On Axially Symmetric Incompressible Magnetohydrodynamics in Three Dimensions

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Abstract

In this short article, we prove the global regularity of axially symmetric solutions to the systems of incompressible ideal magnetohydrodynamics and resistive magnetohydrodynamics in three dimensions in the case that the magnetic fields are purely swirling and perpendicular to the velocity fields.

Keywords: Magnetohydrodynamics, global regularity, axially symmetry.

1 Introduction

Magnetohydrodynamics (MHD) is to study the behavior of an electrically-conducting fluids. Examples of such fluids include plasmas, liquid metals, salt water, etc. The field of MHD was initiated by Hannes Alfvén, for which he received the Nobel Prize in Physics in 1970. However, the mathematical theory on MHD is still very little known until today.

The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn creates forces on the fluid and also changes the magnetic field itself. MHD owes its peculiar interest and difficulty to this interaction between the field and the fluid motion. The set of equations which describe MHD are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism.

Our main result concerns the following incompressible three-dimensional ideal MHD:

\[
\begin{align*}
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mu \Delta \mathbf{u} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}, \\
\partial_t \mathbf{B} &= \nabla \times (\mathbf{u} \times \mathbf{B}), \\
\nabla \cdot \mathbf{u} &= 0, \quad \nabla \cdot \mathbf{B} = 0,
\end{align*}
\]

where \( \mathbf{B} \) denotes the magnetic field, \( \mathbf{u} \) the bulk plasma velocity and \( p \) the plasma pressure. The magnetic constant \( \mu_0 \) and the fluid viscosity \( \mu \) are both positive. We will set all the constants to be 1 since they play no role in this paper. The ideal MHD is used when the electrically-conducting fluid has so little resistivity that it can be treated as a perfect conductor. This is the limit of infinite magnetic Reynolds number. For applications of ideal MHD, see, for instance, [5].

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The following theorem shows that if the magnetic field is purely swirling and is perpendicular to the velocity field, then the 3D incompressible ideal MHD (1.1) is globally well-posed in the axially symmetric case.

**Theorem 1.1.** Suppose that \( u_0 \) and \( B_0 \) are both axially symmetric divergence-free vectors with \( u_0^\theta = 0 \) and \( B_0^r = B_0^\theta = 0 \). Moreover, we assume that \((u_0, B_0) \in H^2 \) and \( \frac{B_0^r}{r} \in L^\infty \). Then there exists a unique global solution \((u, B)\) for the ideal MHD (1.1) with the initial data \((u_0, B_0)\) which satisfies

\[
\|u(t, \cdot)\|_{H^2}^2 + \|B(t, \cdot)\|_{H^2}^2 + \int_0^t \|\nabla u\|_{H^2}^2 ds \lesssim e^{ct^2}.
\]

The notations used here will be introduced in section 2. Note that the Faraday’s equation for \( B \) in (1.1) is exactly the same as the vorticity equation for the 3D incompressible Euler equations (by identifying \( B \) and \( \nabla \times u \)). This may lead to an essential difficulties for the global well-posedness of the ideal MHD (1.1) in general case. Indeed, the global regularity of (1.1) is widely open in the even two-dimensional case if the magnetic field is non-trivial. We achieve Theorem 1.1 by exploring the underlying special structures of the MHD system in axially symmetric case. The magnetic stretching term \( B \cdot \nabla u \) in Faraday’s equation can be absorbed into the convection term by dividing the equation by \( r \). This yields that \( \Pi = \frac{B^r}{r} \) is only transported by the velocity field \( u \). On the other hand, by dividing \( r \) in the vorticity equation, one can absorb the vortex stretching term into the convection term, leaving only one term involving \( \Pi \) as a forcing one in \( \Omega \) equation. See section 2 and 3 for more details.

Similar result in Theorem 1.1 is of course expected to hold for the following resistive MHD:

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u + \nabla p &= \Delta u + (\nabla \times B) \times B, \\
\partial_t B &= \nu \Delta B + \nabla \times (u \times B), \\
\nabla \cdot u &= 0, \quad \nabla \cdot B = 0.
\end{aligned}
\]

(1.2)

Again, we will set the resistivity constant \( \nu > 0 \) to be 1 since it plays no role here. We have the following theorem:

**Theorem 1.2.** Suppose that \( u_0 \) and \( B_0 \) are both axially symmetric divergence-free vectors with \( u_0^\theta = 0 \) and \( B_0^r = B_0^\theta = 0 \). Moreover, we assume that \((u_0, B_0) \in H^1 \) and \( \frac{B_0^r}{r} \in L^\infty \). Then there exists a unique global solution \((u, B)\) for the resistive MHD (1.2) with the initial data \((u_0, B_0)\). Moreover, \((u, B)\) is smooth in the sense that \((u(t, \cdot), B(t, \cdot)) \in H^s \) for any \( s \geq 0 \) and \( t > 0 \).

Our motivation of the above results is a novelty observation on the connections between MHD and axially symmetric Navier-Stokes equations. If we rewrite the 3D incompressible axially symmetric Navier-Stokes equations as (2.9), in terms of \( u = u^r e_r + u^\theta e_\theta \) and \( b = u^\theta e_\theta \), then there is only a sign difference\(^1\) between the Navier-Stokes equations (2.9) for \((u, b)\) and the resistive MHD (1.2) for \((u, B)\) (see Remark 2.1 in section 2 for details). However, this difference of sign significantly changes the difficulties in solving 3D axially symmetric incompressible equations of MHD.

\(^1\)In fact, the pressure is also changed. But the pressure is not a troublesome term for our purpose due to the divergence-free condition.
We remark that the perfect resistive case will be treated in a forthcoming paper \[11\]. It is also interesting to consider the case when \(u^\theta = B^\theta = 0\).

