Pairs of Bloch electrons and magnetic translation groups

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I. INTRODUCTION

The first attempts to describe movement of electrons in the presence of a constant external magnetic field had been done by Landau and Peierls. In the fifties many authors dealt with similar problems, but a crystalline periodic potential was also included. The pioneering works of Brown and Zak were preceded by Wannier’s paper. The first two authors independently introduced and investigated the so-called magnetic translations, i.e. nonunitary, mutually noncommuting, operators which commute with the Hamiltonian. For more than thirty years these operators have been applied in many problems concerning movement of electrons in crystal lattice. Recently, much attention has been paid to two-dimensional systems in the external magnetic field due their relations with high-\(T_c\) superconductors, anyons, the Hall effect etc.

From the group-theoretical point of view magnetic translations can be considered as a projective (ray) representation of the translation group \(T\) of a crystal lattice (this is Brown’s approach). However, projective representations of any group can be found as vector representations of its covering group (the so-called magnetic translation group \(MTG\)). This later group can be constructed as a central extension of a given group by the group of factors, in general \(U(1) \subset \mathbb{C}^*\) or its subgroup. This construction is a basis of Zak’s considerations and is very close related to the Mac Lane method for determination of all inequivalent (abelian) extensions of two groups. In this paper irreducible representations (irreps) of \(MTG\)’s are considered. In fact they were determined by Brown and Zak but both authors rejected most of them as ‘nonphysical’. It is here shown that all representations are ‘physical’ and a very simple example of their applications is presented. Moreover, the Clebsch–Gordan coefficients are calculated in this case.

II. MAGNETIC TRANSLATIONS

The \(MTG\) appears in a natural way when one considers an electron in a periodic potential \(V(r)\) and a uniform magnetic field \(H\) determined by a vector potential (a gauge) \(A\). This system is described by the well-known Hamiltonian

\[
\mathcal{H} = \frac{1}{2m}(p + eA/c)^2 + V(r),
\]

which does not commute with the usual translation operators

\[
\hat{T}_0(R) = \exp(-iR \cdot p/\hbar)
\]

and

\[
\hat{T}_0(R)\psi(r) = \psi(r - R)
\]

(introduced by Brown). It is easy to check that in this way a projective representation of the translation group is defined; the corresponding factor system is given as

\[
m(R, R') = \exp \left[ -\frac{i}{2\hbar c} (R \times R') \cdot H \right].
\]

It has to be stressed that these operators commute with the Hamiltonian if the vector potential \(A\) fulfills the condition

\[
\partial A_j/\partial x_k + \partial A_k/\partial x_j = 0 \quad \text{for} \quad j, k = 1, 2, 3.
\]

This relation holds, for example, for the (global) gauge \(\tilde{A}(r) = \frac{1}{c}(H \times r)\), which was used by both authors. It is worth noting that introducing a local gauge one can consider any vector potential \(A\).

Projective representations of a given group are related with vector representations of the covering group, which can be determined as a central extension. In the considered case, one deals with representations of the translation group \(T \simeq \mathbb{Z}^3\) and the magnetic translation group \(\mathcal{T}\) is its covering group, i.e. \(\mathcal{T}\) is included in a central extension of \(T\) by \(U(1)\). Let \(\mathcal{T}\) consist of pairs \((u, R)\), \(u \in U(1), R \in T\), with the multiplication rule

\[
(u, R)(u', R') = (uu' m(R, R'), R + R')
\]

\[
\hat{T}_0(R, R') = \exp(-i(R + R') \cdot p/\hbar)
\]
with $m: T \times T \to U(1)$ being a factor system, and let $\Xi$ be an irrep of $U(1)$. An irrep of $T$ is given as a product

$$\Gamma(u, R) = \Xi(u)\Lambda(R),$$  

where $\Lambda$ is a projective representation of $T$ with a factor system

$$\nu(R, R') = \Xi(m(R, R')).$$  

Therefore, the representation $\Lambda$ has the factor system

$$\nu_\alpha(R, R') = \Xi(m(R, R'))^\alpha.$$  

A product of two 'physical' representations gives a 'nonphysical' one. However, there are no a priori rules to exclude (as 'nonphysical') a product of two ('physical') representations. Therefore, it has to be assumed that also $\Gamma$ is relevant for physics.

