The time distribution of quantum events

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We develop a general theory of the time distribution of quantum events, applicable to a large class of problems such as arrival time, dwell time and tunneling time. The basic assumption is that there is a stopwatch that ticks until an awaited event is detected, at which time the stopwatch stops. The awaited event is represented by a projection operator \( \pi \), while the ideal stopwatch is modeled as a series of ideal projective measurements at different times, so that, at each of those times, the quantum state suffers an instantaneous projection with either \( \bar{\pi} = 1 - \pi \) (when the awaited event does not happen) or \( \pi \) (when the awaited event eventually happens). In the approximation in which the time \( \delta t \) between the subsequent measurements is sufficiently small (but not zero!), we find a fairly simple general formula for the time distribution \( P(t) \), representing the probability density that the awaited event will be detected at time \( t \).

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I. INTRODUCTION

In the standard formulation of quantum mechanics (QM) time is not an operator \( \mathbb{H} \), which leads to various incarnations of the problem of time in QM \( \mathbb{H}^2 \). From an operational point of view, particularly important is the large class of problems in which one asks what is the probability that a given event will happen at time \( t \). Some members of this class of problems are the arrival time \( \mathbb{H}^{10} \), dwell time \( \mathbb{H}^{17} \) and tunneling time \( \mathbb{H}^{20,26} \). Several inequivalent theoretical approaches have been proposed for each member in this class (see e.g. the review \( \mathbb{H}^6 \)) and there is no consensus which of those approaches, if any, should be the correct one. Moreover, although the different problems in this class are all related to each other, in the literature each of those problems is usually treated separately from the other problems. A satisfying general theory that treats all such problems on an equal footing seems to be missing.

In this paper we develop such a general theory. Our approach is strictly operational in the sense that we study the probability that an event will be detected at time \( t \). But in addition to being operational, our approach is also very general, in the sense that the theory does not depend on details of the detector. All essential quantum ingredients of the theory are formulated in the Hilbert space \( \mathcal{H} \) of the studied system, without a need to study explicitly the Hilbert space of detector states. (Nevertheless, the states of the detector can also be included in the description, which we discuss too.) With such an approach the detector is specified by only two quantities: the time resolution \( \delta t \) of the detector and the projector \( \pi \) acting in \( \mathcal{H} \) that represents the detected event. A typical example useful to have in mind is a detector that determines whether the particle has appeared inside the spatial region \( V_{\text{det}} \), in which case

\[
\pi = \int_{V_{\text{det}}} d^3x |x\rangle\langle x|,
\]

where \( |x\rangle \) are position eigenstates of the considered particle.

In addition to \( \pi \) and \( \delta t \), our general theory involves also the initial state \( |\psi_0\rangle \) and the intrinsic Hamiltonian \( H \) of the studied system, where by “intrinsic” we mean that \( H \) acts on the states in \( \mathcal{H} \) and does not involve interaction with the detector. In the absence of detector, the time evolution of the state is given by \( |\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle \). When the detector is working and when \( \delta t \) is sufficiently small (the precise meaning of “sufficiently small” will be specified later) the main new result of this paper can be summarized by a concise formula for the probability density \( P(t) \) that the event will be detected at time \( t \):

\[
P(t) = w(t)e^{-\int_0^t dt' w(t')},
\]

where

\[
w(t) = \frac{1}{\delta t} \langle \psi_c(t)|\pi|\psi_c(t)\rangle,
\]

\[
|\psi_c(t)\rangle = e^{-iH\delta t/\hbar} e^{-\mathcal{P}(t-\delta t)/\hbar}|\psi_0\rangle,
\]

\[
\mathcal{P} = \bar{\pi}H\bar{\pi}, \quad \bar{\pi} = 1 - \pi.
\]

The derivation of these formulas and their physical meaning is explained in the rest of the paper. We outline the final formulas above so that a reader more interested in applications than in abstract theory can, in principle, skip abstract theory presented in Secs. \( \mathbb{II} \) and \( \mathbb{III} \) and jump to Secs. \( \mathbb{IV} \), \( \mathbb{V} \) and \( \mathbb{VI} \) where we explain the general principles of how can those formulas be applied.

The paper is organized as follows. The main principles of the theory, giving rise to a derivation of the formulas above, are presented in Sec. \( \mathbb{I} \) while various additional aspects of the theory that may be needed for a deeper understanding.
conceptual understanding are discussed in Sec. [III]. After that, we outline general principles of how to apply the theory to a decay of an unstable state in Sec. [V] to arrival time in Sec. [VI] and to dwell time and tunneling time in Sec. [VII]. The conclusions are drawn in Sec. [VIII].

