Well posedness of a stochastic hyperviscosity-regularized 3D Navier-Stokes equation

B. Ferrario
Dipartimento di Matematica - Universit`a di Pavia

1 Introduction

We call stochastic Navier–Stokes problem the following:

\[
\begin{align*}
\frac{du}{dt} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f + n \\
\text{div} u &= 0 \\
u u |_{t=0} &= u_0
\end{align*}
\]  

(1)

Here \( u = u(t, x) \) is the 3-dimensional velocity vector field defined for \( t \geq 0 \) and \( x \in D \subseteq \mathbb{R}^3 \), \( p = p(t, x) \) is the scalar pressure field, \( \nu > 0 \) is the coefficient of kinematic viscosity, \( u_0 = u_0(x) \) is the initial velocity, \( f = f(t, x) \) and \( n = n(t, x) \) are, respectively, the deterministic and stochastic forcing terms. If the spatial domain has a boundary, we assume that \( u \) vanishes on \( \partial D \).

When there is no noise term \( n \), this reduces to the deterministic Navier–Stokes problem which models the motion of viscous fluids. For the 3-dimensional setting, both in the deterministic and in the stochastic case we know the existence of a weak solution but uniqueness is proved in a smaller class, where existence is not known. The question of proving existence and uniqueness on any finite time interval and with any initial data for the deterministic Navier–Stokes equation is one of the Millennium Prize problems (see, e.g., [3]). However, for the 3-dimensional problem there are results for small initial data or locally in time, whereas the 2-dimensional problem is well posed (see, e.g., [12, 13] for the deterministic problem and [7, 4] for the 2-dimensional stochastic problem with additive noise).

There have been many attempts to modify the 3-dimensional Navier–Stokes equation in order to prove an existence and uniqueness result. The first models go back to Lions [8]. For more recent results, we focus on two particular cases: [9] and [11]. In [9] the Laplacian operator \( -\Delta \) is replaced with \( (-\Delta)^\alpha \) (for \( \alpha > 1 \)) in the deterministic Navier–Stokes equation; the stochastic problem with a similar modification \( (-\nu \Delta \text{ replaced with } -\nu_0 \Delta + \nu_1 (-1)^{\alpha} \Delta^\alpha) \) is studied in [11]. Setting \( \alpha > 1 \) we obtain a model for hyperviscous fluids (see [11] and references therein).
Our aim is to analyse the well posedness of the stochastic version of the modified Navier–Stokes equation considered by Mattingly–Sinai in [9], that is
\[
\begin{aligned}
\frac{du}{dt} + \nu(-\Delta)u + (u \cdot \nabla)u + \nabla p &= n \\
\text{div } u &= 0 \\
u \alpha |u|^{\alpha-1} u &= 0
\end{aligned}
\]
(2)

We shall consider the model of additive noise, i.e. $n$ is independent of $u$. This is the simplest case, which reduces the technicalities. However, the case of multiplicative noise can be treated in a similar way.

In Section 3, we shall prove an existence and uniqueness result for $\alpha \geq \frac{5}{4}$, as conjectured in [5]. The bound $\alpha > \frac{5}{4}$ appeared first in [9] for the deterministic problem. Regularity results will be given in Section 4.

Finally, we point out that our technique is different from that of [9] or [11]; indeed, we use tools from [12] and [7].

2 Notations and preliminaries

Let the spatial domain be a torus, i.e. the spatial variable $x$ belongs to $\mathcal{T} = [0, L]^3$ and periodic boundary conditions are assumed.

We introduce the classical spaces for the Navier–Stokes equation (see, e.g., [13, 12] for all the results in this section). $D^{\infty}$ is defined as the space of infinitely differentiable divergence free periodic fields $u : \mathcal{T} \to \mathbb{R}^3$, with zero mean ($\int_{\mathcal{T}} u(x) dx = 0$). Let $H^m$ be the closure of $D^{\infty}$ in the $[L^2(\mathcal{T})]^3$-topology; it is the subspace of $[L^2(\mathcal{T})]^3$ of all fields $u$ such that div $u = 0$, the normal component of $u$ on the boundary is periodic, $\int_{\mathcal{T}} u(x) dx = 0$. We endow $H^0$ with the inner product $(u, v) = \int_{\mathcal{T}} u(x) \cdot v(x) dx$ and the associated norm $|\cdot|$.

