A DISCRETIZED VERSION OF KALUZA-KLEIN THEORY
WITH TORSION AND MASSIVE FIELDS

Nguyen Ai Viet and Kameshwar C. Wali

Department of Physics, Syracuse University,
Syracuse, NY 13244-1130, U.S.A.

Abstract
We consider an internal space of two discrete points in the fifth dimension of the
Kaluza-Klein theory by using the formalism of noncommutative geometry developed
in a previous paper of a spacetime supplemented by two discrete points. With the
nonvanishing internal torsion 2-form there are no constraints implied on the vielbeins.
The theory contains a pair of tensor, a pair of vector and a pair of scalar fields. Using
the generalized Cartan structure equation we are able not only to determine uniquely
the hermitian and metric compatible connection 1-forms, but also the nonvanishing
internal torsion 2-form in terms of vielbeins. The resulting action has a rich and
complex structure, a particular feature being the existence of massive modes. Thus
the nonvanishing internal torsion generates a Kaluza-Klein type model with zero and
massive modes.

* On leave of absence from the High Energy Division, Centre of Theoretical Physics, P.O.Box 429
Bo Ho 10000, Hanoi Vietnam.
† Address after September 1, 1995: Physics Department, Rockefeller University, New York, NY, USA
1 Introduction

It is generally believed that our current description of spacetime underlying both classical physics as well as quantum field theories is unsatisfactory and inadequate to deal with the description of phenomena at short distances. One is seeking a mathematical formalism that provides a quantum description of natural phenomena that, a priori, does not speak about spacetime in its basic formulation, but spacetime of classical physics as well as quantum field theories emerges in certain limiting regimes—just as classical behaviour of quantum systems can emerge in certain limiting regimes [2]. The recent proposal of Connes [3] and the so called noncommutative geometry (NCG) appears very promising towards the achievement of such a goal. It has given rise to the description of the Standard Model [4] with new insights as regards spontaneous symmetry breaking and quark and lepton masses. It is natural to ask whether and how the classical general relativity fits into the scheme of NCG.

The first step in answering this question was taken by Chamseddine et al [5], whose starting point was an abstract two-sheeted continuum that could be considered as the direct product of a single spacetime continuum and two discrete points. This led to gravity coupled to a Brans-Dicke scalar field. The scalar field can be interpreted as the distance between the two sheets [6].

Similarly, it is always extremely tempting to give geometrical meaning to other physical fields. Thus, in the traditional Kaluza-Klein theory massless tensor, vector and scalar fields together with their massive excitations appear as result of extending the physical four-dimensional spacetime by an additional continuous fifth dimension.

† More recently, other authors using different approaches have obtained essentially the same result. See Ref. [6] for references to related work.
Unfortunately, the massive modes are infinite in number. In a previous paper [1], we have developed the formalism for a discretized version of Kaluza-Klein theory within the framework of NCG. The starting point, as in Ref.[1], is an extended spacetime that includes two discrete points of the continuous internal fifth dimension of the Kaluza-Klein theory. We presented a generalization of the usual Riemannian geometry in the new context that demanded a vielbein consisting, to begin with, a pair of tensor, a pair of vector and a pair of scalar fields. Following the usual steps in building a theory of gravitation with the new geometry, we imposed torsion free, metric compatibility conditions on the connection 1-forms from which we constructed the action through the Ricci scalar curvature. We found that the imposed conditions altered the field content of the theory in a dramatic way, requiring in addition to the tensor, vector and scalar fields, new dilaton-like dynamical fields. The connection 1-forms and hence the Ricci scalar curvature were determined uniquely in terms of these fields. The resulting action provided a rich structure that lent itself to intriguing interpretations. One of the dilaton fields, for instance, could give rise to masses and cosmological constant. Moreover by imposing a reality condition on the vielbein 1-forms we could make the dilaton fields disappear leading to the zero-mode sector of the Kaluza-Klein theory as in Ref.[6]. The previous NCG models that contain gravity coupled to the Brans-Dicke scalar can be considered as a particular case when the vector field is set to zero.

While these interpretations are interesting in themselves to merit further study, we seek in this paper a formulation that does not alter the initial field content of the theory of two independent tensor, vector and scalar fields. From the viewpoint of the underlying mathematical framework of NCG, this is a reasonable requirement:
the vielbein 1-forms should be free of constraints, retaining their most general form. The problem is how to achieve this. Now for physical reasons, it is necessary that we impose the metric compatibility condition. We recall that in the ordinary Cartan-Riemannian geometry the vanishing of torsion yields unique connection 1-forms in terms of the metric coefficients and their derivatives. Non-vanishing torsion requires additional information besides the metric. In our formulation, we find a way to avoid this situation. We impose a reality condition on the connection 1-forms and release the strict torsion free condition. In order to keep as close as possible to the usual Riemannian geometry, we assume that the usual spacetime indexed torsion 2-forms do vanish. However, we do not assume that the discrete internal space indexed torsion 2-form vanishes. This results in the unique determination of the related connection 1-forms. As we shall see, the nonvanishing internal torsion 2-form can be also determined in terms of the given vielbeins. This way we have an action that describes the general field content that we started with initially. The most remarkable result is that this discrete version of Kaluza-Klein theory contains a finite number of massive modes.

The paper is organized as follows: In the next section we will review briefly the basic formalism and give the necessary formulas in order to make this paper self-contained. In Sect.3, we discuss how we compute the connection 1-forms, internal torsion and the Ricci scalar curvature. In Sect.4, we present the general structure of the action and consider special cases. The final section is devoted to a summary and conclusions.

2 Two-point internal space and vielbein
2.1 Algebra of smooth functions and generalized derivatives

We consider a physical space-time manifold $M$ extended by a discrete internal space of two points to which we assign a $\mathbb{Z}_2$-algebraic structure. With this extended space-time, the customary algebra of smooth functions $C^\infty(M)$ is generalized to $\mathcal{A} = C^\infty(M) \oplus C^\infty(M)$ and any generalized function $F \in \mathcal{A}$ can be written as

$$F(x) = f_+(x)e + f_-(x)r,$$

where $e, r \in \mathbb{Z}_2 = \{e, r \mid e^2 = e, r^2 = e, er = re = r\}$. We adopt a $2 \times 2$ matrix representation for $e, r$:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2.2)

Then the function $F(x)$ assumes a $2 \times 2$ matrix form,

$$F = f_+(x)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + f_-(x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix},$$

(2.3)

where

$$f_\pm(x) = 1/2.(f_1(x) \pm f_2(x)).$$

(2.4)

In this paper we will use small letters to denote the quantities of ordinary geometry and capital letters for generalized quantities of NCG.

