Fast dynamics and spectral properties of a multilongitudinal-mode semiconductor laser: evolution of an ensemble of driven, globally coupled nonlinear modes

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We analyze the fast transient dynamics of a multi-longitudinal mode semiconductor laser on the basis of a model with intensity coupling. The dynamics, coupled to the constraints of the system and the below-threshold initial conditions, imposes a faster growth of the side modes in the initial stages of the transient, thereby leading the laser through a sequence of states where the modal intensity distribution dramatically differs from the asymptotic one. A detailed analysis of the below-threshold, deterministic dynamical evolution allows us to explain the modal dynamics in the strongly coupled regime where the total intensity peak and relaxation oscillations take place, thus providing an explanation for the modal dynamics observed in the slow, hidden evolution towards the asymptotic state (cf. Phys. Rev. A \textbf{85}, 043823 (2012)). The dynamics of this system can be interpreted as the transient response of a driven, globally coupled ensemble of nonlinear modes evolving towards an equilibrium state. Since the qualitative dynamics do not depend on the details of the interaction but only on the structure of the coupling, our results hold for a whole class of globally, bilinearly coupled oscillators. (All figures in color online).

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\section{I. INTRODUCTION}

The transient turn-on dynamics of lasers is a topic which has received a considerable amount of attention over the years (cf., e.g., \cite{1,2,3}) not only for the fundamental understanding of its evolution but also, and especially, for the importance that it represents in data encoding in telecommunications with directly-modulated semiconductor lasers. While Multiple Quantum Well \cite{9,10} (or even Quantum Dot \cite{11}) lasers have been proven to possess a peculiar modal dynamics – modelled with the help of mutual nonlinear coupling and noise \cite{9,12,13} with a Complex Ginzburg-Landau approach \cite{14} or based on multiscale analysis \cite{15} where the resulting model provides proof for a true phase instability \cite{16} –, inexpensive, edge-emitting devices show a dynamics \cite{17} which is more appropriately characterized by cooperation, rather than competition \cite{18}.

For these latterdevices, we have recently shown \cite{19} that a realistic model for a multimode semiconductor laser, with experimentally matched parameters \cite{17}, predicts a slow, hidden dynamical mode-evolution governed by a master mode. An overall agreement exists between our recent predictions and a wealth of experimental data (e.g. \cite{8,4}) since it has been time and again shown that a gradual line narrowing exists in the progress of the multimode laser transient (cf. also previous theoretical calculations, e.g., \cite{20,22}). However, no specific experiments seem to have been carried out to quantify the physical features of the multimode dynamics in edge-emitting semiconductor lasers (beyond its more technical aspects), and the physical origin of our predictions concerning the slow components of the dynamics remain so far unexplained.

In this paper, we analyze the deterministic dynamics of the multimode laser transient and determine the modal intensity distribution at the end of the fast transient starting from a below-threshold condition and leading into the slow dynamics \cite{19} with the characteristic oscillations of a Class-B laser \cite{23}. The apparent equilibrium solution, taking place at the end of the fast transient, in reality amounts to an inherent out-of-equilibrium one, when analyzed in terms of modal intensity distribution. Our interest for answering the questions related to this non-equilibrium intensity distribution is of a fundamental nature and extends, beyond strongly multimode semiconductor lasers, to all those other types of laser – e.g. fiber lasers \cite{24} or solid state lasers \cite{25} – which are characterized by the simultaneous operation, at least in transient, of a large number of longitudinal modes.

From the point of view of dynamical systems, our model can be viewed as an ensemble of globally coupled modes possessing a common, stable attractor. Globally coupled systems have been, and are still, a topic receiving a large amount of attention given their potential for application to various fields (cf., e.g., \cite{24,25}). Multimode lasers and laser arrays have been recognized very early on as interesting physical examples of globally coupled systems (cf., e.g., \cite{31,33}). At variance with most other investigations, we do not study synchronization or the appearance of collective states, but the transient evolution towards a fixed point (attractor) by a collection of

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modes with unequal degrees of coupling to a global energy source. However, transient dynamics can become part of self-sustained dynamics if the system parameters are modulated on a time scale comparable to (at least one) of its internal constants.

The results are non-trivial and show the existence of multiple time scales and of collective dynamics which contribute to a fast evolution (studied in this paper), which precedes the slow, hidden dynamics already presented elsewhere [19]. The discussion is entirely cast in terms of laser physics, but, making abstraction from the direct physical meaning of the variables – i.e., identifying the carrier number as a global coupling field and the laser modes as oscillators nonlinearly coupled to that (mean) field –, the results can be transposed to a generic dynamical system consisting of a large number of oscillators each individually coupled to a global field.

The model is briefly discussed in Section II and the transient evolution is analyzed in Section III which is divided into three subsections. The first two provide a detailed analysis of the below-threshold, deterministic dynamics for the population – where an analytical solution can be found for the carrier density (Section IIIA) – and for the modal intensity distribution – describable in terms of approximate iterative analytical solutions (Section IIIB). This detailed analysis paves the way for the central point of the paper: the analysis of the strongly coupled (multimode) transient in the oscillatory regime (Section IIIC). This regime, which displays the characteristic damped oscillations both in the carrier number and in the total intensity, connects the below-threshold dynamics with the slow mode one previously reported [10]. In particular, its final state – corresponding to the disappearance of oscillations in the carrier density and in the total laser intensity – is responsible for the ensuing, hidden modal dynamics [19]. The paper concludes with the analysis of the characteristics of the frequency spectrum emitted by the laser during the fast transient (Section IV). Comments, a summary and conclusions are offered in Section V.

II. MODEL

We resort to a standard model, where the physical constants are determined by comparing its predictions to experimental results [17], for the study of the fast transient response of a semiconductor laser to the sudden switch-on of its pump (control parameter). The details, numerical values and labeling choices for the simulations we perform have been published in [10].

