DECAY OF SEMIGROUP FOR AN INFINITE INTERACTING
PARTICLE SYSTEM ON CONTINUUM CONFIGURATION SPACES

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Abstract. We show the heat kernel type variance decay $t^{-d/2}$, up to a logarithmic correction, for the semigroup of an infinite particle system on $\mathbb{R}^d$, where every particle evolves following a divergence-form operator with diffusivity coefficient that depends on the local configuration of particles. The proof relies on the strategy from [30], and generalizes the localization estimate to the continuum configuration space introduced by S. Albeverio, Y.G. Kondratiev and M. Röckner.

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1. INTRODUCTION

In this work, we study an interacting diffusive particle system in $\mathbb{R}^d$ and the heat kernel type estimate for its semigroup. Let us give an informal introduction to the model and main result at first. We denote by $\mathcal{M}_\delta(\mathbb{R}^d)$ the set of point measures of type $\mu = \sum_{i=1}^{\infty} \delta_{x_i}$ on $\mathbb{R}^d$, which we call configurations of particles, by $\mathcal{F}_U$ the $\sigma$-algebra generated by $\mu(V)$ tested with all the Borel set $V \subseteq U$, and use the shorthand $\mathcal{F} := \mathcal{F}_{\mathbb{R}^d}$. Let $\mathbb{P}_\rho$ be the Poisson point process of density $\rho \in (0, \infty)$ as the law for the configuration $\mu$, with $\mathbb{E}_\rho, \mathbb{V}_\rho$ the associated expectation and variance. We have $a_\rho : \mathcal{M}_\delta(\mathbb{R}^d) \to [1, \Lambda]$ an $\mathcal{F}_{B_1}$-measurable function, i.e. it only depends on the configuration in the unit ball $B_1$, and let $a(\mu, x) := a_\rho(\tau_{-x}\mu)$ be the diffusive coefficient with local interaction at $x$, where $\tau_{-x}$ represents the transport operation by the direction $-x$. Denoting by $\mu_t := \sum_{i=1}^{\infty} \delta_{x_{i,t}}$ the configuration at time $t \geq 0$, our model can be informally described as an infinite-dimensional system with local interaction such that every particle $x_{i,t}$ evolves as a diffusion associated to the divergence-form operator $-\nabla \cdot a(\mu_t, x_{i,t}) \nabla$.

More precisely, it is a Markov process $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_\rho)$ defined by the Dirichlet form

$$
E^\rho(f, f) := \mathbb{E}_\rho \left[ \int_{\mathbb{R}^d} a(\mu, x) \nabla f(\mu, x) \cdot \nabla f(\mu, x) \, d\mu(x) \right],
$$

where the directional derivative $e_k \cdot \nabla f(\mu, x) := \lim_{\lambda \to 0} \frac{1}{\lambda}(f(\mu - \delta_x + \delta_{x+\lambda e_k}) - f(\mu))$ along the canonical direction $\{e_k\}_{1 \leq k \leq d}$ is defined for a family of suitable functions and $x \in \text{supp}(\mu)$.

One may expect that the diffusion follows the heat kernel estimate established by the pioneering work of John Nash [38], as every single particle is a diffusion of divergence type. This is the object of our main theorem. Let $u : \mathcal{M}_\delta(\mathbb{R}^d) \to \mathbb{R}$ be an $\mathcal{F}$-measurable function, depending only on the configuration in the cube $Q_{t_0} := \left[-\frac{t_0}{2}, \frac{t_0}{2}\right]^d$, and smooth with respect to the transport of every particle (i.e. $u$ belongs to the function space $C^\infty_c(\mathcal{M}_\delta(\mathbb{R}^d))$ defined in Section 2.1.2), and let $u_t := \mathbb{E}_\rho[u(\mu_t)|\mathcal{F}_0]$. Denoting $L^\infty := L^\infty(\mathcal{M}_\delta(\mathbb{R}^d), \mathcal{F}, \mathbb{P}_\rho)$, we have the following estimate.

Theorem 1.1 (Decay of variance). There exists two finite positive constants $\gamma := \gamma(\rho, d, \Lambda)$, $C := C(\rho, d, \Lambda)$ such that for any $u \in C^\infty_c(\mathcal{M}_\delta(\mathbb{R}^d))$ supported in $Q_{t_0}$, then we have

$$
\mathbb{V}_\rho[u_t] \leq C(\log(1 + t))^{-d} \left(\frac{t}{\sqrt{t}}\right)^d \|u\|_{L^\infty}^2.
$$

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Interacting particle systems remain an active research topic, and it is hard to list all
the references. We refer to the excellent monographs [32, 33, 34, 42] for a panorama of
the field. In recent years, many works in probability and stochastic processes illustrate the
diffusion universality in various models: a well-understood model is the random conductance
model, see [13] for a survey, and especially the heat kernel bound and invariance principle
is established for the percolation clusters in [12, 37, 41, 36, 11, 28, 40]; from the view point
of stochastic homogenization, the quantitative results are also proved in a series of work
[9, 6, 10, 7, 25, 26, 22, 23, 24], and the monograph [8], and these techniques also apply on
the percolation clusters setting, as shown in [5, 18, 27, 19]; for the system of hard-spheres,
Bodineau, Gallagher and Saint-Raymond prove that Brownian motion is the Boltzmann-Grad
limit of a tagged particle in [14, 15, 16]. All these works make us believe that the model in
this work should also have diffusive behavior in large scale or long time.

Notice that our model is of non-gradient type, and our result is established in the continuum
configuration space rather than a function space on \( \mathbb{R}^d \). In previous works, the construction
of similar diffusion processes is studied by Albeverio, Kondratiev and Röckner using Dirichlet
forms in [1, 2, 3, 4]; see also the survey [39]. To the best of our knowledge, we do not find
Theorem 1.1 in the literature. While in the lattice side, let us remark one important work
[30] by Janvresse, Landim, Quastel and Yau, where the decay of variance is proved in the \( \mathbb{Z}^d \)
zero range model, which is of gradient type. Since our research is inspired by [30] and also
uses some of their techniques, we point out our contributions in the following.

Firstly, we give an explicit bound with respect to the size of the support of the local
function \( u \), that is uniform over \( t \); the bound \( \left( \frac{l_u}{\sqrt{t}} \right)^d \) captures the correct typical scale. For
comparison, [30, Theorem 1.1] states the result

\[
\text{Var}_\rho[u_t] = \frac{[\bar{\varphi}'(\rho)]^2 \chi(\rho)}{8\pi \phi'(\rho)t} + o\left(t^{-\frac{d}{2}}\right),
\]

which should be considered as the asymptotic behavior in long time, and the term \( o\left(t^{-\frac{d}{2}}\right) \) is
of type \( (l_u)^{5d}t^{-\left(\frac{d}{2}+\varepsilon\right)} \) if one tracks carefully the dependence of \( l_u \) in the steps of the proof of
[30, Theorem 1.1]. To get the typical scale \( \left( \frac{l_u}{\sqrt{t}} \right)^d \), we do some combinatorial improvement in
the intermediate coarse-graining argument in eq. (3.14); see also Figure 1 for illustration. On
the other hand, we also wonder if we could establish a similar result as eq. (1.3) to identify the
diffusive constant in the long time behavior. This an interesting question and one perspective
in future research, but a major difficulty here is to characterize the effective diffusion constant,
because the zero range model satisfies the gradient condition while our model does not. We
believe that it is related to the bulk diffusion coefficient and the equilibrium density fluctuation
in the lattice nongradient model as indicated in [42, eq.(2.14), Proposition 2.1].

Secondly, we extend a localization estimate to the continuum configuration space: under
the same context of Theorem 1.1, and recalling that \( \mathcal{F}_{Q_K} \) represents the information of
\( \mu \) in the cube \( Q_K = [-K, K]^d \), we define \( A_Ku_t := \mathbb{E}_\rho[u_t|\mathcal{F}_{Q_K}] \), and show that for every
\( t \geq \max\{(l_u)^2, 16\Lambda^2\} \) and \( K \geq \sqrt{t} \)

\[
\mathbb{E}_\rho\left[\left( u_t - A_Ku_t \right)^2 \right] \leq C(\Lambda) \exp\left(-\frac{K}{\sqrt{t}}\right)\mathbb{E}_\rho[u_t^2].
\]

This is a key estimate appearing in [30, Proposition 3.1], and is also natural as \( \sqrt{t} \) is the
typical scale of diffusion, thus when \( K \gg \sqrt{t} \) one get very good approximation in eq. (1.4).
Its generalization in the continuum configuration space is non-trivial, since in the proof of
[30, Proposition 3.1], one tests the Dirichlet form with \( A_Ku_t \), but in our model it is not in
the domain of Dirichlet form \( \mathcal{D}(\mathcal{E}^u) \) and one cannot put \( A_Ku_t \) directly in the Dirichlet form
Continuum configuration space, which serves as the

where Section 3.1 gives its outline and we fix the minor error in [30] mentioned above. The
we have to apply some regularization steps which we present in Theorem 4.1.

\( (2.3) \)

(2.2)

\( (2.1) \)

continuum configuration space.

In the case

For a

algebra such that for every Borel subset

\( \mathcal{V} \)

\( \mathcal{U} \)

set of Radon measures on

\( \mathcal{U} \)

\( \mathcal{V} \)

2.1. Notations. In this part, we introduce the notations used in this paper. We write \( \mathbb{R}^d \)

for the \( d \)-dimensional Euclidean space, \( B_r(x) \) for the ball of radius \( r \) centered at \( x \), and

\( Q_s(x) := x + \left[ -\frac{s}{2}, \frac{s}{2} \right]^d \)

as the cube of edge length \( s \) centered at \( x \). We also denote by \( B_r \) and

\( Q_s \)

respectively short for \( B_r(0) \) and \( Q_s(0) \). The lattice set is defined by \( \mathbb{Z}_s := \mathbb{Z}^d \cap Q_s \).

2.1.1. Continuum configuration space. For any metric space \( (E, d) \), we denote by \( \mathcal{M}(E) \) the

set of Radon measures on \( E \). For every Borel set \( U \subseteq E \), we denote by \( \mathcal{F}_U \) the smallest \( \sigma \)-algebra such that for every Borel subset \( V \subseteq U \), the mapping \( \mu \in \mathcal{M}(E) \mapsto \mu(V) \) is measurable.

For a \( \mathcal{F}_U \)-measurable function \( f : \mathcal{M}(E) \to \mathbb{R} \), we say that \( f \) supported in \( U \) i.e. \( \text{supp}(f) \subseteq U \).

In the case \( \mu \in \mathcal{M}(E) \) is of finite total mass, we write

\[
\int f \, d\mu := \frac{\int f \, d\mu}{\int d\mu}.
\]

We also define the collection of point measure \( \mathcal{M}_\delta(E) \in \mathcal{M}(E) \)

\[
\mathcal{M}_\delta(E) := \left\{ \mu \in \mathcal{M}(E) : \mu = \sum_{i \in I} \delta_{x_i} \text{ for some } I \text{ finite or countable, and } x_i \in E \text{ for any } i \in I \right\},
\]

which serves as the continuum configuration space where each Dirac measure stands the


d

position of a particle. In this work we will mainly focus on the Euclidean space \( \mathbb{R}^d \) and its

associated point measure space \( \mathcal{M}_\delta(\mathbb{R}^d) \), and use the shorthand notation \( \mathcal{F} := \mathcal{F}_{\mathbb{R}^d} \).

We define two operations for elements in \( \mathcal{M}_\delta(\mathbb{R}^d) \): restriction and transport.

- For every \( \mu \in \mathcal{M}_\delta(\mathbb{R}^d) \) and Borel set \( U \subseteq \mathbb{R}^d \), we define the restriction operation \( \mu \mathcal{L} U \),

such that for every Borel set \( V \subseteq \mathbb{R}^d \), \( \mu(V) = \mu(U \cap V) \). Then for a function

\( f : \mathcal{M}_\delta(\mathbb{R}^d) \to \mathbb{R} \)

which is \( \mathcal{F}_U \)-measurable, we have \( f(\mu) = f(\mu \mathcal{L} U) \).

- The transport on the set is defined as

\[
\forall h \in \mathbb{R}^d, U \subseteq \mathbb{R}^d, \tau_h U := \{ y + h : y \in U \}.
\]

Then for every \( \mu \in \mathcal{M}_\delta(\mathbb{R}^d) \) and \( h \in \mathbb{R}^d \), we define the transport operation \( \tau_h \mu \) such that for every Borel set \( U \), we have

\[
(2.2)
\]

\[
\tau_h \mu(U) := \mu(\tau_{-h} U).
\]

For \( f \) an \( \mathcal{F}_V \)-measurable function, we also define the transport operation \( \tau_h f \) as a

pullback that

\[
(2.3)
\]

which is an \( \mathcal{F}_{\tau_h V} \)-measurable function.
Notice that the restriction operation can be defined similarly in \( \mathcal{M}(E) \) for a metric space, but the transport operation requires that \( E \) is at least a vector space.

We fix \( \rho > 0 \) once and for all, and define \( \mathbb{P}_\rho \) a probability measure on \( (\mathcal{M}_\delta(\mathbb{R}^d), \mathcal{F}) \), to be the Poisson measure on \( \mathbb{R}^d \) with density \( \rho \) (see [31]). We denote by \( \mathbb{E}_\rho \) the expectation, \( \text{Var}_\rho \) the variance associated with the law \( \mathbb{P}_\rho \), and by \( \mu \) the canonical \( \mathcal{M}_\delta(\mathbb{R}^d) \)-valued random variable on the probability space \( (\mathcal{M}_\delta(\mathbb{R}^d), \mathcal{F}, \mathbb{P}_\rho) \). In the case \( U \subseteq \mathbb{R}^d \) a bounded Borel set, we can rewrite the expectation \( \mathbb{E}_\rho[f] \) in an explicit expression

\[
\mathbb{E}_\rho[f] = \sum_{N=0}^{\infty} e^{-\rho|U|} \frac{(\rho|U|)^N}{N!} \int_{U^N} f\left(\sum_{i=1}^{N} \delta_{x_i}\right) \, dx_1 \cdots dx_N.
\]

For instance, for every bounded Borel set \( U \subseteq \mathbb{R}^d \) and bounded measurable function \( g : U \rightarrow \mathbb{R} \), we can write

\[
\mathbb{E}_\rho\left[ \int_U g(x) \, d\mu(x) \right] = \rho \int_U g(x) \, dx.
\]

Notice that the measure \( \mu \) is a Poisson point process under \( \mathbb{P}_\rho \). In particular, the measures \( \mu \ll U \) and \( \mu \ll (\mathbb{R}^d \setminus U) \) are independent, and the conditional expectation \( \mathbb{E}_\rho[\mu|\mathcal{F}(\mathbb{R}^d \setminus U)] \) can thus be described equivalently as an averaging over the law of \( \mu \ll U \).

