CATEGORIES OF ORBIT TYPES FOR PROPER LIE
GROUPOIDS

JACK MORAVA

Abstract. It is widely understood that the quotient space of a topo-
logical group action can have a complicated combinatorial structure,
indexed somehow by the isotropy groups of the action [3 II §2.8]; but
how best to record this structure seems unclear. This sketch defines a
database category of orbit types for a proper Lie groupoid (based on
recent work [13-15] with roots in the theory of geometric quantization)
as an attempt to capture some of this information.

1. Introduction and background

A topological groupoid or stack [12]

\[ X := s, t : X_1 \xrightarrow{\approx} X_0 \]

is proper if the map \( s \times t : X_1 \to X_0 \times X_0 \) is proper; such an object in the
category of smooth manifolds and maps is a proper Lie groupoid. The
quotient

\[ X \to \mathcal{X} \]

of \( X_0 \) by the equivalence relation thus defined is a Hausdorff topological
space, sometimes called the coarse moduli space of \( X \).

Examples:

- Orbifolds [5]
- A topological transformation group, defined by a group action

\[ G \times X \to X \]

has an associated topological groupoid \([X/G]\) with \( X_0 = X, X_1 = G \times X \);
I'll write \( X/G \) for its quotient space. For instance

- Toric varieties, eg \( G = \mathbb{T}^{n+1}/\mathbb{T} \cong \mathbb{T}^n \) acting on \( X = \mathbb{C}P^n \) by

\[ (u_0, \ldots, u_n) \cdot [z_0 : \cdots : z_n] = [u_0z_0 : \cdots : u_nz_n], \]

form a particularly accessible class of examples. Their quotient objects
are polytopes: in the case above \( X/G \cong \Delta^n \) is a simplex. The faces of
the polytope define a stratification [see §4.1 below] of the quotient, with the
interiors of the faces as strata. This defines an interesting poset, or category, associated to the groupoid: in this example it is the category of subsets of \{0, \ldots, n\} under inclusion.

An earlier paper [11] attempted to capture the sort of information encoded by the face poset of a toric variety, for more general group actions. The present note uses recent work on proper Lie groupoids [16], cf also [1] to propose a more general construction.

Acknowledgement I am indebted to the organizers of the September 2013 Barcelona conference on homotopy type theory [2] for inspiration and hospitality, and for the opportunity to pursue these questions. I hope I will not be misunderstood by suggesting that classification problems of the sort considered here have a deep and nontrivial history in philosophy [17].

2. Some technical preliminaries

2.1 Definition A reasonable space \(X\) has a universal map \(X \to \pi_0 X\) to a discrete set, defined by the adjoint to the inclusion of the category of sets into that of topological spaces. The diagram

\[
\begin{array}{ccc}
X & \xrightarrow{X_1} & X_0 \\
\downarrow & & \downarrow \\
\pi_0 X & \xrightarrow{\pi_0 X_1} & \pi_0 X_0 \\
\end{array}
\]

extends \(\pi_0\) to a functor from topological to discrete groupoids, such that

\[
\pi_0[X/G] \cong [\pi_0(X)/\pi_0(G)],
\]

with \(\pi_0(X)/\pi_0(G) \cong \pi_0(X/G)\) for reasonable actions. The natural transformation

\[
\pi_0(X \times Y) \to \pi_0 X \times \pi_0 Y
\]

is an isomorphism in such cases.

2.2 Regarding groups as categories with a single object defines a two-category \((\text{Gps})\) of groups. The set of homomorphisms from \(G_0\) to \(G_1\) has an action of \(G_1\) by conjugation, defining a groupoid

\[
\text{Hom}_{\text{Gps}}(G_0, G_1) := [\text{Hom}(G_0, G_1)/G_1^{\text{conj}}]
\]

of morphisms from \(G_0\) to \(G_1\).

There are many variations on this theme, eg the topological two-category \((\text{Gps}_c)\) defined by compact groups and continuous homomorphisms. I will

\[\text{See [8. 21] for approaches based on } \pi_1 \text{ rather than } \pi_0\]
write \((\text{Gps}^+)\) (resp. \((\text{Gps}_c^+)\)) for the subcategories with such groups as objects, and spaces \(\text{Hom}_c^+(G_0, G_1)\) of continuous \textbf{one-to-one} homomorphisms as maps.

The construction which is the identity on objects, and is the functor
\[
[\text{Hom}_c(G_0, G_1)/G^\text{conj}_1] \to \text{Hom}_{\pi_0\text{Gps}}(G_0, G_1) := \pi_0[\text{Hom}_c(G_0, G_1)/G^\text{conj}_1]
\]
on morphism categories, defines a monoidal two-functor
\[
(\text{Gps}_c) \to (\pi_0\text{Gps}_c)
\]
[and similarly for \((\text{Gps}_c^+)\)].

