Moduli Space Dimensions of Multi-Pronged Strings

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Abstract

The numbers of bosonic and fermionic zero modes of multi-pronged strings are counted in $\mathcal{N} = 4$ super-Yang-Mills theory and compared with those of the IIB string theory. We obtain a nice agreement for the fermionic zero modes, while our result for the bosonic zero modes differs from that obtained in the IIB string theory. The possible origin of the discrepancy is discussed.

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I. INTRODUCTION

The recent development of non-perturbative string theories has provided new powerful tools to understand the supersymmetric gauge theories. The low energy dynamics of the D-branes are described by the supersymmetric gauge theories. The BPS spectrum of the supersymmetric theory will then correspond to the BPS configurations of strings and branes ending on the background brane configurations.

Various known properties of the $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory have been studied based on $N$ parallel D3-branes. The BPS state spectrum of the massive gauge bosons, monopoles and dyons preserving half of the supersymmetry are identified with the $(p, q)$ strings connecting two separated D3-branes. With more than two D3-branes, we can have the string junction configurations [1] that preserve only 1/4 of the supersymmetries [2,3]. The condition for the string junction configurations gives the set of field equations describing the corresponding BPS states of the gauge theory [3–8]. In addition to the first order differential equations describing the 1/2 BPS states of monopoles, the string junction needs a second order equation of the Gauss law. The field theoretic solutions corresponding to multi-pronged strings were explicitly constructed for $SU(3)$ theory [5,6] and generalized to $SU(N)$ theory [7,8].

To study the quantum properties of this string junction, one needs to understand the zero modes around the classical configurations. As in the case of monopoles [3,10], the bosonic zero modes will correspond to the collective coordinates of the moduli space, while the fermionic zero modes correspond to the spin structures of the supermultiplets. The number of zero modes of monopoles with an arbitrary gauge group is well known [11].

In this paper, we will count the number of bosonic and fermionic zero modes of the multi-pronged strings in the $SU(N)$ field theory and compare it with that of IIB string picture [12]. The number of fermionic zero modes was already discussed for a specific $SU(3)$ solution in Ref. [6].

In section 2, we briefly describe the BPS equations describing the string junction and the zero mode equations. In section 3, we count the zero modes. The equations of the bosonic zero modes consist of those for the magnetic monopole and one more second order equation. Usually, the index, i.e., the number of the zero modes is evaluated by asymptotic expansion of the field configurations [1]. However, for the string junctions, the method of evaluating the index by the expansion of the electric fields fails. Instead, we will count the bosonic and fermionic zero modes based on considering the constraints imposed on the zero modes of multi-monopoles. The detail of mathematical arguments are in the Appendix A. In section 4, this counting is shown to be different from that based on the Type II string theory. In section 5, we summarize our main results, and indicate future directions.

II. MULTI-PRONGED STRINGS AND THEIR MODULI SPACE

We begin by recapitulating the basic properties of 1/4 BPS states in the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory, whose Lagrangian, for the bosonic part only, reads

$$\mathcal{L} = -\frac{1}{4g_{\text{YM}}^2} \text{Tr} \left[ F^{\mu\nu} F_{\mu\nu} + 2D_\mu \phi^I D^\mu \phi^I - 2\sum_{I<J} [\phi^I, \phi^J]^2 \right]$$  \hspace{1cm} (1)
where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, $D_\mu \phi^I \equiv \partial_\mu \phi^I - i[A_\mu, \phi^I]$ and the indices $I$ and $J$ run from 1 to 6. It is well known that, in this theory, the monopoles, the dyons and the $W$-particles preserve half of sixteen supersymmetries of the theory. It is shown recently that there may be also $1/4$ BPS states that preserve a quarter of the total supersymmetry. As shown in Refs. [3,12], these states describe multi-pronged strings connecting $N$ D3-branes in the type IIB string picture. Examples of closed-form solutions corresponding to the pronged strings were found in the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory [5–8]. We shall investigate the number of both bosonic and fermionic zero modes of the multi-pronged strings in the $\mathcal{N} = 4$ super-Yang-Mills theory for the gauge group $SU(N)$. This will be the first step to extract the structure or geometry of the moduli space involved with the multi-pronged strings. Thereby, one is ultimately interested in obtaining the low-energy effective dynamics of the multi-pronged strings.

