A new localization mechanism and Hodge duality for $q$–form field

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Abstract

In this paper, through a general Kaluza-Klein decomposition we investigate a new localization mechanism for a massless $q$–form field on the $p$–brane world with codimension one. We obtain two Schrödinger-like equations for the Kaluza-Klein (KK) modes, from which we can find the mass spectra of the KK modes and analyze their characters. It is found that there are two types of massless KK modes, a massless $q$–form KK mode and a massless $(q-1)$–form one, which cannot be localized on the brane at the same time. Because of this the Hodge duality on the brane can be naturally satisfied, which indicates a duality between a localized massless $q$–form mode and a $(p-q-1)$–form one. While if there exist some bound massive KK modes, they will couple with a massless $(q-1)$–form mode with different coupling constants, and so does the bulk $(p-q)$–form field. In this case the effective $q$–form field is still dual to the $(p-q)$–form one, just as in the bulk, although the mass spectra of the KK modes for both fields are not the same.

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I. INTRODUCTION

When the Arkani-Hamed-Dimopoulos-Dvali (ADD) [1] and Randall-Sundrum (RS) [2, 3] brane-world models were brought up, they opened a new avenue to solve the long-standing hierarchy problem and the cosmology problem [4–11]. And since then the brane-world and extra dimension theories have received more and more attention [10–27].

In the brane-world theory, one of the most important and interesting work is to investigate the Kaluza-Klein (KK) modes of various fields [28–48], which are the codes of the extra dimensions. In this work we are interested in the KK modes of a higher dimensional massless $q$–form field on the $p$–brane world with codimension one. As it is known the $0$–form and $1$–form fields are the scalar and vector fields, respectively, and the usual $2$–form field is the Kalb-Ramond field, which is used to describe the torsion of the space-time in the Einstein-Cartan theory. While the higher-form fields are new types of particles in a higher-dimensional space-time, which are useful for some unknown problems such as the cosmological constant problem or dark energy problem [7, 49].

To investigate the KK modes of a higher-dimensional field, we should propose a localization mechanism for the field. There are some work about the localization of $q$–form field [34, 50–63], where the authors usually first chose a gauge to make the localization mechanism be simpler. But these gauge choices make us only see parts of the whole localization information. In this paper, we will try to find a new localization mechanism for the $q$–form field without any gauge choice in order to give a whole view of the field’s localization.

To this end, we first give a general KK decomposition for a higher-dimensional $q$–form field $X_{M_1M_2\cdots M_q}$ without any gauge choice:

\begin{align}
X_{\mu_1\mu_2\cdots\mu_q}(x, z) &= \sum_n \hat{X}^{(n)}_{\mu_1\mu_2\cdots\mu_q}(x^\mu) U_1^{(n)}(z) e^{a_1 A(z)}, \\
X_{\mu_1\mu_2\cdots\mu_{q-1}z}(x, z) &= \sum_n \hat{X}^{(n)}_{\mu_1\mu_2\cdots\mu_{q-1}}(x^\mu) U_2^{(n)}(z) e^{a_2 A(z)},
\end{align}

where the $\hat{\cdot}$ denotes the effective quantities on the brane, the index $n$ marks different KK modes, $U_i(z)$ are only the function of the extra dimension coordinate $z$, $a_i$ are constants, and $A(z)$ will be explained latter. Here we have classified the higher-dimensional $q$–form field into two types, i.e., $X_{\mu_1\mu_2\cdots\mu_{q-1}z}$ containing the index of $z$ and $X_{\mu_1\mu_2\cdots\mu_q}$ not containing, because they have different effective fields on the brane. The effective fields on the brane for $X_{\mu_1\mu_2\cdots\mu_{q-1}z}$ and $X_{\mu_1\mu_2\cdots\mu_q}$ are the $(q - 1)$–form and $q$–form fields, respectively.
Then through the dimensional reduction we will get the effective action for the \( q \)-form field, the orthonormality conditions for the KK modes, and the equations of motion of \( U_1^{(n)}(z) \) and \( U_2^{(n)}(z) \), which are found to be two Schrödinger-like equations. With the orthonormality conditions and the two Schrödinger-like equations we can get the mass spectra of the KK modes and analyze their characters in any \( p \)-brane world model, where the line element of the space-time is assumed as a RS-like one

\[
ds^2 = e^{2A(z)} (\hat{g}_{\mu \nu}(x^\lambda) dx^\mu \ dx^\nu + dz^2),
\]

where \( A(z) \) is the warp factor that is only the function of \( z \), and \( \hat{g}_{\mu \nu}(x^\lambda) \) is the induced metric on the brane.

