Strain-induced kinetics of intergrain defects as the mechanism of slow dynamics in the nonlinear resonant response of humid sandstone bars

Oleksiy O. Vakhnenko 1, Vyacheslav O. Vakhnenko 2, Thomas J. Shankland 3, and James A. Ten Cate 3
1 Bogolyubov Institute for Theoretical Physics, 14-B Metrologichna Str., Kyiv 03143, Ukraine
2 Institute of Geophysics, 63-B Bohdan Khmel’nyts’kyi Str., Kyiv 01054, Ukraine
3 Los Alamos National Laboratory, Earth and Environment Sciences Division, Los Alamos, New Mexico 87545

A closed-form description is proposed to explain nonlinear and slow dynamics effects exhibited by sandstone bars in longitudinal resonance experiments. Along with the fast subsystem of longitudinal nonlinear displacements we examine the strain-dependent slow subsystem of broken intergrain and inter-lamina cohesive bonds. We show that even the simplest but phenomenologically correct modelling of their mutual feedback elucidates the main experimental findings typical for forced longitudinal oscillations of sandstone bars, namely, (i) hysteretic behavior of a resonance curve on both its up- and down-slopes, (ii) linear softening of resonant frequency with increase of driving level, and (iii) gradual recovery (increase) of resonant frequency at low dynamical strains after the sample was conditioned by high strains. In order to reproduce the highly nonlinear elastic features of sandstone grained structure a realistic non-perturbative form of strain potential energy was adopted. In our theory slow dynamics associated with the experimentally observed memory of peak strain history is attributed to strain-induced kinetic changes in concentration of ruptured inter-grain and inter-lamina cohesive bonds causing a net hysteretic effect on the elastic Young’s modulus. Finally, we explain how enhancement of hysteretic phenomena originates from an increase in equilibrium concentration of ruptured cohesive bonds that are due to water saturation.

PACS numbers: 05.45.-a, 62.40.+i, 83.80.Fg, 46.05.+b

Apart from their excellent static characteristics as building materials, sandstones have been shown to demonstrate a number of unexpected and even surprising dynamical properties [1–5]. Here we consider the numerous experimental results on nonlinear resonant response exhibited by sandstone rods in forced longitudinal oscillations that appear even at extremely small forcing levels and consequently at small dynamic strains [1–5]. The most intriguing nonlinear feature is slow dynamics, which is defined here as long-term (minutes to hours) change of elastic properties in response to a disturbance such as dynamic and static strain or temperature.

Specifically, we have to underline that in the vicinity of bar resonant frequency the longitudinal alternating drive produces strong essentially non-trivial nonlinear responses: 1) At high drive levels the effective width of resonance curves depends on the direction of frequency sweep; it is narrower for upward sweeps (i.e., from lower to higher frequencies) than at downward sweeps (i.e., from higher to lower frequencies) [1–5]. This effect proves to be a typical manifestation of slow dynamics and can be treated as hysteresis both on low- and high-frequency slopes of a resonance curve. 2) The resonance peak is shifted toward lower frequency almost linearly with increase of driving amplitude [1, 4]. 3) Other evidence of slow dynamics comprises gradual recovery (increase) of resonant frequency to its original value as defined at extremely low drive level after the sample has been conditioned at high drive level [3,5].

These facts cannot be understood in the framework of standard theories of resonant nonlinear response [6] and imply memory of peak strain history [2]. Some aspects of the problem have been explained by the interpretation of Guyer, McCall and Van Den Abeele [7] in the framework of a McCall-Guyer quasistatic model [8]. This approach uses the concept of auxiliary hysteretic elements that allow the introduction of an additional nontrivial nonlinear term into the dynamical equation for the field of longitudinal displacements. However, this theoretical treatment lacks completeness in that it initially neglects the dynamics of hysteretic elements and postulates temporal evolution of amplitude-frequency characteristic (the key point of claimed results) to be developed afterwards. Although Capogrosso-Sansone and Guyer recently suggested a dynamical realization of the McCall-Guyer quasistatic model [9], evaluating its adequacy to explain experimental data turns out to be difficult.

In this communication we omit the idea of auxiliary hysteretic elements as the sole approach for treating all peculiar hysteretic phenomena and call attention to an alternative notion used by Davydov and Ermakov for the description of bistability in nonlinear resonant tunneling of electrons through a set of potential barriers [10]. Their approach consists of explicit but physically motivated separation of given physical system into two nonlinear subsystems, namely fast and slow subsystems with mutual coupling taken into account.

