Augmented Biracks and their Homology

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Abstract

We introduce augmented biracks and define a (co)homology theory associated to augmented biracks. The new homology theory extends the previously studied Yang-Baxter homology with a combinatorial formulation for the boundary map and specializes to \( N \)-reduced rack homology when the birack is a rack. We introduce augmented birack 2-cocycle invariants of classical and virtual knots and links and provide examples.

Keywords: Biracks, rack homology, enhancements of counting invariants, cocycle invariants

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1 Introduction

Quandles, an algebraic structure associated to oriented knots and links, were introduced by Joyce and Matveev independently in 1982 in [11, 13]. Racks, the analogous structure associated to framed knots and links, were introduced by Fenn and Rourke in 1992 in [8]. Soon thereafter, biracks were introduced in [9] and later the special cases known as biquandles were studied in papers such as [12, 7, 15].

A homology and cohomology theory associated to racks was introduced in [10]. In [4], a subcomplex of the rack chain complex was identified in the case when our rack is a quandle, and cocycles in the quotient complex (known as quandle cocycles) were used to enhance the quandle counting invariant, yielding CJKLS quandle cocycle invariants. In [6], the degenerate subcomplex was generalized to the case of non-quandle racks with finite rack rank, yielding an analogous enhancement of the rack counting invariant via \( N \)-reduced cocycles.

In [3], Yang-Baxter (co)homology was defined as a natural generalization of the quandle (co)homology for biquandles, but the boundary map was difficult to define combinatorially for arbitrary dimensions, making it impossible to define the degenerate subcomplex in general. Nevertheless, in the special case of biquandles, reduced 2–cocycles were defined which allowed enhancement of the biquandle counting invariant, generalizing the CJKLS cocycle invariants.

In this paper, we give a reformulation of the birack structure in terms of actions of a set by an augmentation group generalizing the augmented quandle and augmented rack structures defined in [11] and [8]. Our reformulation also restores the original approach taken in [9] of using the “sideways operations” as the primary operations, as opposed to the more usual approach of using the “direct operations” as primary. This approach enables us to define a (co)homology theory for biracks with a fully combinatorial formula for the boundary map, which we are able to employ to identify the degenerate subcomplex associated to \( N \)-phone cord moves for arbitrary biracks of finite characteristic, generalizing the previous cases of quandle,
The paper is organized as follows. In Section 2 we introduce augmented biracks. In Section 3 we define augmented birack (co)homology and discuss relationships with previously studied (co)homology theories. Section 4 deals with augmented birack cocycles and enhancement of the counting invariant. In Section 5 we give some examples of the new cocycle invariants and their computation. We end in Section 6 with some questions for future research.

2 Augmented Biracks

Definition 1 Let $X$ be a set and $G$ be a subgroup of the group of bijections $g : X \to X$. An augmented birack structure on $(X, G)$ consists of maps $\alpha, \beta, \pi, \overline{\alpha}, \overline{\beta} : X \to G$ (i.e., for each $x \in X$ we have bijections $\alpha_x : X \to X$, $\beta_x : X \to X$, $\overline{\alpha}_x : X \to X$ and $\overline{\beta}_x : X \to X$) and a distinguished element $\pi \in G$ satisfying

(i) For all $x \in X$, we have
\[ \alpha_{\pi(x)}(x) = \beta_x \pi(x) \quad \text{and} \quad \overline{\beta}_{\pi(x)}(x) = \overline{\alpha}_x \pi(x), \]
(ii) For all $x, y \in X$ we have
\[ \overline{\alpha}_{\beta_x(y)}(\alpha_y(x)) = x, \quad \overline{\beta}_{\alpha_x(y)}(\beta_y(x)) = x, \quad \alpha_{\overline{\beta}_x(y)}(\overline{\alpha}_y(x)) = x, \quad \text{and} \quad \beta_{\overline{\alpha}_x(y)}(\overline{\beta}_y(x)) = x, \]
and
(iii) For all $x, y \in X$, we have
\[ \alpha_{\alpha_x(y)} \alpha_x = \alpha_{\beta_y(x)} \alpha_y, \quad \beta_{\alpha_x(y)} \alpha_x = \alpha_{\beta_y(x)} \beta_y, \quad \text{and} \quad \beta_{\alpha_x(y)} \beta_x = \beta_{\beta_y(x)} \beta_y. \]

Remark 1 Alternatively, in definition 1 we could let $G$ be an arbitrary group with an action $\cdot : G \times X \to X$ and maps $\alpha, \beta, \pi, \overline{\alpha}, \overline{\beta} : X \to G$ satisfying the listed conditions where $g(y)$ means $g \cdot y$.

Example 1 Let $\tilde{\Lambda} = \mathbb{Z}[t^{\pm 1}, s, r^{\pm 1}] / (s^2 - (1 - t^{-1}r)s)$, let $X$ be a $\tilde{\Lambda}$-module and let $G$ be the group of invertible linear transformations of $X$. Then $(G, X)$ is an augmented biquandle with
\[ \alpha_x(y) = ry, \quad \beta_y(x) = tx - tsy, \quad \overline{\alpha}_x(r) = r^{-1}x, \quad \overline{\beta}_x(y) = sr^{-1}x + t^{-1}y, \quad \text{and} \quad \pi(x) = (t^{-1}r + s)x. \]

For example, we have
\[ \beta_{\alpha_x(y)} \alpha_x(y) = \beta_{r} (ry) = tr y - ts rz = r(ty - ts z) = \alpha_{\beta_y(x)} \beta_x(y). \]

An augmented birack of this type is known as a $(t, s, r)$-birack.

