$-parametrized class $U$, and let $a_1, a_2, \ldots \in R$. Then there is a parametrized Tutte polynomial $T$ on $\Gamma$ with $T(E_n) = a_n$ for all $n$ with $E_n \in \Gamma$ if and only if the following identities are satisfied:

1. Whenever $e_1$ and $e_2$ appear together in a circuit of a $k$-component graph $G \in \Gamma$, then $a_k (x_{e_1} Y_{e_2} + y_{e_1} X_{e_2}) = a_k (x_{e_2} Y_{e_1} + y_{e_2} X_{e_1})$.

2. Whenever $e_1$, $e_2$ and $e_3$ appear together in a circuit of a $k$-component graph $G \in \Gamma$, then $a_k X_{e_3} (x_{e_1} Y_{e_2} + y_{e_1} X_{e_2}) = a_k X_{e_1} (x_{e_2} Y_{e_3} + x_{e_2} Y_{e_2})$.

3. Whenever $e_1$, $e_2$ and $e_3$ are parallel to each other in a $k$-component graph $G \in \Gamma$, then $a_k Y_{e_3} (x_{e_1} Y_{e_2} + y_{e_1} x_{e_2}) = a_k Y_{e_1} (Y_{e_2} x_{e_3} + Y_{e_2} y_{e_3})$.

A most general parametrized Tutte polynomial, what possibly could be called the parametrized Tutte polynomial, might begin with the polynomial ring on independent variables $\{x_e, y_e, X_e, Y_e: e \in U\} \cup \{a_i : i \geq 1\}$. However, the resulting function is not technically a polynomial, in that it must take its values not in the polynomial ring, but has as $R$ the polynomial ring modulo the ideal generated by the identities in Theorem 17.

The question also arises as to whether “the most general” parametrized Tutte polynomial should be multiplicative on disjoint unions and one point joins of graphs, as this introduces additional relations among the $a_i$’s. This is because a parametrized Tutte polynomial is not necessarily multiplicative. A sufficient condition is the following.

**Proposition 4** Suppose $T$ is a parametrized Tutte polynomial on a minor-closed class of graphs that contains at least one graph with $k$ components for every $k$ and that is closed under one-point unions and the removal of isolated vertices. Then $T$ is multiplicative with respect to both disjoint unions and one-point joins if and only if the $\alpha_1 = T(E_1)$ is idempotent, and $\alpha_k = \alpha_1$ for all $k \geq 1$.
Bollobás and Riordan [BR99] emphasize graph invariants, and hence require that the parametrization be a coloring of the graph. That is, graphs are edge-colored (not necessarily properly), with edges of the same color having the same parameter sets. This enables consideration of parametrized Tutte polynomials that are invariants of colored graphs, but requires the following additional constraints. For every \( e_1 \in U \), there are \( e_2, e_3 \in U \) with \( e_1 \neq e_2 \neq e_3 \neq e_1 \) such that \( x_{e_1} = x_{e_2} = x_{e_3} \), \( y_{e_1} = y_{e_2} = y_{e_3} \), \( X_{e_1} = X_{e_2} = X_{e_3} \), and \( Y_{e_1} = Y_{e_2} = Y_{e_3} \).

Proofs of the above results and further details may be found in [Zas92, BR99, E-MT06]. We note that any relation between this Tutte polynomial generalization with its edge parameters, and the W- and U-polynomials of Subsection 4.1 with their vertex weights, has not yet been studied.

Interestingly, although the parametrized Tutte polynomial has an activities expansion analogous to that of the classical Tutte polynomial, it does not necessarily have an analog of the rank-nullity formulation. However, under modest assumptions involving non-zero parameters and some inverses, the parametrized Tutte polynomial may be expressed in a rank-nullity form. This is fortunate, because significant results for the zeros of the chromatic and Tutte polynomial have arisen from such a multivariable realization. Examples may be found in Sokal [Sok01a], Royle and Sokal [RS04], and Choe, Oxley, Sokal, and Wagner [COSW04].

### 4.3 The Generalized Transition Polynomial

A number of state model polynomials, for example the circuit partition polynomials, Penrose polynomial, the Kauffman bracket for knots and links, and the transition polynomials of Jaeger [Jae90], that are not specializations of the Tutte polynomial, are specializations of the multivariable generalized transition polynomial of [E-MS02] which we describe here. This multivariable extension is a Hopf algebra map, which leads to structural identities that then inform its various specializations. The medial graph construction that relates the circuit partition polynomial and the classical Tutte polynomial extends to similarly relate the generalized transition polynomial and the parametrized Tutte polynomial when it has a rank-nullity formulation.

The graphs here are Eulerian, although not necessarily connected, with loops and multiple edges allowed. A vertex state, or transition, is a choice of local reconfiguration of a graph at a vertex by pairing the half edges incident with that vertex. A graph state, or transition system, \( S(G) \), is the result of choosing a vertex state at each vertex of degree greater than 2, and hence is a union of disjoint cycles. We will write \( St(G) \) for the set of graph states of \( G \), and throughout we assume weights have values in \( R \), a commutative ring with unit.

A **skein relation** for graphs is a formal sum of weighted vertex states, together with an evaluation of the terminal forms (the graph states). See [E-M98, E-MS02] for a detailed discussion of these concepts, which are appropriated from knot theory, in their most general form, and Yetter [Yet90] for a general theory of invariants given by linear recursion relations. A **skein type**, (or state model,
or transition) polynomial is one which is computed by repeated applications of skein relations. See Jaeger [Jae90] for a comprehensive treatment of these in the case of 4-regular graphs.

For brevity, we elide technical details such as free loops and isomorphism classes of graphs with weight systems which may be found in [E-MS02].

