Six-dimensional time-space crystals

Giedrius Žlabys,¹ Chu-hui Fan,² Egidijus Anisimovas,¹ and Krzysztof Sacha²

¹Institute of Theoretical Physics and Astronomy,
Vilnius University, Saulėtekio 3, LT-10257 Vilnius, Lithuania
²Instytut Fizyki Teoretycznej, Uniwersytet Jagielloński,
ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland

(Dated: December 7, 2020)

Time crystals are characterized by regularity that single-particle or many-body systems manifest in the time domain, closely resembling the spatial regularity of ordinary space crystals. Here we show that time and space crystalline structures can be combined together and even six-dimensional time-space crystals can be realized. As an example, we demonstrate that such time-space crystals can reveal the six-dimensional quantum Hall effect quantified by the third Chern number.

Ordinary space crystals are characterized by a spatially periodic distribution of atoms observed at a fixed instance of time, i.e. the moment of the experimental detection. In time crystals [1–6], the roles of time and space are inverted. One fixes the position in space, which corresponds to the location of a detector, and asks if the probability of clicking of the detector behaves periodically in time. Time crystal behavior can emerge spontaneously in many-body systems [7–29] and can also be engineered by suitable external time-periodic driving [3, 5, 6, 30, 31]. In the latter case we deal with a situation similar to photonic crystals [32] where periodic modulation of the refractive index in space has to be imposed externally. Analogs of various condensed-matter phases have been already investigated in time crystals: Anderson and many-body localization, Mott insulator and topological phases can be observed in the time domain [31, 33–38]. Periodically driven systems can also reveal crystalline structures in the phase space [30, 39, 40]. See [3, 5, 6] for recent reviews.

In the present letter, we demonstrate the notion of time-space crystals (TSC) that support both spatial and temporal periodic structures simultaneously [41–47]. We begin with a single particle moving in a one-dimensional (1D) spatially periodic potential. If such a potential is periodically and resonantly driven in time, a time crystalline structure can be created and combined with the periodic structure in space to form a 2D TSC. Similar shaking can be realized in all three orthogonal directions and allows one to create a 6D TSC. Realization of a 6D crystalline structure paves the way towards investigation of 6D condensed-matter phases. Here we demonstrate how these structures can be endowed with synthetic gauge fields [48–54], and this development completes the toolbox necessary for the realization of the 6D quantum Hall effect (QHE). Here, in addition to the first Chern number [55], the nonlinear quantized response is characterized by the third Chern number [56, 57]. A previous proposal [58] focused on the 4D QHE [59] realized by introducing a single extra dimension to a 3D lattice. From the perspective of accessing higher-dimensional physics with the help of extra or synthetic dimensions [60–64] our work seeks to promote the time as a resource suitable for doubling of the number of dimensions.

2D time-space crystals.—Let us start with a particle in a 1D spatially periodic potential which is periodically shaken in time. For ultracold atoms, such a potential can be realized by modulating two counter-propagating laser beams [54]. Switching to the frame moving with the lattice the scaled Hamiltonian (in the recoil units [65]) reads

$$H(x,p,t) = p_x^2 + V_0 \sin^2 x + p_x \lambda \omega \sin \omega t.$$  

(1)

Here $V_0$ is the depth of the optical lattice, and the last term in (1) results from the transition to the moving frame, see [65]. The position of the optical lattice is harmonically modulated with a small amplitude $\lambda$ and frequency $\omega$ which is chosen resonant with the motion of a particle. Classically, this means that the particle of energy $E < V_0$ is moving periodically in a single lattice site with the frequency $\Omega(E) = \omega/s$, with an integer $s$. In order to understand how time crystalline structure is created let us apply the classical secular approximation. First, we introduce the canonical variables where the position of the particle on a resonant orbit is described by the angle $\theta \in [-\pi, \pi]$. Then, we switch to the frame moving along the orbit, $\Theta = \theta - \omega t/s$, and finally average the resulting Hamiltonian over time. This yields the time-independent effective Hamiltonian, $H_i = P^2/2m_{\text{eff}} - V_{\text{eff}} \cos(s\Theta)$ that describes the motion of a particle in the $i$-th lattice site in the vicinity of the resonant orbit — $m_{\text{eff}}$ and $V_{\text{eff}}$ are constant effective parameters [65]. Thus, in the frame moving along the resonant orbit, a particle behaves like an electron in a spatially periodic potential with $s$ sites. Quantizing the Hamiltonian $H_i$, we obtain its eigenstates in the form of Bloch waves and for $s \gg 1$ the corresponding eigenenergies form energy bands. Focusing on the first energy band, Wannier states $w_{i,\alpha}(\Theta)$ can be defined; they are localized [66] in individual sites $(\alpha = 1, \ldots, s)$ of the effective potential $-V_{\text{eff}} \cos(s\Theta)$. In the original variables, these Wannier states are localized wave packets $w_{i,\alpha}(\theta - \omega t/s)$ moving along the resonant orbit. On a detector located close to the resonant orbit (i.e. at fixed
(\theta), the probability of detection of a particle prepared in, e.g., a Bloch wave \( e^{iK(\theta - \omega t/s)} \sum \psi_{i,\alpha}(\theta - \omega t/s) \) changes periodically in time and reflects a crystalline structure in the time domain — \( K \) can be interpreted as the time-quasimomentum.

