I Introduction

Noncommutative (NC) physics has became an integral part of present-day high energy physics theories. It reflects a structure of space-time which is modified in comparison to space-time structure underlying the ordinary commutative physics. This modification of space-time structure is a natural consequence of the appearance of a new fundamental length scale known as Planck length \([1, 2]\). There are two major motivations for introducing a Planck length. The first motivation comes from loop quantum gravity in which the Planck length plays a fundamental role. There, a new fundamental length scale emerges as a consequence of the fact that the area and volume operators in loop quantum gravity have discrete spectra, with minimal value proportional to the square and cube of Planck length, respectively. The second
motivation stems from some observations of ultra-high energy cosmic rays which seem to contradict the usual understanding of some astrophysical processes like, for example, electron-positron production in collisions of high energy photons. It turns out that deviations observed in these processes can be explained by modifying dispersion relation in such a way as to incorporate the fundamental length scale $\ell$. NC space-time has also been revived in the paper by Seiberg and Witten [4] where NC manifold emerged in a certain low energy limit of open strings moving in the background of a two form gauge field.

As a new fundamental, observer-independent quantity, Planck length is incorporated in kinematical theory within the framework of the so called doubly special relativity theory (DSR) [5, 6]. The idea that lies behind DSR is that there exist two observer-independent scales, one of velocity, identified with the speed of light, and the other of mass, which is expected to be of the order of Planck mass. It can also be considered as a semi-classical, flat space limit of quantum gravity in a similar way special relativity is a limit of general relativity and, as such, reveals a structure of space-time when the gravitational field is switched off.

Following the same line of reasoning, the symmetry algebra for doubly special relativity can be obtained by deforming the ordinary Poincaré algebra to get some kind of a quantum (Hopf) algebra, known as $\kappa$-Poincaré algebra [7], [8], so that $\kappa$-Poincaré algebra is in the same relation to DSR theory as the standard Poincaré algebra is related to special relativity.

$\kappa$-Poincaré algebra is an algebra that describes in a direct way only the energy-momentum sector of the DSR theory. Although this sector alone is insufficient to set up physical theory, the Hopf algebra structure makes it possible to extend the energy-momentum algebra to the algebra of space-time. It is shown in [9] that different representations (bases) of $\kappa$-Poincaré algebra correspond to different DSR theory. However, the resulting space-time algebra, obtained by the extension of energy-momentum sector, is independent of the representation, i.e. energy-momentum algebra chosen [9] [10].

It is also shown in [10] that there exists a transformation which maps $\kappa$-Minkowski space-time into space-time with noncommutative structure described by the algebra first introduced by Snyder [11]. In [10], the use of Snyder algebra provided NC space-time structure of Minkowski space with undeformed Lorentz symmetry. In the
same paper it is argued that the algebra introduced by Snyder provides a structure of configuration space for DSR and thus can be used to construct the second order particle Lagrangian, what would make it possible to define physical four-momenta determined by the particle dynamics. This would be significant step forward because the theoretical development in this area has been mainly kinematical so far. One such dynamical picture has been given recently in [12] where it was shown that reparametrisation symmetry of the proposed Lagrangian, together with the appropriate change of variables and conveniently chosen gauge fixing conditions, leads to an algebra which is a combination of $\kappa$-Minkowski and Snyder algebra. This generalized type of algebra describing noncommutative structure of Minkowski space-time is shown to be consistent with the Magueijo-Smolin dispersion relation. This type of NC space is also considered in [13]. It has to be stressed that NC spaces in neither of these papers are of Lie-algebra type.

In order to fill this gap, in the present paper we unify $\kappa$-Minkowski and Snyder space in a more general NC space which is of a Lie-algebra type and, in addition, is characterized by the undeformed Poincaré algebra and deformed coalgebra. In other words, we shall consider a type of NC space which interpolates between $\kappa$-Minkowski space and Snyder space in a Lie-algebraic way and has all deformations contained in the coalgebraic sector. First such example of NC space with undeformed Poincaré algebra, but with deformed coalgebra is given by Snyder [11]. Some other types of NC spaces are also considered within the approach in which the Poincaré algebra is undeformed and coalgebra deformed, in particular the type of NC space with $\kappa$-deformation [9],[10],[14],[15],[16]. Here we present a broad class of Lie-algebra type deformations with the same property of having deformed coalgebra, but undeformed algebra. The investigations carried out in this paper are along the track of developing general techniques of calculations, applicable for a widest possible class of NC spaces and as such are a continuation of the work done in a series of previous papers [14],[15],[16],[17],[18],[19],[20]. The methods used in these investigations were taken over from the Fock space analysis carried out in [21],[22].

The plan of paper is as follows. In section 2 we introduce a type of deformations of Minkowski space that have a structure of a Lie algebra and which interpolate between $\kappa$-type of deformations and deformations of the Snyder type. In section 3 we analyze realizations of NC space in terms of operators belonging to the undeformed
Heisenberg-Weyl algebra. In section 4 we tackle the issue of the way in which general invariants and tensors can be constructed out of NC coordinates. Section 5 is devoted to an analysis of the effects which these deformations have on the coalgebraic structure of the symmetry algebra and after that, in section 6 we specialize the general results obtained to some interesting special cases, such as $\kappa$-Minkowski space and Snyder space.

