Approximate solution for the minimal length case of Klein Gordon equation for trigonometric cotangent potential using Asymptotic Iteration Method

C Cari, A Suparmi, and Isnaini Lilis Elviyanti
Physics Department, Graduate Program, Sebelas Maret University
Email: cari@staff.uns.ac.id

Abstract. The approximate solution of Klein Gordon equation within minimal length formalism for trigonometric cotangent potential was solved using Asymptotic Iteration Method. Asymptotic Iteration Method was used to obtain relativistic energy and wave function of alternative solution Klein Gordon equation within minimal length formalism. The relativistic energy was calculated numerically using Matlab software, and unnormalized wave function was expressed in hypergeometric terms. The minimal length parameter which was used in this research was very small value. The results show that the relativistic energy within minimal length formalism increased by the increase of minimal length parameter.

1. Introduction
The dynamic of spin-zero particles in relativistic quantum mechanics was described using the Klein Gordon equation [1,2]. The vector potential ($V$) and scalar potential ($S$) of Klein Gordon equation had an exact solution in the case of the exact symmetric spin condition and symmetric pseudospin condition. In these two cases, Klein Gordon equation was reduced to Schrodinger like equation thus a state solution was obtained using various methods in non-relativistic quantum mechanics [3]. The Klein Gordon equation has been solved by Nugraha et al.[4,5], Ikot et al.[6], and Poszwa A [7].

In quantum mechanics, the minimal length idea was based on the following deformed commutation relations between position and momentum operators in Generalized Uncertainty Principle (GUP) [8]. The minimal length may be viewed as an intrinsic scale characterizing the system [9]. The Klein Gordon equation within minimal length can be solved using Heun’s function for Coulomb potential [10] and approach Algebraic for trigonometric potential [11]. The approximate solution of minimal length in Bohr-Mottelson equation has been investigated by Chabab et al. [12].

In this paper, we solved a radial part of Klein Gordon equation within minimal length for symmetric spin condition using approximate solution for trigonometric cotangent function potential. The trigonometric cotangent potential was used to explain nucleon excitation [13]. The Asymptotic Iteration Method was used to obtain the relativistic energy and radial wave functions in an approximate solution of minimal length. The work was organized as follows. In Section 2, the approximate solution of Klein Gordon equation with minimal length formalism was introduced. In Section 3, trigonometric cotangent potential was presented. We described the Asymptotic Iteration Method which was used to solve the approximate solution of Klein Gordon equation was presented in section 4. In Section 5, the results and discussion were presented. The conclusion was presented in Section 6.
2. Klein Gordon Equation with Minimal Length Formalism

The uncertainty of particle position was explained using commutation relations between position and momentum operators as expressed in Heisenberg's uncertainty principle. The modified Heisenberg's uncertainty principle was given as [14,15],

\[ [X, P] \geq i\hbar \left(1 + \alpha_{ML} \left(\Delta P\right)^2\right) \quad (1) \]

where \(X\) was a position, \(P\) was a corresponding momentum and \(\alpha_{ML}\) was a minimal length parameter that has very small positive values. By equation (1) the position and momentum operators [14-16] were defined as,

\[ \hat{X}_i = \hat{x}_i \]
\[ \hat{P}_i = \left(1 + \alpha_{ML} \hat{p}^2\right) \hat{p}_i \quad (2) \]

Klein Gordon equation was given by [10,16]

\[ \left(E - V(r)\right)^2 \psi(r) = \left[P^2c^2 + \left(M_o c^2 + S(r)\right)^2\right] \psi(r) \quad (4) \]

Where \(V(r)\) and \(S(r)\) were vector and scalar potentials. \(E\) and \(M_o\) were energy and rest mass. By setting \(S(r) = V(r)\) for case symmetric spin condition in equation (4) and substituting equation (3) into equation (4) with \(\hat{p} = -i\hbar \nabla\) and \(c = \hbar = 1\). By setting \(2V(r) \rightarrow V(r)\), we had

\[ -\left(\Delta - 2\alpha_{ML} \Delta^2\right) \psi(r) - \left(E^2 - M_o^2 - (E + M_o) V(r)\right) \psi(r) = 0 \quad (5) \]

Equation (5) was Klein Gordon equation within minimal length which can be solved using an approximate solution by substituting a new wave function [12], was given by

\[ \psi(r) = (1 + 2\alpha_{ML}) \phi(r) \quad (6) \]

