Zeta-regularization for exact-WKB resolution of a general 1D Schrödinger equation

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Abstract
We review an exact analytical resolution method for general one-dimensional quantal anharmonic oscillators: stationary Schrödinger equations with polynomial potentials. It is an exact form of WKB treatment involving ‘spectral’ (usual) versus ‘classical’ (newer) zeta-regularizations in parallel. The central results are a set of Bohr–Sommerfeld-like but exact quantization conditions, directly drawn from Wronskian identities, and appearing to extend the Bethe-ansatz formulae of integrable systems. Such exact quantization conditions do not just select the eigenvalues; some evaluate the spectral determinants, and others the wavefunctions, for the spectral parameter in general position.

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Introduction
It is a great honour and pleasure to write in this volume for Professor Dowker as a tribute to his prominent contributions to mathematical physics. Because of our common interests in functional (i.e. zeta-regularized) determinants in spectral theory, we stay in this area. However, instead of the more usual applications in geometry (e.g., [18, 9]), quantum fields (e.g., [12, 14]) and analytic number theory (e.g., [10, 26]), here we propose a foray into analysis, with an exact treatment for Sturm–Liouville equations or 1D stationary Schrödinger equations in quantum mechanics.

We specifically review our exact-WKB resolution of that problem, trying to draw the main lines in a global perspective, which is blurred by many technicalities in our earlier works. Therefore, we make no attempt at completeness here and may discard even important
details for which we refer to earlier presentations ([20, 21, 23, 25] on the technical side; [22, 24] on the survey side). Likewise, the bibliography here is partial and largely meant to refer to those in our earlier works, most developed in [21, 24, 25]. Inversely, two points previously scattered among several articles are revisited here in greater detail: the classical side of zeta-regularization in section 2, and the proof of the basic exact-WKB identities (32) in section 3 and the appendix.

1. General setting

We review an analytical resolution method, through an exact-WKB approach assisted by zeta-regularization, for the stationary 1D Schrödinger equation (or Sturm–Liouville problem):

\[ -\psi''(q) + [V(q) + \lambda] \psi(q) = 0, \quad \left( \frac{d}{dq} \right), \]  

with \( V(q) \) (the potential) a polynomial function, here normalized as

\[ V(q) = +q^N + v_1 q^{N-1} + \cdots + v_{N-1} q \quad (N \geq 1). \]

Key parameters will be

\[ N \text{ (degree)}, \quad \mu \equiv \frac{1}{2} + \frac{1}{N} \text{ (order)}, \quad \varphi \equiv \frac{4\pi}{N+2} \text{ (symmetry angle)}, \]

plus the ‘number of conjugates’ \( L \) to be determined in (5). Often we will simplify results by excluding \( N = 2 \), which is most singular as a special case within general-\( N \) theory.

Other quantum-mechanical notation will also help: \( \hat{H} \) (Schrödinger operator), \( E \) (energy variable), condensing (1) to \( \hat{H} \psi = E \psi, \) with \( \hat{H} \equiv -\frac{d^2}{dq^2} + V(q), \) \( E \equiv -\lambda, \)

but this continues to mean the bare differential equation (1): \( E \) is not restricted \textit{a priori} to eigenvalues, nor \( \psi \) to some operator domain.

Being motivated by self-adjoint bound-state calculations in quantum mechanics, we envision both \( q \) and \( V \) as real, or \( \vec{v} \equiv (v_1, \ldots, v_{N-1}) \in \mathbb{R}^{N-1}. \) For such spectral problems, the best setting is a real half-line, say \( q \in [0, +\infty), \) with ultimately a Dirichlet or Neumann boundary condition at the endpoint \( q = 0. \) Still, our exact resolution will involve

– the whole complex \( q \)-plane;

– all ‘conjugate equations’ of the form (1) but complex, that restore (1) under some analytic dilation \( q \mapsto e^{i\varphi} q; \) see [15, section 2.7]; there are finitely many, precisely

\[ L = \begin{cases} N+2 & \text{in general} \\ \frac{1}{2}N+1 & \text{for an even polynomial } V(q) \end{cases} \]

\[ \text{distinct conjugates :} \]

\[ V^{(\ell)}(q) \equiv e^{-i\varphi} V(e^{-i\varphi}/2 q), \quad \lambda^{(\ell)} \equiv e^{-i\varphi} \lambda \quad \text{for } \ell = 0, 1, \ldots, L - 1 \quad (\text{mod } L), \]

with the angle \( \varphi = 4\pi/(N+2) \) as announced in (3).

