Asymptotic Uncorrelation for Mexican Needlets

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Abstract

We recall Mexican needlets from [5]. We derive an estimate for certain types of Legendre series, which we apply to the statistical properties of Mexican needlets. More precisely, we shall show that, under isotropy and Gaussianity assumptions, the Mexican needlet coefficients of a random field on the sphere are asymptotically uncorrelated, as the frequency parameter goes to infinity. This property is important in the analysis of cosmic microwave background radiation.

Keywords. Spherical harmonics, spherical Laplacian operator, wavelets, Mexican needlets, angular power spectrum, Legendre (Gegenbauer) polynomials.

AMS Subject Classification (2000). 62M40, 33C55, 65T60

1 Introduction

Let $\Delta$ denote the spherical Laplacian. Let $\{Y^m_l\}$ for $l \in \mathbb{N}$ and $m = -l, \ldots, l$ be the usual spherical harmonics on $S^2$. They constitute an orthonormal basis for $L^2(S^2)$ and are eigenfunctions for the Laplacian operator with eigenvalues $l(l+1)$, i.e., $\Delta Y^m_l = l(l+1)Y^m_l$. We shall denote $\lambda_l = l(l+1)$.

By the spectral theorem for the spherical Laplacian $\Delta$, for any $f \in \mathcal{S}(\mathbb{R}^+)$ and $t > 0$, the operator $f(t^2\Delta)$ is a bounded operator on $L^2(S^2)$. Let $K_t$ denote the kernel of operator $f(t^2\Delta)$ defined on $S^2 \times S^2$. In [5] we observed that $f(t^2\Delta)$ has an explicit kernel $K_t$, given by

$$K_t(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f(t^2\lambda_l)Y^m_l(x)Y^m_l(y).$$

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It is easy to see that $K_t$ is smooth in $(t,x,y)$ for $t > 0$. 
In applications to wavelets, one needs to assume $f(0) = 0$, and we do so in what follows. 

The localization properties of these kinds of kernels have already been studied in detail in [5]; briefly, we showed that for every pair of $C^\infty$ differential operators $X$ (in $x$) and $Y$ (in $y$) on $S^2$, and for every integer $N \in \mathbb{N}_0$, there exists $c := c_{N,X,Y}$ such that for all $t > 0$ and $x,y \in S^2$

$$|XYK_t(x,y)| \leq c \frac{t^{-(2l+I)}}{(d(x,y)}^N, \quad (2)$$

where $I := \deg X$ and $J := \deg Y$. (In the case where $f$ has compact support away from 0, and if one replaces $\lambda_l = l(l+1)$ by $l^2$ in the formula for $K_t$, this was earlier shown by Narcowich, Petrushev and Ward, in [7] and [9].) In fact our argument worked on general smooth compact oriented Riemannian manifolds, not just the sphere.

Moreover, on the sphere $S^2$, the kernel $K_t(x,y)$ in (1) is rotationally invariant, i.e., $K_t(x,y) = K_t(\rho x,\rho y)$ for any rotation $\rho$ defined on the sphere. Intuitively, we define $\psi_{t,x}(y) := K_t(x,y)$, then $\psi_{t,x}(y)$ can be thought of as analogous to “$t$-dilation” and “$x$-translation” of the single function $\psi(y) := K_t(1,y)$, where $1 = (1,0,0)$ is the “north pole”. For more about the analogues of “dilation” and “translation” we refer the reader to our earlier work [5].

In [5] we also proved that if $f \in S(\mathbb{R}^+)$ with $f(0) = 0$ satisfies a discrete version of Calderón’s formula, then the associated kernel is a wavelet, i.e., for $a > 1$ and for a carefully chosen discrete set $\{x_{j,k}\}_{(j,k) \in \mathbb{Z}^2}$ on the sphere and certain weights $\mu_{j,k}$, the collection of $\{\psi_{j,k} := \mu_{j,k}K_{a^j}(x_{j,k},\cdot)\}$ constitutes a wavelet frame. By this we mean that there exist constants $0 < A \leq B < \infty$, such that for any $F \in L^2(S^2)$ the following holds:

$$A \| F \|_2^2 \leq \sum_{j,k} | \langle F, \psi_{j,k} \rangle |^2 \leq B \| F \|_2^2. \quad (3)$$

