Filtering precipitation through soil

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Abstract. The article studies the process of filtering precipitation through soil. Two mathematical models are obtained and used, specifically, deterministic and stochastic ones. The basis of these tasks is a diffusion parabolic partial differential equation supplemented by initial-boundary conditions. The findings are used in solving the stochastic problem. The mathematical analytical formulation of the problem using the finite difference method is reduced to an inhomogeneous linear algebraic system of equations, which is implemented in the environment of the Matlab computing complex. Consideration of the task operator linearity and the principle of superposition reduced the amount of computation significantly. A number of conclusions were drawn on the basis of solutions.

1. Introduction

The increased loads posed on the hydrosphere intensify the large-scale processes occurring in it. Today the catastrophic changes in the natural cycle of water whose consequences are difficult to predict and evaluate are being intensively discussed. The requirements for environmental protection and environmental management are constantly being restricted. In this regard, one of the priority scientific tasks is the rational and environmentally safe use of natural resources, diagnostics of the state of the hydrosphere, forecasting the development of technogenic processes and managing them. And the attention to it is constantly increasing. These and other reasons lead to the necessity to clarify the models of practical problems and methods for solving them.

The interconnected issues concerning filtering water through soil and predicting groundwater levels are currently becoming critical. The reason is the global climate change of the Earth and the extremely and commonly increased intensity of precipitation in particular. Studying the pattern of moisture distribution in the upper layers of the soil is relevant for successful land use in agriculture and construction industry.

Groundwater is formed mainly due to the infiltration of precipitation and the seepage of water from rivers, lakes and reservoirs. Groundwater causing catastrophic floods everywhere is very sensitive to all atmospheric changes. These issues and the related ones were studied in [1–4].

Mathematical models of precipitation filtration and groundwater level contain many parameters of statistical origin, which makes it difficult to accurately predict them. In such cases, one has to invoke stochastic models [5].

Depending on the amount of precipitation and the depth of groundwater, their surface experiences seasonal and long-term fluctuations. The magnitudes of seasonal and perennial amplitudes of fluctuations in groundwater levels can reach 20 meters or more, which must be taken into account...
when constructing various kinds of water management facilities and conducting reclamation agricultural work.

2. Mathematical models

2.1. Deterministic task

The one-dimensional advancement of the wetting boundary in a distributed system such as soil under precipitation in a linearized version is described by the diffusion equation in partial derivatives of the parabolic type

\[ LH(x, t) = \frac{\partial}{\partial t}H(x, t) - a \frac{\partial^2}{\partial x^2}H(x, t) + b \]

(1),

\[ (x, t) \in G = \{ x, t : 0 < x < l, 0 < t \leq T \}. \]

Here, \( L \) is a linear deterministic operator, \( H(x, t) \) is the boundary of moistened soil, \( a \) is the moisture conductivity coefficient, \( b \) is a coefficient that takes into account the loss of soil permeability, for example, due to an increase in soil density in proportion to depth, \( G \) is the domain of problem definition, \( f_1(x, t) \) is the precipitation intensity. The boundary and initial conditions must be added to equation (1)

\[ H(0, t) = f_2(t), \quad H(l, t) = f_3(t), \quad 0 \leq t \leq T, \]

(2)

\[ H(x, 0) = f_4(x), \quad 0 \leq x \leq l. \]

(3)

The conditions for matching the initial and boundary conditions must also be met.

\[ f_2(0) = f_4(0), \quad f_3(l) = f_4(l). \]

(4)

This means that the function \( H(x, t) \) is inextricable at the points \((0,0), \quad (l, 0)\). Often at the beginning of the process, there is already a certain layer on the surface of the soil saturated with liquid. Its lower boundary can be set by the function \( f_4(x) \). The boundary conditions at all points are equal to zero. This means that the boundaries of the problem definition domain coincide with the boundary of a thundercloud. Let us take it as a rectangle

\[ (x, t) \in G = \{ x, t : 0 \leq x \leq l, 0 \leq t \leq T \}. \]

In this case, all boundary conditions (2)–(4) will be satisfied automatically with a suitable choice of function \( f_1(x, t) \). For example, as follows

\[ f_1(x, t) = F \sin \frac{\pi x}{l} \sin \frac{\pi t}{T}. \]

(5)

It is worth noting that the actually observed precipitation process approximately corresponds to such a description, when at the beginning and end of precipitation along the time and edges through the rectangle, the rain intensity is equal to zero, and in the center it is the highest.

