A HIGHER GROTHENDIECK CONSTRUCTION

AMIT SHARMA

Abstract. In this note we present a Grothendieck construction for functors taking values in quasi-categories. We construct a simplicial space from such a functor whose zeroth row is the desired construction. Using our construction we give a new proof of rectification theorem for coCartesian fibrations of simplicial sets.

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1. Introduction

The Grothendieck construction is ubiquitous in category theory. This construction associates to a (pseudo) functors $F : D \to \text{Cat}$, a (op)fibration over the (small) category $D$. The construction establishes an equivalence and therefore allows us to switch between $\text{Cat}$-valued functors and fibrations. In this note we want to extend the classical Grothendieck construction to quasi-category valued functors with the aim of establishing an equivalence between a category of $\mathcal{S}$-valued functors and appropriately defined (simplicial)-fibrations over the nerve of the domain category of functors. In this note we will be primarily working with the adaptation of Joyal model category structure on $\mathcal{S}$, $\text{Joy08b}$, $\text{Joy08a}$, to marked simplicial sets.

It is well known that a left fibration of simplicial sets over the nerve of a (small) category $N(D)$ is determined up to equivalence by a homotopy coherent diagram taking values in Kan complexes, see $\text{Cis19}$, 5.3]. The same holds for coCartesian fibration of simplicial sets over $N(D)$ with respect to homotopy coherent diagrams taking values in quasi-categories, see $\text{Lur09}$, Ch. 3]. The main goal of this note is to show that the aforementioned homotopy coherent diagrams can be rectified i.e. up to equivalence they can be replaced by an honest functor. More precisely, we will show that for each coCartesian fibration $p : X \to N(D)$, there exists a (honest) functor $Z : D \to \mathcal{S}$, taking values in quasi-categories whose Grothendieck construction, denoted $\int_{d \in D} Z$, is equivalent to the fibration $p$ in a suitably defined model category structure on $\mathcal{S}/N(D)$. Such a result first appeared in $\text{Lur09}$, Ch. 3] where the author defines an extension of the classical Grothendieck construction called relative nerve which determines a functor $N_{\bullet}(D) : [D, \mathcal{S}] \to \mathcal{S}/N(D)$. The author goes further to present another version of the relative nerve for marked simplicial sets which is a functor $N^+_{\bullet}(D) : [D, \mathcal{S}^+] \to \mathcal{S}^+/N(D)$. This functor is shown to be the right Quillen functor of a Quillen adjunction between the coCartesian model category structure on $\mathcal{S}^+/N(D)$ and the projective model category structure on $[D, (\mathcal{S}^+, \mathcal{Q})]$. The guiding principle of our Grothendieck construction is that homotopy colimit of a functor $H : D \to \mathcal{S}$ taking values in quasi-categories should be obtained upon inverting the coCartesian edges of the total space of the Grothendieck construction. We recall that a homotopy colimit of a functor $G : D \to \text{Cat}$ is obtained in this way. Our Grothendieck construction is isomorphic to the relative nerve of a functor $H : D \to \mathcal{S}$ but our construction is a part of a larger structure, namely a simplicial space, which we extract out of the functor $H$. The main objective of this note is to establish a Quillen equivalence whose left Quillen functor has a (total left) derived functor which is isomorphic to a (total right) derived functor of (a marked simplicial sets version of) our Grothendieck construction.

In section 2 of this note we describe a (higher) Grothendieck construction for functors taking values in $\mathcal{S}$. In the same section we define a simplicial space (or a bisimplicial set) for each functor $F : D \to \mathcal{S}$ which encodes the information in the functor as a fibration. We define the Grothendieck construction of $F$ to be the zeroth row of the aforementioned simplicial space. This defines a functor $\int^{d \in D} - : [D, \mathcal{S}] \to \mathcal{S}/N(D)$. In section 3 we define a version of our Grothendieck construction functor for marked simplicial sets $\int^{+}_{d \in D} - : [D, \mathcal{S}^+] \to \mathcal{S}^+/N(D)$. In the same section we establish a Quillen equivalence $(\mathcal{Q}_{D}, \mathcal{R}_{D}^{+})$ between the projective model category structure on $[D, \mathcal{S}^+]$ and the coCartesian model category structure on $\mathcal{S}^+/N(D)$. This result implies that a (total right) derived functor
of our Grothendieck construction functor is isomorphic to a (total left) derived functor of $2^D$.

A version of this result for left fibrations has been proved in [HM15] where the authors establish a Quillen equivalence between the covariant model category structure on $S/N(D)$, see [Joy08b, Ch. 8] and the projective model category structure on the functor category $\mathcal{D}, (\mathcal{S}, \text{Kan})$.

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2. A Grothendieck construction

In this section we will describe a Grothendieck construction for quasi-categories. The classical Grothendieck construction defines a functor

\[ \int^{d \in D} - : [D; \text{Cat}] \to \text{Cat}/D \]

The construction described in this note also defines a functor, which we denote by \( f^{d \in D} - \), which is a left Kan extension of the above functor along the Nerve functor \( [D; N] : [D; \text{Cat}] \to [D; \text{S}] \).

Let \( X : D \to \text{S} \) be a functor. We recursively define a collection of simplicial sets as follows:

\[ G_0^X(d) := X(d). \]

For a map \( f : d_1 \to d_2 \) in \( D \), we define a simplicial set \( G_1(f) \) by the following pullback square:

\[
\begin{array}{ccc}
G_1^X(f) & \xrightarrow{p_2(f)} & [\Delta[1]; X(d_2)] \\
p_1(f) \downarrow & & \downarrow [d_1; X(d_2)] \\
[\Delta[0]; X(d_1)] \times [\Delta[0]; X(d_2)] & \xrightarrow{\Delta[0]; X(f) \times \text{id}} & [\Delta[0]; X(d_2)] \times [\Delta[0]; X(d_2)]
\end{array}
\]

Remark 1. For each object \( d \in D \)

\[ G_1^X(id_d) = [\Delta[1], X(d)]. \]