II. DERIVATION OF THE MAIN FORMULA

Suppose that initially, at time $t_0 = 0$, a quantum system is prepared in the state $|\psi_0\rangle$. Let $H$ be the intrinsic Hamiltonian of the system so that, in the absence of detection, the state of the system evolves as

$$|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle.$$  \hfill (6)

(To save writing, unless specified otherwise we work in units $\hbar = 1$.)

Now consider a stopwatch with a time resolution $\delta t$, so that it ticks at times $t_1 = \delta t$, $t_2 = 2\delta t$, $t_3 = 3\delta t$, etc. Furthermore, suppose that the stopwatch is coupled to a detector, so that, at each $t_k$, the detector checks whether the system has an awaited property defined by a projector $\pi$. If the awaited property is detected at a given time $t_k$, then the effect of the detector is to induce the “wave function collapse”

$$|\psi\rangle \rightarrow \frac{\pi|\psi\rangle}{|\pi|\psi\rangle|}.$$  \hfill (7)

At that time, the stopwatch stops and the experiment is over. The final state of the stopwatch (e.g. the final spatial position of the clock’s needle) records the time $t_k$ of detection. Our goal is to determine the probability $P(t_k)$ that the experiment will be over at the time $t_k$.

From a practical point of view, the collapse corresponds to a gain of new information, that is, the information that the property has been detected at $t_k$. Our analysis will not depend on whether the collapse is interpreted as a real physical event, or just as an update of information. The equations that we shall write will not depend on the interpretation of QM. (But some notes on interpretations are given in Sec. [III-D].)

Furthermore, we assume that the detector has a perfect efficiency. Hence, if the awaited property is not detected at $t_k$, then we also gain a new information - the information that the system does not have the property $\pi$ at time $t_k$. Hence an absence of detection also induces a “wave function collapse”, namely

$$|\psi\rangle \rightarrow \frac{\bar{\pi}|\psi\rangle}{|\bar{\pi}|\psi\rangle|},$$  \hfill (8)

where

$$\bar{\pi} = 1 - \pi.$$  \hfill (9)

For example, suppose that the event has not been detected at $t_1$. Then the state evolution form $t_0$ to $t_1$ is

$$|\psi_0\rangle \rightarrow e^{-iH\delta t}|\psi_0\rangle \rightarrow \frac{\bar{\pi}e^{-iH\delta t}|\psi_0\rangle}{|\bar{\pi}e^{-iH\delta t}|\psi_0\rangle|} = \frac{V|\psi_0\rangle}{|V|\psi_0\rangle|},$$  \hfill (10)

where

$$V \equiv \bar{\pi}e^{-iH\delta t}.$$  \hfill (11)

Likewise, if the event has not been detected at $t_1$ and $t_2$, then the state evolution from $t_0$ to $t_1$, and then from $t_1$ to $t_2$, can be written more succinctly as

$$|\psi_0\rangle \rightarrow \frac{V|\psi_0\rangle}{|V|\psi_0\rangle|} \rightarrow \frac{VV|\psi_0\rangle}{|VV|\psi_0\rangle|}.$$  \hfill (12)

By induction, if the event has not been detected at all times from $t_0$ to $t_k$, then the state evolution from $t_0$ to $t_k$ can be written compactly as

$$|\psi_0\rangle \rightarrow \frac{V^k|\psi_0\rangle}{|V^k|\psi_0\rangle|}.$$  \hfill (13)

Now suppose that $\delta t$ is a sufficiently short time, so that the state is not changed much during $\delta t$ by the evolution governed by $H$. This means that $|\langle\psi|e^{-iH\delta t}|\psi\rangle| = 1 - \epsilon$, with $\epsilon \ll 1$. Then $V^k$ can be approximated as

$$V^k = \bar{\pi}e^{-iH\delta t}\bar{\pi}e^{-iH\delta t} \cdots \bar{\pi}e^{-iH\delta t} \approx \bar{\pi}(1 - iH\delta t)\bar{\pi}(1 - iH\delta t) \cdots \bar{\pi}(1 - iH\delta t) = \bar{\pi}(1 - iH\delta t)\bar{\pi}\bar{\pi}(1 - iH\delta t) \cdots \bar{\pi}(1 - iH\delta t),$$  \hfill (14)

where we have used $\bar{\pi} = \bar{\pi}^2$. We assume that the initial state $|\psi_0\rangle$ lies in the Hilbert space $\overline{\mathcal{H}} = \bar{\pi}\mathcal{H}$, i.e. that

$$|\psi_0\rangle = \bar{\pi}|\psi_0\rangle.$$  \hfill (15)

