The quark strange star in the enlarged Nambu-Jona-Lasinio model

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Abstract. The strange quark star is investigated within the enlarged SU(3) Nambu-Jona-Lasinio (NJL). The stable quark star exists till maximal configuration with \( \rho_m = 3.1 \times 10^{15} \text{g/cm}^3 \) with \( M_m = 1.61 M_\odot \) and \( R_m = 8.74 \text{km} \) is reached. Strange quarks appear for density above \( \rho_c = 9.84 \text{g/cm}^3 \) for the quark star with radius \( R_c = 8.003 \text{km} \) and \( M_c = 0.77 M_\odot \). The comparison of a quark star properties obtained in the Quark Mean Field (QMF) approach to a neutron star model constructed within the Relativistic Mean Field (RMF) theory is presented.

PACS numbers: 26.60+c, 21.65+f, 24.10.Jv, 21.30.Fe

Submitted to: *New J. Phys.*

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Introduction

At sufficiently high density, the transition to deconfined strange quark matter is widely expected. Chiral symmetry is spontaneously broken in the QCD vacuum. Lattice QCD simulations at nonzero temperature $T$ and zero baryon-chemical potential $\mu_B$ indicate that chiral symmetry is restored above a temperature $T \sim 150\,\text{MeV}$ [1]. The NJL [2] model is an effective theory which is believed to be related to QCD at low energies, when one has integrated out the gluon fields. The NJL model might yield reasonable results at the density range where confinement is no longer crucial but chiral symmetry as a symmetry of full QCD remains to be important. The NJL model has proved to be very successful in the description of the spontaneous breakdown of chiral symmetry exhibited by the true (nonperturbative) QCD vacuum. This model has been extensively used over the past years not only to describe hadron properties [3] (see for reviews [4, 5]) and phase transitions in dense matter [6, 7, 8] but to describe the quark strange stars as well [9, 10, 11, 12, 13]. The detailed properties of the quark phase in compact stars has been a topic of recent interest [14] (for a review see [15, 16]).

Quark strange stars are astrophysical compact objects which are entirely made of deconfined $u,d,s$ quark matter (strange matter) staying in $\beta$-equilibrium. The possible existence of strange stars is a direct consequence of the conjecture [17] that strange matter may be the absolute ground state of strongly interacting matter.

The three-flavor NJL model has been discussed by many authors, e.g. [18]. For the quark phase we follow Buballa and Oertel [19] in using the three-flavor version of the NJL model.

The aim of this paper is to investigate a strange quark star within the enlarged SU(3) Nambu-Jona-Lasinio (NJL) model. The comparison of a quark star properties obtained in the Quark Mean Field (QMF) approach to a neutron star model constructed in framework the Relativistic Mean Field theory (RMF) will be made. This paper is organized as follows.

In Section 1 there are presented general properties of the NJL model in the mean field approach based on the Feynman - Bogolubov inequality for free energy of the system. In this section the employed equation of state (EoS) is calculated for NJL model. The EoS is used then to determine the equilibrium configurations of the quark star in Section 2. Finally, in Section 3 the main implications of the results are summarized.

Nambu-Jona-Lasinio model

The NJL[2] model was widely used for describing hadron properties [20] and the chiral phase transition [21]. The enlarged (ENJL)[22] simplest version of the model is given by the Lagrangian:

$$L = \bar{q}(i\gamma^\mu\partial_\mu - m_0)q + \frac{1}{2}G_s \sum_{\alpha=0}^{8} [(\bar{q}\lambda^\alpha q)^2 + (\bar{q}\lambda^\alpha i\gamma_5 q)^2] - 2K \prod_{f=\{u,d,s\}} (\bar{q}_f q_f)$$

$$- \frac{1}{2}G_v \sum_{\alpha=0}^{8} [(\bar{q}\gamma^\mu\lambda^\alpha q)(\bar{q}_\mu\lambda^\alpha q) + (\bar{q}\gamma^\mu\gamma_5\lambda^\alpha q)(\bar{q}\gamma_5\gamma_5\lambda^\alpha q)]$$  \hspace{0.5cm} (1)
+ i \sum_{f=1}^{2} \overline{L}_f \gamma^\mu \partial_\mu L_f - \sum_{f=1}^{2} m_f \overline{L}_f L_f + B_0

The first term contains the free kinetic part, including the current quark \( q_f = \{ q_u, q_d, q_s \} \) masses \( m_0 \) which break explicitly the chiral symmetry of the Lagrangian and the term representing the free relativistic leptons \( L_f = \{ e^-, \mu^- \} \). The fermion fields are composed of quarks and leptons (electrons, muons). Here \( q \) denotes a quark field with three flavors, \( u, d \) and \( s \), and three colors. We restrict ourselves to the isospin SU(2) unbroken symmetric case, \( m_0^u = m_0^d \), thus

\[
m_0 = m_0, f \delta_{f, f'} = \begin{pmatrix} m_{0,u} & m_{0,d} \\ m_{0,d} & m_{0,s} \end{pmatrix}
\]