For any \( 1 \leq p < \infty \), we denote by \( L^p(\mathcal{M}_\delta(U)) \) the set of \( \mathcal{F}_U \)-measurable functions \( f : \mathcal{M}_\delta(U) \rightarrow \mathbb{R} \) such that the norm

\[
\|f\|_{L^p(\mathcal{M}_\delta(U))} := \left( \mathbb{E}_\rho[|f|^p] \right)^{\frac{1}{p}}
\]

is finite and \( L^p \) short for \( L^p(\mathcal{M}_\delta(\mathbb{R}^d)) \). We denote by \( L^\infty(\mathcal{M}_\delta(U)) \) the norm defined by essential upper bound under \( \mathbb{P}_\rho \).

2.1.2. Derivative and \( C^\infty_0(\mathcal{M}_\delta(U)) \). We define the directional derivative for a \( \mathcal{F}_U \)-measurable function \( f : \mathcal{M}_\delta(U) \rightarrow \mathbb{R} \). Let \( \{e_k\}_{1 \leq k \leq d} \) be \( d \) canonical directions, for \( x \in \text{supp}(\mu) \), we define

\[
\partial_k f(\mu, x) := \lim_{h \to 0} \frac{1}{h}(f(\mu - \delta_x + \delta_x + he_k) - f(\mu)),
\]

if the limit exists, and the gradient as a vector

\[
\nabla f(\mu, x) := (\partial_1 f(\mu, x), \partial_2 f(\mu, x), \ldots, \partial_d f(\mu, x)).
\]

One can define the function with higher derivative iteratively, but here we use a more natural way: for every Borel set \( U \subseteq \mathbb{R}^d \) and \( N \in \mathbb{N} \), let \( \mathcal{M}_\delta(U, N) \subseteq \mathcal{M}_\delta(E) \) be defined as

\[
\mathcal{M}_\delta(U, N) := \left\{ \mu \in \mathcal{M}_\delta(\mathbb{R}^d) : \mu = \sum_{i=1}^{N} x_i, x_i \in U \text{ for every } 1 \leq i \leq N \right\}.
\]

Then a function \( f : \mathcal{M}_\delta(U, N) \rightarrow \mathbb{R} \) can be identified with a function \( \tilde{f} : U^N \rightarrow \mathbb{R} \) by setting

\[
\tilde{f}(x) = \tilde{f}(x_1, \ldots, x_N) := f\left(\sum_{i=1}^{N} \delta_{x_i}\right).
\]

The function \( \tilde{f} \) is invariant under permutations of its \( N \) coordinates. Conversely, any function satisfying this symmetry can be identified with a function from \( \mathcal{M}_\delta(U, N) \) to \( \mathbb{R} \). We denote by \( C^\infty(\mathcal{M}_\delta(U, N)) \) the set of functions \( f : \mathcal{M}_\delta(U, N) \rightarrow \mathbb{R} \) such that \( \tilde{f} \) is infinitely differentiable. For every \( f \in C^\infty(\mathcal{M}_\delta(U, N)) \) and \( x_1, \ldots, x_N \in U \), the gradient at \( x_1 \) coincide with the its canonical sense for the coordinate \( x_1 \).

\[
\nabla f\left(\sum_{i=1}^{N} \delta_{x_i}, x_1\right) = \nabla_{x_1} \tilde{f}(x_1, \ldots, x_N).
\]

We denote by \( C^\infty_0(\mathcal{M}_\delta(U)) \) the set of functions \( f : \mathcal{M}_\delta(U) \rightarrow \mathbb{R} \) that satisfy:

1. there exists a compact Borel set \( V \subseteq U \) such that \( f \) is \( \mathcal{F}_V \)-measurable;
for every \( N \in \mathbb{N} \),

\[
\begin{align*}
\mathcal{M}_\delta(U, N) & \to \mathbb{R} \\
\mu & \mapsto f(\mu)
\end{align*}
\]

belongs to \( C^\infty(\mathcal{M}_\delta(U, N)) \).

(3) the function is bounded.

A more heuristic description for \( f \in C^\infty_c(\mathcal{M}_\delta(U)) \) is a function uniformly bounded, depending only on the information in a compact subset \( V \subseteq U \), and when we do projection \( f(\mu) = f(\mu \wedge V) \) it can be identified as a function \( C^\infty \) with finite coordinate, and also smooth when the number of particles in \( V \) changes.

2.1.3. Sobolev space on \( \mathcal{M}_\delta(U) \). We define the \( H^1(\mathcal{M}_\delta(U)) \) norm by

\[
\|f\|_{H^1(\mathcal{M}_\delta(U))} := \left( \|f\|^2_{L^2(\mathcal{M}_\delta(U))} + \mathbb{E}_\rho \left[ \int_U |\nabla f|^2 \, d\mu \right] \right)^{1/2},
\]

and let \( H^1_0(\mathcal{M}_\delta(U)) \) denote the completion with respect to this norm of the space

\[
\left\{ f \in C^\infty_c(\mathcal{M}_\delta(U)) : \|f\|_{H^1(\mathcal{M}_\delta(U))} < \infty \right\}.
\]

2.2. Construction of model.

2.2.1. Diffusion coefficient. In this part, we define the coefficient field of the diffusion. We give ourselves a measurable function \( a_\circ : \mathcal{M}_\delta(\mathbb{R}^d) \to \mathbb{R} \) which satisfies the following properties:

- uniform ellipticity: there exists \( \Lambda \in [1, +\infty) \) such that for every \( \mu \in \mathcal{M}_\delta(\mathbb{R}^d) \),

\[
1 \leq a_\circ(\mu) \leq \Lambda;
\]

- locality: for every \( \mu \in \mathcal{M}_\delta(\mathbb{R}^d) \), \( a_\circ(\mu) = a_\circ(\mu \wedge B_1) \).

We extend \( a_\circ \) by stationarity using the transport operation defined in eq. (2.3): for every \( \mu \in \mathcal{M}_\delta(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \),

\[
a(\mu, x) := \tau_x a_\circ(\mu) = a_\circ(\tau_x \mu).
\]

A typical example of a coefficient field \( a \) of interest is \( a_\circ(\mu) := 1 + 1_{\mu(B_1)=1} \) whose extension is given by \( a(\mu, x) := 1 + 1_{\mu(B_1(x))=1} \). In words, for \( x \in \text{supp}(\mu) \), the quantity \( a(\mu, x) \) is equal to \( 2 \) whenever there is no other point than \( x \) in the unit ball around \( x \), and is equal to \( 1 \) otherwise.

2.2.2. Markov process defined by Dirichlet form. In this part, we construct our infinite particle system on \( \mathcal{M}_\delta(\mathbb{R}^d) \) by Dirichlet form (see [20, 35] for the notations). We define at first the non-negative bilinear symmetric form

\[
\mathcal{E}^a(f, g) := \mathbb{E}_\rho \left[ \int_{\mathbb{R}^d} a(\mu, x) \nabla f(\mu, x) \cdot \nabla g(\mu, x) \, d\mu \right],
\]

on its domain \( \mathcal{D}(\mathcal{E}^a) \) that

\[
\mathcal{D}(\mathcal{E}^a) := H^1_0(\mathcal{M}_\delta(\mathbb{R}^d)).
\]

We also use \( \mathcal{E}^a(f) := \mathcal{E}^a(f, f) \) for short. It is clear that \( \mathcal{E}^a \) is closed and Markovian thus it is a Dirichlet form, so it defines the correspondence between the Dirichlet form and the generator \( \mathcal{L} \) that

\[
\mathcal{E}^a(f, g) = \mathbb{E}_\rho \left[ f(\mathcal{L} g) \right], \quad \mathcal{D}(\mathcal{E}^a) = \mathcal{D}(-\mathcal{L}).
\]

and a \( L^2(\mathcal{M}_\delta(\mathbb{R}^d)) \) strongly continuous Markov semigroup \( (P_t)_{t \geq 0} \). We denote by \((\mathcal{F}_0)_{t \geq 0}\) its filtration and \((\mu_t)_{t \geq 0}\) the associated \( \mathcal{M}_\delta(\mathbb{R}^d) \)-valued Markov process which stands the configuration of the particles, then for any \( u \in L^2(\mathcal{M}_\delta(\mathbb{R}^d)) \),

\[
u_t(\mu) := P_t u(\mu) = \mathbb{E}_\rho[u(\mu_t) | \mathcal{F}_0],
\]
is an element in $\mathcal{D}(\mathcal{E}^a)$ and is characterized by the parabolic equation on $\mathcal{M}_{\rho}(\mathbb{R}^d)$ that for any $v \in \mathcal{D}(\mathcal{E}^a)$

\begin{equation}
(2.8) \quad \mathbb{E}_\rho[u_t v] - \mathbb{E}_\rho[v u] = - \int_0^t \mathcal{E}^a(\mu_s, v) \, ds.
\end{equation}

Finally, we remark that the average is conserved for $u_t$ as we test eq. (2.8) by constant $1$ that

\begin{equation}
(2.9) \quad \mathbb{E}_\rho[u_t] - \mathbb{E}_\rho[u] = - \int_0^t \mathbb{E}_\rho \left[ \int_{\mathbb{R}^d} \mathbf{a}(\mu, x) \nabla \cdot \nabla u_a(\mu, x) \, d\mu \right] \, ds = 0.
\end{equation}

In this work, we focus more on the quantitative property of $P_t$; see [39] for more details about the trajectory property of similar type of process.

2.3. A solvable case. We propose a solvable model to illustrate that the behavior of this process is close to the diffusion and the rate of decay is the best one that we can expect.

In the following, we suppose that $a = \frac{1}{2}$ which means that in fact every particle evolves as a Brownian motion i.e. $\mu = \sum_{i=1}^{\infty} \delta_{x_i}$, $\mu_t = \sum_{i=1}^{\infty} \delta_{B_t^{(i)}}$ that $(B_t^{(i)})_{t \geq 0}$ is a Brownian motion issued from $x_i$ and $(B^{(i)})_{i \in \mathbb{N}}$ is independent and independent with $\mu$.

**Example 2.1.** Let $u(\mu) := \int_{\mathbb{R}^d} f \, d\mu$ with $f \in C^\infty_c(\mathbb{R}^d)$. In this case, we have

$$u_t(\mu) = P_t u(\mu) = \mathbb{E}_\rho[u(\mu_t) | \mathcal{F}_0] = \mathbb{E}_\rho\left[ \sum_{i \in \mathbb{N}} f \left( B_t^{(i)} \right) \right] = \int_{\mathbb{R}^d} f_t(x) \, d\mu(x),$$

where $f_t \in C^\infty_c(\mathbb{R}^d)$ is the solution of the Cauchy problem of the standard heat equation: let $\Phi_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left( -\frac{|x|^2}{2t} \right)$, then $f_t(x) = \Phi_t \ast f(x)$. Then we use the formula of variation for Poisson process

$$\text{Var}_{\rho}[u] = \int_{\mathbb{R}^d} f^2(x) \, dx = \|f\|_{L^2(\mathbb{R}^d)}^2,$$

$$\text{Var}_{\rho}[u_t] = \int_{\mathbb{R}^d} f_t^2(x) \, dx = \|f_t\|_{L^2(\mathbb{R}^d)}^2.$$

By the heat kernel estimate for the standard heat equation, we known that $\|f_t\|_{L^2(\mathbb{R}^d)} \leq C(t) t^{-\frac{d}{2}} \|f\|_{L^2(\mathbb{R}^d)}$, thus the scale $t^{-\frac{d}{2}}$ is the best one that we can obtain. Moreover, if we take $f = 1_{Q_r}$, and $t = r^{2(1-\varepsilon)}$ for a small $\varepsilon > 0$, then we see that the typical scale of diffusion is a ball of size $r^{1-\varepsilon}$.

So for every $x \in Q_r(1-r^{-\varepsilon})$, the value $f_t(x) \approx 1 - e^{-r^{2(1-\varepsilon)}}$ and we have

$$\text{Var}_{\rho}[u_t] \approx \int_{\mathbb{R}^d} f_t^2(x) \, dx \geq r^d (1 - r^{-2\varepsilon}) = (1 - r^{-2\varepsilon}) \text{Var}_{\rho}[u].$$

It illustrates that before the scale $t = r^2$, the decay is very slow so in the Theorem 1.1 the factor $\left( \frac{1}{\sqrt{t}} \right)^d$ is reasonable.

3. Strategy of proof

In this part, we state the strategy of the proof of Theorem 1.1. We will give a short outline in Section 3.1, which can be see as an “approximation-variance decomposition”, and then focus on the term approximation in Section 3.2. Several technical estimates will be used in this procedure and their proofs will be postponed in Section 4 and Section 5.
3.1. Outline. As mentioned, this work is inspired from \cite{30}, and we revisit the strategy here. We pick a centered \( u \in C^\infty_c(M_d(\mathbb{R}^d)) \) supported in \( Q_{l_u} \) such that \( \mathbb{E}_\rho[u] = 0 \) and this implies \( \mathbb{E}_\rho[u_t] = 0 \) from eq. (2.9). Then we set a multiscale \( \{ t_n \}_{n \geq 0}, t_{n+1} = R t_n, \) where \( R > 1 \) is a scale factor to be fixed later. It suffices to prove that eq. (1.2) for every \( t_n, \) then for \( t \in [t_n, t_{n+1}], \) one can use the decay of \( L^2 \) that

\[
\mathbb{E}_\rho[(u_t)^2] \leq \mathbb{E}_\rho[(u_{t_n})^2] \leq C(\log(1 + t_n)) \frac{1}{t_n^d} \| u \|^2_{L^\infty} \leq CR^d \gamma(\log(1 + t)) \frac{l_n}{\sqrt{t_n}} \| u \|^2_{L^\infty},
\]

then by resetting the constant \( C \) one concludes the main theorem. Another ingredient of the proof is an “approximation-variance type decomposition”:

\[
u_t = v_t + w_t,
\]
\[
v_t := u_t - \frac{1}{|Z_K|} \sum_{y \in Z_K} \tau_y u_t,
\]
\[
w_t := \frac{1}{|Z_K|} \sum_{y \in Z_K} \tau_y u_t,
\]

where we recall \( Z_K = Q_K \cap \mathbb{Z}^d \) is the lattice set of scale \( K \). The philosophy of this decomposition is that in long time, the information in a local scale \( K \) is mixed, thus \( w_t \) is a spatial average is a good approximation of \( u_t \) and \( v_t \) is the error term. Thus, the following control Proposition 3.1 and Proposition 5.2 of the two terms \( w_t \) and \( v_t \) proves the main theorem Theorem 1.1.

