The Confinement Property in SU(3) Gauge Theory

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(Dated: May 10, 2009)

We study confinement property of pure SU(3) gauge theory, combining in this effort the non-perturbative gluon and ghost propagators obtained as solutions of Dyson–Schwinger equations with solutions of an integral ladder diagram summation type equation for the Wilson loop. We obtain the string potential and effective UV coupling.

PACS numbers: 12.38.Aw,11.15.Pg,11.15.Tk

I. OVERVIEW

The problem of explaining quark confinement has been of foremost importance since the formulation of quantum chromodynamics (QCD). The principal manifestation of confinement is the linear growth of QCD potential between color charges. This is known to be the property of the Wilson loop \(W\). However, it has been impossible so far to use this in an analytic \(ab\) initio calculation in 3 + 1 dimensional QCD. We want to deal with this challenging problem by combining:

a) the Erickson–Semenoff–Szabo–Zarembo (ESSZ) \[2, 3\] formulation for Bethe–Salpeter type equation for Wilson loops, with

b) Dyson–Schwinger equations (DSE) for the gluon and ghost propagator in Landau gauge \[4, 5\].

We solve DSE for gluons and ghosts in the pure glue two-point sector. Then we insert the resulting QCD coupling \(\alpha = g^2/4\pi\) and the gluon propagator into ESSZ equation for a rectangular (non-supersymmetric) Wilson loop, and solve this integral equation, which yields the Wilson potential.

The ESSZ type ladder (or rainbow) diagram summation has long been a major tool for extracting non-perturbative information about dynamics of a gauge theory. However the strength of this method is more evident in \(\mathcal{N} = 4\) supersymmetric Yang–Mills due to higher order vertex correction cancelation. In principle the use of ESSZ ladder summation in our context of non-supersymmetric QCD is doubtful, and we will make several efforts to establish this approach: we will study the vertex correction terms by comparing the leading order (LO) contribution to the next to LO (NLO) contribution of the three-gluon vertex, and we will consider convergence of the entire procedure by evaluating the string tension at different DSE scale fixing points.

Within QCD, the DSE for propagators and vertex functions have been studied in great depth, for review see \[4, 5\] and references therein. Relation of DSE to lattice results is discussed in \[6\]. An alternative related method of functional renormalization group has been discussed in \[7\]. The relevant results on three-point functions are seen in \[8\], and on quark propagator in \[9\], the question of confinement inherent alone in DSE are discussed in \[9, 10, 11\], the uniqueness of the infrared (IR) scaling of Green functions established and gluon propagator IR non-singularity strictly supported in \[12, 13\], the IR universality established in \[14\].

Below in section II we describe the ESSZ equations in a pure Yang–Mills theory with an arbitrary propagator (form-factor). In section III we present DSE and our solution, our results are in agreement with the standard state-of-the art calculations of ghost and gluon propagators in Landau gauge. In section IV we evaluate the ESSZ truncated Wilson loop employing the DSE propagators from section III and check the significance of the NLO vertex correction. In section V we discuss the reasons why confining potential is not observed either in pure-gluon two-point sector of DSE, or ESSZ solely, yet it is seen in the combination thereof.

II. ESSZ EQUATION

The Wilson loop

\[
W(C) = \langle \text{tr} \exp \left\{ \int_C A_\mu(x)dx^\mu \right\} \rangle
\]

(1)

offers information about the behaviour of quarks in the theory, and the quark-antiquark potential is

\[
V(L) = -\lim_{T \to \infty} \frac{1}{T} \ln W(C_{T,L}),
\]

(2)

where \(C_{T,L}\) is a rectangular Wilson loop in the \((x^0, x^1)\) plane, with \(T\) being loop temporal length, and \(L\) loop spatial length, \(T \gg L\).
The Wilson loop Eq. (1) can be represented in terms of a perturbative expansion, which can be found e.g. in the review [13]. A set of Feynman rules for Wilson loops can be found in [16], which will be of use to us below. Perturbative treatment of Wilson loops is not useful in the non-Abelian case, and especially in the present context, as it yields obviously wrong results for Yang–Mills theory, for which it predicts a Coulomb-type potential [13]. A large-$N_c$ partial summation of ladder diagrams has been proposed in [2] and performed for a circular and a rectangular loop in $\mathcal{N} = 4$ supersymmetric model (SUSY). This method is adapted here to the case of a non-SUSY theory, for the case that the partial summation of perturbation theory (PT) series for propagators has already been performed in terms of solving DSE.

