QUADRATIC ISOPERIMETRIC INEQUALITY FOR 7-LOCATED SIMPLICIAL COMPLEXES

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Abstract. We show that 7-located simplicial complexes satisfy a quadratic isoperimetric inequality.

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1. Introduction

Curvature can be expressed both in metric and combinatorial terms. Metrically, one can refer to nonpositively curved (respectively, negatively curved) metric spaces in the sense of Aleksandrov, i.e. by comparing small triangles in the space with triangles in the Euclidean plane (hyperbolic plane). These are the CAT(0) (respectively, CAT(-1)) spaces.

Combinatorially, one looks for local combinatorial conditions implying some global features typical for nonpositively curved metric spaces. A very important combinatorial condition of this type was formulated by Gromov [Gro87] for cubical complexes, i.e. cellular complexes with cells being cubes. Namely, simply connected cubical complexes with links (that can be thought as small spheres around vertices) being flag (respectively, 5-large, i.e. flag-no-square) simplicial complexes carry a canonical CAT(0) (respectively, CAT(-1)) metric. Another important local combinatorial condition is local \(k\)-largeness, introduced independently by Chepoi [Che00] (under the name of bridged complexes), Januszkiewicz-Świątkowski [JS06] and Haglund [Hag03]. A flag simplicial complex is \(locally\ k\text{-large}\) if its links do not contain 'essential' loops of length less than \(k\).

In [Osa13b,CO15,BCC+13,CCHO14] some other curvature conditions are studied – they form a way of unifying CAT(0) cubical and systolic theories. On the other hand, Osajda [Osa15] introduced a local combinatorial condition called \(m\)-location, and used it, for \(m = 8\), to provide a new solution to Thurston’s problem about hyperbolicity of some 3-manifolds. In [Laz15] and [Laz15b] a systematic study of a version of \(m\)-location, suggested in [Osa15], is undertaken. This version is in a sense more natural than the original one (tailored to Thurston’s problem), and neither of them is implied by the other. Roughly, the new \(m\)-location says that essential loops of length at most \(m\) admit filling diagrams with at most one internal vertex. In [Laz15] (Theorem 4.3) it is shown that \(8\)-location is a negative-curvature-type condition. Namely, it is proven that simply connected, 8-located simplicial complexes are Gromov hyperbolic. In [Laz15b] we introduce another combinatorial curvature condition, called the \(5/9\)-condition, and we show that the complexes which fulfill it, are also Gromov hyperbolic.
Isoperimetric inequalities relate the length of closed curves to the infimal area of the discs which they bound. It is well-known that every closed loop of length $L$ in the Euclidean plane bounds a disc whose area is less than $\frac{\pi L^2}{4}$, and this bound is optimal. Thus one has a quadratic isoperimetric inequality for loops in Euclidean space. In contrast, loops in real hyperbolic space satisfy a linear isoperimetric inequality: there is a constant $C$ such that every closed loop of length $L$ in hyperbolic space bounds a disc whose area is less than or equal to $C \cdot L$. It is known that (with a suitable notion of area) a geodesic space $X$ is $\delta$-hyperbolic if and only if loops in $X$ satisfy a linear isoperimetric inequality (see [BH99], chapter III.H, page 417 and page 419). Both 8-located complexes and 5/9-complexes satisfy therefore, under the additional hypothesis of simply connectedness, a linear isoperimetric inequality (see [Laz15], [Laz15b]). For loops in arbitrary CAT(0) spaces, however, there is a quadratic isoperimetric inequality (see [BH99], chapter III.H, page 414).

It is known that cycles in systolic complexes satisfy a quadratic isoperimetric inequality (see [JS06]). In [Els09] explicit constants are provided presenting the optimal estimate on the area of a systolic disc. In systolic complexes the isoperimetric function for 2-spherical cycles (the so called second isoperimetric function) is linear (see [JS06]). In [CCHO14] it is shown that meshed graphs (thus, in particular, weakly modular graphs) satisfy a quadratic isoperimetric inequality.

The purpose of the current paper is to show that for cycles in 7-located complexes, there is a quadratic isoperimetric inequality. We prove that the disc in the diagram associated to a cycle in a simply connected, 7-located complex, is itself 7-located. Then we show that such a disc satisfies a quadratic isoperimetric inequality. To prove this, we use a method introduced in [CCHO14].

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2. Preliminaries

Let $X$ be a simplicial complex. We denote by $X^{(k)}$ the $k$-skeleton of $X$, $0 \leq k < \dim X$. A subcomplex $L$ in $X$ is called full as a subcomplex of $X$ if any simplex of $X$ spanned by a set of vertices in $L$, is a simplex of $L$. For a set $A = \{v_1, ..., v_k\}$ of vertices of $X$, by $\langle A \rangle$ or by $\langle v_1, ..., v_k \rangle$ we denote the span of $A$, i.e. the smallest full subcomplex of $X$ that contains $A$. We write $v \sim v'$ if $\langle v, v' \rangle \in X$ (it can happen that $v = v'$). We write $v \not\sim v'$ if $\langle v, v' \rangle \notin X$. We call $X$ flag if any finite set of vertices which are pairwise connected by edges of $X$, spans a simplex of $X$.

A cycle (loop) $\gamma$ in $X$ is a subcomplex of $X$ isomorphic to a triangulation of $S^1$. A full cycle in $X$ is a cycle that is full as a subcomplex of $X$. A $k$-wheel in $X$ ($v_0; v_1, ..., v_k$) (where $v_i, i \in \{0, ..., k\}$ are vertices of $X$) is a subcomplex of $X$ such that $\langle v_1, ..., v_k \rangle$ is a full cycle and $v_0 \sim v_1, ..., v_k$. The length of $\gamma$ (denoted by $|\gamma|$) is the number of edges in $\gamma$.

We define the metric on the 0-skeleton of $X$ as the number of edges in the shortest 1-skeleton path joining two given vertices and we denote it by $d$. A ball (sphere) $B_i(v, X)$ ($S_i(v, X)$) of radius $i$ around some vertex $v$ is a full subcomplex of $X$ spanned by vertices at distance at most $i$ (at distance $i$) from $v$.

**Definition 2.1.** A simplicial complex is $m$-located if it is flag and every full homotopically trivial loop of length at most $m$ is contained in a 1-ball.
Let σ be a simplex of X. The link of X at σ, denoted X_σ, is the subcomplex of X consisting of all simplices of X which are disjoint from σ and which, together with σ, span a simplex of X.

**Definition 2.2.** A simplicial map \( f : X \to Y \) between simplicial complexes X and Y is a map which sends vertices to vertices, and whenever vertices \( v_0, ..., v_k \in X \) span a simplex σ of X then their images span a simplex τ of Y and we have \( f(σ) = τ \). Therefore a simplicial map is determined by its values on the vertex set of X. A simplicial map is called **nondegenerate** if it is injective on each simplex.

