Computing optimal k-regret minimizing sets with top-k depth contours

Sean Chester, Alex Thomo, S. Venkatesh, and Sue Whitesides
Computer Science Department
University of Victoria
PO Box 1700 STN CSC
Victoria, Canada
{schester, sue}@uvic.ca, {thomo, venkat}@cs.uvic.ca

ABSTRACT
Regret minimizing sets are a very recent approach to representing a dataset \( D \) with a small subset \( S \) of representative tuples. The set \( S \) is chosen such that executing any top-1 query on \( S \) rather than \( D \) is minimally perceptible to any user. To discover an optimal regret minimizing set of a predetermined cardinality is conjectured to be a hard problem. In this paper, we generalize the problem to that of finding an optimal \( k \)-regret minimizing set, wherein the difference is computed over top-\( k \) queries, rather than top-1 queries.

We adapt known geometric ideas of top-\( k \) depth contours and the reverse top-\( k \) problem. We show that the depth contours themselves offer a means of comparing the optimality of regret minimizing sets with \( L_2 \) distance. We design an \( O(cn^2) \) plane sweep algorithm for two dimensions to compute an optimal regret minimizing set of cardinality \( c \). For higher dimensions, we introduce a greedy algorithm that progresses towards increasingly optimal solutions by exploiting the transitivity of \( L_2 \) distance.

Categories and Subject Descriptors
H.3.3 [Information Storage and Retrieval]: Information Search and Retrieval; F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems—geometrical problems and computations

General Terms
Algorithms, Theory

Keywords
regret, representative databases, top-k, arrangement of lines, plane sweep

1. INTRODUCTION

For a user navigating a large dataset, the availability of a succinct representative subset of the data is crucial. For example, consider Table 1, a toy, but real, dataset consisting of the top eight scoring NBA players from the 2009 basketball season. A user viewing this data would typically be curious which of these eight players were “top of the class” that season. That is, he is curious which few tuples best represent the entire dataset, without his having to peruse it in entirety.

A well-established approach to representing a dataset is with the skyline operator which returns all pareto-optimal points. The intention of the skyline operator is to reduce the dataset down to only those tuples that are guaranteed to best suit the preferences or interests of somebody. If the toy dataset in Table 1 consisted only of the attributes points and rebounds, then the skyline would consist only of the players Kevin Durant, Amare Stoudemire, and Zach Randolph, so these three players would represent well what are the most impressive combinations of point-scoring and rebounding statistics. The skyline is a powerful summary operator only on low dimensional datasets, however; even on this toy example, everybody is in the skyline if we consider all four attributes. In general, there is no guarantee that the skyline is an especially succinct representation of the dataset.

Regret
A promising new alternative is the regret minimizing set, introduced by Nanongkai et al. [4], which hybridizes the skyline operator with top-\( k \) queries. A top-\( k \) query takes as input a utility function \( f \) and evaluates each tuple according to \( f \), reporting the \( k \) tuples with highest values. Figure 1 shows how highly three of the points rank for a user utility function of \( f(pts, reb) = (pts + reb)/2 \), if the attributes are normalized. The distance from the orthogonal line of each point is proportional to the point’s score for that user weight. This reveals that Randolph earns the highest normalized

| id  | player name    | points | rebs | steals | fouls |
|-----|----------------|--------|------|--------|-------|
| 1   | Kevin Durant   | 2472   | 623  | 112    | 171   |
| 2   | LeBron James   | 2258   | 554  | 125    | 119   |
| 3   | Dwyane Wade    | 2045   | 373  | 142    | 181   |
| 4   | Dirk Nowitzki  | 2027   | 520  | 70     | 208   |
| 5   | Kobe Bryant    | 1970   | 391  | 113    | 187   |
| 6   | Carmelo Anthony| 1943   | 454  | 88     | 225   |
| 7   | Amare Stoudemire| 1896  | 732  | 52     | 281   |
| 8   | Zach Randolph  | 1681   | 950  | 80     | 226   |

Table 1: Statistics for the top eight NBA point scorers from the 2009 regular season, taken from databasebasketball.com. The top score in each statistic is bolded.

1Pareto-optimal points are those for which no other point is higher ranked with respect to every attribute.
score (0.840), compared to Kevin Durant (0.828) and then Kobe Bryant (0.604).

To evaluate whether a subset effectively represents the entire dataset well, Nanongkai et al. introduce regret ratio as the ratio of how far from the best score in the dataset is the best score in that subset, with respect to a given utility function. Graphically, this is proportional to how much smaller than the largest arrow is the largest arrow in the subset. For the subset \{Bryant, Durant\}, the regret ratio is:

\[
\frac{(0.840 - 0.828)}{0.840} = 0.0143,
\]

since the score for Randolph is the best in the dataset at 0.840, and the score for Durant is the best in the subset at 0.828.

Figure 1: A utility function \( f(\text{pts}, \text{rebs}) = (\text{pts} + \text{rebs})/2 \) represented as a vector \( \vec{f} = (0.5, 0.5) \), and three data points from Table 1 shown with their scores being proportional to the distance from the line orthogonal to \( \vec{f} \).

![Figure 1](image)

Motivated to derive a succinct representation of a dataset, one with fixed cardinality, Nanongkai et al. introduce regret minimizing sets [14], posing the question, “Does there exist one set of \( c \) tuples that makes every user at least \( x\% \) happy?” A regret minimizing set is a subset of a dataset that minimizes the regret ratio.

A linear top-\( k \) query can be considered as a problem of projection [6], where each tuple is regarded as a vector, as is the utility function. The score of a tuple is proportional to the size of its projection onto the utility function vector and its scores for every possible utility function trace a (hyper-)sphere emanating from the point. Minimizing regret ratio is equivalent to finding a subset that minimizes the maximum distance between the “best” spheres in the subset and the “best” spheres in the entire dataset, as illustrated in Figure 2. Of the eight basketball players of Table 1, Zach Randolph achieves this criteria, so he is the optimal regret minimizing set of order (i.e., size) 1.

