Quantum Matrix Pairs

J. E. Nelson\textsuperscript{1}  
Dipartimento di Fisica Teorica  
Università degli Studi di Torino  
via Pietro Giuria 1, 10125 Torino  
Italy  

and  

R. F. Picken\textsuperscript{2}  
Departamento de Matemática and Centro de Matemática Aplicada  
Instituto Superior Técnico  
Avenida Rovisco Pais, 1049-001 Lisboa  
Portugal

Abstract

The notion of quantum matrix pairs is defined. These are pairs of matrices with non-commuting entries, which have the same pattern of internal relations, q-commute with each other under matrix multiplication, and are such that products of powers of the matrices obey the same pattern of internal relations as the original pair. Such matrices appear in an approach by the authors to quantizing gravity in 2 space and 1 time dimensions with negative cosmological constant on the torus. Explicit examples and transformations which generate new pairs from a given pair are presented.

\textsuperscript{1}email: nelson@to.infn.it  
\textsuperscript{2}email: picken@math.ist.utl.pt
1 Introduction

The notion of quantum matrix pairs arose in the context of our recent work [1] on quantum gravity in 2 space and 1 time dimensions with negative cosmological constant on the torus. As shown by Witten [2], this model is equivalent to a Chern-Simons theory with non-compact structure group $SL(2, R) \times SL(2, R)$. After imposing the constraints the classical geometry may be encoded, up to equivalence, by two commuting $SL(2, R)$ matrices $U_1$ and $U_2$ (together with an identical second pair for the other $SL(2, R)$ factor of the structure group), which represent the holonomies of a flat connection around two generating cycles of the (abelian) fundamental group of the torus. The usual approach to quantizing this theory, e.g. [3], is to work with gauge-invariant variables, namely the traces of the holonomies, but in [1] we chose instead to use the gauge covariant holonomy matrices themselves as variables. There we also argued that, after quantization, these matrices obey a q-commutation relation

$$U_1 U_2 = q U_2 U_1,$$  \hfill (1)

where $q \to 1$ in the classical limit. Using the gauge covariance to put the matrices in standard form, the simplest solution of (1) is to take both matrices to be diagonal, in which case the diagonal elements of $U_1$ and $U_2$ obey standard q-commutation relations. However, when studying more general solutions with both matrices of upper-triangular form, we found matrices which had non-trivial internal relations, in addition to the “mutual” relations involving elements of both matrices. This feature of having internal relations is characteristic of quantum groups, and indeed the algebraic structure which emerges has many similarities with quantum groups, as we will explain below. The main purpose of this article is to describe this new algebraic structure, and present some examples.

As mathematical objects quantum matrix pairs may be thought of as a simultaneous generalization of two familiar notions of “quantum mathematics”, namely the quantum plane and quantum groups. [6] To make this statement more precise we first recall the quantum plane (over the field $k$), described by two non-commuting coordinates $x$ and $y$, satisfying the relation

$$xy = qyx$$ \hfill (2)

for some invertible $q \in k$, $q \neq 1$. The algebra of polynomial functions on the quantum plane is then given by $k\{x, y\}/(xy - qyx)$, where $k\{x, y\}$ is the free algebra with coefficients in $k$ and $(xy - qyx)$ is the ideal generated by $xy - qyx$. This algebra is a deformation of the algebra of polynomial functions in two commuting variables $k[x, y]$.

\footnote{For background material on noncommutative geometry and quantum groups, see [4], [5], [6].}
Quantum groups may be presented in a similar way. For example, consider a $2 \times 2$ matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with non-commuting entries satisfying the relations (R)

$$ab = qba; \quad ac = qca; \quad ad - da = (q - q^{-1})bc;$$

$$bc = cb; \quad bd = qdb; \quad cd = qdc.$$  

for some invertible $q \in k, q \neq 1$. The algebra of polynomial functions of these entries, denoted $M_q(2)$, is given by the quotient of the free algebra $k\{a, b, c, d\}$ by the ideal generated by the relations (R), and is a deformation of the algebra of polynomial functions in the commuting variables $a, b, c, d$.