Definition 19 Pair, vertex, and state weights:

1. A pair weight is an association of a value \( p(e_v, e'_v) \) in a unitary ring \( R \) to a pair of half edges incident with a vertex \( v \) in \( G \). A weight system, \( W(G) \), of an Eulerian graph \( G \) is an assignment of a pair weight to every possible pair of adjacent half edges of \( G \).

2. The vertex state weight of a vertex state is \( \prod p(e_v, e'_v) \) where the product is over the pairs of half edges comprising the vertex state.

3. The state weight of a graph state \( S \) of a graph \( G \) with weight system \( W \) is

\[
\omega(S) = \prod \omega(v, S),
\]

where \( \omega(v, S) \) is the vertex state weight of the vertex state at \( v \) in the graph state \( S \), and where the product is over all vertices of \( G \).

When \( A \) is an Eulerian subgraph of an Eulerian graph \( G \) with weight system \( W(G) \), then \( A \) inherits its weight system \( W(A) \) from \( G \) in the obvious way, with each pair of adjacent edges in \( A \) having the same pair weight as it has in \( G \). When \( A \) is a graph resulting from locally replacing the vertex \( v \) by one of its \( \prod_{i=0}^{n-1} (2n - (2i + 1)) \) vertex states, then all the pair weights are the same as they are in \( G \), except that all the pairs of half edges adjacent to the newly formed vertices of degree 2 in \( A \) have pair weight equal to 1, the identity in \( R \).

The generalized transition polynomial \( N(G; W, x) \) has several formulation, and we give two of them, a linear recursion formula and a state model formula, here.

Definition 20 The generalized transition polynomial, \( N(G; W, x) \), is defined recursively by repeatedly applying the skein relation

\[
N(G; W, x) = \sum \beta_i N(G_i; W(G_i), x)
\]

at any vertex \( v \) of degree greater than 2. Here the \( G_i \)'s are the graphs that result from locally replacing a vertex \( v \) of degree \( 2n \) in \( G \) by one of its vertex states. The \( \beta_i \)'s are the vertex state weights. Repeated application of this relation reduces \( G \) to a weighted (formal) sum of disjoint unions of cycles, (the graph states). These terminal forms are evaluated by identifying each cycle with the variable \( x \), weighted by the product of the pair weights over all pairs of half edges in the cycle.

Definition 21 The state model definition of the generalized transition polynomial is:

\[
N(G; W, x) = \sum_{St(G)} \left( \left( \prod \omega(v, S) \right) x^{k(S)} \right) = \sum_{St(G)} \omega(S) x^{k(S)}.
\]
Note that vertex states commute, that is, if $G_{uv}$ results from choosing a vertex state at $u$, and then at $v$, we have $G_{uv} = G_{vu}$. Thus, Definition 20 gives a well-defined function, and Definitions 20 and 21 are equivalent.

Several of the polynomials we have already seen are specializations of this generalized transition polynomial. For example, if all the pair weights are 1, then the circuit partition polynomial for an unoriented Eulerian graph results. If $\vec{G}$ is an Eulerian digraph, and $G$ is the underlying undirected graph with pair weights of 1 for pairs half edges corresponding to one inward and one outward oriented half edge of $\vec{G}$ and 0 otherwise, then the oriented version of the circuit partition polynomial results.

In the special case that $G$ is 4-regular, the polynomial $N(G; W, x)$ is essentially the same as the transition polynomial $Q(G, A, \tau)$ of Jaeger [Jae90], where $G$ is a 4-regular graph and $A$ is a system of vertex state weights (rather than pair weights). If the vertex state weight in $(G, A)$ is $w$, then define $W(G)$ by letting the pair weights for each of the two pairs of edges determined by the state be $\sqrt{w}$. The two polynomials then just differ by a factor of $x$, so $N(G; W, x) = xQ(G, A, x)$, and here we retain vertices of degree 2 in the recursion while they are elided in [Jae90]. Thus $N(G; W, x)$ gives a generalization of Jaeger’s transition polynomials to all Eulerian graphs.

Because $Q(G, A, \tau)$ assimilates the original Martin polynomial for 4-regular graphs and digraphs, the Penrose polynomial and the Kauffman bracket of knot theory (see [Jae90]), and $N(G; W, x)$ assimilates $Q(G, A, \tau)$, we have that the Penrose polynomial and Kauffman bracket are also specializations of $N(G; W, x)$. Specifically, if $G$ is a planar graph with face 2-colored medial graph $G_m$, and we give a weight system to $G_m$ by assigning a value of 1 to pairs of edges that either cross at a vertex or bound the same black face and 0 otherwise, then $N(G_m; W, x) = P(G; x)$. Similarly, if $L$ is a link, and $G_L$ is the signed, face 2-colored universe of $L$, then a weight system can be assigned to $G_L$ so that $N(G_L; W, a^2 + a^{-2}) = (a^2 + a^{-2})K[L]$ where $K[L]$ is the Kauffman bracket of the link.

Because of these specializations, the algebraic properties of the generalized transition polynomial are available to inform these other polynomials as well. In particular, $N(G; W, x)$ is a Hopf algebra map from the freely generated (commutative) hereditary Hopf algebra of Eulerian graphs with weight systems to the binomial bialgebra $R[x]$ (details may be found in [E-MS02]). This leads to two structural identities, the first from the comultiplication in the Hopf algebra, the second from the antipode.

