Topologically non-trivial configurations in 3-dimensional Yang-Mills theory

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Recently Anishetty, Majumdar and Sharatchandra have proposed a way of characterizing topologically non-trivial configurations for 2+1-dimensional Yang-Mills theory in a local and manifestly gauge invariant manner. In this paper we develop criteria to locate such objects in lattice gauge theory and find them in numerical simulations.

1. Introduction

Monopoles are expected to play an important role in confining quarks in QCD. In lattice simulations, one usually looks for U(1) monopoles by fixing an Abelian gauge \([1]\). However it is important to have a gauge invariant way of detecting monopoles in non-Abelian gauge theories. In a recent paper \([2]\) Anishetty, Majumdar and Sharatchandra have given a criterion for characterizing topologically non-trivial configurations in 3-dimensional SU(2) Yang-Mills theory. This criterion is local and manifestly gauge invariant. This was achieved by formulating the theory in terms of gauge invariant variables closely related to gravity. This rewriting also turns out to be a duality transformation as it neatly separates the “spin waves” from the “topological degrees of freedom”. We see this explicitly from simulation data (see fig.1). Reformulation of the theory, even if it gives us the criterion, does not tell us whether such configurations actually occur. That is an important dynamical question and we attempt to answer it by simulating 2+1-dimensional SU(2) lattice gauge theory and checking for the topologically non-trivial configurations. From now on, motivated by the fact that finally we want to look at the 3+1-dimensional theory, we will call these topologically non-trivial objects in three dimensions as monopoles, even though they are actually instantons of the 3-dimensional theory.

2. Method

The partition function of 3-dimensional Yang-Mills theory is

\[
Z = \int DA_a^i \exp \left( - \frac{1}{2\kappa^2} \int d^3 x F_{ij}^a(x) F_{ij}^a(x) \right) (1)
\]

where \(i, j\) run over 1, 2, 3 and

\[
F_{ij}^a(x) = \partial_i A_{j}^a - \partial_j A_{i}^a + \epsilon^{abc} A_{i}^b A_{j}^c (2)
\]

is the usual field strength.

To identify the monopoles of the theory one can use the orthogonal set of eigenfunctions of a positive symmetric matrix. For that purpose we consider the eigenvalue equation of the matrix

\[
I_{ik}^a(x) = \delta_{ik}^j A_{j}^a - \partial_k A_{i}^a + \epsilon^{abc} A_{k}^b A_{i}^c (3)
\]

Here \(A\) is not summed over but labels the eigenvalues.

Isolated points where \(I_{ik}\) have triply degenerate eigenvalues are special, and have topological significance. At such points, the vector fields formed by the eigenvectors of \(I_{ij}\) are singular. The index of the vector field at the singular point is the monopole number. Thus the monopoles in any Yang-Mills configuration \(A_{i}^a(x)\) can be located in terms of the eigenvectors \(\chi_{i}^A(x)\). One can also construct coordinate system using \(\chi_{i}^A(x)\). Integral curves of this vector field are equivalent to
the $r$-coordinate. In that case, the coordinate singularities of this coordinate system correspond to the monopoles.

For our simulations, we choose the usual Wilson action for SU(2) gauge theory

$$S_W = \frac{\beta}{2} \sum_{\text{plaquettes}} \text{tr}(U_{ij} U_{jk} U_{kl} U_{li})$$

(4)

and measure the basic plaquette at every site. The plaquette variable can be written as

$$\exp i F^a_{ij} \sigma^a = \cos(|F_{ij}|) + i \hat{F}^a_{ij} \sigma^a \sin(|F_{ij}|)$$

(5)

where $\hat{F}^a_{ij}$ is the unit vector corresponding to $F^a_{ij}$. Therefore $F^a_{ij} = \hat{F}^a_{ij} \cos^{-1}(\cos(|F_{ij}|))$. Once we get $F^a_{ij}$, we construct $I_{ij}$ as $I_{ij} = F^a_{ik} F^a_{jk}$.

