Quantum Superintegrability and Exact Solvability in N Dimensions

Miguel A. Rodríguez
Dept. de Física Teórica II, Facultad de Físicas, Universidad Complutense, 28040-Madrid, Spain

Pavel Winternitz†
Centre de Recherches Mathématiques and Département de Mathématiques et de Statistique
Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, Québec H3C 3J7, Canada

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A family of maximally superintegrable systems containing the Coulomb atom as a special case is constructed in n-dimensional Euclidean space. Two different sets of n commuting second order operators are found, overlapping in the Hamiltonian alone. The system is separable in several coordinate systems and is shown to be exactly solvable. It is solved in terms of classical orthogonal polynomials. The Hamiltonian and n further operators are shown to lie in the enveloping algebra of a hidden affine Lie algebra.

I. INTRODUCTION

We shall consider the stationary Schrödinger equation

$$H\psi = E\psi, \quad H = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + V(x_1, \ldots, x_n)$$

(1)
in an n-dimensional Euclidean space $E_n$. By analogy with classical Hamiltonian mechanics, such a system is called integrable if there exist $n - 1$ algebraically independent linear operators $X_a$ satisfying

$$[H, X_a] = 0, \quad [X_a, X_b] = 0, \quad a, b = 1, \ldots, n - 1.$$  

(2)
The system is called “superintegrable” if there exist $k$ further operators $\{Y_1, \ldots, Y_k\}$ commuting with the Hamiltonian

$$[H, Y_j], \quad j = 1, \ldots, k,$$

(3)
such that the set $\{H, X_1, \ldots, X_n, Y_1, \ldots, Y_k\}$ is algebraically independent. Note that the additional operators $Y_i$ need not commute with the operators $X_a$ nor amongst each other. The number of additional operators satisfies

$$1 \leq k \leq n - 1.$$  

(4)

For $k = 1$ we call the system “minimally superintegrable”, for $k = n - 1$ it is “maximally superintegrable”.

The best known superintegrable systems in $E_3$ (and also in $E_n$ for any $n \geq 2$) are the harmonic oscillator and the hydrogen atom (or Kepler system in classical mechanics). The harmonic oscillator is superintegrable because of the $su(n)$ algebra of first and second order operators commuting with the Hamiltonian. The hydrogen atom in $E_n$ is superintegrable, because of the $o(n + 1)$ Lie algebra of linear operators commuting with the Hamiltonian. In both cases it is possible to choose different subsets of $n$ operators commuting with each other and overlapping only in the Hamiltonian. Each subset corresponds to the separation of variables in the Schrödinger equation in a different system of coordinates.

Characteristic features of these two superintegrable systems are:

1. In classical mechanics all finite (bounded) trajectories are periodic. Moreover, Bertrand’s theorem tells us that $\gamma/r$ and $\gamma r^2$ are the only spherically symmetric potentials for which all finite trajectories are periodic.

*Electronic mail: rodrigue@eucmos.sim.ucm.es
†Electronic mail: wintern@crm.umontreal.ca
2. In quantum mechanics these two systems are exactly solvable: their energy levels can be calculated algebraically, as can the degeneracies of these levels. Their eigenfunctions are polynomials in the appropriate variables, multiplied by some overall factor.

3. These systems are extremely important in physical applications, both in classical and quantum physics.

It makes sense to search systematically for superintegrable systems in classical and quantum mechanics, specially for maximally superintegrable ones. It can be safely assumed that they will all have the above properties 1 and 2 and hoped that they will also, to some degree, share property 3.

In searches for superintegrable systems restrictions are imposed on the form of the commuting operators $X_a$ and $Y_i$. A systematic search in $E_2$ and $E_3$ was conducted some time ago. The restriction was that all operators involved should be at most of second order. All superintegrable systems satisfying this restriction in $E_2$ and $E_3$ were found. Four classes of them exist in $E_2$, 5 maximally superintegrable ($2n - 1 = 5$ operators commuting with $H$) and 8 minimally superintegrable ones ($n + 1 = 4$ operators) in $E_3$. These results have been recently extended to two and three dimensional spaces of constant curvature and to complex spaces and also to certain two dimensional spaces of nonconstant curvature.

With the restriction to second order operators all superintegrable systems turned out to be multiseparable, that is, separable in at least two different coordinate systems. In two dimensional spaces they also turned out to be exactly solvable. By this we mean that their energy spectra can be calculated algebraically (by solving algebraic equations only). It was also shown that superintegrable systems are obtained by considering non-Abelian algebras of generalized Lie symmetries.

The purpose of this article is to consider a family of integrable systems in $n$ dimensional Euclidean space for any $n$. The family, containing the $n$ dimensional hydrogen atom as a special case, is introduced in Section 2, together with a set of $2n - 1$ algebraically independent operators, commuting with the Hamiltonian. In Section 3 we solve the Schrödinger equation in parabolic and spherical coordinates and show that it is exactly solvable in a precise and well defined sense. Finally, in Section 4 we introduce parabolic rotational coordinates in $E_n$ and solve the Schrödinger equation in these coordinates and also in spherical ones. We also prove the exact solvability in this case. Some conclusions are drawn in Section 5.