For a pair of maps \( f_1 : d_1 \to d_2, f_2 : d_2 \to d_3 \) in \( D \), we define a simplicial set \( G_2^X(f_1, f_2) \) by the following pullback square:

\[
\begin{array}{ccc}
G_2^X(f_1, f_2) & \xrightarrow{p_2((f_1, f_2))} & [\Delta[2]; X(d_3)] \\
p_1((f_1, f_2)) \downarrow & & \downarrow ([d_0; X(d_3)], [d_1; X(d_3)], [d_2; X(d_3)]) \\
G_1^X(f_2) \times G_1^X(f_2 f_1) \times G_1^X(f_1) & \xrightarrow{F_1 \times F_2 \times F_1} & \prod X([\Delta[1]; X(d_3)])
\end{array}
\]

where \( F_1 \) is the composite map:

\[ G_1^X(f_2) \xrightarrow{p_2(f_2)} [\Delta[1]; X(d_2)] \xrightarrow{[\Delta[1]; X(f_2)]} [\Delta[1]; X(d_3)], \]

\( F_3 = p_2(f_2), F_2 = p_2(f_2 f_1) \) and \( p_1((f_1, f_2)) = (p_1(f_1), p_1(f_2), p_1(f_2 f_1)). \)

Remark 2. For each \( f \in \text{Mor}(D) \), the simplicial sets \( G_2^X((f, id)) \) and \( G_2^X((id, f)) \) are given by the following two pullback squares respectively:

\[
\begin{array}{ccc}
G_2^X(f, id) & \xrightarrow{p_2((f, id))} & [\Delta[2]; X(d_2)] \\
p_1((f, id)) \downarrow & & \downarrow ([d_1; X(d_3)], [d_2; X(d_3)]) \\
G_1^X(f) \times G_1^X(id) & \xrightarrow{F_2 \times F_1} & [\Delta[1]; X(d_2)] \times [\Delta[1]; X(d_2)]
\end{array}
\]
and

\[ \mathcal{G}^X_n(id, f) \xrightarrow{p_2((id, f))} [\Delta[2]; X(d_2)] \]

\[ \mathcal{G}^X_n(f) \times \mathcal{G}^X_n(f) \times [\Delta[1], X(d_1)] \xrightarrow{F_2 \times F_2 \times i} [\Delta[1]; X(d_2)] \times [\Delta[1]; X(d_2)] \times [\Delta[1]; X(d_2)] \]

For an \( n \)-tuple \( \sigma = (f_1, f_2, \ldots, f_n) \in (N(D))_n \), we define a simplicial set \( \mathcal{G}^X_n(\sigma) \) by the following pullback square:

\[
\begin{array}{ccc}
\mathcal{G}^X_n(\sigma) & \xrightarrow{p_2(\sigma)} & [\Delta[n]; X(d_{n+1})] \\
p_1(\sigma) \downarrow & & \downarrow H \\
\prod_{i=0}^n \mathcal{G}^X_{n-1}(d_i(\sigma)) & \xrightarrow{F_{n+1} \times \cdots \times F_1} & \prod_{i=0}^n [\Delta[n-1]; X(d_{n+1})]
\end{array}
\]

where \( H = ([d_0; X(d_{n+1})], [d_1; X(d_{n+1})], \ldots, [d_n; X(d_{n+1})]) \) and for \( 2 \leq i \leq n+1 \) the simplicial map \( F_i \) is the following composite:

\[ \mathcal{G}^X_{n-1}(d_i(\sigma)) \xrightarrow{p_2(d_i(\sigma))} [\Delta[n-1]; X(d_{n+1})] \]

The map \( F_1 \) is the following composite

\[ \mathcal{G}^X_{n-1}(d_n(\sigma)) \xrightarrow{p_2(d_n(\sigma))} [\Delta[n-1]; X(d_n)] \xrightarrow{[\Delta[n-1]; X(f_n)]} [\Delta[n-1]; X(d_{n+1})] \]

**Remark 3.** For the canonical simplex \( \sigma = \text{id}(d_n) \), see definition ??, the simplicial set

\[ \mathcal{G}^X_n(\text{id}(d_n)) = [\Delta[n], X(d_{n+1})]. \]

**Definition 2.1.** For a pair consisting of an \( n \)-simplex \( \sigma \in N(D)_n \) and a functor \( X : D \to S \), we will refer to \( \mathcal{G}^X_n(\sigma) \) as the 1-Gerbe over \( \sigma \) determined by \( X \).

**Proposition 2.2.** For each \((n - 1)\)-simplex \( \rho \) in \( N(D) \) there is an inclusion map

\[ i^\rho_j : \mathcal{G}^X_{n-1}(\rho) \to \mathcal{G}^X_n(s_j(\rho)) \]

where \( s_j \) is the \( j \)th degeneracy operator of \( N(D) \) for \( 1 \leq j \leq n \).

**Proof.** The simplicial map \( i^\rho_j \) is the unique map into the pullback shown in the following diagram:

\[
\begin{array}{ccc}
\mathcal{G}^X_{n-1}(\rho) & \xrightarrow{p_2(\rho)} & [\Delta[n-1]; X(d_{n+1})] \\
\mathcal{G}^X_n(s_j(\rho)) & \xrightarrow{p_2(s_j(\rho))} & [\Delta[n]; X(d_{n+1})] \\
\prod_{i=0}^n \mathcal{G}^X_{n-1}(d_i(s_j(\rho))) & \xrightarrow{F_{n+1} \times \cdots \times F_1} & \prod_{i=0}^n [\Delta[n-1]; X(d_{n+1})]
\end{array}
\]

where \( i_j \) is the inclusion into the \( j \)th component namely \( \mathcal{G}^X_{n-1}(d_j s_j(\rho)) = \mathcal{G}^X_{n-1}(\rho) \).