II Noncommutative coordinates and Poincaré algebra

We are considering a Lie algebra type of noncommutative (NC) space generated by the coordinates $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1}$, satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i (a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + s M_{\mu\nu},$$

where indices $\mu, \nu = 0, 1, \ldots, n-1$ and $a_0, a_1, \ldots, a_{n-1}$ are components of a four-vector $a$ in Minkowski space whose metric signature is $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$. The quantities $a_\mu$ and $s$ are deformation parameters which measure a degree of deviation from standard commutativity. They are related to length scale characteristic for distances where it is supposed that noncommutative character of space-time becomes important. When parameter $s$ is set to zero, noncommutativity \((1)\) reduces to covariant version of $\kappa$-deformation, while in the case that all components of a four-vector $a$ are set to 0, we get the type of NC space considered for the first time by Snyder \([11]\). The NC space of this type has been analyzed in the literature from different points of view \([23]\), \([24]\), \([25]\), \([26]\), \([27]\).

The symmetry of the deformed space \((1)\) is assumed to be described by an undeformed Poincaré algebra, which is generated by generators $M_{\mu\nu}$ of the Lorentz algebra and generators $D_\mu$ of translations. This means that generators $M_{\mu\nu}$, $M_{\mu\nu} = -M_{\nu\mu}$, satisfy the standard, undeformed commutation relations,

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda},$$

(2)
with the identical statement as well being true for the generators $D_{\mu}$,

$$[D_{\mu}, D_{\nu}] = 0, \quad (3)$$

$$[M_{\mu\nu}, D_{\lambda}] = \eta_{\nu\lambda} D_{\mu} - \eta_{\mu\lambda} D_{\nu}. \quad (4)$$

The undeformed Poincaré algebra, Eqs.(2),(3) and (4) define the energy-momentum sector of the theory considered. However, full description requires space-time sector as well. Thus, it is of interest to extend the algebra (2),(3) and (4) so as to include NC coordinates $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1}$, and to consider the action of Poincaré generators on NC space $\mathbb{I}$,

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda} - i (a_{\mu} M_{\nu\lambda} - a_{\nu} M_{\mu\lambda}). \quad (5)$$

The main point is that commutation relations (1),(2) and (5) define a Lie algebra generated by Lorentz generators $M_{\mu\nu}$ and $\hat{x}_\lambda$. The necessary and sufficient condition for consistency of an extended algebra, which includes generators $M_{\mu\nu}$, $D_{\mu}$ and $\hat{x}_\lambda$, is that Jacobi identity holds for all combinations of the generators $M_{\mu\nu}$, $D_{\mu}$ and $\hat{x}_\lambda$. Particularly, the algebra generated by $D_{\mu}$ and $\hat{x}_\nu$ is a deformed Heisenberg-Weyl algebra and we require that this algebra has to be of the form,

$$[D_{\mu}, \hat{x}_\nu] = \Phi_{\mu\nu}(D), \quad (6)$$

where $\Phi_{\mu\nu}(D)$ are some functions of generators $D_{\mu}$, which are required to satisfy the boundary condition $\Phi_{\mu\nu}(0) = \eta_{\mu\nu}$. This condition means that deformed NC space, together with the corresponding coordinates, reduces to ordinary commutative space in the limiting case of vanishing deformation parameters, $a_{\mu}, s \to 0$.

One certain type of noncommutativity, which interpolates between Snyder space and $\kappa$-Minkowski space, has already been investigated in \[12,13\] in the context of Lagrangian particle dynamics. However, in these papers algebra generated by NC coordinates and Lorentz generators is not linear and is not closed in the generators of the algebra. Thus, it is not of Lie-algebra type. In contrast to this, here we consider an algebra (1),(2),(5), which is of Lie-algebra type, that is, an algebra which is linear in all generators and Jacobi identities are satisfied for all combinations of generators of the algebra. Besides that, it is important to note that, once having relations (1) and (2), there exists only one possible choice for the commutation relation between
$M_{\mu \nu}$ and $\hat{x}_\lambda$, which is consistent with Jacobi identities and makes Lie algebra to close, and this unique choice is given by the commutation relation (5).

In subsequent considerations we shall be interested in possible realizations of the space-time algebra (1) in terms of canonical commutative space-time coordinates $X_\mu$,

$$[X_\mu, X_\nu] = 0,$$  \hspace{1cm} (7)

which, in addition, with derivatives $D_\mu$ close the undeformed Heisenberg algebra,

$$[D_\mu, X_\nu] = \eta_{\mu \nu}.$$  \hspace{1cm} (8)

From the beginning, the generators $D_\mu$ are considered as deformed derivatives conjugated to $\hat{x}$ through the commutation relation (6). However, in the whole paper we restrict ourselves to natural choice [16] in which deformed derivatives are identified with the ordinary derivatives, $D_\mu \equiv \frac{\partial}{\partial X_\mu}$.

Thus, our aim is to find a class of covariant $\Phi_{\alpha \mu}(D)$ realizations,

$$\hat{x}_\mu = X^\alpha \Phi_{\alpha \mu}(D),$$  \hspace{1cm} (9)

satisfying the boundary conditions $\Phi_{\alpha \mu}(0) = \eta_{\alpha \mu}$, and commutation relations (1) and (5). In order to complete this task, we introduce the standard coordinate representation of the Lorentz generators $M_{\mu \nu}$,

$$M_{\mu \nu} = X_\mu D_\nu - X_\nu D_\mu.$$  \hspace{1cm} (10)

All other commutation relations, defining the extended algebra, are then automatically satisfied, as well as all Jacobi identities among $\hat{x}_\mu$, $M_{\mu \nu}$, and $D_\mu$. This is assured by the construction (9) and (10).