II. A FAMILY OF MAXIMALLY SUPERINTEGRABLE SYSTEMS IN $E_n$ CONTAINING THE HYDROGEN ATOM

Let us first consider the hydrogen atom in $n$ dimensional Euclidean space $E_n$

$$H = -\frac{1}{2} \Delta - \frac{\gamma}{r}, \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad r = (x_1^2 + \cdots + x_n^2)^{1/2}. \tag{5}$$

This Hamiltonian commutes with $n(n + 1)/2$ linear operators, namely

$$L_{ik} = x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i}, \quad 1 \leq i < k \leq n$$

$$A_i = \frac{1}{2} \sum_{a=1}^{n} (p_a L_{ia} + L_{ia} p_a) + \gamma \frac{x_i}{r}, \quad 1 \leq i \leq n \tag{6}$$

with $p_i = \partial x_i$. The operators $L_{ik}$ correspond to angular momentum, $A_i$ to the $n$ dimensional Laplace-Runge-Lenz vector, characterizing the Coulomb or Kepler problem. Only $2n - 1$ of the operators $\{H, L_{ik}, A_i\}$ can be, and are, algebraically independent. They satisfy the commutation relations

$$[H, L_{ik}] = [H, A_i] = 0$$

$$[L_{ij}, L_{ab}] = \delta_{ja} L_{ib} + \delta_{ib} L_{ja} - \delta_{ia} L_{jb} - \delta_{ib} L_{ia}$$

$$[L_{ij}, A_k] = \delta_{jk} A_i - \delta_{ik} A_j$$

$$[A_i, A_j] = -2i H L_{ij} \tag{7}$$

The commutation relations in general correspond to a Kac-Moody algebra. For a fixed energy, $H = E$ they correspond to the Lie algebra of the rotation group $O(n + 1)$, the Lorentz group $O(n, 1)$ and the Euclidean group $E(n)$ for $E < 0$, $E > 0$ and $E = 0$, respectively. These symmetries for $n = 3$ were discovered implicitly by Pauli and explicitly by Fock and Bargmann.
According to the operator approach to the separation of variables, separation of variables in Schrödinger equation is achieved by looking for eigenfunctions of a complete set of $n$ commuting second order operators $\{H, X_1, \ldots, X_{n-1}\}$

$$H \psi = E \psi \quad X_a \psi = \lambda_a \psi, \quad a = 1, \ldots, n - 1$$

The operators $X_a$ will be at most linear in $A_i$ and bilinear in $L_{ik}$: If more than one inequivalent set of commuting operators exists, the system is multiseparable, i.e., separable in more than one coordinate system.

In view of the commutation relations \((\ref{eq:commutation_relations})\) any set of commuting operators $\{X_i\}$ can contain at most one operator involving $A_i$:

$$X = \sum_i a_i A_i + \sum_{i,k,j,m} b_{ik,jm} L_{ik} L_{jm}, \quad \sum_{i=1}^n a_i^2 \neq 0$$

The complete sets of commuting operators can be classified under the action of $O(n)$: in particular we can rotate and normalize so as to have $a_n = 1$, $a_k = 0$ for $k = 1, \ldots, n - 1$. Here we just give the example of the case $n = 3$. It is easy to verify by a direct calculation that in this case, precisely four inequivalent sets exist: $\{H, X_1, X_2\}$ with

$$\begin{align*}
X_1 &= A_3, \quad X_2 = L_{12}^2 \\
X_1 &= A_3 + a(L_{12}^2 + L_{23}^2 + L_{31}^2), \quad X_2 = L_{12}^2 \\
X_1 &= L_{12}^2 + L_{23}^2 + L_{31}^2, \quad X_2 = L_{12}^2 \\
X_1 &= L_{12}^2 + L_{23}^2 + L_{31}^2, \quad X_2 = L_{23}^2 + fL_{31}^2
\end{align*}$$

They correspond to the separation of variables in parabolic rotational coordinates, shifted spheroidal coordinates, spherical coordinates and sphericoconical coordinates, respectively.

In each coordinate system it is possible to add further terms to the potential $-\gamma/r$ in such a manner that the Schrödinger equation still separates. The system will remain integrable and the corresponding operators $X_1$ and $X_2$ will only be modified by the addition of a scalar function. It is also possible to preserve superintegrability and to require that the extended potentials should allow separation of variables in at least two coordinate systems.

Here we will be interested in the most general potential allowing separation of variables in the same four coordinate systems as the hydrogen atom itself. In $E_3$ there is, up to equivalence, only one such Hamiltonian, namely (see Ref.\cite{Ref1,Ref2,Ref3})

$$H = -\frac{1}{2} \Delta - \frac{\gamma}{r} + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}$$

One triplet of commuting operators for the Hamiltonian (\ref{eq:Hamiltonian_E3}) consists of

$$\begin{align*}
X &= \frac{1}{2} (p_1 L_{31} + L_{31} p_1 + p_2 L_{32} + L_{32} p_2) + 2x_3 \left( \frac{\gamma}{2r} - \frac{\beta_1}{x_1^2} - \frac{\beta_2}{x_2^2} \right) \\
Z &= L_{12}^2 - 2r^2 \left( \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} \right)
\end{align*}$$

Another triplet can be chosen to be $H$ and

$$\begin{align*}
Y_1 &= L_{12}^2 + L_{23}^2 + L_{31}^2 - 2r^2 \left( \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} \right) \\
Y_2 &= L_{23}^2 - 2\frac{x_2^2}{x_3^2} \frac{x_3^2}{x_2^2}
\end{align*}$$

It is the set of 5 algebraically independent operators $\{H, X_1, X_2, Y_1, Y_2\}$ which guarantees that the Hamiltonian (\ref{eq:Hamiltonian_E3}) is maximally superintegrable.

The generalization to the $n$ dimensional Euclidean space $E_n$ is immediate. Thus, the Hamiltonian will be

$$H = -\frac{1}{2} \Delta - \frac{\gamma}{r} + \sum_{i=1}^{n-1} \frac{\beta_i}{x_i^2}$$
We see immediately that the variables separate and we can solve the corresponding ODE’s to obtain eigenvalues and common eigenfunctions of the systems:

\[
X = \frac{1}{2} \sum_{k=1}^{n-1} (L_{nk} p_k + p_k L_{nk}) + 2x_n \left( \frac{\gamma}{2r} - \sum_{i=1}^{n-1} \frac{\beta_i}{x_i} \right)
\]

\[
Z_l = \sum_{1 \leq i < k \leq l+1} L_{ik}^2 - 2 \left( \sum_{i=1}^{l+1} x_i \right) \left( \sum_{k=1}^{l+1} \frac{\beta_k}{x_k} \right), \quad 1 \leq l \leq n - 2
\]  \hspace{1cm} (18)

Another complete set of commuting operators is again \(H\) and

\[
Y_p = \sum_{p \leq i < k \leq n} L_{ik}^2 - 2 \left( \sum_{i=p}^{n} x_i^2 \right) \left( \sum_{k=p}^{n-1} \frac{\beta_k}{x_k} \right), \quad 1 \leq p \leq n - 1
\]  \hspace{1cm} (19)

The two sets (18) and (19) are disjoint. If we set \(\beta_i = 0, 1 \leq i \leq n - 1\), then the operator \(Z_l\) will be a Casimir operator of the group \(O(l + 1)\) acting on the coordinates \(\{x_1, \ldots, x_{l+1}\}\). The operator \(Y_p\) will be a Casimir operator of \(O(n + 1 - p)\) acting on the coordinates \(\{x_p, \ldots, x_n\}\).

