Research Article

An Algorithm to Compute the H-Bases for Ideals of Subalgebras

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Received 24 April 2021; Accepted 16 June 2021; Published 7 July 2021

Academic Editor: Qamar Din

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The concept of H-bases, introduced long ago by Macaulay [1], has become an important ingredient for the treatment of various problems in computational algebra. The concept of H-bases is for ideals in polynomial rings, which allows an investigation of multivariate polynomial spaces degree by degree. Similarly, we have the analogue of H-bases for subalgebras, termed as SH-bases. In this paper, we present an analogue of H-bases for finitely generated ideals in a given subalgebra of a polynomial ring, and we call them “HSG-bases.”

1. Introduction

The concept of H-bases, introduced long ago by Macaulay [1], is based solely on homogeneous terms of a polynomial. In [2], an extension of Buchberger’s algorithm is presented to construct H-bases algorithmically. Some applications of H-bases are given in [3]; in addition, many of the problems in applications which can be solved by the Gröbner technique can also be treated successfully with H-bases. The concept of H-basis for ideals of a polynomial ring over a field $K$ can be adopted in a natural way to $K$-subalgebras of a polynomial ring. In [4], SH-basis (Subalgebra Analogue to H-basis for Ideals) for the $K$-subalgebra of $K[x_1, \ldots, x_n]$ is defined. The properties of SH-bases are typically similar to H-basis results [3]. Like H-bases, the concept of SH-basis is also tied to homogeneous polynomials. In this paper, we will present an analogue to H-bases for ideals in a given subalgebra of a polynomial ring, and we call them “HSG-bases.”

The paper is organized as follows. In Section 2, we briefly describe the underlying concept of grading which leads to SAGBI-Gröbner bases and HSG-basis. Then, we give the notion of si-reduction, which is one of the key ingredients for the characterization and construction of HSG-basis.

After setting up the necessary notation, we present the si-reduction algorithm (see Algorithm 1). Also, here we present some properties characterizing HSG-basis (Theorem 1). In Section 3, we present a criterion through which we can check that the given system of polynomials is an HSG-basis of the subalgebra it generates (Theorem 2), and further on the basis of this theorem, we present an algorithm for the construction of HSG-basis (Algorithm 2).

2. HSG-Bases and SAGBI-Gröbner Bases

Here and in the following sections we consider polynomials in $n$ variables $x_1, \ldots, x_n$ with coefficients from a field $K$. For short, we write $P := K[x_1, \ldots, x_n], \quad (1)$

If $G$ is a subset of subalgebra $\mathcal{A}$ in $K[x_1, \ldots, x_n]$, then the set

$I := \left\{ \sum_{g \in G} h_g g | h_g \in \mathcal{A} \text{ and only finitely many } h_g \neq 0 \right\}, \quad (2)$

where $h_g$ are coefficients.
is the ideal of A in P generated by G and we write it shortly as \( \langle G \rangle_A \). In this section, we want to introduce HSG-bases and discuss some of their properties. This concept is very similar to the concept of SAGBI-Gröbner bases. Therefore, we will briefly explain the underlying common structure. Let \( \Gamma \) denote an ordered monoid, i.e., an abelian semigroup under an operation \(+\), equipped with a total ordering \( > \) such that, for all \( \alpha, \beta, \gamma \in \Gamma \), \( \alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma \). (3)

A direct sum,

\[
P = \bigoplus_{\gamma \in \Gamma} P_{\gamma}^{(\gamma)},
\]

is called grading (induced by \( \Gamma \)) or briefly a \( \Gamma \)-grading if for all \( \alpha, \beta \in \Gamma \),

\[
f \in P_{\alpha}^{(\alpha)}, g \in P_{\beta}^{(\beta)} \Rightarrow f \cdot g \in P_{\alpha \beta}^{(\alpha \beta)}.
\]

Since the decomposition above is a direct sum, each polynomial \( f \neq 0 \) has a unique representation.

