ON A CHARACTERIZATION OF SPACES SATISFYING OPEN MAPPING AND EQUIVALENT THEOREMS

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Abstract. For classes of topological vector spaces, we analyze under which conditions open-mapping, bounded-inverse, and closed-graph properties are equivalent. We show that closure under quotients with closed subspaces and closure under closed graphs are sufficient.

We show that the class of barreled Pták spaces is exactly the largest class of locally-convex topological vector spaces, which contains all Banach spaces, is closed under quotients with closed subspaces, is closed under closed graphs, is closed under continuous images, and for which an open-mapping theorem, a bounded-inverse theorem, and a closed-graph theorem holds.

1. Introduction

This is a short paper in the field of topological vector spaces (t.v.s.), concerned with theorems on open mappings, bounded inverses, and closed graphs. These theorems have a long history, with many applications in different branches of functional analysis, e.g., [Wer97, Mat98, Alt06, AV05]. Initially only formulated for Banach spaces, one line of research was to extend these theorems to very general classes of spaces, e.g., see [Sch71, Pták58, Hus62] and the references therein. While this research states closed-mapping theorems for e.g., linear mappings $u : E \to F$ with $E$ taken from one class $A$ of t.v.s. and $F$ taken from a possibly different class $B$, e.g., see [HM62, Kri71], we approach the topic differently. We only allow $E$ and $F$ to come from the very same class of t.v.s. $C$, and we ask, under which conditions on $C$, the open-mapping theorem, the bounded-inverse theorem, and the closed-graph theorem are actually equivalent and hold. For the equivalence of these theorems for a class $C$, the crucial insight is that $C$ needs closure properties weaker than expected. Besides closure under quotients under closed subspaces, additionally, only closure under closed graphs is needed, not closure under closed finite products or closed subspaces. This insight leads to a characterization result, showing that the class of barreled Pták spaces is the natural habitat of these theorems, and that at least for locally-convex spaces, the barrier of being barreled and Pták cannot be overcome. As research in the 1960s considered Pták and barreled spaces already, this paper thus may explain, why research on these topics faded out in the 1970s.

2. Equivalences

In this work, we use notation, definitions, and results from the excellent textbook of Schaefer [Sch71]. Throughout, w.l.o.g. we assume that all topological vector spaces (t.v.s.) are $T_0$. Hence, they are fully regular. In particular, they are $T_2$.

Recall that a map $u : E \to F$ is called closed if the set $\text{Graph}(u) = \{(e, u(e)) \mid e \in E\}$ is a closed subset of $E \times F$.

We define three properties for a class $C$ of t.v.s.
(O) **Open-mapping property:** For every pair of t.v.s. $E$ and $F$ in $C$ it holds that every surjective, linear, continuous map $u : E \to F$ is open.

(B) **Bounded-inverse property:** For every pair of t.v.s. $E$ and $F$ in $C$ it holds that every bijective, linear map $u : E \to F$ is continuous if its inverse $u^{-1}$ is continuous.

(C) **Closed-graph property:** For every pair of t.v.s. $E$ and $F$ in $C$ it holds that every linear map $u : E \to F$ is closed if it is continuous.

We say that a class $C$ of t.v.s. is **closed under closed graphs**, if for every $E$ and $F$ in $C$ and every linear, closed map $u : E \to F$ its graph $\text{Graph}(u)$ is in $C$. Furthermore, we say that a class $C$ of t.v.s. has the **OBC-equivalence property**, if it is closed under quotients with closed subspaces (with $E$ in $C$ and $A$ a closed subspace of $E$, the quotient space $E/A$ is in $C$), and if it is closed under closed graphs.

**Theorem 1.** Let $C$ be a class of (T$_0$) t.v.s. satisfying the OBC-equivalence property. Then properties (O), (B), and (C) are equivalent for $C$.

The following arguments in the proof of the above theorem are well-known and thus not new. Presenting them needs justification. We give three reasons: (1) emphasis on where exactly the closure-properties of the class $C$ are needed, (2) first-time crystal-clear presentation of these equivalences in this general setting, not found in textbooks in functional analysis, and (3) for the sake of completeness.

**Proof.** (O) implies (C): Let $E$ and $F$ be t.v.s. in $C$, and let $u : E \to F$ be bijective, linear, and continuous. By (O), $u$ is open. Hence, $u^{-1}$ is continuous. Analogously, argue for $u^{-1}$.

(C) implies (O): Let $E$ and $F$ be t.v.s. in $C$, and let $u : E \to F$ be surjective, linear, and continuous. Subspace $N = u^{-1}(0)$ is closed by continuity of $u$. As $C$ is closed by quotients with closed subspaces, $E/N$ is in $C$. The induced map $u_0 : E/N \to F$ is bijective and continuous. By (C), $u_0^{-1}$ is continuous. Hence, $u_0$ is open. Then finally, the map $u = p \circ u_0$ is open as composition of open maps, where $p : E \to E/N$ denotes the linear, continuous, and open projection.