It is the Hamiltonian (17) that we shall study in the following sections, first for \(n = 3\), then for arbitrary \(n\).

### III. Exact Solvability of the Superintegrable System for \(N = 3\)

#### A. Solution by separation of variables

Let us first consider the Hamiltonian (14) and the complete set of commuting operators (15). We are looking for eigenvalues and common eigenfunctions of the systems:

\[
H \psi = E \psi, \quad X \psi = \lambda \psi, \quad Z \psi = k \psi
\]  \hspace{1cm} (20)

To do this we introduce parabolic rotational coordinates, putting

\[
x_1 = \mu \nu \cos \phi, \quad x_2 = \mu \nu \sin \phi, \quad x_3 = \frac{1}{2}(\mu^2 - \nu^2)
\]  \hspace{1cm} (21)

In these coordinates the operators in (17) are

\[
H = \frac{1}{2(\mu^2 + \nu^2)} \left( \frac{\partial^2}{\partial \mu^2} + \frac{1}{\mu} \frac{\partial}{\partial \mu} + \frac{\partial^2}{\partial \nu^2} + \frac{1}{\nu} \frac{\partial}{\partial \nu} + 4\gamma \right) - \frac{1}{2\mu^2 \nu^2} \left( \frac{\partial^2}{\partial \phi^2} - \frac{2\beta_1}{\cos^2 \phi} - \frac{2\beta_2}{\sin^2 \phi} \right)
\]

\[
X = \frac{1}{2(\mu^2 + \nu^2)} \left( -\nu^2 \left( \frac{\partial^2}{\partial \mu^2} + \frac{1}{\mu} \frac{\partial}{\partial \mu} \right) + \mu^2 \left( \frac{\partial^2}{\partial \nu^2} + \frac{1}{\nu} \frac{\partial}{\partial \nu} \right) + 2\gamma(\mu^2 - \nu^2) \right) + \frac{\mu^2 - \nu^2}{2\mu^2 \nu^2} \left( \frac{\partial^2}{\partial \phi^2} - \frac{2\beta_1}{\cos^2 \phi} - \frac{2\beta_2}{\sin^2 \phi} \right)
\]

\[
Z = \frac{\partial^2}{\partial \phi^2} - \frac{2\beta_1}{\cos^2 \phi} - \frac{2\beta_2}{\sin^2 \phi}
\]

We see immediately that the variables separate and we can solve the corresponding ODE’s to obtain

\[
\psi_{N_1, N_2, J} = (\sin \phi)^{p_2} (\cos \phi)^{p_1} (\mu \nu)^{m} e^{-\frac{\sqrt{-2E}}{\sqrt{2}(\mu^2 + \nu^2)}} \times \]

\[
P_{J}^{(p_2-1/2, p_1-1/2)}(\cos 2\phi) L_{N_1}^{m} (\sqrt{-2E} \mu^2) L_{N_2}^{m} (\sqrt{-2E} \nu^2)
\]  \hspace{1cm} (23)

where \(P_{J}^{(\alpha, \beta)}(z)\) and \(L_{N}^{m}(x)\) are Jacobi and Laguerre polynomials respectively. We have put

\[
\beta_i = \frac{1}{2} p_i (p_i - 1), \quad m = 2J + p_1 + p_2
\]
and the eigenvalues in Equation (20) are equal to

\[ E = -\frac{\gamma^2}{2(N_1 + N_2 + 2J + p_1 + p_2 + 1)^2} \]

\[ \lambda = -\frac{\gamma(N_1 - N_2)}{N_1 + N_2 + 2J + p_1 + p_2 + 1} \]

\[ k = -m^2 = -(2J + p_1 + p_2)^2 \]

(24)

We see that the bound state energy is given by a shifted Balmer formula and the only effect of the \( \beta_i/x_i^2 \) terms in the potential is to add a constant \( p_1 + p_2 \) to the principal quantum number. The solutions (23) are square integrable and correspond to bound states when \( J, N_1 \) and \( N_2 \) are integers. They are polynomials multiplied by a factor which, however, is not “universal”. It depends on the energy \( E \) and also on the angular quantum number \( J \) (since we have \( m = 2J + p_1 + p_2 \)).

The second set of commuting operators, namely (14) also corresponds to the separation of variables, this time in spherical coordinates, chosen as

\[ x_1 = r \cos \theta, \quad x_2 = r \sin \theta \cos \alpha, \quad x_3 = r \sin \theta \sin \alpha \]

(25)

In these coordinates we have

\[ H \psi = -\frac{1}{2} \left[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right] + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \alpha^2} \right) + \]

\[ \frac{2\gamma}{r} \frac{1}{r^2} \left( \frac{\beta_1}{\cos^2 \theta} + \frac{\beta_2}{\sin^2 \theta \cos^2 \alpha} \right) \psi = E \psi, \]

\[ Y_1 \psi = \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{2\beta_1}{\cos^2 \theta} + \frac{k_2}{\sin^2 \theta} \right) \psi = k_1 \psi, \]

\[ Y_2 \psi = \left( \frac{\partial^2}{\partial \alpha^2} - \frac{2\beta_2}{\cos^2 \alpha} \right) \psi = k_2 \psi. \]

(26)

The coordinates separate and we obtain \( \psi = R(r)F(\theta)G(\alpha) \) where \( F \) and \( G \) can be expressed in terms of Jacobi polynomials and \( R \) in terms of Laguerre ones.

