On centralizers of interval diffeomorphisms in critical (intermediate) regularity

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To the memory of Sergio Plaza Salinas

Abstract. We extend to the critical (intermediate) regularity several results concerning rigidity for centralizers and group actions on the interval.

Introduction

Group actions on 1-dimensional manifolds is a well-developed subject that takes its source from the theory of codimension-1 foliations (see [8] for a general panorama). When these actions are by smooth-enough (namely, \( C^2 \)) diffeomorphisms, the general picture is well-understood essentially by the classical works of Denjoy, Sacksteder, and Kopell, among others. The interest in considering actions of lower regularity comes from different sources (see, for example, [5, 6, 11]). It appears that many interesting phenomena from both the group theoretical and the dynamical viewpoints arise in intermediate regularity, that is, for actions by diffeomorphisms of differentiability classes between \( C^1 \) and \( C^2 \). This is the main subject of [3, 4, 7, 9], where several relevant problems have already been settled. Nevertheless, due to technical reasons, in many cases it was necessary to avoid certain critical regularities for which the so far existing arguments do not apply. Despite of this, it is conjectured that the corresponding rigidity phenomena should still hold in these critical cases. In this work, we confirm this intuition for centralizers and group actions on the interval by providing concrete proofs. According to the (methods of) construction of [11] for Theorem A below, [7] for Theorem B, and [3] for Theorem C, our results are optimal (in the Hölder scale). Unfortunately, one of our main arguments does not apply in the most important context, namely, that of the generalized Denjoy Theorem in critical regularity, although it provides important evidence for its validity.

The generalized Kopell Lemma in critical regularity. Our first result is the extension to the critical regularity of [4, Theorem B]. Actually, this may be considered as our main result, as all the next ones are based on similar ideas but using more involved combinatorial constructions.

Theorem A. Let \( \{I_{i_1,\ldots,i_{d+1}} : (i_1,\ldots,i_{d+1}) \in \mathbb{Z}^{d+1}\} \) be a family of subintervals of \([0,1]\) that are disposed respecting the lexicographic order. Assume that \( f_1,\ldots,f_{d+1} \) are diffeomorphisms such that \( f_j(I_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{d+1}}) = I_{i_1,\ldots,i_{j-1},1+i_{j+1},i_{j+2},\ldots,i_{d+1}} \), for all \( 1 \leq j \leq d+1 \). Then \( f_1,\ldots,f_d \) cannot be all of class \( C^{1+1/d} \) provided that \( f_{d+1} \) is of class \( C^{1+\alpha} \) for some \( \alpha > 0 \) and commutes with \( f_1,\ldots,f_d \).

Compared to [4, Theorem B], this result is an improvement in what concerns the hypothesis of regularity for \( f_1,\ldots,f_d \). Nevertheless, we impose an extra regularity assumption for \( f_{d+1} \) (in [4], we just require \( f_{d+1} \) to be \( C^1 \)). Moreover, [4, Theorem B] holds in the case of noncommuting maps (see [8, Exercise 4.1.36]), and here we strongly use the fact that \( f_{d+1} \) commutes with the other \( f_i \)'s (although these \( f_i \)'s are not assumed to commute between them).

The generalized Kopell Lemma in critical but different regularities. Our second result is an extension of Theorem A inspired by [7].
Theorem B. Assume we are in the context of Theorem A but each $f_i$ is of class $C^{1+\alpha_i}$, with $0 < \alpha_i \leq 1$, for non-necessarily equal values of $\alpha_i$. Then $\alpha_1 + \cdots + \alpha_d < 1$.

No smoothing of the Farb-Franks action in the critical regularity. Finally, we extend [Theorem A] to the critical regularity. The details for the statement below are provided in [3].

Theorem C. Farb-Franks’ action of $N_d$ is not topologically semiconjugated to an action by $O^{1+\alpha}$ diffeomorphisms for $\alpha = \frac{2}{d(d-1)}$.

As in the case of Theorem A, Theorem C above should have a version for different regularities for the elements in a canonical generating set, the proof of which should be a combination of the techniques of [2] and [3] below. Moreover, a combination of the ideas of [3] §3 and [7] §3 should allow showing that such an extended result is in fact optimal. The reader will certainly agree in that including all of this would have artificially overloaded this already very technical article.

About the proofs. Roughly, the proof of all the results above proceeds as follows. Assume that $g$ is a diffeomorphism of the interval that commutes with many diffeomorphisms $f_1, \ldots, f_k$, and let $h_n = f_{i_n} \circ \cdots \circ f_{i_1}$ be a “random composition” of $n$ factors among these $f_i$’s. Taking derivatives in the equality $g^k = h_n^{-1} \circ g^k \circ h_n$, we obtain

$$Dg^k(x) = \frac{Dh_n(x)}{Dh_n(g^k h_n(x))} \cdot Dg^k(h_n(x)) = \frac{Dh_n(x)}{Dh_n(g^k(x))} \cdot Dg^k(h_n(x)).$$

As shown in [3] [4] [7], whenever the regularity is strictly larger than the corresponding critical one, it is possible to estimate (uniformly on $n$) the value of the distortion of $h_n$, that is, an expression of type $Dh_n(x)/Dh_n(y)$ as above. This allows showing that the derivatives of the iterates of $g$ are uniformly bounded, which is impossible unless $g$ is trivial. However, as it was already noticed in the aforementioned works, this is no longer possible for the critical regularity because of the failure of convergence of a certain series. The main new idea consists in noticing that despite of the absence of uniform control for the distortion, elementary estimates show that its growth (in $n$) is slow (actually, sublinear). By choosing $n = n(k)$ appropriately, this allows showing that the growth of the derivatives of $g$ is sublinear, which is impossible. This last issue was cleverly noticed in [10], and we reproduce it below (as a lemma) with its original proof for the convenience of the reader.

Lemma [Polterovich-Sodin] If $g: I \to I$ is a nontrivial $C^1$ diffeomorphism of a bounded interval, then there exists an infinite increasing sequence of positive integers $k_j$ such that

$$\max_{x \in I} |Dg^{k_j}(x)| > k_j.$$

Proof. Let $x_0$ be a point such that $g(x_0) \neq x_0$. Denoting by $J$ the open interval of endpoints $x_0$ and $g(x_0)$, we have that $J, g^{-1}(J), g^{-2}(J), \ldots$ are two-by-two disjoint. Therefore,

$$\sum_{k \geq 1} |g^{-k}(J)| < \infty. \quad (1)$$

If the conclusion of the lemma does not hold, then there exists $k_0$ such that for all $k \geq k_0$ and all $y \in I$, we have $Dg^{-k}(y) \geq 1/k$. Since $|g^{-k}(J)| = |Dg^{-k}(y_k)|J|$ for a certain $y_k \in J$, this would imply that

$$\sum_{k \geq k_0} |g^{-k}(J)| \geq \sum_{k \geq k_0} \frac{|J|}{k} = \infty,$$

which contradicts (1). □
In order to apply our main argument, we will crucially use the following result of Borichev [1], which extends prior results of Polterovich and Sodin (valid in the $C^2$ context) to the $C^{1+\alpha}$ category (see also [2]).

**Theorem [Borichev]** Let $g$ be a $C^{1+\alpha}$ diffeomorphism of a closed interval $I$ without hyperbolic fixed points, with $0 < \alpha < 1$. Then letting $C_g$ be the $\alpha$-Hölder constant of $\log(Dg)$, for every $k \geq 0$, we have

$$\max_{x \in I} Dg^k(x) \leq \exp \left( 3C_g |I|^{\alpha} k^{1-\alpha} \right).$$

(2)

It is important to point out that although this result is not stated in this way in [1], it readily follows from it (and its proof). Indeed, Theorem 4 of [1] only claims that for $I = [0, 1]$, one has the estimate

$$\max_{x \in [0,1]} Dg^k(x) \leq \exp \left( Ak^{1-\alpha} \right)$$

for a certain constant $A = A(C_g)$. However, a careful reading of the proof therein shows that $A = 3C_g$ actually works. As the reader will notice, having such a nice quantitative version will be important in our proof. Moreover, the introduction of the factor $|I|^{\alpha}$ will be also important. This factor comes from an easy renormalization argument. Indeed, if we denote by $\tilde{g}$ the renormalization of $g$ to the unit interval (more precisely, $\tilde{g} := \varphi_I \circ g \circ \varphi_I^{-1}$, where $\varphi_I$ is the unique orientation-preserving affine homeomorphism sending $I$ into $[0, 1]$), then (the straightened version of) Borichev’s theorem (for the unit interval) yields

$$\max_{x \in I} D\tilde{g}^k(x) = \max_{y \in [0,1]} D\tilde{g}^k(y) \leq \exp(3C_g k^{1-\alpha}).$$

(3)

Since $C_g = C_g |I|^{\alpha}$, this implies (2).

1 **Proof of the critical generalized Kopell lemma via a random walk argument**

To prove Theorem A, we denote $g := f_{d+1}$ and we consider a composition $h_n = f_i \circ \cdots \circ f_2 \circ f_1$ of the $f_i$’s. Then for each $k \geq 1$, we have

$$g^k = h_n^{-1} \circ g^k \circ h_n,$$

which yields

$$Dg^k(x) = \frac{Dh_n(x)}{Dh_n(h_n^{-1} g^k h_n(x))} \cdot Dg^k(h_n(x)) = \frac{Dh_n(x)}{Dh_n(g^k(x))} \cdot Dg^k(h_n(x)).$$

We will restrict this equality to $x$ in the interval $I$ defined as the convex closure of $\bigcup_{i \in \mathbb{Z}} I_{0,0,\ldots,0,i_{d+1}}$. (Notice that $I$ is invariant under $g$.) Let $C$ be a simultaneous $\frac{1}{d}$-Hölder constant for $\log(Df_i)$, where $1 \leq i \leq d$. Denoting $h_j := f_{i_j} \circ \cdots \circ f_{i_1}$ whenever $0 \leq j \leq n$ and letting $y = y_k := g^k(x) \in I$, we have

$$\left| \log \left( \frac{Dh_n(x)}{Dh_n(g^k(x))} \right) \right| = \left| \log \left( \prod_{j=1}^n Df_{i_j}(h_{j-1}(x)) \right) \right| \left| \log \left( \prod_{j=1}^n Df_{i_j}(h_{j-1}(y)) \right) \right|$$

$$\leq \sum_{j=1}^n \left| \log Df_{i_j}(h_{j-1}(x)) - \log Df_{i_j}(h_{j-1}(y)) \right|$$

$$\leq C \sum_{j=1}^n \left| h_{j-1}(x) - h_{j-1}(y) \right|^{1/d}$$

$$\leq C \sum_{j=0}^{n-1} \left| h_j(I) \right|^{1/d}.$$
If we denote by $M_n$ the value of the last sum, this implies that

$$Dg^k(x) \leq \exp(CM_n) \cdot Dg^k(h_n(x)).$$

In order to control the growth of $S_n$, we will use the first of the two properties provided by the next

**Lemma 1.1.** Let $\ell: \mathbb{N}_0^d \to \mathbb{R}$ be a function taking only positive values. Assume that

$$\sum_{(i_1, \ldots, i_d) \in \mathbb{N}_0^d} \ell(i_1, \ldots, i_d) < \infty. \tag{5}$$