We also showed that, if the points $\{x_{j,k}\}$ were selected carefully enough, $B$ could be made arbitrarily close to $B_a$ and $A$ could be made arbitrarily close to $A_a$, where $B_a/A_a \to 1$ almost quadratically as $a \to 1$. The frame is therefore a “nearly tight frame”. Like before, $\psi_{j,k}$ is analogous to $a^j$-dilation and $x_{j,k}$-translation of the function $K(1,\cdot)$. We shall call such a frame needlets based on $f$. (The term “needlets” was used earlier for the frames of Narcowich, Petrushev and Ward.) The present work involves the needlets based on the Schwartz functions

$$f(s) = s^r f_0(s), \quad (4)$$

where $0 \neq f_0 \in S(\mathbb{R}^+)$ and $r \in \mathbb{N}$. For $f_0(s) = e^{-s}$ and $r \in \mathbb{N}$, we shall call these needlets Mexican needlets.

Let $\psi^{t,x} = K_t(x,\cdot)$, so that $\psi_{j,k} = \mu_{j,k}\psi^{a^j,x_{j,k}}$. By the spherical harmonic expansion of the kernel of $f(t^2\Delta)$ in (1), for any $F \in L^2(S^2)$ at every scale $t$ and every pixel $x$ we have

$$\beta_{t,x} := \langle F, \psi^{t,x} \rangle = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f(t^2\lambda_l) \langle F, Y^{l} \rangle Y^{l,m}(x). \quad (5)$$

It is standard to denote $\hat{F}(l,m) = \langle F, Y^{l,m} \rangle$ and call it the spherical harmonic coefficient of the function $F$ at $(l,m)$.
The main goal of this work is to study the behavior of the following quantity, for different \( P \)
and \( x \) and \( \rho \) and \( t \), when \( t \) is close to zero:

\[
\beta_{t,x} = \langle F, \psi^t(x) \rangle = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f(t^2 \lambda_l) a_{l,m} Y^m_l(x).
\]

Note \( \beta_{t,x} \) is real, since \( \sum_{m} (-1)^m Y_{l,-m} \). Taking \( t = a^j \) and \( x = x_{j,k} \), \( \mu_{j,k} \beta_{t,x} \) become the random needlet coefficients \( \langle F, \psi_{j,k} \rangle \).

The isotropy assumption implies that \( E(a_{t,m} a_{t,m'}) = 0 \) for \( l \neq l' \), \( m \neq m' \) and \( E(a^2_{t,m}) = c_l \) (independent of \( m \)). Then for any \( t > 0 \) and \( x, y \in S^2 \), the expectation of \( \beta_{t,x} \beta_{t,y} \) is given by

\[
E(\beta_{t,x} \beta_{t,y}) = \sum_{l} f(t^2 \lambda_l)^2 c_l (2l + 1) P_l^{1/2}(x \cdot y),
\]

where \( P_l^{1/2} \) is the Legendre polynomial of degree \( l \) (or Legendre polynomial of index 1/2 and degree \( l \)) and \( x \cdot y \) is the usual inner product of \( x \) and \( y \). The \( P_l^{1/2} \) may be defined in terms of the generating function

\[
(1 - 2 \xi \eta + \xi^2)^{-1/2} = \sum_{i=0}^{\infty} P_i^{1/2}(\eta) \xi^i.
\]

The main goal of this work is to study the behavior of the following quantity, for different \( x, y \) on the sphere, when \( t \) is close to zero:

\[
\Cor(\beta_{t,x}, \beta_{t,y}) = \frac{E(\beta_{t,x} \beta_{t,y})}{\sqrt{E(\beta^2_{t,x})} \sqrt{E(\beta^2_{t,y})}}
\]

From the statistical point of view, \( \langle F, \psi^t(x) \rangle \) is referred to the correlation of the \( \beta_{t,x} \) and \( \beta_{t,y} \) at scale \( t \) and at positions \( x, y \) respectively and is denoted by \( \Cor(\beta_{t,x}, \beta_{t,y}) \). Using this statistical terminology, we shall show that the correlation approaches zero as \( t \to 0 \), if \( x \) and \( y \) are fixed. That is, they are asymptotically uncorrelated. For this, first we prove in Lemma 1.1 that the term in (7) is well localized near \( x = y \). Then applying this lemma, we present the results concerning the asymptotic uncorrelation in Theorem 1.2. Note that if \( p, q > 0 \), then \( \Cor(p \beta_{t,x}, q \beta_{t,y}) = \Cor(\beta_{t,x}, \beta_{t,y}) \). Thus if \( t = a^j \), \( x = x_{j,k} \), \( y = x_{j,k'} \), for certain \( k, k' \), the correlation in (8) gives the correlation between needlet coefficients.

