ENTROPY OF FOLIATIONS WITH LEAFWISE FINSLER STRUCTURE

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Abstract. We extend the notion of the geometric entropy of foliation to foliated manifolds equipped with leafwise Finsler structure. We study the relation between the geometric entropy and the topological entropy of the holonomy pseudogroup. The case of foliated manifold with leafwise Randers structure. In this case the estimates for one dimensional foliation defined by a vector field in term of topological entropy of a flow are presented.

1. Introduction

The notion of the topological entropy was introduced by Adler, Konheim and McAndrew in 1965 in [1]. Another approach was presented by Bowen [2] in early 70’s. Ghys, Langevin and Walczak, in [3], extended this notion to the topological entropy for finitely generated groups and pseudogroups of continuous transformations, as well as the geometric entropy of foliation on a compact foliated Riemannian manifold. The entropy of foliation has more geometric nature, because it depends on a Riemannian metric chosen for foliated manifold.

The aim of this paper is to extend the notion of the geometric entropy of foliations to the foliated manifolds equipped with leafwise Finsler structure. In paragraphs 2 and 3, one can find all necessary definitions and properties related to entropy and foliations with leafwise Finsler metric. Next paragraph describes relations between geometric
and topological entropy. Fifth part of the paper refers to foliations with leafwise Randers norm. Last paragraph describes the entropy of one dimensional foliations defined by a unit vector field with leafwise Randers metric.

2. LEAFWISE FINSLER STRUCTURES

Let us recall that a Minkowski norm on a vector space $V$ is a non-negative function $F : V \to [0, \infty)$ such that

1. $F$ is $C^\infty$ on $V \setminus \{0\}$,
2. $F(\lambda v) = \lambda F(v)$ for any $\lambda > 0$ and $v \in V$,
3. for every $v \in V \setminus \{0\}$, the symmetric bilinear form

$$g_y(u,v) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(y + su + tv)|_{t=s=0}$$

is positively defined.

Now, let $M$ be a smooth manifold. A function $F : TM \to [0, \infty)$ is called a Finsler norm if

1. $F$ is $C^\infty$ on the tangent bundle with removed the zero section $TM \setminus \{0\}$,
2. for any $x \in M$ the restricted norm $F_x = F|_{T_x M}$ is a Minkowski norm. The pair $(M, F)$ is called a Finsler space.

**Example 2.1.** Let $(M, g)$ be a Riemannian manifold, and let $\beta : TM \to \mathbb{R}$ be a 1-form. Let $\alpha : TM \to [0, \infty)$ be the norm defined by $g$, that is, $\alpha(v) = \sqrt{g_x(v,v)}$ for all $v \in T_x M$. Suppose that the $g$-norm of $\beta$ satisfies $||\beta||_g < 1$. We set $F(v) = \alpha(v) + \beta(v)$. $F$ is a Finsler norm and it is called a Randers norm.

Note that the Finsler norm induces a function $d : M \times M \to [0, \infty)$ by the formula

$$d(x,y) = \inf_{\gamma} \int_0^1 F(\dot{\gamma}(t)) dt$$

where the infimum is taken over all curves $\gamma : [0, 1] \to M$ linking $x$ and $y$. Function $d$ is a quasi-metric, that is,

$$(d(x,y) = 0 \text{ iff } x = y) \text{ and } d(x,y) + d(y,z) \geq d(x,z).$$
Let \((M, \mathcal{F}, g)\) be a foliated Riemannian manifold. Having \(g\), we decompose the tangent bundle to the orthogonal sum of the bundle tangent to \(\mathcal{F}\) and the orthogonal bundle, that is, \(TM = T\mathcal{F} \oplus T\mathcal{F}^\perp\). We replace the norm induced in \(T\mathcal{F}\) by the Riemannian structure \(g|_{T\mathcal{F}}\) by a Finsler norm \(F_{\mathcal{F}}\). Denote by \(\pi_1 : TM \to T\mathcal{F}\) and \(\pi_2 : TM \to T\mathcal{F}^\perp\) the natural projections. We set

\[
F(v) = \sqrt{F_{\mathcal{F}}^2(\pi_1(v)) + g(\pi_2(v), \pi_2(v))}.
\]

\(F\) is a Finsler norm on \(TM\) and coincides with \(\sqrt{g(v, v)}\) for \(v \in T\mathcal{F}^\perp\) and with \(F_{\mathcal{F}}\) for \(v \in T\mathcal{F}\). We call \(F\) a leafwise Finsler structure on \((M, \mathcal{F})\).

3. Geometric entropy of foliations with leafwise Finsler metric

Let \((M, \mathcal{F}, F)\) be a foliated manifold with leafwise Finsler structure. Let \(U\) be a nice covering, i.e., a covering by the domains \(D_\varphi\) of the charts of a nice foliated atlas \(A\), that is an atlas satisfying

1. the covering \(\{D_\varphi : \varphi \in A\}\) is locally finite,
2. for any \(\varphi \in A\), the set \(R_\varphi = \varphi(D_\varphi) \subset \mathbb{R}^n\) is an open cube,
3. if \(\varphi, \psi \in A\), and \(D_\varphi \cap D_\psi \neq \emptyset\), then there exists a chart \(\chi\) such that for any leaf \(L\) of \(\mathcal{F}\) the connected components of \(L \cap D_\chi\) are given by the equation \(\chi'' = \text{const}\), and \(R_\chi\) is an open cube, \(D_\chi\) contains the closure of \(D_\varphi \cup D_\psi\) and \(\varphi = \chi|_{D_\varphi}\).

Let \(U \in U\). Equip the space of plaques \(T_U = U/\mathcal{F}|_U\) with the quotient topology. The disjoint union \(T = \bigsqcup\{T_U; U \in U\}\) is called a complete transversal for \(\mathcal{F}\). Note that each of \(T_U\) can be mapped homeomorphically onto a \(C^r\)-submanifold \(T'_U \subset U\) transverse to \(\mathcal{F}\).