**Theorem 18** Let $G$ be an Eulerian graph. Then

\[
N(G; W, x + y) = \sum N(A_1; W(A_1), x) N(A_2; W(A_2), y)
\]

where the sum is over all ordered partitions of $G$ into two edge-disjoint Eulerian subgraphs $A_1$ and $A_2$, and

\[
N(G; W, -x) = N(\zeta(G; W), x),
\]

26
where \( \zeta \) is the antipode \( \zeta(G,W) = \sum (-1)^{|P|}(A_1 \ldots A_{|P|}), \) with the sum over all ordered partition \( P \) of \( G \) into \( |P| \) edge-disjoint Eulerian subgraphs each with inherited weight system. Here \( N(G;W,x) \) is extended linearly over such formal sums.

This type of Hopf algebraic structure has already been used to considerably extend the known combinatorial interpretations for evaluations of the Martin, Penrose, and Tutte polynomials implicitly by Bollobás [Bol02], and explicitly by Ellis-Monaghan and Sarmiento [E-M98, E-MS01, Sar01, E-M04a, E-M04b]. The first identity has been used to find combinatorial interpretations for the Martin polynomials for all integers, where this was previously only known for 2, -1, 0, 1 in the oriented case, and 2, 0, 2 in the unoriented case. This then led to combinatorial interpretations for the Tutte polynomial (and its derivatives) of a planar graph for all integers along the line \( x = y \), where previously 1, 3 were the only known non-trivial values. These results for the Tutte polynomial were mentioned in the preceding chapter and for the circuit partition polynomial in Subsection 3.4. The second identity has been used to determine combinatorial interpretations for the Penrose polynomial for all negative integers, where this was previously only known for positive integers.

5 Two Applications

Graph polynomials have a wide range of applications throughout many fields. We have already seen some examples of this with various applications of the classical Tutte polynomial in the previous chapter. Here we present two representative important applications (out of many possible) and show how they may be modeled by graph polynomials.

5.1 DNA Sequencing

We begin with string reconstruction, a problem that may be modeled by the interlace and circuit partition polynomials (and hence indirectly in special cases by the Tutte polynomial). String reconstruction is the process of reassembling a long string of symbols from a set of its subsequences together with some (possibly incomplete, redundant, or corrupt) sequencing information. While we focus on DNA sequencing, which was original the motivation for the development of the interlace polynomial, the methods here apply to any string reconstruction problem. For example, fragmenting and reassembling messages is a common network protocol, and reconstruction techniques might be applied when the network protocol has been disrupted, yet the original message must be reassembled from the fragments.

DNA sequences are typically too long to read at once with current laboratory techniques, so researchers probe for shorter fragments (reads) of the strand. They then are faced with the difficulty of recovering the original long sequence from the resulting set of subsequences. DNA sequencing by hybridization is a
method of reconstructing the nucleotide sequence from a set of short substrings (see Waterman [Wat95] for an overview). The problem of determining the number of possible reconstructions may be modeled using Eulerian digraphs, with a correct sequencing of the original strand corresponding to exactly one of the possible Eulerian circuits in the graph. The probability of correctly sequencing the original strand is thus the reciprocal of the total number of Euler circuits in the graph.

The most basic (two-way repeats only) combinatorial model for DNA sequencing by hybridization uses an Eulerian digraph with two incoming and two outgoing edges at each vertex (see Pevzner [Pev89] and Arratia, Martin, Reinert, Waterman [AMRW96]). The raw data consists of all subsequences of the DNA strand of a fixed length L, called the L-spectrum of the sequence. As L increases, the statistical probability that the beginning and end of the DNA strand are the same approaches zero, as does the likelihood of three or more repeats of the same pattern of length L or more in the strand (see Dyer, Frieze, and Suen [DFS94]). Thus, this model assumes that the only consideration in reconstructing the original sequence is the appearance of interlaced two-way repeats, that is, alternating patterns of length L or greater, for example, \(\ldots\) ACTG \(\ldots\) CTCT \(\ldots\) ACTG \(\ldots\) CTCT \(\ldots\).

From the multiset (duplicates are allowed) of subsequences of length L, create a single vertex of the de Bruijn graph for each subsequence of length L-1 that appears in one of the subsequences. For example, if L = 4 and ACTG appears as a subsequence, create two vertices, one labeled ACT and one labeled CTG. Edges are directed from head to tail of a subsequence, e.g. there would be a directed edge from ACT to CTG labeled ACTG. If there is another subsequence ACTT, we do not create another vertex ACT, but rather draw an edge labeled ACTT from the vertex ACT to a new vertex labeled CTT. If, in the multiset of subsequences, ACTG appears twice, then we draw two edges from ACT to CTG.

The beginning and end of the strand are identified to be represented by the same vertex, and, since by assumption no subsequence appears more than twice, the result is an Eulerian digraph of maximum degree 4. Tracing the original DNA sequence in this graph corresponds to an Eulerian circuit that starts at the vertex representing the beginning and end of the strand. All other possible sequences that could be (mis)reconstructed from the multiset of subsequences correspond to other Eulerian circuits in this graph. Thus (up to minor reductions for long repeats and forced subsequences), finding the number of DNA sequences possible from a given multiset of subsequences corresponds to enumerating the Eulerian circuits in this directed graph.

The generalized transition polynomial models this problem directly: when the pair weights are identically 1, it reduces to the circuit partition polynomial. This is a generating function for families of circuits in a graph, so the coefficient of \(x\) is the number of Eulerian circuits. The interlace polynomial informs the problem as follows. Consider an Eulerian circuit through the de Bruijn graph, which gives a sequence of the vertices visited in order. Now construct the interlace graph by placing a vertex for each symbol and an edge between symbols that
are interlaced (occur in alternation) in the sequence. The interlace polynomial of the interlace graph is then a translation of the circuit partition polynomial of the original de Bruijn graph, as in Theorem 12, where again the coefficient of $x$ is the number of Eulerian circuits (see Arratia, Bollobás, Coppersmith, and Sorkin [ABCS00], and Arratia, Bollobás, and Coppersmith [ABS00, ABS04a]).