**Definition 2.3.** Let γ be a cycle in X. A **filling diagram** for γ is a simplicial map \( f : D \to X \) where D is a triangulated 2-disc, and \( f|_{∂D} \) maps ∂D isomorphically onto γ. We denote a filling diagram for γ by \((D, f)\) and we say it is:

- **minimal** if D has minimal area i.e. it consists of the least possible number of 2-simplices among filling diagrams for γ;
- **nondegenerate** if f is a nondegenerate map;

**Lemma 2.1.** Let X be a simplicial complex and let γ be a homotopically trivial loop in X. Then:

1. there exists a filling diagram \( (D, f) \) for γ (see [Che00] - Lemma 5.1, [JS06] - Lemma 1.6 and [Pry18] - Theorem 2.7);
2. any minimal filling diagram for γ is simplicial and nondegenerate (see [Che00] - Lemma 5.1, [JS06] - Lemma 1.6, Lemma 1.7 and [Pry18] - Theorem 2.7).

**Lemma 2.2.** Let X be a simplicial complex and let γ be a homotopically trivial loop in X. Let \((D, f)\) be a minimal filling diagram for γ. Then adjacent 2-simplices of D have distinct images under f (see [Che00] - Lemma 5.1).

Let D be a simplicial disc. We denote by C the cycle bounding D and by AreaC the area of D. We denote by \( V_i \) and \( V_b \) the numbers of internal and boundary vertices of D, respectively. Then: \( \text{AreaC} = 2V_i + V_b - 2 = |C| + 2(V_i - 1) \) (Pick’s formula). In particular, the area of a simplicial disc depends only on the numbers of its internal and boundary vertices.

**Definition 2.4.** Given a path \( γ = (v_0, v_1, ..., v_n) \) in a simplicial complex X, one can **tighten** it to a full path \( γ' \) with the same endpoints by repeatedly applying the following operations:

- if \( v_i \) and \( v_j \) are adjacent in X for some \( j > i + 1 \), then remove from the sequence all \( v_k \) where \( i < k < j \);
- if \( v_i \) and \( v_j \) coincide in X for some \( j > i \), then remove from the sequence all \( v_k \) where \( i < k \leq j \).

The tightening of a full loop is the loop itself.

### 3. Quadratic isoperimetric inequality for 7-located complexes

We start with a useful lemma.

**Lemma 3.1.** Let X be a simplicial complex and let γ be a homotopically trivial loop in X. Let \((D, f)\) be a minimal filling diagram for γ. We consider in D an interior vertex v such that \( D_v \leq k, 4 \leq k \leq 7 \). Then the map f is injective on \( X_{f(v)} \).
Proof. Let $D = (v_1, v_2, \ldots, v_k)$, $4 \leq k \leq 7$. Because $(D, f)$ is a minimal filling diagram, Lemma 2.1 implies that the map $f$ is simplicial and nondegenerate. Therefore, since in $D$ there are simplices $v, v_j, (v, v_j), 1 \leq j \leq k, (v_{j-1}, v_j), 2 \leq j \leq k, (v_k, v_1)$, in $X$ there are simplices $f(v), f(v_j), (f(v), f(v_j)), 1 \leq j \leq k, (f(v_{j-1}), f(v_j)), 2 \leq j \leq k, (f(v_k), f(v_1))$. Lemma 2.2 implies that adjacent 2-simplices of $D$ have distinct images under $f$. Hence $f(v_{i \mod k + 1}) \neq f(v_{(i+2) \mod k + 1}), 1 \leq i \leq k$.

We show further that $f(v_{j \mod k + 1}) \neq f(v_{(j+3) \mod k + 1}), 1 \leq j \leq k$. Suppose by contradiction there exists $i$ such that $f(v_{i \mod k + 1}) = f(v_{(i+3) \mod k + 1}), 1 \leq i \leq k$. We choose a filling diagram $(D', f')$ for $\gamma$ such that in $D'$ we have $v_{i \mod k + 1} \sim v_{(i+j) \mod k + 1}, 2 \leq j \leq 4$. We triangulate $D'$ with the same simplices like $D$ except for the triangles $(v, v_{(i+j) \mod k + 1}, v_{(i+j+1) \mod k + 1}), 0 \leq j \leq 3$ in $D'$ which are replaced in $D$ by the triangles $(v, v_{i \mod k + 1}, v_{(i+4) \mod k + 1}), (v_{i \mod k + 1}, v_{(i+j) \mod k + 1}, v_{(i+j+1) \mod k + 1}), 1 \leq j \leq 3$. We define $f'$ such that it coincides with $f$ on all simplices which are common to $D$ and $D'$. We define $f'$ such that $f'((v_{i \mod k + 1}) = f((v_{i+3} \mod k + 1)) = f(v_{i \mod k + 1}), f'((v_{i \mod k + 1}, v_{(i+3) \mod k + 1})) = f(v_{(i+3) \mod k + 1}).$ As argued above $f(v_{j \mod k + 1}) \neq f(v_{(j+2) \mod k + 1})$, then because $f'(v_{(i+2) \mod k + 1}) = f(v_{(i+3) \mod k + 1}), 1 \leq j \leq k.$ Since in $X$ we have $f(v_{(i+3) \mod k + 1}) \sim f((v_{i \mod k + 1}) = f'(v_{i \mod k + 1}), we define $f'$ such that $f'(v_{i \mod k + 1}) = f'(v_{(i+3) \mod k + 1}).$ We define $f'$ such that $f'(v_{i \mod k + 1}) = f(v_{i \mod k + 1}), f'(v_{i+1 \mod k + 1}) \leq j \leq k, f'(v_{i \mod k + 1}) = f((v_{i \mod k + 1}), f'(v_{i+1 \mod k + 1})) = f(v_{i \mod k + 1}), f'(v_{i+1 \mod k + 1}) = f(v_{(i+4) \mod k + 1}).$ We define $f'$ such that $f'(v_{i \mod k + 1}) = f'(v_{i+1 \mod k + 1}, v_{(i+4) \mod k + 1})), 1 \leq j \leq 3.$ Hence, since $f$ is simplicial, $f'$ is also simplicial. So $(D', f')$ is indeed a filling diagram for $\gamma$. Note that $D$ and $D'$ have the same area. Therefore $D'$ has minimal area. Then Lemma 2.1 implies that the map $f'$ is nondegenerate. But since $f'(v_{i \mod k + 1}) = f'(v_{(i+3) \mod k + 1}), f'$ is degenerate. Because we have reached a contradiction, $f(v_{i \mod k + 1}) \neq f(v_{(i+3) \mod k + 1}), 1 \leq i \leq k.$

In conclusion the map $f$ is injective on $X_{f(v)}$. 

Next we prove the minimal filling diagrams lemma for 7-located simplicial complexes.

**Lemma 3.2.** Let $X$ be a 7-located simplicial complex and let $\gamma$ be a homotopically trivial loop in $X$. Let $(D, f)$ be a minimal filling diagram for $\gamma$. Then $D$ is 7-located.

**Proof.** Because $(D, f)$ is a minimal filling diagram, Lemma 2.1 implies that the map $f$ is simplicial and nondegenerate. Therefore, since in $D$ there are simplices $v, v_j, (v, v_j), 1 \leq i \leq k, (v_{i-1}, v_i), 2 \leq i \leq k, (v_1, v_k)$, in $X$ there are simplices $f(v), f(v_j), (f(v), f(v_j)), 1 \leq i \leq k, (f(v_{i-1}), f(v_i)), 2 \leq i \leq k, (f(v_k), f(v_1))$.