It will be finally found that for any \( q \)-form field there are two types of massless KK modes, a \( q \)-form mode and a \((q-1)\)-form mode, which insures the satisfactory of the Hodge duality on the brane. What is interesting is that if the \( q \)-form field has a localized \( q \)-form zero mode, then its dual \((p - q)\)-form field (the duality is built through the Hodge duality in the bulk) must have a massless \((p - q - 1)\)-form one. This means that in a 3-brane, we can not know where a localized 1-form field is from. It may be from a bulk 1-form one or a 2-form one. In fact, for any effective \( q \)-form field dual to itself on the brane, it may be from a bulk \( q \)-form or \((p - q)\)-form one.

It will also be seen that if there exist bound massive KK modes for the \( q \)-form field, they will couple with a massless \((q-1)\)-form mode. For example, if there are bound massive KK modes on a 3-brane for a bulk 1-form field, they will couple with a massless 0-form mode. Remarkably, we will find that the effective \( q \)-form field is still dual to the \((p - q)\)-form one on the brane, just as in the bulk, even though the mass spectra for the two fields are different.

This paper is organized as follows. We first investigate the new localization mechanism in Sec. II and then discuss the massless and bound massive KK modes respectively in subsections II A and II B. Finally, we give a brief conclusion in Sec. III.

II. A NEW LOCALIZATION MECHANISM AND HODGE DUALITY

In a brane-world background, there are usually three steps to investigate the localization of a higher-dimensional \( q \)-form field:
• Firstly, do a preparation, i.e., choose a simple gauge for the field, such as

\[ X_{\mu_1 \mu_2 \cdots \mu_{q-1}}(x^\mu, z) = \partial_{\mu_1} X^{\mu_1 \mu_2 \cdots \mu_q}(x^\mu, z) = 0, \quad (3) \]

and make a KK decomposition for other components of the field:

\[ X_{\mu_1 \mu_2 \cdots \mu_q}(x^\mu, z) = \sum_n \hat{X}^{(n)}_{\mu_1 \mu_2 \cdots \mu_q}(x^\mu) \, U_1^{(n)}(z) \, e^{a_1 A}. \quad (4) \]

• Secondly, substitute the KK decomposition into the equations of motion for the \( q \)-form field:

\[ \partial_{\mu_1} (\sqrt{-g} \, Y_{\mu_1 \mu_2 \cdots \mu_{q+1}}) + \frac{1}{q+1} \partial_z (\sqrt{-g} \, Y_{z \mu_2 \cdots \mu_{q+1}}) = 0, \quad (5a) \]

\[ \partial_{\mu_1} (\sqrt{-g} \, Y_{\mu_1 \cdots \mu_q z}) = 0, \quad (5b) \]

then a Schrödinger-like equation for the KK modes \( U_1^{(n)}(z) \) can be obtained, where the gauge choice \((3)\) has been considered. And at the same time, use the KK decomposition to reduce the action of the bulk field into the effective one, from which the orthonormality conditions for the KK modes will be found.

• Lastly, with a background solution, by solving the equation of \( U_1^{(n)}(z) \), the localization condition and the mass spectrum of the KK modes are obtained, and their characters can also be analyzed.

The reason for choosing the gauge \((3)\) is to obtain a simpler equation of \( U_1^{(n)}(z) \) from \((5a)\), which is used to find the mass spectrum and the wave functions of the KK modes satisfying the orthonormality conditions.

If we substitute the general KK decompositions \((1)\) into the field equations \((3)\), we will get complex field equations for the KK modes \( U_1^{(n)}(z) \) and \( U_2^{(n)}(z) \). However, we indeed need these equations to discuss the localization of the \( q \)-form field.

