CONTINUITY OF GEVREY-HÖRMANDER PSEUDO-DIFFERENTIAL OPERATORS ON MODULATION SPACES

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Abstract. Let \( s \geq 1, \omega, \omega_0 \in P_{\omega,s}, a \in \Gamma_{s}^{(\omega_0)} \), and let \( \mathcal{B} \) be a suitable invariant quasi-Banach function space. Then we prove that the pseudo-differential operator \( Op(a) \) is continuous from \( M(\omega, \mathcal{B}) \) to \( M(\omega_0, \mathcal{B}) \).

0. Introduction

In the paper we consider continuity properties for a class of pseudo-differential operators introduced in [2] when acting on a broad class of modulation spaces. The symbols of the pseudo-differential operators are smooth, should obey strong ultra-regularity of Gevrey or Gelfand-Shilov types, and are allowed to grow exponentially or subexponentially.

Related questions were considered in the framework of the usual distribution theory in [31], where pseudo-differential operators were considered, with symbols in \( S^{(\omega_0)} \), the set of all smooth \( a \) which satisfies

\[
|\partial^{\alpha}a| \leq C_\alpha \omega_0.
\]

(See [18] and Section \([1]\) for notations.) In [31, Theorem 3.2] it was deduced that if \( \mathcal{B} \) is a translation invariant BF-space, \( \omega \) and \( \omega_0 \) belong to \( \mathcal{P} \), i.e. moderate and polynomially bounded weights, and \( a \in S^{(\omega_0)} \), then corresponding pseudo-differential operator, \( Op(a) \) is continuous from the modulation space \( M(\omega_0, \mathcal{B}) \) to \( M(\omega, \mathcal{B}) \). The obtained result in [31] can also be considered as extensions of certain results in the pioneering paper [24] by Tachizawa. For example, for suitable restrictions on \( \omega, \omega_0 \) and \( \mathcal{B} \), it follows that [31, Theorem 3.2] covers [24, Theorem 2.1].

Several classical continuity properties follows from [31, Theorem 3.2]. For example, since \( \mathcal{I} \) and \( \mathcal{I}' \) are suitable intersections and unions, respectively, of modulation spaces at above, it follows that \( Op(a) \) is continuous on \( \mathcal{I} \) and on \( \mathcal{I}' \) when \( a \in S^{(\omega_0)} \) with \( \omega_0 \in \mathcal{P} \).

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Some further conditions on the symbols in $S^{(\omega_0)}$ are required if corresponding pseudo-differential operators should be continuous on Gelfand-Shilov spaces, because of the imposed Gevrey regularity on the elements in such spaces. For elements $a$ in $\Gamma^{(\omega_0)}$ and $\Gamma^{(\omega_0)}_{0,s}$, the condition (0.1) is replaced by refined Gevrey-type conditions of the form

$$|\partial^\alpha a| \leq C h^{|\alpha|} \alpha! \omega_0,$$

(0.2)

involving global constants $C$ and $h$ which are independent of the derivatives (cf. [2]). More precisely, $\Gamma^{(\omega_0)}_{s}$ consists of all smooth $a$ such that (0.2) holds for some constants $C > 0$ and $h > 0$, and $a$ belongs to $\Gamma^{(\omega_0)}_{0,s}$, whenever for every $h > 0$ there is a constant for some $C > 0$ (which depends on both $a$ and $h$) such that (0.2) holds. In the case $s \geq 1$, the set $\mathcal{P}$ in [31] of weight functions are essentially replaced by the broader classes $\mathcal{P}^{E,s}_{0}$ and $\mathcal{P}^{E,s}_{E,s}$ in [2]. Here $\omega_0 \in \mathcal{P}^{E,s}_{E,s}$ whenever $\omega$ is $v_r$-moderate for some $r > 0$, where

$$v_r = e^{r|\cdot|^1},$$

(0.3)

and $\omega_0 \in \mathcal{P}^{E,s}_{E,s}$ whenever $\omega$ is $v_r$-moderate for every $r > 0$.

We notice that

$$\mathcal{P} \subseteq \mathcal{P}^{0}_{E,s_1} \subseteq \mathcal{P}^{0}_{E,s_2} \subseteq \mathcal{P}^{0}_{E,s_1}, \quad s_1 < s_2.$$

Hence, despite that $\Gamma^{(\omega_0)}_{0,s} \subseteq \Gamma^{(\omega_0)}_{s} \subseteq S^{(\omega_0)}$ holds for every $\omega_0$, we have

$$\Gamma^{(\omega)}_{0,s} \not\subset \bigcup_{\omega_0 \in \mathcal{P}} S^{(\omega_0)}$$

for some $\omega \in \mathcal{P}^{0}_{E,s}$, and

$$\Gamma^{(\omega)}_{0,s} \not\subset \bigcup_{\omega_0 \in \mathcal{P}^{0}_{E,s}} \Gamma^{(\omega_0)}_{s}$$

for some $\omega \in \mathcal{P}^{0}_{E,s}$.

In [2] it is proved that if $\omega_0 \in \mathcal{P}^{0}_{E,s}$ and $a \in \Gamma^{(\omega_0)}_{0,s}$, then corresponding pseudo-differential operators $Op(a)$ is continuous on the Gelfand-Shilov space $\Sigma_s$ of Beurling type, and its distribution space $\Sigma'_s$. If instead $\omega_0 \in \mathcal{P}^{0}_{E,s}$ and $a \in \Gamma^{(\omega_0)}_{s}$, then $Op(a)$ is continuous on the Gelfand-Shilov space $S_s$ of Roumieu type, and its distribution space $S'_s$. (Cf. Theorems 4.10 and 4.11 in [2].)

In Section 2 we complement these continuity properties by deducing continuity properties for such pseudo-differential operators when acting on a broad family of modulation spaces. More precisely, if $\omega_0, \omega \in \mathcal{P}^{0}_{E,s}$, $\mathcal{B}$ is a suitable invariant quasi-Banach-Function space (QBF-space), $M(\omega, \mathcal{B})$ is the modulation space with respect to $\omega$ and $\mathcal{B}$, and $a \in \Gamma^{(\omega)}_{s}$, then we show that $Op(a)$ is continuous from $M(\omega_0, \mathcal{B})$ to $M(\omega, \mathcal{B})$, and that the same holds true with $\mathcal{P}^{0}_{E,s}$ and $\Gamma^{(\omega)}_{0,s}$ in place.
of $\mathcal{P}_E^0$ and $\Gamma_s^{(\omega)}$ (cf. Theorems 2.5, 2.8 and 2.10 and Corollary 2.11). In the case when $\mathcal{B}$ is a Banach space, then the restrictions on $\mathcal{B}$ are given in Definition 1.4 while if $\mathcal{B}$ fails to be a Banach space, then suitable Lebesgue quasi-norm estimates are imposed on the elements in $\mathcal{B}$.

Evidently, by replacing $\mathcal{P}_E^0$ and $\Gamma_s^{(\omega)}$ with $\mathcal{P}$ and $S^{(\omega)}$, our results in Section 2 when $\mathcal{B}$ is a Banach space, take the same form as the main result Theorem 3.2 in [31]. Some of the results in Section 2 can therefore be considered as analogies of the results in [31] in the framework of ultra-distribution theory. We also remark that using the fact that Gelfand-Shilov spaces and their distribution spaces equal suitable intersections and unions of modulation spaces, the continuity results for pseudo-differential operators in [2] are straight-forward consequences of Theorems 2.5 and 2.8. We also refer to [16,19,20,25,26,32,35] and the references therein for more facts about pseudo-differential operators in framework of Gelfand-Shilov and modulation spaces.

The (classical) modulation spaces $M^{p,q}$, $p,q \in [1,\infty]$, as introduced by Feichtinger in [5], consist of all tempered distributions whose short-time Fourier transforms (STFT) have finite mixed $L^{p,q}$ norm. It follows that the parameters $p$ and $q$ to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}$. The theory of modulation spaces was developed further and generalized in [8–10,13], where Feichtinger and Gröchenig established the theory of coorbit spaces. In particular, the modulation space $M^{p,q}_{(\omega)}$, where $\omega$ denotes a weight function on phase (or time-frequency shift) space, appears as the set of tempered (ultra-) distributions whose STFT belong to the weighted and mixed Lebesgue space $L^{p,q}_{(\omega)}$.

1. Preliminaries

In this section we discuss basic properties for modulation spaces and other related spaces. The proofs are in many cases omitted since they can be found in [3,5,10,11,27,30].