**Proposition 3.1.** There exists a finite positive number \( C := C(d) \) such that for any \( u \in C^\infty_c(M_d(\mathbb{R}^d)) \) supported in \( Q_{l_u} \) and \( K \geq l_u, \) we have

\[
\text{Var}_\rho \left[ \left( \frac{1}{|Z_K|} \sum_{y \in Z_K} \tau_y u_t \right)^2 \right] \leq C(d) \frac{l_u}{K} \mathbb{E}_\rho[u^2].
\]

**Proof.** Then we can estimate the variance simply by \( L^2 \) decay that

\[
\mathbb{E}_\rho[(w_t)^2] = \mathbb{E}_\rho \left[ \left( P_t \left( \frac{1}{|Z_K|} \sum_{y \in Z_K} \tau_y u \right) \right)^2 \right] \leq \mathbb{E}_\rho \left[ \left( \frac{1}{|Z_K|} \sum_{y \in Z_K} \tau_y u \right)^2 \right] = \frac{1}{|Z_K|^2} \sum_{x, y \in Z_K} \mathbb{E}_\rho[(\tau_{x-y} u) u].
\]

We know that for \( |x - y| \geq l_u, \) then the term \( \tau_{x-y} u \) and \( u \) is independent so \( \mathbb{E}_\rho[(\tau_{x-y} u) u] = 0. \) This concludes eq. (3.2). \( \square \)

**Proposition 3.2.** There exists two finite positive numbers \( C := C(d, \rho), \gamma := \gamma(d, \rho) \) such that for any \( u \in C^\infty_c(M_d(\mathbb{R}^d)) \) supported in \( Q_{l_u} \), \( K \geq l_u \) and \( v_t \) defined in eq. (3.1), for \( \{ t_n \}_{n \geq 0}, t_{n+1} = R t_n, R > 1 \) we have

\[
(t_n + 1)^{\frac{d+2}{2}} \mathbb{E}_\rho[(v_{t_n+1})^2] - (t_n)^{\frac{d+2}{2}} \mathbb{E}_\rho[(v_{t_n})^2] \leq C(\log(t_{n+1}))\gamma K^{-2}(l_u)^d \| u \|^2_{L^\infty} \mathbb{E}_\rho[u^2].
\]
Proof of Theorem 1.1 from Proposition 3.2 and Proposition 3.1. Without loss of generality, we can suppose that \( t \geq 1 \). We put eq. (3.3) into eq. (1.2) by setting \( K := \sqrt{t_{n+1}} \) that
\[
\mathbb{E}_\rho((u_{t_{n+1}})^2) \leq 2\mathbb{E}_\rho([v_{t_{n+1}}]^2) + 2\mathbb{E}_\rho([w_{t_{n+1}}]^2) \\
\leq 2\left( \frac{t_n}{t_{n+1}} \right)^{\frac{d+2}{2}} \mathbb{E}_\rho([v_{t_n}]^2) + 2(t_{n+1})^{-\frac{d+2}{2}} (C(\log(t_{n+1}))^\gamma t_{n+1} (l_u)^d \| u \|_{L_\infty}^2 + \mathbb{E}_\rho[u^2]) \\
\quad + 2\mathbb{E}_\rho([w_{t_{n+1}}]^2) \\
\leq 4\left( \frac{t_n}{t_{n+1}} \right)^{\frac{d+2}{2}} \mathbb{E}_\rho([u_{t_n}]^2) + 2(t_{n+1})^{-\frac{d+2}{2}} (C(\log(t_{n+1}))^\gamma t_{n+1} (l_u)^d \| u \|_{L_\infty}^2 + \mathbb{E}_\rho[u^2]) \\
\quad + 4\left( \frac{t_n}{t_{n+1}} \right)^{\frac{d+2}{2}} \mathbb{E}_\rho([w_{t_n}]^2) + 2\mathbb{E}_\rho([w_{t_{n+1}}]^2).
\] (3.4)

We set \( U_n = (t_n)^{\frac{d}{2}} \mathbb{E}_\rho([u_{t_n}]^2) \) and put eq. (3.2) into the equation above, we have
\[
U_{n+1} \leq \theta U_n + C_2 \left( (\log(t_{n+1}))^\gamma (l_u)^d \| u \|_{L_\infty}^2 + \mathbb{E}_\rho[u^2] \right) + C_3 (l_u)^d \mathbb{E}_\rho[u^2],
\]
where \( \theta = 4R^{-1} \). By choose \( R \) large such that \( \theta \in (0,1) \) and \( t_0 = (l_u)^2 \), we do a iteration for the equation above to obtain that
\[
U_{n+1} \leq \sum_{k=1}^{n} \left( C_2 \left( (\log(t_{k+1}))^\gamma (l_u)^d \| u \|_{L_\infty}^2 + \mathbb{E}_\rho[u^2] \right) + C_3 (l_u)^d \mathbb{E}_\rho[u^2] \right) \theta^{n-k} + U_0 \theta^{n+1} \\
\leq \frac{1}{1-\theta} \left( C_2 \left( (\log(t_{n+1}))^\gamma (l_u)^d \| u \|_{L_\infty}^2 + \mathbb{E}_\rho[u^2] \right) + C_3 (l_u)^d \mathbb{E}_\rho[u^2] \right) + (l_u)^d \mathbb{E}_\rho[u^2] \\
\Longrightarrow \mathbb{E}_\rho((u_{t_{n+1}})^2) \leq C_4 (\log(t_{n+1}))^\gamma \left( \frac{l_u}{\sqrt{t_{n+1}}} \right)^d \| u \|_{L_\infty}^2.
\]

Remark 3.3. We remark that there is a small error in the similar argument in [30, Proof of Proposition 2.2]: the authors apply eq. (3.3) from \( t_0 \) to \( t_n \), and they neglect the change of scale in \( K \) at the endpoints \( \{t_n\}_{n>0} \). However, it does not harm the whole proof and we fix it here: we add one more step of decomposition in eq. (3.4), and put the iteration directly in \( u_t \) instead of \( v_t \), which avoids the problem of the changes of \( K \).

3.2. Error for the approximation. In this part, we prove Proposition 3.2. The proof can be divided into 6 steps.

Proof of Proposition 3.2. Step 1: Setting up. To shorten the equation, we define
\[
\Delta_n := (t_{n+1})^{\frac{d+2}{2}} \mathbb{E}_\rho([v_{t_{n+1}}]^2) - (t_n)^{\frac{d+2}{2}} \mathbb{E}_\rho([v_{t_n}]^2),
\]
and it is the goal of the whole subsection. In the step setting up, we do derivative for the flow \( t^{\frac{d+2}{2}} \mathbb{E}_\rho([v(t)]^2) \) that
\[
\Delta_n = \int_{t_n}^{t_{n+1}} \left( \frac{d+2}{2} \right) t^{\frac{d}{2}} \mathbb{E}_\rho([v(t)]^2) - 2t \frac{d+2}{2} \mathbb{E}_\rho(v_t(-\mathcal{L}v_t)) dt.
\] (3.6)

Step 2: Localization. We set \( \mathcal{A}_t v := \mathbb{E}[v_t|\mathcal{F}_{t_n}] \) and use it to approximate \( v_t \) in \( L^2 \). Since it is a diffusion process, one can guess naturally a scale larger than \( \sqrt{t} \) will have enough information for this approximation. In Theorem 4.1 we prove an estimate
\[
\mathbb{E}_\rho\left[ (v_t - \mathcal{A}_t v_t)^2 \right] \leq C(A) \exp\left( -\frac{L}{\sqrt{t}} \right) \mathbb{E}_\rho\left[ (v_0)^2 \right].
\]
and we choose $L = [\gamma \log(t_{n+1})]^{1/2}$ here, and put it back to eq. (3.6) to obtain
\begin{equation}
\Delta_n \lesssim \int_{t_n}^{t_{n+1}} (d + 2)t^{\frac{d}{2}}E_\rho[(A_L v_t)^2] + (d + 2)t^{\frac{d}{2} - \gamma}E_\rho[(v_0)^2] - 2t^{\frac{d+2}{2}}E_\rho[v_t(-\mathcal{L} v_t)]\,dt \tag{3.7}
\end{equation}

where we obtain the last line by choosing a scale $l$ and we choose $c = (L/l)^d$ and $M_{L,l} = (M_1, M_2, \ldots, M_q)$ a random vector, where $M_i$ is the number of the particle in $i$-th cube of scale $l$. Then we define an operator
\begin{equation}
B_{L,l}v_t := E_\rho[v_t|M_{L,l}].
\end{equation}

The main idea here is that the random vector $M_{L,l}$ captures the information of convergence, once we know the density in every cube of scale $l$ converges to $\rho$. In Proposition 5.1 we will prove a spectral inequality that
\begin{equation}
E_\rho[(A_L v_t - B_{L,l}v_t)^2] \leq R_0l^2E_\rho[v_t(-\mathcal{L} v_t)].
\end{equation}

We put this estimate into eq. (3.7)
\begin{equation}
\Delta_n \lesssim E_\rho[(u_0)^2] + \int_{t_n}^{t_{n+1}} 2(d + 2)t^{\frac{d}{2}}E_\rho[(B_{L,l}v_t)^2] + 2t^{\frac{d}{2}}((d + 2)R_0l^2 - t)E_\rho[v_t(-\mathcal{L} v_t)]\,dt \tag{3.8}
\end{equation}

where we obtain the last line by choosing a scale $l = c\sqrt{t_{n+1}}$ such that $(d + 2)R_0l^2 \leq t_n$ and $L/l \in \mathbb{N}$.

It remains to estimate how small $E_\rho[(B_{L,l}v_t)^2]$ is. The typical case is that the density is close to $\rho$ in every cube of scale $l$ in $Q_L$. Let us define $M = (M_1, M_2, \ldots, M_q)$, and we have
\begin{equation}
B_{L,l}v_t(M) = E_\rho[v_t|M_{L,l} = M].
\end{equation}

Then we call $C_{L,l,\rho,\delta}$ the $\delta$-good configuration that
\begin{equation}
C_{L,l,\rho,\delta} := \left\{ M \in \mathbb{N}^q \mid \forall 1 \leq i \leq q, \frac{M_i}{\rho|Q_l|} - 1 \leq \delta \right\}.
\end{equation}

We can use standard Chernoff bound and union bound to prove the upper bound of $\mathbb{P}_\rho[M_{L,l} \notin C_{L,l,\rho,\delta}]$: for any $\lambda > 0$, we have
\begin{equation}
\mathbb{P}_\rho\left[ \exists 1 \leq i \leq q, \frac{M_i}{\rho|Q_l|} \geq 1 + \delta \right] \leq \left( \frac{L}{l} \right)^d \exp(-\lambda(1 + \delta))E_\rho\left[ \exp\left( \frac{\lambda\mu(Q_l)}{\rho|Q_l|} \right) \right]
= \left( \frac{L}{l} \right)^d \exp\left( -\lambda(1 + \delta) + \rho|Q_l|\left( e^{\frac{\lambda}{\rho|Q_l|}} - 1 \right) \right)
\leq \left( \frac{L}{l} \right)^d \exp\left( -\lambda\delta + \frac{\lambda^2}{\rho|Q_l|} \right).
\end{equation}

In the second line we use the exact Laplace transform for $\mu(Q_l)$ as we know $\mu(Q_l)$ law Poisson($\rho|Q_l|$). Then we do optimization by choosing $\lambda = \frac{\delta\rho|Q_l|}{2}$. The other side is similar and we conclude
\begin{equation}
\mathbb{P}_\rho\left[ M_{L,l} \notin C_{L,l,\rho,\delta} \right] \leq (\gamma \log(t_{n+1}))^d \exp\left( -\frac{\rho|Q_l|\delta^2}{4} \right). \tag{3.9}
\end{equation}
For the case \( M \notin C_{L,l,p,\delta} \), we can bound \( B_{L,l}v_t(M) \) naively by \( |B_{L,l}v_t(M)| \leq C \| u_0 \|_{L^\infty} \), thus we have

\[
\mathbb{E}_\rho \left[ \left( B_{L,l}v_t \right)^2 \right] \leq \sum_{M \in C_{L,l,p,\delta}} \mathbb{P}_\rho[M_{L,l} = M](B_{L,l}v_t(M))^2 + (\gamma \log(t_{n+1}))^d \exp \left( -\frac{\rho |Q| \delta^2}{4} \right) \| u_0 \|_{L^\infty}^2
\]

and we finish this step by

\[
\Delta_n \leq \mathbb{E}_\rho[(u_0)^2] + (t_{n+1})^{\frac{d+2}{2}} (\gamma \log(t_{n+1}))^d \exp \left( -\frac{\rho |Q| \delta^2}{4} \right) \| u_0 \|_{L^\infty}^2
\]

\[(3.10)\]

We remark that the parameter \( \delta > 0 \) will be fixed at the end of the proof.

**Step 4: Perturbation estimate.** It remains to estimate the term \( (B_{L,l}v_t(M))^2 \) for the the \( \delta \)-good configuration. Now we put the expression of \( v_t \) in and obtain

\[
(B_{L,l}v_t(M))^2 = \left( \frac{1}{|Z_K|} \sum_{y \in Z_K} (B_{L,l}(u_t - \tau_y u_t))(M) \right)^2,
\]

and our aim is to control

\[(3.11)\]

\[
\int_{t_n}^{t_{n+1}} 2(d + 2) t^\frac{d}{2} \left( \frac{1}{|Z_K|} \sum_{y \in Z_K} (B_{L,l}(u_t - \tau_y u_t))(M) \right)^2 dt.
\]

To treat eq. (3.11), we calculate the Radon-Nikodym derivative that

\[(3.12)\]

\[g_M := \frac{d\mathbb{P}_\rho[M_{L,l} = M]}{d\mathbb{P}_\rho} = \frac{1}{\mathbb{P}_\rho[M_{L,l} = M]} \mathbb{1}_{\{M_{L,l} = M\}}.
\]

Then we use the reversibility of the semigroup \( P_t \) and denote by \( g_{M,l} := P_t g_M \)

\[B_{L,l}(u_t - \tau_y u_t)(M) = \mathbb{E}_\rho[g_M(u_t - \tau_y u_t)] = \mathbb{E}_\rho[g_{M,l}(u - \tau_y u)].
\]

Then we would like to apply the a perturbation estimate Proposition 5.2 to control it: let \( l_k := l_u + 2k \) then for any \( |y| \leq k \), we have

\[
\mathbb{E}_\rho[g_M(u_t - \tau_y u_t)] \leq C(d)(l_k \| u \|_{L^\infty})^2 \mathcal{E}_{Q_{l_k}}(\sqrt{g_M}).
\]

where \( \mathcal{E}_{Q_{l_k}}(\sqrt{g_M}) \) is a localized Dirichlet form defined in eq. (5.4). A heuristic analysis of order is \( \mathcal{E}_{Q_{l_k}}(\sqrt{g_M}) \approx O \left( \frac{d}{(l_k)^d} \right) \) since it is a Dirichlet form on \( Q_{l_k} \). If we choose \( k = K \) here to cover all the term, the bound will be of order \( O(K^d) \), which is big when \( K \approx \sqrt{t} \geq l_u \).