With the algebra $\mathcal{A}$ of smooth functions, we have what we may consider as the algebra of the generalized 0-forms $\Omega^0(M) = C^\infty(M) \oplus C^\infty(M)$. To build the corresponding generalized higher forms, we need an exterior derivative or the Dirac operator $D \mathbb{Z}_2$ in the language of NCG. For this purpose, as in Ref.[3], let us define derivatives...
\(D_N(N = \mu, 5)\) by

\[
D_\mu = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix}, \quad \mu = 0, 1, 2, 3,
\]

\[
D_5 = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \quad (2.5)
\]

where \(m\) is a parameter with dimension of mass. We specify the action of the derivatives on the 0-form elements as given by

\[
D_N(F) = [D_N, F], \quad N = \mu, 5, \quad (2.6)
\]

satisfying the Newton-Leibnitz rule,

\[
D_N(FG) = D_N(F)G + FD_N(G). \quad (2.7)
\]

Then the exterior derivative operator \(D\) is given by

\[
D \equiv (DX^\mu D_\mu + DX^5 \sigma^\dagger D_5), \quad (2.8)
\]

where

\[
\sigma^\dagger = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.9)
\]

\(DX^M\) are in general 2\(\times\)2 matrices that form a basis of the generalized 1-forms. They are direct generalizations of the differential elements. When spacetime becomes curved, as in general relativity (GR), \(DX^M\) denote a generalized curvi-linear differential elements. Their concrete form can be given in a concrete basis. The explicit form of \(DX^M\) in the orthonormal basis will be given in the next subsection.
2.2 General and orthonormal basis of 1-forms

The possible metric structure is guaranteed by the existence of a local orthonormal basis: the vielbein $E^A$. Analogously to GR, if we work in the locally flat basis the vielbein $E^A$ can be chosen to be orthonormal. In Ref.[1], we chose a diagonal representation for the curvi-linear basis $DX^\mu$ and $DX^5\sigma^\dagger$ to construct generalized one- and higher forms in analogy with the usual Riemannian geometry. However, it is more convenient to work in a representation in which the vielbeins $E^A(A = a, \dot{5})$ are diagonal. Locally, $E^A$ is given as follows

\[
E^a = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix},
E^{\dot{5}} = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix},
\]

(2.10)

where $e^a$ is some ordinary vierbein 1-forms and $\theta$ is some hermitian Clifford element§.

In this basis the wedge product can be defined as follows

\[
E^A \wedge E^B = -E^B \wedge E^A.
\]

(2.11)

In the orthonormal and locally flat basis $E^A$, the curvi-linear differential elements $DX^M$ are in general not diagonal any more. Conversely, we can choose to work in the representation in which $DX^M$ are diagonal. Then $E^A$ is not diagonal anymore as discussed in Ref.[1]. Both basis span the space of generalized 1-forms; hence an

§ Completely, in analogy with GR, we can represent $e^a$ and $\theta$ as the locally flat Dirac matrices $\gamma^a$ and $\gamma^5$ as in the spinorial representation of Connes-Lott model.( This representation is used widely in literature. See for example [6, 7, 8] for details. However, in our formalism the two sheets are not necessarily the ones of different chiralities. Hence $\theta$ in general will be kept as an abstract Clifford element).
arbitrary 1-form $U$ in NCG is given by

$$U = E^A U_A = DX^M U_M,$$  \hfill (2.12)

where $U_A$ and $U_M$ are the components of the 1-form $U$ in the $E^A$ and $DX^M$ basis respectively. As $E^A$ and $DX^M$ themselves are also 1-forms, we can express them in terms of each other as follows

$$E^A = DX^M E^M_A, \quad DX^M = E^A E^M_A,$$  \hfill (2.13)

where $E^M_A$ and $E^A_M$ are generalized functions satisfying

$$E^A_N E^B_N = \delta^A_B, \quad E^A_N E^M_A = \delta^M_N.$$  \hfill (2.14)

Without any loss of generality we can choose $E^A_M$ as follows:

$$E^a_\mu = \begin{pmatrix} e^1_\mu(x) & 0 \\ 0 & e^2_\mu(x) \end{pmatrix}, \quad E^a_5 = 0$$

$$E^5_\mu = \begin{pmatrix} a_1(\varphi_1(x)) & 0 \\ 0 & a_2(\varphi_2(x)) \end{pmatrix} = A_\mu, \quad E^5_5 = \begin{pmatrix} \varphi_1(x) & 0 \\ 0 & \varphi_2(x) \end{pmatrix} = \Phi,$$  \hfill (2.15)

(We use a 5 index in the orthonormal basis to distinguish it from the index 5 in the curvi-linear basis). Thus

$$DX^\mu = E^a_\mu E^a_\mu, \quad DX^5 = (E^5 - E^a_\mu A_\mu)\Phi^{-1},$$  \hfill (2.16)

where $A_\mu = E^a_\mu A_a$.  

