SUPERALGEBRAS, THEIR QUANTUM DEFORMATIONS AND THE INDUCED REPRESENTATION METHOD

(On the occasion of the 30-th anniversary of the Vietnam Mathematical Society)

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ABSTRACT

Some introductory concepts and basic definitions of the Lie superalgebras and their quantum deformations are exposed. Especially the induced representation methods in both cases are described. Based on the Kac representation theory we have succeeded in constructing representations of several higher rank superalgebras. When representations of quantum superalgebras are concerned, we have developed a method which can be applied not only to the one-parametric quantum deformations but also to the multi-parametric ones. As an intermediate step the Gel’fand–Zetlin basis description is extended to the case of superalgebras and their quantum deformations. Our approach also allows us to establish in consistent ways defining relations of quantum (super)algebras. Some illustrations are given.

Running title: Superalgebras and quantum superalgebras.

Mathematical subject classification 1991: 17A70, 81R50

Key words: Superalgebras, quantum deformations, quantum superalgebras, induced representations, Gel’fand–Zetlin basis.

1 Work based on preprint IC/96/130 and an invited talk given at the ”V-th National Mathematics Conference” (Hanoi, 17–20 September 1997).
1. Introduction

The symmetry principles [17, 77, 78], in particular, the supersymmetry idea [23, 72, 75], occupy central places in constructions of different physics theories. They are described by (super) symmetry groups or infinitesimally by corresponding (super) algebras. Especially, superalgebras [28, 29, 30, 62] play an important role in supersymmetry and supergravity theories [71, 73, 74]. They have various applications in quantum physics, superdynamical symmetry (superquantum mechanics), nuclear physics, etc. Usually, as in the case of ordinary algebras, applications of superalgebras lead us to finding explicit expressions for matrix elements of their generators. Therefore, constructing representations of superalgebras is an actual topic. Unfortunately, the purely mathematical problem is solved only partially. Irrespective of the fact that all finite–dimensional irreducible representations of the basic classical Lie superalgebras are classified, the questions concerning indecomposable finite–dimensional representations and constructions of explicit (indecomposable and irreducible) representations are less understood and solved. Especially for the orthosymplectic superalgebras it is not known how to construct all such indecomposable representations and modules. These (indecomposable and irreducible) representations of the basic superalgebras and the structure of the corresponding modules were subjects of investigations of several authors, who succeeded in constructing explicit representations and modules only for lower rank superalgebras [43, 57, 64], while explicit representations of larger superalgebras were known very partially, besides some general expects [43, 44, 70]. Later, some movements forward were made in Refs. [31, 32, 41, 54] where all finite–dimensional representations and a wide class of infinite–dimensional representations of several higher rank superalgebras have already been investigated in detail and constructed explicitly.

The quantum deformations [16, 19, 26, 27, 47, 79], originated from intensive investigations on the quantum scattering problems and Yang–Baxter equations, represent an extension of the symmetry concept. Since they became a subject of great interest, many algebraic and geometric structures and some representations of quantum (super-) groups and algebras have been obtained and understood (see in this context, for example, Refs. [7, 9, 15, 16, 19, 26, 27, 33, 35, 47, 49, 76, 79, 80, 81]). In particular, the quantum algebra $U_q[sl(2)]$ is very well studied [6, 36, 46, 58, 59, 66]. As in the non–deformed case for applications of quantum groups and algebras we often need their explicit representa-
tions. However, although the progress in this direction is remarkable the problem is still far from being satisfactorily solved. Especially, representations of quantum superalgebras \cite{[8],[14],[21],[37],[38],[39],[40],[42],[55],[82]} are presently under development. Explicit representations are known mainly for quantum superalgebras of lower ranks and of particular types like $U_q[gl(n/1)]$, $U_q[osp(1/2)]$, etc., while for higher rank quantum superalgebras of nonparticular type, only some general structures \cite{[82]}, $q$–oscillator representations (see, for example, Refs. \cite{[14],[21]}) and a class of representations of $U_q[gl(m/n)]$ (Ref. \cite{[55]}) have been well investigated. In general, representations, including the finite–dimensional ones, of quantum superalgebras have not been explicitly constructed and completely investigated (at neither generic $q$ nor $q$ being roots of unity). Recently, in Ref. \cite{[37]} we proposed an induced representation method by which we can construct representations of higher rank quantum superalgebras such as $U_q[gl(m/n)]$ for large $m$, $n$.

Here, in the framework of this paper, more precisely in the next section, we shall make an introduction to the superalgebras and briefly describe the induced representation method which is based on the representation theory developed by Kac \cite{[28]}. Then, in Sect. III, we give a construction procedure for finite–dimensional representations of the superalgebras $gl(m/n)$. The induced representation method allowed us to construct explicitly all finite–dimensional representations and a wide class of infinite–dimensional representations of several higher rank superalgebras like $gl(2/2)$, $gl(3/2)$ and $osp(3/2)$. Sect. IV is devoted to some introductory concepts of the quantum superalgebras. Due to the method proposed in Ref. \cite{[37]} and described in Sect. V we succeeded for the first time in finding all finite–dimensional representations, including the irreducible ones, of a higher rank quantum superalgebra, namely $U_q[gl(2/2)]$ (see Refs. \cite{[37]} and \cite{[42]}). It is clear that our method is applicable not only to other one–parametric deformations but also to the multi–parametric ones \cite{[38],[39],[40]}.

Let us list some of the abbreviations and notations which will be used throughout the present paper:

- $\text{fidirmod}(s)$ – finite–dimensional irreducible module(s),
- $GZ$ basis – Gel’fand–Zetlin basis,
- $\text{lin. env.}\{X\}$ – linear envelope of $X$,
$q$ – the deformation parameter,

$[x]_r = (r^x - r^{-x})/(r - r^{-1})$, $r = r(q)$, where $x$ is some number or operator,

$[x] ≡ [x]_q$,

$[E, F] – supercommutator between $E$ and $F$,

$[E, F]_r ≡ EF ± rFE – r-deformed supercommutator between $E$ and $F$,

$[m] – a highest weight in a (GZ, for example,) basis $(m)$,

$I^q_k – the maximal invariant subspace in $W^q([m])$, corresponding to the class $k$,

$W^q_k([m]) = W^q([m])/I^q_k – the class $k$ nontypical module,

$(m)^{±ij} – a pattern obtained from $(m)$ by shifting $m_{ij} → m_{ij} ± 1$,

Note that we must not confuse the quantum deformation $[x] ≡ [x]_q$ of $x$ with the highest weight (signature) $[m]$ in the GZ basis $(m)$ or with the notation $[,]$ for commutators.

2. Superalgebras and their representations

There exist several good references on Lie superalgebras and their representations (see, for example, Refs. [28, 29, 30] and [62]). Let us give here some introductory concepts and basic definitions from the topic. A Lie superalgebra (from now on, only superalgebras) $A$, endowed with a $Z$-gradation, by definition, is a vector space which

1) is a direct sum of vector subspaces $A_i$, where $i ∈ Z$:

$$A = \bigoplus_{i ∈ Z} A_i,$$  \hspace{1cm} (2.1)

and

2) has a bilinear product (supercommutator) $[,]$ such that

$$[x_i, x_j] := x_i x_j - (-1)^{ij} x_j x_i ∈ A_{i+j}, \text{ for } x_{i(j)} ∈ A_{i(j)},$$  \hspace{1cm} (2.2a)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$  \hspace{1cm} (2.2b)
One sees that the Lie superalgebra $A$ admits the following $Z_2$-graded structure decomposition:

$$A = A_0 \bigoplus A_1,$$  \hspace{1cm} (2.3a)

where

$$A_{0\,\left(1\right)} = \bigoplus_{i=\text{even(odd)}} A_i \hspace{1cm} (2.3b)$$

Here $A_0$, called the even subalgebra of $A$, is an ordinary Lie algebra, while $A_1$ is a subspace of the odd generators and represents an $A_0$–module as the supercommutator (2.2) defines in $A_1$ a homomorphism:

$$A_1 \rightarrow A_0 \hspace{1cm} (2.4)$$

We say the above $Z$–gradation is consistent with the $Z_2$–one.

Rewriting the decomposition (2.1a) in the form:

$$A = A_- \bigoplus A_0 \bigoplus A_+ \hspace{1cm} (2.1a')$$

where

$$A_- = \bigoplus_{i<0} A_i, \hspace{1cm} A_+ = \bigoplus_{i>0} A_i \hspace{1cm} (2.1b')$$

we see that $A_0$, referred to as a stability subalgebra, is either the even algebra $A_0$ or its subalgebra and $A_1$'s are the adjoint representation spaces of $A$ restricted to $A_0 : [A_0, A_i] \subseteq A_i$. The Cartan subalgebra is contained in $A_0$, while $A_+$ and $A_-$ are subspaces of the creation and annihilation generators, respectively. One can construct a representation of $A$ induced from a representation of the stability subalgebra $A_0$ by expressing the generators from $A_i$ in a basis of the corresponding $Ad(A_0)$–module.

Let us denote by $V_B$ a module of a subalgebra $B$ of $A$. This $B$–module $V_B$ can be extended to a $U(A)$–module, where $U(A)$ is the universal enveloping algebra of $A$. An $A$–module $\tilde{V} := Ind_B^A V_B$ induced from the $B$–module is a $Z_2$–graded space obtained from $U(A) \otimes V_B$ factorized by all the elements of the form $ab \otimes v - a \otimes b(v)$, $a \in A$, $b \in B$, $v \in V_B$ and endowed with the structure $a(u \otimes v) = au \otimes v$, $a \in A$, $u \in U(A)$, $v \in V_B$. 

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If \( B = A_0 \oplus A_+ \) we can start from an \( A_0 \)-module \( V_0(\Lambda) \), where \( \Lambda \) is a signature characterizing the corresponding representation of \( A_0 \) in \( V_0 \). The latter becomes a \( B \)-module \( V_B(\Lambda) \) by setting
\[
A_+ V_0 = 0 \text{ (2.5)}
\]
The induced module \( \tilde{V}(\Lambda) = Ind^A_B V_B \), in general, contains a (unique) maximal submodule \( I(\Lambda) \).

**Definition 2.1:** An irreducible representation of a Lie superalgebra \( A \) with the signature \( \Lambda \) is called the factor–module
\[
W(\Lambda) = \tilde{V}(\Lambda)/I(\Lambda),
\]
where \( I \) is the maximal submodule.

Let
\[
V = V_0 \oplus V_1 \text{ (2.7)}
\]
be a \( \mathbb{Z}_2 \)-graded vector space of the dimension
\[
dim V = (\dim V_0, \dim V_1) = (m, n) \text{ (2.8)}
\]
and \( End(V)_L \) is a Lie superalgebra of endomorphism mappings \( End(V) \) endowed with the multiplications (2.2).