The physical description is based on an ensemble of $M$ lasing modes, intensity-coupled to the carrier number $N$:

\[
\frac{dI_j(t)}{dt} = \left[\Gamma G_j(N) - \frac{1}{\tau_p}\right]I_j + \beta_j BN(N + P_0), \quad (1a)
\]

\[
\frac{dN(t)}{dt} = \frac{J}{q} - R(N) - \sum_j \Gamma G_j(N) I_j, \quad (1b)
\]

where $I_j(t)$ is the intensity of each longitudinal mode of the electromagnetic (e.m.) field (1 ≤ $j$ ≤ $M$), $N(t)$ is the number of carriers as a function of time, $\Gamma$ is the optical confinement factor, $G_j(N)$ is the optical gain for the $j$-th lasing mode, $\tau_p$ is the photon lifetime in the cavity, $\beta_j$ is the fraction of spontaneous emission coupled in the $j$-th lasing mode, $B$ is the band-to-band recombination constant, $P_0$ is the intrinsic hole number in the absence of injected current, $J$ is the current injected into the active region, $q$ is the electron charge, and $R(N)$ is the incoherent recombination term (including radiative and nonradiative recombination) which represents the global loss terms for the carrier number (i.e., the population inversion).

The bracket multiplying $I_j$ in eq. (1a) represents the global gain (effective modal gain minus losses) for the $j$-th mode, while $\beta_j BN(N + P_0)$ represents the effective mean contribution of the spontaneous emission to each mode. $\frac{J}{q}$ represents the normalized current (i.e. carrier number injected into the junction – energy provided to the laser) in eq. (1b), and the last term accounts for the global carrier number depletion due to the stimulated emission into all lasing modes.

The optical gain function, $G_j(N)$, contains information about the carrier number, $N_0$, necessary to achieve transparency (i.e., no absorption: $N = N_0$):

\[
G_j(N) = g_p (N - N_0)(1 - \epsilon I_j)
\left[1 - 2 \left(\frac{\lambda_j - \lambda_p}{\Delta \lambda_G}\right)^2\right]. \quad (2)
\]

The parameters are: $g_p$ differential gain, $\epsilon$ gain compression factor (multiplying the total intensity $I_t \equiv \sum_j I_j$), the individual mode wavelength $\lambda_j$, the wavelength at the peak of the gain curve $\lambda_p$, and the Full-Width-at-Half-Maximum (FWHM) of the gain curve itself, $\Delta \lambda_G$. See [10] for further details.

As in [12], we have allowed for $M = 113$ modes to take part in the dynamics in order to include in the simulations modes extending over approximately about 1.5 times the FWHM of the gain curve (cf. [10] for additional details). This choice allows us to give a good description of the transient regime – the focus of this paper – since the comparatively large contribution of the spontaneous emission to the side modes renders them an important element in the initial phases of the transient and plays a crucial role in the determination of the non-equilibrium distribution at the end of the fast transient, thereby considerably enlarging the transient power spectrum.

We numerically integrate the model equations, eqs. (1), in response to a sudden switch of the injected current $J$.
(in form of a Heaviside function), obtaining for the total laser intensity $I_t$ the response shown in Fig. 1 (curve (b)).

We notice the standard delay at turn-on for $I_t$, relative to the application of the pump-switch at $t = 0$, followed by the usual relaxation oscillations with rapid convergence towards steady state at $t = T \approx 1.5\,\text{ns}$. The transient, however, composed of the sum of all individual transients and its apparent usual behaviour – i.e., that of a single-longitudinal mode laser [34] – is far from trivial. Fig. 1 also shows a selection of lasing modes. The behaviour, contributing to the global oscillation for $I_t$ (and $N$). However, we remark that $\frac{I_{t}(0)}{I_{t}(\infty)} \neq \frac{I_{t}(t = \infty)}{I_{t}(t \to \infty)} \equiv \frac{7}{11}$ for all modes (even though $I_{t}(7) = I_{t} = I_{t}(t \to \infty)$, which represents the starting point for the slow dynamics [19].

![Image](image.png)

FIG. 1: Time evolution of the total laser intensity, $I_t$ (curve (b)), following a sudden switch-on of the injected current (pump rate) in form of a Heaviside function centered at $t = 0\,\text{ns}$. Initial current value: $I_0(t \leq 0^{-}) = 1.0\,\text{mA}$, final current value $J_f(t \geq 0^{+}) = 37.5\,\text{mA}$. The threshold current for this laser is $J_{th} = 17.5\,\text{mA}$. Model with $M = 113$ modes. All other parameters as in [19]. All parameter values are kept constant throughout the paper. With these values of $J_f$ and $J_t$ the intensity peak is reached at $t_p \approx 0.72\,\text{ns}$, while the steady state is attained at $T \approx 1.5\,\text{ns}$. Here and in all subsequent figures (unless otherwise specifically noted in the figure caption) the total intensity $I_t$ is divided by $2.5 \times 10^5$, the individual mode intensities by $1 \times 10^5$ and the carrier number $N$ is divided by $10^5$. Curve (a) shows the carrier number as a function of time. The other curves show the individual mode intensities for the central mode ($j = 57$, (c)), and side modes close to line center (blue side): $j = 56$ (d), $j = 54$ (e), $j = 52$ (f) and $j = 50$ (g). The symmetric modes, relative to the central one, are not shown but are identical. Notice that in the first phases of the transient (peak) curves (c) and (d) are superposed on this scale: they separate only at the second oscillation.

III. ANALYSIS

The fast transient, leading to the intensity peak and the usual damped oscillations, can be divided into two parts. A first one, where the carrier number $N$ can be decoupled from the modal intensities $I_j$’s, and a second one where – due to the strong coupling – the fully non-linear system must be retained.

In the decoupled regime, the carrier number plays the role of an energy reservoir, virtually unaffected by the presence of the laser modes (too weak to have an impact on the carriers). Thus, it is possible to find an analytical solution for its evolution – starting from an initial condition – towards a transient final state defined by the breakdown of this approximation (Section III A). In this regime, the modal intensities are decoupled from one another (Section III B) and evolve under the action of the (time-dependent) energy reservoir, $N$, whose behaviour has been analytically obtained with good accuracy in Section III A. This approximate but accurate treatment allows us to define a “threshold” for the multimode laser (including spontaneous emission) in analogy with a basic single mode laser model, where spontaneous emission is traditionally neglected [55]. The strongly coupled regime is analyzed in detail in Section III C on the basis of the knowledge acquired in Sections III A and III B and ad hoc considerations.

The analysis of the fast transient is best conducted by choosing an initial state with the laser (almost) entirely off. A small amount of prebias ($J_i = 1.0\,\text{mA}$, well below the lasing threshold [50] which is placed around $J_{th} \approx 17.5\,\text{mA}$) is useful since it provides an average contribution of the spontaneous emission to each mode, without having to wait for the carrier number to grow. This choice is not restrictive and does not limit the validity of our results. Our analysis holds any time the laser starts from an injection current value below threshold, which corresponds to an initial energy repartition among modes which strongly differs from the final one, above threshold. If instead the laser is already pumped above threshold and we suddenly change its pump to another value above threshold [55, 55], the ensuing dynamics will reflect the slow modal intensity redistribution examined in [19].