The pattern of relations (R) seems somewhat arbitrary at first sight, although there are deep reasons for this particular form. One might also ask why the symbols $a, b, c, d$ are displayed as entries of a matrix, instead of, say, as components of a 4-vector belonging to some non-commutative 4-dimensional space. A very nice, and not widely known, explanation of the $2 \times 2$ matrix form was given by Vokos, Zumino and Wess [7], who showed that the internal relations are preserved under matrix multiplication in the following sense:

1. If $U$ is as above the entries of $U^n$ satisfy the relations (R) with $q$ substituted by $q^n$ for all positive integers $n$.

   This result can be extended to all integers by making a minor modification. The quantum determinant $D_q = ad - qbc$ is central in $M_q(2)$. Formally adjoining a new generator $D^{-1}_q$, which commutes with $a, b, c, d$ and satisfies the relations $(ad - qbc)D^{-1}_q = 1$, gives rise to an algebra, denoted $GL_q(2)$, for which $U$ has the matrix inverse:

$$U^{-1} = D^{-1}_q \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}.$$  

2. If $U$ is as above with entries belonging to $GL_q(2)$, the entries of $U^n$ satisfy the relations (R) with $q$ substituted by $q^n$ for all integers $n$.

Finally suppose that

$$U' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

is a second matrix with entries satisfying the same relations as those of $U$, denoted $(R')$, and commuting with the entries of $U$. Make $U$ and $U'$ invertible by adjoining the generators $D^{-1}_q$ and $(D')^{-1}_q$, as above. Thus the entries of $U$ and $U'$ belong to the
quotient of the free algebra $k \left\{ a, b, c, d, D_q^{-1}, a', b', c', d', (D'_q)^{-1} \right\}$ by the ideal generated by $(R), (R')$, commutativity relations between primed and unprimed generators, commutativity relations between $D_q^{-1}, (D'_q)^{-1}$ and all other generators, and the relations $(ad - qbc)D_q^{-1} = 1$ and $(a'd' - qb'c')(D'_q)^{-1} = 1$.

3. If $U$ and $U'$ are as above, the entries of $U^n U'^m$ satisfy the relations $(R)$ with $q$ replaced by $q^n$ for all integers $n$.

These multiplicative properties of $2 \times 2$ quantum matrices are very striking, and provide the inspiration for the definition of quantum matrix pairs which follows.

Suppose, then, that we have two invertible matrices

$$U_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad U_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

whose entries take values in a non-commutative algebra $\mathcal{A}$ over $k$, and satisfy relations of two different types: internal relations $(I)$, involving the entries of one matrix only and having the same structure for both matrices, and mutual relations $(M)$, involving the entries of both matrices at the same time. Both types of commutation relation may involve $q$, as well as possibly other scalar parameters.

**Definition 1** If the relations $(I)$ and $(M)$ for the matrices $U_1$ and $U_2$ are such that

a) $U_1 U_2 = q U_2 U_1,$

for some invertible $q \in k$, $q \neq 1$, where $q$ acts by scalar multiplication on the right-hand-side, and

b) the entries of $U_1^n U_2^m$ obey internal relations with the same structure $(I)$, for all integers $n$ and $m$, up to possible substitution of scalar parameters,

we call $(U_1, U_2)$ a quantum matrix pair.

If condition a) holds, but condition b) only holds for $U_1^n U_2^m$, for all integers $n$, we call $(U_1, U_2)$ a restricted quantum matrix pair.

Requirement a) is a natural generalization of the quantum plane relation (2), with $x$ and $y$ being replaced by the $2 \times 2$ matrices $U_1$ and $U_2$. Requirement b), in either its restricted or unrestricted form, is analogous to the multiplicative properties of quantum matrices described above.

We will proceed to give examples of quantum matrix pairs, both restricted and unrestricted. These examples, based on our previous work [1], are relatively simple in
that both matrices are upper triangular, but still display novel features not found in diagonal examples. In particular, the internal relations are not standard Weyl-type relations like (2), but involve three matrix entries simultaneously. We expect that quantum matrix pairs for groups other than $SL(2, R)$ can be found and will have an important role to play in the context of Chern-Simons theory.

