Clifford Algebra with *Mathematica*

G. Aragón-Camarasa\textsuperscript{a}, G. Aragón-González\textsuperscript{b}, J.L. Aragón\textsuperscript{\ast,c,d}, M.A. Rodríguez-Andrade\textsuperscript{e}

\textsuperscript{a}Department of Computing Science, Room: G151, Sir Alwyn Williams Building, Lilybank Gardens, University of Glasgow, Glasgow, G12 8QQ, Scotland, UK
\textsuperscript{b}PDPA, Universidad Autónoma Metropolitana, Azcapotzalco, San Pablo 180, Colonia Reynosa-Tamaulipas, 02200 D.F. México
\textsuperscript{c}Centre for Mathematical Biology, Mathematical Institute, University of Oxford, 24-29 St. Giles', Oxford OX1 3LB, UK.
\textsuperscript{d}Centro de Física Aplicada y Tecnología Avanzada, Universidad Nacional Autónoma de México, Apartado Postal 1-1010, Querétaro 76000, México
\textsuperscript{e}Departamento de Matemáticas, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional. Unidad Profesional Adolfo López Mateos, Edificio 9. 07300 D.F. México

**Abstract**

The Clifford algebra of a \(n\)-dimensional Euclidean vector space provides a general language comprising vectors, complex numbers, quaternions, Grassman algebra, Pauli and Dirac matrices. In this work, a package for Clifford algebra calculations for the computer algebra program *Mathematica*\textsuperscript{1} is introduced through a presentation of the main ideas of Clifford algebras and illustrative examples. This package can be a useful computational tool since allows the manipulation of all these mathematical objects. It also includes the possibility of visualize elements of a Clifford algebra in the 3-dimensional space.

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\textsuperscript{\ast}Corresponding author

\textit{Email addresses: gerardo@dcs.gla.ac.uk} (G. Aragón-Camarasa),
\textit{gag@correo.azc.uam.mx} (G. Aragón-González), \textit{aragon@fata.unam.mx} (J.L. Aragón),
\textit{marco@polaris.esfm.ipn.mx} (M.A. Rodríguez-Andrade)

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1. Introduction

The importance of Clifford algebra was recognized for the first time in quantum field theory but there has recently been a tendency to exploit their power in many others fields. These fields include projective geometry \[13\], electrodynamics \[14\], analysis on manifolds and differential geometry \[12\], crystallography \[2, 3\] and others. A recent account on applications of Clifford algebra in fields such as robotics, computer vision, computer graphics, engineering, neural and quantum computing, etc., can be found in \[3\] and \[20\].

Libraries, packages and specialized programs for doing Clifford algebra already exist; \textit{CLIFFORD/Bigebra} is a \textit{Maple} package which includes additional specialized packages such as \textit{SchurFkt} (for the Hopf Algebra of Symmetric Functions) and \textit{GfG - Groebner for Grassmann} (for Computing Groebner Bases for Ideals in Grassmann Algebra) \[1\]; \textit{TCliffordAlgebra} is an add-on application for the \textit{Tensorial Mathematica} package that implements some Clifford algebra operations \[8\]. \textit{Clifford} is a \textit{Mathematica} package for doing Clifford algebra, oriented to differential geometry \[18\] and \textit{CLICAL} is a stand-alone calculator-type computer program for \textit{MS-DOS} \[17\].

While the first two packages requires more specialized knowledge on Clifford algebras, \textit{CLICAL} is easy to use and can be used by non mathematicians. It has the advantage of being a stand-alone program (does not require a proprietary program to run) but does not allow symbolic computations and, as far as we know, is currently not under maintaining. The \textit{Mathematica} package \textit{Clifford} described in \[18\] (which is not easily available) performs symbolic calculations and multivectors notation is quite similar to the used in our package. The main difference lies in the implementation; while the package by Parra and Roselló \[18\] is based on matrix representations of the Clifford algebra, \textit{clifford.m} is based on isomorphisms between blades and \(n\)-tuples. As we shall see, this approach enables a more efficient and elegant computational scheme, providing also further flexibility concerning the dependence on the dimension of the space under work. In most of the package functions, for instance, the dimension of the space where a multivector is embedded is calculated on the fly, so it is not necessary to specify since the very beginning the dimension of the space under work.

The \textit{Mathematica} package herein described has the main purpose to implement general operations of a Clifford algebra on the language of the computer algebra program \textit{Mathematica}. In order to introduce the package and
its main algorithms, we first present a gentle introduction to the Clifford algebra of $\mathbb{R}^n$, with examples thought to show the generality of this algebra in the sense of providing a general language comprising vectors, complex numbers, quaternions, Pauli and Dirac matrices. These examples are worked out with clifford.m with the aim of describing the use of the main predefined functions. For more general introductions to Clifford algebras, readers are referred to books, such as Refs. [19] and [9]).

We must point out that an early version of clifford.m, intended for applications in crystallography, was given in Ref. [7]. The package presented here, however, is more general in purpose; the code has been rewritten and it has been enriched with functions to draw multivectors in $\mathbb{R}^3$. The program has been tested in several ways. For instance, calculations and graphical examples in the work by Zhang et al. [25] were repeated and the resulting notebook is available for download (see below). The package has been also used in works by our group [2, 3, 4] and also in Ref. [24]. The package clifford.m, a user guide (UserGuide.nb), a Palette with the most common predefined functions (Clifford.nb) as well as the notebook with the calculations by Zhang et al. [25] (ZhangEtAl.nb) are available for download at:

\protect\vrule width0pt\protect\href{http://www.fata.unam.mx/aragon/software/}{http://www.fata.unam.mx/aragon/software/}

2. The Clifford algebra of $\mathbb{R}^n$

Let us consider the vector space $\mathbb{R}^n$ with the inner product $\langle \cdot , \cdot \rangle$ and an orthonormal basis $\{e_1, e_2, \ldots , e_n\}$. One of the most simple and direct ways to define a Clifford Algebra is by making use of their generators [6].

Definition 2.1. The real associative and distributive algebra generated by the Euclidean space $\mathbb{R}^n$ with the product rules

\begin{align}
\quad e_i^2 &= 1, \quad i = 1, 2, \ldots , n, \\
\quad e_i e_j + e_j e_i &= 0, \quad i \neq j,
\end{align}

where $\{e_1, e_2, \ldots , e_n\}$ is the canonical basis of $\mathbb{R}^n$, is called universal Clifford algebra of the space $\mathbb{R}^n$ and is denoted by $\mathbb{R}_n$.