1.1. Weight functions. A weight or weight function on $\mathbb{R}^d$ is positive function in $L^{\infty}_{\text{loc}}(\mathbb{R}^d)$. Let $\omega$ and $v$ be weights on $\mathbb{R}^d$. Then $\omega$ is called $v$-moderate or moderate, if

$$\omega(x_1 + x_2) \lesssim \omega(x_1)v(x_2), \quad x_1, x_2 \in \mathbb{R}^d. \quad (1.1)$$

Here $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \leq cg(\theta)$ for some constant $c > 0$ which is independent of $\theta$ in the domain of $f$ and $g$. If $v$ can be chosen as polynomial, then $\omega$ is called a weight of polynomial type. The function $v$ is called submultiplicative, if it is even and (1.1) holds for $\omega = v$.

We let $\mathcal{P}_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$, and $\mathcal{P}(\mathbb{R}^d)$ be the subset of $\mathcal{P}_E(\mathbb{R}^d)$ which consists of all polynomially moderate
functions on $\mathbb{R}^d$. We also let $\mathcal{P}_{E,s}(\mathbb{R}^d)$ ($\mathcal{P}^0_{E,s}(\mathbb{R}^d)$) be the set of all weights $\omega$ in $\mathbb{R}^d$ such that
\[
\omega(x_1 + x_2) \lesssim \omega(x_1)e^{r|x_2|^\frac{1}{2}}, \quad x_1, x_2 \in \mathbb{R}^d.
\] (1.2)
for some $r > 0$ (for every $r > 0$). We have
\[
\mathcal{P} \subseteq \mathcal{P}^0_{E,s_1} \subseteq \mathcal{P}_{E,s_1} \subseteq \mathcal{P}^0_{E,s_2} \subseteq \mathcal{P}_E \quad \text{when} \quad s_1 < s_2
\]
and
\[
\mathcal{P}_{E,s} = \mathcal{P}_E
\]
when $s \geq 1$,
where the last equality follows from the fact that if $\omega \in \mathcal{P}_E(\mathbb{R}^d) \ (\omega \in \mathcal{P}^0_E(\mathbb{R}^d))$, then
\[
\omega(x + y) \lesssim \omega(x)e^{r|y|^\frac{1}{2}} \quad \text{and} \quad e^{-r|x|} \leq \omega(x) \lesssim e^{r|x|}, \quad x, y \in \mathbb{R}^d
\] (1.3)
hold true for some $r > 0$ (for every $r > 0$) (cf. [15]).

1.2. Gelfand-Shilov spaces. Let $0 < h, s, t \in \mathbb{R}$ be fixed. Then $S_{s,h}(\mathbb{R}^d)$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that
\[
\|f\|_{S_{t,h}} \equiv \sup_{h^{|\alpha|+|\beta|}} \frac{|x^\alpha \partial^\beta f(x)|}{h^{|\alpha|+|\beta|+s t!}}
\] (1.4)
is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

Obviously $S_{s,h}$ is a Banach space, contained in $\mathcal{S}$, and which increases with $h$, $s$ and $t$ and $S_{s,h} \subset \mathcal{S}$. Here and in what follows we use the notation $A \subset B$ when the topological spaces $A$ and $B$ satisfy $A \subset B$ with continuous embeddings. Furthermore, if $s, t > \frac{1}{2}$, or $s = t = \frac{1}{2}$ and $h$ is sufficiently large, then $S_{s,h}$ contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathcal{S}$ and in $S_{t,h}$, it follows that the dual $(S_{s,h})' \ (\mathbb{R}^d)$ of $S_{t,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}''(\mathbb{R}^d)$, for such choices of $s$ and $t$.

The Gelfand-Shilov spaces $S^t_1(\mathbb{R}^d)$ and $\Sigma_t(\mathbb{R}^d)$ are defined as the inductive and projective limits respectively of $S_{s,h}(\mathbb{R}^d)$. This implies that
\[
S^s_1(\mathbb{R}^d) = \bigcup_{h > 0} S_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_t(\mathbb{R}^d) = \bigcap_{h > 0} S_{s,h}(\mathbb{R}^d),
\] (1.5)
and that the topology for $S^s_1(\mathbb{R}^d)$ is the strongest possible one such that the inclusion map from $S_{s,h}(\mathbb{R}^d)$ to $S^s_1(\mathbb{R}^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma_t(\mathbb{R}^d)$ is a Fréchet space with seminorms $\| \cdot \|_{S_{s,h}}$, $h > 0$. Moreover, $\Sigma_t(\mathbb{R}^d) \neq \{0\}$, if and only if $s + t \geq 1$ and $(s, t) \neq \left(\frac{1}{2}, \frac{1}{2}\right)$, and $S^s_1(\mathbb{R}^d) \neq \{0\}$, if and only if $s + t \geq 1$. 


The Gelfand-Shilov distribution spaces \((\mathcal{S}_s^t)'(\mathbb{R}^d)\) and \((\Sigma^t_s)'(\mathbb{R}^d)\) are the projective and inductive limit respectively of \((\mathcal{S}_{t,h}^s)'(\mathbb{R}^d)\). This means that
\[
(\mathcal{S}_s^t)'(\mathbb{R}^d) = \bigcap_{h>0}(\mathcal{S}_{t,h}^s)'(\mathbb{R}^d) \quad \text{and} \quad (\Sigma^t_s)'(\mathbb{R}^d) = \bigcup_{h>0}(\mathcal{S}_{t,h}^s)'(\mathbb{R}^d).
\]

We remark that in [12] it is proved that \((\mathcal{S}_s^t)'(\mathbb{R}^d)\) is the dual of \(\mathcal{S}_s^t(\mathbb{R}^d)\), and \((\Sigma^t_s)'(\mathbb{R}^d)\) is the dual of \(\Sigma^t_s(\mathbb{R}^d)\) (also in topological sense). For convenience we set
\[
\mathcal{S}_s = \mathcal{S}_s^s, \quad \mathcal{S}_s' = (\mathcal{S}_s^s)', \quad \Sigma_s = \Sigma_s^s \quad \text{and} \quad \Sigma_s' = (\Sigma_s^s)'.
\]

For every admissible \(s, t > 0\) and \(\varepsilon > 0\) we have
\[
\Sigma_s^t(\mathbb{R}^d) \hookrightarrow \mathcal{S}_s^t(\mathbb{R}^d) \hookrightarrow \Sigma_{s+t+\varepsilon}(\mathbb{R}^d)
\]
and
\[
(\Sigma_{s+t+\varepsilon}^t)'(\mathbb{R}^d) \hookrightarrow (\mathcal{S}_s^t)'(\mathbb{R}^d) \hookrightarrow (\Sigma_s^t)'(\mathbb{R}^d).
\]

From now on we let \(\mathcal{F}\) be the Fourier transform which takes the form
\[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)} \, dx
\]
when \(f \in L^1(\mathbb{R}^d)\). Here \((\cdot, \cdot)\) denotes the usual scalar product on \(\mathbb{R}^d\). The map \(\mathcal{F}\) extends uniquely to homeomorphisms on \(\mathcal{S}'(\mathbb{R}^d)\), from \((\mathcal{S}_s^t)'(\mathbb{R}^d)\) to \((\mathcal{S}_s^s)'(\mathbb{R}^d)\) and from \((\Sigma_s^t)'(\mathbb{R}^d)\) to \((\Sigma_s^s)'(\mathbb{R}^d)\). Furthermore, \(\mathcal{F}\) restricts to homeomorphisms on \(\mathcal{S}^t_s(\mathbb{R}^d)\), from \(\mathcal{S}_s^t(\mathbb{R}^d)\) to \(\mathcal{S}_s^s(\mathbb{R}^d)\) and from \(\Sigma_s^t(\mathbb{R}^d)\) to \(\Sigma_s^s(\mathbb{R}^d)\), and to a unitary operator on \(L^2(\mathbb{R}^d)\). Similar facts hold true when \(s = t\) and the Fourier transform is replaced by a partial Fourier transform.

Gelfand-Shilov spaces and their distribution spaces can in convenient ways be characterized by means of estimates of short-time Fourier transforms, (see e.g. [17, 32, 34]). We here recall the details and start by giving the definition of the short-time Fourier transform.