So we got,

\[ \left[-\Delta - \frac{\left(E^2 - M_o^2 - (E + M_o) V(r)\right)}{1 + 2\alpha_{ML} \left(E^2 - M_o^2 - (E + M_o) V(r)\right)\right} \phi(r) = 0 \quad (7) \]

Equation (7) was an approximate solution of Klein Gordon equation within the minimal length. It can be solved using Binomial expansion, so equation (7) becomes

\[ \Delta \phi(r) + \left(E^2 - M_o^2 - (E + M_o) V(r) - 2\alpha_{ML} \left(E^2 - M_o^2 - (E + M_o) V(r)\right)\right) \phi(r) = 0 \quad (8) \]

From equation (8) was obtained \(\alpha_{ML}^2\) that had very small value for the approximate solution, so \(\alpha_{ML}^2\) was ignored. By applying spherical Laplacian operator, was given as,

\[ \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}\right) \quad (9) \]

And by setting \(\phi(r) = Q(r) Y_{LM}(\theta, \phi)\), where we have used

\[ \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}\right) Y_{LM}(\theta, \phi) = -L(L + 1) Y_{LM}(\theta, \phi) \quad (10) \]

Then by inserting equations (9) and (10) into equation (8) we get
The equation
\[ Q(r) = 0 \] (11)
Equation (11) was an approximate solution for the radial part of Klein Gordon equation with the minimal length for the case symmetric spin condition.

3. Asymptotic Iteration Method
Asymptotic Iteration Method is used to solve the second order differential equation which is expressed as [17-19],
\[ y_n^\prime(x) = \lambda_0(x) y_n(x) + s_0(x) y_n^\prime(x) \] (12)
where \( n \) is a quantum number, \( \lambda_0(x) \neq 0 \) and \( s_0(x) \) are the coefficient of a differential equation. By writing the \((k+1)^{th}\) derivative of \( y \) which is obtained from equation (12), we get
\[ y_{n,k+1}(x) = \hat{\lambda}_{k-1}(x) y_n(x) + s_{k-1}(x) y_n^\prime(x) \] (13)
where,
\[ \lambda_k(x) = \hat{\lambda}_{k-1}(x) + s_{k-1}(x) + \hat{\lambda}_0(x) \hat{\lambda}_{k-1}(x) \] (14)
\[ s_k(x) = s_{k-1}(x) + s_0(x) \hat{\lambda}_{k-1}(x) \] (15)
\[ k = 1,2,3... \] (16)
The eigenvalue was obtained from the quantization condition, is given by
\[ \Delta_k(x) = \hat{\lambda}_k(x) s_{k-1}(x) - \hat{\lambda}_{k-1}(x) s_k(x) = 0 \] (17)
The one-dimensional Schrodinger like equation was reduced to a hypergeometric type differential equation which is given as,
\[ y_n^\prime(x) = 2 \left( \frac{ax^{N+1}}{1-bx^{N+2}} - \frac{t+1}{x} \right) y_n(x) - \frac{wx^N}{1-bx^{N+2}} y_n(x) \] (18)
The eigenfunction was obtained from equation (9), is given as,
\[ y_n(x) = (-1)^t C \left( N+2 \right)^{\frac{t}{2}} \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} \frac{2t+N+3}{N+2} F_1 \left( -n, \rho+n; \sigma; bx^{N+2} \right) \] (19)
where
\[ \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} = \frac{2t+N+3}{N+2} \frac{(2t+1)b+2a}{(N+2)b} \] (20)
\( C \) is normalization constant and \( _1F_1 \) is a hypergeometric function. The wave function of Klein Gordon equation can be obtained by using equations (18-20) [17-19].

4. Trigonometric Cotangent Potential
The equation of trigonometric cotangent potential [12] was given as,
\[ V(r) = V_o \cot(\mu r) + V_i \] (21)
where \( V_o \) and \( V_i \) were potential constants. \( \mu \) was a range of potential. By setting \( V_o = 0.1, V_i = 0.01 \) and \( \mu = 0.1 \), the trigonometric cotangent potential was given by...
Figure 1. The trigonometric cotangent potential

Figure 1 was a visualization of trigonometric cotangent potential in \( r \) function. In various value of \( r \), we can see in Figure 1, a value of \( r \) was approximately from 0 \( \text{fm} \) until 14 \( \text{fm} \), the trigonometric cotangent potential had the different value. The trigonometric cotangent potential comes to infinity value in a very small value of \( r \), while for the higher value of \( r \), the trigonometric cotangent potential incline to be constant.