As a final remark in this section, it is interesting to observe that the trilinear commutation relation among NC coordinates has a particularly simple form,

$$[[\hat{x}_\mu, \hat{x}_\nu], \hat{x}_\lambda] = a_\lambda(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + s(\hat{x}_\mu \eta_{\nu \lambda} - \hat{x}_\nu \eta_{\mu \lambda}),$$  \hspace{1cm} (11)

which shows that on the right hand side Lorentz generators completely drop out. In the next section we turn to problem of finding an explicit $\Phi_{\mu \nu}(D)$ realizations (9).
III Realizations of NC coordinates

Let us define general covariant realizations:

\[ \hat{x}_\mu = X_\mu \varphi + i(aX) \left( D_\mu \beta_1 + ia_\mu D^2 \beta_2 \right) + i(XD) \left( a_\mu \gamma_1 + i(a^2 - s)D_\mu \gamma_2 \right), \tag{12} \]

where \( \varphi, \beta_i \) and \( \gamma_i \) are functions of \( A = ia_\alpha D^\alpha \) and \( B = (a^2 - s)D_\alpha D^\alpha \). We further impose the boundary condition that \( \varphi(0,0) = 1 \) as the parameters of deformation \( a_\mu \to 0 \) and \( s \to 0 \). In this way we assure that \( \hat{x}_\mu \) reduce to ordinary commutative coordinates in the limit of vanishing deformation.

It can be shown that Eq. (6) requires the following set of equations to be satisfied,

\[ \frac{\partial \varphi}{\partial A} = -1, \quad \frac{\partial \gamma_2}{\partial A} = 0, \quad \beta_1 = 1, \quad \beta_2 = 0, \quad \gamma_1 = 0. \]

Besides that, the commutation relation (11) leads to the additional two equations,

\[ \varphi \left( \frac{\partial \varphi}{\partial A} + 1 \right) = 0, \tag{13} \]

\[ (a^2 - s)[2(\varphi + A)\frac{\partial \varphi}{\partial B} - \gamma_2(A\frac{\partial \varphi}{\partial A} + 2B\frac{\partial \varphi}{\partial B}) + \gamma_2 \varphi] - a^2 \frac{\partial \varphi}{\partial A} - s = 0. \tag{14} \]

The important result of this paper is that all above required conditions are solved by a general form of realization which in a compact form can be written as

\[ \hat{x}_\mu = X_\mu (-A + f(B)) + i(aX)D_\mu - (a^2 - s)(XD)D_\mu \gamma_2, \tag{15} \]

where \( \gamma_2 \) is necessarily restricted to be

\[ \gamma_2 = -\frac{1 + 2f(B)d_B f(B)}{f(B) - 2B d_B f(B)}. \tag{16} \]

From the above relation we see that \( \gamma_2 \) is not an independent function, but instead is related to generally an arbitrary function \( f(B) \), which has to satisfy the boundary condition \( f(0) = 1 \). Also, it is readily seen from the realization (15) that \( \varphi \) in (12) is given by \( \varphi = -A + f(B) \). Various choices of the function \( f(B) \) lead to different realizations of NC space-time algebra (11). The particularly interesting cases are for \( f(B) = 1 \) and \( f(B) = \sqrt{1 - B} \).

It is now straightforward to show that the deformed Heisenberg-Weyl algebra (6) takes the form

\[ [D_\mu, \hat{x}_\nu] = \eta_{\mu\nu}(-A + f(B)) + ia_\mu D_\nu + (a^2 - s)D_\mu D_\nu \gamma_2 \tag{17} \]
and that the Lorentz generators $M_{\mu\nu}$ can be expressed in terms of NC coordinates as

$$M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu)\frac{1}{\mathcal{C}}.$$  \hspace{1cm} (18)

We also point out that in the special case when realization of NC space (1) is characterized by the function $f(B) = \sqrt{1 - B}$, there exists a unique element $Z$ satisfying:

$$[Z^{-1}, \hat{x}_\mu] = -ia_\mu Z^{-1} + sD_\mu, \quad [Z, D_\mu] = 0.$$  \hspace{1cm} (19)

From these two equations it follows

$$[Z, \hat{x}_\mu] = ia_\mu Z - sD_\mu Z^2, \quad \hat{x}_\mu Z\hat{x}_\nu = \hat{x}_\nu Z\hat{x}_\mu.$$  \hspace{1cm} (20)

The element $Z$ is a generalized shift operator [15] and its expression in terms of $A$ and $B$ can be shown to have the form

$$Z^{-1} = -A + \sqrt{1 - B}.$$  \hspace{1cm} (21)

As a consequence, the Lorentz generators can be expressed in terms of $Z$ as

$$M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu)Z,$$  \hspace{1cm} (22)

and one can also show that the relation

$$[Z^{-1}, M_{\mu\nu}] = -i(a_\mu D_\nu - a_\nu D_\mu)$$

holds. In the rest of paper we shall only be interested in the realizations determined by $f(B) = \sqrt{1 - B}$.

\section*{IV Invariants under Lorentz algebra}

As in the ordinary commutative Minkowski space, here we can also take the operator $P^2 = P_\alpha P^\alpha = -D^2$ as a Casimir operator, playing the role of an invariant in noncommutative Minkowski space. In doing this, we introduced the momentum operator $P_\mu = -iD_\mu$. In this case, arbitrary function $F(P^2)$ of Casimir also plays the role of invariant, namely $[M_{\mu\nu}, F(P^2)] = 0$. However, unlike the ordinary Minkowski
space, in NC case we have a freedom to introduce still another invariant by generalizing the standard notion of d’Alambertian operator to the generalized one required to satisfy

\[ [M_{\mu\nu}, \square] = 0, \]
\[ [\square, \hat{x}_\mu] = 2D_\mu. \]  

The general form of the generalized d’Alambertian operator \( \square \), valid for the large class of realizations (15), which are characterized by an arbitrary function \( f(B) \), can be written in a compact form as

\[ \square = \frac{1}{a^2 - s} \int_0^B \frac{dt}{f(t) - t\gamma_2(t)}, \]  

where \( \gamma_2(t) \) is given in (16). Due to the presence of the Lorentz invariance in NC Minkowski space (1), the basic dispersion relation is undeformed, i.e. it reads \( P^2 + m^2 = 0 \) for all \( f(B) \). Specially, for \( f(t) = \sqrt{1-t} \), we have \( \gamma_2(t) = 0 \) and, consequently, the generalized d’Alambertian is given by

\[ \square = \frac{2(1 - \sqrt{1 - B})}{a^2 - s}. \]  

It is easy to check that in the limit \( a, s \to 0 \), we have the standard result, \( \square \to D^2 \), valid in undeformed Minkowski space.