Traditionally, the ordinary differential equation (1) only admitted asymptotic (hence approximate) solutions in an analytic form: \textit{WKB expressions}, valid for \( q \) or \( \lambda \) large. Then, parts of WKB theory have been turned fully exact [6, 3, 15, 17, 1]. What we will outline here is a \textit{global exact-WKB} treatment for (1), which just falls short of self-sufficiency; curiously, it still needs explicit inputs of prior asymptotic-WKB results; hence, we review these first.

\footnote{Errata for Voros 2000 [21]: (1) we misused the term ‘quasi-exactly solvable’ for ‘supersymmetric’ (system) throughout—with no substantial consequences; (2) in the ‘Airy zeros’ part of table 1, \( Z_1^{+1}(0) \approx 0.0861126, \) \( e^{-Z_1^{+1}(0)} \approx 0.9174909, \) \( e^{-Z_1^{+1}(0)} \approx 1.2585417. \)
2. Asymptotic ingredients

2.1. Known spectral properties

To streamline this part, we phrase it for even polynomials \( V(q) \). (The non-even case adds minor, but tedious, complications [21].) Then, the Schrödinger operator \( \hat{H} \) over all of \( \mathbb{R} \), with \( V(q) \uparrow +\infty \) at both ends \( q \to \pm \infty \), has a discrete countable \( E \)-spectrum \( E \) defined as \( E_k = \{E_k\}_{k=0,1,...} \) with \( E_k \uparrow +\infty \). By parity-symmetry, moreover, \( E^+_\mu = \{E_k\}_{k \text{ even}} \) is the Neumann spectrum for (4) over the half-line \( q \in [0, +\infty) \), while \( E^-_\mu = \{E_k\}_{k \text{ odd}} \) is the Dirichlet spectrum.

2.1.1. Semiclassical asymptotics. The spectral semiclassical regime is \( k \to +\infty \), for which the eigenvalues \( E_k \) asymptotically obey a Bohr–Sommerfeld condition shaped by the classical Hamiltonian \( p^2 + V(q) \) over the phase space \( \mathbb{R}^2 \), as [5, section 10.5]

\[
\oint_{p^2 + V(q) = E} \frac{p \, dq}{2\pi} \sim k + \frac{1}{2} \text{ for integer } k \to +\infty;
\]

the leading power behaviour in \( E \to +\infty \) is

\[
\oint_{p^2 + V(q) = E} \frac{p \, dq}{2\pi} \sim b_\mu E^\mu \text{ with } \mu = \frac{1}{2} + \frac{1}{N} \left( \text{and } b_\mu = \oint_{p^2 + q^2 = 1} \frac{p \, dq}{2\pi} \right),
\]

introducing an essential parameter, the order \( \mu \in (\frac{1}{2}, \frac{3}{2}] \). However, a stronger result needs to be known: that the complete Bohr–Sommerfeld formula, which incorporates the semiclassical corrections to (7) to all orders [5, section 10.7], admits a complete \( E \to +\infty \) expansion as a descending power series,

\[
\sum_{a \in A} b_a E^a \sim k + \frac{1}{2} \text{ for integer } k \to +\infty, \quad A = \{\mu, \mu - \frac{1}{N}, \mu - \frac{2}{N}, \ldots\}
\]

(each \( b_a \) is a polynomial expression in the \( \{v_j\}_{j \leq (\mu - a)N} \)).

2.1.2. Spectral functions. The form (9) implies the following statements, issued for real \( \lambda > -\min V \) initially (other \( \lambda \) are to be reached by analytical continuation).

- The pair of series

\[
Z_{\pm}(s, \lambda) \overset{\text{def}}{=} \sum_{k \text{ even/odd}} (E_k + \lambda)^{-s} \quad \text{for } \Re{s} > \mu
\]

converge, defining analytic functions \( Z_{\pm} \): (generalized) spectral zeta-functions.

- The summations in (10) can be handled by Euler–Maclaurin formulae, whereby \( Z_{\pm} \) continue to meromorphic functions in the whole complex \( s \)-plane, pole-free (i.e. regular) at \( s = 0 \) [20, section 1.2].