The easiest way to describe the entire system beyond a single site is to apply a quantum version of the secular approximation [67]. Due to the periodicity of the system in space and in time we may look for time-periodic Floquet states (i.e., eigenstates of the Floquet Hamiltonian \( \mathcal{H} = H - i\partial_t \) [68, 69]) that describe resonant motion of a particle and are Bloch waves of the form \( e^{ikx}u_{k,\alpha}(x,t) \) where \( k \) is the usual quasimomentum while \( \alpha = 1, \ldots, s \) labels resonant Floquet states. The wave functions \( u_{k,\alpha}(x,t) = u_{k,\alpha}(x,t) = u_{k,\alpha}(x + \pi, t) \) fulfill the eigenvalue problem

\[
[H^{(0)} + (p_x + k)\lambda \sin \omega t - i\partial_t)]u_{k,\alpha} = E_{k,\alpha}u_{k,\alpha},
\]

where the unperturbed part of the Hamiltonian reads \( H^{(0)} = (p_x + k)^2 + V_0 \sin^2 x \) and \( E_{k,\alpha} \) are quasienergies. We choose the eigenstates \( \psi_{k,\alpha}(x) \) of the unperturbed Hamiltonian, \( H^{(0)}\psi_{k,\alpha} = E_{k,\alpha}\psi_{k,\alpha} \), as the basis for the Hilbert space, perform the time-dependent unitary transformation \( \psi'_{k,\alpha} = e^{i\omega t/k}\psi_{k,\alpha} \) (which is the quantum analog of the classical transformation to the frame moving along the resonant orbit) and finally neglect all time-oscillating terms [65]. This yields the following matrix elements of the effective Floquet Hamiltonian

\[
\langle\psi'_{k',\alpha}'|H|\psi'_{k,\alpha}\rangle \approx \left(\frac{E_{k',\alpha}^{(0)} - E_{k,\alpha}^{(0)}}{\omega}\right) \delta_{n',n} - \langle\psi_{k',\alpha}'|\hat{p}_x|\psi_{k,\alpha}\rangle \times \frac{i\omega}{2} \left(\delta_{n'-n-s} - \delta_{n',n+s}\right).
\]

Diagonalization of (3) allows us to obtain the resonant Floquet-Bloch states \( e^{ikx}u_{k,\alpha}(x,t) \). Proper superposition of \( e^{ikx}u_{k,\alpha}(x,t) \) form time-periodic Wannier states \( w_{i,\alpha}(x,t) \) which are localized wave packets moving along the resonant orbit with the period \( 2\pi\omega/\omega \), see Fig. 1. In the subspace spanned by the Wannier states, the quasienergy of the system takes the form of the tight-binding model,

\[
\mathcal{E} = \int_0^{sT} \frac{dt}{sT} \langle\psi|H|\psi\rangle = -\frac{1}{2} \sum_{i,\alpha,j,\beta} J_{i,\alpha}^{j,\beta} a_{i,\alpha}^\dagger a_{j,\beta} + \chi_{i,\alpha}^\dagger \chi_{i,\alpha},
\]

where \( \psi = \sum_{i,\alpha} a_{i,\alpha} w_{i,\alpha}(x,t) \) and \( T = 2\pi/\omega \). The tunneling amplitudes \( J_{i,\alpha}^{j,\beta} = -(2/sT) \int_0^{sT} dt \langle w_{j,\beta}|H|w_{i,\alpha}\rangle \) describe hopping transitions between sites of the optical lattice (Latin labels) and between the time-lattice sites (Greek labels) and are dominant for nearest-neighbor hopping (note that the coefficients \( J_{i,\alpha}^{j,\beta} \) are constant and are the on-site energies). The range of the validity of the quantum secular Hamiltonian (3) can be examined by checking if the classical secular Hamiltonian reproduces quantitatively the exact classical motion [65]. If it is so, the quantum counterpart is also valid. This implies that we are interested in the regime of large amplitude \( V_0 \) where eigenenergies of \( H^{(0)} \) support multiple bands.

Experimentally, this regime is attainable but in order to deal with non-negligible hopping between sites of the optical lattice, the resonant condition for time-periodic shaking of the lattice must correspond to a highly excited band with energy \( E \lesssim V_0 \) and this is the regime we explore here [74]. In order to load ultracold atoms to the proper resonant manifold, the atomic cloud must be initially prepared in an auxiliary static optical lattice consisting of narrow wells so that the width of the corresponding Wannier states matches the width of the Wannier wave packets \( w_{i,\alpha} \). Next the auxiliary lattice should be turned off and the vibrating optical lattice (slightly displaced with respect to the auxiliary lattice) should be turned on. This results in a state where for each \( i \) only one site of the time lattice is occupied, see Fig. 2.