Generators of the \( U(3) \) algebra \( \lambda^a = \{ \lambda^0 = \sqrt{2/3} I, \lambda^i \} \) (where \( I \) is an identity matrix, \( \lambda^i \) are Gell-Mann matrices of the \( SU(3) \) algebra) obeying \( Tr(\lambda^a \lambda^b) = 2 \delta_{ab} \). Due to this normalization of this algebra the coupling constants \( G_s \) and \( G_v \) can be redefined and written as \( \tilde{G}_s = 2/3 G_s \), \( \tilde{G}_v = 2/3 G_v \).

| TABLE 1: Parameter sets of the NJL models. |
|-------------------------------------------|
|                                           |
| \( m_u = m_d \)                          | 5.5 MeV | 5.5 MeV | 5.5 MeV | 3.61 MeV |
| \( m_s \)                                | 0       | 140.7 MeV | 132.9 MeV | 88.0 MeV |
| \( \Lambda \)                             | 631 MeV | 602.3 MeV | 631.4 MeV | 750.0 MeV |
| \( G_s \Lambda^2 \)                      | 2.19    | 3.67    | 3.67    | 3.624    |
| \( K \)                                  | 0       | 12.36   | 9.40    | 9.40     |
| \( \tilde{G}_v \Lambda^2 \)              | 0       | 0       | 0       | 3.842    |

The NJL model is non-renormalizable, thus it is not defined until a regularization procedure has been specified. This cut-off limits the validity of the model to momenta well below the cut-off. In most of our calculations we will adopt the parameters presented in Table 1. With \( \Lambda, G_s \) specified above, chiral symmetry is spontaneously broken in vacuum.

The model contains eight parameters of the standard NJL model (the current mass \( m_{0,u}, m_{0,s} \) of the light and strange quarks, the coupling constants \( G_s \), determinant coupling \( K \) and the momentum cut-off \( \Lambda \)) and additional constant \( G_v \) (\( G_v = x_v G_s, x_v = 1.06 \) [12]). In the quark massless limit the system has a \( U(3)_L \times U(3)_R \) chiral symmetry. The system has following global currents: the baryon current

\[
J_B^\nu = \frac{1}{3} \overline{q} \lambda^0 q
\]

and the isospin current which exists only in the asymmetric matter

\[
J_3^\nu = \frac{1}{2} \overline{q} \gamma^\nu \lambda^3 q, \quad J_8^\nu = \frac{1}{2} \overline{q} \gamma^\nu \lambda^8 q
\]

The conserved baryon and isospin charges are given by the relations:

\[
Q_i = \frac{1}{3} \int d^3x q^+ \lambda^i q, \quad i = \{0, 3, 8\}
\]
which are connected to commuting Cartan algebra. The physical system is defined by the thermodynamic potential [23]
\[ \Omega = -kT \ln \text{Tr}(e^{-\beta(H - \mu^i Q_i)}) \]  
where \( H \) stands for the Hamiltonian. More convenient is to use chemical potentials connected to the quark flavor \( f \) in the such way that
\[ \mu^i Q_i = \sum_f \mu_f \bar{Q}_f \]

(5)

Quarks and electrons are in \( \beta \)-equilibrium which can be described as a relation among their chemical potentials
\[ \mu_d = \mu_u + \mu_e = \mu_s \]
\[ \mu_\mu = \mu_\mu \]

where \( \mu_u, \mu_d, \mu_s \) and \( \mu_e, \mu_\mu \) stand for quarks and lepton chemical potentials, respectively.

These conditions means that matter is in equilibrium with respect to the weak interactions. If the electron Fermi energy is high enough (greater then the muon mass) in the neutron star matter muons start to appear as a result of the following reactions
\[ d \to u + e^- + \bar{\nu}_e \]
\[ s \to u + \mu^- + \bar{\nu}_\mu \]

The neutron chemical potential is
\[ \mu_n \equiv \mu_u + 2\mu_d. \]

In a pure quark state the star should to be charge neutral. This gives us an additional constraint on the chemical potentials
\[ \frac{2}{3} n_u - \frac{1}{3} n_d - \frac{1}{3} n_s - n_e - n_\mu = 0. \]

(7)

where \( n_f \) (\( f \in u, d, s \)), \( n_e \) are the particle densities of quarks and electrons, respectively. The EoS can now be parameterized by only one parameter, namely the dimensionless \( u \) quark Fermi momentum \( x \) (\( k_{F,u} = M x \) \( M = 939 \) MeV is the nucleon mass)).

