Equilibration of a hard-disks system

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Abstract

The process of equilibration of a colliding hard-disks system is studied in the framework of classical mechanic. The method consists of dividing the nonequilibrium system into the interacting subsystems; the evolution one of these subsystems is analyzed employing generalized Lagrange and Liouville equations. The subsystem-subsystem interaction force is considered as an evolution parameter. The mechanism by which its system equilibrates is described.

Keywords: nonequilibrium; irreversibility; many-body systems; entropy; evolution.

1. Introduction

The existing methods of analyzing many-body systems based on the models of disks and spheres have been shown to be highly effective, see e.g. [Kryilov, 1950; Sinai, 1970; Beijeren & Dorfman, 1995]. Nevertheless, it is not clear whether it is possible to prove rigorously, within the framework of classical mechanics laws, that equilibrium is established in a sum system, see [Petrosky & Prigogine, 1997; Zaslavsky, 1984,1999]. Such an enterprise would demand new methods and new approaches.

Some features of the nonequilibrium evolution of colliding disks were studied and reported in [Somsikov, 1996, 1998, 2001a]. This approach enables one to carry out analytical and numerical
study of the evolution process within the framework of classical mechanics on the basis of equation of motion of hard disks. These equations are written down for a matrix of colliding disks. It was found that the establishment of equilibrium was determined by the rate of decrease of forces between disks group, [Somsikov, 1996,2001b].

In the present work the process of establishment of equilibrium state for a hard-disks system within the framework of the laws of the classical mechanics is studied. The principal feature of this study is that the interaction force of subsystems into which a disks-system is divided, is used as evolution parameter.

This allows us to avoid application of probabilistic concepts for an explanation of the mechanism of equilibration. Moreover it helps us understand the nature of the probabilistic laws in a systems of the classical mechanics.

The work is constructed as follows. Using D’Alambert principle and equations of motion for disks, we construct generalized Lagrange, Hamiltonian and Liouville equations for a subsystem, selected from the full system. With the help of these equations, we analyze peculiarities of evolution of the disks-system from a nonequilibrium state to equilibrium.

The study is based on the following method. First we prepare a nonequilibrium disks system. A macroscopic subsystem of disks is selected from this system. The evolution of forces on this subsystem by other subsystems is analyzed.

2. Generalized equations of the classical mechanics

It is known that the methods, based on canonical Lagrange or Hamilton equations, can be used successfully for investigating many-body systems, which is not faraway from an equilibrium state see [Lanczos, 1962; Landau, 1976; Landau & Lifshits, 1974]. At the same time these methods have inherent problem when we attempt to use them for the study of strongly nonequilibrium systems, especially ones with nonholonomic connections and polygenic forces (see [Lanczos, 1962]). But in the real world, the majority of systems are of this type. Indeed, even a disks-systems in the space with ideally reflecting walls are nonholonomic [Goldstein, 1975]. Therefore for considering the process of evolution to equilibrium state in a hard-disks system, we should modify the equations of the classical mechanic so that they are applicable to nonequilibrium systems. For deriving these equations, we must take into account the fact that the forces acting between subsystems, in general, cannot be expressed through a potential function; the work done in going from one point to the other in a configuration space depends
on the path taken. These equations will allow us to consider process equilibration, by using the interaction force of subsystems, as a dynamic evolutionary parameter. In accord with Lanczos [1962], we shall name these as "generalized equations".

Let us take a system, which consist of \( N \) disks. Divide this system into \( R \) subsystems, so that in each subsystem will be of \( T \) disks. Therefore, \( N = RT \). The energy of the system is constant. It is equal to the sum of internal energies of all subsystems and interaction energies between subsystems. Let us selected one of them, which we call \( p \)-subsystem. Let \( \delta W_d^p \), be the virtual work of active forces in \( p \)-subsystem. In the general case, this work can be express as follows: \( \delta W_d^p = \sum_{k=1}^{T} \sum_{s=1}^{N-T} F_{ks}^p \delta r_k = \sum_{k=1}^{T} F_{ks}^p \delta r_k \), where \( k = 1, 2, 3...T \) are disks of \( p \)-subsystem, \( s = 1, 2, 3...N - T \) are external disks that interacts with the \( p \)-disks. \( F_{ks}^p \) is the interaction force of \( k \)- and \( s \)-disks, \( \delta r_k \) is the virtual displacement of the \( k \)-disk, \( F_{k}^p = \sum_{s=1}^{N-T} F_{ks}^p \). The virtual work of the interaction force of internal disks of \( p \)-subsystem is equal to zero.

