Relatively divisible and relatively flat objects in exact categories: applications

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Abstract
For a Quillen exact category $\mathcal{C}$ endowed with two exact structures $\mathcal{D}$ and $\mathcal{E}$ such that $\mathcal{E} \subseteq \mathcal{D}$, an object $X$ of $\mathcal{C}$ is called $\mathcal{E}$-divisible (respectively $\mathcal{E}$-flat) if every short exact sequence from $\mathcal{D}$ starting (respectively ending) with $X$ belongs to $\mathcal{E}$. We continue our study of relatively divisible and relatively flat objects in Quillen exact categories with applications to finitely accessible additive categories and module categories. We derive consequences for exact structures generated by the simple modules and the modules with zero Jacobson radical.

Keywords
Exact category · Divisible object · Flat object · Cotorsion pair · Finitely accessible additive category · Module category · Pure short exact sequence · Simple module · Jacobson radical

Mathematics Subject Classification 18E10 · 18G50 · 16D90

1 Introduction

Relatively divisible and relatively flat objects in (Quillen) exact categories were introduced and studied in our paper [14]. Let $(\mathcal{C}, \mathcal{D})$ denote an exact category, and let $\mathcal{E} \subseteq \mathcal{D}$ be a coarser exact structure on $\mathcal{C}$. Recall that an object $X$ of $\mathcal{C}$ is called $\mathcal{E}$-divisible (respectively $\mathcal{E}$-flat) if every short exact sequence from $\mathcal{D}$ (called conflation) starting (respectively ending) with $X$ belongs to $\mathcal{E}$. The usual ambient exact structure $\mathcal{D}$ will be that given by the class of all short exact sequences in the category. In module categories there are numerous examples of $\mathcal{E}$-divisible objects: injective, pure-injective, absolutely...
pure (i.e., every short exact sequence starting with it is pure [20, 22]), finitely injective (i.e., every short exact sequence starting with it is finitely split [2]), finitely pure-injective (i.e., every pure short exact sequence starting with it is finitely split [2]), absolutely neat (i.e., every short exact sequence starting with it is neat [13]), absolutely coneat (i.e., every short exact sequence starting with it is coneat [5]), absolutely s-pure (i.e., every short exact sequence starting with it is s-pure [10]), weak injective (i.e., every short exact sequence starting with it is closed [34]) objects etc. Each of the above \( E \)-divisible objects has a corresponding notion of \( E \)-flat object, namely projective, pure-projective, flat, finitely projective [2], finitely pure-projective [2], neat-flat [7], coneat-flat [5], max-flat [31] and weak flat [35] object respectively. For further motivation and background the reader is referred to [14].

In the present paper, which is a companion of [14], we illustrate how our general results in exact categories allow us to obtain in an easy and unified way corresponding results for modules, which have been usually treated separately in the literature. There are some unifications in module categories, but they concern some other aspects, e.g. [12, 33]. We start in Sect. 2 with an application to finitely accessible additive categories, and we prove that an object of a finitely accessible additive category is absolutely pure if and only if its pure-injective envelope is injective. Section 3 contains our main applications to module categories. We consider flatly generated exact structures, and we show and use their relationship with projectively generated ones. We establish several results connecting purity-related notions with relatively divisible and relatively flat properties. We also obtain some characterizations of right IF rings and right FGF rings. Motivated by some results by Fuchs [17], Sect. 4 discusses the case of exact structures projectively, injectively and flatly generated by the class of simple modules, and we deduce and complete a series of known properties. Section 5 deals with exact structures projectively, injectively and flatly generated by the class of modules with zero Jacobson radical, and all results are new.

### 2 Applications to finitely accessible additive categories

We recall some terminology on finitely accessible additive categories, mainly from [25]. An additive category \( \mathcal{C} \) is called finitely accessible if it has direct limits, the class of finitely presented objects is skeletally small, and every object is a direct limit of finitely presented objects. Recall that an object \( C \) of \( \mathcal{C} \) is finitely presented if the representable functor \( \text{Hom}_\mathcal{C}(C, -) : \mathcal{C} \to \text{Ab} \) commutes with direct limits, where \( \text{Ab} \) is the category of abelian groups.

Let \( \mathcal{C} \) be a finitely accessible additive category. By a sequence \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) in \( \mathcal{C} \) we mean a pair of composable morphisms \( f : X \to Y \) and \( g : Y \to Z \) such that \( gf = 0 \). The above sequence in \( \mathcal{C} \) is called pure exact if it induces an exact sequence of abelian groups

\[
0 \to \text{Hom}_\mathcal{C}(P, X) \to \text{Hom}_\mathcal{C}(P, Y) \to \text{Hom}_\mathcal{C}(P, Z) \to 0
\]
for every finitely presented object $P$ of $\mathcal{C}$. This implies that $f$ and $g$ form a kernel-cokernel pair, that $f$ is a monomorphism and $g$ an epimorphism. In such a pure exact sequence $f$ is called a pure monomorphism and $g$ a pure epimorphism.

Any finitely accessible additive category $\mathcal{C}$ may be embedded as a full subcategory of the functor category $(\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})$ of all contravariant additive functors from the full subcategory $\text{fp}(\mathcal{C})$ of finitely presented objects of $\mathcal{C}$ to the category $\text{Ab}$ of abelian groups. In this way the pure exact sequences in $\mathcal{C}$ are those which become exact sequences in $(\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})$ through the embedding, and $\mathcal{C}$ may be seen as being equivalent to the full subcategory of flat objects from $(\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})$. The relevant functor inducing the equivalence is the covariant Yoneda functor $H : \mathcal{C} \to (\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})$, given on objects by $X \mapsto H_X = \text{Hom}_{\mathcal{C}}(-, X) \mid_{\text{fp}(\mathcal{C})}$ and correspondingly on morphisms. Then $H$ preserves and reflects purity. Moreover, the pure exact sequences define an exact structure in any finitely accessible additive category $\mathcal{C}$, because $\mathcal{C}$ is equivalent to an extension closed full subcategory of an abelian category (e.g., see [4, Lemma 10.20]). Note that every category of modules over a ring with identity, and more generally, every functor category is finitely accessible.

Recall some purity-related concepts in finitely accessible additive categories [25] and Grothendieck categories [28, 32].

**Definition 1** Let $\mathcal{C}$ be a finitely accessible additive category. An object $X$ of $\mathcal{C}$ is called:

1. **pure-injective** if it is injective with respect to every pure monomorphism in $\mathcal{C}$.
2. **absolutely pure** if every monomorphism $X \to Y$ in $\mathcal{C}$ is pure.

**Definition 2** Let $\mathcal{G}$ be a Grothendieck category. An object $M$ of $\mathcal{G}$ is called:

1. **flat** if every epimorphism $A \to M$ in $\mathcal{G}$ is pure.
2. **cotorsion** if $\text{Ext}^1_{\mathcal{G}}(A, M) = 0$ for every flat object $A$ of $\mathcal{G}$.
3. **weakly absolutely pure** if every monomorphism $M \to A$ in $\mathcal{G}$ with $A$ flat is pure.