Similarly, for $m \in \mathbb{N}$ let $H^m$ be the closure of $D^{\infty}$ in the $[H^m(\mathcal{T})]^3$-topology.

Let $A : D(A) \subset H^0 \to H^0$ be the operator $Au = -\Delta u$ (componentwise) with $D(A) = H^2$. This is called Stokes operator and it is a strictly positive unbounded self-adjoint operator in $H^0$, whose eigenvectors $h_j$ form a complete orthonormal basis of the space $H^0$; the eigenvalues $\lambda_j$ are strictly positive and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ with $\lambda_j \sim j^{2/3}$ for $j \to \infty$. Since the spatial domain is the torus, we know the expressions of the eigenvectors with their eigenvalues (see, e.g., [6]).

The power operators $A^\alpha$ are defined for any $\alpha \in \mathbb{R}$. If $u = \sum_j u_j h_j$, then

\[ A^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha u_j h_j \quad \text{and} \quad |A^\alpha u|^2 = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |u_j|^2. \]

We set $|u|_{2\alpha} = |A^\alpha u|$ and the space $H^s$ can be defined (for any $s \in \mathbb{R}$) as the closure of $D^{\infty}$ in the $|\cdot|_s$-metric. For $s < 0$ the space $H^s$ is the dual space of $H^{-s}$ with respect to the $H^0$-topology. The space $H^{s+r}$ is dense and compactly embedded in $H^s$ for any $r > 0$. 

2
Notice that \(|u|_m\) is equivalent to the usual \([H^m(T)]^3\)-norm. In particular
\[
|u|^2_1 = \langle u, Au \rangle = \sum_{i,j=1}^{3} \int_T (\partial_j u_i(x))^2 \, dx.
\]

We have \(H^0 = \text{span}\{h_k\}\) and we set \(H_n = \text{span}\{h_k : |k| \leq n\}\); moreover, we denote by \(\pi_n\) the projection operator from \(H^0\) onto \(H_n\). The operators \(A\) and \(\pi_n\) commute. By \(\Pi\) we denote the projector operator from \([L^2(T)]^3\) onto \(H^0\). The operator \(-A\) generates in \(H^0\) (and in any \(H^s\)) an analytic semigroup of negative type \(\{e^{-tA}\}_{t \geq 0}\) of class \(C_0\).

Let \(B(\cdot, \cdot) : H^1 \times H^1 \rightarrow H^{-1}\) be the bilinear operator defined as
\[
\langle w, B(u, v) \rangle = \sum_{i,j=1}^{3} \int_T u_i(\partial_i v_j)w_j \, dx
\]
for every \(u, v, w \in H^1\). By the incompressibility condition, we have
\[
\langle B(u^1, u^2), u^2 \rangle = 0, \quad \langle B(u^1, u^2), u^3 \rangle = -(\langle B(u^1, u^3), u^2 \rangle).
\]
We shall use the following estimates (see Lemma 2.1 in [13]):
\[
|\langle B(u^1, u^2), u^3 \rangle| \leq c |u^1| |u^2|_\alpha |u^3|_\alpha \quad \text{for } \alpha \geq \frac{5}{4}, \quad (5)
\]
\[
|\langle B(u^1, u^2), Au^3 \rangle| \leq c |u^1|_\alpha |u^2|^1 |Au^3|_{\alpha - 1} \quad \text{for } \alpha \geq \frac{5}{4}
= c |u^1|_\alpha |u^2|^1 |u^3|_{\alpha + 1}
\]
and similarly
\[
|\langle B(u^1, u^2), Au^3 \rangle| \leq c |u^1|_1 |u^2|_\alpha |Au^3|_{\alpha - 1} \quad \text{for } \alpha \geq \frac{5}{4}
= c |u^1|_1 |u^2|_\alpha |u^3|_{\alpha + 1}.
\]
Here and in the following, we denote by \(c\) a positive constant, which may vary from place to place.

