Robust utility maximization problem in a discontinuous filtration

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Abstract

We study a problem of utility maximization under model uncertainty with information including jumps. We prove first that the value process of the robust stochastic control problem is described by the solution of a quadratic-exponential backward stochastic differential equation with jumps. Then, we establish a dynamic maximum principle for the optimal control of the maximization problem. The characterization of the optimal model and the optimal control (consumption-investment) is given via a forward-backward system which generalizes the result of Duffie and Skiadas [14] and El Karoui et al. [17] in the case of maximization of recursive utilities including model with jumps.

Keywords: Robust maximization problem, preferences, model uncertainty, stochastic control, recursive utility, stochastic differential utility, backward stochastic differential equations, forward-backward system, maximum principle, jump model.

1 Introduction

The utility maximization is a basic problem in mathematical finance which was introduced by Merton [38]. Using stochastic control methods, he has exhibited a closed formula for the value function and the optimal portfolio/consumption when the risky asset follows a geometric Brownian motion and the utility function is of CRRA type. There exists a huge literature on this problem based on two approaches: the Bellman approach and the martingale one. Karatzas et al. [28] have studied a consumption-investment problem in a more general case, taking into consideration the inherent non-negativity of consumption and bankruptcy constraint. When the risky assets are modeled by geometric Brownian motions, the value function is determined explicitly, as well as the optimal consumption and the investment strategy, by solving the Bellman Equation and using a verification theorem. Martingale methods were introduced by Karatzas et al. [29] and Cox and Huang [10], who characterized the optimal consumption-portfolio policies when there are non-negativity constraints on both consumption and final wealth. Unlike the non-linear Bellman equation, they gave a verification theorem which involves a linear partial

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Our paper deals with the problem of utility maximization from a terminal value and an intermediate control under model uncertainty. Uncertainty refers to the case in which a decision marker does not know the probability distribution governing the stochastic nature of the problem she/he is facing. This uncertainty is captured by using capacities or sets of probability measures over the space of state of the world. The set of such probability measures on some measurable space $(\Omega, \mathcal{F})$ is called by economics *objectively rational beliefs*, and each element of such set is called beliefs on $\mathcal{F}$ that the decision maker is able to justify on the basis of the available information. The incompleteness of information is then captured by the fact that one considers a non singleton set of probability measures.

In the mathematical finance literature, there are two approaches to solve robust utility maximization problems. The first one relies on duality methods and are presented in Quenez [42], Gundel [20], Shied and Wu [45], and Shied [44]. The second approach, which is the one followed in this paper, is based on the penalization method and the minimization is taken over all possible models as in Anderson, Hansen and Sargent [2] and Hansen et al. [21]. These authors have introduced and discussed the robust utility maximization problem when the model uncertainty $\mathcal{Q}$ is penalized by a relative entropy term with respect to a given reference probability measure $\mathbb{P}$. Both papers are cast in a Markovian setting and use mainly formal manipulations of Hamilton-Jacobi-Bellman (HJB) equations to provide insights about the optimal investment behaviour in these situations. In [48], Skiadas follows the same point of view and gives the dynamics of the control problem via BSDE in the Markovian context. More precisely, Skiadas [48] points out that the BSDE coincides with the one describing a stochastic differential utility; hence, working with a standard expected utility under (a particular form of) model uncertainty is equivalent to working with a corresponding stochastic differential utility under a fixed model (see also Duffie and Epstein [13] and Duffie and Skiadas [14] for more about stochastic differential utilities). We have to mention the interesting works of Maccheroni, Marinacci and Rustichini [35, 36] who have studied preferences and robustness by using variational technics (principle).

More recently, Bordigoni, Matoussi and Schweizer [7] have studied this robust problem in more general setting (non Markovian approach) by using stochastic control technics. They have considered the robust maximization problem:

$$\sup_{\psi} \inf_{c} \inf_{Q \in \mathcal{Q}} U((\psi, c), Q)$$

where $\psi$ runs through a set of random variables, $c$ through a set of processes and $\mathcal{Q}$ through a set of models (measures), and where the criteria $U((\psi, c), Q)$ is the sum of a $Q$-expected utility and (*a penalization term*) associated with the relative entropy. They have solved only the minimization problem and proved the existence of a unique $Q^*$ optimal minimizer model. Moreover, in the case of *continuous filtration*, they have used the dynamic programming Bellman principle to show that the value function of the stochastic control problem is the unique solution of a generalized BSDE with quadratic driver.

Bordigoni [6] studied partially the maximization problem by using classical optimization arguments and assuming some conjectures. She derived the Gâteaux differential of the Lagrangian associated to the optimization problem. She has obtained necessary and
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sufficient conditions that must be fulfilled for the optimal strategy in a complete market in the case of consumption/investment problem.

Faidi, Matoussi and Mnif [15] have studied the maximization part of problem (1) by using the BSDE approach as in Duffie and Skiadas [14] and El Karoui, Peng and Quenez [17] in the case of continuous filtration. We mention also that there is another approach based on Ekeland Variational principle to obtain a dynamic maximum principle for recursive utility optimization problem (see Ji and Zhou [25, 26]).

In our paper, we prove first that in the case of discontinuous filtration (information including jumps), the value process \( V \) of the stochastic minimization problem in (1) is described by a class of quadratic-exponential BSDE with jumps (QBSDEJs in short). Moreover, we characterize the minimal optimal probability measure by means of the martingale part solution appearing in these QBSDEJs and we prove existence and uniqueness of solution of this class of equations, by using the related stochastic control technics. We stress that for a given unbounded terminal condition, the study of QBSDEs is a difficult problem, see for instance Briand and Hu [8, 9] and Barrieu and El Karoui [4] in the continuous framework and we emphasize that adding jumps in [7] involves significant difficulties in solving the related BSDEs. Then, in order to tackle the maximization problem, we prove a comparison theorem for this class of QBSDEJs with unbounded final condition which allows us to prove a dynamic maximum principle for the stochastic control problem in term of the minimal optimal probability measure (see Proposition 8 and Theorem 5). These results may be considered as a generalization of the maximum principle proven in [17, Theorem 4.2]. Moreover, characterization of the optimal control \((\psi^*, c^*, Q^*)\) as the solution of a forward-backward system is given. Finally, we prove in the case of logarithmic utility function of the control process \( c \) that the optimal control \((0, c^*, Q^*)\) is characterized via the unique solution of some forward-backward stochastic differential equation. This part of our work is a generalization of the dynamic maximum principle obtained by El Karoui, Quenez and Peng [17], in a framework including jumps and under model uncertainty. Our results may also be considered as a generalization of the works of [14, 46, 47, 15].

Finally, we have to stress that some class of QBSDEJs was studied by Becherer [5] and Morlais [39, 40] only in the case of bounded terminal condition (see also [1]). More recently, using the forward approach introduced in [4], El Karoui, Matoussi and Ngoupeyou [19] have obtained an existence result of a general class of QBSDEJs with unbounded final condition.

The paper is organized as follows. In Section 2, we describe the model and the stochastic control problem. In Section 3, we characterize the optimal model measure for the minimization problem via a solution of QBSDEJs. We prove a comparison and a concavity results for the solution of our QBSDEJ with respect to control parameters. In Section 4, we derive the necessary and sufficient conditions that must satisfy the optimal control and then we establish the dynamic maximum principle which characterizes implicitly the optimal strategy \((\psi^*, c^*, Q^*)\) as a solution of a forward-backward system. For a specific choice of utility functions, the value function is given in Section 5 in terms of the optimal plan. Finally, Section 6 contains a technical proof concerning a regularity result of QBSDEJs.
2 The Model and the stochastic control problem

We consider a filtered probability space \((\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})\). All the processes are \(\mathbb{G}\)-adapted, and defined on the time interval \([0,T]\) where \(T\) is the finite horizon. We recall that any special \(\mathbb{G}\)-semimartingale \(Y\) admits a canonical decomposition \(Y = Y_0 + A + M^{Y,c} + M^{Y,d}\) where \(A\) is a predictable finite variation process, \(M^{Y,c}\) is a continuous martingale and \(M^{Y,d}\) is a discontinuous martingale.

**Assumption A 1.** We make the following assumptions:

1. For each \(i = 1, \ldots, d\), \(H^i\) is a counting process and there exists a positive adapted process \(\lambda^i\), called the \(\mathbb{P}\) intensity of \(H^i\), such that the process \(N^i\) with

\[
N^i_t := H^i_t - \int_0^t \lambda^i_s ds
\]

is a martingale. We assume that the processes \(H^i, i = 1, \ldots, d\) have no common jumps.

2. Any discontinuous martingale \(M^d\) admits a representation of the form \(dM^d_t = \sum_{i=1}^d y^i_t dN^i_t\) where \(y^i, i = 1, \ldots, d\) are predictable processes.

This hypothesis is satisfied in the case where the filtration is generated by a continuous martingale and an inhomogeneous Poisson process, or in the case where the counting processes are stopped after the first jump, as it is done in credit risk, as in the following example:

**Example 1.** For each \(i = 1, \ldots, d\), let \(H^i_t = 1_{\{\tau_i \leq t\}}\), where \(\tau_i\) is a random time and \(\mathbb{P}(\tau_i = \tau_j) = 0, i \neq j\). Let \(\mathbb{G}\) be the smallest right-continuous filtration which contains the filtration \(\mathbb{F}^B\) generated by a \(p\) dimensional Brownian motion \(B\) and the filtration generated by the processes \(H^i\). Under the assumption that \(\mathbb{P}(\tau_i \in d\theta_i; i = 1, \ldots, d|\mathbb{F}^B) \sim \mathbb{P}(\tau_i \in d\theta_i; i = 1, \ldots, d)\), then any local \(\mathbb{G}\)-martingale \(\zeta = (\zeta_t)_{t \geq 0}\) admits the following decomposition: \(\mathbb{P}\)-a.s.,

\[
\zeta_t = \zeta_0 + \int_0^t Z_s \cdot dW_s + \int_0^t U_s \cdot dN_s \quad \forall t \geq 0
\]

where \(W\) is the martingale part of the \(\mathbb{G}\)-semimartingale \(B\) (see [24]) , \(Z := (Z^1, \ldots, Z^p)\) and \(U := (U^1, \ldots, U^d)\) are \(\mathbb{G}\) predictable processes. Furthermore, if \(\zeta\) is square integrable

\[
\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty, \quad \sum_{i=1}^d \mathbb{E}\left[\int_0^T |U^i_s|^2 \lambda^i_s ds\right] < \infty.
\]

We denote by \(|X|\) the Euclidean norm of a vector or a row vector \(X\). We give now some notations and definitions:

**Definition 1.**

\(L^{\exp}\) is the space of all \(\mathcal{G}_T\)-measurable random variables \(X\) with \(\mathbb{E}^\mathbb{P}[\exp(\gamma |X|)] < \infty, \quad \forall \gamma > 0\).

\(D_0^{\exp}\) is the space of progressively measurable processes \(X = (X_t)_{t \leq T}\) with

\[
\mathbb{E}^\mathbb{P}\left[\exp(\gamma \text{ ess sup}_{0 \leq t \leq T} |X_t|)\right] < \infty, \quad \forall \gamma > 0.
\]
$D_{1}^{\text{exp}}$ is the space of progressively measurable processes $X = (X_{t})_{t \leq T}$ such that

$$
\mathbb{E}^{P} \left[ \exp \left( \gamma \int_{0}^{T} |X_{s}| ds \right) \right] < \infty, \quad \forall \gamma > 0.
$$

$\mathcal{M}_{0}^{P}(\mathbb{P})$ is the space of $\mathbb{P}$-martingales $M = (M_{t})_{t \leq T}$ with $M_{0} = 0$ and $\mathbb{E}^{P} \left[ \sup_{0 \leq t \leq T} |M_{t}|^{p} \right] < \infty$.

$\mathcal{L}^{2}(\lambda, \mathbb{P})$ is the space of $\mathbb{R}^{d}$-valued predictable processes $X$ such that $\sum_{i=1}^{d} \mathbb{E}^{P} \left[ \int_{0}^{T} (X_{s}^{i})^{2} \lambda_{s}^{i} ds \right] < \infty$.

$\mathcal{H}^{2}(\mathbb{P})$ is the space of $\mathbb{R}$-valued predictable processes $X$ such that $\mathbb{E}^{P} \left[ \int_{0}^{T} X_{s}^{2} ds \right] < \infty$.

$\mathcal{S}^{2}(\mathbb{P})$ is the space of all $\mathbb{R}$-valued predictable processes $X$ such that $\mathbb{E}^{P} \left[ \sup_{0 \leq t \leq T} |X_{t}|^{2} \right] < \infty$.

$\mathcal{M}_{0, \text{loc}}^{\beta}(\mathbb{P})$ is the set of continuous local martingales.

**Definition 2.**

For any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_{T})$,

$$
H(\mathbb{Q} || \mathbb{P}) := \begin{cases} 
\mathbb{E}^{\mathbb{Q}} \left[ \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right] & \text{if } \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{G}_{T} \\
+\infty & \text{otherwise}
\end{cases}
$$

is the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. We denote by $\mathcal{Q}_{f}$ (resp. $\mathcal{Q}_{f}^{c}$) the space of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{G}_{T})$ with $\mathbb{Q} \ll \mathbb{P}$ (resp. equivalent to $\mathbb{P}$) on $\mathcal{G}_{T}$ and $H(\mathbb{Q} || \mathbb{P}) < +\infty$. Note that the reference probability measure $\mathbb{P}$ belongs to $\mathcal{Q}_{f}^{c}$.

### 2.1 The robust optimization problem

We define a discounting process $S_{t}^{\delta} := e^{-\int_{0}^{t} \delta s ds}$ for all $t \in [0, T]$ where $\delta$ is a non-negative adapted process. For $\mathbb{Q} \in \mathcal{Q}_{f}$, we denote by $Z^{\mathbb{Q}} = (Z_{t}^{\mathbb{Q}})_{0 \leq t \leq T}$ (a càdlàg $\mathbb{P}$-martingale) its Radon-Nikodym density with respect to $\mathbb{P}$. Let $U$ be a given process (the cost process) and $\tilde{U}_{T}$ a given random variable (the terminal target). The robust utility maximization problem $\mathcal{P}(U, \tilde{U}_{T}, \beta)$ is to find the infimum of $\Gamma(\mathbb{Q})$ over the set $\mathcal{Q}_{f}$ where

$$
\Gamma(\mathbb{Q}) = \mathbb{E}^{\mathbb{Q}} \left[ \int_{0}^{T} S_{s}^{\delta} U_{s} ds + S_{T}^{\delta} \tilde{U}_{T} \right] + \beta \mathbb{E}^{\mathbb{Q}} \left[ \int_{0}^{T} \delta s S_{s}^{\delta} \ln Z_{s}^{\mathbb{Q}} ds + S_{T}^{\delta} \ln Z_{T}^{\mathbb{Q}} \right] 
$$

$$
= : \mathbb{E}^{\mathbb{Q}} [U_{0,T}] + \beta \mathbb{E}^{\mathbb{Q}} [\mathcal{R}_{0,T}(\mathbb{Q})] 
$$

(4)

The first term in the right-hand side of (4) will be linked, in the following section, to the $\mathbb{Q}$-expected discounted utility from target and cost process. The second term is a discounted relative entropy term and $\beta > 0$ is a given positive constant which determines the strength of this penalty term. Note that the optimal probability $\mathbb{Q}$ for the problem $\mathcal{P}(U, \tilde{U}_{T}, \beta)$ is optimal for the minimization problem $\mathcal{P}(U^{\beta}, \tilde{U}_{T}^{\beta}, 1)$ where $U^{\beta} = U / \beta$, $\tilde{U}_{T}^{\beta} = \tilde{U}_{T} / \beta$, therefore, we shall restrict our attention to the problem $\mathcal{P}(U, \tilde{U}_{T}) := \mathcal{P}(\mathbb{U}, \tilde{U}_{T}, 1)$.

**Assumption A 2.** For a more precise formulation of our problem, we make the following further assumptions:

i) the discount rate $\delta$ is a non-negative bounded process, more precisely there exist two constants
\( \varepsilon > 0 \) and \( c > 0 \) such that for any \( t \geq 0, 0 < \varepsilon \leq \delta_t \leq \| \delta \| \infty \leq c, \) a.s.

ii) the cost process \( U \) belongs to \( D^1 \) and the terminal target \( \bar{U}_T \) is in \( L^\infty. \)

iii) the process \( \Lambda_t^i := \int_0^t \lambda_s^i \, ds \) is uniformly bounded, i.e., \( \Lambda_T^i \leq C, \) a.s.