Lorentz symmetry provides us with the possibility of constructing the invariants. In most general situation, for a given realization \( \Phi_{\mu\nu} \), Eq.(15), Lorentz invariants can as well be constructed out of NC coordinates \( \hat{x}_\mu \). In order to show how is this possible, it is convenient to introduce the vacuum state \( \hat{\phi}(\hat{x}) \) in NC coordinates, with vacuum having the properties,

\[ \hat{\phi}(\hat{x})|\hat{0}\rangle \equiv \hat{\phi}(\hat{x}) \cdot \hat{1} = \hat{\phi}(\hat{x}), \]
\[ D_\mu|\hat{0}\rangle \equiv D_\mu \hat{1} = 0, \quad M_{\mu\nu}|\hat{0}\rangle = 0. \]  

To be more precise, we are looking at the formal series expansions of functions \( \hat{\phi}(\hat{x}) \), which constitute the ring of polynomials in \( \hat{x} \). The vacuum state \( |\hat{0}\rangle \) belongs to \( D \)-module over this ring of polynomials in \( \hat{x} \). It is also understood that NC coordinates
\(\hat{x}\), appearing in (28), refer to some particular realization (15), i.e. they are assumed to be represented by (15).

Analogously to relations (28) and (29), we introduce the vacuum state \(|0>\equiv 1\) as a unit element in the space of ordinary functions \(\phi(X)\) in commutative coordinates, with vacuum \(|0>\) having the properties,

\[\phi(X)|0>\equiv \phi(X),\quad D_{\mu}|0>\equiv D_{\mu}1 = 0,\quad M_{\mu\nu}|0> = 0.\quad (31)\]

The introduced objects are then mutually related by the following relations,

\[\hat{\phi}(\hat{x})|0> = \phi(X),\quad (32)\]
\[\phi(X)|\hat{0}> = \hat{\phi}(\hat{x}).\quad (33)\]

To proceed further with the construction of invariants in NC coordinates for a given realization \(\Phi_{\mu\nu}\), it is also of interest to write down the inverse of realization (15), namely,

\[X_{\mu} = [\hat{x}_{\mu} - i(a\hat{x})\frac{1}{f(B) - B\gamma_2}D_{\mu} + (a^2 - s)(\hat{x}D)\frac{1}{f(B) - B\gamma_2}D_{\mu}\gamma_2] - A + f(B).\quad (34)\]

Since we know how to construct invariants out of commutative coordinates and derivatives, namely, \(X_{\mu}\) and \(D_{\mu}\), relation (34) ensures that the similar construction can be carried out in terms of NC coordinates \(\hat{x}_{\mu}\). The same construction also applies to tensors. All that is required is that the general invariants and tensors, expressed in terms of \(X_{\mu}\) and \(D_{\mu}\), have to be transformed into corresponding invariants and tensors in NC coordinates \(\hat{x}_{\mu}\) and \(D_{\mu}\) with the help of the inverse transformation (34), which, in accordance with Eq. (2), can compactly be written as \(X_{\mu} = \hat{x}_{\mu}(\Phi^{-1})_\alpha^\mu\).

General tensors in NC coordinates can now be built from tensors \(X_{\mu_1}\cdots X_{\mu_k}D_{\nu_1}\cdots D_{\nu_l}\) in commutative coordinates by making use of the inverse transformation (34),

\[X_{\mu_1}\cdots X_{\mu_k}D_{\nu_1}\cdots D_{\nu_l} = \hat{x}_{\beta_1}(\Phi^{-1})^\beta_1_\mu_1\cdots \hat{x}_{\beta_k}(\Phi^{-1})^\beta_k_\mu_kD_{\nu_1}\cdots D_{\nu_l}.\]

The same holds for the invariants. For example, following the described pattern, we can construct the second order invariant in NC coordinates in a following way. Knowing that the object \(X^2 = X_\alpha X^\alpha\) is a Lorentz second order invariant, \([M_{\mu\nu}, X_\alpha X^\alpha] = 0\), the corresponding second order invariant \(\hat{I}_2\) in NC coordinates can be introduced as \(\hat{I}_2 = X_\alpha X^\alpha|\hat{0}>\). After use has been made of (34), simple calculation gives \(\hat{I}_2\)
expressed in terms of NC coordinates, $\hat{I}_2 = \hat{x}_\alpha \hat{x}^\alpha - i(n-1)\alpha_\alpha \hat{x}^\alpha$. It is now easy to check that the action of Lorentz generators on $\hat{I}_2$ gives $M_{\mu\nu}\hat{I}_2|0> = 0$, confirming the validity of the construction.