In any case, quantum matrix pairs constitute an interesting algebraic structure, worthy of study in its own right. The fact that they closely resemble quantum groups could lead to novel insights and perspectives on the latter subject. Here we should point out that there is a similar construction which arises in the context of Majid’s braided matrices [8, Section 10.3]. He gives examples of pairs of $2 \times 2$ matrices with internal and mutual relations between their entries, such that the product of the first matrix with the second obeys the same internal relations (but not the other way round). By comparison the internal relations in our examples seem very robust, since they carry over to the product of the original matrices in either order, as well as to other monomials in the two matrices or their inverses. Furthermore, two different products of the original matrices may themselves form a new quantum matrix pair.

Perhaps the most intriguing feature about our examples is that their geometric origin reemerges from the algebra, when we find an action of the modular group on spaces of quantum matrix pairs. A fuller understanding of the role of quantum matrix pairs in physical models will undoubtedly involve notions of noncommutative non-local geometry.

Our material is organized as follows. In section 2 the three types of example of quantum matrix pairs are given, and their individual features are discussed. In section 3 several ways are described of obtaining new quantum matrix pairs from a given one for the examples of section 2, which then leads to an action of the modular group on spaces of quantum matrix pairs. Section 4 contains some comments.

2 Examples of quantum matrix pairs

As pointed out in the introduction, the holonomy matrices are not gauge invariant. Under gauge transformations they transform by simultaneous conjugation with an element of $SL(2, R)$. Classically this means that the matrices can be simultaneously diagonalized, when they are diagonalizable, but there is also a sector where both matrices are (upper) triangular in form. We will concentrate on the corresponding sector in the quantum theory, since it has much more interesting behaviour than the diagonal case.

Thus from now on we will take both matrices $U_1$ and $U_2$ to be of upper triangular
form and will denote their entries as follows:

\[ U_i = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \gamma_i \end{pmatrix}, \quad i = 1, 2 \]  

(3)

Also, for brevity, we will henceforth adopt the following convention: when the index \( i \) appears in a statement, \( i = 1, 2 \) is understood. Since we require both matrices \( U_i \) to be invertible, we take \( \alpha_i \) and \( \gamma_i \) to be invertible, which is formally achieved by adjoining new generators \( \alpha_i^{-1}, \gamma_i^{-1} \), and corresponding relations, to the free algebra generated by \( \alpha_i, \beta_i \) and \( \gamma_i \). The inverse of \( U_i \) is then given by

\[ U_i^{-1} = \begin{pmatrix} \alpha_i^{-1} & -\alpha_i^{-1}\beta_i \gamma_i^{-1} \\ 0 & \gamma_i^{-1} \end{pmatrix}. \]

(4)

In accordance with the definition of a quantum matrix pair, we must specify internal (I) and mutual (M) relations between the generators. These we will subdivide further into diagonal (D) relations when they involve only diagonal entries, and non-diagonal (ND) relations, when they also involve non-diagonal entries. We will list various options for the relations below, and different combinations of these options will furnish the three types of example we want to present.

For the internal diagonal relations there are two choices, namely

(ID1) \( \alpha_i \gamma_i = \gamma_i \alpha_i = 1 \)

(ID2) \( \alpha_i \gamma_i = \gamma_i \alpha_i \)

The first choice implies \( \gamma_i = \alpha_i^{-1} \), whereas the second merely requires the diagonal entries to commute. For the non-diagonal internal relations there are also two choices:

(IND1) \( \alpha_i \beta_i = \beta_i \gamma_i \),

(IND2) \( \alpha_i \beta_i = r \beta_i \gamma_i \),

(6)  

(7)

where in the second choice \( r \) is an invertible element of \( k \), \( r \neq 1 \). Of course, \( r \) and \( q \) need not be independent. For instance, they may be equal, or \( r \) may be a power of \( q \).

The mutual diagonal relations appear in the following two forms:

(MD1) \( \alpha_1 \alpha_2 = q \alpha_2 \alpha_1 \),

(MD2) \[ \begin{align*}
\alpha_1 \alpha_2 &= q \alpha_2 \alpha_1, \\
\alpha_1 \gamma_2 &= q^{-1} \gamma_2 \alpha_1, \\
\alpha_2 \gamma_1 &= q \gamma_1 \alpha_2, \\
\gamma_1 \gamma_2 &= q \gamma_2 \gamma_1.
\end{align*} \]
where \( q \in k \) is the parameter for the fundamental \( q \)-commutation relation (1). In practice these two choices are the same, since we will always combine (MD1) with (ID1), which implies (MD2), according to Proposition 2 below.

