Deferred statistical order convergence in Riesz spaces

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Abstract

Some types of statistical convergence such as statistical order and deferred statistical convergences have been studied and investigated in Riesz spaces, recently. In this paper, we introduce the concept of deferred statistical convergence in Riesz spaces with order convergence. Moreover, we give some relations between deferred statistical order convergence and other kinds of statistical convergences.

Keywords: deferred statistical convergence, order convergence, deferred statistical order convergence, Riesz space

1 Introduction and Preliminaries

Statistical convergence is a generalization of the ordinary convergence of a real or complex sequence. It was introduced by Steinhaus in [18]. Maddox discussed the statistical convergence in more general abstract spaces such as locally convex spaces in [15]. Küçükaslan and Yılmaztürk introduced and investigated the deferred statistical convergence in [12]. It is enough to mention the theory of statistical convergence (cf. [11, 11, 11, 15, 15]). On the other hand, Riesz space (or, vector lattice) is another concept of functional analysis that was introduced by Riesz [16]. Then, many authors developed the subject. Riesz space is an ordered vector space that has many applications in measure theory, Banach space, operator theory, and applications in economics (cf. [2, 3, 4, 14, 20]). The present paper aims to combine the concepts of deferred statistical convergence of real sequences and order convergence in Riesz spaces.
A real-valued vector space $E$ with an order relation is said to be ordered vector space if, for each $x, y \in E$ with $x \leq y$, we have $x + z \leq y + z$ and $\alpha x \leq \alpha y$ for all $z \in E$ and $\alpha \in \mathbb{R}_+$. An ordered vector space $E$ is called Riesz space or vector lattice if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf \{x, y\} \quad \text{and} \quad x \vee y = \sup \{x, y\}$$

exist in $E$, respectively. For an element $x$ in a vector lattice $E$, the positive part, the negative part, and module of $x$ are respectively defined as follows:

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x).$$

Thus, in the present paper, the vertical bar $|\cdot|$ of elements in vector lattices will stand for the module of the given elements. A subset $A$ of a vector lattice $E$ is called solid if, for each $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. A solid vector subspace of a vector lattice is referred to as an ideal. A vector lattice is called $\sigma$-order complete if every nonempty bounded above countable subset has a supremum (or, equivalently, whenever every nonempty bounded below countable subset has an infimum) (cf. [2]).

A sequence $(x_n)$ in a Riesz space $E$ is said to be increasing whenever $x_1 \leq x_2 \leq \cdots$ and is decreasing if $x_1 \geq x_2 \geq \cdots$ holds. Then, we denote them by $x_n \uparrow$ and $x_n \downarrow$, respectively. Moreover, if $x_n \uparrow$ and $\sup x_n = x$, then we write $x_n \uparrow x$. Similarly, if $x_n \downarrow$ and $\inf x_n = x$, then we write $x_n \downarrow x$. Then, we call that $(x_n)$ is increasing or decreasing as monotonic.

On the other hand, order convergence is crucial for this paper, and so, we continue with its definition.

**Definition 1.1.** Let $(x_n)$ be a sequence in a vector lattice $E$. Then, it is called order convergent to $x \in E$ if there exists another sequence $y_n \downarrow 0$ (i.e., $\inf y_n = 0$ and $y_n \downarrow$) such that $|x_n - x| \leq y_n$ holds for all $n \in \mathbb{N}$, and abbreviated it as $x_n \rightarrow^{o} x$.

For the definition of statistical convergence, the important point is the natural density of subsets of natural numbers. Recall that the density of a subset $K$ of $\mathbb{N}$ is the limit $\lim_{n \to \infty} \frac{1}{n} \{|k \leq n : k \in K\}$ whenever this unique limit exists, and it is mostly abbreviated by $\delta(K)$, where $|\{k \leq n : k \in K\}|$ is the cardinality of $K$, and it does not exceed $n$. A sequence $(x_n)$ of real numbers is called statistical convergent to a real number $l$ if, for every $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \frac{1}{n} \{|k : n \geq k, \ |x_n - l| > \varepsilon\} = 0.$$

Now, consider a sequence $x := (x_k)$ and take the sequences $(p_n)$ and $(q_n)$ of non-negative integers such that $p_n < q_n$ for each $n$ and $q_n$ divergent to infinity. Then, define a new sequence

$$(D_{p,q}x)_n := \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k,$$

for each $n \in \mathbb{N}$. The sequence $(D_{p,q}x)_n$ is called the deferred Cesáro mean as a generalization of Cesáro mean of real (or complex) valued sequence;
see [1]. On the other hand, $x$ is said to be strong $D_{p,q}$-convergent to $l$ if the following limit exists

$$
\lim_{n \to \infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} |x_k - l| = 0.
$$

Then, we abbreviate it as $x_k \xrightarrow{D_{p,q}} l$. In this article, unless otherwise, when we mention $p$ and $q$ sequences, they always hold the above properties, and also, these properties are said to be the deferred property.