Let \(\phi \in \mathcal{S}_s^t(\mathbb{R}^d)\) be fixed. Then the short-time Fourier transform \(V_\phi f\) of \(f \in \mathcal{S}_s^t(\mathbb{R}^d)\) with respect to the window function \(\phi\) is the Gelfand-Shilov distribution on \(\mathbb{R}^{2d}\), defined by
\[
V_\phi f(x, \xi) \equiv (\mathcal{F}_2(U(f \otimes \phi)))(x, \xi) = \mathcal{F}(f \phi(\cdot - x))(\xi),
\]
where \((UF)(x, y) = F(y, x - y)\). Here \(\mathcal{F}_2 F\) is the partial Fourier transform of \(F(x, y) \in \mathcal{S}_s^t(\mathbb{R}^{2d})\) with respect to the \(y\) variable. If \(f, \phi \in \mathcal{S}_s(\mathbb{R}^d)\), then it follows that
\[
V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y)\overline{\phi(y - x)}e^{-i(y, \xi)} \, dy.
\]

We have now the following characterisations of Gelfand-Shilov functions and their distributions.

**Proposition 1.1.** Let \(s, t, s_0, t_0 > 0\) be such that \(s_0 + t_0 \geq 1, s_0 \leq s\) and \(t_0 \leq t\). Also let \(\phi \in \mathcal{S}_s^t(\mathbb{R}^d) \setminus \{0\}\) and \(f \in \mathcal{S}_s^{s_0}(\mathbb{R}^d)\). Then the following is true:
(1) \( f \in \mathcal{S}_0^s(\mathbb{R}^d) \), if and only if
\[
|V_\phi f(x, \xi)| \lesssim e^{-r(|x|^p + |\xi|^q)} ,
\] (1.7)
holds for some \( r > 0 \);

(2) if in addition \( (s_0, t_0) \neq \left( \frac{1}{p}, \frac{1}{q} \right) \) and \( \phi \in \Sigma_0^{s_0}(\mathbb{R}^d) \), then \( f \in \Sigma_0^{s_0}(\mathbb{R}^d) \), if and only if (1.7) holds for every \( r > 0 \).

A proof of Theorem 1.1 can be found in e.g. [17, 34] (cf. [17, Theorem 2.7]). The corresponding result for Gelfand-Shilov distributions is the following. We refer to [32, 33] for the proof.

**Proposition 1.2.** Let \( s, t, s_0, t_0 > 0 \) be such that \( s_0 + t_0 \geq 1 \), \( s_0 \leq s \) and \( t_0 \leq t \). Also let \( \phi \in \mathcal{S}_0^s(\mathbb{R}^d) \setminus 0 \) and \( f \in (\mathcal{S}_0^{s_0})'(\mathbb{R}^d) \). Then the following is true:

(1) \( f \in (\mathcal{S}_t^s)'(\mathbb{R}^d) \), if and only if
\[
|V_\phi f(x, \xi)| \lesssim e^{r(|x|^p + |\xi|^q)} ,
\] (1.8)
holds for every \( r > 0 \);

(2) if in addition \( (s_0, t_0) \neq \left( \frac{1}{p}, \frac{1}{q} \right) \) and \( \phi \in \Sigma_t^s(\mathbb{R}^d) \), then \( f \in (\Sigma_t^s)'(\mathbb{R}^d) \), if and only if (1.8) holds for some \( r > 0 \).

1.3. **Modulation spaces.** Let \( \phi \in \Sigma_1(\mathbb{R}^d) \setminus 0 \), \( p, q \in (0, \infty] \) and \( \omega \in \mathcal{D}_E(\mathbb{R}^{2d}) \) be fixed. Then the modulation space \( \mathcal{M}_{\omega}^{p,q}(\mathbb{R}^d) \) consists of all \( f \in \Sigma_1^e(\mathbb{R}^d) \) such that
\[
\|f\|_{\mathcal{M}_{\omega}^{p,q}} \equiv \left( \int \left( \int |V_\phi f(x, \xi)\omega(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty \quad (1.9)
\]
(with the obvious modifications when \( p = \infty \) and/or \( q = \infty \)). We set \( \mathcal{M}_\omega^p = \mathcal{M}_{\omega}^{p,p} \), and if \( \omega = 1 \), then we set \( \mathcal{M}_\omega^{p,q} = \mathcal{M}_{\omega}^{p,q} \) and \( \mathcal{M}_\omega^p = \mathcal{M}_\omega^p \).

The following proposition is a consequence of well-known facts in [14, 33]. Here and in what follows, we let \( p' \) denotes the conjugate exponent of \( p \), i.e.
\[
p' = \begin{cases} 
\infty & \text{when } p \in (0, 1] \\
p & \text{when } p \in (1, \infty) \\
p - 1 & \text{when } p = \infty.
\end{cases}
\]

**Proposition 1.3.** Let \( p, q, p_j, q_j, r \in (0, \infty] \) be such that \( r \leq \min(1, p, q) \), \( j = 1, 2 \), let \( \omega, \omega_1, \omega_2, v \in \mathcal{D}_E(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, \( \phi \in \mathcal{M}_v^e(\mathbb{R}^d) \setminus 0 \), and let \( f \in \Sigma_1^e(\mathbb{R}^d) \). Then the following is true:

(1) \( f \in \mathcal{M}_{\omega}^{p,q}(\mathbb{R}^d) \) if and only if (1.9) holds, i.e. \( \mathcal{M}_{\omega}^{p,q}(\mathbb{R}^d) \) is independent of the choice of \( \phi \). Moreover, \( \mathcal{M}_{\omega}^{p,q}(\mathbb{R}^d) \) is a Banach space under the norm in (1.9), and different choices of \( \phi \) give rise to equivalent norms.
(2) if $p_1 \leq p_2$, $q_1 \leq q_2$ and $\omega_2 \leq \omega_1$ for some constant $C$, then
$$
\Sigma_1(R^d) \subseteq M^{p_1,q_1}_{\omega_1}(R^d) \subseteq M^{p_2,q_2}_{\omega_2}(R^d) \subseteq \Sigma'_1(R^d).
$$

Proposition 1.3 (1) permits us to be rather vague about the choice of $\phi \in M_{r(v)} \setminus 0$ in (1.9). For example, if $C > 0$ is a constant and $\Omega$ is a subset of $\Sigma'_1$, then $\|a\|_{M_{r(v)}} \leq C\|a\|_{M_{r(v)}}$ for every $a \in \Omega$, means that the inequality holds for some choice of $\phi \in M_{r(v)} \setminus 0$ and every $a \in \Omega$. Evidently, for any other choice of $\phi \in M_{r(v)} \setminus 0$, a similar inequality is true although $C$ may have to be replaced by a larger constant, if necessary.

Let $s, t \in \mathbb{R}$. Then the weights $(x, \xi) \mapsto \langle x \rangle^s \langle \xi \rangle^t$ and $(x, \xi) \mapsto \langle (x, \xi) \rangle^s$, $x, \xi \in \mathbb{R}^d$, (1.10) are common in the applications. It follows that they belong to $\mathcal{P}(R^{2d})$ for every $s, t \in \mathbb{R}$. If $\omega \in \mathcal{P}(R^{2d})$, then $\omega$ is moderated by any of the weights in (1.10) provided $s$ and $t$ are chosen large enough.

We refer to [5, 8–11, 14, 22, 33] for more facts about modulation spaces.

1.4. A broader family of modulation spaces. As announced in the introduction we consider in Section 2 mapping properties for pseudo-differential operators when acting on a broader class of modulation spaces, which are defined by imposing certain types of translation invariant solid BF-space norms on the short-time Fourier transforms. (Cf. [5–9].)

**Definition 1.4.** Let $B \subseteq L^1_{\text{loc}}(R^d)$ be a quasi-Banach of order $r \in (0, 1]$, and let $v \in \mathcal{P}_E(R^d)$. Then $B$ is called a translation invariant Quasi-Banach Function space on $R^d$ (with respect to $v$), or invariant QBF space on $R^d$, if there is a constant $C$ such that the following conditions are fulfilled:

1. if $x \in \mathbb{R}^d$ and $f, g \in B$, then $f(\cdot - x) \in B$, and
$$
\|f(\cdot - x)\|_B \leq C_{v(x)}\|f\|_B; \quad (1.11)
$$

2. if $f, g \in L^1_{\text{loc}}(R^d)$ satisfy $g \in B$ and $|f| \leq |g|$, then $f \in B$ and
$$
\|f\|_B \leq C\|g\|_B.
$$

If $v$ belongs to $\mathcal{P}_{E,s}(R^d)$, then $B$ in Definition 1.4 is called an invariant BF-space of Roumieu type (Beurling type) of order $s$.