Although the motivation is from statistics and cosmology, our arguments to prove Theorem 1.2 will be purely mathematical and will not use this motivation. However, in order to prove this theorem, we shall assume some regularity assumptions for the expected values \( c_l \). (These values are known as the angular power spectrum in astrophysics.) Our assumptions will be reasonable, based on the astrophysical literature. More precisely, we suppose that \( c_l \) is given by formula of a function \( u \in C^\infty(\mathbb{R}^+) \), \( c_l := u(l) \), with the following properties: For some real number \( \alpha > 2 \):

(a) for any \( k \in \mathbb{N}_0 \) there exists a constant \( c_k \) such that
and (b) there exist positive constants $c_0, c_1$ such that for any $s \in (1, \infty)$,\]
$c_0 s^\alpha \leq u(s) \leq c_1 s^\alpha$ holds.

With these assumed conditions for the angular power spectrums $c_l$, in Theorem 4.2 we will show that for any fixed points $x$ and $y$ the expression for the correlation of $\beta_{t,x}$ and $\beta_{t,y}$ satisfies\]
$|\text{Cor}(\beta_{t,x}, \beta_{t,y})| = \frac{|E(\beta_{t,x}\beta_{t,y})|}{\sqrt{E(\beta_{t,x}^2)\sqrt{E(\beta_{t,y}^2)}}} \leq Ct^{4r-\alpha+2} \frac{(d(x,y))^{2N}}{\lambda_l \leq l(l+1)}$ in the definition (see (1)), which is a minor distinction.

Asymptotic uncorrelation was first studied in [4]. There the authors assumed that $c_l := 2^j(\alpha + j)$ for every $l$ such that $2^j < l < 2^{j+1}$; here $(g_j)_j$ is a sequence of functions (they can also be constants), which have a uniform bounded differentiability condition up to order $M$, for some large $M$. The investigation of needlets from a stochastic point of view is due to [2, 3, 4]. Needlets have been used by astrophysicists to study cosmic microwave background radiation (CMB). (See for instance [1, 6] and references therein.)

Needlets and Mexican needlets each have their own advantages. Needlets have these advantages: for appropriate $f$ and $x_{i,k}$, needlets are a tight frame on the sphere (i.e., $A = B$ in (3)). The frame elements at non-adjacent scales are orthogonal. The random needlet coefficients are asymptotically uncorrelated (there is no “$r$” that needs to be assumed large in relation to $\alpha$).

Mexican needlets (as developed in [5], for which $f(s) = s^r e^{-s^2}$) have their own advantages. We write down an approximate formula for them which can be used directly on the sphere in [5]. (This formula,\]
[1] While finishing this paper, we learned by personal communication that simultaneously X. Lan and D. Marinucci [8] have obtained an analogous result to Theorem 4.2. The assumptions are not equivalent and the approaches are entirely different. We believe both are of independent interest and should be utilized.
which arises from computation of a Maclaurin series, has been checked numerically. It is work in progress, expected to be completed soon, to estimate the remainder terms in this Maclaurin series.) Assuming this formula, Mexican needlets have Gaussian decay at each scale. They do not oscillate (for small $r$), so they can be implemented directly on the sphere, which is desirable if there is missing data (such as the “sky cut” of the CMB). Finally, as the proofs in this article will show, the constant $C$ in (11) depends on finitely many derivatives of $f$; so in order to be sure that this constant is small as possible, it is desirable to use real analytic functions, such as $s^r e^{-s}$, as $f(s)$.

In our opinion, both needlets and Mexican needlets should be utilized in the analysis of CMB, and the results should be compared.