**6D time-space crystals.—**Generalization of 2D TSC to 4D or 6D TSC is straightforward. Indeed, let us consider a 3D optical lattice vibrating along all three orthogonal directions. Then, the Hamiltonian of a particle in the frame vibrating with the lattice is the sum \( H(x, p_x, t) + H(y, p_y, t) + H(z, p_z, t) \) where \( H \) is given in (1). Because there is no coupling between different degrees of freedom of a particle, in order to describe 3D resonant motion of a particle one may use the results obtained in the case of the 2D TSC. We define the Wannier states \( W_{I,\alpha}(z, t) =

![FIG. 1: Probability densities of a Floquet-Bloch state](image-url)
monic vibration must now be modulated according to
the same way. Also, the previously considered har-
lattice potential of Eq. (1) in the presence of an ad-
hanced by tilting the TSC along the time direction.

The last stage of the realization of the QHE in the
2D TSC is to reestablish the suppressed hopping but
with complex tunneling amplitudes \( J_{i}^{\alpha} = e^{i\varphi_{i}} \) where
the phase \( \varphi_{i} \) is a linear function of the site index \( i \). To
this end we apply two laser beams with slightly differ-
ent frequencies (i.e. the difference of the photon ener-
gies matches the difference of the on-site energies, \( 2U_{c} \) per site)

\[
H(x, p_{x}, t) = \frac{p_{x}^{2}}{2m_{\text{eff}}} + \sum_{x} H_{\text{eff}}^{(x)} w_{\alpha} w_{\beta} \cos(s\Theta) + U_{c} \Theta, \tag{6}
\]

and assume that \( U_{c} 2\pi/s \) is smaller than the energy
gap between the first and second energy bands of \( \hat{H}_{i} \) with
\( f_{x}(t) = 0. \) This requires that the additional modu-
lation \( f_{x}(t) \) of the lattices’ motion is chosen in such a way that
\( f_{x}^{(x)} p_{n}(E) \) are the Fourier components of the linear po-
tential \( U_{c} \Theta \). Here \( p_{n}(E) \) are Fourier components of \( p_{x}(t) \)
corresponding to the motion of a particle on the unper-
turbed resonant orbit, i.e. \( p_{x}(t) = \sum_{n} p_{n}(E)e^{in\hbar\omega t} \)
with \( \Omega(E) = \omega/s \) [65]. Note that for \( V_{1} = 0 \), the even
components \( p_{2m}(E) = 0 \), therefore in order to realize the
linear potential (6) we have to add the secondary optical
lattice, cf. (5) to provide \( p_{2m}(E) \neq 0 \). The presence of the
linear potential in the effective Hamiltonian \( \hat{H}_{i} \)
indicates that in the frame moving along the resonant or-
bit a particle experiences a potential tilt. Eigenstates of
the quantized version of \( \hat{H}_{i} \) are localized in \( \alpha = 1, \ldots, s \)
sites of the potential \( -V_{\text{eff}} \cos(s\Theta) \) and they are Wan-
nier states \( w_{i,\alpha}(\Theta) \) and hopping between different sites is
suppressed. The quantized version of the classical secular Hamiltonian \( \hat{H}_{i} \) allows us to find how to shake the optical lattices
in order to turn off hopping of a particle between Wan-
nier states \( w_{i,1} \) and \( w_{i,\beta} \). The predictions based on the
classical secular approach are confirmed by the results of
the quantum secular approximation method which yields the
tight-binding Hamiltonian of the form of (4) where
\( J_{i}^{\beta} = 0 \) for \( \beta \neq \alpha \) and \( J_{i}^{\alpha} = \alpha 2\pi U_{c}/s \). It means we are
able to suppress hopping between time-lattice sites by
tilting the TSC along the time direction.

The optical potentials of the two beams act as a
superlattice (dashed line) that consists of a periodic
sequence of narrow potential wells — their widths are cho-
sen such that the lowest-band Wannier functions (shown
by the colored full lines) approximate the targeted Wannier wave
packets \( w_{i,\alpha} \) at the classical turning points. Next the auxiliary
lattice is turned off and the targeted shaken lattice (solid
black line) is turned on.

\[
w_{i,\alpha}(x, t)w_{y,\gamma}(y, t)w_{z,\zeta}(z, t), \text{ where } w_{i,\alpha} \text{ are Wan-
nier wave packets constructed in the previous paragraph, and derive the 6D tight-binding model of the same form as (4) but with the indices } \vec{i} \text{ and } \alpha \text{ generalized to three-
component vectors } \vec{i} = (i_{x}, i_{y}, i_{z}) \text{ and } \vec{\alpha} = (\alpha_{x}, \alpha_{y}, \alpha_{z}).
\]

In the following, we show how 6D QHE can be realized
in such a 6D TSC by first setting the stage with the re-
alization of synthetic gauge fields that support the 2D
QHE in a 2D TSC.

Quantum Hall effect in 2D time-space crystals.—In ul-
tracold atoms prepared in a 2D time-independent opti-
cal lattice, the QHE was realized with the help of the
photon-assisted tunneling against a potential gradient
created along one of the two spatial directions [51, 52],
alternatively — along a synthetic dimension [61] encoded
by the atom’s internal state [62, 63]. Importantly, the
hopping amplitude acquired a phase that could be con-
trolled by changing the angle between the laser beams.
It allowed one to realize a system where tunneling of an
atom around an elementary plaquette resulted in a
Aharonov-Bohm phase [53] that defines an artificial flux.

The energy bands of such a system are characterized by
the first Chern number that determines the quantization of
the Hall conductivity. Here we show that the con-
cept of photon-assisted tunneling can extended to TSC,
however, the potential tilt is now realized along the time
direction.