**Remark 3** (1) Absolutely pure objects of a Grothendieck category $\mathcal{G}$ coincide with $\mathcal{E}$-divisible objects, where the ambient structure $\mathcal{D}$ is the exact structure given by all short exact sequences and $\mathcal{E}$ is the exact structure given by all pure exact sequences.

(2) Let $(\mathcal{C}, \mathcal{D})$ be an exact category, let $\mathcal{E} \subseteq \mathcal{D}$ be an exact structure on $\mathcal{C}$, and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$. Recall that an object $X$ of $\mathcal{C}$ is called: $\mathcal{E}$$-\mathcal{A}$-*divisible* if every inflation $X \twoheadrightarrow A$ with $A \in \mathcal{A}$ is an $\mathcal{E}$-inflation, and $\mathcal{E}$$-\mathcal{A}$-*flat* if every deflation $A \twoheadrightarrow X$ with $A \in \mathcal{A}$ is an $\mathcal{E}$-deflation [14]. Weakly absolutely pure objects of a Grothendieck category $\mathcal{G}$ coincide with its $\mathcal{D}$$-\mathcal{E}$$-\mathcal{A}$-divisible objects, where $\mathcal{D}$ is the exact structure given by all short exact sequences, $\mathcal{E}$ is the exact structure given by all pure exact sequences and $\mathcal{A}$ is the class of flat objects.

Now we may deduce the following known characterizations.
Proposition 4 [25, Proposition 5.6] Let \( \mathcal{C} \) be a finitely accessible abelian category. The following are equivalent for an object \( X \) of \( \mathcal{C} \):

(i) \( X \) is absolutely pure.

(ii) The pure-injective envelope of \( X \) is injective.

(iii) \( X \) is a pure subobject of an injective object.

(iv) \( X \) is a pure subobject of an absolutely pure object.

Proof Consider the exact structures on \( \mathcal{C} \) given by the classes \( \mathcal{D} \) of all short exact sequences and \( \mathcal{E} \subseteq \mathcal{D} \) of pure exact sequences, and the perfect cotorsion pair \((\mathcal{A}, \mathcal{B})\) in \( \mathcal{C} \), where \( \mathcal{A} \) is the class of all objects and \( \mathcal{B} \) is the class of pure-injective objects of \( \mathcal{C} \). Then the class of \( \mathcal{E} \)-\( \mathcal{A} \)-divisible objects of \( \mathcal{C} \) coincides with the class of absolutely pure objects. Now use [14, Theorem 4.9] (see also [14, Proposition 4.6]). □

We are interested in showing a similar result in the case of arbitrary finitely accessible additive categories. In the above proposition we have essentially used the fact that all short exact sequences yield an exact structure on any abelian category. Such a result is no longer true if the category is not (at least) quasi-abelian (see [14, Example 2.3]). In order to be able to remove the hypothesis on \( \mathcal{C} \) to be abelian, we need the following proposition on weakly absolutely pure objects in functor categories, which can be immediately deduced from our results.

Proposition 5 Let \( S \) be a small preadditive category. The following are equivalent for an object \( K \) of \( (S^{\text{op}}, \text{Ab}) \):

(i) \( K \) is weakly absolutely pure.

(ii) The cotorsion envelope of \( K \) is weakly absolutely pure.

(iii) There is a pure exact sequence \( 0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0 \) with \( M \) weakly absolutely pure cotorsion and \( N \) flat.

(iv) There is a pure exact sequence \( 0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0 \) with \( M \) weakly absolutely pure and \( N \) flat.

Proof Consider the exact structures on \( (S^{\text{op}}, \text{Ab}) \) given by the classes \( \mathcal{D} \) of all short exact sequences and \( \mathcal{E} \subseteq \mathcal{D} \) of pure exact sequences, and the perfect cotorsion pair \((\mathcal{F}, \mathcal{C})\) in \( (S^{\text{op}}, \text{Ab}) \) [3, Theorem 3] (also see [15, Corollary 3.3]), where \( \mathcal{F} \) is the class of flat objects and \( \mathcal{C} \) is the class of cotorsion objects. Note that \( \mathcal{F} \) coincides with the class of \( \mathcal{E} \)-flat objects in \( (S^{\text{op}}, \text{Ab}) \) and weakly absolutely pure objects are the same as \( \mathcal{E} \)-\( \mathcal{A} \)-divisible objects, where \( \mathcal{A} = \mathcal{F} \). Then use [14, Theorem 4.9]. □

The following theorem relates purity-related properties of objects of a finitely accessible additive category and of objects of the associated functor category. It follows from [9, Theorem 1.4], [18, Lemma 3] and the definition of weakly absolutely pure objects in functor categories.

Theorem 6 Let \( \mathcal{C} \) be a finitely accessible additive category. The equivalence induced by the covariant Yoneda functor \( H : \mathcal{C} \rightarrow (\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab}) \) between \( \mathcal{C} \) and the full
subcategory of \((\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})\) consisting of the flat objects restricts to equivalences between the following full subcategories:

(i) pure-injective objects of \(\mathcal{C}\) and cotorsion flat objects of \((\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})\).

(ii) absolutely pure objects of \(\mathcal{C}\) and weakly absolutely pure flat objects of \((\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})\).

(iii) injective objects of \(\mathcal{C}\) and cotorsion weakly absolutely pure flat objects of \((\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})\).

Now we are in a position to deduce characterizations of absolutely pure objects in finitely accessible additive categories, which generalizes [25, Proposition 5.6] from finitely accessible abelian categories to finitely accessible additive categories. Note that every object of a finitely accessible additive category does have a pure-injective envelope [18, Theorem 6].

**Corollary 7** Let \(\mathcal{C}\) be a finitely accessible additive category. Then the following are equivalent for an object \(X\) of \(\mathcal{C}\):

(i) \(X\) is absolutely pure.

(ii) The pure-injective envelope of \(X\) is injective.

(iii) \(X\) is a pure subobject of an injective object.

(iv) \(X\) is a pure subobject of an absolutely pure object.

**Proof** This follows from Proposition 5 and Theorem 6, using the fact that purity is preserved and reflected by the covariant Yoneda functor \(H : \mathcal{C} \to (\text{fp}(\mathcal{C})^{\text{op}}, \text{Ab})\). \(\square\)

## 3 Applications to module categories

Throughout \(\text{Mod}(R)\) denotes the category of right modules over a ring \(R\) with identity.

**Proposition 8** ([14, Proposition 2.4], [24, Section 4], [27, Proposition 2.1]) Let \((\mathcal{C}, \mathcal{D})\) be an exact category, let \(\mathcal{M}\) be a class of objects of \(\mathcal{C}\), and let \(\text{Ab}\) be the category of abelian groups. For any object \(M\) of \(\mathcal{M}\), consider the additive functors \(H_M = \text{Hom}_\mathcal{C}(M, -)\), \(G_M = \text{Hom}_\mathcal{C}(-, M) : \mathcal{C} \to \text{Ab}\), and denote by \(\mathcal{E}^M_H\) and \(\mathcal{E}^M_G\) respectively the exact structures on \(\mathcal{C}\) whose conflations are the short exact sequences in \(\mathcal{C}\) whose exactness is preserved by \(H_M\) and \(G_M\) respectively. Then

\[
\mathcal{E}^M_H = \bigcap \{\mathcal{E}_{H_M} \mid M \in \mathcal{M}\} \quad \text{and} \quad \mathcal{E}^M_G = \bigcap \{\mathcal{E}_{G_M} \mid M \in \mathcal{M}\}
\]

are exact structures on \(\mathcal{C}\), which are called the exact structures projectively generated by \(\mathcal{M}\) and injectively cogenerated by \(\mathcal{M}\) respectively.

