Abstract

There are two variants of the classical multi-armed bandit (MAB) problem that have received considerable attention from machine learning researchers in recent years: contextual bandits and simple regret minimization. Contextual bands are a sub-class of MABs where, at every time step, the learner has access to side information that is predictive of the best arm. Simple regret minimization assumes that the learner only incurs regret after a pure exploration phase. In this work, we study simple regret minimization for contextual bandits. Motivated by applications where the learner has separate training and autonomous modes, we assume that the learner experiences a pure exploration phase, where feedback is received after every action but no regret is incurred, followed by a pure exploitation phase in which regret is incurred but there is no feedback. We present the Contextual-Gap algorithm and establish performance guarantees on the simple regret, i.e., the regret during the pure exploitation phase. Our experiments examine a novel application to adaptive sensor selection for magnetic field estimation in interplanetary spacecraft, and demonstrate considerable improvement over algorithms designed to minimize the cumulative regret.

1 Introduction

The multi-armed bandit (MAB) is a framework for sequential decision making where, at every time step, the learner selects (or “pulls”) one of several possible actions (or “arms”), and receives a reward based on the selected action. The regret of the learner is the difference between the maximum possible reward and the reward resulting from the chosen action. In the classical MAB setting, the goal is to minimize the sum of all regrets, or cumulative regret, which naturally leads to an exploration/exploitation trade-off problem (Auer et al., 2002a). If the learner explores too little, it may never find an optimal arm which will increase its cumulative regret. If the learner explores too much, it may select sub-optimal arms too often which will also increase its cumulative regret. There are a variety of algorithms that solve this exploration/exploitation trade-off problem (Auer et al., 2002a; Auer, 2002; Auer et al., 2002b; Agrawal and Goyal, 2012; Bubeck et al., 2012).

The contextual bandit problem extends the classical MAB setting, with the addition of time-varying side information, or context, made available at every time step. The best arm at every time step depends on the context, and intuitively the learner seeks to determine the best arm as a function of context. To date, work on contextual bandits has studied cumulative regret minimization, which is motivated by applications in health care, web advertisement recommendations and news article recommendations (Li et al., 2010). The contextual bandit setting is also called associative reinforcement learning (Auer, 2002) and linear bandits (Agrawal and Goyal, 2012; Abbasi-Yadkori et al., 2011). In classical (non-contextual) MABs, the goal of the learner isn’t always to minimize the cumulative regret. In some applications, there is a pure exploration phase during which the learning incurs no regret (i.e., no penalty for sub-optimal decisions), and performance is
measured in terms of simple regret, which is the regret assessed at the end of the pure exploration phase. For example, in top-arm identification, the learner must guess the arm with highest expected reward at the end of the exploration phase. Simple regret minimization clearly motivates different strategies, since there is no penalty for sub-optimal decisions during the exploration phase. Fixed budget and fixed confidence are the two main theoretical frameworks in which simple regret is generally analyzed (Gabillon et al., 2012; Jamieson and Nowak, 2014; Garivier and Kaufmann, 2016; Carpentier and Valko, 2015).

In this paper, we extend the idea of simple regret minimization to contextual bandits. In this setting, there is a pure exploration phase during which no regret is incurred, followed by a pure exploitation phase during which regret is incurred, but there is no feedback so the learner cannot update its policy. To our knowledge, previous work has not addressed novel algorithms for this setting. Guan and Jiang (2018) provide simple regret guarantees for the policy of uniform sampling of arms in the i.i.d setting. The contextual bandit algorithm of Tekin and van der Schaar (2015) also has distinct exploration and exploitation phases, but unlike our setting, the agent has control over which phase it is in, i.e., when it wants to receive feedback. In the work of Hoffman et al. (2014); Soare et al. (2014); Libin et al. (2017); Xu et al. (2018) there is a single best arm even when contexts are observed (directly or indirectly). Our algorithm, Contextual-Gap, generalizes the idea of Gabillon et al. (2012) and Hoffman et al. (2014) to the contextual bandits setting.

We make the following contributions: 1. We formulate a novel problem: that of simple regret minimization for contextual bandits. 2. We develop an algorithm, Contextual-Gap, for this setting. 3. We present performance guarantees on the simple regret in the fixed budget framework. 4. We present experimental results for multiclass online classification with partial feedback, and for adaptive sensor selection in nano-satellites.

The paper is organized as follows. In section 2, we motivate the new problem based on the real-life application of magnetometer selection in spacecraft. In section 3, we state the problem formally, and to solve this new problem, we present the Contextual-Gap algorithm in section 4. In section 5, we present the learning theoretic analysis and in section 6, we present and discuss experimental results. Section 7 concludes.

2 Motivation

Our work is motivated by autonomous systems that go through an initial training phase (the pure exploration phase) where they learn how to accomplish a task without being penalized for sub-optimal decisions, and then are deployed in an environment where they no longer receive feedback, but regret is incurred (the pure exploitation phase).

Figure 1: Scientific measurement: magnetic field lines of the Earth (Credit: NASA/Goddard Scientific Visualization Studio)

An example scenario arises in the problem of estimating weak interplanetary magnetic fields (Figure 1) in the presence of noise using resource-constrained spacecraft known as nano-satellites or CubeSats. Spacecraft systems generate their own spatially localized magnetic field noise due to large numbers of time-varying current paths in the spacecraft. Historically, with large spacecraft, such noise was minimized by physically separating the sensor from the spacecraft using a rigid boom. In highly resource-constrained satellites such as nano-satellites, however, structural constraints limit the use of long rigid booms, requiring sensors to be close to or inside the CubeSat (Figure 2). Thus, recent work has focused on nano-satellites equipped with multiple magnetic field sensors (magnetometers) (Sheinker and Moldwin, 2016).

A natural problem in nano-satellites is that of determining the best sensor to actuate at any given time. Power constraints motivate the selection of a single sensor at each time step. Furthermore, the best sensor changes with time. This stems from the time-varying localization of noise in the spacecraft, which in turn results from different operational events such as data transmission, spacecraft maneuvers, and power generation. This dynamic sensor selection problem is readily cast as a contextual bandit problem. The context is given by the spacecraft’s telemetry system which provides real-time measurements related to spacecraft operation, including solar panel currents, temperatures, momentum wheel information, and real-time current consumption (Springmann and Cutler, 2012).

In this application, however, conventional contextual bandit algorithms are not applicable because feedback is not always available. Feedback requires knowledge
of sensor noise, which in turn requires knowledge of the true magnetic field. Yet the true magnetic field is known only during certain portions of a spacecraft’s orbit (e.g., when the satellite is near other spacecraft, or when the earth shields the satellite from sun-induced magnetic fields). Moreover, when the true magnetic field is known, there is no need to estimate the magnetic field in the first place! This suggests a learning scenario where the agent (the sensor scheduler) operates in two phases, one where it has feedback but incurs no regrets (because the field being estimated is known), and another where it does not receive feedback, but nonetheless needs to produce estimates. This is precisely the problem we study.

In the magnetometer problem defined above, the exploration and exploitation times occur in phases, as the satellite moves into and out of regions where the true magnetic field is known. For simplicity, we will address the problem in which the first $T$ time steps belong to the exploration phase, and all subsequent time steps to the exploitation phase. Nonetheless, the algorithm we introduce can switch between phases indefinitely, and does not need to know in advance when a new phase is beginning.

Figure 2: TBEx Small Satellite with Multiple Magnetometers (Tsunoda, 2016; England et al., 2018)

Sensor management, adaptive sensing, and sequential resource allocation have historically been viewed in the decision process framework where the learner takes actions on selecting the sensor based on previously collected data. There have been many proposed solutions based on Markov decision processes (MDPs) and partially observable MDPs, with optimality bounds for cumulative regret (Hero and Cochran, 2011; Castanon, 1997; Evans and Krishnamurthy, 2001; Krishnamurthy, 2002; Chong et al., 2009). In fact, sensor management and sequential resource allocation was one of the original motivating settings for the classical MAB problem (Mahajan and Teneketzis, 2008; Bubeck et al., 2012; Hero and Cochran, 2011), again with the goal of cumulative regret minimization. We are interested in an adaptive sensing setting where the optimal decisions and rewards also depend on the context, but where the actions can be separated into a pure exploration and pure exploitation phases, with no regret during exploration, and with no feedback during pure exploitation.

3 Formal Setting

We denote the context space as $\mathcal{X} = \mathbb{R}^d$. Let $\{x_t\}_{t=1}^{\infty}$ denote the sequence of observed contexts. Let the total number of arms be $A$. For each $x_t$, the learner is required to choose an arm $a \in [A]$, where $[A] := \{1, 2, \ldots, A\}$.

For arm $a \in [A]$, let $f_a : \mathcal{X} \rightarrow \mathbb{R}$ be a function that maps context to expected reward when arm $a$ is selected. Let $a_t$ denote the arm selected at time $t$, and assume the reward at time $t$ obeys $r_t := f_{a_t}(x_t) + \zeta_t$, where $\zeta_t$ is noise (described in more detail below). We assume that for each $a$, $f_a$ belongs to a reproducing kernel Hilbert space (RKHS) defined on $\mathcal{X}$. The first $T$ time steps belong to the exploration phase where the learner observes context $x_t$, chooses arm $a_t$, and obtains reward $r_t$. The time steps after $T$ belong to an exploitation phase where the learner observes context $x_t$, chooses arm $a_t$ and earns an implicit reward $r_t$ that is not returned to the learner.

For the theoretical results below, the following general probabilistic framework is adopted, following Abbasi-Yadkori et al. (2011) and Durand et al. (2018), assume that $\zeta_t$ is a zero mean, $\rho$-conditionally sub-Gaussian random variable, i.e., $\zeta_t$ is such that for some $\rho > 0$ and $\forall \gamma \in \mathbb{R}$,

$$
\mathbb{E}[e^{\gamma \zeta_t} | \mathcal{H}_{t-1}] \leq \exp \left( \frac{\gamma^2 \rho^2}{2} \right).
$$

Here $\mathcal{H}_{t-1} = \{x_1, \ldots, x_{t-1}, \zeta_1, \ldots, \zeta_{t-1}\}$ is the history at time $t$ (see supplementary material for additional details).

