Group Classification and Conservation Laws for a
Two-dimensional Generalized Kuramoto-Sivashinsky
Equation

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Abstract
The two-dimensional anisotropic Kuramoto-Sivashinsky equation is a forth-
order nonlinear evolution equation in two spatial dimensions that arises in
sputter erosion and epitaxial growth on vicinal surfaces. A generalization
of this equation is proposed and studied via group analysis methods. The com-
plete group classification of this generalized Kuramoto-Sivashinsky equation is
carried out, it is classified according to the property of the self-adjointness and
the corresponding conservation laws are established.

Keywords: anisotropic Kuramoto-Sivashinsky equation, nonlinear
self-adjointness, group classification, conservation laws, computer algebra

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1. Introduction

The celebrated Kuramoto-Sivashinsky Equation (KSE)

\[ u_t + u_{xxxx} + u_{xx} + \frac{1}{2} u_x^2 = 0, \]  \hspace{1cm} (1)

where \( u = u(x,t) \), is an equation that for nearly half a century has attracted
the attention of many researchers from various areas due to its simple and yet
rich dynamics. It first appeared in mid-1970s by Kuramoto in the study of
angular-phase turbulence for a system of reaction–diffusion equations model-
ing the Belousov–Zhabotinskii reaction in three spatial dimensions \cite{24,25,26}.

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and independently by Sivashinsky in the study of hydrodynamic instabilities in laminar flame fronts \cite{40, 33, 41}.

In a physical context equation (1) is used to model continuous media that exhibits chaotic behavior such as weak turbulence on interfaces between complex flows (quasi-planar flame front and the fluctuation of the positions of a flame front, fluctuations in thin viscous fluid films flowing over inclined planes or vertical walls, dendritic phase change fronts in binary alloy mixtures), small perturbations of a metastable planar front or interface (spatially uniform oscillating chemical reaction in a homogeneous medium) and physical systems driven far from the equilibrium due to intrinsic instabilities (instabilities of dissipative trapped ion modes in plasmas and phase dynamics in reaction-diffusion systems) \cite{5, 27, 31, 42, 14, 7}.

As a dynamical system the KSE is known for its chaotic solutions and complicated behavior due to the terms that appear. Namely, the \( u_{xx} \) term acts as an energy source and has a destabilizing effect at large scale, the dissipative \( u_{xxxx} \) term provides dumping in small scales and, finally, the nonlinear term provides stabilization by transferring energy between large and small scales. Because of this fact, equation (1) was studied extensively as a paradigm of finite dynamics in a partial differential equation (PDE). Its multi-modal, oscillatory and chaotic solutions have been investigated \cite{16, 17, 23, 29, 32}, its non-integrability was established via its Painlevé analysis \cite{6, 36, 45} and due to its bifurcation behavior a connection to low finite-dimensional dynamical systems is established \cite{44, 47}.

The generalization of KSE to two dimensions comes naturally, the two-dimensional Kuramoto-Sivashinsky Equation,

\[
    u_t + \nabla^4 u + \nabla^2 u + (\nabla u) \cdot (\nabla u) = 0, \tag{2}
\]

where now \( u = u(x, y, t) \) and \( \nabla^2 = \nabla \cdot \nabla \), \( \nabla^4 = \nabla \cdot \nabla (\nabla \cdot \nabla) \). Eq. (2) has equally attracted much attention because of the same spatiotemporal chaos properties that exhibits and its applications in modeling complex dynamics in hydrodynamics \cite{3, 12, 13, 28}. Nevertheless, due to the additional spatial dimension equation (2) is very challenging and even its well-posedness is still an open problem \cite{8, 22}.

One generalization of equation (2) of much interest is the anisotropic two-dimensional Kuramoto-Sivashinsky Equation,

\[
    u_t = \frac{1}{2} u_x^2 + \frac{\beta}{2} u_y^2 - u_{xx} - \alpha u_{yy} - u_{xxxx} - 2u_{xxyy} - u_{yyyy}, \tag{3}
\]

where the two real parameters \( \alpha, \beta \) control the anisotropy of the linear and the nonlinear term, respectively. In other words, the stability of the solutions of equation (3). The anisotropic two-dimensional Kuramoto-Sivashinsky Equation, due to the fact that it describes linearly unstable surface dynamics in the presence of in-plane anisotropy, has a wide range of applications. For instance, as a model for the nonlinear evolution of sputter-eroded surfaces and describing the epitaxial growth of a vicinal surface destabilized by step edge barriers, for further details see \cite{39} and the references therein, in particular \cite{30}. 

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This paper focuses in the following generalization of the anisotropic KSE (3):

$$u_t = \frac{1}{2}u_x^2 + h(u)u_y^2 + r(u)u_{xx} + g(u)u_{yy} - u_{xxxx} - 2u_{xxyy} - u_{yyyy} + f(u)$$  (4)

and its study under the prism of Lie point symmetries and conservation laws. (Here, \(f, h, g\) and \(r\) are considered smooth functions of \(u = u(x, y, t)\).)

The symmetries of a differential equation are of fundamental importance since they are a structural property of the equation. In addition, finding the symmetries of a differential equation is an analytic method that can be applied to integrable and non-integrable equations alike. Nevertheless the symmetry analysis is constrained to rudimentary generalizations of equation  (2) [34, 35]. Things are worse when one looks for conservation laws. For the complexity of the calculations involved, the research is constrained to generalizations of the one dimensional Kuramoto-Sivashinsky Equation [4].

In this frame two different classifications are performed: the complete group classification and a classification with respect to the property of self-adjointness. Having the symmetries for each possible case and the self-adjoint cases at hand, the conservation laws for that system by using the Noether operator \(\mathcal{N}\) are obtained, see also [19, 20, 21].

Calculating the symmetries of the system (3), obtaining its adjoint system and applying the Noether operator to obtain the conserved vectors are well-defined algorithmic procedures. Nevertheless, the calculations involved are usually very difficult and extensive even for the simplest equations. Thus, it may become very tedious and error prone. For that reason the use of computer algebra systems like Mathematica, Maple, Reduce, etc. and of special symbolic packages that are build based on them is very crucial. For this work the Mathematica package SYM [10, 11, 9] was extensively used for all the results that follow. Namely, for obtaining the symmetries of the system, to get and simplify the adjoint system and the conserved vectors that emerge from the use of the Noether operator.

In Section 2 the definitions and the analytical tools used are introduced. Section 3 explores the self-adjointness of equation (4). Then, the complete group classification of eq. (3) is carried out in Section 4 followed by Section 5 where the conservation laws are established. Finally in Section 6 some comments and concluding remarks are presented.

2. Notation and methodology

We shall employ in this work the two analytical tools: the symmetry analysis and the use of the Noether operator identity for the explicit construction of conservation laws. For both the necessary definitions of the notions that will be encountered in the main body of this work are illustrated below, adapted accordingly to the needs of the present paper.
For brevity, we denote:
\[
\Delta(x, y, t, u, u_x, \ldots, u_{yyyy}) =
\]
\[
\frac{1}{2}u_x^2 + h(u)u_y^2 + r(u)u_{xx} + g(u)u_{yy} - u_{xxxx} - 2u_{xxyy} - u_{yyyy} - u_t + f(u).
\]

2.1. Modern group analysis

The symmetry or modern group analysis is a valuable analytic tool for the investigation of differential equations. For a full treatise of the subject there is a wealth of classical texts that encompass all aspects of the theory \[1, 2, 15, 18, 37, 38, 43\].

Definition 2.1. Let the differential operator,
\[
X = \xi^1(x, y, t, u)\partial_x + \xi^2(x, y, t, u)\partial_y + \xi^3(x, y, t, u)\partial_t + \eta(x, y, t, u)\partial_u.
\]

This operator, from now on called infinitesimal generator, determines a Lie point symmetry of equation (4), if and only if, its action on the equation will be, modulo the equation itself, identically zero, that is:
\[
X(\Delta(x, y, t, u, u_x, \ldots, u_{yyyy})) \equiv 0,
\]

where \(X(4)\) is the fourth order prolongation of the operator \(X\) given by
\[
X^{(4)} = X + \sum_{s=1}^{4} \eta_i^{(s)} \partial_{u_{i_1} \ldots i_s} , \ i_n = 1, 2, 3
\]

with
\[
\eta_i^{(1)} = D_i\eta - (D_i\xi^j)u_j, \quad \eta_i^{(s)} = D_i\eta_i^{(s-1)} - (D_i\xi^j)u_{i_1 \ldots i_{s-1} j}
\]

and the partial derivatives denoted by
\[
u_i = \frac{\partial u}{\partial x^i} , (x^1, x^2, x^3) = (x, y, t).
\]

From the condition (6), called linearized symmetry condition, an overdetermined system of linear partial differential equations emerges. By solving this system, called the determining equations, we determine the coefficients \(\xi^i, \eta\) of the infinitesimal generator. Hence, the point symmetries of the equation. The group classification occurs in that phase: the determining equations contain also the functions \(f, g, h, r\). The group classification is performed by investigating each case where specific relations among the unknown elements remove equations from the set of determining equations, and by doing that enlarge the set of solutions.
2.2. The adjoint and self-adjoint concept and conservation laws

In accordance to [19, 20, 21] we introduce the required notions that will enable us to construct conservation laws for the equation (4).

**Definition 2.2.** The adjoint equation to equation (4) is

\[ \frac{\delta L}{\delta u} = 0, \]  

(8)

where \( L \) is the formal Lagrangian given by

\[ L = \nu(x, y, t) \Delta(x, y, t, u, u_x, \ldots, u_{yyyy}). \]

Here \( \nu \) is a new dependent variable, called also nonlocal variable, and \( \delta/\delta u \) is the Euler-Lagrange operator

\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{i_x+i_y+i_t=1}^\infty (-1)^{i_x+i_y+i_t} D_x^{i_x} D_y^{i_y} D_t^{i_t} \frac{\partial}{\partial u_{i_x+i_y+i_t}}, \]

with \( u_{i_x+i_y+i_t} \) denoting \( \partial^{i_x+i_y+i_t} u/\partial x^{i_x} \partial y^{i_y} \partial t^{i_t}, i_x, i_y, i_t \geq 0. \)

**Definition 2.3.** We say that the equation (4) is strictly self-adjoint if the adjoint equation (8) becomes equivalent to the equation (4) after the substitution \( \nu = u \):

\[ \frac{\delta L}{\delta u} = \lambda \Delta(x, y, t, u, u_x, \ldots, u_{yyyy}), \]

with \( \lambda \) a generic coefficient.

**Definition 2.4.** We say that the equation (4) is quasi self-adjoint if the adjoint equation (8) becomes equivalent to the equation (4) after the substitution \( \nu = \phi(u) \), where \( \phi(u) \neq 0 \).

**Definition 2.5.** We say that the equation (4) is nonlinearly self-adjoint if the adjoint equation (8) becomes equivalent to the equation (4) after the substitution \( \nu = \phi(x, y, t, u) \), where \( \phi(x, y, t, u) \neq 0 \).

**Remark 2.6.** The concept of the nonlinear self-adjointness can be further extended by considering differential substitutions of the form

\[ \nu = \phi(x, y, t, u, u_{(1)}, \ldots, u_{(r)}), \]

where \( u_{(r)} \) are the derivatives of \( u \) of order \( r \).

**Remark 2.7.** From the above definitions it is obvious that if an equation is strictly or quasi self-adjoint then it is also and nonlinearly self-adjoint.
Theorem 2.1 (Explicit formula for conserved vectors). Let equation (4) be nonlinearly self-adjoint and a its Lie point symmetry. A conserved vector can be constructed by the following formula:

\[ C^i = \xi^i \mathcal{L} + \sum_{i_x + i_y + i_t = 0}^{\infty} D_{x_{i_x}} D_{y_{i_y}} D_{t_{i_t}} (W) \delta^* \mathcal{L}, \quad i = x, y, t, \quad i_i \geq 0 \quad (9) \]

where \( W = \eta - \xi^1 u_x - \xi^2 u_y - \xi^3 u_t \), \( \delta^* / \delta^* u \) is the “weighted” Euler-Lagrange operator

\[
\frac{\delta^* \mathcal{L}}{\delta^* u_{i_x, i_y, i_t}} = \frac{\partial}{\partial u} + \sum_{s=j_x+j_y=1}^{\infty} (-1)^s \left( \frac{s}{i_x+i_y+i_t} \right) D_{x_{j_x}} D_{y_{j_y}} D_{t_{j_t}} \frac{\partial}{\partial u(i_x+j_x)(i_y+j_y)(i_t+j_t)}
\]

with \( \binom{N}{i_x, i_y, \ldots, i_t} = \frac{N!}{i_x!i_y!\cdots i_t!} \), \( N = i_1 + i_2 + \cdots + i_r \), the multinomial and \( \mathcal{L} \) the formal Lagrangian after substituting with \( \psi = \phi(x, y, t, u) \).

Proof. For a proof see [19]. □

Remark 2.8. From a conserved vector \((C^x, C^y, C^t)\) the conservation law has the form

\[ D_x(C^x) + D_y(C^y) + D_t(C^t) = 0 \]

satisfied on the solutions of equation (4).

Definition 2.9. A conserved vector is called trivial if,

- its divergence is identically zero and,
- the components of the vector vanish on the solutions of equation (4).

Remark 2.10. Only the nontrivial conserved vectors will be considered.