We thus consider an atom in the vibrating optical
lattice potential of Eq. (1) in the presence of an ad-
ditional secondary optical lattice vibrating in time in the
same way. Also, the previously considered har-
monic vibration must now be modulated according to
\( f_{x}(t) = \sum_{n \neq 0} f_{n}^{(x)} e^{in\omega t}/s \). The total Hamiltonian of the system reads

\[
\hat{H}(x, p_{x}, t) = H(x, p_{x}, t) + V_{1} \sin^{2}(2x + \phi) + p_{x} f_{x}(t), \tag{5}
\]

where \( H \) is given in (1), \( V_{1} \ll V_{0} \) and \( \phi \neq 0 \). The classical
secular approximation now yields the effective Hamil-
tonian with an additional linear term, \( \hat{H}_{i} = \hat{P}^{2}/(2m_{\text{eff}}) - 
\]

\[
\text{FIG. 2: Loading ultracold atoms to the resonant manifold.}
The atoms are first loaded into the ground state of an aux-
iliary superlattice (dashed line) that consists of a periodic
sequence of narrow potential wells — their widths are cho-
sen such that the lowest-band Wannier functions (shown
by the colored full lines) approximate the targeted Wannier wave
packets \( w_{i,\alpha} \) at the classical turning points. Next the auxiliary
lattice is turned off and the targeted shaken lattice (solid
black line) is turned on.
FIG. 3: Panel (a): Energy dispersion obtained from a 2D tight-binding model of a three-leg ribbon with hopping parameters \( J = |J^{i+1,\alpha}| \) and \( |J^{i,\alpha+1}| = 0.4J \) in the long and short directions, respectively, and \( \pi/2 \) flux per plaquette encoded by Peierls phases. Panel (b): The lowest eigenstate in the Brillouin zone (BZ) where \( \mathbf{k} \) momentum is plotted as a function of the time-lattice indices \( \mathbf{\alpha} = (\alpha_x, \alpha_y, \alpha_z) \) and illustrates its localized nature with respect to all three directions, i.e. on the edge of the 6D ribbon. Parameters that can realize such a ribbon are \( \omega = 236.2, V_0 = 4320, V_1 = 0.1V_0, \phi = \pi/8 \) and \( U_z = 1 \). The first ten harmonics for the tilted potential are included. This gives the hopping parameter \( J \sim 10^{-3} \).

on-site energies of the \( \alpha = 1 \) and \( \alpha = s \) sites. Thus, we deal with the ribbon geometry characterized by open boundary conditions along the time direction. Importantly the complex phase of \( J^{i,\alpha+1} \) changes with \( i \) and it can be controlled by changing the angle between the laser beams. Thus, an atom acquires a phase when it tunnels around a plaquette of the TSC. Such a phase can be associated with a magnetic-like flux and the system can reveal the 2D quantum Hall effect. That is, in the limit of \( s \rightarrow \infty \), the tight-binding model describes a bulk system and depending on the value of the magnetic-like flux different number of energy bands form which can be characterized by the non-vanishing first Chern number \( \nu_1 = (2\pi)^{-1} \int_{BZ} d\mathbf{k} d\mathbf{K} \Omega^{xt} \) which is the integral of the Berry curvature, \( \Omega^{xt} = \partial_x A_K - \partial_K A_x \), over the Brillouin zone (BZ) where \( A_k = \langle n_{k,K}\delta_h|n_{k,K}\rangle \) and \( A_K = \langle n_{k,K}\delta_h|n_{k,K}\rangle \) define the Berry connection and \( e^{i(k_j + K_{\alpha})} |n_{k,K}(j,\alpha)\rangle \) is a Bloch wave belonging to the \( n \)-th band [55–57].

We would like to stress that the topological TSC considered here is described by a 2D tight-binding model and is distinct from a system that realizes a 1D topological insulator [57, 59, 70, 71] based on a 1D model whose parameter changes periodically in time.

Quantum Hall effect in 6D time-space crystals.—The idea for the realization of the topological 2D TSC can be generalized to higher dimensions. Instead of a 1D optical lattice we can consider a particle in a 3D lattice vibrating in three orthogonal directions which is described by the Hamiltonian \( \hat{H}(x,p_x,t) + \hat{H}(y,p_y,t) + \hat{H}(z,p_z,t) \) where \( \hat{H} \) is given in (5). For each degree of freedom of a particle one can apply the same idea of the photon-assisted tunneling as in the case of the 2D TSC and realize a 6D counterpart of the tight-binding model (4) where the complex phase \( \phi_{\mathbf{i}} \) of the tunneling amplitudes \( J^{i,\mathbf{\alpha}} \propto e^{i\phi_{\mathbf{i}}} \) changes with a change of the optical lattice indices \( \mathbf{i} \). The 6D quantum Hall effect can be observed if the third Chern numbers \( \nu_3 \) of energy bands of such a tight-binding model do not vanish. The spatial degrees of freedom of a particle are decoupled which implies that eigenstates of the 6D tight-binding model are products of 2D Bloch waves, i.e. \( e^{i(k_j + K\cdot\mathbf{\alpha})} |n_{k,K}(j,\alpha_x)\rangle |n_{k,K}(j,\alpha_y)\rangle |n_{k,K}(j,\alpha_z)\rangle \), and consequently the third Chern number is a product of the first Chern numbers \( \nu_3 = \nu_1^{(y)}\nu_1^{(x)}\nu_1^{(z)} \). The third Chern number determines how the third order current response depends on an external electromagnetic perturbation [57]. The topological character of the 6D TSC can be illustrated by the presence of topologically protected edge states. For a finite resonance number \( s \), the photon-assisted tunneling reestablishes hopping between time lattice sites but not between the first \( \alpha_x, \alpha_y, \alpha_z = 1 \) and the last \( \alpha_x, \alpha_y, \alpha_z = s \) sites which means we deal with the open boundary conditions and edges in the TSC. The model supports eigenstates that localize at the edges — Fig. 3 shows an example [75]. Such edge states can be prepared experimentally by means of the method illustrated in Fig. 2.