A. Decoupled carrier dynamics

A good understanding of the initial phases of the laser turn-on requires a description, even though approximate, of its evolution away from noise. A good deal of work has been dedicated to an analytical or semi-analytical description of this question [55, 55]. The following, we briefly outline the (standard) derivation of an approximate analytical solution for ease of comparison with the full, numerical integration of the model. The quantitative comparison will show that the analytical approximation provides excellent results even in the initial stages of the
transient when the e.m. field intensity reaches relatively high values, i.e., exactly where one would expect the approximate solution to already fail.

The contribution of the spontaneous emission (intrinsic noise) to each mode is quite small. As a first approximation, we can therefore consider that the initial phases of the transient are well described by the set of eqs. [4] without the coupling term between individual modal intensities $I_j$ and carrier number $N$ in eq. [10]. In this approximation, we start by integrating the carrier number, eq. [11], by variable separation:

$$\int_{N_i}^{N_f} \frac{dN'}{p(N')} = \int_0^t dt', \quad (3)$$

where

$$p(N) = aN^3 + bN^2 + cN + d, \quad (4)$$

$$a = -C, \quad (5)$$

$$b = -(B + 2CP_0), \quad (6)$$

$$c = -(A + BP_0 + CP_0^2), \quad (7)$$

$$d = \frac{J}{q}. \quad (8)$$

with the constants given in [10].

Formal integration of the left-hand-side (l.h.s.) of eq. [3] gives [17]

$$\int_{N_i}^{N_f} \frac{dN'}{p(N')} = \sum_{\text{roots } \mu_k \text{ of } p(N)} \frac{\log(N - \mu_k) - \log(N_i - \mu_k)}{3d\mu_k^2 + 2c\mu_k + b}, \quad (9)$$

With the parameter values of the problem, eqs. [4][3], only one root (which we will denote $\mu_1$) of $p(N)$ is real, while the other two are complex conjugate of each other

$$\mu_3 = \mu_2^* = a - ib. \quad (10)$$

Defining

$$z = N - \mu_1, \quad (12)$$

$$z^* = N - \mu_3, \quad (13)$$

and equivalently for $z_i$, we expand the denominator of the formal solution (r.h.s. of eq. [9])

$$3d\mu_k^2 + 2c\mu_k + b =$$

$$[3d(a^2 - b^2) + 2ca + b] \pm i[6dab + 2cb]. \quad (14)$$

$$\equiv \Omega \pm i\Omega_i, \quad (15)$$

$$k = 2, 3. \quad (16)$$

where the $+$ ($-$) sign in eqs. [14][15] corresponds to $k = 2$ ($k = 3$).

Thus, the contributions to eq. [9] coming from the two complex roots can be rewritten as

$$\sum_{\mu_k=2,3} \frac{\log(N - \mu_k) - \log(N_i - \mu_k)}{3d\mu_k^2 + 2c\mu_k + b} =$$

$$(\log(z) - \log(z_i)) \frac{\Omega - i\Omega_i}{|\Omega|^2} +$$

$$(\log(z^*) - \log(z_i^*)) \frac{\Omega + i\Omega_i}{|\Omega|^2}, \quad (17)$$

with $z_i \equiv N_i - \mu_2$ (cf. eqs. [12][13]).

Substituting into the full expression, we finally obtain the solution for eq. [3]:

$$t = \frac{\log(N - \mu_1) - \log(N_i - \mu_1)}{3d\mu_1^2 + 2c\mu_1 + b} +$$

$$2[\log |z| - \log |z_i|] \frac{\Omega_1}{|\Omega|^2} +$$

$$2[\text{Arg } \{z\} - \text{Arg } \{z_i\}] \frac{\Omega_i}{|\Omega|^2}. \quad (18)$$

This relationship implicitly defines the approximate solution for the carrier number and gives the most practical way of numerically representing the analytical solution: treating time as the dependent variable, $t(N)$, the function can be straightforwardly plotted (exchanging the horizontal and vertical axes allows one to restore the $N(t)$ appearance of the function). Alternately, simple handling allows for a mathematical expression for $N(t)$ which takes the form:

$$N(t) - \mu_1 \left( N^2(t) - 2\text{Re } \{\mu_2\} N(t) + |\mu_2|^2 \right)$$

$$e^{2\text{Arg}(N(t) - \mu_2)} =$$

$$\mathcal{K} e^t, \quad (19)$$

where

$$\mathcal{K} = (N_i - \mu_1) \left( N_i^2 - 2\text{Re } \{\mu_2\} N_i + |\mu_2|^2 \right) e^{2\text{Arg}(N_i - \mu_2)}, \quad (20)$$

which can be used as an alternative definition of $N(t)$ [48].

Fig. 2 compares the analytical solution $N(t)$ to the computed $N(t)$ obtained from the numerical integration of the full system, eqs. [1]. The agreement between the approximate, analytical solution and the full integration is excellent up until values of time which are quite close to the intensity peak (occurring at $t \approx 0.72s$), as shown in the inset. The deviation is initially positive and remains below 0.5% (in absolute value) until $t \approx 0.62ns$.

The initial positivity of the relative deviation $\frac{N(t) - N(t)}{N(t)}$, resides in the fact that the full solution requires a somewhat larger initial carrier number to support the different field modes, while in the approximate form no energy is lost to support them. The switch in sign in the difference between the exact and the approximate value for
the carrier number comes from the fact that once the different lasing modes grow sufficiently large, the carrier number saturates, instead of continuing its growth, as in the approximate expression. The difference curve (inset) is numerically computed by integrating the full set of equations, eqs. (1), and a set equivalent to the analytical solution, obtained by removing from eq. (1b) the last term \( \sum_j \Gamma G_j(N)I_j \).

Notice that at \( t \equiv t_{th} \approx 0.62\,\text{ns} \), the carrier number \( N(t) \) crosses its asymptotic value \( N(t_{th}) = N(\infty) \approx N(t \to \infty) \), cf. Fig. B. In a laser model without spontaneous emission such a value would define the laser threshold, separating the decoupled dynamics from the strongly coupled regime (above threshold) \[35\]. Here, this value sets the upper limit to the carrier number for which we can consider its dynamics decoupled from that of the modal intensities (when growing out of noise), thus we can extend the concept of “threshold” even in the presence of spontaneous emission.