**Remark 1.** The assumption \( U \) belongs to \( D^\infty \) implies that \( \mathbb{E}^{Q} \left[ \int_0^T |U_s| \, ds \right] < \infty \) for all \( Q \in \mathcal{Q}_f. \)

Indeed, applying the following estimate:

\[
xy \leq y \ln y - y + e^x, \quad \text{for all } x \in \mathbb{R}, \ y \geq 0,
\]

we get

\[
\mathbb{E}^{Q} \left[ \int_0^T |U_s| \, ds \right] = \mathbb{E}^{P} \left[ Z_T^Q \right] = H(Q|P) - \mathbb{E}^{P} \left[ Z_T^Q \right] + \mathbb{E}^{P} \left[ \exp \left( \int_0^T |U_s| \, ds \right) \right].
\]

**Remark 2.** The assumption iii) is a technical hypothesis needed only in the proof of Theorem 4.

We recall the existence result of the optimal probability measure for the minimization problem \( \inf_{Q \in \mathcal{Q}_f} \Gamma(Q) \) which was given in Theorem 9 and Theorem 12 in Bordigoni, Matoussi et Schweizer [7]:

**Proposition 1.** Under Assumptions A1-A2, there exists a unique \( Q^* \) which minimizes \( \Gamma(Q) \) over all \( Q \in \mathcal{Q}_f: \)

\[
\Gamma(Q^*) = \inf_{Q \in \mathcal{Q}_f} \Gamma(Q) \tag{5}
\]

Furthermore, \( Q^* \) is equivalent to \( \mathbb{P}, \) i.e \( Q^* \in \mathcal{Q}^e_f. \)

### 3 The Optimal Model Measure and BSDE

#### 3.1 A BSDE description of the value process

We use stochastic control techniques to describe the dynamics of the value process \( V \) associated with our robust optimization problem, via BSDEs. In a markovian framework with continuous filtration (see Skiadas [48]) or in a continuous semimartingale setting (see Bordigoni, Matoussi and Schweizer [7]), the authors have established that \( V \) is the unique solution of a backward stochastic differential equation (BSDE) with a quadratic driver.

In our paper, the BSDE associated with our control problem (in a framework including jumps) will contain quadratic and exponential terms and will be of the following form:

**Definition 3.** A triple of processes \((Y, M^{Y,c}, y)\) such that \( Y \) is a \( \mathbb{P} \)-semimartingale, \( M^{Y,c} \) is a locally square-integrable continuous local \( \mathbb{P} \)-martingale null at 0 and \( y = (y^1, \ldots, y^d) \) an \( \mathbb{R}^d \)-valued predictable locally bounded process, is called solution of BSDEJ, if it satisfies:

\[
\begin{align*}
    dY_t &= \left[ \sum_{i=1}^d g(y^i_t) \lambda^i_t - U_t + \delta_t Y_t \right] dt + \frac{1}{2} d(M^{Y,c})_t + dM^{Y,c}_t + \sum_{i=1}^d y^i_t dN^i_t \\
    Y_T &= \bar{U}_T
\end{align*}
\]

where \( g \) is the convex function \( g(x) = e^{-x} + x - 1. \) Note that \( Y \) is a special \( \mathbb{P} \)-semimartingale.
Remark 3. In the case where the filtration \( G \) is generated by a Brownian motion \( W \) and the jump process \( N \), the BSDEJ involves a quadratic term:

\[
dY_t = \left[ \sum_{i=1}^{d} g(y^i_t)i_t - U_t + \delta_t Y_t + \frac{1}{2} |Z_t|^2 \right] dt + Z_t dW_t + \sum_{i=1}^{d} g^i_t dN^i_t
\]

\( Y_T = \bar{U}_T \)

Such BSDEs have been studied recently in the case where the terminal condition is bounded and typically appear in problems from pricing-hedging derivative options by indifference pricing of and maximization of expected exponential utility including jumps on the wealth portfolio; see for instance Becherer [5], Morlais [39, 40], Lim and Quenez [34], Ankirchner, Blanchet-Scalliet and Eyraud-Loisel [1] and Schroder and Skiadas [47] for some recent references. However, all existence and comparison results for such equations assume that the terminal value \( Y_T \) is bounded; here, we relax this condition. In a recent work, El Karoui, Matoussi and Ngoupeyou [19] have obtained an existence result of a general class of QBSDEJs and unbounded final condition.

We first establish a recursion relation for solutions of (6) which implies the uniqueness of the solution:

**Proposition 2.** Let \((Y, M^{Y,c}, y) \in D^\text{exp}_0 \times \mathcal{M}_0,\text{loc}^c(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})\) be a solution of the BSDEJ (6). Then, \( Y \) satisfies the following recursion equality: for any stopping time \( \tau \) valued in \([t, T]\),

\[
Y_t = -\ln \mathbb{E}^P \left[ \exp \left( -Y_\tau + \int_t^\tau (\delta_s Y_s - U_s) ds \right) \right] \bigg| \mathcal{G}_t.
\]  

(7)

Moreover the BSDE (6) admits at most one solution which belongs to \( D^\text{exp}_0 \times \mathcal{M}_0,\text{loc}^c(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})\).

**Proof:** (i) Assume that \((Y, M^{Y,c}, y)\) is a solution of (6), and define \(X_t := Y_t - Y_0 - \int_0^t (\delta_s Y_s - U_s) ds\) and \(Z_t := e^{-X_t}\). Itô’s formula leads to \(dZ_t = Z_t \left[ -dM^{Y,c}_t + \sum_{i=1}^{d} \left( e^{-y^i_t} - 1 \right) dN^i_t \right]\).

Hence, \(Z\) is a non-negative local martingale. Assuming that \(Z\) is a martingale, one obtains, for \(t < \tau < T\):

\[
e^{-Y_t} = \mathbb{E}^P \left[ \exp \left( -Y_\tau + \int_t^\tau (\delta_s Y_s - U_s) ds \right) \right] \bigg| \mathcal{G}_t.
\]  

(8)

Otherwise, using a localizing sequence \(\tau_n\) such that the stopped process \(Z^{\tau_n}\) is a martingale, we obtain (8) with \(\tau_n \wedge \tau\) instead of \(\tau\). By the integrability Assumption 2 and the assumption that \(Y \in D^\text{exp}_0\), we obtain a \(\mathbb{P}\)-integrable upper bound for the right-hand side of (8) and letting \(n\) go to infinity, we obtain (7) for \(\tau\) by dominated convergence.

(ii) **Uniqueness of the solution of the BSDE (6):** Assume that \((Y, M^{Y,c}, y)\) and \((\bar{Y}, M^{\bar{Y},c}, \bar{y})\) are two solutions of (6) in \(D^\text{exp}_0 \times \mathcal{M}_0,\text{loc}^c(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})\). Suppose that, for some \(t \in [0, T]\), the set \(A = \{Y_t > \bar{Y}_t\} \in \mathcal{G}_t\) satisfies \(\mathbb{P}(A) > 0\) and define \(\tau = \inf\{s \geq t \mid Y_s \geq \bar{Y}_s\}\), so that \(\bar{Y}_\tau \geq Y_\tau\). Since \(Y_T = \bar{Y}_T\), one has \(\tau \leq T\), and:

\[
\int_t^\tau (\delta_s Y_s - U_s) ds - Y_\tau > \int_t^\tau (\delta_s \bar{Y}_s - U_s) ds - \bar{Y}_\tau \text{ on } A,
\]

then from the recursion relation (7), it follows that

\[
\exp (-Y_t) = \mathbb{E}^P \left[ \exp \left( \int_t^\tau \delta_s Y_s - U_s ds - Y_t \right) \right] \bigg| \mathcal{G}_t > \exp (-\bar{Y}_t) \text{ on } A
\]

which implies that \(Y_t < \bar{Y}_t\) on \(A\) in contradiction with the definition of \(A\); therefore \(Y\) and \(\bar{Y}\) are indistinguishable. Therefore \(M^{Y,c} = M^{\bar{Y},c}\) and \(y = y'\). \(\Box\)
Remark 4. In the case $\delta = 0$, the process $Y$, part of the solution of (6), is given in a closed form as

$$Y_t = -\ln \mathbb{E}^\mathbb{P} \left[ \exp \left( -\bar{U}_T - \int_t^T U_s ds \right) \bigg| \mathcal{G}_t \right].$$

In the case $U \equiv 0$, we recognize the dynamic entropic risk measure studied, in particular, by Barrieu and El Karoui [3].

The main result of this section gives the BSDE description of the value process of our robust control problem. It extends earlier works by Skiadas [48, Theorem 5, pp. 482] and Bordigoni, Matoussi and Schweizer [7, Theorem 12] (see also Lazrak and Quenez [33] and Schroder and Skiadas [46]).

**Theorem 1.** Assume (A1) and (A2). Then, there exists a unique triple $(Y, M^{\gamma_c}, \nu) \in D^\text{exp}_0 \times \mathcal{M}^p_0(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$ solution of (6). Furthermore, the optimal measure $\mathbb{Q}^*$ solution of (5) admits the Radon-Nikodym density $Z^{\mathbb{Q}^*} = \mathcal{E}(L)$ w.r.t. $\mathbb{P}$ where

$$dL_t = -dM_t^{\gamma_c} + \sum_{i=1}^d \left( e^{-y_i} - 1 \right) dN^i_t, \quad L_0 = 0. \tag{9}$$

**Proof:** We divide the proof in three steps. We first prove that the value process $V$ of our control problem is a $\mathbb{P}$-special semimartingale, i.e., $V = V_0 + M^\gamma + A^V$ with $dM^\gamma_t = dM^{\gamma_c}_t + \sum_{i=1}^d \nu_i dN^i_t$. Secondly, we prove that $(V, M^{\gamma_c}, \nu)$ is a solution of the BSDE (6). Finally, we show that $(V, M^{\gamma_c}, \nu) \in D^\text{exp}_0 \times \mathcal{M}^p_0(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$.

**Step 1:** We embed the minimization of $\Gamma(\mathbb{Q})$ in a stochastic control problem and we use mainly the martingale optimality principle from El Karoui [16] (Theorem 1.15, Theorem 1.17 and Theorem 1.21) to get our result. To that end, we introduce a few more notation. We define the minimal conditional cost

$$J(\tau, \mathbb{Q}) := \mathbb{Q} - \text{ess inf}_{\mathbb{Q}' \in D(\mathbb{Q}, \tau)} \Gamma(\tau, \mathbb{Q}')$$

with $\Gamma(\tau, \mathbb{Q}') := \mathbb{E}_\mathbb{Q} \left[ U^\delta_{0,T} + R^\delta_{0,T}(\mathbb{Q}') \bigg| \mathcal{G}_\tau \right]$ and $D(\mathbb{Q}, \tau) = \{ Z^{\mathbb{Q}'} | \mathbb{Q}' \in \mathbb{Q}_f \text{ and } \mathbb{Q}' = \mathbb{Q} \text{ on } \mathcal{G}_\tau \}$. So, we can write our minimization problem as

$$\inf_{\mathbb{Q} \in \mathbb{Q}_f} \Gamma(\mathbb{Q}) = \mathbb{E}^\mathbb{P} \left[ J(0, \mathbb{Q}) \right]$$

by using the dynamic programming equation and the fact that $\mathbb{Q} = \mathbb{P}$ on $\mathcal{G}_0$ for every $\mathbb{Q} \in \mathbb{Q}_f$. A measure $\mathbb{Q} \in \mathbb{Q}_f$ is called optimal if it minimizes $\mathbb{Q} \mapsto \Gamma(\mathbb{Q})$ over $\mathbb{Q} \in \mathbb{Q}_f$. We know from Proposition 1 (or Theorem 9 and Theorem 12 in [7]) that there exists an optimal $\mathbb{Q}^*$ which belongs to $\mathbb{Q}^*_f$, hence, w.l.o.g., we restrict our attention to minimize $\mathbb{Q} \mapsto \Gamma(\mathbb{Q})$ over $\mathbb{Q} \in \mathbb{Q}^*_f$. For each $\mathbb{Q} \in \mathbb{Q}^*_f$ and $\tau \in \mathcal{I}$, where $\mathcal{I}$ is the set of $\mathbb{G}$-stopping times valued in $[0, T]$, we define

$$V(\tau, \mathbb{Q}) := \mathbb{Q} - \text{ess inf}_{\mathbb{Q}' \in D(\mathbb{Q}, \tau)} \mathbb{E}_{\mathbb{Q}'} \left[ U^\delta_{\tau,T} + R^\delta_{\tau,T}(\mathbb{Q}') \bigg| \mathcal{G}_\tau \right]$$

which is the value of the control problem started at time $\tau$ and assuming one has used the model $\mathbb{Q}$ up to time $\tau$. By using the Bayes formula and the definition of $R^\delta_{\tau,T}(\mathbb{Q}')$, one can easily prove that $V(\tau, \mathbb{Q}) = V(\tau)$ does not depend on $\mathbb{Q} \in \mathbb{Q}^*_f$. Moreover, comparing the definitions of $V(\tau)$ and $J(\tau, \mathbb{Q})$ yields for $\mathbb{Q} \in \mathbb{Q}^*_f$
Robust utility maximization problem

\[ J(\tau, Q) = S^Q V(\tau) + \int_0^\tau S^Q_s U_s ds + \int_0^\tau \delta_s S^Q_s \ln Z^Q_s ds + S^Q_t \ln Z^Q_T \]

because we can also take the ess inf for \( J(\tau, Q) \) under \( P \sim Q \). From the martingale optimality principle proved in [7] (Proposition 13 pp.140), there exists an adapted \( \mathbb{Q} \)-local martingale process \( J^Q = \langle J^Q \rangle_{0 \leq t \leq T} \) which is a right closed \( \mathbb{Q} \)-submartingale such that \( J^Q = J(\tau, Q) \). Thus we can choose an adapted \( \mathbb{Q} \)-local martingale \( V = (V_t)_{0 \leq t \leq T} \) such that \( V_\tau = V(\tau) = V(\tau, Q) \), \( \mathbb{P} \)-a.s. for \( \tau \in \mathcal{I} \) and \( Q \in \mathbb{Q}^e_f \), and then we get, for each \( Q \in \mathbb{Q}^e_f \),

\[ J^Q = S^Q V + \int S^Q_s U_s ds + \int \delta_s S^Q_s \ln Z^Q_s ds + S^Q_t \ln Z^Q_T. \] (10)

As \( P \in \mathbb{Q}^e_f \) and \( J^P \) is a \( \mathbb{P} \)-submartingale (from Proposition 13 pp. 140 in [7]), equation (10) yields that \( J^P = S^Q V + \int S^Q_s U_s ds \). Thus \( V \) is a \( \mathbb{P} \)-special semimartingale, i.e., its canonical decomposition can be written as

\[ V = V_0 + M^V + A^V. \]

Since \( S^Q \) is uniformly bounded from below and \( J^P \) is a \( \mathbb{P} \)-submartingale, Assumption (A2) implies that \( M^V \) is a true \( \mathbb{P} \)-martingale and that \( dM^V_t = dM^V_{t,c} + \sum_{i=1}^d v_i^t dN^i_t \) where \( M^V_{t,c} \) is a continuous \( \mathbb{P} \)-martingale.

**Step 2:** We now prove that \((V, M^V_{t,c}, v)\) is solution of the BSDE (6) where \( v := (v^1, \ldots, v^d) \).

For \( Q \in \mathbb{Q}^e_f \), we denote by \( L^Q \) the stochastic logarithm of \( Z^Q \), i.e., the \( \mathbb{P} \)-local martingale such that \( dZ^Q_t = Z^Q_{t-} dL^Q_t \).