Then there exists a constant $B > 0$ such that for each $n \in \mathbb{N}$ there is a geodesic path of length $n$ in $\mathbb{N}_0^d$, say $\{(i_1(j), \ldots, i_d(j)) : 0 \leq j \leq n\}$, satisfying $i_1(0) = \ldots = i_d(0) = 0$ and

$$\sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^{1/d} \leq B \left( \log(n+1) \right)^{1/d} \tag{6}$$

and

$$\ell(i_1(n), \ldots, i_d(n)) \leq \frac{B}{(n+1)^{d-1}}. \tag{7}$$

**Proof.** Denote by $L$ the sum in (5) above. As in [4], we consider the Markov process on $\mathbb{N}_0^d$ with transition probabilities

$$p((i_1, \ldots, i_d) \mapsto (i_1, \ldots, i_{j-1}, 1 + i_j, i_{j+1}, \ldots, i_d)) := \frac{1 + i_j}{i_1 + \ldots + i_d + d}.$$

For this process, the transition probabilities in $n$ steps are equidistributed along the $n$-sphere $S_n$ for every $n \geq 1$:

$$i_1 + \ldots + i_d = n \implies \mathbb{P}_n((0, \ldots, 0) \mapsto (i_1, \ldots, i_d)) = \frac{1}{|S_n|}.$$

Letting $A_d > 0$ be such that $|S_n| \geq A_d(n + 1)^{d-1}$ for all $n \geq 0$, a direct application of Hölder’s inequality yields, for every $0 < \tau < 1$,

$$\mathbb{E} \left( \sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^\tau \right) \leq \sum_{j=0}^{n-1} \mathbb{E} \left( \ell(i_1(j), \ldots, i_d(j))^\tau \right)$$

$$= \sum_{j=0}^{n-1} \frac{1}{|S_j|} \sum_{(i_1, \ldots, i_d) \in S_j} \ell(i_1, \ldots, i_d)^\tau$$

$$\leq \left( \sum_{j=0}^{n-1} \sum_{(i_1, \ldots, i_d) \in S_j} \ell(i_1, \ldots, i_d) \right)^\tau \left( \sum_{j=0}^{n-1} \sum_{(i_1, \ldots, i_d) \in S_j} \left( \frac{1}{|S_j|} \right)^{1/\tau} \right)^{1-\tau}$$

$$\leq L^\tau \left( \sum_{j=0}^{n-1} \frac{1}{|S_j|} \left( \frac{1}{|S_j|} \right)^{1/\tau} \right)^{1-\tau}$$

$$= L^\tau \left( \sum_{j=0}^{n-1} \left( \frac{1}{|S_j|} \right)^{1/\tau} \right)^{1-\tau}$$

$$\leq \frac{L^\tau}{A_d^\tau} \left( \sum_{j=0}^{n-1} \frac{1}{(j + 1)^{(d-1)/\tau}} \right)^{1-\tau}.$$
Now, for $\tau = \frac{1}{d}$, we have $(d - 1)\tau = 1$, hence
\[
\mathbb{E}\left(\sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^{1/d}\right) \leq \frac{L^{1/d}}{A_d^{1/d}} \left(\sum_{j=1}^{n} 1\right)^{1-1/d} \leq \frac{L^{1/d}}{A_d^{1/d}} (\log(n + 1))^{1-1/d}.
\]

A direct application of Chebyshev’s inequality then shows that with probability larger than 2/3, we must have
\[
\sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^{1/d} \leq \frac{3L^{1/d}}{A_d^{1/d}} (\log(n + 1))^{1-1/d}.
\]
Moreover, since \(\sum_{(i_1, \ldots, i_d) \in S_n} \ell(i_1, \ldots, i_d) \leq L\) and the arrival probabilities in \(n\) steps are equidistributed along \(S_n\), with probability larger than 2/3 we must have
\[
\ell(i_1(n), \ldots, i_d(n)) \leq \frac{3L}{|S_n|} \leq \frac{3L}{A_d(n + 1)^{d-1}}.
\]
Thus, letting
\[
B := \max\left\{\frac{3L^{1/d}}{A_d^{1/d}}, \frac{3L}{A_d}\right\},
\]
we have that (6) and (7) simultaneously hold with probability larger than 1/3. This ensures the existence of the desired geodesic path. \(\square\)

Coming back to the proof of Theorem A, we let \(\ell\) be the function that associates to \((i_1, \ldots, i_d)\) the length of the convex closure of \(U_{i_{d+1} \in \mathbb{Z}} I_{i_1, \ldots, i_{d+1}}\). (Notice that this interval coincides with \(f_1^{i_1} \circ \cdots \circ f_d^{i_d}(I)\).) Let \(h_n := f_{i_n} \circ \cdots \circ f_{i_1}\) be a random composition for which (6) and (7) hold, and let \(g_n\) be the restriction of \(g\) to \(h_n(I)\). We claim that \(g_n\) has no hyperbolic fixed point. Otherwise, since \(g\) commutes with each \(f_i\), it would have a sequence of hyperbolic fixed points (with the same derivative) accumulating at a limit point, which is clearly impossible.

We are hence under the hypothesis of Borichev’s theorem, and an application of (2) in (4) yields
\[
\max_{x \in I} D g^k(x) \leq \exp(CM_n) \cdot D g^k_h(h_n(x)) \leq \exp\left( CB \left(\log(n + 1)\right)^{1-1/d}\right) \exp(3C g_h|h_n(I)|^{\alpha} k^{1-\alpha}) \leq \exp\left( CB \left(\log(n + 1)\right)^{1-1/d}\right) \exp\left(\frac{3C g B_k^{1-\alpha}}{(n + 1)^{d-1}\alpha}\right).
\]
Taking \(n = n_k\) so that \(k^{1-\alpha} \sim n^{(d-1)\alpha}\), hence \(\log(k) \sim \log(n_k)\), we obtain for a certain constant \(A\) (independent of \(k\)),
\[
\max_{x \in I} D g^k(x) \leq \exp\left( A \left(\log(k)\right)^{1-1/d}\right).
\]
However, since the last expression is of order \(o(k)\), this turns out to be impossible because of the Polterovich-Sodin lemma.

2 Proof of the critical generalized Kopell lemma for different regularities via a deterministic argument

The proof of Theorem B consists in a combination of the ideas of [1] and [7]. The case \(d = 2\) is relatively straightforward. Nevertheless, for larger \(d\), we need a slight but nontrivial modification
of the concatenation argument of [7]. Just for pedagogical reasons, we independently develop the cases \(d = 2, d = 3,\) and the general case \(d \geq 3,\) so that to introduce the necessary new ideas in a progressive manner. (Although the reader should have no problem in passing directly from the case \(d = 2\) to the general case \(d \geq 3.\))

For all cases, we will proceed by contradiction. We assume that \(\alpha_1 + \ldots + \alpha_d = 1,\) and we let again \(\ell\) be the function that associates to \((i_1, \ldots, i_d)\) the length of the convex closure of \(\bigcup_{i_{d+1} \in \mathbb{Z}} I_{i_1, \ldots, i_d, i_{d+1}}.\)

Then we consider parallelepipeds \(Q(n)\) in \(\mathbb{N}_0^d\) whose \(s\)th-side has length of order \(2^{n\alpha_s}.\) For such a \(Q(n),\) we set

\[
L_n := \sum_{(i_1, \ldots, i_d) \in Q(n)} \ell(i_1, \ldots, i_d).
\]

Our task consists in showing that, for an appropriately chosen sequence \((Q(n))\) of finite multiplicity \(M\) (that is, such that no point is contained in more than \(M\) of these parallelepipeds), there is a positive constant \(B\) for which there exist (non-necessarily nonempty) geodesic segments \(\gamma_1, \gamma_1^2, \ldots, \gamma_1^{d_1}, \gamma_2^1, \gamma_2^2, \ldots, \gamma_2^d, \ldots,\) with \(d_n \leq d,\) satisfying the following properties:

- For each \(n \geq 1\) and \(1 \leq k \leq d_n,\) the segment \(\gamma_n^k\) is contained in \(Q(n).\)
- Each of these segments intersects the next nonempty one in the sequence above.
- For certain positive constants \(\alpha, D\) and each \(n \geq 1,\) at least one of the segments \(\gamma_n^1, \ldots, \gamma_n^{d_n}\) contains no less than \(2^{n\alpha}/D\) points.
- Each \(\gamma_n^k\) is an unidirectional path pointing in a \(s\)-direction, with \(s = s_{n,k},\) and

\[
\sum_{(i_1, \ldots, i_d) \in \gamma_n^k} \ell(i_1, \ldots, i_d)^{\alpha_s} \leq B \max\{L_n^{\alpha_s}, L_{n+1}^{\alpha_s}\}.
\]

We next explain how such a sequence of geodesic segments allows proving Theorem B. The next paragraphs will be devoted to the constructions of the parallelepipeds as well as the sequences of geodesics segments satisfying the desired properties in the corresponding cases.

First of all, notice that the concatenation of the geodesic segments along intersecting points produces an infinite (non-necessarily geodesic) path \(\gamma : \mathbb{N}_0 \to \mathbb{N}_0^d.\) We will assume that \(\gamma\) starts at the origin: if this is not the case, adding an initial segment \(\gamma_0 \subset Q(1)\) from the origin to the initial point of \(\gamma_1^1,\) one may apply the same arguments below modulo slightly changing the constant \(B.\)

For each \(n \geq 1,\) we denote by \(N = N(n)\) the entry-time of \(\gamma\) into \(Q(n+1).\) If we denote by \(s(m)\) the direction corresponding to the jump from \(\gamma(m)\) to \(\gamma(m+1),\) then \([9]\) combined with Hölder’s inequality yields

\[
\sum_{m=0}^{N} \ell(\gamma(m))^{\alpha_{s(m)}} \leq B \sum_{m=1}^{n+1} \sum_{k=1}^{d} L_m^{\alpha_{s,m,k}} \leq B \sum_{k=1}^{d} \left(2 \sum_{m=1}^{n+1} L_m\right)^{\alpha_{s,m,k}}(n+1)^{1-\alpha} \leq 2^{\alpha'} dB M^{\alpha'} (n+1)^{1-\alpha},
\]

where \(\alpha := \min\{\alpha_1, \ldots, \alpha_d\}\) and \(\alpha' := \max\{\alpha_1, \ldots, \alpha_d\}.\) Moreover, the assumption on the size of \(Q(n)\) easily implies the asymptotic equivalence \(n \sim \log(N).\) (This equivalence will be even more transparent for the explicit choice of \(Q(n)\) further on.) As a consequence, there is a constant \(A' > 0\) such that the previous estimate becomes

\[
\sum_{m=0}^{N} \ell(\gamma(m))^{\alpha_{s(m)}} \leq A'(\log(N))^{1-\alpha}.
\]
The path \( \gamma \) induces a sequence \((h_n)\) of compositions of maps from \(\{f_1, f_1^{-1}, \ldots, f_d, f_d^{-1}\}\) such that if \(I\) denotes the convex closure of \(\bigcup_{i \in \mathbb{Z}} I_0, \ldots, i_{d+1}\), then
\[
\sum_{m=0}^{N} |h_m(I)|^{\alpha_{i(m)}} \leq A'(\log(N))^{1-\alpha}.
\] (10)

Let us write \( h_m = f_{i_m} \circ \cdots \circ f_{i_1}\), where each \( f_{i_j}\) lies in \(\{f_1, f_1^{-1}, \ldots, f_d, f_d^{-1}\}\), and let \(C\) be a common upper bound for the \(\alpha_i\)-Hölder constants of \(\log(Df_{i_1}), \log(Df_{i_1}^{-1})\), where \(1 \leq i \leq d\). Given \(n > 1\), let \(N'\) be such that \(N(n-1) \leq N' \leq N(n) = N\). For each \(x, y\) in \(I\), estimate (10) yields
\[
\left| \log \left( \frac{Dh_{N'}(x)}{Dh_{N'}(y)} \right) \right| = \left| \log \left( \frac{\prod_{m=1}^{N'} Df_{i_m}(h_{m-1}(x))}{\prod_{m=1}^{N'} Df_{i_m}(h_{m-1}(y))} \right) \right|
\]
\[
\leq \sum_{m=1}^{N'} |\log Df_{i_m}(h_{m-1}(x)) - \log Df_{i_m}(h_{m-1}(y))|
\]
\[
\leq C \sum_{m=1}^{N'} |h_{m-1}(x) - h_{m-1}(y)|^{\alpha_{i(m-1)}}
\]
\[
\leq C \sum_{m=0}^{N} |h_m(I)|^{\alpha_{i(m)}}
\]
\[
\leq CA'(\log(N))^{1-\alpha}.
\]

Moreover, by the third property of our sequence, we may choose \(k\) such that \(\gamma_k \) contains at least \(2^{n \alpha}/D\) points. Since the sum of the values of \(\ell\) along these points is \(\leq 1\), such a segment must contain a point at which the value of \(\ell\) is \(\leq D/2^{n \alpha}\). In other words, we may choose \(N'\) above so that \(|h_{N'}(I)| \leq D/2^{n \alpha}\).