The Bogomol’nyi bound for the $1/4$ BPS states can be found by considering the energy functional of the $\mathcal{N} = 4$ system,

$$M = \frac{1}{2g_{YM}^2} \int d^3x \text{Tr} \left[ E \cdot E + B \cdot B + D \phi^I \cdot D \phi^I + D_0 \phi^I D_0 \phi^I - \sum_{I<J} \left[ \phi^I, \phi^J \right]^2 \right].$$

(2)

Introducing two orthonormal six vectors $e^I$ and $b^I$, we present the energy equivalently by

$$M = \frac{1}{2g_{YM}^2} \int d^3x \text{Tr} \left[ |e^I E + b^I B - D \phi^I|^2 + |D_0 \phi^I e^I|^2 + |D_0 \phi^I b^I + i \left[ \phi^I e^I, \phi^I b^I \right]|^2 \right.$$

$$\left. + \left| D_0 \phi^I \right|^2 - \sum_{I<J} \left[ \phi^I e^I, \phi^J e^I \right]^2 + 2 \nabla \cdot (E \phi^I e^I) + 2 \nabla \cdot (B \phi^I b^I) \right],$$

(3)

where $\phi^I$ refers to components of the six-vector perpendicular to the unit vectors, $e^I$ and $b^I$, and we have used the Gauss law constraint

$$D \cdot E + i \left[ \phi^I e^I, D_0 \phi^I e^I \right] + i \left[ \phi^I b^I, D_0 \phi^I b^I \right] = 0,$$

(4)

and the Bianchi identity $D \cdot B = 0$ to perform the integration by part\footnote{The Gauss law is, in fact, given by $D \cdot E + i[\phi^I, D_0 \phi^I] = 0$ but, with a restriction set by the Bogomol’nyi equation $D_0 \phi^I = 0$, one may consistently use the form given here.}. This, then, implies that the energy is bounded from below by

$$g_{YM}^2 M \geq Q^I_E e^I + Q^I_M b^I,$$

(5)

where we define the charge six vectors as

$$Q^I_E \equiv \int d^3x \nabla \cdot \text{Tr} E \phi^I = Q^p_E h^{Ip},$$

$$Q^I_M \equiv \int d^3x \nabla \cdot \text{Tr} B \phi^I = Q^p_M h^{Ip}.$$

(6)

For the equalities, we used the asymptotic condition,
\[ \langle \phi^I \rangle = h^{lp} H_p , \]  
(7)

where \( H_p \) is the \( N - 1 \) mutually commuting operators that span the Cartan subalgebra. The raising and lowering generators \( E_\alpha \),

\[ [H_p, E_\alpha] = \alpha_p E_\alpha , \]  
(8)

are normalized by

\[ [E_\alpha, E_{-\alpha}] = \alpha_p H_p , \]  
(9)

where \( \alpha_p \) are the roots. We will choose the simple roots \( \beta_p \) by requiring \( h^{lp}b^l/\beta_p > 0 \) for the maximal symmetry breaking\(^2\) case along \( \phi^I b^I \).

The saturation of the bound occurs if \[ D_\phi^I = e^I E + b^I B , \quad D_0 \phi^I e^I = 0 , \quad D_0 \phi^I b^I + i[\phi^I e^I, \phi^J b^J] = 0 \]  
(10)

and

\[ [\phi^I_\perp, \phi^J_\perp] = 0 , \quad D_0 \phi^I_\perp = 0 . \]  
(11)

There are two types of BPS states that may be classified by considering two \( \mathcal{N} = 4 \) central charges given by \[ Z_\pm = \sqrt{\| Q^I_E \|^2 + \| Q^I_M \|^2 \pm \| Q^I_E \| \| Q^I_M \| \sin \chi} \left( \chi \in [0, \pi) \right) . \]  
(12)

where \( \chi \) denotes the angle between the charges vectors. For \( \chi = 0 \), the state preserves the eight supersymmetries and these are described field-theoretically by the monopole, the dyons and W-particles. With non-vanishing \( \chi \), the two charge-vectors are no longer parallel, and the state preserves only 1/4 of the supersymmetry. We will be mainly concerned on these 1/4 BPS states.

Owing to the tracelessness of the charges, this, in fact, guarantees the balance of tension of the corresponding multi-pronged junction. Moreover, one finds a restriction on the electric and the magnetic charges

\[ Q^p_E h^{lp} b^l - Q^p_M h^{lp} e^l = 0 \]  
(13)

which turns out to be the balance condition of the torque applied on the associated D3-branes by the pronged string. This restriction follows from

\[ Q^p_E h^{lp} b^l = Q^I_E b^l = \int d^3 x \text{Tr} E \cdot D \phi^I b^l = \int d^3 x \text{Tr} E \cdot B = \int d^3 x \text{Tr} B \cdot D \phi^I e^l = Q^p_M h^{lp} e^l \]  
(14)

\(^2\)We consider only the case of the maximal symmetry breaking in the direction of \( \phi^I b^I \) for simplicity. Obviously, this condition may be lifted to study the effect of nonabelian symmetry breaking.
where we used the Bogomol'nyi equation and the definitions of charges.