It is important to realize that NC space with the type of noncommutativity (1) can be mapped to Snyder space with the help of transformation

$$\hat{\tilde{x}}_\mu = \hat{x}_\mu - ia^\alpha M_{\alpha\mu}, \quad (35)$$

generalizing the transformation used in [10] to map $\kappa$-deformed space to Snyder space. After applying this transformation, we get

$$[\hat{\tilde{x}}_\mu, \hat{\tilde{x}}_\nu] = (s - a^2)M_{\mu\nu}, \quad (36)$$

$$[M_{\mu\nu}, \hat{\tilde{x}}_\lambda] = \eta_{\nu\lambda}\hat{\tilde{x}}_\mu - \eta_{\mu\lambda}\hat{\tilde{x}}_\nu. \quad (37)$$

The Lorentz generators are expressed in terms of this new coordinates as

$$M_{\mu\nu} = (\hat{\tilde{x}}_\mu D_\nu - \hat{\tilde{x}}_\nu D_\mu) \frac{1}{f(B)}, \quad (38)$$

and $\hat{\tilde{x}}_\mu$ alone, allows the representation

$$\hat{\tilde{x}}_\mu = X_\mu f(B) - (a^2 - s)(XD)D_\mu \gamma_2, \quad (39)$$

in accordance with (15). These results, starting with the mapping (35) and all down through Eq.(39), are valid not only for the choice $f(B) = \sqrt{1 - B}$, but instead are valid for an arbitrary function satisfying the boundary condition $f(0) = 1$.

V Leibniz rule and coalgebra

The symmetry underlying deformed Minkowski space, characterized by the commutation relations (11), is the deformed Poincaré symmetry which can most conveniently be described in terms of quantum Hopf algebra. As was seen in relations (2), (3) and (4), the algebraic sector of this deformed symmetry is the same as that of undeformed Poincaré algebra. However, the action of Poincaré generators on the deformed Minkowski space is deformed, so that the whole deformation is contained in the coalgebraic sector. This means that the Leibniz rules, which describe the
action of $M_{\mu\nu}$ and $D_\mu$ generators, will no more have the standard form, but instead will be deformed and will depend on a given $\Phi_{\mu\nu}$ realization.

Generally we find that in a given $\Phi_{\mu\nu}$ realization we can write \[e^{i\hat{k}\hat{x}}|0\rangle = e^{iK_\mu(k)X^\mu}\] (40) and 
$e^{i\hat{k}\hat{x}}(e^{iqX}) = e^{iP_\mu(k,q)X^\mu}$, (41)
where the vacuum $|0\rangle$ is defined in (30),(31) and $k\hat{x} = k^\alpha X^\beta \Phi_{\beta\alpha}(D)$. As before, NC coordinates $\hat{x}$, refer to some particular realization (15). The quantities $K_\mu(k)$ are readily identified as $K_\mu(k) = P_\mu(k,0)$ and $P_\mu(k,q)$ can be found by calculating the expression 
$P_\mu(k,-iD) = e^{-ik\hat{x}}(-iD_\mu)e^{ik\hat{x}},$ (42)
where it is assumed that at the end of calculation the identification $q = -iD$ has to be made. One way to explicitly evaluate the above expression is by using the BCH expansion perturbatively, order by order. To avoid this tedious procedure, we can turn to much more elegant method to obtain the quantity $P_\mu(k,-iD)$. This consists in writing the differential equation 
$\frac{dP^{(t)}_\mu(k,-iD)}{dt} = \Phi_{\mu\alpha}(iP^{(t)}(k,-iD))k^\alpha,$ (43)
satisfied by the family of operators $P^{(t)}_\mu(k,-iD)$, defined as 
$P^{(t)}_\mu(k,-iD) = e^{-itk\hat{x}}(-iD_\mu)e^{itk\hat{x}}, \quad 0 \leq t \leq 1,$ (44)
and parametrized with the free parameter $t$ which belongs to the interval $0 \leq t \leq 1$. The family of operators (44) represents the generalization of the quantity $P_\mu(k,-iD)$, determined by (12), namely, $P_\mu(k,-iD) = P^{(1)}_\mu(k,-iD)$. Note also that solutions to differential equation (43) have to satisfy the boundary condition $P^{(0)}_\mu(k,-iD) = -iD_\mu \equiv q_\mu$. The function $\Phi_{\mu\alpha}(D)$ in (43) is deduced from (15) and reads as 
$\Phi_{\mu\alpha}(D) = \eta_{\mu\alpha}(-A + f(B)) + ia_\mu D_\alpha - (a^2 - s)D_\mu D_\alpha \gamma_2.$ (45)
In all subsequent considerations we shall restrict ourselves to the case where $f(B) = \sqrt{1 - B}$. Then we have $\gamma_2 = 0$ and consequently Eq.(43) reads 
$\frac{dP^{(t)}_\mu}{dt} = k_\mu \left[aP^{(t)} + \sqrt{1 + (a^2 - s)(P^{(t)})^2}\right] - a_\mu kP^{(t)},$ (46)
where we have used an abbreviation $P^{(t)}_{\mu} \equiv P^{(t)}_{\mu}(k, -iD)$. The solution to differential equation (46), which obeys the required boundary conditions, looks as

$$P^{(t)}_{\mu}(k, q) = q_{\mu} + (k_{\mu}Z^{-1}(q) - a_{\mu}(kq)) \frac{\sinh(tW)}{W} + \left[(k_{\mu}(ak) - a_{\mu}k^2) Z^{-1}(q) + a_{\mu}(ak)(kq) - sk_{\mu}(kq)\right] \frac{\cosh(tW) - 1}{W^2}. \tag{47}$$