Randolph, the optimal regret minimizing tuple, however, is a peculiar choice to represent the dataset of Table 1 since he is the worst rated with respect to points. This exposes a weakness of regret minimizing sets: they are based on assuming that a “happy” user is one who obtains their absolute top choice. However, for an analyst curious to know what is a high point-scoring player, is he really dissatisfied with Lebron James as a query response rather than Kevin Durant?

To change the scenario a bit, consider a dataset of hotels and a user searching for one that suits his preferences. The absolute top theoretical choice may not suit him especially well at all. It could be fully booked. Or, he might know that the manager reminds him of his ex-wife. Regardless, it makes sense to present him a few, say \( k \), options, any of which he would be happy.

We generalize the concept of regret and of regret minimizing sets to that of \( k \)-regret, analogous to the difference between top-\( k \) queries and top-1 queries, because top-\( k \) is often a better threshold for “happiness”. The analogous problem is to find a subset \( S \) of points in the dataset that minimize the distance from the best point in \( S \) to the \( k \)'th best point in the entire dataset. This relaxation prevents having to fit an outlier tuple like Randolph.

Optimality

A fundamental open question remains with regards to both the problem introduced by Nanongkai et al. and our generalisation of it. How can one efficiently compute the optimal \( k \)-regret minimizing set of a predetermined cardinality \( c \), the subset that achieves the minimal regret ratio of all size \( c \) subsets of the dataset? This is a problem conjectured by Nanongkai et al. [14] to be \( \mathcal{NP} \)-Hard for \( k = 1 \): it involves searching for the best among \( \mathcal{O}(n^c) \) different subsets.

We introduce algorithms that strive to compute optimal \( k \)-regret minimizing sets. Towards this end, we relate the recent work on top-\( k \) depth contours of Chester et al. [2] for the reverse top-\( k \) problem of Vlachou et al. [19]. The top-\( k \) depth contours are a dual space, geometric idea that succinctly represent exactly the \( k \)'th ranked tuple for all utility functions. We demonstrate that these ideas are directly applicable to discovering optimal \( k \)-regret minimizing sets. For instance, if the cardinality restriction is lifted, then the contours are precisely the optimal solutions (Lemma 3).

In the presence of cardinality restrictions, the contours still aid in finding optimal solutions (Theorem 4).

1.1 Contributions and Outline

In this paper we propose the first algorithms for computing optimal \( k \)-regret minimizing sets. In particular, we:

- generalize regret and regret minimizing sets, top-1 concepts, to those of \( k \)-regret and \( k \)-regret minimizing sets, top-\( k \) concepts (Section 2);
- identify that apparently unrelated work on top-\( k \) depth contours and reverse top-\( k \) queries [7] sheds insight into the problem of identifying optimal \( k \)-regret minimizing sets (Section 5);
- introduce an \( \mathcal{O}(n^c) \) algorithm to compute the optimal size-\( c \) \( k \)-regret minimizing subset \( S \) of a two-dimensional dataset \( D \) for the family of positive linear functions \( \mathcal{L}^+ \), despite a conjecture by Nanongkai et al. [14] that the general dimension problem is intractable for \( k = 1 \) (Section 4).
introduce a greedy algorithm for general dimensions that leverages the relationship between top-k depth contours and optimality in order to progress towards more optimal solutions (Section 3), and

- relate our work within the context of other literature (Section 5).

2. PRELIMINARIES

In the following sections, we will demonstrate how to compute optimal k-regret minimizing sets by equating the problem to one in dual space. Within the dual space, the optimal solution is the top-k depth contour if it small enough, or else the convex chain through the arrangement of lines in dual space that minimizes a particular distance ratio. Before embarking on these objectives, however, we introduce some concepts formally in four subsections: first, k-regret and k-regret minimizing sets (Section 2.1); next, the transformation of the data into an arrangement of lines in dual space and some of the tools therein that we use (Section 2.2); penultimately, the top-k depth contours that exist in the arrangement of lines and are fundamentally connected to finding optimal k-regret minimizing sets (Section 2.3); and, finally, the problem definition under study (Section 2.4). Throughout the paper, we consider the family of positive linear functions under study (Section 2.4). We recall the definitions from that family of positive unit linear functions produced by Nanongkai et al. [14].

DEFINITION 2.1 (GAIN [14]). The gain for a subset \( S \subseteq D \) on \( f \in \mathcal{F} \) is:

\[
gain(S, f) = \max_{p \in S} f(p).
\]

That is to say, the gain of a subset \( S \), given a utility function \( f \), is simply the highest score achievable in \( S \) for the function \( f \). Recalling the example of Table 4 and the utility function \( f(\text{pts}, \text{rebs}) = (\text{pts} + \text{rebs})/2 \), and assuming the data is normalized, the gain of \{Bryant, Durant, Wade\} is 0.828. The generalisation of gain is to k-gain:

DEFINITION 2.2 (K-GAIN). Consider a descending order list of \( f(p) \) for all \( p \in S \subseteq D \), given \( f \in \mathcal{F} \). Then, the k-gain of \( S \) on \( f \) is simply the k’th value in the list.

In other words, the k-gain for a subset \( S \subseteq D \) is the k’th best score achieved by a point in \( S \) on the utility function \( f \). For the subset \( S = \{\text{Bryant, Durant, Wade}\} \) and the same function \( f \), the 2-gain is the second best score, 0.748, the score for Durant. For \( k = 1 \), this definition is equivalent to Definition 2.1.

Regret ratio, then, is a reflection of how well the gain of a subset approaches that of the entire dataset.

DEFINITION 2.3 (REGRET AND REGRET RATIO [14]). The regret for a subset \( S \subseteq D \) on \( f \in \mathcal{F} \) is:

\[
\text{regret}(S, f) = \text{gain}(D, f) - \text{gain}(S, f).
\]

The regret ratio is:

\[
\text{regret ratio}(S, f) = \frac{\text{regret}(S, f)}{\text{gain}(D, f)}.
\]

Since the best score for \( f \) is 0.840 the regret for the running example \( S \) is \((0.840 - 0.828) \) and the regret ratio is \((0.840 - 0.828)/0.840 \).