The geometric shape of the junction and the meaning of the charges may be clearly found by fixing our six coordinate system. For later purpose, we shall denote $\phi^I b^I \equiv A_4$, and $\phi^I e^I \equiv X$ and further set $\phi^I_\perp = 0$ without loss of generality.

The BPS equations are rewritten as

\[ B = DA_4, \]  
\[ E = DX, \]  
\[ D_0 X = 0, \quad D_0 A_4 + i[X, A_4] = 0, \]

with the Gauss law,

\[ D \cdot DX - [A_4, [A_4, X]] = 0. \]

Let us now choose a gauge $A_0 = X$. Eqs. (16) and (17) then lead to a relation $\dot{A}_m = \dot{X} = 0$ ($m = 1, 2, 3, 4$), which implies any solutions of the BPS equation are static with this gauge choice.

Eq. (15) is the usual BPS monopole equation. Hence the junction BPS state is a kind of monopole surrounded by W-boson cloud, which is determined by Eq. (18). Topological argument leads to the quantization of the magnetic charge by

\[ Q_M = 4\pi \sum_{a=1}^{N-1} m_a \beta^a_p h^p b^I, \]

where the integer $m_a$ counts the number of each fundamental monopole.

The moduli space of a given multi-pronged string with fixed D-brane positions is defined by the solution space of the above BPS equation modulo gauge transformation with fixed vacuum expectation values of scalars and electric/magnetic charges.

The tangent vectors of the moduli space with the gauge $A_0 = X$, will satisfy the zero-mode equation

\[ \epsilon^{ijk} D_j \delta A_k - D_4 \delta A^i + D^i \delta A_4 \equiv \eta_{mn}^a \delta A_m = 0, \]
\[ D_m D_m \delta A_0 + 2i[D_m A_0, \delta A_m] = 0, \]
\[ D_m \delta A_m = 0, \]

where we introduced a notation $-i[A_4, (\cdot)] \equiv D_4 (\cdot)$ and the 't Hooft symbol

\[ \eta_{mn}^a = \begin{cases} 
\varepsilon_{mn}, & m, n = 1, 2, 3, \\
-\delta_{mn}, & m = 4, \\
\delta_{mn}, & n = 4, \\
0, & m = n = 4,
\end{cases} \]

and Eq. (22) is the gauge condition. The zero modes are then normalizable solutions of the above equations with a norm,

\[ ||\delta A||^2 = \int d^3 x \left( \delta A_0 \delta A_0 + \delta A_m \delta A_m \right). \]
The number of the zero modes will agree with the dimensions of the moduli space or the tangent space at a given point of the moduli space. In dealing with the above zero-mode equations, we note that a multi-pronged string of \( SU(N) \) can be always embedded into the \( SU(N) \) theory with a larger \( \tilde{N} \). Then in the \( SU(N) \) theory, the field components of the multi-pronged solution other than the \( SU(N) \) component are zero by construction. Then with this background solution, the fluctuations of components other than the \( SU(N) \) component will be dynamically decoupled in the above zero-mode equations and, hence, can be trivially set to vanish from the beginning. Hereafter, we shall choose such an \( SU(N) \) subgroup with the minimum rank, where one can embed the pronged string, and work within such \( SU(N) \). This \( \tilde{N} \) is then the number of D-branes where the prongs end, which will be denoted by \( \tilde{N} \).

To expose the structure of the low-energy effective Lagrangian resulted from the moduli-space approximation, let us suppose the zero modes are given by \( \delta_s A (s = 1, 2, \cdots, \# \text{zero-mode}) \). Let us further denote the moduli-space element as \( A(r; \xi) \) with \( \xi^s \) being the coordinate of the moduli space.