One may also consider another particular generated exact structure on \(\text{Mod}(R)\) (e.g., see [24, Section 4] and [27]), as follows.
Proposition 9 Let $\mathcal{M}$ be a class of right $R$-modules. For any right $R$-module $M$ in $\mathcal{M}$, consider the additive functor $T_M = M \otimes_R - : \text{Mod}(R^{\text{op}}) \to \text{Ab}$, and denote by $\mathcal{E}_T$ the exact structure on $\text{Mod}(R)$ whose conflations are the short exact sequences in $\text{Mod}(R)$ whose exactness is preserved by $T_M$. Then

$$\mathcal{E}^M_T = \bigcap \{ \mathcal{E}_{T_M} \mid M \in \mathcal{M} \}$$

is an exact structure on $\text{Mod}(R^{\text{op}})$, called the exact structure flatly generated by $\mathcal{M}$.

Proposition 10 Let $\mathcal{E}$ be an exact structure on $\text{Mod}(R^{\text{op}})$ such that $\mathcal{E}$ is flatly generated by a class $\mathcal{M}$ of right $R$-modules. Then a left $R$-module $Z$ is $\mathcal{E}$-flat if and only if $\text{Tor}_1^R(M, Z) = 0$.

Proof Consider a short exact sequence $0 \to X \to Y \to Z \to 0$ of left $R$-modules with $Y$ projective. For every $M \in \mathcal{M}$, it induces the exact sequence

$$0 = \text{Tor}_1^R(M, Y) \to \text{Tor}_1^R(M, Z) \to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0$$

Assume first that $Z$ is $\mathcal{E}$-flat. Then $0 \to X \to Y \to Z \to 0$ is an $\mathcal{E}$-conflation. Since $\mathcal{E} = \mathcal{E}^M_T$, it induces a short exact sequence $0 \to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0$ for every $M \in \mathcal{M}$. From the above exact sequence, it follows that $\text{Tor}_1^R(M, Z) = 0$ for every $M \in \mathcal{M}$.

Conversely, assume that $\text{Tor}_1^R(\mathcal{M}, Z) = 0$. Then we have the short exact sequence $0 \to M \otimes_R X \to M \otimes_R Y \to M \otimes_R Z \to 0$ for every $M \in \mathcal{M}$. This shows that the short exact sequence $0 \to X \to Y \to Z \to 0$ is an $\mathcal{E}_T^M$-conflation. This implies that $Z$ is $\mathcal{E}$-flat. \qed

Let us see some of the most important examples of exact structures on module categories.

Example 11 (1) Let $\mathcal{M}$ be the class of finitely presented right $R$-modules.

The $\mathcal{M}$-pure short exact sequences in $\text{Mod}(R)$ are called pure [30, p. 281]. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by short exact sequences and pure short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-$\mathcal{E}$-divisible and $\mathcal{D}$-$\mathcal{E}$-flat objects coincide with absolutely pure and flat right $R$-modules respectively. Note that the exact structure of pure short exact sequences is also injectively generated by the class of pure-injective modules [30, 34.7].

The $\mathcal{M}$-copure short exact sequences in $\text{Mod}(R)$ are called copure (note the different terminology with respect to [30, p. 322], where $\mathcal{M}$ is the class of finitely copresented right $R$-modules). Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by short exact sequences and copure short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-$\mathcal{E}$-divisible and $\mathcal{D}$-$\mathcal{E}$-flat objects will be called absolutely copure and coflat right $R$-modules respectively.

The short exact sequences in $\text{Mod}(R^{\text{op}})$ which give the exact structure flatly generated by the class of finitely presented (or all) right $R$-modules coincide with pure short exact sequences [30, 34.5].
(2) Let $\mathcal{M}$ be the class of finitely generated right $R$-modules.

The $\mathcal{M}$-pure short exact sequences in $\text{Mod}(R)$ are called *finitely split* [2]. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by pure (all) short exact sequences and finitely split short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-cofinitely divisible and $\mathcal{D}$-cofinitely flat objects coincide with finitely pure-injective (finitely injective) and finitely pure-projective (finitely projective) right $R$-modules respectively [2, p. 115].

The $\mathcal{M}$-copure short exact sequences in $\text{Mod}(R)$ will be called *cofinitely split*. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by pure (all) short exact sequences and cofinitely split short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-cofinitely divisible and $\mathcal{D}$-cofinitely flat objects will be called *cofinitely pure-injective* (cofinitely injective) and *cofinitely pure-projective* (cofinitely projective) right $R$-modules respectively.

The short exact sequences in $\text{Mod}(R^{\text{op}})$ which give the exact structure flatly generated by the class of finitely generated (or all) right $R$-modules coincide with pure short exact sequences [30, 34.5].

(3) Let $\mathcal{M}$ be the class of (semi)simple right $R$-modules.

The $\mathcal{M}$-pure short exact sequences in $\text{Mod}(R)$ are called *neat* [17]. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by short exact sequences and neat short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-cofinitely divisible and $\mathcal{D}$-cofinitely flat objects coincide with absolutely neat [13] and neat-flat [6] right $R$-modules respectively. Note that by [13, Theorem 3.4] absolutely neat right $R$-modules coincide with $m$-injective right $R$-modules in the sense of [11], that is, right $R$-modules which are injective with respect to every short exact sequence of the form $0 \to I \to R \to R/I \to 0$ with $I$ a maximal right ideal of $R$.

The $\mathcal{M}$-copure short exact sequences in $\text{Mod}(R)$ are called *coneat* by Fuchs [17]. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by short exact sequences and coneat short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-cofinitely divisible and $\mathcal{D}$-cofinitely flat objects coincide with absolutely coneat [13] and coneat-flat [5] right $R$-modules respectively.

The short exact sequences in $\text{Mod}(R^{\text{op}})$ which give the exact structure flatly generated by $\mathcal{M}$ are called *s-pure* [10]. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by short exact sequences and s-pure short exact sequences in $\text{Mod}(R^{\text{op}})$ respectively. Then $\mathcal{D}$-cofinitely divisible and $\mathcal{D}$-cofinitely flat objects coincide with absolutely s-pure [10] and max-flat [31] left $R$-modules respectively. Note that if $R$ is commutative, then coneat short exact sequences in the sense of Fuchs are the same as s-pure short exact sequences [17, Proposition 3.1].

(4) Let $\mathcal{M}$ be the class of right $R$-modules $M$ with Jacobson radical $\text{Rad}(M) = 0$.