The explicit expression for the eigenfunctions in this type of spherical coordinates is:

\[ \psi_{N_1,J_1,J_2}(r, \theta, \alpha) = r^{m_1 - 1/2} e^{-\sqrt{-2E} r} L_{N_1}^{2m_1}(2\sqrt{-2E} r) (\sin \theta)^{m_1} (\cos \theta)^{p_1} (\cos \alpha)^{p_2} \times 
\]

\[ P_{J_1}^{(m_2 - 1/2)}(\cos \theta) P_{J_2}^{(1/2, p_2 - 1/2)}(\cos 2\alpha) \]

(27)

with eigenvalues equal to:

\[ E = -\frac{\gamma^2}{2(N + 2J + 1)} \]

\[ k_1 = \frac{1}{4} - m_1^2, \quad k_2 = -m_2^2, \quad m_1 = 2J_1 + 2J_2 + p_1 + p_2 + \frac{1}{2}, \quad m_2 = 2J_2 + p_2 \]

(28)

**B. Exact solvability and underlying affine Lie algebra**

We have established that the potential (14) provides a Hamiltonian that is maximally superintegrable and multiseparable. Let us now turn to the question of exact solvability. A Hamiltonian is exactly solvable if its spectrum can be calculated algebraically. This occurs if it can be explicitly transformed into block diagonal form where each block is finite dimensional. This in turn means that there exists an infinite flag of finite dimensional subspaces in the Hilbert space \( \mathcal{H} \) of bound state solutions that is preserved by the Hamiltonian:

\[ \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_i \subset \mathcal{H} \]

Typically this occurs under the following circumstances
We see here a phenomenon which has been called "metamorphosis" or "migration" of the coupling constant. In equation (33) the energy \( E \) plays the role of the frequency of a harmonic oscillator whereas the Coulomb coupling constant \( \gamma \) plays the role of an eigenvalue of \( Q_0 \). The other eigenvalues, \( \lambda \) and \( m^2 \), remain eigenvalues (of \( Q_1 \) and \( Z \) respectively).

Similarly as in the case of potentials containing the Coulomb atom as a special case in 2 dimensions, it is the system \( g_{P_1} \) (rather than the original system \( \{20\} \)) that is exactly solvable in the sense defined above. Indeed, let us gauge rotate the operators \( Q_0, Q_1 \) and \( Z \) and transform to the variables \( s, t \) and \( z \). We obtain

\[
\begin{align*}
\tilde{Q}_0 &= g^{-1}Q_0g = -2\sqrt{2E}(s\partial_s^2 + (m + 1 - s)\partial_s + t\partial_t^2 + (m + 1 - t)\partial_t - m - 1), \\
\tilde{Q}_1 &= g^{-1}Q_1g = -2\sqrt{2E}(s\partial_s^2 + (m + 1 - s)\partial_s - t\partial_t^2 - (m + 1 - t)\partial_t), \\
\tilde{Q}_2 &= g^{-1}Q_2g = 4(1 - z^2)\partial_z^2 + 4(p_1 - p_2 - (p_1 + p_2 + 1)z)\partial_z - (p_1 + p_2)^2.
\end{align*}
\]

We see that \( \tilde{Q}_\mu, \mu = 0, 1, 2 \) lie in the enveloping algebra of the direct sum of three affine Lie algebras, \( saff(1, \mathbf{R}) \oplus saff(1, \mathbf{R}) \oplus saff(1, \mathbf{R}) \), realized by

\[
\{\partial_s, s\partial_s, \partial_t, t\partial_t, \partial_z, z\partial_z\}.
\]

Now let us consider the two remaining operators \( Y_1 \) and \( Y_2 \) of \( \{24\} \). They are not diagonal in the basis that we use (where \( H, X, Z \) and equivalently \( Q_0, Q_1, Q_2 \) are diagonal). They do however commute with the Hamiltonian so they can only mix states of equal energy. We also have

\[ h = g^{-1}Hg, \quad hP = EP, \]

that is, there exists a gauge transformation and a change of variables to a new Hamiltonian \( h \) that has polynomial eigenfunctions.
In parabolic coordinates the operators $H$ only. Indeed, we have

$$Y_1 = g^{-1}Y_1g$$

corresponding to the set (19). The parabolic coordinates $(\mu, \nu, \theta)$ depend not only on the energy, but also on $m$, the eigenvalue of $Z$. Thus, $Y_1 = g^{-1}Y_1g$ should transform polynomials into polynomials, whereas $Y_2 = g^{-1}Y_2g$ is not obliged to. Performing the gauge transformation and change of variables we find

$$\tilde{Y}_1 = g^{-1}Y_1g = s(t(\partial_s - \partial_t)^2 - (m + 1)(s - t)(\partial_s - \partial_t) - m(m + 1)).$$

The algebra underlying this expression includes $t\partial_s$ and $s\partial_t$, in addition to the elements listed in (25). We recognize this to be the Lie algebra $\text{saff}(2, \mathbf{R}) \oplus \text{saff}(1, \mathbf{R})$

Superintegrable systems, including the hydrogen atom as special case, in 2 dimensions were associated with the algebra $\text{saff}(2, \mathbf{R})$. The extension to $n = 3$ is seen to lead to $\text{saff}(2, \mathbf{R}) \oplus \text{saff}(1, \mathbf{R})$, not to $\text{saff}(3, \mathbf{R})$ as one might have expected. It follows from expression (37) that $\tilde{Y}_1$ will take polynomials into polynomials and indeed we have

$$\tilde{Y}_1 P_{N_1, N_2, j} = -[2N_1N_2 + (N_1 + N_2 + m)(m + 1)]P_{N_1, N_2, j} + (N_1 + m)(N_2 + 1)P_{N_1 - 1, N_2 + 1, j} + (N_1 + 1)(N_2 + m)P_{N_1 + 1, N_2 - 1, j}.$$  

The operator $\tilde{Y}_2$ does not take polynomials into polynomials and cannot be written as an element of the enveloping algebra of an affine Lie algebra.