Therefore, we apply a coarse-graining argument: let \( [0,y]_k := \{z_i\}_{0 \leq i < k} \) be a lattice path that of scale \( k \); \( z_0 = 0, z_n(u) = y, \{z_i\}_{1 \leq i < n(u)} \in (kZ)^d \) so the length of path is the shortest one. (See Figure 1 for illustration.) Then we have

\[
(u - \tau_y u) = \sum_{i=0}^{n(y)-1} (\tau_{z_i} u - \tau_{z_i+1} u) = \sum_{i=0}^{n(y)-1} \tau_{z_i} (u - \tau_{h_i} u),
\]
where $h_{z_i} = z_{i+1} - z_i$ the vector connecting the two and $|h_{z_i}| \leq k$. This expression with the transport invariant law of Poisson point process, Cauchy-Schwartz inequality

$$\left( \mathcal{B}_{L,t} (u_t - \tau_y u_t) (M) \right)^2 = \left( \sum_{z \in [0,y]_k} \mathbb{E}_p [ g_{M,t} \tau_z (u - \tau_{h_z} u) ] \right)^2$$

(3.13)

$$\leq C(d) n(y) \sum_{z \in [0,y]_k} \left( \mathbb{E}_p [ (\tau_z g_{M,t}) (u - \tau_{h_z} u) ] \right)^2 .$$

This term appears a perturbation estimate, which will be proved in Proposition 5.2 that

$$(\mathbb{E}_p [ (\tau_z g_{M,t}) (u - \tau_{h_z} u) ] )^2 \leq C(d)(l_k | u|_{L^\infty})^2 \mathcal{E}_{Q_{l_k}} \left( \sqrt{\mathcal{E}_{Q_{l_k}}} \right),$$

where in the last step we use the transport invariant property of Poisson point process. Now we turn to the choice of the scale $k$. By the heuristic analysis that every $\mathcal{E}_{Q_{l_k}}$ contributes order $O((l_k)^d)$ and taking in account $n(y) \leq K/k$ we have in eq. (3.13)

$$\left( \mathcal{B}_{L,t} (u_t - \tau_y u_t) (M) \right)^2 \approx O \left( \left( \frac{K}{k} \right)^2 (l_k)^{d+2} \right) \approx O \left( \left( \frac{K}{l_u} \right)^2 (l_u + 2k)^{d+2} \right).$$

From this we see that a good scale should be $k = l_u$ so the term above is of order $O(K^2(l_u)^d)$. We put these estimate back to eq. (3.11)

$$\text{(3.14) eq. (3.11)} \quad \leq \ |u|^2_{L^\infty} \int_{l_u}^{l_{n+1}} 2(d+2) t^2 K l_u \left( \frac{1}{|Z_K|} \sum_{y \in Z_K} \sum_{z \in [0,y]_l} \mathcal{E}_{\tau_z Q_{l_u}} \left( \sqrt{g_{M,t}} \right) \right) dt .$$

**Figure 1.** The illustration of the coarse-graining argument, where we take a lattice path of scale $k$ to connect 0 and $y$. The ball in blue is the support of $u$ and the box in red is $Q_{l_k}$. For the one on the left, the scale is $k = l_u$; the one on the right the scale is finer and we see that the coarse-graining is too dense.

**Step 5: Covering argument.** In this step, we calculate the right hand side of eq. (3.14), where we notice one essential problem: there are totally about $K^{d+1}/l_u$ terms of Dirichlet form
\[ E_{\tau_z Q M_u}(\sqrt{g M_t}) \text{ in the sum } \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} E_{\tau_z Q M_u}(\sqrt{g M_t}), \text{ but the one with } z \text{ close to 0 are counted of of order } K^d \text{ times, while the one with } z \text{ near } \partial \mathcal{Z}_K \text{ are counted only constant times.} \]

To solve this problem, we have to reaverage the sum: by the transport invariant property of Poisson point process, at the beginning of the Step 1, we can write

\[ \Delta_n = \frac{1}{|Z|} \sum_{x \in \mathcal{Z}_t} (t_{n+1}) \frac{d}{d\tau} E_\tau[(\tau_z v_{t_{n+1}})^2] - (t_n) \frac{d}{d\tau} E_\tau[(\tau_z v_{t_n})^2] \].

Then all estimates works in Step 1, Step 2 and Step 3 work by replacing \( v_t \mapsto \tau_x v_t \) and \( u_t \mapsto \tau_x u_t \). In the Step 4, this operation will change our object term eq. (3.11)

\[ \text{eq. (3.11)-avg } = \int_{t_n}^{t_{n+1}} 2(d + 2) t^d \left( \frac{1}{|Z|} \sum_{w \in \mathcal{Z}_t} \frac{1}{|Z_K|} \sum_{y \in \mathcal{Z}_K} (B_{L,t} u_t - \gamma_{y u_t}(M))^2 \right) dt, \]

and the perturbation argument Proposition 5.2 reduces the problem as

\[ (3.15) \quad \text{eq. (3.11)-avg } \leq \|u\|_{L^\infty}^2 \int_{t_n}^{t_{n+1}} 2(d + 2) t^d K_{l_u} \]

\[ \times \left( \frac{1}{|Z|} \sum_{w \in \mathcal{Z}_t} \frac{1}{|Z_K|} \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} E_{\tau_{wz} Q M_u}(\sqrt{g M_t}) \right) dt. \]

Now we can apply the Fubini’s lemma

\[ \frac{1}{|Z|} \sum_{w \in \mathcal{Z}_t} \frac{1}{|Z_K|} \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} E_{\tau_{wz} Q M_u}(\sqrt{g M_t}) \]

\[ = \frac{1}{|Z|} \frac{1}{|Z_K|} E_p \left[ \int_{\mathcal{R}^d} \left( \sum_{w \in \mathcal{Z}_t} \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} 1_{x \in \tau_{wz} Q M_u} \right) \nabla \sqrt{g M_t}(\mu, x) \cdot \nabla \sqrt{g M_t}(\mu, x) d\mu(x) \right] , \]

while we notice that

\[ \sum_{w \in \mathcal{Z}_t} \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} 1_{x \in \tau_{wz} Q M_u} = \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} \sum_{w \in \mathcal{Z}_t} 1_{\tau_{wz} Q M_u} \]

\[ \leq |Q_{l_u}| \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} (3|l_u|)^d \leq C(d)(l_u)^{d-1} K^{d+1}, \]

so we have

\[ \frac{1}{|Z|} \sum_{w \in \mathcal{Z}_t} \frac{1}{|Z_K|} \sum_{y \in \mathcal{Z}_K} \sum_{z \in [0,y]_{l_u}} E_{\tau_{wz} Q M_u}(\sqrt{g M_t}) \leq \frac{C(d)(l_u)^{d-1} K}{|Z|} E(\sqrt{g M_t}). \]

We put this estimate to eq. (3.15) and use \( \ell = c\sqrt{t_{n+1}}, \)

\[ \text{eq. (3.11)-avg } \leq C(d)\|u\|_{L^\infty}^2 K^2(l_u)^d \int_{t_n}^{t_{n+1}} \left( \frac{t^{\frac{d}{2}}}{\ell} \right) \frac{d}{dt} \left( \sqrt{g M_t} \right) dt \]

\[ \leq C(d)\|u\|_{L^\infty}^2 K^2(l_u)^d \int_{t_n}^{t_{n+1}} E(\sqrt{g M_t}) dt. \]
We put eq. (3.16) back to eq. (3.11) and eq. (3.10) and conclude
\[
\begin{align*}
\Delta_n & \leq \mathbb{E}_\rho[(u_0)^2] + (t_{n+1})^{\frac{d+2}{2}} (\gamma \log(t_{n+1}))^d \exp\left(-\frac{\rho |Q| \delta^2}{4}\right) \|u_0\|_{L^\infty}^2 \\
& \quad + C(d)\|u\|_{L^\infty}^2 K^2(t_u)^d \sum_{M \in \mathcal{C}_{L,L',\rho,\delta}} \mathbb{P}_\rho[M_{L,L'} = M] \int_{t_n}^{t_{n+1}} \mathcal{E}(\sqrt{g_{M,t}}) \, dt.
\end{align*}
\] (3.17)

**Step 6: Entropy inequality.** In this step, we analyze the quantity \( \int_{t_n}^{t_{n+1}} \mathcal{E}(\sqrt{g_{M,t}}) \, dt \). We recall the definition of the entropy inequality: let \( H(g_M) = \mathbb{E}_\rho[g_M \log(g_M)] \), then
\[
H(g_{M,t}) = H(g_M) - 4 \int_0^t \mathbb{E}_\rho[\sqrt{g_{M,s}}(-L\sqrt{g_{M,s}})] \, ds,
\] (3.18)

we have
\[
\int_{t_n}^{t_{n+1}} \mathcal{E}(\sqrt{g_{M,t}}) \, dt \leq \int_{t_n}^{t_{n+1}} \mathbb{E}_\rho[\sqrt{g_{M,t}}(-L\sqrt{g_{M,t}})] \, dt \leq H(g_{M,t_{n+1}}) \leq H(g_M).
\]

For any \( M \in \mathcal{C}_{L,L',\rho,\delta} \), one can calculate the bound of the entropy and we prove it in Lemma 5.4
\[
H(g_M) \leq C(d,\rho) \left( \frac{L}{t} \right)^d \left( \log(l) + t^d \delta^2 \right).
\]

This helps us conclude that
\[
\begin{align*}
\Delta_n & \leq \mathbb{E}_\rho[(u_0)^2] \\
& \quad + \|u_0\|_{L^\infty}^2 (\gamma \log(t_{n+1}))^d (t_{n+1})^{\frac{d+2}{2}} \exp\left(-\frac{\rho |Q| \delta^2}{4}\right) + K^2(t_u)^d \left( \log(l) + t^d \delta^2 \right).
\end{align*}
\]

To make the bound small, we choose a parameter \( \delta = c(d,\rho)(\log t_{n+1})^{\frac{3}{2}}(t_{n+1})^{-\frac{d}{2}} \), where \( c(d,\rho) \) is a positive number large enough to compensate the term \( (t_{n+1})^{\frac{d+2}{2}} \) and this proves eq. (3.3) \( \square \)

4. **Localization estimate**

In this part, we prove the key localization estimate: we recall our notation of conditional expectation here that \( \mathcal{A}_s f = \mathbb{E}[f|\mathcal{F}_{Q_s}] \) for \( Q_s \) a closed cube \( \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]^d \).

**Theorem 4.1.** For \( u \in L^2(\mathcal{M}_\delta(\mathbb{R}^d)) \) of compact support that \( \text{supp}(u) \subseteq Q_t \), any \( t \geq \max\{L_t^2, 16A^2\} \), \( K \geq \sqrt{t} \), and \( u_t \) the function associated to the generator \( L \) at time \( t \), then we have the estimate
\[
\mathbb{E}_\rho\left[(u_t - \mathcal{A}_K u_t)^2\right] \leq C(\Lambda) \exp\left(-\frac{K}{\sqrt{t}}\right) \mathbb{E}_\rho\left[u_t^2\right].
\] (4.1)

This is an important inequality which allows us to pay some error to localize the function, and it is introduced in [30] and also used in [21]. The main idea to prove it is to use a multi-scale functional and analyze its evolution with respect to the time. Let us introduce its continuous version: for any \( f \in H^1_0(\mathcal{M}_\delta(\mathbb{R}^d)) \), \( f \mapsto (\mathcal{A}_s f)_{s \geq 0} \) is a càdlàg \( L^2 \)-martingale with respect to \( (\Omega, (\mathcal{F}_{Q_s})_{s \geq 0}, \mathbb{P}) \).

Our multiscale functional for \( f \in H^1_0(\mathcal{M}_\delta(\mathbb{R}^d)) \) is defined as
\[
S_{k,K,\beta}(f) = \alpha_K \mathbb{E}_\rho\left[(\mathcal{A}_k f)^2\right] + \int_k^K \alpha_s d\mathbb{E}_\rho\left[(\mathcal{A}_s f)^2\right] + \alpha_K \mathbb{E}_\rho\left[(f - \mathcal{A}_K f)^2\right],
\] (4.2)
with \( \alpha_s = \exp\left(\frac{s}{\beta}\right) \), \( \beta > 0 \). We can apply the integration by part formula for the Lebesgue-Stieltjes integral and obtain

\[
S_{k,K,\beta}(f) = \alpha_K \mathbb{E}_\rho \left[ f^2 \right] - \int_k^K \alpha_s' \mathbb{E}_\rho \left[ (A_s f)^2 \right] \, ds,
\]

where \( \alpha_s' \) is the derivative with respect to \( s \). The main idea is to put \( u_t \) in eq. (4.3) and then study its derivative \( \frac{d}{dt} S_{k,K,\beta}(f) \) and use it to prove Theorem 4.1. In this procedure, we will use the Dirichlet form for \( A_s f \), but we have to remark that in fact we do not know à priori this is a function in \( H^1_0(M_\delta(\mathbb{R}^d)) \). We will give a counter example to make it more clear in the next section and introduce a regularized version of \( A_s f \) to pass this difficulty.

### 4.1. Conditional expectation, spatial martingale and its regularization.

\( (A_s f)_{s \geq 0} \) has nice property: we can treat it as a localized function or a martingale. Thus we use the notation

\[
M^f_s := A_s f,
\]

which is a more canonical notation in martingale theory. In this subsection, we would like to understand the regularity of the closed martingale \((M^f_s)_{s \geq 0}\). We will see it is a càdlàg \( L^2 \)-martingale and the jump happens when there is particles on the boundary \( \partial Q_s \). At first, we remark a useful property for Poisson point process.

**Lemma 4.2.** With probability 1, for any \( 0 < s < \infty \), there is at most one particle one the boundary \( \partial Q_s \).