Following [3], let us recall that for a given nice covering \(U\) of \((M, \mathcal{F})\) there exists an \(\varepsilon_0 > 0\) such that any point \(x \in U\), \(U \in U\) can be projected orthogonally in the unique way to the plaque \(P_y \subset U\) passing through a point \(y \in U\) if only \(d(x, y) < \varepsilon_0\) and \(d(y, x) < \varepsilon_0\).

Let \(\gamma : [0, 1] \to L\) be a leafwise curve beginning at \(x \in U\). For any \(y \in U\) lying within the distance \(\varepsilon < \varepsilon_0\), we can project orthogonally an
initial part of the curve $\gamma$ to the plaque $P_y$ passing through $y$. Replacing $x$ and $y$ by the endpoints of the already projected piece and its image $\gamma_1$, we can continue this process as long as the distance between $\gamma$ and $\gamma'$ does not exceed $\varepsilon_0$. We will denote the projection of $\gamma$ by $p_y\gamma$.

Let $U$ be a nice covering and let $T$ be the complete transversal for $U$. Let $\varepsilon \in (0, \varepsilon_0)$.

**Definition 3.1.** We say that $x, y \in T$ are $(R, \varepsilon)$-separated by $F$ with respect to $F$ if either

- $d(x, y) \geq \varepsilon_0$ or $d(y, x) \geq \varepsilon_0$

or

- there exists a leaf curve $\gamma : [0, 1] \to \mathcal{L}_x$ such that $\gamma(0) = x$, $l(\gamma) = \int_0^1 F(\dot{\gamma}(t))dt \leq R$ and

$$d(\gamma(1), p_y\gamma(1)) \geq \varepsilon \text{ or } d(p_y\gamma(1), \gamma(1)) \geq \varepsilon.$$ 

(or a leaf curve $\gamma : [0, 1] \to \mathcal{L}_x$ such that $\gamma(0) = y$, $l(\gamma) \leq R$, and $d(\gamma(1), p_x\gamma(1)) \geq \varepsilon \text{ or } d(p_x\gamma(1), \gamma(1)) \geq \varepsilon$).

A subset $A \subset T$ is called $(R, \varepsilon)$-separated if any two points $x, y \in A$, $x \neq y$, are $(R, \varepsilon)$-separated. Let $s(R, \varepsilon, F)$ denote the maximum cardinality of an $(R, \varepsilon)$-separated subset of $T$. We set $s(\varepsilon, F) = \limsup_{R \to \infty} \frac{1}{R} \log s(R, \varepsilon, F)$, and

$$h(F, F) = \lim_{\varepsilon \to 0^+} s(\varepsilon, F).$$

**Remark 3.2.** The number $h(F, F)$ does not depend on the choice of the nice covering $U$. Let $U'$ and $T'$ be another nice covering and complete transversal. Let $\varepsilon > 0$ be small enough, and let us denote by $d_F$ the leafwise metric induced by the Finsler structure $F$. Since $M$ is compact, the geometry of $F$ is bounded. Hence, one can project $T$ onto $T'$ in such a way that any $(R, \varepsilon)$-separated points $x, y \in T$ are projected to $x', y' \in T'$, respectively, which are $(R + R_0, \varepsilon)$-separated with $R_0$ being the maximum of the numbers $d_F(x, x')$ and $d_F(x', x)$, $x \in T \cap U$, $x' \in T' \cap U'$, $U \in U$, $U' \in U'$, and the plaques $P_x \subset U$ and
$P_{x'} \subset U'$ intersects. Thus

$$s'(R - R_0, \varepsilon, \mathcal{F}) \leq s(R, \varepsilon, \mathcal{F}) \leq s'(R + R_0, \varepsilon, \mathcal{F}),$$

and both numbers $h(\mathcal{F}, F)$ and $h'(\mathcal{F}, F)$ are equal.

**Remark 3.3.** Since any two Riemannian structures $g$ and $g'$ on a compact manifold satisfies

$$c^{-2}g(v, w) \leq g'(v, w) \leq c^2g(v, w)$$

for some constant $c > 1$, then the number $h(\mathcal{F}, F)$ does not depend on the choice of the Riemannian part of $F$. Indeed, there exists a constant $a > 1$ such that for any leaf curve $\gamma$ and its orthogonal projections $p_g\gamma$ and $p'_{g'}\gamma$, with respect to $g$ and $g'$ respectively, satisfies

$$d(\gamma(t), p_g\gamma(t)) \leq a \cdot d'(\gamma(t), p'_{g'}\gamma(t)),$$

if $d(\gamma(t), p'_{g'}\gamma(t)) < \varepsilon$ for sufficiently small $\varepsilon > 0$. Thus any two $(R, \varepsilon)$-separated points with respect to $F' = \sqrt{F^2 + g'}$ are $(R, \varepsilon/a)$-separated with respect to $F = \sqrt{F'^2 + g}$, and $h(\mathcal{F}, F') \leq h(\mathcal{F}, F)$. Analogically we show that $h(\mathcal{F}, F) \leq h(\mathcal{F}, F')$.

The number $h(\mathcal{F}, F)$ is called the **geometric entropy of foliation with leafwise Finsler structure.** In further consideration we will denote by $F$ both, the structure $F_\mathcal{F}$ and the leafwise Finsler structure $F = \sqrt{F^2 + g}$.

### 4. Relation between geometric entropy and topological entropy of holonomy pseudogroup

Let $(M, \mathcal{F})$ be a compact foliated manifold. Following [3] or [6], one can define the topological entropy of the holonomy pseudogroup $\mathcal{H}_\mathcal{U}$ defined by the nice covering $\mathcal{U}$. The symbol $D_f$ denotes here the domain of a map $f$.