The one-dimensional equations appearing above are easily shown to be related to the standard types in the classification of exact and quasi-exact solvable one-dimensional systems.[35][34]

IV. EXACT SOLVABILITY OF THE SUPERINTEGRABLE SYSTEM IN $E_N$

A. Solution by separation of variables

We now consider the Hamiltonian (35) for arbitrary $n$. It allows separation of variables in many coordinate systems. We shall use parabolic rotational coordinates corresponding to the set of operators (36) and spherical ones corresponding to the set (37). The parabolic coordinates $(\mu, \nu, \theta_1, \ldots, \theta_{n-2})$ are defined by the relations

$$x_1 = \mu \nu \cos \theta_1 \cos \theta_2 \ldots \cos \theta_{n-3} \cos \theta_{n-2},$$

$$x_2 = \mu \nu \cos \theta_1 \cos \theta_2 \ldots \cos \theta_{n-3} \sin \theta_{n-2},$$

$$x_3 = \mu \nu \cos \theta_1 \cos \theta_2 \ldots \sin \theta_{n-3},$$

$$\vdots$$

$$x_{n-2} = \mu \nu \cos \theta_1 \sin \theta_2,$$

$$x_{n-1} = \mu \nu \sin \theta_1,$$

$$x_n = \frac{1}{2}(\mu^2 - \nu^2).$$

We put $\beta_i = p_i(p_i - 1)/2$ in equation (37). The eigenvalue problem that we have to solve is

$$H \psi = E \psi, \quad X \psi = \lambda \psi, \quad Z_l \psi = k_l \psi, \quad 1 \leq l \leq n - 2.$$  

In parabolic coordinates the operators $H$ and $X$ will involve all variables, the operators $Z_l$ will involve the angles only. Indeed, we have

$$H = -\frac{1}{2(\mu^2 + \nu^2)} \left( \frac{\partial^2}{\partial \mu^2} + \frac{n - 2}{\mu} \frac{\partial}{\partial \mu} + \frac{\partial^2}{\partial \nu^2} + \frac{n - 2}{\nu} \frac{\partial}{\partial \nu} \right) - \frac{2\gamma}{\mu^2 + \nu^2} + \frac{1}{2\mu^2 \nu^2} \left[ \Delta(S_{n-2}) - \frac{p_1(p_1 - 1)}{\cos^2 \theta_1 \ldots \cos^2 \theta_{n-2}} \right] \right),$$

$$X = \frac{1}{2(\mu^2 + \nu^2)} \left( -\nu^2 \left( \frac{\partial^2}{\partial \mu^2} + \frac{n - 2}{\mu} \frac{\partial}{\partial \mu} + \frac{\partial^2}{\partial \nu^2} + \frac{n - 2}{\nu} \frac{\partial}{\partial \nu} \right) \right) + \gamma \frac{\mu^2 - \nu^2}{\mu^2 + \nu^2} + \frac{\mu^2 - \nu^2}{2\mu^2 \nu^2} \left[ \Delta(S_{n-2}) - \frac{p_1(p_1 - 1)}{\cos^2 \theta_1 \ldots \cos^2 \theta_{n-2}} \right].$$

$$\vdots$$

$$\frac{p_1(p_1 - 1)}{\cos^2 \theta_1 \ldots \cos^2 \theta_{n-2}} - \frac{p_2(p_2 - 1)}{\cos^2 \theta_1 \ldots \cos^2 \theta_{n-3} \sin^2 \theta_{n-2}} - \cdots - \frac{p_{n-1}(p_{n-1} - 1)}{\sin^2 \theta_1}. \right)$$

(42)
where $\Delta(S_{n-2})$ is the Laplace operator on an $n - 2$ dimensional sphere.

The operators $Z_l$ satisfy

$$Z_1 \psi = \left( \frac{\partial^2}{\partial \theta_{n-2}^2} - \frac{p_1(p_1 - 1)}{\cos^2 \theta_{n-2}} - \frac{p_2(p_2 - 1)}{\sin^2 \theta_{n-2}} \right) \psi = k_1 \psi,$$

$$Z_2 \psi = \left( \frac{\partial^2}{\partial \theta_{n-3}^2} - \tan \theta_{n-3} \frac{\partial}{\partial \theta_{n-3}} + \frac{k_1}{\cos^2 \theta_{n-3}} - \frac{p_3(p_3 - 1)}{\sin^2 \theta_{n-3}} \right) \psi = k_2 \psi,$$

and in general

$$Z_l \psi = \left( \frac{\partial^2}{\partial \theta_{n-l-1}^2} - (l - 1) \tan \theta_{n-l-1} \frac{\partial}{\partial \theta_{n-l-1}} + \frac{k_{l-1}}{\cos^2 \theta_{n-l-1}} - \frac{p_l(p_l - 1)}{\sin^2 \theta_{n-l-1}} \right) \psi = k_l \psi$$

for $1 \leq l \leq n - 2$, $k_0 = -p_1(p_1 - 1)$.