Denote by \(g_{N'}\) the restriction of \(g := f_{d+1}\) to \(h_{N'}(I)\). As in (11) the map \(g\) (hence \(g_{N'}\)) cannot have hyperbolic fixed points. Therefore, taking derivatives in the equality \(g^k = h_{N'}^{-1} \circ g^k \circ h_{N'}\) and using the previous estimate for \(y := g^k(x) \in I\), Borichev’s theorem yields
\[
Dg^k(x) = \frac{Dh_{N'}(x)}{Dh_{N'}(g^k(x))} \cdot Dg^k(h_{N'}(x)) = \frac{Dh_{N'}(x)}{Dh_{N'}(g^k(x))} \cdot Dg^k(h_{N'}(x))
\]
\[
\leq \exp \left( C A'(\log(N))^{1-\alpha} \right) \exp \left( 3Cg_h N' \alpha k \right) \alpha_{d+1} k^{1-\alpha_{d+1}}
\]
\[
\leq \exp \left( C A'(\log(N))^{1-\alpha} \right) \exp \left( 3Cg_h N' \alpha k^{1-\alpha_{d+1}} \right) k_{2\alpha_{d+1}}
\]

Take \(n = n_k\) so that \(k^{1-\alpha_{d+1}} < 2^{n\alpha_{d+1}}\), hence \(n_k \sim \log(k)\). Using the fact that \(\log(N(n)) \sim n_k\), we obtain for a certain constant \(A\) (independent of \(k\)),
\[
\max_{x \in I} Dg^k(x) \leq \exp \left( A (\log(k))^{1-\alpha} \right).
\]
However, the last expression is of order \(o(k)\), which is impossible by the Polterovich-Sodin lemma.

2.1 The case \(d = 2\)

Following [11], let \(Q(n)\) be the rectangle defined as \(Q(2n+1) := [[4^{n\alpha_1}, 4^{(n+1)\alpha_1}]] \times [[4^{n\alpha_2}, 4^{(n+2)\alpha_2}]]\) and \(Q(2n + 2) := [[4^{n\alpha_1}, 4^{(n+2)\alpha_1}]] \times [[4^{(n+1)\alpha_2}, 4^{(n+2)\alpha_2}]]\), where \([[x, y]]\) stands for the set of integers between \(x\) and \(y\). Notice that the multiplicity of the sequence \((Q(n))\) is 4.
A set of the form \( Q(n) \cap \{ j = \text{const} \} \) (resp. \( Q(n) \cap \{ i = \text{const} \} \)) is said to be an horizontal (resp. vertical) segment in \( Q(n) \). Notice that the cardinality of this set \( \mathcal{H}_n \) (resp. \( \mathcal{V}_n \)) of horizontal (resp. vertical) segments is \( \geq 2^{n\alpha_2} / D_1 \) (resp. \( \geq 2^{n\alpha_1} / D_1 \)) for a certain constant \( D_1 > 0 \) (independent of \( n \)). Moreover, there exists a positive constant \( D_2 \) such that the number of points in each of these horizontal (resp. vertical) segments is \( \geq D_2 2^{n\alpha_1} \) (resp. \( \geq D_2 2^{n\alpha_2} \)).

Say that an horizontal segment \( \gamma \) in \( Q(2n + 2) \) is good if

\[
\sum_{(i, j) \in \gamma} \ell(i, j) \leq \frac{L_{2n+2}}{|\mathcal{H}_{2n+2}|}.
\]

Clearly, there must be at least one good horizontal segment. For such a segment \( \gamma = \gamma_{2n+2} \), Hölder’s inequality yields

\[
\sum_{(i, j) \in \gamma} \ell(i, j)^{\alpha_1} \leq \left( \frac{L_{2n+2}}{|\mathcal{H}_{2n+2}|} \right)^{\alpha_1} |\gamma|^{1-\alpha_1} \leq L_{2n+2}^{\alpha_1} \left( \frac{D_1}{2^{(2n+2)\alpha_2}} \right)^{\alpha_1} (D_2 2^{(2n+2)\alpha_1})^{1-\alpha_1} = D_1^{\alpha_1} D_2^{\alpha_2} L_{2n+2}^{\alpha_1}.
\]

Similarly, say that a vertical segment \( \gamma \) in \( Q(n) \) is good if

\[
\sum_{(i, j) \in \gamma} \ell(i, j) \leq \frac{L_n}{|\mathcal{V}_n|}.
\]

Again, there must exist a good vertical segment \( \gamma = \gamma_{2n+1} \subset Q(2n + 1) \), and for this segment we have

\[
\sum_{(i, j) \in \gamma} \ell(i, j)^{\alpha_2} \leq \left( \frac{L_{n+1}}{|\mathcal{V}_{n+1}|} \right)^{\alpha_2} |\gamma|^{1-\alpha_2} \leq L_{n+1}^{\alpha_2} \left( \frac{D_1}{2^{(2n+1)\alpha_1}} \right)^{\alpha_2} (D_2 2^{(2n+1)\alpha_2})^{1-\alpha_2} = D_1^{\alpha_2} D_2^{\alpha_1} L_{n+1}^{\alpha_2}.
\]

Thus, the segments \( \gamma_1, \gamma_2, \ldots, \gamma_n, \ldots \) satisfy \([\text{4}]\) for \( B \geq \max\{D^{\alpha_1}D_2^{\alpha_2}, D_1^{\alpha_2}D_2^{\alpha_1}\} \). Each of these segments intersects the next one, and it is easy to check that between the concatenating points, \( \gamma_n \) contains at least \( 2^{n\alpha}/D \) points for a certain constant \( D > 0 \), where \( \alpha := \min\{\alpha_1, \alpha_2\} \). Therefore, all the conditions from \([\text{2}]\) are fulfilled, and this concludes the proof of Theorem B in the case \( d = 2 \).

2.2 The case \( d = 3 \)

In the case \( d \geq 3 \), we let \( Q(n) := \prod_{i=1}^d [x_i/n, y_i/n] \) be inductively defined by \( Q(1) := [(1, 4^d)]^d \) and \( Q(n+1) := \cdots \times [1 + 2^{d\alpha}(x_n^m - 1), y_n^m] \times [x_n^{m+1} + 1 + 2^{d\alpha}(y_n^{m+1} - 1)] \times \cdots \), where for each \( n \geq 1 \), we denote by \( m = m(n) \in \{1, \ldots, d\} \) the residue class of \( n \) modulo \( d \). (The dots mean that the corresponding factors remain untouched.) Notice that the sequence \( (Q(n)) \) has multiplicity \( d + 2 \).

One easily checks the asymptotic equivalence \( y_n^k - x_n^k \sim 2^{n\alpha_k} \). We let \( D_1 \) be a constant such that \( 2^{n\alpha_1}/D_1 \leq y_n^k - x_n^k \leq 2^{n\alpha_1} - 1 \). Moreover, we may fix a constant \( D_2 > 0 \) such that

\[
y_n^{m+1} - x_n^{m+1} \geq D_2 (y_n^m - x_n^m) \quad \text{and} \quad y_n^{m+1} - x_n^{m+1} \geq D_2 (y_n^{m+1} - x_n^{m+1}). \tag{11}
\]

We now specialize to the case \( d = 3 \). A plane \( P \) of the form \( P = Q(n) \cap \{ i_{m+2} = \text{const} \} \) will be said to be an \( h \)-plane, and the family of \( h \)-planes in \( Q(n) \) will be denoted by \( \mathcal{P}_n \). Notice that the cardinality of \( \mathcal{P}_n \) is \( \geq 2^{n\alpha_{m+2}}/D_1 \).

Given an \( h \)-plane \( P \in \mathcal{P}_n \), an horizontal (resp. vertical) segment in \( P \) is a set of the form \( P \cap \{ i_{m+1} = \text{const} \} \) (resp. \( P \cap \{ i_m = \text{const} \} \)). The cardinality of the family \( \mathcal{H}_n \) (resp. \( \mathcal{V}_n \)) of horizontal (resp. vertical) segments is \( \geq 2^{n\alpha_{m+1}}/A \) (resp. \( \geq 2^{n\alpha_m}/A' \)) for certain positive constants \( A, A' \).
Let $\lambda \geq 1$ be fixed. Say that an $h$-plane $P$ in $\mathcal{P}_n$ is $\lambda$-good if (see (4))

$$\sum_{(i_1, i_2, i_3) \in P} \ell(i_1, i_2, i_3) \leq \frac{\lambda L_n}{|\mathcal{P}_n|}.$$ 

By Chebyshev’s inequality, the proportion of $h$-planes that are $\lambda$-good is larger than $1 - 1/\lambda$. Similarly, say that an horizontal segment $\gamma$ of $P \in \mathcal{P}_n$ is $\lambda$-good relatively to $P$ if

$$\sum_{(i_1, i_2, i_3) \in \gamma} \ell(i_1, i_2, i_3) \leq \frac{\lambda \sum_{(i_1, i_2, i_3) \in P} \ell(i_1, i_2, i_3)}{|\mathcal{H}_n|}.$$ 

As before, horizontal directions are relatively $\lambda$-good in a proportion larger than $1 - 1/\lambda$. Finally, say that a vertical segment $\gamma$ in $P \in \mathcal{P}_n$ is $\lambda$-good if

$$\sum_{(i_1, i_2, i_3) \in \gamma} \ell(i_1, i_2, i_3) \leq \frac{\lambda \sum_{(i_1, i_2, i_3) \in P} \ell(i_1, i_2, i_3)}{|\mathcal{V}_n|}.$$ 

Once again, vertical segments are $\lambda$-good in a proportion larger than $1 - 1/\lambda$.