### 3 Main theorem

We apply the projection operator \(\Pi\) to the first equation in (2). We get an Itô equation in an infinite dimensional Hilbert space:
\[
\begin{cases}
\frac{du(t)}{dt} + \left[ \nu A^\alpha u(t) + B(u(t), u(t)) \right] dt = A^{-\gamma} dw(t) \\
u A^\alpha u(0) = u_0
\end{cases}
\]
assuming the noise is of white type in time and with spatial covariance independent of \(u\). This means that \(w\) is a cylindrical Wiener process in \(H^0\) defined
on a complete probability space with filtration \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) (i.e. given a sequence \(\{\beta_j\}_j\) of i.i.d. standard Wiener processes, we represent the Wiener process in series as \(w(t) = \sum_j \beta_j(t)h_j\)). For simplicity we consider the operator in front of \(w\) to be a power of the Stokes operator; this is the model studied in [5].

For \(\alpha = 1\), this corresponds to the stochastic Navier–Stokes equation as analysed for instance in [7] for the 2-dimensional setting.

The technique to study equation (8) comes from [1, 14, 2, 7]. First we consider the linear equation, that is the modified stochastic Stokes equation

\[
\begin{aligned}
\frac{dz(t)}{dt} + \nu A^\alpha z(t) dt &= A^{-\gamma} dw(t) \\
z(0) &= 0
\end{aligned}
\] (9)

Then, we define \(v := u - z\); this unknown solves the following equation, obtained subtracting equation (9) to equation (8) and bearing in mind the bilinearity of the operator \(B\):

\[
\begin{aligned}
\frac{dv(t)}{dt} + \left[ \nu A^\alpha v(t) + B(v(t), v(t)) + B(z(t), v(t)) + B(v(t), z(t)) \right] &= -B(z(t), z(t)) \\
v(0) &= u_0
\end{aligned}
\] (10)

The noise term \(A^{-\gamma} dw(t)\) has disappeared.

Let \([0, T]\) be any finite time interval. We now state our main result.

**Theorem 3.1** Let \(\alpha \geq \frac{\theta}{4}\).

For any \(u_0 \in H^1\), if \(\gamma > \frac{3}{4}\) then there exists a unique process \(u\) which is a strong solution of (8) such that

\[
u \in C([0, T]; H^1) \cap L^{\frac{2\alpha}{2\alpha - 1}}(0, T; H^\alpha) \quad \mathbb{P} - a.s.;
\]

\(u\) is progressively measurable in these topologies and is a Markov process in \(H^1\).

### 3.1 Existence

We study pathwise the problems for the unknowns \(z\) and \(v\). This will imply an existence result for \(u\).

For the linear problem we have (see, e.g., Proposition 4.1 in [5], based on [2])

**Lemma 3.2** If

\[\alpha + 2\gamma > \theta + \frac{3}{2},\] (11)

then equation (9) has a unique strong solution \(z\) such that

\[\mathbb{P}\{z \in C([0, T]; H^\theta)\} = 1.\] (12)

Now, we work pathwise for the equation satisfied by \(v\) and therefore also for \(u\).
Proposition 3.3 Let $\alpha \geq \frac{5}{4}$.
For any $u_0 \in H^1$, if $\gamma > \frac{3}{4}$ then there exists a process $v$ which is a strong solution of (10) such that

$$v \in C([0, T]; H^1) \cap L^2(0, T; H^{1+\alpha}) \quad P \text{- a.s.}$$

$v$ is progressively measurable in these topologies.