**Definition 2.2:** A linear representation of \( A \) in \( V \) is defined as a homomorphism
\[
\phi : A \rightarrow End(V)_L := gl(m/n). \text{ (2.9)}
\]

We call \( gl(m/n) \) a general linear Lie superalgebra which is a super–analogue of the ordinary general linear Lie algebra \( gl(m) \). Any superalgebra is a subalgebra of \( gl(m/n) \) and has the following matrix representation:
\[
\begin{bmatrix}
A & C \\
D & B
\end{bmatrix} \text{ (2.10)}
\]
where $A$, $B$, $C$ and $D$ are matrices of dimensions $m \times m$, $n \times n$, $m \times n$ and $n \times m$, respectively. The even subalgebra $A_0$ is spanned by $A \oplus B \subseteq gl(m) \oplus gl(n)$, while $C$ and $D$ are respectively the spaces of the positive– and the negative odd root generators. For a basis of $gl(m/n)$ we can choose the Weyl matrices $e_{ij}$,

$$(e_{ij})_{kl} = \delta_{ik} \delta_{jl}, \quad i, j = 1, 2, \ldots, m + n, \quad (2.11a)$$

satisfying the supercommutation relations:

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-1)^{(i)+(j)(k)+(l)} \delta_{il} e_{jk}, \quad 1 \leq i, j, k, l \leq m + n, \quad (2.11b)$$

where the gradation index $(i)$ is 0 for $1 \leq i \leq m$ and 1 for $m + 1 \leq i \leq m + n$.

3. Representations of $gl(m/n)$ in a $gl(m) \oplus gl(n)$ basis

Here we shall outline a construction procedure for a representation of the superalgebra $gl(m/n)$ induced from a representation of the even subalgebra $A_0 = gl(m/n)_0 \equiv gl(m) \oplus gl(n) = A_0$ (3.1) in a module $V_0(\Lambda)$, where $\Lambda$ is some signature characterizing the considered representation and being a highest weight in the case of a finite–dimensional representation. The highest weight $\Lambda$ represents an ordered set $(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n})$ of the eigen–values $\lambda_i$ of the Cartan generators $e_{ii}$, $i = 1, 2, \ldots, m + n$, on the so–called highest weight vector $M(\Lambda)$ which is defined as a vector from $V_0(\Lambda)$ and annihilated by the creation generators $e_{ij}$, $1 \leq i < j \leq m$ or $m + 1 \leq i < j \leq m + n$,

$$e_{ii}M = \lambda_i M, \quad \Lambda := (\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{m+n}), \quad (3.2a)$$

$$e_{ij}M = 0, \quad \text{for } 1 \leq i < j \leq m \quad \text{or} \quad m + 1 \leq i < j \leq m + n \quad (3.2b)$$

Identifying the subspaces $A_\pm$ as

$$A_+ = \{e_{ij} \parallel m + n \geq j > m \geq i \geq 1\}, \quad (3.3a)$$

$$A_- = \{e_{ij} \parallel m + n \geq i > m \geq j \geq 1\} \quad (3.3b)$$

we demand the condition

$$A_+ V_0(\Lambda) = 0 \quad (3.4)$$
which turns the module $V_0(\Lambda)$ in a $B$–module, where

$$B = gl(m) \oplus gl(n) \oplus A_+.$$  \hfill (3.5)

The $gl(m/n)$–module $W(\Lambda)$ induced from the $gl(m) \oplus gl(n)$–module $V_0(\Lambda)$ is the factor–space

$$W(\Lambda) = (U \otimes V_0(\Lambda)/I(\Lambda))$$  \hfill (3.6)

where $U$ is the universal enveloping algebra of $gl(m/n)$, while $I(\Lambda)$ is the subspace

$$I(\Lambda) = \text{lin.env.}\{ ub \otimes v - u \otimes bv \mid u \in U, b \in B \subset U, v \in V_0(\Lambda) \}$$  \hfill (3.7)

The Poincaré–Birkhoff–Witt theorem states that $U$ is a linear span of all the elements

$$g = \prod_{e_{ij} \in A_-} (e_{ij})^{\theta_{ij}} b := a_{(-)} b, \quad b \in B, \quad \theta_{ij} = 0, 1,$$  \hfill (3.8)

where

$$a_{(-)} := \prod_{e_{ij} \in A_-} (e_{ij})^{\theta_{ij}}$$  \hfill (3.9)

are ordered sequences of the odd generators belonging to $A_-$. Now, considering $g \otimes v$ as an element of $W(\Lambda)$, from (3.9) we have

$$g \otimes v = a_{(-)} b \otimes v = a_{(-)} \otimes w, \quad w = bv \in V_0(\Lambda)$$  \hfill (3.10)

Therefore

$$W(\Lambda) = \text{lin.env.}\left\{ \prod_{e_{ij} \in A_-} (e_{ij})^{\theta_{ij}} \otimes v \mid v \in V_0(\Lambda) \right\}, \quad \theta_{ij} = 0, 1$$  \hfill (3.11a)

or

$$W(\Lambda) = T \otimes V_0(\Lambda)$$  \hfill (3.11b)

where

$$T = \text{lin.env.}\left\{ \prod_{e_{ij} \in A_-} (e_{ij})^{\theta_{ij}} \mid \theta_{ij} = 0, 1 \right\} \subset U$$  \hfill (3.12)

Since $T$, considered as an $Ad(A_0)$–module, is $2^{mn}$–dimensional, the module $W$ can be decomposed in a direct sum of a number ($2^{mn}$, at most) of $A_0$–modules $V_k(\Lambda_k)$ with highest weights $\Lambda_k$, $0 \leq k \leq (2^{mn} - 1)$, i.e.,

$$W(\Lambda) = \bigoplus_{0}^{2^{mn}-1} V_k(\Lambda_k).$$  \hfill (3.13)

where the notation

$$\Lambda_0 \equiv \Lambda.$$  \hfill (3.14)
is used. According to formulas (3.11), the vectors

$$|\theta_{ij}; (m,n)\rangle := \prod_{e_{ij} \in A} (e_{ij})^{\theta_{ij}(m,n)} = a_{(-)}(m,n) \quad (3.15)$$

altogether span a basis of $W(\Lambda)$, where $(m,n)$ is a basis of $V_0(\Lambda)$. Therefore, when $V_0$ is finite–dimensional, the module $W$ and all other $gl(m/n)_{0}$–submodules $V_k$ are finite–dimensional, as well. For a basis of such a finite–dimensional $gl(m/n)_{0}$–module we can choose the Gel’fand–Zetlin (GZ) tableaux also called GZ (basis) vectors or patterns [1, 3, 22]:

$$(m, n)_k \equiv \left(\begin{array}{cccc}
m_{1m} & m_{2m} & \ldots & m_{m-1, m} \\
m_{1m-1} & m_{2m-1} & \ldots & m_{m-1, m-1} \\
\vdots & & & \\
m_{12} & m_{22} & \\
m_{11} & \\
\end{array}\right) \otimes \left(\begin{array}{cccc}
n_{1n} & n_{2n} & \ldots & n_{n-1, n} \\
n_{1n-1} & n_{2n-1} & \ldots & n_{n-1, n-1} \\
\vdots & & & \\
n_{12} & n_{22} & \\
n_{11} & \\
\end{array}\right)_k \quad (3.16)$$

where $m_{ij}$ and $n_{ij}$ are complex numbers, satisfying the conditions:

$$m_{ij} - m_{kl} \in \mathbb{Z}, \quad m_{ij} \geq m_{ij-1} \geq m_{i-1, j} \quad (3.17a)$$

and

$$n_{ij} - n_{kl} \in \mathbb{Z}, \quad n_{ij} \geq n_{ij-1} \geq n_{i-1, j}, \quad (3.17b)$$

as for $k = 0$ we take

$$(m, n)_0 \equiv (m, n). \quad (3.18)$$

When there does not exist any threat of degenerations, the other subscripts $k$ are also not necessary and therefore can be skipped. Thus, for every $V_k$ the highest weight (signature) is characterized by the first row in (3.16)

$$\Lambda := [\Lambda_r, \Lambda_l] := [m_{1m}, m_{2m}, \ldots, m_{mm}, n_{1n}, n_{2n}, \ldots, n_{nn}] := [m, n] \quad (3.19)$$

combining the highest weights

$$\Lambda_r := [m] = [m_{1m}, m_{2m}, \ldots, m_{mm}] \quad (3.20a)$$

and

$$\Lambda_l := [n] = [n_{1n}, n_{2n}, \ldots, n_{nn}] \quad (3.20b)$$

of $gl(m)$ and $gl(n)$, respectively.
As vectors from an $Ad(A_0)$–module, $a_{(-)}$ can be expressed in terms of a $gl(m) \oplus gl(n)$–GZ basis:

$$(m', n')_k := \left(\begin{array}{cccccc}
m'_{1m} & m'_{2m} & \ldots & m'_{m-1m} & m'_{mm} \\
m'_{1m-1} & m'_{2m-1} & \ldots & m'_{m-1m-1} & \\
\vdots & & & & \\
m'_{12} & m'_{22} & \\
m'_{11} & \end{array}\right) \otimes \left(\begin{array}{cccccc}
n'_{1n} & n'_{2n} & \ldots & n'_{n-1n} & n'_{nn} \\
n'_{1n-1} & n'_{2n-1} & \ldots & n'_{n-1n-1} & \\
\vdots & & & & \\
n'_{12} & n'_{22} & \\
n'_{11} & \end{array}\right)$$

Then, the basis (3.15) takes the form

$$|\theta_{ij}; (m, n)\rangle = \mathcal{N}.(m', n') \odot (m, n)$$

where $\mathcal{N}$ is a norm.

The induced representations obtained, in general, are reducible in the latter basis (3.22) referred to as an induced basis. In order to single out all its irreducible subrepresentations we have to pass to another basis, namely the reduced basis which is the union of all the GZ basis vectors (3.16) for $k$ running from 0 to $2^{mn} - 1$. The reduced basis vectors (3.16) are connected with the induced basis ones (3.22) by the Clebsch–Gordan decompositions written formally as follows

$$(m, n)_k = \sum_{(m', n')\parallel (m, n)} C[(m, n)_k|(m', n'); (m, n)] (m', n') \odot (m, n)$$

and vice versa

$$(m', n') \odot (m, n) = \sum_{(m, n)_k} C^{-1}[(m', n'); (m, n)|(m, n)_k] (m', n')_k$$

where $C$ and $C^{-1}$ are short hands for the Clebsch–Gordan coefficients and its invert expressions, respectively. The sums in (3.23) and (3.24) spread over the Gel’fand–Zetlin ranges (3.17) for all possible GZ patterns concerned. In Ref. [32] we proposed a modified GZ basis description which can be extended later to the case of quantum superalgebras [37, 38, 40, 42].