\[
f = \sum_{i=1}^{s} f_{\gamma_i}, \quad 0 \neq f_{\gamma_i} \in P_{\gamma_i}^{(\gamma_i)}.
\]

Assuming that \( \gamma_1 > \gamma_2 > \cdots > \gamma_s \), the \( \Gamma \)-homogeneous term \( f_{\gamma_i} \) is called the maximal part of \( f \), denoted by \( M^{(\Gamma)}(f)_{\gamma_i} = f_{\gamma_i} \), and \( f - M^{(\Gamma)}(f) \) is called the \( d \)-reductum of \( f \). For \( G \subset A \), \( M^{(\Gamma)}(G) = \{ M^{(\Gamma)}(g) | g \in G \} \).

There are two major examples of gradings. The first one is grading by degrees:

\[
P_d^{(\Gamma)} = \{ p \in P | p \text{ is homogeneous of degree } d \}, \quad \forall d \in \mathbb{N}.
\]

Here, \( \Gamma = \mathbb{N} \) with the natural total ordering. This grading is called the \( H \)-grading because of the homogeneous polynomials. Therefore, we also write \( H \) in place of this \( \Gamma \). The space of all polynomials of degree at most \( d \) can now be written as

\[
P_d = \bigoplus_{k=0}^{d} P_H^{(k)}.
\]

The maximal part of a polynomial \( f \neq 0 \) is its homogeneous form of highest degree, \( M^{(\Gamma)}(f) \). For simplicity, let \( M^{(\Gamma)}(0) = 0 \).

**Definition 1.** A subset \( G = \{ g_1, \ldots, g_s \} \subset A \) (subalgebra) is called HSG-basis for the ideal \( I_A \subset A \), if for all \( 0 \neq f \in I_A \),

\[
\exists h_1, \ldots, h_s \in A : f = \sum_{i=1}^{s} h_i g_i, \deg(f) = \max_{i=1}^{s} \{ \deg(h_i g_i) \} \\text{(Note that this condition is not obvious, } -x^3 y^3 + x^4
\]

\[
= (x^2)(x^3 y + x^2) + (-xy)(x^4 + x^2 y^2) \text{ see in } K[x^2, xy]).
\]

The representation for \( f \) in (9) is also called its HSG representation with respect to \( G \).

Note that HSG-basis for ideal in a subalgebra is also a generating set of it. To obtain more insights into HSG-bases, we will give some equivalent definitions. First, we need a more technical notion.

**Definition 2.** For given \( f, f_1, \ldots, f_m \), we say that \( f \) si-reduces to \( f \) with respect to \( F = \{ f_1, \ldots, f_m \} \) in \( A \) if

**Algorithm 1:** Algorithm to compute si-reduction

\[
\text{Input: a subalgebra } A \text{ and a finite subset } G \subset A \text{ and a polynomial } f \in A.
\]

\[
\text{Output: a polynomial } h \text{ such that } f \longrightarrow_{si}^{} h.
\]

(1) \( h_1 = f \).

(2) While \( (h_\neq 0 \text{ and } G_h = \{ \sum a_i g_i | M^{(H)}(\sum a_i g_i) = M^{(H)}(h) \neq \emptyset \} \) do

(3) Choose \( \sum a_i g_i \in G_h \).

(4) \( h_\neq = h - \sum a_i g_i \) and continue at 2.

**Algorithm 2:** Algorithm for the construction of HSG basis.
\[ \tilde{f} = f - \sum_{i=1}^{m} a_i f_i, \text{deg}(\tilde{f}) < \text{deg}(f) \]  
(10)

holds with polynomials \( a_i \in \mathcal{A} \) satisfying \( \text{deg}(a_i f_i) \leq \text{deg}(f) \). We write it as \( f \rightarrow_{F,E} \tilde{f} \). By \( \rightarrow_{F,E,*} \), we denote the transitive closure of the binary relation \( \rightarrow_{F,E} 1 \).