A sequence $x := (x_k)$ is called deferred statistical convergent to $l \in \mathbb{R}$ whenever, for all $\varepsilon > 0$, we have

$$
\lim_{n \to \infty} \frac{1}{q_n - p_n} \left| \left\{ p_n < k \leq q_n : |x_k - l| \geq \varepsilon \right\} \right| = 0
$$

holds; see [12]. In this case, we write $x_k \xrightarrow{DS_{p,q}} l$.

A characterization of statistical convergence on vector lattices was introduced by Erkan in [9], and also, some kinds of statistical convergence in Riesz spaces were introduced and studied by Aydin in [3, 5, 6, 7, 8].

**Definition 1.2.** Let $(x_n)$ be a sequence in a Riesz space $E$. Then, $(x_n)$ is called

- statistical order decreasing to 0 if there exists a set $K = \{k_1 < k_2 < \cdots \} \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that $(x_{k_n})$ is decreasing and $\inf_{n \in K} (x_{k_n}) = 0$, i.e., $(x_{k_n})_{k_n \in K} \downarrow 0$, and it is abbreviated as $x_n \xrightarrow{\text{st}} 0$;

- statistical order convergent to $x \in E$ if there exists a sequence $q_n \xrightarrow{\text{st}} 0$ with an index set $K = \{k_1 < k_2 < \cdots \} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and

$$
|x_{k_n} - x| \leq q_{k_n}
$$

for every $k_n \in K$, and so, we write $x_n \xrightarrow{\text{st}} x$.

It is clear that every order convergent sequence is statistical order convergent to the same point.

# 2 Deferred statistical decreasing

Tripathy [19] introduced the statistical monotonicity for real sequences, and also, statistically monotone sequences in Riesz spaces were investigated. We extend it to deferred statistical decreasing in Riesz spaces.

**Definition 2.1.** Let $(p_n)$ and $(q_n)$ be sequences of nonnegative integers satisfying the deferred property. Then, a sequence $(z_n)$ in a Riesz space $E$ is called deferred statistical order decreasing to 0 if there exists a set $K \subseteq \mathbb{N}$ such that the deferred density of $K$

$$
\delta_{p,q}(K) := \lim_{n \to \infty} \frac{1}{q_n - p_n} \left| \left\{ p_n < k \leq q_n : k \in K \right\} \right| = 1
$$

and $(z_{k_n})_{k_n \in K} \downarrow 0$ holds on $K$. Then, we abbreviate it as $z_n \xrightarrow{D_{p,q}} 0$. 3
Remark 2.2.

(i) If $q(n) = n$ and $p(n) = 0$, then Definition 2.7 coincides with the definition of statistical order decreasing.

(ii) If $(z_n)$ is monotone decreasing to zero, then it is deferred statistical order decreasing to zero. But, the converse does not need to be true in general. To see this, consider the Euclidean space $\mathbb{R}^2$ with the coordinatewise ordering and the sequences $q(n) = n$ and $p(n) = 0$ and $(z_n)$ denoted by

$$
    z_n := \begin{cases} 
        (0, n^3) & \text{if } n = k^3 \\
        (0, \frac{1}{n^2}) & \text{if } n \neq k^3 
    \end{cases},
$$

where $k \in \mathbb{N}$. Hence, we get $z_n \downarrow_{p,q} (0,0)$. But, observe that the whole sequence $(z_n)$ is not monotonic.

(iii) A deferred statistical order decreasing to zero sequence may contain a subsequence of decreasing or incomparable elements of $E$ but the index set of such a subsequence has deferred density zero.

(iv) In Riesz spaces, it is well known that $z_n \downarrow 0$ implies $z_{k_n} \downarrow 0$ for every subsequence $(z_{k_n})$ of $(z_n)$. However, this may not hold in the setting of deferred statistical monotone decreasing sequences. For example, take the sequences in the (ii) with the subsequence $(z_{k_n})$, where $k_n = j^3$ for some $j \in \mathbb{N}$, does not have a supremum.