Finally, we restrict ourselves to a single choice of mutual non-diagonal relations:

\[
\begin{align*}
\text{(MND)} & \quad \left\{ \begin{array}{l}
\alpha_1 \beta_2 = q \beta_2 \gamma_1, \\
\beta_1 \gamma_2 = q \alpha_2 \beta_1
\end{array} \right.
\end{align*}
\]

\( (8) \)

Before giving the examples, we need to derive a few results, starting with a simple but important proposition, which underlies all the subsequent calculations.

**Proposition 1** Let \( A \) be an algebra over the field \( k \), and let \( \alpha, \beta \) and \( \gamma \) be elements of \( A \), with \( \alpha \) and \( \gamma \) invertible. Let \( q \) and \( r \) be invertible elements of \( k \). Then

\( a) \ \alpha \gamma = q \gamma \alpha \Rightarrow \alpha^n \gamma^m = q^m \gamma^m \alpha^n, \ \forall n, m \in \mathbb{Z} \)

\( b) \ \alpha \beta = r \beta \gamma \Rightarrow \alpha^n \beta = r^n \beta \gamma^n, \ \forall n \in \mathbb{Z}. \)

**Proof:**

\( a) \) This is trivial for \( n, m \geq 0 \), and follows from the relations \( \alpha^{-1} \gamma = q^{-1} \gamma \alpha^{-1} \), \( \alpha^{-1} \gamma^{-1} = q^{-1} \gamma^{-1} \alpha \) and \( \alpha^{-1} \gamma^{-1} = q \gamma^{-1} \alpha^{-1} \) when \( n \) or \( m \) or both are negative. \( b) \) This is trivial for \( n \geq 0 \), and follows from \( \alpha^{-1} \beta = r^{-1} \beta \gamma^{-1} \) when \( n \) is negative. \( \square \)

An immediate corollary of part \( a) \) is the result mentioned after (2).

**Proposition 2**

\( (ID1) \land (MD1) \Rightarrow (MD2) \)

The first requirement for \( (U_1, U_2) \) to constitute a quantum matrix pair is the relation \( a) \) of the definition. This will be guaranteed in all examples by the mutual relations (MD2) and (MND):

\[
U_1 U_2 = \begin{pmatrix}
\alpha_1 \alpha_2 & \alpha_1 \beta_2 + \beta_1 \gamma_2 \\
0 & \gamma_1 \gamma_2
\end{pmatrix} = q \begin{pmatrix}
\alpha_2 \alpha_1 & \alpha_2 \beta_1 + \beta_2 \gamma_1 \\
0 & \gamma_2 \gamma_1
\end{pmatrix} = q U_2 U_1.
\]

\( (9) \)

To analyse the requirement \( b) \) of the definition we need expressions for powers of the matrices \( U_i \). These depend on the choice of internal non-diagonal relations.

**Proposition 3**

\( a) \ (IND1) \Rightarrow U_i^n \overset{\text{def}}{=} \begin{pmatrix}
\alpha_i(n) & \beta_i(n) \\
0 & \gamma_i(n)
\end{pmatrix} = \begin{pmatrix}
\alpha_i^n & n \beta_i \gamma_i^{n-1} \\
0 & \gamma_i^n
\end{pmatrix}, \ \forall n \in \mathbb{Z} \)

\( b) \ (IND2) \Rightarrow U_i^n \overset{\text{def}}{=} \begin{pmatrix}
\alpha_i(n) & \beta_i(n) \\
0 & \gamma_i(n)
\end{pmatrix} = \begin{pmatrix}
\alpha_i^n & \bar{n} \beta_i \gamma_i^{n-1} \\
0 & \gamma_i^n
\end{pmatrix}, \ \forall n \in \mathbb{Z} \)

where \( \bar{n} = (1 - r^n)/(1 - r). \)
Proof: For \( n \geq 0 \) the formulae are proved by induction. The induction step uses the equalities:

\[
\alpha_i^n \beta_i + n \beta_i \gamma_i^n = (n + 1) \beta_i \gamma_i^n \\
\alpha_i^n \beta_i + \bar{n}_r \beta_i \gamma_i^n = (r^n + \bar{n}_r) \beta_i \gamma_i^n = (n + 1) \beta_i \gamma_i^n
\]

for (IND1) and (IND2) respectively. The final equality is the calculation:

\[
\alpha_i^n \beta_i + \bar{n}_r \beta_i \gamma_i^n = (r^n + \bar{n}_r) \beta_i \gamma_i^n = (n + 1) \beta_i \gamma_i^n
\]

for (IND1) and (IND2) respectively. These equalities follow from Proposition [2].