To conclude, we have shown that by combing time and space crystalline structures it is possible to realize 6D time-space crystals. Six-dimensional condensed matter physics is attainable if a 3D spatially periodic system is resonantly driven in time. As an example we have described a route for the realization of gauge fields and observation of the 6D quantum Hall effect. Is 6D time-space electronics around the corner?

G.Z. and C.-h.F. made an equal contribution to the letter. We thank Daria Cegielska for the collaboration at the early stage of the project and Tomasz Kawalec for the helpful discussion. Support of the National Science Centre, Poland via Project No. 2018/31/B/ST2/00349 (C.-h.F.) and QuantERA programme No. 2017/25/Z/ST2/03027 (K.S.) is acknowledged. The work of G.Z. and E.A. was supported by the European Social Fund under Grant No. 09.3.3-LMT-K-712-01-0051.

[1] A. Shapere and F. Wilczek, Phys. Rev. Lett. 109, 160402 (2012), URL http://link.aps.org/doi/10.1103/PhysRevLett.109.160402.
SUPPLEMENTAL MATERIAL

In this Supplemental Material, we present the classical and quantum versions of the secular approximation for the dynamics of a particle in periodically shaken optical lattice potential and show how one derives the tight-binding model for the time-space crystalline structure. We start with the introduction of the basic Hamiltonian used in the letter.

HAMILTONIAN FOR A PARTICLE IN A PERIODICALLY SHAKEN OPTICAL LATTICE POTENTIAL

Let us consider an atom in a periodically shaken optical lattice potential described by the scaled dimensionless Hamiltonian

\[ H(x, p_x, t) = p_x^2 + V_0 \sin^2(x - \lambda \cos \omega t). \]  

(8)

Here we choose to work in the recoil units for the energy \( h^2k_L^2/2m \) and length \( 1/k_L \), with \( k_L \) being the wave number of laser beams that create the optical lattice. The lattice depth is \( V_0 \) and its position is modulated periodically in time with the amplitude \( \lambda \) and frequency \( \omega \). It is convenient to switch to the frame moving with the lattice; in the classical case this is done by means of the time-dependent canonical transformation \( x' = x - \lambda \cos \omega t \) and \( p_x' = p_x \) while in the quantum case using the unitary transformation \( e^{i p_x \lambda \cos \omega t} \). This leads to the Hamiltonian

\[ H(x, p_x, t) = p_x^2 + V_0 \sin^2 x + p_x \lambda \omega \sin \omega t, \]  

(9)

where we drop the primes and use the same symbols \( x \) and \( p_x \) for the transformed position and momentum variables. The Hamiltonian (9) is the starting point for the entire analysis performed in the letter.

We note that by means of a further transformation the momentum shift can be traded for a term \(-xF(t)\) describing a homogeneous inertial force [1]. However, we choose not to follow this course in order to preserve the explicit spatial translational symmetry.

CLASSICAL SECULAR APPROXIMATION APPROACH

For the sake of reference, let us start with the unperturbed situation, i.e., a stationary lattice (\( \lambda = 0 \)) and a classical particle of energy \( E < V_0 \) undergoing periodic oscillations in the vicinity of a single potential minimum with an energy-dependent frequency \( \Omega(E) \). To simplify the classical description it is convenient to perform the canonical transformation to the action-angle variables [2],

\[ I = \frac{1}{2\pi} \int p_x(E, x) dx, \]  

(10a)

\[ \theta = \frac{\partial}{\partial t} \int_{x_0}^{x} p_x(E, x') dx', \]  

(10b)

with \( x_0 \) set to the left classical turning point. Then the unperturbed Hamiltonian depends on the new momentum \( I \) (the action) alone, i.e., \( H_0 = p_x^2 + V_0 \sin^2 x = H_0(I) \), and the solution of the Hamilton equations of motion is trivial: \( I = \text{const.} \), while the new coordinate (the angle) is changing at a constant rate \( \theta(t) = \Omega(E)t + \theta(0) \) with \( \Omega(E(I)) = dH_0(I)/dI \).