2 Notations and Preliminaries

Let $S^2$ denote the unit sphere in $\mathbb{R}^3$ and let $1 = (1, 0, 0)$ denote the “north pole”. We shall consider $S^2$ with its rotationally invariant measure $\mu$. We may write

$$L^2(S^2) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l,$$

where $\mathcal{H}_l$ is the space of spherical harmonic of degree $l$. In fact, $\mathcal{H}_l$ is the restriction of homogeneous harmonic polynomials of degree $l$ on $\mathbb{R}^3$. The Laplacian operator for the sphere is in a sense the “restriction” of the Laplacian operator for $\mathbb{R}^3$ on $S^2$. We shall denote it by $\Delta$. A precise formula for $\Delta$ in spherical coordinates is given by

$$\Delta = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

If $P \in \mathcal{H}_l$, then

$$\Delta P = l(l+1)P,$$

which means that the spherical harmonics of degree $l$ are eigenfunctions for $\Delta$ with eigenvalues $\lambda_l = l(l+1)$.

Within each space $\mathcal{H}_l$ is a unique zonal harmonic $Z_l$, which has the property that for all $P \in \mathcal{H}_l$, $P(1) = \langle P, Z_l \rangle$. In particular, $P$ is orthogonal to $Z_l$ if and only if $P(1) = 0$. Obviously, $Z_l(1) = \langle Z_l, Z_l \rangle$ and $Z_l(y)$ is known explicitly in terms of the Legendre polynomials. In fact, if $\omega_2$ is the area of $S^2$, then for $c = \frac{1}{\omega_2}$,

$$Z_l(y) = c(2l+1)P_l^\lambda(y_1),$$

where $y = (y_1, y_2, y_3)$, $\lambda = 1/2$, and $P_l^\lambda$ is the Legendre polynomial of degree $l$ associated with $\lambda$ \cite{10}. In the sequel we shall avoid the notation $\lambda$ as $\lambda = 1/2$. Since

$$P_l(1) = \binom{l}{l} = 1,$$

then

$$Z_l(1) = c(2l+1).$$

$\mathcal{H}_l$ is a finite dimensional vector space in $L^2(S^2)$ with dimension

$$\dim \mathcal{H}_l = 2l + 1.$$
We shall choose an orthonormal basis for each $\mathcal{H}_l$, one of whose elements is $Z_l/\|Z_l\|_2$. For $l \in \mathbb{N}_0$, let $\{Y_l^m\}_{m=-l}^l$ be an orthonormal basis of $2l+1$ elements for $\mathcal{H}_l$ with central element $Y_l^0 = Z_l/\|Z_l\|_2$, and with $Y_l^m = Y_{-m}^l$. Therefore any $F \in L^2(S^2)$ has a spherical harmonic expansion $\{Y_l^m\}_{m=-l}^l$:

\begin{equation}
F = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{F}(l, m) \ Y_l^m
\end{equation}

where $\hat{F}(l, m) := \langle F, Y_l^m \rangle$ are called the spherical harmonic coefficients.

Next we introduce notation concerning the discrete version of derivation on a sequence:

**Definition 2.1.** Let $\Delta^+$ and $\Delta^-$ denote the difference operators defined on any sequence $\{a_l\}_{l \in \mathbb{Z}}$ as follows: For all $l$,

\begin{align*}
\Delta^+ a_l &= a_{l+1} - a_l \\
\Delta^- a_l &= a_l - a_{l-1}.
\end{align*}

Obviously, the operators $\Delta^-$ and $\Delta^+$ commute.

Suppose $\{b_l\}_{l \in \mathbb{Z}}$ and $\{c_l\}_{l \in \mathbb{Z}}$ are any pair of sequences. Then the product rules for the difference operators are given as follows:

\begin{align*}
\Delta^+ (b_l c_l) &= (\Delta^+ b_l) c_{l+1} + b_l (\Delta^+ c_l), \\
\Delta^- (b_l c_l) &= (\Delta^- b_l) c_{l} + b_{l-1} (\Delta^- c_l).
\end{align*}

For two sequences $\{a_l\}_{l \in \mathbb{Z}}$ and $\{b_l\}_{l \in \mathbb{Z}}$, we shall say $a_l$ is uniformly bounded by $b_l$ from above, and write $a_l = O(b_l)$, if for some positive constant $c$ we have

\[ |a_l| \leq c |b_l| \ \forall \ l \in \mathbb{Z}. \]

### 3 Motivation

Following the notations of our earlier work in [5], for any $f \in \mathcal{S}(\mathbb{R}^+)$ with $f(0) = 0$ and $t > 0$, let $K_t$ be the associated kernel of the operator $f(t^2 \Delta)$, i.e., for any $F \in L^2(S^2)$

\begin{equation}
f(t^2 \Delta) F(x) = \int_{S^2} K_t(x, y) F(y) \ d\mu.
\end{equation}