We notice that the quasi-norm $\| \cdot \|_B$ in Definition 1.4 should satisfy
$$
\|f + g\|_B \leq 2^{1-s}\|f\|_B + \|g\|_B \quad f, g \in B. \quad (1.12)
$$

By Akira and Rolewicz in [1, 21] it follows that there is an equivalent quasi-norm to the previous one which additionally satisfies
$$
\|f + g\|'_B \leq \|f\|'_B + \|g\|'_B \quad f, g \in B. \quad (1.13)
$$
From now on we suppose that the quasi-norm of $B$ has been chosen such that both (1.12) and (1.13) hold true.

It follows from (2) in Definition 1.4 that if $f \in B$ and $h \in L^\infty$, then $f \cdot h \in B$, and

$$\|f \cdot h\|_B \leq C\|f\|_B\|h\|_{L^\infty}.$$  (1.14)

If $r = 1$, then $B$ in Definition 1.4 is a Banach space, and the condition (2) means that a translation invariant QBF-space is a solid BF-space in the sense of (A.3) in [6]. The space $B$ in Definition 1.4 is called an invariant BF-space (with respect to $v$) if $r = 1$, and Minkowski’s inequality holds true, i.e.

$$\|f \ast \varphi\|_B \leq C\|f\|_B\|\varphi\|_{L^1(v)}, \quad f \in B, \varphi \in C^\infty_0(\mathbb{R}^d)$$  (1.15)

for some constant $C$ which is independent of $f \in B$ and $\varphi \in C^\infty_0(\mathbb{R}^d)$.

**Example 1.5.** Assume that $p, q \in [1, \infty]$, and let $L^{p,q}_1(\mathbb{R}^{2d})$ be the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^{2d})$ such that

$$\|f\|_{L^{p,q}_1} \equiv \left( \int_\mathbb{R}^d \left( \int_\mathbb{R}^d |f(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q}$$

is finite. Also let $L^{p,q}_2(\mathbb{R}^{2d})$ be the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^{2d})$ such that

$$\|f\|_{L^{p,q}_2} \equiv \left( \int_\mathbb{R}^d \left( \int_\mathbb{R}^d |f(x, \xi)|^q d\xi \right)^{p/q} dx \right)^{1/p}$$

is finite. Then it follows that $L^{p,q}_1$ and $L^{p,q}_2$ are translation invariant BF-spaces with respect to $v = 1$.

For translation invariant BF-spaces we make the following observation.

**Proposition 1.6.** Assume that $v \in \mathcal{P}_E(\mathbb{R}^d)$, and that $B$ is an invariant BF-space with respect to $v$ such that (1.15) holds true. Then the convolution mapping $(\varphi, f) \mapsto \varphi \ast f$ from $C^\infty_0(\mathbb{R}^d) \times B$ to $B$ extends uniquely to a continuous mapping from $L^1_{(v)}(\mathbb{R}^d) \times B$ to $B$, and (1.15) holds true for any $f \in B$ and $\varphi \in L^1_{(v)}(\mathbb{R}^d)$.

The result is a straight-forward consequence of the fact that $C^\infty_0$ is dense in $L^1_{(v)}$.

Next we consider the extended class of modulation spaces which we are interested in.

**Definition 1.7.** Assume that $B$ is a translation invariant QBF-space on $\mathbb{R}^{2d}$, $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, and that $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$. Then the set $M_{(\omega)} = M(\omega, B)$ consists of all $f \in \Sigma_1(\mathbb{R}^d)$ such that

$$\|f\|_{M_{(\omega)}} = \|f\|_{M(\omega, B)} \equiv \|V_\phi f \omega\|_B$$

is finite.
Obviously, we have $M_{(\omega)}^{p,q}(\mathbb{R}^d) = M(\omega, \mathcal{B})$ when $\mathcal{B} = L_1^{p,q}(\mathbb{R}^{2d})$ (cf. Example 1.5). It follows that many properties which are valid for the classical modulation spaces also hold for the spaces of the form $M(\omega, \mathcal{B})$. For example we have the following proposition, which shows that the definition of $M(\omega, \mathcal{B})$ is independent of the choice of $\phi$ when $\mathcal{B}$ is a Banach space. We omit the proof since it follows by similar arguments as in the proof of Proposition 11.3.2 in [14].

**Proposition 1.8.** Let $\mathcal{B}$ be an invariant BF-space with respect to $v_0 \in \mathcal{P}_E(\mathbb{R}^{2d})$ for $j = 1, 2$. Also let $\omega, v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $v$-moderate, $M(\omega, \mathcal{B})$ is the same as in Definition 1.7, and let $\phi \in M_{(v_0)}^{1}(\mathbb{R}^d) \setminus 0$ and $f \in \Sigma_1'(\mathbb{R}^d)$. Then $f \in M(\omega, \mathcal{B})$ if and only if $V_0^f \omega \in \mathcal{B}$, and different choices of $\phi$ gives rise to equivalent norms in $M(\omega, \mathcal{B})$.

The quasi-Banach spaces here above is usually a mixed quasi-normed Lebesgue space, given as follows. Let $E$ be a non-degenerate parallelepiped in $\mathbb{R}^d$ which is spanned by the ordered basis $\kappa(E) = \{e_1, \ldots, e_d\}$. That is,

$$E = \{x_1e_1 + \cdots + x_de_d; (x_1, \ldots, x_d) \in \mathbb{R}^d, 0 \leq x_k \leq 1, k = 1, \ldots, d\}.$$  

The corresponding lattice, dual parallelepiped and dual lattice are given by

$$\Lambda_E = \{j_1e_1 + \cdots + j_de_d; (j_1, \ldots, j_d) \in \mathbb{Z}^d\},$$

$$E' = \{\xi_1e'_1 + \cdots + \xi_de'_d; (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, 0 \leq \xi_k \leq 1, k = 1, \ldots, d\},$$

and

$$\Lambda'_E = \Lambda_{E'} = \{\iota_1e'_1 + \cdots + \iota_de'_d; (\iota_1, \ldots, \iota_d) \in \mathbb{Z}^d\},$$

respectively, where the ordered basis $\kappa(E') = \{e'_1, \ldots, e'_d\}$ of $E'$ satisfies

$$\langle e_j, e'_k \rangle = 2\pi\delta_{jk} \quad \text{for every} \quad j, k = 1, \ldots, d.$$

Note here that the Fourier analysis with respect to general biorthogonal bases has recently been developed in [23].

The basis $e'_1, \ldots, e'_d$ is called the **dual basis** of $e_1, \ldots, e_d$. We observe that there is a matrix $T_E$ such that $e_1, \ldots, e_d$ and $e'_1, \ldots, e'_d$ are the images of the standard basis under $T_E$ and $T_{E'} = 2\pi(T_E^{-1})^t$, respectively.

In the following we let

$$\max q = \max(q_1, \ldots, q_d) \quad \text{and} \quad \min q = \min(q_1, \ldots, q_d)$$

when $q = (q_1, \ldots, q_d) \in (0, \infty]^d$, and $\chi_\Omega$ be the characteristic function of $\Omega$.

**Definition 1.9.** Let $E$ be a non-degenerate parallelepiped in $\mathbb{R}^d$ spanned by the ordered set $\kappa(E) \equiv \{e_1, \ldots, e_d\}$ in $\mathbb{R}^d$, $p = (p_1, \ldots, p_d) \in (0, \infty]^d$.
and \( r = \min(1, p) \). If \( f \in L^p_{\text{loc}}(\mathbb{R}^d) \), then

\[
\|f\|_{L^p_{\text{loc}}(\mathbb{R}^d)} \equiv \|g_{d-1}\|_{L^p_{\text{loc}}(\mathbb{R})}
\]

where \( g_k(z_k), z_k \in \mathbb{R}^{d-k}, k = 0, \ldots, d - 1 \), are inductively defined as

\[
g_0(x_1, \ldots, x_d) \equiv |f(x_1e_1 + \cdots + x_de_d)|,
\]

and

\[
g_k(z_k) \equiv \|g_{k-1}(\cdot, z_k)\|_{L^p_{\text{loc}}(\mathbb{R})}, \quad k = 1, \ldots, d - 1.
\]

The space \( L^p_{\kappa(E)}(\mathbb{R}^d) \) consists of all \( f \in L^p_{\text{loc}}(\mathbb{R}^d) \) such that \( \|f\|_{L^p_{\kappa(E)}} \) is finite, and is called \( E \)-split Lebesgue space (with respect to \( p \) and \( \kappa(E) \)).

