Triangle Sides for Congruent Numbers less than 10,000

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Abstract

We have computed a table of the triangle sides of all congruent numbers less than 10,000, which improves and extends the existing public table. We give some background on properties of the triangle sides, and explain how we computed our table, which is available on https://github.com/dgpaloalto/Congruent-Numbers

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1 Introduction

A congruent number is a positive integer that is the area of a right triangle with rational sides. For example 5 is congruent, being the area of the triangle with sides 3/2, 20/3 and 41/6. A basic reference is the textbook [5]. We know of only a single table giving the rational sides of congruent numbers, appearing at [1] and [2]. One is likely a copy of the other, since they share the same glitch at \(N = 559\), as will be explained later. This table gives the sides for all congruent numbers under 1000, but there is no information about how the numbers were computed. The purpose of this note is to introduce an extended table for congruent numbers up to 10000, together with a detailed explanation of how they were computed.

When \(N\) is congruent, there are infinitely many representations by right triangles. If the sides are \(\alpha, \beta\) and \(\gamma\), with \(\gamma\) the hypotenuse, we define the "height" of the representation as \(\max(\alpha_1, \alpha_2, \beta_1, \beta_2)\) where \(\alpha = \alpha_1/\alpha_2, \beta = \beta_1/\beta_2\) are the rational sides as the quotients of relatively prime integers. We strive to give the representation with the triangle of minimum height, although we do not claim to have always achieved that. In particular, our new table has representations with smaller heights than the existing table for seven different congruent numbers. Figure 1 has the details.

*Also on [3], \(n = 199\) has \(Q = 27368486201\) instead of the correct value 273684876201.
Table 1: The seven numbers less than 1000 that have a lower height in our new table.

| n   | α_1 | α_2 | β_1 | β_2 | height | α_1 | α_2 | β_1 | β_2 | height |
|-----|-----|-----|-----|-----|--------|-----|-----|-----|-----|--------|
| 219 | 1752| 55  | 55  | 4   | 1752   | 264 | 13  | 945 | 44  | 949    |
| 333 | 12920| 297 | 297 | 20  | 12920  | 1785| 92  | 3496| 105 | 3496   |
| 330 | 60  | 1   | 11  | 1   | 60     | 24  | 1   | 55  | 2   | 55     |
| 410 | 1640| 9   | 9   | 2   | 1640   | 451 | 18  | 360 | 11  | 451    |
| 434 | 496 | 3   | 21  | 4   | 496    | 279 | 10  | 280 | 9   | 280    |
| 609 | 20  | 1   | 609 | 10  | 609    | 28  | 3   | 261 | 2   | 261    |
| 915 | 3660| 11  | 11  | 2   | 3660   | 244 | 63  | 945 | 2   | 945    |

Figure 1: A visualization of our table. For each congruent number $n$, it plots the base-10 logarithm of the height, $\max(\alpha_1, \alpha_2, \beta_1, \beta_2)$.

As explained in [5], assuming the Birch-Swinnerton-Dyer conjecture, there is an easily computable test due to Tunnell for whether a number is congruent. We use that test when selecting numbers for representation by triangle sides. Figure 1 shows a visualization of our new table.

The table and the code that generated it is available on GitHub as https://github.com/dgpaloalto/Congruent-Numbers

2 Properties of the Representation

We might as well assume that $N$ is square free, since $N$ is congruent if and only if its square free factor is congruent, and the sides of one are easily deduced from the other.

**Theorem 1.** If $N$ is a square-free congruent number represented with perpendicular sides $\alpha_1/\alpha_2$ and $\beta_1/\beta_2$ both in reduced form, then $(\alpha_1, \beta_1) = 1$, $(\alpha_2, \beta_2) = 1$ and $\alpha_1$ and $\beta_1$ have opposite parities.
Proof. The definition of congruent number means that

(1) \[ \alpha_1 \beta_1 = 2N\alpha_2\beta_2 \]

If \( \alpha_2 \) and \( \beta_2 \) have a common prime factor \( p \), then \( p^2 | \alpha_1\beta_1 \), so \( p \) divides at least one of them, contradicting that the fractions are in reduced form. This gives

(2) \( (\alpha_2, \beta_2) = 1 \)

From equation (1) and \( N \) square free, it follows that \( \alpha_1 \) and \( \beta_1 \) have no common odd prime factor. To finish the proof, we only need to show that \( \alpha_1 \) and \( \beta_1 \) are of opposite parity. First, they can’t both be odd, since \( 2N\alpha_2\beta_2 \) is even. And they can’t both be even. If they were, then \( \alpha_2 \) and \( \beta_2 \) are both odd. Also, from the definition of congruent number,

\[
\frac{\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2}{\beta_1^2\beta_2^2} = \frac{\alpha_1^2}{\alpha_2^2} + \frac{\beta_1^2}{\beta_2^2} = \delta_1^2
\]

which shows that numerator on the left is a square:

