Twitter as an innovation process with damping effect

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In the existing literature about innovation processes, the proposed models often satisfy the Heaps’ law, regarding the rate at which novelties appear, and the Zipf’s law, that states a power law behavior for the frequency distribution of the elements. However, there are empirical cases far from showing a pure power law behavior and such a deviation is mostly present for elements with high frequencies. We explain this phenomenon by means of a suitable “damping” effect in the probability of a repetition of an old element. We introduce an extremely general model, whose key element is the update function, that can be suitably chosen in order to reproduce the behaviour exhibited by the empirical data. In particular, we explicit the update function for some Twitter data sets and show great performances with respect to Heaps’ law and, above all, with respect to the fitting of the frequency-rank plots for low and high frequencies. Moreover, we also give other examples of update functions, that are able to reproduce the behaviors empirically observed in other contexts.

In our lives we continuously perform actions and these actions can be the repetition of something we have already done in the past or they can be a new experience: we can employ a technology that we already know or we can decide to try a new one, we can listen again a song that we already listened to in the past or we can decide to listen a new one, we can see old friends or we can decide to meet new people and so on. As a consequence, with our actions, we can contribute to diffuse an existing word or idea or product, or we can create a new trend. In particular, thinking about social platform, like Twitter, users can diffuse an existing post by means of a “retweet” or a “quote” of it, or they can write a new one.

Understanding the innovation process, that is the underlying mechanisms through which novelties emerge, diffuse and trigger further novelties is undoubtedly of fundamental importance in many areas (biology, linguistics, social science and others1–12). Novelties can be viewed as first time occurrences of some event and the mathematical object used to model an innovation process is an urn model with infinitely many colors, also known as species sampling sequence13–15. Let X1 be the first observed color, then, given the colors X1, . . . , Xn of the first n extractions, the color of the (n + 1)-th extracted ball is a new one with a probability b1 which is a function of X1, . . . , Xn (sometimes called “birth probability”) and it is equal to the already observed color c with probability pc = ∑rn=1 qrnI[Xn=c], where qrn is a function of X1, . . . , Xn. The quantities b1 and qrn specify the model: precisely, b1 describes the probability of having a new color (that is a novelty) at time-step t + 1 and qrn is the weight at time-step t associated to the event n, with 1 ≤ n ≤ t, so that the probability of having at time-step t + 1 the “old” color c is proportional to the total weight at time-step t associated to that color (a reinforcement mechanism, called “weighted preferential attachment” principle). Note that the number of possible colors is not fixed a priori, but new colors continuously enter the system. We can see the urn with infinitely many colors as the space of possibilities, while the sequence of extracted balls with their colors represents the history which has been actually realized.

Although there are only a few explicit prediction rules which give rise to exchangeable sequences, this kind of prediction rules are widely used, because exchangeability is a natural assumption in many statistical problems, in particular from the Bayesian viewpoint, and many theoretical results are known for exchangeable sequences13–17. We recall that a sequence is said exchangeable if its joint distribution is invariant with respect to permutations that act on only finitely many indices, with the rest fixed. Exchangeability is a powerful assumption, but, in some situations, it could be too restrictive and unrealistic, because it does not take into account the possible causality in the data. Therefore the introduction and study of species sampling sequences, which are not exchangeable, but which still have interesting theoretical properties, is welcome.

The Blackwell-MacQueen urn scheme14,18 provides the most famous example of exchangeable prediction rule. According to this prediction rule, a new color is observed with probability b1 = θ/(θ + t), where θ > 0, and an old color is observed with a probability proportional to the number Kt,c of times that color was extracted in the previous extractions: qt,c = 1/(θ + t), i.e. pt,c = Kt,c/(θ + t). This is the “simple” preferential attachment rule,

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also called “popularity” principle. This urn model is also known as Dirichlet process\textsuperscript{19} or as Hoppe’s model\textsuperscript{20} and, in terms of random partitions, it corresponds to the so-called Chinese restaurant process\textsuperscript{17}. Afterwards, it has been extended introducing an additional parameter and it has been called Poisson-Dirichlet model\textsuperscript{17,21–23}. More precisely, for the Poisson-Dirichlet model, we have
\[
b_t = \frac{\theta + \gamma D_t}{\theta + t}, \quad q_{0,t} = \frac{1 - \gamma/K_{c,t}}{\theta + t}, \quad \text{and so} \quad p_{c,t} = \frac{K_{c,t} - \gamma}{\theta + t},
\]
where \(0 \leq \gamma < 1, \theta > -\gamma\) and \(D_t\) denotes the number of distinct extracted colors until time-step \(t\). This model again generates an exchangeable sequence. In\textsuperscript{24}, the authors introduce and study a generalization of the Poisson-Dirichlet urn, introducing some random weights so that an old color is observed with a probability proportional to the total weight associated to that color during the previous extractions. The model so obtained does not give rise to an exchangeable sequence anymore, but the generated sequence is a conditionally identically distributed sequence\textsuperscript{25} and so properties usually required in Bayesian statistics are preserved.

From an applicative point of view, as an innovation process, the Poisson-Dirichlet process has the merit to reproduce in many cases the correct basic statistics, namely the Heaps’\textsuperscript{26–28} and the (generalized) Zipf’s laws\textsuperscript{29–30}, which quantify, respectively, the rate at which new elements appear and the frequency distribution of the elements.