The mean field approach means that the quantum correlations
\[ (A - < A >)(B - < B >) = AB - A < B > - B < A > + < A > < B > \sim 0 \]
may be neglected. It allows us to substitute \( AB \) by
\[ AB \sim A < B > + B < A > - < A > < B > \]

Using this approximation the Lagrange function \( \mathcal{L} \) may be expressed as
\[ \mathcal{L} = \bar{q}(i\gamma^\mu D_\mu - m_0)q + g_s \sum_{a=0}^{8} \sigma^a (\bar{q}\lambda^a q) - \frac{1}{2} m_s^2 \sigma^a \sigma^a + \frac{1}{2} m_{\nu}^2 V_{\mu} V^{\mu \nu} + \\
- 2K \sum_f (\bar{q}_f q_f) \prod_{f' \neq f} < \bar{q}_f q_{f'} > + 4K \sum_f < \bar{q}_f q_f > \]

(8)
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where the covariant derivative is given by

\[ D_\mu = \partial_\mu + \frac{1}{2}ig_\nu V^{\alpha}_\mu \lambda^\alpha \] (9)

Here, the meson fields first appear as non-dynamical variables

\[ G_s < \bar{q} \lambda^a q > = g_s \sigma^a \] (10)
\[ G_v < \bar{q} \lambda^a \gamma_\mu q > = g_v V^a_\mu \] (11)

The similar pattern may be extended to the axial mesons

\[ G_s < \bar{q} \lambda^a i \gamma_5 q > = g_s \phi^a \] (12)
\[ G_v < \bar{q} \lambda^a \gamma_\mu i \gamma_5 q > = g_v A^a_\mu \] (13)

Mesons masses are defined as

\[ m_\sigma = g_s / \sqrt{G_s} \] (14)
\[ m_v = g_v / 2 \sqrt{G_v} \] (15)

This is a process of bosonization in which the NJL model produce essentially the \( u(3) \) linear sigma model as an approximate effective theory for the scalar and pseudoscalar meson sector [24].

In this paper the variational method based on the Feynman-Bogolubov inequality [25] is incorporated (see more details in [26])

\[ \Omega \leq \Omega_1 = \Omega_0(m_{eff}) + < H - H_0 >_0 . \] (16)

with the trial Lagrange function described by

\[ L_0(m_{eff}) = \overline{\sigma}(i\gamma^\mu \bar{D}_\mu - m_{eff})q \] (17)

and suggested by the mean field form of the Lagrange function (8). The covariant derivative

\[ \bar{D}_\mu = \partial_\mu + \frac{1}{2}ig_\nu V^{\alpha}_\mu \lambda^\alpha \] (18)

is limited to the commuting Cartan subalgebra \( \lambda^i = \{ \lambda^0, \lambda^3, \lambda^8 \} \). This approach introduces the fermion interaction with homogeneous boson condensates \( \sigma^a, V^a_\mu \) which together with the effective masses \( m_{eff} \) will be treated as variational parameters. \( \Omega_0 \) is the thermodynamic potential of the effectively free quasiparticle system

\[ \Omega_0(m_{eff}) = E_0 - k_B T \frac{N_q}{2\pi^2} \sum_f \int_0^\Lambda dk k^2 \left( \ln \left(1 + e^{-\beta(\sqrt{k^2 + m_{eff,f}^2} - \mu_f)} \right) + \ln \left(1 + e^{-\beta(\sqrt{k^2 + m_{eff,f}^2} + \mu_f)} \right) \right) \] (19)

We could not forget about energy of quantum fluctuations because it depends on the quark effective mass. As fermions give \(-1/2 \hbar \omega\) to the vacuum energy, we get

\[ \mathcal{E}_0 = -\frac{N_q}{2\pi^2} \sum_{f=\{u,d,s\}} \int_0^\Lambda dk k^2 \sqrt{k^2 + m_{eff,f}^2} \] (20)
assuming that if $m_{eff} = m_0$ the energy of quantum fluctuations may be neglected. The effective quark masses entering into the Lagrangian function $L_0(m_{eff})$ of the trial system are calculated from the extremum conditions

$$\frac{\partial \Omega_1}{\partial m_{eff,f}} = 0$$

which give

$$(m_{eff})_{f,f'} = m_{eff,f} \delta_{f,f'} = m_e \delta_{f,f'} - G_s \sum_{a=0}^{8} < \bar{q}_f \lambda^a q_f > 0 + 2K \delta_{f,f'} \prod_{f' \neq f} < \bar{q}_{f'} q_{f'} > 0$$

or

$$m_{eff,f} = m_{c,f} - G_s < \bar{q}_f q_f > 0 + 2K \prod_{f' \neq f} < \bar{q}_{f'} q_{f'} > 0,$$  

where

$$< \bar{q}_f q_f > 0 = \frac{m_{eff,f} N_q}{\pi^2} \int_0^\Lambda \frac{k^2 dk}{\sqrt{k^2 + m_{eff,f}^2}} \left\{ \frac{1}{\exp(\beta(\sqrt{k^2 + m_{eff,f}^2} - \mu_f)) + 1} - 1 \right\}. \tag{23}$$

At $T = 0$ we have only

$$< \bar{q}_f q_f > 0 = -m_{eff,f} \frac{N_q}{2\pi^2} \int_{k_F}^\Lambda dk \frac{k^2}{\sqrt{k^2 + m_{eff,f}^2}}. \tag{24}$$

In vacuum we get the constituent quarks with mass

$$m_c = m_0 - 2G_s < \bar{q} q >_0 + 2K \prod_{f' \in \{u,d,s\}} < \bar{q}_{f'} q_{f'} >_0$$  

where

$$< \bar{q}_f q_f >_0 = -m_{v,f} \frac{N_q}{2\pi^2} \int_{k_F}^\Lambda dk \frac{k^2}{\sqrt{k^2 + m_{v,f}^2}}. \tag{26}$$