Example 2 We can define an augmented birack structure symbolically on the finite set $X = \{1, 2, 3, \ldots, n\}$ by explicitly listing the maps $\alpha_x, \beta_x : X \to X$ for each $x \in X$. This is conveniently done by giving a $2n \times n$ matrix whose upper block has $(i, j)$ entry $\alpha_j(i)$ and whose lower block has $(i, j)$ entry $\beta_j(i)$, which we might denote by $M_{G, X} = \begin{bmatrix} \alpha_j(i) \\ \beta_j(i) \end{bmatrix}$. Such a matrix defines an augmented birack with $G$ being the symmetric group $S_n$ provided the maps thus defined satisfy the augmented birack axioms; note that if the axioms are satisfied, then the maps $\pi, \overline{\alpha}$ and $\overline{\beta}$ are determined by the maps $\alpha_x, \beta_x$. For example, the matrix
\[ M_{G, X} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \]
encodes the $(t, s, r)$-birack structure on $X = \{1, 2, 3\} = \mathbb{Z}_3$ with $t = 1, s = 2, r = 2$. 

\[ 2 \]
An augmented birack defines a birack map $B : X \times X \to X \times X$ as defined in previous work by setting

$$B(x, y) = (\beta^{-1}_x(y), \alpha^{-1}_{\beta^{-1}_x(y)}(x)).$$

The $G$-actions $\alpha_x, \beta_y$ are the components of the sideways map in the notation of previous papers.

The geometric motivation for augmented biracks come from labeling semi-arcs in an oriented framed link diagram with elements of $X$; each Reidemeister move yields a set of necessary and sufficient conditions for labelings before and after the move to correspond bijectively.

The names are chosen so that if we orient a crossing, positive or negative, with the strands oriented upward, then the unbarred actions go left-to-right and the barred actions go right-to-left, with $\alpha$ and $\beta$ standing for “above” and “below”\footnote{Thanks to Scott Carter for this observation!}. Thus, $\alpha_x(y)$ is the result of $y$ going above $x$ left-to-right and $\beta_y(x)$ is the result of $x$ going below $y$ from right-to-left.

The element $\pi \in G$ is the \textit{kink map} which encodes the change of semiarc labels when going through a positive kink.

In particular, each $(G, X)$-labeling of a framed oriented knot or link diagram before a framed type I move corresponds to a unique $(G, X)$-labeling after the move. If $\pi = 1$ is the identity element in $G$, our augmented birack is an \textit{augmented biquandle}; labelings of an oriented link by an augmented biquandle are independent of framing.

Axiom (ii) is equivalent to the condition that the map $S : X \times X \to X \times X$ defined by

$$S(x, y) = (\alpha_x(y), \beta_y(x))$$

is a bijection with inverse

$$S^{-1}(y, x) = (\beta^{-1}_y(x), \alpha^{-1}_{\beta^{-1}_y(x)}(y)).$$

Note that the condition that the components of $S$ are bijective is not sufficient to make $S$ bijective; for instance, if $X$ is any abelian group, the map $S(x, y) = (\alpha_x(y), \beta_y(x)) = (x + y, x + y)$ has bijective component maps but is not bijective as a map of pairs. The maps $\alpha_x, \beta_y$ are the components of the inverse of the sideways map; we can interpret them as labeling rules going right to left. At negatively oriented crossings, the top and bottom labels are switched.
Note that \((G,X)\)-labelings of a framed knot or link correpond bijectively before and after both forms of type II moves: \textit{direct type II} moves where both strands are oriented in the same direction and \textit{reverse type II} moves in which the strands are oriented in opposite directions.

Axiom (iii) encodes the conditions arising from the Reidemeister III move:

Thus by construction we have

\textbf{Theorem 1} If \(L\) and \(L'\) are oriented framed links related by oriented framed Reidemeister moves and \((G,X)\) is an augmented birack, then there is a bijection between the set of labelings of \(L\) by \((G,X)\), denoted \(\mathcal{L}(L,(G,X))\), and the set of labelings of \(L'\) by \((G,X)\), denoted \(\mathcal{L}(L',(G,X))\).

\textbf{Remark 2} Augmented biracks include several previously studies algebraic structures as special cases.

- As mentioned above, an augmented birack is a birack with birack map
  \[ B(a,b) = (a^b, b_a) = (\beta_a^{-1}(b), \alpha_{\beta_a^{-1}(b)}(a)) \]
  and is a \textit{biquandle} if \(\pi = \text{Id} : X \to X\),

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• An augmented birack in which \( \alpha_x = \text{Id} : X \to X \) for all \( X \) is an augmented rack with augmentation group \( G \) and augmentation map \( \delta(y) = \beta_y^{-1} \), as well as a rack with rack operation \( x \triangleright y = \beta_y^{-1}(x) \).

