IMEX schemes for a Parabolic-ODE system of European Options with Liquidity Shocks

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Abstract

The coupled system, where one is a degenerate parabolic equation and the other has not a diffusion term arises in the modeling of European options with liquidity shocks. Two implicit-explicit (IMEX) schemes that preserve the positivity of the differential problem solution are constructed and analyzed. Numerical experiments confirm the theoretical results and illustrate the high accuracy and efficiency of the schemes in combination with Richardson extrapolation

Key words Parabolic-ordinary system, European options, finite difference scheme, comparison principle, positivity

1 Introduction

We study numerically a parabolic-ODE system modelling option pricing liquidity shocks. The presence of liquidity shocks is a source of non-liquidity risk and makes this market incomplete. Ludkowski and Shen [5] investigate a nonlinear pricing mechanism based on utility maximization. They consider the investor whose utility is described by an exponential utility function

\[ U(x) = -e^{-\gamma x}, \]  

where \( \gamma > 0 \) is the coefficient of risk aversion. The investor seeks to maximise utility of both terminal wealth and option payoff at time horizon \( T < \infty \), which is chosen to coincide with the expiration date of all securities in market model. Properties of the exponential utility function (1) imply that the value functions can be expressed as

\[ \hat{U}^i(t, X, S) = -e^{-\gamma X} e^{-\gamma R^i(t, S)}, i = 0, 1, \]

where \( X = X_t \) is the wealth process and the functions \( R^i(t, S) \) are related to the price of options in the two states, see (7) below. Then the pair \( \{ R^i(t, S), i = 0, 1 \} \) is the unique viscosity solutions of the coupled semi-linear system,

\[ R_t^0 + \frac{1}{2} \sigma^2 S^2 R_{SS}^0 - \frac{\nu_0}{\gamma} e^{-\gamma (R^1 - R^0)} + \frac{(d_0 + \nu_0 \lambda)}{\gamma} = 0, \]

\[ R_t^1 - \frac{\nu_1}{\gamma} e^{-\gamma (R^0 - R^1)} + \frac{\nu_1 \lambda}{\gamma} = 0. \]
The terminal conditions are:

\[ R^i(T, S) = h(S), \quad i = 0, 1. \] (4)

Here \( \sigma \) is volatility of the underlying, \( \nu_{01}, \nu_{10} \) are transition intensities from state (0) to state (1) and vice versa, respectively, \( \mu \) is drift of the underlying and \( d_0 = \mu^2/2\sigma^2 \), see [5] for more details.

Using \( \hat{U}^i \) and \( \hat{V}^i \), the buyer’s indifference price \( p \) (initial state 0) and \( q \) (initial state 1) are defined via

\[ \hat{U}^0(t, X - p, S) = \hat{V}^0(t, X), \quad \hat{U}^1(t, X - q, S) = \hat{V}^1(t, X), \] (5)

where \( \hat{U}, \hat{V} \) are the optimal solutions for terminal wealth with and without options respectively. The value functions \( \hat{V}^i, i = 1, 2 \) are given by

\[ \hat{V}^i = e^{-\gamma S} F_i(t) \quad \text{and} \quad \hat{V}^i(t, X, S) = e^{-\gamma R^i(t, S)}, \quad i = 1, 2 \] (6)

and the functions \( F_0(t), F_1(t) \) by,

\[
F_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \\
F_1(t) = \frac{1}{\nu_{01}} \{c_1 (d_0 + \nu_{01} - \lambda_1)e^{\lambda_1 t} + c_2 (d_0 + \nu_{01} - \lambda_2)e^{\lambda_2 t},
\]

where

\[
\lambda_{1,2} = \frac{d_0 + \nu_{01} + \nu_{10} \pm \sqrt{(d_0 + \nu_{01} + \nu_{10})^2 - 4d_0\nu_{10}}}{2}, \\
c_1 = \frac{\lambda_2 - d_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 T} \quad \text{and} \quad c_2 = \frac{\lambda_1 - d_0}{\lambda_1 - \lambda_2} e^{-\lambda_2 T}.
\]

Then, we obtain from (2), (5), (6)

\[ p = R^0 + \gamma^{-1} \ln F_0(t), \quad q = R^1 + \gamma^{-1} \ln F_1(t) \] (7)

and from (3), (4) the parabolic-ordinary system for \( p \) and \( q \)

\[
p_t + \frac{1}{2} \sigma^2 S^2 p_{SS} - \frac{\nu_{01}}{\gamma} \frac{F_0}{F_0} e^{-\gamma (q - p)} + \frac{(d_0 + \nu_{01})}{\gamma} - \frac{1}{\gamma} \frac{F_0'}{F_0} = 0, \\
q_t - \frac{\nu_{10}}{\gamma} \frac{F_1}{F_1} e^{-\gamma (q - p)} + \frac{2\nu_{10}}{\gamma} - \frac{1}{\gamma} \frac{F_1'}{F_1} = 0
\] (8)

with terminal conditions

\[ p(T, S) = q(T, S) = h(S). \] (9)

The numerical solution of the system (8) is the main object of the present paper. The numerical treatment of the boundary layer effect for small values of \( \sigma_0 \) and \( \gamma \), the degeneracy at \( S = 0 \) of the parabolic equation and the exponential nonlinearity lead to challenging problems [10]. The introduction of exponential nonlinear terms is an available assumption based on the financial nature of the model system (8). There are many numerical schemes to solve nonlinear parabolic and hyperbolic equations. However, very few have dealt with an exponential nonlinear term. The special nature of the nonlinear exponential term for a hyperbolic problem is discussed in [10]. A possible way to build an efficient numerical solution of (8), (9) is to implement an IMEX method [1,9]
In this procedure the diffusion term is discretized implicitly in time and the reaction terms are discretized explicitly.