In the general case, the example in Remark 2.2(ii) shows that a subsequence of deferred statistical monotone decreasing sequence need not be deferred statistical monotone decreasing. But, we give a positive result in the following theorem.

Theorem 2.3. Let $(z_n)$ be a sequence in a Riesz space $E$. If $z_n \downarrow_{p,q} 0$, then any subsequence $(z_{k_n})$ of $(z_n)$ with index set $\delta_{p,q}(K) = 1$ such that $(z_{k_n})$ is decreasing on $K$ is deferred statistical order decreasing to 0.

Proof. Suppose that $z_n \downarrow_{p,q} 0$ holds in $E$. Then, there exists a set $K \subseteq \mathbb{N}$ such that $\delta_{p,q}(K) = 1$ and $(z_{k_n})_{k_n \in K} \downarrow 0$ on $K$. Let us consider any arbitrary index set $M \subseteq \mathbb{N}$ such that $K \neq M$, $\delta_{p,q}(M) = 1$ and $(z_n)$ is decreasing on $M$. It can be observed that if there is not such a set $M$, then the proof is complete. It follows from $z_{k_n} \downarrow 0$ that $0 \leq z_{k_n}$ for all $k_n \in K$. Also, we have $\delta_{p,q}(K \cap M) = 1$. Thus, for some $k_m \in K$ and $m_n \in M$, we have $k_m = m_n$. Hence, we have $z_{m_1} \geq z_{m_2} \geq \cdots \geq z_{m_n} = z_{k_n} \geq 0$. We can find infinitely many of such pairs of indices. By continuing this way, we obtain $z_{m_n} \geq 0$ for every $m_n \in M$, i.e., zero is a lower bound of $(z_{m_n})$. Take another lower bound $u$ of $(z_{m_n})$. Therefore, we have $u \leq z_{m_n}$ for every $m_n \in M$. Then, we can find some $z_{n_{k_t}}$ such that $z_{n_{k_t}} = z_{m_k} \geq u$ for some $m_k \in M$. By following this way, we can construct a subsequence $(z_{n_{k_1}}, z_{n_{k_2}}, \cdots)$ of $(z_{k_n})$ such that $u$ is a lower bound of $(z_{n_{k_t}})$ for $t \in \mathbb{N}$. It follows from $z_{n} \downarrow 0$ that the infimum of every subsequence of $(z_{k_n})$ is zero. Hence, we get $u = 0$. Therefore, we get the desired result, $z_{m_n} \downarrow_{p,q} 0$. □
In the next results without proof, we give the linear property of deferred statistical order decreasing sequences.

**Proposition 2.4.** Let \( x_n \downarrow_{D, p, q} 0 \) and \( y_n \downarrow_{D, p, q} 0 \) be a sequence in a Riesz space \( E \) and \( \lambda \in \mathbb{R} \). Then, we have

\[
\begin{align*}
(i) \quad & (x_n + y_n) \downarrow_{D, p, q} 0; \\
(ii) \quad & \lambda x_n \downarrow_{D, p, q} 0.
\end{align*}
\]

### 3 Deferred statistical order convergence

**Definition 3.1.** Let \( p \) and \( q \) be sequences of positive integers satisfying the deferred property. Then, a sequence \( (x_n) \) in a Riesz space \( E \) is called deferred statistical order convergent to \( x \) if there exists a sequence \( z_n \downarrow_{D, 0} \) with an index set \( K \subseteq \mathbb{N} \) such that \( \delta_{p, q}(K) = 1 \) and

\[
|x_{k_n} - x| \leq z_{k_n}
\]

holds for all \( k_n \in K \). Then, we write \( x_n \downarrow_{D, p, q}(x) \).

**Remark 3.2.** It can be seen that, in the case of \( x_n \downarrow_{D, p, q}(x) \), we have

\[
\delta_{p, q}(\{ n \in \mathbb{N} : |x_n - x| \geq z_n \}) = 0.
\]

**Remark 3.3.** It can be observed that the deferred statistical order convergence of the sequence \( (x_n) \) in Definition 3.1 with sequence \( (z_n) \) to \( x \) implies that \( x_{k_n} \downarrow_{D, p, q}(x) \) with the same sequence \( (z_n) \). The converse is also true, i.e., if there exists a subsequence \( x_{k_n} \downarrow_{D, p, q}(x) \) of a sequence \( (x_n) \) with a sequence \( z_n \downarrow_{D, 0} 0 \), then \( x_n \downarrow_{D, p, q}(x) \) with the same sequence \( (z_n) \).