\( \square \)
Proposition 2.3. There is a simplicial space i.e. a functor \( (\int_{d \in D} X) \), \( \Delta^{op} \to S \) whose degree \( n \) simplicial-set is defined as follows:

\[
\left( \int_{d \in D} X \right) ([n]) := \bigcup_{\sigma \in (N(D))_n} \{\sigma\} \times G_{n}^{X}(\sigma)
\]

Proof. We will define the degeneracy and face operators. Each \( G_{n}^{X}(\sigma) \) is equipped with a projection map

\[
d_{i}(p_{1}(\sigma)) : G_{n}^{X}(\sigma) \to G_{n-1}(d_{i}(\sigma))
\]

For \( i \in \{0, 1, 2, \ldots, n\}, \) this map is given by the following composite:

\[
g_{n}^{X}(\sigma) \xrightarrow{p_{1}(\sigma)} \prod_{i=0}^{n} G_{n-1}(d_{i}(\sigma)) \xrightarrow{pr_{i}} G_{n-1}(d_{i}(\sigma)),
\]

where \( f_{n} : d_{n} \to d_{n+1} \) is the last map in \( \sigma = (f_{1}, \ldots, f_{n}) \) and \( pr_{i} \) are the obvious projections from the product. The maps \( d_{i}(p_{1}(\sigma)) \) join together to form a map

\[
d_{i} : \bigcup_{\sigma \in (N(D))_n} G_{n}^{X}(\sigma) \to \bigcup_{\rho \in (N(D))_{n-1}} G_{n-1}^{X}(\sigma)
\]

which is our \( i \)th face operator for \( 0 \leq i \leq n \).

The maps \( i_{p}^{\rho} \) from proposition 2.2 gives us the \( i \)th degeneracy map

\[
s_{j} : \bigcup_{\rho \in (N(D))_{n-1}} G_{n-1}^{X}(\rho) \to \bigcup_{\sigma \in (N(D))_n} G_{n}^{X}(\sigma)
\]

\( \square \)

Notation 2.4. Each pair \( (K, L) \) of simplicial sets defines a bisimplicial sets i.e. a functor

\[
K \square L : \Delta^{op} \times \Delta^{op} \to \text{Sets}
\]
as follows:

\[
K \square L([m], [n]) := K_{m} \times L_{n}
\]

Remark 4. The simplicial space \( (\int_{d \in D} X) \), is equipped with a map of simplicial spaces:

\[
p_{1}^{X} : \left( \int_{d \in D} X \right) \to N(D) \square \Delta[0].
\]

Notation 2.5. Each simplicial space \( Z : \Delta^{op} \to S \) determines a bisimplicial set, also denoted by \( Z \)

\[
Z : \Delta^{op} \times \Delta^{op} \to \text{Sets}
\]
by \( Z([m], [n]) = (Z[m])_{n} \). Further we denote the following simplicial set by \( i_{1}^{*}(Z) \):

\[
\Delta \stackrel{(-[0])}{\rightarrow} \Delta \times \Delta \stackrel{Z}{\rightarrow} \text{Sets}
\]

Now we can define the (total space of) the Grothendieck construction of \( X : D \to S \) as follows:

\[
\int_{d \in D} X = i_{1}^{*} \left( \left( \int_{d \in D} X \right) \right)
\]
Remark 5. The set of n-simplices of $\int^{d \in D} X$ can be represented as follows:

$$\left( \int^{d \in D} X \right)_n = \bigcup_{\sigma \in N(D)_n} \sigma \times \left( G^X_n(\sigma) \right)_0.$$

Remark 6. An n-simplex $\delta$ of $\int^{d \in D} X$ is a pair $\delta = (\sigma, \beta)$ where $\sigma = (f_1, f_2, \ldots, f_n) \in N(D)_n$ and $\beta \in G^X_n(\sigma)$ i.e. $\beta = (\beta_1, \beta_2)$. This pair consists of $(\beta_{n-1}, \beta_{n-1}) = \beta \in G^X_n(d_n(\sigma))$ and $\beta \in X(d_{n+1})_n$, where $f_n : d_n \to d_{n+1}$. The n-simplex $\delta$ satisfies the following two conditions:

1. $X(f_n)(p_2(\delta)) = d_n(\beta)$.
2. For $0 \leq i \leq n - 2$

$$(d_i(\beta), d_i(\beta)) \in G^X_{n-1}(d_i(\sigma)).$$

Remark 7. Let $\beta = (\beta, \beta) \in G^X_n(\sigma)$, where $\sigma \in N(D)_n$ as in remark 6. We observe that $d_n(\beta) = \beta = (\beta_{n-1}, \beta_{n-1})$. Further, $d_{n-1}(\beta) = \beta_{n-1} = (\beta_{n-2}, \beta_{n-2}) \in G^X_{n-2}(d_{n-2}(\sigma))$. Since $n$ is finite, there exists a $\beta_0 \in G^X_0(d_1)$ such that

$$\beta_0 = d_1 \circ \cdots \circ d_{n-1} \circ d_n(\beta).$$

The notion of relative nerve was introduced in [Lur09, 3.2.5.2]. Next we will review this notion:

**Definition 2.6.** Let $D$ be a category, and $f : D \to S$ a functor. The nerve of $D$ relative to $f$ is the simplicial set $N_f(D)$ whose n-simplices are sets consisting of:

(i) a functor $d : [n] \to D$; We write $d(i, j)$ for the image of $i \leq j$ in $[n]$.

(ii) for every nonempty subposet $J \subseteq [n]$ with maximal element $j$, a map $\tau^J : \Delta^J \to f(d(j))$.

(iii) such that for nonempty subsets $I \subseteq J \subseteq [n]$ with respective maximal elements $i \leq j$, the following diagram commutes:

$$\begin{array}{ccc}
\Delta^I & \xrightarrow{\tau^I} & f(d(i)) \\
\downarrow & & \downarrow \\
\Delta^J & \xrightarrow{\tau^J} & f(d(j))
\end{array}$$

For any $f$, there is a canonical map $p_f : N_f(D) \to N(D)$ down to the ordinary nerve of $D$, induced by the unique map to the terminal object $\Delta^0 \in S$ [Lur09, 3.2.5.4]. When $f$ takes values in quasi-categories, this canonical map is a coCartesian fibration.

Remark 8. A vertex of the simplicial set $N_f(D)$ is a pair $(c, g)$, where $c \in \text{Ob}(D)$ and $g \in f(c)_0$. An edge $\xi : (c, g) \to (d, k)$ of the simplicial set $N_f(D)$ consists of a pair $(e, h)$, where $e : c \to d$ is an arrow in $D$ and $h : f(e)_0(g) \to k$ is an edge of $f(d)$.

An immediate consequence of the above definition is the following proposition:

**Proposition 2.7.** Let $f : D \to S$ be a functor, then the fiber of $p_f : N_f(D) \to N(D)$ over any $d \in \text{Ob}(D)$ is isomorphic to the simplicial set $f(d)$.