In case of the pair interaction the virtual work of external forces is of the form: \( \delta W_p = \sum_{k=1}^{T} F_k^p \delta r_k = \sum_{k=1}^{T} F_{ks}^p \delta r_k \), because in this case \( F_{k}^p = F_{ks}^p \). The inertial force can be presented as \( \delta W_{in} = \sum_{k=1}^{T} V_k \delta r_k \). The sum of the active and the inertial forces constitute as effective forces. The principle of D’Alambert asserts that the work of effective forces is always zero [Lanczos, 1962], i.e.

\[
\delta W_{p} = \delta W_{in} - \delta W_d^p.
\]

(1)

The feature of the virtual work above implies, that in general it does not get reduced to complete differential.

From the equations of motion for hard disks see [Somsikov, 2001b], it follows: \( \sum_{p=1}^{R} \sum_{k=1}^{T} F_{ks}^p \delta r_k = 0 \). Therefore total active and inertial work for all subsystems at any moment of time is equal to zero, i.e. \( \sum_{p=1}^{R} \delta W_{in} = \sum_{p=1}^{R} \delta W_d^p = 0 \). This equality can take place in two cases: when the sum of nonzero terms is equal to zero, or when each term of the sum is equal to zero. Obviously the second case, appropriate for an equilibrium state, takes place when \( T \to \infty \). For this case, with the help of the equations of motion for hard disks [Somsikov, 2001b], it is possible to record:

\[
\sum_{k=1}^{T} \varphi_{ks} \delta (\psi_{ks}(t)) \Delta k_s(t) |\Delta k_s(t)| = \sum_{k=1}^{T} F_{ks}^p = \sum_{k=1}^{T} \dot{V}_k = 0,
\]

(2)

where \( V_k \) is a \( k \)-disks velocity; \( \varphi_{ks} = i\beta e^{i\vartheta_{ks}}; \) \( i \) is an imaginary unit; \( \beta = \sin \vartheta_{ks}; \) \( \vartheta_{ks} \) is scattering angle for \( k \) and \( s \) colliding disks, which varies from 0 to \( \pi \); \( \delta(x) \) is a delta function; \( \psi_{ks} = 1 - |l_{ks}|; l_{ks}(t) = l_{ks}^0 + \int_0^t \Delta k_s dt \) are distances between centers of colliding disks; \( r_{ks}^0 = r_k^0 - r_s^0 \)
are initial values of disks coordinates; $\Delta_{k_s} = V_k - V_s$ are relative velocities. The impact parameter $d_{k_s} = \cos \vartheta_{k_s}$ is determined by the distance between centers of the colliding disks in the complex plane with real axis $x$ and imaginary axis $y$. The value of the impact parameter is: $d_{k_s} = Im(l_{k_s}\Delta_{k_s})/|l_{k_s}\Delta_{k_s}|$. The $k$-disk swoops on the $j$- disk lying along the $x$ - axis. Mass and diameter of each disk is set equal to 1. Boundary conditions are either periodical or hard walls. Equating the right-hand side of Eq. (2) to zero implies, that the selected $p$-subsystem is in a stationary state.