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Now we can derive the transformation rules for the components of an arbitrary 1-form $U$ between the two basis

$$
U_a = E_a^\mu(U_\mu + A_\mu U_5) , \quad U_5 = \Phi^{-1} U_5 ,
$$

$$
U_\mu = E_\mu^a U_a - A_\mu \Phi U_\dot{5} , \quad U_\dot{5} = \Phi U_\dot{5} . \quad (2.17)
$$

To this end we note that the exterior derivative of a general 1-form $U = DX^M U_M = E^A U_A$ is given by

$$
DU = (DX^\mu + DX^5 \sigma^\dot{5} D_\dot{5}) U ,
$$

$$
= E^a \wedge E^b (DU)_{ab} + E^a \wedge E^5 2(DU)_{a\dot{5}} . \quad (2.18)
$$

Using Eq.(2.16), we find

$$(DU)_{bc} = \frac{1}{2} E^\mu_b E^\nu_c (\partial_\mu E^a_\nu - \partial_\nu E^a_\mu) U_a - \frac{1}{2} E^\mu_b E^\nu_c (\partial_\mu A_\nu - \partial_\nu A_\mu) \Phi U_\dot{5}
$$

$$
+ \frac{1}{2} (E^\mu_b \partial_\mu U_c - E^\nu_c \partial_\nu U_b) + \frac{m}{2} ((A_\mu E^\nu_c - A_c E^\nu_\mu) E^a_\nu U_a + (A_\mu U_\dot{5} - A_\mu \Phi) U_\dot{5}) ,
$$

$$(DU)_{b\dot{5}} = \frac{1}{2} \tilde{E}^\mu_b \left( \frac{\partial_\mu \Phi}{\Phi} U_\dot{5} + \partial_\mu U_\dot{5} \right) + \frac{m}{2} (\Phi^{-1}(\tilde{U}_b - \tilde{E}^\mu_b E^a_\mu U_a) + (A_\mu \Phi - A_\mu) U_\dot{5}) ,
$$

(2.19)

where we have redefined $A_\mu$ in Eq.(2.13) as $-A_\mu \Phi^{-1}$.

In the $E^A$ basis the hermitian conjugate of an arbitrary 1-form $U = E^a U_a + E^5 U_\dot{5}$ is the 1-form $U^\dagger = E^a U_a + E^5 \tilde{U}_\dot{5}$ where

$$
\tilde{F} = \begin{pmatrix} f_2 & 0 \\ 0 & f_1 \end{pmatrix} , \quad \text{for any function } F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} . \quad (2.20)
$$
In the orthonormal basis, we have chosen $E^A$ to be hermitian, the general 1-forms need not be hermitian, neither does the $DX^M$ basis.

### 2.3 Generalized metric

Following Ref.[1, 3], we define the metric $G$ as the sesquilinear inner product of two 1-forms $U$ and $V$ satisfying

\[
< U F , V G > = F < U , V > G ,
\]
\[
< U \otimes R , V \otimes S > = R^\dagger < U , V > S ,
\]
\[ (2.21) \]

where $F, G$ are functions and $R, S$ are 1-forms. Assuming the existence of the local orthonormal basis $E^A$, we have

\[
< E^A , E^B > = \eta^{AB} ,
\]
\[ (2.22) \]

where $\eta^{AB} = signature( - , + + + + )$.

From Eqs. (2.21) and (2.22) we obtain the generalized metric tensor in the familiar form

\[
G_{MN} = E^A_M \eta_{AB} E^B_N ,
\]
\[
G^{MN} = E^M_A \eta^{AB} E^N_B .
\]
\[ (2.23) \]

With the vielbeins given in Eq.(2.13), the components of the metric tensors $G^{MN}$ and $G_{MN}$ turn out to be

\[
G^{\mu\nu} = G^{\mu\nu} \triangleq \left( \begin{array}{cc} g^{\mu\nu}_1 & 0 \\ 0 & g^{\mu\nu}_2 \end{array} \right) ,
\]
\[ G^{\mu 5} = A^\mu = G^{5\mu} , \]
\[ G^{55} = \Phi^{-2} + A^2 , \]
\[ G_{\mu\nu} = G_{\mu\nu} + A_\mu A_\nu \]
\[ G_{\mu 5} = G_{5\mu} = A_\mu \Phi , \]
\[ G_{55} = \Phi^2 . \] 

(2.24)

where \( g_{i\mu
u} = e_{ia}^\mu e^\nu_{ib} \), \( i = 1, 2 \) are the metric tensors on the two sheets.

In passing we note that the components of the metric tensor are identical in form with those in the 5-dimensional Kaluza-Klein theory except that in the present case, the usual continuous \( x^5 \)-dependence is replaced by the matrix form.

3 Connection, torsion and curvature

3.1 Hermitian and metric compatible connection 1-forms

We have shown in Ref.[1, 6] that the metric compatible or Levi-Civita connection 1-form \( \Omega_{AB} \) satisfies

\[ \Omega_{AB} = - \Omega_{BA} . \] 

(3.1)

In this paper we will impose an additional reality condition on the connection,

\[ \Omega_{AB} = \Omega_{AB} . \] 

(3.2)

Together with the metric compatibility condition (3.1) the reality condition implies the following conditions on the components \( \Omega_{ABC} \) of the connection 1-form \( \Omega_{AB} \) in the \( E^A \)

\footnote{The interested reader might see Ref.[1] for the relation between the connection and the covariant derivative.}
basis

\[ \Omega_{abc} = - \Omega^{bac}, \]
\[ \Omega_{ab\hat{5}} = - \Omega_{ba\hat{5}} = \omega_{ab\hat{5}}e, \]
\[ \Omega_{a\hat{5}b} = - \Omega_{\hat{5}ab}, \]
\[ \Omega_{a\hat{5}\hat{5}} = - \Omega_{\hat{5}a\hat{5}} = \omega_{a\hat{5}\hat{5}}e, \]
\[ \Omega_{\hat{5}\hat{5}a} = \Omega_{\hat{5}\hat{5}\hat{5}} = 0. \] (3.3)

That is to say, the reality condition (3.2) requires that the internal indexed components of the connection 1-forms are ordinary functions.