Now fix $\lambda \geq 2/D_2$, and let $P \in \mathcal{P}_n$ be a $\lambda$-good $h$-plane. By the first inequality in (11), more than a half of the vertical segments of $P$ contained in $Q(n+1)$ are $\lambda$-good. Since $\lambda \geq 2$, more than a half of the $h$-planes in $Q(n+1)$ are $\lambda$-good. Therefore, there must exist a $\lambda$-good $h$-plane $P' \in \mathcal{P}_{n+1}$ such that $P \cap P'$ is a $\lambda$-good vertical segment of $P$. Moreover, by the second inequality in (11), more than a half of the vertical segments of $P'$ contained in $Q(n)$ are $\lambda$-good relatively to $P'$.

We may hence fix a sequence $(P_n)$ of $\lambda$-good $h$-planes such that $P_n \cap P_{n+1}$ is a $\lambda$-good vertical segment $\gamma_n^2$ of $P_n$, for each $n \geq 1$. (See Figure 1.) Each $P_n$ must contain a relatively $1$-good horizontal segment $\gamma_n^1$. Finally, let $\gamma_n^3$ be a $\lambda$-good vertical segment of $P_{n+1}$ contained in $Q(n)$. (See Figure 2.) We have thus constructed an infinite sequence of geodesic segments $\gamma_1^1, \gamma_1^2, \gamma_1^3, \gamma_2^1, \gamma_2^2, \gamma_2^3, \ldots$, each of which intersects the next one in the sequence. Moreover, since $P_n$ is $\lambda$-good and $\gamma_n^1$ is a relatively $1$-good horizontal segment in $P_n$,

$$\sum_{(i_1, i_2, i_3) \in \gamma_n^1} \ell(i_1, i_2, i_3) \leq \frac{\sum_{(i_1, i_2, i_3) \in P_n} \ell(i_1, i_2, i_3)}{|\mathcal{H}_n|} \leq \frac{1}{|\mathcal{H}_n| |\mathcal{P}_n|} \leq \frac{AD_1 \lambda L_n}{2^{n\alpha_m+1} 2^{n\alpha_m+2}}.$$ 

By Hölder’s inequality, this implies that

$$\sum_{(i_1, i_2, i_3) \in \gamma_n^1} \ell(i_1, i_2, i_3)^{\alpha_m} \leq \left( \frac{AD_1 \lambda L_n}{2^{n\alpha_m+1} 2^{n\alpha_m+2}} \right)^{\alpha_m} |\gamma_n^1|^{1-\alpha_m} \leq \left( \frac{AD_1 \lambda L_n}{2^{n\alpha_m+1} 2^{n\alpha_m+2}} \right)^{\alpha_m} (D_1 2^{n\alpha_m})^{1-\alpha_m},$$

hence

$$\sum_{(i_1, i_2, i_3) \in \gamma_n^1} \ell(i_1, i_2, i_3)^{\alpha_m} \leq (AD_1 \lambda)^{\alpha_m} D_1^{1-\alpha_m} L_n^{\alpha_m}. \tag{12}$$

Similarly, since $P_n$ is $\lambda$-good and $\gamma_n^2$ is a $\lambda$-good vertical segment of $P_n$,

$$\sum_{(i_1, i_2, i_3) \in \gamma_n^2} \ell(i_1, i_2, i_3) \leq \frac{\lambda \sum_{(i_1, i_2, i_3) \in P_n} \ell(i_1, i_2, i_3)}{|\mathcal{V}_n|} \leq \frac{\lambda \lambda L_n}{|\mathcal{V}_n| |\mathcal{P}_n|} \leq \frac{A' D_1 \lambda^2 L_n}{2^{n\alpha_m} 2^{n\alpha_m+2}}.$$ 

Again, by Hölder’s inequality,

$$\sum_{(i_1, i_2, i_3) \in \gamma_n^2} \ell(i_1, i_2, i_3)^{\alpha_{m+1}} \leq \left( \frac{A' D_1 \lambda^2 L_n}{2^{n\alpha_m} 2^{n\alpha_m+2}} \right)^{\alpha_{m+1}} |\gamma_n^2|^{1-\alpha_{m+1}} \leq \left( \frac{A' D_1 \lambda^2 L_n}{2^{n\alpha_m} 2^{n\alpha_m+2}} \right)^{\alpha_{m+1}} (D_1 2^{n\alpha_{m+1}})^{1-\alpha_{m+1}}.$$
hence
\[ \sum_{(i_1, i_2, i_3) \in \gamma_n^3} \ell(i_1, i_2, i_3)^{\alpha_{m+1}} \leq (A'D_1 \lambda^2)^{\alpha_{m+1}} D_1^{1-\alpha_{m+1}} L_{n+1}^{\alpha_{m+1}}. \tag{13} \]

Finally, \( \gamma_n^3 \) is a \( \lambda \)-good vertical segment of \( P_{n+1} \), which is a \( \lambda \)-good horizontal plane of \( Q(n+1) \), hence
\[ \sum_{(i_1, i_2, i_3) \in \gamma_n^3} \ell(i_1, i_2, i_3) \leq \frac{\lambda \sum_{(i_1, i_2, i_3) \in P_{n+1}} \ell(i_1, i_2, i_3)}{|V_{n+1}|} \leq \frac{\lambda L_{n+1}}{|V_{n+1}|} \leq \frac{A'D_1 \lambda^2 L_{n+1}}{2(n+1)\alpha_{m+1}2(n+1)\alpha_m}, \]
and Hölder’s inequality yields
\[ \sum_{(i_1, i_2, i_3) \in \gamma_n^3} \ell(i_1, i_2, i_3)^{\alpha_{m+2}} \leq \left( \frac{A'D_1 \lambda^2 L_{n+1}}{2(n+1)\alpha_{m+1}2(n+1)\alpha_m} \right)^{\alpha_{m+2}} \left( \frac{\gamma_n^3}{\lambda} \right)^{1-\alpha_{m+2}} \]
\[ \leq \left( \frac{A'D_1 \lambda^2 L_{n+1}}{2(n+1)\alpha_{m+1}2(n+1)\alpha_m} \right)^{\alpha_{m+2}} (D_1 2(n+1)\alpha_{m+2})^{1-\alpha_{m+2}}, \]
that is
\[ \sum_{(i_1, i_2, i_3) \in \gamma_n^3} \ell(i_1, i_2, i_3)^{\alpha_{m+2}} \leq (A'D_1 \lambda^2)^{\alpha_{m+2}} D_1^{1-\alpha_{m+2}} L_{n+1}^{\alpha_{m+2}}. \tag{14} \]

By (12), (13) and (14), condition (9) holds for \( B \geq \max_k \max \{ (AD_1 \lambda)^\alpha D_1^{1-\alpha_k}, (A'D_1 \lambda^2)^\alpha D_1^{1-\alpha_k} \} \). Finally, it is easy to see that for a certain constant \( D > 0 \), each \( \gamma_n^1 \) contains at least \( 2^{n\alpha}/D \) points between the concatenating points, where \( \alpha := \min \{ \alpha_1, \alpha_2, \alpha_3 \} \). This concludes the checking of the properties from (2).

2.3 The general case \( d \geq 3 \)

For the proof in the general \( (d \geq 3) \) case, it is better to isolate and inductively extend a concatenation argument that has already been used (in a weak form) in the cases \( d = 2 \) and \( d = 3 \).

The \( d \)-dimensional black box. Let \( Q \) be a \( d \)-dimensional parallelepiped, with \( d \geq 3 \), and let \( \ell \) be a positive function defined on \( Q \). Set
\[ L_Q := \sum_{(i_1, \ldots, i_d) \in Q} \ell(i_1, \ldots, i_d). \]
For $1 \leq m \leq d$, an $m$-segment is a set of the form $Q \cap \{ i_j = const_j \}$ except for $j = m$. Denote by $S(m)$ the family of all $m$-segments. The elements in the family $S := \bigcup_m S(m)$ will be said to be unidirectional segments. Given $\lambda \geq 1$, say that that $\gamma \in S(m)$ is $\lambda$-good if

$$\sum_{(i_1, \ldots, i_d) \in \gamma} \ell(i_1, \ldots, i_d) \leq \frac{\lambda L_Q}{|S(m)|}.$$

More generally, let us consider a $d'$-dimensional parallelepiped $Q'$ and a $d''$-dimensional parallelepiped $Q''$, both contained in $Q$, such that $d' < d''$ and $Q' \subset Q''$. Denote by $c(Q', Q'')$ the number of (disjoint) translates of $Q'$ that fill $Q''$ (that is, the number of copies of $Q'$ in $Q''$). For example, $c(\gamma, Q) = |S(m)|$ for any $m$-segment $\gamma$. Say that $\gamma$ is $\lambda$-good relatively to $Q''$ whenever

$$\sum_{(i_1, \ldots, i_d) \in Q'} \ell(i_1, \ldots, i_d) \leq \frac{\lambda}{c(Q', Q'')} \sum_{(i_1, \ldots, i_d) \in Q''} \ell(i_1, \ldots, i_d).$$

Now, say that $\gamma \in S(m)$ is fully $\lambda$-good if there exists a flag of $d'$-dimensional parallelepipeds $Q'^{d'}$ that are $\lambda$-good in $Q$, have the form $Q'^{d'} = Q \cap \{ i_j = \text{const}_j \}$ except for $j = m, m+1, \ldots, m + d' - 1$, and satisfy $\gamma = Q^1 \subset Q^2 \subset \cdots \subset Q^{d-1}$.

**Lemma 2.1.** Given $0 < \kappa < 1$ and $\mu \geq 1$, there exists $\lambda' = \lambda'(\mu, \kappa, d)$ (not depending on $Q$) such that the following holds: If $\gamma$ is a fully $\mu$-good 1-segment of $Q$, then for a proportion larger than $\kappa$ of the points $(i_1, \ldots, i_d) \in \gamma$, there exists a finite sequence of $\lambda'$-good unidirectional segments $\gamma_1, \ldots, \gamma_{d'} - 1$, with $d' \leq d$, such that each $\gamma_i$ intersects $\gamma_{i+1}$, with $\gamma_1$ starting at a point of $\gamma$ and $\gamma_{d'-1}$ ending at $(i_1, \ldots, i_d)$.

**Proof.** We leave the case $d = 3$ to the reader (it uses similar arguments to those below; compare also [2.2]).

Assume inductively that the claim holds in dimension $d$, and let us deal with the $(d + 1)$-dimensional case. Let $\gamma = Q^1 \subset \cdots \subset Q^d$ be the flag of $\mu$-good parallelepipeds associated to $\gamma$. Fix $\lambda \geq 1$ large-enough so that $\kappa < (d-1)(1-1/\lambda) - (d-2)$. For each $2 \leq m \leq d$, Chebyshev’s inequality implies that the proportion of the set of integers $i$ in the projection of $Q$ along the first coordinate for which $Q^m(i) := Q^m \cap \{ i_1 = i \}$ is $\lambda$-good relatively to $Q^m$ is larger than $1 - 1/\lambda$. Therefore, for a proportion larger than $(d-1)(1-1/\lambda) - (d-2)$ of this set of integers $i$, these properties hold simultaneously, which means that the segment $Q^2(i)$ is fully $\lambda$-good in $Q^d(i)$. On each such a $Q^d(i)$, the inductive procedure yields a proportion larger than $\frac{\kappa}{[(d-1)(1-1/\lambda) - (d-2)]}$ of points in $Q^d(i)$ that may be reached by concatenating no more than $d$ unidirectional $\lambda'$-good segments $\gamma_2, \ldots, \gamma_{d'}$ of $Q^d(i)$ (with $\gamma_2$ starting at a point of $\gamma_1 := Q^2(i)$), where $\lambda' = \lambda' \left( \lambda, \frac{\kappa}{[(d-1)(1-1/\lambda) - (d-2)]}, d \right)$. Notice that each of these segments is $\mu\lambda'$-good in $Q$. Thus we have a proportion larger than

$$\frac{\kappa}{[(d-1)(1-1/\lambda) - (d-2)]} \cdot \frac{\kappa}{[(d-1)(1-1/\lambda) - (d-2)]} = \kappa$$

of points in $Q$ that may be reached by a sequence of $d$ unidirectional segments that are $\mu \lambda'$-good, the first of which intersects $\gamma$. This concludes the inductive proof. $\Box$

**Proof of Theorem B (for $d \geq 3$).** We consider the sequence of parallelepipeds $Q(n)$ from the beginning of [2.2]. Fix $\lambda > 2(d-1)$, and let $\lambda' := \lambda'(\lambda, 1/2, d)$ be the constant defined in the statement of Lemma [2.2]. We will perform a process that starts by arbitrarily choosing a fully $\lambda$-good 1-segment $\gamma^1_1$ of $Q(1)$. (Since $\lambda > 2(d-1)$, we have $(d-1)(1-1/\lambda) - (d-2) > 1/2 > 0$, and an application of Chebyshev’s inequality ensures the existence of such a segment.)