Zak had rejected irreps with $\Xi(u) \neq u$ since 'representations with the correspondence $\epsilon \to \epsilon^n$ with $n \neq 1$ are nonphysical'. However, the above mentioned constant contains the electric charge (of an electron). If one assumes that representations with $\Xi(u) = u^2$ describe movement of a particle (or a system of particles) with a charge $Q = -2e$ then all formulae will be consistent. The simplest interpretation says that such representations describe a pair of electrons. This agrees with the way in which they have been obtained: $\Gamma$ describing a pair of electrons is a product of two one-electron representations $\Gamma^1$ and $\Gamma^2$. Writing the Hamiltonian $H$ in the form

$$H = \frac{1}{2\beta m}(p + \alpha e A/c)^2 + V(r)$$

one can say that for $\alpha = \beta$ it describes movement of $\alpha$ electrons in the magnetic field and the periodic potential. If $\beta \neq \alpha = 0$ then this Hamiltonian corresponds to a particle of a mass $\beta m$ without electric charge. Since $\alpha = 0$ then both factor systems (for the central extension $T$ and the projective representation $\tilde{T}$) are trivial and the original translation group $T$ and its vector irreps are appropriate to describe dynamics of the system. (The magnetic field is irrelevant if one considers classical or spinless particles, of course.)

### III. IRREDUCIBLE REPRESENTATIONS

Let $\Gamma^1$ and $\Gamma^2$ be irreps of $T$ satisfying the condition $\Xi(u) = u$. Matrix elements of their Kronecker product $\Gamma = \Gamma^1 \otimes \Gamma^2$ can be found as

$$\Gamma_{jk,lm}(u, R) = \Gamma^1_{jk,l}(u, R)\Gamma^2_{km}(u, R).$$

Taking into account the definition of an irrep one obtains

$$\Gamma_{jk,lm}(u, R) = u^2\Lambda^1_{jk,l}(R)\Lambda^2_{km}(R).$$

The last product in this formula determines a product $\Lambda = \Lambda^1 \otimes \Lambda^2$ of two projective representations, which is a projective representation itself. To determine its factor system one has to calculate a product $\Lambda(R_1)\Lambda(R_2)$:

$$(\Lambda(R_1)\Lambda(R_2))_{jk,lm} = \sum_{n,p} \Lambda_{jk,np}(R_1)\Lambda_{np,lm}(R_2) = m(R_1, R_2)^2\Lambda_{jk,lm}(R_1 + R_2).$$

Therefore, the representation $\Lambda$ has the factor system $\nu(R, R') = m(R, R')^2$, which means that it corresponds to the irrep $\Xi(u) = u^2$ [cf. Eq. (8)]. In the other words — a product of two 'physical' representations gives a 'nonphysical' one. However, there are no a priori rules to

### IV. FINITE TWO-DIMENSIONAL MTG’S

One can introduce finite representations of MTG’s imposing the periodic boundary conditions in the form $T(Na_j) = 1$, where $a_j$, $j = 1, 2, 3$, are the unit vectors of a crystal lattice. This is equivalent to considerations of a finite translation group $T = Z^3_N$ (identical periods in each direction are assumed). Both approaches yield that the magnetic field should be parallel to a lattice vector. It is convenient to assume that $H \parallel a_2$ and is perpendicular to $a_1$ and $a_2$. This allows to consider $T = Z^2_N$ and a factor group to be $C_N$ (the multiplicative group of the $N$th roots of 1). Therefore, a finite two-dimensional magnetic translation group is a central extension of a direct product $Z_N \otimes Z_N$ by the cyclic group $C_N$. This group, denoted as above by $\tilde{T}$, consists of elements $(\omega^j, [k,l])$, where $\omega = \exp(2\pi i/N)$ and $j, k, l = 0, 1, \ldots, N - 1$. The multiplication rule is given by the following formula (all additions modulo $N$)

$$(\omega^j, [k,l])(\omega^{j'}, [k',l']) = (\omega^{j+j'+hkl'}, [k + k', l + l']).$$