3. The self-adjointness classification

As it can be seen by the definitions of self-adjointness the first step is to obtain the adjoint equation. For equation (4) the adjoint equation is

\[
2(h(u) - g')u_y v_y - g(u)v_{yy} - r(u)v_{xx} + (1 - 2r')u_x v_x - v \left( f' + (g'' - h')u_y \right) u_y + 2(g' - h(u))u_{yy} + r'' u_x^2 + (2r' - 1)u_{xx} + \nu v_{yyyy} + 2v_{xyy} + v_{xxy} - \nu_t = 0
\]

where \( \nu = \nu(x, y, t) \) is the new dependent variable.

Theorem 3.1. Equation (4) has no strictly self-adjoint subcase.
Proof. By making the substitution \( v = u \) in the adjoint equation (10) and then substituting \( u \) from equation (11), we get

\[
(h(u) - 2g') u_y^2 - 2g(u)u_{yy} + 2u_{yyyy} + \frac{1}{2} (1 - 4r') u_x^2 - 2r(u)u_{xx}
- u \left( f' + \left( g'' - h' \right) \right) u_y^2 + 2 \left( g' - h(u) \right) u_{yy} + r''u_x^2 + \left( -1 + 2r' \right) u_{xx}
+ 4u_{xxyy} + 2u_{xxxy} - f(u) = 0.
\]

It is obvious that there is no choice of the functions \( f, h, g, r \) satisfying the above condition. In other words there is no choice of the functions \( f, h, g, r \) that will make equation (11) strictly self-adjoint.

Theorem 3.2. Equation (11) is quasi self-adjoint when \( f = c_1, r = u/2 + c_2, h = g' \).

Proof. After making the substitution \( v = \phi(u) \) in the adjoint equation (10) and then substituting \( u \) from equation (11), we get

\[
((h(u) - 2g') \phi' + \phi(u) (h' - g'') - g(u) \phi'' + 6 \phi'''u_y) + 2 \phi''''u_{xx}) u_y^2
+ \phi(u) \left( 1 - 2r' \right) - 2r(u) \phi' + 2 \phi''u_{yy} \right) u_{xx} + 4 \phi'u_{xxyy} + 2 \phi''''u_y^2 u_x^2
+ \left( \frac{1}{2} \left( (1 - 4r') \phi' - 2 \left( \phi(u) r'' + r(u) \phi'' \right) \right) + 2 \phi'''(u_{yy} + 3u_{xx}) \right) u_x^2
+ 3 \phi''u_{yy}^2 + 2 \phi'u_{yyyy} + \phi''''u_x^4 + 4 \phi''u_{xy}^2 + 3 \phi''u_{xx}^2 + 2 \phi'u_{xxxx}
- \left( \phi(u) f' + f(u) \phi' \right) + \phi''''u_y^4 + 2 \phi(u) (h(u) - g') - 2g(u) \phi') u_{yy}
+ 4 \phi''''(u_{yyyy} + u_{xxyy}) u_{yy} + 4 \phi''''(u_{yyyy} + u_{xxyy} + \phi''''u_{xxxy} + \phi''''u_{xxxx}) u_x = 0.
\]

This condition must vanish for every solution \( u \) of equation (11). Hence we arrive at the system:

\[
\phi(u)f' + f(u)\phi' = 0,
\phi(u) \left( 1 - 2r' \right) - 2r(u) \phi' = 0,
\frac{\partial}{\partial x} (u\phi(u) = \phi(u) g' + g(u) \phi'),
\phi' = 0, \phi'' = 0, \phi''' = 0, \phi'''' = 0,
(1 - 4r') \phi' - 2 \left( \phi(u) r'' + r(u) \phi'' \right) = 0,
\phi(u) h(u) - 2g' \phi' + \phi(u) (h' - g'') - g(u) \phi'' = 0.
\]

Solving the above system we get that \( \phi(u) = c \neq 0 \) and as stated, \( f = c_1, r = u/2 + c_2, h = g' \).

Theorem 3.3. Equation (11) is nonlinear self-adjoint if and only if:

1. \( r = u/2 + \alpha, g = \beta u + \gamma, h = \beta, f = \delta u^2 + \epsilon u + \zeta, \)
2. \( r = u/2 + \alpha, h = g', f = \beta u^2 + \gamma u + \delta + c \int g(u) du, \beta,c,g' \neq 0, \)
3. \( r = u/2 + \alpha, h = g', f = \beta u + \gamma + c \int g(u) du, \quad c,g' \neq 0, \)
Looking at the above system it is obvious that

\( u = \frac{g'}{2} + \alpha, h = g', f = \beta u^2 + \gamma u + \delta, \beta, g'' \neq 0, \)

4. \( r = u/2 + \alpha, h = g', f = \beta u + \gamma, g'' \neq 0, \)

Proof. After making the substitution \( v = \phi(x,y,t,u) \) in the adjoint equation \((10)\) and then substituting \( u_t \) from equation \((11)\), we get

\[
\begin{align*}
- \phi(x,y,t,u)f' - f(u)\phi_u &+ 2u_{yyyy}\phi_u + 4u_{xxyy}\phi_u + 2u_{xxxx}\phi_u + 3u_{yy}\phi_u + 4u_{xyy}\phi_u + 3u_{xxy}\phi_u + u_{xxx}\phi_u + 2u_{uuu} - \phi_t + 4u_{yyy}\phi_y + 4u_{xy}\phi_y + 4u_{yy}\phi_{uu} + 4u_{yyyy}\phi_{uu} - g(u)\phi_{yy} + \phi_{yyyy} + 4u_{xxyy}\phi_{xy} + 4u_{xxxy}\phi_{xx} + 4u_{xxyy}\phi_{uu} - r(u)\phi_{xx} \\
+ u_{xx}(\phi(x,y,t,u)(1 - 2r') - 2r(u)\phi_u + 2\phi_{yy} + 6\phi_{xx}u) + 2\phi_{xxxx} + \phi_{xxxxx} + u_{yy} &+ u_{uu} - r(u)\phi_u + 4u_{yyyy}\phi_{uu} + 4u_{xxyy}\phi_{xy} + 2(\phi(x,y,t,u)(h(u) - g') - g(u)\phi_{uu} + 6u_{yy}\phi_{uu} + 2u_{xx}\phi_{uuu} + 2u_{xx}^2\phi_{uuu} + 6u_{yyyy}\phi_{uu} + 4u_{xxyu}\phi_{uu} + 2\phi_{xxxx}u + 6\phi_{xxxx}\phi_{uu}) \\
&+ u_{xx}(4u_{yyyy}\phi_{uu} + 4u_{xxyy}\phi_{uu} + 2(h(u) - g')\phi_y - 2g(u)\phi_{yy} + 12u_{yy}\phi_{yy}u + 4u_{xx}\phi_{yyyy} + 4u_{xxyy}\phi_{xy} + 8u_{xy}\phi_{xx} + 8u_{xxy}\phi_{uu} + 8u_{xyy}\phi_{uuu} + 8u_{xyy}\phi_{uuu} + 4(\phi_{yyyy} + \phi_{xyy})) \\
&+ u_{xx}(4u_{xyy}\phi_{ uu} + 4u_{xxxx}\phi_{ uu} + 8u_{xyy}\phi_{ uu} + (1 - 2r')\phi_x - 2r(u)\phi_x + 12u_{xxxx}\phi_{ uu} + 4(\phi_{yyyy} + \phi_{xxxx})) = 0.
\end{align*}
\]

This condition must vanish for every solution \( u \) of equation \((11)\). Hence we attain the system:

\[
\begin{align*}
(1 - 2r')\phi_x - 2r(u)\phi_xu + 4(\phi_{xyy} + \phi_{xxx}) & = 0, \\
2(h(u) - g')\phi_y - 2g(u)\phi_{yy} + 4\phi_{yyyy} + 4\phi_{xyy} & = 0, \\
(h(u) - g')\phi(x,y,t,u) + 3\phi_{yy} + \phi_{xxu} - g(u)\phi_u & = 0, \\
(1 - 2r')\phi(x,y,t,u) - 2r(u)\phi_u + 2\phi_{yy} + 6\phi_{xxu} & = 0, \\
\phi_{yy} & = 0, \phi_{uu} = 0, \phi_{xxuu} = 0, \phi_{yyyy} = 0, \phi_{uuu} = 0, \\
\phi_u & = 0, \phi_{xx} = 0, \phi_{yyyy} = 0, \phi_{uu} = 0, \phi_{xxxx} = 0, \\
(1 - 4r')\phi_x - 2r(u)\phi_{uu} - 2r'\phi(x,y,t,u) + 4\phi_{yyyy} + 12\phi_{xxu} & = 0, \\
f'\phi + f(u)\phi_u + \phi_t + g(u)\phi_{yy} - \phi_{yyyy} + r(u)\phi_{xx} - 2\phi_{xxxy} - \phi_{xxxx} & = 0, \\
(h' - g'')\phi(x,y,t,u) + h(u)\phi_u - 2g'\phi_u - g(u)\phi_{uu} + 6\phi_{yyyy} + 2\phi_{xxuu} & = 0.
\end{align*}
\]

Looking at the above system it is obvious that

\[
\phi = \phi(x,y,t), \quad r(u) = u/2 + c, \quad h = g'.
\]
By substituting (11) in the system, we arrive to the following equation:

$$\phi' + \phi,_{t} + g(u)\phi,_{yy} - \phi,_{yyyy} + \frac{1}{2} (u + 2c) \phi,_{xx} - 2\phi,_{xxyy} - \phi,_{xxxx} = 0. \quad (12)$$

By differentiating equation (12) two times with respect to $u$ we have in addition

$$\phi'' + g'' \phi,_{yy} = 0,$$

$$\phi'' + g' \phi,_{yy} + \frac{\phi,_{xx}}{2} = 0.$$

By using these two equations in combination with equation (12) we arrive at the five cases stated.

Therefore having obtained the specific self-adjoint subcases of equation (4) it remains to know their symmetries in order to construct the corresponding conserved vectors. In the next Section the complete group classification for equation (4) is given.

4. The Group Classification

From the invariant surface condition (11) for equation (4) the system of the determining equations is obtained, see Appendix. By solving the subsystem not containing any of the functions $f, g, h, r$ we obtain:

$$\xi^1(x, y, t, u) = F_{11}(t) - y F_{13}(t) + \frac{x}{4} F'_1(t),$$

$$\xi^2(x, y, t, u) = F_{12}(t) + x F_{13}(t) + \frac{y}{4} F'_1(t),$$

$$\xi^3(x, y, t, u) = F_{1}(t),$$

$$\eta(x, y, t, u) = F_{15}(y, t) - \frac{u}{2} F'_1(t) - \frac{1}{8} x (8F'_{11}(t) - 8y F'_{13}(t) + x F''_{1}(t)), $$

and the remaining determining equations are:

$$r' F''_{1} = 0,$$

$$h' F''_{1} = 0,$$

$$g' F''_{1} = 0,$$

$$r' F'_{13} = 0,$$

$$h' F'_{13} = 0,$$

$$g' F'_{13} = 0,$$

$$r' F'_{11} = 0,$$

$$h' F'_{11} = 0,$$

$$g' F'_{11} = 0.$$
\[ f'F_1'' - F_1''' = 0, \]
\[ f'F_1'' - F_1''' = 0, \]
\[ f'F_1' - F_1'' = 0, \]
\[ (1 + 2h(u))F_1' = 0, \]
\[ (2h(u) - 1)F_1 = 0, \]
\[ h'(2F_1 - uF_1') = 0, \]
\[ (g(u) - r(u))F_1 = 0, \]
\[ 4F_1' + yF_1'' + 8h(u)F_1y = 0, \]
\[ 8F_1r' + 4(r(u) - ur')F_1' = 0, \]
\[ 8F_1g' + 4(g(u) - u')F_1' = 0, \]
\[ 4F_1f' + 6f(u)F_1' + 2uF_1'' - r(u)F_1'' - 4F_1t \]
\[ + 4g(u)F_1y - 4F_1yyy = 0. \]

The resulted group classification is summarized in Table 1. The first row gives the symmetries that occur for every possible choice of the functions \( f, g, h \) and \( r \). In each subsequent row a special case appears along with the additional symmetries it has.

5. Conservation Laws

By consulting table 1 one can observe that the only point symmetries admitted by the self-adjoint classification given in Section 3 are \( X_1 = \partial_t, X_2 = \partial_x, X_3 = \partial_y \). The only exception is the nonlinear self-adjoint case, \( u_t = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \left( \frac{u}{2} + \alpha \right) (u_{xx} + u_{yy}) - u_{xxxx} - u_{yyyy} - 2u_{xyyy} + \delta u^2 + \epsilon u + \zeta \) which also admits the symmetry \( X_4 = x\partial_y - y\partial_x \). The nontrivial conserved vectors for each case follow.

**Case 1 (Quasi self-adjoint).** According to Theorem 3.2 the quasi self-adjoint case is

\[ u_t = \alpha + g' u_x^2 + g(u)u_{yy} - u_{yyyy} + \frac{u_x^2}{2} + \left( \beta + \frac{1}{2} u \right) u_{xx} - 2u_{xyyy} - u_{xxxx}, \]

with formal Lagrangian \( L = c\Delta(x, y, t, u, u_x, \ldots, u_{yyyy}) \). Using the formula 4, for each one of the above mentioned three point symmetries, three trivial conserved vectors are obtained.

