THE HAKE-MCSHANE AND HAKE-HENSTOCK-KURZWEIL INTEGRALS
OVER m-DIMENSIONAL UNBOUNDED SETS

SOKOL BUSH KALIAJ

Abstract. In this paper, we extend the Hake-McShane and Hake-Henstock-Kurzweil integrals of
Banach space valued functions from m-dimensional open and bounded sets to m-dimensional sets
G such that |G \ G^n| = 0. We will prove the full descriptive characterizations of new integrals in
terms of the locally McShane and locally Henstock-Kurzweil integrals.

1. Introduction and Preliminaries

In this paper, we continue the investigation of characterizations of the Hake-McShane and
Hake-Henstock-Kurzweil integrals in terms of the McShane and Henstock-Kurzweil integrals, started in
[13]. At first, we define the Hake-McShane and Hake-Henstock-Kurzweil integrals of Banach space
valued functions defined on a subset \( E \subset \mathbb{R}^m \) such that \(|E \setminus E^n| = 0\), see Definition [13]. If \( E \) is a
bounded set and \( E = E^n \), then Definition [13] is the same with the corresponding definition in [13]. In
the paper [13] are proved full descriptive characterizations of the Hake-McShane and Hake-Henstock-
Kurzweil integrals of Banach space valued functions defined on a bounded and open subset \( G \subset \mathbb{R}^m \)
in terms of the McShane and Henstock-Kurzweil integrals, see Theorems 3.1 and 3.2 in [13]. Here,
we will prove full descriptive characterizations of the Hake-McShane and Hake-Henstock-Kurzweil
integrals of Banach space valued functions defined on a subset \( G \subset \mathbb{R}^m \) such that \(|G \setminus G^n| = 0\) in
terms of the locally McShane and locally Henstock-Kurzweil integrals, see Theorems 2.8 and 2.9.

Throughout this paper \( X \) denotes a real Banach space with the norm \( || \cdot || \). The Euclidean space
\( \mathbb{R}^m \) is equipped with the maximum norm. \( B_m(t, r) \) denotes the open ball in \( \mathbb{R}^m \) with center \( t \) and
radius \( r > 0 \). We denote by \( \mathcal{L}(\mathbb{R}^m) \) the \( \sigma \)-algebra of Lebesgue measurable subsets of \( \mathbb{R}^m \) and by \( \lambda \)
the Lebesgue measure on \( \mathcal{L}(\mathbb{R}^m) \). \( |A| \) denotes the Lebesgue measure of \( A \in \mathcal{L}(\mathbb{R}^m) \).

The subset \( \prod_{j=1}^m [a_j, b_j] \subset \mathbb{R}^m \) is said to be a closed non-degenerate interval in \( \mathbb{R}^m \), if \(-\infty < a_j < b_j < +\infty \), for \( j = 1, \ldots, m \). Two closed non-degenerate intervals \( I \) and \( J \) in \( \mathbb{R}^m \) are said to be non-overlapping if \( I^o \cap J^o = \emptyset \), where \( I^o \) denotes the interior of \( I \). By \( \mathcal{I} \) the family of all closed non-degenerate subintervals in \( \mathbb{R}^m \) is denoted and by \( \mathcal{I}_E \) the family of all closed non-degenerate subintervals in \( E \subset \mathbb{R}^m \). A function \( F : \mathcal{I}_E \to X \) is said to be an additive interval function, if for
each two non-overlapping intervals \( I, J \in \mathcal{I}_E \) such that \( I \cup J \in \mathcal{I}_E \), we have
\[ F(I \cup J) = F(I) + F(J). \]

A pair \( (t, I) \) of a point \( t \in E \) and an interval \( I \in \mathcal{I}_E \) is called an \( \mathcal{M} \)-tagged interval in \( E \), \( t \) is the
tag of \( I \). Requiring \( t \in I \) for the tag of \( I \) we get the concept of an \( \mathcal{HK} \)-tagged interval in \( E \). A
finite collection \( \{(t_i, I_i) : i = 1, \ldots, p\} \) of \( \mathcal{M} \)-tagged intervals (\( \mathcal{HK} \)-tagged intervals) in \( E \) is called
an \( \mathcal{M} \)-partition (\( \mathcal{HK} \)-partition) in \( E \), if \( \{I_i : i = 1, \ldots, p\} \) is a collection of pairwise non-overlapping
intervals in \( \mathcal{I}_E \). Given \( Z \subset E \), a positive function \( \delta : Z \to (0, +\infty) \) is called a gauge on \( Z \). We say that
an \( \mathcal{M} \)-partition (\( \mathcal{HK} \)-partition) \( \pi = \{(t_i, I_i) : i = 1, \ldots, p\} \) in \( E \) is
\begin{itemize}
  \item \( \mathcal{M} \)-partition (\( \mathcal{HK} \)-partition) of \( E \), if \( \cup_{i=1}^p I_i = E \),
  \item \( Z \)-tagged if \( \{t_1, \ldots, t_p\} \subset Z \),
\end{itemize}