We also define the following terms. Let $D_{a,t}$ be the set of all time indices when arm $a$ was selected up to time $t - 1$ and set $N_{a,t} = |D_{a,t}|$. Let $X_{a,t}$ be the data matrix whose columns are $\{x_\tau\}_{\tau \in D_{a,t}}$, and similarly let $Y_{a,t}$ denote the column vector of rewards $\{r_\tau\}_{\tau \in D_{a,t}}$. Thus, $X_{a,t} \in \mathbb{R}^{d \times N_{a,t}}$ and $Y_{a,t} \in \mathbb{R}^{N_{a,t}}$.

3.1 Problem Statement

At every time step $t$, the learner observes context $x_t$. During the exploration phase $t \leq T$, the learner chooses a series of actions to explore and learn the mappings $f_a$
from context to reward. During the exploitation phase \( t > T \), the goal is to select the best arm as a function of context. We define the simple regret associated with choosing arm \( a \in [A] \), given context \( x \), as:

\[
R_a(x) := f^*(x) - f_a(x),
\]

where \( f^*(x) := \max_{a \in [A]} f_a(x) \) is the expected reward for the best arm for context \( x \). The learner aims to minimize the simple regret for \( t > T \). To be more precise, let \( \Omega \) be the fixed policy mapping context to arm during the exploitation phase. The goal is to determine policies for the exploration and exploitation phases such that for all \( \epsilon > 0 \) and \( t > T \)

\[
\Pr(\Omega(x_t) \neq \epsilon(x_t)) \leq b_\epsilon(T),
\]

where \( b_\epsilon(T) \) is an expression that decreases to 0 as \( T \to \infty \).

The following section presents an algorithm to solve this problem.

## 4 Algorithm

We propose an algorithm that extends the Bayes Gap algorithm (Hoffman et al., 2014) to the contextual setting. Note that Bayes Gap itself is originally motivated from UGapE (Gabillon et al., 2012).

### 4.1 Estimating Expected Rewards

A key ingredient of our extension is an estimate of \( f_a \) for each \( a \), based on the current history. We use kernel methods to estimate \( f_a \). Let \( k : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) be a symmetric positive definite kernel function on \( \mathcal{X} \), \( \mathcal{H} \) be the corresponding RKHS and \( \phi(x) = k(\cdot, x) \) be the associated canonical feature map. Let \( \phi(X_{a,t}) := [\phi(x_j)]_{j \in D_{a,t}} \). We define the kernel matrix associated with \( X_{a,t} \) as \( K_{a,t} := \phi(X_{a,t})^T \phi(X_{a,t}) \in \mathbb{R}^{N_{a,t} \times N_{a,t}} \) and the kernel vector of context \( x \) as \( k_{a,t}(x) := \phi(X_{a,t})^T \phi(x) \). Let \( I_{a,t} \) be the identity matrix of size \( N_{a,t} \). We estimate \( f_a \) at time \( t \), via kernel ridge regression, i.e.,

\[
\hat{f}_{a,t}(x) = \arg \min_{f_a \in \mathcal{H}} \sum_{j \in D_{a,t}} (f_a(x_j) - r_j)^2 + \lambda \|f_a\|^2.
\]

The solution to this optimization problem is \( \hat{f}_{a,t}(x) = k_{a,t}(x)^T (K_{a,t} + \lambda I_{a,t})^{-1} y_{a,t} \). Furthermore, Durand et al. (2018) establish a confidence interval for \( f_a(x) \) in terms of \( \hat{f}_{a,t}(x) \) and the “variance” \( \hat{\sigma}_{a,t}^2(x) := k(x, x) - k_{a,t}(x)^T (K_{a,t} + \lambda I_{a,t})^{-1} k_{a,t}(x) \).

\textbf{Theorem 4.1} (Restatement of Theorem 2.1 in Durand et al. (2018)). \textit{Consider the contextual bandit scenario described in section 3. For any } \beta > 0, \textit{with probability at least } 1 - e^{-\beta^2}, \textit{it holds simultaneously over all } x \in \mathcal{X} \textit{ and all } t \leq T,

\[
|f_a(x) - \hat{f}_{a,t}(x)| \leq (C_1 \beta + C_2) \hat{\sigma}_{a,t}(x)/\sqrt{\lambda}, \tag{3}
\]

where \( C_1 = \rho \sqrt{2} \) and \( C_2 = \rho \sqrt{\sum_{t=2}^{T} \ln(1 + \frac{1}{\lambda} \hat{\sigma}_{a,\tau-1}(x)\beta)} + \sqrt{\|f_a\|_H} \).

In the supplementary material we show that \( C_2 = O(\rho \sqrt{\ln T}) \). For convenience, we denote the width of the confidence interval \( s_{a,t}(x) := 2(C_1 \beta + C_2) \hat{\sigma}_{a,t}(x)/\sqrt{\lambda} \).

Thus, the upper and lower confidence bounds of \( f_a(x) \) are

\[
U_{a,t}(x) := \hat{f}_{a,t}(x) + \frac{s_{a,t}(x)}{2} \text{ and } L_{a,t}(x) := \hat{f}_{a,t}(x) - \frac{s_{a,t}(x)}{2}.
\]

The upper confidence bound is the most optimistic estimate of the reward and the lower confidence bound is the most pessimistic estimate of the reward.

### 4.2 Contextual-Gap Algorithm

#### Algorithm 1 Contextual-Gap

\textbf{Input:} Number of arms \( A \), Time Steps \( T \), parameter \( \beta \), regularization parameter \( \lambda \), burn-in phase constant \( N_\lambda \).

// Exploration Phase I: Burn-in Period //

\textbf{for} \( t = 1, \ldots, AN_\lambda \) \textbf{do}

\hspace{1cm} Observe context \( x_t \)

\hspace{1cm} Choose \( a_t = t \mod A \)

\hspace{1cm} Receive reward \( r_t \in \mathbb{R} \)

\textbf{end for}

// Exploration Phase II: Contextual-Gap Policy //

\textbf{for} \( t = AN_\lambda + 1, \ldots, T \) \textbf{do}

\hspace{1cm} Observe context \( x_t \)

\hspace{1cm} Learn reward estimators \( \hat{f}_{a,t}(x_t) \) and confidence interval \( s_{a,t}(x_t) \) based on history

\hspace{1.5cm} \( U_{a,t}(x_t) := \hat{f}_{a,t}(x_t) + \frac{s_{a,t}(x_t)}{2} \)

\hspace{1.5cm} \( L_{a,t}(x_t) := \hat{f}_{a,t}(x_t) - \frac{s_{a,t}(x_t)}{2} \)

\hspace{1cm} \( B_{a,t}(x_t) := \max_{a \neq a_t} U_{a,t}(x_t) - L_{a,t}(x_t) \)

\hspace{1cm} \( J_t(x_t) = \arg \min_{a} B_{a,t}(x_t) \)

\hspace{1cm} Choose \( a_t = \arg \max_{a \in \{J_t(x_t), J_t(x_t)\}} s_{a,t}(x_t) \)

\hspace{1cm} Receive reward \( r_t \in \mathbb{R} \)

\textbf{end for}

// Exploitation Phase //

\textbf{for} \( t > T \) \textbf{do}

\hspace{1cm} Observe context \( x_t \)

\hspace{1.5cm} for \( \tau = AN_\lambda + 1, \ldots, T \) \textbf{do}

\hspace{2cm} Evaluate and collect \( J_t(x_t), B_{J_t(x_t)}(x_t) \)

\hspace{1.5cm} \textbf{end for}

\hspace{1cm} Choose \( \Omega(x_t) = J_t(x_t) \)

\textbf{end for}

During the exploitation phase, the Contextual-Gap algorithm proceeds as follows. First, the algorithm has
a burn-in period where it cycles through the arms (ignoring context) and pulls each one $N_\lambda$ times. Following this burn-in phase, when the algorithm is presented with context $x$ at time $t \leq T$, the algorithm identifies two candidate arms, $J_l(x)$ and $J_r(x)$, as follows. For each arm $a$ the contextual gap is defined as $B_{a,t}(x) := \max_{i \neq a} U_{i,t}(x) - L_{a,t}(x)$. $J_l(x)$ is the arm that minimizes $B_{a,t}(x)$ and $J_r(x)$ is the arm (excluding $J_l(x)$) whose upper confidence bound is maximized. Among these two candidates, the one with the widest confidence interval is selected. Note that one can rewrite $J_l(x) = \arg \max_a L_{a,t}(x) - \max_{i \neq a} U_{i,t}(x)$ which clearly shows that $J_l(x)$ is the best arm considering a pessimistic estimate of the reward.

In the exploitation phase, for a given context $x$, the contextual gap for all time steps in the exploration phase are evaluated. The arm with the smallest gap over the entire exploration phase for the given context $x$ is chosen as the best arm associated with context $x$. Because there is no feedback during the exploitation phase, the algorithm moves to the next exploitation step without modification to the learning history. The exact description is presented in Algorithm 1.

During the exploitation phase, looking back at all history may be computationally prohibitive. Thus, in practice, we just select the best arm as $J_T(x_t), \forall t > T$. As described in the experimental section, this works well in practice. Theoretically, $N_\lambda$ has to be bigger than a certain number defined in Lemma 5.2, but for experimental results we keep $N_\lambda = 1$.

### 4.3 Comparison of Contextual-Gap and Kernel-UCB

In this section, we illustrate the difference between the policies of Kernel-UCB (which minimizes cumulative regret) and exploration phase of Contextual-Gap (which aims to minimize simple regret). At each time step, Contextual-Gap selects one of two arms: $J_l(x)$, the arm with highest pessimistic reward estimate, or $J_r(x)$, the arm excluding $J_l(x)$ with highest optimistic reward estimate. Kernel-UCB, in contrast, selects the arm with the highest optimistic reward estimate (i.e., with the maximum upper confidence bound).