In order to investigate this problem, we will compare the equations of motion for the KK modes \( U_1^{(n)}(z) \) and \( U_2^{(n)}(z) \) derived from two ways. One is from the effective action, which is obtained by KK reduction from the fundamental action for the \( q \)-form field. Another is from \((5)\) as well as the KK decomposition \((1)\). Let us show the details.
We first would like to get the effective action for the $q$–form field. With the KK decomposition, the corresponding components of the field strength become

\[ Y_{\mu_1\mu_2\cdots\mu_{q+1}} = \sum_n \hat{Y}^{\mu_1\mu_2\cdots\mu_{q+1}}(x^\mu) U_1^{(n)}(z) e^{(a_1-2(q+1)) A}, \tag{6} \]

\[ Y^{\mu_1\mu_2\cdots\mu_q x} = \frac{q}{q+1} \sum_n \hat{Y}^{\mu_1\mu_2\cdots\mu_q}(x^\mu) U_2^{(n)}(z) e^{(a_2-2(q+1)) A} \]

\[ + \frac{1}{q+1} \sum_n \hat{X}^{\mu_1\mu_2\cdots\mu_q}(x^\mu) \partial_z \left( U_1^{(n)}(z) e^{a_1 A} \right) e^{-2(q+1) A}, \tag{7} \]

where the indices of the quantities with $\hat{\cdot}$ are raised or lowered by the four-dimensional metric $\hat{g}^{\mu\nu}(x)$ or $\hat{g}_{\mu\nu}(x)$. Substituting the above expressions into the action for the $q$–form field, we have:

\[ S = \int d^D x \sqrt{-g} Y^{M_1M_2\cdots M_{q+1}} Y_{M_1M_2\cdots M_q}, \]

\[ = \int d^D x \sqrt{-g} \left( Y^{\mu_1\mu_2\cdots\mu_q} Y_{\mu_1\mu_2\cdots\mu_q} + Y^{\mu_1\mu_2\cdots\mu_q z} Y_{\mu_1\mu_2\cdots\mu_q z} \right), \]

\[ = \sum_n \sum_{n'} \left[ I_{nn'}^{(1)} \int d^{p+1} x \sqrt{-\hat{g}} \hat{Y}^{\mu_1\mu_2\cdots\mu_q+1} \hat{Y}^{(n')}_{\mu_1\mu_2\cdots\mu_q+1} \right] + I_{nn'}^{(2)} \int d^{p+1} x \sqrt{-\hat{g}} \hat{Y}^{\mu_1\mu_2\cdots\mu_q} \hat{Y}^{(n')}_{\mu_1\mu_2\cdots\mu_q} \]

\[ + I_{nn'}^{(3)} \int d^{p+1} x \sqrt{-\hat{g}} \hat{X}^{\mu_1\mu_2\cdots\mu_q} \hat{X}^{(n')}_{\mu_1\mu_2\cdots\mu_q} \]

\[ + 2I_{nn'}^{(4)} \int d^{p+1} x \sqrt{-\hat{g}} \hat{X}^{\mu_1\mu_2\cdots\mu_q} \hat{X}^{(n')}_{\mu_1\mu_2\cdots\mu_q} \right], \tag{8} \]

where we have let $a_1 = a_2 = (2q-p)/2$, and supposed that $U_1^{(n)}(z)$ and $U_2^{(n)}(z)$ satisfy the following orthonormality conditions:

\[ I_{nn'}^{(1)} \equiv \int dz \ U_1^{(n)} U_1^{(n')} = \delta_{nn'}, \tag{9a} \]

\[ I_{nn'}^{(2)} = \frac{q^2}{(q+1)^2} \int dz \ U_2^{(n)} U_2^{(n')} = \delta_{nn'}, \tag{9b} \]

such that the effective four-dimensional fields $\hat{X}^{(n)}_{\mu_1\mu_2\cdots\mu_q}(x^\mu)$ and $\hat{X}^{(n')}_{\mu_1\mu_2\cdots\mu_{q-1}}(x^\mu)$ have canonical kinetic terms, and $I_{nn'}^{(3)}$ and $I_{nn'}^{(4)}$ are given by

\[ I_{nn'}^{(3)} \equiv \frac{1}{(q+1)^2} \int dz \ e^{(p-2q) A} \partial_z (U_1^{(n)} e^{a_1 A}) \partial_z (U_1^{(n')} e^{a_1 A}), \tag{10} \]

\[ I_{nn'}^{(4)} \equiv \frac{q}{(q+1)^2} \int dz \ e^{(p-2q) A/2} U_2^{(n)} \partial_z (U_1^{(n')} e^{a_1 A}). \tag{11} \]
For the effective action (8), it is necessary to analyze the mass dimensions of the constants $I_{nn'}^{(3)}$ and $I_{nn'}^{(4)}$ in the natural units with $\hbar = c = 1$. From the following result