To derive the general Lagrange equation for $p$-subsystem, let us transform D’Alambert equation (1) by multiplying it by $dt$, and integrating it over an interval from $t_1$ to $t_2$. In general we have,

$$
\int_{t_1}^{t_2} \delta \bar{W}_q^p dt = \int_{t_1}^{t_2} \sum_{k=1}^{T} \left[ \frac{d}{dt} V_k - \sum_{j \neq k} F_{kj}^p - F_{k}^p \right] \delta r_k dt = \\
\delta \int_{t_1}^{t_2} \left[ \sum_{k=1}^{T} V_k^2 dt - \int_{t_1}^{t_2} \sum_{k=1}^{T} \left( F_{k}^p + \sum_{j \neq k} F_{kj}^p \right) \delta r_k \right] dt - \left[ \sum_{k=1}^{T} V_k \delta r_k \right]_{t_1}^{t_2} \tag{2a}
$$

In Eq. (2a) the term $\sum_{j \neq k} F_{kj}^p$ determines the force of interaction in the $p$-subsystem, $k$ and $j$ are the colliding disks of the $p$-subsystem. The term $\sum_{k=1}^{T} F_{k}^p$ is the force on the $p$-subsystem from the rest. On demanded at the ends of the interval $[t_1, t_2]$, the virtual displacements is zero. Then the last term in (2a) will be zero.

Let us assume that a subsystem has equilibrated. Then for internal forces of interaction in a subsystem we can set a function $U(r_1, r_2, ...r_T)$, for which the following condition is satisfied: $\int_{t_1}^{t_2} \left[ \sum_{k=1}^{T} \sum_{j \neq k} F_{kj}^p \delta r_k \right] dt = -\delta \int_{t_1}^{t_2} U(r_1, r_2, ...r_T)dt$. Here $r_1, r_2, ...r_T$ are the coordinates of the $p$-subsystem disks. In the general case it is impossible to express forces on $p$-subsystem, as a gradient see [Lanczos, 1962]. In this case the Eq. (2a), can be written as

$$
\int_{t_1}^{t_2} \delta \bar{W}_q^p dt = \int_{t_1}^{t_2} \left[ \sum_{k=1}^{T} \left( \frac{d}{dt} \frac{\partial L_p}{\partial V_k} - \frac{\partial L_p}{\partial \delta r_k} - F_{k}^p \right) \delta r_k \right] dt = 0 \tag{3}
$$

In Eq. (3) we denote $L_p = \sum_{k=1}^{T} \frac{v_k^2}{2} + U(r_1, r_2, ...r_T)$. If the interaction of disks is potential, then $L_p$ will include also internal potential energy of the $p$-subsystem - $U(r_1, r_2, ...r_T)$. Since any variations of integral in equation (3) will be zero,we can set:

$$
\sum_{k=1}^{T} \left( \frac{d}{dt} \frac{\partial L_p}{\partial V_k} - \frac{\partial L_p}{\partial \delta r_k} - F_{k}^p \right) = \sum_{k=1}^{T} F_{k}^p = F_p \tag{4}
$$

In the above $\sum_{k=1}^{T} F_{k}^p = F_p$. 

4
Eq. (4) is a generalized equation of Lagrange for a $p$-subsystem. $F_p$ is the polygenic force acting on the $p$-subsystem. When $F_p = 0$, Eq. (4) transforms to a canonical equation of Lagrange for equilibrium, conservative system. The equality, $F_p = 0$, is a sufficient condition for a stationary state.

Let us now derive Hamilton equation for the chosen $p$-subsystem. The differential for $L_p$ can be written as,

$$dL_p = \sum_{k=1}^{T} \left( \frac{\partial L_p}{\partial r_k} dr_k + \frac{\partial L_p}{\partial p_k} dV_k \right) + \frac{\partial L_p}{\partial t} dt,$$

where $\frac{\partial L_p}{\partial V_k} = p_k$ is disks momentum. With the help of Lagrange transformation, it is possible to write:

$$d\left[ \sum_{k=1}^{T} p_k V_k - L_p \right] = \sum_{k=1}^{T} \left( - \frac{\partial L_p}{\partial r_k} dr_k + V_k dp_k \right) = \sum_{k=1}^{T} \frac{\partial L_p}{\partial p_k} F_p^k.$$

Since $\frac{\partial H_p}{\partial t} = - \frac{\partial L_p}{\partial t}$, where $H_p = \sum_{k=1}^{T} p_k V_k - L_p$, we have from (4),

$$\frac{\partial H_p}{\partial r_k} = - \dot{p}_k + F_p^k. \quad (5)$$

$$\frac{\partial H_p}{\partial p_k} = V_k. \quad (6)$$

The above is a general Hamilton equations for the selected $p$-subsystem. The right-hand side of Eq. (5) denote the external forces, which act on $p$-subsystem.