**Case 2 (Nonlinear self-adjoint).** Following Theorem 3.3 to each one of the five possible nonlinear self-adjoint cases formula 4 is applied using each of its admitted point symmetries. The conserved vector is given in the form \( (C^1, C^2, C^3) \) yielding the conservation law \( D_xC^1 + D_yC^2 + D_tC^3 = 0 \) for every solution of the case of equation 4 studied.

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| $f$  | $g$  | $h$  | $r$  | Symmetries                                                                 |
|-------|------|------|------|---------------------------------------------------------------------------|
| $\forall$ | $\forall$ | $\forall$ | $\forall$ | $\mathcal{X}_1 = \partial_t, \mathcal{X}_2 = \partial_x, \mathcal{X}_3 = \partial_y$ |
| $\forall$ | $r$  | $\frac{1}{2}$ | $\forall$ | $\mathcal{X}_4 = x \partial_y - y \partial_x$ |
| $\alpha u + \beta, \alpha \neq 0$ | $r$  | $\frac{1}{2}$ | $c$ | $\mathcal{X}_4 = e^{\alpha t} \partial_u, \mathcal{X}_5 = x \partial_y - y \partial_x, \mathcal{X}_6 = \frac{e^{\alpha t}}{\alpha} \partial_y - e^{\alpha t} y \partial_u, \mathcal{X}_7 = \frac{e^{\alpha t}}{\alpha} \partial_x - e^{\alpha t} x \partial_u$ |
| $\beta$ | $r$  | $\frac{1}{2}$ | $c \neq 0$ | $\mathcal{X}_4 = \partial_u, \mathcal{X}_5 = t \partial_x - x \partial_u, \mathcal{X}_6 = t \partial_y - y \partial_u, \mathcal{X}_7 = x \partial_y - y \partial_x$ |
| $\gamma \alpha$ | $r$  | $\frac{1}{2}$ | $\alpha u + \beta, \alpha \neq 0$ | $\mathcal{X}_4 = x \partial_y - y \partial_x, \mathcal{X}_5 = 4 \alpha t \partial_h + \alpha x \partial_u + \alpha y \partial_y - 2(\alpha u + \beta) \partial_u$ |
| $\zeta (\alpha u + \beta)^4, \alpha \neq 0$ | $\gamma (\alpha u + \beta)^4$ | $\frac{1}{4}$ | $\delta (\alpha u + \beta), \delta \neq \gamma$ | $\mathcal{X}_4 = 4 \alpha t \partial_h + \alpha x \partial_u + \alpha y \partial_y - 2(\alpha u + \beta) \partial_u$ |
| $\delta$ | $\alpha$ | $\gamma \neq 0$ | $\beta \neq \alpha$ | $\mathcal{X}_4 = \partial_u, \mathcal{X}_5 = t \partial_x - x \partial_u, \mathcal{X}_6 = 2 t \partial_y - \frac{u}{\alpha} \partial_u$ |
| $\gamma$ | $\alpha$ | $0$ | $\beta \neq \alpha$ | $\mathcal{X}_5 = \mathcal{F}(y,t) \partial_h, \mathcal{F}_t - \alpha \mathcal{F}_{yy} + \mathcal{F}_{yyyy} = 0$ |
| $\zeta (\alpha u + \beta)^3 \gamma (\alpha u + \beta)^4 \epsilon \neq \frac{1}{2}$ | $\delta (\alpha u + \beta)^4$ | $\frac{1}{2}$ | | $\mathcal{X}_4 = 4 \alpha t \partial_h + \alpha x \partial_u + \alpha y \partial_y - 2(\alpha u + \beta) \partial_u$ |
| $\delta u + \epsilon, \delta \neq 0$ | $\alpha^4$ | $\gamma \neq 0, \frac{1}{2}$ | $\beta^4$ | $\mathcal{X}_4 = e^{\beta} \partial_u, \mathcal{X}_5 = \frac{e^{\beta}}{\delta} \partial_x - e^{\beta} x \partial_u, \mathcal{X}_6 = \frac{2 e^{\beta}}{\beta} \partial_y - \frac{e^{\beta}}{\delta} \partial_u$ |
| $\gamma u + \delta, \gamma \neq 0$ | $\alpha^4$ | $0$ | $\beta^4$ | $\mathcal{X}_5 = \mathcal{F}(y,t) \partial_h, \mathcal{F}_t - \alpha \mathcal{F}_{yy} - \gamma \mathcal{F} + \mathcal{F}_{yyyy} = 0$ |
| $\delta$ | $\alpha^4$ | $\gamma \neq 0, \frac{1}{2}$ | $\beta^4$ | $\mathcal{X}_4 = t \partial_x - x \partial_u, \mathcal{X}_5 = 2 t \partial_y - \frac{u}{\alpha} \partial_u$ |
| $\gamma$ | $\alpha^4$ | $0$ | $\beta^4$ | $\mathcal{X}_4 = t \partial_x - x \partial_u, \mathcal{X}_5 = \mathcal{F}(y,t) \partial_h, \mathcal{F}_t - \alpha \mathcal{F}_{yy} + \mathcal{F}_{yyyy} = 0$ |
| $\beta u + \gamma, \beta \neq 0$ | $\alpha \neq 0, \frac{1}{2}$ | $0$ | $\alpha \neq 0, \frac{1}{2}$ | $\mathcal{X}_4 = e^{\alpha \gamma} \partial_u, \mathcal{X}_5 = \frac{e^{\alpha \gamma}}{\gamma} \partial_x - e^{\alpha \gamma} x \partial_u, \mathcal{X}_6 = \frac{2 e^{\alpha \gamma}}{\beta} \partial_y - \frac{e^{\alpha \gamma}}{\gamma} \partial_u$ |
| $\alpha u + \beta, \alpha \neq 0$ | $0$ | $0$ | $0$ | $\mathcal{X}_4 = \frac{e^{\alpha \beta}}{\alpha \beta} \partial_x - e^{\alpha \beta} x \partial_u, \mathcal{X}_5 = \mathcal{F}(y,t) \partial_h, \mathcal{F}_t - \alpha \mathcal{F} + \mathcal{F}_{yyyy} = 0$ |
| $\beta$ | $0$ | $\alpha \neq 0, \frac{1}{2}$ | $0$ | $\mathcal{X}_4 = \partial_u, \mathcal{X}_5 = t \partial_x - x \partial_u, \mathcal{X}_6 = 2 t \partial_y - \frac{u}{\alpha} \partial_u, \mathcal{X}_7 = 4 t \partial_h + x \partial_x + y \partial_y - 2(4 u - 3 \beta t) \partial_u$ |
| $\alpha$ | $0$ | $0$ | $0$ | $\mathcal{X}_4 = t \partial_x - x \partial_u, \mathcal{X}_5 = 4 t \partial_h + x \partial_x + y \partial_y - 2(4 u - 3 \beta t) \partial_u, \mathcal{X}_6 = \mathcal{F}(y,t) \partial_h, \mathcal{F}_t - \alpha \mathcal{F} + \mathcal{F}_{yyyy} = 0$ |

Table 1: Complete group classification for equation (1)
Subcase 2.1 ($r = u/2 + \alpha, g = \beta u + \gamma, h = \beta, f = \delta u^2 + \epsilon u + \zeta$). For this case equation (4) takes the form:

\[ u_t = \zeta + \epsilon u + \delta u^2 + \beta u_y^2 + (\gamma + \beta u) u_{yy} - u_{yyyy} + \frac{u_x^2}{2} + \left(\alpha + \frac{u}{2}\right) u_{xx} - 2u_{xxyy} - u_{xxxx}. \tag{13} \]

The new dependent variable \( v = F_1(x, y, t) \), where \( F_1(x, y, t) \) is any solution of the system

\[ 2\delta F_1 + \beta F_{1yy} + \frac{F_{1xx}}{2} = 0, \]

\[ \epsilon F_1 + \gamma F_{1yy} - F_{1yyyy} + \alpha F_{1xx} - 2F_{1xxyy} - F_{1xxxx} = 0. \]

\( \partial_t \):

\[ C^1 = \frac{1}{12} (4(1 - 6\beta)F_{1yty}u_x + 3u_x^2F_{1yt} - 8F_{1yt}u_{xy} + 8u_yF_{1xy} - 6F_{1t} ((2\alpha + 8\delta + u)u_x - 2(u_{xy} + u_{xxx}) - 4F_{1xt} (u_{yy} + 3u_{xx})) + u ((\alpha + 4\delta)F_{1xt} - F_{1yxt}) + 2\beta u F_{1yyt}, \]

\[ C^2 = \frac{1}{6} (3\beta u^2 F_{1yt} + F_{1t} (-2(3\gamma + 4\delta + 3\beta u)u_y + 6(u_{yyy} + u_{xx})) - 2 ((-3 + 2\beta)u_y F_{1yty} + 2F_{1xt}u_{xy} - 2u_x F_{1yxt} + F_{1yt} (3u_{yy} + u_{xx}))) + u ((\gamma + 4\delta)F_{1yt} + (-1 + 2\beta)F_{1yyy}), \]

\[ C^3 = -\zeta F_1(x, y, t) + u F_{1t}. \]

This conserved vector will be nontrivial if and only if \( F_{1t} \neq 0 \).

\( \partial_x \):

\[ C^1 = \frac{1}{6} (3u_x (2(1 - 2\beta)F_{1yxy} + (2\alpha + 8\delta + u)F_{1x}) - 2F_{1y}u_{xx} + 2 ((6\beta - 1)F_{1yxy} + 3u_x u_{xx} - 2u_{xy} F_{1xy}) - F_{1y} u_{yy}, \]

\[ - F_1(x, y, t) \left( \zeta - u_t + \beta u_y^2 + \gamma u_{yy} + u (\epsilon + \delta u + \beta u_{yy}) - 4\delta u_{xx} \right) \]

\[ C^2 = \frac{1}{3} \left( (-3 + 2\beta)F_{1yy} + F_1 (x, y, t) (3\beta u_x + (3\gamma + 4\delta + 3\beta u)u_{xy}) - 2F_{1xy}u_{xx} + F_{1x} (3u_{yy} + 5u_{xy}) + F_{1y} u_{xx} + (1 - 2\beta)F_{1yy} u_x + F_{1y} (u_{xy} + (\gamma + 4\delta + \beta u) u_x), \right. \]

\[ C^3 = -F_1(x, y, t) u_x. \]

This conserved vector will be nontrivial if and only if \( F_{1x} \neq 0 \).
\[ \partial_y: \]

\[ C^1 = \frac{1}{6} \left( 2 \left( -2u_{yy}F_{1yy} + (1 + 6\beta)F_{1yy}u_{xx} + F_{1x} (u_{xyy} + 3u_{xxx}) \right) 
- 3(2\alpha + 8\delta + u)u_xF_{1y} - 2F_{1y} (3u_{yy} + u_{xxy}) \right) 
- (1 - 2\beta)F_{1xy}u_x 
- F_1(x, y, t) \left( \zeta - u_x + \beta y^2 + \gamma u_{yy} + u (\epsilon + \delta u + \beta u_y) - 4\delta u_{xx} \right), \]

\[ C^2 = \frac{1}{3} \left( (2\beta - 3)F_{1yy}u_{xy} + (3\gamma + 4\delta + 3\beta u)u_{xy}F_1(x, y, t) - 2F_{1xy}u_{xx} 
+ 5u_{xyy}F_{1x} + u_{xx}F_{1y} \right) 
+ (1 - 2\beta)F_{1yy}u_x 
+ \beta u_x F_1(x, y, t) + u_{yy}F_{1x} + F_{1y} \left( u_{xyy} - (\gamma + 4\delta + \beta u)u_x \right), \]

\[ C^3 = -F_1(x, y, t)u_x. \]

This conserved vector will be nontrivial if and only if \( F_{1y} \neq 0. \)

\[ y\partial_x - x\partial_y: \]

If, in addition, \( \beta = 1/2 \) and \( \gamma = \alpha \) then equation (13) admits also the symmetry \( x\partial_y - y\partial_x. \) Employing it we have:

\[ C^1 = \frac{1}{6} \left( 2u_{yy}F_{1x} + 3uyu_xF_{1y} + u_y \left( 4F_{1yy} - 3x(2\alpha + 8\delta + u)F_{1x} \right) 
+ 4xF_{1yy}u_{xx} + 8F_{1xy}u_{xy} - 4uxyF_{1xy} - 4y_{xy}F_{1xy} 
- 2yF_{1x}u_{xyy} - 4yF_{1yy}u_{xx} + 2F_{1y} (2u_{yy} + 2uxy - 2ux - yu_{xyy}) 
+ 3F_1(x, y, t) \left( u_y^2 + xuy + u (uy + yu_{yy} + xu_{xy}) \right) \right) 
+ y\alpha u_xF_{1x} 
+ 4\delta u_xF_{1x} + xF_{1x}u_{xyy} - yF_{1y}u_{xx} + F_{1y} (yu_{yy} + xu_{xx}) 
+ F_1(x, y, t) \left( u\gamma + y\delta u^2 - yu - (\alpha + 4\delta)u_y + y\alpha u_{yy} + x\alpha u_y 
+ 4\delta u_x + cyu - xu_{yy} + 4\delta u_{xx} \right), \]

\[ C^2 = \frac{1}{6} \left( -4u_{yy}F_{1yy} + 3yuF_{1y}u_x + 4F_{1yy}u_x + 4uyF_{1x} - 8F_{1y}u_{xy} 
+ 4F_{1yy}u_{xy} - u_y \left( 3x(2\alpha + 8\delta + u)F_{1y} + 4F_{1xy} \right) - 4uxyF_{1xy} 
+ 4xF_{1x}u_{xyy} - 4F_{1xy}u_{xx} + 4yF_{1xy}u_{xx} + 2F_{1y}u_{xyy} - 10yF_{1x}u_{xyy} 
- 2yF_{1y}u_{xxx} - F_1(x, y, t) \left( 8\delta u - 8xu_{yy} + 3yu_{xx} + 3xu^2 
+ 8y\delta u_x + 3u (u_x + yu_{xy} + xu_{xx}) \right) \right) 
+ y\alpha u_xF_{1y}u_x 
+ 4y\delta u_F_{1y}u_x - yu_{yy}F_{1x} - yF_{1y}u_{xyy} - xF_{1x}u_{xx} 
- F_1(x, y, t) \left( x\zeta + x\delta u^2 - xu + \alpha u_x + y\alpha u_{xy} - u_{xyy} + x\alpha u_x 
+ cxu - xu_{xyy} \right), \]

\[ C^3 = F_0(x, y, t) (yu_x - xu_y). \]