At minimum the effective free energy has the form

$$\Omega_{eff} = \Omega_1|_{\text{min}} = \Omega_0(m_{eff}) + B_{eff}$$

with the effective bag constant

$$B_{eff} = \frac{1}{2} G_s < \bar{q} \lambda^a q >^2 - 4K \prod_{f \in \{u,d,s\}} < \bar{q}_f q_f >$$

$$- \frac{1}{2} G_v < \bar{q} \gamma^\mu \lambda^a q > < \bar{q} \gamma^\mu \lambda^a q > - B_0 \tag{28}$$

Quarks as effectively free quasiparticles in vacuum with nonvanishing bag 'constant’. The constant $B_0$ was chosen in this way to have massive ($m_{v,u,d} = 367.61 \text{ MeV}, m_{v,s} = 549.45 \text{ MeV}$ for the NJL (I) parameters set and $m_{v,u,d} = 366.13 \text{ MeV}, m_{v,s} = 504.13 \text{ MeV}$ for enlarged NJL model). However, in high density medium they are less massive (Figure 1) but the effective bag constant (Figure 2) grows up to $B_{eff}^{1/4} \div 150 - 180 \text{ MeV}$. The frequently
The quark strange star in the enlarged Nambu-Jona-Lasinio model used case with current quarks and bag constant is valid only in very high density limit. This is a justification when the quark matter phase is modeling in the context of the MIT bag model [9, 27, 28] as a Fermi gas of $u$, $d$, and $s$ quarks. In this model the phenomenological bag constant $B_{\text{MIT}}$ is introduced to mimic QCD interactions. In the original MIT bag model the bag constant was constant and the value $B = B_c = (154.5 \text{ MeV})^4$ makes the strange matter absolutely stable.

To avoid quantum fluctuations the meson fields may be redefined to produce the phenomenological sigma field as

$$g_s \varphi_a = G_s(\langle \bar{q}\lambda^a q \rangle_0 - \langle \bar{q}\lambda^a q \rangle_{0v})$$

(29)

so, the effective quark mass can be rewritten as

$$m_{\text{eff}} = m_c - g_s \varphi_a \lambda^a$$

(30)

Thus, the effective quark mean field (QMF) theory appears. The Lagrange function in the mean field approximation may be written in the following form

$$\mathcal{L}_{\text{QMF}} = \bar{q} (i\gamma^\mu D_\mu - m_c) q + \frac{1}{2} \partial_\mu \varphi_a \partial^\mu \varphi_a - \frac{1}{2} m_\sigma^2 \varphi_a \varphi_a$$

$$- \frac{1}{4} F_{\mu\nu}^a F_{a,\mu\nu} + \frac{1}{2} m_v^2 V_{a} V_{a,\mu} V_{a,\mu}$$

(31)

(32)

Unfortunately, the vacuum quantum fluctuations are missed, but meson fields gain the dynamical character. Restricted ourself only to $u(2) \times u(1)$ subalgebra ($a = \{0,1,2,3,8\}$) case we have the simplest version of the QMF theory. Defining new base with $\tau^a$, $a = \{0,1,2,3,4\}$ as

$$\tau^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tau^i = \begin{pmatrix} \sigma^i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad \tau^4 = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(33)

($i = \{1,2,3\}$) the meson fields may be decomposed as follows

$$\varphi = \varphi_a \tau^a = \sigma \tau^0 + \delta_i \tau^i + \sigma_s \tau^4$$

and

$$V_\mu = V_\mu^a \lambda^a = \omega_\mu \tau^0 + \omega_{s,\mu} \tau^4 + b_i \tau^i$$

Now the new meson fields are denoted by

$$\omega_\mu = \sqrt{\frac{2}{3}} V_\mu^0 + \frac{1}{\sqrt{2}} V_\mu^8,$$

$$\omega_{s,\mu} = \sqrt{\frac{1}{3}} V_\mu^0 - V_\mu^8,$$

and $b^i_\mu = V_{\mu}^i$ (with $i = \{1,2,3\}$), respectively. The simplest $u(2)$ version ($\sigma_s = 0$, $\omega_{s,\mu} = 0$) has the Lagrange density function with the following form

$$\mathcal{L}_{\text{QMF}} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - U(\sigma) - \frac{1}{4} \Omega_{\mu\nu} \Omega^{\mu\nu} + \frac{1}{2} M_\omega^2 \omega_\mu \omega^\mu +$$

$$\frac{1}{2} \partial_\mu \delta_i \partial^\mu \delta_i - \frac{1}{2} m_s^2 \delta_i^2 - \frac{1}{4} R_{\mu\nu}^a R_{a,\mu\nu} + \frac{1}{2} M_{b,a}^2 b^a_b$$

$$\bar{q} (i\gamma^\mu D_\mu - m_c) q + g_\sigma^a \sigma \bar{q} q + g_\delta^a \delta_i \bar{q} \tau^i q$$

(34)
The field tensors $R_{\mu\nu}^a$, $\Omega_{\mu\nu}$ and the covariant derivative $D_\mu$ are given by

$$R_{\mu\nu}^a = \partial_\mu b_\nu^a - \partial_\nu b_\mu^a + g_\rho \varepsilon_{abc} b_\mu^b b_\nu^c,$$

(35)

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu,$$

(36)

$$D_\mu = \partial_\mu + \frac{i}{2} g_\rho b_\mu^a \tau^a + \frac{i}{2} g_\omega \omega_\mu.$$

(37)

The $\delta = 0$ limit gives the simplest version of the Quark Mean Field (QMF) model.