Let $\beta = (w_1, \ldots, w_7)$ be a full cycle in $X$. Because $X$ is 7-located and $\beta$ has length 7, it is contained in the link of a vertex $x$. Since $f$ is simplicial and nondegenerate, there are vertices $v_i \in D, 1 \leq i \leq 7$ such that $f((v_i, v_{i+1})) = (w_i, w_{i+1}), 1 \leq i \leq 6$. So the loop $\alpha = (v_1, \ldots, v_7)$ in $D$ also has length 7. We show that $\alpha$ is full. Suppose,
by contradiction, that \( v_1 \sim v_4 \). Then, due to Lemma 3.1 in \( X \) we have \( w_1 \sim w_4 \). Since \( \beta \) is full, this implies a contradiction. So \( v_1 \sim v_4 \). One can similarly show that \( v_1 \sim v_3 \). Hence \( \alpha \) is full.

Suppose by contradiction that \( \alpha \) is not contained in the link of a vertex. Because \( \alpha \) is full, there are at least two vertices in the interior of \( \alpha \). Assume at first there are two such vertices, say \( z \) and \( y \). Obviously \( z \sim y \). Assume w.l.o.g. \( D_z = (y, v_3, v_2, v_1, v_7) \) and \( D_y = (v_3, v_4, v_5, v_6, v_7, z) \). For any other triangulation of \( D \), we proceed similarly. We consider a minimal filling diagram \((D', f')\) for \( \gamma \) such that \( D' \) is triangulated with the same simplices like \( D \) except for the triangles \( \langle z, v_i, v_{i+1} \rangle \), \( 1 \leq i \leq 2 \), \( \langle z, v_7, y \rangle \), \( \langle z, v_7, v_1 \rangle \) in \( D \) which are replaced in \( D' \) by the triangles \( \langle y, v_1, v_7 \rangle \), \( \langle y, v_i, v_{i+1} \rangle \), \( 1 \leq i \leq 2 \). So in \( D' \) the cycle \( \alpha = (v_1, \ldots, v_7) \) has a single interior vertex \( y \). We define \( f' \) such that it coincides with \( f \) on all simplices which are common to \( D \) and \( D' \). We define \( f' \) such that \( f'(y) = x, f'((y, v_i)) = (x, v_i), 1 \leq i \leq 2 \), \( f'((y, v_i, v_{i+1})) = (x, v_i, v_{i+1}), 1 \leq i \leq 2 \), \( f'(y, v_1, v_7) = (x, v_1, v_7) \). Since \( f \) is simplicial, \( f' \) is also simplicial. Hence \((D', f')\) is indeed a filling diagram for \( \gamma \). Note that the area of \( D' \) is less than the area of \( D \). Because \( D' \) has less interior vertices than \( D \), this holds also due to Pick’s formula. Based on the minimality of the area of \( D \), we have reached a contradiction. Similarly, if there are at least three vertices inside \( \alpha \), arguments similar to those above or Pick’s formula, also imply a contradiction. Therefore \( \alpha \) is contained in the link of a vertex. One can similarly show that any loop in \( D \) of length less than 7 but at least 4, is also contained in the link of a vertex. Then \( D \) is 7-located.

The proof of the main result of the paper relies on the following lemma. The proof follows closely, even up to the notations, the one given in [CCH04] (Lemma 9.2) for meshed graphs.

**Lemma 3.3.** Let \( X \) be a simply connected, 7-located simplicial complex. Let \( \gamma \) be a loop in \( X \) and let \((D, f)\) be a minimal filling diagram for \( \gamma \). Then for any three vertices \( u, v, w \) of \( D \) such that \( v \sim w \) and for any shortest \((u, v)\)-path \( P \), there is a shortest \((u, w)\)-path \( Q \) such that \( \text{Area} C \leq \text{const} \cdot d(u, v) \) where \( C \) is the cycle formed by the paths \( P, Q \) and the edge \( \langle v, w \rangle \). We denote by \( \text{const} \) any natural number such that \( \text{const} > 2 \).

**Proof.** Because \( D \) is minimal, due to Lemma 3.1, the simplicial map \( f \) is nondegenerate. Also the previous lemma implies that \( D \) is 7-located.

Let \( k = d(u, v) \) and let \( l = d(u, w) \). Let \( v' \) be a vertex of \( P \) such that \( v \sim v' \). Let \( P' \) be a shortest \((u, v')\)-path such that \( P = P' \cup \langle v', v \rangle \). There are three cases to be analyzed: \( l = k + 1, l = k, l = k - 1 \).

3.1. **Case 1.** We consider the case when \( d(u, w) > d(u, v) \). Then \( l = k + 1 \). Let \( Q = P \cup \langle v, w \rangle \) be a shortest \((u, w)\)-path. Then \( \text{Area} C = 0 \leq \text{const} \cdot d(u, v) \). This completes the proof in this case.

3.2. **Case 2.** We consider the case when \( d(u, w) = d(u, v) \). Hence \( l = k \). We prove by induction on \( k \) the existence of a shortest \((u, w)\)-path \( Q \) such that \( \text{Area} C \leq \text{const} \cdot k \) where \( C \) is the cycle formed by the paths \( P, Q \) and the edge \( \langle v, w \rangle \).

We consider the case when \( w \sim v' \). Also if \( k = 1 \), then \( w \sim v' \). In both cases, let \( Q = P' \cup \langle v', w \rangle \). Then \( \text{Area} C = 1 \leq \text{const} \cdot k \) what completes the proof in these cases.
From now on we assume that \( w \sim v' \). We consider a \((v', w)\)-path \((v', w_1, \ldots, w_n, w)\), \( n \geq 1 \) that does not pass through \( v \) but, except for that, it is tightened. Let \( \alpha = (v, v', w_1, \ldots, w_n, w, v) \).

(1) Case 2.1. The cycle \( \alpha \) is full. Depending on the value of \( n \), there are two cases to be analyzed. We treat them below.

(a) If \( n \leq 4 \), then \( 4 \leq |\alpha| \leq 7 \). Because \( \alpha \) is full and its length is at most 7, by 7-location, there is a vertex \( z \) of \( D \) such that \( \alpha \subseteq D_z \). Note that \( d(v', u) = d(z, u) = d(w_n, u) = k - 1 \). Because \( v' \sim z \), by induction on \( P' \), there is a shortest \((u, z)\)-path \( Z \) such that \( \text{Area}C_1 \leq \text{const} \cdot (k - 1) \) where \( C_1 \) is the cycle formed by the paths \( P', Z \) and the edge \((v', z)\).

Because \( z \sim w_n \), by induction on \( Z \), there is a shortest \((u, w_n)\)-path \( Q' \) such that \( \text{Area}C_2 \leq \text{const} \cdot (k - 1) \) where \( C_2 \) is the cycle formed by the paths \( Z, Q' \) and the edge \((z, w_n)\). Let \( Q = Q' \cup \langle w_n, w \rangle \) be a shortest \((u, w)\)-path. In conclusion \( \text{Area}C = \text{Area}C_1 + \text{Area}C_2 + \text{Area}(v, v', z) + \text{Area}(v, w, z) + \text{Area}(w, z, w_n) \leq 2 \cdot \text{const} \cdot (k - 1) + 3 \leq \text{const} \cdot k \). The last inequality holds because \( \text{const} > 2 \).