The $\mathcal{M}$-pure short exact sequences in $\text{Mod}(R)$ will be called *rad-neat*. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by short exact sequences and rad-neat short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-cofinitely divisible and $\mathcal{D}$-cofinitely flat objects will be called *absolutely rad-neat* and rad-neat-flat right $R$-modules respectively.

The $\mathcal{M}$-copure short exact sequences in $\text{Mod}(R)$ will be called *rad-coneate*. They were studied under the name of coneat short exact sequences by Mermut [23]. Let $\mathcal{D}$ and $\mathcal{E}$ be the exact structures given by short exact sequences and rad-coneate short exact sequences in $\text{Mod}(R)$ respectively. Then $\mathcal{D}$-cofinitely divisible and $\mathcal{D}$-cofinitely flat objects will be called *absolutely rad-coneate* and rad-coneate-flat right $R$-modules.
respectively. Note that if \( R \) is semilocal, then coneat short exact sequences are the same as rad-coneaut short exact sequences [23, Theorem 3.8.7].

The short exact sequences in \( \text{Mod}(R^{\text{op}}) \) which give the exact structure flatly generated by \( \mathcal{M} \) will be called \textit{rad-pure} in this paper. Let \( \mathcal{D} \) and \( \mathcal{E} \) be the exact structures given by short exact sequences and rad-pure short exact sequences in \( \text{Mod}(R^{\text{op}}) \) respectively. Then \( \mathcal{D} \)-\( \mathcal{E} \)-divisible and \( \mathcal{D} \)-\( \mathcal{E} \)-flat objects will be called \textit{absolutely rad-pure} and \textit{rad-pure-flat} left \( R \)-modules respectively. Note that if \( R \) is commutative semilocal, then coneat (equivalently, rad-coneaut) short exact sequences are the same as rad-pure short exact sequences.

\textbf{Remark 12} Our next results and applications will refer to the above examples of exact structures and to the case when \( \mathcal{D} \) is the exact structure on module categories given by all short exact sequences. We show some properties relating relative divisibility and relative flatness with absolute purity and flatness respectively. But with some exceptions of less known or unknown results, we do not intend to deduce corollaries for the pure exact structure on module categories, which has been extensively studied. We rather insist on exact structures generated by the simple modules and the modules with zero Jacobson radical in the next sections. There are also some other relevant examples of exact structures in module categories, for which the interested reader may obtain consequences of our properties. For instance, complement (or closed), supplement and coclosed submodules induce exact structures on module categories (e.g., see [24, Section 5]).

For a left \( R \)-module \( X \), we denote by \( X^+ = \text{Hom}_\mathbb{Z}(X, \mathbb{Q}/\mathbb{Z}) \) its character module, which is a right \( R \)-module. For a short exact sequence \( E : 0 \to X \to Y \to Z \to 0 \) of left \( R \)-modules, we denote by \( E^+ : 0 \to Z^+ \to Y^+ \to X^+ \to 0 \) the induced short exact sequence of right \( R \)-modules.

The following property shows that we may study flatly generated exact structures in module categories by means of projectively generated exact structures.

\textbf{Theorem 13} [26, Theorem 8.1] Let \( \mathcal{M} \) be a class of right \( R \)-modules and \( E \) a short exact sequence of left \( R \)-modules. Then \( E \in \mathcal{E}^\mathcal{M}_T \) if and only if \( E^+ \in \mathcal{E}^\mathcal{M}_H \).

\textbf{Proposition 14} Let \( \mathcal{M} \) be a class of right \( R \)-modules and \( Z \) a left \( R \)-module. Then \( Z \) is \( \mathcal{E}^\mathcal{M}_T \)-flat if and only if \( Z^+ \) is an \( \mathcal{E}^\mathcal{M}_H \)-divisible right \( R \)-module.

\textbf{Proof} Use [14, Proposition 3.3], Proposition 10 and the fact that \( \text{Ext}^1_R(\mathcal{M}, Z^+) \cong \text{Tor}^R_1(\mathcal{M}, Z)^+ \). \( \square \)

Recall that a right \( R \)-module \( Z \) is called \textit{FP-projective} if \( \text{Ext}^1_R(Z, X) = 0 \) for every absolutely pure right \( R \)-module \( X \) [21].

\textbf{Proposition 15} Let \( \mathcal{E} \) be an exact structure on \( \text{Mod}(R) \).
(1) Assume that \( \mathcal{E} \) is projectively generated by a class \( \mathcal{M} \) of right \( R \)-modules. Then every right \( R \)-module in \( \mathcal{M} \) is FP-projective if and only if every absolutely pure right \( R \)-module is \( \mathcal{E} \)-divisible.

(2) Assume that \( \mathcal{E} \) is injectively generated by a class \( \mathcal{M} \) of right \( R \)-modules. Then every right \( R \)-module in \( \mathcal{M} \) is cotorsion if and only if every flat right \( R \)-module is \( \mathcal{E} \)-flat.

**Proof** (1) Note that \((\mathcal{A}, \mathcal{B})\) is a cotorsion pair in \( \text{Mod}(R) \) for \( \mathcal{A} \) and \( \mathcal{B} \) the classes of FP-projective and absolutely pure right \( R \)-modules respectively [21, Theorem 2.14].

Assume first that every module in \( \mathcal{M} \) is FP-projective. Let \( X \) be an absolutely pure right \( R \)-module. Then \( \text{Ext}_R^1(F, X) = 0 \) for every FP-projective right \( R \)-module \( F \), hence \( \text{Ext}_R^1(M, X) = 0 \) for every \( M \in \mathcal{M} \). Then \( X \) is \( \mathcal{E} \)-divisible by [14, Proposition 3.3].

Conversely, assume that every absolutely pure right \( R \)-module is \( \mathcal{E} \)-divisible. Let \( M \in \mathcal{M} \). By hypothesis and [14, Proposition 3.3], \( \text{Ext}_R^1(M, X) = 0 \) for every absolutely pure right \( R \)-module \( X \). Then \( M \) is FP-projective.

(2) Note that \((\mathcal{A}, \mathcal{B})\) is a cotorsion pair in \( \text{Mod}(R) \) for \( \mathcal{A} \) and \( \mathcal{B} \) the classes of flat and cotorsion right \( R \)-modules respectively [3, Theorem 3], and proceed in a similar way as for (1).

\[\square\]

**Corollary 16** Let \( \mathcal{E} \) be an exact structure on \( \text{Mod}(R) \) projectively generated by a class \( \mathcal{M} \) of finitely generated right \( R \)-modules. Then every right \( R \)-module in \( \mathcal{M} \) is finitely presented if and only if every absolutely pure right \( R \)-module is \( \mathcal{E} \)-divisible.

**Proof** Note that a finitely generated right \( R \)-module is FP-projective if and only if it is finitely presented [16, Proposition].

\[\square\]

**Proposition 17** Let \( \mathcal{E} \) be an exact structure on \( \text{Mod}(R) \) projectively generated by a class \( \mathcal{M} \) of finitely generated right \( R \)-modules. Then the following are equivalent:

(i) Every right \( R \)-module from \( \mathcal{M} \) embeds into a (finitely generated) projective right \( R \)-module.

(ii) Every absolutely pure right \( R \)-module is \( \mathcal{E} \)-flat.

