FRACTIONAL SOBOLEV SPACES WITH POWER WEIGHTS

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Abstract. We investigate the form of the closure of the smooth, compactly supported functions $C^\infty_0(\Omega)$ in the weighted fractional Sobolev space $W^{s,p;w,v}(\Omega)$ for bounded $\Omega$. We focus on the weights $w$, $v$ being powers of the distance to the boundary of the domain. Our results depend on the lower and upper Assouad codimension of the boundary of $\Omega$. For such weights we also prove the comparability between the full weighted fractional Gagliardo seminorm and the truncated one.

1. Introduction and preliminaries

Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $0 < s < 1$ and $1 \leq p < \infty$. We recall that the fractional Sobolev space is defined as

$$W^s,p(\Omega) = \left\{ f \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dy \, dx < \infty \right\}.$$ 

This is a Banach space endowed with the norm

$$\|f\|_{W^s,p(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^s,p(\Omega)},$$

where $[f]_{W^s,p(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dy \, dx \right)^{1/p}$ is called the Gagliardo seminorm.

In this paper we consider weighted fractional Sobolev spaces. For weights $w, v$ (i.e. measurable nonnegative functions on $\Omega$) we define the weighted Gagliardo seminorm as

$$[f]_{W^s,p;w,v(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} w(y)v(x) \, dy \, dx \right)^{1/p}$$

and the weighted fractional Sobolev space as

$$W^{s,p;w,v}(\Omega) = \left\{ f \in L^p(\Omega) : [f]_{W^{s,p;w,v}(\Omega)} < \infty \right\}.$$

For bounded $\Omega$ the space defined above is always nonempty, because it contains constant functions. Moreover, if $w_\alpha(x) = \text{dist}(x, \partial \Omega)^{-\alpha}$ and $v_\beta(y) = \text{dist}(y, \partial \Omega)^{-\beta}$ for $\alpha, \beta \in \mathbb{R}$, we denote

$$W^{s,p;w_\alpha,v_\beta}(\Omega) = W^{s,p;\alpha,\beta}(\Omega).$$

The space $W^{s,p;w,v}(\Omega)$ is equipped with the natural norm

$$\|f\|_{W^{s,p;w,v}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{s,p;w,v}(\Omega)}.$$ 

We remark here that all results of the paper remain true if we replace the space $L^p(\Omega)$ appearing in the definition of $W^{s,p;w,v}(\Omega)$ by the weighted analogue $L^p(\Omega,W)$ for any almost everywhere positive weight $W$, which is locally comparable to a constant (see Definition 18) or continuous and satisfies $\int_{\Omega} W(x) \, dx < \infty$. Notice that the last condition ensures that the constant function $1_\Omega$ is in $L^p(\Omega,W)$. However, for simplicity we consider only the unweighted case.

For an open set $\Omega$ we use the notation $d_\Omega(x) = \text{dist}(x, \partial \Omega)$.

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Definition 1. By $W_0^{s,p;w,v}(\Omega)$ we denote the closure of $C_c^\infty(\Omega) \cap W^{s,p;w,v}(\Omega)$ (smooth functions with compact support in $\Omega$) in $W^{s,p;w,v}(\Omega)$ with respect to the weighted fractional Sobolev norm and by $W_0^{s,p;w,v}(\Omega)$ we denote the closure of all compactly supported, measurable functions in $\Omega$ (not necessarily smooth) in $W^{s,p;w,v}(\Omega)$ with respect to the weighted fractional Sobolev norm. We also denote $W_0^{s,p;\alpha,\beta}(\Omega)$, $W_c^{s,p;\alpha,\beta}(\Omega)$.

We refer to Section 3 for a discussion on the cases when $C_c^\infty(\Omega)$ is or is not a subset of $W_0^{s,p;\alpha,\beta}(\Omega)$. In general, it may occur that the space $W_0^{s,p;\alpha,\beta}(\Omega)$ is empty.

The main result of this paper is a generalization of the density result for unweighted fractional Sobolev spaces, which can be found in [9, Theorem 2]. We present some necessary and sufficient conditions for the space $C_c^\infty(\Omega)$ to be dense in $W^{s,p;\alpha,\beta}(\Omega)$. In the negative case, under some additional assumptions we also find explicitly the form of the space $W_0^{s,p;\alpha,\beta}(\Omega)$. The necessary geometrical and technical definitions are contained in Section 2. In Section 3 we present Lemmas, most of them being generalization of these from [9] and [10] for the weighted case.

Let us remark that the weighted fractional Sobolev spaces related to the weighted Sobolev-type norm $\| \cdot \|_{W^{s,p;\alpha,\beta}(\Omega)}$ and the problem of density of $C_c^\infty(\Omega)$ were investigated before by Díaz and Valdinoci in [6] for the case $\Omega = \mathbb{R}^d \setminus \{0\}$, $\alpha = \beta \in [0,(d - sp)/2)$, $p^* = dp/(d - sp)$ and $W(x) = |x|^{\frac{2d}{d - sp}}$. However, this problem is not directly comparable to ours, because we consider only bounded sets $\Omega$. Similar weighted fractional Sobolev spaces were an object of study in [1] in connection with weighted Caffarelli–Kohn–Nirenberg and fractional Hardy inequalities. Moreover, related results for unweighted Sobolev-type spaces can be found for example in [12], where the authors considered spaces of functions vanishing on the boundary of cones, by Prats and Saksman [23] in a more general context of Triebel–Lizorkin spaces and generalized later by Rutkowski [24] for the kernels of the form $|x - y|^{-d}\varphi((x - y))^{-\ell}$, with $\varphi$ satisfying certain technical assumptions. Some versions of the reduction of the integration theorems can also be found in [4], [5] and [16]. We want to point out that a variant of comparability is nonexplicitly contained in the early work of Seeger [25]. We prove a weighted analogue of the reduction of the integration theorem for the space $W^{s,p;\alpha,\beta}(\Omega)$, provided that $0 \leq \alpha, \beta < \text{co dim}_A(\partial \Omega)$. This result is stated below.

Theorem 2. Let $\Omega$ be a nonempty, bounded, uniform domain, let $0 < s < 1$ and $1 \leq p < \infty$. Moreover, let $0 < \theta \leq 1$. Suppose that $0 \leq \alpha, \beta < \text{co dim}_A(\partial \Omega)$. Then the full seminorm $[f]_{W^{s,p;\alpha,\beta}(\Omega)}$ and the truncated seminorm

$$\left( \int_{\Omega} \int_{B(x,\theta d_\Omega(x))} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha} dy dx \right)^{\frac{1}{p}}$$

are comparable, that is there exists a constant $C = C(\theta, d, s, p, \alpha, \beta, \Omega) > 0$ such that

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d_\Omega(y)^{\beta} d_\Omega(x)^{\alpha} \leq C \int_{\Omega} \int_{B(x,\theta d_\Omega(x))} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d_\Omega(y)^{\beta} d_\Omega(x)^{\alpha} dy dx,$$

for all $f \in L_{loc}^{\infty}(\Omega)$. 
It is clear that the reverse inequality is trivial with constant equal to one, hence we indeed obtain the comparability between the full and the truncated weighted Gagliardo seminorms. Moreover, when \( p = 1 \), the comparability can be formulated in a more general setting, for all \( A_1 \) class Muckenhoupt weights, see Theorem 23.

Section 5 contains proofs of our main results, Theorems 3 and 4. Theorem 3 is a generalization of [9, Theorem 2] and Theorem 4 is a generalization of [9, Theorem 3], provided that \( \Omega \) is a uniform domain.

**Theorem 3.** Let \( \Omega \subset \mathbb{R}^d \) be a nonempty, bounded, open set, let \( 0 < s < 1, 1 \leq p < \infty \) and \( \alpha, \beta \geq 0 \).

(I) If \( sp + \alpha + \beta < d - \operatorname{co dim}_A(\partial \Omega) \), then \( W^{s,p; \alpha, \beta}_0(\Omega) = W^{s,p; \alpha, \beta}(\Omega) \).

(II) If \( \Omega \) is \( (d - sp - \alpha - \beta) \)-homogeneous, \( p > 1 \) and \( sp + \alpha + \beta = \operatorname{co dim}_A(\partial \Omega) \), then \( W^{s,p; \alpha, \beta}_0(\Omega) = W^{s,p; \alpha, \beta}(\Omega) \).

(III) If \( \Omega \) is \( k \)-plump and \( sp + \alpha + \beta > \operatorname{co dim}_A(\partial \Omega) \), then \( W^{s,p; \alpha, \beta}_0(\Omega) \neq W^{s,p; \alpha, \beta}(\Omega) \).

