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COMMENT ON: PSEUDO-EFFECTIVITY OF THE RELATIVE CANONICAL DIVISOR ... BY ZSOLT PATAKFALVI

JÁNOS KOLLÁR

We discuss a more elementary approach to the existence of non-uniruled ramified covers of varieties, which was one of the steps in the proof of the main theorem in [Pat20]. This approach gives reasonably good control of the degree of the ramification divisor. A disadvantage is that we get non-uniruledness only for very general members of a family of covers, whereas both [Pat20] and [PZ20] give an open subset of non-uniruled members.

We work over an algebraically closed field \( k \) of arbitrary characteristic. The main observation is the following.

**Proposition 1.** Let \( X^n \subset \mathbb{P}^N \) be an integral \( k \)-variety and \( H \subset \mathbb{P}^N \) a very general hypersurface of degree \( \geq n + 1 \). Then \( X \cap H \) is not uniruled.

**Proof.** As a direct consequence of a result of Matsusaka [Mat68], it is sufficient to find one hypersurface \( H_0 \) of degree \( n + 1 \) such that \( X \not\subset H_0 \) and at least one of the irreducible components of \( X \cap H_0 \) is not uniruled; see [Kol96, IV.1.7–8] for details.

Choose now a linear projection \( \pi : \mathbb{P}^N \to \mathbb{P}^n \) that is finite on \( X \). By [RW18, Thm.1.1] there is a hypersurface \( W \subset \mathbb{P}^n \) of degree \( n + 1 \) that is not uniruled. We can then take \( H_0 \) to be the closure of \( \pi^{-1}(W) \).

**Corollary 2.** Let \( Y \) be a smooth, projective \( k \)-variety of dimension \( n \). Then there is a non-uniruled, smooth, projective \( k \)-variety \( Y' \) of dimension \( n \), and a finite morphism \( p : Y' \to Y \) of degree \( n + 2 \).

**Proof.** Choose an embedding \( Y \to \mathbb{P}^N \) and let \( X \subset \mathbb{P}^{N+1} \) be the cone over \( Y \). Set \( Y' := X \cap H \), where \( H \) is a very general hypersurface of degree \( n + 2 \). Then \( Y' \) is non-uniruled by Proposition 1 and we can take \( p \) to be the projection from the vertex of the cone.

Note that Proposition 1 is sharp in case \( X \) is a linear subspace. However one expects to do better in all other cases. Mainly to encourage further study of this question, let me state the strongest variant that could be true.

**Conjecture 3.** Let \( X \) be a smooth projective variety and \( |H| \) a very ample linear system on \( X \) such that \( K_X + H \) is effective. Then a general \( H \in |H| \) is not uniruled.

This is true in characteristic 0, but there is very little evidence for this in positive characteristic. It is not even clear that it holds for \( \dim X = 3 \). Note that the smoothness of \( X \) is essential here; cones over uniruled varieties with ample canonical class give singular counterexamples.

**Remark 4.** The proof of [RW18] is quite subtle, but it leads to simple, explicit examples defined over finite fields; see [RW18, 3.11].

Still it may be worthwhile to find an elementary argument, giving possibly worse degree bounds. As the most naive example, let \( (g(x, y) = 0) \subset \mathbb{A}^2 \) be a nonrational
cubic. Then
\[ (g(x_1, y_1) = \cdots = g(x_n, y_n) = 0) \subset \mathbb{A}^{2n} \]
is a complete intersection of degree \(3^n\) that contains no rational curves. A birational projection of it to \(\mathbb{P}^{n+1}\) gives a non-uniruled hypersurface of degree \(3^n\).

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Princeton University, Princeton NJ 08544-1000, kollar@math.princeton.edu