Application of Sharafutdinov’s Ray Transform in Integrated Photoelasticity

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Abstract. We explain the main concepts centered around Sharafutdinov’s ray transform, its kernel, and the extent to which it can be inverted. It is shown how the ray transform emerges naturally in any attempt to reconstruct optical and stress tensors within a photoelastic medium from measurements on the state of polarization of light beams passing through the strained medium. The problem of reconstruction of stress tensors is crucially related to the fact that the ray transform has a nontrivial kernel; the latter is described by a theorem for which we provide a new proof which is simpler and shorter as in Sharafutdinov’s original work, as we limit our scope to tensors which are relevant to Photoelasticity. We explain how the kernel of the ray transform is related to the decomposition of tensor fields into longitudinal and transverse components. The merits of the ray transform as a tool for tensor reconstruction are studied by walking through an explicit example of reconstructing the $\sigma_{33}$-component of the stress tensor in a cylindrical photoelastic specimen. In order to make the paper self-contained we provide a derivation of the basic equations of Integrated Photoelasticity which describe how the presence of stress within a photoelastic medium influences the passage of polarized light through the material.

Keywords: Photoelasticity, stress tensor reconstruction, ray transform, tensor tomography, anisotropic media, birefringence, generalization of Radon transform

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1. Introduction

A transparent optical medium which initially is isotropic and homogeneous with respect to light propagation may become birefringent when subjected to external strain, an effect which is known as Photoelasticity [1, 2]. The spatially varying tensor of refraction then reflects the presence of stresses within the material, opening up the possibility of examining internal stresses by means of the change in the state of polarization of light beams passing through the strained medium. The reconstruction of local optical and stress tensors from the set of data collected for all possible directions, and locations, of light beams passing through the specimen is called Integrated Photoelasticity [3].

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Numerous efforts have been made in the past to tackle the problem of Integrated Photoelasticity [3–12]. While the reconstruction of the optical tensors in the two-dimensional problem is well-established (for an overview see [5]), the three-dimensional (3D) case so far has been solved only for special cases involving a priori assumptions about the symmetry of the 3D stress distribution, the rotation of the principal axes of the stress tensor, or the strength of the anisotropy of the dielectric tensor. The most general case of 3D tensor tomography with arbitrarily shaped bulk matter and no restriction on the degree of anisotropy is unsolved. The source of difficulty is the fact that the Radon transform which provides a well-established formalism for the reconstruction of scalar fields can no longer be utilized if tensor fields are involved.

In [13], Sharafutdinov has proposed a generalization of the Radon transform to the case of symmetric tensor fields of arbitrary degree $m$ defined on a, possibly curved, Riemannian manifold of arbitrary dimension $n$; he has termed his construction the ray transform. Amongst many other aspects, his work encompasses the examination of the formal structure of the ray transform, its relation to the Fourier transform, its non-trivial kernel and the problem of inversion of the ray transform given the fact that the kernel is non-zero. Sharafutdinov’s contribution undoubtedly opens up the right path to the goal of solving the general problem of Integrated Photoelasticity; however, since his work aims to tackle the most general case of symmetric tensors of arbitrary degree $m$ in $n$ dimensions, his formalism necessarily exhibits a degree of complexity which may be undesirable for practitioners who wish to put his theory into a specific physical or engineering context. It is therefore an important task to lay out his results in a way which focuses on the special case of symmetric ($m = 2$) tensor fields, and vector ($m = 1$) fields, defined in a three-dimensional Euclidean space which is to be identified with the bulk material of the medium, and to study the merits of the ray transform as a tool for tensor reconstruction within the specific framework of Photoelasticity.

In the present paper we have attempted to provide some steps in this task: We explain the main concepts related to the ray transform, its kernel, and the extent to which it can be inverted, for the special case of vector- and symmetric tensor fields in $\mathbb{R}^3$. Our goal is on the one hand to provide an overview of the mathematical structure of the ray transform, and on the other hand to illustrate the method by discussing a simple application within the field of Integrated Photoelasticity. With these objectives in mind we have performed a new proof on one of the central theorems in Sharafutdinov’s theory, namely the kernel of the ray transform; this theorem provides essential insight
into the degree to which optical or stress tensors can be reconstructed from photoelastic data. Our proof focuses on vector- and symmetric 2-tensors only, which moreover are reconstructed plane-wise, so that they are effectively defined on a two-dimensional \( \mathbb{R}^2 \). This allows us to take alternative routes in the various steps of the proof which are not available in the most general case. This proof is given along the way of explaining the emergence of the ray transform in photoelastic stress reconstruction.

The plan of the paper is as follows: In section 2 we derive the basic equations of Integrated Photoelasticity governing the evolution of the components of the electric field vector of a polarized light beam passing through a medium with spatially varying dielectric tensor. In section 3 we show how these equations can be used to describe the impact of stress in a transparent material on polarized light. In section 4 we show how the concept of the ray transform emerges as we try to reconstruct optical and stress tensors from photoelastic measurement data. In section 5 we characterize the kernel of the ray transform of vector and symmetric tensor fields, providing a new proof, albeit with a narrower scope, to Sharafutdinov’s general theorem. In section 6 we show how the kernel of the ray transform is related to the decomposition of tensor fields into longitudinal and transverse components and explain how this knowledge can be utilized to reconstruct stress tensor components in a photoelastic specimen. Finally, in section 7 we speculate about the extension of the formalism at hand into the high-stress regime using the Newton-Kantarovich method. In section 8 we summarize our results.

2. Basic equations of Integrated Photoelasticity

We now derive the basic equations of Integrated Photoelasticity under the straight-line assumption, i.e., light rays passing through photoelastic media are assumed to propagate along straight lines. This assumption is justified as long as anisotropy is sufficiently weak.