**Theorem 4.** Let \( \Omega \subset \mathbb{R}^d \) be a nonempty, bounded, uniform and open set, let \( 0 < s < 1, 1 \leq p < \infty \) and \( 0 \leq \alpha, \beta \leq \operatorname{co dim}_A(\partial \Omega) \). If \( sp + \alpha + \beta > \operatorname{co dim}_A(\partial \Omega) \), then

\[
W^{s,p; \alpha, \beta}_0(\Omega) = \left\{ f \in W^{s,p; \alpha, \beta}(\Omega) : \int_{\Omega} \frac{|f(x)|^p}{d_{\Omega}(x)^{sp+\alpha+\beta}} \, dx < \infty \right\}.
\]

Theorem 4 reveals the property known partially also for classical (unweighted) Sobolev spaces \( W^{1,p}(\Omega) \), see [18, Example 9.12] or [17].

**Remark 5.** In the proof of the case II in the Theorem 3 we use a reflexivity property of the space \( W^{s,p; \alpha, \beta}(\Omega) \) (see Proposition 25). This explains why \( p = 1 \) is excluded from the assumptions. It is not clear if the density property holds in this case and we leave it as an open problem.

To prove the case (III) of the Theorem 3, we use a (weak) weighted fractional Hardy inequality, which can be easily derived from the (weak) fractional \( (s,p,a) \)-Hardy inequality, given in [11, Corollary 3] and also in [9, Theorem 5] in the case \( (T') \) of the result below. It suffices to take the function \( \phi(x) = x^{sp+\alpha+\beta} \) and notice that \( \operatorname{dist}(y, \partial \Omega) \lesssim \operatorname{dist}(x, \partial \Omega) \) on the ball \( B(x, R \operatorname{dist}(x, \partial \Omega)) \). We present this version below.

**Theorem 6.** ([11, Corollary 3], [9, Theorem 5]) Let \( 0 < p < \infty, 0 < s < 1 \) and \( \alpha, \beta \geq 0 \). Suppose that \( \Omega \neq \emptyset \) is an open, \( k \)-plump set so that either condition (T), or condition \( (T') \), or condition \( (F) \) holds

(\( T \)) \( sp + \alpha + \beta < \operatorname{co dim}_A(\partial \Omega) \), \( \Omega \) is unbounded and \( \xi = 0 \),

(\( T' \)) \( sp + \alpha + \beta < \operatorname{co dim}_A(\partial \Omega) \), \( \Omega \) is bounded and \( \xi = 1 \),

(\( F \)) \( sp + \alpha + \beta > \operatorname{co dim}_A(\partial \Omega) \), \( \Omega \) is bounded or \( \partial \Omega \) is unbounded and \( \xi = 0 \).

Then there exist constants \( c \) and \( R \) such that the following inequality

\[
(1) \quad \int_{\Omega} \frac{|u(x)|^p}{d_{\Omega}(x)^{sp+\alpha+\beta}} \, dx \leq c \int_{\Omega} \int_{\Omega \cap B(x, R d_{\Omega}(x))} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \frac{dy}{d_{\Omega}(y)^{\beta}} \frac{dx}{d_{\Omega}(x)^{\alpha}} + c \|u\|_{L^p(\Omega)}^p,
\]

holds for all measurable functions \( u \) for which the left-hand side is finite.

As an easy corollary in the case \( (T') \), deriving directly from Theorem 3 and Theorem 6, we obtain the embedding \( W^{s,p; \alpha, \beta}(\Omega) \subset L^p(\Omega, \operatorname{dist}(\cdot, \partial \Omega)^{-sp-\alpha-\beta}) \).

**Theorem 7.** Let \( 1 \leq p < \infty \) and \( 0 < s < 1 \). Suppose that \( \Omega \neq \emptyset \) is an open, uniform, bounded set such that \( 0 \leq \alpha, \beta < \operatorname{co dim}_A(\partial \Omega) \) and \( sp + \alpha + \beta < \operatorname{co dim}_A(\partial \Omega) \). Then there
exists a constant $c$ such that
\[ \int_{\Omega} \left| f(x) \right|^p \, d\Omega(x)^{s+p+\alpha+\beta} \, dx \leq c \| f \|^p_{W^{s,p,\alpha,\beta}(\Omega)} < \infty, \]
for all $f \in W^{s,p,\alpha,\beta}(\Omega)$.

Theorem 7 is a generalization of the unweighted case from [9, Theorem 4], provided that $\Omega$ is a uniform domain.

**Notation.** Having two nonnegative functions $A$ and $B$ we use a symbol $\lesssim$ if there exists a constant $c > 0$ such that $A \leq cB$. The constant $c$ usually depends on some parameters, like $\alpha, \beta, d, s, p, \Omega$, but not on the arguments of the functions $A, B$ and the set of these parameters arises from context. Moreover, we write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

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## 2. Definitions

We will use the same definitions as in [9, Section 2]; for Reader’s convenience we repeat them below.

### 2.1. Assouad and Minkowski dimensions

Recall that we denote the distance from $x \in \mathbb{R}^d$ to a set $E \subset \mathbb{R}^d$ by $\text{dist}(x, E) = \inf_{y \in E} |x - y|$.

**Definition 8.** Let $r > 0$. For open sets $\Omega \subset \mathbb{R}^d$ we define the **inner tubular neighbourhood** of $\Omega$ as
\[ \Omega_r = \{ x \in \Omega : d_\Omega(x) \leq r \}, \]
and for arbitrary sets $E \subset \mathbb{R}^d$ we define the **tubular neighbourhood** of $E$ as
\[ \tilde{E}_r = \{ x \in \mathbb{R}^d : \text{dist}(x, E) \leq r \}. \]

**Definition 9.** [15, Section 3] Let $E \subset \mathbb{R}^d$. The **lower Assouad codimension** $\co \dim_A(E)$ is defined as the supremum of all $q \geq 0$, for which there exists a constant $C = C(q) \geq 1$ such that for all $x \in E$ and $0 < r < R < \text{diam} E$ it holds
\[ \frac{|\tilde{E}_r \cap B(x, R)|}{|B(x, R)|} \leq C \left( \frac{r}{R} \right)^q. \]

Conversely, the **upper Assouad codimension** $\co \dim_A(E)$ is defined as the infimum of all $s \geq 0$, for which there exists a constant $c = c(s) > 0$ such that for all $x \in E$ and $0 < r < R < \text{diam} E$ it holds
\[ \frac{|\tilde{E}_r \cap B(x, R)|}{|B(x, R)|} \geq c \left( \frac{r}{R} \right)^s. \]

We remark that having strict inequality $R < \text{diam} E$ above makes the definitions applicable also for unbounded sets $E$; for bounded sets $E$ we could have $R \leq \text{diam} E$.

In Euclidean space $\mathbb{R}^d$ (more general - in Ahlfors $d$-regular measure metric spaces) it holds
\[ \dim_A(E) + \co \dim_A(E) = \dim_A(E) + \co \dim_A(E) = d, \]
where $\dim_A(E)$ and $\dim_A(E)$ denote respectively the well known lower and upper Assouad dimension – see for example [15, Section 2]. Moreover, if $\co \dim_A(E) = \co \dim_A(E)$, we simply denote both of these values by $\co \dim_A(E)$.

We recall a notion of $\sigma$-homogenity, coming from [20, Theorem A.12].
Definition 10. Let $E \subset \mathbb{R}^d$ and let $V(E, x, \lambda, r) = \{ y \in \mathbb{R}^d : \text{dist}(y, E) \leq r, |x-y| \leq \lambda r \}$. We say that $E$ is $\sigma$-homogeneous, if there exists a constant $L$ such that
\[
|V(E, x, \lambda, r)| \leq Lr^d \lambda^\sigma
\]
for all $x \in E$, $\lambda \geq 1$ and $r > 0$.

If $0 < r < R < \text{diam}(E)$, then taking $\lambda = R/r$ in the definition gives
\[
\left| \tilde{E}_r \cap B(x, R) \right| = \left| V\left( E, x, \frac{R}{r}, r \right) \right| \leq C |B(x, R)| \left( \frac{r}{R} \right)^{d-\sigma},
\]
where $C = C(d, E)$ is a constant. This means that if $\text{co dim}_A(E) = s$, then $(d-s)$-homogeneous sets are precisely these sets $E$, for which the supremum in the definition of the lower Assouad codimension is attained. For the definition of the concept of homogeneity from a different point of view the Reader may also see [20, Definition 3.2].