• An augmented birack in which \( \alpha_x = \text{Id} : X \to X \) for all \( X \) and \( \pi = \text{Id} : X \to X \) is an augmented quandle with augmentation group \( G \) and augmentation map \( \delta(y) = \beta_y^{-1} \), as well as a quandle with rack operation \( x \triangleright y = \beta_y^{-1}(x) \).

Let us now consider the case when \( X \) is a finite set. For any framed oriented link \( L \) with \( n \) crossings, there are at most \( |X|^{2n} \) possible \((G, X)\)-labelings of \( L \), so \( \mathcal{L}(L, (G, X)) \) is a positive integer-valued invariant of framed oriented links. More generally, if we choose an ordering on the components of a \( c \)-component link \( L \), then framings on \( L \) correspond to elements \( \vec{w} \in \mathbb{Z}^c \) and we have a \( c \)-dimensional integral lattice of framings of \( L \).

If \( X \) is a finite set, then \( G \) is a subgroup of the symmetric group \( S_{|X|} \); in particular there is a unique smallest positive integer \( N \) such that \( \pi^N = 1 \in G \). This \( N \) is called the characteristic or birack rank of the augmented birack \((G, X)\). The value of \( \mathcal{L}(L, (G, X)) \) is unchanged by \( N \)-phone cord moves:

In particular, framed oriented links which are equivalent by framed oriented Reidemeister moves and \( N \)-phone cord moves have the same \( \mathcal{L}(L, (G, X)) \)-values, and links which differ only by framing with framing vectors equivalent mod \( N \) have the same \( \mathcal{L}(L, (G, X)) \)-values. Hence, the \( c \)-dimensional lattice of values of \( \mathcal{L}(L, (G, X)) \) is tiled with a \( c \)-dimensional tile of side length \( N \). We can thus obtain an invariant of the unframed link by summing the \( \mathcal{L}(L, (G, X)) \)-values over a single tile.

**Definition 2** Let \( L \) be a link of \( c \) components and \((G, X)\) a finite augmented birack. Then the integral augmented birack counting invariant of \( L \) is the sum over one tile of framings mod \( N \) of the numbers of \((G, X)\)-labelings of \( L \). That is,

\[
\Phi_{(G, X)}^Z(L) = \sum_{\vec{w} \in (\mathbb{Z}_N)^c} \mathcal{L}(L_{\vec{w}}, (G, X)).
\]

where \( L_{\vec{w}} \) is \( L \) with framing vector \( \vec{w} \).

### 3 Augmented Birack Homology

Let \((X, G)\) be an augmented birack. Let \( C_n = \mathbb{Z}[X^n] \) be the free abelian group generated by ordered \( n \)-tuples of elements of \( X \) and let \( C^n(X) = \{f : C_n \to \mathbb{Z} \mid f \text{ is } \mathbb{Z} \text{-linear transformation}\} \). For \( k = 1, 2, \ldots, n \), define maps \( \partial_k', \partial_k'' : C_n(X) \to C_{n-1}(X) \) by

\[
\partial_k'(x_1, \ldots, x_n) = (x_1, \ldots, \widehat{x_k}, \ldots, x_n)
\]
and
\[ \partial_k'(x_1, \ldots, x_n) = (\beta x_k(x_1), \ldots, \beta x_k(x_{k-1}), \hat{x}_k, \alpha x_k(x_{k+1}), \ldots, \alpha x_k(x_n)) \]
where the \( \hat\) indicates that the entry is deleted, i.e.
\[ (x_1, \ldots, \hat{x}_k, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \]

**Theorem 2** The map \( \partial_n : C_n(X) \to C_{n-1}(X) \) given by
\[ \partial_n(x) = \sum_{k=1}^{n} (-1)^k (\partial_k'(x) - \partial_k''(x)) \]
is a boundary map; the map \( \delta^n : C^n(X) \to C^{n+1}(X) \) given by \( \delta^n(f) = f \partial_{n+1} \) is the corresponding coboundary map. The quotient group \( H_n(X) = \text{Ker} \partial_n / \text{Im} \partial_{n+1} \) is the \( n \)th augmented birack homology of \( (X, G) \), and the quotient group \( H^n(X) = \text{Ker} \delta^n / \text{Im} \delta^{n-1} \) is the \( n \)th augmented birack cohomology of \( (X, G) \).

To prove theorem 2, we will find it convenient to first prove a few key lemmas.

**Lemma 3** Let \( j < k \). Then \( \partial_j' \partial_k'(\vec{x}) = \partial_{k-1}'' \partial_j' \).

**Proof.** We compute
\[ \partial_j' \partial_k'(\vec{x}) = \partial_j'(x_1, \ldots, \hat{x}_k, \ldots, x_n) = \partial_j'(x_1, \ldots, \hat{x}_j, \ldots, x_n) \]
obtaining the input vector with the entries in the \( j \)th and \( k \)th positions deleted. On the other hand, if we first delete the \( j \)th entry, each entry with subscript greater than \( j \) is now shifted into one lower position; in particular, \( x_k \) is now in the \((k-1)\)st position and we have
\[ \partial_{k-1}'' \partial_j'(\vec{x}) = \partial_j'(x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_n) \]
as required. □

**Corollary 4** The map \( \partial' : C_n \to C_{n-1} \) defined by \( \sum_{k=1}^{n} (-1)^k \partial'(\vec{x}) \) is a boundary map.