Inserting the solutions to the Lagrangian with time-dependent moduli-coordinates, one finds that

\[
L_{\text{eff}} = \frac{1}{2g^2_{\text{YM}}} \int d^3x \Tr \left[ \dot{A}_0 \dot{A}_0 + \dot{A}_m \dot{A}_m - 2\partial_i (A_0 \dot{A}_i) \right],
\]

where we have used the gauge condition \( A_0 = X \) and have dropped a constant term. The effective Lagrangian may be rewritten, to the quadratic order in velocities, as

\[
L_{\text{eff}} = \frac{1}{2} g_{ss'}(\xi) \dot{\xi}^s \dot{\xi}^{s'} - A_s(\xi) \dot{\xi}^s,
\]

with an appropriate gauge transformation,

\[
\dot{A} \rightarrow \dot{\xi}^s (\partial_s A - D\epsilon_s) \equiv \dot{\xi}^s \delta_s A,
\]

that insures the gauge condition \( (22) \) of the zero mode. The metric and the vector potential can be expressed, in terms of the zero mode:

\[
g_{ss'}(\xi) = \frac{1}{2g^2_{\text{YM}}} \int d^3x \Tr [\delta_s A_0 \delta_s A_0 + \delta_s A_m \delta_s A_m],
\]

\[
A_s(\xi) = \frac{1}{g^2_{\text{YM}}} \int_{r=\infty} dS^i \Tr A_0 \delta_s A_i = \int d\Omega h_p e^i \lim_{r\rightarrow \infty} \left[ r^2 \Tr (\hat{r}^i \delta_s A^i H_p) \right],
\]

where, in the last equation, we have used the normalizability condition \( \delta_s A_i = O(1/r^2) \) for a large \( r \). The vector potential term in Eq. \( (26) \) is in fact a total time derivative term. This can be easily seen from the last term of Eq. \( (25) \), where \( A_0 \) can be replaced by its asymptotic value. Although this total time derivative term does not affect the classical dynamics of the moduli space, it is relevant in its quantum version especially when some of the directions of the moduli space are compact. The motion in these compact directions is indeed involved with the quantization of the electric charges.
III. COUNTING THE NUMBER OF ZERO MODES

The equations of the bosonic zero modes and the gauge fixing condition are given by Eqs. (20), (21) and (22). In this section we shall analyze the number of the normalizable solutions of these equations. Here and below, all vector potential and the covariant derivative denote, respectively, the background solutions of pronged strings and the covariant derivative with respect to the background. The normalizable solutions of Eqs. (20) and (22) are the zero modes of a BPS monopole. The problem here is whether the equation (21) gives a normalizable solution for $\delta A_0$ or not. As discussed in appendix (A), the condition that the solution of Eq. (21) be normalizable is given by

$$\int d^3 x \text{Tr} (\Lambda_a [D_m A_0, \delta A_m]) = 0,$$

(30)

where the trace is over the color indices and $\Lambda_a$ ($a = 1, \cdots, \tilde{N} - 1$) are the zero modes of the operator $-D_m D_m$. From the junction BPS equation, $D_m D_m A_0 = D_m D_m A_4 = 0$ holds. They are independent for a 1/4 BPS state, and hence they are two of $\Lambda_a$. As shown in appendix (A), the conditions (30) are equivalent to the condition that the electric charges should not change under the infinitesimal changes of the configuration.

Apparently these conditions seem to give $\tilde{N} - 1$ conditions on the tangent moduli space of the monopole, but this is not. In fact, one of the conditions

$$\int d^3 x \text{Tr}(A_4 D_m A_0, \delta A_m) = 0$$

(31)

is satisfied for any monopole zero mode $\delta A_m$. To show this, we can use an identity resulting from the simple fact that the magnetic charges do not change under the infinitesimal change. As shown in appendix (A), this implies that

$$\int d^3 x \text{Tr}(A_0 D_m A_4, \delta A_m) = 0.$$  

(32)

One can easily show that the partial integration of this equation gives Eq. (31). Alternatively, one may understand Eq. (31) in terms of the torque balance identity (13) by $\delta \text{Tr}[A_4 (r = \infty) Q_E] = \delta \text{Tr}[A_0 (r = \infty) Q_M]$ and $\delta Q_M = 0$.) Hence the number of the constraints is in fact $\tilde{N} - 2$. The number of the bosonic zero modes of the monopole BPS equations is given by $\#\text{monopole} \times 4$ [11], where $\#\text{monopole}$ is the total number of the fundamental monopoles, $\sum_a m_a$ (see Eq. (19)). Therefore, taking into account the constraints, the total number of the bosonic zero modes (BZM) of the junction solution is given by

$$\#\text{BZM} = \#\text{monopole} \times 4 - \tilde{N} + 2.$$  

(33)

Let us now discuss the number of the fermionic zero modes of the junction solution. The equations for the fermionic zero modes are given by

$$D^{(-)} \psi^{(-)} = \begin{pmatrix} -2D_0 & \tau_m^+ D_m \\ \tau^-_m D_m & 0 \end{pmatrix} \psi^{(-)} = 0,$$

(34)

$$D^{(+)} \psi^{(+)} = \begin{pmatrix} 0 & \tau^-_m D_m \\ \tau^+_m D_m & -2D_0 \end{pmatrix} \psi^{(+)} = 0,$$