We wish to examine the propagation of a plane polarized electromagnetic wave through a weakly birefringent photoelastic medium. The birefringence, i.e., the anisotropy of the dielectric tensor \( \epsilon_{ij}, i, j = 1, 2, 3 \), is assumed to be entirely resulting from the stress within the material, such that, in the unloaded state, the material is homogeneous and isotropic with a real dielectric tensor \( \epsilon_{ij} = \epsilon \delta_{ij} \), and \( \epsilon = \text{const} \). The material is assumed to be unmagnetic with vacuum permeability \( \mu_0 \), and non-absorbing at least in the optical range of wavelengths, so that the material is transparent to visible light. The permittivity of vacuum will be denoted as \( \epsilon_0 \).
We start with the source-free Maxwell equations in the dielectric medium,
\begin{align}
\nabla \times \mathbf{H} &= \dot{\mathbf{D}}, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \\
\nabla \cdot \mathbf{E} &= 0, \quad \nabla \cdot \mathbf{B} = 0,
\end{align}
\tag{1a, 1b}
where we have ignored the microscopic sources of the dielectric; they are taken into account phenomenologically by the spatially varying dielectric tensor $\epsilon_{ij}$, which links the electric field $\mathbf{E}$ and the displacement field $\mathbf{D}$ according to
\begin{equation}
D_i = \epsilon_{ij} E_j, \tag{2}
\end{equation}
where the convention of summing over double indices is adopted. Furthermore, the effective charge density due to polarization effects is ignored,
\begin{equation}
\rho_{\text{eff}} = -\nabla \cdot \mathbf{P} = 0. \tag{3}
\end{equation}
If the second equation in (1a) is inserted into the first, we obtain the wave equation
\begin{equation}
-\Delta \mathbf{E} + \frac{1}{\mu_0} \ddot{\mathbf{D}} = 0, \tag{4}
\end{equation}
where $\Delta$ is the three-dimensional Laplace operator in Cartesian coordinates.

We now assume a harmonic time dependence $\sim e^{-i\omega t}$ of all fields, and a wave propagation in the $z$-direction. We may neglect the $x$- and $y$-dependence of the fields, since the electromagnetic energy in the geometrical-optics limit propagates along narrow tubes of light rays in such a way that, along the cross-section of a given tube, the fields may be treated as independent of $x$ and $y$. Then (4) becomes a Helmholtz equation
\begin{equation}
\frac{d^2 \mathbf{E}}{dz^2} + \mu_0 \omega^2 \mathbf{D} = 0. \tag{5}
\end{equation}
Provided that birefringence is weak, i.e., the load on the material is not too strong, the component of the electric field in the direction $\mathbf{e}_z$ of the wave vector can be neglected. This leads to an Ansatz for the electric field as follows:
\begin{align}
\mathbf{E}(z,t) &= \Re \{ A_1(z) \mathbf{e}_1 + A_2(z) \mathbf{e}_2 \} e^{ikz-i\omega t}, \tag{6a}
k &= \frac{\omega}{u} = \frac{1}{\sqrt{\mu_0 \epsilon}}, \tag{6b}
\end{align}
This Ansatz contains the assumption that the phase velocity $u$ of the wave is homogeneous and isotropic, and is determined by the permittivity $\epsilon$ of the unstressed medium; this is true under the same conditions under which the $z$-component of the electric field can be neglected, i.e.
sufficiently weak anisotropy. Eqs. (6) express the straight-line assumption mentioned at the beginning of this section.

Now the second derivative in \( z \) of the complex electric field in (6) is

\[
E''_i = A''_i e^{ikz} + 2ik A'_i e^{ikz} - k^2 A_i e^{ikz}, \quad i = 1, 2, \quad (7)
\]

where a prime indicates a derivative \( \partial/\partial z \). If the medium were homogeneous, the amplitudes \( A_i \) were independent of \( z \), just as within vacuum. The inhomogeneity of the medium makes them \( z \)-dependent, but as long as birefringence is weak, this dependence is sufficiently weak to neglect the second derivative \( A''_i \) in (7). If we then insert (7) into (5) and divide by \( 2ik \), we obtain

\[
A'_i = \frac{k}{2\epsilon} A_i - \frac{k}{2i\epsilon} \epsilon_{ij} A_j, \quad i = 1, 2, \quad (8)
\]

We introduce a constant

\[
C_0 = \frac{k}{2\epsilon} = \frac{\omega}{2c\sqrt{\epsilon_0\epsilon}}, \quad (9)
\]

with the help of which (8) can be written as

\[
\frac{dA_i}{dz} = A'_i = i C_0 (\epsilon_{ij} A_j - \epsilon A_i), \quad i = 1, 2, \quad (10a)
\]

or as a matrix equation

\[
\frac{d}{dz} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = i M(z) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad M(z) = C_0 \begin{pmatrix} \epsilon_{11} - \epsilon & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} - \epsilon \end{pmatrix}. \quad (10b)
\]

The complex two-component vector \( A \equiv (A_1, A_2)^T \) is called the Jones vector of the light beam. Eq. (10b) can be converted into an operator equation as follows: We may partition the photoelastic medium into many thin slices perpendicular to the direction of the wave propagation \( e_z \). Each slice acts on the Jones vector of the light beam either as a retarder, i.e., introducing a phase shift between two orthogonally polarized components of the wave, or as a rotator, i.e., rotating the plane of polarization of a plane-polarized wave, or both. This implies that each slice acts on the local Jones vector as a unitary matrix, and hence the same must be true for the total bulk of the photoelastic specimen. If \( z_i \) is some point before the medium (possibly \( z_i = -\infty \)) and \( z_f \) is some point behind the medium (possibly \( z_f = +\infty \)), then the Jones vectors \( A_i \) and \( A_f \) at these points must be related by a unitary transformation \( U(z_f, z_i) \). In fact, such a transformation must exist for any pair of points \( z, z' \), so that we can write

\[
A_z = U(z, z') A_{z'} \quad \text{for all } z, z'. \quad (11)
\]
The matrix $U(z, z')$ must satisfy an equation analogous to (10b),
\[
\frac{d}{dz} U(z, z') = i M(z) U(z, z') .
\] (12)

The matrices $U(z, z')$ obviously must have the group property,
\[
U(z_3, z_1) = U(z_3, z_2) U(z_2, z_1) ,
\]
\[
U(z_1, z_2) = U(z_2, z_1)^{-1} ,
\]
\[
U(z, z) = \mathbb{1}_2 .
\] (13)