**Proof.** If we apply \( \partial' \) twice, each term with first summation index less than the second summation index is matched by an equal term with first summation index greater than the second summation index but of opposite sign:
\[
\partial'(\partial'(\vec{x})) = \sum_{j<k} (-1)^{j+k} \partial_j' \partial_k'(\vec{x}) + \sum_{j>k} (-1)^{j+k} \partial_j' \partial_k'(\vec{x}) \\
= \sum_{j>k} (-1)^{j+k+1} \partial_j' \partial_k'(\vec{x}) + \sum_{j<k} (-1)^{j+k} \partial_j' \partial_k'(\vec{x}) \\
= 0.
\]
□
Lemma 5 If \( j < k \) we have \( \partial^j_1 \partial^j_k(\bar{x}) = \partial^j_{k-1} \partial^j_1(\bar{x}) \).

Proof. On the one hand,

\[
\partial^j_1 \partial^j_k(\bar{x}) = \partial^j_1(\beta_{x_1}(x_1), \ldots, \beta_{x_k}(x_{k-1}), \bar{x}_k, \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n))
\]

\[
= (\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{j-1}), \bar{x}_j, \beta_{x_k}(x_{j+1}), \ldots, \beta_{x_k}(x_{k-1}), \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n))
\]

On the other hand, applying \( \partial^j_1 \) first shifts \( x_k \) into the \( (k-1) \) position and we have

\[
\partial^j_{k+1} \partial^j_1(\bar{x}) = \partial^j_{k+1}(x_1, \ldots, \bar{x}_j, \ldots, x_n)
\]

\[
= (\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{j-1}), \bar{x}_j, \beta_{x_k}(x_{j+1}), \ldots, \beta_{x_k}(x_{k-1}), \bar{x}_k, \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n))
\]

as required.

Lemma 6 If \( j < k \) we have \( \partial^j_1 \partial^j_k(\bar{x}) = \partial^j_{k-1} \partial^j_1(\bar{x}) \).

Proof. On the one hand,

\[
\partial^j_1 \partial^j_k(\bar{x}) = \partial^j_1(x_1, \ldots, \bar{x}_k, \ldots, x_n)
\]

\[
= (\beta_{x_j}(x_1), \ldots, \beta_{x_j}(x_{j-1}), \bar{x}_j, \beta_{x_j}(x_{j+1}), \ldots, \beta_{x_j}(x_{k-1}), \bar{x}_k, \alpha_{x_j}(x_{k+1}), \ldots, \alpha_{x_j}(x_n))
\]

As above, applying \( \partial^j_{k-1} \) shifts the positions of the entries with indices greater than \( j \), and we have

\[
\partial^j_{k-1} \partial^j_1(\bar{x}) = \partial^j_{k-1}(\beta_{x_j}(x_1), \ldots, \beta_{x_j}(x_{j-1}), \bar{x}_j, \alpha_{x_j}(x_{j+1}), \ldots, \alpha_{x_j}(x_n))
\]

\[
= (\beta_{x_j}(x_1), \ldots, \beta_{x_j}(x_{j-1}), \bar{x}_j, \alpha_{x_j}(x_{j+1}), \ldots, \alpha_{x_j}(x_{k-1}), \bar{x}_k, \alpha_{x_j}(x_{k+1}), \ldots, \alpha_{x_j}(x_n))
\]

as required.

The final lemma depends on the augmented birack axioms.

Lemma 7 If \( j < k \) we have \( \partial^j_1 \partial^j_k(\bar{x}) = \partial^j_{k-1} \partial^j_1(\bar{x}) \).

Proof. We have

\[
\partial^j_1 \partial^j_k(\bar{x}) = \partial^j_1(\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{k-1}), \bar{x}_k, \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n))
\]

\[
= (\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{j-1}), \bar{x}_j, \beta_{x_k}(x_{j+1}), \ldots, \beta_{x_k}(x_{k-1}), \bar{x}_k, \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n))
\]

while again applying \( \partial^j_1 \) first shifts the positions of the entries with indices greater than \( j \), and we have

\[
\partial^j_{k-1} \partial^j_1(\bar{x}) = \partial^j_{k-1}(\beta_{x_j}(x_1), \ldots, \beta_{x_j}(x_{j-1}), \bar{x}_j, \alpha_{x_j}(x_{j+1}), \ldots, \alpha_{x_j}(x_n))
\]

\[
= (\beta_{x_j}(x_1), \ldots, \beta_{x_j}(x_{j-1}), \bar{x}_j, \beta_{x_j}(x_{j+1}), \ldots, \beta_{x_j}(x_{k-1}), \bar{x}_k, \alpha_{x_j}(x_{k+1}), \ldots, \alpha_{x_j}(x_n))
\]

and the two are equal after application of the augmented birack axioms.
Corollary 8 The map $\partial'' : C_n \to C_{n-1}$ defined by $\partial''(\vec{x}) = \sum_{k=1}^{n} (-1)^k \partial_k''(\vec{x})$ is a boundary map.