Before ending the introduction, let us mention some important results in the field of incompressible MHD. The local well-posedness of the resistive MHD (1.2) was established in \[15\] where the authors also proved the global well-posedness in 2D case. A nontrivial blowup criterion for the perfect resistive MHD was established in terms of only \(L^1_t(BMO)\) norm of vorticity of the velocity field in \[12\]. Recently, Lin, Xu and Zhang \[13\] obtained the global well-posedness of classical solutions for the 2D ideal MHD (1.1) under the assumption that the initial velocity field and the displacement of the magnetic field from a non-zero constant is sufficiently small in appropriate Sobolev spaces. Cao and Wu \[2\] proved the global regularity of 2D resistive MHD with partial viscosity and resistivity (see also \[3\] and the references therein). We also emphasize the partial regularity theory and Serrin type criterions in \[6, 7\], and various blowup criterions in \[1, 4\] (see also the reference therein).

The remaining of this paper is simply organized as follows: In section 2 we will derive the axisymmetric MHD in cylindrical coordinate. We will also make a comment on the difference between the resistive MHD (1.2) and the axially symmetric Navier-Stokes equations and prove a maximum principle for \(\Pi\). We will prove Theorem 1.1 in section 3. Then in section 4 we present the proof of Theorem 1.2.

### 2 Axially Symmetric MHD and A Maximum Principle

In this section we will first derive the incompressible axially symmetric MHD in cylindrical coordinate. Then we will show that the quantity \(\Pi\) satisfies a maximum principle. We also present an interesting connection between the axisymmetric MHD studied in Theorem 1.2 and the axisymmetric Navier-Stokes equations with non-trivial swirl \(u^\theta\) (see (2.6) and (2.9)).

Let us begin with some notations. A point in \(\mathbb{R}^3\) is denoted by \(x = (x_1, x_2, z)\). Let \(r = \sqrt{x_1^2 + x_2^2}\) and

\[
\begin{align*}
e_r &= \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), & e_\theta &= \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), & e_z &= \left(0, 0, 1\right),
\end{align*}
\]

be the three orthogonal unit vectors along the radial, the angular, and the axial directions respectively. An axially symmetric solution to the 3D incompressible MHD (1.2) is a solution \((u, B, p)\) which takes the following form

\[
\begin{align*}
u(t, x) &= u_r(t, r, z)e_r + u^\theta(t, r, z)e_\theta + u_z(t, r, z)e_z, \\
B(t, x) &= B_r(t, r, z)e_r + B^\theta(t, r, z)e_\theta + B_z(t, r, z)e_z, \\
p(t, x) &= p(t, r, z).
\end{align*}
\]

We will also write the vorticity field \(\nabla \times u\) in cylindrical coordinate:

\[
\nabla \times u(t, x) = \omega_r(t, r, z)e_r + \omega^\theta(t, r, z)e_\theta + \omega_z(t, r, z)e_z,
\]

where

\[
\omega_r = -\partial_z u^\theta, \quad \omega^\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \frac{1}{r} \partial_r (ru^\theta).
\]
Define

\[ \Pi = \frac{B^\theta}{r}, \quad \Omega = \frac{\omega^\theta}{r}, \quad \Gamma = ru^\theta. \]  

(2.1)

By expanding the Lorentz force term as

\[ (\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{B} - \nabla \frac{|\mathbf{B}|^2}{2}, \]

and then taking the inner product of \( \mathbf{u} \) and \( \mathbf{B} \) equations with \( \mathbf{e}_r, \mathbf{e}_\theta \) and \( \mathbf{e}_z \), respectively, we can derive the resistive MHD in cylindrical coordinate:

\[
\begin{align*}
\partial_t u^r + u^r \partial_r u^r + u^z \partial_z u^r - \frac{(u^\theta)^2}{r} + \partial_r P &= (\Delta - \frac{1}{r^2}) u^r + B^r \partial_r B^r + B^z \partial_z B^r - \frac{(B^\theta)^2}{r}, \\
\partial_t u^\theta + u^r \partial_r u^\theta + u^z \partial_z u^\theta + \frac{u^r u^\theta}{r} &= (\Delta - \frac{1}{r^2}) u^\theta + B^r \partial_r B^\theta + B^z \partial_z B^\theta + \frac{B^\theta B^\theta}{r}, \\
\partial_t u^z + u^r \partial_r u^z + u^z \partial_z u^z + \partial_z P &= \Delta u^z + B^r \partial_r B^z + B^z \partial_z B^z, \\
\partial_t B^r + u^r \partial_r B^r + u^z \partial_z B^r &= (\Delta - \frac{1}{r^2}) B^r + B^r \partial_r u^r + B^z \partial_z u^r, \\
\partial_t B^\theta + u^r \partial_r B^\theta + u^z \partial_z B^\theta &= (\Delta - \frac{1}{r^2}) B^\theta + B^r \partial_r u^\theta + B^z \partial_z u^\theta + \frac{u^r B^\theta}{r}, \\
\partial_t B^z + u^r \partial_r B^z + u^z \partial_z B^z &= \Delta B^z + B^r \partial_r u^z + B^z \partial_z u^z,
\end{align*}
\]

(2.2)

where the pressure is given by

\[ P = p + \frac{|\mathbf{B}|^2}{2}. \]

(2.3)

The incompressible constraints are

\[ \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \quad \partial_r B^r + \frac{B^r}{r} + \partial_z B^z = 0. \]

(2.4)

The general axially symmetric resistive MHD is governed by (2.2) and (2.4). In this paper, we consider a family of solutions with the form

\[ \mathbf{u}(t, \mathbf{x}) = u^r(t, r, z)\mathbf{e}_r + u^z(t, r, z)\mathbf{e}_z, \quad \mathbf{B}(t, \mathbf{x}) = B^\theta(t, r, z)\mathbf{e}_\theta. \]

(2.5)