Assume now that there is a concatenating sequence of unidirectional segments $\gamma_1^1, \ldots, \gamma_1^d, \ldots, \gamma_{n-1}^1, \ldots, \gamma_{n-1}^{d_{n-1}}, \gamma_n^1$, with each $d_j \leq d$, such that:
• For $1 \leq n' \leq n - 1$ and $2 \leq k \leq d$, the segment $\gamma_{n'}^k$ is $\lambda'$-good in $Q(n') \cap Q(n' + 1)$.

• For $1 \leq n' \leq n$, the segment $\gamma_{n'}^1$ is a fully $\lambda'$-good $m(n')$-segment in $Q(n')$.

We would like to extend this sequence by appropriately choosing $\gamma_{n_0}^2, \ldots, \gamma_{n_0}^{d_0}$, and $\gamma_{n_0+1}^1$. To do this, we first invoke Lemma 2.4.2, which ensures that more than a half of the points of $Q(n) \cap Q(n + 1)$ may be reached starting at a point of $\gamma_{n}^1$ by concatenating $\lambda'$-good segments $\gamma_{n}^2, \ldots, \gamma_{n}^{d'}$, with $d' \leq d$.

On the other hand, since $(d - 1)(1 - 1/\lambda) - (d - 2) > 1/2 > 0$, an application of Chebyshev’s inequality ensures that for more than a half of the points of $Q(n) \cap Q(n + 1)$ lie in a fully $\lambda$-good $m(n + 1)$-segment $\gamma_{n+1}^1$ of $Q(n + 1)$. These two sets must necessarily intersect, and this allows to define the desired concatenating segments.

Checking the properties from 2.4 now mimics the cases $d = 2$ and $d = 3$. Indeed, let $A > 0$ be a constant such that for every $s$-segment $\gamma$ in $Q$,

$$2^{n(1-\alpha_s)}/A \leq c(\gamma, Q) \leq A 2^{n(1-\alpha_s)}.$$

If $\gamma = \gamma_{n}^k$ and $k \neq 1$, then letting $s := s_{n,k}$ be its direction, Hölder’s inequality yields

$$\sum_{(i_1,\ldots,i_d) \in \gamma_{n}^k} \ell(i_1,\ldots,i_d)^{\alpha_s} \leq \left( \sum_{(i_1,\ldots,i_d) \in \gamma_{n}^k} \ell(i_1,\ldots,i_d) \right)^{\alpha_s} |\gamma_{n}^k|^{1-\alpha_s} \leq \left( \frac{\lambda'LQ(n)\cap Q(n+1)}{c(\gamma_{n}^k, Q_n)} \right)^{\alpha_s} (A' 2^{\alpha_s})^{1-\alpha_s} \leq (\lambda'A)^{\alpha_s} (A')^{1-\alpha_s} L_{n}^{\alpha_s}.$$

In the case of $\gamma_{n}^1$, a similar argument applies, so that 3 holds for $B \geq \max_k \{ (\lambda'A)^{\alpha_s} (A')^{1-\alpha_s} \}$, where $\lambda' = \max \{ \lambda, \lambda' \}$. Finally, it is easy to see that for a certain constant $D > 0$, each segment $\gamma_{n}^1$ contains at least $2^{n\alpha}/D$ points, where $\alpha := \min \{ \alpha_1, \ldots, \alpha_d \}$. \hfill $\square$

3 Proof of the non-smoothability of the Farb-Franks action in critical regularity

We now deal with the group $N_d$ of $(d+1) \times (d+1)$ lower-triangular matrices with integer entries, all of which are equal to 1 on the diagonal. For $i > j$, we denote by $f_{i,j}$ the element represented by a matrix whose only nonzero entry outside the diagonal is the $(i,j)$-entry and equals 1. Notice that the $f_{i,j}$’s generate $N_d$.

Let us briefly remind Farb-Franks’ action of $N_d$ on $[0,1]$. First, notice that $N_d$ acts linearly on $\mathbb{Z}^{d+1}$ with the affine hyperplane $1 \times \mathbb{Z}^d$ remaining invariant. The thus-induced action on $\mathbb{Z}^d$ allows producing an action on the interval as follows. Let $\{I_{i_1,\ldots,i_d} : (i_1,\ldots,i_d) \in \mathbb{Z}^d\}$ be a family of intervals such that the sum $\sum_{i_1,\ldots,i_d} |I_{i_1,\ldots,i_d}|$ is finite, say equal to 1 after normalization. We join these intervals lexicographically on the closed interval $[0,1]$, and we identify $f \in N_d$ to the (unique) homeomorphism of $[0,1]$ that sends affinely the interval $I_{i_1,\ldots,i_d}$ into $I_{f(i_1,\ldots,i_d)}$, where $f(i_1,\ldots,i_d)$ stands for the action of $f \in N_d$ on $\mathbb{Z}^d \sim \{1\} \times \mathbb{Z}^d$.

As is shown in 3, if we let $\alpha = \alpha(d) := \frac{2}{d(d-1)}$, then for every $\varepsilon > 0$, this action is conjugated to an action by $C^{1+\alpha-\varepsilon}$ diffeomorphisms, but it cannot be (semi-)conjugated to an action by $C^{1+\alpha+\varepsilon}$ diffeomorphisms. Our aim is to extend the last result to the critical regularity $C^{1+\alpha'}$. To do this, we will follow a similar strategy to that of the generalized Kopell’s lemma. We assume that a topological conjugacy exists, and for simplicity we continue denoting by $I_{i_1,\ldots,i_d}$ the image of the corresponding interval under this conjugacy. We let $I$ be the convex closure of $\bigcup_{i_d \in \mathbb{Z}} I_{0,\ldots,0,i_d}$. Notice that the element $g := f_{d+1,1}$ lies in the center of $N_d$ and fixes the interval $I$. Moreover, every element in $N_d$
sends $I$ into either itself or a disjoint interval. If we consider the isomorphic copy $N_{d-1}^* \subset N_d$ of $N_{d-1}$ formed by all elements whose last row and column coincide with those of the identity, then the orbit of $I$ under $N_d$ coincides with that under $N_{d-1}^*$. Moreover, the stabilizer of $I$ under the $N_{d-1}^*$-action corresponds to the subgroup formed by the elements whose first column coincides with those of the identity. Since this subgroup is naturally isomorphic to $N_{d-2}$, the orbit-graph of $I$ identifies to a coset space $N_{d-1}/N_{d-2}$, and has $\mathbb{Z}^{d-1}$ as set of vertices. (See [3, Figure 2] for an illustration in the case $d=3$.)

### 3.1 From sublinear distortion to the proof of Theorem C

As in previous sections, we will decompose (part of) the orbit of $I$ (which identifies to $\mathbb{Z}^{d-1}$) into parallelepipeds. Following [3, §2.4], we define $Q(n)$ by induction. We first let $Q(0):=[1,1+4d+1]^{d-1}$. Now, assuming that $Q(n):=[x_{1,n},y_{1,n}] \cdots \times [x_{d-1,n},y_{d-1,n}]$ has been already defined, we let $i(n) \in \{1, \ldots, d-1\}$ be the residue class (mod. $d-1$) of $n$, and we set

$$Q(n+1):=\cdots \times \left[1+4^{i(n)}(x_{i(n),n}-1),y_{i(n),n}\right] \times \left[x_{i(n)+1,n},1+4^{i(n)+1}(y_{i(n)+1,n}-1)\right] \times \cdots,$$

where the dots mean that the corresponding factors remain untouched. Notice that all $x_{i,n}$, $y_{i,n}$, and $y_{i,n}-x_{i,n}$, are asymptotically equivalent to $4^{\frac{m}{d-1}}$.

For each $(i_1, \ldots, i_{d-1}) \in \mathbb{Z}^{d-1}$, we let $\ell(i_1, \ldots, i_{d-1})$ be the length of the interval $I_{i_1,\ldots,i_{d-1}}$ defined as the convex closure of $\bigcup_{i_d \in \mathbb{Z}}I_{i_1,\ldots,i_{d-1},i_d}$. We also set

$$L_n := \sum_{(i_1, \ldots, i_{d-1}) \in Q(n)} \ell(i_1, \ldots, i_{d-1}).$$

Our task now consists in showing that there exists a sequence of paths (segments) $\gamma_0, \gamma_1, \ldots, \gamma_{k_1}, \ldots, \gamma_n, \ldots, \gamma_{k_m}, \ldots$, with each $k_j \leq K_d$ for a certain constant $K_d$, such that

- Each $\gamma_{k_j}$ is contained in $Q(n)$, whereas $\gamma_0$ is contained in $Q(0)$.
- For each $n, k$, there exists a generator $f_{i,j}$ of $N_{d-1}^*$ such that two consecutive points in $\gamma_{k}^n$ differ by the action of either $f_{i,j}$ or its inverse.
- There exists a constant $D > 0$ such that for each $n$, at least one of the $\gamma_{k}^n$ has no less than $4^{\frac{m}{d-1}}/D$ points.
- There exists a constant $B > 0$ such that for all $n, k$,

$$\sum_{(i_1, \ldots, i_{d-1}) \in \gamma_{k}^n} \ell(i_1, \ldots, i_{d-1})^\alpha \leq BL_n^\alpha. \quad (15)$$

Assuming this, we next explain how to complete the proof of Theorem C along the lines of the arguments given for Theorem B. (Showing the existence of the desired sequences of parallelepipeds and segments will be postponed to the next two sections.)