DEFINITION 2.4 (K-REGRET AND K-REGRET RATIO). The k-regret ratio is:

\[
\text{k-regret ratio}(S, f) = \frac{\text{regret}(S, f)}{\text{gain}(D, f)}.
\]

The k-regret ratio is:

\[
\text{k-regret ratio}(S, f) = \frac{\text{regret}(S, f)}{\text{gain}(D, f)}.
\]

Since Durant is the second highest scoring tuple in \( D \) for \( f \), the 2-regret ratio of \( S = \{\text{Bryant, Durant, Wade}\} \) is \((0.828 - 0.828)/0.828 = 0 \). The subset \( S \) perfectly matches the top-2 requirement for utility function \( f \). Finally,

DEFINITION 2.5 (MAXIMUM K-REGRET RATIO). The maximum k-regret ratio for a subset \( S \subseteq D \) with respect to a family of utility functions \( \mathcal{F} \) is:

\[
\text{maximum k-regret ratio}(S, \mathcal{F}) = \max_{f \in \mathcal{F}} \text{k-regret ratio}(S, f).
\]

The maximum k-regret ratio is the largest observable k-regret ratio for any utility function in an entire family. For \( S \) the 1-regret ratio is maximized for \( g(\text{pts}, \text{rebs}) = \text{rebs} \), at which the best score obtainable is \( S \) is 0.655 and the 1-regret ratio is \((1.000 - 0.655)/1.000 \) and the 2-regret ratio is \((0.771 - 0.655)/0.771 \).

Finally, a k-regret minimizing set of order \( c \) is simply one with cardinality \( c \) that minimizes the maximum k-regret ratio. There exist optimal k-regret minimizing sets of order \( c \), which are those that achieve minimal maximum k-regret ratio of all subsets of size \( c \).

DEFINITION 2.6 (OPTIMAL K-REGRET MINIMIZING SET). An optimal k-regret minimizing set of order \( c \) on a dataset \( D \) given a family of utility functions \( \mathcal{F} \) is:

\[
\text{optimal k-regret minimizing set}(D, \mathcal{F}) = \text{argmin}_{S \subseteq D, |S| \leq c} \text{maximum k-regret ratio}(S, \mathcal{F}).
\]

As well, Definition 2.6 reduces to that of Nanongkai et al. [14] if \( k = 1 \).

2.2 Arrangements of Lines

The algorithms that we propose in this paper are geometric in nature and operate on arrangements of hyperplanes in dual space. Arrangements of hyperplanes (or lines, in two dimensions), are well studied in Computational Geometry and are induced by the intersections of a set of hyperplanes.

DEFINITION 2.7 (ARRANGEMENT). An arrangement of a set of d-dimensional hyperplanes \( \mathcal{H} \), denoted \( \mathcal{A}_d \), is a partitioning of \( \mathbb{R}^d \) into cells, edges, and vertices. Each cell is a connected component of \( \mathbb{R}^d \setminus \mathcal{H} \). Each vertex is an intersection point of some d hyperplanes in \( \mathcal{H} \). An edge is a line segment between two vertices of \( \mathcal{A} \).

We arrive at an arrangement of hyperplanes by applying the duality transform introduced by Chester et al. [7], which fixes an arbitrary positive real \( \tau \) and converts every point \( p_i \in D \) to a hyperplane \( h_i \) by considering \( p \) as a vector \( \vec{p} \) and constructing the hyperplane \( h_i \) to be all vectors \( \vec{x} \) that solve \( \vec{p} \cdot \vec{x} = \tau \).
The tuples are transformed into translated nullspace equations, and the resulting arrangement of lines is shown. Also depicted in thicker, light magenta lines is the second top-k contour \( k \), a succinct representation of the 2nd ranked tuples for any top-k query.

**Definition 2.8 (Translated Nullspace Transform)**. Given a fixed positive real, \( \tau \), and a dataset of d-dimensional points \( D \), the translated nullspace transform each primal space point \( p_i \in D \) into a dual space \((d-1)\)-hyperplane \( L_i \) (or line \( l_i \) in two dimensions) composed of all vector solutions to the equation \( \vec{p} \cdot \vec{x} = \tau \).

For the basketball example, considering only the attributes points and rebounds, which have first been normalized, the arrangement of lines produced by the translated nullspace duality transform is illustrated in Figure 3. Note that the intersection points of two lines \( l_i \) and \( l_j \) occur exactly in the direction of the vector \( \vec{f} \) for which \( f(p_i) = f(p_j) \).

Two other important concepts that are central ideas in Computational Geometry and of high relevance to this paper are lower envelopes of arrangements of lines and convex chains within arrangements of lines.

**Definition 2.9 (Lower Envelope)**. The lower envelope of an arrangement of lines is the set of edges under which no other edges exist.

For the purposes of this paper, in which we consider only the positive quadrant of Euclidean space, the lower envelope is the set of edges closest to the origin, \( O \).

**Definition 2.10 (Convex Chain)**. A convex chain in an arrangement of lines \( A_C \) is the lower envelope in the arrangement of some subset \( L' \subseteq L \) of lines, \( A_{C'} \).

Alternatively, a convex chain can be considered to be any set of edges in the arrangement that form a convex polygon with \( O \).

### 2.3 Top-k Depth Contours

We also recall here two definitions to establish what are top-k depth contours, since they form the basis of our algorithms.

**Definition 2.11 (Top-k Depth)**. The top-k rank depth of a point \( p \) within an arrangement \( A \) is the number of edges of \( A \) between \( p \) and the origin. That is to say, the depth of \( p \) is the number of intersections between edges of \( A \) and \( \{O,p\} \).

We remark that this is identical to the more familiar concept of a level if all the lines pass through the positive quadrant, as we assume here. Nonetheless, we adopt this definition because there is no reason why the techniques described in this paper cannot be extended easily to handle attributes that range into negative values.

top-k rank depth of a cell or edge of \( A \) is the top-k rank depth of every point within that cell.

