Extended Self-Similarity and Hierarchical Structure in Turbulence

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It is shown that the two remarkable properties of turbulence, the Extended Self-Similarity (ESS) [R. Benzi et al., Phy. Rev. E 48, R29, (1993)] and the She-Leveque Hierarchical Structure (SLHS) [Z.S. She and E. Leveque, Phy. Rev. Lett. 72, 336, (1994)] are related to each other. In particular, we have shown that a generalized hierarchical structure together with the most intense structures being shock-like give rise to ESS. Our analysis thus suggests that the ESS measured in turbulent flows is an indication of the shock-like intense structures. Results of analysis of velocity measurements in a pipe-flow turbulence support our conjecture.

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Fully developed turbulence is characterized by power-law dependence of the moments of velocity fluctuations. It was suggested by Kolmogorov in 1941 (K41) [1] that there is a constant rate of energy transfer from large to small scales and that the statistical properties of the velocity difference across a separation \( r \), \( \delta v_r \equiv v(x + r) - v(x) \), depend only on the mean energy transfer or equivalently the mean energy dissipation rate \( \epsilon \) and the scale \( r \) when \( r \) is within an inertial range. Dimensional considerations then lead to the prediction that the velocity structure functions, which are moments of the magnitude of the velocity difference, have simple power-law dependence on \( r \) within the inertial range:

\[
S_p(r) \equiv \langle |\delta v_r|^p \rangle \sim \epsilon^{p/3} r^{p/3} \quad (1)
\]

Experiments [2] have indicated that there is indeed power law scaling in the inertial range but the scaling exponents are different from \( p/3 \):

\[
S_p(r) \sim r^{\zeta_p} \quad (2)
\]

where \( \zeta_p \) has a nonlinear dependence on \( p \). Such a deviation implies that the functional form of the probability density function (pdf) of \( \delta v_r \) depends on \( r \), that is, the velocity fluctuations have scale-dependent statistics. Understanding this deviation from K41 is essential to our fundamental understanding of the small scale statistical properties of turbulence.

Recently, Benzi et al. [3] have discovered a remarkable new scaling property: \( S_p(r) \) has a power-law dependence on \( S_3(r) \) over a range substantially longer than the scaling range obtained by plotting \( S_p(r) \) as a function of \( r \). This behavior was named Extended Self-Similarity (ESS); its discovery has enabled more accurate determination of the scaling exponents \( \zeta_p \), particularly at moderately high Reynolds numbers assessible experimentally and numerically. It was later reported that ESS is invalid for anisotropic turbulent flows such as atmospheric boundary layer and channel flow [4, 5, 6]. This inspires the study of a generalized ESS (GESS), a scaling behavior of the normalized structure functions when plotted against each other [7, 8], which is still valid in these anisotropic flows. The validity of ESS suggests that the different order structure functions have the same dependence on \( r \) when \( r \) is near the dissipative range [9, 10, 11]. Very recently, Yakhot argued that some mean-field approximations of the pressure contributions in the Navier-Stokes equation would lead to ESS [2].

A number of phenomenological models have been proposed to explain the anomalous scaling exponents \( \zeta_p \). Among them, a recent successful one is from She and Leveque [13] with a hypothesis of a hierarchical structure (HS). When stated for the velocity structure functions, the HS hypothesis reads

\[
\frac{S_{p+2}(r)}{S_{p+1}(r)} = A_{p+1} \left[ \frac{S_{p+1}(r)}{S_p(r)} \right]^\beta [S^\infty(r)]^{1-\beta} \quad (3)
\]

Here \( S^\infty(r) \equiv \lim_{p \to \infty} S_{p+1}(r)/S_p(r) \) and \( 0 < \beta < 1 \) is a constant. This hypothesis (and the similar version for the local energy dissipation) was supported by experimental velocity measurements taken in turbulent jets and wakes [14, 15, 16]. Note that the boundedness of the velocity will ensure the existence of \( S^\infty(r) \). In fact, one can show that \( S^\infty(r) \equiv \lim_{p \to \infty} S_{p+1}^{1/p} \), and is equal to \( |\delta v_r|^{\max} \), the maximum magnitude of \( \delta v_r \) [4].

It was later reported that the passive scalar structure functions [13] and the local passive scalar dissipation [17], the temperature structure functions [20] and the local temperature dissipation [21] in turbulent convection, and the velocity structure functions in a class of shell models [22, 23, 24] were all found to possess similar hierarchical structures.