From Assumption 1, the local martingale \( L^Q \) admits the decomposition \( dL^Q_t = dL^Q_{t,c} + \sum_{i=1}^d \ell^i_t dN^i_t \), where \( L^Q_{t,c} \) is a continuous \( \mathbb{P} \)-local martingale, and \( \ell^i_t \) are predictable processes, and one has

\[ d \ln Z^Q_t = dL^Q_{t,c} - \frac{1}{2} d(L^Q_{t,c})_t + \sum_{i=1}^d \ln(1 + \ell^i_t) dN^i_t + \sum_{i=1}^d (\ln(1 + \ell^i_t) - \ell^i_t) \lambda^i_t dt. \] (11)

Using integration by parts formula, we obtain after some simple computations and using (10) and (11):

\[ dJ^Q_t = S^Q_t \left( (-\delta_t V_t + U_t)dt + (dV_t + d \ln Z^Q_t) \right) \]

\[ = S^Q_t \left( (-\delta_t V_t + U_t)dt + dM^V_{t,c} + dA^V_t + dL^Q_{t,c} - \frac{1}{2} d(L^Q_{t,c})_t \right) \]

\[ + \sum_{i=1}^d (v^i_t + \ln(1 + \ell^i_t)) dN^i_t + \sum_{i=1}^d (\ln(1 + \ell^i_t) - \ell^i_t) \lambda^i_t dt. \]

From Girsanov’s theorem, the processes \( \tilde{N}^i_t \) and \( \tilde{M}^c_t \) defined as \( d\tilde{N}^i_t = dN^i_t - \ell^i_t \lambda^i_t dt, \) and \( d\tilde{M}^c_t = d(M^V_{t,c} + L^Q_{t,c}) - d(M^V_{t,c} + L^Q_{t,c}), \) are \( \mathbb{Q} \)-local martingales, and:

\[ dJ^Q_t = S^Q_t \left[ (-\delta_t V_t + U_t)dt + d\tilde{M}^c_t + dA^V_t + d(M^V_{t,c} + L^Q_{t,c})_t - \frac{1}{2} d(L^Q_{t,c})_t \right] \]

\[ + \sum_{i=1}^d (v^i_t + \ln(1 + \ell^i_t)) d\tilde{N}^i_t + \sum_{i=1}^d (\ell^i_t (v^i_t - 1) + (1 + \ell^i_t) \ln(1 + \ell^i_t)) \lambda^i_t dt. \]
In order that the process $J^Q$ is a $Q$-submartingale for each $Q \in \mathcal{Q}_f$, we impose that its finite variation part is a non-decreasing process.

$$A^Y = -\text{ess inf}_{Q^f} \int_0^t (U_s - \delta_s V_s) ds + \langle M^{V,c} + L^{Q,c}, L^{Q,c} \rangle_t - \frac{1}{2} \langle L^{Q,c} \rangle_t$$

$$+ \sum_{i=1}^d \int_0^t (\ell^i_s(v^i_s - 1) + (1 + \ell^i_s) \ln(1 + \ell^i_s)) \lambda^i_s ds. \quad (12)$$

To find the $\text{ess inf}$, we divide (12) in two parts, the continuous part and the discontinuous part; hence we have two optimization problems:

$$A^Y_t = \int_0^t (\delta_s V_s - U_s) ds - \text{ess inf}_{Q^f} \{ \langle M^{V,c}, L^{Q,c} \rangle_t + \frac{1}{2} \langle L^{Q,c} \rangle_t \}$$

$$- \text{ess inf}_{Q^f} \sum_{i=1}^d \int_0^t (\ell^i_s(v^i_s - 1) + (1 + \ell^i_s) \ln(1 + \ell^i_s)) \lambda^i_s ds. \quad (13)$$

It is proved in [7] that the first infimum is obtained for $L^{Q,c} = -M^{V,c}$ and $-\text{ess inf}_{Q^f} \{ \langle M^{V,c}, L^{Q,c} \rangle + \frac{1}{2} \langle L^{Q,c} \rangle \} = \frac{1}{2} \langle M^{V,c} \rangle$. The second part of the optimization problem reduces to find the optimal $\ell^i$, solution of $\text{ess inf} (\ell^i_s(v^i_s - 1) + (1 + \ell^i_s) \ln(1 + \ell^i_s))$ which is an easy task, the solution being $\ell^i_e = e^{-v^i_e} - 1$, which leads to $-\text{ess inf} (\ell^i_s(v^i_s - 1) + (1 + \ell^i_s) \ln(1 + \ell^i_s))$

$$= e^{-v^i_e} + v^i_e - 1 = g(v^i_e) \text{ where } g(x) = e^{-x} + x - 1. \text{ Therefore,}$$

$$A^Y = \int_0^t (\delta_s V_s - U_s) ds + \frac{1}{2} \langle M^{V,c} \rangle_t + \int_0^t \sum_{i=1}^d g(v^i_s) \lambda^i_s ds. \quad (14)$$

It follows that $(V, M^{V,c}, v)$ is a solution of

$$dV_t = \left(\delta_t V_t - U_t + \sum_{i=1}^d g(v^i_t) \lambda^i_t \right) dt + \frac{1}{2} d\langle M^{V,c} \rangle_t + dM^{V,c}_t + \sum_{i=1}^d v^i_t dN^i_t, \quad V_T = \bar{U}_T \quad (13)$$

Furthermore there exists a solution of the QBSDE (6) and the optimal probability measure $Q^*$ is characterized by its Radon-Nikodym density

$$dZ^*_t = Z^*_t dL_t, \quad dL_t = -dM^{V,c}_t + \sum_{i=1}^d \left(e^{-v^i_t} - 1 \right) dN^i_t.$$

**Step 3:** In this step we prove that the solution $(Y, M^{Y,c}, y)$ of the BSDE (6) belongs to the required spaces.

From Lemma 19 and Proposition 20 in [7], we have that $Y$ belongs to $D^{exp}_0$. Let now study the space of the process $M^{Y,c}$. We introduce the $\mathbb{P}$-martingale:

$$K_t := \mathbb{E}^\mathbb{P} \left[ \exp \left( \int_0^T (\delta_s Y_s - U_s) ds - \bar{U}_T \right) \right] G_t.$$

Using the fact that $Y \in D^{exp}_0$, we obtain that the process $K$ belongs to $\mathcal{M}^p(\mathbb{P})$. Now, the recursive property leads to $K_t = \exp \left( -Y_t + \int_0^t (\delta_s Y_s - U_s) ds \right)$ and from Itô’s formula and the canonical decomposition of $Y$,

$$dM^{Y,c}_t = -\frac{dK^c_t}{K_t}. \quad (14)$$
From Assumption 1, there exists \( k^i \) and \( M^{K,c} \) such that \( K_t = K_0 + M_t^{K,c} + \sum_{i=1}^d \int_0^t k^i_s dN_s^i \). Hence, from (14):

\[
\langle M^{Y,c} \rangle_T \leq \int_0^T \frac{1}{K_t^2} d\langle K^c \rangle_t \leq \langle K^c \rangle_T \sup_{0 \leq t \leq T} \frac{1}{K_t^2}
\]

\[
\leq \langle K^c \rangle_T \exp \left( 2 \sup_{0 \leq t \leq T} \|Y_t\| (1 + \|\delta\|_\infty T) + 2 \int_0^T |U_s| ds \right)
\]

By BDG’s inequalities, there exists a constant \( C \) such that for every \( p \in [1, +\infty) \):

\[
\mathbb{E}^p \left[ \langle K^c \rangle_T + \int_0^T (k^i_t)^2 dH^i_t \right]^{\frac{p}{2}} \leq C \mathbb{E}^p \left( \sup_{0 \leq t \leq T} |K_t|^p \right)
\]

(15)

Since \( K \in \mathcal{M}^p(\mathbb{P}) \), we conclude that \( M^{Y,c} \) lies in the space \( \mathcal{M}_0^p(\mathbb{P}) \) for every \( p \in [1, +\infty) \). We conclude, using again BDG’s inequalities. Finally, let now characterize the space of process \( y \). Using the recursive relation and the decomposition of the process \( K \) we get \( \ln(K_{t^-} + k^i_t) - \ln(K_{t^-}) = -y^i_t \), hence:

\[
\mathbb{E}^p \left[ \int_0^T (e^{-y^i_t} - 1)^2 dH^i_t \right]^{\frac{p}{2}} = \mathbb{E}^p \left[ \int_0^T \left( \frac{k^i_t}{K_t} \right)^2 dH^i_t \right]^{\frac{p}{2}} \leq \mathbb{E}^p \left[ \left( \sup_{0 \leq t \leq T} \frac{1}{K_t^2} \right) \left( \int_0^T (k^i_t)^2 dH^i_t \right) \right]^{\frac{p}{2}}
\]

Since \( \sup_{0 \leq t \leq T} \left( \frac{1}{K_t} \right) \in \mathbb{L}^p(\mathbb{P}) \) for any \( p \in [1, +\infty) \), using (15) and Cauchy inequalities, we conclude:

\[
\mathbb{E}^p \left[ \int_0^T (e^{-y^i_t} - 1)^2 dH^i_t \right]^{\frac{p}{2}} < \infty.
\]

(16)

In particular

\[
\mathbb{E}^p \left[ \int_0^T (e^{-y^i_t} - 1)^2 \lambda^i_t dt \right] = \mathbb{E}^p \left[ \int_0^T \left( \frac{k^i_t}{K_t} \right)^2 dH^i_t \right] \leq \mathbb{E}^p \left[ \left( \sup_{0 \leq t \leq T} \frac{1}{K_t^2} \right) \int_0^T (k^i_t)^2 dH^i_t \right] < \infty.
\]

(17)

By using similar arguments, one proves that:

\[
\mathbb{E}^p \left[ \int_0^T (e^{y^i_t} - 1)^2 \lambda^i_t dt \right] < \infty.
\]

(18)

Moreover, by using the inequality \( |y|^2 \leq 2(|e^{-y} - 1|^2 + |e^y - 1|^2) \), \( \forall y \in \mathbb{R} \) and (17)-(18) we conclude that the process \( y \) belongs to \( \mathcal{L}^2(\lambda, \mathbb{P}) \). \( \square \)

**Remark 5.** The martingale part of the BSDE solution, i.e., \( M = -M^{Y,c} + \sum_{i=1}^d \int_0^T (e^{-y^i_t} - 1) dN^i_t \) belongs to \( \mathcal{M}_0^p(\mathbb{P}) \) for any \( p \in [1, +\infty) \). Indeed, since \( M^{Y,c} \in \mathcal{M}_0^p(\mathbb{P}) \), and (16)

\[
\mathbb{E}^p \left[ \langle M^{Y,c} \rangle_T + \sum_{i=1}^d \int_0^T (e^{-y^i_t} - 1)^2 dH^i_t \right]^{\frac{p}{2}} < \infty,
\]

and using BDG inequality, we obtain \( \mathbb{E}^p \left( \sup_{0 \leq t \leq T} |M_t|^p \right) < \infty. \)
3.2 Properties of the value process

In this part, we establish a comparison theorem for the class of QBSDEJs (6) which is a key point to derive the dynamic maximum principle for the maximization problem.

**Definition 4.** For two random variables $X$ and $Y$, we write $X \leq Y$ for $X \leq Y$ a.s. For two processes $A$ and $B$, we write $A \leq B$ for $A_t \leq B_t, \forall t \in [0,T], a.s.$ We write $(X, A) \leq (Y, B)$ if $X \leq Y$ and $A \leq B$.

**Theorem 2.** Assume that for $k = 1, 2$, $(Y^k, M^{k,c}, y^k)$ is the solution of the BSDE (6) associated with $(U^k, \bar{U}_T^k)$. We denote $Y^{12} := Y^1 - Y^2$, $U^{12} := U^1 - U^2$ and $\bar{U}_T^{12} := \bar{U}_T^1 - \bar{U}_T^2$. Then,

$$S_t^d Y_t^{12} \leq \mathbb{E}^{Q^*} \left[ \int_t^T S_s^d U_s^{12} ds + S_T^d \bar{U}_T^{12} | \mathcal{G}_t \right]$$

(19)

where $Q^*$ is the solution of $\mathcal{P}(U^2, \bar{U}_T^2)$, i.e., the probability measure equivalent to $\mathbb{P}$ with Radon Nikodým density $Z^{Q^*}$ given by

$$dZ_t^{Q^*} = Z_t^{Q^*} \left( -dM_t^{2,c} + \sum_{i=1}^d \left( e^{-y_t^{i,2}} - 1 \right) dN_t^i \right).$$

(20)

In particular, if $(U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)$, one obtains $Y_t^1 \leq Y_t^2$, $dP \otimes dt$-a.e.

**Proof:** We denote $y^{i,12} := y^{i,1} - y^{i,2}$ and $M^{12,c} := M^{1,c} - M^{2,c}$, then we find that:

$$Y_t^{12} = \bar{U}_T^{12} + \int_t^T (U_s^{12} - \delta_s Y_s^{12}) ds - \sum_{i=1}^d \int_t^T y_s^{i,12} dN_s^i - \sum_{i=1}^d \int_t^T \left[ g(y_s^{i,1}) - g(y_s^{i,2}) \right] \lambda_s^i ds + \frac{1}{2} \int_t^T d\langle M^{2,c} \rangle_s - d\langle M^{1,c} \rangle_s - \int_t^T dM_s^{12,c}$$

Note that, since $M^{k,c}$ are continuous martingales,

$$-\langle M^{2,c}, M^{12,c} \rangle + \frac{1}{2} \langle M^{2,c} \rangle + \frac{1}{2} \langle M^{1,c} \rangle = \frac{1}{2} \langle M^{12,c} \rangle$$

(21)

Using the fact that the process $\langle M^{12,c} \rangle$ is increasing and that the function $g$ is convex we get:

$$Y_t^{12} \leq \bar{U}_T^{12} + \int_t^T (U_s^{12} - \delta_s Y_s^{12}) ds + \sum_{i=1}^d \int_t^T (e^{-y_s^{i,2}} - 1) y_s^{i,12} \lambda_s^i ds + \int_t^T d\langle M^{2,c}, M^{12,c} \rangle_s - \int_t^T dM_s^{12,c} - \sum_{i=1}^d \int_t^T y_s^{i,12} dN_s^i.$$

Let $N^*$ and $M^{*,c}$ be the $Q^{*,2}$-martingales obtained by Girsanov’s transformation from $N$ and $M^{12,c}$, where $dQ^{*,2} = Z^{Q^{*,2}} d\mathbb{P}$ and where $Z^{Q^{*,2}}$ is given by (20). Then:

$$Y_t^{12} \leq \bar{U}_T^{12} + \int_t^T (U_s^{12} - \delta_s Y_s^{12}) ds - \sum_{i=1}^d \int_t^T y_s^{i,12} dN_s^i - \int_t^T dM_s^{*,c}$$
which implies that \( Y_t^{12} \leq \mathbb{E}^{\mathbb{Q}^{*2}} \left[ \int_t^T e^{-\int_t^s \delta_r \, dr} U_s^{12} \, ds + e^{-\int_t^T \delta_r \, dr} \bar{U}_T^{12} \, \mathcal{G}_t \right] \). In particular, if \((U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)\), then \( Y_t^1 \leq Y_t^2 \) \( \mathbb{P} \otimes dt \)-a.e.

We now prove standard a priori estimates for the solution of BSDE (6).

**Proposition 3.** (A priori estimates) Let \((Y^k, M^k, c, g^k)\) be the solution associated with \((U^k, \bar{U}_T^k)\) for \(k = 1, 2\) where we assume that \((U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)\); then, there exists a constant \(C > 0\) such that

\[
\mathbb{E}^{\mathbb{Q}^{*2}} \left[ \sup_{0 \leq t \leq T} |Y_t^{12}|^2 + \langle M^{12} \rangle_t + \sum_{i=1}^d \int_0^T |g_t^{i,1}|^2 \lambda_i^t \, dt \right] \leq C \mathbb{E}^{\mathbb{Q}^{*2}} \left[ |\bar{U}_T^{12}|^2 + \int_0^T |U_t^{12}|^2 \, dt \right]
\]

where \( \lambda_i^* = \lambda_i e^{-y_i^2} \) is the intensity process of \( H_i \) under the probability \( \mathbb{Q}^{*2} \). In the case \((U^2, \bar{U}_T^2) \leq (U^1, \bar{U}_T^1)\), the same inequality holds with \( \mathbb{Q}^{*1} \).

**Proof:** Using Itô’s formula:

\[
d(Y_t^{12})^2 = 2Y_t^{12} \left( \delta_t Y_t^{12} - U_t^{12} \right) \, dt + \frac{1}{2} d\langle M^{12} \rangle_t - \frac{1}{2} d\langle M^{2, c} \rangle_t + d\langle M^{12} \rangle_t
\]

\[
+ \sum_{i=1}^d \left[ \left( g_t^{y_i^1,1} - g_t^{y_i^1,2} \right) \lambda_t^i \right] \, dt + \sum_{i=1}^d \left( y_t^{i,12} \right)^2 \lambda_t^i \, dt + d\text{mart}_t
\]

where \( d\text{mart}_t = 2Y_t^{12} \left[ dM^{12, c} + \sum_{i=1}^d y_t^{i,12} dN_t^i \right] + \sum_{i=1}^d \left( y_t^{i,12} \right)^2 dN_t^i \) corresponds to a local martingale. Assuming \((U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)\), it follows from the comparison Theorem 2 that \( Y^1 \leq Y^2 \). Using the relation (21) and the convexity property of the function \( g \), we get

\[
(Y_t^{12})^2 + \int_t^T d\langle M^{12, c} \rangle_s \leq (\bar{U}_T^{12})^2 + 2 \int_t^T Y_s^{12} \left[ -\delta_s Y_s^{12} + U_s^{12} \right] \, ds
\]

\[
+ \sum_{i=1}^d \int_t^T Y_s^{12} \left( e^{-y_s^{i,2}} - 1 \right) y_s^{i,12} \lambda_s^i \, ds - 2 \int_t^T Y_s^{12} d\langle M^{1, c}, M^{2, c} \rangle_s
\]

\[
+ \int_t^T Y_s^{12} d\langle M^{2, c} \rangle_s - \sum_{i=1}^d \int_t^T \left( y_s^{i,12} \right)^2 \lambda_s^i \, ds + \int_t^T d\text{mart}_s
\]

Hence we finally obtain the following inequality:

\[
(Y_t^{12})^2 + \int_t^T d\langle M^{12, c} \rangle_s \leq (\bar{U}_T^{12})^2 + 2 \int_t^T Y_s^{12} \left[ -\delta_s Y_s^{12} + U_s^{12} \right] \, ds - \sum_{i=1}^d \int_t^T \left( y_s^{i,12} \right)^2 \lambda_s^i \, ds
\]

\[
+ \int_t^T d\text{mart}_s^*
\]

where \( \text{mart}^* \) is a \( \mathbb{Q}^{*2} \) local martingale and \( \lambda_s^* := \lambda_s e^{-y_s^2} \) is the intensity of \( H_i \) under \( \mathbb{Q}^{*2} \).