One of the original motivating goals of Arratia, Bollobás, Coppersmith, and Sorkin [ABCS00] was classifying Eulerian digraphs with a given number of Eulerian circuits. The BEST theorem, a formula for the number of the circuits of an Eulerian graph in terms of its Kirchhoff matrix (see Fleischner [Fle91] for good exposition) gives only a tautological classification: the Eulerian digraphs with $m$ Eulerian circuits are those where BEST theorem formula gives $m$ circuits. Critically, all of the above graph polynomials encode much more information than is available from the BEST theorem, and all of them are embedded in broader algebraic structures that provide tools for extracting information from them. Thus, they better serve the goal of seeking structural characterizations of graph classes with specified Eulerian circuit properties.

5.2 The Potts Model of Statistical Mechanics

Here we have an important physics model that remarkably was found to be exactly equivalent to the Tutte polynomial.

Complex systems are networks in which very simple interactions at the microscopic level determine the macroscopic behavior of the system. The Potts model of statistical mechanics models complex systems whose behaviors depend on nearest neighbor energy interactions. This model plays an important role in the theory of phase transitions and critical phenomena, and has applications as widely varied as magnetism, adsorption of gases on substrates, foam behaviors, and social demographics, with important biological examples including disease transmission, cell migration, tissue engulfment, diffusion across a membrane, and cell sorting.

Central to the Potts model is the Hamiltonian,

$$h(\omega) = -J \sum_{\{i,j\} \in E(G)} \delta(\sigma_i, \sigma_j),$$

a measure of the energy of the system. Here a spin, $\sigma_i$, at a vertex $i$, is a choice of condition (for example, healthy, infected or necrotic for a cell represented by the vertex). $J$ is a measure of the interaction energy between neighboring vertices, $\omega$ is a state of a graph $G$ (that is, a fixed choice of spin at each vertex), and $\delta$ is the Kronecker delta function.

The Potts model partition function is the normalization factor for the Boltzmann probability distribution. Systems such as the Potts model, following Boltzmann distribution laws, will have the number of states with a given energy (Hamiltonian value) exponentially distributed. Thus, the probability of the system being in a particular state $\omega$ at temperature $t$ is:

$$\Pr (\omega, \beta) = \frac{\exp (-\beta h(\omega))}{\sum \exp (-\beta h(\varpi))}.$$
Here, the sum is over all possible states \( \varpi \) of \( G \), and \( \beta = \frac{1}{\kappa t} \), where \( \kappa = 1.38 \times 10^{-23} \) joules/Kelvin is the Boltzmann constant. The parameter \( t \) is an important variable in the model, although it may not represent physical temperature, but some other measure of volatility relevant to the particular application (for example ease of disease transmission/reinfection). The denominator of this expression, \( P(G; q, \beta) = \sum \exp (-\beta h(\varpi)) \), called the Potts model partition function, is the most critical, and difficult, part of the model.

Remarkably, the Potts model partition function is equivalent to the Tutte polynomial:

\[
P(G; q, \beta) = q^{k(G)} v^{\nu(G) - k(G)} T \left( G, \frac{q + v}{v}, v + 1 \right),
\]

where \( q \) is the number of possible spins, and \( v = \exp(J\beta) - 1 \). See Fortuin and Kasteleyn [FK72] for the nascent stages of this theory, later exposition in Tutte [Tut84], Biggs [Big96], Bollobás [Bol98], Welsh [Wel93], and surveys by Welsh and Merino [WM00], and Beaudin, Ellis-Monaghan, Pangborn and Shrock [BE-MPS].

One common extension of the Potts model involves allowing interaction energies to depend on individual edges. With this, the Hamiltonian becomes \( h(\omega) = \sum_{e \in E(G)} J_e \delta(\sigma_i, \sigma_j) \), where \( J_e \) is the interaction energy on the edge \( e \). The partition function is then

\[
P(G) = \sum_{A \subseteq E(G)} q^{k(A)} \prod_{e \in A} v_e,
\]

where \( v_e = \exp(\beta J_e) - 1 \) again see Fortuin and Kasteleyn [FK72], and more recently Sokal [Sok00, Sok01b]. As we have seen in Subsection 3.4, the Tutte polynomial has also been extended to parametrized Tutte functions that incorporate edge weights. The generalized partition function given above satisfies the relations of Theorem 17, however, and thus is a special case of a parametrized Tutte function.

This relationship between the Potts model partition function and the Tutte polynomial has led to a remarkable synergy between the fields, particularly for example in the areas of computational complexity and the zeros of the Tutte and chromatic polynomials. For overviews, see Welsh and Merino [WM00], and Beaudin, Ellis-Monaghan, Pangborn and Shrock [BE-MPS].

6 Conclusion

There are a great many other graph polynomials equally interesting to those surveyed here, including for example the F-polynomials of Farrell, the Hosaya or Wiener polynomial, the clique/ independence and adjoint polynomials, etc. In particular, Farrell [Far79b] has a circuit cover polynomial (different from the circuit partition polynomial of Subsection 3.4) with noteworthy interrelations with the characteristic polynomial. Also, Chung and Graham developed
a ‘Tutte-like’ polynomial for directed graphs in [CG95]. The resultant cover polynomial is extended to a symmetric function generalization, like those in section 4.1 by Chow [Cho96]. Similarly, Courcelle [Cou08] and Traldi [Tra] have also very recently developed multivariable extensions of the interlace polynomial. Some surveys of graph polynomials with complementary coverage to this one include: Pathasarthy [Par89], Jaeger [Jae90], Farrell [Far93], Fiol [Fio97], Godsil [God84], Aigner [Aig01], Noy [Noy03] and Levit and Mandrescu [LM05].