With this definition, we can easily establishes the relationship between the canonical scalar product in $\mathbb{R}^n$ and the so-called geometric product of the algebra $\mathbb{R}_n$. 

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Proposition 2.2. If $x, y \in \mathbb{R}^n$ then $x^2 \geq 0$ and $xy + yx \in \mathbb{R}$. Even more

$$x^2 = \langle x, x \rangle$$
$$\frac{xy + yx}{2} = \langle x, y \rangle$$

where $\langle \cdot, \cdot \rangle$ is the canonical inner product in $\mathbb{R}^n$.

**Proof.** It is enough to write $x$ and $y$ in terms of the canonical basis $\{e_1, e_2, \ldots, e_n\}$ and use Equation (1).

In a Clifford algebra the inverse of a nonzero vector can be defined [6]. In particular, the inverse of any vector $x \in \mathbb{R}^n$, $x \neq 0$, is given by

$$x^{-1} = \frac{x}{x^2} = \frac{x}{\|x\|^2}.$$  \hfill (2)

The Clifford algebra $\mathbb{R}_n$ is itself a vector space of dimension $\sum_{p=0}^{n} \binom{n}{p} = 2^n$, with basis

$$\{1, e_1, e_2, \ldots, e_n, e_1e_2, e_1e_3, \ldots, e_1e_n, \ldots, e_1e_2\cdots e_n\},$$

such that an element $A$ in $\mathbb{R}_n$ is written as

$$A = a_0 + a_{11}e_1 + \cdots + a_{1i}e_i + a_{21}e_1e_2 + \cdots + a_{2i}e_1e_n + \cdots + a_{di}e_1e_2\cdots e_n,$$  \hfill (3)

where $d = 2^n - 1$ and $i = \binom{n}{p}$ for the real numbers $a_{pi}$. Consequently, the vector space $\mathbb{R}_n$ can be decomposed, as direct sum, in $n+1$ subspaces:

$$\mathbb{R}_n = \Lambda^0\mathbb{R}^n \oplus \Lambda^1\mathbb{R}^n \oplus \cdots \oplus \Lambda^n\mathbb{R}^n.$$  \hfill (4)

where $\Lambda$ corresponding to the exterior product [6], and each subspace is of dimension $\binom{n}{p}$.

The elements $A$ (Eqn. 3) of the Clifford algebra $\mathbb{R}_n$ are called multivectors, and those of $\Lambda^p\mathbb{R}^n$, $p$-vectors. In particular, 0-vectors are real numbers and dim($\Lambda^0\mathbb{R}^n$) = 1. $\Lambda^1\mathbb{R}^n$ has the basis $\{e_1, e_2, \ldots, e_n\}$, so 1-vectors are simply vectors and dim($\Lambda^1\mathbb{R}^n$) = $n$. $\Lambda^2\mathbb{R}^n$ has the basis $\{e_1e_2, e_1e_3, \ldots, e_1e_n\}$ and their elements (2-vectors) are also called bivectors. Finally, $\Lambda^n\mathbb{R}^n$ has as basis $\{e_1e_2\cdots e_n\}$ and since dim($\Lambda^n\mathbb{R}^n$) = 1, the $n$-vectors of $\mathbb{R}_n$ are also referred as pseudoscalars.
In what follows, arbitrary multivectors will be denoted by non bold upper-case characters without ornamentation such as \( A \). \( p \)-vectors will be denoted by \( A_p \), with the exception of vectors (1-vectors), that will be denoted by bold lowercase characters such as \( \mathbf{a} \).

Bearing in mind the decomposition of the vector space \( \mathbb{R}^n \) as the direct sum of the subspaces \( \Lambda^p \mathbb{R}^n \), \( 0 \leq p \leq n \), given in (4), any multivector \( A \) can be written as
\[
A = \langle A \rangle_0 + \langle A \rangle_1 + \cdots + \langle A \rangle_n, \tag{5}
\]
where \( \langle A \rangle_p \), the \( p \)-vector part of \( A \), is the projection of \( A \in \mathbb{R}^n \) into \( \Lambda^p \mathbb{R}^n \). \( \langle \cdot \rangle \) is called the grade operator.

2.1. Inner and outer products

Given the decomposition (4), an important property of a Clifford algebra is the existence of products that allows us to move from one subspace of \( \mathbb{R}^n \) to another. Let us first consider the product of two 1-vectors. For all \( u, v \in \Lambda^1 \mathbb{R}^n \), their product \( uv \) can be written as
\[
uv = \frac{1}{2} (uv + vu) + \frac{1}{2} (uv - vu).
\]
Now define the “inner” and “outer” products as follows
\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= \frac{1}{2} (uv + vu) = \langle \mathbf{a}, \mathbf{b} \rangle, \\
\mathbf{u} \wedge \mathbf{v} &= \frac{1}{2} (uv - vu).
\end{align*}
\]
The inner product is symmetric and notice that vectors \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal if and only if \( uv = -vu \). The outer product \( \mathbf{u} \wedge \mathbf{v} \) is antisymmetric (and associative) and vanishes whenever the two vectors are collinear, that is, \( \mathbf{u} \) and \( \mathbf{v} \) are collinear (or linearly dependent) if and only if \( uv = vu \). Thus, the product \( uv \) provides information about the relative directions of the vectors. Anticommutativity means orthogonality and commutativity means collinearity. Notice that for the basis vectors of \( \mathbb{R}^n \), we have
\[
\begin{align*}
e_i e_j &= e_i \wedge e_j, \quad i \neq j, \\
e_i \cdot e_i &= e_i^2 = 1.
\end{align*}
\]
From the following equality
\[
\begin{align*}
uv &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}, \\
&= \langle uv \rangle_0 + \langle uv \rangle_2,
\end{align*}
\]
we can extend the notions of inner and outer product to the case of $p$- and $q$-vectors in the following way. For a $p$-vector $A_p \in \Lambda^p \mathbb{R}^n$ and a $q$-vector $B_q \in \Lambda^q \mathbb{R}^n$, the inner product $A_p \cdot B_q$ is defined by

$$A_p \cdot B_q = \begin{cases} \langle A_p B_q \rangle_{|p-q|} & \text{if } p, q > 0, \\ 0 & \text{if } p = 0 \text{ or } q = 0. \end{cases}$$