The following lemma is a consequence of this definition and the above discussion:
Lemma 2.8. For each functor $X : D \to S$, we have the following isomorphism in the category $\mathcal{S}/N(D)$:

$$\int^{d \in D} X \cong N_X(D).$$

Proof. An $n$-simplex in $\int^{d \in D} X$ is a pair $(\sigma, \beta)$, where $\sigma \in N(D)_n$. This $n$-simplex $\sigma$ can be viewed as a functor $\sigma : [n] \to D$. The inclusion of each non-empty subposet $i_J : J \subseteq [n]$ gives a map

$$\left( \int^{d \in D} X \right)(i_J) : \left( \int^{d \in D} X \right)_n \to \left( \int^{d \in D} X \right)_J.$$

We are using the fact that $J$ is isomorphic to an object of $\Delta$ which we also denote by $J$. The inclusion map can now be seen as a map in $\Delta$. This map gives us a map $\beta$.

We define $\beta$ to be the n-simplex of $X(d(n))$ which represents the above map. We have an inclusion $[n - 1] \subseteq [n]$ in $\Delta$. The n-simplex $\gamma$ contains another simplicial map

$$\Delta[n - 1] \to X(d(n - 1)).$$

We define $\beta$ to be the pair $(\alpha, (\beta_{n-1}, \beta_{n-1}))$ consisting of an $(n-1)$-simplex $\beta_{n-1}$ of $X(d(n-1))$ which represents the above map and $\alpha = d_n(\sigma)$. The first condition of remark 6 follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
\Delta[n - 1] & \xrightarrow{\beta_{n-1}} & X(d(n-1)) \\
\downarrow & & \downarrow \\
\Delta[n] & \xrightarrow{\beta} & X(d(n))
\end{array}$$

The second condition of remark 6 follows from definition 2.6 (iii). \qed

Next we will define a function object for the category $\mathcal{S}/D$. We shall denote by $[X,Y]_D$ the simplicial set of maps from $X$ to $Y$ in $\mathcal{S}/D$. An $n$-simplex in $[X,Y]_D$ is a map $\Delta[n] \times X \to Y$ in $\mathcal{S}/D$, where $\Delta[n] \times (X,p) = (\Delta[n] \times X, pp_2)$, where $p_2$ is the projection $\Delta[n] \times X \to X$. The enriched category $\mathcal{S}/D$ admits tensor and cotensor products. The tensor product of an object $X = (X,p)$ in $\mathcal{S}/D$ with a simplicial set $A$ is the objects

$$A \times X = (A \times X, pp_2).$$
The cotensor product of $X$ by $A$ is an object of $\mathcal{S}/D$ denoted $(X)^{[A]}$. If $q : (X)^{[A]} \to N(\Gamma^{op})$ is the structure map, then a simplex $x : \Delta[n] \to (X)^{[A]}$ over a simplex $y = qx : \Delta[n] \to N(D)$ is a map $A \times (\Delta[n], y) \to (X, p)$. The object $((X)^{[A]}, q)$ can be constructed by the following pullback square in $\mathcal{S}$:

\[
\begin{array}{ccc}
(X)^{[A]} & \longrightarrow & [A, X] \\
\downarrow{q} & & \downarrow{[A, p]} \\
N(D) & \longrightarrow & [A, D]
\end{array}
\]

where the bottom map is the diagonal. There are canonical isomorphisms:

\[
[\mathcal{A} \times X, Y]_D \cong [A, [X, Y]_D] \cong [X, (Y)^{[A]}]_D
\]

We now define a functor $\mathfrak{R}_D : \mathcal{S}/D \to [D, \mathcal{S}]$. For each $Y \in \mathcal{S}/N(D)$, the functor $\mathfrak{R}_D(Y)$ is defined as follows:

\[
\mathfrak{R}_D(Y)(d) := [N(d/D), Y]_D
\]

The contravariant functor $N(\cdot/D)$, see (??), ensures that this defines a functor $\mathfrak{R}_D(Y) : D \to \mathcal{S}$.

**Notation 2.9.** For a simplicial map $p : X \to B$, we denote the fiber of $p$ over an $n$-simplex $\sigma \in B_n$ by $X(\sigma)$. In other words, the simplicial set $X(\sigma)$ is defined by the following pullback square:

\[
\begin{array}{ccc}
X(\sigma) & \longrightarrow & X \\
\downarrow & & \downarrow{p} \\
\Delta[n] & \longrightarrow & B
\end{array}
\]
3. Rectification of coCartesian fibrations

In this section we will prove a rectification theorem for coCartesian fibrations of simplicial sets over the nerve of a small category $D$. We will do so along the lines of a similar theory developed in appendix ?? for left fibrations. More precisely, we will show that a marked version of our Grothendieck construction functor is a left Quillen functor of a Quillen equivalence between the coCartesian model category $S^+ / N(D)$ and the projective model category $[D, (S, Q)]$. We begin with a review of coCartesian fibrations over the simplicial set $N(D)$. We will also review a model category structure on the category $S^+ / N(D)$ in which the fibrant objects are (essentially) coCartesian fibrations.

**Definition 3.1.** Let $p : X \to S$ be an inner fibration of simplicial sets. Let $f : x \to y \in (X)_1$ be an edge in $X$. We say that $f$ is $p$-coCartesian if, for all $n \geq 2$ and every (outer) commutative diagram, there exists a (dotted) lifting arrow which makes the entire diagram commutative:

\[
\begin{array}{c}
\Delta^{(0,1)} \\
\downarrow f \\
\Lambda^0[n] \rightarrow X \\
\downarrow p \\
\Delta[n] \rightarrow S
\end{array}
\]

**Remark 9.** Let $M$ be a (ordinary) category equipped with a functor $p : M \to I$, then an arrow $f$ in $M$, which maps isomorphically to $I$, is coCartesian in the usual sense if and only if $f$ is $N(p)$-coCartesian in the sense of the above definition, where $N(p) : N(M) \to \Delta[1]$ represents the nerve of $p$.

