A Natural Framing for Asymptotically Flat Integral Homology 3-Sphere

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Abstract
For an integral homology 3-sphere embedded asymptotically flatly in an Euclidean space, we find a natural framing extending the standard trivialization on the asymptotically flat part.

Suppose $\overline{M}$ is a 3-dimensional closed smooth manifold which has the same integral homology groups as the 3-sphere $S^3$. $x_0$ is a fixed point in $\overline{M}$. Embed $\overline{M}$ in a Euclidean space $\mathbb{R}^n$ such that $x_0$ is the infinite point of the 3-dimensional flat space $\mathbb{R}^3 \times \{0\}$ of $\mathbb{R}^n$ and a neighborhood of $x_0$ contains the whole flat space $\mathbb{R}^3 \times \{0\}$ except a compact set. Precisely, for any positive number $r$, let $B_r$ denote the closed ball of radius $r$ in $\mathbb{R}^3$ and $N_r = (\mathbb{R}^3 - B_r) \times \{0\}$; there exists $r_0$, a positive number, such that $N_{r_0}$ is contained in $\overline{M}$ and $N_{r_0} \cup \{x_0\}$ is an open neighborhood of $x_0$ in $\overline{M}$.

Let $M = \overline{M} - \{x_0\}$, it is an asymptotically flat 3-dimensional manifold with acyclic homology. The main purpose of this article is to define a natural framing for $M$. If we identify the tangent spaces of points in the flat part $N_{r_0}$ with $\mathbb{R}^3 \times \{0\}$, then the tangent bundle of $M$ can be thought as a 3-dimensional vector bundle over the closed manifold $M_0 = M/N_s$, where $s$ is a number greater than $r_0$ and $\overline{N_s}$ is the closure of $N_s$; we shall call this vector bundle the tangent bundle $T(M_0)$ of $M_0$. And our natural framing is just a trivialization of $T(M_0)$, which corresponds to a trivialization of the tangent bundle $T(M)$ whose restriction to the flat part is the standard trivialization on $\mathbb{R}^3$. Because $M_0$ is a closed 3-manifold, there are countably infinite many
choices of framings associated with the infinite elements in \([M_0, SO(3)]\). ( When \(H_*(M_0) \approx H_*(S^3), [M_0, SO(3)] \approx [S^3, SO(3)] \approx \mathbb{Z}.\) Therefore, our natural framing is a special choice from the infinite many.

On the other hand, this natural framing for \(T(M_0)\) can also provide a special one-to-one correspondence between the infinite framings of \(S^3\) and that of \(\overline{M}\). ( Note: Here, we do not think that \(\overline{M}\) and \(M_0\) have the same tangent bundle. Conversely, we may think that the tangent bundle of \(\overline{M}\) is equal to the connected sum of the tangent bundles of \(M_0\) and \(S^3\). )

There are two main steps to the natural framing on \(T(M_0)\).

**Step 1  A special map from \(C_2(M)\) to \(S^2\)**

We define \(C_2(M)\) at first.

For any set \(X\), \(\Delta(X)\) denote the diagonal subset \(\{(x, x) \in X \times X, x \in X\}\) of \(X \times X\) and \(C_2(X) = X \times X - \Delta(X)\). Thus \(C_2(M)\) is the configuration space of all pairs of distinct two points in \(M\).

Fix some large number \(s\) such that \(M \subset (B_s \times \mathbb{R}^{n-3}) \cup N_s\).

For any \(r \geq s\), let \(B_r = \{x \in \mathbb{R}^3 : |x| \leq r\},\) \(N_r = (\mathbb{R}^3 - B_r) \times \{0\}\) and \(M_r = M - N_r\).

Let \(Y\) denote the union of the following three subsets of \(C_2(M)\):

\[
\begin{align*}
(i) & \quad Y_0 = C_2(N_s) \\
(ii) & \quad Y_1 = \bigcup_{r \geq s} (N_{r+s} \times M_r) \\
(iii) & \quad Y_2 = \bigcup_{r \geq s} (M_r \times N_{r+s})
\end{align*}
\]

Let \(\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^3\) denote the projection

\[
\pi(t_1, t_2, \cdots, t_n) = (t_1, t_2, t_3)
\]
and $f : Y \longrightarrow S^2$ denote the map

$$f(x, y) = \frac{\pi(y - x)}{|\pi(y - x)|}$$

for $(x, y) \in Y$, $x, y \in M$.