Proof. From Lemma 3.2 we have that $z \in C([0, T]; H^\alpha)$ $P$-a.s., since $\gamma > \frac{3}{4}$. We take the scalar product of equation (10) with $v$ and use (6) and Young inequality to obtain

$$\frac{1}{2} \frac{d}{dt} |v|^2 + \nu |v|^2 = -\langle B(v, v) + B(v, z) + B(z, v) + B(z, z), v \rangle$$

$$= -\langle B(v, z) + B(z, v), v \rangle$$

$$\leq c |v| |z|_\alpha |v|_\alpha + c |z|_\alpha^2 |v|_\alpha$$

$$\leq \frac{\nu}{2} |v|^2 + c \nu (|z|_\alpha^2 |v|^2 + |z|_\alpha^4).$$

Then

$$\frac{d}{dt} |v|^2 \leq c |z|_\alpha^2 |v|^2 + c |z|_\alpha^4$$

and from Gronwall lemma: $\sup_{0 \leq t \leq T} |v(t)|^2 < \infty$. Moreover, integrating in time $0 \leq t \leq T$ the first inequality above, we have $\int_0^T |v(t)|^2 dt < \infty$.

Now we take the scalar product of equation (10) with $Av$:

$$\frac{1}{2} \frac{d}{dt} |v|^2 + \nu |v|^2 = \frac{1}{2} \langle v, Av \rangle - \langle B(v, v) + B(v, z) + B(z, v) + B(z, z), Av \rangle.$$

We use (10) and Young inequality to obtain

$$\frac{1}{2} \frac{d}{dt} |v|^2 + \nu |v|^2 \leq \frac{\nu}{2} |v|_{1+\alpha}^2 + c \nu (|z|_\alpha^2 + |v|_\alpha^2) |v|_1^2 + c \nu |z|_\alpha^4.$$

(13)

As usual, from

$$\frac{d}{dt} |v|_1^2 \leq 2 c \nu (|z|_\alpha^2 + |v|_\alpha^2) |v|^2 + 2 c \nu |z|_\alpha^4,$$

Gronwall inequality, with the result $v \in L^2(0, T; H^\alpha)$ proved before, implies $\sup_{0 \leq t \leq T} |v(t)|_\alpha^2 < \infty$ and integrating in time (13) we get $\int_0^T |v(t)|_{1+\alpha}^2 dt < \infty$.

The technique to prove existence is classical (see [12]). We consider first the finite dimensional problem in the unknown $v_n$, obtained projecting equation (10) onto $H_n$. This is the Galerkin approximation, for any $n = 1, 2, \ldots$. The above estimates hold uniformly also for the Galerkin sequence: for any $n$

$$\sup_{0 \leq t \leq T} |v_n(t)|_1^2 < c_1, \quad \int_0^T |v_n(t)|_{1+\alpha}^2 dt < c_2$$

for constants $c_1$ and $c_2$ independent of $n$.

Any finite dimensional (Galerkin) problem has a solution. By passing to the
limit as \( n \to \infty \) we get an existence result for \([10]\). We also need that \( \frac{d y}{d t} \) is uniformly bounded in \( L^2(0, T; H^{1-\alpha}) \). We verify easily that \( \frac{d y}{d t} \in L^2(0, T; H^{1-\alpha}) \) and the norm is bounded uniformly for all \( n \), since \( A^\alpha y \in L^2(0, T; H^{1-\alpha}) \) and according to \([10]\) all the bilinear terms in \([10]\) are in \( L^2(0, T; H^\alpha) \). Therefore, we have a compact embedding (see Theorem 2.1, Ch. III in \([12]\)) so the Galerkin sequence stays in a compact subset of \( L^2(0, T; H^1) \). Therefore there exists a subsequence converging to \( v \) as follows:

\[
\begin{align*}
  v_m & \to v \quad \text{weakly in } L^2(0, T; H^{1+\alpha}), \\
  v_m & \to v \quad \text{*-weakly in } L^\infty(0, T; H^1), \\
  v_m & \to v \quad \text{strongly in } L^2(0, T; H^1).
\end{align*}
\]

The strong convergence allows one to pass to the limit in the bilinear term (see Lemma 3.2, Ch. III in \([12]\)). Finally, \( v \in C([0, T]; H^1) \) (see Lemma 1.2, Ch. III in \([12]\)). The limit \( v \) fulfills all the estimates found above: \( v \in C([0, T]; H^1) \cap L^2(0, T; H^{1+\alpha}) \).