(iii) Every injective right \( R \)-module is \( \mathcal{E} \)-flat.

(iv) The injective envelope of every right \( R \)-module from \( \mathcal{M} \) is \( \mathcal{E} \)-flat.

(v) For every free left \( R \)-module \( Z \), \( Z^+ \) is an \( \mathcal{E} \)-flat right \( R \)-module.

**Proof** (i)\( \Rightarrow \) (ii) Let \( X \) be an absolutely pure right \( R \)-module. Let \( f : M \to E \) be a homomorphism with \( M \in \mathcal{M} \). By hypothesis, \( M \) embeds into some finitely generated projective right \( R \)-module \( P \). Since \( P/M \) is finitely presented, \( f \) can be extended to a homomorphism \( P \to X \). Then \( X \) is \( \mathcal{E} \)-flat by [14, Proposition 4.6].

(ii)\( \Rightarrow \) (iii) and (iii)\( \Rightarrow \) (iv) These are clear.

(iv)\( \Rightarrow \) (i) Let \( M \in \mathcal{M} \). Since the injective envelope \( E(M) \) of \( M \) is \( \mathcal{E} \)-flat, the inclusion homomorphism \( i : M \to E(M) \) factors through a projective right \( R \)-module \( P \) by [14, Proposition 4.6]. It follows that \( M \) embeds into \( P \).
(iii)⇒(v) Let $Z$ be a free left $R$-module. Then $Z^+$ is an injective right $R$-module, hence $Z^+$ is $\mathcal{E}$-flat by hypothesis.

(v)⇒(iii) Let $X$ be an injective right $R$-module. Consider an epimorphism $Y \to X^+$ for some free left $R$-module $Y$. Then $Y^+$ is $\mathcal{E}$-flat by hypothesis and we have an induced monomorphism $X \to X^{++} \to Y^+$. But this splits, since $X$ is injective, hence $X$ is $\mathcal{E}$-flat. \hfill \Box

Recall that a ring $R$ is called a right IF ring if every injective right $R$-module is flat [8].

Corollary 18 [8, Theorem 1] The following are equivalent:

(i) Every finitely presented right $R$-module embeds into a (finitely generated) projective right $R$-module.
(ii) Every absolutely pure right $R$-module is flat.
(iii) $R$ is a right IF ring.
(iv) The injective envelope of every finitely presented right $R$-module is flat.
(v) For every free left $R$-module $Z$, $Z^+$ is a flat right $R$-module.

Proof This follows from Proposition 17, considering the exact structure given by pure short exact sequences of right $R$-modules. \hfill \Box

Recall that a ring $R$ is called a right FGF ring if every finitely generated right $R$-module embeds into a free (projective) right $R$-module.

Corollary 19 The following are equivalent:

(i) $R$ is a right FGF ring.
(ii) Every absolutely pure right $R$-module is finitely projective.
(iii) Every injective right $R$-module is finitely projective.
(iv) The injective envelope of every finitely generated right $R$-module is finitely projective.
(v) For every free left $R$-module $Z$, $Z^+$ is a finitely projective right $R$-module.

Proof This follows from Proposition 17, considering the exact structure given by finitely split short exact sequences of right $R$-modules. \hfill \Box

Corollary 20 Let $\mathcal{E}$ be an exact structure on $\text{Mod}(R)$.

1. Assume that $\mathcal{E}$ is projectively generated by a class $\mathcal{M}$ of right $R$-modules. Then the following are equivalent:
   (i) For every short exact sequence $0 \to X \to Y' \to Z' \to 0$ of right $R$-modules with $X$ absolutely pure (respectively cotorsion) and $Y'$ $\mathcal{E}$-divisible, $Z'$ is $\mathcal{E}$-divisible.
   (ii) For every short exact sequence $0 \to Z \to U \to V \to 0$ of right $R$-modules with $V \in \mathcal{M}$ and $U$ projective, $Z$ is FP-projective (respectively flat).
Relatively divisible and relatively flat objects in exact...

(2) Assume that \( \mathcal{E} \) is injectively generated by a class \( \mathcal{M} \) of right \( R \)-modules. Then the following are equivalent:

(i) For every short exact sequence \( 0 \to Z \to U \to V \to 0 \) of right \( R \)-modules with \( V \) \( FP \)-projective (respectively flat) and \( U \) \( \mathcal{E} \)-flat, \( Z \) is \( \mathcal{E} \)-flat.

(ii) For every short exact sequence \( 0 \to X \to Y' \to Z' \to 0 \) of right \( R \)-modules with \( X \in \mathcal{M} \) and \( Y' \) injective, \( Z' \) is absolutely pure (respectively cotorsion).

**Proof** Use [14, Proposition 4.3] for the cotorsion pair \((\mathcal{A}, \mathcal{B})\), where \( \mathcal{A} \) is the class of \( FP \)-projective (respectively flat) right \( R \)-modules and \( \mathcal{B} \) is the class of absolutely pure (respectively cotorsion) right \( R \)-modules [3, Theorem 3], [21, Theorem 2.14]. \(\square\)

**Corollary 21** Let \( \mathcal{E} \) be an exact structure on \( \text{Mod}(R) \).

(1) Assume that \( \mathcal{E} \) is projectively generated by a class \( \mathcal{M} \) of right \( R \)-modules. Then the following are equivalent:

(i) The class of \( \mathcal{E} \)- divisible right \( R \)-modules is closed under homomorphic images.

(ii) For every short exact sequence \( 0 \to Z \to U \to V \to 0 \) of right \( R \)-modules with \( V \in \mathcal{M} \) and \( U \) projective, \( Z \) is projective.

(2) Assume that \( \mathcal{E} \) is injectively generated by a class \( \mathcal{M} \) of right \( R \)-modules. Then the following are equivalent:

(i) The class of \( \mathcal{E} \)- flat right \( R \)-modules is closed under submodules.

(ii) For every short exact sequence \( 0 \to X \to Y' \to Z' \to 0 \) of right \( R \)-modules with \( X \in \mathcal{M} \) and \( Y' \) injective, \( Z' \) is injective.

**Proof** This follows from [14, Proposition 4.4]. \(\square\)

**Corollary 22**

(i) [29, Theorem 3.2] The class of absolutely pure right \( R \)-modules is coresolving if and only if \( R \) is right coherent.

(ii) [22, Theorem 2] The class of absolutely pure right \( R \)-modules is closed under homomorphic images if and only if \( R \) is right semihereditary.

**Proof** This follows from Corollaries 20 and 21, considering the exact structure given by pure short exact sequences of right \( R \)-modules. \(\square\)

We leave the interested reader to deduce consequences of the above results in the case of the exact structure given by all pure short exact sequences of modules. We end this section with some comments on pure Xu exact structures on module categories.