For the well-defining of the map $f$, we should check that $|\pi(y - x)|$ is a non-zero value. When $(x, y)$ is in $Y_0$, $|\pi(y - x)| = |y - x|$, it is non-zero. When $(x, y)$ is in $Y_1$, $(x, y)$ is in $N_{r+s} \times M_r$ for some $r \geq s$; thus $\pi(x)$ is outside of $B_{r+s}$ and $\pi(y)$ is in $B_r$, and hence $\pi(y - x) = \pi(y) - \pi(x)$, it has also a non-zero norm. It is similar for the case that $(x, y)$ is in $Y_2$.

The following proposition describes some homology properties for the space $Y$ and the map $f$.

**Proposition 1**

(i) $H_*(Y) \approx H_*(S^2)$

(ii) $f_*: H_2(Y) \longrightarrow H_2(S^2)$ is an isomorphism.

(iii) Let $j : Y \longrightarrow C_2(M)$ denote the inclusion map.

$$j_* : H_i(Y) \longrightarrow H_i(C_2(M))$$

is isomorphic, for all integer $i \geq 0$.  

In the proof of the proposition, we strongly use the assumption that $H_*(M)$ is acyclic.

**Remark:** All the homologies in this article are with integral coefficients.

By Proposition 1, the continuous map $f : Y \longrightarrow S^2$ uniquely extends to a continuous map $\overline{f} : C_2(M) \longrightarrow S^2$ up to homotopy relative to the subspace $Y$. (That is, if both $\overline{f}_1$ and $\overline{f}_2$ are the extensions of $f$ to the whole space $C_2(M)$, then there is a homotopy $F : C_2(M) \times [0, 1] \longrightarrow S^2$ such that
\[ F(\xi, 0) = \mathcal{F}_1(\xi), \quad F(\xi, 1) = \mathcal{F}_2(\xi), \text{ for all } \xi \in C_2(M), \text{ and } F(\xi', t) = f(\xi') \text{ for all } \xi' \in Y \text{ and } t \in [0, 1]. \]

Usually, the homotopy class of a map from \( C_2(M) \) to \( S^2 \) cannot give any framing on \( T(M_0) \). But the extension of \( f \) does give a framing on \( T(M_0) \) as shown in Step 2.

**Step 2 The framing determined by the map \( \mathcal{F} \) on \( C_2(M) \)**

The normal bundle of \( \Delta(M) \) in \( M \times M \) can be identified as the tangent bundle \( T(M) \) of \( M \). Consider a suitable compactification of \( C_2(M) \), the spherical bundle \( S(TM) \) become a part of boundary of \( C_2(M) \). Let \( h : S(TM) \longrightarrow S^2 \) denote the restriction of \( \mathcal{F} \) to \( S(TM) \). On the flat part \( N_s \) of \( M \), the spherical bundle \( S(TN_s) = N_s \times S^2 \) and \( h \) on \( S(TN_s) \) is equal to the map restricted from \( f \) which is exactly the projection from \( N_s \times S^2 \) to \( S^2 \). Thus \( h \) induces a map \( h_0 : S(TM_0) \longrightarrow S^2 \).

\( S(TM_0) \) is a \( SO(3) \)-bundle over \( M_0 \).

Can \( h_0 : S(TM_0) \longrightarrow S^2 \) determine uniquely an orthogonal map, that is, a fibrewise orthogonal map? (An orthogonal map is exactly a framing for the vector bundle.) There is also an interesting question that can \( h_0 \) be homotopic to an orthogonal map; if such an orthogonal map exists, is it unique up to homotopy? We shall answer the questions partially.