Besides, (term by term) all sums of the form (10) obey the general identity

\[
\partial_\lambda Z(s, \lambda) = -sZ(s + 1, \lambda).
\]

Next, the regularity of \( Z_{\pm} \) at \( s = 0 \) allows us to specify spectral determinants, not just formally as

\[
D_{\pm}(\lambda) = \prod_{k \text{ even/odd}} (\lambda + E_k)^{-1} \quad \text{(severely divergent)},
\]

but rigorously, as well-defined entire functions of \( \lambda \), through

\[
\log D_{\pm}(\lambda) \overset{\text{def}}{=} -\partial_\lambda Z(s, \lambda)_{s=0} \quad \text{(zeta-regularization)}.
\]
2.2.1. Known WKB results.

The differential equation (1) on the half-line \([0, +\infty )\).

\[
\psi (q) \propto \Pi_\lambda (q)^{-1/2} \exp \left\{ \pm \int^q \Pi_\lambda (\tilde{q}) \, d\tilde{q} \right\}, \quad \Pi_\lambda (q) \overset{\text{def}}{=} (V(q) + \lambda)^{1/2}. \tag{17}
\]
i.e. \(i\Pi_\lambda (q)\) is the momentum in the classically forbidden region; primitives (as in the exponent) are the action functions; \(\Pi_\lambda (q) > 0\) for \(q \to +\infty\) fixes the branch. Differentiations of (17) (with respect to \(q, \lambda\)) are also valid.

An explicit \(q \to +\infty\) form for \(\psi\) stems from the classical expansion

\[
[V(q) + \lambda]^{-s+1/2} \sim \sum_\sigma \beta_\sigma (s; \vec{v}, \lambda) q^{\sigma - N s}, \quad \sigma = \frac{N}{2}, \frac{N}{2} - 1, \frac{N}{2} - 2, \ldots \tag{18}
\]
(in which the \(\beta_\sigma\) come out as polynomials in \(\vec{v}\) and \(s\)), as

\[
\psi (q) \propto q^{-N/4 + \beta_{-1}(s)} \exp \left\{ \pm \sum_{\sigma = 0}^N \beta_{\sigma - 1}(q) \frac{q^\sigma}{\sigma} \right\} \tag{19}
\]

[15, theorem 6.1(ii)], where \(\beta_{\sigma - 1}(\vec{v}) \overset{\text{def}}{=} \beta_{\sigma - 1}(0; \vec{v}, \lambda)\) is independent of \(\lambda\) for \(\sigma \geq 0\) when \(N \neq 2\).
2.2.2. Classical zeta-regularization. The overall constant factor implied in (19) depends on the primitive of $\pm \Pi_\lambda(q)$ selected within (17). Against common practice and for later simplicity, here we opt for an ‘improper action’, formally meant as [21]

$$I_\lambda(q) = \int_q^{+\infty} \Pi_\lambda(q) \, d\bar{q} \quad \text{(a highly divergent integral).}$$  \hspace{1cm} (20)

A natural regularization for (20) would be the analytical continuation of

$$I_\lambda(q, s) \overset{\text{def}}{=} \int_q^{+\infty} (V(q) + \lambda)^{1/2-s} \, d\bar{q} \quad \text{(Re } s > \mu)$$  \hspace{1cm} (21)

to $s = 0$, but the singular structure of $I_\lambda(q, s)$, which follows from expansion (18) term by term, shows (for even $N$) a generic simple pole at $s = 0$ of residue $N^{-1} \beta_{-1}(\bar{v})$; just for definiteness, $\beta_{-1}(\bar{v})$ is the (finite) sum [20]

$$\beta_{-1}(\bar{v}) = \sum_{\{r_j \neq 0\}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - \sum_{j=1}^{N-1} r_j)} \frac{\beta_1^{r_1} \cdots \beta_{N-1}^{r_{N-1}}}{r_1! \cdots r_{N-1}!} \quad \text{(for } N \neq 2).$$  \hspace{1cm} (22)

Thus (even though the Kronecker $\delta$ in (22) can often suppress the whole sum), a fully general specification of $I_\lambda(q, s)$ must accommodate $\beta_{-1}(\bar{v}) \neq 0$; this we pursue again at $q = 0$ and for an even $V(q)$ (still for simplicity only).

