Loose Hamilton Cycles in Random 3-Uniform Hypergraphs

Alan Frieze*

Submitted: Jan 4, 2010; Accepted: Feb 10, 2010; Published: XX
Mathematics Subject Classification: 05C65, 05C80

Abstract

In the random hypergraph $H = H_{n,p;3}$ each possible triple appears independently with probability $p$. A loose Hamilton cycle can be described as a sequence of edges $\{x_i, y_i, x_{i+1}\}$ for $i = 1, 2, \ldots, n/2$ where $x_1, x_2, \ldots, x_{n/2}, y_1, y_2, \ldots, y_{n/2}$ are all distinct. We prove that there exists an absolute constant $K > 0$ such that if $p \geq K \log n / n^2$ then

$$\lim_{n \to \infty} \frac{4}{n} \Pr(H_{n,p;3} \text{ contains a loose Hamilton cycle}) = 1.$$ 

1 Introduction

The threshold for the existence of Hamilton cycles in the random graph $G_{n,p}$ has been known for many years, see [7], [1] and [3]. There have been many generalisations of these results over the years and the problem is well understood. It is natural to try to extend these results to Hypergraphs and this has proven to be difficult. The famous Pósa lemma fails to provide any comfort and we must seek new tools. In the graphical case, Hamilton cycles and perfect matchings go together and our approach will be to build on the deep and difficult result of Johansson, Kahn and Vu [6], as well as what we have learned from the graphical case.

A $k$-uniform Hypergraph is a pair $H = (V, E)$ where $E \subseteq \binom{V}{k}$. We say that a $k$-uniform sub-hypergraph $C$ of $H$ is a Hamilton cycle of type $\ell$, for some $1 \leq \ell \leq k$, if there exists a cyclic ordering of the vertices $V$ such that every edge consists of $k$ consecutive vertices and for every pair of consecutive edges $E_{i-1}, E_i$ in $C$ (in the natural ordering of the edges) we have $|E_{i-1} \setminus E_i| = \ell$. When $\ell = k - 1$ we say that $C$ is a loose Hamilton cycle and in this paper we will restrict our attention to loose Hamilton cycles in the random 3-uniform hypergraph $H = H_{n,p;3}$. In this hypergraph, $V = [n]$ and each of the $\binom{n}{3}$ possible edges (triples) appears independently with probability $p$. While $n$ needs to be even for $H$ to contain a loose Hamilton cycle, we need to go one step further and assume that $n$ is a multiple of 4. Extensions to other $k, \ell$ and $n \equiv 2 \mod 4$ pose problems. We will prove the following theorem:

*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, U.S.A. Supported in part by NSF grant DMS-0753472.
Theorem 1 There exists an absolute constant $K > 0$ such that if $p \geq \frac{K \log n}{n^2}$ then
\[
\lim_{n \to \infty} \Pr(H_{n,p,3} \text{ contains a loose Hamilton cycle}) = 1.
\]

Thus $\frac{\log n}{n^2}$ is the threshold for the existence of loose Hamilton cycles, at least for $n$ a multiple of 4. This is because if $p \leq \frac{(1-\epsilon) \log n}{2n^2}$ and $\epsilon > 0$ is constant, then whp $H_{n,p,3}$ contains isolated vertices.

The proof of Theorem 1 will follow fairly easily from the following three theorems.

We start with a special case of the theorem of [6]: Let $X$ and $Y$ be a disjoint sets. Let $\Omega = \binom{X}{2} \times Y$. Let $\Gamma = \Gamma(X,Y,p)$ be the random 3-uniform hypergraph where each triple in $\Omega$ is independently included with probability $p$. Assuming that $|X| = 2|Y| = 2m$, a perfect matching of $\Gamma$ is a set of $m$ triples $(x_{2i-1}, y_i, x_{2i})$, $i = 1, 2, \ldots, m$ such that $X = \{x_1, \ldots, x_{2m}\}$ and $Y = \{y_1, \ldots, y_m\}$.

Theorem 2 [6] There exists an absolute constant $K > 0$ such that if $p \geq \frac{K \log n}{n^2}$ then whp $\Gamma$ contains a perfect matching.

This version is not actually proved in [6], but can be obtained by straightforward changes to their proof.

Our next theorem concerns rainbow Hamilton cycles in random regular graphs. If we edge colour a graph then a set $S$ of edges is rainbow if all edges in $S$ are a different colour. Janson and Wormald [5] proved the following: Let $G_{2r}$ be a random 2$r$-regular multi-graph on vertex set $[n]$. The distribution is not uniform, it is the one induced by the configuration model, see e.g. Bollobás [2]. We can condition on there being no loops.

Theorem 3 If the edges of $G_{2r}$ are coloured randomly with $n$ colours so that each colour is used exactly $r$ times, $r \geq 4$, then whp it contains a rainbow Hamilton cycle.

This of course implies the result for random 2$r$-regular graphs.

We partition $[n = 4m]$ into $X = [2m]$ and $\bar{X} = [2m + 1, n]$. The (multi-)graph $G^*$ has vertex set $X$ and an edge $(x, x')$ of colour $y$ if $(x, y, x')$ is an edge of $H$. If $G^*$ contains a rainbow Hamilton cycle, then $H$ contains a loose Hamilton cycle. We will use Theorem 2 to show that whp $G^*$ contains an edge coloured graph that is close to satisfying the conditions of Theorem 3.