The concept of \( s_i \)-reduction plays an important role in the characterization and construction of HSG-basis. For \( f \in \mathcal{A} \) and \( G \subseteq \mathcal{A} \), the following algorithm computes \( h \) such that \( f \rightarrow_{G_{s_i}} h \) (i.e., \( f \) reduces to \( h \) completely).

We note that such an element \( a_i \) in the subalgebra \( \mathcal{A} \) can easily be determined as in the case of reduction in polynomial ring. We also note that \( \text{deg}(h - \sum a_i g_i) \) is strictly smaller than \( \text{deg}(h) \) (by the choice of \( \sum a_i g_i \)). This shows that Algorithm 1 always terminates.

**Theorem 1.** Let \( G = \{g_1, \ldots, g_s\} \subseteq \mathcal{A} \) (subset of subalgebra \( \mathcal{A} \)) and \( I_{\mathcal{A}} \) be an ideal of \( \mathcal{A} \). Then, the following conditions are equivalent:

1. \( G \) is an HSG-basis for the ideal \( I_{\mathcal{A}} \).
2. \( (M^{(H)}(g_1), \ldots, M^{(H)}(g_s)) \subseteq K[M^{(H)}] = (M^{(H)}(f)) \), \( f \in I_{\mathcal{A}} \).
3. For all \( f \in I_{\mathcal{A}}, f \rightarrow_{G_{s_i}}, 0 \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( M^{(H)}(p) \in \langle M^{(H)}(f) | f \in I_{\mathcal{A}} \rangle \) for some \( p \in I_{\mathcal{A}} \). Since \( G \) is an HSG-basis, by (9), there are some \( h_1, \ldots, h_s \in \mathcal{A} \) so that

\[
\text{deg} \left( \sum_{j=1}^{s} M^{(H)}(h_{ij}) M^{(H)}(g_j) \right) > \text{deg} \left( \sum_{j=1}^{s} M^{(H)}(h_{i+1,j}) M^{(H)}(g_j) \right), \quad i = 1, 2, \ldots, d.
\]

Hence,

\[
\text{deg}(f) = \max \left( \text{deg} \left( \sum_{j=1}^{s} M^{(H)}(h_{ij}) M^{(H)}(g_j) \right) \right).
\]

(11) and (15) give the HSG representation.

The second major example of gradings leads to the SAGBI-Gröbner basis concept. Here, \( \Gamma = \mathbb{N}^n \) with component-wise addition equipped with a total ordering satisfying (11). In addition, \( \gamma \geq 0, \forall \gamma \in \Gamma \). For arbitrary \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma \), the space \( p^{(\Gamma)} \) is a vector space of dimension 1, namely,

\[ p^{(\Gamma)} = \{ c \cdot x^{\gamma_1} \cdots x^{\gamma_n} | c \in K \}. \]

The maximal part \( M^{(\Gamma)}(f) \) of a polynomial \( f \) is a product of a leading coefficient \( \text{LC}(f) \) and a leading monomial \( \text{LM}(f) \), that is \( M^{(\Gamma)}(f) = \text{LC}(f) \cdot \text{LM}(f) \), where \( \text{LC}(f) \in K \). The si-reduction \( f \rightarrow_{G_{s_i}} \tilde{f} \) is defined if there exists a polynomial \( g \in G \) and \( a \in \mathcal{A} \) such that \( \text{LM}(f) = \text{LM}(g) \text{LM}(a) \) and then we set

\[ \tilde{f} = f - \sum_{i=1}^{m} a_i f_i, \text{deg}(\tilde{f}) < \text{deg}(f). \]

(10)

A SAGBI-Gröbner basis \( G \) (with respect to a given monomial ordering and a given ideal \( I_{\mathcal{A}} \) in a subalgebra \( \mathcal{A} \)) is a set of polynomials generating \( I_{\mathcal{A}} \) and satisfying one of the following equivalent conditions:

(i) Every \( f \in I_{\mathcal{A}} \) has a representation:

\[ f = \sum_{i=1}^{s} h_i g_i, \]

(17)

where \( h_i \in \mathcal{A} \) and \( g_i \in G \).