All matrix elements of $gl(m/n)$–generators in the induced basis or in the reduced basis can be obtained by using formulas (2.11), (3.15), (3.16), and (3.21)–(3.24). The main problem here is to find the Clebsch–Gordan coefficients which are not always known.
explicitly, especially for higher rank cases. For now, using the general method described above, we can find all finite–dimensional representations of the superalgebras $gl(2/2)$ and $gl(3/2)$, while the results for higher rank $gl(m/n)$ are still partial.

As an example we can consider the superalgebra $gl(2/1)$ generated by the generators $e_{ij}$, $i, j = 1, 2, 3$ satisfying (2.11) for $m = 2$, $n = 1$. Now the space $T$ in (3.12) takes the following form

$$T = \text{lin.env} \left\{ (e_{31})^{\theta_1} (e_{32})^{\theta_2}, \theta_i = 0, 1 \right\}$$

(3.25)

Then the module $W(\Lambda)$ (3.13) induced from a finite–dimensional irreducible module (fidirmod) $V_0(\Lambda)$ of $gl(2/1)_0$ can be decomposed into four $gl(2/1)_0$–fidirmods $V_k$, $k = 0, 1, 2, 3$,

$$W(\Lambda) = \bigoplus_{0}^{3} V_k(\Lambda_k).$$

(3.26)

Now, the GZ basis (3.16)–(3.18) for a finite–dimensional $gl(2/1)_0$–module $V_k$ represents a tensor product

$$\begin{bmatrix} m_{12} & m_{22} & m_{32} = m_{31} \\ m_{11} & m_{11} & m_{31} \end{bmatrix} \equiv \begin{bmatrix} [m]_2 \\ [m]_1 \end{bmatrix} \equiv (m)_{gl(2)} \otimes m_{31} \equiv (m)_k$$

(3.27a)

between the GZ basis $(m)_{gl(2)}$ of $gl(2)$ and the $gl(1)$–factors $m_{31}$, where $m_{ij}$ are complex numbers such that

$$m_{12} - m_{11}, \ m_{11} - m_{22} \in \mathbb{Z}_+$$

(3.27b)

and

$$m_{32} = m_{31}.$$  

(3.27c)

Then $T$ as an $\text{Ad}(gl(2/1)_0)$–module is spanned on the following basis vectors

$$1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(3.28a)

$$e_{3i} = (-1)^i \begin{bmatrix} 0 & -1 & 1 \\ i - 2 & 1 & 1 \end{bmatrix}, \ i = 1, 2,$$

(3.28b)

$$e_{31}e_{32} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}.$$  

(3.28c)

Using (3.11), (3.23), (3.24), (3.27) and (3.28) we can describe all the basis vectors of the module $W$ and their transformations under the actions of $gl(2/1)$–generators and then we can investigate the irreducible and indecomposable structure of the module $W$. In such a way all finite-dimensional irreducible representations of $gl(2/1)$ are found. In order to
make the present paper more compact we do not expose here these results which represent classical limits of those of $U_q[gl(2/1)]$ when $q \rightarrow 1$. The structure of the $gl(2/1)$–module $W$ is similar to that of the module $W^q$ of $U_q[gl(2/1)]$. The latter quantum superalgebra and its representations will be considered (however, in a different approach) in section 5.

As far as the orthosymplectic Lie superalgebras $osp(m/n)$ are concerned, the induced representation method is also applicable. However, this case has some specific features which deserve to be mentioned. The orthosymplectic Lie superalgebras $osp(m/n)$ which are a subclass of $gl(m/n)$ have various applications in superfield theories [24, 71], superquantum mechanics [1, 4, 11, 13, 18], nuclear physics [2, 65], etc. Unfortunately, the mathematical problem to determine the representations (or, say, only the finite–dimensional representations) of the orthosymplectic Lie superalgebras is, at present, far from being solved. It is much less developed even in comparison to the other big class of the basis Lie superalgebras, namely $sl(m/n)$. Here, as an example, the superalgebra $osp(3/2)$ is taken [41].

In Ref. [41] we constructed explicitly all finite–dimensional representations and a wide class of infinite–dimensional ones of $osp(3/2)$ induced from finite–representations of the stability algebra $A_0 \equiv so(3) \oplus gl(1)$ which is a subalgebra of the even subalgebra $so(3) \oplus sp(2)$. The method depending on the representations of the even algebras leads to an infinite–irreducible or indecomposable $osp(3/2)$–module $\bar{W}(p, q)$ labeled by a number pair $(p, q)$. Any such module has, as mentioned above, a natural induced basis, in which one easily writes transformations under the actions of the generators. However, we need another basis, called reduced, in order to easily single out and describe the invariant subspace $\bar{W}_{inv}(p, q)$ of the module $W(p, q)$ carrying infinite–irreducible or indecomposable representations of $osp(3/2)$, the finite Kac module $\bar{W}_{Kac} = \bar{W}/\bar{W}_{inv}$ (also carrying an irreducible or indecomposable representation of $osp(3/2)$ and, finally, the irreducible $osp(3/2)$ submodule $W_{Kac}(p, q)$ (which differs from the Kac module only in the case of nontypical representations).

4. Quantum superalgebras and their representations

As mentioned in the Introduction the quantum superalgebras and their representations
are subjects of intensive investigations in both physics and mathematics. The quantum superalgebras as quantum deformations can be introduced and defined in different ways. Here, we shall give some introductory concepts and basic definitions exposed mostly in [37] where an induced representation method was proposed and showed to be useful in constructing explicit representations of quantum superalgebras. Then, in the next section, for an illustration of our method we shall consider the quantum superalgebra $Uq[gl(2/1)]$.

Let $g$ be a rank $r$ (semi-) simple superalgebra, for example, $sl(m/n)$ or $osp(m/n)$. The quantum superalgebra $U_q(g)$ as a quantum deformation ($q$-deformation) of the universal enveloping algebra $U(g)$ of $g$, is completely defined by the Cartan-Chevalley canonical generators $h_i, e_i$ and $f_i, i = 1, 2, ..., r$ which satisfy

1) the quantum Cartan–Kac supercommutation relations

\[
[h_i, h_j] = 0, \\
[h_i, e_j] = a_{ij}e_j, \\
[h_i, f_j] = -a_{ij}f_j, \\
[e_i, f_j] = \delta_{ij}[h_i]_{q^i},
\]

(4.1)

2) the quantum Serre relations

\[
(ad_q\mathcal{E}_i)^{1-\tilde{a}_{ij}}\mathcal{E}_j = 0, \\
(ad_q\mathcal{F}_i)^{1-\tilde{a}_{ij}}\mathcal{F}_j = 0
\]

(4.2)

where $(\tilde{a}_{ij})$ is a matrix obtained from the non-symmetric Cartan matrix $(a_{ij})$ by replacing the strictly positive elements in rows with 0 on the diagonal entry by $-1$, while $ad_q$ is the $q$–deformed adjoint operator given by the formula (4.8) and

3) the quantum extra–Serre relations (for $g$ being $sl(m/n)$ or $osp(m/n)$)

\[
\{[e_{m-1}, e_m]_{q^2}, [e_m, e_{m+1}]_{q^2}\} = 0,
\]

(4.3)
\[\{[f_{m-1}, f_m]q^2, [f_m, f_{m+1}]q^2\} = 0, \quad (4.3)\]

being additional constraints on the unique odd Chevalley generators \(e_m\) and \(f_m\). In the above formulas we denoted \(q_i = q^{d_i}\) where \(d_i\) are rational numbers symmetrizing the Cartan matrix \(d_{ij} = d_{ji} \), \(1 \leq i, j \leq r\). For example, in the case \(g = sl(m/n)\) we have
\[
d_i = \begin{cases} 
1 & \text{if } 1 \leq i \leq m, \\
-1 & \text{if } m + 1 \leq i \leq r = m + n - 1.
\end{cases} \quad (4.4)
\]

The above–defined quantum superalgebras form a subclass of a special class of Hopf algebras called by Drinfel’d quasitriangular Hopf algebras \([16]\). They are endowed with a Hopf algebra structure given by the following additional maps:

1) **coproduct** \(\Delta : U \to U \otimes U\)

\[
\Delta(1) = 1 \otimes 1,
\]

\[
\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,
\]

\[
\Delta(e_i) = e_i \otimes q_i^{h_i} + q_i^{-h_i} \otimes e_i,
\]

\[
\Delta(f_i) = f_i \otimes q_i^{h_i} + q_i^{-h_i} \otimes f_i, \quad (4.5)
\]

2) **antipode** \(S : U \to U\)

\[
S(1) = 1,
\]

\[
S(h_i) = -h_i,
\]

\[
S(e_i) = -q_i^{a_{ii}} e_i,
\]

\[
S(f_i) = -q_i^{-a_{ii}} f_i \quad (4.6)
\]

and

3) **counit** \(\varepsilon : U \to \mathbb{C}\)

\[
\varepsilon(1) = 1,
\]

\[
\varepsilon(h_i) = \varepsilon(e_i) = \varepsilon(f_i) = 0, \quad (4.7)
\]

Then the quantum adjoint operator \(ad_q\) has the following form \([8, 61]\)

\[
ad_q = (\mu_L \otimes \mu_R)(id \otimes S)\Delta \quad (4.8)
\]
with \( \mu_L \) (respectively, \( \mu_R \)) being the left (respectively, right) multiplication: \( \mu_L(x) y = xy \) (respectively, \( \mu_R(x) y = (-1)^{deg_x \cdot deg_y} y x \)).