Hence we can write

$$V^k|\psi_0\rangle \approx \bar{\pi}(1 - iH\delta t)\bar{\pi}\cdot \bar{\pi}(1 - iH\delta t)\bar{\pi} \cdots \bar{\pi}(1 - iH\delta t)\bar{\pi}|\psi_0\rangle = (\bar{\pi} - i\overline{H}\delta t)(\bar{\pi} - i\overline{H}\delta t) \cdots (\bar{\pi} - i\overline{H}\delta t)|\psi_0\rangle = (1 - i\overline{H}\delta t)(1 - i\overline{H}\delta t) \cdots (1 - i\overline{H}\delta t)|\psi_0\rangle = (1 - i\overline{H}\delta t)^k|\psi_0\rangle,$$  \hfill (16)

where

$$\overline{H} \equiv \bar{\pi}H\bar{\pi}.$$  \hfill (17)

Finally, defining $t = k\delta t$ and assuming that $k \gg 1$ (so that $t$ is not small even though $\delta t$ is small), we can use the formula

$$\lim_{k \to \infty} (1 - i\overline{H}t/k)^k = e^{-i\overline{H}t}$$  \hfill (18)

to conclude that (16) can be approximated with

$$V^k|\psi_0\rangle \approx e^{-i\overline{H}t}|\psi_0\rangle.$$  \hfill (19)

Since $\overline{H}$ is a hermitian operator, $e^{-i\overline{H}t}$ is unitary. Hence (18) can be approximated with

$$|\psi_0\rangle \approx e^{-i\overline{H}t}|\psi_0\rangle.$$  \hfill (20)
Eq. (20) is quite remarkable; it tells us that the evolution governed by $H$ and interrupted with a large number $k$ of discrete non-unitary collapses can be approximated with a continuous unitary evolution governed by a modified Hamiltonian $\overline{\mathcal{H}}$. To compare it with (3), the content of (20) can be written as the evolution

$$|\psi_c(t)\rangle \simeq e^{-i\overline{\mathcal{H}}\delta t}|\psi_c(t - \delta t)\rangle,$$

(21)

where the label $c$ indicates that it is the conditional state, namely the state valid for the case when no detection has happened up to time $t$. The bar in $|\psi_c(t)\rangle$ indicates that this state lies in the Hilbert space $\overline{\mathcal{H}} = \pi \mathcal{H}$, which is a subspace of the full $\mathcal{H}$.

Now what if the detection has not happened up to time $t - \delta t$ but finally happened at the time $t$? Then the state at time $t$ is $\pi|\psi_c(t)\rangle/|\pi|\psi_c(t)\rangle|$, where

$$|\psi_c(t)\rangle \equiv e^{-iH\delta t}|\overline{\psi}(t - \delta t)\rangle.$$

So if the detection has not happened up to time $t - \delta t$, then the conditional probability that the detection will happen at time $t$ is

$$p(t) = \langle\psi_c(t)|\pi|\psi_c(t)\rangle.$$  

(23)

This can also be written as

$$p(t) = \langle\psi_c(t - \delta t)|e^{iH\delta t}\pi e^{-iH\delta t}|\overline{\psi}(t - \delta t)\rangle,$$

(24)

which shows that in the limit $\delta t \to 0$ we have

$$p(t) \bigg|_{\delta t \to 0} \to \langle\overline{\psi}(t)|\pi|\overline{\psi}(t)\rangle = 0.$$  

(25)

But what is the probability that the detection will not happen up to the time $t - \delta t$? This is the probability $1 - p(t_1)$ that the detection will not happen at $t_1 = \delta t$, times the probability $1 - p(t_2)$ that the detection will not happen at $t_2 = 2\delta t$ (given that it has not happened at $t_1$), $...$, times the probability $1 - p(t_{k-1})$ that the detection will not happen at $t_{k-1} = (k-1)\delta t$ (given that it has not happened at $t_{k-2}$). Hence the overall probability that the detection will happen at time $t = t_k$ is

$$P(t) = p(t_k)[1 - p(t_{k-1})] \cdots [1 - p(t_2)][1 - p(t_1)]$$

$$= p(t_k)e^{\sum_{i=1}^{k-1} \ln[1 - p(t_i)]}$$

$$\simeq p(t_k)e^{-\sum_{i=1}^{k-1} p(t_i)},$$

(26)

where in the last line we have used the approximation $\ln[1 - p(t_i)] \simeq -p(t_i)$, valid because the last line in $\Delta$ shows that $p(t_i)$ is a small quantity for small $\delta t$. For small $\delta t$ we can approximate the sum with the integral

$$\sum_{i=1}^{k-1} \frac{\delta t}{\delta t} \simeq \int_0^{t-\delta t} \frac{dt}{\delta t} \simeq \int_0^{t} \frac{dt}{\delta t},$$

(27)

so (26) can be written in the final form as

$$P(t) \simeq w(t)e^{-\int_0^t dt'w(t')}.$$  

(28)

where

$$P(t) = \frac{P(t)}{\delta t}, \quad w(t) = \frac{p(t)}{\delta t}$$

(29)

are probability densities.