By expansion in (17) and the spectral theorem, one has:

\begin{equation}
\psi_{t,x}(y) := K_t(x, y) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f(t^2 \lambda_l) Y_l^m(x) Y_l^m(y)
\end{equation}

\begin{equation}
= \frac{1}{\omega_2} \sum_{l=1}^{\infty} f(t^2 \lambda_l) (2l+1)P_l(x,y).
\end{equation}

From now on, we shall neglect the factor $\frac{1}{\omega_2}$ in the rest of the work, since it will not affect our main results.
Since $⟨Y^m_l, Z_l⟩ = Y^m_l(1) = 0$, then
\[ \psi_t(y) := \psi^{+1}(y) = \sum_{l=1}^{\infty} f(t^2 \lambda_l) Z_l(y) = \sum_{l=1}^{\infty} f(t^2 \lambda_l)(2l + 1) P_l(y). \]  

(25)

As we explained in the introduction, the $\psi_t$, generates a needlet frame (based on $f$) for $L^2(S^2)$.

For the scale $t > 0$ and position $x \in S^2$,
\[ \beta_{t,x} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f(t^2 \lambda_l) \langle F, Y^m_l \rangle Y^m_l(x). \]  

(26)

Now, suppose that $\{a_{l,m}\}$ are Gaussian random variables. For $t > 0$ and $x \in S^2$, we set
\[ \beta_{t,x} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f(t^2 \lambda_l) a_{l,m} Y^m_l(x) \]  

(27)

For $t > 0$ and any other point $y \in S^2$, we define the correlation:
\[ \text{Cor}(\beta_{t,x}, \beta_{t,y}) = \frac{E(\beta_{t,x} \beta_{t,y})}{\sqrt{E(\beta_{t,x}^2)} \sqrt{E(\beta_{t,y}^2)}} \]  

(28)

where for any random variable $w$, $E(w)$ is its expectation.

As motivated in the introduction, we assume that there exist $\{c_l\}$ such that
\[ E(a_{l,m} a_{l',m'}) = c_l \delta_{l,l'} \delta_{m,m'} \quad \forall \: l, l', m, m' \in \mathbb{Z}. \]  

(29)

We first calculate $E(\beta_{t,x} \beta_{t,y})$ for any $t > 0$ and $x, y \in S^2$:
By the linearity of $E$ and using (29) one obtains:
\[ E(\beta_{t,x} \beta_{t,y}) = \sum_{l,m} E(a_{l,m}^2) f(t^2 \lambda_l)^2 Y^m_l(x) Y^m_l(y) \]  

(30)

Surely $c_l := E(a_{l,m}^2)$ (which is called the angular power spectrum). Therefore
\[ E(\beta_{t,x} \beta_{t,y}) = \sum_{l} (2l + 1) c_l f(t^2 \lambda_l)^2 P_l(x,y). \]  

(31)

and, hence, for $x = y$ we obtain
\[ E(\beta_{t,x}^2) = \sum_{l} (2l + 1) c_l f(t^2 \lambda_l)^2 P_l(1). \]  

(32)

Therefore the correlation formula is given by
\[ \text{Cor}(\beta_{t,x}, \beta_{t,y}) = \frac{\sum_{l} (2l + 1) c_l f(t^2 \lambda_l)^2 P_l(x,y)}{\sum_{l} (2l + 1) c_l f(t^2 \lambda_l)^2 P_l(1)}. \]  

(33)
A similar formula was derived in [4].

We will estimate (33) for small $t > 0$ with following assumptions on the angular power spectrums $c_l$:

Assume that $c_l$ is given by

$$c_l = u(l)$$

(34)

where $u$ is a smooth function on $(0, \infty)$ which satisfies the following conditions, for some $\alpha \in \mathbb{R}_{>2}$:

- (i) for all $k \in \mathbb{N}_0$ there exists a constant $C_k$ such that $|\partial^k u(s)| \leq C_k s^{-\alpha-k}$ (uniformly for all $s \geq 1$), and
- (ii) there exist $k_0, k_1 > 0$ such that

$$k_0 s^{-\alpha} \leq u(s) \leq k_1 s^{-\alpha} \quad \forall s \geq 1.$$  

(35)