It is easy to check that \((u^\theta, B^r, B^z)\) can be zero for all time if they are zero initially. In this case, \((\mathbf{u}, \mathbf{B}, P)\) in (2.5) and (2.3) is governed by

\[
\begin{align*}
\partial_t u^r + u^r \partial_r u^r + u^z \partial_z u^r + \partial_r P &= (\Delta - \frac{1}{r^2}) u^r - \frac{(B^\theta)^2}{r}, \\
\partial_t u^z + u^r \partial_r u^z + u^z \partial_z u^z + \partial_z P &= \Delta u^z, \\
\partial_t B^\theta + u^r \partial_r B^\theta + u^z \partial_z B^\theta &= (\Delta - \frac{1}{r^2}) B^\theta + \frac{u^r B^\theta}{r},
\end{align*}
\]

(2.6)
together with the incompressible constraint
\[
\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0. \tag{2.7}
\]
To avoid the explicit presence of pressure, we also need the vorticity formula of (2.6):
\[
\begin{aligned}
\partial_t B^\theta + u^r \partial_r B^\theta + u^z \partial_z B^\theta &= \left( \Delta - \frac{1}{r^2} \right) B^\theta + \frac{u^\theta}{r} , \\
\partial_t \omega^\theta + u^r \partial_r \omega^\theta + u^z \partial_z \omega^\theta - \frac{u^\theta}{r} &= \left( \Delta - \frac{1}{r^2} \right) \omega^\theta - \frac{\partial_r (B^\theta)^2}{r}.
\end{aligned} \tag{2.8}
\]

**Remark 2.1.** It is well-known that the axially symmetric Navier-Stokes equations (in the case of \( B \equiv 0 \)) are (see, for instance, [14])
\[
\begin{aligned}
\partial_t u^r + u^r \partial_r u^r + u^z \partial_z u^r + \partial_r p &= \left( \Delta - \frac{1}{r^2} \right) u^r + \frac{(u^\theta)^2}{r} , \\
\partial_t u^z + u^r \partial_r u^z + u^z \partial_z u^z + \partial_z p &= \Delta u^z , \\
\partial_t u^\theta + u^r \partial_r u^\theta + u^z \partial_z u^\theta &= \left( \Delta - \frac{1}{r^2} \right) u^\theta - \frac{u^r u^\theta}{r}.
\end{aligned} \tag{2.9}
\]
If we denote \( u = u^r e_r + u^z e_z \) and \( b = u^\theta e_\theta \), we can rewrite the above axially symmetric Navier-Stokes equations as
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= \Delta u - b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b &= \Delta b - b \cdot \nabla u, \\
\nabla \cdot u &= \nabla \cdot b = 0.
\end{aligned} \tag{2.9}
\]

If we compare the MHD equations (1.2) with the Navier-Stokes equations (2.9), we find that we can recover (1.2) from (2.9) by changing the sign of the terms \( b \cdot \nabla b \) and \( b \cdot \nabla u \). The significance of the tiny difference, especially, the sign of \( b \cdot \nabla u \), yields a much stronger a priori estimate in the MHD case.

**Proposition 2.1 (Maximum Principle).** Assume that \( (u, B, P) \) is a smooth bounded solution to (2.6) with or without resistivity. Then the quantity \( \Pi \) satisfies the maximum principle
\[
\| \Pi(t, \cdot) \|_{L^\infty} \leq \| \Pi(0, \cdot) \|_{L^\infty}, \quad \forall \ t \geq 0.
\]

**Proof.** In the case of zero resistivity, by dividing the equation for \( B^\theta \) by \( r \), one has
\[
\partial_t \Pi + u^r \partial_r \Pi + \frac{u^\theta}{r} \partial_z \Pi = 0, \tag{2.10}
\]
which gives the maximum principle for \( \Pi \) in the case of zero resistivity.

Similarly, in the resistive case, we have
\[
\partial_t \Pi + u^r \partial_r \Pi + \frac{u^\theta}{r} \partial_z \Pi = \left( \partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2 \right) \Pi. \tag{2.11}
\]
Then the maximum principle follows by interpreting \( \left( \partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2 \right) \Pi \) as a five-dimensional Laplacian operator.
3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Throughout this paper, we will use $A_1 \lesssim A_2$ to denote that $A_1 \leq C_0 A_2$ and $A_1 \sim A_2$ to denote that $C_0^{-1} A_2 \leq A_1 \leq C_0 A_2$ for a generic positive constant $C_0 > 1$ and two positive quantities $A_1$ and $A_2$.

**Proof of Theorem 1.1.** Let us rewrite the vorticity equation in (2.8) in terms of $\Omega$:

$$\partial_t \Omega + u^r \partial_r \Omega + u^z \partial_z \Omega = (\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \Omega - \partial_z \Pi^2.$$ 

By taking the $L^2$ inner product of the above equation with $\Omega$ and preforming the standard energy estimate, one has

$$\frac{1}{2} \frac{d}{dt} \|\Omega\|_{L^2}^2 - \int \Omega (\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \Omega dx = -\frac{1}{2} \int (u^r \partial_r \Omega^2 + u^z \partial_z \Omega^2) dx - \int \Omega \partial_z \Pi^2 dx.$$ 

Using the incompressibility condition (2.7) and the fact of $dx = 2\pi r dr dz$, one has

$$\int (u^r \partial_r \Omega^2 + u^z \partial_z \Omega^2) dx = 0$$

and

$$-\int \Omega (\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \Omega dx = \|\nabla \Omega\|_{L^2}^2 + 2\pi \int_{\mathbb{R}} |\Omega(t,0,z)|^2 dz.$$ 

By integration by part and interpolation, we have

$$\int \Omega \partial_z \Pi^2 dx \leq \|\Pi\|_{L^4}^2 \|\partial_z \Omega\|_{L^2} \leq \frac{1}{2} \|\Pi\|_{L^2}^2 \|\Pi\|_{L^\infty}^2 + \frac{1}{2} \|\partial_z \Omega\|_{L^2}^2.$$ 

Consequently, one has

$$\frac{d}{dt} \|\Omega\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 \leq \|\Pi\|_{L^2}^2 \|\Pi\|_{L^\infty}^2. \quad (3.1)$$