This definition leads us to the notion of a coCartesian fibration of simplicial sets:

**Definition 3.2.** A map of simplicial sets $p : X \to S$ is called a coCartesian fibration if it satisfies the following conditions:

1. $p$ is an inner fibration of simplicial sets.
2. For each edge $p : x \to y$ of $S$ and each vertex $x$ of $X$ with $p(x) = x$, there exists a $p$-coCartesian edge $\tilde{f} : x \to y$ with $p(\tilde{f}) = f$.

A coCartesian fibration roughly means that it is up to weak-equivalence determined by a functor from $S$ to a suitably defined ∞-category of ∞-categories. This idea is explored in detail in [Lur09, Ch. 3].

**Notation 3.3.** To each coCartesian fibration $p : X \to N(D)$ we can associate a marked simplicial set denoted $X^\natural$ which is composed of the pair $(X, E)$, where $E$ is the set of $p$-coCartesian edges of $X$.

**Notation 3.4.** Let $(X, p), (Y, q)$ be two objects in $S^+ / N(D)$. We denote by $[X, Y]^+_D$, the full (marked) simplicial subset of $[X, Y]^+_D$ spanned by maps in $S^+ / N(D)(X, Y)$, namely spanned by maps in $[X, Y]^+_D$ which are compatible with the projections $p$ and $q$. We denote by $[X, Y]^+_D$, the full simplicial subset of $[X, Y]^+_D$ spanned by maps in $S^+ / N(D)(X, Y)$. We denote by $[X, Y]^+_D \subseteq [X, Y]^+_D$ the simplicial subsets spanned by maps in $S^+ / N(D)$. 
Definition 3.5. A morphism \( F : X \to Y \) in the category \( S^+/N(D) \) is called a coCartesian-equivalence if for each coCartesian fibration \( p : Z \to N(D) \), the induced simplicial map

\[
[F, Z^b]_D : [Y, Z^b]_D \to [X, Z^b]_D
\]

is a categorical equivalence of simplicial-sets (quasi-categories).

Proposition 3.6. Let \( u : X \to Y \) be a map in \( S^+/N(D) \), then the following are equivalent.

1. \( u \) is a coCartesian equivalence.
2. For each functor \( Z : D \to S^+ \), such that \( Z(d) \) is a quasi-category whose marked edges are equivalences, the following (simplicial) map is a categorical equivalence:

\[
\left[ u, \int^D_+ Z \right]_D^b : \left[ Y, \int^D_+ Z \right]_D^b \to \left[ X, \int^D_+ Z \right]_D^b
\]

3. For each functor \( Z : D \to S^+ \), such that \( Z(d) \) is a quasi-category whose marked edges are equivalences, the following map is a bijection:

\[
\pi_0 \left[ u, \int^D_+ Z \right]_D^\sharp : \pi_0 \left[ Y, \int^D_+ Z \right]_D^\sharp \to \pi_0 \left[ X, \int^D_+ Z \right]_D^\sharp
\]

Proof. (1 \( \Rightarrow \) 2) Follows from the definition of coCartesian equivalence because \( \int^D_+ Z \) is a coCartesian fibration under the given hypothesis.

Let us assume that \( \left[ u, \int^D_+ Z \right]_D^b \) is a categorical equivalence of quasi-categories for each functor \( Z \) satisfying the given hypothesis. This implies that \( \left[ u, T^\sharp \right]_D^b \) is a categorical equivalence if and only if \( \left[ u, \int^D_+ Z(T) \right]_D^b \) is one.

(2 \( \Rightarrow \) 3) We recall from \cite{Lur09} Prop. 3.1.3.3 and \cite{Lur09} Prop. 3.1.4.1 that, for any coCartesian fibration \( T^\sharp \in S^+/N(D) \), the simplicial map \( \left[ u, T^\sharp \right]_D^\sharp \) is a categorical equivalence if and only if the map \( \left[ u, T^\sharp \right]_D^\sharp \) is a homotopy equivalence of Kan complexes. This implies that \( \pi_0 \left[ u, \int^D_+ Z \right]_D^\sharp \) is a bijection.

(3 \( \Rightarrow \) 1) We recall from \cite{Lur09} Cor. 3.1.4.4 that the coCartesian model category is a simplicial model category with simplicial function object given by the bifunctor \([\cdot, \cdot]_D^\sharp\). This implies that \( u \) is a coCartesian equivalence if and only if \( \pi_0 \left[ u, W^\sharp \right]_D^\sharp \) is a bijection for each fibrant object \( W \) of the coCartesian model category. By \cite{Lur09} Prop. 3.1.4.1 we may replace \( W \) by a coCartesian fibration \( W \cong T^\sharp \). Further, it follows from \cite{Lur09} Prop. 3.2.5.18(2) that for each cocartesian fibration \( T^\sharp \) there exists a functor \( Z(T) : D \to S^+ \), which satisfies the assumptions of the functor in the statement of the proposition, such that there is map

\[
F_T : T^\sharp \to \int^D_+ Z(T)
\]

which is a coCartesian equivalence. Now it follows that \( u \) is a coCartesian equivalence if and only if \( \pi_0 \left[ u, \int^D_+ Z(T) \right]_D^\sharp \) is a bijection for each functor \( Z \) satisfying the conditions mentioned in the statement of the proposition.
Next we will recall a model category structure on the overcategory \( S^+/N(D) \) from [Lur09, Prop. 3.1.3.7.] in which fibrant objects are (essentially) coCartesian fibrations.

**Theorem 3.7.** There is a left-proper, combinatorial model category structure on the category \( S^+/N(D) \) in which a morphism is

1. a cofibration if it is a monomorphism when regarded as a map of simplicial sets.
2. a weak-equivalences if it is a coCartesian equivalence.
3. a fibration if it has the right lifting property with respect to all maps which are simultaneously cofibrations and weak-equivalences.