A quantum superalgebra \( U_q[gl(m/n)] \) is generated by the generators \( k_i^{\pm 1} \equiv q_i^{\pm E_{ii}}, e_j \equiv E_{j,j+1} \) and \( f_j \equiv E_{j+1,j}, i = 1, 2, ..., m + n, j = 1, 2, ..., m + n - 1 \) such that the following relations hold [37]

1) the super–commutation relations

\[
\begin{align*}
k_i k_j &= k_j k_i, & k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad k_i e_j k_i^{-1} &= q_i^{(\delta_{ij} - \delta_{i,j+1})} e_j, & k_i f_j k_i^{-1} &= q_i^{(\delta_{i,j+1} - \delta_{i,j})} f_j, \\
[e_i, f_j] &= \delta_{ij} [h_i]_{q^2}, & \text{where } q_i^{h_i} &= k_i k_i^{-1}. \quad (4.9)
\end{align*}
\]

2) the Serre relations (4.2) taking now the explicit forms

\[
\begin{align*}
[e_i, e_j] &= [f_i, f_j] = 0, \text{ if } |i - j| \neq 1, \\
e_i^2 &= f_i^2 = 0, \\
[e_i, [e_i, e_j]_{q^{\pm 2}}]_{q^{\mp 2}} &= [f_i, [f_i, f_j]_{q^{\pm 2}}]_{q^{\mp 2}} = 0, \text{ if } |i - j| = 1 \quad (4.10)
\end{align*}
\]

and

3) the extra–Serre relations (4.3)

\[
\begin{align*}
\{[e_{m-1}, e_m]_{q^2}, [e_m, e_{m+1}]_{q^2}\} &= 0, \\
\{[f_{m-1}, f_m]_{q^2}, [f_m, f_{m+1}]_{q^2}\} &= 0. \quad (4.3)
\end{align*}
\]

Here, besides \( d_i, 1 \leq i \leq r = m + n - 1 \) given in (4.4), we introduced

\[
d_{m+n} = -1. \quad (4.11)
\]

The Hopf structure on \( k_i \) looks like

\[
\begin{align*}
\Delta(k_i) &= k_i \otimes k_i, \\
S(k_i) &= k_i^{-1}, \\
\varepsilon(k_i) &= 1. \quad (4.12)
\end{align*}
\]
The generators $E_{ii}$, $E_{i,i+1}$ and $E_{i+1,i}$ together with the generators defined in the following way

$$E_{ij} := [E_{ik}E_{kj}]_{q^2} - q^{-2}E_{kj}E_{ik}, \ i < k < j,$$

$$E_{ji} := [E_{jk}E_{ki}]_{q^2} - q^2E_{ik}E_{jk}, \ i < k < j,$$

play an analogous role as the Weyl generators $e_{ij}$,

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl}, \ i, j = 1, 2, ..., m + n$$

of the superalgebra $gl(m/n)$ whose universal enveloping algebra $U[gl(m/n)]$ represents a classical limit of $U_q[gl(m/n)]$ when $q \to 1$.

The quantum algebra $U_q[gl(m/n)] \cong U_q[gl(m) \oplus gl(n)]$ generated by $k_i$, $e_j$ and $f_j$, $i = 1, 2, ..., m + n$, $m \neq j = 1, 2, ..., m + n - 1$,

$$U_q[gl(m/n)] = \text{lin.env.}\{E_{ij}\} \ 1 \leq i, j \leq m \ \text{and} \ m + 1 \leq i, j \leq m + n$$

is an even subalgebra of $U_q[gl(m/n)]$. Note that $U_q[gl(m/n)]_0$ is included in the largest even subalgebra $U_q[gl(m/n)]_0$ containing all elements of $U_q[gl(m/n)]$ with even powers of the odd generators.

In Ref. [37] we describe the construction method for induced representations of $U_q[gl(m/n)]$ and give in detail a procedure of how to construct all finite–dimensional representations of $U_q[gl(2/2)]$ (see Refs. [37] and [42]). Let us briefly explain why and how we can use the proposed induced representation method, which can be applied not only to one–parametric deformations but also to multi–parametric ones [38, 39, 40].

Indeed, our method is based on the fact [15, 60] that a finite–dimensional representation of a Lie algebra $g$ can be deformed to a finite–dimensional representation of its quantum analogue $U_q(g)$. In particular, finite–dimensional representations of $U_q[gl(m) \oplus gl(n)]$ are quantum deformations of those of $gl(m) \oplus gl(n)$. This means that a finite–dimensional irreducible representation of $U_q[gl(m) \oplus gl(n)]$ is again highest weight. On the other hand, as we can see from (4.3), (4.9)–(4.11), (4.13) and (4.14), $U_q[gl(m) \oplus gl(n)]$ is the stability subalgebra of $U_q[gl(m/n)]$. Therefore, we can construct representations of $U_q[gl(m/n)]$ induced from finite–dimensional irreducible representations of $U_q[gl(m) \oplus gl(n)]$. Let $V^0_\Lambda$
be a $U_q[gl(m) \oplus gl(n)]$-fidirmod characterized by some highest weight $\Lambda$. The module $V_0^q(\Lambda)$ represents a tensor product between a $U_q[gl(m)]$-fidirmod $V_{0,m}^q(\Lambda_m)$ of a highest weight $\Lambda_m$ and a $U_q[gl(n)]$-fidirmod $V_{0,n}^q(\Lambda_n)$ of a highest weight $\Lambda_n$

$$V_0^q(\Lambda) = V_{0,m}^q(\Lambda_m) \otimes V_{0,n}^q(\Lambda_n) \quad (4.15)$$

where $(\Lambda_m)$ and $(\Lambda_n)$ are defined respectively as the left and right components of $\Lambda$

$$\Lambda = [\Lambda_m, \Lambda_n]. \quad (4.16)$$

For a basis of each of $V_{0,m}^q(\Lambda_m)$ and $V_{0,n}^q(\Lambda_n)$, i.e., of $V_0^q(\Lambda)$ we can choose the Gel’fand-Zetlin (GZ) tableaux $[3, 1, 22]$, since the latter are invariant under the quantum deformations $[10, 25, 60, 68, 69]$. Therefore, the highest weight $\Lambda$ is described again by the first rows of the GZ tableaux called from now on as the GZ (basis) vectors.

Demanding

$$E_{m,m+1}V_0^q(\Lambda) \equiv e_mV_0^q(\Lambda) = 0 \quad (4.17)$$

i.e.

$$U_q(A_+)V_0^q(\Lambda) = 0 \quad (4.18)$$

we turn $V_0^q(\Lambda)$ into a $U_q(B)$-module, where

$$A_+ = \{E_{ij} \mid 1 \leq i \leq m < j \leq m+n\}, \quad (4.19)$$

$$B = A_+ \oplus gl(m) \oplus gl(n). \quad (4.20)$$

The $U_q[gl(m/n)]$-module $W^q$ induced from the $U_q[gl(m) \oplus gl(n)]$-module $V_0^q(\Lambda)$ is the factor-space

$$W^q = W^q(\Lambda) = [U_q \otimes V_0^q(\Lambda)]/I^q(\Lambda) \quad (4.21)$$

where $U_q \equiv U_q[gl(m/n)]$, while $I^q(\Lambda)$ is the subspace

$$I^q(\Lambda) = \text{lin.env.} \{ub \otimes v - u \otimes bv \mid u \in U_q, \ b \in U_q(B) \subset U_q, \ v \in V_0^q(\Lambda)\}. \quad (4.22)$$

Any vector $w$ from the module $W^q$ has the form

$$w = u \otimes v, \quad u \in U_q, \ v \in V_0^q \quad (4.23)$$

Then $W^q$ is a $U_q[gl(m/n)]$-module in the sense

$$gw \equiv g(u \otimes v) = gu \otimes v \in W^q \quad (4.24)$$
for \( g, u \in U_q, w \in W^q \) and \( v \in V_0^q \).

As we can see from (4.17) the modules \( W^q(\Lambda) \) and \( V_0^q(\Lambda) \) have one and the same highest weight vector. Therefore, they are characterized by one and the same highest weight \( \Lambda \).

5. Induced representations of \( U_q[\mathfrak{gl}(2/1)] \)

Although the general expressions of the finite–dimensional representations of the quantum superalgebra \( U_q[\mathfrak{gl}(2/1)] \) can be found from [50, 82] for \( n = 2 \), the irreducible representations, however, have not yet been considered in detail. Now, as an illustration of the method described above [37], we investigate and construct explicitly all irreducible (i.e., typical and nontypical) finite–dimensional representations of \( U_q[\mathfrak{gl}(2/1)] \). Here, we assume that the quantum deformation parameter \( q \) is generic. It means that there does not exist any positive integer \( N \in \mathbb{Z}^+ \) such that \( q^N = 1 \). We can construct directly and explicitly representations of the quantum superalgebra \( U_q[\mathfrak{gl}(2/1)] \) induced from some (usually, irreducible) finite–dimensional representations of the even subalgebra \( U_q[\mathfrak{gl}(2) \oplus \mathfrak{gl}(1)] \). Since the latter is a stability subalgebra of \( U_q[\mathfrak{gl}(2/1)] \) we expect that the constructed induced representations of \( U_q[\mathfrak{gl}(2/1)] \) are decomposed into finite–dimensional irreducible representations of \( U_q[\mathfrak{gl}(2) \oplus \mathfrak{gl}(1)] \). For this purpose we shall introduce a \( U_q[\mathfrak{gl}(2/1)] \)–basis (i.e., a basis within a \( U_q[\mathfrak{gl}(2/1)] \)–module or shortly, a basis of \( U_q[\mathfrak{gl}(2/1)] \)) convenient for us in investigating the module structure. This basis (see (5.29)) can be expressed in terms of some basis of the even subalgebra \( U_q[\mathfrak{gl}(2) \oplus \mathfrak{gl}(1)] \) which in turn represents a (tensor) product between a \( U_q[\mathfrak{gl}(2)] \)–basis and a \( \mathfrak{gl}(1) \)–factor. It will be shown that the finite–dimensional representations of \( U_q[\mathfrak{gl}(2)] \), i.e., of \( U_q[\mathfrak{gl}(2) \oplus \mathfrak{gl}(1)] \) can be realized in the Gel’fand–Zetlin (GZ) basis. The finite–dimensional representations of \( U_q[\mathfrak{gl}(2/1)] \) constructed are irreducible and can be decomposed into finite–dimensional irreducible representations of the subalgebra \( U_q[\mathfrak{gl}(2) \oplus \mathfrak{gl}(1)] \).

The quantum superalgebra \( U_q[\mathfrak{gl}(2/1)] \) is completely defined through the Cartan–Chevalley generators \( E_{12}, E_{21}, E_{23}, E_{32}, \) and \( E_{ii}, i = 1, 2, 3, \) satisfying the relations (4.9) and (4.10) which now read

\[1) \text{ the super–commutation relations } (1 \leq i, i + 1, j, j + 1 \leq 3): \]
\[ [E_{ii}, E_{jj}] = 0, \quad (5.1a) \]
\[ [E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1})E_{j,j+1}, \quad (5.1b) \]
\[ [E_{ii}, E_{j+1,j}] = (\delta_{i,j+1} - \delta_{ij})E_{j+1,j}, \quad (5.1c) \]
\[ [E_{12}, E_{32}] = [E_{21}, E_{23}] = 0, \quad (5.1d) \]
\[ [E_{12}, E_{21}] = [h_1], \quad (5.1e) \]
\[ \{E_{23}, E_{32}\} = [h_2], \quad (5.1f) \]
\[ h_i = (E_{ii} - \frac{d_{i+1}}{d_i}E_{i+1,i+1}), \quad (5.1g) \]

with \( d_1 = d_2 = -d_3 = 1, \)

2) the Serre–relations:

\[ E_{23}^2 = E_{32}^2 = 0, \]
\[ [E_{12}, E_{13}]_q = 0, \]
\[ [E_{21}, E_{31}]_q = 0, \quad (5.2) \]

respectively, where

\[ E_{13} := [E_{12}, E_{23}]_{q^{-1}} \quad (5.3a) \]

and

\[ E_{31} := -[E_{21}, E_{32}]_{q^{-1}}. \quad (5.3b) \]

are defined as new odd generators which have vanishing squares. Now the extra–Serre relations are not necessary, unlike higher rank cases [20, 34, 37, 42, 63].