### B. Decoupled modal dynamics

We begin this section with an overview of the modal dynamics during the transient evolution, regardless of the degree of coupling with the carrier number. This allows us to get a general picture, which we then refine in the analysis of the decoupled regime (this section) and of the strongly coupled one (next section).

The individual mode traces in Fig. B show that at short times the lateral modes contribute more than their steady state share. Their peak (cf., e.g., curves (f) and (g) corresponding to modes 5 and 7 places away from line center) is much higher than the asymptotic intensity towards which they tend, contrary to those modes close to line center (for the central mode even the peak intensity is lower than its steady state value) \[10\]. This strongly suggests that a mechanism must exist for these modes to grow, in transient, well beyond their final value, while the very few central modes do not. Indeed, the accumulated intensity of the modes farthest away in the wings can exceed, in transient, that of the central mode, thus highlighting the impact that the side modes have on the initial phases of the dynamics.

A more complete illustration of this point is given by Fig. C which compares the time-dependent intensity of the central mode – curve (a) –, to the cumulative contribution of the most distant twenty modes on each side of line center, \( I_{far} \), (cf. text for details) and (b) the sum of all side modes, \( I_{M-17} \), save for those near line center (cf. text for details). The inset shows the initial phases of the transient and shows that the sum of the far modes (c) exceeds the intensity of the central mode (a) up until \( t \approx t_{th} \).

![Figure 2](image1.png)

**FIG. 2:** Comparison between the approximate analytical solution (solid line) \( N(t) \) and the complete carrier number (dashed line) \( N(t) \) resulting from the integration of the full model, eqs. (1). The inset shows the difference between the full solution \( N(t) \) and the analytical solution \( \bar{N}(t) \) until the latter diverges away (at \( t \approx 0.72\,\text{ns} \)). Notice that even extremely near the intensity peak (cf. Fig. B) the difference between the approximate and full solution remains modest (\( \lesssim 8\% \)).

![Figure 3](image2.png)

**FIG. 3:** Temporal evolution of modes or groups of modes. Curve (a) traces the evolution of the central mode, (c) the sum of the most distant 20 modes on each side of line center, \( I_{far} \), (cf. text for details) and (b) the sum of all side modes, \( I_{M-17} \), save for those near line center (cf. text for details). The inset shows the initial phases of the dynamics. Considering, though, that these modes are very far out in the wings and that their asymptotic value is order of magnitudes lower than that of the central mode, their strong influence on the dynamics up to threshold is an indicator of the essential difference between the transient intensity distribution and the asymptotic one!

The picture changes more dramatically if we include the lateral modes, \( I_{M-17} \), which carry little energy at
steady state (globally of the order of 3% of the total intensity), but which in transient give a cumulative peak larger than the one of the central mode (curve (b)). This contribution is rather striking. Indeed, it displays a faster growth than the central mode and deformed (incomplete) oscillations. The latter are explained by the fact that the effective relaxation frequency is not constant over the whole set of modes (cf. [21] for discussion on a slightly different model and Section III C in this paper), thus the sum over the ensemble deforms and attenuates the oscillations.

The substantial contribution of \(I_{M-17}\) to the peak in transient is a consequence of the observation that in the initial phases of the dynamics the lateral modes contribute, relatively, much more than they do at steady state (or at least on the long time scales). In addition, we remark that the anticipated peak for \(I_{M-17}\) contributes to a faster growth of the total intensity. We now analyze the initial portion of the transient, i.e., the decoupled regime.

The remarks concerning the transient leading to, but excluding, the intensity peaks (and oscillations) can be better understood with the help of the following approximate analysis. Using the analytical (approximate) solution for \(N(t)\) (eq. (15)) we can now decouple the modal intensities to obtain an approximate dynamical evolution:

\[
\begin{align*}
\frac{d^dI_j}{dt^d} & \approx \left[ \Gamma G_j(N) - \frac{1}{\tau_p} \right] dI_j + \beta_j BN(t) [N(t) + P_0], \\
\end{align*}
\]

where the superscript \(d\) denotes the approximate, decoupled variable.

The following auxiliary quantities simplify the notations:

\[
\begin{align*}
\alpha_j &= \Gamma g_0 \left[ 1 - 2 \left( \frac{\lambda_j}{\Delta \omega} \right)^2 \right], \\
\lambda_j &= \alpha_j N_0 + \frac{1}{\tau_p}, \\
\beta_j &= \beta_j B,
\end{align*}
\]

where we have neglected the saturation term \(\epsilon I_j\) in the expression for \(G_j(N)\), eq. (2), since we are examining the transient portions in which the intensity is very small, \(\epsilon\) being also very small (cf. Table III in [19]). This way, the modal intensities obey the set of M decoupled ODEs

\[
\begin{align*}
\frac{d^dI_j}{dt^d} & \approx [\alpha_jN(t) - \lambda_j] dI_j + B_jN(t) [N(t) + P_0],
\end{align*}
\]

which can be globally analyzed.

Over very short time intervals \(\Delta t\) (\(\Delta t: N(t + \Delta t) \approx N(t), \forall t : (0 \leq t \leq t_{th})\)) we can consider the two terms \([\alpha_jN(t) - \lambda_j]\) and \(B_jN(t) [N(t) + P_0]\) as being constants, thus reducing eq. (23) to an ensemble of ODEs with constant coefficients. In such a case, formal integration provides immediately a solution for the approximate problem, eq. (23):

\[
\begin{align*}
\frac{dI_j(t_k + t')}{dt} & \approx - \frac{B_jN(t_k) [N(t_k) + P_0]}{\alpha_jN(t_k) - \lambda_j} + \left[ dI_j(t_k) + B_jN(t_k) [N(t_k) + P_0] \right] e^{[\alpha_jN(t_k) - \lambda_j]t'}, \\
& = \frac{B_j(t_k)}{\gamma_j(t_k)} + \left[ dI_j(t_k) + B_j(t_k) \right] e^{\gamma_j(t_k)t'}, \\
\gamma_j(t_k) & \equiv \alpha_jN(t_k) - \lambda_j (< 0), \\
B_j(t_k) & \equiv B_jN(t_k) [N(t_k) + P_0], \\
t_k & \equiv k \times \Delta t, \\
k & = 0 \ldots q, \\
0 & \leq t' \leq \Delta t, \\
\frac{dI_j(t_0)}{dt} & = I_j(t_0),
\end{align*}
\]

are the initial conditions for each modal intensity. The inequality in parenthesis (eq. (26)) holds strictly in the range of validity of the current approximation. Here \(t'\) is a local time defined in each interval \(\Delta t\) and is zeroed at the beginning of each time step.