**Definition 1.10.** Let \( E_0 \subseteq \mathbb{R}^d \) be a non-degenerate parallelepiped with dual parallelepiped \( E'_0 \), and \( E \subseteq \mathbb{R}^{2d} \) be a parallelepiped spanned by the ordered set \( \kappa(E) = \{e_1, \ldots, e_{2d}\} \). Then \( E \) is called a phase-shift split parallelepiped (with respect to \( E_0 \)) if \( E \) is non-degenerate and \( d \) of the vectors \( \{e_1, \ldots, e_{2d}\} \) spans \( E_0 \) and the other \( d \) vectors is the corresponding dual basis which spans \( E'_0 \).

1.5. **Pseudo-differential operators.** Next we recall some facts on pseudo-differential operators. Let \( A \in \mathcal{M}(d, \mathbb{R}) \) be fixed and let \( a \in \Sigma_1(\mathbb{R}^{2d}) \). Then the pseudo-differential operator \( \text{Op}_A(a) \) is the linear and continuous operator on \( \Sigma_1(\mathbb{R}^d) \), defined by the formula

\[
(\text{Op}_A(a)f)(x) = (2\pi)^{-d} \int \int a(x - A(y),\xi) f(y) e^{i(x-y,\xi)} dyd\xi. \quad (1.16)
\]

The definition of \( \text{Op}_A(a) \) extends to any \( a \in \Sigma'_1(\mathbb{R}^{2d}) \), and then \( \text{Op}_A(a) \) is continuous from \( \Sigma'_1(\mathbb{R}^d) \) to \( \Sigma'_1(\mathbb{R}^d) \). Moreover, for every fixed \( A \in \mathcal{M}(d, \mathbb{R}) \), it follows that there is a one to one correspondence between such operators, and pseudo-differential operators of the form \( \text{Op}_A(a) \).

(See e.g. [13].) If \( A = 2^{-1}I \), where \( I \in \mathcal{M}(d, \mathbb{R}) \) is the identity matrix, then \( \text{Op}_A(a) \) is equal to the Weyl operator \( \text{Op}_w(a) \) of \( a \). If instead \( A = 0 \), then the standard (Kohn-Nirenberg) representation \( \text{Op}(a) \) is obtained.

If \( a_1, a_2 \in \Sigma'_1(\mathbb{R}^{2d}) \) and \( A_1, A_2 \in \mathcal{M}(d, \mathbb{R}) \), then

\[
\text{Op}_{A_1}(a_1) = \text{Op}_{A_2}(a_2) \iff a_2(x, \xi) = e^{i((A_1-A_2)D_\xi, D_x)} a_1(x, \xi). \quad (1.17)
\]

(Cf. [13].)

The following special case of [35, Theorem 3.1] is important when discussing continuity of pseudo-differential operators when acting on quasi-Banach modulation spaces.
Proposition 1.11. Let $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ be such that

$$\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \lesssim \omega_0(x, \eta - \xi, y - x).$$

(1.18)

Also let $p \in (0, \infty]^{2d}$, $E$ be a phase-shift split parallelepiped in $\mathbb{R}^{2d}$ and let $a \in M^{\infty, 1}_{(\omega_0)}(\mathbb{R}^{2d})$. Then $\text{Op}_p(a)$ is continuous from $M(L^p, \omega_1)$ to $M(L^p, \omega_2)$.

In the next section we discuss continuity for pseudo-differential operators with symbols in the following definition. (See also the introduction.)

Definition 1.12. Let $\omega_0$ be a weight on $\mathbb{R}^d$, and let $s \geq 0$.

(1) The set $\Gamma^{(\omega_0)}_{s}(\mathbb{R}^d)$ consists of all $a \in C^\infty(\mathbb{R}^d)$ such that

$$|\partial^\alpha f(x)| \lesssim h^{\alpha!} \omega_0(x), \quad \alpha \in \mathbb{N}^d,$$

for some constant $h > 0$;

(2) The set $\Gamma^{(\omega_0)}_{0,s}(\mathbb{R}^d)$ consists of all $a \in C^\infty(\mathbb{R}^d)$ such that

$$|\partial^\alpha f(x)| \lesssim h^{\alpha!} \omega_0(x), \quad \alpha \in \mathbb{N}^d,$$

holds for every $h > 0$.

2. CONTINUITY FOR PSEUDO-DIFFERENTIAL OPERATORS WITH SYMBOLS IN $\Gamma^{(\omega)}_{s}$ AND $\Gamma^{(\omega)}_{0,s}$

In this section we discuss continuity for operators in $\text{Op}(\Gamma^{(\omega)}_{s}(\mathbb{R}^d))$ and $\text{Op}(\Gamma^{(\omega)}_{0,s}(\mathbb{R}^d))$ when acting on a general class of modulation spaces. In Theorem [2, 25] below it is proved that if $\omega, \omega_0 \in \mathcal{P}_E\omega_0, A \in \mathcal{M}(d, \mathbb{R})$ and $a \in \Gamma^{(\omega_0)}_{0,s}$, then $\text{Op}_A(a)$ is continuous from $M(\omega_0, \mathcal{B})$ to $M(\omega_0, \mathcal{B})$. This gives an analogy to [31, Theorem 3.2] in the framework of operator theory and Gelfand-Shilov classes.

We need some preparations before discussing these mapping properties. The following result shows that for any weight in $\mathcal{P}_E$, there are equivalent weights that satisfy strong Gevrey regularity.

Proposition 2.1. Let $\omega \in \mathcal{P}_E(\mathbb{R}^d)$ and $s > 0$. Then there exists a weight $\omega_0 \in \mathcal{P}_E(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that the following is true:

1. $\omega_0 \asymp \omega$;

2. $|\partial^\alpha \omega_0(x)| \lesssim \omega_0(x)h^{\alpha!} \alpha!^s < \omega(x)h^{\alpha!} \alpha!^s$ for every $h > 0$.

Proof. We may assume that $s < \frac{1}{2}$. It suffices to prove that (2) should hold for some $h > 0$. Let $\phi_0 \in \Sigma_s^\omega(\mathbb{R}^d) \setminus 0$, and let $\phi = |\phi_0|^2$. Then $\phi \in \Sigma_s^\omega(\mathbb{R}^d)$, giving that

$$|\partial^\alpha \phi(x)| \lesssim h^{\alpha!} e^{-c|x|^\frac{1}{s}} \alpha!^s,$$

for every $h > 0$ and $c > 0$. Now let $\omega_0 = \omega * \phi.$
We have
\[ |\partial^\alpha \omega_0(x)| = \left| \int \omega(y) (\partial^\alpha \phi)(x - y) \, dy \right| \]
\[ \lesssim h^{[\alpha]} \alpha! \int \omega(y) e^{-c |x-y|^s} \, dy \]
\[ \lesssim h^{[\alpha]} \alpha! \int \omega(x + (y - x)) e^{-c |x-y|^s} \, dy \]
\[ \lesssim h^{[\alpha]} \alpha! \omega(x) \int e^{-\frac{c}{2} |x-y|^s} \, dy \approx h^{[\alpha]} \alpha! \omega(x), \]
where the last inequality follows (1.2) and the fact that \( \phi \) is bounded by a super exponential function. This gives the first part of (2).

The equivalences in (1) follows in the same way as in [30, 32]. More precisely, by (1.2) we have
\[ \omega_0(x) = \int \omega(y) \phi(x - y) \, dy = \int \omega(x + (y - x)) \phi(x - y) \, dy \]
\[ \lesssim \omega(x) \int e^{c |x-y|^s} \phi(x - y) \, dy \approx \omega(x). \]
In the same way, (1.3) gives
\[ \omega_0(x) = \int \omega(y) \phi(x - y) \, dy = \int \omega(x + (y - x)) \phi(x - y) \, dy \]
\[ \gtrsim \omega(x) \int e^{-c |x-y|^s} \phi(x - y) \, dy \approx \omega(x), \]
and (1) as well as the second part of (2) follow. \( \square \)

The next result shows that \( \Gamma_s(\omega) \) and \( \Gamma_{0,s}(\omega) \) can be characterised in terms of estimates of short-time Fourier transforms.

**Proposition 2.2.** Let \( s \geq 1, \phi \in S_s(\mathbb{R}^d) \setminus 0 \), and let \( f \in S'_\eta(\mathbb{R}^d) \).