It is clear that deferred statistical order decreasing sequence is deferred statistical order convergent. But, the converse does not hold in general.

**Remark 3.4.** Let \( q(n) = n \) and \( p(n) = 0 \). Then, we have the following observations:

\[
\begin{align*}
(i) \quad & \text{an order convergent sequence is deferred statistical order convergent to its order limit;} \\
(ii) \quad & \text{the statistical order convergence and deferred statistical order convergence coincide.}
\end{align*}
\]

One can observe that a subsequence of a deferred statistical order convergent sequence need not be deferred statistical order convergent.

**Proposition 3.5.** Let \( (x_n) \) be a sequence in a Riesz space \( E \). Then, \( x_n \downarrow_{D, p, q}(x) \) holds if and only if there exists another sequence \( (y_n) \) in \( E \) such that \( \delta_{p, q}(\{ n \in \mathbb{N} : x_n = y_n \}) = 1 \) and \( y_n \downarrow_{D, p, q}(x) \).
Proof. Suppose that there exists a sequence \( \{y_n\} \) in \( E \) such that \( \delta_{p,q}(\{n \in \mathbb{N} : x_n = y_n\}) = 1 \) and \( y_n \xrightarrow{D_{st_e}}(p,q) x \). Then, there is another sequence \( z_n \xrightarrow{D_{st_e}} 0 \) in \( E \) with \( \delta_{p,q}(K) = 1 \) such that \( |x_{k_n} - x| \leq z_{k_n} \) for each \( k_n \in K \). Thus, it follows from the including

\[
\{p_n + 1 \leq m \leq q_n : |x_m - x| \leq z_m\} \subseteq \{p_n + 1 \leq m \leq q_n : x_m \neq y_m\}
\]

that we have

\[
\lim_{n \to \infty} \frac{1}{q_n - p_n} \{p_n + 1 \leq m \leq q_n : |x_m - x| \leq z_m\} \leq \lim_{n \to \infty} \frac{1}{q_n - p_n} \{p_n + 1 \leq m \leq q_n : x_m \neq y_m\}
\]

because of \( \delta_{p,q}(\{p_n + 1 \leq m \leq q_n : |y_m - x| \leq z_m\}) = 0 \). Thus, we obtain

\[
\lim_{n \to \infty} \frac{1}{q_n - p_n} \{p_n + 1 \leq m \leq q_n : |x_m - x| \leq z_m\} = 0.
\]

Therefore, we get the desired result, \( x_n \xrightarrow{D_{st_e}(p,q)} x \). The other part of proof is obvious, and so, we omit it. \( \square \)

**Proposition 3.6.** The deferred statistical order limit is linear and uniquely determined.

**Proof.** Assume that \( x_n \xrightarrow{D_{st_e}(p,q)} x \) and \( x_n \xrightarrow{D_{st_e}(p,q)} y \) hold in a Riesz space \( E \). Then, there are sequences \( z_n \xrightarrow{D_{st_e}} 0 \) with \( \delta_{p,q}(K) = 1 \) and \( t_n \xrightarrow{D_{st_e}} 0 \) with \( \delta_{p,q}(M) = 1 \) such that \( |x_{k_n} - x| \leq z_{k_n} \) and \( |x_{m_n} - y| \leq t_{m_n} \) for all \( k_n \in K \) and \( m_n \in M \). Thus, it follows that

\[
|x - y| \leq |x - x_j| + |x_j - y| \leq z_{j_n} + t_{j_n}
\]

for every \( j_n \in J := K \cap M \). By using \( (z_{j_n} + t_{j_n})_{j_n \in J} \), we obtain that \( |x - y| = 0 \). Thus, we get the equality of \( x \) and \( y \).