This conserved vector will be nontrivial if and only if \( F_1 \neq F(x^2 + y^2, t). \)

\( \bullet \) Subcase 2.2 \((r = u/2 + \delta, h = g', f = \frac{3\alpha^2}{2} + \beta u + \gamma + c fg(u) du, \alpha, c, g' \neq 0). \)
For this case equation (4) takes the form:

\[ u_t = \frac{\alpha u^2}{2} + \beta u + \gamma + c \int g(u) \, du + g(u) u_y^2 + g(u) u_{yyy} + \frac{u_x^2}{2} + \left( \frac{u}{2} + \delta \right) u_{xx} - 2u_{xxyy} - u_{xxxx}. \]

The new dependent variable is

\[ u = e^{t(4\alpha^2 - \beta + e^2 + 2\alpha(\delta + 2c))} \left( \cos(\sqrt{2\alpha}x) \left( c_1 \cos(\sqrt{c}y) + c_3 \sin(\sqrt{c}y) \right) + \sin(\sqrt{2\alpha}x) \left( c_2 \cos(\sqrt{c}y) + c_4 \sin(\sqrt{c}y) \right) \right). \]

\[ \delta_t: \]

\[ C_1 = \frac{1}{4} e^{Dt} D \left( A\sqrt{2\alpha} (8\alpha + 4\delta + 8c + u) \, u - 2B (4\alpha + 2\delta + 4c + u) \, u_x - 4A\sqrt{2\alpha} Au_{xx} + 4Bu_{xxx} \right), \]
\[ C_2 = e^{Dt} D \left( E\sqrt{c} \int g(u) \, du + Eu^{3/2} \, u - (\mathcal{A}c + \mathcal{A}g(u)) \, u_y - E\sqrt{c}u_{yy} + Bu_{yyy} - 2E\sqrt{c}u_{xx} + 2Bu_{xyy} \right), \]
\[ C_3 = -e^{Dt} B(\gamma - Du). \]

This conserved vector will be nontrivial if and only if \( D \neq 0 \).

\[ \delta_x: \]

\[ C_1 = \frac{1}{2} e^{Dt} \left( 4B\alpha u_{xx} + 2\sqrt{2\alpha} Au_{xxx} - 2B\gamma - 4B\alpha (2\alpha + \delta + 2c) \, u - B\alpha u^2 - \sqrt{2\alpha}A (4\alpha + 2\delta + 4c + u) \, u_x \right), \]
\[ C_2 = \sqrt{2\alpha} e^{Dt} \left( E\sqrt{c} \int g(u) \, du + Eu^{3/2} \, u - \mathcal{A}c + \mathcal{A}g(u) \, u_y - E\sqrt{c}u_{yy} + Au_{yyy} - 2E\sqrt{c}u_{xx} + 2Au_{xyy} \right), \]
\[ C_3 = \sqrt{2\alpha} e^{Dt} u. \]

\[ \delta_y: \]

\[ C_1 = \frac{1}{4} e^{Dt} \left( 4\mathcal{C} u_{xxx} + \sqrt{2\mathcal{E}} \sqrt{\mathcal{C}} (u (8\alpha + 4\delta + 8c + u) - 4u_{xx}) - 2\mathcal{C} (4\alpha + 2\delta + 4c + u) \, u_x \right), \]
\[ C_2 = -e^{Dt} \left( B\gamma + Bc \int g(u) \, du + Bc^2 u + C\sqrt{c} (c + g(u)) \, u_y - \sqrt{c} \left( B\sqrt{c} \left( u_{yy} + 2u_{xx} \right) + C \left( u_{yyy} + 2u_{xyy} \right) \right) \right), \]
\[ C_3 = \mathcal{C} \sqrt{c} e^{Dt} u. \]
In the above formulae

\[ A = \sin(\sqrt{2\alpha}x) \left( c_1 \cos(\sqrt{\gamma}y) + c_3 \sin(\sqrt{\gamma}y) \right) \]
\[ - \cos(\sqrt{2\alpha}x) \left( c_2 \cos(\sqrt{\gamma}y) + c_4 \sin(\sqrt{\gamma}y) \right), \]
\[ B = \cos(\sqrt{2\alpha}x) \left( c_1 \cos(\sqrt{\gamma}y) + c_3 \sin(\sqrt{\gamma}y) \right) \]
\[ + \sin(\sqrt{2\alpha}x) \left( c_2 \cos(\sqrt{\gamma}y) + c_4 \sin(\sqrt{\gamma}y) \right), \]
\[ C = \cos(\sqrt{2\alpha}x) \left( c_3 \cos(\sqrt{\gamma}y) - c_1 \sin(\sqrt{\gamma}y) \right) \]
\[ + \sin(\sqrt{2\alpha}x) \left( c_2 \cos(\sqrt{\gamma}y) - c_4 \sin(\sqrt{\gamma}y) \right), \]
\[ D = 2\alpha(2\alpha + \delta) + 4\alpha + c^2 - \beta \]
\[ \text{and} \]
\[ E = \sin(\sqrt{2\alpha}x) \left( c_1 \sin(\sqrt{\gamma}y) - c_3 \cos(\sqrt{\gamma}y) \right) \]
\[ + \cos(\sqrt{2\alpha}x) \left( c_4 \cos(\sqrt{\gamma}y) - c_2 \sin(\sqrt{\gamma}y) \right). \]

**Subcase 2.3** \((r = u/2 + \gamma, h = g', f = \alpha u + \beta + c \int g(u) \, du, \, c, g'' \neq 0)\). For this case equation (4) assumes the form:

\[ u_t = \beta u + \gamma + c \int g(u) \, du + g'u_y + g(u)u_{yy} - u_{yyy} + \frac{u_x^2 + 2\alpha + u}{2} u_xx - 2u_{xxx} \]

The new dependent variable \( v \) is,

\[ v = e^{(c^2-\alpha)t} \left( (c_1 + c_2x) \cos(\sqrt{\gamma}y) + (c_3 + c_4x) \sin(\sqrt{\gamma}y) \right). \]

\( \partial_t: \)

\[ C^1 = \frac{1}{4} e^{Dt} D (Bu(4\gamma + 8c + u) - 2A(2\gamma + 4c + u)u_x - 4Bu_{xx} + 4Au_{xxx}), \]
\[ C^2 = e^{Dt} \left( C \sqrt{c} \int g(u) \, du + C^{3/2} u - Acu_y - Ag(u)u_y - C \sqrt{c}u_{yy} + Au_{yy} \right) \]
\[ - 2G \sqrt{c}u_{xx} + 2Au_{xy}) , \]
\[ C^3 = -e^{Dt} A(\beta - Du). \]

This conserved vector will be nontrivial if and only if \( D \neq 0 \).

\( \partial_x: \)

\[ C^1 = \frac{1}{2} e^{Dt} B (u(2Dx - u) + 2(uxu_t - \beta x - \gamma u_x + 2u_{xy} + u_{xx})), \]
\[ C^2 = e^{Dt} \left( E \sqrt{c} \left( \int g(u) \, du + cu \right) - B(c + g(u))u_y - E \sqrt{c}u_{yy} + Bu_{yy} \right), \]
\[ C^3 = -e^{Dt} B u_x. \]
This conserved vector will be nontrivial if and only if \( c_2^2 + c_4^2 \neq 0 \).

\[ \partial_y: \]

\[
C^1 = \frac{1}{4} e^{Dt} \cosh \left( 2\beta x (2c_1 + c_2) \sin(\sqrt{c_2}) + \sqrt{c_4} u^2 - 4\sqrt{c_2} (2u_{yy} + u_{xx}) \right) \\
- 2\beta x (2c_3 + c_4 x) \cos(\sqrt{c_2}) + u (4\sqrt{c_4} - 2C u_x) \\
+ 4\sqrt{c_2} (u_{xxx} - \gamma u_x + 2u_{xyy})),
\]

\[
C^2 = \sqrt{c_4} e^{Dt} \left( A\sqrt{c_4 u_{yy}} + C u_{yxyy} - A\sqrt{c_4} \left( \int g(u) du + cu \right) \right) \\
- C \left( c + g(u) \right) u_y),
\]

\[
C^3 = C\sqrt{c_4} e^{Dt}.
\]

For the above conserved vectors we have used the following notation:

\[
A = (c_1 + c_2) \cos(\sqrt{c_2}) + (c_3 + c_4) \sin(\sqrt{c_2}),
\]

\[
B = c_2 \cos(\sqrt{c_2}) + c_4 \sin(\sqrt{c_2}),
\]

\[
C = (c_3 + c_4) \cos(\sqrt{c_2}) - (c_1 + c_2) \sin(\sqrt{c_2}),
\]

\[
D = c_2 - \alpha
\]

and

\[
E = c_4 \cos(\sqrt{c_2}) - c_2 \sin(\sqrt{c_2}).
\]

• **Subcase 2.4** (*r = u/2 + \( \alpha \), \( h = g' \), \( f = \alpha u^2 + \beta u + \gamma \), \( \alpha, g'' \neq 0 \)). For this case equation (4) takes the form:

\[
u_t = u_t = \alpha u^2 + \beta u + \gamma + g' u_y^2 + g(u) u_{yy} - u_{yyyy} + \frac{u^2}{2} + \left( \delta + \frac{1}{2} u \right) u_{xx} \\
- 2u_{xyy} - u_{xxxx}.
\]

The new dependent variable \( \upsilon \) is,

\[
\upsilon = e^{(16\alpha^2 - \beta + 4\alpha \delta)} t \left( (c_1 + c_2) \cos(2\sqrt{\alpha} x) + (c_3 + c_4) \sin(2\sqrt{\alpha} x) \right).
\]

\[ \partial_t: \]

\[
C^1 = \frac{1}{2} e^{Dt} D \left( B\sqrt{\alpha} u^2 - 2 \left( A\sqrt{\alpha} + \delta \right) u_x + 2B\sqrt{\alpha} u_{xx} - A u_{xxx} \right) \\
+ (4B\sqrt{\alpha} (4\alpha + \delta) u - A u_x)\right).
\]

\[
C^2 = -e^{Dt} \left( g(u) (Cu_t + ADu_y) + D (Cu_{yy} - Au_{yy} + 2 (Cu_{xy} - Au_{xy})) \right).
\]

\[
C^3 = e^{Dt} \left( ADu + C g(u) u_y - A \gamma \right);
\]

this conserved vector will be nontrivial if and only if \( D \neq 0 \).
\[ \partial_x: \]

\[ C^1 = e^{Dt} \left( Cg(u)u_y - A\alpha u^2 - u \left( 4\alpha(4\alpha + \delta) + B\sqrt{\alpha}u_x \right) 
-2\sqrt{\alpha} \left( B(4\alpha + \delta)u_x - 2A\sqrt{\alpha}u_{xx} - Bu_{xxx} \right) \right), \]
\[ C^2 = e^{Dt} \left( \sqrt{\alpha} \left( 2Bu_{yy} + \sin(2\sqrt{\alpha}x) \left( \gamma(c_1 y + 2c_2) y - 4(c_1 y + c_2)u_{xyy} 
+ 2c_1 (u_{yy} + 2u_{xx}) \right) - \cos(2\sqrt{\alpha}x) \left( 2c_3 (u_{yy} + 2u_{xx}) 
+ \gamma(c_3 y + 2c_4) y - 4(c_3 y + c_4)u_{xyy} \right) 
- g(u) (2B\sqrt{\alpha}u_y + Cu_x) \right) \right), \]
\[ C^3 = 2e^{Dt}B\sqrt{\alpha}u. \]

\[ \partial_y: \]

\[ C^1 = \frac{1}{2} e^{Dt} \left( \mathcal{E}\sqrt{\alpha}u^2 + (4\mathcal{E}\sqrt{\alpha}(4\alpha + \delta) - \mathcal{C}u_x) u 
-2 \left( \mathcal{C}(4\alpha + \delta)u_x + 2\mathcal{E}\sqrt{\alpha}u_{xx} - \mathcal{C}u_{xxx} \right) \right), \]
\[ C^2 = Ce^{Dt} \left( u_{yyy} + 2u_{xxy} - \gamma y - g(u)u_y \right), \]
\[ C^3 = e^{Dt}C u, \]

where

\[ A = (c_1 y + c_2) \cos(2\sqrt{\alpha}x) + (c_3 y + c_4) \sin(2\sqrt{\alpha}x), \]
\[ B = (c_3 y + c_4) \cos(2\sqrt{\alpha}x) - (c_1 y + c_2) \sin(2\sqrt{\alpha}x), \]
\[ C = c_1 \cos(2\sqrt{\alpha}x) + c_3 \sin(2\sqrt{\alpha}x), \]
\[ D = 16\alpha^2 - \beta + 4\alpha\delta \]

and

\[ \mathcal{E} = c_3 \cos(2\sqrt{\alpha}x) - c_1 \sin(2\sqrt{\alpha}x). \]