The concatenation of the segments above produces an infinite path $\gamma: \mathbb{N}_0 \to \mathbb{N}_0^d$, which we may assume to start at the origin. (Otherwise, we add an extra initial segment and we slightly change the constant $B$.) For each $m \geq 0$, we let $f_m$ be the element of the form $f_{i,j}^{m+1}$ that moves the $m^{th}$ point of $\gamma$ into the $(m+1)^{th}$ one, and we denote $h_m := f_m \circ \cdots \circ f_1$, with $h_0 := Id$. Moreover, for each $n \geq 1$, we denote by $N = N(n)$ the entry-time of $\gamma$ into $Q(n+1)$. Due to the asymptotics of the lengths of the sides of $Q(n)$, we have $n \sim \log(N)$. By (15) and Hölder’s inequality, for a certain constant $A' > 0$, we have

$$\sum_{m=0}^{N} |h_m(I)|^\alpha \leq B \sum_{m=0}^{n} \sum_{k=1}^{K_d} L_m^\alpha \leq B \sum_{k=1}^{K_d} \left( \sum_{m=0}^{n} L_m \right)^\alpha (n+1)^{1-\alpha} \leq BK_{d}(d+2)^\alpha(n+1)^{1-\alpha} \leq A'(\log(N))^{1-\alpha}.$$
where the factor \((d+2)\) comes from the multiplicity of the sequence \((Q(n))\).

Now, for every \(x \in I\), the equality \(g^k = h_m^{-1} \circ g^k \circ h_m\) yields

\[
Dg^k(x) = \frac{Dh_m(x)}{Dh_m(y)} \cdot Dg^k(h_m(x)),
\]

(16)

where \(y := y_k = g^k(x)\). Since \(y\) belongs to \(I\), letting \(C\) be a common upper bound for the \(\alpha_i\)-Hölder constants of \(\log(Df_{i,j}), \log(Dg_{i,j}^{-1})\), where \(i > j\), for each \(N'\) such that \(N(n-1) \leq N' \leq N(n) = N\), we have

\[
\left| \log \left( \frac{Dh_{N'}(x)}{Dh_{N'}(y)} \right) \right| = \left| \log \left( \frac{\prod_{m=1}^{N'} Df_{im}(h_{m-1}(x))}{\prod_{m=1}^{N'} Df_{im}(h_{m-1}(y))} \right) \right|
\]

\[
\leq \sum_{m=1}^{N'} \left| \log Df_{im}(h_{m-1}(x)) - \log Df_{im}(h_{m-1}(y)) \right|
\]

\[
\leq C \sum_{m=1}^{N'} |h_{m-1}(x) - h_{m-1}(y)|^\alpha
\]

\[
\leq C \sum_{m=0}^{N} |h_m(I)|^\alpha
\]

\[
\leq CA'(\log(N))^{1-\alpha}.
\]

Moreover, since at least one of the \(\gamma_k\)'s is assumed to have no less than \(4^{\frac{n}{4d-1}}\) points, we may choose such an \(N'\) so that \(|h_{N'}(I)| \leq D/4^{\frac{n}{4d-1}}\). Using Borichev’s theorem, this yields

\[
Dg^k(h_{N'}(x)) \leq \exp(3Cg|h_{N'}(I)|^\alpha k^{1-\alpha}) \leq \exp \left( \frac{3D^\alpha Cg k^{1-\alpha}}{4^{\frac{n}{4d-1}}} \right),
\]

which due to \([16]\) and the previous estimate implies

\[
Dg^k(x) \leq \exp(CA'(\log(N))^{1-\alpha}) \exp \left( \frac{3D^\alpha Cg k^{1-\alpha}}{4^{\frac{n}{4d-1}}} \right).
\]

Choosing \(n = n_k\) such that \(k^{1-\alpha} \sim 4^{\frac{n}{4d-1}}\), hence \(n \sim \log(N) \sim \log(k)\), this yields, for a certain constant \(A > 0\),

\[
\max_{x \in I} Dg^k(x) \leq \exp \left( A (\log(k))^{1-\alpha} \right).
\]

However, the last expression is of order \(o(k)\), which is impossible by the Polterovich-Sodin lemma.

### 3.2 The case \(d = 3\)

Again for pedagogical reasons, we first deal with the case \(d = 3\), though the reader should have no problem in passing directly to the general case treated in the next section. Notice that for \(d=3\), the critical value of \(\alpha\) is 1/3. In analogy to \([2.2]\) let us introduce some terminology.

An horizontal set in \(Q(2n+1)\) is a subset \(P = P_r\) of the form

\[
Q(2n+1) \cap \{(i,j): i \in [x_{1,2n+1}, y_{1,2n+1}], j \in [x_{2,2n+1} + (r-1)y_{1,2n+1}, x_{2,2n+1} + ry_{1,2n+1}]\},
\]

where \(r \in \{1, 2, \ldots, r_{2n+1}\}\), with \(r_{2n+1} \sim (y_{2,2n+1} - x_{2,2n+1})/y_{1,2n+1}\) chosen as the smallest possible index so that \(Q(2n+1)\) is the union of the \(P_r\)'s. Given \(\lambda \geq 1\), such a set is said to be \(\lambda\)-good whenever \(r < r_{2n+1}\) and

\[
\sum_{(i,j) \in P} \ell(i,j) \leq \frac{\lambda L_{2n+1}}{r_{2n+1}}.
\]

(17)
An horizontal segment in $Q(2n + 1)$ is a subset of the form $Q(2n + 1) \cap \{(i, j) : j = \text{const}\}$. Such a segment $\gamma$ will be said to be $\lambda$-good relatively to the horizontal set $P$ containing it whenever

$$
\sum_{(i, j) \in \gamma} \ell(i, j) \leq \frac{\lambda}{y_{1, 2n+1}} \sum_{(i, j) \in P} \ell(i, j).
$$

(18)

A vertical set in $Q(2n) \cap Q(2n + 1)$ is a set of type $P_{2n+1}^2(k) := Q(2n) \cap Q(2n + 1) \cap \{(i, j) : i = k\}$. A vertical segment in $Q(2n) \cap Q(2n + 1)$ is a set of the form $\gamma_{2n+1}^2(k, r) := P_r \cap P_{2n+1}(k)$. This segment is $\lambda$-good relatively to the vertical set $P_{2n+1}(k)$ in $Q(2n) \cap Q(2n + 1)$ containing it whenever

$$
\sum_{(i, j) \in \gamma_{2n+1}^2(k, r)} \ell(i, j) \leq \frac{\lambda}{r_{2n+1}} \sum_{(i, j) \in P_{2n+1}(k)} \ell(i, j).
$$

(19)

A vertical set in $Q(2n + 1) \cap Q(2n + 2)$ is a set $P_{2n+2}^2(k) := Q(2n + 1) \cap Q(2n + 2) \cap \{(i, j) : i = k\}$. A vertical segment in $Q(2n + 1) \cap Q(2n + 2)$ is a set of the form $\gamma_{2n+2}^2(k, r) := P_r \cap P_{2n+2}^2(k)$. This segment is $\lambda$-good relatively to the horizontal set $P_r$ in $Q(2n + 1)$ containing it whenever

$$
\sum_{(i, j) \in \gamma_{2n+2}^2(k, r)} \ell(i, j) \leq \frac{\lambda}{1 + y_{1, 2n+2} - x_{1, 2n+2}} \sum_{(i, j) \in P_r} \ell(i, j).
$$

(20)

Finally, a vertical set in $Q(2n + 2)$ is a set of type $P(k) := Q(2n + 2) \cap \{(i, j) : i = k\}$. Such a set $P$ is $\lambda$-good provided

$$
\sum_{(i, j) \in P} \ell(i, j) \leq \frac{\lambda L_{2n+2}}{1 + y_{1, 2n+2} - x_{1, 2n+2}}.
$$

(21)

Now, for each $k \in [x_{1, 2n+2}, y_{1, 2n+2}]$, we decompose $\{k\} \times [x_{2, 2n+2}, y_{2, 2n+2}] \sim [x_{2, 2n+2}, y_{2, 2n+2}]$ into $k$ paths, each of which has consecutive points at distance $k$. The resulting paths will be said to be vertical segments in $Q(2n + 2)$\(^1\). Such a vertical segment $\gamma$ is said to be $\lambda$-good relatively to the vertical set $P = P(i)$ in $Q(2n + 2)$ containing it if

$$
\sum_{(i, j) \in \gamma} \ell(i, j) \leq \frac{\lambda}{i} \sum_{(i, j) \in P} \ell(i, j).
$$

(22)

(Notice that vertical segments in $Q(2n + 2)$ naturally arise from the action of $f_{3, 2}$.)

Assume we are given a $\lambda$-good vertical set $P = P(k)$ in $Q(2n)$ and a 1-good vertical segment $\gamma_{2n}^1$ relatively to $P$. For at least a half of the $r \in \{1, 2, \ldots, r_{2n+1} - 1\}$, the vertical segment $\gamma_{2n+1}^2(k, r)$ is 2-good relatively to $P_{2n+1}(k)$. Similarly, at least a half of the horizontal sets in $Q(2n + 1)$ are 2-good. Consequently, there must be some $r \in \{1, 2, \ldots, r_{2n+1} - 1\}$ such that the corresponding vertical segment $\gamma_{2n+1}^2(k, r) \subset Q(2n) \cap Q(2n + 1)$ and horizontal set $P_r \subset Q(2n + 1)$ are 2-good. Let $\gamma_{2n+1}$ be a 1-good horizontal segment in $P_r$. The segments $\gamma_{2n}^1$ and $\gamma_{2n+1}$ do not necessarily intersect, but using the vertical segment $\gamma_{2n} := \gamma_{2n+1}^2(k, r)$, we can concatenate them.

Assume now that we are given $r$ such that $P_r$ is a $\lambda$-good horizontal set in $Q(2n + 1)$ together with a 1-good horizontal segment $\gamma_{2n+1}^1$ relatively to $P$. For more than a half of the $k' \in [x_{1, 2n+2}, y_{1, 2n+2}]$, the vertical segment $\gamma_{2n+2}^2(k', r)$ is 2-good relatively to $P_r$. Similarly, for more than a half of these $k'$, the vertical set $P(k') \subset Q(2n + 2)$ is 2-good. Take $k'$ lying simultaneously in both sets, and choose any vertical segment $\gamma_{2n+2}^1$ that is 1-good relatively to $P(k')$. Again, the segments $\gamma_{2n+1}^1$ and $\gamma_{2n+2}^1$ do not necessarily intersect, but using $\gamma_{2n+1} := \gamma_{2n+1}^2(k', r)$, we can concatenate them.

\(^1\)Rather surprisingly, there is no need of the intricate decomposition procedure of [3, §2.5] here.
Thus, starting with any vertical segment \( \gamma_0 \) that is 1-good relatively to a 1-good vertical set in \( Q_0 \), we can produce a concatenating sequence \( \gamma_0, \gamma_1, \gamma_1^1, \gamma_1^2, \gamma_2, \ldots \). We claim that this induces a good-enough sequence of segments in that they satisfy the properties of §3.1, thus concluding the proof of Theorem C for the case \( d = 3 \). Indeed, the first two properties are clear from the construction, whereas the third one is easily seen to hold for \( \gamma_0^1 \). To check the fourth property, that is, (15), we will use throughout the asymptotics of \( x_{i,n}, y_{i,n}, y_{i,n} - x_{i,n} \) (which are all of order \( 4^\frac{n}{n-1} \)). Recall also that \( \alpha = 1/3 \).