For sandstone bars we identify the fast subsystem with the field of rapid longitudinal displacements while the slow subsystem represents the concentration of defects in intergrain contact bonds. In doing this we bear in mind that, because of preferable vapor condensation onto surfaces with greater concave curvature [11], the sandstone pore structure [4, 11] retains some residual pore water [11], and its impact on the resonant properties of rock is crucial [12, 13]. Thus, thermodynamical estimations ap-
plied to porous rocks show that intergrain cohesive forces become weaker in the presence of water [14], that agrees with an alternative conception of swelling pressure [13, 15]. This treatment is supported by recent experiments [13] establishing an abrupt decrease in Young’s modulus within the first twenty percent interval of water saturation (i.e., until the degree that void surfaces become completely wet). We could additionally invoke ordinary capillary forces [13] or hydrolysis of silicon-oxygen-silicon bond-chains [16] in our consideration. However, either of these mechanisms also leads to softening of Young’s modulus with saturation increase. Here the significant issue is apparently not in excessive (presumably unclaimed) detailing all plausible mechanisms that might modify the Young’s modulus in a qualitatively similar way, but in their reasonable concise formalization by means of a minimal number of slow fields.

According to Kosevich [17] the equilibrium concentration of defects associated with a stress \( \sigma \) is given by the formula

\[
c_c = c_0 \exp(\nu \sigma / kT),
\]

where \( k \) and \( T \) are the Boltzmann constant and the absolute temperature, respectively, and the parameter \( \nu > 0 \) stands for a typical volume accounting for a single defect and characterizes the intensity of dilatation. The equilibrium concentration of defects in an unstrained bar \( c_0 \) has to be some function of both temperature \( T \) and water saturation \( s \). In order to describe strain-induced changes in nonequilibrium concentrations of defects \( c \), we assume that at any instant of time \( t \) the concentration \( c \) must evolve to its would-be equilibrium value \( c_\sigma \), where the stress \( \sigma \) in (1) is applied at the same instant. Supposing the distributions of activation barriers for defect annihilation \( U \) and activation barriers for defect creation \( W \) to be uniform respectively over the ranges \( U_0 \leq U \leq U_0 + U_+ \) and \( W_0 \leq W \leq W_0 + W_+ \) with \( U_0, U_+ \) and \( W_0, W_+ \) being insensitive to the choice of bar cross-section, we will deal with the density of defect concentration \( g \) governed by the following kinetic equation

\[
\frac{\partial g}{\partial t} = -[\mu \theta(g - g_\sigma) + \nu \theta(g_\sigma - g)](g - g_\sigma). \tag{2}
\]

Here \( \mu = \mu_0 \exp(-U/kT) \) and \( \nu = \nu_0 \exp(-W/kT) \) are the rates of defect annihilation and defect creation respectively, \( g_\sigma = c_\sigma / U_+ W_+ \), and \( \theta(z) \) designates the Heaviside step-function. The quantities \( g \) and \( c \) are related by the simple definition

\[
c = \int_{U_0}^{U_0 + U_+} dU \int_{W_0}^{W_0 + W_+} dW \cdot g. \tag{3}
\]

Under the tensile load there is an immense number of spatial ways for the intergrain cementation contact to be cleaved with the same basic result: the creation of crack. The similar scenario is true also for the already existed balanced crack to be further expanded. On the contrary under the compressive load the crack ones emerged has only one spatial way to be annihilated or contracted. These observations are the principal ones and imply the huge disparity \( \nu_0 \gg \mu_0 \) between the priming rates \( \nu_0 \) and \( \mu_0 \) notwithstanding the generic cohesive properties of cementation material. Moreover, because of possible fragmentation of cementation material and/or water intercalation between the opposite faces of crack we can expect the typical value of barrier \( U \) to exceed that of barrier \( W \). In combination all these factors might sustain predominantly even the more immense disparity \( \nu \gg \mu \) between the actual rates \( \nu \) and \( \mu \) of defect creation and defect annihilation, apparently comprising many orders, and as a result provide the physical mechanism that breaks the symmetry of system response to an alternating external drive and acts as a sort of soft ratchet or leaky diode.