Choose a framing for \( S(TM_0) \) and we may think \( h_0 \) as a map from \( M_0 \times S^2 \) to \( S^2 \). Let \( y_0 \) denote the point in \( M_0 \) representing the set \( N_s \). Then the restriction of \( h_0 \) to \( y_0 \times S^2 \) is the identity map of \( S^2 \). Thus the restriction of \( h_0 \) to each fibre \( x \times S^2, x \in M_0, \) is also a homotopy equivalence; and hence, \( h_0 \) induces a map \( \hat{h}_0 \) from \( M_0 \) to \( G(3) \), the space of all homotopy equivalences of \( S^2 \) to itself. Choose a base point \( z_0 \) in \( S^2 \), and consider the subspace \( F(3) \) of \( G(3) \) consisting of all the homotopy equivalences which fix the base point \( z_0 \). Then \( F(3) \) is the fibre of the fibration \( G(3) \) over \( S^2 \), it is the key fact for the homotopic computations.
For any two spaces $X_1$ and $X_2$ with base points $x_1$ and $x_2$, respectively, $[X_1, X_2]$ denotes the set of homotopy classes of continuous maps from $X_1$ to $X_2$ and sending $x_1$ to $x_2$. In the following, $M_0$ is with base point $y_0$ representing the set $\mathbb{N}_s$; $SO(3)$, $G(3)$ and $F(3)$ are with the base point the identity of $S^2$. We shall consider only the maps sending the base point to base point and consider only the homotopies which keep the base point fixed.

$M_0$ has the same homology as $S^3$. Usually, we can not expect they also have the same homotopy behavior. But we still have the following proposition.

**Proposition 2** Suppose $\phi : M_0 \rightarrow S^3$ is a degree 1 map. Then the homotopy classes $[M_0, SO(3)]$, $[M_0, G(3)]$, $[M_0, F(3)]$ are all groups, and the group homomorphisms induced by $\phi$,

$$[S^3, SO(3)] \xrightarrow{\phi^\ast} [M_0, SO(3)]$$

$$[S^3, G(3)] \xrightarrow{\phi^\ast} [M_0, G(3)]$$

$$[S^3, F(3)] \xrightarrow{\phi^\ast} [M_0, F(3)]$$

$$[S^3, S^2] \xrightarrow{\phi^\ast} [M_0, S^2]$$

are all isomorphisms of groups. □

There are further relations between these homotopy classes.

**Proposition 3** Let $p : SO(3) \rightarrow G(3)$ and $q : F(3) \rightarrow G(3)$ denote the inclusions. Then, for any integral homology 3-sphere $M_0$, the homomorphism

$$p_* \oplus q_* : [M_0, SO(3)] \oplus [M_0, F(3)] \rightarrow [M_0, G(3)]$$

is an isomorphism.

Especially, when $M_0 = S^3$, we have

$$\pi_3(G(3)) \approx \pi_3(SO(3)) \oplus \pi_3(F(3))$$

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Furthermore, the group isomorphism
\[ q_*^{-1} : [M_0, G(3)]/q_*([M_0, SO(3)]) \longrightarrow [M_0, F(3)] \]
induces a group homomorphism
\[ Q : [M_0, G(3)] \longrightarrow [M_0, F(3)] \cong \mathbb{Z}_2 \ . \]

For a continuous map \( g : M_0 \times S^2 \longrightarrow S^2 \), let \( \tilde{g} \) denote the map from \( M_0 \) to \( G(3) \) defined by \( \tilde{g}(x)(y) = g(x, y) \), for \( x \in M_0 \) and \( y \in S^2 \) and let \( Q(g) = Q([\tilde{g}]) \).

**Theorem 4** A continuous map \( g : M_0 \times S^2 \longrightarrow S^2 \) is homotopic to an orthogonal map, if and only if, \( Q(g) = 0 \) in \([M_0, F(3)]\). ■

Now, \( h_0 \) still denotes the map from \( S(TM_0) \) to \( S^2 \) given by the map \( \overline{f} : C_2(M) \longrightarrow S^2 \). Choose a framing for \( TM_0 \), \( \psi : S(TM_0) \longrightarrow M_0 \times S^2 \), it is a fibre map and fibrewise orthogonal. Then \( h_0 \circ \psi^{-1} \) is a map from \( M_0 \times S^2 \) to \( S^2 \) and the value \( Q(h_0 \circ \psi^{-1}) \) is independent of the choice of the framing \( \psi \). Therefore, \( Q(h_0 \circ \psi^{-1}) \) is an invariant of the integral homology 3-sphere \( \overline{M} \), it is the obstruction for \( h_0 \) to be homotopic to an orthogonal map. We hope that this is not really an obstruction.