We have defined a function object for the category \( S^+/N(D) \) above. The simplicial set \([X, Y]_D^\circ\) has verices, all maps from \( X \) to \( Y \) in \( S^+/N(D) \). An \( n \)-simplex in \([X, Y]_D^\circ\) is a map \( \Delta[n]^\circ \times X \to Y \) in \( S^+/N(D) \), where \( \Delta[n]^\circ \times (X, p) = (\Delta[n]^\circ \times X, pp_2) \), where \( p_2 \) is the projection \( \Delta[n]^\circ \times X \to X \). The enriched category \( S^+/N(D) \) admits tensor and cotensor products. The tensor product of an object \( X = (X, p) \) in \( S^+/N(D) \) with a simplicial set \( A \) is the objects

\[
\{A^\circ \times_X = (A^\circ \times X, pp_2).
\]

The cotensor product of \( X \) by \( A \) is an object of \( S^+/N(D) \) denoted \( X^{[A]} \). If \( q : X^{[A]} \to N(D)^\circ \) is the structure map, then a simplex \( x : \Delta[n]^\circ \to X^{[A]} \) over a simplex \( y = qx : \Delta[n] \to N(D)^\circ \) is a map \( A^\circ \times (\Delta[n]^\circ, y) \to (X, p) \). The object \((X^{[A]}, q)\) can be constructed by the following pullback square in \( S^+\):

\[
\begin{array}{ccc}
X^{[A]} & \rightarrow & [A^\circ, X]^+ \\
q \downarrow & & \downarrow [A^\circ, p]^+ \\
N(D)^\circ & \rightarrow & [A^\circ, N(D)^\circ]^+
\end{array}
\]

where the bottom map is the diagonal. There are canonical isomorphisms:

\[(8) \quad [A^\circ \times_X Y]^+_D \cong [A, [X, Y]^+_D] \cong [X, Y^{[A]}]^+_D \]

**Remark 10.** The coCartesian model category structure on \( S^+/N(D) \) is a simplicial model category structure with the simplicial Hom functor:

\([-, -]^\circ_D : S^+/N(D)^{op} \times S^+/N(D) \to S.\]

This is proved in [Lur09, Corollary 3.1.4.4.]. The coCartesian model category structure is a \((S, Q)\)-model category structure with the function object given by:

\([-, -]^\circ_D : S^+/N(D)^{op} \times S^+/N(D) \to S.\]

This is remark [Lur09 3.1.4.5.].

**Remark 11.** The coCartesian model category is a \((S^+, Q)\)-model category with the Hom functor:

\([-, -]^+_D : S^+/N(D)^{op} \times S^+/N(D) \to S^+.\]

This follows from [Lur09 Corollary 3.1.4.3] by taking \( S = N(D) \) and \( T = \Delta[0] \), where \( S \) and \( T \) are specified in the statement of the corollary.
Definition 3.8. Let $F : D \to S^+$ be a functor. We can compose it with the forgetful functor $U$ to obtain a composite functor $F : D \xrightarrow{F} S^+ \xrightarrow{U} S$. The marked Grothendieck construction of $F$, denoted $\int^+_{d \in D} F$, is the marked simplicial set $\left( \int^+_{d \in D} F, E \right)$, where the set $E$ consists of those edges $e = (c, h)$ of $\int^+_{d \in D} F$, see remark [8] which determines a marked edge of the marked simplicial set $F(d)$, where $e : c \to d$ is an arrow in $D$.

The above construction of the marked Grothendieck construction determines a functor
\begin{equation}
\int^+_{d \in D} - : [D, S^+] \to S^+/N(D).
\end{equation}

Now we will define a marked version of the functor $\mathcal{R}_D$, denoted $\mathcal{R}_D^+$:
\[ \mathcal{R}_D^+(X)(d) := \left[ N(d/D)^\sharp, X \right]_D^+ \]
where $X$ is an object of $S^+/N(D)$. This functor has a left adjoint which we denote by $L_D^+$.

Definition 3.9. Let $X : D \to S^+$ be a functor. For each $d \in D$ we define a map of marked simplicial sets
\[ \eta^+_X(d) : X(d) \to \left[ N(d/D), \int^+_{d \in D} X \right]_D^+. \]
Let $x \in X(d)_n$ be an $n$-simplex in $X(d)$. This $n$-simplex defines a canonical map $\eta^+_X(d)(x) : N(d/D) \times \Delta[n] \to \int^+_{d \in D} X$ in $S^+/N(D)$ whose value on $(id(d)_n, id_n) \in (N(d/D) \times \Delta[n])_n$ is the image of $x$ in $\int^+_{d \in D} X$, namely the $n$-simplex $(\underline{c}, x)$, where $\underline{c} = (x_{n-1}, d_n(x))$. We recall that a $k$-simplex in $\Delta[n]$ is a map $\alpha : [k] \to [n]$ in the category $\Delta$ and therefore it can be written as $\Delta[n](\alpha)(id_n)$. For a $k$-simplex $((g, f_1, \ldots, f_k), \alpha)$ in $N(d/D) \times \Delta[n]$, we define
\[ \eta^+_X(d)(x)((g, f_1, f_2, \ldots, f_k), \alpha) := X(f_{k+1} \circ f_k \circ \cdots \circ g)(X(d)(\alpha)(x)). \]
This defines the (simplicial) map $\eta^+_X(d)(x)$. These simplicial maps glue together into a natural transformation $\eta^+_X$.

Now we define a map $\iota^+_d$ in $S^+/N(D)$:
\begin{equation}
\Delta[0]^\sharp \xrightarrow{\iota^+_d} N(d/D)^\sharp \xrightarrow{id_d} N(D)^\sharp
\end{equation}

Lemma 3.10. For each $d \in D$ the morphism $\iota^+_d$ defined in (10) is a coCartesian equivalence.

Proof. We will show that for each functor $Z : D \to S^+$ such that, for each $d \in D$, $Z(d)$ is a quasi-category whose marked edges are equivalences, we have the following bijection:
\[ \pi_0\left[ \int^+_d \int^+_{d \in D} Z \right]_D^\sharp : \pi_0\left[ N(d/D), \int^+_{d \in D} Z \right]_D^\sharp \to \pi_0\left[ \Delta[0], \int^+_{d \in D} Z \right]_D^\sharp \cong \pi_0(J(Z(d))). \]
where \(J(Z(d))\) is the largest Kan complex contained in \(Z(d)\). Let \(z \in J(Z(d))_0\) be a vertex of \(J(Z(d))^\sharp\). We will construct a morphism \(F_z : N(d/D) \to \int_{d \in D} Z\) in the category \(S^+/N(D)\). The vertex \(z\) represents a natural transformation
\[
T_z : D(d, -) \Rightarrow Z
\]
such that \(T_z(id_d) = z\). Since \(N(d/D) \cong \int_{d \in D} D(d, -)\) therefore we have a map
\[
F_z : N(d/D) \cong \int_{d \in D} D(d, -) \Rightarrow \int_{d \in D} T_z \Rightarrow \int_{d \in D} Z
\]
in \(S^+/N(D)\) such that \(F_z(id_d) = z\). Thus we have shown that the map \(\pi_0\left[\int_{d \in D} Z\right]_D\) is a surjection.