Some years ago Guichon proposed an interesting model concerning the change of the nucleon properties in nuclear matter (quark-meson coupling model (QMC)) [30]. The model construction mimics the relativistic mean field theory, where the scalar $\sigma$ and the vector meson $\omega$ fields couple not with nucleons but directly with quarks. The quark mass has to change from its bare current mass due to the coupling to the $\sigma$ meson. More recently, Shen and Toki [31] have proposed a new version of the QMC model, where the interaction takes place between constituent quarks and mesons. They refer the model as the quark mean field model (QMF). In this work we shall also investigate the quark matter within the QMF theory motivated by parameters coming from the enlarged Nambu-Jona-Lasinio (ENJL). Enlargement of the NJL model is based on inclusion of vector mesons while QMF model includes vector mesons at the beginning.

Here the QMF model is a bit generalized by the inclusion of the isovector $\delta$ ($a_0(980)$) meson. It splits $u$ and $d$ masses (or proton and neutron masses in the case of RMF approach [35, 36]). Both $\delta^i$ and $b_i^\mu$ mesons may be neglected in the case of symmetric nuclear matter. Its role in the asymmetric nuclear matter of the neutron star is significant and is a subject of our current interest.

The QMF model is more flexible. The $SU(3)$ symmetry restrict $g_\sigma^q = \sqrt{2/3} g_\sigma$, $g_\omega^q = g_\omega$, $g_\rho^q = g_\rho$ to $g_\rho = g_\omega = g_\sigma$ and $m_\rho = m_\omega = m_\sigma$.

The effective quarks mass $m_{F,f}$ (or $\delta_f = m_{F,f}/M$) dependence on the dimensionless Fermi momentum $x = k_{F,u}/M$ is presented on the Figure 1.

There is not quarks confinement both in the NJL and QMF models. There is no mechanism (except for the NJL solvable model [13]) to prevent hadrons to decay into free constituent quarks. Free constituent quark will produce nearly the same density and pressure like free nucleons $3m_{v,u,d} \sim M$. Without any mechanism of confinement the quark star for small densities will have properties very similar to those of neutron stars (the case $x_v > 0.65$ in paper [12]) or even white dwarfs. However, this is rather unphysical artefact. It is natural to assume that quarks are not allowed to propagate over the distance $\lambda \sim m_{e_f}^{-1}$. In this language the confinement mechanism introduces infrared cut-off $\lambda$ [40]. The quarks confinement mechanism in the form of the harmonic oscillator potential [31] may give the nucleon mass $M = M(\sigma) = M - g_{N\sigma} \sigma + ...$ and generate the Relativistic Mean Field (RMF) approach.

Relativistic Mean Field Theory [32] is very useful in describing nuclear matter and finite nuclei. Recent theoretical studies show that the properties of nuclear matter can be described nicely in terms of the Relativistic Mean Field Theory. Properties of the neutron star in this model were also examined [10, 9, 33, 26, 34].
Figure 1. The effective quark masses for different NJL models as a function of the $u$ quark dimensionless Fermi momentum $x = k_{F,u}/M$ ($M = 939 MeV$ is the nucleon mass)
Figure 2. The effective bag constant as a function of the $u$ quark dimensionless Fermi momentum $x = k_{F,u}/M$ ($M = 939\text{ MeV}$ is the nucleon mass).

Its extrapolation to large charge asymmetry is of considerable interest in nuclear astrophysics and particularly in constructing neutron star model where extreme conditions of isospin is realized. The construction of neutron star model is based on various realistic equations of state and results in the general picture of neutron stars interiors. Thus the proper form of the equation of state is essential in determining neutron star properties such as the mass-range, mass-radius relation or the crust thickness. However, the complete and more realistic description of a neutron star requires taking into consideration not only the interior region of a neutron star but also the remaining layers, namely the inner and outer crust and the surface. The Lagrangian of the RMF theory which tends towards the construction of neutron star model contains baryon and meson degrees of freedom and as input quantities coupling constants of the mesons and parameters of the potential $U(\sigma)$ which are determined from
Figure 3. The binding energy $E_b$ for the quark and nuclear system. The pure quark system ($usd$ quarks with the bag constant $B$) and QMF have nonrealistic behavior for small densities as $B \neq 0$. To compare the symmetric nuclear matter (the RMF approach with TM1[39] parameterization) is presented.
Figure 4. The form of equation of state (EoS) for quark (NJL(I), enlarged NJL and QMF model) matter and nucleon one (TM1[39]). The blue ($n_B^s = 3.94 n_0^B$) points the strange $s$ quark appearance in the strange star.

nuclear matter properties.

\[ \mathcal{L}_{QMF} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - U(\sigma) - \frac{1}{4} \Omega_{\mu\nu} \Omega^{\mu\nu} + \frac{1}{2} M_\omega^2 \omega_\mu \omega^\mu + \]
\[ - \frac{1}{4} R^{\mu\nu}_\alpha R^{\alpha\mu\nu} + \frac{1}{2} M_\rho^2 \rho_\mu \rho^\mu + \frac{1}{4} c_3 (\omega_\mu \omega^\mu)^2 \]
\[ \overline{\psi} (i \gamma^\mu D_\mu - M) \psi + g_{N\sigma} \sigma \overline{\psi} \psi \]

Now $\psi$ describes nucleons and $g_{N\sigma} = 3 g_\sigma^q$, $g_{N\omega} = 3 g_\omega^q$ and $g_{N\rho} = g_\rho^q$.