(3) \( (\alpha_1\beta_2)^2 + (\alpha_2\beta_1)^2 = C^2 \)

for some integer \( C \). If both \( \alpha_1 = 2 \mod 4 \), then when you divide equation (3) by 4, each summand on the left hand side is congruent to 1 mod 4, so their sum is congruent to 2 mod 4, which is not a possible residue for \( C^2 \). Therefore one of the \( \alpha_i \) must be divisible by 4 and their product is divisible by 8. That means both sides of equation (1) are divisible by 8, forcing \( N \) to be divisible by 4, contradicting that \( N \) is square free. This contradiction shows that \( \alpha_1 \) and \( \beta_1 \) are of opposite parity.

We strive to produce the representation of minimal height. Note that

**Theorem 2.** The height \( \max(\alpha_1, \alpha_2, \beta_1, \beta_2) \) is equal to \( \max(\alpha_1, \beta_1) \).

**Proof.** From equation (1), the product \( (\alpha_1/\alpha_2)(\beta_1/\beta_2) \) is an integer and the fractions are relatively prime, so \( \alpha_2 | \beta_1 \) and therefore \( \alpha_2 \leq \beta_1 \). Similarly \( \beta_2 \leq \alpha_1 \).

## 3 Representation using \( P \) and \( Q \)

For a fixed \( N \), there is a 1-1 mapping between a pair of rational sides and a pair of relatively prime integers \( (P, Q) \). The existing tables [1] and [2] give the representation in terms of these \( P \) and \( Q \).

To go from \( \alpha \) and \( \beta \) to \( P \) and \( Q \), rewrite equation (3) as:

\[
A = \alpha_1\beta_2 \\
B = \alpha_2\beta_1 \\
A^2 + B^2 = C^2
\]

3
Therefore \((A, B, C)\) can be represented as a pythagorean triple \((5)\), page pp. 1–2

\[
A = P^2 - Q^2 \\
B = 2PQ \\
C = P^2 + Q^2
\]

It is an easy consequence of Theorem \([1]\) that \((A, B, C)\) is a primitive pythagorean triple, so the representation exists. The above gets you from \(\alpha = \frac{\alpha_1}{\alpha_2}\) and \(\beta = \frac{\beta_1}{\beta_2}\) to integers \(A, B, C\) and thus to \(P, Q\).

To go from \(P\) and \(Q\) to \(\alpha\) and \(\beta\), use the above to get \(A\) and \(B\) and use the following to compute \(D = \alpha_2\beta_2\) and then \(\alpha_1\) and \(\alpha_2\).

\[
\frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} = 2N \\
\frac{A}{\alpha_2 \beta_2} = 2N \\
\frac{AB}{2} = \alpha_2^2 \beta_2^2 N = D^2 N \\
D = \sqrt{\frac{AB}{2N}} \\
\frac{\alpha_1}{\alpha_2} = A \frac{D}{\alpha_2} \\
d = \gcd(A, D) \\
\alpha_1 = A/d \\
\alpha_2 = D/d
\]

And similarly for \(\beta_1, \beta_2\). The integers \(A/d\) and \(D/d\) have a ratio of \(\alpha_1/\alpha_2\). But are they equal to the original \(\alpha_1, \alpha_2\)? Yes, because both pairs \((\alpha_1, \alpha_2)\) and \((A/d, D/d)\) are relatively prime and have the same quotient.

We can now explain the glitch for \(N = 559\) in the tables \([1]\) and \([2]\). For that \(N\) they give \(P = 2608225\) and \(Q = 4489\), which are both odd. But the \(P\) and \(Q\) coming from a primitive pythagorean triple always have opposite parities.

There is an even more compact representation of the triangle sides than using \(P\) and \(Q\).