For negative \( n \), set \( n = -p \) with \( p \) positive. \( U_i^{-1} \) has the internal relations:

\[
\alpha_i(-1) \beta_i(-1) = \alpha_i^{-1}(-\alpha_i^{-1} \beta_i \gamma_i^{-1}) = (-\alpha_i^{-1} \beta_i \gamma_i^{-1}) \gamma_i^{-1} = \beta_i(-1) \gamma_i(-1), \\
\alpha_i(-1) \beta_i(-1) = \alpha_i^{-1}(-\alpha_i^{-1} \beta_i \gamma_i^{-1}) = r^{-1}(-\alpha_i^{-1} \beta_i \gamma_i^{-1}) \gamma_i^{-1} = r^{-1} \beta_i(-1) \gamma_i(-1)
\]

for (IND1) and (IND2) respectively, using (4) and Proposition 1. Since \( U^n = (U^{-1})^p \) one derives:

\[
\beta_i(n) = p \beta_i(-1) \gamma_i(-1)^{p-1} = p(-\alpha_i^{-1} \beta_i \gamma_i^{-1}) \gamma_i(-1)^{-1} = -p \beta_i \gamma_i^{-1} = n \beta_i \gamma_i^{n-1}
\]

and

\[
\beta_i(n) = \bar{p}_{r^{-1}} \beta_i(-1) \gamma_i(-1)^{p-1} = \bar{p}_{r^{-1}}(-\alpha_i^{-1} \beta_i \gamma_i^{-1}) \gamma_i(-1)^{-1} = -r^{-1} \bar{p}_{r^{-1}} \beta_i \gamma_i^{-p-1} = \bar{n}_r \beta_i \gamma_i^{n-1}
\]

for (IND1) and (IND2) respectively. The final equality is the calculation:

\[
-r^{-1} \bar{p}_{r^{-1}} = -r^{-1} \frac{(1 - (r^{-1})^p)}{1 - r^{-1}} = -\frac{(1 - r^{-p})}{r - 1} = \frac{1 - r^n}{1 - r} = \bar{n}_r.
\]

As a corollary we obtain formulae for the entries of \( U_1^n U_2^m \) for \( n, m \in \mathbb{Z} \). We remark that, in view of the \( q \)-commutation relation (4) and Proposition 1, any word in \( U_1 \), \( U_2 \) and their inverses is proportional to \( U_1^n U_2^m \) for some \( n, m \in \mathbb{Z} \). Setting

\[
U_1^n U_2^m = \begin{pmatrix} \alpha(n, m) & \beta(n, m) \\ 0 & \gamma(n, m) \end{pmatrix},
\]

we obtain the following formulae:

\[
\alpha(n, m) = \alpha_1^n \alpha_2^m
\]

\[
\gamma(n, m) = \gamma_1^n \gamma_2^m
\]

(IND1) \( \implies \) \( \beta(n, m) = m \alpha_1^n \beta_2 \gamma_2^{m-1} + n \beta_1 \gamma_1^{n-1} \gamma_2^m \)

(IND2) \( \implies \) \( \beta(n, m) = \bar{m}_r \alpha_1^n \beta_2 \gamma_2^{m-1} + \bar{n}_r \beta_1 \gamma_1^{n-1} \gamma_2^m \)
With these preliminary calculations out of the way, we are in a position to present our three types of example, and prove that they are quantum matrix pairs. We do this in the form of a theorem.