Now, for the linearity of the deferred statistical order limit, take sequences \( x_n \xrightarrow{D_{st_e}(p,q)} x \) and \( y_n \xrightarrow{D_{st_e}(p,q)} y \) in a Riesz space \( E \). Then, there are sequences \( z_n \xrightarrow{D_{st_e}} 0 \) and \( t_n \xrightarrow{D_{st_e}} 0 \) such that \( \delta_{p,q}(\{n \in \mathbb{N} : |x_n - x| \leq z_n\}) = 0 \) and \( \delta_{p,q}(\{n \in \mathbb{N} : |y_n - y| \leq t_n\}) = 0 \). It follows from the triangular inequality in Riesz spaces that

\[
\{n \in \mathbb{N} : |(x_n + y_n) - (x + y)| \leq z_n + t_n\} \subseteq \{n \in \mathbb{N} : |x_n - x| \leq z_n\}
\]

\[
\cup\{n \in \mathbb{N} : |y_n - y| \leq t_n\}.
\]

Therefore, we obtain \( \delta_{p,q}(\{n \in \mathbb{N} : |(x_n + y_n) - (x + y)| \leq z_n + t_n\}) = 0 \), i.e., we obtain \( x_n + y_n \xrightarrow{D_{st_e}(p,q)} x + y \). \( \square \)

In the following result, we observe some relations between deferred statistical order convergence and lattice properties.
**Theorem 3.7.** Let $x_n \xrightarrow{D_{sta}(p,q)} x$ and $y_n \xrightarrow{D_{sta}(p,q)} y$ in a Riesz space $E$. Then, we have the following statement:

(i) $x_n \lor y_n \xrightarrow{D_{sta}(p,q)} x \lor y$;

(ii) $x_n \land y_n \xrightarrow{D_{sta}(p,q)} x \land y$;

(iii) $x_n^+ \xrightarrow{D_{sta}(p,q)} x^+$;

(iv) $x_n^- \xrightarrow{D_{sta}(p,q)} x^-$;

(v) $|x_n| \xrightarrow{D_{sta}(p,q)} |x|$.

**Proof.** It is enough to show the first statement because the other case can be obtained by applying [3, Thm.1.7] and Proposition 3.6. Now, from $x_n \xrightarrow{D_{sta}(p,q)} x$ and $y_n \xrightarrow{D_{sta}(p,q)} y$, we have sequences $z_n \downarrow_{D_{sta}} 0$ and $t_n \downarrow_{D_{sta}} 0$ with indexes sets $\delta_{p,q}(K) = \delta_{p,q}(M) = 1$ such that $|x_{n}\!\!-\!\!x| \leq z_n$ and $|y_{n}\!\!-\!\!y| \leq t_m$ hold for all $k_n \in K$ and $m_n \in M$. On the other hand, by applying [3, Thm.1.9] and by taking $J := N \cap M$, we get

$$|x_{j_n} \lor y_{j_n} - x \lor y| \leq |x_{j_n} - x| + |y_{j_n} - y| \leq z_{j_n} + t_{j_n}$$

for every $j_n \in J$. Hence, we obtain

$$\delta_{p,q}\left(\{n \in \mathbb{N} : |x_{j_n} \lor y_{j_n} - x \lor y| \leq z_{j_n} + t_{j_n}\}\right) = 0.$$ 

Therefore, we get the desired result, $x_n \lor y_n \xrightarrow{D_{sta}(p,q)} x \lor y$. $\square$

**Corollary 3.8.** The positive cone $E_+ = \{x \in E : 0 \leq x\}$ of a Riesz space $E$ is closed under the deferred statistical order convergence.

**Proposition 3.9.** If $x_n \xrightarrow{D_{sta}(p,q)} x$, $y_n \xrightarrow{D_{sta}(p,q)} y$ and $x_n \geq y_n$ satisfy for every $n \in \mathbb{N}$ in a Riesz space, then $x \geq y$ holds.

**Proof.** Assume that $y_n \leq x_n$ holds for each $n \in \mathbb{N}$. Then, we have $0 \leq x_n - y_n \in E_+$ for each $n \in \mathbb{N}$. It follows from Corollary 3.8 that $x_n - y_n \xrightarrow{D_{sta}(p,q)} x - y \in E_+$ because of $(x_n - y_n) \in E_+$. Thus, we get $x - y \geq 0$, i.e., $x \geq y$. $\square$

**Theorem 3.10.** If $(x_n)$ is a monotone and deferred statistical order convergent in a Riesz space, then it is order convergent.