Consider a three arm scenario at some time $\tau$ with context $x_\tau$. Suppose that the estimated rewards and confidence intervals are as in Figures 3 and 4, reflecting two different cases.

- **Case 1 (Figure 3):** In this case, Kernel-UCB would pick arm 1, because it has the maximum upper confidence bound. Kernel-UCB’s policy is designed to be optimistic in the case of uncertainty. In the Contextual-Gap, we first calculate $J_r(x_\tau)$ which minimizes $B_{a_\tau,t}(x_\tau)$. Note that $B_{1_\tau,t}(x_\tau) = U_{2_\tau,t}(x_\tau) - L_{1_\tau,t}(x_\tau) = 7 - 2 = 5$, $B_{2_\tau,t}(x_\tau) = 3$ and $B_{3_\tau,t}(x_\tau) = 7$. In this case, $J_r(x_\tau) = 2$ and hence $J_r(x_\tau) = 1$. Finally, Contextual-Gap would choose among arm 1 and arm 2, and would finally choose arm 1 because it has the largest confidence interval. Hence, in case 1, Contextual-Gap chooses the same arm as that of Kernel-UCB.

- **Case 2 (Figure 4):** In this case, Kernel-UCB would pick arm 1. Note that $B_{1_\tau,t}(x_\tau) = U_{2_\tau,t}(x_\tau) - L_{1_\tau,t}(x_\tau) = 7 - 4 = 3$, $B_{2_\tau,t}(x_\tau) = 7$ and $B_{3_\tau,t}(x_\tau) = 4$. Then $J_r(x_\tau) = 1$ and hence $J_r(x_\tau) = 2$. Finally, Contextual-Gap chooses arm 2, because it has the widest confidence interval. Hence, in case 2, Contextual-Gap chooses a different arm compared to that of Kernel-UCB.

Clearly, the use of the lower confidence bound along with upper confidence bound allows Contextual-Gap to explore more than kernel-UCB. However, Contextual-Gap doesn’t explore just any arm, but rather it explores only among arms with some likelihood of being optimal. The following section details high probability bounds on the simple regret of the Contextual-Gap algorithm.
5 Learning Theoretic Analysis

We now analyze high probability simple regret bounds which depend on the gap quantity \( \Delta_a(x) := |\max_{i \neq a} f_i(x) - f_a(x)| \). The bounds are presented in the non-i.i.d setting described in Section 3. For the confidence interval to be useful, it needs to shrink to zero with high probability over the feature space as each arm is pulled more and more. This requires the smallest non-zero eigenvalue of the sample covariance matrix of the data for each arm to be lower bounded by a certain value. We make an assumption that allows for such a lower bound, and use it to prove that the confidence intervals shrink with high probability under certain assumptions. Finally, we bound the simple regret using the result of shrinking confidence interval, the gap quantity, and the special exploration strategy described in Algorithm 1. We now make additional assumptions to the problem setting.

A I \( \{ \mathcal{X}_t \}_{t \geq 1} \subset \mathbb{R}^d \) is a random process on compact space endowed with a finite positive Borel measure.

A II Kernel \( k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \) is bounded by a constant \( L \), the canonical feature map \( \phi : \mathcal{X} \rightarrow \mathcal{H} \) of \( k \) is a continuous function, and \( \mathcal{H} \) is separable.

We denote \( \mathbb{E}_{t-1}[\cdot] := \mathbb{E}[\cdot|x_1, x_2, \ldots, x_{t-1}] \) and by \( \lambda_r(A) \) the \( r \)th largest eigenvalue of a compact self-adjoint operator \( A \). For a context \( x \), the operator \( \phi(x)\phi(x)^T : \mathcal{H} \rightarrow \mathcal{H} \) is a compact self-adjoint operator. Based on this notation, we make the following assumption:

A III There exists a subspace of dimension \( d^* \) with projection \( P \), and a constant \( \lambda_x > 0 \), such that \( \forall t, \lambda_r((P^T \mathbb{E}_{t-1}\phi(x_1)\phi(x_1)^T)(I-P)) = 0, \forall r > d^* \).

Assumption A III facilitates the generalization of Bayes gap (Hoffman et al., 2014) to the kernel setting with non-i.i.d, time varying contexts. It allows us to lower bound, with high probability, the \( r \)th eigenvalue of the cumulative second moment operator \( S_t := \sum_{s=1}^t \phi(x_s)\phi(x_s)^T \) so that it is possible to learn the reward behavior in the low energy directions of the context at the same rate as the high energy ones with high probability.

We now provide a lower bound on the \( r \)th eigenvalue of a compact self-adjoint operator. There are similar results in the setting where reward is a linear function of context, including Lemma 2 in Gentile et al. (2014) and Lemma 7 in Li and Zhang (2018) which provides lowest eigenvalue bounds with the assumption of linear reward and full rank covariance, and Theorem 2.2 in Tu and Recht (2017) which assumes more structure to the contexts generated. We extend these results to the setting of a compact self-adjoint operator scenario with data occupying a finite dimensional subspace. Let \( W_t := \sum_{s=1}^t \mathbb{E}_{s-1}[\phi(x_s)\phi(x_s)^T]^2 - (\mathbb{E}_{s-1}[\phi(x_s)\phi(x_s)^T])^2 \). By construction and Assumption A III we can show that \( W_t \) has \( d^* \) non-zero eigenvalues (See Section 4.1 in the supplementary material).

Lemma 5.1 (Lower bound on \( r \)th Eigen-value of compact self-adjoint operators). Let \( x_t \in \mathcal{X}, t \geq 1 \) be generated sequentially from a random process. Assume that conditions A I-A III hold. Let \( p(t) = \min(-1,1) \) and \( 0 < a > \frac{1}{2}(L^2 + \sqrt{L^4 + 36t}) \) let \( \tilde{d} := 50\sum_{r=1}^{d^*} p(-\frac{\alpha_r(W_t)}{L^2p}) \leq 50d^* \). Let

\[
A(t, \delta) = \log \frac{(tL^4 + 1)(tL^4 + 3)\tilde{d}}{\delta},
\]

and

\[
h(t, \delta) = \left( t\lambda_x - \frac{L^2}{3} \sqrt{18tA(t, \delta)} + A(t, \delta)^2 \right) - \frac{L^2}{3} A(t, \delta).\]

Then for any \( \delta > 0 \),

\[\lambda_r(S_t) \geq h(t, \delta)\]

holds for all \( t > 0 \) with probability at least \( 1 - \delta \). Furthermore, if \( L = 1, r \leq d^* \) and \( 0 < \delta \leq \frac{1}{8} \), then the event

\[\lambda_r(S_t) \geq \frac{t\lambda_x^2}{2}, \forall t \geq \frac{256}{\lambda_x^2} \log \left( \frac{128d^*}{\lambda_x^2\lambda_x^2} \right),\]

holds with probability at least \( 1 - \delta \).

Lemma 5.1 provides high probability lower bounds on the minimum nonzero eigenvalue of the cumulative second moment operator \( S_t \). Using the preceding lemma and the confidence interval defined in Theorem 4.1, it is possible to provide high probability monotonic bounds on the confidence interval widths \( s_{a,t}(x) \).

Lemma 5.2 (Monotonic upper bound of \( s_{a,t}(x) \)). Consider a contextual bandit simple regret minimization problem with assumptions A I-A III and fix \( T \). Assume \( \|\phi(x)\| \leq 1, \lambda > 0 \) and \( \forall a \in [A], N_{a,t} > N_\lambda := \max \left( \frac{2(1-\lambda)}{\lambda}, d^*, \frac{256}{\lambda_x^2} \log \left( \frac{128d^*}{\lambda_x^2\lambda_x^2} \right) \right) \). Then, for any \( 0 < \delta \leq \frac{1}{8} \),

\[s_{a,t}(x_t)^2 \leq g_{a,t}(N_{a,t})\]

with probability at least \( 1 - \delta \), for the monotonically decreasing function \( g_{a,t} \) defined as \( g_{a,t}(N_{a,t}) := 8(C_1\beta + C_2)^2 \left( \frac{1}{x + N_{a,t}\lambda_x/2} \right) \).

The condition \( N_{a,t} > N_\lambda \) results in a minimum number of tries that arm \( a \) has to be selected before any bound
will hold. In \( N_\lambda := \max \left( \frac{\lambda_0(1-\lambda)}{\lambda_0}, d^*, \frac{256}{\lambda_0^2} \log(\frac{128d}{\lambda_0}) \right) \), the first and third term in the max are needed so that we can give concentration bounds on eigenvalues and prove that the confidence width shrinks. The second term is needed because one has to get at least \( d^* \) contexts for every arm so that at least some energy is added to the lowest eigenvalues. These high probability monotonic upper bounds on the confidence estimate can be used to upper bound the simple regret. The upper bound depends on a context-based hardness quantity defined for each arm \( a \) (similar to Hoffman et al. (2014)) as

\[
H_{a,\epsilon}(x) = \max \left( \frac{1}{2}(\Delta_a(x) + \epsilon), \epsilon \right). \tag{4}
\]

Denote its lowest value as \( H_{a,\epsilon} := \inf_{x \in X} H_{a,\epsilon}(x) \). Let total hardness be defined as \( H_t := \sum_{a \in [A]} H_{a,\epsilon}^2 \) (Note that \( H_t \leq \frac{A}{2} \)). The recommended arm after time \( t \geq T \) is defined as

\[
\Omega(x) = \arg \min_{a, t \leq T} H_{x+a,\epsilon}(x)(x_t)
\]

from Algorithm 1. We now upper bound the simple regret as follows:

**Theorem 5.3.** Consider a contextual bandit problem as defined in Section 3 with assumptions \( A, A^\text{I}, A^\text{II} \). For \( 0 < \delta \leq \frac{1}{3}, \epsilon > 0 \) and \( N_\lambda := \max \left( \frac{\lambda_0(1-\lambda)}{\lambda_0}, d^*, \frac{256}{\lambda_0^2} \log(\frac{128d}{\lambda_0}) \right) \), let

\[
\beta = \sqrt{\frac{\lambda_0(T - N_\lambda(A - 1)) + 2A\lambda}{16C_1^2H_c}C_2 - C_1} \tag{5}
\]

For all \( t > T \) and \( \epsilon > 0 \),

\[
\mathbb{P}(R_{\Omega(x_t)}(x_t) < \epsilon | x_t) \geq 1 - A(T - AN_\lambda)e^{-\beta^2} - A\delta. \tag{6}
\]

Note that the term \( C_2 \) in (5) grows logarithmically in \( T \) (see supplementary material). For \( \beta \) to be positive, \( T \) should be greater than \( \frac{16C_1^2C_2 \lambda_0^2 - 2A\lambda}{\lambda_0} + N_\lambda(A - 1) \) We compare the term \( e^{-\beta^2} \) in our bound with the uniform sampling technique in Guan and Jiang (2018) which leads to a bound that decay like \( Ce^{cT(\frac{1}{\epsilon} - \frac{1}{\epsilon'})} \geq C e^{-cT(\frac{1}{\epsilon'_0})} \), where \( d_1 \geq 2, d \) is the context dimension, and \( C \) and \( c \) are constants. In our case, the decay rate has the form \( C'Te^{-c'T} \) for constants \( C', c' \). Clearly, our bound is superior for \( \forall d \geq 1 \).

6 Experimental Results and Discussion

We present results from two different experimental set-ups, first from online multiclass classification with partial feedback, and second from a lab generated non-i.i.d spacecraft magnetic field as described in Section 2. The datasets were split into cross-validation and evaluation datasets and each of those datasets were further split into exploration and exploitation phases. Cross validation was performed to minimize *average simple regret* for the exploitation phase while training with the exploration phase, both from the cross validation dataset. The value of \( T \) selected in both the cross validation and evaluation datasets were of similar magnitude. Evaluation of the algorithm for average simple regret behavior is performed with the evaluation dataset.

We present average simple regret comparisons of the Contextual-Gap algorithm against four baselines:

1. Uniform Sampling: We equally divide the exploration budget \( T \) among arms and learn a reward estimating function \( f_a : \mathcal{X} \rightarrow \mathbb{R} \) for each of the arm during the exploration phase. During the exploitation phase, we select the best arm based on estimated reward function \( f_a \).

2. Epsilon Greedy: At every step, we select the best arm (according to estimated \( f_a \)) with probability \( 1 - \epsilon_t \) and other arms with probability \( \epsilon_t \). We use \( \epsilon_t = 0.99^t \), where \( t \) is the time step.

3. Kernel-UCB: We implement kernel-UCB from Valko et al. (2013).

4. Kernel-TS: We use kernelized version of Thompson Sampling from Chowdhury and Gopalan (2017).

For all the algorithms, we use the Gaussian kernel and tune the bandwidth of the kernel, and the regularization parameter. The exploration parameter \( \alpha := C_1 \beta + C_2 \) is set to 1 for the results in this section and we show results for different values of \( \alpha \) in the supplementary material.

6.1 Multi-class Classification

We present results of contextual simple regret minimization for multiclass datasets. At every time step, we observe a feature vector and need to select the class to which the example belongs. Each class is treated like an arm or action. If we select the best arm (true class) we get a reward of one, otherwise we get a reward of zero. This setting is different from standard online multiclass classification, because we don’t learn the true class if our selection is wrong. We present results over three multiclass datasets: MNIST (LeCun 1988). The code to reproduce our results is available at https://www.dropbox.com/sh/0f6ycz6x9kaprl3/AACUFHyNgT6eSB15s2YhuM5ga?dl=0
et al., 1998), USPS (Hull, 1994) and Letter (Hsu and Lin, 2002). Figure 5 shows the variation of the average simple regret with increasing exploration phase for five algorithms. The dataset for evaluation of simple regret was kept constant. Since the datasets are i.i.d in nature, multiple simple regret evaluations are performed by shuffling the evaluation datasets, and the average curves are reported. Note that the algorithms have been cross validated for simple regret minimization. The plots are generated by varying the length of the exploration phase and keeping the exploitation dataset constant for evaluation of simple regret. It can be seen that the simple regret of the Contextual-Gap converges faster than the simple regret of other baselines.

6.2 Experimental Spacecraft Magnetic Field Dataset

We present the experimental setup and results associated with a lab generated, realistic spacecraft magnetic field dataset with non-i.i.d contexts. In spacecraft magnetic field data, we are interested in identifying the least noisy sensor for every time step (see Section 2). The dataset was generated with contexts $x_t$ consisting of measured variables associated with the electrical behavior of the GRIFEX spacecraft (Norton et al., 2012; Cutler et al., 2015), and reward is the negative of the magnitude of the sensor noise measured at every time step.

Data were collected using 3 sensors (arms), and sensor readings were downloaded for all three sensors at all times steps, although the algorithm does not know these in advance and must select one sensor at each time step. The context information was used in conjunction with a realistic simulator to generate spacecraft magnetic field, and hence a realistic model of sensor noise, as a function of context. The true magnetic field was computed using models of the earth’s magnetic field.

Figure 5d shows the simple regret minimization curves for the spacecraft data-set and even in this case Contextual-Gap converges faster compared to other algorithm. Note that, in addition to the non-i.i.d nature, there exists large variability in reward for certain regions of the context space.

7 Conclusion

In this work, we present a novel problem: that of simple regret minimization in the contextual bandit setting.
We propose the Contextual-Gap algorithm, give a regret bound for the simple regret, and show empirical results on three multiclass datasets and one lab-based spacecraft magnetometer dataset. It can be seen that in this scenario persistent and efficient exploration of the best and second best arms with the Contextual-Gap algorithm provides improved results compared against algorithms designed to optimize cumulative regret.

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8 Remarks

- In Section 9, we formalize the notion of History $H_t$ in the main paper.
- Detailed comments about constant $C_2$ (Theorem 4.1 of the main paper) are at the end of Subsection 12.1.
- Detailed experiments and results about different $\alpha$ are in Section 13.

9 Probabilistic Setting and Martingale Lemma

For the theoretical results, the following general probabilistic framework is adopted, following Abbasi-Yadkori et al. (2011) and Durand et al. (2018). We formalize the notion of history $H_t$ defined in the Section 3 of the main paper using filtration. A filtration is a sequence of $\sigma$-algebras $\{\mathcal{F}_t\}_{t=1}^\infty$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots$. Let $\{\mathcal{F}_t\}_{t=1}^\infty$ be a filtration such that $x_t$ is $\mathcal{F}_{t-1}$ measurable, and $\zeta_t$ is $\mathcal{F}_1$ measurable. For example, one may take $\mathcal{F}_t := \sigma(x_1, x_2, \cdots, x_{t+1}, \zeta_1, \zeta_2, \cdots, \zeta_t)$, i.e., $\mathcal{F}_t$ is the $\sigma-$algebra generated by $x_1, x_2, \cdots, x_{t+1}, \zeta_1, \zeta_2, \cdots, \zeta_t$. 


We assume that $ζ_i$ is a zero mean, $ρ$-conditionally sub-Gaussian random variable, i.e., $ζ_i$ is such that for some $ρ > 0$ and $∀γ ∈ ℝ$,
\[ E[e^{γζ_i}|F_{t-1}] ≤ \exp\left(\frac{γ^2ρ^2}{2}\right). \] (7)

**Definition 9.1** (Definition 4.11 in Motwani and Raghavan (1995)). Let $(Σ,F,Pr)$ be a probability space with filtration $F_0,F_1,...$. Suppose that $Z_0,Z_1,...$ are random variables such that for all $i > 0$, $Z_i$ is $F_i$ measurable. The sequence $Z_0,Z_1,...$ is a martingale provided for all $i ≥ 0$,
\[ E[Z_{i+1}|F_i] = Z_i. \]

**Lemma 9.2** (Theorem 4.12 in Motwani and Raghavan (1995)). Any subsequence of a martingale is also a martingale (relative to the corresponding subsequence of the underlying filter).

The above Lemma is important because we construct confidence intervals for each arm separately. Note that we define a subset of time indices $(D_{a,t})$ of each arm $a$, when the arm $a$ was selected. Based on these indices we can form sub-sequences of the main context $(x_t)_{t=1}^∞$ and noise sequence $(ζ_t)_{t=1}^∞$ such that the assumptions on the main sequence hold for subsequences.

### 9.1 Theorem 4.1 in Main Paper

Theorem 4.1 is a slight modification of Theorem 2.1 in Durand et al. (2018). In the contextual bandit setting in Durand et al. (2018), for any $δ ∈ (0,1]$, Theorem 2.1 in Durand et al. (2018) establishes that with probability at least $1 − δ$, it holds simultaneously over all $x ∈ X$ and $t ≥ 0$,
\[ |f_a(x) - ˆf_{a,t}(x)| ≤ \frac{δ_{a,t}(x)}{√λ} \left(√X\|f_a\|_H + ρ√2\ln(1/δ) + 2γ_t(λ)\right), \]
where $γ_t(λ) = \frac{1}{2} ∑_{τ=1}^t \ln(1 + \frac{1}{2}δ_{a,τ-1}(x_τ))$.