The Heaps’ law states that the number \(D_t\) of distinct observed elements (i.e., colors, according the metaphor of the urn) when the system consists of \(t\) elements (i.e., after \(t\) extractions from the urn) follows a power law: \(D_t \propto t^b\), \(0 < b < 1\). Recently, Tria et al.\textsuperscript{31–33} have introduced and studied a new model, called urn with triggering, that includes the Poisson-Dirichlet process as a particular case. This model is based on Kauffman’s principle of the adjacent possible\textsuperscript{34}; indeed, the model starts with an urn with a finite number \(N_0 > 0\) of balls with distinct colors and, whenever a color is extracted for the first time, a set of balls with new colors is added to the urn. This represents Kauffman’s idea that, when a novelty occurs, it triggers further novelties. Therefore, in the urn with triggering, the space of possible colors expands and it can be seen as an urn with infinitely many colors, where
\[
b_t = \frac{N_0 + \nu D_t}{N_0 + pt + aD_t}, \quad \text{and} \quad p_{c,t} = \frac{\rho K_{c,t} + a - \nu}{N_0 + pt + aD_t},
\]
where \(\nu > 0, \rho > 0\) and \(a = \langle \rho + v - \rho + 1 \rangle \) with \(\bar{\rho} = 1\). The Poisson-Dirichlet model corresponds to the case \(a = 0\) (taking \(\theta = N_0/\rho\) and \(\gamma = \nu/\rho \in [0, 1]\)). In general, the sequences generated by the urn with triggering are not exchangeable. Moreover, while the Poisson-Dirichlet process can predict only a sub-linear power law behavior for the number of distinct observed colors/elements (i.e. Heaps’ law with \(\gamma < 0\)), the urn with triggering is able to provide also a linear growth for it (i.e. Heaps’ law with \(\gamma = 1\)):

\[
\text{Zipf’s law}\quad \gamma = 1,
\]

essentially the number of distinct observed elements (i.e. colors, according the metaphor of the urn) when the system consists of \(t\) elements (i.e., after \(t\) extractions from the urn) follows a power law: \(D_t \propto t^b\), \(0 < b < 1\). Recently, Tria et al.\textsuperscript{31–33} have introduced and studied a new model, called urn with triggering, that includes the Poisson-Dirichlet process as a particular case. This model is based on Kauffman’s principle of the adjacent possible\textsuperscript{34}; indeed, the model starts with an urn with a finite number \(N_0 > 0\) of balls with distinct colors and, whenever a color is extracted for the first time, a set of balls with new colors is added to the urn. This represents Kauffman’s idea that, when a novelty occurs, it triggers further novelties. Therefore, in the urn with triggering, the space of possible colors expands and it can be seen as an urn with infinitely many colors, where
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b_t = \frac{N_0 + \nu D_t}{N_0 + pt + aD_t}, \quad \text{and} \quad p_{c,t} = \frac{\rho K_{c,t} + a - \nu}{N_0 + pt + aD_t},
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where \(\nu > 0, \rho > 0\) and \(a = \langle \rho + v - \rho + 1 \rangle \) with \(\bar{\rho} = 1\). The Poisson-Dirichlet model corresponds to the case \(a = 0\) (taking \(\theta = N_0/\rho\) and \(\gamma = \nu/\rho \in [0, 1]\)). In general, the sequences generated by the urn with triggering are not exchangeable. Moreover, while the Poisson-Dirichlet process can predict only a sub-linear power law behavior for the number of distinct observed colors/elements (i.e. Heaps’ law with \(\gamma < 1\)), the urn with triggering is able to provide also a linear growth for it (i.e. Heaps’ law with \(\gamma = 1\)):

\[
\text{Zipf’s law}\quad \gamma = 1.
\]
the previous extractions) that also incorporates a long-term memory or aging component: precisely, at a time step \( t \), a new element appears with probability \( b \), whereas with probability \((1-b)\), an old element is chosen, going back in time by \( n \) time steps with a probability \( Q_n(n) \) that decays as a power law. This generative stochastic model is able to fit with high accuracy the observed frequency-rank plots in the collaborative tagging context. Finally, in\(^{47}\), the authors observe an exponential decay in the frequency-rank plot and this fact is ascribed to the model is able to fit with high accuracy the observed frequency-rank plots in the collaborative tagging context.

More precisely, we generalize the urn with triggering model by the introduction of a function \( F \) that drives the update mechanism of the number of balls of the same color of the extracted one when it is of an “old” color. In the standard model, this function is linear so that it generates a power law behavior of the frequency-rank plot (Zipf’s law), that usually matches the empirical ones only in the part of rare elements (i.e. large ranks). Instead, if we take the function \( F \) linear until a certain point and then still linear but with a smaller slope or sub-linear (for instance, the square root), then we obtain a frequency-rank plot closer to the empirical ones also in the part of high frequencies. This fact can be seen as a damping effect on the old elements: the number of balls of an old color increases linearly with the number of times it is extracted until a certain threshold, then it increases slower. Our unique general model is able to reproduce the empirically observed power law behavior with two different scaling exponents mentioned above and also other kinds of curves, observed in real data sets. Indeed, given the function \( g \) that fits the empirical frequency-rank plot (see eq. (9)), we are able to find the corresponding function \( F \) of the proposed generative model (see eq. (3)). This is a very useful result for applicative purposes, since in applications one usually observes and tries to fit the empirical frequency-rank plot. Further, we have shown how to obtain the asymptotic behavior of the number \( D_t \) of distinct observed elements starting from the function \( g \) and we have employed this methodology with some specific functions \( g \) coming from empirical studies. The obtained theoretical results are supported by simulations.