Using Eqs. (5) and (6), we can find the Liouville equation for $p$-subsystem. For this purpose, let us take a generalized current vector - $J_p = (\dot{r}_k, \dot{p}_k)$ of the $p$-subsystem in a phase space [Zaslavsky, 1984]. From Eqs. (5) and (6), we find:

$$\text{div} J_p = \sum_{k=1}^{T} \left( \frac{\partial}{\partial r_k} V_k + \frac{\partial}{\partial p_k} \dot{V}_k \right) = \sum_{k=1}^{T} \frac{\partial}{\partial p_k} F_p^k \quad (7)$$

The differential form of particles number conservation law in the subsystem is a continuity equation: $\frac{df_p}{dt} + \text{div}(J_p f_p) = 0$, where $f_p = f_p(r_k, p_k, t)$ is a normalized distribution function of disks in the $p$-subsystem. With the help of continuity equation and Eq. (7) for divergence of a generalized current vector in a phase space, we can show that:

$$\frac{df_p}{dt} = \frac{\partial f_p}{\partial t} + \sum_{k=1}^{T} \left( V_k \frac{\partial f_p}{\partial r_k} + p_k \frac{\partial f_p}{\partial p_k} \right) = \frac{\partial f_p}{\partial t} + \text{div}(J_p f_p) - f_p \text{div} J_p = - f_p \sum_{k=1}^{T} \frac{\partial}{\partial p_k} F_p^k.$$  

Thus, we have:

$$\frac{df_p}{dt} = - f_p \sum_{k=1}^{T} \frac{\partial}{\partial p_k} F_p^k \quad (8)$$

Equation (8) is a Liouville equation for $p$-subsystem. It has a formal solution:

$$f_p = \text{const} \cdot \exp \left[ - \int \left( \sum_{k=1}^{T} \frac{\partial}{\partial p_k} F_p^k \right) dt \right] \quad (8a)$$

From this solution it follows, that the $p$-subsystem will be in a stationary state when the external forces disappear, i.e. $\int \left( \sum_{k=1}^{T} \frac{\partial}{\partial p_k} F_p^k \right) dt = 0$. So, the exponent index of the solution of the Eq. (8)
determines the characteristic relaxation time of the system to the equilibrium state. Hence, the change of the phase volume of the $p$-subsystem will last during time of aspiration of the force, $F_p$, to zero. How this comes about is discussed below. It is possible for all points of the phase space except for "islands", filled by periodic and quasiperiodic orbits of some Hamiltonian systems, see [Loskytov & Mihailov, 1990; Zaslavsky, 1999]. If at the time of preparation, the system is in such an island, it will return to it periodically. For periodic points we do not have mixing, and the correlations do not disappear. We can then say that the probability of return to the initial periodic point is determined by the probability of preparing the system at its periodic point.

Let us consider the question: how is the description of selected subsystems connected to disks system description as a whole.

As, $\sum_{p=1}^{R} \sum_{k=1}^{T} F_k^p = 0$, the next equation for the full system Lagrangian, $L_R$, will be:

$$\frac{d}{dt} \frac{\partial L_R}{\partial V_k} - \frac{\partial L_R}{\partial r_k} = 0$$

(9)

and the appropriate Liouville equation is

$$\frac{\partial f_R}{\partial t} + V_k \frac{\partial f_R}{\partial r_k} + \dot{p}_k \frac{\partial f_R}{\partial p_k} = 0$$

(10)

The function, $f_R$, corresponds to the full system that is conservative. Therefore, we have: $\sum_{p=1}^{R} \text{div}J_p = 0$. This expression is equivalent to the next equality: $\frac{d}{dt}(\sum_{p=1}^{R} \ln f_p) = \frac{d}{dt}(\ln \prod_{p=1}^{R} f_p) = (\prod_{p=1}^{R} f_p)^{-1} \frac{d}{dt}(\prod_{p=1}^{R} f_p) = 0$. So, $\prod_{p=1}^{R} f_p = \text{const}$. In equilibrium state we have $\prod_{p=1}^{R} f_p = f_R$. Because the equality $\sum_{p=1}^{R} F_p = 0$ is fulfilled at all times, we have the equality, $\prod_{p=1}^{R} f_p = f_R$, as the integral of motion. This is in agreement with Liouville theorem about conservation of phase space [Landau & Lifshits, 1973].