Proof. As with $\partial'$, we observe that every term in $\partial''_{n-1} \partial''_n(\vec{x})$ with $j < k$ is matched by an equal term with $j > k$ but with opposite sign. □

Remark 3 Corollary 8 shows that the conditions in augmented birack axiom (iii) are precisely the conditions required to make $\partial''$ a boundary map. This provides a non-knot theoretic alternative motivation for the augmented birack structure.

Proof. (of theorem) We must check that $\partial_{n-1} \partial_n = 0$. Our lemmas show that each term in the sum with $j < k$ is matched by an equal term with opposite sign with $j > k$. We have

$$\partial_{n-1}(\partial_n(x_1, \ldots, x_n)) = \sum_{k=0}^{n-1} (-1)^k (\partial_k'(\vec{x}) - \partial_k''(\vec{x}))$$

$$= \sum_{j=0}^{n-1} \sum_{k=0}^{n} (-1)^{k+j} (\partial_j' \partial_k'(\vec{x}) - \partial_j'' \partial_k''(\vec{x}))$$

$$= \sum_{j<k} (-1)^{k+j} (\partial_j' \partial_k'(\vec{x}) - \partial_j'' \partial_k''(\vec{x}))$$

$$+ \sum_{j>k} (-1)^{k+j} (\partial_j' \partial_k'(\vec{x}) - \partial_j'' \partial_k''(\vec{x}))$$

$$= \sum_{j<k} (-1)^{k+j} (\partial_j' \partial_k'(\vec{x}) - \partial_j'' \partial_k''(\vec{x}))$$

$$+ \sum_{j<k} (-1)^{k+j-1} (\partial_j' \partial_k'(\vec{x}) - \partial_j'' \partial_k''(\vec{x}))$$

$$= 0.$$

Definition 3 Let $(G, X)$ be an augmented birack of characteristic $N$. Say that an element $\vec{v}$ of $C_n(X)$ is $N$-degenerate if $\vec{v}$ is a linear combination of elements of the form

$$\sum_{k=1}^{N} (x_1, \ldots, x_{j-1}, \pi^k(x_j), \pi^{k-1}(x_j), x_{j+2}, \ldots, x_n).$$

Denote the set of $N$-degenerate $n$-chains and $n$-cochains as $C_n^D(X)$ and $C^n_D(X)$ and the homology and cohomology groups, $H_n^D$ and $H^n_D$.

Theorem 9 The sets of $N$-degenerate chains form a subcomplex of $(C_n, \partial)$.

Proof. We must show that $\vec{v} \in C_n^D(X)$ implies $\partial(\vec{v}) \in C_{n-1}^D(X)$. Using linearity it is enough to prove that

$$\partial \left( \sum_{k=1}^{N} (x_1, \ldots, x_{j-1}, \pi^k(x_j), \pi^{k-1}(x_j), x_{j+2}, \ldots, x_n) \right)$$

is $N$-degenerate. Let $\vec{u} = \sum_{k=1}^{N} (x_1, \ldots, x_{j-1}, \pi^k(x_j), \pi^{k-1}(x_j), x_{j+2}, \ldots, x_n)$, we have:
\[ \partial(u) = \partial \left[ \sum_{k=1}^{N} (x_1, \ldots, x_{j-1}, \pi^k(x_j), \pi^{k-1}(x_j), x_{j+2}, \ldots, x_n) \right] \]

\[ = \sum_{k=1}^{N} \partial(x_1, \ldots, x_{j-1}, \pi^k(x_j), \pi^{k-1}(x_j), x_{j+2}, \ldots, x_n) \]

\[ = \sum_{k=1}^{N} \left\{ \sum_{i=1}^{j-1} (-1)^i [(x_1, \ldots, \hat{x}_i, \ldots, \pi^k(x_j), \pi^{k-1}(x_j), x_{j+2}, \ldots, x_n) \]

\[ - \{\beta_{\pi^k(x_j)}(x_1), \ldots, \beta_{\pi^k(x_j)}(x_{i-1}), \alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)), \alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)), \ldots, \alpha_{\pi^k(x_j)}(x_n)\} \]

\[ + \{(-1)^j [(x_1, \ldots, x_{j-1}, \pi^k(x_j), x_{j+2}, \ldots, x_n) \]

\[ - \{\beta_{\pi^{k-1}(x_j)}(x_1), \ldots, \beta_{\pi^{k-1}(x_j)}(x_{j-1}), \alpha_{\pi^{k-1}(x_j)}(\pi^k(x_j)), \alpha_{\pi^{k-1}(x_j)}(\pi^k(x_j)), \ldots, \alpha_{\pi^{k-1}(x_j)}(x_n)\} \}

\[ + \sum_{k=1}^{N} \left\{ \sum_{i=j+2}^{n} (-1)^i [(x_1, \ldots, \pi^k(x_j), \pi^{k-1}(x_j), \ldots, \hat{x}_i, \ldots, x_n) \]

\[ - \{\beta_{\pi^k(x_j)}(x_1), \ldots, \beta_{\pi^k(x_j)}(x_{i-1}), \beta_{\pi^k(x_j)}(\pi^{k-1}(x_j)), \beta_{\pi^k(x_j)}(\pi^{k-1}(x_j)), \ldots, \alpha_{\pi^k(x_j)}(x_n)\} \} \right\} \]

where as usual \((x_1, \ldots, \hat{x}_i, \ldots, x_n)\) means \((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\). Now the rest of the proof is based on the following two facts: \((1)\) \(\pi^N = 1\) and \((2)\) \(\alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)) = \beta_{\pi^{k-1}(x_j)}(\pi^k(x_j))\) which is obtained by induction from axiom (i) in the definition (1) of augmented birack.