Similarly, using equation (2.11) and preforming the $L^2$ energy estimate, one has

$$\|\Pi(t,\cdot)\|_{L^2} \leq \|\Pi_0\|_{L^2}, \quad \forall \ t \geq 0. \quad (3.2)$$

Consequently, by Proposition 2.1 and using (3.1), (3.2), we have

$$\|\Omega(t,\cdot)\|_{L^2} \leq 1 + \sqrt{t}, \quad \int_0^t \|\nabla \Omega\|_{L^2}^2 dt \leq 1 + t, \quad \forall \ t \geq 0. \quad (3.3)$$

Here we used that $\Omega_0 \in L^2$ which is due to the fact that $u_0 \in H^2$ and

$$|\nabla (\nabla \times u)| = |(e_r \partial_r + \frac{1}{r}e_\theta \partial_\theta + e_z \partial_z)\omega \omega^\theta e_\theta|^2 = |\nabla \omega^\theta|^2 + |\Omega|^2.$$ 

Similarly, one also has $\Pi_0 \in L^2$ since $B_0 \in H^1$. 

6
To proceed, we need a technical lemma regarding the property of a Riesz operator on \( \mathbb{R}^3 \). We first recall the following weighted \( \text{Calderon-Zygmund} \) inequality for a singular integral operator with a weight function which is in the \( A_p \) class (see Stein [16] pp. 194-217 for details). Let \( K \) be a Riesz operator in \( \mathbb{R}^n \) and \( w(x) \) be a weight in the \( A_p \) class (see page 194 of [16] for definition). One can extend the Calderon-Zygmund inequality for the singular integral operator with the integral having weight function \( w(x) \). Specifically, for \( 1 < p < \infty \), there holds
\[
\|Kf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall \ f \in L^p(\mathbb{R}^n).
\]

The following lemma plays an essential role in our global regularity analysis.

**Lemma 3.1.** There holds
\[
\int_0^T \|r^{-1}u^r(t, \cdot)\|_{L^\infty} dt \lesssim \sup_{0 \leq t \leq T} \|\Omega(t, \cdot)\|_{L^q} \int_0^T \|\nabla \Omega(t, \cdot)\|_{L^r} dt.
\]

**Remark 3.2.** We pointed out that in [8] the authors have established an inequality \( \|r^{-1}\partial_z u^r\|_{L^p} \lesssim \|\Omega\|_{L^p} \) for \( 1 < p < \infty \) for \( \mathbb{R}^2 \times T^1 \), where \( T^1 \) is a one-dimensional torus.

**Proof.** We follow the proof in [8]. By the incompressible constraint (2.7), we can introduce the angular stream function \( \psi^\theta \) such that
\[
-(\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} + \partial_\theta^2) \psi^\theta = \omega^\theta,
\]
and
\[
u^r = -\partial_\theta \psi^\theta, \quad u^z = \frac{1}{r} \partial_r (r \psi^\theta).
\]

We divide by \( r \) in (3.4), which gives that
\[
-(\partial_r^2 + \frac{3}{r} \partial_r + \partial_\theta^2) \frac{\psi^\theta}{r} = \frac{\omega^\theta}{r}.
\]

Following [8], we interpret the Laplace operator in (3.5) as a five-dimensional one. We formally write
\[
y = (y_1, y_2, y_3, y_4, z), \quad r = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}, \quad \Delta_y = (\partial_r^2 + \frac{3}{r} \partial_r + \partial_\theta^2).
\]

This way we have \( \frac{\psi^\theta}{r} = (-\Delta_y)^{-1} \omega^\theta \). In the remaining part of the proof of this lemma, we will use a subscript \( y \) to denote the derivatives with respect to \( y \).

It is clear that
\[
\nabla_y^2 \frac{\psi^\theta}{r} = (e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta + e_z \partial_z)(e_r \partial_r \frac{\psi^\theta}{r}) + \nabla \partial_\theta \frac{\psi^\theta}{r} \otimes e_z
\]
\[
= e_r \otimes e_r \partial_r^2 \frac{\psi^\theta}{r} + e_\theta \otimes e_\theta \frac{1}{r} \partial_\theta \frac{\psi^\theta}{r} + (e_z \otimes e_r + e_r \otimes e_z) \partial_\theta^2 \frac{\psi^\theta}{r} + e_z \otimes e_z \partial_\theta^2 \frac{\psi^\theta}{r}.
\]

Consequently, one has
\[
|\nabla_y^2 \frac{\psi^\theta}{r}|^2 \simeq |\partial_r^2 \frac{\psi^\theta}{r}|^2 + \left| \frac{1}{r} \partial_\theta \frac{\psi^\theta}{r} \right|^2 + |\partial_\theta^2 \frac{\psi^\theta}{r}|^2 + |\partial_\theta^2 \frac{\psi^\theta}{r}|^2.
\]
On the other hand, one also has

$$\nabla^2 \psi^\theta = \left( \bar{e}_r \partial_r + \nabla_{\theta} + \bar{e}_z \partial_z \right) \left( \bar{e}_r \partial_r \psi^\theta \right) + \nabla_y \partial_z \psi^\theta \frac{\partial^2}{r} \bar{e}_z$$

$$= \bar{e}_r \otimes \bar{e}_r \partial_r^2 \psi^\theta + \nabla_{\theta} \bar{e}_r \partial_r \psi^\theta + (\bar{e}_z \otimes \bar{e}_r + \bar{e}_r \otimes \bar{e}_z) \partial_r^2 \psi^\theta \frac{\partial^2}{r} + \bar{e}_z \bar{e}_z \partial_r^2 \psi^\theta \frac{\partial^2}{r}$$

$$= \bar{e}_r \otimes \bar{e}_r \partial_r^2 \psi^\theta + (I_0 - \bar{e}_r \otimes \bar{e}_r) \frac{1}{r} \partial_r \psi^\theta$$

$$+ (\bar{e}_z \otimes \bar{e}_r + \bar{e}_r \otimes \bar{e}_z) \partial_r^2 \psi^\theta \frac{\partial^2}{r} + \bar{e}_z \bar{e}_z \partial_r^2 \psi^\theta \frac{\partial^2}{r}.$$ 

where $I_0 = \left( \begin{array}{cc} I_{4 \times 4} & 0 \\ 0 & 0 \end{array} \right)$ and $\nabla_{\theta}$ is defined by

$$\nabla_{\theta} = \nabla - \bar{e}_r (\bar{e}_r \cdot \nabla_y) - \bar{e}_z \partial_z, \quad \bar{e}_r = \frac{1}{r} \left( \begin{array}{ccc} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \right), \bar{e}_z = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right).$$