3. Stress-optical relations

The stress-optical equations linking the dielectric tensor $\epsilon_{ij}$ with the stress tensor $\sigma_{ij}$,
\[
\epsilon_{ij} = \epsilon \delta_{ij} + C_1 \sigma_{ij} + C_2 \text{tr} \sigma \delta_{ij} ,
\] (14)

were discovered long ago by Maxwell [14]. Since the material is assumed homogeneous with respect to its stress-optical properties, the quantities $C_1$ and $C_2$ are constants, characterising the stress-optical properties of the specimen. In (14), $\epsilon_{ij}$ and $\sigma_{ij}$ depend on the spatial coordinates $(x, y, z)$, but $\epsilon$ is constant throughout the material, as mentioned in section 2. Using (14) we can express the equations (10) in terms of the stress tensor $\sigma_{ij}$ rather than the dielectric tensor $\epsilon_{ij}$. We start again with (10b) and substitute (14) for the components of the dielectric tensor in the matrix $M(z)$; this gives
\[
M(z) = C_0 \begin{pmatrix} C_1 \sigma_{11} + C_2 \text{tr} \sigma & C_1 \sigma_{12} \\ C_1 \sigma_{21} & C_1 \sigma_{22} + C_2 \text{tr} \sigma \end{pmatrix} ,
\] (15)

where the trace $\text{tr} \sigma$ runs over all three indices,
\[
\text{tr} \sigma \equiv \sum_{n=1}^{3} \sigma_{nn} .
\] (16)

We now perform the split
\[
C_1 \sigma_{11} + C_2 \text{tr} \sigma = \frac{1}{2} C_1 (\sigma_{11} - \sigma_{22}) + B ,
\] (17a)
\[
C_1 \sigma_{22} + C_2 \text{tr} \sigma = -\frac{1}{2} C_1 (\sigma_{11} - \sigma_{22}) + B ,
\] (17b)

where
\[
B = \frac{1}{2} C_1 (\sigma_{11} + \sigma_{22}) + C_2 \text{tr} \sigma .
\] (17c)
When inserted into (15), the matrix $M$ can be written as

$$M(z) = C_0 C_1 \left[ \frac{1}{2}(\sigma_{11} - \sigma_{22}) \quad \frac{\sigma_{12}}{\sigma_{21}} \quad -\frac{1}{2}(\sigma_{11} - \sigma_{22}) \right] + C_0 B \mathbb{1}_2 = (18a)$$

$$= N(z) + C_0 B \mathbb{1}_2 . \quad (18b)$$

We see that the term $C_0 B$ just introduces a phase common to both components $A_1$ and $A_2$ of the Jones vector. This phase is physically immaterial and can be absorbed by defining a "reduced" Jones vector $A_r$,

$$A_r \equiv A \exp \left( -i C_0 \int_{-\infty}^{z} dz' B(z') \right) . \quad (19)$$

The dynamical equations (10) expressed in terms of $A_r$ then take the simpler form

$$\frac{d}{dz} A_r(z) = i N(z) A_r(z) , \quad (20a)$$

$$N(z) = \frac{1}{2} C_0 C_1 \left[ \frac{\sigma_{11} - \sigma_{22}}{\sigma_{21}} \quad \frac{2 \sigma_{12}}{\sigma_{21}} \quad -(\sigma_{11} - \sigma_{22}) \right] . \quad (20b)$$

Since Jones vectors at different points are unitarily related, see (11), eq. (20a) gives rise to an operator differential equation analogous to (10b),

$$\frac{d}{dz} U(z, z') = i N(z) U(z, z') , \quad U(z, z) = \mathbb{1}_2 . \quad (21)$$

This can be converted into an integral equation

$$U(z, z') = \mathbb{1}_2 + i \int_{z'}^{z} dz_1 N(z_1) U(z_1, z') . \quad (22)$$

A formal solution is given by the Born-Neumann series

$$U(z, z') = \mathbb{1}_2 + \int_{z'}^{z} dz_1 i N(z_1) + \int_{z'}^{z_1} dz_1 i N(z_1) \int_{z'}^{z_2} dz_2 i N(z_2) + \cdots . \quad (23)$$

If the specimen is located such that the $z$-axis intersects the medium, and $z, z'$ refer to points before and behind the medium, respectively, then the matrix $U(z, z')$ contains all the data acquired from measurements of the phase retardation between orthogonal components of plane-polarized light, and the rotation of the plane of polarized light, along that line of intersection.
For weak birefringence, (23) can be truncated after the first-order term,

\[ U(z, z') = 1 + i \int_{z'}^{z} \frac{dz}{ BD^2 + i z} N(z_1) . \] (24)

Since the medium occupies a bounded region in space, the limits \( z \) and \( z' \) in the integral in (24) may be taken as \( \pm \infty \). On inserting (20b) into (24) we see that the evolution of the state of polarization along the light ray is determined by the two integrals

\[ I_w = \int_{-\infty}^{\infty} dz \sigma_{12} , \] (25a)

\[ I_u = \int_{-\infty}^{\infty} dz (\sigma_{11} - \sigma_{22}) . \] (25b)

The significance of the notation \( I_w, I_u \) will become clear shortly.

4. The ray transform in Photoelasticity

We now wish to examine the stress within a cylindrical object which occupies a cylindrical domain \( G = D \times (a, b) \), where \( D \) is a two-dimensional region in the \( xy \)-plane and the boundary \( \partial D \) of \( D \) is a strictly convex smooth curve. By \( B = \partial D \times (a, b) \) we denote the lateral surface of the cylinder \( G \). The stress tensor is supposed to be smooth in \( G \) and on \( B \), and satisfies the equilibrium conditions

\[ \frac{\partial}{\partial x_j} \sigma_{ij} = 0 \] (26)

everywhere. Furthermore, the absence of external forces on the lateral boundary \( \partial B \) implies that

\[ \sigma_{ij} n_j = 0 \] ,

where \((n_1, n_2, 0)\) is the unit outward normal to \( \partial B \) on the lateral surface \( B \) of the cylinder. Strain is therefore applied only at the top \( (z = b) \) and the bottom \( (z = a) \) of the object.