For $T ≥ t$, one can replace $t$ in the log terms with $T$. Then $∀x,∀t ≥ 1$, we have
\[ 1 − δ ≤ \mathbb{P}\left(|f_a(x) - ˆf_{a,t}(x)| ≤ \frac{δ_{a,t}(x)}{√λ} \left(√X\|f_a\|_H + ρ√2\ln(1/δ) + 2γ_T(λ)\right)\right). \]
Let $δ = e^{-β^2}$. In that case,
\[ 1 − e^{-β^2} ≤ \mathbb{P}\left(|f_a(x) - ˆf_{a,t}(x)| ≤ \frac{δ_{a,t}(x)}{√λ} \left(√X\|f_a\|_H + ρ√2β^2 + 2γ_T(λ)\right)\right). \]

Using triangle inequality $√p + q ≤ √p + √q$ for any $p,q ≥ 0$,
\[ 1 − e^{-β^2} ≤ \mathbb{P}\left(|f_a(x) - ˆf_{a,t}(x)| ≤ \frac{δ_{a,t}(x)}{√λ} \left(√X\|f_a\|_H + ρ√2β^2 + 2ρ√2γ_T(λ)\right)\right). \]
Let $C_1 = ρ\sqrt{2}$ and $C_2 = √X\|f_a\|_H + ρ\sqrt{2γ_T(λ)}$. Hence, we have
\[ 1 − e^{-β^2} ≤ \mathbb{P}\left(|f_a(x) - ˆf_{a,t}(x)| ≤ \frac{δ_{a,t}(x)}{√λ} [C_1β + C_2]\right). \]

### 10 Lower Bound on $r^{th}$ Eigenvalue

First we state the Lemmas that we use to prove Lemma 5.1 in main paper.

**Lemma 10.1** (Lemma 9 in Li and Zhang (2018)). If $a > 0,b > 0,ab ≥ e$, then for all $t ≥ 2a\log(ab)$,
\[ t ≥ a\log(bt). \] (8)

**Lemma 10.2** (Lemma 1.1 in Zi-Zong (2009)). Let $A ∈ ℝ^{n×n}$ be a symmetric positive definite matrix partitioned according to
\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}, \]
where $A_{11} ∈ ℝ^{(n-1)×(n-1)}$, $A_{12} ∈ ℝ^{(n-1)×1}$ and $A_{22} ∈ ℝ^1$. Then $\det(A) = \det(A_{11})/|A_{22} - A_{12}^T A_{11}^{-1} A_{12}|$.

**Lemma 10.3** (Special case of extended Horn’s inequality (Theorem 4.5 of Bercovici et al. (2009))). Let $A,B$ be compact self-adjoint operators. Then for any $p ≥ 1$,
\[ λ_p(A + B) ≤ λ_1(A) + λ_p(B). \] (9)

**Theorem 10.4** (Freedman’s inequality for self adjoint operators, Thm 3.2 & section 3.2 in Minsker (2017)). Let $(\{Φ_i\}_{i=1,...})$ be a sequence of self-adjoint Hilbert Schmidt operators $Φ_i : H → H$ acting on a separable Hilbert space $(EΦ$ is an operator such that
We now define a few terms on the fixed difference sequence of self-adjoint operators such that \( \|\Phi_t\| \leq L^2 \) almost surely for all \( 1 \leq t \leq T \) and some positive \( L \in \mathbb{R} \). Denote by \( W_t = \sum_{s=1}^{t} E_{s-1}[\Phi_s^2] \) and \( p(t) = \min(-t, 1) \). Then for any \( a \geq \frac{1}{9} (L^2 + \sqrt{L^4 + 36b}), b \geq 0,
\[
\mathbb{P} \left( \| \sum_{j=1}^{t} \Phi_j \| > a \text{ and } \lambda_1(W_t) \leq b \right) \leq \tilde{d} \cdot \exp \left( -\frac{a^2/2}{b + aL^2/3} \right),
\]
where \( \| \| \) is the operator norm and \( \tilde{d} := 50 \sum_{r=1}^{\infty} (p(-\frac{a\lambda_r(E_{W_t})}{L^2b})). \)

Note that \( \tilde{d} \) is a function of \( t \) but it’s upper bounded by \( d^* \) which is the rank of \( E_{s-1}[\phi(x)\phi(x)^T] \).

### 10.1 Proof of Lemma 5.1 in main paper

Lemma 7 in Li and Zhang (2018) gives the lower bound on minimum eigenvalue (finite dimensional case) when reward depends linearly on context. We extend it to \( t \)th largest eigenvalue (infinite dimensional case) and the case when reward depends non-linearly on context.

**Proof.** \( \mathcal{X} \subset \mathbb{R}^d \) is a compact space endowed with a finite positive Borel measure. For a continuous kernel \( k \) the canonical feature map \( \phi \) is a continuous function \( \phi : \mathcal{X} \rightarrow \mathcal{H} \), where \( \mathcal{H} \) is a separable Hilbert space (See section 2 of Michielli et al. (2006) for a construction such that \( \mathcal{H} \) is separable). In such a setting \( \phi(\mathcal{X}) \) is also compact space with a finite positive Borel measure (Michielli et al., 2006). We now define a few terms on \( \phi(\mathcal{X}) \).

Define the random variable \( \Phi_t := E_{s-1}[\phi(x_s)\phi(x_s)^T] - \phi(x_t)\phi(x_t)^T \). Let \( Z_t := \sum_{s=1}^{t} \Phi_s = \sum_{s=1}^{t} E_{s-1}[\phi(x_s)\phi(x_s)^T] - S_t = V_t - S_t \).

By construction, \( \{Z_t\}_{t=1,2,...} \) is a martingale and \( \{\Phi_s\}_{s=1,2,...} \) is the martingale difference sequence. Notice that \( \lambda_1(\Phi_t) \leq L^2 \). To use the Freedman’s inequality, we lower bound the operator norm of \( Z_t \), \( \| Z_t \| \) and upper bound the largest eigenvalue of \( W_t, \lambda_1(W_t) \). Let \( \nu(A) = \max_\lambda |\lambda(A)| \) be the spectral radius of operator \( A \). We work with the spectral radius because it is not necessary that \( Z_t \) is a positive definite operator. It is well known that
\[
\nu(A) \leq \| A \|. \tag{10}
\]

By assumption A III, \( E_{s-1}[\phi(x)\phi(x)^T] \) lies in a fixed \( d^* \) dimensional subspace with its eigenvalues \( \lambda_r(E_{s-1}[\phi(x)\phi(x)^T]) > \lambda_2 \) for \( r \leq d^* \). Thus, for \( V_t = \sum_{s=1}^{t} E_{s-1}[\phi(x)\phi(x)^T] \), \( \lambda_r(V_t) \geq t\lambda_2 \).

**Bound on \( \| Z_t \| \):** By definition, \( V_t = Z_t + S_t \). Hence, \( \lambda_r(V_t) \leq \lambda_1(Z_t) + \lambda_r(S_t) \) by using Horn’s inequality (Lemma 10.3).

\[
\begin{align*}
\lambda_1(Z_t) & \geq \lambda_r(V_t) - \lambda_r(S_t) \\
\lambda_1(Z_t) & \geq t\lambda_2 - \nu(S_t) \\
\nu(Z_t) & \geq t\lambda_2 - \nu(S_t),
\end{align*}
\]

where the second step is due to A III and the third step is by definition of spectral radius. By Eqn. (10), we have
\[
\| Z_t \| \geq t\lambda_2 - \nu(S_t). \tag{11}
\]

**Bound on \( \lambda_1(W_t) \):** To bound the term \( \lambda_1(W_t) \), write
\[
W_t = \sum_{s=1}^{t} E_{s-1}[\Phi_s^2] = \sum_{s=1}^{t} E_{s-1}[\phi(x_s)\phi(x_s)^T - \phi(x_s)\phi(x_s)^T]^2.
\]

By using square expansion,
\[
W_t = \sum_{s=1}^{t} E_{s-1}[\phi(x_s)\phi(x_s)^T]^2 + (\phi(x_s)\phi(x_s)^T)^2 - E_{s-1}[\phi(x_s)\phi(x_s)^T](\phi(x_s)\phi(x_s)^T)
\]
\[
- (\phi(x_s)\phi(x_s)^T)E_{s-1}[\phi(x_s)\phi(x_s)^T]
\]
\[
= \sum_{s=1}^{t} E_{s-1}[\phi(x_s)\phi(x_s)^T]^2 - E_{s-1}[\phi(x_s)\phi(x_s)^T]^2.
\]

Taking norm on both sides,
\[
\| W_t \| = \| \sum_{s=1}^{t} E_{s-1}[\phi(x_s)\phi(x_s)^T]^2 - E_{s-1}[\phi(x_s)\phi(x_s)^T]^2 \|.
\]

As both terms on the right hand side are positive semi-definite matrices,
\[
\| W_t \| \leq \| \sum_{s=1}^{t} E_{s-1}[\phi(x_s)\phi(x_s)^T]^2 \|.
\]

Next, we use convexity properties of norms to get the
Therefore, we can bound the norm \( \|W_t\| \) as
\[
\|W_t\| \leq L^2 \sum_{s=1}^{L} [\|E_{s-1}[(\phi(x_s)\phi(x_s)^T)]\|] \\
= L^2 \sum_{s=1}^{L} [\|E_{s-1}[(\phi(x_s)\phi(x_s)^T)])\|] \\
\leq L^2 \sum_{s=1}^{L} [\|E_{s-1}[(\phi(x_s))]\|] \\
\leq L^2 \sum_{s=1}^{L} [\|\phi(x_s)\|] \leq L^2
\]
where the first step is due to the triangle inequality and the third step is due to the upper bound \( \|\phi(x)\| \leq L \), the fourth step is due to the convexity of the operator norm and Jensen’s inequality. Using the properties of Hilbert Schmidt operators, we can write
\[
E_{s-1}[\|\phi(x_s)\phi(x_s)^T]\|] \leq E_{s-1}[\||\phi(x_s)\phi(x_s)^T||]_{HS} \\
= E_{s-1}[\||\phi(x_s)||^2] \leq L^2
\]
Therefore, we can bound the norm \( \|W_t\| \) as
\[
\|W_t\| \leq L^2 \sum_{s=1}^{L} L^2 \\
= tL^4,
\]
Again, by using Eqn. (10), we have
\[
\lambda_1(W_t) \leq tL^4, \quad (12)
\]
Now, we shall construct a parameter \( A \) such that
\[
\frac{\alpha^2/2}{b + \alpha L^2/3} \geq A.
\]
For this inequality to hold, one can see, by its quadratic solution, \( a \geq f(A,b) := \frac{3}{4} AL^2 + \sqrt{\frac{1}{4} A^2 L^4 + 2Ab} \). Note that for \( A > 1 \), the condition of \( a \geq f(A,b) \) also satisfies the conditions of Friedman’s inequality in Theorem 10.4.