To express the evolution equation

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} + \frac{\partial}{\partial x} \left[ \frac{\partial F}{\partial (\partial^2 u / \partial x \partial t)} \right] \tag{4}
\]

for the field of longitudinal displacements \( u \) we choose the stress-strain relation in the form

\[
\sigma = \frac{E \text{sech} \eta}{(r - a)[\cosh \eta \partial u / \partial x + 1]^{v+1}} - \frac{E \text{sech} \eta}{(r - a)[\cosh \eta \partial u / \partial x + 1]^{\gamma+1}} \tag{5}
\]

which at \( r > a > 0 \) allows one to block the bar compressibility at strains \( \partial u / \partial x \) tending to \( +0 - \text{sech} \eta \). To put it differently the parameter \( \text{sech} \eta \) is reserved for the typical thickness of intergrain cementation contact divided by the typical distance between the centers of neighbouring grains. The dissipative function \( F \) we take in the form

\[
F = (\gamma / 2) \left[ \frac{\partial^2 u}{\partial x \partial t} \right]^2 \tag{6}
\]

giving rise to Stokes internal friction [18]. Here \( x \) denotes the longitudinal Lagrange coordinate of the bar sample. The quantities \( \rho \) and \( \gamma \) are respectively the mean density of sandstone and the coefficient of internal friction in an elastic subsystem. We ignore their dependences on temperature and water saturation assuming that the main effect is manifested through the linear decrease of Young’s modulus \( E \) with the concentration of defects

\[
E = (1 - c / c_cr) E_+. \tag{7}
\]

Here \( c_cr \) and \( E_+ \) are the critical concentration of defects and the maximal possible value of Young’s modulus, respectively. Both of these parameters we also take to be independent of temperature and water saturation.

Typical resonant response experiments [1–5] correspond to the kinematic excitation [19] of a bar sample, which we associate with the following boundary conditions

\[
u(x = 0|t) = D(t) \cos \left( \varphi + \int_0^t d\tau \omega(\tau) \right) \tag{8}
\]

\[
\frac{\partial u}{\partial x}(x = L|t) = 0 \tag{9}
\]
where \( L \) is the sample length and \( t > 0 \). The driving amplitude \( D(t) \) is assumed to be basically constant except for the moments when the driving system is switched on, is switched into another constant driving level, or is switched off. The time dependence of cyclic driving frequency \( \omega(t) \) in turn is prescribed by the chosen regime of frequency sweep. Initial conditions are given in the form

\[
u(x|t = 0) = 0, \quad \frac{\partial u}{\partial t}(x|t = 0) = 0 \quad (0 < x < L) \quad (10)
\]

\[
g(x|t = 0) = c_0/U_+W_+ \quad (0 < x < L). \quad (11)
\]

When experimental data for Young’s modulus in unstrained samples are obtainable from the resonant response experiments by the use of a low amplitude protocol (driving amplitude \( D \) very small and negligible strain-induced defects), we can compare them with values taken from equation (7) at \( c = c_0 \) in order to fit the equilibrium concentration of defects \( c_0 \) as a function of \( T \) and \( s \) by some extrapolation formula. In particular, relying upon the Sutherland temperature extrapolation [20] and analyzing temperature dependent data at zero saturation [21] and saturation dependent data at room temperature [13] for Berea sandstone, we suggest the formula

\[
c_0 = c_{cr} \left( \frac{T}{T_{cr}} \right)^2 \left[ \cosh^2 \alpha - \exp \left( -\frac{\beta s}{1 - s} \right) \sinh^2 \alpha \right] \quad (12)
\]

with the following fitting parameters \( T_{cr} = 1475 \) K, \( \cosh^2 \alpha = 16 \), \( \beta = 10 \). Here the saturation can vary within the interval \( 0 \leq s \leq 1 \). At \( s \neq 0 \) this approximation is expected to work at least for temperatures exceeding the freezing-point of pore water.

Computer modelling of nonlinear and slow dynamics effects was performed in the vicinity of the resonant frequency \( f_0(2) \), which we understand as the second frequency \( f = 2 \omega \) in the fundamental set

\[
f_0(l) = \frac{2l - 1}{4L} \sqrt{\frac{1 - c_0}{c_{cr}}} \frac{E_+}{\rho} \quad (l = 1, 2, 3, \ldots) \quad (13)
\]

given by the linear theory of kinematic excitation for zero dissipation \( \gamma = 0 \).