(b) If \( n > 4 \), then \( |\alpha| > 7 \). Because \( \alpha \) is full, there are vertices inside \( \alpha \). Because the area of \( D \) is minimal, based on Pick's formula, the number of vertices inside \( \alpha \) is also minimal. Let \( z_j, 1 \leq j \leq r \) be the vertices inside \( \alpha \) such that \( v' \sim z_1, z_j \sim z_{j+1}, 1 \leq j \leq r - 1 \), \( z_r \sim w_n \).

Besides, for \( 1 \leq j \leq r \), either \( v \sim z_j \) or \( v \sim z_j \) or \( v \sim z_j \) or \( v \sim z_j \). Because \( D \) is flat, there is a unique vertex \( z_q, 1 \leq q \leq r \) such that \( v \sim z_q \sim w \). Note that \( d(v', u) = d(z_j, u) = k - 1, 1 \leq j \leq q \).

Because \( v' \sim z_1 \), by induction on \( P' \), there is a shortest \((u, z_1)\)-path \( Z_1 \) such that \( \text{Area}C_0 \leq \text{const} \cdot (k - 1) \) where \( C_0 \) is the cycle formed by \( P', Z_1, \langle v', z_1 \rangle \). For \( j \in \{1, \ldots, q - 1\} \), because \( z_j \sim z_{j+1} \), by induction on \( Z_j \), there is a shortest \((u, z_{j+1})\)-path \( Z_{j+1} \) such that \( \text{Area}C_j \leq \text{const} \cdot (k - 1) \) where \( C_j \) is the cycle formed by \( Z_j, Z_{j+1}, \langle z_j, z_{j+1} \rangle \).

Let \( Q = Z_q \cup \langle z_q, w \rangle \) be a shortest \((u, w)\)-path. Note that there are \( q + 1 \) triangles contained in the cycle \((v, v', z_1, \ldots, z_q, w)\). In conclusion \( \text{Area}C = \sum_{j=0}^{n-1} \text{Area}C_j + \text{Area}(v, v', z_1) + \ldots + \text{Area}(v, z_q, w) \leq q \cdot \text{const} \cdot (k - 1) + q + 1 \leq \text{const} \cdot k \). The last inequality holds due to the fact that \( \text{const} > 2 \).

(2) Case 2.2. The cycle \( \alpha \) is not full. Because \( w \sim v' \) and because the path \((v', w_1, \ldots, w_n, w)\) is tightened (except for the fact that it does not pass through \( v \)), the possible diagonals of \( \alpha \) are \( \langle v, w_i \rangle, 1 \leq i \leq n \), \( \langle w, w_j \rangle, 1 \leq i \leq n - 1 \). Depending on this, there are several cases to be analyzed. We treat them below.

Case 2.2.1. Suppose \( v \sim w_1 \). Note that \( d(v', u) = d(w_1, u) = k - 1 \). Because \( v' \sim w_1 \), by induction on \( P' \), there is a shortest \((u, w_1)\)-path \( R' \) such that \( \text{Area}C_1 \leq \text{const} \cdot (k - 1) \). We denoted by \( C_1 \) the cycle formed by the paths \( P', R', \langle v', w_1 \rangle \). Then \( \text{Area}C' = \text{Area}C_1 + \text{Area}(v, v', w_1) \leq \text{const} \cdot (k - 1) + 1 \leq \text{const} \cdot k \). We denoted by \( C' \) the cycle formed by \( P, R', \langle v, w_1 \rangle \).

Case 2.2.2. Let \( w_i, 2 \leq i \leq n \) such that \( v \sim w_i \) and \( v \sim w_{i-j}, 1 \leq j \leq i - 1 \). Let \( \delta = (v, v', w_1, \ldots, w_i) \). Note that, due to the choice of \( w_i \), \( \delta \) is full. Depending on the value of \( i \), there are two cases to be analyzed. We present them below.
(a) If \( i \leq 5 \) then \( 4 \leq |\delta| \leq 7 \). Then, by 7-location, there is a vertex \( z \) such that \( \delta \subset D_z \). Note that \( d(v', u) = d(z, u) = d(w_i, u) = k - 1 \). Because \( v' \sim z \), by induction on \( P' \), there is a shortest \((u, z)\)-path \( Z \) such that \( \text{Area} C_1 \leq \text{const} \cdot (k - 1) \) where \( C_1 \) is the cycle formed by the paths \( P', Z \) and the edge \( \langle v', z \rangle \). Because \( z \sim w_i \), by induction on \( Z \), there is a shortest \((u, w_i)\)-path \( R' \) such that \( \text{Area} C_2 \leq \text{const} \cdot (k - 1) \) where \( C_2 \) is the cycle formed by the paths \( Z, R' \) and the edge \( \langle z, w_i \rangle \). Then we have \( \text{Area} C' = \text{Area} C_1 + \text{Area} C_2 + \text{Area}(\langle v', v \rangle) + \text{Area}(\langle v, w_i, z \rangle) \leq 2 \cdot \text{const} \cdot (k - 1) + 2 \leq \text{const} \cdot k \). We denoted by \( C' \) the cycle formed by the paths \( P, R' \) and the edge \( \langle v, w_i \rangle \).

(b) If \( i > 5 \) then \( |\delta| > 7 \). Because \( \delta \) is full, there are vertices inside \( \delta \). Because the area of \( D \) is minimal, based on Pick's formula, the number of vertices inside \( \delta \) is also minimal. Let \( z_j, 1 \leq j \leq r \) be the vertices inside \( \delta \) such that \( v' \sim z_j, z_j \sim z_{j+1}, 1 \leq j \leq r - 1, z_r \sim w_i \). Besides, for \( 1 \leq j \leq r, v \sim z_j \). Note that \( d(v', u) = d(z_j, u) = d(w_i, u), 1 \leq j \leq r \). By induction on \( P' \), there is a shortest \((u, z_1)\)-path \( Z_1 \) such that \( \text{Area} C_0 \leq \text{const} \cdot (k - 1) \) where \( C_0 \) is the cycle formed by \( P', Z_1 \) and the edge \( \langle v', z_1 \rangle \). For \( j \in \{1, ..., r - 1\} \), because \( z_j \sim z_{j+1} \), by induction on \( Z_j \), there is a shortest \((u, z_{j+1})\)-path \( Z_{j+1} \) such that \( \text{Area} C_j \leq \text{const} \cdot (k - 1) \) where \( C_j \) is the cycle formed by \( Z_j, Z_{j+1}, (z_j, z_{j+1}) \). Because \( z_r \sim w_i \), by induction on \( Z_r \), there is a shortest \((u, w_i)\)-path \( R' \) such that \( \text{Area} C_r \leq \text{const} \cdot (k - 1) \) where \( C_r \) is the cycle formed by \( Z_r, R', (z_r, w_i) \). Note that there are \( r + 1 \) triangles contained in the cycle \( (v, v', z_1, ..., z_r, w_i) \). In conclusion \( \text{Area} C' = \sum_{j=0}^{r} \text{Area} C_j + \text{Area}(\langle v, v', z_1 \rangle) + \cdots + \text{Area}(\langle v, w_i, z_r \rangle) \leq (r + 1) \cdot \text{const} \cdot (k - 1) + (r + 1) \leq \text{const} \cdot k \). We denoted by \( C' \) the cycle formed by the paths \( P, R' \) and the edge \( \langle v, w_i \rangle \).