An IMEX method for numerical solution of reaction-diffusion equation with pure Neumann boundary conditions is developed in [3]. IMEX schemes, by applying explicit approximation both integral term and the convection term and an implicit approximation for the second differential term are developed for integro-differential equations of finance in [2].

The rest of the paper is organized as follows. In the next section some results concerning well-posedness of Cauchy problem for (3) and a comparison principle, obtained in [4], are discussed. Also, two lemmas concerning discrete maximum principle [6,7] are formulated. In Section 3 an implicit-explicit linear scheme is introduced. Comparison discrete principle and convergence of the scheme are proved. Similar results are obtained for an IMEX linearized scheme in Section 4. The computational experiments in Section 5 confirm the applicability of our schemes and the theoretical results. Finally, Section 6 summarises our conclusions.

Notation Let Ω be a bounded interval in \( \mathbb{R}^+ = (0, \infty) \) and let \( C^0(\Omega) \) denote the space of continuous functions on \( \Omega \) with the norm of any \( w \in C^0(\Omega) \) defined by \( \| w \|_0,\Omega = \sup_{x \in \Omega} |w(x)| \). For each integer \( k \geq 1 \), let \( C^k(\Omega) \) denote the space of \( k \)-times differentiable functions on \( \Omega \), with continuous derivatives up to and including those of order \( k \), with the norm of any \( w \in C^k(\Omega) \) defined by \( \| w \|_{k,\Omega} = \max_{0 \leq l \leq k} \| w^{(l)}(x) \|_\Omega \). The notational conventions \( \| w \|_{0,\Omega} = \| w \|_\Omega = \| w \| \) are adopted. The explicit reference to \( \Omega \) is dropped whenever the domain question is evident. For any mesh functions on arbitrary mesh \( \Omega_N = \{x_i\}^N_{i=1} \), \( \Omega_N = \{x_i\}^N_{i=0} \) the discrete maximum norm is defined by \( \| w \|_{C(\Omega_N)} = \max_{0 \leq i \leq N} |w_i| \).

Maximum norms and semi-norms for smooth functions of two variables are introduced in a similar way. Let \( Q_T = (0, T) \times \Omega \). Then \( \| w \|_{Q_T} = \sup_{(x,\tau) \in Q_T} |w(x,\tau)| \) and if \( C^0(Q_T) \) is the space of all functions on \( Q_T \) with continuous derivatives then

\[
C^k(Q_T) = \left\{ w : \frac{\partial^{i+j} w}{\partial x^i \partial \tau^j} \in C^0(Q_T) \text{ for } i, j = 0, 1, 2, \ldots \text{ with } 0 \leq i + 2j \leq k \right\}.
\]

2 Preliminaries

In this section we will describe some properties of the solution to system (8) using results obtained in [4]. Also, following [6,7], two lemmas, concerning discrete maximum principle (DM) are formulated.

We will consider solutions of (8) satisfying

\[
|p|, |q|, |h| \leq A \exp \left( \alpha \ln^2 S \right) = AS^{\alpha \ln S} \tag{10}
\]

for some positive constants \( A \) and \( \alpha \). In [4], well-posedness in weighted Sobolev spaces and comparison principle for the corresponding Cauchy problem (8), (9) are established. With sufficient smoothness of the initial data the weak solutions are classical ones.

In this paper we use the comparison principle for classical solutions \( p(S,t), q(S,t) \) of the problem (8), (9), i.e \( p \in C((0, +\infty) \times (0, T)] \cap C^2((0, +\infty) \times (0, T)), \) \( q \in \)
$C((0, +\infty) \times (0, T]), \ q_i \in ((0, +\infty) \times (0, T))$.

**Proposition 1** ([4]) Let $(p_1, q_1)$ and $(p_0, q_0)$ be two classical solutions of problem (8), (9) corresponding to terminal data $h = h_1(S)$ and $h = h_0(S)$, respectively. If there exists some positive constants $A$ and $\alpha$ such that $p_i(S, t)$ and $h_i(S)$, $i = 0, 1$ satisfy conditions (10), then

\[
\begin{align*}
\inf(h_1 - h_0) \leq p_1(S, t) - p_0(S, t) \leq \sup(h_1 - h_0), \\
\inf(h_1 - h_0) \leq q_1(S, t) - q_0(S, t) \leq \sup(h_1 - h_0),
\end{align*}
\]

(11)

In particular, let $h(S)$ be bounded from below (or from above) by a constant $h(S) \geq h_*$ (resp. $h(S) \leq h^*$ and the pair $p(S, t), q(S, t)$ be a classical solution of the terminal problem (8), (9). Then $p(S, t) \geq h_*$ and $q(S, t) \geq h_*$ (respectively $p(S, t) \leq h^*$ and $q(S, t) \leq h^*$).

for any $S \in (0, +\infty)$ and any $t \in (0, T]$.

By making the substitutions $\tau = T - t$, $u = \gamma R^0$ and $v = \gamma R^1$, the system (3) becomes

\[
\begin{align*}
L^p(u, v) &\equiv u_\tau - \frac{1}{2}\sigma^2 S^2 u_{SS} + ae^u v - b = 0, \\
L^q(u, v) &\equiv v_\tau + ce^v u - c = 0,
\end{align*}
\]

(12)

where $a = \nu_{01}$, $b = d_0 + \nu_{01}$, $c = \nu_{10}$. In accordance with (9) we take the initial conditions to be

\[
u(0, S) = u_0(0, S) = \gamma h(S), \quad v(0, S) = v_0(0, S) = \gamma h(S).
\]

(13)

For a call option,

\[
h(S) = \max(S - K, 0).
\]

(14)

We assume ground conditions for $u, v$ of the form (10). In the next sections, the analysis of the difference approximations of problem (12)-(14) will use the following comparison principle that follows from those one for $(p, q)$:

**Proposition 2** Let $(\Pi, \sigma), (u, v) \in C((0, T) \times (0, +\infty)) \cap C^2((0, T) \times (0, +\infty))$ be two pairs of classical solutions of (12)-(14) corresponding to the initial data $h = \overline{h}$ and $h = h$, respectively and such that conditions of the type (10) hold. If the following inequalities also hold:

\[
L^p(\Pi, \sigma) \geq L^p(u, v), \quad L^q(\Pi, \sigma) \geq L^q(u, v) \quad \text{and} \quad \overline{h} \geq \underline{h},
\]

(15)

then

\[
\Pi \geq u, \sigma \geq v.
\]

Hereinbelow we will use the following canonical form of writing a 3-point difference scheme

\[
\begin{align*}
A_i y_{i-1} - C_i y_i + B_i y_{i+1} &= -F_i, \quad i = 1, 2, \ldots, N - 1, \\
y_0 &= \mu_1, \quad y_N = \mu_2.
\end{align*}
\]

(16)

The discrete comparison principle for problem (16) was proved in [6,7] and is formulated in the following way.
Lemma 2.1. Let the conditions

\[ A_i > 0, \ B_i > 0, \ D_i = C_i - A_i - B_i \geq 0, \ i = 1, 2, \ldots, N - 1 \] (17)

be fulfilled. Then the solution of the difference scheme (15) satisfies the inequalities

\[ y_i \geq 0, \ i = 0, \ldots, N, \ if \ F_i \geq 0, \ i = 1, \ldots, N - 1, \ \mu_1 \geq 0, \ \mu_2 \geq 0; \]

\[ y_i \leq 0, \ i = 0, \ldots, N, \ if \ F_i \leq 0, \ i = 1, \ldots, N - 1, \ \mu_1 \leq 0, \ \mu_2 \leq 0. \]

Lemma 2.2. Let the conditions

\[ |A_i| \geq 0, \ |B_i| \geq 0, \ D_i = |C_i| - |A_i| - |B_i| > 0, \ i = 1, \ldots, N - 1 \]

be met. Then for the solution the problem (16) the estimate holds

\[ \|y\|_{C(Q)} \leq \max \left\{ |\mu_1|, |\mu_2|, \left\| \frac{F}{D} \right\|_{C(Q)} \right\}. \]

3 Implicit-Explicit Linear Scheme

In this section, we develop a linear IMEX scheme to solve the coupled semi-linear parabolic-ordinary system problem (11)-(12).

For call option one possible pair of boundary conditions is, see e.g. [10,11]

\[ u(\tau, 0) = \varphi_l(\tau) = 0, \quad u(\tau, S) = \varphi_r(\tau) \approx S_{\text{max}} \text{ for large } S. \] (18)

The left natural boundary condition for \( u \) is

\[ u_{\tau}(\tau, 0) = -ae^{-(\nu(\tau, 0) - u(\tau, 0))} + b. \] (19)

On the discrete domain \( w_{S\tau} = w_S \times w_\tau \):

\[ w_S = \{ S_i = i\Delta S, \ \Delta S > 0, \ i = 0, 1, \ldots, I; \ I\Delta S = S_{\text{max}} \}, \ w_S = w_S \cup \{ S_0, S_I \}; \]

\[ w_\tau = \{ \tau_j = j\Delta \tau, \ \Delta \tau > 0, \ j = 0, 1, \ldots, J; \ J\Delta \tau = T \}, \ w_\tau = w_\tau \cup \{ \tau_0, \tau_J \}. \]

On the discrete domain \( w_{S\tau} \) we approximate the problem (12)-(14) by the difference scheme

\[ L^p(U, V) = \frac{U_{j+1}^i - U_j^i}{\Delta \tau} - \frac{1}{2} \sigma^2 S_i^2 \frac{U_{j+1}^{i+1} - 2U_j^{i+1} + U_{j+1}^{i+1}}{(\Delta S)^2} + ae^{-\nu \tau} e^{U_j^i} - b = 0, \] (20)
\[ i = 1, 2, \ldots, I - 1; \]

\[ L^0(U, V) = \frac{V_{i+1}^j - V_i^j}{\Delta \tau} + ce^{-U_i^j} e^{V_i^j} - c = 0, \quad i = 0, 1, \ldots, I, \quad (21) \]

\[ j = 0, 1, \ldots, J - 1; \]

\[ U_i^0 = U_0(S_i), \quad i = 0, 1, \ldots, I; \quad (22) \]

\[ U_0^j = \varphi_l(\tau_j), \quad U_I^j = \varphi_r(\tau_j), \quad j = 0, 1, \ldots, J; \quad (23) \]

\[ V_i^0 = V_0(S_i), \quad i = 1, \ldots, I. \quad (24) \]

The natural boundary condition can be approximated as follows

\[ U_{j+1}^0 = U_0(S_j), \]

On the \((j + 1)\)-th, \(j = 0, 1, \ldots, J - 1\) time level the scheme (20)-(23) has the form

\[ -A_i U_{i-1}^{j+1} + C_i U_{i}^{j+1} - B_i U_{i+1}^{j+1} = F_i, \]

\[ V_{i+1}^{j+1} = V_i^j - \Delta \tau c e^{-U_i^j} e^{V_i^j} + c, \quad (25) \]

where

\[ A_i = B_i = \frac{1}{2} \sigma^2 \frac{S_i^2}{(\Delta S)^2}, \quad C_i = \frac{1}{\Delta \tau} + A_i + B_i, \]

\[ F_i = \frac{1}{\Delta \tau} U_i^j - ae^{-V_i^j} e^{U_i^j} + b, \quad i = 1, \ldots, I - 1; \]