To begin, let $\mathcal{G}$ be a pseudogroup (see [6]) on a metric space $(X, d)$ generated by a good symmetric finite set $\mathcal{G}_1$, that is
(1) for any $g \in \mathcal{G}$ and any $x \in D_g$ there exist $g_1, \ldots, g_n \in \mathcal{G}_1$ and open subset $U \subset D_g$ containing $x$ such that

$$g|_U = g_1 \circ \cdots \circ g_n|_U,$$

(2) for any $g \in \mathcal{G}_1$ the exits a compact set $K_g \subset D_g$ such that $g|_{\text{int} K_g}$ generate $\mathcal{G}$.

We say that $x, y \in X$ are $(n, \varepsilon)$-separated by $\mathcal{G}$ if there exists

$$g \in \mathcal{G}_n^* := \{g_1|_{K_1} \circ \cdots \circ g_n|_{K_n}; g_i \in \mathcal{G}_1\}$$

such that $\{x, y\} \subset D_g$ and

$$d(g(x), g(y)) \geq \varepsilon \text{ or } d(g(y), g(x)) \geq \varepsilon.$$

A subset $A$ of $X$ is called $(n, \varepsilon)$-separated if any two distinct points of $A$ are $(n, \varepsilon)$-separated. Let $s(n, \varepsilon, \mathcal{G}_1)$ be the maximal cardinality of an $(n, \varepsilon)$-separated subset of $X$. We set

$$s(\varepsilon, \mathcal{G}_1) = \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \mathcal{G}_1).$$

The number $h(\mathcal{G}, \mathcal{G}_1) = \lim_{\varepsilon \to 0^+} s(\varepsilon, \mathcal{G}_1)$ is called the topological entropy of the pseudogroup $\mathcal{G}$ with respect to $\mathcal{G}_1$.

Let $\mathcal{U}$ be a nice covering of $(M, f)$ and let $T$ be a complete transversal. Given two sets $U, V \in \mathcal{U}$ such that $U \cap V \neq \emptyset$, one can define the holonomy map $h_{VU} : D_{VU} \to T_V$ with $D_{VU}$ being the open subset of $U$ consisting of all plaques $P \subset U$ such that $P \cap V \neq \emptyset$ by

$$h_{VU}(P) = P' \text{ iff } P \subset U \text{ and } P' \subset V \text{ intersects}.$$

The mappings $h_{VU}$ generates the holonomy pseudogroup $\mathcal{H}_U$ on $T$. We will denote by $\mathcal{H}^1_{U}$ the set of $\{h_{VU}\}$ of the generators of $\mathcal{H}_U$.

One of the main results of [3] is Theorem 3.4 about the relation between the geometric entropy $h(\mathcal{F}, g)$ of a foliation on a Riemannian manifold and the topological entropy of the holonomy pseudogroup $\mathcal{H}_U$ defined by the nice covering $\mathcal{U}$. We here extends this result to class of foliations with leafwise Finsler structures.
Theorem 4.1. Let $U$ be a nice covering, and let $\text{diam}(U)$ be the diameter of the nice covering $U$, that is,

$$\text{diam}(U) = \max_{U \in U} \max_{P \subset U} \max_{x,y \in P} d_F(x,y),$$

where $P$ denotes a plaque of a chart $U$, and $d_F$ is the leafwise distance defined by the leafwise Finsler structure. Then

$$h(\mathcal{F}, F) = \sup_U \left\{ \frac{1}{\text{diam}(U)} h(H_U, H^1_U) \right\}$$

Here we repeat the proof of Theorem 3.4 of [3] with the necessary changes due to the fact that the metric induced by the Finsler structure is asymmetric. In the proof, $\Lambda = \max_{v \in T_1F} F(v) - F(-v)$.

Lemma 4.2. For $\Delta$ and $\rho$ small enough, there exists $\tilde{\beta} > 0$ such that $\rho > \tilde{\beta}$ and the following is satisfied:

Let $x_1, x_2$ be two points lying on the same leaf $L$ and such that $d_F(x_1, x_2) = \frac{2\Delta}{\Lambda} - \alpha$ for some $\alpha > 0$. Let $y_1, y_2$ be two points of transversals $T_1$ and $T_2$ passing through $x_1$ and $x_2$, respectively, and lying on the same plaque with diameter not exceeding $4\Delta$. If

$$d(x_1, y_1) < \tilde{\beta}, \quad d(y_1, x_1) < \tilde{\beta}, \quad d(x_2, y_2) < \tilde{\beta}, \quad \text{and} \quad d(y_2, x_2) < \tilde{\beta},$$

then $d_F(y_1, y_2) \leq \frac{2\Delta}{\Lambda} - \frac{\alpha}{2}$.

Proof. Let $\gamma : [0, 1] \to L$ be a curve linking $x_1$ with $x_2$ such that $l(\gamma) = \frac{2\Delta}{\Lambda} - \alpha$. Let $p_{y_1\gamma}$ be the orthogonal projection of $\gamma$ onto the plaque $P_{y_1}$. Since $M$ is compact, there exists $\tilde{\beta} > 0$ such that $|l(\gamma) - l(p_{y_1\gamma})| < \frac{\alpha}{4}$, $d_F(y_2, p_{y_1\gamma}(1)) < \frac{\alpha}{4}$, and $d_F(p_{y_1\gamma}(1), y_2) < \frac{\alpha}{4}$. Thus

$$d_F(y_1, y_2) \leq l(p_{y_1\gamma}) + d_F(y_2, p_{y_1\gamma}(1)) + d_F(p_{y_1\gamma}(1), y_2) \leq \frac{2\Delta}{\Lambda} - \frac{\alpha}{2}.$$ 

This ends our proof.