References

[ABCS00] Arratia, R., Bollobás, B., Coppersmith, D., Sorkin, G.: Euler circuits and DNA sequencing by hybridization. Discrete Appl. Math., 104, 63–96 (2000)

[ABS00] Arratia, R., Bollobás, B., Sorkin, G.: The interlace polynomial: a new graph polynomial. Proceedings of the eleventh annual ACM-SIAM symposium on discrete algorithms. San Francisco, CA (2000)

[ABS04a] Arratia, R., Bollobás, B., Sorkin, G.: The interlace polynomial of a graph. J. Combin. Theory Ser. B, 92, 199-233 (2004)

[ABS04b] Arratia, R., Bollobás, B., Sorkin, G.: A two-variable interlace polynomial. Combinatorica, 24, 567-584 (2004)

[Aig97] Aigner, M.: The Penrose polynomial of a plane graph. Math. Ann., 307, 173–189 (1997)

[Aig00] Aigner, M.: Die Ideen von Penrose zum 4-Farbenproblem. Jahresber. Deutsch. Math.-Verein. 102, 43–68 (2000)

[Aig01] Aigner, M.: The Penrose polynomial of graphs and matroids. In: Hirschfeld, J. W. P. (ed) Surveys in Combinatorics, 2001. Cambridge University Press, Cambridge (1997)

[Aih76] Aihara, J.: A New Definition of Dewar-Type Resonance Energies. Journal of the American Chemical Society, 98, 2750–2758 (1976)

[All94] Allys, L.: Minimally 3-connected isotropic systems. Combinatorica, 14, 247–262, (1994)

[AM00] Aigner, M., Mielke, H.: The Penrose polynomial of binary matroids. Monatsh. Math., 131, 1-13, (2000)

[AMRW96] Arratia, R., Martin, D., Reinert, G., Waterman, M.: Poisson process approximation for sequence by hybridization. J. of Computational Biology, 3, 425–463 (1996)
[AT92]  Alon, N., Tarsi, M.: Colorings and orientations of graphs. Combinatorica, 12, 125-134 (1992)

[AvdH04]  Aigner, M., van der Holst, H.: Interlace polynomials. Linear Algebra Appl. 377, 11–30 (2004)

[Bar94]  Barvinok, A. I.: Computing the Ehrhart polynomial of a convex lattice polytope. Discrete Comput. Geom., 12, 35–48 (1994)

[BBCP02]  Balister, P. N., Bollobás, B., Cutler, J., Pebody, L.: The interlace polynomial of graphs at $-1$. European J. Combin., 23, 761–767 (2002)

[BBRS01]  Balister, P. N., Bollobás, B., Riordan, O. M., Scott, A. D.: Alternating knot diagrams, Euler circuits and the interlace polynomial. European J. Combin., 22, 1–4 (2001)

[BE-MPS]  Beaudin, L., Ellis-Monaghan, J., Pangborn, G., Shrock, R.: A Little Statistical Mechanics for the Graph Theorist. Preprint, arXiv:0804.2468

[BG96]  Bouchet, A., Ghier, L.: Connectivity and $\beta$ invariants of isotropic systems and 4-regular graphs. Discrete Math., 161, 25–44 (1996)

[Big96]  Biggs, N.: Algebraic Graph Theory. Cambridge University Press, Cambridge, second edition (1996)

[BK85]  Betke, U., Kneser, M.: Zerlegungen und Bewertungen von Gitterpolytopen. J. Reine Angew. Math., 358, 202–208 (1985)

[Bol98]  Bollobás, B.: Modern Graph Theory, Graduate Text in Mathematics. Springer-Verlag New York, New York (1998)

[Bol02]  Bollobás, B.: Evaluations of the circuit partition polynomial. J. Combin. Theory Ser. B, 85, 261–268 (2002)

[Bou85]  Bouchet, A.: Characterizing and recognizing circle graphs. In: Tošić, R., Acketa, D., Petrović, V. (eds) Graph Theory. Proceedings of the sixth Yugoslav seminar held in Dubrovnik. University of Novi Sad, Novi Sad (1986)

[Bou87a]  Bouchet, A.: Isotropic systems. European J. Combin., 8, 231–244 (1987)

[Bou87b]  Bouchet, A.: Reducing prime graphs and recognizing circle graphs. Combinatorica, 7, 243–254 (1987)

[Bou87c]  Bouchet, A.: Unimodularity and circle graphs. Discrete Math., 66, 203-208 (1987)
Bouchet, A.: Graphic presentations of isotropic systems. J. Combin. Theory Ser. B, 45, 58–76 (1988)

Bouchet, A.: Connectivity of isotropic systems. Combinatorial Mathematics, Proc. 3rd int. conf, New York, NY, USA 1985, Ann. N. Y. Acad. Sci., 555, 81–93 (1989)

Bouchet, A.: Tutte-Martin polynomials and orienting vectors of isotropic systems. Graphs Combin., 7, 235–252 (1991)

Bouchet, A.: Compatible Euler tours and supplementary Eulerian vectors. European J. Combin., 14, 513–520 (1993)

Bouchet, A.: Circle graph obstruction. J. Combin. Theory Ser. B, 60, 107–144 (1994)

Bouchet, A.: Multimatroids. I: Coverings by independent sets. SIAM J. Discrete Math., 10, 626–646 (1997)

Bouchet, A.: Multimatroids. II. Orthogonality, minors and connectivity. Electron. J. Combin. 5 (1998)

Bouchet, A.: Multimatroids. IV. Chain-group representations. Linear Algebra Appl., 277, 271–289 (1998)