As mentioned earlier, these generators \( E_{ij}, \ i, j = 1, 2, 3, \) are quantum deformation analogues (q–analogues) of the Weyl generators \( e_{ij} \)

\[ (e_{ij})_{kl} = \delta_{ik}\delta_{jl}, \ i, j, k, l = 1, 2, 3, \quad (5.4) \]

of the classical (i.e., non–deformed) superalgebra \( gl(2/1) \) whose universal enveloping algebra \( U[gl(2/1)] \) is a classical limit of \( U_q[gl(2/1)] \) when \( q \to 1. \)

>From the relations (5.1)–(5.3) we see that every of the odd spaces \( A_{\pm} \)

\[ A_{\pm} = \text{lin.env.}\{E_{13}, E_{23}\}, \quad (5.5) \]
as always, is a representation space of the even subalgebra $U_q[gl(2/1)] \equiv U_q[gl(2) \oplus gl(1)]$ which, generated by the generators $E_{12}$, $E_{21}$, and $E_{ii}$, $i = 1, 2, 3$, is a stability subalgebra of $U_q[gl(2/1)]$. Therefore, we can construct representations of $U_q[gl(2/1)]$ induced from some (usually, irreducible) representations of $U_q[gl(2/1)]_0$ which are realized in some representation spaces (modules) $V_0^q$ being tensor products of $U_q[gl(2)]$–modules $V_{0,gl_2}^q$ and $gl(1)$–modules (factors) $V_{0,gl_1}^q$. Following Ref. [37] we demand

$$E_{23}V_0^q = 0$$

that is

$$U_q(A_+)V_0^q = 0.$$  

In such a way we turn the $U_q[gl(2/1)]_0$–module $V_0^q$ into a $U_q(B)$–module where

$$B = A_+ \oplus gl(2) \oplus gl(1).$$  

The $U_q[gl(2/1)]$–module $W^q$ induced from the $U_q[gl(2/1)]_0$–module $V_0^q$ is the factor–space

$$W^q = [U_q \otimes V_0^q]/I^q$$

where

$$U_q \equiv U_q[gl(2/1)],$$

while $I^q$ is the subspace

$$I^q = \text{lin.env.}\{ub \otimes v - u \otimes bv| u \in U_q, b \in U_q(B) \subset U_q, v \in V_0^q\}.  \tag{5.12}$$

Any vector $w$ from the module $W^q$ is represented as

$$w = u \otimes v, \quad u \in U_q, \quad v \in V_0^q.$$  \tag{5.13}

Then $W^q$ is a $U_q[gl(2/1)]$–module in the sense

$$gw \equiv g(u \otimes v) = gu \otimes v \in W^q$$

for $g, u \in U_q$, $w \in W^q$ and $v \in V_0^q$. Using the commutation relations (5.1)–(5.2) and the definitions (5.3) we can prove the following analogue of the Poincaré–Birkhoff–Witt theorem
Lemma 5.1: The quantum deformation $U_q := U_q[gl(2/1)]$ is spanned on all possible linear combinations of the elements

$$g = (E_{23})^m (E_{13})^{n_2} (E_{31})^{\theta_1} (E_{32})^{\theta_2} g_0,$$

where $\eta_i, \theta_i = 0, 1$ and $g_0 \in U_q[gl(2/1)_0] \equiv U_q[gl(2) \oplus gl(1)].$

Taking into account (5.10)–(5.12) and (5.15) we arrive at the following assertion

Lemma 5.2: The induced $U_q[gl(2/1)]$–module $W^q$ is the linear span

$$W^q = \text{lin.env.} (E_{31})^{\theta_1} (E_{32})^{\theta_2} \otimes v \mid v \in V_q^0, \theta_1, \theta_2 = 0, 1,$$

and, consequently, all the vectors of the form

$$|\theta_1, \theta_2; (m) := (E_{31})^{\theta_1} (E_{32})^{\theta_2} \otimes (m), \theta_1, \theta_2 = 0, 1,$$

constitute a basis in $W^q$, where (m) is a (GZ, for example,) basis in $V_0^q$.

Therefore, if $V^q_0$ is a finite–dimensional $U_q[gl(2/1)_0]$–module, the $U_q[gl(2/1)]$–module $W^q$ is finite–dimensional, as well. Moreover, $W^q$ is a highest weight module due to the condition (5.7) imposed on $V^q_0$ which, as a finite–dimensional $U_q[gl(2/1)_0]$–module, is always highest weight. Then, based on the latest Lemma 5.2 and the fact that $U_q[gl(2/1)_0]$ is a stability subalgebra of $U_q[gl(2/1)]$ we conclude that $W^q$ can be decomposed in a number of $U_q[gl(2/1)_0]$–modules which in turn are finite–dimensional and, therefore, highest weight. Such a highest weight module is characterized by a signature (referred otherwise to as a highest weight), being an ordered set of the eigen–values of the Cartan generators on the so–called highest weight vector defined as a vector annihilated by the creation generators (see (3.4) and (3.5)). Thus, the condition (5.7) means that $W^q$ and $V^q_0$ have one and the same highest weight vector, i.e., one and the same highest weight. Let $V^q_0$ be a finite-dimensional irreducible module (fidirmod) of $U_q[gl(2/1)_0]$.

Lemma 5.3: The $U_q[gl(2/1)]$–module $W^q$ is decomposed into (4 or less) finite–dimensional irreducible modules $V^q_k$ of the even subalgebra $U_q[gl(2/1)_0]$

$$W^q([m]) = \bigoplus_{0 \leq k \leq 3} V^q_k([m]_k),$$

where $[m]$ and $[m]_k$ are some signatures (highest–weights) characterizing the module $W^q \equiv W^q([m])$ and the modules $V^q_k \equiv V^q_k([m]_k)$, respectively.
The proof of this lemma follows from the very construction of $W^q$ and the same argument used for deriving (3.13).

Each of the fidirmods $V_k^q$, $0 \leq k \leq 3$, is spanned on a basis, say $(m)_k$, which can be taken as a tensor product between a GZ basis of $U_q[gl(2)]$ and $gl(1)$–factors. In this case, we also call $(m)_k$ as a GZ basis. It is clear that

$$(m)_0 \equiv (m)$$

(5.19a)

and

$$[m]_0 \equiv [m]$$

(5.19b)

in our notations. We refer to the basis (5.17) as the induced $U_q[gl(2/1)]$–basis (or simply, the induced basis) in order to distinguish it from the reduced one introduced later in the next subsection.

5.a. Finite–dimensional representations of $U_q[gl(2/1)]$

We can show that finite–dimensional representations of $U_q[gl(2/1)]_0$ can be realized in some spaces (modules) $V_k^q$ spanned by the (tensor) products

$$\begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{31} \end{bmatrix}, \quad m_{32} = m_{31} \equiv (m)_k \quad (\text{5.20a})$$

between the (GZ) basis vectors $(m)_{gl(2)}$ of $U_q[gl(2)]$ and the $gl(1)$–factors $m_{31}$, where $m_{ij}$ are complex numbers such that

$$m_{12} - m_{11}, \quad m_{11} - m_{22} \in \mathbb{Z}_+$$

(5.20b)

and

$$m_{32} = m_{31}.$$ 

(5.20c)

Indeed, any finite–dimensional representation of (not only) $U_q[gl(2)]$ is always highest weight and if the generators $E_{ij}$, $i, j = 1, 2$ and $E_{33}$ are defined on (5.20) as follows

$$E_{11}(m)_k = (l_{11} + 1)(m)_k,$$

$$E_{22}(m)_k = (l_{12} + l_{22} - l_{11} + 2)(m)_k,$$

$$E_{12}(m)_k = ([l_{12} - l_{11}][l_{11} - l_{22}])^{1/2}(m)_k^{+11},$$

$$E_{21}(m)_k = ([l_{12} - l_{11} + 1][l_{11} - l_{22} - 1])^{1/2}(m)_k^{-11},$$

$$E_{33}(m)_k = (l_{31} + 1)(m)_k,$$

(5.21a)
where
\[ l_{ij} = m_{ij} - (i - 2\delta_{i,3}), \tag{5.2b} \]
while \((m)_k^{\pm ij}\) is a vector obtained from \((m)\) by replacing \(m_{ij}\) with \(m_{ij} \pm 1\), they really satisfy the commutation relations (5.1a)–(5.1e) for \(U_q[gl(2/1)_0]\). The highest weight described by the first row (signature)
\[ [m]_k = [m_{12}, m_{22}, m_{32}] \tag{5.22} \]
of the patterns (5.20) is nothing but an ordered set of the eigen–values of the Cartan generators \(E_{ii}, \ i = 1, 2, 3\), on the highest weight vector \((M)_k\) defined as follows
\[ E_{12}(M)_k = 0, \tag{5.23} \]
\[ E_{ii}(M)_k = m_{i2}(M)_k, \tag{5.24} \]
The highest weight vector \((M)_k\) can be obtained from \((m)_k\) by setting \(m_{11} = m_{12}\)
\[ (M)_k = \left[ \begin{array}{ccc} m_{12} & m_{22} & m_{32} = m_{31} \\ m_{12} & m_{31} & m_{31} \end{array} \right]. \tag{5.25} \]
A lower weight vector \((m)_k\) can be derived \textit{vice versa} from \((M)_k\) by the formula
\[ (m)_k = \left( \frac{[m_{11} - m_{22}]!}{[m_{12} - m_{22}]![m_{12} - m_{11}]!} \right)^{1/2} (E_{21})^{m_{12} - m_{11}}(M)_k. \tag{5.26} \]
Especially, for the case \(k = 0\), instead of the above notations, we skip the subscript 0, i.e.,
\[ (m)_0 \equiv (m); \ [m]_0 \equiv [m]; \ (M)_0 \equiv (M), \tag{5.27} \]
putting
\[ m_i = m_{i3}, \quad i = 1, 2, 3, \tag{5.28} \]
where \(m_{i3}\) are some of the complex values of \(m_{i2}\), therefore, \(m_{13} - m_{11}, \ m_{11} - m_{23} \in \mathbb{Z}_+\).
We emphasize that \([m]\) and \((M)\), because of (5.7), are also, respectively, the highest weight and the highest weight vector in the \(U_q[gl(2/1)]\)–module \(W^q = W^q([m])\). Characterizing the latter module as the whole, \([m]\) and \((M)\) are, respectively, referred to as the global highest weight and the global highest weight vector, while \([m]_k\) and \((M)_k\) are, respectively, the local highest weights and the local highest weight vectors characterizing only the submodules \(V^q_k = V^q_k([m]_k)\).
Following the arguments of Ref. [37], for an alternative with (5.17) basis of $W^q$, we can choose the union of all the bases (5.20) which are denoted now by the patterns

\[
\begin{bmatrix}
m_{13} & m_{23} & m_{33} \\
m_{12} & m_{22} & m_{32} \\
m_{11} & 0 & m_{31}
\end{bmatrix}_k \equiv \begin{bmatrix}
m_{12} & m_{22} & m_{32} = m_{31} \\
m_{11} & m_{31} \\
m_{12} & 0 & m_{31}
\end{bmatrix}_k \equiv (m)_k,
\]

where the first row $[m] = [m_{13}, m_{23}, m_{33}]$ is simultaneously the highest weight of the submodule $V_0^q = V_0^q([m])$ and the whole module $W^q = W^q([m])$, while the second row $[m]_k = [m_{12}, m_{22}, m_{32}]$ is the local highest weight of some $U_q gl[(2/1)_0]$–module $V_k^q = V_k^q([m]_k)$ containing the considered vector $(m)_k$. The basis (5.29) of $W^q$ is called the $U_q gl[(2/1)]$–reduced basis or simply the reduced basis. The latter representing a modified GZ basis description is convenient for us in investigating the module structure of $W^q$.