Then the following is true:

1. If \( \omega \in \mathcal{P}_{E,s}(\mathbb{R}^d) \), then \( f \in C^\infty(\mathbb{R}^d) \) and satisfies
   \[ |\partial^\alpha f(x)| \lesssim \omega(x) \epsilon^{[\alpha]} \alpha! \]
   for some \( \epsilon > 0 \), if and only if
   \[ |V_\phi f(x, \xi)| \lesssim \omega(x) e^{-\epsilon |\xi|^s}, \]
   for some new \( \epsilon > 0 \);

2. If \( \omega \in \mathcal{P}_{E,s}(\mathbb{R}^d) \) and in addition \( \phi \in \Sigma_\epsilon(\mathbb{R}^d) \), then \( f \in C^\infty(\mathbb{R}^d) \) and satisfies (2.1) for every \( \epsilon > 0 \) (resp. for some \( \epsilon > 0 \)), if and only if (2.2) holds true for every \( \epsilon > 0 \) (resp. for some new \( \epsilon > 0 \)).
Proof. We shall follow the proof of Proposition 3.1 in [2]. We only prove (2), and then when (2.1) or (2.2) are true for every \( \varepsilon > 0 \). The other cases follow by similar arguments and are left for the reader.

Assume that \( \phi \in \Sigma_s(R^d) \), \( \omega \in \mathcal{P}_{E,A}(R^d) \) and that (2.1) holds for every \( \varepsilon > 0 \). Then for every \( x \in R^d \) the function

\[
y \mapsto F_x(y) = f(y + x)\bar{\phi}(y)
\]

belongs to \( \Sigma_s \), and \( \omega(x + y) \lesssim e^{h_0|y|^\frac{1}{2}} \omega(x) \) for some \( h_0 > 0 \). By a straight-forward application of Leibnitz formula and the facts that

\[
|\partial^s \phi(x)| \lesssim e^{\varepsilon|\alpha|} e^{-h|x|^\frac{1}{2}} \quad \text{and} \quad \omega(x + y) \lesssim \omega(x)e^{h_0|y|^\frac{1}{2}}
\]

for some \( h_0 > 0 \) and every \( \varepsilon, h > 0 \) we get

\[
|\partial_y^s F_x(y)| \lesssim \omega(x)e^{-h|y|^\frac{1}{2}} e^{\varepsilon|\alpha|},
\]

for every \( \varepsilon, h > 0 \). In particular,

\[
|F_x(y)| \lesssim \omega(x)e^{-h|y|^\frac{1}{2}} \quad \text{and} \quad |\hat{F}_x(\xi)| \lesssim \omega(x)e^{-h|\xi|^\frac{1}{2}},
\]

(2.3)

for every \( h > 0 \). Since \( |V_{\phi}f(x, \xi)| = |\hat{F}_x(\xi)| \), the estimate (2.2) follows from the second inequality in (2.3). This shows that if (2.1) holds for every \( \varepsilon > 0 \), then (2.2) holds for every \( \varepsilon > 0 \).

Next suppose that (2.2) holds for every \( \varepsilon > 0 \). By the inversion formula we get

\[
f(x) = (2\pi)^{-\frac{d}{2}}\|\phi\|_2^2 \int_{R^{2d}} V_{\phi}f(y, \eta)\phi(x - y)e^{i(x, \eta)} \, dyd\eta.
\]

By differentiation and the fact that \( \phi \in \Sigma_s \) we get

\[
|\partial^\alpha f(x)| \preceq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\eta^{\beta} V_{\phi}f(y, \eta)(\partial^{\alpha - \beta}\phi)(x - y)e^{i(x, \eta)}| \, dyd\eta
\]

\[
\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{R^{2d}} |\eta^{\beta} V_{\phi}f(y, \eta)(\partial^{\alpha - \beta}\phi)(x - y)| \, dyd\eta
\]

\[
\lesssim \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{R^{2d}} |\eta^{\beta} \omega(y)e^{-\varepsilon_3|y|^\frac{1}{2}}(\partial^{\alpha - \beta}\phi)(x - y)| \, dyd\eta
\]

\[
\lesssim \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \varepsilon_2^{\alpha - \beta} |\alpha - \beta|!^s \int_{R^{2d}} |\eta^{\beta} \omega(y)e^{-\varepsilon_3|y|^\frac{1}{2}} e^{-\varepsilon_1|x - y|^\frac{1}{2}}| \, dyd\eta,
\]

for every \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \). Since

\[
|\eta^{\beta} e^{-\varepsilon_3|y|^\frac{1}{2}}| \lesssim \varepsilon_2^{\frac{\beta}{2}} \beta!^s e^{-\varepsilon_3|y|^\frac{1}{2}/2},
\]

we get

\[
|\partial^\alpha f(x)| \preceq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\eta^{\beta} V_{\phi}f(y, \eta)(\partial^{\alpha - \beta}\phi)(x - y)e^{i(x, \eta)}| \, dyd\eta
\]

\[
\lesssim \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \varepsilon_2^{\alpha - \beta} |\alpha - \beta|!^s \int_{R^{2d}} |\eta^{\beta} \omega(y)e^{-\varepsilon_3|y|^\frac{1}{2}} e^{-\varepsilon_1|x - y|^\frac{1}{2}}| \, dyd\eta,
\]

for every \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \). Since

\[
|\eta^{\beta} e^{-\varepsilon_3|y|^\frac{1}{2}}| \lesssim \varepsilon_2^{\frac{\beta}{2}} \beta!^s e^{-\varepsilon_3|y|^\frac{1}{2}/2},
\]
when $\varepsilon_3$ is chosen large enough compared to $\varepsilon_2^{-1}$, we get

$$|\partial^\alpha f(x)| \lesssim \varepsilon_2^{[\alpha]} \sum_{\beta \leq \alpha} \left(\frac{\alpha}{\beta}\right) (\beta! (\alpha - \beta)!)^s \int_{\mathbb{R}^d} \omega(y) e^{-\varepsilon_3|\eta|^{1/2}} e^{-\varepsilon_1|x-y|^{1/2}} \, dy \, d\eta$$

$$\lesssim (2\varepsilon_2)^{[\alpha]} \varepsilon_1^s \int_{\mathbb{R}^d} \omega(y) e^{-\varepsilon_1|x-y|^{1/2}} \, dy$$

Since $\omega(y) \leq \omega(x) e^{h_0|x-y|^{1/2}}$ for some $h_0 \geq 0$ and $\varepsilon_1$ can be chosen arbitrarily large, it follows from the last estimate that

$$|\partial^\alpha f(x)| \lesssim (2\varepsilon_2)^{[\alpha]} \varepsilon_1^s \omega(x),$$

for every $\varepsilon_2 > 0$, and the result follows. \hfill $\square$

The following result is now a straight-forward consequence of the previous proposition and the definitions.

**Proposition 2.3.** Let $s \geq 1$, $q \in (0, \infty)$, $\omega \in \mathcal{P}_{E,s}(\mathbb{R}^d)$ and let $\omega_r(x, \xi) = \omega(x) e^{-r|\xi|^{1/2}}$ when $x, \xi \in \mathbb{R}^d$. Then

$$\bigcup_{r>0} M^{\infty,q}_{(1/\omega_r)}(\mathbb{R}^d) = \Gamma^{(\omega)}_s(\mathbb{R}^d) \quad \text{and} \quad \bigcap_{r>0} M^{\infty,q}_{(1/\omega_r)}(\mathbb{R}^d) = \Gamma^{(\omega)}_{0,s}(\mathbb{R}^d).$$

The following lemma is a consequence of Theorem 4.6 in [2].

**Lemma 2.4.** Let $s \geq 1 \omega \in \mathcal{P}_{E}(\mathbb{R}^{2d})$, $A_1, A_2 \in \mathcal{M}(d, \mathbb{R})$, and that $a_1, a_2 \in \Sigma^1(\mathbb{R}^{2d})$ are such that $\text{Op}_{A_1}(a_1) = \text{Op}_{A_2}(a_2)$. Then

$$a_1 \in \Gamma^{(\omega)}_{s}(\mathbb{R}^{2d}) \quad \iff \quad a_2 \in \Gamma^{(\omega)}_{s}(\mathbb{R}^{2d})$$

and

$$a_1 \in \Gamma^{(\omega)}_{0,s}(\mathbb{R}^{2d}) \quad \iff \quad a_2 \in \Gamma^{(\omega)}_{0,s}(\mathbb{R}^{2d}).$$

We have now the following result.