The coefficients \(B_j(t_k)\) and \(\gamma_j(t_k)\) are constant over a small time interval \(\Delta t\) and are “updated”, as time evolves, at the next value \(t_k\). This, in itself, is not a limitation since their functional dependence is known and explicitly given by eqs. (19), (20), (22). Thus, eq. (24) is a “piecewise”, recursive solution over a discrete ensemble of times \(t_k (k = 0 \ldots q) : t_0 = 0, t_q \approx t_{th}\). At each time step, each \(dI_j\) starts to converge, for a short time, towards its locally asymptotic solution before being updated to the next time step. The form of the approximate solution, eq. (24), highlights the role of the different coefficients: the spontaneous emission term, \(B_j\), controls the amplitude (together with \(\gamma_j\) in the prefactor of eq. (23)), while gain, \(\alpha_j\), and cavity losses, \(\lambda_j\), combined into \(\gamma_j\) play the role of an effective relaxation constant. Notice that \(\gamma_j\) results from the composition, eq. (29), of the two functions shown in the inset of Fig. 4 (notice the shift \(\gamma_A\) (curve A) and \(\gamma_C\) (curve B)). The different local curvatures, combined with the multiplication of curve A by the carrier number \(N(t)\) (or equivalently \(N(t)\) when using the approximate solution), differentiate the coefficient \(\gamma_j\) for the individual modes (with the possibility of obtaining, in transient, positive values when curve \(A \times N\) is larger than curve \(B\) – possible only outside the range of validity of the current approximation).

In order to better understand the behaviour of this approximate solution, we trace the coefficients \(\gamma_j\) and \(\gamma_j\) for selected modes in Figs. 4 and 5, respectively. Fig. 4 shows the evolution of \(\gamma_j\) as a function of time for a representative sample of modes. Aside from the initial portions of the transient (\(t \lesssim 0.24\,ns\)) the larger \(\gamma_j\), the
The temporal evolution of the individual modes includes the contributions of the substitution, eqs. (26, 27)) discussed so far, the temperature, which becomes effective relaxation constants, the figure immediately shows that the wing modes, with their larger (negative) values of $\gamma_j$, contribute to the full set of equations, eqs. (19, 20). The inset shows the modal dependence of $I_j$ on $t$ for selected values of $\gamma_j$, which is nothing else than the ratio $I_j^{(t)} / I_j^{(0)}$ computed from the full set of equations, eqs. (19, 20). The dashed lines (which separate only for $t > t_{th}$ from the solid lines) are computed for the analytical solution $N(t)$, eq. (18)). The inset shows the modal dependence of the central mode(s) react(s), the energy is transferred back, allowing the side modes to relax to their asymptotic values. Notice that the agreement between the coefficients calculated from the analytical solution for the carrier number and from the full numerical solution holds extremely well before threshold: in Figs. 4, 5 one can distinguish solid and dashed curves only because the side modes maintain a level which is considerably larger than their asymptotic contribution.

For the latter mode we remark that $\gamma_57 \approx 57$ – somewhat less for the plotted mode farthest away from line center ($j = 10$). For the latter mode we remark that $\gamma_57 \approx \frac{57}{10} \approx 5.7$, thus maintaining a nearly constant, fast relaxation throughout the decoupled regime. The fast, nearly constant, reaction time explains the sizeable growth of the wing modes, in spite of the smaller contribution which they receive from the spontaneous emission. The moderate change in $\gamma_57$ (and equivalent from the wing modes) ensures a slight growth in the prefactor of eq. (25), thus a rather stable energy contribution of these modes below threshold. For comparison, in the units of Fig. 4 $\gamma_57 (t \rightarrow \infty) \approx -10^{-3}$ (cf. [19]). Hence, the effective relaxation rate changes considerably for the center mode and is eventually responsible for their predominance at the end of the below-threshold regime, even though the side modes maintain a level which is considerably larger than their asymptotic contribution.

The approximate solution, eq. (24), shows the role played by $\gamma_57(t_{th})$, which is nothing else than the ratio between the coefficients shown in Figs. 4, 5. Thus, another way of interpreting the dynamics below threshold is to plot, Fig. 6, the r.h.s. of eq. (24) – thus the intensity increment – for the whole ensemble of modes at selected times (i.e., for selected values of $N(t)$), from the initial condition up to threshold.

When the carrier number is well below transparency (curves a-c) the gain is nearly equal for all modes, with a somewhat higher level in the wings for the first two curves a-d. This implies that the modes far from line center start growing faster than the central ones. Near
transparency, curve (d), the central modes start dominating and the combined effect of their somewhat larger intensity and of their increasing coefficients (cf. Figs. 11) favors them. For the parameters of curve (e) the laser is already beyond transparency ($N_0 = 7.8 \times 10^7$, $t \approx 0.24\text{ms}$) and the tendency for a strong differential growth becomes obvious and sharpens itself with growing values of $N$. Equivalent curves showing the ratio (cf. eq. 25) $\frac{I_j}{I_{j0}}$ (each component plotted in Figs. 11) look qualitatively very similar to those of Fig. 4 since they differ only for the intensity $dI_j$ which multiplies $\gamma_j$ (the relative deformation is most visible for curve (h), which is more strongly reshaped by the larger modal intensity variations).

In spite of the clear increase in differential intensity growth between the central and the side modes displayed in Fig. 6 the ratio remains finite up until threshold, as shown in Fig. 4 where the amplitude of each intensity mode is displayed normalized to the intensity of the central mode. Fig. 7 shows again that the gain is largest for the wing modes at the start of the transient (curve (h), $j = 10$, is the highest at $t = 0$), due to their lower absorption and that during the whole evolution in the decoupled regime the relative side of the mode largely exceeds their asymptotic value ($\frac{dI_j}{dI_{j0}} |_{t=0.6\text{ns}} \approx 0.05$ – cf. Fig. 7) to be compared to $\frac{7}{10^2} \approx 1.5 \times 10^{-4}$ from Table II. This result is consistent with the numerical observation – obtained from the integration of the full model, eqs. (1) – showing that at $t = 0.6\text{ns}$ the cumulative intensity of the 20 modes farthest away from line center, $I_{far}$, exceeds that of the central mode (cf. Fig. 3). It also explains why at $t = 7$ the intensity distribution is far from the asymptotic one (Fig. 1 and 11): the wing modes at the beginning of the strongly coupled regime hold a very large relative amount of energy ($\approx 300$ times its relative asymptotic value for $j = 10$).