It is now assumed that measurements analogous to (24) are performed along all horizontal straight lines in \( \mathbb{R}^3 \) defined by \( t \mapsto X + t \xi(\alpha) \), where \( X = (x, y, z) \), \((x, y)\) varies through the points in the \( xy \)-plane, \( z \) takes values in the interval \((a, b)\), the angle \( \alpha \) varies in \((0, 2\pi)\), and

\[ \xi(\alpha) = (\cos \alpha, \sin \alpha, 0) \] (28)
is the horizontal unit vector in the direction of the line. These lines generalize the measurements along the \( z \)-axis which were discussed in section 2. For the straight line passing through the point \( X \) along the direction \( \xi \) we now choose an adapted right-handed coordinate system \((\eta, z, t)\) in which the direction of propagation of the light beam is along the positive \( t \)-axis (the \( \eta \)-coordinate line is perpendicular to the \( t \)-coordinate line, and the new \( z \)-direction coincides with the old one). On performing a measurement along the line so defined we then obtain quantities analogous to (25),

\[
I_w(X, \xi) = \int_{-\infty}^{\infty} dt \, \sigma_{\eta z} ,
\]

\[
I_u(X, \xi) = \int_{-\infty}^{\infty} dt \, (\sigma_{\eta\eta} - \sigma_{zz}) ,
\]

the notation on the left-hand side indicating that these lines pass through the point \( X \) and are directed along the unit vector \( \xi \). The tensor components in these integrals can be expressed in terms of the laboratory-frame components \( \sigma_{ij} \) as

\[
(\sigma_{\eta\eta} - \sigma_{zz}) (t, 0, 0) = \sin^2 \alpha \, (\sigma_{11} - \sigma_{33})(X + t\xi) - 2 \sin \alpha \cos \alpha \, \sigma_{12}(X + t\xi) + \cos^2 \alpha \, (\sigma_{22} - \sigma_{33})(X + t\xi) ,
\]

\[
\sigma_{\eta z}(t, 0, 0) = -\sin \alpha \, \sigma_{13}(X + t\xi) + \cos \alpha \, \sigma_{23}(X + t\xi) ,
\]

\[
\sigma_{zz}(t, 0, 0) = \sigma_{33}(X + t\xi) .
\]

We now define a vector field \( w \) in \( \mathbb{R}^2_z \), where

\[
\mathbb{R}^2_z \equiv \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2 , \, z \in (a, b) \text{ fixed}\} ,
\]

by

\[
(w_1, w_2) = (\sigma_{23}, -\sigma_{13}) ,
\]

and a symmetric tensor field \( u \) by

\[
(u_{11}, u_{12}, u_{21}, u_{22}) = (\sigma_{22} - \sigma_{33}, -\sigma_{12}, -\sigma_{21}, \sigma_{11} - \sigma_{33}) .
\]

If (30) is expressed in terms of the components of the tensors \( u \) and \( w \) and inserted into (29) we obtain

\[
I_w(X, \xi) = \int dt \, w_m \, \xi_m ,
\]

\[
I_u(X, \xi) = \int dt \, u_{mn} \, \xi_m \, \xi_n .
\]
According to [13], the collection of quantities $Iw(X, \xi)$, taken for all $X \in \mathbb{R}^2$ and $\xi = \xi(\alpha)$ with $\alpha \in [0, 2\pi]$, constitutes the ray transform of the two-dimensional vector field $w$; whilst the collection of all $Iu(X, \xi)$, with the same range of variables, constitutes the ray transform for the two-dimensional symmetric tensor $u$.

The general definition of the ray transform of a symmetric tensor field $T_{i_1\ldots i_m}$ of degree $m$ defined in $\mathbb{R}^n$, which includes both $Iw$ and $Iu$ in (33) as special cases, is given by

$$IT(X, \xi) = \int dt \, T_{i_1\ldots i_m}(X + t\xi) \xi_{i_1} \ldots \xi_{i_m},$$

where $X, \xi \in \mathbb{R}^n$ with $|\xi| = 1$, i.e. $\xi$ lies on the unit sphere $S_{n-1}$ in $\mathbb{R}^n$. Since the ray transform is constant for all $X'$ lying on the line $t \mapsto X + t\xi$,

$$IT(X + t\xi, \xi) = IT(X, \xi),$$

we can restrict the range of the variable $X$ to the subspace

$$\xi_\perp = \{ x \in \mathbb{R}^n | x \perp \xi \}$$

of $\mathbb{R}^n$ orthogonal to $\xi$, since any component of $X$ in the direction of $\xi$ may be set to zero. It follows then that the ray transform is really a function on the tangent bundle $TS_{n-1}$ of the unit sphere $S_{n-1}$,

$$TS_{n-1} = \{ (X, \xi) \in \mathbb{R}^n \times S_{n-1} | X \perp \xi \}.$$  

As a consequence, the Fourier transform with respect to $X$ may be restricted to the subspace $\xi_\perp$,

$$\hat{IT}(k, \xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\xi_\perp} dV^{n-1}(X') \, IT(X', \xi) \, e^{-i\langle k, X' \rangle}, \quad k \in \xi_\perp.$$  

The Fourier transform (38) of the ray transform is related to the Fourier transform $\hat{T}$ of the tensor $T$ as follows:

$$\hat{IT}(k, \xi) = (2\pi)^{1/2} \hat{T}_{i_1\ldots i_m}(k) \xi_{i_1} \ldots \xi_{i_m}, \quad k \in \xi_\perp.$$  

This can be proven easily: Inserting (34) into (38) we find

$$\hat{IT}(k, \xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\xi_\perp} dV^{n-1}(X') \int dt \, T_{i_1\ldots i_m}(X') \xi_{i_1} \ldots \xi_{i_m} \, e^{-i\langle k, X' \rangle}.$$  

Since $|\xi| = 1$ and $X' \perp \xi$, we can introduce coordinates $X$ on $\mathbb{R}^n$ by $X \equiv (X', t)$, in which case $dV^{n-1}(X') dt = dV^n(X)$. Furthermore,
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since \( k \perp \xi \) by definition (38) of the Fourier transform, the argument of the exponential is

\[
\langle k, X \rangle = \langle k, X' + t \xi \rangle = \langle k, X' \rangle.
\]

If these relations are inserted into (40), (39) follows.