From the obvious inequality

\[
(Y_t^{12})^2 - \frac{1}{\delta_t} (U_t^{12}) Y_t^{12} \geq \frac{1}{4\delta_t^2} (U_t^{12})^2
\]
and the non-negativity of \( \delta \), we deduce easily that

\[
-Y_t^{12} \left( \delta_t Y_t^{12} - U_t^{12} \right) \leq \frac{1}{4 \delta_t} (U_t^{12})^2
\]  

(23)

Plotting relation (23) and using the fact that the process \( \delta \) is bounded below, there exists a constant \( C > 0 \) such that:

\[
\mathbb{E}^{Q^*} \left[ \sup_{t \in [0,T]} |Y_t^{12}|^2 + \langle M^{12,c} \rangle_T + \sum_{i=1}^d \int_0^T |y_t^{12,i}|^2 \lambda_t^{i,s} dt \right] \leq C \mathbb{E}^{Q^*} \left[ |\bar{U}_T|_2^2 + \int_0^T |U_t^{12}|^2 dt \right].
\]

Permuting \( Y^1 \) and \( Y^2 \) and assuming \( (U^1, \bar{U}_T^1) \geq (U^2, \bar{U}_T^2) \) leads to similar inequality.

As a direct consequence of the comparison theorem, we prove the concavity property for the BSDE solution.

**Theorem 3.** (Concavity property) Define the map \( F : D_1^{\text{exp}} \times L^{\text{exp}} \rightarrow D_0^{\text{exp}} \) as

\[
F(U, \bar{U}) = V
\]

where \( (V, M^{V,c}, v) \) is the solution associated with \((U, \bar{U})\). Then \( F \) is concave, namely, for all \( \theta \in (0,1) \) and \((U^1, \bar{U}_T^1), (U^2, \bar{U}_T^2) \in D_1^{\text{exp}} \times L^{\text{exp}} :\)

\[
F \left( \theta U^1 + (1-\theta)U^2, \theta \bar{U}_T^1 + (1-\theta)\bar{U}_T^2 \right) \geq \theta F(U^1, \bar{U}_T^1) + (1-\theta)F(U^2, \bar{U}_T^2).
\]

**Proof:** Let \((V^k, M^{V,c}, v^k)\) be the solution of BSDE (6) associated with \((U^k, \bar{U}_T^k) \in D_1^{\text{exp}} \times L^{\text{exp}}\), then for any \( \theta \in (0,1) :\)

\[
d(\theta V_t^1 - (1-\theta)V_t^2) = \left[ \delta_t (\theta V_t^1 + (1-\theta)V_t^2) - (\theta U_t^1 + (1-\theta)U_t^2) \right] dt
+ \theta d(M^{1,c})_t + (1-\theta)d(M^{2,c})_t + d(\theta M^{1,c}_t + (1-\theta)M^{2,c}_t)
+ \sum_{i=1}^d [\theta v_t^{1,i} + (1-\theta)v_t^{2,i}] dN_t^i + \sum_{i=1}^d [\theta g(v_t^{1,i}) + (1-\theta)g(v_t^{2,i})] \lambda_t^i dt.
\]

We recall the following general result: Let \( X \) and \( Y \) be two continuous martingales. Then, for all \( \theta \in (0,1) \), \( \theta \langle X \rangle + (1-\theta)\langle Y \rangle - \langle \theta X + (1-\theta)Y \rangle \) is an increasing process. Indeed, \( X \) and \( Y \) are increasing processes. Indeed, we have:

\[
\langle \theta X + (1-\theta)Y \rangle - \theta \langle X \rangle - (1-\theta)\langle Y \rangle
= (\theta^2 - \theta)\langle X \rangle + ((1-\theta)^2 - (1-\theta))\langle Y \rangle + 2\theta(1-\theta)\langle X, Y \rangle
= \theta(\theta - 1)\left[ \langle X \rangle + \langle Y \rangle - 2\langle X, Y \rangle \right] = \theta(\theta - 1)\langle X - Y \rangle.
\]

Therefore, using the convexity property of the function \( g \), we get:

\[
\theta V_t^1 + (1-\theta)V_t^2 \leq \left( \theta \bar{U}_T^1 + (1-\theta)\bar{U}_T^2 \right) - \int_t^T \left[ \delta_s (\theta V_s^1 + (1-\theta)V_s^2) - (\theta U_s^1 + (1-\theta)U_s^2) \right] ds
- \int_t^T d(\theta M^{1,c}_s + (1-\theta)M^{2,c}_s) - \int_t^T d(\theta M_s^{1,c} + (1-\theta)M_s^{2,c})
- \sum_{i=1}^d \int_t^T (\theta v_s^{1,i} + (1-\theta)v_s^{2,i}) dN_s^i - \sum_{i=1}^d \int_t^T g(\theta v_s^{1,i} + (1-\theta)v_s^{2,i}) \lambda_s^i ds. \tag{24}
\]
Let \((V^\theta, M^{\theta,c}, v^\theta)\) be the solution of the BSDE associated with \((\theta U^1 + (1 - \theta)U^2, \theta \bar{U}^1 + (1 - \theta)\bar{U}^2)\) and set \(M^{V,c,\theta} = \theta M^{1,c} + (1 - \theta)M^{2,c}\) and for \(i = 1, \ldots, d\), \(v^{\theta,i} = \theta v^{1,i} + (1 - \theta)v^{2,i}\). Then, using (24):

\[
\theta V^1_t + (1 - \theta)V^2_t - V^\theta_t \leq \int_t^T \delta_s(V^\theta_s - (\theta V^1_s + (1 - \theta)V^2_s)) \, ds - \int_t^T d(M^{V,c,\theta})_s + \int_t^T d(M^{\theta,c})_s \\
- \int_t^T d(M^{V,c,\theta} - M^{\theta,c}) - \sum_{i=1}^d \int_t^T (g(v^{\theta,i}_s) - g(\bar{v}^{\theta,i}_s)) \lambda^i_s \, ds - \sum_{i=1}^d \int_t^T (\bar{v}^{\theta,i}_s - v^{\theta,i}_s) \, dN^i_s.
\]

Using (21) and the convexity property of the function \(g\), we get:

\[
\theta V^1_t + (1 - \theta)V^2_t - V^\theta_t \leq \int_t^T \left[ \delta_s(V^\theta_s - (\theta V^1_s + (1 - \theta)V^2_s)) \right] \, ds \\
+ \sum_{i=1}^d \int_t^T \left( e^{-v^{\theta,i}_s} - 1 \right) (\bar{v}^{\theta,i}_s - v^{\theta,i}_s) \lambda^i_s \, ds - \int_t^T d(M^{\theta,c})_s + \int_t^T d(M^{V,c,\theta})_s + \int_t^T d(M^{\theta,c})_s \\
- \int_t^T d(M^{V,c,\theta} - M^{\theta,c}) - \sum_{i=1}^d \int_t^T (\bar{v}^{\theta,i}_s - v^{\theta,i}_s) \, dN^i_s.
\]

Therefore, we find the following inequality:

\[
\theta V^1_t \leq \int_t^T \left[ \delta_s(V^\theta_s - (\theta V^1_s + (1 - \theta)V^2_s)) \right] \, ds \quad \sum_{i=1}^d \int_t^T \left( e^{-v^{\theta,i}_s} - 1 \right) \lambda^i_s \, ds \\
- \int_t^T d\left( (M^{V,c,\theta} - M^{\theta,c}) + (M^{V,c,\theta} - M^{\theta,c}) \right).
\]

Let \(Q^{*,\theta}\) be the probability measure equivalent to \(\mathbb{P}\) with Radon-Nikodym density given by

\[
dZ^Q_t = Z^Q_t - dM^{\theta,c} + \sum_{i=1}^d \left( e^{-v^{\theta,i}_s} - 1 \right) dN^i_t
\]

then using integration by parts and Girsanov’s theorem, taking \(Q^{*,\theta}\)-conditional expectations, we have \(S^\theta_t (\theta V^1_t + (1 - \theta)V^2_t - V^\theta_t) \leq 0\), which gives the result. 

\[\square\]

4 The second optimization problem

In this section, we assume that \(U_s = U(c_s)\) and \(\bar{U}_T = \bar{U}(\psi)\) where \(U\) and \(\bar{U}\) are given utility functions, \(c\) is a given non-negative \(G\)-adapted process and \(\psi\) a \(G_T\)-measurable non-negative random variable. We fix a probability \(\mathbb{P}\) equivalent to \(\mathbb{P}\) with a Radon-Nikodym density \(\tilde{Z}\) with respect to \(\mathbb{P}\) given by:

\[
d\tilde{Z}_t = \tilde{Z}_{t-} (\theta_t dM^c + \sum_{i=1}^n (e^{-z^i_t} - 1) dN^i_t), \quad \tilde{Z}_0 = 1.
\]

(25)
4.1 Formulation of the problem

We study the following optimization problem of the robust maximization initial problem (1):

\[
\sup_{(c,\psi)\in A(x)} \mathbb{E}^{Q^*} \left[ \int_0^T S_s^c U(c_s) ds + S_T^c \bar{U}(\psi) \right] + \mathbb{E}^{Q^*} \left[ \int_0^T \delta_s \ln Z_s^Q ds + Z_T^Q \right]
\]

where \( A(x) \) is the set of admissible control parameters, \( V_0 \) is the value at initial time of the value process \( V \), part of the solution \( (V, M^{V,c}, \nu) \) of the BSDE (6) in the case \( U_s = U(c_s) \) and \( \bar{U}_T = \bar{U}(\psi) \). Here, \( Q^* \) is the optimal measure for \( \mathcal{P}(U(c), \bar{U}(\psi)) \), and depends on \( (c, \psi) \).

The preferences are modeled by the utility functions \( U \) and \( \bar{U} \) which satisfy the following conditions:

**Assumption A 3.** The utility functions \( U \) and \( \bar{U} \) satisfy the usual conditions:

i) Strictly increasing and concave.

ii) Continuous differentiable on the set \( \{U > -\infty\} \) and \( \{\bar{U} > -\infty\} \), respectively,

iii) \( U'(\infty) := \lim_{x\to\infty} U'(x) = 0 \) and \( \bar{U}'(\infty) := \lim_{x\to\infty} \bar{U}'(x) = 0 \),

iv) \( U'(0) := \lim_{x\to0} U'(x) = +\infty \) and \( \bar{U}'(0) := \lim_{x\to0} \bar{U}'(x) = +\infty \),

v) Asymptotic elasticity \( AE(U) := \lim_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1 \).

**Definition 5.** \( A(x) \) is the set of control parameters \( (c, \psi) \in \mathcal{H}^2([0, T]) \times \mathbf{L}^2(\Omega, \mathcal{G}_T) \) such that

\[
\mathbb{E}^{\bar{P}} \left[ \int_0^T c_t dt + \psi \right] \leq x
\]

and \( (U(c), \bar{U}(\psi)) \in D^{\exp,1} \times L^{\exp} \) and \( (eU'(c), \psi \bar{U}'(\psi)) \in D^{\exp,1} \times L^{\exp} \) for any pair \( (e, \psi) \in \mathcal{H}^2([0, T]) \times \mathbf{L}^2(\Omega, \mathcal{G}_T) \), and the process \( \exp(\gamma \int_0^t |U(c_t)| dt) \) (resp. \( \exp(\gamma \int_0^t |c_t||U'(c_t)| dt) \)) belongs to the class \([D]\) (see Dellacherie and Meyer, pp.89, Chapter VI [12] for definition).

**Remark 6.** Under our assumptions, the set \( A(x) \) is convex and closed in the topology of convergence in measure (see, Cuoco [11] Lemma B3., pp.70).

In order to clarify and motivate the constraint (26) satisfied by the control parameters \( (c, \psi) \), we present a generic example in a financial market where the process \( c \) can be interpreted as a consumption and \( \psi \) as a terminal wealth:

**Example 2 (Consumption-investment problem).** We assume the same model as in Example 1, and we consider a financial market consisting of \( d + p \) assets. The savings account is assumed to be constant equal to 1, the prices of the \( d + p \) risky assets are \( \mathcal{G} \)-semi-martingales given by

\[
dS^i_t = S^i_t \left[ \mu^i_t dt + \sum_{j=1}^d \phi^i_j dN^j_t + \sum_{k=1}^p \sigma^i_k dW^k_t \right], \quad i = 1, \ldots, d + p
\]

where \( \sigma \) is a \( d + p \times d \) volatility matrix \((\sigma^{i,k}, i = 1, \ldots, d + p; k = 1, \ldots, p)\) and \( \phi \) is a \( d + p \times d \) matrix \((\phi^{i,j}, i = 1, \ldots, d + p; j = 1, \ldots, d)\). We note \( \Sigma \) the \( d + p \times (d + p) \) matrix \( \Sigma = [\sigma, \phi]. \)
Given an initial wealth $x$ and a policy $(c, \pi)$, the wealth process $(X_t^{x,c,\pi})_{0 \leq t \leq T}$ associated to the triple $(x, c, \pi)$ where $x$ is the initial wealth, $\pi$ is the portfolio strategy and $c$ the consumption plan, follows the dynamics given by:

$$dX_t^{x,c,\pi} = \pi_t dS_t - c_t dt, \quad X_0^{x,c,\pi} = x,$$

(28)

The set of consumption-investment strategies $(c, \pi)$ satisfying the following no-bankruptcy condition is called the admissible strategies set and is denoted by $\mathcal{A}(x)$:

$$\mathbb{P} - \text{a.s.,} \quad X_t^{x,c,\pi} \geq 0, \quad \forall t \in [0, T].$$

(29)

For this model, one assumes that:

- The appreciation rates $(\mu^i, i = 1, \ldots, d + p)$ are bounded predictable processes.
- The processes $(\varphi^{i,j}, 1 \leq i \leq d + p, 1 \leq j \leq d)$ are bounded and predictable and satisfy $\varphi^{i,j} \leq -1$ a.s.
- The processes $(\sigma^{i,k}, 1 \leq i \leq d + p, 1 \leq j \leq d)$ are bounded and predictable.
- The matrix $\Sigma$ is invertible. This condition ensures that the market is arbitrage free.

Then, the pair consumption-terminal wealth satisfies the budget constraint

$$\mathbb{E}^\mathbb{P} \left[ \int_0^T c_t dt + X_T^{x,c,\pi} \right] \leq x,$$

where $\mathbb{P}$ is the unique equivalent martingale measure with density $\bar{Z}$ given by (25), where $\theta_t := \Sigma_t^{-1} \mu_t$.

### 4.2 Properties of the value process

In this section, we derive necessary conditions satisfied by the optimal control parameters. We start by showing regularity properties for the value process $V^{x,c,\psi}$ with respect to $(c, \psi)$.

**Proposition 4.** Define the map $G : \mathcal{A}(x) \longrightarrow D_0^{\text{exp}}$ as $G(c, \psi) = V$, where $(V, M^{V,c,\psi})$ is the solution of the BSDE (6) associated with $(U(c), \bar{U}(\psi))$. Then

(i) $G$ is concave, i.e., for all $\theta \in (0, 1)$ and $(c^1, \psi^1), (c^2, \psi^2) \in \mathcal{A}(x)$:

$$G \left( \theta c^1 + (1 - \theta) c^2, \theta \psi^1 + (1 - \theta) \psi^2 \right) \geq \theta G(c^1, \psi^1) + (1 - \theta) G(c^2, \psi^2).$$

(ii) Let $G_0(c, \psi)$ be the value of $G(c, \psi)$ at time 0, i.e., $G_0(c, \psi) = V_0$. If $(c^n, \psi^n) \in \mathcal{A}(x)$ converges decreasingly to $(c, \psi) \in \mathcal{A}(x)$, then $G_0(c^n, \psi^n)$ converges decreasingly to $G_0(c, \psi)$. Moreover $G_0$ is upper continuous with respect to the control parameters.

**Proof:** Let $(V_k, M^{k,c,\psi}_k)$ be the solution of the BSDE (6) associated with $(U(c^k), \bar{U}(\psi^k))$ for $k = 1, 2$. For any $\theta \in (0, 1)$, let $(\tilde{V}^\theta, \tilde{M}^{\theta,c,\psi})$ be the solution of (6) associated with $(U(\theta c^1 + (1 - \theta) c^2), \bar{U}(\theta \psi^1 + (1 - \theta) \psi^2))$ and $(\tilde{V}^\theta, M^{\theta,c,\psi})$ be the solution of (6) associated with $\theta U(c^1) + (1 - \theta) U(c^2), \theta \tilde{U}(\psi^1) + (1 - \theta) \bar{U}(\psi^2))$ and set $\tilde{V}^\theta := \theta V^1 + (1 - \theta) V^2$. Then, by using both the concavity properties of $(U, \tilde{U})$ and Theorem 2, we get $\tilde{V}^\theta \geq V^\theta$. Moreover,
as consequence of Theorem 3, we obtain \( V^\theta \geq V^\theta \), which gives the assertion (i).