The simultaneous fulfillment of conditions of phase volume preservation for full system and validation of time-dependence solution (b) of Eq. (8) for nonequilibrium subsystems is correct only when the centre of mass of a system moves on trajectory, reversible in time. It should take place irrespective of, whether all subsystems are in equilibrium or not. Therefore, for the considered $p$-subsystem only those irreversible redistribution of phase volume and energy that lead to reversibility of motion of the centre of mass of all system, is possible.

3. The mechanism of equilibration
Let us show that for a hard-disks system, the external force decreases due to the mixing properties, i.e. \( \int \left( \sum_{k=1}^{T} \frac{\partial}{\partial p_k} F_p^{(p)} \right) dt \to 0 \) when \( t \to \infty \). This means that because of mixing, the system will go to a stationary state irrespective of where it was at the initial time (except the periodic points).

The mixing properties of two hard disks was proved by Sinai [1970] and Zaslavsky [1984]. It was shown that the correlation function for colliding disks is: \( R(t) \exp(n \ln K) \), where \( K = \rho/2 \): \( \rho \)-length of free run; \( n \)-the number of collisions. Therefore the characteristics time of the decay of correlations, \( t_d \), for regular collision disks with unit diameter through a time interval \( \tau \) is determined by equation: \( t_d = \tau/\ln(\rho/2) \). The condition \( K = \rho/2 > 1 \) is satisfied by rare gas. Hence, two hard disks are mixed. As will be shown bellow, the mixing property is both, necessary and sufficient for the equilibration. Although in the situation when we have several disks, the strict mathematical proof of a mixing property is absent; nevertheless the existence of the mixing property is usually accepted a priori as well.

For proving the property of aspiration of the resulting force to zero for a hard-disks system, we shall simplify Eq. (2) for the \( p \)-subsystem. Let us assume, that all disks collide simultaneously in equal, short enough intervals of time, \( \tau \). It is clear, that if for this condition the system goes to equilibrium then does so definitely for the general case. After such simplification, Eq. (2) for a \( p \)-subsystem, can be written as

\[
\dot{V}_k^n = \varphi^n_{k\sigma(n)} \Delta_{k\sigma(n)}^{n-1}
\]

(11)

Here, to each "\( k \)" disk from \( p \)-subsystem in moment of time, \( n\tau \) corresponds to "\( s(n) \)", disk from other subsystems; \( n = 1, 2, 3, \ldots \).

The evolution of the \( p \)-subsystem is determined by the vector, \( \vec{V}_p^T \), with components denoting the velocities of disks of the \( p \)-subsystem: \( \vec{V}_p^T = \{V_k^p\}; k = 1, 2, 3, \ldots, T \). Some of the time-dependent properties of this subsystem will be determined by studying the sum of it components. Let us designate this sum as \( \Upsilon_p \). Carrying out the summation in (11) over all disks of the \( p \)-subsystem, we obtain:

\[
\dot{\Upsilon}_p^n = \sum_{k=1}^{T} \varphi^n_{k\sigma(n)} \Delta_{k\sigma(n)}^{n-1} = F_{p}^{(p)}
\]

(12)

Equation (12) describes the change of total momentum, acting on the \( p \)-subsystem as a result of collisions at time \( n\tau \). The aspiration of a total momentum to zero is equivalent to aspiration to zero of the force, \( F_{p}^{(p)} \).
Now let us show, if the mixing property for a disks system is assumed, the homogeneous distribution of impact parameters of disks occur as well.