(6)

Analogously, the outer product $A_p \wedge B_q$ is defined by

$$A_p \wedge B_q = \langle A_p B_q \rangle_{p+q}.$$  

(7)

Since arbitrary multivectors can be decomposed as in (5), inner and outer product can be extended by linearity to $\mathbb{R}^n$. Then, given $A, B \in \mathbb{R}^n$, we have

$$A \cdot B = \sum_{k,l=1}^n \langle A \rangle_k \cdot \langle B \rangle_l,$$

(8)

$$A \wedge B = \sum_{k,l=1}^n \langle A \rangle_k \wedge \langle B \rangle_l,$$

(9)

2.2. Geometric interpretation

Bivectors have an interesting geometric interpretation. Just as a vector describes an oriented line segment, with the direction of the vector represented the oriented line and the magnitude of the vector is equal to the length of the segment; so a bivector $a \wedge b$ describes an oriented plane segment, with the direction of the bivector represented the oriented plane and the magnitude of the bivector measuring the area of the plane segment (Figure 1). The same interpretation is extended to high-order terms: $a \wedge b \wedge c$ represents an oriented volume. In general, multivectors contain information about orientation of subspaces.

2.3. The reverse of a multivector

The magnitude or modulus of a multivector $A$ is defined by the equation

$$|A| = \langle \widetilde{AA} \rangle_0^{1/2},$$

(10)

where $\sim$ denotes the operation reverse defined as

$$(e_1 e_2 \cdots e_p)^\sim = e_p \cdots e_2 e_1.$$  

The operation reverse is distributive [12] so that the reverse of an arbitrary multivector $A$ can be easily calculated.

If the inverse of a multivector $A$ exists, it is denoted by $A^{-1}$ or $1/A$, and is defined by the equation $AA^{-1} = 1$.  

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2.4. General metrics

In many applications one considers real vector spaces \( \mathbb{R}^n \) with metrics that are not positive definite with the bilinear form \( \langle \cdot, \cdot \rangle \), such that
\[
\langle x, x \rangle = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2,
\]
where \( n = p + q \). In this case, the vector space is denoted as \( \mathbb{R}^{p,q} \), giving rise to the Clifford algebra \( \mathbb{R}_{p,q} \) [6]:

**Definition 2.3.** The real associative and distributive algebra generated by the space \( \mathbb{R}^{p,q} \) \((p + q = n)\) with the product rules
\[
e_i^2 = 1, \quad i = 1, 2, \ldots, p,
\]
\[
e_i^2 = -1, \quad i = 1, 2, \ldots, q
\]
\[
e_i e_j + e_j e_i = 0, \quad i \neq j, i, j = 1, 2, \ldots, n
\]
where \( \{e_1, e_2, \ldots, e_n\} \) is the canonical basis of \( \mathbb{R}^{p,q} \), is called universal Clifford algebra of the space \( \mathbb{R}^{p,q} \) and is denoted by \( \mathbb{R}_{p,q} \).

As in Proposition 2.2, the relationship between the canonical scalar product in \( \mathbb{R}^{p,q} \) and the geometric product of the Clifford algebra \( \mathbb{R}_{p,q} \) can be established.

**Proposition 2.4.** If \( x, y \in \mathbb{R}^{p,q} \) then \( x^2 \) and \( xy + yx \in \mathbb{R} \). Even more
\[
x^2 = \langle x, x \rangle
\]
\[
\frac{xy + yx}{2} = \langle x, y \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) is the canonical inner product in \( \mathbb{R}^{p,q} \).

**Proof.** If \( x = \sum_{i=1}^{n} x_i e_i \) and \( y = \sum_{i=1}^{n} y_i e_i \) then, by (11) we have
\[
x^2 = \sum_{i=2}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2 \in \mathbb{R}
\]
\[
xy + yx = \sum_{i=2}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i \in \mathbb{R}.
\]

Now, an element \( a \in \mathbb{R}^{p,q} \) is said to be invertible if \( a^2 \neq 0 \). The element
\[
a^{-1} = \frac{a}{a^2}
\]
is called the inverse of \( a \).
3. Clifford algebra calculations with *Mathematica*

According to Eqns. (5), (6), (7), (8), (9) and (10), all that we should need in order to manipulate multivectors in a computer algebra program such as *Mathematica*, would be to define the two fundamentals operations (algorithms): geometric product and grade operator.

### 3.1. Fundamentals algorithms

The first fundamental algorithm for the computation of the geometric product between multivectors can be devised by noticing that a general multivector $A$ in $\mathbb{R}_{p,q}$, such as (3), is formed by a linear combination of terms of the form

$$e_{m_1} e_{m_2} \cdots e_{m_n},$$

where $m_i = 1, 0, (i = 1, \ldots, n)$, that we call blades. The geometric product of two of these blades (which is also a blade) is:

$$(e_{m_1} e_{m_2} \cdots e_{m_n}) (e_{r_1} e_{r_2} \cdots e_{r_n}) = (-1)^s e_{m_1 + r_1} e_{m_2 + r_2} \cdots e_{m_n + r_n},$$

where the sum $m_i + r_i$ is evaluated modulus two, and

$$s = \sum_{1 \leq i < j \leq n} r_i m_j.$$

If $m_i + r_i = 2$ then, in order to have the right hand side in the form (13) when considering the signature of the bilinear form, $e_{m_i + r_i}^i$ will be replaced with $\langle e_i, e_i \rangle e_i^0$, and in this case we have:

$$e_{m_1 + r_1} \cdots e_{m_i + r_i} \cdots e_{m_n + r_n} = \langle e_i, e_i \rangle e_{m_1 + r_1} \cdots e_i^0 \cdots e_{m_n + r_n}.$$

Equation (13) enables us to establish an isomorphism between blades and n-tuples

$$e_{m_1} e_{m_2} \cdots e_{m_n} \longleftrightarrow (m_1, m_2, \ldots, m_n),$$

that can be more easily manipulated from a computational point of view, represented by lists of 0s and 1s.