Case 2.2.3. Let \( w_i, 1 \leq i \leq n \) such that \( w \sim w_i, w \sim w_{i-1}, 1 \leq j \leq i - 1 \).

We consider the cycle \( \delta = (v, v', w_1, ..., w_i, w, v) \).

Case 2.2.3.a Assume \( \delta \) is full. Depending on the value of \( i \), there are two cases to be analyzed. We discuss them below.

(a) If \( i \leq 4 \) then \( 4 \leq |\delta| \leq 7 \). Then, by 7-location, there is a vertex \( z \) such that \( \delta \subset D_z \). Note that \( d(v', u) = d(z, u) = d(w_i, u) = k - 1 \). Because \( v' \sim z \), by induction on \( P' \), there is a shortest \((u, z)\)-path \( Z \) such that \( \text{Area} C_1 \leq \text{const} \cdot (k - 1) \) where \( C_1 \) is the cycle formed by the paths \( P', Z \) and the edge \( \langle v', z \rangle \). Because \( z \sim w_i \), by induction on \( Z \), there is a shortest \((u, w_i)\)-path \( Q' \) such that \( \text{Area} C_2 \leq \text{const} \cdot (k - 1) \) where \( C_2 \) is the cycle formed by the paths \( Z, Q' \) and the edge \( \langle z, w_i \rangle \). Let \( Q = Q' \cup (w_i, w) \) be a shortest \((u, w)\)-path. Then we have \( \text{Area} C = \text{Area} C_1 + \text{Area} C_2 + \text{Area}(\langle v, v', z \rangle) + \text{Area}(\langle v, w, z \rangle) + \text{Area}(\langle w, w_i, z \rangle) \leq 2 \cdot \text{const} \cdot (k - 1) + 3 \leq \text{const} \cdot k \).

(b) If \( i > 4 \) then \( |\delta| > 7 \). Because \( \delta \) is full, there are vertices inside \( \delta \). Because the area of \( D \) is minimal, Pick's formula implies that the number of vertices inside \( \delta \) is also minimal. Let \( z_j, 1 \leq j \leq r \) be the vertices inside \( \delta \) such that \( v' \sim z_j, z_j \sim z_{j+1}, 1 \leq j \leq r - 1, z_r \sim w_i \). Besides, for \( 1 \leq j \leq r \), either \( v \sim z_j \) or \( w \sim z_j \) or \( v \sim z_j \sim w \). Because \( D \) is flat, there is a unique vertex \( z_q, 1 \leq q \leq r \) such that \( v \sim z_q \sim w \). Note that \( d(v', u) = d(z_j, u) = k - 1, 1 \leq j \leq q \). Because
\( v' \sim z_1, \) by induction on \( P' \), there is a shortest \((u, z_1)\)-path \( Z_1 \) such that Area\(C_0\) \(\leq\) const \((-k - 1)\) where \(C_0\) is the cycle formed by \(P', Z_1\) and the edge \((v', z_1)\). For \(j \in \{1, \ldots, q - 1\} \), because \(z_j \sim z_{j+1}\), by induction on \(Z_j\), there is a shortest \((u, z_{j+1})\)-path \(Z_{j+1}\) such that Area\(C_j\) \(\leq\) const \((-k - 1)\) where \(C_j\) is the cycle formed by \(Z_j, Z_{j+1}, (z_j, z_{j+1})\).

Let \(Q = Z_q \cup \langle z_q, w \rangle\) be a shortest \((u, w)\)-path. Note that there are \(q + 1\) triangles contained in the cycle \((v, v', z_1, \ldots, z_q, w)\). In conclusion Area\(C = \sum_{i=0}^{q-1} \) Area\(C_i + \) Area\((\langle v, v', z_1 \rangle + \ldots + \) Area\((\langle v, z_q, w \rangle) \leq q \cdot \) const \((-k - 1)\) + \(q + 1 \leq\) const \(\cdot k\).

Case 2.2.3.b Assume \(\delta\) is not full.

Suppose \(v \sim w_1\). Then \(d(v', u) = d(w_1, u) = k - 1\). Because \(v' \sim w_1\), by induction on \(P'\), there is a shortest \((u, w_1)\)-path \(R'\) such that Area\(C_1\) \(\leq\) const \((-k - 1)\) where \(C_1\) is the cycle formed by \(P', R', \langle v, w_1 \rangle\). Then Area\(C'\) = Area\(C_1 + \) Area\((\langle v, v', w_1 \rangle) \leq\) const \((-k - 1)\) + \(1 \leq\) const \(\cdot k\). We denoted by \(C'\) the cycle formed by \(P, R', \langle v, w_1 \rangle\).

Let \(w_s, 2 \leq s \leq i\) such that \(v \sim w_s, v \sim w_{s-j}, 1 \leq j \leq s - 1\). Because the cycle \((v, v', w_1, \ldots, w_s)\) is full and of length at least 4, Case 2.2.2. implies that there is a shortest \((u, w_s)\)-path \(R''\) such that Area\(C'\) \(\leq\) const \(\cdot k\). We denoted by \(C''\) the cycle formed by \(P, R'', \langle v, w_s \rangle\).

If \(s = i\), let \(Q = R'' \cup \langle w_i, w \rangle\) be a shortest \((u, w)\)-path. Then Area\(C =\) Area\(C' + \) Area\((\langle v, w, w_i \rangle) \leq\) const \((-k - 1)\) + \(1 \leq\) const \(\cdot k\).

From now on assume that \(s \neq i\). Let \(\beta = (v, w_s, \ldots, w_i, w, v)\). In case 2.2.3.b.1 we treat the situation when \(\beta\) is full. In case 2.2.3.b.2 we discuss the case when \(\beta\) is not full.

Case 2.2.3.b.1 Depending on the value of \(i - s\), there are two cases to be analyzed. We present them below.

(a) If \(i - s \leq 4\), then \(4 \leq |\beta| \leq 7\). By 7-location, there is a vertex \(z\) such that \(\beta \subseteq D_z\). Note that \(d(w_s, u) = d(z, u) = d(w_1, u) = k - 1\). Because \(w_s \sim z\), by induction on \(R''\), there is a shortest \((u, z)\)-path \(Z\) such that Area\(C_1\) \(\leq\) const \((-k - 1)\) where \(C_1\) is the cycle formed by the paths \(R'', Z\) and the edge \(\langle w_s, z \rangle\). Because \(z \sim w_i\), by induction on \(Z\), there is a shortest \((u, w_i)\)-path \(Q'\) such that Area\(C_2\) \(\leq\) const \((-k - 1)\) where \(C_2\) is the cycle formed by the paths \(Z, Q'\) and the edge \(\langle z, w_i \rangle\). Let \(Q = Q' \cup \langle w_i, w \rangle\) be a shortest \((u, w)\)-path. Then we have Area\(C'' = \) Area\(C_1 + \) Area\(C_2 + \) Area\((\langle w_s, v, z \rangle) + \) Area\((\langle v, w, z \rangle) + \) Area\((\langle w_i, z \rangle) \leq 2 \cdot\) const \((-k - 1)\) + \(3 \leq\) const \(\cdot k\). We denoted by \(C''\) the cycle formed by the paths \(R'', Q\) and the edges \(\langle w_s, v \rangle, \langle v, w \rangle\).