(ii) \( \langle M^{(\Gamma)}(g) | g \in G \rangle = \langle M^{(\Gamma)}(f) | f \in I_{\mathcal{A}} \rangle \).

(iii) Every \( f \in I_{\mathcal{A}} \) si-reduces to 0 with respect to \( G \).

The proof of this equivalence and many other equivalent conditions can be found in [5]. If a monomial ordering is compatible with the semiordering by degrees,

\[ \text{deg}(x^\gamma) > \text{deg}(x^\delta) \Rightarrow \gamma > \delta, \gamma, \delta \in \mathbb{N}^n, \]

(18)
then any SAGBI-Gröbner representation as given in (i) is an HSG representation; in other words, a SAGBI-Gröbner basis with respect to a degree compatible ordering is an HSG-basis as well. The converse is false, as the following example shows.

**Example 1.** Let \( f_1 = x^4 + 2x^2y^2 + y^4 - 1 \), \( f_2 = x^2y^2 + y^4 - 2 \), \( f_3 = 2x^2 + y^2 \). These polynomials belong to the subalgebra \( \mathcal{A} = \mathbb{Q}[x^2, y^2] \). Then, we can see that \( f_1, f_3 \), and \( f_4 \) already constitute an HSG-basis for ideal \( \mathcal{I} = \langle f_1, f_2, f_3 \rangle \) in \( \mathcal{A} \). If we order the monomials by degree lexicographical ordering, then \( \langle M^{(H)}(H) \rangle f = \{ f_1, f_2, f_3 \} \) in \( \mathcal{A} \). Every SAGBI-Gröbner basis \( G \) with respect to this ordering contains at least four elements, for instance, \( G = \{ g_1, g_2, g_3, g_4 \} \). We may as-

**Theorem 2.** (HSG-basis criterion). Let \( G = \{ g_1, \ldots, g_s \} \) be the subset of a subalgebra \( \mathcal{A} \). Let \( Q \) be \( M^{(H)} \)-generating set for the \( \text{syz}(M^{(H)}(G)) \). Then, \( G \) is an HSG-basis for \( \langle G \rangle \) if and only if for every \( \bar{q}_j = (q_{j1}, \ldots, q_{jn}) \in Q \), we have

\[
\sum_{i=1}^{m} q_{ji}g_i \longrightarrow_{G} 0.
\]

**Proof.** \( \Rightarrow \): The statement is a direct result of Theorem 1.

\( \Leftarrow \): Take \( f \in (G)_f \). We need to show that \( M^{(H)}(f) \in \langle M^{(H)}(G) \rangle \). For this, we write \( f = \sum_{i=1}^{m} a_i g_i \) such that \( p_0 = \max \{ M^{(H)}(a_i) \} \) (degree wise) is minimal among all such representations of \( f \). We have \( M^{(H)}(f) \leq p_0 \). Suppose that \( M^{(H)}(f) < p_0 \). Assume that \( a_1 g_1, \ldots, a_m g_m \) are contributing to \( p_0 \), i.e., \( M^{(H)}(a_i) = p_0 \) for all \( 1 \leq i \leq m \). If we set \( \bar{a} = (a_1, \ldots, a_m, 0, \ldots, 0) \), we can see that \( M^{(H)}(\bar{a}) \in \text{syz}(M^{(H)}(G)) \), which implies there are \( b_1, \ldots, b_n \in \mathcal{A} \) and \( \bar{Q}_1, \ldots, \bar{Q}_n \in \bar{Q} \) such that \( M^{(H)}(\bar{a}) = \sum_{i=1}^{n} M^{(H)}(b_i) \bar{Q}_i \). We may assume that \( M^{(H)}(b_i)M^{(H)}(q_{ji})M^{(H)}(g_i) = p_0 \) for each \( j \) by homogeneity of the syzygies. Now,