With above assumptions on $c_l$, in the next section we show that if $4r + 2 > \alpha$, then for two different points $x, y$ on the sphere and $t > 0$ one has:

$$|\text{Cor}(\beta t, x, \beta t, y)| = \left| \frac{\sum_{l=1}^{\infty} f(t^2 \lambda_l)^2 c_l f(Z_l(x \cdot y))}{\sum_{l=1}^{\infty} f(t^2 \lambda_l)^2 c_l f(Z_l(1))} \right| \leq C t^{4r-\alpha+2} d(x, y)^{2N},$$

(36)

where $C := C(r, \alpha)$ is independent of choice of $x, y$ and $t$ and $N$ is the least integer greater than $2r-\alpha/2+1$.

Here we shall present an example for $u$ for which the conditions (i)-(ii) are satisfied:

**Example:** Say $u(s) = \frac{F(\log s) P(s)}{s^{\beta + q - p}}$ on $(0, \infty)$, where: $P$ and $Q$ are polynomials of degree $p$ and $q$ respectively, and $\beta + q - p = \alpha$; $F$ is a smooth function defined on $\mathbb{R}$ with all bounded derivatives; and $P, Q$ and $F$ are all positive on $[1, \infty)$, and $F > \tau > 0$ on $[0, \infty)$. Then, surely, $u$ satisfies (i) and (ii).

### 4 The Key Estimate

For the study of the behavior of

$$\frac{\sum_l (2l+1) c_l f(t^2 \lambda_l)^2 P_l(x, y)}{\sum_l (2l+1) c_l f(t^2 \lambda_l)^2 P_l(1)}$$

(37)

when $t$ is close to zero, we first prove the next lemma, which has a crucial role in the proof of our main theorem, Theorem 4.2.

**Lemma 4.1.** Suppose that for $\{a_l\} \subset \mathbb{Z}$ and $\mu \in \mathbb{R}$ the following hold:

- (i) $a_l = O(l^\mu) \quad \forall l \in \mathbb{N}$, $a_0 = 0$, and
- (ii) for all $k_1, k_2 \in \mathbb{N}_0$, $(\Delta^-)^{k_1}(\Delta^+)^{k_2} a_l = O(l^{\mu-2(k_1+k_2)}).$
Define \( f(\cos \theta) := \sum_{l=1}^{\infty} a_l Z_l(\cos \theta) \). If \( \mu + 2 > 0 \), then for some positive constant the following inequality holds uniformly:

\[
|f(\cos \theta)| \leq \frac{C}{|\theta|^{2N}} \quad \forall \ 0 < \theta \leq \pi,
\]

when \( N \) is the least integer greater than \( \mu/2 + 1 \). (Observe that here \( Z_l(\cos \theta) \) is understood as \( (2l + 1)P_l(\cos \theta) \) for the Legendre polynomial of degree \( l \).)

**Proof.** In order to verify the estimation (38), first we shall examine the spherical harmonics expansion of \( (\cos \theta - 1)^N f(\cos \theta) \). That is, for any \( N \in \mathbb{N} \), we will find the coefficients \( d_l \) in

\[
(\cos \theta - 1)^N f(\cos \theta) = \sum_{l \in \mathbb{N}_0} d_l Z_l(\cos \theta).
\]

(It will follow from our arguments that only the zonal functions appear in the spherical harmonic expansion of function \( (\cos \theta - 1)^N f \).)

Using the following recursion formula for the Legendre polynomials

\[
(2l + 1)(x - 1)P_l(x) = (l + 1)P_{l+1}(x) - (2l + 1)P_l(x) + lP_{l-1}(x), \quad \forall x \in [-1, 1], \quad l \in \mathbb{N}_0,
\]

where we put \( P_{-1} \equiv 0 \), and with the convention \( a_{-1} \equiv 0 \), we obtain the following equalities for \( N = 1 \):

\[
(\cos \theta - 1) f(\cos \theta) = \sum_{l=0} a_l (\cos \theta - 1) Z_l(\cos \theta)
\]

\[
= \sum_{l=0} a_l \left\{ \frac{l + 1}{2l + 1} Z_{l+1} - \frac{2l + 1}{2l + 1} Z_l + \frac{l}{2(2l - 1) + 1} Z_{l-1} \right\} (\cos \theta)
\]