3. Groupoids of fixed-points with level structure

3.1 Definition: If \(X\) is a proper topological groupoid, and \(H\) is a compact Lie group, let
\[
X(H)_0 := \{(x, \phi) \mid x \in X_0, \phi : H \to \text{Iso}(x) \in \text{Gps}_c^+\}
\]
and let \(X(H)_1\) be the set of commutative diagrams of the form
\[
\begin{array}{ccc}
H & \xrightarrow{\phi} & \text{Iso}(x') \\
\downarrow{\gamma} & & \downarrow{g-\text{conj}} \\
H & \xrightarrow{\phi} & \text{Iso}(x)
\end{array}
\]
(with \(g : x' \to x \in X_1\)). The resulting proper topological groupoid \(X(H)\) is a model for the subgroupoid of \(X\) defined by points fixed by a group isomorphic to \(H\). There is a forgetful morphism \(X(H) \to X\), but it can’t be expected to be the inclusion of a subgroupoid.

Proposition: \(H \mapsto X(H)\) defines a (two-)functor \(X(\bullet)\) from \((\text{Gps}_c^+)\) to the two-category \((\text{Gpoids}_c)\) of proper topological groupoids.

Proof: First of all, if \(\alpha : H_0 \xrightarrow{\alpha} H_1 \in (\text{Gps}_c^+)\) then
\[
\begin{array}{ccc}
H_0 & \xrightarrow{\alpha} & H_1 \\
\downarrow{\gamma^\alpha} & & \downarrow{g-\text{conj}} \\
H_0 & \xrightarrow{\phi_1} & \text{Iso}(x)
\end{array}
\]
defines a functor
\[
\alpha^{H_0}_{H_1} : X(H_1) \to X(H_0)
\]
Moreover, if \(\alpha : H \to H\) is an inner automorphism of \(H\) (ie \(\alpha\) is conjugation by \(a \in H\)) then there is a natural equivalence
\[
\alpha^H_H \cong 1_{X(H)}
\]
defined by the commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\phi} & \text{Iso}(x) \\
\downarrow & & \downarrow \\
H & \xrightarrow{\phi(a^{-1})} & \text{Iso}(x)
\end{array}
\]

\[1_H H \rightarrow H \rightarrow H \rightarrow \phi \rightarrow \text{Iso}(x).
\]

\[
\square
\]

3.2 Claim: For any \(X\) as above, there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Iso}} & (\text{Gpoids}_{c^*}) \\
\downarrow & & \downarrow \\
(\text{Gps}_c) & \xrightarrow{X[\cdot]} & (\text{Gpoids}_c)
\end{array}
\]

(with the category of pointed proper groupoids in the upper right corner, and the forgetful map to the category of proper groupoids along the right-hand edge). The left-hand vertical map sends \(x \in X_0\) to its isotropy group, and the top horizontal map sends \(x\) to \(X(\text{Iso}(x))\), with \(x\) as distinguished point.

**Corollary** The universal property of a fiber product defines a continuous functor

\[
X \rightarrow \Phi_0(X)
\]

to the category defined by the pullback

\[
\begin{array}{ccc}
\Phi_0(X) & \xrightarrow{\pi_0} & (\text{Gpoids}_c) \\
\downarrow & & \downarrow \\
(\text{Gps}_c) & \xrightarrow{X[\cdot]} & (\text{Gpoids}_c)
\end{array}
\]

(where the two right vertical arrows are the obvious forgetful functors). \(\square\)

It’s natural to think of \(\Phi_0(X)\) as a database category [11, 18]. However I don’t know how to characterize \(\Phi_0\) by some universal property (such as being an adjoint).

3.3 Example: A proper transformation group \([X/G]\) defines a functor

\[
G > H \mapsto X^H = \{x \in X \mid \text{Iso}(x) \subset H\}
\]

from the topological category \((G - \text{Orb})\) [with closed subgroups of \(G\) as objects, and

\[
\text{Mor}_{G-\text{Orb}}(H_0, H_1) = \text{Maps}_G(G/H_0, G/H_1) = \{g \in G \mid gH_0g^{-1} \subset H_1\}/H_1^{\text{conj}}
\]

as morphism objects [3 I §10], to spaces.
This extends to a functor
\[ S^0[X^*] : (G - \text{Spaces}) \ni X \mapsto S^0[X^H] \in \text{Func}(G - \text{Orb}, S^0 - \text{Mod}) \]
which provides a model \( [6 \text{ V } \S 9, 10, 20] \) for the \( G \)-equivariant stable category in terms of sheaves of spectra (ie \( S^0 \)-modules) over \( (G - \text{Orb}) \).