**Remark 23** Let \( \mathcal{E} \) be the exact structure given by pure short exact sequences in \( \text{Mod}(R) \). Then \( \mathcal{E} \) is both projectively and injectively generated (see Example 11). With the notation from [14, Section 3], we have \( \Psi(\mathcal{E}) = (\mathcal{A}_1, \mathcal{B}_1) \), where \( \mathcal{A}_1 \) and \( \mathcal{B}_1 \)
are the classes of \textit{FP}-projective and absolutely pure right \(R\)-modules respectively, and \(\Phi(\mathcal{E}) = (\mathcal{A}_2, \mathcal{B}_2)\), where \(\mathcal{A}_2\) and \(\mathcal{B}_2\) are the classes of flat and cotorsion right \(R\)-modules respectively. Then \(\mathcal{E}\) is a projectively generated Xu exact structure if and only if every \textit{FP}-projective right \(R\)-module is pure-projective. Note that the hypothesis from [14, Theorem 3.10] that every right \(R\)-module has a pure-projective cover is equivalent to \(R\) being a right pure-semisimple ring (e.g., see [1, Theorem 6.18]). In this case, every right \(R\)-module is pure-projective, and so \(\mathcal{E}\) is clearly a projectively generated Xu exact structure. On the other hand, over an arbitrary ring \(R\), every right \(R\)-module has a pure-injective envelope. Then by [14, Theorem 3.10], \(\mathcal{E}\) is an injectively generated Xu exact structure if and only if every cotorsion right \(R\)-module is pure-injective if and only if the class of pure-injective modules is closed under extensions, that is, \(R\) is a right Xu ring [19].

4 Exact structures generated by the simple modules

In this section we deduce and complete a series of known results concerning relatively divisible and relatively flat modules with respect to the exact structures projectively, injectively and flatly generated by the class of simple modules. Some of them have been previously given only for modules over commutative rings.

\textbf{Corollary 24}

\begin{enumerate}
\item (i) The classes of absolutely neat (respectively absolutely coneat, absolutely s-pure) modules are closed under extensions and neat (respectively coneat, s-pure) submodules.
\item (ii) The class of absolutely neat modules is closed under direct products.
\item (iii) The classes of absolutely neat, absolutely coneat and absolutely s-pure modules are closed under direct sums.
\item (iv) Every module is absolutely neat (respectively absolutely coneat, absolutely s-pure) if and only if every short exact sequence of modules is neat (respectively coneat, s-pure).
\item (v) Every module is absolutely neat if and only if \(R\) is semisimple.
\end{enumerate}

\begin{enumerate}
\item (i) The classes of neat-flat (respectively coneat-flat, max-flat) modules are closed under extensions and neat (respectively coneat, s-pure) quotients.
\item (ii) The classes of neat-flat, coneat-flat and max-flat modules are closed under direct sums.
\item (iii) The class of max-flat modules is closed under direct limits. If every simple module is finitely presented, then the class of neat-flat modules is closed under direct limits.
\item (iv) Every module is neat-flat (respectively coneat-flat, max-flat) if and only if every short exact sequence of modules is neat (respectively coneat, s-pure).
\item (v) Every right \(R\)-module is coneat-flat if and only if \(R\) is a right \(V\)-ring.
\end{enumerate}
Proof This mainly follows from [14, Proposition 4.1], considering the exact structures given by neat (respectively coneat, s-pure) short exact sequences of modules, which are projectively (respectively injectively, flatly) generated by the class of simple modules. We only add some details.

(1) (ii) Using the isomorphism $\text{Hom}_R(S, \prod_{i \in I} M_i) \cong \prod_{i \in I} \text{Hom}_R(S, M_i)$ for every (simple) module $S$ and every family $(M_i)_{i \in I}$ of modules, it is straightforward to show that the class of neat short exact sequences is closed under direct products.

(iii) Note that every simple module $S$ is finitely generated. Hence we have an isomorphism $\text{Hom}_R(S, \bigoplus_{i \in I} M_i) \cong \bigoplus_{i \in I} \text{Hom}_R(S, M_i)$ for every family $(M_i)_{i \in I}$ of modules, from which it is straightforward (or use [30, 18.2]) to show that the class of neat short exact sequences is closed under direct sums. One can easily show that the classes of coneat and s-pure short exact sequences are closed under direct sums by [30, 16.2] and [30, 12.15] respectively.

(v) By [14, Proposition 4.1], every module is absolutely neat if and only if every simple module is projective if and only if $R$ is semisimple.

(2) (ii) The classes of neat (respectively coneat, s-pure) short exact sequences are closed under direct sums by the proof of (1) (iii).

(iii) Since the tensor functor commutes with direct limits, it is clear that the class of s-pure short exact sequences is closed under direct limits. If every simple module $S$ is finitely presented, then we have an isomorphism $\text{Hom}_R(S, \lim_{\to} M_i) \cong \lim_{\to} \text{Hom}_R(S, M_i)$ for every family $(M_i)_{i \in I}$ of modules, from which it is straightforward to show that the class of neat short exact sequences is closed under direct limits.

(v) By [14, Proposition 4.1], every right $R$-module is coneat-flat if and only if every simple right $R$-module is injective if and only if $R$ is a right $V$-ring. □

Corollary 25 [5, Proposition 3.4], [13, Theorem 3.3], [7, Theorem 3.2]

(1) The following are equivalent for a right $R$-module $X$:

(i) $X$ is absolutely neat.
(ii) $X$ is a neat submodule of an injective right $R$-module.
(iii) $X$ is a neat submodule of an absolutely neat right $R$-module.
(iv) $\text{Ext}^1_R(S, X) = 0$ for every simple right $R$-module $S$.

(2) The following are equivalent for a right $R$-module $Z$:

(i) $Z$ is neat-flat.
(ii) $Z$ is a neat quotient module of a projective right $R$-module.
(iii) $Z$ is a neat quotient module of a neat-flat right $R$-module.
(iv) For every simple right $R$-module $S$, every morphism $S \to Z$ factors through a projective right $R$-module.

Proof This follows from [14, Proposition 4.6] and [14, Proposition 3.3], considering the exact structure given by neat short exact sequences of right $R$-modules. □

Corollary 26 [5, Theorem 3.1]

(1) The following are equivalent for a right $R$-module $X$:

(i) $X$ is absolutely coneat.
(ii) $X$ is a coneat submodule of an injective right $R$-module.
(iii) $X$ is a coneat submodule of an absolutely coneat right $R$-module.
(iv) For every simple right $R$-module $S$, every morphism $X \to S$ factors through an injective right $R$-module.

(2) The following are equivalent for a right $R$-module $Z$:
(i) $Z$ is coneat-flat.
(ii) $Z$ is a coneat quotient module of a projective right $R$-module.
(iii) $Z$ is a coneat quotient module of a coneat-flat right $R$-module.
(iv) $\text{Ext}^1_R(Z, S) = 0$ for every simple right $R$-module $S$.

Proof This follows from [14, Proposition 4.6] and [14, Proposition 3.3], considering the exact structure given by coneat short exact sequences of right $R$-modules.

Corollary 27 [6, Lemmas 3.3, 3.4], [10, Theorem 2.2], [13, Theorem 3.3]

(1) The following are equivalent for a left $R$-module $X$:
(i) $X$ is absolutely $s$-pure.
(ii) $X$ is an $s$-pure submodule of an injective left $R$-module.
(iii) $X$ is an $s$-pure submodule of an absolutely $s$-pure left $R$-module.