Clearly, $\bar{e}_r \otimes \bar{e}_r, I_0 - \bar{e}_r \otimes \bar{e}_r, \bar{e}_z \otimes \bar{e}_r, \bar{e}_r \otimes \bar{e}_z$ and $\bar{e}_z \otimes \bar{e}_z$ are all mutually orthogonal. Consequently, one also has

$$|\nabla^2 \psi^\theta|^2 \sim |\partial_r \psi^\theta|^2 + \frac{1}{r} |\partial_r \psi^\theta|^2 + |\partial_z \psi^\theta|^2 + \frac{1}{r} |\partial_z \psi^\theta|^2. \quad (3.7)$$

By (3.6) and (3.7), we have

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \left| \nabla^2 \psi^\theta \right|^p r \, dr \, dz \simeq \left( \int_{-\infty}^{\infty} \int_{0}^{\infty} \left( |\partial_r \psi^\theta|^2 + \frac{1}{r} |\partial_r \psi^\theta|^2 + |\partial_z \psi^\theta|^2 + \frac{1}{r} |\partial_z \psi^\theta|^2 \right)^\frac{p}{2} r \, dr \, dz \right)^\frac{2}{p} w(r)r^3 \, dr \, dz$$

$$\simeq \int_{-\infty}^{\infty} \int_{0}^{\infty} \left| \nabla^2 \psi^\theta \right|^p w(r)r^3 \, dr \, dz$$

Consequently, one has

$$\int |\nabla^2 \psi^\theta|^p r \, dr \lesssim \int |\omega^\theta|^p r \, dr.$$ 

Let $1 < p < \infty$. Using Lemma 2 in [8] (see also a general version in Lemma 5.1 in Appendix of this paper), we have

$$\int |\nabla^2 (\Delta_y)^{-1} \omega^\theta|^p w(r) \, dy \lesssim \int |\omega^\theta|^p w(r) \, dy$$

Consequently, one has

$$\int |\nabla^2 \psi^\theta|^p r \, dr \lesssim \int |\omega^\theta|^p r \, dr. \quad (3.8)$$
Repeating the above procedure, one also has
\[ \int \left| \nabla \frac{\partial z \psi^0}{r} \right|^p dx \lesssim \int \left| \frac{\partial z \omega^0}{r} \right|^p dx. \] (3.9)

Taking \( p = 2 \) in (3.8) and (3.9) and using the interpolation inequality \( \|f\|_{L^\infty} \lesssim \|\nabla f\|_{L^2}^{1/2} \|\nabla^2 f\|_{L^2}^{1/2} \) in \( \mathbb{R}^3 \), one has
\[ \int_0^T \| r^{-1} u(t, \cdot) \|_{L^\infty} dt = \int_0^T \| r^{-1} \partial_z \psi^0(t, \cdot) \|_{L^\infty} dt \lesssim \int_0^T \| \nabla \partial_z (r^{-1} \psi^0(t, \cdot)) \|_{L^2} \left\| \nabla^2 (r^{-1} \psi^0(t, \cdot)) \right\|_{L^2} dt \lesssim \sup_{0 \leq t \leq T} \| \Omega(t, \cdot) \|_{L^2} \int_0^T \| \partial_z \Omega(t, \cdot) \|_{L^2} dt. \]

This finishes the proof of the lemma. \( \square \)

Now we derive an \( L^\infty \) estimate for \( B^\theta \). Ignoring the viscosity in the equation of \( B^\theta \) in (2.6), one has
\[ \| B^\theta(t, \cdot) \|_{L^\infty} \leq \| B^\theta_0 \|_{L^\infty} + \int_0^t \| B^\theta(s, \cdot) \|_{L^\infty} \frac{u^r}{r} \|_{L^\infty} ds. \]

By Gronwall’s inequality and using (3.3) and Lemma 3.1, we have
\[ \| B^\theta(t, \cdot) \|_{L^\infty} \leq \| B^\theta_0 \|_{L^\infty} e^{\int_0^t \| r^{-1} u^r(s, \cdot) \|_{L^\infty} ds} \lesssim e^{t^{5/4}}. \] (3.10)

Let us coming back to (2.8) and estimate that
\[ \frac{1}{2} \frac{d}{dt} \int |\omega^0|^2 dx + \int \left( |\nabla \omega^0|^2 + \frac{|\omega^0|^2}{r^2} \right) dx \leq \| \frac{u^r}{r} \|_{L^\infty} \int (\omega^0)^2 dx + \| \Pi \|_{L^\infty} \| B^\theta \|_{L^2} \| \partial_z \omega^0 \|_{L^2} \leq \| \frac{u^r}{r} \|_{L^\infty} \int (\omega^0)^2 dx + \frac{1}{2} \| \Pi \|_{L^\infty}^2 \| B^\theta \|_{L^2}^2 + \frac{1}{2} \| \partial_z \omega^0 \|_{L^2}^2. \]

Recalling the following basic energy law
\[ \frac{1}{2} \frac{d}{dt} \left( \| u \|_{L^2}^2 + \| B \|_{L^2}^2 \right) + \int_0^t \| \nabla u \|_{L^2}^2 ds = 0, \] (3.11)

and using the \textit{a priori} estimate in (3.3) and Lemma 3.1 one has
\[ \| \nabla \times u(t, \cdot) \|_{L^2} \lesssim e^{t^{5/4}}, \quad \int_0^t \| \nabla \times u \|_{L^2}^2 dt \lesssim e^{t^{5/4}}, \quad \forall \ t \geq 0. \] (3.12)