Eq. (28) is our main final result, with $w(t)$ being defined through Eqs. (29), (23), (22) and (21). It coincides with Eqs. (21-23) in the Introduction.

Finally, let us briefly present an alternative derivation of (28), by a reasoning that can be viewed as complementary to (20). We want to determine the probability $P(t_k)$ that the event will be detected at the time $t_k$. The probability that the event will be detected up to the time $t_{k-1}$ is $\sum_{i=1}^{k-1} P(t_i)$, so the probability that it will not be detected up to the time $t_{k-1}$ is $1 - \sum_{i=1}^{k-1} P(t_i)$. Therefore, instead of the first line in (26), alternatively we can write

$$P(t) = p(t_k) \left[1 - \sum_{i=1}^{k-1} P(t_i)\right].$$

(30)

Approximating the sum with the integral and defining $P(t) = P(t)/\delta t$, $w(t) = p(t)/\delta t$ as before, it becomes an integral equation

$$P(t) \simeq w(t) \left[1 - \int_0^t dt' P(t')\right].$$

(31)

To transform the integral equation into a differential one, we take the time derivative of (31)

$$\dot{P} \simeq \dot{w} \left[1 - \int_0^t dt' P(t')\right] - wP \simeq \frac{\dot{w}}{w} - wP,$$

(32)

where the dot denotes the time derivative. The resulting differential equation can then be written as

$$\frac{dP}{dt} \simeq \left[\frac{\ln w}{dt} - w\right] dt,$$

(33)

which is easily integrated to yield (28).

III. GENERAL NOTES

A. Can we let $\delta t \to 0$?

The results in the previous section were obtained in the approximation of sufficiently small $\delta t$. Can we just take the limit $\delta t \to 0$ and say that the equations are exact in that limit? The answer is that we cannot. Not because the limit wouldn’t exist (mathematically it exists!), but because the limit would be trivial and physically uninteresting. This is seen as follows. By expanding $e^{iH\delta t}$ and $e^{-iH\delta t}$ in (24) into powers of $\delta t$ and using $\langle\psi_c(t - \delta t)|\pi = 0$, $\pi|\psi_c(t - \delta t)\rangle = 0$, one finds that the terms proportional to $(\delta t)^0$ and $(\delta t)^1$ do not contribute and that the lowest non-vanishing contribution is

$$p(t) = (\delta t)^2 \langle\psi_c(t - \delta t)|H\pi H|\psi_c(t - \delta t)\rangle + O((\delta t)^4).$$

(34)
Producing a new variable may happen at any time but at certain time it happens, usually called the quantum Zeno effect [27–30].

In other words, if one checks with infinite frequency whether an event has happened, then the event will never happen. This seemingly paradoxical result is in fact well known as the quantum Zeno effect [27–30].

In the unitary quantum description of the stopwatch, the wave function of the clock has many branches, corresponding to many different positions $x_{\text{clock}}$ at which the clock may stop.

\[
V_{V}(\psi_0) \otimes |d, \rangle \rightarrow \cdots
\]

\[
V V_{V}(\psi_0) \otimes |d, \rangle \rightarrow \pi V_{V}(\psi_0) \otimes |d, 3 \rangle \rightarrow \cdots
\]

\[
V(\psi_0) \otimes |d, \rangle \rightarrow \pi V(\psi_0) \otimes |d, 2 \rangle \rightarrow U \pi V(\psi_0) \otimes |d, 2 \rangle \rightarrow \cdots
\]

\[
|\psi_0 \rangle \otimes |d, \rangle \rightarrow \pi U|\psi_0 \rangle \otimes |d, 1 \rangle \rightarrow U \pi U|\psi_0 \rangle \otimes |d, 1 \rangle \rightarrow \cdots
\]

FIG. 1: Left: The stopwatch modeled as a clock with a needle at the position $x_{\text{clock}}$ as a function of time $t$. Initially the clock is running with $x_{\text{clock}} = vt$, but at certain time the clock stops, after which $x_{\text{clock}}$ does not longer change with time. The trajectory $x_{\text{clock}}(t)$ can be thought of as a classical trajectory, or, alternatively, as a (coarse grained) trajectory of clock particles in the Bohmian interpretation. Right: In the unitary quantum description of the stopwatch, the wave function of the clock has many branches, corresponding to many different positions $x_{\text{clock}}$ at which the clock may stop.