### 3.2 The first structure equation and torsion 2-forms

The first Cartan structure equation defines the torsion 2-forms \( T^A \) as given by:

\[ T^A = DE^A - E^B \wedge \Omega^A_B, \] (3.4)

In Ref. [1], we had assumed \( T^A = 0 \ (A = a, \hat{5}) \) to determine the connection \( \Omega \). As noted before, we were lead to a theory with a tensor, vector and scalar fields and also additional dilaton-like fields. In the present paper, we shall assume

\[ T_{abc} = T_{ab\hat{5}} = 0, \]
\[ T_{\hat{5}AB} = t_{\hat{5}AB}^r. \] (3.5)

In other words, the torsion 2-forms involving the external physical spacetime index vanish while the torsion 2-form involving the internal index \( \hat{5} \) as in Eq.(3.5) does not vanish. Then we can determine \( t_{\hat{5}AB} \) as well as the hermitian and metric compatible connection 1-forms \( \Omega_{AB} \) in terms of the vielbeins.
Using the general formula (2.19), it is straightforward to compute the exterior derivatives $DE^A$ needed to calculate $\Omega_{ABC}$ in Eq.(3.4). We omit the details and give only the results.

\[
(DE_a)_{bc} = -(DE_a)_{cb} = \frac{1}{2} \left[ (E^\mu_b E^\nu_c - E^\mu_c E^\nu_b) \partial_\mu E_{av} 
\right. \\
\left. + m \left( (A_b \tilde{E}^\nu_c - A_c \tilde{E}^\nu_b) E_{av} + (A_c \eta_{ab} - A_b \eta_{ac}) \right) \right],
\]

\[
(DE_a)_{b\dot{5}} = -(DE_a)_{\dot{5}b} = \frac{m}{2} \Phi^{-1} (\eta_{ab} - \tilde{E}^\mu_b E_a^\mu),
\]

\[
(DE_b)_{bc} = -(DE_b)_{cb} = -\frac{1}{2} \left[ (E^\mu_b E^\nu_c - E^\mu_c E^\nu_b) \Phi \partial_\mu A_{\nu} 
\right. \\
\left. + m (A_b \tilde{E}^\nu_c - A_c \tilde{E}^\nu_b) (\tilde{A}_\nu - A_\nu) \Phi \right],
\]

\[
(DE_{\dot{5}})_{b\dot{5}} = -(DE_{\dot{5}})_{\dot{5}b} = \frac{1}{2} \tilde{E}^\mu_b \left[ \frac{\partial_\mu \Phi}{\Phi} + m (A_\mu - \tilde{A}_\mu \Phi \tilde{\Phi}^{-1}) \right].
\] (3.6)

In component form, the first Cartan structure equation reduces to

\[
T_{ABC} = (DE_A)_{BC} - \frac{1}{2} (\Omega_{ABC} - \Omega_{ACB}).
\] (3.7)

With the condition (3.5) on the torsion 2-forms we obtain

\[
\Omega_{abc} = (DE_a)_{bc} + (DE_b)_{ca} - (DE_c)_{ab},
\] (3.8)

which in conjunction with Eqs.(3.6) determines $\Omega_{abc}$ in terms of vielbeins.

The condition (3.5) together with Eq.(3.3) leads to the following equation

\[
\Omega_{a\dot{5}b} - T^t_{\dot{5}ab} = (DE_a)_{b\dot{5}} + (DE_b)_{\dot{5}a} - (DE_{\dot{5}})_{ab},
\] (3.9)

from which we can determine $\Omega_{a\dot{5}b} = \omega_{a\dot{5}b} e$ and $T^t_{\dot{5}ab} = t^t_{\dot{5}ab} r$. 

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Finally, from
\[ \Omega_{5\delta c} = 2(T_{5\delta c} - (DE_{5})_{5c}) \]  \hspace{1cm} (3.10)
we can determine \( \Omega_{5\delta c} = \omega_{5\delta c}e \) and \( T_{5\delta c} = t_{5\delta c}r \).

The final results are as follows:

The components of the torsion 2-form \( T^\delta \) are

\[
T_{5ab} = \frac{1}{2} \tilde{E}_a^\mu \left[ \tilde{E}_b^\nu \tilde{F}_{\mu \nu} - E_b^\nu \Phi F_{\mu \nu} \right] + \frac{m}{4} \left[ (E'_a E_{b \mu} - E'_b E_{a \mu}) \tilde{\Phi}^{-1} \right. \\
+ \Phi^{-1}(\tilde{E}_a^\mu - E_b^\mu E_{a \mu}) + 2(\tilde{A}_\mu \left( (A_b \tilde{E}_b^\mu - A_b \tilde{E}_a^\mu) \Phi - (\tilde{A}_a E_b^\mu - A_b E_a^\mu) \tilde{\Phi} \right),
\]

\[
T_{5a\delta} = \frac{1}{4} \left[ (\tilde{E}_b^\nu \partial_\mu \Phi - E_b^\nu \partial_\mu \tilde{\Phi}) + m(\tilde{E}_b^\nu A_\mu - E_b^\nu \tilde{A}_\mu \\
+ A_b \tilde{\Phi} \Phi^{-1} - \tilde{A}_b \Phi \tilde{\Phi}^{-1}) \right]. \hspace{1cm} (3.11)
\]

The components of the connection 1-forms \( \Omega_{AB} \) are given by:

\[
\Omega_{abc} = \frac{1}{2} \left[ E_b^\mu E_c^\nu (\partial_\mu E_{a \nu} - \partial_\nu E_{a \mu}) + E_c^\mu E_a^\nu (\partial_\mu E_{b \nu} - \partial_\nu E_{b \mu}) - E_a^\mu E_b^\nu (\partial_\mu E_{c \nu} - \partial_\nu E_{c \mu}) \right] + \frac{m}{2} \left[ (A_b \tilde{E}_c^\nu - A_c \tilde{E}_b^\nu) E_{a \nu} + (A_c \tilde{E}_a^\nu - A_a \tilde{E}_c^\nu) E_{b \nu} - (A_a \tilde{E}_b^\nu - A_b \tilde{E}_a^\nu) E_{c \nu} \\
+ 2(A_a \eta_{cb} - A_b \eta_{ac}) \right],
\]