Let us now consider \((c^n, \psi^n)\) a decreasing sequence of control parameters in \(A(x)\) such that \(c^n_t \rightarrow c_t\) a.s and \(\psi^n \rightarrow \psi\) a.s; then, by using inequality (19), and the fact that the functions \(U\) and \(\bar{U}\) are non-decreasing, we get

\[
|V_0^{c^n, \psi^n} - V_0^{c, \psi}| = \mathbb{E}^Q^* \left[ \int_0^T (U(c^n_t) - U(c_t)) \, ds + (\bar{U}(\psi^n) - \bar{U}(\psi)) \right] \tag{30}
\]

where \(Q^*\) is the optimal measure associated with \((U(c), \bar{U}(\psi))\). Thus, by using the monotone convergence theorem and the a priori estimate (22), \(V_0^{c^n, \psi^n}\) converges decreasingly to \(V_0^{c, \psi}\). Let \((c^n, \psi^n) \in A(x)\) be a sequence of control parameters such that \(c^n \rightarrow c\) a.s and \(\psi^n \rightarrow \psi\) a.s where \((c, \psi) \in A(x)\) and denote \(c^n = \sup_{m \geq n} c^m, \bar{c}^n = \sup_{m \geq n} \psi^m\). Then, \(c^n \rightarrow c\) a.s. decreasingly and \(\bar{c}^n \rightarrow \psi\) a.s decreasingly. It follows that \(V_0^{c^n, \psi^n}\) converges to \(V_0^{c, \psi}\) decreasingly and therefore:

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} V_0^{c^n, \psi^n} \leq \lim_{n \to \infty} V_0^{c^n, \bar{c}^n} = V_0^{c, \psi}.
\]

Hence, \(G_0\) is upper semicontinuous with respect to the control parameters. \(\square\)

**Definition 6.** The pairs \((c^1, \psi^1), (c^2, \psi^2) \in A(x)\) are comparable if either \((c^1, \psi^1) \geq (c^2, \psi^2)\) or \((c^1, \psi^1) \leq (c^2, \psi^2)\) with the order introduced in Definition 4.

**Proposition 5.** Assume that Assumption A. 3 holds and let \((c^1, \psi^1), (c^2, \psi^2)\) be two comparable plans in \(A(x)\). Then, the function \(\Psi\), defined on \((0, 1)\) and valued in \(D_0^{sp}\),

\[
\Psi(\epsilon) = G(c^1 + \epsilon(c^2 - c^1), \psi^1 + \epsilon(\psi^2 - \psi^1))
\]

is right-continuous at 0.

**Proof:** Assume first that \((c^1, \psi^1) \leq (c^2, \psi^2)\). Let, for \(\epsilon \in \mathbb{R}\), \(V^\epsilon = G(c^1 + \epsilon(c^2 - c^1), \psi^1 + \epsilon(\psi^2 - \psi^1))\) and \(V = G(c^1, \psi^1)\). From Proposition 3 and the obvious inequalities \(U(c^1 + \epsilon(c^2 - c^1)) \geq U(c^1)\) and \(\bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) \geq \bar{U}(\psi^1)\), we obtain

\[
\mathbb{E}^Q^* \left[ \sup_{0 \leq t \leq T} |V_t - V^\epsilon_t| \right]^2 \leq C\mathbb{E}^Q^* \left[ \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1) \right]^2 + \int_0^T \left[ U(c^1_t + \epsilon(c^2_t - c^1_t)) - U(c^1_t) \right]^2 \, ds.
\]

Using now the concavity properties of \(U\) and \(\bar{U}\), we obtain

\[
0 \leq U(c^1_t + \epsilon(c^2_t - c^1_t)) - U(c^1_t) \leq \epsilon U'(c^1_t)(c^2_t - c^1_t) \\
0 \leq \bar{U}(\psi^1_t + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1_t) \leq \epsilon \bar{U}'(\psi^1_t)(\psi^2 - \psi^1).
\]

Thus, we have

\[
\mathbb{E}^Q^* \left[ \sup_{0 \leq t \leq T} |V_t - V^\epsilon_t| \right]^2 \leq C\mathbb{E}^Q^* \left[ (\bar{U}'(\psi^1))^2(\psi^2 - \psi^1)^2 + \int_0^T (U'(c^1_t))^2(c^2_t - c^1_t)^2 \, ds \right].
\]

Assume now that \((c^1, \psi^1) \geq (c^2, \psi^2)\). Then, using the fact that \(G\) is concave with respect to the control parameters, one has \(V^\epsilon \geq (1 - \epsilon)V^1 + \epsilon V^2\) where the \(V^k\) are associated with
Finally, we conclude there exists a constant $C > 0$ such that: $E^{Q^*} \left[ \sup_{0 \leq t \leq T} \frac{V^1_t - V^2_t}{\epsilon} \right] \leq C$, where $Q^* = Q^{*,1}$ if $(c^1, \psi^1) \geq (c^2, \psi^2)$ and $Q^* = Q^{*,2}$ if $(c^1, \psi^1) \leq (c^2, \psi^2)$, then by Kolmogorov’s criteria, we deduce that $\Psi$ is right-continuous at 0. \qed

We give now a differentiability result for our BSDE (6). We note that Imkeller et al [23] showed a differentiability result for a quadratic BSDE’s driven by a continuous martingale, but the paper does not cover our case, so the proof of the result will given in the Appendix.

**Theorem 4.** Let $(c^1, \psi^1)$ and $(c^2, \psi^2)$ be two comparable plans in $A(x)$. Let $(V^\epsilon, M^\epsilon,c, \nu^\epsilon)$ be the solution of (6) associated with $(U(c^1 + \epsilon(c^2 - c^1)), \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)))$ and $(V^1, M^{1,c}, \nu^1)$ the solution of (6) associated with $(U(c^1), \bar{U}(\psi^1))$. Then, $V^\epsilon$ is right-differentiable with respect to $\epsilon$ at 0. Moreover, if we denote by $\partial_\epsilon V := \lim_{\epsilon \to 0} \frac{V^\epsilon - V^1}{\epsilon}$, then there exists $\partial_\epsilon \bar{M}^{1,c}, \partial_\epsilon \nu \in L^2(Q^{1,*}) \times L^2(\bar{\lambda}, Q^{1,*})$ such that the triple $(\partial_\epsilon V, \partial_\epsilon \bar{M}^{1,c}, \partial_\epsilon \nu)$ is the solution of the following BSDEJ:

\[
\begin{cases}
  d\partial_\epsilon V_t = (\delta_t \partial_\epsilon V_t - U'(c^1_t)(c^2_t - c^1_t)) dt + d\partial_\epsilon \bar{M}^{1,c}_t + \sum_{i=1}^d \partial_\epsilon v^i_t d\bar{N}^i_t, & \text{Q}^{1,*}-a.s. \\
  \partial_\epsilon V_T = \bar{U}'(\psi^1)(\psi^2 - \psi^1),
\end{cases}
\]

(31)

where $\bar{\lambda}^i := \lambda^i e^{v^i - 1}$ and $\bar{N}^i := N^i - \int_0^t (e^{v^i - 1} - 1)\lambda^i_t dt$ is a $Q^{1,*}$-martingale.

Moreover, we obtain for all $t \leq T$:

\[
\partial_\epsilon V_t = E^F \left[ \frac{Z^{Q^{1,*}}_t}{Z^{Q^{1,*}}_t} S^\delta \frac{Z^{Q^{1,*}}_t}{Z^{Q^{1,*}}_t} U'(\psi^1)(\psi^2 - \psi^1) \right. \left. + \int_t^T \frac{Z^{Q^{1,*}}_s}{Z^{Q^{1,*}}_s} S^\delta \frac{Z^{Q^{1,*}}_s}{Z^{Q^{1,*}}_s} U'(c^1_s)(c^2_s - c^1_s) ds \right] G_t. \]

(32)

### 4.3 The Dynamic maximum principle

We recall that we are interested in the following optimization problem: we associate with a pair $(c, \psi) \in A(x)$ the quantity $X^{c,\psi}_0 = E^F \left[ \int_0^T c_s ds + \psi \right]$ and we study

\[
u(x) = \sup_{X^{c,\psi}_0 \leq x} V^{(c,\psi)}_0.
\]

(33)

Here $V^{(c,\psi)}_0 = V_0$, where $(V, M^{1,c}, \nu)$ is the solution of the BSDE (6) associated with $(U(c), \bar{U}(\psi))$. Note that if we are in the setting of Example 2, our problem correspond to a maximization of recursive utility function over consumption-investment strategy where $X_0$ is the initial wealth associated with the consumption $c$ and terminal wealth $\psi$. 

Proposition 6. There exists an optimal pair \((c^0, \psi^0)\) which solves (33).

**Proof:** The uniqueness is a consequence of the strictly concavity property of \(V_0\). We shall prove the existence by using Komlòs theorem.

**First step:** Let us first prove that \(\sup_{(c, \psi) \in A(x)} V_0^{c, \psi} < +\infty\). Because \(P \in \mathcal{Q}_f^e\), we have:

\[
\sup_{(c, \psi) \in A(x)} V_0^{c, \psi} \leq \sup_{(c, \psi) \in A(x)} E^P \left[ \bar{U}(\psi) + \int_0^T U(c_x)ds \right] =: \bar{u}(x)
\]

Using the elasticity assumption on \(U\) and \(\bar{U}\), we can find \(\gamma \in (0, 1)\) and \(x_0 \in \mathbb{R}\) such that, for any \(\theta > 1\), one has:

\[
U(\theta x) < \theta^\gamma U(x), \quad \bar{U}(\theta x) < \theta^\gamma \bar{U}(x) \quad \forall x > x_0,
\]

hence, for any \(x > x_0\):

\[
\bar{u}(\theta x) = E^P \left[ \bar{U}(\theta \frac{\psi x}{\theta}) + \int_0^T U(\theta \frac{c_x}{\theta})ds \right] \leq \theta^\gamma \bar{u}(x).
\]

Then, \(AE(\bar{u}) < 1\), which permits to conclude that, for any \(x > 0\), \(\bar{u}(x) < +\infty\) (see [31] and [41] chap. 3, Lemma 3).

**Second step:** Let \((c^n, \psi^n) \in A(x)\) be a maximizing sequence such that:

\[
\sup_{n \to +\infty} \lim V_0^{c^n, \psi^n} = \sup_{(c, \psi) \in A(x)} V_0^{c, \psi} < +\infty,
\]

where the RHS is finite thanks to step 1. Using Komlòs criterion, we can find a convex combination \((\tilde{c}, \tilde{\psi}) \in \text{conv}\left((c^n, \psi^n), (c^{n+1}, \psi^{n+1}), \ldots\right)\) which converges \(P\text{-a.s.}\). We denote by \((c^*, \psi^*)\) this limit, which belongs to \(A(x)\) since it is a closed convex set. Moreover, there exists \(N_n \geq n\) and a positive sequence \((\theta^m)_{m \in \mathbb{N}}\) satisfying \(\sum_{m=n}^{N_n} \theta^m = 1\) such that \((\tilde{c}_n, \tilde{\psi}_n) = (\sum_{m=n}^{N_n} \theta^m c^m, \sum_{m=n}^{N_n} \theta^m \psi^m)\). Therefore, by using the concavity and the increasing properties of the functional \(V_0\) which respect to the control plan we get:

\[
V_0^{c^n, \psi^n} = V_0^{\sum_{m=n}^{N_n} \theta^m c^m, \sum_{m=n}^{N_n} \theta^m \psi^m} \geq \sum_{m=n}^{N_n} \theta^m V_0^{c^m, \psi^m} \geq V_0^{c^*, \psi^*}.
\]

Moreover, using the upper semi-continuous property of the functional \(V_0\) which respect to the control plan we get:

\[
\sup_{(c, \psi) \in A(x)} V_0^{c, \psi} = \limsup_n V_0^{c^n, \psi^n} \leq \limsup_n V_0^{\tilde{c}_n, \tilde{\psi}_n} = V_0^{c^*, \psi^*}.
\]

In order to characterize the optimal solution, we recall the classical convex analysis result.

**Proposition 7.** There exists a constant \(\nu^* > 0\) such that:

\[
u(x) = \sup_{(c, \psi)} \left\{ V_0^{c, \psi} + \nu^* \left( x - X_0^{(c, \psi)} \right) \right\} \quad (34)
\]
and if the maximum is attained in (33) by \((c^*, \psi^*)\), then it is attained in (34) by \((c^*, \psi^*)\) with \(X_0^{(c^*, \psi^*)} = x\). Conversely, if there exists \(\nu^0 > 0\) and \((c^0, \psi^0)\) such that the maximum is attained in
\[
\sup_{(c, \psi)} \left\{ V_0^{(c, \psi)} + \nu^0 \left( x - X_0^{(c, \psi)} \right) \right\}
\]
with \(X_0^{(c^0, \psi^0)} = x\), then the maximum is attained in (34) by \((c^0, \psi^0)\).

Let \(\nu > 0\) be fixed and \(L\) be the map given by \(L(c, \psi) = V_0^{(c, \psi)} - \nu X_0^{(c, \psi)}\). We now study the following optimization problem:
\[
\sup_{(c, \psi)} L(c, \psi) \tag{35}
\]

**Proposition 8.** The optimal plan \((c^0, \psi^0)\) which solves (35) satisfies the following (implicit) equations:
\[
U'(c^0) = \frac{Z_T^2 c^0}{\nu S_T^2} \, dt \otimes d\mathbb{P} \text{ a.s.}, \quad \bar{U}'(\psi^0) = \frac{Z_T^2}{\nu S_T^2} \, d\mathbb{P} \text{ a.s.}
\]
where \(Z_T^0\) is the Radon-Nikodym density of the probability measure \(\mathbb{Q}^0\) associated with the optimal plan \((c^0, \psi^0)\).

**Proof:** Consider the optimal plan \((c^0, \psi^0)\) which solves (35) and another plan \((c, \psi)\). For \(\epsilon \in (0, 1)\), one has \(L(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0)) \leq L(c^0, \psi^0)\), then
\[
\frac{1}{\epsilon} \left( V_0^{(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0))} - V_0^{(c^0, \psi^0)} \right) - \nu \frac{1}{\epsilon} \left( X_0^{(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0))} - X_0^{(c^0, \psi^0)} \right) \leq 0
\]
\[
\tag{37}
\]
From the definition, we obtain that
\[
\partial_\epsilon X_0^{(c^0, \psi^0)} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( X_0^{(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0))} - X_0^{(c^0, \psi^0)} \right) = \mathbb{E}^\bar{\mathbb{P}} \left[ \int_0^T (c_s - c_s^0) ds + (\psi - \psi^0) \right].
\]
Taking the limit when \(\epsilon\) goes to 0 in (37), we obtain:
\[
\partial_\epsilon V_0^{(c^0, \psi^0)} - \nu \partial_\epsilon X_0^{(c^0, \psi^0)} \leq 0
\]
\[
\tag{38}
\]
where \(\partial_\epsilon V^{(c^0, \psi^0)}\) exists and is given explicitly by Theorem 4. Note that:
\[
\partial_\epsilon V_0^{(c^0, \psi^0)} - \nu \partial_\epsilon X_0^{(c^0, \psi^0)} = \mathbb{E}^\bar{\mathbb{P}} \left[ S_T^0 Z_T^2 U'(\psi^0)(\psi - \psi^0) + \int_0^T S_s^0 Z_s^2 U'(c_s^0)(c_s - c_s^0) ds \right] - \nu \mathbb{E}^\bar{\mathbb{P}} \left[ Z_T^2 (\psi - \psi^0) + \int_0^T Z_s^2 (c_s - c_s^0) ds \right]
\]
It follows from the equality (38) that
\[
\mathbb{E}^\bar{\mathbb{P}} \left[ (S_T^0 Z_T^2 U'(\psi^0) - \nu Z_T^2)(\psi - \psi^0) + \int_0^T \left( S_s^0 Z_s^2 U'(c_s^0) - \nu Z_s^2 \right)(c_s - c_s^0) ds \right] \leq 0
\]
The end of the proof is the same as in El Karoui et al. [17] (proof of Theorem 4.2, p. 677). In particular, for any \( \psi \), \( \mathbb{E}^{\tilde{P}} \left[ \left( S_t^0 Z_t^0 U'(\psi^0) - \nu Z_t^{\tilde{P}} \right) (\psi - \psi^0) \right] \leq 0 \), hence

\[
S_t^0 Z_t^0 U'(\psi^0) - \nu Z_t^{\tilde{P}} = 0 \quad \text{a.s}
\]

We find the optimal \( c \) with similar arguments.

**Theorem 5.** Let \( I \) and \( \tilde{I} \) be the inverse of the functions \( U' \) and \( \tilde{U}' \). The optimal plan \( (c^0, \psi^0) \) which solve the problem (34) is given by:

\[
c_t^0 = I \left( \frac{\nu^0 Z_t^{\tilde{P}}}{S_t^0 Z_t^0} \right) dt \otimes d\tilde{P} \quad \text{a.s.}, \quad \psi^0 = \tilde{I} \left( \frac{\nu^0 Z_t^{\tilde{P}}}{S_t^0 Z_t^0} \right) \quad \text{\( \tilde{P} \) a.s.}
\]

where \( \nu^0 > 0 \) satisfies:

\[
\mathbb{E}^{\tilde{P}} \left[ \int_0^T I \left( \frac{\nu^0 Z_t^{\tilde{P}}}{S_t^0 Z_t^0} \right) dt + \tilde{I} \left( \frac{\nu^0 Z_t^{\tilde{P}}}{S_t^0 Z_t^0} \right) \right] = x.
\]

**Proof:** Define the map: \( f : (0, +\infty) \rightarrow (0, +\infty) \) as

\[
f(\nu) = \mathbb{E}^{\tilde{P}} \left[ \int_0^T I \left( \frac{\nu Z_t^{\tilde{P}}}{S_t^0 Z_t^0} \right) dt + I \left( \frac{\nu Z_t^{\tilde{P}}}{S_t^0 Z_t^0} \right) \right].
\]

Then, using assumption A.3, \( f \) is monotone and satisfies \( \lim_{\nu \rightarrow 0} f(\nu) = +\infty \) and \( \lim_{\nu \rightarrow +\infty} f(\nu) = 0 \). For any initial wealth \( x \in (0, +\infty) \), there exists a unique \( \nu^0 \) such that \( f(\nu^0) = x \).