In this Letter, we shall show that the She-Leveque hierarchical structure (SLHS) leads to GESS. In other words, the SLHS is a special form of GESS. We then give a generalized form of SLHS, which is equivalent to GESS, with the constant \( \beta \) replaced by a function of \( p \). Furthermore,
it is shown that \( |\delta v_r|^{\text{max}} \) is independent of \( r \) for \( r \) within the GESS range (which will be the case when the most intense structures are shock-like), the generalized SLHS (and hence GESS) will give rise to ESS. We then conjecture that the observed ESS in turbulent flows is an indication of the most intense structures being shock-like. Consequently, we predict that in the anisotropic flows where GESS but not ESS holds, \( |\delta v_r|^{\text{max}} \) has a dependence on \( r \). Finally, we present a systematic procedure of analysis of experimental turbulent signals. Application of this analysis to turbulent velocity fluctuations in a pipeline flow demonstrates that \( |\delta v_r|^{\text{max}} \) indeed depends on \( r \) in the near-wall strong-shear regions where only GESS but not ESS is valid but is consistent with being \( r \)-independent in the centerline of the pipe where ESS is valid.

First, rewrite Eq. (3) as

\[
\frac{S_{k+1}(r)}{S_k(r)S^{(\infty)}(r)} = A_k \left[ \frac{S_k(r)}{S_{k-1}(r)S^{(\infty)}(r)} \right]^\beta
\]

for integer \( k \), which implies

\[
\frac{S_k(r)}{S_{k-1}(r)S^{(\infty)}(r)} = \prod_{j=0}^{k-1} A_j^{\beta_{k-1-j}} \frac{S_1(r)}{S^{(\infty)}(r)}
\]

where \( B_n \equiv \prod_{k=1}^n A_k^{\beta_{k-1}} \), which further gives

\[
\frac{S_n(r)}{S_{n/3}^{(\infty)}(r)} = \frac{B_n}{B_{n/3}^{n/3}} \frac{S_1(r)}{S^{(\infty)}(r)} \left[ \frac{S^{(\infty)}(r)}{S_1(r)} \right]^{3(1-\beta_n)/(3(1-\beta))}
\]

Equation (6) then implies the GESS property:

\[
T_n(r) \sim T_m(r)^{\rho(n,m)}
\]

which is a scaling behavior for the normalized structure functions, \( T_n(r) \equiv S_n(r)/S_3(r)^{n/3} \), with the normalized exponents \( \rho(n,m) \) even when \( S_n(r) \) does not have a scaling behavior in \( r \). For SLHS,

\[
\rho(n,m) = \frac{3(1-\beta_n) - m(1-\beta_m)}{3(1-\beta_m) - m(1-\beta_m)}
\]

We have thus proved that SLHS implies GESS.

Earlier, Benzi et al. [8] showed that GESS holds if the structure functions are of the form

\[
S_p(r) \sim g_1(r)^p g_2(r)^{H(p)}
\]

for any functions \( g_1(r) \), \( g_2(r) \), and \( H(p) \). On the other hand, if Eq. (8) with any \( \rho(n,m) \) is valid, we can always write the structure functions in the form of Eq. (10), say, with \( g_1(r) = S_3(r)^{1/3} \), \( g_2(r) = S_q(r)/S_3(r)^{1/3} \), and \( H(p) = \rho(p,q) \) for any chosen value of \( q^* \). Thus, Eq. (8) is equivalent to GESS.

Our demonstration here gives the functions \( g_1(r) \), \( g_2(r) \) a meaning. Although the choice of \( g_1(r) \) and \( g_2(r) \) is not unique, the following form can be obtained from Eq. (4):

\[
g_1(r) = S^{(\infty)}(r) \quad \text{and} \quad g_2(r) = S_3(r)/[S^{(\infty)}(r)]^{3/5}
\]

Here, \( g_1(r) \) describes the r-dependence of very strong fluctuations \( S^{(\infty)}(r) \) and \( g_2(r) \) describes the normalized r-dependence of (typical) weak fluctuations [e.g. \( S_3(r) \)] by \([S^{(\infty)}(r)]^{-3/5}\). Express \( g_1 \) and \( g_2 \) this way, we have

\[
S_p(r) \sim \left[ S^{(\infty)}(r) \right]^p \left\{ \frac{S_3(r)}{[S^{(\infty)}(r)]^{3/5}} \right\}^{f(p)}
\]

where \( f(p) \) is a function to be discussed below. Note that when \( g_2(r) \) is constant, or the weak fluctuations have the same r-dependence as the strong fluctuations, we have the K41 scaling. The function \( f(p) \) has to satisfy various conditions. By definition, \( f(0) = 0 \), \( f(3) = 1 \), and \( \lim_{p \to \infty} f(p+1) - f(p) = \lim_{p \to \infty} f(p)/p = 0 \). Furthermore, the boundedness of the velocity restricts that \( df(p)/dp \geq 0 \).