In accordance with the mixing condition, we have the following [Loskytov & Mihailov, 1990],

$$\mu(\delta)/\mu(d) = \delta/d$$  \hspace{1cm} (13)

Here, \(\mu(d)\), is a measure corresponding to the total value of impact parameter - "\(d\)"; \(\delta\) is an arbitrary interval of the impact parameter and, \(\mu(\delta)\), is a corresponding measure. Equation (13) implies the number of collisions is proportion to the interval "\(\delta\)". It also implies that the distribution of the impact parameters is homogeneous.

As is well known, see e.g. [Loskytov & Mihailov, 1990; Zaslavsky, 1984], for mixed systems of correlations decay. For Eq. (12) this condition can be written down as $$\langle \varphi_{ks(n)}^{n} \Delta_{ks(n-1)}^{n-1} \rangle = \langle \varphi_{ks(n)}^{n} \rangle \langle \Delta_{ks(n-1)}^{n-1} \rangle$$ i.e. the average from two multiplied functions is equal to the multiplication of the average of these functions. \(\varphi_{ks(n)}^{n}\) come from impact parameters, and \(\Delta_{ks(n-1)}^{n-1}\) come from relative velocities of colliding disks. Therefore this condition is similar to the condition of independence of coordinates and momenta, widely used in statistical physics see e.g. [Rumer & Ryvkin, 1977].

Thus, it is possible to carry out the summation in the multiplier, \(\varphi_{ks(n)}^{n}\), over impact parameters, independent of the summation of \(\Delta_{ks(n-1)}^{n-1}\) over relative velocities of colliding disks. Then, under the condition of the homogeneous distribution of impact parameters and when \(T >> 0\), we can go from summation to integration. We will have, see [Somsikov, 2001a]:

$$\phi = 1/T \lim_{T \to \infty} \sum_{k=1}^{T} \varphi_{ks}^{n} = \frac{1}{G} \int_{0}^{\pi} \varphi_{ks}^{n} d(\cos \vartheta) = -\frac{2}{3},$$ \hspace{1cm} (14)

where \(G = 2\) is the normalization factor.

Taking into account (14), we have from Eq. (12):

$$\dot{\Upsilon}_{p} = -\frac{2}{3} \sum_{k=1}^{T} \Delta_{ks(n)}.$$ \hspace{1cm} (15)

The negative sign in the right-hand side Eq. (15) means, that the force, \(F_{p}\), decreases.

The next question is about the stability of a stationary point. The stability of a stationary point of \(p\)-subsystem can be established with the help of the Eq. (15). We set the initial deviation from stationary point, and then consider, how this deviation changes with time.

Let the point, \(Z_{0}\), be a stationary point, so that, \(F_{p}\), acting on \(p\)-subsystem, is zero. From the Lyapunov’s theorem about stability it follows that the point, \(Z_{0}\), is asymptotically stable if any deviation from it gets attenuated.
Let us expand the left and right-hand sides of the equation (15) in a series by small parameter, \( \nu \), of perturbation of velocities of disks of the \( p \)-subsystem, near point, \( Z_0 \), and keep terms up to first-order infinitesimal. The expansion of the left-hand side of the Eq. (15) gives:

\[
\dot{\nu} = \sum_{k=1}^{T} \dot{\varepsilon}_k,
\]

where the summation is carried out on the components of the variation, \( \nu \). In the expansion of the right-hand side, there remains only \(( -\frac{2}{3} ) \sum_{k=1}^{T} \varepsilon_k = -2/3 \nu \). A contribution into the expansion is given by collisions of disks of the \( p \)-subsystem, with disks of its complement. We have:

\[
\dot{\nu}^n = -\frac{2}{3} \nu^{n-1}.
\]  

Equation (16) means, that any deviation from an equilibrium state will decay. Hence, the stationary point in the presence of mixing is steady. Stability is provided by emergence of returning force, \( F_p \), at a deviation of a subsystem from an equilibrium point. We shall note, that the Eq. (16) also follows from the theory of fluctuations see [Landau, 1976].

Let us show, that the emergency of force due to deviation of system from its equilibrium, provides restriction of spontaneous fluctuations.