Bouchet, A.: Multimatroids. III. Tightness and fundamental graphs. European J. Combin., 22, 657–677 (2001)

Bouchet, A.: Graph polynomials derived from Tutte-Martin polynomials. Discrete Math., 302, 32–38 (2005)

Bollobás, B., Riordan, O.: A Tutte polynomial for coloured graphs. Comb. Probab. Comput., 8, 45–93 (1999)

Bollobás, B., Riordan, O.: A polynomial invariant of graphs on orientable surfaces. Proc. London Math. Soc., 83, 513–531 (2001)

Bollobás, B., Riordan, O.: A polynomial of graphs on surfaces. Math. Ann., 323, 81–96 (2002)

Brylawski, T. H.: Intersection theory for graphs. J. Combin. Theory Ser. B, 30, 233–246 (1981)

Cvetković, D. M., Doob, M., Sachs, H.: Spectra of Graphs: Theory and Applications. Academic Press, New York, 1980.

Chung, F. R. K., Graham, R. L.: On the cover polynomial of a digraph. J. Combin. Theory Ser. B, 65, 273–290 (1995)

Chmutov, S.: Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial. Preprint.
[Cho96] Chow, T.: The path-cycle symmetric function of a digraph. Adv. Math., 118, 71–98 (1996)

[CL98] Cvetković, D. M., Lepović, M.: Seeking counterexamples to the reconstruction conjecture for characteristic polynomials of graphs and a positive result. Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., Sci. Math., 116, 91–100 (1998)

[Cou08] Courcelle, B.: A multivariate interlace polynomial and its computation for graphs of bounded clique-width. The Electronic Journal of Combinatorics, 15(1), R69 (2008)

[COSW04] Choe, Y-B., Oxley, J., Sokal, A., Wagner, D.: Homogeneous multivariate polynomials with the half-plane property. Adv. Appl. Math., 32, 88–187 (2004)

[CP07] Chmutov, S., Pak, I.: The Kauffman bracket of virtual links and the Bollobás-Riordan polynomial. Mosc. Math. J., 7, 409–418, 573 (2007)

[DFS94] Dyer, M., Frieze, A., Suen, S.: The probability of unique solutions of sequencing by hybridization. J. of Computational Biology, 1, 105–110 (1994)

[DR97] Diaz, R., Robins, S.: The Ehrhart polynomial of a lattice polytope. Annals of Mathematics, 145, 503–518 (1997)

[EG96] Ellingham, M. N., Goddyn, L.: List edge colourings of some 1-factorable multigraphs. Combinatorica, 16, 343–352 (1996)

[Ehr67a] Ehrhart, E.: Sur un problème de géométrie diophantienne linéaire I. J. Reine Angew. Math., 226, 1–29 (1967)

[Ehr67b] Ehrhart, E.: Sur un problème de géométrie diophantienne linéaire II. J. Reine Angew. Math., 227, 25–49 (1967)

[Ehr67c] Ehrhart, E.: Démonstration de la loi de réciprocité du polyèdre rationnel. C. R. Acad. Sci. Paris Sér. A-B, 265, A91–A94 (1967)

[Ehr77] Ehrhart, E.: Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire. International Series of Numerical Mathematics, Vol. 35. Birkhäuser Verlag, Basel-Stuttgart (1977)

[E-M98] Ellis-Monaghan, J. A.: New results for the Martin polynomial. J. Combin. Theory Ser. B, 74, 326–352 (1998)

[E-M00] Ellis-Monaghan, J. A.: Differentiating the Martin polynomial. Cong. Numer., 142, 173–183, (2000)

[E-M04a] Ellis-Monaghan, J. A.: Exploring the Tutte-Martin connection. Discrete Math., 281, 173–187 (2004)
[E-M04b] Ellis-Monaghan, J. A.: Identities for circuit partition polynomials, with applications to the Tutte polynomial. Adv. in Appl. Math., 32, 188–197 (2004)

[E-MS02] Ellis-Monaghan, J., Sarmiento, I.: Generalized transition polynomials. Congr. Numer., 155, 57–69 (2002)

[E-MS01] Ellis-Monaghan, J., Sarmiento, I.: Medial graphs and the Penrose polynomial. Congr. Numer., 150, 211–222 (2001)

[E-MS07] Ellis-Monaghan, J., Sarmiento, I.: Distance hereditary graphs and the interlace polynomial. Comb. Probab. Comput., 16, 947–973 (2007)

[E-MT06] Ellis-Monaghan, J., Traldi, L.: Parametrized Tutte polynomials of graphs and matroids. Comb. Probab. Comput., 15, 835–834 (2006)

[Far79a] Farrell, E. J.: An introduction to matching polynomials. J. Combin. Theory Ser. B, 27, 75–86 (1979)

[Far79b] Farrell, E. J.: On a class of polynomials obtained from the circuits in a graph and its application to characteristic polynomials of graphs. Discrete Math., 25, 121–133, (1979)

[Far88] Farrell, E. J.: On the matching polynomial and its relation to the rook polynomial. J. Franklin Inst., 325, 527–543 (1988)

[Far93] Farrell, E. J.: The impact of $F$-polynomials in graph theory. Quo vadis, graph theory? Ann. Discrete Math., 55, North-Holland, Amsterdam, 173–178 (1993)

[Fio97] Fiol, M. A.: Some applications of the proper and adjacency polynomials in the theory of graph spectra. Electron. J. Combin., 4(1), R21 (1997)

[FK72] Fortuin C. M., Kasteleyn, P. W.: On the random cluster model. Physica, 57, 536–564 (1972)

[Fle91] Fleischner, H.: Eulerian Graphs and Related Topics, Part 1, Volume 2. Annals of Discrete Mathematics, 50. North-Holland Publishing Co., Amsterdam (1991)