Definition 11. The upper Minkowski dimension of a set $E \subset \mathbb{R}^d$ is defined as
\[
\overline{\text{dim}}_M(E) = \inf \{ s \geq 0 : \limsup_{r \to 0} \left| \tilde{E}_r \right| r^{d-s} = 0 \},
\]
see for example [14, Section 2].

It is not hard to see that $\text{co dim}_A(E) \leq d - \overline{\text{dim}}_M(E)$ and the equality holds if $E$ is $(d - \text{co dim}_A(E))$-homogeneous. Moreover (considering again open, bounded sets $\Omega$), the distance zeta function
\[
\zeta_{\Omega}(q) := \int_{\Omega} \frac{dx}{d_{\Omega}(x)^q}
\]
is finite if $q < d - \overline{\text{dim}}_M(\partial \Omega)$ and infinite if $q > d - \overline{\text{dim}}_M(\partial \Omega)$ (see [14, Lemma 3.3 and Lemma 3.5]).

We recall a geometric notion from [27], appearing among other assumptions in Theorem 6.

Definition 12. A set $E \subset \mathbb{R}^d$ is $\kappa$-plump with $\kappa \in (0, 1)$ if, for each $0 < r < \text{diam}(E)$ and each $x \in E$, there is $z \in \overline{B}(x, r)$ such that $B(z, \kappa r) \subset E$.

2.2. Whitney decomposition and operator $P^n$. Let $\Omega$ be an open, nonempty, proper subset of $\mathbb{R}^d$. Let $Q$ be any closed cube in $\mathbb{R}^d$. We denote by $l(Q)$ the length of the side of $Q$ and by $x_Q$ the center of $Q$. Following [23], there exists a family of dyadic cubes $\mathcal{W} = \{ Q_n \}_{n \in \mathbb{N}}$, called the Whitney decomposition, satisfying for all $Q, S \in \mathcal{W}$ the conditions:

- $\Omega = \bigcup_n Q_n$;
- if $Q \neq S$, then $\text{Int} Q \cap \text{Int} S = \emptyset$;
- there exists a constant $C = C(\mathcal{W})$ such that $C \text{diam} Q \leq \text{dist}(Q, \partial \Omega) \leq 4C \text{diam} Q$;
- if $Q \cap S \neq \emptyset$, then $l(Q) \leq 2l(S)$;
- if $Q \subset 5S$ then $l(S) \leq 2l(Q)$.

The dilation of the cube $Q$, $cQ$ for $c > 0$, is always taken with respect to its center, that is $cQ$ is a cube with the same center as $Q$, but the length of the side $cl(Q)$.

Inspired by [23] we define a shadow of a cube $Q \in \mathcal{W}$ as
\[
\text{Sh}_\theta(Q) = \{ S \in \mathcal{W} : S \subset B(x_Q, \theta l(Q)) \}.
\]
The „realization" of $\text{Sh}_\theta$ is $\text{SH}_\theta(Q) = \bigcup \text{Sh}_\theta(Q)$. When $\theta$ is fixed, we abbreviate the notation as $\text{Sh}_\theta(Q) =: \text{Sh}(Q)$ and $\text{SH}_\theta(Q) =: \text{SH}(Q)$.

For all $Q, S \in \mathcal{W}$ we define their long distance $D$ as
\[
D(Q, S) = l(Q) + \text{dist}(Q, S) + l(S).
\]
We say that a sequence of cubes \((Q, R_1, R_2, \ldots, R_n, S)\) is a chain, if all two adjacent cubes have nonempty intersection. We denote \((Q, R_1, R_2, \ldots, R_n, S) = [Q, S]\) and \([Q, S] = [Q, S] \setminus S\).

The Whitney decomposition is admissible, if there exists \(a > 0\) such that for all \(Q, S \in \mathcal{W}\) there exists a chain \([Q, S] = (Q_1, Q_2, \ldots, Q_n)\) satisfying

- \(\sum_{i=1}^n l(Q_i) \leq \frac{1}{a} D(Q, S)\);
- there exists \(1 \leq i_0 \leq n\) such that \(l(Q_i) \geq a D(Q, Q_i)\) for all \(1 \leq i \leq i_0\) and \(l(Q_i) \geq a D(Q_i, S)\) for all \(i_0 \leq i \leq n\). We denote \(Q_{i_0} =: Q_S\). This is the so-called central cube in the chain \([Q, S]\).

As stated in [23], for a \(\gamma\)-admissible Whitney decomposition we can always take sufficiently large \(\rho = \rho_\gamma > 1\) such that for every \(\gamma\)-admissible chain of cubes \([Q, S]\) we have \(Q \in \mathbf{Sh}_\rho(P)\) for \(P \in [Q, Q_S]\) and \(5Q \subset \mathbf{SH}_\rho(Q)\) for every Whitney cube \(Q \in \mathcal{W}\).

Next, we recall the definition and basic properties of the operator \(P^n\), defined in [10]. From now on we fix a Whitney decomposition \(\mathcal{W}\) such that \(C(\mathcal{W}) = 1\) (see [26]) and \(0 < \varepsilon < \sqrt{5/4} - 1 < \frac{1}{4}\). If \(Q\) is a cube, we denote by \(Q^*\) the cube \(Q\) "blown up" (1 + \(\varepsilon\)) times, that is the cube with the same center \(x_{Q^*} = x_Q\), but the length of the side \(l(Q^*) = (1 + \varepsilon)l(Q)\). The cube \(Q_{n^*}\) is defined in a similar way, that is \(Q_{n^*} = (Q_{n}^*)^*\). Notice that our choice of \(\varepsilon\) guarantees that \((1 + \varepsilon)^2 < \frac{5}{2}\) and in consequence \(Q_{n^*} \subset \frac{3}{2} Q_n\). Moreover, each point \(x \in \Omega\) belongs to at most \(12^d\) cubes \(Q_{n^*}\).

Let \(\{\psi_n\}_{n \in \mathbb{N}}\) be a partition of unity adjusted to the Whitney decomposition \(\mathcal{W} = \{Q_n\}_{n \in \mathbb{N}}\) of \(\Omega\), that is a family of functions satisfying \(0 \leq \psi_n \leq 1\), \(\psi_n = 1\) on \(Q_n\), \(\text{supp}\psi_n \subset Q_{n^*}\), \(\psi_n \in C^\infty(\Omega)\), \(\sum_n \psi_n = 1\) and \(|\psi_n(x) - \psi_n(y)| \leq C|x - y|/l(Q_n)\) for some positive constant \(C\) independent of \(Q_n\). Let us also fix a nonnegative function \(h: \mathbb{R}^d \to \mathbb{R}\) with the following properties: \(\sup h = B(0, 1), \int_{B(0, 1)} h(x) \, dx = 1\), \(h \in C^\infty(\mathbb{R}^d)\). For \(\delta > 0\) we define its dilation as \(h_\delta(x) = \delta^{-d}h(x/\delta)\). Moreover, let \(\eta: \mathcal{W} \to (0, \infty)\) be any function satisfying \(\eta(Q) < \frac{\varepsilon}{4} l(Q)\) for all \(Q \in \mathcal{W}\) (a typical example is \(\eta(Q) = \delta l(Q)\) for any \(\delta < \varepsilon/2\)). For \(f \in L_{loc}(\Omega)\), extended by 0 on \(\mathbb{R}^d \setminus \Omega\), we define the operator \(P^n\) as

\[
P^n f = \sum_{n=1}^\infty (f \psi_n) * h_\eta(Q_n).
\]

Here \(f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y) \, dy\) is the standard convolution operation. It was proved in [10] that \(P^n\) is well defined, \(P^n f \in C^\infty(\Omega)\) and \(P^n\) maps the space of all compactly supported, locally integrable functions into \(C^\infty(\Omega)\) (see [10], Propositions 1 and 2).

2.3. Uniform domains. There are two equivalent ways to define the notion of uniform domain. The first one comes from [27], and the second one uses the Whitney decomposition and chains of cubes and can be found for example in [23]. We present both definitions here.

**Definition 13.** A domain (i.e. connected, open set) \(\Omega \subset \mathbb{R}^d\) is uniform, if there exists a constant \(C \geq 1\) such that for all points \(x, y \in \Omega\) there is a curve \(\gamma: [0, l] \to \Omega\) joining them, parameterized by arc length and satisfying \(l \leq C|x - y|\) and \(\text{dist}(z, \partial\Omega) \geq \frac{1}{C} \min\{|z - x|, |z - y|\}\) for all \(z \in \gamma\). Equivalently, a domain \(\Omega \subset \mathbb{R}^d\) is uniform, if there exists an admissible Whitney decomposition of \(\Omega\).