Lemma 4.3. Let $Z = \{z_1, \ldots, z_N\}$ be a $\beta$-dense subset of $M$, $\beta < \frac{\tilde{\beta}}{10}$. Let $x_1$ and $x_2$ be two points of the same leaf with $d_F(x_1, x_2) < \frac{2\Delta}{\Lambda} - \alpha$. Let $z \in Z$ (resp. $z' \in Z$) be an $\beta$-close point of $x_1$ (resp. $x_2$). Then the subsets $U_z$ and $U_{z'}$ have the following property:
If \( \xi_1 \in T_z \) and \( \xi_2 \in T_z' \) lie on the same plaque with diameter not exceeding \( 4\Delta \) and
\[
d(\xi_1, x_1) \leq \beta, \quad d(x_1, \xi_1) < \beta, \quad d(\xi_2, x_2) \leq \beta, \quad d(x_2, \xi_2) < \beta,
\]
then the minimal leaf geodesic in \( L_{\xi_1} = L_{\xi_2} \) linking \( \xi_1 \) and \( \xi_2 \) is contained in the sum
\[
B_F(\xi_1, \frac{\Delta}{\Lambda}) \cup B_F(\xi_2, \Delta).
\]

**Proof.** By Lemma 4.2, \( d_F(\xi_1, \xi_2) < \frac{2\Delta}{\Lambda} - \frac{\alpha}{2} \). So, there exists a curve \( \gamma : [0, 1] \to L_{\xi_1} \) such that \( \gamma(0) = \xi_1, \gamma(0) = \xi_2, \) and \( l(\gamma) = d_F(\xi_1, \xi_2) \). Let \( t \in [0, 1] \) be such a number that \( d_F(\xi_1, \gamma(t)) = \frac{\Delta}{\Lambda} \). Then \( d_F(\gamma(t), \xi_2) = \frac{\Delta}{\Lambda} \). Since \( d_F \) is asymmetric then \( d_F(\xi_2, \gamma(t)) \leq \frac{\Delta}{\Lambda} \). \( \square \)

**Proof of Theorem 4.1.** To begin, let \( x, y \in T_U, U \in \mathcal{U} \) be \((n, \varepsilon)\)-separated with respect to \( h(\mathcal{H}_U, \mathcal{H}^1_U) \). Then there exists a chain of maps \((U_1, \ldots, U_n)\) such that the corresponding chains of plaques \((P_1, \ldots, P_n)\) and \((Q_1, \ldots, Q_n)\) with \( x \in P_1, y \in Q_1, P_i, Q_i \in U_i, P_i \cap P_{i+1} \neq \emptyset, Q_i \cap Q_{i+1} \neq \emptyset \) satisfy
\[
d(x_n, y_n) \geq \varepsilon \text{ or } d(y_n, x_n) \geq \varepsilon
\]
where \( x_n \in P_n \cap T_{U_n} \) and \( y_n \in Q_n \cap T_{U_n} \) is the images in the holonomy map determined by \((U_1, \ldots, U_n)\) of \( x \) and \( y \), respectively. Let \( x_0 = x \) and let us choose points \( x_i \in P_i \cap P_{i+1}, i = 1, \ldots, n-1 \). Link the points \( x_i \) and \( x_{i+1} \) by a leaf geodesic \( \gamma_i, i = 1, \ldots, n-1 \). Then the length of every \( \gamma_i \) is smaller than \( \text{diam}(\mathcal{U}) \), and the length of a curve \( \gamma \) built of \( \gamma_i \)'s and linking \( x_0 \) with \( x_n \) is smaller than \( n \cdot \text{diam}(\mathcal{U}) \). Shortening \( \gamma \), if necessary, we can assume that the distance between \( \gamma \) and its orthogonal projection \( p_y \gamma \) is always smaller than \( \varepsilon_0 \), and the whole \( \gamma \) can be projected to \( L_y \).

Since \( T = \bigsqcup T_U \) is compact, there exists a constant \( C > 0 \) such that
\[
\frac{1}{C}d(z, w) \leq d(z, p(z)) \leq Cd(z, w)
\]
and
\[
\frac{1}{C}d(w, z) \leq d(p(z), z) \leq Cd(w, z)
\]
if only $z, w \in T_U, U \in \mathcal{U}$, and $p(z)$ is the orthogonal projection of $z$ to the plaque $P_w$ passing through $w$. Hence $d(\gamma(1), p_y \gamma(1)) \geq \frac{\varepsilon}{C}$. This gives that $x$ and $y$ are $(n \cdot \text{diam}(U), \frac{\varepsilon}{C})$-separated with respect to $\mathcal{F}$. Thus,

$$s(n, \varepsilon, \mathcal{H}_{\mathcal{U}}^1) \leq s(n \cdot \text{diam}(U), \frac{\varepsilon}{C}, \mathcal{F})$$

for all $n \in \mathbb{N}$, and $\varepsilon \in (0, \varepsilon_0)$. Finally,

$$h(\mathcal{H}_{\mathcal{U}}, \mathcal{H}_{\mathcal{U}}^1) \leq \text{diam}(U) \cdot h(\mathcal{F}, F).$$

Let $\eta > 0$, and $\Delta > 0$ be such that the leafwise exponential mapping $\exp^F$ maps the balls $B^F(0, 4\Delta)$, where $B^F(0, r) = \{ v \in T_x \mathcal{F} : F^F(v) < r \}$, diffeomorphically onto strictly convex balls

$$B_F(x, 4\Delta) = \{ y \in L_x : d_F(x, y) < 4\Delta \}, \quad x \in M.$$ 

Let $\rho > 0$, and let $S_x = \exp B_\perp(0_x, 2\rho)$ be the image in the exponential map on $M$ where $B_\perp$ is a ball centered in $0_x$ and contained in the orthogonal complement $T_x \mathcal{F}_\perp$ of $T_x \mathcal{F}$. Set

$$T_x = \exp B_\perp(0, \rho), \quad U_x = \bigcup_{y \in T_x} B_F(y, \frac{\Delta}{\Lambda}).$$

Note that for small enough $\rho$ and $\Delta$, the sets $U_x$ are the domains of distinguished charts. Moreover, for any plaque $P \subset U_x$, the diameter $\text{diam}(P_x) \leq (1 + \frac{1}{\Lambda})\Delta$.