In their paper, Chester et al. \(^1\) show that the rank of a point in a dataset \( D \) is precisely its top-k rank depth, and that top-k rank depth creates a series of \( n \) contours in \( \mathbb{R}^d \), the \( i \)’th of which is comprised of the transformed points that had rank exactly \( i \) in \( D \).

**Definition 2.12 (Top-k Rank Depth Contour)**. A top-k rank depth contour is the set of edges in an arrangement \( A_C \) that have top-k rank depth exactly \( k \).

### 2.4 Problem Definition

Now, we can formally describe the problem under study in this paper:

**Problem Definition 1**. Given any integer \( c \) and set \( D \) of \( n \) d-dimensional points, find an optimal \( k \)-regret minimizing set of order \( c \), \( S_c(D, U^+) \), for the family of positive unit linear functions \( U^+ \).

### 3. A Contour View of Regret

In this section, we show that the concept of regret that was introduced by Nanongkai et al. \(^5\) and also the generalisation we introduce in this paper—are strongly connected to the dual space concept of top-k depth contours introduced by Chester et al. \(^1\). More precisely, we prove Theorem 1 which equates the problem of finding an optimal \( k \)-regret minimizing set to a dual space problem of finding a set of lines that are “closest” to the top-k depth contour. This alternative formulation of the problem facilitates designing algorithms in the dual space for two dimensions (Section 4) and general dimension (Section 5) to find optimal regret minimizing sets.

The argument proceeds by showing that the contour itself, \( C_k \), is the optimal solution, provided that it is small enough, \( |C_k| \leq c \) (Lemma 3.1). In the dual space, the regret ratio of a line relative to another line, given a utility function \( f \), is given by the relative distances of the lines from the origin in the direction indicated by \( f \) (Lemma 3.2). So, the evaluation of regret in dual space is a scaled Euclidean distance computation.

We also show that the best options available to users within a set of points \( S \subseteq D \) is exactly given in the the set of dual lines of \( S \), their lower envelope (Lemma 3.4). So, minimizing the scaled distance of that envelope from the contour yields an optimal solution (Theorem 1).

**Lemma 3.1**. The set of points contributing to \( C_b \) is a \( k \)-regret minimizing set \( S_b(D, U^+) \) if \( |C_b| \leq c \).

**Proof**. \( C_b \) is constructed such that, for any linear function \( f \in U^+ \), a point \( p \) on \( C_b \) has rank exactly \( k \). Therefore, \( k_{FT}(\{p\}, f) = 0, \forall f \in U^+ \).

To summarize Lemma 3.1 the contour is necessarily an optimal solution for any \( c \geq |C_b| \) because it is a representation of the \( k \)’th ranked tuple for any linear utility function, and the \( k \)-regret ratio of the \( k \)’th ranked tuple is 0.

Since the \( k \)-contour represents the barrier of no \( k \)-regret, any points transformed to lines farther from the origin \( O \) than the contour have positive regret proportional to the distance from \( O \). Conversely, any points transformed to lines closer to the origin than \( C_b \) with respect to \( f \) have \( k \)-regrets=0, because they are within the top-k on \( f \).
Lemma 3.2. For any utility function \( f \in U^+ \) and tuple \( p_i \in D \)
transformed to line \( l_i \in \mathcal{L} \), let \( \Delta_{C_k} \) denote the distance of \( C_k \) from \nO with respect to \( f \). \( \Delta_k \), the distance of \( l_i \) from \( O \), and \( \Delta' \geq 0 \) denote the distance of \( l_i \) from \( C_k \). Then, \( krr_D \{p_i\} = \Delta'/\Delta_k \).

Proof. Recall that each line \( l_i \in \mathcal{L} \) is constructed of vectors \( \vec{x} \) such that \( \vec{p}_i : \vec{x} = \tau \). So, since \( \|f\| = 1 \), the distance of \( l_i \) from \( O \) in the direction of \( f \) is \( \tau / f(p_i) \).

Thus, if \( p_{C_k} \in D \) represents a point on \( C_k \) in the direction of \( f \),
\[
\Delta' = \Delta_k - \Delta_{C_k} = \frac{\tau}{f(p_i)} - \frac{\tau}{f(p_{C_k})} = \frac{\tau(f(p_{C_k}) - f(p_i))}{f(p_i)f(p_{C_k})} = \frac{\Delta_k(krr_D\{p_i\})}{f(p_i)f(p_{C_k})}
\]

\square

Corollary 3.3. For a fixed \( f \in U^+ \), let \( \Delta_k \) denote the non-negative distance for some line \( l_i \in \mathcal{L} \) from \( C_k \) in the direction of \( f \). Then, \( \Delta_k \propto krr_D \{p_i\}, f \).

Lemma 3.2 establishes that the regret for a singleton set \( \{p_i\} \) on a function \( f \) in dual space is related to the Euclidean distance of the transformed line \( l_i \) to \( O \) and to \( C_k \). Corollary 3.3 notes that, if we consider only a single utility function, then the regret for different singleton sets can be straightforwardly compared by the distance from \( C_k \), since the distance of \( C_k \) to \( O \) is static. Next, we show that, given a non-singleton set, the maximum k-regret can be evaluated efficiently, because it is to be observed on the lower envelope in the dual space.

Lemma 3.4. For a set \( S \subseteq D \), let \( \mathcal{L} \) denote the set of lines produced by transforming every point \( p_i \in S \) into its translated nullspace, \( l_i \). The lower envelope of \( \mathcal{L} \) captures the maximum gain (and, ergo, minimum regret) of \( S \) for any \( f \in U^+ \).

Proof. For any \( f \in U^+ \), the nearest line to \( O \) is that line \( l \) which is on the lower envelope in the direction of \( f \). Since \( l \) has the smallest distance to \( O \) of all lines in \( \mathcal{L} \), it also has the smallest distance (possibly negative) to \( C_k \) of all lines in \( \mathcal{L} \). By Corollary 3.3, \( p_i \) then also has the minimum regret with respect to \( f \) of all \( p \in S \).