Let \( (c, \psi) \in \mathcal{A}(x) \) and \( (V(c, \psi), M^{V,c}) \) (resp. \( (V(c^0, \psi^0), M^{V^0,c}) \)) the solution of the BS-DEJ (6) associated with \( (U(c), \tilde{U}(\psi)) \) (resp. \( (U(c^0), \tilde{U}(\psi^0)) \)) then from the inequality (19) (see the comparison theorem), we get:

\[
V_0^{(c, \psi)} - V_0^{(c^0, \psi^0)} \leq \mathbb{E}^{Q_0} \left[ S_t^0 (\tilde{U}(\psi) - \tilde{U}(\psi^0)) + \int_0^T S_s^0 (U(c_s) - U(c^0_s)) ds \right]
\]

\[
\leq \mathbb{E}^{Q_0} \left[ S_t^0 \tilde{U}'(\psi^0)(\psi - \psi^0) + \int_0^T S_s^0 \tilde{U}'(c^0_s)(c_s - c^0_s) ds \right].
\]

It follows that:

\[
V_0^{(c, \psi)} - V_0^{(c^0, \psi^0)} \leq \nu^0 \mathbb{E}^{Q_0} \left[ \frac{Z_t^{\tilde{P}}}{Z_t^0} (\psi - \psi^0) + \int_0^T \frac{Z_s^{\tilde{P}}}{Z_s^0} (c_s - c^0_s) ds \right]
\]

\[
\leq \nu^0 \left( \mathbb{E}^{\tilde{P}} \left( \psi + \int_0^T c_s ds \right) - \mathbb{E}^{\tilde{P}} \left( \psi^0 + \int_0^T c^0_s ds \right) \right)
\]

Since \( (c, \psi) \in \mathcal{A}(x) \), then \( \mathbb{E}^{\tilde{P}} \left[ \psi + \int_0^T c_s ds \right] \leq x \). Using that \( \mathbb{E}^{\tilde{P}} \left[ \psi^0 + \int_0^T c^0_s ds \right] = x \), we conclude:

\[
V_0^{(c, \psi)} \leq V_0^{(c^0, \psi^0)}.
\]

\( \square \)
5 Logarithm Case

In this section, we assume that the process $\delta$ is deterministic and that $U(x) = \ln(x)$ and $\bar{U}(x) = 0$, hence $I(x) = \frac{1}{x}$ for all $x \in (0, +\infty)$. We introduce, as in Theorem 5, the optimal process $c_t^* = I \left( \frac{\nu}{\nu J_t^*} \right) \frac{S_t^\beta Z_t^a}{\frac{S_t^\beta}{Z_t^a}}$. Recall that the Radon-Nikodym density $\bar{Z}$, and the Radon-Nikodym density of the optimal probability measure $Z^*$ (given in (9)) satisfy

\[ d\bar{Z}_t = \bar{Z}_{t-} (\theta_t dM_t^c + \sum_{i=1}^n (e^{-z_i^t} - 1) dN_i^t), \quad \bar{Z}_0 = 1 \]  
\[ dZ_t^a = Z_{t-}^a (-dM_t^Y + \sum_{i=1}^n (e^{-y_i^t} - 1) dN_i^t), \quad Z_0^a = 1. \]  

(39) (40)

For any deterministic function $\alpha$ such that $\alpha(T) = 0$, $V$ admits a decomposition as

\[ V_t = \alpha(t) \ln(c_t^*) + \beta_t \]

where $\beta$ is a process such that $\beta_T = 0$. Our goal is to characterize the process $\beta$. As in [6], we introduce $J_t = \frac{1}{1 + \alpha(t)} \beta_t$ in order to obtain a simple BSDEJ. Note that, even if $Z^*$ is implicit (the coefficients depend on the solution $c^*$), the BSDEJ for $J$ is explicitly determined in terms of the given parameters $\lambda^i$ and of the given probability $\bar{P}$.

Proposition 9. The value function $V$ has the form $V_t = \alpha(t) \ln(c_t^*) + (1 + \alpha(t)) J_t$ where $\alpha(t) = -\int_t^T e^{\int_s^t \delta(u) du} ds$ and $(J, \bar{M}^J, \bar{J})$ is the unique solution of the following BSDEJ:

\[ \left\{ \begin{array}{l}
  dJ_t = \left((1 + \delta(t))(1 + k(t)) J_t - k(t) \delta(t)\right) dt + d\bar{M}_t^J + \frac{1}{2} d\langle \bar{M}^J \rangle_t + \frac{1}{2} k(t) (1 + k(t)) \theta_t ^2 d\langle M^c \rangle_t \\
  + \sum_{i=1}^d j_i d\bar{N}_i^t + \sum_{i=1}^d \left(g(j_i) \bar{\lambda}_t^i + \left(k(t) (e^{-z_i^t} - 1) + e^{k(t) z_i^t} \right) \right) \lambda_t^i) dt \\
  J_T = 0
\end{array} \right. \]

where $k(t) = \frac{\alpha(t)}{1 + \alpha(t)}$. Here, the processes $\bar{M}^J$ and $d\bar{N}_i^t = dH_t^i - \bar{\lambda}_t dt$ are $\bar{P}$-martingales where $d\bar{P}|_{\mathcal{G}_t} = Z_t d\bar{P}|_{\mathcal{G}_t}$, $\bar{\lambda}_t^i = e^{k(t) z_i^t} \lambda_t^i$ and

\[ d\bar{Z}_t = -\bar{Z}_{t-} \left( k(t) \theta_t dM_t^c - \sum_{i=1}^d \left(e^{k(t) z_i^t} - 1\right) dN_i^t \right) \]  

(41)

Note that, in a complete market, one obtains a forward backward system for the pair $J$-optimal wealth.

Proof: Using the fact that $V$ satisfies the BSDE (6) and the assumed form of $V$ in terms of $(\alpha, \beta)$, one obtains $dV_t = (\delta(t)V_t - \ln(c_t^*)) dt - d(\ln Z_t^*) = \alpha(t) d(\ln c_t^*) + (\ln c_t^*) \alpha'(t) dt + d\beta_t$. Therefore

\[ d\beta_t = \delta(t)(V_t + \alpha(t)) dt - (1 + \alpha'(t)) \ln(c_t^*) dt + \alpha(t) d\ln \bar{Z}_t + (\alpha(t) + 1) d\ln Z_t^* \]
\[ = \left( \delta(t) \alpha(t) - 1 - \alpha'(t) \right) \ln(c_t^*) + \delta(t) \beta_t + \alpha(t) \delta(t) \right) dt + \alpha(t) d\ln \bar{Z}_t + (\alpha(t) + 1) d\ln Z_t^* \]
We choose $\alpha$ so that $\delta(t)\alpha(t) = 1 + \alpha'(t)$. It follows that
\[ d\beta_t = \delta(t)(\beta_t + \alpha(t))dt + \alpha(t)d\ln \tilde{Z}_t + (\alpha(t) + 1)d\ln Z^*_t. \]
After some obvious computations taking into account the form of $\tilde{Z}$ and $Z^*$, one obtains
\[ d\beta_t = \delta(t)(\beta_t + \alpha(t))dt + \sum_{i=1}^d \left( (\alpha(t) + 1)(e^{-\gamma_i^t} - 1) - \alpha(t)(e^{-z_i^t} - 1) \right) \lambda^i_t dt + \sum_{i=1}^d \left( (\alpha(t) + 1)y^i_t - \alpha(t)z^i_t \right) dH^i_t. \]
We now define $J_t := \frac{1}{1+\alpha(t)} \beta_t$ and set $k(t) = -\frac{\alpha(t)}{1+\alpha(t)}$, then we find the following dynamics:
\[ \frac{dJ_t}{J_t} = \left( 1 + \delta(t) \right) dt + \sum_{i=1}^d \left( g(y^i_t) + k(t)g(z^i_t) \right) \lambda^i_t dt + \frac{1}{2} \left( k(t)\theta^2_t d\langle M^c \rangle_t + d\langle M^{V,c} \rangle_t \right) + \sum_{i=1}^d \left( y^i_t + k(t)z^i_t \right) dN^i_t. \]
We introduce the martingale $M^{J,c}$ as $dM^{J,c}_t = dM^{Y,c}_t - k(t)\theta_t dM^c_t$. It is easy to check that $d\langle M^{J,c} \rangle_t = d\langle M^{Y,c} \rangle_t - k^2(t)\theta^2_t d\langle M^c \rangle_t - 2k(t)\theta_t d\langle M^{J,c}, M^c \rangle_t$ and we denote $j^i_t = y^i_t + k(t)z^i_t$. Using the fact that, due to the form of $g$, for any $x, k, z, \lambda$,
\[ x dN_i + \lambda g(x-kz) + k g(z) dt = x(dN_i - (e^{kz} - 1)\lambda dt) + \left( g(x) e^{kz} + (e^{-z} - 1)k + e^{kz} - 1 \right) \lambda dt \]
one obtains
\[ \frac{dJ_t}{J_t} = \left( (1 + \delta(t))(1 + k(t)) \right) dt + \sum_{i=1}^d \left( g(j^i_t) e^{k(t)z^i_t} + k(t)(e^{-z^i_t} - 1) + e^{k(t)z^i_t} - 1 \right) \lambda^i_t dt + \frac{1}{2} \left( k(t)d\langle M^{J,c} \rangle_t + k(t)\theta_t d\langle M^{J,c}, M^c \rangle_t + \frac{1}{2} k(t)(k(t) + 1)\theta^2_t d\langle M^c \rangle_t \right) + \sum_{i=1}^d j^i_t dN^i_t - (e^{k(t)z^i_t} - 1)\lambda^i_t dt \]
We define $\bar{P}$ as $d\bar{P} = \tilde{Z}dP$, where $d\tilde{Z}_t = -\tilde{Z}_t \left[ k(t)\theta_t dM^c_t - \sum_{i=1}^d (e^{k(t)z^i_t} - 1)dN^i_t \right]$. The processes $\tilde{M}^{J,c}$ and $\tilde{N}^{c}$ defined as $d\tilde{M}^{J,c}_t = d\tilde{M}^{Y,c}_t + k(t)\theta_t d\langle M^{J,c}, M^c \rangle_t$ and $d\tilde{N}^{c}_t = dN^{c}_t - (e^{k(t)z^i_t} - 1)\lambda^i_t dt = dH^i_t - \lambda^i_t dt$ are $\bar{P}$ martingales. The result follows.

6 Appendix : Proof of Theorem 4

Let $(V^c, M^{c,e}, v^c)$ be the solution of (6) associated with $(U(c^1 + \epsilon(c^2 - c^1)), \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)))$ and $(V^1, M^{1,e}, v^1)$ be the solution of (6) associated with $(U(c^1), \bar{U}(\psi^1))$ and denote
\[
\Delta \epsilon V := \frac{V^c - V^1}{\epsilon}, \quad \Delta \epsilon M^{c,e} := \frac{M^{c,e} - M^{1,e}}{\epsilon}, \quad \Delta \epsilon v^i := \frac{v^{i,e} - v^{i,1}}{\epsilon}, \\
\Delta \epsilon U := \frac{U(c^1 + \epsilon(c^2 - c^1)) - U(c^1)}{\epsilon}, \quad \Delta \epsilon \bar{U} := \frac{\bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1)}{\epsilon}
\]
then, \((\Delta V, \Delta M^c, \Delta v)\) satisfies the following equation:

\[
\Delta V_t - \int_0^t (\delta_s \Delta s - \Delta U_s) ds = \frac{1}{2\epsilon} \langle (M^{c,e})_t - (M^{1,e})_t \rangle \\
+ \frac{1}{\epsilon} \sum_{i=1}^d \int_0^t (g(v_i^{c,e}) - g(v_i^{1,e})) \lambda_i^s ds + \Delta_v M^c_t + \sum_{i=1}^d \int_0^t \Delta_v v_i^s dN_i^s,
\]

with final condition \(\Delta V_T = \Delta \tilde{U}_T\). We start first to give the following a priori estimates:

**Lemma 1.** Assume the same conditions as in Theorem 4. Then, there exists a constant \(C > 0\) such that: \(\forall i = 1, \ldots, d, \forall p \in \mathbb{N}^*, \forall \epsilon > 0\),

\[
\mathbb{E}^{Q^{*\epsilon}} \left[ \sup_{0 \leq t \leq T} |\Delta V_t|^2 + \langle \Delta M^c \rangle_T + \sum_{i=1}^d \int_0^T \frac{|\Delta_v v_i^p|^p}{p!} \lambda_i^s ds \right] \leq C,
\]

where \(\Delta M^c\) is the \(Q^{*,\epsilon}\) martingale part of the \(Q^{*,\epsilon}\) semimartingale \(\Delta_v M^c\), and \(\lambda^i := \lambda^i e^{-v^{1,i}}\) is the intensity process of the process \(H^i\) under the probability measure \(Q^{*,\epsilon}\).

**Proof:** Let \((c^1, \psi^1)\) and \((c^2, \psi^2)\) be two comparable plans. We introduce the processes

\[
K_t^e := \mathbb{E}^p \left[ \exp \left( \int_0^T (\delta_s V^e_s - U(c^1_s + \epsilon (c^2_s - c^1_s))) ds - \bar{U}(\psi^1 + \epsilon (\psi^2 - \psi^1)) \right) \big| \mathcal{G}_t \right]
\]

\[
K_t^1 := \mathbb{E}^p \left[ \exp \left( \int_0^T (\delta_s V^1_s - U(c^1_s)) ds - \bar{U}(\psi^1) \right) \big| \mathcal{G}_t \right].
\]

Obviously, for all \(t \in [0, T]\), one has:

\[
V_t^e = -\ln(K_t^e) + \int_0^t (\delta_s V^e_s - U(c^1_s + \epsilon (c^2_s - c^1_s))) ds
\]

\[
V_t^1 = -\ln(K_t^1) + \int_0^t (\delta_s V^1_s - U(c^1_s)) ds,
\]

hence,

\[
\frac{V_t^e - V_t^1}{\epsilon} = -\ln \left( \left( \frac{K_t^e}{K_t^1} \right)^{1/\epsilon} \right) + \int_0^t \left[ \delta_s \frac{1}{\epsilon} (V^e_s - V^1_s) - \frac{1}{\epsilon} (U(c^1_s + \epsilon (c^2_s - c^1_s)) - U(c^1_s)) \right] ds.
\]

For \(t \in [0, T]\), we define \(\tilde{K}_t^e := \frac{K_t^e}{K_t^1}\) and \(\tilde{K}_t^1 = \left(\tilde{K}_t^e\right)^{1/\epsilon}\). The processes \(K^e\) and \((\tilde{K}^e)^{-1}\) are positive semi-martingales which belong to \(L^p(\mathbb{P})\) since:

\[
(K_t^1)^{-p} = \mathbb{E} \left[ p \Delta_v V_t + \int_0^t p(\Delta_s U(c^1_s) - \delta_s \Delta_v V_s) ds \right].
\]

In the other hand, by using the dynamics of \(K^e\) and \(K^1\) under the probability measure \(\mathbb{P}\):

\[
dK_t^e = K_t^e \left[ -dM_t^{c,e} + \sum_{i=1}^d (e^{-v_i^{c,e}} - 1) dN_t^i \right]
\]

\[
dK_t^1 = K_t^e \left[ -dM_t^{1,e} + \sum_{i=1}^d (e^{-v_i^{1,e}} - 1) dN_t^i \right]
\]
and applying integration by parts formula, we get the dynamics of $\tilde{K}^\epsilon$ given by:

$$d\tilde{K}^\epsilon_t = \tilde{K}^\epsilon_{t-} \left[ -d(M^c_t - M^{1,c}_t) - (M^{c,c}_t - M^{1,c}_t) \right] dt + \sum_{i=1}^d \left( e^{-v_i^c x_i} - 1 \right) [dH^c_i - e^{-v_i^c x_i} \chi_i^c dt].$$

(46)

Clearly, $\tilde{K}^\epsilon$ is $Q^{*,1}$-local martingale. Then, the processes $\tilde{K}^\epsilon$ and $(\tilde{K}^\epsilon)^{-1}$ are positive $Q^{*,1}$-submartingales. We now split the study into two cases.

First case: $(c^1, \psi^1) \leq (c^2, \psi^2)$. Using the inequality (19), for all $t \in [0, T]$:

$$|\Delta_t V| \leq \mathbb{E}^{Q^{*,1}} \left[ \frac{S^\delta_t}{S^\delta_t} \tilde{U}'(\psi^1)(\psi^2 - \psi^1) + \int_t^T \frac{S^\delta_s}{S^\delta_t} U'(c^1_s)(c^2_s - c^1_s) ds \right] G_t.$$

$$\sup_{0 \leq t \leq T} \left( \frac{1}{\tilde{K}^\epsilon_t} \right)^p \leq \exp \left( \kappa \sup_{0 \leq t \leq T} \mathbb{E}^{Q^{*,1}} \left[ \tilde{U}'(\psi^1)(\psi^2 - \psi^1) + \int_t^T U'(c^1_s)(c^2_s - c^1_s) ds \right] G_t \right)^p$$

$$+ \int_0^T pU'(c^1_s)(c^2_s - c^1_s) ds.$$

Setting $\kappa = p(c + 1)$ where $c$ is the constant given in Assumption A2, we obtain:

$$\sup_{0 \leq t \leq T} \left( \frac{1}{\tilde{K}^\epsilon_t} \right)^p \leq \sup_{0 \leq t \leq T} \mathbb{E}^{Q^{*,1}} \left[ \exp \left( \tilde{U}'(\psi^1)(\psi^2 - \psi^1) + \int_t^T U'(c^1_s)(c^2_s - c^1_s) ds \right) G_t \right]^\kappa$$

$$\times \exp(\int_0^T pU'(c^1_s)(c^2_s - c^1_s) ds).$$

(47)

Thanks to the assumption $(c^1, \psi^1) \in A(x)$, we conclude that $\sup_{0 \leq t \leq T} \left( \frac{1}{\tilde{K}^\epsilon_t} \right) \in \mathcal{L}^p(\mathbb{P})$.