Uniform domains and various reformulations of the definitions above appear also in [13], [21] and [22]. To give a concrete, nontrivial example, we remark here that the Koch snowflake is known to be uniform, despite the highly irregular behaviour of its boundary. It is also \(\sigma\)-homogeneous with \(\sigma = \log_3 4\), according to [19, Theorem 1.1].
2.4. Muckenhoupt class $A_1$ and Hardy-Littlewood maximal operator.

**Definition 14.** For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ the (non-centered) maximal Hardy-Littlewood operator is defined as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y) \, dy,$$

where supremum is taken over all cubes $Q$ containing $x$. Equivalently, $M$ can be defined using balls containing $x$ instead of cubes (up to a multiplicative constant). It is well known that this operator is bounded on $L^p(\mathbb{R}^d)$, whenever $1 < p \leq \infty$.

**Definition 15.** We say that a positive weight $w$ belongs to the Muckenhoupt class $A_1$, if there exists a constant $A > 0$ such that for all cubes $Q \subset \mathbb{R}^d$ it holds

$$\frac{1}{|Q|} \int_Q w(x) \, dx \leq A \inf_{y \in Q} w(y). \tag{2}$$

Notice that by (2) we can easily see that if $w \in A_1$, then the maximal Hardy-Littlewood operator acting on the function $w$ satisfies

$$Mw(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q w(y) \, dy \leq Aw(x), \tag{3}$$

where $A$ depends on $w$. This property will be important for us later in the proof of Theorem 2. Moreover, it was proved in [8, Theorem 1.1 (B)] that the weight $d_\alpha^{-\alpha}$ belongs to the Muckenhoupt class $A_1$ if and only if $0 \leq \alpha < \text{co dim}_A(\partial \Omega)$. Hence, by (3), $Md_\alpha^{-\alpha}$ satisfies

$$Md_\alpha^{-\alpha}(x) \leq A d_\alpha(x)^{-\alpha}, \tag{4}$$

where the constant $A$ depends on $\Omega$ and $\alpha \in [0, \text{co dim}_A(\partial \Omega))$.

### 3. Lemmas

We start with showing that under some assumptions $C_c^\infty(\Omega)$ is a subset of $W^{s,p;\alpha,\beta}(\Omega)$ and in consequence the latter is not trivial. This is an analogue of [6, Lemma 2.1], where the same fact was established for $\Omega = \mathbb{R}^d \setminus \{0\}$. Although we consider bounded domains, it agrees with the cited result in some aspects, as we have $\text{co dim}_A(\{0\}) = d$. Noteworthy, if one of the exponents $\alpha, \beta$ is nonpositive, then the corresponding weight is bounded and this case is trivial.

**Lemma 16.** Let $\Omega \subset \mathbb{R}^d$ be a bounded, uniform domain. Suppose that $0 < s < 1$, $1 \leq p < \infty$, $0 \leq \alpha, \beta < \text{co dim}_A(\partial \Omega)$ and $\alpha + \beta < d - \text{dim}_M(\partial \Omega) + p(1-s)$. Then $C_c^\infty(\Omega) \subset W^{s,p;\alpha,\beta}(\Omega)$.

**Proof.** Let $\varphi \in C_c^\infty(\Omega)$. Then $\varphi$ is Lipschitz and locally integrable, so, by Theorem 2 with $\theta = \frac{1}{2}$ we have

$$\|\varphi\|^p_{W^{s,p;\alpha,\beta}(\Omega)} \lesssim \int_{\Omega} \int_{B(x, \frac{1}{2}d_\Omega(x))} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta} \, dy \, dx$$

$$\lesssim \int_{\Omega} d_\Omega(x)^{-\alpha - \beta} \, dx \int_{B(x, \frac{1}{2}d_\Omega(x))} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{d+sp}} \, dy$$

$$\lesssim \int_{\Omega} d_\Omega(x)^{-\alpha - \beta} \, dx \int_{B(x, \frac{1}{2}d_\Omega(x))} \frac{dy}{|x - y|^{d+sp - p}}$$

$$\lesssim \int_{\Omega} d_\Omega(x)^{-\alpha - \beta + p(1-s)} \, dx < \infty,$$

where the last inequality follows from the properties of the distance zeta function. □
Remark 17. We make an easy observation that for any Borel subset $A \subset \Omega$ and $\alpha, \beta \geq 0$ it holds
\begin{equation}
\int_A \int_A \frac{|f(x) - f(y)|^p}{|x - y|^{d + sp}} \, d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} \, dy \leq 2 \int_A \int_A \frac{|f(x) - f(y)|^p}{|x - y|^{d + sp}} \, d\Omega(x)^{-\alpha - \beta} \, dy \, dx.
\end{equation}
Indeed, to prove (5) it suffices to split the inner integral into integrals over $A \cap \{d\Omega(x) \geq d\Omega(y)\}$ and $A \cap \{d\Omega(x) < d\Omega(y)\}$ and use the symmetry between variables $x$ and $y$.

According to above, if we abandon the assumption about the uniformity of $\Omega$ in Lemma 16, then, using (5), if $\Omega$ is bounded, we can analogously show that $C^\infty_c(\Omega) \subset W^{s,p,\alpha,\beta}(\Omega)$ for $\alpha + \beta < d - \text{dim}_M(\partial \Omega)$. Interestingly, this is a different range of parameters than in the Lemma 16.

Moreover, if $C^\infty_c(\Omega) \subset W^{s,p,\alpha,\beta}(\Omega)$ and $\Omega$ is bounded, then we cannot have $\alpha, \beta > d - \text{dim}_M(\partial \Omega)$. Indeed, if $\varphi \in C^\infty_c(\Omega)$, then simple calculation shows that
\begin{align*}
[\varphi]_{W^{s,p,\alpha,\beta}(\Omega)}^p & \geq \text{diam}(\Omega)^{-d - sp} \int_\Omega \int_\Omega |\varphi(x) - \varphi(y)|^p d\Omega(y)^{-\beta} d\Omega(x)^{-\alpha} \, dy \\
& \geq \text{diam}(\Omega)^{-d - sp} \int_{\text{supp } \varphi} \int_{\text{supp } \varphi} |\varphi(x)|^p d\Omega(y)^{-\beta} d\Omega(x)^{-\alpha} \, dy.
\end{align*}
The inner integral $\int_{\text{supp } \varphi} d\Omega(y)^{-\beta} \, dy$ is infinite if $\beta > d - \text{dim}_M(\partial \Omega)$. The case when $\alpha > d - \text{dim}_M(\partial \Omega)$ can be obtained similarly.

Definition 18. A weight $w: \Omega \to \mathbb{R}^d$ is locally comparable to a constant if for every compact subset $K \subset \Omega$ there exists $C_K > 0$ such that $\frac{1}{C_K} \leq w(x) \leq C_K$ for almost all $x \in K$.

The following Theorem is a generalization of [10, Theorem 12], where the same fact was proved for $w = v$.

Theorem 19. Let $\Omega \subset \mathbb{R}^d$ be an nonempty open set, $0 < s < 1$, $p \in [1, \infty)$. Denote
\begin{equation}
\widetilde{W}^{s,p,w,v}(\Omega) = \left\{ f: \Omega \to \mathbb{R}^d \text{ measurable} : \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{d + sp}} \, w(x) \, v(y) \, dx \, dy < \infty \right\}.
\end{equation}
We understand $\widetilde{W}^{s,p,w,v}(\Omega)$ as a semi-normed space. If $w$ and $v$ are locally bounded and satisfy the integral condition
\begin{equation}
\int_\Omega \frac{w(x)}{(1 + |x|)^{d + sp}} \, dx < \infty, \quad \int_\Omega \frac{v(x)}{(1 + |x|)^{d + sp}} \, dx < \infty,
\end{equation}
then $C^\infty(\Omega) \cap \widetilde{W}^{s,p,w,v}(\Omega)$ is dense in $\widetilde{W}^{s,p,w,v}(\Omega)$. Moreover, we have
\begin{equation}
W_0^{s,p,w,v}(\Omega) = \tilde{W}_c^{s,p,w,v}(\Omega).
\end{equation}

Proof. The proof follows the proof of [10, Theorem 12]. First, we fix a Whitney decomposition $\mathcal{W} = \{Q_n\}_{n \in \mathbb{N}}$ of $\Omega$ with a constant $C(\mathcal{W}) = 1$. We extend $w$ and $v$ by 0 outside $\Omega$. If $w$ or $v$ take the value zero on $\Omega$, then we can artificially augment them by adding a positive, locally comparable to a constant weights $w'$, $v'$, which in addition satisfy (6). New weights $w + w'$ and $v + v'$ are also locally comparable to a constant, positive and satisfy (6). In this case $w$ and $v$ should be replaced by $w + w'$ and $v + v'$ in all the computations below.