$$
\begin{align*}
[Y_{M_1M_2\cdots M_{q+1}}] &= [M^{(p+2)/2}] = (p + 2)/2, \\
[\hat{Y}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)}] &= [\hat{Y}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)}] = (p + 1)/2, \\
[\hat{X}_{\mu_1\mu_2\cdots \mu_{q}}^{(n)}] &= [\hat{X}_{\mu_1\mu_2\cdots \mu_{q-1}}^{(n)}] = (p - 1)/2, \\
[U_1^{(n)}] &= [U_2^{(n)}] = 1/2,
\end{align*}
$$

we have

$$
[I_{nn'}^{(3)}] = 2, \quad [I_{nn'}^{(4)}] = 1.
$$

Further from the action (8), the equations of motion for the effective fields can be obtained as

$$
\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left( \sqrt{-\hat{g}} \hat{Y}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)} \right) - \sum_n \left( I_{nn'}^{(3)} \hat{X}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)} + I_{nn'}^{(4)} \hat{Y}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)} \right) = 0,
$$

and

$$
\partial_{\mu_1} \left( \sqrt{-\hat{g}} \hat{Y}_{\mu_1\mu_2\cdots \mu_{q}}^{(n)} \right) + \sum_n I_{nn'}^{(4)} \partial_{\mu_1} \left( \sqrt{-\hat{g}} \hat{X}_{\mu_1\mu_2\cdots \mu_{q}}^{(n)} \right) = 0.
$$

On the other hand, substituting the KK decomposition (11) into eqs. (5a) and (5b), we get

$$
\frac{1}{\sqrt{-\hat{g}}} \partial_{\mu_1} \left( \sqrt{-\hat{g}} \hat{Y}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)} \right) + \lambda_1 \hat{X}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)} + \lambda_2 \hat{Y}_{\mu_1\mu_2\cdots \mu_{q+1}}^{(n)} = 0,
$$

and

$$
\partial_{\mu_1} \left( \sqrt{-\hat{g}} \hat{Y}_{\mu_1\mu_2\cdots \mu_{q}}^{(n)} \right) + \lambda_3 \partial_{\mu_1} \left( \sqrt{-\hat{g}} \hat{X}_{\mu_1\mu_2\cdots \mu_{q}}^{(n)} \right) = 0,
$$

where

$$
\begin{align*}
\lambda_1 &= \frac{e^{-(a_1+p-2q)A}}{(q + 1)^2 U_1^{(n)}} \partial_z \left( e^{(p-2q)A} \partial_z \left( U_1^{(n)} e^{a_1 A} \right) \right), \\
\lambda_2 &= \frac{q e^{-(a_1+p-2q)A}}{(q + 1)^2 U_1^{(n)}} \partial_z \left( U_2^{(n)} e^{(a_2+p-2q)A} \right), \\
\lambda_3 &= \frac{\partial_z \left( U_1^{(n)} e^{a_1 A} \right)}{q U_2^{(n)} e^{a_2 A}}.
\end{align*}
$$
It is clear that eqs. (14), (15) and (16), (17) must be consistent with each other, which results in that

\[ I^{(3)}_{nn'} = \frac{m_n^2}{(q+1)^2} I^{(1)}_{nn'} = \frac{m_n^2}{(q+1)^2} \delta_{nn'}, \]  
(19)

\[ I^{(4)}_{nn'} = \bar{m}_n \delta_{nn'}, \]  
(20)

and

\[ \lambda_1 = -\frac{m_n^2}{(q+1)^2}, \]  
(21a)
\[ \lambda_2 = -\bar{m}_n, \]  
(21b)
\[ \lambda_3 = \bar{m}_n, \]  
(21c)

where \([m_n] = [\bar{m}_n] = 1\).