We then argue that (20) (at $q = 0$) can be fully defined in complete classical parallelism with the zeta-regularization of (12) to $\log D(\lambda)$ in the quantal setting, when $D(\lambda) = \det(\hat{H} + \lambda)$ over the whole real line or $D(\lambda) = D^{+}(\lambda)D^{-}(\lambda)$, all under $V(q)$ even. Indeed, the classical limits of the Schrödinger operator $\hat{H}$ and of the trace operation $\text{Tr}$ are the Hamiltonian function $p^2 + V(q)$ and the phase-space integration $(2\pi)^{-1} \int dp \, dq$, respectively. Hence, the classical analogue of the zeta-function $Z(s, \lambda) = \text{Tr}(\hat{H} + \lambda)^{-s}$ has to be

$$Z_{cl}(s, \lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} dq \, dp \, (p^2 + V(q) + \lambda)^{-s} \quad \text{(Re } s > \mu)$$

$$= \frac{\Gamma(s - \frac{1}{2})}{\sqrt{\pi} \Gamma(s)} \int_0^{+\infty} (V(q) + \lambda)^{1/2-s} \, dq = \frac{\Gamma(s - \frac{1}{2})}{\sqrt{\pi} \Gamma(s)} I_\lambda(q = 0, s),$$  \hspace{1cm} (23)

and the last expression yields a regular continuation at $s = 0$ (where the simple pole of $I_\lambda(q, s)$ gets cancelled by $\Gamma(s)^{-1}$). We can thus define

$$\log D_{cl}(\lambda) \overset{\text{def}}{=} -\partial_s Z_{cl}(s, \lambda)_{s=0}. \quad \text{(24)}$$

Now when $I_\lambda$ is regular at $s = 0$, $-\frac{1}{2} \partial_s Z_{cl}(s, \lambda)_{s=0}$ evaluated through (23) coincides with $I_\lambda(0, 0) = \int_0^{+\infty} \Pi_\lambda(q) \, dq$ formally). Inversely then,

$$\int_0^{+\infty} \Pi_\lambda(q) \, dq \overset{\text{def}}{=} \frac{1}{\pi} \log D_{cl}(\lambda) \quad \text{(always finite)} \quad \text{(25)}$$

extends the improper action to the singular-$I_\lambda$ case as well. Equivalently, in parallel to the quantal case (although $D_{cl}$ has a branch cut instead of being entire) $\log D_{cl}$ has a canonical large-$\lambda$ expansion; now only one primitive of $\int_0^{+\infty} \partial_s \Pi_\lambda(q) \, dq$ can have that property; it then specifies $\int_0^{+\infty} \Pi_\lambda(q) \, dq$.

Improper actions benefit WKB theory in many other ways [25], and they are no less accessible than conventional action integrals; e.g.,

$$\int_0^{+\infty} (q^N + \lambda)^{1/2} \, dq = \frac{\Gamma(1 + \frac{1}{N})}{2\sqrt{\pi}} \Gamma(-\frac{1}{2} - \frac{1}{N}) \lambda^{\frac{1}{4} + \frac{1}{2N}} \quad \text{(N \neq 2)}$$

$$= -\frac{1}{2} \lambda (\log \lambda - 1) \quad \text{(N = 2)}$$

$$\int_0^{+\infty} (q^4 + vq^2 + \lambda)^{1/2} \, dq = \text{case } v, \lambda \geq 0$$

5
\[
\begin{align*}
\frac{1}{2} (v + 2 \sqrt{\lambda})^{1/2} & \left[ 2 \sqrt{\lambda} K(k) - v E(k) \right], \\
\frac{1}{2} \lambda^{1/2} & \left[ (2 \sqrt{\lambda} + v) K(\tilde{k}) - 2 v E(\tilde{k}) \right],
\end{align*}
\]

\[ k \overset{\text{def}}{=} \frac{\left( \frac{\nu}{\nu - \sqrt{\lambda}} \right)^{1/2}}{\sqrt{\nu - 2 \sqrt{\lambda}}} \quad (v \geq 2 \sqrt{\lambda})
\]

\[ k \overset{\text{def}}{=} \frac{(\sqrt{\nu} - \sqrt{\lambda})^{1/2}}{\sqrt{\nu - 2 \sqrt{\lambda}}} \quad (v \leq 2 \sqrt{\lambda}),
\]

where \( E(\cdot), K(\cdot) \) are the usual complete elliptic integrals [11, vol II, section 13.8].