5. The kernel of the ray transform

Our task is now to reconstruct the components of the stress tensor from the ray transforms (33) as far as possible. We shall see that, for the given experimental setup, involving horizontal light rays only, we can only retrieve one component of the stress tensor, namely \( \sigma_{33} \). In order to proceed we need to understand that the ray transform, regarded as an operator \( I \) sending symmetric tensor fields \( T \) (including fields of degree \( m = 1 \), i.e. vector fields) to the set of quantities \( IT \), has a nontrivial kernel: Given two tensor fields \( T_{i_1 \ldots i_m}^{(1)} \) and \( T_{i_1 \ldots i_m}^{(2)} \), does \( IT^{(1)} = IT^{(2)} \) imply \( T^{(1)} = T^{(2)} \)? The answer is no, as proved in a theorem in [13].

The original version of this theorem is designed to encompass the most general case of symmetric \( m \)-tensors defined on \( \mathbb{R}^n \), with no reference to a specific physical application or context. Accordingly, the proof, encumbered with this degree of generality, occupies a substantial number of pages. However, this generality is not required for the particular purposes of reconstructing vector- and symmetric 2-tensor fields within the context of Photoelasticity. If we limit our scope to these types of fields, the proof of the theorem can be performed along alternative routes which are shorter than in the original work. In this section we provide this new proof for the two special cases of a vector field \( w \) and a symmetric tensor field \( u \) defined in \( \mathbb{R}^2 \) as discussed in (32, 33):

THEOREM 5.1 (Kernel of ray transform).

A. Let \( w \) be a smooth vector field in \( \mathbb{R}^2 \) with compact support. Then the following statements are equivalent:

a) \( Iw = 0 \) .

b) There exists a compactly supported scalar field \( \phi \) on \( \mathbb{R}^2 \) such that its support is contained in the convex hull of the support of \( w \), and

\[
w_i = \partial_i \phi , \quad i = 1, 2 .
\]

(42a)

c) The identity

\[
\partial_1 w_2 - \partial_2 w_1 = 0
\]

holds.

(42b)
**B.** Let $u$ be a symmetric tensor field on $\mathbb{R}^2$ with compact support. Then the following statements are equivalent:

a) $Iu = 0$ .

b) There exists a compactly supported vector field $v$ in $\mathbb{R}^2$ such that its support is contained in the convex hull of the support of $u$, and

$$u_{ij} = \frac{1}{2} (\partial_i v_j + \partial_j v_i) , \quad i = 1, 2 . \quad (42c)$$

c) The identity

$$\partial_1^2 u_{22} + \partial_2^2 u_{11} - 2 \partial_1 \partial_2 u_{12} = 0 , \quad (42d)$$

holds.

**Proof.**

5.1.1. *Case A.*

The equivalence $(A.b) \Leftrightarrow (A.c)$ is well-known in standard Vector Analysis, where $(42b)$ expresses the fact that the vector field $w_i$ must be curl-free in order for a potential $\phi$ to exist. We therefore shall not prove this equivalence here.

We now prove $(A.b) \Rightarrow (A.a)$: Inserting $(42a)$ into the first integral of $(33)$ yields

$$Iw(X, \xi) = \int dt \, \xi_m \partial_m \phi(X + t \xi) = \int dt \, \frac{d}{dt} \phi(X + t \xi) = 0 , \quad (43)$$

since $\phi$ has compact support and therefore must vanish at $t = \pm \infty$.

Finally we prove $(A.a) \Rightarrow (A.c)$: To this end we observe that the function $\partial_1 w_2 - \partial_2 w_1$ is a scalar with respect to rotations in the plane $\mathbb{R}^2$. This is obvious if we extend the field $w_i, i = 1, 2$, smoothly to a three-dimensional vector field $\tilde{w}_i, i = 1, 2, 3$, defined in $\mathbb{R}^3$, such that $\tilde{w}_1 = w_1$ and $\tilde{w}_2 = w_2$ on the plane $z = \text{const}$. Then $(42b)$ is just the $z$-component of the curl of $\tilde{w}$, which transforms like a scalar under $SO(2)$-rotations in this plane. It follows that the collection of integrals

$$\int dt \,(\partial_1 w_2 - \partial_2 w_1)(X + t \xi) , \quad (44)$$

taken for all directions $\xi$ in $\mathbb{R}^2$, is just the two-dimensional *Radon transform* of this scalar field. If all of the integrals (44) were zero, the
invertibility of the Radon transform would imply that the integrand must vanish and (42b) must hold. But this is indeed the case: Let us take the derivative of the equation $Iw = 0$ with respect to $x_1$,

$$\partial_1 Iw = \int dt \{ \cos \theta w_{1,1} + \sin \theta w_{2,1} \} = 0 \quad , \quad (45)$$

where we use the abbreviation $w_{1,1} \equiv \partial_1 w_1$, etc. On account of the fact that $w_1$ has compact support and therefore must vanish at infinity we have

$$\int dt \{ \cos \theta w_{1,1} + \sin \theta w_{1,2} \} = \int dt \frac{d}{dt} w_1 = 0 \quad , \quad (46)$$

so that (45) gives

$$\partial_1 Iw = \sin \theta \int dt \{ w_{2,1} - w_{1,2} \} = 0 \quad . \quad (47)$$

From $\partial_2 Iw = 0$ we infer a similar equation, with $\sin \theta$ replaced by $\cos \theta$ in front of the integral. These equations must hold for all $\theta$, so that the line integrals (44) all vanish. This proves the statement.

This finishes the proof of case (A.).

5.1.2. Case B.

Our strategy here is different, since we cannot employ standard techniques from Vector Analysis.

We first prove the equivalence (B.b) $\Leftrightarrow$ (B.c): The implication (B.b) $\Rightarrow$ (B.c) is trivial, and follows immediately by inserting (42c) into (42d).