**Conjecture 5** \( Q(h_0 \circ \psi^{-1}) = 0 \), for any integral homology 3-sphere \( \overline{M} \). ■

On the other hand, the group isomorphism
\[ p_*^{-1} : [M_0, G(3)]/p_*([M_0, F(3)]) \longrightarrow [M_0, SO(3)] \]
induces a group homomorphism
\[ P : [M_0, G(3)] \longrightarrow [M_0, SO(3)] \ . \]
For a continuous map \( g : M_0 \times S^2 \to S^2 \), let \( P(g) = P([\hat{g}]) \).

For the map \( h_0 \) and the corresponding element \( P(h_0 \circ \psi^{-1}) \) in \([M_0, SO(3)]\), choose an orthogonal map \( g_0 : M_0 \times S^2 \to S^2 \) such that the associated map \( \hat{g}_0 \) is in the homotopy class \( P(h_0 \circ \psi^{-1}) \). Then we get an orthogonal map \( g_0 \circ \psi : S(TM_0) \to S^2 \) which represents a homotopy class of framings determined by \( h_0 \), also by the map \( \overline{f} : C_2(M) \to S^2 \). This framing can also be characterized by the following theorem.

**Theorem 6** There exists a framing \( \psi_0 : S(TM_0) \to M_0 \times S^2 \) unique up to homotopy such that \( P(h_0 \circ \psi_0^{-1}) = 0 \).

**Proofs**

**Outline of Proof of Proposition 1**

\( N_s \) is a subset of \( \mathbb{R}^3 \times \{0\} \). In \( N_s \), we choose a subspace \( S_3 \) which is a deformation retract of \( N_s \) and a point \( x_1 \) in the bounded component of \( \mathbb{R}^3 \times \{0\} - S_3 \). Let \( S = \{x_1\} \times S_3 \), it is a subspace of \( Y \). We show that the three maps, the inclusion of \( S \) in \( Y \), the restriction of \( f \) to \( S \), and the restriction of \( j \) to \( S \), all induce isomorphisms of homology groups of the corresponding spaces. That is, \( H_*(S) \to H_*(Y) \), \( (f|_S)_* : H_*(S) \to H_*(S) \), and \( (j|_S)_* : H_*(S) \to H_*(C_2(M)) \) all are isomorphisms.

**Proof of Proposition 1**

First we compute the homology of \( Y_0, Y_1, Y_2 \), separately.

\[ Y_0 = C_2(N_s) = N_s \times N_s - \Delta(N_s) \subset N_s \times N_s. \] \( N_s \) is homeomorphic to \( S^2 \times (s, \infty) \). Thus \( H_*(N_s \times N_s) \approx H_*(S^2 \times S^2) \). By Thom Isomorphism, \( H_*(N_s \times N_s, Y_0) \approx H_{i-3}(N_s) \).

Now, we use the long exact sequence of the pair \( (N_s \times N_s, Y_0) \) to determine \( H_*(Y_0) \).
\[ \rightarrow H_{i+1}(N_s \times N_s, Y_0) \xrightarrow{\partial_*} H_i(Y_0) \rightarrow H_i(N_s \times N_s) \rightarrow \]
\[ \rightarrow H_i(N_s \times N_s, Y_0) \rightarrow \cdots \]

When \(i\) is odd, both \(H_{i+1}(N_s \times N_s, Y_0)\) and \(H_i(N_s \times N_s)\) are the trivial group \(\{0\}\). Thus we have

\[ H_4(Y_0) \approx H_4(N_s \times N_s) \oplus \partial_*(H_5(N_s \times N_s, Y_0)) \approx \mathbb{Z} \oplus \mathbb{Z} \]
\[ H_2(Y_0) \approx H_2(N_s \times N_s) \oplus \partial_*(H_3(N_s \times N_s, Y_0)) \approx \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \]

and \(H_i(Y_0)\) is trivial, if \(i\) is odd.

(\(\mathbb{Z}\) denotes the group of integers.)

To find the generators of \(H_4\) and \(H_2\) of \(Y_0\), we choose three 2-spheres \(S_1, S_2, S_3\) in \(\mathbb{R}^3 \times \{0\}\) of radius \(2s, 4s, 6s\), respectively, all with center the origin. (\(S_i\) is the boundary of \(N_{2s \times i}\), \(i = 1, 2, 3\).) For each \(i\), \(i = 1, 2, 3\), choose a point \(x_i\) in \(S_i\). The 2-spheres are also oriented in the same way, that is, the natural diffeomorphisms of the 2-spheres are orientation-preserving. Then \(S_i \times x_j\) and \(x_j \times S_i, 1 \leq i \neq j \leq 3\), are 2-cycles in \(Y_0\), also in \(N_s \times N_s\); \(S_i \times S_j, 1 \leq i \neq j \leq 3\), are 4-cycles in \(Y_0\), also in \(N_s \times N_s\).