This conserved vector will be nontrivial if and only if \( c_1^2 + c_3^2 \neq 0. \)

- **Subcase 2.5** \( (r = u/2 + \gamma, h = g', f = \alpha u + \beta, g'' \neq 0). \) For this case equation (14) assumes the form:

\[ u_t = u_t = \alpha u + \beta + g'u_y^2 + g(u)u_{yy} - u_{yy}^2 + \frac{u_x^2}{2} + \left( \gamma + \frac{1}{2}u \right) u_{xx} - 2u_{xx}u_y 
- u_{xxx}. \]

The new dependent variable \( v \) is,

\[ v = e^{-\alpha t} \left( (c_1 + c_2)x y + c_3 + c_4x \right). \]
∂ₜ:

\[ C_1 = \frac{1}{4} e^{-\alpha t} \left( 2 (A u_x - 2 \gamma B) + 4 \gamma A u_x + 4 B u_{xx} - B u^2 - 4 A u_{xxx} \right), \]
\[ C_2 = e^{-\alpha t} \left( g(u) (\alpha A u_y - C u_t) + \alpha (C u_{yy} - A u_{gyy} + 2 C u_{xx} - 2 A u_{xyy}) \right), \]
\[ C_3 = e^{-\alpha t} \left( C g(u) u_y - A(\beta + \alpha u) \right). \]

This conserved vector will be nontrivial if and only if \( \alpha \neq 0. \)

∂ₓ:

\[ C_1 = \frac{1}{2} e^{-\alpha t} \left( 2 c_2 x g(u) u_y - B (2 \beta x + (2 \gamma + u) u_x - 4 u_{xyy} - 2 u_{xxx}) \right), \]
\[ C_2 = e^{-\alpha t} \left( B u_{yy} - c_2 u_{yy} - g(u) (B u_y + c_2 u_x) \right), \]
\[ C_3 = e^{-\alpha t} B u. \]

This conserved vector will be nontrivial if and only if \( c_2^2 + c_4^2 \neq 0. \)

∂ᵧ:

\[ C_1 = \frac{1}{4} e^{-\alpha t} \left( c_2 u^2 - 4 \gamma C u_x + 2 u (2 \gamma c_2 - C u_x) - 4 c_2 u_{xx} + 4 C u_{xxx} \right), \]
\[ C_2 = e^{-\alpha t} C (u_{yy} + 2 u_{xx} - \beta y - g(u) u_y), \]
\[ C_3 = e^{-\alpha t} C u. \]

This conserved vector will be nontrivial if and only if \( c_1^2 + c_2^2 \neq 0. \)

In the last three conserved vectors
\[ A = (c_1 + c_2 x) y + c_3 + c_4 x, \]
\[ B = c_2 y + c_4, \]
and
\[ C = c_1 + c_2 x. \]

6. Conclusions

In the present work a generalization of the anisotropic two-dimensional Kuramoto-Sivashinsky Equation was studied under the prism of the modern group analysis. Specifically, two distinct classifications were performed; one with respect to the Lie point symmetries and a second with respect to the property of self-adjointness. The wealth of information obtained by those two classifications not only shed light to the structure of the generalization studied, by highlighting the interesting subcases, but provides us also with ways for attaining analytical nontrivial solutions. A fact that will be the subject of a future work.
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Appendix A. The Determining Equations