The next step is to bootstrap the regularity of \( u \) and \( B \). We are going to show the \( L^1([0, T], \text{Lip}(\mathbb{R}^3)) \) estimate of \( u \). We will make use of the structure of the ideal MHD in (2.6)
to avoid some possible technical complications. The key observation is that we can write the vorticity equation as
\[ \partial_t(\nabla \times u) + \nabla \times [(\nabla \times u) \times u] = \Delta(\nabla \times u) - \partial_z(\Pi B^\theta e_\theta). \]

Here by the maximum principle in Proposition 2.1 and (3.10), one has \( \Pi B^\theta \in L^\infty([0, t], L^\infty(\mathbb{R}^3)). \) Moreover, we can apply (3.12) to bootstrap the regularity of \( (\nabla \times u) \times u \). Then we may apply the standard parabolic estimate to get the \( L^1([0, t], L^\infty(\mathbb{R}^3)) \) estimate for \( \nabla \times u \).

We first perform \( L^4 \) energy estimate for (2.8) and derive that
\[
\frac{1}{4} \frac{d}{dt} \int |\omega^\theta|^4 dx + \int (|\nabla|\omega^\theta|^2 + |\omega^\theta|^4 r^{-2}) dx \\
\leq \|\frac{u^r}{r}\|_{L^\infty} \int (\omega^\theta)^4 dx + \|\Pi\|_{L^\infty} \|B^\theta\|_{L^\infty} \|\partial_z|\omega^\theta|^2\|_{L^2} \|\omega^\theta\|_{L^2} \\
\leq \|\frac{u^r}{r}\|_{L^\infty} \int (\omega^\theta)^4 dx + \frac{1}{2} \|\Pi\|_{L^\infty}^2 \|B^\theta\|_{L^2}^2 \|\omega^\theta\|_{L^2}^2 + \frac{1}{2} \|\partial_z|\omega^\theta|^2\|_{L^2}^2.
\]

Using the a priori estimate in (3.3) and Lemma 3.1, one has
\[
\|\omega^\theta\|_{L^\infty([0, t], L^2(\mathbb{R}^3))} + \|\nabla|\omega^\theta|^2\|_{L^2([0, t], L^2(\mathbb{R}^3))} \lesssim e^{t^\frac{5}{4}}.
\]
By Sobolev imbedding inequality, one has
\[
\|\omega^\theta\|_{L^\infty([0, t], L^4(\mathbb{R}^3))} + \|\omega^\theta\|_{L^4([0, t], L^{12}(\mathbb{R}^3))} \lesssim e^{t^\frac{4}{3}}.
\]
On the other hand, by Sobolev imbedding, one also has
\[
\|u\|_{L^\infty([0, t], L^\infty(\mathbb{R}^3))} \lesssim \|u\|_{L^\infty([0, t], L^2(\mathbb{R}^3))} + \|\omega^\theta\|_{L^\infty([0, t], L^4(\mathbb{R}^3))} \lesssim e^{t^\frac{4}{3}}.
\]
Hence, we have
\[
\|((\nabla \times u) \times u)\|_{L^4([0, t], L^{12}(\mathbb{R}^3))} \lesssim e^{t^\frac{4}{3}}.
\]
Write
\[
\nabla \times u = e^{t\Delta} \nabla \times u_0 - \int_0^t e^{(t-s)\Delta} \left( \nabla \times ((\nabla \times u) \times u) + \partial_z(\Pi B^\theta e_\theta) \right) ds.
\]
A standard parabolic estimate gives that
\[
\|\nabla \nabla \times u\|_{L^4([0, t], L^{12}(\mathbb{R}^3))} \lesssim e^{t^\frac{4}{3}}.
\]
By Sobolev imbedding, we have
\[
\|\nabla u\|_{L^4([0, t], L^\infty(\mathbb{R}^3))} \lesssim e^{t^\frac{4}{3}}. \tag{3.13}
\]
Now let us derive the \( L^1([0, T], \text{Lip}(\mathbb{R}^3)) \) estimate of \( B \). We first write
\[
\partial_t B + u \cdot \nabla B = \frac{u^r}{r} B.
\]
Applying $\nabla$, one has
\[
\partial_t \nabla B + u \cdot \nabla \nabla B = -\nabla u \cdot \nabla B + \frac{u^r}{r} \nabla B + \nabla u' \Pi e_r + (\nabla \frac{1}{r}) u' B.
\]

Note that
\[
(\nabla \frac{1}{r}) u' B = -\frac{u^r}{r} \Pi e_r,
\]
once has
\[
\|\nabla B(t, \cdot)\|_{L^\infty} \lesssim \|\nabla B_0\|_{L^\infty} + \int_0^t \left( \|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty} \right) \|\nabla B(s, \cdot)\|_{L^\infty} \, ds
+ \int_0^t \left( \|\nabla u\|_{L^\infty} + \|\frac{u^r}{r}\|_{L^\infty} \right) \|\Pi(s, \cdot)\|_{L^\infty} \, ds.
\]

We can use \eqref{3.3}, \eqref{3.13}, Lemma 3.1 and Gronwall’s inequality to estimate that
\[
\|\nabla B(t, \cdot)\|_{L^\infty} \lesssim e^{\frac{\bar{F}}{2}}.
\tag{3.14}
\]