\[
\Omega_{ab\delta} = \frac{1}{4} \left( E_a^\mu E_b^\nu F_{\mu \nu} \Phi + \tilde{E}_a^\mu \tilde{E}_b^\nu \tilde{F}_{\mu \nu} \tilde{\Phi} \right) + \frac{m}{4} \left[ \Phi^{-1}(E_a^\mu E_{b \mu} - \tilde{E}_a^\mu E_{b \nu}) \\
+ \Phi^{-1}(E_a^\nu \tilde{E}_{b \mu} - E_b^\nu \tilde{E}_{a \mu}) + (\tilde{A}_\mu - A_\mu) \left( (\tilde{A}_b E_{b \nu} - \tilde{A}_b E_{a \nu}) \tilde{\Phi} \\
- (A_b \tilde{E}_b^\nu - A_b \tilde{E}_a^\nu) \tilde{\Phi} \right) \right] \\
- \frac{1}{4} \left( E_a^\nu E_b^\nu F_{\mu \nu} \Phi + \tilde{E}_a^\nu \tilde{E}_b^\nu \tilde{F}_{\mu \nu} \tilde{\Phi} \right) + \frac{m}{4} \left[ \Phi^{-1} \left( 4\eta_{ab} - (3 \tilde{E}_b^\nu E_{a \nu} + \tilde{E}_a^\nu E_{b \nu}) \right) \right] \\
\hspace{1cm} (3.11)
\]
\[ \Phi^{-1}(E^\mu_b \bar{E}_{a\mu} + E^\mu_a \bar{E}_{b\mu}) - (\bar{A}_\nu - A_\nu)(\bar{A}_a E^\nu_b - \bar{A}_b E^\nu_a) \Phi \\
- (A_a \bar{E}^\nu_b - A_b \bar{E}^\nu_a) \Phi \] \\
\Omega_{\hat{5}\hat{5}} = - \Omega_{\hat{5}\hat{5}} = 1/2 \left[ \bar{E}^\mu_b \partial_\mu \Phi + E^\mu_b \partial_\mu \bar{\Phi} \right] + \frac{m}{2} \left[ \bar{E}^\mu_b (A_\mu - \bar{A}_\mu \Phi \bar{\Phi}^{-1}) \right. \\
+ \left. E^\mu_b (\bar{A}_\mu - A_\mu \Phi \bar{\Phi}^{-1}) \right]. \quad (3.12) \\

3.3 Second structure equation, curvature and the action

The second Cartan structure equation defines curvature 2-forms as follows

\[ R_{AB} = D\Omega_{AB} + \Omega_{AC} \wedge \Omega^C_B \]  \\
(3.13)

It is straightforward to use the expressions for the connection 1-forms given in Eq. (3.12) to compute the components \( R_{ABCD} \) of the curvature 2-forms. We recall from Ref. [1] the expression of the Ricci scalar curvature

\[ R = \eta^{AC} R_{ABCD} \eta^{BD}. \]  \\
(3.14)

After a lengthy but straightforward calculation we obtain the final expression of the generalized Ricci scalar curvature in the form

\[ R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} = R^{(0)} + R^{(1)} + R^{(2)}, \]  \\
(3.15)

where \( R^{(0)}, R^{(1)}, R^{(2)} \) represent terms proportional to \( m^0, m, m^2 \) respectively. The explicit expressions of \( R^{(0)}, R^{(1)} \) and \( R^{(2)} \) are given as follows:

\[ R^{(0)} = \frac{1}{2} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} - \frac{1}{32} \left( 3\Phi^2 F^2 + 2\Phi \bar{\Phi} \bar{E}^{ab\mu} E^\rho_a \bar{E}^{b\rho\nu} \bar{E}^{e\tau}_b \bar{F}_{\mu\nu} \bar{F}_{\rho\tau} - \bar{\phi}^2 \bar{F}^2 \right) \]

\[ \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \]  \\
\[ (3.15) \]