2010 Mathematics Subject Classification. Primary 28B05, 46B25; Secondary 46G10.
Key words and phrases. Hake-Henstock-Kurzweil integral, locally Henstock-Kurzweil integral, Hake-McShane inte-
gral, locally McShane integral, Banach space, m-dimensional Euclidean space.
\begin{itemize}
  \item $\delta$-fine if for each $i = 1, \ldots, p$, we have $I_i \subset B_m(t_i, \delta(t_i))$.
\end{itemize}

We now recall the definitions of the McShane and Henstock-Kurzweil integrals of a function $f : J \to X$, where $J$ is a fixed interval in $\mathcal{I}$. The function $f$ is said to be \textit{McShane (Henstock-Kurzweil) integrable} on $J$ if there is a vector $x_f \in X$ such that for every $\varepsilon > 0$, there exists a gauge $\delta$ on $J$ such that for every $\delta$-fine $\mathcal{M}$-partition (HK-partition) $\pi$ of $J$, we have

$$|\sum_{(t, I) \in \pi} f(t)|I| - x_f| < \varepsilon.$$ 

In this case, the vector $x_f$ is said to be the \textit{McShane (Henstock-Kurzweil) integral} of $f$ on $J$ and we set $x_f = (M) \int_J f d\lambda$ ($x_f = (HK) \int_J f d\lambda$). The function $f$ is said to be \textit{McShane (Henstock-Kurzweil) integrable} over a subset $A \subset J$, if the function $f\chi_A : J \to X$ is McShane (Henstock-Kurzweil) integrable on $J$, where $\chi_A$ is the characteristic function of the set $A$. The McShane (Henstock-Kurzweil) integral of $f$ over $A$ will be denoted by $(M) \int_A f d\lambda$ ((HK) $\int_A f d\lambda$). If $f : J \to X$ is McShane integrable on $J$, then by Theorem 4.1.6 in [18] the function $f$ is the McShane integrable on each Lebesgue measurable subset $A \subset J$, while by Theorem 3.3.4 in [18], if $f$ is Henstock-Kurzweil integrable on $J$, then $f$ is the Henstock-Kurzweil integrable on each $I \in \mathcal{I}_J$. Therefore, we can define an additive interval function $F: \mathcal{I}_J \to X$ as follows

$$F(I) = (M) \int_I f d\lambda, \quad (F(I) = (HK) \int_I f d\lambda), \quad \text{for all } I \in \mathcal{I}_J,$$

which is called the primitive of $f$.

The basic properties of the McShane integral and the Henstock-Kurzweil integral can be found in [1], [2], [3], [5]-[7], [8]-[10], [11], [12], [14], [15], [16] and [18]. We do not present them here. The reader is referred to the above mentioned references for the details.

\textbf{Definition 1.1.} Assume that an open subset $W \subset \mathbb{R}^m$, a function $f : W \to X$ and an additive interval function $F : \mathcal{I}_W \to X$ are given. For each $I \in \mathcal{I}_W$, we denote

$$f_I = f|_I \text{ and } F_I = F|_I.$$

The function $f$ is said to be \textit{locally McShane (locally Henstock-Kurzweil) integrable} on $W$ with the primitive $F$, if for each $I \in \mathcal{I}_W$, $f_I = f|_I$ is McShane (Henstock-Kurzweil) integrable on $I$ with the primitive $F_I = F|_I$.

We now fix a subset $E \subset \mathbb{R}^m$ such that $|E \setminus E^\circ| = 0$, where $E^\circ$ is the interior of $E$. A sequence $(I_k)$ of pairwise non-overlapping intervals in $\mathcal{I}_E$ is said to be a \textit{division} of $E^\circ$, if

$$E^\circ = \bigcup_{k=1}^{+\infty} I_k.$$

We denote by $\mathcal{D}_{E^\circ}$ the family of all divisions of $E^\circ$. By Lemma 2.43 in [4], the family $\mathcal{D}_{E^\circ}$ is not empty.

\textbf{Definition 1.2.} An additive interval function $F : \mathcal{I}_E \to X$ is said to be a \textit{Hake-function}, if given a division $(I_k) \in \mathcal{D}_{E^\circ}$, we have

\begin{itemize}
  \item the series
    $$\sum_{k : |I \setminus I_k| > 0} F(I \cap I_k)$$
    is unconditionally convergent in $X$, for each $I \in \mathcal{I}$,
  \item the equality
    $$F(I) = \sum_{k : |I \setminus I_k| > 0} F(I \cap I_k),$$
    holds for all $I \in \mathcal{I}_E$.
\end{itemize}
Definition 1.3. We say that the additive interval function \( F : I_E \to X \) has \( M \)-negligible (\( HK \)-negligible) variation over a subset \( Z \subset \mathbb{R}^m \), if for each \( \varepsilon > 0 \) there exists a gauge \( \delta_v \) on \( Z \) such that for each \( Z \)-tagged \( \delta_v \)-fine \( M \)-partition (\( HK \)-partition) \( \pi_v \) in \( \mathbb{R}^m \), we have

- the series
  \[
  \sum_{k: |I \cap I_k| > 0} F(I \cap I_k)
  \]
  is unconditionally convergent in \( X \), for each \((t, I) \in \pi_v \),

- the inequality
  \[
  \left\| \sum_{(t, I) \in \pi} \left( \sum_{k: |I \cap I_k| > 0} F(I \cap I_k) \right) \right\| < \varepsilon,
  \]
  holds, whenever \((I_k) \in \mathcal{P}_E \). We say that \( F \) has \( M \)-negligible (\( HK \)-negligible) variation outside of \( E^o \) if \( F \) has \( M \)-negligible (\( HK \)-negligible) variation over \((E^o)^c = \mathbb{R}^m \setminus E^o \).