The calculation as above but for the ratio \( D^+ / D^- \) in place of the product \( D^+ D^- \) is now regular, and its output, \([D^+_q / D^-_q](\lambda) \equiv \Pi_\lambda(0)\), completes the results we need to rewrite the WKB formulæ suggestively for later analogy. Together with (25), and with translation covariance restored (thanks to \( \int_I \Pi_\lambda(\tilde{q}) \, dq \) being canonical for finite intervals \( I \)), all of that converts the canonically normalized recessive WKB solution (17) plus its \( q \)-derivative partner, namely

\[
\begin{align*}
\psi_{\text{WKB}}(q) & = -\Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(\tilde{q}) \, d\tilde{q}, \\
[\psi]'_{\text{WKB}}(q) & = -\Pi_\lambda(q)^{+1/2} \exp \int_q^{+\infty} \Pi_\lambda(\tilde{q}) \, d\tilde{q},
\end{align*}
\]

\[ (32) \]

to the form (also valid for non-even \( V \))

\[ \psi_{\text{WKB}}(q) \equiv D^-_{q;cl}(\lambda), \quad [\psi]'_{\text{WKB}}(q) \equiv -D^+_{q;cl}(\lambda), \]

\[ (30) \]

where \( D^-_{\boldsymbol{q};cl}(\lambda) \) are the ‘classical determinants’ defined like \( D^\pm_{\boldsymbol{q}}(\lambda) \) but over the half-line \([q, +\infty)\) in place of \([0, +\infty)\).

3. Exact analysis of the Schrödinger equation

The differential equation (1) is known to admit exact solutions \( \psi(q) \) which are recessive, i.e. decaying for \( q \rightarrow +\infty \), according to the asymptotic WKB form (17). We specifically select the exact solution \( \psi_\lambda(q) \) that is canonically recessive: i.e. asymptotic to our normalized (decaying) WKB form (29),

\[ \psi_\lambda(q) \sim \psi_{\text{WKB}}(q) \equiv \Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(\tilde{q}) \, d\tilde{q}, \quad \text{for} \quad q \rightarrow +\infty. \]

\[ (31) \]

3.1. Basic exact-WKB identities

The exact solution \( \psi_\lambda \), given the normalization (31) for \( q \rightarrow +\infty \), is then expressed at finite \( q \) by the exact identities [21]

\[ \psi_\lambda(q) \equiv D^-_q(\lambda), \quad \psi'_\lambda(q) \equiv -D^+_q(\lambda), \]

\[ (32) \]

in terms of the zeta-regularized Dirichlet (−), resp. Neumann (+), spectral determinants over the half-line \([q, +\infty)\).

The crucial feature of (32) is the unseen one! No fudge factor whatsoever in-between the solution (\( \psi_\lambda, \psi'_\lambda \)) and the spectral determinants \( D^\pm \), thanks to careful normalizations, is what will allow for exact solution algorithms.

In view of the alternative form (30) exhibited by the usual asymptotic-WKB solutions, we see (32) as exact-WKB formulæ for the exact solution \( \psi_\lambda \). Clearly, these do not appear as explicit as the asymptotic-WKB solutions, expressible by quadratures as in (29). Nevertheless, our main point is this: equation (32) too will end up ‘solving’ the Schrödinger problem analytically, and in a more roundabout but now exact manner.

Proof of (32) (adapted from [17, appendices A and B]). It has two steps: (1) prove the first logarithmic derivatives of the identities (32) with respect to \( \lambda \); (2) do one controlled
integration thereupon to attain (the logarithm of) (32) itself. (All that for \( N > 2 \); here we skip the particular cases \( N = 1 \) and 2, which need two differentiations/integrations.) We denote \( q \equiv \frac{\delta}{\delta q} \) and, without loss of generality, set \( q = 0 \) again.

(1) Applying to (13) the basic functional relation (11) tuned to \( s = 0 \), \([\partial s Z(\lambda, \nu, \nu_0)] = -Z(1, \nu)\), we find the logarithmic derivative of (32) with respect to \( \lambda \), at \( q = 0 \), to be

\[
\frac{\dot{\psi}_0}{\psi_0} = Z^{-1}(1, \lambda), \quad \frac{\dot{\psi}_1}{\psi_1} = Z^+(1, \lambda).
\]  

(33)

We then refer to the appendix for a trace-formula proof of (33).

(2) One integration upon (33) with respect to \( \lambda \) gives

\[
\log \psi_0(0) \equiv \log D^- (\lambda) + C^-, \quad \log[\psi_1(0)] \equiv \log D^+ (\lambda) + C^+, \quad (34)
\]

where \( C^\pm \) are integration constants. (For sign-fixing, every argument of \( \log(\cdot) \) has to be positive for large \( \lambda > 0 \).) Now, WKB asymptotics like (31) also hold for \( \lambda \rightarrow +\infty \) at fixed \( q \), and we have painstakingly normalized earlier to ensure canonical large-\( \lambda \) behaviour on all sides; therefore, any extra additive constants like \( C^\pm \) are banned by (16), and (32) results.