Denote by $\tau_y$ the translation operator, that is $\tau_y f(x) = f(x - y)$, $x, y \in \mathbb{R}^d$, and let $M = 12^{d(p - 1)}$. Moreover, let $f \in \tilde{W}^{s,p,w,v}(\Omega)$ and
\begin{equation}
g_n(x, y) = \frac{f(x)\psi_n(x) - f(y)\psi_n(y)}{|x - y|^{\frac{d + s}{p}}} \mathbb{1}_{\Omega \times \Omega}(x, y).
\end{equation}
We have
\[
\left[ P^{n_k} f - f \right]_{W^{s,p,\nu,\nu}(\Omega)} \leq M \sum_{n=1}^{\infty} \left\| \tau_{n_k}(Q_n) u - g_n \right\|_{L^p(\mathbb{R}^{2d} \times \mathbb{R}^d)} h(u) \, du
\]
and, for \( t < \eta_k(Q_n) \),
\[
\left\| \tau_n u - g_n \right\|_{L^p(\mathbb{R}^{2d} \times \mathbb{R}^d)} \leq M \sum_{n=1}^{\infty} \left\| \tau_{n_k}(Q_n) u - g_n \right\|_{L^p(\mathbb{R}^{2d} \times \mathbb{R}^d)} h(u) \, du
\]
\[
\leq \int_{Q_n^*} \int_{Q_n^*} \frac{|f(x-t)\psi_n(x-t) - f(y-t)\psi_n(y-t) + f(x)\psi_n(x) + f(y)\psi_n(y)|^p}{|x-y|^{d+sp}} w(x) \, v(y) \, dx \, dy \\
+ \int_{Q_n^*} \int_{Q_n^*} \frac{|f(x)\psi_n(x) - f(x)\psi_n(x-t)|^p}{|x-y|^{d+sp}} w(x) \, v(y) \, dx \, dy \\
+ \int_{Q_n^*} \int_{Q_n^*} \frac{|f(x)\psi_n(x) - f(y)\psi_n(y)|^p}{|x-y|^{d+sp}} v(x) \, w(y) \, dx \, dy
\]
\[=: I_1 + I_2 + I_3.\]

The estimates of the integrals \( I_1, I_2 \) and \( I_3 \) and completely analogous to these from [9, Proof of Theorem 12]. Notice that the properly modified version of [10, Proposition 9] also holds. The equality between \( W_0^{s,p,\nu,\nu}(\Omega) \) and \( W^{s,p,\nu,\nu}(\Omega) \) is a consequence of [10, Proposition 2] and the fact that the approximating functions are of the form \( P^{n_k} f \).

**Remark 20.** Suppose that \( \Omega \) is bounded. Then we trivially have
\[
1 \leq (1 + |x|)^{d+sp} \leq M := \sup_{x \in \Omega} (1 + |x|)^{d+sp} < \infty,
\]
hence, the condition (6) is equivalent to \( w, v \in L^1(\Omega) \). Moreover, if \( w(x) = d_\Omega(x)^{-\alpha} \), \( v(x) = d_\Omega(x)^{-\beta} \), then (6) is satisfied when \( 0 \leq \alpha, \beta < d - \dim_M(\partial \Omega) \) (we refer again to [14]). Of course, the function \( d_\Omega(x)^{-\alpha} \) is locally comparable to a constant on \( \Omega \) for every \( a \in \mathbb{R} \).

**Lemma 21.** Let \( \Omega \subset \mathbb{R}^d \) be a nonempty, open set such that \( |\Omega| < \infty \). Then we have
\[
W_0^{s,p,\nu,\nu}(\Omega) = W^{s,p,\nu,\nu}(\Omega) \iff 1_\Omega \in W_0^{s,p,\nu,\nu}(\Omega).
\]

**Proof.** Using the result of Theorem 19 about the equality between \( W_0^{s,p,\nu,\nu}(\Omega) \) and \( W^{s,p,\nu,\nu}(\Omega) \), the proof is a copy of [9, Lemma 13].

**Lemma 22.** Let \( \Omega \) be an open, uniform, bounded domain and let
\[
v_n(x) = \max \{ \min \{ 2 - nd_\Omega(x), 1 \}, 0 \} = \begin{cases} 1 & \text{when } d_\Omega(x) \leq 1/n, \\ 2 - nd_\Omega(x) & \text{when } 1/n < d_\Omega(x) \leq 2/n, \\ 0 & \text{when } d_\Omega(x) > 2/n. \end{cases}
\]

There exists a constant \( C = C(d, s, p, \alpha, \beta, \Omega) > 0 \) such that the following inequality holds for all functions \( f \in W^{s,p,\alpha,\beta}(\Omega) \) and \( 0 \leq \alpha, \beta < \codim_M(\partial \Omega) \),
\[
[fv_n]_{W^{s,p,\alpha,\beta}(\Omega)} \leq C n^{sp} \int_{\Omega^*} \frac{|f(x)|^p}{d_\Omega(x)^{\alpha+\beta}} \, dx + C \int_{\Omega^*} \int_{\Omega^*} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta} \, dx \, dy.
\]

Moreover, without assuming the uniformity of \( \Omega \), the following weaker inequality is satisfied for all \( \alpha, \beta \geq 0, \alpha + \beta < d - \dim_M(\partial \Omega) \) and \( f \in L^\infty(\Omega) \),
\[
[fv_n]_{W^{s,p,\alpha,\beta}(\Omega)} \leq C \| f \|_{L^\infty} n^{sp} \int_{\Omega^*} \frac{dx}{d_\Omega(x)^{\alpha+\beta}} + C \int_{\Omega^*} \int_{\Omega^*} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta} \, dx \, dy.
\]
Proof. The following proof is a modification of [9, Lemma 10]. By Theorem 2, taking $\theta = \frac{1}{2}$ we have

\[
[fv_n]^p_{W^{s,p},\alpha,\beta}(\Omega) \lesssim \int_{\Omega} \int_{B(x,\frac{1}{2}d_\Omega(x))} \frac{|f(x)v_n(x) - f(y)v_n(y)|^p}{|x-y|^{d+sp}} d\Omega(x) - \alpha d\Omega(y)^{-\beta} dy dx
\]

\[
= \int_{\Omega_1} \int_{B(x,\frac{1}{2}d_\Omega(x)) \cap \Omega_1} \frac{|f(x)v_n(x) - f(y)v_n(y)|^p}{|x-y|^{d+sp}} d\Omega(x) - \alpha d\Omega(y)^{-\beta} dy dx
\]

\[
+ \int_{\Omega \setminus \Omega_1} \int_{B(x,\frac{1}{2}d_\Omega(x)) \cap (\Omega \setminus \Omega_1)} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx
\]

\[
= J_1 + J_2 + J_3.
\]

Starting with estimating the integral $J_1$, we obtain

\[
J_1 \lesssim \int_{\Omega_1} \int_{B(x,\frac{1}{2}d_\Omega(x)) \cap \Omega_1} \frac{|v_n(y)|^p|f(x) - f(y)|^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx
\]

\[
+ \int_{\Omega_1} \int_{B(x,\frac{1}{2}d_\Omega(x)) \cap (\Omega \setminus \Omega_1)} \frac{|f(x)|^p|v_n(x) - v_n(y)|^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx
\]

\[
=: K_1 + K_2.
\]

The integral $K_1$ can be trivially bounded from above by the remainder of the weighted Gagliardo seminorm, that is

\[
K_1 \leq \int_{\Omega_1} \int_{\Omega_1} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx.
\]

Moreover, using the bound $|v_n(x) - v_n(y)| \leq \min\{1, |x-y|\}$ and the fact that $d\Omega(x) \approx d\Omega(y)$ on the ball $B(x,\frac{1}{2}d_\Omega(x))$ we can estimate $K_2$ as follows,

\[
K_2 \lesssim \int_{\Omega_1} \int_{\Omega_1} \frac{|f(x)|^p (\min\{1, |x-y|\})^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx.
\]

Splitting the inner integral over $dy$ into $|x-y| > 1/n$ and $|x-y| \leq 1/n$ gives the first term in (7).