\square

Furthermore, while the lower envelope captures all the interestingness of a set \( S \), there are only particular points on the envelope where the maximum regret can occur: the points where the distance ratio could be maximized. Lemma 3.5 confirms that these points are exactly the vertices of the lower envelope and the vertices of the contour.

Lemma 3.5. Consider a contour \( C_k \) and a convex chain of line segments \( C \). Let \( \Delta_k \) denote the distance of \( C_k \) from \( O \) in the direction of \( f \) and, similarly, \( \Delta' \), the distance of \( C \) to \( C_k \). The expression \( \Delta'/\Delta_k \) is maximized either at a vertex of \( C_k \) or a vertex of \( C \).

Proof. Both \( C_k \) and \( C \) are piecewise linear, so the expression \( \Delta'/\Delta_k \) can only be maximized at some junction point.

\square

Since Lemmata 3.4 and 3.5 permit evaluating regret in the dual space of translated nullspaces, we can derive an alternative view of the problem of finding an optimal k-regret minimizing set, as shown in Theorem 1 and illustrated in Figure 4.

Theorem 1. Let \( g \) denote the function which transforms any point \( p \in D \) to its translated nullspace and let \( C_k \) denote the top-k depth contour of \( D \). The optimal k-regret minimizing set \( S \subseteq D \), with size at most \( c \) on the family of positive utility linear functions \( U^+ \) is exactly the set of lines \( \mathcal{L} = \{g(p), p \in S\}, |\mathcal{L}| \leq c \), the lower envelope \( \mathcal{E} \) of which minimizes the maximum ratio of distances from \( \mathcal{E} \) to \( C_k \) and \( \mathcal{E} \) to \( O \) at any vertex of \( \mathcal{E} \) and of \( C_k \).

Proof. From Lemma 3.2, the regret ratio for each point in \( S \) is equivalent to the ratio of \( E \) to \( C_k \) and \( E \) to \( O \), and from Lemma 3.4, the best such ratio is on the lower envelope of \( \mathcal{E} \). From Lemma 3.5, this must occur at either a vertex of \( C_k \) or a vertex of \( \mathcal{E} \).

\square

A final remark is with regard to the two dimensional case, in particular. We note that any lower envelope is, in fact, a convex chain, so the two dimensional problem can be viewed rather as searching for the best convex chain.

Lemma 3.6. Let \( g \) denote the function which transforms any point \( p \in D \) to its translated nullspace and let \( C_k \) denote the top-k depth contour of \( D \). The optimal k-regret minimizing set \( S \subseteq D \subseteq \mathbb{R}^2 \) with size at most \( c \) is exactly the convex chain \( C \) through the arrangement of lines \( \mathcal{L} = \{g(p), p \in S\} \) that has at most \( c \) turns and that minimizes the maximum ratio of the distance from \( C_k \) and the distance of \( C \) to \( O \).

Proof. Note that any lower envelope of a set of lines is, in fact, a convex chain and that, by convexity, any line can appear at most once on the lower envelope. So, a convex chain with \( c \) lines will have \( c - 1 \) turns. Also, any convex chain with \( c - 1 \) turns that is optimal must be a lower envelope of some set of lines, otherwise the sequence of turns that follows the lower envelope of the same set of lines will be more optimal. By Theorem 1, this is the optimal k-regret minimizing set of size \( c \) for \( D \).

\square

4. AN ALGORITHM FOR TWO DIMENSIONS

As we showed in Lemma 3.6 solving the regret minimizing problem reduces to finding the best convex chain \( C \) with fewer than \( c \) turns through an arrangement. The optimal solution is the one which minimizes the distance ratio of \( C \) to \( C_k \) and \( C \) to \( O \). There are potentially \( \binom{n}{c} \) different convex chains with at most \( c \) – 1 turns; however, so we need to improve upon the \( \mathcal{O}(n^c) \) naive algorithm which tries every combination.

We offer here a plane sweep, dynamic programming algorithm which runs \( \mathcal{O}(cn^2) \) and independently of \( k \). The algorithm follows each of the \( n \) translated nullspace lines in \( \mathcal{L} \) radially from the
Consider an intersection point \( p_{i,j} \), the intersection of lines \( l_i \) and \( l_j \). Because the lines are intersecting, we know they are immediately adjacent in \( \mathcal{L} \). We swap \( l_i \) and \( l_j \) in \( \mathcal{L} \) to reflect the fact that immediately after \( p_{i,j} \), they will have opposite order as compared to beforehand.

\[ \mathcal{Q} \]

The priority queue contains all those intersection points that are between \( r \) and the positive \( x \)-axis that feature two lines which have been adjacent at some point between the positive \( y \)-axis and \( r \). Again, consider an intersection point \( p_{i,j} \). Immediately thereafter, lines \( l_i \) and \( l_j \) have been swapped. As such, there are potentially two new intersection points to add to \( \mathcal{Q} \), namely \( l_i \) and his new neighbour (should one exist) and \( l_j \) and his new neighbour (again, should one exist). Both these intersection points are added to the appropriate place in \( \mathcal{Q} \), provided that they are between \( r \) and the positive \( x \)-axis. The point \( p_{i,j} \) is removed.

\[ \mathcal{P} \]

Again, consider an intersection point \( p_{i,j} \) featuring lines \( l_i \) and \( l_j \). Let \( l_i \) be farther from the origin than \( l_j \) in the direction of a ray after \( r \). There are three valid paths through \( p_{i,j} \), as illustrated in Figure 6.

First consider the line, \( l_i \), that emerges above after \( p_{i,j} \) (line \( l_2 \) in Figure 6). Consider also the row of \( c \) cells of \( \mathcal{P} \) that describe best paths for \( l_i \). The \( h \)‘th such cell, describes the chain with optimum distance to \( C_h \) that uses at most \( h - 1 \) turns and emerges from \( p_{i,j} \) along line \( l_i \). Because the turn \((l_j,l_i)\) is invalid, paths for \( l_i \) cannot change, only their costs. The cost is updated to the larger of what the value was before and the distance from \( p_{i,j} \) to \( C_h \) in the direction of \( r \).