\[ \square \]

3.2. Explicit Wronskian identity

A second solution of (1) besides \( \psi_2 \), now recessive for \( q \rightarrow +e^{-i\varphi/2}\infty \), can be specified through the first conjugate equation, (6) for \( \ell = 1 \), [15, section 2.7] as

\[
\Psi_\ell(q) \equiv \psi_{\ell,0}(e^{i\varphi/2} q).
\]  

(35)

Because the asymptotic form (19) for \( \psi_2 \) and its analogues for \( \Psi_\ell(q) \) and the \( q \)-derivatives all hold over the positive direction, the constant Wronskian of \( \Psi_\ell \) and \( \psi_{\ell,0} \), \( W(\Psi_\ell, \psi_{\ell,0}) \equiv \Psi_\ell \psi_{\ell,0}' - \psi_{\ell,0} \Psi_\ell' \), can be deduced explicitly, just from their \( q \rightarrow +\infty \) asymptotics, see [15, theorem 21.1(iii)]. Under the canonical normalizations and recalling (22) for \( \beta_{-1}(\vec{v}) \), it is [20, including corrigendum]

\[
W(\Psi_\ell, \psi_{\ell,0}) \equiv 2i e^{i\varphi/4} e^{i\varphi_{-1} (\vec{v})/2} \quad (N \neq 2). \quad (36)
\]

This basic Wronskian identity is the other essential piece of the framework: if we now use the exact-WKB identities (32) within the alternative form \( W(\Psi_\ell, \psi_{\ell,0}) = \Psi_\ell(0)\psi_{\ell,0}'(0) - \Psi_{\ell}'(0)\psi_{\ell,0}(0) \), this then translates (36) into a functional relation between spectral determinants, which for \( N \neq 2 \) reads

\[
e^{i\varphi/4} D^{[11]^+} (e^{-i\varphi} \lambda) D^{-} (\lambda) - e^{-i\varphi/4} D^{[11]} (\lambda) D^{[11]^+} (e^{-i\varphi} \lambda) \equiv 2i e^{i\varphi_{-1} (\vec{v})/2}. \quad (37)
\]

3.3. Exact quantization conditions

At first glance, the original Wronskian identity (36), or its sibling (37), looks grossly underdetermined: one equation for two unknown functions. This view however proves a delusion here, thanks to the structure formulae (14). Their first effect is to reduce quests for determinants to quests for their zeros (the eigenvalues): in other words, we are to seek exact quantization conditions. The latter problem then becomes central to the whole framework; we now sketch our solution for it (as valid for \( N > 2 \)) [21].

To analyse the eigenvalues \( \{E_k\} \) of the Schrödinger operator \( \hat{H} \) on the half-line, we may divide the functional relation (37) at \( \lambda = -E_k \) by its first conjugate partner, to obtain

\[
2 \arg D^{[11]} (- e^{-i\varphi} E) - \varphi \beta_{-1}(\vec{v}) \big|_{E = E_k} = \pi \left( k + \frac{1}{2} \pm \frac{N - 2}{2(N + 2)} \right), \quad \text{for } k \text{ even}. \quad \]  

(38)
The formulae (38) have the outer form of Bohr–Sommerfeld quantization conditions (smooth left-hand-side functions of $E$, sampled at integers). Now they are exact, albeit not closed (the left-hand sides invoke the determinants of the first conjugate spectra, equally unknown).

However, in either (Neumann / Dirichlet) sector—there is complete decoupling between them—the set of all quantization conditions for the $L$ conjugate problems plus the $L$ structure formulae (now used to reconstruct each determinant from its spectrum) add up to a formally closed system:

$$i^{-1} \left[ \log D^{(l+1)\pm}(-e^{-i\varphi}E_{k}^{(l)}) - \log D^{(l-1)\pm}(-e^{i\varphi}E_{k}^{(l)}) \right] = (-1)^{l}\varphi\beta_{-1}(\bar{\varphi})$$

$$= \pi \left[ k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \right], \quad \text{(39)}$$

$$\log D^{(l)\pm}(\lambda) \equiv \lim_{K \to +\infty} \left\{ \sum_{k<K} \log \left( E_{k}^{(l)} + \lambda \right) + \frac{1}{2} \log \left( E_{k}^{(l)} + \lambda \right) \right. - \frac{1}{2} \sum_{|a|>0} b_{a}^{(l)}(E_{k}^{(l)})^{2} \left[ \log E_{k}^{(l)} - \frac{1}{a} \right] \bigg\}, \quad \text{(40)}$$