We write

$$\psi = M(\mu) N(\nu) \prod_{l=1}^{n-2} F_l(\theta_{n-l-1})$$

and solve (43) to obtain $F_l$ in terms of Jacobi polynomials

$$F_l(\theta_{n-l-1}) = (\sin \theta_{n-l-1})^{p_{l+1}} (\cos \theta_{n-l-1})^{m_{l+1}} e^{l/2} P_{l}^{(p_{l+1}, 1/2, m_{l+1})} (\cos 2 \theta_{n-l-1})$$

with

$$m_l = 2 \sum_{i=1}^{l} J_i + \sum_{i=1}^{l+1} p_i + \frac{l - 1}{2}, \quad k_l = \frac{(l - 1)^2}{4} - m_l^2$$

The equations for $M(\mu)$ and $N(\nu)$ are obtained from equation (41) and (42) once the angular part is replaced by $k_{n-2}$. The final result is that the wave functions are:

$$\psi_{N_1, \ldots, N_{n-2}, J_1, \ldots, J_{n-2}, \mu, \nu, \theta_1, \ldots, \theta_{n-2}} = (\mu \nu)^{\sigma} e^{-\sqrt{-E/2} (\mu^2 + \nu^2)} \times \prod_{l=1}^{n-2} (\sin \theta_{n-l-1})^{p_{l+1}} (\cos \theta_{n-l-1})^{m_{l+1}} e^{l/2} P_{l}^{(p_{l+1}, 1/2, m_{l+1})} (\cos 2 \theta_{n-l-1}) \times \prod_{l=1}^{n-2} L_{J_l}^{(p_{l+1}, 1/2, m_{l+1})} (\cos 2 \theta_{n-l-1}), \quad \sigma = 2 \sum_{i=1}^{n-2} J_i + \sum_{i=1}^{n-2} p_i$$

The energy is given by a shifted Balmer formula

$$E = -\frac{\gamma^2}{2(N_1 + N_2 + 2 \sum_{i=1}^{n-2} J_i + \sum_{i=1}^{n-1} p_i + \frac{n - 1}{2})^2}$$

and the remaining quantum number is

$$\lambda = -\frac{\gamma(N_1 - N_2)}{N_1 + N_2 + 2 \sum_{i=1}^{n-2} J_i + \sum_{i=1}^{n-1} p_i + \frac{n - 1}{2}}$$

We see that the case of $n$ arbitrary is a straightforward generalization of $n = 3$ and involves the same functions, namely, Jacobi and Laguerre polynomials.

Obviously, one can also solve in spherical coordinates. In fact, formulas (38) can be written as

$$x_a = \mu \nu s_a, \quad x_n = \frac{1}{2} (\mu^2 - \nu^2), \quad a = 1, \ldots, n - 1, \quad \sum_{a=1}^{n-1} s_a^2 = 1$$

and we could introduce any coordinates on the $S_{n-2}$ sphere that allow separation of variables in the Laplace-Beltrami equation. For a discussion of such coordinate systems see (38) and (39).
We will write for the sake of completeness the explicit expression of the eigenfunctions in the following set of spherical coordinates on the $S_{n-1}$ sphere (which are a generalization to dimension $n$ of those used in the case $n = 3$, see Equation (27)):

\[
\begin{align*}
  x_1 &= r \cos \theta_1 \\
  x_1 &= r \sin \theta_1 \cos \theta_2 \\
  &\vdots \\
  x_{n-1} &= r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\
  x_n &= r \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}
\end{align*}
\]

and the Hamiltonian can be written in these coordinates as

\[
H = -\frac{1}{2} \left[ \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{2n}{r^2} \right] - \frac{1}{2r^2} \left\{ \frac{\partial^2}{\partial \theta_1^2} + (n-2) \cot \theta_1 \frac{\partial}{\partial \theta_1} - \frac{p_1(p_1-1)}{\cos^2 \theta_1} + \right.
\]

\[
\left. \frac{1}{\sin^2 \theta_1} \left[ \frac{\partial^2}{\partial \theta_2^2} + (n-3) \cot \theta_2 \frac{\partial}{\partial \theta_2} - \frac{p_2(p_2-1)}{\cos^2 \theta_2} \right] + \right.
\]

\[
\cdots + \frac{1}{\sin^2 \theta_{n-3}} \left[ \frac{\partial^2}{\partial \theta_{n-2}^2} + \cot \theta_{n-2} \frac{\partial}{\partial \theta_{n-2}} - \frac{p_{n-2}(p_{n-2}-1)}{\cos^2 \theta_{n-2}} \right] + \right.
\]

\[
\frac{1}{\sin^2 \theta_{n-2}} \left[ \frac{\partial^2}{\partial \theta_{n-1}^2} - \frac{p_{n-1}(p_{n-1}-1)}{\cos^2 \theta_{n-1}} \right] \left\} \right.
\]

The set of $Y_l$, $l = 1, \ldots, n-1$ operators are:

\[
Y_l = \partial^2_{\theta_l} + (n-l-1) \cot \theta_l \frac{\partial}{\partial \theta_l} - \frac{p_l(p_l-1)}{\cos^2 \theta_l} + \frac{k_{l+1}}{\sin^2 \theta_l}, \quad l = 1, \ldots, n-2
\]

\[
Y_{n-1} = \partial^2_{\theta_{n-1}} - \frac{p_{n-1}(p_{n-1}-1)}{\cos^2 \theta_{n-1}}
\]

and the eigenvalue equations:

\[
H \psi = E \psi, \quad Y_l G_l(\theta_l) = k_l G_l(\theta_l), \quad l = 1, \ldots, n-1, \quad \psi = R(r) \prod_{l=1}^{n-1} G_l(\theta_l)
\]

can be easily solved. The solution for the angular part is ($m_n = -1/2$):

\[
\prod_{l=1}^{n-1} G_l(\theta_l) = \prod_{l=1}^{n-1} (\sin \theta_l)^{m_l+1/2-(n-l)/2} (\cos \theta_l)^{p_l} P_j^{(m_{l+1}; p_l - 1/2)}(\cos \theta_l)
\]

and for the radial part:

\[
R(r) = r^{m_1-(n-2)/2} e^{-\sqrt{2E}r} L_{N_r}^{2m_1}(2\sqrt{2E}r)
\]

The energy is written as:

\[
E = -\frac{\gamma^2}{2(N_r + 2 \sum_{i=1}^{n-1} J_i + \sum_{i=1}^{n-1} p_n - \frac{1}{2}(n-1))^2}
\]

and the eigenvalues of the operators $Y_l$ are:

\[
k_l = \frac{1}{4}(n-l-1)^2 - m_l^2, \quad m_l = 2 \sum_{i=l}^{n-1} J_i + \sum_{i=1}^{n-1} p_i + \frac{1}{2}(n-l-1), \quad l = 1, \ldots, n-1
\]