**Proof.** Suppose that $(x_n) \downarrow$ and $x_n \xrightarrow{D_{sta}(p,q)} x$ in a Riesz space $E$. Fix any $k \in \mathbb{N}$. Then, we have $x_k - x_n \geq 0$ for all $n \geq k$. It follows that $x_k - x_n \xrightarrow{D_{sta}(p,q)} x_k - x \geq 0$, i.e., $x_k \geq x$. Thus, $x$ is an lower bound of $(x_n)$ because $k$ is arbitrary. Choose another lower bound $z$ of $(x_n)$. Hence, we have $x_n - z \xrightarrow{D_{sta}(p,q)} x - z \geq 0$, i.e., $x \geq z$. Therefore we get the desired result, $x_n \downarrow x$. $\square$
Remark 3.11. Let $A$ be an ideal in a vector lattice $E$ and $(a_n)$ be a sequence in $A$. One can observe that if $a_n \not\rightarrow 0$ in $A$, then $a_n \not\rightarrow 0$ in $E$. Hence, it is clear that $a_n \downarrow_{D^{st}a} 0$ in $A$ implies $a_n \downarrow_{D^{st}a} 0$ in $E$. For the converse, if $a_n \not\rightarrow 0$ in $E$ and order bounded, then $a_n \not\rightarrow 0$ in $A$, and so, $a_n \downarrow_{D^{st}a} 0$ in $A$ for order bounded sequences.

Thanks to Remark 3.11 we give the following two results.

Theorem 3.12. Let $A$ be an ideal in an $\sigma$-order complete vector lattice and $(x_n)$ be a sequence in $A$. Then, $x_n \xrightarrow{D^{st}_a(p,q)} 0$ in $A$ if and only if $x_n \xrightarrow{D^{st}_a(p,q)} 0$ in $E$.

Proof. Assume that $x_n \xrightarrow{D^{st}_a(p,q)} 0$ in $A$. Then, there exists a sequence $z_n \downarrow_{D^{st}a} 0$ in $A$ with index set $\delta_{p,q}(K) = 1$ such that $|x_{k_n}| \leq z_{k_n}$ for all $k_n \in K$. Now, by using Remark 3.11 it follows from $(z_{k_n})_{k_n \in K} \downarrow 0$ in $A$ that $(z_{k_n})_{k_n \in K} \downarrow 0$ in $E$, i.e., we get $z_n \downarrow_{D^{st}a} 0$ in $E$. Therefore, we have $x_n \xrightarrow{D^{st}_a(p,q)} 0$ in $E$.

Conversely, assume $x_n \xrightarrow{D^{st}_a(p,q)} 0$ in $E$. Then, there is a sequence $z_n \downarrow_{D^{st}a} 0$ in $E$ with index set $\delta_{p,q}(K) = 1$ such that $|x_{k_n}| \leq z_{k_n}$ for all $k_n \in K$. Thus, Remark 3.11 implies that $z_n \downarrow_{D^{st}a} 0$ in $A$. Therefore, we get $x_n \xrightarrow{D^{st}_a(p,q)} 0$ in $A$.

The following result is similar to [13 Thm.3.1].

Theorem 3.13. Let $(x_n)$ be a sequence in a Riesz space $E$ and $(x_{k_n})_{k_n \in K}$ be a subsequence of $(x_n)$. If the limit

$$\lim_{n \rightarrow \infty} \frac{1}{q_n - p_n} \left| \left\{ p_n < k_n \leq q_n : k_n \in K \right\} \right| > 0$$

holds and $x_n \xrightarrow{D^{st}_a(p,q)} x$ for some sequences $p$ and $q$ satisfying the deferred property, then $x_{k_n} \xrightarrow{D^{st}_a(p,q)} x$.

Proof. Assume that $x_n \xrightarrow{D^{st}_a(p,q)} x$ satisfies in $E$. Then, there is a sequence $z_n \downarrow_{D^{st}a} 0$ in $E$ such that $\delta_{p,q}(\{ n \in \mathbb{N} : |x_n - x| \not\geq z_n\}) = 0$. It can be seen that

$$\{ p_n < k_n \leq q_n : k_n \in K, |x_{k_n} - x| \not\geq z_n\} \subseteq \{ p_n < n \leq q_n : |x_n - x| \not\geq z_n\}.$$

Thus, by taking $H_n := \{ p_n < k_n \leq q_n : k_n \in K \}$ for all $n$, we have

$$\frac{1}{|H_n|} \left| \left\{ p_n < k_n \leq q_n : k_n \in K, |x_{k_n} - x| \not\geq z_n\right\} \right|$$

$$\leq \frac{1}{|H_n|} \left| \left\{ p_n < n \leq q_n : |x_n - x| \not\geq z_n\right\} \right|.$$