Definition 1.4. We say that a function \( f : E \to X \) is Hake-McShane (Hake-Henstock-Kurzweil) integrable on \( E \) with the primitive \( F : I_E \to X \), if we have

- for each \( \varepsilon > 0 \) there exists a gauge \( \delta_\varepsilon \) on \( Z = J \cap (G \setminus G^o) \) such that for each \( \delta_\varepsilon \)-fine \( M \)-partition (\( HK \)-partition) \( \pi \) in \( E^o \), we have
  \[
  \left\| \sum_{(t, I) \in \pi} (f(t)|I| - F(I)) \right\| < \varepsilon,
  \]
- \( F \) is a Hake-function,
- \( F \) has \( M \)-negligible (\( HK \)-negligible) variation outside of \( E^o \).

Clearly, if \( E \) is a bounded set and \( E = E^o \), then Definition 1.4 is the same with the corresponding definition in [13].

2. The Main Results

From now on \( G \) will be a subset of \( \mathbb{R}^m \) such that \( G^o \neq \emptyset \) and \( |G \setminus G^o| = 0 \). The main results are Theorems 2.8 and 2.9. Let us start with a few auxiliary lemmas.

Lemma 2.1. Let \( f : G \to X \) be a function and let \( J \in \mathcal{I} \). Then, given \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \( Z = J \cap (G \setminus G^o) \) such that for each \( \delta \)-fine \( Z \)-tagged \( M \)-partition \( \pi \) in \( J \), we have

\[
\left\| \sum_{(t, I) \in \pi} f(t)|I| \right\| < \varepsilon.
\]

Proof. Define a function \( g_J : J \to X \) as follows

\[
g_J(t) = \begin{cases} 
  f(t) & \text{if } t \in Z \\
  0 & \text{otherwise}.
\end{cases}
\]

Then, by Theorem 3.3.1 in [18], \( g_J \) is McShane integrable on \( J \) and

\[
(M) \int_I g_J \, d\lambda = 0, \text{ for all } I \in \mathcal{I}_J.
\]

Therefore, by Lemma 3.4.2 in [18], given \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \( Z \) such that for each \( \delta \)-fine \( Z \)-tagged \( M \)-partition \( \pi \) in \( J \) we have

\[
\left\| \sum_{(t, I) \in \pi} g_J(t)|I| \right\| < \varepsilon,
\]

and since \( g_J(t) = f(t) \) for all \( t \in Z \), the last result proves the lemma. \( \square \)
Lemma 2.2. Let \( f : G \to X \) be a function, let \( F : \mathcal{I}_G \to X \) be an additive interval function and let \( J \in \mathcal{I} \). If \( F \) has \( \mathcal{M} \)-negligible variation outside of \( G^o \), then given \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \( Z = J \cap (G \setminus G^o) \) such that for each \( \delta \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( J \) we have

\[
\| \sum_{(t,I) \in \pi} \left( f(t)|I| - \sum_{k: |I \cap I_k| > 0} F(I \cap I_k) \right) \| < \varepsilon,
\]

whenever \((I_k) \in \mathcal{D}_{G^o}\).

Proof. Since \( F \) has \( \mathcal{M} \)-negligible variation outside of \( G^o \), given \( \varepsilon > 0 \) there exists a gauge \( \delta_o \) on \((G^o)^c\) such that for each \( \delta_o \)-fine \((G^o)^c\)-tagged \( \mathcal{M} \)-partition \( \pi_o \) in \( J \), we have

\[
\| \sum_{(t,I) \in \pi_o} \left( \sum_{k: |I \cap I_k| > 0} F(I \cap I_k) \right) \| < \frac{\varepsilon}{2},
\]

whenever \((I_k) \in \mathcal{D}_{G^o}\).

By Lemma 2.1 there exists a gauge \( \delta_0 \) on \( Z \) such that for each \( \delta_0 \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( J \), we have

\[
\| \sum_{(t,I) \in \pi} f(t)|I| \| < \frac{\varepsilon}{2}.
\]

Define a gauge \( \delta \) on \( Z \) by \( \delta(t) = \min\{\delta_o(t), \delta_0(t)\} \) for all \( t \in Z \). Let \( \pi \) be a \( \delta \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( J \). Then,

\[
\| \sum_{(t,I) \in \pi} \left( f(t)|I| - \sum_{k: |I \cap I_k| > 0} F(I \cap I_k) \right) \| \leq \| \sum_{(t,I) \in \pi} f(t)|I| \| + \| \sum_{(t,I) \in \pi} \left( \sum_{k: |I \cap I_k| > 0} F(I \cap I_k) \right) \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

and this ends the proof. \( \square \)

The next lemma can be proved in the same way as Lemma 2.2.