Besides the numerical evidence of Fig. 7, eq. 25 shows the reason for the not-so-large differences among modes, below threshold. Expanding the exponential in eq. 25 to first order, we obtain an approximate form for the modal intensity:

$$dI_j(t_k + t') \approx dI_j(t_k)(1 + \gamma_j(t_k) \cdot t') + B_j(t_k) \cdot t' \tag{32}$$

thus its increment relative to the previous time step takes the form:

$$dI_j(t_k + t') - dI_j(t_k) \approx dI_j(t_k)\gamma_j(t_k) \cdot t' + B_j(t_k) \cdot t' \tag{33}$$

The first term on the r.h.s. of eq. 33 represents a relaxation ($\gamma_j < 0$, $\forall (j,t_k)$), while the second one is the modal intensity increment brought about by the (average) spontaneous emission contribution. Thus, the differential growth between central and wing modes remains moderate (cf. Fig. 5), which explains why even close to threshold the ratio of the wing modes to the center mode (Fig. 7) remains very large compared to its asymptotic value. Notice that eq. 33 indirectly determines a time step $\Delta t$, since $\Delta t \ll \frac{1}{\gamma_j(t_k)} \forall (j,t_k)$.

Fig. 8 shows the temporal evolution of a selection of modes (those close to line center) for the full integration (eqs. 1, solid lines) and for the decoupled system (eqs. 18–20, dashed lines). We notice that over the whole range of validity ($t \leq t_{th}$) of the analytical solution for the carrier number, eq. 18, the decoupled approximation is very good. Inset A shows a detail of the temporal evolution for the same modes close to threshold: all curves lie extremely close to one another, confirming the validity of our approximation. Inset B shows the evolution for three wing modes (cf. caption): the
which shows how the intensity distribution among all modes at the end of the decoupled regime (i.e., the instant when the population carrier crosses its asymptotic value – threshold – for the first time) does not differ very substantially from the initial distribution. Curve a (black online) shows the equilibrium intensity distribution at the initial condition – the side modes carry more intensity thanks to their reduced absorption (below transparency). Curve b (red online) shows the intensity distribution at threshold (cf. above) – i.e., \( t \approx 0.62\text{ns} \). Even though the central modes carry a larger amount of intensity (and the shape of the curve is inverted, compared to the initial condition, when comparing to the asymptotic, equilibrium, distribution (curve d, blue online), the quantitative differences among modes at the first threshold crossing are nearly negligible. Anticipating on the next section, we remark that even at the peak in the total intensity the intensity distribution among modes (curve c, green online) differs less from the threshold condition than from the asymptotic one.

In conclusion, in this section we have shown that the decoupled model gives a very good representation of the evolution of the modal content of the laser output in the below threshold region, i.e., over a large part of the transient. We have also seen, both numerically and analytically, how the side modes carry an amount of energy much larger than their asymptotic share. Once the system overcomes this threshold, the analytic approximation for the carrier number breaks down, the coupling becomes strong and only the full numerical solution can account for the remainder of the fast evolution towards the steady state for the two global variables \((N, \text{and } I)\).

\hspace{1cm}

\textbf{C. Strongly coupled regime}

The two preceding sections have analyzed in detail the portion of the dynamical evolution below the threshold value for the carrier number. Here we are going to investigate the next portion of the dynamics, which corresponds to the full nonlinear coupling between modes and which leads, eventually, into the time-independent total intensity with the residual slow evolution of the modal intensity distribution. Our analysis of the multimode high-intensity regime is based on a direct integration of the model equations and on the potential for the interpretation of the resulting observations which comes directly from the detailed analysis of the previous sections. A direct-solution approach, such as the one adopted in for an analytical construction of the solution for short pulses in a single-mode device, would be very hard to adopt in our strongly multimode model.

\hspace{1cm}

\textbf{Fig. 10} clearly shows the transition between the decoupled regime, where the deviation between the approximate and exact modal intensities is at most 5\% (at \( t \approx 0.6\text{ns} \)), and the fully nonlinear regime, where this difference rapidly diverges. The transition is clearly il-
illustrated by Fig. 4 which shows that during their growth some $\gamma_j$’s become positive for a subensemble of modes (near line center) and oscillate together with the carrier number to relax towards their (negative) asymptotic value (not shown). When some $\gamma_j > 0$ the nature of eq. (25) changes, since the growth of the $dI_j$’s is no longer exclusively due to the growth of the prefactor but also to the exponential growth of some $dI_j$’s, which eventually leads to a divergence from the solution of the full model, eqs. (11).

From the point of view of the solution of eqs. (11) the positivity of some $\gamma_j$’s bears no weight. Indeed, throughout the growth of the modal intensities ($t \leq 0.72$ ns, cf. Fig. 4), the r.h.s. of eq. (10) (or, equivalently, eq. (21)) remains positive due to the contribution of the spontaneous emission ($B_j$), independently on the sign of $\gamma_j$. Thus, the latter contributes quantitatively, but not qualitatively, to the growth of the modal intensities $dI_j$.

A closer look at the modal dynamics (solid lines in Fig. 5) shows that all individual modes display the same generic behaviour (peak with damped oscillations) but that some differences appear in the time at which the peaks occur and in the shape (frequency and damping) of the oscillations. Table I gives the time $t_p$ at which the intensity maximum ($p I_j$) is reached for a selection of modes. A clear anticipation in the first peak appears as the mode index moves away from line center ($j = 57$), not unexpectedly after the discussion of the previous section based on the behaviour of the $\gamma_j$’s (Fig. 4). An inversion in this tendency, however, appears comparing $p I_{50}$ and $p I_{10}$, hinting to a more complex behaviour in the strongly-coupled regime.

Fig. 10 shows the ratio between the maximum peak amplitude for each mode and its corresponding asymptotic value (circles, left scale). For the central mode the peak amplitude $p I_{57}$ is about half its steady state, $I_{57}$, while in the far wings the peak (e.g., $p I_{10}$) is only slightly higher than its steady state ($I_{10}$), showing that equilibrium for these modes is reached rather smoothly (i.e., with little overshoot and oscillations). However, the peak amplitude grows very large for modes off line-center, but not too far from it, the largest ratio occurring for $j = 50$ (six places away from line center) whose peak intensity $p I_{50}$ is $14 \times I_{50}$. This highlights the role of the (close) side modes in the strongly coupled regime, which temporarily carry a large amount of energy.