As an example, let $A_2 = 7e_1 e_2$ and $B_2 = e_3 e_4$ be two blades in $\mathbb{R}_4$. They can be represented as $e_1^1 e_2^0 e_3^0 e_4^0$, $(1, 1, 0, 0)$, and $e_1^0 e_2^1 e_3^1 e_4^1$, $(0, 0, 1, 1)$, respectively. The geometric product $A_2 B_2$ is given by:

$$A_2 B_2 = (-1)^2 7e_1^{1+1} e_2^{0+1} e_3^{1+0} e_4^{0+1} = 7e_2 e_3 e_4.$$
or (0, 1, 1, 1).

Once we are able to calculate the geometric product between two arbitrary blades $A_p$ and $B_q$ in $\mathbb{R}_n$, the product between two multivectors $A$ and $B$ in $\mathbb{R}_n$ can be evaluated by using the decomposition (6). Indeed, if

$$A = \sum_{i=0}^{n} \langle A \rangle_i = \sum_{i=0}^{n} \alpha_i \left( \prod_{k=1}^{n} e_{ik}^{m_{ik}} \right),$$

$$B = \sum_{j=0}^{n} \langle A \rangle_j = \sum_{j=0}^{n} \beta_j \left( \prod_{k=1}^{n} e_{jk}^{r_{jk}} \right),$$

the geometric product $AB$ is given by

$$AB = \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{s} \alpha_i \beta_j \left( \prod_{k=1}^{n} e_{ik}^{m_{ik}+r_{jk}} \right),$$

where $s = \sum_{1 \leq p < q \leq n} r_{jp} m_{iq}$ and $m_{iq} + r_{jp} \equiv 0 \mod 2$.

Finally, according to (8), (9), and (10), to calculate the inner and outer product between general multivectors and the magnitude of a multivector, the grade operator must be implemented (also the reverse operation but it turns to be trivial). Given a multivector $A \in \mathbb{R}_{p,q}$, to calculate, for instance $\langle A \rangle_r$, we proceed as follows

1. Decompose $A$ into a sum of blades as in (5).
2. According (13), the grade of the $i$-th blade of the above decomposition is simply $m_{i1} + m_{i2} + \cdots + m_{in}$.
3. Keep the terms of the decomposition with grade equal to $r$. If no such terms exist then $\langle A \rangle_r = 0$.

### 3.2. Implementation

We denote, in Mathematica code, the $j$-th basis vector $e_j$ as $e[j]$, a blade such as $e_1 e_3 e_4$, ( $e_1^2 e_3^2 e_4$ using the nomenclature of Eqn. 13) is written as

$$e[1] e[3] e[4],$$

and can be internally represented simply by (1, 0, 1, 1). Care must be taken in preserving the canonical order of the expression, since for instance $e[1] e[3]$ is a geometric product of two vectors and $e[1] e[3] \neq e[3] e[1]$. 

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Let us consider the Clifford algebra $\mathbb{R}_{p,q}$. If $\dim = p + q$ is the dimension of the vector space, the following Mathematica code implements the transformation of a blade, such as (15), onto a $n$-tuple:

```mathematica
ntuple[x_, dim_] := ReplacePart[Table[0, {dim}], 1, List @@ x /. e[k_] -> {k}]
```

The signature of the bilinear form can be set by using $\text{SetSignature} = p$ and Equations (11) can be coded as:

```mathematica
bilinearform[e[i_], e[i_]] := If[i <= $\text{SetSignature}$, 1, -1]
```

Except for some functions, it is not necessary to define the dimension of the vector space since it can be calculated directly. The maximum dimension of the space where a blade is embedded can be extracted from the list:

```mathematica
dimensions[x_] := List @@ x /. e[k_?Positive] -> k
```

To calculate the dimension of the space where a general multivector is embedded, the previous function must be generalized by including the distributivity of addition:

```mathematica
dimensions[x_Plus] := List @@ Distribute[f[x]] /. f -> dimensions
dimensions[a_] := {0} /; FreeQ[a, e[?Positive]]
dimensions[a_ x_] := dimensions[x] /; FreeQ[a, e[?Positive]]
```

### 3.2.1. The geometric product

From (14), the geometric product between two blades with a bilinear form of signature $p$ can therefore be evaluated with help of the function `geoprod`:

```mathematica
geoprod[x_, y_] := Module[{q=1, s, r={},
p1=ntuple[x, Max[dimensions[x], dimensions[y]]],
p2=ntuple[y, Max[dimensions[x], dimensions[y]]],
s=Sum[p2[[m]]*p1[[n]], {m,Length[p1]-1}, {n,m+1,Length[p2]}];
  r=Mod[p1+p2,2];
  Do[If[r[[i]] == 2, q*=bilinearform[e[i],e[i]]];
    If[r[[i]]==1, q*=e[i], {i, Length[r]}]
  (-1)^s*q ]
```

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This function evaluates the geometric product of two blades in $\mathbb{R}_{p,q}$. One step further consists in to calculate the geometric product of two arbitrary multivectors such as (3). This can be achieved from geoprod by providing the transformation rules which contains the properties of the geometric product under multiplication of blades by real numbers and addition of blades. Here is the behavior under scalar multiplication:

geoprod[a_, y_] := a y /; FreeQ[a, e[_?Positive]]
geoprod[x_, a_] := a x /; FreeQ[a, e[_?Positive]]
geoprod[a_ x_, y_] := a geoprod[x, y] /; FreeQ[a, e[_?Positive]]
geoprod[x_, a_ y_] := a geoprod[x, y] /; FreeQ[a, e[_?Positive]]

and the distributivity of addition:

geoprod[x_, y_Plus] := Distribute[f[x, y, p], Plus] /. f -> geoprod
geoprod[x_Plus, y_] := Distribute[f[x, y, p], Plus] /. f -> geoprod

Therefore, the function to calculate the geometric product of arbitrary multivectors is defined as

GeometricProduct[___] := $Failed
GeometricProduct[m1_, m2_, m3___] := f[ GeometricProduct[m1, m2], m3] /. f -> GeometricProduct
GeometricProduct[m1_, m2_] := geoprod[Expand[m1], Expand[m2]]