So, considering first a chain that leaves along \( l_j \), the best possible route to the next intersection point of \( l_j \) is exactly whatever was

Figure 6: An illustration of the three possible paths through an intersection point. Either line \( l_1 \) or \( l_2 \) could simply pass through. Because an envelope of lines must form a convex chain, on the other hand, only \( l_2 \) has the luxury of turning onto \( l_1 \). The path \((l_2,l_1)\) requires an “illegal” concave turn.
Figure 7: The processing of the tenth event point, the intersection of the lines corresponding to Stoudemire and Randolph. The distance along r of the intersection point from the contour is 0.034. In this case, the newly discovered length-2 chain, (Stoudemire, Randolph), has cost max(0.105, 0.034), which does not improve on the value already found for Randolph, 0.083.

| Lines               | slope of r |
|---------------------|------------|
| (Stoudemire, Randolph) | 2.64       |
| (Nowitzki, Randolph)  | 2.48       |
| (Nowitzki, Stoudemire)| 2.22       |
| (Wade, Anthony)      | 2.07       |
| (James, Nowitzki)    | 0.74       |
| (Wade, Bryant)       | 0.62       |

(b) The priority queue, including all intersection points between r and the x-axis which have been discovered between the y-axis and r. The grayed entry is the one being processed. The bolded entry is the one that is newly added by processing this point.

(c) The best cost values for each line as of the processing of the darker point in (a). The bold indicates the sustained value in the table after processing the point.

the best possible route along \( l_j \) to \( p_{i,j} \). The cost of that route is the larger of the distance from \( p_{i,j} \) to \( C_k \) and the cost of the best possible route to the previous intersection point of \( l_j \). This value is updated for each of the \( m \) cells in row \( i \).

On the other hand, for a chain that emerges along \( l_j \), there are two possibilities, depending on the best route to get to \( p_{i,j} \). Specifically, the best route of \( h \) turns to \( p_{i,j} \) is the cheaper of the best route to the previous intersection point along \( l_j \) that used \( h \) turns and the best route to the previous intersection point along \( l_j \) that used \( h - 1 \) turns. The final cost is then the larger of the distance from \( p_{i,j} \) to \( C_k \) and the minimum cost to \( p_{i,j} \) as just described.

4.2 Algorithm Description

As we have hinted, the algorithm is a plane sweep through the arrangement of lines, searching for the minimal cost convex chain with fewer than \( c \) turns. The sweep features a ray \( r \), originally positioned on the positive y-axis, moving radially through the positive quadrant. This plane sweep approach is appropriate as a consequence of Lemma 3.5 which reveals that the cost of any convex chain is maximized at an event point.

To initialize the algorithm, the intersection point of every line with the positive y-axis is processed, populating the data structures as in Figure 5. We only add to the priority queue the intersection points of lines that are immediate neighbours with respect to a sort on y-intercept, and for which the intersection point occurs in the positive quadrant. These points are maintained in the queue in descending order of angle from the positive x-axis (i.e., in the order in which the ray \( r \) will “sweep” them). The array \( P \) is initialised with empty paths for every cell, with a cost set to the distance of the relevant line to the contour.

The algorithm proceeds simply by popping the next event from \( Q \), updating the data structures as per Section 4.1.1 and pushing the new event points onto the queue. For the running basketball example, this is illustrated in Figure 7. For event points that correspond to vertices of the contour (since these, too, are intersection points of lines that will eventually be discovered by the plane sweep), we update every cell of \( P \) with new maximum costs for each line that has become more distant from \( C_k \).

Eventually, \( Q \) will be exhausted as \( r \) reaches the positive x-axis (see Figure 6). Every cell is updated with new maximum costs at this last contour vertex (the contour of the intersection with the \( x \)-axis). The final step is to scan through all of \( P \) and determine the smallest cost. This is the optimal solution, which is reported along with the path used to obtain it.

Algorithm 1 describes the algorithm with greater detail.

4.3 Asymptotic Complexity

Theorem 2. Algorithm 1 for the two-dimensional case finds an optimal \( k \)-regret minimizing set of order \( c \) in \( \mathcal{O}(cn^2) \) time with \( \mathcal{O}(n^2) \) space.

Proof. First consider space used. The size of the contour is bounded by \( n \). Of the three data structures, \( L \) is of size exactly \( n \), \( P \) is of size exactly \( n \times c \leq n^2 \), and \( Q \) is proportional to the largest number of intersection points that have been discovered but not processed, clearly less than \( n \times (n - 1) \). Therefore, the total space is \( \mathcal{O}(n^2) \).

Regarding time, for each non-contour event point, of which there may be up to \( n^2 \), up to \( 2c \) cells of \( P \) are updated. For each contour event point, of which there are \( |C_k| \leq n \), each of the \( nc \) cells of \( P \) could potentially be updated. The initialisation requires a sort of \( n \) lines and then an initialisation of up to \( n - 1 \) insertions into \( Q \) and \( nc \) values for \( P \), which can be computed in constant time. At the conclusion of the plane sweep, all \( nc \) cells of \( P \) must be scanned. Therefore, the entire procedure is \( \mathcal{O}(n^2c) \).

5. AN ALGORITHM FOR GENERAL DIMENSION

In Section 4 we gave an efficient plane sweep, dynamic programming algorithm to find optimal \( k \)-regret minimizing sets in two dimensions. Unfortunately, plane sweep algorithms do not readily generalize to higher dimensions. So, in this section, we offer a greedy algorithm which exploits Lemma 5.1 in order to progress towards an optimal solution.