The trajectory $x_{\text{clock}}(t)$ is a probability "density" in a consequence is that the corresponding probability density $w(t)$ in (29) vanishes in the limit $\delta t \rightarrow 0$, in which case (28) reduces to the trivial result

\[
P(t) = 0.
\]

(35)

In other words, if one checks with infinite frequency whether an event has happened, then the event will never happen. This seemingly paradoxical result is in fact well known as the quantum Zeno effect [27–30].

B. Total probability

What is the probability that the detection will eventually happen at any time? It is simply $\int_0^t dt P(t)$. Introducing a new variable

\[
u(t) = \int_0^t dt' w(t'),
\]

from (28) we see that

\[
\int_0^\infty dt P(t) = \int_{u(0)=0}^{u(\infty)} du' e^{-u'} = 1 - e^{-u(\infty)} \leq 1. \quad (37)
\]

In particular, the detection will sooner or later happen with certainty if and only if $u(\infty) = \infty$.

Note that $\int_0^\infty dt w(t)$ can be larger than 1, which is consistent because $w(t)$ is a probability "density" in a different sense than $P(t)$. While different $t$'s in $P(t)$ label different random events, different $t$'s in $w(t)$ label different conditions under which an event happens. Let us illustrate it by an example in classical probability. Suppose that a player plays roulette every minute, each time putting money on a single number. Then $\delta t = 1$ minute, each time the probability of winning is $p = \frac{1}{18}$ and $w(t) = \frac{p}{\delta t} = \frac{1}{18}$ per minute is $t$-independent. Hence $\int_0^t dt' w(t') = wt$, which is larger than 1 when $t > 37$ minutes. Now suppose that the player decides to play until he wins. Then (24) gives the probability density that he
will win at the time $t$, so the average time needed for winning is $\langle t \rangle = \int_0^\infty dt t \mathcal{P}(t) = \int_0^\infty dt t e^{-wt} = \frac{1}{w} = 37$ minutes.

### C. POVM

The most general measurements in QM can be described as POVM measurements \[31–37\]. The measurement of the time of detection as described in this paper is not an exception. Eqs. \(2\)–\(4\) can be written as

$$\mathcal{P}(t) = |\psi_0 \rangle \mathcal{E}(t) |\psi_0 \rangle,$$  

where $\mathcal{E}(t)$ is a POVM operator

$$\mathcal{E}(t) = \mathcal{K}^\dagger(t) \mathcal{K}(t),$$  

$$\mathcal{K}(t) = \frac{e^{-\int_0^t dt' w(t')/2}}{\sqrt{\delta t}} \pi e^{-i\delta t H} e^{-i\mathcal{P}(t-\delta t)},$$  

(and we still use units $\hbar = 1$.) Those operators, together with $\overline{E} \equiv 1 - \int_0^\infty dt \mathcal{E}(t)$, make a resolution of the identity

$$\overline{E} + \int_0^\infty dt \mathcal{E}(t) = 1.$$  

A peculiar (yet consistent) feature of the POVM operator \[39\] is that it depends on the state $|\psi_0 \rangle$, through the dependence on $w(t')$ in \[10\] which depends on $|\psi_0 \rangle$ by \[39–41\].

### D. Unitarity, branching and Bohmian mechanics

So far we formulated the theory in terms of “wave function collapses” induced by measurements. But measurement can also be described in a fully unitary manner, without an explicit collapse, provided that the quantum state of the measuring apparatus is also taken into account. Such a formulation is particularly important in the theory of decoherence \[28–38\], as well as in the many world \[34–41\] and Bohmian \[42–46\] interpretations of QM. In the unitary description, every measurement with more than one possible outcomes is associated with a branching of the full quantum state (describing the measured system and the apparatus), with one branch for each possible outcome.

Figs. \[1\] and \[2\] show how such a branching looks like for the theory developed Sec. \[1\] with possible detections at discrete times $t_1 = \delta t$, $t_2 = 2\delta t$, etc. Fig. \[1\] depicts the branching of the clock wave function, while Fig. \[2\] shows a branching diagram for the full quantum state. Fig. \[2\] shows that, at time $t_1$, the full state is a superposition of $i + 1$ terms, namely

$$|\Psi(t_0)\rangle = |\psi_0 \rangle \otimes |d, \rangle,$$
$$|\Psi(t_1)\rangle = V |\psi_0 \rangle \otimes |d, \rangle + \pi U |\psi_0 \rangle \otimes |d, 1\rangle,$$
$$|\Psi(t_2)\rangle = V V |\psi_0 \rangle \otimes |d, \rangle + \pi UV |\psi_0 \rangle \otimes |d, 2\rangle + U \pi U |\psi_0 \rangle \otimes |d, 1\rangle,$$
$$|\Psi(t_3)\rangle = V V V |\psi_0 \rangle \otimes |d, \rangle + \pi U V V |\psi_0 \rangle \otimes |d, 3\rangle + \pi U U \pi U |\psi_0 \rangle \otimes |d, 1\rangle,$$

and so on.