[GC75] Gutman, I., Cvetković, D. M.: The reconstruction problem for the characteristic polynomial of graphs. Publ. Electrotehn, Fac. Ser. Fiz. No.488–541, 45–48 (1975)

[GMT76] Gutman, I., Milun, M., Trinajstić, N.: Graph theory and molecular orbitals. XVIII. On topological resonance energy. Croatica Chemica Acta, 48, 87–95 (1976)
[GMT77] Gutman, I., Milun, M., Trinajstić, N.: Graph theory and molecular orbitals. 19. Nonparametric resonance energies of arbitrary conjugated systems. Journal of the American Chemical Society, 99, 1692–1704 (1977)

[GP06] Glantz, R., Pelillo, M.: Graph polynomials from principal pivoting. Discrete Math., 306, 3253–3266 (2006)

[God81b] Godsil, C. D.: Matchings and walks in graphs. J. Graph Theory, 5, 285–297 (1981)

[God84] Godsil, C. D.: Real graph polynomials. In: Bondy, J. A., Murty, U. S. R., (eds) Progress in graph theory. Academic Press, Toronto (1984)

[God93] Godsil, C. D.: Algebraic Combinatorics. Chapman & Hall, New York (1993)

[God95] Godsil, C. D.: Tools from linear algebra. In: Graham, R. L., Grötschel M., Lovász L.(eds) Handbook of Combinatorics, Vol. 2. Elsevier, Amsterdam (1995)

[GT76] Gutman, I., Trinajstić, N.: Graph theory and molecular orbitals, XIV. On topological definition of resonance energy. Acta Chimica Academiae Scientiarum Hungaricae, 91, 203–209 (1976)

[Gut91] Gutman, I.: Polynomials in graph theory. In: Bonchev, D., Rouvray, D. H. (eds) Chemical Graph Theory: Introduction and Fundamentals. Abacus Press, NY (1991)

[Har62] Harary, F.: The determinant of the adjacency matrix of a graph. SIAM Rev., 4, 202–210 (1962)

[HJ90] Horn, R. A., Johnson, C. R.: Matrix Analysis. Cambridge University Press, Cambridge (1990)

[HL70] Heilmann, O. J., Lieb, E. H.: Monomers and Dimers. Phys. Rev. Lett., 24, 1412–1414 (1970)

[HL72] Heilmann, O. J., Lieb, E. H.: Theory of monomer-dimer systems. Commun. Math. Phys., 25, 190–232 (1972)

[Hos71] Hosaya, H.: Topological Index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons. Bulletin of the Chemical Society of Japan, 44, 2332–2339 (1971)

[Kau89] Kauffman, L. H.: A Tutte polynomial for signed graphs. Discrete Appl. Math., 25, 105–127 (1989)
[Kel57] Kelly, P. J.: A congruence theorem for trees. Pacific J. Math., 7, 961–968 (1957)

[Kel65] Kel’mans, A. K.: The number of trees in a graph. I. Automat. i. Telemech., 26, 2194–2204 (1965) (in Russian); transl. Automat. Remote Control, 26, 2118–2129 (1965)

[KK93] Kantor, J. M., Khovanskii, A.: Une application du Théorème de Riemann-Roch combinatoire au polynôme d’Ehrhart des polytopes entier de $\mathbb{R}^d$. C. R. Acad. Sci. Paris, 1317, 501–507 (1993)

[Koc81] Kocay, W. L.: An extension of Kelly’s lemma to spanning subgraphs. Congr. Numer., 31, 109–120 (1981)

[Kun70] Kunz, H.: Location of the zeros of the partition function for some classical lattice systems. Physics Letters A, 32, 311–312 (1970)

[Jac91] Jackson, B.: Supplementary Eulerian vectors in isotropic systems. J. Combin. Theory Ser. B, 53, 93–105 (1991)

[Jae90] Jaeger, F.: On Transition polynomials of 4-regular graphs, Cycles and Rays. NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 301. Kluwer Acad. Publ., Dordrecht (1990)

[Las79] Las Vergnas, M.: On Eulerian partitions of graphs. In: Wilson, R. J. (ed) Graph Theory and Combinatorics. Pitman, Boston London (1979)

[Las83] Las Vergnas, M.: Le polynôme de Martin d’un graphe eulérien. In: Berge, C., Bresson, D., Camion, P., Maurras J.-F., Sterboul, F. (eds) Combinatorial Mathematics. North-Holland, Amsterdam (1983)

[Las88] Las Vergnas, M.: On the evaluation at (3,3) of the Tutte polynomial of a graph. J. Combin. Theory Ser. B, 44, 367–372 (1988)

[LM05] Levit, V. E., Mandrescu, E.: The independence polynomial of a graph—a survey. In: Bozapalidis, S., Kalampakas, A., Rahonis, G. (eds) Proceedings of the 1st International Conference on Algebraic Informatics. Aristotle Univ. Thessaloniki, Thessaloniki (2005)

[LP86] Lovász, L., Plummer, M. D.: Matching Theory. Annals Discrete Math. 29, Amsterdam (1986)

[Mac71] Macdonald, I. G.: Polynomials associated with finite cell complexes. J. London Math. Soc., 4, 181–192 (1971)

[Mar77] Martin, P.: Enumérations eulériennes dans le multigraphes et invariants de Tutte-Grothendieck. Thesis, Grenoble (1977)
[Mar78] Martin, P.: Remarkable valuation of the dichromatic polynomial of planar multigraphs. J. Combin. Theory Ser. B, 24, 318–324 (1978)

[McM71] McMullen, P.: On zonotopes. Trans. Amer. Math. Soc., 159, 91–109 (1971)

[MN] Merino, C., Noble, S. D.: The equivalence of two graph polynomials and a symmetric function. Preprint, arXiv:0805.4793

[Mof] Moffatt, I.: Unsigned state models for the Jones polynomial. Preprint.