**Theorem 3.** \(P = P_0 P_1^2\) and \(Q = Q_0 Q_1^2\) where \(P_0 Q_0 \ | \ N\).

**Proof.**

\[
2PQ(P^2 - Q^2) = AB \\
= 2N\alpha_2^2 \beta_2^2 \\
(4) \quad PQ(P - Q)(P + Q) = N\alpha_2^2 \beta_2^2
\]

If a prime \(p\) divides \(P\), then it either divides \(N\) or \(\alpha_2^2 \beta_2^2\). If the latter, then \(p^2\) divides both sides of \((4)\). But \((P, Q) = 1\) means that \(p \nmid Q\) and so \(p^2 \mid P\).
| k | occurrences | examples of N          |
|---|-------------|------------------------|
| 2 | 237         | 205, 221, 438          |
| 3 | 60          | 517, 751, 1103         |
| 4 | 4           | 3805, 6198, 9118, 9143 |
| 5 | 2           | 3093, 6887             |

Table 2: Frequency of Heegner points $P$ that have a $k$-division point $Q$, that is $kQ = P$ on the curve $y^2 = x^3 - N^2x$. For each $k < 10$, we list the number of occurrences of a congruent number $N$ less than 10000 that have $k$-division points, together with some sample $N$.

Consequently, for each divisor $p | P$, either $p | N$ or $p^2 | P$. Rewrite (4) by dividing out all the factors of $P$, and then repeat the argument for $Q$.

As an example of the compression, for $N = 53$, instead of $P = 1873180325$ and $Q = 1158313156$ you have $P = 53 \cdot 5945^2$ and $Q = 34034^2$.

4 Strategy for Generating the Table

You can get sides $\alpha$ and $\beta$ for a congruent number $N$ by finding a rational point on the elliptic curve $y^2 = x^3 - N^2x$. See [5], Proposition 19 on page 47. So we compute the sides by using SageMath [6] to get rational points on an elliptic curve. Details are in the next section.

In SageMath, we create an elliptic curve using $e = \text{EllipticCurve}([-N*N, 0])$.

Then we use either $e.gens$ or $\text{pari}(e).\text{ellheegner}()$ to get rational points. The first uses Cremona’s mwrank C library, the second uses the Heegner point method, described in [3].

We need both methods because Heegner only works for points of rank one, but e.gens is extremely slow for some $N$. Fortunately for each $N < 10000$, one of the two methods will produce a rational side in a reasonable amount of elapsed computation time. The use of Heegner points is mentioned in [4] for $N = 1063$.

We use two heuristics to try and get sides of minimal height. For points of rank greater than one, if $g_j$ are generators, we get sides for the rational point $g_i$ and $g_i \pm g_j$ for $i < j$, and then pick the set of sides of minimal height. For the Heegner point $g$, we try dividing $g$ by $k = 10, 9, \ldots 2$. If one of those divisions succeeds, we use the sides from $g/k$. Because the group of rational points for these curves have torsion elements, we actually test for divisibility of $g+t$ where $t$ ranges over the torsion subgroup.

The effectiveness of these heuristics was tested by an exhaustive computation that computed, for each $N$, the smallest height representation with $Q < P < 25000$. These never had less height than the representation using our heuristics.

Here are a few numbers related to the heuristics. For mwrank, there are 447 congruent numbers $N$ where $y^2 = x^3 - N^2x$ has two generators, and 15 with three generators. For Heegner points, see table 2. But note that this table only considers divisors $k \leq 10$. 


5 Details on Generating the Table

If \((x, y)\) is a point on the elliptic curve \(y^2 = x^3 - N^2x\), write

\[
x = \frac{s}{t}, \quad y = \frac{u}{v}
\]

Then \((x', y') = 2(x, y)\) is given by

\[
x' = -2\frac{s}{t} + \left(\frac{3(s^2 - N^2)}{v^2}\right)^2
\]
\[
= -2\frac{s}{t} + \left(\frac{3s^2v - t^2vu^2}{2ut^2}\right)^2
\]
\[
= -2\frac{s}{t} + \frac{v^2}{4u^2t^4} \left(3s^2 - t^2n^2\right)^2
\]
\[
= \frac{8su^2t^3}{4u^2t^4} + \frac{v^2}{4u^2t^4} \left(3s^2 - t^2n^2\right)^2
\]
\[
= \frac{-8su^2t^3 + v^2 \left(3s^2 - t^2n^2\right)^2}{(2ut^2)^2}
\]
\[
= \frac{s'}{t'} = \frac{s'}{r^2}
\]

Next, use Koblitz, Proposition 19 on page 47, which says that if \((x', y')\) is the sum of a point on \(y^2 = x^3 - N^2x\) with itself, then you get rational sides with area \(N\) for sides \(\alpha, \beta\) with

\[
\alpha = \sqrt{x' + N} - \sqrt{x' - N}
\]
\[
\beta = \sqrt{x' + N} + \sqrt{x' - N}
\]

so

\[
\alpha = \sqrt{x' + N} - \sqrt{x' - N} = \frac{\sqrt{s' + Nt'} - \sqrt{s' - Nt'}}{\tau}
\]
\[
\beta = \sqrt{x' + N} + \sqrt{x' - N} = \frac{\sqrt{s' + Nt'} + \sqrt{s' - Nt'}}{\tau}
\]