for $l \in \mathbb{Z}/L\mathbb{Z}$ and $k, K$ even and $K$ odd. The self-consistency brought by (40) makes this a fixed-point problem, with parities decoupled, hence abstractly of the form $\mathcal{M}^{\pm}\{\mathcal{E}^{\pm}\} = \mathcal{E}^{\pm}$, where $\mathcal{E}^{\pm}$ (‘compound spectra’) denote the disjoint unions of all the $L$ conjugate spectra $\mathcal{E}^{(l)\pm}$ of a given parity (figure 1), and $\mathcal{M}^{\pm}$ are some spectra-to-spectra mappings. Note that as a completely general fact, such concrete forms and mappings are highly non-unique; we just hope to find some such explicit form(s) of (39)–(40) that behave nicely.

The equations (40) furthermore require asymptotically correct arguments $\mathcal{E}^{\pm}$, for the $K \to +\infty$ convergence to take place.

In the simplest case, all homogeneous potentials $V(q) = q^{2}$ [19, 7], the compound spectra amount to the original ones $\mathcal{E}^{\pm} \subset \mathbb{R}_{+}$, mappings $\mathcal{M}^{\pm}$ exist in the real form: $|E_{k}^{(l)} > 0 \leftrightarrow |E_{k}^{(l)} > 0 \rangle (k \text{ even } \lambda \text{ odd})$, given by the formulæ

$$A_{\pm}(E_{k}^{(l)}) = k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \quad \text{for } k \text{ even } \lambda \text{ odd},$$

$$A_{\pm}(E^{(l)}) \equiv \sum_{\ell} \arg(E_{\ell}^{(l)} - e^{-i\varphi}E^{(l)}) \quad \text{for } \ell \text{ even } \lambda \text{ odd}, E^{(l)} > 0, \quad (N \neq 2), \quad \text{(41)}$$

2 Errata for Voros 1999 [19b]: (1) in equations (18), $D^{\pm}(e^{-i\varphi}\lambda)$ should read $D^{\pm}(-e^{-i\varphi}\lambda)$ (twice) and just underneath, $[0, e^{-i\varphi}\infty)$ should read $[0, -e^{-i\varphi}\infty)$; (2) in equations (42) and (43), we were unaware of a related reference [27].
and these have now been proven globally contracting [2]. Next, for moderately inhomogeneous potentials, we have only tested quartic and sextic cases numerically, see [20, section 3], and still—but without proof to date—observed contracting maps. As long as this holds, the unknowns in (39)–(40) (the eigenvalues and the determinants) get uniquely determined by the fixed-point equations, and effectively reached as limits under iterations of the contracting maps; thus, the fixed-point equations qualify as (exact) quantization conditions. Only in some larger-\(v\)-regions do our current schemes become numerically unstable—a non-elucidated issue. (Note that all numerical mappings have to be finite dimensional as well, which entails some approximation.) Again for homogeneous potentials, moreover, (41) mysteriously coincides with the Bethe-ansatz equations for certain solvable models of 2D statistical mechanics. This has fostered a new research area, the ODE/IM correspondence, relating our exact-WKB solvability to complete integrability in the modern sense [7, 4, 16, 8].

We may then briefly categorize 1D Bohr–Sommerfeld quantization conditions for the eigenvalues \(E_k\), in the common general form \(A(E_k) = k + \frac{1}{2}\), as follows:

- old exact-WKB setting, special: the \(E_k\) get specified exactly via an explicit function \(A(E)\), but only for exceptional systems (e.g., harmonic oscillator);
- old general WKB approaches, semiclassical: the \(E_k\) get specified via an explicit function or expansion \(A(E)\), but only asymptotically for \(k \to +\infty\);
– our general exact-WKB formalism: the $E_k$ get specified exactly via a function $\Lambda(E)$, itself specified exactly, but implicitly through a fixed-point equation which, while infinite-dimensional, is explicit.

3.4. Solving for the unknown functions $\psi$

We focus on finding the canonical recessive solution $\psi_\lambda$ for arbitrary $\lambda$ (once a particular solution is known, others trivially follow).

Equations (32) have specified $\psi_\lambda(q)$ (and $\psi'_\lambda(q)$) as particular spectral determinants over the half-line $[q, +\infty)$; now the latter are of the type fully resolved by the exact quantization conditions above. Consequently, the unknown wavefunctions $\psi$ are computable as well in the exact approach (as numerical tests confirm: figure 2) [20, 22].