In particular, Fig. \[1\] can be used to understand why is the formula \[20\] for probability $P(t_k)$ of detection at time $t_k$, derived with standard QM, valid also in the Bohmian interpretation. Essentially, this is because Fig. \[1\] shows how the measurement of time is reduced to a measurement of the position of something \[47\], while probabilities of positions in the Bohmian interpretation are the same as probabilities of positions in the standard QM \[42–47\]. It should also be noted that in the literature \[48–50\] the arrival time has been calculated with Bohmian mechanics in a different way, without taking into account the behavior of the measuring apparatus. Such a calculation may give a time distribution of particle arrivals that differs from that in our theory (see Sec. \[V\]). However, a result obtained without taking into account the behavior of the measuring apparatus is not directly relevant for making measurable predictions. In general, Bohmian mechanics makes the same measurable predictions as standard QM only when the behavior of the measuring apparatus is taken into account, as in Figs. \[1\] and \[2\] and Eq. \[12\].

### IV. DECAY OF AN UNSTABLE STATE

#### A. Decay from the general theory

Now we want to understand how the general formalism developed in Sec. \[1\] can be applied to study a decay of an unstable system. Hence we assume that the initial state $|\psi_0 \rangle$ is an unstable state that can decay into many different states, called decay states, orthogonal to $|\psi_0 \rangle$. Furthermore, we assume that the detector (which may be comprised of many small detectors) can detect any of those decay states. Hence we can take

$$\pi = 1 - |\psi_0 \rangle \langle \psi_0 |,$$  

so $\overline{\pi} = |\psi_0 \rangle \langle \psi_0 |$ and the evolution governed by $\overline{\mathcal{T}}$ just keeps the state in the initial state $|\psi_0 \rangle$, up to an irrelevant time-dependent phase. Therefore we can write $|\psi_c(t)\rangle = |\psi_0 \rangle$ and \[21\] reduces to

$$p = \langle \psi_0 | e^{i\mathcal{H} \delta t} [1 - |\psi_0 \rangle \langle \psi_0 |] e^{-i\mathcal{H} \delta t} |\psi_0 \rangle$$
$$= 1 - |\langle \psi_0 | e^{-i\mathcal{H} \delta t} |\psi_0 \rangle|^2,$$  

where we have used the normalization $\langle \psi_0 | \psi_0 \rangle = 1$. 


B. Decay for very small $\delta t$

By expanding $e^{-iH\delta t}$ in (33) up to the term quadratic in $\delta t$ and restoring the units $\hbar \neq 1$, we obtain

$$p \simeq \frac{(\delta t)^2}{\hbar^2} - \frac{1}{\hbar^2} [\langle \psi_0 | H^2 | \psi_0 \rangle - \langle \psi_0 | H | \psi_0 \rangle^2] = \frac{(\delta t)^2 (\Delta H)^2}{\hbar^2},$$

(45)

where $\Delta H$ is the uncertainty of energy defined by $H$, in the state $|\psi_0\rangle$. Eq. (45) is just a special case of (43) and is a well-known result in the study of the quantum Zeno effect for decays [28, 30].

C. Quasi-spontaneous decay

What is usually called a “spontaneous” decay in the literature is in fact a quasi-spontaneous decay. One often thinks of a decay as a process in which a micro system, say an atom, randomly jumps into a more stable state. Intuitively, one often imagines that this jump is spontaneous, in the sense that nothing outside of the micro system influences it. But in standard QM, a random jump is in fact a “wave function collapse” induced by some kind of “measurement”, where “measurement” always involves decoherence induced by a large number of environment degrees of freedom [28, 38]. So there can be no random jump without environment. According to standard QM, in a hypothetic universe containing only one excited atom and nothing else, a random jump should never happen. In this sense there is no such thing as spontaneous decay.

At best we can have a decay which does not depend on details of the environment, creating an illusion that the environment is not important at all, which we refer to as a quasi-spontaneous decay.