We apply the proposed model to some data sets from the social platform Twitter. The main mechanism in Twitter leading content diffusion is the possibility for users to “re-tweet”, reply or quote the post sent by others. Therefore, ordered sequences of posts can be seen as generated by an urn model with infinitely many colors: a new color is associated to a new tweet, while the extraction of an old color corresponds to a re-sharing (by means of a re-tweet or a quote or a reply to) of an old post. The update function \( F \) rules the probability of a generic posted tweet to be re-shared. For all the considered data sets, we observe the same damping effect on the old elements: the update function \( F \) grows linearly until a certain threshold, then it increases sub-linearly, precisely according to the square root. Moreover, we empirically verify the uniform growth of the variable \( D_t \), which agrees with the proven theoretical result. We refer to\(^{50}\) for a survey on information diffusion in on-line social networks. In\(^{50}\) the authors face the problem to predict the future time evolution of the popularity of a certain tweet and, in particular, to estimate the final number of re-tweets of that given tweet (see also the reference therein for other works related to the same question). In\(^{50,52}\) the focus is instead on the “retweet graph”, which is the graph of users who participated in the discussion of a specific topic and where a directed edge indicates that a user re-tweeted a tweet of another user.

Finally, we explain our choice of the term “damping” with respect to the other terms “saturation” and “aging” employed in literature. The term “saturation” is typically used when the probability of having the repetition of a certain old element decreases with the difference between the present time and the last time of observation of that old element\(^{44}\). The effect that we model refers to a damping factor in the probability of a repetition of an old element: using the metaphor of the urn, the number of balls of a given color in the urn increases with the number of times that color has been extracted according to a suitable update function that exhibits two different speeds, one before a certain threshold and a lower one after the threshold. Therefore, it obviously differs from the above saturation effect and it is related in some sense to the age of the elements, not directly and so it is not exactly the same of the above recalled aging effect.

Summing up, the contribution of the present paper is twofold. First, we introduce a general and flexible model for innovation processes that is able to generate frequency-rank plots that are not pure power law. Hence, this work could result interesting for researchers who work in various contexts of innovation theory. Second, we analyze some Twitter data sets, that shows the presence of a damping effect, perfectly explained by means of the proposed model with a suitable update function. Therefore, this work could be of interest for who are involved in the study of Twitter activity.

**Results**

We firstly introduce the model, using the metaphor of the urn, and we state the main related results. Then we show the empirical results.

**Model.**- Given an increasing function \( F \) defined on \( \mathbb{N} \setminus \{0\} \) with \( F(1) > 0 \), the model works as follows. An urn initially contains balls of \( N_0 > 0 \) different colors. For each color, we have one ball. Then, at each time step \( t \), a ball is drawn at random from the urn, its color is registered in a sequence \( S \) and:

- if the color of the extracted ball is a new one, that is it appears for the first time in \( S \) (it corresponds to the realization of a novelty), then the number of balls of the extracted color in the urn becomes \( F(1) > 0 \) and we add \( v + 1 \) (with \( v \geq 0 \)) distinct balls of different new colors, that is of colors not yet present in the urn (precisely, we add one ball for each new color);
• if the color $c$ of the extracted ball is already present in $S$ and $K_{c,t}$ denotes the number of times the color $c$ was extracted until time step $t$ (included), the number of balls of color $c$ in the urn becomes $F(K_{c,t}) > 0$.

The fact that the update function $F$ is increasing means that we have a reinforcement mechanism: the larger the number of times the color $c$ has been extracted, the larger the number of balls of color $c$ in the urn. Moreover, the addition of a set of balls with new colors in the urn whenever a color is extracted for the first time represents Kauffman’s principle of the adjacent possible\(^n\), i.e. the idea that, when a novelty occurs, it triggers further novelties.

If $X_{t+1}$ is the color of the extracted ball at time step $t+1$, $D_t$ is the number of different colors extracted until time step $t$, $N_t$ is the number of different colors in the urn until time step $t$ and $T_t$ is the total number of balls in the urn at time step $t$, we have:

$$N_t = N_0 + (\nu + 1)D_t$$

$$T_t = \sum_{i=1}^{N_t} F(K_{i,t}) = N_0 + \nu D_t + \sum_{j \in S} F(K_{j,t})$$

(the sum $\sum_{j \in S}$ denotes the sum on the $D_t$ different colors in $S$) and

$$b_t = P(X_{t+1} = \text{new} \mid X_1, \ldots, X_t) = \frac{(N_t - D_t)}{T_t} = \begin{cases} 1 & \frac{N_0 + \nu D_t}{N_0 + \nu D_t + \sum_{j \in S} F(K_{j,t})} \text{ for } t = 0 \\ \frac{N_0 + \nu D_t}{N_0 + \nu D_t + \sum_{j \in S} F(K_{j,t})} \text{ for } t \geq 1 \end{cases} \quad (1)$$