Let \(f : y \to z\) be an edge of \(J(Z(d))\), then by the (enriched) Yoneda’s lemma followed by an application of the Grothendieck construction functor, this edge uniquely determines a map
\[
T_f : N(d/D) \times D[1] \to \int_{d \in D} Z
\]
in \(S^+/N(D)\) such that \(F_z((id_d, id_1)) = f\). Thus we have shown that the map \(\pi_0\left[\int_{d \in D} Z\right]_D\) is also an injection.

\[\square\]

Lemma 3.11. For any projectively fibrant functor \(X : D \to S^+\), the map \(\eta_X^+\) defined in (3.9) is an objectwise categorical equivalence of marked simplicial sets.

Proof. Under the hypothesis of the lemma, it follows from [Lur09 Prop. 3.2.5.18(2)] and lemma 3.9 that \(\int_{d \in D} X\) is a fibrant object in the coCartesian model category. Now lemma 3.10 and remark 11 gives us, for each \(d \in D\), the following homotopy equivalence in \((S^+, Q)\):
\[
\left[\int_{d \in D} X\right]_D^+ : [N(d/D)^\sharp, \int_{d \in D} X]_D^+ \to [\Delta[0]^\flat, \int_{d \in D} X]_D^+ \cong X(d)
\]
such that \(c \circ [id_d, \int_{d \in D} X]_D \circ \eta_X(d) = id_{X(d)}\), where \(c\) is the canonical isomorphism between the fiber of \(p : \int_{d \in D} X \to N(D)\) over \(d \in D\) and \(X(d)\) i.e. the value of the functor \(X\) on \(d\). Now the 2 out of 3 property of weak equivalences in a model category tells us that \(\eta_X(d)\) is a homotopy equivalence for each \(d \in D\) therefore \(\eta_X^+\) is an objectwise homotopy equivalence in \([D, (S^+, Q)]\). \[\square\]

An immediate consequence of the definition of the right adjoint functor \(R_D^+\) is the following lemma:

Lemma 3.12. The adjunction \((L_D^+, R_D^+)\) is a Quillen adjunction between the projective model category structure on \([D, (S^+, Q)]\) and coCartesian model category \(S^+/N(D)\).

Proof. The coCartesian model category is a \((S^+, Q)\)-model category, see remark 11. This implies that \(R_D^+\) maps (acyclic) fibrations in the coCartesian model category to (acyclic) projective fibrations in \([D, S^+]\) which are objectwise (acyclic) fibrations of marked simplicial sets. \[\square\]

Now we get to the main result of this note:
Theorem 3.13. The Quillen pair $(\mathcal{L}_D^+, \mathcal{R}_D^+)$ is a Quillen equivalence.

Proof. We will prove this proposition by showing that the (right) derived functor of $\mathcal{R}_D^+$ induces an equivalence of categories between the two homotopy categories in context. It follows from [Lur09 Prop. 3.2.5.18(2)] that each fibrant object $Z$ in the coCartesian model category is equipped with a coCartesian equivalence

$$\delta(Z) : Z \rightarrow \int^t_{+} \mathcal{F}_+(D)(\mathcal{R}_D^+(Z)).$$

We observe that $\mathcal{R}_D^+(Z)$ is a fibrant object of $[D, (S^+, Q)]$.

Let $T$ denote the full subcategory of $S^+/N(D)$ in which every object $W$ is fibrant in the coCartesian model category and lies in the image of $\int^d_{+} D$, i.e. for any $W \in T$, there exists a fibrant $V$ in $[D, (S^+, Q)]$ such that $W = \int^d_{+} V$. We denote the full subcategory of $Ho_{S^+/N(D)}$ whose objects are those of $T$ by $Ho_T$. We will refer to it as the homotopy category of $T$. We observe that the inclusion map $i : Ho_T \rightarrow Ho_{S^+/N(D)}$ is an equivalence of categories. Now it is sufficient to show that the (right) derived functor of $\mathcal{R}_D^+$ induces an equivalence of categories between $Ho_T$ and $Ho_{[D,(S^+, Q)]}$. It follows from [Lur09 Prop. 3.2.5.18(2)], lemma 3.11 and lemma [23] that for each fibrant $X$ in the projective model category $[D, (S^+, Q)]$ we have the following composite map which is a weak-equivalence in $[D, (S^+, Q)]$:

$$\mathcal{F}_+(D) \left( Q \left( \int^d_{+} X \right) \right) \xrightarrow{q} \mathcal{F}_+(D) \left( \int^d_{+} X \right) \xrightarrow{\epsilon} X \xrightarrow{\eta} \mathcal{R}_D^+ \left( \int^d_{+} X \right),$$

where $\epsilon$ is the counit map of the Quillen equivalence [Lur09 Prop. 3.2.5.18(2)], which is a weak equivalence, $r : id \Rightarrow R$ is a chosen fibrant replacement functor and $q : Q \Rightarrow id$ is a chosen cofibrant replacement functor. We claim that this defines a natural isomorphism between the (restriction to $Ho_T$ of) left derived functor of $\mathcal{F}_+(D)$, which we also denote $(\mathcal{F}_+(D))^L : Ho_T \rightarrow Ho_{[D,(S^+, Q)]}$ and the (restriction to $Ho_T$ of) right derived functor of $\mathcal{R}_D^+$, which we also denote $(\mathcal{R}_D^+)^R : Ho_T \rightarrow Ho_{[D,(S^+, Q)]}$. The Quillen equivalence [Lur09 Prop. 3.2.5.18(2)] implies that the right derived functor of $\int^d_{+} D$ is fully-faithful. Since $X$ and $Y$ are fibrant, this means that for each equivalence class $[u]$ representing an arrow $[u] : R \left( \int^d_{+} D \right) \rightarrow R \left( \int^d_{+} E \right)$ in $Ho_T$, there exists an arrow $v : X \rightarrow Y$ in $[D, (S^+, Q)]$ such that the arrow $R \left( \int^d_{+} v \right)$ is a representative of $[u]$. This implies that we have defined the desired natural isomorphism. Since $(\mathcal{F}_+(D))^L : Ho_T \rightarrow Ho_{[D,(S^+, Q)]}$ is an equivalence of categories, the above natural isomorphism implies that the functor $(\mathcal{R}_D^+)^R : Ho_T \rightarrow Ho_{[D,(S^+, Q)]}$ is also one.