The parameter $h = 0, 1, \ldots, N - 1$ labels inequivalent extensions and corresponds to the magnetic field $H$ in Eq. (10). It is evident that algebraic properties of this group depend on $h$ or, strictly speaking, on the greatest common divisor gcd($h, N$) since for gcd($h, N$) = gcd($h', N$)
groups labeled by $h$ and $h'$ are isomorphic. In the further considerations we assume $h = 1$ in order to reduce a number of parameters and of different cases. It is worthwhile to mention that for $\gcd(h, N) > 1$ the extension of $\mathbb{Z}_N \otimes \mathbb{Z}_N$ by $C_{N/\gcd(h, N)}$ with the multiplication rule parameterized by $h/\gcd(h, N)$ should be taken into account.

It follows from Eq. (8) that irreps of $T$ are labeled by $\xi = 0, 1, \ldots, N - 1$ corresponding to the irreps of $C_N$, i.e. we have $\Xi(\omega^j) = \omega^{\xi j}$. For each $\xi$ we have to find all (inequivalent) projective representations $\Lambda^\xi$ of $\mathbb{Z}_N \otimes \mathbb{Z}_N$. These representations satisfy the following conditions: (i) a factor system of $\Lambda^\xi$ is given as [see Eq. (3)]

$$\nu^\xi([k, l], [k', l']) = \omega^{\xi k l'};$$

(ii) for a given factor system $\nu^\xi$ we have

$$\sum |\Lambda^\xi|^2 = N^2$$

(the sum is taken over all inequivalent projective irreps with the factor system $\nu^\xi$). It can be shown that for given $\xi$ there are $\gcd(\xi, N)^2$ projective representations, each of dimension $N/\gcd(\xi, N)$. These representations are labeled by numbers $\kappa, \lambda = 0, 1, \ldots, \gcd(\xi, N) - 1$, corresponding to irreps of $\mathbb{Z}_{\gcd(\xi, N)} \otimes \mathbb{Z}_{\gcd(\xi, N)}$. (Thus for given $\xi$ the crystal lattice is ‘scaled’ $N/\gcd(\xi, N)$ times.)

To make a long story short an actual form of matrix elements will not be discussed but only some general properties will be presented. (In fact so used irreps are similar to those considered by Brown and Zak.)

It follows from the previous considerations that the representations (vector ones of $T$ or projective ones of $T$) with $\xi > 1$ describe the movement of a particle with a charge $-\xi e$. Note that the periodic boundary conditions imply that particles with charge $q$ and $q + N$ behave in the same way. In particular it also applies to products of irreps: a product of two representations labeled by $\xi_1$ and $\xi_2$, respectively, decomposes into a sum of representations labeled by $\xi_1 + \xi_2$ (modulo $N$). Thus a system of two particles with charges $-\xi_1 e$ and $-\xi_2 e$ has total charge $-(\xi_1 + \xi_2)e$. This relation follows from the form of the first factor in Eq. (3):

$$(\Xi_1 \otimes \Xi_2)(\omega^j) = \omega^{(\xi_1 + \xi_2)j}.$$ 

In particular, a square of the $N$-dimensional representation $\Gamma^1$ (determined by the unique projective irrep $\Lambda$) corresponds to a pair of electrons. A number of terms and the multiplicity coefficients $f(\kappa, \lambda)$ in the decomposition