Consider a trapezoidal loop $W(C) = \Gamma(T_1, T_2; L)$ with long parallel temporal sides of lengths $T_1, T_2$, separated by a spatial distance $L$. Then the requirement that adding a propagator to the summed expression does not change it leads to the following integral equation for the sum of all ladder diagrams:

\[ \Gamma(T_1, T_2, L) = 1 + \frac{g^2 N_c}{4\pi^2} \int_{T_1}^{T_2} dt_1 \int_{T_2}^{T_1} dt_2 \times \]
\[ \times \Gamma(t_1, t_2, L) D_{\mu\nu}((x_1 - x_2)^2) \dot{x}_1^\mu \dot{x}_2^\nu, \]

dots denote derivatives in $t_{1,2}$ respectively, where $x_1^\mu(t_1)$, $x_2^\nu(t_2)$ are paths running over the Wilson loop as functions of $t_1, t_2$. For a rectangular loop $x_1 = (-L/2, t_1, 0, 0), x_2 = (L/2, t_2, 0, 0)$. Configuration space propagator is related to the momentum-space form-factor $F(p^2)$, introduced in the next Section III by:

\[ D_{\mu\nu}(x^2) = \frac{1}{(2\pi)^4} \int \frac{d^4 p e^{-ipx}}{p^2} F(p^2) \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \]

(4)

For simplicity we write $D_{\mu\nu}(x^2) \dot{x}_1^\mu \dot{x}_2^\nu = D(x^2)$. Boundary conditions imposed upon $\Gamma$ are

\[ \Gamma(T, 0; L) = \Gamma(0, T; L) = 1. \]

(5)

The potential is related to $\Gamma(T_1, T_2; L)$ in the following way

\[ V(L) = -\lim_{T \to \infty} \frac{1}{T} \log \Gamma(T, T; L). \]

(6)

Equation (3) is depicted symbolically in Fig. [11]. Obviously, if we write down the first term for $\Gamma(T_1, T_2; L)$ in the $g^2$ expansion of the solution, we shall reproduce the perturbative result for the Wilson loop.

The central filled square in Fig. [11] symbolizes an irreducible kernel, containing (potentially) all the possible loop corrections. A convenient way of solving Eq. (3) is to consider the equivalent differential equation:

\[ \frac{\partial^2 \Gamma(t_1, t_2; L)}{\partial t_1 \partial t_2} = \frac{g^2 N_c}{4\pi^2} D \left( (t_1 - t_2)^2 + L^2 \right) \Gamma(t_1, t_2; L), \]

(7)

We now introduce the variables $x = (t_1 - t_2)/L$, $y = (t_1 + t_2)/L$. With this Ansatz the separation of variables becomes possible, and using the form:

\[ \Gamma = \sum_n \psi_n(x) e^{\frac{\Omega_n}{2} x} \]

(8)

we will be solving the 1d-equation

\[ -\frac{d^2}{dx^2} \psi_n(x) + U(x; L) \psi_n(x) = -\frac{\Omega_n^2}{4} \psi_n(x), \]

(9)

with the effective potential

\[ U(x; L) = -\frac{g^2 N_c}{4\pi^2} L^2 \left( L^2(1 + x^2) \right) \]

(10)

We are solely interested in the unique ground state solution of Eq. (9), since the Wilson quark-quark potential is

\[ V(L) = -\lim_{T \to \infty} \frac{1}{T} \log \sum_n \psi_n(x) e^{\frac{\Omega_n}{2} x} = -\frac{\Omega_0}{L}. \]

(11)

A degeneracy in solutions of Eq. (9) may arise and thus complicate the situation, however, we have never observed it in our numeric calculations shown below in section IV. It is now evident, that in order to complete the Wilson potential evaluation we need the propagator $D$ and the coupling $\alpha = g^2/4\pi$ derived from DSE in order to be able to evaluate $V = \Omega_0(L)/L$.

III. DYSON–SCHWINGER EQUATIONS

We now obtain the nonperturbative input to ESSZ equations, i.e. the Dyson–Schwinger improved gluon propagator and coupling $\alpha$. The difference between DSE and the simple renormalization group (RG) improved quantity is in the IR and medium momentum ranges, their ultra violet (UV) behaviour being identical (up to 1 loop at least). Our DSE procedure uses the technique described in [12, 20], the reader familiar with this may skip the current section where we demonstrate that the results of [14, 20] are independently reproduced by us.