Lemma 5.1. Consider the envelope produced by some set \( L \). For another set \( L' \) to better minimize regret, it is necessary that some line of \( L' \setminus L \) either passes through the area between the contour and the envelope or remains entirely under the contour at the angle for which the distance ratio for \( S \) is maximized.
The idea of representing an entire dataset with a few representative tuples for multi-criteria decision making has drawn much research attention in the past decade, since the introduce of the Skyline operator by Börzsönyi et al. [2]. However, its susceptibility to the curse of dimensionality is well-known. Chan et al. [4] made a compelling case for this, demonstrating that on the NBA basketball dataset (as it was at the time), more than 1 in 20 tuples appear in the skyline in high dimensions. Consequently, there have been numerous efforts to derive a smaller cardinality representative subset (e.g., [3,13,20]), especially one that presents very distinct tuples (e.g., [9,16]).

Regret and regret minimizing sets are relatively new in the lineage of these efforts. When introduced by Nanongkai et al. [14], the emphasis was on proving that the maximum regret ratio is bounded by:

\[
\frac{d - 1}{(c - d + 1)^{d - 1} + d - 1}.
\]

Naturally, this bound holds for our generalisation introduced in this paper. As far as we know, no research has yet concerned itself with finding optimal regret minimizing sets.

The top-\(k\) query off which regret is based is well studied. The pareto-dominance graph [21] uses ideas of pareto-optimality to index for top-\(k\) queries. The Onion Technique [5] is a depth-based approach, but suffers the same curse of dimensionality as the skyline. Ilyas offers a nice survey on the topic of top-\(k\) queries [12].

Duality transforms are pervasive in this research area (e.g., [6,17]). We use the duality transform of Chester et al. [6] because of the immediate results on top-\(k\) depth contours it provides in answering the reverse top-\(k\) queries of Vlachou et al. [18,19].

Using duality transforms on data points casts the problem into the context of arrangements. In two dimensions, plane sweep algorithms [10] are a typical approach to solving problems on arrangements of lines. Agarwal et al. [11] give bounds on the the number of edges and vertices that can exist at each level (or depth) of an arrangement. Top-\(k\) depth contours are not the only notion of contours or depth in arrangements of lines: Hugg et al. [11] evaluate several depth measures and Zuo et al. [22] derive general statistical results that apply to many of them and could be useful in extending this work. Rousseuw and Hubert consider depth in arrangements for dimensions greater than two [15], which could present deeper insight into the greedy algorithm presented here.
7. CONCLUSION

Regret minimizing sets are a nice alternative to skyline as a succinct representative subset of a dataset, but suffer from a very strict assumption that users expect to see their top-1 choice for their queries. We generalised the notion to that of k-regret minimizing sets, which evaluates how representative a subset of a dataset is not by how closely it approximates every users’ top-1 choice, but their top-k choice. We showed that in dual space, the top-k depth contour of a dataset is exactly the optimal k-regret minimizing set. If the cardinality of the k-regret minimizing set is specified as an input parameter, then the convex chain that minimizes the ratio of distances from itself to the contour and the contour to the origin is precisely the optimal solution. We used these ideas to construct an $O(n^2c)$ optimal algorithm for two dimensions and a greedy algorithm for general dimension.

8. REFERENCES

[1] P. K. Agarwal, B. Aronov, and M. Sharir. On levels in arrangements of lines, segments, planes, and triangles. In Proc. Symposium on Computational Geometry, pages 30–38, New York, NY, USA, 1997. ACM.

[2] S. Börzsönyi, D. Kossmann, and K. Stocker. The skyline operator. In Proc. International Conference on Data Engineering (ICDE), pages 421–430, Washington, DC, USA, 2001. IEEE Computer Society.

[3] C.-Y. Chan, H. V. Jagadish, K.-L. Tan, A. K. H. Tung, and Z. Zhang. Finding k-dominant skylines in high dimensional space. In Proc. ACM Special Interest Group on Management of Data (SIGMOD), pages 503–514, New York, NY, USA, 2006. ACM.

[4] C.-Y. Chan, H. V. Jagadish, K.-L. Tan, A. K. H. Tung, and Z. Zhang. On high dimensional skylines. In Proc. International Conference on Extending Database Technology (EDBT), pages 478–495, Berlin, Heidelberg, 2006. Springer-Verlag.

[5] Y.-C. Chang, L. Bergman, V. Castelli, C.-S. Li, M.-L. Lo, and J. R. Smith. The onion technique: indexing for linear optimization queries. In Proc. ACM Special Interest Group on Management of Data (SIGMOD), pages 391–402, New York, NY, USA, 2000. ACM.

[6] S. Chester, A. Thomo, S. Venkatesh, and S. Whitesides. Indexing for vector projections. In Proc. Database Systems for Advanced Applications (DASFAA), pages 367–376. Springer-Verlag, April 2011.

[7] S. Chester, A. Thomo, S. Venkatesh, and S. Whitesides. Indexing reverse top-k queries. 2012. arXiv:1205.0837v1 [cs.DB].

[8] G. Das, D. Gunopulos, N. Koudas, and N. Sarkas. Ad-hoc top-k query answering for data streams. In Proc. Very Large Databases (PVLDB), pages 185–194. VLDB Endowment, 2007.

[9] A. Das Sarma, A. Lall, D. Nanongkai, R. J. Lipton, and J. Xu. Representative skylines using threshold-based preference distributions. In Proceedings of the 2011 IEEE 27th International Conference on Data Engineering, ICDE ’11, pages 387–398, Washington, DC, USA, 2011. IEEE Computer Society.

[10] H. Edelsbrunner and L. J. Guibas. Topologically sweeping an arrangement. In Proc. 18th ACM Symposium on Theory of Computing, pages 389–403, New York, NY, USA, 1986. ACM.

[11] J. Hugg, E. Rafalin, K. Seyboth, and D. Souvaine. An experimental study of old and new depth measures. In In Proc. Workshop on Algorithm Engineering and Experiments (ALENEX06), Lecture Notes in Computer Science, pages 51–64, Springer, 2006.

[12] I. F. Ilyas, G. Beskales, and M. A. Soliman. A survey of top-k query processing techniques in relational database systems. ACM Computing Surveys, 40(4):11–1158, Oct. 2008.