In the above expression we have introduced the following abbreviations,

$$W = \sqrt{(ak)^2 - sk^2}, \tag{48}$$
$$Z^{-1}(q) = (aq) + \sqrt{1 + (a^2 - s)q^2} \tag{49}$$

and it is understood that quantities like $(kq)$ mean the scalar product in a Minkowski space with signature $\eta_{\mu\nu} = \text{diag}(-1, 1, \cdots, 1)$. Now that we have $P^{(t)}_{\mu}(k, q)$, the required quantity $P_{\mu}(k, q)$ simply follows by setting $t = 1$ and finally we also get

$$K_{\mu}(k) = \left[k_{\mu}(ak) - a_{\mu}k^2\right] \frac{\cosh W - 1}{W^2} + k_{\mu} \frac{\sinh W}{W}. \tag{50}$$

Furthermore, we define the star product by the relation,

$$e^{ikX} \star e^{iqX} \equiv e^{iK^{-1}(k)\hat{x}}(e^{iqX}) = e^{iD_{\mu}(k, q)X_{\mu}}, \tag{51}$$

where

$$D_{\mu}(k, q) = P_{\mu}(K^{-1}(k), q), \tag{52}$$

with $K^{-1}(k)$ being the inverse of the transformation (50). It is possible to show that quantities $Z^{-1}(k)$ and $\Box(k)$ can be expressed in terms of quantity $K^{-1}(k)$ as

$$Z^{-1}(k) \equiv (ak) + \sqrt{1 + (a^2 - s)k^2} = \cosh W(K^{-1}(k)) + aK^{-1}(k) \frac{\sinh W(K^{-1}(k))}{W(K^{-1}(k))}, \tag{53}$$
$$\Box(k) = \frac{2}{a^2 - s} \left[1 - \sqrt{1 + (a^2 - s)k^2}\right] = 2(K^{-1}(k)) \frac{1 - \cosh W(K^{-1}(k))}{W^2(K^{-1}(k))}, \tag{54}$$

where $W(K^{-1}(k))$ is given by (48), or explicitly

$$W(K^{-1}(k)) = \sqrt{(aK^{-1}(k))^2 - s(K^{-1}(k))^2}. \tag{55}$$

The function $D_{\mu}(k, q)$ determines the deformed Leibniz rule and the corresponding coproduct $\Delta D_{\mu}$. Relations (53) and (54) are useful in obtaining the expression
for the coproduct. However, in the general case of deformation, when both parameters \( a_\mu \) and \( s \) are different from zero, it is quite a difficult task to obtain a closed form for \( \Delta D_\mu \), so we give it in a form of a series expansion up to second order in the deformation parameter \( a \),

\[
\Delta D_\mu = D_\mu \otimes 1 + 1 \otimes D_\mu - iD_\mu \otimes aD + i a_\mu D_\alpha \otimes D^\alpha - \frac{1}{2} (a^2 - s) D_\mu \otimes D^2 - a_\mu (a D) D_\alpha \otimes D^\alpha + \frac{1}{2} a_\mu D^2 \otimes aD + \frac{1}{2} s D_\mu D_\alpha \otimes D^\alpha + \mathcal{O}(a^3). \tag{56}
\]

Now that we have a coproduct, it is a straightforward procedure \([14],[16]\) to construct a star product between arbitrary two functions \( f \) and \( g \) of commuting coordinates, generalizing in this way relation (51) that holds for plane waves. Thus, the general result for the star product, valid for the NC space (1), has the form

\[
(f \star g)(X) = \lim_{\substack{Y \to X \\ Z \to X}} e^{X_\alpha [iD_\alpha (-iD_Y - iD_Z)-D^\alpha - D_2^\alpha]} f(Y)g(Z). \tag{57}
\]

Although star product is a binary operation acting on the algebra of functions defined on the ordinary commutative space, it encodes features that reflect noncommutative nature of space (1).

In the following section we shall specialize the general results obtained so far to three particularly interesting special cases.

VI Special cases

VI.1 1. case \((s = a^2)\)

In this case, NC commutation relations take on the form

\[
[\hat{x}_\mu, \hat{x}_\nu] = i (a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + a^2 M_{\mu\nu}. \tag{58}
\]

Since we now have \( f(B) = f(0) = 1 \), the generalized shift operator becomes \( Z^{-1} = 1 - A \) and the realizations \([15]\) and \([22]\) for NC coordinates and Lorentz generators, respectively, take on a simpler form, namely,

\[
\hat{x}_\mu = X_\mu (1 - A) + i (aX) D_\mu, \tag{59}
\]
\[ M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) \frac{1}{1 - A}. \] 

(60)

In addition, the generalized d’Alambertian operator becomes a standard one, \( \Box = D^2 \), and deformed Heisenberg-Weyl algebra (17) reduces to

\[ [D_\mu, \hat{x}_\nu] = \eta_{\mu\nu}(1 - A) + ia_{\mu}D_{\nu}. \] 

(61)

Relations (19) and (20), that include generalized shift operator, also change in an appropriate way. Particularly we have

\[ [1 - A, \hat{x}_\mu] = -ia_{\mu}(1 - A) + a^2D_{\mu}. \] 

(62)

We see from Eq.(56) that the coproduct for this case also simplifies since the term with \((a^2 - s)\) drops out.