Going back to the integral $J_2$ and remembering that $v_n = 0$ on $\Omega_1 \setminus \Omega_2$ we have

\[
J_2 = \int_{\Omega_1} \int_{B(x,\frac{1}{2}d_\Omega(x)) \cap (\Omega_1 \setminus \Omega_2)} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx
\]

\[
\lesssim \int_{\Omega_2} \int_{B(x,\frac{1}{2}d_\Omega(x)) \cap (\Omega_1 \setminus \Omega_2)} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx
\]

\[
\leq \int_{\Omega_2} \int_{\Omega_1 \cap B(x,\frac{1}{2}d_\Omega(x))} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} dy dx
\]

\[
\leq \int_{\Omega_2} |f(x)|^p d\Omega(x)^{-\alpha} \beta dx \int_{B(x,1/n) \cap \Omega_1} \frac{dy}{|x-y|^{d+sp}}
\]

\[
\lesssim n^{sp} \int_{\Omega_2} |f(x)|^p d\Omega(x)^{-\alpha - \beta} dx.
\]
The integral $J_3$ can be estimated in the similar way as $J_2$. That ends the proof of (7).
We note that the proof of (8) is analogous to the previous part. $K_1$ estimates by (5) and in the integrals $J_2$ and $J_3$ we use the fact that $d_\Omega(y) \geq d_\Omega(x)$ for $y \notin \Omega_\frac{n}{p}$ and $x \in \Omega_\frac{n}{p}$, hence, the comparability is not necessary here. The only thing that essentially changes is the estimation of $K_2$. In this case we bound $|f(x)|$ from above by its $L^\infty$-norm, use (5) and then proceed similarly as before to obtain the desired result. That proves (8). \hfill \square

4. PROOF OF THE COMPARABILITY

Proof of Theorem 2. In the proof of this Theorem we use techniques coming from [23]. We start with fixing sufficiently fragmented Whitney decomposition $\mathcal{W} = \mathcal{W}(\theta)$, so that for $(x, y) \in Q \times 5Q$ it holds $y \in B(x, \theta d_\Omega(x))$. Suppose first that $p > 1$. Let $q = \frac{p}{p-1}$ be the Hölder conjugate exponent to $p$. Using the duality between spaces $L^p(\Omega \times \Omega)$ and $L^q(\Omega \times \Omega)$ we can write the weighted Gagliardo seminorm wherewithal dual norm, that is

$$
\left( \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}}d_\Omega(y)^{-\beta}d_\Omega(x)^{-\alpha} \, dx \right)^{\frac{1}{p}} = \sup \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x - y|^{\frac{d}{p}+s}}d_\Omega(x)^{-\frac{\alpha}{q}}d_\Omega(y)^{-\frac{\beta}{q}}g(x, y) \, dy \, dx,
$$

where the supremum is taken over all nonnegative $g \in L^q(\Omega \times \Omega)$ satisfying $\|g\|_{L^q(\Omega \times \Omega)} \leq 1$. For now on we fix such a function $g$. Now, we split the integration range as follows,

$$
\int_\Omega \int_\Omega = \sum_Q \int_Q \int_{2Q} + \sum_{Q,S} \int_Q \int_{S \setminus 2Q} =: S_1 + S_2.
$$

Thanks to our assumption about the Whitney decomposition, the first sum can be immediately estimated by the truncated seminorm with making use of the Hölder inequality,

$$
S_1 \leq \left( \sum_Q \int_Q \int_{2Q} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}}d_\Omega(x)^{-\alpha}d_\Omega(y)^{-\beta} \, dy \, dx \right)^{\frac{1}{p}} \|g\|_{L^q(\Omega \times \Omega)}
$$

$$
\leq \left( \sum_Q \int_Q \int_{2Q} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}}d_\Omega(x)^{-\alpha}d_\Omega(y)^{-\beta} \, dy \, dx \right)^{\frac{1}{p}}
$$

$$
\leq \left( \int_\Omega \int_{B(x, \theta d_\Omega(x))} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}}d_\Omega(x)^{-\alpha}d_\Omega(y)^{-\beta} \, dy \, dx \right)^{\frac{1}{p}}.
$$

Hence, we only need to estimate the second part, $S_2$. We denote by $f_Q$ the average value of $f$ on the cube $Q$, that is $f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$ (the latter is finite by assumption). Using similar arguments as in [23, Section 4] we observe that for $x \in Q$ and $y \in S \setminus 2Q$
it holds $|x - y| \approx D(Q, S)$, hence, triangle inequality yields
\[
S_2 \lesssim \sum_Q \sum_S \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q, S)^{d+sp}} d\Omega(x)^{-\frac{q}{p}} d\Omega(y)^{-\frac{q}{p}} g(x, y) \, dy \, dx
\]
\[
\leq \sum_Q \sum_S \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q, S)^{d+sp}} d\Omega(x)^{-\frac{q}{p}} d\Omega(y)^{-\frac{q}{p}} g(x, y) \, dy \, dx
\]
\[
+ \sum_Q \sum_S \int_Q \int_S \frac{|f(y) - f(S)|}{D(Q, S)^{d+sp}} d\Omega(x)^{-\frac{q}{p}} d\Omega(y)^{-\frac{q}{p}} g(x, y) \, dy \, dx
\]
\[
+ \sum_Q \sum_S \int_Q \int_S \frac{|f(S) - f(y)|}{D(Q, S)^{d+sp}} d\Omega(x)^{-\frac{q}{p}} d\Omega(y)^{-\frac{q}{p}} g(x, y) \, dy \, dx
\]
\[
+ \sum_Q \sum_S \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q, S)^{d+sp}} d\Omega(x)^{-\frac{q}{p}} d\Omega(y)^{-\frac{q}{p}} g(x, y) \, dy \, dx
\]
=: (A) + (B) + (C) + (D).

Let us estimate (A) first. By Hölder inequality and Fubini-Tonelli theorem we get
\[
(A) \leq \sum_Q \int_Q |f(x) - f_Q| d\Omega(x)^{-\frac{q}{p}} \left( \sum_S \int_S g(x, y)^q \, dy \right) \left( \sum_S \int_S \frac{d\Omega(y)^{-\beta}}{D(Q, S)^{d+sp}} \, dx \right)^{\frac{1}{p}} \, dx
\]
\[
\leq \left( \sum_Q \int_Q |f(x) - f_Q|^p d\Omega(x)^{-\alpha} \sum_S \int_S \frac{d\Omega(y)^{-\beta}}{D(Q, S)^{d+sp}} \, dx \right)^{\frac{1}{p}}.
\]

By [23, Lemma 2.7] with $r = l(Q)$ and the Muckenhoupt condition (4) we have
\[
\sum_S \int_S \frac{d\Omega(y)^{-\beta}}{D(Q, S)^{d+sp}} \, dy \lesssim l(Q)^{-sp} \inf_{y \in Q} Md_{\Omega}^{-\beta}(y)
\]
\[
\lesssim l(Q)^{-sp} \inf_{y \in Q} d\Omega(y)^{-\beta}
\]
\[
\lesssim l(Q)^{-sp} d\Omega(y)^{-\beta}
\]
for any $y \in Q$, where $M$ is the Hardy-Littlewood maximal function. Hence, by Jensen inequality and Whitney decomposition properties, (A) can be bounded from above as follows,
\[
(A)^p \lesssim \sum_Q \int_Q \frac{1}{|Q|} \int_Q |f(x) - f(y)|^p d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} l(Q)^{-sp} \, dy \, dx
\]
\[
\lesssim \sum_Q \int_Q \int_Q |f(x) - f(y)|^p d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} l(Q)^{-sp-d} \, dy \, dx
\]
\[
\lesssim \sum_Q \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} \, dy \, dx
\]
\[
\lesssim \int \int_{D(x, \theta d\Omega(x))} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d\Omega(x)^{-\alpha} d\Omega(y)^{-\beta} \, dy \, dx,
\]
as it holds $|x - y| \lesssim l(Q)$ for $x, y \in Q$.

Now, we face the estimation of the component (B). We denote by $\mathcal{N}(P)$ the successor of the cube $P$ in the chain $[Q, S]$. It holds $\mathcal{N}(P) \subseteq 5\, P$ and $Q \in \text{Sh}(P)$ for $P \in [Q, Q_S]$. 