Take the system in a nonequilibrium condition. As follows from the previous statement, any nonequilibrium condition is characterized by the force, \( F_p \), which acts on the \( p \)-subsystem. This force is determined by Eq. (12). The time decrease of the force, \( F_p \), is determined by equation:

\[
t_{\text{din}} = \int \frac{d\Omega}{F_p}.
\]

Thus, if the system somehow appears at a nonequilibrium point, over a characteristic time, \( t_{\text{din}} \), then it should return to equilibrium.

For the further consideration we shall accept two statements which follow from the mechanism of equilibration.

First: the degree nonequilibrium is defined by the force, \( F_p \). So, there is a mutual unique conformity between \( F_p \) and a phases space points.

Second: we shall consider, that the spontaneous deviations increase of system from an equilibrium condition occurs under increase of \( F_p \). The deviation is proportional to force.

If these conditions are fulfilled, it is possible to prove that such fluctuations are realized only, if \( t_{\text{din}} > t_{\text{prob}} \). Time, \( t_{\text{prob}} \) is determined by probabilistic principles. According to the formula Smoluhovsky [Zaslavsky, 1984], for the case of an ergodic system, average resetting time, \( t \), or Poincare’ cycle time, is equal to \( t_{\text{prob}} = t(1 - P_0)/(P_0 - P_1) \), were \( P_1 \) is the probability of reversibility during the time \( t \), \( P_0 \) is the probability of initial phase region. Suppose that the system in a probabilistic way begins to deviate from equilibrium. The characteristic time of a
deviation to any point, \( Z_p \), should be: \( \sim t_{\text{prob}} \). But because in this point the force, \( F_p \), acts on subsystem as returning, the system will approach to the point, \( Z_p \), in a probabilistic way if only \( t_{\text{prob}} < t_{\text{din}} \).

Thus, within the scope of assumption made in this work, the dynamics of a hard-disks system is completely determined by the deterministic classical mechanics. Therefore, the need for probabilistic principles in the description of evolution of the system and its area of their use are determined by roughness of transition from summation to integration on impact parameters, and also from the periodic or quasiperiodic points at the time of preparation.

4. Conclusion

Splitting conservative, nonequilibrium system as a set of interacting subsystems, and the analysis of the evolution of forces of interaction between these subsystems is a basic idea proposed in this paper. This approved allows us to take into account the exchange of energy between subsystems and a feedback between active and inertial forces of their interaction. The feedback provides relaxation of the system to equilibrium. Therefore this approach is applicable for studying evolutionary processes in open systems with polygenic forces and nonholonomic connections.

The analysis nonequilibrium evolution is based on generalized equation of Lagrange set for a selected subsystem. The right-hand side of this equation is a polygenic force of interaction of subsystems. Because of mixing, this force tends to zero. It causes evolution of system to equilibrium state. The establishment of equilibrium is possible starting from all the points of the phase space except for periodic points. In the vicinity of equilibrium, where, \( F_p \) is near zero, the statistical theory of fluctuations constructed on the basis of the canonical Hamilton equations is applicable.

Stability of an equilibrium state is ensured by the emergence of the returning force when a system deviates from equilibrium. It imposes appropriate restrictions on amplitudes of probable fluctuations of system.

Within the framework of this study, the dynamics of the system is deterministic. Probabilistic principles enter only by the uncertainty of initial conditions, and coarse graining of transition from the discrete to the continuous.

Let us compare our explanation of irreversibilities with existing one, see, for example [Zaslavsky, 1984]. In accordance with this explanation, irreversibility is a consequence of "coarse
graining" of the phase space. The mixing implies and institutes average procedure. As a consequence, the information on separate phase trajectories is lost. This is equivalent to the irreversibility. The shortcoming of such an explanation is that the nature of averaging in a phase space because the dynamic equations do not contain the mechanism of coarse graining.

Though the mechanism offered here is also based on mixing property, using a force, $F_p$, as evolution parameter, allows getting rid of explicit use of "coarse graining" idea. In contrast to directly "coarse graining" of the phase space, the transition to integration on impact parameters does not deform the nature of aspiration of the system to an equilibrium state. Here, the role of replacement of summation on integration on impact parameters consist in transition from discrete functions to continuous functions convenient for differential calculus.

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