Finally, the eigenfunctions are:

\[
\psi_{n,J_1,\ldots,J_{n-1}}(r, \theta_1, \ldots, \theta_{n-1}) = r^{m_1-(n-2)/2} e^{-\sqrt{2E}r} L_{N_r}^{2m_1}(2\sqrt{2E}r) \times
\]

\[
\prod_{l=1}^{n-1} \left[ (\sin \theta_l)^{m_{l+1}+1-(n-l)/2} (\cos \theta_l)^{p_l} P_j^{(m_{l+1}; p_l - 1/2)}(\cos \theta_l) \right]
\]
B. Exact solvability

The exact solvability of the system \cite{17} for general \( n \) can be treated in the same way as for \( n = 3 \). We can gauge transform each of the operators in the set \cite{18} separately and transform to the variables

\[
s = \sqrt{-2E \mu^2}, \quad t = \sqrt{-2E \nu^2}, \quad z_{n-l+1} = \cos 2\theta_{n-l+1}, \quad l = 1, \ldots, n - 2
\]

Before doing this, we again introduce \( Q_0 \) and \( Q_1 \) as in equation \cite{18}.

The final result is

\[
\begin{align*}
\hat{Q}_0 + \hat{Q}_1 &= g^{-1}(Q_0 + Q_1)g = -2\sqrt{-2E}(2s\partial_s^2 + 2(1 + m_{n-2} - s)\partial_s - m_{n-2} - 1) \\
\hat{Q}_0 - \hat{Q}_1 &= g^{-1}(Q_0 - Q_1)g = -2\sqrt{-2E}(2t\partial_t^2 + 2(1 + m_{n-2} - t)\partial_t - m_{n-2} - 1) \\
\hat{Z}_l &= g^{-1}Z_lg = 4(1 - z_{n-l-1}^2)\partial_z^2 z_{n-l-1} + \\
&\quad \frac{l(l-2)}{4} - (m_{l-1} + p_{l+1})(m_{l-1} + p_{l+1} + 1)
\end{align*}
\]

We see that the entire set of operators \( \{Q_0, Q_1, Z_1, \ldots, Z_{n-2}\} \) lies in the enveloping algebra of direct product of \( n \) special affine Lie algebras \( \text{saff}(1,\mathbb{R}) \).

Finally let us turn to the other complete set of commuting operators \cite{19}, associated with the separation of variables in spherical coordinates. Among these operators there is just one, namely \( Y_1 \), that commutes with all the operators \( Z_l \). We have

\[
[Y_1, Z_l] = 0, \quad [Y_1, X] \neq 0, \quad [Y_p, Z_l] \neq 0, \quad 1 \leq l \leq n - 2, \quad 2 \leq p \leq n - 1
\]

Thus, \( \hat{Y}_l \) will take polynomials into polynomials but \( \{\hat{Y}_2, \ldots, \hat{Y}_{n-1}\} \) will not. We have

\[
\hat{Y}_l = g^{-1}\hat{Y}_lg = st(\partial_s - \partial_t)^2 - (m_{n-2} + 1)(s - t)(\partial_s - \partial_t) - m_{n-2}(m_{n-2} + 1) + \frac{(n-3)(n-1)}{4}
\]

Finally we see that the “hidden Lie algebra” that is not a symmetry algebra of the problem, but underlies its exact solvability is \( \text{saff}(2,\mathbb{R}) \oplus [\text{saff}(1,\mathbb{R})]_1 \oplus \cdots \oplus [\text{saff}(1,\mathbb{R})]_{n-2} \) generated by

\[
\{\partial_s, \partial_t, s\partial_s, t\partial_t, s\partial_t, t\partial_s, \partial_{z_1}, z_1\partial_{z_1}, \ldots, \partial_{z_{n-2}}, z_{n-2}\partial_{z_{n-2}}\}
\]

V. CONCLUSIONS

Superintegrability and exact solvability were defined in completely different ways, though both have a group theoretical underpinning. Superintegrability for a Hamiltonian system is defined by the requirement that there be more integrals of motion than degrees of freedom.\cite{14} It can be characterized by the fact that the corresponding Schrödinger equation allows a nonabelian algebra of generalized symmetries, containing an \( n \)-dimensional Abelian subalgebra.\cite{17} Exact solvability is defined by the requirement that the energy spectrum can be calculated algebraically.\cite{18,19} It can be characterized by the fact that the Hamiltonian lies in the enveloping algebra of a certain type of finite dimensional affine Lie algebra. It was conjectured that all maximally superintegrable systems are exactly solvable. In this article we have confirmed the conjecture for the considered integrable system in \( E_n \).

The exact connection between superintegrability and exact solvability remains an open problem.