The following sum vanishes:

\[ \sum_{k=1}^{N} \left\{ [(x_1, \ldots, x_{j-1}, \pi^k(x_j), x_{j+2}, \ldots, x_n) \]

\[ - \{\beta_{\pi^k(x_j)}(x_1), \ldots, \beta_{\pi^k(x_j)}(x_{j-1}), \alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)), \alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)), \ldots, \alpha_{\pi^k(x_j)}(x_n)\} \]

\[ - [(x_1, \ldots, x_{j-1}, \pi^k(x_j), x_{j+2}, \ldots, x_n) \]

\[ - \{\beta_{\pi^{k-1}(x_j)}(x_1), \ldots, \beta_{\pi^{k-1}(x_j)}(x_{j-1}), \alpha_{\pi^{k-1}(x_j)}(\pi^k(x_j)), \alpha_{\pi^{k-1}(x_j)}(\pi^k(x_j)), \ldots, \alpha_{\pi^{k-1}(x_j)}(x_n)\} \} \]

because \(\alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)) = \beta_{\pi^{k-1}(x_j)}(\pi^k(x_j))\) and \(\pi^N = 1\). The rest of the sums can be written as combination of degenerate elements as in the proof of theorem 2 in [4] by the authors.

**Definition 4** The quotient groups \(H_n^{NR}(X) = H_n(X)/H_n^{B}(X)\) and \(H_n^{NR}(X) = H_n(X)/H_n^{D}(X)\) are the \(N\)-Reduced Birack Homology and \(N\)-Reduced Birack Cohomology groups.

### 4 Augmented Birack Cocycle Invariants

In this section we will use augmented birack cocycles to enhance the augmented birack counting invariant analogously to previous work.
Let $L_{\vec{w}}$ be an oriented framed link diagram with framing vector $\vec{w}$ and a labeling $f \in \mathcal{L}(L_{\vec{w}}, (G, X))$ by an augmented birack $(G, X)$ of characteristic $N$. For a choice of $\phi \in H^2_{NR}$, we define an integer-valued signature of the labeling called a **Boltzmann weight** by adding contributions from each crossing as pictured below. Orienting the crossing so that both strands are oriented upward, each crossing contributes $\phi$ evaluated on the pair of labels on the left side of the crossing with the understrand label listed first.

Then as we can easily verify, the Boltzmann weight $BW(f) = \sum_{\text{crossings}} \pm \phi(x, y)$ is unchanged by framed oriented Reidemeister moves and $N$-phone cord moves. Starting with move III, note that $\phi \in H^2(x)$ implies that

\[
(\delta^2 \phi)(x, y, z) = \phi(\partial_2(x, y, z)) \\
= \phi((y, z) - (\alpha_x(y), \alpha_x(z)) - (x, z) + (\beta_y(x), \alpha_y(z)) + (x, y) - (\beta_z(x), \beta_z(y))) \\
= \phi(y, z) - \phi(\alpha_x(y), \alpha_x(z)) - \phi(x, z) + \phi(\beta_y(x), \alpha_y(z)) + \phi(x, y) - \phi(\beta_z(x), \beta_z(y)) \\
= 0
\]

and in particular we have

\[
\phi(y, z) + \phi(\beta_y(x), \alpha_y(z)) + \phi(x, y) = \phi(\alpha_x(y), \alpha_x(z)) + \phi(x, z) + \phi(\beta_z(x), \beta_z(y)).
\]

Then both sides of the Reidemeister III move contribute the same amount to the Boltzmann weight:
Both sides of both type II moves contribute 0 to the Boltzmann weight:

\[
\phi(x, y) - \phi(x, y) = 0
\]

Similarly, both sides of the framed type I moves contribute zero; here we use the alternate form of the framed type I move for clarity, with \( y = \alpha_x \pi(x) = \beta_{\pi(x)}(x) \):

\[
\phi(y, y) - \phi(y, y) = 0 = \phi(x, \pi(x)) - \phi(x, \pi(x))
\]
Finally, the $N$-phone cord move contributes a degenerate $N$-chain:

Putting it all together, we have our main result:

**Theorem 10** Let $L$ be an oriented unframed link of $c$ components and $(G, X)$ a finite augmented birack of characteristic $N$. For each $\phi \in H^{2}_{NR}(X)$, the multiset $\Phi^{M}_{\phi}(L)$ and polynomial $\Phi^{\phi}(L)$ defined by

$$\Phi^{M}_{\phi}(L) = \{BW(f) \mid f \in \mathcal{L}(L_{\vec{w}}, (G, X)), \vec{w} \in (\mathbb{Z}_{N})^{c}\}$$

and

$$\Phi^{\phi}(L) = \sum_{\vec{w} \in (\mathbb{Z}_{N})^{c}} \left( \sum_{f \in \mathcal{L}(L_{\vec{w}}, (G, X))} u^{BW(f)} \right)$$

are invariants of $L$ known as the augmented birack 2-cocycle invariants of $L$.