In conclusion Area\(C =\) Area\(C' +\) Area\(C'' \leq\) const \(\cdot k\).

(b) If \(i - s > 4\), then \(|\beta| > 7\). Because \(\beta\) is full, there are vertices inside \(\beta\). Because the area of \(D\) is minimal, based on Pick’s formula, the number of vertices inside \(\beta\) is also minimal. Let \(z_j, 1 \leq j \leq r\) be the vertices inside \(\beta\) such that \(w_s \sim z_1, z_j \sim z_{j+1}, 1 \leq j \leq r - 1, z_r \sim w_i\). Besides, for \(1 \leq j \leq r\), either \(v \sim z_j\) or \(w \sim z_j\) or \(v \sim z_j \sim w\). Because \(D\) is flat, there is a unique vertex \(z_q, 1 \leq q \leq r\) such that \(v \sim z_q \sim w\). Note that \(d(w_s, u) = d(z_j, u) = k - 1, 1 \leq j \leq q\). By induction on \(R''\), because \(w_s \sim z_1\), there is a shortest \((z_1, u)\)-path \(Z_1\) such that Area\(C_0\) \(\leq\) const \((-k - 1)\) where \(C_0\) is the cycle formed by \(R'', Z_1\) and the edge \(\langle w_s, z_1 \rangle\). For \(j \in \{1, \ldots, q - 1\}\), by induction on \(Z_j\), because
$z_j \sim z_{j+1}$, there is a shortest $(u, z_{j+1})$-path $Z_{j+1}$ such that $\text{AreaC}_j \leq \text{const} \cdot (k-1)$ where $C_j$ is the cycle formed by $Z_j, Z_{j+1}, (z_j, z_{j+1})$. Let $Q = Z_q \cup \{z_q, w\}$ be a shortest $(u, w)$-path. Note that there are $q+1$ triangles contained in the cycle $(v, w, z_1, \ldots, z_q, w)$. In conclusion $\text{AreaC}_{q'} = \sum_{j=1}^{q-1} \text{AreaC}_j + \text{Area}(\langle v, w, z_1 \rangle) + \ldots + \text{Area}(\langle v, z_q, w \rangle) \leq q \cdot \text{const} \cdot (k-1) + q+1 \leq \text{const} \cdot k$. We denoted by $C''$ the cycle formed by the paths $P', Q$ and the edges $\langle w, v \rangle, \langle v, w \rangle$.

In conclusion $\text{AreaC} = \text{AreaC'} + \text{AreaC''} \leq \text{const} \cdot k$.

Case 2.2.3.b.2 If $\beta$ is not full, we split $\beta$ into full cycles $\beta_1, \ldots, \beta_j, j \geq 2$ containing each at least one edge of $\beta$ and at least one of its diagonals. We argue for each of these full cycles the same way we argued in one of the cases discussed above.

3.3. Case 3. We consider the case when $d(u, w) < d(u, v)$. Hence $l = k - 1$.

We prove by induction on $k$ the existence of a shortest $(u, w)$-path $Q$ such that $\text{AreaC} \leq \text{const} \cdot k$ where $C$ is the cycle formed by the paths $P, Q$ and the edge $\langle v, w \rangle$.

If $k = 1$ or more generally if $w = v'$, then let $Q = P'$ be a shortest $(u, w)$-path. In this case we get $\text{AreaC} = \emptyset \leq \text{const} \cdot k$.

If $v' \sim w$, note that $d(v', u) = d(v, u) = k - 1$. By induction on $P'$, there is a shortest $(u, w)$-path $Q$ such that $\text{AreaC'} \leq \text{const} \cdot (k - 1)$ where $C'$ is the cycle formed by the paths $P', Q$ and the edge $\langle v', w \rangle$. Then we have $\text{AreaC} = \text{AreaC'} + \text{Area}(\langle v', v' \rangle, w) \leq \text{const} \cdot (k - 1) + 1 \leq \text{const} \cdot k$.

From now on assume that $w \neq v'$ and that $w \sim v'$.

We consider a $(v', w)$-path $(v', w_1, \ldots, w_n, w)$ that does not pass through $v$ but, except for that, it is tightened. Let $\alpha = (v, v', w_1, \ldots, w_n, w, v), n \geq 1$.

1. Case 3.1. The cycle $\alpha$ is full. Depending on the value of $n$, there are two cases to be analyzed.

(a) If $n \leq 4$, then $4 \leq |\alpha| \leq 7$. By 7-location, there is a vertex $z$ such that $z \in D_z$. Note that $d(v', u) = d(z, u) = d(v, u) = k - 1$. Because $v' \sim z$, by induction on $P'$, there is a shortest $(u, z)$-path $Z$ such that $\text{AreaC}_1 \leq \text{const} \cdot (k-1)$ where $C_1$ is the cycle formed by $P', Z, (v', z)$. Because $z \sim w$, by induction on $Z$, there is a shortest $(u, w)$-path $Q$ such that $\text{AreaC}_2 \leq \text{const} \cdot (k-1)$ where $C_2$ is the cycle formed by $Z, P, (z, w)$. In conclusion $\text{AreaC} = \text{AreaC}_1 + \text{AreaC}_2 + \text{Area}(\langle v', v, z \rangle) + \text{Area}(\langle w, v, z \rangle) \leq 2 \cdot \text{const} \cdot (k - 1) + 2 \leq \text{const} \cdot k$.

(b) If $n > 4$, then $|\alpha| > 7$. Because $\alpha$ is full, there are vertices inside $\alpha$. Because the area of $D$ is minimal, due to Pick's formula, the number of vertices inside $\alpha$ is also minimal. Let $z_j, 1 \leq j \leq r$ be the vertices inside $\alpha$ such that $v' \sim z_1, z_j \sim z_{j+1}, 1 \leq j \leq r - 1, z_r \sim w_n$. Besides, for $1 \leq j \leq r$, either $v \sim z_j$ or $w \sim z_j$ or $v \sim z_j \sim w$. Because $D$ is flat, there is a unique vertex $z_q, 1 \leq q \leq r$ such that $v \sim z_q \sim w$. Note that $d(v', u) = d(z_j, u) = d(v, u) = k - 1, 1 \leq j \leq q$. Because $v' \sim z_1$, by induction on $P'$, there is a shortest $(u, z_1)$-path $Z_1$ such that $\text{AreaC}_0 \leq \text{const} \cdot (k - 1)$ where $C_0$ is the cycle formed by $P', Z_1$ and the edge $\langle v', z_1 \rangle$. For $j \in \{1, \ldots, q - 1\}$, because $z_j \sim z_{j+1}$, by induction on $Z_j$, there is a shortest $(u, z_{j+1})$-path $Z_{j+1}$ such that $\text{AreaC}_j \leq \text{const} \cdot (k - 1)$ where $C_j$ is the cycle
formed by $Z_i, Z_{j+1}, (z_j, z_{j+1})$. Because $z_q \sim w$, by induction on $Z_q$, there is a shortest $(u, w)$-path $Q$ such that $\text{AreaC}_q \leq \text{const} \cdot (k - 1)$ where $C_q$ is the cycle formed by $Z_q, Q, (z_q, w)$. Note that there are $q + 1$ triangles contained in the cycle $(v, v', z_1, \ldots, z_q, w)$. In conclusion $\text{AreaC} = \sum_{j=0}^q \text{AreaC}_j + \text{Area}(\langle v, v', z_1 \rangle) + \ldots + \text{Area}(\langle v, z_q, w \rangle) \leq q \cdot \text{const} \cdot (k - 1) + q + 1 \leq \text{const} \cdot k$.