Lemma 2.3. Let \( f : G \to X \) be a function, let \( F : \mathcal{I}_G \to X \) be an additive interval function and let \( J \in \mathcal{I} \). If \( F \) has \( \mathcal{HK} \)-negligible variation outside of \( G^o \), then given \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \( Z = J \cap (G \setminus G^o) \) such that for each \( \delta \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( J \) we have

\[
\| \sum_{(t,I) \in \pi} \left( f(t)|I| - \sum_{k: |I \cap I_k| > 0} F(I \cap I_k) \right) \| < \varepsilon,
\]

whenever \((I_k) \in \mathcal{D}_{G^o}\).

Lemma 2.4. Let \( W \) be an open subset of \( \mathbb{R}^m \), let \( v : W \to X \) be a function and let \( V : \mathcal{I}_W \to X \) be an additive interval function. Then, the following statements are equivalent:

(i) \( v \) is locally McShane integrable on \( W \) with the primitive \( V \),

(ii) for each \( \varepsilon > 0 \) there exists a gauge \( \delta_\varepsilon \) on \( W \) such that for each \( \delta_\varepsilon \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( W \), we have

\[
\| \sum_{(t,I) \in \pi} \left( v(t)|I| - V(I) \right) \| < \varepsilon.
\]

Proof. (i) \( \Rightarrow \) (ii) Assume that \( v \) is locally McShane integrable on \( W \) with the primitive \( V \). Fix a division \((C_k)\) of \( W \) and let \( \varepsilon > 0 \) be given. For each \( k \in \mathbb{N} \), we denote

\[
v_k = v|_{C_k} \text{ and } V_k = V|_{2C_k}.
\]
By hypothesis each function \( v_k \) is McShane integrable on \( C_k \) with the primitive \( V_k \). Hence, by Lemma 3.4.2 in [18], there exists a gauge \( \delta_k \) on \( C_k \) such that for each \( \delta_k \)-fine \( \mathcal{M} \)-partition \( \pi_k \) in \( C_k \), we have

\[
\| \sum_{(t,I) \in \pi_k} (v_k(t)|I| - V_k(I)) \| \leq \frac{1}{2k} \varepsilon.
\] (2.1)

Note that for any \( t \in W = \bigcup_k C_k \), we have the following possible cases:

- there exists \( i_0 \in \mathbb{N} \) such that \( t \in (C_{i_0})^o \);
- there exists \( j_0 \in \mathbb{N} \) such that \( t \in C_{j_0} \setminus (C_{j_0})^o \). In this case, there exists a finite set \( \mathcal{N}_t = \{ j \in \mathbb{N} : t \in C_j \setminus (C_j)^o \} \) such that \( t \in \bigcap_{j \in \mathcal{N}_t} C_j \) and \( t \notin C_k \), for all \( k \in \mathbb{N} \setminus \mathcal{N}_t \). Hence, \( t \in \bigcup_{j \in \mathcal{N}_t} C_j \).

For each \( k \in \mathbb{N} \), we can choose \( \delta_k \) so that for any \( t \in W \), we have

\[
t \in (C_k)^o \Rightarrow B_m(t, \delta_k(t)) \subset C_k
\]
and

\[
t \in C_k \setminus (C_k)^o \Rightarrow B_m(t, \delta_k(t)) \subset \bigcup_{j \in \mathcal{N}_t} C_j.
\]

Define a gauge \( \delta_\varepsilon : W \to (0, +\infty) \) as follows. For each \( t \in W \), we choose

\[
\delta_\varepsilon(t) = \begin{cases} 
\delta_{i_0}(t) \\
\min\{\delta_j(t) : j \in \mathcal{N}_t\} & \text{if } t \in (C_{i_0})^o \\
\text{otherwise.}
\end{cases}
\]

Let \( \pi \) be an arbitrary \( \delta_\varepsilon \)-fine \( \mathcal{M} \)-partition in \( W \). Then, \( \pi = \pi_1 \cup \pi_2 \), where

\[
\pi_1 = \{(t, I) \in \pi : (\exists i_0 \in \mathbb{N}[t \in (C_{i_0})^o]\}
\]
\[
\pi_2 = \{(t, I) \in \pi : (\exists j_0 \in \mathbb{N}[t \in C_{j_0} \setminus (C_{j_0})^o]\}.
\]

Hence,

\[
\| \sum_{(t,I) \in \pi} (v(t)|I| - V(I)) \| \leq \| \sum_{(t,I) \in \pi_1} (v(t)|I| - V(I)) \| + \| \sum_{(t,I) \in \pi_2} (v(t)|I| - V(I)) \| \\
(2.2)
\]