Moreover, if $c$ denotes an old color, we have for each $t \geq 1$

$$p_{c,t} = P(X_{t+1} = c \mid X_1, \ldots, X_t) = \frac{F(K_{c,t})}{N_0 + \nu D_t + \sum_{j \in S} F(K_{j,t})} \quad (2)$$

The quantities $b_t$ and $p_{c,t}$ drive the model. The model parameters $N_0$ and $\nu$ can be any real positive numbers with $\max(N_0, \nu) > 0$. The update function $F$ can be any increasing function defined on the strictly positive integer numbers and with values in $(0, +\infty)$. For instance, the standard urn model with triggering\(^{31-33}\) corresponds to the update function $F$ defined as $F(x) = \hat{\rho} + \rho(x - 1)$, with $\hat{\rho} = \hat{\rho} + 1 > 0$ and $\rho > 0$.

For the case $\nu > 0$, when we assume a dependence in the frequency-rank plot of the form $g(z(r)) = -a \ln(r) + b$, with an invertible differentiable function $g$ and a constant $a > 0$, we obtain (see Section 3.2 for the computations) the relation

$$F(x) \approx \frac{ax}{g'(x)} \quad (3)$$

This relation is not an exact equality, because it is due to (8) and (10), which have been obtained by approximation. From the applicative point of view, relation (3) is of fundamental utility because it allows us to guess the right update function $F$ in the model by means of the function $g$ that fits well the empirical frequency-rank plot. For instance, for the standard urn with triggering, we have $g = \ln$ with $a = \rho/\nu$ and, indeed, we have $F(x) = \hat{\rho} + \rho(x - 1) \approx ax = ax$ for large $x$ and so (3) is satisfied. More generally, whenever we have a Zipf’s law, that is $g = \ln$ in (9), we get that, according to (3), we have to choose a linear update function $F$ in the model.

Regarding the asymptotic behavior of the number $D_t$ of distinct elements in $S$ as a function of its length $t$, we note that

$$t = \sum_{r=1}^{D_t} z(r) \approx \int_1^{D_t} z(r)dr \quad (4)$$

and so the behavior of $D_t$ can be obtained starting from the frequency-rank function $r \mapsto z(r)$. In the Section 3.3, we derive the asymptotic behavior of $D_t$ starting from different kinds of function $g$.

In Figs. 1, 2 and 3, we show the simulations of the model with $N_0 = 2, \nu = 0.75$ and different kinds of update function $F$. The relation (3) and the other theoretical results described in the Methods Section are supported by these simulations. More precisely, in Fig. 1 we exhibit the results for

$$F(x) = \begin{cases} \rho x & \text{for } 1 \leq x < z_* \\ \rho \sqrt{x} & \text{for } x \geq z_* \end{cases} \quad (5)$$

for different values of $\rho$ and $z_*$. The model with this function $F$ gives rise to a dependence structure in the frequency-rank plot described by $g(z) = \ln(z)$ before $z_*$ and $g(z) = \sqrt{z}$ after $z_*$ and to a linear growth of $D_t$, that is a Heaps’ law with exponent 1. (See Subsec. 3.3.1 for technical details).

In Fig. 2 we exhibit the results for

$$F(x) = \begin{cases} \rho_1 x & \text{for } 1 \leq x < z_* \\ \rho_1 x + (\rho_1 - \rho_2)z_* & \text{for } x \geq z_* \end{cases} \quad (6)$$

for different values of $\rho_1, \rho_2$ and $z_*$. The model with this function $F$ gives rise to a Zipf’s law with two different coefficients and to different kinds of behavior for $D_t$, depending on the value $\rho_1/\nu$. (See Subsec. 3.3.2 for technical details).
we verify (see the right panels in Figs. 4 and 5) the linear growth of the number of epidemics, the number of observed distinct colors is... the period from February 21st to April 20th 2020, including tweets in Italian language. The keywords used for the query are relative to the COVID-19 epidemic. The total number of posts is $T = 4,580,781$. More details on the data set can be found in 53.

- **Italy, Migration debate**
  Data were collected through the Filter API since 23rd of January to 22nd of February 2019 and targeted the Italian debate on migration. The total number of posts is $T = 1,066,677$. More details on the data set can be found in 53.

- **Italy, COVID-19 epidemic**
  The data set covers the period from February 21st to April to 20th 2020, including tweets in Italian language. The keywords used for the query are relative to the COVID-19 epidemic. The total number of posts is $T = 4,580,781$. More details on the data set can be found in 54.

Using the metaphor of the urn, the extractions correspond to the publication of posts on Twitter. Therefore, the sequence $S$ of colors is constructed looking at the posts ordered by their time-stamps. The color of the ball tells if the post is a new tweet (that is a novelty) or a re-tweet/quote/reply (that is a repetition): a new color is associated to a new tweet; while an old color corresponds to a re-sharing (by a re-tweet or a quote or a reply) of an old post. More precisely, in the latter case, we register in $z_{	ext{old}}$ the color of the original message: for instance, given $t_3$ we have the quote of the post published at time $t_2$, which is a retweet of the post at time $t_1$, we register at positions $t_2$ and $t_3$ the same color of $t_1$. For the sequence obtained from the data set regarding the migration debate, the number of observed distinct colors is $D_T = 210,190$ and the maximum number of times a given color is repeated is $z_{\text{max}} = 3,694$. For the sequence obtained from the data set regarding the COVID-19 epidemic, the number of observed distinct colors is $D_T = 1,447,623$ and the maximum number of times a given color is repeated is $z_{\text{max}} = 4,818$.