Associated with this strong variation in the peak intensity, a remarkable, nearly anti-correlated, variation in the damping appears. The ratio between the second and
the first peak $\left(\frac{\nu_j}{\nu_j} \right)$ is represented by squares (Fig. 11, right scale)). While at line center and far off in the wings, the second peak is almost as large as the first ($\gtrsim 80\%$), for the strongest overshoot we notice (almost) the largest damping (with a slight shift, $\frac{\nu_j}{\nu_j} \approx 0.2$).

Assuming a simple dependence for the damped oscillations, of the form $I_j(t) = I_{0,j} e^{-(\sigma_j + i2\pi \nu_j)t}$, we can estimate the modal damping constant $\sigma_j$ in the strongly coupled regime, together with the oscillation frequency $\nu_j$, directly from the data, using the peak amplitudes $P_j I_j$, $P_i I_j$ and times $P_j \nu_j \Delta t_j \equiv P_j \nu_j t_j$ at which they occur. With the help of the standard expression for the undamped frequency $\nu_0$ of a damped oscillator, we can further estimate $\nu_{0,j} = \sqrt{\left(\frac{\nu_j}{\nu_j}\right)^2 + \nu_j^2}$. The corresponding results are shown in Fig. 12 for the damping $\sigma_j$ (diamonds, left scale) and for the damped (down-triangles), $\nu_j$, and undamped (up-triangles), $\nu_{0,j}$, frequencies (right scale for both).

A resonance-like behaviour appears clearly in the relaxation oscillation frequency, with a maximum at mode $j = 48$ for $\nu_0$ (at $j = 49$ for $\nu$), and a sizeable variation in frequency $\Delta \nu_0 = \nu_{48} - \nu_{47} \approx 1.2\,GHz$ (i.e., $\Delta \nu_0 \approx 25\%$). The contribution due to the modal damping $\sigma_j$ is clearly visible in the graph and increases the frequency, at its maximum, by $\nu_{48} - \nu_{47} \approx 4\%$, while shifting it away from line center (by one mode). The combination of the contribution of all modes lends the total intensity (Fig. 1) features (cf. captions of Figs. 11, 12), which are a combination of those of different modes. Looking at the damping, $\sigma_{tot}$ or at the relative height of the first peak, $\frac{\nu_j}{\nu_j}$, the values are similar to those of mode $j = 53$, the relaxation oscillation frequencies, $\nu_{0,j}$, $\nu_{tot}$, are consistent with mode $j = 54$, while the second peak is less strongly damped, as for a mode placed between $j = 55$ and $j = 56$.

We therefore conclude that the dynamics in the portion of the fast transient corresponding to the strongly-coupled regime is dominated by a group of modes close to, but not at, line-center.

FIG. 12: Modal damping coefficient, $\sigma_j$, (diamonds, left scale), modal damped (down-triangles), $\nu_j$, and undamped (up-triangles), $\nu_{0,j}$, relaxation oscillation frequency (right scale). Notice that the central mode is at the far right, thus for picturing the full set of modes one should mirror-image the figure with respect to vertical axis passing through $j = 57$. The equivalent quantities for the total intensity are: $\sigma_{tot} = 3.10 \times 10^9\,s^{-1}$, $\nu_{0,tot} = 4.885\,GHz$, $\nu_{tot} = 4.910\,GHz$.

An equivalent graphical illustration is given by Fig. 13, which shows the transient evolution for each mode, normalized to its asymptotic value. The modes far in the wings show a very early and gradual growth with little overshoot and oscillations (the decay is masked by the normalization to their asymptotic state, cf. 15). Coming closer to line center we notice a gradual delay in the growth, accompanied by a sharp, and very large, peak and a gradual relaxation towards the asymptotic intensity level. The last mode to grow is the central one, whose peak, together with the first side mode, is lower than its steady state. The relaxation for the central modes is extremely slow and extends well beyond the figure range.

Fig. 13 together with the analysis of the previous sections, convincingly demonstrates the reason for the slow dynamics discussed in 10. The transient evolution from the initial, below-threshold condition takes place first through a regime where the modal growth is not strongly differentiated between mode center and wings (i.e., below threshold). When the laser finally passes threshold, the central (slower) modes have to grow away from an intensity distribution which is far from the asymptotic one (Fig. 9). This intensity configuration favours those lateral modes which are sufficiently close to line center to have a strong growth, but which at the same time possess faster time constants, which allow them to more rapidly exploit the excess population inversion. Thus, we find a strongly out-of-equilibrium intensity distribution at the peak (curve c in Fig. 9), and it is not surprising that when the total intensity and the carrier number have relaxed to their respective asymptotic values (at $t \approx 1.5\,ns$)
for the initial conditions chosen in this paper, Fig. 11, the modal repartition of the intensity be still strongly very far from its asymptotic configuration. In this way, we clearly understand the reason for the slow dynamics reported in [19].

IV. TRANSIENT SPECTRA

The analytical and numerical treatment of the transient has given us a good description of the modal evolution from the initial state (laser – nearly – off) to the point where the two global variables (N and I_t) have attained equilibrium. We now look at the time-resolved transient emission spectrum, defined as the frequency interval over which the modal intensity passes a chosen percentage of the total output intensity, W_{FS}^x, where FS stands for Full Spectral and W for Width, while x fixes the chosen level in dB. We remark that, contrary to what found for the slow dynamics [19], the transient spectra are only marginally affected by the modal arrangement under the gain line (even vs. odd number of modes in the simulation – cf. [19] for a discussion). We will therefore discuss only the modal placement used so far.