Note that, if we define

\[
\pi_1^k = \{(t, I) \in \pi_1 : t \in (C_k)^o\},
\]
\[
\pi_2^k = \{(t, I \cap C_k) : (t, I) \in \pi_2, t \in C_k \setminus (C_k)^o, |I \cap C_k| > 0\}
\]
then \( \pi_1^k \) and \( \pi_2^k \) are \( \delta_k \)-fine \( \mathcal{M} \)-partitions in \( C_k \). Therefore, by (2.1), it follows that

\[
\| \sum_{(t,I) \in \pi_1} (v(t)|I| - V(I)) \| = \| \sum_k \left( \sum_{(t,I) \in \pi_1^k} (v(t)|I| - V(I)) \right) \| \leq \sum_k \| \left( \sum_{(t,I) \in \pi_1^k} (v_k(t)|I| - V_k(I)) \right) \| \leq \sum_{k=1}^{+\infty} \frac{1}{2k} \varepsilon = \frac{\varepsilon}{2}
\]
and
\[
\| \sum_{(t,I) \in \pi_2} (v(t)|I| - V(I)) \| = \| \sum_{(t,I) \in \pi_2} \left( \sum_{j \in \mathcal{N}_I, |I \cap C_j| > 0} (v_j(t)|I \cap C_j| - V_j(I \cap C_j)) \right) \| \\
= \| \sum_{(t,I) \in \pi_2} \left( \sum_{j \in \mathcal{N}_I, |I \cap C_j| > 0} (v_j(t)|I \cap C_j| - V_j(I \cap C_j)) \right) \| \\
\leq \sum_{k} \| \left( \sum_{(t,I) \in \pi_k^2} (v_k(t)|I \cap C_k| - V_k(I \cap C_k)) \right) \| \\
\leq \sum_{k=1}^{+\infty} \frac{1}{2^k} = \frac{\varepsilon}{2}.
\]

The last results together with (2.2) yield
\[
\| \sum_{(t,I) \in \pi} (v(t)|I| - V(I)) \| < \varepsilon.
\]

(ii) ⇒ (i) Assume that (ii) holds. Then, given \( \varepsilon > 0 \) there exists a gauge \( \delta_{\varepsilon} \) on \( W \) such that for each \( \delta_{\varepsilon}\)-fine \( \mathcal{M} \)-partition \( \pi \) in \( W \), we have
\[
\| \sum_{(t,I) \in \pi} (v(t)|I| - V(I)) \| < \varepsilon.
\]

Fix an arbitrary \( J \in \mathcal{I}_W \). We will prove that \( v_J = v|_J \) is McShane integrable on \( J \) with the primitive \( V_J = V|_J \). Let \( \pi_J \) be a \( \delta_J\)-fine \( \mathcal{M} \)-partition of \( J \), where \( \delta_J = \delta_{\varepsilon}|_J \). Then, \( \pi_J \) is a \( \delta_{\varepsilon}\)-fine \( \mathcal{M} \)-partition in \( W \) and, therefore,
\[
\| \sum_{(t,I) \in \pi_J} (v_J(t)|I| - V_J(I)) \| = \| \sum_{(t,I) \in \pi_J} (v(t)|I| - V(I)) \| < \varepsilon.
\]

This means that \( v_J \) is McShane integrable on \( J \) with the primitive \( V_J \). Since \( J \) was arbitrary, the last result means that \( v \) is locally McShane integrable on \( W \) with the primitive \( V \), and this ends the proof. \( \square \)

Using Lemma 3.4.1 in [18], the next lemma can be proved in the same manner as Lemma 2.4.

**Lemma 2.5.** Let \( W \) be an open subset of \( \mathbb{R}^m \), let \( v : W \to X \) be a function and let \( V : \mathcal{I}_W \to X \) be an additive interval function. Then, the following statements are equivalent:

(i) \( v \) is locally Henstock-Kurzweil integrable on \( W \) with the primitive \( V \),

(ii) for each \( \varepsilon > 0 \) there exists a gauge \( \delta_{\varepsilon} \) on \( W \) such that for each \( \delta_{\varepsilon}\)-fine \( \mathcal{HK} \)-partition \( \pi \) in \( W \), we have
\[
\| \sum_{(t,I) \in \pi} (v(t)|I| - V(I)) \| < \varepsilon.
\]

Given a function \( f : G \to X \), we denote by \( h : \mathbb{R}^m \to X \) the function defined as follows
\[
h(t) = \begin{cases} 
  f(t) & \text{if } t \in G \\
  0 & \text{if } t \in \mathbb{R}^m \setminus G.
\end{cases}
\]
Lemma 2.6. Let $f : G \to X$ be a function. If $h$ is locally McShane integrable on $\mathbb{R}^m$ with the primitive $H$, then given $\varepsilon > 0$ there exists a gauge $\delta_v$ on $(G_0)^c$ such that for each $\delta_v$-fine $(G_0)^c$-tagged $\mathcal{M}$-partition $\pi_v$ in $\mathbb{R}^m$, we have

$$\| \sum_{(t,I) \in \pi_v} H(I) \| < \varepsilon.$$ 

Proof. By Lemma 2.4, given $\varepsilon > 0$ there exists a gauge $\delta_v$ on $\mathbb{R}^m$ such that for each $\delta_v$-fine $\mathcal{M}$-partition $\pi$ in $\mathbb{R}^m$, we have

$$(2.3) \quad \| \sum_{(t,I) \in \pi} (h(t)|I| - H(I)) \| < \frac{\varepsilon}{3}.$$ 