The interested reader can see the Ref. [1] for the expression of \( R \) in an inner product form.
\[ R^{(1)} = m \left[ \frac{1}{4} \left( \partial_{\mu} E_{a\nu} - \partial_{\nu} E_{a\mu} \right) \left( 3 E^{a\nu} E_{b\mu} E^{b\rho} A_{\rho} - 4 E^{a\nu} A^{\mu} \left( \tilde{E}^{b\rho} E_{b\rho} \right) - 4 E^{a}_{\mu} A^{\rho} \tilde{E}^{b\rho} \right) + 4 \tilde{E}^{a\mu} E^{b\rho} E^{b\nu} + 8 A^{\mu} \tilde{E}^{a\nu} + 12 A^{\nu} E^{a\mu} + E^{a\mu} G^{\mu\nu} \tilde{A}_{\rho} - E^{b\nu} E^{a\mu} \tilde{A}_{b} \tilde{\Phi} \tilde{\Phi}^{-1} \\
- A^{\nu} E^{a\mu} \tilde{\Phi} \tilde{\Phi}^{-1} \right) + \frac{1}{4} \left( \partial_{\mu} E_{a\nu} - \partial_{\nu} E_{a\mu} \right) \left( 4 \tilde{E}^{a\mu} A^{b} \tilde{E}^{b\nu} + \tilde{E}^{a\mu} G^{\mu\nu} A_{\rho} \right) \right. \\
\left. - \tilde{A}^{\nu} \tilde{E}^{a\mu} \tilde{\Phi} \tilde{\Phi}^{-1} + \tilde{E}^{a\mu} \tilde{E}^{b\nu} E^{a\rho} \tilde{A}_{\rho} - \tilde{E}^{b\nu} \tilde{E}^{a\mu} A_{b} \tilde{\Phi}^{-1} \tilde{\Phi} \right) - \frac{1}{4} F_{\mu\nu} \left( \frac{5}{2} \tilde{E}^{a\mu} E^{a\nu} \right) + \frac{3}{2} \Phi \tilde{\Phi}^{-1} G^{\mu\nu} E^{b\mu} \tilde{E}_{b\rho} + (\tilde{A}_{\rho} - A_{\rho}) \left( 2 \Phi \tilde{\Phi} G^{\mu\nu} \tilde{A}_{a} E^{a\nu} + 3 \Phi^{2} A^{\mu} E^{a\nu} \tilde{E}_{a} \right) \right] \\
+ \frac{1}{4} \tilde{F}_{\mu\nu} \left( \frac{3}{2} (\tilde{A}_{\rho} - A_{\rho}) \tilde{\Phi} A^{\mu} G^{\mu\nu} \tilde{E}^{a\nu} b_{b} \tilde{\Phi} - \frac{1}{2} \tilde{\Phi} \tilde{\Phi}^{-1} \tilde{G}^{\mu\nu} E^{a\mu} \tilde{E}_{a\rho} + \frac{1}{2} \tilde{E}^{a\mu} E^{a\nu} \tilde{E}^{b\mu} \tilde{E}^{b\nu} \tilde{E}_{a\rho} \right) \\
+ \tilde{\Phi}^{2} (\tilde{A}_{\rho} - A_{\rho}) \tilde{A}^{\mu} \tilde{E}^{a\nu} E^{a\rho} b_{b} \tilde{\Phi} + \tilde{\Phi}^{2} (\tilde{A}_{\rho} - A_{\rho}) \tilde{\Phi} A^{\mu} \tilde{G}^{\mu\nu} \tilde{E}^{a\nu} b_{b} \tilde{\Phi} - \frac{1}{2} \tilde{\Phi} \tilde{\Phi}^{-1} \tilde{G}^{\mu\nu} E^{a\mu} \tilde{E}_{a\rho} + \frac{1}{2} \tilde{E}^{a\mu} E^{a\nu} \tilde{E}^{b\mu} \tilde{E}^{b\nu} \tilde{E}_{a\rho} \right) \right] \\
- E^{a\mu} \partial_{\mu} \left( A_{\rho} \tilde{E}^{a\rho} - A^{a} (\tilde{E}^{b\rho} E_{b\rho}) + 3 A^{a} \right) + \frac{1}{4} \frac{\partial_{\mu} \tilde{\Phi}}{\tilde{\Phi}} \left( 3 A^{a} \tilde{E}^{a\mu} - A^{a} \tilde{E}^{a\mu} (\tilde{E}^{b\rho} E_{b\rho}) \right) + \tilde{E}^{a\mu} E_{a\rho} A_{\rho} - \tilde{A}^{\mu} (\tilde{E}^{b\rho} E_{b\rho}) + 3 A^{a} + \tilde{G}^{\mu\rho} A_{\rho} - 2 \tilde{E}^{a\mu} A_{a} \tilde{\Phi} \tilde{\Phi}^{-1} \right) \right] \\
+ \frac{1}{4} \frac{\partial_{\mu} \tilde{\Phi}}{\tilde{\Phi}} \left( 3 A^{a} + A^{\mu} (\tilde{E}^{b\rho} E_{b\rho}) + G^{\mu\rho} A_{\rho} - \tilde{A}^{a} E^{b\rho} E_{b\rho} E^{a\rho} + 3 \tilde{A}^{a} E^{a\mu} \right) \\
- \tilde{E}^{a\rho} E_{a\mu} A_{\rho} + 2 \tilde{\Phi}^{-1} E^{a\mu} A^{a} \right),
\[ R^{(2)} = \frac{m^2}{16} \left[ \Phi^{-2} \left( -32 + 48 \tilde{E}^{bp} E_{bp} - 7 \tilde{E}^{bp} E_{ap} \tilde{E}^{ap} E_{bp} - \tilde{G}^{\mu \nu} G_{\mu \nu} - 8(\tilde{E}^{bp} E_{bp})^2 \right) \right. \]
\[ + \tilde{\Phi}^{-2} \left( E^{bp} \tilde{E}_{a \rho} E_{a \nu} \tilde{E}_{b \mu} - G^{\mu \nu} \tilde{G}_{\mu \nu} \right) - 2 \Phi^{-1} \tilde{\Phi}^{-1} \left( 4 - \tilde{E}^{ap} E_{bp} \tilde{E}_b \nu E^\nu_a \right) \]
\[ + \frac{m^2}{4} \left( \tilde{A}_\mu - A_\mu \right) \left[ A_\rho \tilde{G}^{\mu \rho} - A_a \tilde{E}^{ap} E_{bp} \tilde{E}^{bp} + \Phi \tilde{\Phi}^{-1} (E^b \tilde{E}_{a \mu} A_b \tilde{E}^b_\nu - A^n) \right] \]
\[ + \frac{m^2}{4} \left[ -6 A_\nu \tilde{E}^{bp} E_{bp} \tilde{E}^{ap} E_{b \nu} - 2 A^2 (\tilde{E}^{bp} E_{bp})^2 + 4 A^2 (\tilde{E}^{bp} E_{bp}) \right. \]
\[ - 12 A^2 + 3 A_\nu A_a \tilde{E}^{a \mu} (\tilde{E}^{bp} E_{bp}) + 3 A^2 \tilde{E}^{bp} E_{b \mu} \tilde{E}^{a \mu} E_{a \nu} - 3 A_\nu \tilde{E}^{ap} A_b \tilde{E}^{bp} G_{\mu \nu} \]
\[ + 3 A^2 \tilde{G}^{\mu \nu} G_{\mu \nu} + 7 \tilde{A}_\rho A_\rho - 4 \tilde{A}_\rho A_\rho (\tilde{E}^{bp} E_{bp}) + 10 \tilde{A}_a A_\nu - \tilde{A}_a A_\nu (\tilde{E}^{bp} E_{bp}) \]
\[ - A_\mu \tilde{A}_\mu (\tilde{E}^{bp} E_{bp}) + 3 \tilde{A}_\mu A_\mu + \tilde{A}_\mu G^{\mu \nu} \tilde{A}_\mu - \tilde{A}_\mu E^{a \mu} A_a (\tilde{E}^{bp} E_{bp}) \]
\[ + 3 \tilde{A}_a E^{a \mu} \tilde{A}_a + 4 A_\mu \tilde{A}_\mu \tilde{\Phi}^{-1} - 2 \tilde{A}^2 \tilde{\Phi}^{-2} + 2 A_\nu E^{a \mu} \tilde{A}_a \tilde{\Phi}^{-1} \]
\[ + 2 A_\nu \tilde{E}^{a \mu} A_a \tilde{\Phi}^{-1} \right] - \frac{m^2}{4} \left( \tilde{A}_a \tilde{\Phi}^{-1} + A_a \tilde{\Phi}^{-1} \right) \left[ A_\nu \tilde{E}^{a \nu} - A_a (\tilde{E}^{bp} E_{bp}) \right. \]
\[ + 3 A^a + \tilde{A}_\nu E^{a \nu} - \tilde{A}_a (\tilde{E}^{bp} E_{bp}) + 3 \tilde{A}_a \right] \]
\[ + \frac{m^2}{16} \left( \tilde{A}_\mu - A_\mu \right) \left( \tilde{A}_\nu - A_\nu \right) \left[ 6 \tilde{\Phi} (A_\mu \tilde{A}_\nu - \tilde{A}_a A_a \tilde{E}^{a \mu} E^\nu_a) \right. \]
\[ + 5 \tilde{\Phi}^2 (A^2 \tilde{G}^{\mu \nu} - A_b \tilde{E}^{b \mu} \tilde{E}^{a \nu} A_a) + \tilde{\Phi}^2 (A^2 \tilde{G}^{\mu \nu} - \tilde{A}_a \tilde{A}_b \tilde{E}^{a \mu} \tilde{E}^{b \nu}) \right) , \quad (3.18) \]