We employ in our work the Newton-method based numerical technique described in [21]. We solve a system...
for ghost and gluon propagators, corresponding to the
representation seen in figure Fig. [2]. Here bulbs denote
dressing of the propagators, and transparent bulbs — dressing of vertices.

\begin{align*}
\text{Ghost} & \quad = \quad \text{Gluon} \\
\begin{array}{c}
p \quad \text{p} \\
\text{p} \quad \text{q} \\
\text{q} \quad \text{p} \\
\text{p} \quad \text{q}
\end{array}
\end{align*}

FIG. 2: Diagrammatic representation of DSE.

These equations can be written in the form:

\begin{align*}
\begin{cases}
\frac{1}{G(p^2)} - \frac{1}{G(\mu^2)} = -(\Sigma(p^2) - \Sigma(\mu^2)), \\
\frac{1}{F(p^2)} - \frac{1}{F(\mu^2)} = - (\Pi(p^2) - \Pi(\mu^2)),
\end{cases}
\end{align*}

(12)

where vacuum polarization is

\[ \Pi(p^2) = \Pi^2c(p^2) + \Pi^2g(p^2), \]

(13)

and self-energy is

\[ \Sigma(p^2) = N_c g^2 \int \frac{d^d q}{(2\pi)^d} M_0(p^2, q^2, r^2) G(q^2) G(r^2), \]

(15)

Here \( \bar{\mu}_{g,c} \) are subtraction points, \( \bar{\mu}_c = 0 \), \( \bar{\mu}_g = \bar{\mu} \), \( \bar{\mu} \) is the limit of the interval \( p^2 \in (0, \bar{\mu}^2) \) in the momentum space where we solve DSE, coupling \( g^2 \) is meant to be \( g^2(\bar{\mu}^2) \). \( F \) is gluon propagator form-factor in Landau gauge, defined via relation

\[ D_{\mu \nu}^{F, ab}(p) = \delta^{ab} \left( g_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{F(p^2)}{p^2 + i\epsilon}. \]

(16)

and the ghost propagator non-trivial behaviour is described by the form-factor \( G \)

\[ D^{G, ab}(p) = \frac{\delta^{ab}}{p^2 + i\epsilon} G(p^2). \]

(17)

Variable \( z \) is the logarithmic variable

\[ z = \ln \frac{p^2}{\mu^2}, \]

(18)

and scale \( \mu \) is yet to be defined upon solving Dyson–Schwinger equations from comparing the obtained coupling \( \alpha_{DSE}(z) \) to the known values of \( \alpha_{PDG}(p^2) \) at point \( M \):

\[ \alpha_{DSE}(\ln(M^2/\mu^2)) = \alpha_{PDG}(M^2). \]

(19)

The coupling constant \( g^2/4\pi \equiv \alpha \) is expressed in terms of \( G, F \) solely [17,18], as vertex is finite in Landau gauge (at one-loop level)

\[ \alpha_{DSE}(\ln(p^2)) = \alpha_{DSE}(\sigma) F(p^2) G(p^2). \]

(20)

In our case, we shall use varying scale fixing point \( M \) so that we can prove that our results are independent of scale fixing point choice within the error margin of our procedure.

The kernels \( M_0, K_0, Q_0 \) are known in literature, but for self-containedness of the paper we show them here:

\[ K_0(x, y, \theta) = \frac{g^2 \sin(\theta)}{(-2 \cos(\theta) \sqrt{xy} + x + y)^2}, \]

\[ M_0(x, y, \theta) = - \frac{g^2 \sin(\theta)}{3x(-2 \cos(\theta) \sqrt{xy} + x + y)}. \]

\[ Q_0(x, y, \theta) = - \frac{1}{12x} \left[ \begin{array}{c}
2 \cos(2\theta) (6x^2 + 31xy + 6y^2) - \ln \sqrt{\bar{\mu}^2} \sqrt{\bar{\mu}^2} + xy \cos(4\theta) - 48 \cos(\theta) \sqrt{\bar{\mu}^2} (x + y) - 12y \cos(3\theta) \sqrt{\bar{\mu}^2} + 3x^2 + 27xy + 3y^2)
\end{array} \right]. \]

(22)

For convenience, variables \( x = p^2, y = q^2 \) are introduced; variable \( \theta \) is defined via \( (p - q)^2 = x + y - 2\sqrt{xy} \cos \theta \).