For each $k \in \mathbb{N}$, let

$$N_k = \{ t \in G \setminus G^o : k - 1 \leq ||h(t)|| < k \}.$$ 

Since $|G \setminus G^o| = 0$, for each $k \in \mathbb{N}$, we have $|N_k| = 0$ and, therefore, there exists an open set $G_k$ such that

$$(2.4) \quad G_k \supset N_k \quad \text{and} \quad |G_k| < \frac{\varepsilon}{3k^2}.$$ 

Define a gauge $\delta_v$ on $(G_0)^c$ in such a way that

$$t \in G^c \Rightarrow \delta_v(t) = \delta_v(t)$$

and

$$t \in N_k \subset G_k \Rightarrow B_m(t, \delta_v(t)) \subset G_k \quad \text{and} \quad \delta_v(t) \leq \delta_v(t).$$

Suppose that $\pi_v$ is an arbitrary $\delta_v$-fine $(G_0)^c$-tagged $\mathcal{M}$-partition in $\mathbb{R}^m$. Then, $\pi_v = \pi_v^1 \cup \pi_v^2$ where

$$\pi_v^1 = \{(t,I) \in \pi_v : t \in G^c\} \quad \text{and} \quad \pi_v^2 = \{(t,I) \in \pi_v : t \in G \setminus G^o\}.$$ 

Hence,

$$(2.5) \quad \| \sum_{(t,I) \in \pi_v} H(I) \| \leq \| \sum_{(t,I) \in \pi_v^1} H(I) \| + \| \sum_{(t,I) \in \pi_v^2} H(I) \|.$$ 

Since $\pi_v^1$ and $\pi_v^2$ are also $\delta_v$-fine $\mathcal{M}$-partitions in $\mathbb{R}^m$, we obtain by (2.3) that

$$(2.6) \quad \| \sum_{(t,I) \in \pi_v^1} H(I) \| = \| \sum_{(t,I) \in \pi_v^1} (h(t)|I| - H(I)) \| < \frac{\varepsilon}{3}$$

and

$$(2.7) \quad \| \sum_{(t,I) \in \pi_v^2} H(I) \| \leq \| \sum_{(t,I) \in \pi_v^2} (h(t)|I| - H(I)) \| + \| \sum_{(t,I) \in \pi_v^2} h(t)|I| \|< \frac{\varepsilon}{3} + \| \sum_{(t,I) \in \pi_v^2} h(t)|I| \| = \frac{\varepsilon}{3} + \| \sum_{(t,I) \in \pi_v^2} f(t)|I| \|.$$ 

By (2.4), we have also

$$\| \sum_{(t,I) \in \pi_v^2} h(t)|I| \| = \sum_{k=1}^{+\infty} \left( \sum_{t \in N_k} h(t)|I| \right) \|\leq \sum_{k=1}^{+\infty} \left( \sum_{t \in N_k} h(t)|I| \right) \| < \sum_{k=1}^{+\infty} \left( \sum_{t \in N_k} |I| \right) \| \leq \sum_{k=1}^{+\infty} k|G_k| < \sum_{k=1}^{+\infty} \frac{\varepsilon}{32k} = \frac{\varepsilon}{3}.$$
The last result together with (2.5), (2.6) and (2.7) yields
\[ \| \sum_{(t, I) \in \pi_v} H(I) \| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \]
and since \( \pi_v \) was arbitrary, this ends the proof. \( \square \)

Using Lemma 2.5, the next lemma can be proved in the same manner as Lemma 2.6.

**Lemma 2.7.** Let \( f : G \to X \) be a function. If \( h \) is locally Henstock-Kurzweil integrable on \( \mathbb{R}^m \) with the primitive \( H \), then given \( \varepsilon > 0 \) there exists a gauge \( \delta_0 \) on \( (G^o)^c \) such that for each \( \delta_0 \)-fine \( (G^o)^c \)-tagged HK-partition \( \pi_v \) in \( \mathbb{R}^m \), we have
\[ \| \sum_{(t, I) \in \pi_v} H(I) \| < \varepsilon. \]

We are now ready to present the main results.

**Theorem 2.8.** Let \( f : G \to X \) be a function and let \( F : \mathcal{I}_G \to X \) be an additive interval function. Then, the following statements are equivalent:

(i) \( f \) is Hake-McShane integrable on \( G \) with the primitive \( F \),

(ii) \( h \) is locally McShane integrable on \( \mathbb{R}^m \) with the primitive \( H \) such that \( H(I) = F(I) \), for all \( I \in \mathcal{I}_G \), and

\[ H(I) = \sum_{k : |I \cap C_k| > 0} F(I \cap C_k), \quad \text{for all } I \in \mathcal{I}, \]

whenever \( (C_k) \in \mathcal{D}_{G^o} \).