Let \( A(m,\delta) = \log \frac{(m+1)(m+3)}{\delta} \) and \( P \) be the probability of event \( \exists t : \lambda_r(S_t) \leq t\lambda_x - f(A(tL^4,\delta), tL^4) \).

\[
P = P\left[ \exists t : \lambda_r(S_t) \leq t\lambda_x - f(A(tL^4,\delta), tL^4) \right] \quad (14)
\]
\[
\leq \sum_{m=0}^{\infty} P\left[ \exists t : \lambda_r(S_t) \leq t\lambda_x - f(A(m,\delta), m) \right] \quad (15)
\]
\[
\leq \sum_{m=0}^{\infty} \exp (-A(m,\delta)) \quad (16)
\]
\[
= \tilde{d} \sum_{m=0}^{\infty} \frac{\delta}{(m+1)(m+3)} \quad (17)
\]
where \( (15) \) is because \( A \) is increasing in \( m, f \) is increasing in \( A,b \), and Eqn. (12). Eqn. (16) is by application of the union bound over all the events for which \( \lambda_1(W_t) \leq m \). Also, Eqn. (17) is due to Eqn. (11) and Eqn. (18) is due to Theorem 10.4.

The result is obtained by replacing \( \delta \) by \( \frac{\delta}{4} \).

For the second part. Let \( \tilde{\lambda}_x := \frac{\lambda_x}{2} \). By definition of \( L, \tilde{\lambda}_x \leq 1 \). Let \( t \geq \frac{256}{\lambda_x^2} \log t \delta \frac{128 \tilde{d}}{\Lambda^2} \). Then by using the Lemma 10.1,
\[
t \geq \frac{128}{\lambda_x^2} \log \frac{t \delta \tilde{d}}{\Lambda^2} \quad (20)
\]
Rearranging the terms, we get
\[
\frac{t \tilde{\lambda}_x^2}{4} \geq 32 \log \frac{t \delta \tilde{d}}{\delta}
\]
Taking square root and then multiplying by \( \sqrt{t} \) on both sides
\[
\frac{t \tilde{\lambda}_x}{2} \geq \sqrt{32t \log \frac{t \delta \tilde{d}}{\delta}} \\
= \frac{2}{3} \sqrt{72t \log \frac{t \delta \tilde{d}}{\delta}} \\
= \frac{2}{3} \sqrt{36t \log \frac{t \delta \tilde{d}}{\delta} + 36t \log \frac{t \delta \tilde{d}}{\delta}}.
\]
Using equation (20),
\[
\frac{t\lambda_x}{2} \geq \frac{2}{3} \sqrt{36t \log \frac{t\lambda}{\delta} + 36 \cdot 12 \cdot \frac{(\log \frac{t\lambda}{\delta})^2}{\lambda_x^2}}.
\]
\[
= \frac{2}{3} \sqrt{36t \log \frac{t\lambda}{\delta} + 36 \cdot 32 \cdot 4 \cdot \frac{(\log \frac{t\lambda}{\delta})^2}{\lambda_x^2}}.
\]
Since \( \lambda_x^2 \leq 1 \) we have
\[
\frac{t\lambda_x}{2} \geq \frac{2}{3} \sqrt{36t \log \frac{t\lambda}{\delta} + (36 \cdot 32) \cdot \frac{(\log \frac{t\lambda}{\delta})^2}{\lambda_x^2}}.
\]
Taking log of both sides,
\[
2 \log \frac{t\lambda}{\delta} \geq \log \left( \frac{(t+1)(t+3)}{\delta} \right) = A(t, \delta).
\]
Without loss of generality, we will assume that \( L = 1 \).
From Eqn. (23) and Eqn. (21), we have
\[
\frac{t\lambda_x}{2} \geq \frac{2}{3} \sqrt{18t \cdot A(t, \delta) + A(t, \delta)^2}.
\]
\[
= \frac{1}{3} \sqrt{18t \cdot A(t, \delta) + A(t, \delta)^2}.
\]
\[
\geq \frac{1}{3} \sqrt{18t \cdot A(t, \delta) + A(t, \delta)^2} + \frac{1}{3} A(t, \delta)
\]
Therefore,
\[
\frac{t\lambda_x}{2} \geq f(A(t, \delta), t).
\]
Equations (14) and (24) complete the proof.

11 Monotonic Upper bound of \( s_{a,t}(x) \)

Lemma 11.1. [Arithmetic Mean-Geometric Mean Inequality (Steele, 2004)] For every sequence of non-negative real numbers \( a_1, a_2, ... a_n \) one has
\[
\left( \prod_{i=1}^{n} a_i \right)^{1/n} \leq \frac{\sum_{i=1}^{n} a_i}{n}
\]
with equality if and only if \( a_1 = a_2 = ... = a_n \).

Lemma 11.2. If \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 \), and \( \mu_1 \geq 0, \mu_2 \geq 0 \cdots \mu_d \geq 0 \) such that \( \sum_j \mu_j = L \) and \( \lambda_d \geq L \) then
\[
\prod_{i=1}^{d} \left( 1 + \frac{\mu_i}{\lambda_i} \right) - 1 \leq \frac{2L}{\lambda_d}.
\]
Proof. By replacing each \( \lambda_i \) with the smallest element \( \lambda_d \) we get,
\[
\prod_{i=1}^{d} \left( 1 + \frac{\mu_i}{\lambda_i} \right) - 1 \leq \prod_{i=1}^{d} \left( 1 + \frac{\mu_i}{\lambda_d} \right) - 1 = \prod_{i=1}^{d} \frac{\lambda_d + \mu_i}{\lambda_d} - 1 = \left( \prod_{i=1}^{d} (\lambda_d + \mu_i) \right)^{-1} - 1 \leq \left( \sum_{i=1}^{d} (\lambda_d + \mu_i) \right)^{-d} - 1 = \frac{(d \lambda_d + L)^d}{d \lambda_d} - 1 = \left( 1 + \frac{L}{d \lambda_d} \right)^d - 1 \leq e^{L/d \lambda_d} - 1,
\]
where the fourth inequality is by Lemma 11.1 and last inequality holds because \((1 + \frac{a}{x})^x \) approaches \( e^a \) as \( x \to \infty \) and \( (1 + \frac{a}{x})^x \) is a monotonically increasing function of \( x \).

By \( e^x \leq 1 + 2x \) for \( x \in [0,1] \) and the assumption that \( \lambda_d \geq L \),
\[
\prod_{i=1}^{d} \left( 1 + \frac{\mu_i}{\lambda_i} \right) - 1 \leq e^{L/d \lambda_d} - 1 = 1 + 2L \lambda_d - 1 = \frac{2L}{\lambda_d}.
\]

11.1 Proof of Lemma 5.2 in main paper
Proof. We will assume that \( L = 1 \). We write
\[
K_{a,t+1} + \lambda_{I_{t+1}} = \left[ \frac{K_{a,t} + \lambda_{I_{t}}}{K_{a,t}(x)} - \frac{k_{a,t}(x)}{K(x, x) + \lambda} \right].
\]
Let \( \mu_i = \lambda_i \left( K_{a,t+1} + \lambda_{I_{t+1}} \right) - \lambda_i \left( K_{a,t} + \lambda_{I_{t}} \right) \).
Note that $\lambda_d^{*,a,t} = \lambda_d^{*}(K_{a,t+1} + L_{a,t+1}) = \lambda_d^{*}(K_{a,t+1} + \lambda)$. By Lemma 5.1 in main paper $\lambda_d^{*}(K_{a,t+1}) \geq N_{a,t}\lambda_x$. We can apply Lemma 11.2 only when
\[
 \frac{1}{\lambda + N_{a,t}\lambda_x/2} < 1
\]
or
\[
 N_{a,t} > \frac{2(1 - \lambda)}{\lambda_x}.
\]
The assumption in the statement of the lemma satisfies the above equation. Hence, we have
\[
 s_{a,t}(x)^2 \leq 4(C_1\beta + C_2)^2 \left( \frac{2}{\lambda + N_{a,t}\lambda_x/2} \right)
 = 8(C_1\beta + C_2)^2 \left( \frac{1}{\lambda + N_{a,t}\lambda_x/2} \right)
 = g_{a,t}(N_{a,t}).
\]
This concludes the proof. 