**Theorem 2.5.** Let $A \in \mathcal{M}(d, \mathbb{R})$, $s \geq 1$, $\omega, \omega_0 \in \mathcal{P}_{E_0,s}(\mathbb{R}^{2d})$, $a \in \Gamma^{(\omega_0)}_{s}(\mathbb{R}^{2d})$, and that $\mathcal{B}$ is an invariant BF-space on $\mathbb{R}^{2d}$. Then $\text{Op}_{A}(a)$ is continuous from $M(\omega, \mathcal{B})$ to $M(\omega, \mathcal{B})$.

We need some preparations for the proof, and start by recalling Minkowski’s inequality in a somewhat general form. Assume that $d\mu$ is a positive measure, and that $f \in L^1(d\mu; \mathcal{B})$ for some Banach space $\mathcal{B}$. Then Minkowski’s inequality asserts that

$$\left\| \int f(x) \, d\mu(x) \right\|_{\mathcal{B}} \leq \int \|f(x)\|_{\mathcal{B}} \, d\mu(x).$$

We also need some lemmas.

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Lemma 2.6. Suppose $s \geq 1$, $\omega \in \mathcal{P}_E(R^d)$ and that $f \in C^\infty(R^{d+d_0})$ satisfies
\begin{equation}
|\partial^\alpha f (x, y)| \lesssim h|\alpha|! s e^{-r|x|^\frac{1}{2}} \omega(y), \alpha \in \mathbb{N}^{d+d_0}
\end{equation}
for some $h > 0$ and $r > 0$. Then there are $f_0 \in C^\infty(R^{d+d_0})$ and $\psi \in \mathcal{S}_r(R^d)$ such that (2.4) holds with $f_0$ in place of $f$ for some for some $h > 0$ and $r > 0$, and $f(x, y) = f_0(x, y)\psi(x)$.

Proof. By Proposition 2.1 there is a submultiplicative weight $v_0 \in \mathcal{P}_{E, s}(R^d) \cap C^\infty(R^d)$ such that
\begin{equation}
v_0(x) \asymp e^{\frac{s}{2}|x|^\frac{1}{2}}
\end{equation}
and
\begin{equation}
|\partial^\alpha v_0(x)| \lesssim h|\alpha|! s v_0(x), \quad \alpha \in \mathbb{N}^d
\end{equation}
for some $h > 0$. Since $s \geq 1$, a straight-forward application of Faà di Bruno’s formula on (2.6) gives
\begin{equation}
\left| \partial^\alpha \left( \frac{1}{v_0(x)} \right) \right| \lesssim h|\alpha|! s \frac{1}{v_0(x)}, \quad \alpha \in \mathbb{N}^d
\end{equation}
for some $h > 0$. It follows from (2.5) and (2.6) that if $\psi = 1/v$, then $\psi \in \mathcal{S}_r(R^d)$. Furthermore, if $f_0(x, y) = f(x, y)v_0(x)$, then an application of Leibnitz formula we get
\begin{equation}
|\partial^\alpha \partial^\alpha f_0(x, y)| \lesssim \sum_{\gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right) |\partial^\beta \partial^\gamma f(x, y)||\partial^\alpha - \gamma v_0(x)|
\end{equation}
\begin{equation}
\lesssim h|\alpha|! s \sum_{\gamma \leq \alpha} \left( \begin{array}{c} \alpha \\ \gamma \end{array} \right) (\gamma! \alpha_0)! s e^{-r|x|^\frac{1}{2}} \omega(y)(\alpha - \gamma)! s v_0(x)
\end{equation}
\begin{equation}
\lesssim (2h)^{2|\alpha|! s} \omega(y)(\alpha! \alpha_0)! s e^{-r|x|^\frac{1}{2}} v_0(x)\omega(y) \asymp (2h)^{|\alpha|! s} e^{-\frac{s}{2}|x|^\frac{1}{2}} v_0(x)\omega(y)
\end{equation}
for some $h > 0$, which gives the desired estimate on $f_0$. The result now follows since it is evident that $f(x, y) = f_0(x, y)\psi(x)$.

\begin{flushright}$\square$
\end{flushright}

Lemma 2.7. Let $s \geq 1$, $\omega \in \mathcal{P}_{E, s}(R^{2d})$, $\vartheta \in \mathcal{P}_{E, s}(R^d)$ and $\nu \in \mathcal{P}_{E, s}(R^d)$ be such that $\nu$ is submultiplicative, $\omega \in \Gamma_{0, s}(R^{2d})$ is $v \otimes \nu$-moderate, $\varphi = \nu^{\frac{1}{2}}$ and $\vartheta \in \Gamma_{0, s}(R^d)$. Also let $a \in \Gamma_{s}(R^{2d})$, $f \in \mathcal{S}_s(R^d)$, $\phi \in \Sigma_s(R^d)$, $\phi_2 = \phi\nu$,
\begin{equation}
\Phi(x, \xi, z, \zeta) = \frac{a(x + z, \xi + \zeta)}{\omega(x, \xi)v(z)v(\zeta)}
\end{equation}
and
\begin{equation}
H(x, \xi, y) = \int\int \Phi(x, \xi, z, \zeta)\phi_2(z)v(\zeta)e^{i(y-x-z,\zeta)} dz d\zeta.
\end{equation}
Then
\[
V_{\phi}(\text{Op}(a)f)(x, \xi) = (2\pi)^{-d}(f, e^{i\langle \cdot, \xi \rangle}H(x, \xi, \cdot))\omega(x, \xi). \tag{2.8}
\]

Furthermore, the following is true:

1. \(H \in C^\infty(\mathbb{R}^{3d})\) and satisfies
   \[
   |\partial^\alpha H(x, \xi, y)| \lesssim h_0 |\alpha|! e^{-r_0 |x-y|^s}, \tag{2.9}
   \]
   for some \(h_0, r_0 > 0\);

2. there are functions \(H_0 \in C^\infty(\mathbb{R}^{3d})\) and \(\phi_0 \in \mathcal{S}_s(\mathbb{R}^d)\) such that
   \[
   H(x, \xi, y) = H_0(x, \xi, y)\phi_0(y - x), \tag{2.10}
   \]
   and such that (2.9) holds for some \(h_0, r_0 > 0\), with \(H_0\) in place of \(H\).

**Proof.** When proving the first part, we will mainly follow the proof of [31, Lemma 3.3]. By straight-forward computations we get
\[
V_{\phi}(\text{Op}(a)f)(x, \xi) = (a(\cdot, D)f, \phi(\cdot - x) e^{i\langle \cdot, \xi \rangle})
\]
\[
= (f, \text{Op}(a)^*(\phi(\cdot - x) e^{i\langle \cdot, \xi \rangle}))
\]
\[
= (2\pi)^{-d}(f, e^{i\langle \cdot, \xi \rangle}H_1(x, \xi, \cdot))\omega(x, \xi), \tag{2.11}
\]
where
\[
H_1(x, \xi, y) = (2\pi)^d e^{-i\langle y, \xi \rangle}(\text{Op}(a)^*(\phi(\cdot - x) e^{i\langle \cdot, \xi \rangle}))(y)/\omega(x, \xi)
\]
\[
= \int \int \frac{a(z, \zeta)}{\omega(x, \xi)} \phi(z - x) e^{i\langle y - z, \zeta - \xi \rangle} dz d\zeta
\]
\[
= \int \int \Phi(x, \xi, z - x, \zeta - \xi)\phi_2(z - x) e^{i\langle y - z, \zeta - \xi \rangle} dz d\zeta.
\]

If \(z - x\) and \(\zeta - \xi\) are taken as new variables of integrations, it follows that the right-hand side is equal to \(H(x, y, \xi)\). This gives the first part of the lemma.

In order to prove (1), let
\[
\Phi_0(x, \xi, z, \zeta) = \Phi(x, \xi, z, \zeta)\phi(z),
\]
and let \(\Psi = \mathcal{F}_\zeta \Phi_0\), where \(\mathcal{F}_\zeta \Phi\) is the partial Fourier transform of \(\Phi_0(x, \xi, z, \zeta)\) with respect to the \(z\) variable. Then it follows from the assumptions that
\[
|\partial^\alpha \Phi_0(x, \xi, z, \zeta)| \lesssim h_0 |\alpha|! e^{-r_0 |z|^s},
\]
for some \(h_0, r_0 > 0\), which shows that \(z \mapsto \Phi_0(x, \xi, z, \zeta)\) is an element in \(\mathcal{S}_s(\mathbb{R}^d)\) with values in \(\Gamma_s^{(1)}(\mathbb{R}^{3d})\). As a consequence, \(\Psi\) satisfies
\[
|\partial^\alpha \Psi(x, \xi, \eta, \zeta)| \lesssim h_0 |\alpha|! e^{-r_0 |\eta|^s},
\]
for some \(h_0, r_0 > 0\).
for some $h_0, r_0 > 0$. Hence
\[ |\partial^\alpha (\Psi(x, \xi, \zeta, \zeta) v(\zeta))| \lesssim h_0^{[\alpha]} \alpha! e^{-r_0|\zeta|^\frac{1}{2}} \]
for some $h_0, r_0 > 0$.