\[
f = \sum_{i=1}^{m} a_i g_i - \sum_{j=1}^{m} b_j q_{ji} g_i + \sum_{j=1}^{m} b_j \left( \sum_{i=1}^{n} q_{ji} g_i \right)
\]

\[
= \sum_{j=1}^{m} \left( a_i - \sum_{j=1}^{n} b_j q_{ji} g_i + \sum_{j=1}^{m} b_j \left( \sum_{i=1}^{n} p_{j,i} g_i \right) \right),
\]

where \( \sum_{j=1}^{m} p_{j,i} g_i \) is an HSG representation for \( \sum_{i=1}^{m} q_{ji} g_i \) since \( \sum_{i=1}^{m} q_{ji} g_i \longrightarrow G_0 \). If we define \( H_j = \max \{ M^{(H)}(p_{j,i}, g_i) \} \), then

\[
H_j = M^{(H)}(\sum q_{ji} g_i) < \max(M^{(H)}(q_{ji} g_i)) \quad \text{for all } j,
\]

because \( M^{(H)}(\bar{Q}_i) \in \text{syz}(M^{(H)}(G)) \). Consider the first sum of equation (20). For \( i \leq m_0 \), we have \( M^{(H)}(a_i) = M^{(H)}(\sum_{j=1}^{n} b_j q_{ji}) \), so by the cancellation of highest terms,

\[
M^{(H)} \left[ a_i - \sum_{j=1}^{n} b_j q_{ji} g_i \right] < M^{(H)}(a_i) = p_0.
\]

For \( i > m_0 \), \( M^{(H)}(a_i) < p_0 \) and \( \sum_{j=1}^{m} M^{(H)}(b_j)M^{(H)}(q_{ji}) = 0 \) implies that

\[
M^{(H)} \left[ \sum_{j=1}^{n} b_j q_{ji} g_i \right] < \max(M^{(H)}(b_j q_{ji} g_i)) = p_0.
\]

Since

\[
M^{(H)} \left[ a_i - \sum_{j=1}^{n} b_j q_{ji} \right] \leq \max \left\{ M^{(H)}(a_i), M^{(H)} \left( \sum_{j=1}^{n} b_j q_{ji} g_i \right) \right\} < p_0 \quad \forall i.
\]
So, first sum of equation (20) is less than \( p_0 \). For the second sum of equation (20), we have
\[
M^{(H)} \left( \sum_{j=1}^{n} b_j \pi_j g_i \right) \leq \max_{j} M^{(H)}(b_j) \pi_j g_i ) 
\]
(25)
\[
\leq \max_j \left[ M^{(H)}(b_j) H \right] 
< \max_i \left[ M^{(H)}(b_i q_i g_i) \right] = p_0. 
\]

Hence, equation (20) does provide a new representation for \( f \) such that \( \max(M^{(H)}(a_i g_i)) \leq p_0 \), a contradiction. Therefore, \( M^{(H)}(f) = p_0 \) and \( M^{(H)}(f) = \sum_{i=1}^{m} M^{(H)}(a_i g_i) \in \langle M^{(H)}(G) \rangle \).

On the basis of Theorem 2, now we present an algorithm which computes HSG-basis from a given set of generators. This algorithm is not necessarily terminating but does terminate, if and only if, the considered ideal in the subalgebra has a finite HSG-basis.

Now we present some examples which show the computation of HSG-basis through Algorithm 2.

Example 2. Let the subalgebra \( \mathcal{A} = Q[x^2, xy] \) and \( G = \{ x^3 y, x^2 y + 2 \} \subseteq \mathcal{A} \). Consider \( H = G \); then, \( M^{(H)}(H) = \{ x^3 y, x y \} \).