For \( \gamma_{2n}^1 \), using the adapted version of (21) and (22) together with Hölder’s inequality, we get

\[
\sum_{(i,j) \in \gamma_{2n}^1} \ell(i,j)^\alpha \leq \left( \frac{2}{i} \sum_{(i,j) \in P(k)} \ell(i,j) \right)^\alpha |\gamma_{2n}^1|^{1-\alpha}
\]

\[
\leq \left( \frac{2}{i} \frac{2L_{2n}}{1 + y_{1,2n} - x_{1,2n}} \right)^\alpha C \left( \frac{y_{2,2n} - x_{2,2n}}{i} \right)^{1-\alpha}
\]

\[
\leq \frac{C}{x_{1,2n} \left( y_{1,2n} - x_{1,2n} \right)^\alpha} \cdot \frac{L_{2n}^\alpha}{x_{1,2n} \left( y_{1,2n} - x_{1,2n} \right)^\alpha} \cdot L_{2n}^\alpha
\]

\[
\leq \frac{B^{4^{2n(1-\alpha)}} L_{2n}^\alpha}{4^{4\alpha} 4^{n\alpha}}
\]

\[
= \frac{B^{4^{n/3}} L_{2n}^\alpha}{4^{n\alpha/3}}
\]

\[
= BL_{2n}^\alpha.
\]
For $\gamma_{2n}^2$, using (19) and Hölder’s inequality, we get

$$\sum_{(i,j)\in\gamma_{2n}^2} \ell(i,j)^\alpha \leq \left( \frac{2}{r_{2n+1}} \sum_{(i,j)\in B_{2n+1}^2} \ell(i,j) \right)^\alpha |\gamma_{2n}^2|^{1-\alpha}$$

$$\leq \left( \frac{2C_{y1,2n+1}}{y_{2,2n+1} - x_{2,2n+1}} \cdot \frac{2L_{2n}}{1 + y_{1,2n} - x_{1,2n}} \right)^\alpha y_{1,2n+1}^{1-\alpha}$$

$$\leq C' \frac{4^n}{(4^{(2n+1)}4^n)^\alpha} \cdot L_{2n}^\alpha$$

$$\leq B \frac{4^n}{4^{(2n+1)}4^n} L_{2n}^\alpha$$

$$= BL_{2n}^\alpha.$$ 

For $\gamma_{2n+1}^1$, using the appropriate versions of (17) and (18), Hölder’s inequality yields

$$\sum_{(i,j)\in\gamma_{2n+1}^1} \ell(i,j)^\alpha \leq \left( \frac{2}{y_{1,2n+1}} \sum_{(i,j)\in P_r} \ell(i,j) \right)^\alpha |\gamma_{2n+1}^1|^{1-\alpha}$$

$$\leq \left( \frac{2}{y_{1,2n+1}} \cdot \frac{2L_{2n+1}}{r_{2n+1}} \right)^\alpha C (1 + y_{1,2n+1} - x_{1,2n+1})^{1-\alpha}$$

$$\leq C' \frac{(1 + y_{1,2n+1} - x_{1,2n+1})^{1-\alpha}}{y_{1,2n+1}^\alpha} \cdot L_{2n+1}^\alpha$$

$$\leq B \frac{4^n}{4^{(2n+1)}4^n} L_{2n+1}^\alpha$$

$$= BL_{2n+1}^\alpha.$$ 

Finally, for $\gamma_{2n+1}^2$, using (20) we obtain

$$\sum_{(i,j)\in\gamma_{2n+1}^2} \ell(i,j)^\alpha \leq \left( \frac{2}{1 + y_{1,2n+1} - x_{1,2n+1}} \sum_{(i,j)\in P_r} \ell(i,j) \right)^\alpha |\gamma_{2n+1}^2|^{1-\alpha}$$

$$\leq \left( \frac{C L_{2n+1}}{(1 + y_{1,2n+1} - x_{1,2n+1})r_{2n+1}} \right)^\alpha y_{1,2n+1}^{1-\alpha}$$

$$\leq C' \frac{y_{1,2n+1}^{1-\alpha}}{(1 + y_{2,2n+1} - x_{2,2n+1})^{\alpha} (1 + y_{1,2n+1} - x_{1,2n+1})^{\alpha}} \cdot L_{2n+1}^\alpha$$

$$\leq B \frac{4^n}{4^{(2n+1)}4^n} L_{2n+1}^\alpha$$

$$= BL_{2n+1}^\alpha.$$ 

### 3.3 The general case

As in the case of Theorem B, to prove Theorem C we will argue by induction.

**The vertical subdivision procedure.** Given $d \geq 3$, we let $Q := \prod_{k=1}^{d-1}[x_k, y_k]$ be a parallelepiped in $\mathbb{Z}^{d-1}$, where all $x_1, y_1, \ldots, x_{d-1}, y_{d-1}$ are integers. Given $A \geq 1$, we say that $Q$ is $A$-round if

$$\frac{(1 + y_1 - x_1)^i}{A} \leq x_i \leq y_i \leq A(1 + y_1 - x_1)^i, \quad \frac{(1 + y_1 - x_1)^i}{A} \leq 1 + y_i - x_i \leq A(1 + y_1 - x_1)^i. \quad (23)$$
By cutting along the last coordinate, every $A$-round parallelepiped $Q$ may be divided into disjoint parallelepipeds $Q_1, \ldots, Q_{M_1}$, each of which has $(d-1)^{th}$-side of length $y_{d-2} - 1$ possibly excepting the last one. By (23), we have

$$\frac{1 + y_1 - x_1}{A^2} \leq M_1 \leq 1 + \frac{1 + y_{d-1} - x_{d-1}}{y_{d-2} - 1} \leq 1 + A^2(1 + y_1 - x_1).$$

Similarly, each $Q_{m_1}$ satisfying $m_1 < M_1$ may be subdivided into disjoint parallelepipeds $Q_{m_1,1}, Q_{m_1,2}, \ldots, Q_{m_1,M_2}$, each of which has $(d-1)^{th}$-side of length $y_{d-3} - 1$ possibly excepting the last one. Again, (23) implies that

$$\frac{1 + y_1 - x_1}{A^2} \leq M_2 \leq 1 + A^2(1 + y_1 - x_1).$$

In general, for $k \leq d - 2$, each parallelepiped $Q_{m_1,\ldots,m_{k-1}}$ satisfying $m_j \neq M_j$ for all $j \leq k - 1$ may be divided into $Q_{m_1,\ldots,m_{k-1},1}, Q_{m_1,\ldots,m_{k-1},2}, \ldots, Q_{m_1,\ldots,m_{k-1},M_k}$, each of which has $(d-1)^{th}$-side of length $y_{d-k-2}$ possibly excepting the last one. Moreover, (23) implies that

$$\frac{1 + y_1 - x_1}{A^2} \leq M_k \leq 1 + A^2(1 + y_1 - x_1). \quad (24)$$

Here, for $k = 0$, we interpret $Q_{m_1,\ldots,m_k}$ as $Q$.

A level in $Q$ is a set of the form $H_i \cap Q$, where $H_i := \{(i_1, \ldots, i_{d-2}, i_{d-1}) \in \mathbb{Z}^{d-1} : i_{d-1} = i\}$. To each level there is a unique associated sequence

$$H_i \cap Q \subset Q_{m_1,\ldots,m_{d-2}} \subset Q_{m_1,\ldots,m_{d-3}} \subset \ldots \subset Q_{m_1} \subset Q. \quad (25)$$

Say that the level is admissible if each of the $m$'s above differs from the corresponding $M$. Using (23) and (24), one easily checks that for a certain constant $A' = A'(A, d)$, the proportion of non-admissible levels is no larger than

$$\frac{1}{(1 + y_{d-1} - x_{d-1})} \left[ (y_{d-2} - 1) + (y_{d-3} - 1)M_1 + (y_{d-4} - 1)M_1M_2 + \ldots \right] \leq \frac{A'}{1 + y_1 - x_1}.$$

A vertical section in $Q$ is a set of the form $V_{j_1,\ldots,j_{d-2}} \cap Q$, where $V_{j_1,\ldots,j_{d-2}} := \{(j_1, \ldots, j_{d-2}, i) : i \in \mathbb{Z}\}$.

**Very good points and levels.** Assume now we are given a positive function $\ell$ defined on $\mathbb{Z}^{d-1}$. For each parallelepiped $Q' \subset \mathbb{Z}^{d-1}$, denote

$$L_{Q'} := \sum_{(i_1, \ldots, i_{d-1}) \in Q'} \ell(i_1, \ldots, i_{d-1}),$$

and let $\langle \ell_{Q'} \rangle$ be the value above divided by the cardinality of $Q'$.

Given $\lambda \geq 1$, we say that a level $H_i \cap Q$ with associated sequence (25) is fully $\lambda$-good if for all $k \geq 1$,

$$\langle \ell_{Q_{m_1,\ldots,m_k}} \rangle \leq \lambda \langle \ell_{Q} \rangle.$$

Notice that the proportion of fully $\lambda$-good levels is larger than $(1 - \frac{d-1}{\lambda})$. Analogously, we say that the point $p := (j_1, \ldots, j_{d-2}, i_{d-1}) = V_{j_1,\ldots,j_{d-2}} \cap H_{i_{d-1}}$ is fully $\lambda$-good whenever for all $k \geq 1$,

$$\langle \ell_{Q_{m_1,\ldots,m_k} \cap V_{j_1,\ldots,j_{d-2}}} \rangle \leq \lambda \langle \ell_{Q} \rangle. \quad (26)$$

For each $\lambda' \geq 1$, the proportion of fully $\lambda'\lambda$-good points in any fully $\lambda$-good level is larger than $(1 - \frac{d-1}{\lambda'} - \frac{d-1}{\lambda' \lambda})$. 

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Reaching points from very good points along good vertical sections. A segment in $Q$ is a sequence of points for which there exists a generator $f_{i,j} \in N_d$ such that each point is obtained from the preceding one by the action of either $f_{i,j}$ or its inverse. Such a segment $\gamma$ is said to be horizontal if this generator is $f_{2,1}$ and $\gamma$ contains $(1 + y_1 - x_1)$ points. The segment is said to be vertical if the generator involved is one among $f_{d,1}, \ldots, f_{d,d-1}$ (with no hypothesis on the number of points).

Given $\lambda \geq 1$, say that a segment $\gamma$ in $Q$ is $\lambda$-good if

$$\langle \ell_\gamma \rangle \leq \lambda \langle \ell_Q \rangle.$$

Lemma 3.1. Given $0 < \kappa < 1$ and $\mu \geq 1$, there exist constants $\lambda = \lambda_1(\kappa, \mu, A, d)$ and $D' > 0$ such that the following holds: If $p := (j_1, \ldots, j_{d-2}, t) = V_{j_1, \ldots, j_{d-2}} \cap H_i$ is a fully $\mu$-good point in a $A$-round parallelepiped $Q$ such that the level $H_i \cap Q$ is admissible, then starting from $p$ one can reach a proportion at least $\kappa$ of the points in $V_{j_1, \ldots, j_{d-2}} \cap Q$ by concatenating no more than $d - 2$ vertical segments that are $\lambda$-good and have no more than $D'(1 + y_{1,n} - x_{1,n})$ points.