The commutative diagram
\[
\begin{array}{c}
[X/G] \xrightarrow{\text{Iso}} \text{(Sets)}_* \\
\downarrow \quad \downarrow \\
(G - \text{Orb})_{\pi_0X^*} \xrightarrow{\Phi_0(X)} \text{(Sets)}
\end{array}
\]

defines a functor from \([X/G]\) to a fiber product category \( \Phi_0[X/G] \) (with objects, pairs consisting of a subgroup \( H \) of \( G \), and a component of \( X^H \)), analogous to the construction in the previous paragraph \( [11 \S 2.2] \). The sheaf \( S^\infty X^* \) of spectra pulls back to a sheaf of spectra over \( \Phi_0(X) \).

One might hope for an unstable version of this construction, applicable in the theory of \( \infty \)-categories (cf eg \( [7 \S 5.5.6.18] \)); but because it depends on a presentation of \([X/G]\) as a global quotient, it does not seem to be homotopy-invariant.

4. Proper Lie groupoids, after Pflaum et al

4.1 A stratification \( S \) of a (paracompact, second countable) topological space \( X \) assigns to each \( x \in X \), the germ of a closed subset \( S_x \) (containing \( x \)) of \( X \). With suitably defined morphisms \( [9 \S 1.8, 16 \S 1] \), stratified spaces form a category. A stratification defines a locally finite partition
\[ X = \coprod_{S \in \Sigma(S)} X_S \]
of \( X \) into locally closed subsets (called its strata), such that if \( x \in X_S \) then \( S_x \) is the associated set germ.

Very interesting recent work of M. Pflaum et al [building on earlier work of Weinstein and Zung ([22]; cf also [13]) shows that

Theorem [16 Theorem 5.3, Cor 5.4] The quotient space \( X \) of a proper Lie groupoid \( X \) has a canonical Whitney stratification. The associated decomposition of \( X_0 \) into locally closed submanifolds
\[ X_{0(H)} = \{ x \in X_0 \mid \text{Iso}(x) \cong H \} \]
is indexed [16 Theorem 5.7] by (isomorphism classes of) compact Lie groups \( H \).
4.2 Definition The orbit groupoid $O(x) \subset X$ of $x \in X_0$

$$O_0(x) = \{y \in X_0 \mid \exists g : y \to x \in X_1\}$$

$$O_1(x) = \{g \in X_1 \mid s(g), t(g) \in O_0(x)\}$$

reduces, in the case of a transformation groupoid $[X/G]$, to the groupoid

$$[(G/Iso(x))/G] = [*/Iso(x)]$$

A slice at $x \in X_0$ is (very roughly [13 §3.3-4.3.8-9]) the germ of an $Iso(x)$-invariant submanifold of $X_0$ containing $x$, transverse to $O_0(x)$; for a transformation group it is something like the image of an exponential map

$$[N_x/Iso(x)] = [(N_x \times_{Iso(x)} G)/G] \to [X/G]$$

(where $N_x \in (Iso(x) - \mathrm{Mod})$ is the linear representation

$$0 \to T_xG \to T_xX_0 \to N_x \to 0$$

defining the normal bundle to the orbit of $x$).

**Theorem** [16 §3.11] There is an (essentially unique) slice at every object of a proper Lie groupoid $X$; the corresponding set germs define the canonical stratification [16 §5.4] of $X$.

**Definition** The normal orbit type of $x \in X_0$ is the equivalence class of its normal $Iso(x)$-representation $N_x$. More precisely, $x_0 \sim x_1$ if there are isomorphisms

$$\phi : Iso(x_0) \to Iso(x_1), \Phi : N_{x_0} \to \phi^*(N_{x_1})$$

of groups and representations. The connected components $\nu \in \pi_0(X_0)$ of the normal orbit types of $X$ are [16 §5.7] the strata of the canonical partition of $X_0$.