At the location of the monopole, all three eigenvalues of $I_{ij}$ should be degenerate. However on the lattice we do not expect the eigenvalues to become exactly degenerate, but we look for sites where the difference between the eigenvalues are less than some small but non-zero number. Henceforth we shall refer to this number as cut-off. For spherically symmetric monopoles in continuum, around the location of the monopole, one of the eigenvalues will become non-degenerate with the other two which would still be degenerate. The eigenvector corresponding to this eigenvalue will show a radial behaviour. On the lattice we choose the eigenvector corresponding to the largest eigenvalue and plot it at the site of the monopole and its nearest neighbors to check for this radial behavior.

3. Results

In our simulation, we look at various lattice sizes and couplings $\beta$. In three dimensions since $\beta$ has dimension of length (to leading order), we keep the ratio between $\beta$ and the lattice size fixed. This increases the equilibration time for the larger lattices. For lattice size $64^3$ (the largest lattice we consider) the equilibration time is of the order of 240 updates. We ignore the first 300 updates and after that take measurements in every successive update for 300 updates.

The number of monopoles is very sensitive to the choice of the cut-off. A small variation in the cut-off can change the number of monopoles detected by an order of magnitude. To choose the cutoff we look at the distribution of the smallest eigenvalues for the various lattice sizes. Then we choose the cut-off to be half the mean minimum eigenvalue for each lattice size. This minimizes the chance that the eigenvalues become degenerate purely due to statistical effects. The various values for which we take data are shown in Table 1.

| $\beta$ | lattice size | mean lowest eigenvalue | cut-off |
|--------|--------------|------------------------|---------|
| 1.5    | 16           | 0.253                  | 0.1265  |
| 2.25   | 24           | 0.129                  | 0.0646  |
| 3      | 32           | 0.079                  | 0.0397  |
| 3.75   | 40           | 0.0521                 | 0.026   |
| 4.5    | 48           | 0.0368                 | 0.0184  |
| 5.25   | 56           | 0.0264                 | 0.0132  |
| 6      | 64           | 0.0217                 | 0.0109  |

Table 1
The choice of cut-off for various $\beta$

With these parameters, typically we find one lattice site in every two or three measurements which has the difference of eigenvalues less than the cut-off. For a few configurations (roughly 3 or 4 out of the 300 probed), in every lattice size, we find more than one site in a single configuration which meets the eigenvalue criterion. After this we look at the eigenvectors corresponding to the largest eigenvalue around the lattice site which satisfies the criterion for the degenerate eigenvalues. Among them we look for sites that have at least three non-coplanar eigenvectors which converge to or diverge from a point. Figure 2 shows a typical configuration we are looking for. In order to make sure that the eigenvectors really converge, we rotate the eigenvectors and check that they remain convergent from all angles. In our formulation, the eigenvectors only specify the rays and not the direction of the vector. So we do not distinguish between monopoles and antimonopoles. Our results are shown in Table 1.
Table 2
Total number of sites where the eigenvalues are degenerate and around which the eigenvectors show radial behavior.

| $\beta$ | lattice size | degenerate eigenvalues | radial behavior of eigenvector |
|---------|-------------|------------------------|-------------------------------|
| 1.5     | 16          | 59                     | 22                            |
| 2.25    | 24          | 58                     | 12                            |
| 3       | 32          | 67                     | 6                             |
| 3.75    | 40          | 63                     | 14                            |
| 4.5     | 48          | 78                     | 16                            |
| 5.25    | 56          | 52                     | 7                             |
| 6       | 64          | 69                     | 18                            |

Note that the sites which show radial behavior also have degenerate eigenvalues.

4. Conclusions

We have seen that it is indeed possible to detect topologically non-trivial configurations using the criterion presented in [2]. Our data also indicates that configurations extended over many cells or plaquettes are favored compared to the ones over a single cell or plaquettes. Moreover only a fraction of the monopoles detected have a spherical symmetry.

REFERENCES

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2. Ramesh Anishetty, Pushan Majumdar and H. S. Sharatchandra, Phys. Lett. B478 (2000) 373.

Figure 1. Snapshot of a x-y plane with eigenvectors projected to the plane. Lattice size 16.

Figure 2. Eigenvector configuration we are looking for. The eigenvectors shown here are scaled to twice their size to show their crossing explicitly. Lattice size is 56.