Proof. Starting from $p$ and using $f_{d,1}^{\pm 1}$, one can reach all points in $Q_{m_1, \ldots, m_{d-2}} \cap V_{j_1, \ldots, j_{d-2}}$ (via the segment $\gamma := Q_{m_1, \ldots, m_{d-2}} \cap V_{j_1, \ldots, j_{d-2}}$). Using (26) with $k = d - 2$ together with (23) and (24), we obtain

$$\sum_{(i_1, \ldots, i_{d-1}) \in \gamma} \ell(i_1, \ldots, i_{d-1}) = L_{Q_{m_1, \ldots, m_{d-2}} \cap V_{j_1, \ldots, j_{d-2}}} \leq \frac{\mu L_Q}{(M_{d-2} - 1) \cdots (M_1 - 1) \prod_{j=1}^{d-2} (1 + y_j - x_j)} \leq \frac{\mu 2^{d-2} A^{3d-6}}{(1 + y_1 - x_1)^{d-2} + \frac{|d-2|(d-1)}{2} L_Q} = \frac{\mu 2^{d-2} A^{3d-6}}{(1 + y_1 - x_1)^{\alpha(d-1)}} L_Q.$$

Hence,

$$\langle \ell_\gamma \rangle \leq \mu 2^{d-2} A^{d-4} \langle \ell_Q \rangle.$$ 

Now, the action of $f_{d,2}$ divides $Q_{m_1, \ldots, m_{d-3}} \cap V_{j_1, \ldots, j_{d-2}}$ into $j_1$ segments. Given $\lambda' \geq 1$, for a proportion larger than $(1 - \frac{1}{\lambda'})$ of these segments $\gamma$, we have

$$\sum_{(i_1, \ldots, i_{d-1}) \in \gamma} \ell(i_1, \ldots, i_{d-1}) \leq \frac{\lambda'}{j_1} L_{Q_{m_1, \ldots, m_{d-3}} \cap V_{j_1, \ldots, j_{d-2}}} \leq \frac{A\lambda'}{(1 + y_1 - x_1)^{d-2} + \frac{|d-2|(d-1)}{2} L_Q} \leq \frac{\mu 2^{d-2} A^{3d-7} \lambda'}{(1 + y_1 - x_1)^{\alpha(d-1)}} L_Q.$$

By concatenating these segments with the previous ones, we may reach from $p$ a proportion larger than $(1 - \frac{2}{\lambda'})$ of the points of $Q_{m_1, \ldots, m_{d-3}} \cap V_{j_1, \ldots, j_{d-2}}$.

Similarly, the action of $f_{d,3}$ divides $Q_{m_1, \ldots, m_{d-4}}$ into $j_2$ paths; from these, a proportion larger than $(1 - \frac{2}{\lambda'})$ is $\lambda''$-good for $\lambda'' := \mu 2^{d-4} A^{3d-8} \lambda'$. By concatenating these paths to the preceding ones, we may reach from $p$ a proportion larger than $(1 - \frac{2}{\lambda'})$ of the points in $Q_{m_1, \ldots, m_{d-4}} \cap V_{j_1, \ldots, j_{d-2}}$.

Continuing this procedure and choosing appropriately $\lambda'$, the concatenation property follows. Moreover, it is clear from the construction that the claim concerning the cardinality of each of the $\lambda$-good segments holds for a certain constant $D' = D'(A)$. We leave the details to the reader. □

2The extra factor 2 comes from that the segments in consideration may differ in number of points in $Q_{m_1, \ldots, m_{d-3}}$ by 1.
Concatenating sequences along finitely many parallelepipeds. We let $\mathcal{F}_d$ be the family of finite sequences $Q^1, \ldots, Q^{d-1}$ of parallelepipeds in $\mathbb{Z}^{d-1}$ such that if $Q^j = \prod_{i=1}^{d-1} [x_{i,j}, y_{i,j}]$, then

$$Q^{j+1} = \cdots \times [x', y_{j,j}] \times [x_{j+1,j}, y'] \times \cdots,$$

where $x' > x_{j,j}$, $y' > y_{j+1,j}$ (the dots mean that the corresponding entries remain untouched). Given $A \geq 1$, we denote by $\mathcal{F}_{d,A}$ the subfamily of all sequences made of $A$-round parallelepipeds.

Given $\mu \geq 1$, say that an horizontal segment $\gamma := \{ (i, j_2, \ldots, j_{d-1}) : i \in [x_{1,1}, y_{1,1}] \}$ in $Q^1$ is fully $\mu$-good with respect to $Q^1, \ldots, Q^{d-1}$ if the following properties hold:

- The level $H_{j_{d-1}} \cap Q^{d-1}$ is admissible and fully $\mu$-good in $Q^{d-1}$.
- The level $\{ (i_1, \ldots, i_{d-3}, j_{d-2}, j_{d-1}) : i_k \in \mathbb{Z} \} \cap Q^{d-2}$ is admissible and fully $\mu$-good in the parallelepiped $Q^{d-2} \cap \{ (i_1, \ldots, i_{d-3}, j_{d-2}, j_{d-1}) : i_k \in \mathbb{Z} \}$ (where the last intersection is understood as being contained in $\mathbb{Z}^{d-2} \sim \mathbb{Z}^{d-2} \times \{ j_{d-1} \}$).

Notice that for a certain constant $A'' = A''(A, d)$, in a proportion larger than $(1 - \frac{d-3}{\mu} \frac{A''}{1 + 2n - x_{1,1}})$, horizontal segments in $Q^1$ are fully $\lambda$-good.

Given a sequence $Q^1, \ldots, Q^{d-1}$ in $\mathcal{F}_d$, a concatenating sequence from $Q^1$ to $Q^{d-1}$ is a sequence of segments $\gamma^1, \ldots, \gamma^k$ such that:

- Each $\gamma^t$ is a segment in one of the $Q^j$'s.
- Each $\gamma^t$ intersects $\gamma^{t+1}$.
- The segment $\gamma^1$ is horizontal in $Q^1$, whereas $\gamma^k$ is vertical in $Q^{d-1}$.

We say that such a sequence is $\lambda$-good for $\lambda \geq 1$ if each of its segments is $\lambda$-good in one of the $Q^j$'s containing it.

**Lemma 3.2.** Given $A > 0$, $\mu \geq 1$, and $0 < \kappa < 1$, there exists $\lambda = \lambda_2(\kappa, \mu, A, d)$ such that the following holds: Let $Q^1, \ldots, Q^{d-1}$ be a sequence in $\mathcal{F}_d(A)$ and $\gamma := \{ (i, j_2, \ldots, j_{d-1}) : i \in [x_{1,1}, y_{1,1}] \}$ a fully $\mu$-good horizontal segment for this sequence. Then one can reach a proportion at least $\kappa$ of the points in $Q^{d-1}$ via a $\lambda$-good concatenating sequence from $Q^1$ to $Q^{d-1}$ that starts with $\gamma^1 := \gamma$ and is formed by no more than $K_d$ segments for a certain constant $K_d \geq 1$.

**Proof.** We proceed by induction. The argument for $d = 3$ is similar to that of the general case. It also corresponds to a more accurate quantitative version of that given in the previous section. For this reason, we leave it as an exercise to the reader.

Assume that the claim holds up to $d$, and let us consider the case of a sequence $Q^1, \ldots, Q^d$ in $\mathcal{F}_{d+1}(A)$. The inductive hypothesis applies to the sequence $Q^1 \cap H_{d+1}, \ldots, Q^{d-1} \cap H_{d+1}$, where each of these intersections is understood as a parallelepiped in $\mathbb{Z}^{d-1}$. Indeed, the definition above is made so that $\gamma$ is also fully $\mu$-good with respect to this sequence. Accordingly, if we fix $0 < \kappa' < 1$, then starting with $\gamma^1 := \gamma$ and using no more than $K_d$ segments that are $\lambda_2(\kappa', \mu, A, d)$-good, we may reach a proportion larger than $\kappa'$ of the points in $H_{d+1} \cap Q^d$. By (24), these correspond to a proportion larger than $\kappa' + 1/\lambda - 1$ of the points in $H_{d+1} \cap Q^d$. This last level is fully $\mu$-good in $Q^d$, hence in a proportion larger than $(1 - (d - 2)(1 - \kappa'))$, its points are fully $\mu/(1 - \kappa')$-good. By Lemma 3.1, every such a point may reach a proportion at least $\kappa'$ of the points in its vertical set in $Q^d$ by concatenating no more than $d$ vertical segments that are $\lambda_1(\kappa, \mu/(1 - \kappa'), A, d+1)$-good. Therefore,
the concatenation of these two sequences of segments allows reaching a proportion of points in \( Q^d \) larger than
\[
1 - [(1 - (\kappa' + 1/2^d - 1)) + (1 - \kappa')] = 2\kappa' + 1/2^d - 2.
\]
Choosing \( \kappa' \) appropriately (very close to 1), this allows concluding the proof for \( K_{d+1} := K_d + d \) and
\[
\lambda_2(\kappa, \mu, A, d + 1) := \text{max}\{\lambda_2(\kappa', \mu, A, d), \lambda_1(\kappa, \mu/(1 - \kappa'), A, d + 1)\}.
\]

**Proof of Theorem C.** We come back to the sequence of parallelepipeds \((Q(n))\) introduced at the beginning of \([3.1]\). By the asymptotics of the lengths of the their sides, there exists a constant \( A = A_d \) such that the following holds: For each \( l \geq 0 \), the finite sequence \( Q^l(d-1)+1, Q^l(d-1)+2, \ldots, Q^l(d-1)+d-1 \) belongs to the family \( F_d(A) \).

Fix \( \mu \geq 1 \) such that for all \( n \) bigger than or equal to a certain fixed \( N_0 \),
\[
1 - \frac{d - 3}{\mu} - \frac{A''(A, d)}{1 + y_{1,n(d-1)} - x_{1,n(d-1)}+1} > \frac{1}{2}.
\]
Then more than a half of the horizontal segments of \( Q(n(d-1) + 1) \) are fully \( \mu \)-good. We may hence fix \( \lambda' \geq 1 \) so that horizontal segments are not only fully \( \mu \)-good but also \( \lambda' \)-good in \( Q(n(d-1) + 1) \) in a proportion larger than \( 1/2 \). By the preceding lemma, starting with any fully \( \mu \)-good horizontal segment in \( Q((d-1)N_0 + 1) \), we may find an infinite concatenating sequence of \( \lambda \)-good segments for
\[
\lambda := \text{max}\{\lambda', \lambda_2(1/2, \mu, A_d, d)\}.
\]
Moreover, according to Lemma \([3.1]\) each of these segments contained in \( Q(n) \) have no more than \( D'(1 + y_{1,n} - x_{1,n}) \) points.

Modulo slightly changing the constant \( \lambda \) above, we may actually assume that the sequence begins at \( Q(0) \). We claim that the sequence that remains after cutting along concatenation points satisfies the properties from \([3.1]\). The first two ones are obvious, while the third one easily follows from that the segments lying in a parallelepiped \( Q(n) \) must communicate between \( Q(n-1) \) and \( Q(n+1) \), and these two last parallelepipeds are at distance which is comparable with the length of one of the sides of \( Q(n) \). To conclude the proof, we need to check the appropriate version of \([15]\). To do this, just notice that if \( \gamma \) is \( \lambda \)-good in \( Q(n) \) and contains at least \( 4^{\frac{\lambda}{D'}} \) points, then by Hölder’s inequality,
\[
\sum_{(i_1, \ldots, i_{d-1}) \in \gamma} \ell(i_1, \ldots, i_{d-1})^\alpha \leq \left( \sum_{(i_1, \ldots, i_{d-1}) \in \gamma} \ell(i_1, \ldots, i_{d-1}) \right)^\alpha |\gamma|^{1-\alpha}
\]
\[
\leq \left( \frac{\lambda L_Q}{(1 + y_{1,n} - x_{1,n})^{1-\frac{1}{\alpha}}} \right)^\alpha (D'(1 + y_{1,n} - x_{1,n}))^{1-\alpha}
\]
\[
= BL_Q^\alpha,
\]
where the last equality defines \( B \).

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