(2) The following are equivalent for a left $R$-module $Z$:
(i) $Z$ is max-flat.
(ii) $Z$ is an $s$-pure quotient module of a projective left $R$-module.
(iii) $Z$ is an $s$-pure quotient module of a max-flat left $R$-module.
(iv) $\text{Tor}^1_R(S, Z) = 0$ for every simple right $R$-module $S$.
(v) $Z^+$ is an absolutely neat right $R$-module.

Proof This follows from [14, Proposition 4.6] and Propositions 10 and 14, considering the exact structure given by $s$-pure short exact sequences of left $R$-modules.

Corollary 28 [5, Proposition 4.9] Every simple right $R$-module is finitely presented if and only if every absolutely pure right $R$-module is absolutely neat.

Proof This follows from Corollary 16, considering the exact structure given by neat short exact sequences of right $R$-modules.

Corollary 29 [6, Theorem 4.9] The following are equivalent:
(i) $R$ is a right Kasch ring.
(ii) Every absolutely pure right $R$-module is neat-flat.
(iii) Every injective right $R$-module is neat-flat.
(iv) The injective envelope of every simple right $R$-module is neat-flat.
(v) For every free left $R$-module $Z$, $Z^+$ is a neat-flat right $R$-module.

Proof This follows from Proposition 17, considering the exact structure given by neat short exact sequences of right $R$-modules.
**Corollary 30**

(1) **The following are equivalent:**
   (i) For every short exact sequence $0 \to X \to Y' \to Z' \to 0$ of right $R$-modules with $X$ absolutely pure (respectively cotorsion) and $Y'$ absolutely neat, $Z'$ is absolutely neat.
   (ii) For every short exact sequence $0 \to Z \to U \to V \to 0$ of right $R$-modules with $V$ simple and $U$ projective, $Z$ is FP-projective (respectively flat).

(2) **The following are equivalent:**
   (i) The class of absolutely neat right $R$-modules is closed under homomorphic images.
   (ii) For every short exact sequence $0 \to Z \to U \to V \to 0$ of right $R$-modules with $V$ simple and $U$ projective, $Z$ is projective.

*Proof* This follows from Corollaries 20 and 21, considering the exact structure given by neat short exact sequences of right $R$-modules.  

**Corollary 31**

(1) **The following are equivalent:**
   (i) For every short exact sequence $0 \to Z \to U \to V \to 0$ of right $R$-modules with $V$ FP-projective (respectively flat) and $U$ coneat-flat, $Z$ is coneat-flat.
   (ii) For every short exact sequence $0 \to X \to Y' \to Z' \to 0$ of right $R$-modules with $X$ simple and $Y'$ injective, $Z'$ is absolutely pure (respectively cotorsion).

(2) **The following are equivalent:**
   (i) The class of coneat-flat right $R$-modules is closed under submodules.
   (ii) For every short exact sequence $0 \to X \to Y' \to Z' \to 0$ of right $R$-modules with $X$ simple and $Y'$ injective, $Z'$ is injective.

*Proof* This follows from Corollaries 20 and 21, considering the exact structure given by coneat short exact sequences of right $R$-modules.  

5 **Exact structures generated by the modules with zero Jacobson radical**

In this section we deduce a series of results concerning relatively divisible and relatively flat modules with respect to the exact structures projectively, injectively and flatly generated by the class of modules with zero Jacobson radical. We start with a series of examples.

**Example 32** (1) For $k \in \mathbb{N}$, denote $Z_k = \mathbb{Z}/k\mathbb{Z}$. Let $p$ be a prime, $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}$ and $B = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$. By [32, Example, p. 75], the short exact sequence $0 \to A \to B \to B/A \to 0$ of $\mathbb{Z}$-modules is rad-pure.

(2) Since $Z_2 \otimes \mathbb{Z} \cong Z_2$ and $Z_2 \otimes \mathbb{Q} = 0$, the sequence $0 \to Z_2 \otimes \mathbb{Q} \to Z_2 \otimes \mathbb{Q} \to Z_2 \otimes (\mathbb{Q}/\mathbb{Z}) \to 0$ of $\mathbb{Z}$-modules is not exact. So the short exact sequence $0 \to Z \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of $\mathbb{Z}$-modules is not rad-pure.
(3) Let $k$ be a field and $R = k[x, y]$ the polynomial ring in two indeterminates $x$ and $y$. Let $M = \langle x, y \rangle$. Note that $\text{Rad}(M) = 0$. Since $M \otimes M \cong M^2, M \otimes R \cong M, M \otimes (R/M) = 0$ and $M \not\cong M^2$, the sequence $0 \to M \otimes M \to M \otimes R \to M \otimes (R/M) \to 0$ of $\mathbb{Z}$-modules is not exact. Therefore the short exact sequence $0 \to M \to R \to R/M \to 0$ of $R$-modules is not rad-pure.

Clearly, every rad-pure short exact sequence is an $s$-pure short exact sequence. But the converse is not true in general, as we may see in the following example.

**Example 33** Let $R$ be an integral domain having a simple module $S$ with projective dimension $p.d.(S) > 1$. Then there exists a non-splitting short exact sequence $0 \to D \to M \to S \to 0$ such that $D$ is an $h$-divisible torsion $R$-module [17, Example 3.3]. As indicated in [13, Example 3.1], $D$ is absolutely coneat (hence absolutely $s$-pure, because $R$ is commutative). Since $D/\text{Rad}(D)$ is again an $h$-divisible torsion $R$-module, we have $T \otimes (D/\text{Rad}(D)) = 0$ for every simple $R$-module $T$. Then the sequence

$$0 \to (D/\text{Rad}(D)) \otimes D \to (D/\text{Rad}(D)) \otimes M \to (D/\text{Rad}(D)) \otimes S \to 0$$

is not exact. So $D$ is not an absolutely rad-pure $R$-module.

**Corollary 34**

(1) (i) The classes of absolutely rad-neat (respectively absolutely rad-coneal, absolutely rad-pure) modules are closed under extensions and rad-neat (respectively rad-coneal, rad-pure) submodules.

(ii) The class of absolutely rad-neat modules is closed under direct products.

(iii) The classes of absolutely rad-coneal and absolutely rad-pure modules are closed under direct sums. If every module with zero Jacobson radical is finitely generated, then the class of absolutely rad-neat modules is closed under direct sums.

(iv) Every module is absolutely rad-neat (respectively absolutely rad-coneal, absolutely rad-pure) if and only if every short exact sequence of modules is rad-neat (respectively rad-coneal, rad-pure).

(v) Every module is absolutely rad-neat if and only if every module with zero Jacobson radical is projective.

(2) (i) The classes of rad-neat-flat (respectively rad-coneal-flat, rad-pure-flat) modules are closed under extensions and rad-neat (respectively rad-coneal, rad-pure) quotients.

(ii) The classes of rad-coneal-flat and rad-pure-flat modules are closed under direct sums. If every module with zero Jacobson radical is finitely generated, then the class of rad-neat flat modules is closed under direct sums.