3.2.2. The grade operator

To complete the basic operations of a Clifford algebra, it remains to implement the grade operator. The following auxiliary function calculates the grade of a blade in $\mathbb{R}_{p,q}$

gradblade[a_] := 0 /; FreeQ[a, e[_?Positive]]
gradblade[x_] := Plus @@ ntuple[x, Max[dimensions[x]]]
gradblade[a_ x_] := gradblade[x] /; FreeQ[a, e[_?Positive]]

Now, Grade[x, n] should extract the term of grade n from the multivector X. Firstly, we consider the case when the multivector X is a blade of grade, say r: $\langle X \rangle_n = 0$ if $r \neq n$ and $\langle X \rangle_n = X$ if $r = n$. The code reads:

Grade[x_, n_?NumberQ] := If[gradblade[x] == n, x, 0]]

For a general multivector X, we have:
Functions GeometricProduct and Grade enable us to construct all the operations which can be defined in a Clifford algebra, such as outer product (OuterProduct[v,w,...]), inner product (InnerProduct[v,w,...]), magnitude (Magnitude[v]), reverse (Turn[v]), inverse (MultivectorInverse[v]), dual (Dual[v,dim]), and many others, all included in the package Clifford.m. This package works with general multivectors of the form (3), but particular cases can help to envisage the power of multivector calculus, as is made explicit in what follows.

4. Vectors

Let us consider the Clifford algebra \( \mathbb{R}_{p,0} \). \( n \)-dimensional vectors are 1-vectors and lie in the subspace \( \Lambda^1 \mathbb{R}^{p,0} \). A vector \( \mathbf{a} \) is therefore \( \mathbf{a} = \langle A \rangle_1 \), where \( A \) is a general multivector.

The inner product defined in (6) becomes now the standard “dot product” between vectors. The “cross product” \( \mathbf{a} \times \mathbf{b} \) of vector calculus is defined in \( \mathbb{R}^3 \) and is related to the outer product of vectors \( \mathbf{a} \) and \( \mathbf{b} \) as follows. The vector \( \mathbf{a} \times \mathbf{b} \) is perpendicular to \( \mathbf{a} \wedge \mathbf{b} \) and with the same magnitude: \(|\mathbf{a} \wedge \mathbf{b}| = |\mathbf{a} \times \mathbf{b}|\). The explicit algebraic relation between them is (11):

\[
\mathbf{a} \times \mathbf{b} = (-e_1e_2e_3)(\mathbf{a} \wedge \mathbf{b}),
\]

where \( e_1e_2e_3 \) is the pseudoscalar of \( \mathbb{R}_{3,0} \). We may actually take this as a definition of the cross product.

From (16) it is easy to define the function crossprod[v,w] that gives the cross product between two three-dimensional vectors \( \mathbf{v} \) and \( \mathbf{w} \):

\[
crossprod[v_,w_]:= \text{GeometricProduct}[-e[1]e[2]e[3],\text{OuterProduct}[v,w]]
\]

The associativity of the geometric product allows algebraic manipulations typical of real numbers that are no possible in the Gibbs’ vector algebra since the cross and dot products are not generally associative. For example

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.
\]

Even more, many products such as \( \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) \) are not even defined. With Clifford algebra all products are not only well defined but associative making simpler many algebraic manipulations and allowing to define derivatives.
and integrals just as they are defined for real functions of real variables, provided that we are careful to maintain the order of the factors since geometric product is not commutative.

One specific example that shows the simplicity of some expressions if geometric product is used, is concerning rotations. Consider a vector \( \mathbf{v} \) in \( \mathbb{R}^n \) which is rotated by an angle \( \theta \) in the oriented plane characterized by the bivector \( \mathbf{a} \wedge \mathbf{b} \). After the rotation, the vector \( \mathbf{v} \) is transformed into \( \mathbf{v}' \), given by [11]:

\[
\mathbf{v} = \tilde{U} \mathbf{v} U,
\]

(17)

where

\[
U = \cos(\theta/2) + \frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|} \sin(\theta/2).
\]

The direction of the rotation (clockwise or counterclockwise) is specified by the orientation of the bivector \( \mathbf{a} \wedge \mathbf{b} \). Eqn. 17 gives the rotated vector \( \mathbf{v}' \) with no matter the dimension of the space in which it is embedded. No corresponding simple expression exists in vector algebra.

As a quite simple example, which can be easily visualized, consider the vector \( \mathbf{v} = (1,1,1) \), to be rotated 90° maintaining invariant the plane \( xy \). To characterize the plane \( xy \), we can use \( \mathbf{a} \wedge \mathbf{b} \) where \( \mathbf{a} = (1,0,0) \) and \( \mathbf{b} = (0,1,0) \), in which case we get a rotation counterclockwise and is easy to see that \( \mathbf{v}' = (-1,1,1) \), but if we use \( \mathbf{b} \wedge \mathbf{a} \) then the rotation is clockwise and \( \mathbf{v}' = (1,-1,1) \). Here is this example solved with Clifford:

\[
\text{In}[1] := \text{<< Clifford.m}
\]

\[
\text{In}[2] := \text{v} = \text{e}[1]+\text{e}[2]+\text{e}[3];
\]

\[
\text{In}[3] := \text{plane} = \text{OuterProduct[}\text{e}[1],\text{e}[2]\text{]}/\text{Magnitude[OuterProduct[}\text{e}[1],\text{e}[2]\text{]}]
\]

\[
\text{Out}[3] = \text{e}[1]\text{e}[2]
\]

The operator \( U \) is now defined (\( \tilde{U} \) is the reverse of \( U \)):

\[
\text{In}[4] := \text{u} = \text{Cos[Pi/4]} + \text{plane Sin[Pi/4]};
\]

\[
\text{In}[5] := \text{vprime} = \text{GeometricProduct[}\text{Turn[}\text{u}\text{]}\text{,GeometricProduct[}\text{v},\text{u}\text{]}\text{]}
\]

\[
\text{Out}[5] = -\text{e}[1]\text{e}[2]+\text{e}[3]
\]

So we get a counterclockwise rotation where the vector \( \mathbf{v} = (1,1,1) \) becomes \( \mathbf{v}' = (-1,1,1) \). The implementation in \textit{Mathematica} of this rotation is the function \texttt{Rotation[v,a,b]}. 