We conclude noting that by interpolation \( L^\infty(0, T; H^1) \cap L^2(0, T; H^{1+\alpha}) \subset L^{\frac{2}{1+\alpha}}(0, T; H^\alpha) \). Since the paths of \( z \) belong to \( C([0, T]; H^\alpha) \) and those of \( v \) to \( L^\infty(0, T; H^1) \cap L^2(0, T; H^{1+\alpha}) \), then \( u = v + z \in C([0, T]; H^1) \cap L^{\frac{2}{1+\alpha}}(0, T; H^\alpha) \) \( \mathbb{P} \)-a.s. This concludes the existence result of Theorem 3.1.

The measurability property is inherited by from the Galerkin sequence.

**Remark 3.4** The spatial covariance of the noise can be taken of a more general form. Indeed, what is needed is that pathwise we have \( z \in C([0, T]; H^\alpha) \). Hence, we can prove the same result of Theorem 3.1 when instead of \( A^\gamma dw(t) \) the noise is \( Gdw(t) \) assuming that the linear operator \( G : H^0 \to H^0 \) is a Hilbert–Schmidt operator. This allows to consider any finite noise, that is acting on a finite number of components \( h_k \) of the space \( H^0 \).

More generally, Lemma 3.2 is true if the range of the operator \( G \) is a subset of \( D(A^\gamma) \) with \( \gamma \) fulfilling \([11]\).

### 3.2 Pathwise uniqueness

We consider two solutions \( u_1 \) and \( u_2 \) of \([8]\) obtained in the previous section; we have that, for \( \alpha \geq \frac{2}{3} \),

\[
u_1, u_2 \in C([0, T]; H^1) \cap L^2(0, T; H^\alpha) \mathbb{P} \text{-a.s.}
\]

since \( \frac{2\alpha}{\alpha-1} > 2 \). The difference \( U = u_1 - u_2 \) satisfies

\[
\frac{d}{dt}U(t) + \nu A^\alpha U(t) + B(u_1(t), u_1(t)) - B(u_2(t), u_2(t)) = 0. \tag{14}
\]

Since the operator \( B \) is bilinear, this becomes

\[
\frac{d}{dt}U(t) + \nu A^\alpha U(t) + B(u_1(t), U(t)) + B(U(t), u_2(t)) = 0. \tag{15}
\]
Taking the scalar product of (15) with $AU$ in $H^0$, we get
\[ \frac{1}{2} \frac{d}{dt}|U(t)|^2_{1} + \nu|U(t)|^2_{1+\alpha} = -\langle B(u_1(t), U(t)) + B(U(t), u_2(t)), AU(t) \rangle \]
with $U(0) = 0$.

We estimate the r.h.s. according to (6)-(7)
\[ \frac{1}{2} \frac{d}{dt}|U(t)|^2_{1} + \nu|U(t)|^2_{1+\alpha} \leq c|u_1(t)|^2_{\alpha} + c|u_2(t)|^2_{\alpha}|U(t)|^2_{1} \]
Thus, by Young inequality:
\[ \frac{d}{dt}|U(t)|^2_{1} \leq c(|u_1(t)|^2_{\alpha} + |u_2(t)|^2_{\alpha})|U(t)|^2_{1} \]
and, by Gronwall inequality
\[ |U(t)|^2_{1} \leq |U(0)|^2_{1} e^{\int_0^t c(|u_1(s)|^2_{\alpha} + |u_2(s)|^2_{\alpha})ds}. \]
Then $U(t) = 0$ for all $t \geq 0$, because $U(0) = 0$. This means that pathwise we have $u_1(t) = u_2(t)$ for all $t \geq 0$.

**Remark 3.5**

i) The Markov property of $u$ comes from the same properties for the Galerkin approximations and from the uniqueness result (see, e.g., [7]).

ii) The pathwise estimate (16) implies also the Feller property in $H^1$, that is, given a sequence of solutions $u^j$ with initial data $u_0^j$, if $\lim_{j} u_0^j = u_0$ in $H^1$ then
\[ \lim_{j} \mathbb{E} \phi(u^j(t)) = \mathbb{E} \phi(u(t)) \] for any $t \in [0, T]$ and any continuous bounded function $\phi : H^1 \to \mathbb{R}$.