(B) implies (C): Let $E$ and $F$ be t.v.s. in $C$, and let $u: E \to F$ be linear. Define the bijective, linear map $v: E \to \text{Graph}(u)$ by $v(e) = (e, u(e))$. Let $p_F$ and $p_E$ denote the linear, continuous projections from $E \times F$, respectively. If $u$ is continuous, then by Prop. [1] below, $\text{Graph}(u)$ is closed. And if $\text{Graph}(u)$ is closed, then it is in $C$ by closure under closed graphs. As $v^{-1} = p_F : \text{Graph}(u) \to E$ is bijective, linear, and continuous, the map $v$ is continuous by application of (B).

(C) implies (B): Let $E$ and $F$ be t.v.s. in $C$. Define $s : E \times F \to F \times E$ by $s(x, y) = (y, x)$. Clearly, $s$ is a topological isomorphism. Let $u : E \to F$ be bijective and linear. By (C), the map $u$ is continuous iff $\text{Graph}(u)$ is closed. This holds iff $\text{Graph}(u^{-1}) = s(\text{Graph}(u))$ is closed. Again by (C), the former holds iff $u^{-1}$ is continuous.

As we could only find proofs of the following proposition in the context of Banach spaces, we give a proof in full generality for the sake of completeness.

**Proposition 1.** If a map between topological $T_2$ spaces is continuous, then it is closed.

**Proof.** Let $E$ and $F$ be topological $T_2$ spaces, and let $u : E \to F$ be a continuous map. Define map $v : E \times F \to E \times F$ by $v(e, y) = (e, u(e))$. Then $v$ is continuous and $v(E \times F) = \text{Graph}(u)$. Consider an arbitrary point $(e, f)$ in the closure $\text{Graph}(u)$. Then there exists a filter $C$ containing $\text{Graph}(u)$ and converging to $(e, f)$. By continuity of $v$, the image filter $v(C)$ converges to $v(e, f) = (e, u(e))$. As $E \times F$ is in $C$, we have $\text{Graph}(u)$ in $v(C)$. The set of intersections of sets from $C$ and $v(C)$
Closure under finite products or closure under closed subspaces is not necessary. Above theorem is that the weaker property of closure under closed graphs suffices. Spaces (l.c.s.), complete metrizable t.v.s. (Fréchet), Banach spaces, and nuclear spaces, respectively, see [Sch71, IV.8.2, IV.8.3 Cor. 3].

It is well-known that the classes (all assumed T₀) of complete locally-convex spaces (l.c.s.), complete metrizable t.v.s. (Frechet), Banach spaces, and nuclear spaces all satisfy the OBC-equivalence property.

In contrast, it is unclear if subclasses of barreled spaces, Pták spaces, or Baire spaces satisfy the property of OBC-equivalence, because in general, barreled spaces and Baire spaces are not closed under closed subspaces, and Pták spaces are not closed under finite products. At least, barreled spaces are closed under finite products and quotients with closed subspaces, see [Sch71, II.7.1 comment and Cor. 1], and Pták spaces are closed under closed subspaces and quotients with closed subspaces, respectively, see [Sch71] IV.8.2, IV.8.3 Cor. 3].

3. Characterization

Recall that a linear map $e: E \to F$ is called nearly-open, if for each 0-neighborhood $U \subseteq E$, $u(U)$ is dense in some 0-neighborhood in $u(E)$.

We say that a class $C$ of t.v.s. is closed under continuous images, if for every $E$ in $C$, every l.c.s. $F$, and every injective, linear, continuous, and nearly-open map $u: E \to F$, its image $u(E)$ is in $C$.

**Proposition 2.** The classes of barreled Pták spaces and Banach spaces are closed under continuous images.

**Proof.** Let $F$ be an arbitrary l.c.s., and let $u: E \to F$ be an arbitrary injective, linear, continuous, and nearly-open map. Space $u(E)$ is l.c.s. as a subspace of $F$.

If $E$ is a Banach space, then it is a Fréchet space, and thus a Pták space by the theorem of Krein-Šmulian, see [Sch71] IV.6.4, Thm.]. By [Sch71] IV.8.3, Thm.], its image $u(E)$ is isomorphic to the Banach space $E$ itself. Hence, $u(E)$ is a Banach space.