The above three equations are interesting. First, eq. (21a) is in fact a Schrödinger-like equation of \(U^{(n)}_1\). And with eq. (32) the expressions (9b) and (11) are found to have the following relationship:

\[ I^{(2)}_{nn'} = \frac{1}{m_n^2 (q+1)^2} \delta_{nn'} = \delta_{nn'}, \]  
(22)
\[ I^{(4)}_{nn'} = \frac{1}{\bar{m}_n (q+1)^2} \delta_{nn'}. \]  
(23)

The consistency of \(I^{(4)}_{nn'}\) in eqs. (20) and (23) results in

\[ \bar{m}_n = \frac{m_n}{q+1} \quad \text{or} \quad I^{(4)}_{nn'} = \frac{m_n}{q+1} \delta_{nn'}. \]  
(24)

Then eqs. (18a)-(18c) and (21a)-(32) are equivalent to the following coupled equations of \(U^{(n)}_1\) and \(U^{(n)}_2\):

\[ \partial_z U^{(n)}_2(z) + \frac{p - 2q}{2} A'(z) U^{(n)}_2(z) = -\frac{q+1}{q} m_n U^{(n)}_1(z), \]  
(25)
\[ \partial_z U^{(n)}_1(z) - \frac{p - 2q}{2} A'(z) U^{(n)}_1(z) = +\frac{q+1}{q} m_n U^{(n)}_2(z), \]  
(26)

which can also be written as a Schrödinger-like equation for each mode:

\[ [ - \partial_z^2 + V_1^q(z)] U^{(n)}_1(z) = m_n^2 U^{(n)}_1(z), \]  
(27a)
\[ [ - \partial_z^2 + V_2^q(z)] U^{(n)}_2(z) = m_n^2 U^{(n)}_2(z). \]  
(27b)
where the effective potentials are given by

\begin{align}
V_1^q(z) &= \frac{(p-2q)^2}{4} A^2(z) + \frac{p-2q}{2} A''(z), \quad (28) \\
V_2^q(z) &= \frac{(p-2q)^2}{4} A^2(z) - \frac{p-2q}{2} A''(z). \quad (29)
\end{align}

It is worth to note that the above two equations (27a) and (27b) can be rewritten alternatively as

\begin{align}
\mathcal{Q} \mathcal{Q}^\dagger \ U_1^{(n)}(z) &= m_n^2 U_1^{(n)}(z), \quad (30) \\
\mathcal{Q}^\dagger \mathcal{Q} \ U_2^{(n)}(z) &= m_n^2 U_2^{(n)}(z), \quad (31)
\end{align}

with the operator \( \mathcal{Q} \) given by \( \mathcal{Q} = \partial_z + \frac{p-2q}{2} A'(z) \). So we have the following conclusions: (1) There is no eigenstate with negative eigenvalue, namely, we always have \( m_n^2 \geq 0 \). (2) There is only one zero mode with \( m_0 = 0 \), \( U_1^{(0)} \) or \( U_2^{(0)} \), can survive with the boundary condition \( U_{1,2}^{(0)}(|z| \to \infty) \to 0 \). (3) The two base functions \( U_1^{(n)} \) and \( U_2^{(n)} \) share the same mass spectrum except for \( m_0 = 0 \).

Now it is clearly that by solving the two Schrödinger-like equations (27a) and (27b) with the orthonormality conditions (9a) and (9b), we can find the mass spectrum and the KK modes that localized on the brane.

We usually classify the KK modes into massless and massive ones, as the former is regarded as the field has been on the brane, and the latter are carrying the information of extra dimensions, which can be distinguished from the ones that have been on the brane. For a realistic brane world, these two types KK modes are expected to be localized on the brane. For the massless mode, its analytical wave function can be easily got, so we can check whether it can be localized on the brane through the normalization condition. While for the massive KK modes, we usually have to use numerical method to solve them from the Schrödinger-like equations. There may also exist bound massive KK modes. In the following we will discuss the massless and massive ones separately.
A. Massless KK modes

For the massless KK modes $U^{(0)}_{1,2}$ with $m_0 = 0$, the solutions can be obtained from eqs. (27a) and (27b) or eqs. (25) and (26):

\[ U^{(0)}_1(z) = N_1 e^{+ (p-2q)A/2}, \]
\[ U^{(0)}_2(z) = N_2 e^{- (p-2q)A/2}, \]

where $N_1$ and $N_2$ are the normalization constants. Their effective action reads

\[ S_{\text{eff}}^{(0)} = \int d^{p+1}x \sqrt{-g} \left( I^{(1)}_{00} \hat{Y}^{\mu_1 \mu_2 \cdots \mu_{q+1}}(x^\mu) N_1, \right. \]
\[ \left. + I^{(2)}_{00} \hat{Y}^{\mu_1 \mu_2 \cdots \mu_{q}}(x^\mu) \frac{q}{q+1} N_2 \right) e^{(2q-p)A(z)}. \]

where

\[ I^{(1)}_{00} = N_1^2 \int dz \ e^{(p-2q)A(z)}, \]
\[ I^{(2)}_{00} = N_2^2 \int dz \ e^{-(p-2q)A(z)}. \]

It can be seen that $I^{(1)}_{00}$ and $I^{(2)}_{00}$ cannot be finite at the same time for a brane with infinite extra dimension.