Note once again that the condition

\[m_{32} = m_{31}\]

has to be kept always.

The highest weight vectors $(M)_k$, now, in notation (5.29) have the form

\[
(M)_k = \begin{bmatrix}
m_{13} & m_{23} & m_{33} \\
m_{12} & m_{22} & m_{32} \\
m_{12} & 0 & m_{31}
\end{bmatrix}_k,
\]

as for $k = 0$ the notations (5.27) and (5.28) are also taken into account.

**Lemma 5.4:** The highest weight vectors $(M)_k$ are expressed in terms of the induced basis (5.17) as follows

\[
(M)_0 = a_0 |0, 0; (M)\rangle, \quad a_0 \equiv 1,
\]

\[
(M)_1 = a_1 |0, 1; (M)\rangle,
\]

\[
(M)_2 = a_2 \left\{ |1, 0; (M)\rangle + q^{2l} [2 l]^{-1/2} |0, 1; (M)^{-1}\rangle \right\}
\]

\[
(M)_3 = a_3 \left\{ |1, 1; (M)\rangle \right\},
\]

where $a_i$, $i = 0, 1, 2, 3$, are some numbers depending, in general, on $q$, while $l$ is

\[l = \frac{1}{2} (m_{13} - m_{23}).\]
Proof: Indeed, all the vectors \((M)_k\) given above satisfy the condition (3.4).

From formulae (5.24) and (5.31) the highest weights \([m]_k\) can be easily identified

\[
[m]_0 = [m_{13}, m_{23}, m_{33}], \\
[m]_1 = [m_{13}, m_{23} - 1, m_{33} + 1], \\
[m]_2 = [m_{13} - 1, m_{23}, m_{33} + 1], \\
[m]_3 = [m_{13}, m_{23}, m_{33} + 2]
\]  
(5.32)

Using the rule (5.26) we obtain all the basis vectors \((m)_k\):

\[
(m)_0 \equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} & m_{23} & m_{33} \\ m_{11} & 0 & m_{33} \end{bmatrix} = |0, 0; (m)\rangle,
\]

\[
(m)_1 \equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} & m_{23} - 1 & m_{33} + 1 \\ m_{11} & 0 & m_{33} + 1 \end{bmatrix} = a_1 \left\{ - \left( \frac{[l_{13} - l_{11}]}{[2l + 1]} \right)^{1/2} |1, 0; (m)^{+11}\rangle + q^2(l_{11} - l_{13}) \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} |0, 1; (m)\rangle \right\},
\]

\[
(m)_2 \equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} - 1 & m_{23} & m_{33} + 1 \\ m_{11} & 0 & m_{33} + 1 \end{bmatrix} = a_2 \left\{ \left( \frac{[l_{11} - l_{23}]}{[2l]} \right)^{1/2} |1, 0; (m)^{+11}\rangle + q^{l_{11} - l_{23}} \left( \frac{[l_{13} - l_{11}]}{[2l]} \right)^{1/2} |0, 1; (m)\rangle \right\},
\]

\[
(m)_3 \equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} - 1 & m_{23} - 1 & m_{33} + 2 \\ m_{11} & 0 & m_{33} + 2 \end{bmatrix} = a_3 |1, 1; (m)\rangle,
\]  
(5.33)

where \(l_{ij}\) and \(l\) are given in (3.21b) and (5.31b), respectively. Here, we skip the subscript \(k\)
in the patterns given above since there are no degenerations between them. The formulae (5.33), in fact, represent the way in which the reduced basis (5.29) is written in terms of the induced basis (5.16). From (5.33) we can derive their invert relation

\[ |0, 0; (m)\rangle = (m)_0 \equiv (m) \]
\[ |1, 0; (m)\rangle = -\frac{1}{a_1}q^{l_{11} - l_{23} - 1} \left( \frac{[l_{13} - l_{11} + 1]}{2l + 1} \right)^{1/2} (m)_1^{-11} \]
\[ + \frac{1}{a_2}q^{l_{11} - l_{13} - 1} \left( \frac{[l_{11} - l_{23} - 1][2l]}{2l + 1} \right)^{1/2} (m)_2^{-11}, \]
\[ |0, 1; (m)\rangle = \frac{1}{a_1} \left( \frac{[l_{11} - l_{23}]}{2l + 1} \right)^{1/2} (m)_1 \]
\[ + \frac{1}{a_2} \left( \frac{[l_{13} - l_{11}][2l]}{2l + 1} \right)^{1/2} (m)_2, \]
\[ |1, 1; (m)\rangle = \frac{1}{c_3} (m)_3^{-11}. \] (5.34)

Now we are ready to compute all the matrix elements of the generators in the basis (5.29). As we shall see, the latter basis allows an evident description of a decomposition of a \( U_q[gl(2/1)] \)–module \( W^q \) in irreducible \( U_q[gl(2/1)_{0}] \)–modules \( V^q_k \). Since the finite–dimensional representations of the \( U_q[gl(2/1)] \) in some basis are completely defined by the actions of the even generators and the odd Weyl–Chevalley ones \( E_{23} \) and \( E_{32} \) in the same basis, it is sufficient to write down the matrix elements of these generators only. For the even generators the matrix elements have already been given in (5.21), while for \( E_{23} \) and \( E_{32} \), using the relations (5.1)–(5.3), (5.33) and (5.34) we have

\[ E_{23}(m) = 0, \]
\[ E_{23}(m)_1 = a_1 \left( \frac{[l_{11} - l_{23}]}{2l + 1} \right)^{1/2} [l_{23} + l_{33} + 3](m), \]
\[ E_{23}(m)_2 = a_2 \left( \frac{[l_{13} - l_{11}]}{2l} \right)^{1/2} [l_{13} + l_{33} + 3](m), \]
\[ E_{23}(m)_3 = a_3 \left\{ \frac{1}{a_1 q} \left( \frac{[l_{13} - l_{11}]}{2l + 1} \right)^{1/2} [l_{13} + l_{33} + 3](m)_1 \right. \]
\[ \left. - \frac{1}{a_2 q} \left( [l_{11} - l_{23}][2l] \right)^{1/2} \frac{[l_{23} + l_{33} + 3]}{2l + 1}(m)_2 \right\} \] (5.35)
and

\[
E_{32}(m) = \frac{1}{a_1} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} (m)_1 \\
+ \frac{1}{a_2} \left( \frac{[l_{13} - l_{11}] [2l]}{[2l + 1]} \right)^{1/2} (m)_2
\]

\[
E_{32}(m)_1 = \frac{a_1}{a_3} q \left( \frac{[l_{13} - l_{11}]}{[2l + 1]} \right)^{1/2} (m)_3,
\]

\[
E_{32}(m)_2 = -\frac{a_2}{a_3} q \left( \frac{[l_{11} - l_{23}]}{[2l]} \right)^{1/2} (m)_3,
\]

\[
E_{32}(m)_3 = 0.
\]  

(5.35b)

**Lemma 5.5:** The finite–dimensional representations (5.35) of $U_q[gl(2/1)]$ are irreducible and called typical if and only if the condition

\[
[l_{13} + l_{33} + 3][l_{23} + l_{33} + 3] \neq 0
\]

holds.

*Proof:* By the same argument used in Ref. k4 we can conclude that $W^q$ is irreducible if and only if

\[
E_{14} E_{23} E_{13} E_{31} E_{32} E_{41} \otimes (M) \neq 0.
\]

The latest condition in turn can be proved, after some elementary calculations, to be equivalent to

\[
[E_{11} + E_{33} + 1][E_{22} + E_{33}](M) \neq 0,
\]

which is nothing but the condition (5.36).

In case the condition (5.36) is violated, i.e. one of the following condition pairs

\[
[l_{13} + l_{33} + 3] = 0
\]  

(5.37a)

and

\[
[l_{23} + l_{33} + 3] \neq 0
\]  

(5.37b)

or

\[
[l_{13} + l_{33} + 3] \neq 0
\]  

(5.38a)
and
\[ l_{23} + l_{33} + 3 = 0 \]  \hspace{1cm} (5.38b)

(but not both (5.37a) and (5.38b) simultaneously) holds, the module \( W^q \) is no longer irreducible but indecomposable. However, there exists an invariant subspace, say \( I^q_k \), of \( W^q \) such that the factor–representation in the factor–module
\[
W^q_k := W^q / I^q_k
\]
is irreducible. We say that is a nontypical representation in a nontypical module \( W^q_k \).

Then, as in Ref. [42], it is not difficult for us to prove the following assertions

**Lemma 5.6:**
\[
V^q_3 \subset I^q_k,
\]
and
\[
V^q_0 \cap I^q_k = \emptyset.
\]

From (5.35)–(5.38) we can easily find all nontypical representations of \( U_q[gl(2/1)] \) which are classified in 2 classes.