For the truncation error corresponding to (20) we find

\[ Tr1 = \frac{1}{2} \Delta \tau \frac{\partial^2 u}{\partial \tau^2}(\tau_{j+1} - \theta_1 \Delta \tau, S_i) \]

\[ + \Delta \tau \left( \frac{\partial u}{\partial \tau}(\tau_{j+1} - \rho^\Delta \tau, S_{i-1}) + \frac{\partial u}{\partial \tau}(\tau_{j+1} - \rho \Delta \tau, S_i) + \frac{\partial u}{\partial \tau}(\tau_{j+1} - \rho^+ \Delta \tau, S_{i+1}) \right) \]

\[ + \frac{1}{24} (\Delta S)^2 \left\{ \frac{\partial^4 u}{\partial \tau^4}(\tau_{j+1}, S_i + \theta_1 \Delta S) + \frac{\partial^4 u}{\partial \tau^4}(\tau_{j+1}, S_i - \theta_1 \Delta S) \right\} \]

\[ + \Delta \tau \frac{\partial V}{\partial \tau}(\tau_{j+1} - \tilde{\rho} \Delta \tau, S_i) e^{n(\tau_{j+1}, S_i)} = \frac{\partial u}{\partial \tau}(\tau_{j+1} - \tilde{\rho} \Delta \tau, S_i) e^{-v(\tau_{j+1}, S_i)} \]

\[ = O(\Delta \tau) + (\Delta S)^2. \]
For the truncation error corresponding to (21) we get

\[
T_{r2} = \frac{1}{2} \Delta \tau^2 \frac{\partial^2 u}{\partial \tau^2} (\tau_{j+1} - \theta_2 \Delta \tau, S_i) + \Delta \tau \frac{\partial u}{\partial \tau} (\tau_{j+1} - \eta \Delta \tau, S_i) e^{u(\tau_{j+1}, S_i)}
\]

\[\] - \frac{\partial v}{\partial \tau} (\tau_{j+1} - \eta \Delta \tau, S_i) e^{-u(\tau_{j+1}, S_i)} = O(\Delta \tau).

0 < \rho, \rho^-, \rho^+, \tilde{\rho} \leq 1, 0 < \theta_1, \theta_1^- < 1.

In accordance with Notation we define the strong norms on the meshes \(w_S\) and \(w_{S\tau}\), respectively,

\[\|z\|_{C(w_S)} = \max_{0 \leq i \leq I} |z_i|, \quad \|z\|_{C(w_{S\tau})} = \max_{0 \leq i \leq I} |z^i_j| .\]

Let denote\(C_u = \sup_{(\tau, S) \in Q_T} |u(\tau, S)|, \ C_v = \sup_{(\tau, S) \in Q_T} |v(\tau, S)|.\)

**Theorem 1** Suppose that there exists classical solution \((u, v) \in C^{2,4}(Q_T)\) of problem (10)-(14). Then for sufficiently small \(\Delta S\) and \(\Delta \tau\) the following error estimate holds:

\[\|u - U\|_{C(ws\tau)} + \|v - V\|_{C(ws\tau)} \leq C(\Delta \tau + (\Delta S)^2); \tag{26} \]

where the constant \(C\) doesn’t depend of \(\Delta S\) and \(\Delta \tau\).

**Proof** Define the errors \(\varepsilon^i_j, \mu^i_j\) by

\[\varepsilon^i_j = U^i_j - u(\tau_j, S_i), \quad \mu^i_j = V^i_j - v(\tau_j, S_i), \quad i = 1, \ldots, I.\]

Then \(\{\varepsilon^i_j\}, \{\mu^i_j\}\) satisfy the linear system of algebraic equations:

\[A_i \varepsilon^{i+1}_{i-1} - C_i \varepsilon^j_{i+1} + B_i \mu^j_{i+1} = F_i, \quad i = 1, \ldots, I - 1,\]

\[\varepsilon^j_0 = 0, \quad \varepsilon^j_I = 0,\]

where

\[F^j_i = \frac{1}{\Delta \tau} \varepsilon^j_i + \alpha_i^j\]

and

\[\mu^j_{i+1} = \mu^j_i + \Delta \tau \beta_i^j.\]

Here \(\alpha_i^j\) and \(\beta_i^j\) are the local truncation errors corresponding to the difference equations (20) and (21), respectively. They will be estimated as follows.

Let us derive the truncation error corresponding to nonlinear (right) part:

For the nonlinear, right hand side of the first equation we obtain

\[e^{-v^j_i} = e^{-\mu^j_i - v(\tau_j, S_i)} e^{\varepsilon^j_i + u(\tau_j, S_i)}\]
\[
\begin{align*}
\alpha_i^j & = O(\Delta \tau) + (\Delta S)^2 + (\varepsilon_i^j - \mu_i^j) e^{-u(\tau_j, S_i)} + O(\varepsilon_i^j \mu_i^j) + O((\varepsilon_i^j)^2) + O(\mu_i^j)^2).
\end{align*}
\]

In a similar way, we find
\[
\beta_i^j = O(\Delta \tau) + (\mu_i^j - \varepsilon_i^j) e^{-u(\tau_j, S_i)} + O(\varepsilon_i^j \mu_i^j) + O((\varepsilon_i^j)^2) + O(\mu_i^j)^2).
\]

Applying Lemma 2.1 we get
\[
\|\varepsilon_i^{j+1}\|_C \leq \Delta \tau \|\tilde{F}\|,
\]
where \(\|\cdot\|_C\) is the strong norms \(C(\mathbb{T}_S)\) as defined above.