$$\Gamma^{\xi_1}_{\kappa_1, \lambda_1} \otimes \Gamma^{\xi_2}_{\kappa_2, \lambda_2} = \bigoplus_{\kappa, \lambda} f(\kappa, \lambda) \Gamma^{\xi_1 + \xi_2}_{\kappa, \lambda}$$

depend on the arithmetic relations between $\xi_1$, $\xi_2$ and $N$. In the case $\Gamma^1 \otimes \Gamma^1$ one obtains different results for $N$ odd and even. In the first case the product decomposes into $N$ copies of the (unique) representation $\Gamma^0$.

If $N$ is a prime number then $\gcd(\xi, N) = 1$ or $N$, and it is easy to determine decomposition of each product:

$$\Gamma^0_{\kappa, \lambda} \otimes \Gamma^0_{\kappa', \lambda'} = \Gamma^0_{\kappa + \kappa', \lambda + \lambda'};$$

$$\Gamma^0_{\kappa, \lambda} \otimes \Gamma^\xi = \Gamma^\xi,$$

for $\xi = 1, 2, \ldots, N - 1$;

$$\Gamma^\xi \otimes \Gamma^N - \xi = \bigoplus_{\kappa, \lambda = 0}^{N-1} \Gamma^0_{\kappa, \lambda};$$

$$\Gamma^{\xi_1} \otimes \Gamma^{\xi_2} = N \Gamma^{\xi_1 + \xi_2},$$

for $\xi_1 + \xi_2 \neq N$.

The first nontrivial case corresponds to $N = 4$. However, this case does not show all the richness of possible products, since there is only one nontrivial divisor $\xi = 2$. The central extension of $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ has 22 irreps:

(i) 16 one-dimensional ones for $\xi = 0$ labeled by $\kappa, \lambda = 0, 1, 2, 3$; they are simply the ordinary vector representations of $\mathbb{Z}_4 \times \mathbb{Z}_4$;

(ii) 2 four-dimensional ones for $\xi = 1$ and $\xi = 3$;

(iii) 4 two-dimensional ones for $\xi = 2$ labeled by $\kappa, \lambda = 1, 2$.

Two-electron states form a 16-dimensional space with the basis vectors $|p_1 p_2\rangle$, where $p_1, p_2 = 0, 1, 2, 3$ label vectors of the representation $\Gamma^1$. This space decomposes into $8$ two-dimensional representations $\Gamma^2_{\kappa, \lambda}$ with $f(\kappa, \lambda) = 2$ for all $\kappa, \lambda = 0, 1$. Hence, the irreducible basis can be denoted as $|\kappa \lambda v q\rangle$, where $v = 0, 1$ is the repetition index and $q = 0, 1$ labels vectors of $\Gamma^1_{\kappa, \lambda}$. The relatively simple form of matrix elements allows determination of the Clebsch–Gordan coefficients. In the presented case they lead to the following formulae:

$$|0000\rangle = (|00\rangle + |22\rangle)/\sqrt{2}, \quad |0001\rangle = (|11\rangle + |33\rangle)/\sqrt{2};$$

$$|0010\rangle = (|13\rangle + |31\rangle)/\sqrt{2}, \quad |0100\rangle = i(|00\rangle - |22\rangle)/\sqrt{2},$$

$$|0110\rangle = (|13\rangle - |31\rangle)/\sqrt{2}, \quad |1000\rangle = i(|01\rangle + |21\rangle)/\sqrt{2}.$$ 