5.b. Nontypical representations of \( U_q[gl(2/1)] \)

1) **Class 1 nontypical representations:**

This class is characterized by the conditions (5.37a) and (5.37b) which for generic \( q \) take the forms
\[
l_{13} + l_{33} + 3 = 0, \] \hspace{1cm} (5.37x)

and
\[
l_{23} + l_{33} + 3 \neq 0, \] \hspace{1cm} (5.37y)

respectively. In other words, we have to replace everywhere all \( m_{33} \) by \(-m_{13} - 1\) and keep (5.37y) valid. Thus we have

**Lemma 5.7:** *The class 1 maximal invariant subspace in \( W^q \) is*
\[
I^q_1 = V^q_3 \oplus V^q_2.
\]

**Proof:** Applying (5.37) to (5.35) we obtain (5.42).
Then the class 1 nontypical representations in

\[ W_1^q = W_1^q([m_{13}, m_{23}, -m_{13} - 1]) \]  \hspace{1cm} (5.43)

are given through (5.35) by keeping the conditions (5.37) (i.e., (5.37x) and (5.37y)) and replacing with 0 all vectors belonging to \( I_1^q \):

\[
E_{23}(m) = 0,
\]

\[
E_{23}(m)_1 = a_1 \left( \frac{l_{11} - l_{23}}{2l + 1} \right)^{1/2} \left[ l_{23} - l_{13}(m) \right] (5.44a)
\]

and

\[
E_{32}(m) = \frac{1}{a_1} \left( \frac{l_{11} - l_{23}}{2l + 1} \right)^{1/2} (m)_1,
\]

\[
E_{32}(m)_1 = 0. \hspace{1cm} (5.44b)
\]

2) Class 2 nontypical representations:

For this class nontypical representations we must keep the conditions

\[ l_{13} + l_{33} + 3 \neq 0, \hspace{1cm} (5.38x) \]

and

\[ l_{23} + l_{33} + 3 = 0. \hspace{1cm} (5.38y) \]

derived, respectively, from (5.38a) and (5.38b) when the deformation parameters \( q \) are generic. Equivalently, we have to replace everywhere all \( m_{33} \) by \(-m_{23}\) and keep (5.38x) valid.

Now the invariant subspace \( I_2^q \) is the following

**Lemma 5.8:** The class 2 maximal invariant subspace in \( W^q \) is

\[ I_2^q = V_3^q \oplus V_1^q. \]  \hspace{1cm} (5.45)

**Proof:** Using (5.38) in (5.35) we derive (5.45).

The class 2 nontypical representations in

\[ W_2^q = W_2^q([m_{13}, m_{23}, -m_{23}]) \]  \hspace{1cm} (5.46)
are also given through (5.35) but by keeping the conditions (5.38) (i.e., (5.38x) and (5.38y)) valid and replacing by 0 all vectors belonging to the invariant subspace $I^q_2$:

$$E_{23}(m) = 0,$$

$$E_{23}(m)_2 = a_1 \left( \frac{[l_{13} - l_{11}]}{[2l]} \right)^{1/2} [2l + 1](m)$$ \hfill (5.47a)

and

$$E_{32}(m) = \frac{1}{a_2 \frac{([l_{13} - l_{11}][2l])^{1/2}}{[2l + 1]}} (m)_2,$$

$$E_{32}(m)_2 = 0.$$ \hfill (5.47b)

We have just considered the quantum superalgebra $U_q[gl(2/1)]$ and constructed all its typical and nontypical representations leaving the coefficients $a_i$, $i = 1, 2, 3$, as free parameters which can be fixed by some additional conditions, for example, the Hermiticity condition. As an intermediate step (which, however, is of an independent interest) we also introduced the reduced basis (5.29) which, as it is an extension of the Gel’fand–Zetlin basis to the present case, is appropriate for an evident description of decompositions of $U_q[gl(2/1)]$–modules in irreducible $U_q[gl(2/1)_0]$–modules. We can prove the following propositions

**Lemma 5.9:** The class of the finite–dimensional representations determined in this paper (Subsects. 5.a and 5.b), contains all finite–dimensional irreducible representations of $U_q[gl(2/1)]$ and $U_q[sl(2/1)]$.

and

**Lemma 5.10:** The finite–dimensional representations of the quantum superalgebra $U_q[gl(2/1)]$ are quantum deformations of the finite–dimensional representations of the superalgebra $gl(2/1)$.

The **Lemma 5.9** is proved by similar arguments as those used in the proofs of Proposition 9 and Proposition 10 in Ref. [42], while **Lemma 5.10** can be verified by direct computations.

Since the nontypical representations have only been well investigated for a few cases of both classical and quantum superalgebras, the present results can be considered as a
small step forward in this direction.

6. Conclusion

Certainly, many questions remain unconsidered in the framework of the present paper but we hope that the latter gives relevant information about the classical and quantum superalgebras as well as an idea on their representations. Based on Kac’s representation theory for the superalgebras \[28, 29, 30, 62\] we succeeded in finding all finite–dimensional representations and a wide class of infinite–dimensional representations of several higher rank superalgebras of nonparticular types \[31, 32, 41, 54\]. Recently, extending that classical theory to the quantum deformation we worked out a method for explicit constructions of representations of quantum superalgebras \[37, 38, 39, 40, 42\]. Our method, avoiding the use of the Clebsch–Gordan coefficients which are usually unknown for higher rank (classical and quantum) algebras, is applicable not only to the one–parametric deformations but also to the multi–parametric ones (see, for example, Refs. \[38, 39, 40\]). Moreover, our approach may have an advantage as it is worthy to mention that the theory of representations and, especially, of the nontypical representations is far from being complete even for the nondeformed superalgebras. In particular, the dimensions of the nontypical representations are unknown unless the ones for \(sl(1/n)\) computed recently in Ref. \[67\]. Based on the generalizations of the concept of the GZ basis (see Refs. \[37, 42\] and \[50\] and references therein) the matrix elements of all nontypical representations were computed only for superalgebras of lower ranks or of particular types like \(sl(1/n)\) and \(gl(1/n)\) in Ref. \[51, 52\]. Later, the essentially typical representations of \(gl(m/n)\) were also constructed \[53\]. So far, however, the GZ basis concept was not defined and presumably cannot be defined for nontypical \(gl(m/n)\)–modules with \(m, n \geq 2\). This was of why to try to describe the nontypical modules in terms of the basis of the even subalgebras. This approach, developed so far for classical superalgebras \[31, 32, 41\] and for quantum superalgebras \[37, 38, 42\] turns out to be appropriate for explicit descriptions of all nontypical modules of \(gl(2/2)\) (see Refs. \[32, 34\]), \(U_q[gl(2/2)]\) (see Refs. \[37\] and \[42\]) and multi–parametric quantum superalgebras \[38, 39, 40\]. Our approach in Refs \[37, 38, 39, 40, 42\], unlike some earlier approaches, avoids, however, the use of the Clebsch–Gordan coefficients which are not always known for higher rank (quantum and classical) algebras. Other extensions were made in Ref. \[56\] for all finite–dimensional representations of \(U_q[gl(1/n)]\) and in Ref. \[55\] for a class of finite–dimensional representations of \(U_q[gl(m/n)]\). To the best of our
knowledge, we gave for the first time [31, 32, 37, 38, 41, 42, 54], explicit expressions for all finite–dimensional representations or a wide class of infinite–dimensional representations of several classical and quantum superalgebras including those of higher ranks.

We hope that our approach allowing to establish in consistent ways defining relations of quantum (super)algebras [37, 39, 40] can be extended to the case of deformation parameters being roots of unity (for representations of quantum groups at roots of unity, see, for example, [12]).

Acknowledgements

I am grateful to Professor J. Tran Thanh Van and "Rencontres du Vietnam" for financial support and Professor S. Randjbar–Daemi for kind hospitality at the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. I would like to thank the organizers of the V–th National Mathematics Conference (Hanoi, 17–20 September 1997) for giving me the opportunity to report on the present topic.

This work was also supported by the Vietnam National Basic Research Programme in Natural Science under the grant KT 4.1.5.
References

[1] G. Baird and L. Biedenharn, On the representations of the semisimple Lie groups. II, *J. Math. Phys.*, 4 (1963) 1449–1466.

[2] A. Balantekin, H. Schmitt and B. Barett, Coherent states for the harmonic oscillator representations of the orthosympletic group $OSp(1/2N, R)$, *J. Math. Phys.* 29 (1988) 1634–1639.

[3] A. Barut and R. Raczka, *Theory of Group Representations and Applications*, Polish Scientific Publishers, Warszawa, 1980.

[4] J. Beckers and J. Cornwell, On the chains of an orthosympletic Lie algebras and the $n$-dimensional quantum harmonic oscillator, *J. Math. Phys.* 30 (1989) 1655–1661.

[5] J. Beckers, D. Dehin and V. Hussin, On the Heisenberg and orthosympletic superalgebras of the harmonic oscillator, *J. Math. Phys.* 29 (1988) 1705–1711.

[6] L. C. Biedenharn, The quantum group $SU_q(2)$ and a q–analogue of the boson operators, *J. Phys. A* 22 (1989) L873–878.

[7] E. Celeghini and M. Tarlini, eds, *Italian workshop on quantum groups*, Florence, February 3–6, 1993, [hep-th/9304160](https://arxiv.org/abs/hep-th/9304160).

[8] M. Chaichian and P. Kulish, Quantum Lie superalgebras and q–oscillators, *Phys. Lett.* B 234 (1990) 72–80.

[9] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge univ. press, Cambridge, 1994).

[10] I.V. Cherednik, A new interpretation of Gel’fand–Tsetlin bases, *Duke Math. Jour.* 5 (1987) 363–377.

[11] N. Debergh, On the harmonic oscillator and its invariance, *J. Phys. A: Math. Gen.* 24 (1991) 147–151.

[12] C. De Concini and V. Kac, Representations of quantum groups at roots of 1, in *Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory*, Progress in Math. 92 (1990) 471–506.
[13] M. de Crombrugghe and V. Rittenberg, Supersymmetric quantum mechanics, *Ann. Phys.* **151** (1983) 99-126.

[14] E. D’Hoker, R. Floreanini and L. Vinet, $q$–oscillator realizations of the metaplectic representation of quantum $osp(3, 2)$, *J. Math. Phys.* **32** (1991) 1427–1429.

[15] H. Doebner and J. Hennig, eds., *Quantum groups*, Lecture Notes in Physics, Vol. 370, Springer–Verlag, Berlin, 1990.

[16] V. Drinfel’d, Quantum groups, in *Proceeding of the International Congress of Mathematicians* (Berkeley 1986), The American Mathematical Society, Providence, RI, 1987, Vol. 1, 798–820.

[17] J. Elliott and P. Dawber, "Symmetry in Physics", Vol. 1 and 2, The Macmillan Press Ltd, London, 1979.

[18] M. Englefield, Superalgebras and supersymmetric harmonic oscillators, *J. Phys. A: Math. Gen.* **21** (1988) 1309–1639.

[19] L. Faddeev, N. Reshetikhin and L. Takhtajan, Quantization of Lie groups and Lie algebras, *Alg. Anal.* **1** (1987) 178.

[20] R. Floreanini, D. Leites and L. Vinet, On the defining relations of quantum superalgebras, *Lett. Math. Phys.* **23** (1991) 127-131.

[21] R. Floreanini, V. Spiridonov and L. Vinet, $q$–oscillator realizations of the quantum superalgebras $sl_q(m, n)$ and $osp_q(m, 2n)$, *Commun. Math. Phys.* **137** (1991) 149–160.