**Proof.** We denote by

\[
\mathcal{N} := \{ \mu : \exists 0 < s < \infty, \text{ there exist more than two particles on } \partial Q_s \}.
\]

Then we choose an increasing sequence \( \{ s^\varepsilon_k \}_{k \geq 0} \) with \( s_0^\varepsilon = 0 \), such that

\[
\mathbb{R}^d = \bigcup_{k=1}^{\infty} C_{s_k^\varepsilon}, \quad C_{s_k^\varepsilon} := Q_{s_{k}^{\varepsilon}} \setminus Q_{s_{k-1}^{\varepsilon}}, \quad |C_{s_k^\varepsilon}| = \frac{\varepsilon}{k}.
\]

Then we have that

\[
\mathbb{P}_\rho[\mathcal{N}] \leq \mathbb{P}_\rho \left[ \exists k, \mu(C_{s_k^\varepsilon}) \geq 2 \right] \\
\leq \sum_{k=1}^{\infty} \mathbb{P}_\rho \left[ \mu(C_{s_k^\varepsilon}) \geq 2 \right] \\
\leq \sum_{k=1}^{\infty} (\rho|C_{s_k^\varepsilon}|)^2 \\
\leq (\rho \varepsilon)^2.
\]

We we let \( \varepsilon \) go down to 0 and prove that \( \mathbb{P}_\rho[\mathcal{N}] = 0 \). \( \square \)

For this reason, in the following, we can do modification of the probability space and always suppose that there is at most one particle on the boundary. This helps us to prove the following regularity property for \((M^f_s)_{s \geq 0}\).

**Lemma 4.3.** After a modification, for any \( f \in C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d)) \) the process \((M^f_s)_{s \geq 0}\) is a càdlàg \( L^2 \)-martingale with finite variation, and the discontinuity point occurs for \( s \) such that \( \mu(\partial Q_s) = 1 \).
Proof. By the classical martingale theory, we know that \( \{ F_{Q_s} \}_{s \geq 0} \) is a right continuous filtration, thus after a modification the process is càdlàg. Moreover, from Lemma 4.2 we can modify the value to 0 on a negligible set so that \( \mu(\partial Q_s) \leq 1 \) for all positive \( s \). It remains to prove that if \( \mu(\partial Q_s) = 0 \), then the process is also left continuous. In this case, there exists a \( 0 < \varepsilon < s \) such that for any \( 0 < \varepsilon < \varepsilon_0 \), we have \( \mu(Q_{s-\varepsilon}) = \mu(Q_s) \). Then

\[
A_s f(\mu) = A_s f(\mu \ll Q_s) = A_s f(\mu \ll Q_{s-\varepsilon}).
\]

We use \( \mu \ll Q_s = (\mu \ll Q_{s-\varepsilon}) + (\mu \ll (Q_s \setminus Q_{s-\varepsilon})) \), then

\[
A_{s-\varepsilon} f(\mu) = \mathbb{E}_\rho [A_s f(\mu \ll Q_s) | F_{Q_{s-\varepsilon}}] \\
= \mathbb{E}_\rho [A_s f(\mu \ll Q_{s-\varepsilon} + \mu \ll (Q_s \setminus Q_{s-\varepsilon})) | F_{Q_{s-\varepsilon}}] \\
= \mathbb{P}_\rho [\mu(Q_s \setminus Q_{s-\varepsilon}) = 0] A_s f(\mu \ll Q_{s-\varepsilon}) \\
+ \mathbb{P}_\rho [\mu(Q_s \setminus Q_{s-\varepsilon}) \geq 1] \mathbb{E}_\rho [A_s f(\mu \ll Q_{s-\varepsilon}) | F_{Q_{s-\varepsilon}}] \\
= e^{-\rho(Q_s \setminus Q_{s-\varepsilon})} A_s f(\mu \ll Q_{s-\varepsilon}) + \left( 1 - e^{-\rho(Q_s \setminus Q_{s-\varepsilon})} \right) \mathbb{E}_\rho [A_s f(\mu \ll Q_{s-\varepsilon}) | F_{Q_{s-\varepsilon}}].
\]

If we suppose that \( \| f \|_{L^\infty} \) is finite, then we have \( \lim_{\varepsilon \downarrow 0} A_{s-\varepsilon} f(\mu) = A_s f(\mu) \). Moreover, we have a estimate that

\[
|A_{s-\varepsilon} f(\mu) - A_s f(\mu)| \leq C \rho \varepsilon d^{-1} \| f \|_{L^\infty}.
\]

This implies that the process is locally Lipschitz, thus of finite variation. \( \Box \)

The following corollaries are simple applications of the result above.

**Corollary 4.4.** For \( f \in C_c^\infty(\mathcal{M}_d(\mathbb{R}^d)) \), we can define a bracket process for \( \{ \mathcal{M}_t^f \}_{s \geq 0} \); we define that

\[
[ \mathcal{M}_s^f ] := \sum_{0 < r < s} (\Delta \mathcal{M}_r^f)^2, \quad \Delta \mathcal{M}_s^f = \mathcal{M}_s^f - \mathcal{M}_{s-}, \quad \tau \text{ is jump point}.
\]

Then \( \{ (\mathcal{M}_s^f)^2 - [\mathcal{M}_s^f] \}_{s \geq 0} \) is a martingale with respect to \((\Omega, (\mathcal{F}_{Q_s})_{s \geq 0}, \mathbb{P}_\rho)\).

**Proof.** This is a direct result from jump process; see [29, Chapter 4e]. \( \Box \)

**Corollary 4.5.** Let \( x \in \text{supp}(\mu) \), and we define a stopping time for \( x \)

\[
\tau(x) := \min \{ s \geq 0 | x \in Q_s \},
\]

and the normal direction \( \overline{n}(x) \) and we define

\[
A_{\tau(x)} f(\mu - \delta_x + \delta_{x-}) := \lim_{\varepsilon \downarrow 0} A_{\tau(x) - \varepsilon} f(\mu - \delta_x + \delta_{x-} \overline{n}(x)).
\]

Then we have almost surely

\[
A_{\tau(x)} f(\mu) = A_{\tau(x)} f(\mu - \delta_x), \quad A_{\tau(x)} f(\mu - \delta_x + \delta_{x-}) = A_{\tau(x)} f(\mu).
\]

**Proof.** The equation \( A_{\tau(x)} f(\mu) = A_{\tau(x)} f(\mu - \delta_x) \) is the result of left continuous: from Lemma 4.2 we know with probability 1 there is only \( x \) on \( \partial Q_{\tau(x)} \) and \( \mu - \delta_x \) does not have particle on the boundary so we apply Lemma 4.3 and obtain this equation.

For the second equation, we have

\[
A_{\tau(x)} f(\mu) = \lim_{\varepsilon_1 \downarrow 0} A_{\tau(x)} f(\mu - \delta_x + \delta_{x-\varepsilon_1} \overline{n}(x)) \\
= \lim_{\varepsilon_1 \downarrow 0} A_{\tau(x)} f(\mu - \delta_x + \delta_{x- \varepsilon_1} \overline{n}(x)) \\
= \lim_{\varepsilon_2 \downarrow 0 \varepsilon_1 \downarrow 0} A_{\tau(x) - \varepsilon_2} f(\mu - \delta_x + \delta_{x- \varepsilon_1} \overline{n}(x)) \\
= \lim_{\varepsilon \downarrow 0} A_{\tau(x) - \varepsilon} f(\mu - \delta_x + \delta_{x- \varepsilon} \overline{n}(x)).
\]
In the last step, we use the uniformly left continuous for \( A_s f \) and the continuity with respect to \( x \).

One important remark about the conditional expectation is that in fact for \( f \in C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d)) \), we may have \( A_L f \notin C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d)) \). The reason is that the conditional expectation creates a small gap at the boundary for the function. Here we give an example of the conditional expectation for \( \mathbb{E}_\rho[f|\mathcal{F}_{B_r}] \), which is easier to state but it shares the same property of \( A_L f \).

**Example 4.6.** Let \( \eta \in C_c^\infty(\mathbb{R}^d) \) be a plateau function:

\[
\text{supp}(\eta) \subseteq B_1, 0 \leq \eta \leq 1, \eta \equiv 1 \text{ in } B_{\frac{1}{2}}, \eta(x) = \eta(|x|) \text{ decreasing with respect to } |x|. 
\]

and we define our function

\[
f(\mu) = \left( \int_{\mathbb{R}^d} \eta(x) \, d\mu(x) \right) \wedge 3.
\]

We define the level set \( B_r \) such that

\[
B_r := \left\{ x \in \mathbb{R}^d \mid \frac{1}{2} \leq \eta(x) \leq 1 \right\}.
\]

Then, we have \( \mathbb{E}_\rho[f|\mathcal{F}_{B_r}] \notin C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d)) \).

**Proof.** Let \( \mu_1 = \mu \mathbb{L} B_r, \mu_2 = \mu \mathbb{L} (B_1 \setminus B_r) \), then since \( \text{supp}(f) \subseteq B_1 \), we have that

\[
\mathbb{E}_\rho[f|\mathcal{F}_{B_r}] = (\mu_1(\eta) + \mu_2(\eta)) \wedge 3
\]

Let us choose a specific configuration to see that \( \mathbb{E}_\rho[f|\mathcal{F}_{B_r}] \) is not even continuous:

\[
\mu_1 = \delta_{x_1} + \delta_{x_2} + \delta_{x_3}, \text{ where } x_1, x_2, x_3 \in B_{\frac{1}{2}} \setminus B_{\frac{1}{2}}.
\]

Then we can calculate that \( 2.5 \leq \mu_1(\eta) < 3 \) and \( 2.5 \leq \mathbb{E}_\rho[f|\mathcal{F}_{B_r}] \mu_1 \leq 3 \). However, if we take another \( \mu_1 \) that

\[
\mu_1 = \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}, \text{ where } x_1, x_2 \in B_{\frac{1}{2}}, x_3, x_4 \in B_r, \setminus B_{\frac{1}{2}}.
\]

Then we see that \( \mu_1(\eta) > 3 \) and we have \( \mathbb{E}_\rho[f|\mathcal{F}_{B_r}] \mu_1 = 3 \). Therefore, once the 4-th particle \( x_4 \) enters the ball \( B_r \), the value of the function will jump to 3. From this we conclude that \( \mathbb{E}_\rho[f|\mathcal{F}_{B_r}] \notin C_c^\infty(\mathcal{M}_\delta(\mathbb{R}^d)) \).

To make the conditional expectation more regular, we introduce its regularized version: for any \( 0 < \varepsilon < \infty \), we define

\[
(4.9) \quad A_{s, \varepsilon} f := \frac{1}{\varepsilon} \int_0^\varepsilon A_{s+t} f \, dt,
\]

Then we have the following properties.

**Proposition 4.7.** The function \( A_{s, \varepsilon} f \in H^1_0(\mathcal{M}_\delta(\mathbb{R}^d)) \) and \( (\mathbb{E}_\rho[(A_{s, \varepsilon} f)^2])_{s \geq 0} \) a \( C^1 \) increasing process.

**Proof.** We calculate the formula for \( \mathbb{E}_\rho[(A_{s, \varepsilon} f)^2] \):

\[
\mathbb{E}_\rho[(A_{s, \varepsilon} f)^2] = \frac{1}{\varepsilon^2} \int_0^\varepsilon \int_0^\varepsilon \mathbb{E}_\rho[A_{s+t_1} f A_{s+t_2} f] \, dt_1 dt_2.
\]

As we know that \( \mathbb{E}_\rho[A_{s+t_1} f A_{s+t_2} f] = \mathbb{E}_\rho[(A_{s+(t_1+t_2)} f)^2] \), we obtain that

\[
(4.10) \quad \mathbb{E}_\rho[(A_{s, \varepsilon} f)^2] = \frac{2}{\varepsilon^2} \int_0^\varepsilon (\varepsilon - t) \mathbb{E}_\rho[(A_{s+t} f)^2] \, dt.
\]
Then we calculate its derivative that for $0 < h < \varepsilon$

$$
\lim_{h \to 0} \frac{1}{h} \left( \mathbb{E}_\rho \left( (A_{s+h,e,f})^2 \right) - \mathbb{E}_\rho \left( (A_{s,e,f})^2 \right) \right)
$$

$$
= \lim_{h \to 0} \frac{2}{\varepsilon^2} \left( \int_0^{\varepsilon+h} (\varepsilon + h - t) \mathbb{E}_\rho \left( (A_{s+t,f})^2 \right) dt - \int_0^{\varepsilon} (\varepsilon - t) \mathbb{E}_\rho \left( (A_{s+t,f})^2 \right) dt + \int_0^{\varepsilon} h \mathbb{E}_\rho \left( (A_{s+t,f})^2 \right) dt \right)
$$

$$
= \frac{2}{\varepsilon^2} \int_0^{\varepsilon} \mathbb{E}_\rho \left( (A_{s+t,f})^2 \right) - \mathbb{E}_\rho \left( (A_{s,f})^2 \right) dt.
$$

In the last step, we use the right continuity and this proves that

$$
\frac{d}{ds} \mathbb{E}_\rho \left( (A_{s,f})^2 \right) = \frac{2}{\varepsilon^2} \int_0^{\varepsilon} \mathbb{E}_\rho \left( (A_{s+t,f})^2 \right) - \mathbb{E}_\rho \left( (A_{s,f})^2 \right) dt.
$$

Then we calculate the partial derivative. We use the formula that

$$
\mathbf{e}_k \cdot \nabla A_{s,e,f}(\mu, x) = \lim_{h \to 0} \frac{1}{h} \left( \int_0^{\varepsilon} A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) - A_{s+t,f}(\mu) dt \right)
$$

We study this derivative case by case.

1. Case $x \in Q^c_{e+\varepsilon}$. In this case, in eq. (4.12), for a $h$ small enough, for any $t \in [0, \varepsilon]$, neither $x$ nor $x + h e_k$ is in $Q_{s+t}$, so we have $A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) = A_{s+t,f}(\mu \lor Q_{s+t})$. This implies that eq. (4.12) is 0 in this case.