Therefore, it is enough to show $\lim_{n \rightarrow \infty} \frac{1}{|H_n|} \left| \left\{ p_n < n \leq q_n : |x_n - x| \not\geq z_n\right\} \right| = 0$ for proving the convergence $x_{k_n} \xrightarrow{D^{st}_a(p,q)} x$. We observe the following inequality

$$\lim_{n \rightarrow \infty} \frac{1}{q_n - p_n} \left| \left\{ p_n < n \leq q_n : |x_n - x| \not\geq z_n\right\} \right| \left| H_n \right|$$
Therefore, we have
\[
\limsup_{n \to \infty} \frac{1}{|H_n|} \left| \left\{ p_n < n \leq q_n : |x_n - x| \not\geq z_n \right\} \right| = 0
\]
because of \( x_n \xrightarrow{\text{Dsto}(p,q)} x \). Thus, we obtain the desired result. \( \square \)

In Remark 3.3, we give a relation between statistical order and deferred statistical order convergences by taking \( q(n) = n \) and \( p(n) = 0 \). We give another relation under a new condition in the next theorem.

**Theorem 3.14.** If the sequence \( \left\{ \frac{p_n}{q_n} \right\} \) is bounded for any \( p \) and \( q \) sequences having the deferred property, then the statistical order convergence implies the deferred statistical order convergence.

**Proof.** Assume that \( x_n \xrightarrow{\text{st}} x \) in a Riesz space \( E \) and \( \left\{ \frac{p_n}{q_n} \right\} \) is a bounded sequence for some sequences \( p \) and \( q \) satisfying the deferred property.

Thus, there exists a sequence \( z_n \downarrow 0 \) such that
\[
\lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : |x_k - x| \not\geq z_k \} \right| = 0.
\]

It follows from the deferred properties of \( \left\{ q_n \right\} \) that we obtain
\[
\lim_{n \to \infty} \frac{1}{q_n} \left| \{ k \leq q_n : |x_k - x| \not\geq z_k \} \right| = 0.
\]

So, by the following inclusion
\[
\{ p_n < k \leq q_n : |x_k - x| \not\geq z_k \} \subseteq \{ k \leq q_n : |x_k - x| \not\geq z_k \},
\]
we obtain
\[
\lim_{n \to \infty} \frac{1}{q_n - p_n} \left| \{ p_n < k \leq q_n : |x_k - x| \not\geq z_k \} \right|
\leq \lim_{n \to \infty} \frac{1}{q_n} (1 + \frac{p_n}{q_n - p_n}) \left| \{ k \leq q_n : |x_k - x| \not\geq z_k \} \right|.
\]

Thus, we get the desired result, \( x_n \xrightarrow{\text{Dsto}(p,q)} x \). \( \square \)

The converse of Theorem 3.14 need not be true in general. To see this, we give the following example.

**Example 3.15.** Consider the Riesz space \( E := \mathbb{R}^2 \) equipped with the coordinatewise ordering and a sequence \( (x_n) \) in \( E \) as follows:

\[
x_n := \begin{cases} 
(0, \frac{n}{2}), & \text{n is odd} \\
(0, -\frac{n}{2}), & \text{n is even}
\end{cases}
\]

for all \( n \). Also, take \( (q_n) := (2n) \) and \( (p_n) := (4n) \). Then, it is clear that the assumption of Theorem 3.14 is fulfilled, and also, \( x_n \xrightarrow{\text{Dsto}(p,q)} (0,0) \). But, it is not statistical order convergent.

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Theorem 3.16. Let \( p', q' \) and \( p, q \) be pairs of sequences satisfying the deferred property such that \( p_n \leq p'_n \) and \( q'_n \leq q_n \) for each \( n \in \mathbb{N} \), and \( (x_n) \) be a sequence in a Riesz space \( E \). Then, \( x_n \overset{D_{st_{(p,q)}}}{\rightarrow} x \) implies \( x_n \overset{D_{st_{(p',q')}}}{\rightarrow} x \) in \( E \) whenever the sets \( \{k : p_n < k \leq p'_n \} \) and \( \{k : q'_n < k \leq q_n \} \) are finite for every \( n \in \mathbb{N} \).