Note that Eq. (11) is a general expression of GESS. Rewriting Eq. (11) in the form of Eq. (4), we have

\[
\frac{S_{p+1}(r)}{S_p(r)S^{(\infty)}(r)} = C_p \left[ \frac{S_p(r)}{S_{p-1}(r)S^{(\infty)}(r)} \right]^{g(p)}
\]

where \( g(p) \equiv [f(p+1) - f(p)]/[f(p) - f(p-1)] \). It is clear that SLHS corresponds to the particular case of \( g(p) = \beta \) or equivalently

\[
f(p) = \frac{1 - \beta^p}{1 - \beta^3}
\]

We can list several additional possible classes of \( f(p) \), which include \( f(p) = [(\sigma + 1)^a - 1]/[(3\sigma + 1)^a - 1] \) and \( f(p) = \ln(\sigma + 1)/\ln(3\sigma + 1) \) with \( \sigma > 0 \) and \( 0 < \alpha < 1 \). The latter is the limit of the former when \( \alpha \to 0 \). These two cases correspond to the two results derived by Novikov, Eqs. (18) and (19) in Ref. [26], using the theory of infinitely divisible distributions.

To study when GESS would further give rise to ESS, we rewrite Eq. (8) in the following form:

\[
\frac{S_p(r)^{1/p}}{S_3(r)^{1/3}} \sim \left[ \frac{S_q(r)^{1/q}}{S_3(r)^{1/3}} \right]^{\rho(p,q)}
\]

with

\[
\rho(p,q) = \frac{f(p) - p/3}{f(q) - q/3}
\]

Thus we see that if there exists an \( p^* \neq 0 \) such that \( S_{p^*}(r)^{1/p^*} \) is independent of \( r \), then we would have a scaling behavior of \( S_p \) vs \( S_3 \) (and thus \( S_q \) for \( q \neq p \)). If \( p^* \) is finite, \( \zeta_{p^*} = 0 \) which implies \( \zeta_m = 0 \) for \( 0 \leq m \leq p^* \).
Because $\zeta_3$ has to be a non-decreasing function of $n$ in order for the velocity field to be bounded\textsuperscript{2}. It further gives $\zeta_3 = 0$ for all values of $p$ if $\zeta_3$ is an analytic function of $p$. This could be avoided only if $p^\ast$ approaches $\infty$, which means that $S^{(\infty)}(r)$ is independent of $r$. Hence, GESS together with the condition that $S^{(\infty)}(r)$ or equivalently $|\delta v_3|^{\max}$ being independent of $r$ would give rise to ESS:

$$S_p(r) \sim S_3(r)^{\eta(p,3)}$$

(16)

with $\eta(p, 3) = f(p)$ even when $S_p(r)$ does not have a power-law dependence on $r$. We note that the presence of shocks with maximum velocity, $v_{\max}$, in opposite directions across the shock discontinuity would give $|\delta v_3|^{\max} = 2v_{\max}$, which is thus independent of $r$.

Using Eq. (14), an independence of $S^{(\infty)}$ on $r$ implies

$$\zeta_3 = \zeta_3 f(p)$$

(17)

For SLHS, $\lim_{p \to \infty} f(p) = 1/(1 - \beta^3)$ hence Eq. (14) further implies a saturation of $\zeta_3$ as $p \to \infty$. A connection of the saturation of the exponents with the existence of shocks was suggested earlier by Chen and Cao\textsuperscript{21}.

Both ESS and SLHS have been observed in a variety of fluctuating flow fields. The above analysis points to a possibility that the two combined may be indicative of a property of the flow field: $S^{(\infty)}(r)$ is independent of $r$. One way to check the plausibility of these ideas is to examine experimental velocity measurements that demonstrate GESS, and to investigate whether $|\delta v_3|^{\max}$ is indeed independent of $r$ when ESS is valid and otherwise dependent on $r$ when ESS is not valid. Note, however, that the detectable $|\delta v_3|^{\max}$ in any finite measurement would almost surely underestimate the true value.