By letting $H_2(x, \xi, \cdot, \cdot)$ be the partial Fourier transform of $\Psi(x, \xi, \zeta) v(\zeta)$ with respect to the $\zeta$ variable, it follows that
\[ |\partial^\alpha H_2(x, \xi, y)| \lesssim h_0^{[\alpha]} \alpha! e^{-r_0|y|^\frac{1}{2}} \]
for some $h_0, r_0 > 0$. The assertion (1) now follows from the latter estimate and the fact that $H(x, \xi, y) = H_2(x, \xi, x - y)$.

In order to prove (2) we notice that (2.12) shows that $y \mapsto H_2(x, \xi, y)$ is an element in $S_h(\mathbb{R}^d)$ with values in $\Gamma_s^2(\mathbb{R}^{2d})$. By Lemma 2.6 there are $H_3 \in C^\infty(\mathbb{R}^{2d})$ and $\phi_0 \in S_0(\mathbb{R}^d)$ such that (2.12) holds for some $h_0, r_0 > 0$ with $H_3$ in place of $H_2$, and
\[ H_2(x, \xi, y) = H_3(x, \xi, y) \phi_0(-y). \]
This is the same as (2), and the result follows.

Proof of Theorem 2.8. Let $g = \text{Op}(a)f$. By Lemma 2.4 we have
\[ V_0 g(x, \xi) = (2\pi)^{-d} \mathcal{F}((f \cdot \overline{\phi_0}(\cdot - x)) \cdot H_0(x, \xi, \cdot))(\xi) \omega(x, \xi) \]
\[ = (2\pi)^{-d} \mathcal{F}((f \cdot \phi_0(\cdot - x)) \ast (\mathcal{F}(H_0(x, \xi, \cdot)))(\xi)) \omega(x, \xi) \]
\[ = (2\pi)^{-d} (V_0 f)(x, \cdot) \ast (\mathcal{F}(H_0(x, \xi, \cdot)))(\xi) \omega(x, \xi) \]

Since $\omega$ and $\omega_0$ belongs to $\mathcal{D}_{E, \alpha}^0(\mathbb{R}^d)$, (2) in Lemma 2.7 gives
\[ |V_0 g(x, \xi) \omega_0(x, \xi)| \lesssim |V_0 f(x, \cdot) \omega(x, \cdot) \omega_0(x, \cdot)| * e^{-r_0|\cdot|^{\frac{1}{2}}} \]

By applying the $B$ norm we get for some $v \in \mathcal{D}_{E, \alpha}^0(\mathbb{R}^d)$,
\[ \|g\|_{M(\omega_0, B)} \lesssim \|V_0 f \cdot \omega_0 \ast e^{-r_0|\cdot|^{\frac{1}{2}}} \otimes \delta_0\|_B \]
\[ \leq \|V_0 f \cdot \omega_0\|_B \|e^{-r_0|\cdot|^{\frac{1}{2}}} v\|_L \times \|f\|_{M(\omega_0, B)}. \]

This gives the result.

By similar arguments as in the proof of Theorem 2.5 and Lemma 2.7 we get the following. The details are left for the reader.

Theorem 2.8. Let $A \in M(d, \mathbb{R})$, $s \geq 1$, $\omega, \omega_0 \in \mathcal{D}_{E, \alpha}^0(\mathbb{R}^{2d})$, $a \in \Gamma_{0, s}^0(\mathbb{R}^{2d})$, and that $B$ is an invariant BF-space on $\mathbb{R}^{2d}$. Then $\text{Op}_A(a)$ is continuous from $M(\omega_0, B)$ to $M(\omega, B)$.

Lemma 2.9. Let $s \geq 1$, $\omega \in \mathcal{D}_{E, \alpha}^0(\mathbb{R}^{2d})$, $\partial \in \mathcal{D}_{E, \alpha}(\mathbb{R}^d)$ and $v \in \mathcal{D}_{E, \alpha}^0(\mathbb{R}^d)$ be such that $v$ is submultiplicative, $\omega \in \Gamma_{0, s}^0(\mathbb{R}^{2d})$ is $v \otimes v$-moderate, $\vartheta = v^{-\frac{1}{2}}$ and $\overline{\vartheta} \in \Gamma_{0, s}^0(\mathbb{R}^d)$. Also let $a \in \Gamma_{0, s}^0(\mathbb{R}^{2d})$, $f, \phi \in$
\[ \Sigma_s(\mathbb{R}^d), \phi_2 = \phi v, \text{ and let } \Phi \text{ and } H \text{ be as in Lemma 2.7. Then (2.8) and the following hold true:} \]

1. \( H \in C^\infty(\mathbb{R}^{3d}) \) and satisfies (2.9) for every \( h_0, r_0 > 0; \)
2. there are functions \( H_0 \in C^\infty(\mathbb{R}^{3d}) \) and \( \phi_0 \in \Sigma_s(\mathbb{R}^d) \) such that (2.10) holds, and such that (2.9) holds for every \( h_0, r_0 > 0, \) with \( H_0 \) in place of \( H. \)

We finish the section by discussing continuity for pseudo-differential operators with symbols in \( \Gamma_{s}^{(\omega_0)} \) or in \( \Gamma_{0,s}^{(\omega_0)} \) when acting on quasi-Banach modulation spaces. More precisely, by straight-forward computations it follows that if \( \omega, \omega_0 \in \mathcal{P}_{E,s}(\mathbb{R}^{2d}) \) (\( \omega, \omega_0 \in \mathcal{P}_{E,s}^{0}(\mathbb{R}^{2d}) \)), then

\[
\frac{\omega(x, \xi)}{\omega(y, \eta) \omega_0(y, \eta)} \lesssim \frac{e^{r(|\xi-\eta|^d + |y-x|^d)}}{\omega_0(x, \eta)}.
\]

holds for some \( r > 0 \) (for every \( r > 0 \)). Hence the following result is a straight-forward consequence of Propositions 1.11 and 2.3 and Lemma 2.4 (Cf. Definition 1.10 for the definition of involved parallelepipeds.)

**Theorem 2.10.** Let \( A \in \mathcal{M}(d, \mathbb{R}), s \geq 1, \omega, \omega_0 \in \mathcal{P}_{E,s}^{0}(\mathbb{R}^{2d}), p \in (0, \infty)^{2d}, \)

\( E \) be a phase-shift split parallelepiped in \( \mathbb{R}^{2d}, \) and let \( a \in \Gamma_{s}^{(\omega_0)}(\mathbb{R}^{2d}). \)

Then \( \text{Op}_{A}(a) \) is continuous from \( M(L^{p,E}(\mathbb{R}^{2d}), \omega_0) \to M(L^{p,E}(\mathbb{R}^{2d}), \omega). \)

The same holds true with \( \mathcal{P}_{E,s} \) and \( \Gamma_{s}^{(\omega_0)} \), or with \( \mathcal{P} \) and \( S^{(\omega_0)} \) in place of \( \mathcal{P}_{E,s}^{0} \) and \( \Gamma_{s}^{(\omega_0)} \), respectively, at each occurence.

**Corollary 2.11.** Let \( A \in \mathcal{M}(d, \mathbb{R}), s \geq 1 \) and \( \omega, \omega_0 \in \mathcal{P}_{E,s}^{0}(\mathbb{R}^{2d}), p, q \in (0, \infty), \) and let \( a \in \Gamma_{s}^{(\omega_0)}(\mathbb{R}^{2d}). \)

Then \( \text{Op}_{A}(a) \) is continuous from \( M^{p,q}_{\omega_0}(\mathbb{R}^{d}) \to M^{p,q}_{\omega}(\mathbb{R}^{d}). \)

The same holds true with \( \mathcal{P}_{E,s} \) and \( \Gamma_{s}^{(\omega_0)} \), or with \( \mathcal{P} \) and \( S^{(\omega_0)} \) in place of \( \mathcal{P}_{E,s}^{0} \) and \( \Gamma_{s}^{(\omega_0)} \), respectively, at each occurence.

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