Minimum spanning trees on random networks

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We show that the geometry of minimum spanning trees (MST) on random graphs is universal. Due to this geometric universality, we are able to characterise the energy of MST using a scaling distribution \( P(\epsilon) \) found using uniform disorder. We show that the MST energy for other disorder distributions is simply related to \( P(\epsilon) \). We discuss the relationship to invasion percolation (IP), to the directed polymer in a random media (DPRM) and the implications for the broader issue of universality in disordered systems.

Universality is an unproven hypothesis in disordered systems, though it has been widely assumed in the development of theories for the random-field Ising model and for spin glasses \( [3] \). The hope has been that scaling exponents in disordered systems should not depend on the nature of the disorder, provided it is uncorrelated and the disorder distribution is not too broad. Percolation \( [2] \) and the directed polymer in a random medium (DPRM) \( [3] \) reassure us that universality does hold. However, the random field Ising model (RFIM) has recently provided a counter example \( [4–6] \). In particular we showed that the mean-field RFIM is non-universal in the ground state as the parameter exponent can depend continuously on the details of the disorder \( [6] \). In contrast we show here that the MST is superuniversal in the sense that the MST geometry is unaltered even if the distribution of disorder is made very broad. Due to this fact, the energy of a MST can be found from a universal function, for a given graph topology. The MST geometry is important in the strong disorder limit \( [7,10,11] \), for example as a model for spin glasses \( [7] \) and for hopping transport at low temperatures \( [9,10,11] \). As we shall discuss below, the paths on the minimum spanning tree are those on which the energy barrier is smallest and for this reason MST paths dominate the kinetics at low temperatures \( [7,11] \).

Perhaps the simplest non-trivial optimisation problem, the minimum spanning tree (MST) is a tree which visits every site in a graph so that the sum of the costs of the edges in the tree is minimal (see Fig. 1). In physics terminology each edge, (\( ij \)) has an energy, \( \epsilon_{ij} \), and the total energy is the sum of the energies of the bonds which make up the minimum spanning tree, ie.

\[
E_{MST} = \sum_{ij \text{ in tree}} \epsilon_{ij}.
\]

Due to its practical applications in a variety of contexts, including image analysis, transportation networks etc, this problem has been heavily studied by the engineering community. This problem is also one of the most fundamental problems in combinatorial optimisation and has been intensively studied in the computer science and applied math communities \( [2] \). The physics community has been less aware of this problem, with notable exceptions \( [3,11] \), though it has close connections to the problem of a fluid invading a porous medium, as modeled by the invasion percolation (IP) process. However invasion percolation is a dynamic process which grows minimum spanning trees, whereas the MST itself is a global minimum of a cost function. The MST must visit every site in the graph and so corresponds to continuing the invasion process until every site in a finite graph is reached. This is not usually studied in invasion percolation, in which the steady state regime in a very large lattice is of most interest \( [10,17] \).

We concentrate on two aspects of minimal spanning trees on random graphs: the geometry of the paths on the minimum spanning tree; and the cost of the minimal spanning tree. First we show that the geometry of minimum spanning trees is universal even when the disorder is broad.

To demonstrate the universality of MST, it is useful to first describe how MST’s are found in practice. For simplicity, consider square and cubic lattices whose edges are assigned costs (energies) drawn from a uniform distribution on the interval \([0,1]\). In order to find the minimal spanning tree on such graphs, we use Prim’s algorithm which is a greedy algorithm (in physics these are called growth, invasion or extremal algorithms) which chooses the best site for advance at each timestep. In the computer science literature \( [2] \) Prim’s method starts by choosing the cheapest bond in the whole graph, and then by growing outward to the cheapest bond which is adjacent to the starting bond. Each bond which is invaded is added to the growing cluster and the process is iterated until every site has been reached. Bonds can only be invaded if they do not produce a cycle, so that the

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tree structure is maintained. However it is not essential to start from the cheapest bond as growth starting from any site leads to the same MST. This latter process is identical to *bond invasion percolation*, which then finds the MST exactly. Intuitively, invasion algorithm finds the exact minimum spanning tree because each site in a MST must be visited at least once and the IP algorithm selects the best way to make this choice at each site. It is a standard exercise in algorithm theory to prove this rigorously.

The property which makes the MST geometry universal is that to produce a unique MST in a graph we need only specify a unique ordering of the bonds of the graph, according to their energies. It does not matter if the bond energies are nearly the same or wildly different, it is only the ordering of the energies that matters. This can be intuitively understood by making a list of the bonds ordered from the smallest in energy to the largest. Now sequentially remove bonds, starting with the largest in energy, however with the rule that the removal of a bond cannot disconnect the graph into two pieces (this ensures that we end up with a spanning tree). Continue this process until no bonds can be removed. This final state is the minimum spanning tree and is very similar to the algorithm suggested by Cieplak et al. for this problem (the invasion method is much more efficient though). All that matters for this bond removal process is the ordering of the bond energies, and hence the geometry of the final tree so formed only depends on that ordering.