Then with (32) and (33), for the bulk $q$–form and its dual fields, their KK decompositions now are:

\[ Y_{\mu_1 \mu_2 \cdots \mu_{q+1}}(x^\mu, z) = \hat{Y}^{\mu_1 \mu_2 \cdots \mu_{q+1}}(x^\mu) N_1, \]
\[ Y_{\mu_1 \mu_2 \cdots \mu_q}(x^\mu, z) = \hat{Y}^{\mu_1 \mu_2 \cdots \mu_q}(x^\mu) \frac{q}{q+1} N_2 \ e^{(2q-p)A(z)}, \]

where we have supposed that there is only the zero mode that is localized on the brane. Then substituting the above decompositions (37) into the below bulk Hodge duality

\[ \sqrt{-g} \hat{Y}^{\mu_1 \mu_2 \cdots \mu_{p-q}} = \epsilon^{\mu_1 \mu_2 \cdots \mu_{p-q} \nu_1 \nu_2 \cdots \nu_{q+1}} Y_{\nu_1 \nu_2 \cdots \nu_{q+1}}, \]
\[ \sqrt{-g} \hat{Y}^{\mu_1 \mu_2 \cdots \mu_p \nu_{q+1}} = \epsilon^{\mu_1 \mu_2 \cdots \mu_{p+1} \nu_1 \nu_2 \cdots \nu_{q}} Y_{\nu_1 \nu_2 \cdots \nu_{q}}, \]

the Hodge duality on the brane is naturally satisfied:

\[ \sqrt{-g} \hat{Y}^{\mu_1 \mu_2 \cdots \mu_{p-q}}(x^\mu) = \epsilon^{\mu_1 \mu_2 \cdots \mu_{p-q} \nu_1 \nu_2 \cdots \nu_{q+1}} Y_{\nu_1 \nu_2 \cdots \nu_{q+1}}(x^\mu), \]
where we have assumed that

\[ N_1 = \frac{p - q}{p - q + 1} \tilde{N}_2, \quad N_2 = \frac{q + 1}{q} \tilde{N}_1. \]  

(40)

And from (39) we see that there is a duality between a \( q \)-form zero mode and \( (p-q-1) \)-form one.

From the discussion about the localization of the zero mode for a \( q \)-form field, we see that there is also a \( (p-q) \)-form or \( (p-q-1) \)-form zero mode for a higher-dimensional \( (p-q) \)-form field. It is interesting to note that for a \( q \)-form field and its dual \( (p-q) \)-form field, with eq. (40), there are some relationships between the normalization constants:

\[
\tilde{I}^{p-q(1)}_{00} = \tilde{N}_1^2 \int dz \ e^{(p-2(p-q))A} = \left( \frac{q}{q+1} \right)^2 \tilde{I}^{q(2)}_{00},
\]

(41)

\[
\tilde{I}^{p-q(2)}_{00} = \tilde{N}_2^2 \int dz \ e^{(2(p-q)-p)A} = \left( \frac{p - q + 1}{p - q} \right)^2 \tilde{I}^{q(1)}_{00},
\]

(42)

where \( \tilde{I}^{p-q(1)}_{00} \) and \( \tilde{I}^{p-q(2)}_{00} \) are the normalization constants appearing in the effective action of the \( (p-q) \)-form field. It is clear that if there is a localized \( q \)-form ( or \( (q-1) \)-form ) zero mode for a bulk \( q \)-form field, there must be a localized \( (p-q-1) \)-form ( or \( (p-q) \)-form ) zero mode for its dual field. And this just satisfies the requirement of the Hodge duality on the brane.

We have known that on some 3-brane models there is a localized 1-form zero mode [58] from a bulk 1-form field, but now it is seen that the localized 1-form also may be from a bulk 2-form one. In fact, for any effective \( q \)-form field dual to itself on the brane, we can not sure it is from a bulk \( q \)-form or \( (p-q) \)-form field.