In the following, if \(c\) is a cycle in \(Y_0\), \([c]\) shall denote the corresponding homology class in \(Y_0\).

**Lemma 7**

(i) \([S_1 \times S_3]\) is the generator of \(H_4(N_s \times N_s)\).

(ii) \([(S_1 - S_3) \times S_2]\) is the generator of the subgroup \(\partial_*(H_5(N_s \times N_s, Y_0))\) in \(H_4(Y_0)\).

We use the lemma to prove Proposition 1, and prove the lemma later.

There are some relations between these classes in \(H_4(Y_0)\):

\[ [(S_1 - S_3) \times S_2] = [S_1 \times S_2] - [S_3 \times S_2], \quad [(S_1 \times S_2) = [S_1 \times S_3] \]
\[ \text{and} \quad [S_3 \times S_2] = [S_3 \times S_1]. \]

Thus \([S_1 \times S_3]\) and \([S_3 \times S_1]\) form the basis of \(H_4(Y_0)\).
Similarly, \([S_1 \times S_3, x_1 \times S_3]\] are the basis of \(H_2(N_s \times N_s)\); 
\([(S_1 - S_3) \times x_2] = \epsilon_0[x_2 \times (S_1 - S_3)]\), \(\epsilon_0\) is 1 or \(-1\) is the generator of the subgroup \(\partial_*(H_3(N_s \times N_s, Y_0))\) in \(H_2(Y_0)\). 

Thus \([S_1 \times x_3, x_1 \times S_3]\) and \([(S_1 - S_3) \times x_2]\) form a basis of \(H_2(Y_0)\).

Now we study the homology of \(Y_1\) and \(Y_2\). 
It is easy to see that the inclusion of \(N_4s \times M_3s\) in \(Y_1\) and the inclusion of \(S_3 \times M_3s\) in \(N_4s \times M_3s\) both are homotopy equivalences. Thus \(H_*(Y_1) \approx H_*(S_3 \times M_3s) \approx H_*(S_3)\). (Recall: \(M_r\) is acyclic, for any \(r \geq s\).) Similarly, \(Y_2\) also has the same homology as 2-sphere.

\(Y_1\) and \(Y_2\) are disjoint, and hence the homology of their union \(Y_1 \cup Y_2\) is also determined. We can use the Mayer-Vietoris Sequence of the triple \((Y, Y_0, Y_1 \cup Y_2)\) to find the homology of \(Y\). In fact, we have

(i) The cycle \(S_1 \times S_3\) is contained in \(Y_2\) and is killed in \(Y_2\).

(ii) The cycle \(S_3 \times S_1\) is contained in \(Y_1\) and is killed in \(Y_1\).

(iii) The cycle \(x_3 \times S_1\) is contained in \(Y_1\) and is killed in \(Y_1\).

(iv) The cycle \(S_1 \times x_3\) is contained in \(Y_2\) and is killed in \(Y_2\).

Therefore, \(H_4(Y) = \{0\}\) and in \(H_2(Y)\), we have \([x_1 \times S_3, S_3 \times x_2]\) left; the equality \([(S_1 - S_3) \times x_2] = \epsilon_0[x_2 \times (S_1 - S_3)]\) become the new equality \(-[S_3 \times x_2] = -\epsilon_0[x_2 \times S_3]\). Thus \([x_1 \times S_3] = [x_2 \times S_3] = \epsilon_0[S_3 \times x_2] = \epsilon_0[S_3 \times x_1]\]. This proves that \(H_*(Y) \approx H_*(S^2)\). Actually, we know more than that: the inclusion of the space \(\{x_1\} \times S_3\) in \(Y\) induces isomorphisms of the homology groups. It is easy to see that the map \(f\), restricted to \(\{x_1\} \times S_3\), is an homotopy equivalence from \(\{x_1\} \times S_3\) to \(S^2\). This proves the second statement that \(f_*\) is an isomorphism.