Given the fact that a MST only depends on the ordering of the bond energies, any transformation $\epsilon \rightarrow f(\epsilon)$ which preserves the ordering of the bond energies (e.g. the bond which has the fiftieth largest energy is the same before and after the transformation) leaves the MST geometry unaltered. Now note that if $f(\epsilon)$ is any non-decreasing function of $\epsilon$, the ordering of a set $\{\epsilon_1, \ldots, \epsilon_n\}$ is unaltered under the transformation to $\{f(\epsilon_1), \ldots, f(\epsilon_n)\}$. This observation is germane to the issue of universality due to the fact that we can use the uniform distribution to sample according to a general distribution $F(x)$, by assigning $F(x)\,dx = dy$ which transforms the interval $dy$ of the uniform distribution to the interval $F(x)\,dx$ of the general distribution. Thus if we randomly choose a number $y$ from the uniform distribution, the corresponding random number from the distribution $F(x)$ is

$$x = G^{-1}(y) \quad \text{where} \quad G(x) = \int_0^x F(x')\,dx'.$$ (2)

Now note that $G(x)$ is a non-decreasing function of $x$ because $F(x)$ is a probability and so is positive which implies that $G^{-1}(y)$ is also a non-decreasing function of $y$ (provided $G$ is invertible). Thus the transformation (2) preserves the ordering of the bond costs and hence the geometry of MST is the same for any probability distribution. This is one of the few non-trivial problems for which universality to disorder can be explicitly demonstrated.

Numerical calculation of the geometry of the paths on a MST is carried out as illustrated in Fig. 1. The number of bonds, $n$, which lie on a MST path between two sites which are separated by Euclidean distance $l$, is found to scale as $l^\nu$, where $y = 1.22 \pm 0.01$ (square lattice) and $y = 1.42 \pm 0.02$ (cubic lattice) (see Fig. 2). The strands in invasion percolation give exactly the same scaling. In fact the paths on MST scale with the same fractal dimension throughout the Prim growth process and only the form of the scaling distribution changes (see Figs. 2a and 2b). A more detailed study of the form of these scaling distributions will be discussed elsewhere.

Having proven that the geometry of MST is fractal and universal, we now show that it is possible to find the energy of MST from one universal function, for a given graph topology. Numerical results for the appropriate universal function are presented in Figs. 3a,b for minimum spanning trees on square and cubic lattices. The function we plot is the probability $P(\epsilon)$ that a bond of cost $\epsilon$ for a uniform distribution (on the interval $[0,1]$), lies on the minimum spanning tree. The dashed lines in Fig. 3 are the cost distributions for the MST. The solid lines are for invasion percolation. In invasion percolation $P(\epsilon)$ is the acceptance function for the case of bond invasion. In this case, it has the interesting property that in the scaling limit $P(\epsilon) \rightarrow 0$ for $\epsilon > p_c$ ($p_c$ is the bond percolation threshold). However, the MST must reach every site of the graph, in which case it is necessary to include bonds which have $\epsilon > p_c$ (see the dashed lines in Figs. 3a,b).

From the acceptance probability $P(\epsilon)$, the total energy of the minimum spanning tree with bonds drawn from a uniform distribution is simply (in the scaling limit),

$$E = \int_0^1 \epsilon P(\epsilon)\,d\epsilon.$$ (3)

As shown above, the geometry of the minimum spanning tree is unaltered if we make a transformation $\epsilon \rightarrow f(\epsilon)$ of the bond costs provided $f(\epsilon)$ is a non-decreasing function of the bond costs. After making this transformation the energy is simply $E = \int_0^1 f(\epsilon) P(\epsilon)\,d\epsilon$. In addition, using the arguments given above we may generalise to the case of an arbitrary distribution $F(\epsilon)$, in which case (from Eq. (2))

$$E_F = \int_0^1 G^{-1}(\epsilon)P(\epsilon)\,d\epsilon,$$ (4)

for any disorder distribution $F(x)$.