If the GESS is of the special form of SLHS, a systematic procedure can be performed to obtain an indirect estimate of the $r$-dependence of $S^{(\infty)}(r)$ as follows. First, one verifies if the SLHS is valid by performing a $\beta$-test. It consists of computing the normalized structure functions $T_p(r)$ and obtaining the normalized exponents $\rho(p, q)$ by measuring the slopes of $\ln(T_p)$ vs $\ln(T_q)$. Let $\Delta \rho(p, q) = \rho(p + 1, q) - \rho(p, q)$. It is easy to derive the following equation when SLHS is valid:

$$\Delta \rho(p + 1, q) = \beta \Delta \rho(p, q) + \frac{(1 - \beta)(1 - \beta^3)}{q(1 - \beta^3) - 3(1 - \beta^2)}$$

(18)

If one finds parallel straight lines when plotting $\Delta \rho(p + 1, q)$ vs $\Delta \rho(p, q)$ for a set of values of $q$, we say that the data passes the $\beta$-test and the turbulent flow field possesses the SLHS property. The slope and intercept provides a double estimate of the constant $\beta$. With the estimated $\beta$, one can then construct $f(p)$ using Eq. (13).

For an indirect estimate of $S^{(\infty)}(r)$, we introduce

$$F_p(r, r_0) = \frac{\ln[S_p(r)/S_p(r_0)] - f(p) \ln[S_3(r)/S_3(r_0)]}{p - 3 f(p)}$$

(19)

From Eq. (14), $F_p(r, r_0)$ should be independent of $p$ and equal to $\log[S^{(\infty)}(r)/S^{(\infty)}(r_0)]$ for $r$ within the GESS range. Thus, one computes $F_p(r, r_0)$ for a fixed value of $r_0$, within the GESS range, and plots it as a function of $r$ for a set of values of $p$ to estimate indirectly the $r$-dependence of $\ln[S^{(\infty)}(r)/S^{(\infty)}(r_0)]$. In particular, if $S^{(\infty)}$ is independent of $r$, then $F_p(r, r_0) = 0$ for $r$ within the GESS range.

We have applied the above procedure to analyze hot-wire measurements of longitudinal velocity fluctuations in a pipe flow\textsuperscript{28}. The pipe is 22.5 m long with an inner diameter of 10.5 cm, and the Reynolds number is $1.35 \times 10^5$. The velocity measurements studied were taken at a cross section at 18.2 m away from the entrance both at the centerline of the pipe and at a distance of 0.1 mm from the pipe-wall. The number of measurements taken at each point is $5.76 \times 10^3$ and the maximum order of moments computed is $p = 10$. We have checked that GESS is valid for both sets of measurements but ESS is valid only at the centerline. The range of GESS (and also ESS when valid) is $r \approx 10 - 500$, in units of the sampling time $1/48$ ms. All $r$‘s quoted below will be in the same units. We have obtained $\rho(p, q)$ for the two locations and $\eta(p, 3)$ for the centerline location. In Fig. 10, we plot $\Delta \rho(p + 1, q)$ vs $\Delta \rho(p, q)$ for some values of $q$ for both locations. The data can be fitted by parallel straight lines showing that the hierarchical structure is indeed of the SL form in both locations. We estimate the value of $\beta$ simultaneously from the slope and the intercept and get $0.93 \pm 0.01$ and $0.85 \pm 0.01$ respectively for the centerline and the near-wall location. In the inset, we see good agreement of $\eta(p, 3)$ with $f(p)$ for the centerline measurements. This implies the $r$-independence of $S^{(\infty)}$, in accord with the validity of ESS in these measurements.

We have next computed $F_p(r, r_0)$ with $r_0 = 69$ for both locations and found that the data indeed collapse when $p$ is large. In Fig. 2, we plot $F_p(r, r_0)$ as a function of $r$. We see that $F_p(r, r_0)$ for the centerline measurements is almost zero for $r \geq 30$ while that for the near-wall measurements shows a clear $r$-dependence throughout. Since ESS is valid for the centerline but not the near-wall measurements (see the inset), these results support our conjecture that the validity of ESS in turbulent flows is an indication of $S^{(\infty)}$ being independent of $r$.

In summary, our analysis proposes a link between a measurable statistical property of turbulence, the ESS, and a property of the most intense structure, the $r$-dependence of $|\delta v_3|^{\max}$ or $S^{(\infty)}(r)$. We have developed a systematic procedure to test this conjecture when the SL hierarchical structure holds. The analysis of experimental pipe flow data support our conjecture: the near-wall strong-shear turbulence contains more complex structures while the centerline fully developed turbulence has shock-like structure for $S^{(\infty)}(r)$. A consequence of the $r$-independence of $S^{(\infty)}(r)$ is the saturation of the scaling exponents of the velocity structure functions at.
very high orders. Further experimental and numerical tests are highly desirable.

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