Let us briefly explain how quasi-spontaneous decay can be understood within our theory. The only quantitative property of the environment in our theory is the time resolution $\delta t$ of the detector. Hence we can say that the decay is quasi-spontaneous when $w$ in (28) does not depend on $\delta t$. From (29) we see that this means that $p/\delta t$ does not depend on $\delta t$, where $p$ is given by (44). Therefore the decay is quasi-spontaneous when (44) is proportional to $\delta t$, i.e., when we can write

$$p = \Gamma \delta t.$$

(46)

In this case we have $w = \Gamma$, so (2) reduces to

$$P(t) = \Gamma e^{-\Gamma t}.$$

(47)

Hence the probability that the decay will not happen at time $t$ or before is

$$P(t) = 1 - \int_0^t dt' P(t') = e^{-\Gamma t},$$

(48)

which is nothing but the usual exponential law for the survival probability.

But how can the linear law (46) be valid without contradicting (15)? It turns out that the exponential law (15), which is equivalent to the linear law (46), is just an approximation (see e.g. [51, 52]), approximately valid for a large range of intermediate times $t$, but completely wrong for very short and very long times. Hence there is no contradiction because (15) is applicable to intermediate times, while (46) is applicable to very short times.

V. ARRIVAL TIME

Suppose that a wide 1-particle wave packet travels towards a detector laying in the $x$-$y$ plane at $z = 0$. One is interested in the probability density that the particle will arrive to the detector at the time $t$. The corresponding measurable quantity is the probability density that the particle will be detected at the time $t$. The corresponding projector can be taken to be

$$\pi = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-l/2}^{l/2} dz |x, y, z\rangle \langle x, y, z|,$$

(49)

where $l$ is the width of the detector in the $z$-direction. Eq. (3) can then be written as

$$w(t) = \frac{1}{\delta t} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-l/2}^{l/2} dz |\psi_c(x, y, z, t)|^2,$$

(50)

where $\psi_c(x, y, z, t) = (x, y, z) |\psi_c(t)\rangle$. By taking $H$ to be the free Hamiltonian $H = p^2/2m$, the rest of the analysis is, in principle, straightforward (but possibly complicated in practice).

There is, however, one conceptual issue that we want to resolve. The exponential factor in (2) seems to suggest that $P(t)$ decreases with time. While such a decrease is something to be expected in the case of a decay (Sec. IV), it is not expected in the case of an arrival time. So what is the physical meaning of the exponential factor in (2)?

We stress that there is nothing inherently quantum about the exponential factor. It arises from the classical multiplication of probabilities in (20). Hence, to understand qualitatively the meaning of the exponential factor in the time of arrival, it is illuminating to study a purely classical setup. So consider a 1-dimensional system in which a classical particle moves with the constant velocity $v$ from $z < 0$ towards the detector at $z = 0$. We consider a classical statistical ensemble in which the velocity is known exactly, while the knowledge of the position is described by a classical probability density $\rho(z, t)$. The classical analog of the wave function is then simply $\psi(z, t) = \sqrt{\rho(z, t)}$. To further simplify the analysis, we assume that $\delta t \to 0$ (which is allowed in the classical setting) and that $\rho(z, t)$ is a rectangular distribution for fixed $t$. Thus we think of $\psi(z, t)$ as a 1-dimensional rectangular wave packet (Fig. 3), with $|\psi(z, t)|^2 = 1/L$ inside the packet and $|\psi(z, t)|^2 = 0$ outside of the packet. The
FIG. 3: The rectangular wave packet $|\psi(z)|^2$ of width $L$ at a time $t$, for $0 < t < T$. The packet moves to the right with the velocity $v$. The corresponding conditional wave packet $|\psi_c(z)|^2$ (gray region) has support only for $z < 0$, so its width is $\alpha L$ with $\alpha = (T - t)/T$. The correct normalization of the conditional wave packet gives $|\psi_c(z)|^2 = \alpha^{-1}|\psi(z)|^2$ for $z < 0$, so in this region $|\psi(z)|^2 = \alpha|\psi_c(z)|^2$.

packet moves with the velocity $v$ from the region $z < 0$ towards the detector at $z = 0$. Assuming that the front end of the packet approaches the detector at $t = 0$, it follows that the rear end approaches the detector at the time $T = L/v$. Clearly, the probability density of detecting the particle at the time $t$ can only be non-zero for $0 < t < T$. For the rectangular wave packet one expects that this probability density should be uniform, i.e. that $\rho(z,t) = 1/T$ for $0 < t < T$. How can that be compatible with the exponential factor in (2)?