**Theorem 1** Matrix pairs of the form (3), satisfying internal and mutual relations as set out in the table below, give rise to three types of quantum matrix pairs, of which the third is a restricted quantum matrix pair.

| Example | Internal relations | Mutual relations |
|---------|--------------------|------------------|
| Type I  | (ID1), (IND1)      | (MD1), (MND)     |
| Type II | (ID2), (IND1)      | (MD2), (MND)     |
| Type III| (ID2), (IND2)      | (MD2), (MND)     |

For type I, $U_1^n U_2^m$, for all $n, m \in \mathbb{Z}$, has internal relations:

$$
\alpha(n, m) \gamma(n, m) = \gamma(n, m) \alpha(n, m) = q^{nm} \quad (11)
$$

$$
\alpha(n, m) \beta(n, m) = \beta(n, m) \gamma(n, m). \quad (12)
$$

For type II, $U_1^n U_2^m$, for all $n, m \in \mathbb{Z}$, has internal relations:

$$
\alpha(n, m) \gamma(n, m) = \gamma(n, m) \alpha(n, m) \quad (13)
$$

$$
\alpha(n, m) \beta(n, m) = \beta(n, m) \gamma(n, m). \quad (14)
$$

For type III, $U_1^n U_2^n$, for all $n \in \mathbb{Z}$, has internal relations:

$$
\alpha(n, n) \gamma(n, n) = \gamma(n, n) \alpha(n, n) \quad (15)
$$

$$
\alpha(n, n) \beta(n, n) = r^n \beta(n, n) \gamma(n, n). \quad (16)
$$

In each case these internal relations have the same structure as those of the corresponding $U_i$, with 1 in (3) replaced with $q^{nm}$ in (11) for type I, and $r$ in (7) replaced with $r^n$ in (16) for type III. Whilst these three types of example are very similar, each of them exhibits some special feature distinguishing it from the others. In the type I case, both $U_1$ and $U_2$ have determinant 1, but mixed products of the $U_i$ have non-unit determinant, as a result of the non-commutativity of the algebra. In the type II case, despite the noncommutativity, the property of having commuting diagonal entries propagates to all products of the $U_i$. Finally, the type III case has internal relations involving a parameter, a feature which is reminiscent of quantum groups. We remark that this parameter cannot simply be removed by a rescaling $\alpha_i \rightarrow r^{1/2} \alpha_i$, $\gamma_i \rightarrow r^{-1/2} \gamma_i$, since the (MND) relations are not preserved under these replacements. The parameter propagates to powers of $U_i$ in a manner again reminiscent of quantum groups (cf. the second Vokos et al result in the introduction).
Theorem 1: Because of Proposition 2 and equation (8), all three types satisfy requirement a) of the definition of a quantum matrix pair. All that remains is to show that the relations (11)-(16) hold. Equations (11), (13) and (15) follow from Proposition 1, as well as Proposition 2 for the type I case. For (16), the calculation is slightly modified due to the presence of the parameter $r$ in the (IND2) relation:

$$(\alpha_1^n \alpha_2^m) (\alpha_1^n \beta_2 \gamma_2^{m-1} + \beta_1 \gamma_1^{n-1} \gamma_2^n)$$

$$= q^{-n^2} \alpha_1^n \alpha_2^m \gamma_2^{n-1} + r^n q^{-n} \beta_1 \gamma_1^{n-1} \gamma_2^n$$

$$= r^n (\alpha_1^n \beta_2 \gamma_2^{n-1} + \beta_1 \gamma_1^{n-1} \gamma_2^n)$$

where we have omitted the factor $n$ in $\beta(n, n)$.

3 Generating new quantum matrix pairs

The examples of the previous section showed how the internal relations are preserved under multiplication of the matrices belonging to a quantum matrix pair. However there is another aspect to these examples. By taking two different products of the $U_i$, in some circumstances it is possible to generate a new quantum matrix pair of the same or a similar type, as we will see in this section. Furthermore the transformations amongst quantum matrix pairs may preserve the type, so that we can regard them as acting on the space of all quantum matrix pairs of a certain type. We show how this can give rise to representations of a discrete group, namely $SL(2, \mathbb{Z})$ (the modular group), on the space of quantum matrix pairs of type I and type II.

We start with a trivial first result in this direction.
Proposition 4 Let \((U_1, U_2)\) be a quantum matrix pair of any of the three types described in the previous section. Then the pair \((\check{U}_1, \check{U}_2)\) with entries \(\check{\alpha}_i = \alpha_i, \check{\beta}_i = \gamma_i\) and \(\check{\gamma}_i = c_i\beta_i\), where \(c_i \in k\) are arbitrary constants, is a new quantum matrix pair of the same type as the original pair.