Now we prove (B.c) $\Rightarrow$ (B.b): To this end we define a two-dimensional vector field $(v_1, v_2)$ as follows:

$$v_i = \frac{1}{\pi} \int_{-\infty}^{0} d\theta \int_0^{2\pi} dt \; u_{mn}(X + t \xi) \; \xi_m \xi_n \xi_i \quad , \quad (48)$$

where $i = 1, 2$, $\xi_1 = \cos \theta$, $\xi_2 = \sin \theta$. We can now compute the expression $\partial_1 \partial_1 v_1 \equiv v_{1,11}$, by using the fact that

$$\frac{d}{dt} u_{11} = \cos \theta u_{11,1} + \sin \theta u_{11,2} \quad . \quad (49)$$

Making use of the assumption (42d) we are able to derive

$$v_{1,11} = u_{11,1} \quad . \quad (50a)$$

In a similar way we find

$$v_{1,12} = u_{11,2} \quad . \quad (50b)$$
Eqs. (50) imply that $\partial_1 v_1 = u_{11} + c$, where $c$ is a constant. However, $u_{11}$ has compact support and therefore vanishes at infinity, whilst $v_1$ by construction behaves like $1/|X|$ for $|X| \to \infty$ and, in particular, vanishes at infinity. It follows that $c = 0$, and hence

$$u_{11} = \partial_1 v_1 \quad ,$$

which proves eq. (42c) for the case $i = j = 1$. The proof for $u_{12}$ and $u_{22}$ proceeds in exactly analogous a manner.

Next we prove (B.b) $\Rightarrow$ (B.a): Inserting (42c) into (33b) yields

$$I u(X, \xi) = \int dt \xi_m \xi_n \partial_m v_n(X + t \xi) = \int dt \xi_n \frac{d}{dt} v_n(X + t \xi) = 0 \quad ,$$

since $v$ has compact support.

Finally we prove (B.c) $\Rightarrow$ (B.a): Using the definition (33b) of $I u$, we derive the equations

$$(I u)_{11} = \int dt \left\{ \cos^2 \theta u_{11,11} + 
\quad + 2 \sin \theta \cos \theta u_{12,11} + \sin^2 \theta u_{22,11} \right\} \quad ,$$

$$(I u)_{12} = \int dt \left\{ \cos^2 \theta u_{11,12} + 
\quad + 2 \sin \theta \cos \theta u_{12,12} + \sin^2 \theta u_{22,12} \right\} \quad ,$$

$$(I u)_{22} = \int dt \left\{ \cos^2 \theta u_{11,22} + 
\quad + 2 \sin \theta \cos \theta u_{12,22} + \sin^2 \theta u_{22,22} \right\} \quad .$$

Starting with (53b) we find on using (42d) that

$$(I u)_{12} = \int dt \left\{ \cos \theta \frac{d}{dt} u_{11,2} + \sin \theta \frac{d}{dt} u_{22,1} \right\} = 0 \quad .$$

Using the same technique we obtain for (53a, 53c)

$$(I u)_{11} = (I u)_{22} = 0 \quad .$$

It follows that

$$(I u)_{1} = f_1(\theta) \quad , \quad (I u)_{2} = f_2(\theta) \quad .$$

The functions $f_1$ and $f_2$ can be evaluated at points $X$ arbitrarily close to infinity, so using the fact that $u$ has compact support implies that
$f_1 = f_2 = 0$. Then, repeating the same argument for $Iu_1$ and $Iu_2$ shows that we must have $Iu = 0$.

This finishes the proof of case (B.).

This finishes the proof of theorem 5.1. ■

From eqs. (42b) and (42d) we learn that elements of the kernel of the ray transform can be characterized by the identical vanishing of a differential expression of the vector/tensor components $w_i, u_{ij}$. The differential operator acting in (42b) and (42d) is generically called the Saint-Venant operator [13]. Its vanishing can be understood as an integrability condition ensuring the existence of the scalar/vector fields $\phi$ and $v$ occurring in eqs. (42a) and (42c). We see that in the simplest case of a vector field $w$, the condition for the vanishing of the ray transform is equivalent to the statement that $w$ is the gradient of a potential $\phi$, and (42b) is nothing but the necessary and sufficient condition for the existence of the potential. The case of a symmetric $m$-tensor in $\mathbb{R}^n$ provides a nontrivial generalization of this scenario.

6. Longitudinal and transverse tensor components

The significance of the kernel of the ray transform is further elucidated by considering the standard decomposition of symmetric tensor fields of degree $m$ (including vector fields) into transverse $T_{\perp i_1...i_m}$ and longitudinal $(dv)_{i_1...i_m}$ components,

$$T_{i_1...i_m} = T_{\perp i_1...i_m} + (dv)_{i_1...i_m} \quad .$$

(57)

The transverse part $T_{\perp i_1...i_m}$ is characterized by its vanishing divergence, $\partial_{i_m} T_{\perp i_1...i_m} = 0$, while the longitudinal part $dv$ is the symmetrized covariant derivative of the symmetric $(m - 1)$ tensor $v$, and is given by

$$(dv)_{i_1...i_m} = \frac{1}{m!} \sum_{\pi \in S_m} v_{i_{\pi(1)}...i_{\pi(m-1)}, i_{\pi(m)}} \quad ,$$

(58)

where $S_m$ is the group of permutations of $m$ elements. The transverse part of $T$ is determined by the projection of the Fourier transform $\hat{T}_{i_1...i_m}(k)$ of $T_{i_1...i_m}(X)$ onto the subspace orthogonal to the direction of propagation $k$ of the Fourier mode $e^{i(k,X)}$,

$$T_{\perp i_1...i_m}(X) = \frac{1}{(2\pi)^{n/2}} \int d^n k \hat{T}_{\perp i_1...i_m}(k) e^{i(k,X)} \quad ,$$

(59a)

$$\hat{T}_{\perp i_1...i_m}(k) = \lambda_{i_1 j_1} \cdots \lambda_{i_m j_m} \hat{T}_{j_1...j_m}(k) \quad ,$$

(59b)
where
\[ \lambda_{ij} = \delta_{ij} - \frac{\delta_{ij}}{k^2} \]

is the projector onto the subspace perpendicular to \( k \). By construction, this projector leaves the Fourier transform \( \hat{T}_\perp \) of the transverse component \( T_\perp \) invariant, while it annihilates the Fourier transform \( \hat{T} - \hat{T}_\perp \) of the longitudinal component \( T - T_\perp \).