The numbers $p_1$ and $p_2$ can be interpreted as quasi-momenta since we have $\vec{T}(a_2)p_1 = |p_1 - 1|$ [cf. Ref. 4 Eq. (25)]. The translation along $a_2$ is distinguished due to the choice of the matrix form of the considered representations. In general, there is always one distinguished direction and the number $p$ labels the corresponding quasi-momentum. Such interpretation of the indices $p_1$ and $p_2$ allows the introduction of a Hamiltonian which
commutes with all operators $\Gamma^1 \otimes \Gamma^4(u, R)$ (matrix elements are given):

$$\mathcal{H}_{p_1 p_2, p'_1 p'_2} = \delta_{p_1 + p_2, p'_1 + p'_2} a_{p_1 + p_2, p_1 - p'_1},$$

where

$$a_{0,0} = a_{2,0} = a_0, \quad a_{1,0} = a_{3,0} = a_1, \quad a_{p,1} = a_{p,3}.$$

All these relations follow from the symmetry requirements. The terms $a_{p,0}$ correspond to the total quasimomentum $p$ and describe the kinetic energy ($a_{p,0} > 0$); the condition $a_{0,0} = a_{2,0}$ is connected with 'rescaling' of the lattice since the representations $\Gamma^2$ are two-dimensional. The terms $a_{p,q}$ for $q \neq 0$ correspond to the interchange of a quasi-particle with the quasi-momentum $q$, or, in the other words, to the interaction of electrons. In the simplest approximation one can assume that $a_{p,q}$ for $q \neq 0$ does not depend on $p$ (so it will be hereafter denoted as $b_0$; recall that $b_1 = b_3$) and is negative. It is also natural to assume that $a_0 < a_1$ and that the probability of interaction with $q = 2$ is smaller than this one for $q = 1$ (to begin with one can assume $b_0 = 0$).

In such an approximation one finds that levels corresponding to $\Gamma^1_{01}$ and $\Gamma^1_{11}$ are four-fold degenerated with energies $a_0 - b_2$ and $a_1 - b_2$, respectively. The representation $\Gamma^2_{10}$ leads to two 2-fold degenerated levels with energies $a_1 + b_2 \pm 2b_1$. Similarly, one obtains that two representations $\Gamma^2_{00}$ describe levels with energies $a_0 + b_2 \pm 2b_1$, respectively. In two later cases the following linear combinations of vectors take the form:

$$\frac{1}{\sqrt{2}} (|\kappa 000\rangle \pm |\kappa 010\rangle), \quad \text{for} \quad \kappa = 0, 1.$$

The ground-state energy is $E = a_0 + b_2 + 2b_1$ and the corresponding eigenvector is

$$\frac{1}{2} (|00\rangle + |22\rangle + |13\rangle + |31\rangle),$$

i.e. it is the sum of states $|p, -p\rangle$. Such a result resembles the BCS state but it is not antisymmetric. However, the performed investigations are semi-classical and electrons have been considered as spinless particles.

VI. FINAL REMARKS

The algebraic analysis of the magnetic translation groups (or, equivalently, of the projective irreducible representations of the translation group) gives us deeper insight into their structure. This relates to many physical problems: movement of charged particles in a magnetic (or an electromagnetic) field and a periodic potential, high $T_c$-superconductors, the Hall effects (especially the fractional quantum Hall effect), anyons, finite phase spaces etc. The above presented considerations indicate the importance of the product of representations. The discussed examples are very simple and the physical interpretation is a bit naive, but they have shown the main (mathematical) properties of the proposed picture.

Let $\varphi = hc/e$ be a fluxon and $H = h\varphi$. Replacing the electron charge $e$ by a charge $Q = -\xi\varphi$ the factor system (3) determined by Brown can be written as

$$m(R, R') = \exp \left[ 2\pi i \frac{\xi}{\varphi} (R \times R') \cdot h \right].$$

This formula shows that physical properties, which depend on this factor, are periodic with respect to the magnetic field, lattice vectors and the charge. The first case had been pointed out by Azebel and noted also by Zak. The second is, in a sense, the basis of introduction of magnetic cells ($\Pi$) ($R$ and $R'$ are linear combinations with integer coefficients of basis vectors $a_j$). This work has shown that also the periodicity with respect to the charge of a particle should be taken into account.

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