\[\begin{align*}
\xi^3_x &= 0, \\
\xi^3_y &= 0, \\
\xi^3_u &= 0, \\
\xi^3_{xy} &= 0, \\
\xi^3_{xu} &= 0, \\
\xi^3_{yu} &= 0, \\
\xi^3_{uu} &= 0, \\
\xi^3_{xuu} &= 0, \\
\xi^3_{xyu} &= 0, \\
\xi^3_{yuu} &= 0, \\
\xi^3_{uuu} &= 0, \\
\xi^3_{xuuu} &= 0, \\
g(u)\xi^3_u &= 0, \\
r(u)\xi^3_u &= 0, \\
3\xi^3_{yy} + \xi^3_{xx} &= 0, \\
\xi^3_{yy} + 3\xi^3_{xx} &= 0, \\
7\xi^3_{yy} + 5\xi^3_{xx} &= 0, \\
5\xi^3_{yy} + 7\xi^3_{xx} &= 0, \\
\xi^2_{uuu} + 2\xi^3_{yuu} &= 0, \\
\xi^2_{uuuu} + 2\xi^3_{yuuu} &= 0, \\
\xi^1_{uuuu} + 2\xi^3_{xuuu} &= 0, \\
(g(u) + r(u))\xi^3_u &= 0, \\
\end{align*}\]
\[\xi_{yyu} + \xi_{xuu} + \xi_{xuu} = 0,\]
\[\xi_{y} + \xi_{x} + 2g(u)\xi_{xy} = 0,\]
\[\xi_{y} + \xi_{x} + 2r(u)\xi_{xy} = 0,\]
\[2h\xi_{u} + (1 + h(u))\xi_{uu} = 0,\]
\[2(1 + r')\xi_{u} + r(u)\xi_{uu} = 0,\]
\[\xi_{yyuu} + \xi_{xuuu} + \xi_{xuuu} = 0,\]
\[8h'\xi_{u} + (1 + 4h(u))\xi_{uu} = 0,\]
\[(11 + 16r')\xi_{u} + 8r(u)\xi_{uu} = 0,\]
\[6\xi_{yyu} + 2\xi_{xuu} - 6g(u)\xi_{uu} = 0,\]
\[6\xi_{yyu} + 2\xi_{xuu} - 2r(u)\xi_{uu} = 0,\]
\[2\xi_{yyu} + 6\xi_{xuu} - 2g(u)\xi_{uu} = 0,\]
\[2\xi_{yyu} + 6\xi_{xuu} - 6r(u)\xi_{uu} = 0,\]
\[6g(u)\xi_{y} - 4(\xi_{yyy} + \xi_{xyy}) = 0,\]
\[6r(u)\xi_{x} - 4(\xi_{xxy} + \xi_{xxx}) = 0,\]
\[6g(u)\xi_{u} - 4(3\xi_{yyu} + \xi_{xuu}) = 0,\]
\[6r(u)\xi_{u} - 4(\xi_{yyu} + 3\xi_{xuu}) = 0,\]
\[4h(u)\xi_{u} + 2g'\xi_{u} + g(u)\xi_{uu} = 0,\]
\[\xi_{u} + (1 + 2g')\xi_{y} + 2g(u)\xi_{yu} = 0,\]
\[\xi_{u} + (1 + 2r')\xi_{y} + 2r(u)\xi_{yu} = 0,\]
\[\xi_{uu} + 2h'\xi_{y} + (1 + 2h(u))\xi_{yu} = 0,\]
\[\xi_{uu} + 2h'\xi_{x} + (1 + 2h(u))\xi_{xu} = 0,\]
\[11h(u)\xi_{u} + 8g'\xi_{u} + 4g(u)\xi_{uu} = 0,\]
\[5\xi_{u} + (6 + 8r')\xi_{x} + 8r(u)\xi_{xu} = 0,\]
\[(3 + 8h(u) + 8g')\xi_{u} + 4g(u)\xi_{uu} = 0,\]
\[(1 + 2h(u) + 4r')\xi_{u} + 2r(u)\xi_{uu} = 0,\]
\[(2 + 3h(u) + 4r')\xi_{u} + 2r(u)\xi_{uu} = 0,\]
\[(5 + 6h(u) + 4r')\xi_{u} + 2r(u)\xi_{uu} = 0,\]
\[(1 + 2h(u) + 4g')\xi_{u} + 2g(u)\xi_{uu} = 0,\]
\[3\xi_{u} + 4((1 + r')\xi_{y} + r(u)\xi_{yu}) = 0,\]
\[\xi_{yu} + \xi_{xu} + 2g'\xi_{xy} + 2g(u)\xi_{xy} = 0,\]
\[\xi_{yu} + \xi_{xu} + 2r'\xi_{xy} + 2r(u)\xi_{xy} = 0,\]
\[(3 + 10h(u) + 4g')\xi^3_u + 2g(u)\xi^3_{uu} = 0,
3h''\xi^3_u + 3h'\xi^3_{uu} + (1 + h(u))\xi^3_{uuu} = 0,
3r''\xi^3_u + (2 + 3r')\xi^3_{uu} + r(u)\xi^3_{uuu} = 0,
3g''\xi^3_u + (1 + 3g')\xi^3_{uu} + g(u)\xi^3_{uuu} = 0,
\xi^1_{yuu} + \xi^2_{xuu} + 2h'\xi^3_{xy} + 2h(u)\xi^3_{xyu} = 0,
\xi^1_u + 2(h(u)\xi^3_x + g'(\xi^3_x + g(u)\xi^3_{xx}) = 0,
\xi^1_u + 2(h(u)\xi^3_x + r'\xi^3_x + r(u)\xi^3_{xx}) = 0,
3\xi^1_u + 8h(u)\xi^3_x + 4g'\xi^3_x + 4g(u)\xi^3_{xx} = 0,
2g(u)\xi^3_y - 4(\xi^3_{yy} + \xi^3_{xxy} - r(u)\xi^3_y) = 0,
12h''\xi^3_u + 12h'\xi^3_{uu} + (1 + 4h(u))\xi^3_{uuu} = 0,
\xi^1_{uuu} + 4h''\xi^3_x + 8h'\xi^3_{xx} + 4h(u)\xi^3_{xuu} = 0,
4g(u)\xi^3_x - 8(\xi^3_{xy} + \xi^3_{xxx} - r(u)\xi^3_x) = 0,
5\xi^2_u + 12h(u)\xi^3_y + 8g'\xi^3_y + 8g(u)\xi^3_{yu} = 0,
8g(u)\xi^3_y + 4r(u)\xi^3_y - 8(\xi^3_{yy} + \xi^3_{xy}) = 0,
4g(u)\xi^3_x + 2r(u)\xi^3_x - 4(\xi^3_{xy} + \xi^3_{xx}) = 0,
(1 + 4g' + 4r')\xi^3_u + 2(g(u) + r(u))\xi^3_{uu} = 0,
2g(u)\xi^3_u - 4(3\xi^3_{yy} + \xi^3_{xu} - r(u)\xi^3_u) = 0,
4g(u)\xi^3_u - 8(\xi^3_{yy} + 3\xi^3_{xu} - r(u)\xi^3_u) = 0,
12r''\xi^3_u + (13 + 12r')\xi^3_{uu} + 4r(u)\xi^3_{uuu} = 0,
3\xi^2_u + (3 + 4h(u) + 2g')\xi^3_y + 2g(u)\xi^3_{yu} = 0,
2g(u)\xi^3_u + 6r(u)\xi^3_u - 4(3\xi^3_{yy} + \xi^3_{xu}) = 0,
8g(u)\xi^3_u + 4r(u)\xi^3_u - 8(3\xi^3_{yy} + \xi^3_{xu}) = 0,
6g(u)\xi^3_u + 2r(u)\xi^3_u - 4(\xi^3_{yy} + 3\xi^3_{xu}) = 0,
4g(u)\xi^3_u + 2r(u)\xi^3_u - 4(\xi^3_{yy} + 3\xi^3_{xu}) = 0,
3\xi^1_u + 2((1 + 3h(u) + r')\xi^3_x + r(u)\xi^3_{xx}) = 0,
(5 + 12g' + 4r')\xi^3_u + 2(3g(u) + r(u))\xi^3_{uu} = 0,
2(1 + 3g' + 2r')\xi^3_u + (3g(u) + 2r(u))\xi^3_{uu} = 0,
\xi^2_u + 2(g'\xi^3_y + r'\xi^3_y + (g(u) + r(u))\xi^3_{yu}) = 0,
\xi^1_u + 2(g'\xi^3_x + r'\xi^3_x + (g(u) + r(u))\xi^3_{xx}) = 0,
4h''\xi^3_u + 6h'\xi^3_{uu} + 4h\xi^3_{uuu} + h(u)\xi^3_{uuuu} = 0,
21}
\[ \xi_{1u}^1 + 2 \left( h' \xi^3_x + g'' \xi^3_x + h(u) \xi^3_{xx} + 2g' \xi^3_{xu} + g(u) \xi^3_{xuu} \right) = 0, \]
\[ \xi_{1u}^1 + 2 \left( h' \xi^3_x + r'' \xi^3_x + h(u) \xi^3_{xu} + 2r' \xi^3_{xu} + r(u) \xi^3_{xuu} \right) = 0, \]
\[ 3\xi_{u}^2 + 4h(u) \xi^3_y + 2g' \xi^3_y + 6r' \xi^3_y + 2g(u) \xi^3_{yu} + 6r(u) \xi^3_{yu} = 0, \]
\[ 4h' \xi^3_u + 6g' \xi^3_u + \xi_{uu} + 2h(u) \xi_{uu} + 6g' \xi_{u}^3 + 2g(u) \xi_{uu} = 0, \]
\[ 4h' \xi^3_u + 6r' \xi^3_u + \xi_{uu} + 2h(u) \xi_{uu} + 6r' \xi_{uu} + 2r(u) \xi_{uu} = 0, \]
\[ \xi_{1uu}^1 + 2g'' \xi^3_x + 6g'' \xi^3_{xu} + \xi_{xuu} + 6g' \xi^3_{xuu} + 2g(u) \xi^3_{xuu} = 0, \]
\[ (1 + 8g') \xi^3_u + 4h(u) \xi^3_{uu} + 2r(u) \xi^3_{uu} - 4\xi_{yuu} - 12\xi_{xxxuu} = 0, \]
\[ 3\xi_{1u}^3 + 3\xi_{2u}^2 + 2 \left( r'' \xi^3_{xy} + (1 + 2r') \xi^3_{xyu} + r(u) \xi^3_{xyuu} \right) = 0, \]
\[ 6h' \xi^3_u + 6r' \xi^3_u + 2c^3_{uu} + 3h(u) \xi^3_{uu} + 6r' \xi^3_{uu} + 2r(u) \xi^3_{uu} = 0, \]
\[ h(u) \xi^3_y + 2r' \xi^3_y + g(u) \xi^3_{yu} + 2r(u) \xi^3_{yu} - 2\xi_{yuu} - 2\xi_{xyuu} = 0, \]
\[ h(u) \xi^3_u + 4r' \xi^3_u + g(u) \xi^3_{uu} + 2r(u) \xi^3_{uu} - 6\xi_{yuu} - 2\xi_{xyuu} = 0, \]
\[ 3\xi_{uu}^2 + 2r'' \xi^3_y + 6r' \xi^3_y + 5\xi_{yu}^3 + 6r' \xi_{yu}^3 + 2r(u) \xi_{yu}^3 = 0, \]
\[ 5\xi_{uu}^2 + 2r'' \xi^3 = 6r' \xi^3 + 9\xi_{xu}^3 + 6r' \xi_{xu}^3 + 2r(u) \xi_{xu}^3 = 0, \]
\[ 12h' \xi^3_u + 6r' \xi^3_u + 5\xi_{uu}^3 + 6h(u) \xi^3_{uu} + 6r' \xi^3_{uu} + 2r(u) \xi^3_{uu} = 0, \]
\[ 5\xi_{uu}^2 + 4 \left( 4h' \xi^3_y + 2g' \xi^3_y + 4h(u) \xi^3_{yu} + 2g' \xi^3_{yu} + g(u) \xi^3_{yu} \right) = 0, \]
\[ \xi_{uu}^2 + 2h'' \xi^3_y + 6h' \xi^3_y + 6h' \xi^3_{yu} + \xi^3_{yuu} + 2h(u) \xi^3_{yuu} = 0, \]
\[ \xi_{uu}^1 + 2h'' \xi^3_x + 6h' \xi^3_x + 6h' \xi^3_{xu} + \xi_{xuu} + 2h(u) \xi_{xuu} = 0, \]
\[ 3\xi_{1u}^2 + 4 \left( 2h' \xi^3_x + g'' \xi^3_x + 2h(u) \xi^3_{xu} + 2g' \xi^3_{xu} + g(u) \xi^3_{xu} \right) = 0, \]
\[ 3\xi_{1u}^1 + 2 \left( 5h' \xi^3_x + g'' \xi^3_x + 5h(u) \xi^3_{xu} + 2g' \xi^3_{xu} + g(u) \xi^3_{xu} \right) = 0, \]
\[ 20h' \xi^3_u + 6g'' \xi^3_u + 3\xi_{uu}^3 + 10h(u) \xi^3_{uu} + 6g' \xi^3_{uu} + 2g(u) \xi^3_{uu} = 0, \]
\[ 5\xi_{uu}^2 + 2 \left( 7h' \xi^3_y + 3g'' \xi^3_y + 7h(u) \xi^3_{yu} + 6g' \xi^3_{yu} + 3g(u) \xi^3_{yu} \right) = 0, \]
\[ 16h' \xi^3_u + 12g' \xi^3_u + 3\xi_{uu}^3 + 8h(u) \xi^3_{uu} + 12g' \xi^3_{uu} + 4g(u) \xi^3_{uu} = 0, \]
\[ (5 + 12g' + 4r') \xi^3_u + 6h(u) \xi^3_{uu} - 4 \left( \xi_{yuu} + 3\xi_{xuu} - r(u) \xi_{xuu} \right) = 0, \]
\[ 3\xi_{uu}^1 + 2 \left( 3h' \xi^3_x + r'' \xi^3_x + 3h(u) \xi^3_{xu} + 2r' \xi^3_{xu} + r(u) \xi^3_{xuu} \right) = 0, \]
\[ 3\xi_{uu}^2 + 2h' \xi^3_y + 3\xi_{yu}^3 + 4h(u) \xi^3_{yu} + 4g' \xi^3_{yu} + 2g(u) \xi^3_{yu} = 0, \]
\[ 2\xi_{uu}^1 + (3 + 6g' + 2r') \xi^3_{u} + 6g(u) \xi^3_{u} + 4r(u) \xi^3_{u} - 4g' \xi_{yu}^3 - 4\xi_{yuu} = 0, \]
\[ 5h(u) \xi^3_{u} + 2g' \xi^3_{u} + 6r' \xi^3_{u} + 2g(u) \xi^3_{u} + 3r(u) \xi^3_{u} - 6\xi_{yu}^3 - 2\xi_{yuu} = 0, \]
\[ 18g' \xi^3_{u} + 6r' \xi^3_{u} + 3\xi_{uu}^3 + 18g' \xi_{uu}^3 + 6r' \xi_{uu}^3 + 2g(u) \xi_{uu}^3 + 2r(u) \xi_{uu}^3 = 0, \]
\[ 3\xi_{uu} + 3\xi_{xuu} + 4h' \xi^3_{xy} + 2g' \xi^3_{xy} + 4h(u) \xi^3_{xy} + 4g' \xi_{xy}^3 + 2g(u) \xi_{xy}^3 = 0, \]
\[ 6h'' \xi^3_u + 4r''' \xi^3_u + 6h' \xi^3_{uu} + 6r'' \xi^3_{uu} + 2h(u) \xi^3_{uu} + 4r' \xi^3_{uu} + r(u) \xi^3_{uu} = 0, \]
\[ \xi_{3u}^3 + 3h(u) \xi^3_y + g' \xi^3_y + 3r' \xi^3_y + 2g(u) \xi^3_{yu} + 3r(u) \xi^3_{yu} - 2\xi_{yuu} - 2\xi_{xyuu} = 0, \]
\[g(u)^2 \xi_y^3 - 2f' \xi_y^3 + 2\eta_yu - 2f(u)\xi_y^3 u - 3\xi_y^y - \xi_x^x - 2g(u) (\xi_y^y - \xi_x^x) = 0,\]
\[\xi_y^y - r(u)^2 \xi_x^x + 2f' \xi_x^x - 2\eta_xu + 2f(u)\xi_x^x u + 3\xi_x^u + 2r(u) (\xi_x^y + \xi_x^x) = 0,\]
\[2 (3g'' \xi_x^x + r'' \xi_x^x + \xi_x^x u + 6g' \xi_x^x u + 2r' \xi_x^x u + 3g(u)\xi_x^x u + r(u)\xi_x^x u) + 3\xi_x^u = 0,\]
\[g(u)\xi_y^y + r(u)\xi_x^x - 4f' \xi_x^x y + 4\eta_xyu - 4f(u)\xi_x^x yu - 2\xi_x^x xy - 2\xi_y^y = 0,\]
\[f(u)\xi_x^x t + \xi_x^x - g(u)\xi_x^x y + 2r(u)\xi_x^x yy + \xi_x^x yyyy - 4\xi_x^x yx - 7r(u)\xi_x^x xx + 2\xi_x^x yxy + \xi_x^x xxxx = 0,\]
\[f(u)\xi_x^x u + \xi_x^x - 4\xi_x^x y + 7g(u)\xi_x^x y + \xi_x^x yyyy - 2g(u)\xi_x^x xx + r(u)\xi_x^x xx + 2\xi_x^x yy + \xi_x^x xxxx = 0,\]
\[9h'' \xi_y^y + g'' \xi_y^y + 18h' \xi_y^y u + 3g'' \xi_y^y u + 9h(u)\xi_y^y uu + 3g' \xi_y^y uu + g(u)\xi_y^y uu + 5 \xi_x^x uu = 0,\]
\[5h'' \xi_x^x + g'' \xi_x^x + 10h' \xi_x^x u + 3g'' \xi_x^x u + 5h(u)\xi_x^x uu + 3g' \xi_x^x uu + g(u)\xi_x^x uu + 3 \xi_x^u uu = 0,\]
\[42h'' \xi_x^x u + 4g'' \xi_x^x u + 42h' \xi_x^x uu + 6g'' \xi_x^x uu + 14h(u)\xi_x^x uu + 4g' \xi_x^x uu + g(u)\xi_x^x uu = 0,\]
\[4f' \xi_x^x u - 2\eta_uu + 2f(u)\xi_x^x uu + 4\xi_y^y u + 3\xi_y^y + 2h(u)\xi_y^y yy + 4\xi_x^x u + \xi_x^x xx + 6h(u)\xi_x^x xx = 0,\]
\[10h' \xi_x^x u + 3g'' \xi_x^x u + 9r'' \xi_x^x u + 5h(u)\xi_x^x uu + 3g' \xi_x^x uu + 9r' \xi_x^x uu + g(u)\xi_x^x uu + 3r(u)\xi_x^x uu = 0,\]
\[g(u) r(u)\xi_y^y - 2f' \xi_y^y + 2\eta_yu - 2f(u)\xi_y^y u - \xi_y^y - 2r(u)\xi_y^y yyyy - 4\xi_y^y - 3\xi_x^x xx - 2r(u)\xi_x^x xxy = 0.\]
\[
\frac{3}{2} \xi_{uuu} + 3h'' \xi_x + r''' \xi_x + 6h' \xi_{xx} + 3r'' \xi_{xx} + \xi_{xxu} + 3h(u) \xi_{xuu} + 3r' \xi_{xuu} \\
+ r(u) \xi_{xuu} = 0, \\
30h'' \xi_u + 8g''' \xi_u + 30h' \xi_{uu} + 12g'' \xi_{uu} + 3\xi_{uuu} + 10h(u) \xi_{uuu} + 8g' \xi_{uuu} \\
+ 2g(u) \xi_{uuu} = 0, \\
3\xi_{uuu} + 4h'' \xi_y + 2g'' \xi_y + 8h' \xi_{yu} + 6g'' \xi_{yu} + 3\xi_{yyu} + 4h(u) \xi_{yyu} + 6g' \xi_{yyu} \\
+ 2g(u) \xi_{yyu} = 0, \\
18h'' \xi_u + 8r''' \xi_u + 18h' \xi_{uu} + 12r'' \xi_{uu} + 5\xi_{uuu} + 6h(u) \xi_{uuu} + 8r' \xi_{uuu} \\
+ 2r(u) \xi_{uuu} = 0, \\
3\xi_{yy} + 2f' \xi^3_x + 4\xi^2_{xy} + \xi^3_{xx} + g(u) \left(-r(u) \xi^3_x + 2 \left( \xi^3_{yy} + \xi^3_{xx} \right) \right) \\
- 2\eta_{yu} + 2f(u) \xi^3_{xu} = 0, \\
2 \left( h'' \xi^3_y + r''' \xi^3_y + 2h' \xi^3_{yu} + 3r'' \xi^3_{yu} + h(u) \xi^3_{yu} + 3r' \xi^3_{yu} \\
+ r(u) \xi^3_{yuu} \right) + \xi^2_{uu} = 0, \\
2h' \xi^3_y + g'' \xi^3_y + 3r'' \xi^3_y + 2h(u) \xi^3_{yu} + 2g' \xi^3_{yu} + 6r' \xi^3_{yu} + g(u) \xi^3_{yu} \\
+ 3r(u) \xi^3_{yu} + \frac{3}{2} \xi^2_{uu} = 0, \\
2f'' \xi^3_y + 4f' \xi^3_{yu} - 2\eta_{yu} + 2f(u) \xi^3_{yuu} + \xi^2_{yyu} + \xi^3_{yyu} + 4\xi^1_{xyu} + 4\xi^2_{xuu} + 4\xi^2_{xyy} \\
- \frac{1}{2} r(u) \xi^2_u - \frac{1}{2} g(u) \xi^3_y = 0, \\
4f' \xi^3_u - r(u) \xi^3_u - 2\eta_{uu} + 2f(u) \xi^3_{uu} + \xi^3_{yyu} + 2r' \xi^3_{yyu} + 8\xi^1_{xuu} + 3\xi^3_{xxx} + 6r' \xi^3_{xx} \\
+ 2r(u) \left( \xi^3_{yyu} + 3\xi^3_{xxx} \right) = 0, \\
3\eta_{uu} - 6f' \xi^3_u - 3f(u) \xi^3_{uu} - 2\xi^3_{yy} - 2r' \xi^3_{yy} - 2r(u) \xi^3_{yyu} - 12\xi^1_{xu} - 6\xi^3_{xx} \\
- 6r' \xi^3_{xx} - 6r(u) \xi^3_{xxx} = 0, \\
f(u) \xi^3_u + \xi^3_t - 2\xi^2_y - 2g(u) \xi^3_{yu} - 3r(u) \xi^3_{yu} + \xi^3_{yyu} - 2\xi^1_x - 3g(u) \xi^3_{xx} \\
- 2r(u) \xi^3_{xx} + 2\xi^3_{xyy} + \xi^3_{xxxx} = 0,
\]
\[ 6 f'' \xi^3_{uu} + 4 f' \xi^3_{u uu} - \eta_{uu uu} + f(u) \xi^3_{uu uu} + \xi^3_{gy uu} + 4 \xi^1_{x uu} + 3 \xi^3_{xxx} \\
\quad - \frac{1}{4} (1 - 16 f''') \xi^3_u - \frac{1}{2} r(u) \xi^3_{uu} = 0, \]