Figure 1 shows typical resonance curves, i.e. dependencies of response amplitude \( R \) (calculated at \( x = L \)) on drive frequency \( f = \omega/2\pi \), at successively higher drive amplitudes \( D \). The continuous lines correspond to the conditioned resonance curves calculated after two frequency sweeps were performed at each driving level in order to achieve repeatable hysteretic curves. The dotted line illustrates an unconditioned curve obtained without any preliminary conditioning. Arrows on the three highest curves indicate sweep directions. For the sake of definiteness the results of the computer simulation were adapted to the experimental conditions for the data obtained by Ten Cate and Shankland for Berea sandstone [2]. In particular, the ratio \( E_+/\rho \) was estimated by means of relationships (13) and (12) with the second order frequency, bar length, temperature and saturation as follows: \( f_0(2) = 3920\text{Hz}, \) \( L = 0.3 \text{m}, \) \( T = 297 \) K and \( s = 0.25 \).

The ratio \( \gamma/\rho \) characterizing internal friction was chosen from the best fit to low amplitude theoretical (Figure 1) and experimental [2] resonance curves using the quality factor \( Q \) from resonance lineshape. The parameters \( \mu_0 \exp(-U_0/kT) = 1s^{-1} \) and \( U_+ = 2925 \) K determining the character of slow relaxation were estimated according to the experimental measurements of decay of acceleration at fixed frequency [2] and the observations of recovering resonant frequency as a function of time [5]. The combination of parameters \( vE_+/k\cosh \eta = 275 \) K was chosen to quantitatively reproduce the hysteretic phenomena in the sweep regimes typical for the actual experiments [2]. The nonlinearity parameter \( c_0 \) was estimated to map the true asymmetry of experimental resonance curves [2]. Other parameters appearing in the stress-strain relation (5) have been adopted as follows: \( r = 4, \) \( a = 2 \).

We would like to stress that through the drop of equilibrium concentration \( c_0 \) our theory is able to catch the drastic suppression of hysteresis with decrease of water saturation (see Eq. (12)). This conclusion has been confirmed by direct computation (not shown). Simultaneously we have observed a monotonic increase in quality factor \( Q \) with saturation decrease, i.e., precisely the well documented tendency in experiments [12].

Figure 2 compares the shifts of resonant frequency as functions of driving amplitude at two essentially different values of dilatation parameter \( \nu \) while other parameters were kept the same as for Figure 1. Thus curve 1 calculated at \( vE_+/k\cosh \eta = 275 \) K, when the strain-induced feedback between the slow and the fast subsystems is substantial, demonstrates the almost linear dependence typical for materials with nonclassical nonlinear response,
i.e., materials possessing all the basic features of slow dynamics. On the other hand, curve 2 calculated at \(v = 0\), when the strain-induced excitation of slow subsystem is absent and hence the mutual feedback between the slow and the fast subsystems is totally broken, demonstrates the almost quadratic dependence typical for the materials with classical nonlinear response.

![Image](image.png)

**FIG. 2.** The shift \(f_r - f_0\) of resonant frequency \(f_r\) from its asymptotic value \(f_0\) as the function of driving amplitude \(D\) for the hysteretic nonlinear material (curve 1) and for the classical nonlinear material with \(v = 0\) (curve 2).

Finally, Figure 3 shows the gradual recovery of resonant frequency \(f_r\) to its maximum limiting value \(f_0\) after the bar has been subjected to high amplitude conditioning and conditioning is stopped. In the computer simulation we have plotted three different curves corresponding to three different saturations with all other model parameters adopted earlier for Figure 1 being preserved. The total shift of resonant frequency \(f_r - f_0\) consists of two physically different parts, namely (i) the traditional dynamic shift caused by strain nonlinearity at high levels of excitation and (ii) the shift caused by the effect of the slow subsystem. However, only the second part might actually be registered during the recovery process, because the first one vanishes almost instantaneously when the conditioning was switched off. Hence, the whole character of recovery should inevitably be governed by the slow kinetics responsible for restoration of intergrain bonds [5]. From Figure 3 we clearly see that the suggested kinetic equation for the density of defect concentration (2) supplemented by the simple definition of total concentration (3) and the reasonable relationship between Young’s modulus and a concentration of defects (7) yields the very wide time interval \(10 \leq (t - t_c)/t_0 \leq 1000\) of logarithmic recovery of the resonant frequency in complete agreement with experimental results [5]. Here \(t_c\) is the moment when conditioning switches off and \(t_0=1\) s.

This work was carried out within the framework of project No 1747 supported by the STCU. O.O.V. acknowledges support from the National Academy of Sciences of Ukraine (Grant 0102U002332). We would like to thank Paul A. Johnson for his unflagging interest to the process of multi-stage amendments of the model.

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