(35)
where \( D_0 \) is understood as \( D_0(\cdot) = -i[A_0(\cdot)] \) and the } \( 2 \times 2 \) \text{ matrices is defined by } \tau_m^- = (\sigma^i, i) \text{ and } \tau_m^+ = (\sigma^i, -i) \text{ with the Pauli matrices } \sigma^i. \text{ To first analyze the equation (35), we decompose it by}

\[
\psi^{(+)} = \begin{pmatrix}
\psi_1^{(+)} \\
\psi_2^{(+)}
\end{pmatrix}.
\]  

(36)

Then the equation (35) is

\[
\begin{align*}
\tau_m^- D_m \psi_2^{(+)} &= 0, \\
\tau_m^+ D_m \psi_1^{(+)} - 2D_0 \psi_2^{(+)} &= 0.
\end{align*}
\]  

(37) \hspace{0.5cm} (38)

Since the operator \( \tau_m^- D_m \) has no zero modes, the solution is given by that \( \psi_2^{(+)} = 0 \). Hence \( \tau_m^+ D_m \psi_1^{(+)} = 0 \), and \( \psi_1^{(+)} \) is just the fermionic zero modes of a BPS monopole. Thus the number of the fermionic zero modes resulting from Eq. (35) is given by \#monopole \times 4.

With a similar decomposition to two-component spinors, Eq. (34) can be rewritten as

\[
\begin{align*}
\tau_m^+ D_m \psi_1^{(-)} &= 0, \\
\tau_m^- D_m \psi_2^{(-)} - 2D_0 \psi_1^{(-)} &= 0.
\end{align*}
\]  

(39) \hspace{0.5cm} (40)

The equation (39) just gives the fermionic zero modes of a BPS monopole and the number of the solutions is \( 4 \times \#\text{monopole} \). Let us discuss the normalizability of \( \psi_2^{(-)} \) resulting from Eq. (40). First act an operator \( \tau_m^+ D_m \) on Eq. (40). Using the BPS equation, we obtain

\[
D_m D_m \psi_2^{(-)} - 2\tau_m^+ D_m D_0 \psi_1^{(-)} = 0.
\]  

(41)

This equation looks very similar to the bosonic one (21). To obtain a normalizable solution of \( \psi_2^{(-)} \), the following constraints on \( \psi_1^{(-)} \) must be satisfied:

\[
\int d^3x \text{ Tr} \left( \Lambda_a \tau_m^+ D_m D_0 \psi_1^{(-)} \right) = 0,
\]  

(42)

where the trace is over the color indices. As similar to the bosonic case, one of the constraints,

\[
\int d^3x \text{ Tr} \left( A_4 \tau_m^+ D_m D_0 \psi_1^{(-)} \right) = 0,
\]  

(43)

is identically satisfied by the monopole fermionic zero modes.

To show this, let us start with Eq. (39). Applying the operator \( \tau_m^- D_m \) on it, one obtains

\[
(D_m D_m + i\tilde{\eta}_{mn}^a \sigma_a D_m D_n) \psi_1^{(-)} = 0 \quad \text{(The symbols } \tilde{\eta}_{mn}^a \text{ differ from } \eta \text{ by a change in the sign of } \delta \text{ in the definition (23))}.
\]  

Using further the BPS equations, we obtain

\[
(D_m D_m - 2i\tau_m^+ D_m D_4) \psi_1^{(-)} = 0.
\]  

(44)

Hence the normalizability of the fermionic zero mode \( \psi_1^{(-)} \) reads

\[
\int d^3x \text{ Tr} \left( \Lambda_a \tau_m^+ D_m D_0 \psi_1^{(-)} \right) = 0,
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Using further the BPS equations, we obtain

\[
(D_m D_m - 2i\tau_m^+ D_m D_4) \psi_1^{(-)} = 0.
\]  

(44)

Hence the normalizability of the fermionic zero mode \( \psi_1^{(-)} \) reads

\[
\int d^3x \text{ Tr} \left( \Lambda_a \tau_m^+ D_m D_0 \psi_1^{(-)} \right) = 0,
\]  

(42)

\[
\int d^3x \text{ Tr} \left( A_4 \tau_m^+ D_m D_0 \psi_1^{(-)} \right) = 0,
\]  

(43)

is identically satisfied by the monopole fermionic zero modes.
FIG. 1. The configuration of D-string ending on D3-branes is depicted. The figure is for magnetic charges \((m_1, m_2) = (1, 1)\) in \(SU(3)\) case. The moduli-space is four dimensional.

\[
\int d^3x \, Tr \left( \Lambda_a \tau^+_m D_m D_4 \psi_1^{(-1)} \right) = 0.
\]

By the partial integration of Eq. (45) with the substitution \(\Lambda_a = A_0\), we obtain Eq. (43).