**Proof.** (i) \( \Rightarrow \) (ii) Assume that \( f \) is Hake-McShane integrable on \( G \) with the primitive \( F \). Then, by Definition 1.4 given \( \varepsilon > 0 \) there exists a gauge \( \delta_0 \) on \( G^o \) such that for each \( \delta_0 \)-fine \( \mathcal{M} \)-partition \( \pi_0 \) in \( G^o \), we have
\[ \| \sum_{(t, I) \in \pi_0} \left( f(t)|I| - F(I) \right) \| < \frac{\varepsilon}{4}. \]

Since \( F \) is a Hake-function, we can define an additive interval function \( H : \mathcal{I} \to X \) as follows
\[ H(I) = \sum_{k : |I \cap C_k| > 0} F(I \cap C_k), \quad \text{for all } I \in \mathcal{I}, \]
where \( (C_k) \) is an arbitrary division of \( G^o \). Clearly, \( H(I) = F(I) \), for all \( I \in \mathcal{I}_G \). We can choose \( \delta_0 \) so that \( B_m(t, \delta_0(t)) \subset G^o \) for all \( t \in G^o \). Hence,
\[ \| \sum_{(t, I) \in \pi_0} \left( h(t)|I| - H(I) \right) \| < \frac{\varepsilon}{4}, \]
whenever \( \pi_0 \) is a \( \delta_0 \)-fine \( \mathcal{M} \)-partition in \( G^o \).

Assume that an interval \( J \in \mathcal{I} \) is given. We are going to prove that \( h_J = h|_J \) is McShane integrable on \( J \) with the primitive \( H_J = H|_J \).

Since \( F \) has \( \mathcal{M} \)-negligible variation outside of \( G^o \), there exists a gauge \( \delta_0 \) on \( J \cap (G^o)^c \) such that for each \( J \cap (G^o)^c \)-tagged \( \delta_0 \)-fine \( \mathcal{M} \)-partition \( \pi_v \) in \( J \), we have
\[ \| \sum_{(t, I) \in \pi_0} H_J(I) \| = \| \sum_{(t, I) \in \pi_v} \left( \sum_{k : |I \cap C_k| > 0} F(I \cap C_k) \right) \| < \frac{\varepsilon}{4}. \]
By Lemma 2.2 we can choose \( \delta_0 \) so that for each \( J \cap (G \setminus G^o) \)-tagged \( \delta_0 \)-fine \( \mathcal{M} \)-partition \( \pi_v \) in \( J \), we have
\[ \| \sum_{(t, I) \in \pi_v} (h_J(t)|I| - H_J(I)) \| = \| \sum_{(t, I) \in \pi_v} \left( f(t)|I| - \sum_{k : |I \cap C_k| > 0} F(I \cap C_k) \right) \| < \frac{\varepsilon}{2}. \]
Define a gauge $\delta_J : J \to (0, +\infty)$ as follows.

$$\delta_J(t) = \begin{cases} 
\delta_0(t) & \text{if } t \in J \cap G^o \\
\delta_v(t) & \text{if } t \in J \cap (G^o)^c.
\end{cases}$$

Let $\pi$ be an arbitrary $\delta_J$-fine $\mathcal{M}$-partition of $J$. Then,

$$\pi = \pi_1 \cup \pi_2 \cup \pi_3,$$

where

$$\pi_1 = \{(t, I) \in \pi : t \in J \cap G^o\},$$
$$\pi_2 = \{(t, I) \in \pi : t \in J \cap (G \setminus G^o)\},$$
$$\pi_3 = \{(t, I) \in \pi : t \in J \setminus \overline{G}\}.$$ 

Note that

$$\sum_{(t, I) \in \pi} (h_J(t)|I| - H_J(I))|| \leq || \sum_{(t, I) \in \pi_1} (h_J(t)|I| - H_J(I))|| + || \sum_{(t, I) \in \pi_2} (h_J(t)|I| - H_J(I))|| + || \sum_{(t, I) \in \pi_3} H_J(I)||.$$ 

Hence, by (2.9), (2.10) and (2.11), it follows that

$$|| \sum_{(t, I) \in \pi} (h_J(t)|I| - H_J(I))|| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon,$$

and since $\pi$ was an arbitrary $\delta_J$-fine $\mathcal{M}$-partition of $J$, we obtain that $h_J$ is McShane integrable on $J$ with the primitive $F_J$.

$(ii) \Rightarrow (i)$ Assume that $(ii)$ holds. Then, by Lemma 2.4 given $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon$ on $\mathbb{R}^m$ such that for each $\delta_\varepsilon$-fine $\mathcal{M}$-partition $\pi$ in $\mathbb{R}^m$, we have

$$(2.12) \quad || \sum_{(t, I) \in \pi} (h(t)|I| - H(I))|| < \varepsilon.$$

We can choose $\delta_\varepsilon$ so that $B_m(t, \delta_\varepsilon(t)) \subset G^o$ for all $t \in G^o$. Let $\pi_0$ be a $\delta_\varepsilon$-fine $\mathcal{M}$-partition in $G^o$, where $\delta_0 = \delta_\varepsilon|G^o$. Then, $\pi_0$ is a $\delta_\varepsilon$-fine $\mathcal{M}$-partition in $\mathbb{R}^m$. Hence, by (2.12) it follows that

$$|| \sum_{(t, I) \in \pi_0} (f(t)|I| - F(I))|| < \varepsilon.$$

By equality

$$H(I) = F(I), \text{ for all } I \in \mathcal{I}_G$$

and by (2.8), it follows that $F$ is a Hake-function.