When a new tweet has been sent, the addition of $\nu + 1$ balls of new distinct colors can be seen as the potential new tweets that the posted tweet may generate. Hence the parameter $\nu$ is related to the ability of a generic new tweet to give rise to future new tweets. On the other hand, the update function $F$ rules the probability of a generic posted tweet to be re-shared (with a retweet or a quote or a reply). For all the considered data sets, we observe the same damping effect on the old elements: the update function $F$ increases linearly until a certain threshold, then it increases sub-linearly as the square root. Indeed, looking at the empirical frequency–rank plot, we observe a dependence structure given by $g(z) = \ln(z)$ before a certain threshold $z_\ast$ and $g(z) = \sqrt{z}$ after $z_\ast$ (see the left panels in Figs. 4 and 5), that corresponds to the update function $F$ in the model described in (13). Moreover, we verify (see the right panels in Figs. 4 and 5) the linear growth of the number $D_t$ of distinct observed tweets, which agrees with the proven theoretical result (see Subsec. 3.3.1). Finally, Fig. 6 shows that for both data sets the frequency distribution $f(D_t)$ of inter-event time steps $\Delta_t$ between pairs of consecutive occurrences of the same color in $S$ exhibits a behavior similar to the one obtained by simulations of the model with $F$ given by (13).

Finally, in Fig. 3 we exhibit the results for

$$F(x) = \rho_1 x \ln(x + 1) \quad (7)$$

for different values of $\rho_1$. The model with this update function $F$ gives rise to an exponential decay of the frequency–rank function $z(t)$ (that corresponds to $g(z) = \ln(\ln(z))$) and to a logarithm growth of $D_t$. (See Subsec. 3.3.3 for technical details). It is worthwhile to note that these behaviors for $z(t)$ and $D_t$ can be achieved also by the standard urn with triggering with $F$ (see 15). However, the two models are completely different: for the standard model, the assumption regards the parameter $\nu$ that rules the probability $b_t$ of a new color, while the number of balls of an old color increases linearly according to a “free” coefficient; whereas, for the proposed new model, the assumption regards the update function $F$ that drives the growth of the number of balls of an old color in the urn and the parameter $\nu$ can be whatever.

**Empirical results.** Two data sets have been collected from the Twitter platform, using the official API to stream the exchange of messages on several topics:

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Figure 2. Simulations for case (6) with $N_0 = 2$, $\nu = 0.75$ and different values of $\rho_1$, $\rho_2$ and $z_\ast = z_0$: (Up) Frequency-rank plot in log-log scale. The different colors of the dots correspond to different quantities of data taken for depicting the plot. (Below) Behavior of $D_t$: the plot on the left shows a linear growth of $D_t$ (Heaps’ law with exponent 1) when $\rho_2/\nu < 1$ and the plot on the right shows a power-law growth (Heaps’ law with exponent smaller than 1) when $\rho_2/\nu > 1$. 
**Discussion**

The innovation models introduced so far satisfy the Heaps' law, regarding the rate at which novelties appear, and the Zipf's law, that states a power law behavior for the frequency distribution of the elements. However, there are empirical cases far from showing a pure power law behavior and such a deviation is present for elements with low ranks, that is with high frequencies. In this work we explain such deviations from the Zipf's law by adding a suitable *damping effect* in the urn with triggering model. More precisely, we generalize the standard urn with triggering\(^3\) to\(^3\) by the introduction of a function \(F\) that drives the update mechanism of the number of balls of the same color of the extracted one when it is of an old color. Indeed, if we take the update function \(F\) linear until a certain point and then still linear but with a smaller slope or sub-linear, then we obtain a frequency-rank plot closer to the empirical ones also in the part of high frequencies. This means that the number of balls of an old

**Figure 3.** Simulations for case (7) with \(N_0 = 2, \nu = 0.75\) and different values of \(\rho_1\): (Left) Frequency-rank plot with \(g(z) = \ln(\ln(z))\). The different colors of the dots correspond to different quantities of data taken for depicting the plot. (Right) Behavior of \(D_t\).

**Figure 4.** Migration: (Left) Frequency-rank plot in two different scales: log-sqrt before a certain \(R_*\) and log-log after \(R_*\). Parameters (see Sections 3.3.1 and 3.4 for details): \(a_1 = 7.07, a_2 = 1.13, R_* \in [-5.97, -5.19]\), that give \(\rho/\nu = a_2 = 1.13\) and \(z_* = 4(a_1/a_2)^2 = 156.58\). The different colors of the dots correspond to different quantities of data taken for depicting the plot. (Right) Behavior of \(D_t\).
color increases linearly with the number of times it is extracted until a certain threshold, then it increases slower, that is we have a damping factor in the updating of the urn.

Given the function $g$ that fits the empirical frequency-rank plot (see eq. (9)), we are able to find the corresponding update function $F$ of the proposed generative model (see eq. (3)). This is a very useful result for applicative purposes, since in applications one usually observes and tries to fit the empirical frequency-rank plot. Further, we have shown how to obtain the asymptotic behavior of the number $D_t$ of distinct observed elements starting from the function $g$ and we have employed this methodology with some specific functions $g$. The obtained theoretical results are supported by simulations.