5. Result and Discussion

By applying equation (21) into equation (11), we got

\[
\left( \frac{1}{r^2} \right) \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \left( E^2 - M_o^2 \right) \left( V_o \cot \left( \mu r \right) + V_i \right) \right] + \left( E + M_o \right) \left( V_o \cot \left( \mu r \right) + V_i \right) \right) \right] 
\]

Then equation (22) becomes,

\[
\frac{d^2 F(r)}{dr^2} - \left( \frac{\mu L(L+1)}{\sin^2 \mu r} \right) F(r) 
+ \left( 4\alpha_{ML} \left( E^2 - M_o^2 \right) \left( E + M_o \right) \left( V_o \right) \cot \left( \mu r \right) \right) \right] 
= 0
\]

Then equation (24) was reduced becomes,
\[
\frac{d^2 F(r)}{dr^2} - \left[ \frac{v(v+1)}{\sin^2(\mu r)} - 2q \cot(\mu r) + K^2 \right] F(r) = 0
\]  
(25)

By setting,
\[
v(v+1) = \left( \mu L(L+1) + 2\alpha_{ML}(E+M_o)^2 V_o^2 \right)
\]  
(26)

\[
2q = \left( \left( 4\alpha_{ML}(E^2-M_o^2)(E+M_o)-(E+M_o) \right) V_o - 2\alpha_{ML}(E+M_o)^2 V_2 \right)
\]  
(27)

\[
-K^2 = \left( \left( 4\alpha_{ML}(E^2-M_o^2)(E+M_o)-(E+M_o) \right) V_o + \left( E^2-M_o^2 - 2\alpha_{ML}(E^2-M_o^2)^2 \right) \right)
\]  
(28)

Equation (24) was a differential equation which had to be reduced to hypergeometric differential equation type. By using the suitable variable change \( \cot(\mu r) = i(1-2u) \), we got,
\[
u(1-u) \frac{d^2 F(r)}{dy^2} + (1-2u) \frac{dF(r)}{dy} + \left[ v' (v' + 1) - \frac{4\alpha^2}{4u} - \frac{4\beta^2}{4(1-u)} \right] F(r) = 0
\]  
(29)

with,
\[
\frac{2qi - K^2}{\mu^2} = 4\alpha^2, \quad \frac{-2qi - K^2}{\mu^2} = 4\beta^2, \quad v'(v' + 1) = \frac{v(v+1)}{\mu^2}
\]  
(30)

Intermediate of the hypergeometric differential equation was shown in equation (29). Equation (29) was reduced by the new wavefunction as
\[
F(r) = u^\alpha (1-u)^\beta g(u)
\]  
(31)

and we got
\[
u(1-u) \frac{g''(u)}{du^2} + \left[ \left( 2\alpha +1 \right) - \left( 2\alpha +2\beta+2 \right) u \right] \frac{g(u)}{du} + \left[ v'(v' + 1) - (\alpha + \beta)(\alpha + \beta +1) \right] g(u) = 0
\]  
(32)

The equation (32) must be reduced to AIM type equation, by dividing it with \( u(1-u) \), then we got
\[
g'(u) + \left[ \frac{(2\alpha +1) - (2\alpha +2\beta+2) u}{u(1-u)} \right] g'(u) + \left[ \frac{v'(v' + 1) - (\alpha + \beta)(\alpha + \beta +1)}{u(1-u)} \right] g(u) = 0
\]  
(33)

By comparing equation (12) and equation (32), we had
\[
\lambda_u = \frac{(2\alpha +1)}{u} + \frac{(2\beta+1)}{1-u}
\]  
(34)

\[
s_u = \frac{(\alpha + \beta)(\alpha + \beta +1) - v'(v' + 1)}{u} + \frac{(\alpha + \beta)(\alpha + \beta +1) - v'(v' + 1)}{1-u}
\]  
(35)

Using equations (18-21) and equation (33), we obtained the eigenvalue, was given was
\[
v'(v' + 1) = (\alpha + \beta + n)(\alpha + \beta + (n+1))
\]  
(36)