There is a minor technical point in that we can only use Theorem 2 to prove the existence of a randomly coloured (multi-)graph $\Gamma_{2r}$ that is the union of 2$r$ independent matchings. Fortunately,

Theorem 4 $\Gamma_{2r}$ is contiguous to $G_{2r}$

By this we mean that if $\mathcal{P}_n$ is some sequence of (multi-)graph properties, then

\[
\Gamma_{2r} \in \mathcal{P}_n \text{ whp } \iff G_{2r} \in \mathcal{P}_n \text{ whp.}
\]

Theorem 4 is proved in Janson [4] (Theorem 11) and in Molloy, Robalewska-Szalat, Robinson and Wormald [8].

\[\text{1An event } \mathcal{E}_n \text{ occurs with high probability, or whp for brevity, if } \lim_{n \to \infty} \Pr(\mathcal{E}_n) = 1.\]
2 Proof of Theorem 1

We begin by letting $\mathcal{Y}$ be a set of size $2rm$ consisting of $r = O(1)$ copies $y_1, y_2, \ldots, y_r$ of each $y \in X$. We will later fix $r$ at 4, but we leave it unspecified for now. Next let $Y_1, Y_2, \ldots, Y_{2r}$ be a uniformly random partition of $\mathcal{Y}$ into $2r$ sets of size $m$.

Define $p_1$ by $p = 1 - (1 - p_1)^{2r}$. With this choice, we can generate $H_{n,p;3}$ as the union of $2r$ independent copies of $H_{n,p;3}$. Similarly, define $p_2$ by $p_1 = 1 - (1 - p_2)^r$.

Viewing $H_{n,p;3}$ as the union of $r$ independent copies $H_1, H_2, \ldots, H_r$ of $H_{n,p;3}$ we can couple $\Gamma(X, Y_j, p_1)$ with a subgraph of $H_{n,p;3}$ by placing $(x, y, x')$ in $E(H_i)$ whenever $(x, y, x') \in E(\Gamma(X, Y_j, p_1))$. It follows from Theorem 2 that whp $\Gamma(X, Y_j, p_1)$ contains a perfect matching $M_j$. (We need the split into $r$ copies of $H_{n,p;3}$ to allow a “colour” to appear several times in a matching).

Now each perfect matching $M_j$ gives rise to an edge-coloured perfect matching $M^*_j$ of $G^*$ where $(x, y, x')$ gives rise to an edge $(x, x')$ of colour $y$. By symmetry, these matchings are uniformly random and they are independent by construction. Also the edges have been randomly coloured so that each colour appears exactly $r$ times. Indeed to achieve such a random colouring we can take any partition of the edge set of $M^*_1 \cup M^*_2 \cup \cdots \cup M^*_r$ into $2r$ sets $S_1, S_2, \ldots, S_{2r}$ of size $m$ and then colour the edges by using random bijections from $Y_j \to S_j$ for $j = 1, 2, \ldots, 2r$.

We apply Theorems 3 and 4 to finish the proof. For a $2r$-regular graph $G$ let $\Omega_G$ denote the set of equitable edge colourings of $G$. By equitable, we mean that each colour is used $r$ times. Suppose that $\sigma$ is chosen uniformly from $\Omega_G$ and $\pi_G = \Pr(R)$ where $R$ is the event that there is no rainbow Hamilton cycle. Theorem 3 can be expressed as follows: Let $\mathcal{G}_{2r}$ denote the set of $2r$-regular loopless multi-graphs with vertex set $[n]$ and configuration distribution $\kappa_G$. Then,

$$\sum_{G \in \mathcal{G}_{2r}} \kappa_G \pi_G \leq \frac{1}{\omega} \quad (2)$$

where $\omega \to \infty$ as $n \to \infty$. The event $\mathcal{P}_n$ of (1) can now be defined:

$$\mathcal{P}_n = \left\{ \pi_G \leq \frac{1}{\omega^{1/2}} \right\}.$$

Think of $\pi_G$ as a random variable for $G$ chosen from $\mathcal{G}_{2r}$. Then (2) states that $\mathbf{E}(\pi_G) \leq 1/\omega$. The Markov inequality then implies that $\Pr(\pi_G \geq 1/\omega^{1/2}) \leq 1/\omega^{1/2}$ and so $\Pr(\mathcal{P}_n) \geq 1 - 1/\omega^{1/2}$ and this completes the proof of Theorem 1.

Acknowledgement I am grateful to Andrzej Dudek, Michael Krivelevich, Oleg Pikhurko and Andrzej Ruciński for their comments.

References

[1] M. Ajtai, J. Komlós and E. Szemerédi, The first occurrence of Hamilton cycles in random graphs, *Annals of Discrete Mathematics* 27 (1985) 173-178.

[2] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European Journal of Combinatorics* 1 (1980) 311-316.

[3] B. Bollobás, The evolution of sparse graphs, in *Graph Theory and Combinatorics*, Academic Press, *Proceedings of Cambridge Combinatorics, Conference in Honour of Paul Erdős* (B. Bollobás; Ed) (1984) 35-57.
[4] S. Janson, Random regular graphs: asymptotic distributions and contiguity, *Combinatorics Probability and Computing* 4 (1995) 369-405.

[5] S. Janson and N. Wormald, Rainbow Hamilton cycles in random regular graphs, *Random Structures Algorithms* 30 (2007) 35-49.

[6] A. Johansson, J. Kahn and V. Vu, *Factors in random graphs*, Random Structures and Algorithms 33 (2008), 1–28.

[7] J. Komlós and E. Szemerédi, Limit distributions for the existence of Hamilton circuits in a random graph. *Discrete Mathematics* 43 (1983) 55-63.

[8] M. Molloy, H. Robalewska-Szalat, R. Robinson and N. Wormald, *Random Structures and Algorithms* 10 (1997) 305-321. 1-Factorisations of random regular graphs,