We have applied the proposed model to some data sets from the social platform Twitter, where the update function $F$ rules the probability of a generic posted tweet to be re-shared. For all the considered data sets, we observed the same damping effect on the old elements: the update function $F$ grows linearly until a certain threshold, then it increases sub-linearly, precisely according to the square root. Moreover, we empirically verified the linear growth of the variable $D_t$, which agrees with the proven theoretical result. We would like to point out that the analyzed data sets include a huge number of posts, but they cover a short period of time (one or two months). Therefore, a possible research line for the future could be to investigate if data sets collected over longer periods exhibit the same behavior or are well fitted with a different update function $F$. For instance, we can generalize the proposed model taking into account a time-dependent update function. Moreover, both topics for which data were collected are such that there was considerable public interest on the topics during the period of collection. Hence, it could be interesting to consider topics that emerge suddenly (such as a natural disaster) but interest on the topic does not stay for a long period. To this regard, we think that the model will be again able to reproduce the phenomenon taking an update function that increases with two different rates: again, a damping effect, such as in the data sets analyzed in the present paper, but probably with a more heavily damping in the growth rate of the update function.

Finally, we underline that the proposed model provides a general framework that is able to explain also the power law behavior with two different scaling exponents observed in\textsuperscript{43,46,48} and other kinds of empirically observed curves (e.g. in\textsuperscript{8}). Therefore, it results a very flexible generative model that could perfectly reproduce the frequency-rank plot for low and high ranks, together with the behavior of $D_t$, in many contexts. It could be interesting to consider in the future other data sets, related to different contexts and possessing different characteristics, and exploit the proposed model with other update functions.

**Methods**

Take $\nu > 0$ and assume $F$ to be extended with continuity on the whole $[1, +\infty)$ and in such a way that it is differentiable everywhere except in a finite number of points.

**Relation between $F$ and the frequency distribution.** For each $k \geq 1$, we denote by $Q_{k,t}$ the number of colors $c$ in $\mathcal{S}$ with $K_{c,t} = k$ and we set $p_k = \lim_t Q_{k,t}/D_t$. The family $(p_k)_k$ is the (stationary) frequency distribution.

We have $D'_t = b_t = \frac{N_t + 2D_t}{2}$ and so we can write $T_t \approx \frac{vD_t}{T_t}$. Moreover, we have $\sum_{k \geq 1} Q_{k,t} = D_t$ and we can write the following master equation for $Q_{k,t}$:

$$\frac{\partial Q_{k,t}}{\partial t} = -\frac{Q_{k,t}F(k)}{T_t} + \frac{Q_{k-1,t}F(k-1)}{T_t} = \frac{(Q_{k,t} - Q_{k-1,t})(F(k) - F(k-1))}{T_t} \approx -\frac{1}{T_t} \frac{\partial F(k)Q_{k,t}}{\partial k}.$$

Using the asymptotic relations $Q_{k,t} \approx p_kD_t$ and $\frac{\partial Q_{k,t}}{\partial t} \approx D'_t p_k$, from the above relation we get
Since \( d[F(k)p_k] = F'(k)p_k + F(k)p'_k \), we obtain

\( \frac{p'_k}{p_k} \approx -\frac{\nu}{F(k)} - \frac{F'(k)}{F(k)}. \)

When \( 1/F \) has primitive function \( H \), this equation has solution

\( \ln(p_k) \approx -\ln(F(k)) - \nu H(k) + C \),

that is

\( p_k \approx k^{-\nu/\rho} \) for large \( k \) (see also\( ^{18} \)).

Figure 6. Frequency distribution of inter-event time steps between pairs of consecutive occurrences of the same color: (Up, Left) Migration; (Up, Right) COVID-19; (Below) Simulations of the model with \( F \) given by (13), \( N_0 = 2, \nu = 0.75, \hat{\rho} = \rho \) and different values of \( \rho \) and \( z_* = z_0 \).
Relation between \( F \) and the frequency-rank plot. Assume a dependence in the frequency-rank plot of the form

\[
g(z(r)) = -a \ln(r) + b
\]  

(9)

with a strictly increasing function \( g \) and a constant \( a > 0 \). Then, we get

\[
r = \exp\left(-\frac{g(z(r)) - b}{a}\right).
\]

If \( g \) is differentiable, we have

\[
\frac{\partial g(r)}{\partial r} = \frac{g^{-1}(-a \ln(r) + b)}{a} = \frac{1}{g'(g^{-1}(-a \ln(r) + b))} \frac{1}{r}.
\]

Since \( \delta r \approx p(z(r))|\delta z| \), we get

\[
p(z(r)) \propto -\frac{1}{r} = g'(g^{-1}(-a \ln(r) + b)) \frac{r}{a} = a^2 g'(z(r)) \exp\left(-\frac{g(z(r)) - b}{a}\right).
\]

Setting \( y = z(r) \) in the above relation and taking the logarithm, we find

\[
\ln(p(y)) \approx \ln(g'(y)) - \frac{g(y)}{a} + C_1
\]

\[
\approx -\ln\left(\frac{av}{g'(y)}\right) - v \int_{y_0}^{y} \frac{1}{av} g'(s) \, ds + C_2.
\]

(10)

If we compare this last equation with (8), we arrive to the relation (3) between the function \( F \) of the model and the function \( g \) describing the frequency rank plot.