Thus the constraints (12) give \(4(\tilde{N} - 2)\) constraints on the fermionic BPS zero modes. Thus the total number of the fermionic zero modes (FZM) is given by

\[
\#\text{FZM} = \#\text{monopole} \times 8 - 4(\tilde{N} - 2).
\]

In the next section, we will compare the numbers of the zero modes, Eqs. (33) and (46), with those derived from the type IIB description.

Finally we mention the reason why we should take \(\tilde{N}\) instead of \(N\) in our formulae for the numbers of the zero modes. Let us consider a D-string of \(SU(3)\) theory in FIG. 1. The zero mode equations take the same form as Eqs. (20), (21) and (22) with \(D_m A_0 = 0\). Hence the solution of Eq. (21) is simply \(\delta A_0 = 0\). The others possess \(2 \times 4\) zero-modes because the D-string is regarded as two fundamental monopoles in Eqs. (20) and (22). This is incorrect because we know from the beginning, there are only four moduli degrees of freedom around this configuration. This failure of the zero modes analysis may be understood as follows. Around the D-string configuration, the field-theoretic potential is too flat to capture the correct number of zero modes by just considering the linearized fluctuations of Eqs. (20), (21) and (22). The higher order analysis will show that the the relative motions of the two fundamental monopoles drops out of the moduli space, which leaves just four overall translation degrees.

However, our formula, Eqs. (33) and (46), for the zero modes are still valid for the D-string because we are using the minimal embedding of solutions and, hence, \(\tilde{N} = 2\).

IV. COMPARISON WITH IIB STRING THEORY

Our result of the number of the bosonic zero modes does not agree with that from the IIB string theory. To illustrate, let us consider the simplest case of a tree three-pronged string
with two-form charges (1,0), (0,1) and (−1,−1). The magnetic charge of the corresponding field theory solution in SU(3) is \((m_1, m_2) = (1, 1)\). Thus the magnetic part has two fundamental monopoles, and the result \((33)\) tells that there are seven bosonic zero modes. This number looks quite odd, and moreover this does not agree with the IIB result that is just three for the present case \([12]\). However this is a very natural result from the explicit junction solution discussed in Ref. \([6]\). Their solution is composed of two monopole cores which are surrounded by clouds of W-bosons, i.e. electric charges. The monopole part has eight bosonic zero modes which are composed of the zero modes associated to three translations, two gauge, two relative orientations, and one relative distance. The electric part is determined by the monopole part when the vacuum expectation values of the scalar fields take given fixed values. Especially, the electric charges of the junction solution are determined by the relative distance, while the magnetic charges are just topological numbers and stable. Since the electric charges appear in the asymptotic \(1/r\) behavior of the junction solution, the change of the relative distance is not normalizable. Thus we should keep the relative distance fixed. The other monopole zero modes are naturally expected to be normalizable, and hence there are seven bosonic zero modes in the junction solution.

On the other hand, we find a nice agreement for the fermionic part. Let us consider a monopole configuration with magnetic charges \((m_1, m_2, \cdots, m_{\tilde{N}−1})\). The corresponding string configuration is such that \(\tilde{N}\) D3-branes are aligned on a line and that the \(a\)-th and the \((a+1)\)-th D3-brane are connected by \(m_a\) D-strings \((a = 1, \cdots, \tilde{N}−1)\). This configuration may be regarded as the string configuration corresponding to a junction solution in the limit of vanishing electric charges. To recover from the limit, let us now add small NS-NS charges on each string. We do not take care of the quantization of the NS-NS charges, since our treatment in the field theory is just classical and does not care about the quantization of the electric charges. Then the configuration of the string is deformed, for example, to the one in FIG. 3 where there are \(m_a−1\) internal loops between the \(a\)-th and \((a+1)\)-th D3-branes. Although the diagram changes if the assignment of the small NS-NS charges is changed, the number of loops do not change. The total number of the loops in the diagram is given by

\[
\tilde{N}−1 \sum_{a=1}^{\tilde{N}−1} (m_a − 1) = \#\text{monopole} − (\tilde{N}−1).
\]

\hspace{1cm} (47)

\[\text{FIG. 2. The configuration of pronged strings ending on D3-branes which corresponds to our field theory analysis. The long strings have R-R charge one and small NS-NS charges. The figure is for magnetic charges } (m_1, m_2, m_3) = (1, 3, 2) \text{ in } SU(4) \text{ case.} \]