Now, let $\mathcal{U}_\Delta = \{ U_z, z \in \mathcal{Z} \}$. We may assume that the closures $\bar{U}_z$ and $\bar{U}_{z'}$, $z, z' \in \mathcal{Z}$, overlap if only $\bar{U}_z$ and $\bar{U}_{z'}$ do. Thus $\mathcal{U}_\Delta$ is a nice covering of $(M, \mathcal{F})$. Moreover, $\text{diam}(\mathcal{U}_\Delta) \leq (1 + \frac{1}{\Lambda})\Delta$.

Let $\varepsilon > 0$, and let $x, y$ be such that

$$d(x, y) \leq \varepsilon \text{ and } d(y, x) \leq \varepsilon$$

and additionally they are $(R, \varepsilon)$-separated by $\mathcal{F}$ with respect to $F$. Hence, there exists a curve $\gamma : [0, R] \to L_x$ starting at $x$ with $l(\gamma) \leq R$ and such that $p_y \gamma$ is well defined on $[0, r], r < R$, and $d(\gamma(r), p_y \gamma(y)) \geq \varepsilon$. Let us assume that $R = (1 + \frac{1}{\Lambda})(1 - \eta)n\Delta$, and let $x_k = \gamma(\frac{kn}{n}), k = 0, \ldots, n$. For each $x_k$ let us find a point $z_k \in \mathcal{Z}$ which is $\beta$-close (see Lemma 4.3).
The charts \((U_{z_0}, \ldots, U_{z_n})\) form a chain along \(\gamma|_{[0,r]}\), and the corresponding holonomy map \(h \in H_{\Delta}\) is well defined on the plaques \(P, Q \in U_{z_0}\) containing \(x\) and \(y\), respectively. Moreover,

\[
d(h(P), h(Q)) \geq C \cdot \varepsilon,
\]

where \(C\) is the constant from the first part of this proof. We deduce that

\[
s((1 + \frac{1}{\Lambda})(1 - \eta)n \Delta, C\varepsilon, \mathcal{F}) \leq N(\varepsilon) \cdot s(n, \varepsilon, H_{\Delta}),
\]

with \(N(\varepsilon)\) being the minimal cardinality of a covering of \(M\) by balls of radius \(\varepsilon\). Therefore,

\[
s(C\varepsilon, \mathcal{F}) \leq \frac{1}{(1 + \frac{1}{\Lambda})(1 - \eta)\Delta} s(\varepsilon, H_{\Delta}).
\]

Passing with \(\eta\) to zero, we obtain

\[
h(\mathcal{F}, \mathcal{F}) \leq \frac{1}{(1 + \frac{1}{\Lambda})\Delta} h(H_{\Delta}, H_{\Delta}^1) \leq \frac{1}{\text{diam}(\mathcal{U})} h(H_{\Delta}, H_{\Delta}^1).
\]

This ends the proof. \(\square\)

5. Foliations with leafwise Randers norm

Let \((M, \mathcal{F}, g)\) be a foliated Riemannian manifold. Let \(F\) be a leafwise Randers norm, that is the norm given by

\[
F(v) = \sqrt{g(v, v) + \beta(v)}, \quad v \in T\mathcal{F}.
\]

Let \(\|\beta\| = \max_{v \in T^1_{g,F}} \beta(v)\). As in Example 2.1, \(\|\beta\| < 1\).

**Theorem 5.1.** The following inequalities hold:

\[
\frac{1}{1 + \|\beta\|} h(\mathcal{F}, g) \leq h(\mathcal{F}, F) \leq \frac{1}{1 - \|\beta\|} h(\mathcal{F}, g).
\]

**Proof.** Let \(g(v) = \sqrt{g(v, v)}\). Since \(F(v) = g(v) + \beta(v)\) then for any \(v \in T\mathcal{F}\)

\[
F(v) \leq g(v) + \|\beta\| g(v) \quad \text{and} \quad g(v) \leq F(v) + \|\beta\| g(v).
\]

(5.1)

Let \(x, y\) be \((R, \varepsilon)\)-separated with respect to \(g\). So, there exists a curve \(\gamma : [0, 1] \to L_x\) such that \(\gamma(0) = x, l_g(\gamma) \leq R\) and

\[
d(\gamma(1), p_y \gamma(1)) \geq \varepsilon.
\]
Using the first inequality in (5.1), we obtain
\[ l_F(\gamma) = \int_0^1 F(\dot{\gamma}(t))dt \leq \int_0^1 g(\dot{\gamma}(t))dt + \int_0^1 \|\beta\|g(\dot{\gamma}(t))dt \]
\[ \leq R + \|\beta\|R = (1 + \|\beta\|)R. \]
Thus \( x, y \) are \((1 + \|\beta\|)R, \varepsilon\)-separated with respect to \( F \). Hence,
\[ s(R, \varepsilon, g) \leq s((1 + \|\beta\|)R, \varepsilon, F), \]
\[ \frac{1}{R} \log s(R, \varepsilon, g) \leq \frac{1 + \|\beta\|}{1 + \|\beta\|} \frac{1}{R} \log s((1 + \|\beta\|)R, \varepsilon, F), \]
\[ \limsup_{R \to \infty} \frac{1}{R} \log s(R, \varepsilon, g) \leq (1 + \|\beta\|) \limsup_{R \to \infty} \frac{1}{1 + \|\beta\|} \frac{1}{R} \log s((1 + \|\beta\|)R, \varepsilon, F), \]
\[ s(\varepsilon, F, g) \leq (1 + \|\beta\|)s(\varepsilon, F). \]
Finally, \( h(F, g) \leq (1 + \|\beta\|)h(F, F). \)

The second inequality follows directly from the second inequality in (5.1) and from the fact that every two points which are \((R, \varepsilon)\)-separated with respect to \( F \) are \((\frac{R}{1 - \|\beta\|}, \varepsilon)\)-separated with respect to \( g \). \( \square \)

6. Topological entropy of one dimensional foliation

We will now recall the definition (following [2] and [4]) of the topological entropy of a uniformly continuous map on a quasi-metric space.