(2) Case 3.2. The cycle $\alpha$ is not full. Because $w \sim v'$ and because the path $(v', w_1, \ldots, w_n, w)$ is tightened (except for the fact that it does not pass through $v$), the possible diagonals of $\alpha$ are $\langle v, w_i \rangle, 1 \leq i \leq n, \langle w, w_i \rangle, 1 \leq i \leq n - 1$. Depending on this, there are several cases to be analyzed. We treat them below.

Case 3.2.1. Suppose $v \sim w_1$. Case 2.2.1. implies that there is a shortest $(u, w_1)$-path $R'$ such that $\text{AreaC'} \leq \text{const} \cdot k$. We denoted by $C'$ the cycle formed by $P, R', (v, w_1)$.

Case 3.2.2. Let $w_i, 2 \leq i \leq n$ such that $v \sim w_i$ and $v \sim w_{i-1}, 1 \leq j \leq i - 1$. Let $\delta = (v, v', w_1, \ldots, w_i)$. Due to the choice of $w_i$, $\delta$ is full. Case 2.2.2. implies that there is a shortest $(u, w_i)$-path $R'$ such that $\text{AreaC'} \leq \text{const} \cdot k$. We denoted by $C'$ the cycle formed by $P, R', (v, w_i)$.

Case 3.2.3. Let $w_i, 1 \leq i \leq n$ such that $w \sim w_i, w \sim w_{i-1}, 1 \leq i \leq 1$. We consider the cycle $\delta = (v, v', w_1, \ldots, w_i, w, v)$.

Case 3.2.3.a Assume $\delta$ is full. Depending on the value of $i$, there are two cases to be analyzed. We present them below.

(a) If $i \leq 4$ then $4 \leq |\delta| \leq 7$. By 7-locational, there is a vertex $z$ such that $\delta \subset D_z$. Note that $d(v', u) = d(z, u) = d(w, u) = k - 1$. Because $v' \sim z$, by induction on $P'$, there is a shortest $(u, z)$-path $Z$ such that $\text{AreaC}_1 \leq \text{const} \cdot (k - 1)$ where $C_1$ is the cycle formed by the paths $P', Z$ and the edge $\langle v', z \rangle$. Because $z \sim w$, by induction on $Z$, there is a shortest $(u, w)$-path $Q$ such that $\text{AreaC}_2 \leq \text{const} \cdot (k - 1)$ where $C_2$ is the cycle formed by the paths $Z, Q$ and the edge $\langle z, w \rangle$. Then we have $\text{AreaC} = \text{AreaC}_1 + \text{AreaC}_2 + \text{Area}(\langle v, v', z \rangle) + \text{Area}(\langle v, w, z \rangle) \leq 2 \cdot \text{const} \cdot (k - 1) + 2 \leq \text{const} \cdot k$.

(b) If $i > 4$ then $|\delta| > 7$. Because $\delta$ is full, there are vertices inside $\delta$. Because the area of $D$ is minimal, due to Pick’s formula, the number of vertices inside $\delta$ is also minimal. Let $z_j, 1 \leq j \leq r$ be the vertices inside $\delta$ such that $v' \sim z_1, z_j \sim z_{j+1}, 1 \leq j \leq r - 1, z_r \sim w$. Besides, for $1 \leq j \leq r$, either $v \sim z_j$ or $v \sim z_j \sim w$ or $w \sim z_j$. Because $D$ is flat, there is a unique vertex $z_q$ such that $v \sim z_q \sim w, 1 \leq q \leq r$. Note that $d(v', u) = d(z_j, u) = d(w, u) = k - 1, 1 \leq j \leq r$. By induction on $P'$, because $v' \sim z_1$, there is a shortest $(u, z_1)$-path $Z_1$ such that $\text{AreaC}_0 \leq \text{const} \cdot (k - 1)$ where $C_0$ is the cycle formed by $P', Z_1, (v', z_1)$. For $j \in \{1, \ldots, q - 1\}$, by induction on $Z_j$, because $z_j \sim z_{j+1}$, there is a shortest $(u, z_{j+1})$-path $Z_{j+1}$ such that $\text{AreaC}_j \leq \text{const} \cdot (k - 1)$ where $C_j$ is the cycle formed by $Z_j, Z_{j+1}, (z_j, z_{j+1})$. Because $z_q \sim w$, by induction on $Z_q$, there is a shortest $(u, w)$-path $Q$ such that $\text{AreaC}_q \leq \text{const} \cdot (k - 1)$ where $C_q$ is the cycle formed by $Z_q, Q, (z_q, w)$. Note that there are $q + 1$ triangles contained in the cycle $(v, v', z_1, \ldots, z_q, w)$. In conclusion $\text{AreaC} = \sum_{j=0}^q \text{AreaC}_j + \text{Area}(\langle v, v', z_1 \rangle) + \ldots + \text{Area}(\langle v, w, z_q \rangle) \leq (q + 1) \cdot \text{const} \cdot (k - 1) + q + 1 \leq \text{const} \cdot k$. 
Case 3.2.3.\textit{b} Assume \(\delta\) is not full.

Suppose \(v \sim w_1\). Then \(d(v', u) = d(w_1, u) = k - 1\). Because \(v' \sim w_1\), by induction on \(P'\), there is a shortest \((u, w_1)\)-path \(R'\) such that \(\text{Area}_{C} \leq \text{const} \cdot (k - 1)\) where \(C_1\) is the cycle formed by \(P', R', (v', w_1)\). Then \(\text{Area}_{C'} = \text{Area}_{C'} + \text{Area}(\langle v, v', w_1 \rangle) \leq \text{const} \cdot (k - 1) + 1 \leq \text{const} \cdot k\). We denoted by \(C'\) the cycle formed by \(P, R', (v, w_1)\).

Let \(w_s, 2 \leq s \leq i\) such that \(v \sim w_s, v \sim w_{s-j}, 1 \leq j \leq s - 1\). Because \((v, v', w_1, ..., w_s)\) is full and of length at least 4, Case 2.2.2. implies that there is a shortest \((u, w_s)\)-path \(R'\) such that \(\text{Area}_{C'} \leq \text{const} \cdot (k - 1)\). We denoted by \(C''\) the cycle formed by \(P, R', (v, w_s)\).