Second case: $(c^2, \psi^2) \geq (c^2, \psi^2)$. Then, using concavity property, we obtain for all $t \in [0, T]$:

$$|\frac{V^*_t - V^1_t}{\epsilon} - |V^1_t - V^2_t|, \quad |\Delta_t U(c^1_t)| \leq U'(c^1_t)(c^1_t - c^2_t)$$

Now, using the same arguments as in the first step, we get that:

$$\sup_{0 \leq t \leq T} \left( \frac{1}{\tilde{K}^\epsilon_t} \right)^p \leq \sup_{0 \leq t \leq T} \mathbb{E}^{Q^{*,2}} \left[ \exp \left( \tilde{U}'(\psi^2)(\psi^1 - \psi^2) + \int_t^T U'(c^2_s)(c^2_s - c^1_s) ds \right) G_t \right]^\kappa$$

$$\times \exp(\int_0^T pU'(c^2_s)(c^2_s - c^1_s) ds).$$

We use the same arguments to prove $\sup_{0 \leq t \leq T} |\tilde{K}^\epsilon_t| \in \mathcal{L}^p(\mathbb{P})$. From the representation theorem, there exists two continuous martingales $M^{c,c}, \tilde{M}^{c,c}$ and $d$ predictable processes $\tilde{k}^{c,i}, \tilde{k}^{c,i}$ such that:

$$\tilde{K}^\epsilon_t = K^\epsilon_0 + \tilde{M}^{c,c} + \sum_{i=1}^d \int_0^t \tilde{k}^{c,i}_s dN^i_s$$

$$\frac{1}{\tilde{K}^\epsilon_t} = \frac{1}{K^\epsilon_0} + \tilde{M}^{c,c} + \sum_{i=1}^d \int_0^t \tilde{k}^{c,i}_s dN^i_s.$$
These processes being positive $Q^{*1}$-submartingales, using (47) there exists two constants $C_K$ and $\tilde{C}_K$ such that:

$$
\mathbb{E}^{Q^{-1}} \left[ \int_0^T \left( k_s^{i,j} \right)^2 \lambda_s^{i,j} ds \right] \leq \mathbb{E}^{Q^{-1}} \left[ (\tilde{K}^{\epsilon \lambda}_T)^2 \right] \leq C_K
$$

$$
\mathbb{E}^{Q^{-1}} \left[ \int_0^T \left( \tilde{k}_s^{i,j} \right)^2 \tilde{\lambda}_s^{i,j} ds \right] \leq \mathbb{E}^{Q^{*1}} \left[ \left( \frac{1}{K_T^{\epsilon \lambda}} \right)^2 \right] \leq \tilde{C}_K
$$

(48)

From the uniqueness of the representation theorem and (45), we obtain, for $1 \leq i \leq d$,

$$
-\Delta_s v_i^t = \ln \left[ 1 + \frac{k_s^{i,j}}{K_T^{\epsilon \lambda}} \right] \text{ and } \Delta_s v_i^t = \ln \left[ 1 + \tilde{k}_s^{i,j} \tilde{K}_T^{\epsilon \lambda} \right].
$$

Therefore we find $\exp(|\Delta_s v_i^t|) - 1 \leq \frac{|k_s^{i,j}|}{K_T^{\epsilon \lambda}} + |\tilde{k}_s^{i,j} \tilde{K}_T^{\epsilon \lambda}}$. Moreover we have:

$$
\mathbb{E}^{Q^{-1}} \left[ \int_0^T (e^{\Delta_s v_i^t} - 1) \tilde{\lambda}_s^{i,j} ds \right] \leq \mathbb{E}^{Q^{-1}} \left[ \int_0^T \frac{k_s^{i,j}}{K_T^{\epsilon \lambda}} \tilde{\lambda}_s^{i,j} ds + \int_0^T \left| \tilde{k}_s^{i,j} \tilde{K}_T^{\epsilon \lambda} \right| \tilde{\lambda}_s^{i,j} ds \right] \leq \mathbb{E}^{Q^{-1}} \left[ \sup_{t \leq T} \frac{1}{K_T^{\epsilon \lambda}} \int_0^T \frac{k_s^{i,j}}{K_T^{\epsilon \lambda}} \tilde{\lambda}_s^{i,j} ds + \sup_{t \leq T} \tilde{K}_T^{\epsilon \lambda} \int_0^T \left| \tilde{k}_s^{i,j} \tilde{K}_T^{\epsilon \lambda} \right| \tilde{\lambda}_s^{i,j} ds \right]
$$

Using Cauchy-Schwartz inequality, we find that:

$$
\mathbb{E}^{Q^{-1}} \left[ \int_0^T (e^{\Delta_s v_i^t} - 1) \tilde{\lambda}_s^{i,j} ds \right] \leq \left[ \mathbb{E}^{Q^{-1}} \left( \sup_{t \leq T} \left( \frac{1}{K_T^{\epsilon \lambda}} \right)^2 \int_0^T \tilde{\lambda}_s^{i,j} ds \right) \right]^\frac{1}{2} \left( \mathbb{E}^{Q^{-1}} \int_0^T \left| \frac{k_s^{i,j}}{K_T^{\epsilon \lambda}} \right|^2 \tilde{\lambda}_s^{i,j} ds \right)^{\frac{1}{2}}
$$

$$
+ \left[ \mathbb{E}^{Q^{-1}} \left( \sup_{t \leq T} \left( \tilde{K}_T^{\epsilon \lambda} \right)^2 \int_0^T \tilde{\lambda}_s^{i,j} ds \right) \right]^\frac{1}{2} \left( \mathbb{E}^{Q^{-1}} \int_0^T \left| \tilde{k}_s^{i,j} \tilde{K}_T^{\epsilon \lambda} \right|^2 \tilde{\lambda}_s^{i,j} ds \right)^{\frac{1}{2}}.
$$

We prove now that $\int_0^T \tilde{\lambda}_s^{i,j} dt$ is square integrable under the probability $Q^{*1}$. We write first the expression under $P$ using Bayes’s formula:

$$
\mathbb{E}^{Q^{-1}} \left[ \int_0^T \tilde{\lambda}_s^{i,j} dt \right]^2 = \mathbb{E}^P \left[ Z_T^{Q^{-1}} \int_0^T e^{-\nu s_i^{i,j}} \lambda_s^{i,j} ds \right]^2 \leq \mathbb{E}^P \left[ (Z_T^{Q^{-1}})^2 \int_0^T \lambda_s^{i,j} ds \int_0^T e^{-2\nu s_i^{i,j}} \lambda_s^{i,j} ds \right]
$$

then we use Cauchy-Schwartz inequality to find the following estimates:

$$
\mathbb{E}^{Q^{-1}} \left[ \int_0^T \tilde{\lambda}_s^{i,j} dt \right]^2 \leq c_1 \left( \mathbb{E} (Z_T^{Q^{-1}})^4 \right)^\frac{1}{2} \left( \mathbb{E} \left( \int_0^T e^{-4\nu s_i^{i,j}} \lambda_s^{i,j} ds \right) \right)^\frac{1}{2}
$$

where we make use several times of Assumption A2-iii). Moreover, we can see that

$$
\mathbb{E} \left[ \int_0^T e^{-4\nu s_i^{i,j}} \lambda_s^{i,j} ds \right] = \mathbb{E} \left[ \int_0^T (e^{-\nu s_i^{i,j}} - 1 + 1)^4 \lambda_s^{i,j} ds \right] \leq 16 \mathbb{E} \left[ \int_0^T (e^{-\nu s_i^{i,j}} - 1)^4 \lambda_s^{i,j} ds + \int_0^T \lambda_s^{i,j} ds \right].
$$

Therefore, since the martingale $-M^{1,c} + \int_0^T \sum_{i=1}^d (e^{-\nu s_i^{i,j}} - 1) dN_i^{j}$ belongs to $L^p(P)$, and by assumption A2-iii) again, we conclude that $\mathbb{E}^P \left[ \int_0^T (e^{-\nu s_i^{i,j}} - 1)^p \lambda_s^{i,j} ds \right] < +\infty$ for any $p \geq 1$. 


Moreover since $Z^{Q^s_1} \in L^p(\mathbb{P})$, we get that $\mathbb{E}^{Q^s_1} \left[ \int_0^T |\tilde{\lambda}_s^i| ds \right] < \infty$. Then using again Cauchy inequality:

$$
\mathbb{E}^{Q^s_1} \left[ \int_0^T (e^{\Delta_1 v_s^i} - 1) \tilde{\lambda}_s^i ds \right] \leq C \left( \mathbb{E}^{Q^s_1} \left[ \sup_{0 \leq t \leq T} \frac{1}{(K_t^i)^4} \right] \right)^{\frac{1}{2}} \left( \mathbb{E}^{Q^s_1} \left[ \int_0^T |\tilde{K}_s^{i,1} | \tilde{\lambda}_s^i ds \right] \right)^{\frac{1}{2}}
$$

$$
+ C \left( \mathbb{E}^{Q^s_1} \left[ \sup_{0 \leq t \leq T} (K_t^i)^4 \right] \right)^{\frac{1}{2}} \left( \mathbb{E}^{Q^s_1} \left[ \int_0^T |\tilde{K}_s^{i,1} |^2 \tilde{\lambda}_s^i ds \right] \right)^{\frac{1}{2}}
$$

From (47) and (48), we deduce that there exists a constant $C_2 > 0$ such that:

$$
\mathbb{E}^{Q^s_1} \left[ \int_0^T (e^{\Delta_1 v_s^i} - 1) \tilde{\lambda}_s^i ds \right] \leq C_2
$$

and then using the expansion of the functional $x \to e^x$ we get:

$$
\mathbb{E}^{Q^s_1} \left[ \int_0^T |\Delta_1 v_s^i| \tilde{\lambda}_s^i ds \right] \leq C_2 p!
$$

In order to conclude the proof of the lemma, it remains to establish that there exists a constant $C_1$ satisfying:

$$
\mathbb{E}^{Q^s_1} [(\Delta_1 \tilde{M})_T] \leq C_1.
$$

**First case:** $(c^2, \psi^2) \geq (c^1, \psi^1)$, then $U(c^1 + \epsilon(c^2 - c^1)) \geq U(c^1)$ and $\bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) \geq \bar{U}(\psi^1)$. From Proposition 3, it follows that:

$$
\mathbb{E}^{Q^s_1} \left[ \sup_{0 \leq t \leq T} |V_t^e - V_t^1|^2 + \langle \tilde{M}^e, c - \tilde{M}^1, c \rangle_T + \sum_{i=1}^d \int_0^T (v_{s}^{e,i} - v_{s}^{1,i})^2 \tilde{\lambda}_s^i ds \right]
$$

$$
\leq \mathbb{E}^{Q^s_1} \left[ \left( \tilde{U}(\psi + \epsilon(\psi^2 - \psi^1)) - \tilde{U}(\psi^1) \right)^2 + \int_0^T |U(c_s^1 + \epsilon(c_s^2 - c_s^1)) - U(c_s^1)|^2 ds \right]
$$

Since

$$
0 \leq U(c_s^1 + \epsilon(c_s^2 - c_s^1)) - U(c_s^1) \leq \epsilon U'(c_s^1)(c_s^2 - c_s^1)
$$

and

$$
0 \leq \tilde{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) - \tilde{U}(\psi^1) \leq \epsilon \tilde{U}'(\psi^1)(\psi^2 - \psi^1).
$$

we get:

$$
\mathbb{E}^{Q^s_1} \left[ \sup_{0 \leq t \leq T} |\Delta_1 V_t|^2 + \langle \Delta_1 \tilde{M}^e \rangle_T + \int_0^T \langle \Delta_1 v_s^i \rangle^2 \tilde{\lambda}_s^i ds \right] \leq \mathbb{E}^{Q^s_1} \left[ (U'(\psi^1))^2 (\psi^2 - \psi^1)^2 \right]
$$

$$
+ \int_0^T (U'(c_s^1))^2 (c_s^2 - c_s^1)^2 ds
$$

The process $Z^{Q^s_1}$ belongs to $L^p(\mathbb{P})$; moreover $U'(\psi^1)(\psi^2 - \psi^1) \in L^1$ and $U'(c_s^1)(c_s^2 - c_s^1) \in D_1^{\exp}$ since $(c^1, \psi^1), (c^2, \psi^2) \in \mathcal{A}(x)$. It follows that there exists a constant $C > 0$ such that:

$$
\mathbb{E}^{Q^s_1} \left[ \sup_{0 \leq t \leq T} |\Delta_1 V_t|^2 + \langle \Delta_1 \tilde{M}^e \rangle_T + \sum_{i=1}^d \int_0^T (\Delta_1 v_s^i)^2 \tilde{\lambda}_s^i ds \right] \leq C
$$
Second case: \((c^2, \psi^2) \leq (c^1, \psi^1)\). We first prove that for all \(t \in [0, T]\), \(\tilde{K}^\epsilon_t \geq 1\). Let us recall that:

\[
\tilde{K}^\epsilon_t = \exp\left(-\Delta \epsilon V_t + \int_0^t (\delta_s \Delta \epsilon V_s - \Delta \epsilon U_s) ds\right)
\]

Define the process \(X\) as

\[
X_t = -\Delta \epsilon V_t + \int_0^t (\delta_s \Delta \epsilon V_s - \Delta \epsilon U_s) ds, \quad 0 \leq t \leq T.
\]

From integration by part formula, we get:

\[
S^\delta_t X_t = -\Delta \epsilon V_0 - \int_0^t S^\delta_s d\Delta \epsilon V_s - \int_0^t S^\delta_s \Delta \epsilon U_s ds
\]

Since the process \(\delta\) is positive and bounded, there exists a constant \(L > 0\) such that \(S^\delta < L < 1\). It follows that:

\[
S^\delta_t X_t \geq (-1 + L)\Delta \epsilon V_0 - L\Delta \epsilon V_t - \int_0^t S^\delta_s \Delta \epsilon U_s ds
\]

Note that, for all \(t \in [0, T]\), \(\Delta \epsilon U_t \leq 0\) since \((c^2, \psi^2) \leq (c^1, \psi^1)\) and using comparison theorem \(\Delta \epsilon V_t \leq 0\).

Therefore, for all \(t \in [0, T]\), \(X_t \geq 0\). Finally, \(\tilde{K}^\epsilon_t \geq 1\).