We have the following commutative triangle of functors between homotopy categories:
Now the 2 out of 3 property of weak equivalences in the model category \( \text{Cat} \) tells us that the right derived functor of \( R_D^+ \) induces an equivalence of categories between the homotopy categories. Thus we have proved that the Quillen pair \((L_D^+, R_D^+)\) is a Quillen equivalence.

The natural isomorphism constructed in the proof of the above theorem implies the following proposition:

**Proposition 3.14.** The (total) left derived functor of the left Quillen functor \( \mathcal{F}_D^+ \) is naturally isomorphic to the (total) right derived functor of \( R_D^+ \).

The proposition has the following corollary:

**Corollary 3.15.** The (total) left derived functor of \( \mathcal{L}_D^+ \) is naturally isomorphic to the (total) right derived functor of \( \int_{D}^{d \in D} - \).

**Notation 3.16.** The total (total) right derived functor of \( \int_{D}^{d \in D} - \) refers to the total right derived functor of the relative nerve functor for marked simplicial sets, see [Lur09, Prop. 3.2.5.18(2)].

**Proposition 3.17.** For all functor \( X : D \to S^+ \) and \( K \in S \) we have the following isomorphism

\[
\mathcal{L}_D^+(X \otimes K) \cong \mathcal{L}_D^+(X) \otimes K.
\]
Appendix A. A review of marked simplicial sets

In this appendix we will review the theory of marked simplicial sets. Later in this paper we will develop a theory of coherently commutative monoidal objects in the category of marked simplicial sets.

Definition A.1. A marked simplicial set is a pair \((X, E)\), where \(X\) is a simplicial set and \(E\) is a set of edges of \(X\) which contains every degenerate edge of \(X\). We will say that an edge of \(X\) is marked if it belongs to \(E\). A morphism \(f : (X, E) \rightarrow (X', E')\) of marked simplicial sets is a simplicial map \(f : X \rightarrow X'\) having the property that \(f(E) \subseteq E'\). We denote the category of marked simplicial sets by \(S^+\).

Every simplicial set \(S\) may be regarded as a marked simplicial set in many ways. We mention two extreme cases: We let \(S^\# = (S, S_1)\) denote the marked simplicial set in which every edge is marked. We denote by \(S^\flat = (S, s_0(S_0))\) denote the marked simplicial set in which only the degenerate edges of \(S\) have been marked.

The category \(S^+\) is cartesian-closed, i.e. for each pair of objects \(X, Y \in \text{Ob}(S^+)\), there is an internal mapping object \([X, Y]^+\) equipped with an evaluation map \([X, Y]^+ \times X \rightarrow Y\) which induces a bijection:

\[ S^+(Z, [X, Y]^+) \cong S^+(Z \times X, Y), \]

for every \(Z \in S^+\).

Notation A.2. We denote by \([X, Y]^\flat\) the underlying simplicial set of \([X, Y]^+\).

The mapping space \([X, Y]^\flat\) is characterized by the following bijection:

\[ S(K, [X, Y]^\flat) \cong S^+(K^\flat \times X, Y), \]

for each simplicial set \(K\).

Notation A.3. We denote by \([X, Y]^\sharp\) the simplicial subset of \([X, Y]^\flat\) consisting of all simplices \(\sigma \in [X, Y]^\flat\) such that every edge of \(\sigma\) is a marked edge of \([X, Y]^+\).

The mapping space \([X, Y]^\sharp\) is characterized by the following bijection:

\[ S(K, [X, Y]^\sharp) \cong S^+(K^\sharp \times X, Y), \]

for each simplicial set \(K\).

The Joyal model category structure on \(S\) has the following analog for marked simplicial sets:

Theorem A.4. There is a left-proper, combinatorial model category structure on the category of marked simplicial sets \(S^+\) in which a morphism \(p : X \rightarrow Y\) is a

1. cofibration if the simplicial map between the underlying simplicial sets is a cofibration in \((S, Q)\), namely a monomorphism.
2. a weak-equivalence if the induced simplicial map on the mapping spaces

\[ [p, K^\sharp] : [X, K^\sharp] \rightarrow [Y, K^\sharp] \]

is a weak-categorical equivalence, for each quasi-category \(K\).
3. fibration if it has the right lifting property with respect to all maps in \(S^+\) which are simultaneously cofibrations and weak equivalences.

Further, the above model category structure is enriched over the Joyal model category, i.e. it is a \((S, Q)\)-model category.
The above theorem follows from [Lur09, Prop. 3.1.3.7].

**Notation A.5.** We will denote the model category structure in Theorem A.4 by $(S^+, Q)$ and refer to it either as the Joyal model category of marked simplicial sets or as the model category of marked quasi-categories.

**Theorem A.6.** The model category $(S^+, Q)$ is a cartesian closed model category.

**Proof.** The theorem follows from [Lur09, Corollary 3.1.4.3] by taking $S = T = \Delta[0]$. \[\square\]

There is an obvious forgetful functor $U : S^+ \to S$. This forgetful functor has a left adjoint $(-)^\flat : S \to S^+$.

**Theorem A.7.** The adjoint pair of functors $((-)^\flat, U)$ determine a Quillen equivalence between the Joyal model category of marked simplicial sets and the Joyal model category of simplicial sets.

The proof of the above theorem follows from [Lur09, Prop. 3.1.5.3].

**Remark 12.** A marked simplicial set $X$ is fibrant in $(S^+, Q)$ if and only if it is a quasi-category with the set of all its equivalences as the set of marked edges.

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**E-mail address:** asharm24@kent.edu

**Department of mathematical sciences, Kent State University, Kent, OH**