The initial boundary value problem (1)–(5) is a mathematical model for determining the level of groundwater \( H(x, t) \). Its implementation with real components of the vector – function \( f(f_1, f_2, f_3, f_4) \) presents significant complications when using analytical methods. Therefore, we use numerical methods [6, 7], the finite difference algorithm in particular. In order to use it, instead of the two-dimensional continuum \( G \), we introduce a grid domain of discrete points

\[ \Omega = \{ x_i = (i-1)h, \quad i = 1, 2, \ldots, n; \quad t_j = (j-1)\tau, \quad j = 1, 2, \ldots, m \}, \]

\[ h = l/(n-1), \quad \tau = T/(m-1). \]

Let us then take a four-point implicit scheme in the finite difference method. By following [6, 7] instead of equations (1)–(3), we obtain a finite-difference scheme of the problem. Since in this case, only discrete coordinates will be used in contrast to analytical methods and approximate values of the groundwater level will be obtained, we introduce the following notation

\[ y_i^j = H(x_i, t_j), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m. \]

Then the main equation (1) in the finite difference algorithm takes the following form

\[ \frac{-y_i^{j+1} + y_i^{j+1}}{2\tau} - \frac{a}{h^2}(y_i^{j+1} - 2y_i^{j+1} + y_i^{j+1}) + by_i^{j+1} = \phi_i^{j+1}, \]
where

\[
\phi^{i+1}_j = f(x_i, t_j + 0.5\tau).
\]

Let us transform it to a form convenient for calculations, and then we obtain the following

\[
-qy^{i+1}_{i-1} + cy^{i+1}_i - qy^{i+1}_{i+1} = d_i, \quad i = 2, 3, \ldots, n-1, \quad j = 2, 3, \ldots, m-1,
\]

\[
c = 1 + 2q + 2b\tau, \quad q = \frac{2\alpha\tau}{h^2}, \quad d_i = \frac{y^{i-1}_j + 2\tau\phi^{i+1}_j}{h^2}.
\]

Let us note that here the analytic derivatives are replaced by finite-difference with the accuracy \(O(h^2)\).

Instead of conditions (2), (3) let us write the following

\[
y_1^j = f_2, \quad y_n^j = f_3, \quad j = 1, 2, \ldots, m.
\]

Conditions (8) correspond to the case when at the very beginning of the process (\(t = 0\)) there is a soil layer already saturated with moisture at the upper boundary of the soil. Let us choose its lower boundary in the form of a straight line

\[
f_3(x_i) = f_2 + x_i \frac{f_3 - f_4}{l}.
\]

Equations (6) and (7)–(9) form a programmable algorithm for computer calculations.

The final result is that the initial-boundary task for solving a partial differential equation of parabolic type is reduced to the solution of an inhomogeneous system of linear algebraic equations

\[
By = d
\]

on each layer of the grid region in time. Here \(y, d\) are vectors of dimension \(n\), \(B\) is a square matrix of \(n\) order. Such a system is compiled and solved with the gradual movement of the two-layer template one step up.

The matrix and the vector have the following form

\[
B = \begin{bmatrix}
1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
-q & c & -q & \cdots & \cdots & \cdots \\
-q & c & -q & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-q & c & -q & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-q & c & -q & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}, \quad d = (f_2, d_2, d_3, \ldots, d_{n-1}, f_3)^T.
\]

Here the superscript \(T\) corresponds to the vector \(d\) transpose.

There are a large number of well-developed methods for solving system (10), and they are included in the software of all computing systems. In this case, we will use the capabilities of the Matlab computing complex.

**Task 1.** The main units of measurement are a meter and an hour.

\[
l = 1000 \text{ m}, \quad a = 10 \text{ m/h}, \quad b = 0.2 \text{ m^2/h}, \quad T = 12 \text{ h},
\]

\[
n = 1001, \quad m = 1201, \quad f_1(x,t) = F \sin \frac{\pi x}{l} \sin \frac{\pi t}{2T},
\]

\[
F = 0.15 \text{ m/h}, \quad f_2 = 0.03 \text{ m}, \quad f_3 = 0.05 \text{ m}.
\]

It is required to identify function \(H(x,t)\) in the finite difference algorithm

The calculation results obtained in the Matlab environment in the form of curves \(H(x,t)\) are presented in Fig.
Figure 1. Wet soil boundaries

Each curve is accompanied by a value of time $t$. It can be seen that precipitation gradually penetrates into the soil and the boundary of the moistened zone moves inland. The curved lines depict the functions $H(x,t)$. The blue area is discussed in paragraph 2.2 given below.