**Remark 4** We note that if $\phi \in H^{2}(X)$ then the corresponding quantities,

$$\Phi^{M}_{\phi}(L_{\vec{w}}) = \{BW(f) \mid f \in \mathcal{L}(L_{\vec{w}}, (G, X))\} \quad \text{and} \quad \Phi^{\phi}(L_{\vec{w}}) = \sum_{f \in \mathcal{L}(L_{\vec{w}}, (G, X))} u^{BW(f)}$$

are invariants of $L_{\vec{w}}$ as a framed link.

**Remark 5** If $L$ is a virtual link, $\Phi^{M}_{\phi}(L)$ and $\Phi^{\phi}$ are invariants of $L$ under virtual isotopy via the usual convention of ignoring the virtual crossings.

As in quandle homology, we have

**Theorem 11** Let $(G, X)$ be an augmented birack. If $\phi \in H^{2}(X)$ is a coboundary, then for any $(G, X)$-labeling $f$ of a framed link $L_{\vec{w}}$ the Boltzmann weight $BW(f) = 0$.

**Proof.** If $\phi \in H^{2}(X)$ is a coboundary, then there is a map $\psi \in H^{1}$ such that $\psi = \delta^{2} \phi = (\phi \delta_{2})$. Then for any $(x, y)$ we have

$$\phi(x, y) = \psi(\delta_{2}(x, y)) = \psi(y) - \psi(\alpha_{x}(y)) - \psi(x) + \psi(\beta_{y}(x))$$

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and the Boltzmann weight can be pictured at a crossing as below.

\[ \begin{array}{c}
\downarrow \quad -\psi(x) \\
\downarrow \quad +\psi(y)
\end{array} \quad \begin{array}{c}
\downarrow \quad -\psi(\alpha_x(y)) \\
\downarrow \quad +\psi(\beta_y(x))
\end{array} \]

In particular, every semiarc labeled \( x \) contributes a \( +\psi(x) \) at its tail and a \( -\psi(x) \) at its head, so each semiarc contributes zero to the Boltzmann weight.

\[ \square \]

**Corollary 12** Cohomologous cocycles define the same \( \Phi_\phi(L) \) and \( \Phi_\phi^M(L) \) invariants.

### 5 Examples

In this section we collect a few examples of the augmented birack cocycle invariants and their computation.

**Example 3** Let \( X = \{1, 2, 3, 4\} \) be the set of four elements and \( G = S_4 \) the group of permutations of \( X \). The pair \((G, X)\) has augmented birack structures including

\[
M_{(G,X)} = \begin{bmatrix}
2 & 3 & 3 & 2 \\
4 & 1 & 1 & 4 \\
1 & 4 & 4 & 1 \\
3 & 2 & 2 & 3 \\
3 & 2 & 2 & 3 \\
1 & 4 & 4 & 1 \\
4 & 1 & 1 & 4 \\
2 & 3 & 3 & 2
\end{bmatrix}.
\]

This augmented birack has kink map \( \pi = (14)(23) \) and hence characteristic \( N = 2 \). Thus, to find a complete tile of labelings of a link \( L \), we’ll need to consider diagrams of \( L \) with framing vectors \( \vec{w} \in (\mathbb{Z}_2)^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \). Then for instance the Hopf link \( L = L2a1 \) has no labelings in framings \((0, 0)\), \((1, 0)\) and \((0, 1)\) and sixteen labelings in framing \((1, 1)\), for a counting invariant value of \( \Phi_{(G,X)}^\mathbb{Z}(L2a1) = 16 + 0 + 0 + 0 = 16 \).
Note that $C^2(X)$ has $\mathbb{Z}$-basis $\{\chi_{ij} \mid i, j = 1, 2, 3, 4, 5\}$ where

$$\chi_{(i,j)}((i', j')) = \begin{cases} 1 & (i, j) = (i', j') \\ 0 & (i, j) \neq (i', j') \end{cases}.$$ 

The function $\phi : X \times X \to \mathbb{Z}$ defined by

$$\phi = \chi_{(2,1)} + \chi_{(2,4)} + \chi_{(3,1)} + \chi_{(3,4)}$$

is an $N$-reduced 2-cocycle in $H^2_{NR}(G, X)$. We then compute $\Phi_\phi(L)$ by finding the Boltzmann weight for each labeling.