Now let us turn on the shaking, which couples to the momentum \( p_x(\theta, I) \), and assume that the frequency \( \omega \) fulfills the resonant condition \( \omega = s\Omega(E(I)) \) where \( s \) is an integer and \( I_s \) is the resonant value of the action. Such a \( s : 1 \) resonant driving of a particle can be accurately described by the secular approximation provided the time-periodic perturbation is weak [2–5]. In order to obtain the effective Hamiltonian that describes the motion of a particle in the vicinity of the resonant orbit, we first switch to the frame moving along the resonant orbit, \( \Theta = \theta - \omega t/s \), which results in

\[ H(\Theta, I, t) = H_0(I) - \frac{\omega I}{s} + p_x(\Theta + \omega t/s, I) \lambda \omega \sin \omega t. \]  

(11)

As the momentum is periodic with respect to \( \theta \), we perform the Fourier expansion,

\[ p_x(\Theta + \omega t/s, I) = \sum_n p_n(I) e^{in(\Theta + \omega t/s)}, \]  

(12)

and finally average the Hamiltonian over the time keeping \( \Theta \) and \( I \) fixed because they are slowly varying variables if we stay close to the the resonant orbit, i.e. for \( P = I - I_s \approx 0 \) [2–5]. This yields the effective time-independent Hamiltonian

\[ H_i = \left[ H_0(I_s) - \frac{\omega I_s}{s} \right] + \frac{P^2}{2m_{\text{eff}}} - V_{\text{eff}} \cos(s\Theta), \]  

(13)

where \( \Theta \in [-\pi, \pi] \), \( V_{\text{eff}} = \lambda \omega |p_x(I_s)| \) and we have performed the Taylor expansion for \( I \) around \( I_s \) up to the second order, which allowed us to define the effective mass \( m_{\text{eff}} = d^2H_0(I_s)/dI_s^2 \). The constant term in the square brackets in (13) can be omitted because it does not influence the dynamics. The Hamiltonian (13) describes the resonant motion of a particle in the \( i \)-th site of the shaken optical lattice. Its form shows that for \( s \gg 1 \), a particle behaves like an electron moving in a periodic potential created by ions in a solid state crystal. This situation can be interpreted as engineering of a synthetic dimension, related to the time variable rather than atom's
internal degrees of freedom [6–8]. Note that for a particle moving in the sinusoidal potential \( V_0 \sin^2 x \) even Fourier components of the momentum vanish, \( p_{2n}(E) = 0 \). Consequently only odd numbers \( s \) of sites in the periodic potential in (13) can be realized.

In the letter we are also interested in the realization of a tilted periodic effective potential, i.e. when the effective Hamiltonian describing resonant motion of a particle in the \( i \)-th lattice has the form

\[
\hat{H}_i = \frac{\hat{P}^2}{2m_{\text{eff}}} - \hat{V}_{\text{eff}} \cos(s\Theta) + U_x \Theta. \tag{14}
\]

It can be done if one adds a weak secondary optical lattice and starts shaking the entire potential in an appropriate manner in time. With this goal in mind, the Hamiltonian (9) is modified to read

\[
\hat{H}(x, p_x, t) = p_x^2 + V_0 \sin^2 x + V_1 \sin^2(2x + \phi) + p_x [\lambda \sin \omega t + f_x(t)], \tag{15}
\]

with \( V_1 \ll V_0 \) and

\[
f_x(t) = \sum_{n \neq 0} f_n(x)e^{in\omega t/s} \tag{16}
\]

describing a weak modulation of the harmonic vibration of the entire lattice. Introducing the action-angle variables corresponding to the new unperturbed Hamiltonian \( \hat{H}_0 = p_x^2 + V_0 \sin^2 x + V_1 \sin^2(2x + \phi) \) we again rely on the same secular approximation and obtain the effective Hamiltonian (14) where

\[
U_x \Theta = U_x \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{in\Theta} = \sum_{n \neq 0} f_n(x) p_n(E) e^{in\Theta}. \tag{17}
\]

That is, \( f_x(t) \) is chosen so that \( f_n(x)p_n(E) \) are Fourier components of the linear potential. Note, that if \( p_n(E) \neq 0 \) for all \( n \), it is always possible to adjust \( f_n(x) \) so that the products \( f_n(x)p_n(E) \) take values we need. The role of the secondary lattice is now clear because for \( V_1 = 0 \) we would not be able to reproduce the linear potential in (17) due to the fact that the even components \( p_{2n}(E) = 0 \). When the secondary lattice is on, all components \( p_n \) are non-zero and we can engineer any effective potential, in particular the linear one.

The validity of the secular Hamiltonian (13) [or (14)] can be examined by comparison of the phase space picture it generates with the stroboscopic map obtained from numerical integration of the exact equations of motion. Such a comparison is presented in Fig. 4 and shows that for sufficiently weak shaking the secular approximation method leads to accurate quantitative description of the system.

FIG. 4: Top panels: phase space pictures, in the action-angle variables, generated by the classical secular Hamiltonian (13) for \( s = 3 \), \( V_0 = 4320 \), \( \omega = 240 \) and \( \lambda = 0.01 \) (left) and \( \lambda = 0.025 \) (right). Bottom panels: stroboscopic maps obtained by numerical integration of the exact equations of motion — bottom left (right) panel corresponds to the phase space pictures presented in top left (right) panel. The stroboscopic maps are obtained by plotting points \( \{\Theta(nT), I(nT)\} \) where \( T = 2\pi/\omega \) is the period of the lattice shaking and \( n \)'s are integer.