The potential function $U(\sigma)$ may possesses a polynomial form introduced by Boguta and Bodmer [37] in order to get a correct value of the compressibility $K$ of nuclear matter at
The electron and quarks dimensionless Fermi momentum as a function of the $u$ quark one ($x = k_{F,u}/M$, $M = 939 \text{ MeV}$ is the nucleon mass). The muon distribution is not visible in this scale.

saturation density

$$U(\sigma) = \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{3} g_3 \sigma^3 + \frac{1}{4} g_3 \sigma^4.$$  (39)

Different parameter sets give different forms of the equation of state in the high density region above saturation. In this paper the $TM1$ [39] parameter set was exploit. The RMF model leads its own fenomenological life but its parameters should be connected to the enlarged NJL model. The $TM1$ parametrization suggests that $x_v = 0.65$.

The $\rho$ meson plays decisive role in accounting for the asymmetry energy of nuclear matter thus its inclusion in a theory of neutron star matter is essential. Also the proton number density is determined by this meson. The results for the binding enrgy is presented in Figure 3. It shows that the nuclear or quark matter in $\beta$-equilibrium has a larger energy per particle
than symmetric nuclear matter. For neither parameter set are these matters self-bounded. Figure 3 depicts the binding energy for different models. It reproduces the standard results for symmetric nuclear matter too. The asymmetric matter is less bound than the symmetric one. In this model we are dealing with the electrically neutral neutron or quark star matter being in $\beta$-equilibrium. Therefore the imposed constrains, namely the charge neutrality and $\beta$-equilibrium, imply the presence of leptons.

**The quark star properties**

To calculate the properties of the quark star we need the energy-momentum tensor. The energy-momentum tensor can be calculated taking the quantum statistical average

$$\bar{T}_{\mu\nu} = \langle T_{\mu\nu} \rangle,$$

where

$$T_{\mu\nu} = 2 \frac{\partial L}{\partial g_{\mu\nu}} - g_{\mu\nu} L.$$  (40)

In case of the fermion fields it is more convenient to use the reper field $e^a_\mu$ defined as follows $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ where $\eta_{ab}$ is the flat Minkowski space-time matrix. Then

$$T_{\mu\nu} = e^a_\mu \frac{\partial L}{\partial e^a_\nu} - g_{\mu\nu} L.$$  (42)

We define the density of energy and pressure by the energy - momentum tensor

$$\bar{T}_{\mu\nu} = (P + \epsilon)u_\mu u_\nu - Pg_{\mu\nu} = \begin{pmatrix} \epsilon = e^2 \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

where $u_\mu$ is a unite vector ($u_\mu u^\mu = 1$). Both $\epsilon$ and $P$ depend on the quarks chemical potential $\mu$ or Fermi momentum $x_f$. Fermions (quarks and leptons) contribution to the energy and pressure are

$$\epsilon_F = \sum_{f=\{u,d,s\}} \epsilon_0 \chi_B(x_f, T) + \sum_{f=\{e,\mu\}} \epsilon_0 \chi_L(x_f, T)$$  (44)

$$P_F = \sum_{f=\{u,d,s\}} P_0 \phi(x_f, T) + \sum_{f=\{e,\mu\}} P_0 \phi(x_f, T)$$  (45)

The fact that effective quarks mass depends $m_{eff,f} = \delta_f M$ on fermion concentration (or quark chemical potential $\mu_f$) now must be included into $\chi(x_f, T)$ and $\phi(x_f, T)$,

$$\chi(x, T) = \frac{3}{\pi^2} \int^{\Lambda/M}_\Lambda \frac{dz}{z^2} \frac{1}{\exp((\sqrt{\delta^2(x) + z^2}) + \mu')/\tau) + 1}$$

$$+ \frac{1}{\exp((\sqrt{\delta^2(x) + z^2}) + \mu')/\tau) + 1},$$

$$\phi(x, T) = \frac{3}{\pi^2} \int^{\Lambda/M}_\Lambda \frac{dz}{z^2} \frac{1}{\exp((\sqrt{\delta^2(x) + z^2}) + \mu')/\tau) + 1}$$

$$+ \frac{1}{\exp((\sqrt{\delta^2(x) + z^2}) + \mu')/\tau) + 1},$$
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\[ \phi(x, T) = \frac{1}{\pi^2} \int_0^{\Lambda/M} \frac{z^4 dz}{\sqrt{z^2 + \delta^2(x)}} \left\{ \frac{1}{\exp((\sqrt{\delta^2(x) + z^2} - \mu')/\tau) + 1} \right\} \]