## 4 Regularity results

Considering as phase space other Hilbert spaces $H^s$, we get different bounds on $\alpha$ to obtain that the dynamics of the stochastic Navier–Stokes equation is well posed in such $H^s$. As it has been pointed out in [9], the smaller is $s$ (with $u_0 \in H^s$) the bigger is $\alpha$. Theorem 3.1 deals with $s = 1$. In this section, we show that problem (8) is well-posed in the space $H^0$ if $\alpha > \frac{3}{2}$, and in the spaces $H^s$ with $s \geq 2$ if $\alpha \geq \frac{5}{4}$.

### H$^0$-regularity

We have the following result

**Proposition 4.1** Let $\alpha > \frac{3}{2}$.

For any $u_0 \in H^0$, if $\gamma > \frac{3}{4}$ then there exists a unique process $u$ which is a strong solution of (8) such that
\[ u \in C([0, T]; H^0) \cap L^2(0, T; H^\alpha) \quad \mathbb{P} - a.s.; \]
u is progressively measurable in these topologies and a Markov process in $H^0$. 

7
Proof. Existence is proved by means of a priori estimates as in the proof of Proposition 3.3; to be precise, for \( \alpha \geq \frac{3}{2} \) we get that there exists a solution \( u \in C([0,T];H^0) \cap L^2(0,T;H^0) \) \( \mathbb{P} \)-a.s. if \( z \) has paths in \( C([0,T];H^\alpha) \), i.e. if \( \gamma > \frac{3}{4} \).

Pathwise uniqueness is obtained according to the result by Prodi [10], requiring \( u \in L^s(0,T;[L^q(T)]^3) \) \( \mathbb{P} \)-a.s. for \( \frac{2}{3} + \frac{3}{q} \leq 1 \). However, using an interpolation result and Sobolev embedding we have

\[
L^\infty(0,T;H^0) \cap L^2(0,T;H^\alpha) \subset L^s(0,T;H^{2\alpha}) \subset L^s(0,T;[L^q(T)]^3)
\]

for \( 2 < s < \infty \) and \( \frac{1}{q} = \frac{1}{2} - \frac{2\alpha}{3\alpha} \). The condition \( \frac{2}{s} + \frac{3}{q} \leq 1 \) holds if \( \alpha > \frac{3}{2} \). \( \square \)

Remark 4.2 For \( \alpha \geq 1 \) we can prove that there exists a solution of equation (8) such that \( u \in C_w([0,T];H^0) \cap L^\infty(0,T;H^0) \cap L^2(0,T;H^\alpha) \) \( \mathbb{P} \)-a.s. But uniqueness is unknown.

Consider, for instance, \( \alpha = 1 \). We require that \( z \in C([0,T];H^\frac{3}{2}) \) \( \mathbb{P} \)-a.s. and equation (10) is treated as in the deterministic setting.

For this, change the first estimate in the proof of Proposition 3.3 as follows:

\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \nu |v|^2 = -\langle B(v+z), v \rangle + \nu \langle v, v \rangle + c |v| |z|^2 + c \nu |z|^4.
\]

Then \( \sup_{0 \leq t \leq T} |v(t)| < \infty \), \( \int_0^T |v(t)|^2 \, dt < \infty \).

Moreover, from Lemma 2.1 in [13] we have that \( B : H^0 \times H^1 \to H^{-\sigma} \) for any \( \sigma > \frac{3}{2} \). Then \( \dot{v} = -\nu B(v+z) + B(v+z) \in L^2(0,T;H^{-\sigma}) \). This gives a compact embedding and therefore there exists a subsequence of the Galerkin sequence that converges to \( v \) as follows:

\[
v_m \to v \quad \text{weakly in } L^2(0,T;H^1),
\]

\[
v_m \to v \quad \text{weakly in } L^\infty(0,T;H^0),
\]

\[
v_m \to v \quad \text{strongly in } L^2(0,T;H^0).
\]

Finally \( v \in C_w([0,T];H^0) \). Notice that the previous result \( v \in C([0,T];H^0) \) came from \( v \in L^2(0,T;H^\alpha) \), \( \dot{v} \in L^2(0,T;H^{-\alpha}) \) (see Lemma 1.2, Ch. III in [12]).