2. Case $x \in Q^c_{e}$. In this case, for a $h$ small enough, for any $t \in [0, \varepsilon]$, both $x$ and $x + h e_k$ is in $Q_{s+t}$, then we have

$$
\mathbf{e}_k \cdot \nabla A_{s,e,f}(\mu, x) = \lim_{h \to 0} \frac{1}{h} \left( \int_0^{\varepsilon} A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) - A_{s+t,f}(\mu) dt \right)
$$

$$
= \frac{1}{\varepsilon} \int_0^{\varepsilon} \lim_{h \to 0} \frac{1}{h} \left( A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) - A_{s+t,f}(\mu) dt \right)
$$

$$
= A_{s,e,f}(\mathbf{e}_k \cdot \nabla f(\mu, x)).
$$

3. Case $x \in (Q_{s+t} \setminus Q^c_{e})$, $\mathbf{e}_k$ is the normal direction. In this case, we study at first the situation $\mathbf{n}(x)$ and $h \to 0$. We divide eq. (4.12) in three terms:

$$
\mathbf{e}_k \cdot \nabla A_{s,e,f}(\mu, x) = \mathbf{I} + \mathbf{II} + \mathbf{III}
$$

$$
\mathbf{I} = \frac{1}{\varepsilon} \int_0^{\varepsilon} \mathbf{1}_{(s+t < \tau(x))} \frac{1}{h} \left( A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) - A_{s+t,f}(\mu) \right) dt
$$

$$
\mathbf{II} = \frac{1}{\varepsilon} \int_0^{\varepsilon} \mathbf{1}_{(s+t > \tau(x) + h)} \frac{1}{h} \left( A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) - A_{s+t,f}(\mu) \right) dt
$$

$$
\mathbf{III} = \frac{1}{\varepsilon} \int_0^{\varepsilon} \mathbf{1}_{(\tau(x) + s < t < \tau(x) + h)} \frac{1}{h} \left( A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) - A_{s+t,f}(\mu) \right) dt.
$$

The term $\mathbf{I}$ and $\mathbf{II}$ are similar as we have discussed above and we have

$$
\lim_{h \to 0} \mathbf{I} + \mathbf{II} = \frac{1}{\varepsilon} \int_0^{\varepsilon} \mathbf{1}_{(s+t > \tau(x))} A_{s+t} \left( \mathbf{e}_k \cdot \nabla f(\mu, x) \right) dt.
$$

For the term $\mathbf{III}$, since $x + h e_k \notin Q_{s+t}$ under this situation, we have $A_{s+t,f}(\mu - \delta_x + \delta_{x+h,e_k}) = A_{s+t,f}(\mu - \delta_x)$. Then, we use the right continuity of $A_s f$

$$
\lim_{h \to 0} \mathbf{III} = \lim_{h \to 0} \frac{1}{h} \int_{\tau(x) - h}^{\tau(x) - s} A_{s+t,f}(\mu - \delta_x) - A_{s+t,f}(\mu) dt
$$

$$
= \frac{1}{\varepsilon} \left( A_{\tau(x)} f(\mu - \delta_x) - A_{\tau(x)} f(\mu) \right).
$$

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We should also remark that is also the case we do partial derivative from left, in this case we should pay attention on the term III which is

\[
\lim_{h \to 0} \text{III}' = \lim_{h \to 0} \frac{1}{h} \int_0^\epsilon 1_{s(x)-h \leq t \leq s(x)+h} (A_{s+t} f(\mu - \delta_x) - A_{s+t} f(\mu - \delta_x + \delta_x)) \, dt
\]

\[
= \frac{1}{\epsilon} \left( A_{s(x)-} f(\mu - \delta_x) - A_{s(x)-} f(\mu - \delta_x + \delta_x) \right).
\]

In the last step, we use the left continuity of \( A_{s(x)} f \) when the particle on the boundary is removed. Thanks to Corollary 4.5, we know this limit coincide with that of III. In conclusion, we could use the notation eq. (4.5)

\[
\Delta A_{s(x)} f = A_{s(x)} f - A_{s(x)-} f,
\]

to unify the two. Thus we see it is nothing but the jump of the càdlàg martingale.

4. Case \( x \in (Q^{s+} \setminus Q^o_s) \), \( e_k \) is not the normal direction. This case is simpler than \( e_k \) is normal direction, where we do not have to consider the term III in the discussion above.

In conclusion, we obtain the formula that for any \( x \in \text{supp}(\mu) \)

\[
\nabla A_{s,\epsilon} f(\mu, x) = \begin{cases} A_{s,\epsilon} (\nabla f(\mu, x)) & x \in Q^o_s; \\
\frac{1}{\epsilon} \int_{s(x)-}^{s(x)+\epsilon} A_{s+t} (\nabla f(\mu, x)) \, dt - \frac{\tilde{E}(x)}{\epsilon} \Delta A_{s(x)} f & x \in (Q^{s+} \setminus Q^o_s); \\
0 & x \in Q^c_{s+}. \end{cases}
\]

4.2. **Proof of Theorem 4.1.** In this part, we prove Theorem 4.1 in three steps.

**Proof. Step 1: Setting up.** We propose a regularized multi-scale functional of eq. (4.2)

\[
S_{k,K,\beta,\epsilon}(f) = \alpha_k \mathbb{E}_p \left[ (A_{k,\epsilon} f)^2 \right] + \int_k^K \alpha_s (\frac{d}{ds} \mathbb{E}_p \left[ (A_{s,\epsilon} f)^2 \right]) \, ds + \alpha_K \mathbb{E}_p \left[ f^2 - (A_{K,\epsilon} f)^2 \right],
\]

where we recall that \( \alpha_s = \exp \left( \frac{\epsilon}{\beta} \right) \). The advantage is that \( \mathbb{E}_p \left[ (A_{s,\epsilon} f)^2 \right] \) is \( C^1 \) for \( s \) from eq. (4.11), we can treat it as usual Riemann integral and apply integration by part to obtain an equivalent definition

\[
S_{k,K,\beta,\epsilon}(f) = \alpha_K \mathbb{E}_p \left[ f^2 \right] - \int_k^K \alpha_s \mathbb{E}_p \left[ (A_{s,\epsilon} f)^2 \right] \, ds.
\]

Our object is to calculate \( \frac{d}{dt} S_{k,K,\beta,\epsilon}(u_t) \), and we pay attention to \( \frac{d}{dt} \mathbb{E}_p \left[ (A_{s,\epsilon} u_t)^2 \right] \). We use the formula from eq. (4.10)

\[
\frac{d}{dt} \mathbb{E}_p \left[ (A_{s,\epsilon} u_t)^2 \right] = \frac{d}{dt} \left( \frac{2}{\epsilon^2} \int_0^\epsilon (\epsilon - r) \mathbb{E}_p \left[ (A_{s+r} u_t)^2 \right] \, dr \right)
\]

\[
= \frac{d}{dt} \left( \frac{2}{\epsilon^2} \int_0^\epsilon (\epsilon - r) \mathbb{E}_p \left[ (A_{s+r} u_t) u_t \right] \, dr \right).
\]

We define that

\[
\overline{A}_{s,\epsilon} f := \frac{2}{\epsilon^2} \int_0^\epsilon (\epsilon - r) A_{s+r} f \, dr,
\]

and it satisfies similar property as \( A_{s,\epsilon} f \). For example, we have also the formula

\[
\nabla \overline{A}_{s,\epsilon} f(\mu, x) = \begin{cases} \overline{A}_{s,\epsilon} (\nabla f(\mu, x)) & x \in Q^o_s; \\
\frac{2}{\epsilon^2} \left( \int_{s(x)-}^{s(x)+\epsilon} (\epsilon - r) A_{s+r} (\nabla f(\mu, x)) \, dr - (s + \epsilon - \tau(x)) \Delta A_{s(x)} f \tilde{E}(x) \right) & x \in (Q^{s+} \setminus Q^o_s); \\
0 & x \in Q^c_{s+}. \end{cases}
\]
then we have
\begin{equation}
\frac{d}{dt} \mathbb{E}_\rho[(A_{s,\varepsilon}u_t)^2] = \frac{d}{dt} \mathbb{E}_\rho[(\overline{A}_{s,\varepsilon}u_t)u_t] = \mathbb{E}_\rho\left[\left(\frac{d}{dt} \overline{A}_{s,\varepsilon}u_t\right)u_t\right] + \mathbb{E}_\rho\left[\overline{A}_{s,\varepsilon}u_t(Lu_t)\right].
\end{equation}

We study at first the semi-group. For a function \( g \in H_0^1(M_d(\mathbb{R}^d)) \), we recall the definition that

\( g_t(\mu) = P_t g(\mu) := \mathbb{E}_\rho[g(\mu_t) | \mathcal{F}_0] \).

We also know its semi-group that

\( \frac{d}{dt} P_t g(\mu) = \mathcal{L} P_t g(\mu) \Rightarrow \partial_t g_t(\mu) = \mathcal{L} g_t(\mu). \)

Now in our question we propose that \( g = \overline{A}_{s,\varepsilon}u_0 \), then we have

\[ g_t(\mu) = P_t \left( \frac{2}{\varepsilon^2} \int_0^\varepsilon (\varepsilon - r) \mathbb{E}_\rho[u(\mu_t) | \mathcal{F}_{Q_{s,\varepsilon}}] dr \right) = \mathbb{E}_\rho \left[ \left( \frac{2}{\varepsilon^2} \int_0^\varepsilon (\varepsilon - r) \mathbb{E}_\rho[u(\mu_t) | \mathcal{F}_{Q_{s,\varepsilon}}] dr \right) | \mathcal{F}_0 \right] = \frac{2}{\varepsilon^2} \int_0^\varepsilon (\varepsilon - r) \mathbb{E}_\rho [u(\mu_t) | \mathcal{F}_0] | \mathcal{F}_{Q_{s,\varepsilon}}] dr = \overline{A}_{s,\varepsilon}u_0(\mu). \]

Therefore, we have \( \frac{d}{dt} \overline{A}_{s,\varepsilon}u_t(\mu) = \mathcal{L} \overline{A}_{s,\varepsilon}u_t(\mu) \) and put it back to eq. (4.19) and use reversibility to obtain that

\[ \frac{d}{dt} \mathbb{E}_\rho[(A_{s,\varepsilon}u_t)^2] = 2 \mathbb{E}_\rho[\overline{A}_{s,\varepsilon}u_t(Lu_t)]. \]

We conclude that

\begin{equation}
\frac{d}{dt} S_{k,K,\beta,\varepsilon}(u_t) = 2\alpha K \mathbb{E}_\rho[u_t(Lu_t)] + \int_k^K 2\alpha' \mathbb{E}_\rho[\overline{A}_{s,\varepsilon}u_t(-Lu_t)] ds.
\end{equation}

**Step 2: Estimate of a localized Dirichlet energy.** In this step, we will give an estimate for the term \( \mathbb{E}_\rho[\overline{A}_{s,\varepsilon}u_t(-Lu_t)] \) appeared in eq. (4.20). We will establish the following lemma.

**Lemma 4.8.** For any \( f \in H_0^1(M_d(\mathbb{R}^d)) \), we define that

\begin{equation}
I^f_s := \mathbb{E}_\rho\left[ \int_{Q_s} a|\nabla f|^2 d\mu \right].
\end{equation}

then for \( \overline{A}_{s,\varepsilon}f \) introduced in eq. (4.17), for any \( s, \theta, \varepsilon \in (0, \infty) \), we have

\begin{equation}
\mathbb{E}_\rho[\overline{A}_{s,\varepsilon}f(-Lf)] \leq \frac{d}{ds} \mathbb{E}_\rho[(A_{s,\varepsilon}f)^2] + \Lambda \left( I^f_s - I^{f}_{s+\varepsilon} \right) + \Lambda \left( \frac{\theta}{\varepsilon} + 1 \right) \left( I^f_{s+\varepsilon} - I^f_s \right) + \frac{\Lambda}{2\theta} \int_0^\infty \frac{d}{ds} \mathbb{E}_\rho[(A_{s,\varepsilon}f)^2].
\end{equation}

**Proof.** From eq. (4.18), we can decompose the quantity \( \mathbb{E}_\rho[\overline{A}_{s,\varepsilon}f(-Lf)] \) into three terms

\begin{equation}
\mathbb{E}_\rho[\overline{A}_{s,\varepsilon}f(-Lf)] = \mathbb{E}_\rho\left[ \int_{Q_{s+1}} a\nabla (\overline{A}_{s,\varepsilon}f) \cdot \nabla f d\mu \right] + \mathbb{E}_\rho\left[ \int_{Q_{s}\setminus Q_{s+1}} a\nabla (\overline{A}_{s,\varepsilon}f) \cdot \nabla f d\mu \right]
\end{equation}

**eq. (4.23)-a**

\begin{equation}
+ \mathbb{E}_\rho\left[ \int_{Q_{s+1}\setminus Q_s} a\nabla (\overline{A}_{s,\varepsilon}f) \cdot \nabla f d\mu \right].
\end{equation}

**eq. (4.23)-b**

**eq. (4.23)-c**
For the first term eq. (4.23)-a, since \( x \in Q_{s-1} \), then the coefficient is \( \mathcal{F}_{Q_s} \) measurable. We use the formula eq. (4.18), eq. (4.17) and apply Jensen’s inequality for conditional expectation

\[
\text{eq. (4.23)-a} = \frac{2}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{s-1}} \int_{0}^{\varepsilon} (\varepsilon - r) a \mathcal{A}_{s+r}(\nabla f) \cdot \nabla f \, dr \, d\mu \right] \\
= \frac{2}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{s-1}} \int_{0}^{\varepsilon} (\varepsilon - r) \mathbb{E}_\rho [a |\mathcal{A}_{s+r}(\nabla f)|^2 |\mathcal{F}_{Q_{s+r}}] \, dr \, d\mu \right] \\
\leq \frac{2}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{s-1}} \int_{0}^{\varepsilon} (\varepsilon - r) \mathbb{E}_\rho [a |\nabla f|^2 |\mathcal{F}_{Q_{s+r}}] \, dr \, d\mu \right] \\
= \mathbb{E}_\rho \left[ \int_{Q_{s-1}} a |\nabla f|^2 \, d\mu \right]
\]

For the second term eq. (4.23)-b, it is similar but \( a \) is no longer \( \mathcal{F}_{Q_s} \) measurable. We use at first Young’s inequality

\[
\text{eq. (4.23)-b} \leq \frac{2}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{s-1}} \int_{0}^{\varepsilon} (\varepsilon - r) a \mathcal{A}_{s+r}(\nabla f) \cdot \nabla f \, dr \, d\mu \right] \\
\leq \frac{1}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{s-1}} \int_{0}^{\varepsilon} (\varepsilon - r) a (|\mathcal{A}_{s+r}(\nabla f)|^2 + |\nabla f|^2) \, dr \, d\mu \right]
\]

Then for the part with conditional expectation, we use the uniform bound \( 1 \leq a \leq \Lambda \) that

\[
\frac{1}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_s \setminus Q_{s-1}} \int_{0}^{\varepsilon} (\varepsilon - r) a |\mathcal{A}_{s+r}(\nabla f)|^2 \, dr \, d\mu \right] \leq \frac{\Lambda}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_s \setminus Q_{s-1}} \int_{0}^{\varepsilon} (\varepsilon - r) |\mathcal{A}_{s+r}(\nabla f)|^2 \, dr \, d\mu \right] \\
\leq \frac{\Lambda}{2} \mathbb{E}_\rho \left[ \int_{Q_s \setminus Q_{s-1}} |\nabla f|^2 \, d\mu \right] \\
\leq \frac{\Lambda}{2} \mathbb{E}_\rho \left[ \int_{Q_s \setminus Q_{s-1}} a |\nabla f|^2 \, d\mu \right].
\]

This concludes that eq. (4.23)-b \( \leq \Lambda \mathbb{E}_\rho \left[ \int_{Q_s \setminus Q_{s-1}} a |\nabla f|^2 \, d\mu \right] \).