Also, $D(Q, S) \approx D(P, S)$ for such $P$. Hence, analogously to [23], by triangle inequality and Jensen inequality we can estimate (B) as follows,

$$(B) \leq \sum_{Q,S} \int_Q \int_S \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(Q, S)^{\frac{\alpha}{p}+s}} g(x, y) \sum_{P \in \{Q, Q_s\}} |f_P - f_{\mathcal{N}(P)}| \, dy \, dx$$

$$\leq \sum_{Q,S} \int_Q \int_S \frac{1}{|P|} \int_N \left| f(\xi) - f(\zeta) \right| d\xi d\zeta \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(Q, S)^{\frac{\alpha}{p}+s}} g(x, y) \, dy \, dx$$

$$\lesssim \sum_{P} \frac{1}{|P|^5} \int_P \int_S \left| f(\xi) - f(\zeta) \right| d\xi d\zeta \sum_{Q \in \text{Sh}(P)} \int_Q \int_S \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(P, S)^{\frac{\alpha}{p}+s}} g(x, y) \, dy \, dx.$$  

Define

$$G(x) = \left( \int_\Omega g(x, y)^q \, dy \right)^{\frac{1}{q}} \quad x \in \Omega.$$

Using again Hölder inequality, Muckenhoupt condition (4) and Whitney covering properties we have

$$\sum_{Q \in \text{Sh}(P)} \sum_S \int_Q \int_S \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(P, S)^{\frac{\alpha}{p}+s}} g(x, y) \, dy \, dx$$

$$\leq \sum_{Q \in \text{Sh}(P)} \int_Q \frac{d_\Omega(x)^{-\frac{\alpha}{p}}}{\left( \sum_S \int_S \frac{d_\Omega(y)^{-\frac{\beta}{p}}}{D(P, S)^{\frac{\alpha}{p}+sp}} \right)^{\frac{1}{p}}} G(x) \, dx$$

$$\lesssim l(P)^{-\frac{s}{p} - s} \int_{\text{Sh}(P)} G(x) d_\Omega(x)^{-\frac{\alpha}{p}} \, dx.$$  

Let us take small $\varepsilon > 0$, to be established in a moment. We apply Hölder inequality with exponents $q - \varepsilon$ and $\frac{q - \varepsilon}{q - \varepsilon - 1}$ to the integral above to obtain

$$\int_{\text{Sh}(P)} G(x) d_\Omega(x)^{-\frac{\alpha}{p}} \, dx \leq \left( \int_{\text{Sh}(P)} G(x)^{q-\varepsilon}(x) \, dx \right)^{\frac{1}{q - \varepsilon}} \left( \int_{\text{Sh}(P)} d_\Omega(x)^{-\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}} \, dx \right)^{\frac{q - \varepsilon - 1}{q - \varepsilon}}.$$  

Notice that

$$\lim_{\varepsilon \to 0^+} \frac{q - \varepsilon}{p(q - \varepsilon - 1)} = \frac{q}{p(q - 1)} = 1,$$

hence, remembering that by assumption $0 \leq \alpha < \text{co dim}_A(\partial \Omega)$, for sufficiently small $\varepsilon$ we still have $0 \leq \frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)} < \text{co dim}_A(\partial \Omega)$ (this condition defines $\varepsilon$, as well as $q - \varepsilon > 1$). According to this, by [23, Lemma 2.7] and (4), we have

$$\left( \int_{\text{Sh}(P)} G(x)^{q-\varepsilon}(x) \, dx \right)^{\frac{1}{q - \varepsilon}} \left( \int_{\text{Sh}(P)} d_\Omega(x)^{-\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}} \, dx \right)^{\frac{q - \varepsilon - 1}{q - \varepsilon}}$$

$$\lesssim \left( l(P)^d \inf_{x \in P} MG^{q-\varepsilon}(x) \right)^{\frac{1}{q - \varepsilon}} \left( l(P)^d \inf_{x \in P} Md_{\Omega}^{\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}}(x) \right)^{\frac{q - \varepsilon - 1}{q - \varepsilon}}$$

$$\lesssim \left( l(P)^d \inf_{x \in P} MG^{q-\varepsilon}(x) \right)^{\frac{1}{q - \varepsilon}} \left( l(P)^d \inf_{x \in P} Mg_{\Omega}^{\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}}(x) \right)^{\frac{q - \varepsilon - 1}{q - \varepsilon}}$$

$$\lesssim \left( l(P)^d \inf_{x \in P} MG^{q-\varepsilon}(x) \right)^{\frac{1}{q - \varepsilon}} \left( l(P)^d \inf_{x \in P} Mg_{\Omega}^{\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}}(x) \right)^{\frac{q - \varepsilon - 1}{q - \varepsilon}}$$

$$\leq l(P)^d \frac{\hat{\alpha}}{p} \left( MG^{q-\varepsilon}(x) \right)^{\frac{1}{q - \varepsilon}}$$
for any $\zeta \in P$. Finally, summing up all the considerations above, by Jensen inequality, Hölder inequality and boundeness of the Hardy-Littlewood maximal function on $L^{\frac{1}{p+\varepsilon}}(\mathbb{R}^d)$ we get the following result,

\[(B) \lesssim \sum_P \frac{l(P)^{d-\alpha \frac{d-s}{p} - \beta}}{|P|^{\frac{d}{5}}} \int_P \int_{5P} |f(\xi) - f(\zeta)| (MG^{q=\varepsilon}(\zeta))^\frac{1}{q} d\xi d\zeta
\]

\[
= \sum_P \frac{l(P)^{\frac{-\alpha}{p} - \frac{d-s}{p} - \beta}}{|P|^{\frac{d}{5}}} \int_P \int_{5P} |f(\xi) - f(\zeta)| (MG^{q=\varepsilon}(\zeta))^\frac{1}{q} d\xi d\zeta
\]

\[
\leq \sum_P l(P)^{\frac{-\alpha}{p} - \frac{d-s}{p} - \beta} \left( \int_P \left( \frac{1}{|5P|} \int_{5P} |f(\xi) - f(\zeta)| d\xi \right)^p d\zeta \right)^{\frac{1}{p}} \left( \int_P (MG^{q=\varepsilon}(\zeta))^\frac{q}{q} d\zeta \right)^{\frac{1}{q}}
\]

\[
\leq \sum_P l(P)^{\frac{-\alpha}{p} - \frac{d-s}{p} - \beta} \left( \int_P \left( \frac{1}{|5P|} \int_{5P} |f(\xi) - f(\zeta)|^p d\zeta \right) d\xi \right)^{\frac{1}{p}} \left( \int_P (MG^{q=\varepsilon}(\zeta))^\frac{q}{q} d\zeta \right)^{\frac{1}{q}}
\]

\[
\leq \left( \sum_P \int_P \int_{5P} |f(\xi) - f(\zeta)|^p |\xi - \zeta|^{d+sp} d\Omega(\xi)^{-\alpha} d\Omega(\xi)^{-\beta} \right)^{\frac{1}{p}} \left( \int_{\Omega} (MG^{q=\varepsilon}(\zeta))^{\frac{q}{q}} d\zeta \right)^{\frac{1}{q}}
\]

\[
\leq \left( \sum_P \int_P \int_{5P} |f(\xi) - f(\zeta)|^p |\xi - \zeta|^{d+sp} d\Omega(\xi)^{-\alpha} d\Omega(\xi)^{-\beta} d\xi d\zeta \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int_{\Omega} \int_{B(x,\theta d\Omega(x))} \frac{|f(\xi) - f(\zeta)|^p |\xi - \zeta|^{d+sp} d\Omega(\xi)^{-\alpha} d\Omega(\xi)^{-\beta} d\xi d\zeta \right)^{\frac{1}{p}}.
\]

That ends (B). We observe that the case (C) is symmetric to (B) (as we may have $Q_s = S_q$). We will obtain the same estimate as in (B), but with $\alpha$ and $\beta$ changed, however it holds $d\Omega(x) \approx d\Omega(y)$ for $x, y \in 5P$, hence, we will obtain exactly the same bound. The case (D) is symmetric to (A). That ends the proof in the case $p > 1$. When $p = 1$, we proceed similarly and actually this case is simpler and does not require the usage of dual norms.

When $p = 1$, we can formulate even a more general result.

Theorem 23. Let $\Omega$ be a nonempty, bounded, uniform domain and $0 < s < 1$, $0 < \theta \leq 1$. If the weights $w, v$ belong to the Muckenhoupt class $A_{1}$, then the full seminorm $[f]_{W^{s,1,\infty}(\Omega)}$ and the truncated seminorm

\[
\int_{\Omega} \int_{B(x,\theta d\Omega(x))} \frac{|f(x) - f(y)|}{|x - y|^{d+sp}} (w(x)v(y) + w(y)v(x)) \, dy \, dx
\]

are comparable for all $f \in L^1_{\text{loc}}(\Omega)$. The comparability constant depends on $\Omega, s, d, w, v$ and $\theta$.