To prove the third statement that \(j_*\) is an isomorphism, it is also enough to show that the restriction of \(j\) to \(\{x_1\} \times S_3\) induces isomorphisms for the
homology groups. Similar to the computation of the homology of $C_2(N_s)$, we consider the long exact sequence of pair $(M \times M, C_2(M))$

$$H_{i+1}(M \times M) \rightarrow H_{i+1}(M \times M, C_2(M)) \xrightarrow{\partial_*} H_i(C_2(M)) \rightarrow H_i(M \times M).$$

Because $H_*(M)$ is acyclic, $H_*(M \times M)$ is also acyclic.

We have

$$H_i(C_2(M)) \approx H_{i+1}(M \times M, C_2(M)), \text{ for all } i \geq 1.$$

But $H_{i+1}(M \times M, C_2(M)) \approx H_{i-2}(M)$, by the Thom Isomorphism. Thus $C_2(M)$ has the same homology as 2-sphere.

And it is easy to see that the inclusion of $\{x_1\} \times (M, M - x_1)$ in $(M \times M, C_2(M))$ induces isomorphisms of homology groups, and hence, the inclusion of $\{x_1\} \times (M - x_1)$ in $C_2(M)$ also induces isomorphisms of homology groups. The cycle $\{x_1\} \times S_3$ is a generator of $H_2(\{x_1\} \times (M - x_1))$, and hence also a generator of $H_2(C_2(M))$. This proves the third statement that $j_*$ is an isomorphism.

(i) of Lemma 7 is obvious. Now, we are going to prove (ii) in Lemma 7.
Consider the following commutative diagram

\[
\begin{array}{c}
H_2(S^2) \otimes H_3(N_s, N_s - S^2) \xrightarrow{id \otimes \partial_*} H_2(S^2) \otimes H_2(N_s - S^2) \\
\downarrow \tau_1 \hspace{2cm} \downarrow \eta_1 \\
H_5(S^2 \times N_s, S^2 \times N_s - S_2 \times S_2) \xrightarrow{\partial_*} H_4(S^2 \times N_s - S_2 \times S_2) \\
\downarrow \tau_2 \hspace{2cm} \downarrow \eta_2 \\
H_5(S^2 \times N_s, S^2 \times N_s - \Delta(S_2)) \xrightarrow{\partial_*} H_4(S^2 \times N_s - \Delta(S_2)) \\
\downarrow \tau_3 \hspace{2cm} \downarrow \eta_3 \\
H_5(N_s \times N_s, Y_0) \xrightarrow{\partial_*} H_4(Y_0)
\end{array}
\]

The maps \(\tau_1\) and \(\eta_1\) are isomorphisms from Kunneth formula. Other homomorphisms are induced by the corresponding inclusion maps. \(\tau_2\) is an isomorphism by the result of Lefschetz Duality in the 5-dimensional manifold \(S^2 \times N_s\); \(\tau_3\) is an isomorphism by the result of Thom Isomorphism Theorem. Precisely, consider the following commutative diagram

\[
\begin{array}{c}
H_5(S^2 \times N_s, S^2 \times N_s - S_2 \times S_2) \xrightarrow{\sigma_1} H^0(S^2 \times S_2) \\
\downarrow \tau_2 \hspace{2cm} \downarrow \tau_4 \\
H_5(S^2 \times N_s, S^2 \times N_s - \Delta(S_2)) \xrightarrow{\sigma_2} H^0(\Delta(S_2))
\end{array}
\]

where \(\sigma_i, i = 1, 2\), are the isomorphisms of Lefschetz Duality, \(\tau_4\) is the homomorphism induced by the inclusion.

Because \(\tau_4\) is an isomorphism, \(\tau_2\) is also an isomorphism. The proof of isomorphism of \(\tau_3\) is in some sense analogous to that for \(\tau_2\), we omit it.

From the long exact sequence of the pair \((N_s, N_s - S_2)\), it is easy to see that \([S_1 - S_3]\) is the generator of \(\partial_*(H_3(N_s, N_s - S_2))\), and hence, \([S_2 \times (S_1 - S_3)]\)
is the generator of $(id \otimes \partial_*)(H_2(S_2) \otimes H_3(N_s, N_s - S_2))$. By the commutativity of the above diagram, $[S_2 \times (S_1 - S_3)] = -[(S_1 - S_3) \times S_2]$ is the generator of $\partial_*(H_3(N_s \times N_s, Y_0))$. This proves Lemma 7 and completes the long proof of Proposition 1.