The a priori estimates \eqref{3.13} and \eqref{3.14} are enough for the global regularity of the ideal MHD equations \eqref{1.1}. Indeed, applying the standard $H^2$ energy estimate, one has
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \|\nabla^2 B(t, \cdot)\|_{L^2}^2 \right) + \|\nabla^3 u(t, \cdot)\|_{L^2}^2
\leq \int ( - \nabla^2 u \nabla^2 (u \cdot \nabla u) + \nabla^2 u \nabla^2 (B \cdot \nabla B)) \, dx
+ \int ( - \nabla^2 B \nabla^2 (u \cdot \nabla B) + \nabla^2 B \nabla^2 (B \cdot \nabla u)) \, dx
\leq \frac{1}{2} \|\nabla^3 u(t, \cdot)\|_{L^2}^2 + \|u(t, \cdot)\|_{L^\infty} \|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \|B(t, \cdot)\|_{L^\infty} \|\nabla^2 B(t, \cdot)\|_{L^2}^2
+ \left( \|\nabla u(t, \cdot)\|_{L^\infty} + \|\nabla B(t, \cdot)\|_{L^\infty} \right) \left( \|\nabla^2 B(t, \cdot)\|_{L^2}^2 + \|\nabla^2 u(t, \cdot)\|_{L^2}^2 \right).
\]

Here we also used the Gagliardo-Nirenberg’s inequality $\|\nabla f\|_{L^4}^2 \lesssim \|f\|_{L^\infty} \|\nabla^2 f\|_{L^2}$, the integration by parts and $\int \nabla^2 B(u \cdot \nabla) \nabla^2 B \, dx = 0$. Consequently, one has
\[
\left\{ \begin{array}{l}
\|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \|\nabla^2 B(t, \cdot)\|_{L^2}^2 \lesssim e^{\frac{\bar{F}}{2}}, \\
\int_0^t \|\nabla^3 u\|_{L^2}^2 \, dt \lesssim e^{\frac{\bar{F}}{2}},
\end{array} \right. \quad \forall \ t \geq 0.
\]

We finished the proof of Theorem 1.1.

**Remark 3.3.** The proof of Lemma 3.1 can also be proved by using the following Biot-Savart law (see \cite{17}):
\[
|u^r(t, x)| \lesssim \int_{|y-x| \leq 4r} \left| \frac{\omega^0(t,y)}{|x-y|^2} \right| \, dy + r \int_{|y-x| \geq 4r} \left| \frac{\omega^0(t,y)}{|x-y|^3} \right| \, dy,
\]
which, by Young’s inequality, gives that
\[
|u^r(t,r,z)| \lesssim r \frac{1}{|x|^2} \ast \Omega \leq r \| \frac{1}{|x|^2} \ast \Omega \|_{L^\infty} \|\Omega\|_{L^{3,1}}.
\]
Here $L^{p,q}$ denotes the usual Lorentz norm. Then using the real interpolation and Sobolev imbedding, one has
\[
\left| \frac{u'(t,r,z)}{r} \right| \lesssim \|\Omega\|_{L^{1,1}} \leq \|\Omega\|_{L^2}^{\frac{1}{2}} \|\Omega\|_{L^\infty}^{\frac{1}{2}}.
\] (3.15)

### 4 Proofs of Theorem 1.2

In this section we prove Theorem 1.2.

**Proof of Theorem 1.2.** Similarly as in obtaining (3.1), one has
\[
\frac{1}{2} \frac{d}{dt} \|\Omega\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 \lesssim \left| \int \Omega \partial_z \Pi^2 dx \right|
\leq \|\Pi\|_{L^\infty}^\frac{3}{2} \|\Pi\|_{L^2}^{\frac{3}{2}} \|\partial_z \Omega\|_{L^2}
\lesssim \|\Pi\|_{L^\infty}^\frac{3}{2} \|\Pi\|_{L^2}^{\frac{3}{2}} \|\nabla \Pi\|_{L^2} \|\partial_z \Omega\|_{L^2}.
\]

Consequently, one has
\[
\frac{d}{dt} \|\Omega\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 \lesssim \|\Pi\|_{L^\infty}^\frac{3}{2} \|\Pi\|_{L^2}^{\frac{3}{2}} \|\nabla \Pi\|_{L^2}^2.
\] (4.1)

Applying a similar argument to $\Pi$ equation in (2.11), one has
\[
\frac{d}{dt} \|\Pi\|_{L^2}^2 + \|\nabla \Pi\|_{L^2}^2 \leq 0.
\] (4.2)

Clearly, the combination of (4.1), (4.2) and the maximum estimate in Proposition 2.1 gives the following a priori estimate
\[
\|\Pi(t,\cdot)\|_{L^2} + \|\Omega(t,\cdot)\|_{L^2} \lesssim 1 \quad (\forall \ t \geq 0), \quad \int_0^\infty (\|\nabla \Pi\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2) dt \lesssim 1.
\] (4.3)

Now let us come back to the equation of $\omega^\theta$ in (2.8). Applying the standard energy estimate, one has
\[
\frac{1}{2} \frac{d}{dt} \int |\omega^\theta|^2 dx + \int \left( |\nabla \omega^\theta|^2 + \frac{|\omega^\theta|^2}{r^2} \right) dx
= \int \frac{u^r(\omega^\theta)^2}{r} dx - \int \frac{\partial_z (B^\theta)^2}{r} \omega^\theta dx.
\] (4.4)

Using Sobolev imbedding theorem and interpolation, one has
\[
\left| \int \frac{u^r(\omega^\theta)^2}{r} dx \right| \lesssim \|u^r\|_{L^6} \|\Omega\|_{L^6} \|\omega^\theta\|_{L^3}
\lesssim \|u^r\|_{L^2} \|\nabla \Omega\|_{L^2} \|\omega^\theta\|_{L^2} \|\nabla \omega^\theta\|_{L^2}^{\frac{1}{2}}
\lesssim \|u^r\|_{L^2}^2 \|\nabla \Omega\|_{L^2}^2 + \|\omega^\theta\|_{L^2}^2 + \frac{1}{4} \|\nabla \omega^\theta\|_{L^2}^2.
\]
On the other hand, it is clear that one also has
\[
\left| \int \frac{\partial_z (B^\theta)^2}{r} \omega^\theta dx \right| \leq \|\Pi\|_{L^\infty} \|B^\theta\|_{L^2} \|\partial_z \omega^\theta\|_{L^2}
\]
\[
\leq \|\Pi\|_{L^\infty}^2 \|B^\theta\|_{L^2}^2 + \frac{1}{4} \|\partial_z \omega^\theta\|_{L^2}.
\]

Using the \textit{a priori} estimate \eqref{4.3}, the basic energy law \eqref{3.11} and Proposition \ref{2.1}, we have
\[
\|\nabla \times u(t, \cdot)\|_{L^2} \lesssim 1 \quad (\forall \ t \geq 0), \quad \int_0^\infty \|\nabla (\nabla \times u)^2\|_{L^2} dt \lesssim 1.
\]

The \textit{a priori} estimate \eqref{4.5} is enough to get the global regularity of the resistive MHD \eqref{1.2}. Indeed, by using the equation of $B$, one can easily verifies that $\nabla B$ also satisfies \eqref{4.5}. We have finished the proof of Theorem \ref{1.2}.