The zero mode analysis in the IIB framework was done in Ref. \([12]\). The result is

\[
\#FZM(IIB) = 8F_{\text{int}} + 4E_{\text{ext}},
\]

\hspace{1cm} (48)
where \( F_{\text{int}} \) denotes the number of internal loops (faces) of the string diagram and \( E_{\text{ext}} \) is the number of the external strings. Thus, applying to the present case, we obtain

\[
\#FZM(\text{IIB}) = 8 \left( \#\text{monopole} - (\tilde{N} - 1) \right) + 4\tilde{N} = \#\text{monopole} \times 8 - 4\tilde{N} + 8, \quad (49)
\]

which agrees with Eq. (46).

Finally, the discrepancy of the bosonic zero modes from the IIB string picture might be understood as follows. The expression (33) of the bosonic zero modes can be written as

\[
\#BZM = \left( 2(\#\text{monopole} - 1) + \#\text{monopole} \right) + \#BZM(\text{IIB}), \quad (50)
\]

where the last term is the IIB result [12].

\[
\#BZM(\text{IIB}) = F_{\text{int}} + 3. \quad (51)
\]

It is intriguing to note that the discrepancy (50) agrees with the number of the compact directions of the monopole moduli space, i.e. there are 2(\#\text{monopole} − 1) relative spatial orientations among the fundamental monopoles and one \( U(1) \) gauge direction per each fundamental monopole. It might be expected that, when the junction BPS state is treated quantum mechanically, the wave function prevails the compact directions and these directions do not appear as the moduli of the state.

V. CONCLUSION

In this paper, we nonperturbatively identified the numbers of the bosonic and fermionic zero modes of the multi-pronged strings in the context of the \( \mathcal{N} = 4 \) super-Yang-Mills theory. The bosonic zero modes differ from the IIB string picture, but the fermionic zero modes are matching with those in the IIB string picture.

The discrepancy is due to the softness of the field-theoretic configurations. Namely the monopoles of the multi-pronged strings in the field theory can take a relative motion in the parallel space of the D3-branes, whereas the corresponding degrees in the IIB picture cannot be permitted. In the case of the minimal three-pronged strings, the number of bosonic zero modes are seven while there are twelve fermionic zero modes. On the ground of the remaining supersymmetries of the system, the natural number of bosonic degrees would be even due to the complex structure of the remaining supersymmetry. We expect that the analysis of detailed moduli dynamics may be helpful in resolving this issue. The comparison with the M-theory result [13] or the D-string worldsheet approach [14] would also be interesting.

The dynamics of the moduli space is in itself of importance, especially in related with the quantizations of the electric charges. The supersymmetric quantum mechanics of the moduli space has been constructed in case of monopoles [15, 16]. Our work can be used in the identification of the supersymmetric quantum mechanics for the multi-pronged strings. As is also done for the monopoles and dyons [17], the response analysis of the multi-pronged strings to the excitations of unbroken gauge fields will clarify most of leading physical processes around the multi-pronged strings. These require further studies.
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APPENDIX A: THE NORMALIZABILITY CONDITION

In this appendix, we will show that the condition of getting a normalizable \( \delta A_0 \) from Eq. (21) is given by Eq. (30).

To simplify the expressions, we take a gauge where the vacuum expectation value of \( A_4 \) is expressed in a diagonal form. We assume also that the diagonal entries take general distinct values. Then, since the massless fields are associated only to the diagonal entries, the asymptotic behavior of the solutions to the equation \( D_m D_m \Lambda_a = 0 \) should take the form

\[
\Lambda_a = h_a^{(0)} + \frac{h_a^{(1)}}{r} + O\left(\frac{1}{r^2}\right),
\]

where \( h_a^{(0,1)} \) are diagonal matrices. We assume that there are \( \tilde{N} - 1 \) solutions to this equation \( (a = 1, \cdots, \tilde{N} - 1) \), and that the \( h_a^{(0)} \) span the Cartan subalgebra. Note that \( \Lambda_a \) includes \( A_0 \) and \( A_4 \), and we denote \( \Lambda_1 = A_0 \) and \( \Lambda_2 = A_4 \). The diagonal entries of the \( h_1^{(1)} \) are the electric charges of the junction solution, while those of \( h_2^{(1)} \) are the magnetic charges. Since the vacuum expectation values are fixed, the infinitesimal changes caused by the changes of the moduli parameters of a monopole should be in the form

\[
\delta A_0 = \frac{\delta h_1^{(1)}}{r} + O\left(\frac{1}{r^2}\right), \quad \delta A_4 = \frac{\delta h_2^{(1)}}{r} + O\left(\frac{1}{r^2}\right).
\]