Or since \(\alpha = \alpha_1/\alpha_2, \beta = \beta_1/\beta_2\)

\[
\alpha_1 = \sqrt{s' + Nt'} - \sqrt{s' - Nt'}
\]
\[
\alpha_2 = \tau
\]
\[
\beta_1 = \sqrt{s' + Nt'} + \sqrt{s' - Nt'}
\]
\[
\beta_2 = d
\]

The python code using these formulas is available on GitHub as [https://github.com/dgpaloalto/Congruent-Numbers](https://github.com/dgpaloalto/Congruent-Numbers)
6 First Rows of the Table

Here are the first nineteen lines of the table. The columns $P_0$, $P_1$, $Q_0$, and $Q_1$ are defined in Theorem [3].

| $n$ | $P_0$ | $P_1$ | $Q_0$ | $Q_1$ | $\alpha_1$ | $\alpha_2$ | $\beta_1$ | $\beta_2$ | $P_0$ | $P_1$ | $Q_0$ | $Q_1$ | weight |
|-----|-------|-------|-------|-------|-----------|-----------|-----------|-----------|-------|-------|-------|-------|--------|
| 5   | 5     | 4     | 3     | 2     | 20        | 3         | 5         | 1         | 1     | 1     | 2     | 2     | 20     |
| 6   | 2     | 1     | 3     | 1     | 4         | 1         | 2         | 1         | 1     | 1     | 4     | 4     |
| 7   | 16    | 9     | 35    | 12    | 24        | 5         | 1         | 4         | 1     | 3     | 35    | 35    |
| 13  | 325   | 36    | 780   | 323   | 323       | 30        | 13        | 5         | 1     | 6     | 780   | 780   |
| 14  | 8     | 1     | 8     | 3     | 21        | 2         | 2         | 2         | 1     | 1     | 21    | 21    |
| 15  | 4     | 1     | 4     | 1     | 15        | 2         | 1         | 2         | 1     | 1     | 15    | 15    |
| 21  | 4     | 3     | 7     | 2     | 12        | 1         | 1         | 2         | 3     | 1     | 12    |
| 22  | 50    | 49    | 33    | 35    | 140       | 3         | 2         | 5         | 1     | 7     | 140   |
| 23  | 24336 | 17689 | 80155 | 20748 | 41496     | 3485      | 1         | 156       | 1     | 133   | 80155 |
| 29  | 4901  | 4900  | 99    | 910   | 52780     | 99        | 29        | 13        | 1     | 70    | 52780 |
| 30  | 3     | 2     | 5     | 1     | 12        | 1         | 3         | 1         | 2     | 1     | 12    |
| 31  | 1600  | 81    | 720   | 287   | 8897      | 360       | 1         | 40        | 1     | 9     | 8897  |
| 34  | 9     | 8     | 17    | 6     | 24        | 1         | 1         | 3         | 2     | 2     | 24    |
| 37  | 777925| 1764  | 450660| 777923| 777923    | 6090      | 37        | 145       | 1     | 42    | 777923|
| 38  | 1250  | 289   | 1700  | 279   | 5301      | 425       | 2         | 25        | 1     | 17    | 5301  |
| 39  | 13    | 12    | 5     | 2     | 156       | 5         | 13        | 1         | 3     | 2     | 156   |
| 41  | 25    | 16    | 123   | 20    | 40        | 3         | 1         | 5         | 1     | 4     | 123   |
| 46  | 72    | 49    | 253   | 42    | 188       | 11        | 2         | 6         | 1     | 7     | 253   |
| 47  | 14561856 | 2289169 | 11547216 | 2097655 | 98589785 | 5773608 | 1 | 3816 | 1 | 1513 | 98589785 |

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References

[1] Congruum $g : 1|g| g = 999$. [http://www.asahi-net.or.jp/~KC2H-MSM/\texttt{mathland/math10/matb2000.htm}] Accessed: 2021-05-01.

[2] Numeri congruenti minori di 1000. [http://bitman.name/math/table/29] Accessed: 2021-03-21.

[3] H. Cohen. *Number Theory Volume I: Tools and Diophantine Equations*. Springer, New York, 2007.

[4] N. D. Elkies. Heegner point computations. In *International Algorithmic Number Theory Symposium*, pages 122–133. Springer, 1994.

[5] N. Koblitz. *Introduction to Elliptic Curves and Modular Forms*. Springer-Verlag, New York, 1993.

[6] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.2)*, 2020. [https://www.sagemath.org]