We estimate \(\|F^{j+1}\|\):
\[
\begin{align*}
\|F^{j+1}\| & \leq \left(\frac{1}{\Delta \tau} + e^{C_u C_v}\right)\|\varepsilon^j\| + e^{C_u C_v} \|\mu^j\| \\
& + O(\Delta \tau) + O((\Delta S)^2) + O((\varepsilon_i^j)^2) + O(\|\mu_i^j\|^2)).
\end{align*}
\]

Next,
\[
\|\mu_i^{j+1}\| \leq \|\mu_i^j\| + \Delta \tau (e^{C_u C_v} \|\varepsilon_i^j\| + O(\Delta \tau) + O((\varepsilon_i^j)^2) + O(\|\mu_i^j\|^2)).
\]

Therefore,
\[
\|\varepsilon_i^{j+1}\| + \|\mu_i^{j+1}\| \leq (1 + 2\Delta \tau e^{C_u C_v})\|\varepsilon_i^j\|
\]
\[
+ (1 + \Delta \tau C_u C_v)\|\mu_i^j\| + \Delta \tau O(\|\varepsilon_i^j\| + (\Delta S)^2) + O(\|\mu_i^j\|^2) + O(\|\mu_i^j\|^2)).
\]

For \(j = 0\) we have \(\varepsilon_i^0 = 0, \mu_i^0 = 0\) and then
\[
\alpha_i^0 = O(\Delta \tau) + (\Delta S)^2, \quad \beta_i^0 = O(\Delta \tau).
\]

Since \(\|\varepsilon_i^0\| = \|\mu_i^0\| = 0\), we get
\[
\|\varepsilon_i^1\| = C\Delta \tau (\Delta \tau + (\Delta S)^2), \quad \|\varepsilon_i^1\| = C\Delta \tau (\Delta \tau + (\Delta S)^2).
\]

Therefore, by induction we have
\[ \|\varepsilon^{j+1}\| + \|\mu^{j+1}\| \leq (1 + 2\Delta\tau C_n C_v) (\|\varepsilon^j\| + \|\mu^j\|) + \Delta\tau C(\Delta\tau + (\Delta S)^2). \]

which implies that

\[ \|\varepsilon^{j+1}\| + \|\mu^{j+1}\| \leq C \sum_{k=0}^{j} (1 + 2\Delta\tau C_n C_v)^k \Delta\tau (\Delta\tau + (\Delta S)^2) \]

\[ C(\Delta\tau + (\Delta S)^2) \left( (1 + 2\Delta\tau C_n C_v)^j (1 + 2\Delta\tau C_n C_v)^{-1} \right) \leq C(\Delta\tau + (\Delta S)^2) \]

The following discrete comparison principle for the \((U, V)\) is crucial for the positivity of the discrete approximations of the indifference prices \(p\) and \(q\) on the base of the scheme (20)-(24).

**Theorem 2** Let the assumptions of Theorem 1 hold. Let also \((U, V)\) be the grid functions defined on \(W_S\), and the inequalities hold:

\[ L^p(U, V) \geq L^p(U, V), \quad L^0(U, V) \geq L^0(U, V), \]  
\[ U_i^0 \geq U_i^0, \quad V_i^0 \geq V_i^0, \quad i = 0, \ldots, I, \]  
\[ V_M^i \geq V_M^i, \quad T^i_M \geq T^i_M, \quad j = 1, \ldots, J. \]  

Then for sufficiently small \(\Delta S\) and \(\Delta\tau\) we have

\[ U_i^j \geq U_i^j, \quad V_i^j \geq V_i^j, \quad i = 0, 1, \ldots, I, \quad j = 0, 1, \ldots, J. \]  

**Proof.** Let introduce

\[ y_i^j = U_i^j - U_i^j, \quad z_i^j = V_i^j - V_i^j, \quad i = 0, 1, \ldots, I, \quad j = 0, 1, \ldots, J. \]

Then, from (26) we obtain

\[ \frac{y_i^{j+1} - y_i^j}{\Delta\tau} = \frac{1}{2} \sigma^2 S_i^2 \frac{y_i^{j+1} - 2y_i^{j+1} + y_i^{j+1}}{(\Delta S)^2} + a(e^{-\tau_i^j} e^{\tau_i^j} - e^{-\tau_i^j} e^{\tau_i^j}) \geq 0, \]  
\[ \frac{z_i^{j+1} - z_i^j}{\Delta\tau} + c(e^{-\tau_i^j} e^{\tau_i^j} - e^{-\tau_i^j} e^{\tau_i^j}) \geq 0, \quad i = 1, \ldots, I - 1, \quad j = 1, \ldots, J - 1, \]

Using the mean-value theorem we get

\[ e^{-\tau_i^j} e^{\tau_i^j} - e^{-\tau_i^j} e^{\tau_i^j} = e^{-\tilde{\tau}_i^j} e^{\tilde{\tau}_i^j} (y_i^j - z_i^j), \]

\[ U_i^j = U_i^j + \tilde{\theta}(U_i^j - U_i^j), \quad V_i^j = V_i^j + \tilde{\theta}(V_i^j - V_i^j), \quad 0 < \tilde{\theta} < 1, \]

\[ e^{-\tau_i^j} e^{\tau_i^j} - e^{-\tau_i^j} e^{\tau_i^j} = e^{-\tilde{\tau}_i^j} e^{\tilde{\tau}_i^j} (z_i^j - y_i^j), \]

\[ U_i^j = U_i^j + \tilde{\theta}(U_i^j - U_i^j), \quad V_i^j = V_i^j + \tilde{\theta}(V_i^j - V_i^j), \quad 0 < \tilde{\theta} < 1. \]
We rewrite (30) in the form

\[ A_i y_{i+1}^{j+1} - C_i y_i^{j+1} B_i y_{i+1}^{j+1} \geq -F_i, \]

\[ A_i = \frac{1}{2} \sigma^2 S_i^2 \frac{\Delta \tau}{\Delta S}, \quad B_i = \frac{1}{2} \sigma^2 S_i^2 \frac{\Delta \tau}{\Delta S}, \quad C_i = \frac{1}{\Delta \tau} + A_i + B_i, \]

\[ F_i = \frac{1}{\Delta \tau} y_i^j - ae^{-\bar{V}_i^j} e^{U_i^j} (y_i^j - z_i^j). \]

Next, we rewrite (31) in the form

\[ z_{i+1}^{j+1} \geq \left( 1 - ce^{-\bar{V}_i^{j+1}} \right) z_i^j + ce^{-\bar{V}_i^j} e^{U_i^j} y_i^j. \]  (33)

We apply the method of mathematical induction with respect to \( j \) to prove that

\[ y_i^j \geq 0, \quad z_i^j \geq 0, \quad i = 0, 1, \ldots, I, \quad j = 0, 1, \ldots, J. \]  (34)

From (16), (17), we have

\[ y_i^0 \geq 0, \quad z_i^0 \geq 0, \quad i = 0, 1, \ldots, I, \]

Assuming that (32) holds when \( j = k - 1 \), we will show that for \( j = k \) the above inequalities are true.