\[ 3 \eta_{uu} - 6 f' \xi^3_u - 3 f(u) \xi^3_{uu} - 12 \xi^2_{yy} - 12 h(u) \xi^3_{yy} - 6 g' \xi^3_{yy} - 6 g(u) \xi^3_{yy} \\
\quad - 4 h(u) \xi^3_{xx} - 2 g' \xi^3_{xx} - 2 g(u) \xi^3_{xx} = 0, \]

\[ 4 f' \xi^3_u - 2 \eta_{uu} + 2 f(u) \xi^3_{uu} + 4 \xi^2_{yy} + 3 \xi^3_{yy} + 2 g' \xi^3_{yy} + 4 \xi^1_{x uu} + \xi^3_{xx} + 6 g' \xi^3_{xx} \\
\quad + g(u) \left(-r(u) \xi^3_u + 2 \xi^3_{yy} + 6 \xi^3_{xxx}\right) = 0, \]

\[ 2 f' \xi^3_u - \eta_{uu} + f(u) \xi^3_{uu} + 4 \xi^2_{yy} + 3 h(u) \xi^3_{yy} + 3 g' \xi^3_{yy} + h(u) \xi^3_{xx} + g' \xi^3_{xx} \\
\quad + g(u) \left(3 \xi^3_{yy} + \xi^3_{xxx}\right) - \frac{1}{2} g(u)^2 \xi^3_u = 0, \]

\[ 6 \xi^1_{yy} - g(u) \xi^1_u - 2 h(u) r(u) \xi^3_x + 4 f'' \xi^3_x + 8 f' \xi^3_x - 4 \eta_{xx uu} + 4 f(u) \xi^3_{x uu} \\
\quad + 8 \xi^2_{x uu} + 4 h(u) \xi^3_{xx} + 2 \xi^1_{xx} + 4 h(u) \xi^3_{xx} = 0, \]

\[ 2 f' \xi^3_u - \eta_{uu} + f(u) \xi^3_{uu} + 2 \xi^2_{yy} + h(u) \xi^3_{yy} + 3 g' \xi^3_{yy} + 3 r' \xi^3_{yy} + 2 g^3_{yy} + 2 \xi^1_{xx} \\
\quad + 3 h(u) \xi^3_{xx} + r' \xi^3_{xx} + r(u) \xi^3_{xxx} - \frac{1}{2} g(u) r(u) \xi^3_u = 0, \]

\[ 2 f' \xi^3_u - \eta_{uu} + f(u) \xi^3_{uu} + 2 \xi^2_{yy} + g' \xi^3_{yy} + 3 r' \xi^3_{yy} + g(u) \xi^3_{yy} + 3 r(u) \xi^3_{yy} \\
\quad + 2 \xi^1_{xx} + 3 g' \xi^3_{xx} + r' \xi^3_{xx} + 3 g(u) \xi^3_{xx} + r(u) \xi^3_{xxx} = 0, \]

\[ 2 h(u) \xi^1_y + 2 g(u) \xi^1_{yy} + \xi^2_y + 2 r(u) \xi^2_{xu} - 8 f'' \xi^3_y - 16 f' \xi^3_y \\
\quad + 8 \eta_{xy uu} - 8 f(u) \xi^3_{xy uu} - 4 \xi^2_{xy uu} - 4 \xi^1_{xy uu} = 0, \]

\[ \xi^1_u - 4 \xi^1_{y uu} + \xi^3_x - 8 f''' \xi^3_x - 24 f'' \xi^3_{xu} + 2 r(u) \left(\xi^1_{uu} + \xi^3_{xx}\right) - 24 f' \xi^3_{xu} \\
\quad + 8 \eta_{x uu} - 8 f(u) \xi^3_{x uu} - 4 \xi^3_{xx uu} - 12 \xi^3_{x uu} - 4 \xi^3_{xxx uu} = 0, \]

\[ \eta_t - \eta(x, y, t, u) f' - f(u)^2 \xi^3_{uu} - g(u) \eta_{yy} + \eta_{yy} - r(u) \eta_{xx} + 2 \eta_{xy} + \eta_{xxx} \\
\quad + f(u) \left(\eta_u - \xi^3 + g(u) \xi^3_{yy} - \xi^3_{yy} + r(u) \xi^3_{xx} - 2 \xi^3_{xy} = 0, \right. \]

\[ 4 f''' \xi^3_y - r(u) \xi^3_{uu} - h(u) \xi^3_y - g(u) \xi^3_{yy} + 12 f'' \xi^3_y + 12 f' \xi^3_{yy} - 4 \eta_{y uu} \\
\quad + 4 f(u) \xi^3_{y uu} + 2 \xi^2_{yy} + 2 \xi^3_{yy} + 8 \xi^3_{xy uu} + 6 \xi^2_{xy uu} + 2 \xi^3_{x uu} - \frac{1}{2} \xi^2_u = 0, \]
\[6f''\xi^3_u + 6f'\xi^3_{uu} - 2\eta_{uuu} + 2f(u)\xi^3_{uuu} + 4\xi^2_{yuu} + 2h'\xi^3_{yy} + 3\xi^3_{yyu} + 4\xi^1_{xuu} + 6h'\xi^3_{xx} + \xi^3_{xu} + h(u)\left(-r(u)\xi^3_u + 2\xi^3_{yyu} + 6\xi^3_{xxu}\right) - \frac{1}{2}g(u)\xi^3_u = 0,\]

\[2f''\xi^3_x + 4f'\xi^3_{xx} - 2\eta_{xuu} + 2f(u)\xi^3_{xuu} + 4\xi^2_{xyu} + 2g'\xi^3_{xyy} + \xi^1_{xxu} + 2g'\xi^3_{xxx} - r(u)g'\xi^3_x + 3\xi^3_{yyu} - \frac{1}{2}g(u)\left(\xi^3_x + 2r(u)\xi^3_{xx} - 4\left(\xi^3_{yyu} + \xi^3_{xxu}\right)\right) = 0,\]

\[3\xi^1_{yyu} + 6f''\xi^3_{x} + 12f'\xi^3_{xx} - 6\eta_{xuu} + 6f(u)\xi^3_{xuu} + 2\xi^3_{xyu} + 2r'\xi^3_{xyy} + 9\xi^1_{xuu} + 2\xi^3_{xxx} + r(u)^2\xi^3_{xxu} - \frac{1}{2}r(u)\left(2\xi^3_u + (3 + 2r')\xi^3_x - 4\left(\xi^3_{yyu} + \xi^3_{xxu}\right)\right) + 2r'\xi^3_{xxx} = 0,\]

\[2f''\xi^3_y + 4f'\xi^3_{yy} - g(u)\left(\xi^3_y + r(u)\xi^3_{yyu}\right) + 2f(u)\xi^3_{yyu} - h(u)r(u)\xi^3_y - 2\eta_{uuu} + \xi^2_{yuu} + 2r'\xi^3_{yyu} + 2r(u)\xi^3_{yuuu} + 4\xi^2_{xyu} + 3\xi^3_{xxx} + 2r'\xi^3_{xyy} + 2r(u)\xi^3_{xxyu} = 0,\]

\[h(u)^2\xi^3_u + 2g(u)h'\xi^3_{u} - 4f''\xi^3_{uu} - 6f'\xi^3_{uuu} - 4f'\xi^3_{uuuu} + \eta_{uuuu} - f(u)\xi^3_{uuuu} - 12h'\xi^3_{yyu} - 6h''\xi^3_{xx} - 4h'\xi^3_{xuu} + h(u)\left(g(u)\xi^3_{uu} - 2\left(3\xi^3_{yyu} + \xi^3_{xxu}\right)\right) - 6h''\xi^3_{yy} - 4\xi^2_{yuuu} = 0,\]

\[-\xi^1 + g(u)\xi^1_{yy} - \xi^1_{yyu} - \eta_{x} + 2r(u)f'\xi^3_x - 2r(u)\eta_{xu} - 4f'\xi^3_{xy} + 4\eta_{xyu} + f(u)\left(-\xi^1_u + \xi^3_x + 2r(u)\xi^3_{xxu} - 4\xi^3_{xyu} - 4\xi^3_{xxu}\right) - \xi^1_{xuu} + r(u)\xi^1_{xu} - 2\xi^1_{xyy} - 4f'\xi^3_{xxx} + 4\eta_{xxu} = 0,\]

\[18f''\xi^3_u + 18f'\xi^3_{uu} - 6\eta_{uuu} + 6f(u)\xi^3_{uuu} + 2r''\xi^3_{yy} + 5\xi^3_{yyu} + 4r'\xi^3_{yyu} + 12r'\xi^3_{xxu} - r(u)^2\xi^3_{uu} - \frac{1}{2}r(u)\left((5 + 4r')\xi^3_u - 4\left(\xi^3_{yyu} + 3\xi^3_{xxu}\right)\right) + 15\xi^3_{xuu} + 24\xi^1_{xuu} + 6r''\xi^3_{xx} = 0,\]

\[3\xi^1_{yyu} - r(u)h'\xi^3_x + 2f''\xi^3_x + 6f'\xi^3_{xuu} + 6f'\xi^3_{xxu} - 2\eta_{xxuu} + 2f(u)\xi^3_{xuuu} + 4\xi^2_{xyu} - \frac{1}{2}g(u)\xi^1_{uuu} - \frac{1}{2}h(u)\left(\xi^1_u + \xi^3_x + 2r(u)\xi^3_{xx} - 4\xi^3_{xyu} - 4\xi^3_{xxu}\right) + 2h'\xi^3_{xyy} + \xi^1_{xuuu} + 2h'\xi^3_{xxx} = 0,\]
\[6g''\xi^3_{xx} - 2r(u)g'\xi^3_u + 6f''\xi^3_u + 6f'\xi^3_{uu} - 2\eta_{uuu} + 2f(u)\xi^3_{uw} + 4\xi^2_{yyu} + 2g''\xi^3_{yy} + \xi^3_{xux} + 12g'\xi^3_{xuu} - \frac{1}{2}g(u)\left(\xi^3_u + 2r(u)\xi^3_{uu} - 4\xi^3_{yyu} - 12\xi^3_{xxu}\right) + 3\xi^3_{yyu} + 4g'\xi^3_{yyu} + 4\xi^3_{xuu} = 0,\]

\[2g(u)f'\xi^3_y - \xi^3 + 2h(u)\eta_y - 2g(u)\eta_{yu} + g(u)\xi^2_{yy} - 4f'\xi^3_{yuu} + 4h_{yyu} - \xi^2_{yyu} - f(u)\left(\xi^2_u - 2h(u)\xi^3_y + 2g(u)\xi^3_{yu} + 4\xi^3_{yyu} + 4\xi^3_{xxyu}\right) - 2\xi^2_{xxyy} - \xi^2_{xxxx} + r(u)\xi^2_{xx} - 4f'\xi^3_{xy} + 4h_{xxyu} = 0,\]

\[6f''\xi^3_y - g(u)^2\xi^3_{yu} + 12f'\xi^3_{yu} - 6h_{yu} + 6f(u)\xi^3_{yu} + 9\xi^2_{yyu} + 4h(u)\xi^3_{yyu} + 2g'\xi^3_{xy} - g(u)\left(\xi^2_u + 3h(u)\xi^3_y + g'\xi^3 + 2\xi^3_{yyu} - 2\xi^3_{xxu}\right) + 2g'\xi^3_{yyu} + 3\xi^2_{xxu} + 4h(u)\xi^3_{xyy} = 0,\]

\[2f''\xi^3_y + 6f''\xi^3_{yu} + 6f'\xi^3_{yu} - 2\eta_{uuu} + 2f(u)\xi^3_{yu} + 3\xi^2_{yu} + 2h'\xi^3_{yy} + 2h'\xi^3_{xyy} - h(u)^2\xi^3_y - \frac{1}{2}h(u)\left(\xi^2_u + 2g(u)\xi^3_{yu} - 4\xi^3_{yu} + 4\xi^3_{yyu} + 4\xi^3_{xxyu}\right) + \xi^2_{xxu} - \frac{1}{2}g(u)\left(\xi^2_{uu} + 2h'\xi^3_y\right) = 0,\]

