A massless quantum field theory over the p-adics

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(joint work with Ajay Chandra, Gianluca Guadagni)

This talk is in three parts. In Part 1, we briefly outline a general program for the rigorous study of scalar quantum field theory (QFT), in the continuum. We use a probabilistic framework in the spirit of Dobrushin [3]. In Part 2, we explain that everything in Part 1 makes perfect sense when spacetime $\mathbb{R}^d$ is replaced by $\mathbb{Q}_p^d$. Finally, in Part 3, we report on ongoing progress made, in collaboration with A. Chandra and G. Guadagni (research funded by U.Va. and the NSF under grant DMS#0907198), on the $p$-adic analog of the massless model studied by Brydges, Mitter and Scoppola.

1. Outline of a program for Euclidean QFT in the continuum

The goal is to develop a mathematical theory which is a rigorous version of the methods one finds in physics QFT textbooks (e.g., Ch. 8 and Ch. 10 of [7]). We put the emphasis on Symanzik-Nelson positivity rather than reflection positivity. The study of QFT thus becomes that of probability measures $d\mu$ on the space of distributions $S'((\mathbb{R}^d)$ with the cylinder $\sigma$-algebra. We focus on measures which have finite moments and are invariant by translation (stationary processes) and by the orthogonal group $O(d)$. One can also require self-similarity. This program involves the following steps.

Step 0: It is to classify the self-similar Gaussian case. Let $S_n(f_1, \ldots, f_n) = \langle \phi(f_1) \cdots \phi(f_n) \rangle$ denote the moments of the measure $d\mu$ under consideration. $S_n$ can also be thought of as an element of $S'((\mathbb{R}^d)$. In the (centered) Gaussian case, $S_2$ contains all the information. By translation invariance, $S_2(x, y) = S_2(x - y)$ where $S_2 \in S'((\mathbb{R}^d)$. The classification reduces to that of $O(d)$-invariant distributions $S_2$ homogeneous of degree $-2[\phi]$ where $[\phi]$ is the scaling dimension of the field. For any $[\phi] \in \mathbb{R}$, there is a 1-dimensional space of solutions. Adding the positive-type condition entails $[\phi] \geq 0$. We will restrict the discussion to the range $0 < [\phi] < \frac{d}{2}$. One then has a simple expression in both direct and Fourier space for the two-point function $S_2(x, y) \sim |x - y|^{-2[\phi]}$, $\hat{S}_2(k) \sim |k|^{2[\phi] - d}$. Note that this is only part of a bigger picture. One can handle zero-modes, e.g., by restricting $S((\mathbb{R}^d)$ using moment vanishing conditions [3]. See [5] for $d = 2$, $[\phi] = 0$ which is pertinent for conformal QFT. For work related to $d = 1$, $[\phi] < 0$, see the talk by J. Unterberger.

Step 1: Putting cut-offs. One replaces, e.g., the covariance $C = S_2$ by $C_r(x) \sim \int_{\Lambda_s}^{\infty} \frac{d\rho}{\rho} \rho^{-2[\phi]}\mu_r(\rho)$ for some nice function $u$. Here $L$ is the renormalization group (RG) magnification ($L > 1$ is an integer). One also introduces a box $\Lambda_s$ of side length $L^s$.

Step 2: Perturb the cut-off Gaussian $d\mu_{C_r}(\phi)$ to get a new probability measure $d\nu_{r,s}(\phi) = \frac{1}{2} \exp(-\tilde{V}_{r,s}(\phi))d\mu_{C_r}(\phi)$ where $\tilde{V}_{r,s}(\phi) = \int_{\Lambda_s} d^d x \{ \tilde{g}_r : \phi^4 : C_r \}(x) + \tilde{\mu}_r : \phi^2 : C_r \}(x) + \cdots$. Given a bare ansatz, i.e., the germ at $-\infty$ of a sequence
notation from the talk by S. Hollands, we would like to define local field operators. Step 3: Composite fields and operator product expansion (OPE). Borrowing our notion of Borchers class in this probabilistic setting? Also of interest is the massless situation. Note that, for $\nu > 0$, $(|k|^{d - 2|\phi|} + \mu)^{-1}$ has large distance decay $|x|^{2|\phi| - 2d}$ if $d - 2|\phi| \not\in 2\mathbb{N}$. Thus, the appropriate definition of ‘massless’ for general $[\phi]$ is the requirement of non $L^1$ rather than power law decay for $S_2$.

Step 3: Composite fields and operator product expansion (OPE). Borrowing our notation from the talk by S. Hollands, we would like to define local field operators $\mathcal{O}_A[\phi](x)$, e.g., renormalized versions of $\phi(x)^n$. After smearing by $f \in S(\mathbb{R}^d)$, one would like $\phi \to \mathcal{O}_A[\phi](f)$ to be a function $S'(\mathbb{R}^d) \to \mathbb{C}$. This typically fails if $|\phi| > 0$. One should instead define $\phi \to \mathcal{O}_A[\phi](f)$ as a generalized function (or rather functional) in the spirit of Hida’s white noise calculus [4]. One needs a space $\mathcal{D}(S'(\mathbb{R}^d))$ of test functionals $F$ on $S'(\mathbb{R}^d)$ which should at least contain monomials of the form $\phi(f_1) \cdots \phi(f_n)$. Then $\mathcal{O}_A$ should be constructed as a linear map from $S(\mathbb{R}^d)$ to the dual space of generalized functionals $\mathcal{D}'(S'(\mathbb{R}^d))$. The duality pairing is that given by the QFT/measure $d\nu$. Note that the correlations $\langle \mathcal{O}_A[\phi](f) F(\phi) \rangle = \langle \mathcal{O}_A[\phi](f) \phi(f_1) \cdots \phi(f_n) \rangle$ make sense, in the free case, even at coinciding points. Namely, this defines a distribution on all of $\mathbb{R}^{(n+1)d}$. The functional $F$ corresponds to the spectator fields for the OPE. In the case of a single operator insertion, one can then follow the procedure explained in the talk by S. Hollands, in order to study the singularities on the diagonals and inductively define the operator products $\mathcal{O}_A$ from the corresponding short distance asymptotics. For the OPE with several operator insertions, one needs to define the mixed correlations at noncoinciding points, then repeat the procedure.