3.5. Towards 1D quantum perturbation theory

On potentials like $V(q) = q^2 + gq^4$, it is possible to monitor the $g \to 0^+$ limit of the exact-WKB framework, in spite of its extremely singular character (thus the degree $N$, which is the most critical parameter in the framework, jumps from 4 to 2 at $g = 0$). Results are largely governed by the singular behaviour of improper action integrals, like $\int_0^\infty (q^4 + g^{-2/3}q^2 + \lambda)^{1/2} dq (g \to 0^+)$ for the case $V(q) = q^2 + gq^4$ [23, 25].

Appendix. Derivation of (33) as trace formulae

The identities of (33) awaiting proof indeed take the form of trace formulae,

$$\psi_\lambda(0)/\psi_\lambda(0) \equiv \text{Tr}(\hat{H} + \lambda)^{-1}, \quad \psi'_\lambda(0)/\psi'_\lambda(0) \equiv \text{Tr}(\hat{H} + \lambda)^{-1},$$

(A.1)

where $\psi_\lambda$ is the solution (31) of $(\hat{H} + \lambda)\psi = 0$ recessive for $q \to +\infty$, and $(\hat{H} + \lambda)^{-1}$ are the resolvent operators (Green’s functions) on the half-line $q \in [0, +\infty)$ with the corresponding boundary conditions at $q = 0$ (− = Dirichlet and + = Neumann). The resolvents are trace-class under our assumption $N > 2$.

Note that the $\psi$-normalizations, so essential in the main text, inversely become irrelevant here; already in (A.1), any recessive solution $c\psi_\lambda$ will do.

If now $\psi_\lambda^\pm(q)$ is the Neumann solution of $(\hat{H} + \lambda)\psi = 0$, also unique up to a factor, then the integral kernels $G_\lambda^\pm(q, \tilde{q})$ of the Green functions are known, in this 1D setting, to be

$$G_\lambda^\pm(q, \tilde{q}) = W(\psi_\lambda, \psi_\lambda^\pm)^{-1} \psi_\lambda^\pm(\min(q, \tilde{q})) \psi_\lambda(\max(q, \tilde{q})) \quad (q, \tilde{q} \in \mathbb{R}_+),$$

(A.2)

where $W(\psi, \phi) \equiv \psi \phi' - \psi' \phi$ denotes the Wronskian of $(\psi, \phi)$ (a constant when $(\psi, \phi)$ is a solution pair). As an intermediate result, then,

$$\text{Tr}(\hat{H} + \lambda)^{-1} = \int_0^\infty G_\lambda^\pm(q, q) dq = W(\psi_\lambda, \psi_\lambda^\pm)^{-1} \int_0^\infty \psi_\lambda^\pm(q) \psi_\lambda(q) dq.$$ 

(A.3)

Next, the subtraction of $\psi_\lambda^\pm[(\hat{H} + \lambda)\psi_\lambda^\pm] = 0$ from $\psi_\lambda^\pm[(\hat{H} + \lambda)\psi_\lambda] = 0$ (two obvious equalities) yields

$$\psi_\lambda^\pm \psi_\lambda = W(\psi_\lambda^\pm, \psi)_\lambda;$$

(A.4)

this reveals an explicit primitive for the product $\psi_\lambda^\pm \psi_\lambda$, reducing (A.3) to

$$\text{Tr}(\hat{H} + \lambda)^{-1} = W(\psi_\lambda^\pm, \psi_\lambda^\pm)^{-1} W(\psi_\lambda^\pm, \psi_\lambda)_\lambda^\infty.$$ 

(A.5)
But as \(q \rightarrow +\infty\), using (17) and its derivatives, 
\[
W(\psi_\pm^\lambda, \dot{\psi}_\lambda)(q) = O\left[\int_q^\infty \frac{d\tilde{q}}{\Pi_\lambda(\tilde{q})}\right]
\]
which tends to 0 under \(N > 2\); hence
\[
\text{Tr}(\hat{H} + \lambda)^{-1} = -W(\psi_\lambda, \psi_\pm^\lambda)^{-1}W(\psi_\pm^\lambda, \dot{\psi}_\lambda)_{q=0}.
\] (A.6)

Finally, this collapses to the identity pair (A.1) when \(W(\psi_\lambda, \psi_\pm^\lambda)\) (constant) is itself expressed at \(q = 0\) and the boundary conditions for \(\psi_\pm^\lambda\), resp. \(\dot{\psi}_\lambda\), are applied.

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