On the base of Theorem 1 we can confirm that for sufficiently small \( \Delta \tau, \Delta S \) there exists constants \( C_u, C_v \), such that

\[ \max(\|U\|, \|U\|) \leq 2C_u, \quad \max(\|V\|, \|V\|) \leq 2C_v. \]

Then, if it necessary, we choose \( \Delta \tau \) in additional smaller such that

\[ \Delta \tau < \min(a, c) e^{2C_u} e^{2C_v}. \]  (35)

By induction, \( y_i^{k-1} \geq 0, \quad z_i^{k-1} \geq 0 \) and using (Lemma 2.1) we conclude that \( F_i \geq 0, \quad i = 0, 1, \ldots, I - 1 \). Now Lemma 2.1 implies \( y_i^k \geq 0, \quad i = 0, 1, \ldots, I \). It is clear from (32) and (34) that \( z_i^k \geq 0, \quad i = 1, \ldots, I - 1 \). \[ \square \]

4 Implicit-Explicit Linearised Scheme

Let us consider first the implicit scheme:

\[ \frac{U_i^{j+1} - U_i^j}{\Delta \tau} - \frac{1}{2} \sigma^2 S_i^2 \frac{U_i^{j+1} - 2U_i^{j+1} + U_i^{j+1}}{\Delta S} + ae^{-V_i^{j+1}} e^{U_i^{j+1}} - b = 0, \]  (36)

\[ \frac{V_i^{j+1} - V_i^j}{\Delta \tau} + ce^{-U_i^{j+1}} e^{V_i^{j+1}} - c = 0, \]  (37)

\[ i = 1, 2, \ldots, I - 1; \quad j = 0, 1, \ldots, J - 1. \]

with boundary and initial approximations (22)-(24).

By Taylor expansion we get

\[ e^{U_i^{j+1} - V_i^{j+1}} = e^{-V_i^j} e^{U_i^j} (1 + V_i^j - U_i^j) + e^{V_i^j} e^{U_i^j} (U_i^{j+1} - V_i^{j+1}). \]
+O((U_i^{j+1} - U_i^j)^2) + O((V_i^{j+1} - V_i^j)^2),

e^{V_i^{j+1} - U_i^{j+1}} = e^{-U_i^j} e^{V_i^j} (1 - V_i^j + U_i^j) + e^{-U_i^j} e^{V_i^j} (V_i^{j+1} - U_i^{j+1})

+O((U_i^{j+1} - U_i^j)^2) + O((V_i^{j+1} - V_i^j)^2).

We drop the $O$-terms and the results we insert in (36) and (37) to obtain:

\[-\hat{A}_i U_i^{j+1} + \hat{C}_i U_i^{j+1} - \hat{B}_i U_i^{j+1} + \hat{D}_i V_i^{j+1} = \hat{F}_i, \quad (38)\]

\[\hat{E}_i U_i^{j+1} + \hat{K}_i V_i^{j+1} = \hat{G}_i, \quad (39)\]

where

\[\hat{A}_i = B_i = \frac{1}{2} \sigma^2 \frac{S_i^2}{(\Delta S)^2}, \quad \hat{C}_i = \frac{1}{\Delta \tau} + A_i + B_i + a e^{U_i^{j+1} - V_i^j},\]

\[\hat{D}_i = -a e^{U_i^{j+1} - V_i^j}, \quad \hat{E}_i = \frac{1}{\Delta \tau} U_i^j - a e^{U_i^{j+1} - V_i^j} (1 + V_i^j - U_i^j) + b \Delta \tau,\]

\[\hat{K}_i = \frac{1}{\Delta \tau} + c e^{V_i^{j+1} - U_i^j},\]

\[\hat{G}_i = \frac{1}{\Delta \tau} V_i^j - c e^{V_i^{j+1} - U_i^j} (1 - V_i^j + U_i^j) + c.\]

Since $ae^{U_i^{j+1} - V_i^j} > 0$, the diagonal domination can significally increase in comparison with IMEX linear scheme, see system (20),(21).

**Theorem 3** Let the assumptions of Theorem 1 hold. Then suppose that there exists classical solution $(u, v) \in C^{2,1}(Q_T)$ of problem (10). Then for sufficiently small $\Delta S$ and $\Delta \tau$ the following error estimate holds:

\[\| u - U \|_{C(\omega S)} + \| v - V \|_{C(\omega S)} \leq C (\Delta \tau + (\Delta S)^2),\]

where the constant $C$ doesn’t depend of $\Delta S$ and $\Delta \tau$.

**Proof.** Substituting $V_i^{j+1}$ from (37) into (36) the first one we get

\[-\hat{A}_i U_i^{j+1} + \left( \hat{C}_i - \frac{\hat{D}_i \hat{E}_i}{\hat{K}_i} \right) U_i^{j+1} - \hat{B}_i U_i^{j+1} + \hat{D}_i V_i^{j+1} = \hat{F}_i - \frac{\hat{D}_i}{\hat{K}_i} \hat{F}_i,\]

\[V_i^{j+1} = \frac{\hat{G}_i}{\hat{K}_i} - \frac{\hat{E}_i}{\hat{K}_i} U_i^{j+1}, \quad i = 1, \ldots, I - 1\]

with $U_i^0$, $i = 0, 1, \ldots, I$ and $U_i^j$, $U_i^j$, $j = 0, 1, \ldots, J$ given by (22),(23) and (24).