For the third term eq. (4.23)-c, we use eq. (4.18) and obtain

\[
\text{eq. (4.23)-c} \leq \text{eq. (4.23)-c1} + \text{eq. (4.23)-c2} \\
\text{eq. (4.23)-c1} = \frac{2}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{s+1} \setminus Q_{s}} \int_{\tau(x)-s}^{\varepsilon} (\varepsilon - r) a \mathcal{A}_{s+r}(\nabla f) \cdot \nabla f \, dr \, d\mu \right] \\
\text{eq. (4.23)-c2} = \frac{2}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{s+1} \setminus Q_{s}} a (\varepsilon - \tau(x)) \Delta \mathcal{A}_{\tau(x)} f \nabla \mathbf{n} (x) \cdot \nabla f \, d\mu \right].
\]

The part of eq. (4.23)-c1 is similar as that of eq. (4.23)-b and we have that

\[
\text{eq. (4.23)-c1} \leq \Lambda \mathbb{E}_\rho \left[ \int_{Q_{s+1} \setminus Q_{s}} a |\nabla f|^2 \, d\mu \right].
\]
We study the part eq. (4.23)-c2 with Young's inequality

\[
\frac{2}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{t+x} \cap Q_x} a(\varepsilon - \eta(x)) \Delta A_r(x) f \nabla f \, d\mu \right] \leq \frac{\Lambda}{\theta \varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{t+x} \cap Q_x} (s + \varepsilon - \eta(x)) \Delta |A_r(x)|^2 \, d\mu \right] + \frac{\theta \Lambda}{\varepsilon^2} \mathbb{E}_\rho \left[ \int_{Q_{t+x} \cap Q_x} (s + \varepsilon - \eta(x)) |\nabla f|^2 \, d\mu \right]
\]

The first part is in fact the bracket process defined in Corollary 4.4

\[
\mathbb{E}_\rho \left[ \int_{Q_{t+x} \cap Q_x} (s + \varepsilon - \eta(x)) \Delta |A_r(x)|^2 \, d\mu \right] = \mathbb{E}_\rho \left[ \frac{\Lambda}{\theta \varepsilon^2} \sum_{s \subseteq r \subseteq s + \varepsilon} (s + \varepsilon - \eta) |\Delta \mathcal{A}_t^f|^2 \right].
\]

Then we develop it with Fubini theorem and the \( L^2 \)-isometry that \( \mathbb{E}_\rho [(\mathcal{M}_s)^2] = \mathbb{E}_\rho [(\mathcal{M}_s)^2] = \mathbb{E}_\rho [(A_s,f)^2] \)

\[
\mathbb{E}_\rho \left[ \int_{Q_{t+x} \cap Q_x} (s + \varepsilon - \eta(x)) \Delta |A_r(x)|^2 \, d\mu \right] = \mathbb{E}_\rho \left[ \frac{\Lambda}{\theta \varepsilon^2} \sum_{s \subseteq r \subseteq s + \varepsilon} \int_{s \subseteq r \subseteq s + \varepsilon} |\Delta \mathcal{A}_t^f|^2 \, dr \right]
\]

\[
= \mathbb{E}_\rho \left[ \frac{\Lambda}{\theta \varepsilon^2} \int_{s \subseteq r \subseteq s + \varepsilon} \Delta \mathcal{A}_t^f \, dr \right]
\]

\[
= \mathbb{E}_\rho \left[ \frac{\Lambda}{\theta \varepsilon^2} \int_{s \subseteq r \subseteq s + \varepsilon} \Delta \mathcal{A}_t^f \, dr \right]
\]

\[
= \mathbb{E}_\rho \left[ \frac{\Lambda}{2 \theta} ds \right]
\]

In the last step, we use the identity eq. (4.11). This concludes that

\[
\text{eq. (4.23)-c} \leq \left( \frac{\theta \Lambda}{\varepsilon} + \Lambda \right) \mathbb{E}_\rho \left[ \int_{Q_{t+x} \cap Q_x} a|\nabla f|^2 \, d\mu \right] + \frac{\Lambda}{2 \theta} \mathbb{E}_\rho \left[ (A_{s+\varepsilon}, f)^2 \right],
\]

and we combine all the estimate for the three terms eq. (4.23)-a, eq. (4.23)-b, eq. (4.23)-c to obtain the desired result in eq. (4.22).

\text{Step 3: End of the proof.} We take } k = \sqrt{t}, K > k \text{ and and put the estimate eq. (4.23) into eq. (4.20) with } \theta, \varepsilon, \beta > 0 \text{ to be fixed,}

\[
\frac{d}{dt} S_{k,K,\varepsilon} (u_t) \leq 2 \alpha K \mathbb{E}_\rho [u_t (L u_t)] + \int_k^K 2 \alpha' s \mathbb{E}_\rho \left[ A_{s+\varepsilon} u_t (-L u_t) \right] \, ds
\]

\[
\leq -2 \alpha K I_{s+\varepsilon}^{u_t} + \int_k^K 2 \alpha' \left( I_{s+\varepsilon}^{u_t} + \Lambda \left( I_{s+\varepsilon}^{u_t} - I_{s+\varepsilon}^{u_t} \right) + \Lambda \left( \frac{\theta}{\varepsilon} + 1 \right) (I_{s+\varepsilon}^{u_t} - I_{s+\varepsilon}^{u_t}) + \frac{\Lambda}{\theta} ds \mathbb{E}_\rho \left[ (A_{s+\varepsilon}, f)^2 \right] \right) \, ds.
\]

We recall that \( \alpha' = \frac{\alpha}{\beta} \), then we do some calculus and obtain that

\[
\frac{d}{dt} S_{k,K,\varepsilon} (u_t) \leq \int_{k-1}^{K+\varepsilon} \left( -2 \alpha K \lambda (s+1) + 2 \Lambda (\alpha s+1 - \alpha_s) + 2 \Lambda \left( \frac{\theta}{\varepsilon} + 1 \right) (\alpha_s - \alpha_{s-\varepsilon}) \right) \, ds
\]

\[
+ \int_{k-1}^{K+\varepsilon} -2 \alpha K I_{s+\varepsilon}^{u_t} + \int_k^\infty -2 \alpha K I_{s+\varepsilon}^{u_t} + \frac{\Lambda}{\beta \theta} \int_k^K \alpha_s \left( \frac{d}{ds} \mathbb{E}_\rho \left[ (A_{s+\varepsilon}, f)^2 \right] \right) \, ds.
\]
We see that the term \(2\Lambda (\alpha_{s+1} - \alpha_s) \simeq \frac{2\Lambda}{\beta} \alpha_s\) and \(2\Lambda \left(\frac{\theta}{\varepsilon} + 1\right) (\alpha_s - \alpha_{s-\varepsilon}) \simeq 2\Lambda \left(\frac{\theta}{\beta} + \frac{1}{\beta}\right) \alpha_s\). One can choose the parameters \(\theta = \frac{\beta}{\varepsilon}, \varepsilon = \frac{1}{\frac{\beta}{\gamma}}\), then for \(\beta > 4\Lambda\), the part of integration with respect to \(I_n^\ast\) is negative. We use the definition eq. (4.15) and obtain that

\[
\frac{d}{dt} S_{k,K,\beta,\varepsilon}(u_t) \leq \frac{\Lambda}{\beta} \int_k^K \alpha_s \left(\frac{d}{ds} E_p[(A_{s,x} u_t)^2]\right) ds \leq \frac{2\Lambda^2}{\beta^2} S_{k,K,\beta,\varepsilon}(u_t),
\]

which implies that for \(k = \sqrt{t} \geq l\), \((l\) the diameter of support of \(u_0\) in Theorem 4.1)

\[
\alpha_K E_p\left[(u_t)^2 - (A_{K,\varepsilon} u_t)^2\right] \leq S_{k,K,\beta,\varepsilon}(u_t) \leq \exp\left(\frac{2\Lambda^2 t}{\beta^2}\right) S_{k,K,\beta,\varepsilon}(u_0) = \exp\left(\frac{2\Lambda^2 t}{\beta^2}\right) \alpha_k E_p\left[(u_0)^2\right].
\]

Finally we remark that

\[
E_p\left[(u_t - A_{K,\varepsilon} u_t)^2\right] = E_p\left[(u_t)^2 - (A_{K,\varepsilon} u_t)^2\right] \leq E_p\left[(u_t)^2 - (A_{K,\varepsilon} u_t)^2\right],
\]

and choose \(\beta = \sqrt{t}\) and it gives us the desired result, after shrinking a little the value of \(K\). \(\square\)

5. Spectral inequality, perturbation and perturbation

In this section, we collect several other estimates used in the proof of the main result. They can also be read for independent interests.

5.1. Spectral inequality. The spectral inequality is an important topic in probability theory and Markov process, and it has its counterpart in analysis known as Poincaré’s inequality.

Let \(L > l > 0\) and \(q = L/l \in \mathbb{N}\), and denote by \(\{Q_i\}_{i \in \mathbb{N}}\) the partition of \(Q_L\) by the small cube by scale \(l\). Let \(M_{L,l} = (M_1,M_2,\ldots,M_q)\), be a random vector that \(M_i = \mu(\hat{Q}_i)\), and we define \(B_{L,l}f := E_p[f|M_{L,l}]\), then we have the following estimate.

**Proposition 5.1** (Spectral inequality). There exists a finite positive number \(R_0(d)\), such that for any \(0 < l < L < \infty, L/l \in \mathbb{N}\), we have an estimate for any \(f \in H^1(M_d(\mathbb{R}^d))\),

\[
E_p\left[(A_L f - B_{L,I} f)^2\right] \leq R_0 l^2 E_p\left[\int_{Q_L} |\nabla f|^2 d\mu\right].
\]

**Proof.** We prove at first a simple corollary from Efron-Stein inequality [17, Chapter 3]: let \(f_n \in C^1(\mathbb{R}^{d\times n})\) and \(X = (X_1, X_2, \ldots, X_n)\), where \((X_i)_{i \in \mathbb{N}}\) a family independent \(\mathbb{R}^d\)-valued random variables following uniform law in \(Q_l\), then Efron-Stein inequality states

\[
\text{Var}[f_n(X)] \leq \frac{1}{2} \sum_{i=1}^n E\left[(f_n(X) - f_n(X^i))^2\right],
\]

where \(f_n(X^i) := E[f_n(X)|X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]\). From this, we calculate the expectation with respect to \(X_i\) for \(f_n(X) - f_n(X^i))^2\), and apply the standard Poincaré’s inequality for \(X_i\)

\[
E_{X_i}\left[(f_n(X) - f(X^i))^2\right] = \int_{Q_l} \left(f_n(x_1,x_2,\ldots,x_n) - \int_{Q_l} f_n(x_1,x_2,\ldots,x_n) dx_i\right)^2 dx_i \leq C(d)l^2 \int_{Q_l} |\nabla f_n|^2(x_1,x_2,\ldots,x_n) dx_i,
\]

\[
\implies E\left[(f_n(X) - f(X^i))^2\right] \leq C(d)l^2 E[|\nabla f_n(X)|^2].
\]

We combine the sum of all the term and obtain

\[
\text{Var}[f_n(X)] \leq C(d)l^2 \sum_{i=1}^n E[|\nabla f_n(X)|^2].
\]
We then apply eq. (5.3) in eq. (5.1).
\[
\mathbb{E}_\rho[\langle A_L f - B_L f \rangle^2] = \sum_{M \in \mathbb{N}^d} \mathbb{P}_\rho[M_{L,l} = M] \mathbb{E}_\rho[\langle A_L f - B_L f \rangle^2 | M_{L,l} = M].
\]
Conditioned \( \{ M_{L,l} = M \} \), we know that the expectation of \( A_L f \) is \( B_L f(M) \) and all the particles are distributed uniformly in its small cubes of size \( l \), thus we can apply eq. (5.3) that
\[
\mathbb{E}_\rho[\langle A_L f - B_L f \rangle^2 | M_{L,l} = M] = \text{Var}_\rho[\langle A_L f | M_{L,l} = M \rangle] \\
\leq C(d)\mathbb{E}_\rho[|\nabla A_L f|^2 | M_{L,l} = M] \\
\leq C(d)\mathbb{E}_\rho[|A_L \nabla f|^2 | M_{L,l} = M].
\]
Then we do the sum and concludes eq. (5.1).

5.2. Perturbation. A similar version of the following lemma appears in [30], where the authors give some sketch and here we prove it in our model with some more details. We define a localized Dirichlet form for Borel set \( U \subseteq \mathbb{R}^d \) that
\[
\mathcal{E}_U(f,g) = \mathbb{E}_\rho(g(-\Delta_U f)) = \mathbb{E}_\rho\left[ \int_U \nabla g(\mu,d) \cdot \nabla f(\mu,x) \, d\mu(x) \right],
\]
and we use \( \mathcal{E}_U(f) := \mathcal{E}_U(f,f) \) and \( \mathcal{E}(f) := \mathcal{E}_{\mathbb{R}^d}(f) \) for short.