Behavior of \( D_t \). Since (4), the behavior of \( D_t \) can be obtained starting from the frequency-rank function \( r \mapsto z(r) \). For instance, in the case of a pure (generalized) Zipf’s law with \( \alpha \neq 1 \), we have

\[
t \approx \int_{1}^{D_t} z_{\max} r^{-\alpha} \, dr = z_{\max} \left(\frac{D_t^{1-\alpha} - 1}{1-\alpha}\right)
\]

and so, taking into account that \( z(D_t) \propto \xi_1 \), i.e. \( z_{\max} \propto D_t^\alpha \), we get \( D_t \propto t^{1/\alpha} \) for \( \alpha < 1 \) and \( D_t \propto t \) for \( \alpha > 1 \). When \( \alpha = 1 \) we find \( t \approx D_t \ln(D_t) \) and so \( D_t \propto t / \ln(t) \).

When \( z(r) \) exhibits two different behaviors, one for small ranks, say for \( r < \xi_1 \), and the other for large ranks, say for \( r > \xi_1 \), the above relation becomes

\[
t \approx \int_{1}^{\xi_1} z(r) \, dr + \int_{\xi_1}^{D_t} z(r) \, dr.
\]

(11)

In the following, we will study different cases: the first one is observed in the real data sets that we discuss in the present work and the other two have been observed in other papers\(^{45,46-48}\).

\[
g(z) = \ln z \text{ for } z < z_\ast \text{ and } g = \sqrt{z} \text{ for } z > z_\ast
\]

Suppose that the frequency-rank plot identifies the following dependence structure:

\[
\begin{align*}
\text{for } \ln \left(\frac{r}{D_t}\right) < R_\ast & : \text{ } a_1(\ln \left(\frac{r}{D_t}\right) - R_\ast) \\
\text{for } \ln \left(\frac{r}{D_t}\right) > R_\ast & : \text{ } a_2(\ln \left(\frac{r}{D_t}\right) - R_\ast)
\end{align*}
\]

(12)

where \( a_1, a_2, R_\ast \) and \( z_\ast \) are constants such that \( a_1 > 0, R_\ast < 0 \) and \( z_\ast = z(\xi_1) \) with \( \xi_1 = D_t e^{R_\ast} \). This corresponds to

\[
g(z) = \begin{cases} 
\ln(z) & \text{for } z < z_\ast \\
\sqrt{z} & \text{for } z > z_\ast
\end{cases}
\]

and, leveraging (3), we detect the function \( F \) in the model as

\[
F(x) = \begin{cases} 
\hat{\rho} - \rho + \rho x & \text{for } 1 \leq x < z_\ast \\
\hat{\rho} - \rho + \rho \sqrt{x} & \text{for } x \geq z_\ast
\end{cases}
\]

(13)

where \( \hat{\rho} > 0, \sqrt{a_1} = 2v a_1, \rho = v a_2 > 0 \) and so

\[
z_\ast = 4 \left(\frac{a_1}{a_2}\right)^2.
\]

(14)

From (9) and (12), we get
\[ z(r) = \begin{cases} \frac{g^{-1}(-a_1 \ln r + b_1)}{g^{-1}(-a_2 \ln r + b_2)} & \text{for } \ln \left(\frac{r}{\xi} \right) < R_s \\ \frac{g^{-1}(-a_2 \ln r + b_2)}{g^{-1}(-a_1 \ln r + b_1)} & \text{for } \ln \left(\frac{r}{\xi} \right) > R_s \\ \end{cases} \]

where

\[-a_1 \ln \xi + b_1 = \sqrt{z_s} \quad \text{and} \quad -a_2 \ln \xi + b_2 = \ln(z_s). \quad (15)\]

Now, we recall that

\[ \int_1^x (-a_1 \ln(r) + b_1)^2 \, dr = x \left[-a_1 \ln(x) + b_1 \right]^2 + 2a_1(-a_1 \ln(x) + b_1) - b_1^2 - 2a_1b_1. \]

Taking in the above integral \( x = \xi \) and using the first equality in (15), we obtain

\[ \int_1^{\xi} z(r) \, dr = \xi(z_s + 2a_1^2 + 2a_1\sqrt{z_s}) + O(\ln^2(\xi)) = D_t e^{R_s} \left[ 1 + \rho^2/(2\nu^2) + \rho/\nu \right] + O(\ln^2(D_t)). \]

For the second integral, assuming \( a_2 = \rho/\nu \neq 1 \) and using the second equality in (15), we get

\[ \int_{\xi}^{D_t} z(r) \, dr = \frac{e^{b_1}}{1 - a_2} \left[ D_t^{1-a_2} - \xi^{1-a_2} \right] = D_t^{\frac{\rho}{\nu - 1}} \left( 1 - e^{R_s}(\rho/\nu - 1) \right), \]

while for \( a_2 = \rho/\nu = 1 \), we get

\[ \int_{\xi}^{D_t} z(r) \, dr = e^{b_1} \left[ \ln(D_t) - \ln(\xi) \right] = -z_s R_s D_t e^{R_s}. \]

From (11), (16), (17) and (18), we can conclude that \( D_t \propto t \). This means that we have the Heaps’ law with exponent \( \gamma = 1 \).