Let \( f : X \to X \) be a uniformly continuous transformation of a quasi-metric space \( X \), that is, for any \( \varepsilon > 0 \) and any \( x \in X \) there exists \( \delta > 0 \) such that for any \( y \in X \)
\[ (d(x, y) < \delta \text{ and } d(y, x) < \delta) \]
\[ \Rightarrow (d(f(x), f(y)) < \delta \text{ and } d(f(y), f(x)) < \delta). \]

For any \( n \in \mathbb{N} \) and \( x, y \in X \) let
\[ d_n(x, y) = \max_{0 \leq k \leq n-1} \{\max\{d(f^k(x), f^k(y)), d(f^k(y), f^k(x))\}\}, \ k \in \mathbb{N}. \]
Let \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). A subset \( A \) of \( X \) is said to be \((n, \varepsilon)\)-separated if \( d_n(x, y) > \varepsilon \) for every \( x, y \in A, x \neq y \). A set \( B \subset X \) is said to
\((n, \varepsilon)\)-span another set \(K\) if for every \(x \in K\) there is \(y \in B\) such that \(d_n(x, y) \leq \varepsilon\).

We set \(s(n, \varepsilon, K) = \max\{\#A : A \subset X \text{ is } (n, \varepsilon) - \text{separated}\}\), and \(r(n, \varepsilon, K) = \min\{\#A : A \subset K \text{ is } (n, \varepsilon) - \text{spanning}\}\).

**Lemma 6.1.** The following inequalities hold

1. \(r(n, \varepsilon, K) \leq s(n, \varepsilon, K) \leq r(n, \frac{\varepsilon}{2}, K) < \infty\),
2. for \(\varepsilon' < \varepsilon\)
   \[r(\varepsilon', K) \geq r(\varepsilon, K) \text{ and } s(\varepsilon', K) \geq s(\varepsilon, K)\].

**Proof.** If \(A\) is maximal \((n, \varepsilon)\)-separated subset of \(K\), then \(A\) also \((n, \varepsilon)\)-spans \(K\). Thus \(r(n, \varepsilon, K) \leq s(n, \varepsilon, K)\).

Let \(A \subset K\) be an \((n, \varepsilon)\)-separated set and let \(B(n, \frac{1}{2} \varepsilon)\)-spans \(K\). For any \(x \in K\), there exists \(g(x) \in B\) such that \(d_n(x, g(x)) < \frac{\varepsilon}{2}\). Moreover, if \(g(x) = g(y)\) then \(d_n(x, y) < \varepsilon\). Thus \(g\) is injective on \(A\) (since \(A\) is \((n, \varepsilon)\)-separated), and \(s(n, \varepsilon, K) \leq r(n, \frac{\varepsilon}{2}, K)\).

As \(f\) is uniformly continuous on \((X, d)\) there is a \(\delta > 0\) such that \(d_n(x, y) < \frac{\varepsilon}{2}\) if only \(d(x, y) < \delta\) and \(d(y, x) < \delta\). Thus \(r(n, \frac{\varepsilon}{2}, K)\) does not exceed the number of \(\delta\)-balls \(B_\delta(z) = \{z' \in X : d(z, z') < \delta \text{ and } d(z', z) < \delta\}\) needed to cover \(K\). So, \(r(n, \frac{\varepsilon}{2}, K)\) is finite, as \(K\) is compact.

The inequalities in (2) are obvious. \(\square\)

Finally, we define
\[s(\varepsilon, K) = \lim_{n \to \infty} \sup \frac{1}{n} \log s(n, \varepsilon)\]
and
\[r(\varepsilon, K) = \lim_{n \to \infty} \sup \frac{1}{n} \log r(n, \varepsilon)\].

**Definition 6.2.** For any uniformly continuous map \(f : X \to X\) on a quasi-metric space \((X, d)\) and any compact set \(K \subset X\) define

\[h_{\text{top}}(f, K) = \lim_{\varepsilon \to 0^+} s(\varepsilon, K) = \lim_{\varepsilon \to 0^+} r(\varepsilon, K)\]
and
\[h_{\text{top}}(f) = \sup_{K \text{ compact}} h_{\text{top}}(f, K)\].
The number $h_{\text{top}}(f)$ is called the *topological entropy of* $f$.

Let us now study the geometrical entropy of a foliation given by the integral curves of a vector field $X$ on a compact manifold $M$. Let $F$ be a leafwise Finsler structure $F$ for which $X$ is a unit vector field, that is $F(X(p)) = 1$ for all $p \in M$. Let $\varphi = (\varphi_t : M \to M)_{t \in \mathbb{R}}$ denote the flow of $X$. We recall [6] that the topological entropy of a flow is equal to $h_{\text{top}}(\varphi_1)$.

Let $\Lambda = \max_{v \in T_x F} \frac{F(v)}{F(-v)}$.

**Theorem 6.3.** $(1 + \frac{1}{\Lambda}) h_{\text{top}}(\varphi) \leq h(F, F) \leq (1 + \Lambda) h_{\text{top}}(\varphi)$.

**Proof.** By Theorem 3.4.3 in [6], there exists a Riemannian structure, for which $h(F, g) = 2 h_{\text{top}}(\varphi)$. Note that every leafwise Finsler structure along one-dimensional foliation is a Randers norm for some leafwise Riemannian structure. Indeed, let $v \in T_x F$ be $F$-unit vector such that $F(-v) = a(x) < 1$. We choose these leafwise Riemannian metric $g$ for which $g(v) = \sqrt{g(v, v)} = \frac{1}{2} + a(x)$, and 1-form $\beta$ such that $\beta(v) = \frac{1}{2} - \frac{a(x)}{2}$.