It remains to prove that $F$ has $\mathcal{M}$-negligible variation outside of $G^o$. By Lemma 2.6 there exists a gauge $\delta_v$ on $(G^o)^c$ such that for each $\delta_v$-fine $(G^o)^c$-tagged $\mathcal{M}$-partition $\pi_v$ in $\mathbb{R}^m$, we have

$$|| \sum_{(t, I) \in \pi_v} H(I)|| < \varepsilon.$$ 

Hence,

$$|| \sum_{(t, I) \in \pi_v} \left( \sum_{k|F(I \cap C_k) > 0} F(I \cap C_k) \right)|| \leq \varepsilon,$$

whenever $(C_k) \in \mathcal{G}_{G^o}$. This means that $F$ has $\mathcal{M}$-negligible variation outside of $G^o$, and this ends the proof. $\square$
Theorem 2.9. Let $f : G \to X$ be a function and let $F : \mathcal{I}_G \to X$ be an additive interval function. Then, the following statements are equivalent:

(i) $f$ is Hake-Henstock-Kurzweil integrable on $G$ with the primitive $F$,
(ii) $h$ is locally Henstock-Kurzweil integrable on $\mathbb{R}^m$ with the primitive $H$ such that $H(I) = F(I)$, for all $I \in \mathcal{I}_G$, and

$$H(I) = \sum_{k : |I \cap C_k| > 0} F(I \cap C_k), \text{ for all } I \in \mathcal{I}, \text{ whenever } (C_k) \in \mathcal{D}_G.$$

Proof. In the same manner as the proof of Theorem 2.8, by using Lemma 2.3, it can be proved that

$(i) \Rightarrow (ii)$ and, by using Lemmas 2.5 and 2.7, it can be proved that $(ii) \Rightarrow (i)$.

References

[1] Bongiorno, B., The Henstock-Kurzweil integral, Handbook of measure theory, Vol. I, II, 587-615, North-Holland, Amsterdam, (2002).
[2] Cao, S. S., The Henstock integral for Banach-valued functions, SEA Bull. Math., 16 (1992), 35-40.
[3] Di Piazza, L. and Musial, K., A characterization of variationally McShane integrable Banach-space valued functions, Illinois J. Math., 45 (2001), 279-289.
[4] Folland, G. B., Real Analysis, Modern techniques and their applications. Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, (1999).
[5] Fremlin, D. H., The Henstock and McShane integrals of vector-valued functions, Illinois J. Math., 38 (1994), 471-479.
[6] Fremlin, D. H., Mendoza, J., The integration of vector-valued functions, Illinois J. Math. 38 (1994) 127-147.
[7] Fremlin, D. H., The generalized McShane integral, Illinois J. Math. 39 (1995), 39-67.
[8] Gordon, R. A., The McShane integral of Banach-valued functions, Illinois J. Math. 34 (1990), 557-567.
[9] Gordon, R. A., The Denjoy extension of the Bochner, Pettis, and Dunford integrals, Studia Math. T.XCII (1989), 73-91.
[10] Gordon, R. A., The Integrals of Lebesgue, Denjoy, Perron, and Henstock, Amer. Math. Soc., (1994).
[11] Guoju, Y., and Schwabik, Š., The McShane integral and the Pettis integral of Banach space-valued functions defined on $\mathbb{R}^m$, Illinois J. Math. 46 (2002), 1125-1144.
[12] Guoju, Y., On Henstock-Kurzweil and McShane integrals of Banach space-valued functions, J. Math. Anal. Appl. 330 (2007), 753-765.
[13] Kaliaj, S. B., The New Extensions of the Henstock-Kurzweil and the McShane Integrals of Vector-Valued Functions, Mediterr. J. Math. (2018) 15: 22. https://doi.org/10.1007/s00009-018-1067-2.
[14] Kurzweil, J., Schwabik, Š., On the McShane integrability of Banach space-valued functions, Real Anal. Exchange 2 (2003-2004), 763-780.
[15] Lee P. Y, Lanzhou Lectures on Henstock Integration, Series in Real Analysis 2, World Scientific Publishing Co., Inc., (1989).
[16] Lee P. Y, and Výborný, R., The Integral: An Easy Approach after Kurzweil and Henstock, Australian Mathematical Society Lecture Series 14, Cambridge University Press, Cambridge, (2000).
[17] McShane, E. J., Unified integration, Academic Press, San Diego, 1983.
[18] Schwabik, Š. and Guoju, Y., Topics in Banach Space Integration, Series in Real Analysis, vol. 10, World Scientific, Hackensack, NJ, (2005).
[19] Schwabik, Š. and Guoju, Y., On the strong McShane integral of functions with values in a Banach space, Czechoslovak Math. J. 51 (2001), 819-830.

Mathematics Department, Science Natural Faculty, University of Elbasan, Elbasan, Albania.

E-mail address: sokolkaliaj@yahoo.com