From the eigenvalue in equation (36), we obtained the relativistic energy equation by using equations (30) and (36).The relativistic energy equation of Klein Gordon equation for the minimal length was given as
\begin{equation}
(E^2 - M_o^2) = \omega^2 \left[ \left( \chi_{a_{ua}} \right)^2 - \frac{\left( 2\delta_{a_{ua}} \right)^2}{4 \left( \mu^2 \chi_{a_{ua}} \right)^2} \right] \left( 4\alpha_{ML} \left( E^2 - M_o^2 \right)(E + M_o) - (E + M_o) \right)V_1 + \gamma_{a_{ua}} \tag{37} \end{equation}

where,

\begin{equation}
\chi_{a_{ua}} = \sqrt{\frac{\mu L(L+1) + 2\alpha_{ML} \left( E_0 + M_o \right)^2 V_o^2 + 1}{4} - n - \frac{1}{2}} \tag{38} \end{equation}

\begin{equation}
\delta_{a_{ua}} = \left( \left( 4\alpha_{ML} \left( E^2 - M_o^2 \right)(E + M_o) - (E + M_o) \right)V_o - 2\alpha_{ML} \left( E + M_o \right)^2 V_2 \right) \tag{39} \end{equation}

\begin{equation}
\gamma_{a_{ua}} = 2\alpha_{ML} \left( E^2 - M_o^2 \right)^2 \left( E + M_o \right)^2 V_1 - 2\alpha_{ML} \left( E + M_o \right)^2 V_o^2 \tag{40} \end{equation}

and \( n \) was a quantum number which was expressed in equation (38). Equation (37) was the relativistic energy of Klein Gordon equation with the minimal length using an approximate solution. Equation (36) was calculated numerically by using the Matlab software. The result was shown in Table 1.

Table 1. The relativistic energy of Klein Gordon equation for \( M_o=1, \ V_o=0.01, \ V_1=0.1, \ V_2=0.001, \ \mu = 0.1 \) and \( L=0 \)

| \( \alpha_{ML} \) | E(eV) | \( n=1 \) | \( n=2 \) | \( n=3 \) |
|---|---|---|---|---|
| 0 | 0.0100 | 0.6277 | 0.8527 |
| 0.001 | 0.6282 | 0.8533 | 0.9688 |
| 0.003 | 0.6293 | 0.8546 | 0.9700 |
| 0.005 | 0.6303 | 0.8558 | 0.9712 |
| 0.007 | 0.6313 | 0.8570 | 0.9724 |
| 0.009 | 0.6322 | 0.8582 | 0.9735 |
| 0.010 | 0.6327 | 0.8588 | 0.9740 |

Table 1 shown that the relativistic energy of an approximate solution Klein Gordon equation with the value of minimal length parameter was very small. The relativistic energy within the minimal length was higher than relativistic energy without the minimal length. The increasing of minimal length parameter caused the increase of the relativistic energy value. The relativistic energy increased by the increase of quantum number \( n \).

The general un-normalized wave function of Klein Gordon equation was obtained by inserting equations (30-31,36) and \( \cot (\mu r) = i \left( 1 - 2u \right) \), so we got wave functions that were shown in Table 2.

Table 2. The wave function of Klein Gordon equation within the minimal length

| \( n \) | Wavefunction |
|---|---|
| 0 | \( F_0 (r) = C' \left( \frac{1 + i \cot \mu r}{2} \right)^{\alpha} \left( \frac{1 - i \cot \mu r}{2} \right)^{\beta} \) |
| 1 | \( F_1 (r) = -C' \left( \frac{1 + i \cot \mu r}{2} \right)^{\alpha} \left( \frac{1 - i \cot \mu r}{2} \right)^{\beta} \left( 2\alpha + 1 \right) \left[ \frac{(-1)(2\alpha + 2\beta + 2) \left( \frac{1 + i \cot \mu r}{2} \right)}{(2\alpha + 1)} \right] \) |
6. Conclusion
We investigated the approximate solution of Klein Gordon equation within the minimal length for trigonometric cotangent potential using Asymptotic Iteration method. Asymptotic Iteration method was used to obtain the relativistic energy and radial wave functions for Klein-Gordon equation within the minimal length. The approximate solution was used to minimal length parameter that had very small value. The increasing of minimal length parameter caused the increase of the relativistic energy value. The relativistic energy increased by the increase of quantum number $n$.

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