The condition of the frontier [16 Prop 5.15] asserts that if $\nu' \cap \vartheta \neq \emptyset$ then $\vartheta \supset \nu'$. This implies the existence of a partial order $(\nu > \nu')$ on the set $\Sigma(X)$ of connected components of normal orbit types for $X$, which can thus be regarded as the objects of a category [4 II §2.8]. Thus

$$X_{0(K)} = \bigcup_{\pi_0(X_0(K)) \vartheta \nu \nu'} \nu'$$

This gives us some control of the functor $\Phi_0$ on proper Lie groupoids:

4.3 Proposition For a proper Lie groupoid $X$, we have isomorphisms

$$\bigcup_{H \subset K} X_{0(K)} \times \text{Hom}^+_c(H, K) \xrightarrow{\cong} X(H)_0$$

$$\bigcup_{H \subset K, x \in X_{0(K)}} O_1(x) \times K \xrightarrow{\cong} X(H)_1$$
and consequently
\[ \bigcup_{H<K} X_K \times \text{Hom}_{\mathfrak{c}}^+(H,K)/K_{\text{conj}} \xrightarrow{\sim} X(H) \]
(where \(X_K \subset X\) is the space of orbits with isotropy group isomorphic to \(K\)).

### 4.4 Closing remarks

i) When \(X = [X/G]\) this all simplifies a little. In particular, since
\[ X^H = \bigcup_{H<K<G} X_K, \]
\(\Phi_0[X/G]\) is essentially just the quotient of \(\Phi_0[X/G]\) which collapses the morphism spaces \(\text{Hom}_{\mathfrak{c}}^+(K,H)/K_{\text{conj}}\).

ii) The subspaces \(X_0(K)\) are disjoint unions of strata \(\nu\) indexed by slice representations
\[ K \to \text{Aut}(N_\nu). \]
The resulting family of vector spaces over \(X(H)\) pulls back to a fibered category
\[ N(X) \to \Phi_0(X). \]
This seems to provide a natural repository for Noether’s theorem (which associates conserved quantities to elements of the Lie algebra of symmetries of states of a physical system) [11 §4.1].

iii) I don’t know how generally one can associate a stratification to a topological groupoid. There are many interesting examples, coming from locally compact groupoids (eg the Thom-Boardman theory of singularities of smooth maps [18]), or from infinite-dimensional examples (Ebin’s category of Riemannian metrics up to diffeomorphism, Vassiliev’s finite-type invariants of immersions, . . . ), where a more general theory would be very interesting. The existence and good behavior of slices seem to be an essential requirement for such a theory.
REFERENCES

1. P Cartier, Groupoïdes de Lie et leurs algébroïdes, Sem. Bourbaki 987 (2007-8), Astérisque 326 (2009) 165 - 196
2. A Collective, Homotopy type theory ..., http://homotopytypetheory.org/book/
3. T tom Dieck, Transformation groups, de Gruyter Studies 8 (1987)
4. JJ Duistermaat, JA Kolk, Lie groups, Universitext. Springer (2000)
5. D Gepner, A Henriques, Homotopy theory of orbispaces, arXiv:math/07019
6. LG Lewis, JP May, M Steinberger, JE McClure, Equivariant stable homotopy theory, Springer LNM 1213 (1986)
7. J Lurie, Higher topos theory, Annals of Math Studies 170 (2009)
8. W Lück, Transformation groups and algebraic K-theory, Springer LNM 1408 (1989)
9. J Mather, Stratifications and mappings, in Dynamical systems 195 - 232, ed MM Peixoto, Academic Press (1973)
10. H Miller, The Burnside bicategory of groupoids, arXiv:1208.2360
11. J Morava, Theories of anything, arXiv:1202.0684
12. B Noohi, Homotopy types of topological stacks, arXiv:0808.3799
13. R Palais, On the existence of slices for actions of non-compact Lie groups, Ann. Math. 73 (1961) 295 - 323
14. M Pflaum, Analytic and geometric study of stratified spaces, Springer LNM 1768 (2001)
15. ——, Smooth structures on stratified spaces, in Quantization of singular symplectic quotients 231 - 258, Prog Math (Birkhäuser) 2001
16. ——, H Posthuma, X. Tang, Geometry of orbit spaces of proper Lie groupoids, arXiv:1101.0180
17. G Scholem, On the Kabbalah and its symbolism, Schocken (1996), re http://en.wikipedia.org/wiki/Sephirot
18. DJ Spivak, R Wisnesky, On The relational foundations Of functorial data migration, arXiv:1212.5303
19. R Thom, Singularities of differentiable mapping (notes by H Levine), in Proceedings of Liverpool Singularities I 1-89, Springer LNM 1920
20. S Schwede, Global homotopy theory, www.math.uni-bonn.de/~schwede/global.pdf
21. J Woolf, The fundamental category of a stratified space, arXiv:0811.2580
22. NT Zung, Proper groupoids and momentum maps ..., Ann Sci École Norm Sup 39 (2006) 841 - 869

THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218
E-mail address: jack@math.jhu.edu