For our vector field \( w \) and tensor field \( u \), the decompositions (57) take the special form
\[ w_i = w_\perp i + \partial_i \phi , \quad \partial_i w_\perp i = 0 , \]
\[ u_{ij} = u_\perp ij + \frac{1}{2} (\partial_i v_j + \partial_j v_i) , \quad \partial_j u_\perp ij = 0 . \]

On comparing (42a, 42c) with (60a, 60b) we see that the kernel of the ray transforms \( Iw, Iu \) consists precisely of the longitudinal components \( d\phi, dv \) of the vector/tensor field \( w, u \)! It follows that these components cannot be retrieved from the ray transforms of these fields. This result is obviously of central importance for any attempt to reconstruct optical and stress tensors from photoelastic measurements.

Endowed with this knowledge we can now turn to the question to which extent the ray transform may be inverted: To this end we introduce the integral moments \((\mu^m IT)_{j_1...j_m}(X)\) of the ray transform \((IT)(X, \xi)\) of the symmetric tensor \( T_{i_1...i_m}(X)\) of degree \( m \) on \( \mathbb{R}^n\),
\[ (\mu^m IT)_{j_1...j_m}(X) = \frac{1}{\omega_{n-1}} \int_{S_{n-1}} d\Omega_{n-1}(\xi) \xi_{j_1} \cdots \xi_{j_m} (IT)(X, \xi) , \]

where \( S_{n-1} \) is the unit sphere in \( \mathbb{R}^n \) with \((n-1)\)-dimensional volume \( \omega_{n-1} \) and volume element \( d\Omega_{n-1}(\xi) \). For our vector field \( w \) and symmetric tensor field \( u \) in \( \mathbb{R}^2 \), the integral moments of the ray transform \( Iw(X, \xi) \) are
\[ (\mu^1 Iw)_i(X) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \xi_i (Iw)(X, \xi) , \]
\[ (\mu^2 Iu)_{ij}(X) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \xi_i \xi_j (Iu)(X, \xi) . \]

On taking (two-dimensional) Fourier transforms of eqs. (62a) and (62b) we arrive after a lengthy calculation at
\[ (\hat{\mu}^1 Iw)_i(k) = \frac{2}{|k|} \hat{w}_m(k) \lambda_{im}(k) , \]
The presence of the projector \( \lambda_{ij} \) in these formulas cancels out the longitudinal components of \( \hat{w}_m \) and \( \hat{u}_{mn} \), while leaving their transverse components \( \hat{w}_\perp m \) and \( \hat{u}_\perp mn \) invariant. It follows that we can replace \( \hat{w}_m \) and \( \hat{u}_{mn} \) by \( \hat{w}_\perp m \) and \( \hat{u}_\perp mn \) in formulas (63). On taking the trace of the resulting equations we can then invert them for \( \hat{w}_\perp \) and \( \hat{u}_\perp \),

\[
\hat{w}_\perp i(k) = \frac{|k|}{2} \left( \hat{\mu}^1 Iw \right)_i, \tag{64a}
\]

\[
\hat{u}_\perp ij(k) = \frac{3}{4} |k| \left( \hat{\mu}^2 Iu \right)_{ij} - \frac{1}{4} |k| \sum_{n=1}^2 \left( \hat{\mu}^2 Iu \right)_{nn} \lambda_{ij}. \tag{64b}
\]

Eqs. (64) provide the explicit answer to the question to which degree the components of a vector field \( w \) and a symmetric tensor field \( u \) on \( \mathbb{R}^2 \) can be reconstructed from their ray transforms \( Iw \) and \( Iu \): It is precisely the transverse components of these fields which can be retrieved in full, whereas the longitudinal components, these being symmetric covariant derivatives of lower-rank tensors, must remain undetermined. Performing the inverse Fourier transform on (64) we find that

\[
w_\perp i(X) \equiv W_i \left[ \mu^1 Iw \right](X) = \frac{1}{2\pi} \int d^2 k \frac{|k|}{2} \left( \hat{\mu}^1 Iw \right)_i, \tag{65a}
\]

\[
u_\perp ij(X) \equiv U_{ij} \left[ \mu^2 Iu \right](X) = \frac{1}{2\pi} \int d^2 k \frac{|k|}{2} \left\{ \frac{3}{4} |k| \left( \hat{\mu}^2 Iu \right)_{ij} - \frac{1}{4} |k| \sum_{n=1}^2 \left( \hat{\mu}^2 Iu \right)_{nn} \lambda_{ij} \right\}. \tag{65b}
\]

Here, \( W_i \left[ \mu^1 Iw \right](X) \) indicates a linear functional with respect to \( \mu^1 Iw \), being parametrized by the spatial coordinates \( X \); and similarly for \( U \). The right-hand sides of (65) contain the ray transforms \( Iw \) and \( Iu \) of the vector- and tensor field, and can therefore be regarded as known functions of the data collected from measurements. Now, from theorem 5.1 we know that the Saint-Vernant operator (42b, 42d) annihilates precisely the longitudinal component of a vector- or symmetric tensor field [and therefore must be related to products of the projectors (59c)]