11.2 Closed form of $g_{a,t}^{-1}(s)$

Now we calculate a closed form expression of $N_{a,t}$. Setting the upper bound on confidence in the Theorem 4.1 in main paper to $s$, we calculate the inverse in terms of $N_{a,t}$.
\[
 8(C_1\beta + C_2)^2 \left( \frac{1}{\lambda + N_{a,t}\lambda_x/2} \right) = s^2.
\]
Rearranging all the terms, we get
\[
 8(C_1\beta + C_2)^2 = s^2(\lambda + N_{a,t}\lambda_x/2)
 = \frac{(\lambda + N_{a,t}\lambda_x/2)^2}{s^2} = \frac{16(C_1\beta + C_2)^2}{s^2}\lambda_x
 = \frac{16(C_1\beta + C_2)^2}{s^2\lambda_x} - \frac{2\lambda}{\lambda_x},
\]
Define
\[
 g_{a,t}^{-1}(s) = \frac{16(C_1\beta + C_2)^2}{s^2\lambda_x} - \frac{2\lambda}{\lambda_x}.
\]

12 Simple Regret Analysis

Lemma 12.1 (Value of $\beta$). Assume the conditions in Theorem 4.1 and Lemma 4.2 in main paper. If
\[
 \sum_{a \in [A]} g_{a,t}^{-1}(H_{aa}) = T - N_x(A - 1),
\]
then
\[
 \beta = \sqrt{\frac{\lambda_x(T - N_x(A - 1)) + 2AX}{16C_2^2 H_e}} - \frac{C_2}{C_1}.
\]
Proof. We have
\[ \sum_{a \in [A]} g_{a,t}(H_{a,t}) = T - N_\lambda(A - 1). \]
By using Eqn. (26),
\[ \sum_{a \in [A]} \frac{16(C_1 \beta + C_2)^2}{\lambda_x} - \frac{2\lambda}{\lambda_x} = T - N_\lambda(A - 1) \]
By using definition of $H_e$,
\[ \frac{16(C_1 \beta + C_2)^2 H_e}{\lambda_x} - \frac{2\lambda}{\lambda_x} = T - N_\lambda(A - 1) \]
Rearranging the terms,
\[ \frac{16(C_1 \beta + C_2)^2 H_e}{\lambda_x} = \frac{\lambda_x(T - N_\lambda(A - 1)) + 2\lambda}{\lambda_x} \]
\[ (C_1 \beta + C_2)^2 = \frac{\lambda_x(T - N_\lambda(A - 1)) + 2\lambda}{16H_e} \]
\[ \beta = \sqrt{\frac{\lambda_x(T - N_\lambda(A - 1)) + 2\lambda}{16C_1^2 H_e}} \]
\[ \mathbb{P}(\mathcal{E}) \geq 1 - A(T - AN_\lambda)e^{-\beta^2}. \]

The next part of the proof works by contradiction.

Let $\epsilon > 0$. The recommended arm at the end of time $T$ for context $x$ is defined as follows: let $t^* := \arg \min_{A^\prime} B_{h(x),t^*}(x)$ then the recommended arm is $\Omega := J_{t^*}(x)$.

Conditioned on event $\mathcal{E}$, we will assume that the event $R_{\Omega}(x) > \epsilon$ is true and arrive at a contradiction with high probability. Note that if $R_{\Omega}(x) > \epsilon$, the recommended arm $\Omega$ is necessarily sub-optimal (regret is zero for the optimal arm).

Define $M_{a,T}(x)$ as number of times arm $a \in [A]$ would be selected in $AN_\lambda < t < T$, if we had seen context $x$ at all those times. Hence, $\sum_{a \in [A]} M_{a,T}(x) = T - AN_\lambda$. Also, note that $N_{a,T}(x) = M_{a,T}(x) + N_\lambda$ for $a \in [A]$ and $N_{a,T}(x) = N_\lambda$ otherwise. Let $t_a = t_a(x)$ be the last time instant for which arm $a \in [A]$ may have been selected using the Contextual-Gap algorithm if context $x$ was observed throughout.

The following holds for the recommended arm $\Omega$ with context $x$:
\[ \min(0, s_{a,t_a}(x) - \Delta_a(x)) + s_{a,t_a}(x) \geq B_{t_a}(x) \]
\[ \geq B_{\Omega, T+1}(x) \]
\[ \geq R_{\Omega}(x) > \epsilon. \]

Where the first inequality holds due to Lemma 12.5, the second inequality holds by definition of $B_{\Omega, T+1}$, the third inequality holds due to Lemma 12.2 and the last inequality holds due to the event $R_{\Omega} > \epsilon$. The preceding inequality can also be written as
\[ s_{a,t_a}(x) > 2s_{a,t_a}(x) - \Delta_a(x) > \epsilon, \quad \text{if } \Delta_a(x) > s_{a,t_a}(x). \]
\[ 2s_{a,t_a}(x) - \Delta_a(x) > s_{a,t_a}(x) > \epsilon, \quad \text{if } \Delta_a(x) < s_{a,t_a}(x). \]

This leads to the following bound on the confidence diameter of $a \in [A]$,
\[ s_{a,t_a}(x) > \max(\frac{1}{2} (\Delta_a(x) + \epsilon), \epsilon) := H_{a}(x). \]

For any arm $a$, we consider the final number of arm pulls $M_{a,T}(x) + N_\lambda$. From Lemma 5.2 of the main paper we can write, using the strict monotonicity and there by invertibility of $g_{a,T}$, with probability at least $1 - \delta$ as

12.1 Proof of Theorem 5.3 in main paper

Let $[A] = \{1, ..., A\}$. We define a feasible set $A'(x) \subseteq [A]$ such that elements of $A'(x)$ contain possible set of arms that may be pulled if context $x$ was observed at all times $AN_\lambda < t < T$. The set $A'(x)$ is used to discount the arms that will never be pulled with context $x$.

Proof. The proof broadly follows the same structure presented in Theorem 2 of Hoffman et al. (2014). We will provide the simple regret bound at the recommendation of time $T + 1$, since the algorithm operates in a pure exploitation setting, the recommended arm $\Omega_{T+2}$ will follow the same properties.

Fix $x \in \mathcal{X}$ such that $x$ can be generated from the filtration. We define the event $E_{a,t}(x)$ to be the event in which for arm $a \leq A$, $f_a(x)$ lies between the upper and lower confidence bounds given $x_1, x_2, ..., x_{t-1}$ More precisely,
\[ E_{a,t}(x) = \{L_{a,t}(x_t) \leq f_a(x) \leq U_{a,t}(x_t)|x_1, x_2, ..., x_{t-1}\}. \]
For events $E_{a,t}$, from Theorem 4.1 of the main paper,
\[ \mathbb{P}(E_{a,t}(x)) \geq 1 - e^{-\beta^2}. \]
Let $N_{a,T}$ denote the number of times each arm has been tried upto time $T$. Clearly $\sum_{a=1}^{A} N_{a,T} = T$. Also, note that we try each arm at least $N_\lambda$ number of times before we run our algorithm. We define event $\mathcal{E}$ as $\mathcal{E} := \bigcup_{s_\lambda \leq A, AN_\lambda < t \leq T} E_{a,t}(x)$. By the union bound we can show that
\[ \mathbb{P}(\mathcal{E}) \geq 1 - A(T - AN_\lambda)e^{-\beta^2}. \]
We can make RHS even bigger by adding terms $a \in A'(x)$ we can write

$$T - AN_\lambda + |A'(x)|N_\lambda < \sum_{a \in A'(x)} g_{a,T}^{-1}(H_{ae}),$$

We can make RHS even bigger by adding terms $a \in [A] \setminus A'(x)$. Hence, we get

$$T - (A - |A'(x)|)N_\lambda < \sum_{a \in [A]} g_{a,T}^{-1}(H_{ae}).$$

We can make LHS even smaller by noting that minimum value of $|A'(x)|$ is one.

$$T - AN_\lambda + N_\lambda < \sum_{a \in [A]} g_{a,T}^{-1}(H_{ae}).$$

Rearranging the terms, we get

$$T - AN_\lambda + N_\lambda < \sum_{a \in [A]} g_{a,T}^{-1}(H_{ae})$$

$$T - N_\lambda(A - 1) < \sum_{a \in [A]} g_{a,T}^{-1}(H_{ae}).$$

which contradicts our definition of $g_{a,T}$ in the theorem statement. Therefore $R_{\Omega_T}(x) \leq \epsilon$.

From the preceding argument we have that if $\sum_{a \in [A]} g_{a,T}^{-1}(H_{ae}) \leq T - N_\lambda(A - 1)$, then for any $x \in \lambda$ generated from the filtration, $\mathbb{P}(R_{\Omega_T} < \epsilon|x) \geq 1 - A(T - AN_\lambda)e^{-\beta^2} - A\delta$.

In the above equation, $1 - A(T - AN_\lambda)e^{-\beta^2}$ is from the event $\mathcal{E}$ and $1 - A\delta$ is due to the fact that the monotonic upper bounds holds only with probability $1 - \delta$ for each of the arms. Setting $\beta$ such that $\sum_{a \in [A]} g_{a,T}^{-1}(H_{ae}) = T - N_\lambda(A - 1)$ (See Lemma 12.1), we have for

$$\beta = \sqrt{\frac{\lambda_\epsilon(T - N_\lambda(A - 1)) + 2A\lambda}{16C_1^2H_\epsilon}} - C_2$$

that

$$\mathbb{P}(R_{\Omega_T} < \epsilon|x) \geq 1 - A(T - AN_\lambda)e^{-\beta^2} - A\delta,$$

for $C_1 = \rho \sqrt{2}$ and $C_2 = \rho \sqrt{\sum_{\tau=2}^{T} \ln(1 + \frac{1}{\lambda_\epsilon}\sigma_{a,T-1}(x_\tau))} + \sqrt{\lambda\|f_a\|_H}$. Since $C_2$ depends on $T$, to complete the proof and validity of the bound, we will show that $C_2$ grows logarithmically in $T$. When assumption A III holds and $\|\phi(x)\| \leq 1$, similar to the analysis in Abbasi-Yadkori et al. (2011); Durand et al. (2018), we have

$$C_2 = \rho \sqrt{\sum_{\tau=2}^{T} \ln(1 + \frac{1}{\lambda_\epsilon}\sigma_{a,T-1}(x_\tau)) + \sqrt{\lambda\|f_a\|_H}}$$

$$= \rho \sqrt{\sum_{\tau=2}^{T} \ln(1 + \frac{1}{\lambda_\epsilon}\phi(x_\tau)(I + \frac{1}{\lambda}K_{a,T-1})^{-1}\phi(x_\tau))} + \sqrt{\lambda\|f_a\|_H}$$

$$= \rho \sqrt{\ln(\det(I + \frac{1}{\lambda}K_{a,T})) + \sqrt{\lambda}\|f_a\|_H}$$

$$\leq \rho \sqrt{d\ln \left(\frac{1}{1 - \frac{1}{\lambda}}\right)} + \sqrt{\lambda\|f_a\|_H}.$$
Lemma 12.3. Consider the contextual bandit setting proposed in the main paper. Over event $\mathcal{E}$, for any time $t$ and context $x \in \mathcal{X}$, the following statements hold for the arm $a = a_t$ to be selected:

- If $a = j_t(x)$, then $L_{j_t(x), t}(x) \leq L_{j_t(x), t}(x)$.
- If $a = J_t(x)$, then $U_{j_t(x), t}(x) \leq U_{j_t(x), t}(x)$.