where

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} f_{1 \mu \nu} & 0 \\ 0 & f_{2 \mu \nu} \end{pmatrix} , \quad (3.19) \]

and \( r_1 \) and \( r_2 \) are the ordinary Ricci scalar curvatures on the first and second copies of spacetime, respectively.
The volume element is given by

\[ D^5X = D^4X \sqrt{-\det|\mathcal{G}|} \]  \hspace{1cm} (3.20)

Here \( \det|\mathcal{G}| \) denotes the determinant of our generalized metric defined in Eq. (2.24) and is given by

\[ \det|\mathcal{G}| = \frac{1}{5!} \epsilon_{N_1 N_2 N_3 N_4 N_5} \epsilon_{M_1 M_2 M_3 M_4 M_5} \mathcal{G}^{N_1 M_1} \mathcal{G}^{N_2 M_2} \mathcal{G}^{N_3 M_3} \mathcal{G}^{N_4 M_4} \mathcal{G}^{N_5 M_5} \]

\[ = \frac{1}{4!} \epsilon_{\nu_1 \nu_2 \mu_2 \mu_4} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \mathcal{G}^{\nu_1 \mu_1} \mathcal{G}^{\nu_2 \mu_2} \mathcal{G}^{\mu_3 \mu_3} \mathcal{G}^{\nu_4 \mu_4} \mathcal{G}^{05} \equiv \det|\mathcal{G}|\Phi \ 1, \]  \hspace{1cm} (3.21)

where \( \epsilon \)'s are the fully antisymmetric Levi-Civita tensors and

\[ \det|\mathcal{G}| = \begin{pmatrix} \det|g_1| & 0 \\ 0 & \det|g_2| \end{pmatrix}. \]  \hspace{1cm} (3.22)

The action then is defined as

\[ S = \frac{1}{m\kappa} \text{Tr} \left( \int dx^4 \sqrt{-\det \mathcal{G}} \ R \right), \]

\[ = S_1 + S_2, \]

\[ S_1 = \sqrt{-\det|g_1|} \varphi_1 R_1, \]

\[ S_2 = \sqrt{-\det|g_2|} \varphi_2 R_2, \]  \hspace{1cm} (3.23)

where \( \kappa = 16\pi^2 G^{-2}/m \) and \( G \) is the Newton constant.

The integration over the discrete space follows naturally to be \( \frac{1}{m} \text{Tr} \).
4 Mass terms:

The full action of our model (3.23) contain six independent interacting fields $e_{1\mu}^a$, $e_{2\mu}^a$, $a_{1\mu}$, $a_{2\mu}$, $\varphi_1$ and $\varphi_2$. Since the full expression for the Ricci scalar curvature $R$ in Eqs.(3.15)-(3.18) is obviously extremely complex, here we will concentrate on the massive modes in our model. We will concentrate on the gravity sector first.

4.1 Gravity and massive tensor field

To find the mass content of the tensor field we consider the part of the action that contains only tensor fields. It turns out to be

$$R_t = \int dx^4 \sqrt{-\det |G|} \left[ \frac{1}{2} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} + \frac{m^2}{16} \left( -40 + 48 \tilde{E}^{b\rho} E_{b\rho} \right) - 7 \tilde{E}^{b\mu} E_{a\mu} \tilde{E}^{a\rho} E_{b\rho} - \tilde{G}^{\mu\nu} G_{\mu\nu} - 8 (\tilde{E}^{b\rho} E_{b\rho})^2 + E^{b\mu} \tilde{E}_{ad} E^{a\nu} \tilde{E}_{b\nu} - G^{\mu\nu} \tilde{G}_{\mu\nu} + 2 \tilde{E}^{a\rho} E_{b\rho} \tilde{E}^{b\nu} E_{a\nu} \right] \right]. \quad (4.1)$$

From the terms proportional to $m^2$, we can see that $e_{1\mu}^a$ and $e_{2\mu}^a$ are not the fields corresponding to mass eigenstates since their products appear in these terms giving rise to mixing. To find mass eigenstates we write

$$E^\mu_a = \frac{1}{2} \left( e_{+a}^\mu + e_{-a}^\mu r \right),$$

$$\tilde{E}^\mu_a = \frac{1}{2} \left( e_{+a}^\mu + e_{-a}^\mu r \right), \quad (4.2)$$

and substitute for them in Eq.(4.1). We note that a proper mass term has the general form $m^2 b^{a\mu} b_{a\mu}$, where $b^{a\mu}$ represents the massive tensor field.
With this in mind, we find two possibilities for identifying the massive fields:

i) If we choose \( e^\mu_a \) as the vielbein for the metric that represents gravity, we find the mass term for the tensor field \( e^a_{-\mu} \) as \( \sim 15/16m^2e^a_{-\mu}e_{-\mu} \) in Eq.(4.1). The terms in pure \( e^a_{+\mu} \) give a cosmological constant. In the case we are considering, these terms and the constant term cancel and consequently there is no cosmological constant. Further we note that, in the vacuum \( e_+ \) is a physical field as \( e_- \to 0 \) and \( e^\mu_{+a} \to \delta^\mu_a \).

ii) If we choose \( e^a_{-\mu} \) as the vielbein for the gravity metric. The same terms that give a mass to \( e^a_{-\mu} \) in the previous case, now becomes the mass terms for \( e^a_{+\mu} \). Since the terms in pure \( e^a_{+\mu} \) and the constant terms do not cancel, there is a cosmological constant in this case. In vacuum, \( e^a_{+\mu} \to 0 \) and \( e^{\mu}_{+a} \to \delta^{\mu}_a \). The mass term for \( e^{\mu}_{+a} \) in this case is \(-9/16m^2e^0_{+a}e_{+\mu} \). There are also quartic terms in \( e^{\mu}_{+a} \). It would be interesting to see whether this negative mass terms lead to spontaneous symmetry breaking patterns.