5. Drawing multivectors in the 3-dimensional space.

The elements of the Clifford algebra \( \mathbb{R}_{3,0} \) can be visualized in the 3-dimensional space, providing geometrical insights. The package include a function to draw multivectors belonging to \( \mathbb{R}_{3,0} \), called \texttt{GADraw}, that will be described in what follows.

\texttt{GADraw} includes embedded functions to draw vectors, bivectors and the pseudoscalar of \( \mathbb{R}_{3,0} \). For example let us draw the multivector \( e_3 + e_1 e_2 + e_2 e_3 + e_1 e_2 e_3 \): 

\begin{verbatim}
In[1]:= << Clifford.m
In[2]:= A = e[3]+e[1]e[2]+e[2]e[3]+e[1]e[2]e[3];
In[3]:= GADraw[A];
Out[3]=
\end{verbatim}

Figure 1: A vector, a bivector (plane) and the pseudoscalar (cube) drawn with the aid of the function \texttt{GADraw}.

The result is shown in Fig. 1. The bivector is represented by an area and the pseudoscalar as a scalable cube. In the particular case of the vector \((0,0,1)\), the arrow in its tip was generated by the following code:

\begin{verbatim}
\end{verbatim}
mat[1] = Sin[t]*(e[1]/14) + Cos[t]*(e[2]/14),
mat[2] = Sin[t + 0.25]*(e[1]/14) + Cos[t + 0.25]*(e[2]/14),
mat[3] = e[3]/5,

This arrow is then translated and rotated (in this case to the tip of the vector (0, 0, 1)) with the aid of the function rotation[mat, w, p], where w = e_3 and p is the vector that points the site where the tip of the arrow must be located. If $sc = \sqrt{p[[1]]^2 + p[[2]]^2 + p[[3]]^2}/2$ is a scale factor then all this procedure can be encoded as:

```math
If[OuterProduct[ToBasis[p], e[3]] === 0,
    cone = Table[Array[ToVector[mat[#],3] &,#,3]+p-ToVector[mat[3],3],
     {t,0.25,2*Pi,0.25}],
    elms = Array[sc*ToVector[Grade[Rotation[mat[#],e[3],ToBasis[p]],1],3]&,3];
    cone = Table[elms+p-res[[3]], {t,0.25,2*Pi,0.25}] ];

arrow = Graphics3D[{FaceForm[color], EdgeForm[], Polygon /@ cone}]
```

Where ToBasis and ToVector are functions, defined on the same package Clifford; the former changes from the coordinates of a vector to the basis notation $[15]$ and the last one does the opposite.

It is worth to mention that the above coded was included with the aim to show that symmetry operations, rotations in this case, can be carried out by using the elements of the Clifford algebra without requiring matrices. This algebra provides a consistent computational framework with significant applications in computer graphics, vision and robotics $[10]$. 

6. Complex numbers

Let us consider the Clifford algebra of the most simple space that has a geometrical structure: the plane $\mathbb{R}^{2,0}$. Taking the canonical basis $\{e_1, e_2\}$, a basis for the Clifford algebra $\mathbb{R}_{2,0}$ is $\{1, e_1, e_2, e_1 e_2\}$ and a general multivector $A \in \mathbb{R}_{2,0}$ has the form

$$A = k_0 + k_1 e_1 + k_2 e_2 + k_3 e_1 e_2.$$ 

We can decompose $\mathbb{R}_{2,0}$ as $\mathbb{R}_{2,0} = \mathbb{R}_{2,0}^+ \oplus \mathbb{R}_{2,0}^-$, such that $\mathbb{R}_{2,0}^+$ contains even grade elements and $\mathbb{R}_{2,0}^-$ contains odd grade elements. Therefore, $A$ can be
expressed as the sum of two multivectors: \( A = A^+ + A^- \), where \( A^+ \in \mathbb{R}^+_{2,0} \) and \( A^- \in \mathbb{R}^-_{2,0} \). That is

\[
A = A^+ + A^-,
\]
\[
A^+ = k_0 + k_3 e_1 e_2,
\]
\[
A^- = k_1 e_1 + k_2 e_2.
\]

We focus our attention into \( \mathbb{R}^+_{2,0} \); it is itself an algebra so that it is called the even subalgebra of \( \mathbb{R}_{2,0} \). By taking

\[
i = e_1 e_2,
\]

an element of \( \mathbb{R}^+_{2,0} \) can be written as

\[
z = k_0 + k_3 i.
\]

Since \( i^2 = (e_1 e_2)(e_1 e_2) = -(e_1 e_2 e_2 e_1) = -1 \), and \( i \) is itself a generator of rotations \[11\], we see that \( \mathbb{R}^+_{2,0} \) is equivalent to the algebra of complex numbers.

The algebraic operations of complex numbers can therefore be worked out with \texttt{Clifford}. In order to get a more standard notation we firstly define the function

\[
\text{In}[6]:= \text{transform}[x] := x /. i \rightarrow e[1] e[2]
\]

which makes the identification \[18\]. Here we have two complex numbers:

\[
\text{In}[7]:= w = a + b i;
\]
\[
\text{In}[8]:= z = c + d i;
\]

The product \((a + bi)(c + di)\) becomes

\[
\text{In}[9]:= \text{GeometricProduct[transform[w],transform[z]] /. e[1] e[2] \rightarrow i}
\]
\[
\text{Out}[9]= a c - b c + b c i + a d i
\]

The equivalence between some built-in basic operations of complex numbers in \texttt{Mathematica} and those of a Clifford algebra defined the package is shown in the following table:

\[
16
\]
Built-in Objects  Clifford.m Objects

Re[z]          Grade[z,0]
Im[z]          GeometricProduct[Grade[z,2],-e[1]e[2]]
Conjugate[z]  Turn[z]
Abs[z]         Magnitude[z]

The inverse of the complex number \( w \) can be calculated with MultivectorInverse:

\[
\text{In[10]} := \text{MultivectorInverse[transform[w]] /. e[1]e[2] -> i}
\]

\[
\frac{a - bi}{2} = a + b
\]

7. Quaternions

Taking further the same procedure, we now develop the Clifford algebra of \( \mathbb{R}^{3,0} \), equipped with the canonical basis \{\( e_1, e_2, e_3 \)\}. The Clifford algebra \( \mathbb{R}^{3,0} \) is an eighth-dimensional vector space with the basis

\[
\{1, e_1, e_2, e_3, e_1 e_2, e_2 e_3, e_1 e_3, e_1 e_2 e_3\}.
\]