In the second step of the proof, we give the dynamics of the process \(\tilde{K}^\epsilon\) using Itô’s calculus:

\[
d\tilde{K}^\epsilon_t = \tilde{K}^\epsilon_t \left(-d\Delta \epsilon \tilde{M}^\epsilon_t + \sum_{i=1}^d (e^{-(\psi^1_i - \psi^\epsilon_i)} - 1)d\tilde{N}^\epsilon_i + dA_t\right)
\]

where \(A\) is an increasing process. Since \(\tilde{K}^\epsilon\) is a positive \(\mathbb{Q}^{*,1}\)-submartingale, we obtain from (47) and \(\tilde{K}^\epsilon_t \geq 1\) :

\[
\mathbb{E}^{\mathbb{Q}^{*,1}}\left[\langle \Delta \epsilon \tilde{M}^\epsilon \rangle_T\right] \leq \mathbb{E}^{\mathbb{Q}^{*,1}}\left[\int_0^T (\tilde{K}^\epsilon_t)^2 d\langle \Delta \epsilon \tilde{M}^\epsilon \rangle_t\right] \leq \mathbb{E}^{\mathbb{Q}^{*,1}}\left[(\tilde{K}^\epsilon_t)^2\right] \leq C_K
\]

then we conclude:

\[
\mathbb{E}^{\mathbb{Q}^{*,1}}\left[\langle \Delta \epsilon \tilde{M}^\epsilon \rangle_T\right] \leq C_K.
\]

Finally, by using concavity property we have shown that: \(|\Delta \epsilon V_t| \leq |V^2_t - V^1_t|\), for all \(t \in [0, T]\), then:

\[
\mathbb{E}^{\mathbb{Q}^{*,1}}\left[\sup_{t \in [0, T]} |\Delta \epsilon V_t|^2\right] \leq \mathbb{E}^{\mathbb{Q}^{*,1}}\left[\sup_{t \in [0, T]} |V^2_t - V^1_t|^2\right] \leq 2 \mathbb{E}^{\mathbb{Q}^{*,1}}\left[\sup_{t \in [0, T]} |V^1_t|^2 + \sup_{t \in [0, T]} |V^2_t|^2\right]
\]

Therefore, since the process \(V^1, V^2 \in D^{\text{exp}}_0\) and \(Z^{\mathbb{Q}^{*,1}}\) belongs to \(L^p\), we get by using Cauchy Schwarz inequality that there exists a constant \(C\) such that:

\[
\mathbb{E}^{\mathbb{Q}^{*,1}}\left[\sup_{t \in [0, T]} |\Delta \epsilon V_t|^2\right] \leq C.
\]
Proof of Theorem 4: Let recall first the equality:

\[
\frac{1}{2} \langle (M^{\epsilon,c}) - (M^{1,c}) \rangle = \frac{1}{2} (M^{\epsilon,c} - M^{1,c}) + (M^{\epsilon,c}, M^{1,c}) - (M^{1,c}),
\]

then the equation (43) may be written as:

\[
\Delta \epsilon V_t - \int_0^t (\delta_s \Delta \epsilon V_s - \Delta \epsilon U_s) ds = \frac{1}{\epsilon} \frac{1}{2} (M^{\epsilon,c} - M^{1,c})_t + (M^{\epsilon,c}, M^{1,c})_t - (M^{1,c})_t
\]

\[
+ \sum_{i=1}^{d} \int_0^t \left[ \frac{1}{\epsilon} (e^{-v_s^{i,i}} - e^{-v_s^{1,i}}) + e^{-v_s^{1,i}} \Delta \epsilon v_s^{i} \lambda_s^i ds + \Delta \epsilon M_t^c + \sum_{i=1}^{d} \int_0^t \Delta \epsilon v_s^{i} (dN_s^i - (e^{v_s^{i,i}} - 1) \lambda_s^i ds
\]

\[
= \frac{1}{2\epsilon} (M^{\epsilon,c} - M^{1,c})_t + \sum_{i=1}^{d} \int_0^t \left[ \frac{1}{\epsilon} (e^{-v_s^{i,i}} - e^{-v_s^{1,i}}) + e^{-v_s^{1,i}} \Delta \epsilon v_s^{i} \lambda_s^i ds
\]

\[
+ (\Delta \epsilon M_t^c + (\Delta \epsilon M^c, M^{1,c})_t) + \sum_{i=1}^{d} \int_0^t \Delta \epsilon v_s^{i} (dN_s^i - (e^{v_s^{i,i}} - 1) \lambda_s^i ds).
\]

By Girsanov's theorem, the processes \( \Delta \epsilon \tilde{\lambda}_s^c := \Delta \epsilon M^c + (\Delta \epsilon M^c, M^{1,c})_t \) and \( \tilde{N}_s^i := N_i - \sum_{i=1}^{d} \int_0^t (e^{v_s^{i,i}} - 1) \lambda_s^i ds \) are \( \mathbb{Q}^{1,*} \)-martingales. It follows that the process:

\[
(\Delta \epsilon V_t - \int_0^t (\delta_s \Delta \epsilon V_s - \Delta \epsilon U_s) ds)_{t \geq 0}
\]

is a \( \mathbb{Q}^{1,*} \)-submartingale and we have the following decomposition:

\[
\Delta \epsilon V_t - \int_0^t (\delta_s \Delta \epsilon V_s - \Delta \epsilon U_s) ds = \frac{e}{2} (\Delta \epsilon \tilde{\lambda}_s^c)_t + \sum_{i=1}^{d} \int_0^t \left[ \frac{1}{\epsilon} (e^{-v_s^{i,i}} - e^{-v_s^{1,i}})
\]

\[
+ e^{-v_s^{1,i}} \Delta \epsilon v_s^{i} \lambda_s^i ds + \Delta \epsilon \tilde{\lambda}_s^c + \sum_{i=1}^{d} \int_0^t \Delta \epsilon v_s^{i} d\tilde{N}_s^i.
\]

Moreover, using the uniform estimate (44), we get:

\[
\lim_{\epsilon \to 0} \mathbb{E}^{\mathbb{Q}^{1,*}} \left( \frac{e}{2} (\Delta \epsilon \tilde{\lambda}_s^c)_T \right) \leq C_p \lim_{\epsilon \to 0} \frac{e}{2} = 0,
\]

(50)

and using the expansion of the functional \( x \to e^x \), we get:

\[
0 \leq \lim_{\epsilon \to 0} \mathbb{E}^{\mathbb{Q}^{1,*}} \left( \int_0^T \frac{e^{-v_s^{i,i}} - e^{-v_s^{1,i}}}{\epsilon} + e^{-v_s^{1,i}} \Delta \epsilon v_s^{i} \lambda_s^i ds \right) = \lim_{\epsilon \to 0} \mathbb{E}^{\mathbb{Q}^{1,*}} \left( \int_0^T \frac{e^{-1}}{p!} (\Delta \epsilon v_s^{i})^p \tilde{\lambda}_s^i ds \right)
\]

\[
\leq \sum_{p=2}^{\infty} e^{p-1} \mathbb{E}^{\mathbb{Q}^{1,*}} \left( \int_0^T \frac{|\Delta \epsilon v_s^{i}|^p}{p!} \tilde{\lambda}_s^i ds \right) \leq \sum_{p=2}^{\infty} Ce^{p-1} = \frac{Ce}{1 - e},
\]

thus, passing to the limit as \( \epsilon \to 0 \), we conclude that:

\[
\lim_{\epsilon \to 0} \mathbb{E}^{\mathbb{Q}^{1,*}} \left( \int_0^T \frac{1}{\epsilon} (e^{-v_s^{i,i}} - e^{-v_s^{1,i}}) + e^{-v_s^{1,i}} \Delta \epsilon v_s^{i} \lambda_s^i ds \right) = 0, \quad 1 \leq i \leq d.
\]

(51)
Moreover, the estimate (44) ensures that the sequence $(\Delta_k, V, \Delta_k \bar{M}^c, \Delta_k v)_{k>0}$ is bounded in $\mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P}) \times L^2(\lambda, \mathbb{P})$. As a consequence, we can extract a subsequence $(\Delta_k, V, \Delta_k \bar{M}^c, \Delta_k v)_{k \in \mathbb{N}}$ which converges weakly in $\mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P}) \times L^2(\lambda, \mathbb{P})$ and by Banach-Mazur Lemma, one may construct a sequence $(\tilde{V}^\epsilon, \tilde{M}^c, \tilde{v}^\epsilon)_{\epsilon>0}$ of convex combinations of elements in $(\Delta_k, V, \Delta_k \bar{M}^c, \Delta_k v)_{k \in \mathbb{N}}$ of the form

$$\tilde{V}^\epsilon := \sum_{j=1}^{N_\epsilon} \alpha_j^\epsilon \Delta_j V, \quad \tilde{M}^c := \sum_{j=1}^{N_\epsilon} \alpha_j^\epsilon \Delta_j \bar{M}^c, \quad \tilde{v}^\epsilon := \sum_{j=1}^{N_\epsilon} \alpha_j^\epsilon \Delta_j v$$

such that $(\tilde{V}^\epsilon, \tilde{M}^c, \tilde{v}^\epsilon)_{\epsilon>0}$ converges strongly in $\mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P}) \times L^2(\lambda, \mathbb{P})$ to $(\partial_t V, \partial_t \bar{M}^c, \partial_t v)$. Moreover, the triple $(\tilde{V}^\epsilon, \tilde{M}^c, \tilde{v}^\epsilon)$ satisfies the BSDE (49) associated with $(\tilde{U}^\epsilon, \tilde{U}^\epsilon)$ where

$$\tilde{U}^\epsilon := \sum_{j=1}^{N_\epsilon} \alpha_j^\epsilon \Delta_j U, \quad \tilde{U} := \sum_{j=1}^{N_\epsilon} \alpha_j^\epsilon \Delta_j \tilde{U}.$$

Therefore, passing to the limit in this equation, thanks to (50), (51) and the dominated convergence theorem, we get that $(\partial_t V, \partial_t \bar{M}^c, \partial_t v)$ solves the BSDE

$$d\partial_t V_t = (\partial_t \partial_t V_t - U'(c_t^1)(c_t^2 - c_t^1))dt + d\partial_t \bar{M}^c_t + \sum_{i=1}^d \partial_t c_i^t d\bar{N}_t^i, \quad \partial_t V_T = U'(\psi^1)(\psi^2 - \psi^2).$$

Therefore $(S_t^\delta \partial_t V_t + \int_0^t S_s^\delta U'(c_s^1)(c_s^2 - c_s^1)ds)_{t \geq 0}$ is a $\mathbb{Q}^\epsilon, 1$ martingale which can be written as:

$$S_t^\delta \partial_t V_t + \int_0^t S_s^\delta U'(c_s^1)(c_s^2 - c_s^1)ds = \mathbb{E}^{\mathbb{Q}^\epsilon, 1} \left[ S_T^\delta \partial_t V_T + \int_0^T S_s^\delta U'(c_s^1)(c_s^2 - c_s^1)ds \big| \mathcal{G}_t \right].$$

Hence we get:

$$\partial_t V_t = \mathbb{E}^{\mathbb{Q}^\epsilon, 1} \left[ \frac{S_T^\delta}{S_t^\delta} U'(\psi^1)(\psi^2 - \psi^1) + \int_t^T \frac{S_s^\delta}{S_T^\delta} U'(c_s^1)(c_s^2 - c_s^1)ds \big| \mathcal{G}_t \right].$$

References

[1] Ankirchner, S., Blanchet-Scalliet, C., Elyraud-Loisel, A.: A Credit Risk Premia and Quadratic BSDE’s with a single jump, International Journal of Theoretical and Applied Finance, 13, 7, 1103-1129, (2010).

[2] Anderson E., Hansen L. P., and Sargent T.: A quartet of semigroups for model specification, robustness, prices of risk, and model detection. Journal of the European Economic Association, 1, 68–123 (2003).

[3] Barrieu, P. and El Karoui, N.: Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures, in the book ”Indifference Pricing: Theory and Applications” edited by René Carmona, Springer-Verlag (2008).
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[4] Barrieu, P. and El Karoui, N.: Monotone stability of quadratic semimartingales with applications to general quadratic BSDE's and unbounded existence. The Annals of Probability, to appear.

[5] Becherer, D.: Bounded solutions to Backward SDE's with jump for utility optimization and indifference hedging, Annals of Applied Probability, 16, 2027-2054 (2006).

[6] Bordigoni G.: Stochastic control and BSDEs in a robust utility maximization problem with an entropic penalty term. PhD Thesis of Politecnico di Milano (2005).

[7] Bordigoni, G., Matoussi, A., Schweizer, M.: A Stochastic control approach to a robust utility maximization problem. F. E. Benth et al. (eds.), Stochastic Analysis and Applications. Proceedings of the Second Abel Symposium, Oslo, 2005, Springer, 125-151 (2007).

[8] Briand, Ph. and Hu, Y.: BDSE with quadratic growth and unbounded terminal value. Probab. Theor. and Related Fields, 136, 604-618 (2006).

[9] Briand, Ph. and Hu, Y.: BDSE with convex coefficient and unbounded terminal value. Probab. Theor. and Related Fields, 141, 3-4, 543–567 (2008).

[10] Cox, J. and Huang, C. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory, 49, 33-83 (1989).

[11] Cuoco, D.: Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income. Journal of Economic Theory, 72, 33-73 (1997).

[12] Dellacherie, C., Meyer, P.A.: Probabilités et Potentiel. Chap. V-VIII. Hermann, Paris (1980).

[13] Duffie, D. and Epstein, L.G.: Stochastic differential utility. Econometrica 60, 353-394 (1992).

[14] Duffie, D., Skiadas, C.: Continuous-time security pricing: a utility gradient approach. J. Math. Econom., 23, 107-131 (1994).

[15] Faidi, W., Matoussi, A., Mnif M.: Maximization of Recursive Utilities: A Dynamic Maximum Principle Approach. SIAM J. Financial Math., Vol. 2, 1014-1041 (2011).

[16] El Karoui, N.: Les aspects probabilites du controle stochastique, in Ecole d’ete de Saint-Flour, Lecture Notes in Mathematics 876, 73-238. Springer Verlag Berlin (1982).

[17] El Karoui, N., Peng, S. and Quenez, M.-C.: A dynamic maximum principle for the optimization of recursive utilities under constraints. Annals of Applied Probability 11, 664-693 (2001).

[18] El Karoui, N., Hamadène, S.: BSDEs and risk-sensitive control, zero-sum and non-zero-sum game problems of stochastic functional differential equations. Stochastic Processes and their Applications 107, 145-169 (2003).

[19] El Karoui, N., Matoussi, A. and Ngoupeyou, N.: Quadratic-Exponential semimartingales and applications for Quadratic BSDE. Forthcoming paper.

[20] Gundel, A.: Robust utility maximization for complete and incomplete market models. Finance and Stochastics 9, 151-176 (2005).

[21] Hansen, L.P., Sargent, T.J., Turmuhambetova, G.A., Williams, N.: Robust control and model misspecification, Journal of Economic Theory, 128, 45-90 (2006).
[22] Hu, B. Y, Imkeller, P. and Muller, M.: Utility maximization in incomplete markets. *The Annals of Applied Probability*, 3, 1691-1712 (2005).

[23] Imkeller, P., Réveillac, A. and Richter, A.: Differentiability of quadratic BSDEs generated by continuous martingales. *The Annals of Applied Probability*, 22, 285-336 (2012).

[24] Jeanblanc, M. and Le Cam; Y.: Immersion Property and Credit Risk Modelling, in *Optimality and Risk - Modern Trends in Mathematical Finance: The Kabanov Festschrift* F. Delbaen, M. Rasonyi, and C. Stricker éditeurs (2009), p. 99-131.

[25] Ji S. and Zhou X. Y.: A maximum principle for stochastic optimal control with terminal state constraints and its applications. A special issue dedicated Tyrone Duncan on the occasion of his 65th birthday, *Communications in Information and Systems*, 6, 4, 321-338 (2006).

[26] Ji S. and Zhou X. Y.: A generalized Neyman-Pearson lemma under g-probabilities, *Probability theory and related fields*, 148, 645-669 (2010).

[27] Karatzas, I. and Shreve, S.E.: Methods of Mathematical Finance. *Applications of Mathematics, Stochastic Modelling and Applied Probability*. Springer-Verlag (1991).

[28] Karatzas I., Lehoczky J.P., Sethi S. and Shreve, S.: Explicit solution of a general consumption-investment problem. *Mathematics of Operation Research*, 11, 261-294 (1986).

[29] Karatzas, I., Lehoczky, J.P. and Shreve S.: Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM Journal on Control and Optimization*, 25, 1557-1586 (1987).

[30] Kobylanski, M.: Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth, *Annals of Probability* 28, 558-602 (2000).

[31] Kramkov, D. and Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9, 904-950 (1999).

[32] Kusuoka, S.: A remark on default risk models, *Adv. Math. Econ.*, 1, 69-82, 1999.

[33] Lazrak, A. and Quenez, M.-C.: A generalized stochastic differential utility. *Mathematics of Operations Research*, 28, 154-180 (2003).

[34] Lim, Th. and Quenez, M.-C.: Exponential utility maximization in an incomplete market with defaults. *Electronic Journal of Probability*, vol. 16, p 1434-1464 (2011).

[35] Maccheroni, F., Marinacci, M., and Rustichini, A.: Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74, 1447-1498 (2006).

[36] Maccheroni, F., Marinacci, M., and Rustichini, A.: Portfolio selection with monotone mean-variance preferences. *Mathematical Finance*, 19, 74, 487-521 (2009).

[37] Mania, M. and Schweizer, M.: Dynamic exponential utility indifference valuation. *Annals of Applied Probability*, 15, 2113-2143 (2005).

[38] Merton R.: Optimum Consumption and Portfolio Rules in a Continuous-time Model. *Journal of Economic Theory*, 3, 373-413 (1971).
[39] Morlais, M.-A. : Utility maximization in a jump market model. Stochastics, Vol. 81, 1, 1-27 (2009).

[40] Morlais, M.-A. : Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. Finance Stoch., Vol. 13, 1, 121-150 (2009).

[41] Mrad, M. : Utilités progressives. Thèse de Doctorat de l’École Polytechnique (2009).

[42] Quenez, M.-C. : Optimal portfolio in a multiple-priors model. in: R. Dalang, M. Dozzi and F. Russo (eds.), Seminar on Stochastic Analysis, Random Fields and Applications IV, Progress in Probability 58, Birkhäuser, p. 291-321 (2004).

[43] Royer, M. : Backward stochastic differential equation with jump and related nonlinear expectation, Stochastic Processes and their Applications, 116, 1358-1376 (2006).

[44] Schied, A. : Optimal investments for risk- and ambiguity-averse preferences: a duality approach. Finance and Stochastics, 11, 107-129 (2007).

[45] Schied, A. and Wu, C.-T. : Duality theory for optimal investments under model uncertainty. Sat. Decisions 23, 199-217 (2005).

[46] Schroder, M. and Skiadas, C. : Optimal consumption and portfolio selection with stochastic differential utility. Journal of Economic Theory 89, 68-126 (1999).

[47] Schroder, M., Skiadas, C. : Optimality and state pricing in constrained financial markets with recursive utility under continuous and discontinuous information. Mathematical Finance, Vol. 18, 2, 199-238 (2008).

[48] Skiadas C. : Robust control and recursive utility. Finance and Stochastics 7, 475-489 (2003).

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