QUANTUM SECULAR APPROXIMATION METHOD

Quantization of the classical secular Hamiltonian (13) [or (14)] allows one to obtain quantum description of resonant motion of a particle in a single site of the optical lattice. To incorporate also tunneling transitions between different real-space sites of the optical lattice potential one can switch to the quantum version of the secular approximation method [4, 5, 9] which we introduce in the present section.

The Hamiltonian (9) is periodic in time, \( H(x, p_x, t + T) = H(x, p_x, t) \) where \( T = 2\pi/\omega \). Thus, we may look for time-periodic Floquet states which are eigenstates of the Floquet Hamiltonian \( \hat{H} = \hat{H} - i\partial_t \). The Hamiltonian (9) is also periodic in space \( H(x + \pi, p_x, \cdot) = H(x, p_x, \cdot) \) and consequently the Floquet states have the form of Bloch waves \( e^{ikx}u_{k,\alpha}(x, t) \) where \( k \) is a quasi-momentum, \( u_{k,\alpha}(x + \pi, t) = u_{k,\alpha}(x, t) = u_{k,\alpha}(x, t + T) \). The wavefunctions \( u_{k,\alpha}(x, t) \) fulfill the Floquet eigenvalue equation

\[
\left[ H^{(0)}(k) + (p_x + k)\lambda \sin \omega t - i\partial t \right] u_{k,\alpha} = E_{k,\alpha} u_{k,\alpha}, \tag{18}
\]

where

\[
H^{(0)}(k) = (p_x + k)^2 + V_0 \sin^2 x, \tag{19}
\]

and \( E_{k,\alpha} \)'s are quasi-energies of the system. The index \( \alpha \) labels different Floquet states corresponding to the same quasi-momentum \( k \).
Solutions of (18) related to Floquet states that describe resonant dynamics of a particle can be obtained in a simple way by applying the quantum secular approximation approach. To this end we choose the eigenstates $\psi_{k,n}(x)$ of the unperturbed Hamiltonian,

$$H^{(0)}(k)\psi_{k,n}(x) = E^{(0)}_{k,n}\psi_{k,n}(x),$$

(20)
as the basis for the Hilbert space of a particle. Next we perform the time-dependent unitary transformation $\psi'_{k,n} = e^{-in\omega t/s}\psi_{k,n}$ which is a quantum analog of the canonical transformation to the frame moving along a resonant orbit. It results in the following matrix elements of the Floquet Hamiltonian

$$\langle \psi'_{k,n'}|H|\psi'_{k,n}\rangle = \left(E^{(0)}_{k,n} - n\frac{\omega}{s} + k\lambda\omega \sin \omega t\right)\delta_{n'n} - \langle \psi_{k,n'}|p_x|\psi_{k,n}\rangle \frac{i\lambda\omega}{2}\times \left(e^{i(n'-n+s)\frac{\pi}{\omega}} - e^{i(n'-n-s)\frac{\pi}{\omega}}\right).$$

(21)

Averaging the above Hamiltonian over time while keeping all quantities fixed (which is valid in the resonant subspace where $|E^{(0)}_{k,n+1} - E^{(0)}_{k,n}| \approx \omega/s$) we get the desired time-independent effective Floquet Hamiltonian

$$\langle \psi'_{k,n'}|H|\psi'_{k,n}\rangle \approx \left(E^{(0)}_{k,n} - n\frac{\omega}{s}\right)\delta_{n'n} - \langle \psi_{k,n'}|p_x|\psi_{k,n}\rangle \frac{i\lambda\omega}{2}\times (\delta_{n'n-s} - \delta_{n'n+s}).$$

(22)

Diagonalization of (22) allows us to obtain $s$ resonant Floquet states

$$e^{ikx}u_{k,\alpha}(x,t) = e^{ikx} \sum_n c^{(\alpha)}_{k,n} e^{-in\omega t/s}\psi_{k,n}(x),$$

(23)

where $\alpha = 1, \ldots, s$ and $c^{(\alpha)}_{k,n}$ are constants. To identify the resonant Floquet states among all eigenstates of (22) it is helpful to compare the spectrum of (22) with the spectrum of the quantized version of the classical secular Hamiltonian (13). Figure 5 presents such comparison and demonstrates also consistency and validity of the approximation methods we use. That is, the classical secular Hamiltonian reproduces the exact classical dynamics very well for sufficiently weak time-periodic driving (cf. Fig. 4) and the spectrum of its quantized version match the resonant energy levels of the quantum secular Hamiltonian (22).

**TIGHT-BINDING MODEL OF TIME-SPACE CRYSTALS**

Let us illustrate how to derive a tight-binding model for a time-space crystal by focusing on the system described by the Hamiltonian (9).