[13] J. Lee, G.-w. You, and S.-w. Hwang. Personalized top-k skyline queries in high-dimensional space. Information Systems, 34(1):45–61, Mar. 2009.

[14] D. Nanongkai, A. D. Sarma, A. Lall, R. J. Lipton, and J. Xu. Regret-minimizing representative databases. volume 3, pages 1114–1124. VLDB Endowment, 2010.

[15] P. J. Rousseeuw and M. Hubert. Depth in an arrangement of hyperplanes, 1999.

[16] Y. Tao, L. Ding, X. Lin, and J. Pei. Distance-based representative skyline. In Proc. International Conference on Data Engineering (ICDE), pages 892–903, Washington, DC, USA, 2009. IEEE Computer Society.

[17] P. Tsaparas, N. Koudas, and T. Palpanas. Ranked join indices. In Proc. International Conference on Data Engineering (ICDE), pages 277–288, 2003.

[18] A. Vlachou, C. Doulkeridis, Y. Kotidis, and K. Norvag. Reverse top-k queries. In Proc. International Conference on Data Engineering (ICDE), pages 365–376. IEEE, March 2010.

[19] A. Vlachou, C. Doulkeridis, Y. Kotidis, and K. Norvag. Monochromatic and bichromatic reverse top-k queries. Transactions on Knowledge and Data Engineering (TKDE), 23(8):1215–1229, 2011.

[20] M. L. Yiu and N. Mamoulis. Multi-dimensional top-k dominating queries. 18(3):695–718, June 2009.

[21] L. Zou and L. Chen. Pareto-based dominant graph: An experimental study of old and new depth measures. In In Proc. Workshop on Algorithm Engineering and Experiments (ALENEX06), Lecture Notes in Computer Science, pages 51–64, Springer, 2006.

[22] Y. Zuo and R. Serfling. Structural properties and convergence results for contours of sample statistical depth functions. Annals of Statistics, 28:483–499, 2000.
A. ALGORITHM PSEUODOCODE

Algorithm 1 Calculating an optimal k-max-regret minimizing set S from \( D \subseteq \mathbb{R}^2 \) with \(|S| \leq m\)

1: Input: \( C_k; m; \) \( \mathcal{L} \), sorted by y-intercept
2: Output: \( S \subseteq \mathcal{L} \), the lines that together form an optimal solution \( S \) with \(|S| \leq c\)
3: if \(|C_k| \leq c\) then
4: Return \( C_k \)
5: end if
6: Initialize \( \mathcal{Q} \) as an empty priority queue; priority is angle of points, desc.
7: for all \( l \in \mathcal{L} \) do
8: Set \( \mathcal{P}(l) = k \ast [(y-intercept, \max(y-intercept - y-intercept of \mathcal{C}_k, 0)] \)
9: Add to \( \mathcal{Q} \) intersect\((l \text{ and next } l)\) if not last \( l \)
10: end for
11: while \( \mathcal{Q} \) is not empty do
12: Let \( p \) be next point in \( \mathcal{Q} \)
13: Let \( \Delta \) be distance ratio of \( p \) to \( C_k \) and \( C_k \) to \( O \)
14: if \( p \in C_k \) then
15: for all \( (v \in \mathcal{P}) \) do
16: Let \( \Delta' \) be distance of ratio of line to \( C_k \) and \( C_k \) to \( O \)
17: Let \( v = \max(v, \Delta') \)
18: end for
19: end if
20: Retrieve adjacent \( l_i, l_{i+1} \) that intersect at \( p \)
21: Add intersect\((l_{i-1}, l_{i+1})\) if angle less than that of \( p \)
22: Add intersect\((l_i, l_{i+2})\) if angle less than that of \( p \)
23: for all \( j \in [0, c) \) do
24: Let \( P(l_i) = \max(P(l_i), \Delta) \)
25: if \( j \geq 0 \) then
26: if \( P(l_{i+1})_{j-1} < P(l_{i+1})_j \) then
27: Add \( p \) to path of \( P(l_{i+1})_j \)
28: Let \( P(l_{i+1})_j = \max(P(l_{i+1})_{j-1}, \Delta) \)
29: else
30: Let \( P(l_{i+1})_j = \max(P(l_{i+1})_j, \Delta) \)
31: end if
32: else
33: Let \( P(l_{i+1})_j = \max(P(l_{i+1})_j, \Delta) \)
34: end if
35: end for
36: Swap \( l_i \) and \( l_{i+1} \)
37: end while
38: for all \( l \in \mathcal{L}, j \in [0, c) \) do
39: Remember \( v = P(l)_j \) if smallest yet seen, breaking ties with smaller \( j \)
40: end for
41: RETURN set of lines generating path corresponding to \( v \)

Algorithm 2 Greedy algorithm to compute k-max-regret minimizing set S from \( D \subseteq \mathbb{R}^d \) with \(|S| \leq m\)

1: Input: \( C_k; m; \mathcal{L} \)
2: Output: \( S \subseteq \mathcal{L} \), the lines that together form a solution \( S \) with \(|S| \leq m\)
3: if \(|C_k| \leq m\) then
4: Return \( C_k \)
5: end if
6: Select an arbitrary set \( S \subseteq \mathcal{L} \), such that \(|S| = m\)
7: Let \( p \) be point at which distance ratio from lower envelope of \( S \) to \( C_k \) is maximized
8: Place all lines \( l_i \notin S \) into unsorted queue, \( \mathcal{Q} \).
9: while \( \mathcal{Q} \) is not empty do
10: Let \( l \) be next line in \( \mathcal{Q} \)
11: if \( l \) intersects \([O, p]\) then
12: for all \( (l' \in S) \) do
13: if Distance ratio of \( S \setminus \{l'\} \cup \{l\} \) to \( C_k \) < distance ratio of \( S \) to \( C_k \) then
14: Let \( S = S \setminus \{l'\} \cup \{l\} \)
15: Restore to \( \mathcal{Q} \) all \( l_i \notin S \)
16: Break
17: end if
18: end for
19: end if
20: end while
21: RETURN \( S \)