(iii) The class of rad-pure-flat modules is closed under direct limits. If every module with zero Jacobson radical is finitely presented, then the class of rad-neat-flat modules is closed under direct limits.
(iv) Every module is rad-neat-flat (respectively rad-coneat-flat, rad-pure-flat) if and only if every short exact sequence of modules is rad-neat (respectively rad-coneat, rad-pure).

(v) Every module is rad-coneat-flat if and only if every module with zero Jacobson radical is injective.

Proof This mainly follows from [14, Proposition 4.1], considering the exact structures given by rad-neat (respectively rad-coneat, rad-pure) short exact sequences of modules, which are projectively (respectively injectively, flatly) generated by the class of modules with zero Jacobson radical. We only add some details.

(1) (ii) Using the isomorphism $\text{Hom}_R(M, \prod_{i \in I} M_i) \cong \prod_{i \in I} \text{Hom}_R(M, M_i)$ for every module $M$ and every family $(M_i)_{i \in I}$ of modules, it is straightforward to show that the class of rad-neat short exact sequences is closed under direct products.

(iii) One can easily show that the classes of rad-coneat and rad-pure short exact sequences are closed under direct sums by [30, 16.2] and [30, 12.15] respectively.

If every module $M$ with zero Jacobson radical is finitely generated, then we have an isomorphism $\text{Hom}_R(M, \bigoplus_{i \in I} M_i) \cong \bigoplus_{i \in I} \text{Hom}_R(M, M_i)$ for every family $(M_i)_{i \in I}$ of modules, from which it is straightforward (or use [30, 18.2]) to show that the class of rad-neat short exact sequences is closed under direct sums.

(2) (ii) By the proof of (1) (iii), the classes of rad-coneat and rad-pure short exact sequences are always closed under direct sums, while the class of rad-neat short exact sequences is closed under direct sums if every module with zero Jacobson radical is finitely generated.

(iii) Since the tensor functor commutes with direct limits, it is clear that the class of rad-pure short exact sequences is closed under direct limits. If every module $M$ with zero Jacobson radical is finitely presented, then we have an isomorphism $\text{Hom}_R(M, \lim_{\to} M_i) \cong \lim_{\to} \text{Hom}_R(M, M_i)$ for every family $(M_i)_{i \in I}$ of modules, from which it is straightforward to show that the class of rad-neat short exact sequences is closed under direct limits.

Corollary 35

(1) The following are equivalent for a right $R$-module $X$:

(i) $X$ is absolutely rad-neat.

(ii) $X$ is a rad-neat submodule of an injective right $R$-module.

(iii) $X$ is a rad-neat submodule of an absolutely rad-neat right $R$-module.

(iv) $\text{Ext}_R^1(M, X) = 0$ for every right $R$-module $M$ with zero Jacobson radical.

(2) The following are equivalent for a right $R$-module $Z$:

(i) $Z$ is rad-neat-flat.

(ii) $Z$ is a rad-neat quotient module of a projective right $R$-module.

(iii) $Z$ is a rad-neat quotient module of a rad-neat-flat right $R$-module.

(iv) For every right $R$-module $M$ with zero Jacobson radical, every morphism $M \to Z$ factors through a projective right $R$-module.
Proof This follows from [14, Proposition 4.6] and [14, Proposition 3.3], considering the exact structure given by rad-neat short exact sequences of right $R$-modules.

Corollary 36
(1) The following are equivalent for a right $R$-module $X$:
   (i) $X$ is absolutely rad-coneats.
   (ii) $X$ is a rad-coneats submodule of an injective right $R$-module.
   (iii) $X$ is a rad-coneats submodule of an absolutely rad-coneats right $R$-module.
   (iv) For every right $R$-module $M$ with zero Jacobson radical, every morphism $X \to M$ factors through an injective right $R$-module.

(2) The following are equivalent for a right $R$-module $Z$:
   (i) $Z$ is rad-coneats-flat.
   (ii) $Z$ is a rad-coneats quotient module of a projective right $R$-module.
   (iii) $Z$ is a rad-coneats quotient module of a rad-coneats-flat right $R$-module.
   (iv) $\text{Ext}^1_R(Z, M) = 0$ for every right $R$-module $M$ with zero Jacobson radical.

Proof This follows from [14, Proposition 4.6] and [14, Proposition 3.3], considering the exact structure given by rad-coneats short exact sequences of right $R$-modules.

Corollary 37
(1) The following are equivalent for a left $R$-module $X$:
   (i) $X$ is absolutely rad-pure.
   (ii) $X$ is a rad-pure submodule of an injective left $R$-module.
   (iii) $X$ is a rad-pure submodule of an absolutely rad-pure left $R$-module.

(2) The following are equivalent for a left $R$-module $Z$:
   (i) $Z$ is rad-pure-flat.
   (ii) $Z$ is a rad-pure quotient module of a projective left $R$-module.
   (iii) $Z$ is a rad-pure quotient module of a rad-pure-flat left $R$-module.
   (iv) $\text{Tor}^n_R(M, Z) = 0$ for every right $R$-module $M$ with zero Jacobson radical.
   (v) $Z^+$ is an absolutely rad-coneats right $R$-module.

Proof This follows from [14, Proposition 4.6] and Propositions 10 and 14, considering the exact structure given by rad-pure short exact sequences of left $R$-modules.

Corollary 38
(1) The following are equivalent:
   (i) For every short exact sequence $0 \to X \to Y' \to Z' \to 0$ of right $R$-modules with $X$ absolutely pure (respectively cotorsion) and $Y'$ absolutely rad-neat, $Z'$ is absolutely rad-neat.
   (ii) For every short exact sequence $0 \to Z \to U \to V \to 0$ of right $R$-modules with $V$ having zero Jacobson radical and $U$ projective, $Z$ is FP-projective (respectively flat).
(2) The following are equivalent:

(i) The class of absolutely rad-neat right \( R \)-modules is closed under homomorphic images.
(ii) For every short exact sequence \( 0 \to Z \to U \to V \to 0 \) of right \( R \)-modules with \( V \) having zero Jacobson radical and \( U \) projective, \( Z \) is projective.

Proof This follows from Corollaries 20 and 21, considering the exact structure given by rad-neat short exact sequences of right \( R \)-modules.

Corollary 39

(1) The following are equivalent:

(i) For every short exact sequence \( 0 \to Z \to U \to V \to 0 \) of right \( R \)-modules with \( V \) FP-projective (respectively flat) and \( U \) rad-coneat-flat, \( Z \) is rad-coneat-flat.
(ii) For every short exact sequence \( 0 \to X \to Y' \to Z' \to 0 \) of right \( R \)-modules with \( X \) having zero Jacobson radical and \( Y' \) injective, \( Z' \) is absolutely pure (respectively cotorsion).

(2) The following are equivalent:

(i) The class of rad-coneat-flat right \( R \)-modules is closed under submodules.
(ii) For every short exact sequence \( 0 \to X \to Y' \to Z' \to 0 \) of right \( R \)-modules with \( X \) having zero Jacobson radical and \( Y' \) injective, \( Z' \) is injective.

Proof This follows from Corollaries 20 and 21, considering the exact structure given by rad-coneat short exact sequences of right \( R \)-modules.

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