\[
= \sum_{l=0} a_l \left\{ \frac{l + 1}{2l + 3} Z_{l+1} - Z_l + \frac{l}{2l - 1} Z_{l-1} \right\} (\cos \theta)
\]

\[
= \sum_{l=1} a_{l-1} \frac{l}{2l + 1} Z_l(\cos \theta) - \sum_{l=0} a_l Z_l(\cos \theta) + \sum_{l=-1} a_{l+1} \frac{l + 1}{2l + 1} Z_l(\cos \theta)
\]

\[
= \sum_{l=0} a_{l-1} \frac{l}{2l + 1} Z_l(\cos \theta) - \sum_{l=0} a_l Z_l(\cos \theta) + \sum_{l=0} a_{l+1} \frac{l + 1}{2l + 1} Z_l(\cos \theta)
\]

\[
= \sum_{l=0} \left\{ \frac{l}{2l + 1} a_{l-1} - a_l + \frac{l + 1}{2l + 1} a_{l+1} \right\} Z_l(\cos \theta)
\]

\[
= \sum_{l=0} \frac{l}{2l + 1} (a_{l-1} - 2a_l + a_{l+1}) Z_l(\cos \theta) + \sum_{l=0} \frac{1}{2l + 1} (a_{l+1} - a_l) Z_l(\cos \theta)
\]

\[
= \sum_{l=0} a_l^l Z_l(\cos \theta),
\]

where

\[
a_l^l := \frac{l}{2l + 1} (a_{l-1} - 2a_l + a_{l+1}) + \frac{1}{2l + 1} (a_{l+1} - a_l).
\]
In fact $a_l^1$ can be written as follows:

$$a_l^1 := \{ R(l)\Delta^+\Delta^- + S(l)\Delta^- \} a_l = P(l)a_l,$$

for the sequence elements $R(l)$ and $S(l)$ given as

$$R(l) = \frac{l}{2l+1}, \quad S(l) = \frac{1}{2l+1},$$

and the operator $P(l)$ defined as

$$P(l) := R(l)\Delta^+\Delta^- + S(l)\Delta^-.$$

This implies that $a_l^1$ comes from the operation of operator $P(l)$ on $a_l$; more precisely, say $a := \{a_l\}$, then

$$a_l^1 = P(l)a_l = \{ R(l)\Delta^+\Delta^- + S(l)\Delta^- \} a_l,$$

and $a_{l-1}^l = 0$. For $N > 1$, analogously, by iteration one gets

$$(\cos \theta - 1)^N f(\cos \theta) = \sum_{l=0}^{N} a_l^N Z_l(\cos \theta),$$

where $a_l^N = P(l)^N(a_l)$. To find a upper estimate for the coefficients $a_l^N$, we shall use the following estimates for the formulas $R$ and $S$ in (51):

$$R(l) = \frac{l}{2l+1} = O(1), \quad S(l) = \frac{1}{2l+1} = O(l^{-1}),$$

and in general:

$$(\Delta^-)^{k_1}(\Delta^+)^{k_2} R(l) = O(l^{-(k_1+k_2+1)}),$$

$$(\Delta^-)^{k_1}(\Delta^+)^{k_2} S(l) = O(l^{-(k_1+k_2+1)}).$$

Using the preceding estimates for different exponents $k_i$’s, the product rules for difference operators, and the estimates in (i) and (ii), one obtains the following:

$$a_l^N = O(l^{\mu-2N}).$$

Now let $N$ be the least integer larger than $\mu/2 + 1$. The equality (57) implies

$$| \sum_l a_l^N Z_l(\cos \theta) | \leq c \sum_l (2l+1) l^{-2N+\mu} \leq C.$$

Therefore, since $| \cos \theta - 1 | \geq c_1 \theta^2$ for $0 < \theta \leq \pi$ and some constant $c_1$, we have

$$\left| \sum_l a_l Z_l(\cos \theta) \right| = \frac{1}{| \cos \theta - 1 |^N} \left| \sum_l a_l^N Z_l(\cos \theta) \right| \leq \frac{c_2}{| \cos \theta - 1 |^N} \leq \frac{c_3}{\theta^{2N}},$$

as desired.
We are now ready to state our main result in the next theorem. We shall suppose that the sequence \( \{c_l\}_{l \in \mathbb{N}} \) satisfies (ii) of section 3. Then, we have:

**Theorem 4.2.** Suppose \( \alpha > 2 \) and \( f(s) = s^rf_0(s) \) for a Schwartz function \( f_0 \) on \( \mathbb{R}^+ \) and \( r \in \mathbb{N} \). Assuming that \( 4r + 2 > \alpha \), for any \( t > 0 \) and any two different points \( x, y \) on the sphere one has

\[
\left| \sum_{l=1}^{\infty} f(t^2 \lambda_l)^2 c_l Z_l(x \cdot y) \right| \leq \frac{C t^{4r-\alpha+2}}{d(x, y)^{2N}},
\]

where \( N \) is the least integer greater than \( 2r - \alpha/2 + 1 \) and \( C \) is independent of \( x, y \) and \( t \).

**Proof.** Choose an interval \( I = [a^2, b^2] \subseteq (0, \infty) \) such that \( f^2 \geq c > 0 \) on \( I \). Choose \( b_t \) with \( a < b_t < b \). Then, if \( t \) is sufficiently small, \( f(t^2 \lambda_l) \geq c \) whenever \( a/t \leq \lambda_l \leq b_t/t \). With the assumptions on the coefficients \( c_l \) we have the following inequality for the denominator of (62), if \( t \) is sufficiently small: Since \( Z_l(1) = (2l + 1) \) up to a positive constant, we have

\[
\sum_{l=1}^{\infty} f(t^2 \lambda_l)^2 c_l Z_l(1) \geq \sum_{a/t \leq \lambda_l \leq b_t/t} f(t^2 \lambda_l)^2 c_l Z_l(1)
\]

\[
\geq c \sum_{a/t \leq \lambda_l \leq b_t/t} \frac{(2l + 1)}{t^\alpha}
\]

\[
\geq c t^{-1} t^{-1+\alpha} = c t^{\alpha-2} > 0.
\]

To find a upper estimate for the numerator of (62), for \( t > 0 \), define

\[
g_t(s) := f(t^2 s(s + 1)) = (t^2 s(s + 1))^r f_0(t^2 s(s + 1)) \quad \text{and}
\]

\[
G_t(s) := g_t(s)^2 u(s).
\]

By induction on \( n \), one easily sees that \( \partial_n^a f_0(t^2 s(s + 1)) \) is a finite linear combination of terms of the form \( t^{2i} F(t^2 s(s + 1)) \), where \( F \in \mathcal{S}(\mathbb{R}^+) \), and where \( n = 2i - j \). Since \( F \in \mathcal{S}(\mathbb{R}^+) \), there is a constant \( C \) with \( |F(t^2 s(s + 1))| \leq C |t^2 s(s + 1)|^{-1} \). Hence, for \( s \geq 1 \)

\[
|\partial_n^a f_0(t^2 s(s + 1))| \leq c s^{-n}.
\]

Thus, for \( i \in \mathbb{N}_0 \) the inequality

\[
|\partial_i^a g_t(s)| \leq t^{4r} s^{4r-i} \quad \forall \ s \geq 1
\]

holds uniformly up to a constant which is independent of \( t \). The estimation (69) and the assumptions on \( u \), that for any \( j \in \mathbb{N}_0 \) the inequality \( |\partial^j u(s)| \leq c s^{-\alpha-j} \) holds uniformly up to a constant \( c \), imply that for \( t > 0 \) and \( k \in \mathbb{N}_0 \) the estimation

\[
|\partial^k G_t(s)| \leq t^{4r} s^{4r-k-\alpha}
\]

holds uniformly up to a constant, independent of the parameter \( t \).

By Lemma 4.1 for \( \mu = 4r - \alpha \) we have

\[
\left| \sum_{l=1}^{\infty} G_t(l) Z_l(x \cdot y) \right| = \left| \sum_{l=1}^{\infty} f(t^2 \lambda_l)^2 c_l Z_l(x \cdot y) \right| \leq c \frac{t^{4r}}{(d(x, y))^{2N}}.
\]


Using the estimate (65) for the denominator, we therefore have that
\[
\left| \sum_{l=1} f(t^2 \lambda l^2) c_l Z_l(x \cdot y) \right| \leq t^{4r-\alpha+2} \frac{d(x,y)^2N}{d(x,y)^2N},
\]
(73)
as desired. \[\square\]

**Remark 4.3.** Note that the result of Theorem 4.2 is true for \( r = 1 \) if \( \alpha < 6 \).

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