Proof. Assume that \( x_n \overset{D_{st_{(p,q)}}}{\rightarrow} x \) holds in \( E \). Then, there exists sequence \( z_n \downarrow D_{st_{(p,q)}} 0 \) such that
\[
\delta_{p,q}(\{n \in \mathbb{N} : |x_n - x| \not\leq z_n\}) = 0.
\]
On the other hand, we have the following equality
\[
\{k : p_n < k \leq q_n, |x_n - x| \not\leq z_n\} = \{k : p_n < k \leq p'_n, |x_n - x| \not\leq z_n\}
\]
\[
\cup\{k : p'_n < k \leq q'_n, |x_n - x| \not\leq z_n\} \cup \{k : q'_n < k \leq q_n, |x_n - x| \not\leq z_n\}.
\]
It follows that
\[
\lim_{n \to \infty} \frac{1}{q_n - p_n} |\{k : p_n < k \leq q_n, |x_n - x| \not\leq z_n\}| = 0.
\]
Hence, we get \( x_n \overset{D_{st_{(p,q)}}}{\rightarrow} x \).

Corollary 3.17. Let \( p', q' \) and \( p, q \) be pairs of sequences satisfying the deferred property such that \( \lim_{n \to \infty} \frac{1}{q_n - p_n} = t > 0 \), and \( (x_n) \) be a sequence in a Riesz space \( E \). Then, \( x_n \overset{D_{st_{(p,q)}}}{\rightarrow} x \) implies \( x_n \overset{D_{st_{(p',q')}}}{\rightarrow} x \) in \( E \).

Now, take the set \( C_{(z_n)}^{p,q} := \{(x_n) : \exists x \in E, x_n \overset{D_{st_{(p,q)}}}{\rightarrow} x \text{ with } (z_n)\} \) for a fixed sequence \( z_n \downarrow D_{st_{(p,q)}} 0 \). Then, it is clear that \( C_{(z_n)}^{p,q} \subseteq C_{(w_n)}^{p,q} \) whenever \( z_n \leq w_n \) holds for all \( n \in \mathbb{N} \).

Proposition 3.18. If \( \delta_{p,q}(\{n \in \mathbb{N} : z_n \neq w_n\}) = 0 \), then \( C_{(z_n)}^{p,q} = C_{(w_n)}^{p,q} \).

Proof. Suppose that \( \delta_{p,q}(\{n \in \mathbb{N} : z_n \neq w_n\}) = 0 \) holds for some sequences \( z_n \downarrow D_{st_{(p,q)}} 0 \) and \( w_n \downarrow D_{st_{(p,q)}} 0 \). Take \( (x_n) \in C_{(z_n)}^{p,q} \). Then, we have \( \delta_{p,q}(\{n : |x_n - x| \not\leq z_n\}) = 1 \). It follows from the following inclusion
\[
\{n : |x_n - x| \not\leq w_n\} \subseteq \{n : |x_n - x| \not\leq z_n\} \cup \{n : z_n \neq w_n\}
\]
that we obtain \( (x_n) \in C_{(w_n)}^{p,q} \), and so, \( C_{(z_n)}^{p,q} \subseteq C_{(w_n)}^{p,q} \). Similarly, we can get \( C_{(w_n)}^{p,q} \subseteq C_{(z_n)}^{p,q} \). Therefore, we obtain \( C_{(z_n)}^{p,q} = C_{(w_n)}^{p,q} \).

It is clear that \( C_{(z_n)}^{p,q} \subseteq C_{(z_n)}^{p,q} \) for any subsequence \( z_{k_n} \downarrow D_{st_{(p,q)}} 0 \) of sequence \( z_n \downarrow D_{st_{(p,q)}} 0 \). For the converse, we give the following result.

Proposition 3.19. If \( z_n \downarrow 0 \) holds, then \( C_{(z_n)}^{p,q} \subseteq C_{(z_n)}^{p,q} \) satisfies for each subsequence \( (z_{k_n})_{k_n \in K} \) of \( (z_n) \) with \( \delta_{p,q}(K) = 1 \).

Proof. Assume that \( z_n \downarrow 0 \) and \((z_{k_n})_{k_n \in K}\) be a subsequence of \( (z_n) \) with \( \delta_{p,q}(K) = 1 \). Take any element \( (x_n) \in C_{(z_{k_n})}^{p,q} \). Then, we have \( x \in E \) and an index set \( \delta_{p,q}(M) = 1 \) such that \( |x_{m_n} - x| \leq z_{m_n} \) holds for all \( m_n \in M \).

On the other hand, by taking \( J := M \cap K \), we have \( |x_{j_n} - x| \leq z_{j_n} \) for each \( j_n \in J \). Since \( (z_{j_n}) \) is a subsequence of \( (z_{k_n}) \), we obtain \( (x_n) \in C_{(z_{j_n})}^{p,q} \).
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