2.2. Stochastic task

Variability of moisture in soils and stochasticity are caused by many reasons, specifically, the intensity of rainfall, their duration and repeatability, season, geographical location, type of rain (drizzling, widespread, rainfall), heterogeneity of the soil, etc. Simultaneous consideration of all random factors in the mathematical models is almost impossible, since the necessary statistical information is missing.

In this case, we will be interested in the random nature of the rain intensity, which has the greatest effect on soil moisture saturation. Intensity is defined as the layer or amount of rainfall that falls per unit of time. It usually ranges from 0.25 mm/h to 100 mm/h. Its registration and calculation of indicators are necessary in the construction of many economic systems and structures. The monthly average rainfall is taken into account when designing sewer systems, engineering structures, and drainage of farmland.

The data published in the specialized literature and design standards allow a wide range of parameter values, which clearly shows the relevance of stochastic models for an adequate description of soil moisture.

The input function of the problem in the form of precipitation intensity is actually a spatio-temporal non-stationary random process. Consequently, the output process $H(x,t)$ will be the same. Next, we will be interested in the most important parameters $H(x,t)$ in the form of the mathematical expectation $m_H(x,t)$ and the standard deviation $\sigma_H(x,t)$. The methods of probability theory and mathematical statistics will be used when calculating them [8, 9].

Task 2. Numerical initial parameters of problem 1 except function $f_i(x,t)$ are kept. The rainfall intensity formula now is as follows

$$F(x,t) = m_F \sin \frac{\pi x}{l} \sin \frac{\pi t}{2T}.$$  

Here $m_F = 0.15 \text{ m/h}$, is a random number expectation, $\sigma_F = 0.1m_F \text{ m/h}$ is a mean-square deviation, $f_2 = 0.03 \text{ m}, \ f_3 = 0.05 \text{ m}.$

The choice of such parameter values and the function $F$ significantly reduce the amount of computation. Let us take into account that this system is a linear distributed one, that is, it satisfies the condition of the superposition principle [11]

$$A(u_1 + \alpha_1 u_2 + \alpha_2 u_2 ) = \alpha_1 A u_1 + \alpha_2 A u_2$$

for any state functions $u_1(x,t), \ u_2(x,t)$ and numerical factors $\alpha_1, \ \alpha_2$. Let us use the principle of superposition two times: to determine the mathematical expectation $mH(x,t)$ and mean-square
deviation $\sigma_{H(x,t)}$. Herewith, the operator $A$ includes the main equation and all initial boundary conditions. The first task of determining mathematical expectations takes the following form

$$Am_{H}=m.$$

The expectation function coincides with the function of precipitation intensity of the Example 1. In this case, the values calculated in example 1 $y_i(x_i)$ coincide with expectations $m_{H}(x_i)$. Therefore, two graphs are combined in fig. 1. A similar procedure is repeated with respect to the standard deviation $\sigma_{H}$ and the results shown in Fig. 2. Comparison of fig. 1 and fig. 2 confirms the validity of applying the superposition principle, i.e., the ordinates of Fig. 2 decreased exactly 10 times.

![Figure 2. Wet standards deviations](image)

In all cases, the form of the distribution function of random variables does not affect the calculation results. However, if the boundary of the movement of the moistened soil zone $H(x,t)$, as a random variable, has a Gaussian distribution, we can calculate the probability of a given deviation from the mathematical expectations shown in Fig. 2. It can be used to predict soil moisture.

For this purpose, we use the widespread “three sigma rule” [9]. The task is to find the probability of deviation of a random variable $H(x,t)$. In absolute value it would be less than a given positive number $\delta$, that is, it is required to find the probability of inequality $|H - m_{H}| < \delta$. In this case, $\delta = 3\sigma_{H}$, and desired probability is equal to

$$P(|H - m_{H}| < 3\sigma_{H}) = 2F(3) = 0.9973.$$

Where $F$ is a Laplace tabulated function. The obtained result means that the probability of a random deviation $H(x,t)$ from the expectation in absolute value will be less than three sigma. This area is shaded in blue in Fig. 1.

The probability of a random value ejecting beyond the specified boundaries is 0.0027, i.e. 0.27 %. According to the principle of the impossibility of improbable events, such an event is considered almost impossible.

3. Conclusion
1. The problems having important economic results for several sectors of the economy are resolved.
2. Mathematical models of water filtration through the soil under the influence of precipitation in deterministic and stochastic versions are created.
3. Algorithms for solving problems by numerical methods that are easily implemented in the environments of popular computing complexes are proposed.
4. It is shown that the preliminary solution of the deterministic problem in a linear distributed system significantly simplifies the stochastic problem solution.

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