![FIG. 5: Black lines show energy levels obtained by diagonalization of the quantum secular Hamiltonian (22) versus $\lambda$ for $s = 3$, $V_0 = 4320$ and $\omega = 240$. Red dashed lines present resonant energy levels obtained by diagonalization of the quantized version of the classical secular Hamiltonian (13). Comparison of these two sets of data allows one to identify the resonant energy levels among many eigenenergies of the quantum secular Hamiltonian — the highest energy levels in the plot correspond to the $s = 3$ resonant Floquet states we are interested in. The inset shows a zoomed out version of the spectrum of the quantum secular Hamiltonian with the vertical energy axis covering a broader range while the range of $\lambda$s is the same.](image)

We are interested in resonant dynamics of a particle in the shaken optical lattice potential, i.e. we restrict to the Hilbert subspace spanned by the Floquet-Bloch states (23). Within this subspace one can define Wannier-like states $w_{i,\alpha}(x,t)$ which are localized wave-packets moving along classical resonant orbits in each $i$-site of the optical lattice potential. That is, in each $i$-site one can define $\alpha = 1, \ldots, s$ wave-packets which are propagating along the resonant orbit one by one with the period $2\pi/\omega$ and are delayed with respect to each other by $2\pi/\omega$.

In practice to obtain $w_{i,\alpha}$ one can diagonalize the position operator $\hat{O} = x$ (or $\hat{O} = e^{ix}$) in the subspace spanned by the resonant Floquet-Bloch states,

$$\langle k',\alpha'|\hat{O}|k,\alpha\rangle = \int dx \left[e^{ik'z}u_{k',\alpha}(x,t)\right]^* \hat{O} e^{ikx}u_{k,\alpha}(x,t).$$

(24)

Any $t$ in (24) can be chosen except moments of time when wave-packets strongly overlap because then eigenvalues of $\hat{O}$ become degenerate and the diagonalization can result in linear combinations of the Wannier states we look for. The obtained Wannier states read

$$w_{j,\alpha}(x,t) = \sum_{k,\beta} b^{j,\alpha}_{k,\beta} e^{ikx}u_{k,\beta}(x,t),$$

(25)

where $b^{j,\alpha}_{k,\beta}$ are constants. Having the Wannier state basis one can derive a tight-binding model by calculating...
matrix elements of the Floquet Hamiltonian in such a basis,
\[
J_{j,\alpha}^i,\beta = -2 \int_0^{sT} \frac{dt}{sT} \langle w_{j,\beta} | \mathcal{H} | w_{i,\alpha} \rangle = -2 \sum_{k,\gamma} E_{k,\gamma} \left( b_{k,\gamma}^{j,\beta} \right)^* b_{k,\gamma}^{i,\alpha},
\]
where \(E_{k,\gamma}\) are quasi-energies, cf. (18). Finally we obtain a tight-binding model which corresponds to quasi-energy \(\mathcal{E}\) of a particle prepared in a state \(\psi(x,t) = \sum_{i,\alpha} a_{i,\alpha} w_{i,\alpha}(x,t)\),
\[
\mathcal{E} = \int_0^{sT} \frac{dt}{sT} \langle \psi | \mathcal{H} | \psi \rangle = -\frac{1}{2} \sum_{i,\alpha,j,\beta} J_{j,\alpha}^i,\beta a_{j,\beta}^* a_{i,\alpha},
\]
where \(J_{j,\alpha}^i,\beta\)'s have the meaning of tunneling amplitudes and they are dominant for hopping of a particle between nearest neighbor optical lattice sites.

[1] E. Arimondo, D. Ciampini, A. Eckardt, M. Holthaus, and O. Morsch, in Advances in Atomic, Molecular, and Optical Physics, edited by P. Berman, E. Arimondo, and C. Lin (Academic Press, 2012), vol. 61 of Advances In Atomic, Molecular, and Optical Physics, pp. 515–547, 1203.1259, URL http://www.sciencedirect.com/science/article/pii/B9780123964823000107.
[2] A. Lichtenberg and M. Lieberman, Regular and chaotic dynamics, Applied mathematical sciences (Springer-Verlag, 1992), ISBN 9783540977452, URL https://books.google.pl/books?id=2saPAQAAAMAAJ.
[3] A. Buchleitner, D. Delande, and J. Zakrzewski, Physics reports 368, 409 (2002), URL http://www.sciencedirect.com/science/article/pii/S0370157302002703.
[4] K. Sacha and J. Zakrzewski, Rep. Prog. Phys. 81, 016401 (2018), URL https://doi.org/10.1088/1361-6633/aa8b38.
[5] K. Sacha, Time Crystals (Springer International Publishing, Cham, 2020), ISBN 978-3-030-52523-1, URL https://doi.org/10.1007/978-3-030-52523-1.
[6] B. K. Stuhl, H.-I. Lu, L. M. Aycock, D. Genkina, and I. B. Spielman, Science 349, 1514 (2015), 1502.02496, URL http://arxiv.org/abs/1502.02496.
[7] M. Mancini, G. Pagano, G. Cappellini, L. Livi, M. Rider, J. Catani, C. Sias, P. Zoller, M. Inguscio, M. Dalmonte, et al., Science 349, 1510 (2015), ISSN 0036-8075, 1502.02495, URL https://science.sciencemag.org/content/349/6255/1510.
[8] A. Celi, P. Massignan, J. Ruseckas, N. Goldman, I. B. Spielman, G. Juzeliūnas, and M. Lewenstein, Phys. Rev. Lett. 112, 043001 (2014), URL http://arxiv.org/abs/1307.8349.
[9] G. Berman and G. Zaslavsky, Physics Letters A 61, 295 (1977), ISSN 0375-9601, URL http://www.sciencedirect.com/science/article/pii/0375960177906181.