Proof. The proof is similar to the proof of Theorem 2. The additional term in the truncated seminorm above follows from the fact, that components $(B)$ and $(C)$ are symmetric with respect to $w$ and $v$, but we cannot use the comparability of $w(x)$ and $v(y)$ on the cube $5P$, as for the distance weights.

Remark 24. Interestingly, the result of Theorem 2 allow to deduce in some cases another comparability property, between weighted Gagliardo seminorms $[f]_{W^{s,p,\alpha \beta}(\Omega)}$ and
FRACTIONAL SOBOLEV SPACES WITH POWER WEIGHTS 15

Suppose that $\Omega$ is a nonempty, bounded, uniform domain and the parameters $\alpha, \beta$ satisfy $0 \leq \alpha, \beta, \alpha + \beta < \operatorname{codim}_A(\partial \Omega)$. Take $f \in L^1_{\text{loc}}(\Omega)$. By (5) we have

$$[f]_{W^{s,p;\alpha,\beta}(\Omega)} \leq 2^{\frac{1}{p}} [f]_{W^{s,p;0,0}(\Omega)}.$$

To obtain a converse inequality, we use Theorem 2 with $\theta = \frac{1}{2}$ and get

$$[f]^p_{W^{s,p;\alpha,\beta}(\Omega)} \geq \int \int_{B(x,\frac{1}{2}d\Omega(x))} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d\Omega(y)^{-\beta} d\Omega(x)^{-\alpha} dy \, dx$$

$$\approx \int \int_{B(x,\frac{1}{2}d\Omega(x))} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} d\Omega(y)^{-\alpha-\beta} \, dy \, dx$$

$$\geq [f]^p_{W^{s,p;0,0}(\Omega)}.$$

Overall, we indeed get that

$$[f]_{W^{s,p;\alpha,\beta}(\Omega)} \approx [f]_{W^{s,p;0,0}(\Omega)}.$$

5. PROOFS OF MAIN RESULTS

Before we proceed to prove our main results, we need the following Proposition.

**Proposition 25.** Let $\Omega$ be a nonempty open set. Then the space $W^{s,p;w,v}(\Omega)$ is reflexive for $0 < s < 1$, $1 < p < \infty$ and all weights $w$ and $v$.

**Proof.** The proof is a modification of the proof of the reflexivity of the classical Sobolev space $W^{1,p}(\Omega)$ from [3, Proposition 8.1]. We define the isometry $T: W^{s,p;w,v}(\Omega) \rightarrow L^p(\Omega) \times L^p(\Omega \times \Omega, w \times v)$ (the latter endowed with the natural product norm) by

$$T(u) = \left( u, \frac{u(x) - u(y)}{|x - y|^{\frac{d}{2}+s}} \right).$$

The reflexivity of $W^{s,p;w,v}(\Omega)$ is a consequence of reflexivity of $L^p(\Omega) \times L^p(\Omega \times \Omega, w \times v)$.

**Proof of Theorem 3, case I.** By Lemma 21 and Theorem 19 to prove the density of $C_c^\infty(\Omega)$ in $W^{s,p;\alpha,\beta}(\Omega)$ it suffices to approximate the function $f = \mathbb{1}_\Omega$ by functions with compact support. By (8) (keeping the same notation), we have

$$[fv_n]^p_{W^{s,p;\alpha,\beta}(\Omega)} \leq Cn^{sp} \int_{\Omega_\frac{1}{n}} \frac{dx}{d\Omega(x)^{\alpha+\beta}}.$$

We have

$$n^{sp} \int_{\Omega_\frac{1}{n}} d\Omega(x)^{-\alpha-\beta} \, dx \lesssim \int_{\Omega_\frac{1}{n}} d\Omega(x)^{-\alpha-\beta-sp} \, dx \rightarrow 0,$$

when $n \to \infty$, because $\int_{\Omega} d\Omega(x)^{-\alpha-\beta-sp} \, dx = \zeta_\Omega(\alpha + \beta + sp) < \infty$. 

□
Proof of Theorem 3, case II. Recall that in this case we assume that \( \Omega \) is \((d-sp-\alpha-\beta)\)-homogeneous. Define the layers \( \Omega_{i,n} = \{ x \in \Omega : \frac{3}{2^{i+1}} < d_{\Omega}(x) \leq \frac{3}{2^i} \} \). We observe that

\[
n^{sp} \int_{\Omega_{i,n}} d_{\Omega}(x)^{-\alpha-\beta} \, dx = n^{sp} \sum_{i=0}^{\infty} \int_{\Omega_{i,n}} d_{\Omega}(x)^{-\alpha-\beta} \, dx \\
\approx n^{sp+\alpha+\beta} \sum_{i=0}^{\infty} 2^{-i(\alpha+\beta)} |\Omega_{i,n}| \\
\lesssim n^{sp+\alpha+\beta-\beta} \sum_{i=0}^{\infty} 2^{-i(\alpha+\beta+\beta)} = C,
\]

where \( C \) is a constant independent of \( n \). That means that the sequence \( \{fv_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{s,p;\alpha,\beta}(\Omega) \). Now, the proof follows [9, Proof of Theorem 2, case II]: we use Banach–Alaoglu and Eberlein–Šmulian theorems to conclude that there exists a subsequence \( \{fv_{n_k}\}_{k \in \mathbb{N}} \) convergent to \( \mathbb{I}_\Omega \) in \( W^{s,p;\alpha,\beta}(\Omega) \). The reflexivity of \( W^{s,p;\alpha,\beta}(\Omega) \) is essential here.

Proof of Theorem 3, case III. We proceed analogously as in the unweighted case in [9, Proof of Theorem 2, case III]. In this case we just need to use the fractional weighted Hardy inequality (1) in the case (F) and Fatou’s lemma to prove that the function \( f = \mathbb{I}_\Omega \) cannot be approximated by \( C^\infty_c(\Omega) \) functions in \( W^{s,p;\alpha,\beta}(\Omega) \).

Remark 26. Notice that in the proof of the case III we use the fact that if \( u_n \to \mathbb{I}_\Omega \) in \( L^p(\Omega) \), then there exists a subsequence \( u_{n_k} \) convergent to \( \mathbb{I}_\Omega \) almost everywhere; the same fact holds if we replace \( L^p(\Omega) \) by the weighted space \( L^p(\Omega, W) \) for almost everywhere positive \( W \in L^1(\Omega) \).

Proof of Theorem 4. If \( \int_\Omega |f(x)|p d_{\Omega}(x)^{-sp-\alpha-\beta} \, dx < \infty \), then \( f \in W^{s,p;\alpha,\beta}_0(\Omega) \), because by Lemma 22 we have

\[
[fv_n]_{W^{s,p;\alpha,\beta}(\Omega)}^p \lesssim n^{sp} \int_{\Omega_+} \frac{|f(x)|p}{d_{\Omega}(x)^{\alpha+\beta}} \, dx + \int_{\Omega_+} \int_{\Omega_+} \frac{|f(x) - f(y)|p}{|x-y|^{d+sp}} d_{\Omega}(x)^{-\alpha} d_{\Omega}(y)^{-\beta} \, dy \, dx \\
\lesssim \int_{\Omega_+} \frac{|f(x)|p}{d_{\Omega}(x)^{sp+\alpha+\beta}} \, dx + \int_{\Omega_+} \int_{\Omega_+} \frac{|f(x) - f(y)|p}{|x-y|^{d+sp}} d_{\Omega}(x)^{-\alpha} d_{\Omega}(y)^{-\beta} \, dy \, dx \to 0,
\]

when \( n \to \infty \). On the other side, if \( f \in W^{s,p;\alpha,\beta}_0(\Omega) \), then by the fractional Hardy inequality (1) and Fatou’s lemma we obtain that \( \int_\Omega |f(x)|p d_{\Omega}(x)^{-sp-\alpha-\beta} \, dx < \infty \). That proves the desired characterization of \( W^{s,p;\alpha,\beta}_0(\Omega) \).

Proof of Theorem 7. This is a straightforward consequence of the fractional Hardy inequality (1) in the case \((T')\), case I of the Theorem 3 and Fatou’s lemma. We can easily see that uniform domains are \( \kappa \)-plump, so (6) is applicable.

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