Step 4: Instead of perturbing, in Step 2, around a solution of Step 0, one can also consider similar perturbations of nontrivial RG fixed points along relevant directions.

2. The same over $\mathbb{Q}_p$

The message here is that everything in Part 1 works perfectly if one considers random fields $\phi : \mathbb{Q}_p^d \to \mathbb{R}$. Besides, the RG is much simpler and cleaner than in the real case. Indeed, it reduces to the hierarchical RG. For $p$ a prime number, the field $\mathbb{Q}_p$ is defined as the completion of the field $\mathbb{Q}$ with respect to the $p$-adic norm/absolute value $|p^n \mathbb{Z}_p| = p^{-n}$, for $n, a, b \in \mathbb{Z}$ such that $b \not= 0$ and $p$ does not divide $ab$. A $p$-adic number $x \in \mathbb{Q}_p$ has a unique convergent representation $\sum_{j \in \mathbb{Z}} a_j p^j$, with only finitely many negative powers of $p$, where the ‘digits’ $a_j$ are in $\{0, 1, \ldots, p - 1\}$. The polar part $\{x\}_p = \sum_{j < 0} a_j p^j$ is a rational number. The valuation is given by $\text{val}_p(x) = \min\{|j, a_j \not= 0\}$. The extension of the previous norm is $|x|_p = p^{-\text{val}_p(x)}$. The unit ball $\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x|_p \leq 1\}$ is a compact additive subgroup. In dimension $d$, the norm of a point $x = (x_1, \ldots, x_d)$ in $\mathbb{Q}_p^d$ is defined by $|x| = \max |x_i|_p$. We take $L = p^L$ for the RG zooming ratio. The lattice of mesh $L'$ is...
given by $\mathbb{Q}_p^d/(L^{-r}\mathbb{Z}_p)^d$. The big volume is $\Lambda_s = (L^{-s}\mathbb{Z}_p)^d = \{x, |x| \leq L^s\}$. For the space of test functions we take the Schwartz-Bruhat space $S(\mathbb{Q}_p^d)$ of locally constant functions $f : \mathbb{Q}_p^d \to \mathbb{R}$ of compact support, with the finest locally convex topology. The Fourier transform is $\hat{f}(k) = \int f(x)e^{-2\pi i (x \cdot k)}d^dx$ where $x \cdot k = \sum x_i k_i$ and the additive Haar measure $d^dx$ gives measure 1 to $\mathbb{Z}_p^d$. The analog of $O(d)$ is the maximal compact subgroup $GL_d(\mathbb{Z}_p)$ of $GL_d(\mathbb{Q}_p)$, defined by fixing the norm $|x|$.

For Step 1, the cut-off covariance $C_r$ is obtained from $C_r(x) \sim \sum_{j \in \mathbb{Z}_p} -2j\|x\|_{\mathbb{Z}_p}^d(p^jx)$ by imposing $j \geq rl$. The RG map corresponds to integrating over fluctuations with covariance $C_0 - C_1$. With these modifications, Part 1 works in the $p$-adic setting too. For other work on $p$-adic QFT see [6] and references therein.

3. The $p$-adic BMS model

The BMS model corresponds to $d = 3$ and $[\phi] = \frac{3-\epsilon}{4}$ for some small positive bifurcation parameter $\epsilon$. The bare ansatz only contains $\phi^4$ and $\phi^2$ couplings $\tilde{g}_r, \tilde{\mu}_r$. We rescale to unit lattice $\tilde{V}_{r,s} \to V_r^{(0)}$ and produce new bulk potentials $V_r^{(0)} \to V_r^{(1)} \to \cdots$ by iterating the RG map (we suppressed $s$ in the notation). Constructing a QFT morally amounts to establishing the transverse convergence criterion (TCC): $\forall q \in \mathbb{Z}, \lim_{r \to -\infty} V_r^{(q-r)}$ exists (the effective theory at log-scale $q$). This produces an ideal RG trajectory $(P_q)_{q \in \mathbb{Z}}$. One conjectures that TCC $\Rightarrow \lim d \nu_r = d\nu$ exists. Together with A. Chandra and G. Guadagni we adapted the proofs in [2, 1] to the $p$-adic case and rigorously constructed (in suitable Banach spaces) the nontrivial infrared (IR) fixed point, together with its stable and unstable manifolds. We constructed ideal trajectories as well as established the TCC, starting from a bare ansatz, for two massless theories: one which should be self-similar, at the IR fixed point, and another one which joins the Gaussian and the IR fixed points. Modulo the previous conjecture, we completed all previous steps except Step 3. We are also making rapid progress towards proving this conjecture. This hinges on extending our RG tools to nonuniform local perturbations of the massless Gaussian. This should also help for Step 3.

References

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