Since the magnetic charges are topological and do not change, \( \delta A_m = O(1/r^2) \). We define generalized electric central charges by

\[
Q^E_{\Lambda_a} \equiv \int_{r=\infty} dS_i \text{Tr}(\Lambda_a E_i) = \int_{r=\infty} dS_i \text{Tr}(\Lambda_a D_i A_0) = \text{Tr} \left( h_a^{(0)} h_1^{(1)} \right).
\]

Under the assumption that the \( h_a^{(0)} \) span the Cartan subalgebra, the invariance of the electric charges is equivalent to the invariance of these central charges. Since \( h_a^{(0)} \) is fixed in the zero-mode analysis, the deformation of these central charges by the presence of the zero modes is

\[
\delta Q^E_{\Lambda_a} = \int_{r=\infty} dS_i \text{Tr}[\Lambda_a \delta(D_i A_0)] = \int d^3 x \text{Tr} \left[ D_m \Lambda_a \delta(D_m A_0) + \Lambda_a D_m \delta(D_m A_0) \right].
\]

Using the above asymptotic behaviors together with a little further manipulation of Eq. (A4), one obtains

\[
\delta Q^E_{\Lambda_a} = \int d^3 x \text{Tr}[\Lambda_a D_m D_m \delta A_0] = -2i \int d^3 x \text{Tr} \left( \Lambda_a [D_m A_0, \delta A_m] \right).
\]

\[\text{[5]}\] We cannot take this gauge globally. In the following discussions, we just need to take this gauge for a certain solid angle less than \( 4\pi \) outside a sphere of sufficiently large radius, since the solid angle can be chosen arbitrary.

\[\text{[6]}\] This is explicitly shown for the solutions in [5].
where we have used Eq. (21) for the last equality. Similarly, defining

\[ Q_M^\Lambda_a \equiv \int_{r=\infty} dS_i \text{Tr}(\Lambda_a B_i) = \int_{r=\infty} dS_i \text{Tr}(\Lambda_a D_i A_4) = \text{Tr} \left( h^{(0)}_a h^{(1)}_2 \right), \quad (A6) \]

the deformation by the zero mode can be expressed as

\[ \delta Q_M^\Lambda_a = -2i \int d^3x \text{Tr} \left( \Lambda_a [D_m A_4, \delta A_m] \right), \quad (A7) \]

which is in fact automatically vanishing due to \( \delta Q_M = 0 \) or \( \delta h^{(1)}_2 = 0 \). Thus, recalling the fact that there are \( \tilde{N} - 1 \) degrees of freedom of the electric charges and from Eq. (21), the equation (21) is equivalent to the condition that the electric charges do not change under the infinitesimally small changes of the monopole moduli. Since the electric charges appear in the asymptotic \( 1/r \) behavior of \( \delta A_0 \), this is a necessary condition for the infinitesimal change to be normalizable under the measure \( \int d^3x \text{Tr}(\delta A_0^2) \). This necessary condition becomes a sufficient condition if the order next to \( 1/r \) is \( 1/r^2 \) as in the expansion (A1) and (A2). In the spherically symmetric solution discussed in Refs. [5,7,8], the next order is exponentially damping. In general non-spherical cases [6], the next order is expected to behave as \( 1/r^2 \) from dipole contributions.

There is another way to see the condition (30). Since the operator \(-D_m D_m\) is a semi positive definite Hermitian operator, one may expand \( \delta A_0 \) in terms of the eigenfunctions. The equation (21) is now

\[ PC(P; \Omega) - 2i \int d^3x \text{Tr}(f(P; \Omega)\dagger[D_m A_0, \delta A_m]) = 0, \quad (A8) \]

where \( f(P; \Omega) \) denotes the eigenfunction with eigenvalue \( P \) with \( \Omega \) parameterizing the degeneracies of the eigenfunctions with the same eigenvalue, and \( \delta A_0 \) is expanded as \( \sum_{P,\Omega} C(P; \Omega)f(P; \Omega) \). If the second term of Eq. (A8) is non-zero at \( P = 0 \), the \( C(P; \Omega) \) will behave in \( 1/P \) near \( P = 0 \) (We assume that the eigenvalues of the operator \(-D_m D_m\) exist continuously around \( P = 0 \)). This behavior may violate the normalizability. Therefore the condition that the second term vanishes for \( P \to 0 \) is related to the normalizability. This is the condition (B0). But to conclude this we need the measure near \( P = 0 \) from more knowledge on the spectrum of the eigenvalues and the eigenvectors.
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