**B. Bound massive KK modes**

Furthermore, for some brane backgrounds there may be bound massive KK modes except for the localized zero mode. In this case we are wondering if we do a KK reduction for the bulk field and keep the Hodge duality on the brane, what will happen for the bound massive KK modes for the \( q \)-form and its dual fields. Let us consider this issue in the following discussion.

For the bound massive KK modes, as the two Schrödinger-like equations (27a) and (27b) are not independent to each other, we can get the mass spectra from anyone of them with the
corresponding orthonormality condition (9a) or (9b). The effective action for these \( n \)-level bound KK modes can be written as

\[
S_{\text{massive}}^q = \int d^{p+1}x \sqrt{-g} \left[ \hat{Y}^{\mu_1 \mu_2 \ldots \mu_{q+1}}_{(n)} \hat{Y}^{(n)}_{\mu_1 \mu_2 \ldots \mu_q} + \frac{m_n^2}{(q + 1)^2} \hat{X}^{\mu_1 \mu_2 \ldots \mu_q}_{(n)} \hat{X}^{(n)}_{\mu_1 \mu_2 \ldots \mu_q} 
+ \hat{Y}^{\mu_1 \mu_2 \ldots \mu_q}_{(n)} \hat{Y}^{(n)}_{\mu_1 \mu_2 \ldots \mu_q} + \frac{2m_n}{q + 1} \hat{X}^{\mu_1 \mu_2 \ldots \mu_q}_{(n)} \hat{X}^{(n)}_{\mu_1 \mu_2 \ldots \mu_q} \right].
\] (43)

Different with the zero modes, which have two types \((q-1)\)-form and \((q-1)\)-form fields, the bound massive KK modes are all four-dimensional \( q \)-form fields with mass \( m_n \), and each \( n \)-level bound massive \( q \)-form KK mode couples with the \( n \)-level massless \((q-1)\)-form mode with the coupling constant \( \frac{m_n}{q + 1} \).

Because the effective potentials of the \( q \)-form and its dual \((p - q)\)-form fields have the following relationships:

\[
V_1^{p-q}(z) = V_2^{q}(z), \quad V_2^{p-q}(z) = V_1^{q}(z),
\] (44)

there will also exist bound massive KK modes for the \((p - q)\)-form field with mass \( \frac{m_n}{p - (p+1)} \).

Although the mass spectra for \( q \)- and \((p - q)\)-form fields are not the same, they are in fact dual to each other, which can be discovered through the KK reduction for the Hodge duality in the bulk.

As now the bulk fields are considered as the sum of a localized zero mode and a series of bound massive KK modes, the KK decompositions of the field strengths for the bulk \( q \)-form
and its dual fields can be written as

\[
Y_{\nu_1\nu_2\ldots\nu_{q+1}}(x^\mu, z) = N_1 \hat{Y}^{(0)}_{\nu_1\nu_2\ldots\nu_{q+1}}(x^\mu) + e^{a_1 A} \sum_{n \geq 1} \hat{Y}^{(n)}_{\nu_1\nu_2\ldots\nu_{q+1}}(x^\mu) U_1^{(n)}(z),
\]

\[
Y_{\nu_1\nu_2\ldots\nu q z}(x^\mu, z) = \hat{Y}^{(0)}_{\nu_1\nu_2\ldots\nu q}(x^\mu) \hat{N}_1 e^{(2q-p)A(z)}
+ \frac{q}{q+1} e^{2qA} \sum_{n \geq 1} \left( \hat{Y}^{(n)}_{\nu_1\nu_2\ldots\nu q}(x^\mu) + \frac{m_n}{q+1} \hat{X}^{(n)}_{\nu_1\nu_2\ldots\nu q}(x^\mu) \right) U_2^{(n)}(z),
\]

\[
\sqrt{-g} \tilde{Y}^{\mu_1\mu_2\ldots\mu_{p-q+1}}(x^\mu, z) = \sqrt{-g} \left[ \tilde{Y}^{(0)}_{\mu_1\mu_2\ldots\mu_{p-q+1}}(x^\mu) \hat{N}_1 e^{2qA(z)}
+ e^{(a_1+2q-p)A} \sum_{n \geq 1} \tilde{Y}^{(n)}_{\mu_1\mu_2\ldots\mu_{p-q+1}}(x^\mu) \hat{U}_1^{(n)}(z) \right],
\]