**Proposition 5.2** (Perturbation). Let \( u \in C_c^\infty(M_\delta(\mathbb{R}^d)) \) and \( l_k := l_u + 2k \) be the minimal scale such that for any \( |h| \leq k, \text{supp}(\tau_h u) \subseteq Q_k \), then for any \( g \) such that \( \mathbb{E}_\rho[g] = 1, \sqrt{g} \in H_0^1(M_\delta(\mathbb{R}^d)) \), we have
\[
(\mathbb{E}_\rho[g(u - \tau_h u)])^2 \leq C(d)\mathbb{E}_\rho[|u|_{L^\infty}^2]^{\mathcal{E}_{Q_k}(\sqrt{g})}.
\]

**Proof.** The proof of this proposition relies on the following lemma:

**Lemma 5.3** (Lemma 4.2 of [30]). Let \( \Omega, \mathbb{P}, \mathcal{F} \) be a probability space and let \( (f,g) = \int_\Omega fg \, d\mathbb{P} \) denote the standard inner product on \( L^2(\Omega, \mathbb{P}, \mathcal{F}) \). Let \( A \) be a non-negative definite symmetric operator on \( L^2(\Omega, \mathbb{P}, \mathcal{F}) \), which has 0 as a simple eigenvalue with corresponding eigenfunction the constant function 1, and second eigenvalue \( \delta > 0 \) (the spectral gap). Let \( V \) be a function of means zero, \( \langle 1,V \rangle = 0 \) and assume that \( V \) is essential bounded. Denote by \( \lambda_\varepsilon \) the principal eigenvalue of \( -A + \varepsilon V \) given by the variational formula
\[
\lambda_\varepsilon = \sup_{\|f\|_{L^2} = 1} \langle f, (-A + \varepsilon V) f \rangle.
\]
Then for \( 0 < \varepsilon < \delta \|V\|_{L^\infty}^{-1} \),
\[
0 \leq \lambda_\varepsilon \leq \varepsilon^2 \langle V, A^{-1}V \rangle \frac{1}{1 - 2\|V\|_{L^\infty} \delta^{-1}}.
\]

In our context, we should look for a good frame for this lemma. Since for any \( |h| \leq k, (u - \tau_h u) \in \mathcal{F}_{Q_k} \), we have
\[
\mathbb{E}_\rho[g(u - \tau_h u)] = \mathbb{E}_\rho[(A_{Q_k} g)(u - \tau_h u)] \\
= \sum_{n=0}^\infty \mathbb{P}_\rho[\mu(Q_k) = n] \mathbb{E}_\rho[(A_{Q_k} g)(u - \tau_h u) | \mu(Q_k) = n].
\]
Then, we focus on the estimate of \( \mathbb{E}_\rho[(A_{Q_k} g)(u - \tau_h u) | \mu(Q_k) = n] \): to shorten the notation, we use \( \mathbb{P}_{\rho,n} \) for the probability \( \mathbb{P}_\rho[\mu(Q_k) = n] \) and \( \mathbb{E}_{\rho,n} \) for its associated expectation. Then
we apply Lemma 5.3 on the probability space \((\Omega, \mathcal{F}_{Q_{lk}}, \mathbb{P}_{\rho,n})\), where we set \(V = u - \tau_h u\) and the symmetric non-negative operator \(A\) is \(-\Delta_{Q_{lk}}\) defined for any \(f \in H^1(\mathcal{M}_\delta(Q_{lk}))\)

\[
\mathbb{E}_{\rho,n}[f(-\Delta_{Q_{lk}} f)] := \mathbb{E}_{\rho,n}\left[ \int_{Q_{lk}} |\nabla f|^2 \, d\mu \right].
\]

We should check that this setting satisfies the condition of Lemma 5.3:

- Spectral gap for \(A = -\Delta_{Q_{lk}}\): by eq. (5.2) we have the spectral gap \(\delta = (l_k)^{-2}\) for any function \(f \in H^1(\mathcal{M}_\delta(Q_{lk}))\) with \(\mathbb{E}_{\rho,n}[f] = 0\)

\[
\mathbb{E}_{\rho,n}[f^2] \leq (l_k)^2 \mathbb{E}_{\rho,n}[f(-\Delta_{Q_{lk}} f)].
\]

- Mean zero for \(V = u - \tau_h u\): under the probability \(\mathbb{P}_\rho\) this is clear by the transport invariant property of Poisson point process, while under \(\mathbb{P}_{\rho,n}\) this requires some calculus. By the definition of \(l_k\), we know that \(\text{supp}(u) \subseteq Q_{lk}\), thus we denote by the projection \(u(\mu) = \tilde{u}_m(x_1, x_2, \cdots x_m)\) under the case \(\mu \ll Q_{lk} = \sum_{i=1}^n \delta_{x_i}\). Then we have

\[
\mathbb{E}_{\rho,n}[u] = \sum_{m=0}^n \mathbb{P}_{\rho,n}[\mu(Q_{lk}) = m] \mathbb{E}_{\rho,n}[u|\mu(Q_{lk}) = m] = \sum_{m=0}^n \left( \frac{|Q_{lk}|}{|Q_{lk}|} \right)^m \left( 1 - \frac{|Q_{lk}|}{|Q_{lk}|} \right)^{n-m} \int_{Q_{lk}}^m \tilde{u}_m(x_1, \cdots x_m) \, dx_1 \cdots dx_m,
\]

because under \(\mathbb{P}_{\rho,n}\), the number of particles in \(Q_{lk}\) follows the law \(\text{Bin}(n, \frac{|Q_{lk}|}{|Q_{lk}|})\) and they are uniformly distributed conditioned the number. We use the similar argument for the expectation of \(\tau_h u\), where we should study the case for particles in \(\tau_h Q_{lk} \subseteq Q_{lk}\)

\[
\mathbb{E}_{\rho,n}[\tau_h u] = \sum_{m=0}^n \mathbb{P}_{\rho,n}[\mu(\tau_h Q_{lk}) = m] \mathbb{E}_{\rho,n}[\tau_h u|\mu(\tau_h Q_{lk}) = m] = \sum_{m=0}^n \left( \frac{|\tau_h Q_{lk}|}{|Q_{lk}|} \right)^m \left( 1 - \frac{|\tau_h Q_{lk}|}{|Q_{lk}|} \right)^{n-m} \times \int_{\tau_h Q_{lk}}^m \tilde{u}_m(x_1 + h, \cdots x_m + h) \, dx_1 \cdots dx_m = \sum_{m=0}^n \left( \frac{|Q_{lk}|}{|Q_{lk}|} \right)^m \left( 1 - \frac{|Q_{lk}|}{|Q_{lk}|} \right)^{n-m} \int_{Q_{lk}}^m \tilde{u}_m(x_1, \cdots x_m) \, dx_1 \cdots dx_m.
\]

Thus establish \(\mathbb{E}_{\rho,n}[\tau_h u] = \mathbb{E}_{\rho,n}[u]\) and \(V\) has mean zero.

Now we can apply the lemma: for any \(0 < \varepsilon < \frac{1}{8}(|u|_{L^\infty}(l_k)^2)^{-1}\), we put \(\sqrt{A_{Q_{lk}} g}/\mathbb{E}_{\rho,n}[A_{Q_{lk}} g]\) at the place of \(f\) in eq. (5.6) and combine with eq. (5.7) to obtain that

\[
\mathbb{E}_{\rho,n}[A_{Q_{lk}} g(u - \tau_h u)] \leq 2\varepsilon \mathbb{E}_{\rho,n}[(u - \tau_h u)((-\Delta_{Q_{lk}})^{-1}(u - \tau_h u))] \mathbb{E}_{\rho,n}[A_{Q_{lk}} g] + \frac{1}{\varepsilon} \mathbb{E}_{\rho,n}\left[ \sqrt{A_{Q_{lk}} g}((-\Delta_{Q_{lk}})^{-1} A_{Q_{lk}} g) \right].
\]

Notice that \((-\Delta_{Q_{lk}})^{-1} \in L^2 \rightarrow H^1\) well-defined thanks to the Lax-Milgram theorem and the spectral bound, we get

\[
\sum_{\rho,n}[A_{Q_{lk}} g(u - \tau_h u)] \leq 8\varepsilon(l_k)^2 |u|_{L^\infty}^2 \mathbb{E}_{\rho,n}[A_{Q_{lk}} g] + \frac{1}{\varepsilon} \mathbb{E}_{\rho,n}\left[ \sqrt{A_{Q_{lk}} g}((-\Delta_{Q_{lk}})^{-1} A_{Q_{lk}} g) \right].
\]
For the case $\varepsilon > \frac{1}{8} (\|u\|_{L^\infty}(l_k)^2)^{-1}$, we have $1 \leq 8\varepsilon \|u\|_{L^\infty}(l_k)^2$, thus we use a trivial bound
\begin{equation}
\mathbb{E}_{\rho,n}[A_{Q_{i_k}} g(u - \tau_h u)] \leq 2\|u\|_{L^\infty} \mathbb{E}_{\rho,n}[A_{Q_{i_k}} g] \leq 16\varepsilon (l_k)^2 \|u\|_{L^\infty}^2 A_{Q_{i_k}} g.
\end{equation}
We combine eq. (5.9), eq. (5.10) and do optimization with for $\varepsilon$ to obtain that
\begin{equation}
\mathbb{E}_{\rho,n}[A_{Q_{i_k}} g(u - \tau_h u)] \leq 4l_k \|u\|_{L^\infty} \left( \mathbb{E}_{\rho,n}[A_{Q_{i_k}} g] \mathbb{E}_{\rho,n} \left[ \sqrt{A_{Q_{i_k}} g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{A_{Q_{i_k}} g}} \right) \right] \right)^{\frac{3}{2}}.
\end{equation}
Here the term $\mathbb{E}_{\rho,n} \left[ \sqrt{A_{Q_{i_k}} g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{A_{Q_{i_k}} g}} \right) \right]$ is not the desired term and we should remove the conditional expectation here. For any $x \in Q_{i_k}$, using Cauchy-Schwartz inequality we have
\begin{equation}
A_{Q_{i_k}} \left( \frac{\nabla g(\mu, x)}{g(\mu)} \right)^2 A_{Q_{i_k}} g(\mu) \geq \left( A_{Q_{i_k}} \nabla g(\mu, x) \right)^2 \geq \left| A_{Q_{i_k}} \nabla g(\mu, x) \right|^2.
\end{equation}
Thus, in the term $\mathbb{E}_{\rho,n} \left[ \sqrt{A_{Q_{i_k}} g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{A_{Q_{i_k}} g}} \right) \right]$ we have
\begin{equation}
\mathbb{E}_{\rho,n} \left[ \sqrt{A_{Q_{i_k}} g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{A_{Q_{i_k}} g}} \right) \right] = \frac{1}{4} \mathbb{E}_{\rho,n} \left[ \int_{Q_{i_k}} \frac{A_{Q_{i_k}} \nabla g(\mu, x)}{A_{Q_{i_k}} g(\mu)} \Delta_{Q_{i_k}} g(\mu) \right] \leq \frac{1}{4} \mathbb{E}_{\rho,n} \left[ \int_{Q_{i_k}} A_{Q_{i_k}} \left( \frac{\nabla g(\mu, x)}{g(\mu)} \right)^2 \right] = \mathbb{E}_{\rho,n} \left[ \sqrt{g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{g}} \right) \right].
\end{equation}
Using the transpose invariant property for $\mu$, we obtain
\begin{equation}
\mathbb{E}_{\rho,n}[A_{Q_{i_k}} g(u - \tau_h u)] \leq 4l_k \|u\|_{L^\infty} \left( \mathbb{E}_{\rho,n}[A_{Q_{i_k}} g] \mathbb{E}_{\rho,n} \left[ \sqrt{g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{g}} \right) \right] \right)^{\frac{3}{2}}.
\end{equation}
and put it back to eq. (5.8) and use Cauchy-Schwartz inequality
\begin{equation}
\left( \mathbb{E}_{\rho}[g(u - \tau_h u)] \right)^2 \leq \left( \sum_{n=0}^\infty \mathbb{P}_{\rho}(\mu(Q_{i_k}) = n) \mathbb{E}_{\rho,n}[A_{Q_{i_k}} g] \mathbb{E}_{\rho,n} \left[ \sqrt{g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{g}} \right) \right] \right)^2 \leq \left( \sum_{n=0}^\infty \mathbb{P}_{\rho}(\mu(Q_{i_k}) = n) \mathbb{E}_{\rho,n}[A_{Q_{i_k}} g] \mathbb{E}_{\rho,n} \left[ \sqrt{g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{g}} \right) \right] \right)
\end{equation}
\begin{equation}
( \mathbb{E}_{\rho}[g(u - \tau_h u)])^2 \leq \left( \sum_{n=0}^\infty \mathbb{P}_{\rho}(\mu(Q_{i_k}) = n) \mathbb{E}_{\rho,n}[A_{Q_{i_k}} g] \mathbb{E}_{\rho,n} \left[ \sqrt{g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{g}} \right) \right] \right)^2
\end{equation}
\begin{equation}
= \left( \sum_{n=0}^\infty \mathbb{P}_{\rho}(\mu(Q_{i_k}) = n) \mathbb{E}_{\rho,n}[A_{Q_{i_k}} g] \mathbb{E}_{\rho,n} \left[ \sqrt{g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{g}} \right) \right] \right)^2
\end{equation}
\begin{equation}
= (l_k \|u\|_{L^\infty})^2 \mathbb{E}_{\rho} \left[ \sqrt{g} \left( -\frac{\Delta_{Q_{i_k}}}{\sqrt{g}} \right) \right].
\end{equation}

5.3. Entropy. We recall the definition of $\delta$-good configuration for $\frac{L}{l} \in \mathbb{N}$
\begin{equation}
C_{L,l,\rho,\delta} = \left\{ M \in \mathbb{N} \left( \frac{L}{l} \right)^d \left| \forall i \leq \left( \frac{L}{l} \right)^d, \left| \frac{D_i}{\rho|Q_i|} - 1 \right| \leq \delta \right\}.
\end{equation}

Lemma 5.4 (Bound for entropy). Given $l \geq 1, \frac{L}{l} \in \mathbb{N}, 0 < \delta < \frac{l}{2}$ for any $M \in C_{L,l,\rho,\delta}$, we have a bound for the entropy of $g_M$ defined in eq. (3.12) that
\begin{equation}
H(g_M) \leq C(d, \rho) \left( \frac{L}{l} \right)^d \left( \log(l) + l^d \delta^2 \right).
\end{equation}
Proof.

\[ H(g_M) = \mathbb{E}_\rho[g_M \log(g_M)] = -\mathbb{E}_\rho[g_M \log(\mathbb{P}_\rho[M_{L,t} = M])]. \]

It suffices to prove a upper bound for \(-\log(\mathbb{P}_\rho[M_{L,t} = M])\), which is

\[
-\log(\mathbb{P}_\rho[M_{L,t} = M]) = -\log \left( \prod_{i=1}^{(\frac{n}{M_i})} e^{-\rho|Q_i| (\frac{M_i}{M_i!})} \right) = -\log \left( e^{-\rho|Q_i| (\frac{M_i}{M_i!})} \right).
\]

For every term \(M_i\), we set \(\delta_i := \frac{M_i}{\rho|Q_i|} - 1\), and use Stirling’s formula upper bound \(n! \leq e\sqrt{n} \left( \frac{n}{e} \right)^n\) for any \(n \in \mathbb{N}\)

\[
-\log \left( e^{-\rho|Q_i| (\frac{M_i}{M_i!})} \right) = \rho|Q_i| - M_i \log(\rho|Q_i|) + \log (M_i!)
\]

\[
\leq \rho|Q_i| - M_i \log(\rho|Q_i|) + \log \left( e\sqrt{M_i} \left( \frac{M_i}{e} \right)^{M_i} \right)
\]

\[
\leq \rho|Q_i| \left( \frac{M_i}{\rho|Q_i|} \log \left( \frac{M_i}{\rho|Q_i|} \right) + 1 - \frac{M_i}{\rho|Q_i|} \right) + \frac{1}{2} \log (M_i)
\]

\[
= \rho|Q_i| \left( 1 + \delta_i \right) \log (1 + \delta_i) - \delta_i \frac{1}{\epsilon \delta_i} \log (l) + \frac{1}{2} \log (M_i)
\]

\[
\leq \rho|Q_i| \delta_i^2 + C \log (l).
\]

We use \(|\delta_i| \leq \delta\) and put it back to eq. (5.12) and obtain the desired result. \(\square\)

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