**Proof of Proposition 2**

We need to show the isomorphisms between $[M_0, X]$ and $[S^3, X]$, for $X = SO(3), G(3), F(3)$ and $S^2$.

For the case of $SO(3)$, we consider the classifying space $BSO(3)$ of the $SO(3)$-bundles. Then

$$[M_0, SO(3)] \approx [SM_0, BSO(3)] \text{ and } [S^3, SO(3)] \approx [S^4, BSO(3)] ,$$

where $SM_0$ is the suspension of $M_0$. On the other hand, because $SM_0$ is simply connected and the map $S(\phi) : SM_0 \rightarrow SS^3(= S^4)$ induces isomorphisms of homology groups, $S(\phi)$ is a homotopy equivalence. Thus $S(\phi)^* : [SM_0, BSO(3)] \rightarrow [S^4, BSO(3)]$ is isomorphic, and hence,

$$[M_0, SO(3)] \approx [S^3, SO(3)] .$$

For the cases of $G(3)$ and $F(3)$, we may also consider the corresponding classifying spaces, by the result of Fuchs [2]; and the proof is completely similar.

The group property of the associated homotopy classes is a result of Dold and Lashof [1]; for the convenience of interested reader, we give a proof in the appendix.

For the case of $S^2$, it is enough to note that $[M_0, S^2] \approx [M_0, S^3]$ ( $\approx H^3(M_0)$ ), which implies the isomorphism we need.

**Proof of Proposition 3**

By Proposition 2, it is enough to prove the result for the case that $M_0 = S^3$. 

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Consider the commutative diagram of fibrations over $S^2$

\[
\begin{array}{ccc}
S^1 & \longrightarrow & SO(3) & \longrightarrow & S^2 \\
\downarrow & & \downarrow p & & \downarrow id \\
F(3) & \longrightarrow & G(3) & \longrightarrow & S^2
\end{array}
\]

and the associated commutative diagram of exact sequences of homotopy groups

\[
\begin{array}{ccc}
\pi_i(S^1) & \longrightarrow & \pi_i(SO(3)) & \longrightarrow & \pi_i(S^2) \\
\downarrow & & \downarrow p_* & & \downarrow id \\
\pi_i(F(3)) & \longrightarrow & \pi_i(G(3)) & \longrightarrow & \pi_i(S^2)
\end{array}
\]

For $i \geq 3$, $\pi_i(S^1) = \pi_{i-1}(S^1) = \{0\}$, and hence

\[\pi_i(SO(3)) \approx \pi_i(S^2).\]

Thus $p_* : \pi_i(SO(3)) \longrightarrow \pi_i(G(3))$ can be thought as the right-inverse of $\alpha : \pi_i(G(3)) \longrightarrow \pi_i(S^2)$. This implies that $\alpha$ is an epimorphism, $q_*$ is a monomorphism, and $p_*$ supplies the necessary homomorphism for splitting. Therefore,

\[\pi_i(G(3)) = q_*(\pi_i(F(3)) \oplus p_*(\pi_i(SO(3))), \text{ for all } i \geq 3\]
Appendix

The proof of the appendix is essentially from the proof of the main result in Dold and Lashof [1]. The author just write it for self-interesting.

Suppose $H$ is a path-connected space and has an associative multiplication which has a two-sided unit $e$. For $h_1, h_2 \in H$, $h_1 \cdot h_2$ denotes the product of $h_1$ and $h_2$. Thus $h \cdot e = e \cdot h = h$, for all $h \in H$. Furthermore, assume $X$ is a polyhedron. The purpose of this appendix is to show that the homotopy classes in $[X,H]$ form a group under the following multiplication:

For any two maps $f, g : X \to H$, $(f \cdot g)(x) = f(x) \cdot g(x)$.

The associative law of this multiplication in $[X,H]$ is obvious. It is enough to show that for any $f : X \to H$, there is a map $g : X \to H$ such that $f \cdot g$ is homotopic to the constant map $e : X \to H$, $e(x) = e$, for all $x \in X$.

We shall construct the map $g : X \to H$ and the homotopy $D : X \times I \to H$ satisfying $D(x,0) = e$, $D(x,1) = f(x) \cdot g(x)$, inductively on the skeleton of $X$. ($I$ is the unit interval $[0,1]$.)