\[
\left( \hat{\mu}^2 Iu \right)_{ij} = \frac{2}{3} \hat{u}_{mn}(k) |k|^{-1} \times \left\{ \lambda_{ij} \lambda_{mn} + \lambda_{im} \lambda_{jn} + \lambda_{in} \lambda_{jm} \right\}. \tag{63b}
\]
by Fourier transformation]. Thus, if the Saint-Venant operator acts on \( w \) and \( u \) in the manner of (42b) and (42d), only the transverse components \( w_\perp \) and \( u_\perp \) of \( w \) and \( u \) survive. For these transverse components we now substitute expressions (65); thus, we obtain quantities
\[
\mathcal{W} \equiv \partial_1 W_2 - \partial_2 W_1 , \\
\mathcal{U} \equiv \partial_1^2 U_{22} + \partial_2^2 U_{11} - 2 \partial_1 \partial_2 U_{12} ,
\]
where \( W \) and \( U \) represent known functions of measurement data. On the other hand, we can insert the values (32a) and (32b) for the components of the vector \( w \) and tensor \( u \) into the left-hand sides of (65), so that
\[
\mathcal{W} = -\partial_1 \sigma_{13} - \partial_2 \sigma_{23} , \\
\mathcal{U} = \partial_1^2 (\sigma_{11} - \sigma_{33}) + \partial_2^2 (\sigma_{22} - \sigma_{33}) + 2 \partial_1 \partial_2 \sigma_{12} .
\]
We now regard (67) as a system of equations for the unknown \( \sigma_{ij} \), where the left-hand sides \( \mathcal{W} \) and \( \mathcal{U} \) are known through (66); furthermore, the equilibrium conditions (26) and the boundary conditions (27) are added to this system. The third equation in (26) implies that
\[
\mathcal{W} = \partial_3 \sigma_{33} .
\]
Furthermore, differentiating the first equation in (26) with respect to \( x_1 \), the second with respect to \( x_2 \) and adding the results gives, on using (67a),
\[
\partial_3 \mathcal{W} - \mathcal{U} = \left( \partial_1^2 + \partial_2^2 \right) \sigma_{33} .
\]
This last equation defines a Dirichlet problem in \( \mathbb{R}^2 \) for the unknown \( \sigma_{33} \), the left-hand side containing a known function of measurement data.

The value of \( \sigma_{33} \) on the lateral surface \( B \) of the cylindrical region \( G \) is determined by the ray transform \( Iu \): Let \( Iu(X, X') \) denote the ray transform of \( u \) along a line which connects two points \( X = (x, y, z) \) and \( X' = (x', y', z) \) at the same height \( z \) on the cylindrical specimen, where both \( X \) and \( X' \) belong to the boundary \( B \). By taking the limit \( X' \to X \), the integrand in \( Iu \) is nonvanishing only on an infinitesimally short interval in the neighbourhood of the point \( X \), which becomes tangential to \( B \) in the limit. Using the stress-free condition (27) of the lateral surface then yields
\[
\lim_{X' \to X} \frac{Iu(X, X')}{|X - X'|} = -\sigma_{33} ,
\]
at every point \( X \in B \). Relations (69, 70) are sufficient to uniquely determine the component \( \sigma_{33} \) from the measured data, i.e., \( \mathcal{W} \) and \( \mathcal{U} \),
since the value of $\sigma_{33}$ on the boundary is given by (70), hence the Dirichlet problem (69) can be solved uniquely for any given $z \in (a, b)$. Alternatively, the Dirichlet problem may be solved for just one $z_0 \in (a, b)$ and then (68) may be integrated to obtain $\sigma_{33}$ everywhere.

Finally, we might wonder whether there is any more information to be retrieved from the ray transforms $Iw$ and $Iu$, or alternatively $\tilde{W}$ and $\tilde{U}$, as so far we could only reconstruct one tensor component from these data. The answer is no: As proven in [13], if any two stress tensors $\sigma^{(1)}$ and $\sigma^{(2)}$ satisfy (26, 27) and $\sigma^{(1)}_{33} = \sigma^{(2)}_{33}$ then they produce the same ray transforms, $Iw^{(1)} = Iw^{(2)}$ and $Iu^{(1)} = Iu^{(2)}$. As a consequence, the retrieval of one tensor component is the maximum information to be gleaned from a ray transform which is performed for horizontal lines only. If we wish to reconstruct other tensor components, the orientation of the incident light beam must be changed accordingly.

7. Newton-Kantarovich method

Sharafutdinov’s theory shows that in the linearised problem of photoelastic tomography the transverse part of the stress tensor can be recovered from complete photoelastic data, and that this inversion is stable under suitable smoothness assumptions on $\sigma$. The forward problem represented by the solution of the ODE (21) along each ray is non-linear in $\sigma$, its Fréchet derivative at $\sigma = 0$ being the ray transform (24). Unlike the linearization about non-zero stress, this transform has an explicit Fourier inversion formula. It might therefore be possible to extend the reconstruction method outlined above into the high-stress regime by using the Newton-Kantarovich method with fixed derivative, which means that while the forward problem is solved successively for each updated stress, the difference between predicted and measured optical data is used as boundary data for the same ray transform at each iteration. While the convergence of this method is linear compared with the quadratic convergence when the derivative is updated, it is expected that the efficient computation of the linear step will make the former more efficient numerically. These ideas will be investigated in upcoming papers.

8. Summary

We have described the main ideas centered around Sharafutdinov’s ray transform which represents a generalization of the Radon transform to tensor-field integrands. The tensorial character of the field to be
reconstructed manifests in a non-trivial kernel of the ray transform, as a consequence of which the ray transform can be inverted only with respect to the transverse components of the tensors, while the longitudinal components are lost. We have provided an alternative proof to the main theorem about the kernel of the ray transform as given in Sharafutdinov’s original work, which is simpler since the scope of our proof is limited to those cases which are relevant for the field of Integrated Photoelasticity. In order to illustrate the merits of the ray transform as a tool for tensor reconstruction, a simple example in Photoelasticity, namely the reconstruction of the $\sigma_{33}$ component of the stress tensor inside a birefringent medium from its horizontal ray transform, has been discussed. The inversion of the ray transform for the transverse components of the stress tensor together with the equilibrium conditions on $\sigma_{ij}$ and the stress-free condition of the lateral boundary, resulted in a well-posed Dirichlet problem for the component $\sigma_{33}$ of the stress tensor which admits a unique solution. Finally we suggested a non-linear inversion algorithm based on the Newton-Kantarovich method with fixed derivative which might be capable of utilizing the ray transform in the high-stress regime.

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