Let $F(v) = g(v) + \beta(v)$. With results of Theorem 5.1, we have

$$h(F, F) \leq \frac{1}{1 - \|\beta\|} h(F, g) \leq \frac{2}{1 - \|\beta\|} h_{\text{top}}(\varphi) \leq (1 + \frac{1 + \|\beta\|}{1 - \|\beta\|}) h_{\text{top}}(\varphi).$$

However, $\Lambda = \frac{1 + \|\beta\|}{1 - \|\beta\|}$. This gives the second inequality.

To prove the opposite, it is enough to observe that

$$h(F, F) \geq \frac{1}{1 + \|\beta\|} h(F, g) \geq \frac{2}{1 + \|\beta\|} h_{\text{top}}(\varphi) \geq (1 + \frac{1 - \|\beta\|}{1 + \|\beta\|}) h_{\text{top}}(\varphi).$$

This ends our proof. \qed

Now let us return to the case of Randers norms.

**Corollary 6.4.** Let $\mathcal{F}$ be a one-dimensional foliation given by a vector field $X$. Let $F$ be a leafwise Randers norm given by $F(v) = \sqrt{g(v, v)} + \beta(v)$. With results of Theorem 5.1, we have

$$h(F, F) \leq \frac{1}{1 - \|\beta\|} h(F, g) \leq \frac{2}{1 - \|\beta\|} h_{\text{top}}(\varphi) \leq (1 + \frac{1 + \|\beta\|}{1 - \|\beta\|}) h_{\text{top}}(\varphi).$$

However, $\Lambda = \frac{1 + \|\beta\|}{1 - \|\beta\|}$. This gives the second inequality.

To prove the opposite, it is enough to observe that

$$h(F, F) \geq \frac{1}{1 + \|\beta\|} h(F, g) \geq \frac{2}{1 + \|\beta\|} h_{\text{top}}(\varphi) \geq (1 + \frac{1 - \|\beta\|}{1 + \|\beta\|}) h_{\text{top}}(\varphi).$$

This ends our proof. \qed

Now let us return to the case of Randers norms.
\( \beta(v) \) where \( \beta(X(p)) = a, \ a < \frac{1}{2}, \ p \in M, \) and such that \( F(X(p)) = 1. \) Then \( h(\mathcal{F}, F) = (1 + \frac{1}{1 - 2a})h_{\text{top}}(\varphi) \) where \((\varphi)_{t \in \mathbb{R}}\) denotes the flow of \( X. \)

**Proof.** Obviously,

\[
\Lambda = \max_{v \in \mathcal{F}} \frac{F(v)}{F(-v)} = \begin{cases} 
\frac{1}{1 - 2a} & \text{if } a \in [0, \frac{1}{2}) \\
1 - 2a & \text{if } a < 0 
\end{cases}
\]

Thus,

\[
h(\mathcal{F}, F) \leq (1 + \frac{1}{1 - 2a})h_{\text{top}}(\varphi) \text{ for } a \in (0, \frac{1}{2})
\]

and

\[
(1 + \frac{1}{1 - 2a})h_{\text{top}}(\varphi) \leq h(\mathcal{F}, F) \text{ for } a < 0.
\]

Let \( a < 0. \) Since

\[
\int_{0}^{\frac{n}{\Lambda}} F(-X(\varphi_{t-x})(x))dt = n
\]

then \( \varphi_{-\frac{n}{\Lambda}}A \) is \(((1 + \frac{1}{\Lambda})n, \epsilon)\)-separated by \( \varphi. \) Thus

\[
h(\mathcal{F}, F) \leq (1 + \frac{1}{\Lambda})h_{\text{top}}(\varphi) = (1 + \frac{1}{1 - 2a})h_{\text{top}}(\varphi).
\]

Let \( a > 0. \) We shall repeat the second part of the proof of Theorem 3.4.3 in [6].

Let \( \eta > 0. \) and let us consider the fiber bundle \( \pi : \tilde{M}_{\eta} \to M \) built out of orthogonal balls \( B_{\perp}(0, \eta) \subset T_{\perp}M, \ x \in M. \) For every \( x \in M, \) let \( \text{Tub}_{\eta}(x) = \varphi_{x}^{-1}\tilde{M}_{\eta} \) be the bundle over \( \mathbb{R} \) induced by a map \( \varphi_{x} : t \mapsto \varphi_{t}(x). \) It is known, that for \( \eta \) small enough, the exponential map on \( M \) defines a natural immersion \( \iota_{x} : \text{Tub}_{\eta}(x) \to M, \) and one can equip \( \text{Tub}_{\eta}(x) \) with the induced leafwise Finsler structure and with the induced vector field \( \tilde{X}, \) which generates a local flow \((\tilde{\varphi}_{t})\). As mentioned in [6], the family \( \pi^{-1}(s), \ s \in \mathbb{R}, \) of fibers of \( \text{Tub}_{\eta}(x) \) is not invariant under the flow \((\tilde{\varphi}_{t})\).

Let us fix \( \epsilon > 0. \) Since \( M \) is compact, the family \( \pi^{-1}(s), \ s \in \mathbb{R}, \) of fibers of \( \text{Tub}_{\eta}(x) \) satisfies the following:

For any \( \tau \in (0, 1) \) there exists \( \eta > 0 \) such that for any \( x \in M \) and \( y \in \text{Tub}_{\eta}(x) \cap \pi^{-1}(0) \) with defined local flow \((\tilde{\varphi}_{t})\) and \( \pi(\tilde{\varphi}_{t}(y)) = 1 \) (respectively \( \pi(\tilde{\varphi}_{t}(y)) = -1 \)) we have \( t > \tau \) \((t < -\tau)\). Moreover, if \( \tau \)
and $\eta$ are as above, then $t \geq n \tau$ ($t \leq -n \tau$) whenever $(\tilde{\varphi}_t)$ is defined and $\pi(\tilde{\varphi}_t(y)) = n$ (respectively, $-n$), $n \in \mathbb{N}$.