[22] I. Gel’fand and M. Zetlin, Finite–dimensional representations of the group of unimodular matrices, *Dokl. Akad. Nauk USSR*, **71** (1950) 825–828 (in Russian).

[23] Y. Gol’fand and E. Likhtman, An extension of Poincare group generators algebra and breaking of the P–invariance, *Lett. J. Exp. Theor. Phys.*, **13**, (1971) 452–455 (in Russian).

[24] M. Gunaygin and N. Warner, Unitary supermultiplets of $OSp(8/4, R)$ and the spectrum of the $S^7$ compactification of 11-dimensional supergravity., *Nucl. Phys. B* **272** (1986) 99–124.
[25] M. Jimbo, Quantum R–matrix related to the generalized Toda system: an algebraic approach, in Field Theory, Quantum Gravity and Strings, Lecture Notes in Physics 246, Springer–Verlag, Berlin (1985) 335–361.

[26] M. Jimbo, A $q$–difference analogue of $U(\{}$) and the Yang–Baxter equation, Lett. Math. Phys. 10 (1985) 63–69.

[27] M. Jimbo, A $q$–analogue of $U(gl(N + 1))$, Hecker algebra and the Yang–Baxter equation, Lett. Math. Phys. 11 (1986) 247–252.

[28] V. Kac, A sketch of Lie superalgebra theory, Commun. Math. Phys. 53 (1977) 31–64.

[29] V. Kac, Lie superalgebras, Adv. Math. 26 (1977) 8–96.

[30] V. Kac, in Representations of classical Lie superalgebras, Differential geometrical methods in mathematical physics II: proceedings, Bonn, July 13-16, 1977 (Edited by K. Bleuler, H. Petry and A. Reetz), Lecture Notes in Mathematics, Springer–Verlag, Berlin, 1978, Vol. 676, pp. 597–626.

[31] A. Kamupingene and Nguyen Anh Ky, Finite–dimensional representations of the superalgebra $gl(3/2)$. I., preprint VITP 96 – 03.

[32] A. Kamupingene, Nguyen Anh Ky and T. Palev, Finite–dimensional representations of Lie superalgebra $gl(2/2)$.I. Typical representations, J. Math. Phys. 30, (1989) 553–570.

[33] Ch. Kassel, Quantum groups (Springer–Verlag, New York, 1995).

[34] S. M. Khoroshkin and V. N. Tolstoy, Universal R–matrix for quantized (super) algebras, Comm. Math. Phys. 141 (1991) 599–617.

[35] P. Kulish, ed., Quantum groups, Lecture Notes in Mathematics, Vol. 1510, Springer–Verlag, Berlin, 1992.

[36] P. P. Kulish and N. Yu. Reshetikhin, Quantum linear problem for the sine–Gordon equation and higher representations, J. Soviet Math. 23 (1983) 2436–1441.

[37] Nguyen Anh Ky, Finite dimensional representations of the quantum superalgebra $U_q[gl(2/2)]$. I. Typical representations at generic $q$, J. Math. Phys. 35 (1994) 2583–2606, hep-th/9305183.
[38] Nguyen Anh Ky, Two–parametric deformations $U_{p,q}[gl(2/1)]$ and its induced representations, *J. Phys. A: Math. Gen.* 29 (1996) 1541–1550.

[39] Nguyen Anh Ky, On the algebraic relation between one–parametric and multi–parametric quantum superalgebras, in *Proceedings of 22-th national workshop on theoretical physics, Doson, 3–5 August 1997*, p. 24–28.

[40] Nguyen Anh Ky, *Quantum superalgebra $U_{p,q}[gl(2/2)]$ and its representations*, in preparation.

[41] Nguyen Anh Ky, T. Palev and N. Stoilova, Transformations of some induced $osp(3/2)$ modules in an $so(3) \oplus sp(2)$ basis, *J. Math. Phys.* 33 (1992) 1841–1863.

[42] Nguyen Anh Ky and N. Stoilova, Finite dimensional representations of the quantum superalgebra $U_q[gl(2/2)]$. II. Nontypical representations at generic $q$, *J. Math. Phys.* 36 (1995) 5979–6003, [hep-th/9411098](http://arxiv.org/abs/hep-th/9411098).

[43] R. Le Blanc and D. Rowe, Highest weight representations for $gl(m/n)$ and $gl(m+n)$, *J. Math. Phys.* 30 (1989) 1415–1432.

[44] R. Le Blanc and D. Rowe, Superfield and matrix realizations of highest weight representations for $osp(m/n)$, *J. Math. Phys.* 31 (1990) 14–36.

[45] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, *Adv. Math.* 70 (1988) 273–249.

[46] A. J. Macfarlane, On $q$–analogues of the quantum harmonic oscillators and the quantum group $SU(2)_q$, *J. Phys. A: Math. Gen.* 22 (1989) 4581–4588.

[47] Yu. Manin, *Quantum groups and non–commutative geometry*, Centre des Recherchers Mathématique, Montréal, 1988.

[48] Yu. Manin, Multiparameter quantum deformation of the general linear supergroup, *Commun. Math. Phys.* 123 (1989) 163–175.

[49] Yu. Manin, *Topics in non–commutative geometry*, Princeton Univ. Press, Princeton, New Jersey, 1991.

[50] T. Palev, Finite-dimensional representations of the Lie superalgebra $sl(1/n)$. I. Typical representations, *J. Math. Phys.* 28, (1987) 2280–2303.
[51] T. Palev, Finite-dimensional representations of the Lie superalgebra $sl(1/n)$. II. Non-
typical representations, J. Math. Phys. 29 (1988) 2589–2598.

[52] T. Palev, Irreducible finite-dimensional representations of $gl(1/n)$ in a Gel’fand-Zetlin
basis, J. Math. Phys. 30 (1989) 1433–1442.

[53] T. Palev, Essentially typical representations of the Lie superalgebras $gl(n/m)$ in a
Gel’fand-Zetlin basis, Funkt. Anal. Prilozh. 23 (1989) 69–70 (in Russian), Funct. Anal.
Appl. 23 (1989) 141–142 (English translation).

[54] T. Palev and N. Stoilova, Finite–dimensional representations of Lie superalgebra
$gl(2/2)$.II. Nontypical representations, J. Math. Phys. 31 (1990) 953–998.

[55] T. Palev, N. Stoilova and J. Van der Jeugt, Comm. Math. Phys. 166 (1994) 367–387.

[56] T. Palev and V. Tolstoy, Finite–dimensional irreducible representations of the quantum
superalgebra $U_q(gl(n/1))$, Commun. Math. Phys. 141 (1991) 549–558.

[57] F. Pan and Y. Cao, Matrix representation of $OSp(2, 2)$ in the $U(1/1)$ basis, J. Phys.
A: Math. Gen. 24 (1991) 603–612.

[58] V. Pasquier and H. Saleur, Common structure between finite systems and conformal
field theories through quantum groups, Nucl. Phys. B 330 (1990) 523–556.

[59] P. Roche and D. Arnaudon, Irreducible representations of quantum analogue of
$SU(2)$, Lett. Math. Phys. 17 (1989) 295–300.

[60] M. Rosso, Finite dimensional representations of the quantum analog of the enveloping
algebra of a complex simple Lie algebra, Commun. Math. Phys. 117 (1987) 581–593.

[61] M. Rosso, An analogue of P.B.W. theorem and the universal R–matrix for $U_h sl(N +
1)$, Commun. Math. Phys. 124 (1989) 307–318.

[62] M. Scheunert, The theory of Lie superalgebras, Lecture Notes in Mathematics,
Vol.716, Springer–Verlag, Berlin, 1979.

[63] M. Scheunert, Serre–type relations for Special Linear Lie superalgebras, Lett. Math.
Phys. 24 (1992) 173–181.

[64] M. Scheunert, W. Nahm and V. Rittenberg, Irreducible representations of the
$OSp(2, 1)$ and $Spl(2, 1)$ graded Lie algebras, J. Math. Phys. 18 (1977) 155–162.
[65] H. Schmitt, P. Halse, A. Balantekin and B. Barett, Noncompact orthosympletic supersymmetry in $^6\text{Ni}$ and $^6\text{Ni}$, *Phys. Rev. C* **39** (1989) 2419–2425.

[66] E. K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation, *Funct. Anal. Appl.* **16** (1982) 263–270.

[67] H. Schlosser, Atypical representations of the Lie superalgebra $sl(1,n)$, *Seminar Sophus Lie* **3** (1993) 15–24.

[68] V. Tolstoy, Extremal projectors for quantized Kac–Moody superalgebras and some of their applications, in *Quantum groups* (H.-D. Doebner and J.-D. Hennig, eds.), Lecture Notes in Physics, Vol. 370, Springer–Berlin, 1990, pp. 118–125.

[69] K. Ueno, T. Takebayashi and Y. Shibukawa, Gel’fand–Zetlin basis for $U_q(gl(N+1))$ modules, *Lett. Math. Phys.* **18** (1989) 215–221.

[70] J. Van der Jeugt, Finite– and infinite–dimensional representations of the orthosympletic superalgebra $OSP(3,2)$, *J. Math. Phys.* **25** (1984) 3334–3349.

[71] P. van Nieuwenhuizen, Supergravity, *Phys. Rep.* **68** (1981) 189–398.

[72] D. Volkov and Y. Akulov, On possible universal interactions of neutrino, *Lett. J. Exp. Theor. Phys.* **16** (1972) 621–623 (in Russian), 438–440 (in English).

[73] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton Univ. Press, Princeton, New Jersey, 1983.

[74] P. West, *Introduction to supersymmetry and supergravity*, World Scientific, Singapore, 1986.

[75] J. Wess and B. Zumino, Supergauge transformations in 4 dimensions, *Nucl. Phys. B* **70** (1974) 39–50.

[76] J. Wess and B. Zumino, Covariant differential calculus on the quantum hyperplane, *Nucl. Phys.B Proc. Suppl.* **18** (1990) 302–312.

[77] E. Wigner, Relativistic invariance and quantum phenomena, *Rev. Mod. Phys.* **29** (1957) 255–268.

[78] E. Wigner, *Group theory and its applications to the quantum mechanics of atomic spectra*, Academic Press, New York, 1959.
[79] S. Woronowicz, Compact matrix pseudogroups, *Commun. Math. Phys.* **111** (1987) 613–665.

[80] S. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), *Commun. Math. Phys.* **122** (1989) 125–170.

[81] C. Yang and M. Ge, eds., *Braid groups, knot theory and statistical mechanics*, World Scientific, Singapore, 1989.

[82] R. Zhang, Finite–dimensional irreducible representations of the quantum supergroups $U_q(gl(m/n))$, *J. Math. Phys.*, **34** (1993) 1236–1254.