As stated above, paths on the MST are those on which the bond of maximum energy is minimal. In physical terms MST paths are those on which the barrier is minimal. The barrier on such a path is the largest cost bond which lies on that path. In the steady state limit

$$\epsilon_{\text{barrier}} \rightarrow p_c,$$ (5)
that is, the barrier on invasion percolation (IP) paths, in a graph with edge costs drawn from a uniform distribution, takes on a value equal to the percolation threshold on that graph. For other distributions of disorder, the barrier on IP paths becomes \( \epsilon_{\text{barrier}} = G^{-1}(p_c) \) (from Eq. (2)). However the barrier on typical MST paths has \( \epsilon_{\text{barrier}} > p_c \) for \( l \to \infty \), and it is only on the fractal IP subset of paths on which Eq. (5) holds. Thus hopping transport at low temperatures will typically occur on IP paths, which justifies the use of percolation models in the calculation of diffusivity and conductivity in the strong disorder limit.

It is interesting to compare the behavior of paths on the MST, with the behavior of the directed polymer in a random medium (DPRM) \(^3\). The DPRM, and the associated Kardar-Parisi-Zhang growth process \(^5\), has become a paradigm in the study of disordered systems. More recently it has been noted that DPRM is a subset of the shortest path (SP) problem in computer science and engineering \(^2\). The statement of the SP problem is deceptively similar to that of the MST problem discussed above, however its properties are radically different. The shortest path problem seeks to find a path between two sites in a graph such that the \textit{sum of the bond costs is minimal}, so that,

\[
E_{\text{SP}} = \sum_{(ij) \text{ in path}} \epsilon_{ij}, \quad (6)
\]

If one seeks the shortest path from a starting source site to all other sites in a graph, then a \textit{shortest path tree} (SPT) is formed. The total cost of the shortest path tree is the sum of the costs of all of the paths in the tree. Note however that in this sum it is inevitable that some of the bond costs will appear more than once. In fact bonds near the source site will be counted many times. The SP problem is in the DPRM universality class except for the limit of strong disorder when it approaches the MST problem \(^8\). The crossover to the strong disorder limit can be analysed explicitly using the generalised energy, \( \sum m \epsilon_{ij} \) instead of Eq. (6). In the limit \( m \to \infty \) the largest energy dominates and hence the \textit{largest barrier} is all that matters. It is possible to show, using (2) that \( m \to \infty \) corresponds to the strong disorder limit. The SPT is also distinguished by the fact that there is a \textit{different SPT} for each starting site in the graph, whereas there is only one MST for a finite graph with continuous disorder.

In summary, the geometry of minimum spanning trees (MST) is universal for all disorder distributions because a MST is invariant under the transformation of its edge costs \( \epsilon \to f(\epsilon) \) where \( f \) is a non-decreasing function of the edge cost \( \epsilon \). This universality enabled us to find a universal cost function for the uniform distribution (see Fig. 3) which can be used to calculate the cost of minimum spanning trees, on the same graph, for any other distribution (using Eq. (4)). Paths on the MST are those with minimal barrier and in the steady state IP process this barrier approaches \( p_c \) for a uniform distribution and \( G^{-1}(p_c) \) for an arbitrary distribution (where \( G \) is given in Eq. (2)). The MST geometry underlies physics in the strong disorder limit, implying that in that limit there is a strong universality.

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FIG. 1. The minimum spanning tree (MST) for a $30 \times 30$ square lattice. Each edge in the square lattice is assigned an energy drawn from the uniform distribution on the interval $[0,1]$. Only the bonds on the minimum spanning tree are drawn in the figure. The wandering heavy line is one path on the MST, starting at the center of the square lattice. The Euclidean distance between the two ends of this path is also indicated.

FIG. 2. The scaled distributions of pathlengths, $g(s)$, on minimum spanning trees on a) square and b) cubic lattices. The scaling variable is $s = n/l^{D_f}$ where $n$ is the number of bonds in the MST path, $l$ is the Euclidean distance and $D_f$ is the scaling dimension ($D_f = 1.22 \pm 0.01$ (square lattice) and $D_f = 1.42 \pm 0.02$ (cubic lattice)). The dotted line in these figures is the scaling distribution on the MST. For comparison we also give the scaling distribution for the steady state during growth of the MST, i.e. invasion percolation (solid line). In both cases the paths scale with the same fractal dimension, in fact this holds at all stages of growth of the MST. The results are found from averaging over 2000 realisations of $401 \times 401$ square lattices and over 1500 realisations of $101 \times 101 \times 101$ cubic lattices.

FIG. 3. The probability, $P(x)$, that a bond with energy $x$ lies on the minimum spanning tree for a) square and b) cubic lattices. The case considered here is the uniform distribution of bond disorder. The curving dashed lines are for the MST while the solid lines are for the steady state during growth of the MST, i.e. invasion percolation. The results are found from averaging over 1000 realisations of $401 \times 401$ square lattices and over 1000 realisations of $101 \times 101 \times 101$ cubic lattices.