For the error we have the linear system of algebraic equations

\[-A_i \epsilon_i^{j+1} + \left( C_i - \frac{D_i E_i}{K_i} \right) \epsilon_i^{j+1} - B_i \epsilon_i^{j+1} = \tilde{F}_{i+1} = \frac{\epsilon_i}{\Delta \tau} + \alpha_i^j, \]

\[\epsilon_0^{j+1} = 0, \quad \epsilon_i^{j+1} = 0\]
\[
\mu_{T+1}^{i} = \frac{E_i}{K_i^{i+1}} + \frac{\mu_i^j}{\Delta \tau} + \beta_i^j, \quad i = 1, \ldots, I - 1.
\]

Further we follow the line of Theorem 1 to complete the proof. \(\square\)

The scheme (36), (37) also has similar comparison properties of the linear IMEX scheme described in Theorem 2.

5 Numerical Experiments

In the section we perform numerical experiments to illustrate the accuracy, effectiveness and convergence of the implicit-explicit linear scheme (20)-(24) (Scheme 1) and implicit-explicit linearized scheme (38),(39) (Scheme 2) developed in this article. We provide experiments both with uniform and non-uniform meshes. Also, we present results of numerical experiments using Richardson extrapolation in time.

The Tables (presented results) show the accuracy in maximal discrete norm \(\| \cdot \|\) and convergence rate at final time \(T\), using two consecutive meshes with formulas

\[
\text{Ratio} = \log_2(E_{1/2}^w/E_t^w), \quad E_t^w = \| w_{ex} - W \|_\infty,
\]

where \(w_{ex}\) and \(W\) are the exact and the corresponding numerical solutions, respectively. In our case \(w_{ex}\) is \(R^0\) or \(R^1\).

In Tables 1, 2 we give the results from the computations IMEX linear Scheme 1.

Table 1: Convergence results for at the money \((S = 2, K = 2, S_{\text{min}} = 0 \text{ and } S_{\text{max}} = 5)\) and \(\Delta \tau = \Delta S/2\) based on Scheme 1

| \(I\) | \(R^0\) Value | Difference | Ratio | \(R^1\) Value | Difference | Ratio |
|-----|-------------|------------|-------|-------------|------------|-------|
| 30  | 0.246669    | 0.235165   |       | 0.246669    | 0.235165   |       |
| 60  | 0.247438    | 0.70e-04   | 2.48  | 0.247438    | 0.70e-04   | 2.48  |
| 120 | 0.247749    | 3.11e-04   | 2.48  | 0.247749    | 3.11e-04   | 2.48  |
| 240 | 0.247887    | 1.38e-04   | 2.25  | 0.247887    | 1.38e-04   | 2.25  |
| 480 | 0.247952    | 6.50e-05   | 2.12  | 0.247952    | 6.50e-05   | 2.12  |
| 960 | 0.247983    | 3.10e-05   | 2.10  | 0.247983    | 3.10e-05   | 2.10  |

Table 2 is based on a non-uniform grid and also shows that the scheme is first order in time. Here we use Tavella-Randal [8] mesh:

\[
S_i = K + \alpha \left( c_1^i \frac{2}{T} + c_1 \left( 1 - \frac{i}{7} \right) \right),
\]

\[
c_1 = \sinh^{-1} \left( \frac{S_{\text{min}} - K}{\alpha} \right), \quad c_2 = \sinh^{-1} \left( \frac{S_{\text{max}} - K}{\alpha} \right).
\]

In this case, we choose to concentrate mesh points around the strike price \(K\) since we expect the error to be largest there. In Table 3 we list the results from computation with Scheme 2 that for this non-uniform grid the results are still first order accurate in time as in the uniform case.
Table 2: Convergence results for at the money ($S = 2, K = 2, S_{\text{min}} = 0$ and $S_{\text{max}} = 5$) and taking $\Delta \tau = \Delta S_i/2$ and using nonuniform Tavella-Randal grid with $\alpha = 15$ based on Scheme 1

| I  | Value | Difference | Ratio   | $I$  | Value | Difference | Ratio   |
|----|-------|------------|---------|----|-------|------------|---------|
| 30 | 0.247196 | 0.235660 |          | 60 | 0.247863 | 6.67e-04  | 2.47e-04 |
| 120| 0.248124 | 5.30e-05  | 2.15 (1.10) | 240| 0.248238 | 1.14e-04  | 2.56 (1.35) |
| 480| 0.248291 | 5.30e-05  | 2.15 (1.10) | 120| 0.248124 | 2.61e-04  | 2.29 (1.20) |
| 960| 0.248322 | 3.10e-05  | 1.71 (0.77) | 60 | 0.247863 | 6.45e-04  | 2.61 (1.38) |

Table 3: Convergence results for at the money ($S = 2, K = 2, S_{\text{min}} = 0$ and $S_{\text{max}} = 5$) and taking $\Delta \tau = \Delta S/2$ based on Scheme 2

| I  | Value | Difference | Ratio   | $I$  | Value | Difference | Ratio   |
|----|-------|------------|---------|----|-------|------------|---------|
| 30 | 0.246685 | 0.234952 |          | 60 | 0.247444 | 7.59e-04  | 2.46 (1.30) |
| 120| 0.247752 | 3.08e-04  | 2.46 (1.30) | 240| 0.247889 | 1.37e-04  | 2.25 (1.17) |
| 480| 0.247953 | 6.40e-05  | 2.14 (1.10) | 120| 0.247752 | 3.53e-04  | 2.44 (1.28) |
| 960| 0.247984 | 3.10e-05  | 2.06 (1.05) | 60 | 0.247444 | 8.60e-04  | 2.23 (1.16) |