**VI.2 2. case \((a = 0)\)**

When \(a^2 = 0\), we have a Snyder type of noncommutativity,

\[ [\hat{x}_\mu, \hat{x}_\nu] = sM_{\mu\nu}. \] 

(63)

In this situation, our realization (15) reduces precisely to that obtained in [23]. For a special choice when \(f(B) = 1\), we have the realization

\[ \hat{x}_\mu = X_\mu - s(XD)D_\mu, \] 

(64)

which is the case that was also considered in [28]. In other interesting situation when \(f(B) = \sqrt{1 - B}\), the general result (15) reduces to

\[ \hat{x}_\mu = X_\mu \sqrt{1 + sD^2}. \] 

(65)

This choice of \(f(B)\) is the one for which most of our results, through all over the paper, are obtained and which is the main subject of our investigations. It is also considered by Maggiore [29]. For this case when \(f(B) = \sqrt{1 - B}\), the exact result for the coproduct (52) can be obtained and it is given by

\[ \Delta D_\mu = D_\mu \otimes Z^{-1} + 1 \otimes D_\mu + sD_\mu D_\alpha \frac{1}{Z^{-1} + 1} \otimes D^\alpha, \] 

(66)

where

\[ Z^{-1} = \sqrt{1 + sD^2}. \] 

(67)
VI.3 3. case \((s = 0)\)

The situation when parameter \(s\) is equal to zero corresponds to \(\kappa\)-deformed space investigated in [15]. The generalized d’Alambertian operator is now given as

\[
\Box = \frac{2}{a^2} (1 - \sqrt{1 - a^2 D^2}),
\]

(68)

and the general form (15) for the realizations now reduces to

\[
\hat{x}_\mu = X_\mu (-A + \sqrt{1 - B}) + i(aX) D_\mu,
\]

(69)

where \(B = a^2 D^2\). The Lorentz generators can be expressed as

\[
M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) Z
\]

(70)

and deformed Heisenberg-Weyl algebra (17) takes on the form

\[
[D_\mu, \hat{x}_\nu] = \eta_{\mu\nu} Z^{-1} + ia_\mu D_\nu.
\]

(71)

In the case of \(\kappa\)-deformed space, we can also write the exact result for the coproduct, which in a closed form looks as

\[
\triangle D_\mu = D_\mu \otimes Z^{-1} + 1 \otimes D_\mu + ia_\mu (D_\alpha Z) \otimes D^\alpha - \frac{ia_\mu}{2} \Box Z \otimes iaD,
\]

(72)

where the generalized shift operator (21) is here specialized to

\[
Z^{-1} = -iaD + \sqrt{1 - a^2 D^2}.
\]

(73)

This operator has the following useful properties, with first of them expressing the coproduct for the operator \(Z\),

\[
\triangle Z = Z \otimes Z,
\]

(74)

\[
\hat{x}_\mu Z \hat{x}_\nu = \hat{x}_\nu Z \hat{x}_\mu.
\]

(75)

VII Conclusion

In this paper we have investigated a Lie-algebraic type of deformations of Minkowski space and analyzed the impact of these deformations on some particular issues, such
as the construction of tensors and invariants in terms of NC coordinates and the modification of coalgebraic structure of the symmetry algebra underlying Minkowski space. By finding a coproduct, we were able to see how coalgebra, which encodes the deformation of Minkowski space, modifies and to which extent the Leibniz rule is deformed with respect to ordinary Leibniz rule. Since the coproduct is related to the star product, we were also able to write how star product looks like on NC spaces characterized by the general class of deformations of type (1). We have also found many different classes of realizations of NC space (1) and specialized obtained results to some specific cases of particular interest.

The deformations that we have considered are characterized by the common feature that the algebraic sector of the quantum (Hopf) algebra, which is described by the Poincaré algebra, is undeformed, while, on the other hand, the corresponding coalgebraic sector is affected by deformations.

There is a vast variety of possible physical applications which could be expected to originate from the modified geometry at the Planck scale, which in turn reflects itself in a noncommutative nature of the configuration space. Which type of noncommutativity is inherent to configuration space is still not clear, but it is reasonable to expect that more wider is the class of noncommutativity taken into account, more likely is that it will reflect true properties of geometry and relevant features at Planck scale. In particular, NC space considered in this paper is an interpolation of two types of noncommutativity, \( \kappa \)-Minkowski and Snyder, and as such is more likely to reflect geometry at small distances than are each of these spaces alone, at least, it includes all features of both of these two types of noncommutativity, at the same time. As was already done for \( \kappa \)-type noncommutativity, it would be as well interesting to investigate the effects of combined \( \kappa \)-Snyder noncommutativity on dispersion relations [30],[31], black hole horizons [32], Casimir energy [33] and violation of CP symmetry, the problem that is considered in [34] in the context of Snyder-type of noncommutativity.

Work that still remains to be done includes an elaboration and development of methods for physical theories on NC space considered here, particularly, the calculation of coproduct for the Lorentz generators, \( \triangle M_{\mu\nu} \), S-antipode, differential forms [20],[35], Drinfeld twist [36],[37],[38],[39], twisted flip operator [39],[40],[41] and \( R \)-matrix [42],[41]. We shall address these issues in the forthcoming papers,
together with a number of physical applications, such as the field theory for scalar fields \cite{13,14} and its twisted statistics properties, as a natural continuation of our investigations put forward in previous papers \cite{39,41}.

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References

[1] S. Doplicher, K. Fredenhagen and J. E. Roberts, Phys. Lett. B 331 (1994) 39.

[2] S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. 172 (1995) 187.

[3] G. Amelino-Camelia and T. Piran, Phys. Rev. D 64 (2001) 036005.

[4] N. Seiberg and E. Witten, JHEP 9909 (1999) 032.

[5] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 35.

[6] G. Amelino-Camelia, Phys. Lett. B 510 (2001) 255.

[7] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, Phys. Lett. B 264 (1991) 331.

[8] J. Lukierski, H. Ruegg and W. J. Zakrzewski, Annals Phys. 243 (1995) 90.

[9] J. Kowalski-Glikman and S. Nowak, Phys. Lett. B 539 (2002) 126.