**Zipf’s law with two different coefficients.** Suppose that the frequency-rank plot identifies a “double” Zipf’s law, that is the following dependence structure:

\[ \begin{cases} \ln z(r) - \ln z_s = -a_1 \ln \left(\frac{r}{\xi} \right) - R_s \quad \text{for } \ln \left(\frac{r}{\xi} \right) < R_s \\ \ln z(r) - \ln z_s = -a_2 \ln \left(\frac{r}{\xi} \right) - R_s \quad \text{for } \ln \left(\frac{r}{\xi} \right) > R_s \end{cases} \]

where \( a_1, a_2 \) and \( R_s \) are constants such that \( a_1 > 0, a_1 \neq a_2 \) (typically \( a_1 < a_2 \)), \( R_s < 0 \) and \( z_s = z(\xi) \) with \( \xi = D_t e^{R_s} \). This kind of dependence was observed in and, according to our model (see (3)), it corresponds to

\[ F(x) = \begin{cases} \hat{\rho} - \rho_1 + \rho_1 x & \text{for } 1 \leq x < z_s \\ \hat{\rho} - \rho_1 + \rho_2 x + (\rho_1 - \rho_2)z_s & \text{for } x \geq z_s, \end{cases} \]

where \( \rho_1 = \nu a_1 \neq \rho_2 = \nu a_2 \) (typically \( \rho_2 < \rho_1 \)) and \( \hat{\rho} > 0 \).

From (9) and (19), we get

\[ z(r) = \begin{cases} \frac{g^{-1}(-a_1 \ln r + b_1)}{g^{-1}(-a_2 \ln r + b_2)} & \text{for } \ln \left(\frac{r}{\xi} \right) < R_s \\ \frac{g^{-1}(-a_2 \ln r + b_2)}{g^{-1}(-a_1 \ln r + b_1)} & \text{for } \ln \left(\frac{r}{\xi} \right) > R_s \\ \end{cases} \]

where \( b_1 = \ln z_s + a_1 \ln \xi \) and \( b_2 = \ln z_s + a_2 \ln \xi \). Therefore, we have

\[ \int_1^{\xi} z(r) \, dr = \frac{z_s}{z_s^{\rho/a_1 - 1}} (D_t e^{R_s} - D_t^{\rho/a_1} e^{R_s}) \]

for \( a_1 = \rho_2/\nu \neq 1 \) and

\[ \int_{\xi}^{D_t} z(r) \, dr = D_t^{\frac{\rho}{\nu - 1}} \left( 1 - e^{R_s}(\rho/\nu - 1) \right) \]

for \( a_2 = \rho_1/\nu = 1 \).

From (11), (20), (21), we can conclude that the asymptotic behavior of \( D_t \) is ruled by the value of \( a_1 = \rho_2/\nu \): \( D_t \propto t \) when \( a_1 < 1 \), \( D_t \propto t^{1/a_1} \) when \( a_1 > 1 \).
Frequency-rank plot with exponential decay. Suppose that the frequency-rank plot identifies the following dependence structure:

$$\ln(z(r)) = -ar + b$$

that is

$$z(r) = Ce^{-ar},$$

where $a > 0$ and $C = e^{b} \propto e^{D_{i}}$. This is the case observed in [47]. The corresponding behavior of $D_{i}$ is given by the relation (11), that is

$$t \approx -\frac{C}{a}(e^{-aD_{i}} - e^{-a}) = \frac{1}{a} \left( e^{aD_{i} - 1} - 1 \right) \approx \frac{1}{a} e^{aD_{i}},$$

that implies $D_{i} \propto \ln(t)$. These behaviors for $z(r)$ and $D_{i}$ can be achieved by the introduced model with $F(x) = \rho(x + 1) \ln(x + 1)$ and $\rho > 0$. Indeed, starting from (22) and employing an adaption of the argument used in Section 3.2, we find $p(z(r)) \propto \frac{1}{\sqrt{D_{T}}}$ and so $\ln(p(y)) \approx -\ln(y)$. On the other hand, when inserting the above function $F$ in (8), we find $\ln(p(k)) \approx -\ln(k + 1) - \frac{1}{v/\rho} \ln(\ln(k + 1)) + C' \propto -\ln(k)$. The relation (3) is satisfied with $g(z) = \ln(\ln(z))$ and $\rho = av$.

Empirical analyses: parameters estimation. For each $z$, we fit

$$\begin{cases} \sqrt{z(r)} = -a_{1} \left[ \ln \left( \frac{1}{D_{T}} \right) + b_{1} \right] & \text{for } z(r) > \bar{z} \\ \ln z(r) = -a_{2} \left[ \ln \left( \frac{1}{D_{T}} \right) + b_{2} \right] & \text{for } z(r) < \bar{z} \end{cases}$$

and we compute the corresponding quantity $4(a_{1}/a_{2})^{2}$ (see (14)). Finally, as shown in Fig. 7, we choose $z_{a} = \bar{z}$ such that $\bar{z} = 4(a_{1}/a_{2})^{2}$ and we set $\rho/v$ equal to the corresponding $a_{2}$.

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