Remark 4.3 To compare our result with [11], we have that [11], for its model, deals with the phase space \( H^0 \) assuming \( \alpha \geq 2 \).
\[ H^\alpha \text{-regularity with } s \geq 2 \]

We need the following estimates:

**Lemma 4.4**

\[ |B(u, \tilde{u})|_m \leq c|u|_{m+1}|\tilde{u}|_{m+1} \quad \text{for } m = 1, 2, 3, \ldots \quad (17) \]

**Proof.** First, consider \( (17) \) for \( m = 1 \). We have

\[
|B(u, \tilde{u})|^2 = |(u \cdot \nabla)\tilde{u}|^2 = \sum_{k,l=1}^3 \left| \partial_k \left( \sum_{i=1}^3 u_i \partial_l \tilde{u}_i \right) \right|^2 \leq 2 \sum_{k,l=1}^3 \left| \partial_k u_l \partial_l \tilde{u}_l \right|^2 + 2 \sum_{k,l=1}^3 \left| \sum_{i=1}^3 u_i \partial_i \tilde{u}_i \right|^2 \leq 6 \sum_{k,l,i} \left| \partial_k u_l \right|^2 \left| \partial_l \tilde{u}_l \right|^2 + 6 \sum_{k,l,i} \left| u_i \right|^2 \left| \partial_k \partial_l \tilde{u}_l \right|^2.
\]

Then use the continuous embeddings \( H^1(T) \subset L^4(T) \) and \( H^2(T) \subset L^\infty(T) \).

For \( m = 2, 3, \ldots \) we use that \( H^m \) is a multiplicative algebra if \( m > \frac{3}{2} \); then

\[ |B(u, \tilde{u})|_m \leq c|u|_m |\tilde{u}|_{m+1} \quad \text{for } m = 2, 3, \ldots \]

which is even stronger than \( (17) \). \( \square \)

We have the following result

**Proposition 4.5** Let \( \alpha \geq \frac{s}{2} \) and \( s \geq 2 \).

For any \( u_0 \in H^s \), if \( \alpha + 2\gamma > s + \frac{3}{2} \) then there exists a unique process \( u \) which is a strong solution of \( \mathcal{L} \) such that

\[ u \in C([0, T]; H^\alpha) \]

\( \mathbb{P}\text{-a.s.} \).

\( u \) is progressively measurable in these topologies and is a Markov process in \( H^\alpha \).

For simplicity, we provide the proof for \( s = 2 \). In this way we show the difference with respect to the case \( s = 1 \) considered in the previous section. However, the proof would go along the same lines for \( s > 2 \) using \( (17) \).

**Proof.** Set \( s = 2 \). Then almost every path of \( z \) is in \( C([0, T]; H^2) \).

We prove existence for \( \alpha \geq \frac{s}{4} \). We use \( (17) \) with \( m = 1 \) and take the scalar product of equation \( (10) \) with \( A^2 v \):

\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \nu |v|^{2+\alpha} = -\langle B(v + z, v + z), A^2 v \rangle = -\langle A^* B(v + z, v + z), A^* v \rangle \\
\leq |B(v + z, v + z)|_1 |v|_3 \\
\leq c |v + z|_2^2 |v|_3 \\
\leq c |v + z|_2^2 |v|_{2+\alpha} \\
\leq \frac{\nu}{2} |v|^{2+\alpha} + c_\nu |v|_2^2 + c_\nu |z|_2^2.
\]
Since we already know from Proposition 3.3 that $v \in L^2(0, T; H^{1+\alpha}) \subset L^2(0, T; H^2)$, it follows as usual by Gronwall lemma that $v \in L^\infty(0, T; H^2) \cap L^2(0, T; H^{2+\alpha})$.

From now on, the proof goes as in Proposition 3.3.