[10] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys. D 12 (2003) 299.

[11] H. S. Snyder, Phys. Rev. 71 (1947) 38.

[12] S. Ghosh, Phys. Rev. D 74 (2006) 084019; S. Ghosh, Phys. Lett. B 648 (2007) 262.

[13] C. Chatterjee and S. Gangopadhyay, Europhys. Lett. 83 (2008) 21002.
[14] S. Meljanac and M. Stojic, Eur. Phys. J. C 47 (2006) 531, arXiv:hep-th/0605133.

[15] S. Kresic-Juric, S. Meljanac and M. Stojic, Eur. Phys. J. C 51 (2007) 229, arXiv:hep-th/0702215.

[16] S. Meljanac, A. Samsarov, M. Stojic and K. S. Gupta, Eur. Phys. J. C 53 (2008) 295, arXiv:0705.2471 [hep-th].

[17] L. Jonke and S. Meljanac, Eur. Phys. J. C 29 (2003) 433, arXiv:hep-th/0210042; I. Dadic, L. Jonke and S. Meljanac, Acta Phys. Slov. 55 (2005) 149, arXiv:hep-th/0301066.

[18] N. Durov, S. Meljanac, A. Samsarov and Z. Skoda, J. Algebra 309 (2007) 318, arXiv:math/0604096 [math.RT].

[19] S. Meljanac and S. Kresic-Juric, J. Phys. A 41 (2008) 235203, arXiv:0804.3072 [hep-th].

[20] S. Meljanac and S. Kresic-Juric, J. Phys. A 42 (2009) 365204, arXiv:0812.4571 [hep-th].

[21] M. Doresic, S. Meljanac and M. Milekovic, Fizika B 3 (1994) 57, arXiv:hep-th/9402013; S. Meljanac, M. Mileković and S. Pallhua, Phys. Lett. B 328 (1994) 55, arXiv:hep-th/9404039; S. Meljanac and M. Mileković, Int. J. Mod. Phys. A 11 (1996) 1391; S. Meljanac, M. Mileković and M. Stojić, Eur. Phys. J. C 24 (2002) 331, arXiv:math-ph/0201061.

[22] V. Bardek and S. Meljanac, Eur. Phys. J. C 17 (2000) 539, arXiv:hep-th/0009099; V. Bardek, L. Jonke, S. Meljanac and M. Mileković, Phys. Lett. B 531 (2002) 311, arXiv:hep-th/0107053; 5; L. Jonke and S. Meljanac, Phys. Lett. B 526 (2002) 149, arXiv:hep-th/0106135.

[23] M. V. Battisti and S. Meljanac, Phys. Rev. D 79 (2009) 067505, arXiv:0812.3755 [hep-th].

[24] H. Y. Guo, C. G. Huang and H. T. Wu, Phys. Lett. B 663 (2008) 270.
[25] J. M. Romero and A. Zamora, Phys. Rev. D 70 (2004) 105006.
[26] R. Banerjee, S. Kulkarni and S. Samanta, JHEP 0605 (2006) 077.
[27] L. A. Glinka, Apeiron 16 (2009) 147.
[28] A. L. Licht, arXiv:hep-th/0512134.
[29] M. Maggiore, Phys. Lett. B 304 (1993) 65; M. Maggiore, Phys. Rev. D 49 (1994) 5182.
[30] G. Amelino-Camelia and L. Smolin, arXiv:0906.3731 [astro-ph.HE].
[31] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88 (2002) 190403; J. Magueijo and L. Smolin, Phys. Rev. D 67 (2003) 044017.
[32] H. C. Kim, M. I. Park, C. Rim and J. H. Yee, JHEP 0810 (2008) 060.
[33] H. C. Kim, C. Rim and J. H. Yee, arXiv:0710.5633 [hep-th].
[34] L. A. Glinka, arXiv:0902.4811 [hep-ph].
[35] H. C. Kim, Y. Lee, C. Rim and J. H. Yee, Phys. Lett. B 671 (2009) 398; J. G. Bu, J. H. Yee and H. C. Kim, arXiv:0903.0040 [hep-th].
[36] A. Borowiec, J. Lukierski and V. N. Tolstoy, Eur. Phys. J. C 44 (2005) 139; A. Borowiec, J. Lukierski and V. N. Tolstoy, Eur. Phys. J. C 48 (2006) 633.
[37] A. Borowiec and A. Pachol, Phys.Rev.D 79 (2009) 045012.
[38] J. G. Bu, H. C. Kim, Y. Lee, C. H. Vac and J. H. Yee, Phys. Lett. B 665 (2008) 95.
[39] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac and D. Meljanac, Phys. Rev. D 77 (2008) 105010, arXiv:0802.1576 [hep-th].
[40] C. A. S. Young and R. Zegers, Nucl. Phys. B 797 (2008) 537; C. A. S. Young and R. Zegers, Nucl. Phys. B 804 (2008) 342.
[41] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac and D. Meljanac, Phys. Rev. D 80 (2009) 025014, arXiv:0903.2355 [hep-th].
[42] C. A. S. Young and R. Zegers, Nucl. Phys. B 809 (2009) 439.

[43] M. Daszkiewicz, J. Lukierski and M. Woronowicz, Mod. Phys. Lett. A 23 (2008) 653; M. Daszkiewicz, J. Lukierski and M. Woronowicz, Phys. Rev. D 77 (2008) 105007; M. Daszkiewicz, J. Lukierski and M. Woronowicz, J. Phys. A 42 (2009) 355201.

[44] L. Freidel, J. Kowalski-Glikman and S. Nowak, Phys. Lett. B 648 (2007) 70.