5 Appendix

In this appendix we first prove that $w(y) = r^\alpha$ is a $A_p$ for Riesz operator in $\mathbb{R}^5$ under $-4 < \alpha < 4p(1 - \frac{1}{p})$. The case of $\alpha = -2$ has been studied in \cite{8}.

\textbf{Lemma 5.1 ($A_p$, Weight).} Let $1 < p < \infty$ and $w(y) = r^\alpha, \ y \in \mathbb{R}^5$. Then $w(x)$ is in $A_p$ class if $-4 < \alpha < 4p(1 - \frac{1}{p})$.

\textbf{Proof.} Recall that a real valued non-negative function $w(x)$ is said to be in $A_p(\mathbb{R}^n)$ class if it satisfies
\[
\sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty.
\]

Here $p$ and $q$ are conjugate indices with $1 < p < \infty$.

For any ball $B \subset \mathbb{R}^5$, denote $B = B(y_0, R)$. It is easy to see that if $r_0 > 2R$, one has $r \simeq r_0$ for any $x \in B$. Consequently, for any $\alpha \in \mathbb{R}$, one has
\[
\left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{q}{p}} dx \right)^{\frac{p}{q}} \lesssim \left( \frac{1}{|B|} \int_B r_0^\alpha dx \right) \left( \frac{1}{|B|} \int_B r_0^{-\frac{q}{p}} dx \right)^{\frac{p}{q}} \lesssim 1.
\]

On the other hand, if $r_0 \leq 2R$, then for $\alpha + 3 > -1$ and $-\frac{qa}{p} + 3 > -1$, one has
\[
\left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{q}{p}} dx \right)^{\frac{p}{q}} \lesssim \left( \frac{1}{R^5} \int_{r_0-R}^{r_0+R} dz \int_0^{3R} r_\alpha+3 dr \right) \left( \frac{1}{R^5} \int_{r_0-R}^{r_0+R} dz \int_0^{3R} r^{-\frac{qa}{p}+3} dr \right)^{\frac{p}{q}} \lesssim R^{\alpha} R^{-\alpha} = 1.
\]

Noting that the condition on $\alpha$ is $-4 < \alpha < 4p(1 - \frac{1}{p})$, we in fact have completed the proof of the lemma.
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References

[1] R. Caflisch, I. Klapper and G. Steele, Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD, Comm. Math. Phys. 184 (1997) 443–455.

[2] C. Cao and J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. 226 (2011), no. 2, 1803–1822.

[3] C. Cao, J. Wu and B. Yuan, The 2D Incompressible Magnetohydrodynamics Equations with only Magnetic Diffusion, [http://arxiv.org/abs/1306.3629](http://arxiv.org/abs/1306.3629)

[4] Q. Chen, C. Miao and Z. Zhang, On the regularity criterion of weak solution for the 3D viscous magneto-hydrodynamics equations. Comm. Math. Phys. 284 (2008), no. 3, 919–930.

[5] C. Chiuderi and F. Califano, Resistivity-independent dissipation of magnetohydrodynamic waves in an inhomogeneous, Phys. Rev. E 60, 4701–4707 (1999).

[6] C. He and X. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, J. Funct. Anal., 227 (2005), 113–152.

[7] C. He and X. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, J. Differential Equations, 213 (2005), 235–254.

[8] T. Hou, Z. Lei and C. Li, Global regularity of the 3D axi-symmetric Navier-Stokes equations with anisotropic data. Comm. Partial Differential Equations 33 (2008), no. 7-9, 1622–1637.

[9] O. A. Ladyzhenskaya, Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry. Zapiski Nauchnykh Seminarov Leningrad Otdelenie. Matematicheski. Institut im. V. A. Steklova (LOMI) 7 (1968): 155–77 (Russian).

[10] L. D. Landau and E. M. Lifshitz: Electrodynamics of Continuous Media, 2nd ed. Pergamon, New York, 1984.

[11] Z. Lei and D. Li, Global well-posedness of axially symmetric perfect resistive MHD. Preprint.
[12] Z. Lei and Y. Zhou, BKM’s criterion and global weak solutions for magnetohydrodynamics with zero viscosity, DCDS-A, 25 (2009), no. 2, 575–583.

[13] F.-H. Lin, L. Xu and P. Zhang, Global small solutions to 2-D incompressible MHD system, preprint.

[14] Chiun-Chuan Chen, Robert M. Strain, Tai-Peng Tsai, and Horng-Tzer Yau, Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations, Int. Math Res. Notices (2008), vol. 8, artical ID rnn016, 31 pp.

[15] M. Sermange and R. Temam, Some mathematical questions related to the mhd equations, Comm. Pure Appl. Math., 36 (1983), 635–664.

[16] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton, NJ: Princeton University Press.

[17] T. Shirota and T. Yanagisawa, Note on global existence for axially symmetric solutions of the Euler system. Proc. Japan Acad. Ser. A Math. Sci. 70 (1994), no. 10, 299–304.

[18] R. Temam, Navier-Stokes equations. Theory and numerical analysis. Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001.

[19] M. R. Ukhovskii and V. I. Iudovich. Axially symmetric flows of ideal and viscous fluids filling the whole space. Journal of Applied Mathematics and Mechanics 32 (1968): 52–61.