Let us decompose $\text{Tub}_n(x)$ into the cylinders $C_n(x) = \pi^{-1}([(2n - 1)\varepsilon, (2n + 1)\varepsilon])$, $n \in \mathbb{Z}$. Since $\varepsilon$ is fixed, there exists $\eta$ independent of $x \in M$ such that the sets $\tilde{\varphi}_1(C_0(x))$ and $\tilde{\varphi}_{-1}(C_0(x))$ intersect at most three cylinders of the form $C_n(x)$. For every $y \in C_0(x)$, we consider the sequences $(n_k)_{k \in \mathbb{N}}$ of integers such that $\tilde{\varphi}_k(y) \in C_{n_k}(x)$ (we set $\infty$ if $\tilde{\varphi}_k(y)$ is undefined). The number of such sequences of length $2n - 1$ do not exceed $3^{2n}$. So, we can decompose all cylinders $C_n(x)$ into the unions of sets $C_n(x) = K_1(x) \cup \cdots \cup K_{m(x)}(x)$, $m(x) \leq 3^{2n+2}$ consisting of these points $z \in C_0(x)$ for which $\tilde{\varphi}_k(z)$ is defined, $-\lfloor n\Lambda \rfloor - 1 \leq k \leq n$, and all $\tilde{\varphi}_k(z)$ belongs to the same cylinder $C_{n_k}(x)$.

Let $A \subset T$ be a maximal $(n, \frac{\Lambda}{3\varepsilon})$-separated by $\mathcal{F}$ with respect to $F$, that is, $\sharp A = s(n, \frac{\eta}{3\Lambda}, F)$. Since $A$ is maximal, then it is $(n, \frac{\eta}{3\Lambda})$-spanning for $T$, and the sets

$$A(x) = \{y \in T : \sup_{-\lfloor \frac{\Lambda}{\varepsilon} \rfloor - 1 \leq t \leq n} d(\varphi_t(x), p_y \varphi_t(x)) \leq \frac{\eta}{3\Lambda} \}$$

$$\text{and} \quad \sup_{-\lfloor \frac{\Lambda}{\varepsilon} \rfloor - 1 \leq t \leq n} d(p_y \varphi_t(x), \varphi_t(x)) \leq \frac{\eta}{3\Lambda}, \quad x \in A$$

cover $T$. Moreover, $\max \text{diam} A(x) \leq \frac{2n}{\varepsilon}$. Therefore, $A(x) \subset \eta_x(C_0(x))$.

We choose one point $y^x_j$ in each nonempty piece of $A(x) \cap K_j(x)$. Let $B = \{y^x_j\}$. We have

$$\sharp B \leq 3^{2n+2} s(n, \frac{\eta}{3\Lambda}, F).$$

Finally, let $y \in M$. There exists $R_0 > 0$ independent of $y$ such that $\varphi_t(y) \in T$ for some $t \in (-R_0, R_0)$. So there exists $x \in A$ and $j \leq m(x)$ for which $\varphi_t(y) \in A(x) \cap \eta_x K_j(x)$. Thus $\tilde{\varphi}_{t+i}(y)$ and $\varphi_i(y^x_j)$ belong to the same cylinder $C_{n(i)}(x)$, and

$$d(\varphi_{t+i}(y), \varphi_i(y^x_j)) \leq 2\Lambda\varepsilon + 2\eta \text{ and } d(\varphi_i(y^x_j), \varphi_{t+i}(y)) \leq 2\Lambda\varepsilon + 2\eta,$$

for all $i = 0, \pm 1, \ldots, \pm n\tau$. Note that there exists a constant $\omega$ such that for small $\eta$ and any $z, z' \in M$ the inequalities $d(z, z') < \eta$ and $d(z', z) < \eta$ implies the relations $d(\varphi_t(z), \varphi_t(z')) \leq c\eta$ and $d(\varphi_t(z'), \varphi_t(z)) \leq c\eta$. 

Therefore, the set $\varphi_{-r_n\Lambda}B$ is $(2\tau n, 2\omega(\eta + \Lambda \epsilon))$-spanning with respect to $\varphi$, and
\[
r((1 + \Lambda)\tau n, 2\omega(\eta + \Lambda \epsilon), \phi) \leq 3^{2n+2}s(n, \frac{\eta}{3\Lambda}, F).
\]
Thus
\[
(1 + \Lambda)r(2\omega(\eta + \Lambda \epsilon), \phi) \leq \frac{1}{\tau} \log 3 + \frac{1}{\tau} s(\frac{\eta}{3\Lambda}, F).
\]
This gives
\[
(1 + \Lambda)h_{\text{top}}(\varphi) \leq \log 3 + f(F, F),
\]
when we tend with $\eta$ and $\epsilon$ to zero, and choose $\tau$ arbitrarily close to 1.

Replacing $F$ by $\lambda F$, $\lambda > 0$ we obtain
\[
(1 + \Lambda)h_{\text{top}}(\varphi) \leq \lambda \log 3 + f(F, F).
\]
Since $\lambda$ can be arbitrarily small, we get the required inequality.

The case of $a = 0$ is obvious. This ends our proof. \qed

**Remark 6.5.** Let $\mathcal{F}$ be a one-dimensional foliation given by a vector field $X$. If $F$ is Riemannian, then we get the exact result as in Theorem 3.4.3 of [6], that is,
\[
h(\mathcal{F}, F) = 2h_{\text{top}}(\varphi).
\]

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