If \(s = i\), let \(Q = R' \cup \langle w_i, w \rangle\) be a shortest \((u, w)\)-path. Then \(\text{Area}_C = \text{Area}_{C'} + \text{Area}(\langle v, w, w_i \rangle) \leq \text{const} \cdot (k - 1) + 1 \leq \text{const} \cdot k\).

From now on assume that \(s \neq i\). Let \(\beta = (v, w_s, ..., w_1, w, v)\). In case 3.2.3.\textit{b.1} we treat the situation when \(\beta\) is full. In case 3.2.3.\textit{b.2} we consider the case when \(\beta\) is not full.

Case 3.2.3.\textit{b.1} Depending on the value of \(i - s\), there are two cases to be analyzed. We present them below.

(a) If \(i - s \leq 4\) then \(4 \leq |\beta| \leq 7\). By 7-location, there is a vertex \(z\) such that \(\beta \subset D_z\). Note that \(d(w_s, u) = d(z, u) = d(w, u) = k - 1\). Because \(w_s \sim z\), by induction on \(R'\), there is a shortest \((u, z)\)-path \(Z\) such that \(\text{Area}_{C1} \leq \text{const} \cdot (k - 1)\) where \(C_1\) is the cycle formed by \(R', Z, (w_s, z)\). When \(z \sim w\), by induction on \(Z\), there is a shortest \((u, w)\)-path \(Q\) such that \(\text{Area}_{C2} \leq \text{const} \cdot (k - 1)\) where \(C_2\) is the cycle formed by the path \(Z, P\) and the edge \(\langle z, w \rangle\). Then we have \(\text{Area}_{C'''} = \text{Area}_{C1} + \text{Area}_{C2} + \text{Area}(w_s, v, z) + \text{Area}(v, w, z) \leq 2 \cdot \text{const} \cdot (k - 1) + 2 \leq \text{const} \cdot k\). We denoted by \(C''\) the cycle formed by the path \(R', Q\) and the paths \((u, v, w), (v, w)\).

In conclusion \(\text{Area}_C = \text{Area}_{C'} + \text{Area}_{C''} \leq \text{const} \cdot k\).

(b) If \(i - s > 4\) then \(|\beta| > 7\). Because \(\beta\) is full, there are vertices inside \(\beta\).

Because the area of \(D\) is minimal, based on Pick’s formula, the number of vertices inside \(\beta\) is also minimal. Let \(z_j, 1 \leq j \leq r\) be the vertices inside \(\beta\) such that \(w_s \sim z_1, z_j \sim z_{j+1}, 1 \leq j \leq r, z_r \sim w\). Besides, \(v \sim z_j, 1 \leq j \leq r - 1, v \sim z_r \sim w\). Note that \(d(w_s, u) = d(z_j, u) = d(w, u) = k - 1, 1 \leq j \leq r\). Because \(w_s \sim z_1\), by induction on \(R'\), there is a shortest \((u, z_1)\)-path \(Z_1\) such that \(\text{Area}_{C0} \leq \text{const} \cdot (k - 1)\) where \(C_0\) is the cycle formed by \(R', Z_1, (w_s, z_1)\). For \(j \in \{1, ..., r - 1\}\), because \(z_j \sim z_{j+1}\), by induction on \(Z_j\), there is a shortest \((u, z_{j+1})\)-path \(Z_{j+1}\) such that \(\text{Area}_{C_j} \leq \text{const} \cdot (k - 1)\) where \(C_j\) is the cycle formed by \(Z_j, Z_{j+1}, (z_j, z_{j+1})\). Because \(z_r \sim w\), by induction on \(Z_r\), there is a shortest \((u, w)\)-path \(Q\) such that \(\text{Area}_{C_r} \leq \text{const} \cdot (k - 1)\) where \(C_r\) is the cycle formed by \(Z_r, Q, \langle z_r, w \rangle\). Note that there are \(r + 1\) triangles contained in the cycle \((v, w_s, z_1, ..., z_r, w)\). In conclusion \(\text{Area}_{C'''} = \sum_{j=0}^{r} \text{Area}_{C_j} + \text{Area}(v, w_s, z_1) + ... + \text{Area}(v, z_r, w) \leq (r + 1) \cdot \text{const} \cdot (k - 1) + r + 1 \leq \text{const} \cdot k\). We denoted by \(C''\) the cycle formed by the paths \(R', Q\) and \(\langle w_s, v, \rangle, (v, w)\).

In conclusion \(\text{Area}_C = \text{Area}_{C'} + \text{Area}_{C''} \leq \text{const} \cdot k\).

Case 3.2.3.\textit{b.2} If \(\beta\) is not full, we split \(\beta\) into full cycles \(\beta, ..., \beta_j, j \geq 2\) containing each at least one edge of \(\beta\) and at least one of its diagonals.
We argue for each of these full cycles the same way we argued in one of the cases discussed above.

\[ \square \]

**Theorem 3.4 (quadratic isoperimetric inequality).** Let \( X \) be a simply connected, 7-located simplicial complex. Let \( \gamma \) be a loop in \( X \). Let \((D, f)\) be a minimal filling diagram for \( \gamma \). Let \( D \) be triangulated such that it contains a cycle \( \alpha = (v_0, v_1, \ldots, v_{n-1}, v_0) \) of length \( n \). Let \( D_0 \) be the subdisc of \( D \) bounded by \( \alpha \). Then Area\((D_0) < \text{const} \cdot n^2 \) where \( \text{const} \) is any natural number such that \( \text{const} > 2 \).

**Proof.** For each \( i \in \{0, \ldots, n-1\} \), we define a shortest \((v_0, v_i)\)-path \( P_i \) such that Area\(C_i \leq \text{const} \cdot d(v_0, v_i) \) where \( C_i \) is the cycle formed by the paths \( P_i, P_{i+1} \) and the edge \((v_i, v_{i+1})\). Let \( P_0 \) be the one vertex path \( v_0 \). Assume that we have already constructed \( P_i \) such that it intersects \( P_{i-1} \) only in \( v_0 \). According to the previous lemma, there exists a path \( P_{i+1} \) in \( D_0 \) from \( v_0 \) to \( v_{i+1} \) that intersects \( P_i \) only in \( v_0 \) such that Area\(C_i \leq \text{const} \cdot d(v_0, v_i) \leq \text{const} \cdot \frac{n}{i} < 2 \cdot \text{const} \cdot \frac{n}{i} = \text{const} \cdot n, 0 \leq i \leq n-1 \). We denote by \( C_i \) the concatenation of the paths \( P_i, P_{i+1} \) and the edge \((v_i, v_{i+1})\). Hence one can fill \( D_0 \) using the collection of \( n \) cycles \( C_0, C_1, \ldots, C_{n-1} \) such that it intersects \( P_i \) only in \( v_0 \) such that Area\(C_i \leq \text{const} \cdot n, 0 \leq i \leq n-1 \). In conclusion we have Area\(D_0 \leq n \cdot \text{Area}C_i < \text{const} \cdot n^2, 0 \leq i \leq n-1 \).

\[ \square \]

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