Proof: The non-diagonal relations (3), (7) and (8) are linear in \(\beta_1\) or \(\beta_2\). \(\square\)

The main result of this section is stated in the following theorem:

Theorem 2

a) Let \((U_1, U_2)\) be a quantum matrix pair of type I. Then
\[
(q^{-nm/2}U_1^nU_2^m, q^{-st/2}U_1^sU_2^t)\] is a quantum matrix pair of type I, with \(q\) replaced with \(q^{nt-ms}\), for all \(n, m, s, t \in \mathbb{Z}\).

b) Let \((U_1, U_2)\) be a quantum matrix pair of type II. Then \((U_1^nU_2^m, U_1^sU_2^t)\) is a quantum matrix pair of type II, with \(q\) replaced with \(q^{nt-ms}\), for all \(n, m, s, t \in \mathbb{Z}\).

c) Let \((U_1, U_2)\) be a quantum matrix pair of type III. Then \((U_1^n, U_2^m)\) is a quantum matrix pair of type III, with \(q\) replaced with \(q^n\) and \(r\) replaced with \(r^m\), for all \(n \in \mathbb{Z}\).

Proof: (The statements in the proof hold for all \(n, m, s, t \in \mathbb{Z}\).) First we prove the internal relations. For a), the (ID1) relations follow from (11), since this equation implies that \(q^{-nm/2}\alpha(n, m)\) and \(q^{-nm/2}\gamma(n, m)\) are each other’s inverses. The (IND1) relations follow from (12), since all matrix entries are multiplied by the same factor. For b), the internal relations are equations (13), (14) of the previous section. For c), using the notation and result of Proposition 3 b), \(a_i\gamma_i = \gamma_i\alpha_i\) (ID2) for \(U_i\) implies \(\alpha_i(n)\gamma_i(n) = \gamma_i(n)\alpha_i(n)\), and \(\alpha_i\beta_i = r\beta_i\gamma_i\) (IND2) for \(U_i\), together with Proposition 4 implies \(\alpha_i(n)\beta_i(n) = r^n\beta_i(n)\gamma_i(n)\).

To simplify the proof of the mutual relations, we use the relation
\[
(U_1^nU_2^m)(U_1^sU_2^t) = q^{nt-ms}(U_1^nU_2^m)(U_1^sU_2^t),
\] which follows from the \(q\)-commutation relation (10) satisfied by \(U_i\) and Proposition 4. This implies the equations
\[
\alpha(n, m)\alpha(s, t) = q^{nt-ms}\alpha(s, t)\alpha(n, m) \quad (17)
\]
\[
\gamma(n, m)\gamma(s, t) = q^{nt-ms}\gamma(s, t)\gamma(n, m) \quad (18)
\]
\[
\alpha(n, m)\beta(s, t) + \beta(n, m)\gamma(s, t) = q^{nt-ms}(\alpha(s, t)\beta(n, m) + \beta(s, t)\gamma(n, m)) \quad (19)
\] where we are using the notation of (10).

Now, starting with b), the first and fourth (MD2) relations, with \(q\) replaced with \(q^{nt-ms}\), are equations (17) and (18), and the second and third (MD2) relations
\[
\alpha(n, m)\gamma(s, t) = q^{-(nt-ms)}\gamma(s, t)\alpha(n, m)
\]
\[
\alpha(s, t)\gamma(n, m) = q^{nt-ms}\gamma(n, m)\alpha(s, t)
\]
follow from the second and third (MD2) relations for $U$, and Proposition II. In view of (19), it is enough to show the first of the (MND) relations:

$$
\alpha(n,m)\beta(s,t) = (\alpha_1^n \alpha_2^m)(\alpha_1^s \alpha_2^t \gamma_2^{t-1} + s \beta_1 \gamma_1^{s-1} \gamma_2^t)
$$

For a), the (MD1) relation follows from (17), and the (MND) relation is proved as follows. First:

$$
\alpha_1(n)\beta_2(n) = \alpha_1^n \bar{n}_r \beta_2 \gamma_2^{n-1} = \bar{n}_r q^n \beta_2 \gamma_1^n \gamma_2^{n-1} = \bar{n}_r q^n \beta_2 \gamma_2^{n-1} \gamma_1^n = q^n \beta_2(n) \gamma_1(n).
$$

The modular group $SL(2,\mathbb{Z})$ has a presentation in terms of two generators $S$ and $T$, with relations $S^4 = (ST)^3 = 1$. We have a representation of $SL(2,\mathbb{Z})$, if we can find automorphisms, $S$ and $T$, of a space $X$, which satisfy these relations. There are natural $SL(2,\mathbb{Z})$ representations associated with spaces of quantum matrix pairs, as the following theorem shows.