To solve Dyson–Schwinger equations we use the Ansatz [19,20]:

\begin{align*}
\exp \left( \sum_i \bar{a}_i T_i(z) \right), \quad z \in (\ln \epsilon, \ln \bar{\mu}^2), & \\
F(z) = \exp \left( \sum_i \bar{a}_i T_i(z) \right), \quad z > \ln \bar{\mu}^2, & \\
G(z) = \exp \left( \sum_i \bar{b}_i T_i(z) \right), \quad z \in (\ln \epsilon, \ln \bar{\mu}^2), & \\
Bz^{-\kappa}, \quad z < \epsilon, &
\end{align*}

(23)

Here \( T_i \) are Tschebyschev polynomials, \( \bar{a}_i, \bar{b}_i \) are unknown coefficients yet to be determined from the numerical solution, \( \bar{n} \) is the number of polynomials used.
The coupling $\alpha$ obtained from DSE Eq. (20) is shown in Fig. (4). We compare it to the standard coupling from Particle Data Group [22], and note that the both coincide very well in the UV. We also note here that the IR fixed point seen in the Figure is

$$\alpha(0) \approx 2.9$$

for $N_c = 3$, which is consistent with the up-to-date Dyson–Schwinger results reported by other groups [4, 5].

We have to find the lowest eigenvalue of a Schrödinger equation Eq. (9)

$$\left( -\frac{1}{2} \frac{d^2}{dx^2} + U(x; L) \right) \psi(x) = \mathcal{E} \psi(x)$$

where the auxiliary potential $U(x)$ is related by an linear integral transform to gluon form-factor as

$$U(x) = \frac{2\pi\alpha N_c}{(1+x^2)^2} \int \frac{du}{u} \times \left( uJ_1(u) - (1-3x^2)J_2(u) \right) F \left( \ln \left( \frac{u^2}{L^2 \mu^2 (1+x^2)} \right) \right),$$

where $\mu$ is defined at point $M$ as given in [19], $M$ varying from 1 to 10 GeV, $u$ is dummy scalar dimensionless integration variable. The coupling $\alpha$, in the sense of DSE approach, is taken here at the scale of $\frac{2\pi}{\mu}$, rather than bare. We solve the Schrödinger equation with shooting method and find its ground state. Special care is taken to make sure this state is not degenerate. As a result we get the QCD potential $V(L) = -\frac{2\sqrt{2\pi}}{L}$. The potential is defined up to additive constant, so we shift it to provide convenient comparison to existent results. It is shown in Fig. (5) below, and is compared with lattice results by Gubarev et al. [26] and Necco [27]. Linear IR behaviour of the potential can be clearly seen from Figure. We fit the potential by the standard expression

$$V(L) = -\frac{4\alpha_0}{3} \frac{\sigma}{L} + c_0 + \sigma L.$$  

Dependence on string tension $\sigma$ on the scale fixing point choice is shown in Fig. (5). We see that the variance of $\sigma$ does not exceed that of different lattice results, shown in the table (I). The error we quote arises from an average of results obtained at different scale fixing points. This yields $\alpha_0 = 0.24$ and $\sigma = 1.07 \pm 0.1$. 

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TABLE I: Comparison of string tension from different sources

| Author             | Year | $\sigma$, GeV/fm |
|--------------------|------|------------------|
| Bali et al. [28]   | 2000 | 1.27             |
| Necco [27]         | 2003 | 1.19             |
| Gubarev et al. [26]| 2007 | 0.978            |
| Weise et al. [29]  | 2009 | 1.07             |
| Present work       | 2009 | 1.07 $\pm$ 0.1   |

The key result, the linear confining potential comes as a surprise. It invites the question, how great are the corrections coming from three-point vertex? One actually shouldn’t have thought that QCD can be described with ESSZ partial summation structure. Considering the vertex, the auxiliary potential is then modified:

$$U(x) = U^{(1)}(x) + 4\pi\alpha N_c U^{(2)}(x),$$  (30)

where $U^{(2)}(x)$ comes, in the leading $1/N_c$ order, from the Wilson loop diagram shown in Fig. (7).