\[ + \frac{1}{\exp((\sqrt{\delta^2(x) + z^2} + \mu')/\tau) + 1} \]

where \( \tau = (k_B T)/M \),

\[ \mu' = \mu/M = \sqrt{\delta^2(x) + x^2} \]

and

\[ x = k/M \]

for each flavor \( f \). Similar to paper [33] we have introduced (48,49) the dimensionless “Fermi” momentum even at finite temperature which exactly corresponds to the Fermi momentum at zero temperature. To avoid free quark contributions to the equation of state coming from small densities the infrared cut-off \( \lambda = \delta [40] \) was introduced. The case \( \lambda = 0 \) with the NJL (I) parameters set nicely reproduce the result of paper [12].

The parametric dependence on \( \mu \) (or \( x_f \)) defines the equation of state. The various equations of state for different parameters sets are presented in Figure 4. The binding energies

\[ E_b = \rho/n_B - Mc^2 \]

for the bulk nuclear (\( n_B \) is the baryon number density, \( n_B = (n_p + n_n) \)) and quark matter (\( n_B = (n_u + n_d + n_s)/3 \)) are presented in Figure 3.

The metric is static, spherically symmetric and asymptotically flat

\[ g_{\mu\nu} = \begin{pmatrix}
  e^{\nu(r)} & 0 & 0 & 0 \\
  0 & -e^{\lambda(r)} & 0 & 0 \\
  0 & 0 & -r^2 & 0 \\
  0 & 0 & 0 & -r^2 \sin^2 \theta
\end{pmatrix} \]

(50)

(51)

(52)

(53)
The equation (52) determines the function $\lambda(r)$

$$e^{-\lambda(r)} = 1 - \frac{2Gm(r)}{r}. $$

Having solved the OTV equation the pressure $P(r)$, mass $m(r)$ and density $\rho(r)$ profile is obtained. To obtain the total radius $R$ of the star the fulfillment of the condition $P(R) = 0$ is necessary. Introducing the dimensionless variable $\xi$, which is connected with the star radius $r$ by the relation $r = a\xi$, enables to define the functions $P(r)$, $\rho(r)$ and $m(r)$

$$\rho(r) = \rho_0\chi(x(\xi)) \hspace{1cm} (54)$$

$$P(r) = P_0\varphi(x(\xi)) \hspace{1cm} (55)$$

$$m(r) = M_\odot u(\xi) \hspace{1cm} (56)$$

by $\xi$. Dimensionless functions defined as

$$\alpha_0 = \frac{GM_\odot \rho_c}{P_0a}, \hspace{0.1cm} \beta_0 = 3\frac{M_s}{M_\odot}, \hspace{0.1cm} M_s = \frac{4}{3}\pi \rho_0 a^3 \hspace{1cm} (57)$$

are needed to achieve the OTV equation of the following form

$$\frac{d\varphi}{d\xi} = -\alpha_0(\chi(x(\xi)) + \varphi(x(\xi))) \frac{u(\xi) + \beta_0\varphi(x(\xi))\xi^3}{\xi^2(1 - \frac{r_a}{a}\frac{u(\xi)}{\xi})} \hspace{1cm} (58)$$

$$\frac{du}{d\xi} = \beta_0\chi(x(\xi))\xi^2 \hspace{1cm} (59)$$

with $r_g$ being the gravitational radius. The equations (58,59) are easy integrated numerically. These are equations for dimensionless mass $u(\xi) = m(r)/M_\odot$ up to radius dimensionless $\xi$ and the quark $u$ dimensionless Fermi momentum $x = k_{F,u}/M$. Knowing the variable $x$ all star properties can be calculated. Quarks and electron dimensionless Fermi momenta dependences on $x$ is presented in Figure 5.

Both nuclear and quark matter being in $\beta$-equilibrium are not bound (Figure 3). For quark matter at moderate densities is bound due to the presence the bag constant $B$ which acts as a negative pressure $P = -B + ...$ (Figure 4). Higher density matter is bound only gravity. The gravitational binding energy of the star is defined as

$$E_{b,g} = (M_p - m(R))c^2 \hspace{1cm} (60)$$

where

$$M_p = 4\pi \int_0^R drr^2(1 - \frac{2Gm(r)}{c^2r})^{-\frac{1}{2}}\rho(r) \hspace{1cm} (61)$$

is the proper star mass.