A general multivector \( A \in \mathbb{R}^{3,0} \) is written as

\[
A = k_0 + k_1 e_1 + k_2 e_2 + k_3 e_3 + k_4 e_1 e_2 + k_5 e_2 e_3 + k_6 e_1 e_3 + k_7 e_1 e_2 e_3,
\]

which can be also expressed as \( A = A^+ + A^- \), where

\[
A^+ = k_0 + k_4 e_1 e_2 + k_5 e_2 e_3 + k_6 e_1 e_3
\]

\[
A^- = k_1 e_1 + k_2 e_2 + k_3 e_3 + k_7 e_1 e_2 e_3.
\]

The even-grade elements \( A^+ \) form the subalgebra \( \mathbb{R}^{3,0}_+ \) of \( \mathbb{R}^{3,0} \), equivalent to the algebra of quaternions. This can be seen by making the identifications

\[
i = -e_2 e_3,
\]

\[
j = e_1 e_3
\]

\[
k = -e_1 e_2,
\]

(19)
leading to the famous equations

\[ i^2 = j^2 = k^2 = -1, \]  
\[ ijk = -1. \]  

(20)

With the identifications given in (19), an element of \( \mathbb{R}_{3,0}^+ \) can be written now as

\[ Q = q_0 + q_1 i + q_2 j + q_3 k, \]

which, in view of the properties (20), is a quaternion.

The algebra of quaternions is therefore comprised in the same package. The basic operations of this algebra are carried out by the function already defined, such as GeometricProduct, MultivectorInverse, Magnitude and Turn. To simplify the operations, we have incorporated the definitions (19) and redefined some functions to work only with quaternions and complex numbers. These new functions begin with the word Quaternion, namely, QuaternionProduct, QuaternionInverse, QuaternionMagnitude and QuaternionTurn. So, for instance the inverse of the quaternion \( q = a + 3i + 6j - 10k \) is

\[
\text{In[11]} := q = a + 3 \, i + 6 \, j - 10 \, k; \\
\text{In[12]} := \text{QuaternionInverse}[q] \\
\text{Out[12]} = \frac{a - 3 \, i - 6 \, j + 10 \, k}{2} \\
\]

18

8. Grassmann algebra

The outer product (OuterProduct) defined in (7) is associative and the identity \(1\), in a Clifford algebra \( \mathbb{R}_{n,0} \), is also the identity for the outer product. Consequently, the vector space \( \mathbb{R}^{p,0} \) with the outer product already defined is an algebra of dimension \( 2^n \), which is called the Grassmann algebra of \( \mathbb{R}^{p,0} \). Notice that it does not depend on the inner product of the vector space, but just on the alternation of the outer product.

Grassmann algebra has relevance in modern theoretical physics \( [16] \) and has the right structure for the theory of determinants. That the structure of this algebra is contained in Clifford algebra allows to reformulate Grassmann calculus \( [16] \), and to give an extensive treatment of determinants \( [12] \), both in terms of Clifford algebra.
9. The hyperbolic plane

Perhaps the simplest example of a problem involving non positive definite metrics is the Minkowsky model of the hyperbolic plane. Here we shall develop some basic ideas and calculations just to give a flavor of this kind of application of Clifford algebra and the use of the package Clifford for non positive definite metrics. In the next section, one extra dimension is introduced and the resulting algebras have great relevance in relativity theory and quantum mechanics.

One can always visualize a surface of constant positive Gaussian curvature \(1/R^2\) (see for instance \[21\] for concepts related to differential geometry) as a sphere, with radius \(R\), embedded in a three dimensional Euclidean space. The surface is described by the equation \(x_1^2 + x_2^2 + x_3^2 = R^2\). A surface of constant negative curvature, however, can not be embedded in a Euclidean space, so alternative possibilities must be developed to visualize such surfaces. The simplest surface of constant negative curvature is often called the hyperbolic plane, the Bolyai-Lobachevsky plane, or the pseudosphere. This surface can be globally embedded in a space equipped with the Minkowsky metrics instead of the Euclidean one. A three-dimensional Minkowsky space can be identified by the fact that if \((x_1, x_2, x_3)\) are the coordinates of a vector \(x\) in this space, then the distance to the origin is \(|x|^2 = x_1^2 + x_2^2 - x_3^2\).

The equation

\[ x_1^2 + x_2^2 - x_3^2 = -R^2, \tag{21} \]

defines a hyperboloid of two sheets intersecting the \(x_3\) axis at the points \(\pm 1\). Either sheet (upper or lower) models an infinite surface without a boundary (the Minkowsky metric becomes positive definite upon it) that, as we shall see, has constant Gaussian curvature \(-1/R^2\).

We can easily convince ourselves that \(\mathbb{R}_{2,1}\) is indeed the three-dimensional Minkowsky space (the signature of the bilinear form is 2). We shall proceed to calculate the Gaussian curvature of the hyperboloid \((21)\) with the standard formulas of differential geometry but with the metrics of \(\mathbb{R}_{2,1}\). In three dimensions, the Gaussian curvature of a surface \(f(x_1, x_2, x_3) = 0\) can be written as \[22\]:

\[ k = \frac{1}{2} \left[ \mathbf{n} \cdot \nabla^2 \mathbf{n} + (\nabla \cdot \mathbf{n})^2 \right], \]

where \(\mathbf{n} = \nabla f(x_1, x_2, x_3)/|\nabla f(x_1, x_2, x_3)|\) is the normal to the surface. In a three-dimensional space with a non positive definite metric, the gradient
of a scalar function $\phi$ and the divergence and Laplacian of a vector function $f = (f_1, f_2, f_3)$ are defined as

\[
\nabla \phi = (e_1 \cdot e_1) \frac{\partial \phi}{\partial x_1} e_1 + (e_2 \cdot e_2) \frac{\partial \phi}{\partial x_2} e_2 + (e_3 \cdot e_3) \frac{\partial \phi}{\partial x_3} e_3,
\]
\[
\nabla \cdot f = (e_1 \cdot e_1)^2 \frac{\partial f_1}{\partial x_1} + (e_2 \cdot e_2)^2 \frac{\partial f_2}{\partial x_2} + (e_3 \cdot e_3)^2 \frac{\partial f_3}{\partial x_3},
\]
\[
\nabla^2 f = \left[ (e_1 \cdot e_1)^3 \frac{\partial^2}{\partial x_1^2} + (e_2 \cdot e_2)^3 \frac{\partial^2}{\partial x_2^2} + (e_3 \cdot e_3)^3 \frac{\partial^2}{\partial x_3^2} \right] (f_1 e_1 + f_2 e_2 + f_3 e_3).
\]