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1 J. Jauch and E. Hill, On the problem of degeneracy in quantum mechanics. Phys. Rev. 51, 641–645 (1940).
2 M. Moshinsky and Yu. F. Smirnov, The Harmonic Oscillator in Modern Physics, Harwood, Amsterdam, (1996).
3 W. Pauli, Über das Wasserstoffspektrum von Standpunk der neuen Quanten-mechanik. Zeits. Physik 36, 336–363 (1926).
4 V. A. Fock, Zur Theorie des Wasserstoffatoms, Zeits. Physik 98, 145–154 (1935).
5 V. Bargmann, Zur Theorie des Wasserstoffatoms, Zeits. Physik 99, 576–582 (1936).
6 M. J. Englefield, Group Theory and the Coulomb Problem. Wiley, New York (1972).
7 H. V. MacIntosh, Symmetry and Degeneracy, in Group Theory and its Applications, edited by E.M. Loebl, (Academic Press, New York, 1971), Vol. II, pp. 75-144.
8 E. G. Kalnins, W. Miller Jr, and P. Winternitz, The group $O(4)$, separation of variables and the hydrogen atom. SIAM J. Appl. Math. 30, 630–664 (1976).
9 J. Bertrand, Théorème relatif au mouvement d’un point attiré vers un centre fixe. Comptes Rendus Ac. Sci 77, 849–853 (1873).
10 H. Goldstein, Classical Mechanics (Addison-Wesley, Reading, MA, 1990).
11 I. Friš, V. Mandrosov, J. Smorodinsky, M. Uhlíř, and P. Winternitz, On higher symmetries in quantum mechanics. Phys. Lett. 16, 354–356 (1965).
12 P. Winternitz, J. Smorodinsky, and M. Uhlíř, Symmetry groups in classical and quantum mechanics. Yad. Fiz. 4, 625 (1966), Sov. J. Nucl. Phys. 4, 1326 (1967).
13 A. Makarov, J. Smorodinsky, Kh. Valiev, and P. Winternitz, A systematic search for nonrelativistic systems with dynamical symmetries. Nuovo Cimento A 52, 1061–1084 (1967).
14 N. W. Evans, Superintegrability in classical mechanics. Phys. Rev. A 41, 5666–5676 (1990).
15 N. W. Evans, Group theory of the Smorodinsky-Winternitz system. J. Math. Phys. 32, 3369–3375 (1991).
16 N. W. Evans, Superintegrability of the Winternitz system. Phys. Lett. A 147, 483–486 (1990).
17 E. G. Kalnins, W. Miller Jr, and G. S. Pogosyan, Superintegrability and associated polynomial solutions: Euclidean space and the sphere in two dimensions. J. Math. Phys. 37, 6439–6467 (1996).
18 E. G. Kalnins, W. Miller Jr, and G. S. Pogosyan, Completeness of multiseparability in $E_{2,C}$. J. Phys. A 33, 4105–4120 (2000).
19 E. G. Kalnins, W. Miller Jr, and G. S. Pogosyan, Completeness of multiseparability on the complex 2-sphere. J. Phys. A 33, 6791–6806 (2000).
20 E. G. Kalnins, W. Miller Jr, and G. S. Pogosyan, Completeness of multiseparability in two-dimensional constant curvature spaces. J. Phys. A 34, 4705–4720 (2001).
21 E. G. Kalnins, J. M. Kress, and P. Winternitz, Superintegrability in a two-dimensional space of non-constant curvature. (LANL preprint archives math-ph/0108013). J. Math. Phys. (to appear).
22 P. Tempesta, A. V. Turbiner, and P. Winternitz, Exact solvability of superintegrable systems. J. Math. Phys. 42, 4248–4257 (2001).
23 A. V. Turbiner, Lie algebras and linear operators with invariant subspaces, in Lie Algebras, Cohomologies and New Findings in Quantum Mechanics, edited by N. Kamran and P.J. Olver, (AMS, Providence, 1994), Vol. 160, pp. 263–310.
24 A. V. Turbiner, Quasi-exactly solvable problems and $sl(2)$ algebra. Commun. Math. Phys. 118, 467–474 (1988).
25 M. B. Sheftel, P. Tempesta, and P. Winternitz, Superintegrable systems in quantum mechanics and classical Lie theory. J. Math. Phys. 42, 659–673 (2001).
26 J. Daboul, P. Słodowy, and C. Daboul, The hydrogen algebra as a centerless twisted Kac-Moody algebra. Phys. Lett. B 317, 321–328 (1993).
27 C. Daboul and J. Daboul, From hydrogen atom to generalized Dynkin diagrams. Phys. Lett. B 425 135–144 (1998).
28 P. Winternitz and I. Friš, Invariant expansions of relativistic amplitudes and subgroups of the proper Lorentz group. Sov. J. Nucl. Phys. 1, 636–643 (1965).
29 P. Winternitz, I. Lukac, and Ya. Smorodinskii, Quantum numbers in the little groups of the Poincaré group. Sov. J. Nucl. Phys 7, 139–145 (1968).
30 J. Patera and P. Winternitz, A new basis for representations of the rotation group. Lamé and Heun polynomials. J. Math. Phys. 14, 1130–1139 (1973).
31 W. Miller Jr, J. Patera, and P. Winternitz, Subgroups of Lie groups and separation of variables. J. Math. Phys. 22, 251–260 (1991).
32 W. Miller Jr, Symmetry and the Separation of Variables (Addison-Wesley, Reading, MA, 1997).
33 E. G. Kalnins, *Separation of Variables for Riemannian Spaces of Constant Curvature* (Longman, Burnt Mill, 1986).
34 J. Hietarinta, B. Grammaticos, B. Dorizzi, and A. Ramani, Coupling-constant metamorphosis and duality between integrable Hamiltonian systems. Phys. Rev. Lett. 53, 1707–1710 (1984).
35 A. González-López, N. Kamran, and P.J. Olver, Normalizability of One-dimensional Quasi-Exactly Solvable Schrödinger Operators. Commun. Math. Phys. 153, 117–146 (1993).
36 M. Shifman, New findings in quantum mechanics (Partial algebraization of the spectral problem). Int. J. Mod. Phys. A 4, 2897–2852 (1989).
37 N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations* (AMS, Providence, R.I., 1968).
38 N. Ya. Vilenkin, G. I. Kuznetsov, and Ya. A. Smorodinskii, Eigenfunctions of the Laplace operator realizing representations of the groups $U(2)$, $SU(2)$, $SO(3)$, $U(3)$ and $SU(3)$ and the symbolic method. Sov. J. Nucl. Phys. 2 645–655 (1965).
39 A. A. Izmest’ev, G. S. Pogosyan, A. N. Sissakian, and P. Winternitz, Contractions of Lie algebras and the separation of variables. The n-dimensional sphere. J. Math. Phys. 40, 1549–1573 (1999).
40 G. S. Pogosyan and P. Winternitz, Separation of variables and subgroup bases on n-dimensional hyperboloids (to be published).