Calculating the diagram in Landau gauge with rules as defined in [16] we obtain:

$$U^{(2)}(t_1, t_2) = \int d^4y \int_0^1 dt_3 \frac{1}{(y-x_1)^2(y-x_2)^2(y-x_3)^2} \times \left[ (u_1 u_2)(u_3 y) \left( \frac{1}{(y-x_1)^2} - \frac{1}{(y-x_2)^2} \right) + \text{cyclic permut.} \right]$$  (31)

with

$$\begin{align*}
  x_1 &= (-L/2, t_1, 0, 0) \\
  x_2 &= (L/2, t_2, 0, 0) \\
  x_3 &= (-L/2 + L t_3, t_1 + t_3(t_2 - t_1), 0, 0) \\
  u_1 &= (0, 1, 0, 0) \\
  u_2 &= (0, 1, 0, 0) \\
  u_3 &= (L, t_2 - t_1, 0, 0)
\end{align*}$$  (32)

Numerical evaluation of this integral shows that within the whole range of values of $t_1, t_2$ with which we work, $U^{(2)}(t_1, t_2) = U^{(2)}(t_1 - t_2) \approx U^{(2)}(x)$. This makes separation of variables still possible and provides an extra test for reasonability of our model. Numerical values of $U^{(2)}(x)$ are such that $U^{(2)}(x)/U^{(1)}(x) \approx 10^{-3}$, which makes its contribution to auxiliary potential ground state negligible. This allows us to justify validity of ESSZ equation application in the non-SUSY case: vertex correction is present but numerically suppressed.

V. DISCUSSION AND CONCLUSION

ESSZ approach to SUSY Wilson loops has worked very well in [2, 3]. The reason for that is absence of NLO corrections in the maximally supersymmetric theory. At small coupling their result has restored the perturbatively known IR singularity structure. Moreover, the calculation originally performed in the small coupling limit, could be continued into large coupling limit. At large
coupling the solution to ESSZ equation reproduces almost exactly the gravity dual result \[34, 35, 36\] (up to an overall numerical factor very close to unity). Actually this result, though in a different theory, has been a guide for our QCD treatment: as we are dealing with the IR strongly coupled theory, we are certainly out of order of applicability of any perturbative treatment, and even summation of diagrams would be suspicious.

The reason why the ESSZ equation has never been applied to non-SUSY contents is obvious. It is clear from \[2, 3\] that when a perturbative propagator input is being used only a non-confining Wilson loop, with a Coulomb-type potential may be obtained. This follows from the fact that dependence on Wilson loop spatial size \(L\) may be scaled out of the ESSZ equation, so that any potentials one gets from it are Coulombic, varying from each other by coupling rather than distance dependence. Thus such a result would have been \textit{a priori} useless in understanding anything about strong coupling IR regime of gauge theory, where confinement governs the dynamics. This maybe the reason why summation à la ESSZ has not been before employed in pure Yang–Mills theory.

A direct perturbative calculation of Wilson loop with a Dyson–Schwinger propagator yields no confinement whatsoever. Only quark-gluon vertex functions \[4, 10, 11\] coming from DSE can render something looking like confinement, which is then related to the singular behaviour of the quark-gluon vertex in the IR.

In our opinion, this could not be one of the possible ways to approach within the DSE the confinement problem, since it requires quark coupling to be singular, gluon one regular, which constitutes a severe violation of Ward identities. Rather than to involve three-particle functions, we apply ESSZ summation with DSE solutions which are possessing intrinsic scale, distance \(L\) is no more possible to scale out of ESSZ equations. Thus the resulting Wilson potential is no more necessarily being Coulombic.

A description of a single Wilson loop, from which one can obtain the QCD potential and provide a criterion of confinement, has not been done so far in terms of the two-point sector of DSE hierarchy. Thus our work closes an essential gap in the literature. The main reason for this gap was the theorem by West \[33\], stating that confinement is provided by a very IR-singular propagator \(D(q^2) \sim 1/q^4\). We know however that gluon propagator is regular in the IR in the DSE approach.

Our work is based on combined analysis of Green functions and Wilson loops, allowing thus a study of the spatial QCD potential. This distinguishes our approach from several earlier papers where gluon non-propagation was considered instead of confinement and related to the analytic properties of Green functions, in particular, to the IR scaling \(\kappa\), Eq. \[23\]. These other works use the word “confinement” as in the original paper \[1\] when they mean to say of “non-propagation”. Known are the so-called Kugo–Ojima criterion for colour non-propagation \(\kappa > 0\) \[30\], Zwanziger criterion of ghost non-propagation \(\kappa > 0\) and gluon non-propagation \(\kappa > 1/2\) \[31\]. A claim has been made \[32\] for \(\kappa > 1/4\) to be quark confinement criterion by analysis of the Polyakov loop and effective QCD action in an external field. All these results are about gluon non-propagation rather than the properties of a colour charge-colour charge confining interaction.