In the two limiting cases, when the massive tensor field is set to zero we have the usual Einstein theory with the vielbein \( e^\mu_a \) or the theory with the vielbein \( e^a_{+a} \) together with a cosmological constant.

Now we will consider the mass terms of the vector and scalar fields with the above two choices.

**4.2 Mass terms of vector and scalar fields**

At classical level the tensor fields do not alter the mass terms of vector and scalar fields. Hence we will turn off the tensor fields and consider two limiting cases \( E^\mu_a = \delta^\mu_a \) and \( E^\mu_a = \delta^\mu_a r \). After inserting the particular \( E^\mu_a \) into the the expression for \( R^{(2)} \) we find:

i) \( E^\mu_a = \delta^\mu_a \): There is no mass terms for the scalar fields. However, the mass term
for vector fields is $4m^2a^2_{\mu}$ where $a_{\pm\mu} = 1/2(a_{1\mu} - a_{2\mu})$. This means that in this case $a_{+\mu}$ is massless and $a_{-\mu}$ is massive.

ii) $E^\mu_a = \delta^\mu_a r$: The mass terms in this case are given by

$$R^{(2)} = -96m^2\varphi_+^2 - 36m^2a_+^2,$$

where $\varphi_\pm = \varphi_1 \pm \varphi_2$.

The action for this part is

$$S_m \sim \int dx^4 - 96m^2(\varphi_+^2 + \varphi_+\varphi_-^2) - 36m^2(\varphi_+ + \varphi_-)a_-^2.$$

Note that in vacuum $\varphi_+ = 1$. Therefore $\varphi_-$ is the physical mode while we have to expand $\varphi_+$ in terms of the physical field $\sigma$ as follows:

$$\varphi_+ = 2\exp(-\sigma),$$

where in vacuum $\sigma \to 0$.

Using this expansion, the mass terms of $a_{-\mu}$ and $\varphi_-$ are $-36m^2a_-^2$ and $-96m^2\varphi_-^2$ respectively. These mass terms as well as the mass term for the tensor field $e^\mu_{+\alpha}$ in this case are negative. It would be interesting to include the quartic terms to see whether these negative mass terms lead to some spontaneous symmetry breaking patterns. The quartic potential for vector fields are already there in Eq.(3.18). To have the quartic potential for the scalar field, however, one has to modify the wedge product of forms in Eq.(2.11). Such modifications will be discussed elsewhere.
5 Summary and Conclusions:

We have in the previous papers [1, 6] developed a discretized version of Kaluza-Klein theory by replacing the continuous fifth dimension by two discrete points. In the language of NCG, we may speak of two copies of spacetime instead of an infinite number of them in the standard Kaluza-Klein theory (For every internal point in the fifth dimension we have a four-dimensional spacetime). The geometry of the extended spacetime permitted us to introduce a generalized vielbein consisting of a pair of tensor, a pair of vector and a pair of scalar fields. When we imposed the standard metric compatibility and torsion free conditions to determine the connection 1-form, we found constraints on the vielbeins in the form of dynamical dilaton fields that implied new and interesting consequences.

In the present paper we have pursued the investigation further to see whether we can eliminate the constraints on the vielbeins by relaxing the torsion free condition. In order to remain as close to the Riemannian geometry as possible, we still require that the torsion 2-forms corresponding to the physical spacetime do vanish. However, by making an ansatz about torsion 2-form corresponding to the internal space, we determine uniquely not only all the connection 1-form coefficients, but also the nonvanishing torsion components in terms of the assumed vielbeins. This is in contrast to the usual Riemannian geometry where nonvanishing torsion does not lead to a unique determination of the connection coefficients.

With the unique determination of the connection coefficients, we obtain a Lagrangian and an action that has a rich and complex structure with interacting tensor, vector and scalar fields. It appears as sum of two terms $S_1$ and $S_2$, each consisting of
all the six independent fields and each representing a generally covariant action. In $S_1(S_2)$, the vierbein $e^\mu_{1a} (e^\mu_{2a})$ acts as the metric field with appropriate kinetic term while the other $e^\mu_{2a} (e^\mu_{1a})$ coupled to $e^\mu_{1a} (e^\mu_{2a})$ in quadratic and quartic terms. This suggests that $e^\mu_{1a}$ and $e^\mu_{2a}$ are not eigenstates of mass. Instead we have two mass eigenstates as $e^\mu_{\pm a} = e^\mu_{1a} \pm e^\mu_{2a}$. We have two possibilities of choosing $e^\mu_{+a}$ or $e^\mu_{-a}$ as representing the gravity field. In the first case, $e^\alpha_{-a}$ and $a_{-\mu}$ are massive fields while the scalar fields and $a_{+\mu}$ are massless. There is no cosmological constant in this case. In the second case, there is a cosmological constant and negative mass terms for tensor, vector and scalar fields.

In conclusion, we like to observe that our discretized version of Kaluza-Klein theory within the framework of NCG demonstrates an extremely promising approach to internal structure of elementary particles. If the internal space is discrete, one obtains only a finite number of massive modes and thus avoids the problem of infinite number of massive modes and of the necessity of truncation. In addition to having mass, the fields have interactions proportional to the mass parameter $m$ and the Newton constant $G$. It is extremely interesting to explore the consequences of such theory on gravity. The highly correlated interactions also suggest strong quantum implications that are fascinating to study.

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