We can use \textit{Clifford} to evaluate all these expressions:

\texttt{In[1]:= << Clifford.m}

The adequate metrics is defined

\texttt{In[2]:= \$SetSignature = 2}

Here are the differential operators:

\texttt{In[3]:= var = \{x1,x2,x3\};}
\texttt{In[4]:= \texttt{GeoGrad}[g\_]:= Sum[InnerProduct[e[k],e[k]]*D[g,var[[k]]]*e[k],\{k,3\}];}
\texttt{In[5]:= \texttt{GeoDiv}[v\_]:= Sum[InnerProduct[e[k],e[k]]^2 D[Coeff[v,e[k]],var[[k]]],\{k,3\}];}
\texttt{In[6]:= \texttt{GeoLap}[v\_]:= Sum[(InnerProduct[e[k],e[k]]^3)*D[v,\{var[[k]],2\}],\{k,3\}];}

The function \texttt{Coeff[m,b]}, extracts the coefficient of the blade $b$ in the multivector $m$. The surface and their normal are:

\texttt{In[7]:= f = x1^2 + x2^2 - x3^2 + R^2;}
\texttt{In[8]:= norm = GeoGrad[f] / Magnitude[GeoGrad[f]];}

and, finally, the Gaussian curvature is:

\texttt{In[9]:= kGauss = (InnerProduct[norm,GeoLap[norm]]+(GeoDiv[norm])^2)/2 // Simplify}

\texttt{Out[9]= -------------------------}
\texttt{2 2 2}
\texttt{x1 + x2 - x3}
Points that lie in the surface fulfills $x_3^2 = x_1^2 + x_2^2 + R^2$, therefore:

\[ \text{In}[11] := \text{kGauss} \rightarrow \text{Sqrt}[x_1^2 + x_2^2 + R^2] \]

\[-2\]

\[ \text{Out}[11] = -R \]

which is the Gaussian curvature of the hyperboloid.

Some considerations concerning the isometries (distance-preserving transformations) of the hyperbolic plane are pertinent before we leave this section. Like the Euclidean ones, the isometries of the hyperbolic plane can be described in terms of reflections about given axes. All these isometries have simple expressions in terms of Clifford algebra. For instance, Eqn. (17) for rotations in a given plane remain valid in $\mathbb{R}_{2,1}$ but now is called a Lorentz transformation. In general, given the vectors $u_1, u_2, \ldots, u_k$ in $\mathbb{R}_{p,q}$, such that $(u_1, u_2, \ldots, u_k)(u_k, u_{k-1}, \ldots, u_1) = 1$, the transformation

\[ \mathbf{v} \mapsto (-1)^k (u_k u_{k-1} \cdots u_1) \mathbf{v} (u_1 u_2 \cdots u_k) = (-1)^k \tilde{U} \mathbf{v} U, \]

is an isometry [12]. If $k$ is even, the isometry is a rotation, if $k$ is odd, it is a reflection [3]. Now, if $q = 0$ transformations such as the previous one are orthogonal. For $q = 1$ (as the case of the Minkowski space) they are Lorentz transformations.

10. Dirac and Pauli algebras

Now let us add one extra dimension to the space of the previous example and consider $\mathbb{R}_{3,1}$. This vector space is fifteen-dimensional with basis

\[ \{1, e_1, e_2, e_3, e_4, e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3, e_2 e_4, e_3 e_4, e_1 e_2 e_3 e_4\} \]

The basis vectors $e_i$ satisfy the relations:

\[ e_1^2 = e_2^2 = e_3^2 = 1, \quad e_4^2 = -1, \]

which is the algebra of the Dirac matrices in relativistic quantum theory. Due to this isomorphism, $\mathbb{R}_{3,1}$ is often referred as the Dirac algebra.

The even subalgebra $\mathbb{R}_{3,1}^+$ has a basis $\{1, e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3, e_2 e_4, e_3 e_4, e_1 e_2 e_3 e_4\}$ and is equivalent to the algebra of the Pauli matrices used in quantum mechanics of spin-$\frac{1}{2}$ particles. This can be seen with the identifications

\[ \sigma_1 = e_1 e_4, \quad \sigma_2 = e_2 e_4, \quad \sigma_3 = e_3 e_4. \]
By taking $i = e_1 e_2 e_3 e_4 = \sigma_1 \sigma_2 \sigma_3$, we get
\[
\begin{align*}
\sigma_i^2 &= 1, \quad i = 1, 2, 3 \\
\sigma_1 \sigma_2 &= i \sigma_3 \\
\sigma_2 \sigma_3 &= i \sigma_1 \\
\sigma_3 \sigma_1 &= i \sigma_2,
\end{align*}
\]
which are the familiar Pauli matrix relations. $\mathbb{R}^+_{3,1}$ is also called the Pauli algebra.

Following the same reasoning, we can prove that the even subalgebra of the Pauli algebra is isomorphic to the quaternions. The even subalgebra of the quaternions is isomorphic to the complex numbers. The even subalgebra of the complex numbers is $\mathbb{R}$.

11. Summary

The basic ideas of the Clifford algebra of a vector space are presented and a Mathematica package for calculations within this algebra is developed and used in some examples. The relevance of Clifford algebra in physics and mathematics lies in the fact that it provides a complete algebraic expression of geometric concepts such as directed lines, areas, volumes, etc. (For this reason, Clifford algebra is also referred as Geometric algebra \[12\]). Quantities such as vectors, complex numbers, quaternions, Pauli and Dirac matrices, have been normally described by physicists with a potpourri of disjoint mathematical systems. All of them are naturally contained in a Clifford algebra. It becomes therefore an efficient mathematical language in a vast domain of physics.

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