To answer that question, it is essential to have in mind that (3) depends on the conditional state $|\psi_c(t)|$, and not on the non-conditional state $|\psi(t)|$. In our classical case, for $t > 0$, the non-conditional wave function $\psi(z,t)$ has a part that does not vanish for some $z > 0$. But this part originates from parts of the wave function that traveled through $z = 0$, which in $\psi_c(z,t)$ are absent because the conditional probability density $\rho_c(z,t) = |\psi_c(z,t)|^2$ is the probability density conditioned on the assumption that the particle has not been detected at $z = 0$ before the time $t$, so there can be no particle in the region $z > 0$. (In the quantum case, formally, those parts of the wave function are absent because they are removed by a series of $V$-operators that involve the projections $\pi$.) Hence $|\psi_c(z,t)|^2 = 0$ for $z > 0$ and $|\psi_c(z,t)|^2 = [T/(T - t)]|\psi(z,t)|^2$ for $z < 0$, as illustrated by Fig. 3. So given that the particle has not been detected before $t$, the probability density that it will be detected at time $t$ is

$$w(t) = \begin{cases} \frac{1}{T - t} & \text{for } 0 < t < T \\ 0 & \text{otherwise}. \end{cases} \tag{51}$$

Hence

$$w(t) = \int_0^t dt' w(t') = \begin{cases} \frac{\ln T}{T - t} & \text{for } 0 < t < T \\ \frac{\ln \infty}{\infty - t} & \text{for } t \geq T, \end{cases} \tag{52}$$

so (2) finally gives

$$P(t) = \frac{1}{T - t} e^{-\ln \frac{1}{T - t}} = \frac{1}{T} \tag{53}$$

for $0 < t < T$, as expected. Eq. (53) represents the simplest demonstration of how the exponential factor in (2) may give raise to a probability density that does not decrease with time $t$.

VI. DWELL TIME AND TUNNELING TIME

In the case of dwell time one asks how much time a particle will spend in a spatial region $V$, starting from $t = 0$ and assuming that the particle is in $V$ initially at $t = 0$. Operationally we can say that the particle is in the region $V$ until it gets detected in its complement $\overline{V}$. Hence the relevant projector is

$$\pi = \int_V d^3x |x\rangle \langle x|. \tag{54}$$

The probability that the particle will leave $V$ at time $t$ is the probability that it will be detected in $\overline{V}$ at time $t$. Hence the average time needed to leave $V$, that is the average dwell time, is

$$\langle t \rangle = \int_0^\infty dt tP(t), \tag{55}$$

where $P(t)$ is given by (2).

In the case of tunneling, the full space consists of one classically forbidden region $V_{\text{forb}}$ and two classically allowed regions $V_1$ and $V_2$. In the tunneling time problem, one asks how much time the particle will spend in $V_{\text{forb}}$, given that initially the particle is in $V_1$. Assuming that there is no detector in $V_{\text{forb}}$, the problem can be solved by having two detectors, one in $V_1$ and the other in $V_2$. One first uses a detector in $V_1$ to frequently check whether the particle is still in $V_1$. When, at a certain time, it happens that the particle is no longer in $V_1$, we call this time $t_0 = 0$ and study the response of the second detector in $V_2$. The relevant projector associated with the second detector is

$$\pi = \int_{V_2} d^3x |x\rangle \langle x|. \tag{56}$$

Hence $P(t)$ given by (2) is the probability density that the particle will be detected in $V_2$ at time $t$. Between $t_0 = 0$ and $t$ the particle can be considered to be in $V_{\text{forb}}$, so the average time spent there is again given by a formula of the form (55).

VII. CONCLUSION

In this paper we have developed a general theory of computing probability density that an awaited event will
be detected at the time $t$, in the approximation in which the time resolution of the detector $\delta t$ is sufficiently small, so that the quantum state of the measured system does not change much during $\delta t$ by the evolution governed by the intrinsic Hamiltonian $H$ of the measured system. The theory does not depend on any details of the detector, except on its time resolution $\delta t$. The awaited event is defined by a projector $\pi$ in the Hilbert space of the measured system. Time is treated as a classical parameter and no “time operator” is needed. The measurement of time is reduced to an observation of a position of a macroscopic pointer, such as the position of the needle of a clock. The theory is based on the usual “collapse” postulate induced by quantum measurements, but the predictions of the theory do not depend on whether the “collapse” is interpreted as a real physical event or merely as an information update. In particular, the predictions of the theory are also consistent with the many world and the Bohmian interpretation, in which no real collapse is present.

Being concentrated on general theory, in this paper we have outlined the general principles of how the theory can be applied to some more specific problems (decay, arrival, dwell and tunneling time), but we have not analyzed any such realistic problem in detail. Concrete applications of the theory are left as a project for the future work.

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