If $E$ is a barreled Pták space, then $u(E)$ is a Pták space by [Sch71] IV.8.3, Cor. 2]. Let $B$ be an arbitrary Banach space, and let $v: u(E) \to B$ be an arbitrary linear and closed map. Then the composition map $v \circ u: E \to B$ is linear and closed, the latter because $u: E \to u(E)$ is an isomorphism. As $E$ is barreled, $B$ is $r$-complete, and $v \circ u$ is closed, map $v \circ u$ is continuous by the Thm. of Robertson-Robertson, [Sch71] IV.8.5, Thm.]. Hence, $v = (v \circ u) \circ u^{-1}$ is continuous. Space $u(E)$ is barreled by the Thm. of Mahowald, [Sch71] IV.8.6]. Thus, $u(E)$ is a barreled Pták space. □

**Proposition 3.** The classes of barreled Pták spaces and Banach spaces are closed under closed graphs.

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1For complete l.c.s.: finite products, [Sch71] II.5.2]; closed subspaces, [Sch71] I.2.1, II.6.1]; quotients under closed subspaces, [Sch71] I.2.3, II.6.1]. For complete metrizable spaces: finite products, [Sch71] I.2 Ex. 1(b), [Bou95] Ch. II, §3.5, §3.9]; closed subspaces, [Sch71] I.2.1], [Bou95] Ch. II, §3.4, §3.9]; quotients under closed subspaces, [Sch71] I.2.3, I.6.3]. For Banach spaces: finite products, [Sch71] II.2.2], [Bou95] Ch. IX, §3.4]; closed subspaces, trivial; quotients under closed subspaces, [Sch71] I.2.3, II.2.3], [Bou95] Ch. IX, §3.4]. For nuclear spaces, see [Sch71] III.7.4]
Proof. This holds for Banach spaces, because Banach spaces are closed under finite products and closed subspaces.

Let $E$ and $F$ be arbitrary barreled Pták spaces, and let $u: E \to F$ be an arbitrary linear and closed map. By the theorem of Robertson-Robertson, $\text{Graph}(u)$ is an l.c.s. as a closed subset of $E \times F$. Define the bijective and continuous map $v: E \to \text{Graph}(u)$ by $v(e) = (e, u(e))$. The map $v$ is open and thus nearly-open, because its inverse $v^{-1} = p_E: \text{Graph}(u) \to E$ is continuous. Now, the statement follows from Prop. 2. □

Main Theorem. The class of barreled Pták spaces is exactly the largest ($T_0$) class of l.c.s., which contains all Banach spaces, is closed under quotients with closed subspaces, is closed under closed graphs, is closed under continuous images, and for which an open-mapping theorem (O), a bounded-inverse theorem (B), or a closed-graph theorem (C) holds (and thus all of them).

Proof. First of all, the classes of Banach spaces and of barreled Pták spaces both have the mentioned closure properties: they contain all Banach spaces, are closed under quotients with closed subspaces, are closed under closed graphs (Prop. 3), and are closed under continuous images (Prop. 2). It is well-known that property (O) holds for Banach spaces, and it also holds for barreled Pták spaces by [Sch71, IV.8.3, Cor. 1]. Consequently, for both of these classes, properties (O), (B) and (C) are equivalent (Thm. 1) and hold.

Let $C$ be a maximal class of l.c.s. satisfying the assumed closure properties of the theorem. First of all, $C$ satisfies all properties (O), (B), and (C), because it satisfies OBC-equivalence. Let $E$ be an arbitrary l.c.s. in $C$.

We want to show that $E$ is barreled. Let $B$ be an arbitrary Banach space. We have $B$ in $C$. Let $u: E \to B$ be an arbitrary linear, closed map. By (C), $u$ is continuous. Then by the theorem of Mahowald, $E$ is barreled.

We want to show that $E$ is a Pták space. Let $F$ be an arbitrary l.c.s., and let $u: E \to F$ be an arbitrary linear, continuous, and nearly-open map. Subspace $N = u^{-1}(0)$ is closed, because $u$ is continuous. Hence, $E/N$ is in $C$ by closure under quotients with closed subspaces. The map $u_0: E/N \to F$, associated with $u$, is injective, linear, continuous, and nearly-open. Thus, image $u(E)$ is in $C$ by closure under continuous images. Applying (B) to bijective and continuous map $u_0: E/N \to u(E)$ yields that $u_0$ is open. Hence, $u$ is open. By [Sch71, IV.8.3, Thm.], $E$ is a Pták space.

Consequently, every space in $C$ is a barreled Pták space. Finally, $C$ must equal the class of barreled Pták spaces by maximality. □

Acknowledgements

Unfortunately, we cannot proceed here in the usual fashion, showing gratitude by thanking a researcher for careful proof-reading or helpful comments. We found it difficult to find a person willing to review this paper and being knowledgeable in the field of l.c.s. (As mentioned before, the paper may explain a bit, why not much research is done in this field nowadays.) Thus, this paper has not been reviewed yet. In addition, this is the first paper of the author in this field, probably with weaknesses at least in the style common to this field. Consequently, the paper should be seen as preliminary and improvable. We encourage the reader to give us feedback. Any help is appreciated very much!

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