Returning to the discussion of our results we note that the reliability thereof may be questioned in what concerns the DSE input. The first issue is the truncation of the DSE system we solve to only two-point functions. The truncation is justified by ghost-gluon vertex not acquiring acquire one-loop corrections in Landau gauge. It has been proved that the three-point gluon and quark-gluon functions don’t change ghost dominance property \[8\], even though they are important for bound states \[37\]. In this sense, vertex functions are unimportant for our particular context.

Another question is whether Green functions obtained from DSE are physically relevant within the Wilson loop context we are discussing. We note that it is mostly medium-energy range that provides the important contribution into the auxiliary potential \(U(x; L)\), rather than the perhaps more model dependent IR piece. The Wilson loop thus depends on medium energy range values of the propagators where the DSE behaviour is the same as in lattice. There are unresolved questions regarding comparison of IR scaling \[38\] within lattice and DSE. These issues have yet to be understood and resolved, although they do not affect our results materially.

The observables \(\sigma, \alpha\) we compute are in principle gauge invariant. Our results are obtained in Landau gauge, which, as noted, is a convenient choice. It should be possible to check gauge-invariance explicitly at one loop level, we however do not do that here, since this transcends the scope of the present paper. We think that the possibility for the observable we consider to be gauge invariant at one-loop level comes from the fact that several gauge-dependent objects are combined.

We speculate here à propos that a nonperturbative summation à la ESSZ could improve significantly the properties of a correlator of gluon strengths with Wilson lines

\[
\mathcal{F}(x) = \langle \text{tr} F_{\mu\nu}(x) U(C) F^{\mu\nu}(0) U^+(C) \rangle
\]

\[U(C)\] being a phase factor

\[
U(C) = \text{Pexp} \left\{ i g \int_C A_\mu dx^\mu \right\},
\]

which differs from Wilson loop since the path is connecting the arguments in Eq. \[39\] i.e. points \(x\) and \(0\). Eq. \[39\] had recently been of great interest \[39\] as it represents an important vacuum property. As far as we know, a Bethe–Salpeter equation for this kind of correlator has not been developed yet. We attempted to evaluate it perturbatively \[40\]. The present effort arose from this earlier one but should have actually anteceded it, for then a framework for ESSZ summation may have been closer or even in hand.
A hypothesis should be considered that using a relevant component of the non-perturbative input from Dyson–Schwinger equations one may be able to obtain a self-consistent picture of the QCD vacuum with all higher correlation functions, colour confinement and condensates, which is supported at the simplest LO level by the presented calculation.

To conclude, combining Dyson-Schwinger summation for gluon and ghost propagators with the Ericson–Semenoff–Szabo–Zarembo summation (truncation) for Wilson loop, we have obtained the string tension and have further demonstrated that its value is nearly not dependent on the selection of the DSE scale fixing point, thus establishing the internal consistency of this novel description of confinement. The string tension determined by our method for the pure SU(3) gauge theory is \( \sigma = 1.07 \pm 0.1 \). The UV Coulomb behaviour is governed by \( a_0 \approx 0.24 \).

One can actually be quite amazed that our method has worked so well in QCD, without supersymmetry, thus with vertices non-compensated. One can speculate that the two truncated summations are complementary, ESSZ taking care of ladders and DSE taking care of rainbows in the vertices. Among interesting further steps in the development of this framework we recognize the formulation and evaluation of a similar ESSZ equation or a correlator of two gluons, having in mind its application to gluon non-local condensate Eq. (33). Another, perhaps more challenging further development could be to solve ESSZ and DSE jointly, without the separation into partial systems.

Acknowledgments

We thank Prof. Dr. D. Habs for hospitality at the Physics Department at LMU Garching. One of us (A.Z.) thanks K.Zarembo for useful correspondence. This work was supported by the DFG Cluster of Excellence MAP (Munich Centre of Advanced Photonics), by RFBR Grant 07-01-00526, and by a grant from the U.S. Department of Energy DE-FG02-04ER4131.
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