$X^{(k)}$ denotes the $k$-skeleton of $X$.

Assume $g$ is defined on $X^{(k)}$ and $D$ is defined on $X^{(k)} \times I$ such that $D(x,0) = e$ and $D(x,1) = f(x) \cdot g(x)$, for all $x \in X^{(k)}$. If necessary, we may ask that the base point $x_0$ of $X$ is in $X^{(0)}$ and $f(x_0) = g(x_0) = D(x_0, t) = e$, for any $t \in I$.

For any $(k+1)$-simplex $\Delta$ in $X^{(k+1)}$, we want to extend $g$ to the part $\Delta$ and $D$ to the part $\Delta \times I$. Let $S$ denote the boundary of $\Delta$, it is a $k$-sphere. $S$ is in $X^{(k)}$, $g$ is defined on $S$ and $D$ is defined on $S \times I$.

Claim $g|_S : S \to H$ is null-homotopic.

Proof $\Delta$ is a simplex, there is a contraction map $\gamma : \Delta \times I \to \Delta$, $\gamma(x,0) = x$ and $\gamma(x,1) = x_1$, for all $x \in \Delta$. $x_1$ is some fixed point in $S$. Let $\beta : S \times I \to H$ denote the map $\beta(x,t) = f(\gamma(x,t)) \cdot g(x)$, for $x \in S$. Let
$y_1 = f(x_1)$ and $\overline{y}_1 : S \rightarrow H$ denote the constant map sending the points of $S$ to $y_1$. Then $\beta$ is a homotopy between $f \cdot g$ and $\overline{y}_1 \cdot g$ on $S$. $H$ is path-connected, $\overline{y}_1 \cdot g$ is homotopic to $\overline{\tau} \cdot g = g$. Thus $g$ is homotopic to $f \cdot g$ on $S$. On the other hand, the restriction of $D$ to $S \times I$ provides a homotopy between the restrictions of $f \cdot g$ and $\overline{\tau}$. This proves that $g|_S$ is null-homotopic.

Therefore, we can extend $g|_S$ to the part $\Delta$, say, $g' : \Delta \rightarrow H$, and we can also extend $D|_{S \times I}$ to the whole boundary of $\Delta \times I$ as follows:

We use $D' : \partial(\Delta \times I) \rightarrow H$ to denote the extension. $\partial(\Delta \times I) = \Delta \times \{0\} \cup \Delta \times \{1\} \cup S \times I$.

$D'(x,0) = e$ and $D'(x,1) = f(x) \cdot g'(x)$, for all $x \in \Delta$;

$D'(y,t) = D(y,t)$, for all $y \in S$ and $t \in I$.

The map $D'$ may not be extended to $\Delta \times I$. We shall find a map $g_1 : \Delta \rightarrow H$ with $g_1|_S = \overline{e}|_S$ and modify the map $D'$ by multiplying $D'$ with $g_1$ on the part $\Delta \times \{1\}$ such that the new map is null-homotopic. Precisely, let $D'' : \partial(\Delta \times I) \rightarrow H$ denote the map, $D''(\xi) = D'(\xi)$, for all $\xi \in \Delta \times \{0\} \cup S \times I$, $D''(x,1) = D'(x,1) \cdot g_1(x)$, for all $(x,1) \in \Delta \times \{1\}$.

We may think the map $g_1$ as a map on $\Delta \times \{1\}$ and extend it trivially to the whole boundary $\partial(\Delta \times I)$, that is, sending all points undefined to $e$. Then $D''$ is just equal to $D' \cdot g_1$. To let $D''$ be null-homotopic, we can choose $g_1$ such that $[g_1] = [D']^{-1}$ in $\pi_{k+1}(H)$. Of course, $g'$ should be changed to the new map $g' \cdot g_1$. Therefore, $D''$ is null-homotopic and its extension to $\Delta \times I$ also gives the homotopy between $f \cdot (g' \cdot g_1)$ on $\Delta$. This finishes the extension of $g$ and $D$ to $\Delta$.

References

[1] A. Dold and R. Lashof, Principal quasi-fibrations and fibre homotopy equivalence of bundles, Illi. J. Math. 3 (1959), 285-305.

[2] M. Fuchs, Verallgemeinerte Homotopie-Homomorphismen und klassi-