**Theorem 3** Let QMP1 and QMP2 be the spaces of all quantum matrix pairs of type I and II respectively, with entries in an algebra $\mathcal{A}$, and with a fixed parameter $q$. Then the following definitions give rise to representations of $SL(2,\mathbb{Z})$ on QMP1 and QMP2:

**QMP1**: $S(U_1, U_2) = (U_2, U_1^{-1})$, $T(U_1, U_2) = (q^{-1/2} U_1 U_2, U_2)$

**QMP2**: $S(U_1, U_2) = (U_2, U_1^{-1})$, $T(U_1, U_2) = (U_1 U_2, U_2)$

**Proof**: From a) and b) of the previous theorem, the transformations $S$ and $T$ map type I or II quantum matrix pairs into quantum matrix pairs of the same type, and with the same $q$ parameter. $S^2(U_1, U_2) = S(U_2, U_1^{-1}) = (U_1^{-1}, U_2^{-1})$, and thus $S^4(U_1, U_2) = (U_1, U_2)$. For the type I case, the second relation $(ST)^3 = 1$ is proved as follows. First:

$$(ST)(U_1, U_2) = S(q^{-1/2} U_1 U_2, U_2) = (U_2, q^{1/2} U_2^{-1} U_1^{-1}).$$

\(^2\)For a study of the modular group in the context of $(2+1)$ quantum gravity by Carlip and one of the authors, see [3].
Thus

\[
(ST)^3(U_1, U_2) = (ST)^2(U_2, q^{1/2} U_2^{-1} U_1^{-1})
\]

\[
= (ST)(q^{1/2} U_2^{-1} U_1^{-1}, q^{1/2} (q^{-1/2} U_1 U_2) U_2^{-1})
\]

\[
= (ST)(q^{1/2} U_2^{-1} U_1^{-1}, U_1)
\]

\[
= (U_1, q^{1/2} U_1^{-1} (q^{-1/2} U_1 U_2)) = (U_1, U_2).
\]

For the type II case, set \(q = 1\) in this calculation. 

\[\square\]

4 Final comments

Quantum matrix pairs combine the preservation of internal relations under multiplication, a quantum-group-like feature, with the fundamental \(q\)-commutation relation which holds between the two matrices. We have presented three types of example of this construction, all involving upper-triangular matrices, but with slightly differing features.

It is interesting to make some comparisons between quantum matrix pairs and quantum groups. The internal relations in our examples of quantum matrix pairs differ in structure from the Weyl-type \(q\)-commutation relations normally found in quantum groups, as they involve three matrix elements at the same time. Related to this is the fact that the entries of each matrix do not commute in the limit \(q \to 1\), which also distinguishes them from Majid’s braided matrices \[8\]. Nonetheless, when the internal relations depend on a parameter, as in the third type of example, quantum integers with that parameter appear in the powers of the matrices, which is a feature strongly reminiscent of quantum groups.

An obvious question for further study is to see whether other examples can be found, e.g. \(2 \times 2\) matrices but not of triangular form, or examples involving other groups.

According to theorem \[4\], not only do products of powers of the matrices have the same structure of internal relations, but taking two different products gives rise to new quantum matrix pairs of the same type. This shows that, in a sense, it is the whole quantum matrix pair structure, rather than just the internal relations, which is preserved under multiplication in these examples.

It is striking that the action of the modular group on pairs of commuting matrices extends to quantum matrix pairs. This reveals that the construction, which could be taken on a purely algebraic level, actually has a geometric interpretation as well. In future work we hope to arrive at a deeper understanding of quantum matrix pairs in terms of non-local non-commutative geometry.
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