First pass through the while loop:

(i) \( M^{(H)}(q_1)(x^3 y) + M^{(H)}(q_3)(x y) = 0 \) implies \( Q = \{ -1, x^2 \} \). Then, \( -1(x^3 y + x^2) + (x^2)(x y + 2) = -x^3 y - x^2 + x^2 y + 2x^2 = x^2 \) gives \( P = \{ x^2 \} \).

(ii) As \( x^2 \) is si-reduced with respect to \( H \), \( \text{red}(P) = \{ x^2 \} \).

(iii) Define: \( \text{Old}(H) = H \cup \{ x^2 \} \).

Since \( H \neq \text{Old}(H) \), we repeat the whole process. Now we have \( M^{(H)}(H) = \{ x^3 y, x y, x^2 \} \).

Second pass through the while loop:

(i) \( M^{(H)}(q_1)(x^3 y) + M^{(H)}(q_3)(x y) + M^{(H)}(q_3)(x^2) = 0 \) implies \( Q = \{ -1, x^2, 0, (0, x y, 0), (-1, 0, 0) \} \).

Then, \( (-1)(x^3 y + x^3 y) + (0)(x y + 2) + (x^2)(x^2) = -x^3 y - x^2 + 0 + x^3 y = -x^2 \) gives \( P = \{ x^2, -x^2 \} \).

(ii) Now, \( \text{red}(P) = \emptyset \).

Since \( \text{Old}(H) = H \), we stop here. The HSG-basis for \( \langle G \rangle_{\mathcal{A}} \) is \( \{ x^3 y + x^2 y + 2, x y + 2, x^2 \} \).

Example 3. Let \( \mathcal{A} = Q[x^2, xy] \) and \( G = \{ x^3 y + x^2 y^2 + x^2, x y + 2 \} \subseteq \mathcal{A} \). Consider \( H = G \); then, \( M^{(H)}(H) = \{ x^3 y + x^2 y^2, xy \} \).

First pass through the while loop:

(i) \( M^{(H)}(q_1)(x^3 y + x^2 y^2) + M^{(H)}(q_3)(x y) = 0 \) gives \( Q = \{ -1, x^2 + x y \} \). Then, \( (x^3 y + x^2 y^2) + (x^2 + x y)(x y + 2) = -x^3 y - x^2 + x^2 y + x^2 y + 2x^2 + 2x y = x^2 + 2x y \), \( \text{red}(P) = \{ x^2 - 4 \} \).

(ii) Define: \( \text{Old}(H) = H \cup \{ x^2 - 4 \} \).

As \( H \neq \text{Old}(H) \), we repeat the whole process. Now we have \( M^{(H)}(H) = \{ x^3 y + x^2 y^2, x y, x^2 \} \).

Second pass through the while loop:

(i) From the equation \( M^{(H)}(q_1)(x^3 y + x^2 y^2) + M^{(H)}(q_3)(x y) + M^{(H)}(q_3)(x^2) = 0 \), we have \( Q = \{ -1, x^2 + x y, (0, x y, 0) \} \).

We can compute \( P \) from \( (x^3 y + x^2 y^2 + x^2) + (x y)(x y + 2) + (x y)(x + 2) = -x^3 y - x^2 + x^2 y + 2x^2 + 2x y = x^2 + 2x y \).

(ii) Now, \( \text{red}(P) = \emptyset \).

Since \( \text{Old}(H) = H \), we stop here. The HSG-basis for \( \langle G \rangle_{\mathcal{A}} \) is \( \{ x^3 y + x^2 y^2 + x^2, x y + 2, x^2 - 4 \} \).

4. Conclusion

In this paper, we presented the theory of HSG-bases, which are a good basis of an ideal in a subalgebra of a polynomial ring. We can further develop this theory for an arbitrary grading for which HSG-bases would be a special case for degree-based grading.

Data Availability

No data are required to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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