[Mof08] Moffatt, I.: Knot invariants and the Bollobás-Riordan Polynomial of embedded graphs. European J. Combin., 29, 95–107 (2008)

[Noy03] Noy, M.: Graphs determined by polynomial invariants. Theoretical Computer Science, 307, 365–384 (2003)

[NW99] Noble, S. D., Welsh, D. J. A.: A weighted graph polynomial from chromatic invariants of knots. Annales de l’institute Fourier, 49, 1057–1087 (1999)

[Par89] Pathasarthy, K. R.: Graph Polynomials. In: Kulli, V.R. (ed) Recent Studies in Graph Theory. Vishwa International Publications, Gulbarga (1989)

[Pen69] Penrose, R.: Applications of negative dimensional tensors. In: Welsh, D. A. J. (ed) Combinatorial Mathematics and its Applications: Proceedings of a Conference Held at the Mathematical Institute, Oxford, 1969. Academic Press, London New York (1971)

[Pet91] Peterson, J.: Die theorie der regularen graphs. Acta Math., 15, 193–220 (1891)

[Pev89] Pevzner, P. A.: l-tuple DNA sequencing: computer analysis. J. Biomolec. Struct. and Dynamics, 7, 63–73 (1989)

[Poi01] Poincaré, H.: Second complément à l’analysis situs. Proc. London Math. Soc., 65, 23–45 (1901)

[Pom93] Pommersheim, J.: Toric varieties, lattice points, and Dedekind sums. Math. Ann., 295, 1–24 (1993)

[Rio58] Riordan, J.: An Introduction to Combinatorial Analysis. Wiley, New York (1958)

[RS04] Royle, G., Sokal, A.: The Brown-Colbourn conjecture on zeros of reliability polynomials is false. J. Combin. Theory Ser. B, 91, 345–360 (2004)
[Sar00] Sarmiento, I.: The polychromate and a chord diagram polynomial. Ann. Comb., 4, 227–236 (2000)

[Sar01] Sarmiento, I.: Hopf algebras and the Penrose polynomial. European J. Combin., 22, 1149–1158 (2001)

[She74] Shephard, G. C.: Combinatorial properties of associated zonotopes. Can. J. Math., 26, 302–321 (1974)

[Sok00] Sokal, A. D.: Chromatic polynomials, Potts models and all that. Physica A, 279, 324–332 (2000)

[Sok01a] Sokal, A. D.: Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions. Comb. Probab. Comput., 10, 41–77 (2001)

[Sok01b] Sokal, A. D.: A personal list of unsolved problems concerning lattice gases and antiferromagnetic Potts models. Markov Process Related Fields, 7, 21–38 (2001)

[Sta98] Stanley, R. P.: Graph colourings and related symmetric functions: ideas and applications. A description of results, interesting applications, & notable open problems. Discrete Mathematics, 193, 267–286 (1998)

[Sta95] Stanley, R. P.: A symmetric function generalization of the chromatic polynomial of a graph. Advances in Math., 111, 166–194 (1995)

[Sta96] Stanley, R.: Enumerative Combinatorics, vol. 1. Cambridge University Press, Cambridge (1996)

[Sta76] Stanley, R. P.: Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings. Duke Math. J., 43, 511–531 (1976)

[Sta80] Stanley, R. P.: Decompositions of rational convex polytopes. Ann. Discrete Math., 6, 333–342 (1980)

[Syl78] Sylvester, J. J.: On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics, with three appendices. Amer. J. Math., 1, 64–125 (1878)

[Thi87] Thistlethwaite, M. B.: A spanning tree expansion of the Jones polynomial. Topology, 26, 297–309 (1987)

[Tra] Traldi, L.: A note on pendant-twin reductions and weighted interlace polynomials. Preprint.

[Tut67] Tutte, W. T.: On dichromatic polynomials. J. Combin. Theory, 2, 301–320 (1967)
[Tut79] Tutte, W. T.: All the kings horses. In: Bondy, J. A., Murty, U. S. R. (eds) Graph Theory and Related Topics. Academic Press, London (1979)

[Tut84] Tutte, W. T.: Graph Theory. Addison-Wesley, New York (1984)

[Ula60] Ulam, S.: A Collection of Mathematical Problems. Wiley (Interscience), New York (1960)

[Wat95] Waterman, M. S.: Introduction to Computational Biology: Maps, Sequences and Genomes. Chapman Hall, New York (1995)

[Wel93] Welsh, D. J. A.: Complexity: Knots, Colorings and Counting. Cambridge University Press, Cambridge (1993)

[Wel97] Welsh, D. A. J.: Approximate counting. In: Bailey, R. (ed) Surveys in Combinatorics, 1997. Cambridge University Press, Cambridge (1997)

[Wel99] Welsh, D. J. A.: The Tutte polynomial. Statistical physics methods in discrete probability, combinatorics, and theoretical computer science. Random Structures Algorithms, 15, 210–228 (1999)

[WM00] Welsh, D. J. A., Merino, C.: The Potts model and the Tutte polynomial. Journal of Mathematical Physics, 41, 1127–1152 (2000)

[Yet90] Yetter, David N.: On graph invariants given by linear recurrence relations. J. Combin. Theory Ser. B, 48, 6–18 (1990)

[Zas92] Zaslavsky, T.: Strong Tutte functions of matroids and graphs. Trans. Amer. Math. Soc., 334, 317–347 (1992)
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