Now, we improve the convergence in time applying Richardson extrapolation [4]. To this aim we use the formula

\[ Y_n = \frac{2^p W_n - Z_n}{2^p - 1} \]

where $p$ is order of numerical solution (1 in our case) and $W_n$ is the solution obtained using time step $\Delta \tau/2$ and $Z_n$ is the solution obtained using time step $\Delta \tau$. The resulting solution $Y_n$ has order of accuracy $p + 1$ [4]. Table 5 shows the result of applying this technique to the Scheme 1. The order of accuracy in time is now two. Similarly this technique is applied to Scheme 2, see Table 6. Hence the convergence is much slower but smoother compared to the explicit based Scheme 1 due the error of linearisation. The tables shows second order in time.
Table 4: Convergence results for at the money \((S = 2, K = 2, S_{\text{min}} = 0 \text{ and } S_{\text{max}} = 5)\) and taking \(\Delta \tau = \Delta S_i / 2\) and using nonuniform Tavella-Randal grid with \(\alpha = 15\) based on scheme 2

| \(I\) | Value | Difference | Ratio   | Value | Difference | Ratio   |
|------|------|------------|---------|------|------------|---------|
| 30   | 0.248722 | 7.10e-04 |         | 0.237005 |           |         |
| 60   | 0.249432 | 2.83e-04 | 2.51 (1.33) | 0.238139 | 3.27e-04 | 2.47 (1.30) |
| 120  | 0.249839 | 1.24e-04 | 2.28 (1.19) | 0.238283 | 1.44e-04 | 2.27 (1.18) |
| 240  | 0.249897 | 5.80e-05 | 2.14 (1.10) | 0.238351 | 6.80e-05 | 2.12 (1.08) |
| 480  | 0.249928 | 3.10e-05 | 1.87 (0.90) | 0.238387 | 3.60e-05 | 1.89 (0.92) |
| 960  | 0.249940 | 1.56e-05 | 1.80 (0.90) | 0.238405 | 3.00e-05 | 1.82 (0.91) |

Table 5: Convergence results for at the money \((S = 2, K = 2, S_{\text{min}} = 0 \text{ and } S_{\text{max}} = 5)\) and taking \(\Delta \tau = \Delta S / 2\) based on Scheme 1 using Richardson extrapolation

| \(I\) | \(Z_n, W_n\) | \(Y_n\) | Difference | Ratio (order) |
|------|--------------|--------|------------|--------------|
| 10   | 0.2451080    |        |            |              |
| 20   | 0.2465578    | 0.2480075 |            |              |
| 40   | 0.2472811    | 0.2480045 | 3.02e-6    |              |
| 80   | 0.2476431    | 0.2480051 | 5.79e-7    | 5.22 (2.38)  |
| 160  | 0.2478242    | 0.2480053 | 1.96e-7    | 2.96 (1.56)  |
| 320  | 0.2479148    | 0.2480053 | 5.13e-8    | 3.82 (1.93)  |
| 640  | 0.2479600    | 0.2480053 | 1.27e-8    | 4.05 (2.02)  |
| 1280 | 0.2479827    | 0.2480053 | 3.08e-9    | 4.12 (2.04)  |
| 2560 | 0.2479940    | 0.2480053 | 7.45e-10   | 4.13 (2.05)  |

In Figure 1 we compare options values \(p\) and \(q\) at issue and maturity in the liquid and illiquid states using the parameters \(\mu = 0.06, \sigma = 0.3, \nu_{10} = 1, \nu_{10} = 12, K = 2, T = 1, S_{\text{max}} = 5\) and \(\gamma = 1\) using the Scheme 1. Figure 2 illustrates the linearized scheme, using the same parameters. Figures 1,2 illustrate the positivity of the solution \((p, q)\), using both schemes.

6 Conclusions

In this work we have considered one-dimensional problem of European options with liquidity shocks. We have constructed and analyzed two IMEX finite difference schemes that preserve the positivity property of the differential solution. The second one (the IMEX linearized scheme) has better diagonal domination, respectively monotonicity. It would be interesting to consider extensions of the IMEX schemes to the American options with liquidity shocks. In this case one has to solve a free boundary problem. It could be written as a linear complementary problem which could be discretized using the schemes given here. The extension is beyond the scope of this paper, and we leave it for further work.

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Table 6: Convergence results for at the money \((S = 2, K = 2, S_{\text{min}} = 0\) and \(S_{\text{max}} = 5)\) and taking \(\Delta \tau = \Delta S/2\) based on Scheme 2 using Richardson extrapolation.

| \(T\) | \(Z_n, W_n\) | \(Y_n\) | Difference | Ratio (order) |
|------|--------------|---------|------------|---------------|
| 10   | 0.2451717    |         |            |               |
| 20   | 0.2465832    | 0.2479947 | 7.64e-6    |               |
| 40   | 0.2472928    | 0.2480023 | 2.14e-6    | 3.57 (1.84)   |
| 80   | 0.2476486    | 0.2480045 | 6.22e-7    | 3.44 (1.78)   |
| 160  | 0.2478269    | 0.2480051 | 1.78e-7    | 3.49 (1.81)   |
| 320  | 0.2479161    | 0.2480053 | 4.93e-8    | 3.62 (1.85)   |
| 640  | 0.2479607    | 0.2480053 | 1.32e-8    | 3.73 (1.90)   |
| 1280 | 0.2479830    | 0.2480053 | 3.46e-9    | 3.81 (1.93)   |
| 2560 | 0.2479942    | 0.2480053 | 8.95e-10   | 3.87 (1.95)   |
| 5120 | 0.2479998    | 0.2480053 | 2.29e-10   | 3.91 (1.97)   |
| 10240| 0.2480026    | 0.2480053 | 5.99e-11   | 3.95 (1.99)   |

Figure 1: Comparing European option values at issue and maturity in the liquid and illiquid states for the IMEX Linear scheme

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