Proof. We consider two cases based on which of the two candidate arms $j_t(x), J_t(x)$ is selected.

Case 1: $a = j_t(x)$ is selected. The proof works by contradiction. Assume that $L_{j_t(x), t}(x) > L_{j_t(x), t}(x)$. From the arm selection rule we have $s_{j_t(x), t}(x) \geq s_{j_t(x), t}(x)$. Based on this we can deduce that $U_{j_t(x), t}(x) \geq U_{j_t(x), t}(x)$. As a result,

$$B_{j_t(x), t}(x) = \max_{i \neq j_t(x)} U_{i, t}(x) - L_{j_t(x), t}(x) < \max_{i \neq j_t(x)} U_{i, t}(x) - L_{j_t(x), t}(x) = B_{j_t(x), t}(x).$$

The above inequality holds because the arm $j_t(x)$ must necessarily have the highest upper bound over all the arms. However, this contradicts the definition of $B_{j_t(x), t}(x)$ and as a result it must hold that $L_{j_t(x), t}(x) \leq L_{j_t(x), t}(x)$.

Case 2: $a = J_t(x)$ is selected. The proof works by contradiction. Assume that $U_{j_t(x), t}(x) > U_{j_t(x), t}(x)$. From the arm selection rule we have $s_{j_t(x), t}(x) \geq s_{j_t(x), t}(x)$. Based on this we can deduce that $L_{J_t(x), t}(x) \leq L_{J_t(x), t}(x)$. As a result, similar to Case 1,

$$B_{J_t(x), t}(x) = \max_{j \neq J_t(x)} U_{j, t}(x) - L_{J_t(x), t}(x) < \max_{j \neq J_t(x)} U_{j, t}(x) - L_{J_t(x), t}(x) = B_{J_t(x), t}(x).$$

The above inequality holds because the arm $j_t(x)$ must necessarily be the highest upper bound over all the arms. However, this contradicts the definition of $B_{J_t(x), t}(x)$ and as a result it must hold that $U_{J_t(x), t}(x) \leq U_{J_t(x), t}(x)$.

Corollary 12.4. For context $x$, if arm $a = a_t(x)$ is pulled at time $t$, then $B_{a_t(x), t}(x)$ is bounded above by the uncertainty of arm $a$, i.e.,

$$B_{a_t(x), t}(x) \leq s_{a_t}(x).$$

Proof. By construction of the algorithm $a \in \{j_t(x), J_t(x)\}$. If $a = j_t(x)$, then using the definition of $B_{j_t(x), t}(x)$ and Lemma 12.3, we can write

$$B_{j_t(x), t}(x) = U_{j_t(x), t}(x) - L_{j_t(x), t}(x) \leq U_{j_t(x), t}(x) - L_{j_t(x), t}(x) = s_{a_t}(t).$$

Similarly, for $a = J_t(x)$,

$$B_{J_t(x), t}(x) = U_{J_t(x), t}(x) - L_{J_t(x), t}(x) \leq U_{J_t(x), t}(x) - L_{J_t(x), t}(x) = s_{a_t}(t).$$

Lemma 12.5. On event $\mathcal{E}$, for any time $t \leq T$ and for arm $a = a_t(x)$ the following bounds hold for the minimal gap

$$B_{a_t(x), t}(x) \leq \min(0, s_{a_t}(x) - \Delta_a(x)) + s_{a_t}(x).$$

Proof. The arm to be pulled is restricted to $a \in \{j_t(x), J_t(x)\}$. The optimal arm for the context $x$ at time $t$ can either belong to $\{j_t(x), J_t(x)\}$ or be equal to some other arm. This results in 6 cases:

1. $a = j_t(x), a^* = j_t(x)$
2. $a = j_t(x), a^* = J_t(x)$
3. $a = j_t(x), a^* \notin \{j_t(x), J_t(x)\}$
4. $a = J_t(x), a^* = j_t(x)$
5. $a = J_t(x), a^* = J_t(x)$
6. $a = J_t(x), a^* \notin \{j_t(x), J_t(x)\}$

We define $f^*(x) := f_a(x)$ as the expected reward associated with the best arm and $f_{\{a\}}(x)$ as the expected reward of the $a^{th}$ best arm.

Case 1: The following sequence of inequalities holds:

$$f_{\{2\}}(x) \geq f_{J_t(x)}(x) \geq L_{J_t(x), t}(x) \geq f_{a_t}(x) - s_{a_t}(t).$$

The first inequality follows from the assumption that $a = a^* = j_t(x)$, the chosen and optimal arm has the highest upper confidence bound, and therefore, the expected reward of arm $J_t(x)$ can be at most that of the second best arm. The second inequality follows from event $\mathcal{E}$, the third inequality follows from 12.3. The last inequality follows from event $\mathcal{E}$. Using the above string of inequalities and the definition of $\Delta_a(x)$, we can write

$$s_{a_t}(t) - (f_a(x) - f_{\{3\}}(x)) = s_{a_t} - \Delta_a(x) \geq 0.$$
The first inequality follows from event $\mathcal{E}$ and the second inequality holds because the selected arm has a larger uncertainty. From the definition of $\Delta_a(x)$,

$$B_{J_t,(x),t}(x) \leq 2s_{a,t}(x) - \Delta_a(x)$$

$$\leq s_{a,t}(x) + \min(0, s_{a,t} - \Delta_a(x)).$$

Where the inequality follows from Corollary 12.4.

**Case 3:** $a = j_t(x), a^* \notin \{j_t(x), J_t(x)\}$. We can write the following sequence of inequalities

$$f_{j_t(x)}(x) + s_{j_t(x),t}(x) \geq U_{j_t(x),t}(x) \geq U_{a^*,t}(x) \geq f^*.$$

The first and third inequalities hold due to event $\mathcal{E}$, the second inequality holds by definition as $J_t(x)$ has the highest upper bound on any arm other than $J_t(x)$ neither of which is the optimal arm in this case. From the first and last inequalities, we obtain

$$s_{a,t}(x) - (f^* - f_{a,t}(x)) \geq 0,$$

or $s_{a,t}(x) - \Delta_a(x) \geq 0$. The result follows from Corollary 12.4.

**Case 4:** $a = J_t(x), a^* = j_t(x)$. We can write

$$B_{J_t,(x),t}(x) = U_{J_t,(x),t}(x) - L_{J_t,(x),t}(x)$$

$$\leq f_{j_t(x)}(x) + s_{j_t(x),t}(x)$$

$$- f_{j_t(x)}(x) + s_{J_t(x),t}(x)$$

$$\leq f_a(x) - f^*(x) + 2s_{a,t}(x).$$

The first inequality follows from event $\mathcal{E}$ and the second inequality holds because the selected arm has a larger uncertainty. From the definition of $\Delta_a(x)$,

$$B_{J_t,(x),t}(x) \leq 2s_{a,t}(x) - \Delta_a(x)$$

$$\leq s_{a,t}(x) + \min(0, s_{a,t} - \Delta_a(x)).$$

Where the inequality follows from Corollary 12.4.

**Case 5:** $a = J_t(x), a^* = J_t(x)$. The following sequence of inequalities holds:

$$f_a(x) + s_{a,t}(x) \geq U_{J_t,(x),t}(x)$$

$$\geq U_{j_t(x),t}(x)$$

$$\geq f_{j_t(x)}(x)$$

$$\geq f_{(2)}(x).$$

The first and third inequalities follow from event $\mathcal{E}$, the second inequality is a consequence of Lemma 12.3, the fourth inequality follows from the fact that since $J_t(x)$ is the optimal arm, the upper bound and the arm selected should be as good as the second arm. Using the above chain of inequalities, we can write

$$s_{a,t}(x) - (f_{(2)}(x) - f_a(x)) = s_{a,t}(x) - \Delta_a(x) \geq 0.$$

**Case 6:** $a = J_t(x), a^* \notin \{j_t(x), J_t(x)\}$. We can write the following sequence of inequalities

$$f_{J_t(x)}(x) + s_{J_t(x),t}(x) \geq U_{J_t(x),t}(x) \geq U_{a^*,t}(x) \geq f^*.$$

The first and third inequalities hold due to event $\mathcal{E}$, the second inequality holds by definition as $J_t(x)$ has the highest upper bound on any arm when $a = J_t(x)$ due to Lemma 12.3 and $J_t(x)$ is not optimal in this case. From the first and last inequalities, we obtain

$$s_{a,t}(x) - (f^* - f_{a,t}(x)) \geq 0,$$

or $s_{a,t}(x) - \Delta_a(x) \geq 0$. The result follows from Corollary 12.4.

\[\square\]

### 13 Experimental Details and Additional Experimental Results

The algorithm was implemented with the best arm chosen at the previous time step. For speed and scalability in implementation, the kernel inverse for arm $a$, $(K_{a,t} + \lambda I_{N_{a,t}})^{-1}$ and the kernel vector $K_{a,t}(x)$ updates were implemented as rank one updates. Cross validation was performed for simple regret minimization in the online setting, where the average simple regret for the valuation set was used. The following plots (Figures 6, 7 and 8) provide results for different values of $\alpha$. Note that, the hyper parameter computed with cross validation for $\alpha = 1$ were retained for the evaluation runs for different values of $\alpha$. It can be seen that Contextual Gap performs consistently better for all the datasets under consideration.

A comparison of the average simple regret variation of Contextual gap with a history of the past 25 data points (instead of 1) is shown in Figure 9. It can be seen that there exists only minor differences in contextual gap runs with history.
Figure 6: Simple Regret evaluation with $\alpha = 0.1$

Figure 7: Simple Regret evaluation with $\alpha = 0.5$
Figure 8: Simple Regret evaluation with $\alpha = 2$

Figure 9: Comparison with Contextual Gap algorithm with recent history for Spacecraft dataset