![FIG. 14: Full spectral width W_{FS}^x of the laser evolution during the transient. Solid lines x = 40dB, dashed lines x = 20dB. The level in dB is defined, as in [19], as the modal intensity which passes the preset level compared to the instantaneous total intensity.](image)

Fig. 14 shows the Full Spectral Width (W_{FS}^x), resolved in time, calculated at -40 dB (W_{40}^{FS}, solid line) and at -20 dB (W_{20}^{FS}, dashed line). The frequency “edge”, marked by each line in the figure, is obtained by linear interpolation between the frequencies of the two modes between which the W_{FS}^x criterion is satisfied. The -40 dB level captures the full evolution of the spectral content during the transient; we recognize that in the very first phases of the transient (< 0.2ns) none of the modes pass this mark (the spectral width starts at this point) [58] because the intensity is widely distributed over a large number of modes, rather than being carried by a few modes around line center. In other words, no frequency component is sufficiently strong to reach one-hundredth of the total intensity (equivalent to -40 dB). Since we are integrating over slightly more than 100 modes, this implies that the intensity distribution, in this first phase of the transient, is sufficiently homogeneous that no spectral feature emerges from the spectrum. This is in agreement with our previous findings which show a rather homogeneous intensity distribution among modes in the decoupled regime. The maximum width is obtained between 0.4ns and 0.5ns and corresponds to approximately 47 modes passing the 0.01 x I_t(t) level (i.e., the -40 dB mark).

The W_{40}^{FS} continues to decrease beyond t = t_p and W_{20}^{FS} makes its appearance at t ≈ 0.8ns, grows until t ≈ 1ns and then initiates its slow convergence to its asymptotic value, as discussed in [19]. In light of the results of the previous section the wide spectral width (W_{40}^{FS}) does not come as a surprise and quantitatively shows the degree of modal spread of the laser intensity in the initial phases of the transient. On the other hand, the large modal coverage has a direct impact on the validity of models which are obtained by truncating the full one, M = 113, to a subset of modes around line center [50].

V. COMMENTS, SUMMARY AND CONCLUSIONS

The results of this paper have been obtained on a semiconductor laser model where specific choices have been made to describe relaxation processes to match a specific experimental device (a Trench Buried Heterostructure bulk semiconductor laser). We have selected this model to have a tested, realistic comparison with an actual device [17], but the relevance of the understanding that we have gained in the process extends beyond the specific details of this particular system.

Numerical tests have shown [60] that only minor quantitative details change when replacing the fixed gain line with a variable, dynamic one [61], or when substituting the parabolic profile (eq. 2) with a Gaussian one [20], or even when exchanging the linear gain (eq. 2) with a logarithmic one [62]. More sophisticated modeling choices can be made for the gain line [63, 64], but on the basis of the previous remarks, we expect the general results we have obtained in this paper to hold at least for most kinds of multi-longitudinal mode semiconductor lasers.

The kind of interaction among modes (coupling to a mean field) is at the origin of our numerical observations and of the understanding we have gained, independently of the modeling details. Thus, we expect our predictions to apply to any other kind of laser where an intensity-based intermodal interaction dominates. Candidates for such behaviour are all those lasers where diffusion in the gain medium washes out any trace of spatial modulation resulting from the interference among the modal fields and/or lasers with a partial, but not dominant, inhomogeneous broadening.

Our modeling of the below threshold region (decoupled
In summary, we have numerically investigated the fast transient dynamics of a multimode semiconductor laser, in response to a sudden turn-on by a switch of a parameter (pump), modeled by M ODEs for the modal intensities, with coupling occurring through the population inversion (i.e., the carrier number). The dynamics have been proven to separate into two regimes: (a) one where the carrier number and the modal intensities can be decoupled and for which approximate analytical solutions can be found; and (b) the strongly coupled (or fully nonlinear) one where all variables are interdependent in their evolution. Our analysis gives a clear physical explanation and a quantitative illustration of the origin of the strong deviation for the modal intensity distribution at the onset of equilibrium ($t = \bar{t}$) for the global variables (carrier number $N$ and total intensity $I_{tot}$) which gives rise to the slow dynamics $\bar{t}$.

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Recall that we consider the time interval (0 ≤ t ≤ 0.6) ns (cf. Fig. 3), thus this statement does not cover the strongly coupled regime (t ≈ 0.62 ns) where the modal intensities show a peak.

The larger intensity increment below transparency is explained, for the wing modes, by their lower absorption. Notice that for the given lineshape, eq. (4), for a couple of modes at the extrema of the interval there may be a (temporary) sign inversion, which bears, however, no practical consequences on the numerics.

Since the radius of convergence of the exponential function is infinite, one could use a less restrictive condition. However, the magnitude of ΔI is appropriately set this way and is consistent with the one used in Fig. 7.

An alternative way of defining the upper limit for the range of parameters (or time interval) for which the dynamical decoupling between carrier number and modal intensities holds is the following: evaluate steady states for the variables N and Ij in the decoupled model (eqs. 18, 23) and impose the condition that Ij ≥ 0 ∀ j. While well below transparency three distinct real solutions exist for the steady state condition (eq. (3b) in 19), adapted to the decoupled system, close to threshold only one real one, which we will call N0, can be found. Thus, the steady state intensity takes the form (for all modes) \( I_j = \frac{B_j n_0 K_j - \alpha_j N}{K_j - \alpha_j N} \), thus \( I_j ≥ 0 \) iff \( K_j - \alpha_j N = -\gamma_j > 0 \). Hence, the decoupled regime holds only for \( \gamma_j < 0, \forall j \) and the instant \( t \) for which \( \gamma_j(t) > 0 \) (in particular \( j = 57 \)) defines the upper limit to this regime (either by defining \( t \) or the corresponding value \( N(t) \)). Thus, for the decoupled regime to hold, the effective relaxation constant must remain negative at all times for all modes. As soon as one \( \gamma_j > 0 \) (\( j = 57 \), specifically) the approximation can no longer be used.

One should define two different kinds of coefficients for \( \gamma \) and \( B_j \), depending on whether they are defined as a function of \( N \) or of \( N_0 \). Due to the practically identical results (cf. Figs. 4, 5), originating from the almost coincidence of \( N \) and \( N_0 \) over the whole interval of validity of the decoupled approximation (cf. Fig. 6), for the sake of simplicity we refrain from introducing additional symbols.

The point at which all curves for \( \gamma_j \) cross corresponds to \( N = N_0 \), which represents the carrier number for which the gain medium is bleached (transparency). Below this value, the wing modes experience less absorption, since they are far off-resonance, thus their corresponding \( \gamma_j \)'s are larger. For \( N > N_0 \) the situation is reversed, since the gain is larger near line center.

Recall that we consider the time interval (0 ≤ t ≤ 0.6) ns (cf. Fig. 3), thus this statement does not cover the strongly coupled regime (t ≈ 0.62 ns) where the modal intensities show a peak.

The larger intensity increment below transparency is explained, for the wing modes, by their lower absorption. Notice that for the given lineshape, eq. (4), for a couple...