\[
\sqrt{-g} \tilde{Y}^{\mu_1\mu_2\ldots\mu_{p-q}}(x^\mu, z) = \sqrt{-g} \left[ N_1 \tilde{Y}^{(0)}_{\mu_1\mu_2\ldots\mu_{p-q}}(x^\mu) + \frac{p-q}{p-q+1} e^{(a_2+2q-p)A}
\times \sum_{n \geq 1} \left( \tilde{Y}^{(n)}_{\mu_1\mu_2\ldots\mu_{p-q}}(x^\mu) + \frac{m_n}{p-q+1} \tilde{X}^{(n)}_{\mu_1\mu_2\ldots\mu_{p-q}}(x^\mu) \right) \hat{U}_2^{(n)}(z) \right].
\]

Substituting the above decompositions into the Hodge duality and considering the Hodge duality on the brane (but with the index 0 replaced by \(n\)), we have

\[
U_1^{(n)}(z) \varepsilon^{\mu_1\mu_2\ldots\mu_{p-q}\nu_1\nu_2\ldots\nu_{q+1}} Y^{(n)}_{\nu_1\nu_2\ldots\nu_{q+1}}(x^\mu)
= \sqrt{-g} \frac{p-q}{p+q+1} \hat{U}_2^{(n)}(z) \left( \tilde{Y}^{(n)}_{\mu_1\mu_2\ldots\mu_{p-q}}(x^\mu) + \frac{m_n}{p-q+1} \tilde{X}^{(n)}_{\mu_1\mu_2\ldots\mu_{p-q}}(x^\mu) \right),
\]

and

\[
\frac{q}{q+1} U_2^{(n)}(z) \varepsilon^{\mu_1\mu_2\ldots\mu_{p-q+1}\nu_1\nu_2\ldots\nu q} \left( \hat{Y}^{(n)}_{\nu_1\nu_2\ldots\nu q}(x^\mu) + \frac{m_n}{q+1} \hat{X}^{(n)}_{\nu_1\nu_2\ldots\nu q}(x^\mu) \right)
= \sqrt{-g} \hat{U}_1^{(n)}(z) \tilde{Y}^{(n)}_{\mu_1\mu_2\ldots\mu_{p-q+1}}(x^\mu),
\]

with which we finally find

\[
S_{q \text{ massive}}^q = S_{p-q \text{ massive}}^{p-q}.
\]

This means that the bound massive \(q\)-form KK mode coupling with a massless \((q-1)\)-form mode is dual to the \((p-q)\)-form one coupling with a massless \((p-q-1)\)-form mode. This duality must be satisfied at the same time with the Hodge duality as long as there are bound massive KK modes for the bulk \(q\)- and \((p-q)\)-form fields, because they are just the results of the KK reduction for the Hodge duality in the bulk.
Note that the coupling between a massive $q$–form mode and a massless $(q - 1)$–form one with level $n$ can be eliminated by setting $\hat{X}^{(n)\mu_1\mu_2\cdots\mu_{q-1}} = 0$ ($n \geq 1$), and it is the same for the $(p - q)$–form field. In this case the two effective fields are still dual to each other.

III. CONCLUSION AND DISCUSSION

In this work, we investigated a new localization mechanism for a massless $q$–form field with a general KK decomposition without any gauge choice. It was found that for the KK modes of the $q$–form field, there are two Schrödinger-like equations. By solving these two equations we can obtain the mass spectra of the KK modes and analyze their characters.

We found that there are two types of massless modes for the $q$–form field, a $q$–form mode and a $(q - 1)$–form one, which cannot be localized on the brane at the same time. Therefore, the Hodge duality on the brane can be naturally satisfied. While if there are bound massive KK modes for the $q$–form field, they must couple with a massless $(q - 1)$–form mode. It is the same for the $(p - q)$–form field. For the bound massive KK modes, although the mass spectra for the effective $q$–form and $(p - q)$–form fields are not the same, they are still dual to each other on the brane, just as in the bulk.

We can take an example to see our conclusion clearly. According to our localization mechanism, on some RS-like 3–brane models, if there is a localized massless vector (1–form) mode, it may be from a higher-dimensional 1–form field or its dual 2–form field. While if there are bound massive KK modes for the bulk vector field, they must couple with a scalar field. This is similar to the Higgs mechanism for a massless vector field getting mass, the difference is that here the scalar field is a part of the higher-dimensional vector field.

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