Pathwise uniqueness: the estimates hold for any $\alpha \geq 1$ but the regularity required on $u_i$ holds for $\alpha \geq \frac{5}{4}$. This shows that in $H^2$ it is "easier" to prove uniqueness than existence.

Set $U = u_1 - u_2$ as in Section 3.2 now $u_1, u_2 \in C([0, T]; H^2)$. Taking the scalar product of (15) with $A^2 U$ in $H^0$, we get

$$\frac{1}{2} \frac{d}{dt} |U(t)|^2_2 + \nu |U(t)|^2_{2+\alpha} = -\langle B(u_1(t), U(t)) + B(U(t), u_2(t)), A^2 U(t) \rangle$$

with $U(0) = 0$. As before, we estimate the r.h.s. by means of (17), and get

$$\frac{1}{2} \frac{d}{dt} |U(t)|^2_2 + \nu |U(t)|^2_{2+\alpha} \leq c|u_1(t)|_2|U(t)|_2|U(t)|_3 + c|u_2(t)|_2|U(t)|_2|U(t)|_3$$

$$\leq c|u_1(t)|_2|U(t)|_2|U(t)|_2 + c|u_2(t)|_2|U(t)|_2|U(t)|_2 + c|u_1(t)|_{2+\alpha} + c|u_2(t)|_{2+\alpha}$$

$$\leq \nu \frac{1}{2} |U(t)|^2_{2+\alpha} + c\nu (|u_1(t)|^2_2 + |u_2(t)|^2_2)|U(t)|^2_2.$$ 

From

$$\frac{d}{dt} |U(t)|^2_2 \leq 2c\nu (|u_1(t)|^2_2 + |u_2(t)|^2_2)|U(t)|^2_2$$

we conclude that $|U(t)|_2 = 0$ for any $t \in [0, T]$.  

\textbf{References}

[1] Bensoussan, A., Temam, R.: Équations stochastiques du type Navier-Stokes, in: J. Funct. Anal. \textbf{13} (1973), 195–222.

[2] Da Prato, G., Zabczyk, J.: \textit{Stochastic Equations in Infinite Dimensions}, Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 1992.

[3] Fefferman, C.L.: Existence and smoothness of the Navier–Stokes equation, in \textit{The millennium prize problems}, Clay Math. Inst., Cambridge MA, 57-67, 2006.

[4] Ferrario, B.: Uniqueness result for the 2D Navier–Stokes equation with additive noise, in: \textit{Stoch. Stochastic Reports} \textbf{75} (2003), 435–442.

[5] Ferrario, B.: Absolute continuity of laws for semilinear stochastic equations with additive noise, in: \textit{Communications on Stochastic Analysis} \textbf{2} (2008), 209–227.

[6] Ferrario, B., Flandoli, F.: On a stochastic version of Prouze model in fluid dynamics, in: \textit{Stochastic Processes Appl.} \textbf{118} (2008), no. 5, 762–789.
[7] **Flandoli, F.**: Dissipativity and invariant measures for stochastic Navier-Stokes equations, in: *NoDEA* **1** (1994), 403–423.

[8] **Lions, J.-L.**: *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[9] **Mattingly, J.C., Sinai, Ya.G.**: An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations, in: *Commun. Contemp. Math.* **1** (1999), no. 4, 497–516.

[10] **Prodi, G.**: Un teorema di unicità per le equazioni di Navier–Stokes, in: *Ann. Mat. Pura Appl. (4)* **48** (1959), 173–182.

[11] **Sritharan, S.S.**: Deterministic and stochastic control of Navier-Stokes equation with linear, monotone, and hyperviscosities, in: *Appl. Math. Optim.* **41** (2000), no. 2, 255–308.

[12] **Temam, R.**: *Navier-Stokes Equations, Theory and Numerical Analysis*, 3rd ed., North-Holland, Amsterdam, 1984.

[13] **Temam, R.**: *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1983.

[14] **Vishik, M.J., Fursikov, A.V.**: *Mathematical Problems of Statistical Hydromechanics*, Kluwer Academic Publishers, Dordrecht, 1988.