In the labeling on the left, we have

$$BW(f) = \phi(1, 1) + \phi(1, 1) + \phi(3, 2) + \phi(3, 2) = 0 + 0 + 0 + 0 = 0$$

and in the labeling on the right we have

$$BW(f) = \phi(1, 2) + \phi(2, 1) + \phi(2, 3) + \phi(1, 4) = 0 + 1 + 0 + 0 = 1.$$

Repeating for all 14 other labelings, we get $\Phi_\phi(L_{2a1}) = 8 + 8u$. Similarly the unlink $L_{0a1}$ and $(4, 2)$-torus link $L_{4a1}$ have counting invariant value $\Phi_{\phi}^{Z}(G, X)(L_{0a1}) = \Phi_{\phi}^{Z}(G, X)(L_{4a1}) = 16$ with respect to $(G, X)$ but augmented birack cocycles invariant values $\Phi_\phi(L_{0a1}) = 16$ and $\Phi_\phi(L_{4a1}) = 8 + 8u^2$ respectively.

**Example 4** Now let $X$ be the augmented birack with matrix

$$\begin{bmatrix} 4 & 1 & 4 & 4 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 5 & 4 & 4 & 5 \\ 1 & 5 & 1 & 1 \\ 4 & 1 & 4 & 4 \\ 2 & 3 & 2 & 2 \\ 3 & 2 & 2 & 3 \\ 5 & 4 & 4 & 5 \\ 1 & 5 & 1 & 1 \end{bmatrix}.$$

Augmented birack cocycle invariants are defined for virtual knots by the usual convention of ignoring virtual crossings. Using our Python code available at [http://www.esotericka.org](http://www.esotericka.org), we computed the values of $\Phi_\phi(K)$ for all virtual knots $K$ with up to four crossings as collected in the knot atlas [1] with the cocycle
\( \phi = \chi(1,4) + \chi(1,5) + \chi(5,4); \) the results are collected in the table below.

| \( \Phi_{\phi}(K) \) | \( K \) |
|-----------------|----------------|
| 2 + 3u^{-2}     | 4.1, 4.3, 4.7, 4.25, 4.37, 4.43, 4.48, 4.53, 4.73, 4.81, 4.82, 4.87, 4.89, 4.100 |
| 2 + 3u^{-1}     | 3.2, 3.3, 3.4, 3.5, 3.7, 4.4, 4.5, 4.9, 4.11, 4.15, 4.18, 4.19, 4.20, 4.23, 4.27, 4.29, 4.30, 4.33, 4.34, 4.35, 4.40, 4.42, 4.44, 4.47, 4.52, 4.54, 4.60, 4.61, 4.62, 4.63, 4.65, 4.69, 4.74, 4.78, 4.79, 4.80, 4.83, 4.85, 4.86, 4.91, 4.93, 4.94, 4.96, 4.97, 4.102, 4.106 |
| 5               | 3.1, 3.6, 4.2, 4.6, 4.8, 4.10, 4.12, 4.13, 4.14, 4.16, 4.17, 4.22, 4.24, 4.28, 4.31, 4.32, 4.38, 4.39, 4.41, 4.45, 4.46, 4.49, 4.50, 4.51, 4.55, 4.56, 4.57, 4.58, 4.59, 4.64, 4.66, 4.67, 4.68, 4.70, 4.71, 4.72, 4.75, 4.76, 4.77, 4.84, 4.88, 4.90, 4.92, 4.95, 4.98, 4.99, 4.101, 4.103, 4.104, 4.105, 4.107, 4.108 |
| 2 + 3u          | 2.1, 4.21, 4.26, 4.36 |

Our Python computations, confirmed independently by Maple, show that for this augmented birack and cocycle, \( \Phi_{\phi}(K) = 5 \) for classical knots \( K \) with up to 8 crossings; it seems likely that that this is true for all classical knots \( K \). For classical links \( L \), however, \( \Phi_{\phi}(L) \) is quite nontrivial. Our values for \( \Phi_{\phi}(L) \) for prime classical links with up to 7 crossings as listed in the knot atlas are below.

| \( \Phi_{\phi}(L) \) | \( L \) |
|-----------------|----------------|
| 6u^{-2} + 19    | \( L7a1 \) |
| 12u^{-1} + 41 + 30u + 6u^2 | \( L7a7 \) |
| 25              | \( L5a1, L7a1, L7a3, L7a4, L7a2 \) |
| 125             | \( L6a4 \) |
| 7 + 6u          | \( L2a1, L7a5, L7a6 \) |
| 19 + 6u^2       | \( L8a1, L6a1, L7a2 \) |
| 7 + 6u^3        | \( L6a2, L6a3 \) |
| 29 + 36u + 18u^2 + 6u^3 | \( L6a5, L6a1 \) |

We also note that this example demonstrates that the invariant is sensitive to orientation, as for instance the Hopf link oriented to make both crossings positive has \( \Phi_{\phi} \) value \( 7 + 6u \), while reversing one component yields a \( \Phi_{\phi} \) value of \( 7 + 6u^{-1} \).

### 6 Questions

In this section we collect a few open questions for future research.

In the case of quandle homology, many results are known involving the long exact sequence, the delayed Fibonacci sequence in the dimensions of the homology groups for certain quandles, etc. Which of these results extend to augmented birack homology?

In [5], Yang-Baxter (co)homology was paired with \( S \)-(co)homology to define an enhancement of the virtual biquandle counting invariant. A future paper will consider the relationship between augmented birack (co)homology and \( S \)-cohomology.

Can 3-cocycles in augmented birack homology be used to define invariants of knotted surfaces in \( \mathbb{R}^4 \), analogously to the quandle case?

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