\[2r(u)\xi^1_x - \eta(x, y, t, u)r' - r(u)\xi^3_t + g(u)r(u)\xi^3_{yu} - 2f'\xi^3_{yu} + 2h_{yu},\]

\[g(u)\xi^2_{yy} - \eta(x, y, t, u)g' - g(u)\xi^2_y + 2g(u)\xi^2_{yu} - 6f'\xi^2_{yu} + 6h_{yu} - 4\xi^2_{yyu} + g(u)r(u)\xi^3_{xx} - 2f'\xi^3_{xx} + 2h_{xxu} - f(u)\left(g(u)\xi^3_u + 6\xi^3_{yu} + 2\xi^3_{xxu}\right) - g(u)\xi^3_{yyu} + 4\xi^2_{xy} - 2g(u)\xi^3_{xyy} - g(u)\xi^3_{xxxx} = 0,\]

\[9f''\xi^3_u + 9r^2\xi^3_{uu} + 3f(u)\xi^3_{uuu} + 12\xi^2_{yu} + 15h^2\xi^3_{yu} + 3g^2\xi^3_{yu} + 6g'\xi^3_{yu} - 3h_{uuu} + 15h(u)\xi^3_{yu} + 5h^2\xi^3_{xx} + 5h(u)\xi^3_{xxu} + 2g^2\xi^3_{xxu} - \frac{1}{2}g(u)^2\xi^3_{uu} - \frac{1}{2}g(u)\left(5h(u)\xi^3_u + 2g'\xi^3_u - 6\xi^3_{yu} - 2\xi^3_{xxu}\right) = 0,\]

\[4r(u)h'\xi^3_u - 16h''\xi^3_u + g(u)\xi^3_{uu} - 24f''\xi^3_u - 16f'\xi^3_{uu} + 4h_{uuu} - 8\xi^2_{yu},\]

\[-4f(u)\xi^3_{uu} + 4h''\xi^3_{yu} - 8h'\xi^3_{yu} - 8\xi^3_{yu} - 8\xi^3_{xxu} - 12h''\xi^3_{xx} - 2\xi^3_{xxu} - 24h\xi^3_{xx} + 2h(u)\left(\xi^3_u + r(u)\xi^3_{uu} - 2\xi^3_{yu} - 6\xi^3_{xxu}\right) = 0,\]

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3f''\xi^3_u + 3f'\xi^3_{uu} - \eta_{uuu} + f(u)\xi^3_{uuu} + 2\xi^2_{yyu} + h'\xi^3_{yy} + 3r''\xi^3_{yy} + 6r'\xi^3_{yuu} + 3h'\xi^3_{xx} + r''\xi^3_{xx} + 2r'\xi^3_{xuu} + r(u)\xi^3_{xuuu} - \frac{1}{2}g(u)\left(2r'\xi^3_u + r(u)\xi^3_{uu}\right) - \frac{1}{2}h(u)\left(r(u)\xi^3_u - 2\left(\xi^3_{yyu} + 3\xi^3_{xuu}\right)\right) + 3r(u)\xi^3_{yyuu} + 2\xi^1_{xuu} = 0,

2\xi^1_x - \eta_u - \xi^3_t + g(u)\xi^3_{yy} - 4f''\xi^3_{yy} - 8f'\xi^3_{yuu} + 4\eta_{yyuu} - 4f(u)\xi^3_{yyuu} - \xi^3_{yyy} - 8\xi^1_{xyyu} - 12f''\xi^3_{xx} + r(u)\left(4f'\xi^3_u - 2\eta_{uu} + 2f(u)\xi^3_{uu} + 4\xi^1_{xx} + \xi^3_{xx}\right) - 24f'\xi^3_{xxu} + 12\eta_{xxuu} - 12f(u)\xi^3_{xxuu} - 2\xi^3_{xxyy} - 8\xi^1_{xxu} - \xi^3_{xxxx} = 0,

2g(u)\xi^2_{yy} - \eta(x, y, t, u)h' + 2g(u)f'\xi^3_u - g(u)\eta_{uu} + f(u)g(u)\xi^3_{uu} - 6f''\xi^3_{yy} - 12f'\xi^3_{yyu} + 6\eta_{yyuu} - 6f(u)\xi^3_{yyuu} - 4\xi^2_{yyuu} - 2f''\xi^3_{xx} + 4f'\xi^3_{xxu} + 2\eta_{xxuu} - h(u)\left(\eta_u + \xi^3_t - 2\xi^2_y - g(u)\xi^3_{yy} + \xi^3_{yyuu} - r(u)\xi^3_{xx} + 2\xi^3_{xxyy} + \xi^3_{xxxx}\right) - 2f(u)\xi^3_{xxuu} - 4\xi^2_{xxyu} = 0.

References

[1] G. W. Bluman, A. F. Cheviakov, and S. C. Anco. Applications of symmetry methods to partial differential equations. Applied Mathematical Sciences. Springer, New York, 2009.

[2] G. W. Bluman and S. Kumei. Symmetries and differential equations. Springer, New York, 1989.

[3] B. M. Boghosian, C. C. Chow, and T. Hwa. Hydrodynamics of the Kuramoto-Sivashinsky equation in two dimensions. Phys. Rev. Lett., 83(25):5262–5265, 1999.

[4] M. S. Bruzón, M. L. Gandarias, and N. H. Ibragimov. Self-adjoint subclasses of generalized thin film equations. J. Math. Anal. Appl., 357:307–313, 2009.

[5] B. Cohen, J. Krommes, W. Tang, and M. Rosenbluth. Nonlinear saturation of the dissipative trapped-ion mode by mode coupling. Nucl. Fis., 16:971–992, 1976.

[6] R. Conte and M. Musette. Painlevé analysis and Backlund tranformation in the Kuramoto-Sivashinsky equation. J. Phys. A: Math. Gen., 22:169–177, 1989.

[7] M. C. Cross and P. C. Hohenberg. Pattern formation outside of equilibrium. Rev. Mod. Phys., 65(3):851–1123, July 1993.
[8] A. Demirkaya. The existence of a global attractor for a Kuramoto-Sivashinsky type equation in 2d. *Discrete Contin. Dyn. Syst.*, pages 198–207, 2009.

[9] S. Dimas. Partial differential equations, algebraic computing and nonlinear systems. Ph.D. Thesis, University of Patras, Patras, Greece, October 2008.

[10] S. Dimas and D. Tsoubelis. SYM: A new symmetry-finding package for Mathematica. In N.H. Ibragimov, C. Sophocleous, and P.A. Damianou, editors, *The 10th International Conference in MODern GRoup ANalysis*, pages 64–70, Nicosia, 2005. University of Cyprus.

[11] S. Dimas and D. Tsoubelis. A new Mathematica-based program for solving overdetermined systems of PDEs. In Y. Papegay, editor, *Applied Mathematica, Electronic Proceedings of the Eighth International Mathematica Symposium (IMS’06)*, Avignon, France, 2006. France: INRIA. ISBN 2-7261-1289-7.

[12] Jason T. Drotar, Y.-P. Zhao, T.-M. Lu, and G.-C. Wang. Numerical analysis of the noisy Kuramoto-Sivashinsky equation in 2 + 1 dimensions. *Phys. Rev. E*, 59(1):177–185, 1999.

[13] V. A. Galaktionov, E. Mitidieri, and S. I. Pohozaev. On global solutions and blow-up for Kuramoto-Sivashinsky-type models, and well-posed Burnett equations. *Nonlinear Anal.*, 70:2930–2952, 2009.

[14] T. Halpin-Healy and Y.-C. Zhang. Kinetic roughening phenomena, stochastic growth, directed polymers and all that. Aspects of multidisciplinary statistical mechanics. *Phys. Rep.*, 254:215–414, 1995.

[15] P. E. Hydon. *Symmetry Methods for Differential Equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1st edition, 2000.

[16] J. M. Hyman and B. Nicolaenko. The Kuramoto-Sivashinsky equation: A bridge between pdes and dynamical systems. *Physica D*, 18:113–126, 1986.

[17] J. M. Hyman, B. Nicolaenko, and S. Zaleski. Order and complexity in the Kuramoto-Sivashinsky model of weakly turbulent interfaces. *Physica D*, 23:265–292, 1986.

[18] N. H. Ibragimov. *Transformation Groups Applied to Mathematical Physics*. Mathematics and its Applications. Springer, 1st edition, November 2001.

[19] N.H. Ibragimov. A new conservation theorem. *J. Math. Anal. App*, 333:311–328, 2007.

[20] N.H. Ibragimov. Nonlinear self-adjointness and conservation laws. *J. Phys. A: Math. Theor.*, 44:432002, 2011.
[21] N.H. Ibragimov. Nonlinear self-adjointness in constructing conservation laws. *Archives of ALGA*, 7/8:1–90, 2011.

[22] C. Jayaprakash, F. Hayot, and R. Pandit. Universal properties of the two-dimensional Kuramoto-Sivashinsky equation. *Phys. Rev. Lett.*, 71(1):12–15, 1993.

[23] I. G. Kevrekidis, B. Nicolaenko, and J. C. Scovel. Back in the saddle again: A computer assisted study of the Kuramoto-Sivashinsky equation. *SIAM J. Appl. Math.*, 50:760–790, 1990.

[24] Y. Kuramoto. Diffusion-induced chaos in reactions systems. *Prog. Theor. Phys.*, 64:346–367, 1978.

[25] Y. Kuramoto and T. Tsuzuki. On the formation of dissipative structures in reaction–diffusion systems. *Prog. Theor. Phys.*, 54:687–699, 1975.

[26] Y. Kuramoto and T. Tsuzuki. Persistent propagation of concentration waves in dissipative media far from thermal equilibrium. *Prog. Theor. Phys.*, 55(2):356–369, 1976.

[27] R. LaQuey, S. Mahajan, P. Rutherford, and W. Tang. Nonlinear saturation of the trapped-ion mode. *Phys. Rev. Lett.*, 34:391–394, 1975.

[28] J. Lundbek Hansen and T. Bohr. Fractal tracer distributions in turbulent field theories. *Physica D*, 118:40–48, July 1998.

[29] V. S. L’vov, V. V. Lebedev, M. Paton, and I. Procaccia. Proof of scale invariant solutions in the Kardar-Parisi-Zhang and Kuramoto-Sivashinsky equations in $1+1$ dimensions: analytical and numerical results. *Nonlinearity*, 6:25–47, 1993.

[30] M. A. Makeev and A. L. Barabási. Ion-induced effective surface diffusion in ion sputtering. *Appl. Phys. Lett.*, 71:2800–2802, 1997.

[31] P. Manneville. The Kuramoto-Sivashinsky equation: a progress report. In J. Wesfreid, H. R. Brand, P. Manneville, G. Albinet, and N. Boccara, editors, *Propagation in Systems Far from Equilibrium: Proceedings of the Workshop*, Springer Series in Synergetics, pages 265–280, Les Houches, France, March 1988. Springer.

[32] D. Michelson. Steady solutions of the Kuramoto-Sivashinsky equation. *Physica D*, 19:89–111, 1986.

[33] D. M. Michelson and G. I. Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flames – II. Numerical experiments. *Acta Astronaut.*, 4:1207–1221, 1977.

[34] Mehdi Nadjafikhah and Fatemeh Ahangari. Symmetry reduction of two-dimensional damped Kuramoto-Sivashinsky equation. *Commun. Theor. Phys.*, 56:211–217, 2011.
[35] Mehdi Nadjafikhah and Fatemeh Ahangari. Lie symmetry analysis of the two-dimensional generalized Kuramoto-Sivashinsky equation. *Math. Sci.*, 6(1):3, 2012.

[36] B. Nicolaenko, B. Scheurer, and R. Temam. Attractors for the Kuramoto-Sivashinsky equations. *Physica D*, 16:155–183, 1985.

[37] P. J. Olver. *Applications of Lie Groups to Differential Equations*, volume 107 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 2000.

[38] L. Ovsiannikov. *Group Analysis of Differential Equations*. Academic Press, 1st edition, June 1982. 432 pages.

[39] M. Rost and J. Krug. Anisotropic Kuramoto-Sivashinsky equation for surface growth and erosion. *Phys. Rev. Lett.*, 75:3894–3897, 1995.

[40] G. I. Sivashinsky. Nonlinear analysis of hydrodynamic instability in laminar flames – I. Derivation of basic equations. *Acta Astronaut.*, 4:1177–1206, June 1977.

[41] G. I. Sivashinsky. On flame propagation under conditions of stoichiometry. *SIAM J. Appl. Math.*, 39:67–82, 1980.

[42] G. I. Sivashinsky and D. Michelson. On irregular wavy flow of a liquid film down a vertical plane. *Prog. Theor. Phys.*, 63:2112–2114, 1980.

[43] H. Stephani. *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press, Cambridge, 1st edition, 1990. Editor: MacCallum, Malcolm.

[44] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1997.

[45] O. Thuai and U. Frisch. *Natural boundary in the Kuramoto-Sivashinsky model*, pages 327–336. Les Ulis : Ed. de Physique, Les Houches, March 1986.

[46] R. W. Wittenberg. *Encyclopaedia of Mathematics, Supplement III*, pages 230–233. Kluwer, 2002.

[47] R. W. Wittenberg and P. Holmes. Scale and space localization in the Kuramoto–Sivashinsky equation. *Chaos*, 9(2):452–465, 1999.