Strange quarks star in the NJL model are rather small in comparison to the neutron ones. To compare these strange stars to neutron stars models obtained in the RMF approach the mass - radius relations are also presented in Figure 6. (the black solid line (TM1) for pure npe nuclear matter and the star with a crust (TM1 + Bonn + Negele + Vautherin [42] - dotted line). The smaller sizes of quarks star is due to the fact that the pressure (Figure 4) reaches zero for high densities ($n_B^m \sim 0.26 \text{ fm}^{-3} \sim 1.75 n_B^0$) then to the symmetric nuclear matter
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$(n_B^0 \sim 0.15 \text{ fm}^{-3})$. On the surface such a star has higher density than the saturated nuclear matter. This makes the star smaller and denser. The same situation exists in the QMF approach when the bag constant is included. In the NJL model the effective bag constant has dynamical origin (eq. 28). Its main contributions come from quantum vacuum fluctuations and scalar mesons. The strange quark $s$ appears only for densities high enough ($n_B > n_B^s = 3.94 n_B^0$). For smaller densities the quark star is built only from $u$ and $d$ quarks. It is important to stress that even then the quantum vacuum fluctuation come from all flavors including $s$ quark and all antiparticles. There are no $s$ quarks for small densities in the strange quark star only its quantum fluctuations. In the QMF approach there is no quantum fluctuations at all. This is a significant difference between the NJL and QMF approach. The another one is that the bag constant in the QMF model must be added by hand. This make the QMF approach unreliable for smaller densities (Figure 4).

The gravitational binding energy of a strange quark star and QMF approach) and neutron (RMF) star is presented in Figure 10. The arrow shows a possible transition from unstable neutron star to the strange one with conservation of the baryon number $M_B = M_{\odot}$, $(M_B = m_n c^2 N_B)$.

In the enlarged NJL model [22] vector mesons are included. There contribution to the effective bag constant are positive (Figure 2). This makes that a strange quark star possesses a bit bigger radius and mass then in the NJL (I) model.

However, a maximum stable strange quark star is obtained for $\rho_2 = 3.11 10^{15} \text{ g/cm}^3$ and has the following parameters $M = 1.61 M_\odot$ and $R = 8.74 \text{ km}$. The baryon number is this star is the same as in the case of pure neutron star with $M_B = 2.126 M_\odot$. Below the density $\rho_s = 3.94 \rho_0 (\rho_0 = 2.5 10^{14} \text{ g/cm}^3)$, there are no strange quarks and quark stars. Stable stars are those with $\frac{dM}{dp_c} > 0$ [38] (Figure 7). A gravitational binding energy for a strange quark is lower then a neutron one for $\rho > 1.6 10^{15} \text{ g/cm}^3$. The $M(\rho)$ dependence for the quark star is presented in Figure 7. For the quark star with the maximal central density $\rho_c = 3.11 10^{15} \text{ g/cm}^3$ the star profile is presented on the Figure 8. Quark and electron mass distributions inside the star are presented in Figure 8. The quark partial fractions defined as

$$X_f = \frac{n_f}{(n_u + n_d + n_s)} = \frac{3n_f}{n_B},$$

where $f = (u, d, s)$ are presented in Figure 5.

**Conclusion**

The properties of the strange quark star in the bag model with $B = B_c$ and the current quark masses $(m_u = m_d = 0, m_s = 150 \text{ MeV})$ are presented in [9]. The star model base on the QMC is very similar (the dotted green line in Figure 6) and close to the one for the ENJL model parametrized by the $TM1$ parameter set ($x_v = 0.65$, the solid violet curve 6). The properties of strange stars with quark masses changing continuously from constituent quark mass to the small current (see Figure 1) are presented in [43]. All these stars are more compact then neutron stars (see the Figure 6) and are similar to those of the NJL (I) model.
In this paper the enlarged NJL model is used to construct the EoS and properties of the strange quark star. The stable strange quark star exists from minimal central density till the maximal one $\rho_2 = 3.1 \times 10^{15}$ g/cm$^3$ which gives the following star parameters $M = 1.61 \, M_\odot$ and $R = 8.74 \, km$. Its baryon number is the same as for the pure neutron star with $M_B = 2.126 \, M_\odot$. The very similar strange star but less compact is obtained in the solvable NJL model [13]. The gravitational binding energy for a strange quark star is lower then the neutron one for densities $\rho > 1.6 \times 10^{15}$ g/cm$^3$. The conversion of an unstable neutron star into a strange star is an exciting subject which may help explain the gamma-ray bursts enigma [44].
Similarly to the QMF model the enlarged NJL one includes also the coupling to vector mesons. This is crucial for the quark star properties. Also the quark $s$ mass is important. The mass of a quark $s$ is also relevant because its smaller mass causes that a strange quarks appear for lower densities. Nonzero strangeness of the matter gives as a result a strange star. It is fascinating that the neutron and strange star properties are strictly connected to inner structure of nuclei and nucleons.
Figure 8. The profile for effective quark Fermi momentum of the maximal quark star (ENJL model) with the central density $\rho_c = 3.11 \times 10^{15} \text{g/cm}^3$.

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Figure 9. The quark effective mass profile inside the maximal quark star (ENJL model) with the central density $\rho_c = 3.11 \times 10^{15} \text{g/cm}^3$.

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Figure 10. The gravitational binding energy of the quark strange (ENJL and QMF approach) and neutron (RMF) star. The arrow shows a possible transition from unstable neutron star to the strange one with conservation of the baryon number $M_B = m_n c^2 N_B$. 
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$\varepsilon$ (MeV/fm$^3$)

$P$ (MeV/fm$^3$)

- --- su(2)
- --- su(3) (I)
- --- Fe w. dwarf
- --- su(3) (ENJL)
- --- nucl (RMF)